BOUNDARY VALUE PROBLEMS FOR INTEGRABLE EQUATIONS COMPATIBLE WITH THE SYMMETRY ALGEBRA

Burak Gürel\textsuperscript{a}, Metin Gürses\textsuperscript{a} and Ismagil Habibullin\textsuperscript{b}

\textsuperscript{a}: Department of Mathematics, Faculty of Science
Bilkent University, 06533 Ankara, Turkey

\textsuperscript{b}: Mathematical Institute, Ufa Scientific Center
Russian Academy of Sciences, Chernishevski str. 112
Ufa, 450000, Russia

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Abstract

Boundary value problems for integrable nonlinear partial differential equations are considered from the symmetry point of view. Families of boundary conditions compatible with the Harry-Dym, KdV and MKdV equations and the Volterra chain are discussed. We also discuss the uniqueness of some of these boundary conditions.
1 Introduction

In our previous paper [1] we have briefly discussed a method to construct boundary value problems of the form

\[ u_t = f(u, u_1, u_2, \ldots, u_n), \tag{1} \]

\[ p(u, u_1, u_2, \ldots, u_k)|_{x=0} = 0, \tag{2} \]

completely compatible with the integrability property of Eq.(1). Here \( u = u(x, t) \), \( u_i = \frac{\partial^i u}{\partial x^i} \) and \( f \) is a scalar (or vector) field. The aim of the present paper is to expound detailly our scheme and also extend it to the integrable differential-difference equations.

Let the equation

\[ u_{\tau} = g(u, u_1, \ldots, u_m), \tag{3} \]

for a fixed value of \( m \), be a symmetry of the equation (1). Let us introduce some new set of dynamical variables, consisting of the variable \( v = (u, u_1, u_2, \ldots u_{n-1}) \), and its \( t \)-derivatives \( v_t, v_{tt}, \ldots \). One can express the higher \( x \)-derivatives of \( u \), i.e., \( u_i \) for \( i \geq n \) and their \( t \)-derivatives, by the utility of the equation (1), in terms of the dynamical variable \( v \) and their \( t \)-derivatives. Here \( n \) is the order of the equation (1). In these terms the symmetry (3) may be written as

\[ v_{\tau} = G(v, v_t, v_{tt}, \ldots v_{tt\ldots t}). \tag{4} \]

We call the boundary value problem, Eqs.(1) and (2), as compatible with symmetry (3) if the constraint \( p(v) = 0 \) (or constraints \( p^a(v) = 0 \) where
\( a = 1, 2, \ldots N \) and \( N \) is the number of constraints) is consistent with the \( \tau \)-evolution

\[
\frac{\partial p}{\partial \tau} = 0, \quad (\text{mod} \quad p = 0)
\]  

Eq.(5), by virtue of the equations in (4), must be automatically satisfied. In fact (5) means that the constraint \( p = 0 \) defines an invariant surface in the manifold with local coordinates \( v \). This definition of consistency of boundary value problem with symmetry is closer to the one introduced in [3], but not identical. For instance, let us examine whether the boundary value problem \( u_t = u_{xx}; \ u_x = c u, \ x = 0 \), is compatible with the symmetry \( u_\tau = u_{xxx} \). To this end one has to check if the equation \( w = c u \) defines an invariant surface for the system of equations \( u_\tau = w_t, \ w_\tau = u_{tt} \) (here \( w \) is \( u_1 \)). Evidently the answer is negative. To check the validity of compatibility condition in the sense of [3] one has to compare two sets of equations \( u_{2n+1} = c u_{2n} \) and \( u_{3n+1} = c u_{3n}, \ n \geq 0 \). These equations are obtained by differentiation of the constraint equation \( u_x = c u \) with respect to \( t \) and \( \tau \) variables respectively. In this sense the boundary condition is compatible with the symmetry because the sets of equations don’t contradict each other.

We call the boundary condition (3) is compatible with the equation if it is compatible at least with one of its higher order symmetries.

Our main observation is that if the boundary condition is compatible with one higher symmetry then it is compatible with infinite number of symmetries. We define a set \( S \) with infinite number of elements where its elements are symmetries of the Eq.(1). \( S \) may or may not contain the whole symmetries of (1). For instance , \( S \) contains the even numbered time independent symmetries for the Burgers equation.
We note that all the known boundary conditions of the form (2) consistent with the inverse scattering method are indeed compatible with the infinite series of generalized symmetries. On the other hand, stationary solutions of the symmetries compatible with (2) allow one to construct an infinite dimensional set of "exact" (finite gap) solutions of the corresponding boundary value problem (1) and (2). However, in this work we do not discuss analytical aspects of this problem. We note also that, in this paper we shall deal with boundary conditions of the form given in (2). An effective investigation of boundary conditions involving an explicit $t$-dependence is essentially more complicated. Such a problem has been studied, for instance, in [16].

The article is organized as follows. In Section 2 we introduce the necessary notations and give the Burgers equation as an illustrative example. We prove that if a boundary condition is compatible at least with one higher order symmetry then it is compatible with every one of even order. In Section 3 we consider the nonlinear Schrödinger, Harry-Dym, Korteweg de Vries and modified KdV equations. Using the symmetry approach we find a boundary condition compatible with the symmetry algebra of the Harry-Dym equation

\begin{align}
  u_t &= u^3 u_{xxx}, \\
  u_x &= c u, \quad x = 0 \\
  u_{xx} &= c^2 u/2, \quad x = 0,
\end{align}

where $c$ is an arbitrary real constant. Actually one has here two constraints. Although we are taking the boundary conditions at $x = 0$, one can shift this point to an arbitrary point $x = x_0$ without losing any generality. We conjecture that the boundary value problem given in (3) is compatible with the Hamiltonian integrability and solvable by the inverse scattering technique. In addition we conjecture that (using the idea in [2]) one can prove that on
the finite interval $x_1 \leq x \leq x_2$ the Harry-Dym equation with the boundary conditions $u_x = c_0 u, u_{xx} = c_0^2 u/2$ for $x = x_1$ and $u_x = c_1 u, u_{xx} = c_1^2 u/2$ for $x = x_2$ is a completely integrable Hamiltonian system.

Section 4 is devoted to the differential-difference equations. In the last section we propose further generalization of the compatibility and discuss on some open questions.

## 2 Boundary Conditions Compatible with Symmetries

In the sequel we suppose that eq. (1) admits a recursion operator of the form (see [4-6])

\[
R = \sum_{i=0}^{i_1} \alpha_i D^i + \sum_{i=0}^{k_1} \alpha_{-1,i} D^{-1} \alpha_{-2,i}, \quad i_1 \geq 0, \quad k_1 \geq 0
\]

(7)

where $\alpha_i$, $\alpha_{-1,i}$, $\alpha_{-2,i}$ are functions of the dynamical variables, $D$ is the total derivative with respect to $x$. Recursion operators when applied to a symmetry produce new symmetries. Passing to the new dynamical variables $v, v_t, v_{tt}, ...$ one can obtain, from (7), the recursion operator of the system of equations (4) (we don’t prove that every recursion operator may be rewritten in the matrix form, but we will give below the matrix forms of the recursion operators for the Burgers, KdV, MKdV and Harry-Dym equations)

\[
R = \sum_{i=0}^{M} a_i (\partial_t)^i + \sum_{i=0}^{K} a_{-1,i} (\partial_t^{-1}) a_{-2,i}, \quad M > 0, \quad K \geq 0
\]

(8)

where $a_i$ depends on $v$ and on a finite number of its $t$-derivatives , $\partial_t$ is the operator of the total derivative with respect to $t$. If (4) is a scalar equation,
$R$ is a scalar operator, then $R$ is an $n \times n$ matrix valued operator. Our further considerations are based on the following proposition, which really affirms that if an equation admits an invariant surface, then a kind of its higher symmetries admits also the same invariant surface.

**Proposition 2.1.** Let the equation (4) is of the form $v_\tau = H(R)v_t$, where $R$ is the recursion operator (8) and $H$ is a polynomial function with scalar constant coefficients. If this equation is consistent with the constraint $p(v) = 0$, where rank of $p$ equals $n-1$ (here $n$ is the dimension of the vector $v$) then every equation of the form $v_\tau = L(H(R))v_t$, where $L$ is arbitrary chosen scalar polynomial with constant coefficients, is also compatible with this constraint.

**Proof:** Introduce new variables $w = (w^1, w^2, ..., w^n)$ in the following way: $w^1 = p^1, w^2 = p^2, ..., w^{n-1} = p^{n-1}$ and $w^n = p^n$ is a function of $v$, here $p^i$ is a coordinate of the vector $p$. Then one obtains the equation $w_\tau = Pw_t$ from (5), where $P = A^{-1}H(R)A$ and $A = \partial v/\partial w$ is the Jacobi matrix of the mapping $w \rightarrow v$. Notice that under this change of variables the constraint $p(v) = 0$ turns into the equation $w^i = 0$ for $i = 1, 2, ..., n-1$. Imposing this constraint reduces the equation $w_\tau = Pw_t$ to the form

$$
\begin{pmatrix}
0 \\
\vdots \\
0 \\
w^n_\tau
\end{pmatrix} =
\begin{pmatrix}
P_{11} & \ldots & P_{1n} \\
\ldots & \ldots & \ldots \\
P_{n-1,1} & \ldots & P_{n-1,n} \\
P_{n,1} & \ldots & P_{n,n}
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
w^n_t
\end{pmatrix}
$$

Let us show that elements of the last column of the matrix $P$ are equal zero except maybe $P_{n,n}$: $P_{i,n} = 0$ for $1 \leq i \leq n-1$. Really, letting $P_{j,n} \neq 0$ for some $j \leq n-1$ the equation $P_{j,n}w^n_t = 0$ gives a connection between variables $w^n, w^n_t, ...$ which are supposed to be independent. Since the set
of such block triangular matrices constitutes a subalgebra in the algebra of the all squared matrices hence one can easily conclude that the operator $L(P)(\mod w^i = 0, i \leq n - 1)$ is also block triangular so the equation $w_\tau = L(P)w_\xi$ is consistent with the constraint $w^i = 0, i \leq n - 1$. It completes the proof of the Proposition 2.1.

Proving later a kind of uniqueness theorem (see below the Proposition 2.2) we will use the following statement

**Proposition 2.1**: Suppose that the constraint $p(v) = 0$ of the rank equal $n - 1$ is compatible with the equation (5) having the form $v_\tau = H(R^{n_0})v_\xi$, where $H = H(z)$ is a scalar polynomial function of $z$ with constant coefficients. Assume that $n_0 \geq 1$ is an integer and the leading term $b_N$ (the coefficient before the highest derivative) in the expression $R^{n_0} = b_N(\partial_t)^N + b_{N-1}(\partial_t)^{N-1} + \ldots$ is a scalar matrix, i.e. it is proportional the unit matrix. Then the constraint is consistent with the equation

$$v_\tau = R^{n_0} v_\xi \tag{9}$$

**Proof**: In terms of the variable $w$ we have introduced proving the previous proposition the equation $v_\tau = H(R^{n_0})v_\xi$ takes the form $w_\tau = H(R^{n_0}_1)w_\xi$. Owing the fact that the point transformation preserves the commutativity property of flows, the operator $R_1 = ARA^{-1}$ is the recursion operator in the new variables. Again, just in the previous proposition one has that the operator $P = H(R^{n_0}_1)$ under the substitution $p = 0$ (or really, $w^i = 0, i \leq n - 1$) is a lower block triangular matrix valued one. Our aim now is to prove that the operator $Q = R^{n_0}_1(\mod w^i = 0, i \leq n - 1)$ is also block triangular. Setting $H(Q) = \alpha_n Q^n + \alpha_{n-1} Q^{n-1} + \ldots + \alpha_0$ and representing $Q$ as formal series
\[ \sum_{k=-\infty}^{M} c_k \partial_t^k \] using the famous Campbell-Hausdorf formula one obtains that

\[ H(Q) = \alpha_n(c^n_Q(\partial_t)^nM + nc^{n-1}_Qc_{M-1}(\partial_t)^{nM-1} + ...) + ... + \alpha_0 \]

One has that \( H(Q) \) belongs the set \( M_- \) consisting of all lower block triangular matrices. By looking at the coefficients of different power of the operator \( \partial_t \) one can show that the matrices \( c_i, i = M - 1, M - 2, ... \) satisfy the equations

\[ c^{n-1}_c M + T_i \in M_- \]

where \( T_i \) are polynomials with scalar coefficients on variables \( c_{i+1}, c_{i+2}, ...c_M \) and their derivatives. So, because of assumptions \( c_M = b_M \in M_0 \) and \( detb_M \neq 0 \) it is easy to prove by induction that \( c_i \in M_0 \) for all \( i \leq M \).

For illustration let us give the Burgers equation as an example. The Burgers equation

\[ u_t = u_{xx} + 2u u_x \]  \hspace{1cm} (10)

which possesses the recursion operator of the form

\[ R = D + u + u_x D^{-1} \]  \hspace{1cm} (11)

(see, for instance, [7]). The simplest symmetry of this equation is \( u_{\tau} = u_x \).
In terms of the new dynamical variables this symmetry equation takes the form

\[ u_{\tau} = u_1 \]

\[ u_{1,\tau} = u_t - 2u u_1 \]  \hspace{1cm} (12)

This equation does not admit any invariant surface of the form \( p(u, u_1) = 0 \).
Really, differentiating this constraint with respect to \( \tau \) one obtains
\[ \frac{\partial p}{\partial u} u_1 + \frac{\partial p}{\partial u_1} (u_t - 2 u u_1) = 0 \] (13)

Because of independence of the variables \( u_t \) and \( u_1 \) we have

\[ \frac{\partial p}{\partial u_1} = \frac{\partial p}{\partial u} = 0 \] (14)

which leads to a trivial solution \( p = \text{constant} \). As a conclusion we don’t have any invariant surface (curve) in \((u, u_1)\) - plane. Similarly the third order symmetry \( u_\tau = u_3 + 3 uu_2 + 3u_1^2 + 3u^2u_1 \) rewritten in the new variables \((u, u_1)\) gives the following system of two equations

\[
\begin{align*}
  u_\tau &= u_{1,t} + u u_t + (u^2 + u_1) u_1, \\
  u_{1,\tau} &= u_{tt} - u u_{1,t} + (u^2 + u_1) u_t - 2 u u_1 (u^2 + u_1)
\end{align*}
\] (15)

This system also does not admit any invariant surface of the form \( p(u, u_1) = 0 \). It may be easily proved that the same is true for every symmetry of the odd order, i.e., \( u_\tau = u_{2m+1} + h(u_{2m}, ..., u) \). Because the correspondent system of equations has different orders in the highest \( t \)-derivatives

\[
\begin{align*}
  u_\tau &= \partial^m_t u_1 + ... , \\
  u_{1,\tau} &= \partial^{m+1}_t u + ...
\end{align*}
\] (16)

Unlike the symmetries of odd order, for the symmetries of even order the correspondent system of equations has the same orders in the highest \( t \)-derivatives. This fact leads us to show that the symmetries of even order admit an invariant surface \( p(u, u_1) = 0 \), depending upon two arbitrary parameters.

**Proposition 2.2:** If the boundary condition \( p(u, u_1)|_{x=0} = 0 \) is compatible with a higher symmetry of the Burgers equation, then it is of the form (see [3]) \( c(u_1 + u^2) + c_1 u + c_2 = 0 \) and is compatible with every symmetry of
the form $u_r = P(R^2) u_t$ where $P$ denotes polynomials with scalar constant coefficients.

**Proof:** The Frechet derivative of (10) gives the symmetry equation of the Burgers equation

$$\partial_t \sigma = (D^2 + 2 u D + 2 w) \sigma$$  \hspace{1cm} (17)

where $w$ stands for $u_1$. As the operators acting on symmetries we may take

$$D^{-1} = \partial^{-1}_t (D + 2 u)$$  \hspace{1cm} (18)

in the recursion operator (11). Consequently the recursion formula $u_{r_{i+1}} = R u_{r_i}$ becomes

$$u_{r_{i+1}} = (u + 2 w \partial^{-1}_t u) u_{r_i} + (1 + w \partial^{-1}_t) w_{r_i}$$  \hspace{1cm} (19)

Differentiating it with respect to $x$ and replacing $w_x = u_2 = u_t - 2 u w$ one obtains

$$w_{r_{i+1}} = [\partial_t + 2 (u_t - 2 u w) \partial^{-1}_t u] u_{r_i} + [-u + (u_t - 2 u w) \partial^{-1}_t] w_{r_i}$$  \hspace{1cm} (20)

for $i = 1, 2, \ldots$. Thus the matrix form of the recursion operator $R$ is given by

$$R = \begin{pmatrix} u + 2 w \partial^{-1}_t u & 1 + w \partial^{-1}_t \\ \partial_t + 2 (u_t - 2 u w) \partial^{-1}_t u & -u + (u_t - 2 u w) \partial^{-1}_t \end{pmatrix}$$  \hspace{1cm} (21)

It is well known that every higher order local polynomial symmetry may be represented as a polynomial operator $P_0(R)$ applied to the simplest classical symmetry $u_r = u_x$. It is more convenient to use the following equivalent representation
\[
\begin{pmatrix} u \\ w \end{pmatrix}_\tau = P(R^2) \begin{pmatrix} u \\ w \end{pmatrix}_t + P_1(R^2) \begin{pmatrix} w \\ u_t - 2uw \end{pmatrix} \tag{22}
\]

where \( P \) and \( P_1 \) are polynomials with scalar constant coefficients and \( P_0 \) mentioned above may taken as \( P_0(R) = P(R^2)R + P_1(R^2) \).

Note that one could not apply immediately the Proposition 2.1' to this because the coefficient of \( \partial_t \) in the representation (21) is not diagonal. On the other hand the operator \( R^2 \) has scalar leading part. First we will prove that if the symmetry (22) admits an invariant surface then \( P_1 \) in this equation vanishes. Let us take the invariant surface as \( u = q(w) \). Suppose that the function \( q(w) \) is differentiable at some point \( w = w_0 \). Linearizing \( q \) around the point \( w_0 \) (or as \( w \to w_0 \)) we obtain

\[
u - q(w_0) = q'(w_0)(w - w_0) + o(w - w_0).
\]

It follows from (21) that in this case \( R^2 \) reduces to a scalar operator: \( R^2 \to (\partial_t - w_0 + q^2(w_0))I \) as \( w \to w_0 \), where \( I \) is the unit matrix. Thus in the linear approximation the Eq.(22) takes the form

\[
\begin{pmatrix} u \\ w \end{pmatrix}_\tau = P(\partial_t - w_0 + q^2(w_0)) \begin{pmatrix} u \\ w \end{pmatrix}_t + P_1(\partial_t - w_0 + q^2(w_0)) \begin{pmatrix} w \\ u_t \end{pmatrix} \tag{23}
\]

where now \( P(\partial_t - w_0 + q^2(w_0)) \) and \( P_1(\partial_t - w_0 + q^2(w_0)) \) are scalar operators. It is clear that the linearized equation is consistent with the linearized boundary condition \( u - q(w_0) = q'(w_0)(w - w_0) \), provided \( P_1 = 0 \). Supposing that the equation (22) is compatible with the constraint \( w = c \) where \( c \) is a constant and then linearizing about the point \( (u = 0, w = c) \) one can easily obtain that \( P_1 \) vanishes in this case also.
It is evident now that in Proposition 2.1' one should put \( n_0 = 2 \), because \( R^2 = I \partial_t + \ldots \). With this choice the constraint \( p(u, w) \) describes an invariant surface for the following system

\[
\begin{pmatrix}
  u \\
  w
\end{pmatrix}_\tau = R^2 \begin{pmatrix}
  u \\
  w
\end{pmatrix}_t
\]

which is exactly the coupled Burgers type integrable system (see [8])

\[
\begin{align*}
  u_\tau &= u_{tt} + 2(w + u^2) u_t \\
  w_\tau &= w_{tt} + 2u_t^2 + 2(w + u^2) w_t
\end{align*}
\]

It is straightforward to show that the above system (25) is compatible with the constraint \( p(u, w) = 0 \) only if \( p = w + u^2 + c_1 u + c_2 \) or \( u = \text{const} \).

The above uniqueness proof of the boundary condition \( p = w + u^2 + c_1 u + c_2 \) can be more easily shown if we use a new property of the Burger’s hierarchy.

We have the following proposition:

**Proposition 2.3:** The function \( u(t, x, \tau_n) \) \((n \geq 1)\) satisfy infinitely many Burgers like equations

\[
\begin{align*}
  u_{\tau_i, \tau_i} - u_{\tau_i, \tau_{2i+2}} &= -2 u_{\tau_i} D^{-1} u_{\tau_i}
\end{align*}
\]

Here \( i = -1, 0, 1, 2, \ldots \). Burgers equation corresponds to \( i = -1 \), \( \tau_{-1} = x \), and \( \tau_0 = t \). All \( u_{\tau_i} \) for \( i > -1 \) correspond to higher symmetries. It is straightforward to determine the even numbered symmetries of the Burgers equation from (26). It is very interesting that \( u \) satisfies the Burgers like equations with respect to the variables \((\tau_i, \tau_{2i+2})\) for all \( i = -1, 0, 1, 2, \ldots \).

The proof of this proposition depend crucially on definition of the higher symmetries of the Burgers equation. They are defined through the equation
\[ u_{\tau n} = R^{n+1} u_x \]  \hspace{1cm} (27)

where \( R \) is the recursion operator given in eq. (11) and \( n \geq -1 \). Eq. (27) can also be written as \( u_{\tau n} = R u_{\tau n-1} \). Differentiating this equation once by \( \tau_n \) and using (27) one arrives at (26).

If we let the most general boundary condition of the form \( p = f(u, u_x) = 0 \) at \( x = x_0 \) and take \( \tau_i \) and \( \tau_{2i+2} \) derivatives (for \( i \geq 0 \)) of the function \( p \) and use the equation (26) we obtain

\[ f^2, u, u + f^2 u_x, u_x - 2 f^3_{,u_x} - 2 f_{,u} f_{,u_x} f_{,u_x} u_x = 0 \]  \hspace{1cm} (28)

Letting \( u = x_1 \) and \( u_x + u^2 + c_1 u + c_2 = x_2 \) then eq. (28) becomes

\[ f^2_{,x_2} f_{,x_1,x_1} + f^2_{,x_2} f_{,x_2,x_2} - 2 f_{,x_1} f_{,x_2} f_{,x_1,x_2} = 0, \]  \hspace{1cm} (29)

Assuming \( f_{,x_2} \neq 0 \) and letting \( q = f_{,x_1} / f_{,x_2} \) we find that

\[ q_{,x_1} = q q_{,x_2} \]  \hspace{1cm} (30)

This is a very simple equation and its general solution can be found. We shall not follow this direction to determine \( f(x_1, x_2) \) rather change the form of equation \( p(u, u_x) = 0 \) at \( x = x_0 \). This equation (in principle) implies either

a) \( u_x = h(u) \)

which implies \( f = u_x - h(u) \) at \( x = x_0 \). Or

b) \( u = g(u_x) \)

which implies \( f = u - g(u_x) \) at \( x = x_0 \). It is now very easy to show that the cases a and b when the corresponding \( f \)'s are inserted in (28) we respectively obtain
a) \( h'' + 2 = 0 \)
which implies \( u_x + u^2 + c_1 u + c_2 = 0 \) at \( x = x_0 \)
b) \( g'' + 2 (g')^3 = 0 \)
which implies \( u = \text{constant} \) (for \( g' = 0 \)) and a special case of a (for \( g' \neq 0 \)).
Hence we found all possible boundary conditions.

Remark 2.3: On the invariant surface \( p(u, w) = 0 \) the system \((25)\) turns into the Burgers like equation \( u_t = u_{tt} - 2(c_1 u + c_2) u_t \) which is also integrable \([5]\).

### 3 Application to Other Partial Differential Equations

In this section we shall apply our method to obtain compatible boundary conditions of some nonlinear partial differential equations. Let us start with the following system of equations

\[
\begin{align*}
    u_t &= u_2 + 2u^2 v \\
    -v_t &= v_2 + 2u v^2
\end{align*}
\]

Letting \( v = u^* \), \( t \rightarrow i t \) the above system becomes the well known nonlinear Schrödinger equation. Suppose that it admits a boundary condition of the following form

\[
\begin{align*}
    u_x|_{x=0} &= p^1(u, v) \\
    v_x|_{x=0} &= p^2(u, v)
\end{align*}
\]

compatible with the fourth order symmetry. It means that the constraint \((32)\) defines an invariant surface for this symmetry, presented as a system of four equations with two independent variables.
\[ u_\tau = u_{tt} - 2u^2 v_t - 4u v_1 u_1 + 2v u_1^2 - 2u^3 v^2, \]
\[ v_\tau = -v_{tt} - 2v^2 u_t + 4v u_1 v_1 - 2u v_1^2 + 2v^3 u^2, \]
\[ u_{1,\tau} = u_{1,tt} - 2u^2 v_{1,t} - 2u_1^2 v_1 - 6u^2 v^2 u_1 - 4u v_1 u_1 + 4v u_1 u_t + 4v u^3 v_1, \]
\[ v_{1,\tau} = -v_{1,tt} - 2v^2 u_{1,t} + 2v_1^2 u_1 + 6v_1 v^2 u^2 - 4v u_1 v_t + 4u v_1 v_t - 4v^3 u u_1 \]
(33)

One can check that the system (33) is compatible with the constraint \( u_1 = p^1(u, v), v_1 = p^2(u, v) \) only if \( p^1 = c u \) and \( p^2 = c v \). Since the system (33) is of the form

\[ (u, u_1, v, v_1)^T = R^2 (u, u_1, v, v_1)^T \]
(34)

hence it follows from the Proposition 2.1 that the constraints \( u_1 = c u, v_1 = c v \) are compatible with every symmetry of even order. So the boundary conditions \( u_x|_{x=0} = c u, v_x|_{x=0} = c v \) are compatible with such symmetries. Analytical properties of this boundary value problem are studied previously (see [2],[9-10]) by means of the inverse scattering method.

**Remark 3.1:** On the invariant surface \( u_1 = c u, v_1 = c v \) the system (33) is reduced to a system of two equations:

\[ u_\tau = u_{tt} - 2u^2 v_t - 2c^2 u^2 v - 2u^3 v^2, \]
\[ v_\tau = -v_{tt} - 2v^2 u_t + 2c^2 v^2 u + 2v^3 u^2, \]

The integrability of these equations is shown in [4] (see p.175). Under a suitable change of variables in it this system of two equations becomes the famous derivative nonlinear Schrödinger equation.

Among the nonlinear integrable equations the Harry-Dym equation
\[ u_t + u^3 u_3 = 0 \] (35)

is of special interest. It is not quasilinear and because of this reason its analytical properties are not typical. Using the symmetry approach we find a boundary condition of the form

\[ p(u, u_1, u_2) = 0, \] (36)

compatible with Harry-Dym equation. One has to notice that because of non-quasilinearity of (35) the transformation from the standard set of variables \( u, u_1, u_2, u_3 \) to \( u, u_1, u_2, u_t, u_{1,t}, u_{2,t}, \ldots \) is not regular. For instance \( u_3 = -\frac{u_t}{u^3} \).

It has singular surface given by the equation \( u = 0 \). So one should examine this surface separately. Since the Harry-Dym equation (35) as well as its higher order symmetries possesses the reflection symmetry \( x \rightarrow -x, u \rightarrow -u, t \rightarrow t \) the trivial boundary condition \( u(t, 0) = 0 \) is consistent with the integrability.

Suppose that the boundary value problem (35) and (36) is compatible with the ninth order symmetry \( u_\tau = u^9 u_9 + \ldots \). It means that the constraint \( p(u, v, w) \) is consistent with following system of equations, equivalent to the ninth symmetry

\[
\begin{align*}
    u_\tau &= f_1 \\
    v_\tau &= f_2 \\
    w_\tau &= f_3
\end{align*}
\] (37)
where \( v = u_x \), \( w = u_{xx} \) and \((f_1, f_2, f_3)^T = R^3(u_t, v_t, w_t)^T\),

\[
R = \begin{pmatrix}
uw + u_t \partial_t^{-1} w & \quad \quad -uv - u_t \partial_t^{-1} v & \quad \quad u^2 + u_t \partial_t^{-1} u \\
\frac{1}{u} \partial_t + vw - \frac{w_t}{u_t} + v_t \partial_t^{-1} w & \quad \quad -v^2 - v_t \partial_t^{-1} v & \quad \quad uv + v_t \partial_t^{-1} u \\
w^2 + w_t \partial_t^{-1} w & \quad \quad \frac{1}{u} \partial_t - vw - \frac{w_t}{u_t} - w_t \partial_t^{-1} v & \quad \quad uw + w_t \partial_t^{-1} u
\end{pmatrix}.
\]

The explicit expressions for \( f_2, f_3 \) are very long. Hence we give the explicit form only for the function \( f_1 \):

\[
f_1 = -u_{ttt} + 3u_{tt}u_t + \frac{3}{2}u_{tt}u_1h - \frac{3}{2}u u_3 + \frac{3}{2}uu_{1,tt}h + \frac{3}{2}uu_{1,h} + \frac{3}{2}u_{1,h}t - \frac{15}{16}uh^2h_t - \frac{5}{16}h^3u_t - \frac{3}{2}u_1u_t.
\] (38)

Where \( h = 2u_2 - u_1^2 \). Here one has two choices for the rank of the equation (36). It is either one or two. The first choice does not lead to any regular invariant surface. The second gives

\[
u_x|_{x=0} = cu, \quad u_{xx}|_{x=0} = \frac{c^2 u}{2}
\] (39)

**Remark 3.3**: On the invariant surface \( v = cu, w = c^2u/2 \) the first equation in the system (37) takes the form

\[
u_t = -u_{ttt} + 3u_{tt}u_t/u - 3u_1^3u^2/2
\] (40)

equivalent to the MKdV equation.

Since the symmetry under consideration is of the form \( u_t = R^3u_x \) where \( R = u^3 D^3 u D^{-1} \frac{1}{u^2} \) the recursion operator for the Harry-Dym equation (see [11]) , the Proposition 2.1 implies the following

**Proposition 3.1**: The boundary value problem (35) and (36) is compati-
ble with every symmetry of the form \( u_t = L(R^3)u_x \), where \( L \) is a polynomial with scalar constant coefficients.
The Korteweg de Vries equation $u_t = u_{xxx} + 6u_1u$ admits a recursion operator $R = D^2 + 4u + 2u_1D^{-1}$ which may be represented in the form:

$$
R = \begin{pmatrix}
4u + 12v \partial_t^{-1} u & 0 & 1 + 2v \partial_t^{-1} \\
\partial_t + 12w \partial_t^{-1} u & -2u & 2w \partial_t^{-1} \\
2w + 12(u_t - 6uv) \partial_t^{-1}u & \partial_t - 2v & -2u + 2(u_t - 6uv) \partial_t^{-1}
\end{pmatrix}.
$$

It is not difficult to show that the system of equations $(u, v, w)_τ = R^3(u, v, w)_t$ admits an invariant surface $u = 0$, $w = 0$ on which the equation turns into the MKdV equation. It means that the boundary condition $u(t, x = 0) = 0$, $u_{xx}(t, x = 0) = 0$ is compatible with all symmetries of the form $u_τ = R^3u_x$. Similarly, the MKdV equation $u_t = u_{xxx} + 6u^2u_x$ is compatible with the boundary condition $u(t, x = 0) = 0$, $u_x(t, x = 0) = 0$.

4 Application to Discrete Chains

Consider an integrable nonlinear chain of the form

$$
u_t(n) = f(u(n-1), u(n), u(n+1)) \quad (41)$$

with unknown function $u = u(n, t)$ depending on integer $n$ and real $t$. The natural set of dynamical variables serving the hierarchy of higher symmetries for the chain is the set $u(0), u(±1), u(±2), ...$. However, it is more convenient for our aim to use the following unusual one, consisting of the variables $u(0), u(1)$ and all their $t$-derivatives. Transformations of these sets to each other are given by the equation (42) itself and its differential consequences. In terms of new basic variables every higher order symmetry of this chain

$$
u_τ(n) = g(u(n-m), u(n-m-1), ...u(n+m)), \quad (42)$$
could be presented as a system of two partial differential equations

\[ \begin{align*}
  v_\tau &= G_1(v, w, v_1, w_1, ..., v_s, w_s), \\
  w_\tau &= G_2(v, w, v_1, w_1, ..., v_s, w_s),
\end{align*} \tag{43} \]

where \( v = u(0, t, \tau), \ w = u(1, t, \tau), \ v_i = \frac{\partial v}{\partial t^i}, \ w_i = \frac{\partial w}{\partial x^i}. \)

Prescribe some boundary condition of the form

\[ u(0) = p(u(1), u(2), ..., u(k)) \tag{44} \]

to the equation (41) to hold for all moments \( t. \) We shall call the boundary value problem (41), (44) consistent with the symmetry (42) if the constraint (44) defines an invariant surface for the system (43). Note that interconnection between the hierarchies of the commuting discrete chains and integrable partial differential equations is well-known (see survey [4]). An illustrative example of such a kind connection is related to the famous Volterra chain

\[ u_t(n) = u(n)(u(n + 1) - u(n - 1)) \tag{45} \]

for which the next symmetry

\[ u_\tau(n) = u(n)u(n + 1)(u(n) + u(n + 1) + u(n + 2)) - u(n)u(n - 1)(u(n) + u(n - 1) + u(n - 2)) \]

which might be represented as ([4], p.123)

\[ \begin{align*}
  v_\tau + v_{tt} &= (2vw + v^2)_t, \\
  w_\tau - w_{tt} &= (2vw + w^2)_t \tag{46}
\end{align*} \]

under the substitution \( u(0) = v, \ u(1) = w, \ u(-1) = w - \frac{w}{v}, \)

\( u(2) = v + \frac{w}{w}, \ u(-2) = v - \frac{\ln u(-1)}{\partial t}. \) Moreover, the full hierarchy of the
Volterra chain is completely described by the hierarchy of the last system. According to the definition above the boundary value problem (44), (45) will be consistent with a symmetry of the Volterra chain if the constraint (44) describes an invariant surface for the same symmetry, represented as a system of partial differential equations. Let us examine invariant surfaces of the following system of partial differential equations

\[
\begin{align*}
v_\tau &= v_{ttt} + (3vH^2 - 3v_tH - 2v^3)_t, \\
w_\tau &= w_{ttt} + (3wH^2 + 3w_tH - 2w^3)_t,
\end{align*}
\]

\[H = v + w,\]

which is exactly the higher order symmetry for the Volterra chain (45) of the form

\[
u_\tau(n) = u(n)u(n + 1)(u(n + 2)u(n + 3) + u(n)u(n + 2) + u(n)u(n - 1) + \\
u^2(n) + 2u(n + 1)u(n + 2) + u^2(n + 2) + 2u(n)u(n + 1) + \\
u^2(n + 1)) - u(n)u(n - 1)(u(n)u(n + 1) + u(n)u(n - 2) + u(n - 2)u(n - 3) + \\
u^2(n - 2) + 2u(n)u(n - 1) + u^2(n) + 2u(n - 1)u(n - 2) + u^2(n - 1)).
\]

It is easy to check that the only invariant surface of the form \(v = \text{const}\) admissible by the system (44) is \(v = 0\). The corresponding boundary condition \(u(0) = 0\) is well studied (see [12], [13]).

**Remark 4.1:** On the invariant surface \(v = 0\) the system (47) reduces to the scalar equation

\[
w_\tau = w_{ttt} + 3wttw + 3w^2 + 3w_tw^2;
\]

which is nothing else but the next symmetry of the Burgers equation. Moreover, the constraint is compatible with every generalized polynomial symmetry. On the invariant surface they are all reduced to the symmetries of the
Burgers equation. It is evident for instance, that the system (47) turns into the Burgers equation itself.

Suppose now that \( v = p(w) \). Then one obtains that \( p(w) = -w \). It gives rise to a boundary \( u(0) = -u(1) \) compatible with the Volterra chain (see [14]).

**Remark 4.2:** Under the constraint \( v = -w \) the system (47) turns into the Modified KdV equation

\[
v_t = v_{ttt} + 6v^2 v_t.
\]

It is not difficult to show that there is no any invariant surface of the form \( v = p(w, w_t) \) such that \( \frac{\partial p}{\partial w_t} = 0 \) admissible with the system (47).

For the case \( v_t = p(v, w, w_t) \) calculations become very long so that here we utilized Matematica 2.1 (we thank G.Alekseev for his help with this calculations). Here \( p \) has a form \( p = \frac{\sqrt{w}}{w} w_t + 2v(v + w) \) which produces the boundary condition

\[
u(-1) = -u(0) - u(1) - u(2).
\]

The slight difference with the (44) is overcomed by the simple shift of the discrete variable \( n \).

Using the proposition 2.1 it is easy to check that the invariant surface \( v_t = \frac{\sqrt{w}}{w} w_t + 2v(v + w) \) is compatible with every odd order polynomial generalized symmetry of the system (47). It means that the boundary condition \( u(-1) = -u(0) - u(1) - u(2) \) is compatible with the corresponding symmetries of the Volterra chain.

The well-known boundary condition \( u^2(0) = 1 \) for the modified Volterra chain

\[
u_t(n) = (1 - u^2(n))(u(n + 1) - u(n - 1))
\]
defines the invariant surface $v^2 = 1$ for the following systems of equations

\[ \begin{align*}
v_t + v_{tt} &= 2((1 - v^2)w)_t, \\
w_t - w_{tt} &= 2((1 - w^2)v)_t
\end{align*} \tag{48} \]

and

\[ \begin{align*}
v_t + v_{ttt} &= 2(v(1 - v^2)(3w^2 - 1) - 3vwv_t), \\
w_t + w_{ttt} &= 2(w(1 - w^2)(3v^2 - 1) + 3vww_t)\tag{49}
\end{align*} \]

which are equivalent to the next symmetries of this chain:

\[ u_t(n) = (1 - u^2(n))(D_+ - D_-)(1 - u^2(n))(D_+ - D_-)u(n) \]

and

\[ u_t(n) = (1 - u^2(n))(D_+ - D_-)(1 - u^2(n))(-D_+^2 - D_-^2)u(n) + (D_+ + D_-)(u^2(n)u(n + 1) + u^2(n)u(n - 1) + 2u(n - 1)u(n)u(n + 1)), \]

here $D_+, D_-$ are the shift operators: $D_+ u(n) = u(n + 1), D_- u(n) = u(n - 1)$; $v = u(0), w = u(1)$ and other variables $u(n)$ are expressed through $v, w$ and their $t$-derivatives by means the chain and its differential consequences.

**Remark 4.3:** On the invariant surface $v^2 = 1$ the systems (48), (49) are reduced to the Burgers equation and its third order symmetry.

### 5 Condition of weak compatibility

It is easy to notice that any symmetry of the equation (1.1) rewritten in terms of the non-standard set of the dynamical variables turns into the equation containing $m - 1$ extra variables $u_1, u_2, \ldots, u_{m-1}$. For instance, the fourth order symmetry of the Burgers equation
\[ u_{\tau} = u_{4} + 4u_{3}u + 10u_{2}u_{1} + 6u_{2}u^2 + 12u_{1}^2u + 4u_{1}u^3 \]

takes the following form
\[ u_{\tau} = u_{tt} + 2(w + u^2)u_{t}, \]

where \( w = u_{1} \). To extend it to the closed form it is enough to add one more equation obtained from the above equation by the differentiation with respect to \( x \) and replacing \( u_{2} = u_{t} - 2uw \). This is the general rule for integrable equations: one has to add \( m - 1 \) more equations (to have a closed system of equations), expressing variables \( u_{i\tau}, \ 1 \leq i \leq m - 1 \) through dynamical ones. But from the other hand side one may consider the single symmetry equation alone and suppose the extra variables are expressed in terms of \( u \) and its lower derivatives. Let us pose the question, for which choice of such expressions the symmetry under consideration turns into an integrable equation? As an example let us consider the Burgers equation How should we choose the dependence \( w = w(u) \), such that the equation \( u_{\tau} = u_{tt} + 2(w + u^2)u_{t} \) would be integrable? The only choice is \( w = -u^2 + c_{1}u + c_{2} \) (see [6]). We will call the boundary conditions \( u_{i} = u_{i}(u), \ x = 0 \) (obtained this way) for the equation (1.1) as weakly compatible with the symmetry if these constraints are chosen to satisfy the requirement above: i.e. the equation for the \( nth \) symmetry written down in terms of the introduced variables turns into some integrable equation after replacing \( u_{i} = u_{i}(u), \ u_{it} = u_{t}\frac{\partial u}{\partial u} \). So in the above case of the Burgers equation only the condition \( w(u) = -u^2 + c_{1}u + c_{2} \) is weakly compatible with the fourth order symmetry. As the remarks given above indicate, the compatibility of the condition with a symmetry implies the weak compatibility with it, but not vice versa. However, we conjecture that if the boundary condition is weakly compatible with at least three higher
symmetries then the corresponding initial boundary value problem will be solvable by a suitable generalization of the inverse scattering method.

The following example for the Harry-Dym equation (35) seems to be intriguing. Let us represent the fifth order symmetry

$$u_{\tau_5} = -\frac{1}{2}u^3(2u_5u^2 + 10u_4u_1u + 10u_3u_2u + 5u_3u_1^2)$$

in the form $u_{\tau_5} = \frac{1}{2}(hu)_t$, where $h = 2u_2u - u_1^2$. Represent also the next two symmetries in the similar form:

$$u_{\tau_7} = u_4u_1 - \frac{3}{2}u_4u_1uh + \frac{3}{8}u_1[3(h + u_1^2)^2 - 4u_1^2(h + u_1^2) + u_1^4] - uu_1tt + \frac{3}{8}u_2uh$$

and $u_{\tau_9} = f_1$ (see above the first equation of the system (37)). It is evident that for arbitrary function $F = F(u)$ the constraint $h = 0$, $u_1 = F(u)$ is weakly consistent with fifth and ninth symmetries, because the former takes the trivial form $u_{\tau_5} = 0$ and the latter turns into the integrable equation (40). The seventh order symmetry becomes $u_{\tau_7} = (Su_t)_t$, where $S = F - uF'$. Thus, if for instance, $S = a = const$ or $S = \frac{1}{(\gamma u + \beta)^2}$ one will have the equation $u_{\tau_7} = (Su_t)_t$, to be integrable (see [5], p.129). Supposing $S(u) = a$ one can easily find that $u_1 = cu + a$, $u_2 = \frac{c^2u}{2} + ac + \frac{a^2}{2u}$. It leads to the following boundary condition $u_x = cu + a$, $u_{xx} = \frac{a^2}{2u}$, $x = 0$ for the Harry-Dym equation, which coincides with (39) if $a = 0$. In the case $S = \frac{1}{(\gamma u + \beta)^2}$ to find $F$ one has to integrate the ordinary differential equation $F(u) - uF'(u) = S$.

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