Extensions of Billingsley’s Theorem via Multi-Intensities

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Abstract

Let \( p_1 \geq p_2 \geq \ldots \) be the prime factors of a random integer chosen uniformly from 1 to \( n \), and let

\[
\frac{\log p_1}{\log n}, \frac{\log p_2}{\log n}, \ldots
\]

be the sequence of scaled log factors. Billingsley’s Theorem (1972), in its modern formulation, asserts that the limiting process, as \( n \to \infty \), is the Poisson-Dirichlet process with parameter \( \theta = 1 \).

In this paper we give a new proof, inspired by the 1993 proof by Donnelly and Grimmett, and extend the result to factorizations of elements of normed arithmetic semigroups satisfying certain growth conditions, for which the limiting Poisson-Dirichlet process need not have \( \theta = 1 \). We also establish Poisson-Dirichlet limits, with \( \theta \neq 1 \), for ordinary integers conditional on the number of prime factors deviating from the usual value \( \log \log n \).

At the core of our argument is a purely probabilistic lemma giving a new criterion for convergence in distribution to a Poisson-Dirichlet process, from which the number-theoretic applications follow as straightforward corollaries. The lemma uses ingredients similar to those employed by Donnelly and Grimmett, but reorganized so as to allow subsequent number theory input to be processed as rapidly as possible.

A by-product of this work is a new characterization of Poisson-Dirichlet processes in terms of multi-intensities.

1 Introduction

In this paper we provide a new criterion for convergence in distribution to PD(\( \theta \)) — the Poisson-Dirichlet process with parameter \( \theta \) — and then apply it to extend Billingsley’s theorem on the asymptotic distribution of log prime factors of a random integer to much more general number theoretic contexts.
The new criterion is an application of an existing general weak convergence lemma from [3], restated here as Proposition 1, which supplements Alexandrov’s Portmanteau Theorem on equivalent conditions for weak convergence [4]. That lemma governs, in particular, the convergence of a sequence of discrete nonlattice random variables to a continuum limit possessing a smooth density. Here, we adapt it for direct application to a sequence of random multisubsets of \((0,1]\), with a hypothesis yielding the limiting \(PD(\theta)\).

The following is the simplest version of our new criterion: Given a sequence \(A_n\) of multisubsets of \((0,1]\), let \(T_n\) denote the sum of the elements of \(A_n\), counting multiplicities, and for any set \(S \subseteq (0,1]\) let \(|A \cap S|\) denote the cardinality of the intersection, also counting multiplicities. Also, let \(L(n) = (L_1(n), L_2(n), \ldots)\), where \(L_i(n) := \text{the } i\text{th largest element of } A_n \text{ if } i \leq |A_n|, \text{ and } L_i(n) := 0 \text{ if } i > |A_n|\).

Lemma 2 then asserts the following: Suppose that \(T_n \leq 1\) almost surely, for all \(n\), and that for any collection of disjoint closed intervals \(I_i = [a_i, b_i] \subset (0,1], i = 1, \ldots, k\) satisfying \(b_1 + \cdots + b_k < 1\), for any \(k \geq 1\), we have

\[
\liminf_{n \to \infty} E |A_n \cap I_1| \cdots |A_n \cap I_k| \geq \prod_{i=1}^{k} \log \frac{b_i}{a_i}
\]  

as \(n \to \infty\). Then \(L(n)\) converges in distribution to a \(PD(1)\).

The more general version, Lemma 3, allows a limiting \(PD(\theta)\) with \(\theta \neq 1\), and has a somewhat more complicated expression on the right-hand side of the inequality. Nonetheless both versions are easy to use in our applications.

In proving Billingsley’s original theorem, for instance, the multiset \(A_n\) appearing above consists of the log prime factors, \(\log n p_i\), of a random integer in \([1,n]\); and so each number \(|A_n \cap I_i|\) is simply the number of prime factors, counting multiplicities, falling into \([n^{a_i}, n^{b_i}]\). The main step of the proof, confirmation of the hypothesis (1), reduces to scarcely more than a citation of Mertens’ formula [9]

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + B + o(1),
\]

where \(B\) is a constant.

In the later sections of this paper, we apply our new criterion to give

- A reproof of the original Billingsley’s theorem;
- a generalization to a class of normed arithmetic semigroups for which an analogue of Landau’s prime ideal theorem is valid, still yielding a \(PD(1)\) limit;
- a generalization to a class of normed arithmetic semigroups satisfying the growth hypotheses of Bredikhin’s theorem ([16], Section 2.5), yielding \(PD(\theta)\) limits with \(\theta \neq 1\); and
- a final generalization to ordinary integers, conditional on the number of prime factors in the selected integer deviating unusually from the prescription of Turán’s theorem; here too the limiting \(PD(\theta)\) has \(\theta \neq 1\).
Our work was inspired by the proof of Billingsley’s theorem given by Donnelly and Grimmett \[6\], whose ingredients are included amongst our own. Our initial motivation was to place as much of the burden as possible on self-contained probability tools and isolate the use of number theoretic input. One difference in our approach is that, internal to the new convergence lemma, we work directly with the density function of the GEM distribution instead of aiming for a limit process of the component uniforms, as they do.

2 Probability Background

2.1 PD(\(\theta\)) and GEM(\(\theta\))

For our present purposes the Poisson-Dirichlet point process PD(\(\theta\)) with parameter \(\theta > 0\) can be characterized in the following two equivalent ways:

I) PD(\(\theta\)) is the Poisson point process with scale invariant intensity measure \(\theta \, dx/x\) on \((0,1)\), conditioned on the sum of the arrivals being 1;

II) Let \(U_1, U_2, \ldots\) be a sequence of independent uniforms on \((0,1)\), let \(V_i = U_i^{1/\theta}\) for \(i \geq 1\), and let \(G_1 = 1-V_1, G_2 = V_1(1-V_2), G_3 = V_1V_2(1-V_3), \ldots\). Let \(L_1 \geq L_2 \geq L_3 \geq \ldots\) be the outcome of sorting \(G_1, G_2, G_3, \ldots\) into descending order, or ranking them, as we will say. Then \(L_1, L_2, L_3, \ldots\) are the arrivals of the PD(\(\theta\)) in \((0,1)\).

For information on this process and its role in combinatorial modeling, as well as alternative characterizations with proofs of equivalence see, e.g., \[1, 2, 11\].

The GEM(\(\theta\)) process \(G = (G_1, G_2, \ldots)\), appearing in the second characterization, has itself been well studied (see, e.g., \[1\]). It is easy to see that with probability one we have \(G_1 + \cdots + G_k < 1\), for all \(k\). So, in particular, there are no positive accumulation points and hence ranking \(G\) is actually possible.

We will exploit the following result from \[7\]; see also \[4\], p. 42–43.

**Lemma 1.** Let 
\[ G(n) = (G_1(n), G_2(n), \ldots) \]
be a sequence of processes of nonnegative numbers, each with almost surely finite sum; and for each \(n\) let 
\[ L(n) = (L_1(n), L_2(n), \ldots) \]
be the ranked version of \(G(n)\). Suppose \(G(n)\) converges in distribution to GEM(\(\theta\)). Then \(L(n)\) converges in distribution to PD(\(\theta\)).

For each \(k\), the first \(k\) coordinates \(G_1, \ldots, G_k\) of the GEM(\(\theta\)) possess a joint probability density function \(f_G\), with the formula (see (5.28) in Section 5.4 of \[1\])

\[ f_G(x_1, \ldots, x_k) = \frac{\theta^k(1-x_1-\cdots-x_k)^{\theta-1}}{1(1-x_1)(1-x_1-x_2)\cdots(1-x_1-\cdots-x_{k-1})} \quad (3) \]
for \((x_1, x_2, \ldots, x_k) \in U\), where \(U \subset \mathbb{R}^k\) is the open set

\[
U := \{x \in (0,1)^k : 0 < x_1 + \cdots + x_k < 1 \text{ and } x_i \neq x_j \text{ for } 1 \leq i < j \leq k\}. \quad (4)
\]

While exclusion of the subdiagonals is not usually imposed in the definition of \(U\), excluding a set of Lebesgue measure 0 does no harm, and it definitely finesse a technical issue arising in the proof of our main lemma.

2.2 A Proposition on Weak Convergence

For the reader’s convenience, we quote Proposition 2.1 from [3], upon which the new result will depend. Here we let \(X\) be a random element of \(\mathbb{R}^k\) with density \(f\) of the form \(f = f_U 1_U\), where \(U \subset \mathbb{R}^k\) is open, the function \(f_U : U \to (0, \infty)\) is continuous, and \(1_U : \mathbb{R}^k \to \{0, 1\}\) is the indicator function of \(U\).

**Proposition 1.** Let \(X\) be defined as above, and let \(X_n, n = 1, 2, \ldots\), be arbitrary random elements of \(\mathbb{R}^k\).

Suppose that, for every \(\varepsilon > 0\), there exists \(R < \infty\) for which every closed coordinate box \(B \subset U\) satisfying

\[
\text{diam} (B) < d(B, U^c)/R \quad (5)
\]

also satisfies

\[
\liminf_n \mathbb{P}(X_n \in B) \geq (1 - \varepsilon) \text{ vol } B \inf_B f. \quad (6)
\]

Then \(X_n \Rightarrow X\). That is, as \(n \to \infty\), \(X_n\) converges in distribution to \(X\).

3 Convergence to PD(\(\theta\))

3.1 Preliminaries

Our convergence criterion is most conveniently cast in the language of random multisubsets\(^1\) of the interval \((0, 1]\). In the course of the proof we will also need to consider size-biased permutations.

We consider only multisets whose multiplicities are all finite.

Informally, a process that generates random countable (or finite) multisets \(A \subset (0, 1]\) has been fully specified provided that for any finite collection \(S_1, \ldots, S_k\) of Borel subsets of \((0, 1]\), the cardinalities \(|A \cap S_1|, \ldots, |A \cap S_k|\), including multiplicities, have well-defined joint probability distributions\(^2\). Though infinite cardinalities may occur, for any singleton \(\{x\} \subset (0, 1]\) we must have

\(^1\) By saying \(A\) is a multisubset of \(U\), we mean that \(U\) is a universal set, and for each \(u \in U\), the multiplicity \(m_u = m_u(A)\) of \(u\) as an element of \(A\) is a nonnegative integer. There is of course an alternate reading of the phrase, with “\(A\) is a multisubset of \(B\)” to mean that both \(A, B\) are multisets, and for each \(u\) in the underlying universal set, \(m_u(A) \leq m_u(B)\).

\(^2\) This induces a probability measure on the space whose points are countable subsets of \((0, 1]\). For further information, including the identification of random multisets with random \(\sigma\)-finite integer-valued measures on the ambient space, see, e.g., [10], Chapter 12.
\[|A \cap \{x\}| < \infty.\] The joint distributions must obey any constraints implied by set inclusions.

Given an at most countable fixed multiset \(A\) of numbers in \((0, 1]\) (or, indeed, lying anywhere in the positive reals) with finite sum \(t\) (where each summand is included according to its multiplicity), a size-biased permutation is an ordered list generated by the following process: The first element selected \(\sigma_1\) equals \(s\) with probability proportional to \(m_s s\) where \(m_s\) is the multiplicity of \(s\) in \(A\); explicitly, \(\mathbb{P}(\sigma_1 = s) = m_s s / t\). Thereafter, conditional on selections already made, for any element \(s\) remaining in \(A\) the next element selected is \(s\) with probability proportional to \(m'_s s\), where \(m'_s\) is the multiplicity of \(s\) among those elements yet remaining to be selected. If \(|A| < \infty\), we explicitly set \(\sigma_k = 0\) for all \(k > |A|\). (For multisets the count \(|A|\) includes multiplicities.)

We may also take size-biased permutations of random multisets, with sum \(T < \infty\): The probability \(P(\sigma_1 = s_1, \ldots, \sigma_k = s_k)\), say, that the first \(k\) selections are \(s_1, \ldots, s_k\), is calculated by first conditioning on the random multiset \(A\) and calculating \(P(\sigma_1 = s_1, \ldots, \sigma_k = s_k | A)\) recursively, as above, and then taking the expectation as \(A\) varies.

### 3.2 The Main Lemma, \(\theta = 1\)

Since the special case \(\theta = 1\) is a bit less complicated than the general case, yet already suffices for the classical version of Billingsley’s result as well as for our first extension, we state and prove the result for this case first.

Given an arbitrary sequence \(A_n\) of random multisubsets \(A_n\) of \((0, 1]\), for each \(n\) define \(L(n)\) to be the sequence of elements of \(A_n\), including multiple occurrences, ranked by decreasing size, and padded with an infinite string of 0’s if \(A_n\) is finite. That is, we let \(L(n) = (L_1(n), L_2(n), \ldots)\) where \(L_i(n) := \) the \(i\)th largest element of \(A_n\) if \(i \leq |A_n|\), and \(L_i(n) := 0\) if \(i > |A_n|\). Also, define \(T_n := \sum_{a \in A_n} a\), where the defining sum is taken with multiplicities.

**Lemma 2.** Given an arbitrary sequence \(A_1, A_2, \ldots\) of random multisubsets of \((0, 1]\), with associated ranked sequences \(L(1), L(2), \ldots\) of elements, assume the following: first,

\[P(T_n \leq 1) = 1,\]  

and second, for any collection of disjoint closed \(I_i = [a_i, b_i) \subset (0, 1]\), \(i = 1, \ldots, k\) satisfying the hypothesis

\[b_1 + \cdots + b_k < 1,\]  

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\(^3\) The probability distribution of \(P(\sigma_1 = s_1, \ldots, \sigma_k = s_k | A)\) is itself determined by the joint distributions of cardinalities of intersections, together with the sum \(T\). The occurrence or non-occurrence of the event \(\{P(\sigma_1 = s_1, \ldots, \sigma_k = s_k | A) < x\}\) is determined by the multiplicities of \(s_1, \ldots, s_k\) and the sum of all remaining elements, taken with multiplicities; so the probability of that event is a function of the joint distribution of those quantities, i.e., the joint distribution of the intersection cardinalities \(|A \cap \{s_1\}|, \ldots, |A \cap \{s_k\}|\), and \(T\). So the expectation of \(P(\sigma_1 = s_1, \ldots, \sigma_k = s_k | A)\) is taken, by definition, with respect to this latter joint distribution.
we have

\[
\liminf_{n \to \infty} \prod_{i=1}^{k} |A_n \cap I_i| \geq \prod_{i=1}^{k} \frac{b_i - a_i}{b_i}.
\] (9)

Then \( L(n) \Rightarrow (L_1, L_2, \ldots)_{\theta=1} \), the Poisson-Dirichlet distribution with \( \theta = 1 \).

If in place of (9) we assume

\[
\liminf_{n \to \infty} \prod_{i=1}^{k} |A_n \cap I_i| \geq \prod_{i=1}^{k} \log \frac{b_i}{a_i},
\] (10)

then the same conclusion holds.

**Proof.** Since for \( b \geq a > 0 \) we have \( \log(b/a) \geq (b - a)/a \), it suffices to consider only (9). Taking our cue from Donnelly and Grimmett [6], for each \( n \) define a process,

\[ G(n) = (G_1(n), G_2(n), \ldots), \]

whose components are the successive elements of a size biased permutation of \( A_n \), padded with zeros if \( A_n \) is finite. We will use Lemma 1 with \( X \) equal to the first \( k \) coordinates of the GEM(1), in conjunction with (3) and (4), to show that as \( n \to \infty \), the first \( k \) coordinates of \( G(n) \) converge in distribution to the first \( k \) coordinates of a GEM(1), for each \( k \). Since this implies that \( G(n) \) converges to a GEM(1), we will then conclude by Lemma 1 that \( L(n) \), the ranked version of \( G(n) \), converges to a PD(1).

So let \( B = \prod_{i=1}^{k} [a_i, b_i] \) be a coordinate box whose component intervals satisfy our hypotheses. Conditional on \( A_n \), we see that

\[ \mathbb{P}(G_1(n) \in I_1) = \frac{\sum_{a \in A_n \cap I_1} a}{T_n} \geq |A_n \cap I_1| a_1. \]

Conditional also on the first \( j \) selections lying in \( I_1, \ldots, I_j \), respectively, since their sum must be at least \( a_1 + \cdots + a_j \), the conditional probability that

\[ G_{j+1} \in I_{j+1} \]

is at least

\[ \frac{|A_n \cap I_{j+1}| a_{j+1}}{T_n - (a_1 + \cdots + a_j)} \geq \frac{|A_n \cap I_{j+1}| a_{j+1}}{1 - (a_1 + \cdots + a_j)}, \]

where we have used the disjointness of the intervals to infer that all of \( A_n \cap I_{j+1} \) remains available.

Hence, we find that

\[ \mathbb{P}((G_1(n), \ldots, G_k(n)) \in B) \geq \mathbb{E} \left( \frac{|A_n \cap I_1| a_1}{1} \frac{|A_n \cap I_2| a_2}{1 - a_1} \cdots \frac{|A_n \cap I_k| a_k}{1 - (a_1 + \cdots + a_{k-1})} \right). \] (11)
Combining hypothesis (9) with the fact that \( \text{vol} B = \prod (b_i - a_i) \), and using formula (3) with \( \theta = 1 \), we see that

\[
\liminf_{n \to \infty} \mathbb{P}( (G_1(n), \ldots, G_k(n)) \in B ) \geq \prod_{i=1}^k \frac{a_i}{b_i} \text{vol } (B) f_1(a_1, \ldots, a_k)
\]

\[
\geq \prod_{i=1}^k \frac{a_i}{b_i} \text{vol } (B) \inf_B f_1.
\]

To apply Proposition 1 given \( \epsilon > 0 \) it will suffice to find \( R \) large enough so that for all coordinate boxes satisfying (5), we have \( \prod_{i=1}^k \frac{a_i}{b_i} \geq 1 - \epsilon \). Then (9) will imply (5). Note that while nothing in the statement of Lemma 1 explicitly allows us to restrict attention to boxes whose defining intervals are disjoint, as required for the invocation of (9), our crafty choice of domain \( U \) in (4) makes that automatic. Any closed box lying in \( U \) also satisfies \( b_1 + \cdots + b_k < 1 \), the other requirement.

Without harm we may restrict to \( \epsilon < 1 \). Since given any \( R > 0 \), a box \( B \) satisfying (5) satisfies

\[
b_i - a_i \leq \frac{\text{diam}(B)}{\sqrt{k}} < \frac{1}{R\sqrt{k}} d(B, U^c) \leq \frac{1}{R\sqrt{k}} a_i,
\]

it suffices to take

\[
R = \left[ \sqrt{k} \left( (1 - \epsilon)^{-1/k} - 1 \right) \right]^{-1},
\]

to get \( a_i/b_i > (1 - \epsilon)^{1/k} \) and hence \( \prod a_i/b_i > (1 - \epsilon) \).

With this choice of \( R \) we have satisfied (9), so Lemma 1 applies; and by the discussion beginning the proof we are done. \( \square \)

### 3.3 Characterization of PD(1) via Multi-intensity

Lemma 2 above gives a sufficient condition for convergence to the Poisson-Dirichlet distribution, with parameter \( \theta = 1 \). We now explain how this gives a new characterization of the PD distribution.

A standard concept for point processes is the intensity measure; in our setup with a random multiset set \( A \subset (0, 1] \) this is the deterministic measure \( \nu \) on the Borel subsets, defined by \( \nu(S) = \mathbb{E} |A \cap S| \). A standard result in measure theory, the \( \pi - \lambda \) theorem, implies that \( \nu \) is determined by its values on closed intervals, \( \nu(I) \) for \( I = [a, b], 0 < a < b \leq 1 \). At this level, both the Poisson-Dirichlet process PD(1) and the scale invariant Poisson process with intensity \( dx/x \) on \( (0, 1] \), have the same intensity with \( \nu([a, b]) = \int_a^b dx/x = \ln b/a \).

Second-order intensity has been considered, for example in [5]. It is natural to generalize, and define multi-intensity or \( k \)-fold intensity for \( k = 1, 2, \ldots \), by taking arbitrary choices of \( k \) disjoint closed intervals \( I_i = [a_i, b_i] \), setting \( B = \prod_{i=1}^k [a_i, b_i] \), and defining

\[
\mu(B) = \mathbb{E} |A \cap I_1| \ldots |A \cap I_k|.
\]
In case there is a function $f_k$ on $(0,1)^k \setminus \cup_{1 \leq i < j \leq k} \{x_i = x_j\}$, such that $\mu(B) = \int_B f_k(x_1, \ldots, x_k) \, dx_1 \cdots dx_k$, we say that the random set $A$ has multi-intensity density $f_k(x_1, \ldots, x_k)$ at $(x_1, \ldots, x_k)$. For example, any Poisson process with intensity $f(x)\, dx$ has multi-intensity density $f_k(x_1, \ldots, x_k) = f(x_1) \times \cdots \times f(x_k)$.

In particular, the scale invariant Poisson process with intensity $dx/x$ on $(0,1]$ has multi-intensity density

$$PP: f_k(x_1, \ldots, x_k) = \frac{1}{x_1 \cdots x_k}, \quad (13)$$

for all choices of distinct $x_1, x_2, \ldots, x_k \in (0,1]$.

The multi-intensity for the Poisson-Dirichlet is easily derived from I) in Section 2.1, the characterization of PD as PP conditional on $T = 1$, where $T$ is the sum of all the points of the Poisson process with intensity $dx/x$ on $(0,1]$. A special simplification arises from the property that the density $p(t)$ for $T$, given explicitly by $p(t) = e^{-\gamma} \rho(t)$ where $\rho(\cdot)$ is Dickman’s function, satisfies $p(u)/p(1) = 1$ for all $u \in (0,1]$. For distinct $x_1, \ldots, x_k \in (0,1]$ with $t := x_1 + \cdots + x_k$, by conditioning the Poisson process on having $T = 1$ we have for the PD

$$PD: f_k(x_1, \ldots, x_k) = \frac{1}{x_1 \cdots x_k} \frac{p(1-t)}{p(1)} = \begin{cases} \frac{1}{x_1 \cdots x_k} & \text{if } t < 1 \\ 0 & \text{if } t > 1 \end{cases}. \quad (14)$$

To summarize, by comparing (13) with (14), we see that for $k \geq 2$, the Poisson process and the Poisson-Dirichlet don’t have the same multi-intensity densities, but their densities agree, when restricted to $(x_1, \ldots, x_k)$ with $t := x_1 + \cdots + x_k < 1$.

**Corollary 1.** View the Poisson-Dirichlet process $(X_1, X_2, \ldots)$ as the random multisubset of $(0,1]$ given by $A = \{X_1, X_2, \ldots\}^\mathbb{N}$. Then the PD is the unique random $A$ for which both

$$T := \sum_{x \in A} x \text{ has } \mathbb{P}(T \leq 1) = 1,$$

where the sum is taken with multiplicities, and for each $k = 1, 2, \ldots$, the multi-intensity density of $A$ is given by the right side of (14).

**Proof.** If a multiset $A$ satisfies the given hypotheses, then we can apply Lemma 2 with $A_n = A$, for each $n = 1, 2, \ldots$. Conversely, we have already noted that starting with the PD, we have $1 = \mathbb{P}(T = 1)$, and multi-intensity density as given by (14). \qed

### 3.4 The Main Lemma, Arbitrary $\theta > 0$

We keep the notation of Section 3.2.

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4The condition $T < \infty$ implies that the multiset $A$ can be reconstructed from the sequence $(X_1, X_2, \ldots)$. 

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Lemma 3. Let $\theta > 0$. Given an arbitrary sequence $A_1, A_2, \ldots$ of random multisubsets of $(0, 1]$, with the associated ranked sequences $L(1), L(2), \ldots$ of elements, make the following assumption: Suppose that for some $-\infty < \alpha, \beta < \infty$ with $\alpha + \beta = 1 - \theta$, it is the case that for any collection of disjoint closed $I_i = [a_i, b_i] \subset (0, 1]$, $i = 1, \ldots, k$ satisfying the hypothesis

$$b_1 + \cdots + b_k < 1,$$

(15)

we have both

$$\liminf_{n \to \infty} \mathbb{E} \prod_{i=1}^{k} |A_n \cap I_i| \geq$$

$$\frac{\theta^k}{(1 - a_1 - \cdots - a_k)^{1-\theta}(1 - b_1 - \cdots - b_k)^{\beta}} \prod_{i=1}^{k} \frac{b_i - a_i}{b_i},$$

(16)

and

$$P(T_n \leq 1) = 1.$$  

(17)

Then $L(n) \Rightarrow (L_1, L_2, \ldots)$, the Poisson-Dirichlet process with parameter $\theta$.

If in place of (16) we assume

$$\liminf_{n \to \infty} \mathbb{E} \prod_{i=1}^{k} |A_n \cap I_i| \geq$$

$$\frac{\theta^k}{(1 - a_1 - \cdots - a_k)^{1-\theta}(1 - b_1 - \cdots - b_k)^{\beta}} \prod_{i=1}^{k} \log \frac{b_i}{a_i},$$

(18)

then the same conclusion holds.

Proof. As in the proof of Lemma 2, we appeal to Lemma 1, this time using (3) with $\theta > 0$ to specify the target limit density. Also, it suffices to consider only (16) and not (18). If the coordinates of $G(n) = (G_1(n), G_2(n), \ldots)$ are generated as a size-biased permutation of the elements of $A_n$, padded with zeros if necessary, then (11) gives a lower bound on $P((G_1(n), \ldots, G_k(n)) \in B)$. This combines with hypothesis (16) and formula (3) to give

$$\liminf_{n \to \infty} P((G_1(n), \ldots, G_k(n)) \in B) \geq$$

$$\frac{(1 - a_1 - \cdots - a_k)^{1-\theta}}{(1 - a_1 - \cdots - a_k)^{1-\theta}(1 - b_1 - \cdots - b_k)^{\beta}} \prod_{i=1}^{k} \frac{a_i}{b_i} \text{ vol}(B) f_\theta(a_1, \ldots, a_k)$$

$$= \left(\frac{1 - b_1 - \cdots - b_k}{1 - a_1 - \cdots - a_k}\right)^{-\beta} \prod_{i=1}^{k} \frac{a_i}{b_i} \text{ vol}(B) f_\theta(a_1, \ldots, a_k)$$

$$\geq \left(\frac{1 - b_1 - \cdots - b_k}{1 - a_1 - \cdots - a_k}\right)^{-\beta} \prod_{i=1}^{k} \frac{a_i}{b_i} \text{ vol}(B) \inf_B f_\theta$$

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where we have written $\gamma$ for $\max(0, -\beta)$.

We now show the preliminary factors can be replaced with $1 - \epsilon$: Given $1 > \epsilon' > 0$, setting

$$R = R_1 = \left[ \sqrt{k} \left( (1 - \epsilon')^{-1/k} - 1 \right) \right]^{-1}$$

will serve, via (12) for boxes complying with (5), to ensure that

$$\prod_{i=1}^{k} \frac{a_i}{b_i} \geq 1 - \epsilon'.$$

As for bounding

$$\left( 1 - b_1 - \cdots - b_k \right)^\gamma \left( 1 - a_1 - \cdots - a_k \right)^\gamma$$

when $\gamma > 0$, complying with (5) for a given $R$ also means

$$b_i - a_i \leq \frac{\text{diam}(B)}{\sqrt{k}} < \frac{1}{R\sqrt{k}} d(B, U^c) \leq \frac{1 - b_1 - \cdots - b_k}{kR}$$

for each $i$, where in the rightmost member we have measured distance from the hyperplane $x_1 + \cdots + x_k = 1$. Since $(1 - a_1 - \cdots - a_k) - (1 - b_1 - \cdots - b_k) = (b_1 - a_1) + \cdots + (b_k - a_k)$ we get

$$(1 - a_1 - \cdots - a_k) - (1 - b_1 - \cdots - b_k) \leq \frac{1 - b_1 - \cdots - b_k}{R},$$

and so for $R = R_2$ sufficiently large we have

$$\left( 1 - b_1 - \cdots - b_k \right)^\gamma \left( 1 - a_1 - \cdots - a_k \right)^\gamma \geq 1 - \epsilon'$$

as well. Thus when $\gamma > 0$, given $\epsilon > 0$ pick $\epsilon'$ small enough so that $(1 - \epsilon')^2 \geq 1 - \epsilon$, and then choose $R$ to be the larger of $R_1$ and $R_2$. Proposition 1 now applies, completing the argument. \qed

3.5 Characterization of PD($\theta$) via Multi-intensity

We now treat the situation for general $\theta > 0$, thereby extending the results of Section 3.3. We will be brief and highlight only the changes.

The scale invariant Poisson process with intensity $\theta \, dx/x$ on $(0, 1]$ has multi-intensity density

$$\text{PP}: f_k(x_1, \ldots, x_k) = \frac{\theta^k}{x_1 \cdots x_k},$$

(19)
for all choices of distinct \(x_1, x_2, \ldots, x_k \in (0, 1]\). For any \(\theta > 0\), the density \(p(t)\) for \(T\), restricted to \((0, 1]\), is given (see for example [1], formula (4.20)) by

\[
p(t) = \frac{e^{-\gamma \theta \frac{t}{\theta} - 1}}{\Gamma(\theta)} \quad \text{for } 0 < t \leq 1.
\]

Hence, by conditioning the Poisson \(\theta dx/x\) process on the event \(T = 1\), for distinct \(x_1, \ldots, x_k \in (0, 1]\) with \(t := x_1 + \cdots + x_k\), we have multi-intensity density

\[
PD: f_k(x_1, \ldots, x_k) = \frac{\theta^k}{x_1 \cdots x_k} \frac{p(1 - t)}{p(1)} = \begin{cases} 
\frac{\theta^k (1 - t)^{\theta - 1}}{x_1 \cdots x_k} & \text{if } t < 1 \\
0 & \text{if } t > 1
\end{cases}.
\]

(20)

For \(k = 1\), this gives the intensity measure of the PD,

\[
\nu(dx) = \theta(1 - x)^{\theta - 1} dx/x \quad \text{on } (0, 1],
\]

which differs, when \(\theta \neq 1\), from the intensity \(\theta dx/x\) of the corresponding Poisson process.

**Corollary 2.** Let \(\theta > 0\). View the Poisson-Dirichlet process with parameter \(\theta\), \((X_1, X_2, \ldots)\), as the random multisubset of \((0, 1]\) given by \(A = \{X_1, X_2, \ldots\}\). Then the PD is the unique random \(A\) for which both

\[
T := \sum_{x \in A} x \quad \text{has } P(T \leq 1) = 1,
\]

where the sum is taken with multiplicities, and for each \(k = 1, 2, \ldots\), the multi-intensity density of \(A\) is given by the right side of (20).

**Proof.** If a multiset \(A\) satisfies the given hypotheses, then we can apply Lemma 3 with \(A_n = A\), for each \(n = 1, 2, \ldots\). Conversely, we have already noted that starting with the PD, we have \(1 = P(T = 1)\), and multi-intensity density as given by (20). \(\square\)

## 4 Classic Billingsley

We reprove Billingsley’s original theorem, even though it becomes a special case of a later result.

**Theorem 1.** Given \(n > 1\), let \(p_1 \geq p_2 \geq \ldots\) be the prime factors, including multiple factors, of a random integer \(N\) chosen uniformly from 1 to \(n\); and for \(i > 0\) let

\[
L_i(n) = \log p_i / \log n,
\]

where we set \(L_i(n) = 0\) if \(N = 1\) or \(i\) exceeds the total number of prime factors, including multiplicities. Define

\[
L(n) = (L_1(n), L_2(n), \ldots).
\]

Then \(L(n)\) converges to a \(PD(1)\) as \(n \to \infty\). Equivalently, for each \(k > 0\) the \(k\)-tuple \((L_1(n), \ldots, L_k(n))\) converges in distribution to the first \(k\) coordinates of a \(PD(1)\).
Proof. Define the multiset $A_n$ to contain the non-zero coordinate entries of $L(n)$, and let $A_n^\circ$ be the set underlying $A_n$, i.e., with no multiple copies. We want to use Lemma 2. Since $|A_n \cap I| \geq |A_n^\circ \cap I|$ for any $I = (a, b) \subset (0, 1]$ and we want lower bounds, it suffices to investigate $A_n^\circ$.

Let $I_1 = [a_1, b_1], \ldots, I_k = [a_k, b_k]$ be disjoint subintervals of $(0, 1]$ with $b_1 + \cdots + b_k < 1$. The key step is to see that
\[
E \prod_{i=1}^k |A_n^\circ \cap I_i| = \sum_{q_1, \ldots, q_k} \frac{1}{q_1 \cdots q_k} + o(1) \quad (21)
\]
where the sum is over all $k$-tuples of primes lying in $[n^{a_1}, n^{b_1}] \times \cdots \times [n^{a_k}, n^{b_k}]$.

To verify (21), for integers $m, N > 0$ let $J_N(m)$ be the indicator function of the event $m | N$. If $N$ is our random integer we then have
\[
\prod_{i=1}^k |A_n^\circ \cap I_i| = \prod_{i=1}^k \sum_{q_i} \prod_{i=1}^k J_N(q_i) = \sum_{q_1, \ldots, q_k} J_N(q_1 \cdots q_k)
\]
where each $q_i$ ranges over the primes in $[n^{a_i}, n^{b_i}]$, and where we have used the disjointness of the intervals in the last step. Note that since $b := b_1 + \cdots + b_k < 1$ and each $q_i \leq n^{b_i}$, each product $q_1 \cdots q_k$ in the final sum, and hence also the total number of summands, cannot exceed $n^b = o(n)$.

Since $N$ is uniform random in $[1, n]$ we have
\[
P(J_N(m) = 1) = \frac{1}{n} \left\lfloor \frac{n}{m} \right\rfloor = \frac{1}{m} + \frac{1}{n} O(1),
\]
uniformly in $m, n$. Then
\[
E \prod_{i=1}^k |A_n^\circ \cap I_i| = \sum_{q_1, \ldots, q_k} E J_N(q_1 \cdots q_k) = \sum_{q_1 \cdots q_k} \frac{1}{q_1 \cdots q_k} + \frac{1}{n} \sum \sum O(1)
\]
\[
= \sum_{q_1 \cdots q_k} \frac{1}{q_1 \cdots q_k} + O(n^b/n),
\]
establishing (21).

Putting it all together then yields
\[
\liminf_{n \to \infty} \prod_{i=1}^k |A_n \cap I_i| \geq \liminf_{n \to \infty} \prod_{i=1}^k |A_n^\circ \cap I_i| = \lim_{n \to \infty} \prod_{i=1}^k \sum_{q_i} \frac{1}{q_i} = \prod_{i=1}^k \log \frac{b_i}{a_i},
\]
where we have used Mertens’ formula (2) in the last step. This confirms hypothesis (10). As for (7), since for each $N$ we have $p_1 p_2 \cdots = N \leq n$, it is always the case that $T_n = L_1(n) + L_2(n) + \cdots \leq 1$. Therefore, Lemma 2 applies, completing the proof. \(\square\)
In this section we extend Billingsley’s theorem to the context of normed arithmetic semigroups satisfying growth conditions general enough to allow a PD(θ) limit with \( \theta \neq 1 \).

### 5.1 Normed Arithmetic Semigroups

Following the terminology of [12], a normed arithmetic semigroup \( S \) is a commutative semigroup whose only unit is 1, admitting unique factorization into prime elements and equipped with a nonnegative multiplicative norm function \( s \mapsto |s| \) for which any set of elements with bounded norm is finite. It follows at once that any element \( s \neq 1 \) must have norm \( |s| > 1 \).

Certain growth conditions have been studied, with a view towards providing abstract settings for classical analytic number theory. For \( x > 0 \), let \( \nu_S(x) \) be the number of elements \( s \in S \) with \( |s| \leq x \), and let \( \pi_S(x) \) be the number of primes \( p \in S \) with \( |p| \leq x \). The asymptotic linear growth condition, that for some positive constants \( A \) and \( \delta \), we have

\[
\nu_S(x) = Ax \left(1 + O\left(\frac{1}{x^\delta}\right)\right), \tag{22}
\]

has been studied by, e.g., Knopfmacher in [12] (as well as by Beurling before him) who shows, among many other things, that given (22) we have a generalized Mertens formula ([12], Lemma 2.5) asserting that for positive \( x \),

\[
\sum_{|p| \leq x} \frac{1}{|p|} = \log \log x + B_S + O\left(\frac{1}{\log x}\right) \tag{23}
\]

where the constant \( B_S \) depends on the semigroup. He also proves a prime element theorem, based on Landau’s prime ideal theorem, asserting that for \( x > 0 \) we have

\[
\pi_S(x) \sim x / \log x, \tag{24}
\]

though we will not need that here.

Apart from the ordinary positive integers, of course, the semigroup of integral ideals in a number field is a standard example, with growth condition (22) derived, e.g., in [15], Theorem 39; and many other natural examples are given in [12]. For some additional examples of contemporary interest, see [13, 14].

B. M. Bredikhin has studied normed arithmetic semigroups in which \( \pi_S(x) \) satisfies \( \pi_S(x) \sim \theta x / \log x \) for fixed \( \theta \neq 1 \) and has shown, in particular, that if

\[
\pi_S(x) = \theta x \log x \left(1 + O\left(\frac{1}{(\log x)^\epsilon}\right)\right) \tag{25}
\]
for some $\epsilon > 0$, then
\[
\nu_S(x) = \frac{Ax}{\log^{1-\sigma} x} \left( 1 + O\left( \frac{1}{(\log \log x)^\epsilon} \right) \right)
\]
(26)
for some positive $A$ depending on $S$ and where $\epsilon' = \min(1, \epsilon)$. See [16], Section 2.5 for a complete account.

A generalized Mertens formula is given for this case as well, in passing, on p. 93 of [16], namely
\[
\sum_{|p| \leq x} \frac{1}{|p|} = \theta \log x + O(1),
\]
but we will need the stronger form
\[
\sum_{|p| \leq x} \frac{1}{|p|} = \theta \log x + B_S + o(1),
\]
(27)
for some constant $B_S$ depending only on $S$. This follows, however from (25) via a standard Stieltjes integral argument: Write $G(t) = \pi_S(t) - \theta t / \log t$, so that $G(t) = O(t/(\log t)^{1+\epsilon})$; also $G$ is clearly of bounded variation. Let $r > 1$ be less than the minimum norm value of any prime element. Then we have
\[
\sum_{|p| \leq x} \frac{1}{|p|} = \int_r^x \frac{1}{t} \ d\pi_S(t) = \theta \int_r^x \frac{1}{t} \ dt \left( \frac{t}{\log t} \right) + \int_r^x \frac{1}{t} \ dG(t).
\]
The first integral on the right is
\[
\log \log x - \log \log r + 1/\log x - 1/\log r.
\]
As for the second integral, knowing that $G(t) = O(F(t))$, where
\[
F(t) = t/(\log t)^{1+\epsilon}
\]
entitles us to write, via formula (4.67) on p. 57 of [8],
\[
\int_r^x \frac{1}{t} \ dG(t) = O\left( \frac{1}{r} F(r) \right) + O\left( \frac{1}{x} F(x) \right) + O\left( \int_r^x \frac{1}{t} \ dF(t) \right),
\]
an integration by parts trick well-known to analytic number theorists, but not often derived in textbooks. Thus the integral with respect to $dG(t)$ converges as $x \to \infty$, and so we may write
\[
\int_r^x \frac{1}{t} \ dG(t) = \int_r^x \frac{1}{t} \ dG(t) - \int_x^\infty \frac{1}{t} \ dG(t).
\]
Collecting terms gives us (27), with
\[
B_S = \int_r^\infty \frac{1}{t} \ dG(t) - \theta (\log \log r + 1/\log r).
\]
Examples of semigroups $S$ satisfying Bredikhin’s condition include semigroups of positive integers all of whose prime factors range among a union of disjoint arithmetic sequences with the same increment; by Dirichlet theory, the constant $\theta$ is then the sum of the densities of the primes from each sequence, amongst all the primes.

For an example with $\theta > 1$, let $S$ be the commutative multiplicative semigroup freely generated by two disjoint copies of the usual primes, where the norm of an element is its ordinary value. In this $S$, two elements of the same norm are distinct unless the primes in their respective factorizations can be matched by type. Then $S$ satisfies (25) with $\theta = 2$, with remainder term as derived from various versions of the usual prime number theorem.

### 5.2 New Billingsley

We will state and prove a version of Billingsley’s theorem for normed arithmetic semigroups $S$ of the two types discussed in the preceding subsection. But first we isolate, as lemmas, two pieces of an argument that we will use more than once. We write $\Omega(s)$ for the number of prime factors of $s$, including multiplicities.

**Lemma 4.** Let $S$ be a normed arithmetic semigroup. Given $n > 1$ let $s$ be chosen according to some probability distribution $P_n$ from the elements with norm not exceeding $n$. If $|s| > 1$ let $s = p_1 p_2 \ldots$ be a decomposition into prime factors, with $|p_1| \geq |p_2| \geq \ldots$. For each $n$, define a process $L(n) = (L_1(n), L_2(n), \ldots)$ where each $L_i(n) = 0$ if $s = 1$ but otherwise

$$L_i(n) = \frac{\log |p_i|}{\log n}$$

for $i \leq \Omega(s)$, and

$$L_i(n) = 0$$

for $i > \Omega(s)$.

Let $A_n$ be the multiset of non-zero elements of $L(n)$, and let $I_i = [a_i, b_i] \subset (0, 1)$, $i = 1, \ldots, k$, be disjoint closed intervals. Then

$$\mathbb{E} \prod_{i=1}^{k} |A_n \cap I_i| \geq \sum_{q_1, \ldots, q_k} \mathbb{P}_n((q_1 \cdots q_k)|s)$$

where the primes $q_1, \ldots, q_k$ range over the respective sets $\{p : |p| \in [n^{a_i}, n^{b_i}]\}$.

**Proof.** For each $n$ let $A_n^\circ$ be the set underlying $A_n$, and for $s \in S$, let $J_s(\cdot)$ be the indicator function for “divides $s$.”

Given $s$ we have

$$\prod_{i=1}^{k} |A_n^\circ \cap I_i| = \prod_{i=1}^{k} \sum_{q_i} J_s(q_i) = \sum_{q_1, \ldots, q_k} J_s(q_1 \cdots q_k) = \sum_{q_1, \ldots, q_k} J_s(q_1 \cdots q_k) \quad (28)$$
where each \( q_i \) runs through the primes with \( |q_i| \in [n^{a_i}, n^{b_i}] \), and we are using the disjointness of the intervals in the last equality. Since \( |A_n \cap I| \geq |A_n^c \cap I| \) for any interval and \( \mathbb{E} J_s(q_1 \cdots q_k) = \mathbb{P}_n((q_1 \cdots q_k)|I) \), the result follows.

Also, to estimate certain sums where the terms are to be approximated, with uniformly small relative error, we will need the following elementary fact which takes more space to state than to prove:

**Lemma 5.** For \( n = 1, 2, \ldots \), let \( I_n \) be an arbitrary finite set. For \( i \in I_n \), let \( t(i, n), a(i, n) \in \mathbb{R} \), with \( a(i, n) \geq 0 \), and let \( T_n := \sum_{i \in I_n} t(i, n) \) and \( A_n := \sum_{i \in I_n} a(i, n) \). Assume that \( c := \lim_n A_n \) exists and \( c \geq 0 \). Assume that for all \( n \), for all \( i \in I_n \),

\[
t(i, n) = a(i, n)(1 + e(i, n)),
\]

with

\[
E_n := \sup_{i \in I_n} |e(i, n)| \text{ satisfying } E_n \to 0.
\]

Then \( T_n \to c \).

**Proof.**

\[
|T_n - A_n| \leq \sum_{i \in I_n} |t(i, n) - a(i, n)| = \sum_{i \in I_n} |e(i, n)| a(i, n)
\]

\[
\leq \sum_{i \in I_n} E_n a(i, n) = E_n A_n \to 0 \times c = 0.
\]

We now prove the main theorem of this section.

**Theorem 2.** Let \( S \) be a normed arithmetic semigroup satisfying either (22) or (25). Given \( n > 1 \) let \( s \) be chosen uniformly from the elements with norm not exceeding \( n \). Let \( s = p_1 p_2 \ldots \) be a decomposition into prime factors, with \( |p_1| \geq |p_2| \geq \ldots \). For each \( n \), define a process \( L(n) = (L_1(n), L_2(n), \ldots) \) where each \( L_i(n) = 0 \) if \( s = 1 \) but otherwise,

\[
L_i(n) = \frac{\log |p_i|}{\log n}
\]

for \( i \leq \Omega(s) \), and

\[
L_i(n) = 0
\]

for \( i > \Omega(s) \). Then as \( n \to \infty \)

\[
(L_1(n), L_2(n), \ldots) \Rightarrow (L_1, L_2, \ldots)_{\theta},
\]
a PD(\( \theta \)) limit where \( \theta = 1 \) if (22) holds, while otherwise \( \theta \) takes the same value as in (25) if that formula holds.
Proof. We have $0 \leq L_i(n) \leq 1$ for each such term, and $T_n := \sum_i L_i(n) = \log |s|/\log n \leq 1$, so hypothesis (17) of Lemma 3 is satisfied.

As for hypothesis (16), given $k > 0$ let $I_i = [a_i, b_i]$, $i = 1, \ldots, k$ be disjoint closed subintervals of $[0, 1]$ with $b_1 + \cdots + b_k < 1$.

Retaining the notation of Lemma 4 if $s$ is chosen uniformly from the elements with $|s| \leq n$, then since

$$\mathbb{P}_n((q_1 \cdots q_k)|s) = \frac{\nu_S(n/|q_1 \cdots q_k|)}{\nu_S(n)},$$

it will suffice by that lemma to investigate $\lim_{n \to \infty}$ (or at least $\lim \inf$) of

$$\sum_{q_1, \ldots, q_k} \frac{\nu_S(n/|q_1 \cdots q_k|)}{\nu_S(n)}$$

(29)

where each $q_i$ runs through the primes with $|q_i| \in [n^{a_i}, n^{b_i}]$.

Write $b := b_1 + \cdots + b_k < 1$. If (22) is in effect, then (29) becomes

$$\sum_{|q_1| \cdots |q_k|} \frac{1}{|q_1| \cdots |q_k|} \left(1 + e(|q_1| \cdots |q_k|, n)\right)$$

(30)

where the condition $b < 1$ ensures that $e(|q_1| \cdots |q_k|, n) = o(1)$ as $n \to \infty$, uniformly over choices of $q_1, \ldots, q_k$. By Lemma 5 coupled with the generalized Mertens formula (23) we thus have

$$\lim_{n \to \infty} \sum_{q_1, \ldots, q_k} \mathbb{P}_n((q_1 \cdots q_k)|s) = \lim_{n \to \infty} \sum_{|q_1| \cdots |q_k|} \frac{1}{|q_1| \cdots |q_k|}$$

$$= \lim_{n \to \infty} \prod_{i=1}^k \sum_{q_i \in [n^{a_i}, n^{b_i}]} \frac{1}{|q_i|} = \prod_{i=1}^k \log \left(\frac{b_i}{a_i}\right).$$

Thus by Lemma 4, together with Lemma 3.2 with $\theta = 1$ (or Lemma 2, for that matter), we are done when (22) is in effect.

If Equation (25) and hence (26) are in effect instead, then (29) becomes

$$\sum_{|q_1| \cdots |q_k|} \frac{1}{1 - \log |q_1|/\log n - \cdots - \log |q_k|/\log n}^{1-\theta} \left(1 + e(|q_1| \cdots |q_k|, n)\right)$$

where we use $n/|q_1| \cdots |q_k| \geq n^{1-b}$ to ensure that $e(|q_1| \cdots |q_k|, n) = o(1)$ as $n \to \infty$, uniformly over choices of $q_1, \ldots, q_k$.

Also since, writing $a := a_1 + \cdots + a_k$ we have

$$\frac{1}{1 - \log |q_1|/\log n - \cdots - \log |q_k|/\log n} \geq \frac{1}{(1 - a)},$$

we find again from Lemma 5 that

$$\lim \inf \sum_{q_1, \ldots, q_k} \frac{\nu_S(n/|q_1 \cdots q_k|)}{\nu_S(n)} \geq \frac{\rho^k}{(1 - a)^{1-\theta}} \prod_{i=1}^k \log \left(\frac{b_i}{a_i}\right),$$

(31)
this time using the Mertens formula \(27\).

Once again we have satisfied hypothesis \(18\) of Lemma \(3\) now with \(\alpha = 1 - \theta\) and \(\beta = 0\). This completes the proof.

5.3 A Pair of Examples, \(\theta = 1/2\) and \(\theta = 1\)

The positive integers representable as sums of two squares form a normed arithmetic semigroup \(S\) whose generating primes \(p\) consist of the prime \(p = 2\), the primes \(p = p \equiv 1 \mod(4)\), and the “square primes” \(p = p^2\) where \(p \equiv 3 \mod(4)\).

See standard texts for this theory. We take \(|p| = p\), of course. From Dirichlet theory we know that

\[
\pi_S(x) = \frac{1}{2} \frac{x}{\log x} \left(1 + O \left(\frac{1}{\log x}\right)\right),
\]

which is \(24\) with \(\theta = 1/2\), \(\epsilon = 1\). Therefore, Theorem \(2\) applies, giving us a limiting PD(\(\theta\)) with \(\theta = 1/2\).

Also, the Gaussian integers \(\{r + is : r, s \in \mathbb{Z}\}\) form a principal ideal domain, with unique factorization up to multiplicative units. Mod out by the unit group \(\{\pm 1, \pm i\}\), to get a normed arithmetic semigroup \(S\) with norm \(|r + si| = r^2 + s^2\).

This semigroup satisfies

\[
\nu_S(x) = \frac{\pi}{4} x \left(1 + O \left(\frac{1}{\sqrt{x}}\right)\right),
\]

which is \(22\) with \(A = \pi/4\) and \(\delta = 1/2\). Therefore, Theorem \(2\) applies here as well, but with a limiting PD(\(\theta\)) having \(\theta = 1\).

It is well-known, however, that the positive integers appearing as norms of primes in the two cases are identical. Therefore, the numbers appearing in the respective sequences

\(L(n) = (L_1(n), L_2(n), \ldots)\)

are also identical; but we get different limiting behaviors. One could unify this pair of examples by saying that in both cases we are actually selecting random \(N = r^2 + s^2\) with \(N \leq n\), i.e., from the first semigroup; but in the first case the selection is uniform, while in the second case the probabilities are proportional to the number of representations as sums of two squares (or as norms of Gaussian integers).

6 Integers with Unusual Numbers of Prime Factors

In this final section we derive another extension of Billingsley’s theorem, for a situation that does not seem to be covered by Theorem \(2\)

\[\text{6 Defined in Section 5.1}\]
6.1 The Turán and Erdős-Kac Theorems and Selberg’s Formulas

Given a positive integer $N$, let $\omega(N)$ denote the number of distinct prime factors of $N$ and let $\Omega(N)$ denote the number of prime factors counted with multiplicities. The Erdős-Kac theorem asserts that if $N$ is picked uniformly at random from 1 to $n$, then as $n \to \infty$, the quantities

$$\frac{\Omega(N) - \log \log n}{\sqrt{\log \log n}}$$

and

$$\frac{\omega(N) - \log \log n}{\sqrt{\log \log n}}$$

converge in distribution to standard Gaussian variables. Furthermore, Turan’s theorem from 1934 gives an asymptotic bound for the probability of certain large deviation events. Namely, if $\xi(n) \to \infty$ with $n$, then the probability that the absolute value of either quantity exceeds $\xi(n)$ is $O(1/\xi^2)$. (See, e.g., [17], Section III.3.) So asymptotically, the events

$$|\Omega(N) - \log \log n| \geq \epsilon \log \log n$$

and

$$|\omega(N) - \log \log n| \geq \epsilon \log \log n,$$

for any fixed $\epsilon > 0$, become vanishingly rare, and one would expect such integers $N$ to be atypical in the distribution of their large prime factors, as well as in the number of prime factors.

In the next section we will consider random integers $N$ picked uniformly from those positive integers not exceeding $n$ and for which either $\Omega(N)$ or $\omega(N)$ is required, roughly speaking, to stay vanishingly close to $\tau \log \log x$ for some $\tau > 0$. We will show that a version of Billingsley’s theorem is once again valid, with a PD($\theta$) limit as $n \to \infty$, where $\theta$ is sometimes, but not always, equal to $\tau$.

The proof will exploit three growth formulas due to Selberg and Delange, which we record here: Write

$$\nu_j(x) := \{m \leq x : \Omega(m) = j\}$$

for the count of positive integers, not exceeding $x$, and having exactly $j$ prime factors including multiplicity. Theorems II.6.5 and 6 in [17] describe the growth of these counts involving $\Omega(N)$, as follows. Given $\delta > 0$, we have uniformly over $x \geq 3$ and $1 \leq j \leq (2-\delta) \log \log x$,

$$\nu_j(x) = \frac{x}{\log x} \cdot \frac{(\log \log x)^{j-1}}{(j-1)!} \left\{ \kappa \left( \frac{j - 1}{\log \log x} \right) + O \left( \frac{j}{(\log \log x)^2} \right) \right\}$$

(34)
where \( \kappa \) is
\[
\kappa(z) = \frac{1}{\Gamma(z+1)} \prod_p \left( 1 - \frac{z}{p} \right)^{-1} \left( 1 - \frac{1}{p} \right)^z,
\]
with product taken over the primes. Note that \( \kappa \) is continuous, apart from poles at 2, 3, \ldots. Also for \( \delta > 0 \), uniformly over \( x \geq 3 \) but, this time, \( j \log \log x \in (2 + \delta, A) \) for any fixed \( A \in (2 + \delta, \infty) \), we have
\[
\nu_j(x) = C \frac{x \log x}{2^j} \left\{ 1 + O \left( \frac{1}{(\log x)^{\delta/\delta}} \right) \right\}
\]
where \( C \) is the constant \( C := (1/4) \prod_{p>2} (1 + 1/(p(p-2))) \approx 0.378694 \).

Here is the corresponding formula when \( \Omega \) is replaced with \( \omega \). Write
\[
\nu^\omega_j(x) := \{m \leq x : \omega(m) = j\}
\]
for the count of positive integers, not exceeding \( x \), and having exactly \( j \) distinct prime factors, not counting multiplicity. Theorem II.6.4 in [17] states that given \( A > 0 \), we have uniformly over \( x \geq 3 \) and \( 1 \leq j \leq A \log \log x \),
\[
\nu^\omega_j(x) = \frac{x \log x}{(j-1)!} \left\{ \lambda \left( \frac{j-1}{\log \log x} \right) + O \left( \frac{j}{(\log \log x)^2} \right) \right\}
\]
where \( \lambda \) is
\[
\lambda(z) = \frac{1}{\Gamma(z+1)} \prod_p \left( 1 + \frac{z}{p-1} \right) \left( 1 - \frac{1}{p} \right)^z.
\]
Note that \( \lambda \) is continuous at all non-negative values of the argument \( z \).

For later use we state an approximation involving the leading factors of (36).

Lemma 6. Given \( \theta > 0 \), let \( k = k(x) \) be nonnegative integers with \( k \sim \theta \log \log x \) as \( x \to \infty \). Then
\[
\frac{x}{\log x} \left( \frac{\log \log x}{k!} \right)^k \sim \frac{x}{(\log x)^{\theta'} \sqrt{2\pi \theta} \log \log x}
\]
where \( \theta' = 1 - \theta(1 - \log \theta) \geq 0 \).

Proof. This follows from Stirling’s formula.

Remark. Note that
\[
\frac{1}{\log x} \left( \frac{\log \log x}{k!} \right)^k
\]
is the probability of the integer \( k \) for a Poisson distribution having mean \( \mu = \log \log x \). Thus Selberg’s formula asserts, in particular, that the number of distinct prime factors of a random integer picked uniformly from 1 to \( n \) is distributed, asymptotically, like \( Z-1 \) where \( Z \) is a Poisson random variable with mean \( \log \log n \), conditional on \( Z \geq 1 \).

Also, Reference [13] proves an extension of the Erdős-Kac theorem to factorizations in normed arithmetic semigroups satisfying (22). This raises the possibility that Selberg’s formulas (34), (35), and (36) might also extend to that context, which would then lead to similar extensions of the theorems proved below.
6.2 Generalized Billingsley for Unusual $\Omega(N)$

**Theorem 3.** Fix $\tau \neq 2 \in (0, \infty)$, and let $g(n)$ be any sequence of integers, with $1 \leq g(n) \leq \log_2(n)$, such that as $n \to \infty$

$$\frac{g(n)}{\log \log n} \to \tau. \quad (37)$$

Pick $N$ uniformly from the set of positive integers $\{m : m \leq n, \Omega(m) = g(n)\}$, and let $p_1 \geq p_2 \geq \ldots$ be the sequence of prime factors, including multiplicities. Define $L(n) = (L_1(n), L_2(n), \ldots)$ where $L_i(n) = \log p_i / \log n$ for $i \leq g(n)$ and $L_i(n) = 0$ for $i > g(n)$. Then, with

$$\theta := \min(\tau, 2),$$

we have convergence in distribution to the Poisson-Dirichlet with parameter $\theta$:

$$L(n) \Rightarrow PD(\theta).$$

**Proof.** We retain the multiset notation of Theorems 3 and 2.

To show that the hypothesis (16) is satisfied we observe that if $P_n$ is the measure where $N$ is picked uniformly from $\{m : m \leq n, \Omega(m) = g(n)\}$, then Lemma 4 applies, so that for fixed disjoint subintervals $[a_i, b_i]$, with $0 < a := a_1 + \cdots + a_k < b := b_1 + \cdots + b_k < 1$, and with $p_i$ ranging over $[n^{a_i}, n^{b_i}]$, we have

$$\mathbb{E} \prod_{i=1}^k |A_n \cap I_i| \geq \sum_{p_1, p_2, \ldots, p_k} \mathbb{P}_n(p_1 p_2 \ldots p_k | N). \quad (38)$$

Next, we will establish inequalities of the following sort:

$$\mathbb{P}_n(p_1 p_2 \ldots p_k | N) \geq \frac{1}{p_1 p_2 \ldots p_k} \frac{1}{(1-a)^{\alpha} (1-a)^{\beta}} \theta^k (1 + o(1)) \quad (39)$$

as $n \to \infty$, with error term uniform over choices of $p_1, p_2, \ldots, p_k$, with $\alpha + \beta = 1 - \theta$. This will lead to the required lower bound for

$$\liminf \mathbb{E} \prod_{i=1}^k |A_n \cap I_i|.$$

Now, specifying that we are to pick $N \leq n$ uniformly from the integers with exactly $g(n)$ prime factors, including multiplicity, means that we can write

$$\mathbb{P}_n(p_1 p_2 \ldots p_k | N) = \frac{\nu_{g(n)-k}(n/(p_1 \cdots p_k))}{\nu_{g(n)}(n)}. \quad (40)$$

For the case $\tau < 2$, we apply Selberg’s approximation (34) to both the numerator and denominator of the right side of (40), switching in the numerator from $n$ to $n/(p_1 \cdots p_k)$ in the role of $x$, and from $g(n)$ to $g(n) - k$ in the role of $j$. The first factor $x$ on the right side of (34) yields, as a factor on the right
side of (40), the ratio of \( n/(p_1 \cdots p_k) \) to \( n \), which is exactly exactly \( 1/(p_1 \cdots p_k) \).

From the next factor, \( 1/\log x \), we get the ratio

\[
\frac{\log n}{\log(n/(p_1 \cdots p_k))} \geq 1/(1-a).
\]

From the next factor, \( (\log \log x)^{j-1}/(j-1)! \), using \( g(n)/\log \log n \to \tau \), we get a ratio \( \tau^k(1+o(1)) \) which comes from changing the power of \( \log \log x \) by \( k \), along with changing the base of the factorial by \( k \); and we get an additional factor due to changing the \( x \) inside \( (\log \log x)^j \), namely

\[
\left( \frac{\log \log(n/(p_1 \cdots p_k))}{\log \log n} \right)^g \geq \left( \frac{\log n^{1-b}}{\log \log n} \right)^g = \left( 1 + \frac{\log(1-b)}{\log \log n} \right)^g = (1-b)^\tau(1+o(1)).
\]

The last factor of (34), shown in large braces, contributes another \( 1+o(1) \) to the product of ratios: one must check that \( \kappa\left(\frac{z}{\nu}\right) \), evaluated at the two specified arguments, yields the claimed asymptotic ratio, and it is here that the continuity of \( \kappa \) is invoked.

The net result is that for the right side of (40) we have

\[
\frac{\nu_{g(n)-k}(n/(p_1 \cdots p_k))}{\nu_{g(n)}(n)} \geq \frac{1}{p_1 \cdots p_k} \frac{1}{1-a} \tau^k(1-b)^\tau(1+o(1)),
\]

with the \( o(1) \) relative error term uniformly small over choices of \( p_1, \ldots, p_k \). Summing both sides over \( p_i \in [n^{a_i}, n^{b_i}] \) and appealing to the classical Mertens’ formula (2) as well as Lemma 3 and coupled with (38) and (40), we see that (16) is satisfied with \( \alpha = 1 \) and \( \beta = \tau \). Also (17) is trivially confirmed. Therefore, Theorem 3 applies, completing the argument for \( \tau < 2 \).

For the case with \( \tau > 2 \), we proceed as above, but substituting the use of the much simpler (35) for (34) in deriving a lower bound for (40). Again we get a product of ratios: The first factor, \( x/2^j \), yields the ratio \( 2^k/(p_1 \cdots p_k) \). The next factor, \( \log x \), yields the ratio

\[
\frac{\log(n/(p_1 \cdots p_k))}{\log n} \geq 1-b.
\]

The final factor yields a ratio which is \( 1+o(1) \) as \( n \to \infty \), uniformly over choices of \( p_1, \ldots, p_k \). Once again, summing over \( p_i \in [n^{a_i}, n^{b_i}] \) and applying Mertens shows that Theorem 3 applies, with \( \alpha = 0 \) and \( \beta = -1 \), leading to a PD(2) limit.
6.3 Generalized Billingsley for Unusual $\omega(N)$

**Theorem 4.** Fix $\theta \in (0, \infty)$, and let $g(n)$ be any sequence of integers, with $1 \leq g(n) \leq \log_2(n)$, such that as $n \to \infty$

$$\frac{g(n)}{\log \log n} \to \theta. \quad (41)$$

Pick $N$ uniformly from the set of positive integers $\{m : m \leq n, \omega(m) = g(n)\}$, and let $p_1 > p_2 > \ldots$ be the sequence of distinct prime factors, i.e., not including multiplicities. Define $L(n) = (L_1(n), L_2(n), \ldots)$ where $L_i(n) = \log p_i / \log n$ for $i \leq g(n)$ and $L_i(n) = 0$ for $i > g(n)$. Then, we have convergence in distribution to the Poisson-Dirichlet with parameter $\theta$:

$$L(n) \Rightarrow PD(\theta).$$

**Proof.** The plan of proof is similar to that of the first part of Theorem 3 using $(40)$ in place of $(39)$, but there is a new technical issue to confront: if $p_1 \cdots p_k | N$ then we have $\Omega(N/(p_1 \cdots p_k)) = \Omega(N) - k$ always, a fact exploited in writing $(40)$. However, $\omega(N/(p_1 \cdots p_k))$ may lie anywhere from $\omega(N) - k$ to $\omega(N)$, depending on the multiplicities of the $p_i$ in $N$, a fact which complicates the adaptation of $(40)$ to $\omega$.

Fortunately, large prime factors with multiplicities greater than 1 are sufficiently rare that $(40)$ can still be used with $\nu^\circ$ in place of $\nu$, after an asymptotically negligible tweak. Define the set of positive integers

$$S = \{m \leq n/(p_1 \cdots p_k) : \omega(m) = g(n) - k, \text{ and } p|m \Rightarrow p \notin \{p_1, \ldots, p_k\}\}. $$

Then via multiplication by $p_1 \cdots p_k$, $S$ maps injectively to a subset of

$$T = \{m \leq n : \omega(m) = g(n), \text{ and } (p_1 \cdots p_k)|m\},$$

where $T$ is the set whose (relative) cardinality we wish to bound fairly sharply from below. Also we have

$$|S| \geq \nu_{g(n)-k}^\circ(n/(p_1 \cdots p_k)) - \sum_{p_i} (n/(p_1 \cdots p_k))/p_i$$

$$\geq \nu_{g(n)-k}^\circ(n/(p_1 \cdots p_k)) - kn^{1-a_1}/(p_1 \cdots p_k),$$

and so

$$\mathbb{P}_n(p_1 \cdots p_k|N) = \frac{|T|}{\nu_{g(n)}^\circ(n)} \geq \frac{\nu_{g(n)-k}^\circ(n/(p_1 \cdots p_k))}{\nu_{g(n)}^\circ(n)} - \frac{kn^{1-a_1}/(p_1 \cdots p_k)}{\nu_{g(n)}^\circ(n)}.$$

$$= \frac{\nu_{g(n)-k}^\circ(n/(p_1 \cdots p_k))}{\nu_{g(n)}^\circ(n)}(1 + o(1))$$

where the $o(1)$ is uniform over choices of $p_1, \ldots, p_k$ as $n \to \infty$, using Lemma 6.
Also comparison of (34) with (36) makes it plain that exactly as for the right-hand side of (40), we get

\[
\frac{\nu^\theta_{g(n)^n} (n/p_1 \cdots p_k)}{\nu^{\theta}_{g(n)} (n)} \geq \frac{1}{p_1 \cdots p_k} \frac{1}{1-a} \theta^k (1-b)^\theta (1 + o(1)),
\]

with the \( o(1) \) relative error term uniform over choices of \( p_1, \ldots, p_k \). So once again we get a PD(\( \theta \)) limit, as claimed.

\[\square\]

References

[1] Richard Arratia, A. D. Barbour, and Simon Tavaré. Logarithmic Combinatorial Structures: a probabilistic approach. EMS Monographs in Mathematics. European Mathematical Society, 2003.

[2] Richard Arratia, A. D. Barbour, and Simon Tavaré. The Poisson–Dirichlet distribution and the scale-invariant Poisson process. Combinatorics, Probability and Computing, 8(5):407–416, 1999.

[3] Richard Arratia and Fred Kochman. A simple direct proof of Billingsley’s theorem, to appear.

[4] Patrick Billingsley. Convergence of probability measures. John Wiley & Sons Inc., New York, second edition, 1999.

[5] David R. Brillinger. Estimation of the second-order intensities of a bivariate stationary point process. J. Roy. Statist. Soc. Ser. B, 38(1):60–66, 1976.

[6] Peter Donnelly and Geoffrey Grimmett. On the asymptotic distribution of large prime factors. J. London Math. Soc. (2), 47(3):395–405, 1993.

[7] Peter Donnelly and Paul Joyce. Continuity and weak convergence of ranked and size-biased permutations on the infinite simplex. Stochastic Process. Appl., 31(1):89–103, 1989.

[8] Daniel H. Greene and Donald E. Knuth. Mathematics for the analysis of algorithms, volume 1 of Progress in Computer Science and Applied Logic. Birkhauser Boston, Boston, MA, third edition, 1990.

[9] A. E. Ingham. The distribution of prime numbers. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1932.

[10] Olav Kallenberg. Foundations of Modern Probability. Probability and its Applications (New York). Springer-Verlag, New York, 1997.

[11] J.F.C. Kingman. Poisson processes, volume 3 of Oxford Studies in Probability. The Clarendon Press Oxford University Press, New York, 1993. Oxford Science Publications.
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