Bipolar-valued hesitant fuzzy graph and its application

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Abstract
In the real-world scenario, one has to find a dominating person in a social network, conferences, meetings or any group discussion. The fuzzy graph (network) is one of the most powerful tools to find the strongest influential person in a network. This paper aims to develop a concept of fuzzy graphs (FGs) in the setup of bipolar-valued hesitant fuzzy sets (BVHFs). The concept of bipolar-valued hesitant fuzzy graph (BVHFG) is different from the concept of bipolar fuzzy graph (BFG). BVHFG is the generalization of hesitant fuzzy graph (HFG), which not only considers the satisfaction degree of units in a network but also considers the satisfaction degree to some implicit counter property of units with several bipolar fuzzy values. We first introduce the definition of BVHFG, represented by another class of imprecise membership grades that refers to BVHF membership grades. We shall subsequently see the scope of BVHF membership grades in BVHFG is greater than the scope of bipolar-valued membership grades in BFG. In addition, we also discuss the basic operations and functional properties of BVHFGs. Finally, we propose a numerical method to find the most dominating person using our proposed work. As the proposed method of ranking considers the degree of hesitation as well as bipolarity, this method has the edge over earlier work. To establish the importance of our method, we also find domination degrees for HFG and BVHFG using the same example and show that there is a significant change in the ranking of dominating persons.

Keywords Bipolar-valued hesitant fuzzy sets · Bipolar-valued hesitant fuzzy graph · Directed bipolar-valued hesitant fuzzy graph · Domination degree

1 Introduction
Fuzzy set theory introduced by Klir and Yuan (1996) can be seen as a body of concepts that deal with the kinds of uncertainties which arise when the class boundary is not properly defined. A fuzzy set is basically represented by its membership function which is a generalization of the characteristic function. After the development of fuzzy sets, it has become the most popular research field in various disciplines. Several generalizations of fuzzy sets have been introduced in the field of research, for example, Zhang (1994) presented bipolar-valued fuzzy sets, as an extension of fuzzy sets whose membership lies within [−1, 1]. In the bipolar-valued fuzzy set, membership degree 0 indicates that the element has no relevance to the given condition, membership degree (0, 1] indicates the satisfaction degree of the element corresponding to some given properties, membership degree [−1, 0) indicates the degree of satisfaction of an element for some inherent inverse properties of the given condition. In this generalization, we assume that it is dealing with the degree of satisfaction of the elements that satisfy the corresponding property and some inherent counter property of the given condition, but there is no consideration for hesitation on this satisfaction degree. This generalization is still not more-convenient with real-world problems. Therefore, Mandal and Ranadive (2019) propose the concept of bipolar-valued hesitant fuzzy set, and it deals the situation in bipolar membership degree with hesitation index.

Graph theory originated in 1736 for Euler’s solution to the Königsberg bridge puzzle, provides a powerful tool for solving problems in various disciplines such as computer...
analysis, data structure, mathematical modeling, economics, engineering, social sciences, and statistics. In many cases, some aspects of the graph-theory problems are uncertain, for example, ranking of attributes, group decision-making problems, consistency of preference relation problems, etc. In order to solve such problems, Rosenfeld gave the concept of fuzzy graph, inspired by Kauffman’s (1973) fundamental ideas. Currently, a lot of work is being done in the research field using fuzzy graphs, such as, link prediction in social networks (Mahapatra et al. 2019, 2020), generalized neutrosophic planar graphs (Mahapatra et al. 2021), coloring of covid-19 affected regions (Mahapatra et al. 2021, 2020), radio fuzzy graphs (Mahapatra et al. 2019), and packaged food smart traceability and communication (Mahapatra et al. 2020). The concept of fuzzy homomorphism, fuzzy isomorphism, fuzzy co-weak isomorphism was given by Bhutani (1989). After the concept of fuzzy graphs, this area has been variously generalized. Based on the basic idea of Zhang (1994) bipolar fuzzy set theory, Akram (2011, 2013, 2016), Akram and Dudek (2012) proposed the notion of bipolar fuzzy graphs to deal with the bipolarity of real-world problems and gave some basic operations such as Cartesian product, composition and union of two bipolar fuzzy graphs. The book by Akram, Sarwar and Dudek (2021) entitled Graphs for the Analysis of Bipolar Fuzzy Information is an excellent source for research in fuzzy graphs. The features of bipolar fuzzy graph are very useful in the application of decision-making problems (Akram and Waseem 2018). More information on BFGs, irregular BFGs, and bipolar fuzzy hypergraphs can be found in Samanta and Pal (2012a), (b); Samanta et al. (2014); Rashmanlou et al. (2015); Akram and Akmal (2016); Alghamdi et al. (2018). In fuzzy graph theory, each vertex is characterized by membership degrees only, but this theory is not more convenient for modelling some real-life problems. Therefore, Karaaslan (2019) provided the definition of hesitant fuzzy graph whose basic idea was introduced by Torra (2010).

In this paper, we establish the concept of BVHFG with a different approach from BFG. Here we formulate the many real-world problems in a network with several bipolar-valued hesitant membership grades, more reliable with the decision process and many other problems. In this sequence, we also introduce some basic operations such as Cartesian product, strong product and union of two BVHFGs, we also investigate the concept of homomorphism, isomorphism, weak isomorphism, co-weak isomorphism of these graphs. In the final section, we illustrate by mean of an example, finding the most dominating person in the meeting and compare the usage of BVHFG and HFG.

The content of the paper is organized as follows. Section two reviews some basic knowledge on graphs, fuzzy graphs and BVHFSs. Section three refers to the definition and example of BVHFGs, then some common operational laws along with mapping relationships of BVHFGs and propositions are explored. Section four develops a two-stage algorithmic rule by virtue of BVHFGs, and then, a numerical instance concerning finding the most influential person is given. In section five, we tend to study the comparative analysis between BVHFG and HFG to demonstrate the relevancy and effectiveness of the conferred notion of BVHFG. At last, Section six concludes the total paper and points out the significance of BVHFG.

2 Preliminaries

In this segment, we cover some basic definitions based on graphs, digraphs, fuzzy graphs and bipolar-valued hesitant fuzzy sets (BVHFSs).

2.1 Basic definitions on graphs and fuzzy graphs

Definition 1 (Bollobás 2013) A graph is an ordered pair \( G = (V, E) \), where \( V \) is the set of vertices and \( E \) is the set of edges of \( G \) formed by a pair of vertices \( E \subset \{(x, y) | x, y \in V \} \).

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs, then the following operations are defined as:

- Cartesian product of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1 \times G_2 = \{V_1 \times V_2, E_1 \times E_2\} \) is defined as \( V_1 \times V_2 = \{(v_1, v_2) | v_1 \in V_1 \text{ and } v_2 \in V_2\} \) and \( E_1 \times E_2 = \{(a, a_2)(b_1, b_2) | a \in V_1, a_2 \in V_2 \} \cup \{(a_1, c)(b_1, c) | a_1 \in V_1, c \in V_2\} \).

- Strong product of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1 \otimes G_2 = (V_1 \otimes V_2, E_1 \otimes E_2) \) is defined as \( V_1 \otimes V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\} \) and \( E_1 \otimes E_2 = \{(a, a_2)(b_1, b_2) | a \in V_1, a_2 \in V_2 \} \cup \{(a_1, c)(b_1, c) | a_1 \in V_1, c \in V_2\} \cup \{(a_1, a_2)(b_1, b_2) | a_1 \in E_1, a_2 \in E_2\} \).

Definition 2 A graph \( G = (V, E) \) whose edges have the orientations and arrows on edges are used to represent the directions is called directed graph (digraph), denoted by \( \overrightarrow{G}_d = (V, \overrightarrow{E}) \).

Definition 3 (Rosenfeld 1975) A fuzzy graph \( G^* = (V, \mu, \nu) \) over the graph \( G = (V, E) \) is characterized by membership functions \( \mu : V \rightarrow [0, 1] \) and \( \nu : V \times V \rightarrow [0, 1] \) such that \( \nu(v_1, v_2) \leq \mu(v_1) \land \mu(v_2) \forall v_1, v_2 \in V \), where \( \mu(v_1) \) and \( \nu(v_1, v_2) \) represent the membership value of the vertex \( v_1 \) and edge \( v_1v_2 \) in \( G^* \), respectively.
Definition 4 (Mordeson and Nair 2012) Let $G^* = (V, \mu, \nu)$ be a fuzzy graph, then complimentary fuzzy graph $G^*$ of a fuzzy graph $G^*$ has the same vertices as $G^*$, two vertices are adjacent in $G^*$ if and only if they are not adjacent in $G^*$.

Definition 5 (Poulak and Ghorai 2020) Let $G^* = (V, \mu, \nu)$ be a fuzzy graph, then degree of vertex $v_i$ in a fuzzy graph $G^*$ is defined as $\text{deg}(v_i) = \sum_{v_j \neq v_i} \nu(v_i, v_j)$.

2.2 Bipolar-valued hesitant fuzzy set

Definition 6 (Mandal and Ranadive 2019). Let $X$ be frame of reference. A bipolar-valued hesitant fuzzy set (BVHFS) $A$ on $X$ is defined as:

$$A = \{< x, A(x) > | x \in X\},$$

where $A(x)$ is a set of some values in $[0, 1] \times [-1, 0]$. For convenience, we call $A(x)$ a bipolar-valued hesitant fuzzy element (BVHFE) expressed as:

$$A(x) = \{a_x | a_x \in [0, 1] \times [-1, 0]\},$$

here $a_x = (a^p_x, a^N_x)$ is a bipolar-valued fuzzy number (BVFN) such that $a^p_x \in [0, 1]$ and $a^N_x \in [-1, 0]$.

Definition 7 (Mandal and Ranadive 2019) Let $X$ be universe of discourse and for $x \in X$ let $A(x), A_1(x), A_2(x)$ be the BVHFE’s, then

- $A_1(x) \cup A_2(x) = \{(\max(a^p_{1x}, a^p_{2x}), \min(a^N_{1x}, a^N_{2x})) | a_{1x} \in A_1(x), a_{2x} \in A_2(x)\}$;
- $A_1(x) \cap A_2(x) = \{(\min(a^p_{1x}, a^p_{2x}), \max(a^N_{1x}, a^N_{2x})) | a_{1x} \in A_1(x), a_{2x} \in A_2(x)\}$;
- $A(x)^c = \{(1 - a^p_x, -1 - a^N_x) | a_x \in A(x)\}$.

Definition 8 (Mandal and Ranadive 2019) Let $a_x = (a^p_x, a^N_x) \in A(x)$ be a BVFN, and the value of score $s(a_x)$ is defined as:

$$s(a_x) = \frac{1}{2} (a^p_x - a^N_x),$$

which is the mean of the satisfaction degree corresponding to some given property of attribute and satisfaction degree to some implicit counter property of attribute of an element.

Definition 9 (Mandal and Ranadive 2019) Let $A(x)$ be a BVHFE, and the score function $s(A(x))$ is defined as:

$$s(A(x)) = \frac{1}{l(A(x))} \sum_{a_x \in A(x)} s(a_x),$$

where $l(A(x))$ denote the number of bipolar values in $A(x)$, and $a_x$ is the element in $A(x)$, taken as the form of BVFN.

Inspired by the basic idea of Deepak and John (2014), we define score-based intersection and union of two BVHFSs as follows:

Definition 10 Let $A$ and $B$ be two BVHFSs over $X$. Then, score-based intersection and union of two BVHFE’s $A(x)$ and $B(x)$ are represented by $A(x) \wedge B(x)$ and $A(x) \vee B(x)$, respectively, which is characterized by

$$A(x) \wedge B(x) = \begin{cases} A(x), & \text{if } s(A(x)) < s(B(x)), \\ B(x), & \text{if } s(B(x)) < s(A(x)), \\ A(x) \text{ or } B(x), & \text{if } s(A(x)) = s(B(x)), \end{cases}$$

and

$$A(x) \vee B(x) = \begin{cases} A(x), & \text{if } s(A(x)) > s(B(x)), \\ B(x), & \text{if } s(B(x)) > s(A(x)), \\ A(x) \text{ or } B(x), & \text{if } s(A(x)) = s(B(x)). \end{cases}$$

The following is a trivial consequence of our definition.

Proposition 1 Let $A$ and $B$ be two BVHFSs over the non-empty set $X$. Then, $s(A(x) \wedge B(x)) = s(A(x)) \land s(B(x))$ and $s(A(x) \vee B(x)) = s(A(x)) \lor s(B(x))$.

2.3 Bipolar-valued hesitant fuzzy relation

Definition 11 Let $A$ and $B$ be two BVHFSs over the non-empty set $X$. Then, the score-based Cartesian product of two BVHFSs $A$ and $B$ is represented by $A \times B$ and characterized by

$$A \times B = \{(x, y) | (A(x) \wedge B(y)) > (x, y) \in X \times X\},$$

$$= \{(x, y), (A(x) \times B(y)) > (x, y) \in X \times X\}.$$ 

Definition 12 Let $X$ be a non-empty set. Let $A$ and $B$ be two BVHFSs on $X$, for $x, y \in X$, let $A(x, y) : X \times X \rightarrow \mathbb{P}([0, 1] \times [-1, 0])$ be bipolar-valued hesitant fuzzy relation on $X$, and then, we call $A$ is score-based bipolar-valued hesitant fuzzy relation on $B$ if $s(A(x, y)) \leq s(B(x) \land s(B(y)))$ for all $x, y \in X$.

3 Bipolar-valued hesitant fuzzy graph

In this section, we define the notion of bipolar-valued hesitant fuzzy graph (BVHFG) using the score function, which is the mean of satisfaction degree of an element corresponding to some given attribute property and some implicit counter property of attribute. In addition, we give some examples, basic operations and define the isomorphisms on BVHFGs.
Definition 13 Let $G = (V,E)$ be a graph. A bipolar-valued hesitant fuzzy graph (BVHFG) with $V$ as a reference set is a pair $G^* = (A,B)$ where $A$ and $B$ are BVHFSs in $V$ and $V^2$ respectively, which is characterized by membership functions $A : V \rightarrow P([0,1] \times [-1,0])$ and $B : V^2 \rightarrow P([0,1] \times [-1,0])$ with the condition

$$s(B(xy)) \leq s(A(x)) \land s(A(y)) \quad \forall xy \in V^2$$

and $s(B(xy)) = 0 \forall xy \in (V^2 - E)$,

here $B(xy)$ and $A(x)$ are BVHFEs defined as:

$$B(xy) = \{(b^p_x, b^N_x) \mid (b^p_x, b^N_x) \in [0,1] \times [-1,0]\}$$

and

$$A(x) = \{(a^p_x, a^N_x) \mid (a^p_x, a^N_x) \in [0,1] \times [-1,0]\}.$$

Remark 1

- If we neglect the negative membership degrees of vertices in a network, which represents the satisfaction degrees to some implicit counter property of attribute, then BVHFG reduces to HFG (Karaaslan 2019).
- The concept of bipolar-valued hesitant fuzzy graph differs from the concept of bipolar fuzzy graph in its approach to the definition of 13.
- From definition 13, we can easily understand that the class of imprecise membership grades in BVHFG is greater than the class of imprecise membership grades in BFG. Membership grades of bipolar fuzzy graphs also satisfy the condition of BVHFG, but the converse is not always true. Here in Fig. 1 all membership grades satisfy the condition of BVHFG as in definition 13. However, for the edge between vertex $A$ and vertex $B$, the positive satisfaction degree $0.5 \leq \min(0.4, 0.8)$ and negative satisfaction degree $-0.4 \leq \max(-0.6, -0.3)$, i.e., it does not satisfy the condition of BFG. Hence, the class of membership grades of BVHFG is larger than the class of membership grades in BFG.

Example 1 Suppose there are six participants in the debate competition program and board members select the one participant as a winner according to the four functional properties, that is, influence power, conversation style, knowledge of the given topic, speech ingenuity. Instead of providing a traditional decision matrix, board members comprehensively evaluate the correlation of four properties between six participants. Let $V$ be a set of six participants $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_1v_4, v_2v_3, v_2v_5, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$ be the correlation of four properties among participants, then BVHFG and BVHFSs $A$ and $B$ over $V$ and $V^2$ are given in Fig. 2 and Table 1, respectively:

Definition 14 Let $G^* = (A,B)$ be a BVHFG over the graph $G = (V,E)$, score-based degree of a vertex $v_i \in V$ in BVHFG is denoted by $\mathcal{D}(v_i)$ and defined as $\mathcal{D}(v_i) = \sum_{v \neq v_i \in V} s(B(v_i,v))$.

Example 2 For Example 1, we calculate the score-based degree of each vertex in BVHFG and find that $\mathcal{D}(v_1) = 1.200$, $\mathcal{D}(v_2) = 1.233$, $\mathcal{D}(v_3) = 0.425$, $\mathcal{D}(v_4) = 1.275$, $\mathcal{D}(v_5) = 1.35$, $\mathcal{D}(v_6) = 0.892$.

Definition 15 Let $G^*_1 = (A_1,B_1)$ and $G^*_2 = (A_2,B_2)$ be two BVHFGs over the graph $G = (V,E)$ then we say that $G^*_1$ is score-based BVH- subgraph of $G^*_2$ if it satisfies the conditions

$$s(A_1(x)) \leq s(A_2(x)), \quad s(B_1(xy)) \leq s(B_2(xy))$$

$\forall x \in V, \forall xy \in V^2$.

3.1 Basic operations on BVHFGs

Let $G^*_1 = (A_1,B_1)$ and $G^*_2 = (A_2,B_2)$ be two BVHFGs over the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. We now give some operations and related results for BVHFGs:

1. Cartesian product: Cartesian product of two BVHFGs denoted by $G^*_1 \times G^*_2 = (A_1 \times A_2, B_1 \times B_2)$ is defined as:

![Fig. 1 BVHFG Vs BFG](image1)

![Fig. 2 Bipolar-valued hesitant fuzzy graph (BVHFG)](image2)
To abbreviate, let $\tilde{A}$ denote $\tilde{A}(x_1, x_2) = A_1(x_1) \tilde{A}_2(x_2) \forall (x_1, x_2) \in V_1 \times V_2$.

(1) Let $G_1^* \times G_2^*$ be two BVHFGs, then $G_1^* \times G_2^*$ is a BVHFG.

**Proof** For any $x \in V_1$ and $x_2 \in V_2$ we have,

\[
s((B_1 \times B_2)(x, x_2))(x_1, x_2) \subseteq \{ 0, 0.2, 0.4, 0.5, 0.6, 0.7, 0.8, 1 \}
\]

2. **Strong product**; Strong product of two BVHFGs denoted by $G_1^* \times G_2^* = (A_1 \otimes A_2, B_1 \otimes B_2)$ is defined as:

\[
\begin{align*}
\text{Let Proposition 2 } & \text{ Let } G_1^* \text{ and } G_2^* \text{ be two BVHFGs, then } G_1^* \otimes G_2^* \\
& \text{is a BVHFG.}\end{align*}
\]

**Proof** For any $(x, y) \in V_1 \times V_2$ we have,

\[
s((B_1 \otimes B_2)(x, y))(x_1, y_1) \subseteq \{ 0.2, -0.7, 0.4, -0.6 \}
\]

\[
\begin{align*}
\text{Let Proposition 3 } & \text{ Let } G_1^* \text{ and } G_2^* \text{ be two BVHFGs, then } G_1^* \otimes G_2^* \\
& \text{is a BVHFG.}\end{align*}
\]

**Proof** The proof of this proposition is similar as that of proposition 2 and hence, is omitted.

3. **Union**; Union of two BVHFGs denoted by $G_1^* \cup G_2^* = (A_1 \cup A_2, B_1 \cup B_2)$ s. t.

\[
s((B_1 \cup B_2)(x, y))(x_1, y_1) \subseteq \{ 0.2, -0.7, 0.4, -0.6 \}
\]

\[
\begin{align*}
\text{Let Proposition 4 } & \text{ Let } G_1^* \text{ and } G_2^* \text{ be two BVHFGs, then } G_1^* \cup G_2^* \\
& \text{is a BVHFG.}\end{align*}
\]

**Proof** For any $(x, y) \in E_1 \cup E_2$ we have,

\[
s((B_1 \cup B_2)(x, y))(x_1, y_1) \subseteq \{ 0.2, -0.7, 0.4, -0.6 \}
\]
3.2 Isomorphism between bipolar-valued hesitant fuzzy graphs

Definition 16 Let $G_1^*$ and $G_2^*$ be the BVHFGs. A homomorphism $f : G_1^* \rightarrow G_2^*$ is a mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) \leq s(A_2(f(x_1))) \forall x_1 \in V_1$,
- $s(B_1(x_1,y_1)) \leq s(B_2(f(x_1),f(y_1))) \forall x_1,y_1 \in V_2^2$.

Definition 17 Let $G_1^*$ and $G_2^*$ be the BVHFGs. An isomorphism $f : G_1^* \rightarrow G_2^*$ is a bijective mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) = s(A_2(f(x_1))) \forall x_1 \in V_1$,
- $s(B_1(x_1,y_1)) = s(B_2(f(x_1),f(y_1))) \forall x_1,y_1 \in V_2^2$.

Definition 18 Let $G_1^*$ and $G_2^*$ be the BVHFGs. Then, a weak isomorphism $\tilde{f} : G_1^* \rightarrow G_2^*$ is a bijective mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) = s(A_2(f(x_1))) \forall x_1 \in V_1$,
- $s(B_1(x_1,y_1)) = s(B_2(f(x_1),f(y_1))) \forall x_1,y_1 \in V_2^2$.

Definition 19 Let $G_1^*$ and $G_2^*$ be the BVHFGs. A co-weak isomorphism $\tilde{f} : G_1^* \rightarrow G_2^*$ is a bijective mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) \leq (A_2(f(x_1))) \forall x_1 \in V_1$,
- $s(B_1(x_1,y_1)) = s(B_2(f(x_1),f(y_1))) \forall x_1,y_1 \in V_2^2$.

Remark 2

- If $G_1^* = G_2^* = G^*$, then the homomorphism $f$ over itself is called an endomorphism. An isomorphism $f$ over $G$ is called an automorphism.
- Let $G^*$ be a BVHFG of a graph $G$. Let $Aut(G^*)$ be the set of all bipolar-valued hesitant automorphism of $G^*$. Let $e : G^* \rightarrow G^*$ be a map defined by $e(x) = x \forall x \in V$. Clearly $e \in Aut(G)$.

Proposition 5 If $G_1^*$, $G_2^*$, $G_3^*$ are BVHFGs, then the isomorphism between these graphs is an equivalence relation.

Proof For reflexivity we can use identity mapping between BVHFGs, and it is trivial. For symmetry, we assume a function $h : V_1 \rightarrow V_2$ is an isomorphism on $G_1^*$ onto $G_2^*$ such that $h(m_1) = m_2 \forall m_1 \in V_1$ with conditions

$$s(A_1(m_1)) = s(A_2(h(m_1))), \quad s(B_1(m_1,n_1)) = s(B_2(h(m_1),h(n_1))) \forall m_1 \in V_1, \forall m_1,n_1 \in V_1^2;$$

Further, since $h$ is isomorphism, we have

$$h^{-1}(m_2) = m_1 \forall m_2 \in V_2,$$ satisfizes condition (1), we have

$$s(A_1(h^{-1}(m_2))) = s(A_2(m_2)), \quad s(B_1(h^{-1}(m_2),h^{-1}(n_2))) = s(B_2(m_2,n_2)) \forall m_2 \in V_2, \forall m_2,n_2 \in V_2^2.$$

Thus, a mapping $h^{-1} : V_2 \rightarrow V_1$ is an isomorphism from $G_2$ onto $G_1$. For transitivity, we assume $h_1 : V_1 \rightarrow V_2$ such that $h_1(m_1) = m_2 \forall m_1 \in V_1$, and $h_2 : V_2 \rightarrow V_3$ such that $h_2(m_2) = m_3 \forall m_2 \in V_2$ are isomorphisms between $G_1$ onto $G_2$ and $G_2$ onto $G_3$, respectively. Further $h_2oh_1 : V_1 \rightarrow V_3$ is a composition of $h_1$ and $h_2$ such that

$$h_2oh_1(m_1) = h_2(h_1(m_1)) \forall m_1 \in V_1.$$

Since the map $h_1 : V_1 \rightarrow V_2$ is an isomorphism, we have

$$s(A_1(m_1)) = s(A_2(h_1(m_1))) = s(A_2(m_2)) \forall m_1 \in V_1;$$

$$s(B_1(m_1,n_1)) = s(B_2(h_1(m_1),h(n_1))) = s(B_2(m_2,n_2)) \forall m_1,n_1 \in V_1^2.$$ Again since the map $h_2 : V_2 \rightarrow V_3$ is an isomorphism, we have

$$s(A_2(m_2)) = s(A_3(h_2(m_2))) = s(A_3(m_3)) \forall m_2 \in V_2;$$

$$s(B_2(m_2,n_2)) = s(B_3(h_2(m_2),h(n_2))) = s(B_3(m_3,n_3)) \forall m_2,n_2 \in V_2^2.$$ From expressions (2) and (4), we have

$$s(A_3(m_3)) = s(A_3(h_2(m_3))) = s(A_3(m_3)) \forall m_3 \in V_3.$$ From expressions (3) and (5), we have

$$s(A_3(h_2oh_1(m_3))) \forall m_3 \in V_3.$$
$s(B_1(m_1n_1)) = s(B_2(h_1(m_1)h_1(n_1)))$
$= s(B_2(m_2n_2)) = s(B_3(h_2(m_2)h_2(n_2)))$
$= s(B_3(h_2(h_1(m_1))h_2(h_1(n_1))))$
$= s(B_3((h_2o_1h_1)(m_1)(h_2o_1h_1)(n_1))) \forall m_1n_1 \in \bar{V}_2$.

Hence $h_2o_1h_1$ is an isomorphism between $G_1^*$ and $G_3^*$.

**Proposition 6** If $G_1^*$, $G_2^*$, $G_3^*$ are BVHFGs, then the weak isomorphism between these graphs is an partial order relation.

**Proof** Reflexivity is trivial. For anti-symmetry we assume a function $h : V_1 \rightarrow V_2$ is a weak isomorphism on $G_1^*$ onto $G_2^*$ such that $h_1(m_1) = m_2 \ \forall m_1 \in V_1$ with conditions

$$s(A_1(m_1)) = s(A_2(h_1(m_1)))$$

$$s(B_1(m_1n_1)) \leq s(B_2(h_1(m_1)h_1(n_1))) \forall m_1 \in V_1, \forall m_1n_1 \in \bar{V}_2;$$

suppose $h_2 : V_2 \rightarrow V_1$ is a weak isomorphism between $G_1^*$ and $G_2^*$ such that $h_2(a_2) = a_1 \ \forall a_2 \in V_2$ with condition $s(A_1(a_2)) = s(A_2(h_2(a_2))), s(B_2(a_2b_2))$

$$\leq s(B_2(h_2(a_2)h_2(b_2))) \forall a_2 \in V_2,$$

$$\forall a_2b_2 \in \bar{V}_1^2.$$ From expressions (6) and (7), we conclude that both these inequalities hold if and only if the BVHFGs have the same number of edges and corresponding edges have the same weights, which shows that $G_1^*$ and $G_2^*$ are similar. Furthermore, we can show that the transitivity among the graphs $G_1^*, G_2^*$ and $G_3^*$ is the same as previous proposition.

**Proposition 7** If $G^* = (A, B)$ is a BVHFG and $Aut(G^*)$ is the set of all automorphisms of $G^*$. Then, $(Aut(G^*), o)$ forms a group.

**Proof** For any $\phi, \psi \in Aut(G^*)$ and $x, y \in V$. Then, we have

$$s(B((\phi \psi)(x))(\phi \psi)(y)))$$

$$= s(B(\phi(\psi(x)))(\phi(\psi(y))))$$

$$= s(B(\psi(x))(\psi(y)))$$

$$= s(B(xy)),$$

$$s(A((\phi \psi)(x))) = s(A(\phi(\psi(x))))$$

$$= s(A(\psi(x)))$$

$$= s(A(x)),$$

clearly $\phi \psi \in Aut(G^*)$. Also $Aut(G^*)$ satisfies associative law under the mapping composition, let $I : G^* \rightarrow G^*$ be an identity mapping such that $\phi I = I \phi = \phi \forall \phi \in Aut(G^*)$, also for each $\phi \in Aut(G^*)$ there exists $\phi^{-1} \in G^*$ such

$$t h a t \ s(A(\phi^{-1}(x))) = s(A(\phi(\phi^{-1}(x)))) = s(A(x)) ,$$

$$s(B(\phi^{-1}(x)\phi^{-1}(y))) = s(B(\phi(\phi^{-1}(x))\phi(\phi^{-1}(y)))) = s(B(xy)).$$

Hence, $(Aut(G^*), o)$ forms a group.

### 4 Directed-BVHFGs and their application in decision making

In this section, we present the definition of directed-BVHFGs and develop a method for finding in-degrees and out-degrees of vertices in directed-BVHFGs with the help of bipolar-valued hesitant fuzzy weighted averaging (BVHFWA) operator. Along with this, we have given a numerical example to find the domination degree of people in a meeting based on directed graph.

**Definition 20** A directed-BVHFG $G^*_d = (A, \tilde{B})$ of the graph $G = (V, E)$ is given by the pair of membership functions $A : V \rightarrow \mathcal{P}([0, 1] \times [-1, 0])$ and $\tilde{B} : V \rightarrow \mathcal{P}([0, 1] \times [-1, 0])$ with condition

$$s(\tilde{B}(xy)) \leq s(A(x)) \wedge s(A(y)) \forall xy \in \bar{V}^2$$

$$and \ s(\tilde{B}(xy)) = 0 \ \forall xy \in (\bar{V}^2 - E).$$

**Definition 21** Let $X$ be universe of discourse and $\{A_i(x)| x \in X, i = 1, 2 \ldots n\}$ be a collection of BVHFE's and $w = (w_1, w_2 \ldots w_n)^T$ be the weight vector of $A_i(x) = (a_{i1}^0, a_{i1}^1, a_{i1}^2)(i = 1, 2 \ldots n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, bipolar-valued hesitant fuzzy weighted averaging (BVFWA) operator is a mapping $BVFWA : A^n \rightarrow A$, where

$$BVFWA(A_1(x), A_2(x) \ldots A_n(x)) = \Theta^n_w(\hat{A}_i(x))$$

$$= \left\{ 1 - \frac{1}{n} \frac{1}{n} \ldots \frac{1}{n} \right\}, \text{if } w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T,$$

here if $w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T$, then BVFWA operator will become the bipolar-valued hesitant fuzzy weighted averaging (BVFWA) operator. Equation (8) can be written as:

$$BVFWA(A_1(x), A_2(x) \ldots A_n(x)) = \Theta^n_w(\frac{1}{n} A_i(x))$$

$$= \left\{ 1 - \frac{1}{n} \frac{1}{n} \ldots \frac{1}{n} \right\}, \text{if } w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T,$$

**Definition 22** Let $G^*_d = (A, \tilde{B})$ be a directed-BVHFG over the graph $G = (V, E)$ and $v_r, r = 1, 2, \ldots n$ be adjacent BVH-vertices of $v_k \in V$. With the help of Eq. (9), we
define out-degree and in-degree of a vertex \( v_k \) denoted by \( O_d(v_k) \) and \( I_d(v_k) \), respectively, and defined:

\[
O_d(v_k) = \left\{ 1 - \prod_{i=1}^{n} (1 - b_{v_i,v_k}^P), -1 + \prod_{i=1}^{n} (1 + b_{v_i,v_k}^N) \right\} |v_k \in B \},
\]

(10)

\[
I_d(v_k) = \left\{ 1 - \prod_{i=1}^{n} (1 - b_{v_i,v_k}^N), -1 + \prod_{i=1}^{n} (1 + b_{v_i,v_k}^P) \right\} |v_k \in B \}.
\]

(11)

After finding the in-degree and out-degree of each vertex, we represent its score value by \( s(O_d(v)) \) and \( s(I_d(v)) \), respectively, and calculate it from definition 9.

To find the domination degree of each vertex \( v_k \), we use \( s(O_d(v_k)) - s(I_d(v_k)) \) where \( s(O_d(v_k)) \) and \( s(I_d(v_k)) \) represent the score value of out-degree and in-degree of vertex \( v_k \), respectively, and denoted by \( \omega(v_k) \).

**Algorithm 1:** Based on directed BVHFGs

**Input:** A directed bipolar-valued hesitant fuzzy graph which describes the correlation among elements.

**Output:** Most dominating element.

1. Input the BVHFS defined on the set \( V \).
2. Input the edge set represents correlation among elements.
3. Each edge represented by BVHF membership degrees with the condition

\[
s(B(xy)) \leq s(A(x)) \land s(A(y))
\]

4. By using the Eqn. (10) and Eqn. (11) calculate the in-degree and out-degree of each element.
5. Find the score value of in-degree and out-degree of each element.
6. Calculate the domination degree of elements.
7. Ranking of elements according to their domination degree.
8. Pick the most dominating element.

### 4.1 Numerical example

We cannot measure the impact power of person’s properly, so we are always hesitant to evaluate the impact power of person’s. Apart from this, if we do not know about a person properly, then we can also have a negative impact on him. In this part, we present a directed bipolar-valued hesitant fuzzy graph for such case.

Let us consider the directed-BVHFG of potential power of seven person’s \( P = \{ p_1, p_2 \ldots p_7 \} \) in a business meeting. The potential power of person’s determined in positive and negative membership degrees, positive hesitant membership degree denotes satisfaction degree of person corresponding to potential power, while negative hesitant membership degree denotes satisfaction degree of person to some implicit counter property of potential power.

\[
\{(3,-2),(4,-3),(5,-1),(8,-1)\} \}
\]

be the set of bipolar-valued directed hesitant edges as in Table 3, which demonstrates the impact power of one person onto another person in a meeting as in Fig. 3.

**Step 3:** With the help of Eqs. (10) and (11), we find the out-degree and in-degree of each person as in Table 4.

**Step 4:** Now we calculate the score value of in-degree and out-degree of each person by with the help of definition 8 and definition 9 as in Table 5.

**Step 5:** Finally, we calculate the domination degree using (Karaaslan 2019) of each person in the meeting as in Table 6.

**Step 6:** The ranking of each person present in the meeting based on their domination degree is as follows:

\[
P_2 > P_7 > P_4 > P_3 > P_6 > P_5.
\]
it is clear that the most dominating person in the meeting is $P_2$.

**5 Comparative analysis**

In this section, we evaluate the importance of HFG and BVHFG in decision making methods.

Following Karaaslan (2019), the hesitant fuzzy graph (HFG) $\tilde{G} = (\mathcal{A}, \tilde{G})$ of the graph $G = (V, E)$ is characterize by membership functions $\mathcal{A} : V \to P([0, 1])$ and $\tilde{G} : \tilde{V}^2 \to P([0, 1])$ with the condition

\[ \delta(\tilde{G}(xy)) \leq \delta(\mathcal{A}(x)) \land \delta(\mathcal{A}(y)) \forall xy \in \tilde{V}^2 \]

and $\delta(\tilde{G}(xy)) = 0 \forall xy \in (\tilde{V}^2 - E)$.
Based on the potential power of individuals, we establish the hesitant degree of impact power of one person to another person; here, the directed edge \( P_iP_j \) represents impact power of \( P_i \) on \( P_j \) as in Table 8.

Following Karaaslan (2019), we calculate the out-degree and in-degree of each person, by using the equations

\[
O_d(v_k) = \{(1 - \prod_{i=1}^{n}(1 - a_{v_iv_i})) | v_i \in A \}
\]

\[
I_d(v_k) = \{(1 - \prod_{i=1}^{n}(1 - a_{v_iv_i})) | v_i \in B \}
\]

which is given in Table 9.

Score value of in-degree and out-degree of each person is given in Table 10.

Now we evaluate the domination degree of each person in this meeting is ranked as follows:

\[ P_7 > P_2 > P_4 > P_1 > P_6 > P_5 > P_3, \]

clearly the person \( P_7 \) is the most dominating person in the meeting.

When the results of HFG and BVHFG are examined, we realize that the domination degree and ranking of dominating persons change significantly in two cases. In the BVHFG, \( P_2 \) is the most dominating person in the meeting, while in HFG the person \( P_7 \) is the most dominating person in the meeting. Furthermore, when we examine the ranking of persons in two cases, significant difference between two results is observed.

The main reason for this difference is the capability of BVHFG, and it is simultaneously considering the positive and negative membership degree with no restriction while the HFG considering only positive membership values.

Table 4 Out-degree and in-degree of each person

| Person | Out-degree | In-degree |
|--------|------------|-----------|
| \( P_1 \) | \{(0.79, -0.832), (0.8, -0.72), (0.832, -0.79)\} | \{(0.865, -0.664), (0.832, -0.784)\} |
| \( P_2 \) | \{(0.865, -0.664), (0.832, -0.784)\} | \{(0.45, -0.8), (0.55, -0.95)\} |
| \( P_3 \) | \{(0.496, -0.72)\} | \{(0.784, -0.496), (0.7, -0.52)\} |
| \( P_4 \) | \{(0.916, -0.546), (0.28, -0.55), (0.904, -0.496), (0.6, -0.44)\} | \{(0.958, -0.546)\} |
| \( P_5 \) | \{(0.958, -0.546)\} | \{(0.865, -0.664)\} |
| \( P_6 \) | \{(0.7, -0.52), (0.6, -0.44)\} | \{(0.79, -0.832), (0.832, -0.784), (0.916, -0.546)\} |
| \( P_7 \) | \{(0.55, -0.95), (0.784, -0.496), (0.28, -0.55)\} | \{(0.958, -0.664)\} |
| \( P_8 \) | \{(0.8, -0.72)\} | \{(0.832, -0.79), (0.45, -0.8), (0.496, -0.72), (0.904, -0.496)\} |
### Table 5 Score value of in-degree and out-degree

| Persons | $I_o - O_o$ | Dominant degree |
|---------|-------------|-----------------|
| $\alpha(P_1)$ | 0.784–0.764 | 0.030 |
| $\alpha(P_2)$ | 0.786–0.566 | 0.220 |
| $\alpha(P_3)$ | 0.687–0.783 | -0.096 |
| $\alpha(P_4)$ | 0.608–0.602 | 0.006 |
| $\alpha(P_5)$ | 0.625–0.811 | -0.186 |
| $\alpha(P_6)$ | 0.591–0.760 | -0.169 |
| $\alpha(P_7)$ | 0.752–0.686 | 0.066 |

### Table 6 Dominant degree of each person

| Persons | $I_o - O_o$ | Dominant degree |
|---------|-------------|-----------------|
| $\alpha(P_1)$ | 0.784–0.764 | 0.030 |
| $\alpha(P_2)$ | 0.786–0.566 | 0.220 |
| $\alpha(P_3)$ | 0.687–0.783 | -0.096 |
| $\alpha(P_4)$ | 0.608–0.602 | 0.006 |
| $\alpha(P_5)$ | 0.625–0.811 | -0.186 |
| $\alpha(P_6)$ | 0.591–0.760 | -0.169 |
| $\alpha(P_7)$ | 0.752–0.686 | 0.066 |

### Table 7 Hesitant fuzzy membership of each person

| Person | Hesitant fuzzy membership | Score value |
|--------|---------------------------|-------------|
| $P_1$ | {0.4, 0.5, 0.9} | 0.6 |
| $P_2$ | {0.3, 0.6} | 0.45 |
| $P_3$ | {0.1, 0.5, 0.6, 0.7} | 0.457 |
| $P_4$ | {0.35, 0.40} | 0.357 |
| $P_5$ | {0.45, 0.8, 0.75} | 0.667 |
| $P_6$ | {0.6, 0.9} | 0.75 |
| $P_7$ | {0.2, 0.55, 0.8} | 0.517 |

### Table 8 Influence power of one person onto another

| Edge | Hesitant fuzzy membership | Score value |
|------|---------------------------|-------------|
| $P_1 P_3$ | {0.3, 0.4, 0.5} | 0.4 |
| $P_1 P_6$ | {0.5, 0.6} | 0.55 |
| $P_1 P_7$ | {0.2, 0.3, 0.7} | 0.4 |
| $P_1 P_4$ | {0.1, 0.5, 0.7} | 0.433 |
| $P_1 P_3$ | {0.2, 0.3, 0.4, 0.5} | 0.350 |
| $P_1 P_4$ | {0.45} | 0.450 |
| $P_1 P_7$ | {0.1, 0.5} | 0.300 |
| $P_1 P_7$ | {0.1, 0.2, 0.3} | 0.200 |
| $P_1 P_4$ | {0.1, 0.4, 0.6} | 0.367 |
| $P_1 P_2$ | {0.4, 0.5} | 0.450 |
| $P_1 P_3$ | {0.2, 0.3, 0.5, 0.7} | 0.425 |
| $P_1 P_4$ | {0.1, 0.2} | 0.15 |
| $P_1 P_2$ | {0.2, 0.6, 0.7} | 0.500 |
| $P_1 P_7$ | {0.2, 0.5} | 0.350 |
| $P_1 P_3$ | {0.3, 0.4, 0.5, 0.8} | 0.500 |

### 6 Conclusion

In this paper, we have introduced the concept of BVHFG, some operations and propositions related to BVHFGs, such as Cartesian product, strong product, union, homomorphism, and isomorphism between BVHFGs. We have also investigated the concept of directed-BVHFG, which is a more helpful tool to express the various decision making problems. Furthermore, we have proposed a problem to find most dominating person in a meeting through directed-BVHFG. In the process of solving the problem, we construct a way to find the in-degree and out-degree of vertices, and then, we find out the score value of in-degree and out-degree of vertices, and then we calculate domination degree of persons.

In the final section, we point to comparative analyses between BVHFG and HFG. When we compare the outcome of hesitant fuzzy graph and bipolar-valued hesitant fuzzy graph, we notice the valuable difference between these two results. From the numerical illustrations, we conclude that bipolar-valued hesitant fuzzy graph provides more accurate results.
Table 9  Out-degree and in-degree of each person

| P_i | O_d(P_i) | I_d(P_i) |
|-----|----------|----------|
| 1   | [0.79, 0.8, 0.832] | [0.865] |
| 2   | [0.865, 0.832] | [0.7, 0.6] |
| 3   | [0.45, 0.55] | [0.79, 0.832, 0.916] |
| 4   | [0.496] | [0.55, 0.784, 0.28] |
| 5   | [0.784, 0.7] | [0.958] |
| 6   | [0.916, 0.28, 0.904, 0.6] | [0.8] |
| 7   | [0.958] | [0.832, 0.45, 0.496, 0.904] |

Table 10  Score value of out-degree and in-degree

| P_i | δ(O_d(P_i)) | δ(I_d(P_i)) |
|-----|-------------|-------------|
| 1   | 0.807       | 0.865       |
| 2   | 0.849       | 0.650       |
| 3   | 0.500       | 0.846       |
| 4   | 0.496       | 0.538       |
| 5   | 0.742       | 0.958       |
| 6   | 0.675       | 0.800       |
| 7   | 0.958       | 0.671       |

Table 11  Dominant degree of each person

| P_i | α(P_i) |
|-----|--------|
| 1   | 0.807–0.865, -0.058 |
| 2   | 0.849–0.650, 0.199 |
| 3   | 0.500–0.846, -0.034 |
| 4   | 0.807–0.865, -0.042 |
| 5   | 0.807–0.865, -0.216 |
| 6   | 0.807–0.865, -0.125 |
| 7   | 0.807–0.865, 0.287 |

References

Akram M (2011) Bipolar fuzzy graphs. Inf Sci 181(24):5548–5564
Akram M (2013) Bipolar fuzzy graphs with applications. Knowl-Based Syst 39:1–8
Akram M, Dudek WA (2012) Regular bipolar fuzzy graphs. Neural Comput Appl 21(1):197–205
Akram M, Akmal R (2016) Application of bipolar fuzzy sets in graph structures. Appl Comput Intell Soft Comput
Akram M, Waseem N (2018) Novel applications of bipolar fuzzy graphs to decision making problems. J Appl Math Comput 56(1):73–91
Akram M, Sarwar M, Dudek WA (2021) Graphs for the analysis of bipolar fuzzy information, vol 401. Springer, Berlin
Akram M, Alshehri N, Davbaz A, Ashraf A (2016) Bipolar fuzzy digraphs in decision support systems. J Multip Val Log Soft Comput 27(5–6):553–572
Alghamdi M, Alshehri NO, Akram M (2018) Multi-criteria decision-making methods in bipolar fuzzy environment. Int J Fuzzy Syst 20(6):2057–2064
Bhutani KR (1989) On automorphisms of fuzzy graphs. Pattern Recogn Lett 9(3):159–162
Bollabás B (2013), Modern graph theory, Vol. 184, Springer Science & Business Media, Berlin
Deepak D, John SJ (2014) Homomorphisms of hesitant fuzzy subgroups. Int J Sci Eng Res 59:9–14
Karaaslan F (2019) Hesitant fuzzy graphs and their applications in decision making. J Intell Fuzzy Syst 36(3):2729–2741
Kauffman A (1973) Introduction a La Theorie Des Sous-ensembles Flous, Masson et Cie Editures

Klir GJ, Yuan B (1996) Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh, vol 6. World Scientific, Singapore
Mahapatra R, Samanta S, Allahviranloo T, Pal M (2019) Radio fuzzy graphs and assignment of frequencies in radio stations. Comput Appl Math 38(3):1–20. https://doi.org/10.1007/s40314-019-0888-3
Mahapatra R, Samanta S, Pal M, Xin Q (2019) Rsm index: a new way of link prediction in social networks. J Intell Fuzzy Syst 37(2):2137–2151. https://doi.org/10.3233/JIFS-181452
Mahapatra R, Samanta S, Bhadoria RS, Pal M, Allahviranloo T, Pandey B (2020) A graph networks based quality control model for packaged food smart traceability & communication. Eur J Mol Clin Med 7(6):2830–2848
Mahapatra R, Samanta S, Pal M, Xin Q (2020) Link prediction in social networks by neusroscopic graph. Int J Comput Intell Syst 13:1699–1713
Mahapatra R, Samanta S, Pal M (2020) Applications of edge colouring of fuzzy graphs. Informatica 31(2):313–330
Mahapatra R, Samanta S, Pal M (2021) Generalized neusroscopic planar graphs and its application. J Appl Math Comput 65:693–712
Mahapatra R, Samanta S, Pal M, Lee J-G, Khan SK, Naseem U, Bhadoria RS (2021) Colouring of covid-19 affected region based on fuzzy directed graphs. Comput Mater Continua 68(1):1219–1233
Mandal P, Ranadive AS (2019) Hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets and their applications in multi-attribute group decision making. Granular Comput 4(3):559–583
Mordeson JN, Nair PS (2012) Fuzzy graphs and fuzzy hypergraphs. Physica Vol 46
Poulik S, Ghorai G (2020) Note on “bipolar Fuzzy Graphs with Applications”. Knowl-Based Syst 192:105315
Rashmanlou H, Samanta S, Pal M, Borzooeei RA (2015) A study on bipolar fuzzy graphs. J Intell Fuzzy Syst 28(2):571–580
Rosenfeld A (1975), Fuzzy graphs. In: Fuzzy sets and their applications to cognitive and decision processes, Elsevier, pp 77–95
Samanta S, Pal M (2012a) Bipolar fuzzy hypergraphs. Int J Fuzzy Log Syst 2(1):17–28
Samanta S, Pal M (2012b) Irregular bipolar fuzzy graphs, arXiv preprint arXiv:1209.1682
Samanta S, Pal M (2014) Some more results on bipolar fuzzy information aggregation in decision making. Int J Approx Reason 52(3):395–407
Zhang WR (1994) Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In: NAFIPS/IFIS/NASA’94. Proceedings of the first international joint conference of the North American fuzzy information processing society biannual conference. The Industrial Fuzzy Control and Intellige, IEEE, pp 305–309

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