Research Article

Global Existence and Extinction Singularity for a Fast Diffusive Polytropic Filtration Equation with Variable Coefficient

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In this article, we deal with an inhomogeneous fast diffusive polytropic filtration equation. By using the energy estimate approach, Hardy–Littlewood–Sobolev inequality, and a series of ordinary differential inequalities, we prove the global existence result and obtain the conditions on the occurrence of the extinction phenomenon of the weak solution.

1. Introduction

Our main objectives in this article are to deal with the global existence and the extinction phenomenon of the inhomogeneous fast diffusive polytropic filtration equation:

\[
\begin{align*}
|x|^{-s}u_t - \text{div}\left(|\nabla u|^m|\nabla u|^m\right) &= u^q, \quad (x, t) \in \Omega \times (0, +\infty), \\
u(x, t) &= 0, \quad (x, t) \in \partial\Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^N (N > p)\) is a bounded domain with smooth boundary \(\partial\Omega\), \(x = (x_1, \ldots, x_N) \in \Omega\), \(|x| = \sqrt{x_1^2 + \cdots + x_N^2}\), \(u_0(x)\) is a nonnegative and bounded function with \(u_0 \in W_0^{1,p}(\Omega)\), and the parameters \(m, s, p,\) and \(q\) satisfy

\[
0 < m \leq 1, \\
0 < m(p - 1) < 1, \\
1 - \frac{m}{2} < q \leq 1, \\
0 \leq s < \frac{Nq}{q + m}
\]

Inhomogeneous parabolic problems arise in a wide range of physical contexts (see for instance [1–3] and the references therein, where a more detailed physical background can be found). Problem (1) can be used to describe the compressible fluid flows in a homogeneous isotropic rigid porous medium with \(u(x, t)\) being the density of the fluid and \(\alpha(x) = |x|^{-s}\) acting as the volumetric moisture content. On the other hand parabolic models like (1), together with differential equation models, stochastic differential equations, and linear systems, are regarded as the powerful tools to solve lots of problems from control engineering, image processing, and other areas (see [4–8]). Because of the degeneracy and the singularity, problem (1) might not have classical solution in general, and hence, we introduce definition of the weak solution as follows.

**Definition 1.** By a local weak solution to problem (1), we understand a function \(u \in C_{\text{loc}}(\Omega \times (0, T)) \cap L^2(\Omega \times (0, T)), \nabla u \in L^2(\Omega \times (0, T)), \nabla u \in L^2(\Omega \times (0, T))\) for some \(T > 0\), which moreover satisfies the following assumptions:

(i) For any \(0 \leq \phi \in C_{\text{loc}}(\Omega \times (0, T)), \phi|_{\partial\Omega} = 0\) and \(0 < t_1 < t_2 < T\), one has
In order to state well our results, we first introduce some definitions, fundamental facts, and useful symbols. Since \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), then there is a ball \( B(0, R) \subset \mathbb{R}^N \) centered at 0 with radius
\[
R = \sup_{x \in \Omega} \sqrt{x_1^2 + \cdots + x_N^2},
\]
such that \( \Omega \subset B(0, R) \).

We denote the norm of \( L^r(\Omega) \) by \( \| \cdot \|_r \) and the norm of \( W^{1,r}(\Omega) \) by \( \| \cdot \|_{W^{1,r}(\Omega)} \) that is, for any \( \phi \in L^r(\Omega) \),
\[
\| \phi \|_r = \begin{cases} 
\left( \int_\Omega |\phi(x)|^r \, dx \right)^{1/r} & \text{if } 1 \leq r < +\infty, \\
\text{ess sup} |\phi(x)| & \text{if } r = +\infty,
\end{cases}
\]
and for any \( \phi \in W^{1,r}(\Omega) \), \( \| \phi \|_{W^{1,r}(\Omega)} = \sqrt{\|\phi\|_r^r + \|\nabla \phi\|_r^r} \). According to Poincaré's inequality, one can see that \( \|\phi\|_r \) is equivalent to \( \| \phi \|_{W^{1,r}(\Omega)} \) in \( W^{1,r}(\Omega) \), and hence, we equip \( W^{1,r}_{0}(\Omega) \) with the norm \( \|\phi\| = \|\nabla \phi\| \).

Let \( u(x, t) \) be a weak solution of problem (1). Define an energy functional as the following form:
\[
E(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{m}{m + q} \int_\Omega u^{m+q} \, dx.
\]

Then, by (3), one can easily show that
\[
\frac{\partial E(u)}{\partial t} = -m \int_\Omega u^{m-1}(u_t)^2 \, dx \leq 0,
\]
which tells us that \( E(u) \) is nonincreasing with respect to \( t \).

We state our main results as follows.

**Theorem 1.** Suppose that the parameters \( m, p, q, \) and \( s \) satisfy (2), and the initial data \( u_0(x) \) is a nonnegative and bounded function with \( u_0 \in W^{1,r}_{0}(\Omega) \). Let \( u(x, t) \) be a solution of problem (1). Then, the maximal existence time of \( u(x, t) \) is \( T = +\infty \); that is, \( u(x, t) \) is a global solution. Moreover,

1. If \( 0 \leq \frac{m(p - 1)}{p} < q \leq \frac{m(p - 1)}{p} \), \( 0 \leq s \leq \min\{\frac{p}{(N - p)/(N - s)}, \frac{q}{(N - p)/(N - s)}\} \), and there is a constant \( a > \max\{(m + 1 - am)/(ma - mp), (1/p)\} \) with \( a = p(N - s)/(N - p) \) such that
   \[
   \int_\Omega |x|^{-s} u_0^{m+1} \, dx > 0 \quad \text{and} \quad E(u_0) \leq 0,
   \]
   when \( q = m(p - 1) \),
   \[
   \int_\Omega |x|^{-s} u_0^{m+1} \, dx > 0 \quad \text{and} \quad E(u_0) \leq 0,
   \]
   when \( q < m(p - 1) \),

   (ii) If \( 0 < \frac{(1 - m)/2}{q} \leq m(p - 1) < 1 \) and
where
\[ E(u_0) = \frac{1}{p} \int_{\Omega} |\nabla u_0|^p \, dx - \frac{m}{m+q} \int_{\Omega} u_0^{m+q} \, dx, \quad (11) \]

then the solution \( u(x,t) \) of problem (1) does not possess extinction phenomenon.

The rest of this article is organized as follows. In Section 2, we collect some useful auxiliary lemmas. The last section is mainly focused on the global existence and the conditions on the occurrence of the extinction phenomenon of the solution. By Hardy–Littlewood–Sobolev inequality and some ordinary differential inequalities, the proof of Theorem 1 will be given in Section 3.

### 2. Preliminaries

In this section, as preliminaries, we collect some well-known results, which play an important role in our proof of Theorem 1.

**Lemma 1** (see [37]). Suppose \( N > s \) and \( \Omega \subset \mathbb{R}^N \) is a bounded domain. Then, we have
\[
\kappa_1 \overset{\text{def}}{=} \int_{\Omega} |x|^{-s} \, dx \leq \int_{\Omega} |x|^{-s} \, dx = \int_0^R \int_{B_0 (0,r)} |x|^{-s} \, dS(x) \, dr \\
= \omega_N \int_0^R r^{-s} r^{N-1} \, dr = \frac{\omega_N}{N-s} R^{N-s} \bigg|_0^\infty, \quad (12)
\]

where \( B(0, R) \) is the ball in \( \mathbb{R}^N \) centered at 0 with radius \( R = \sup_{x \in \Omega} \sqrt{x_1^2 + \cdots + x_N^2} \) satisfying \( \Omega \subset B(0, R) \) and
\[
\omega_N = \frac{N \pi^{N/2}}{\Gamma((N/2) + 1)} \quad (13)
\]
denotes the surface area of the unit sphere \( \partial B(0, 1) \), and \( \Gamma \) is the usual Gamma function.

**Lemma 2** (see [38]). Suppose \( N \geq 2, 1 < \mu < N, \ 0 < \theta \leq \mu, \) and \( \sigma = \mu (N-\theta)/(N-\mu). \) Then, there is a positive constant \( \kappa_2 = \kappa_2 (\mu, \theta, N) \) such that
\[
\int_{\Omega} |u(x)|^p \, dx \leq \kappa_2 \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{(N-\theta)/(N-\mu)}, \quad (14)
\]
holds for any \( u \in W_0^{1,p} (\Omega) \), where \( \Omega \subset \mathbb{R}^N \) is a bounded domain.

**Lemma 3** (see [39]). Assume \( \theta, \delta, \) and \( \beta \) are positive constants. Let \( y(t) \) be a nonnegative absolutely continuous function satisfying
\[
\frac{dy}{dt} + \delta y^\theta (t) \geq \beta, \quad t > 0. \quad (15)
\]
Then, we have
\[
y(t) \geq \min \left\{ y(0), \left( \frac{\beta}{\delta} \right)^{1/\theta} \right\}. \quad (16)
\]

**Lemma 4** (see [40]). Suppose \( 0 < k < r \leq 1 \). Let \( y(t) \) be the solution of the ordinary differential inequality:
\[
\frac{dy}{dt} + Cy^k \leq y^r, \quad t > 0, \quad (17)
\]
\[
y(0) = y_0 > 0,
\]
where \( C > 0 \) and \( 0 < y < (Cy_0^{-k/r}/2) \). Then, there are two positive constants \( \eta \) and \( \xi \) such that, for \( t \geq 0, \)
\[
0 \leq y(t) \leq \xi e^{-\eta t}. \quad (18)
\]

### 3. Proof of Theorem 1

In this section, we will give the proof of the global existence result and the conditions on the occurrence of the extinction phenomenon of the solution \( u(x,t) \).

**Case 1.** If \( ((1-m)/2) - q < 1 \). Taking the test function \( \phi = u^m(x,t) \) in (3), and using Hölder’s inequality, one has
\[
\frac{1}{m+1} \frac{d}{dt} \int_{\Omega} |x|^{-s} u^{m+1} \, dx + \|u^m\|^p = \int_{\Omega} |x|^{-s} u^{m+1} \, dx \leq \left( \int_{\Omega} |x|^{-s} u^{m+1} \, dx \right)^{m+q/(m+1)} \left( \int_{\Omega} |x|^{(m+q)/(1-q)} \, dx \right)^{1-q/(m+1)} \quad (19)
\]
which implies that
\[
\frac{d}{dr} \int_{\Omega} |x|^{-s} u^{m+1} \, dx \leq \kappa_3 (m + 1) \left( \int_{\Omega} |x|^{-s} u^{m+1} \, dx \right)^{(m+q)/(m+1)},
\]
where
\[
\kappa_3 = \left( \int_{B(0,R)} |x|^{s(q+m)/(1-q)} \, dx \right)^{(1-q)/(1+m)} = \left( \frac{\omega_N (1-q)}{s(q+m) + N(1-q)/(1-q)} \right)^{(1-q)/(1+m)}.
\]

Integrating (20) from 0 to \( t \), one gets
\[
\int_{\Omega} |x|^{-s} u^{m+1} \, dx \leq \kappa_3 (1-q)t + \left( \int_{\Omega} |x|^{-s} u_0^{m+1} \, dx \right)^{(1-q)/(m+1)} \left( \int_{\Omega} u^{q+m} \, dx \right)^{(m+q)/(1-q)}.
\]

From (19) and (22), it follows that
\[
\|u\|_{q+m}^{q+m} \leq \kappa_3 (1-q)t + \left( \int_{\Omega} |x|^{-s} u_0^{m+1} \, dx \right)^{(1-q)/(m+1)} \left( \int_{\Omega} u^{q+m} \, dx \right)^{(m+q)/(1-q)}.
\]

On the other hand, taking the test function \( \phi = (u^m) \), in (3), then by using Cauchy’s inequality with \( \varepsilon \) and Hölder’s inequality, one can obtain
\[
\frac{4m}{(m+1)^2} \int_{\Omega} |x|^{-s} \left[ \left( u^{(m+1)/2} \right)^2 \right] \, dx + \frac{1}{p} \frac{d}{dr} \|u\|_p^p
\]
\[
= \frac{2m}{m+1} \int_{\Omega} |x|^{s/2} u^{(m+2q-1)/2} |x|^{-s/(2)} \left( u^{(m+1)/2} \right)^2 \, dx
\]
\[
\leq \frac{2me}{m+1} \int_{\Omega} |x|^{-s} \left[ \left( u^{(m+1)/2} \right)^2 \right] \, dx + \frac{m}{2e(m+1)} \int_{\Omega} |x|^{s} u^{m+2q-1} \, dx
\]
\[
\leq \frac{2me}{m+1} \int_{\Omega} |x|^{-s} \left[ \left( u^{(m+1)/2} \right)^2 \right] \, dx + \frac{m}{2e(m+1)} \left( \int_{\Omega} |x|^{s(q+m)/(1-q)} \, dx \right)^{(1-q)/(q+m)} \left( \int_{\Omega} u^{q+m} \, dx \right)^{(m+2q-1)/(q+m)}.
\]

Let \( \varepsilon \) be sufficiently small to ensure that \( (4m/(m+1)^2) - (2me/(m+1)) \geq 0 \), then by (21), (23), and (24), one has
\[
\frac{d}{dr} \|u\|_p^p \leq \frac{mp \kappa_3^2}{2e(m+1)} \left[ \kappa_3 (1-q)t + \left( \int_{\Omega} |x|^{-s} u_0^{m+1} \, dx \right)^{(1-q)/(m+1)} \left( \int_{\Omega} u^{q+m} \, dx \right)^{(m+2q-1)/(q+m)} \right].
\]
which means that the solution $u(x,t)$ of the problem (1) is global.

Case 2. If $q = 1$, taking the test function $\phi = u^m(x,t)$ in (3), then we can see that

$$\frac{1}{m+1} \frac{d}{dt} \int_{\Omega} |x|^{-s} u_{m+1}^m dx + \|u^m\|^p$$

$$= \int_{\Omega} |x|^{-s} u_{m+1}^m |x|^m dx \leq R \int_{\Omega} |x|^{-s} u_{m+1}^m dx,$$

which tells us that

$$\int_{\Omega} |x|^{-s} u_{m+1}^m dx \leq e^{(m+1)Rt} \int_{\Omega} |x|^{-s} u_{0}^m dx,$$

$$\|u^m\|_{m+1} \leq R e^{(m+1)Rt} \int_{\Omega} |x|^{-s} u_{0}^m dx.$$

On the other hand, taking the test function $\phi = (u^m)$, in (3), then Cauchy’s inequality with $\varepsilon$ leads to

$$\frac{4m}{(m+1)^2} \int_{\Omega} |x|^{-s} [(u^{(m+1)/2})_t]_t^2 dx + \frac{1}{p} \frac{d}{dt} \|u^m\|^p$$

$$= \frac{2m}{m+1} \int_{\Omega} |x|^{1/2} u^{(m+1)/2} |x|^{-s/2} (u^{(m+1)/2})_t dx$$

$$\leq \frac{2m}{m+1} \int_{\Omega} |x|^{-s} [(u^{(m+1)/2})_t]_t^2 dx + \frac{m}{2e(m+1)} \int_{\Omega} |x|^s u_{m+1}^m dx$$

$$\leq \frac{2m}{m+1} \int_{\Omega} |x|^{-s} [(u^{(m+1)/2})_t]_t^2 dx + \frac{mR^s}{2e(m+1)} \int_{\Omega} u_{m+1}^m dx.$$

Choosing $\varepsilon \in (0, (2/(m+1))^2)$ to guarantee that $(4m/(m+1)^2) - (2\alpha e/(m+1)) \geq 0$, then by (29), one has

$$\frac{d}{dt} \|u^m\|^p \leq \frac{mpR^s}{2e(m+1)^2} e^{(m+1)\varepsilon t} \int_{\Omega} |x|^{-s} u_{0}^m dx,$$

which implies that

$$\|u^m\|^p \leq \|u^m_0\|^p + \frac{mpR^s}{2e(m+1)^2} (e^{(m+1)\varepsilon t} - 1) \int_{\Omega} |x|^{-s} u_{0}^m dx.$$

Then, the proof of the global existence result is complete. Now, we take our attention to the extinction singularity of the solution $u(x,t)$ to problem (1). We denote $\alpha = \rho(N-s)/(N-p)$. Noticing that $0 \leq s < p$, we can verify that $\alpha \leq p$. Let $a$ be a constant satisfying

$$a > \max \left\{ \frac{m+1-\alpha m - p}{m-mp} \right\}.$$

From (32), it follows that

$$0 < \frac{m(pa+1)+1}{am(a+1)} < 1.$$

Selecting the test function $\phi = u^{m(pa+1)}(x,t)$ in (3), one has

$$\frac{1}{m(pa+1)+1} \frac{d}{dt} \int_{\Omega} |x|^{-s} u^{m(pa+1)+1} dx + \frac{pa+1}{(a+1)^2} \int_{\Omega} |\nabla u^{m(pa+1)+1}|^p dx = \int_{\Omega} u^{m(pa+1)+q} dx.\ (34)$$

Making use of Hölder’s inequality, one can find that
\[
\int_{\Omega} |x|^{-s} u^{m(p+1)+1} \, dx = \int_{\Omega} |x|^{-s(m(p+1)+1)/am(a+1)} u^{m(a+1)-(m(p+1)+1)/am(a+1)} |x|^{-s(1-(m(p+1)+1)/am(a+1))} \, dx \\
\leq \left( \int_{\Omega} |x|^{-s} u^{am(a+1)} \, dx \right)^{m(p+1)+1/am(a+1)} \left( \int_{\Omega} |x|^{-s} \, dx \right)^{1-(m(p+1)+1)/am(a+1)} \\
= \kappa_1^{1-(m(p+1)+1)/am(a+1)} \left( \int_{\Omega} |x|^{-s} u^{am(a+1)} \, dx \right)^{(m(p+1)+1)/am(a+1)},
\]

where \(\kappa_1\) is the same as that in (12). Since \(0 < m(p-1) < 1\) and \(0 < m \leq 1\), one can deduce that \(1 < p < 1 + (1/m)\). This together with the assumption \(N > p\) one has \(1 < p < \min\{N, 1 + (1/m)\}\). Meanwhile, recalling that \(0 \leq s < \min\{p, (Nq/(q + m))\}\), then it follows from Lemma 2 that

\[
\int_{\Omega} |x|^{-s} u^{am(a+1)} \, dx = \int_{\Omega} |x|^{-s} u^{m(a+1)-(p(N-s)/(N-p))} \, dx \leq \kappa_2 \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{(N-s)/(N-p)},
\]

where \(\kappa_2\) is given in Lemma 2. Combining (35) with (32), one sees

\[
\int_{\Omega} |x|^{-s} u^{m(p+1)+1} \, dx \leq \kappa_1^{1-(m(p+1)+1)/am(a+1)} \kappa_2^{\left(\left(\frac{(N-p)(N-s)}{(N-s)^2}\right)\frac{m(p+1)}{m(a+1)}\right)} \left( \int_{\Omega} |\nabla u|^m \, dx \right)^{m(p+1)/am(a+1)}.
\]

Exploiting (34) and (37), one can arrive at

\[
\frac{1}{m(p+1) + 1} \frac{d}{dt} \int_{\Omega} |x|^{-s} u^{m(p+1)+1} \, dx \\
+ \frac{p(a+1)}{(a+1)^p} |x|^{-s} u^{m(p+1)+1} \left( \int_{\Omega} |\nabla u|^m \, dx \right)^{m(p+1)/am(a+1)} \\
\leq \int_{\Omega} |x|^s u^{mp(a+1)} \, dx.
\]

In what follows, for the sake of simplicity, we denote \(y(t) = \int_{\Omega} |x|^{-s} u^{m(p+1)+1} \, dx\), and \(C_1 = \left\{ \frac{(m(p+1))}{(a+1)^p} \right\}_{(N-s)^2/N}^{\left(\frac{(N-p)(N-s)}{(N-s)^2}\right)\frac{m(p+1)}{m(a+1)}} \kappa_2^{\left(\left(\frac{(N-p)(N-s)}{(N-s)^2}\right)\frac{m(p+1)}{m(a+1)}\right)} \left( \int_{\Omega} |\nabla u|^m \, dx \right)^{m(p+1)/am(a+1)}
\)

If \(\max\{(1-m)/2, m(p-1)\} < q = 1\), then from (38), one can immediately know that

\[
\frac{dy}{dt} + C_1 t y^{mp(a+1)/(m(p+1)+1)} (t) \leq C_2 y(t),
\]

where \(C_2 = \left\{ \frac{m(p+1)}{(a+1)^p} \right\}_{(N-s)^2/N}^{\left(\frac{(N-p)(N-s)}{(N-s)^2}\right)\frac{m(p+1)}{m(a+1)}} \kappa_2^{\left(\left(\frac{(N-p)(N-s)}{(N-s)^2}\right)\frac{m(p+1)}{m(a+1)}\right)} \left( \int_{\Omega} |\nabla u|^m \, dx \right)^{m(p+1)/am(a+1)}
\)

which together with (39) yields

\[
0 \leq y(t) \leq \xi_1 e^{-\eta_1 t}.
\]

Putting \(T_0 = \max\{0, ((m+p)/2)_{(N-s)^2/N}^{\left(\frac{(N-p)(N-s)}{(N-s)^2}\right)\frac{m(p+1)}{m(a+1)}} \} \), then for any \(t > T_0\), (40) leads to

\[
2C_2 t^{(1-m(p-1))/(m(p+1)+1)} (t) < C_1,
\]

or

\[
\frac{dy}{dt} + \frac{C_1}{2} t^{mp(a+1)/(m(p+1)+1)} \leq 0, \quad t \geq T_0.
\]

Integrating above inequality from \(T_0\) to \(t\) leads to
\[
y^{(1 - m(p - 1))/(m(p + 1) + 1)}(t) \leq y^{(1 - m(p - 1))/(m(p + 1) + 1)}(T_0) - \frac{C_1|1 - m(p - 1)|}{2[m(p + 1) + 1]}(t - T_0), \quad t \geq T_0.
\]

The above inequality means that
\[
\lim_{t \to T_1^-} y(t) = \lim_{t \to T_1^-} \int_\Omega |x|^{-\frac{m}{m+1}} u^{m(p+1)+1} \, dx = 0,
\]
where
\[
\int_\Omega u^{m(p+1)+1} \, dx \leq |\Omega|^{(1-q)/(m(p+1)+1)} \left( \int_\Omega u^{m(p+1)+1} \, dx \right)^{\frac{m(p+1)+q}{(m+1)(m(p+1)+1)}}
\]
\[
\leq |\Omega|^{(1-q)/(m(p+1)+1)} R^{m(p+1)+q/(m(p+1)+1)} \left( \int_\Omega |x|^{-\frac{m}{m+1}} u^{m(p+1)+1} \, dx \right)^{\frac{m(p+1)+q}{(m+1)(m(p+1)+1)}}.
\]

Combining (38) with (46), one can conclude that
\[
\frac{dy}{dt} + C_1 y^{m(p+1)/(m(p+1)+1)}(t) \leq C_3 y^{m(p+1)+q/(m(p+1)+1)}(t),
\]
where
\[
C_3 = \lambda [m(p + 1) + 1]|\Omega|^{(1-q)/(m(p+1)+1)} R^{m(p+1)+q/(m(p+1)+1)}.
\]

Recalling that \(0 < m(p - 1) < q < 1\) and (32), one can check that
\[
0 < \frac{mp(a + 1)}{m(p + 1) + 1} < \frac{m(p + 1) + q}{m(p + 1) + 1} < 1.
\]

Then, by (47) and Lemma 4, one knows that there are two positive constants \(\eta_2\) and \(\xi_2\) satisfying
\[
0 \leq y(t) \leq \xi_2 e^{\eta_2 t},
\]
provided that \(2C_3 y^{(q-m(p-1))/(m(p+1)+1)}(0) < C_1\). Setting \(T_2 = \max\{0, [(m(p + 1) + 1)/\eta_2, (q - m(p - 1))/\ln((2C_3)/C_1)k_2(q-m(p-1))/(m(p+1)+1)]\}\), then for any \(t > T_2\), (50) leads to
\[
2C_3 y^{(q-m(p-1))/(m(p+1)+1)}(t) < C_1,
\]
which together with (47) yields
\[
\frac{dy}{dt} + C_1 y^{m(p+1)/(m(p+1)+1)} \leq 0, \quad t \geq T_2.
\]

The remainder proof is the same as the previous one in the case \(q = 1\), and we omit it here. Up to now, the proof of
\[
\frac{q - m(p - 1)}{m + q} \int_\Omega u^{m+q} \, dx \geq \left( \frac{q - m(p - 1)}{m + q} \right) M^{(m+q)/(m+1)}(t) \left( \int_\Omega |x|^{-\frac{m+q}{m+1}} u^{m+q} \, dx \right)^{1/(m+1)}.
\]
Exploiting (54) and (57), one can claim that
\[ M'(t) - C_M (m+q) l[(m+1)] (t) \geq - p E(u_0), \] (58)
where
\[ 0 > C_M = \frac{q - m(p - 1)}{m + q} \left( \int_\Omega |x|^q l[(m+q)](1-q)/(m+1) \right)^{(1-q)/(m+1)} \]
\[ > \frac{q - m(p - 1)}{m + q} R^{(m+q)l[(m+1)]}(1-q)/(m+1) > -\infty. \] (59)

Since \( M(0) > 0 \) and \( E(u_0) < 0 \), then from (58) and Lemma 3, it follows that
\[ M(t) \geq \min \left\{ M(0), \left( \frac{p E(u_0)}{C_M} \right)^{(m+1)/(m+q)} \right\} > 0, \] (60)
which means that the solution \( u(x,t) \) of problem (1) does not possess extinction phenomenon.

Data Availability
The data used to support the findings of this study are included within the article.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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