A Hybrid Fluid-Kinetic Theory for Plasma Physics

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Abstract

We parameterize the phase space density by time dependent diffeomorphic, Poisson preserving transformations on phase space acting on a reference density solution. We can look at these as transformations which fix time on the extended space of phase space and time. In this formulation the Vlasov equation is replaced by a constraint equation for the above maps. The new equations are formulated in terms of hamiltonian generators of one parameter families of diffeomorphic, Poisson preserving maps e.g. generators with respect to time or a perturbation parameter. We also show that it is possible to parameterize the space of solutions of the Vlasov equation by composition of maps subject to certain compatibility conditions on the generators. By using this composition principle we show how to formulate new equations for a hybrid fluid kinetic theory. This is done by observing that a certain subgroup of the group of phase space maps with generators which are linear in momentum correspond to the group of diffeomorphic maps parameterizing the continuity equation in fluid theory.

1 Introduction

The development of collisionless plasma physics in recent years have had some interesting breakthroughs. Among these should especially be mentioned the
recently found Eulerian action principles for plasma physics (Larsson\textsuperscript{1,2}, Ye and Morrison\textsuperscript{3}, and Flå\textsuperscript{4}). The intention of this work is to demonstrate that some of the substantial problems in developing good models in plasma physics come partially from purely formal problems with how to formulate model equations in an efficient, invariant language. A key problem is how to discriminate between the incoherent (containing heat fluctuations and resonant particle interactions) and coherent, fluid part of the plasma fluctuations. For this purpose we will use the flexibility of the generator approach to separate between generators of the fluid motion to first order in momentum and an incoherent part of the generators to higher order in momentum.

We will be interested in reformulating the continuity equation and the Vlasov equation in terms of the action of infinite dimensional transformations on space and phase space on the density and phase space density. By parameterizing the densities by these transformations with respect to reference densities, we find that the continuity equation for densities in fluid and kinetic theory are consistent with that the action of the transformations on reference fluid velocity and Hamiltonian vectorfield plus a timelike generator of the transformations are constrained to be equal to the actual fluid velocity and Hamiltonian vectorfield. We thus replace the continuity equations by constraint equations for the time dependent transformations. Moreover, we demonstrate in App. A that the continuity fluid equation and the Vlasov equation can be looked upon as the defining equations for an infinite dimensional pseudogroup on space and phase space extended with time, but constrained to transformations which fixes time. The moment we have realized this an interesting composition principle pops up. Namely, we have not only discovered one new equation, but an infinitely many corresponding to different compositions of transformations which are compatible with the above density structures. Our philosophy is that we encode a priori information into the choice of composition transformations. The a priori information restrict the class of experiments or processes which the constructed theory is intended to describe, and every composition will indeed give new equations. In this respect our philosophy is analogous to the theory of measurements in quantum mechanics where every measurement is related to a new class of operators. In our case, for every measurement or class of experiments there correspond a composition principle and corresponding equations. In our opinion this principle put the construction of invariant wave equations and field theory into a new light (and we believe not only in plasma physics) since to every a priori information encoded there correspond new wave oper-
ators with corresponding new spectrum and dispersion relations.

In this work we will try this new principle by making a hybrid fluid-kinetic theory for collisionless plasma physics. Thus we restrict our attention to a class of experiments where it makes sense to measure fluid density and velocity. If these quantities are not reasonable to measure, the theory of course does not apply. The moment we have decided on this extra bit of a priori information, we have a new theory since we can explicitly construct and separate out the fluid generator from the incoherent generator containing heat fluctuations and resonant particle interactions. It must be noted that in the ordinary Vlasov equation the heat fluctuations are not explicitly described as mode of fluctuation like e.g. in fluid theory. Even in the constant background case the heat fluctuations will be hidden in the continuous spectrum. With our invention of hybrid fluid kinetic theory the heat fluctuations will appear as a mode in any background or size of fluctuations due to the new operators appearing.

The plan of the paper is that we will use canonical coordinates to separate the fluid generators from the the incoherent kinetic generators. This is done to obtain as simple presentation as possible. Then we introduce what we call interaction physical coordinates in which the Poisson bracket is noncanonical, but still fixed by reference electromagnetic fields. These coordinates are essential since otherwise we would have to perturb the brackets also. Then we introduce kinetic fluid generators which is related to the above, but which can be interpreted as near identity transformations and moreover coincide with the fluid generators when we integrate the phase space density over momentum.

\[1\] This notation is however not quite precise since for an experimental situation where it is reasonable to define and measure also the stress tensor as a fluid variable, one would try to separate out a fluid equation also for this variable by an additional transformation with a generator which is second order in momentum. Similar developments and introduction of a priori information can in principle be done to any order in momentum and thus put tighter and tighter constraints on the class of experiments and measurements which can be described by the corresponding theories. The resonant particle effects will for such theories be described by higher order generators (in momentum) in interaction with the the corresponding fluid variables.
2 Parameterization of the Vlasov phase space density

For a given reference distribution $f^0$, it turns out that the accessible part of the space of distributions can be traced by canonical displacements, i.e. $\frac{\partial f}{\partial \epsilon} = \{S, f\}$, where $S$ is any hamiltonian (e.g. Larsson). These are the only allowable displacements in a collisionless plasma. Another way to parameterize such displacements are by near identity transformations (see the references).

$$f = \exp(\mathcal{L}_w)f^0,$$

$$\mathcal{L}_w \equiv \{w, \cdot\}.$$

As we will see a more general way of expressing the above result is in terms of the action of Poisson preserving maps $f = \psi \cdot f^0$. (The notation is explained in Appendix A and in the text below.)

In the litterature one has often considered this parameterization of solutions of the Vlasov equation to be a result of the canonical transformations resulting from the underlying particle orbits. A different, may be more natural point of view, is to consider the above parameterization as a result of that the Vlasov equation is a Lie equation\(^5,6\) having an infinite dimensional symmetry group preserving density on extended phase space. In App. A we have elaborated on this point of view.

The Vlasov equation in canonical coordinates has the form

$$\frac{\partial f}{\partial t} + \{f, H\} = 0. \quad (1)$$

This equation simply expresses conservation of phase space density, $\omega(t) = f(z, t)d^6z$, in a Hamiltonian flow. In this report we will basically use canonical phase space coordinates to derive our hybrid fluid kinetic theory since this leads to that the brackets are not perturbed and the Jacobian is unity for canonical transformations. This approach give simple derivations, but leads to no loss of generality. We will show how to apply the method in physical euclidean coordinates\(^2\) also by introducing a fixed Poisson bracket in physical coordinates which is not perturbed (see App. A and below). In effect,
if one restricts to canonical coordinates it is possible to use the canonical
distribution function instead of the density volumeform as the basic entity.
We demonstrate in App. A that eq. (1) is equivalent to that distribution
function is parameterized by canonical transformations on the phase space
with respect to a reference solution of the Vlasov equation

\[ f = \psi^{-1*} f^0 \equiv f^0 \circ \psi^{-1}. \] (2)

Here \( \psi \) is a canonical (i.e. Poisson bracket preserving) transformation of the
phase space \( P \), i.e. \( \psi^{-1*}\{g, h\} = \{\psi^{-1*}g, \psi^{-1*h}\} \). The infinitesimal version
eq. (3)

\[ f,_{t} = \{\psi_{t}, f\} + \psi^{-1*}(f^0,_{t}) = \{\psi_{t} + \psi^{-1*}H_{0}, f\} \] (3)

Here we have assumed that the distribution function and that the canonical
transformation depend parameterically on \( t \), i.e. \( f(t) \) and \( \psi(t) \). If we in
addition assume that they depend on one (or several) additional parameter
\( \epsilon \), i.e. \( f(t, \epsilon) \) and \( \psi(t, \epsilon) \), we can also define a hamiltonian generator, \( \psi_{\epsilon} \),
with respect to \( \epsilon \) as

\[ f,_{\epsilon} = \{\psi_{\epsilon}, f\} \] (4)

We could think of this additional parameter as a formal perturbation
parameter which vary say between 0 to 1 corresponding to \( f^0, H_0 \) and \( f(t), H \)
respectively. It could, however, have other interpretations (e.g. describing a
one parameter symmetry). In the case that the transformation is composed
of two canonical transformations we have

\[ \psi = \tilde{\psi} \circ \tilde{\psi}, \]
\[ \psi^{*} = (\tilde{\psi} \circ \tilde{\psi})^{-1*} = \tilde{\psi}^{-1*} \circ \tilde{\psi}^{-1*}, \]
\[ f = \tilde{\psi}^{-1*} \circ \tilde{\psi}^{-1*} f^0. \] (5)

For composite transformations it is realized that the generators has to
obey the following rule since they are derived from derivatives with respect
why is that the above physical division of fluctuations are related to Hodge decomposition
which transforms in a nontrivial way with respect to diffeomorphisms. We will return to
this invariant presentation of the fluid and electromagnetic theory in coming papers.
to parameters (see App. A)

\[
\psi_t = \psi_t + \psi_t^{-1} \psi_t^\gamma, \quad (6)
\]

\[
\psi_\epsilon = \psi_\epsilon + \psi_\epsilon^{-1} \psi_\epsilon^\gamma.
\]

We also know from the assumption that we can interchange the \(\epsilon, t\) derivatives for the distribution function that the following Maurer-Cartan relation must hold [App. A]

\[
f,_{t\epsilon} = f,_{\epsilon t},
\]

\[
\psi_{t,\epsilon} - \psi_{\epsilon,t} + \{\psi_t, \psi_\epsilon\} + k_{t\epsilon} = 0,
\]

\[
k_{t\epsilon} = \psi_{\epsilon}^{-1} k_{t\epsilon}^0,
\]

\[
\{k_{t\epsilon}, f\} = 0, \{k_{t\epsilon}^0, f^0\} = 0.
\]

Here we obviously can extend our notation by treating the phase space coordinates \(z_i, i = 1, \ldots, 6\) as parameters and define the hamiltonian generators \(\psi_i,^\epsilon\) (Similarly one can generalize to generators for other parameters like the noncanonical guiding center coordinates or oscillation center coordinates. See a brief discussion of presentation in other coordinates in App. B).

\[
f,_{i\epsilon} = \{\psi_i, f\} + \psi_i^{-1} f,^{0, i}, \quad i = 1, \ldots, 6.
\]

The compatibility condition or the Maurer-Cartan relation then takes the form for the seven coordinates \((z, t)\) (here we don’t use the \(\epsilon\) coordinate)

\[
\psi_{i,j} - \psi_{j,i} + \{\psi_i, \psi_j\} + k_{ij} = 0, \quad i, j = 1, \ldots, 7,
\]

\[
k_{ij} = \psi_{i}^{-1} k_{ij}^{0}, \quad k_{ij} = -k_{ji},
\]

\[
\{k_{ij}, f\} = 0, \{k_{ij}^{0}, f^0\} = 0.
\]

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Notice that the \(t\) and \(\epsilon\) index in e.g. \(\psi_t\) and \(\psi_\epsilon\) are not derivatives. Rather these are the hamiltonian generators corresponding to the hamiltonian vectorfields \(\psi_t\) and \(\psi_\epsilon\) defined in App. A. Derivatives will be distinguished from an index by a comma or by explicit derivation symbols.

Our definition of hamiltonian generators with respect to different parameter variations reflects the fact that our parameterization of the Vlasov equation are Poisson preserving maps. Only in the case that the evolution of the background distribution is hamiltonian or independent with respect to the parameter in question. Hamiltonian evolution will therefore be realized for time and perturbation parameters which we already have observed. We also realize that in case the background distribution has an ignorable coordinate- i.e. a symmetry, the evolution with respect to this coordinate of the phase space density will be hamiltonian.
The Hamiltonian generators and the Maurer-Cartan relation can only be understood invariantly in the language of forms. We will not describe this more general formalism here.

3 Hybrid fluid-kinetic theory

In an earlier paper\(^4\) we have elaborated on the formal expansions of the distribution function with respect to near identity symplectomorphisms parameterized by exponential maps. We purposely did not use the term pull back map in that paper, but all the results can be verified by interchanging the exponential map with the pull back map\(^5\). In this work we found the following equation for the the Hamiltonian generator in the time direction and consequently the compatibility condition if we have a additional parameter \(\epsilon\)

\[
\psi_t = H - \psi^{-1*}H_0, \quad (9)
\]

\[
\psi_{\epsilon,t} + \{\psi_{\epsilon},H\} - H_{\epsilon} = 0
\]

Let us now use our formalism for composite transformations to try to develop interacting equations for fluid and kinetic degrees of freedom. We now assume that our distribution are described by

\[
f = \overline{\psi}^{-1*}\tilde{f} = \overline{\psi}^{-1*} \circ \overline{\psi}^{-1*}\tilde{f}^0, \quad (10)
\]

\[
f^0 = (\overline{\psi}^0)^{-1*}\tilde{f}^0. \quad (11)
\]

Here \(\overline{\psi}\) and \(\tilde{\psi}\) is supposed to contain the fluid degrees of freedom (i.e. mass density and momentum) and the 'incoherent' kinetic degrees of freedom respectively. \(\overline{\psi}_0\) is the map \(\overline{\psi}\) in the reference state. Notice that the order of the composition of the fluctuating and the averaged transformations are in the opposite order than usually used in passive coordinate transformations.

\(^{5}\)In ref.4 we used partial integration in the variational functionals to invert the action of the exponential symplectic transformations from one object to another. This trick cannot be done with generic pull back maps instead of exponential symplectic maps, but the end results are still valid. The reason is that one obtain variational equivalent functionals after doing partial integration with respect to the exponential symplectic transformations. We define variational equivalent functionals to mean all functionals which gives the same result after variation. The trick of partial integration can still be performed after performing variations (see App. B).
where one want to define hypothetical averaged coordinates. (c.f. Flå where we briefly discussed the opposite ordering.) The reason for our choice is our goal of separating out the fluid generators from the resonant distribution \( \tilde{f} \) and obtain a new Liouville equation for this distribution. With another goal in mind other choices could very well be preferrable. To avoid additional tranformations due to the perturbations of the generators this is more suitable.

In App. A we have discussed a similar formalism as the above for ideal fluid theory and time dependent volume preserving transformations on space-time which fixes time (see discussion in App. A). In this case one has to use vectorfield generators \( \psi_t \) and \( \psi_\epsilon \). The parameterisation of the mass density and the constraint equation for the fluid generator given reference density and velocity are

\[
\hat{u} = \psi_t + \psi_* u_0 ,
\]

\[
\rho_t = -\nabla \cdot (\hat{u} \rho) ,
\]

\[
\rho_\epsilon = -\nabla \cdot (\psi_\epsilon \rho) ,
\]

\[
\delta \rho = -\nabla \cdot (\delta \psi \rho) ,
\]

\[
\psi_t = \frac{\partial \psi}{\partial t} \circ \psi^{-1} , \quad \psi_\epsilon = \frac{\partial \psi}{\partial \epsilon} \circ \psi^{-1} ,
\]

\[
\delta \tilde{\psi} = \delta \psi \circ \psi^{-1} .
\]

The parameterized velocity field \( \hat{u} \) give a constraint equation for the above diffeomorphisms when we give the velocity field \( u = \hat{u} \), e.g. as in our case from the momentum equation. Here the invariant object is not mass density, but the density form \( \omega = \rho d^3 x dt \) which the pull back map acts properly on. The volume density preserving pull back maps can also be presented as an action directly on the density by taking into account the Jacobian of the mapping. We introduce the symbol \( \psi \bullet \) for this action which also can be parameterized by a volume density preserving near identity transformation in the following way (see App.A)

\[
\rho(x, t) = \psi \bullet \rho^0(x, t) \equiv \rho^0(\psi^{-1}(x, t), t) J .
\]

In the case of near identity transformations one can further express the action in terms of the near identity generator \( w \) as

\[
\psi \bullet (\cdot) \equiv \exp(-\nabla \cdot (w \cdot))(\cdot) ,
\]
\[ \psi_\ast = \exp(ad(w)), \quad (18) \]
\[ \psi = \exp(w). \]

Here \( ad(w) \equiv [\cdot, w] \) in terms of the standard bracket for vectorfields (see definition in App. A). The proof of these expressions follows from using \( \psi = \exp(w) \) in the above definitions of the actions of diffeomorphisms on densities and vectorfields and compare it with the action of the above operators. Notice, that the fundamental operators in the exponential is Lie derivatives of the corresponding object with respect to the near identity generator vectorfield. Further, we can express the vectorfield generators in terms of near identity generators as (see the definition in App.A)

\[ \psi_t = (\psi_\ast - Id) \frac{\partial}{\partial t} = i \exp(ad(w)) \frac{\partial w}{\partial t}, \quad (19) \]
\[ \psi_\epsilon = (\psi_\ast - Id) \frac{\partial}{\partial \epsilon} = i \exp(ad(w)) \frac{\partial w}{\partial \epsilon}, \]
\[ \delta \bar{\psi} = i \exp(ad(w)) \delta w, \quad (20) \]
\[ i \exp(x) \equiv \frac{\exp(x) - 1}{x}. \quad (21) \]

If we use the identities established in App.A to transform from the above type of expressions on phase space and Hamiltonian vectorfields to Hamiltonian near identity generators, \( w \), we obtain that

\[ \psi = \exp(X_w), \quad (22) \]
\[ \psi_\ast = \exp(-L(X_w)), \]
\[ \psi_t = i \exp(L_w) \frac{\partial w}{\partial t}, \]
\[ \psi_\epsilon = i \exp(L_w) \frac{\partial w}{\partial \epsilon}, \]
\[ \delta \psi = i \exp(L_w) \delta w, \quad (23) \]
\[ \{w, \cdot\} \equiv L_w. \]

Here, the Lie derivative \( L(X_w) = -L_w \); act as the operator \( X_w = -L_w \) on functions, but act of course differently on other objects. The above interpretation corresponds exactly to the point of view we proposed at an earlier stage in Flå^4.
The above formal expansions in terms of near identity generators can be thought of to correspond to Larssons perturbation expansion when an ordering is given to operators (see App.C and Larsson\textsuperscript{1,2}). However, because of our group composition concept, we have considerable freedom when it comes to modelling of specific physical processes. At all steps in our theory it will be possible to specialize to near identity generators, but we will not stress this below.

We can also think of the above maps as a family of diffeomorphisms on space parameterized by time. Here $J$ is the Jacobian of the transformation. The negative sign in the near identity transformation is used to obtain an adequate sign in the continuity equation. We used the same reason for positive sign in the parametrization of the phase space density which corresponds to negative sign in the corresponding Hamiltonian vectorfield. The composition of the above maps is similar as for symplectic pull back maps. The compatibility condition with respect to an additional parameter $\epsilon$, is given by (see App. A)

$$
\psi_{\epsilon,t} - \psi_{t,\epsilon} - [\psi_{\epsilon}, \psi_{t}] + k_{\epsilon t} = 0 ,
\nabla \cdot (k_{\epsilon t} \rho) = 0, \nabla \cdot (k_{\epsilon t}^0 \rho^0) = 0 ,
k_{\epsilon t} = \psi_{t} k_{\epsilon t}^0 .
$$

(24)

Since the velocity field can be parameterized as in eq. (12), we obtain that the compatibility relation can also be written as

$$
\psi_{\epsilon,t} - \hat{u}_{\epsilon} - [\psi_{\epsilon}, \hat{u}] = 0 \mod(k_{\epsilon t}) .
$$

(25)

It turns out that it is possible to lift the fluid generating maps to kinetic theory and consider them as a subgroup of kinetic,canonical transformations through the definition of the following canonical generators \textsuperscript{6}

$$
\overline{\psi}_1(z, t) = \overline{\psi}_1(z, t) + \overline{\psi}_2(x, t) = p \cdot \hat{u} + \overline{\psi}_2 ,
\n\overline{\psi}_{2t} = -m \frac{\hat{u}^2}{2} - \frac{e}{c} A \cdot \hat{u} ,
$$

(26)

(27)

\textsuperscript{6}These symplectic transformations cannot be defined through near identity transformations since the resonant particle distribution will be following fluid orbits. This is done for mathematical convenience of the separation procedure. Later on when we linearize the hybrid fluid kinetic equations, we will relate these symplectic transformations to the fluid generators with respect to a reference state.
Note that we have assumed that the generator \( \psi_{2a} \), \( a = t, \epsilon \) depend only on \( \mathbf{x}, t \). With the above definitions one easily convince oneself that the compatibility conditions for the barred symplectic transformation, \( \bar{\psi}_{1a}, a = t, \epsilon \) is consistent with the fluid compatibility condition in inner product with \( \mathbf{p} \).

The explicit form of the above kinetic fluid generating maps can be described in terms of cotangent lift \( \bar{\Psi}(t)(\mathbf{v}) \equiv T^*\Phi(t) \) in composition with the fibertranslation by an exact form described below. In Abraham and Marsden it is proven that the cotangent lift is a symplectic map which preserves the canonical oneform \( \theta \) on phase space \( T^*M \) \( (= \mathbf{p} \cdot d\mathbf{x} \) in euclidean canonical coordinates). For a point \( \alpha_q \in T^*M = \mathbf{P} \) and \( \mathbf{v} \in T_{\Phi(t)^{-1}(q)}M \) one find that \( T^*\Phi(t)\alpha_q(\mathbf{v}) = \alpha_q(T\Phi(t)\mathbf{v}) \). Define the coordinate functions on \( T^*M \) by \( \pi_\mathbf{P} : \alpha_q \rightarrow \pi_\mathbf{P}(\alpha_q) = z = (\mathbf{x}, \mathbf{p}) \). The action on coordinate functions is therefore \( (T^*\Phi(t))^*\pi_\mathbf{P} \) which for a euclidean metric means that the action of the cotangent lift is \( \bar{\Psi}(t)(z) \equiv \bar{Z} = T^*\Phi(t)(z) = (\Phi(t)^{-1}(\mathbf{x}), (\nabla\Phi(t)(\mathbf{x})) \cdot \mathbf{p}) \).

A simple way to find invariant properties of the above map is to use that the canonical oneform is preserved under cotangent lift, i.e. (here we treat \( p_i, x^i \) as coordinate functions)

\[
\theta = \bar{\Psi}(t)^{-1}\theta,
\]

\[
\Rightarrow p_j dx^j = \bar{P}_j d\bar{\Psi}(t)^{-1}x^j,
\]

\[
\bar{P}_j = \bar{\Psi}(t)^{-1}p_j = ((\nabla\Phi(t)(\mathbf{x}) \cdot \mathbf{p})_j).
\]

Another map which will interest us is fibertranslations by oneforms \( \mathbf{A}^{(1)}(t) \) on \( M \), defined by \( \psi^{-1}_{\mathbf{A}}(t)\theta = \theta - \pi_M^*\mathbf{A}^{(1)}(t) \) where \( \pi_M : T^*M \rightarrow M \). In this case the map is only preserving the symplectic tensor \( \omega = -\frac{d\theta}{\Theta} \) if the oneform \( \mathbf{A}^{(1)}(t) \) is exact since \( \psi^{-1}_{\mathbf{A}}(t)\omega = \omega + \pi_M^*d\mathbf{A}^{(1)}(t) \). In euclidean canonical coordinates the fibertranslations correspond to the map \( \psi_{\mathbf{A}} : (\mathbf{x}, \mathbf{p}) \rightarrow (\mathbf{x}, \mathbf{p} - \mathbf{A}(\mathbf{x}, t)) \) which up to a numerical factor is the transformation to euclidean physical coordinates. The fibertranslation give rise to a generator vectorfield on phase space which can be described as \( \mathbf{X}_{\mathbf{A}}^t = -\frac{\partial \mathbf{A}}{\partial t} \) in euclidean coordinates, but has an invariant description given below (put the fourth component of the four oneform \( \mathbf{A}^{(1)} \) to zero and specialize to Euclidean coordinates in the invariant description).

If we extend the canonical oneform to a oneform \( \Theta \) on extended phase space \( T^*X \), \( X = M \times \mathbb{R} \), \( \Theta |_{T^*M} \equiv \theta \), the extended map \( \psi \) which fixes time will generate a component in the time direction.
Theorem 1 \[ \psi^{-1*}\Theta = \Theta + \hat{d}S + J_tdt = \Theta + dS + \psi_tdt, \]

\[
\begin{align*}
  i_{\partial_t}(\psi^{-1*}\Omega) & = -i_{\chi_{\psi_t}}(\psi^{-1*}\Omega) \equiv \hat{d}\psi_t \\
  \psi^{-1*}\Omega & = \Omega - \hat{d}\psi_t \wedge dt, \quad \Omega \equiv -d\Theta, \\
  i_{\partial_t}(\psi^{-1*}\Theta) & = -i_{\chi_{\psi_t}}(\psi^{-1*}\Theta) \equiv J_t \\
  \frac{\partial S}{\partial t} & = J_t - \psi_t.
\end{align*}
\]

Proof. The proof of the above lemma is simply by observing that

\[
i_{\partial_t}(\psi^{-1*}\Omega) = -i_{\chi_{\psi_t}}(\psi^{-1*}\Omega) = 0 \text{ since the extended oneform by definition has no time components. Similarly, the above relation between the generators and the gauge-fields is a consistency requirement coming from} \quad L(\frac{\partial}{\partial t})(\psi^{-1*}\Omega) = -L(\chi_{\psi_t})(\psi^{-1*}\Omega) = -d(\hat{d}\psi_t) = -L(\frac{\partial}{\partial t})\hat{d}\psi_t \wedge dt.
\]

In coordinates \[ J_t = p \cdot \frac{\partial}{\partial p}. \]

The above extension of the canonical one and twoforms to extended phase space, give an opportunity to consider the action of noncanonical transformations on these forms. Therefore we can e.g. consider transformations generated by oneforms on extended space \[ X. \]

\[
\Theta_A \equiv \psi^{-1*}_A\Theta = \Theta - \pi_X^* A^{(1)},
\]

\[
\Omega_A \equiv \psi^{-1*}_A\Omega = \Omega - \pi_X^* dA^{(1)}.
\]

The generating vectorfield corresponding to the above generalized fibertranslaiton is given by \[ X_t^A = -J_A \cdot \pi_X^*(Y^{(1)}) \], \[ Y^{(1)} \equiv i_{\theta}(dA^{(1)}) \] where \[ J_A = \psi^*_A J \]

and \[ J \] is the Poisson tensors corresponding to \[ \omega_A = \Omega_A \vert_{T^*M} \] and \[ \omega = \Omega \vert_{T^*M}. \]

In the special case of translations by an exact form \[ A^{(1)} = df \] (here \[ f \] is a function on space/time not phase space,we find that the generating vectorfield is hamiltonian and the transformation trivially symplectic, i.e. \[ X_t^{df} = -J_t(J_t df) \]. In fact, a large class of transformations which is not necessarily pre-

serving the symplectic tensor, but is still divergencefree, is generated as

\[ \begin{align*}
  \bar{J}_t & = p \cdot \frac{\partial}{\partial p} = \bar{\psi}_t = p \cdot \phi_t, \quad \phi_t \equiv (\phi - Id) \frac{\partial}{\partial t}.
\end{align*} \]

We can consider our results for \[ J_t \] to be an extension to infinite dimensional pseudogroups of the momentum map for finitely generated groups and Banach Lie groups (see e.g. Abraham and Marsden). Earlier results has been directed towards reduction while we concentrate our efforts towards the composition principle for in principle infinitely generated groups. Similar results with respect to reduced actions also holds for pseudogroups, but we will not study it here.
\[ X^a_t = J_a \cdot (A^a), \quad J_a = \psi_a \cdot J, \quad \Omega_a = \psi^{-1}_a \Omega \quad \text{and} \quad A^a \text{ is some oneform given on } T^*M. \]

It is possible to extend the above theorem to a more general setting based on noncanonical twoforms \( \Omega_n \equiv \psi^{-1}_n \Omega \) and oneforms \( \Theta_n \equiv \psi^{-1}_n \Theta \) by introducing the convective derivative \( \frac{d}{dt} \equiv \frac{\partial}{\partial t} + X_t^a \cdot X_t^b \equiv (\psi_{ns} - Id) \frac{\partial}{\partial t} \).

**Theorem 2** \( (\psi^n)^{-1} \Theta^n = \Theta^n + \hat{d}S^n + J^n \cdot dt, \)

\[
(\psi^n)^{-1} \Omega^n = \Omega^n - \hat{d}(\psi^n - \frac{dS^n}{d\tau}) \wedge dt, \quad \Omega^n \equiv -d\Theta^n, \\
i_d(\Omega^n)^{-1} = -i_{X^n}(\psi^n)^{-1} \Theta^n \equiv J^n, \\
\frac{\partial S^n}{d\tau} = J^n - \psi^n, \\
\psi^n \equiv \psi_n \circ \psi^{-1}_n, \quad X^n = (\psi_{ns} - Id) \frac{d}{d\tau}, \quad S^n \equiv \psi^{-1}s. 
\]

We are now in a position to explicitly describe the above fluid kinetic generators in canonical, euclidean coordinates as the composition of the following cotangent lift and translation by an exact form \( df \), i.e. \( \tilde{\psi} = \tilde{\Psi} \circ \psi_{df} \) with generator \( \tilde{\psi}_t = \tilde{\Psi}_t + \tilde{\Psi}^{-1} f_{,t} = \phi_t \cdot \hat{p} + f_t \), \( f_t \equiv \Psi^{-1} f_{,t} \). Here we must choose \( \phi_t \equiv \hat{u} = \tilde{\psi}_t + \psi_{,u_0} \) which for \( u_0 \equiv \psi_{0t} \) defines the unperturbed \( \phi_0 \) such that \( \phi = \psi \circ \psi_0 \). The gaugefunction has to be chosen as \( f_t = \tilde{\psi}_{2t} \) to agree with the above. Since we have assumed \( u_0 \) to be unperturbed, we find that \( \tilde{\psi}_t = \psi_t \cdot \hat{p} + f_t \), \( f_t \equiv \Psi^{-1} f_{,t} \).

We now have the following theorem for the action of fluid kinetic maps in canonical coordinates defined as a composition between a cotangent lift and translation by an exact form

**Theorem 3** Let \( f = \tilde{\psi}^{-1} \tilde{f} \) such that \( \tilde{\rho}_0 = \int \tilde{f} \hat{d}^3 \hat{p} \) and \( \rho = \int f \hat{d}^3 \hat{p} \). Then one has that

\[ \rho = \phi \cdot \tilde{\rho}_0 = \psi \cdot \rho_0, \quad \rho_0 \equiv \psi_0 \cdot \tilde{\rho}_0. \]

**Proof.** The proof follows from that \( \rho = \int \tilde{f} \tilde{\psi}^* \hat{d}^3 \hat{p} = \int \tilde{f} \tilde{\psi} \cdot \hat{d}^3 \hat{p} = J(\phi^{-1}) \int \tilde{f} \hat{d}^3 \hat{p} \equiv J(\phi^{-1}) \phi^{-1} \hat{\rho}_0 = \phi \cdot \tilde{\rho}_0 = \psi \cdot \rho_0 \cdot \tilde{\rho}_0 = \psi \cdot \rho_0. \]

With the above relation between the Hamiltonian generator and \( \tilde{\psi}_t \) and macroscopic fields we obtain the reduced Hamiltonian which can be identified with the Hamiltonian in the fluid reference frame. We also find a reduced Vlasov equation for the fictive phase space density \( \tilde{f} \)

\[
\tilde{H} = H - \tilde{\psi}_t = (\hat{p} - m \hat{u})^2 / 2m \\
\tilde{p} = \hat{p} - \frac{e}{c} A ,
\]
\[
\frac{\partial \tilde{f}}{\partial t} + \{\tilde{f}, \tilde{H}\} = 0,
\]
\[
\tilde{H} \equiv \tilde{\psi}^* \tilde{H}.
\]
(30)

The motivation for the special choice of \(\psi_2(x,t)\) is to obtain a reduced Hamiltonian of the above form corresponding to the fluid reference frame.

We will now discuss the physical consequences of the compatibility conditions for the distribution function and mass density. The fluid density is given by \(\rho = \int f d^3p\) such that the relation between the fluid and kinetic generators becomes

\[
\rho_{,t} = \int (\{\tilde{\psi}_t, f\} + \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial t}) d^3p = -\nabla \cdot (\tilde{\psi} \rho) + \int \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial t} d^3p ,
\]
(31)

\[
\rho_{,\epsilon} = \int (\{\tilde{\psi}_t, f\} + \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial \epsilon}) d^3p = -\nabla \cdot (\tilde{\psi} \epsilon \rho) + \int \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial \epsilon} d^3p .
\]

Here we have used the fact that \(\int \{\phi(x, t), f(z, t)\} d^3p = 0\) with suitable decay properties on \(f\) for large momenta. We immediately conclude by comparing with the above fluid theory that the following additional restrictions on the hypothetical density \(\tilde{f}\) has to be imposed if the barred symplectic generators should correspond to a fluid subgroup with respect to its moments

\[
\int \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial t} d^3p = 0 ,
\]
(32)

\[
\int \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial \epsilon} d^3p = 0.
\]

It is now obvious that we could add any additional term \(\phi_t(x, t)\) to the generator \(\tilde{\psi}_t\) without changing the form of eq.(31). The trick of using the above form of barred generators is however that it is possible to obey eq. (32) easily.

We find the following theorem

**Theorem 4** With the earlier definitions one obtains \(\int \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial t} d^3p = 0\). Moreover, the definition of the fluid density from the composed fluid density is compatible with fluid theory if we select \(\tilde{f}\) such that \(\int \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial \epsilon} d^3p = 0\). If we select the reduced distribution function such that \(\int \tilde{\psi}^{-1*} \frac{\partial \tilde{f}}{\partial \epsilon} d^3p \mid_{t=0} = 0\) and \(\bar{\rho}(x, t) \mid_{t=0} = \rho_0(x)\), it will also be an identity at any other time.
**Proof.** We need the following lemma

**Lemma 5** For symplectomorphisms, $\psi$, with generators which are not more than linear in momentum the following applies for a phase space density $g$ with suitable decay properties for large momentas and $\int \psi^{-1*} \frac{\partial g}{\partial t} = 0$

\[ \hat{g} = \int \psi^{-1*} g d^3p = \hat{\psi} \bullet \hat{g}, \] (33)

\[ \bar{g} = \int gd^3p. \]

**Proof.** In the case of near identity transformations the lemma can be proved by using near identity generators which are linear in momentum. Since it follows that the deviation from identity is also linear in momentum, one find the above relation after integration over momentum. If one does not assume near identity mappings the proof is a little more involved. The infinitesimal version of the above equation is since the generators are supposed to linear in momentum and with suitable decay properties of $g$ in momentum such that the following holds

\[ \hat{g}(t) = \int \{\psi_t, \psi^{-1*}(t)g\} d^3p + \int \psi^{-1*} \frac{\partial g}{\partial t} d^3p \]

\[ = -\nabla \cdot (\int \hat{u} \psi^{-1*}(t) g d^3p) + \int \psi^{-1*} \frac{\partial g}{\partial t} d^3p, \]

\[ \hat{g}(t) = -\nabla \cdot (\hat{u} \hat{g}(t)). \]

Therefore if $\int \psi^{-1*} \frac{\partial g}{\partial t} = 0$, we can parameterize $\hat{g}(t) = \hat{\psi}(t) \circ \hat{g}$. With this parameterization it also follows that $\hat{\psi} \circ \frac{\partial \hat{g}}{\partial t} = 0$ from which we find that $\frac{\partial \hat{g}}{\partial t} = 0$ since $\hat{\psi}$ is invertible and takes zero to zero.

**Remark 1** The above parameterization is performed with respect to a reference $\hat{g}$ such that $\frac{\partial \hat{g}}{\partial t} = 0$. We can change this reference by doing an active transformation (or alternatively a passive coordinate transformation) $\hat{g} = (\hat{\psi}_0^{-1}) \circ \hat{g}_0$ such that $\frac{\partial \hat{g}}{\partial t} + \nabla \cdot (u_0 \hat{g}_0) = 0$. Now it is possible to define transformations $\psi = \hat{\psi} \circ \hat{\psi}_0^{-1}$ which is more suitable for near identity transformations with respect to reference state. This is what we will eventually do in section 3.2.
Using the above lemma we immediately find that
\[ \int \tilde{\psi}^{-1} \tilde{f} a d^3 p = \tilde{\psi} \bullet \tilde{\rho} a, \quad a = t, \epsilon, \] (34)
\[ \int \tilde{\psi}^{-1} \tilde{f} d^3 p = \tilde{\psi} \bullet \tilde{\rho}. \]

Now we use the reduced Vlasov equation and find
\[ \int \tilde{\psi}^{-1} \tilde{f} a d^3 p = \tilde{\psi} \bullet \tilde{\rho}, \] \(a, \epsilon = t, \epsilon\), (35)
\[ \int \tilde{\psi}^{-1} \frac{\partial \tilde{f}}{\partial t} d^3 p = \int \{\tilde{H}, f\} d^3 p. \]

By partial integration in momentum we find that
\[ \int \tilde{\psi}^{-1} \frac{\partial \tilde{f}}{\partial t} d^3 p = -\nabla \cdot \left( \int \frac{\partial \tilde{H}}{\partial p} f d^3 p \right) \equiv 0 \] (35)
since by definition \(\int (\tilde{p} - m \tilde{u}) f d^3 p = \rho (u - \tilde{u}) = 0\). (I.e., we naturally will restrict the parametrization of the velocity to be equal to the velocity moment of the distribution. In fact this restriction is the defining equation for the volume density preserving transformation.)

The compatibility condition for the mass density parameterized by the above composition of symplectic transformations acting on phase space density leads by eq.’s (31, 32) to the constraint
\[ \frac{\partial^2 \rho}{\partial t \partial \epsilon} = \frac{\partial^2 \rho}{\partial \epsilon \partial t}, \]
\[ \Rightarrow \frac{\partial}{\partial t} \left( \int \tilde{\psi}^{-1} \frac{\partial \tilde{f}}{\partial \epsilon} d^3 p \right) = 0. \] (36)

Therefore we conclude that \(\int \tilde{\psi}^{-1} \frac{\partial \tilde{f}}{\partial t} d^3 p = \psi \bullet \int \frac{\partial \tilde{f}}{\partial \epsilon} d^3 p\) is constant with respect to time. Consequently if the \(\epsilon\) parameterization is such that the \(\int \tilde{\psi}^{-1} \frac{\partial \tilde{f}}{\partial t} d^3 p\) \(|\epsilon_t = 0\) = 0, it will continue to be so at all times. We assume that both the transformations \(\psi^{-1}\) and \(\psi\) exists. Since the action of the symplectomorphisms is such that it preserves volume density forms, it is realized that the zero density must be transported to zero density by the action of all symplectomorphisms. It follows that \(\int \frac{\partial \tilde{f}}{\partial \epsilon} d^3 p = \frac{\partial \rho}{\partial \epsilon} = 0\) if it is initially chosen in such a way. Moreover, it is implied that \(\tilde{\rho} = \rho_0\) is given by the reference density independent of \(\epsilon\) even if \(\tilde{f}\) is depending on \(\epsilon\). By a similar argument
we deduce that $\frac{\partial \tilde{\rho}}{\partial t} = 0$, and consequently corresponds to a spatial reference density given in the fluid frame of reference.

The interpretation of the above result is that to obtain a composition symplectomorphism where the volume density preserving transformation is described as a subgroup of the group of all symplectomorphisms lead to that density perturbations are parameterized in phase space by the fluid subgroup consisting of the barred symplectomorphisms.

Let us now study the continuity and momentum equation more explicitly. The Vlasov equation can be written

$$\frac{\partial f}{\partial t} + \{ f, -\tilde{\psi} + \hat{H} \} = 0$$

$$H = -\tilde{\psi} + \hat{H}.$$  

The zeroth order moment integrated over momentum space now gives the momentum equation with no contribution from the $\{ f, \hat{H} \}$ term. The momentum equation can now be found as

$$\frac{\partial}{\partial t} \int p f d^3p + \int p(-\{\tilde{\psi}, f\} + \{\hat{H}, f\}) d^3p = 0,$$

(37)

$$\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho \hat{u} + \mathbb{P}) - \rho \mathbf{f}_L = 0,$$

$$\mathbf{f}_L = \frac{e}{m} (E + \hat{u} \times B),$$

$$\mathbb{P} \equiv \int \frac{1}{m} (p_p - m\hat{u})(p_p - m\hat{u}) f d^3p.$$  

(38)

Here $\mathbf{f}_L$ and $\mathbb{P}$ are the Lorentz force and the stress tensor of the fluid and the physical momentum is related to the canonical momentum by $p_p = p - \frac{e}{m} A$. The reduced Vlasov equation is given in eq.(29) and form together with the continuity equation and the above momentum equation a new set of equations for collisionless plasma physics, the hybrid fluid-kinetic theory.

3.1 Parameterization of the hybrid fluid-kinetic theory

Let us briefly discuss the parameterization of the continuity equation and the reduced Vlasov equation. We recall that fluid generating vector is related to the parameterized velocity by $\hat{u} = \psi_t + \psi_0 u_0$. We purposely have
been operating with the parameterized velocity $\hat{u}$ generated by the diffeomorphisms different from the velocity vector derived from the moment of the distribution. The reason why is that they are not a priori equal. Indeed, the equality of these two quantities is the constraint equation which together with the above fluid momentum and reduced Vlasov equations determines diffeomorphism

$$u = \hat{u}.$$  \hspace{1cm} (39)

This equation replaces the continuity equation since the density can immediately be mapped the moment we have e.g. the near identity representation of the diffeomorphism. The reduced Vlasov equation can be parameterized in a similar way as the Vlasov equation by

$$\tilde{\psi}_t = \tilde{H} - \tilde{\psi}^{-1}\ast H_0, \hspace{1cm} (40)$$  

$$H_0 = \tilde{H}_0 + \tilde{\psi}_t.$$  

If we parameterize our diffeomorphism by the perturbation parameter $\epsilon$, we use the compatibility equation to obtain the determining equation for the generating function $\tilde{\psi}_\epsilon$

$$\tilde{\psi}_{\epsilon,t} - \{\tilde{H}, \tilde{\psi}_\epsilon\} - \tilde{H}_\epsilon = 0.$$  \hspace{1cm} (41)

Eq.’s (37-41) now form a complete set of equations as an alternative to the continuity and reduced Vlasov equation together with the momentum equation.

We now have to discuss the momentum equation more carefully to obtain an invariant description of the parameterization. To obtain such an invariant description it is useful to describe the velocity field as a oneform $u^{(1)}$ and the stress tensor as a symmetric, covariant twotensor $P^{(2)}$. It is now possible to parametrize the velocity field by using Hodge decomposition with respect to the threedimensional metric $g$ to split it in a rotational and divergent part, $u^{(1)} = -d\eta + \ast_g \tilde{d}A^{(1)} = u^{(1)}_d + u^{(1)}_r$. Here we have used the Hodge decomposition with respect to the transformed metric (see below) in agreement with the philosophy that the decomposition should be given with respect to the standard metric when pulled back to the reference level, i.e. $u^{(1)} = \psi(t)^{-1}\ast \bar{u}^{(1)}$, $\bar{u}^{(1)} = -d\bar{\eta} + \ast_{g_0} \tilde{d}\bar{A}_r^{(1)}$. It can be useful to further describe the rotational gaugefield by Pfaff decomposition as $A^{(1)}_r = \alpha d\beta + d\gamma$,
where the gaugefield does not depend on the gaugepotential $\gamma$. We find this parameterization more convenient than the standard Clebsch decomposition which does not separate into rotational and divergent parts and does not use the metric structure. To split the equation with respect to the above structure we write the momentum equation as

\[
\frac{\partial}{\partial t} + \mathcal{L}(\hat{u}))u^{(1)} = -\frac{1}{\rho} D_g \hat{P}^{(2)} + \hat{d}(\frac{1}{2} i_{\hat{u}} u^{(1)}) + f_L^{(1)},
\]

\[
D_g \hat{P}^{(2)} = \sum_j *g \hat{d} *g (\hat{P}_j^{(1)}) \hat{d}(\psi(t)^{-1} x) = \sum_j \text{div}_g(\hat{P}_j) \hat{d}(\psi(t)^{-1} x),
\]

\[
\hat{P}^{(2)} = J(t) \hat{P}^{(2)}_j \equiv \psi(t)^{-1} \hat{P}^{(1)}_j
\]

\[
\hat{P}^{(2)} = \psi(t)^{-1} \hat{P}^{(2)} = \sum_j \hat{P}_j^{(1)} \otimes \hat{d}(\psi(t)^{-1} x)
\]

\[
g = \psi(t)^{-1} g_0
\]

\[
\hat{\rho}(t) \equiv \psi(t)^{-1} \rho_0(t),
\]

\[
f_L^{(1)} = -\frac{e}{c} \hat{d},
\]

\[
\hat{u} = \psi(t)^{-1} \hat{u}_0
\]

\[
\hat{u}_0 \equiv \hat{u} + \frac{\partial}{\partial t}, \quad u_0 \equiv u + \frac{\partial}{\partial t}
\]

This equation can also be written in an invariant way as an equation for the momentum $M \equiv \omega \otimes u^{(1)}$ and/or written in terms of covariant derivatives. In addition we can embed this it in extended space by taking the wedge product with $dt$ as we did for the density form, but we prefer to defer this formulation to another paper. Here we have used that

\[
D_g(\cdot) = \frac{1}{J(t)} D_{g_0}(J(t) \cdot)
\]

to obtain an invariant form of the equations suitable for transformations by time-dependent diffeomorphisms. We need the following natural properties of the diffeomorphism action

\[
\frac{\partial}{\partial t} + \mathcal{L}(\hat{u})) \circ \psi(t)^{-1} = \psi(t)^{-1} \circ (\frac{\partial}{\partial t} + \mathcal{L}(u_0))
\]

\[
* g \circ \psi(t)^{-1} = \psi(t)^{-1} \circ * g_0,
\]

\[
\hat{d} \circ \psi(t)^{-1} = \psi(t)^{-1} \circ \hat{d}, \quad d \circ \psi^{-1} = \psi^{-1} \circ d
\]

\[
i_{\hat{u}} \circ \psi(t)^{-1} = \psi(t)^{-1} \circ i_{\hat{u}}, \quad i_{\hat{u}} \circ \psi^{-1} = \psi^{-1} \circ i_{u_0}
\]

\[
\hat{u} = \psi(t), \hat{u}_0 = \hat{u} + \frac{\partial}{\partial t}, \quad u_0 \equiv u + \frac{\partial}{\partial t}
\]

\[
D_g(\hat{P}^{(2)}) = \psi(t)^{-1} D_{g_0}(\hat{P}^{(2)}).
\]
four space which fixes time. We now observe that our momentum equation can be pulled back to the reference level where in fact the Lie derivative and the fourdimensional interior multiplication will be linear operators with respect to the background velocity field and all quantities transform in a natural way.

\[
\frac{\partial}{\partial t} + \mathcal{L}(u_0) \tilde{u}^{(1)} = -\frac{1}{\rho_0} D_{g_0} \tilde{P}^{(2)} + \frac{1}{2} i_{\tilde{u}} \tilde{u}^{(1)} + \tilde{f}_L^{(1)},
\]

where

\[
\tilde{f}_L^{(1)} = -\frac{e}{c} i_{u_0} \tilde{F}^{(2)} = -\frac{e}{c} (\tilde{E}^{(1)} + *_{g_0} (u_0^{(1)} \wedge \tilde{B}^{(1)})) ,
\]

\[
F^{(2)} = \psi^{-1} \tilde{F}^{(2)}, \quad E^{(1)} = \psi(t)^{-1} \tilde{E}^{(1)}, \quad B^{(1)} = \psi(t)^{-1} \tilde{B}^{(1)},
\]

\[
B^{(1)} = *_{g} B^{(2)} = *_{g} dA^{(1)}, \quad \tilde{B}^{(1)} = *_{g_0} \tilde{B}^{(2)},
\]

In addition after the pullback all operators and physical quantities according to the above will be specified with respect to the background metric \(g_0\). Our philosophy is that all equations and physical fields should be represented such that they can be pulled back to the reference level. The pulled back equation can now be compared with the equation for the reference solution and an equation for deformations from the reference fields can be formulated in a strikingly simple way in which many of the terms are linear. This make our theory especially attractive from a perturbation theory and complexity point of view, but we believe this also has implications for the interpretation, predictions and formulation of measurements for physical fields. E.g., according to us a linearized equation and fields pulled back to the reference level in no way is linear at the original level which even obtain an induced, nonlinear metric structure.

We now introduce the deviations at the pullback level of the physical fields from the background fields. Since the equation for the reference fields also obey eq. 48 taken at the reference metric \(g_0\), we can subtract the background equation from eq. 48. We then obtain an equation for the fluctuating quantities suitable for perturbation theory

\[
\frac{\partial}{\partial t} + \mathcal{L}(u_0) \tilde{u}^{(1)} = -\frac{1}{\rho_0} D_{g_0} \tilde{P}^{(2)} + \frac{1}{2} d(i_{\tilde{u}} \tilde{u}^{(1)} - i_{u_0} u_0^{(1)}) + \tilde{f}_L^{(1)},
\]

\[
\tilde{u}^{(1)} = u_0^{(1)} + \tilde{u}^{(1)}, \quad \tilde{P}^{(2)} = P_0^{(2)} + \tilde{P}^{(2)}, \quad \tilde{F}^{(2)} = F_0^{(2)} + \tilde{F}^{(2)},
\]

\[
\tilde{f}_L^{(1)} \equiv i_{u_0} \tilde{F}^{(2)}, \text{ e.t.c.}
\]

Notice that even if we were only interested in linearization and linear quantities at the original level, the distinction between the pulled back equations
and the equations at the original level will still be essential since the fluctuations in the pullbacked metric and the pullback map itself will not affect first order quantities, but they will affect background quantities to linear order presented at the original level.

Let us complete our discussion of the momentum equation by showing how elegant it separates into equations for the acoustic and rotational potentials. We split the right hand side of eq. (46,48) into rotational and divergent parts by using Hodge theorem to define the potentials
\begin{align}
-\hat{d}\kappa + *_{\hat{g}}\hat{d}\mathbf{R}^{(1)} &\equiv \hat{f}^{(1)}_L - \frac{1}{\rho}D_{\hat{g}}\mathbf{P}^{(2)}, \\
-\hat{d}\hat{\kappa} + *_{\hat{g}_0}\hat{d}\mathbf{R}^{(1)} &\equiv \hat{f}^{(1)}_L - \frac{1}{\rho_0}D_{\hat{g}_0}\mathbf{P}^{(2)}.
\end{align}

If we now take the exterior derivative of eq.(14), we obtain the vorticity equation
\begin{align}
\left(\frac{\partial}{\partial t} + \mathcal{L}(\hat{u})\right)\hat{d}\mathbf{u}^{(1)}_c = \hat{d} *_{\hat{g}}\hat{d}\mathbf{R}^{(1)},
\end{align}
\begin{align}
\mathbf{u}^{(1)}_c = *_{\hat{g}}\hat{d}\mathbf{A}^{(1)}_c = *_{\hat{g}}(\hat{d}\alpha \wedge \hat{d}\beta).
\end{align}

In fact, up to a potential we can even give the equation for the rotational part of the velocity oneform (relative to the fluctuation metric) or alternatively the vorticity twoform, $\mathbf{\pi}^{(2)}$ as
\begin{align}
\left(\frac{\partial}{\partial t} + \mathcal{L}(\hat{u})\right)\hat{d}\mathbf{u}^{(1)}_c = *_{\hat{g}}\hat{d}\mathbf{R}^{(1)},
\end{align}
\begin{align}
\left(\frac{\partial}{\partial t} + \mathcal{L}(\hat{u})\right)\mathbf{\pi}^{(2)} = \hat{d} *_{\hat{g}}\hat{d}\mathbf{R}^{(1)},
\end{align}
\begin{align}
\mathbf{\pi}^{(2)} = \hat{d}\mathbf{u}^{(1)}_c.
\end{align}

Moreover, if we subtract this equation from the original momentum equation, we obtain an equation for the acoustic potential
\begin{align}
\left(\frac{\partial}{\partial t} + \mathcal{L}(\hat{u})\right)\kappa = \frac{1}{2}i_{\hat{u}}\mathbf{u}^{(1)}.
\end{align}

All this equations transform in a natural way with respect to diffeomorphisms, i.e. we can obviously formulate pullbacked equations and equations for fluctuations with respect to corresponding background.
Let us now describe our diffeomorphisms more explicitly and relate them to the above potentials. We do this by studying the constraint equation for velocity vectorfield reformulated as a constraint on the momentum lifted to a oneform; \( \rho \mathbf{u}^{(1)} = \hat{\rho} \mathbf{u}^{(1)} \). The compatibility equation for the perturbed vectorfields now become a relation between oneforms(lifted by the fixed metric)

\[
(\rho \psi^{(1)}_\epsilon)_t - (\rho \mathbf{u}^{(1)})_\epsilon + *_{g_0} \hat{d}( *_{g_0} (\rho \psi^{(1)}_\epsilon \wedge \mathbf{u}^{(1)})) = 0,
\]

which can be pullbacked to

\[
(\rho_0 \hat{\psi}^{(1)}_\epsilon)_t - (\rho_0 \hat{\mathbf{u}}^{(1)})_\epsilon + *_{g_0} \hat{d}( *_{g_0} (\rho_0 \hat{\psi}^{(1)}_\epsilon \wedge \hat{\mathbf{u}}^{(1)})) = 0.
\]

If we now take the divergence of the first equation, we obtain that

\[
( *_{g_0} \hat{d} *_{g_0} (\rho \psi^{(1)}_\epsilon))_t = -\rho,_{\epsilon t} = ( *_{g_0} \hat{d} *_{g_0} (\rho \mathbf{u}^{(1)}))_\epsilon .
\]

This is in fact the perturbed, parameterized continuity equation.

We can also study the compatibility equation with respect to the oneform \( J \mathbf{u}^{(1)} = \hat{J} \mathbf{u}^{(1)} \). In this case we obtain

\[
(J \psi^{(1)}_\epsilon)_t - (J \mathbf{u}^{(1)})_\epsilon + *_{g_0} \hat{d}( *_{g_0} (J \psi^{(1)}_\epsilon \wedge \mathbf{u}^{(1)})) = 0,
\]

\[
( *_{g_0} \hat{d} *_{g_0} (J \psi^{(1)}_\epsilon))_t = -J,_{\epsilon t} = ( *_{g_0} \hat{d} *_{g_0} (J \mathbf{u}^{(1)}))_\epsilon .
\]

Here we have used the continuity equations for the Jacobian \( J_t + \nabla \cdot (J \psi_t) = 0, J,_{\epsilon t} + \nabla \cdot (J \psi_\epsilon) = 0 \) lifted up to forms. In fact it is possible to find material coordinates such that the Jacobian and the density is equal up to a constant density, i.e. \( \rho = J^t \rho^t_0 \). The parameterization of the velocity oneform if written with respect to reference metric is given by

\[
\mathbf{u}^{(1)} = -d\eta + \frac{1}{J} g \circ g_0^{-1} \circ *_{g_0} \hat{d} \hat{A}_r^{(1)}.
\]

This means that \( J \mathbf{u}^{(1)} \) will only contribute to the above divergence term in the continuity equation for the Jacobian through the acoustic potential. If we want a parameterization where the rotational part does not contribute in the divergence term of the mass continuity equation, we have to use \( J_{id}, g_{id} \). Anyhow, we pull back the velocity oneform and relate it to the generator as a oneform at the reference level and find

\[
\hat{\psi}^{(1)}_t = -\hat{d}\hat{\eta} + *_{g_0} \hat{d}\hat{A}_r^{(1)} - \mathbf{u}^{(1)}_0 = -\hat{d}\hat{\eta} + *_{g_0} \hat{d}\hat{A}_r^{(1)};
\]

\[
\hat{\psi}_t^d \equiv -\hat{d}\hat{\eta}, \hat{\psi}_t^r \equiv *_{g_0} \hat{d}\hat{A}_r^{(1)};
\]

\[
\hat{\psi}_t = \hat{\psi}_t^d + \hat{\psi}_t^r.
\]

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Here we have to choose which generator is first and last corresponding to the composition of e.g. \( \psi = \psi^r \circ \psi^d \) or vice versa.

The above decomposition into divergent and rotational generators shows us the need to close our system of equations by representing the generators according to the type chosen. Obviously, the divergent diffeomorphisms should only have one parameter related to the density structure while the rotational diffeomorphisms should have only two degrees of freedom related to the vorticity structure. We notice that if e.g. \( \psi = \exp(\epsilon \psi_1) \), the generator \( \psi_\epsilon = \psi_1 \) and similar relations can be worked out for other perturbation parameter relations. Therefore the key is to study the decomposition \( \hat{\psi}_\epsilon = \hat{\psi}_\epsilon^d + \hat{\psi}_\epsilon^r \) represented in terms the density and vorticity structure respectively. We define the density structures \( \omega = \rho dV_0, \omega_\epsilon = \rho_\epsilon dV_0, \hat{\omega}_\epsilon = \hat{\rho}_\epsilon dV_0 \equiv \psi^* \omega_\epsilon \).

The pullbacked perturbed density structure is therefore explicitly related to the density perturbation as \( \hat{\rho}_\epsilon = \psi^* \left( \frac{\partial}{\partial \psi} \right) \). Above we have decomposed the velocity oneform into a divergent and rotational part with respect to an invariant volume element. Here we find it more natural to consider the dual decomposition of \( \hat{\rho} \psi_\epsilon, \hat{\rho} \psi_1 \) and \( \hat{\rho} \hat{u} \) (corresponding to \( \omega \otimes \hat{u} \) e.t.c.). In terms of the perturbed, pullbacked density structure we find that we can represent \( \rho_0 \hat{\psi}_\epsilon = \psi_\epsilon \left( \hat{\rho} \psi_\epsilon \right) \) as \( \text{div}_{\hat{g}_0}(\rho_0 \hat{\psi}_\epsilon) = -\hat{\rho}_\epsilon \). Therefore we have that modulo a rotational part we can define \( \rho_0 \hat{\psi}_\epsilon^{d(1)} = -\nabla_{\hat{g}_0}^{-2} \hat{\rho}_\epsilon \) and the rotational part can be represented as \( \rho_0 \hat{\psi}_\epsilon^{r(1)} = *_{\hat{g}_0} \hat{\omega} \hat{A}_\epsilon^{(1)} \). In terms of this parameterization the divergent part of the compatibility equation which we found to be equivalent with the perturbed continuity equation can be formulated at the pullbacked level as

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}^{(3)}(v_0) \right) \hat{\rho}_\epsilon + \text{div}_{\hat{g}_0}(\hat{v}_\epsilon \rho_0) = 0, \quad (58)
\]

\[
\hat{v}_\epsilon \equiv \psi_\epsilon^{-1} v_\epsilon \text{ or } \hat{v}_\epsilon^{(1)} \equiv \psi^* v^{(1)}_\epsilon.
\]

Equation’s (58) and (??) (can be given in perturbed form, but that deserves a separate study) contains the description of the acoustic mode and the interaction with the rotational mode and the kinetic fluctuations. In the linearized case this correspond to a longitudinal wave equation with kinetic and rotational effects in any background fluid and kinetic state. What has to be done is a detailed study of the deformation properties of the stress tensor discussed below with respect to the divergent and rotational diffeomorphisms and the incoherent kinetic transformations.

Let us now discuss the rotational mode in more detail. We define \( \rho \equiv \frac{\partial}{\partial \psi} \).
\[ \psi^d \circ \rho_r, \quad \rho_r \equiv \psi^r \circ \rho_0 \] and \[ \frac{\partial \psi^d}{\partial t} + \nabla \cdot (\mathbf{u}_r, \rho_r) = 0, \quad \mathbf{u} = \psi^d_t + \psi^d_r \mathbf{u}_r, \quad \mathbf{u}_r \equiv \psi^r_t + \psi^r_s \mathbf{u}_0 = \hat{\mathbf{u}}_c + \psi^r_u \mathbf{u}_c, \quad \mathbf{u}_c = \psi^d_u \hat{\mathbf{u}}_c. \] We now find the following equivalent compatibility equations for the rotational generators with respect to the pseudogroup connected to the density \( \rho_r \):

\[ \psi^r_{e,t} - \mathbf{u}_r, \epsilon + [\mathbf{u}_r, \psi^r_e] = 0, \quad (59) \]

equivalent to \[ \psi^r_{e,t} - \hat{\mathbf{u}}_e, \epsilon + [\hat{\mathbf{u}}_e, \psi^r_e] = 0, \]

or \( (\rho_r \psi^r_e), t - (\rho_r \mathbf{u}_r), \epsilon + \ast_{g_0} \hat{d}(\ast_{g_0} (\rho_r \psi^r_e, \mathbf{u}_r)) = 0, \)

or \( \psi^r_e, t - \hat{\mathbf{u}}_e, \epsilon + \ast_{g_0} \hat{d}(\ast_{g_0} (\psi^r_e, \mathbf{u}_e)) = 0. \)

Here \( g_c \equiv \psi^{rs-1} g_0 \) is the fluctuation metric with respect to the rotational part of the diffeomorphism in contrast to the total fluctuation metric used earlier.\(^8\) The above equivalence is due to that \( (\psi^r_u \mathbf{u}_0^d), \epsilon = [\psi^r_u \mathbf{u}_0^d, \psi^r_e] \) which is valid in general for any background vectorfield. The last equality is valid assuming by definition that \( \text{div}_{g_c} (\hat{\mathbf{u}}_e^{(1)}) = \text{div}_{g_c} (\psi^r_e^{(1)}) = 0. \)

In App. C we introduce a description of the rotational vectorfield on a family of level surfaces in threespace given by \( \beta \) defined as

\[ \hat{\mathbf{u}}_c^{(1)} = X^{(1)}_\beta, \quad \psi^r_e^{(1)} = X^{(1)}_\beta, \quad \psi^r_t^{(1)} = X^{(1)}_\beta, \quad X^{(1)}_\beta = g_c^{-1}(\ast_{g_c} \hat{d}(\alpha \hat{d} \beta)) + \text{e.t.c.}. \]

We define a new rotational bracket structure with respect to vorticity situated at the foliations in threespace of \( \beta \) for given metric \( g \) and invariant volumeelement \( dV \) by

\[ \{\alpha, f\}_\beta \ dV \equiv \ast_{g_c} \hat{d}(\alpha \hat{d} \beta) \wedge \ast_{g_c} \hat{d}f = \ast d(f, X^{\beta}_\alpha) \ dV; \quad (63) \]

\[ \{\alpha, f\}_\beta = X^{\beta}_\alpha (f) = \ast_{g_c} (\ast_{g_c} (\hat{d} \alpha \wedge \hat{d} \beta) \wedge \ast_{g_c} \hat{d}f). \quad (64) \]

In App. C we prove that for purely rotational vectorfields one have that

\[ [X^{\beta}_{\alpha_1}, X^{\beta}_{\alpha_2}] = X^{\beta}_{\{\alpha_1, \alpha_2\}}, \]

\(^8\)All the quantities and equations introduced here related to the rotational part can be transformed by acting with \( \psi^d \) to obtain the actual measured quantities. The reason why we do this pullback by \( \psi^d \) is that many of the relations we present will be very complicated without it. But of course all the relations and quantities can be transported to the real measured quantities by the inverse action.
\[ X_\beta^\alpha(f(\beta)) = \{\alpha, f(\beta)\} = 0. \quad (66) \]

This shows that one can think about the rotational bracket as a noncanonical Poisson bracket with functions of type \( f(\beta) \) as Casimirs.

From the above splitting of the constraint equation for the velocity field, we find for the rotational part \( \tilde{u}_c = \psi_t^r + \psi_r^u u_{0c} = (\psi_t^r \circ \phi_{0*} - Id) \frac{\partial}{\partial \tau}, u_{0c} = (\phi_{0*} - Id) \frac{\partial}{\partial \tau} \). Here we will have one representation for each component in the constraint equation. A different representation which concentrates on deformations of the rotational bracket structure of the reference state is \( \tilde{u}_c = X_\alpha^\beta = X_\alpha^\beta + X_\alpha^\beta \equiv u_0^\alpha + \tilde{\psi}^0 \circ (\tilde{\phi}_* - Id) u_{0c} + \tilde{\psi}_c, \beta^0 \equiv \tilde{\psi}^{-1*} \beta_0 \)

\[
(\tilde{\psi}^r \circ \tilde{\phi}_* - Id) \frac{\partial}{\partial \tau} \equiv (\tilde{\psi} - Id) \frac{\partial}{\partial \tau} \equiv \tilde{\psi}_{\tau c} = X_\beta^\alpha \psi_{\tau c} \text{ such that } u_0^\alpha = X_\alpha^\beta_0 = \psi_{\tau c} + \tilde{\psi}_c u_{0c}. \]
Here \( \frac{\partial}{\partial \tau} + \tilde{\psi}_c, \tilde{\psi}_c \equiv (\tilde{\phi}_* - Id) \frac{\partial}{\partial \tau} \equiv \tilde{\phi}_t = X_\beta^\alpha - \psi_{\tau c} \circ (\tilde{\phi}_* - Id) u_{0c}. \)

We parameterize the reference rotational velocity as \( u_{0c} = X_\alpha^\beta_0 \) and find that \( X_\alpha^\beta - \psi_{\tau c} - \tilde{\psi}^{-1*} \alpha_0 \equiv 0 \), i.e., \( \alpha = \tilde{\psi}_{\tau c} + \tilde{\psi}^{-1*} \alpha_0 \text{ mod } (f(\beta^0)) \). The perturbational aspects of this constraint equation with respect to a parameter \( \epsilon \) can be explored by defining \( (\tilde{\psi}^r \circ \tilde{\phi}_* - Id) \frac{\partial}{\partial \epsilon} \equiv (\tilde{\psi}^r - Id) \frac{\partial}{\partial \epsilon} \equiv X_\beta^\alpha \frac{\partial}{\partial \epsilon} \equiv \frac{\partial}{\partial \epsilon} + \tilde{\phi}_c, \tilde{\phi}_c \equiv (\tilde{\phi}_* - Id) \frac{\partial}{\partial \epsilon} \). To avoid perturbations in the bracket structure we should pull back the constraint equation for the rotational potential and obtain \( \phi = \tilde{\psi}_{\tau c} + \alpha_0, \alpha = \tilde{\psi}^{-1*} \phi, \tilde{\psi}_{\tau c} = \tilde{\psi}^{-1*} \tilde{\psi}_{\tau c} \).

Here we refer the potentials to the bracket structure derived from the background foliation \( \beta_0 \) and corresponding vector fields \( X_\beta^\alpha_0 \). We therefore find that analogous to the theory we have developed before for density equations

\[
\begin{align*}
\tilde{\phi}_{\tau c} & = [\tilde{\phi}_c, \tilde{u}_c] + \tilde{u}_{c, \epsilon}, \\
\tilde{\psi}_{\tau c} & = -\{\hat{\alpha}, \tilde{\psi}_c\}_{\beta_0} + \hat{\alpha}_{, \epsilon}, \\
\frac{d}{d\tau_0} & = \frac{d}{d\tau_c} + \{\alpha_0, \cdot\}_{\beta_0}, \\
\tilde{\psi}_{\epsilon} & = \tilde{\psi}^{-1*} \tilde{\psi}_{\epsilon}.
\end{align*}
\]

Equation’s (67) and (52) (can also be perturbed with respect to \( \epsilon \)) give now a description of both the vorticity structure and the related diffeomorphism in interaction with kinetic fluctuations and the longitudinal fluctuations. The above equation for the vorticity (52) with zero right hand side for the purely rotational case is the direct generalization of the potential description of the vorticity equation on fixed two dimensional surfaces to convected level surfaces foliating threespace defined by \( \beta_0 \) and the rotational potential \( \alpha^0 \). In
this case  \( \tilde{u}_c = u_0^c = X^c_\beta \) and  \( \tilde{\phi} = \text{Id} \) since the rotational equation becomes the defining equation for a Lie pseudogroup related to the rotational structure

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}(\tilde{u}) \right) u_c^{(1)} = 0, \quad (68)
\]

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}(\tilde{u}) \right) \pi^{(2)} = 0.
\]

The rotational diffeomorphisms are still described by eq. (67), but with trivial  \( \tilde{\phi} = \text{Id} \). In this case the vorticity equation is analogous to the Vlasov equation with  \( \alpha \) playing the role of the Hamiltonian given by

\[
\ast_{g_0} d_{\ast_{g_0}} d_{\ast_{g_0}} d(X^c_\beta) = \tilde{\alpha} = \tilde{d}(\hat{\alpha} \tilde{d} \beta_0).
\]

If we invert this operator we find an expression for the rotational potential  \( \tilde{\alpha} \) as a functional of the vorticity analogous to noncanonical Hamiltonians on constrained surfaces in classical mechanics. This also explains why this type of models pops up in many applications in fluid and plasma physics. Notice that our composition principle for the velocity induces in principle a finite or infinite series of diffeomorphisms and divergent and rotational potentials and a corresponding series of longitudinal and rotational compatibility equations and bracket structures. The same comments apply to the kinetic generator structure and the deformations of the electromagnetic fields and vector potentials and indeed the fluctuation metric itself. This completes our discussion of the parameterization of the hybrid fluid kinetic theory.

3.1.1 Transformation properties of electromagnetism

To be complete we also have to formulate Maxwell’s equations in such a way that they are invariant with respect to diffeomorphisms in four space which fixes time and transform in a natural way. The standard formulation of Maxwell’s equations in four space is with respect to a fixed metric

\[
dF^{(2)} = 0, \quad (69)
\]

\[
\ast_{g_0} d_{\ast_{g_0}} F^{(2)} = 4\pi j^{(1)},
\]

\[
j^{(1)} = \frac{1}{c} j_k dx^k + \rho dt \equiv J j^{(1)}, \quad (70)
\]

\[
\ast_{g_0} d_{\ast_{g_0}} j^{(1)} = 0. \quad (71)
\]

Notice that the homogenous equation transforms in a natural way with respect to diffeomorphisms in four space while the inhomogenous part of the
We have that \( dF^{(2)} = d\psi^{-1} \hat{F}^{(2)} = \psi^{-1} d\hat{F}^{(2)} = 0 \). We rectify this in a similar way as for the momentum equation by introducing \( \frac{1}{2} *_{g_0} d *_{g_0} J = *_{g_0} d *_{g_0} J *_{g_0} d *_{g_0} J \) id. Here \( g_{id} \) is the metric presented in a frame of reference where -det(\( g_{id} \)) = 1 and \( g_0 = \psi_0^{-1} g_{id}, J_{id} = - \text{det}(g) \).

We now find the following form of inhomogenous Maxwell’s equations

\[
\begin{align*}
*_{g} d *_{g} \hat{F}^{(2)} &= 4\pi \hat{j}^{(1)}, \\
\hat{F}^{(2)} &= F^{(2)}/J, \quad \hat{j}^{(1)} = j^{(1)}/J \\
*_{g} d *_{g} \hat{j}^{(1)} &= 0.
\end{align*}
\] (72)

(73)

When the Maxwell equations are presented in this way (or even better if one tensor it with the four volumeform \( dV_g \) to take into account that it is relation between densities) they transform in the natural way under diffeomorphisms as

\[
\begin{align*}
F^{(2)} &= \psi^{-1} \hat{F}^{(2)}, \quad \hat{F}^{(2)} = \psi^{-1} (\hat{F}^{(2)}), \quad \hat{F}^{(2)} \equiv \hat{F}^{(2)} J(\psi), \\
\mathcal{F} &= \hat{F}^{(2)} \otimes dV_g = F^{(2)} \otimes dV_0 = F_{id}^{(2)} \otimes dV_{id}; \\
\mathcal{J} &= \hat{j}^{(1)} \otimes dV_g = j^{(1)} \otimes dV_0; \\
\mathcal{F} &= \psi^{-1} \hat{F}, \quad \hat{F} = \hat{F}^{(2)} \otimes \psi^* dV_0, \quad \psi^* dV_0 = J(\psi) dV_0, \\
\hat{j}^{(1)} &= \psi^{-1} \hat{j}^{(1)}. \quad \hat{j}^{(1)}.
\end{align*}
\] (74)

(75)

(76)

The upshot is that the new current and field quantities correspond to current density form and the field density form, \( \mathcal{J} \) and \( \mathcal{F} \) and they transform according to these. In case we do not have a euclidean reference metric, it can be an advantage to refer to an orthogonal frame instead with \( F^{(2)} = F_{id}^{(2)} / (- \text{det}(g_0))^{1/2} \) and \( \hat{F}^{(2)} = \hat{F}_{id}^{(2)} / (- \text{det}(g))^{1/2} \). It is possible to introduce electromagnetic equations with respect to \( \mathcal{J} \) and \( \mathcal{F} \), but it requires covariant derivatives which will be beyond the scope of our presentation. We are now ready to pullback our electromagnetic equations to the reference level and we find

\[
\begin{align*}
d\hat{F}^{(2)} &= 0, \\
*_{g_0} d *_{g_0} \hat{F}^{(2)} &= 4\pi \hat{j}^{(1)}, \\
*_{g_0} d *_{g_0} \hat{j}^{(1)} &= 0.
\end{align*}
\] (77)

\[\text{From a measurement point of view it also make sense to study the transformation properties of density forms since we always have to measure with respect to some volume and timeinterval.}\]
The same type of equation will be fulfilled for the reference fields and currents, $F_0^{(2)}$ and $j_0^{(1)}$. If we define $F^{(2)} = \psi^{-1*}(F_0^{(2)} + \tilde{F}^{(2)})$, $\tilde{F}^{(2)} = F^{(2)} - F_0^{(2)}$, $\tilde{F}^{(2)} = \tilde{F}^{(2)} - \tilde{F}_0^{(2)}$, $\tilde{j}^{(1)} = \tilde{j}^{(1)} - j_0^{(1)}$, we find electromagnetic equations suitable for perturbation with respect to one diffeomorphism

\begin{align}
    d\tilde{F}^{(2)} &= 0, \\
    *_{g_0}d*_{g_0} \tilde{F}^{(2)} &= 4\pi \tilde{j}^{(1)}, \\
    *_{g_0}d*_{g_0} \tilde{j}^{(1)} &= 0.
\end{align}

In case we have multiple fluids, we would could perform the above procedure with respect to several diffeomorphisms and fluctuation metrics. Notice, that relations like $A^{(1)} = \Psi^{-1*}A^{(1)}$, e.t.c. are not trivial when written out componentwise, especially since we propose to use a fluctuating metric to obtain contravariant tensors. We will come back to a more elaborate study of the transformation and perturbation properties of electromagnetism in a separate paper elsewhere.

### 3.1.2 Transformation properties of the stress tensor and current density

Let us write our expression for the stress tensor and current density as a covariant twotensor and oneform respectively and study their transformation properties with respect to kinetic theory.

\begin{align}
    \bar{P}^{(2)} &= \int \frac{1}{Jm}(p^{(1)} - mw^{(1)}) \otimes (p^{(1)} - mw^{(1)}) f(z,t) d^3p, \\
    \bar{j}^{(1)} &= \int \frac{-e}{J}(p^{(1)} - mw^{(1)}) f(z,t) d^3p, \\
    w^{(1)} &= \hat{u}^{(1)} + \frac{e}{c}A^{(1)}.
\end{align}

We now find the pullbacked expressions

\begin{align}
    \hat{P}^{(2)} &= \psi^*\bar{P}^{(2)} = \int \frac{1}{m}(p^{(1)} - m\hat{w}^{(1)}) \otimes (p^{(1)} - m\hat{w}^{(1)}) \bar{f}(z,t) d^3p, \tag{79} \\
    \hat{j}^{(1)} &= \psi^*\bar{j}^{(1)} = \int \frac{1}{m}(p^{(1)} - m\hat{w}^{(1)}) \bar{f}(z,t) d^3p.
\end{align}

\footnote{The same remark applies to the stress tensor which transformation properties is discussed below. Although we will not explicitly discuss multiple fluids it is not a major complication from a formal point of view only for the complexity of the presentation. Therefore we have decided to leave this point for explicit, future applications.}
3.2 Physical coordinates

Our theory will not be complete before we formulate the kinetic theory in physical variables also. It is possible to work in canonical coordinates at least when there is no background, magnetic field as we have indicated in our work on OC theory. But it requires doing the transformation from canonical distribution function to physical distribution function as an active symplectic transformation on distributions (see App.A). However, with a magnetic field this transformation is not so valuable since it is not a perturbation. Moreover, there is the question for which coordinates one should specify the reference distribution. The reason why I have been reluctant to give up the canonical formal approach is that in physical coordinates the Poisson bracket itself will be perturbed. However, in section A6.3, we have described new physical coordinates which gives the Poisson bracket in terms of background magnetic fields only. We call this the interaction picture for the Vlasov equation. A direct extension of our work on OC theory would be to specify the OC coordinates in interaction variables. The obvious advantage is that now it is possible to do perturbation expansions without perturbing the invariant bracket and the transformation from interaction distribution function to physical distribution function is now really a perturbation. The relation between the interaction distribution and the physical and canonical distributions is given by $f = \phi_{c1}^{-1}\ast f^i$, $f^i = \phi_{c0}^{-1}\ast f$, where the the transformation $\phi_{c1}^{-1}\ast$ and $\phi_{c0}^{-1}\ast$ are described in eq.(115). The relation between the interaction physical coordinates and canonical or physical coordinates is given by $\dot{p} = p - \epsilon A_0$, $p_{p} = \dot{p} - \epsilon A_1$ which is nothing else than the shift transformations described by $\phi_{c0}^{-1}\ast$ and $\phi_{c1}^{-1}\ast$. Here we use the gauge $\phi_1 = 0$ so $A_1 = -c \int E_1(t') dt'$ can be given a gauge invariant meaning as proportional to the accumulated, perturbed electric field vector referred to some fixed time.

The standard Vlasov equation for the distribution function, $\hat{f}$, in euclidean physical coordinates may be rewritten in a form more suitable for our purposes,

$$\frac{df}{d\tau'} + \{\hat{f}, H^p\} = 0,$$

$$\frac{d}{d\tau'} \equiv \frac{\partial}{\partial t} + \mathbf{J} \cdot (-\frac{e}{c} \partial \mathbf{A}^{(1)}),$$
Here $\mathbf{J}$, $\{ , \}$, $\mathbf{A}^{(1)}$, and $\mathbf{B}^{(2)} = d\mathbf{A}^{(1)}$ are the contravariant Poisson tensor in euclidean physical coordinates, the Poisson bracket in physical coordinates, the covariant vector potential, the covariant magnetic field twoform. The Poisson bracket is written in standard form in eq. (114). $\frac{d}{d\tau}$, $d_P$ and $d$ are a convective derivative with respect to the vector potential part of the electric field, the exterior derivative in six dimensional phase space and the exterior derivative in three dimensional space. In App. A we show that $\mathbf{J} = \phi_c \ast \mathbf{J}_c$ is the pushforward map of the standard canonical Poisson tensor. The definition of the pushforward map is given in App. A. Acting on a vector field, but the action is trivially extended to higher degree contravariant tensors by applying the given action on each tensorindices. We have chosen to represent our tensors with respect to the standard basis in euclidean coordinates, i.e. $dx_i$, $dp_i$ for covariant forms and $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial p_i}$ for contravariant vectorfields with obvious extensions for higher degree tensors. With this notation the antisymmetric covariant twotensor $\mathbf{B}^{(2)} = d\mathbf{A}^{(1)} = \frac{\partial A_k}{\partial x_j} dx_j \wedge dx_k$ is pulled down to a contravariant, antisymmetric multivector identified with the corresponding multivector in momentum space $\mathbf{B}^{(2)}_p = \frac{\partial A_k}{\partial p_j} \frac{\partial}{\partial p_j} \wedge \frac{\partial}{\partial p_k}$. Notice that with a time dependent vector potential it is not possible to treat single particle dynamics as generated by a hamiltonian in six dimensional euclidean physical phase space except for in a convected sense. However, in eight dimensional extended phase space or in six dimensional canonical coordinates the dynamics is described by hamiltonian generators. We show in App. B6.3 that the parameterization can still be presented in the new interaction picture by the following analogous generator equation if we take time independent background electromagnetic fields $\mathbf{B}_0, \mathbf{E}_0 = -\nabla \phi_0$

\[
\frac{\partial f^i}{\partial t} + \{ f^i, H^i \} = 0,
\]

\[11\]

In a noneuclidan space we would have to be careful with the metric. All this can be worked out with respect to an underlying Riemannian metric space, but we have chosen to postpone the description of this more general theory.
\[ \{ f^i, H^i \} = J_0 : (d_P f^i, d_P H^i) \]

\[ H^i = \psi^i + (\psi^i)^{-1} H^i_0 \]

\[ H^i = \left( \dot{\mathbf{p}} - \frac{\mathbf{e}}{c} \mathbf{A}_0 \right)^2 - \gamma_0, \quad H^i_0 = \frac{\dot{\mathbf{p}}^2}{2m} - \gamma_0, \]

\[ \gamma_0 = -e\phi_0. \]

If the background electromagnetic fields are time-dependent, we can transform our Vlasov equation and canonical generator equation to interaction coordinates by

\[ \frac{df^i}{d\tau_0} + \{ f^i, H^i \}_{n0} = 0, \quad \{ f^i, H^i \}_{n0} \equiv J_0 : (d_P f^i, d_P H^i), \]

\[ H^i = \psi^i_{\tau_0} + (\psi^i)^{-1} H^i_0, \]

\[ \psi^i_{\tau_0} \equiv (\psi^i - Id) \frac{d}{d\tau_0}, \]

\[ \frac{d}{d\tau_0} \equiv \frac{\partial}{\partial t} + J_0 \cdot (-e \frac{\partial A_0^{(1)}}{c}), \]

\[ J_0 = \phi_0^{-1} \mathbf{J}_0 = \frac{\partial}{\partial \mathbf{x}} \wedge \frac{\partial}{\partial \mathbf{p}} + \frac{e}{c} B_0^{(2)} \]

\[ \mathbf{J}_c = \frac{\partial}{\partial \mathbf{x}} \wedge \frac{\partial}{\partial \mathbf{p}} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \]

Here \( B_0^{(2)} = dA_0^{(1)}, A_0^{(1)} \), \( J_0 \) and \( \mathbf{J}_c \) are the reference magnetic field twoform, the vector potential oneform, the interaction Poisson tensor and the canonical Poisson tensor.

For those who worry about the gauge invariance of the above equations, we formulate equivalent gauge invariant equations by taking out the potential in the Hamiltonian generators and add a corresponding term to the convective derivatives. We thereby obtain a convective derivative corresponding to the acceleration of the electric field with obvious form of the Vlasov equation which is gauge invariant\(^\dagger\)

\[ \frac{d}{d\tau} = \frac{\partial}{\partial t} + \mathbf{J} \cdot (-e \mathbf{E}^{(1)}), \quad H_p \equiv \frac{p_p^2}{2m}, \]

\(^\dagger\)In a general metric we must study covariant derivatives and parallel translation to give this an invariant meaning.
\[
\frac{d}{d\tau} \equiv \frac{\partial}{\partial t} + J_0 \cdot (-eE_0^{(1)}), \quad \bar{H}^i \equiv \frac{(\hat{\mathbf{p}} - eA_1)^2}{2m},
\]

\[
\bar{H}^i = \psi^i_{\tau_0} + (\psi^i)^{-1}\bar{H}_0^i
\]

\[
\psi^i_{\tau_0} \equiv (\psi^i - \text{Id}) \frac{d}{d\tau_0}.
\]

To interpret the above introduction of new noncanonical coordinates we now give a theorem which show in which sense we can generalize our results in canonical coordinates to any coordinates independent if they are noninertially, convected in phase space with respect to the canonical coordinates or not. Since the definition of what is a hamiltonian flow or not cannot depend on the coordinate system, we suggest to call the flow hamiltonian if it can be transferred to a a hamiltonian flow by a noncanonical (not preserving the Poisson tensor) map as indicated in the theorem below.

**Theorem 6** The transformation of the hamiltonian generator, \( \psi_t \), in canonical coordinates given by exact oneform \( d_P\psi_t \equiv \omega_c \cdot (\psi^*_t - \text{Id}) \frac{d}{d\tau} \) to other noncanonical coordinates (in general time-dependent) \( \phi(t) : z \rightarrow Z = \phi(t)(z) \) is given by the action of the corresponding inverse seven dimensional map which fixes time (c.f. discussion in App. A) \( \phi^{-1}(Z,t) \equiv (\psi^t)(Z,t) \) as

\[
\frac{d}{d\tau} \equiv \phi_* \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + X_t, \quad X_t \equiv (\phi^*_t - \text{Id}) \frac{\partial}{\partial \tau},
\]

\[
\omega \equiv \phi^{-1}*\omega_c,
\]

\[
\bar{\psi} \equiv \text{Ad}(\phi^t) \psi = \phi \circ \psi \circ \phi^{-1}.
\]

The corresponding noncanonical Poisontensor and Poissonbracket are given by

\[
\mathbf{J} = \phi_* \mathbf{J}_c,
\]

\[
\{f,g\}_n = \mathbf{J} : (d_Pf, d_Pg).
\]

Moreover, if \( \psi \) is a canonical, Poisson preserving map with respect \( \mathbf{J}_c, \{,\} \) such that \( \psi_* \mathbf{J}_c = \mathbf{J}_c \) and \( \psi^{-1}*\{f,g\} = \{\psi^{-1}*f, \psi^{-1}*g\} \), then \( \bar{\psi} \equiv \text{Ad}(\phi(t))\psi \) is Poisson preserving with respect to \( \mathbf{J}, \{,\}_n \) such that \( \bar{\psi}_* \mathbf{J} = \mathbf{J} \) and \( \bar{\psi}^{-1}*\{f,g\}_n = \{\bar{\psi}^{-1}*f, \bar{\psi}^{-1}*g\}_n \).
Proof. Let us assume that the vectorfield \( X_{\psi_t} \equiv J_c \cdot (d_P \psi_t) = (\psi_* - Id) \frac{\partial}{\partial t} \) is hamiltonian generated with generator \( \psi_t \). Then transform the exterior derivative of the hamiltonian by pullback of the above map either as a result of an active transformation of the oneform or a passive coordinate transformation (c.f. the discussion below) is given by

\[
\dot{\phi}^{-1} d_P \psi_t = d_P(\phi^{-1} \psi_t) \equiv d_P \dot{\psi}_\tau,
\]

\[
= \phi^{-1} \omega_*((\psi_* - Id) \frac{\partial}{\partial t}) = \omega(\phi_*(\psi_* - Id) \frac{\partial}{\partial t})
\]

\[
= \omega((\phi_* \circ \psi_* \circ \phi_*^{-1} - Id) \phi_* \frac{\partial}{\partial t}) = \omega((Ad(\phi)\psi_* - Id) \frac{d}{d\tau})
\]

\[
= \omega((\bar{\psi}_* - Id) \frac{d}{d\tau}).
\]

\[
\frac{d}{d\tau} \equiv \phi_* \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + X_t,
\]

\[
X_t \equiv (\phi_* - Id) \frac{\partial}{\partial t},
\]

\[
\omega = \phi^{-1} \omega_c = \phi^{-1} \omega_c,
\]

\[
\Rightarrow J = \phi_* J_c = \phi_* J_c.
\]

Here we are treating tensors in phase space as embedded in phase space extended with time such that we freely interchange the action of seven dimensional map \( \phi \) fixing time and the timedependent six dimensional map \( \dot{\phi} \) on such tensors. This will also compress notation and proofs. The action of tensors with even a fixed time component fixes such components, but give extra contributions to the phase space components through terms like \((\phi_* - Id) \frac{\partial}{\partial t}\). Notice that for the case that \( \dot{\phi} \) is not time dependent, we have that \( X_t = 0 \) and \( \frac{d}{d\tau} = \frac{\partial}{\partial t}, \dot{\psi}_\tau = \dot{\psi}_t \) while the Poissontensor, Poisson bracket e.t.c. is still as prescribed above.

We now observe that vectorfield with only phase space components

\[
(\bar{\psi}_* - Id) \frac{d}{d\tau} = X_{\bar{\psi}_*} = J \cdot (d_P \dot{\psi}_\tau),
\]

is indeed hamiltonian with respect to the Poissontensor \( J \) since \( J \cdot \omega = Id \).

The Poisson bracket is transformed by a non Poissonpreserving map as

\[
\phi^{-1} \{f, g\} = \phi^{-1} (J_c : (d_P f, d_P g) = \phi^{-1} \circ J_c \circ \phi^* : (\phi^{-1} \cdot d_P f, \phi^{-1} \cdot d_P g)
\]

\[
= J : (d_P \hat{f}, d_P \hat{g}) = \{\hat{f}, \hat{g}\}_n, \hat{f} = \phi^{-1} f, \hat{g} = \phi^{-1} g.
\]
since $\phi_*T = \phi^{-1*} \circ T \circ \phi^*$ for a contravariant tensor [6].

If $\psi$ is a canonical Poisson preserving map such that $\psi_*J_c = J_c$. Then it follows that a non Poisson preserving map $\phi$ acts as

$$\phi_* \circ \psi_* J_c = (\phi_* \circ \psi_* \circ \phi^{-1*}) \circ \phi_* J_c,$$

$$\Rightarrow \psi_\epsilon J = J, \quad \psi = Ad(\phi).$$

Remark 2 All the formal expansions we introduced earlier is valid simply by replacing $\psi = \exp(X_{\epsilon}), X_{\epsilon} \equiv J_c \cdot d_P w \rightarrow \tilde{\psi} = \exp(X_{\epsilon}), X_{\epsilon} \equiv J \cdot d_P \bar{w},$

$$\psi_t = \int \exp(L w), L w = \{w, \cdot\} \rightarrow \tilde{\psi}_\epsilon t = \int \exp(L_{\tilde{\epsilon}}), L_{\tilde{\epsilon}} = \{\tilde{w}, \cdot\}$$

$$H \rightarrow \tilde{H} = \phi^{-1*} H = H \circ \phi^{-1*}, f \rightarrow \tilde{f} = \phi^{-1*} f = \psi^{-1*} \tilde{f}_0, \tilde{f}_0 \equiv \phi^{-1*} f_0$$

$$\frac{\partial f}{\partial \epsilon} + \{f, H\} = 0 \rightarrow \frac{\partial \tilde{f}}{\partial \epsilon} + \{\tilde{f}, \tilde{H}\} = 0;$$

$$H = \psi_t + \psi^{-1*} H_0, \tilde{H} = \tilde{\psi}_\epsilon + \tilde{\psi}^{-1*} \tilde{H}_0$$

$$\psi_{\epsilon,t} + \{\psi_{\epsilon}, H\} + H_{\epsilon \epsilon} = 0, \rightarrow \tilde{\psi}_{\epsilon,t} + \{\tilde{\psi}_{\epsilon}, \tilde{H}\} = 0; \tilde{H}_{\epsilon \epsilon} = 0$$

Here the notation $\tilde{\epsilon}$ is used to take care of situations where the map $\phi$ also depend on the perturbation parameter $\epsilon$. Therefore, analogous to the above notation for the time generator we have that in a phase space extended by both time and $\epsilon$

$$\frac{d}{d\epsilon} \equiv \phi_* \frac{\partial}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} + (\phi_* - Id) \frac{\partial}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} + X_{\epsilon}.$$  

In the case that the coordinate map does not depend the parameter $\epsilon$, we can replace $\tilde{\epsilon}$ by $\epsilon$ above. This is in agreement with the above interaction picture philosophy where we suggest to let the transformation with respect to canonical variables and consequently the Poisson tensor be unperturbed. After the transformation to the interaction picture we suggest to introduce additional transformations to e.g. gyrokinetic, driftkinetic or oscillation center variables due to adiabatic or exact symmetries in the problem. These applications and description of nonlinear perturbation theory in general is outside the scope of this article since it requires the introduction of noneuclidean metric.

The hybrid fluid kinetic theory can therefore now be presented in interaction physical coordinates by doing the following changes in the canonical theory

$$f^i = (\psi^i)^{-1*} f^i = (\tilde{\psi}^i)^{-1*} \circ (\tilde{\psi}^i)^{-1*} \tilde{f}_0,$$

$$\tilde{f}^i_0 = \tilde{\psi}_0^i f^i, \tilde{\psi}_0^i f^i,$$
\[ \psi_{i\tau_0}(z,t) = \psi_{1\tau_0}(z,t) + \psi_{2\tau_0}(x,t) = \hat{p} \cdot \hat{u} + \psi_{2\tau_0} , \quad (80) \]

\[ \bar{\psi}_{i\tau_0} = -m \frac{\hat{u}^2}{2} - \frac{e}{c} A_1 \cdot \hat{u} , \quad (81) \]

\[ \psi_{i\varepsilon}(z,t) = \psi_{1\varepsilon}(z,t) + \psi_{2\varepsilon}(x,t) = \hat{p} \cdot \psi_{\varepsilon} + \psi_{2\varepsilon} , \quad (82) \]

The hybrid fluid kinetic theory can now be repeated and we find analogously to our earlier derivation in canonical coordinates

\[ \hat{H}^i = H^i - \bar{\psi}_{i\tau_0} = \frac{(\hat{p} - m \hat{w}_1)^2}{2m} - \gamma_0 , \quad (83) \]

\[ \bar{\psi}_t = \bar{H}^i - (\psi_i)^{-1*} \bar{H}_0 , \quad \bar{H}^i \equiv \bar{\psi}^{i*} \hat{H}^i , \quad \bar{H}_0 \equiv \bar{\psi}_0 \hat{H}_0 , \]

\[ \bar{f}^i = (\bar{\psi}_i)^{-1*} \bar{f}_0 , \quad \bar{w}_1 = \hat{u} + \frac{e}{cm} A_1 . \]

The fluid part of the hybrid fluid kinetic theory are of course the same as before while in interaction physical coordinates we have the resonant particle kinetic equation with the above parameterization given as

\[ \frac{d\bar{f}_i}{d\tau_0} + \{\bar{f}_i, \bar{H}^i\}_{n0} = 0 . \quad (84) \]

The corresponding action principle for the resonant particle part will follow trivially from App. C by doing the above replacements in interaction physical coordinates.

We are now ready to replace the above fluid symplectic and resonant particle generators by generators in laboratory frame which are more suitable for perturbations and linearization which it is our aim to do. We write our new parameterization as \[ f^i = (\hat{\psi}_i)^{-1*} \bar{f}_i = (\hat{\psi}_i)^{-1*} (\bar{\psi}_i)^{-1*} \bar{f}_0 . \] Here \[ \bar{f}_i \] is the resonant particle distribution in the laboratory frame defined such that \[ \int \bar{f}_i d^3 \hat{p} = \rho_0 \quad (\frac{\partial \rho}{\partial t} + \nabla \cdot (u_0 \rho_0) = 0) . \] The discussion of the properties of this representation of the follows trivially from what we have proved before. The fluid kinetic generator we now define as

\[ \hat{\psi}_{i\tau_0} \equiv \hat{p} \cdot \psi_i + \psi_{2\tau_0}(x,t) , \quad (85) \]
\[ \hat{\psi}_2(x,t) = -\left( \frac{m}{2} \hat{\psi}_t^2 + \frac{e}{c} A_1 \cdot \hat{\psi}_t \right), \quad (86) \]
\[ \hat{\psi}_\epsilon \equiv \hat{p} \cdot \hat{\psi}_\epsilon + \hat{\psi}_\epsilon(x,t). \]

The resonant particle distribution in the laboratory frame is then described by the following Liouville equation

\[ \frac{d\tilde{f}_i}{d\tau_i'} + \{\tilde{f}_i, \tilde{H}_i\}_{n_0} = 0, \quad (87) \]
\[ \frac{df_i^0}{d\tau_i'} + \{f_i^0, H_i^0\}_{n_0} = 0, \]
\[ \tilde{\psi}_i' = \tilde{H}_i' + \tilde{\psi}_r'_{r_0} - (\tilde{\psi}_i')^{-1} H_i^0, \]
\[ \tilde{H}_i' \equiv \tilde{\psi}_i' \tilde{H}_i', \]
\[ \tilde{H}_i' \equiv \frac{(\tilde{p} + \frac{e}{c} A_1 - \psi_\epsilon)^2}{2m} - \gamma_0. \]

The above form of the equations and parameterization are the one which is suitable for perturbation since everything is now really expressed in interaction physical coordinates and a near identity, symplectic fluid generator which is directly related to the fluid generator \( \psi_t \). Moreover, there is no reason to change the fluid part of the theory earlier developed since it is already expressed in terms of the fluid generator.

We can relate our new kinetic fluid generator to the earlier one by putting \( \hat{\psi} \equiv \tilde{\psi} \circ \tilde{\psi}^{-1} \). We observe that for the reference state \( \tilde{\psi}_0 = \tilde{\psi}_0 \). We now find that one can express

\[ \hat{\psi}_i'^{-1} \psi_r'_{r_0} = -\hat{\psi}_i' + \hat{\psi}_r'_{r_0}, \]
\[ = (\tilde{p} - (m \psi_t + \frac{e}{c} A_1)) \cdot \psi_* u_0 - \frac{m}{2} (\psi_* u_0)^2, \]
\[ \hat{\psi}_i'^{-1} \psi_\epsilon' = -\hat{\psi}_i' + \hat{\psi}_\epsilon' = -\hat{\psi}_{2\epsilon}(x,t) + \hat{\psi}_{2\epsilon}(x,t) = 0. \]

The reason why we can take \( \tilde{\psi}_i' = \hat{\psi}_i' \) is that the fluid compatibility relations for \( \psi_t, \psi_r \) or \( \psi_t, \hat{u} \) are consistent with the kinetic compatibility relations \( \tilde{\psi}_e, \tilde{r}_0 \) are consistent with the fluid compatibility relations \( \tilde{\psi}_e, \tilde{r}_0 \) and \( \tilde{\psi}_e, \tilde{r}_0' \) are consistent with the fluid compatibility relations. Also the fluid description of \( \tilde{\psi}_{2\epsilon} \) can not depend on which momentum coordinates we are using.
Some remarks about the form of the action principle developed in App.B with canonical coordinates is needed. The fluid part of the action principle, $A_F$, can be kept unchanged. In interaction physical coordinates the resonant particle action principle in the fluid frame takes a form analogous to the one in App.C

$$A_r = \int f^i(\bar{\psi}^i_{\tau_0} + (\bar{\psi}^i)^{-1}\bar{\psi}^i_{\tau_0} - H^i)d^6\hat{z}dt = \int \tilde{f}^i(\bar{\psi}^i_{\tau_0} - \bar{\psi}^i_{\hat{H}})d^6\hat{z}dt. \quad (88)$$

We can develop an extended action principle for interaction physical coordinates in the way we did in App.C or equivalently in terms of the new laboratory frame canonical generators. The variation which gives the stress tensor for the fluid theory still has to be performed with respect to the fluid-kinetic action term $A_I \equiv -\int f^i\hat{H}^i d^6\hat{z}dt$ which we observe correspond to the negative time integrated internal energy. However, we can vary the resonant particle distribution with respect to the parameterization in the laboratory frame if we so wish. A variational equivalent form is therefore if one explicitly want to use the laboratory frame representation

$$A_r = \int \tilde{f}^i(\bar{\psi}^i_{\tau_0} - \bar{\psi}^i_{\hat{H}})d^6\hat{z}dt. \quad (89)$$

4 Conclusion

We have described how to parameterize the solutions of the Vlasov-Maxwell equations by using canonical transformations with respect to a reference state. The canonical transformations are described by an equation for the timelike Hamiltonian generator relating it to the local Hamiltonian and the reference Hamiltonian transformed by the Poisson preserving transformations. Our main emphasis is on that the solutions of the Vlasov equation has got a composition principle for the transformations parameterizing the solutions with respect to a given reference solution of the Vlasov equation. In particular there is a specific infinite dimensional group of transformations on phase space extended by time- i.e. a pseudogroup, leaving the reference distribution invariant. To see that the Vlasov equation defines a pseudogroup one has to choose transformations on space/time which fixes time, i.e. a family of canonical transformations on phase space parametrized by time. The pseudogroup which parameterize the solution of the Vlasov equation is
then given by the pseudogroup of all canonical transformations defined modulo the transformations which leave the particle density invariant. We also show how Maxwell’s equations can be formulated such that it transforms in a natural way with respect to diffeomorphisms.

We suggest that the composition principle coming from the underlying pseudogroup are of fundamental importance. The principle is such that it is possible to specify new a priori information in the mathematical structure of the Vlasov equation. Although, the introduction of such a priori information constrains the experimental situations and the physical processes for which the theory is applicable, the theory has a gain in the possibility of modelling kinetic processes in complicated background states. The model is such that any perturbation theory based on it will preserve the number of particles in an invariant way since we are basically using mathematical entities which are not coordinate dependent. Our philosophy is therefore that in any modelling effort such introduction of new a priori information is a necessary first step. In fact, since we have replaced the continuity equation by a constraint equation for Poisson maps on phase space parameterized by time, the introduction of such a priori information will change the structure of our equations. Here we specifically use the composition principle to make the Vlasov-Maxwell equations into a hybrid fluid kinetic theory and in addition to formulate the equations for the divergent and rotational modes in any background. (This application by no means exhaust the vast number of applications for this fundamental principle in averaging, separation and perturbation techniques. We are currently exploring a few of these both in plasma and fluid theory.) The success of the technique in this case is due to that we are able to identify a canonical transformation which after integration over the momentum coordinates coincides with the parameterization of the fluid density. By this trick we are able to define new particle density coordinates and corresponding parameterization which contains no deviation on average from the density and momentum of the reference state. However, the new density contains resonant particle effects and higher order fluid moments. We succeed in presenting the theory in new physical coordinates which we call interaction physical coordinates and the fluid kinetic generators in a form which is such that it is suitable for near identity transformations. Thus we have obtained a theory where kinetic and fluid effects are naturally separated.

Such a decomposition of kinetic theory into hybrid fluid kinetic theory is bound to influence the way one define and think about physical phenom-
ena in plasmas like resonant particle distribution, dispersion relation, wave-resonant particle distribution effects, instability e.t.c. Work are under way to separate higher order fluid moments like stress. The linearized equations can be presented by Hermitian operators with respect to an indefinite inner product, but now with a mixed fluid kinetic inner product. If the system contain some extra symmetries (exact or approximate) either in the reference state or in an intermediate state it might be convenient to introduce other coordinates than the interaction physical, euclidean coordinates we have introduced. Examples of such symmetries are gyrophase and wave phase symmetry or simply an ignorable coordinate in the plasma description. This is outside the scope of the present work since one has to think carefully about noneuclidean, invariant descriptions for kinetic theory.

5 Appendix A The pseudogroup connected to continuity equations

Pseudogroups are infinite dimensional groups. We will not give the detailed definition of infinite dimensional Lie groups since we would then have to introduce much more mathematical machinery than we intend to do here, but the interested reader may find it in the references (5,6). For our purpose we will only notice that some equations in physics define two type pseudogroups: one type which fixes the physical field in question and another one which deforms and parameterize solutions of the the corresponding physical equation. Mathematically this means that the equations in question allows an infinite dimensional symmetry taking solutions into solutions.

We define pseudogroups with smooth structure in the following way:

**Def.** The totality \( \Gamma \), of smooth maps on a space \( M \) form a pseudogroup \( \Gamma \equiv \{ \phi : M \to M \mid \phi \in C^\infty(M,M) \} \) if

i) \( f \circ g \in \Gamma \) when \( f, g \in \Gamma \),

ii) \( id \), the identity map is in \( \Gamma \),

iii) \( f \circ g^{-1} \in \Gamma \) when \( f, g \in \Gamma \).
The continuity equation in fluid mechanics
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) = 0 \]
can be replaced by:

**Theor.** The continuity equation can be parameterized by a family (with respect to time parameter \( t \)) of transformations on space \( M \), \( \psi(t) : M \to M \),
\[ \rho(t) = \psi(t) \rho_0(t) \equiv J(t) \rho_0(t) \circ \psi^{-1}(t). \] (90)

Here \( J(t) \equiv \left| \frac{\partial \psi(t)}{\partial s} \right| \) is the Jacobian of the map \( \psi^{-1}(t) \). The velocity field defines an equation for the parameterization with respect to the reference velocity field by

\[ \mathbf{v}(t) \equiv \psi(t) \bullet \mathbf{v}_0(t) + \mathbf{k} \equiv \psi_t + \psi(t) \ast \mathbf{v}_0(t) + \mathbf{k} \] (91)
\[ \nabla \cdot (\mathbf{k} \rho(t)) = 0, \quad \mathbf{k} = \psi(t) \ast \mathbf{k}_0, \] (92)
\[ \psi_t \equiv \frac{\partial \psi(t)}{\partial t} \circ \psi(t)^{-1}, \]
\[ \psi(t) \ast \mathbf{v}_0(t) \equiv (\mathbf{v}_0 \cdot \nabla \psi(t)) \circ \psi(t)^{-1}. \]

**Proof:**
Here the map \( \psi(t) \), defined above is the standard pushforward map. We parameterized the density at a shifted time by
\[ \rho(t + s) = J_{\psi_0(t,s) \circ \psi(t)}(\psi_0(t,s)^{-1} \circ \psi(t)^{-1}) \rho_0(t + s), \]
where we have decomposed \( \psi(t + s) = \psi_0(t,s) \circ \psi(t) \) so that \( \psi_t = \frac{\partial \psi_0(t,s)}{\partial s} |_{s=0} \). The following relations follows
\[ \frac{\partial J_{\psi_0(t,s)}}{\partial s} |_{s=0} = -\nabla \cdot \psi_t, \]
\[ J_{\psi_0(t,s) \circ \psi(t)} = (J_{\psi(t)} \circ \psi_0(t,s)^{-1}) J_{\psi_0(t,s)}, \]
\[ \frac{\partial \rho_0(t + s)}{\partial s} |_{s=0} = -\psi_t \cdot \nabla \rho(t), \]
\[ \frac{\partial \rho_0(t)}{\partial s} \bigg|_{s=0} = -\nabla \cdot (\mathbf{v}_0 \rho_0(t)). \]

These relations immediately gives us that
\[ \frac{\partial \rho(t)}{\partial t} = \frac{\partial \rho(t + s)}{\partial s} \bigg|_{s=0} = -\rho(t) \nabla \cdot \psi_t - \psi_t \cdot \nabla \rho(t) - \psi(t) \bullet (\nabla \cdot (\mathbf{v}_0 \rho_0(t))). \]
Since we have the identity
\[
\psi(t)^{-1} \nabla \cdot (\nabla \cdot \mathbf{u}) = \frac{1}{J_{\psi(t)}} \nabla \cdot (J_{\psi(t)} \psi(t) \mathbf{u}) ,
\]
one obtains that
\[
\frac{\partial \rho(t)}{\partial t} = -\nabla \cdot (v \rho(t)) = -\nabla \cdot \left( (\psi_t + \psi(t)^* v_0) \rho(t) \right) ,
\]
\[
v = \psi_t + \psi(t)^* v_0 + k .
\]
Here \( k \) is any vectorfield such that \( \nabla \cdot (k \rho(t)) = 0 \). Such vectorfields can also be parameterized with respect to the reference density according to the above identity for divergences
\[
k = \psi(t)^* k_0 , \quad \nabla \cdot (k_0 \rho_0(t)) = 0 .
\]
In the following we will often not write this additional freedom explicit.

**End of proof.**

We see that in the case that the reference velocity is zero the above parametrization is equivalent to the Lagrangian description of fluids, but transported back to the Eulerian velocity \( v(t) \) by the inverse mapping. One can verify that the parameterization of the continuity equation is compatible with composition of families of smooth maps since

\[
J_{\phi(t) \circ \psi(t)} = (J_{\psi(t) \circ \phi(t)}^{-1}) J_{\phi(t)} , \quad (\phi(t) \circ \psi(t))^* \rho_0(t) = \psi(t) \bullet (\phi(t)^* \rho_0(t)) ,
\]
\[
v(t) = (\phi(t) \circ \psi(t))^* v_0(t) = \phi(t) \bullet (\psi(t) \bullet v_0(t)) = \psi_t + \phi(t)^* (\psi_t + \psi(t)^* v_0(t)) .
\]

The family of transformations defined above does not conform with the definition of pseudogroups because of the time-dependence. However, if we look at our family of transformations as transformations on space-time \( X = M \times \mathbb{R} \) which fixes time \( \phi(x,t) = (\phi(t)(x),t) \) they can be viewed as member of a pseudogroup. Moreover, the velocity in this extended space is naturally defined as \( v = (v(\cdot),1) \) while the family of density maps in three space is transcribed to the density \( \rho_0 \) in fourspace which has the same value evaluated
at corresponding points and time. Now it is possible to express the above parameterizations in a more compressed form as

\[ \rho = \phi \cdot \rho_0 \equiv J (\rho_0 \circ \phi^{-1}) , \]
\[ v = \phi_\ast v_0 \equiv (v_0 \cdot \nabla \phi) \circ \phi^{-1} , \]
\[ J = | \frac{\partial \phi^{-1}}{\partial (x,t)} | = | \frac{\partial \phi(t)^{-1}}{\partial x} | = J(t) . \]

Here we have used \( \nabla \) as the gradient operator both in three and four space. When we in addition identify the continuity equation in four space \( \nabla \cdot (v \rho) = 0 \) as an infinitesimal Lie equation, it is clear how the density structure define a pseudogroup \( \Gamma_d \) which leaves the density in four space invariant\(^5,^6\)

\[ \Gamma_d \equiv \{ \phi \in C^\infty(X,X) \mid \phi \cdot \rho = \rho , \phi(x,t) = (\phi(t)(x),t) \} . \]

The above pseudogroup is a finite Lie equation and the infinitesimal version of it corresponds to the continuity equation obtained by taking the infinitesimal map \( \phi = Id + \epsilon v \cdot \nabla \ldots \) in the finite Lie equation. The parametrization of the solution space of the continuity equation as given by eq.(94) with respect to a reference distribution is generated by the pseudogroup of all smooth transformations on space-time which fix time \( \Gamma_t \equiv \{ \phi \in C^\infty(X,X) \mid \phi(x,t) = (\phi(t)(x),t) \} \).

**Def.** We define the density leaf fixed by a density \( \rho_0 \) as \( P_0 = \{ \rho \mid \rho = \phi \cdot \rho_0 , \forall \phi \in \Gamma_t \} . \)

We notice that it is also possible to define a pseudogroup of smooth transformations which leaves \( \rho_0 \) invariant \( \Gamma_{0d} \) which in fact can be transported to every element in the density leaf by \( \Gamma_{0d} \equiv \{ \tilde{\psi} \equiv \phi \circ \psi \circ \phi^{-1} \mid \psi \in \Gamma_{0d} , \phi \in \Gamma_t \} . \)

In this sense it is possible to look at \( \Gamma_{0d} \) as a type of generalized gauge transformations with respect to a given leaf and moreover it will be sufficient to generate the leaf by \( \tilde{\Gamma} \equiv \Gamma_t \) modulo \( \Gamma_{0d} \) with the above transportation. (In fact it corresponds to that one has to introduce a semidirect product as group product in \( \tilde{\Gamma} \).) We will not pursue the generalization of gauge transformations on the density leaf any further in this paper as it also will need more mathematical background than we intend to show here.
5.1 Geometric interpretation of the fluid generator

In our derivation of the parameterization the new vectorfield \( \psi_t \) appeared as the derivative with respect to the near identity map \( \psi_0(t, s) \). This map have the property that \( \psi_0(t, 0) = Id \). Now, we shall think of our transformations with respect to a specific density leaf \( P_0 \) given by a reference density \( \rho_0 \). Then it is realized that \( \rho(t + s) = \psi_0(t, s) \cdot \rho(t) \). Therefore for \( s = 0 \), \( \psi_0 \) is the identity map at the point \( \rho \in P_0 \). Consequently, one can also think about the vectorfield \( \psi_t \) as a point in the space of vectorfields \( X_\rho \) at the point \( \rho \).

At the reference density we have the reference space of vectorfields \( X_{\rho_0} \). It turns out that all vectorfields can be pulled back to the space of reference vectorfields. This is done by defining the related near identity map \( \hat{\psi}_0 \) by

\[
\psi(t + s) = \psi(t) \circ \hat{\psi}_0(t, s).
\]

Again we see that \( \hat{\psi}_0(t, 0) = Id \). This time we have that

\[
\psi(t)^{-1} \cdot \rho(t + s) \equiv \hat{\rho}_0(t, s) = \hat{\psi}_0(t, s) \cdot \rho_0(t).
\]

Therefore one realizes that the corresponding vectorfield \( \hat{\psi}_t = \left( \frac{\partial \hat{\psi}_0(t, s)}{\partial s} \right) |_{s=0} \in X_{\rho_0} \). Moreover, it is verified that

\[
\psi_t = \psi(t) \cdot \hat{\psi}_t = (\hat{\psi}_t \cdot \nabla \psi(t)) \circ \psi(t)^{-1}, \quad (96)
\]

\[
\mathbf{v} = \psi(t) \ast (\hat{\psi}_t + \mathbf{v}_0).
\]

Notice that a sum of vectorfields at the reference space of vectorfields ordered according to the connected composition, corresponds to a total vectorfield at \( \rho \) by

\[
\hat{\psi}_t = \sum_{i=1}^{n} \hat{\psi}_{t,i}, \quad (97)
\]

\[
\psi(t) = \prod_{i=1}^{n} \psi_i(t) \circ, \quad (98)
\]

\[
\psi_t = \psi_{t,1} \ast \psi_{1}(t) \ast \psi_{t,2} + ... + (\prod_{i=1}^{n-1} \psi_i(t) \circ) \psi_{t,n},
\]

\[
\psi_{t,i} = (\prod_{j=1}^{n-1} \psi_j(t) \circ) \hat{\psi}_{t,i}.
\]
This is in fact our fundamental relation which tells us how to extend the above parameterization of the solution of the continuity equation to a composition of several and in principle also infinitely many transformations.

### 5.2 Variational relations and perturbations

We need to define perturbations and variations of our quantities with respect to generators in the space of vectorfields $X_\rho$ which as we have demonstrated can be pulled back to $X_{\rho_0}$. We do that by first defining the perturbed family of maps with respect to one parameter $\epsilon$, $\psi(t, \epsilon)$ such that $\psi(t, 0) = \psi(t)$. From this map we define the near identity maps $\psi_0(t, \epsilon, \delta) \equiv \psi(t, \epsilon + \delta) \circ \psi(t, \epsilon)^{-1}$ and $\hat{\psi}_0(t, \epsilon, \delta) \equiv \psi(t, \epsilon)^{-1} \circ \psi(t, \epsilon + \delta)$. One realizes as before that on the density leaf $P_0$ these maps are deformations from densities $\rho(t, \epsilon)$ and $\rho_0(t)$ respectively. This follows by defining the perturbed density $\rho_0(t, \epsilon + \delta) \equiv \psi(t, \epsilon + \delta) \cdot \rho_0(t) = \psi_0(t, \epsilon, \delta) \cdot \rho(t, \epsilon)$ and $\hat{\rho}_0(t, \epsilon, \delta) \equiv \psi(t, \epsilon)^{-1} \cdot \rho(t, \epsilon + \delta) = \hat{\psi}_0(t, \epsilon, \delta) \cdot \rho_0(t)$. We can now define the generating vectorfields for the defined $\epsilon$ perturbation as

$$
\psi_\epsilon \equiv \left. \frac{\partial \psi_0(t, \epsilon, \delta)}{\partial \delta} \right|_{\delta=0} \in X_\rho , \quad (99)
$$

$$
\hat{\psi}_\epsilon \equiv \left. \frac{\partial \hat{\psi}_0(t, \epsilon, \delta)}{\partial \delta} \right|_{\delta=0} \in X_{\rho_0} .
$$

Instead of one parameter families of deformations, one could define many parameter families of deformations or simply deformed maps $\tilde{\psi}(t)$ of $\psi(t)$ without any reference to any parameters. We define the corresponding near identity maps $\tilde{\psi}_0(t) \equiv \tilde{\psi}(t) \circ \psi(t)^{-1}$ and $\hat{\tilde{\psi}}_0(t) \equiv \psi(t)^{-1} \circ \tilde{\psi}(t)$ and the deformed densities $\tilde{\rho}(t) \equiv \tilde{\psi}_0(t) \cdot \rho(t)$, $\hat{\tilde{\rho}}_0(t) \equiv \psi_0(t) \cdot \rho_0(t)$. The corresponding generating variations of these maps are then defined as the infinitesimal vectorfields

$$
\delta \psi \equiv (\delta \tilde{\psi}(t)) \circ \psi(t)^{-1} \in X_\rho , \quad (100)
$$

$$
\delta \hat{\psi} \equiv \psi(t)^{-1} \circ \delta \tilde{\psi}(t) \in X_{\rho_0} .
$$

We understand from the above that to formulate variational principles on a density leaf has certain nonconventional aspects due to that the fields involved are generated by the action of maps on densities. We give the following identities needed to do variations on density and the pushforward velocity.
These can be verified by doing infinitesimal variations of the corresponding parameterized fields.

\[
\begin{align*}
\delta \rho(t) &= -\nabla \cdot (\delta \psi \rho(t)) , \\
\delta(\psi(t) \cdot v_0) &= [\delta \psi, \psi(t) \cdot v_0] , \\
\frac{\partial \rho(t, \epsilon)}{\partial \epsilon} &= -\nabla \cdot (\psi \rho(t, \epsilon)) , \\
\frac{\partial(\psi(t, \epsilon) \cdot v_0)}{\partial \epsilon} &= [\psi, \psi(t, \epsilon) \cdot v_0] .
\end{align*}
\] (101)

Here the symbol \([\cdot, \cdot]\) is the usual vectorfield bracket which is defined as \([X, Y] \equiv X \cdot \nabla Y - Y \cdot \nabla X\). The deformation generating vectorfields and the generating vectorfields in the timelike direction satisfy a certain compatibility condition in the space \(X_\rho\).

**Theor.** When \(\delta \psi, \psi_\epsilon, \psi_t, k \in X_\rho\), we have the compatibility conditions

\[
\begin{align*}
\delta \psi_t - \delta \psi - [\delta \psi, \psi_t] + k &= 0 , \\
\psi_\epsilon t - \psi_t - [\psi_\epsilon, \psi_t] + k &= 0 \\
\nabla \cdot (k \rho) &= 0.
\end{align*}
\] (103)

For \(\delta \hat{\psi}, \hat{\psi}_\epsilon, \hat{\psi}_t \in X_{\rho_0}\), we have the compatibility conditions

\[
\begin{align*}
\delta \hat{\psi}_t - \delta \hat{\psi} + [\delta \hat{\psi}, \hat{\psi}_t] \rho_0 + k_0 &= 0 , \\
\hat{\psi}_\epsilon t - \hat{\psi}_t + [\hat{\psi}_\epsilon, \hat{\psi}_t] \rho_0 + k_0 &= 0 , \\
\nabla \cdot (k_0 \rho) &= 0, \quad k = \psi(t) \cdot k_0 .
\end{align*}
\] (105)

**Proof:** The proof of the above result follows simply by writing out the compatibility conditions for the two equal variations \(\delta \frac{\partial}{\partial t} \hat{\rho}(t) \big|_{\hat{\rho} = \rho} = \frac{\partial}{\partial t} \delta \hat{\rho}(t) \big|_{\hat{\rho} = \rho}\).

If we use the above formulas, it is obtained that

\[
\nabla \cdot ((\delta \psi_{t, \epsilon} - \delta \psi_t - [\delta \psi, \psi_t]) \rho(t)) = 0 ,
\]
from which the first identity follows. The second compatibility condition follows similarly from that \(\frac{\partial}{\partial t} \delta \rho(t, \epsilon) = \frac{\partial}{\partial t} \delta \rho(t, \epsilon)\).

The last compatibility conditions in \(X_{\rho_0}\) either follow from pulling back the above compatibility conditions in \(X_\rho\) to \(X_{\rho_0}\) or by studying the compatibility conditions for the deformed density at \(\rho_0\), i.e.\(\delta \frac{\partial}{\partial t} \hat{\rho}_0(t) \big|_{\hat{\rho}_0 = \rho_0} = \frac{\partial}{\partial t} \delta \hat{\rho}_0(t) \big|_{\hat{\rho}_0 = \rho_0}\). In either case one finds that

\[
\nabla \cdot ((\delta \hat{\psi}_{t, \epsilon} - \delta \hat{\psi}_t + [\delta \hat{\psi}, \hat{\psi}_t]) \rho_0(t)) = 0.
\]

The derivation for the \(\epsilon\)-parameterized case is similar.
5.3 Parameterization of the Vlasov equation

A special case of continuity equations are the Liouville equation and the Vlasov equation on the phase space of space and momentum. In this case we will have to deal with Hamiltonian vectorfields $X_H$ and reference Hamiltonian vectorfields $X_{H_0}$. The conservation law for the particle density $f$ on phase space in this case is given by the Vlasov equation (must be supplied by the definition of the Hamiltonian in question and the Maxwell’s equations$^8$)

\[
\frac{\partial f}{\partial t} + X_H \cdot \nabla f = 0, \text{ or } \frac{\partial f}{\partial t} + \{f, H\} = 0.
\]

where $\{,\}$ is the Poisson bracket.

The volumeform in 6 dimensional phase space $P$ with coordinates $Z$ is given by the expression $dV = J_V d^6 Z$ where $J_V \equiv | \frac{\partial \phi^{-1}_V}{\partial Z} |$ is the Jacobian of the map $\phi^{-1}_V : Z \to z = \phi^{-1}_V(Z)$ to a standard system where $dV = d^6 z$. For our purposes we will only use standard Euclidean space with physical or canonical momentum coordinates which both will have volumeelement in the standard form. For physical and canonical coordinates with respect to Euclidean space we have since the vectorfields $X_H$ preserve phase space volume that $J_V = 1$ and $\nabla \cdot (X_H) = 0$. In more general coordinate systems, which is needed in gyrokinetic and oscillation center kinetic theory the conservation of phase space volume can be expressed by the conservation law

\[
\frac{\partial J_V}{\partial t} + \nabla \cdot (X_H J_V) = 0.
\]

If we combine these two equations we obtain the continuity equation in phase space for the quantity $\rho = J_V f$

\[
\frac{\partial (J_V f)}{\partial t} + \nabla \cdot (X_H J_V f) = 0.
\]

In an analogous way as above we can now parameterize

\[
\rho(t) = \psi(t) \cdot \rho_0(t), \psi(t) \in C^\infty(P, P),
\]

\[
\rho_0(t) = J_{V_0}(t)f_0(t),
\]
\[ J_V(t) = \psi(t) \cdot J_{V_0}(t), \]
\[ f(t) = f_0(t) \circ \psi(t)^{-1} \equiv (\psi(t)^{-1})^* f_0(t), \]
\[ X_H(t) = \frac{\partial \psi(t)}{\partial t} \circ \psi(t)^{-1} + \psi(t)^* X_{H_0}(t). \]

The Vlasov equation has additional structure for situations when the flow is described by Hamiltonian vectorfields. The maps which is generated by Hamiltonian flows are Poisson preserving maps and we must therefore take this into account.

1. Define the space of Poisson preserving maps as \( \mathcal{F} = \{ \psi \in C^\infty(P, P) | \psi^* \{ f, g \} = \{ \psi^* f, \psi^* g \} \text{ for any } f, g \in C^\infty(P, P) \} \).

We then have the following theorem:

**Theorem** For \( \psi(t) \in \mathcal{F} \) we have

\[ \psi(t)^* X_{H_0} = X_{\psi(t)^{-1} H_0}, \quad (111) \]
\[ \frac{\partial \psi(t)}{\partial t} \circ \psi(t)^{-1} = \psi(t)^* X_{\dot{\psi}_t} = X_{\psi(t)^{-1} \dot{\psi}_t} = X_{\dot{\psi}_t}, \]
\[ \dot{\psi}_t \equiv \psi^{s-1} \dot{\psi}_t, \]
\[ X_{\dot{\psi}_t} \equiv \frac{\partial \dot{\psi}_0(t, s)}{\partial s} \bigg|_{s=0}, \]
\[ X_{\dot{\psi}_t} \equiv \frac{\partial \dot{\psi}_0(t, s)}{\partial s} \bigg|_{s=0}. \]
\[ \psi(t + s) = \psi(t) \circ \dot{\psi}_0(t, s) = \psi_0(t, s) \circ \psi(t), \]
\[ H = \dot{\psi}_t + \psi(t)^{-1} H_0. \quad (112) \]

**Remark:**

Notice that both the map \( \psi_0 \) and \( \dot{\psi}_0 \) are identity maps for \( s = 0 \). Therefore we can regard the Hamiltonian vectorfield \( X_{\dot{\psi}} \) as an element of the Lie algebra at the reference density structure corresponding to the pseudogroup \( \Gamma_t \) restricted to Poisson preserving maps. It is not our purpose here to study this Lie algebra and its correspondence to our generalized gaugegroup (i.e. the Lie pseudogroup keeping the density fixed) since it is best formulated with some more exact mathematical machinery available than we have presently assumed.

**Proof:**
Let the vectorfield \( \psi(t)_* X_{H_0} \) act on a function on phase space \( f \in C^\infty(P, \mathbb{R}) \). One can convince oneself that in this case one has the alternative expression (see [6]) for this vectorfield when one think of it as an operator acting as directional derivative, i.e. \( X(f) \equiv (X \cdot \nabla)f \) as is commonly done

\[
(\psi(t)_* X_{H_0}) f \equiv \psi(t)^{-1} \circ X_{H_0} \circ \psi(t) \circ f = \psi(t)^{-1} \{ \psi(t)^* f, H_0 \}.
\]

Since the map is Poisson preserving we immediately get the result

\[
(\psi(t)_* X_{H_0}) f = \{ f, \psi(t)^{-1} H_0 \} = (X_{\psi(t)^{-1} H_0}) f.
\]

Consequently we have proven the first of the above results up to a possible Casimir generated vectorfield \( X_C \) such that \( \{ C, g \} = 0 \), \( \forall g \in C^\infty(P, \mathbb{R}) \). For symplectic maps and canonical coordinates there are no Casimir for the Poisson bracket while in general noncanonical coordinates there will be Casimirs. However, in our case this present no problem since we are only interested in functions restricted to the Poisson leaf generated by a reference \( f^0 \). On such a leaf the Casimir is fixed and there will be no loss in generality to assume the above identity up to any function commuting with the Poisson leaf density \( f \) related to a reference density \( f^0 \).

One can easily prove that \( \psi_0(t, s) \) and \( \hat{\psi}_0(t, s) \) are Poisson maps since \( \psi(t) \) and \( \psi(t + s) \) are Poisson maps. The phase space functions \( \hat{\psi}_t \) and \( \psi_t \) are the Hamiltonians corresponding to the Hamilonian vectorfields defined by

\[
\frac{\partial \hat{\psi}_0(t, s)}{\partial s} \bigg|_{s=0} = \frac{\partial \psi(t)^{-1} \circ \psi(t + s)}{\partial s} \bigg|_{s=0} = X_{\hat{\psi}_t},
\]

\[
\frac{\partial \psi_0(t, s)}{\partial s} \bigg|_{s=0} = \frac{\partial \psi(t + s) \circ \psi(t)^{-1}}{\partial s} \bigg|_{s=0} = X_{\psi_t}.
\]

From these definitions we derive that

\[
\frac{\partial \psi(t)}{\partial t} = \frac{\partial \psi(t + s)}{\partial s} \bigg|_{s=0} = (X_{\hat{\psi}_t} \cdot \nabla)\psi(t).
\]

and therefore one deduce that \( X_{\psi_t} = \psi(t)_* X_{\hat{\psi}_t} \). Together with the first equality we then obtain that \( X_{\psi_t} = X_{\psi(t)^{-1}(\hat{\psi}_t)} \). It then follows that we can put \( \psi_t = \psi(t)^{-1} \hat{\psi}_t \) up to any function poisson commuting with the leaf density \( f \) corresponding to a reference density \( f^0 \). End of proof.

In canonical coordinates \( z = (x, p) \), where \( p = p_p + \frac{e}{\epsilon} A \) and \( p_p, A \) are the physical momentum and vectorpotential, the Hamiltonian vectorfield is
given by \( X_H = J_c \cdot d_P H = \left( \frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x} \right) \) in canonical, euclidean coordinates. The transformed Hamiltonian is given in the same form since the symplectic and the Poisson tensor does not change by canonical transformations, \( X_{\psi(t)^{-1}H_0} = J_c \cdot d_P (\psi(t)^{-1}H_0) = \left( \frac{\partial \psi(t)^{-1}H_0}{\partial p}, -\frac{\partial \psi(t)^{-1}H_0}{\partial x} \right) \). In physical coordinates based on Euclidean space the symplectic tensor depends on the magnetic field which change also has to be specified, i.e. \( B_0 \to B \). Therefore we have that

\[
X_{H_0} = J_0 \cdot d_P H_0 = \left( \frac{\partial H_0}{\partial p}, -\frac{\partial H_0}{\partial x} - \frac{B_0}{c} \times \frac{\partial H_0}{\partial p} \right),
\]

\[
X_{\psi(t)^{-1}H_0} = J \cdot d_P (\psi(t)^{-1}H_0) = \left( \frac{\partial \psi(t)^{-1}H_0}{\partial p}, -\frac{\partial \psi(t)^{-1}H_0}{\partial x} - \frac{B}{c} \times \frac{\partial \psi(t)^{-1}H_0}{\partial p} \right).
\]

We have not given explicitly how the magnetic field changes under the action of the pseudogroup of smooth transformations on space time here. The answer to this question follows from the same infinite dimensional symmetry for electromagnetic fields which are responsible for the usual gauge parameterizations. We will explore this more general parameterization of the electromagnetic fields in a forthcoming paper.

The connection between the canonical distribution function and the physical distribution function in physical coordinates \((x, \hat{p} = mv)\) is given by a canonical tranformation in the radiation gauge as

\[
\hat{f} = \phi_c^{-1*} f, \\
\phi_c^{-1*} = \exp(X_c), \\
X_c = \frac{e}{c} A \cdot \frac{\partial}{\partial \hat{p}} = -J \cdot \left( \frac{e}{c} A^{(1)} \right).
\]

Here \(A^{(1)}\) is the vectorfield \(A\) lifted to a oneform. Notice that the vectorfield \(X_c\) is phase space volume and Poisson preserving, but it is not generated by a hamiltonian. The explicit form of this vectorfield in other coordinates will follow from the tranformation properties of the Poisson tensor, \(J\). The above transformation is nothing else than the shift transformation from physical \(p_p\) to canonical \(p\).

- **Lemma** The canonical transformation \(\phi_c\) transforms the bracket between two functions \(f, g\) on canonical phase space to the physical

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bracket between the corresponding functions \( \hat{f}, \hat{g} \) on physical phase space.

\[
\phi_c^{-1*} \{ f, g \} = \{ \hat{f}, \hat{g} \}_n = J : (d_P \hat{f}, d_P \hat{g}).
\]

Here the physical bracket is given in it's standard euclidean form

\[
\{ \hat{f}, \hat{g} \}_n \equiv \frac{\partial \hat{f}}{\partial x_p} \cdot \frac{\partial \hat{g}}{\partial p_p} + \frac{eB}{c} \cdot \left( \frac{\partial \hat{f}}{\partial p_p} \times \frac{\partial \hat{g}}{\partial x_p} \right). \tag{114}
\]

**Proof.** The Poisson tensor in canonical coordinates can be expressed by the multivector \( J_c = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial p} \). The action of the pullback map gives \( \phi_c^{-1*} \{ f, g \} = \phi_c^{-1*} (J_c : (d_P f, d_P g)) = (\phi_c^{-1*} \circ J_c \circ \phi_c^*) : (\phi_c^{-1*} d_P f, \phi_c^{-1*} d_P g) \). The exterior derivative operator commutes with the pullback operator \( \phi_c^{-1*} d_P f = d_P \hat{f} \) and for contravariant tensors \( \phi_* T = \phi_c^{-1*} \circ T \circ \phi_c^* \). Therefore one have that

\[
\phi_c^{-1*} \{ f, g \} = J : (d_P \hat{f}, d_P \hat{g}), \quad J = \phi_{c*} J_c = \frac{\partial}{\partial x_p} \wedge \frac{\partial}{\partial p_p} + \frac{eB}{c} \left( \frac{\partial \hat{f}}{\partial p_p} \times \frac{\partial \hat{g}}{\partial x_p} \right) \wedge \frac{\partial}{\partial p_p}.
\]

This expression is identical to the standard particle Poisson tensor in euclidean physical phase space variables given above.

\[ \blacksquare \]

We immediately notice two major problems with this bracket. It is not compatible with the reference distribution, \( f^0 \) since the Vlasov equation for that has to be expressed with respect to background electromagnetic fields. Secondly, it is not compatible with perturbation theory either since then one would have to do a perturbation expansion of the bracket itself. This completely destroys the ideas we advocated for above using canonical fixed brackets as a tool for invariant expansions. To resolve this in our opinion fundamental problem in plasma physics, we suggest to define a new physical distribution function given by the background fields \( f^i \equiv \phi_{c0}^{-1*} f \). The bracket for these kind of distribution functions are now transformed to the same form as in eq.(114), but with \( B \to B_0 \). This distribution function is still a gaugeinvariant distribution function since we will use the gauge, \( \phi_1 = 0 \) where

\[
A_1 = -c \int f \mathbf{E}_1(t') dt' \]

has a physical meaning in terms of the timeintegrated perturbed electric field, \( \mathbf{E}_1 \equiv \mathbf{E} - \mathbf{E}_0 \). The relation between the physical distribution function and the interaction distribution function is given by

\[
\hat{f} = \phi_{c1}^{-1*} f^i, \tag{115}
\]
\[ \phi^{-1*}_{c1} = \exp(X_{c1}), \]
\[ X_{c1} = \frac{e}{c} A_1 \cdot \frac{\partial}{\partial p} = -J_0 \cdot \left( \frac{e}{c} A_1^{(1)} \right) \]

\( J_0 \) is the euclidean physical coordinates Poisson tensor with \( B \to B_0 \). It seems fitting to call this description of the Vlasov fields the interaction picture since it is now possible to separate background and fluctuating quantities in an invariant way suitable for perturbation theory.

### 6 Appendix B Hybrid fluid-kinetic action principle

We will in this appendix study the action principle for the hybrid fluid-kinetic theory. In two other works\(^4,8\) we have elaborated on the action principles for the Vlasov equation and the ideal fluid equations respectively. Our approach is based on varying the generators of the underlying infinite dimensional group acting on the respective densities. The basic method is quite different from the approach of Larsson\(^1,2\) which is using canonical conjugate variables on the accessible leaf. However, our method can be revised to introduce canonical conjugate variables with certain differences since our action also explicitly takes into account the group composition law and the compatibility conditions. The action principle for the Vlasov equation is (the Maxwell equation has its own action principle which in fact also can be parameterized by an infinite dimensional group)

\[ A_p = \int f(\psi, H) d^6 z dt. \]  

(117)

The compatibility condition for the parameterized phase space density can then be used to formulate a revised action principle for densities which depend on an additional formal perturbation parameter \( \epsilon \), i.e. \( f(t, \epsilon) \)

\[ A_p^{(1)} = \int_{0}^{1} \int f(\psi_{\epsilon t} + \{ \psi_{\epsilon}, H \} - H_{\epsilon}) d^6 z dt d\epsilon \]

(118)

In eq.\(^117\) we treat \( f \) as parameterized by symplectic transformations with respect to a reference state \( f^0 \). The variation is nonstandard in the sense that the action is varied and sought stationary with respect to the the infinitesimal Hamiltonian generator \( \delta \psi \) such that \( \delta f = \{ \delta \psi, f \} \). (However, this
action imply standard variational principles by the introduction of the revised variational principle through one parameter variations.) Moreover, the variation of the generator $\psi_t$ is determined through the compatibility relation for the variation $\delta \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \delta f$ which leads to the compatibility condition for variations

$$\delta \psi - \delta \psi_t + \{\delta \psi, \psi_t\} \equiv 0 \mod k,
\quad k = \psi^{-1} k^0, \quad \{k, f\} = 0.$$  

(119)

The function $k$ has no influence on the variations. Note that if the variations are restricted to a one parameter group, the above compatibility condition is equivalent to the one we have derived before since then $\delta \psi = \psi_\epsilon \delta \epsilon$. This means that for both action principles one obtain the Vlasov equation by the variations

$$\frac{\delta A_p^{(1)}}{\delta \psi} = \frac{\delta A_p}{\delta \psi_t} = -f_t - \{f, H\} = 0. $$

In the revised action principle it is possible to introduce a variation with respect to $\delta f$ keeping $\psi_\epsilon$ fixed since variation with respect to the one parameter generator $\psi_\epsilon$ is only a subvariation. The variation with respect to $f$ then gives the compatibility condition as one of the variational equations.

$$\frac{\delta A_p^{(1)}}{\delta f} = \psi_{\epsilon,t} - H_\epsilon + \{\psi_\epsilon, H\} = 0.$$

In fact, there is no reason why one could not introduce many (even infinite) parameter groups if this is suitable for the problem at hand. If we want, we could also give up the explicit parameterization of $f$ through symplectic transformations in the revised action principle and formulate a canonical field theory for canonical conjugate variables $\psi_\epsilon, f$ as Larsson$^{1,2}$ do. In this case one would have to introduce $f_{\epsilon} = \{\psi_\epsilon, f\}$ as an additional constraint. In our paper$^4$ we do this by introducing a Lagrange multiplier, but also by embedding the problem in a larger double symplectic space which includes also the $\epsilon$-dynamics. For some purposes this might be a somewhat restrictive point of view, e.g. if one want to derive model equations based on several layers of transformations as we want to do.

We are now in a position to formulate a new hybrid fluid kinetic action principle where we restrict one part of the symplectomorphism, $\overline{\psi}$, to correspond to what we have found in section 3 to be equivalent to volume density
preserving transformations in space (both parameterized by time even if we
do not explicitly indicate it). The second part of the composition corresponds
to an incoherent kinetic transformation $\tilde{\psi}$ due to higher order Hamiltonian
generators than linear in the momentum coordinate. The total hybrid fluid-
kinetic action restricted to such a composition takes the form

$$A_H = A_r + A_F,$$

$$A_r = \int f(\overline{\psi}_t + \overline{\psi}^{-1}\tilde{\psi}_t - H)d^6zdt = \int f(\overline{\psi}^{-1}\tilde{\psi}_t - \tilde{H})d^6zdt.$$

Here the Hamiltonian $\tilde{H}$ and the related $\hat{H}$ is defined in eq.(29). Up
to variational equivalence (which after all is what is important in a var-
ritional principle), we can freely move the action of a symplectic tran-
sformation between a phase space density $f = \phi^{-1}\hat{f}$ with suitable decaying
properties in infinity and a multiplying phase space function $\int fgd^6zdt \iff
\int \hat{f}(\phi^*g)d^6zdt$. Therefore equivalently the incoherent part of the action prin-
ciple restricted to fluid orbits can be written

$$A_r = \int \tilde{f}(\tilde{\psi}_t - \tilde{\psi}^*\tilde{H})d^6zdt \quad (120)$$

Such changes between variational equivalent forms of variational principles
will later on be freely done without further mentioning. The fluid part of the
action principle has the form$^8$(here the density parameterization is given in
the action principle)

$$A_F = \int \rho(\mathbf{w} \cdot \hat{\mathbf{u}} - \frac{\mathbf{u}^2}{2})d^4x, \quad (121)$$

$$\mathbf{w} = \mathbf{u} + \frac{e}{mc}\mathbf{A},$$

$$\rho = \psi\rho^0, \quad (122)$$

$$\hat{\mathbf{u}} = \psi_t + \psi^0 \mathbf{u}_0. \quad (123)$$

**Remark 3** The fluid action, $A_F = \overline{A}_F + A_I$, can further devided into one
part which is simply the momentum space integrated $\overline{A}_F = \int(\int f\overline{\psi}_td^3p)d^4x$
if we identify $\frac{\mathbf{u}^2}{2}$ with $\text{u}^2_0$ and one part which could be identified as the elec-
tromagnetic/fluid interaction part, $A_I = \int \rho\frac{e}{mc}\mathbf{A} \cdot \tilde{\mathbf{u}}d^4x$. 

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Lemma 7  The above fluid action is again nonstandard in the sense that the variations has to be done with respect to a infinitesimal variation of the fluid generator \( \delta \psi \) for the quantities which are parameterized with respect to the reference fluid state \( \rho^0, \mathbf{u}^0 \)

\[
\begin{align*}
\delta \rho &= -\nabla \cdot (\delta \psi \rho) , \\
\delta \psi_t &= \delta \psi_t + [\delta \psi, \psi_t] , \\
\delta \mathbf{u} &= \delta \psi_t - [\delta \psi, \mathbf{u}] .
\end{align*}
\] (124)

The variations with respect to \( \mathbf{u} \) and the electromagnetic potential are standard. We have in our earlier work, found by using the above relations that

\[
\frac{\delta A_F}{\delta \psi} = -\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{\hat{u}} \cdot \nabla \mathbf{u} - \frac{1}{m} f_L \right) , \\
f_L \equiv \frac{e}{m} (\mathbf{E} + \frac{1}{c} \mathbf{\hat{u}} \times \mathbf{B}) , \\
\mathbf{E} \equiv -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} , \mathbf{B} = \nabla \times \mathbf{A} , \\
\frac{\delta A_F}{\delta \mathbf{u}} = \rho (\mathbf{\hat{u}} - \mathbf{u}) = 0 \Rightarrow \mathbf{u} = \mathbf{\hat{u}} .
\]

It also seems natural to call the term \( A_I = -\int f \mathbf{H} d^6 z dt \) which correspond to the internal energy for the fluid-kinetic interaction part of the action. The variation of this part of the action with respect to the fluid generator gives the divergence of the stress tensor needed to complete the fluid momentum equation. In this formulation of the action principle the mass density continuity equation is implicitly given by parameterization of density.

\[\frac{\delta A_I}{\delta \psi} = \int \frac{\delta \mathbf{p}}{\delta \psi} \{ f, \mathbf{H} \} d^3 \mathbf{p} = -\int p \{ f, \mathbf{H} \} d^3 \mathbf{p} = -\nabla \cdot (\mathbf{P}) .\]

Here we have used the obvious lemma valid for phase space densities with a suitable decay in infinity and an appropriate class of phase space observables which \( g(z,t) \) belongs to

Lemma 8 \( \int G(x,t) \{ g, f \} d^3 \mathbf{p} = -\nabla \cdot \left( \int \frac{\partial g}{\partial p} f d^3 \mathbf{p} \right) G(x,t) .\)
The variation with respect to the infinitesimal generator $\delta \tilde{\psi}$ gives us the reduced Liouville equation on fluid orbits with respect to similar variational rules as we discussed in eq. (119)

$$\frac{\delta A_H}{\delta \tilde{\psi}} = -\frac{\partial \tilde{f}}{\partial t} - \{\tilde{f}, \tilde{\psi}\} = 0.$$  (125)

### 6.1 Revised hybrid fluid-kinetic action principle

From the parameterized version of the hybrid fluid-kinetic action principle it is possible to derive a revised action principle in the same way as we did above for the Vlasov action principle. This is done simply by assuming that $f$, $\tilde{f}$, $\psi$ and $\tilde{\psi}$ depend on an additional formal parameter $\epsilon$. We then find the revised hybrid fluid-kinetic action principle

\[
A_H^{(1)} = \int_0^1 \int (\psi_\epsilon \cdot (-\rho \frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla u - \frac{1}{m} f_L - \nabla \cdot P)) d^4 x d\epsilon 
- (\psi_\epsilon \cdot w) \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\tilde{u} \rho) \right) 
+ \rho (\tilde{u}_\epsilon + \psi_\epsilon \cdot \nabla u) \cdot (\tilde{u} - u) + \rho \frac{e}{cm} A_{\epsilon} \cdot \tilde{u} d^4 x d\epsilon 
+ \int_0^1 \int \tilde{f} (\tilde{\psi}_\epsilon, t + \{\tilde{\psi}_\epsilon, \tilde{\psi}\} - \tilde{\psi} \frac{\partial \tilde{H}}{\partial \epsilon}) d^6 z dtd\epsilon.
\]  (126)

The underlined terms in the fluid and kinetic part of the action are interchangeable forms of the same term. With this revised action principle we obtain the same equations as above by varying with respect to $\psi_\epsilon$, $u_\epsilon$ and $\tilde{\psi}_\epsilon$.

### 7 App. C Rotation and divergence defined in an invariant way

One way to define the rotation and divergence of a fluid element in an invariant way is through the Hodge star operation relative to a metric \([\mathbb{3}]\). Another more intrinsic way is through the Lie derivative of a volume element. Our
interest in this is motivated by the need to formulate physical equations
and here fluid dynamics in such a way that they transform naturally with
respect to diffeomorphisms. From a practical point of view this is needed to
formulate perturbation theory with low complexity and new models. How-
ever, from a more fundamental point of view there is a need for an intrinsic
description of observable quantities like rotation and divergence of the flow
of a fluid element. The definition of the Hodge star operator with respect to
an invariant volume element $dV = JdV_0 = \psi^{-1}*dV_0$, give that one can easily
check that it must behave naturally with respect to diffeomorphisms

$$
\alpha \wedge *_g \beta = \psi^{-1}(\alpha, \beta)dV = \langle \alpha, \psi^{-1}(\beta) \rangle dV \quad (129)
$$

$$
*_g = \psi^{-1} \circ *_{g_0} \circ \psi^*, \quad (130)
$$

$$
g = \psi^{-1}_0 \circ *_g \circ \psi_0. \quad (128)
$$

Here $*_g$ and $*_g_0$ is the Hodge star operation with respect to $g$ and $g_0$ respectively and $\alpha$ and $\beta$ are forms in $\Lambda^k T^* M$ for some $k (k = 1, 2, 3$ for us in three
dimensional space). By abuse of notation we will use the same notation for
the metric $g$ and its inverse $g^{-1}$ as for the maps induced by them, e.g. here
$g^{-1} \beta = g^{-1}(\cdot, \beta)$ is a contravariant multivectorfield in $\Lambda^k T^* M$. Moreover,
$\langle \cdot, \cdot \rangle$ is the standard contraction between $\Lambda^k T^* M$ and it’s dual space
$\Lambda^k TM$. A more direct description of the action of the Hodge star operation
is formulated by

**Lemma**

$$
*_g v = *_{g_0}(J g_0 \circ g^{-1}(v)),
$$

$$
*_g v = \frac{1}{2} g \circ g^{-1}_0 \circ *_{g_0} v;
$$

where $v$ is a form in $\Lambda^k T^* M , k = 1, 2, 3$.

**Proof.** With respect to the metric $g_0$ we have that $\alpha \wedge *_{g_0} \beta = \\
\langle \alpha, g_0^{-1}(\beta) \rangle dV_0$. Therefore one find that $\alpha \wedge *_g v = \langle \alpha, g_0^{-1}((g_0 \circ g^{-1})v) \rangle dV_0 = \langle \alpha, g_0^{-1}(J(g_0 \circ g^{-1})v) \rangle dV_0$.

$$
= \alpha \wedge *_{g_0}(J(g_0 \circ g^{-1})v). \quad \text{Since this equality is valid for all k-forms } \alpha, \text{ the first part of the lemma is proved. For the second part of the lemma we use that we can write } v = *_g v' = *_{g_0} v'_0 \text{ where } v' = *_g v \text{ and } v'_0 = *_{g_0} v. \text{ Here we have used that in three space (same equality up to sign in some other dimension) } *_g \circ *_g = *_{g_0} \circ *_{g_0} = 1. \text{ The first part of the lemma then implies}
$$

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that $v' = *_{g_0} v = J(g_0 \circ g^{-1}) v'$ and consequently $v' = *_{g} v = \frac{1}{f}(g \circ g_0^{-1}) *_{g_0} v$. This proves the second part of the lemma.

We are now in a position to state Hodge decomposition (we do not consider singular contributions) with respect to a general metric as

**Theorem 9** For $v, A, \eta$ as forms in $\Lambda^k T^* M, \Lambda^{k-1} T^* M, \Lambda^{n-k-1} T^* M$ respectively ($n = 3$ for $M$ three dimensional), we have that

$$v = d\eta + *_{g} dA = d\eta + \frac{1}{f} g \circ g_0^{-1} *_{g_0} A, \quad (131)$$

$$Jg^{-1}(v) = Jg^{-1}(d\eta) + g_0^{-1}(g_0 A),$$

$$v = \psi^{-1} *(d\eta + *_{g_0} dA), \eta = \psi^{-1} \eta, \quad A = \psi^{-1} \hat{A}.$$

The proof follows from direct use of the above lemma.

We will now specialize to oneforms and define divergence and rotation with respect to a transformed metric $g$.

**Definition 10** $\text{div}_g(v) \equiv *_{g} d *_{g} v^{(1)}, \text{curl}_g(v) \equiv g^{-1}(g_0 d v^{(1)}), \quad v^{(1)} = g(v)$. Here $v$ is a vectorfield in $TM, v : M \to TM$.

This definition leads to the following theorem

**Theorem 11**

$$\text{div}_g(v) = \frac{1}{f} \text{div}_{g_0}(Jv) = \psi^{-1}(\text{div}_{g_0}(\hat{v})), \quad v = \psi \hat{v}, \quad (132)$$

$$\text{curl}_g v = \frac{1}{f} \text{curl}_{g_0}(g_0^{-1} \circ g(v)) = \psi \text{curl}_{g_0}(\hat{v}).$$

**Proof.** We prove this by applying the above definition for divergence and curl. $\text{div}_g(v) = *_{g} d *_{g} v^{(1)} = *_{g} d *_{g_0}(Jv_0^{(1)}) = \frac{1}{f} *_{g_0} d *_{g_0}(Jv_0^{(1)}) = \frac{1}{f} \text{div}_{g_0}(Jv), \quad v_0^{(1)} = g_0(v)$. On the other hand, we have that $*_{g} d *_{g} v^{(1)} = \psi^{-1}(*_{g_0} d *_{g_0} \hat{v}_0^{(1)}) = \psi^{-1}(\text{div}_{g_0}(\hat{v})), \quad v_0^{(1)} = g_0(\hat{v})$. Similarly for curl we have that $\text{curl}_g(v) = g^{-1}(g_0 d v^{(1)}) = \frac{1}{g^{-1} \circ g_0^{-1}(g_0 d (g_0^{-1} \circ g(v))))} = \frac{1}{f} \text{curl}_{g_0}(g_0^{-1} \circ g(v)).$ On the other hand we have that $g^{-1}(g_0 d v^{(1)}) = \psi^{(1)}(g_0^{-1}(g_0 d (\hat{v}^{(1)}))) = \psi \text{curl}_{g_0}(\hat{v})$, and the theorem is proved.

A more geometric way to study divergence independent of metric is by Lie derivative of the invariant volumeform with respect to the velocity field $\mathcal{L}(v) dV = di_v dV \equiv div_g(v)dV = Jdiv_g(v)dV_0 = div_v(JdV_0) =$
\[ di(\nu) dV_0 = \mathcal{L}(J\nu) dV_0 = div_{g_0}(J\nu) dV_0. \] On the other hand we have that \( \mathcal{L}(\nu) dV = \psi^{-1*}(\mathcal{L}(\hat{\nu}) dV_0) = \psi^{-1*}(div_{g_0}(\hat{\nu})) JdV_0. \) We therefore observe that a divergence free vector field or purely rotational is simply a vector field \( \nu_c \) in the kernel of the Lie operator, i.e. (by the way an infinitesimal Lie equation by App. A) \( \mathcal{L}(\nu_c) dV = div_{g}(\nu_c) dV = div_{g_0}(J\nu_c) dV_0 = 0. \) This is consistent with a parameterization of a rotational vector field for an invariant volume element as \( \nu_c = \text{curl}_g(A) = \frac{1}{J} \text{curl}_{g_0} A_0 = \psi_*(\text{curl}_{g_0}(\hat{A})) \) where \( A_0 \equiv g_0^{-1} \circ g(A), \ A = \psi_* \hat{A}. \) The Hodge decomposition then give us that the velocity field can be decomposed with respect to an invariant volume element as

\[
\nu^{(1)} = -d\eta + *_g dA^{(1)} = -d\eta + \frac{1}{J} g \circ g_0^{-1}(*_{g_0} dA^{(1)}). \tag{133}
\]

However, with respect to the reference metric \( g_0 \) we could consider the decomposition of \( J\nu^{(1)} \) or \( J\nu. \) In fact with respect to a reference state where the fluid is fixed and homogenous (\( \rho_0^{f} \) is constant and \( J^f \) is the Jacobian with respect to the corresponding diffeomorphism), we could just as well multiply by the constant mass density and obtain since \( \rho = J^f \rho_0^{f} \) a decomposition of \( \rho\nu^{(1)} \) or \( \rho\nu \) as \( \rho \nu = -\rho \nabla_g \eta + \text{curl}_{g_0}(\rho_0^{f} g_0^{-1}(A^{(1)})) \), \( \nabla_g \eta \equiv g_0 \circ g^{-1}(d\eta) \). This decomposition is interesting since it is exactly the one we need in connection with the discussion of the pseudogroup defined by the continuity equation defined in App.A.

### 7.1 Rotational bracket structure

We will represent a rotational vector field \( \nu_c \) (or one form \( \nu^{(1)}_c \)) by a Pfaff decomposition of the one form \( A^{(1)} \) which in the nonsingular case is \( A^{(1)} = \alpha d\beta + d\gamma \). The rotational vector field is then \( \nu^{(1)}_c = *_g d(\alpha d\beta), \) i.e. we can mod out \( \gamma. \) We want to think about \( \beta \) as a family of level surfaces (foliations) which the rotational vector field is situated on. With this interpretation in mind we will use the notation \( \nu_c = X^\beta_\alpha = g^{-1}(\nu^{(1)}_c), \ \nu^{(1)}_c = X^{\beta(1)}_\alpha \) for a given metric \( g. \) Analogous with the Poisson bracket in phase space we define the new rotational bracket on the foliations defined by \( \beta \) as \( \{ \alpha, f \}_\beta = -X^\beta_\alpha(f) = *_g (X^{\beta(1)}_\alpha \wedge *_g df) = < df, g^{-1}(X^{\beta(1)}_\alpha) > = g^{-1}(df, X^{\beta(1)}_\alpha). \) The equivalence of the first two definitions comes through that both of them are by trivial use of the definitions equivalent to the third expression. We can now find the following lemma valid for rotational one forms represented with respect to a given foliation.
Lemma 12

\[ *_g (X^\beta_1 \wedge X^\beta_2) = \{\alpha_1, \alpha_2\}_\beta d\beta, \] (134)

\[ [X^\beta_1, X^\beta_2] = X^\beta_{-(\alpha_1, \alpha_2)} , \]

\[ X^\beta_\alpha(f(\beta)) = 0 \] where \( f \) is differentiable.

Proof. For the proof of the first identity we find that

\[ *_g (*_g(X^\beta_1 \wedge d\beta \wedge X^\beta_2)) = *_g(d\beta \wedge *_g(X^\beta_2 \wedge d\alpha_1)). \]

Now, we use the identity

\[ *_g(a^{(1)} \wedge *_g(b^{(1)} \wedge c^{(1)})) = g^{-1}(a^{(1)}, c^{(1)}) b^{(1)} - g^{-1}(b^{(1)}, c^{(1)}) a^{(1)} (\text{analogue to the classical triple crossproduct}) \]

to derive

\[ *_g(X^\beta_1 \wedge X^\beta_2) = -g^{-1}(X^\beta_1, d\alpha_1)d\beta + g^{-1}(X^\beta_2, d\beta)d\alpha_1. \]

But we have that \( X^\beta_{\alpha_2}(\beta) = -*_g(*_g(d\alpha_2 \wedge d\beta) \wedge d\beta) = 0 \) which implies

\[ *_g(X^\beta_1 \wedge X^\beta_2) = \{\alpha_1, \alpha_2\}_\beta d\beta. \] This statement also proves the third part of the lemma since \( X^\beta_{\alpha_2}(f(\beta)) = f'(\beta)X^\beta_\alpha(\beta) = 0. \)

The second identity is established by that for purely rotational vectorfields (c.f. the section about parametrization of hybrid fluid kinetic theory in the main text)

\[ [X^\beta_1, X^\beta_2]^{(1)} = -*_g d(*_g(X^\beta_1 \wedge X^\beta_2)) = -*_g d(\{\alpha_1, \alpha_2\}d\beta) = X^\beta_{-(\alpha_1, \alpha_2)}. \]

Therefore there is a Lie antihomomorphism between the Lie algebra of rotational vectorfields and the corresponding Lie algebra with respect to the rotational bracket on the space of rotational potentials for a given family of foliations of space. All the usual relations for Lie algebra’s like Jacobi identity e.t.c. follows for the rotational bracket structure through this antihomomorphism. ■

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