A PRESENTATION FOR THE SYMPLECTIC BLOB ALGEBRA

R. M. GREEN, P. P. MARTIN, AND A. E. PARKER

ABSTRACT. The symplectic blob algebra $b_n$ ($n \in \mathbb{N}$) is a finite dimensional algebra defined by a multiplication rule on a basis of certain diagrams. The rank $r(n)$ of $b_n$ is not known in general, but $r(n)/n$ grows unboundedly with $n$. For each $b_n$ we define an algebra by presentation, such that the number of generators and relations grows linearly with $n$. We prove that these algebras are isomorphic.

1. Introduction

The transfer matrix formulation of lattice Statistical Mechanics (see e.g. [1, 11]) is a source for many sequences of algebras and representations — among the best known examples are the Temperley–Lieb algebras [15] and the quantum groups [9]. Physically one seeks to diagonalise the transfer matrix, and this corresponds to computing the irreducible representations of the associated algebras. Statistical Mechanics often provides algebras with a basis of ‘diagrams’ (describing the configuration of physical states), leading to the notion of diagram algebras. The Temperley–Lieb diagram algebra arises in several different Statistical Mechanical models (such as Potts models, $q$-spin chains and vertex models), but in each case the algebra manifests only when specific ‘open’ physical boundary conditions are imposed. It is physically appropriate to consider other boundary conditions, however, and this forces a generalisation in the algebra. For example, periodic boundary conditions necessitate generalisation to the blob diagram algebra [13]. More recently it has been shown [5] that other physically interesting boundary conditions necessitate further generalisation. Both the Temperley–Lieb and blob algebras have alternative definitions by presentation, and each of the diagram- and presentation-based definitions suggest candidates for suitable generalisations. The study of these two generalisations has begun in [5, 6] and [12], but the isomorphism between them was not established (and it does not follow from the isomorphisms for the earlier algebras). We prove the isomorphism here.

In the study of Hecke algebras of arbitrary type, a useful tool is the Temperley–Lieb algebra of the same type (see [7, 12] for references). This is, in each case, a Hecke quotient algebra defined by presentation. Type-A gives the presentational form of the ordinary Temperley–Lieb algebra. Type-B gives the blob algebra; and the presentational form of the new generalisation is a quotient of type-Č (also known as the two-boundary Temperley–Lieb algebra [5]). For this reason, the new diagram algebra is known as the symplectic blob algebra, $b_n$ (in [12] the notation $b^n_s$ is used).

1Corresponding author
In [12] we investigated its generic representation theory and proved various representation theoretically important properties of the algebra, for instance that it has a cellular basis, that it is generically semi-simple (in the Hecke algebra parameters), that the associated sequence $n \to \infty$ of module categories has a ‘thermodynamic limit’, and that it is a quotient of the Hecke algebra of type-$\tilde{C}$. For a number of reasons explained in the original paper (the role of Temperley–Lieb and blob algebras in Statistical Mechanics and in solving the Yang–Baxter equations; the intrinsic interest in the Hecke algebra of type-$\tilde{C}$, and so on) one is interested in the representation theory of this algebra. The representation theory of the ordinary Temperley–Lieb and blob cases is rather well understood, and has an elegant geometrical description, over an arbitrary algebraically closed field [4]. So far here, however, not even the blocks over $\mathbb{C}$ are known. As with finite-dimensional algebras defined as diagram algebras in general (or indeed any algebra), a powerful tool in representation theory is to be able to give an efficient presentation, so this is our objective here.

The paper is structured as follows. We first review the various objects and notations and some of the basic properties of the symplectic blob algebra that will be used in the paper. This is followed by a statement and proof of a presentation for the algebra. The proof occupies the majority of the paper.

It is easy to establish an explicit surjective algebra homomorphism in one direction, and we start with this. However a suitable closed formula for the rank at level $n$ is not presently known for either algebra, so we are motivated to use a method that does not rely on rank bounds. Our method generalises an approach in [8], and so should be of wider interest in the study of Coxeter groups and related algebras.

2. The symplectic blob algebra

We start with a summary of [12, §6]. Fix $n, m \in \mathbb{N}$, with $n + m$ even, and $k$ a field. A Brauer $(n,m)$-partition $p$ is a partition of the set $V \cup V'$ into pairs, where $V = \{1, 2, \ldots, n\}$ and $V' = \{1', 2', \ldots, m'\}$. Following Brauer [2] and Weyl [16] we will depict $p$ as a Brauer $(n,m)$-diagram. A diagram for $p$ is a rectangle with $n$ vertices labelled 1 through to $n$ on the top edge and $m$ vertices labelled $1'$ through to $m'$ on the bottom, and two vertices $a$ and $b$ connected, with an arbitrary line embedded in the plane of the rectangle, if $\{a, b\} \in p$.

Any two rectangles with embeddings coding the same set partition are called equivalent, and regarded as the same Brauer diagram.

Now consider a diagram among whose embeddings (in the above sense) are embeddings with no lines crossing. For such a diagram, we may consider the sub-equivalence class of embeddings that indeed have no crossings. This class (or a representative thereof) is a Temperley–Lieb diagram. Note that such a diagram $d$ defines not only a pair-partition of $V \cup V'$ but also a partition of the open intervals of the frame of the rectangle excluding $V \cup V'$ (two intervals are in the same part if there is a path from one to the other in the rectangle that does not cross a line of $d$).
Our first objective is to define a certain diagram category, that is a \( k \)-linear category whose hom-sets each have a basis consisting of diagrams, and where multiplication is defined by diagram concatenation (the object class is \( \mathbb{N} \) in our case), and simple straightening rules to be applied when the concatenated object is not formally a diagram. For example in the Brauer or Temperley–Lieb diagram category, a concatenation may produce a diagram, as here:

\[
\begin{array}{c}
\begin{array}{c}
\quad \\
\end{array} \\
\begin{array}{c}
\quad \\
\end{array}
\end{array}
\quad \equiv \quad 
\begin{array}{c}
\begin{array}{c}
\quad \\
\end{array} \\
\begin{array}{c}
\quad \\
\end{array}
\end{array}
\]

or not, as here:

\[
\begin{array}{c}
\begin{array}{c}
\quad \\
\end{array} \\
\begin{array}{c}
\quad \\
\end{array}
\end{array}
\quad \equiv \quad 
\begin{array}{c}
\begin{array}{c}
\quad \\
\end{array} \\
\begin{array}{c}
\quad \\
\end{array}
\end{array}
\]

A straightening rule is a way of expressing such products as (1) in the span of basis diagrams.

The resulting diagram in (1) is an example of a pseudo Temperley–Lieb diagram. Simply put, it fails to be a proper Temperley–Lieb diagram because of the loop. The set of pseudo Temperley–Lieb diagrams includes all the Temperley–Lieb diagrams, but we also allow diagrams with loops, which may appear anywhere in the diagram, although still with no crossing lines. Here (in addition to the equivalence of different embeddings of open lines, as before) isotopic deformation of a loop without crossing a line results in an equivalent embedding.

The set of pseudo Temperley–Lieb diagrams with \( m = n \) is closed under concatenation. Thus we can define a straightening rule for multiplication of Temperley–Lieb diagrams by imposing a relation on the \( k \)-space spanned by pseudo Temperley–Lieb diagrams that will remove the loops (and is consistent with concatenation).

**Definition 2.1.** For \( \delta \in k \) and \( n \in \mathbb{N} \), the Temperley–Lieb algebra \( TL_n = TL_n(\delta) \) is the \( k \)-algebra with \( k \)-basis the Temperley–Lieb \( (n,n) \)-diagrams and multiplication defined by concatenation. We impose the relation: each loop that may arise when multiplying is omitted and replaced by a factor \( \delta \).

Next we generalise to decorated Temperley–Lieb diagrams. Here we put elements of a monoid on the lines (like beads on a string). When decorated diagrams are concatenated, two or more line segments are combined in sequence as before. But now we need a rule to combine the monoid elements on these segments to make a new monoid element for the combined line. One such rule
is simply to multiply in the monoid in the indicated order. This gives us a well defined associative
diagram calculus — see section 3 of [12] for a detailed discussion and proof of this.

We will now focus on a particular set of decorated Temperley–Lieb diagrams — the ones used to
define the symplectic blob algebra. To begin, we decorate with the free monoid on two generators. The beads depicting these generators are called blobs: a “left” blob, $L$, (usually a black filled-in
circle on the diagrams) and a “right” blob, $R$, (usually a white filled-in circle on the diagrams).

A line in a (pseudo) Temperley–Lieb diagram is said to be $L$-exposed (respectively $R$-exposed)
if it can be deformed to touch the left hand side (respectively right hand side) of the rectangular
frame without crossing any other lines.

A left-right blob pseudo-diagram is a diagram obtained from a pseudo Temperley–Lieb diagram
by allowing left and right blob decorations with the following constraints. Any line decorated with
a left blob must be $L$-exposed and any line decorated with a right blob must be $R$-exposed. Also
all segments with decorations must be deformable so that the left blobs can touch the left hand
side and the right blobs touch the right hand side of the frame simultaneously without crossing.

Concatenating diagrams cannot change a $L$-exposed line to a non-$L$-exposed line, and similarly
for $R$-exposed lines. Thus the set of left-right blob pseudo-diagrams is closed under diagram
concatenation. (See [12, proposition 6.1.2].)

The set of left-right blob pseudo-diagrams is infinite. For example, if a left blob can appear on a
line in a given underlying pseudo-diagram, then arbitrarily many such blobs can appear. To define
a finite dimensional $k$-algebra, as for the blob algebra (see [14, section 1.1] for a definition) and the
Temperley–Lieb algebra (defined above), we will straighten by certain rules, into the $k$-span of a
finite subset. For example we may identify a pseudo-diagram with certain localised features, such
as multiple blobs on a line, with a scalar multiple of an otherwise identical diagram with other
features in that locale (fewer blobs, possibly none, on that line).

We now proceed to define a specific such straightening (i.e. a finite target set, and a suitable
collection of rules). We have six parameters, $\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR} = k_L = k_R$, which are all
elements in the base field $k$.

Consider the set of eight features drawn on the left-hand sides of the sub-tables of table 1. We
define $B'_n$ to be the set of left-right blob pseudo-diagrams with $n$ vertices at the top and $n$ at the
bottom of the diagram that do not have features from this set.
The set $B'_n$ is finite. We call its elements \textit{left-right blob diagrams}.

Now define a relation on the $k$-span of all left-right blob pseudo-diagrams as follows. If $d,d'$ are scalar multiples of single diagrams, set $d \sim d'$ if $d'$ differs from $d$ by a substitution from left to right in either sub-table. Extend this $k$-linearly.

A moment’s thought makes it clear that to obtain a consistent set of relations we need $RLRL = k_RRL = k_LR.$, i.e., that $k_L = k_R.$

Another (perhaps longer) moment’s thought reveals that the $k_L$ relation is only needed for $n$ odd and the $\kappa_{LR}$ relation is only needed when $n$ is even. It turns out to be convenient to set $\kappa_{LR} = k_L = k_R.$

We have the following result.

**Proposition 2.2** ([12, section 6.3]). The above relations on the $k$-span of left-right blob pseudo-diagrams define, with diagram concatenation, a finite dimensional algebra, $b'_n,$ which has a diagram basis $B'_n.$

We study this algebra by considering the quotient by the “topological relation”:

\[
\kappa_{LR} \quad \Rightarrow \quad k_L \quad = \quad k_R
\]

(2)

where each shaded area is shorthand for subdiagrams that do not have propagating lines (a line is called \textit{propagating} if it joins a vertex on the top of the diagram to one on the bottom of the diagram). (Note that there is no freedom in choosing the scalar multiple, once we require a relation of this \textit{form}.)

We define $B_n$ to be the subset of $B'_n$ that does not contain diagrams with features as in the right hand side of relation (2).
Definition 2.3. We define the symplectic blob algebra, $b_n$ (or $b_n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR})$ if we wish to emphasise the parameters) to be the $k$-algebra with basis $B_n$, multiplication defined via diagram concatenation and relations as in the table above (with $\kappa_{LR} = k_L = k_R$) and with relation (2).

That these relations are consistent and that we do obtain an algebra with basis $B_n$ is proved in [12, section 6.5].

We have the following (implicitly assumed in [12]):

Proposition 2.4. The symplectic blob algebra, $b_n$, is generated by the following diagrams

\[ e := \quad \quad e_1 := \quad \quad e_2 := \quad \quad \cdots, \]

\[ e_{n-1} := \quad f := \quad \cdots \]

Proof. We may argue in a similar fashion as in appendix A of [12] but by now inducing on the number of decorations. If a diagram $d$ has no decorations then the diagram is a Temperley–Lieb diagram and the result follows.

So now assume that we have a diagram $d$ with $m$ decorations and that (for the sake of illustration) there is a left blob — we would use the dual reduction in the case of a right blob. We claim that we may use the same procedure as in the $l = 0$ case of [12, appendix A]. If there is a decorated line starting in the first position, then we can decompose the diagram into a product of $e$ then a diagram with one fewer decoration. If there is no such line then take the first line decorated with a black blob and do the same reduction as in [12, appendix A]. For example, taking a diagram with a line with both a left and right blob on it:

The white blobs can either be moved into the shaded regions or above or below the horizontal dotted lines. The middle region (after “wiggling” the line enough times) is then the product $e_1ee_2e_1$. The outside diagrams have strictly fewer than $m$ decorations and hence the result follows by induction.

3. Presenting the symplectic blob algebra

We start by defining an algebra by a presentation that is a direct generalisation of the well-known presentation for the (ordinary) Temperley–Lieb algebra.
Definition 3.1. Fix $n \geq 1$. Let $S_n = \{E_0, E_1, \ldots, E_n\}$, and let $S_n^*$ be the free monoid on $S_n$. Define the commutation monoid $M_n$ to be the quotient of $S_n^*$ by the relations

$$E_i E_j \equiv E_j E_i \text{ for all } 0 \leq i, j \leq n \text{ with } |i - j| > 1.$$ 

Definition 3.2. Let $P_n = P_n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR})$ be the quotient of the $k$-monoid-algebra of $M_n$ by the following relations:

- $E_0^2 = \delta E_0$, $E_1 E_0 E_1 = \kappa_L E_1$,
- $E_i^2 = \delta E_i$ for $1 \leq i \leq n - 1$, $E_i E_{i+1} E_i = E_i$ for $1 \leq i \leq n - 2$,
- $E_n^2 = \delta R E_n$, $E_{i+1} E_i E_{i+1} = E_{i+1}$ for $1 \leq i \leq n - 2$,
- $I = \kappa_{LR} I$,
- $J = \kappa_{LR} J$,

where

$$I = \begin{cases} E_1 E_3 \cdots E_{2m-1} & \text{if } n = 2m, \\ E_1 E_3 \cdots E_{2m-1} E_{2m+1} & \text{if } n = 2m + 1, \end{cases}$$

$$J = \begin{cases} E_0 E_2 \cdots E_{2m-2} E_{2m} & \text{if } n = 2m, \\ E_0 E_2 \cdots E_{2m} & \text{if } n = 2m + 1. \end{cases}$$

Note $I = E_1$ and $J = E_0$ if $n = 1$. We will sometimes write $E$ for $E_0$ and $F$ for $E_n$.

Remark 3.3. The presentation obtained by omitting the last two relations ($IJI = \kappa_{LR} I$ and $JIJ = \kappa_{LR} J$) generalises the presentation for the blob algebra (sometimes known as the one-boundary Temperley–Lieb algebra, because of its role in modelling two-dimensional Statistical Mechanical systems with variable boundary conditions at one boundary). For this reason the generalisation is sometimes known as the two-boundary Temperley–Lieb algebra. It also coincides, in Hecke algebra representation theory, with a Temperley–Lieb algebra of type-$\tilde{C}$ (see [12]).

Theorem 3.4. Suppose $\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}$ are invertible, then the symplectic blob algebra $b_n$ is isomorphic to the algebra $P_n$ via an isomorphism

$$\phi : P_n \rightarrow b_n$$

induced by $E_0 \mapsto e$, $E_1 \mapsto e_1$, $\ldots$, $E_{n-1} \mapsto e_{n-1}$ and $E_n \mapsto f$.

It is straightforward to check that the generators already given for the symplectic blob algebra satisfy the $P_n$ relations. Thus the map $\phi$ in the theorem is a surjective homomorphism and hence we need only to prove injectivity. The rest of this paper is devoted to proving this theorem.
4. Definitions associated to the monoid $M_n$

Two monomials $u, u'$ in the generators $S_n$ are said to be commutation equivalent if $u \equiv u'$ in $M_n$. The commutation class, $\mathfrak{u}$, of a monomial $u$ consists of the monomials that are commutation equivalent to it.

The left descent set (respectively, right descent set) of a monomial $u$ consists of all the initial (respectively, terminal) letters of the elements of $\mathfrak{u}$. We denote these sets by $L(u)$ and $R(u)$, respectively.

**Definition 4.1.** A reduced monomial is a monomial $u$ in the generators $S_n$ such that no $u' \in u$ can be expressed as a scalar multiple of a strictly shorter monomial using the relations in Definition 3.2. If we have $u = u_1su_2su_3$ for some generator $s$, then the occurrences of $s$ in $u$ are said to be consecutive if $u_2$ contains no occurrence of $s$.

**Definition 4.2.** Two monomials in the generators, $u$ and $u'$, are said to be weakly equivalent if $u$ can be transformed into a nonzero multiple of $u'$ by applying finitely many relations in $P_n$. In this situation, we also say that $D$ and $D'$ are weakly equivalent diagrams, where $D$ and $D'$ are the diagrams equal to $\phi(u)$ and $\phi(u')$, respectively. If $P$ is a property that diagrams may or may not possess, then we say $P$ is invariant under weak equivalence if, whenever $D$ and $D'$ are weakly equivalent diagrams, then $D$ has $P$ if and only if $D'$ has $P$.

**Definition 4.3.** Let $D$ be a diagram. For $g \in \{L, R\}$ and $k \in \{1, \ldots, n, 1', \ldots, n'\}$, we say that $D$ is $g$-decorated at the point $k$ if (a) the edge $x$ connected to $k$ has a decoration of type $g$, and (b) the decoration of $x$ mentioned in (a) is closer to point $k$ than any other decoration on $x$.

In the sequel, we will sometimes invoke Lemma 4.4 without explicit comment.

**Lemma 4.4.** The following properties of diagrams are invariant under weak equivalence:

(i) the property of being $L$-decorated at the point $k$;

(ii) the property of being $R$-decorated at the point $k$;

(iii) for fixed $1 \leq i < n$, the property of points $i$ and $(i + 1)$ being connected by an undecorated edge;

(iv) for fixed $1 \leq i < n$, the property of points $i'$ and $(i + 1)'$ being connected by an undecorated edge.

**Proof.** It is enough to check that each of these properties is respected by each type of diagrammatic reduction, because the diagrammatic algebra is a homomorphic image of the algebra given by the
monomial presentation. This presents no problems, but notice that the term “undecorated” cannot be removed from parts (iii) and (iv), because of the topological relation.

Elements of the commutation monoid $M_n$ have the following normal form, established in [3].

**Proposition 4.5** (Cartier–Foata normal form). Let $s$ be an element of the commutation monoid $M_n$. Then $s$ has a unique factorization in $M_n$ of the form

$$s = s_1s_2 \cdots s_p$$

such that each $s_i$ is a product of distinct commuting elements of $S_n$, and such that for each $1 \leq j < p$ and each generator $t \in S_n$ occurring in $s_{j+1}$, there is a generator $s \in S_n$ occurring in $s_j$ such that $st \neq ts$ or $s = t$.

**Remark 4.6.** The Cartier–Foata normal form may be defined inductively, as follows. Let $s_1$ be the product of the elements in $L(s)$. Since $M_n$ is a cancellative monoid, there is a unique element $s' \in M_n$ with $s = s_1s'$. If $s' = s_2 \cdots s_p$ is the Cartier–Foata normal form of $s'$, then

$$s_1s_2 \cdots s_p$$

is the Cartier–Foata normal form of $s$.

**Definition 4.7.** Let $u$ be a reduced monomial in the generators $E_0, \ldots, E_n$. We say that $u$ is left reducible (respectively, right reducible) if it is commutation equivalent to a monomial of the form $u' = stv$ (respectively, $u' = vts$), where $s$ and $t$ are noncommuting generators and $t \notin \{E_0, E_n\}$. In this situation, we say that $u$ is left (respectively, right) reducible via $s$ to $tv$ (respectively, to $vt$).

## 5. Preparatory lemmas

The following result is similar to [8, Lemma 5.3], but we give a complete argument here because the proof in [8] contains a mistake (we thank D. C. Ernst for pointing this out).

**Lemma 5.1.** Suppose that $s \in M_n$ corresponds to a reduced monomial, and let $s_1s_2 \cdots s_p$ be the Cartier–Foata normal form of $s$. Suppose also that $s$ is not left reducible. Then, for $1 \leq i < p$ and $0 \leq j \leq n$, the following hold:

(i) if $E_0$ occurs in $s_{i+1}$, then $E_1$ occurs in $s_i$;

(ii) if $E_n$ occurs in $s_{i+1}$, then $E_{n-1}$ occurs in $s_i$;

(iii) if $j \notin \{0, n\}$ and $E_j$ occurs in $s_{i+1}$, then both $E_{j-1}$ and $E_{j+1}$ occur in $s_i$. 

Proof. The assertions of (i) and (ii) are almost immediate from properties of the normal form, because \( E_1 \) (respectively, \( E_{n-1} \)) is the only generator not commuting with \( E_0 \) (respectively, \( E_n \)). We need only consider the other alternative of \( E_0 \) (respectively, \( E_n \)) being in \( s_i \). If \( E_0 \) occurs in \( s_i \), then this moves to the end of the \( s_i \) which then cancels with the \( E_0 \) from the \( s_{i+1} \) contradicting the assumption that \( s \) is not reducible.

We will now prove (iii) by induction on \( i \). Suppose first that \( i = 1 \).

Suppose that \( j \not\in \{0, n\} \) and that \( E_j \) occurs in \( s_2 \). By definition of the normal form, there must be a generator \( s \in s_1 \) that does not commute with \( E_j \) or \( s = E_j \). If \( s = E_j \) then \( s \) is reducible as before. If \( s \neq E_j \) then \( s \) cannot be the only generator that does not commute with \( E_j \), or \( s \) would be left reducible via \( s \). Since the only generators not commuting with \( E_j \) are \( E_{j-1} \) and \( E_{j+1} \), these must both occur in \( s_1 \).

Suppose now that the statement is known to be true for \( i < N \), and let \( i = N \geq 2 \). Suppose also that \( j \not\in \{0, n\} \) and that \( E_j \) occurs in \( s_{N+1} \). As in the base case, there must be at least one generator \( s \) occurring in \( s_N \) that does not commute with \( E_j \).

Let us first consider the case where \( j \not\in \{1, n-1\} \), and write \( s = E_k \) for some \( 0 \leq k \leq n \). The restrictions on \( j \) means that \( k \not\in \{0, n\} \) and so that we cannot have \( E_j E_k E_j \) occurring as a subword of any reduced monomial. However, \( E_j \) occurs in \( s_{N-1} \) by the inductive hypothesis, and this is only possible if there is another generator, \( s' \), in \( s_N \) that does not commute with \( E_j \). This implies that \( \{s', E_k\} = \{E_{j-1}, E_{j+1}\} \), as required.

Now suppose that \( j = 1 \) (the case \( j = n-1 \) follows by a symmetrical argument). If both \( E_0 \) and \( E_2 \) occur in \( s_N \), then we are done. If \( E_2 \) occurs in \( s_N \) but \( E_0 \) does not, then the argument of the previous paragraph applies. Suppose then that \( E_0 \) occurs in \( s_N \) but \( E_2 \) does not. By statement (i), \( E_1 \) occurs in \( s_{N-1} \), but arguing as in the previous paragraph, we find this cannot happen, because it would imply that \( s \) was commutation equivalent to a monomial of the form \( v' E_1 E_0 E_1 v'' \), which is incompatible with \( s \) being reduced. This completes the inductive step. \( \square \)

The following is a key structural property of reduced monomials.

**Proposition 5.2.** Suppose that \( s \in M_n \) corresponds to a reduced monomial, and let \( s_1 s_2 \cdots s_p \) be the Cartier–Foata normal form of \( s \), where \( s_p \) is nonempty. Suppose also that \( s \) is neither left reducible nor right reducible. Then either (i) \( p = 1 \), meaning that \( s \) is a product of commuting generators or (ii) \( p = 2 \) and either \( s = IJ \) or \( s = JI \).

**Proof.** If \( p = 1 \), then case (i) must hold, so we will assume that \( p > 1 \).

A consequence of Lemma 5.1 is that if \( s_{i+1} = I \) then \( s_i = J \), and if \( s_{i+1} = J \) then \( s_i = I \). It follows that if \( s_p \in \{I, J\} \) (in \( P_n \)), then \( s \) must be an alternating product of \( I \) and \( J \). Since \( s \) is reduced, this forces \( p = 2 \) and either \( s = IJ \) or \( s = JI \). We may therefore assume that \( s_p \not\in \{I, J\} \).
Since \( s_p \not\in \{ I, J \} \) and \( s_p \) is a product of commuting generators, at least one of the following two situations must occur.

(a) For some \( 2 \leq i \leq n \), \( s_p \) contains an occurrence of \( E_i \) but not an occurrence of \( E_{i-2} \).

(b) For some \( 0 \leq i \leq n - 2 \), \( s_p \) contains an occurrence of \( E_i \) but not an occurrence of \( E_{i+2} \).

Suppose we are in case (a). In this case, Lemma 5.1 means that there must be an occurrence of \( E_{i-1} \) in \( s_{p-1} \); Now \( E_{i-1} \) fails to commute with two other generators (\( E_i \) and \( E_{i-2} \)). However, one of these generators, \( E_{i-2} \) does not occur in \( s_p \). It follows that \( s \) is right reducible (via \( E_i \)), which is a contradiction. Case (b) leads to a similar contradiction, again involving right reducibility, which completes the proof. □

Lemma 5.3. Let \( u = u_1 s u_2 s u_3 \) be a reduced word in which the occurrences of the generator \( s \) are consecutive, and suppose that every generator in \( u_2 \) not commuting with \( s \) is of the same type, \( t \) say. Then \( u_2 \) contains only one occurrence of \( t \), and \( s \in \{ E_0, E_n \} \).

Proof. The proof is by induction on the length, \( l \), of the word \( u_2 \). Note that \( u_2 \) must contain at least one generator not commuting with \( s \), or after commutations, we could produce a subword of the form \( ss \). This means that the case \( l = 0 \) cannot occur.

If \( u_2 \) contains only one generator not commuting with \( s \), then after commutations, \( u \) contains a subword of the form \( sts \). This is only possible if \( s \in \{ E_0, E_n \} \), and this establishes the case \( l = 1 \) as a special case.

Suppose now that \( l > 1 \). By the above paragraph, we may reduce to the case where \( u_2 = u_4 u_5 u_6 \), and the indicated occurrences of \( t \) are consecutive. Since the occurrences of \( s \) were consecutive, \( u_5 \) does not contain \( s \). Thus every generator in \( u_5 \) that does not commute with \( t \) will be the same generator \( u \neq s \). Now since \( u_5 \) is shorter than \( u_2 \), we can apply the inductive hypothesis to show that \( t \in \{ E_0, E_n \} \) and \( u_5 \) contains only one occurrence of the generator, \( u \). But this means that \( t \) fails to commute with two different generators, \( u \) and \( s \) contradicting the fact that \( t \in \{ E_0, E_n \} \) and completing the proof. □

Lemma 5.4. Let \( u \) be a reduced monomial.

(i) Between any two consecutive occurrences of \( E_0 \) in \( u \), there is precisely one letter not commuting with \( E_0 \) (i.e., an occurrence of \( E_1 \)).

(ii) Between any two consecutive occurrences of \( E_n \) in \( u \), there is precisely one letter not commuting with \( E_n \) (i.e., an occurrence of \( E_{n-1} \)).

(iii) Let \( 0 < i < n \). Between any two consecutive occurrences of \( E_i \) in \( u \), there are precisely two letters not commuting with \( E_i \), and they correspond to distinct generators.

Proof. To prove (i), we apply Lemma 5.3 with \( s = E_0 \); the hypotheses are satisfied as we necessarily have \( t = E_1 \). The proof of (ii) is similar.
To prove (iii), write $u = u_1 s u_2 s u_3$ for consecutive occurrences of the generator $s = E_i$. Since $s \not\in \{E_0, E_n\}$, the hypotheses of Lemma 5.3 cannot be satisfied, so $u_2$ must have at least one occurrence of each of $t_1 = E_{i-1}$ and $t_2 = E_{i+1}$. Suppose that $u_2$ contains two or more occurrences of $t_1$. The fact that the occurrences of $s$ are consecutive means that two consecutive occurrences of $t_1$ cannot have an occurrence of $s$ between them. Applying Lemma 5.3, this means that there is precisely one generator $u$ between the consecutive occurrences of $t_1$ such that $t_1 u \neq u t_1$, and furthermore, that $t_1 \in \{E_0, E_n\}$. This is a contradiction, because $t_1$ fails to commute with two different generators ($s$ and $u$).

One can show similarly that $u_2$ cannot contain two or more occurrences of $t_2$. We conclude that each of $t_1$ and $t_2$ occurs precisely once, as required. \hfill \Box

6. The map $\phi$

Recall the map $\phi : P_n \rightarrow b_n$ from Theorem 3.4. Here we will consider the possible diagrams arising from reduced monomials in $P_n$. We let $D_s$ be the ‘concrete’ pseudo-diagram [12] associated to a monomial $s = E_{i_1}E_{i_2}\cdots E_{i_m}$ formed by concatenating $e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ in order but without applying any straightening, and without applying any further isotopies that deform across the bounding frames of the concatenating components. Thus we include the possibility that $D_s$ has loops. So $D_s = \phi(s)$ as (a scalar multiple of) a diagram, after applying any straightening rules, but the shape of the concrete pseudo-diagram $D_s$ allows us to reconstruct $s$. For example, the monomial, $s = E_1E_2E_4E_0E_1$ has (concrete pseudo-)diagram as illustrated in Figure 1.

$$D_s = e_1e_2e_4e_0e_1 = \quad \text{Figure 1. Concrete pseudo-diagram associated to the monomial } s = E_1E_2E_4E_0E_1$$

The non-loop arcs in the concrete pseudo-diagram $D_s$ are made up of vertical line segments, cups and caps. In the following development, we will regard such arcs as having a direction or orientation (as we shall see shortly, this arc orientation can be chosen arbitrarily). Thus each vertical line segment becomes oriented northwards ($N$) or southwards ($S$), and each cup or cap is oriented westwards ($W$) or eastwards ($E$). If an oriented arc contains an occurrence of $E$ after an
occurrence of $W$, then we say that the arc has a \textit{west-east direction reversal}; we define \textit{east-west direction reversal} analogously. If an arc has at least one direction reversal, then we say that the arc \textit{changes direction}.

Consider, for example, the decorated arc in Figure 1. If we orient the topmost vertical line segment in this arc by $S$, then, starting with this line segment and working in the direction of the orientation, the segments of this arc are consecutively labelled

$$S, W, W, S, S, E, N, N, E, S, S, S,$$

where the sixth letter from the left (an occurrence of $S$) corresponds to the decorated segment. On the other hand, if we orient the topmost vertical line segment by $N$, then the segments of the arc are consecutively labelled

$$N, N, N, W, S, S, W, N, N, E, E, N$$
in the direction of the orientation, and the eighth letter from the left (an occurrence of $N$) corresponds to the decorated segment. In both cases, we have a west-east direction reversal, but no east-west direction reversal, and the arc changes direction.

The above example illustrates the basic fact that the property of having a west-east direction reversal does not depend on the orientation chosen for an arc. For similar reasons, the same can be said about east-west direction reversals, and about the property of changing direction.

It will turn out to be significant (see Lemma 6.1 below) that between any occurrence of $W$ and an occurrence of $E$ in the particular arc of Figure 1 studied above, there is a vertical segment corresponding to a decoration in the diagram. Figure 2 contains a west-east reversal in which this does not happen, but the concrete pseudo-diagram in Figure 2 corresponds to the non-reduced monomial $E_1E_2E_3E_1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure2.png}
\caption{West-east direction reversal of an undecorated line.}
\end{figure}

**Lemma 6.1.** Let $D$ be a diagram of the form $\phi(s)$ for some reduced monomial $s$.

(i) If an arc of $D$ contains a west-east direction reversal, then that arc must contain a consecutive sequence $X_1, \ldots, X_k$ of cups, caps and vertical segments with $1 < a < b < k$ and $a < b - 1$, such that
(a) $X_1$ and $X_a$ are labelled $W$;
(b) $X_b$ and $X_k$ are labelled $E$;
(c) for some $P, Q$ with $\{P, Q\} = \{N, S\}$,
    (1) the $X_i$ for $a < i < b$ are all labelled $P$, and exactly one of them carries a left blob;
    (2) the $X_j$ for $1 < j < a$ and for $b < j < k$ are all labelled $Q$;
(d) $X_1$ and $X_k$ form a cup-cap pair corresponding to a single occurrence of $E_1$.

(ii) If an arc of $D$ contains a west-east direction reversal, then that arc must contain a consecutive sequence $X_1, \ldots, X_k$ of cups, caps and vertical segments with $1 < a < b < k$ and $a < b - 1$, such that
(a) $X_1$ and $X_a$ are labelled $E$;
(b) $X_b$ and $X_k$ are labelled $W$;
(c) for some $P, Q$ with $\{P, Q\} = \{N, S\}$,
    (1) the $X_i$ for $a < i < b$ are all labelled $P$, and exactly one of them carries a right blob;
    (2) the $X_j$ for $1 < j < a$ and for $b < j < k$ are all labelled $Q$;
(d) $X_1$ and $X_k$ form a cup-cap pair corresponding to a single occurrence of $E_{n-1}$.

Proof. Recall that Lemma 5.4 constrains what can happen between two occurrences of the same generator $E_i$ in the reduced monomial $s$. Up to commutation of generators, these cases are shown in the next five diagrams, which illustrate the cases $i = 0, i = n, 2 \leq i \leq n - 2, i = 1$ and $i = n - 1$ respectively.

\[
\begin{array}{c}
e e_1 e = \includegraphics[scale=0.5]{diagram1} \\
\end{array}
\quad
\begin{array}{c}
fe_{n-1} f = \includegraphics[scale=0.5]{diagram2} \\
\end{array}
\quad
\begin{array}{c}
e_i e_{i-1} e_{i+1} e_i = \includegraphics[scale=0.5]{diagram3} \\
\end{array}
\quad
\begin{array}{c}
2 \leq i \leq n - 2,
\end{array}
\]
If a direction reversal occurs in an arc of $D$, this must correspond to a consecutive sequence of oriented segments of the form

$$Y_1, Y_2, \ldots, Y_{k-1}, Y_r$$

in which $\{Y_1, Y_r\} = \{W, E\}$, and all the $Y_i$ for $1 < i < k$ are either all equal to $S$ or all equal to $N$. In this case, $Y_1$ and $Y_r$ correspond to distinct letters of $s$; call these $s_1$ and $s_2$. Because $Y_1$ and $Y_2$ are separated only by vertical line segments, it follows that $s_1$ and $s_2$ correspond to consecutive occurrences in $s$ of the same generator, $E_i$. The possibilities enumerated in the diagrammatic version of Lemma 5.4 now force either $i = 1$ or $i = n - 1$, and the conclusions now follow from the corresponding two pictures, with $X_a = Y_1$ and $X_b = Y_r$. □

**Lemma 6.2.** Let $D$ be a diagram representing a reduced monomial $u$ (i.e., $D = \phi(u)$).

(i) The diagram $D$ is $L$-decorated at 1 (respectively, $1'$) if and only if the left (respectively, right) descent set of $u$ contains $E_0$.

(ii) The diagram $D$ is $R$-decorated at $n$ (respectively, $n'$), if and only if the left (respectively, right) descent set of $u$ contains $E_n$.

(iii) Suppose that $1 \leq i < n$. Then points $i$ and $i + 1$ (respectively, $i'$ and $(i + 1)'$) in $D$ are connected by an undecorated edge if and only if the left (respectively, right) descent set of $u$ contains $E_i$.

**Proof.** In all three cases, the “if” statements follow easily from diagram calculus considerations, so we only prove the “only if” statements.

Suppose for a contradiction that $D$ is $L$-decorated at 1, but that the left descent set of $u$ does not contain $E_0$. For this to happen, the line leaving point 1 must eventually encounter an $L$-decoration, but must first encounter a cup corresponding to an occurrence of the generator $E_1$. The only way this can happen and be consistent with Lemma 6.1 is for the line to then travel to the east wall after encountering $E_1$, then change direction and then travel back to the west wall, as shown in Figure 3. (Note that this can only happen if $n$ is odd, and that as before, the thin dotted lines in the diagram indicate pairs of horizontal edges that correspond to the same generator. In the figure we illustrate the element $E_4E_2E_1E_3E_4E_2E_0$ (which is not actually reduced).) So we may obtain another reduced expression for $u$, namely, $u = vE_1u''E_0u'''$ where $u''$ and $u'''$ are reduced, and $vE_1u''$ does not contain $E_0$. 

![Diagram](image-url)
The arc leaving 1 in the diagram for $vE_1u''$ contains an east-west direction reversal, and thus by Lemma 6.1 contains an occurrence of $E_n$, but no occurrence of $E_0$. This arc is therefore $R$-decorated. By Lemma 4.4, neither it, nor the arc leaving 1 from $D$, can be $L$-decorated, which is a contradiction.

The claim regarding $1'$ and the right descent set is proved similarly. This completes the proof of (i), and the proof of (ii) follows by modifying the above proof in the obvious way.

We now turn to (iii). Suppose for a contradiction that points $i$ and $i + 1$ in $D$ are connected by an undecorated edge, but that the left descent set of $u$ does not contain $E_i$.

Since $i$ is connected to $i + 1$, the arc leaving $i$ (respectively, $i + 1$) must (after possibly traversing some vertical line segments) either encounter a cup corresponding to $E_{i-1}$ or $E_i$ (respectively, $E_i$ or $E_{i+1}$). Suppose for a contradiction that the arc leaving $i$ encounters an $E_{i-1}$ first. By Lemma 6.1, the arc leaving $i$ performs a west-east direction reversal, as shown in Figure 4. This implies that the arc is $L$-decorated at $i$, which contradicts Lemma 4.4.
We have shown that the arc leaving $i$ encounters an occurrence of $E_i$ first, and a similar argument shows that the arc leaving $i + 1$ also encounters an occurrence of $E_i$ first. This proves the claim about the left descent set. The claim regarding the right descent set is proved similarly. This completes the proof of (iii).

Lemma 6.3. Let $u$ and $u'$ be reduced monomials that map to the same diagram $D$ under $\phi$.

(i) If $u'$ is a product of commuting generators, then $u$ and $u'$ are equal in $P_n$.

(ii) If $u' = IJ$ or $u' = JI$, then $u$ and $u'$ are equal in $P_n$.

Proof. Note that as $u'$ is the product of commuting generators we have

$$L(u') = R(u') = \{ E_i \mid E_i \text{ occurs in a minimal length expression for } u' \}$$

We first prove (i). By Lemma 6.2 and the fact that $u$ and $u'$ represent the same diagram, we must have

$$L(u) = L(u') = R(u') = R(u).$$

Suppose that $u'$ contains an occurrence of the generator $E_0$. This implies that $u$ must contain an occurrence of $E_0$, because $E_0 \in L(u') = L(u)$. Suppose also (for a contradiction) that $u$ contains two occurrences of the generator $E_0$. By Lemma 5.4, there must be an occurrence of $E_1$ between the first (i.e., leftmost or northernmost) two occurrences of $E_0$.

Since points 1 and $1'$ of $D$ are connected by an $L$-decorated line, (using Lemma 6.2) there must be an occurrence of $E_2$ immediately above the aforementioned occurrence of $E_1$ in order to prevent the line emerging from 1 from exiting the diagram at 2, as illustrated below:

```
  \begin{center}
    \begin{tabular}{|c|}
      \hline
      \cdot
      \hline
    \end{tabular}
  \end{center}
```

(“Immediately above” means that there are no other occurrences of $E_1$ or $E_2$ between the two occurrences mentioned.) In turn, we must have an occurrence of $E_3$ immediately below the aforementioned occurrence of $E_2$ in order to prevent the line from exiting the box at point $3'$. This is only sustainable if the arc between 1 and $1'$ has an east-west direction reversal. Lemma 6.2 then forces the arc to contain a right blob, which in turn implies that $n$ is odd, as shown below:

```
  \begin{center}
    \begin{tabular}{|c|}
      \hline
      \cdot
      \hline
    \end{tabular}
  \end{center}
```
There are two ways this picture can continue to the bottom. Either the line exits the box at point 1′ without encountering further generators, or the line encounters an occurrence of $E_1$. The first situation cannot occur because it contradicts Lemma 6.2 and the hypothesis that $E_0 \in \mathcal{R}(u)$. The second situation cannot occur because it shows (by repeating the argument in the paragraph above) that $u$ is commutation equivalent to a monomial of the form $vJIJv′$, which contradicts the hypothesis that $u$ be reduced.

We conclude that $u$ contains precisely one occurrence of $E_0$, and furthermore that $u$ contains no occurrences of $E_1$.

A similar argument shows that if $u′$ contains an occurrence of the generator $E_n$, then $u$ contains at most one occurrence of $E_n$, and it can only contain $E_n$ if it contains no occurrences of $E_{n-1}$.

It follows that at least one of the three situations must occur:

(a) $u′$ contains $E_0$ and $u = E_0 D_E$, where $D_E$ contains no occurrences of $E_0$ or $E_1$;
(b) $u′$ contains $E_n$ and $u = D_F E_n$, where $D_F$ contains no occurrences of $E_{n-1}$ or $E_n$;
(c) $u′$ contains neither $E_0$ nor $E_n$.

In cases (a) and (b), there is a corresponding factorization of $u′$, and the result claimed now follows from the faithfulness of the diagram calculus for the blob algebra [4, 10]. For example, in case (a), we have $u′ = E_0 D'_E$ and $u = E_0 D_E$. We view $D'_E$ and $D_E$ as elements of the blob algebra, where the blob in this case is identified with $E_n$. Then as $D'_E$ and $D_E$ have the same diagram, they must be also equal in $P_n$ by the faithfulness of the blob algebra. Thus $u′$ and $u$ are equal in $P_n$.

Suppose that we are in case (c), but that $u$ contains an occurrence of $E_0$ or $E_n$. Because the diagram $D$ corresponds to $u′$, it cannot have decorations, so it must be the case that $\phi(u)$ is either $L$-decorated at some point, or $R$-decorated at some point. This contradicts the hypotheses on $u′$, using Lemma 4.4. Since neither $u$ nor $u′$ contains $E_0$ or $E_n$, the result follows by the faithfulness of the diagram calculus for the Temperley–Lieb algebra [11, §6.4]. This completes the proof of (i).

We now prove (ii) in the case where $u′ = IJ$; the case $u′ = JI$ follows by a symmetrical argument. Thus, $u$ maps to the same diagram as $IJ$. The fact that $\mathcal{L}(u)$ is the set of generators in $I$ and $\mathcal{R}(u)$ is the set of generators in $J$ means that $u$ cannot be left or right reducible. By Proposition 5.2 (ii), this immediately means that $u = IJ$. □

7. Proof of the theorem

Lemma 7.1. Let $u$ be a reduced monomial and let $D$ be the corresponding diagram. Then $D$ avoids all the features on the left hand sides of Table 1, (the table in section 2 depicting all the straightening relations). Furthermore, $D$ contains at most one line with more than one decoration.
Proof. The proof is by induction on the length of $u$. If $u$ is a product of commuting generators, or $u = IJ$, or $u = JI$, the assertions are easy to check, so we may assume that this is not the case. (This covers the base case of the induction as a special case.)

By Proposition 5.2, $u$ must either be left reducible or right reducible. We treat the case of left reducibility; the other follows by a symmetrical argument.

By applying commutations to $u$ if necessary, we may now assume that $u = stv$, where $s$ and $t$ are noncommuting generators, and $t \not\in \{E_0, E_n\}$. By induction, we know that the reduced monomial $tv$ corresponds to a diagram $D'$ with none of the forbidden features and at most one edge with two decorations.

Suppose that $t = E_1$ and $s = E_0$. By Lemma 6.2, points 1 and 2 of $D'$ must be connected by an undecorated edge, and the effect of multiplying by $E_0$ is simply to decorate this edge. This does not introduce any forbidden features, nor does it create an edge with two decorations, and this completes the inductive step in this case.

The case where $t = E_{n-1}$ and $s = E_n$ is treated similarly to the above case, so we may now assume that $s, t \not\in \{E_0, E_n\}$. We must either have $s = E_i$ and $t = E_{i+1}$, or vice versa.

Suppose that $s = E_i$ and $t = E_{i+1}$. By Lemma 6.2, this means that points $i+1$ and $i+2$ of $D'$ are connected by an undecorated edge. The effect of multiplying by $s$ is then (a) to remove this undecorated edge, then (b) to disconnect the edge emerging from point $i$ of $D'$ and reconnect it to point $i+2$, retaining its original decorated status, then (c) to install an undecorated edge between points $i$ and $i+1$. This procedure does not create any forbidden features, nor does it create a new edge with more than one decoration.

The case in which $s = E_{i+1}$ and $t = E_i$ is treated using a parallel argument, and this completes the inductive step in all cases.

\[\square\]

Lemma 7.2. Let $u$ be a reduced monomial with corresponding diagram $D$.

(i) If points 1 and 2 (respectively, 1' and 2') are connected in $D$ by an edge decorated by $L$ but not $R$, then $u$ is equal (as an algebra element) to a word of the form $u' = E_0E_1v$ (respectively, $u' = vE_1E_0$).

(ii) If points $n-1$ and $n$ (respectively, $(n-1)'$ and $n'$) are connected in $D$ by an edge decorated by $R$ but not $L$, then $u$ is equal (as an algebra element) to a word of the form $u' = E_nE_{n-1}v$ (respectively, $u' = vE_{n-1}E_n$).

Proof. We first prove the part of (i) dealing with points 1 and 2. By Lemma 6.2, we have $E_0 \in L(u)$, so $u = E_0v'$. Now $v'$ is also a reduced monomial, and by Lemma 7.1, $v'$ corresponds to a diagram $D'$ with no forbidden features. Since multiplication by $e$ does not change the underlying shape of a diagram (ignoring the decorations), it must be the case that points 1 and 2 of $D'$ are connected by an edge with some kind of decoration. Since $D$ has no forbidden features and the corresponding edge in $D$ has no $R$-decoration, the only way for this to happen is if the edge connecting points 1
and 2 in $D'$ is undecorated. By Lemma 6.2, this means that $v'$ is equal as an algebra element to a monomial of the form $E_1v$, and this completes the proof of (i) in this case.

The other assertion of (i) and the assertions of (ii) follow by parallel arguments. □

Proof of Theorem 3.4. As all the parameters are invertible, it is enough to prove the statement when $\kappa_L = \kappa_R = 1$. Indeed the rescaling of $E_0$ and $E_n$:

$$E_0 \mapsto \frac{E_0}{\kappa_L}, \quad E_n \mapsto \frac{E_n}{\kappa_R}$$

has the following effect on the parameters:

$$\delta \mapsto \delta, \quad \delta_L \mapsto \frac{\delta L}{\kappa_L}, \quad \delta_R \mapsto \frac{\delta R}{\kappa_R}, \quad \kappa_L \mapsto 1, \quad \kappa_R \mapsto 1, \quad \kappa_{LR} \mapsto \frac{\kappa_{LR}}{\kappa_L \kappa_R}$$

Thus any $P_n$ (with 6 parameters) is isomorphic to a case with $\kappa_L = \kappa_R = 1$.

It is clear from the generators and relations that the reduced monomials are a spanning set, and that the diagram algebra is a homomorphic image of the abstractly defined algebra. By Lemma 7.1, all reduced monomials map to basis diagrams. The only way the homomorphism could fail to be injective is therefore for two reduced monomials $u$ and $u'$ to map to the same diagram $D$, and yet to be distinct as elements in $P_n$.

It is therefore enough to prove that if $u$ and $u'$ are reduced monomials mapping to the same diagram, then they are equal in $P_n$. Without loss of generality, we assume that $\ell(u) \leq \ell(u')$ (where $\ell$ denotes length).

We proceed by induction on $\ell(u)$. If $\ell(u) \leq 1$, or, more generally, if $u$ is a product of commuting generators, then Lemma 6.3 shows that $u = u'$. Similarly, if $u = IJ$ or $u = JI$, then $u = u'$, again by Lemma 6.3. In particular, this deals with the base case of the induction.

By Proposition 5.2, we may now assume that $u$ is either left or right reducible. We treat the case of left reducibility, the other being similar. By applying commutations if necessary, we may reduce to the case where $u = stv$, $s$ and $t$ are noncommuting generators, and $t \not\in \{E_0, E_n\}$.

Suppose that $s = E_0$, meaning that $t = E_1$. In this case, points 1 and 2 of $D$ are connected by an edge decorated by $L$ but not $R$. By Lemma 7.2 (i), this means that we have $u' = stv$ as algebra elements. Since $u$ and $u'$ share a diagram, the (not necessarily reduced) monomials $tu$ and $tu'$ must also share a diagram. Since $tst = \kappa_Lt = t$, the (reduced) monomials $tv$ and $tv'$ also map to the same diagram, $D'$. However, $tv$ is shorter than $u$, so by induction, $tv = tv'$, which in turn implies that $u = u'$.

Suppose that $s = E_n$, meaning that $t = E_{n-1}$. An argument similar to the above, using Lemma 7.2 (ii), establishes that $u = u'$ in this case too.

We are left with the case where $s = E_i$ and either $t = E_{i+1}$ or $t = E_{i-1}$ (where $t \not\in \{E_0, E_n\}$). We will treat the case where $t = E_{i+1}$; the other case follows similarly. In this case, we have $tst = t$, and so $tu = tsv = tv$. It is not necessarily true that $tu'$ is a reduced monomial, but it maps to the same diagram as $tv$, which is reduced. After applying algebra relations to $tu'$, we may
transform it into a scalar multiple of a reduced monomial, \( r \). Since reduced monomials map to basis diagrams (Lemma 7.1), the scalar involved must be 1. Now the reduced monomials \( tv \) and \( r \) map to the same basis diagram, and \( tv \) is shorter than \( u \), so by induction, we have \( tv = r \) in \( P_n \).

Since \( s \in L(u) \), we have \( s \in L(u') \) by Lemma 6.2, so that \( u' = sv'' \) for some reduced monomial \( v'' \). Since \( sts = s \), we have \( s(tu') = u' \). We have shown that \( tu' = r = tv \), so we have

\[
u' = stu' = stv = u,
\]

which completes the proof. □

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Department of Mathematics, University of Leeds, Leeds, LS2 9JT, UK

E-mail address: ppmartin@maths.leeds.ac.uk
E-mail address: parker@maths.leeds.ac.uk
