A data availability attack on a blockchain protocol based on LDPC codes

Massimo Battaglioni, Paolo Santini, Giulia Rafaiani, Franco Chiaraluce, Marco Baldi

Abstract—In a blockchain Data Availability Attack (DAA), a malicious node publishes a block header but withholds part of the block, which contains invalid transactions. Honest full nodes, which can download and store the full blockchain, are aware that some data are not available but they have no formal way to prove it to light nodes, i.e., nodes that have limited resources and are not able to access the whole blockchain data. A common solution to counter these attacks exploits linear error correcting codes to encode the block content. A recent protocol, called SPAR, employs coded Merkle trees and low-density parity-check (LDPC) codes to counter DAAs. We show that the sparse nature of LDPC matrices and the use of the so-called peeling decoder make the protocol less secure than expected, owing to a new possible attack strategy that can be followed by malicious nodes.

Index Terms—Incorrect-coding proof, blockchain, data availability attacks, LDPC codes, SPAR protocol.

I. INTRODUCTION

A blockchain can be seen as an ordered list of blocks, each containing transactions occurred among the participants of a peer-to-peer network. The recent discovery of Data Availability Attacks (DAAs) represents a new threat against blockchain security. Since the DAA introduction in [1], there has been a growing research interest in finding efficient countermeasures to this type of attacks, possibly leading to new blockchain models with improved scalability and security (e.g., [2]–[5]).

In fact, scalability, which is related to the ability of supporting large transaction rates, represents one of the main issues of most existing blockchains [6]. The straightforward solution of increasing the block size raises a series of further concerns. In fact, the larger the block the smaller the number of nodes able to download the full blockchain and, indeed, to participate in the network as full nodes, verifying the validity of new blocks and of every contained transaction. More peers would rather participate in the network as light nodes, which, due to their limited resources, store only a squeezed version of the blockchain [7] and consequently cannot autonomously verify the validity of transactions.

Light nodes aim at downloading as less data as possible. For instance, they may store only the block headers, which unambiguously identify the content of the blocks. However, in a setting with relatively few full nodes, it is more probable that they collude, which makes light nodes more susceptible to DAAs. In fact, the aim of a DAA is to make at least one light node accept a block which has not been fully disclosed to the network. This can happen if and only if honest full nodes are prevented from preparing fraud proofs, i.e., demonstrations that the block is invalid [2], [8].

One of the most promising countermeasures to DAAs consists in encoding the blocks through some error correcting code. Encoding introduces redundancy and distributes the information of each transaction across all the codeword symbols, so that recovering a small portion of an encoded block may be enough to retrieve the entirety of its contents through decoding. This strategy, combined with a sampling process in which light nodes ask for fragments of an encoded block and then gossip them to full nodes, ensures that malicious block producers are forced to reveal enough pieces of the invalid block [8]. An alternative to transactions encoding is to change the protocol in such a way that a group of light nodes can collaboratively (between themselves) and autonomously (from full nodes) verify blocks [5]. Another option is to decouple the consensus rules from the transaction validity rules [4].

In a recent paper [2], Yu et al. proposed SPAR, a blockchain protocol which uses Low-Density Parity-Check (LDPC) codes to counter DAAs; LDPC codes for this specific application have then been studied in [3], [9]. SPAR comes as an improvement of the protocol in [8] using two-dimensional Reed-Solomon codes, whose parameters have been optimized in [10]. The authors of SPAR study the protection against DAAs in case the adversary aims to prevent honest full nodes from successfully decoding the block, which is a strict requirement to settle a proper fraud proof. In [2], this situation is investigated assuming the adversary operates by withholding pieces of the encoded block; under a coding theory perspective, this gets modeled as a transmission over an erasure channel. They conclude that, unless the adversary is able to find stopping sets (which is a NP-hard problem [11]), SPAR guarantees that the success probability of a DAA is sufficiently small even when light nodes download a small amount of data besides the block header. As a consequence, SPAR claims improvements in all the relevant metrics [2, Table 1].

Our contribution: In this paper we study the security of SPAR considering an adversary acting as a channel that adds a mixture of errors and erasures. We describe a DAA in which malicious nodes add one error and a relatively small amount of erasures, proving that the current version of SPAR is susceptible to such an attack (which requires a relatively small computational effort), and that the attack success probability is significantly larger than that considered in [2].
**Paper organization:** The paper is organized as follows. In Section II we describe the notation and some background. In Section III we introduce a general framework to study DAA.s. In Section IV we introduce a novel DAA against the SPAR protocol and in Section V we draw some conclusions.

II. NOTATION AND BACKGROUND

In this section we establish the notation used throughout the paper, and recall some background notions.

A. Mathematical notation

Given two integers \( a \) and \( b \), we use \([a, b]\) to indicate the set of integers \( x \) such that \( a \leq x \leq b \). For a set \( A \), we use \(|A|\) to denote its cardinality. We denote with \( \mathbb{F}_q \) the finite field with \( q \) elements. Given a vector \( \mathbf{v} \), we use \( \text{supp}(\mathbf{v}) \) to denote its support, i.e., the set containing the positions of its non-zero entries and \( \omega_2(\mathbf{v}) \) to denote its Hamming weight, that is, the size of its support. Given an integer \( l \) and a set \( A \), \( A^l \) is the set of vectors of length \( l \) taking entries in \( A \). Given a matrix \( \mathbf{M} \), \( m_{i,j} \) denotes its entry at row \( i \) and column \( j \), \( \mathbf{M}_i \) denotes the \( i \)-th row, and \( \mathbf{M}_{ij} \) denotes the \( j \)-th column. Given a set \( A \), \( \mathbf{M}, \mathbf{M}_A \) (respectively, \( \mathbf{M}_{A,i} \)) represents the matrix formed by the columns (respectively, rows) of \( \mathbf{M} \) indexed by \( A \).

We denote by Hash a cryptographic hash function, with codomain \( D \). Given some vector \( \mathbf{a} \), we use \( \mathcal{T}(\mathbf{a}) \) to denote a generic hash tree structure constructed from \( \mathbf{a} \) and using Hash as underlying function. The root of the tree is denoted as \( \mathcal{T}.\text{Root}(\mathbf{a}) \); it generically takes values in \( D^d \) and is a one-way function. With analogous notation, \( \mathcal{T}.\text{Proof}(\mathbf{a}, i) \) refers to the proof that the \( i \)-th entry of \( \mathbf{a} \) is a leaf in the base layer of the tree. Notice that, when the hash function Hash is properly chosen, then for any pair of strings \( \mathbf{a} \neq \mathbf{a}' \) we have \( \mathcal{T}.\text{Root}(\mathbf{a}) \neq \mathcal{T}.\text{Root}(\mathbf{a}') \) and, for any index \( i \), \( \mathcal{T}.\text{Proof}(\mathbf{a}, i) \neq \mathcal{T}.\text{Proof}(\mathbf{a'}, i) \) with overwhelming probability; therefore, for the sake of simplicity, in the following we assume the absence of root and proof collisions.

B. LDPC codes

LDPC codes are a family of linear codes characterized by parity-check matrices having a relatively small number of non-zero entries compared to the number of zeros. Namely, if an LDPC \( \mathbf{H} \in \mathbb{F}_q^{n \times n} \) has full rank \( r < n \) and row and column weight in the order of \( \log(n) \) and \( \log(r) \), respectively, then it defines an LDPC code with length \( n \) and dimension \( k = n - r \). The associated code is \( \mathcal{C} = \{ \mathbf{c} \in \mathbb{F}_q^n \mid \mathbf{c} \mathbf{H}^\top = 0 \} \), where \( \top \) denotes transposition. The rows of the parity-check matrix define the code *parity-check equations*, that is,

\[
\sum_{j=1}^n c_j h_{i,j} = 0, \quad \forall i \in [1, r], \quad \forall \mathbf{c} \in \mathcal{C}.
\]

Equivalently, any code can be represented in terms of a generator matrix \( \mathbf{G} \in \mathbb{F}_q^{k \times n} \), which forms a basis for \( \mathcal{C} \).

In an Erasure Channel (EC), some of the codeword symbols are replaced with the erasure symbol \( \epsilon \). To this end, we express the action of an EC as \( \mathbf{c} + \epsilon' \), where \( \epsilon' \) is the input sequence and \( \epsilon' \in \{0, \epsilon\}^n \), with \( \epsilon \) such that \( \epsilon + \alpha = \epsilon \), \( \forall \alpha \in \mathbb{F}_q \). A decoding algorithm for the EC aims to obtain a codeword by substituting each erasure with an element from \( \mathbb{F}_q \). In the case of LDPC codes, the most common decoder used over the EC is the peeling decoder [12]. This algorithm works by expressing \( \mathbf{H} \) as a linear system, where the unknowns are exactly the erased symbols. Due to the sparsity of \( \mathbf{H} \), with large probability the linear system will include several univariate equations, i.e., containing only one erasure. Each of these equations can be solved to compute the corresponding unknown, which is then substituted into all the other equations. This procedure is iterated until all the unknowns are found or, at some point, the linear system does not contain any univariate equation, i.e., all the unsolved equations contain at least two unknowns. In the former case we have a decoding success, while in the latter case we have a failure, due to a *stopping set* [13], i.e., a set of symbols participating to parity-check equations containing at least two unknowns each. If all the symbols forming a stopping set are erased, peeling decoding fails. The *stopping ratio* \( \beta \) of an LDPC code is defined as the minimum stopping set size divided by \( n \).

C. Components of the SPAR protocol

SPAR is based on a novel hash tree called Coded Merkle Tree (CMT), combined with an ad-hoc *hash-aware* peeling decoder.

**Coded Merkle Tree:** A CMT is a hash tree which is constructed from \( \ell \) linear codes \( \{\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(\ell)}\} \) over \( \mathbb{F}_q \); the \( i \)-th code has length \( n_i \) and dimension \( k_i \). Each code \( \mathcal{C}^{(i)} \) is defined by the systematic generator matrix \( \mathbf{G}^{(i)} = [\mathbf{I}_{k_i}, \mathbf{A}_i] \), with \( \mathbf{A}_i \in \mathbb{F}_q^{(n_i-k_i)\times n} \) and \( \mathbf{I}_{k_i} \) being the identity matrix of size \( k_i \). The CMT uses an integer \( b \) which must be a divisor of all blocklength values \( n_1, \ldots, n_{\ell} \). Furthermore, one needs to have partitions for the sets \( [1, n_i] \), for \( i \in [1, \ell - 1] \). Namely, we have \( \mathcal{S}_i = \{S_1^{(i)}, \ldots, S_{k_i+1}^{(i)}\} \) which is a partition of \( [1, n_i] \), such that the \( S_j^{(i)} \) are all disjoint and each one contains \( b \) elements, since \( k_i+1 = n_i/b \). Starting from \( \mathbf{c} \in \mathcal{C}^{(1)} \), we build the associated CMT \( \mathcal{T}'(\mathbf{c}) \) as follows:

1) set \( i = 1 \);
2) for \( j \in \{1, \ldots, k_{i+1}\} \), set \( u_j = \text{Hash}(c_{z_1}^{(i)}, \ldots, c_{z_b}^{(i)}) \), with \( \{z_1, \ldots, z_b\} = S_j^{(i)} \);
3) encode \( \mathbf{u} = [u_1, \ldots, u_{k_{i+1}}] \) as \( \mathbf{u} = \mathbf{c} \mathbf{G}^{(i+1)} \);
4) if \( i < \ell - 1 \), increase \( i \) and restart from step 2), otherwise set \( \mathcal{T}.\text{Root}(\mathbf{c}) = \mathbf{u} \).

**Hash-aware peeling decoder:** A hash aware peeling decoder, described in [2] Section 4.3, is an algorithm that decodes a set of \( \ell \) words which are expected to constitute a CMT. Namely, let \( \{x^{(1)}, \ldots, x^{(\ell)}\} \), where \( x^{(i)} \in \mathbb{F}_q \), be the words to be decoded. The hash-aware peeling decoder works in a top-down fashion and, at every iteration, uses the peeling decoder strategy (i.e., recover erasures that participate

\[1\] Notice that, when LDPC codes are considered, encoding is conveniently performed using the parity-check matrix rather than the generator matrix. This implementation detail does not affect the conclusions of our analysis but, considering encoding with the parity-check matrix, we would unnecessarily burden the notation. Therefore, we stick to encoding with the generator matrix.
in univariate parity-check equations) for any layer of the CMT. Additionally, the hash-aware peeling decoder verifies the consistency between symbols of connected layers of the tree via hash functions, whilst the symbols are recovered. Decoding fails whenever a stopping set or a failed parity-check equation is met just like the conventional peeling decoder; furthermore, the hash-aware peeling decoder fails in case check consistency fails for some layer.

III. A GENERAL FRAMEWORK TO STUDY DAAS

In this section we present a general framework to study DAAs, and then apply it to the SPAR protocol. For brevity, we only give the fundamentals of the model; for further details concerning DAAs, we refer the interested reader to \[2\], \[8\].

A. A general model for DAAs

We consider a game in which an adversary \(A\) exchanges messages with \(m\) players \(P_1, \ldots, P_m\), who cannot communicate one each other. Each player has access to an oracle \(O\), who can only perform polynomial time operations. Every list of transactions is seen as a vector \(u \in \mathbb{F}_q^k\). We assume that the following information is publicly available:

- a validity function \(f : \mathbb{F}_q^k \rightarrow \{\text{False}, \text{True}\}\), which depends on the blockchain rules and on its current status;
- two hash trees \(T, T'\);
- a \(k\)-dimensional code \(C \subseteq \mathbb{F}_q^n\) with generator matrix \(G\).

The game proceeds as follows:

1) \(A\) chooses \(u \in \mathbb{F}_q^k\) such that \(f(u) = \text{False}\) and \(\tilde{c} \in \mathbb{F}_q^n\);
2) \(A\) challenges the players with \((h_u, h_c)\), where \(h_u = T.\text{Root}(u)\), \(h_c = T'.\text{Root}(\tilde{c})\);
3) each player \(P_i\) selects \(J_i \subseteq [1, n]\) with size \(s\);
4) \(A\) receives \(U = \bigcup_{i=1}^{m} J_i\);
5) to reply to a query containing the index \(i\), \(A\) must send \(\{\tilde{c}, T.\text{Proof}(\tilde{c}, i)\}; \ A\) is free to choose which queries to reply and which ones to neglect;
6) if a player does not receive a valid reply for any of his queries, then he discards \((h_u, h_c)\);
7) the players gossip all the valid answers to \(O\), which aims to produce a proof for one of the following facts:
   a) \(\exists \tilde{c} \in C\) such that \(T'.\text{Root}(\tilde{c}) = h_c\);
   b) \(\exists \tilde{c} \in C\) such that \(T'.\text{Root}(\tilde{c}) = h_c, \ \tilde{c} = uG\) and \(T.\text{Root}(u) \neq h_u\);
   c) \(\exists \tilde{c} \in C\) such that \(T'.\text{Root}(\tilde{c}) = h_c, \ \tilde{c} = uG, \ T.\text{Root}(u) = h_u\) and \(f(u) = \text{False}\).

\(A\) wins the game if \(O\) cannot produce any proof, and there is at least a player receiving valid answers to all his queries. We denote by \(\gamma\) the Adversarial Success Probability (ASP), i.e., the probability that \(A\) wins a random execution of the game.

It can be easily seen that, in our model, the players \(P_1, \ldots, P_m\) correspond to the light nodes connected to a malicious node modeled by \(A\). The oracle \(O\) instead represents the fact that any light node must be connected to at least one honest full node wishing to broadcast fraud proofs. We remark that the hypotheses and properties that underlie our model are the same under which DAAs have been studied in the literature \[2\], \[3\], \[8\], \[10\]. Finally, our model does not fix any hash tree, nor code family; thus, it can be used to study several blockchain networks. We now proceed by describing how SPAR adapts to such a model, but it can be easily seen that also the protocol proposed in \[8\] fits into the model.

B. DAAs in the SPAR protocol

In SPAR, the CMT is instantiated using the code design procedure considered in \[12\], which produces an ensemble of LDPC codes whose parity-check matrices have at most column weight \(v\) and at most row weight \(w\). As mentioned in Section \[1\], besides the CMT, SPAR requires the use of another hash tree, denoted by \(T\) and considered as a standard Merkle tree. Let \(u \in \mathbb{F}_q^k\) denote the list of transactions of a new block. Then, a correctly constructed header contains \(h_u = T.\text{Root}(u)\) and \(h_c = T'.\text{Root}(c)\), with \(c = uG(1)\).

However, in case of a DAA, the word \(\tilde{c} = c + e\) upon which \(h_c\) is constructed may be any vector picked from \(\mathbb{F}_q^n\). The authors of SPAR study the protection of the protocol against DAAs; namely, they initially consider the following two cases:

a) if \(\tilde{c}(i) \notin C(i)\), then the proof consists in sending the symbols that participate in a failed parity-check equation, together with their CMT proofs; we refer to such a proof as incorrect-coding proof;

b) if \(\tilde{c} = c\) but \(f(u) = \text{False}\), the adversary succeeds only if the samples received by the oracle are not enough to allow recovering of \(u\) from \(c\) through decoding.

The following bound for the ASP is derived \[2\, Theorem 1\]:

\[
\gamma \leq \min \left\{ 1 - \alpha_{\min}^s, 2^{\max\{b(\alpha_i)n_i + ms\log(1 - \alpha_i)\}} \right\}
\]  

(2)

where \(b(\cdot)\) is the binary entropy function, \(\alpha_i\) is the undecodable ratio of \(C(i)\), that is, the minimum fraction of coded symbols the adversary needs to make unavailable in order to prevent the oracle from full decoding, \(\alpha_{\min} = \min\{\alpha_i\}\), and \(s\) is the number of queries performed by each light node.

IV. A NOVEL DAA TO THE SPAR PROTOCOL

In this section we describe a novel DAA that can be successfully performed against the SPAR protocol. We define an adversarial behavior for which the ASP is significantly larger than that resulting from \[2\]. To this end, we first describe how the behavior of a malicious node can be seen as the effect of a communication channel, and then we present our attack strategy. In doing this, we preliminarily highlight the main difference between this paper and the analysis in \[2\]. Namely, in \[2\] the authors do not consider that, by introducing errors in addition to erasures, an adversary can make the Merkle proofs of the symbols participating in an incorrect parity-check equation unavailable, under certain circumstances, thus preventing the oracle from dispatching fraud proofs.

A. Incorrect-coding proofs in presence of errors

Without loss of generality, we study only the case in which errors are added in the first layer of the CMT. In order to model an adversary that is free to modify any symbol in \(c = uG(1)\), we consider an Error - Erasures Channel (EEC), where
the original codeword might suffer a mixture of errors and erasures. The adversary first behaves as an additive channel that, on input \( c \), outputs \( \tilde{c}^{(1)} = c + e \), where \( e \in \mathbb{F}_q^n \) and is such that \( \tilde{c}^{(1)} \notin C^{(1)} \). Then, \( \tilde{c}^{(1)} \) is used to build the second layer of the CMT by aggregating and hashing the symbols in \( \tilde{c}^{(1)} \) according to \( S^{(1)} \). This way, the adversary obtains a sequence \( u \in \mathbb{F}_q^n \), which is encoded into \( uG^{(2)} \in C^{(2)} \). If no error is added in the second layer, then all the parity-check equations for \( C^{(2)} \) will be satisfied. The same applies to all the upper layers. In other words, we consider a CMT \( T'(\tilde{c}^{(1)}) \) built from \( \tilde{c}^{(1)} \notin C^{(1)} \), with root \( h_c \). Let \( U \subseteq [1, n] \) be the set containing all the positions asked by the players. The adversary selects \( E \subseteq U \), corresponding to the positions for which he does not provide a reply. Notice that this can be modeled as the transmission over an erasure channel, which receives \( \tilde{c}^{(1)} \) as input and outputs \( x = \tilde{c}^{(1)} + e' \), where \( e' \in \{0, \epsilon\}^n \). Clearly, the adversary does not need to disclose to the players symbols which were not requested, that is,

\[
\text{supp}(e') = E \cup ([1, n] \setminus U).
\]

Remember that, in the presence of errors, SPAR is designed such that the oracle provides an incorrect-coding proof by sending the value of all the symbols of \( \tilde{c}^{(1)} \) that participate in a failed parity-check equation, except one of them, together with the CMT proofs of all the symbols in the incorrect parity-check equation. So, the oracle necessarily needs to find a parity-check equation of \( C^{(1)} \) that is not satisfied, along with the CMT proofs of all the involved symbols.

However, in the presence of errors, the oracle may not be able to compute valid CMT proofs for the symbols recovered from erasures through decoding. In fact, the presence of errors in univariate equations may trigger an avalanche effect during decoding, and errors in the estimate of erased symbols may spread as decoding proceeds, leading to failed parity-check equations due to errors. However, the number of wrongly estimated symbols depends on the actual positions of errors and erasures. In order to help visualizing this phenomenon, we report in Figure 1 a toy example in which the same amount of errors and erasures yield different outputs (in terms of correctly recovered erasures), considering \( \mathbb{F}_2 \) and

\[
H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad c = [0, 0, 0, 0, 0].
\]

We also consider the correlation between CMT proofs. Since a CMT proof depends on all the codeword symbols, apparently a proof cannot be provided unless all the symbols in the base layer are known. However, the oracle receives CMT proofs for all the non-erased symbols. Then, the oracle can take advantage of the fact that CMT proofs of different symbols may share some elements in intermediate layers of the CMT, depending on their positions, and reuse parts of the received proofs. So, actually, in order to send the CMT proofs of some symbols, the oracle may not need to know all the codeword symbols. Therefore, the estimate of the actual capacity of the oracle to provide CMT proofs for erased symbols requires involved computations. Yet, we can focus on a specific case, in which no erasure is correctly recovered. This is a conservative point of view, since our attack may work also in the other cases, that is, those in which the available CMT proofs do not allow the oracle to craft all the missing proofs. To this end, we consider the following proposition.

**Proposition 1.** Let \( \mathcal{O} \) be an oracle equipped with an hash-aware peeling decoder \( D \), receiving as input \( x = c + e + e' \), with \( c = uG \), \( e \in \mathbb{F}_q^n \), \( e' \in \{0, \epsilon\}^n \) such that \( \text{supp}(e) \cap \text{supp}(e') = \emptyset \). Let \( i_1 = \min\{\text{supp}(H_{1,:})\} \) and \( i_2 \neq i_1 \in \text{supp}(H_{1,:}) \). Let \( w_{i_1}(e) = 1 \) and \( e_{i_1} \neq 0 \). Let \( e' \) be such that:

1. \( |\text{supp}(H_{1,:}) \cap \text{supp}(e')| \geq 1, \forall l \in \text{supp}(H_{1,:}). \)
2. \( \text{supp}(H_{1,:}) \cap \text{supp}(e') = i_2, \)
3. \( \exists j \in [1, r] \setminus \text{supp}(H_{1,:}) \text{ such that } \text{supp}(H_{j,:}) \cap \text{supp}(e') = i_2; \)

then, the hash-aware peeling decoder exits at the first iteration, and outputs \( x' \) which is identical to \( x \) in all the entries except for that in position \( i_2 \), which is different from \( c_{i_2} \).

**Proof:** We observe that, due to 1), before the first iteration, all the incorrect parity-check equations contain at least an erasure. So, the decoder starts by analyzing the first parity-check equation which, due to 2), contains only one erasure in position \( i_2 \) (affecting \( x_{i_2} \)) and gets incorrectly filled, being \( c_{i_1} \neq 0 \). Due to 3), we have a univariate parity-check equation which does not contain errors, whose unknown is in position \( i_2 \). So, the value of \( x_{i_2} \) is substituted into this parity-check equation, which results as unsatisfied. Then, the decoder stops and outputs a vector identical to \( x \) in all coordinates, apart from that in position \( i_2 \) which is wrongly estimated.

As a consequence of the above proposition, the oracle cannot provide any CMT proof for the symbols which are

\[ e' = [0 \; \epsilon \; \epsilon \; 0 \; 0], \quad e' = [0 \; \epsilon \; \epsilon \; 0 \; 0], \]
\[ e = [1 \; 0 \; 0 \; 0 \; 0], \quad e = [0 \; 0 \; 0 \; 0 \; 1], \]
\[ x = [1 \; \epsilon \; \epsilon \; 0 \; 0], \quad x = [0 \; \epsilon \; 0 \; 0 \; 1]. \]

\[
\begin{align*}
1 + \epsilon + 0 + 0 + 0 &= 0 \quad (0 + \epsilon + 0 + 0 + 0 = 0) \\
1 + 0 + \epsilon + 0 + 0 &= 0 \quad (0 + \epsilon + 0 + 1 + 0 = 0) \\
0 + \epsilon + 0 + 0 + 0 &= 0 \quad (0 + 0 + \epsilon + 0 + 0 = 0) \\
0 + 0 + \epsilon + 0 + 0 &= 0 \quad (0 + 0 + \epsilon + 0 + 0 = 0)
\end{align*}
\]

(a) (b)

Fig. 1: In case (a), the peeling decoder wrongly sets \( x_2 = 1 \) from the first equation, which is then substituted in the third equation, which results in being unsatisfied. Then, it stops and outputs \([1, 1, \epsilon, 0, 0]\). In case (b), the decoder correctly sets \( x_2 = 0 \); in the second iteration it sets \( x_3 = 1 \) from the second equation, and then exits after substitution into the fourth equation. The output in this case is \([0, 0, 1, 0, 1]\).

\footnote{For the sake of simplicity, we have considered a peeling decoder that starts evaluating parity-check equations from the first one. If the peeling decoder starts processing from a generic \( k \)-th parity-check equation, the adversary can easily adapt his strategy by setting \( e_{k_1} \neq 0 \), where \( k_1 = \min\{\text{supp}(H_{k,:})\}. \)
affected by an erasure. In fact, he does not know their values, since the decoder has not been able to correctly recover any erasure. Consequently, the oracle cannot show that some of the parity-check equations of $C^{(1)}$ are unsatisfied, because all the unsatisfied parity-check equations (i.e., the ones in which the $i_j$-th entry participates) contain at least an erasure. According to the discussion in Section III-A this shows that the oracle cannot provide an incorrect-coding proof in such a situation.

We now proceed with the analysis of the proposed DAA by providing an upper bound on the number of erasures $e'$ must contain, in order to be compliant with Proposition 1.

**Lemma 1.** Let $H$ be the parity-check matrix of the code $C^{(1)}$ employed in the SPAR CMT, and let $e, e'$ be compliant with Proposition 7. Then, $w_H(e') \leq v$.

**Proof:** Let $d = w_H(H_{i_1,i_1})$; by construction, we have $d \leq v$. Under the pessimistic assumption that the supports of the parity-check equations involving the non-zero entries of $H_{i_1,i_1}$ are all disjoint, then we need one erased symbol per equation, yielding a total of $d$ erasures. In case of indexes appearing in multiple equations, the adversary can erase the corresponding symbol, which will affect more than one equation. Consequently, it must be $w_H(e') \leq d \leq v$.

So, a procedure the adversary can use to design $e$ and $e'$ meeting the hypotheses of Proposition 1 is as follows:
1. set $e_{i_1} = z$, where $i_1 = \min \{\text{supp}(H_{1,i_1})\}$ and $z \in \mathbb{F}_q \setminus \{0\}$;
2. randomly choose $j \in [1,v] \setminus \text{supp}(H_{i_1,i_1})$ such that $|\text{supp}(H_{j,i_1}) \cap \text{supp}(H_{1,i_1})| \geq 1$;
3. set $e'_{i_2} = \epsilon$, where $i_2 \in \{\text{supp}(H_{j,i_1}) \cap \text{supp}(H_{1,i_1})\}$;
4. randomly draw a position from $\text{supp}(H_{j,i_1}) \cap [1,n] \setminus \{\text{supp}(H_{j,i_1}) \cup \text{supp}(H_{1,i_1})\}$, $\forall l \in \text{supp}(H_{i_1,i_1}) \setminus \{1\}$, and erase the corresponding symbols. If the above set is empty, go back to step 1.

We remark that the probability that the $l$-th parity-check equation does not contain a symbol in $[1,n] \setminus \{\text{supp}(H_{j,i_1}) \cup \text{supp}(H_{1,i_1})\}$, whose cardinality is larger than $n - 2w$, is negligible since, by definition of LDPC code, $w \ll n$. Clearly, the above procedure does not require a significant computational effort from the adversary, with respect to a single random draw and erasure of up to $v$ symbols.

We are now ready to compute a lower bound on the probability with which the newly proposed DAA succeeds.

**Theorem IV.1.** The DAA described above provides ASP

$$\gamma \geq 1 - \left(1 - \frac{(n-v)}{s}\right)^m.$$  

**Proof:** Let us consider $w_H(e') = v$ and $E = \text{supp}(e')$. Since, by construction, incorrect-coding proofs cannot be computed by the oracle, we have that the adversary succeeds if there is at least one player who does not ask for symbols coming from $E$, thus accepting the header. For a single player, this happens with probability $p = \frac{\binom{n-v}{s}}{\binom{n}{s}}$. Considering $m$ players, the ASP is $1 - (1 - p)^m$. According to Lemma 1 the adversary may use a number of erasures which is smaller than $v$, thus we have a lower bound the actual ASP.

It is immediately seen that, since $v \ll n$ by definition of LDPC code, the success probability of our DAA is very large.

**B. Numerical examples**

Let us consider the code parameters proposed in [2] as a benchmark. It is shown in [2] Table 2] that the most favourable value of the stopping ratio of the constructed ensemble ($\beta^*$) is obtained when $w = 8$ and the code rate is $R = 1/4$, from which $v = 6$ easily follows. We consider two cases: a strong adversary able to find stopping sets and erase the corresponding symbols, and a weak adversary unable to find them and hence forced to erase random symbols. For the strong adversary, the undecodable ratio is $\alpha^* = \beta^* = 12.4\%$; in case of weak adversary, we instead have $\alpha^* = 47\%$ [2]. According to [2] Table 2], when $n = 4096$, the probability that the code stopping ratio $\alpha$ is smaller than the ensemble stopping ratio is relatively small ($3.2 \cdot 10^{-4}$). Notice that [3] is not influenced by the stopping ratio that, conversely, influences [2].

In Table I we report the upper bound on the code and the newly assessed lower bound on the ASP, for some values of $s$, considering $n = 4096$ and $m = 1024$: the new lower bound is never smaller than the previously computed upper bound. For the chosen parameters, the sampling procedure is very effective against both weak and strong adversaries, on condition that the DAA presented in Section IV is not considered. However, taking the novel DAA presented in Section IV into account, each light node needs to request a number of symbols which is extremely large, in order to obtain at least the same ASP as in [2]. For example, without the novel DAA, the target $\gamma = 10^{-2}$ (considered in [2]) is achieved with $s \approx 8$ for a weak adversary and $s = 35$ for a strong adversary. Instead, considering the novel DAA, the ASP is not lower than $10^{-2}$ for $s \in [1,3307]$. Moreover, even if light nodes sample up to the 60% of the codeword symbols, the ASP remains close to 1. We have verified that the same general conclusions hold true for several other values of $m \in [10^2,10^5]$.

**V. Conclusion**

We have shown that the SPAR protocol security can be undermined by a novel data availability attack, in which a malicious node introduces errors and erasures, in such a way that honest full nodes are not able to deliver fraud proofs.

| s  | Upper bound on $\gamma$ | Lower bound on $\gamma$ |
|----|-------------------------|-------------------------|
| $\gamma$ | Weak Adv. | Strong Adv. | Novel DAA |
| 8   | $6.23 \cdot 10^{-3}$   | $\approx 1$           | $\approx 1$            |
| 35  | $2.24 \cdot 10^{-10}$  | $9.72 \cdot 10^{-3}$  | $\approx 1$             |
| 200 | $7.16 \cdot 10^{-35}$  | $3.17 \cdot 10^{-12}$ | $\approx 1$               |
| 2000| $\approx 0$            | $1.02 \cdot 10^{-115}$| $\approx 1$             |
| 3500| $\approx 0$            | $\approx 0$            | $9.47 \cdot 10^{-3}$    |
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