Homology groups of spaces of non-resultant quadratic polynomial systems in $\mathbb{R}^3$

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Abstract. We calculate the rational homology groups of spaces of non-resultant (that is, having no non-trivial common zeros) systems of homogeneous quadratic polynomials in $\mathbb{R}^3$.

Keywords: resultant, cohomology, simplicial resolution, configuration space.

To Jean-Pierre Serre with admiration

§ 1. Introduction

The resultant subvariety in the space of systems of homogeneous polynomials in $\mathbb{R}^n$ consists of systems having common zeros besides the origin. Varieties of this kind were considered, in particular, by Smale and his collaborators in connection with the $P \neq NP$ problem over fields of characteristic zero (see [1]–[3]). The topology of complements of resultant varieties in various spaces of systems of non-homogeneous polynomials in one complex variable was studied in [4]–[6]. The homology groups of spaces of non-resultant homogeneous systems in $\mathbb{R}^2$ were calculated in [7]. We shall study a problem at the next level of complexity, that of homogeneous quadratic polynomial systems in $\mathbb{R}^3$.

Let $W \simeq \mathbb{R}^6$ be the space of all real quadratic forms on $\mathbb{R}^3$. For every positive integer $k$ we consider the space $W^k \simeq \mathbb{R}^{6k}$ of all quadratic maps $\mathbb{R}^3 \to \mathbb{R}^k$. Denote by $\Sigma$ the resultant subvariety in $W^k$, that is, the space of all systems of $k$ quadratic forms having common zeros in $\mathbb{R}^3 \setminus 0$.

Theorem 1. For every even $k \geq 4$, the Poincaré polynomial of the group $H^*(W^k \setminus \Sigma, \mathbb{Q})$ is equal to

$$ (1 + t^{k-1})(1 + t^{3k-9} + t^{5k-14}). $$

(1)

For every odd $k \geq 3$, the Poincaré polynomial of $H^*(W^k \setminus \Sigma, \mathbb{Q})$ is equal to

$$ 1 + t^{k-1} + t^{3k-7}(1 + t^{k-5})(1 + t^{2k-3}). $$

(2)

The group $H^i(W^2 \setminus \Sigma, \mathbb{Q})$ is equal to $\mathbb{Q}$ if $i = 0$ or $i = 1$, and is trivial otherwise.

This work is supported by the Russian Science Foundation under grant 14-50-00005.

AMS 2010 Mathematics Subject Classification. 14P25.
Example 1. When \( k = 3 \), the polynomial (2) is equal to \((2 + t^3)(1 + t^2)\). In particular, the space \( W^3 \setminus \Sigma \) consists of two connected components. Indeed, any non-resultant triple of quadratic forms in \( \mathbb{R}^3 \) determines a map \( \mathbb{R}P^2 \to \mathbb{R}^3 \setminus 0 \sim S^2 \). The degree mod 2 of this map is constant along each connected component of \( W^3 \setminus \Sigma \). This degree vanishes for every system containing non-negative quadratic forms and is non-trivial for a triple of quadratic forms whose zero sets in \( \mathbb{R}P^2 \) meet each other like \( \bigcirc \bigcirc \).

§ 2. First reductions

2.1. Alexander duality. Denote by \( \overline{H}_* \) the Borel–Moore homology group, that is, the homology group of the complex of locally finite singular chains of a topological space. In the case of homology groups with coefficients in a group (that is, in a constant local system) \( A \) there is an equivalent definition: \( \overline{H}_*(X, A) \) is the homology group of the one-point compactification of \( X \) reduced modulo the added point. However, the last definition is generally inapplicable in the case of non-constant coefficient systems. We write \( \tilde{H}_* \) and \( \tilde{H}^* \) for the homology and cohomology groups reduced modulo a point.

Following [8], we use the Alexander isomorphism

\[
\tilde{H}^i(W^k \setminus \Sigma) \simeq \Pi_{6k-i-1}(\Sigma).
\]

(3)

2.2. A toy calculation: homology of the Stiefel manifold. To calculate the group on the right-hand side of (3), we consider a simplicial resolution of the space \( \Sigma \). Let us demonstrate this method in a simpler situation, replacing the quadratic maps by linear ones. The space of non-resultant \( k \)-tuples of linear functions \( \mathbb{R}^3 \to \mathbb{R}^1, \ k \geq 3 \), is homotopy equivalent to the Stiefel manifold \( V_3(\mathbb{R}^k) \). Its cohomology group is well known; let us show how to calculate it using our technique.

Proposition 1. For every even \( k \geq 3 \), the Poincaré polynomial of the group \( \tilde{H}^i(V_3(\mathbb{R}^k), \mathbb{Q}) \) is equal to \((1 + t^{k-1})(1 + t^{2k-5})\). For every odd \( k \geq 3 \) it is equal to \((1 + t^{k-3})(1 + t^{2k-3})\).

Proof. Consider the space \( \mathbb{R}^{3k} \) of all \( k \)-tuples of linear functions and its discriminant subvariety \( \Sigma \) consisting of the tuples vanishing on some non-trivial subspace. To construct the simplicial resolution \( \sigma \) of this variety, we list all subsets of \( \mathbb{R}P^2 \) which can be zero loci of systems of linear equations.

These are, first, individual points (the space of these is \( \mathbb{R}P^2 \) itself), second, lines (the space of these is the dual projective plane \( \widetilde{\mathbb{R}P}^2 \)) and, third, the whole of \( \mathbb{R}P^2 \) (this is a single object named \( [\mathbb{R}P^2] \)). We denote these spaces of subsets by \( J_1, J_2, \) and \( J_3 \) respectively.

Consider them as topological spaces with their standard topology. The disjoint union \( J_1 \sqcup J_2 \sqcup J_3 \) is partially ordered by inclusion of the corresponding subsets. Consider the topological order complex \( \Psi \) of this partially ordered set. Namely, take the join \( J_1*J_2*J_3 \) of these three spaces (that is, the naturally topologized union of all triangles and closed intervals whose vertices correspond to points of different spaces).
and define $\Psi$ as the union of those simplices all of whose vertices are incident to each other. For every point $X$ in $J_1$, $J_2$, or $J_3$ we define its subordinate subcomplex $\Psi(X)$ as the union of all such simplices whose vertices correspond to subsets of $\mathbb{RP}^2$ contained in the set $\{X\}$. We also write $\partial \Psi(X)$ for the link of $\Psi(X)$, that is, the union of all simplices participating in $\Psi(X)$ but not containing the vertex $X$. Let $\hat{\Psi}(X)$ be the difference $\Psi(X) \setminus \partial \Psi(X)$. We write $\Psi_p$, $p = 1, 2, 3$, for the union of complexes $\Psi(X)$ over all $X \in J_j$, $j \leq p$. Clearly, $\Psi_3 = \Psi(\mathbb{RP}^2) = \Psi$.

The simplicial resolution $\sigma$ of $\Sigma$ is defined as a subspace in the direct product $\Psi \times \mathbb{R}^{3k}$. Namely, for every $p = 1, 2, 3$ and every element $X \in J_p$ we take the product $\Psi(X) \times L(X)$, where $L(X) \subset \mathbb{R}^{3k}$ is the space of all systems of $k$ linear functions vanishing on the set $\{X\} \subset \mathbb{RP}^2$. $\sigma$ is defined as the union of all such products over all $p$ and all $X \in J_p$. The subspace $\sigma_p \subset \sigma$ is the union of such products only over all $X \in J_j$, $j \leq p$. It is easy to see that the difference $\sigma_p \setminus \sigma_{p-1}$ is the space of a fibre bundle over $J_p$ whose fibre at $X \in J_p$ is equal to $\hat{\Psi}(X) \times L(X)$. In particular, $\sigma_p \setminus \sigma_{p-1}$ is the space of a vector bundle over $\Psi_p \setminus \Psi_{p-1}$ with fibres $L(X)$, $X \in J_p$.

The projection $\sigma \rightarrow \Sigma$ is a proper map with contractible fibres, so it induces an isomorphism $\overline{H}_*(\sigma, \mathbb{Q}) \to \overline{H}_*(\Sigma, \mathbb{Q})$. Now let us calculate the spectral sequence converging to the group $\overline{H}_*(\sigma, \mathbb{Q})$ and generated by the filtration $\sigma_1 \subset \sigma_2 \subset \sigma_3 \equiv \sigma$. By definition, its initial term $E_{p,q}^1$ is equal to $\overline{H}_{p+q}(\sigma_p \setminus \sigma_{p-1}, \mathbb{Q})$.

The space $\sigma_1$ is fibred over $J_1 = \mathbb{RP}^2$. Its fibre over any point $x \in \mathbb{RP}^2$ is isomorphic to $\mathbb{R}^{2k}$ and consists of all linear system vanishing on the corresponding line in $\mathbb{R}^3$. This fibre bundle is orientable if and only if $k$ is even. Hence, by Thom’s isomorphism, we have $\overline{H}_i(\sigma_1, \mathbb{Q}) \simeq H_{i-2k}(\mathbb{RP}^2, \mathbb{Q})$ for even $k$ and $\overline{H}_i(\sigma_1, \mathbb{Q}) \simeq H_{i-2k}(\mathbb{RP}^2, \mathbb{Q} \otimes \mathbb{Q} \text{or})$ for odd $k$, where $\mathbb{Q} \text{or}$ is the orientation bundle of $\mathbb{RP}^2$.

The space $\sigma_2 \setminus \sigma_1$ is fibred over $J_2 = \mathbb{RP}^2$. Its fibre over any point $l \in \mathbb{RP}^2$ is the product of a $k$-dimensional subspace $L(l) \subset \mathbb{R}^{3k}$ and an open two-dimensional disc $\hat{\Psi}(l)$, which is the union of all half-open intervals connecting the point $l \in J_2$ with all points of $J_1$ corresponding to the points of the line $l$. The bundle over $J_2$ with fibres $L(l)$ is orientable if and only if $k$ is even. The bundle with fibres $\hat{\Psi}(l)$ is always non-orientable. Therefore $\overline{H}_i(\sigma_2 \setminus \sigma_1, \mathbb{Q}) \simeq H_{i-k-2}(\mathbb{RP}^2, \mathbb{Q})$ when $k$ is odd, and $\overline{H}_i(\sigma_2 \setminus \sigma_1, \mathbb{Q}) \simeq H_{i-k-2}(\mathbb{RP}^2, \mathbb{Q} \otimes \mathbb{Q} \text{or})$ when $k$ is even. Finally, in all cases, $\sigma_3 \setminus \sigma_2$ is the entire order complex $\Psi$ of non-empty vector subspaces of $\mathbb{R}^3$, from which the order subcomplex $\partial \Psi$ of only proper subspaces is removed. $\Psi$ is a cone over this subcomplex, which is known to be homeomorphic to $S^4$ (see, for example, [9], [10]). Hence the group $\overline{H}_i(\sigma_3 \setminus \sigma_2, \mathbb{Q})$ is equal to $\mathbb{Q}$ when $i = 5$ and is trivial otherwise.

To summarize, we see that the only non-trivial terms $E_{p,q}^1$ (all equal to $\mathbb{Q}$) of our spectral sequence are as follows. For even $k$ these are the terms with $(p, q)$ equal to $(1, 2k - 1)$, $(2, k + 2)$, and $(3, 2)$. For odd $k$ these are the terms with $(p, q)$ equal to $(1, 2k + 1)$, $(2, k)$, and $(3, 2)$. Clearly, all further differentials of this spectral sequence are trivial. We obtain the group $\overline{H}_*(\sigma, \mathbb{Q})$, and Proposition 1 follows by the Alexander duality similar to (3), but with $6k$ replaced by $3k$. $\square$

2.3. Singular sets for the resultant in the spaces of quadratic maps. From now on, we return to the space $W^k$ of systems of $k$ quadratic functions $\mathbb{R}^3 \to \mathbb{R}$,
consider its resultant subvariety $\Sigma$, and construct a simplicial resolution $\sigma$ of $\Sigma$. In this construction we follow Gorinov’s modification (see [11], [12]) of the construction of resolutions of discriminant sets proposed in [13].

We fix the following list of families of compact subsets in $\mathbb{R}P^2$, which are basic for the construction of the resolution (compare [13], [11]):

- $J_1 \sim \mathbb{R}P^2$ consists of single points;
- $J_2$ is the configuration space $B(\mathbb{R}P^2, 2)$ of unordered pairs of distinct points in $\mathbb{R}P^2$;
- $J_3$ is the corresponding space $B(\mathbb{R}P^2, 3)$ of subsets of cardinality 3 in $\mathbb{R}P^2$;
- $J_4$ is the space of quadruples of distinct points all of which belong to the same line in $\mathbb{R}P^2$;
- $J_5$ is the space $\hat{\mathbb{R}}P^2$ of lines;
- $J_6$ is the space $B(\mathbb{R}P^2, 4)$ of quadruples of distinct points in $\mathbb{R}P^2$ which do not belong to the same line;
- $J_7$ is the space of configurations of the form $\{\text{a line, a point outside this line}\}$ in $\mathbb{R}P^2$;
- $J_8$ is the space $B(\hat{\mathbb{R}}P^2, 2)$ of pairs of distinct lines in $\mathbb{R}P^2$;
- $J_9$ is the space of non-singular non-empty conics in $\mathbb{R}P^2$;
- $J_{10}$ consists of a unique point and corresponds to whole of $\mathbb{R}P^2$.

Following [11], we equip the union of these spaces with the Hausdorff metric on the space of compact subsets of $\mathbb{R}P^2$: the distance between compact subsets $X, Y \subset \mathbb{R}P^2$ is equal to

$$\max_{x \in X} \rho(x, Y) + \max_{y \in Y} \rho(y, X), \quad (4)$$

where $\rho$ is the usual metric on the plane $\mathbb{R}P^2$ regarded as a quotient space of the unit sphere. We denote the resulting metric space $J_1 \cup J_2 \cup \cdots \cup J_{10}$ by $\mathfrak{J}$. It is easily seen to be a closed subset of the space of all compact subsets of $\mathbb{R}P^2$ equipped with the metric (4). Since the last space is compact, $\mathfrak{J}$ is compact too.

We again consider the space $\mathfrak{J}$ as partially ordered by incidence of the corresponding subsets. Unlike the situation considered in §2.2, some of its strata $J_p$ are non-closed and adjacent to each other. This slightly complicates the construction of the simplicial resolution. In particular, we use the following notion (see [14], [15]).

**Definition 1.** Given a compact finite-dimensional topological space $X$, we embed it generically into a space $\mathbb{R}^\omega$ of very large dimension and define its $r$th self-join $X^{*r}$ as the union of all simplices of dimensions $1, \ldots, r - 1$ in $\mathbb{R}^\omega$ all of whose vertices belong to the image of $X$ under this embedding. (The genericity of an embedding means that intersections of distinct simplices are their common faces only.)

We easily see that the spaces $X^{*r}$ defined by different generic embeddings of $X$ are canonically homeomorphic to each other.

Define the order complex $\Psi$ of the sets $J_1, \ldots, J_{10}$ as the following subspace of the 10th self-join $\mathfrak{J}^{*10}$: it consists of all possible simplices (of different dimensions)
in $\mathcal{J}^{10}$ all of whose vertices are incident to each other. For every $X \in \mathcal{J}$ we define the space $\Psi(X)$ as the union of simplices in $\Psi$ all of whose vertices correspond to sets contained in $\{X\}$. We also define the link $\partial \Psi(X)$ as the union of all simplices in $\Psi(X)$ not containing the vertex $X$, and the space $\bar{\Psi}(X)$ as the difference $\Psi(X) \setminus \partial \Psi(X)$. By the exact sequence of a pair we have

$$\bar{H}_i(\bar{\Psi}(X)) \equiv \bar{H}_{i-1}(\partial \Psi(X)).$$

The space $\Psi$ is naturally filtered: for every $p = 1, \ldots, 10$ the subspace $\Psi_p \subset \Psi$ is the union of the spaces $\Psi(X)$ over all $X \in J_p, j \leq p$. By construction, $\Psi_p \setminus \Psi_{p-1}$ is the union of the spaces $\Psi(X)$ over all $X \in J_p$.

For every point $X \in J_p$ we define the space $\sigma(X) \subset \Psi \times W^k$ as the direct product $\Psi(X) \times L(X)$, where $L(X)$ is the vector space of all systems of polynomials vanishing on the set $\{X\}$. The space $\sigma_p$ is defined as the union of the spaces $\sigma(X)$ over all $X \in J_p, j \leq p; \sigma \equiv \sigma_{10}$. We obtain the filtrations

$$\Psi_1 \subset \cdots \subset \Psi_{10} \equiv \Psi, \quad \sigma_1 \subset \cdots \subset \sigma_{10} \equiv \sigma. \quad (5)$$

The obvious projection $\Psi \times W^k \to W^k$ induces a map $\Pi: \sigma \to \Sigma$. For every point $F \in \Sigma$ (that is, a resultant system of polynomials), its pre-image $\Pi^{-1}(F)$ is the subspace $\Psi(\Sigma(F)) \times F$, where $X(F) \in \mathcal{J}$ is the set of common zeros of all polynomials in $F$. Every such set belongs to some $J_p$.

**Lemma 1.** 1) The map $\Pi: \sigma \to \Sigma$ induced by the obvious projection $\Psi \times W^k \to W^k$ is proper and its extension to a map between the one-point compactifications of $\sigma$ and $\Sigma$ is a homotopy equivalence. In particular, it induces an isomorphism of the Borel–Moore homology groups: $\bar{H}_*(\sigma) \simeq \bar{H}_*(\Sigma)$.

2) For every $p = 1, \ldots, 10$ the projection $\Psi \times W^k \to \Psi$ maps $\sigma_p$ to $\Psi_p$. Its restriction to $\sigma_p \setminus \sigma_{p-1}$ is a vector bundle over $\Psi_p \setminus \Psi_{p-1}$ whose rank is equal to the dimension of $L(X)$ for all $X \in J_p$.

3) For every $p$, $\Psi_p \setminus \Psi_{p-1}$ is fibred locally trivially over $J_p$ and its fibre over $X \in J_p$ is equal to $\bar{\Psi}(X)$.

**Proof.** The first part follows from Theorem 2.8 in [11]. Indeed, the set of families $J_1, \ldots, J_{10}$ clearly satisfies all the hypotheses listed in [11], p. 399 (see also [12], pp. 13, 14). The second and third parts follow immediately from the construction. $\square$

Thus we can (and will) calculate the group $\bar{H}_*(\sigma)$ instead of $\bar{H}_*(\Sigma)$.

Consider the spectral sequence $E^r_{p,q}$ which calculates the group $\bar{H}_*(\sigma, \mathbb{Q})$ and is generated by the filtration (5) with one exception: we do not distinguish $\sigma_4$ from the rest of $\sigma_5$. Hence the revised filtration is as follows: $\sigma_1 \subset \sigma_2 \subset \sigma_3 \subset \sigma_5 \subset \cdots$. The term $E^r_{p,q}$ is thus canonically isomorphic to the group $\bar{H}_{p+q}(\sigma_p \setminus \sigma_{p-1}, \mathbb{Q})$ if $p = 1, 2$ or $3$, to $\bar{H}_{p+q}(\sigma_5 \setminus \sigma_3, \mathbb{Q})$ if $p = 4$, and to $\bar{H}_{p+q}(\sigma_{p+1} \setminus \sigma_p, \mathbb{Q})$ if $p = 5, \ldots, 9$.

**Theorem 2.** For every even (resp. odd) $k$, all non-trivial groups $E^1_{p,q}$ of the initial term of our spectral sequence are shown in Fig. 1a (resp. Fig. 1b). In the case of odd $k$, the differential $d^1: E^1_{8,k+5} \to E^1_{7,k+5}$ is an isomorphism.
Proof of Theorem 1. Theorem 1 for \( k \geq 3 \) follows immediately from Theorem 2 because no other differentials \( d^r : E^r_{p,q} \to E^r_{p-r,q+r-1} \) of the spectral sequence can connect non-trivial cells of these tables.

When \( k = 2 \) the table of Fig. 1a is as shown in Fig. 2. Since the entire space \( W^2 \) is 12-dimensional, all groups \( E_{p,q} \), \( p+q \geq 12 \), should be killed by some differentials. This can happen in only one way: the differentials \( d^4 : E^4_{8,7} \to E^4_{4,10} \) and \( d^3 : E^3_{9,5} \to E^3_{6,7} \) must be isomorphisms. Hence the only surviving term is \( E^1_{1,9} \simeq \mathbb{Q} \). This yields that \( \overline{H}_{10}(\Sigma, \mathbb{Q}) \simeq \mathbb{Q} \) and, by Alexander duality, proves Theorem 1 in the remaining case \( k = 2 \). □

The proof of Theorem 2 occupies the rest of the paper.

Remark 1. In both spectral sequences shown in Fig. 1, the column \( p = 9 \) consists of the Borel–Moore homology groups of \( \sigma_{10} \setminus \sigma_9 \equiv \Psi_{10} \setminus \Psi_9 \equiv \tilde{\Psi}(\mathbb{P}^2) \). Namely, \( E^1_{9,q} \simeq \overline{H}_{9+q}(\tilde{\Psi}(\mathbb{P}^2), \mathbb{Q}) \equiv \overline{H}_{8+q}(\partial\Psi(\mathbb{P}^2), \mathbb{Q}) \). The space \( \partial\Psi(\mathbb{P}^2) \) is the simplicial resolution of the topological order complex of the partially ordered set of all proper subsets of \( \mathbb{P}^2 \) listed in §2.3. Therefore, in particular, we prove that this order complex has the rational homology group of \( S^{13} \). This order complex is very similar (but not equal) to the order complex of all proper vector subspaces of \( W \) that consist of quadrics vanishing on some subset of \( \mathbb{P}^2 \). The latter complex is studied in more detail in [16]. In particular, it is proved there that this complex also has the rational homology of \( S^{13} \).
Figure 2. $E^3$ when $k = 2$

A similar order complex of sets defined by systems of linear functions on $\mathbb{R}^3$ is homeomorphic to $S^4$. We have used this fact in the proof of Proposition 1.

§ 3. Preliminary facts on the homology of configuration spaces

Given a topological space $M$, we write $B(M, j)$ for the configuration space whose points are the subsets of cardinality $j$ in $M$.

**Lemma 2.** For every $j$ there is a locally trivial fibre bundle $B(S^1, j) \to S^1$ whose fibre is homeomorphic to $\mathbb{R}^{j-1}$. This fibre bundle is orientable (and hence trivial) for odd $j$ and non-orientable for even $j$.

**Proof.** Consider $S^1$ as the unit circle in $\mathbb{C}^1$. Then the projection map of this fibre bundle is given by taking the product of $j$ complex numbers lying on this circle. The fibre of this bundle can be identified in terms of the universal covering $\mathbb{R}^j \to T^j$ with any connected component of the complement in some hyperplane $\{x_1 + \cdots + x_j = \text{const}\} \subset \mathbb{R}^j$ of all affine planes given by the equations $x_m = x_n + 2\pi k$, $m \neq n$, $k \in \mathbb{Z}$. Any such component is convex and, therefore, diffeomorphic to $\mathbb{R}^{j-1}$. The assertion on orientability can be checked directly. □

**Remark 2.** The image of the standard section of this bundle consists of all $j$-tuples of points dividing $S^1$ into equal arcs.

For every manifold $M$ we denote the orientation local system of groups on $M$ by $\mathcal{O}_r$. It is locally isomorphic to the constant $\mathbb{Z}$-system, but the orientation-reversing elements of $\pi_1(M)$ act on it as multiplication by $-1$. For every topological space $M$ we denote by $\pm \mathbb{Z}$ the local system on $B(M, j)$ which is locally isomorphic to the constant $\mathbb{Z}$-sheaf, but the elements of $\pi_1(B(M, j))$ inducing odd permutations of $j$ points act on it as multiplication by $-1$. We also put $\pm \mathbb{Q} \equiv \pm \mathbb{Z} \otimes \mathbb{Q}$.

**Lemma 3.** For every even-dimensional manifold $M$, a closed loop in the manifold $B(M, j)$ is orientation reversing if and only if the union of the traces of all $j$ points of our configuration under the corresponding movement is an orientation-reversing
cycle in $H_1(M, \mathbb{Z}_2)$ (that is, a cycle whose value on the first Stiefel–Whitney class of $M$ is non-zero).

Proof. This is trivial (see, for example, [6]). □

Consider the $j!$-fold covering $\nu: I(\mathbb{R}P^2, j) \to B(\mathbb{R}P^2, j)$, where $I(\mathbb{R}P^2, j)$ is the space or ordered $j$-tuples of distinct points in $\mathbb{R}P^2$.

Lemma 4. The Borel–Moore homology groups $\overline{H}_i(I(\mathbb{R}P^2, j), \mathbb{Q})$, $i \geq 0$, are trivial for every integer $j \geq 2$.

Proof. We have $I(\mathbb{R}P^2, j) = (\mathbb{R}P^2)^j \setminus \Delta$, where $\Delta \subset (\mathbb{R}P^2)^j$ is the set of all sequences of $j$ points in $\mathbb{R}P^2$ at least two of which coincide. By the exact sequence of the pair $((\mathbb{R}P^2)^j, \Delta)$ it suffices to prove that $\overline{H}_i(\Delta, \mathbb{Q}) = 0$ for all $i$. The set $\Delta$ is the union of the $\binom{j}{2}$ spaces $\Delta_{r,s} \simeq (\mathbb{R}P^2)^{j-1}$, $1 \leq r < s \leq j$, which are defined by the coincidence of the $r$th and $s$th terms of sequences of $j$ points in $\mathbb{R}P^2$. Consider the Mayer–Vietoris spectral sequence of this union. All intersections (possibly, multiple) of the spaces $\Delta_{r,s}$ are homeomorphic to $(\mathbb{R}P^2)^m$, $m \in [1, j - 1]$ and, in particular, have the rational homology of a point. Therefore the spectral sequence reduces to the one calculating the homology of the order complex of all such multiple intersections. This order complex has a maximal element (corresponding to the non-empty intersection of all these spaces) and, therefore, is contractible. □

Corollary 1. The groups $\overline{H}_i(B(\mathbb{R}P^2, j), \mathbb{Q})$ and $\overline{H}_i(B(\mathbb{R}P^2, j), \pm \mathbb{Q})$ are trivial for any $j \geq 2$ and $i \geq 0$.

Proof. $\overline{H}_*(I(\mathbb{R}P^2, j), \mathbb{Q}) \simeq \overline{H}_*(B(\mathbb{R}P^2, j), \nu_!(\mathbb{Q}))$, where $\nu_!(\mathbb{Q})$ is the direct image of the constant $\mathbb{Q}$-sheaf on $I(\mathbb{R}P^2, j)$. $\nu_!(\mathbb{Q})$ is a $j!$-dimensional local system on $B(\mathbb{R}P^2, j)$ containing both local systems $\mathbb{Q}$ and $\pm \mathbb{Q}$ as direct summands. Therefore both groups mentioned in Corollary 1 are direct summands of the group $\overline{H}_*(I(\mathbb{R}P^2, j), \mathbb{Q})$, which is trivial by Lemma 4. □

We recall the notation $B^\times(\mathbb{R}P^2, 4)$ for the subset of non-collinear tuples in $B(\mathbb{R}P^2, 4)$.

Lemma 5. The groups $\overline{H}_i(B^\times(\mathbb{R}P^2, 4), \mathbb{Q})$ and $\overline{H}_i(B^\times(\mathbb{R}P^2, 4), \pm \mathbb{Q})$ are trivial for all $i \geq 0$.

Proof. We write $T$ for the difference $B(\mathbb{R}P^2, 4) \setminus B^\times(\mathbb{R}P^2, 4)$. By Corollary 1 and the exact sequence of the pair $(B(\mathbb{R}P^2, 4), T)$ it suffices to prove that the groups $\overline{H}_*(T, \mathbb{Q})$ and $\overline{H}_*(T, \pm \mathbb{Q})$ are trivial. By Poincaré duality, this is equivalent to the triviality of the groups $H_*(T, \mathbb{Q} \otimes \mathcal{O}^r)$ and $H_*(T, \pm \mathbb{Q} \otimes \mathcal{O}^r)$.

$T$ is fibred over the space $\mathbb{R}P^2$ of all lines in $\mathbb{R}P^2$. Its fibre over the point corresponding to a line $l$ is the configuration space $B(l, 4)$. By Lemma 2 this fibre is homotopy equivalent to $S^1$. Retractions of these fibres to the circles described in Remark 2 can be performed uniformly for all $l$. Therefore $T$ contains a three-dimensional deformation retract $\overline{T}$, which is a closed manifold fibred over $\mathbb{R}P^2$. Its fibre over a point $\{l\}$ consists of all quadruples of points dividing the line $l$ (regarded as a circle in the unit sphere with opposite points identified) into
arcs of length $\pi/4$. Hence it remains to prove the triviality of all groups

$$H_i(\widetilde{T}, \mathbb{Q} \otimes \widetilde{O}_r), \quad H_i(\widetilde{T}, \mathbb{Z} \otimes \widetilde{O}_r),$$

where $\widetilde{O}_r$ is the restriction to $\widetilde{T}$ of the orientation sheaf of $T$.

The group $SO(3)$ (and, therefore, its double covering $SU(2)$) acts transitively on $\widetilde{T}$. This action determines a principal 16-fold covering $SU(2) \to \widetilde{T}$. The 16-dimensional direct image of the trivial $\mathbb{Q}$-bundle under this covering contains both sheaves $\mathbb{Q} \otimes \widetilde{O}_r$ and $\mathbb{Z} \otimes \widetilde{O}_r$ as direct summands. Hence both groups (6) are at most one-dimensional when $i = 0$ and $i = 3$ and are trivial for all other values of $i$. However, since both sheaves are inequivalent to the constant one, the groups (6) are trivial when $i = 0$. By Euler characteristic considerations, the same holds for $i = 3$. □

**Lemma 6.** The group $\overline{H}_i(B(\mathbb{R}P^2, 2), \mathbb{O}_r \otimes \mathbb{Q})$ is equal to $\mathbb{Q}$ when $i = 4$ and $i = 1$ and is trivial for all other values of $i$.

**(Proof.** This group is Poincaré dual to $H_{4-i}(B(\mathbb{R}P^2, 2), \mathbb{Q})$. Let us calculate the latter group. Associating with any two-point configuration the line spanned by these points, we obtain a fibration of $B(\mathbb{R}P^2, 2)$ over $\mathbb{R}P^2$. Its fibre is homotopy equivalent to $S^1$. The generator of $\pi_1(\mathbb{R}P^2)$ acts non-trivially on the group $H_1$ of the fibre, whence the term $E^2$ of the homological spectral sequence of this fibration has only two non-trivial cells: $E^2_{0,0} \simeq \mathbb{Q}$ and $E^2_{2,1} \simeq H_2(\mathbb{R}P^2, \mathbb{O}_r \otimes \mathbb{Q}) \simeq \mathbb{Q}$. □

**Proposition 2** (Carathéodory’s theorem; see, for example, [10], [17]). For every $r \geq 1$ the $r$th self-join $(S^1)^*r$ of the circle is homeomorphic to $S^{2r-1}$.

**Lemma 7.** If $r$ is even, then the sphere $(S^1)^*r \sim S^{2r-1}$ has a canonical orientation and, therefore, admits a canonical generator of the group $H_{2r-1}$. If $r$ is odd, then the homeomorphism of $(S^1)^*r$ onto itself induced by an orientation-reversing automorphism of $S^1$ reverses the orientation of $(S^1)^*r$.

**(Proof.** The group $H_{2r-1}((S^1)^*r)$ is generated by the Borel–Moore fundamental class of an open dense subset of $(S^1)^*r$ which is fibred over the configuration space $B(S^1, r)$. Its fibre over any $r$-configuration is the open $(r - 1)$-dimensional simplex whose vertices correspond to the points of this configuration. An orientation of this subset consists of orientations of the base and the fibres. Given an $r$-configuration in $S^1$, we enumerate its points arbitrarily and define the $j$th tangent vector of the base ($j \in \{1, \ldots, r\}$) as a shift of the $j$th point in the direction of the chosen orientation of $S^1$, keeping the remaining $r - 1$ points fixed. The fibre over this configuration is oriented by the same order of these $r$ points. Renumbering these points, we simultaneously reverse or preserve the orientations of the base and the fibre, thus preserving the orientation of the total space. On the other hand, changing the chosen orientation of $S^1$ reverses the $r$ basis vectors of the orienting frame. □

§ 4. Calculation of the spectral sequence

By part 2 of Lemma 1 and Thom’s isomorphism we have

$$\overline{H}_i(\sigma_p \setminus \sigma_{p-1}, \mathbb{Q}) \simeq \overline{H}_{i-d(p)k}(\Psi_p \setminus \Psi_{p-1}, \mathbb{Q} \otimes \mathbb{O})$$

(7) for all $i \geq 0$ and $p$, where $\mathbb{O}$ is the orientation sheaf of the vector bundle whose fibre over any point of $\Psi(X)$, $X \in J_p$, is the space $L(X) \subset W^k$ of all quadratic systems.
vanishing on \{X\}, and \(d(p)\) is the dimension of these spaces \(L(X), X \in J_p\) (that is, \(d(1) = 5, d(2) = 4, d(3) = d(4) = d(5) = 3, d(6) = d(7) = 2, d(8) = d(9) = 1\) and \(d(10) = 0\)). By construction, this vector bundle is the direct sum of \(k\) copies of the bundle whose fibres are the corresponding subspaces of \(W\). Therefore this orientation sheaf is trivial for even \(k\) and coincides with the orientation sheaf of the latter vector bundle for odd \(k\).

Thus we are going to calculate the groups on the right-hand side of (7).

### 4.1. First column.

The first term \(\sigma_1\) of our filtration of \(\sigma\) is the space of a \(5k\)-dimensional vector bundle over \(\mathbb{RP}^2\). This vector bundle is naturally oriented for every \(k\). Indeed, each of its \(k\) summands is a \(5\)-dimensional vector subbundle of the trivial bundle \(\mathbb{RP}^2 \times W \to \mathbb{RP}^2\). Its fibre over a point \(x \in \mathbb{RP}^2\) is the space of all polynomials \(f \in W^k\) vanishing on the line \(\{x\} \subset \mathbb{R}^3\). The canonical orientation of such a fibre in \(W\) is given by increasing the polynomials \(f\) on the line \(\{x\}\) (it is important here that \(f\) is of even degree). By (7) we have

\[
E^1_{1,q} \equiv \overline{H}^{1+q-5k}(\mathbb{RP}^2, \mathbb{Q}).
\]

Hence the group \(E^1_{1,q}\) is equal to \(\mathbb{Q}\) if \(q = 5k - 1\) and is trivial otherwise.

### 4.2. Second column.

**Lemma 8.** The group \(\overline{H}_i(\sigma_2 \setminus \sigma_1, \mathbb{Q})\) is trivial for all \(i \geq 0\).

**Proof.** \(\sigma_2 \setminus \sigma_1\) is the space of a fibre bundle over the configuration space \(B(\mathbb{RP}^2, 2)\). Namely, it is a fibred product of the following factors.

(i) The \(4k\)-dimensional vector bundle whose fibre over a point \(\{x, y\} \in B(\mathbb{RP}^2, 2)\) consists of all systems \(F \in W^k\) vanishing on both lines in \(\mathbb{R}^3\) corresponding to the points \(x\) and \(y\),

(ii) The bundle of open intervals whose endpoints are associated with the points \(x\) and \(y\). Every such interval consists of two half-intervals \((\{x\}, \{x \cup y\}\) and \((\{y\}, \{x \cup y\}\) in \(\overline{\Psi}_2\), where \(\{x\}\) and \(\{y\}\) are the points in \(J_1\) that correspond to the points \(x\) and \(y\), and \(\{x \cup y\}\) \(\in J_2\). The endpoints \(\{x\}\) and \(\{y\}\) are excluded since they belong to the first term \(\Psi_1\) of our filtration.

Any loop in \(B(\mathbb{RP}^2, 2)\) which does not permute the two points of the configuration does not reverse the orientations of these two bundles. A loop permuting these points reverses the orientations of all \(k\) factors of the first bundle and the orientation of the second bundle. Thus the \(i\)th Borel–Moore homology group of the total space of this fibre bundle is equal to \(\overline{H}_{i-4k-1}(B(\mathbb{RP}^2, 2), \mathbb{Q})\) for odd \(k\) and \(\overline{H}_{i-4k-1}(B(\mathbb{RP}^2, 2), \pm \mathbb{Q})\) for even \(k\). All these groups are trivial by Corollary 1. □

Therefore the column \(E^2_{1,*}\) of the main spectral sequence consists of zeros only.

### 4.3. Third column.

The space \(\sigma_3 \setminus \sigma_2\) is fibred over the configuration space \(J_3 = B(\mathbb{RP}^2, 3)\). Its fibre over a 3-point configuration \(X \in J_3\) is equal to the product of the \(3k\)-dimensional vector space \(L(X)\) and an open two-dimensional simplex (whose vertices are associated with the points of the configuration, and the whole simplex is the order complex of non-empty subsets of this configuration).
Every element of $\pi_1(B(\mathbb{RP}^2, 3))$ that induces an odd permutation of three points reverses the orientations of the bundle of triangles and of all $k$ summands of the $3k$-dimensional vector bundle. This yields the following lemma.

**Lemma 9.** For even $k$ we have $\overline{H}_i(\sigma_3 \setminus \sigma_2, \mathbb{Q}) = \overline{H}_{i-3k-2}(B(\mathbb{RP}^2, 3), \pm \mathbb{Q})$. For odd $k$ we have $\overline{H}_i(\sigma_3 \setminus \sigma_2, \mathbb{Q}) = \overline{H}_{i-3k-2}(B(\mathbb{RP}^2, 3), \mathbb{Q})$.

All these groups are trivial by Corollary 1. Hence the column $p = 3$ of the main spectral sequence is also trivial.

**4.4. Fourth column.** The space $\sigma_5 \setminus \sigma_3$ is fibred over the space $\mathbb{RP}^2$ of all lines in $\mathbb{RP}^2$. Its fibre over a line $l$ is equal to the product of the $3k$-dimensional subspace $L(l) \subset W^k$ (consisting of all polynomial systems vanishing on $l$) and the space $\Psi(l) \setminus \Psi_3$.

The space $\Psi(l)$ is a compact cone with vertex $\{l\} \in J_5$. In particular, it is contractible. Its subset $\Psi(l) \cap \Psi_3$ is the order complex of all non-empty subsets of cardinality at most three in the line $l \sim S^1$. It is the union of order subcomplexes $\Psi(\{a, b, c\})$, where $a, b, c$ are arbitrary distinct points of $l$. Every such subcomplex can be identified with the triangle with vertices $\{a\}, \{b\}$ and $\{c\}$. Then the entire space $\Psi(l) \cap \Psi_3$ becomes identified with the union of all such triangles equipped with the natural topology. By Proposition 2, this space is homeomorphic to $S^5$. The exact sequence of the pair $(\Psi(l), \Psi(l) \cap \Psi_3)$ now proves that the $i$th Borel–Moore homology group of the fibre $\Psi(l) \setminus \Psi_3$ is equal to $\mathbb{Q}$ when $i = 6$ and is trivial for all other values of $i$. By Lemma 7, the monodromy over a non-contractible loop in $\mathbb{RP}^2$ reverses the orientation of this factor of the fibre and, therefore, reverses the orientation of this homology group.

On the other hand, the bundle with fibre $L(l)$ splits into the direct sum of $k$ three-dimensional bundles, each of which changes its orientation under the generator of $\pi_1(\mathbb{RP}^2)$. We obtain the following lemma.

**Lemma 10.** For even $k$ we have $\overline{H}_i(\sigma_5 \setminus \sigma_3, \mathbb{Q}) \simeq \overline{H}_{i-3k-6}(\mathbb{RP}^2, \mathbb{Q} \otimes \mathbb{Q}r)$. In particular, this group is equal to $\mathbb{Q}$ when $i = 3k + 8$ and is trivial for all other values of $i$. For odd $k$ we have $\overline{H}_i(\sigma_5 \setminus \sigma_3, \mathbb{Q}) \simeq \overline{H}_{i-3k-6}(\mathbb{RP}^2, \mathbb{Q})$. In particular, this group is equal to $\mathbb{Q}$ when $i = 3k + 6$ and is trivial for all other values of $i$.

This gives us the columns $p = 4$ in Fig. 1a and Fig. 1b.

**4.5. Fifth column.** The space $\sigma_6 \setminus \sigma_5$ is fibred over the restricted configuration space $B^\times(\mathbb{RP}^2, 4)$. Its fibre over a configuration $X \in B^\times(\mathbb{RP}^2, 4)$ is the product of the $2k$-dimensional vector space $L(X)$ and an open three-dimensional simplex whose vertices correspond to the points of $\{X\}$. Arguing in exactly the same way as in §§4.2, 4.3, we obtain the following lemma.

**Lemma 11.** The group $\overline{H}_i(\sigma_6 \setminus \sigma_5, \mathbb{Q})$ is equal to $\overline{H}_{i-2k-3}(B^\times(\mathbb{RP}^2, 4), \mathbb{Q})$ if $k$ is odd, and to $\overline{H}_{i-2k-3}(B^\times(\mathbb{RP}^2, 4), \pm \mathbb{Q})$ if $k$ is even.

All these groups are trivial for all $i$ by Lemma 5. Therefore the columns $p = 5$ of our spectral sequences are equal to zero.
4.6. Sixth column.

**Lemma 12.** If $k$ is odd, then $\overline{H}_i(\sigma_7 \setminus \sigma_6, \mathbb{Q}) \simeq \overline{H}_{i-2k-9}(\mathbb{RP}^2, \mathbb{Q} \otimes \mathbb{Q}r)$. In particular, this group is equal to $\mathbb{Q}$ when $i = 2k + 11$ and is trivial otherwise. If $k$ is even, then $\overline{H}_i(\sigma_7 \setminus \sigma_6, \mathbb{Q}) \simeq \overline{H}_{i-2k-9}(\mathbb{RP}^2, \mathbb{Q})$. In particular, this group is equal to $\mathbb{Q}$ when $i = 2k + 9$ and is trivial otherwise.

*Proof.* $\sigma_7 \setminus \sigma_6$ is the space of a fibre bundle over the space $J_7$ of configurations of the form

$$\{\text{a line } \{l\} \in \mathbb{RP}^2, \text{ a point } u \in \mathbb{RP}^2 \setminus l\}. \quad (9)$$

The fibre of this bundle over such a configuration $X = \{l, u\} \in J_7$ is the product

$$\Psi(X) \times L(X), \quad (10)$$

where $L(X) \subset W^k$ is the $2k$-dimensional space of all systems $F \in W^k$ vanishing on $l$ and at $u$. The space of configurations (9) is obviously diffeomorphic to $T_r\mathbb{RP}^2$. In particular, it is orientable.

**Lemma 13.**

1) The non-trivial element of the fundamental group

$$\pi_1(T_r\mathbb{RP}^2) \simeq \pi_1(\mathbb{RP}^2)$$

of this configuration space reverses the orientation of each of the $k$ two-dimensional summands of the second factor $L(X)$ of the fibre (10).

2) The link $\partial \Psi(X)$ of the first factor of the fibre (10) is homotopy equivalent to $S^6$. In particular, the group $\overline{H}_i(\Psi(X), \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ when $i = 7$ and is trivial for all other values of $i$.

3) The monodromy over the non-trivial element of $\pi_1(T_r\mathbb{RP}^2)$ acts as multiplication by $-1$ on the homology bundle associated with the factors described in part 2.

*Proof.* 1) As a loop that generates the fundamental group of our configuration space, we choose the family of pairs $\{l_\varphi, u_\varphi\}$, $\varphi \in [0, \pi]$, where the line $l_\varphi$ is given by the equation $x \cos \varphi + y \sin \varphi = 0$ and the point $u_\varphi$ is given by its homogeneous coordinates $(\cos \varphi : \sin \varphi : 0)$. For every configuration $X = \{l, u\}$, the fibre of each of our $k$ summands over $X$ consists of all quadratic forms vanishing on $l$ and at $u$. We choose the families

$$(x \cos \varphi + y \sin \varphi)(x \sin \varphi - y \cos \varphi), \quad (x \cos \varphi + y \sin \varphi)z$$

for linear independent sections of the two-dimensional bundle of such forms over the set of pairs $\{l_\varphi, u_\varphi\}$, where $\varphi$ runs over the half-open interval $\varphi \in [0, \pi)$. The first of these sections extends to a continuous section over the entire closed loop, while the limit value of the second section as $\varphi \to \pi$ is opposite to the value at $\varphi = 0$. Hence the monodromy over this loop acts on the fibre as the operator $|_{0}^{1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2) The link $\partial \Psi(X)$, $X = \{l, u\}$, consists of two subspaces: $A$ (the order complex $\Psi(l)$) and $B$ (the union of all order complexes $\Psi(\{a, b, c, u\})$, where $a$, $b$ and $c$ are arbitrary distinct points of the line $l$). The intersection $A \cap B$ is the space $\Psi(l) \cap \Psi_3$, that is, the union of the corresponding order subcomplexes $\Psi(\{a, b, c\})$. We know from §4.4 that this intersection $A \cap B$ is homeomorphic to $S^5$. Since
A is compact and contractible, the pair \((A, A \cap B)\) is homotopy equivalent to the pair \((a \text{ cone over } A \cap B, \text{ its base } A \cap B)\). Each of the order complexes \(\Psi(\{a, b, c, u\})\) constituting the space \(B\) is canonically PL-homeomorphic to the tetrahedron with vertices \(a, b, c\) and \(u\), and hence to the cone over \(\Psi(\{a, b, c\})\) with vertex \(\{u\}\). These homeomorphisms over all triples \(\{a, b, c\}\) determine a homeomorphism between the union \(B\) of all these tetrahedra and another cone over \(\Psi(l) \cap \Psi_3\). Therefore our space \(A \cup B\) is homotopy equivalent to the suspension of \(A \cap B \sim S^5\).

3) By Lemma 7, the generator of \(\pi_1(\mathbb{R}P^2)\) acts on the group \(H_5(\Psi(l) \cap \Psi_3, \mathbb{Q})\) as multiplication by \(-1\). The two parts \(A\) and \(B\) of the space \(\partial \Psi(\{l, u\}) \sim \Sigma(\Psi(l) \cap \Psi_3)\) are of different origin and cannot be permuted by the monodromy. Hence the group \(H_6(\partial \Psi(\{l, u\}), \mathbb{Q})\) is also reversed by this generator. □

**Completion of the proof of Lemma 12.** Lemma 12 follows immediately from Lemma 13 by the spectral sequence of the fibre bundle (10) over the space \(J_6 \sim T_1\mathbb{R}P^2\) of configurations (9) and by the Thom isomorphism for the (non-orientable) fibration of this space over \(\mathbb{R}P^2\). □

This gives us the columns \(p = 6\) of both spectral sequences in Fig. 1a, b.

### 4.7. Seventh column.

**Proposition 3.** For every even \(k\), the group \(\overline{H}_i(\sigma_8 \setminus \sigma_7, \mathbb{Q})\) is trivial for all \(i\). If \(k\) is odd, then the group \(\overline{H}_i(\sigma_8 \setminus \sigma_7, \mathbb{Q})\) is isomorphic to \(\mathbb{Q}\) when \(i = k + 12\) or \(i = k + 9\) and is trivial for all other values of \(i\).

**Proof.** The space \(\sigma_8 \setminus \sigma_7\) is fibred over the space \(J_8 = B(\mathbb{R}P^2, 2)\) of two-point configurations in the dual projective plane. Its fibre over any point \(X \in J_8\) is homeomorphic to \(\hat{\Psi}(X) \times \mathbb{R}^k\).

**Lemma 14.** For every point \(X \in B(\mathbb{R}P^2, 2)\), the link \(\partial \Psi(X)\) of the order complex \(\Psi(X)\) is homology equivalent to the sphere \(S^7\).

**Proof.** Let \(l\) and \(l'\) be the two lines constituting the configuration \(X\), and let \(O\) be their common point. We denote the set

\[
\bigcup_{a \in l} \Psi(\{l', a\}) \cup \bigcup_{a' \in l'} \Psi(\{l, a'\})
\]

by \(\mathcal{N}\) and study the following filtration of the link \(\partial \Psi(X)\):

\[
(\Psi(l) \cup \Psi(l')) \subset \mathcal{N} \subset \partial \Psi(X).
\] (11)

The order complexes \(\Psi(l)\) and \(\Psi(l')\) are contractible by definition, and their intersection is the single point of \(\Psi_1\) corresponding to the point \(O\). Therefore the union \(\Psi(l) \cup \Psi(l')\) is also contractible.

The space \(\mathcal{N} \setminus (\Psi(l) \cup \Psi(l'))\) consists of two identical disjoint pieces, swept out by all order complexes \(\Psi(\{l, a'\}) \setminus \Psi(l)\) (resp. \(\Psi(\{l', a\}) \setminus \Psi(l')\)), where \(a'\) runs over the affine line \(l' \setminus O\) and \(a\) runs over \(l \setminus O\). Clearly, the first piece is homeomorphic to the direct product \((l' \setminus O) \times (\Psi(\{l, a'\}) \setminus \Psi(l))\) for an arbitrary \(a' \in l' \setminus O\). By the proof of part 2 of Lemma 13, the pair \((\Psi(\{l, a'\}), \Psi(l))\) is homotopy equivalent.
to the pair \((D^7, D^6)\), where \(D^7\) is the 7-dimensional ball and \(D^6\) is a closed hemisphere on its boundary \(\partial D^7 \sim S^6\). In particular, the group \(H_i(\Psi(l, a'), \Psi(l)) \equiv \overline{H}_i(\Psi(l, a') \setminus \Psi(l))\) is trivial for every \(i\). Hence, by the Künneth formula, all Borel–Moore homology groups of the first piece of \(\Sigma \setminus (\Psi(l') \cup \Psi(l'))\) are also trivial. The second piece is homeomorphic to the first. Hence the whole second term of the filtration \((11)\) has the homology of a point.

Finally, the space \(\partial \Psi(X) \setminus \Sigma\) is the union of all spaces \(\Psi_l(a, b, c, d)\), where \(a \neq b \in l \setminus O\), and \(c \neq d \in l' \setminus O\). These spaces are homeomorphic to an open tetrahedron, and the base \(B(RP^1, 2) \times B(RP^1, 2)\) of this family is homeomorphic to a 4-dimensional open ball. In particular, the Borel–Moore homology group of this space is equal to \(\mathbb{Q}\) in degree 7 and is trivial in all other degrees. \(\square\)

We now study the action of \(\pi_1(B(\mathbb{RP}^2, 2))\) on the Borel–Moore homology group of the fibres \(\Psi_l(X) \times L_l(X)\) of our fibration of \(\sigma_7 \setminus \sigma_6\) over \(B(\mathbb{RP}^2, 2)\). By Lemma 14, this group is equal to \(\mathbb{Q}\) in degree \(8 + k\) and is trivial in all other degrees.

We choose the following pair of lines for a basepoint in \(B(\mathbb{RP}^2, 2)\). The first (resp. second) line consists of the meridians with longitudes \(0^\circ\) and \(180^\circ\) (resp. \(90^\circ\) and \(270^\circ\)). The group \(\pi_1(B(\mathbb{RP}^2, 2))\) is generated by two elements, \(\theta\) and \(\varpi\), where \(\theta\) rotates the sphere by \(90^\circ\) around the main axis containing the poles, and \(\varpi\) rotates the first line by \(180^\circ\) around the axis through the points of the equator with coordinates \((0^\circ, 0^\circ)\) and \((0^\circ, 180^\circ)\), and keeps the second line fixed.

**Lemma 15.** Both loops \(\theta, \varpi \in \pi_1(B(\mathbb{RP}^2, 2))\) reverse the orientations of \(B(\mathbb{RP}^2, 2)\) and of each of the \(k\) factors of the fibre bundle with fibres \(L_l(X)\). The one-dimensional homology groups \(\overline{H}_8(\Psi_l(X), \mathbb{Q})\) and \(H_7(\partial \Psi_l(X), \mathbb{Q})\) have a canonical orientation for all \(X\), and this orientation depends continuously on \(X\). In particular, the group \(\pi_1(B(\mathbb{RP}^2, 2))\) acts trivially on them.

**Proof.** The assertion on the orientation of the base follows immediately from Lemma 3. Each of the \(k\) summands of the fibre \(L_l(X)\) is the space of quadratic functions vanishing on our pair of lines. These lines divide \(\mathbb{RP}^2\) into two domains. We choose one of them and orient each summand by the increase of our quadratic functions inside this domain. Each of the loops \(\theta, \varpi\) permutes these two domains and, therefore, reverses this orientation.

We now define an invariant orientation of the homology bundle whose fibre over a point \(X \in B(\mathbb{RP}^2, 2)\) is the group \(H_7(\partial \Psi_l(X), \mathbb{Q})\). This group is generated by the fundamental class of the 7-dimensional cell described in the last paragraph of the proof of Lemma 14. To orient this cell, we must choose an orientation of its base \(B(l \setminus O, 2) \times B(l' \setminus O, 2)\) and, for every point on the base (that is, a four-point configuration), an orientation of the fibre over it, that is, of the 3-dimensional simplex whose four vertices correspond to the points of this configuration. To do this, we fix an arbitrary enumeration of these four points and, for every \(j = 1, \ldots, 4\) define the \(j\)th basic tangent vector to the base as a shift of the \(j\)th point towards the intersection point \(O\) of these lines and away from the other point of the configuration in the same line. The orientation of the fibre (that is, a tetrahedron) is determined by the same enumeration of its vertices. Another enumeration of the
four points simultaneously reverses or preserves the orientations of the base and the fibre and, therefore, preserves the orientation of the total space.

The orientation thus defined is obviously invariant and determines a trivialization of the homology bundle over $B(\mathbb{RP}^2, 2)$ with fibre $H_7(\partial \Psi(X), \mathbb{Q})$ or, which is the same by the boundary isomorphism, of the corresponding bundle with fibres $\overline{H}_8(\tilde{\Psi}(X), \mathbb{Q})$. □

**Corollary 2.** If $k$ is even (resp. odd), then the group $\overline{H}_i(\sigma_9 \setminus \sigma_8, \mathbb{Q})$ is equal to $\overline{H}_{i-k}(B(\mathbb{RP}^2, 2), \mathbb{Q})$ (resp. $\overline{H}_{i-k}(B(\mathbb{RP}^2, 2), \mathbb{Q}) \otimes R$) for all $i$.

**Proof.** This follows from Lemma 15 by the Thom isomorphism. □

Proposition 3 is deduced from Corollary 2 using Corollary 1 and Lemma 6. This justifies the columns $p = 7$ in Fig. 1a, b. □

### 4.8. Eighth column.

**Lemma 16.** The space $J_9$ of non-empty non-singular conics in $\mathbb{RP}^2$ is diffeomorphic to the total space of a 3-dimensional vector bundle over $\mathbb{RP}^2$, namely, of the symmetric square $S^2T^*(\mathbb{RP}^2)$ of the cotangent bundle. In particular, this space is an orientable 5-dimensional manifold homotopy equivalent to $\mathbb{RP}^2$.

**Proof.** Every such conic is uniquely (up to multiplication by a positive number) determined by a quadratic form of signature $(1, 2)$. We fix an arbitrary Euclidean structure on $\mathbb{R}^3$. Then the line of positive eigenvectors of these quadratic forms indicates a point $u \in \mathbb{RP}^2$, and the plane of negative ones can naturally be identified with the tangent plane to $\mathbb{RP}^2$ at $u$. We normalize our quadratic form in such a way that it takes the value 1 on its positive eigenvectors of length 1. Then we obtain an isomorphism between the space of all conics of this signature with eigendirection $\{u\}$ (on one side) and the space of negative-definite quadratic forms on the tangent space to $\mathbb{RP}^2$ at $u$ (on the other). The latter space is a convex open subset of $S^2T_u^*(\mathbb{RP}^2)$. Therefore it is diffeomorphic to $S^2T_u^*(\mathbb{RP}^2)$ (and these diffeomorphisms can be performed uniformly over all fibres). □

**Lemma 17.** For every $k$, the group $\overline{H}_i(\sigma_9 \setminus \sigma_8, \mathbb{Q})$ is isomorphic to the group $\overline{H}_{i-k-11}(\mathbb{RP}^2, \mathbb{Q} \otimes \mathbb{Q})$, that is, to $\mathbb{Q}$ when $i = k + 13$ and to the trivial group for all other values of $i$.

**Proof.** The space $\sigma_9 \setminus \sigma_8$ is fibred over $J_9$. Its fibre over a conic $C \in J_9$ is equal to $\tilde{\Psi}(C) \times L(C)$, $L(C) \simeq \mathbb{R}^k$.

The spaces $L(C)$ form a trivial vector bundle over $J_9$. Indeed, each of its one-dimensional summands is canonically oriented since it has a section consisting of quadratic forms of signature $(1, 2)$.

Every non-singular conic $C$ is homeomorphic to $S^1$, and its proper subsets mentioned in the list of §2.3 are all possible subsets of cardinality at most four. Hence the link $\partial \Psi(C)$ is homeomorphic to $S^7$ by Proposition 2. In particular, the Borel–Moore homology groups of the fibres $\tilde{\Psi}(C)$ are isomorphic to $\mathbb{Q}$ in degree 8 and are trivial in all other degrees. By Lemma 7, the fundamental group of the base $J_9$ acts trivially on these homology groups.
Lemma 17 now follows from the Thom isomorphism

\[ \mathcal{H}_i(\sigma_9 \setminus \sigma_8, \mathbb{Q}) \simeq \mathcal{H}_{i-k-8}(J_9, \mathbb{Q}) \]

of our orientable fibre bundle over the space of conics and the Thom isomorphism \( \mathcal{H}_j(J_9, \mathbb{Q}) \simeq H_{j-3}(\mathbb{RP}^2, \mathbb{Q} \otimes \mathcal{O}_r) \) for the fibration of \( J_9 \) over \( \mathbb{RP}^2 \) (see Lemma 16). \( \square \)

This proves the formulae for the columns \( p = 8 \) in Fig. 1a, b. In particular, we see that both groups \( E_{8,5}^1 \equiv \mathcal{H}_{k+13}(\sigma_9 \setminus \sigma_8, \mathbb{Q}) \) and \( E_{7,5}^1 \equiv \mathcal{H}_{k+12}(\sigma_8 \setminus \sigma_7, \mathbb{Q}) \) are isomorphic to \( \mathbb{Q} \) in the case of odd \( k \).

4.9. The differential \( d_1 : E_{8,k+5}^1 \to E_{7,k+5}^1 \) for odd \( k \).

Proposition 4. The boundary homomorphism

\[ \mathcal{H}_{k+13}(\sigma_9 \setminus \sigma_8, \mathbb{Q}) \xrightarrow{\partial} \mathcal{H}_{k+12}(\sigma_8 \setminus \sigma_7, \mathbb{Q}) \]

in the exact sequence of the triple is an isomorphism for every odd \( k \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Different approximations of \( J_8 \) from \( J_9 \)}
\end{figure}

Proof. Let \( X \) be an arbitrary point of the stratum \( J_8 \), that is, a pair of distinct lines in \( \mathbb{RP}^2 \). This stratum has codimension 1 in the space of conics, and the open stratum \( J_9 \) approaches it from two different sides at \( X \) (see Fig. 3). We recall that \( \sigma_9 \setminus \sigma_8 \) is fibred over \( J_9 \), and the fibre over a conic \( C \in J_9 \) is equal to \( \Psi(C) \times L(C) \), \( L(C) \simeq \mathbb{R}^k \). The base \( J_9 \) is oriented as an open subvariety of the space \( \mathbb{RP}^5 \) of all conics, the canonical orientation of the fibres \( L(C) \) (and even of each of their \( k \) summands) was defined in the proof of Lemma 17, and the canonical orientation of the fibres \( \Psi(C) \) was defined in the proof of Lemma 7. In a small neighbourhood \( U \) of the point \( X \), the manifold \( J_8 \) may be regarded as the boundary of either of the two local components into which it divides \( J_9 \). Therefore the restriction of the orientation of \( J_9 \) to either of these components induces the boundary orientation of \( J_8 \cap U \).
Lemma 18. The following assertions hold when the points $C \in J_9$ tend to a point $X$ of the hypersurface $J_8 \subset \mathbb{RP}^5$ from two different local components of $J_9$.

1) The boundary orientations of $J_8$ at $X$ induced from the restrictions of the same orientation of $J_9$ to these components are opposite to each other.

2) The 7-dimensional fundamental classes of canonically oriented fibres $\partial \Psi(C) \sim S^7$ tend to the fundamental class of the fibre $\partial \Psi(X) \setminus \mathbb{R}$, which is equipped with the canonical orientation (described in the last paragraph of the proof of Lemma 14) and generates the Borel–Moore homology group of the whole $\partial \Psi(X)$.

3) The canonical orientations of any of the $k$ summands of the $k$-dimensional vector bundle with fibres $L(C)$ tend to opposite orientations of the corresponding line over $X$.

By the boundary isomorphisms, part 2 of Lemma 18 is equivalent to the following assertion. The 8-dimensional fundamental classes of canonically oriented fibres $\tilde{\Psi}(C)$ (which are homeomorphic to open 8-dimensional balls) tend to the canonically oriented fundamental class of a piece of $\tilde{\Psi}(X)$ that generates its 8-dimensional Borel–Moore homology group.

Proof of Lemma 18. 1) Part 1 follows since the union of $J_9$ and $J_8$ is an open subset in the oriented manifold $\mathbb{RP}^5$ of all conics.

2) All these orientations, as they were defined, depend only on the directions of shifts of the points of 4-configurations. They are directed along the same orientation of our conic in the case of $J_9$ and towards the crossing point in the case of $J_8$ (see Fig. 3). The limit positions of the first directions, when the conic degenerates into a cross, differ from the latter ones an even number of times. Therefore the corresponding orientations coincide.

3) For every point $C \in J_9$ any such one-dimensional summand consists of all quadratic functions vanishing on the corresponding cone. The basis vector defining the canonical orientation of this line is a quadratic function which is positive inside the simply-connected component of the complement of the cone in $\mathbb{R}^3$. The limits of such functions on the left- and right-hand sides of Fig. 3 are functions vanishing on the coordinate cross but having opposite signs outside it. Therefore they define opposite orientations of the corresponding summand of the fibre $L(X)$ over the point $X \in J_8$. □

Thus, the boundary orientations of $\sigma_8 \setminus \sigma_7$ induced from these two sides differ by the factor $(-1) \cdot 1 \cdot (-1)^k$, which is equal to 1 for odd $k$. Hence the boundary operator is non-trivial. This proves Proposition 4. □

4.10. Ninth and last column.

Proposition 5. The group $\overline{H}_i(\sigma_{10} \setminus \sigma_9, \mathbb{Q})$ is equal to $\mathbb{Q}$ for $i = 14$ and is trivial for all other values of $i$.

Proof. The space $J_{10}$ consists of the unique point $[\mathbb{RP}^2]$ specified by the zero quadratic form. Therefore $\sigma_{10} \setminus \sigma_9 = \tilde{\Psi}(\mathbb{RP}^2)$. The order complex $\Psi([\mathbb{RP}^2])$ can be considered as the space $\sigma$ of our simplicial resolution in the case $k = 0$ (that is, with zero-dimensional spaces $L(X)$). Therefore the Borel–Moore homology group of its link $\partial \Psi([\mathbb{RP}^2]) \subset \sigma_9$ can be calculated by the spectral sequence whose term $E_1$
coincides with the columns \( p = 1, \ldots, 8 \) of our main spectral sequence for \( k = 0 \). These columns are calculated above, and the resulting spectral sequence is shown in Fig. 4a.

![Spectral Sequence](image)

Figure 4. a) \( E^1 \) for \( k = 0 \); b) \( E^1 \) for \( k = 1 \)

**Lemma 19.** The differential \( \partial_2 : E_{6,3}^2 \to E_{4,4}^2 \) of the spectral sequence in Fig. 4a is an isomorphism.

**Proof.** Consider our main spectral sequence for \( k = 1 \). Since its columns with \( p = 1, \ldots, 8 \) have already been calculated, it is as shown in Fig. 4b, where both symbols "??" in the ninth column should be replaced by \( \mathbb{Q} \) or zero depending on whether the differential in question is trivial or not. However, the answer is known in the case \( k = 1 \): the space \( W^1 \setminus \Sigma \) consists of two contractible components. Therefore, by Alexander duality, the group \( H_i(\Sigma, \mathbb{Q}) \equiv H_i(\sigma, \mathbb{Q}) \) is equal to \( \mathbb{Q} \) when \( i = 5 \) and is trivial for all other values of \( i \). Thus all cells of our spectral sequence except for \( E_{1,4}^1 \) should be killed by some of its differentials. This is impossible if the group \( E_{0,0}^1 \) is non-trivial. \( \square \)

Thus the group \( H_i(\partial \Psi(\mathbb{R}P^2), \mathbb{Q}) \) is isomorphic to \( \mathbb{Q} \) when \( i = 0 \) or \( i = 13 \) and is trivial for all other values of \( i \). Proposition 5 follows from this fact by the exact sequence of the pair \( (\Psi(\mathbb{R}P^2), \partial \Psi(\mathbb{R}P^2)) \). \( \square \)

This completes the proof of Theorem 2.

§ 5. Concluding remarks

5.1. A problem. Is there a more direct proof of Theorem 1, in the same way as Proposition 1 can be proved using [18], [19] or similar methods?

Of course, the space \( W^k \setminus \Sigma \) is fibred over \( S^{k-1} \): with any non-resultant system we can associate the image of a fixed vector, for example, \((1, 0, 0)\). However, it seems that this structure is of little help in our problem. The Poincaré polynomials of the fibres of these bundles can easily be found using Theorem 1. For odd (resp. even) \( k \geq 3 \) they are equal to \( 1 + t^{3k-7}(1 + t^{k-5})(1 + t^{k-2}) \) (resp. \( 1 + t^{3k-9} + t^{5k-14} \)).
5.2. The construction of resolutions of discriminants in terms of self-joins was stated in full generality (sufficient for our current needs) in [11] (although particular versions appeared in less difficult situations in [15], [6], [20]). It may seem that this construction is less canonical than the one in terms of Hilbert schemes (see, for example, [13], [16]). However in practice, when one needs not only to prove abstract theorems but to accomplish a difficult explicit calculation of homology, the approach of [11] proves to be more effective.

5.3. The formal analogue. The space $W^k$ of quadratic maps $\mathbb{R}^3 \to \mathbb{R}^k$ is a subspace of the infinite-dimensional space of all maps $\mathbb{R}^3 \to \mathbb{R}^k$ whose restriction to every one-dimensional subspace of $\mathbb{R}^3$ is quadratic. The latter space of maps also contains a resultant variety: it consists of the maps having common zeros in $\mathbb{R}^3 \setminus 0$. The corresponding space of non-resultant maps is homotopy equivalent to the space of continuous maps $\mathbb{R}P^2 \to S^{k-1}$. Its rational homology groups for $k > 3$ can easily be calculated (for example, by the methods of [6]) and coincide with those of $S^{k-1}$; the triviality of all other rational homology groups follows from the explicit form of the first term of the spectral sequence described in [6] and from our Corollary 1. In particular, this group is much poorer than its unstable analogue calculated above.

I thank the referee very much for important remarks.

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