A Fresh Look at Generalized Veneziano Amplitudes

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Abstract

The dual resonance model, which was a precursor of string theory was based upon the idea that two-particle scattering amplitudes should be expressible equivalently as a sum of contributions of an infinite number of s channel poles each corresponding to a finite number of particles with definite spin, or as a similar sum of t channel poles. The famous example of Veneziano [1] satisfies all these requirements, and is additionally ghost free. We recall other trajectories which provide solutions to the duality constraints, e.g. the general Mobiüs trajectories and the logarithmic trajectories, which were thought to be lacking this last feature. We however demonstrate, partly empirically, the existence of a regime within a particular deformation of the Veneziano amplitude for logarithmic trajectories for which the amplitude remains ghost free.
1 Introduction.

The origin of String Theory as an interpretation of the celebrated dual 4-point amplitude of Veneziano seems to us in danger of being forgotten. In this article we re-examine the ideas behind the concept of dual amplitudes in the hope of opening further avenues for the exploration of new String Theories. This concept, as applied to 4-point tree amplitudes, required that the amplitude could alternatively be expressed either as a sum of an infinite number of $s$ channel poles at positions $s = \lambda_n$, with residues given by polynomials of maximal degree $n$ in the dual variable $t$, or as a sum of $t$ channel poles with polynomial residues in $s$. There was a further constraint which was technically very difficult to implement; the partial wave decomposition of the residues into appropriate angular functions (Legendre Polynomials in the case of 3+1 dimensional space-time) should have only positive coefficients, otherwise the corresponding particles exchanged would be ghosts.

The original solution of Veneziano to the duality constraints was to take a linear (mass squared) spectrum, i.e. a linearly rising Regge trajectory. He proposed an amplitude of the well known Euler $\beta$-function form:

$$\prod_{n=0}^{\infty} \frac{s + t - \lambda_n - \lambda_0}{(s - \lambda_n)(t - \lambda_n)},$$

which has all the required properties, including freedom from ghosts for the choice

$$\lambda_n = \alpha^{-1}n + \lambda_0,$$

a linear relation with slope $\alpha$ and intercept $\alpha_0 = -\alpha\lambda_0$, provided $\alpha\lambda_0$ lies between -1 and 0. By rescaling $\alpha$ can be put to 1.

This construction rapidly led to the development of string theory in the late sixties and early seventies, together with all the machinery of a Fock space representation in terms of an infinite number of creation and annihilation operators. Another solution to the duality constraint, resulting in the so-called logarithmic trajectories was found independently by several authors. That of is the unique solution for the dual 4-point trajectories when the residues are polynomials of finite extent from $t^n - Q$ (or obviously from $t^0$ when $n < Q$) to $t^n$ where $Q$ is a fixed value independent of $n$. These logarithmic trajectories are of the form

$$\lambda_n = a\sigma^n,$$
a geometric series. A simple scaling and shift for $s, t$, redefining $s \mapsto a(\sigma - 1)s + 1$, $t \mapsto a(\sigma - 1)t + 1$ yields a pole spectrum

$$\lambda_n = l(n)$$

(1.4)

with

$$l(n) = \frac{\sigma^n - 1}{\sigma - 1}, \quad \text{i.e.} \quad 0, 1, 1 + \sigma, 1 + \sigma + \sigma^2 \ldots,$$

(1.5)

showing that in the limit $\sigma \to 1$, the linear mass spectrum is recovered. In the regime $\sigma > 1$ for which these amplitudes were thought applicable, they possess ghosts and this is perhaps the principal reason for which their study was abandoned. In this context it is amusing to note that although Baker and Coon and Cremmer and Nuyts discovered the same trajectory, these pairs of authors had different 4-point functions, neither of which was ghost free! It is not surprising that a satisfactory operator realisation connecting in the appropriate limit with the usual string operator formalism has not been found, despite some attempts [8, 9]. One has too few sets of oscillators, the other too many!

We find empirical evidence that in the region $0 \leq \sigma \leq 1$ the amplitude proposed by Baker and Coon when adjusted to correspond with the Veneziano amplitude in the limit $\sigma \to 1$ is indeed ghost free. In this region the sequence $\lambda_n$ tends to a limit point, and this would give rise to an essential singularity in this approximation to the scattering amplitude. In terms of the thinking in the 1970’s with an interpretation of the amplitude as a hadronic amplitude, this was unacceptable, and Baker and Coon chose the range of their parameter such that $\sigma > 1$, with consequent ghost behaviour. However, in view of much recent discussion reviving the old idea of a limiting distance [11] and the modern re-interpretation of strings as describing elementary particles with a scale appropriate to the Planck mass, it may not be too fanciful to suggest that these models with a spectrum with an accumulation point indicate the possibility of a limit to the energy in an elementary scattering process. Similar ideas have arisen in speculations concerning deformations of the Poincaré group. It should also be borne in mind that many perfectly respectable physical systems, for example, the hydrogen atom and positronium, possess an infinite number of bound states with an accumulation point.

In the course of this investigation we rediscovered a more general solution to the duality constraint than that of the logarithmic trajectories [5]. This
corresponds to a spectrum of the form

\[ \lambda_n = \frac{a\sigma^n + b}{c\sigma^n + d}, \quad a, b, c, d, \sigma \text{ constant}, \quad (1.6) \]

which we call a Möbius trajectory. To our surprise, we found that it had already been found by Coon in his very first paper on the subject [10]! The reason that it was dropped from subsequent papers, we suppose, is that if all pole positions are positive, then, whatever the value of \( \sigma \) the model has an accumulation point, except for the linear and exponential cases (1.3). While the 4-point amplitude is viable, we have not succeeded in constructing a candidate 5-point amplitude with polynomial residues of the correct degree except in the logarithmic \((c = 0)\) case.

The outline of this paper is as follows. In section 2 we justify the Möbius trajectory solution [10] in our own way. In section 3 we extend the analysis to the 5-point function, and show that the Baker and Coon ansatz satisfies recurrence relations similar to those proposed for the linear spectrum by Bardakci and Ruegg [12]. Vice-versa these recurrence relations are sufficient to determine the 5-point amplitude up to an overall multiplication constant.

The question of the absence of ghosts which is equivalent to the positivity of the coefficients of the expansion of the residues in terms of orthogonal Legendre polynomials is addressed in the final section. Since this is an extremely difficult analytic problem and since the result was proved for the Veneziano amplitude only after operator factorisation and the no-ghost theorem [13, 14] we have had to recourse to computer verification using REDUCE. Thus our results on the domain of parameters for which our amplitude is ghost free remain conjectural, and await an operator realisation. On the other hand, our results are sufficiently positive as to encourage belief in the existence of such a representation. Something along the lines of a q-deformed operator realisation could be expected [15].

2 Outline of the problem. The Generalized Linear Case.

We suppose that the dual 4-point amplitude for four incoming particles of momenta \( p_i, \quad i = 1, 2, 3, 4, \sum p_i = 0 \) can be written in the following form
symmetrical in \( s = (p_1 + p_2)^2 \) and \( t = (p_2 + p_3)^2 \) which generalizes in an obvious way the usual solution \((1.1)\)

\[
A(s,t) = N \prod_{r=0}^{\infty} \frac{\gamma_r (s + t) - \alpha_r - \beta_r st}{(s - \lambda_r)(t - \lambda_r)}, \quad (2.1)
\]

where \( N \) is a suitably chosen, possibly infinite normalisation constant, the \( \lambda_r \) are the positions of the poles both in the \( s \) and \( t \) channels, while \( \hat{\alpha}_r = \alpha_r / \gamma_r \) and \( \hat{\beta}_r = \beta_r / \gamma_r \) are infinite sets of parameters which will be restricted by the physical requirements outlined in the introduction. The set of parameters \( \gamma_r \) which can be absorbed in the normalisation constant \( N \) is introduced for later convenience. We call this case the generalized linear case as, when evaluated for one variable at one of its poles, every term in the numerator is of first degree in the other variable. The numerator in the right-hand side of \((2.1)\) is obviously the most general product of polynomials symmetric under the exchange of \( s \) and \( t \) with this property. In agreement with the ideas expressed in the introduction we will now require that when evaluated at a pole \( \lambda_n \) in the \( s \)-variable the remaining \( t \)-dependence of \( A \) is restricted to a polynomial of degree \( n \) at most. This implies that the \( t \)-poles at all the values \( \lambda_r \) should be killed by zeros arising in the numerators at values

\[
t_m = \frac{\hat{\alpha}_m - \lambda_n}{1 - \hat{\beta}_m \lambda_n}, \quad (2.2)
\]

We shall now suppose that this is done in a sequential way. The \( n \) first terms in the product in the numerator survive. The next terms are suppressed sequentially in the numerator and denominator. Hence we require

\[
t_{n+r} = \lambda_r \quad (2.3)
\]

for all \( r \geq 0 \) and \( n \geq 0 \). These relations become

\[
E(n, r) \equiv \hat{\alpha}_{n+r} - \lambda_n - \lambda_r + \hat{\beta}_{r+n} \lambda_r \lambda_n = 0. \quad (2.4)
\]

At first sight the problem appears overdetermined but fortunately many relations are redundant. Let us note first that the equations are symmetrical in \( n \) and \( r \) so that we can restrict ourselves to \( n \geq r \). The sequential solution of this (doubly infinite) set of equations can be obtained easily as follows. Indeed let us take a set of \( n \) and \( r \) values such that \( n + r \) is fixed. The first four values \( n + r = 0, 1, 2, 3 \) are special and have to be treated separately. Then starting at \( n + r = 4 \) we obtain the general regime.
0) For \( n + r = 0 \) we only have one relation between \( \lambda_0, \hat{\beta}_0 \) and \( \hat{\alpha}_0 \). Hence two of them (say \( \lambda_0 \) and \( \hat{\alpha}_0 \)) can be considered as free parameters.

1) For \( n + r = 1 \) there is again only one relation for \( n = 1, r = 0 \) between \( \hat{\beta}_1, \hat{\alpha}_1, \lambda_0 \) and \( \lambda_1 \). Hence there are two new free parameters (say \( \lambda_1 \) and \( \hat{\alpha}_1 \)).

2) For \( n + r = 2 \) there are two relations \( n = 2, r = 0 \) and \( n = 1, r = 1 \) which allow the determination of \( \hat{\beta}_2 \) and \( \hat{\alpha}_2 \) with one new free parameter \( \lambda_2 \).

3) For \( n + r = 3 \) there are again two relations \( n = 3, r = 0 \) and \( n = 2, r = 1 \) which allow the determination of \( \hat{\beta}_3 \) and \( \hat{\alpha}_3 \) with one new free parameter \( \lambda_3 \).

4) For \( n + r = p \geq 4 \) there are three relations or more. Three of them allow the determination of \( \hat{\beta}_p, \hat{\alpha}_p \) and \( \lambda_p \) in terms of \( \lambda \)'s of lower values. These determinations are sequential. We write them down explicitly:

\[
\begin{align*}
\lambda_p &= \frac{\lambda_{p-2}\lambda_{p-1}\lambda_2 - \lambda_{p-2}\lambda_{p-1}\lambda_1 + \lambda_{p-2}\lambda_2\lambda_1 - \lambda_{p-1}\lambda_2\lambda_1 + \lambda_{p-1}\lambda_1\lambda_0}{\lambda_{p-2}\lambda_2 - \lambda_{p-2}\lambda_0 - \lambda_{p-1}\lambda_1 + \lambda_{p-1}\lambda_0 - \lambda_2\lambda_0 + \lambda_1\lambda_0} \\
\hat{\beta}_p &= \frac{\lambda_{p-2} - \lambda_{p-1} + \lambda_2 - \lambda_1}{\lambda_{p-2}\lambda_2 - \lambda_{p-1}\lambda_1} \\
\hat{\alpha}_p &= \frac{\lambda_{p-2}\lambda_{p-1}\lambda_2 - \lambda_{p-2}\lambda_{p-1}\lambda_1 + \lambda_{p-2}\lambda_2\lambda_1 - \lambda_{p-1}\lambda_2\lambda_1}{\lambda_{p-2}\lambda_2 - \lambda_{p-1}\lambda_1}.
\end{align*}
\]

(2.5)

The remaining relations apart from the first three for \( n + r \) fixed are then automatically satisfied. The easiest way to see this is from the general solution to (2.5) which we shall obtain.

Apart from the trajectory, which we shall discuss next, the amplitude (up to the arbitrary normalisation factor \( N \)) depends upon two further parameters which may be taken to be \( \hat{\alpha}_0 \) and \( \hat{\alpha}_1 \).

The Regge trajectory, which is the important physical quantity, has four arbitrary parameters \( \lambda_0, \lambda_1, \lambda_2 \) and \( \lambda_3 \) with the higher poles \( \lambda_p, p \geq 4 \) given
by (2.3). After choosing $\gamma_r$ suitably, we find that the general solution can be written as

$$
\lambda_r = \frac{a \sigma^r + b}{c \sigma^r + d}.
$$

(2.6)

$$
\alpha_r = \frac{a^2 \sigma^r - b^2}{c \sigma^r + d} \quad \text{for } r > 1.
$$

(2.7)

$$
\beta_r = \frac{c^2 \sigma^r - d^2}{c \sigma^r + d} \quad \text{for } r > 1.
$$

(2.8)

$$
\gamma_r = \frac{a c \sigma^r - b d}{c \sigma^r + d} \quad \text{for } r > 1.
$$

(2.9)

we refer to such a spectrum as a M"{o}bius trajectory.

To be complete let us quote the two remaining equations which are associated with the two first values of $n + r$

$$
\hat{\alpha}_0 - 2\lambda_0 + \hat{\beta}_0 \lambda_0^2 = 0 \quad (2.10)
$$

$$
\hat{\alpha}_1 - \lambda_0 - \lambda_1 + \hat{\beta}_1 \lambda_0 \lambda_1 = 0. \quad (2.11)
$$

However to make everything nicely continuous in the discrete variable $r$ we will choose the two free remaining parameters $\hat{\alpha}_0$ and $\hat{\alpha}_1$ in such a way that (2.7),(2.8) and (2.9) are valid even for $r = 0$ and $r = 1$.

The four independent (complex) parameters, three in the $SL(2, C)$ M"{o}bius matrix $M$

$$
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det M = ad - bc = 1 \quad (2.12)
$$

and $\sigma$, are then determined in terms of the first four pole positions (in principle complex also) $\lambda_0, \lambda_1, \lambda_2$ and $\lambda_3$, by solving three linear and one quadratic equation. The parameter $\sigma$ is determined as any solution of the equation

$$
\sigma^2 + \sigma(z + 1) + 1 = 0 \quad (2.13)
$$

where $z$ is the cross ratio

$$
z = \frac{(\lambda_3 - \lambda_0)(\lambda_1 - \lambda_2)}{(\lambda_1 - \lambda_0)(\lambda_3 - \lambda_2)}. \quad (2.14)
$$

Regarding the equations (2.5) for fixed $n + r$ as linear equations for $\hat{\alpha}(n + r)$ and $\hat{\beta}(n + r)$ the solution (2.6) can be verified to ensure that only two of
these equations are linearly independent, and thus that the solution is fully consistent.

The invariance of the manifold of the trajectories is essentially $SL(2, C) \otimes GL(1, C)$ where $GL(1, C)$ corresponds to the multiplication of $\sigma$ by an arbitrary complex number. The trajectories themselves are invariant under the $Z_2$ transformation which interchanges $\sigma$ and $1/\sigma$ while also interchanging $a \leftrightarrow b$ and $c \leftrightarrow d$.

Regarding purely real trajectories, since $\lambda_0$ and $\lambda_1$ can be put to 0 and 1 respectively by a translation (fixing the intercept) and a dilation (fixing the energy scale), the two last parameters $\lambda_2$ and $\lambda_3$ are those which fix the shape of the trajectory. Let us remark here that this shape is very sensitive to the values of the remaining physical parameters $\lambda_2$ and $\lambda_3$. Some trajectories saturate to a finite value when $r$ goes to $\infty$, others go to $\infty$. There are other trajectories which behave as tangents but should probably be excluded physically since they involve tachyons. Allowing some of the parameters to become complex can lead, for example, to trajectories whose few first members are real and hence stable while the higher ones are complex and hence intrinsically unstable.

Remarkably, all three expressions $\lambda_r$, $\hat{\alpha}_r$ and $\hat{\beta}_r^{-1}$ all satisfy the same Möbius type recurrence relation

$$
\lambda_{r+1} = \frac{v\lambda_r + w}{x\lambda_r + y} \overset{\text{def}}{=} \lambda_r^{[1]},
$$

(2.15)

with

$$
v = bc - ad\sigma
$$

$$
w = ab(\sigma - 1)
$$

$$
x = cd(1 - \sigma)
$$

$$
y = bcs - ad.
$$

The perceptive reader may wonder how it comes about that the general solution of a sequence $\lambda_n$ which satisfies a second order recurrence relation (2.3) also satisfies a first order relation (2.15). The reason is that (2.3) also encodes the boundary values to be satisfied by $\lambda_n$. 

7
Armed with these relations, it is possible to show that the amplitude
\( A(s, t) \) defined in equation (2.1) satisfies the functional equation
\[
A(s, t) = \frac{\gamma_0(s + t) - \alpha_0 - \beta_0 st}{s - \lambda_0} A(s^{-1}, t)
\] (2.16)
and
\[
s^{-1} = \frac{-ys + w}{xs - v}.
\] (2.17)
As expected, the transformation between \( s \) and \( s^{-1} \) is the inverse of the Möbius transformation (2.15).

When the trajectory is restricted to be purely logarithmic, say (1.3) with \( a = 1 \), defining \( \tau = 1/\sigma \) and noting that \( s^{-1} = \tau s \), one finds a recurrence relation
\[
A(s, t) = sA(s, \tau t) + A(\tau s, t),
\] (2.18)
which may be re-cast in the form of a \( q \)-derivative-difference equation
\[
D_s A(s, t) = \frac{1}{1 - \tau} A(s, \tau t),
\] (2.19)
where
\[
D_s A(s, t) = \frac{A(\tau s, t) - A(s, t)}{s(\tau - 1)}.
\] (2.20)
Equations of the form (2.19) determine the 4-point function up to a constant and will be used more systematically for the 5-point function.

3 The five point function.

Let as usual \( s_{ij} = (p_i + p_j)^2, i, j = 1, \ldots, 5 \) denote the generalized Mandelstam variables in terms of the incoming momenta of five scalar particles. Consider the usual planar tree diagrams and the particular subset of variables \( s_k \equiv s_{k,k+1} \) where poles at positions \( \lambda_m \) due to the tree diagrams can occur. These poles must obey the duality, factorization and symmetry restrictions:

a) The residue of any pole in any of the \( s_A \) variable, say \( s_1 \) should have no pole in the neighbouring variables i.e. no poles in the variables \( s_5 \) and \( s_2 \). Double poles can occur only in non-neighbouring variables.
b) The residues of the allowed double poles should factorize in the usual way.

c) The amplitude should have the $D_5$ symmetry of the pentagon. It consists in the cyclic $C_5$ symmetry permuting the five legs of the dual diagram. Under this symmetry the five variables $s_1, s_2, s_3, s_4, s_5$ are rotated cyclically. The extra symmetry needed to define $D_5$ by closure under products is generated by, say, the mirror symmetry $Z_2^{(1)}$ generated by the simultaneous interchange of the variables $s_2 \leftrightarrow s_5$ and $s_3 \leftrightarrow s_4$ leaving $s_1$ fixed.

The Baker Coon 5-point function, which is again more conveniently expressed in terms of $\tau = 1/\sigma$, is constructed as a multiple sum

\[
A(s_1, s_2, s_3, s_4, s_5) = \sum_{n_1=0}^{\infty} G(s_5 s_1 \tau^{n_1}) G(s_2 s_3 \tau^{n_4}) \prod_{r=1}^{n_4} \frac{s_1 s_2 - s_4 \tau^r}{1 - \tau^r}.
\] (3.1)

which is obviously $D_5$ invariant and where

\[
f_n = (1 - \tau)(1 - \tau^2) \cdots (1 - \tau^n).
\] (3.2)

They show that 4 of the 5 summations can be performed formally yielding

\[
A(s_1, s_2, s_3, s_4, s_5) = \sum_{n_4=0}^{\infty} \frac{G(s_5 s_1 \tau^{n_4})}{G(s_5 \tau^{n_4}) G(s_1)} \frac{G(s_2 s_3 \tau^{n_4})}{G(s_2) G(s_3 \tau^{n_4})} \prod_{r=1}^{n_4} \frac{s_1 s_2 - s_4 \tau^r}{1 - \tau^r}.
\] (3.3)

Here $G(z)$ denotes the function

\[
G(z) = \prod_{r=0}^{\infty} (1 - z \tau^r).
\] (3.4)

The basic result used is the Euler sum

\[
\sum_{n=0}^{\infty} \frac{z^n}{\prod_{r=1}^{n} (\tau^r - 1)} = \prod_{r=0}^{\infty} \frac{1}{(1 - z \tau^r)} = E_{r}(\frac{z \tau}{1 - \tau}).
\] (3.5)

Here $E_r(z)$ denotes the Jackson $[16]$ form of the $q$-deformed exponential for $q = \tau$. This is meaningful, i.e. the series converges uniformly and absolutely
for all finite \( z \) if \( \tau \leq 1 \), but for \( \tau > 1 \) (\(|\sigma| < 1\)) the series converges uniformly and absolutely for \(|z| < 1\) and diverges otherwise. This is the range for which the amplitudes are found to be ghost free.

In this representation it is easy to see that their amplitude has no simultaneous poles in adjacent variables, but does admit double poles in \( s_3 \) and \( s_5 \) at \( s_3 = \sigma^{k_3} \) and \( s_5 = \sigma^{k_5} \) with residue a polynomial in the variables \( s_1, s_2, s_4 \) built from monomials of the form

\[
s_1^{k_3-\beta} s_2^{k_3-\beta} s_4^{\beta}, \quad 0 \leq \beta \leq \min(k_3, k_5)
\]

(3.6)
as can easily be seen from (3.3). This is in agreement with the standard Feynman tree amplitude.

In their original derivation of the 5-point function for linear trajectories [12], Bardakci and Ruegg developed an integral representation for their amplitude, by analogy with that for the 4-point, which exhibits the correct double poles. They went on on to develop an alternative approach to the 5-point based upon functional recursion relations which it must satisfy and which are derived by partial integration of their 5-point representation. We have derived just such a recurrence relation in section 2 for our 4-point function and now display analogous recurrence relationships which are satisfied by Baker and Coon’s dual 5-point function. After performing permutations of the variables one obtains altogether five equations. A typical member is as follows;

\[
A(s_1, s_2, s_3, s_4, s_5) = s_5 A(\tau s_1, s_2, s_3, \tau s_4, s_5)
+ A(s_1, s_2, s_3, s_4, \tau s_5).
\]

(3.7)

These equations are derived from the representation in the form of equation (3.1). They are analogous to the recurrence equation (2.19) we have found for the 4-point function and may be re-cast in the form of a \( q \)-derivative-difference equation

\[
D_{s_5} A(s_1, s_2, s_3, s_4, s_5) = \frac{1}{1-\tau} A(\tau s_1, s_2, s_3, \tau s_4, s_5).
\]

(3.8)

These equations can be used alternatively to define a 5-point amplitude. Indeed, from these equations it is easy to prove the following relations

\[
\frac{\partial^{m_1+m_2+m_3+m_4+m_5}}{(\partial s_1)^{m_1}(\partial s_2)^{m_2}(\partial s_3)^{m_3}(\partial s_4)^{m_4}(\partial s_5)^{m_5}} A_{s_1=s_2=s_3=s_4=s_5=0}
\]
\[
= m_1 \tau^{m_2+m_5} \frac{\partial^{m_1+m_2+m_3+m_4+m_5-1}}{1 - \tau^{m_1}} (\partial s_1)^{m_1-1}(\partial s_2)^{m_2}(\partial s_3)^{m_3}(\partial s_4)^{m_4}(\partial s_5)^{m_5} A_{s_1=s_2=s_3=s_4=s_5=0}.
\]

(3.9)

From the basic equations (3.8) numerous other identities can be derived. The following ones are particularly interesting in the light they cast upon the pole structure of the amplitude

\[
A(s_1, \tau s_2, s_3, s_4, \tau s_5) = \frac{1}{s_1 - 1} \times
\]

\[
((s_2 - 1)A(\tau s_1, s_2, \tau s_3, s_4, s_5) + (s_5 - s_3)A(\tau s_1, \tau s_2, s_3, \tau s_4, s_5)).
\]

(3.10)

These equations may be used to determine the values of the expansion coefficients in a multiple power series in \( s_i \) for the 5-point amplitude. It is easy to prove from the recurrence relations satisfied by these coefficients that (3.9) and its cyclic partners determine this amplitude uniquely, up to a multiplicative constant, as that of Baker and Coon.

The substitution of the Möbius transformed values of \( s_i^{[M]} \) in the amplitude \( A(s_i) \) of Coon and Baker (3.1) gives rise to a Möbius amplitude \( A^{[M]}(s_i) \)

\[
A^{[M]}(s_i) = A(s_i^{[M]})
\]

(3.11)

where, see (2.6),

\[
s_i^{[M]} = \frac{ds_i - b}{-cs_i + a}
\]

(3.12)

which obviously has poles at the Möbius spectrum (2.15) with no simultaneous neighbouring poles. Unfortunately the residues of these poles are of too high degree to permit an interpretation in terms of the exchange of particles with spin no greater than that associated with the corresponding pole (3.6).

The corresponding identities (3.10) are written

\[
(s_1 - \lambda_0) A(s_1, s_2^{[-1]}, s_3, s_4, s_5^{[-1]}) - (s_2 - \lambda_0)A(s_1^{[-1]}, s_2, s_3^{[-1]}, s_4, s_5) = (s_5 - s_3)A(s_1^{[-1]}, s_2^{[-1]}, s_3, s_4^{[-1]}, s_5)
\]

(3.13)

using the inverse Möbius transformed variables \( s_i^{[-1]} \) defined previously (2.17).
4 Ghost Freedom

One of the most serendipitous properties of the Veneziano amplitudes is the absence of ghosts, i.e. the absence of contributions from intermediate states of integral spin with negative coefficients in dimensions $\leq 26$, when the ground state mass squared lies between -1 and 0. (The former limit corresponds to a ground state tachyon, for which the theory admits an operator factorisation.) Indeed the only analytic way known to us to demonstrate freedom from ghosts is via the operator formalism $[13, 14]$.

It would be remarkable if this property were not to continue to hold when the Veneziano amplitude is slightly deformed. As there is no obvious way to test this analytically, the only way to proceed is to test for a range of cases using an algebraic computation programme (in our case REDUCE).

We have taken the basic Baker and Coon amplitude and performed a rescaling as explained in the introduction to make the $\sigma = 1$ limit coincide exactly with the Veneziano amplitude with slope 1 and a shift in such a way that the scalar external particle be of (rescaled) mass $m$ with $-1 \leq m^2 \leq 0$

$$\lambda_n = \frac{\sigma^n - 1}{\sigma - 1} + m^2. \quad (4.1)$$

This amplitude takes the form

$$A(s, t) = \frac{N}{\prod_{r=0}^{\infty} \left( (\sigma - 1)(s - m^2) + 1 \right) \left( (\sigma - 1)(t - m^2) + 1 \right) - \sigma^r} \cdot (4.2)$$

where $N$ is a (possibly infinite) normalisation constant. In a theory with centre of mass momentum $p$, we have

$$t = -2p^2(1 - \cos \theta) = -(\frac{s}{2} - 2m^2)(1 - \cos \theta). \quad (4.3)$$

We can then write a simple procedure to calculate $R(j, k)$, the coefficient of $P_k(\cos \theta)$ once the residue of the pole at $s = \lambda_j$ has been expanded in orthogonal Legendre polynomials. A negative value of $R(j, k)$, the square of the coupling between the two external scalars and the exchanged particle of mass square $\lambda_j$ and spin $k$, is the signature of an exchanged ghost.
A crucial formula \[17\]

\[
\int_{-1}^{1} (1 - x)^m P_k(x) dx = \begin{cases} 
(-1)^k \frac{2^{m+1}(m!)^2}{(m-k)!(m+k+1)!} & m \geq k, \\
0 & m < k.
\end{cases} \tag{4.4}
\]

enables us to project out the \(k\)th partial wave of the residue at the pole \(s = \lambda_j\) (see (1.5))

\[\lambda_j = l(j) + m^2. \tag{4.5}\]

This residue is, as we know, a polynomial of maximum degree \(j\) in \(t\). One can then write easily a simple symbolic procedure to calculate \(R(j,k)\).

Using this procedure, we have computed analytically the \(R(j,k)\) for quite a number of low values of \(j\) and \(k\). In particular, it is possible to deduce a general formula for cases \(R(j,j)\) and \(R(j,j-1)\) as follows;

\[
R(j,j) = \frac{(j!)^2}{g(j)(2j)!} (l(j) - 3m^2)^j,
\]

\[
R(j,j-1) = \frac{((j-1)!)^2}{2g(j)(2j-2)!} (l(j) - 3m^2)^{j-1} \times
\]

\[
(jm^2 - jl(j) + \frac{2}{\sigma^j} \frac{\partial l(j+1)}{\partial \sigma}), \tag{4.6}
\]

where as usual \(l(j)\) is given by (1.5) and

\[
g(j) = \frac{(\sigma - 1)(\sigma^2 - 1)(\sigma^3 - 1) \cdots (\sigma^j - 1)}{(\sigma - 1)^j \sigma^{\frac{j(j+1)}{2}}}. \tag{4.7}
\]

We have explored systematically these analytical \(R(j,k)\) for all values of \(k\) within a large range of \(j\), looking for patterns inside the interesting domain \(D\) given by \(-1 \leq m^2 \leq 0\) and \(\sigma > 0\). For later reference let us take \(m^2\) as the horizontal axis and \(\sigma\) as the vertical axis. The domain \(D\) is then a semi-infinite vertical strip.

Using the explicit general forms (4.4) and the set of examples we have shown that within the domain \(D\) the behaviour of \(R(j,k)\) for \(j + k\) even is different from the behaviour for \(j + k\) odd. Indeed :

1) For even \(j+k\), \(R(j,k)\) is always positive. The curve given by \(R(j,k) = 0\) does not cross the domain \(D\).
2) For odd \( j + k \), \( R(j, k) = 0 \) is a curve which crosses the domain \( D \) once only and divides it into two parts. In the low \( \sigma \) part \( R(j, k) \) is positive while in the higher \( \sigma \) part it is negative. The curve obviously depends on the values \( j \) and \( k \) but, and this is remarkable, it is always confined to a small region in the domain.

We now discuss the curve \( R(j, k) = 0 \) in a more precise way when \( j + k \) is negative.

First let us examine the crucial \( R(1, 0) = 0 \) curve. Its analytical form is given by

\[
m^2 = 1 - \frac{2}{\sigma}
\]

i.e. a hyperbola passing through the points \( m^2 = -1, \sigma = 1 \) and \( m^2 = 0, \sigma = 2 \), its concavity turned upwards and with a positive slope inside \( D \).

All the other curves \( R(j, k) = 0 \) with \( j + k \) odd are like the \( R(1, 0) = 0 \) curve within the domain \( D \). They pass through the point \( m^2 = -1, \sigma = 1 \). Their concavity is upwards and they have a positive slope inside \( D \), though they are no longer hyperbolas but higher degree polynomials. Finally the intersection point in the vertical axis \( m^2 = 0 \) has a coordinate \( \sigma = \sigma_{jk} \) and

\[
1 < \sigma_{jk} \leq 2.
\]

Numerical estimates show that, when \( j \) increases, \( \sigma_{jk} \), while decreasing, converges to 1. We here quote a few values \( \sigma_{10} = 2, \sigma_{21} = 1.246979, \sigma_{32} = 1.113625, \sigma_{30} = 1.111834, \sigma_{43} = 1.065670, \sigma_{41} = 1.064472, \sigma_{54} = 1.042874, \sigma_{52} = 1.042066, \sigma_{50} = 1.041721. \)

From these considerations we can draw the following conclusions which were also tested by a numerical exploration for much higher values of \( j \) and \( k \) and can easily seen to be correct for analytical \( R(j, j - 1) \) given above in (4.6).

1) for \(-1 \leq m^2 \leq 0 \) and \( 0 < \sigma \leq 1 \) all the \( R(j, k) \) are positive and hence the 4 point amplitude is ghost free.

2) If the point \( m^2, \sigma \) is above the curve (4.8), then the amplitude is not ghost free, but systematically \( R(j, k) > 0 \) if \( j + k \) is even and \( R(j, k) < 0 \) if \( j + k \) is odd. In this case, which would correspond to an exponentially rising \( \lambda_n \) since \( \sigma > 1 \), it is not impossible that a mechanism might be
devised to reinterpret those wrong sign poles, since the pattern of their occurrence is so simple.

3) In the intermediate range i.e. when $\sigma$ lies between the line $\sigma = 1$ and the curve (4.3), the results have a pattern in $R(j, k)$ which varies with $\sigma$ and $m^2$. For all even $j + k$ the sign is systematically positive. For odd $j + k$, on a given point $m^2, \sigma$ in the intermediate range, some $R$’s at low values of $j$ and $k$ are positive while they are all negative at higher values of $j$ and $k$.

4) If the patterns we have found are correct (and we believe they are) they show that $\sigma = 1$, the Veneziano case, is exactly the borderline case of the region where the interpretation of the amplitude in terms of exchanged particles of definite spins holds good. In this limit, when the mass $m^2$ approaches the tachyon value -1, the intermediate range shrinks to zero. At the transition point to the alternating pattern, which occurs at $\sigma = 1$, all the $R(j, k)$ are equal to 0 for odd $j + k$. This is the statement that the odd daughters decouple.

The case envisaged by Baker and Coon and also Cremmer and Nuyts corresponds to the regime where $\sigma > 1$ and thus there is ghost behaviour.

The ghost free case, $0 \leq \sigma \leq 1$, as we have remarked in the introduction implies that the spectrum in mass(squared) approaches a limit point

$$\frac{\sigma}{1 - \sigma} + m^2.$$  \hfill (4.10)

We stress that when $\sigma$ is very close to 1 (i.e. when we are close to the linear case $\sigma = 1$), the limit point may be made arbitrarily high (tentatively the Planck mass ?) and that the low lying resonances lie on a trajectory experimentally indistinguishable from a linear trajectory. This would be appropriate for an interpretation in terms of hadronic amplitudes.

The existence of an accumulation point in the spectrum obviously implies a breakdown of the validity of the expression for the scattering amplitude at this $s$ value. However, this may be reconcilable with current attempts to introduce a fundamental length via string theories. (See, for example [11]).
5 Conclusion

In this paper we have re-opened some old questions concerning the nature of the dual resonance model and its realisation as a theory of first quantized strings. The principal new result is the conjecture, based upon extensive empirical observation, that the Veneziano amplitude admits a deformation which preserves the properties of duality and in particular, is ghost free. The related trajectory has an accumulation point which can be made to occur at an arbitrarily high value of the energy. This strongly suggests the existence of a deformed string theory to give an interpretation to such amplitudes. Is there an operator formalism which reproduces these results, involving, maybe, quantum creation and annihilation operators?

We have also shown that the Möbius trajectory allows the construction of a dual 4-point amplitude describing the exchange of an infinite number of particles or resonances of maximum spin $n$ increasing monotonically along the trajectory. It is a more general solution than the logarithmic trajectories with exponential spectrum, since it depends upon the four parameters which may be taken to be the masses of the first four poles and some of them can even be chosen as complex. A secondary series of questions is concerned with the amplitudes for the Möbius trajectory. Is there anything corresponding to an integral representation for our Möbius amplitudes? There are two additional parameters in our 4-point amplitude; $\hat{\alpha}_0$ and $\hat{\alpha}_1$. Can we exploit this freedom and extend to a viable 5-point function?

In summary, we believe that the miraculous way in which the 4 parameter spectrum works for the Möbius trajectory and the absence of ghosts in certain ranges of the amplitudes for the logarithmic trajectories with an accumulation point suggest a new avenue of exploration in string theory.

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