Ultimate quantum limit for amplification: a single atom in front of a mirror

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Abstract.

We investigate three types of amplification processes for light fields coupling to an atom near the end of a one-dimensional semi-infinite waveguide. We consider two setups where a drive creates population inversion in the bare or dressed basis of a three-level atom and one setup where the amplification is due to higher-order processes in a driven two-level atom. In all cases, the end of the waveguide acts as a mirror for the light. We find that this enhances the amplification in two ways compared to the same setups in an open waveguide. Firstly, the mirror forces all output from the atom to travel in one direction instead of being split up into two output channels. Secondly, interference due to the mirror enables tuning of the ratio of relaxation rates for different transitions in the atom to increase population inversion. We quantify the enhancement in amplification due to these factors and show that it can be demonstrated for standard parameters in experiments with superconducting quantum circuits.
1. Introduction

Amplification of measurement signals is crucial to achieve good signal-to-noise ratios in many experiments in quantum information and quantum optics [1, 2]. Ideally, amplifiers used for such tasks should be compact, add as little noise as possible [3, 4], and produce high gain. To reach the ultimate limit in terms of size, a single atom or other quantum emitter could be used as an amplifier. However, to achieve high gain with a single quantum emitter coupled to an electromagnetic field in free space is extremely challenging, since imperfect spatial mode matching leads to a weak coupling [5–11]. The mode matching, and thus a strong coupling, is much easier to achieve when the propagation of the field is confined to a one-dimensional (1D) waveguide. Such systems are widely studied in waveguide quantum electrodynamics (waveguide QED), which has proven an excellent platform for quantum-optical experiments [12, 13].

In the past two decades, many quantum-optics phenomena have been demonstrated using superconducting circuits [13–16], e.g., lasing [17–20]. Superconducting circuits consist of superconducting qubits [15, 21] coupled to a coplanar waveguide (either open or made into a resonator) [13, 22–24] or three-dimensional cavities [13, 16, 25]. One advantage of superconducting circuits over natural atomic systems is that strong, and even ultrastrong, coupling between the quantum emitters and cavities or open waveguides can be achieved quite easily [26–32]. This advantage has, for example, been demonstrated by Wen et al. [33], who used superconducting circuits to realize a 7% amplification of a weak probe signal on a strongly-driven two-level system coupled to a waveguide. Similar experiments with many natural atoms [34] or a single quantum dot [35] were only able to achieve 0.4% and 0.005% amplification, respectively.

The vast majority of waveguide-QED experiments with superconducting circuits so far were performed with one or more superconducting qubits coupled to an open waveguide [12, 13, 24, 29, 36–49]. However, the waveguide can also be shorted or left open at one end, where an incoming electromagnetic field will be reflected with a phase shift [33, 50–52]. When a superconducting qubit is included [33, 52], this setup is equivalent to putting an atom in front of a mirror, which has been studied experimentally [33, 46, 52, 57] and theoretically [58–70] for both natural and artificial atoms. In this article, we investigate the advantages of using an atom in front of a mirror, instead of an atom in an open waveguide, for signal amplification.

There are several ways to achieve amplification in an atomic system driven by an electromagnetic field. One amplification mechanism is population inversion, where excitations are pumped into higher atomic levels with a finite life time, where they stay long enough to induce amplification through stimulated emission [71, 72]. There are also mechanisms that can lead to amplification and lasing without inversion in the bare-state basis [73]. For instance, if an atom is driven strongly, the energy levels can split and population inversion can occur in the dressed-state basis [40, 74] if the drive is off resonance. If the drive is on resonance, the power spectrum exhibits the so-called Mollow triplet [24, 56, 75, 76]; amplification without population inversion can then be
achieved at frequencies between the triplet peaks due to higher-order processes between the dressed states of the driven atom [33–35, 77, 78].

We study all three amplification mechanisms outlined above for an atom, with either two or three levels, coupled to a 1D waveguide terminated by a mirror (in the form of a short) at one end, as depicted in Fig. 1. We show that this setup has two advantages over the corresponding one in an open waveguide, leading to a doubling or more of the maximum amplification that we can achieve. Firstly, the mirror reflects the electromagnetic field such that we only have one input-output channel, which avoids losing half of the atomic output in one direction, as happens in an open waveguide. The second advantage of the mirror setup is that, due to interference effects, the coupling to the waveguide is set by the position of the atom and its transition frequency. This enables manipulation of the relative coupling strengths for different transitions in a three-level atom, either by changing the atomic frequency, which is possible in superconducting circuits [13–16], or by changing the distance from the atom to the mirror. We note that the coupling strengths can also be made frequency-dependent and tunable using giant atoms [48, 49, 79–91], which couple to the waveguide at multiple
points, but if the giant atom is placed in an open waveguide, the problem of losing half the output remains. With a mirror, a single small atom is simpler to implement than a giant atom, but still sufficient to achieve the advantageous frequency-dependent coupling.

The first system we consider is shown in Fig. 1(b) and discussed in Sec. 2. It is a three-level atom with a strong drive on the transition between the ground state $|0\rangle$ and the second excited state $|2\rangle$. When the decay rate from $|2\rangle$ to the first excited state $|1\rangle$ is larger than the decay rate from $|1\rangle$ to $|0\rangle$, a population inversion between $|1\rangle$ and $|0\rangle$ is created. This leads to amplification of a weak probe signal on the $|0\rangle \leftrightarrow |1\rangle$ transition. We find that with the mirror, a maximum amplitude gain of 25% can be reached, whereas the maximum amplification in an open waveguide is 12.5% [36].

Next, we study, in Sec. 3, the resonantly driven two-level atom depicted in Fig. 1(c). The strong drive splits the energy levels of the atom and enable transitions in the dressed-state basis. By probing the system in the vicinity of the bare resonance frequency, we achieve a maximal amplitude gain of around 6.9% with the mirror. For an open waveguide, we find an amplification of around 3.4% for the same system parameters. In contrast to the previous case, amplification is not due to population inversion, but enabled by higher-order processes between the dressed states [33, 78].

The last system we study, in Sec. 4, is the three-level system shown in Fig. 1(d), driven at half the $|0\rangle \leftrightarrow |2\rangle$ transition frequency. Similarly to the strongly driven two-level system, the energy levels are split by the driving and transitions take place between dressed states. By probing the system, we find a maximal amplification of around 6.2% with a mirror, which exceeds the amplification of the same setup in an open waveguide by more than a factor 2 [40]. Here the amplification is due to hidden inversion — population inversion between the dressed states of the system.

We further show that all these systems can be realized with currently available state-of-the-art technology in experimental waveguide-QED setups with a transmon qubit [92] coupled to a 1D transmission line. The transition frequencies of such a qubit are tunable in situ, which means that the ratio of the decay rates of the transmon can be chosen to reach the optimal amplification settings. Our proposed setups, which represent an ultimate quantum limit for amplification, may thus find applications in superconducting quantum information processing.

2. Amplification with a strongly driven three-level atom in front of a mirror

We begin by studying the setup with a three-level atom in front of a mirror shown in Fig. 1(b). The system is coherently driven with amplitude $\Omega_d$ on the transition between the ground state $|0\rangle$ and the second excited state $|2\rangle$. The aim is to create a population inversion between the states $|0\rangle$ and $|1\rangle$, which can lead to a gain in the reflection of a weak coherent probe resonant with the $|0\rangle \leftrightarrow |1\rangle$ transition. To achieve population inversion, the life-time $1/\Gamma_{10}$ of the first excited state should be much longer than the life-time $1/\Gamma_{21}$ of the second excited state. Here, we assume that the atom is a good
approximation of a ladder-type Ξ system with $\Gamma_{20} \ll \Gamma_{10}, \Gamma_{21}$.

2.1. Hamiltonian and master equation

The Hamiltonian of the system in the frame rotating at the drive frequencies is (we set $\hbar = 1$ throughout this article)

$$H = H_a + H_{\text{int}},$$

$$H_a = \delta \omega_{10} \sigma_{11} + \delta \omega_{20} \sigma_{22},$$

$$H_{\text{int}} = \frac{\Omega_d}{2} (\sigma_{20} + \sigma_{02}) + \frac{\Omega_p}{2} (\sigma_{10} + \sigma_{01}).$$

where $\sigma_{ij} = |i\rangle\langle j|$, the drive amplitude on the $|0\rangle \leftrightarrow |2\rangle$ ($|0\rangle \leftrightarrow |1\rangle$) transition is given by $\Omega_d (\Omega_p)$, and $\delta \omega_{ij} = \omega_{ij} - \omega_{ij}^d$ for $i > j$ is the detuning between the transition frequency $\omega_{ij} = \omega_i - \omega_j$ and the frequency $\omega_{ij}^d$ of the drive on that transition. The dynamics of the system is described by the master equation

$$\dot{\rho} = -i\frac{\hbar}{\hbar} [H, \rho] + L[\rho]$$

for the density matrix $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|$. The Lindbladian term in Eq. (4) is given by

$$L[\rho] = \Gamma_{21} \rho_{22} (-\sigma_{22} + \sigma_{11}) + \Gamma_{10} \rho_{11} (-\sigma_{11} + \sigma_{00}) - \sum_{i\neq j} \gamma_{ij} \rho_{ij} \sigma_{ij},$$

with the dephasing $\gamma_{ij} = \gamma_{ji}$ and the relaxation rates $\Gamma_{ij}$ between the states $|i\rangle$ and $|j\rangle$, $i > j$. Since we assume negligible temperature, we can neglect thermal excitations, i.e., $\Gamma_{01} = \Gamma_{12} = \Gamma_{02} = 0$.

2.2. Steady-state solution

We assume that the probe on the $|0\rangle \leftrightarrow |1\rangle$ is weak, i.e., $\Omega_{10}/\Gamma_{10} \ll 1$. Solving the master equation for the steady state ($\dot{\rho} = 0$), we obtain

$$\rho_{00} = \frac{A}{A + B + 1}, \quad \rho_{11} = \frac{B}{A + B + 1}, \quad \rho_{22} = \frac{1}{A + B + 1},$$

$$\rho_{10} = \frac{i \Omega_p}{2 \lambda_{10}} \left( \frac{\Omega_d^2}{4 \lambda_{02} \lambda_{12}} (\rho_{00} - \rho_{22}) + \rho_{11} - \rho_{00} + \rho_{22} \right),$$

where

$$A = \frac{\rho_{00}}{\rho_{22}} = \frac{2 \Gamma_{21} |\lambda_{02}|^2}{\gamma_{20} \Omega_d^2} + 1, \quad B = \frac{\rho_{11}}{\rho_{22}} = \frac{\Gamma_{21}}{\Gamma_{10}},$$

and $\lambda_{ij} = \lambda_{ji}^*$ with

$$\lambda_{10} = \gamma_{10} + i\delta \omega_{10}, \quad \lambda_{12} = \gamma_{21} - i\delta \omega_{20} + i\delta \omega_{10}, \quad \lambda_{02} = \gamma_{20} - i\delta \omega_{20}. $$
2.3. Amplification and optimal drive strength

We now assume that $\Gamma_{21} \gg \Gamma_{10}$ to ensure population inversion. From Eqs. (6) and (8), we see that the second excited state then is nearly unpopulated. Neglecting terms of order $O(\Gamma_{10}/\Gamma_{21})$, this gives

$$\rho_{10} \approx i\Omega p (\rho_{11} - \rho_{00})/2\lambda_{10} + \frac{\Omega_d^2}{2\lambda_{12}}.$$  \hspace{1cm} (10)

The reflection coefficient is given by \[52, 56\]

$$r = 1 - 2i\frac{\Gamma_{10}}{\Omega_p} \langle \sigma_{01} \rangle = 1 - 2i\frac{\Gamma_{10}}{\Omega_p} \rho_{10} = 1 + 2\Gamma_{10} \frac{(\rho_{11} - \rho_{00})}{2\lambda_{10} + \frac{\Omega_d^2}{2\lambda_{12}}}.$$  \hspace{1cm} (11)

This is where our derivation deviates from that in Ref. [36] for an open waveguide. The mirror adds a factor 2 to the second term of the reflection coefficient compared to the open waveguide.

From Eq. (11), it is clear that amplification requires $\rho_{11} > \rho_{00}$, which leads to

$$\Gamma_{21} > \frac{2\Gamma_{21}\lambda_{02}^2}{\gamma_{20}\Omega_d^2} + 1 \Rightarrow \Omega_d^2 > \frac{2\Gamma_{10}\lambda_{02}^2}{\gamma_{20}} + \frac{\Omega_d^2\Gamma_{10}^2}{\Gamma_{21}} \approx \frac{2\Gamma_{10}\lambda_{02}^2}{\gamma_{20}}.$$  \hspace{1cm} (12)

For a resonant drive, $\delta\omega_{20} = 0$, this reduces to

$$\Omega_d^2 > 2\Gamma_{10}\gamma_{20}.$$  \hspace{1cm} (13)

If we further assume no pure dephasing, we have $\gamma_{20} = \Gamma_{21}/2$ and thus

$$\Omega_d^2 > \Gamma_{10}\Gamma_{21}.$$  \hspace{1cm} (14)

We now calculate the maximal possible amplitude gain of the single-atom amplifier. We consider double resonance, $\delta\omega_{20} = \delta\omega_{10} = 0$, and no pure dephasing, i.e., $\gamma_{10} = \Gamma_{10}/2$, $\gamma_{21} = \Gamma_{21}/2$, and $\gamma_{20} = \Gamma_{21}/2$. We can then rewrite the populations in Eq. (6) as $\rho_{00} = 1/(1 + \nu)$ and $\rho_{11} = \nu/(1 + \nu)$ with $\nu = \Omega_d^2/(\Gamma_{10}\Gamma_{21})$, which leads to the reflection coefficient

$$r = 1 + 2\frac{(\nu - 1)}{(1 + \nu)^2}.$$  \hspace{1cm} (15)

This expression reaches its maximum value for $\nu = 3$, which is achieved when

$$\Omega_d^2 = 3\Gamma_{10}\Gamma_{21}.$$  \hspace{1cm} (16)

With this value for the drive amplitude, the maximum reflection is given by

$$|r| = 1 + \frac{1}{4},$$  \hspace{1cm} (17)

which corresponds to an amplitude gain of 25%. If we include higher orders of $\Gamma_{10}/\Gamma_{21}$ in the calculation, the first-order correction to the maximum value of the reflection becomes

$$r = 1 + \frac{1}{4} - \frac{3\Gamma_{10}}{8\Gamma_{21}} + O\left(\frac{\Gamma_{10}}{\Gamma_{21}}\right)^2.$$  \hspace{1cm} (18)

In Fig. 2(a), we plot the absolute value of the reflection coefficient as a function of the drive amplitude $\Omega_d$ and the detuning $\delta\omega_{10}$. In Fig. 2(b), we further illustrate the effect of non-zero detuning with a few linecuts from Fig. 2(a). It is clear that the maximum reflection is achieved on resonance.
Ultimate quantum limit for amplification: a single atom in front of a mirror

Figure 2. Optimizing the amplitude gain of a strongly driven three-level atom in front of a mirror. (a) The absolute value of the reflection coefficient, $|r|$, i.e., the amplitude gain, as a function of the drive strength $\Omega_d/\Gamma_{21}$ and the probe detuning $\delta\omega_{10}/\Gamma_{21}$. The reflection is calculated for a resonant drive, $\delta\omega_{20} = 0$, no pure dephasing, $\gamma_{21} = \gamma_{20} = \Gamma_{21}/2$ and $\gamma_{10} = \Gamma_{10}/2$, and with $\Gamma_{10}/\Gamma_{21} = 0.01$. The reflection reaches its maximum at $\Omega_{20} = \sqrt{3}\Gamma_{21}\Gamma_{10} \approx 0.17\Gamma_{21}$. (b) Horizontal linecuts from panel (a) showing the amplitude gain as a function of $\Omega_d$ for three different values of the probe detuning: $\delta\omega_{10} = 0$ (purple, solid curve), $\delta\omega_{10}/\Gamma_{21} = 0.003$ (red, dashed), and $\delta\omega_{10}/\Gamma_{21} = 0.01$ (orange, dashed). The maximum values for the orange and red curves are slightly below 1.25. (c) Vertical blue dashed linecut from panel (a) showing the reflection at the optimal driving strength $\Omega_d = \sqrt{3}\Gamma_{21}\Gamma_{10} \approx 0.17\Gamma_{21}$ as a function of the detuning $\delta\omega_{10}/\Gamma_{21}$.

2.4. Optimal population inversion

In the previous subsection, we found that the drive strength $\Omega_d^2 = 3\Gamma_{10}\Gamma_{21}$ gives the highest amplitude gain. Inserting this into the expressions for $\rho_{00}$ and $\rho_{11}$ in Eq. (6), we obtain

$$\rho_{00} = \frac{1}{4} + \frac{\Gamma_{10}}{\Gamma_{21}} \ll \frac{3}{4}, \quad \rho_{11} = \frac{1}{4} + \frac{\Gamma_{10}}{\Gamma_{21}} \approx \frac{1}{4}.$$  

(19)

We see that in order to maximize the amplitude gain, the population is not completely inverted (that would be $\rho_{11} = 1, \rho_{00} = 0$). For a complete population inversion to happen, the following condition has to be fulfilled:

$$\frac{B}{A} = \frac{\rho_{11}}{\rho_{00}} \gg 1 \quad \Rightarrow \quad \frac{\Gamma_{21}}{\Gamma_{10}} \gg \frac{2\Gamma_{21}\gamma_{20}}{\Omega_d^2}.$$  

(20)

This could be achieved by further increasing the drive strength $\Omega_d$. However, if we look at the expression for the reflection in Eq. (11), we see that increasing the pumping...
strength towards infinity would make the reflection revert to 1. This trade-off explains why we do not achieve a maximum amplitude gain of $\sqrt{2}$ (a power gain of 2), which would be the result if an incoming photon would stimulate emission of another photon from a perfectly inverted atom.

2.5. Correction with pure dephasing

Now we discuss the effect of pure dephasing on the previous results. Neglecting terms of order $\mathcal{O}(\Gamma_{10}/\Gamma_{21})$, the reflection coefficient on resonance with pure dephasing included can be written as

$$r = 1 + 2\frac{\Gamma_{10}}{\gamma_{10}} \frac{(\eta - 1)}{(\eta + 1)\left(\frac{\gamma_{10}\gamma_{20}}{\gamma_{21}\gamma_{10}} + 2\right)}$$

with $\eta = \frac{\Omega^2_d}{2\Gamma_{10}\gamma_{20}}$. Maximizing this expression, we find the optimal value for $\eta$:

$$\eta_{\text{max}} = 1 + \eta_c, \quad \eta_c = \sqrt{2 \left(1 + 2\frac{\gamma_{21}\gamma_{10}}{\Gamma_{10}\gamma_{20}}\right)}$$

Hence, the optimal drive strength including pure dephasing is

$$\Omega_d^2 = 2\Gamma_{10}\gamma_{20}(1 + \eta_c),$$

for which we obtain the maximum reflection

$$r = 1 + 2\frac{\Gamma_{10}}{\gamma_{10}} \left[\frac{1}{1 + \sqrt{2 \left(1 + 2\frac{\gamma_{21}\gamma_{10}}{\Gamma_{10}\gamma_{20}}\right)}^{-1/2}}\right] \left[\frac{1}{\sqrt{2 \left(1 + 2\frac{\gamma_{21}\gamma_{10}}{\Gamma_{10}\gamma_{20}}\right) + 1}}\right] \frac{\Gamma_{10}\gamma_{20}}{\gamma_{21}\gamma_{10}} + 2\right].$$

2.6. Experimental feasibility

We now apply the theoretical results above to a typical experimental system of a superconducting transmon qubit [92] to see what the optimal parameters for an experiment would be, and whether they are within reach for currently available devices. By shorting one end of the transmission line to create an effective mirror, the decay rates $\Gamma_{10}$ and $\Gamma_{21}$ become a function of the transition frequency $\omega_{10}$ of the energy levels [52]

$$\Gamma_{10} = 2\Gamma_{10}^{\text{TL}} \cos^2 \left[\frac{L}{v} \omega_{10}\right],$$

$$\Gamma_{21} = 2\Gamma_{21}^{\text{TL}} \cos^2 \left[\frac{L}{v} (\omega_{10} + \alpha)\right],$$

where $\Gamma_{10}^{\text{TL}}/2\pi = 37.5$ MHz and $\Gamma_{21}^{\text{TL}}/2\pi \approx 2\Gamma_{10}^{\text{TL}}/2\pi = 75$ MHz are the bare relaxation rates in an open transmission line, $\alpha = \omega_{21} - \omega_{10}$ is the anharmonicity between the transition frequencies, $L = 33$ mm is the distance between the transmon and the mirror, and $v = 9 \cdot 10^7$ m/s is the speed of light in the transmission line. The given values are typical for this kind of setup [33, 52].

The transition frequencies of the transmon are tunable in situ by an external magnetic flux, so we want to find the resonance frequency $\omega_{10}$ that gives the highest
Ultimate quantum limit for amplification: a single atom in front of a mirror

9

Figure 3. Reflection coefficient and decay rates as functions of transition frequency. (a) Absolute value of the reflection coefficient as a function of transition frequency $\omega_{10}$ at drive strength $\Omega_d/2\pi = 59.5$ MHz and pure dephasing $\Gamma_{10}^\phi/2\pi = 1.65$ MHz, $\Gamma_{21}^\phi/2\pi = \Gamma_{30}^\phi/2\pi = 5$ MHz. (b) The decay rates $\Gamma_{10}/2\pi$ (purple) and $\Gamma_{21}/2\pi$ (red), and their difference $\Gamma_{21}/2\pi - \Gamma_{10}/2\pi$ (orange), as a function of the transition frequency $\omega_{10}$.

possible reflection. We therefore express the reflection as a function of the drive strength $\Omega_d$ and the frequency $\omega_{10}$, using Eqs. (25)–(26), and maximize this function numerically. A plot of the resulting reflection amplitude can be seen in Fig. 3(a). With the above parameters and pure dephasing rates of $\Gamma_{10}^\phi/2\pi = 1.65$ MHz, $\Gamma_{21}^\phi/2\pi = \Gamma_{20}^\phi = 5$ MHz, again chosen from typical values [33, 49, 52] and optimal drive strength $\Omega_d/2\pi = 59.5$ MHz, the reflection reaches a maximum of 1.2 which corresponds to an amplitude gain of 20%. Due to the non-zero dephasing, this is lower than the theoretical limit of 25% calculated above. We note that dephasing and non-radiative decay rates can be lower than what we have assumed here, as shown, e.g., in Refs. [47, 57].

We find that there are two local maxima for the gain in Fig. 3(a), located in the area close to the nodes of the decay rate $\Gamma_{10}$ [see Fig. 3(b)], e.g., between $\omega_{10}/2\pi \approx 4.5$ GHz and $\omega_{10}/2\pi \approx 5.0$ GHz. This is the area where the requirement for amplification $\Omega_d^2 > 3\Gamma_{10}\Gamma_{21}$ is fulfilled. Between the two local maxima, we find a local minimum with a 0% gain. This local minimum occurs at the node of the electromagnetic field, where the decay rate $\Gamma_{21}$ goes to zero and no population inversion is possible [see Fig. 3(b)].

We also note that driving the $|0\rangle \leftrightarrow |2\rangle$ transition directly is hard in a transmon due to selection rules [92]. It is possible to drive the transition with a two-photon drive instead, where the frequencies of two drive photons sum up to $\omega_{20}$. However, if this drive is too strong, the qubit states will be dressed and we will instead have a setup like that discussed in Sec. 4. Another solution is to use another superconducting qubit
which does not suffer from this limitation on allowed transitions, e.g., the flux qubit, as was done in Ref. [36].

Finally, if the $|0\rangle \leftrightarrow |2\rangle$ transition is driven through the waveguide, the strength with which the drive couples to the system will be frequency-dependent in the same way as in Eq. (25), with $\omega_{10}$ replaced by $\omega_{20}$. As long as the drive frequency does not correspond to a node of the field at the atom, a decrease or increase in coupling strength can be compensated by adjusting the input drive power. It would also be possible to avoid any such issues by driving the atom through a separate line not affected by the interference with the mirror.

3. Amplification with a strongly driven two-level atom in front of a mirror

The next setup we consider is a two-level atom in front of a mirror that is driven strongly on resonance, as depicted in Fig. 1(c). The strong driving results in a splitting of the atomic energy levels such that the dynamics are best understood in terms of dressed states. As shown theoretically in Refs. [77, 78] and experimentally in Refs. [33–35], amplification can be achieved in this setup through higher-order processes when probing at frequencies in-between those of the Mollow triplet.

3.1. Hamiltonian and equations of motion

The Hamiltonian of a driven two-level system, with ground state $|0\rangle$ and excited state $|1\rangle$, interacting with the continuum of modes in the semi-infinite waveguide, is, in a frame rotating with the drive frequency $\omega_d$,

$$H = H_a + H_f + H_{\text{int}},$$

$$H_a = \delta \omega_{10} \sigma_{11} + E \left( \sigma_t + \sigma_t^\dagger \right),$$

$$H_f = \int d\omega \ \omega a^\dagger(\omega)a(\omega),$$

$$H_{\text{int}} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega \left( a^\dagger(\omega) \sigma_t + \sigma_t^\dagger a(\omega) \right),$$

where $\delta \omega_{10} = \omega_{10} - \omega_d$, $\Omega_d = 2\sqrt{\Gamma_{10}} E$, $|E|^2$ is the number of incoming drive photons per second, $\sigma_t = \sqrt{\Gamma_{10}} \sigma_{01}$, and $a^\dagger(\omega)$ [$a(\omega)$] are the photon creation [annihilation] operators at frequency $\omega$. We calculate the eigenenergies $\omega_g$, $\omega_e$ and the corresponding dressed eigenstates $|g\rangle$, $|e\rangle$ of the atomic Hamiltonian $H_a$ in Eq. (28), and define the population and transition operators for the dressed states as

$$\sigma_{\mu\nu} = |\mu\rangle \langle \nu|,$$

where $\mu, \nu \in \{g, e\}$. The Heisenberg equations of motion for these operators become

$$\frac{d}{dt} \sigma_{\mu\nu} = i \omega_{\mu\nu} \sigma_{\mu\nu} - \xi_{\mu\nu} - i \xi_{\mu\nu} a_{\text{in}}(t) + i a_{\text{in}}^\dagger(t) \xi_{\nu\mu}.$$
with
\[ \xi_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} \sigma_t^\dagger \sigma_t^\dagger + \frac{1}{2} \sigma_t^\dagger \sigma_t \sigma_{\mu\nu} - \sigma_t^\dagger \sigma_t \sigma_{\mu\nu} \] (33)
\[ \zeta_{\mu\nu} = [ \sigma_{\mu\nu}, \sigma_t^\dagger ] . \] (34)

The derivation of Eq. (32), which closely follows that for a three-level atom in an open waveguide in Ref. [40], is given in Appendix A. The main difference is that, in an open waveguide, the atom couples to two continua of modes in the waveguide, one right-moving and one left-moving, which both enter in Eq. (30). This means that \( \sigma_t = \sqrt{\Gamma_{10}/2} \sigma_{01} \) in the open-waveguide case and leads to a \( \xi_{\mu\nu} \) with the right-hand side in Eq. (33) multiplied by 2. In the end, these differences lead to a twice as large amplification with the mirror than without it.

### 3.2. Steady-state solution and linear response

The expectation values of the operators \( \langle \sigma_{\mu\nu} \rangle \) are divided into steady-state and linear-response components [40]
\[ \langle \sigma_{\mu\nu} \rangle = \langle \sigma_{\mu\nu} \rangle_S + \langle \sigma_{\mu\nu} \rangle_L e^{i(\omega_d - \omega_p)t} , \] (35)
where \( \omega_p \) is the probe frequency. The steady-state component \( \langle \sigma_{\mu\nu} \rangle_S \) of Eq. (35) is calculated from Eq. (32) with the probe turned off (\( \langle a_{\text{in}} \rangle = 0 \)), i.e., by solving
\[ i \omega_{\mu\nu} \langle \sigma_{\mu\nu} \rangle_S - \sum_{\mu'\nu'} \xi_{\mu\nu,\mu'\nu'} \langle \sigma_{\mu'\nu'} \rangle_S = 0 , \] (36)
where \( \xi_{\mu\nu,\mu'\nu'} = \langle \mu' | \xi_{\mu\nu} | \nu' \rangle \), and applying the condition \( \sum_{\mu} \langle \sigma_{\mu\nu} \rangle_S = 1 \). Inserting them into the following equations (see Appendix A), we can calculate the linear response components by solving
\[ i(\omega_{\mu\nu} + \omega_p - \omega_d) \langle \sigma_{\mu\nu} \rangle_L - \sum_{\mu'\nu'} \xi_{\mu\nu,\mu'\nu'} \langle \sigma_{\mu'\nu'} \rangle_L = iF \times \sum_{\mu'\nu'} \zeta_{\mu\nu,\mu'\nu'} \langle \sigma_{\mu'\nu'} \rangle_S \] (37)
with \( \zeta_{\mu\nu,\mu'\nu'} = \langle \mu' | \zeta_{\mu\nu} | \nu' \rangle \) and \( F \) the amplitude of the weak probe (\( |F|^2 \) is the number of incoming probe photons per second).

### 3.3. Amplification

The reflection coefficient is defined by
\[ r = \frac{\langle a_{\text{out}} \rangle}{\langle a_{\text{in}} \rangle} , \] (38)
with
\[ \langle a_{\text{in}} \rangle = Fe^{i(\omega_d - \omega_p)t} \] (39)
\[ \langle a_{\text{out}} \rangle = \langle a_{\text{in}} \rangle - i \langle \sigma_t \rangle = \left( F - i \sum_{\mu\nu} \sigma_{t,\mu\nu} \langle \sigma_{\mu\nu} \rangle_L \right) e^{i(\omega_d - \omega_p)t} \] (40)
Ultimate quantum limit for amplification: a single atom in front of a mirror

Figure 4. Maximizing the reflection coefficient of a strongly driven two-level system in front of a mirror. (a) Reflection coefficient $|r|$ of a weak probe for resonant drive ($\omega_d - \omega_{10} = 0$) as a function of the detuning of the probe frequency and drive amplitude. The maximum possible amplitude gain can be seen in the bright areas around $\Omega_d \approx 2\Gamma_{10}$ (marked by the solid line) and $(\omega_p - \omega_{10}) \approx \pm 1.2\Gamma_{10}$. (b) A plot of the linecut at $\Omega_d = 2\Gamma_{10}$ in (a). It compares the reflection of a two-level system in front of a mirror (blue, the linecut) to that in an open waveguide (purple, dashed). In both cases, we observe a maximum gain around $(\omega_p - \omega_{10}) \approx \pm 1.2\Gamma_{10}$, but the gain for the atom in front of a mirror is around 6.9%, around twice the gain for the atom in an open waveguide, which is around 3.4%.

$\sigma_{t,\mu\nu} = \langle \mu | \sigma_t | \nu \rangle$. For the two-level system with $\delta \omega_{10} = 0$, the reflection coefficient becomes

$$r = 1 - \frac{2\Gamma_{10}^2 (\Gamma_{10}^3 - 3i\Gamma_{10}^2\delta - 2\Gamma_{10}\delta^2 + 2i\delta\Omega_d^2)}{(\Gamma_{10} - 2i\delta) (\Gamma_{10}^2 + 2\Omega_d^2) (\Gamma_{10}^2 - 3i\Gamma_{10}\delta - 2\delta^2 + 2\Omega_d^2)},$$

(41)

where $\delta = \omega_{10} - \omega_p$.

By maximizing $|r|$ using Eq. (41), we find that the maximum possible amplitude gain is $|r| \approx 1.069$; it is achieved for the drive amplitude $\Omega_{10} \approx 2\Gamma_{10}$ and probe detuning of $\delta \approx \pm 1.2\Gamma_{10}$. It is interesting to note that the experiment in Ref. [33] appears to have come very close to this theoretical maximum.

Performing a similar analysis of the reflection coefficient for a two-level system in an open waveguide, we find that the maximum reflection for the same drive amplitude and detuning is only given by $|r|^2 \approx 1.034$, which is only half of the gain for the atom in front of a mirror. This makes sense, since the atomic output is divided between two propagation directions in the open waveguide, while it is collected in a single output channel when a mirror is included.

In Fig. 4(a), we plot the reflection coefficient of the strongly driven two-level system as a function of the detuning $\delta$ of the probe frequency and the drive strength $\Omega_d$ for resonant drive. The bright areas correspond to gain and the dark areas to attenuation. We can see how gain is achieved at probe frequencies in-between the frequencies corresponding to the Mollow triplet. In Fig. 4(b), the linecut of Fig. 4(a)
Ultimate quantum limit for amplification: a single atom in front of a mirror

is depicted. The blue curve, showing the reflection of the two-level system in front of a mirror, has a higher maximum gain than the reflection of the two-level system in an open waveguide, seen as a comparison by the purple dashed curve. This coincides with the analysis of the reflection coefficient above.

4. Amplification with a strongly two-photon-driven three-level atom in front of a mirror

For our last setup, we consider a strongly driven three-level atom in front of a mirror, with the drive at half the $|0⟩ \leftrightarrow |2⟩$ transition frequency, as sketched in Fig. 1(d). As shown experimentally for an open waveguide in Ref. [40], amplification can be achieved in this setup through population inversion among the dressed states of the three-level atom.

4.1. Hamiltonian and equations of motion

We consider the same Hamiltonian as for the two-level case, Eqs. (27)–(30), but including the third atomic level in the bare atomic Hamiltonian and in the Hamiltonian describing the interaction between the atom and the waveguide. The atom Hamiltonian in Eq. (28) is modified to read

$$H_a = \delta_{10}\sigma_{11} + \delta_{20}\sigma_{20} + E\left(\sigma_t + \sigma_t^\dagger\right)$$

where $\delta_{10} = \omega_{10} - \omega_d$, $\delta_{20} = \omega_{20} - 2\omega_d$, and $\sigma_t = \sqrt{\Gamma_{10}}\sigma_{01} + \sqrt{\Gamma_{21}}\sigma_{12}$. This new expression for $\sigma_t$ is the only change required in the interaction Hamiltonian in Eq. (30).

We set up and solve the equations of motion for the dressed-state operators $\sigma_{\mu\nu}$ in the same way as for the two-level system in Sec. 3.1, but with $\mu, \nu \in \{g, m, e\}$, where $\{g, m, e\}$ are the dressed states of the three-level system. As shown in detail in Appendix A, the equations for the steady-state and linear-response components of the reflected probe signal are the same as in Eqs. (36)–(37) in Sec. 3.1 except for the new definitions of variables given here.

4.2. Amplification

In Fig. 5, we plot the numerically computed reflection coefficient for a weak probe as a function of probe frequency $\omega_p$ and drive amplitude $\Omega_d = 2E\sqrt{\Gamma_{10}}$ for typical experimental parameters [40]. We observe a maximum amplitude gain of $\sim 6\%$. The largest gains are observed when the probe is close to resonant with one of the dressed-state transitions $|m, N⟩ \leftrightarrow |e, N + 1⟩$ and $|g, N⟩ \leftrightarrow |m, N + 1⟩$.

Since the population inversion among dressed states is essential for the amplification in this system, we explore further whether we can increase this population inversion by tuning the ratio between relaxation rates for the different atomic transitions. We consider the reflection along the branches for resonant probing, which are shown
Figure 5. Amplification through population inversion among the dressed states of a strongly driven three-level atom. (a) Reflection as a function of the probe frequency $\omega_p$ and drive amplitude $\Omega_d$ for a drive frequency of $\omega_d/2\pi = 7.26$ GHz for a three-level system in front of a mirror with the transition frequencies $\omega_{10}/2\pi = 7.4$ GHz, $\omega_{20} = 2\omega_d$, and the relaxation rates $\Gamma_{10}/2\pi = 40$ MHz, $\Gamma_{21} = 2\Gamma_{10}$. The dashed lines show the possible transitions between dressed states in the system. The dark region at $\omega_p/2\pi \approx 7.58$ GHz is outside of the plot range, since the reflection is low. (b) Sketch of the dressed states. The arrows demonstrating the transitions correspond to the three upper branches in panel (a). (c) Matrix elements of the steady-state solution showing the (non-)inverted population. Population inversion occurs for positive values of $\langle \sigma_{\mu\mu} \rangle_S - \langle \sigma_{\nu\nu} \rangle_S$, $\mu > \nu$, which is indicated by the grey dashed line at 0. The colors of the curves correspond to the colors of the arrows in (b).

by the dashed black lines in Fig. 5. The expression for the reflection on resonance $\omega_p = \omega_d + \omega_\nu - \omega_\mu$ can be simplified to

$$r = 1 + \frac{|\langle \mu | \sigma_\nu | \nu \rangle|^2}{\xi_{\mu\nu\mu'}\nu'} \left( \langle \sigma_{\nu\nu} \rangle_S - \langle \sigma_{\mu\mu} \rangle_S \right).$$ (43)

This equation shows that population inversion among the dressed states leads to a gain in the reflection, whereas we obtain attenuation for non-inverted population.

In Fig. 6, we plot the second term of the right-hand side in Eq. (43), which corresponds to the gain, for the upper branches in Fig. 5 as a function of the drive strength $\Omega_d$ and the ratio of the decay rates $\Gamma_{21}/\Gamma_{10}$. The bright parts of the panels in Fig. 6 correspond to the highest resonant gains. Selecting the values for the drive strength $\Omega_d/\Gamma_{10}$ and the ratio of the decay rates $\Gamma_{21}/\Gamma_{10}$ that give the highest resonant gain, we find a maximum gain of 6.2 % by searching around the resonance frequency of the $|g, N \rangle \leftrightarrow |m, N + 1 \rangle$ [Fig. 6(a)] transition for $\Omega_d/\Gamma_{10} = 8$ and $\Gamma_{21}/\Gamma_{10} = 2.3$. The corresponding gain for a transmon qubit in an open transmission line is around 3 %. Note that for a transmon in an open transmission line, $\Gamma_{21}/\Gamma_{10}$ is always 2 (assuming a flat spectral density for the transmission line). The mirror allows us to tune $\Gamma_{21}/\Gamma_{10}$
to achieve a higher gain, but the increase is small, since the optimal ratio of relaxation rates is close to 2. Thus, the main contribution of the mirror to the increased gain is to direct all atomic output in one direction.

Repeating the same optimization for the \(|m, N\rangle \leftrightarrow |e, N + 1\rangle\) transition in Fig. 6(b), we find a maximum gain of 6.1% for \(\Omega_d/\Gamma_{10} = 8.5\) and \(\Gamma_{21}/\Gamma_{10} = 2.8\). Once again, the optimal ratio of relaxation rates is close to 2, meaning that the improvement in gain compared to the open-transmission-line case is just a little more than a factor 2. The \(|g, N\rangle \leftrightarrow |e, N + 1\rangle\) transition [Fig. 6(c)] differs from the previous two in that the highest amplification is found when \(\Gamma_{21} \gg \Gamma_{10}\). However, the maximum gain around this transition is smaller than that close to the other two transitions.

5. Discussion and conclusion

In this article, we have investigated three different types of single-atom amplifiers, using population inversion, higher-order multi-photon processes, and hidden inversion in the dressed-state basis. For all these schemes, we compared the maximum achievable amplitude gain with the atom placed in front of a mirror versus when the atom was coupled to an open waveguide. The results are summarized in Table I. We note that first setup with a mirror reached an amplitude gain not too far from the absolute theoretical limit of \(\sqrt{2}\), which corresponds to perfect population inversion and perfect stimulated
Table 1. A summary of the results in the article. We compare the highest amplitude gain we found for each of the three setups with a mirror in Fig. 1 to the highest amplitude gain found or observed for the same systems in an open waveguide.

| Setup | 3 levels, $\omega_d = \omega_{20}$ | 2 levels, $\omega_d = \omega_{10}$ | 3 levels, $\omega_d = \omega_{20}/2$ |
|-------|---------------------------------|---------------------------------|---------------------------------|
| Schematic | ![1(b)](image) | ![1(c)](image) | ![1(d)](image) |
| Gain with mirror | 25% | 6.9% | 6.2% |
| Gain in open waveguide | 12.5% | 3.4% | 3% |

We found that for all schemes, the gain is enhanced by the mirror, mainly because of two reasons:

(i) The mirror reduces the number of output channels for the electromagnetic field from two in an open waveguide to one. All output from the atom is thus contributing to the gain instead of only half.

(ii) The mirror creates standing waves of the electromagnetic field through interference such that the strength of the coupling between the atom and the field becomes sensitive to the atomic position and transition frequencies. This makes it possible to tune the ratio of decay rates for different atomic transitions to increase population inversion and thus enhance amplification.

These insights are ready to be demonstrated in experiments with superconducting qubits in waveguide QED. Specifically, we showed that our set-ups can be implemented with a transmon coupled to a 1D semi-infinite transmission line with currently available technology (at least two such experiments have already come close to the limits in Table 1). We believe that this can prove useful for on-chip amplification to improve signal-to-noise ratios in experiments in quantum information and quantum optics.

An important direction for future work is to investigate how the achievable gain changes if more qubits are added to the setups described here. For example, one could imagine a cascaded setup of atoms in front of mirrors with circulators ensuring unidirectional propagation from one mirror to the next, as shown to enhance photon detection with three-level atoms in Ref. [94]. It could also be interesting to check how the anharmonicity of the qubit affects the gain for the three-level system with a two-photon drive.

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Appendix A. Equations of motion for a strongly driven atom in front of a mirror

In this appendix, we derive the equations of motion for the strongly driven atom in front of a mirror. The derivation applies both for the two-level atom in Sec. 3 and three-level atom in Sec. 4. We start with the diagonalised form of the two-level Hamiltonian \( H_2^{\text{dressed}} \) and three-level Hamiltonian \( H_3^{\text{dressed}} \), given by

\[
H_2^a = \omega_g \sigma_{gg} + \omega_e \sigma_{ee},
\]

\[
H_3^a = \omega_g \sigma_{gg} + \omega_m \sigma_{mm} + \omega_e \sigma_{ee}.
\]

The field and interaction Hamiltonians are

\[
H_f = \int d\omega \, \omega a^\dagger(\omega)a(\omega),
\]

\[
H_{a-f} = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \left( a^\dagger(\omega)\sigma_t + \sigma_t^\dagger a(\omega) \right),
\]

with \( \sigma_t = \sqrt{\Gamma_{10}\sigma_{01}} \) for the two-level system, and \( \sigma_t = \sqrt{\Gamma_{10}\sigma_{01} + \Gamma_{21}\sigma_{12}} \) for the three-level system.

In the following, we denote the dressed-state operators by \( \sigma_{\mu\nu} \) and the dressed-state transition frequencies by \( \omega_{\mu\nu} = \omega_\mu - \omega_\nu \), with \( \mu\nu \in \{g, m, e\} \) for the three-level atom and \( \mu\nu \in \{g, e\} \) for the two-level atom. The equation of motion for \( \sigma_{\mu\nu} \) can be calculated by the Heisenberg equation

\[
\frac{d}{dt} \sigma_{\mu\nu} = i[H, \sigma_{\mu\nu}].
\]

We find

\[
i[H_a, \sigma_{\mu\nu}] = i[\omega_g \sigma_{gg} + \omega_m \sigma_{mm} + \omega_e \sigma_{ee}, \sigma_{\mu\nu}]
\]

\[
= \omega_g \left( |g\rangle \langle g||\mu\rangle \langle \nu| - |\mu\rangle \langle \nu||g\rangle \langle g| \right)
\]

\[
+ \omega_m \left( |m\rangle \langle m||\mu\rangle \langle \nu| - |\mu\rangle \langle \nu||m\rangle \langle m| \right)
\]

\[
+ \omega_e \left( |e\rangle \langle e||\mu\rangle \langle \nu| - |\mu\rangle \langle \nu||e\rangle \langle e| \right)
\]

\[
= i(\omega_\mu - \omega_\nu) \sigma_{\mu\nu},
\]

and

\[
i[H_{a-f}, \sigma_{\mu\nu}] = \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega \left( a^\dagger(\omega)[\sigma_t, \sigma_{\mu\nu}] + [\sigma_t^\dagger, \sigma_{\mu\nu}] a(\omega) \right)
\]
Ultimate quantum limit for amplification: a single atom in front of a mirror

\[ = \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega a^\dagger(\omega)[\sigma_t, \sigma_{\mu\nu}] + \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] a(\omega). \quad (A.7) \]

In the same way, we calculate the equation of motion for \( a(\omega) \):

\[ \dot{a}(\omega) = i \int_{-\infty}^\infty d\omega \omega' [a^\dagger(\omega)a(\omega), a(\omega')] + i \int_{-\infty}^\infty d\omega \sqrt{\frac{1}{2\pi}} [a^\dagger(\omega)\sigma_t, a(\omega')] = -i\omega a(\omega) - \frac{i}{\sqrt{2\pi}} \sigma_t. \quad (A.8) \]

The solution of Eq. (A.8) can be written as

\[ a(\omega) = e^{-i\omega(t-t_0)}a_0(\omega) - \frac{i}{\sqrt{2\pi}} \int_{t_0}^t dt' \sigma_t(t')e^{-i\omega(t-t')}. \quad (A.9) \]

Inserting this solution into Eq. (A.7), we find

\[ \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega a^\dagger(\omega)[\sigma_t, \sigma_{\mu\nu}] = \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega e^{i\omega(t-t_0)}a_0^\dagger(\omega)[\sigma_t, \sigma_{\mu\nu}] - \frac{1}{2\pi} \int_{t_0}^t dt' \int_0^\infty d\omega e^{i\omega(t-t_0)}\sigma^\dagger(t')[\sigma_t, \sigma_{\mu\nu}] = -ia^\dagger_{in}[\sigma_t, \sigma_{\mu\nu}] - \int_{t_0}^t dt' \delta(t-t')\sigma^\dagger(t')[\sigma_t, \sigma_{\mu\nu}] = -ia^\dagger_{in}[\sigma_t, \sigma_{\mu\nu}] - \frac{1}{2} \sigma^\dagger_t[\sigma_t, \sigma_{\mu\nu}], \quad (A.10) \]

and

\[ \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega a(\omega) \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] = \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega e^{-i\omega(t-t_0)}a_0(\omega) \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] + \frac{1}{2\pi} \int_{t_0}^t dt' \int_0^\infty d\omega e^{-i\omega(t-t_0)} \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] \sigma_t(t') = i \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] a_{in} + \int_{t_0}^t dt' \delta(t'-t) \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] \sigma_t(t') = i \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] a_{in} + \frac{1}{2} \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] \sigma_t, \quad (A.11) \]

where we used

\[ \int_{t_0}^t dt' \sigma_t(t')\delta(t-t') = \frac{1}{2} \sigma_t(t) \quad (A.12) \]

and defined

\[ a_{in}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty d\omega e^{-i\omega(t-t_0)}a_0(\omega). \quad (A.13) \]

Combining the above results, the equation of motion for \( \sigma_{\mu\nu} \) becomes

\[ \frac{d}{dt} \sigma_{\mu\nu} = i(\omega_\mu - \omega_\nu)\sigma_{\mu\nu} - ia^\dagger_{in}[\sigma_t, \sigma_{\mu\nu}] + \frac{1}{2} \sigma^\dagger_t[\sigma_t, \sigma_{\mu\nu}] + i \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] a_{in} - \frac{1}{2} \left[ \sigma^\dagger_t, \sigma_{\mu\nu} \right] \sigma_t = i\omega_{\mu\nu}\sigma_{\mu\nu} + ia^\dagger_{in}[\sigma_{\mu\nu}, \sigma_t] - i[\sigma_{\mu\nu}, \sigma_t] a_{in} - \frac{1}{2} \sigma_{\mu\nu} \sigma^\dagger_t \sigma_t - \frac{1}{2} \sigma^\dagger_t \sigma_t \sigma_{\mu\nu} + \sigma^\dagger_t \sigma_{\mu\nu} \sigma_t = i\omega_{\mu\nu}\sigma_{\mu\nu} - \xi_{\mu\nu} - i\zeta_{\mu\nu} a_{in}(t) + ia^\dagger_{in}(t)\zeta^\dagger_{\nu\mu}, \quad (A.14) \]
with

$$
\xi_{\mu
u} = \frac{1}{2} \sigma_{\mu\nu} \sigma_t^\dag \sigma_t + \frac{1}{2} \sigma_t^\dag \sigma_t \sigma_{\mu\nu} - \sigma_t^\dag \sigma_{\mu\nu} \sigma_t \quad (A.15)
$$

$$
\zeta_{\mu\nu} = [\sigma_{\mu\nu}, \sigma_t^\dag]. \quad (A.16)
$$

The steady-state solution to Eq. \[A.14\] with \(\langle a_m \rangle = 0\) is given by

$$
\frac{d}{dt} \langle \sigma_{\mu\nu} \rangle_S = 0 \Rightarrow \ i\omega_{\mu\nu} \langle \sigma_{\mu\nu} \rangle_S - \sum_{\mu',\nu'} \xi_{\mu\nu,\mu',\nu'} \langle \sigma_{\mu',\nu'} \rangle_S = 0. \quad (A.17)
$$

Together with the linear-response part \(\langle \sigma_{\mu\nu} \rangle_L e^{i(\omega_d - \omega_p)}\), this gives

$$
\frac{d}{dt} \left( \langle \sigma_{\mu\nu} \rangle_S + \langle \sigma_{\mu\nu} \rangle_L e^{i(\omega_d - \omega_p)} \right) = i\omega_{\mu\nu} \left( \langle \sigma_{\mu\nu} \rangle_S + \langle \sigma_{\mu\nu} \rangle_L e^{i(\omega_d - \omega_p)} \right)
$$

$$
- \sum_{\mu',\nu'} \xi_{\mu\nu,\mu',\nu'} \left( \langle \sigma_{\mu',\nu'} \rangle_S + \langle \sigma_{\mu',\nu'} \rangle_L e^{i(\omega_d - \omega_p)} \right)
$$

$$
- i \sum_{\mu',\nu'} \zeta_{\mu\nu,\mu',\nu'} \left( \langle \sigma_{\mu',\nu'} \rangle_S + \langle \sigma_{\mu',\nu'} \rangle_L e^{i(\omega_d - \omega_p)} \right) F e^{i(\omega_d - \omega_p)}
$$

$$
+ i \sum_{\mu',\nu'} \zeta_{\mu\nu,\mu',\nu'} \left( \langle \sigma_{\mu',\nu'} \rangle_S + \langle \sigma_{\mu',\nu'} \rangle_L e^{i(\omega_d - \omega_p)} \right) F e^{-i(\omega_d - \omega_p)}. \quad (A.18)
$$

Now we use \(\frac{d}{dt} \langle \sigma_{\mu\nu} \rangle_S = 0\), \(\frac{d}{dt} \langle \sigma_{\mu\nu} \rangle_L = \omega_d - \omega_p \langle \sigma_{\mu\nu} \rangle_L\), \(i\omega_{\mu\nu} \langle \sigma_{\mu\nu} \rangle_S = \sum_{\mu',\nu'} \xi_{\mu\nu,\mu',\nu'} \langle \sigma_{\mu',\nu'} \rangle_S\), \(F \times \langle \sigma_{\mu',\nu'} \rangle_L \ll 1\), and neglect fast rotating terms. We then find

$$
i(\omega_{\mu\nu} - \omega_d + \omega_p) \langle \sigma_{\mu\nu} \rangle_L - \sum_{\mu',\nu'} \xi_{\mu\nu,\mu',\nu'} \langle \sigma_{\mu',\nu'} \rangle_L e^{i(\omega_d - \omega_p)} = iF \sum_{\mu',\nu'} \zeta_{\mu\nu,\mu',\nu'} \langle \sigma_{\mu',\nu'} \rangle_S, \quad (A.19)$$

which is used to calculate the results for amplification in Secs. 3 and 4.

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