Some Effects of Classical Feedback on the Classical Capacity of a Memoryless Quantum Channel

Gleb V. Klimovitch
Information System Laboratory
Stanford University
Stanford, CA
gleb@stanford.edu

April 1, 2022

Abstract
Classical feedback is defined here as the knowledge by the transmitter of the quantum state of the qubit received by the receiver. Such classical feedback doubles capacities of certain memoryless quantum channels without preexisting entanglement between transmitter and receiver. The increase in capacity, which is absent on classical memoryless channels, occurs because we can transform an entangled qubit pair into any other entangled state by applying a unitary operator to only one of the qubits.

1 Introduction

For a classical channel, feedback can be naturally defined as the knowledge of received symbol at the transmitter after transmission [1]. Quantum case is more subtle, since physical arrival of a qubit at the receiver does not necessarily mean that the receiver knows the state of the received qubit and/or can copy the unknown state to the transmitter. In this paper, we still use the definition of classical feedback given above for a quantum channel. In particular, we assume that the received state can be known to the transmitter in the presence of channel noise, before it is measured by and therefore known to the receiver. Such an assumption may appear counterintuitive at first glance, but it does work for some channels.

Consider the following (fancy but workable) example. The channel either flips the spin of the qubit or does not affect the qubit at all. There are extra "helper qubits" in the channel that are represented by particles different from "information qubits". The helper qubits are independent identically distributed, each of them being in only two orthogonal states, e.g. spin-up and spin-down,
with equal probabilities one half. The helper qubits are neither sent nor affected by the transmitter, but the transmitter learns the initial state of each helper qubit, e.g. by receiving its "carbon copy". Both helper qubits and information qubits are equally affected by the channel. The receiver can separate the two kinds of qubits and measure only the states of the received helper qubits. The receiver still cannot detect spin flips, since it does not know the initial states of the helper qubits”. The transmitter obviously can, once it gets the feedback from the receiver.

Thus defined, classical feedback increases the classical capacity of certain memoryless quantum channels, as demonstrated by a simple example in Section 2. In Section 3, we argue that increase in capacity due to feedback becomes possible for non-classical (quantum) channels due to distributed nature of information stored by qubits.

2 How it works

We consider a quantum subchannel, whose effect on qubit is described by random unitary operators \( \hat{U}_{\text{channel}}(\mathbf{c}) \) with probability distribution \( p(\mathbf{c}) \) for random vector \( \mathbf{c} \). For memoryless channel, operators \( \hat{U}_{\text{channel}} \) corresponding to different transmission events are statistically independent.

Our quantum channel consists of two quantum subchannels with different noise levels. Let each channel transmit one qubit per unit of time. The main idea is best illustrated by assuming that one subchannel is so noisy that its capacity approaches zero, while the other subchannel is noiseless.

For example let the noisy subchannel be disturbed by random magnetic field which interacts with magnetic moment of qubits, so that qubit states are completely randomized (but not collapsed) during transmission. The effect of the noisy subchannel on the qubit can be modeled by random unitary operators

\[
\hat{U}_{\text{channel}}(\mathbf{c}) = \exp(j \lambda \mathbf{c} \hat{\sigma}),
\]

where vector \( \hat{\sigma} \) has Pauli matrices as components; \( \mathbf{c} \) is a random three-dimensional gaussian vector with zero mean and unit variance per dimension; and \( \lambda \gg 1 \) is a constant proportional to the product of a typical channel field, the magnetic moment of the qubit, and transmission time.

Without feedback, the capacity of the noisy subchannel approaches zero, as \( \lambda \) goes to infinity. Information can be sent only over the quiet subchannel with the maximum load of one bit per transmission (without preexisting entanglement between transmitter and receiver). Thus the channel capacity equals one - the rate of qubit transmission over each subchannel.

Feedback doubles the capacity as follows. Transmitter forms entangled pairs of qubits and sends members of each pair at different times. First, one member of the pair goes over the noisy subchannel. After it is received, the transmitter learns the channel estimate during its transmission (let it be vector \( \mathbf{c} \)). Then the transmitter undoes the effect of the noisy subchannel on the quantum state.
of the pair by applying a unitary operator $\hat{U}_{undo}(c)$ to the second member of the pair (that is still at the transmitter).

For example, if the pair is in the singlet state, then $\hat{U}_{undo}(c)$ is also given by the r.h.s. of equation (1) but is applied to the second rather than the first member of the pair. Indeed, the combined action of channel and "undo" operators on the pair can be rewritten in terms of the (operator of the) total spin of the pair $\hat{S}$ as follows:

$$\hat{U}_{undo}\hat{U}_{channel} = \exp(2j\lambda c\hat{S}) = \hat{1}.$$  

(2)

The last equality holds because the spin of the pair is zero.

After the action of the noisy subchannel on the pair is undone, the transmitter applies the superdense quantum coding [2] of two bits per qubit to the second member of the pair and transmits it over the quiet subchannel (simultaneously with transmitting the first member of another pair over the noisy subchannel). The receiver waits until both members of the pair arrive, then the pair is decoded. Thus the channel capacity doubles to two - the combined rate of qubit transmission over both subchannels. Despite one subchannel being extremely noisy, we achieve the maximum possible capacity (without preexisting entanglement between transmitter and receiver) of one bit per qubit due to feedback.

3 Why it works

Let us start with an insight "why it does not work" for classical channels, i.e. why feedback does not increase the capacity of a classical memoryless channel. The feedback gives the transmitter the knowledge of decoder errors. However, transmission at a higher rate but with errors necessitates transmission of information to correct such errors, which in turn reduces the number of bits available to send new data. As a result, the capacity does not increase due to feedback on a classical memoryless channel. For example, a classical analog of our system consists of one extremely noisy and one quiet subchannel of equal (unit) rates. Without feedback, we would just ignore the noisy subchannel. With feedback, we could use the quiet subchannel to correct all the errors on the noisy subchannel. However, such an error correction would leave no resources to send new data over the quiet subchannel, and the capacity remains the same.

In quantum case, information distributed between two entangled qubits as follows. While both qubits are needed to decode anything, it suffices to access one qubit to encode (or re-encode) two bits, i.e. the maximum amount of information per qubit pair (unless the pair itself is entangled with the receiver). As a result, even after sending the first qubit in the pair and having only the second qubit at its disposal, the transmitter can still cancel the effects of the noisy subchannel on the whole pair and re-encode the whole pair. Speaking informally, it is the miracle of superdense quantum coding [2] that helps to increase the capacity of such a quantum channel by feedback.
4 Summary

Defined as the knowledge of already received state at the transmitter, classical feedback on memoryless quantum channel can increase (double) the channel capacity. The increase is achieved by a method similar to superdense quantum coding.

5 Acknowledgements

The author strongly appreciates discussions with Prof. Thomas Cover on topics in both classical and quantum information theory.

The author will be quite grateful for feedback on this paper, including comments on and references to related work, and hopes that such a feedback could increase the "channel capacity" of the paper.

References

[1] Cover, T.M. and Joy, A.T. Elements of Information Theory, New York: Wiley.

[2] Nielsen, M.A. and Chuang, I.L. Quantum Computation and Quantum Information. New York: Cambridge University Press, 2000.