Postnikov–Stanley Linial arrangement conjecture

Shigetaro Tamura

Abstract
A characteristic polynomial is an important invariant in the field of hyperplane arrangement. For the Linial arrangement of any irreducible root system, Postnikov and Stanley conjectured that all roots of the characteristic polynomial have the same real part. In relation to this conjecture, Yoshinaga obtained an explicit relationship between the characteristic quasi-polynomial and the Ehrhart quasi-polynomial for the fundamental alcove. In this paper, we calculate Yoshinaga’s explicit formula through the decomposition of the Ehrhart quasi-polynomial into several quasi-polynomials and a modified shift operator and obtain new formulas for the characteristic quasi-polynomial of the Linial arrangement. In particular, when the parameter of the Linial arrangement is relatively prime to the period of the Ehrhart quasi-polynomial, we prove the Postnikov–Stanley Linial arrangement conjecture. This generalizes some of the results for the root systems of classical types that have been proved by Postnikov–Stanley and Athanasiadis. For other cases, we verify this conjecture for exceptional root systems using a computational approach.

Keywords Hyperplane arrangement · Linial arrangement · Characteristic quasi-polynomial · Quasi-polynomial · Ehrhart quasi-polynomial · Eulerian polynomial

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1 Introduction

Let $A$ be a hyperplane arrangement, that is, a finite collection of affine hyperplanes in a vector space $V$. One of the most important invariants of $A$ is the characteristic polynomial $\chi(A, t)$ [10]. Let $\Phi$ be an irreducible root system with the Coxeter number $h$. Let $a, b \in \mathbb{Z}$ be integers with $a \leq b$. Let us denote by $A_{\Phi}^{[a,b]}$ the truncated affine Weyl arrangement. In particular, $A_{\Phi}^{[1,n]}$ is called the Linial arrangement. Postnikov and Stanley [11] conjectured that every root $z \in \mathbb{C}$ of the equation $\chi(A_{\Phi}^{[1,n]}, t) = 0$ satisfies $\text{Re } z = \frac{nh}{2}$ (see §2.7 for details).

Postnikov and Stanley proved this conjecture for $\Phi = A_\ell$ [11]. Subsequently, Athanasiadis gave proofs for $\Phi = A_\ell, B_\ell, C_\ell$, and $D_\ell$ using a combinatorial method [2]. Yoshinaga approached the conjecture through the characteristic quasi-polynomial $\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t)$, which was introduced by Kamiya Takemura, and Terao [6–8]. The characteristic quasi-polynomial $\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t)$ has the important property that when $t$ is relatively prime to the period of $\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t)$, the formula $\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = \chi(A_{\Phi}^{[1,n]}, t)$ holds [2, Theorem 2.1]. Yoshinaga has proved the following formula [16] (see Theorem 2.36).

$$\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = R_{\Phi}(S^{n+1})L_{\Phi}(t),$$

where $S$ is the shift operator for the variable $t$ (see §2.1), $L_{\Phi}(t)$ is the Ehrhart quasi-polynomial for the closed fundamental alcove of type $\Phi$ (see §2.5), and $R_{\Phi}(t)$ is the generalized Eulerian polynomial of type $\Phi$, which was introduced by Lam and Postnikov [9] (see §2.6). By using this formula, Yoshinaga verified several cases of the conjecture (see §2.7).

1.1 Main results

Let $\rho$ be the period of the characteristic quasi-polynomial $\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t)$. Let $m$ be an integer with $n + 1 = m \cdot \gcd(n + 1, \rho)$. Let $c_0, \ldots, c_\ell$ be integers that are coefficients of each simple root when the highest root is expressed as a linear combination of simple roots in an irreducible root system $\Phi$ of rank $\ell$ (see §2.4). By
calculating the right-hand side of (1), we prove the formula
\[
\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = \left( \prod_{j=0}^{\ell} \frac{1}{m} [m]_{S_{\ell}^{m,j}, \rho_{n+1}} \right) \chi_{\text{quasi}}(A_{\Phi}^{[1,\rho_{n+1}-1]}, t),
\]
where \( \rho_{n+1} = \gcd(n + 1, \rho) \) and \([m]_{S_{\ell}^{m,j}, \rho_{n+1}} = 1 + t + \cdots + t^{m-1} \) (see Theorem 3.3). Furthermore, the characteristic quasi-polynomial \( \chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) \) has the period \( \gcd(n + 1, \rho) \). In particular, when the parameter \( n + 1 \) is relatively prime to the period \( \rho \) of the Ehrhart quasi-polynomial \( L_{\Phi}(t) \), that is, \( \gcd(n + 1, \rho) = 1 \), we have
\[
\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = \left( \prod_{j=0}^{\ell} \frac{1}{n+1} [n+1]_{S_{\ell}^{m,j}} \right) t^\ell
\]
from (2) (see Theorem 3.5). In this case, from (3) and the technique used by Postnikov and Stanley in [11] (see Lemma 3.4), we see that the conjecture holds. In addition, we prove the formula for the characteristic polynomial
\[
\chi(A_{\Phi}^{[1,\rho_{n+1}-1]}, t) = \left( \prod_{j=0}^{\ell} \frac{1}{\eta} [\eta]_{S_{\ell}^{m,j}, \rho_{n+1}} \right) \chi(A_{\Phi}^{[1,\rho_{n+1}d]}(\rho_{n+1}-1), t),
\]
where \( \eta = \frac{\rho_{n+1}}{\rho_{n+1}d} \) (see Theorem 3.8). From (2) and (4), if all roots of the characteristic polynomial \( \chi(A_{\Phi}^{[1,\rho_{n+1}d]}(\rho_{n+1}-1), t) \) have the same real part \( \frac{(\rho_{n+1}d-1)h}{2} \), then the same method as for (3) can be used to show that \( \chi(A_{\Phi}^{[1,n]}, t) \) satisfies the conjecture. We verify the conjecture for \( \Phi \in \{ E_6, E_7, E_8, F_4 \} \) by computing the real part of all roots of \( \chi(A_{\Phi}^{[1,\rho_{n+1}d]}(\rho_{n+1}-1), t) \) using a computational approach (see §3.2).

1.2 Outline of the proof

To prove the conjecture, we transform the right-hand side of (1) into a suitable form. One of the difficulties in this transformation is that the shift operator \( S \) acts on a quasi-polynomial, not a polynomial. To overcome this difficulty, we introduce the operator \( \overline{S} \), which acts on a constituent of a quasi-polynomial (Definition 2.11). Additionally, we define a quasi-polynomial \( \overline{f}^i(t) \) from a quasi-polynomial \( f(t) \) (Definition 2.4). The quasi-polynomial \( \overline{f}^i(t) \) is like an average of the constituents of the quasi-polynomial \( f(t) \), and its minimal period is a divisor of the integer \( i \). Using a modification of Lemma 2.2 in [2] (Lemma 2.3), for a quasi-polynomial \( f(t) \) of degree \( \ell \) and period \( \rho \), we obtain the formula
\[
[c]_{\overline{S}^m}^{\ell+1} g(S^m) f(t) = [c]_{\overline{S}^m}^{\ell+1} g(\overline{S}^m) \overline{f}^{\gcd(m, \rho)}(t),
\]
where \( g(S) \) is the substituted shift operator \( S \) for a polynomial \( g(t) \) (Proposition 2.15).
The Ehrhart quasi-polynomial \( L_{\Phi}(t) \) decomposes into several quasi-polynomials that have a degree and period that is less than or equal to its own degree and period:

\[
L_{\Phi}(t) = \hat{\ell} \sum_{k=0}^{\hat{\ell}} L^{(\hat{\ell}_{c_k})}(t),
\]

where \( \hat{c}_0, \cdots, \hat{c}_{\hat{\ell}} \) are all the different integers in \( c_0, \cdots, c_{\ell}, \hat{\ell}_{c_k} + 1 \) is the number of multiples of \( \hat{c}_k \) in \( c_0, \cdots, c_{\ell} \) (see §2.5), and \( L^{(\hat{\ell}_{c_k})}(t) \) is a quasi-polynomial of degree \( \hat{\ell}_{c_k} \) with period \( \hat{c}_k \) (Proposition 2.25). This decomposition is well matched with the following decomposition of generalized Eulerian polynomials, which was proved in [9].

\[
R_{\Phi}(t) = [c_0]_1[c_1]_1 \cdots [c_{\ell}]_1 A_{\ell}(t),
\]

where \( A_{\ell}(t) \) is the Eulerian polynomial (Theorem 2.31). The right-hand side of (7) has the divisor \([\hat{c}_k]_{\hat{\ell}_{c_k} + 1} \). Hence, we can apply (5) to each \( L^{(\hat{\ell}_{c_k})}(t) \) of (6). From the above argument, we have the formula

\[
R_{\Phi}(S^{n+1})L_{\Phi}(t) = R_{\Phi}(S^{n+1})L^{\gcd(n+1,\rho)}(t),
\]

(see Theorem 3.1). We can think of the operator \( R_{\Phi}(S^{n+1}) \) as acting on a polynomial, or more precisely, on a constituent of the quasi-polynomial \( L^{\gcd(n+1,\rho)}(t) \). Thus, we can easily calculate the right-hand side of (8) and prove (2).

The remainder of this paper is organized as follows. Section 2 contains some preliminaries required to prove the main results. First, we describe a modification of Athanasiadis’ Lemma [2] in §2.1. In §2.2, we introduce the operator \( S \) and a quasi-polynomial \( \tilde{f}(t) \), and prove (5). In §2.3, we prove that a decomposition of a quasi-polynomial holds using a generating function. In §2.4, §2.5, and §2.6, we prepare several concepts required to explain Yoshinaga’s results [16] for the characteristic quasi-polynomial \( \chi_{\text{quasi}}(A^{[1,n]}_{\Phi}, t) \), which is explained together with the Postnikov–Stanley Linial arrangement conjecture in §2.7. The explanations in §2.1, §2.4, §2.5, §2.6, and §2.7 are based on [16]. We prove (2), (3), and (4) in §3.1. We present a table of the characteristic polynomial \( \chi(A^{[1,\rad(\rho_{n+1})-1]}_{\Phi}, t) \) and the real part of all of its roots for \( \Phi \in \{E_6, E_7, E_8, F_4\} \) in §3.2.

2 Preliminaries

2.1 Shift operator and congruence

Let \( f : \mathbb{Z} \to \mathbb{C} \) be a function. Let \( S : t \mapsto t - 1 \) be the shift operator. For a polynomial \( g(S) = \sum_k a_k S^k \) in \( S \), the action is defined by
\[ g(S)f(t) = \sum_k a_k f(t - k). \] (9)

**Proposition 2.1** ([16], Proposition 2.8) *Let* \( g(S) \in \mathbb{C}[S] \) *and* \( f(t) \in \mathbb{C}[t] \). *Suppose* \( \deg f = \ell \). *Then,* \( g(S)f(t) = 0 \) *if and only if* \((1 - S)^{\ell+1}\) *divides* \( g(S) \).

**Remark 2.2** Note that, because \((1-S)f(t) = f(t) - f(t-1)\) is the difference operator, \(\deg(1-S)f = \deg f - 1\). Hence, inductively, \((1-S)^{\deg f + 1}f(t) = 0\). Proposition 2.1 implies that if polynomials \( g_1(S) \) and \( g_2(S) \) satisfy the congruence

\[ g_1(t) \equiv g_2(t) \mod (1-t)^{\ell+1}, \] (10)

then for any polynomial \( f(t) \) of degree less than or equal to \( \ell \),

\[ g_1(S)f(t) = g_2(S)f(t), \] (11)

since \((1-S)^{\ell+1}f(t) = 0\). Conversely, when \( g_1(S)f(t) = g_2(S)f(t) \) for a polynomial \( f(t) \) of degree \( \ell \), (10) holds.

Lemmas 2.3 can be proved in the very similar way as in Lemma 2.2 in [2]. Let \([c]_t := \frac{1-t^c}{1-t} = 1 + t + \cdots + t^{c-1}\), where \( c \) is a non-negative integer.

**Lemma 2.3** If \( g(t) = \sum_k a_k t^k \) is a polynomial and \( n \) is a positive integer, then a polynomial \( g(t) \) can be divided by \([n]_{\ell+1} \) if and only if the following formulas hold.

\[ \frac{1}{n}g(t) \equiv \sum_{k \equiv 0 \mod n} a_k t^k \equiv \cdots \equiv \sum_{k \equiv n-1 \mod n} a_k t^k \mod (1-t)^{\ell+1}. \] (12)

### 2.2 Shift operator and quasi-polynomial

A function \( f : \mathbb{Z} \rightarrow \mathbb{C} \) is called a quasi-polynomial if there exists a positive integer \( n > 0 \) and polynomials \( f_1(t), \ldots, f_n(t) \in \mathbb{C}[t] \) such that

\[ f(t) = \begin{cases} 
  f_1(t), & t \equiv 1 \mod n, \\
  f_2(t), & t \equiv 2 \mod n, \\
  \vdots \\
  f_{n-1}(t), & t \equiv n - 1 \mod n, \\
  f_n(t), & t \equiv 0 \mod n.
\] (13)

Such a \( n \) is called the period of the quasi-polynomial \( f(t) \). The minimum of the period of \( f(t) \) is called the minimal period. The polynomials \( f_1(t), \ldots, f_n(t) \) are the constituents of \( f(t) \). We define \( \deg f := \max_{1 \leq i \leq n} \deg f_i \) as the degree of a quasi-polynomial \( f(t) \). Moreover, if \( f_r(t) = f_{\gcd(r,n)}(t) \) for any \( r \in \{1, \ldots, n\} \), then we say that the quasi-polynomial \( f(t) \) has the gcd-property.
Definition 2.4 Let \( f(t) \) be a quasi-polynomial with minimal period \( n \). Let \( s \) be a positive integer.

\[
f(t) = \begin{cases} 
  f_1(t), & t \equiv 1 \mod sn, \\
  f_2(t), & t \equiv 2 \mod sn, \\
  \quad \vdots \\
  f_{sn-1}(t), & t \equiv sn - 1 \mod sn, \\
  f_{sn}(t), & t \equiv 0 \mod sn.
\end{cases}
\]

We define the action of the symmetric group \( G_{sn} \) on a quasi-polynomial as follows.

\[
f^\sigma(t) := \begin{cases} 
  f_{\sigma^{-1}(1)}(t), & t \equiv 1 \mod sn, \\
  f_{\sigma^{-1}(2)}(t), & t \equiv 2 \mod sn, \\
  \quad \vdots \\
  f_{\sigma^{-1}(sn-1)}(t), & t \equiv sn - 1 \mod sn, \\
  f_{\sigma^{-1}(sn)}(t), & t \equiv 0 \mod sn,
\end{cases}
\]

where \( \sigma \in G_{sn} \). Let \( \sigma_{sn} \) be the cyclic permutation \( (1, 2, \ldots, sn) \in G_{sn} \). For any positive integer \( s \), we have \( f^{\sigma_{sn}}(t) = f^{\sigma_n}(t) \). In other words, the action of the cyclic permutation \( \sigma_{sn} = (1, \ldots, sn) \) on \( f(t) \) does not depend on \( s \). From now on, we denote a cyclic permutation \( (1, \ldots, n) \) by \( \sigma \), where \( n \) takes the minimal period of a quasi-polynomial on which \( \sigma \) acts in each case. Let \( k \) be an integer. Define the following quasi-polynomial for \( k \):

\[
f^k(t) := f(t) + f^{\sigma^k}(t) + f^{\sigma^{2k}}(t) + \cdots + f^{\sigma^{(n-1)k}}(t).
\]

Remark 2.5 (1) Let \( n \) be a period of \( f(t) \). Let \( k \) be a divisor of \( n \) and \( m := \frac{n}{k} \). The quasi-polynomial \( f^k(t) \) has the period \( k \).

\[
f^k(t) = \begin{cases} 
  f_1(t) + f_{k+1}(t) + f_{2k+1}(t) + \cdots + f_{(m-1)k+1}(t), & t \equiv 1 \mod k, \\
  \frac{f_2(t) + f_{k+2}(t) + f_{2k+2}(t) + \cdots + f_{(m-1)k+2}(t)}{m}, & t \equiv 2 \mod k, \\
  \quad \vdots \\
  \frac{f_{k-1}(t) + f_{2k-1}(t) + f_{3k-1}(t) + \cdots + f_{mk-1}(t)}{m}, & t \equiv k - 1 \mod k, \\
  \frac{f_k(t) + f_{2k}(t) + f_{3k}(t) + \cdots + f_{mk}(t)}{m}, & t \equiv 0 \mod k.
\end{cases}
\]

(2) When a quasi-polynomial \( f(t) \) has a period \( n \), we have that \( f^k(t) = f^{k+n}(t) \).

Lemma 2.6 Let \( f(t) \), \( g(t) \), and \( h(t) \) be quasi-polynomials such that \( f(t) = g(t) + h(t) \) holds. Then, \( f^\sigma(t) = g^\sigma(t) + h^\sigma(t) \), that is, the action of the cyclic permutation \( \sigma \) is linear.

Proof Let \( n \) be the minimal period of \( f(t) \). Let \( sn \) be the least common multiple of the minimal periods of \( g(t) \) and \( h(t) \). Let \( g_j(t) \) and \( h_j(t) \) be constituents of \( g(t) \) and
Let $k$ be an integer. Then,

Let $f$ be a quasi-polynomial with period $n$. Let $\sigma_{sn} := (1, \cdots, sn)$. Note that we use the notation $\sigma$ as the cyclic permutation for the minimal period of a quasi-polynomial on which $\sigma$ acts, and we have

$$f^\sigma(t) = f^{\sigma_{sn}}(t) = \begin{cases}
g^{\sigma_{sn}^{-1}(1)}(t) + h^{\sigma_{sn}^{-1}(1)}(t), & t \equiv 1 \mod sn, 
g^{\sigma_{sn}^{-1}(2)}(t) + h^{\sigma_{sn}^{-1}(2)}(t), & t \equiv 2 \mod sn, \\
\vdots & 
g^{\sigma_{sn}^{-1}(sn-1)}(t) + h^{\sigma_{sn}^{-1}(sn-1)}(t), & t \equiv sn-1 \mod sn, 
g^{\sigma_{sn}^{-1}(sn)}(t) + h^{\sigma_{sn}^{-1}(sn)}(t), & t \equiv 0 \mod sn
\end{cases}$$

$$= g^{\sigma_{sn}}(t) + h^{\sigma_{sn}}(t)$$

$$= g^\sigma(t) + h^\sigma(t).$$

\[\square\]

**Lemma 2.7** Let $f(t), g(t),$ and $h(t)$ be quasi-polynomials such that $f(t) = g(t) + h(t)$ holds. Let $k$ be an integer. Then, $\tilde{f}^k(t) = \tilde{g}^k(t) + \tilde{h}^k(t)$.

**Proof** Let $n_0, n_1, n_2$ be the minimal period of each $f(t), g(t), h(t)$. Note that $n_1n_2$ is a multiple of $n_0$. Then, by Lemma 2.6, Remark 2.5 (2),

$$\tilde{f}^k(t) = \frac{f(t) + f^{\sigma_k}(t) + \cdots + f^{\sigma^{(n_0-1)k}}}{n_0}$$

$$= \frac{f(t) + f^{\sigma_k}(t) + \cdots + f^{\sigma^{(n_1n_2-1)k}}}{n_1n_2}$$

$$= \frac{g(t) + g^{\sigma_k}(t) + \cdots + g^{\sigma^{(n_1n_2-1)k}}(t) + h(t) + h^{\sigma_k}(t) + \cdots + h^{\sigma^{(n_1n_2-1)k}}(t)}{n_1n_2}$$

$$= \frac{n_2(g(t) + g^{\sigma_k}(t) + \cdots + g^{\sigma^{(n_1-1)k}}(t))}{n_1n_2} + \frac{n_1(h(t) + h^{\sigma_k}(t) + \cdots + h^{\sigma^{(n_2-1)k}}(t))}{n_1n_2}$$

$$= \tilde{g}^k(t) + \tilde{h}^k(t).$$

\[\square\]

**Proposition 2.8** Let $f(t)$ be a quasi-polynomial with period $n$. Let $k$ be an integer.

$$\tilde{f}^k(t) = \tilde{f}^{\gcd(k, n)}(t).$$

**Proof** Let $[b] := b + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$. We will prove that

$$\{[\mu k]\}_{\mu=0}^{n-1} = \{[\mu \gcd(k, n)]\}_{\mu=0}^{n-1},$$

\[\square\] Springer
If (16) holds, then from the relation \( f^{\sigma^i}(t) = f^{\sigma^{i+n}}(t) \), we have that
\[
f(t) + f^{n_k}(t) + \cdots + f^{n_{(n-1)_k}}(t) = f(t) + f^{\gcd(k,n)}(t) + \cdots + f^{n_{(n-1)_k}\gcd(k,n)}(t).
\]
Let \( \mu \in \{0, 1, \ldots, n - 1\} \). Then, there exists \( \mu' \in \{0, 1, \ldots, n - 1\} \) with \( [\mu'] = [\mu_{\gcd(k,n)}] \). Hence,
\[
[\mu k] = [\mu \frac{k}{\gcd(k,n)} \gcd(k,n)] = [\mu' \gcd(k,n)] \in \{[\mu \gcd(k,n)]\}_{\mu=0}^{n-1}.
\]
Thus, \( \{[\mu k]_{\mu=0}^{n-1} \subset \{[\mu \gcd(k,n)]\}_{\mu=0}^{n-1} \). Because the map
\[
\phi_{k, \gcd(k,n)} : \{[\mu \gcd(k,n)]\}_{\mu=0}^{n-1} \to \{[\mu k]_{\mu=0}^{n-1}
\]
\[
\psi \quad \psi
\]
\[
x \mapsto \quad x \mapsto \frac{k}{\gcd(k,n)}
\]
is bijective, we have that \( \{[\mu k]_{\mu=0}^{n-1} = \{[\mu \gcd(k,n)]\}_{\mu=0}^{n-1} \}\). \( \square \)

We now prepare a lemma on greatest common divisors that will be used later.

**Lemma 2.9** Let \( n \) and \( d \) be integers. Then, for any integers \( \mu_0 \in \mathbb{Z} \),
\[
\gcd(d + \mu_0 \text{rad}(n) \gcd(d, n), n) = \gcd(d, n),
\]
where \( \text{rad}(n) := \prod_{p: \text{prime}, p|n} p \) is a radical of \( n \).

**Proof** Note that \( \gcd(\frac{d}{\gcd(d,n)} + \frac{n}{\gcd(d,n)}, \frac{n}{\gcd(d,n)}) = \gcd(d, n) \). Hence, for any integer \( \mu_0 \), we have that \( \gcd(\frac{d}{\gcd(d,n)} + \mu_0 \text{rad}(n), \frac{n}{\gcd(d,n)}) = 1 \). Therefore,
\[
\gcd(d + \mu_0 \text{rad}(n) \gcd(d, n), n)
\]
\[
= \gcd(d, n) \gcd(\frac{d}{\gcd(d,n)} + \mu_0 \text{rad}(n), \frac{n}{\gcd(d,n)})
\]
\[
= \gcd(d, n).
\]
\( \square \)

**Proposition 2.10** Let \( f(t) \) be a quasi-polynomial of period \( n \) with the gcd-property. Let \( k \) be a positive integer. Let \( n_k := \gcd(k,n) \). Let \( f^{n_k}(t) \) be the constituent of the quasi-polynomial \( f^{nk}(t) \) for \( t \equiv j \mod n_k \). If \( \gcd(j, n) = 1 \), then
\[
f^{n_k}_j(t) = f^{\text{rad}(n_k)}_j(t).
\]
We will prove that
\[
\gcd(j + \mu n_k, n) \bigm|_{\mu=0}^{n-1} = \gcd(j + \mu \rad(n_k), n) \bigm|_{\mu=0}^{n-1}.
\] (19)

If (19) holds, then from the gcd-property of \( f(t) \),
\[
f_j(t) + f_{j+n_k}(t) + \cdots + f_{j+(n-1)n_k}(t) = f_j(t) + f_{j+\rad(n_k)}(t) + \cdots + f_{j+(n-1)\rad(n_k)}(t).
\]

Let \([b] := b + \mathbb{Z} / n \mathbb{Z}\). For any integer \( \mu \in \{0, 1, \ldots, n - 1\} \) such that \( \mu' = [\mu]c \). Hence, \( \gcd(j + \mu n_k, n) \bigm|_{\mu=0}^{n-1} \subset \gcd(j + \mu \rad(n_k), n) \bigm|_{\mu=0}^{n-1} \). Next, we write \( n = r_1^{s_1} r_2^{s_2} \cdots r_m^{s_m} \) and \( n_k = r_1^{q_1} r_2^{q_2} \cdots r_m^{q_m} \), where \( r_1, r_2, \ldots, r_m \) are primes, \( s_1, \ldots, s_m, q_1, \ldots, q_m \) are positive integers, and \( i_1, \ldots, i_m \in \{0, 1\} \), and then, we define \( \hat{n}_k := r_1^{s_1(1-i_1)} r_2^{s_2(1-i_2)} \cdots r_m^{s_m(1-i_m)} \). Note that \( \gcd(n_k, \hat{n}_k) = 1 \) and any divisor of \( n \) that is relatively prime to \( n_k \) divides \( \hat{n}_k \). We have \( \gcd(\frac{n_k}{\rad(n_k)}), \frac{\rad(n)\hat{n}_k}{\rad(n_k)} \) = 1 since \( \gcd(n_k, \frac{\rad(n)}{\rad(n_k)}) = 1 \) and \( \gcd(n_k, \frac{\hat{n}_k}{\rad(n_k)}) = 1 \). Hence, for any integer \( \mu \in \{0, 1, \ldots, n - 1\} \), there exist integers \( \mu_1, \mu_2 \in \mathbb{Z} \) such that \( \mu = \mu_1 \frac{n_k}{\rad(n_k)} + \mu_2 \frac{\rad(n)\hat{n}_k}{\rad(n_k)} \). We transform the formula
\[
j + \mu \rad(n_k) = j + (\mu_1 \frac{n_k}{\rad(n_k)} + \mu_2 \frac{\rad(n)\hat{n}_k}{\rad(n_k)})\rad(n_k) = j + \mu_1 n_k + \mu_2 \rad(n)\hat{n}_k.
\] (20)

The integer \( \gcd(j + \mu_1 n_k, n) \) is relatively prime to \( n_k \) since \( \gcd(j + \mu_1 n_k, n_k) = 1 \). Since any divisor of \( n \) that is relatively prime to \( n_k \) divides \( \hat{n}_k \), \( \gcd(j + \mu_1 n_k, n) \) divides \( \hat{n}_k \). Let \( \mu_3 := \frac{\hat{n}_k}{\gcd(j + \mu_1 n_k, n)} \in \mathbb{Z} \). From (20), we obtain
\[
j + \mu \rad(n_k) = j + \mu_1 n_k + \mu_2 \mu_3 \rad(n)\gcd(j + \mu_1 n_k, n).
\] (21)

Hence, using Lemma 2.9 for the right-hand side of (21), we have the formula \( \gcd(j + \mu \rad(n_k), n) = \gcd(j + \mu_1 n_k, n) \). Furthermore, since there exists an integer \( \mu_1 \in \{0, 1, \ldots, n - 1\} \) such that \( [\mu_1] = [\mu'] \), we have that \( \{\gcd(j + \mu \rad(n_k), n)\}^{n-1}_{\mu=0} \subset \{\gcd(j + \mu n_k, n)\}^{n-1}_{\mu=0} \). \( \square \)

**Definition 2.11** Let \( f(t) \) be a quasi-polynomial with period \( n \) as follows.
\[
f(t) = \begin{cases}
  f_1(t), & t \equiv 1 \mod n, \\
f_2(t), & t \equiv 2 \mod n, \\
  \vdots \\
f_{n-1}(t), & t \equiv n - 1 \mod n, \\
f_n(t), & t \equiv 0 \mod n.
\end{cases}
\]
We define the operator $\overline{S}$ as follows.

$$\overline{S}f := \begin{cases} f_1(t - 1), & t \equiv 1 \mod n, \\ f_2(t - 1), & t \equiv 2 \mod n, \\ \vdots \\ f_{n-1}(t - 1), & t \equiv n - 1 \mod n, \\ f_n(t - 1), & t \equiv 0 \mod n. \end{cases}$$

**Remark 2.12** The operators $S$ and $\overline{S}$ have the relation

$$(Sf)(t) = \begin{cases} f_n(t - 1), & t \equiv 1 \mod n, \\ f_1(t - 1), & t \equiv 2 \mod n, \\ \vdots \\ f_{n-2}(t - 1), & t \equiv n - 1 \mod n, \\ f_{n-1}(t - 1), & t \equiv 0 \mod n, \end{cases} = (\overline{S}f^\sigma)(t).$$

**Lemma 2.13** (1) Let $f(t)$ and $g(t)$ be quasi-polynomials, which may have different minimal periods. Then, $\overline{S}(f(t) + g(t)) = \overline{S}f(t) + \overline{S}g(t)$, that is, the operator $\overline{S}$ is linear. (2) Let $h(t)$ be a quasi-polynomial. Then, $(1 - \overline{S})^{\deg h + 1}h(t) = 0$.

**Proof** (1) Let $m$ and $n$ be the minimal periods of $f(t)$ and $g(t)$, respectively. Let $k$ be an integer. Let $f_k(t)$ and $g_k(t)$ be constituents of $f(t)$ and $g(t)$ for $t \equiv k \mod \text{lcm}(m, n)$. If $t \equiv k \mod \text{lcm}(m, n)$, then $\overline{S}(f(t) + g(t)) = f_k(t - 1) + g_k(t - 1) = \overline{S}f(t) + \overline{S}g(t)$. (2) By the definition of the operator $\overline{S}$, the inequality $\deg((1 - \overline{S})h) < \deg h$ holds. Hence, inductively, $(1 - \overline{S})^{\deg h + 1}h(t) = 0$.

**Lemma 2.14** Let $f(t)$ be a quasi-polynomial with period $n$. Let $j$ and $m$ be integers. Let $c$ be a multiple of $\frac{n}{\gcd(m, n)}$. Let $\sum_{k \equiv j \mod c} a_k f^{mk}$ be a polynomial. Then,

$$\left( \sum_{k \equiv j \mod c} a_k S^{mk} f \right)(t) = \left( \sum_{k \equiv j \mod c} a_k \overline{S}^{mk} f^{\sigma^m} \right)(t).$$
Proof First, note that \((S^{mj+mc} f)(t) = (S^{mj+mc} f^{\sigma^{mj+mc}})(t) = (S^{mj+mc} f^{\sigma^{mj}})(t)\)
because \(mc\) is a multiple of \(n\).

\[
\left( \sum_{k \equiv j \mod c} a_k S^{mk} f \right)(t) = \left( a_j S^{mj} f^{\sigma^{mj}} \right)(t) + \left( \sum_{k \equiv -c \mod c} a_k S^{mk} f^{\sigma^{mj}} \right)(t) + \cdots
\]

\[
= \left( \sum_{k \equiv j \mod c} a_k S^{mk} f^{\sigma^{mj}} \right)(t).
\]

\[\Box\]

The following proposition concerns an average of a quasi-polynomial using the shift operator.

**Proposition 2.15** Let \(f(t)\) be a quasi-polynomial of degree \(\ell\) with period \(n\). Let \(g(t)\) be a polynomial. Let \(m\) be an integer and \(c\) be a multiple of \(\gcd(m,n)\). Then,

\[
[c]^{\ell+1} S^m g(S^m) f(t) = [c]^{\ell+1} S^m g(S^m) \tilde{f}^{\gcd(m,n)}(t).
\] (23)

**Proof** Let \([c]^{\ell+1} S^m g(S^m) =: \sum_k a_k S^{mk}\). We calculate \([c]^{\ell+1} S^m g(S^m) f(t)\) using Lemma 2.3, Proposition 2.8, Lemma 2.13, and Lemma 2.14.

\[
([c]^{\ell+1} S^m g(S^m) f)(t) = \left( \sum_k a_k S^{mk} f \right)(t)
\]
\[
= \left( \sum_{k \equiv 0 \mod c} a_k S^{mk} f \right)(t) + \cdots + \left( \sum_{k \equiv -c \mod c} a_k S^{mk} f^{\sigma^{m(c-1)}} \right)(t)
\]
\[
= \left( \sum_{k \equiv 0 \mod c} a_k S^{mk} f \right)(t) + \cdots + \left( \sum_{k \equiv -c \mod c} a_k S^{mk} f^{\sigma^{m(c-1)}} \right)(t)
\]
\[
= \left( \sum_{k} \frac{1}{c} a_k S^{mk} \right) \left( f(t) + f^{\sigma^m}(t) + \cdots + f^{\sigma^{m(c-1)}}(t) \right)
\]
\[
= \left( \sum_{k} a_k S^{mk} \right) \left( \frac{ f(t) + f^{\sigma^m}(t) + \cdots + f^{\sigma^{m(c-1)}}(t) }{c} \right)
\]
\[
= \left( [c]^{\ell+1} S^m g(S^m) \tilde{f}^m \right)(t)
\]
\[
= \left( [c]^{\ell+1} S^m g(S^m) \tilde{f}^{\gcd(m,n)} \right)(t).
\]

\[\Box\]
2.3 Decomposition of a quasi-polynomial

First, we summarize the relation between (quasi-)polynomial and rational functions.

**Lemma 2.16** [3, 13, Corollary 4.3.1] If
\[
\sum_{n=0}^{\infty} f(n)x^n = \frac{g(x)}{(1-x)^{\ell+1}},
\]
then \( f(t) \) is a polynomial of degree \( \ell \) if and only if \( g(x) \) is a polynomial of degree at most \( \ell \) and cannot be divided by \((1 - x)\).

**Lemma 2.17** ([3, 13, Proposition 4.4.1]) If
\[
\sum_{n=0}^{\infty} f(n)x^n = \frac{g(x)}{h(x)},
\]
then \( f(t) \) is a quasi-polynomial of degree \( \ell \) with period \( p \) if and only if \( g(x) \) and \( h(x) \) are polynomials such that \( \deg g < \deg h \) and all roots of \( h(x) \) are \( p \)-th roots of unity of multiplicity at most \( \ell + 1 \), and there is a root of multiplicity equal to \( \ell + 1 \) (all of this assuming that \( \frac{g(x)}{h(x)} \) has been reduced to its lowest terms).

The following classical lemma is called partial fraction decomposition.

**Lemma 2.18** Let \( g(x) \) and \( h(x) \) be polynomials with \( \deg g < \deg h \). Let \( h_1(x), \ldots, h_n(x) \) be polynomials with \( h(x) = h_1(x)h_2(x) \cdots h_n(x) \) that are relatively prime to each other. Then, there exist polynomials \( g_1(x), \ldots, g_n(x) \) such that \( \deg g_i < \deg h_i \) for any \( i \in \{1, \ldots, n\} \) and
\[
\frac{g(x)}{h(x)} = \frac{g_1(x)}{h_1(x)} + \cdots + \frac{g_n(x)}{h_n(x)}.
\]

(24)

**Proposition 2.19** Let \( g(x) \) and \( h(x) \) be relatively prime polynomials such that \( \deg g < \deg h \) and all roots of \( h(x) \) are \( p \)-th roots of unity. Let \( X_p \) be the set of divisors of \( p \). Let \( \ell_i + 1 \) be the number of primitive \( i \)-th roots of unity in the roots of \( h(x) \). If
\[
\sum_{n=0}^{\infty} f(n)x^n = \frac{g(x)}{h(x)},
\]
then there exist quasi-polynomials \( f_i^{(\ell_i)}(t) \), \( i \in X_p \) that satisfy
\[
f(t) = \sum_{i \in X_p} f_i^{(\ell_i)}(t). \tag{25}\]

**Proof** Let \( \{h_i(x)\}_{i \in X_p} \) be polynomials such that all roots of \( h_i(x) \) are primitive \( i \)-th roots of unity in the roots of \( h(x) \) and \( h(x) = \prod_{i \in X_p} h_i(x) \). By Lemma 2.18, there exist polynomials \( \{g_i(x)\}_{i \in X_p} \) such that \( g_i(x) \) and \( h_i(x) \) are relatively prime, \( \deg g_i < \deg h_i = \ell_i + 1 \) for any \( i \in X_p \) and
\[
\sum_{n=0}^{\infty} f(n)x^n = \sum_{i \in X_p} \frac{g_i(x)}{h_i(x)}. \tag{26}\]

By Lemma 2.17, for any \( i \in X_p \), there exists the quasi-polynomial \( f_i^{(\ell_i)}(t) \) of degree \( \ell_i \) with period \( i \) such that \( \sum_{n=0}^{\infty} f_i^{(\ell_i)}(n)x^n = \frac{g_i(x)}{h_i(x)} \). Hence,
\[
\sum_{n=0}^{\infty} f(n)x^n = \sum_{i \in X_p} \sum_{n=0}^{\infty} f_i^{(\ell_i)}(n)x^n. \tag{26}\]

By comparing each term of (26), we obtain the formula stated in (25). \( \square \)

### 2.4 Root system

We introduce some concepts that help to explain the results for the characteristic polynomial of the Linial arrangement given by Yoshinaga [16]. Let \( V = \mathbb{R}^\ell \) be the Euclidean space with inner product \((\cdot, \cdot)\). Let \( \Phi \subset V \) be an irreducible root system with Coxeter number \( h \). Fix a positive system \( \Phi^+ \subset \Phi \) and the set of simple roots \( \Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Phi^+ \). The highest root, denoted by \( \tilde{\alpha} \in \Phi^+ \), can be expressed as the linear combination \( \tilde{\alpha} = \sum_{i=1}^{\ell} c_i \alpha_i \) (\( c_i \in \mathbb{Z}_{>0} \)). We also set \( a_0 := -\tilde{\alpha} \) and \( c_0 := 1 \). Then, we have the linear relation
\[
c_0 a_0 + c_1 a_1 + \cdots + c_\ell a_\ell = 0. \tag{27}\]

The integers \( c_0, \ldots, c_\ell \) have the following relation with the Coxeter number \( h \):

**Proposition 2.20** ([4])
\[
c_0 + c_1 + \cdots + c_\ell = h. \tag{28}\]
2.5 Ehrhart quasi-polynomial for the fundamental alcove

The coweight lattice $Z(\Phi)$ and the coroot lattice $Q(\Phi)$ are defined as follows.

$$Z(\Phi) := \{ x \in V \mid (\alpha_i, x) \in \mathbb{Z}, \alpha_i \in \Delta \},$$

$$Q(\Phi) := \sum_{\alpha \in \Phi} \mathbb{Z} \cdot \frac{2\alpha}{(\alpha, \alpha)}.$$

The index $# \frac{Z(\Phi)}{Q(\Phi)} = f$ is called the index of connection. Let $\sigma_i \in Z(\Phi)$ be the dual basis for the simple roots $\alpha_1, \ldots, \alpha_\ell$, that is, $(\alpha_i, \sigma_j) = \delta_{ij}$. Then, $Z(\Phi)$ is a free abelian group generated by $\sigma_1, \ldots, \sigma_\ell$. We also have $c_i = (\sigma_i, \tilde{\alpha})$. A connected component of $V \setminus \bigcup_{\alpha \in \Phi^+} H_{\alpha, k}$ is called an alcove. Let us define the fundamental alcove $F_\Phi$ of type $\Phi$ as

$$F_\Phi := \left\{ x \in V \left| \begin{array}{c} (\alpha_i, x) > 0, \ (1 \leq i \leq \ell) \\ (\tilde{\alpha}, x) < 1 \end{array} \right. \right\}.$$

The closure $\overline{F_\Phi} = \{ x \in V \mid (\alpha_i, x) \geq 0 \ (1 \leq i \leq \ell), \ (\tilde{\alpha}, x) \leq 1 \}$ is the convex hull of $0, \frac{\sigma_1}{c_1}, \ldots, \frac{\sigma_\ell}{c_\ell} \in V$. The closed fundamental alcove $\overline{F_\Phi}$ is a simplex. For a positive integer $q \in \mathbb{Z}_{>0}$, we define the function $L_\Phi : \mathbb{Z}_{>0} \to \mathbb{Z}$ as

$$L_\Phi(q) := \#(qF_\Phi \cap Z(\Phi)).$$

The following statements hold for $L_\Phi(t)$.

**Theorem 2.21 (Suter [14])** Let $\Phi$ be an irreducible root system of rank $\ell$. If $q \in \mathbb{Z}$, then

$$L_\Phi(-q) = (-1)^{\ell}L_\Phi(q - h).$$

The following statements hold for $L_\Phi(t)$.

**Theorem 2.22 (Suter [14])**

1. The Ehrhart quasi-polynomial $L_\Phi(t)$ has the gcd-property.
2. The degree of $L_\Phi(t)$ is the rank of $\Phi$.
3. The minimal period $\rho$ is as given in Table 1.
4. $L_\Phi(-1) = L_\Phi(-2) = \cdots = L_\Phi(-(h - 1)) = 0$.
5. The generating function of $L_\Phi(t)$ is

$$\sum_{n=0}^{\infty} L_\Phi(n)x^n = \frac{1}{(1 - x^{c_0}) \cdots (1 - x^{c_\ell})}.$$
There is a following relation between the Ehrhart quasi-polynomials of type $A_\ell$ and its other types.

**Proposition 2.23** Let $\Phi$ be an irreducible root system of rank $\ell$. The following formula holds.

$$L_{A_\ell}(t) = [c_0]S[c_1]S\cdots[c_\ell]S L_\Phi(t). \quad (32)$$

**Proof** Recall that $c_0 + c_1 + \cdots + c_\ell = h$. The Ehrhart polynomial for the fundamental alcove of any irreducible root system $\Phi$ satisfies $L_\Phi(-1) = L_\Phi(-2) = \cdots = L_\Phi(-(h-1)) = 0$. Hence,

$$[c_0]x \cdots [c_\ell]x \sum_{n=0}^\infty L_\Phi(n)x^n = \sum_{n=0}^\infty ([c_0]S \cdots [c_\ell]S L_\Phi)(n)x^n. \quad (33)$$

Note that $c_0 = c_1 = \cdots = c_\ell = 1$ for type $A_\ell$. On the left-hand side of (33), by Theorem 2.22, we can write

$$x \cdots [c_\ell]x \sum_{n=0}^\infty L_\Phi(n)x^n = [c_0]x \cdots [c_\ell]x \frac{1}{(1-x^{c_0}) \cdots (1-x^{c_\ell})}$$

$$= \frac{1}{(1-x)^{\ell+1}}$$

$$= \sum_{n=0}^\infty L_{A_\ell}(n)x^n.$$ 

Therefore, by comparing each term of (33), we obtain the formula given in (32). \qed

**Remark 2.24** Note that $(1-S)L_{A_\ell}(t) = L_{A_{\ell-1}}(t)$. We obtain the relations between the Ehrhart quasi-polynomials of root systems of different ranks from (32), Proposition 2.1, and Table 1. The following are some examples.

$$(1-S^2)L_{C_\ell}(t) = L_{C_{\ell-1}}(t).$$

$$(1-S^2)L_{D_\ell}(t) = L_{D_{\ell-1}}(t).$$

$$(3[S][4]S(1-S)L_{E_7}(t) = L_{E_6}(t).$$

$$(2[S][5][6]S(1-S)L_{E_8}(t) = L_{E_7}(t).$$

$$(2[S][4]S(1-S)^2L_{F_4}(t) = L_{G_2}(t).$$

$$(1-S)^2L_{E_6}(t) = (1+S^2)L_{F_4}(t).$$

Let $\hat{c}_0, \cdots, \hat{c}_\ell$ be all the different integers in $c_0, \cdots, c_\ell$ and $\ell \hat{c}_k + 1$ be the number of multiples of $\hat{c}_k$ in $c_0, \cdots, c_\ell$. Theorem 2.22 and Proposition 2.19 lead to the following decomposition of the Ehrhart quasi-polynomial $L_\Phi(t)$. 

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Proposition 2.25 For any irreducible root system $\Phi$, there exist quasi-polynomials \( \{ L^{(\ell_{\hat{c}_k})}_{\hat{c}_k} \}_{k=0}^{\hat{\ell}} \) such that

\[
L_{\Phi}(t) = \sum_{k=0}^{\hat{\ell}} L^{(\ell_{\hat{c}_k})}_{\hat{c}_k}(t),
\]

where $L^{(\ell_{\hat{c}_k})}_{\hat{c}_k}$ has period $\hat{c}_k$ and degree $\ell_{\hat{c}_k}$.

Remark 2.26 The decomposition \( \{ L^{(\ell_{\hat{c}_k})}_{\hat{c}_k} \}_{k=0}^{\hat{\ell}} \) are not unique. For example, the Ehrhart quasi-polynomial of $E_6$ has some of the decomposition, \( \{ L^{(6)}_{1} (t), L^{(2)}_{2} (t), L^{(0)}_{3} (t) \} \), \( \{ L^{(6)}_{1} (t) - t, L^{(2)}_{2} (t) + t, L^{(0)}_{3} (t) \} \), and so on.

2.6 Eulerian polynomial

We summarize some facts about the Eulerian polynomial and the generalized Eulerian polynomial with reference to [16].

Definition 2.27 (Eulerian polynomial) For a permutation $\tau \in S_n$, define

\[ a(\tau) := \# \{ i \in \{ 1, \ldots, n-1 \} \mid \tau(i) < \tau(i+1) \}. \]

Then,

\[ A(n, k) := \# \{ \tau \in S_n \mid a(\tau) = k - 1 \} \]

\((1 \leq k \leq n)\) is called the Eulerian number and the generating polynomial

\[
A_n(t) := \sum_{k=1}^{n} A(n, k) t^k = \sum_{\tau \in S_n} t^{1+a(\tau)}
\]
is called the Eulerian polynomial. Define $A_0(t) = 1$.

The Eulerian polynomial $A_\ell(t)$ satisfies the duality $A_\ell(t) = t^{\ell+1} A_\ell(\frac{1}{t})$. The following theorem is the so-called Worpitzky identity.

**Theorem 2.28** (Worpitzky [15]) Note that $L_{A_\ell(t)} = (t+\ell+1) A_\ell(\frac{1}{t})$. Then,

$$t^\ell = A_\ell(S) L_{A_\ell(t)}.$$  \hfill (37)

The Eulerian polynomial also satisfies the following congruence.

**Theorem 2.29** ([5, 16]) Let $\ell \geq 1$, $n \geq 2$. Then,

$$A_\ell(t^n) \equiv \frac{1}{n^{\ell+1}} [n]_{\ell+1} A_\ell(t) \mod (1-t)^{\ell+1}.$$  \hfill (38)

Lam and Postnikov introduced the following generalization of Eulerian polynomials [9].

**Definition 2.30** *(Generalized Eulerian polynomial)* Let $W$ be the Weyl group of an irreducible root system $\Phi$. For $\omega \in W$, the integer $\text{asc}(\omega) \in \mathbb{Z}$ is defined by

$$\text{asc}(\omega) := \sum_{0 \leq i \leq \ell \atop \omega(\alpha_i) > 0} c_i.$$  \hfill (39)

Then,

$$R_\Phi(t) := \frac{1}{f} \sum_{\omega \in W} t^{\text{asc}(\omega)}$$  \hfill (39)

is called the generalized Eulerian polynomial of type $\Phi$.

The generalized Eulerian polynomial $R_\Phi(t)$ can be expressed in terms of the polynomial $[c]_\ell$ and the Eulerian polynomial $A_\ell(t)$.

**Theorem 2.31** (Lam–Postnikov [9], Theorem 10.1) Let $\Phi$ be an irreducible root system of rank $\ell$. Then,

$$R_\Phi(t) = [c_0]_\ell [c_1]_\ell \cdots [c_\ell]_\ell A_\ell(t).$$  \hfill (40)

Some basic properties of the generalized Eulerian polynomial $R_\Phi(t)$ follow from Theorem 2.31 (Lam–Postnikov [9]).

**Proposition 2.32** (1) $\deg R_\Phi = h - 1$.

(2) $t^h R_\Phi(\frac{1}{t}) = R_\Phi(t)$.

(3) $R_{A_\ell}(t) = A_\ell(t)$.

We can obtain the following formula from Theorems 2.29 and 2.31.
Proposition 2.33  Let $\Phi$ be an irreducible root system of rank $\ell$. Let $n$ be a positive integer. Then,

$$R_{\Phi}(t^n) \equiv \left( \prod_{i=0}^{\ell} \frac{1}{n}[n]_{c_i^1} \right) R_{\Phi}(t) \mod (1-t)^{\ell+1}. \quad (41)$$

Proof  Using Theorems 2.29 and 2.31, we calculate the following.

$$R_{\Phi}(t^n) = [c_0]_n[c_1]_n \cdots [c_{\ell}]_n A_{\ell}(t^n)$$

$$\equiv \frac{1}{n^{\ell+1}} [c_0]_n c_1 \cdots [c_{\ell}]_n A_{\ell}(t) \mod (1-t)^{\ell+1}$$

$$\equiv \frac{1}{n^{\ell+1}} [n]_{c_0} [c_1]_n \cdots [n]_{c_{\ell}} [c_0]_n c_1 \cdots [c_{\ell}]_n A_{\ell}(t) \mod (1-t)^{\ell+1}$$

$$\equiv \frac{1}{n^{\ell+1}} [n]_{c_0} [n]_{c_1} \cdots [n]_{c_{\ell}} R_{\Phi}(t) \mod (1-t)^{\ell+1}. \quad \square$$

2.7 Postnikov–Stanley Linial arrangement conjecture

Let $V$ be a vector space with the inner product $(\cdot, \cdot)$. For any integer $k \in \mathbb{Z}$ and $\alpha \in V$, the affine hyperplane $H_{\alpha,k}$ is defined by

$$H_{\alpha,k} := \{ x \in V \mid (\alpha, x) = k \}. \quad (42)$$

Let $a, b \in \mathbb{Z}$ be integers with $a \leq b$. Define the hyperplane arrangement $A_{\Phi}^{[a,b]}$ as follows.

$$A_{\Phi}^{[a,b]} := \{ H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}, a \leq k \leq b \}. \quad (43)$$

Note that we define $A_{\Phi}^{[1,0]}$ as an empty set. The hyperplane arrangement $A_{\Phi}^{[a,b]}$ is called the truncated affine Weyl arrangement. In particular, $A_{\Phi}^{[1,n]}$ is called the Linial arrangement. Let us denote by $\chi(A_{\Phi}^{[a,b]}, t)$ the characteristic polynomial of $A_{\Phi}^{[a,b]}$. Postnikov and Stanley conjectured the following for $\chi(A_{\Phi}^{[a,b]}, t)$.

Conjecture 2.34  (Postnikov–Stanley [11], Conjecture 9.14) Let $a, b \in \mathbb{Z}$ with $a \leq 1 \leq b$. Suppose that $1 \leq a + b$. Then, every root $z \in \mathbb{C}$ of the equation $\chi(A_{\Phi}^{[a,b]}, t) = 0$ satisfies $\Re z = \frac{(b-a+1)h}{2}$.

By the following theorem, Conjecture 2.34 holds if it holds for every Linial arrangement.
Theorem 2.35 (Yoshinaga [16]) Let \( n \geq 0 \) and \( k \geq 0 \). The characteristic quasi-polynomial of the Linial arrangement \( \mathcal{A}_\Phi^{[1,n]} \) is

\[
\chi(\mathcal{A}_\Phi^{[1,n]}, t) = \chi(\mathcal{A}_\Phi^{[1-k,n+k]}, t + kh).
\]

(44)

For classical root systems, the formula in (44) has been proved by Athanasiadis [1, 2]. Conjecture 2.34 was proved by Postnikov and Stanley for \( \Phi = A_\ell \) [11], and by Athanasiadis for \( \Phi = A_\ell, B_\ell, C_\ell, D_\ell \) [2]. Yoshinaga verified Conjecture 2.34 for \( E_6, E_7, E_8, F_4 \) when the parameter \( n > 0 \) of the Linial arrangement \( \mathcal{A}_\Phi^{[1,n]} \) satisfies

\[
n \equiv -1 \quad \text{mod} \ 6, \ \Phi = E_6, E_7, F_4
\]

\[
n \equiv 0 \quad \text{mod} \ 30, \ \Phi = E_8.
\]

(45)

in [16], and also verified it when the parameter \( n \) is a sufficiently large integer in [17]. The case \( \Phi = G_2 \) is easy.

In proving the conjecture for the case in (45), Yoshinaga used the characteristic quasi-polynomial [16], which was introduced by Kamiya Takemura, and Terao [6–8]. One of the most important properties of the characteristic quasi-polynomial is that it coincides with the characteristic polynomial on the integers that are relatively prime to its own period as a quasi-polynomial. Let us denote by \( \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t) \) the characteristic quasi-polynomial of \( \mathcal{A}_\Phi^{[1,n]} \). Yoshinaga proved the explicit formula for \( \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t) \).

Theorem 2.36 (Yoshinaga [16]) Let \( n \geq 0 \). The characteristic quasi-polynomial of the Linial arrangement \( \mathcal{A}_\Phi^{[1,n]} \) is

\[
\chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t) = R_\Phi(S^{n+1})L_\Phi(t).
\]

(46)

From (46), we see that \( \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t) \) has the same period as \( L_\Phi(t) \), namely, the period \( \rho \). Note that \( \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t) = \chi(\mathcal{A}_\Phi^{[1,n]}, t) \) when \( t \equiv 1 \mod \rho \). We will calculate the left-hand side of (46) in the following section.

When \( n = 0 \), that is, \( \mathcal{A}_\Phi^{[1,0]} = \emptyset \), Theorem 2.36 leads to the following generalization of the Worpitzky identity (37) [16, 17].

Theorem 2.37 (Yoshinaga [16])

\[
t^\ell = R_\Phi(S)L_\Phi(t).
\]

(47)
3 Main results

3.1 Postnikov–Stanley Linial arrangement conjecture when the parameter $n + 1$ is relatively prime to the period

**Theorem 3.1** Let $n \geq 0$.

$$
\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = R_{\Phi}(\overline{S^{n+1}})^{-\gcd(n+1, \rho)}(t).
$$

(48)

*In particular, $\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t)$ has the period $\gcd(n+1, \rho)$.***

**Proof** Let $\Phi$ be an irreducible root system of rank $\ell$. We can define the polynomial

$$
g_k(t^{n+1}) := \frac{[c_0]_{n+1}[c_1]_{n+1} \cdots [c_\ell]_{n+1} A_{\ell}(t^{n+1})}{[\hat{c}_k]_{n+1}^{\ell \hat{c}_k + 1}}
$$

for $k \in \{0, \ldots, \hat{\ell}\}$ because $[\hat{c}_k]_{n+1}^{\ell \hat{c}_k + 1}$ divides $[c_0]_{n+1}[c_1]_{n+1} \cdots [c_\ell]_{n+1}$. By Proposition 2.25,

$$
L_{\Phi}(t) = \sum_{k=0}^{\hat{\ell}} L_{\hat{c}_k}^{(\ell \hat{c}_k)}(t).
$$

(49)

Note that $L_{\hat{c}_k}^{(\ell \hat{c}_k)}(t)$ is a quasi-polynomial of degree $\ell \hat{c}_k$ with period $\hat{c}_k$. Because $\rho$ is a multiple of $\hat{c}_k$, by Proposition 2.8, we obtain $L_{\hat{c}_k}^{(\ell \hat{c}_k)}(t) = L_{\hat{c}_k}^{(\ell \hat{c}_k) \gcd(n+1, \hat{c}_k)}(t)$. By Proposition 2.15, for any $k \in \{0, \ldots, \hat{\ell}\}$,

$$
[\hat{c}_k]_{S^{n+1}}^{\ell \hat{c}_k + 1} g_k(S^{n+1}) L_{\hat{c}_k}^{(\ell \hat{c}_k)}(t) = [\hat{c}_k]_{S^{n+1}}^{\ell \hat{c}_k + 1} g_k(S^{n+1}) L_{\hat{c}_k}^{(\ell \hat{c}_k) \gcd(n+1, \rho)}(t).
$$
Therefore, by Lemma 2.7 and Theorem 2.31,

\[ \chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = R_{\Phi}(S^{n+1})L_{\Phi}(t) \]

\[ = R_{\Phi}(S^{n+1})(\sum_{k=0}^{\ell} L_{\hat{\Phi}}^{(\ell_{k})}(t)) \]

\[ = ([c_0]\times [c_1]\times \ldots \times [c_{\ell}]\times S^{n+1})A_{\Phi}(S^{n+1})(\sum_{k=0}^{\ell} L_{\hat{\Phi}}^{(\ell_{k})}(t)) \]

\[ = \left[ \hat{c}_0 \right]_{S^{n+1}} g_0(S^{n+1})L_{\hat{\Phi}}^{(\ell_{0})}(t) + \ldots + \left[ \hat{c}_\ell \right]_{S^{n+1}} g_\ell(S^{n+1})L_{\hat{\Phi}}^{(\ell_{\ell})}(t) \]

\[ = \left[ \hat{c}_0 \right]_{S^{n+1}} g_0(S^{n+1})L_{\hat{\Phi}}^{(\ell_{0})}(t) + \ldots + \left[ \hat{c}_\ell \right]_{S^{n+1}} g_\ell(S^{n+1})L_{\hat{\Phi}}^{(\ell_{\ell})}(t) \]

\[ = \left[ \hat{c}_0 \right]_{S^{n+1}} g_0(S^{n+1})L_{\hat{\Phi}}^{(\ell_{0})}(t) + \ldots + \left[ \hat{c}_\ell \right]_{S^{n+1}} g_\ell(S^{n+1})L_{\hat{\Phi}}^{(\ell_{\ell})}(t) \]

\[ = \left[ \hat{c}_0 \right]_{S^{n+1}} g_0(S^{n+1})L_{\hat{\Phi}}^{(\ell_{0})}(t) + \ldots + \left[ \hat{c}_\ell \right]_{S^{n+1}} g_\ell(S^{n+1})L_{\hat{\Phi}}^{(\ell_{\ell})}(t) \]

\[ = R_{\Phi}(S^{n+1})(\sum_{k=0}^{\ell} L_{\hat{\Phi}}^{(\ell_{k})}(t)) \]

\[ = R_{\Phi}(S^{n+1})L_{\Phi}^{(\ell_{k})}(t). \]

The characteristic quasi-polynomial \( \chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) \) has the period \( \text{gcd}(n + 1, \rho) \) because \( \tilde{L}_{\Phi}^{(\ell_{k})}(t) \) has the period \( \text{gcd}(n + 1, \rho) \). \( \square \)

**Remark 3.2** Using the above results, we can prove Proposition 2.23 again. Note that \( \tilde{L}_{\Phi}^{(\ell_{k})}(t) \) is a polynomial and if a function \( f(t) \) is a polynomial, then \( (Sf)(t) = (\bar{S}f)(t) \).

The following comes immediately from Theorems 2.37 and 3.1.

\[ t^\ell = R_{\Phi}(S)\tilde{L}_{\Phi}^{(\ell_{k})}(t). \] (50)

By (50) and Theorem 2.31,

\[ A_{\ell}(\bar{S})L_{A_{\ell}}(t) = R_{\Phi}(\bar{S})\tilde{L}_{\Phi}^{(\ell_{k})}(t) \]

\[ = [c_0]_{\bar{S}}[c_1]_{\bar{S}} \cdot \ldots \cdot [c_{\ell}]_{\bar{S}}A_{\ell}(\bar{S})\tilde{L}_{\Phi}^{(\ell_{k})}(t) \]

\[ = A_{\ell}(\bar{S})[c_0]_{\bar{S}}[c_1]_{\bar{S}} \cdot \ldots \cdot [c_{\ell}]_{\bar{S}}\tilde{L}_{\Phi}^{(\ell_{k})}(t) \]

\[ = A_{\ell}(\bar{S})[c_0]_{\bar{S}}[c_1]_{\bar{S}} \cdot \ldots \cdot [c_{\ell}]_{\bar{S}}L_{\Phi}(t). \]

Thus,

\[ A_{\ell}(S)(L_{A_{\ell}}(t) - [c_0]_S[c_1]_S \ldots [c_{\ell}]_S L_{\Phi}(t)) = 0. \]

If \( (L_{A_{\ell}}(t) - [c_0]_S[c_1]_S \ldots [c_{\ell}]_S L_{\Phi}(t)) \neq 0 \), then Proposition 2.1 implies that \((1 - S) \) divides \( A_{\ell}(S), \) but \((1 - S) \) does not divide \( A_{\ell}(S). \) Hence,

\[ L_{A_{\ell}}(t) - [c_0]_S[c_1]_S \ldots [c_{\ell}]_S L_{\Phi}(t) = 0. \]

\( \square \) Springer
Theorem 3.3  Let \( m := \frac{n+1}{\gcd(n+1, \rho)} \). Let \( \rho_{n+1} := \gcd(n+1, \rho) \). Then,

\[
\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = \left( \prod_{j=0}^{\ell} \frac{1}{m} [m]_{S^c_j} \rho_{n+1} \right) \chi_{\text{quasi}}(A_{\Phi}^{[1,\rho_{n+1}-1]}, t). \tag{51}
\]

**Proof**  By Theorem 3.1, Lemma 2.13, and Proposition 2.33,

\[
\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = R_{\Phi}(S^{n+1}) \tilde{L}_{\Phi}(t) \\
= \frac{1}{m^{\ell+1}} ([m]_{S^c_0} \rho_{n+1} \cdots [m]_{S^c_{\ell}} \rho_{n+1}) R_{\Phi}(S^{n+1}) \tilde{L}_{\Phi}(t) \\
= \frac{1}{m^{\ell+1}} ([m]_{S^c_0} \rho_{n+1} \cdots [m]_{S^c_{\ell}} \rho_{n+1}) \chi_{\text{quasi}}(A_{\Phi}^{[1,\rho_{n+1}-1]}, t).
\]

\( \square \)

Let \( b \in \mathbb{R} \). Let

\[
L_b := \{ f(t) \in \mathbb{C}[t] \mid \text{All the roots of } f \text{ are on the line with real part } b. \}, \]

\[
U := \{ f(t) \in \mathbb{C}[t] \mid \text{All the roots of } f \text{ are on the unit circle}. \}.
\]

We prove Conjecture 2.35 using the following lemma. This lemma was also used in [2, 11], and [16].

**Lemma 3.4**  (Postnikov–Stanley [11], Lemma 9.13)  Let \( b \in \mathbb{R} \). Let \( g(t) \in U \) and \( f(t) \in L_b \). Then, \( g(S) f(t) \in L_{\deg g + b} \).

Theorem 3.5  Let \( n \) be an integer with \( \gcd(n+1, \rho) = 1 \). Then,

\[
\chi_{\text{quasi}}(A_{\Phi}^{[1,n]}, t) = \left( \prod_{j=0}^{\ell} \frac{1}{n+1} [n+1]_{S^c_j} \right) t^{\ell}. \tag{52}
\]

In particular, the characteristic quasi-polynomial becomes a polynomial and any root \( z \) of the equation \( \chi(A_{\Phi}^{[1,n]}, t) = 0 \) satisfies \( \text{Re } z = \frac{nb}{2} \).

**Proof**  We calculate the characteristic polynomial using Theorems 3.3 and 2.37.
By Lemma 3.4, the real part of any root of the equation \( \chi(A^{[1,n]}_\Phi, t) = 0 \) is
\[
\frac{n(c_1 + c_2 + \cdots + c_2)}{2} = \frac{nh}{2}.
\]
\[\square\]

**Remark 3.6**  Theorem 3.5 is a generalization of the expression of the characteristic polynomial of \( A^{[1,n]}_{\mathcal{A}} \) given by Postnikov and Stanley [11] and the expression of \( A^{[1,n]}_{\mathcal{B}_{\ell}} \), \( A^{[1,n]}_{\mathcal{C}_{\ell}} \), and \( A^{[1,n]}_{\mathcal{D}_{\ell}} \) for even values of \( n \) given by Athanasiadis [2].

**Example 3.7**  (case \( E_6 \)) Let \( n \) be a positive integer. Let \( m := \frac{n+1}{\gcd(n+1, \rho)} \).

If \( \gcd(n + 1, 6) = 1 \), then
\[
\chi_{\text{quasi}}(A^{[1,n]}_{E_6}, t) = \left( \frac{1}{m} [m]_S \right)^3 \left( \frac{1}{m} [m]_{S^2} \right)^3 \left( \frac{1}{m} [m]_{S^3} \right)^6 t^6.
\]

If \( \gcd(n + 1, 6) = 2 \), then
\[
\chi_{\text{quasi}}(A^{[1,n]}_{E_6}, t) = \left( \frac{1}{m} [m]_{S^2} \right)^3 \left( \frac{1}{m} [m]_{S^4} \right)^3 \left( \frac{1}{m} [m]_{S^6} \right)^3 \chi_{\text{quasi}}(A^{[1,1]}_{E_6}, t).
\]

If \( \gcd(n + 1, 6) = 3 \), then
\[
\chi_{\text{quasi}}(A^{[1,n]}_{E_6}, t) = \left( \frac{1}{m} [m]_{S^3} \right)^3 \left( \frac{1}{m} [m]_{S^6} \right)^3 \left( \frac{1}{m} [m]_{S^9} \right)^3 \chi_{\text{quasi}}(A^{[1,2]}_{E_6}, t).
\]

If \( \gcd(n + 1, 6) = 6 \), then
\[
\chi_{\text{quasi}}(A^{[1,n]}_{E_6}, t) = \left( \frac{1}{m} [m]_{S^6} \right)^3 \left( \frac{1}{m} [m]_{S^{12}} \right)^3 \left( \frac{1}{m} [m]_{S^{18}} \right)^3 \chi_{\text{quasi}}(A^{[1,5]}_{E_6}, t).
\]

**Theorem 3.8**  Let \( \rho_{n+1} := \gcd(n + 1, \rho) \) and \( \eta := \frac{\rho_{n+1}}{\text{rad}(\rho_{n+1})} \).

\[
\chi(A^{[1,\rho_{n+1}-1]}_\Phi, t) = \left( \prod_{j=0}^{\ell} \frac{1}{\eta} [\eta]_{S^j \text{rad}(\rho_{n+1})} \right) \chi(A^{[1,\text{rad}(\rho_{n+1})-1]}_\Phi, t). \tag{53}
\]

**Proof**  We set \( t \equiv 1 \mod \rho \). Then, we have that \( \tilde{L}_{\Phi}^{\rho_{n+1}}(t) = \tilde{L}_{\Phi}^{\text{rad}(\rho_{n+1})}(t) \) by Proposition 2.10 and Theorem 2.22. Hence, by Lemma 2.13, Proposition 2.33, and Theorem 3.1,
\[
\chi(A^{[1,\rho_{n+1}-1]}_\Phi, t) = R_{\Phi}(S^{\rho_{n+1}}) \tilde{L}_{\Phi}^{\rho_{n+1}}(t) = \left( \prod_{j=0}^{\ell} \frac{1}{\eta} [\eta]_{S^j \text{rad}(\rho_{n+1})} \right) R_{\Phi}(S^{\text{rad}(\rho_{n+1})}) \tilde{L}_{\Phi}^{\text{rad}(\rho_{n+1})}(t)
\]
\[
= \left( \prod_{j=0}^{\ell} \frac{1}{\eta} [\eta]_{S^j \text{rad}(\rho_{n+1})} \right) \chi(A^{[1,\text{rad}(\rho_{n+1})-1]}_\Phi, t).
\]
\[\square\]
Table 2  Characteristic polynomials for \( F_4 \) (rad(\( \rho \)) = 6)

| \( \chi(\mathcal{A}_{F_4}^{[1,n]}, t) \) | \( n \) | Real part |
|---------------------------------|------|----------|
| \( t^4 - 24t^3 + 258t^2 - 1368t + 2917 \) | 2 – 1 | 6        |
| \( t^4 - 48t^3 + 1000t^2 - 10176t + 41572 \) | 3 – 1 | 12       |
| \( t^4 - 120t^3 + 5986t^2 - 143160t + 1361989 \) | 6 – 1 | 30       |

We now provide examples of Theorem 3.8 for \( E_8 \). Using the notation of Theorem 3.8, in the case of \( E_8 \), \( \eta \) can only take a value of 1 or 2. If \( \eta = 1 \), then Theorem 3.8 is trivial. The following formulas are examples of Theorem 3.8 for \( \eta = 2 \).

Example 3.9  (Case \( E_8 \))

\[
\chi(\mathcal{A}_{E_8}^{[1,4-1]}, t) = \left( \prod_{i=0}^{\ell} \frac{1}{2} [2]_{S^{2c_i}} \right) \chi(\mathcal{A}_{E_8}^{[1,2-1]}, t). \tag{54}
\]

\[
\chi(\mathcal{A}_{E_8}^{[1,12-1]}, t) = \left( \prod_{i=0}^{\ell} \frac{1}{2} [2]_{S^{6c_i}} \right) \chi(\mathcal{A}_{E_8}^{[1,6-1]}, t). \tag{55}
\]

\[
\chi(\mathcal{A}_{E_8}^{[1,20-1]}, t) = \left( \prod_{i=0}^{\ell} \frac{1}{2} [2]_{S^{10c_i}} \right) \chi(\mathcal{A}_{E_8}^{[1,10-1]}, t). \tag{56}
\]

\[
\chi(\mathcal{A}_{E_8}^{[1,60-1]}, t) = \left( \prod_{i=0}^{\ell} \frac{1}{2} [2]_{S^{30c_i}} \right) \chi(\mathcal{A}_{E_8}^{[1,30-1]}, t). \tag{57}
\]

3.2 Verification of the Postnikov–Stanley Linial arrangement conjecture

We verify Conjecture 2.34 for \( \Phi = E_6, E_7, E_8, \) or \( F_4 \). We use the notation of Theorems 3.3 and 3.8. Recall that, according to these theorems, the following formula holds.

\[
\chi(\mathcal{A}_{\Phi}^{[1,n]}, t) = \left( \prod_{j=0}^{\ell} \frac{1}{m} [m]_{S^{j \rho_{n+1}}} \right) \left( \prod_{j=0}^{\ell} \frac{1}{\eta} [\eta]_{S^{j \rho_{n+1}}} \right) \chi(\mathcal{A}_{\Phi}^{[1,\text{rad}(\rho_{n+1})-1]}, t). \tag{58}
\]

If the real part of any root of the equation

\[
\chi(\mathcal{A}_{\Phi}^{[1,\text{rad}(\rho_{n+1})-1]}, t) = 0
\]

is \( \frac{(\text{rad}(\rho_{n+1})-1)\eta}{2} \) for \( \Phi \in \{ E_6, E_7, E_8, F_4 \} \), then Lemma 3.4 implies that Conjecture 2.34 holds. We have computed the characteristic polynomial such that the parameter \( n + 1 \) is a factor of \( \text{rad}(\rho) \) other than 1 and have determined the real part of the roots.
### Table 3  Characteristic polynomials for $E_6$ (rad($\rho$) = 6)

| $\chi (A_{E_6}^{[1,n]}, t)$ | $n$  | Real part |
|-----------------------------|------|-----------|
| $t^6 - 36r^5 + 630r^4 - 6480r^3 + 40185r^2 - 140076r + 211992$ | 2 - 1 | 6         |
| $t^6 - 72r^5 + 2400r^4 - 46080r^3 + 528600r^2 - 3396672r + 9474200$ | 3 - 1 | 12        |
| $t^6 - 180r^5 + 14550r^4 - 666000r^3 + 18019065r^2 - 271143900r + 1762474040$ | 6 - 1 | 30        |
| $n$   | Real part                                      |
|-------|------------------------------------------------|
| 2 − 1 | $17^7 − 631^6 + 1953^5 − 360851^4 + 4463551^3 − 34173091^2 + 151542511^1 − 29798253$ |
| 3 − 1 | $17^7 − 126^6 + 7476^5 − 264600^4 + 59496368^3 + 84088368^2 + 68720272^1 − 24002744^0$ |
| 6 − 1 | $17^7 − 315^6 + 4546^5 − 385075^4 + 454605^3 − 3850750^2 + 204937635^1 − 160911109625$ |
Table 5  Characteristic polynomials for $E_8$ (rad($\rho$) = 30)

| $\chi(A^{[1,n]}_{E_8}, t)$ | $n$ | Real part |
|---------------------------|-----|-----------|
| $r^8 - 120r^7 + 7140r^6 - 264600r^5 + 6540030r^4 - 108901800r^3 + 1181603220r^2 - 7583286600r + 21918282249$ | 2 | 15 |
| $r^8 - 240r^7 + 27440r^6 - 1915200r^5 + 88161360r^4 - 2716963200r^3 + 54385106720r^2 - 643164643200r + 3426392186728$ | 3 | 30 |
| $r^8 - 480r^7 + 107520r^6 - 14515200r^5 + 1281219408r^4 - 75249457920r^3 + 2857900896480r^2 - 6391860253600r + 642465923287416$ | 5 | 60 |
| $r^8 - 600r^7 + 167300r^6 - 28035000r^5 + 3065453790r^4 - 222698637000r^3 + 10449830016500r^2 - 288505461225000r + 3577184806486057$ | 6 | 75 |
| $r^8 - 1080r^7 + 538020r^6 - 160234200r^5 + 31018986558r^4 - 3977954041320r^3 + 328758988903380r^2 - 15957853314798600r + 34737804233610441$ | 10 | 135 |
| $r^8 - 1680r^7 + 1297520r^6 - 597643200r^5 + 178602069408r^4 - 35307879102720r^3 + 4493170619530880r^2 - 335521093135065600r + 11227745283721390816$ | 15 | 210 |
| $r^8 - 3480r^7 + 5550020r^6 - 52665102000r^5 + 3236633286558r^4 - 1314003597910920r^3 + 343011765319289780r^2 - 52494228716611434600r + 3597446896074261934441$ | 30 | 435 |
using a computational method. The case of $\text{rad}(\rho_{n+1}) = \text{rad}(\rho)$ was already verified in [16]. We present the characteristic polynomials for $\Phi \in \{E_6, E_7, E_8, F_4\}$ in Tables 2, 3, 4 and 5.

Below, we will explain the computational method used to determine the real part of the roots. First, we compute the characteristic polynomial using Theorem 3.1 and the computation results of the Ehrhart quasi-polynomial given by Suter [14]. We also compute $\chi(A_{[1,n]}^{[\rho, \sqrt{-1t + \frac{nh}{2}}]}).$ If all the roots of $\chi(A_{[1,n]}^{[\rho, \sqrt{-1t + \frac{nh}{2}}]}, t)$ are real, then all the roots of $\chi(A_{[1,n]}^{[\rho, \sqrt{-1t + \frac{nh}{2}}]}, t)$ have same the real part $\frac{nh}{2}.$ We verify that all the roots of $\chi(A_{[1,n]}^{[\rho, \sqrt{-1t + \frac{nh}{2}}]}, t)$ are real by using the Fourier–Budan Theorem [12]. The computation can be performed using only integers.

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