CONVEX ORDERING OF PÓLYA RANDOM VARIABLES AND MONOTONICITY OF THE ERROR ESTIMATE OF BERNSTEIN-STANCU OPERATORS

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ABSTRACT. In the present paper we show that in Pólya’s urn model, for an arbitrarily fixed initial distribution of the urn, the corresponding random variables satisfy a convex ordering with respect to the replacement parameter. As an application, we show that in the class of convex functions, the absolute value of the error of Bernstein-Stancu operators is a non-decreasing (strictly increasing under an additional hypothesis) function of the corresponding parameter.

The proof relies on two results of independent interest: an interlacing lemma of three sets and the monotonicity of the (partial) first moment of Pólya random variables with respect to the replacement parameter.

1. INTRODUCTION

More than 100 years ago, in a beautiful and short paper Serge Bernstein ([1]) gave a simple, constructive proof of Weierstrass’s theorem on uniform approximation of continuous functions by polynomials, known nowadays as Bernstein polynomials.

About 50 years later, D. D. Stancu noticed that the binomial distribution used by Bernstein is a particular case of the Pólya’s urn distribution (the case of the replacement parameter being equal to zero), and he introduced ([6], [7]) a more general class of polynomials/operators, known in the literature as the Pólya-Stancu operators (the operator $P^c_n$ defined by (2.7)).

Aside from a passing remark that for a particular choice (negative real number) of the replacement parameter $c$ one obtains the Lagrange interpolation polynomial (which cannot be used for uniform approximation), in his work Stancu considered only non-negative values $c \geq 0$ of the replacement parameter in Pólya’s urn model, this choice being adopted by subsequent researchers in the field.

Recently (in [3], see also [4] and [9]), the last two authors introduced the operator $R_n$ (given by (2.8)), corresponding to a negative choice (pointwise minimal) of the replacement parameter of Pólya’s urn distribution, and showed that this leads to better approximation results. To be precise, we showed that the upper bounds of the error estimates (in terms of the first/second order modulus of continuity of the function, etc) are smaller than the corresponding estimates for the Bernstein operator, and we also provided numerical evidence (for various choices of the function, smooth, continuous, even discontinuous - see [3]) which indicated that among all Bernstein-Stancu type operators, the newly introduced operator $R_n$ gives the best approximation.

A criticism received while publishing these results was that even though the upper bounds for the error of the new operator are smaller than the corresponding ones for the Bernstein operator, this is not a proof that the new operator is a better approximation operator.

In the present paper, we fill this gap (at least partially), by showing that for a sufficiently large class of functions (convex functions) the absolute value of the error of approximation of Bernstein-Stancu type operators $P^c_n$ is (pointwise) a monotonically increasing function of the replacement parameter $c$. This shows that in the case of operators considered by Stancu (non-negative replacement parameter), the Bernstein operator $B_n$ gives the best approximation of convex functions, and that the choice of the operator $R_n$ considered by the last two authors (minimal admissible choice of the replacement parameter) further improves this approximation, giving the best approximation of convex functions.

The proof of the main result (Theorem 4.1) relies on a result of independent interest (Theorem 3.3), which shows that the Pólya random variables satisfy a convex ordering with respect to the replacement parameter.

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parameter. In turn, the proof of this result relies on two other results with interest on their own: a result concerning the ordering (interlacing) of three sets (Lemma 3.1) and the monotonicity of the partial centered moment of the Pólya distribution (Lemma 3.2).

The structure of the paper is the following. In Section 2 we introduce the notation and the main results needed in the sequel.

In Section 3 we prove the convex ordering of Pólya random variables, and, as an application, in Section 4 we prove that in the case of convex functions, the absolute value of the error of approximation of the Bernstein-Stancu operators is a non-decreasing function of the replacement parameter (strictly increasing under an additional hypothesis).

2. Preliminary results

Recall that for given integer parameters \(a, b \geq 0\), \(c\), and \(n \geq 1\) satisfying the compatibility condition
\[
a + (n - 1) c \geq 0 \quad \text{and} \quad b + (n - 1) c \geq 0,
\]
Pólya’s urn model gives the number of white balls ("successes") extracted in \(n\) trials from an urn containing initially \(a\) white balls and \(b\) black balls, where after each extraction, the extracted ball is replaced in the urn together with \(c\) balls of the same color. Denoting by \(X_{n}^{a,b,c}\) the random variable representing the number of successes in this experiment (which will be referred to as a Pólya random variable with parameters \(a, b, c\) and \(n\)), we have
\[
P_{n,k}^{a,b,c} = P\left(X_{n}^{a,b,c} = k\right) = \frac{C_{n}^{a} a^{(k,c)} b^{(n-k,c)}}{(a+b)^{(n,c)}}, \quad k \in \{0, 1, \ldots, n\},
\]
where for \(x, h \in \mathbb{R}\) and \(n \in \mathbb{N}\) we denoted by
\[
x^{(n,h)} = x (x + h) (x + 2h) \cdots (x + (n - 1) h)
\]
the generalized (rising) factorial with increment \(h\). We are using the convention that an empty product is equal to 1, that is \(x^{(0,h)} = 1\) for any \(x, h \in \mathbb{R}\). The binomial theorem for the rising factorial shows that (2.2) still defines a distribution \(X_{n}^{a,b,c}\) for real values of the parameters \(a, b \geq 0\) and \(c\), satisfying the compatibility condition (2.1).

It is known (see [2]) that the partial first absolute centered moment of the Pólya distribution \(X_{n}^{x,1-x,c}\) is given by
\[
E\left(\left(nx - X_{n}^{x,1-x,c}\right) 1_{X_{n}^{x,1-x,c} \leq s-1}\right) = \sum_{k=0}^{s-1} (nx - k) p_{n,k}^{x,1-x,c} = s p_{n,s}^{x,1-x,c} (1 - x + (n - s) c)
\]
for all \(s \in \{1, \ldots, n\}\).

Denoting by \(\mathcal{F}([0, 1])\) the set of real-valued functions defined on \([0, 1]\), for an integer \(n \geq 1\) denote by \(P_{n}^{a,b,c} : \mathcal{F}([0, 1]) \to \mathcal{F}([0, 1])\) the operator defined by
\[
P_{n}^{a,b,c}(f; x) = E\left(\frac{1}{n} X_{n}^{a,b,c}\right) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{a,b,c}
\]
where the parameters \(a, b \geq 0\) and \(c \geq -\min\{x, 1-x\}/n-1\) may depend on \(n\) and \(x \in [0, 1]\) and satisfy the compatibility condition (2.1) (see [3]). In particular, consider the Bernstein operator
\[
B_{n}(f; x) = P_{n}^{x,1-x,0}(f; x),
\]
the Bernstein-Stancu operator
\[
P_{n}^{c}(f; x) = P_{n}^{x,1-x,c}(f; x),
\]
and the operator \(R_{n}\) introduced by the last two authors in [3]
\[
R_{n}(f; x) = P_{n}^{x,1-x,-\min\{x, 1-x\}/(n-1)},
\]
which corresponds to the minimal value of the replacement parameter \(c\) (satisfying the compatibility condition (2.1)) in the case \(a = x\) and \(b = 1 - x\).
Finally, recall that a random variable $X$ is said to be smaller in the convex order than a random variable $Y$ (denoted by $X \leq_{cx} Y$) iff

$$E(\phi(X)) \leq E(\phi(Y))$$

for any convex function $\phi : \mathbb{R} \to \mathbb{R}$ for which the above expectations exist.

If $X$ and $Y$ are random variables for which the means $EX$, $EY$ exist and are equal, it is known (see e.g. [5], Theorem 3.1.1) that $X \leq_{cx} Y$ iff

$$E((t - X)_+) \leq E((t - Y)_+)$$

for all $t \in \mathbb{R}$, where $x_+ = \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$.

### 3. Main Results

In order to prove the main result of this section, we begin with the following auxiliary result, of independent interest.

**Lemma 3.1.** For any real number $x \in (0, 1)$, integers $n \geq 3$ and $k \in \{1, \ldots, n - 2\}$, there exists a disjoint partition $\{n_1, \ldots, n_k\} \bigcup \{m_1, \ldots, m_{n-k-1}\}$ of the set $\{1, 2, \ldots, n - 1\}$ such that

$$n_i \leq \frac{i}{x}, \quad i \in \{1, \ldots, k\} \quad \text{and} \quad m_i \leq \frac{i}{1-x}, \quad i \in \{1, \ldots, n-k-1\}.$$

**Proof.** First note that for $i \geq 1$ the interval $(i, i + 1)$ cannot contain two distinct elements of the set $\{\frac{1}{x}, \ldots, \frac{k}{x}\}$. This is so for otherwise, there would exist indices $1 \leq j_1 < j_2 \leq k$ such that $i < \frac{j_1}{x} < \frac{j_2}{x} < i + 1$, and therefore

$$\frac{1}{x} \leq \frac{j_2 - j_1}{x} < i + 1 - i = 1,$$

which implies $x > 1$, a contradiction. A similar proof shows that the interval $(i, i + 1)$ cannot contain two distinct elements of the set $\left\{\frac{1}{1-x}, \ldots, \frac{n-k-1}{1-x}\right\}$.

Secondly, note that the interval $(i, i + 1)$ cannot contain an element of the set $\left\{\frac{1}{x}, \ldots, \frac{k}{x}\right\}$ and an element of the set $\left\{\frac{1}{1-x}, \ldots, \frac{n-k+1}{1-x}\right\}$. This is so for otherwise there would exist indices $1 \leq j_1 \leq k$ and $1 \leq j_2 \leq n - k - 1$ such that $i < \frac{j_1}{x} < i + 1$ and $i < \frac{j_2}{x} < i + 1$, and therefore

$$i = ix + i(1 - x) < j_1 + j_2 < (i + 1)x + (i + 1)(1 - x) = i + 1,$$

which is impossible since $j_1 + j_2$ is an integer.

A similar proof shows that if the interval $[i, i + 1]$ contains an element of the set $\left\{\frac{1}{x}, \ldots, \frac{k}{x}\right\}$ and an element of the set $\left\{\frac{1}{1-x}, \ldots, \frac{n-k-1}{1-x}\right\}$, then they both must be equal either to $i$ or to $i + 1$. Note that in the latter case the interval $(i + 1, i + 2]$ cannot contain any element of the set $\left\{\frac{1}{x}, \ldots, \frac{k}{x}\right\} \cup \left\{\frac{1}{1-x}, \ldots, \frac{n-k-1}{1-x}\right\}$ (since $x < 1$ and $1 - x < 1$).

Define the sequences $(n'_i)_{i=1,k}$ and $(m'_i)_{i=1,n-k-1}$ by $n'_i = j$ if $\frac{i}{x} \in (j, j + 1]$ for some $j \in \mathbb{N}, i \in \{1, \ldots, k\}$, and

$$m'_i = \begin{cases} j, & \text{if } \frac{i}{1-x} \in (j, j + 1] \text{ for some } j \in \mathbb{N}, i \in \{1, \ldots, k\}; \\ j + 1, & \text{if } \frac{i}{1-x} = j + 1 \text{ for some } j \in \mathbb{N}, i \in \{1, \ldots, n - k - 1\}, \end{cases}$$

thus $n'_i \leq \frac{i}{x}$ for $i \in \{1, \ldots, k\}$ and $m'_i \leq \frac{i}{1-x}$ for $i \in \{1, \ldots, n - k - 1\}$.

The first part of the proof shows that $n'_1, n'_2, \ldots, n'_{k-1}, m'_1, m'_2, \ldots, m'_{n-k-2}$ are all distinct, and in particular this shows that the set $N = \{n'_1, \ldots, n'_{k-1}, m'_1, \ldots, m'_{n-k-2}\}$ has $n - 1$ elements.

Denoting by $r$ the number of distinct elements of the set $N \cap (n - 1, \infty)$, it follows that the set $\{1, \ldots, n - 1\} \setminus (N \cap [0, n - 1])$ also has $r$ elements, and therefore there exists a bijection $f : N \cap (n - 1, \infty) \to \{1, \ldots, n - 1\} \setminus (N \cap [0, n - 1])$.

Define

$$n_i = \begin{cases} n_i, & \text{if } n'_i \leq n - 1; \\ f(n'_i), & \text{if } n'_i > n - 1, \end{cases}$$

for $i \in \{1, \ldots, k\}$,
and

\[ m_i = \begin{cases} \frac{i}{x}, & \text{if } m_i \leq n - 1 \\ f(m_i), & \text{if } m_i > n - 1 \end{cases}, \quad i \in \{1, \ldots, n - k - 1\}. \]

It is not difficult to see that \( n_1, \ldots, n_k, m_1, \ldots, m_{n-k-1} \) are distinct, \( \{1, \ldots, n - 1\} = \{n_1, \ldots, n_k, m_1, \ldots, m_{n-k-1}\} \) (recall the definition of the function \( f \), in particular \( f \leq n - 1 \)), and they satisfy

\[ n_i \leq \frac{i}{x}, \quad i \in \{1, \ldots, k\} \quad \text{and} \quad m_i \leq \frac{i}{1 - x}, \quad i \in \{1, \ldots, n - k - 1\}, \]

concluding the proof. \( \square \)

A second auxiliary result of independent interest is the following monotonicity of the (partial) first centered moment of Pólya random variables.

**Lemma 3.2.** For any \( x \in [0, 1] \) and any integers \( n \geq 1 \) and \( k \in \{0, 1, \ldots, n\} \), the sum

\[ \sum_{i=0}^{k} \left( x - \frac{i}{n} \right) p_{n,i}^{x,1-x,c} \]

is a non-decreasing function of \( c \geq -\frac{1}{n-1} \min\{x, 1-x\} \).

Moreover, for \( x \in (0, 1), \quad n \geq 2, \quad \text{and} \quad k \in \{0, \ldots, n-1\} \), the above sum is increasing with respect to \( c \geq -\frac{1}{n-1} \min\{x, 1-x\} \).

**Proof.** For \( k = n \) we have

\[ \sum_{i=0}^{n} \left( x - \frac{i}{n} \right) p_{n,i}^{x,1-x,c} = x - \frac{1}{n} EX_n^{x,1-x,c} = x - x = 0, \]

and for \( k = 0 \) we have

\[ \sum_{i=0}^{0} \left( x - \frac{i}{n} \right) p_{n,i}^{x,1-x,c} = x p_{n,0}^{x,1-x,c} = x (1 - x) (1 - x + c) \ldots (1 - x + (n-1)c), \]

thus the claim of the lemma is true in these cases.

Also, since for \( x = 0 \) and \( x = 1 \) the probabilities \( p_{n,i}^{x,1-x,c} \) are independent on the value of \( c \) (\( p_{n,i}^{0,1,c} = p_{n,n}^{1,0,c} = 1 \), and \( p_{n,i}^{x,1-x,c} = 0 \) for \( i \in \{1, \ldots, n-1\} \)), the claim of the lemma is also true in these cases.

Without loss of generality we may therefore assume that \( x \in (0, 1), \quad k \in \{1, \ldots, n-1\}, \quad \text{and} \quad n \geq 2. \)

Since the expression in (3.2) is a continuous, differentiable function of \( c \geq -\frac{1}{n-1} \min\{x, 1-x\} \), in order to prove the claim of the lemma, using (2.4), it suffices to show that for any \( c > -\frac{1}{n-1} \min\{x, 1-x\} \) we have

\[ \frac{\partial}{\partial c} \sum_{i=0}^{k} (nx - i) p_{n,i}^{x,1-x,c} = \frac{\partial}{\partial c} \left( (k+1) p_{n,k+1}^{x,1-x,c} (1 - x + (n - (k+1))c) \right) > 0. \]

Taking logarithms and using (2.2), we have left to prove that

\[ \frac{\partial}{\partial c} \left( \ln \left( (k+1) C_n^{k+1} x^{(k+1,c)} (1-x)^{(n-k-1,c)} (1 - x + (n - (k-1))c) \right) \right) > 0, \]

or equivalent

\[ \sum_{i=0}^{k} \frac{i}{x + ic} + \sum_{i=0}^{n-k-1} \frac{i}{1 - x + ic} > \sum_{i=0}^{n-1} \frac{i}{1 + ic}, \]

for any \( c > -\frac{1}{n-1} \min\{x, 1-x\} \).

For \( k = n - 1 \) the above inequality is readily satisfied (recall that \( x \in (0, 1) \)), so we have left to consider the case \( n \geq 3 \) and \( k \in \{1, \ldots, n - 2\} \).
For arbitrarily fixed \( n \geq 3, k \in \{1, \ldots, n-2\}, x \in (0,1), \) and \( c > \frac{1}{n-1} \min\{x, 1-x\}, \) the function
\[
\varphi : \left[1, \frac{n-1}{\min\{x, 1-x\}}\right] \to \mathbb{R}
\]
defined by \( \varphi(u) = \frac{u}{1+u}c \) is increasing.

Lemma 3.1 shows that we can find a partition \( \{n_1, \ldots, n_k\} \cup \{m_1, \ldots, m_{n-k-1}\} \) of the set \( \{1, 2, \ldots, n-1\} \) such that \( n_i \leq \frac{n}{2} \) for \( i \in \{1, \ldots, k\} \) and \( m_i \leq \frac{n}{2n-2} \) for \( i \in \{1, \ldots, n-k-1\} \), and therefore we obtain
\[
\sum_{i=1}^{k} \varphi\left(\frac{i}{x}\right) + \sum_{i=1}^{n-k-1} \varphi\left(\frac{i}{1-x}\right) > \sum_{i=1}^{k} \varphi(n_i) + \sum_{i=1}^{n-k-1} \varphi(m_i) = \sum_{i=1}^{n-1} \varphi(i),
\]
which is equivalent to (3.6), thus concluding the proof. \( \square \)

With the above preparation we can now proceed to prove the convex ordering of Pólya random variables \( X_n^{a,b,c} \) with respect to the replacement parameter \( c \). The precise statement of the following.

**Theorem 3.3.** The Pólya random variables \( X_n^{x,1-x,c} \) satisfy the following convex ordering

\[
X_n^{x,1-x,c} \leq_{\text{cx}} X_n^{x,1-x,c'},
\]

for any integer \( n \geq 1, x \in [0,1], \) and any \( c' \geq c \geq \frac{1}{n-1} \min\{x, 1-x\} \).

**Proof.** Since \( EX_n^{x,1-x,c} = nx \) for any value of \( c \) satisfying the compatibility condition \( c \geq \frac{1}{n-1} \min\{x, 1-x\} \), the claim of the theorem is equivalent by (2.10) to

\[
\frac{\partial}{\partial c} E\left((t - X_n^{x,1-x,c})_+\right) \geq 0, \quad t \in \mathbb{R}.
\]

Since \( X_n^{x,1-x,c} \) takes values in \( \{0, 1, \ldots, n\} \) and \( EX_n^{x,1-x,c} = nx \) is independent of \( c \), it is easy to see that the above inequality is satisfied for \( t < 0 \) and \( t > n \), thus it remains to prove it for \( t \in [0, n] \).

Replacing \( t \) by \( nt \) with \( t \in [0, 1] \), the above inequality is thus equivalent to

\[
\frac{\partial}{\partial c} E\left(t - \frac{1}{n} X_n^{x,1-x,c}\right) + \sum_{i=0}^{[nt]} \left(t - \frac{i}{n}\right) \frac{\partial}{\partial c} p_{n,i}^{x,1-x,c} \geq 0, \quad t \in [0, 1].
\]

The above inequality is further equivalent to the apparent weaker inequality

\[
\sum_{i=0}^{k} \left(\frac{k}{n} - \frac{i}{n}\right) \frac{\partial}{\partial c} p_{n,i}^{x,1-x,c} \geq 0, \quad k \in \{0, 1, \ldots, n\},
\]

the reason being the following.

For \( t = 1 = \frac{n}{n} \) the inequality (3.9) follows immediately from (3.10). For \( t \in [0, 1) \), we can write \( t = \alpha \frac{k}{n} + (1 - \alpha) \frac{k+1}{n} \) as a convex combination of \( \frac{k}{n} \) and \( \frac{k+1}{n} \), where \( k = [nt] \) and \( \alpha = k + 1 - nt \in [0, 1) \). If the inequality (3.10) holds true, then

\[
\frac{\partial}{\partial c} \sum_{i=0}^{[nt]} \left(t - \frac{i}{n}\right) p_{n,i}^{x,1-x,c} = \frac{\partial}{\partial c} \left(\alpha \sum_{i=0}^{k} \left(\frac{k}{n} - \frac{i}{n}\right) p_{n,i}^{x,1-x,c} + (1 - \alpha) \sum_{i=0}^{k} \left(\frac{k+1}{n} - \frac{i}{n}\right) p_{n,i}^{x,1-x,c}\right) = \alpha \sum_{i=0}^{k} \left(\frac{k}{n} - \frac{i}{n}\right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} + (1 - \alpha) \sum_{i=0}^{k+1} \left(\frac{k+1}{n} - \frac{i}{n}\right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} \geq 0,
\]

by Lemma 3.2 thus proving (3.9).
In order to prove (3.10), first note that the claim is trivial if $x = 1$, for in this case $p_{n,n}^{1,0,c} = 1$ and $p_{n,i}^{1,0,c} = 0$ for $i \in \{0, \ldots, n-1\}$ (thus $p_{n,i}^{1,0,c}$ is independent of the value of the replacement parameter $c$). Without loss of generality we can therefore assume that $x \neq 1$.

Suppose that (3.10) does not hold for a certain $x \in \{0,1\}$ and $k \in \{0,1,\ldots,n\}$, that is

\[
\sum_{i=0}^{k} \left( \frac{k - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} < 0.
\]

We distinguish the following cases.

i) $nx > k$

Note that in this case we cannot have $k = 0$, for in this case the sum in (3.11) is equal to 0. Since $k \geq 1$ and $x \in [0,1)$ we can write $x = \alpha \left( \frac{k-1}{n} \right) + (1-\alpha) \left( \frac{k+1}{n} \right)$, where $\alpha = k - nx < 0$.

Using again Lemma 3.2 and the same argument as above we obtain

\[
\alpha = \sum_{i=0}^{k-1} \left( \frac{k - 1 - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} + (1-\alpha) \sum_{i=0}^{k} \left( \frac{k - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c}
\]

\[
= \frac{\partial}{\partial c} \sum_{i=0}^{k-1} \left( \alpha \left( \frac{k - 1 - i}{n} \right) + (1-\alpha) \left( \frac{k - i}{n} \right) \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c}
\]

\[
= \sum_{i=0}^{k-1} \left( x - \frac{i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} \geq 0,
\]

and therefore (the second sum on the first line above being assumed to be strictly negative, and since $\alpha < 0$) we deduce that

\[
\sum_{i=0}^{k-1} \left( \frac{k - 1 - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} < 0.
\]

We showed that if the sum in (3.11) is strictly negative, then so is the sum in (3.12). Proceeding inductively on $k$, this implies that

\[
0 \equiv \sum_{i=0}^{0} \left( \frac{0 - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} < 0,
\]

a contradiction.

ii) $nx \leq k < k+1$

Note that in this case we cannot have $k = n$, for

\[
\sum_{i=0}^{n} \left( \frac{n - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} = \frac{\partial}{\partial c} E \left( 1 - \frac{1}{n} X_{n,i}^{x,1-x,c} \right) = \frac{\partial}{\partial c} (1 - x) \equiv 0.
\]

Since $k \leq n - 1$ and $x \in [0,1)$ we can write $x = \alpha \left( \frac{k+1}{n} \right) + (1-\alpha) \left( \frac{k}{n} \right)$, where $k + 1 \leq n$ and $\alpha = k + 1 - nx \geq 1$.

We have

\[
\alpha \sum_{i=0}^{k} \left( \frac{k - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} + (1-\alpha) \sum_{i=0}^{k+1} \left( \frac{k+1 - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c}
\]

\[
= \alpha \sum_{i=0}^{k} \left( \frac{k - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} + (1-\alpha) \sum_{i=0}^{k} \left( \frac{k + 1 - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c}
\]

\[
= \sum_{i=0}^{k} \left( \frac{n - i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} \geq 0,
\]

a contradiction.
for any integer 
\[ n \]
and the definition (2.9) of convex ordering.

To prove the second part of the theorem, note that if for certain values 
\[ 2 \]
we replace \( f \) by \( f + \max_{x \in [0,1]} f(x) \) (being convex, \( f \) is also continuous),
thus without loss of generality we may assume that \( f(x) > 0 \) for \( x \in [0,1] \).

Next note that for \( x \in (0,1) \), from the definition (2.3) of the rising factorial, it follows that 
\[ x^{(k,c)}(1-x)^{(n-k,c)} \]
is a polynomial of degree \( n-2 \) in the variable \( c \) if \( k \in \{1, \ldots, n-1 \} \) (with leading coefficient 
\[ x((1-x)(k-1))^{(n-k-1)} \].

Using (3.11) and the fact that \( \alpha \geq 1 \), from the above inequality we conclude that

\[
\sum_{i=0}^{k+1} \left( \frac{k+1}{n} - \frac{i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} < 0.
\]

We showed that if the sum in (3.11) is strictly negative, then so is the sum in (3.14). Proceeding
inductively on \( k \), we obtain

\[
0 = \frac{\partial}{\partial c} (1-x) = \frac{\partial}{\partial c} E \left( 1 - \frac{1}{n} X_{n,1-x,c} \right) = \sum_{i=0}^{n} \left( \frac{n-i}{n} \right) \frac{\partial p_{n,i}^{x,1-x,c}}{\partial c} < 0,
\]
a contradiction.

In both cases above we showed that (3.11) leads to a contradiction. This shows that (3.10) holds true,
thus concluding the proof of the theorem.

4. AN APPLICATION TO THE ERROR ESTIMATE OF PÓLYA-STANCU OPERATORS

As an application of Theorem 3.3 we have the following.

**Theorem 4.1.** For any convex function \( f : [0,1] \to \mathbb{R} \) the absolute value of the error of approximation of the
Pólya-Stancu operator \( P_n^c \) is a non-decreasing function of \( c \), that is

\[
|P_n^{c_2} f(x) - f(x)| \geq |P_n^{c_1} f(x) - f(x)|,
\]
for any integer \( n \geq 2 \), \( x \in [0,1] \), and any \( c_2 > c_1 \geq -\frac{1}{n-1} \min \{x, 1-x\}. \)

If moreover

\[
B_n f(x) \neq B_1 f(x)
\]
for certain values of \( n \geq 2 \) and \( x \in [0,1] \), then the above monotonicity is strict, that is

\[
|P_n^{c_2} f(x) - f(x)| > |P_n^{c_1} f(x) - f(x)|,
\]
for any \( c_2 > c_1 \geq -\frac{1}{n-1} \min \{x, 1-x\}. \)

**Proof.** Jensen’s inequality shows that

\[
P_n^c f(x) = Ef \left( \frac{1}{n} X_{n,1-x,c} \right) \geq f \left( \frac{1}{n} E X_{n,1-x,c} \right) = f(x), \quad c \geq -\frac{1}{n-1} \min \{x, 1-x\},
\]
thus the claim (4.1) is equivalent to \( P_n^{c_2} f(x) \geq P_n^{c_1} f(x) \), and it follows immediately from Theorem 3.3
and the definition (2.9) of convex ordering.

To prove the second part of the theorem, note that if for certain values \( n \geq 2 \) and \( x \in [0,1] \) we have
\( P_n^{c_2} f(x) = P_n^{c_1} f(x) \) for some \( c_2 > c_1 \geq -\frac{1}{n-1} \min \{x, 1-x\} \), then by the first part of the proof we have that

\[
P_n^c f(x) = C, \quad c \in [c_1, c_2],
\]
is a constant function of \( c \) (the constant \( C \) may depend on \( n \) and \( x \)).

The above still holds if we replace \( f \) by \( f + 1 + \max_{x \in [0,1]} f(x) \) (being convex, \( f \) is also continuous),
thus without loss of generality we may assume that \( f(x) > 0 \) for \( x \in [0,1] \).
and a polynomial of degree \( n - 1 \) in the variable \( c \) if \( k \in \{0, n\} \) (with leading coefficient \( (n - 1)! (1 - x) \) for \( k = 0 \) and \( (n - 1)!x \) for \( k = n \)). Using this and the definition (2.7) of the operator \( P_n^c \), it can be seen that for fixed values \( n \geq 2 \), \( x \in [0, 1] \) and \( f \),

\[
P_n^c f(x) = \sum_{k=0}^{n} \left( f \left( \frac{k}{n} \right) C_n^{k} x^{(k,c)} (1-x)^{(n-k,c)} \right)
\]

is the ratio of two polynomials of degree \( n - 1 \) in the variable \( c \geq -\frac{1}{n-1} \min \{x, 1-x\} \) (recall that \( f > 0 \) and the special cases \( k = 0 \) and \( k = n \) above).

From (4.4) we conclude that these two polynomials (in the variable \( c \)) are a constant multiple of each other; in particular this shows that their leading and free term coefficients are proportional, thus

\[
\frac{(n-1)! f(0)(1-x) + (n-1)! f(1)x}{(n-1)!} = \frac{\sum_{k=0}^{n} f \left( \frac{k}{n} \right) x^{k} (1-x)^{n-k}}{1},
\]

or equivalent

\[
f(0)(1-x) + f(1)x = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) x^{k} (1-x)^{n-k}.
\]

Finally, note that \( \sum_{k=0}^{n} f \left( \frac{k}{n} \right) x^{k} (1-x)^{n-k} = B_n f(x) \) is just the Bernstein polynomial of degree \( n \) corresponding to \( f \) evaluated at \( x \), and \( f(0)(1-x) + f(1)x = B_1 f(x) \), thus we have equivalent (4.5)

\[
B_n f(x) = B_1 f(x).
\]

We have shown that (4.4) implies the condition (4.5). Therefore if (4.5) does not hold, then (4.4) cannot hold, thus concluding the proof of the theorem. \( \square \)

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