The averaged model of layered elastic-creeping composite materials

Tatiana Bobyleva¹ and Alexei Shamaev²,³

¹Moscow State University of Civil Engineering, Yaroslavskoye shosse, 26, Moscow, 129337, Russia
²Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Pr. Vernadskogo, 101-1, Moscow, 119526, Russian Federation
³Lomonosov Moscow State University, Leninskie Gory, GSP-1, Moscow, 119991, Russia

E-mail: ¹tatyana2211@outlook.com

Abstract. Composite materials consisting of several phases are widely used in modern construction. The mechanical characteristics of elastic-creeping layered composite materials are considered in the article. Each of the constituent phases has the properties of elasticity, viscosity or creep. Numerous experiments have shown that the properties of structurally heterogeneous materials can differ significantly from those of the individual components making up the composition. Besides, rapidly changing coefficients of differential equations describing such composite materials greatly complicate the solution of boundary value problems even with the help of computer calculation methods. Therefore, the homogenization method is used to solve such problems. Creep kernels are given by the sum of a finite number of decreasing exponential functions. The use of creep kernels of this type is experimentally justified. In this paper, it is shown that an effective (averaged) model for a composite material with the indicated properties is a homogenized medium that is described by a creep kernel, also represented by a sum of exponential functions. An algorithm for the rapid and accurate calculation of averaged creep kernels of a homogenized material is proposed.

1. Introduction

Problems for heterogeneous materials arise in many areas of construction. Many structural materials are layered. Such heterogeneity is the reason for their specific behavior during deformation. Differential equations that describe the behavior of such materials contain rapidly varying coefficients, so solving these equations takes a long time, even with modern computers. The method of asymptotic averaging transforms such material into a homogeneous one, obeying the averaged equations. This averaging method was developed in [1, 2], one of the first applications was the problems of the theory of elasticity [3, 4]. As an example, the problem of averaging an inhomogeneous elastic medium with the use of theoretical results was solved in [5]. In addition, the stress-strain state of composite materials changes over long time intervals. This should be taken into account when we investigate the strength of such materials. The problem of studying rheological properties is relevant for construction practice, since creep acts on the redistribution of stresses, and in some cases can lead to an unacceptable increase in deformation. In [6-9] the foundations of the theory of the hereditary Boltzmann-Volterra mechanics are presented. This theory describes processes in which the state of a mechanical system depends on the entire history of actions performed on it. Examples of solving problems of linear hereditary elasticity theory and nonlinear creep theory are given. Averaged models
describing the joint motion of the layers consisting of elastic and viscoelastic materials [10], and also of two creeping materials [11], were constructed. In paper [12] effective elastic modules of a layered elastic-creeping medium were obtained.

The article proposes a method for modelling the stress-strain state of layered materials, based on a combination of methods of the theory of averaging and creep theory. As the main rheological relations, the relations of the theory of linear heredity of Boltzmann-Volterra are accepted, according to which the deformation at time \( t \) has the form:

\[
\varepsilon(t) = \frac{1}{E} \left[ \sigma(t) + \int_0^t K(t-\tau)\sigma(\tau)\,d\tau \right]
\]

where \( E \) is the Young’s modulus, \( \sigma \) is the stress, \( K(t-\tau) \) is the creep kernel, which is a decreasing function [8].

The parameters of the rheological model are determined on the basis of laboratory tests. The finite sums of the exponential functions of time are taken in this article as the creep kernels of each layer. Experiments prove that such functions are a good way to describe the creep of materials over a long period of time. Such kernels are called spectral. Moreover, in a number of works other types of kernels are also used, for example, the Abel kernels.

It is clear that models may be very different for the various materials used in engineering practice, such as metals and alloys, polymers, fiber composites (with polymer or metal matrix), concrete and wood. However, to a large extent it is possible to employ the same principles and concepts in establishing constitutive relations for these different materials.

Viscoelastic materials models can be built by combining elastic elements and viscous elements. Each elastic element can be represented by a spring with parameter \( E \) which is its Young’s modulus. An elastic element follows Hooke’s law. Each viscous element can be represented by a dashpot with parameter \( \eta \) which is viscosity of the material. One can build up a model of linear viscoelasticity by considering combinations of the linear elastic spring and the linear viscous dashpot. Such model of viscoelastic body is called the simplest [6].

For example, the Maxwell model can be represented by a purely viscous dashpot and a purely elastic spring connected in series, as shown in the Figure 1. The other two-element model, the Voigt model, consists of the same spring and dashpot in parallel, as shown in the Figure 2.

\[ \text{Figure 1. Maxwell model} \quad \text{Figure 2. Voigt model} \]

The Maxwell model represents a viscoelastic fluid, whereas the Voigt model represents a viscoelastic solid.

So the Voigt and the Maxwell models are the simplest viscoelastic bodies. The Maxwell model exhibits an exponential (reversible) stress relaxation and a linear (non reversible) strain creep; it is also referred to as the relaxation element.

The Voigt model exhibits an exponential (reversible) strain creep, but no stress relaxation; it is also referred to as the retardation element. The model is extremely good with modelling creep in materials.

In order to describe the rheological behavior of a complex material, one can combine in various combinations the models of these simplest ideal bodies. Such elements can be combined in parallel or in series. There were also generalizations of mechanical models allowing description of physical nonlinearity [13, 14]. For example, the nonlinear Maxwell-type constitutive relation with two arbitrary material functions for viscoelastoplastic multi-modulus materials is studied in paper [15] analytically in uniaxial isothermic case. Paper [16] deals with parameter identification for basic and generalized Voigt and Maxwell models. The results were applied to fluid viscous dampers. A family of methods
for identification of the parameters of both the Voigt fractional model and the Maxwell fractional model are presented in the paper [17]. The validity and effectiveness of procedures have been tested using artificial and real experimental data. In the paper [18], a simple nonlinear Maxwell model consisting of a nonlinear spring connected in series with a nonlinear dashpot obeying a power-law with constant material parameters, for representing successfully the time-dependent properties of a variety of viscoelastic materials, is proposed. The paper [19] describes the numerical simulation, designed for simulation of the stress-strain state in the ground subjected to wave processes. Voigt and Maxwell models are used to describe deformations in the soil and a concrete structure immersed in it. As a result authors created their own computer code based on the finite element method (FEM).

It is well known that in the modelling of elastic-creeping or viscoelastic materials with the help of combinations of springs and dashpots, the transfer function (the multiplier that relates the Laplace transforms of stresses and deformations) is the sum of the simplest fractions with coefficients of the same sign [6]. The purpose of this paper is to investigate the problem of saving this transfer function’s property for the averaged model of a layered composite. This study will answer the following question: for which deformations of a layered composite we can use the approximating model consisting of springs and dashpots mentioned above.

2. Problem specification and decision
Consider a region \( \Omega \) consisting of pairwise alternating layers of elastic-creeping isotropic materials. Let \( e \) be the length ratio of two adjacent layers to the characteristic size of the sample, and the thicknesses of each individual layer are respectively equal to: \( e_1 = eh \) and \( e_2 = e(1-h) \), \( 0 \leq h \leq 1 \). Cartesian coordinate system \( Ox_1x_2x_3 \) is chosen. All the layers of the considered region \( \Omega \) are parallel to the \( Ox_1x_2 \) coordinate plane. Components of the stress tensor \( \sigma_{ij} \) are determined not only by deformation at the moment, but also by all the preceding history of the body deformation. Therefore, the equations of state connecting the components of the strain and stress tensors for each layer are as follows [6,7]:

\[
\sigma_{ij}^{(s)} = b_{ikh}^{(s)} \ast \epsilon_{ikh}^{(s)},
\]

(2)

here \( b_{ikh} = c_{ikh} \delta(t) + d_{ikh} \),

\[
\epsilon_{ikh}^{(s)} = \frac{1}{2} \left( \frac{\partial u_{ikh}^{(s)}}{\partial x_h} + \frac{\partial u_{ikh}^{(s)}}{\partial x_k} \right), \quad (k,h = 1-3), \quad (s = 1,2 \text{ is the layer number}), \quad u_k \text{ are components of the displacement vector}, \quad c_{ikh}^{(s)} \text{ are components of the elastic modulus tensor}, \quad \delta(t) \text{ is Dirac-delta}, \quad d_{ikh}^{(s)}(t,\tau) \text{ are Volterra integral operators}, \text{ namely},
\]

\[
d_{ikh} \ast \epsilon_{ikh} = \int_0^t d_{ikh}(t-\tau)\epsilon_{ikh}(\tau) d\tau,
\]

(3)

and variable \( t \) specifies time. (Einstein convention for repeated indices is used.)
Relaxation kernels \( d_{ikh}(t-\tau) \) depend on the difference \( t - \tau \). This follows from the condition of invariance of the quantity \( \sigma_{ij} \) with respect to the origin of time \( t \).

In this article, the relaxation kernels are taken in exponential form, since such kernels can be recommended for analysis of long-term deformation processes.

The ideal contact conditions are assumed on the horizontal surfaces of the layers, namely: components of the displacement and the components of stress parallel the \( x_3 \)-axis are continuous, i.e. \( [u_i] = 0, [\sigma_{ij}] = 0, \quad (i = 1-3) \).

Equilibrium equations in the theory of elasticity have the form [9]:
\[ \frac{\partial \sigma_{ij}(x,t)}{\partial x_j} = f_i(x,t) \]  

(4)  

In (4) we designated: \( x = (x_1, x_2, x_3) \), \( f_i(x,t) \) are components of a vector of external forces. We consider isotropic materials, therefore, the components of the elastic tensor and relaxation kernel tensors in (2) have the form [6]:  

\[ c_{ijkh} = \lambda \delta_{ij} \delta_{kh} + \mu (\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}) \].

\( d_{ijkh} = -D_s(t)(\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}) \). We denote here by \( D_s \) and \( D_t \) the regular part of the shear and the bulk relaxation respectively, by \( \delta_{ij} \) Kronecker symbol. Suppose that the amplitude of a bulk relaxation kernel is proportional to the amplitude of the shear relaxation kernel with a proportionality coefficient \( k_s \) for each layer, that is: \( D_s(t) = k_s (D_m(t)) \), \( D_t(t) = k_t (D_m(t)) \), \( k_s = \text{const} \), \( k_t > 0 \). Further, \( D_m(t) \) is denoted by \( D \).

In this problem, all elastic modulus and relaxation kernels are periodic functions of the coordinate  

\[ y = \frac{x_3}{e} \]  

(\( e \) is the relative cell period, as it was said earlier) and are piecewise constant functions of this variable, i.e., elastic modulus and relaxation kernels have the form [12]:

\[ \lambda(y) = \begin{cases} \lambda_1, & y \in [0; h] \\ \lambda_2, & y \in [1-h; 1] \end{cases} \]

\[ \mu(y) = \begin{cases} \mu_1, & y \in [0; h] \\ \mu_2, & y \in [1-h; 1] \end{cases} \]

\[ D(y,t) = \begin{cases} D_1(t), & y \in [0; h] \\ D_2(t), & y \in [1-h; 1] \end{cases} \]

As usual, we denote here by \( \lambda_s, \mu_s \) Lame parameters for each layer. We choose creep kernel of exponential type for each layer (\( s \) is the layer number, \( s = 1, 2 \)):

\[ D_s(t) = \sum_{j=1}^{N} d_{js} \exp(-\alpha_{js} t), \quad d_{js}, \alpha_{js} \text{ are constants}, \quad d_{js} < 0, \quad \alpha_{js} > 0 \]  

[6]. As experiments show, such kernels describe well the behavior of relaxation materials for long periods of time.

We apply the Laplace transform in the time domain to the equations (4) with the assumption (2)

\[ \tilde{f}(p) = \int_0^\infty f(t)e^{-pt}dt \]  

(5)

The result is the system of elasticity theory with a complex parameter \( p \). We apply homogenization method described in [1, 2] to this system. After this we obtain a homogeneous anisotropic medium with averaged (effective) modules [12].

Let us study some qualitative properties of received effective modules. For this, the following two statements are needed.

Statement 1. Let

\[ f(z) = a_0 + \frac{a_1}{z + z_1} + \cdots + \frac{a_N}{z + z_N}, \]

(6)

where \( a_0 > 0, a_i < 0, z_i < 0 \) (\( i = 1, \ldots, N \)). (As this is known from the theory of viscoelasticity [3]).

Zeros \( y_i \) and poles \( z_i \) of a function \( f(z) \) are shown in Figure 3. Then the converse expression will have a similar form \( f^{-1}(z) = a_i^{-1} + \frac{b_1}{z + y_i} + \cdots + \frac{b_N}{z + y_N} \), where \( y_i \) is the single zero of a function \( f(z) \) on a segment \([z_i, z_{i+1}] \) (\( i = 2, \ldots, N \)) and \( y_1 \) is the single zero on a segment \([z_0, z_1] \), as shown in Figure 4, and \( b_i \) (\( i = 1, \ldots, N \)) are positive constants.
Figure 3. Zeros $y_i$ and poles $z_i$ ($i=1 \div 4$) of the function $f(z)$

Figure 4. Zeros $z_i$ and poles $y_i$ ($i=1 \div 4$) of the function $f^{-1}(z)$

**Statement 2.** Let $f(z), g(z)$ are two functions of the form (10). We consider the function $q(z) = [h f^{-1}(z) + (1 - h) g^{-1}(z)]^{-1}$, where $h$ is constant, $h \in (0,1)$. This function will have the following form $q(z) = \hat{a}_0 + \frac{c_1}{z+y_1} + \cdots + \frac{c_{N+M}}{z+y_{N+M}}$, where $c_i < 0$ ($i=1, \ldots, N+M$), $y_i$ are the zeros of the
function \( r(z) = hf^{-1}(z) + (1-h)g^{-1}(z) \) on segments whose ends are points from the set \( \{y_i\}, \{v_j\} \), \( v_j \) are the zeros of the function \( g(z) \) between its poles, and one of the zeros is between the pole nearest to zero and zero \((i = 1, \ldots, N, j = 1, \ldots, M)\). In doing so, we take segments that do not have internal points from this set \( \{y_i\}, \{v_j\} \). Within each of these segments there is a single root of the function \( r(z) = hf^{-1}(z) + (1-h)g^{-1}(z) \), as shown in Figure 5.

In addition \( \hat{a}_i = [ha_i^{-1} + (1-h)a_i^{-1}]^{-1} \) where \( a_i \) is the analogous constant for the function \( g(z) \) as \( a_0 \) is for \( f(z) \).

We now use the above statements to analyze the qualitative properties of the averaged tensor of an elastic-creeping medium. After applying the Laplace transform, the system of equilibrium equations for the averaged medium takes the form
\[
\frac{\partial}{\partial x_j} \left( \hat{b}_{ijkh}^{\text{hom}} (p) \cdot \tilde{e}_{ij} (p) \right) = \tilde{f}_i (x, p),
\]
where the averaged modules \( \hat{b}_{ijkh}^{\text{hom}} (p) \) are defined in [2]. In particular, if the materials of the two elastic-creeping layers are isotropic, then the module which determines the effective elastic-creeping properties of the composite material in case of displacements, perpendicular to the plane of the layers, has the form [20]:
\[
\hat{b}_{1111}^{\text{hom}} (p) = \left( \frac{1}{L(p) + 2G(p)} \right)^{-1},
\]
where \( L(p), G(p) \) are functions of the form (6) taking constant values in each layer, the variable \( z \) is replaced by \( p \), \( \langle f \rangle = hf_1^1 + (1-h)f_2^2 \) is the average over the layers for the function \( f \), which takes constant values \( f_1 \) and \( f_2 \) in each layer, respectively.

One shear modulus has the form [20]:
\[
\hat{b}_{1212}^{\text{hom}} (p) = \left( \frac{L(p)}{L(p) + 2G(p)} \right)^{-1} \left( \frac{L(p)}{L(p) + 2G(p)} \right)^{-1}.
\]
But if we multiply two functions of the form (6) and decompose the result into simple fractions, then the coefficients of these fractions will have different signs. This is inconsistent with the simplest theory of viscoelasticity.

Direct usage of Statements 1 and 2 leads to the following conclusions.

3. Conclusion
From the statement 2 it is clear that for a layered composite consisting of two isotropic elastic-creeping materials with creep kernels corresponding to the simplest viscoelasticity, the averaged (effective) module of tension or compression in a direction, which is perpendicular to the layers, will also correspond to the simplest viscoelasticity. For such effective modules, new exponent indicators can be easily found by dividing a segment in half, and the coefficients of exponential functions are found by solving systems of linear equations. It should be noted that for other averaged modules the statement about the invariability of the property of simplicity is not true.

References
[1] Oleynik O A, Iosif'yan G A and Shamaev A S 1992 Mathematical problems in elasticity and homogenization (North-Holland: Elsevier)
[2] Bardzokas D I and Zobnin A I 2005 Mathematical modeling of physical processes in composite materials of periodic structure (Moscow: URSS)
[3] Pobedrya B E 1984 Mechanics of composite materials (Moscow: MSU)
[4] Christensen R M 2005 *Mechanics of composite materials* (New York: Dover)
[5] Bobyleva T N 2016 Approximate Method of Calculating Stresses in Layered Array. *Procedia Engineering* 153 103-106
[6] Ilyushin A A and Pobedrya B E 1970 *Foundations of the mathematical theory of thermovisco-elasticity* (Moscow: Nauka)
[7] Christensen R M 2010 *Theory of viscoelasticity* (New York: Dover)
[8] Rabotnov Yu N 1980 *Elements of hereditary solid mechanics* (Moscow: Mir)
[9] Rabotnov Yu N 1991 *Mechanics of Deformable Solids* (Moscow: Nauka)
[10] Shamaev A S and Shumilova V V 2016 Asymptotic behavior of spectrum of one-dimensional fluctuations in the environment of the layers of elastic material and a viscoelastic material of Kelvin-Voigt. *Proceedings of MIAS* 295 (1) 202-212
[11] Shamaev A S and Shumilova V V 2016 Averaging of the state equations for a heterogeneous medium consisting of two layers of creeping materials. *Proceedings of MIAS* 295 (1) 213-224
[12] Bobyleva T N and Shamaev A S 2017 An Efficient Algorithm for Calculating Rheological Parameters of Layered Soil Media Composed from Elastic-Creeping Materials. *Soil Mechanics and Foundation Engineering* 54 (4) 224-230
[13] Ilyushin A A and Ogibalov P M 1966 A certain generalization of the Voigt and Maxwell models. *Polymer Mechanics* 2 190–1966
[14] Samarín Yu P and Klebanov Ya M 1994 *Generalized models in the theory of creep of structures.* (Samara: Samara State Tech. Univ.)
[15] Khokhlov A V 2017 The nonlinear Maxwell-type model for viscoelastoplastic materials: simulation of temperature influence on creep, relaxation and strain-stress curve. *J. Samara State Tech. Univ.: Ser. Phys. Math. Sci.* 21 160–179
[16] Marano G C 2015 Identification of parameters of Maxwell and Kelvin–Voigt generalized models for fluid viscous dampers. *J. Vibration and Control* 21 260-274
[17] Lewandowski R and Chorążyczewski B 2010 Identification of the parameters of the Kelvin–Voigt and the Maxwell fractional models, used to modeling of viscoelastic dampers. *Computers and Structures* 88 1-17
[18] Monsia M D 2011 A Simplified Nonlinear Generalized Maxwell Model for Predicting the Time Dependent Behavior of Viscoelastic Materials. *World Journal of Mechanics* 1 158-167
[19] Sheshenin S V, Zakalyukina I M and Koval’ SV 2014 Numerical Implementation of Voigt and Maxwell Models for Simulation of Waves in the Ground. *Proc. Moscow State Univ. of Civil Eng.* 11 82-89
[20] Bobyleva T N 2016 Method of Calculation of Stresses in the Layered Elastic-Creeping Arrays. *MATEC Web of Conf.* 86 01024