Quantum Double of Heisenberg-Weyl Algebra, its Universal R-Matrix and their Representations

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Abstract

In this paper a new quasi-triangular Hopf algebra as the quantum double of the Heisenberg-Weyl algebra is presented. Its universal R-matrix is built and the corresponding representation theory are studied with the explicit construction for the representations of this quantum double.
1. Introduction

Recently the quantum group theory associated with Yang-Baxter equation for nonlinear integrable system has became a focus of the attention from both theoretical physicists and mathematicians [1]. As a kernal of this theory, Drinfeld’s quantum double construction is a quite powerful tool in constructing the solutions, namely the R-matrices, for quantum Yang-Baxter equation (QYBE) in connection with certain algebraic structures, such as quantum algebras[2], quantum superalgebras[3], quantum affine algebras[2], their multiparameter deformations[4] and the quantum doubles of the Borel subalgebras for universal enveloping algebras (UEAs) of classical Lie algebra[6].

In this paper we will present a different quasi-triangular Hopf algebra that is the quantum double of the Heisenberg-Weyl (HW) algebra based on Drinfeld’s quantum double construction. To construct the explicit R-matrices for QYBE from its universal R-matrix of this quantum double, we study its representation theory and explicitly construct its finite and infinite dimensional representations. A 6-dimensional example of R-matrices for this quantum double is given as an illustration. The studies in this paper shows that, like its q-deformation [6-8], the ordinary HW algebra also realizes a so-called ‘quantum group structure’, quasi-triangular Hopf algebra associated QYBE. This fact shows that the canonical quantization defined by the HW algebra possibly prompts the important role of ‘quantum group structure’ and the QYBE in quantum theory.

2. Quantum Double of HW Algebra

The Heisenberg-Weyl algebra (HW) algebra $A$ is an associative algebra generated by $a, \bar{a}, E$ and the unit 1. These generators satisfy the defining relations

$$[a, \bar{a}] = E, [E, a] = 0 = [E, \bar{a}],$$

(2.1)

If we take a special representation $T$ such that

$$T(a)^+ = T(\bar{a}), T(E) = \text{unit matrix } I$$

then $\bar{a}$ and $a$ can be regarded the creation and annihilation operators of boson states in second quantization. Since the algebra $A$ is the UEA of the HW Lie algebra with
basis \{a, \bar{a}, E\}, \textbf{A} can be endowed with a well-known Hopf algebraic structure

\[ \Delta(x) = x \otimes 1 + 1 \otimes x, S(x) = -x, \epsilon(x) = 0, \]

for \( x = a, \bar{a}, E \) where the algebraic homomorphisms \( \Delta, \epsilon \) and the algebraic antihomomorphism \( S \) defined only for the generators are naturally extended for the whole algebra. According to the PBW theorem, the basis for the algebra \( \textbf{A} \) is chosen as

\[ \{ X(m, n, s) = \bar{a}^m a^n E^s, m, n, s, \in \mathbb{Z}^+ = \{0, 1, 2, ..\} \} \]

Now, let us consider the dual Hopf algebra \( \textbf{B} \) of \( \textbf{A} \). Suppose \( \bar{b}, b \) and \( H \) are the dual generators to \( \bar{a}, a \) and \( E \) respectively and then defined by

\[ < X(m, n, s), \bar{b} > = \delta_{m,1}\delta_{n,0}\delta_{s,0} \]
\[ < X(m, n, s), b > = \delta_{m,0}\delta_{n,1}\delta_{s,0} \]
\[ < X(m, n, s), E > = \delta_{m,0}\delta_{n,0}\delta_{s,1} \]

(2.3)

Since the algebra \( \textbf{A} \) is commutative, its Hopf algebraic dual is Abelian, i.e., the dual generators commute each other. Choosing a basis for \( \textbf{B} \)

\[ Y(m, n, s) = (m!n!s!)^{-1}\bar{b}^m b^n H^s, m, n, s, \in \mathbb{Z}^+ \]

(2.4)

we prove the following proposition

**Proposition 1.** The equations (2.4) define a dual basis \( Y(m, n, s) \) for \( \textbf{B} \) satisfying

\[ < X(m, n, s), Y(m', n', s') >= \delta_{m,m'}\delta_{n,n'}\delta_{s,s'} \]

(2.5)

**Proof.** According to the Hopf algebraic duality between \( \textbf{A} \) and \( \textbf{B} \):

\[ < a, b_1 b_2 > = < \Delta_A(a), b_1 \otimes b_2 >, a \in \textbf{A}, b_1, b_2 \in \textbf{B} \]
\[ < a_1 a_2, b > = < a_2 \otimes a_1, \Delta_B(b) >, a_1, a_2 \in \textbf{A}, b \in \textbf{B} \]
\[ < 1_A, b > = \epsilon_B(b), b \in \textbf{B} \]
\[ < a, 1_B > = \epsilon_A(a), a \in \textbf{A} \]
\[ < S_A(a), S_B(b) > = < a, b >, a \in \textbf{A}, b \in \textbf{B} \]

(2.6)

where for \( C=A,B,\Delta_C, \epsilon_C \) and \( S_C \) are the coproduct, counit and antipode of \( C \) respectively; \( 1_C \) is the unit of \( C \). Without confusion we no longer use the index \( C \) to specify, \( \Delta_C, \epsilon_C \) and \( S_C \). Let \( G = \bar{a}, aE \) corresponding to \( F = \bar{b}, b, E \) respectively. For \( G \neq \bar{a}, \)

\[ < \bar{a}^m G^n, F^n > = < \Delta(\bar{a}^m G^n), F^{n-1} \otimes F > \]
\[
\sum_{k,l=0}^{\infty} \frac{m!s!}{(m-k)!(k-s)!l!} < \bar{a}^{s-l} G^{m-k} \otimes \bar{a}^{l} G^{k}, F^{n-1} \otimes F > \\
= m < \bar{a}^{s} G^{m-1}, F^{n-1} > = m! \delta_{m,n} \delta_{s,0}
\]

For \( F \neq \bar{b} \), similarly, we have

\[
< G^{m}, \bar{b}^{s} F^{n} >= m! \delta_{m,n} \delta_{s,0}
\]

\[
< G^{m}, F^{n} >= m! \delta_{m,n}
\]

Then, we have

\[
< \bar{a}^{m} a^{n}, \bar{b}^{s} b^{l} >= < \Delta(\bar{a}^{m} a^{n}), \bar{b}^{s} \otimes b^{l} >= s! s! \delta_{m,s} \delta_{n,l}
\]

and therefore prove eq.(2.5).

It follows from the above proposition that

\[
< X(m, n, s) \otimes X(k, l, r), \Delta(H) >= < X(k, l, r) X(m, n, s), H >
\]

\[
= \delta_{m,0} \delta_{n,0} \delta_{k,0} \delta_{l,0} (\delta_{s,1} \delta_{r,0} + \delta_{s,0} \delta_{r,1}) + \delta_{m,1} \delta_{n,0} \delta_{k,0} \delta_{l,1} \delta_{r,0}
\]

namely,

\[
\Delta(H) = H \otimes 1 + 1 \otimes H + \bar{b} \otimes b
\]

Similarly, we calculate other operations of the generators for \( B \) under \( \Delta, \epsilon \) and \( S \). The results are summarized as follows.

**Proposition 2.** The dual Hopf algebra \( B \) is generated by \( \bar{b}, b \) and \( H \) and endowed with the following Hopf algebraic structure

\[
\Delta(x) = x \otimes 1 + 1 \otimes x,
\]

\[
S(x) = -x, \epsilon(x) = 0, x = \bar{b}, b
\]

\[
\Delta(H) = H \otimes 1 + 1 \otimes H + \bar{b} \otimes b
\]  

(2.7)

**3. Quantum Double and Universal R-matrix**

It should be noticed that the dual Hopf algebraic structure of \( B \) can also be obtained from the formal group theory [7] of Lie algebra in principle where but the explicit expressions of the e Baker-Comppell-Hausdorff formula for the HW Lie algebra. In this sense the Drinfeld’s theory is not the unique approach to get the dual
Hopf algebraic structure. However, it is important that Drinfeld’s theory can also provide us with a convenient method to ‘combine’ A and B to form a ‘larger’ Hopf algebra D containing A and B as subalgebras. The universal R-matrix for QYBE can be automatically given in this construction.

According to the multiplication formula for the quantum double

\[ ba = \sum_{i,j} <a_i(1), S(b_j(1))> <a_i(3), b_j(3)> a_i(2)b_j(2) \]  \tag{3.1} \]

where \( c_i(k)(k = 1, 2, 3; c = a, b) \) are defined by

\[ \Delta^2(c) = (id \otimes \Delta)\Delta(c) = (\Delta \otimes id)\Delta(c) = \sum_i c_i(1) \otimes c_i(2) \otimes c_i(3) \]

and we use the explicit expressions

\[ \Delta^2(H) = H \otimes 1 \otimes 1 + 1 \otimes H \otimes 1 + 1 \otimes 1 \otimes H + 1 \otimes \bar{b} \otimes b + \bar{b} \otimes b \otimes 1 + \bar{b} \otimes 1 \otimes b \]  \tag{3.2} \]

to prove the following results

**Proposition 3.** The quantum double D is generated by \( a, \bar{a}, b, \bar{b}, E, H \) as an associative algebra with the only nonzero commutators

\[ [a, \bar{a}] = E, [H, a] = \bar{b}, [H, \bar{a}] = -b, \]  \tag{3.3} \]

and as a non-cocomutative Hopf algebra with the structure (2.2) and (2.7). The universal R-matrix, a canonical element intertwining A and B, is

\[ R = \sum_{m,n,s=0}^{\infty} X(m, n, s) \otimes Y(m, n, s) \]
\[ = \exp(\bar{a} \otimes \bar{b}) \exp(a \otimes b) \exp(E \otimes H), \]  \tag{3.4} \]

Using the above commutation relations, we can directly verify the following quasi-triangular relations

\[ \hat{R}\Delta(x) = \sigma\Delta(x)\hat{R}, \]
\[ (\Delta \otimes id)\hat{R} = \hat{R}_{13}\hat{R}_{23}, \]
\[ (id \otimes \Delta)\hat{R} = \hat{R}_{13}\hat{R}_{12}, \]  \tag{3.5} \]
\[ (\epsilon \otimes id)\hat{R} = 1 = (id \otimes \epsilon)\hat{R}, \]
\[ (S \otimes id)\hat{R} = \hat{R}^{-1} = (id \otimes S)\hat{R}, \]
where $\sigma$ is such a permutation that $\sigma(x \otimes y) = y \otimes x, x, y \in D$. The eqs. (3.5) imply that the above constructed universal R-matrix satisfies the abstract QYBE

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23} = \hat{R}_{23}\hat{R}_{13}\hat{R}_{12},$$

(3.6)

where $\hat{R}_{12} = \sum_m a_m \otimes b_m \otimes 1$, $\hat{R}_{13} = \sum_m a_m \otimes 1 \otimes b_m$, $\hat{R}_{23} = \sum_m 1 \otimes a_m \otimes b_m$ and $a_m$ and $b_m$ are the dual bases vectors of $A$ and $B$ respectively. Here, we simply note $\hat{R} = \sum_m a_m \otimes b_m$.

4. On Representations and Realizations of the Quantum Double

In order to obtain the R-matrices for QYBE from new universal R-matrix (3.5), we should consider the representations of the quantum double $D$. For simplicity we by $x$ denote $T(x)$ for a representation of $D$ as follows.

**Proposition 4.** There does not exist a finite dimensional irreducible representation of $D$ besides the trivial representations $T$ for which there at least is one generator $s$ such that $T(x) = 0$.

**Proof.** Thanks to the Schur lemma we know that the representatives of the central elements $E, \bar{b}$ and $b$ must be nonzero scalars for a finite dimensional representation, i.e.,

$$E = \eta \neq 0, b = \xi \neq 0, \bar{b} = \bar{\xi} \neq 0, \eta, \xi, \bar{\xi} \in \text{complex field } C$$

However, taking the trace of $E$, we have

$$tr.(E) = tr.([a, \bar{a}]) = 0$$

that is $\eta = 0$. Then, a contraction appears.

We learn from this proposition and its proof that the finite dimensional representation of $D$ must be neither irreducible nor a sum of some non-trivially irreducible representations. The possible non-trivial finite dimensional representations are only those indecomposable ones, the reducible but not completely reducible representations where $tr.(E) = 0$. For the latter we can give a boson realization

$$a = c, \bar{a} = c^+, b = -\alpha \in C, \bar{b} = -\beta \in C,$$

$$E = 1, H = \alpha c + \beta c^+, \quad (4.1)$$

in terms of the boson operators $c$ and $c^+$ satisfying

$$[c, c^+] = 1, 1x = x1 = x, x = c, c^+, \quad (4.2)$$
Using the Fork representation of $c$ and $c^+$,
\[
c^+ \mid n > = (n+1)^{1/2} \mid n+1 >,
\]
\[
c \mid n > = n^{1/2} \mid n-1 >,  \tag{4.3}
\]
on the Fock space
\[
\{ \mid n > = (n!)^{-1/2}(c^+)^n \mid 0 > \mid c \mid 0 > = 0, n = 0, 1, 2, \ldots \}
\]
, we obtain an infinite irreducible representation of $D$ with explicit matrix elements
\[
(a)_{m,n} = (n+1)^{1/2} \delta_{m,n+1}, (\bar{a})_{m,n} = n^{1/2} \delta_{m,n}
\]
\[
(E)_{m,n} = \delta_{m,n}, (b)_{m,n} = -\alpha \delta_{m,n}, (\bar{b})_{m,n} = -\beta \delta_{m,n}
\]
\[
(H)_{m,n} = \beta(n+1)^{1/2} \delta_{m,n+1} + n^{1/2} \delta_{m,n} \delta_{m,n-1},  \tag{4.4}
\]
In this realization and the corresponding representation, the universal R-matrix (3.4) can be expressed as a generator
\[
R = e^{-\beta C^+} e^{-\alpha C} \otimes e^{\beta C^+ + \alpha C} = e^{-\beta \alpha / 2} D(-\beta, -\alpha) \otimes D(\beta, \alpha),  \tag{4.5}
\]
for the two-mode coherent state
\[
\mid -\beta, \beta > = N^{-1} \sum_{m,n=0}^{\infty} \frac{(c^+)^n \otimes (c^+)^n}{m!n!} \mid 0 >, \tag{4.6}
\]
where
\[
D(\beta, \alpha) = e^{\beta C^+ + \alpha C}
\]
is a non-normalized single coherent state operator.

5. Explicit representations

In this section we consider the explicit construction of representations for the quantum double $D$ and its universal R-matrix.

Since $D$ is the universal enveloping algebra of a Lie algebra with the basis
\[
\{ a, \bar{a}, b, \bar{b}, H, E \},
\]
the PBW theorem determines its basis
\[
X[M] = X(m, n, l, r, s, t) = a^m \bar{a}^n H^l b^r \bar{b}^s E^t,  \tag{4.1}
\]
where \( m, n, l, r, s, t \in \mathbb{Z}^+ \) and \( M \) denotes a 6-vector \( M = (m, n, l, r, s, t) \) in a lattice vector space \( \mathbb{Z}^{+6} \) with the basis

\[
e_1 = (1, 0, 0, 0, 0, 0), e_2 = (0, 1, 0, 0, 0, 0),
\]
\[
e_3 = (0, 0, 1, 0, 0, 0), e_4 = (0, 0, 0, 1, 0, 0),
\]
\[
e_5 = (0, 0, 0, 0, 1, 0), e_6 = (0, 0, 0, 0, 0, 1),
\]

We can construct an explicit representation of \( D \) on the basis \( X[M] \)

**Proposition 5.** The regular representation of \( D \) is

\[
aX[M] = X[M + e_1],
\]
\[
\bar{a}X[M] = X[M + e_2] - mX[M - e_1 + e_6],
\]
\[
EX[M] = X[M + e_6],
\]
\[
bX[M] = X[M + e_4],
\]
\[
\bar{b}X[M] = X[M + e_5],
\]
\[
HX[M] = X[M + e_3] + mX[M - e_1 + e_5] - nX[M - e_2 + e_4],
\]

(5.1)

**Proof.** It follows from the following equations

\[
[\bar{a}, a^n] = -nEa^{n-1}
\]
\[
[a, \bar{a}^n] = nE\bar{a}^{n-1},
\]
\[
[H, a^n] = nE\bar{b}a^{n-1}
\]
\[
[H, \bar{a}^n] = -nEb\bar{a}^{n-1}
\]

which are obtained from eq.(3.3) by induction.

Let \( I \) be a left ideal generated by the element \( H - \mu \), ie.,

\[
L(\mu) = D(H - \mu) = \{ x(H - \mu) \mid x \in D \}
\]

Because the left ideal \( I \) is a left-invariant \( D \)-submodule, on the quotient space \( V(\mu) = D/I(\mu) \):

\[
u(K) = a^m\bar{a}^nb^s\bar{b}^tE^t\text{Mod}I(\mu)
\]

where \( K = (m, n, r, s, t) \), \( m, n, r, s, t \in \mathbb{Z}^+ \), the regular representation induces a infinite dimensional representation
\[
\begin{align*}
au[K] &= u[K + e_1], \\
\bar{au}[K] &= u[K + e_2] - mu[K - e_1 + e_5], \\
Eu[K] &= u[K + e_5], \\
bu[K] &= u[K + e_3], \\
\bar{bu}[K] &= u[K + e_4], \\
Hu[K] &= \mu u[K] + mu[K - e_1 + e_4] - nu[K + e_2 + e_3], \\
\end{align*}
\]

where
\[
\begin{align*}
e_1 &= (1, 0, 0, 0, 0), e_2 = (0, 1, 0, 0, 0), \\
e_3 &= (0, 0, 1, 0, 0), e_4 = (0, 0, 0, 1, 0), \\
e_5 &= (0, 0, 0, 0, 1),
\end{align*}
\]

Now, let us make a key observation from the eq.(5.3) that the sum \(m+n+r+s+t\) for the basis vectors \(u[K] = u(m, n, r, s, t)\) is invariant under the actions of \(D\). This fact tells us that the following vectors
\[
\{u[K] = u(m, n, r, s, t) \mid m + n + r + s + t = N\}
\]
for a fixed \(N \in \mathbb{Z}^+\) span an invariant subspace \(V(\mu, N)\). Then, the quotient space
\[
Q(\mu, N) = V(\mu)/V(\mu, N):
\]
\[
\text{Span}\{v(K) = u[K] \mod V(\mu, N) \mid m + n + r + s + t \leq N - 1\}
\]
is finite dimensional and the dimension is
\[
d(N) = \sum_{k=0}^{N-1} \frac{(k + 4)!}{k!4!}, \quad (5.4)
\]
If we define
\[
f_N(K) = \theta(N - 1 - (m + n + r + s + t))v(k), k = (m, n, r, s, t)
\]
where \(\theta(x) = 1(x \geq 0)\) and \(0(x < 0)\), we can explicitly write out the above finite dimensional representation in the explicit form that is obtained by substituting \(u[K]\) in eq.(5.3) by \(f_N(K)\). Its lowest example is a 6-dimensional representation
\[
\begin{align*}
a &= E_{1,6}, \bar{a} = -E_{5,1} + E_{2,6}, E = E_{5,6}, \\
b &= E_{3,6}, \bar{b} = E_{4,6}, H = \mu \sum_{i=1}^{6} E_{i,i} + E_{4,1} - E_{3,2}
\end{align*}
\]
on an ordered basis

\[ f_1(1,0,0,0,0), f_1(0,1,0,0,0), f_1(0,0,1,0,0) \]

\[ f_1(0,0,0,1,0), f_1(0,0,0,0,1), f_1(0,0,0,0,0) \]

where \( E_{i,j} \) are the matrix units with the corresponding elements

\[ (E_{i,j})_{r,s} = \delta_{i,r} \delta_{j,s} \]

One purpose of building quantum double is to obtain the solutions of the QYBE in terms of its universal R-matrix and matrix representations. In order to find the solutions of QYBE associated with the exotic quantum double \( D \), we have studied representation theory and construct both finite and infinite dimensional representations of \( D \). In fact, for a given representation \( T^{[x]} \) of \( D \):

\[ T^{[x]} : D \rightarrow \text{End}(V) \]

on the linear space \( V \) where \( x \) is a continuous parameter, we can construct a R-matrix

\[ R(x, y) = T^{[x]} \otimes T^{[y]}(\hat{R}) \]

satisfying the QYBE

\[ R_{1,2}(x, y)R_{1,3}(x, z)R_{2,3}(y, z) = R_{2,3}(y, z)R_{1,3}(x, z)R_{1,2}(x, y), \quad (5.6) \]

Here, \( x, y \) and \( z \) appear as the color parameters [24] similar to the non-additive spectrum parameters in QYBE. For example, using the above obtained 6-dimensional representation, we can construct a 36 \( \times \) 36- R-matrix

\[ R = \exp(E_{1,6} \otimes E_{3,6})\exp([-E_{5,1} + E_{2,6}] \otimes E_{4,6})\exp(E_{5,6} \otimes (\mu \sum_{i=1}^{6} E_{i,i} + E_{4,1} - E_{3,2})) \]

\[ = (1 + E_{1,6} \otimes E_{3,6})(1 + [-E_{5,1} + E_{2,6}])(1 + E_{5,6} \otimes (\mu \sum_{i=1}^{6} E_{i,i} + E_{4,1} - E_{3,2})). \quad (5.7) \]

It is need to pointed out that the higher dimensional representations can also be obtained in the same form.

6. Discussions

To conclude this paper, we should give some remarks on our exotic quantum double and its relations to the known results.
From the construction of the exotic quantum double in this paper, we can see that a commutative (Abelian) algebra, e.g., the subalgebra $B$, can be endowed with a non-cocommutative Hopf algebraic structure and its quantum dual $A$ and quantum double $D$ can be deduced as non-commutative algebras in an inverse process of the construction in this paper. Such a process possibly provide us with a scheme of ‘quantization’ from commutative object to non-commutative one. An example of this ‘quantization’ was given [5] recently.

It has to be pointed out that there are some difficulties in the further developments in constructing the general quantum double associated with arbitrary Lie algebra. When one take the subalgebra $B$ to be the whole UEA of an arbitrary Lie algebra, we hardly write down the dual basis explicitly and so the construction scheme of this paper can not work well.

In the formal group theory of Lie algebra [7], the bialgebra structure of the dual to the UEA of a classical Lie algebra can be given abstractly in terms of the formal group. It is not difficult to further define the antipode for this dual bialgebra. So, in this abstract way, the Hopf algebraic structure can be endowed with to the dual Hopf algebra of the UEA. However, writing out the explicit Hopf algebraic structure, namely, the the explicit multiplication relations, coproduct, antipode and counit for the dual generators, completely depends on the explicit evolution of the Baker-Comppell-Hausdorff formula for classical Lie algebra. However, it is much difficult to do it even for the simple case e.g., SU(2). The study in this paper avoids this evolution so that not only the dual Hopf algebraic structure is obtained, but also the corresponding quantum double -the exotic quantum double is built for the Borel subalgebra of the UEA of arbitrary classical Lie algebra by combining the two subalgebras dual to each other.

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