Some virtually abelian subgroups of the group of analytic symplectic diffeomorphisms of $S^2$

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Abstract

We show that if $M$ is a compact oriented surface of genus 0 and $G$ is a subgroup of $\text{Symp}_\nu^\infty(M)$ which has an infinite normal solvable subgroup, then $G$ is virtually abelian. In particular the centralizer of an infinite order $f \in \text{Symp}_\nu^\infty(M)$ is virtually abelian. Another immediate corollary is that if $G$ is a solvable subgroup of $\text{Symp}_\nu^\infty(M)$ then $G$ is virtually abelian. We also prove a special case of the Tits Alternative for subgroups of $\text{Symp}_\nu^\infty(M)$.

1 Introduction

If $M$ is a compact connected oriented surface we let $\text{Diff}_0^r(M)$ denote the $C^r$ diffeomorphisms isotopic to the identity. In particular if $r = \omega$ this denotes the subgroup of those which are real analytic. Let $\text{Symp}_\nu^r(M)$ denote the subgroup of $\text{Diff}_0^r(M)$ which preserve a smooth volume form $\mu$. We will denote by $\text{Cent}^r(f)$ and $\text{Cent}_\mu^r(f)$ the subgroups of $\text{Diff}_0^r(M)$ and $\text{Symp}_\nu^r(M)$, respectively, whose elements commute with $f$. If $G$ is a subgroup of $\text{Symp}_\nu^r(M)$ then $\text{Cent}_\mu^r(f, G)$ will denote the the subgroup of $G$ whose elements commute with $f$. In this article we wish to address the algebraic structure of subgroups of $\text{Symp}_\nu^\infty(M)$ and our results are largely limited to the case of analytic diffeomorphisms when $M$ has genus 0. Our approach is to understand the possible dynamic behavior of such diffeomorphisms and use the structure exhibited to conclude information about subgroups.

Definition 1.1. If $N$ is a connected manifold, an element $f \in \text{Diff}^r(N)$ will be said to have full support provided $N \setminus \text{Fix}(f)$ is dense in $N$. We will say that $f$ has support of finite type if both $\text{Fix}(f)$ and $N \setminus \text{Fix}(f)$ have finitely many components. We will say that a subgroup $G$ of $\text{Diff}^r(N)$ has full support of finite type if every non-trivial element of $G$ has full support of finite type.

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The primary case of interest for this article is analytic diffeomorphisms, but we focus on groups with full support of finite type to emphasize the properties we use, which are dynamical rather than analytic in nature. The following result shows we include $\text{Diff}^{\omega}(M)$.

**Proposition 1.2.** If $N$ is a compact connected manifold, the group $\text{Diff}^{\omega}(N)$ has full support of finite type.

*Proof.* If $f$ is analytic and non-trivial the set $\text{Fix}(f)$ is an analytic set in $N$ and has no interior and the set $N \setminus \text{Fix}(f)$ is a semianalytic set. Hence $\text{Fix}(f)$ has finitely many components and $N \setminus \text{Fix}(f)$ has finitely many components (see Corollary 2.7 of [3] for both these facts).

The following result is due to Katok [11] who stated it only in the analytic case. For completeness we give a proof in section 3 but the proof we give is essentially the same as the analytic case.

**Proposition 1.3** (Katok [11]). Suppose $G$ is a subgroup of $\text{Diff}^2(M)$ which has full support of finite type and $f \in \text{Diff}^2(M)$ has positive topological entropy. Then the centralizer of $f$ in $G$, is virtually cyclic. Moreover, every infinite order element of this centralizer has positive topological entropy.

A corollary of the proof of this result is the following.

**Corollary 1.4.** Suppose $f, g \in \text{Diff}^2(M)$ have infinite order and full support of finite type. If $fg = gf$, then $f$ has positive topological entropy if and only if $g$ has positive entropy.

We remark that in contrast to our subsequent results Proposition 1.3 and its corollary make no assumption about preservation of a measure $\mu$.

We can now state our main result, whose proof is given in section 6.

**Theorem 1.5.** Suppose $M$ is a compact oriented surface of genus 0 and $G$ is a subgroup of $\text{Symp}_\mu^\infty(M)$ which has full support of finite type, e.g. a subgroup of $\text{Symp}_\mu^\omega(M)$. Suppose further that $G$ has an infinite normal solvable subgroup. Then $G$ is virtually abelian.

There are several interesting corollaries of this result. To motivate the first we recall the following result.

**Theorem 1.6** (Farb-Shalen [5]). Suppose $f \in \text{Diff}_0^\omega(S^1)$ has infinite order. Then the centralizer of $f$ in $\text{Diff}_0^\omega(S^1)$, is virtually abelian.

There are a number of instances where results about the algebraic structure of $\text{Diff}(S^1)$ have close parallels for $\text{Symp}_\mu(M)$. By analogy with the result of Farb and Shalen above, it is natural to ask the following question.
Question 1.7. Suppose $M$ is a compact surface and $G$ is a subgroup of $\text{Diff}_0^2(M)$ with full support of finite type and $f \in G$ has infinite order. Is the centralizer of $f$ in $G$ always virtually abelian?

The following result gives a partial answer to this question.

Proposition 1.8. Suppose $M$ is a compact surface of genus 0, $H$ is a subgroup of $\text{Symp}_\infty^\mu(M)$ with full support of finite type and $f \in H$ has infinite order, then $\text{Cent}_\mu^\infty(f,H)$, the centralizer of $f$ in $H$, is virtually abelian. In particular this applies when $H \subset \text{Symp}_\omega^\mu(S^2)$.

Proof. If we apply Theorem 1.5 to the group $G = \text{Cent}_\mu^\infty(f,H)$ and observe that the cyclic group generated by $f$ is a normal abelian subgroup, then we conclude that $G$ is virtually abelian. 

Question 1.9 of [6] asks if any finite index subgroup of $\text{MCG}(\Sigma_g)$ can act faithfully by area preserving diffeomorphisms on a closed surface. The following corollary of Proposition 1.8 and Proposition 1.2 gives a partial answer in the special case of analytic diffeomorphisms and $M$ of genus 0.

Corollary 1.9. Suppose $H$ is a finite index subgroup of $\text{MCG}(\Sigma_g)$ with $g \geq 2$ or $\text{Out}(F_n)$ with $n \geq 3$, and $G$ is a group with the property that every element of infinite order has a virtually abelian centralizer. Then any homomorphism $\nu : H \to G$ has non-trivial kernel. In particular this holds if $G = \text{Symp}_\omega^\mu(S^2)$.

Proof. Suppose at first that $H$ is a finite index subgroup of $\text{MCG}(\Sigma_g)$ with $g \geq 2$. Let $\alpha_1, \alpha_2$ and $\alpha_3$ be simple closed curves in $M$ such that $\alpha_1$ and $\alpha_2$ are disjoint from $\alpha_3$ and intersect each other transversely in exactly one point. Let $T_i$ be a Dehn twist around $\alpha_i$ with degree chosen so that $T_i \in H$. Then $T_1$ and $T_2$ commute with $T_3$ but no finite index subgroup of the group generated by $T_1$ and $T_2$ is abelian. It follows that $\mu$ cannot be injective.

Suppose now that $H$ is a finite index subgroup of $\text{Out}(F_n)$ with $n \geq 3$. Let $A, B$ and $C$ be three of the generators of $F_n$. Define $\Phi \in \text{Aut}(F_n)$ by $B \mapsto BA$, define $\Psi \in \text{Aut}(F_n)$ by $C \mapsto CBAB^{-1}A$ and define $\Theta \in \text{Aut}(F_n)$ by $C \mapsto CABAB^{-1}$, where $\Phi$ fixes all generators other than $B$ and where $\Psi$ and $\Theta$ fix all generators other than $C$. Let $\phi, \psi$ and $\theta$ be the elements of $\text{Out}(F_n)$ determined by $\Phi, \Psi$ and $\Theta$ respectively. It is straightforward to check that $\Phi$ commutes with $\Psi$ and $\Theta$ and that for all $k > 0$, $\Psi^k$ (which is defined by $C \mapsto C(BAB^{-1}A)^k$) does not commute with $\Theta^k$ (which is defined by $C \mapsto C(ABAB^{-1})^k$). Moreover, $[\Theta^k, \Psi^k]$ is not a non-trivial inner automorphism because it fixes both $A$ and $B$. It follows that $[\theta^k, \psi^k]$ is non-trivial for all $k$ and hence that no finite index subgroup of the group generated by $\psi$ and $\theta$, and hence no finite index subgroup of $\text{Cent}(\phi)$, the centralizer of $\phi$ in $\text{Out}(F_n)$, is abelian. The proof now concludes as above. 

Another interesting consequence of Theorem 1.5 is the following result.
Proposition 1.10. Suppose $M$ is a compact surface of genus 0 and $G$ is a solvable subgroup of $\text{Symp}_\mu^\infty(M)$ with full support of finite type, then $G$ is virtually abelian.

The Tits alternative (see J. Tits [16]) is satisfied by a group $G$ if every subgroup (or by some definitions every finitely generated subgroup) of $G$ is either virtually solvable or contains a non-abelian free group. This is a deep property known for finitely generated linear groups, mapping class groups [9], [12], and the outer automorphism group of a free group [1] [2]. It is an important open question for $\text{Diff}^\omega(S^1)$ (see [5]). It is known that this property is not satisfied for $\text{Diff}^\infty(S^1)$ (again see [5]).

This naturally raises the question for surfaces.

Conjecture 1.11 (Tits alternative). If $M$ is a compact surface then every finitely generated subgroup of $\text{Symp}_\mu^\omega(M)$ is either virtually solvable or contains a non-abelian free group.

In the final section of this paper we are able to use the techniques developed to prove an important special case of this conjecture.

Theorem 1.12. Suppose that $M$ is a compact oriented genus zero surface, that $G$ is a subgroup of $\text{Symp}_\mu^\omega(M)$ and that $G$ contains an infinite order element with entropy zero and at least three periodic points. Then either $G$ contains a subgroup isomorphic to $F_2$, the free group on two generators, or $G$ has an abelian subgroup of finite index.

We observe that one cannot expect the virtually abelian (as opposed to solvable) conclusion to hold more generally as the subgroup of $\text{Symp}_\mu^\omega(\mathbb{T}^2)$ generated by an ergodic translation and the automorphism with matrix

\[
\begin{pmatrix}
    1 & 1 \\
    0 & 1
\end{pmatrix}
\]

is solvable, but not virtually abelian.

2 Maximal Annuli for elements of $\text{Symp}_\mu^\infty(M)$.

Definition 2.1. Suppose $M$ is a compact oriented surface. The annular compactification of an open annulus $U \subset M$ is obtained by blowup on an end whose frontier is a single point and by the prime end compactification otherwise. We will denote it by $U_c$ (see 2.7 [6] for details).

For any diffeomorphism $h$ of $M$ which leaves $U$ invariant there is a canonical extension to a homeomorphism $h_c : U_c \to U_c$ which is functorial in the sense that $(hg)_c = h_cg_c$.

Definition 2.2. Suppose $M$ is a compact genus zero surface, $f \in \text{Symp}_\mu^\infty(M)$, and that the number of periodic points of $f$ is greater than the Euler characteristic of $M$. If $f$ has infinite order and entropy 0, we will call it a multi-rotational diffeomorphism. This set of diffeomorphisms will be denoted $Z(M)$.
The rationale for the terminology multi-rotational comes from the following result which is a distillation of several results from [6] (see Theorems 1.2, 1.4 and 1.5 from that paper). In particular, if \( f \in \mathcal Z(M) \) then every point of \( \text{int}(M) \setminus \text{Fix}(f) \) has a well defined rotation number (with respect to any pair of components from \( \partial M \cup \text{Fix}(f) \) and there are non-trivial intervals of rotation numbers.

**Theorem 2.3.** Suppose \( M \) is a compact genus zero surface and \( f \in \mathcal Z(M) \). The collection \( A = A_f \) of maximal \( f \)-invariant open annuli in \( \text{int}(M) \setminus \text{Fix}(f) \) satisfies the following properties:

1. The elements of \( A \) are pairwise disjoint.
2. The union \( \bigcup_{U \in A} U \) is a full measure open subset of \( M \setminus \text{Fix}(f) \).
3. Each \( U \in A \) is essential in \( \text{int}(M) \setminus \text{Fix}(f) \).
4. For each \( U \in A \), the rotation number \( \rho_f : U_c \to S^1 \) is continuous and non-constant. Each component of the level set of \( \rho_f \) which is disjoint from \( \partial U_c \) is essential in \( U \), i.e. separates \( U_c \) into two components, each containing a component of \( \partial U_c \).

We will make repeated use of the properties in the following straightforward corollary.

**Corollary 2.4.** Suppose \( M \) is a compact genus zero surface, \( f \in \mathcal Z(M) \) and \( \text{Cent}_{\mu}^\infty(f) \) denotes its centralizer in \( \text{Symp}_{\mu}^\infty(M) \).

1. If \( g \in \text{Cent}_{\mu}^\infty(f) \) and \( U \in A_f \) then \( g(U) \in A_f \) and the \( \text{Cent}_{\mu}^\infty(f) \)-orbit of \( U \) is finite.
2. If \( U \in A_f \) then any \( g \in \text{Cent}_{\mu}^\infty(f) \) which satisfies \( g(U) = U \) must preserve the components of each level set of \( \rho_f : U_c \to S^1 \).
3. If \( f \) has support of finite type then the set \( A_f \) of maximal annuli for \( f \) is finite.

**Proof.** If \( g \in \text{Cent}_{\mu}^\infty(f) \) then \( g(\text{Fix}(f)) = \text{Fix}(f) \) so \( \text{int}(M) \setminus \text{Fix}(f) \) is \( g \)-invariant. Also \( f(g(U)) = g(f(U)) = g(U) \) so \( g(U) \) is \( f \)-invariant. Clearly \( U \) is a maximal \( f \)-invariant annulus in \( \text{int}(M) \setminus \text{Fix}(f) \) if and only if \( g(U) \) is. This proves \( g(U) \in A_f \). The \( \text{Cent}_{\mu}^\infty(f) \)-orbit of \( U \) consists of pairwise disjoint annuli each with the same positive measure. There can only be finitely many of them in \( M \). This proves (1).

To show (2) we observe that \( g \) is a topological conjugacy from \( f \) to itself and rotation number is a conjugacy invariant. Hence \( g \) must permute the components of the level sets of \( \rho_f \). Clearly those level sets which contain a component of \( \partial U_c \) must be preserved. Since any other such component separates \( U \), if it were not \( g \)-invariant the fact that \( g \) is area preserving would be contradicted.
Since the elements of \( A_f \) are maximal \( f \)-invariant annuli in \( \text{int}(M) \setminus \text{Fix}(f) \), they are mutually non-parallel in \( \text{int}(M) \setminus \text{Fix}(f) \). Let \( E \) be the union of a set of core curves for some given finite subset of \( A_f \). Theorem 2.3-(3) implies that each component of \( M \setminus E \) either contains a component of \( \text{Fix}(f) \cup \partial M \) or has negative Euler characteristic. The dual graph for \( E \) is a tree that has one edge for each element of \( E \) and has at most as many valence one and valence two vertices as there are components of \( \text{Fix}(f) \cup \partial M \). Since the Euler characteristic of the tree is 1 there is a uniform bound on the number of its edges. It follows that the cardinality of \( E \), and hence the number of elements of \( A_f \), is uniformly bounded. This proves (3).

3 The positive entropy case.

Lemma 3.1. Suppose that \( M \) is a closed surface, that \( f \in \text{Diff}^2(M) \) and that \( g \) commutes with \( f \). Suppose further that \( f \) has a hyperbolic fixed point \( p \) of saddle type, and that \( g \) fixes \( p \) and preserves the branches of \( W^s(p,f) \). Then there is a \( C^1 \) coordinate function \( t \) on \( W^s(p,f) \) and a unique number \( \alpha > 0 \) such that in these coordinates \( g(t) = \alpha t \).

Proof. By Sternberg linearization (see Theorem 2 of [15]) there is a \( C^1 \) coordinate function \( t(x) \) on \( W^s(p,f) \) in which the restriction of \( f \) (also denoted \( f \)) satisfies \( f(t) = \lambda t \) where \( \lambda \in (0,1) \) is the eigenvalue of \( Df_p \) corresponding to the eigenvector tangent to \( W^s(p,f) \).

In these coordinates \( g(t) = \lambda^{-n}g(\lambda^n t) \). Applying the chain rule gives \( g'(t) = g'(\lambda^n t) \). Letting \( n \) tend to infinity we get \( g'(t) = g'(0) \) so \( g'(x) \) is constant and \( g(t) = \alpha t \) where \( \alpha = g'(0) \).

Definition 3.2. Suppose that \( M \) is a compact surface and that \( f \in \text{Diff}^r(M) \) has a hyperbolic fixed point \( p \) of saddle type. We define \( \text{Cent}^r_p(f) \) to be the subgroup of elements of \( \text{Diff}^r(M) \) which commute with \( f \), fix the point \( p \), and preserve the branches of \( W^s(f,p) \). Let \( \mathbb{R}^+ \) denote the multiplicative group of positive elements of \( \mathbb{R} \). The expansion factor homomorphism

\[
\phi : \text{Cent}^r_p(f) \to \mathbb{R}^+,
\]

is defined by \( \phi(g) = \alpha \) where \( \alpha \) is the unique number given by Lemma (3.1) for which \( g(x) = \alpha x \) in \( C^1 \) coordinates on \( W^s(f,p) \). It is immediate that \( \phi \) is actually a homomorphism. We also observe that \( \phi(g) \) is just the eigenvalue of \( Dg_p \) whose eigenvector is tangent to \( W^s(f,p) \).

The following result is due to Katok [11] who stated it only in the analytic case, but gave a proof very similar to the one below.

Proposition 1.3 ([Katok [11]]). Suppose \( G \) is a subgroup of \( \text{Diff}^2(M) \) which has full support of finite type and \( f \in \text{Diff}^2(M) \) has positive topological entropy. Then the
centralizer of $f$ in $G$, $\text{Cent}^2(f, G)$, is virtually cyclic. Moreover, every infinite order element of $\text{Cent}^2(f, G)$ has positive topological entropy.

Proof. By a result of Katok [10] there is a hyperbolic periodic point $p$ for $f$ of saddle type with a transverse homoclinic point $q$. Let $\phi : \text{Cent}^2_p(f) \to \mathbb{R}^+$ be the expansion factor homomorphism of Definition 3.2 and let $H = \text{Cent}^2_p(f) \cap G$. Then $H$ has finite index in $\text{Cent}^2(f, G)$ because otherwise the $\text{Cent}^2(f, G)$ orbit of $p$ is infinite and consists of hyperbolic fixed points of $f$ all with the same eigenvalues. This is impossible because a limit point would be a non-isolated hyperbolic fixed point. For the main statement of the proposition, it suffices to show that $H$ is cyclic and for this it suffices to show that the restriction $\phi|_H : H \to \mathbb{R}^+$ is injective and has a discrete image.

To show that $\phi|_H$ is injective it suffices to show that if $g \in H$ and $\phi(g) = 1$ then $g = id$. But $\phi(g) = 1$ implies $W^s(p, f) \subset \text{Fix}(g)$. Note that if $\phi' : \text{Cent}^2_p(f^{-1}) \to \mathbb{R}^+$ is the expansion homomorphism for $f^{-1}$ then $\phi(g)\phi'(g) = \det(Dg_q) = 1$ so we also know that $W^u(p, f) \subset \text{Fix}(g)$. Hence we need only show that this implies $g = id$. Let $J_s$ be the interval in $W^s(p, f)$ joining $p$ to $q$ and define $J_u \subset W^u(p, f)$ analogously. Define $K_n = f^{-n}(J_s) \cup f^n(J_u)$. Then the number of components of the complement of $K_n$ tends to $+\infty$ with $n$. Since $g$ has full support of finite type, each component $V$ of the complement of $K_n$ contains a component of $M \setminus \text{Fix}(g)$, in contradiction to the fact that $M \setminus \text{Fix}(g)$ has only finitely many components. We conclude that $\phi$ is injective.

To show that $\phi|_H$ has discrete image we assume that there is a sequence $\{g_n\}$ of elements of $H$ such that $\lim \phi(g_n) = 1$, and show that $\phi(g_n) = 1$ for $n$ sufficiently large. Let $I_s = f^{-1}(J_s)$ and $I_u = f(J_u)$. Since $I_s$ and $I_u$ are compact intervals and the point $q$ is a point in the interior of each where they intersect transversely, there is a neighborhood $U$ of $q$ such that $U \cap I_s \cap I_u = \{q\}$. But for $n$ sufficiently large $g_n(q) \in U$ and $g_n(q) \in (I_s \cap I_u)$. It follows that $g_n(q) = q$ for $n$ sufficiently large and hence $\phi(g_n) = 1$. This proves that the image of $\phi|_H$ is discrete and hence that $H$ is cyclic.

Each non-trivial element $h \in H$ has $\phi(h) \neq 1$ and $\phi'(h) \neq 1$. Hence $p$ is a hyperbolic fixed point of $h$ and $q$ is a transverse homoclinic point for $h$. It follows that $h$ has positive entropy (see Theorem 5.5 of [14]).

Proposition 3.3. Suppose $G$ is a subgroup of $\text{Diff}^2(M^2)$ which has full support of finite type and $A$ is an abelian normal subgroup of $G$. If there is an element $f \in A$ with positive topological entropy then $G$ is virtually cyclic.

Proof. It follows from Proposition 1.3 that the group $A$ is virtually cyclic. Since $f \in A$ has positive entropy it is infinite order and generates a finite index subgroup $A_f$ of $A$. Hence there exists a positive integer $k$ such that $a^k \in A_f$ for all $a \in A$. In particular, for each $g \in G$, we have $gf^k g^{-1} = (gf g^{-1})^k = f^m$ for some $m \in \mathbb{Z}$. Since $f$ and
$gf g^{-1}$ are conjugate, the topological entropy $\ent(f) = \ent(gfg^{-1})$. Hence
\[
\ent(gfg^{-1})^k = |k| \ent(gfg^{-1}) = |k| \ent(f), \text{ and } \ent(f^m) = |m| \ent(f).
\]
We conclude that $m = \pm k$ and hence that $gf^kg^{-1} = f^{\pm k}$ for all $g \in G$. Let $G_0$ be the subgroup of index at most two of $G$ such that $gf^kg^{-1} = f^k$ for all $g \in G_0$. Then $G_0 \subset \Cent^2(f^k, G)$ and Proposition 1.3 completes the proof.

4 Mean Rotation Numbers

In this section we record some facts (mostly well known) about rotation numbers for homeomorphisms of area preserving homeomorphisms of the closed annulus. In subsequent sections we will want to apply these results when $M$ is a surface, $f \in \Symp^\infty_\mu(M)$, and $U$ is an $f$-invariant open annulus. We will do this by considering the extension of $f$ to the closed annulus $U_c$ which is the annular compactification of $U$. When the annulus $U$ is understood, we will use $\rho_f(x)$ to mean the rotation number of $x \in U$ with respect to the homeomorphism $f_c : U_c \to U_c$ of the closed annulus $U_c$.

**Definition 4.1.** Suppose that $f : A \to A$ is a homeomorphism of a closed annulus $A = S^1 \times [0, 1]$ preserving a measure $\mu$. For each lift $\tilde{f}$, let $\Delta_{\tilde{f}}(x) = p_1(\tilde{f}(\tilde{x})) - p_1(\tilde{x})$ where $\tilde{x}$ is a lift of $x$ and $p_1 : \tilde{A} = \tilde{R} \times [0, 1] \to \tilde{R}$ is projection onto the first factor. As reflected in the notation, $\Delta_{\tilde{f}}(x)$ is independent of the choice of lift $\tilde{x}$ and hence may be considered as being defined on $A$.

If $X \subset A$ is an $f$-invariant $\mu$-measurable set then the mean translation number relative to $X$ and $\tilde{f}$ is defined to be
\[
\mathcal{T}_\mu(\tilde{f}, X) = \int_X \Delta_{\tilde{f}}(x) \, d\mu.
\]
We define the mean rotation number relative to $X$, $\rho_\mu(f, X)$ to be the coset of $\mathcal{T}_\mu(\tilde{f}, X)$ mod $\mathbb{Z}$ thought of as an element of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

The mean rotation number is independent of the choice of lift $\tilde{f}$ since different lifts give values of $\mathcal{T}(\tilde{f}, X)$ differing by an element of $\mathbb{Z}$.

We define the translation number of $x$ with respect to $\tilde{f}$ (see e.g. Definition 2.1 of [6]), by
\[
\tau_{\tilde{f}}(x) = \lim_{n \to \infty} \frac{1}{n} \Delta_{\tilde{f}}^n(x).
\]
A straightforward application of the Birkhoff ergodic theorem implies that $\tau_{\tilde{f}}(x)$ is well defined for almost all $x$ and
\[
\mathcal{T}_\mu(\tilde{f}, X) = \int_X \tau_{\tilde{f}}(x) \, d\mu.
\]
If $X$ is also $g$-invariant and $g$ preserves $\mu$ then
\[ \Delta_{f\tilde{g}}(x) = \Delta\tilde{g}(x) + \Delta\tilde{f}(\tilde{g}(x)) \]
and hence integrating we obtain
\[ T_\mu(f\tilde{g}, X) = T_\mu(f, X) + T_\mu(\tilde{g}, X) \]
\[ \rho_\mu(fg, X) = \rho_\mu(f, X) + \rho_\mu(g, X), \]
i.e. $f \mapsto \rho_\mu(f, X)$ is a homomorphism from the group of $\mu$ preserving homeomorphisms of $A$ to $T = \mathbb{R}/\mathbb{Z}$. Hence if $h = [g_1, g_2]$ for some $g_i : A \to A$ that preserve $\mu$ then $\rho_\mu(h) = 0$.

**Lemma 4.2.** Let $f : A \to A$ be an area preserving homeomorphism of a closed annulus isotopic to the identity. Suppose $X \subset A$ is an $f$-invariant subset with Lebesgue measure $\mu(X) > 0$. If $\rho_\mu(f, X) = 0$ then $f$ has a fixed point in the interior of $A$.

*Proof.* Replacing $X$ with $X \cap \text{int}(A)$ we may assume $X \subset \text{int}(A)$. If $\tau_f(x) = 0$ on a positive measure subset of $X$ then the fixed point exists by Proposition 2.4 of [6]. Otherwise for some $\epsilon > 0$ there is a positive measure $f$-invariant subset $X^+$ on which $\rho_f > \epsilon$ and another $X^-$ on which $\rho_f < -\epsilon$. If $x \in X^+$ is recurrent and not fixed then it is contained in a positively recurring disk which is disjoint from its $f$-image. Similarly if $y \in X^-$ is recurrent and not fixed then it is contained in a negatively recurring disk which is disjoint from its $f$-image. Theorem 2.1 of [7] applied to $f$ on the open annulus $\text{int}(A)$ then implies there is a fixed point in $\text{int}(A)$. \qed

**Lemma 4.3.** Suppose that $f : A \to A$ is an area preserving homeomorphism of a closed annulus which is isometric to the identity. Suppose also that $V \subset A$ is a connected open set which is not essential in $A$ and such that $f^m(V) = V$ for some $m \neq 0$. Then there is a full measure subset $W$ of $V$ on which $\rho_f$ assumes a constant rational value.

*Proof.* Replacing $V$ with $V \cap \text{int}(A)$ we may assume $V \subset \text{int}(A)$. Replacing $f$ with $f^m$ it will suffice to show that $f(V) = V$ implies $\rho_f = 0$ on a full measure subset of $V$. Since $V$ does not contain a simple closed curve that separates the components of $\partial A$, both components of $\partial A$ belong to the same component $X$ of $A \setminus V$ by Lemma 3.1 of [6]. Note that $D := A \setminus X$ contains $V$, is contained in $\text{int}(A)$ and is $f$-invariant. Lemma 3.2 of [6] implies that $D$ is connected and it is simply connected since its complement $X$ is connected. Hence $D$ is an open disk.

By the Brouwer plane translation theorem $D$ contains a point of $\text{Fix}(f)$. Let $\hat{A}$ be the universal cover of $A$ and let $\hat{D} \subset \hat{A}$ be an open disk which is a lift of $D$. Since $f$ has a fixed point in $D$ there is a lift $\hat{f} : \hat{A} \to \hat{A}$ which has a fixed point in $\hat{D}$. Therefore $\hat{f}(\hat{D}) = \hat{D}$. If $p_1(\hat{D})$ is bounded then it is obvious that $\rho_f$ is constant and 0 on $D$. Otherwise, note that by Poincaré recurrence and the Birkhoff ergodic theorem there is a full measure subset $\hat{W}$ of $\hat{D}$ consisting of points which are recurrent and have a well defined translation number. Calculating the translation number of a point $\hat{x} \in \hat{W}$ on a subsequence of iterates which converges to a point of $\hat{A}$ shows that it must be 0. Hence $\hat{W}$, the projection of $\hat{W}$ to $D$, has the desired properties. \qed
5 Pseudo-rotation subgroups of $\text{Symp}_\mu^r(M)$.

**Definition 5.1.** Suppose $M$ is a compact oriented surface. A *pseudo-rotation subgroup* of $\text{Symp}_\mu^r(M)$ with $r \geq 1$, is a subgroup $G$ with the property that every non-trivial element of $G$ has exactly $\chi(M)$ fixed points. An element $f \in \text{Symp}_\mu^r(M)$ will be called a *pseudo-rotation* provided the cyclic group it generates is a pseudo-rotation group.

Our definition may be slightly non-standard in that we consider finite order elements of $\text{Symp}_\mu^r(M)$ to be pseudo-rotations. We observe that if $\chi(M) < 0$ then any pseudo-rotation group is trivial. Since we assume $M$ is oriented we have only the cases that $M$ is $A$, $D^2$, or $S^2$ for which any pseudo-rotations must have precisely 0, 1, or 2 fixed points respectively. By far the most interesting case is $M = S^2$ since we will show in Lemma 5.7 that when $M = D^2$ or $A^2$ any pseudo-rotation group is abelian. This is not the case when $M = S^2$ since $SO(3)$ acting on the unit sphere in $\mathbb{R}^3$ is a pseudo-rotation group.

There are three immediate but very useful facts about pseudo-rotations which we summarize in the following Lemma:

**Lemma 5.2.** Suppose $M$ is a compact surface and $f \in \text{Symp}_\mu^r(M)$ is a pseudo-rotation.

*(1)* Either $f$ has finite order or $\text{Fix}(f) = \text{Per}(f)$.

*(2)* If $\text{Fix}(f)$ contains more than $\chi(M)$ points then $f = \text{id}$. In particular if $f, g$ are elements of a pseudo-rotation group which agree at more than $\chi(M)$ points then $fg^{-1} = \text{id}$ so $f = g$.

*(3)* If $M = D^2$, then the one fixed point of $f$ is in the interior of $D^2$.

**Proof.** Parts (1) and (2) are immediate from the definition of pseudo-rotation and part (3) holds because otherwise the restriction of $f$ to the interior would have recurrent points but no fixed point, contradicting the Brouwer plane translation theorem.

We will make repeated use of these properties.

**Proposition 5.3.** Suppose $f \in \text{Symp}_\mu^r(M)$ is a pseudo-rotation. Then $U = \text{int}(M) \setminus \text{Fix}(f)$ is an open annulus and $\rho_f : U_c \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a constant function. Moreover, if $G \subset \text{Symp}_\mu^r(M)$ is an abelian pseudo-rotation group, with $\text{Fix}(g) = \text{Fix}(f)$ for all $g \in G$, then the assignment $g \mapsto \rho_g$, for $g \in G$, is an injective homomorphism, so, in particular, if $g$ has infinite order then the constant $\rho_g$ is irrational.

**Proof.** The only possibilities for $M$ are $S^2, A$ or $D^2$. If $M = D^2$, as remarked above, its one fixed point must be in the interior of $M$. Hence in all cases $U$ is an open annulus. If $\rho_f$ is not constant on $U_c$ then the Poincaré-Birkhoff Theorem (see Theorem (2.2)})
of [6] for example) implies there are periodic points with arbitrarily large period. We may therefore conclude that $\rho_f$ is constant and equal to $\rho_f(f)$. Suppose now that $G$ is an abelian pseudo-rotation subgroup containing $f$ and $\text{Fix}(f) = \text{Fix}(g)$ for all $g \in G$, so $U$ is $G$-invariant. Since $\rho_\mu : G \to \mathbb{T}$ is a homomorphism $\rho_\mu(g)$ being rational implies $\rho_\mu(g^n) = 0$ for some $n$. Then Lemma 4.2 implies the existence of a fixed point in $U$ for $g^n$. Hence $g^n = id$. We conclude that if $g$ has infinite order then $\rho_\mu(g)$ is irrational. If $\rho_\mu(f) = \rho_\mu(g)$ then $\rho_\mu(fg^{-1}) = 0$ and the same argument shows $fg^{-1} = id$, so the assignment $f \mapsto \rho_\mu(f)$ for $f$ in the abelian group $G$ is injective. \hfill \Box

We observed above that if $f$ is a non-trivial pseudo-rotation then either $f$ has finite order or $\text{Per}(f) = \text{Fix}(f)$. We now prove the converse to this statement.

**Proposition 5.4.** Suppose $M$ is $\mathbb{A}$, $\mathbb{D}^2$ or $S^2$ and $G$ is a subgroup of $\text{Symp}_\infty^\mu(M)$ with the property that every non-trivial element $g$ of $G$ either has finite order or satisfies $\text{Fix}(g) = \text{Per}(g)$. Then $G$ is a pseudo-rotation group.

**Proof.** No element $g \in G$ can have positive entropy since that would imply $g$ has points of arbitrarily high period (see Katok [10]), a contradiction. If $g \in G$, then it must have at least $\chi(M)$ fixed points because the Lefschetz index of a fixed point of $g$ is at most 1 (see [13], for example). If $g$ is non-trivial and has more than $\chi(M)$ fixed points it follows from part (4) of Theorem 2.3 that there is an $g$-invariant annulus $U \subset \text{int}(M)$ for which the rotation number function is continuous and non-constant.

It then follows from Corollary 2.4 of [8] that there are periodic points with arbitrarily high period, contradicting our hypothesis. We conclude that either $g$ has finite order or $\text{card}(\text{Per}(g)) = \chi(M)$. \hfill \Box

**Lemma 5.5.** Suppose $U \subset M$ is an open annulus and that $C_1$ and $C_2$ are disjoint closed connected sets in $U$ each of which separates the ends of $U$ and such that both $U \setminus C_1$ and $U \setminus C_2$ have two components. Then $M \setminus (C_1 \cup C_2)$ has three components: one with frontier contained in $C_1$, one with frontier contained in $C_2$ and an open annulus with frontier contained in $(C_1 \cup C_2)$.

**Proof.** Clearly $M \setminus C_i$ has two components. Let $X_1$ and $X_2$ [resp. $Y_1$ and $Y_2$] be the components of $M \setminus C_1$ [resp. $M \setminus C_2$] labeled so that $C_2 \subset X_2$ and $C_1 \subset Y_1$. Then $X_1$ and $Y_2$ are components of $M \setminus (C_1 \cup C_2)$ with frontiers contained in $C_1$ and $C_2$ respectively and it suffices to show that $V = X_2 \cap Y_1$ is an open annulus. We use the fact that an open connected subset of $S^2$ whose complement has two components is an annulus. Of course the same is true for $M$ which may be considered as a subsurface of $S^2$. But $X_1 \cup C_1 = \text{cl}(X_1) \cup C_1$ is connected and similarly so is $Y_2 \cup C_2 = \text{cl}(Y_2) \cup C_2$. Hence $M \setminus V$ has two components, namely $X_1 \cup C_1$ and $Y_2 \cup C_2$. So $V$ is an annulus. \hfill \Box

Our next result includes a very special case of Proposition (1.8) namely we show that the centralizer of an infinite order pseudo-rotation is virtually abelian. In fact, for later use, we need to consider a slightly more general setting, namely, not just
the centralizer of a single pseudo-rotation, but the centralizer of an abelian pseudo-rotation group all of whose non-trivial elements have the same fixed point set.

**Lemma 5.6.** Suppose \( A \) is an abelian pseudo-rotation subgroup of \( \text{Symp}_\mu^\infty(M) \) containing elements of arbitrarily large order. Suppose also that there is a set \( F \) with \( \chi(M) \) points such that \( \text{Fix}(f) = F \) for all non-trivial \( f \in A \). Then the subgroup \( \mathcal{C}_0 \) of the centralizer, \( \text{Cent}_\mu^\infty(A) \), of \( A \) in \( \text{Symp}_\mu^\infty(M) \) consisting of those elements which pointwise fix \( F \), is abelian and has index at most 2 in \( \text{Cent}_\mu^\infty(A) \).

**Proof.** Elements of \( A \) must have entropy 0 since positive entropy implies the existence of infinitely many periodic points by a result of Katok, [10]. Let \( U \) be the open annulus \( M \setminus (F \cup \partial M) \). The group \( \text{Cent}_\mu^\infty(A) \) preserves \( F \cup \partial M \) and hence \( U \). The subgroup \( \mathcal{C}_0 \subset \text{Cent}_\mu^\infty(A) \), whose elements fix the ends of \( U \), has index at most 2.

We claim that the elements of \( \mathcal{C}_0 \) all have entropy 0. Clearly we need only consider elements of infinite order since all finite order homeomorphisms have entropy 0. Hence the claim follows from Corollary (1.4) if there is an element of infinite order in \( A \). Otherwise if \( \mathcal{C}_0 \) contains an element with positive entropy then it contains an element \( g \) with a hyperbolic fixed point \( p \in U \). Each point in the \( A \)-orbit of \( p \) is in \( \text{Fix}(g) \) and has the same set of eigenvalues for the derivative \( Dg \). It follows that the \( A \)-orbit of \( p \) has finite cardinality, say \( m \), and hence that \( p \in \text{Fix}(f^m) \) for all \( f \in A \). Part (2) of Lemma 5.2 then implies that \( f^m = \text{id} \) for all \( f \in A \), if \( m > 2 \). This contradicts the fact that \( A \) contains elements of arbitrarily high order. This completes the proof that all elements of \( \mathcal{C}_0 \) have entropy zero.

To prove that \( \mathcal{C}_0 \) is abelian, we will show that each commutator \( h \) of two elements in \( \mathcal{C}_0 \) is the identity by assuming that \( h \) is non-trivial and arguing to a contradiction. Since \( h \) is a commutator the map \( h_c : U_c \to U_c \) has mean rotation number \( \rho_\mu(h_c) = 0 \) and hence, by Lemma 4.2, \( h \) has a fixed point in \( U \). Therefore \( \text{Fix}(h) \) contains more than \( \chi(M) \) points. If \( h \) has finite order then in a suitable averaged metric it is an isometry of \( M \). But then each fixed point must have Lefschetz number +1 and hence by the Lefschetz theorem \( h \) has \( \chi(M) \) fixed points, a contradiction. We conclude therefore that \( h \) has infinite order and hence satisfies the hypothesis of Theorem (2.3). By this theorem there exists an element \( V \in \mathcal{A}_h \) and it must be the case that \( V \subset U \) since \( V \) cannot contain a point of \( \text{Fix}(h) \supset F \). We will show this leads to a contradiction, either by showing that \( h \) has a fixed point in \( V \) or by showing that some non-trivial \( f \in A \) has a fixed point in \( U \). We may then conclude \( h = \text{id} \) and hence that \( \mathcal{C}_0 \) is abelian.

Suppose first that \( V \) is inessential in \( U \). Elements of \( A \) commute with \( h \) and hence permute the elements of \( \mathcal{A}_h \) by Corollary 2.4. Since there can be only finitely many elements of \( \mathcal{A}_h \) of any fixed area, the \( A \)-orbit of \( V \) is finite. Hence there is \( m \) such that \( f^m(V) = V \) for all \( f \in A \). The union of \( V \) with the component of its complement which is disjoint from \( F \) is an open disk \( D \subset U \) satisfying \( f^m(D) = D \) for all \( f \in A \). It follows from the Brouwer plane translation theorem that \( f^m \) has a fixed point in \( D \). If the order of \( f \) is greater than \( m \) then \( f^m \) is a non-trivial element of \( A \) with
fixed points not in $F$ contradicting the assumption that $\text{Fix}(f) = F$ for all non-trivial $f \in A$. So the possibility that $V$ is inessential in $U$ is contradicted.

We are now reduced to the case that $V$ is essential in $U$. It follows from part (1) of Corollary 2.4 that each element of $A$ preserves $V$ or maps it to a disjoint element of $\mathcal{A}_h$. Since the annulus $U$ is $A$-invariant and $V$ is essential in $U$ it follows that $V$ is $A$-invariant as the alternative would contradict the fact that $A$ preserves area. We want to replace $V$ with a slightly smaller essential $V_0 \subset V$ which has the property that its frontier (in $M$) lies in $V$ and has measure 0. To do this we observe that $\rho_h$ is non-constant on $V$ and hence has uncountably many level sets. Hence there must be two of its level sets $C_1, C_2$ which have measure 0. Let $V_0$ be the essential open annulus in $V$ whose frontier lies in $C_1 \cup C_2$ (see Lemma 5.5). Then $\mu(V_0) = \mu(\text{cl}_{U_c}(V_0))$ since $\text{cl}_{U_c}(V_0) \setminus V_0 \subset C_1 \cup C_2$. It follows from part (2) of Corollary 2.4 that each $C_i$ is $A$-invariant and hence $V_0$ is also.

As a first subcase, suppose that $\text{cl}_{U_c}(V_0)$ is $g$-invariant for each $g \in \mathcal{C}_0$. Then $h$ is a commutator of elements that preserve $\text{cl}_{U_c}(V_0)$, so
\[
\rho_\mu(h, \text{cl}_{U_c}(V_0)) = 0,
\]
(measured in $U_c$). Since $V_0$ differs from $\text{cl}_{U_c}(V_0)$ in a set of measure zero, $\rho_\mu(h, V_0) = 0$ also. The translation number with respect to a lift $\tilde{h}$ of a point $x \in V_0$ can be measured in either $U_c$ or $(V_0)_c$ giving $\tau_\tilde{h}(x)$ and $\tau_{h|_{\tilde{V}_0}}(x)$. But $x \in V_0$ lies on a compact $h$-invariant level set $C_0$ which is in the interior of both $U_c$ and $(V_0)_c$. This implies $\tau_{h}(x) = \tau_{h|_{\tilde{V}_0}}(x)$. Hence we conclude that $\rho_\mu(h) = 0$, measured in $(V_0)_c$ and by Lemma 4.2, $h$ has a fixed point in $V_0$, a contradiction.

The last subcase is that there exists $g \in \mathcal{C}_0$ such that $g(V_0) \not\subset \text{cl}_{U_c}(V_0) = \text{cl}_{U_c}(V_0)$. Choose a component $W$ of $g(V_0) \cap (U \setminus \text{cl}_{U_c}(V_0))$. Since $g$ leaves $U$ invariant and preserves area and since $g^{-1}(W) \cap W = \emptyset$, $W$ cannot contain a closed curve $\alpha$ that is essential in $U$. Thus $W$ is inessential in $U$. As we noted above $V_0$ is $A$-invariant. Since $g \in \mathcal{C}_0$, it commutes with $A$ so $g(V_0)$ is also $A$-invariant. It follows that $A$ permutes the components of $g(V_0) \cap (U \setminus \text{cl}_{U_c}(V_0))$. In particular for some $m > 0$ we have $f^m(W) = W$ for all $f \in A$. Letting $D$ be the union of $W$ with any components of its complement which do not contain ends of $U$ we conclude that $D$ is an open disk and $f^m(D) = D$ for all $f \in A$. By the Brouwer plane translation theorem there is a point of $\text{Fix}(f^m)$ in $D$. Since $f^m$ is a pseudo-rotation and has more than $\chi(M)$ fixed points, $f^m = id$. Since this holds for any $f \in A$ we have contradicted the hypothesis that $A$ contains elements of arbitrarily high order. \hfill \Box

**Lemma 5.7.** Suppose that $G$ is a pseudo-rotation subgroup of $\text{Symp}_\mu^r(M)$ where $M = \mathbb{A}$ or $\mathbb{D}^2$ and $r \geq 2$. Then $G$ is abelian and $\text{Fix}(g)$ is the same for all non-trivial $g \in G$.

**Proof.** If $M = \mathbb{A}$ then $\text{Fix}(g) = \emptyset$ for all non-trivial $g \in G$. Since $\rho_\mu([f, g]) = 0$ for each $f, g \in G$, Lemma 4.2 implies that each $[f, g]$ has a fixed point in the interior of $\mathbb{A}$ and hence is the identity. Thus each $f$ and $g$ commute and we are done.
We assume for the remainder of the proof that \( M = \mathbb{D}^2 \). For each \( f \in G \) we consider \( \hat{f} \), the restriction of \( f \) to \( \partial \mathbb{D}^2 \). Let \( \hat{G} \) be the image in \( \text{Diff}^r(S^1) \) of \( G \) under the homomorphism \( f \mapsto \hat{f} \).

Lemma 5.2 (3) implies that \( f \) fixes a point in the interior of \( \mathbb{D}^2 \). If \( f^k \) fixes a point in \( \partial \mathbb{D}^2 \) for some \( k \geq 1 \) then \( f^k \) has more than \( \chi(M) \) fixed points and so is the identity. We conclude that if \( \hat{f} \) has a point of period \( k \) then \( f \) is periodic of period \( k \). In particular, if \( \hat{f} : S^1 \to S^1 \) has a fixed point then \( f = \text{id} \). This proves that \( \hat{G} \) acts freely on \( S^1 \). It also proves that the restriction homomorphism \( f \mapsto \hat{f} \) is an isomorphism \( G \to \hat{G} \). Since \( \hat{G} \) acts freely on \( S^1 \) it follows from Hölder’s Theorem (see, e.g., Theorem 2.3 of [5]) that \( \hat{G} \) (and hence \( G \)) is abelian.

Suppose \( f, g \in G \) and let \( \text{Fix}(f) = \{q\} \). Since \( f \) and \( g \) commute, \( g \) preserves \( \{q\} \) and so fixes \( q \). This proves that \( \text{Fix}(f) = \text{Fix}(g) \) and completes the proof.

If \( f \in \text{Diff}^r(S^2) \) and \( p \in \text{Fix}(f) \) we can compactify \( S^2 \setminus \{p\} \) by blowing up \( p \), i.e. adding a boundary component on which we extend \( f \) by letting it be the projectivization of \( Df_p \). More precisely we define \( \tilde{f}_p : S^1 \to S^1 \) by considering the boundary as the unit circle in \( \mathbb{R}^2 \) and letting

\[
\tilde{f}_p(v) = \frac{Df_p(v)}{|Df_p(v)|}.
\]

If \( \text{Fix}(f) \) contains two points then we may blow up these points and obtain the annular compactification \( \tilde{A} \) of \( S^2 \setminus \text{Fix}(f) \).

**Corollary 5.8.** Suppose that \( G \) is a pseudo-rotation subgroup of \( \text{Symp}_\mu^\infty(S^2) \) and that \( G_p \) is the stabilizer of a point \( p \in S^2 \). Then \( G_p \) is abelian and there exists \( q \in S^2 \) such that \( \text{Fix}(g) = \{p, q\} \) for all non-trivial \( g \in G_p \).

**Proof.** Suppose that \( f \in G_p \). Blowing up the point \( p \) and extending \( f \) we obtain \( \tilde{f} : \mathbb{D}^2 \to \mathbb{D}^2 \). This construction is functorial in the sense that if \( h = fg \) then \( \tilde{h} = \tilde{f} \tilde{g} \). Hence the assignment \( f \mapsto \tilde{f} \) is a homomorphism from \( G_p \) to \( \text{Symp}_\mu^\infty(\mathbb{D}^2) \). This homomorphism is injective by part (2) of Lemma 5.2. The result now follows from Lemma 5.7. \( \square \)

It is not quite the case that for abelian pseudo-rotation subgroups of \( \text{Symp}_\mu^\infty(S^2) \) that the fixed point set for non-trivial elements is independent of the element. There is essentially one exception, namely the group generated by rotations of the sphere through angle \( \pi \) about two perpendicular axes.

**Lemma 5.9.** If \( G \) is an abelian pseudo-rotation subgroup of \( \text{Symp}_\mu^\infty(S^2) \) then either \( \text{Fix}(g) \) is the same for every non-trivial element \( g \) of \( G \) or \( G \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). In the latter case the fixed point sets of non-trivial elements of \( G \) are pairwise disjoint and hence their union contains six points.
Proof. Let \( g, h \) be distinct non-trivial elements of \( G \) and suppose \( \text{Fix}(g) \neq \text{Fix}(h) \). Since \( G \) is abelian, \( g(\text{Fix}(h)) = \text{Fix}(h) \). Since \( \text{Fix}(g) \neq \text{Fix}(h) \) and each contains two points, these sets are disjoint. Hence \( g \) switches the two points of \( \text{Fix}(h) \). In this case \( g \) has two fixed points and a period 2 orbit and hence \( g^2 \) has four fixed points and must be the identity. We conclude that \( g \) has order two. Switching the roles of \( g \) and \( h \) we observe that \( h \) has order two.

Let \( G_0 \) be the abelian group generated by \( g \) and \( h \) so \( G_0 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). The possible actions of an element of \( G_0 \) on \( \text{Fix}(g) \cup \text{Fix}(h) \) are: fix all four points, switch the points of one pair and fix the other, or switch both pairs. If \( f \in G \) then \( f \) preserves \( \text{Fix}(g) \) and \( \text{Fix}(h) \) so its action on \( \text{Fix}(g) \cup \text{Fix}(h) \) must be the same as one of the elements of \( G_0 \). Since \( f \) agrees with an element of \( G_0 \) at four points it must, in fact, be an element of \( G_0 \). Hence any abelian pseudo-rotation group containing non-trivial elements whose fixed point sets do not coincide must be isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

Suppose now \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) is a pseudo-rotation subgroup of \( \text{Symp}_\mu(S^2) \). The three non-trivial elements of \( G \) must have pairwise disjoint fixed point sets. To see this note that otherwise there would be two distinct elements, say, \( g \) and \( h \) with a common fixed point. As observed above this implies \( \text{Fix}(g) = \text{Fix}(h) \). But any two non-trivial elements of \( G \) generate it and hence \( \text{Fix}(f) \) is the same for all elements \( f \in G \). Then by Proposition 5.3 there is an injective homomorphism from \( G \) to \( \mathbb{T} \). This is a contradiction since \( \mathbb{T} \) contains a unique element of order two.

\[ \square \]

Remark 5.10. Suppose in the previous lemma \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). Let \( F \) be the union of three sets \( F_i = \{a_i, b_i\} \), \( 1 \leq i \leq 3 \) where \( F_i = \text{Fix}(h_i) \) and \( h_i \) is one of the non-trivial elements of \( G \). The elements of \( G \) preserve each of the sets \( F_i \) and if \( h \) is a non-trivial element of \( G \) it must fix the points of one of the \( F_i \)'s while switching the points of the other two (since \( h \) cannot fix four points).

Suppose \( A \) is a subgroup of the circle \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) and \( \mathcal{G} = A \rtimes_\phi (\mathbb{Z}/2\mathbb{Z}) \) is the semidirect product determined by the homomorphism \( \phi : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(A) \) given by \( \phi(1) = i \in \text{Aut}(A) \) where \( i : A \to A \) is the involution \( i(h) = -h \). Then \( \mathcal{G} \) is called a generalized dihedral group.

Lemma 5.11. Suppose \( G \) is a pseudo-rotation subgroup of \( \text{Symp}_\mu(S^2) \) such that either \( G \) leaves invariant a non-empty finite set \( F \subset S^2 \), or \( G \) has a non-trivial normal solvable subgroup. Then either

1. There is a \( G \)-invariant set \( F_0 \) containing two points in which case \( G \) is isomorphic to a subgroup of the generalized dihedral group \( A \rtimes_\phi (\mathbb{Z}/2\mathbb{Z}) \) for some subgroup \( A \) of the circle \( \mathbb{T} \), or

2. Every \( G \)-invariant set contains more than two points in which case \( G \) is finite.

Moreover, if the set \( F_0 \) in (1) is pointwise fixed then \( G \) is isomorphic to a subgroup of \( \mathbb{T} \).
Proposition 5.3 it follows that $A$ contains in $\text{Cent}_\mu^\infty(A,G)$.

By Lemma 5.9 the set

$$F = \bigcup_{h \in H, h \neq \text{id}} \text{Fix}(h)$$

is finite. If $g \in G$ and $h \in H$ then $ghg^{-1} \in H$. Since $\text{Fix}(ghg^{-1}) = g(\text{Fix}(h))$, it follows that $g(F) = F$. Hence we need only prove the conclusion of our result under the assumption that $G$ leaves invariant a finite set $F$.

Let $A$ be the finite index subgroup of $G$ which pointwise fixes $F$. By Corollary 5.8 $A$ is abelian. If $F$ contains more than two points then $A$ must be trivial and $G = G/A$ is finite since $A$ had finite index. If $F$ is a single point then by Corollary 5.8 there is a set $F'$ containing $F$ and one other point which are the common fixed points for every element of $A$. As above the set $F'$ is $G$-invariant and we replace $F$ by $F'$, so we may assume $F$ contains two points.

In this case $A$ has index at most two and $\text{Fix}(h) = \text{Fix}(h')$ for all $h, h' \in A$. From Proposition 5.3 it follows that $A$ is isomorphic to a subgroup of $\mathbb{T}$.

If $h \in G \setminus A$ then $h$ interchanges the two points of $F$ and reverses the orientation of $H_1(U)$ where $U = S^2 \setminus F$. It follows that $\rho_\mu(h^{-1}gh) = -\rho_\mu(g)$ for all $g \in A$. Also $h$ has two fixed points and $h^2$ fixes the points of $F$ so it has four fixed points and we conclude that $h^2 = \text{id}$. Hence the map $\phi : \mathbb{Z}/2\mathbb{Z} \to G$ given by $\phi(1) = h$ is a homomorphism so $G \cong A \rtimes \phi \mathbb{Z}/2\mathbb{Z}$.

\[ \square \]

Proposition 5.12. If $G$ is a subgroup of $\text{Symp}_\mu^\infty(M)$ which has an infinite normal abelian subgroup $A$ which is a pseudo-rotation group then $G$ has an abelian subgroup of index at most two.

Proof. Since $A$ is non-trivial, $M$ is $A, \mathbb{D}^2$, or $S^2$. Since $A$ is infinite, Lemma 5.7 and Lemma 5.9 imply there is a set $F$ containing $\chi(M) \leq 2$ points such that $F = \text{Fix}(h)$ for all $h \in A$. Let $U = M \setminus F$. Observe that in all cases $U$ is an annulus. The set $F$ must be invariant under $G$ since $F = \text{Fix}(ghg^{-1}) = g(\text{Fix}(h)) = g(F)$ for every element $g$ in $G$. The subgroup $G_0$ of $G$ that pointwise fixes $F$ has index at most two. Also elements of $G_0$ leave $U$ invariant and their restrictions to $U$ are isotopic to the identity.

Since the set $\text{Fix}(h)$ is the same for all $h \in A$, by Proposition 5.3 the homomorphism $\rho_\mu : A \to \mathbb{T}^1$ given by $h \mapsto \rho_\mu(h)$ is injective, where $\rho_\mu(h)$ is the mean rotation number on $U_c$. Since conjugating $h$ by an element $g \in G_0$ does not change its mean rotation number we conclude that $h = ghg^{-1}$ for all $h \in A$. This means that $G_0$ is contained in $\text{Cent}_\mu^\infty(A,G)$, the centralizer of $A$ in $G$.

If $A$ contains elements of arbitrarily large order then it follows from Lemma 5.6 that $G_0$ is abelian and we are done. So we may assume the order of elements in
$A$ is bounded. Since the order of elements of $A$ is bounded there are only finitely many possible values for $\rho_\mu(f)$ with $f \in A$. Hence, since the assignment $f \mapsto \rho_\mu(f)$ is injective, we may conclude $A$ is finite in contradiction to our hypotheses.

Example 5.13. Let $A$ be the subgroup of all rational rotations around the $z$-axis and let $G$ be the $\mathbb{Z}/2\mathbb{Z}$ extension of $A$ obtained by adding an involution $g$ which rotates around the $x$-axis and reverses the orientation of the $z$-axis. More precisely let the homomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(A)$ be given by $\phi(1) = i_g$ where $i_g(h) = g^{-1}hg = h^{-1}$. Every element of $A$ has finite order while $A$ itself has infinite order. Moreover, $G \cong A \rtimes_{\phi} (\mathbb{Z}/2\mathbb{Z})$ is not abelian even though it has an index two abelian subgroup.

We are now able to classify up to isomorphism all pseudo-rotation subgroups of $\text{Symp}_0^\infty(S^2)$ which have a non-trivial normal solvable subgroup. We denote by $\phi$ the homomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \to \mathbb{T}$ defined by $\phi(1) = i$, where $i$ is the automorphism of $\mathbb{T}$, $i(t) = -t$.

Proposition 5.14. If $G$ is a pseudo-rotation subgroup of $\text{Symp}_0^\infty(S^2)$ which has a normal solvable subgroup then $G$ is isomorphic to either a subgroup of the generalized dihedral group $\mathbb{T} \rtimes_{\phi} (\mathbb{Z}/2\mathbb{Z})$ or a subgroup of the group $O$ of orientation preserving symmetries of the regular octahedron (or equivalently the orientation preserving symmetries of the cube).

Proof. Suppose first that there is a $G$-orbit $F_0$ containing two points. Then by Proposition 5.11, $G$ is isomorphic to a subgroup of the generalized dihedral group $\mathbb{T} \rtimes_{\phi} (\mathbb{Z}/2\mathbb{Z})$ and we are done. If this is not the case, then by the same proposition $G$ is finite and every $G$-orbit contains at least three points.

Let $A$ be the last non-trivial element of the derived series of the normal solvable subgroup of $G$, so $A$ is abelian. Since $A$ is a characteristic subgroup of that normal solvable subgroup it is normal in $G$. Hence for any $g \in G$, $g(\text{Fix}(A)) = \text{Fix}(gAg^{-1}) = \text{Fix}(A)$ and we observe $\text{Fix}(A)$ cannot contain only two points since that is the case we already considered. Therefore not all elements of $A$ have the same set of fixed points and we can apply Lemma 5.9 to conclude the group $A$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We observed in Remark 5.10 that there is an $A$-invariant set $F$ which is the union of three sets $F_i = \{ a_i, b_i \}$. $1 \leq i \leq 3$ where $F_i = \text{Fix}(h_i)$ and $h_i$ is one of the non-trivial elements of $A$. We also noted there that the elements of $A$ preserve the pairs $F_i$, fixing the points of one of them and switching the points in each of the other pairs. Since $A$ is normal in $G$, if $h_j = g^{-1}h_ig \in A$ then $F_j = \text{Fix}(h_j) = g(\text{Fix}(h_i)) = g(F_i)$ so the elements of $G$ must permute the three pairs $F_i$.

We define a homomorphism $\theta : G \to O(3)$ as follows. Given $g \in G$ define a matrix $P = P_g$ by $P_{ij} = 1$ if $g(a_j) = a_i$ and $g(b_j) = b_i$, $P_{ij} = -1$ if $g(a_j) = b_i$ and $g(b_j) = a_i$, and $P_{ij} = 0$ otherwise. Each row and column has one non-zero entry which is $\pm 1$ so $P_g \in O(3)$. It is straightforward to see that $\theta(g) = P_g$ defines a homomorphism to $O(3)$. It is also clear that it is injective since $P_g = I$ implies that $g$ fixes the six points

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of $F$. Note that if $h_i$ is a non-trivial element of $A$, then $\theta(h_i)$ must be diagonal since $h_i(F_j) = F_j$, and $\theta(h_i)$ must have two entries equal to $-1$ since $h_i$ switches the points of two of the $F_j$'s. It follows that $\theta(A)$ is precisely the diagonal entries of $O(3)$ with an even number of $-1$'s.

We need to show that $\theta(G)$ lies in $SO(3)$, i.e., all its elements have determinant $1$. Clearly $\theta(A) \subset SO(3)$. We denote the symmetric group on three elements by $S_3$ and think of it as the permutation matrices in $O(3)$. We define a homomorphism $\tilde{\theta} : G \to S_3$ by $\tilde{\theta}(g) = Q \in O(3)$ where $Q_{ij} = |P_{ij}|$ and $P = \theta(g)$. We observe that $A$ is in the kernel of $\tilde{\theta}$. In fact $A$ equals the kernel of $\tilde{\theta}$. To see this note that if $\tilde{\theta}(g) = I$ then $P_g = \theta(g)$ is a diagonal matrix and hence $\ker(\tilde{\theta})$ is abelian. If $g \in \ker(\tilde{\theta}) \setminus A$ then the abelian group generated by $g$ and $A$ is not $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ contradicting Lemma 5.9. We conclude $A = \ker(\tilde{\theta})$.

Elements of $S_3$ have order one, two or three. In case $\tilde{\theta}(g)$ has order one or three, $g^2$ is in $A$ and hence $\det(\tilde{\theta}(g)) = \det(\theta(g))^3 = \det(\theta(g^3)) = 1$ so $\theta(g) \in SO(3)$.

If $\tilde{\theta}(g)$ has order two and $P_g = \theta(g)$ has determinant $-1$ we argue to a contradiction. In this case, without loss of generality we may assume

$$P_g = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$$

where $\det(P_g) = -abc = -1$. Hence $g^2 = id$.

If $a = 1$ then $bc = 1$ and $P_g^2 = I$. We consider the $g$ invariant annulus $S^2 \setminus F_1$ and observe that in this annulus $\rho_{\mu}(g) = 1/2$. But if $h_1 \in A$ is the element with $\theta(h_1) = \text{diag}(1, -1, -1)$ then $h_1(U) = U$ and $\rho_{\mu}(h_1) = 1/2$. Since $g$ and $h$ both fix $F_1$ pointwise they commute by Corollary 5.8 and by Proposition 5.3 $\rho_{\mu}$ is defined and injective on the subgroup generated by $g$ and $h_1$. We conclude $g = h_1$ a contradiction. Finally suppose $a = -1$ and $bc = -1$. If $h_2 \in A$ is the element with $\theta(h_2) = \text{diag}(-1, 1, -1)$ then

$$\theta(h_2 g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & -c & 0 \end{pmatrix},$$

so by the previous argument $\det(\theta(h_2 g)) = 1$. Since $\det(\theta(h_2)) = 1$ this contradicts the assumption that $\det(\theta(g)) = -1$. This completes the proof that $\theta(G) \subset SO(3)$.

Since $\theta(G)$ preserves the set $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ of vertices of the regular octahedron, it follows that $\theta(G)$ is a subgroup of the group $O$ of orientation preserving symmetries of the octahedron.

\[
\square
\]

6 Proof of the Main Theorem

In this section we prove
Theorem (1.5) Suppose $M$ is a compact oriented surface of genus 0 and $G$ is a subgroup of $\text{Symp}_\mu^\infty(M)$ which has full support of finite type, e.g. a subgroup of $\text{Symp}_\omega^\mu(M)$. Suppose further that $G$ has an infinite normal solvable subgroup. Then $G$ is virtually abelian.

Throughout this section $M$ will denote a compact oriented surface of genus 0, perhaps with boundary. We begin with a lemma that allows us to replace the hypothesis that $G$ has an infinite normal solvable subgroup with the hypothesis that $G$ has an infinite normal abelian subgroup.

Lemma 6.1. If $G$ is a subgroup of $\text{Symp}_\mu^\infty(M)$, which contains an infinite normal solvable subgroup, then $G$ contains a finite index subgroup which has an infinite normal abelian subgroup.

Proof. Let $N$ be the infinite normal solvable subgroup. The proof is by induction on the length $k$ of the derived series of $N$. If $k = 0$ then $N$ is abelian and the result holds. Assume the result holds for $k \leq k_0$ for some $k_0 \geq 0$ and suppose the length of the derived series for $N$ is $k = k_0 + 1$. Let $A$ be the abelian group which is the last non-trivial term in the derived series of $N$. The group $A$ is invariant under any automorphism of $N$ and hence is normal in $G$. If $A$ is infinite we are done.

We may therefore assume $A$ is finite and hence that in a suitable metric $A$ is a group of isometries. No non-trivial orientation preserving isometry of $M$ can have infinitely many fixed points so we let $F$ be the finite set of fixed points of non-trivial elements of $A$. Since $A$ is normal $g(F) = F$ for any $g \in G$. Let $G_0$ be the normal finite index subgroup of $G$ which pointwise fixes $F$.

Let $N_1 = G_0 \cap N$ so $N_1$ is an infinite solvable normal subgroup of $G_0$ and the $k$-th term $A_1$ in the derived series of $N_1$ is contained in $A \cap G_0$. We claim that $A_1$ is trivial. If $F$ contains more than $\chi(M)$ points then this follows from the fact that a non-trivial isometry isotopic to the identity fixes exactly $\chi(M)$ points since each fixed point must have Lefschetz index +1.

Otherwise $F$ contains $\chi(M)$ points and we may blow up these points to form an annulus $A$ on which $N_1$ acts preserving each boundary component. Since each element of $A_1$ is a commutator of elements in $N_1$, it acts on $A$ with mean rotation number zero. Each element of $A_1$ therefore contains a fixed point in the interior of $A$ by Lemma 4.2. However, a finite order isometry of the annulus which is isotopic to the identity contains an interior fixed point must be the identity. This is because every fixed point must have Lefschetz index +1 and the Euler characteristic of $A$ is 0. This shows that $A_1$ acts trivially on $A$ and hence on $M$. This verifies the claim that $A_1$ is trivial and hence the length of the derived series for $N_1$ is at most $k_0$. The inductive hypothesis completes the proof.

The next lemma states the condition we use to prove that $G$ is virtually abelian.
Lemma 6.2. Suppose $G_0$ is a subgroup of $\text{Symp}_\mu^\infty(M)$ which has full support of finite type. Suppose further that there is an infinite family of disjoint $G_0$-invariant open annuli. Then $G_0$ is abelian.

Proof. We assume that $[G_0, G_0]$ contains a non-trivial element $h$ and argue to a contradiction. For each $G_0$-invariant open annulus $V$, let $V_c$ be its annular compactification and let $h_c : V_c \to V_c$ be the extension of $h|_V$ (see Definition 2.1 or Definition 2.7 of [6] for details). Since $h$ is a commutator of elements of $G_0$ and since $G_0$ extends to an action on $V_c$, $h_c$ is a commutator and so has mean rotation number zero. Lemma 4.2 therefore implies that $\text{Fix}(h) \cap V \neq \emptyset$.

By assumption, $\text{Fix}(h)$ does not contain any open set and so $\text{Fix}(h|_V)$ is a proper subset of $V$. We claim that $\text{fr}(V) \cup \text{Fix}(h|_V)$ is not connected. To see this, let $S$ be the end compactification of $V$ obtained by adding two points, one for each end of $V$ and let $h_S : S \to S$ be the extension of $h|_V$ that fixes the two added points. If $\text{fr}(V) \cup \text{Fix}(h|_V)$ is connected, then each component $W$ of $S \setminus \text{Fix}(h|_S)$ is an open disk. A result of Brown and Kister [4] asserts $W$ is $h_S$-invariant. However, then the Brouwer plane translation theorem would imply that $h$ has a fixed point in $W$. This contradiction proves the claim.

By passing to a subfamily of the $G_0$-invariant annuli, we may assume that the following is either true for all $V$ or is false for all $V$: some component of $\text{Fix}(h)$ intersects both components of the frontier of $V$. In the former case, the interior of each $V$ contains a component of $\text{Fix}(h)$. In the latter case, no component of $\text{Fix}(h)$ intersects more than two of the annuli in our infinite family. In both cases, $\text{Fix}(h)$ has infinitely many components in contradiction to the assumptions that $h$ is non-trivial and that $G_0$ has full support of finite type. \hfill \square

We need an elementary topology result.

Lemma 6.3. Suppose that $C \subset \text{int}(M)$ is a closed connected set which is nowhere dense and has two complementary components $U_1$ and $U_2$. Then $C' = \text{fr}(U_1) \cap \text{fr}(U_2)$ is a closed connected set with two complementary components, $U'_1$ and $U'_2$, each of which has frontier $C'$ and is equal to the interior of its closure. Moreover $U_i$ is dense in $U'_i$.

Proof. To see that $C'$ separates $U_1$ and $U_2$, suppose that $\sigma$ is a closed path in $M \setminus C'$ with initial endpoint in $U_1$ and terminal endpoint in $U_2$. Then $\sigma \cap \text{fr}(U_2) \neq \emptyset$ and we let $\sigma_0$ be the shortest initial segment of $\sigma$ terminating at a point $x \in \sigma \cap \text{fr}(U_2)$. Each $y \in \sigma_0 \setminus x$ has a neighborhood that is disjoint from $U_2$. Since $C$ has empty interior, every neighborhood of $y$ must intersect $U_1$. It follows that every neighborhood of $x$ must intersect $U_1$ and hence that $x \in C'$. This contradiction proves that $C'$ separates $U_1$ and $U_2$. Since the union of $U_1$ and $U_2$ is dense in $M$, the components $V_1$ and $V_2$ of $M \setminus C'$ that contain them are the only components of $M \setminus C'$. Every neighborhood of a point in $C'$ intersects both $U_1$ and $U_2$ and so intersects both $V_1$ and $V_2$. Thus $C' \subset \text{fr}(V_1) \cap \text{fr}(V_2)$. Since $\text{fr}(V_1), \text{fr}(V_2) \subset C'$ we have that $C' \subset \text{fr}(V_1) \cap \text{fr}(V_2) \subset$
fr(V_i) \subset C' and hence C' = fr(V_1) = fr(V_2) which implies that V_1 and V_2 are the interior of their closures. Since C is nowhere dense in M it follows that U_1 \cup U_2 is dense in M and hence that U_i is dense in U_i'.

**Notation 6.4.** If C and C' are as in Lemma 6.3 then we say that C' is obtained from C by *trimming*. Recall that if f \in \mathcal{Z}(M) and U \in \mathcal{A}_f, then the rotation number function \( \rho = \rho_{f|U} : U \to S^1 \) is well defined and continuous. A component of a point pre-image of \( \rho \) is a level set for \( \rho \) and is said to be *irrational* if its \( \rho \)-image is irrational and to be *interior* if it is disjoint from the frontier of \( U \). If C is an interior level set which is nowhere dense and C' is obtained from C by trimming then we will call C' a trimmed level set. The collection of all trimmed level sets for \( f \) will be denoted \( C(f) \).

**Lemma 6.5.** Suppose \( G \) is a subgroup of \( \text{Symp}_\nu^\infty(M) \) containing an abelian normal subgroup \( A \), that \( f \in \mathcal{Z}(M) \) lies in \( A \) and that \( U \in \mathcal{A}_f \). Suppose further that \( C'_1 \) and \( C'_2 \) are obtained from nowhere dense irrational interior level sets \( C_1 \) and \( C_2 \) for \( \rho = \rho_{f|U} \) by trimming. Letting \( B_i \) be the component of \( M \setminus (C'_1 \cup C'_2) \) with frontier \( C'_i \) and \( V \) be the component of \( M \setminus (C'_1 \cup C'_2) \), with frontier \( (C'_1 \cup C'_2) \), assume that

1. \( \mu(V) < \mu(B_1) < \mu(B_2) \).
2. \( \mu(B_2) > \mu(M)/2 \).

Then for all \( g \in G \), either \( g(V) \cap V = \emptyset \) or \( g(V) = V \). In particular, there is a finite index subgroup of \( G \) that preserves \( V \).

**Proof.** We assume that \( g(V) \cap V \neq \emptyset \) and \( g(V) \neq V \) and argue to a contradiction. Since \( A \) is normal and abelian, \( h = g^{-1} f g \) commutes with \( f \). Corollary 2.4 part (1) implies that \( h \) permutes the elements of \( \mathcal{A}_f \) and hence that \( h^n(U) = U \) for some \( n \geq 1 \). It then follows from Corollary 2.4 part (2) that \( h^n \) preserves each level set for \( \rho = \rho_{f|U} \), and hence each trimmed level set for \( \rho \), and so preserves \( V, B_1 \) and \( B_2 \).

Equivalently, \( g(V), g(B_1) \) and \( g(B_2) \) are \( f^n \)-invariant.

Since \( g(V) \cap V \neq \emptyset \) and \( g(V) \neq V \), there is a component \( W \) of \( g(V) \cap V \) such that \( fr(W) \cap fr(V) \neq \emptyset \). To see this observe that otherwise one of \( V \) and \( g(V) \) would properly contain the other which would contradict the fact that \( g \) preserves \( \mu \). Since \( f^n \) preserves both \( V \) and \( g(V) \), it permutes the components of their intersection. Since \( f \) preserves area, \( f^m(W) = W \) for some \( m > 0 \). If every simple closed curve in \( W \) is inessential in \( V \) then \( \rho \) has a constant rational value on a dense subset of \( W \) by Lemma 4.3. This contradicts the fact that \( \rho \) is continuous on \( U \) and takes only irrational values on \( fr(V) \). We conclude that there is a simple closed curve \( \alpha \subset W \) which is essential in \( V \). Let \( \beta = g^{-1}(\alpha) \subset V \). Item (1) implies that \( \beta \) is also essential in \( V \) because if it were inessential the component of its complement which lies in \( V \) would contain either \( g^{-1}(B_1) \) or \( g^{-1}(B_2) \).

Let \( B'_i \) be the subsurface of \( M \) bounded by \( \beta \) that contains \( B_i \). Item (2) rules out the possibility that \( B_2 \subset g(B'_i) \) so it must be that \( B_1 \subset g(B'_i) \) and \( B_2 \subset g(B'_i) \). Item (1) therefore implies that \( B_i \cap g(B_i) \neq \emptyset \). If there is a simple closed curve in \( B_i \) whose...
$g$-image is in $V$ and essential in $V$ then there is a proper subsurface of $B_i$ whose $g$ image contains $B_i$ in contradiction to the fact that $g$ preserves area. It follows that either $g(B_i) = B_i$ or there is a component $W'_i$ of $g(B_i) \cap V$ which is inessential in $V$ and whose frontier intersects $C'_i$. As above, this implies that $\rho$ is constant and rational on $W'$ in contradiction to the fact that $\rho$ is continuous on $U$ and takes an irrational value on $C'_i$. This contradiction completes the proof.

We are now prepared to complete the proof of our main theorem.

**Theorem (1.5)** Suppose $M$ is a compact oriented surface of genus 0 and $G$ is a subgroup of $\text{Symp}^\infty_{\mu}(M)$ which has full support of finite type, e.g., a subgroup of $\text{Symp}^\mu_{\mu}(M)$. Suppose further that $G$ has an infinite normal solvable subgroup. Then $G$ is virtually abelian.

**Proof.** By Lemma 6.1 it suffices to prove the result when $G$ has an infinite normal abelian subgroup $A$. If $A$ contains an element of positive entropy the result follows by Proposition 3.3. If the group $A$ is a pseudo-rotation group the result follows from Proposition 5.12.

We are left with the case that $A$ contains an element $f \in \mathcal{Z}(M)$. Let $U$ be an element of $\mathcal{A}_f$ and let $\rho = \rho_f|_U$. Since there are only countably many level sets of $\rho$ with positive measure, all but countably many have empty interior or equivalently are nowhere dense. Choose a nowhere dense interior irrational level set $C$ of $\rho$. One component of $M \setminus C$, say, $Y$, will be an open subsurface with $\mu(Y) \leq \mu(M)/2$. Choose a nowhere dense interior irrational level set $C' \subset Y \cap U$ and let $Y'$ be the open subsurface which is the component of the complement of $C'$ satisfying $Y' \subset Y$.

Finally, choose nowhere dense interior irrational level sets $\tilde{C}_1, \tilde{C}_2 \subset (Y \setminus \text{cl}(Y'))$ so that the annulus $\tilde{V}$ bounded by the trimmed sets $\tilde{C}_1$ and $\tilde{C}_2$ has measure less than the measure of $Y'$. Then $\tilde{C}_1$ and $\tilde{C}_2$ satisfy the hypotheses of Lemma 6.5. It follows that the subgroup $G_0$ of $G$ that preserves $\tilde{V}$ has finite index in $G$.

Note that any two trimmed irrational level sets $C'_1$ and $C'_2$ in $\tilde{V}$ satisfy the hypotheses of Lemma 6.5. Moreover if $V$ is the essential open subannulus bounded by $C'_1$ and $C'_2$ then $g(V) \cap V \neq \emptyset$ for each $g \in G_0$ because each such $g$ preserves $\tilde{V}$ and preserves area. Lemma 6.5 therefore implies that each such $V$ is $G_0$-invariant so Lemma 6.2 completes the proof.

### 7 The Tits Alternative

We assume throughout this section that $M$ be a compact oriented surface of genus 0, ultimately proving a special case (Theorem 1.12) of the Tits Alternative.

**Lemma 7.1.** Suppose $f \in \mathcal{Z}(M)$ and $U \in \mathcal{A}(f)$ is a maximal annulus for $f$. If $G$ is a subgroup of $\text{Symp}^\mu_{\mu}(M)$ whose elements preserve each element of an infinite family of trimmed level sets for $\rho_{f|_U}$ lying in $U$, then $G$ is abelian.
Proof. Since infinitely many of the trimmed level sets in $U$ are preserved by $G$ so is each open annulus bounded by two such level sets. It is thus clear that one may choose infinitely many disjoint open annuli in $U$ each of which is $G$-invariant. The result now follows from Lemma 6.2. 

Lemma 7.2. Suppose $f : A \rightarrow A$ is a homeomorphism isotopic to the identity with universal covering lift $\tilde{f} : \tilde{A} \rightarrow \tilde{A}$ which has a non-trivial translation interval and suppose $\alpha \subset A$ is an embedded arc joining the two boundary components of $A$. Then there exists $m > 0$ such that for any $\gamma \subset A$ which is an embedded arc, disjoint from $\alpha$, joining the two boundary components of $A$, every lift of $f^k(\gamma)$, $|k| > m$ must intersect more than one lift of $\alpha$.

Proof. Let $\tilde{\alpha}$ be a lift of $\alpha$ and let $T$ be a generator for the group of covering translations of $\tilde{A}$. We claim that there is an $m > 0$ such that for all $k \geq m$, $\tilde{f}^k(\tilde{\alpha})$ intersects at least 6 adjacent lifts of $\alpha$. To show the claim consider the fundamental domain $D_0$ bounded by $\tilde{\alpha}$ and $T^{-1}(\tilde{\alpha})$. If there are arbitrarily large $k$ for which $\tilde{f}^k(\tilde{\alpha})$ intersects fewer than six translates of $\tilde{\alpha}$, then for such $k$, $\tilde{f}^k(D_0)$ would lie in an adjacent strip of five adjacent translates of $D_0$ and the translation interval for $\tilde{f}$ would have length 0. This contradiction verifies the claim.

There is a lift $\tilde{\gamma}$ of $\gamma$ that is contained in $D_0$. Let $X$ be the region bounded by $T^{-1}(\tilde{\gamma})$ and $T(\tilde{\gamma})$ and note that $\tilde{\alpha} \subset X$. If the lemma fails then we can choose $k \geq m$ and a lift $\tilde{h}$ of $f^k$ such that $\tilde{h}(\tilde{\gamma})$ is disjoint from $T^i(\tilde{\alpha})$ for all $i \neq 0$. It follows that $\tilde{h}(X)$ is contained in $\bigcup_{j=-2}^{2}T^j(D_0)$ in contradiction to the fact that $\tilde{h}(\tilde{\gamma})$ intersects at least six translates of $\tilde{\alpha}$. 

Lemma 7.3. Suppose $f, g \in Z(M)$ and there are trimmed level sets $C'_1 \in C(f)$ and $C'_2 \in C(g)$ such that there exist points $x_i \in M \setminus (C'_1 \cup C'_2)$, $1 \leq i \leq 4$ with the following properties:

(1) $x_1, x_2$ lie in the same component of the complement of $C'_1$ and $x_3, x_4$ lie in the other component of this complement.

(2) $x_1, x_3$ lie in the same component of the complement of $C'_2$ and $x_2, x_4$ lie in the other component of this complement.

Then for some $n > 0$, the diffeomorphisms $f^n$ and $g^n$ generate a non-abelian free group.

Proof. Let $C_i$ be the untrimmed level set whose trimmed version is $C'_i$. We claim that by modifying the points $\{x_i\}$ slightly we may assume that the hypotheses (1) and (2) hold with $C'_i$ replaced by $C_i$. This is because each component of the complement of $C_i$ is a dense open subset of a component of the complement of $C'_i$. Hence each $x_j$ can be perturbed slightly to $\hat{x}_j$ which, for each $i$, is in the component of complement of
There is a dense open subset of the component of the complement of $C_i$ containing $x_j$. Henceforth we will work with $C_i$ and refer to $\hat{x}_j$ simply as $x_j$.

Let $\beta_1$ be a path in $M \setminus C_2$ joining $x_1$ and $x_3$; so $\beta_1$ crosses $C_1$ and is disjoint from $C_2$. Likewise let $\beta_2$ be a path in $M \setminus C_1$ joining $x_1$ and $x_2$; so $\beta_2$ crosses $C_2$ and is disjoint from $C_1$. (See Figure 1.) The level set $C_1$ is an intersection

$$C_1 = \bigcap_{n=1}^{\infty} \text{cl}(B_n),$$

where each $B_n$ is an $f$-invariant open annulus with $\text{cl}(B_{n+1}) \subset B_n$ and the rotation interval $\rho(f, B_n)$ of the annular compactification $f_c$ of $f|_{B_n}$ is non-trivial (see section 15 and the proof of Theorem 1.4 in [6]). For $n$ sufficiently large $\text{cl}(B_n)$ is disjoint from $\beta_2$ and $\{x_i\}_{i=1}^4$. Let $A_1$ be a choice of $B_n$ with this property.

We may choose a closed subarc $\alpha_1$ of $\beta_1$ whose interior lies in $A_1$ and whose endpoints are in different components of the complement of $A_1$. We will use intersection number with $\alpha_1$ with a curve in $A_1$ to get a lower bound on the number of times that curve “goes around” the annulus $A_1$.

Similarly the level set $C_2$ is an intersection

$$C_2 = \bigcap_{n=1}^{\infty} \text{cl}(B'_n),$$

where each $B'_n$ is a $g$-invariant open annulus with $\text{cl}(B'_{n+1}) \subset B'_n$ and the rotation interval $\rho(g, B'_n)$ of the annular compactification $g_c$ of $f|_{B'_n}$ is non-trivial. We construct...
$A_2$ and the arc $\alpha_2$ in a fashion analogous to the construction of $A_1$ and $\alpha_1$. By construction $\alpha_1$ is disjoint from $A_2$ and crosses $A_1$ while $\alpha_2$ is disjoint from $A_1$ and crosses $A_2$. (See Figure 2.)

Note that any essential closed curve in $A_1$ must intersect $\alpha_1$ and must contain points of both components of the complement of $C_2$. To see this latter fact we note that we constructed $\alpha_1$ to lie in one component of the complement of $C_2$ but we could as well have constructed $\alpha_1'$ in the other component of this complement. Any essential curve in $A_1$ must intersect both $\alpha_1$ and $\alpha_1'$. Similarly any essential curve in $A_2$ must intersect $\alpha_2$ and must contain points of both components of the complement of $C_1$.

There is a key consequence of these facts which we now explore. Let $\gamma_0$ be an arc with interior in $A_1$, disjoint from $A_2 \cup \alpha_1$ and with endpoints in different components of the complement of $A_1$. Replace $f$ and $g$ by $f^m$ and $g^{m'}$ where $m$ and $m'$ are the numbers guaranteed by Lemma 7.2 for $f$ and $g$ respectively. Then we know that for $k \neq 0$ the curve $f^k(\gamma_0)$ must intersect more than one lift of the arc $\alpha_1$ in the universal covering $\tilde{A}_1$.

It follows that $f^k(\gamma_0)$ contains a subarc whose union with a subarc of $\alpha_1$ is essential in $A_1$. Hence we conclude that $f^k(\gamma_0)$ contains a subarc crossing $A_2$, i.e. a subarc $\gamma_1$ whose interior lies in $A_1 \cap A_2$ (and hence is disjoint from $\alpha_2$), and whose endpoints are in different components of the complement of $A_2$. (See Figure 2.) Since we replaced $g$ by $g^{m'}$ above we know that for $k \neq 0$ the curve $g^k(\gamma_1)$ must intersect more than one lift of the arc $\alpha_2$ in the universal covering $\tilde{A}_2$. 

Figure 2: The curves $\alpha_1$, $\alpha_2$, $\gamma_0$ and $\gamma_1$
We can now construct \( \gamma_2 \) in a similar manner but switching the roles of \( f \) and \( g \), \( \alpha_1 \) and \( \alpha_2 \), and \( A_1 \) and \( A_2 \). More precisely, for any \( k \neq 0 \) the curve \( g^k(\gamma_1) \) contains a subarc whose union with a subarc of \( \alpha_2 \) is essential in \( A_2 \). It follows that \( g^k(\gamma_1) \) contains a subarc \( \gamma_2 \) whose interior lies in \( A_1 \cap A_2 \) and whose endpoints are in different components of the complement of \( A_1 \). Note that \( \gamma_2 \) is a subarc of \( g^m f^k(\gamma_0) \).

We can repeat this construction indefinitely, each time switching \( f \) and \( g \). Hence if we are given \( h = g^m f^k \ldots g^m f^k \) and \( m_i \neq 0 \), \( k_i \neq 0 \) we can obtain a non-trivial arc \( \gamma_{2n} \) which is a subarc of \( h(\gamma_0) \). Since \( \gamma_{2n} \subset A_1 \cap A_2 \) and \( \gamma_0 \cap A_2 = \emptyset \) it is not possible that \( h = \text{id} \). But every element of the group generated by \( f \) and \( g \) is either conjugate to a power of \( f \), a power of \( g \), or an element expressible in the form of \( h \). Hence we conclude that the group generated by \( f \) and \( g \) is a non-abelian free group.

**Proof of Theorem 1.12.** Suppose \( f \in G \cap \mathcal{Z}(M) \) and \( U \in \mathcal{A}(f) \) is a maximal annulus for \( f \). One possibility is that there is a finite index subgroup \( G_0 \) of \( G \) which preserves infinitely many of the trimmed rotational level sets for \( f \) which lie in \( U \). In this case Lemma 7.1 implies \( G_0 \) is abelian and we are done.

If this possibility does not occur, we claim that there exists a trimmed level set \( C \) in \( U \) and \( h_0 \in G \) such that \( h_0(C) \cap C \neq C \) but \( h_0(C) \cap C \neq \emptyset \). If this is not the case then for every \( h \in G \) and every \( C \) either \( h(C) = C \) or \( h(C) \cap C = \emptyset \). It follows that the \( G \)-orbit of \( C \) consists of pairwise disjoint copies of \( C \). Since elements of \( G \) preserve area this orbit must be finite and it follows that the subgroup of \( G \) which stabilizes \( C \) has finite index. If we now choose \( C_0 \) and \( C_0' \), two trimmed level sets in \( U \) and let \( G_0 \) be the finite index subgroup of \( G \) which stabilizes both of them, then the annulus \( U_0 \) which they bound is \( G_0 \)-invariant. Now if \( C \) is any trimmed level set in \( U_0 \) then its \( G_0 \)-orbit lies in \( U \) and we conclude from area preservation that \( g(C) \cap C \neq \emptyset \) for all \( g \in G_0 \). If for some \( g \) and \( C \) we have \( g(C) \cap C \neq C \) we have demonstrated the claim and otherwise we are in the previous case since \( G_0 \) preserves the infinitely many trimmed level sets lying in \( U_0 \).

So we may assume that the claim holds, i.e., that there is \( h_0 \in G \) and a trimmed level set \( C_1 \) for \( f \) in \( U \) such that \( h_0(C_1) \cap C_1 \neq C_1 \) and \( h_0(C_1) \cap C_1 \neq \emptyset \). Let \( C_2 = h_0(C_1) \) and \( g = h_0 f h_0^{-1} \). Since \( C_1 \neq C_2 \) and each is the common frontier of its complementary components, it follows that each of the complementary components of \( C_1 \) has non-empty intersection with each of the complementary components of \( C_2 \). Hence we may choose points \( x_i \in M \setminus (C_1 \cup C_2) \), \( 1 \leq i \leq 4 \) satisfying the hypothesis of Lemma 7.3 which completes the proof.

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