THE LEAST PRIME NUMBER REPRESENTED BY A BINARY QUADRATIC FORM

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Abstract. Let $D < 0$ be a fundamental discriminant and $h(D)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. Moreover, assume that $\pi_D(X)$ is the number of the split primes with norm less than $X$ in $\mathbb{Q}(\sqrt{D})$ and $R(X, D)$ is the number of the classes of the binary quadratic forms of discriminant $D$ which represents a prime number less than $X$. We prove that

$$\left(\frac{\pi_D(X)}{\pi(X)}\right)^2 \ll \frac{R(X, D)}{h(D)} \left(1 + \frac{h(D)}{\pi(X)}\right),$$

where $\pi(X)$ is the number of the primes less than $X$ and the implicit constant in $\ll$ is independent of $D$. As a result, by assuming the Riemann hypothesis for the Dirichlet L-function $L(s, \chi_D)$, at least $\alpha h(D)$ number of the ideal classes of $\mathbb{Q}(\sqrt{D})$ contain a prime ideal with a norm less than the optimal bound $h(D) \log(|D|)$, where $\alpha > 0$ is an absolute positive constant independent of $D$. More generally, let $K$ be a bounded degree number field over $\mathbb{Q}$ with the discriminant $D_K$ and the class number $h_K$. We conjecture that a positive proportion of the ideal classes of $K$ contain a prime ideal with a norm less than $h_K \log(|D_K|)$.

Contents

1. Introduction 2
  1.1. Motivation 2
  1.2. The generalized Minkowski’s bound for the prime ideals 4
  1.3. Repulsion of the prime ideals near the cusp 6
  1.4. Method of the proof 6
  1.5. Outline of the paper 10
  1.6. Acknowledgements 11
2. Generalized Minkowski’s bound for prime ideals of $\mathcal{O}_\sqrt{D}$ 12
  2.1. Selberg upper bound Sieve 12
  2.2. Proof of Theorem 1.1 21
3. Quantitative equidistribution of integral points on hyperboloid 22
  3.1. Bounding the high frequency contribution 22
  3.2. Maass identity via the Siegel theta kernel 31
  3.3. Bounding the low frequency contribution 36
4. Class number formula with divisibility conditions 38
5. Bounding the $L^2$ norm of the Siegel theta transfer 45
  5.1. The Mellin transform of the theta transfer 45
  5.2. Bounding the $L^2$ norm of the theta transfer 50
References 60

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1. Introduction

1.1. Motivation. In this paper, we consider the problem of giving the optimal upper bound on the least prime number represented by a binary quadratic form in terms of its discriminant. Giving sharp upper bound on the least prime number represented by a binary quadratic form is crucial in the analysis of the complexity of some algorithms in quantum compiling. In particular, Ross and Selinger’s algorithm for the optimal navigation of z-axis rotations in SU(2) by quantum gates [RS14] and its p-adic analogue for finding the shortest path between two diagonal vertices of LPS Ramanujan graphs [Sar17]. In [Sar17], we proved that these heuristic algorithms run in polynomial time under a Cramér type conjecture on the distribution of the inverse image of the integers representable as a sum of two squares by a binary quadratic form; see [Sar17, Conjecture 1.4]. In this paper we show that this Cramér type conjecture holds with a positive probability that depends only on the number of the classes of the binary quadratic forms of discriminant $D$. By assuming the generalized Riemann hypothesis for the zeta function of the Hilbert class field of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, 100% of the binary quadratic forms of discriminant $D$ represent a prime number less than $h(D) \log(|D|)^{2+\epsilon}$ as $D \to -\infty$. In this paper, we remove the GRH assumption and show that unconditionally with probability at least $\alpha \left( \frac{\pi_D(X)}{\pi(X)} \right)^2$ a binary quadratic forms of discriminant $D < 0$ represent a prime number smaller than any fixed scalar multiple of $h(D) \log(|D|)$, where $\alpha$ is an absolute constant independent of $D$. As a result, we prove that if $\left( \frac{\pi_D(X)}{\pi(X)} \right)^2 \gg 1$ for some $X \sim h(D) \log(|D|)$ then a positive proportion of the binary quadratic forms of discriminant $D < 0$ represent a prime number smaller than any fixed scalar multiple of $h(D) \log(|D|))$. More precisely, we prove the following result about the least prime represented by a binary quadratic form of fixed discriminant $D$.

**Theorem 1.1.** Assume that $D < 0$ is a fundamental discriminant, $h(D)$ and $\pi_D(X)$ are the class number and the number of primes in the interval $[X,2X]$ that splits in the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. Moreover, Let $R(X,D)$ be the number of the classes of the binary quadratic forms of discriminant $D$ which represents a prime number less than $X$. Then

$$\left( \frac{\pi_D(X)}{\pi(X)} \right)^2 \ll \frac{R(X,D)}{h(D)} \left( 1 + \frac{h(D)}{\pi(X)} \right),$$

where $\pi(X)$ is the number of the primes less than $X$ and the implicit constant in $\ll$ is independent of $D$.

**Remark 1.2.** Note that by Chebotarev’s density theorem or Dirichlet’s theorem we have $\frac{\pi_D(X)}{\pi(X)} \sim 1/2$ as $X \to \infty$. By assuming Riemann hypothesis or even zero-free region of width $O(\frac{\log \log D}{\log(D)})$ for the Dirichlet L-function $L(s,X_D)$, we have $\frac{\pi_D(X)}{\pi(X)} \sim 1/2$ for any $X \gg D^\epsilon$ where $\epsilon > 0$. Since $h(D) \gg D^{1/2-\epsilon}$ then under GRH we have $\frac{\pi_D(X)}{\pi(X)} \sim 1/2$ for any $X \sim h(D) \log(|D|)$ and it follows that the above proposed algorithms give a probabilistic polynomial time algorithm for navigating SU(2) and $\text{PSL}_2(\mathbb{Z}/q\mathbb{Z})$. 
Next, we show that our result is optimal up to a scalar. Namely, if a positive proportion of the binary quadratic forms of discriminant $D$, represent a prime number less than $X$ then we have the following lower bound on $X$

$$h(D) \log D \ll X.$$  

We give a proof of this claim in what follows. Let $H(D)$ be the genus class of the binary quadratic form of discriminant $D$ and $r(n, D)$ denote the sum of the representation of $n$ by all the classes of binary quadratic forms of discriminant $-D$

$$r(n, D) = \sum_{Q \in H(D)} r(n, Q).$$

By the classical formula due to Dirichlet we have

$$(1.1) \quad r(n, D) = w_D \sum_{d|n} \chi_D(d),$$

where,

$$w_D = \begin{cases} 
6 & \text{if } D = -3 \\
4 & \text{if } D = -4 \\
2 & \text{if } D < -4. 
\end{cases}$$

This means that the multiplicity of representing a prime number $p$ by all the binary quadratic forms of a fixed negative discriminant $D < -4$ is bounded by 4

$$(1.2) \quad r(p, D) \leq 4.$$  

Assume that a positive proportion of binary quadratic forms represent a prime number smaller than $X$. Let $N(X, D)$ denote the number of the pairs $(p, Q)$ such that $p < X$ is a prime number represented by $Q \in H(D)$. We proceed by giving a double counting formula for $N(X, D)$. By our assumption a positive proportion of binary quadratic forms of discriminant $D < -4$ represent a prime number less than $X$, then

$$(1.3) \quad h(D) \ll N(X, D).$$

On the other hand,

$$N(X, D) = \sum_{p < X} r(p, D).$$

By inequality (1.2)

$$N(X, D) \leq 4\pi_D(X),$$

where $\pi_D(X)$ is the number of primes $p < X$ that splits in $\mathbb{Q}(\sqrt{D})$. By the above inequality and inequality (1.3) we obtain

$$h(D) \ll \pi_D(X).$$

By Siegel’s lower bound $D^{1/2-\varepsilon} \ll h(D)$, it follows that

$$h(D) \log(D) \ll X.$$  

This completes the proof of our claim.
1.2. The generalized Minkowski’s bound for the prime ideals. It follows from our result that a positive proportion of the ideal classes of $\mathbb{Q}(\sqrt{D})$ contains a prime ideal with a norm less than the optimal bound $h(D)\log(|D|)$. More precisely, let $D < 0$ be a fundamental discriminant, which means $D$ is square-free and $D \equiv 1 \mod 4$. Let $H_D$ denote the ideal class group of $\mathbb{Q}(\sqrt{D})$ and $N_{\mathbb{Q}(\sqrt{D})}(x + y\sqrt{D}) = x^2 - Dy^2$ be the norm of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. Given an integral ideal $I \subset \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$, let $q_I(x, y)$ be the following class of the integral binary quadratic form defined up to the action of $SL_2(\mathbb{Z})$

$$q_I(x, y) := \frac{N_{\mathbb{Q}(\sqrt{D})}(x\alpha + y\beta)}{N_{\mathbb{Q}(\sqrt{D})}(I)} \in \mathbb{Z},$$

where $x, y \in \mathbb{Z}$, and $I \cong \langle \alpha, \beta \rangle_\mathbb{Z}$ identifies the integral ideal $I$ with $\mathbb{Z}^2$. It follows that $q_I$ only depends on the ideal class $[I] \in H_D$. This gives an isomorphism between $H_D$ and the orbits of the integral binary quadratic forms of discriminant $-D$ under the action of $SL_2(\mathbb{Z})$. Note that if $q_I$ represents the prime number $p$ then $q_I(x, y) = p$ for some $x, y \in \mathbb{Z}$. Then, the principal ideal $(x\alpha + y\beta) = IJ$ factors into the product of $I$ and $J$ where $N_{\mathbb{Q}(\sqrt{D})}(J) = p$ and $J$ belongs to the inverse of the ideal class $[I] \in H_D$. Let $h_D(X)$ denote the number of the ideal classes of the ideal class group of $\mathbb{Q}(\sqrt{D})$ that contains a prime ideal with norm less than $X$. Hence, we have the following corollary from Theorem 1.1.

**Corollary 1.3.** We have

$$\left(\frac{\pi_D(X)}{\pi(X)}\right)^2 \ll \frac{h_D(X)}{h(D)} \left(1 + \frac{h(D)}{\pi(X)}\right),$$

where the implicit constant in $\ll$ is independent of $D$.

More generally, let $K$ be a number field of bounded degree $n$ over $\mathbb{Q}$ with the discriminant $D_K$ and the class number $h_K$. Then we have the following conjecture which generalizes Minkowski’s bound for the prime ideals.

**Conjecture 1.4.** Let $K$ be a number field of bounded degree $n$ over $\mathbb{Q}$ with the discriminant $D_K$ and the class number $h_K$. Then a positive proportion (only depends on $n$) of the ideal classes in the ideal class group of $K$ contain a prime ideal with a norm less than any fixed scalar multiple of $h_K \log(D)$.

Next, we show that these bounds are compatible with the random model for the prime numbers known as Cramér’s model. We cite the following formulation of the Cramér model from [Sou07].

**Cramér’s model 1.5.** The primes behave like independent random variables $X(n)$ ($n \geq 3$) with $X(n) = 1$ (the number $n$ is ‘prime’) with probability $1/\log n$, and $X(n) = 0$ (the number $n$ is ‘composite’) with probability $1 - 1/\log n$.

Note that each class of the integral binary quadratic forms is associated to a Heegner point in $SL_2(\mathbb{Z}) \setminus \mathbb{H}$. By the equidistribution of Heegner points in $SL_2(\mathbb{Z}) \setminus \mathbb{H}$, it follows that almost all classes of the integral quadratic forms has a representative $Q(x, y) := Ax^2 + Bxy + Cy^2$ such that the coefficients of $Q(x, y)$ are bounded by any function growing faster than $\sqrt{D}$:

$$\max(|A|, |B|, |C|) < \sqrt{D} \psi(D),$$
for any function $\psi(D)$ defined on integers such that $\psi(D) \to \infty$ as $D \to \infty$. We show this claim in what follows. We consider the set of representative of the Heegner points inside the Gauss fundamental domain of $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ and denote them by $z_\alpha$ for $\alpha \in H(D)$. They are associated to the roots of a representative of a binary quadratic form in the ideal class group. By the equidistribution of Heegner points in $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ and the fact that the volume of the Gauss fundamental domain decay with rate $y^{-1}$ near the cusp, it follows that for almost all $\alpha \in H(D)$ if $z_\alpha = a + ib$ is the Heegner point inside the Gauss fundamental domain associated to $\alpha$ then

\begin{align}
|a| & \leq 1/2, \\
\sqrt{3}/2 & \leq b \leq \psi(D),
\end{align}

where $\psi(D)$ is any function such that $\psi(D) \to \infty$ as $D \to \infty$. Let $Q_\alpha(x, y) := Ax^2 + Bxy + Cy^2$ be the quadratic forms associated to $\alpha \in H(D)$ that has $z_\alpha$ as its root. Then

$$z_\alpha = \frac{-B \pm i\sqrt{D}}{2A},$$

where $a = \frac{-B}{2A}$ and $b = \frac{\sqrt{D}}{2A}$. By inequality (1.5), we have

$$|b| \leq |a|,$$

(1.6)

$$\frac{\sqrt{D}}{2\psi(D)} \leq A < \sqrt{D}.$$

By the above inequalities and $D = B^2 - 4AC$, it follows that

(1.7) \quad \max(|A|, |B|, |C|) < \sqrt{D\psi(D)}.$$

This concludes our claim. Next, we give a heuristic upper bound on the size of the smallest prime number represented by a binary quadratic forms of discriminant $D$ that satisfies (1.7). Since $D$ is square-free, there is no local restriction for representing prime numbers. So, by the Cramèr’s model and consideration of the Hardy-Littlewood local measures, we expect that for a positive proportion of the classes of the binary quadratic forms $Q$ there exists an integral point $(a, b) \in \mathbb{Z}^2$ such that $|(a, b)|^2 < L(1, \chi_D)\log(D)$ and $Q(a, b)$ is a prime number. We have

$$Q(a, b) = Aa^2 + Bab + Cb^2$$

(1.8)

$$\leq \max(|A|, |B|, |C|)|\{a, b\}|^2$$

$$\ll \sqrt{D}L(1, \chi_D)\psi(D)\log(D).$$

We may take $\psi(D)$ to be any constant in the above estimate. Therefore, we expect that a positive proportion of the quadratic forms of discriminant $D$ represent a prime number less than $h(D)\log(D)$. By a similar analysis, we expect that almost all binary quadratic forms of discriminant $D$ represent a prime number less than $h(D)\log(D)^{2+\epsilon}$. In other words, almost all ideal classes of $Q(\sqrt{D})$ contain a prime ideal with norm less than $h(D)\log(D)^{2+\epsilon}$. In [Sar18], we proved this result by assuming the generalized Riemann hypothesis for the zeta function of the Hilbert class field of the imaginary quadratic field $Q(\sqrt{D})$. We conjectured that this type of generalized Minkowsky’s bounds holds for every number fields.

**Conjecture 1.6.** Let $K$ be a number field of the bounded degree $n$ over $\mathbb{Q}$ with the discriminant $D$ and the class number $h_K$. Then almost all ideal classes in the ideal class group of $K$ contain a prime ideal with norm less than $h_K\log(D)^A$. 

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for some $A > 0$. Note that by the Brauer-Siegel Theorem and GRH, we have $h_K \ll \sqrt{D} \log(D)^c$.

1.3. Repulsion of the prime ideals near the cusp. As we noted above, based on the Cramér’s model we expect that the split prime numbers randomly distributed among the ideal classes of $\mathbb{Q}(\sqrt{D})$, and hence with a positive probability that is independent of $D$, a quadratic form of discriminant $-D$ represent a prime number less than a fixed scalar multiple of $h(D) \log(D)$. We may hope that every ideal class contain a prime ideal of size $h(D) D^{1/2+\epsilon}$. Note that Cramér conjecture states that every short interval of size $\log(X)^{2+\epsilon}$ contains a prime number. By Linnik’s conjecture, every congruence class modulo $q$ contains a prime number less than $q^{1+\epsilon}$. This shows that small prime numbers covers all the short interval and congruence classes. However, we note that the family of binary quadratic forms of discriminant $D < 0$ is different from the family of short intervals and its $p$-adic analogue. Small primes are not covering all the class of binary quadratic forms. For example, the principal ideal class that is associated to the binary quadratic form $Q(x, y) = Dx^2 + y^2$ repels prime number which means the least prime number represented by this form is bigger than $D$ compared to $\sqrt{D} \log(D)^{2+\epsilon}$ that is the upper bound for almost all binary quadratic forms under GRH. This feature is different from the analogues conjectures for the size of the least prime number in a given congruence classes modulo an integer (Linnik’s conjecture) and the distribution of prime numbers in short intervals (Cramér’s conjecture). We call this new feature the repulsion of small primes by the cusp. In fact, the binary quadratic forms with the associated Heegner point near the cusp repels prime numbers. This can be seen in equation (1.8), where $\max(|A|, |B|, |C|)$ could be as large as $D$ near the cusp but for a typical binary quadratic form it is bounded by $D^{1/2+\epsilon}$. This shows that the bound in the Conjecture 1.6 does not hold for every ideal class.

1.4. Method of the proof. Our strategy of the proof is based on our recent work on the distribution of the prime numbers in the short intervals. In [Sar], we proved that a positive proportion of the intervals of any fixed scalar multiple of $\log(X)$ in the dyadic interval $[X, 2X]$ contain a prime number. We also showed that a positive proportion of the congruence classes modulo $q$ contain a prime number smaller than any fixed scalar multiple of $\varphi(q) \log(q)$. These result are compatible with Cramér’s Model.

We briefly describe our method here. We proceed by introducing some new notations and follow the previous ones. Let $w(u)$ be a positive smooth weight function that is supported on $[1, 2]$ and $\int w(u) du = 1$. Let $w_X(u) := w(u/X)$ that is derived from $w(u)$ by scaling with $X$. Let $R(X, D)$ denote the number of the classes of binary quadratic forms of discriminant $D$ that represents a prime number inside the dyadic interval $[X, 2X]$. Let $\pi(Q, w, X)$ denote the number of primes weighted by $w_X$ that are representable by the binary quadratic form $Q$. By the Cauchy-Schwarz inequality, we obtain

\[
\left( \sum_{Q \in H_D} \pi(Q, w, X) \right)^2 \leq R(X, D) \left( \sum_{Q \in H_D} \pi(Q, w, X)^2 \right).
\]

By Dirichlet formula in (1.1), $\sum_{Q \in H_D} \pi(Q, w, X)$ is the weighted number of prime numbers inside the interval $[X, 2X]$ that splits in the quadratic field $\mathbb{Q}(\sqrt{D})$. So,
we have
\[
\pi_D(X) \sim \sum_{Q \in H(D)} \pi(Q, w, X).
\]

Next, we give a double counting formula for the sum \( \left( \sum_{Q \in H(D)} \pi(Q, w, X)^2 \right) \). This sum counts pairs of primes \((p_1, p_2)\) weighted by \(w_X(p_1)w_X(p_2)\) such that \(p_1\) and \(p_2\) are represented by the same binary quadratic form class \([Q] \in H(D)\). Assume that \(Q\) is a representative of that class that represents two prime numbers \(p_1\) and \(p_2\). Without loss of generality we assume that \(Q(x, y) = p_1x^2 + \alpha xy + \beta y^2\) for some integers \(\alpha\) and \(\beta\) such that
\[
D = \alpha^2 - 4p_1\beta.
\]

Since by the action of \(SL_2(\mathbb{Z})\) on the space of the integral binary quadratic forms we can find a representative of \(Q\) with the above form. Since \(Q\) represents \(p_2\) then
\[
p_2 = p_1u^2 + \alpha uv + \beta v^2,
\]
for some integers \(u\) and \(v\). We multiply both side of the above identity by \(4p_1\) and obtain
\[
4p_1p_2 = 4p_1^2u^2 + 4p_1\alpha uv + 4p_1\beta v^2.
\]

We use identity (1.11), and substitute \(\alpha^2 - D = 4p_1\beta\) in the above identity and obtain
\[
4p_1p_2 = 4p_1^2u^2 + 4p_1\alpha uv + (\alpha^2 - D)v^2.
\]

Hence,
\[
4p_1p_2 = (2p_1u + \alpha v)^2 - Dv^2.
\]

We change the variables to \(s := 2p_1u + \alpha v\), and obtain
\[
4p_1p_2 = s^2 - Dv^2.
\]

On the other hand if \((p_1, p_2)\) is a solution to the equation (1.12) for prime numbers \(X < p_1 < 2X\) and \(X < p_2 < 2X\), then \(p_1\) and \(p_2\) are represented by the same binary quadratic form class in \(H(D)\). Heuristically, this number is about \(\pi_D(X)^2\), that is the number of distinct pairs of split primes inside the interval \([X, 2X]\) divided by the number of the classes of binary quadratic forms of discriminant \(D\) plus the contribution of diagonal terms where \(p_1 = p_2\). Therefore, we expect
\[
\left( \sum_{Q \in H(D)} \pi(Q, w, X)^2 \right) \approx \frac{\pi_D(X)^2}{h(D)} + \pi_D(X).
\]

In fact, by applying the Selberg upper bound sieve on the number of the prime solutions \((p_1, p_2)\) to the equation (1.12), we show that
\[
\left( \sum_{Q \in H(D)} \pi(Q, w, X)^2 \right) \ll \frac{\pi(X)^2}{h(D)} + \pi(X).
\]

Therefore, by the inequality (1.9), equation (1.10) and the above inequality, it follows that
\[
\left( \frac{\pi_D(X)}{\pi(X)} \right)^2 \ll \frac{R(X,D)}{h(D)} \left(1 + \frac{h(D)}{\pi(X)} \right),
\]
This gives a proof of Theorem (1.1). Next, we briefly explain how we prove inequality (1.14). We begin by counting the number of the solutions \((p_1, p_2, s, v)\) to the
equation (1.12) weighted by the smooth weight function $w_X$ where $v = 0$. We call them by the diagonal solutions. If $v = 0$ then

$$4p_1p_2 = s^2.$$ 

Hence, $p_1 = p_2 = p$ for some prime number $p < X$ and $s = \pm 2p$. Therefore, the number of diagonal solutions to the equation (1.12) is the number of prime numbers weighted by $w_X$ that is

$$\pi(w_X) \approx \pi(X).$$

Next, we give an upper bound on the number of non-diagonal terms $v \neq 0$ weighted by $w_X(p_1)w_X(p_2)$. Since $D > 0$ and $w_X(p_1)w_X(p_2) \neq 0$ only if $X < p_1, p_2 < 2X$ then

$$|s| \leq 4X,$$

$$|v| \leq \frac{4X}{\sqrt{|D|}}.$$  

(1.15)

We fix $v = v_0$ and apply the Selberg upper sieve for giving a sharp upper bound up to a constant on the number of the prime solutions $(p_1, p_2)$ to the following equation weighted by $w_X(p_1)w_X(p_2)$.

(1.16)

$$s^2 - 4p_1p_2 = Dv_0^2.$$ 

More precisely, we give an upper bound on the weighted number of integral points $(x, y, z)$ lying on the following ternary quadric

(1.17)

$$V_m := \{(x, y, z) : 4xy - z^2 = m\},$$

where $x$ and $y$ do not have any prime divisor smaller than $Y$ where $m = -Dv_0^2$ and $Y \approx D^\delta$ for a small power $\delta > 0$; e.g. $\delta < 1/1036$. In what follows, we explain the Selberg upper sieve. Assume that $d_1, d_2$ and $d$ are square-free integers. Let $\#_{w_X}A_{d_1,d_2}$ denote the number of the integral solutions weighted by $w_X$ to the equation

$$4xy - z^2 = m,$$

where $d_1|x$ and $d_2|y$. Similarly, let $\#_{w_X}A_d$ be the same number where $d|xy$. We write $\#_{w_X}A$ for $\#_{w_X}A_d$ where $d = 1$. It follows from the inclusion exclusion principal that; see [BF94] Lemma 8, Page 79

$$\#_{w_X}A_d = \mu(d) \sum_{[d_1,d_2]=d} \mu(d_1)\mu(d_2)\#_{w_X}A_{d_1,d_2}. $$

(1.18)

Let $S(m, Y)$ denote the weighted number of the integral solutions $(x, y, z)$ to the equation (1.17) where $x$ and $y$ don’t have any prime divisor smaller than $Y$ and $\chi_Y(.)$ denote the indicator function of the integers with no prime divisor less than $Y$. Since $\lambda_1 = 1$ and $\lambda_d$ are real numbers then we have the following upper bound on $\chi_Y(n)$

$$\chi_Y(n) \leq \left( \sum_{d|\gcd(n, \Pi_{p < Y} p)} \lambda_d \right)^2.$$ 

(1.19)
Hence,
\[
S(m, Y) = \sum_{4xy - z^2 = m} \chi_Y(xy)w_X(x)w_X(y)
\leq \sum_{4xy - z^2 = m} \left( \sum_{d \mid \gcd(xy, \prod_{p < Y} p)} \lambda_d \right)^2 w_X(x)w_X(y)
= \sum_d \mu^+(d) \#_{w_X} A_d,
\]
where
\[
\mu^+(d) := \sum_{[d_1, d_2] = d} \lambda_{d_1} \lambda_{d_2}.
\]

In Theorem 3.10, we give an asymptotic formula for \#_{w_X} A_{d_1, d_2} with a power saving error term if \(d_1 d_2 \leq D^{1/518}\). This theorem is a quantitative version of Duke's theorem on the equidistribution of the Heegner points. The proof of this theorem is the main technical part of our work. We apply the Siegel Mass formula on the ternary quadratic form \(z^2 - 4kxy\) in order to give the main term of \#_{w_X} A_{d_1, d_2} as the product of the Hardy-Littlewood local densities. For giving a power saving upper bound on the error term we use Duke's non-trivial bounds on the Fourier coefficients of weight \(1/2\) Maass forms and our bound on the \(L^2\) norm of the theta lift of weight \(1/2\) Maass forms. We give the outline of the proof of Theorem 3.10 in the next section. By assuming this results the main term of the weighted number of integral points comes from the product of the local densities with a power saving error term \(E_r\)
\[
\#_{w_X} A_{d_1, d_2} = \sigma_{\infty, w_X} \prod_p \sigma_p + Er.
\]
where \(\sigma_p := \lim_{k \to \infty} \frac{|V_\omega(Z/p^kZ)|}{p^k}\) and \(\sigma_{\infty, w_X}\) is given by
\[
\sigma_{\infty, w_X} = \lim_{\epsilon \to 0} \int_{m < z^2 - 4d_1 d_2 xy < m + \epsilon} w_X/d_1(x)w_X/d_2(y)dxdydz.
\]
We explicitly compute these local densities in term of the quadratic character \(\chi_D\) and as a result we have an explicit formula for the sieve densities \(w(d)\) where
\[
\#_{w_X} A_d = \#_{w_X} \frac{\omega(d)}{d} + Er.
\]
For a squarefree integer \(l\), define
\[
g(l) := \frac{\omega(l)}{l} \prod_{p \mid l} \left(1 - \frac{\omega(p)}{p}\right)^{-1},
\]
and let
\[
G(Y) := \sum_{l=1}^{Y} g(l),
\]
where the sum is over squarefree variables \(l\). By the fundamental theorem for Selberg sieve \([FT10\, \text{Theorem 7.1}]\), we have
\[
S(m, Y) \leq \frac{\#_{w_X} A}{G(Y)} + Er.
\]
In Lemma 2.7, we show that
\[ L(1, \chi_D)^2 \log(D)^2 \frac{\varphi(v_0)}{v_0} \ll G(Y). \]

Finally, by summing over \(|v_0| \ll \frac{X}{\sqrt{|D|}}\) and proving the analogue of Gallagher’s result on the average size of the Hardy-Littlewood singular series \([Gal76, \text{equation (3)}]\), we prove inequality (1.14) and hence Theorem 1.1.

1.5. Outline of the paper. In Section 2, we give the proof of Theorem 1.1 by assuming the quantitative version of the Duke’s theorem that is equation (2.1). In Lemma 2.1, we compute \(\sigma_{\infty, w} X\) the Hardy-Littlewood measure at the archimedean place. In Lemma 2.3, we give an explicit formula for \(\sigma_p\) in terms of the quadratic character \(\chi_D\) and then an explicit formula for \(#w_d\) involving \(L(1, \chi_D)\) in Lemma 2.4. In Lemma 2.6, we compute the sieve densities \(w(d)\) defined in equation (1.24). In Lemma 2.7, we give a sharp upper bound on the main term of the Selberg sieve. Finally, we average over these bounds and by proving the average size of these singular series is bounded (analogue of the Gallagher’s theorem) we prove Theorem 1.1.

In Section 3, we prove Theorem 3.1 which implies equation (2.1). Let \(Q(x, y, z) = z^2 - 4kxy, V_m := \{(x, y, z) : Q(x, y, z) = m\}\) and \(\Gamma := SO_Q(Z)\) be the integral points of the orthogonal group of \(Q\). Then \(\Gamma\) is a lattice and \(\Gamma \setminus V_m\) has a natural hyperbolic structure with finitely possible elliptic and cusp points. We construct an automorphic function \(W\) defined on \(\Gamma \setminus V_m\) from the smooth function \(w_X\). We spectrally expand \(W\) in the basis of the eigenfunctions of the Laplace-Beltrami operator on \(\Gamma \setminus V_m\). We denote the contribution of the constant function by the main term and the contribution of the non-trivial eigenfunctions (Maass cusp forms and Eisenstein series of \(\Gamma \setminus V_m\)) by \(E_r\). By assuming Theorem 4.4, that we prove in Section 4, the main term is the product of the local densities. Our goal in Section 3 is to give a power saving upper bound on \(E_r\). This power saving in the error term is crucial for the application of the Selberg sieve in Section 2. Let \(T\) be a positive real number. We write \(E_r\) as the sum of the low and the high frequency eigenfunctions in the spectrum
\[ E_r = E_{r, \text{low}, T} + E_{r, \text{high}, T}, \]
where
\begin{align*}
(1.27) & \quad E_{r, \text{low}, T} := \sum_{\lambda < T} \langle f_\lambda, W \rangle R(m, f_\lambda) + cts1/4+t^2 < T(m, W), \\
(1.28) & \quad E_{r, \text{high}, T} := \sum_{\lambda > T} \langle f_\lambda, W \rangle R(m, f_\lambda) + cts1/4+t^2 > T(m, W),
\end{align*}
where \(R(m, f_\lambda)\) is the Weyl sum associated to the eigenfunction \(f_\lambda\); see equation (3.3). In Section 3.1 we give an upper bound on the contribution of \(E_{r, \text{high}}\). The upper bound follows from the integration by parts and showing that \(\langle f_\lambda, W \rangle\), the Fourier coefficients of the smooth function \(W\), decays faster than any polynomial in the spectral parameter \(O_4(\lambda^{-N})\). This implies that if \(T > D^\delta\) for some fixed \(\delta > 0\) then \(E_{r, \text{high}} = O_\delta(1)\). Hence, it suffices to bound the contribution of \(E_{r, \text{low}}\). In Section 3.2 we prove an explicit form of the Maass identity that relates the Weyl sums to the Fourier coefficients of the associated half weight Maass form obtained by the theta transfer using the Siegel theta Kernel. In Section 3.3 we apply Duke’s
non-trivial upper bound on the Fourier coefficients of the weight 1/2 Maass form and the upper bound on the $L^2$ norm of the theta transfer of a Maass form that we prove in Section 5 to give an upper bound on $E_{\text{low}}$. There is a technical issue in using Duke’s result. The bound is exponentially growing in the eigenvalues aspect with the term $\cosh(\pi t/2)$ for the weight half eigenfunctions $\varphi_\lambda$ with norm 1 and eigenvalue $1/4 + t^2$. We show that this term cancels with the exponentially decaying factor $\cosh(-\pi t/2)$ that appears in $|\Theta * \varphi_\lambda|^2$, the $L^2$ norm of the theta transfer of $\varphi_\lambda$. This is the content of Section 5. Our method is based on Katok-Sarnak’s approach [KS93]. Biro [Bir00] generalized the work of Sarnak and Katok in the level aspect for $m > 0$ with a different method. We generalize the work of Sarnak and Katok in the level aspect for $m < 0$. Therefore, we prove a quantitative version of the equidistribution of binary quadratic forms of fixed discriminant $D$ in Theorem 3.1.

In Section 4, we prove a generalized class number formula in Theorem 4.4. This theorem gives the main term of $\#_{w \in \mathcal{A}_{d_1, d_2}}$ defined in the equation (2.1). We briefly describe the proof of Theorem 4.4. The proof uses the Siegel Mass formula that gives a product formula for the sum of the representation number of an integer $n$ by a quadratic form $Q$ averaged over the genus class of $Q$. In the Lemma 4.1, we show that the genus class of $Q(x, y, z) = z^2 - 4kxy$ contains only one element for every $k \in \mathbb{Z}$. In the Lemma 4.2, we show that the representation number of each integral point on $Q(x, y, z) = Dv^2_0$ are equal of $D \gg k^{30}$ where $D$ is squarefree. Finally, Theorem 4.4 shows that in fact the Siegel Mass formula gives a product formula for the number of the integral orbits of the orthogonal group $Q$ on the quadric $Q(x, y, z) = Dv^2_0$.

In Section 5, we give an upper bound on the $L^2$ norm of the theta transfer of a weight 1/2 Maass form $f$ in the eigenvalue and the level aspect up to a polynomial in these parameters. In Lemma 5.1, we compute the Mellin-transform of the theta lift of $f$ by a see-saw identity that is originally due Niwa [Niw75] and used by Sarnak and Katok [KS93]. The see-saw identity in this case identifies the Mellin transform with the inner product of an Eisenstein series against the product of the weight 1/2 modular form $f$ and the complex conjugate of the Jacobi theta series $\bar{\theta}$. The last integral against Eisenstein series is explicitly computable by unfolding the Eisenstein series. Hence, we obtain the Fourier coefficients of the theta transfer at the cusp at infinity. Finally, we bound the $L^2$ norm of a modular form by bounding the truncated sum of the squares of its Fourier coefficients; see [Iwa02a, Page 110, equation 8.17]. Note that the $L^2$ norm of the theta transfer of a new form is given by the Rallis-Inner product formula. Since we also deal with old forms, we rather use a more direct approach. We used the classical Seigel theta kernel in order to lift Maass forms into weight 1/2 modular forms and vice versa.

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2. Generalized Minkowski’s bound for prime ideals of $\mathcal{O}_{\sqrt{D}}$

In this section, we give the proof of Theorem 1.1 by assuming

\[(2.1) \#_{wX}A_{d_1,d_2} = \sigma_{\infty,wX} \prod_p \sigma_p + E_r,\]

with a power saving bound on $E_r$. We prove this identity in Theorem 3.1 which is the quantitative version of the Duke’s theorem. We proceed by computing the local densities $\sigma_{\infty,wX}$ and $\sigma_p$.

2.1. Selberg upper bound Sieve. We begin by computing $\sigma_{\infty,wX}$ explicitly.

**Lemma 2.1.** Let $\sigma_{\infty,wX}$ be as above in equation (1.23). We have

\[\sigma_{\infty,wX} = \frac{X}{d_1d_2} W\left(\frac{m}{X^2}\right),\]

where

\[W(a) := \int_1^2 \int_1^2 \left( \frac{1}{2\sqrt{4x_1x_2 + a}} \right)^+ w(x_1)w(x_2)dx_1dx_2.\]

**Proof.** We change the variables to $u := d_1x$ and $v := d_2y$ then

\[\sigma_{\infty,wX} = \lim_{\epsilon \to 0} \frac{\int_{m < z^2 - 4d_1d_2 x y < m + \epsilon} w_X/d_1(x)w_X/d_2(y)dxdydz}{\epsilon} = \frac{1}{d_1d_2} \lim_{\epsilon \to 0} \frac{\int_{m < z^2 - 4uv < m + \epsilon} w_X(u)w_X(v)dudvdz}{\epsilon}.\]

Next, we scale the coordinates by $1/X$ and define $x_1 = u/X$, $x_2 = v/X$ and $x_3 = z/X$. Hence,

\[\sigma_{\infty,wX} = \frac{1}{d_1d_2} \lim_{\epsilon \to 0} \frac{\int_{m < z^2 - 4uv < m + \epsilon} w_X(u)w_X(v)dudvdz}{\epsilon} = X \frac{1}{d_1d_2} \lim_{\epsilon \to 0} \frac{\int_{m < z^2 - 4uv < m + \epsilon} w_X(u)w_X(v)dudvdz}{\epsilon'} = \frac{X}{d_1d_2} \int_1^2 \int_1^2 \left( \frac{1}{2\sqrt{4x_1x_2 + \frac{m}{X^2}}} \right)^+ w(x_1)w(x_2)dx_1dx_2,
\]

where $\epsilon' := \frac{1}{X}$ and

\[\left( \frac{1}{2\sqrt{a}} \right)^+ := \begin{cases} \frac{1}{2\sqrt{a}} & \text{if } a > 0 \\ 0 & \text{otherwise.} \end{cases}\]

Let

\[W(a) := \int_1^2 \int_1^2 \left( \frac{1}{2\sqrt{4x_1x_2 + a}} \right)^+ w(x_1)w(x_2)dx_1dx_2.\]

Then

\[\sigma_{\infty,wX} = \frac{X}{d_1d_2} W\left(\frac{m}{X^2}\right).\]
It follows that $W$ is a smooth and bounded function where the $L^\infty(W)$ is bounded by a constant that only depends on the smooth function $w$. ■

Next, we compute explicitly, the local density $\sigma_p$ at each odd prime $p$. We have

$$\sigma_p = \sum_{t=0}^{\infty} S(p^t),$$

where $S(1) := 1$ and

$$S(p^t) := \frac{1}{p^{3t}} \sum_{a} \sum_{b} e\left(\frac{a(Q_{d_1d_2}(b) - n)}{p^t}\right),$$

where $a$ runs over integers modulo $p^t$ with $\gcd(a,p) = 1$, and $b$ runs over vectors in $\mathbb{Z}^3$ modulo $p^t$. Since $p$ is an odd prime number, we can diagonalize our quadratic form $Q_{d_1d_2}(X)$ over the local ring $\mathbb{Z}_p$ by changing the variables to $x_1 = z$, $x_2 = x - y$ and $x_3 = x + y$ and obtain

$$Q_{d_1d_2}(x_1, x_2, x_3) = x_1^2 + d_1d_2x_2^2 - d_1d_2x_3^2.$$

We apply the following lemma for the computation of local densities. For another versions for this lemma see; [TS17, Lemma 3.1] and Blomer [Blo08, (1.8)].

**Lemma 2.2.** Let

$$Q(x_1, x_2, x_3) = x_1^2 + p^\alpha dx_2^2 - p^\alpha dx_3^2,$$

where $\alpha \in \mathbb{Z}$ with $\alpha \geq 0$ and $d \in \mathbb{Z}_p$ with $\gcd(d,p) = 1$. Assume that $n = p^\beta n'$ where $n' \in \mathbb{Z}_p$ with $\gcd(n',p) = 1$. Let $V_n$ be the following quadric

$$V_n : Q(x_1, x_2, x_3) = n,$$

defined over $\mathbb{Z}_p$. Then

(2.6)$$\sigma_p(V_n) := \lim_{t \to \infty} \frac{V_n(\mathbb{Z}/p^t\mathbb{Z})}{p^{2t}} = 1 + \sum_{t=1}^{\infty} S(p^t),$$

where

$$S(p^t) := \frac{1}{p^{3t}} \sum_{a} \sum_{b} e\left(\frac{a(Q_{d_1d_2}(b) - n)}{p^t}\right).$$

Moreover if $t$ is odd, then

(2.7)$$S(p^t) = \begin{cases}
\left(\frac{n'}{p}\right)\frac{p^{\min(\alpha+t,2t)}}{p^{3t}} \frac{p^{\frac{t}{2}}}{p^t} & \text{if } \beta = t - 1, \\
0 & \text{otherwise.}
\end{cases}$$

where $\left(\frac{n'}{p}\right)$ denote the Legendre symbol of $n'$ modulo $p$, and if $t$ is even then

(2.8)$$S(p^t) = \begin{cases}
0 & \text{if } \beta < t - 1, \\
-\frac{p^{\min(\alpha+t,2t)}}{p^{3t}} \frac{p^{t/2}}{p^t} & \text{if } \beta = t - 1, \\
\frac{p^{\min(\alpha+t,2t)}}{p^{3t}} \phi(p^t) & \text{if } \beta \geq t.
\end{cases}$$
Proof. We compute

\[ S(p^t) := \frac{1}{p^{3t}} \sum_a \sum_{b \in (\mathbb{Z}/p^3 \mathbb{Z})^3} e \left( \frac{a(Q(b) - n)}{p^t} \right) \]

\[ = \frac{1}{p^{3t}} \sum_a \sum_{b \in (\mathbb{Z}/p^3 \mathbb{Z})^3} e \left( \frac{a(b_1^2 + p^\alpha db_2^2 - p^\alpha db_2^2 - n)}{p^t} \right) \]

\[ = \frac{1}{p^{3t}} \sum_a \sum_{b \in (\mathbb{Z}/p^3 \mathbb{Z})^3} e \left( \frac{(-an)}{p^t} \right) \prod_{i=1}^3 \sum_{b \mod \ p^t} e \left( \frac{aa_i b_i^2}{p^t} \right), \]

where \( a_1 := 1, \ a_2 := d, \ a_3 := -d \) and \( a_3 = \beta \). We note that the last summation is a Gauss sum. Let \( G(h, m) := \sum_{x \mod m} e \left( \frac{hx^2}{m} \right) \) be the Gauss sum, and let \( \varepsilon_m = 1 \) if \( m \equiv 1 \) modulo 4 and \( \varepsilon_m = i \) if \( m \equiv 3 \) modulo 4. Then if \( \gcd(h, m) = 1 \), we have

\[ G(h, m) := \begin{cases} \varepsilon_m \left( \frac{h}{m} \right) m^{1/2} & \text{if } m \text{ is odd,} \\ (1 + \chi_4(h))m^{1/2} & \text{if } m = 4^i, \\ (\chi_8(h) + i\chi_8(h))m^{1/2} & \text{if } m = 2.4^i, \alpha \geq 1, \end{cases} \]

where \( \left( \frac{h}{m} \right) \) is the Jacobi symbol. We define \( G(h, p^{t-\alpha}) := 1 \) when \( t < \alpha \). We have

\[ S(p^t) = \frac{1}{p^{3t}} \sum_a e \left( \frac{-an}{p^t} \right) \prod_{i=1}^3 p^{\min(\alpha, t)} G(aa_i, p^{t-\alpha}). \]

We substitute the values of \( G \) and obtain

\[ S(p^t) = \prod_{i=1}^3 p^{\min(\alpha + t, t)} e \sum_a e \left( \frac{-an}{p^t} \right) \left( \frac{a}{p} \right)^t \left( \frac{-1}{p} \right)^{-t}, \]

By our assumption we have \( n = \beta n' \), where \( \gcd(n', p) = 1 \). If \( t \) is an odd number, then the inner sum is a Gauss sum, and we obtain

\[ \sum_a e \left( \frac{-an}{p^t} \right) \left( \frac{a}{p} \right)^t = \begin{cases} e_p \left( \frac{n'}{p} \right) p^{t-2} & \text{if } \beta = t - 1, \\ 0 & \text{otherwise.} \end{cases} \]

Note \( e_p \left( \frac{1}{p} \right) = 1 \) and \( e_p^2 \left( \frac{1}{p} \right) \left( \frac{n'}{p} \right)^{t-\alpha} = 1 \). Hence if \( t \) is odd, we deduce that

\[ S(p^t) = \begin{cases} \left( \frac{n'}{p} \right)^{\min(\alpha + t, 2t)} p^{t-\alpha} & \text{if } \beta = t - 1, \\ 0 & \text{otherwise.} \end{cases} \]

On the other hand, if \( t \) is even then the inner sum is a Ramanujan sum \( c_{p^t}(n) \):

\[ c_{p^t}(n) = \sum_a e \left( \frac{-an}{p^t} \right) = \begin{cases} 0 & \text{if } \beta < t - 1, \\ -p^{t-1} & \text{if } \beta = t - 1, \\ \phi(p^t) & \text{if } \beta \geq t. \end{cases} \]

Hence if \( t \) is even, it follows that
More generally, we have

\[ S(p^t) = \begin{cases} 
0 & \text{if } \beta < t - 1, \\
\prod_{p \leq t} p^{\min\left(\frac{\alpha + 1}{t}, 1\right)} \cdot p^{t-1} & \text{if } \beta = t - 1, \\
\phi(p^t) \prod_{p \leq t} p^{\min\left(\frac{\alpha + t}{p}, 1\right)} & \text{if } \beta \geq t.
\end{cases} \tag{2.10} \]

Proof. By Lemma 2.2, we have

\[ m = \frac{D_v}{p} \] and \[ V_{\alpha}(x) = \text{Ord}_{\alpha}(m) \] if \( \alpha(d_2) = \text{Ord}_{\alpha}(d_2) \), \( \beta(m) = \text{Ord}_{\beta}(m) \) and \( \sigma_p(V_{d_1,d_2,m}) := \lim_{t \to \infty} \frac{V_{\alpha}(\theta^2/p^t)}{p} \). Then, we have

\[ \sigma_p(V_{d_1,d_2,m}) = \begin{cases} 
1 + \frac{1}{p} + \frac{\chi_D(p)}{p^{k+1}} - \frac{1}{p^{k+1}} & \text{if } \alpha(d_1) = 0 \text{ and } \beta(m) = 2k \\
2 + \frac{\chi_D(p)}{p^{k+1}} - \frac{1}{p^{k+1}} & \text{if } \alpha(d_1) = 1 \text{ and } \beta(m) = 2k \\
\frac{1}{p^k} + \frac{1}{p^{k+1}} & \text{if } \alpha(d_1) = 0 \text{ and } \beta(m) = 2k + 1 \\
\frac{1}{p} & \text{if } \alpha(d_1) = 1 \text{ and } \beta(m) = 2k + 1 \\
p + 1 & \text{if } \alpha(d_1) = 2 \text{ and } \beta(m) = 2k + 1.
\end{cases} \tag{2.11} \]

Proof. By Lemma 2.2, we have

\[ \sigma_p(V_{d_1,d_2,m}) = \sigma_p(\alpha(d_1d_2), \beta(m)), \]

where \( \alpha(d_1d_2) = \text{Ord}_{\alpha}(d_1d_2) \) and \( \beta(m) = \text{Ord}_{\beta}(m) \). If \( \alpha = 0 \) and \( \beta = 0 \), it follows that

\[ \sigma_p(0,0) = 1 + \frac{\chi_D(p)}{p}. \]

More generally, we have

\[ \sigma(0,2k) = 1 + \frac{1}{p} + \frac{\chi_D(p) - 1}{p^{k+1}}. \tag{2.12} \]

Moreover, if \( \alpha = 1 \) or \( 2 \) and \( \beta = 0 \) then

\[ \sigma_p(1,0) = \sigma_p(2,0) = 1 + \chi_D(p). \]

More generally,

\[ \sigma_p(1,2k) = 2 + \frac{\chi_D(p)}{p^k} - \frac{1}{p^k}. \tag{2.13} \]

We also have for \( k \geq 1 \)

\[ \sigma_p(2,2k) = p + 1 + \frac{\chi_D(p)}{p^{k-1}} - \frac{1}{p^{k-1}}. \tag{2.14} \]

Next, we compute the local densities for \( \beta = 2k + 1 \) and \( \alpha = 0, 1, 2 \). We have

\[ \sigma(0,1) = 1 - 1/p^2, \]
\[ \sigma(1,1) = 1 - 1/p, \]
\[ \sigma(2,1) = 0. \]
In general, we have
\[ \sigma(0, 2k + 1) = 1 + \frac{1}{p} - \frac{1}{p^{k+1}} - \frac{1}{p^{k+2}}, \]
\[ \sigma(1, 2k + 1) = 2 - \frac{1}{p^k} + \frac{1}{p^{k+1}}, \]
\[ \sigma(2, 2k + 1) = 1 + p - \frac{1}{p^{k-1}}. \] (2.15)

In the following lemma, we give the asymptotic formula for \( \#_{w_X} A = \#_{w_X} A_{d_1, d_2} \) where \( d_1 = d_2 = 1 \).

**Lemma 2.4.** We have the following formula for \( \#_{w_X} A \)
\[ \#_{w_X} A = X W\left(\frac{m}{X^2}\right) L(1, \chi_D) \frac{6}{\pi^2} \prod_{\beta(p) \geq 2} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{\chi_D(p)}{p}\right) \sigma_p + \text{Er}, \] (2.16)

where \( m = Dv_0^2 \). As a result,
\[ \#_{w_X} A \ll X W\left(\frac{m}{X^2}\right) L(1, \chi_D) \left(\frac{v_0}{\varphi(v_0)}\right)^2. \] (2.17)

**Proof.** By formula 2.1, we have
\[ \#_{w_X} A = \sigma_{\infty, w_X} \prod_p \sigma_p + \text{Er}, \]
where \( \sigma_p = \sigma_p(\alpha, \beta) \) for \( \alpha(p) = 0 \) and \( \beta(p) = \text{Ord}_p(Dv_0^2) \). By Lemma 2.1 and 2.3 we have
\[ \sigma_{\infty, w_X} = X W\left(\frac{m}{X^2}\right), \]
\[ \sigma(0, 0) = \left(1 + \frac{\chi_D(p)}{p}\right) \]
\[ \sigma(0, 1) = \left(1 - \frac{1}{p^2}\right), \]
By substituting the above values in the product formula, we obtain
\[ \#_{w_X} A = X W\left(\frac{m}{X^2}\right) \prod_{\beta(p) = 0} \left(1 + \frac{\chi_D(p)}{p}\right) \prod_{\beta(p) = 1} \left(1 - \frac{1}{p^2}\right) \prod_{\beta(p) \geq 2} \sigma_p + \text{Er}. \]

We simplify the above product formula by applying the following Euler product identities
\[ L(1, \chi_D) = \prod_p \left(1 - \frac{\chi_D(p)}{p}\right)^{-1} \]
\[ \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}. \]
Hence, we have
\[
\#_{w_X} A = X W \left( \frac{m}{X^2} \right) L(1, \chi_D) \prod_p \left( 1 - \frac{\chi_D(p)}{p} \right) \\
\times \prod_{\beta(p) = 0} \left( 1 + \frac{\chi_D(p)}{p} \right) \prod_{\beta(p) = 1} \left( 1 - \frac{1}{p^2} \right) \prod_{\beta(p) \geq 2} \sigma_p + E_r
\]
\[
= X W \left( \frac{m}{X^2} \right) L(1, \chi_D) \prod_{\beta(p) \geq 2} \left( 1 - \frac{1}{p^2} \right)^{-1} \left( 1 - \frac{\chi_D(p)}{p} \right) \sigma_p + E_r.
\]
This completes the proof of the identity (2.16). By Lemma 2.3 if \( \beta(p) \geq 2 \), then
\[
\sigma_p = 1 + 1/p + O(1/p^2).
\]
Hence,
\[
\#_{w_X} A \ll X W \left( \frac{m}{X^2} \right) L(1, \chi_D) \prod_{\beta(p) \geq 2} \left( 1 + \frac{2}{p} \right)
\]
(2.18)
\[
\ll X W \left( \frac{m}{X^2} \right) L(1, \chi_D) \left( \frac{v_0}{\varphi(v_0)} \right)^2.
\]
This completes the proof of our lemma.

Recall that from identity (1.18), we have
\[
\#_{w_X} A_d = \mu(d) \sum_{[d_1, d_2] = d} \mu(d_1) \mu(d_2) \#_{w_X} A_{d_1, d_2}.
\]
In the following lemma, we give the asymptotic formula for \( \#_{w_X} A_d \) and later use it for \( \Lambda^2 \) sieve.

**Lemma 2.5.** We have
\[
(2.19) \quad \#_{w_X} A_d = \#_{w_X} A \frac{\omega(d)}{d} + E_r,
\]
where
\[
(2.20) \quad \omega(d) = \prod_{p|d} \frac{2\sigma_p(1, \beta) - \sigma_p(2, \beta)/p}{\sigma_p(0, \beta)}.
\]

**Proof.** Let \( d_1 \) and \( d_2 \) be two squarefree integers. By product formula (2.1), we have
\[
\#_{w_X} A_{d_1, d_2} = \sigma_{\infty, w_X} \prod_p \sigma_p(\alpha, \beta) + E_r,
\]
where \( \alpha(p) = \text{Ord}_p(d_1 d_2) \) and \( \beta(p) = \text{Ord}_p(Dv_0^2) \). Hence,
\[
\#_{w_X} A_{d_1, d_2} = \frac{\#_{w_X} A}{d_1 d_2} \prod_{p|d_1 d_2} \frac{\sigma_p(\alpha, \beta)}{\sigma_p(0, \beta)} + E_r.
\]
We substitute the above product formula in the identity (1.18) and obtain
\[
\sum_{|d_1,d_2|=d} \mu(d_1) \mu(d_2) \sum_{p|d_1,d_2} \frac{\sigma_p(\alpha,\beta)}{\sigma_p(0,\beta)} + Er
\]
where \(\sigma_p(\alpha,\beta)\) is defined in (2.20).

This completes the proof of our lemma. ■

In the following lemma, we give an explicit formula for \(\omega(p)\) that is defined in (2.20).

**Lemma 2.6.** We have
\[
\omega(p) = \begin{cases}
\frac{2^{\frac{2+2\chi_D(p)-1/p-\chi_D(p)/p}{1+\chi_D(p)/p}}}{1+1/p} & \text{if } \beta(p) = 0 \\
\frac{2}{1+1/p} & \text{if } \beta(p) = 1 \\
\frac{3}{1+1/p} - \frac{1/p}{\chi_D(p)/p^k + 1/p} & \text{if } \beta(p) = 2k \text{ for } k \geq 1 \\
\frac{3}{1+1/p} - \frac{1/p}{\chi_D(p)/p^k + 2/p^{k+1} - 1/p} & \text{if } \beta(p) = 2k + 1 \text{ for } k \geq 1.
\end{cases}
\]

**Proof.** By definition of \(\omega(p)\) given in equation (2.20), we have
\[
\omega(p) = \frac{2\sigma_p(1,\beta) - \sigma_p(2,\beta)/p}{\sigma_p(0,\beta)}.
\]
We substitute the explicit values of \(\sigma_p(\alpha,\beta)\) from Lemma (2.3) and obtain the explicit values of \(w(p)\).

Finally, we give an upper bound on the main term of the \(\Lambda^2\) sieve. For a square-free integer \(l\), define
\[
g(l) := \frac{\omega(l)}{l} \prod_{p|l} \left(1 - \frac{\omega(p)}{p}\right)^{-1},
\]
and let
\[
G(Y) := \sum_{l=1}^{Y} g(l),
\]
where the sum is over square free variables \(l\). In the following lemma, we give an asymptotic formula for \(G(Y)\).

**Lemma 2.7.** Let \(Y = D^\delta\) for some fixed \(\delta > 0\) and \(G(Y)\) be as above. Then
\[
L(1,\chi_D)^2\log(D)^2 \frac{\varphi(v_0)}{v_0} \ll_\delta G(Y).
\]
Proof. First, we estimate the value of $g(p)$ at primes $p$. By equation (2.22), we have
$$g(p) = \frac{\omega(p)}{p - \omega(p)} \geq 0.$$ 

By Lemma 2.6, we have
$$g(p) = \begin{cases} 
\frac{2(1 + \chi_D(p))}{p} + O(1/p^2) & \text{if } \beta(p) = 0 \\
\frac{2}{p} + O(1/p^2) & \text{if } \beta(p) = 1 \\
\frac{3}{p} + O(1/p^2) & \text{if } \beta(p) = 2k \text{ for } k \geq 1 \\
\frac{3}{p} + O(1/p^2) & \text{if } \beta(p) = 2k + 1 \text{ for } k \geq 1,
\end{cases} \quad (2.25)$$

where the implicit constant involved in $O(1/p^2)$ is independent of all variables. Next, we apply the Rankin's trick and relate the truncated sum $G(Y)$ to an Euler product. Note that
$$G(Y) \geq \sum_{n \mid n \implies p \leq Y^{1/10}} \mu(n)^2 g(n) \left( \frac{1}{n^{10/\log(Y)}} - e^{-10} \right).$$

Then
$$G(Y) \geq \prod_{p \leq Y^{1/10}} \left( 1 + \frac{g(p)}{p^{10/\log(Y)}} \right) - e^{-10} \prod_{p \leq Y^{1/10}} \left( 1 + g(p) \right).$$

Since $\frac{\exp(x)}{1+x}$ is a monotone increasing function in $x \geq 0$, then we have
$$\prod_{p \leq Y^{1/10}} \left( 1 + g(p) \right) \left( 1 + \frac{g(p)}{p^{10/\log(Y)}} \right)^{-1} \leq \exp \left( \sum_{p \leq Y^{1/10}} g(p) (1 - \frac{1}{p^{10/\log(Y)}}) \right) \leq \prod_{p \leq Y^{1/10}} \left( 4 \sum_{p \leq Y^{1/10}} \frac{1}{p^{10/\log(Y)}} \right) \sim e^4,$$

where we used the prime number theorem and the fact that $g(p) \leq \frac{4}{p}$. Hence,
$$G(Y) \geq \frac{1}{2} \prod_{p \leq Y^{1/10}} \left( 1 + \frac{g(p)}{p^{10/\log(Y)}} \right).$$

Next, we complete the above Euler product by extending the product over primes $Y^{1/10} < p$. Note that
$$\prod_{Y^{1/10} < p} \left( 1 + \frac{g(p)}{p^{10/\log(Y)}} \right) \leq \exp \left( \sum_{Y^{1/10} < p} \frac{g(p)}{p^{10/\log(Y)}} \right) \leq \exp \left( \sum_{Y^{1/10} < p} \frac{4}{p^{1+10/\log(Y)}} \right) \leq 2 \log(2), \quad (2.26)$$
where we used the fact that $$\sum_{p < N} \frac{1}{p} = \log \log(n) + O(1)$$ and $$g(p) \leq \frac{1}{p}$$. Therefore, we have

$$G(Y) \gg \prod_{p} \left(1 + \frac{g(p)}{p^{\log(Y)}}\right). \quad (2.27)$$

Next, we complexify this Euler product and consider $$G(s)$$, the Dirichlet series associated to the multiplicative function $$g$$

$$G(s) := \sum_{l} \mu(l)^2 g(l) l^{-s} = \prod_{p} \left(1 + \frac{g(p)}{p^s}\right).$$

We write

$$G(s) = \zeta(s + 1)^2 L(s + 1, \chi_D)^2 \eta(s) \tilde{G}(s), \quad (2.28)$$

where

$$\eta(s) = \prod_{\beta(p) \geq 2} \left(1 + \frac{g(p)}{p^s}\right)(1 - \frac{1}{p^{s+1}})^2(1 - \frac{\chi_D(p)}{p^{s+1}})^2, \quad (2.29)$$

and

$$\tilde{G}(s) = \prod_{\beta(p) \leq 1} \left(1 + \frac{g(p)}{p^s}\right)(1 - \frac{1}{p^{s+1}})^2(1 - \frac{\chi_D(p)}{p^{s+1}})^2. \quad (2.30)$$

We analyze the Dirichlet series $$\eta(s)$$ and $$\tilde{G}(s)$$. First, we give an upper bound on $$|\eta(s)|$$. Recall that $$\beta(p) = \text{Ord}_p(Dv_0^2)$$ and $$D$$ is squarefree. Let $$p$$ be a prime number such that $$\beta(p) \geq 2$$. Hence, $$p|v_0^2$$ and by equation (2.25), we have

$$\eta(s) = \prod_{p|v_0} \left(1 + \frac{g(p)}{p^s}\right)(1 - \frac{1}{p^{s+1}})^2(1 - \frac{\chi_D(p)}{p^{s+1}})^2$$

$$= \prod_{p|v_0} \left(1 + \frac{1 - 2\chi_D(p)}{p^{s+1}} + O\left(\frac{1}{p^{s+2}}\right)\right).$$

Hence, for $$\sigma > 0$$ we have

$$\eta(\sigma + it) \gg \prod_{p|v_0} \left(1 - \frac{1}{p}\right) = \frac{\varphi(v_0)}{v_0}. \quad (2.31)$$

In particular,

$$\eta(10/\log(Y)) \gg \frac{\varphi(v_0)}{v_0}. \quad (2.31)$$

Next, we analyze $$\tilde{G}(s)$$. Assume that $$p$$ is a prime number such that $$\beta(p) \leq 1$$. By equation (2.25), it follows that

$$(1 + \frac{g(p)}{p^s})(1 - \frac{1}{p^{s+1}})^2(1 - \frac{\chi_D(p)}{p^{s+1}})^2 = 1 + O\left(\frac{1}{p^{s+2}}\right). \quad (2.32)$$

Hence,

$$\tilde{G}(s) \ll 1, \quad (2.33)$$

$$\tilde{G}(s)^{-1} \ll 1,$$
for \( \Re(s) > -1 + \epsilon \) where the implicit constants depend only on \( \epsilon > 0 \). In particular, we have
\[
\tilde{G}(\frac{10}{\log(Y)}) \ll 1.
\]
By (2.27), (2.28), (2.31) and the above inequality, it follows that
\[
G(Y) \gg \zeta(1 + 10 \log(Y))^{2} \log(D \delta)^{2} L(1 + \frac{10}{\log(Y)}, \chi_{D})^{2} \phi(v_{0}) v_{0}.
\]
Since \( Y = D \delta \) then \( \zeta(1 + 10 \log(Y))^{2} \gg \delta \log(D) \delta \log(Y) \) and it follows that
\[
G(Y) \gg \log(D)^{2} L(1 + \frac{10}{\log(Y)}, \chi_{D})^{2} \phi(v_{0}) v_{0}.
\]
Finally, we make the observation that any completed \( L \)-function is monotone increasing in \( \sigma \geq 1 \). This is a consequence of the fact that all zero are to the left of 1. More precisely, for \( D \) a negative discriminant one looks at
\[
\Lambda(s, \chi_{D}) := \frac{|D|^{s/2}}{\pi} \Gamma\left(\frac{s + 1}{2}\right)L(s, \chi_{D}),
\]
then \( \Lambda(\sigma, \chi_{D}) \) is monotone increasing in \( \sigma \geq 1 \). The proof is an application of the Hadamard factorization formula, which shows that
\[
\Lambda(1, \chi_{D}) \leq \sum_{\rho} |1 - \sigma/\rho|,\]
and since all the zeros have real part in \( (0, 1) \) then each term \( |1 - \sigma/\rho| \) is monotone increasing in \( \sigma \geq 1 \). Therefore,
\[
L(1, \chi_{D}) \ll D^{5/\log(Y)} L(1 + \frac{10}{\log(Y)}, \chi_{D}).
\]
Since \( Y = D \delta \) then \( D^{5/\log(Y)} = e^{5\delta} \). By the above inequality and (2.34), we have
\[
L(1, \chi_{D})^{2} \log(D)^{2} \phi(v_{0}) v_{0} \ll_{\delta} G(Y).
\]
This completes the proof of our lemma.

2.2. Proof of Theorem 1.1

Proof. Recall that \( S(m, Y) \) is the weighted number of the integral solutions \((x, y, z)\)
\[
4xy - z^{2} = m,
\]
where \( x \) and \( y \) do not have a prime divisor smaller than \( Y \) and \( m = Dv_{0}^{2} \). By inequality (2.20) we have
\[
S(m, Y) \leq \sum_{d} \mu(d) \#_{w_{x}} A_{d}.
\]
By the fundamental theorem for Selberg sieve \([FH10, \text{Theorem 7.1}]\), we have
\[
S(m, Y) \leq \frac{\#_{w_{x}} A}{G(Y)} + E r
\]
By Lemma 2.4 and Lemma 2.7, we have
\[
\#_{w_X} A \ll XW\left(\frac{m}{X^2}\right)L(1, \chi_D)\left(\frac{v_0}{\varphi(v_0)}\right)^2,
\]
\[
L(1, \chi_D)^2 \log(D)^2 \frac{\varphi(v_0)}{v_0} \ll \delta G(Y).
\]
Therefore,
\[
S(m, Y) \ll XW\left(\frac{m}{X^2}\right)L(1, \chi_D)\left(\frac{v_0}{\varphi(v_0)}\right)^3,
\]
where \(m = Dv_0^2\). By inequality (1.15), we have \(v_0 \leq 4X/\sqrt{|D|}\). We sum the above inequality for \(0 \leq v_0 \leq 4X/\sqrt{|D|}\) and obtain
\[
(\sum_{Q \in H(D)} \pi(Q, X)^2) \ll \pi(X) + \frac{XW\left(\frac{Dv_0^2}{X^2}\right)}{\log(D)^2 L(1, \chi_D)} \left(\frac{v_0}{\varphi(v_0)}\right)^3 \lesssim \pi(X) + \frac{|X|}{\log(D)^2 L(1, \chi_D)} \sum_{1 \leq v_0 \leq 4X/\sqrt{|D|}} W\left(\frac{Dv_0^2}{X^2}\right)\left(\frac{v_0}{\varphi(v_0)}\right)^3.
\]
By lemma 2.1,
\[
W\left(\frac{Dv_0^2}{X^2}\right) = O(1).
\]
It is easy to check that
\[
\sum_{1 \leq v_0 \leq 4X/\sqrt{|D|}} \left(\frac{v_0}{\varphi(v_0)}\right)^3 = O(X/\sqrt{|D|}).
\]
Therefore, we obtain
\[
(\sum_{Q \in H(D)} \pi(Q, X)^2) \ll \pi(X) + \frac{|X|}{\log(D)^2 L(1, \chi_D)} \frac{X}{\sqrt{|D|}} \lesssim \pi(X) + \frac{\pi(X)^2}{h(D)}.
\]
This proves inequality (1.14) and concludes Theorem 1.1.

3. Quantitative equidistribution of integral points on hyperboloid

Let \(Q(x, y, z) := z^2 - 4xy\), and \(m := -Dv_0^2\) where \(D > 0\) is a square-free integer and \(v_0 \leq \log(D)^A\) for some \(A > 0\). Assume that \(d_1\) and \(d_2\) are integers. Let \(w(u)\) be a positive smooth weight function that is supported on \([1, 2]\) and \(\int_1^2 w(u)du = 1\). Let \(X \gg \sqrt{|m|}\) and \(w_X(u) := w(u/X)\). Recall that \(\#_{w_X} A_{d_1, d_2}(m)\) denote the number of the integral points lying on the quadric \(V_m\)
\[
V_m := \{(x, y, z) : Q(x, y, z) = m\},
\]
and weighted by \(w_X(x)w_X(y)\) such that \(x\) and \(y\) are divisible by \(d_1\) and \(d_2\), respectively. In this section, we show that
\[
\#_{w_X} A_{d_1, d_2}(m) = \sigma_\infty w_X \prod_p \sigma_p(V_m) + E,
\]
where $\sigma_{\infty,w_X}$ and $\sigma_p(V_m)$ are the local densities defined in the equation (1.23) and $E_r$ is the error term that we bound in this section. We briefly explain our method for bounding $E_r$. Let $q(x,y,z) = z^2 - 4kxy$ where $k := d_1d_2$. It follows that $\#_{w_X} A_{d_1,d_2}(m)$ is the number of integral solutions of $q(x,y,z) = m$, weighted by $w_{X/d_1}(x)w_{X/d_2}(y)$. Let $\Gamma := SO_q(\mathbb{Z})$ and consider the hyperbolic surface $\Gamma \backslash V_m$ with the Laplacian operator $\Delta$ (induced from the Casimir operator) defined on $L^2(\Gamma \backslash V_m)$. We assume that the reader is familiar with the spectral theory of $\Gamma \backslash V_m$.

We define the $\Gamma$ periodic function $W$ on $\Gamma \backslash V_m$ by averaging the smooth weight function $w$ on the $\Gamma$ orbits

\[
W(\Gamma h) := \sum_{\gamma \in \Gamma} w(\gamma h).
\]

By Theorem 3.4 the action of $\Gamma$ on $V_m(\mathbb{Z})$ has finitely many orbits. We denote the class of these orbits by $H(m) \subset \Gamma \backslash V_m$. We have

\[
\#_{w_X} A_{d_1,d_2}(m) = \sum_{h \in V_m(\mathbb{Z})} w(h)
= \sum_{\Gamma h \in H(m)} \frac{1}{|\Gamma h|} W(\Gamma h),
\]

where $|\Gamma h|$ denote the order of the stabilizer of $h$ in $\Gamma$. Define the $m$-th Weyl sum associated to a $\Gamma$ periodic function $f$ to be

\[
R(m,f) := \sum_{\Gamma h \in H(m)} \frac{1}{|\Gamma h|} f(\Gamma h).
\]

Hence,

\[
\#_{w_X} A_{d_1,d_2}(m) = R(m,W).
\]

We spectrally expand the smooth weight function $W$ in terms of the eigenfunctions of the the Laplacian operator $\Delta$ and obtain

\[
W = \int_{\Gamma \backslash V_m} Wd\sigma + \sum_{f_\lambda} \langle f_\lambda, W \rangle f_\lambda + \text{cts},
\]

where the $\text{cts}$ term refer to the contribution of the continuous spectrum (Eisenstein series). We use the above expansion and compute $R(m,W)$

\[
R(m,W) = \int_{\Gamma \backslash V_m} Wd\sigma \sum_{\Gamma h \in H(m)} \frac{1}{|\Gamma h|} + \sum_{f_\lambda} \langle f_\lambda, W \rangle R(m, f_\lambda) + \text{cts}.
\]

Note that $\sum_{\Gamma h \in H(m)} \frac{1}{|\Gamma h|}$ is the class number associated to the action of $\Gamma$ on $V_m(\mathbb{Z})$ and this term comes from the contribution of the constant function in the spectral expansion of $W$. By Theorem 3.4 the first term can be written as the product of the local densities

\[
\int_{\Gamma \backslash V_m} Wd\sigma \sum_{\Gamma h \in H(m)} \frac{1}{|\Gamma h|} = \int_{\Gamma \backslash V_m} Wd\sigma \frac{|\Gamma h|}{\text{vol}(\Gamma \backslash V_m)} \times \sigma_\infty \prod_p \sigma_p(V_m),
\]

where $\sigma_p := \frac{|V_m(\mathbb{Z}/p\mathbb{Z})|}{p^2}$ and $\sigma_\infty := \text{vol}(\Gamma \backslash V_m)$. Therefore,
24 NASER T. SARDARI

(3.6) \[ \#_{w, X}A_{d_1, d_2} = \int_{V_m} wd\sigma \times \prod_p \sigma_p(V_m) + Er, \]

where

\[ Er := \sum_{f_{\lambda}} \langle f_{\lambda}, W \rangle R(m, f_{\lambda}) + \text{cts}(m, W). \]

Our goal in this section is to give an upper bound on Er. Let \( T \) be a positive real number. We write Er as the sum of the low and the high frequency eigenfunctions in the spectrum

\[ Er = Er_{low, T} + Er_{high, T}, \]

where

\[ Er_{low, T} := \sum_{\lambda < T} \langle f_{\lambda}, W \rangle R(m, f_{\lambda}) + \text{cts}_{1/4 + \ell^2 < T}(m, W), \]

and

\[ Er_{high, T} := \sum_{\lambda > T} \langle f_{\lambda}, W \rangle R(m, f_{\lambda}) + \text{cts}_{1/4 + \ell^2 > T}(m, W). \]

**Theorem 3.1.** Let \( D \) be a fundamental discriminant and \( m = Dv^2 \) where \( v_0 < \log(D)^A \) for some fixed \( A > 0 \). Let \( \#_{w, X}A_{d_1, d_2}(m) \) be as above. Then, for every \( \epsilon > 0 \) we have

\[ \#_{w, X}A_{d_1, d_2} = \int_{V_m} wd\sigma \times \prod_p \sigma_p(V_m) + O(1 + |m|^{1/2 - 1/2k_{17+1/2} + \epsilon} (\frac{X}{\sqrt{m}})^{3/2} D^\epsilon). \]

As a result, for every \( 0 < \delta \) there exists an \( 0 < \epsilon \) such that if \( k_{518+\delta} \leq D \) and \( X \leq D^{1/2} \log(D)^B \) for some \( B > 0 \) then

\[ \#_{w, X}A_{d_1, d_2} = \int_{V_m} wd\sigma \times \prod_p \sigma_p(V_m) + O(1 + \frac{X}{d_1 d_2} D^\epsilon), \]

where the implied constant in \( O \) depends on \( \epsilon \) and \( w \).

**Proof.** By equation (3.6), we have

\[ \#_{w, X}A_{d_1, d_2} = \int_{V_m} wd\sigma \times \prod_p \sigma_p(V_m) + Er \]

where

\[ Er_{low, T} := \sum_{\lambda < T} \langle f_{\lambda}, W \rangle R(m, f_{\lambda}) + \text{cts}_{1/4 + \ell^2 < T}(m, W), \]

and

\[ Er_{high, T} := \sum_{\lambda > T} \langle f_{\lambda}, W \rangle R(m, f_{\lambda}) + \text{cts}_{1/4 + \ell^2 > T}(m, W). \]

By Lemma 3.10 we have

\[ Er_{low, T} \ll |m|^{1/2 - 1/28k_{17+1/2} + \epsilon} (\frac{X}{\sqrt{m}})^{3/2} D^\epsilon. \]
Let $T = D^{7/2}$, then
\[
\text{E}_{\text{low},T} = O\left(|m|^{1/2-1/28}k^{17+1/2+\varepsilon}\left(\frac{X}{\sqrt{|m|}}\right)^{3/2}D^\varepsilon\right).
\]

By Lemma 3.3,
\[
\text{E}_{\text{high}} = O(1),
\]
where the implied constant in $O$ depends on $\sup_{1 \leq n \leq 70/\varepsilon} d(n)w$. Therefore,
\[
#_{wX} A_{d_1,d_2} = \int_{V_m} w d\sigma \times \prod_p \sigma_p(V_m) + O\left(1 + |m|^{1/2-1/28}k^{17+1/2+\varepsilon}\left(\frac{X}{\sqrt{|m|}}\right)^{3/2}D^\varepsilon\right).
\]

This completes the proof of the equation (3.10). If $k^{518+\delta} \leq D$ then
\[
m^{-1/28}k^{17+1/2} = O\left(D^{-\delta/28}\right).
\]
Moreover if $X \leq D^{1/2}\log(D)^B$, then
\[
|m|^{1/2}\left(\frac{X}{\sqrt{|m|}}\right)^{3/2} = O\left(XD^\varepsilon\right).
\]

By the above inequalities and choosing $\varepsilon$ small enough comparing to $\delta$, we conclude inequality (3.11) and this completes the proof of our Theorem.

3.1. Bounding the high frequency contribution. In this section we give an upper bound on $\text{E}_{\text{high}}$ (3.13). Let $k := d_1d_2$,
\[
A_k := \begin{bmatrix}
0 & -2k & 0 \\
-2k & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
and
\[
C_k := \begin{bmatrix}
1/2\sqrt{k} & 1/2\sqrt{k} & 0 \\
1/2\sqrt{k} & -1/2\sqrt{k} & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
Then
\[
C_k^*A_kC_k = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

We proceed by defining the induced Casimir operator of the orthogonal group $SO(A_k)$ on the quartic $V_m := \{(x,y,z) : z^2 - 4kxy = m\}$ where $m < 0$.

Let $X_1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $X_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. By the definition of the Casimir operator of the orthogonal group $SO(A_k)$:
\[
\Omega := Y_1^2 + Y_2^2 - Y_3^3,
\]
where $Y_1 := CX_1C^{-1}$, $Y_2 := CX_2C^{-1}$ and $Y_3 := CX_3C^{-1}$. We note that $SO(A_k)$ acts transitively on $V_m$ and therefore the Casimir operator $\Omega$ induce a second order operator on $V_m$. In the following lemma, we give a formula in terms of the $(x, y, z)$ coordinates of the the quartic $V_m$ for the Casimir operator $\Omega$. 
Lemma 3.2. Let $\Omega$ be the Casimir operator that is defined as above. Then the induced operator on $V_m$ in $(x, y, z)$ coordinates is given by

\begin{equation}
\Omega = x^2 \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} + \frac{4kxy + 2m}{2D} \frac{\partial^2}{\partial x \partial y} + 2xz \frac{\partial^2}{\partial x \partial z} + y^2 \frac{\partial^2}{\partial y^2} + 2y \frac{\partial}{\partial y} + 2yz \frac{\partial^2}{\partial y \partial z} + (z^2 - m) \frac{\partial^2}{\partial z^2} + 2z \frac{\partial}{\partial z}.
\end{equation}

(3.16)

Proof. We compute the induced first order differential operators associated to $Y_1$, $Y_2$ and $Y_3$ inside the Lie algebra of $SO(A_k)$ on smooth functions defined on $V_m$. Note that

\begin{equation}
Y_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{equation}

This vector is associated to the following first order differential operator

\begin{equation}
Z_1 := x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.
\end{equation}

Similarly

\begin{equation}
Y_2 := \begin{bmatrix} 0 & 0 & 1/2\sqrt{k} \\ 0 & 0 & -1/2\sqrt{k} \\ \sqrt{k} & \sqrt{k} & 0 \end{bmatrix}
\end{equation}

is associated to

\begin{equation}
Z_2 := z/(2\sqrt{k}) \frac{\partial}{\partial x} - y/(2\sqrt{k}) \frac{\partial}{\partial y} + \sqrt{k}(x + y) \frac{\partial}{\partial z},
\end{equation}

and

\begin{equation}
Y_3 := \begin{bmatrix} 0 & 0 & 1/2\sqrt{k} \\ 0 & 0 & -1/2\sqrt{k} \\ -\sqrt{k} & \sqrt{k} & 0 \end{bmatrix},
\end{equation}

is associated to

\begin{equation}
Z_3 := z/(2\sqrt{k}) \frac{\partial}{\partial x} - z/(2\sqrt{k}) \frac{\partial}{\partial y} + (y - x)\sqrt{k} \frac{\partial}{\partial z}.
\end{equation}

The induced Casimir operator is given by

\begin{equation}
Z_1^2 + Z_2^2 - Z_3^2.
\end{equation}

We have

\begin{equation}
Z_1^2 = \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right)^2
\end{equation}

\begin{equation}
= x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y},
\end{equation}

Z_2^2 = (z/(2\sqrt{k}) \frac{\partial}{\partial x} + z/(2\sqrt{k}) \frac{\partial}{\partial y} + \sqrt{k}(x + y) \frac{\partial}{\partial z})^2

\begin{equation}
= z^2/4k \frac{\partial^2}{\partial x^2} + z^2/2k \frac{\partial^2}{\partial x \partial y} + z(x + y) \frac{\partial}{\partial x \partial z} + z/(2 \frac{\partial}{\partial z} + (x + y)/2 \frac{\partial}{\partial x}
\end{equation}

\begin{equation}
+ z^2/4k \frac{\partial^2}{\partial y^2} + z(x + y) \frac{\partial^2}{\partial y \partial z} + z/(2 \frac{\partial}{\partial z} + (x + y)/2 \frac{\partial}{\partial y} + k(x + y)^2 \frac{\partial^2}{\partial z^2},
\end{equation}

(3.20)
and
\[ Z_3^2 = (z/2 \sqrt{k} \frac{\partial}{\partial x} - z/2 \sqrt{k} \frac{\partial}{\partial y} + (y-x)\sqrt{k} \frac{\partial}{\partial z})^2 \]
(3.21)
\[ = z^2/(4k) \frac{\partial^2}{\partial x^2} - z^2/(2k) \frac{\partial^2}{\partial x \partial y} + z(y-x) \frac{\partial^2}{\partial x \partial z} - z/2 \frac{\partial}{\partial z} + (y-x)/2 \frac{\partial}{\partial x} \]
\[ + z^2/(4k) \frac{\partial^2}{\partial y^2} - z(y-x) \frac{\partial^2}{\partial y \partial z} - z/2 \frac{\partial}{\partial z} - (y-x)/2 \frac{\partial}{\partial y} + (y-x)^2 k \frac{\partial^2}{\partial z^2}. \]

By using the formulas in 3.19, 3.20 and 3.21, we have the following formula for the induced Casimir operator on \( V_m \)
\[ \Omega = z^2 \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} + \frac{2k xy + 2m}{2k} \frac{\partial^2}{\partial x \partial y} + (z^2 - m) \frac{\partial^2}{\partial z^2} + \frac{2z}{\partial z}. \]

In the following lemma, we prove an upper bound on the \( L^2 \) norm of \( W \).

**Lemma 3.3.** Let \( W, X \) and \( k \) be as above. Then
\[ |W|_2 \ll X^{3/4} \sqrt{k}. \]
(3.22)

**Proof.** We have,
\[ |W|^2 \Omega = \int_{\Gamma \setminus V_m} |W|^2 d\sigma \]
(3.23)
\[ \leq \sup |W| \int_{\Gamma \setminus V_m} |W| d\sigma. \]

First, we give an upper bound on \( \int_{\Gamma \setminus V_m} |W| d\sigma \). Recall that
\[ W(\Gamma(x, y, z)) := \sum_{\gamma \in \Gamma} w(\gamma(x, y, z)), \]
\[ w(x, y, z) := w(X_1(x)w_{X_2}(y), \]
where \( X_1 = \frac{X_1}{\sigma_1}, X_2 = \frac{X_2}{\sigma_2} \) and \( w_X(u) := w(u/X) \) for fixed smooth function \( w \) with compact support inside the interval \([1, 2]\). Note that the hyperbolic measure defined on \( V_m \) and the Hardy-littlewood measure are different by a factor of \( \frac{1}{\sqrt{m}} \). Hence, by Lemma 2.1 we have
\[ \int_{\Gamma \setminus V_m} |W| d\sigma \leq \int_{V_m} |w| d\sigma \]
(3.24)
\[ \leq \frac{X}{\sqrt{md_1d_2}} = \frac{X}{k \sqrt{m}}. \]

Next, we give an upper bound on \( \sup |W| \). Let
\[ B(X_1, X_2) := \{(x, y, z) \in V_m(\mathbb{R}) : X_1 \leq x \leq 2X_1 \text{ and } X_2 \leq y \leq 2X_2\}. \]

For \( h \in V_m \), define
\[ N(X_1, X_2, h) := \#\{\gamma \in \Gamma : \gamma h \in B(X_1, X_2)\}. \]
(3.25)
Then,
\[
W(h) = \sum_{\gamma \in \Gamma} \Omega^n w(\gamma h) \ll N(X_1, X_2, h).
\]
(3.26)

We give an upper bound on \(N(X_1, X_2, h)\) by applying known results in hyperbolic geometry. Consider the following new variables
\[
\begin{align*}
u_1 &:= \frac{d_1 x_1}{\sqrt{|m|}} \\
u_2 &:= \frac{d_2 x_2}{\sqrt{|m|}} \\
u_3 &:= \frac{x_3}{\sqrt{m}}
\end{align*}
\]
By this change of variables \(V_m\) maps to \(u^2_3 - 4u_1u_2 = -1\) and \(B(X, m)\) maps to
\[
B(X, m) := \{(u_1, u_2, u_3) : u^2_3 - 4u_1u_2 = -1, \frac{X}{\sqrt{m}} \leq u_1 \leq \frac{2X}{\sqrt{m}} \text{ and } \frac{X}{\sqrt{m}} \leq u_2 \leq \frac{2X}{\sqrt{m}}\}.
\]
The quartic \(u^2_3 - 4u_1u_2 = -1\) with its induced metric \((du_1)^2 - 4du_1du_2\) is isomorphic to the hyperbolic plane. The isomorphic is given by the following explicit map
\[
(u_1, u_2, u_3) \rightarrow \frac{u_3 + i}{2u_1}.
\]
It follows that
\[
\text{diam}(B(X, m)) \ll \frac{X}{\sqrt{m}} + 1.
\]
where \(\text{diam}(B(X, m))\) is the largest distance of pairs of points inside \(B(X, m)\) with respect to the hyperbolic metric. For \(h \in \Gamma \setminus V_m\) define the invariant height of \(h\) by
\[
y_{\Gamma}(h) = \max_w \max_{\gamma \in \Gamma} \{\operatorname{Im} \sigma_w \gamma z\},
\]
where \(w\) is a cusp of \(\Gamma\) and \(\sigma_w\) is the associated scaling matrix; see (5.22). It follows that if \(h \in B(X_1, X_2)\) then
\[
y_{\Gamma}(h) \ll \frac{X}{\sqrt{m}} + 1.
\]
(3.28)

By [Iwa02a, Corollary 2.12 Page 52], we have
\[
N(X_1, X_2, h) \ll \text{diam}(B(X, m)) \sup_{h \in B(X, m)} y_{\Gamma}(h).
\]
Therefore, by inequalities (3.27) and (3.28), we have
\[
N(X_1, X_2, h) \ll \frac{X^2}{m} + 1.
\]
(3.29)

By the above inequality and inequalities (3.26), (3.24) and (3.23), we obtain
\[
|W|_2^2 \ll \frac{X^3}{km^{3/2}}.
\]
This concludes our lemma. ■
Next, by applying the integration by parts, we give an upper bound on the inner product of our weight function $W$ with $f_\lambda$, the eigenfunction of the Casimir operator $\Omega$ with eigenvalue $\lambda$ on $\Gamma \setminus V_m$.

**Lemma 3.4.** Let $W$ and $f_\lambda$ be as above. Then we have

\[
(W, f_\lambda) \ll O_A(\frac{X^{3/2}}{m^{3/4} \sqrt{\lambda A}}),
\]

where the implied constant in $O$ depends only on the $\sup_{1 \leq n \leq A} d(n)w$, the supremum of the $n$-th derivative of the smooth weight function $w$.

**Proof.** Since, $f_\lambda$ is an eigenfunction of the Casimir operator $\Omega$ with eigenvalue $\lambda$, then

\[
(W, f_\lambda) = \frac{1}{\lambda^n} \langle W, \Omega^n f_\lambda \rangle
\]

\[
= \frac{1}{\lambda^n} \langle \Omega^n W, f_\lambda \rangle
\]

\[
= \frac{1}{\lambda^n} \int_{\Gamma \setminus V_m} \Omega^n W f_\lambda d\sigma.
\]

\[
\leq \frac{1}{\lambda^n} |\Omega^n W|_2 |f_\lambda|_2
\]

\[
\leq \frac{1}{\lambda^n} \left( \int_{\Gamma \setminus V_m} |\Omega^n W|^2 d\sigma \right)^{1/2},
\]

where we used $|f_\lambda|_2 = 1$. By a similar argument as in the Lemma 3.3, we give an upper bound on $\int_{\Gamma \setminus V_m} |\Omega^n W|^2 d\sigma$. We have

\[
\int_{\Gamma \setminus V_m} |\Omega^n W|^2 d\sigma \leq \sup \frac{|\Omega^n W|}{\lambda^n} \int_{\Gamma \setminus V_m} |\Omega^n W| d\sigma.
\]

We have

\[
\int_{\Gamma \setminus V_m} |\Omega^n W| d\sigma \leq \int_{V_m} |\Omega^n w| d\sigma
\]

\[
\leq \sup |\Omega^n w| \int_{X_1 \leq x \leq 2X_1} \int_{X_2 \leq y \leq 2X_2} d\sigma(x, y, z)
\]

\[
\ll \sup |\Omega^n w| \frac{X}{d_1 d_2 \sqrt{m}}.
\]

Moreover,

\[
\Omega^n W(h) = \sum_{\gamma \in \Gamma} \Omega^n w(\gamma h)
\]

\[
\leq N(X_1, X_2, h) \sup \Omega^n w,
\]

where $N(X_1, X_2, h)$ is defined in (3.25). By inequality (3.29), we have

\[
N(X_1, X_2, h) \ll \frac{X^2}{m} + 1.
\]

Finally, we show that $\sup \Omega^n w = O_A(1)$. Note that $w(x, y, z) := w_{X_1}(x)w_{X_2}(y)$ is independent of the $z$ variable. Therefore, all the partial derivatives that include $\frac{\partial}{\partial z}$
in formula (3.16) vanishes on \( w \) and we obtain:

\[
\Omega^n w(x, y, z) = \left( x^2 \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} + (2xy + m/D) \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2} + 2y \frac{\partial}{\partial y} \right)^n w(x) w(y).
\]

For \( n = 1 \), we check that \( \Omega w \) is bounded by a constant. We have

\[
\Omega w = \frac{x^2}{X_1} w''(\frac{x}{X_1}) w(\frac{y}{X_2}) + \frac{2x}{X_1} w'(\frac{x}{X_1}) w(\frac{y}{X_2}) + \frac{2x}{X_1} \frac{y}{X_2} w'(\frac{x}{X_1}) w'(\frac{y}{X_2}) + \frac{m}{D} X_1 X_2 w'(\frac{x}{X_1}) w'(\frac{y}{X_2}).
\]

We assume that for every \( 0 \leq n \) all the derivatives \( \frac{d^k w}{dt^k} \) for \( 0 \leq k \leq n \) are bounded by a constant \( |w|_{\infty, n} \). Since \( w \) is supported inside \([1, 2]\) then \( 1 \leq X_1, X_2 \leq 2 \), otherwise \( \Omega w = 0 \). Since, \( m < 0 \) and \( z^2 - 4kxy = m \) then \( m \leq 4kX_1 X_2 \) otherwise \( V_m \) does not have any point where \( |x| < 2X_1 \) and \( |y| < 2X_2 \). By these assumptions we can bound each term in equation (3.1) and obtain

\[
|\Omega w| \leq 24|w|^2_{\infty, 2}.
\]

Similarly, for every \( n \), it follows that

\[
\sup |\Omega^n w| \leq (100)^n |w|^2_{\infty, n}.
\]

Therefore, by inequality (3.31), (3.32), and (3.33), we obtain

\[
\langle W, f_{\lambda} \rangle \ll \frac{X^{3/2}}{\sqrt{m^{3/2}k\lambda^{}}}.
\]

where the implied constant depends only on the \( \sup_{1 \leq n \leq A} d^{(n)} w \), the supremum of the \( n \)-th derivative of the smooth weight function \( w \). This completes the proof of our lemma.

Finally, we show that the contribution of the high frequency spectrum is bounded.

**Lemma 3.5.** Let

\[
E_{r_{\text{high}}} := \sum_{\lambda > D^4} \langle f_{\lambda}, W \rangle R(m, f_{\lambda}) + cts_{D^4 \lambda^2}.
\]

Then

\[
E_{r_{\text{high}}} = O(1)
\]

where the implied constant in \( O \) depends on \( \sup_{1 \leq n \leq A} d^{(n)} w \).

**Proof.** First, we give an upper bound on the Weyl sum \( R(m, f_{\lambda}) \). We have

\[
R(m, f_{\lambda}) = \sum_{\Gamma h \in H(m)} \frac{1}{|\Gamma h|} f_{\lambda}(\Gamma h)
\]

\[
|f_{\lambda}|_{\infty} \sum_{\Gamma h \in H(m)} \frac{1}{|\Gamma h|} \leq |f_{\lambda}|_{\infty} h(k, m),
\]

\[
\langle W, f_{\lambda} \rangle \ll \frac{X^{3/2}}{\sqrt{m^{3/2}k\lambda^{}}}.
\]
where $k = d_1 d_2$. By the Weyl law we have the following trivial upper bound on the $L^\infty$ norm of an eigenfunction; see the recent work of Templier for a sharper upper bound \cite{Tem15}

\begin{equation}
|f_\lambda|_\infty \ll \lambda^{1/4} k^{1/2}.
\end{equation}

By Theorem 4.4, Lemma 2.1 and 2.2

\begin{equation}
h(k, m) \ll \frac{X^{1+\epsilon}}{k}.
\end{equation}

Therefore,

\begin{equation}
R(m, f_\lambda) \ll \frac{\lambda^{1/4} X^{1+\epsilon}}{k^{1/2}}.
\end{equation}

By Lemma 3.4 and the above inequality, we have

\begin{equation}
\sum_{\lambda > D^4} \langle f_\lambda, W \rangle R(m, f_\lambda) \ll \sum_{\lambda > D^4} \frac{X^{3/2}}{m^{3/4} \sqrt{kX^A}} \frac{\lambda^{1/4} X^{1+\epsilon}}{k^{1/2}} \ll \frac{X^{4+\epsilon}}{k^{1/2}} \sum_{\lambda > D^4} \lambda^{1/4-A}.
\end{equation}

By Weyl law for $\Gamma \backslash \mathcal{V}_m$, we have

\begin{equation}
\sum_{\lambda > D^4} \lambda^{1/4-A} \ll kD^{6(1+1/4-A)}.
\end{equation}

Recall that $X \ll D^{1/2+\epsilon}$, $k = d_1 d_2 \leq D^{1/10}$. Therefore, by choosing $A$ large enough we obtain

\begin{equation}
\sum_{\lambda > D^4} \langle f_\lambda, W \rangle R(m, f_\lambda) = O(1).
\end{equation}

Similarly, it follows that

$$\sum_{D^4 > t^2} cts \ll O(1).$$

This completes the proof of the lemma. 

\section{Maass identity via the Siegel theta kernel.}

In this section, we write the Weyl sum $R(m, W)$ in terms of the $m$-th Fourier coefficient of the theta transfer of the smooth weight function $W$. We begin by introducing Siegel’s theta kernel associated to the indefinite quadratic form $z^2 - kxy$. Let $H_{A_k}$ denote the majorant space of the symmetric matrix $A_k$ (see \cite{Sie67}):

\[ H_{A_k} := \{ P : P^t = P, P > 0 \text{ and } P^t A_k^{-1} P = A_k \}. \]

For $P \in H_{A_k}$ and $z = x + iy \in \mathbb{C}$ with $y > 0$, define

\[ R(z) := xA + iyP. \]

The Siegel’s theta function is defined for $\alpha \in \mathbb{Q}^3$ with $2A_k \alpha \in \mathbb{Z}^3$ by

\begin{equation}
\Theta_\alpha(z, P) := y^{3/4} \sum_{h \in \mathbb{Z}^3} e(R(z)[h + \alpha]),
\end{equation}

where $R(z)[h + \alpha] := (h + \alpha)^t R(z)(h + \alpha)$ and more generally for matrices $A$ and $B$ we denote $A[B] := B^t A B$. This sum is absolutely convergent for fixed $x$ since $y > 0$ and $P > 0$. We note that the orthogonal group $G := SO(A_k)$ acts transitively on the majorant space $H_A$ by sending $P \in H_{A_k}$ to $P[g] := g^t P g$ for $g \in G$. We extend
the definition of the theta Kernel from $H_A$ to $G$ by fixing an element $P_0 \in H_{A_k}$ and defining

\begin{equation}
\tilde{\Theta}(z, g) := \Theta_\alpha(z, P_0 g^{-1}).
\end{equation}

Note that we used $g^{-1}$ for transforming $P_0$. We pick $P_0$ to be the symmetric positive definite diagonal matrix in $H_{A_k}$, namely:

\begin{equation}
P_0 := \begin{bmatrix}
2k & 0 & 0 \\
0 & 2k & 0 \\
0 & 0 & 1
\end{bmatrix}.
\end{equation}

Next, we cite a theorem that gives the transformation properties of the theta kernel $\Theta_\alpha(z, P_0 [g^{-1}])$ in $z$ variable. This theorem is essentially due to Siegel [Sie51] and is stated in this form in [Duk88, Theorem 3]. It is a consequence of the poisson summation formula for the Weil representation; see [KS93] Proposition 2.2.

**Theorem 3.6** ([Duk88], [KS93]). For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \gamma \in \Gamma_0(4k)$ we have

\begin{equation}
\Theta(\gamma z, g) = \chi(\gamma)(cz + d)(\bar{c}\bar{z} + \bar{d})^{1/2} \Theta(\gamma z, g),
\end{equation}

\begin{equation}
\Omega \Theta(\gamma z, g) = 4\Delta_{z,1/2} \Theta(z, g) + \frac{3}{4} \Theta(z, g)
\end{equation}

where $\chi(\gamma) = \frac{\theta(\gamma z)}{\theta(z)}$ is the theta multiplier, $\Delta_{z,1/2}$ is the laplacian operator defined on weight $1/2$ modular forms and $\Omega$ is the Casimir operator of $G$.

**Remark 3.7.** By the above theorem it follows that if $f_\lambda$ is a cusp form with eigenvalues $\lambda = 1/4 + (2r)^2$, then $(\Theta * f_\lambda)$ is a weight $1/2$ modular form defined on $\Gamma_0(4k)\backslash H$ with eigenvalues $\lambda' = 1/4 + r^2$.

Note that $G$ also acts transitively on the one sheet of the ternary quadric $V_m$. We extend the definition of the smooth weight function $W(x, y, z)$ from $\Gamma \backslash V_m$ to $\Gamma \backslash G$ by fixing a point $x_0 \in V_m$ and defining

\[ \tilde{W}(g) := W(gx_0). \]

We fix

\[ x_0 := \begin{bmatrix} 1/2 \sqrt{m/k} \\ 1/2 \sqrt{m/k} \\ 0 \end{bmatrix}. \]

It is easy to check that $G_{x_0} = G_{P_0}$ where $G_{x_0}$ and $G_{P_0}$ are the stabilizer of $x_0 \in V_m$ and $P_0 \in H_A$ under the action of $G = SO(A_k)$. Let

\[ F(z) := \int_{\Gamma \backslash G} \tilde{\Theta}(z, g) \tilde{W}(g) dg. \]

Theorem 3.6 implies that $F(z)$ is inside $L^2_{1/2}(\Gamma_0(4k)\backslash H)$ the $L^2$ space of weight $1/2$ modular forms of level $N$. By the spectral theory of $L^2_{1/2}(\Gamma_0(4k)\backslash H)$, we write

\begin{equation}
F = \sum_\lambda \langle F, \psi_\lambda \rangle \psi_\lambda + cts
\end{equation}

where $\psi_\lambda$ is an orthogonal basis of $1/2$ Maass forms and $cts$ is the contribution of the Eisenstein series. It is known that $\psi_\lambda(z)$ has a Fourier development at $\infty$ of
the form
\[ \psi_\lambda(u + iv) = c_{\psi, \infty}(v) + \sum_{n \neq 0} \rho_{\psi, \infty}(n) W_1/4_{\text{sgn}}(n, iv)(4\pi|n|v)e(nu), \]
where \(1/4 + t^2 = \lambda\), \(c_{\psi, \infty}(v)\) is a linear combination of \(v^{1/2+it}\) and \(v^{1/2-it}\) and \(W_{\beta, \mu}(v)\) is the Whittaker function normalized so that
\[ W_{\beta, \mu}(v) \approx e^{-v/2} v^{\beta} \text{ as } v \to \infty. \]

We note that the asymptotic of the Whittaker function is independent of the spectral parameter \(\lambda\).

Let
\[ F(u + iv) = c_{F, \infty}(v) + \sum_{n \neq 0} \rho_{F, \infty}(n, v)e(nu), \]
be the Fourier expansion of \(F\) at \(\infty\). Define the \(m\)-th Fourier coefficient of \(F\) to be
\[ \rho_{F, \infty}(m) := \lim_{y \to \infty} \rho_{F, \infty}(m, v)e(4\pi|m|v)\text{sgn}(m)/4. \]

It follows from (3.44) and (3.45) that
\[ \rho_{F, \infty}(m) = \sum_\lambda \langle F, \psi_\lambda \rangle \rho_{\psi, \infty}(m) + cts_{\infty}(m). \]

Maass identity relates \(\rho_{F, \infty}(m)\), the \(m\)-th Fourier coefficient of \(F = \Theta * W\), to the Weyl sum \(R(m, W)\). This identity is stated without proof for the cups forms in [Duk88] [Theorem 6]. We give a proof of this identity for \(W\).

Lemma 3.8 ([Maa59]). Let \(F := W * \Theta\) be the theta transfer of \(W\) via the Siegel theta kernel and \(m < 0\). Then, we have
\[ \rho_{F, \infty}(m) = \frac{\pi^{1/4}}{\sqrt{2}} |m|^{-3/4} R(m, W). \]

Proof. We follow closely the method of Sarnak and Katok [KS93]. We have
\[ \rho_{F, \infty}(m, v) := \int_0^1 F(u + iv)e(-mu)du. \]

We note that
\[ \rho_{F, \infty}(m, v) = \int_{\Gamma \backslash \mathbb{G}} \int_{\mathbb{Z}^2} \Theta(u + iv, P_0[g^{-1}]) \bar{W}(g)e(-mu)dgdu. \]
\[ = \int_{\Gamma \backslash \mathbb{G}} \int_{\mathbb{Z}^2} \sum_{h \in \mathbb{Z}^2} e(uA + ivP_0[g^{-1}][h]) \bar{W}(g)e(-mu)dudg \]
\[ = \int_{\Gamma \backslash \mathbb{G}} \int_{\mathbb{Z}^2} \sum_{h \in \mathbb{Z}^2} e((uA + ivP_0[g^{-1}][h]) \bar{W}(g)e(-mu)dudg \]
\[ = \int_{\Gamma \backslash \mathbb{G}} \sum_{h \in \mathbb{Z}^2, A|h| = m} e(ivP_0[g^{-1}][h]) \bar{W}(g)dg. \]
We unfold the above integral and write it as a finite sum over the integral orbits. Then
\[
\rho_{F,\infty}(m, v) = \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_G e(ivP_0[g^{-1}][l])\tilde{W}(g)dg.
\]

Next, we use Fubini’s theorem and write the above integral over the ternary quadric \(V_m\) with its invariant measure induced from the transitive action of \(G\) on \(V_m\). Recall that
\[
x_0 := \left[\begin{array}{c}
\frac{1/2}{m/k} \\
\frac{1/2}{m/k}
\end{array}\right].
\]

Since \(G\) acts transitively on \(V_m\), for any \(l \in V_m\) there exist \(l_G \in G\) such that
\[
l_Gx_0 = l.
\]

In fact if \(l_Gx_0 = l\) then \(l_Gkx_0 = l\) for any \(k\) inside \(G_{x_0}\), the centralizer of \(x_0\), in \(G\). We write every element of \(g \in G\) as \(l_Gkt\) for \(t \in G_0\backslash G\) and \(k \in G_0\). Since \(dg\) is a Haar measure then \(d(l_Gg) = dg = dkdt\). Note that \(G_{x_0}\) is a compact group, so we normalize the Haar measure so that \(\int_{G_0} dk = 1\). We compute \(M_m(v)\) in terms of the measures defined on \(G_0\) and \(G_0\backslash G\). We use the identity \(k^{-1}l_G^{-1}l = x_0\) in the third line of the following computation:

\[
\rho_{F,\infty}(m, v) = \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_G e(ivP_0[g^{-1}][l])\tilde{W}(g)dg
\]

\[
= \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_{G_0 \backslash G} \int_{G_{x_0}} e(ivP_0([l_Gkt]^{-1})[l])\tilde{W}(l_Gkt)dkdt
\]

\[
= \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_{G_0 \backslash G} \int_{G_{x_0}} e(ivP_0[t^{-1}][k^{-1}l_G^{-1}l])\tilde{W}(l_Gkt)dkdt
\]

\[
= \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_{G_0 \backslash G} \int_{G_{x_0}} e(ivP_0[t^{-1}])\tilde{W}(l_Gkt)dkdt
\]

\[
= \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_{G_0 \backslash G} \int_{G_{x_0}} \tilde{W}(l_Gkt)dkdt.
\]

Recall that \(\tilde{W}(l_Gkt) = W(l_Gktx_0)\). We take the integral over the compact group \(G_{x_0}\) and obtain

\[
(3.51) \quad \rho_{F,\infty}(m, v) = \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_{G_0 \backslash G} e(ivP_0[t^{-1}])V_l(t)dt,
\]

where \(V_l(t) := \int_{G_{x_0}} W(l_Gktx_0)dk\). By our normalization of the Haar measure of \(G_{x_0}\) we obtain

\[
\sup_{t \in G_0 \backslash G} V_l(t) \leq \sup_{x \in V_m} W(x).
\]
So \( V_t \) is a bounded function on \( G_0 \setminus G \). We note that the quotient space \( G_0 \setminus G \) is identified with \( V_m \) by sending \( t \in G_0 \setminus G \) to \( h := t^{-1}x_0 \in V_m \) and we write
\[
h := \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}.
\]
The measure \( dt \) is identified with the invariant measure defined over \( V_m \), that is the hyperbolic measure on \( V_m \). We denote this measure by \( d\sigma \). Next, we change the variables and write the integral (3.52) that is over the quotient space \( G \) that is over the quotient space \( G_0 \setminus G \) in terms of an integral over \( V_m \) and its hyperbolic measure on \( V_m \). We also consider the smooth weight function \( V(t) \) as a function on \( V_m \) by our identification \( t \to t^{-1}x_0 \in V_m \). Hence, we obtain
\[
\rho_{F,\infty}(m,v) = \sum_{l \in C(m)} \frac{v^{3/4}}{|l|} \int_{V_m} e(ivP_0[h])V_i(h)d\sigma.
\]
Let
\[
I(l,v) := v^{3/4} \int_{V_m} e(ivP_0[h])V_i(h)d\sigma.
\]
Then
\[
(3.53) \quad \rho_{F,\infty}(m,v) = \sum_{l \in C(m)} \frac{1}{|l|} I(l,v).
\]
Next, we give an asymptotic formula for \( I(l,v) \) as \( v \to \infty \) for any \( l \in V_m \). We note that
\[
P_0[h] = 2kh_1^2 + 2kh_2^2 + h_3^2.
\]
Then
\[
I(l,v) = v^{3/4} \int_{V_m} \exp(-2\pi v(2kh_1^2 + 2kh_2^2 + h_3^2))V_i(h)d\sigma.
\]
Since \( h \in V_m \) then \( h_3^2 - 4kh_1h_2 = m \), we obtain
\[
I(l,v) = \exp(-2\pi v|m|)v^{3/4} \int_{V_m} \exp(-2\pi v(2k(h_1 - h_2)^2 + 2h_3^2))V_i(h)d\sigma
\]
\[
= \exp(-2\pi v|m|)v^{3/4} \int_{V_m} \exp(-2\pi v(2k(h_1 - h_2)^2 + 2h_3^2))V_i(h)d\sigma.
\]
We change the variables to \( u_1 := \frac{h_1}{\sqrt{|m|}} \), \( u_2 := \frac{h_2}{\sqrt{|m|}} \) and \( u_3 := \frac{h_3}{\sqrt{|m|}} \). Hence, we obtain
\[
I(l,v) = \exp(-2\pi v|m|)v^{3/4} \int_{u_1^2 - u_1u_2 = -1} \exp(-2\pi vm(u_1 - u_2)^2 + 2u_3^2))V_i(u)d\sigma.
\]
We note that as \( v \to \infty \) the above integral localizes around \( u_0 = (1,1,0) \). By stationary phase theorem, it follows that
\[
\lim_{v \to \infty} \int_{u_1^2 - u_1u_2 = -1} \exp(-2\pi vm((u_1 - u_2)^2 + 2u_3^2))V_i(u)d\sigma = (1/2 + O(\frac{1}{\sqrt{v}})) \frac{V(x_0)}{|v|m|}.
\]
where \( x_0 = \left[ \frac{1}{2} \sqrt{\frac{|m|}{k}}, 0 \right] \) is the minimum of the quadratic form \( 2k(h_1 - h_2)^2 + 2h_3^2 \) on \( V_m \). Note that
\[
V_i(x_0) := \int_{G \cdot x_0} W(l_G k x_0) dk = W(l_G x_0) = W(l).
\]
Therefore,
\[
I(l, v) = \exp(-2\pi v |m|) (4\pi |m| v)^{-1/4} W(l) |m|^{-3/4} \pi^{1/4} \frac{1}{\sqrt{2}} (1 + O(1/\sqrt{v})).
\]
We use the above identity in the equation (3.53) and obtain
(3.54)
\[
\rho_{F, \infty}(m, v) = \exp(-2\pi v |m|) (4\pi |m| v)^{-1/4} \sum_{l \in C(m)} \frac{1}{|l|} I(l, v) W(l)
\]
By (3.46), we have
\[
\rho_{F, \infty}(m) = \frac{|m|^{-3/4} \pi^{1/4}}{\sqrt{2}} R(m, W)
\]
This completes the proof of the Maass identity.

3.3. Bounding the low frequency contribution. In this section, we give an upper bound on
\[
E_{\text{low}, T} := \sum_{\lambda < T} \langle f_\lambda, W \rangle R(m, f_\lambda) + cts_{1/4 + t^2 < T} (m, W).
\]
where \( T = D^\delta \) for some fixed power \( \delta > 0 \). In the following lemma, we apply the Maass identity proved in Lemma 3.8 and write \( E_{\text{low}} \) in terms of the Fourier coefficients of the weight 1/2 modular forms.

Lemma 3.9. Let
\[
B_T := \{ \psi_{\lambda'} \in L^2(\Gamma_0(4k) \backslash H) : \Delta_{1/2} \psi_{\lambda'} = \lambda' \psi_{\lambda'} \text{ and } \lambda' < T/4 + 3/16 \},
\]
be an orthonormal basis of weight 1/2 cusp forms of level 4k and eigenvalue less than \( T/4 + 3/16 \). Then we have
(3.55)
\[
E_{\text{low}} = |m|^{3/4} \pi^{-1/4} \sqrt{2} \left( \sum_{\psi_{\lambda'} \in B_T} \langle \Theta * W, \psi_{\lambda'} \rangle \rho_{\psi_{\lambda'}, \infty} (m) + \rho_{cts_{1/4 + t^2 < T/4 + 3/16}} (m, \Theta * W) \right).
\]
Proof. Let \( W_T \) be the spectral projection of \( W \) on the spectrum of \( \Omega \) in the interval \([0, T]\). Then,
(3.56)
\[
W_T = \sum_{0 \leq \lambda \leq T} \langle W, f_\lambda \rangle f_\lambda + cts_{1/4 + t^2 < T}(W),
\]
where \( \{ f_\lambda \} \) is an orthonormal basis of the cusp forms with the \( \Omega \) eigenvalue less than \( T \) and \( cts_{1/4+t^2} < T(W) \) is the projection of \( W \) on the continuous spectrum in interval \([0, T]\). It follows that

\[
\text{Er}_{\text{low}, T} = R(W_T, m).
\]

By Lemma 3.8, we have

\[
R(m, W_T) = |m|^{3/4} \pi^{-1/4} \sqrt{2} \rho_{\Theta^* W, \infty}(m),
\]

where \( \rho_{W_T + \Theta, \infty}(m) \) is the \( m \)-th Fourier coefficient of the theta transfer of \( W_T \) defined in (3.46). It follows from Theorem 3.6, see Remark 3.7, that \( W_T + \Theta \) is spanned by the orthonormal basis \( B_T \) and the continuous spectrum of \( \Delta_{1/2} \) with eigenvalue less than \( T/4 + 3/16 \). Hence,

\[
\Theta \ast W = \sum_{\psi_{\lambda'} \in B_T} (\Theta \ast W, \psi_{\lambda'}) \psi_{\lambda'} + cts_{1/4+t^2 < T/4+3/16}(\Theta \ast W).
\]

By computing the \( m \)-th Fourier coefficient of the both side of the above identity, we have

\[
\rho_{\Theta^* W, \infty}(m) = \sum_{\psi_{\lambda'} \in B_T} (\Theta \ast W, \psi_{\lambda'}) \rho_{\psi_{\lambda'}, \infty}(m) + \rho_{cts_{1/4+t^2 < T/4+3/16}}(m, \Theta \ast W)
\]

By the above and equations (3.57) and (3.58), it follows that

\[
\text{Er}_{\text{low}} = |m|^{3/4} \pi^{-1/4} \sqrt{2} \left( \sum_{\psi_{\lambda'} \in B_T} (\Theta \ast W, \psi_{\lambda'}) \rho_{\psi_{\lambda'}, \infty}(m) + \rho_{cts_{1/4+t^2 < T/4+3/16}}(m, \Theta \ast W) \right).
\]

This completes the proof of the lemma.

Finally, we bound the contribution of \( \text{Er}_{\text{low}, T} \).

**Lemma 3.10.** We have

\[
\text{Er}_{\text{low}, T} \ll |m|^{1/2 - 1/28} k^{17 + 1/2 + \varepsilon} \left( \frac{X}{\sqrt{m}} \right)^{3/2} T^7.
\]

**Proof.** By Lemma 3.10, we have

\[
\text{Er}_{\text{low}, T} = |m|^{3/4} \pi^{-1/4} \sqrt{2} \left( \sum_{\lambda' < T/4+3/16} (\Theta \ast W, \psi_{\lambda'}) \rho_{\psi_{\lambda'}, \infty}(m) + \rho_{cts_{1/4+t^2 < T/4+3/16}}(m, \Theta \ast W) \right),
\]

where, the eigenfunctions \( \psi_{\lambda'} \) has \( L^2 \) norm one. Recall that \( m = Dv_0^2 \) where \( D \) is squarefree and \( v_0 \leq D^r \). By Duke’s upper bound [Duk88 Theorem 5] on the Fourier coefficients of the weight \( 1/2 \) integral forms, we have

\[
|\rho_{\psi_{\lambda'}}(m)| \ll_{\varepsilon} |\lambda|^{3/2} \cosh(\pi t/2) |m|^{-2/7 + \varepsilon}.
\]

Next, we give an upper bound on \( (\Theta \ast W, \psi_{\lambda'}) \). We have

\[
(\Theta \ast W, \psi_{\lambda'}) = \int_{\Gamma_0(4k) \setminus H} \frac{\psi_{\lambda'}(x + iy)}{\Theta(x + iy, h) W(h)} d\sigma(h) d\eta(x).
\]

(3.61)

\[
= \int_{\Gamma \setminus V_m} W(h) \int_{\Gamma_0(4k) \setminus H} \frac{\psi_{\lambda'}(x + iy)}{\Theta(x + iy, h)} \Theta(x + iy, h) d\sigma(h) d\eta(x).
\]

where \( d\eta \) and \( d\sigma \) are invariant measures on \( \Gamma_0(4k) \setminus H \) and \( \Gamma \setminus V_m \), respectively. Let

\[
\varphi_{\lambda}(h) := \int_{\Gamma_0(4k) \setminus H} \psi_{\lambda'}(x + iy) \Theta(x + iy, h) d\eta(x).
\]

(3.62)
It follows from Theorem 3.6 that \( \varphi_\lambda \) is a Maass form of weight zero and eigenvalue \( \lambda = 4\lambda' - 3/4 \). We say \( \varphi_\lambda \) is the theta lift of the weight 1/2 modular form \( \psi_{\lambda'} \). By equation (3.61), we have

\[
\langle \Theta * W, \psi_{\lambda'} \rangle = \int_{\Gamma \backslash \mathcal{V}_m} W(h) \varphi_{\lambda'}(h) d\sigma(h) = \langle W, \varphi_\lambda \rangle.
\]

By the Cauchy-schwarz inequality

\[
\langle \Theta * W, \psi_{\lambda'} \rangle \leq |W|_2 |\varphi_\lambda|_2,
\]

where \(|W|_2\) and \(|\varphi_\lambda|_2\) are the \(L^2\) norm of \(W\) and \(\varphi_\lambda\). By Lemma 3.3, we have

\[
|W|_2 \ll \frac{X^{3/2}}{m^{3/4}}.
\]

By Theorem 5.9, we have

\[
|\varphi_\lambda|_2 \ll \cosh(-\pi r/2)k^{17+\epsilon} \chi^{9/2}.
\]

Therefore,

\[
\langle \Theta * W, \psi_{\lambda'} \rangle \ll \cosh(-\pi r/2)k^{16+1/2+\epsilon} \chi^{9/2} \left( \frac{X}{\sqrt{m}} \right)^{3/2}.
\]

By applying the above and the inequality (3.60) in the equation (3.59), we obtain

\[
\text{Err} \ll |m|^{3/4} \left( \sum_{\lambda' < T/4+3/16} |\lambda|^{3/2} \cosh(\pi t/2)|m|^{-2/7+\epsilon} \cosh(-\pi r/2)k^{16+1/2+\epsilon} \chi^{9/2} \left( \frac{X}{\sqrt{m}} \right)^{3/2} \right).
\]

By the Weyl law the number of eigenvalues \( \lambda' \leq T \) is bounded by \( kT \). Therefore,

\[
\text{Err} \ll |m|^{1/2-1/28} k^{17+1/2+\epsilon} \left( \frac{X}{\sqrt{m}} \right)^{3/2} T^\gamma.
\]

We choose \( T = D^\delta \) for a small fixed \( \delta > 0 \).

4. CLASS NUMBER FORMULA WITH DIVISIBILITY CONDITIONS

Let \( d_1 \) and \( d_2 \) be some integers and \( m = Dv^2 < 0 \) where \( D \) is a fundamental discriminant. Let \( V_{d_1,d_2,m}(\mathbb{Z}) \) denote the set of all integral binary quadratic forms \( F(x,y) := Ax^2 + Bxy + Cy^2 \) with discriminant \( m = B^2 - 4AC \) where \( d_1 | A \) and \( d_2 | C \). Let \( Q(x,y,z) := z^2 - 4d_1d_2xy \) and \( G := SO_Q \) denote the special orthogonal group associated to the quadratic form \( Q \). There is a natural action of \( G(\mathbb{Z}) \) on \( V_{d_1,d_2,m}(\mathbb{Z}) \). In what follows, we briefly describe this action and define the generalized class number associated to this action. We give an explicit formula for it in Theorem 4.1. Let \( g \in G(\mathbb{Z}) \) and \( F(x,y) \in V_{d_1,d_2,m}(\mathbb{Z}) \) then we have

\[
F(x,y) = d_1 A'x^2 + B'xy + d_2 C'y^2,
\]

where \( A', B \) and \( C' \) are integers and

\[
g^T \begin{bmatrix} 0 & -2d_1d_2 & 0 \\ -2d_1d_2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} g = \begin{bmatrix} 0 & -2d_1d_2 & 0 \\ -2d_1d_2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

We denote the action of \( g \) on \( F \) by \( g.F \) which is defined as:

\[
g.F(x,y) := d_1 A''x^2 + B''xy + d_2 C''y^2,
\]

where \( A'', B'' \) and \( C'' \) are integers and

\[
g^T \begin{bmatrix} 0 & -2d_1d_2 & 0 \\ -2d_1d_2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} g = \begin{bmatrix} 0 & -2d_1d_2 & 0 \\ -2d_1d_2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
where
\[
\begin{bmatrix}
A'' \\
B'' \\
C''
\end{bmatrix} := g \times \begin{bmatrix}
A' \\
B \\
C'
\end{bmatrix}.
\]

Note that the discriminant of \( F \) is \( \text{Disc}(F) := B^2 - 4d_1d_2A'C' \) and since \( g \in G(\mathbb{Z}) \), then
\[
B^2 - 4d_1d_2A'C' = B''^2 - 4d_1d_2A''C''.
\]
(4.2)

Therefore, the action of \( G(\mathbb{Z}) \) preserve the discriminant of \( F \). This shows that if \( F \in V_{d_1,d_2,m}(\mathbb{Z}) \) then \( g.F \in V_{d_1,d_2,m}(\mathbb{Z}) \). In the particular, if \( d_1 = d_2 = 1 \) (no divisibility condition on the binary quadratic form \( F(x,y) \)) then \( G(\mathbb{Z}) \) is isomorphic to \( SL_2(\mathbb{Z}) \) and its action is the action of \( PSL_2(\mathbb{Z}) \) on the integral binary quadratic forms with discriminant \( m \). Let \( H(m) \) denote the class of the integral orbits of the above action of \( G(\mathbb{Z}) \) on \( V_{d_1,d_2,m} \). It follows that \( H(m) \) is a finite set. Given an integral orbit \( G(\mathbb{Z})F \in H(m) \) where \( F \in V_{d_1,d_2,m}(\mathbb{Z}) \), we define its representation number by
\[
\frac{1}{|G(\mathbb{Z})_F|}
\]
where \( |G(\mathbb{Z})_F| \) is the order of the stabilizer of \( F \) in \( G(\mathbb{Z}) \). We define the generalized class number \( h(k,m) \) associated to the action of \( G(\mathbb{Z}) \) on \( V_{d_1,d_2,m}(\mathbb{Z}) \) to be the number of its orbits weighted by their representation number
\[
(4.3)
\]
\[
h(k,m) := \sum_{G(\mathbb{Z})F \in H(m)} \frac{1}{|G(\mathbb{Z})_F|}.
\]

In Theorem 4.4 we give a generalized class number formula for \( h(k,m) \). This theorem gives the main term of \( \#_{wX} A_{d_1,d_2} \) defined in the equation (24). In section 5 we give an upper bound on the error term of \( \#_{wX} A_{d_1,d_2} \). We show that the error term is smaller with a factor of \( D^{-\delta} \) compare to this main term. This power saving in the error term is crucial for the application of the Selberg sieve in Section 2. It is a consequence of the Duke’s subconvex bound on the Fourier coefficients of the weight 1/2 modular forms.

We briefly describe the proof of Theorem 4.4. The proof uses the Siegel Mass formula that gives a product formula for the sum of the representation number of an integer \( n \) by a quadratic form \( Q \) averaged over the genus class of \( Q \). In the Lemma 4.1 we show that the genus class of \( Q(x,y,z) = z^2 - 4kxy \) contains only one element for every \( k \in \mathbb{Z} \). In the Lemma 4.3 we show that the representation number of each integral point on \( Q(x,y,z) = Dv_0^2 \) are equal of \( D \gg k^{30} \) where \( D \) is squarefree. Finally, Theorem 4.4 shows that in fact the Siegel Mass formula gives a product formula for the number of the integral orbits of the orthogonal group \( Q \) on the quadric \( Q(x,y,z) = Dv_0^2 \). We begin by showing that the genus class of \( Q(x,y,z) = z^2 - 4kxy \) contains only one element.

**Lemma 4.1.** Let for any \( k \in \mathbb{Z} \). Then the genus of \( Q(x,y,z) \) contains only one class.

**Proof.** We show this by computing the local spinor norms; see [CS99, Chapter 15]. By the work of Kneser [Kne56] on the computation of the local spinor norms for odd primes \( p \) and its improvement by Earnest and Hsia [EH75, EH84] for prime
2, we have the following theorem that implies the genus of an indefinite quadratic forms contains only one class.

**Theorem 4.2** (Due to Kneser, Earnest and Hsia). If \( q \) is an indefinite integral quadratic form with at least 3 variables and the genus of \( q \) contains more than one class, then for some prime number \( p \), \( q \) can be \( p \)-adically diagonalized and the diagonal entries all involve distinct powers of \( p \).

For the proof of this theorem we refer the reader to [CS99, Chapter 15, Theorem 19].

We can diagonalize the quadratic form \( Q(x, y, z) \) over every the local ring \( \mathbb{Z}_p \) where \( p \neq 2 \) by changing the variables to \( x_1 = z \), \( x_2 = x - y \) and \( x_3 = x + y \) and obtain

\[
Q(x_1, x_2, x_3) = x_1^2 + kx_2^2 - kx_3^2.
\]

It is easy to check that \( Q(x, y, z) \) satisfies the conditions of the above theorem and as a result the genus class of \( Q \) contains only one element. This completes the proof of our lemma.

Next, we show that the representation number of the integral points on the quadric \( z^2 - 4kxy = n \) are equal if the squarefree part of \( n \) is large enough comparing to \( k \).

**Lemma 4.3.** Let \( Q(x, y, z) = z^2 - 4kxy \), \( G = SO_Q \) be the special orthogonal group of \( Q \) and \( V_m \) be the following quadric

\[
Q(x, y, z) = m,
\]

where \( m = Dv_0^2 < 0 \), \( D < 0 \) is a fundamental discriminant and \( |k|^{30} < |D| \). Then \( G(\mathbb{Z}) \) acts on \( V_m(\mathbb{Z}) \) and the centralizer of any \( h \in V_m(\mathbb{Z}) \) contains only the identity elements.

**Proof.** We briefly outline the proof here. In the first step by using the fact that the signature of \( Q \) is \((2, 1)\) and \( m < 0 \), we show that the centralizer of \( h \) embeds inside a finite dihedral group of type \( D_2, D_4 \) or \( D_6 \). As a result the order of the nontrivial elements of the centralizer of \( h \) is either 2 or 3. Next, we consider \( \gamma \in G(\mathbb{Z})_h \) inside the centralizer of \( h \) and show that the conjugacy class of \([\gamma] \) inside \( G(\mathbb{Z}) \) contains an element with bounded norm \( k^2 \). Finally, by using the fact that \( m = Dv_0^2 \) where \( D \) is a fundamental discriminant and \(|k|^{30} < |D| \), we show that the only possibility is that \( h = id \). Since we are considering the special orthogonal group we rule out the possibility of reflections which have order 2 in \( O(Q) \). We proceed by giving the details of the proof. Let \( h \in V_m(\mathbb{Z}) \) as in the assumption of our theorem. Let \( h^\perp \subset \mathbb{Z}^3 \) be the orthogonal complement of \( h \) that is a 2 dimensional lattice defined by

\[
h^\perp := \left\{ v \in \mathbb{Z}^3 : v^T \begin{bmatrix} 0 & -2k & 0 \\ -2k & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times h = 0 \right\}.
\]

Let \( Q_h \) denote the restriction of the quadratic from \( Q \) to \( h^\perp \). Since the signature of \( Q \) is \((2, 1)\) and \( Q(h) = m < 0 \) then \( Q_h \) is a positive definite quadratic form on
Let $h \in G(\mathbb{Z})$ denote the centralizer of $h \in V_m(\mathbb{Z})$. Then $H$ acts on the lattice $h^\perp$ and preserve the quadratic form $Q_h$. This gives an embedding of $H$ inside the orthogonal group of $Q_h$. The orthogonal group of a positive definite binary quadratic form $F(x, y)$ is

\begin{equation}
O(F(x, y)) = \begin{cases} 
D_4 & \text{if } F(x, y) \text{ is reduced to } Ax^2 + Bxy + Ay^2 \text{ for some } A, B < 2A \in \mathbb{Z} \\
D_6 & \text{if } F(x, y) \text{ is reduced to } Ax^2 + Axy + Ay^2 \text{ for some } A \in \mathbb{Z} \\
D_2 & \text{Otherwise.}
\end{cases}
\end{equation}

In any case the order of the nontrivial elements of the centralizer of $h$ is either 2 or 3. This shows the first step of our proof.

Next, we identify the orthogonal group $G := SO(Q)$ where $Q(x, y, z) = z^2 - 4kxy$ with $SL_2(\mathbb{R})$ so that the discrete subgroup $G(\mathbb{Z})$ is identified with $\Gamma$ a discrete subgroup of $SL_2(\mathbb{R})$ that contains the congruence subgroup $\Gamma_0(k) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l}
\begin{array}{l}
a, b, c, d \in \mathbb{Z} \\
k | c \end{array}
\end{array}\right\}$. More precisely, $PSL_2(\mathbb{R})$ acts on the space of binary quadratic forms $V := \{ F(x, y) := Ax^2 + Bxy + Cy^2 : A, B, C \in \mathbb{R} \}$ by linear change of variables

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} : F(x, y) \to F([x, y] \times \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = F(ax + cy, bx + dy).
$$

This action preserves the discriminant of the binary quadratic forms. Hence, it identifies $PSL_2(\mathbb{R})$ with $SO(Q_0)$ where $Q_0(x, y, z) = z^2 - 4kxy$ through the map

\begin{equation}
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to g_\gamma = \begin{pmatrix} a^2 & b^2 & ab \\ c^2 & d^2 & cd \\ 2ac & 2bd & ad + bc \end{pmatrix}.
\end{equation}

As a result $PSL_2(\mathbb{Z})$ is isomorphic to the integral points of $SO(Q_0)(\mathbb{Z})$. Let

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$S' \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} S = \begin{pmatrix} 0 & -2k & 0 \\ -2k & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that if $g \in SO(Q_0)$ then $C^{-1}gC \in SO(Q)$. This identifies $PSL_2(\mathbb{R})$ with $SO(Q)$ and we denote this isomorphism by $\psi$

\begin{equation}
\psi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a^2 & kb^2 & ab \\ k^{-1}c^2 & d^2 & k^{-1}cd \\ 2ac & 2kbd & ad + bc \end{pmatrix}.
\end{equation}

We have

$$SO(Q)(\mathbb{Z}) = SL_3(\mathbb{Z}) \cap \text{Image}(\psi).$$

We define $\Gamma := \psi^{-1}(SO(Q)(\mathbb{Z})) \subset PSL_2(\mathbb{R})$. Recall that $\Gamma_0(k) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \pm \text{Id} : \begin{array}{l}
a, b, c, d \in \mathbb{Z} \\
k | c \end{array} \right\}$. Hence, if $\gamma \in \Gamma_0(k)$ then $\psi(\gamma) \in SO(Q)(\mathbb{Z})$ and $\Gamma_0(k) \subset \Gamma$. 

\[\text{(4.7)}\]
We proceed to show the second step of our proof. Let $\gamma \in \Gamma$ be an element with finite order 2 or 3. It follows that

\[
|\text{Trace}(\gamma)| = \begin{cases} 
1 & \text{if } \text{Ord}(h) = 3, \\
0 & \text{if } \text{Ord}(h) = 2.
\end{cases}
\]

Note that $\gamma$ is an elliptic element in $\text{PSL}_2(\mathbb{R})$ and there exists a unique point $z_\gamma$ in the upper half-plane that is fixed by $\gamma$. We find an element $\alpha \in \text{PSL}_2(\mathbb{Z})$ such that $w_\gamma := \alpha z_\gamma \in \mathcal{F}$ where $\mathcal{F}$ is the Gauss fundamental domain for the action of $\text{PSL}_2(\mathbb{Z})$ on the upper-half plane. Next, we show that the imaginary part of $w_\gamma$ is bounded by $2k$. Let

\[
\alpha \gamma \alpha^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},
\]

where $ka_{ij} \in \mathbb{Z}$. Since the order of $\alpha \gamma \alpha^{-1}$ in $\text{PSL}_2(\mathbb{R})$ is 2 or 3. Then it follows that $a_{21} \neq 0$. Hence

\[
\frac{1}{a_{21}} \leq k.
\]

By identity (4.8)

\[
|\text{Trace}(\alpha \gamma \alpha^{-1})| = |a_{11} + a_{22}| \leq 1.
\]

Note that $\alpha \gamma \alpha^{-1}$ fixes $w_\gamma$. Hence,

\[
w_\gamma = \frac{a_{11} w_\gamma + a_{12}}{a_{21} w_\gamma + a_{22}}.
\]

By solving the above quadratic equation we obtain

\[
w_\gamma = \frac{-(a_{22} - a_{11}) \pm \sqrt{(a_{22} - a_{11})^2 + 4a_{21}a_{12}}}{2a_{21}}.
\]

Since $w_\gamma \in \mathcal{F}$ then the real part of $w_\gamma$ is less than $1/2$ and as a result we obtain

\[
\left|\frac{a_{22} - a_{11}}{a_{21}}\right| \leq 1.
\]

We have

\[
\text{Im}(w_\gamma) = \frac{1}{2} \sqrt{\left(\frac{a_{22} - a_{11}}{a_{21}}\right)^2 + 4 \frac{a_{12}}{a_{21}}}.
\]

By inequality (4.12)

\[
\text{Im}(w_\gamma) \leq \frac{1}{2} \sqrt{1 + 4 \frac{a_{12}}{a_{21}}}.
\]

Next, we give an upper bound on the ratio $\frac{a_{12}}{a_{21}}$. From the determinant equation and inequality (4.9), we have

\[
\frac{a_{12}}{a_{21}} = \frac{a_{11} a_{22}}{(a_{21})^2} - \frac{1}{(a_{21})^2} \leq \frac{a_{11} a_{22}}{(a_{21})^2} + k^2.
\]
By inequalities (4.9), (4.10) and (4.12), we have
\[
\frac{a_{11}a_{22}}{(a_{21})^2} = \frac{1}{4} \left( a_{11} + a_{22} \right)^2 - \left( a_{11} - a_{22} \right)^2
\leq \frac{1}{4} \left( \frac{1}{(a_{21})^2} + 1 \right)
\leq \frac{1}{4} (k^2 + 1).
\]
Hence, we have
\[
\text{Im}(w_\gamma) \leq \frac{1}{2} \sqrt{1 + \frac{4a_{12}}{a_{21}}}
\leq \frac{1}{2} \sqrt{1 + 5k^2 + 1}
< 2k.
\]
(4.14)

Let \( w_\gamma = s + it \) and define
\[
W := \begin{bmatrix} \sqrt{t} & s\sqrt{t}^{-1} \\ 0 & \sqrt{t}^{-1} \end{bmatrix}.
\]
Note that
\[
|W| < \sqrt{t} + 1 < \sqrt{2k} + 1.
\]
Then by Mobius transformation \( W \) sends \( i \) to \( w_\gamma \) and we have
\[
\alpha \gamma \alpha^{-1} = WR_\theta W^{-1},
\]
for some \( R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} \). Hence, the norm of \( \alpha \gamma \alpha^{-1} \) is bounded by:
\[
|\alpha \gamma \alpha^{-1}| = |WR_\theta W^{-1}|
\leq 4k.
\]
(4.15)

Next we write \( \alpha \in PSL_2(\mathbb{Z}) \) as:
\[
\alpha = [\alpha] \beta
\]
where \( \beta \in \Gamma_0(k) \) and \([\alpha]\) is a representative in the right cost of \( PSL_2(\mathbb{Z})/\Gamma_0(k) \) that contains \( \alpha \). We can choose a representative
\[
[\alpha] = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right],
\]
where \( |a|, |b|, |c|, |d| < k \); see \[Shi94]. Hence, we have the following upper bound on the norm of \([\alpha]\)
\[
|[\alpha]| < 2k.
\]
(4.16)

By inequalities (4.15) and (4.16), we obtain
\[
|\beta \gamma \beta^{-1}| < 10k^3,
\]
where \( \beta \in \Gamma_0(k) \). This shows that every elliptic element \( \gamma \in \Gamma \) contains an element in its conjugacy class such that its coefficients are bounded by a constant times \( k^3 \). By applying isomorphism \( \psi : \Gamma \to G(\mathbb{Z}) \) defined in (4.7) we have
\[
|\psi(\beta \gamma \beta^{-1})| \ll k^7.
\]
(4.18)
We are ready to finish the proof of our theorem. It is a proof by contradiction. Let \( h \in V_m \) be a point with nontrivial centralizer \( \psi(\gamma) \) for an elliptic element \( \gamma \in \Gamma \). We find \( \beta \in \Gamma_0(k) \) such that satisfies the inequality \( 4.18 \). Let \( h' := \psi(\beta)(h) \). Note that \( \psi(\beta) \in G(\mathbb{Z}) \) and as a result \( h' \in V_m \). Let \( h'' \) be the primitive integral vector parallel to \( h' \). By using the upper bound \( 4.18 \) and the fact that \( h'' \) is the single eigenvector with eigenvalue 1 for \( \psi(\beta \gamma \beta^{-1}) \) it follows that
\[
|h''| \ll k^{14}.
\]
Since we can write a multiple \( h'' \) as the cross product of the row vectors of the \( 3 \times 3 \) matrix \( \psi(\beta \gamma \beta^{-1}) - Id \) where its coordinates are bounded by \( k^{7} \). Next, we give a lower bound on \( h'' \) that contradicts with the above upper bound. By our assumptions \( m = Dv_0^2 < 0 \) and \( k^{30} < D \) where \( D < 0 \) is a fundamental discriminant. Since \( Q(h') = Dv_0^2 \) and \( D \) is squarefree then
\[
|Q(h'')| > D.
\]
Since \( Q(x, y, z) = z^2 - 4kxy \) and \( D > k^{30} \), then
\[
|h''| \gg \sqrt{D/k} \gg k^{14+1/2}.
\]
This contradicts with inequality \( 4.19 \).

Finally, we give a proof for the main theorem of this section.

**Theorem 4.4.** Let \( m = Dv_0^2 < 0 \) where \( D < 0 \) is a fundamental discriminant and \( v_0 \) is any integer.

\[
h(k, m) := \sigma_\infty \prod_p \sigma_p(V_m),
\]

integral orbits where the local densities \( \sigma_p \) are
\[
\sigma_p(V_m) := \lim_{t \to \infty} \frac{|V_m(\mathbb{F}_p^\times) \cap \{Q(x, y, z) - m \equiv \epsilon \mod p^t\}|}{p^{2t}},
\]
and the singular integral \( \sigma_\infty \) is
\[
\sigma_\infty := \lim_{\epsilon \to 0} \frac{|\text{Vol}(G(\mathbb{Z})) \setminus \{(Q(x, y, z) - m \equiv \epsilon \mod p)\}|}{2\epsilon}.
\]

**Proof.** We prove in Lemma 4.1 that the genus class of the indefinite ternary quadratic form \( Q \) contains only one class. Next, we apply Siegel Mass formula to the indefinite ternary quadratic from \( Q(x, y, z) = z^2 - 4kxy \). Let \( X_1, \ldots, X_{h(k, m)} \) be a complete set of \( G(\mathbb{Z}) \)-inequivalent integral points on \( V_m \). Let \( H_j \) be the stabilizer of \( X_j \) by the action of \( G(\mathbb{Z}) \). By Siegel Mass formula, since the genus of \( Q \) contains only one class, we obtain
\[
\frac{1}{\text{vol}(G(\mathbb{Z}) \setminus G(\mathbb{R}))} \sum_{j=1}^{h(d, m)} \text{vol}(H_j(\mathbb{Z}) \setminus H_j(\mathbb{R})) = \prod_p \sigma_p,
\]
where \( \sigma_p \) is the local density
\[
\mu_p = \lim_{a \to \infty} p^{-2a}|\{(x, y, z) \mod p^a : Q_{d_1, d_2}(x, y, z) \equiv m \mod p^a\}|.
\]
Note that in the Siegel Mass formula the integral orbit associated to $X_i$ is weighted by $|\text{vol}(H_j(\mathbb{Z}) \backslash H_j(\mathbb{R}))|$. In Lemma 4.3 by assuming $m = Dv_0^2 < 0$ where $D$ is a fundamental discriminant and $|v_0| < D^{1/10}$, we show that
\[ |H_j(\mathbb{Z})| = 1, \]
and hence
\[ |\text{vol}(H_j(\mathbb{Z}) \backslash H_j(\mathbb{R}))| = 2\pi. \]
Therefore, the individual orbits have the same measure and we can express $h(d, m)$ as the product of local densities in our generalized class number formula in (4.21).

5. Bounding the $L^2$ norm of the Siegel theta transfer

In this section, we give an upper bound on the $L^2$ norm of $\varphi := \Theta * f$ where $\Theta$ is the Siegel theta kernel defined in (3.41) and $f$ is a weight $1/2$ modular form defined on $\Gamma_0(4k) \backslash H$ with $L^2$ norm one and eigenvalue $\lambda'$. In Lemma 5.1 we compute the Mellin-transform of the theta lift $\varphi$ by a see-saw identity that is originally due Niwa [Niw75] and used by Sarnak and Katok [KS93]. The see-saw identity in this case identifies the Mellin transform of $\varphi$ with the inner product of an Eisenstein series against the product of the weight $1/2$ modular form $f$ and the complex conjugate of the Jacobi theta series $\bar{\theta}$. The last integral against Eisenstein series is explicitly computable by unfolding the Eisenstein series. Hence, we obtain the Fourier coefficients of the theta transfer at the cusp at infinity. Finally, we bound the $L^2$ norm of a modular form by bounding the truncated sum of the squares of its Fourier coefficients; see [Iwa02a Page 110, equation 8.17].

5.1. The Mellin transform of the theta transfer. We follow the same notations as in the previous sections. Let $f(z)$ be a weight $1/2$ modular form on $\Gamma_0(4k) \backslash H$ with $L^2$ norm 1 and eigenvalue $\lambda'$. It is known that $f(z)$ has a Fourier development at the cusp $\infty$ of the form
\[ f(z) = c_{f,\infty}(y) + \sum_{n \neq 0} b_{f,\infty}(n)W_{1/4\text{sgn}(n),i\alpha}(4\pi|n|y)e(nx), \]
where $1/4 + t^2 = \lambda'$, $c_{f,\infty}(y)$ is a linear combination of $y^{1/2+it}$ and $y^{1/2-it}$ and $W_{\beta,\mu}(y)$ is the Whittaker function normalized so that
\[ W_{\beta,\mu}(y) \approx e^{-y/2}y^{\beta} \quad \text{as} \quad y \to \infty. \]

For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and $f$ a weight $1/2$ modular form, we define
\begin{equation}
(5.1) \quad f_g(z) := \left( \frac{cz + d}{|cz + d|} \right)^{-1/2} f(gz).
\end{equation}

If $f$ is an eigenfunction of $\Delta_{1/2}$ with eigenvalue $\lambda'$ and invariant under $\Gamma$ then $f_g$ is an eigenfunction of $\Delta_{1/2}$ with eigenvalue $\lambda'$ and is invariant under $g^{-1}\Gamma g$. Let $\varphi(g)$ be the theta transfer of $f$ defined by
\begin{equation}
(5.2) \quad \varphi(g) := \int_{\Gamma_0(N) \backslash H} f(x + iy)\overline{\Theta(x + iy, g)} \frac{dxdy}{y^2}.
\end{equation}
Recall that $\Theta(z, g)$ is $\Gamma$ invariant from the left and $G_{x_0}$ invariant from the right in $g$ variable where $x_0 := \left[ \frac{2\pi i}{m}, \frac{2\pi}{2k}, 0 \right]$. It follows from Theorem 3.6 that $\varphi$ is a Maass form of weight zero and eigenvalue $\lambda = 4\lambda' - 3/4$ on $\Gamma \backslash V_m$. We consider the following torus $G_m$ inside $G$

$$t \in G_m \rightarrow g_t := \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G.$$ 

In the following lemma, we compute the Mellin-transform of $\varphi$ along the above embedding of $G_m$ inside $G$. Let

$$\Omega(s) := \int_0^\infty \varphi(g_t)t^s dt/t,$$

and

$$E(s, z) := \sum_{h \in \mathbb{Z}} \left( \frac{y}{|h_1 + 4h_2 zD|^2} \right)^s,$$

where $\sum_{h_1, h_2}$ is the sum over pairs of co-prime integers.

**Lemma 5.1.** We have

$$\Omega(s) = k^{s/2}2^{s-1} \Gamma \left( \frac{s+1}{2} \right) \pi^{-\frac{s+1}{2}} \int_{\Gamma_0(4k) \backslash H} f(z)\theta(z)E \left( \frac{s+1}{2}, z \right) dx dy.$$ 

**Proof.** We use the integral representation of $\varphi$ in equation (5.2) and obtain:

$$\Omega(s) = \int_0^\infty \left( \int_{\Gamma_0(4k) \backslash H} f(x + iy)\Theta(x + iy, g_t) dx dy \right) t^s dt/t$$

Next, we split $\Theta(z, g_t)$, the Siegel theta Kernel restricted to the embedded $G_m \subset G$, into product of two theta series. By definition (3.42), we have

$$\Theta(x + iy, g_t) := y^{3/4} \sum_{h_1, h_2, h_3 \in \mathbb{Z}} e(x(h_3^2 - 4kh_1h_2)) e(iy(2kt^{-2}h_1^2 + 2kt^2h_2^2 + h_3^2))$$

$$= \left( y^{1/4} \sum_{h \in \mathbb{Z}} e((x + iy)h^2) \right) \left( y^{1/2} \sum_{h_1, h_2 \in \mathbb{Z}} e((-4kxh_1)h_2)e(iy(2kt^{-2}h_1^2 + 2kt^2h_2^2)) \right).$$

We note that the first term in the above equation is the elementary theta series in one variable:

$$\theta(z) := y^{1/4} \sum_{h \in \mathbb{Z}} e((x + iy)h^2).$$

We denote the second term by $\theta_2(z, t)$

$$\theta_2(z, t) := \left( y^{1/2} \sum_{h_1, h_2 \in \mathbb{Z}} e((-4kxh_1)h_2)e(iy(2kt^{-2}h_1^2 + 2kt^2h_2^2)) \right).$$
By the symmetry between $h_1$ and $h_2$ we have

$$\theta_2(z, t) = \theta_2(z, t^{-1}).$$

By equation (5.7), the Siegel theta kernel $\Theta(z, g_t)$ splits into the product of two theta series of dimensions 1 and 2:

$$(5.8) \quad \Theta(z, g_t) := \theta_1(z) \theta_2(z, t).$$

Let

$$(5.9) \quad M(s, z) := \int_0^\infty \theta_2(x + iy, t)t^s dt/t,$$

that is the Mellin-transform of $\theta_2(z, t)$. By the definition of $\Omega(s)$ in (5.6), we obtain

$$(5.10) \quad \Omega(s) = \int_{\Gamma_0(4k) \setminus \mathcal{H}} f(z) \bar{\theta}(z) M(s, z) \frac{dx dy}{y^2}.$$ 

Next, we show that $M(s, z)$ is an Eisenstein series of weight zero and level $4k$. We show this by explicitly computing the integral. Let

$$(5.11) \quad Q_{z, t}(h_1, h_2) := \frac{8\pi k i x h_1 h_2}{4\pi y t^2} + \frac{4\pi k y t^{-2} h_2^2}{2} + \frac{k|z|^2 h_2^2}{y t^2}.$$

Then,

$$\theta_2(z, t) = \theta_2(z, t^{-1}) = y^{1/2} \sum_{h_1, h_2 \in \mathbb{Z}} \exp(-Q_{z, t}(h_1, h_2)).$$

Next, we apply a poisson summation identity on $h_1$ variable. Let $\exp(\xi_1, h_2)$ be the Fourier transform of $\exp(-Q_z(h_1, h_2))$ in $h_1$ variable then:

$$\exp(\xi_1, h_2) := \int_{-\infty}^\infty \exp(-Q_{z, t}(u, h_2) - 2\pi i u \xi_1) du.$$ 

By applying poisson summation in $h_1$ variable, we obtain

$$(5.13) \quad y^{1/2} \sum_{h_1, h_2 \in \mathbb{Z}} \exp(-Q_z(h_1, h_2)) = y^{1/2} \sum_{\xi_1, h_2 \in \mathbb{Z}} \exp(\xi_1, h_2).$$ 

Next, we compute $\exp(\xi_1, h_2)$:

$$(5.14) \quad \exp(\xi_1, h_2) = \int_{-\infty}^\infty \exp \left( -4\pi \left( \sqrt{k y t} u + \frac{ix\sqrt{k} h_2}{t\sqrt{y}} \right)^2 + \frac{k|z|^2 h_2^2}{y t^2} - 2\pi i u \xi_1 \right) du$$

$$= \frac{1}{2t\sqrt{ky}} \exp \left( -\frac{4\pi}{yt^2} \sqrt{k} z h_2 + \frac{\xi_1}{4\sqrt{k}} \right)^2.$$ 

We use the above formula and equation (5.13) to obtain

$$(5.15) \quad \theta_2(z, t^{-1}) = \frac{1}{2t\sqrt{k}} \sum_{h_1, h_2 \in \mathbb{Z}} \exp \left( -\frac{4\pi}{yt^2} \frac{h_1}{4\sqrt{k}} + \sqrt{k} z h_2 \right)^2.$$
Next, we use the above formula in order to simplify $M(s, z)$ that is defined in (5.9). We have
\begin{equation}
M(s, z) = \int_0^\infty \theta_2(z, t) t^s dt / t
\end{equation}
\begin{equation}
= \frac{1}{2\sqrt{k}} \int_0^\infty \sum_{h_1, h_2 \in \mathbb{Z}} \exp \left( - \frac{4\pi t^2}{y} \left| \frac{h_1}{4\sqrt{k}} + \sqrt{k}h_2 \right|^2 \right) t^{s+1} dt / t.
\end{equation}

Therefore,
\begin{equation}
\Omega(s) = \int_{\Gamma_0(4k) \setminus H} f(z) \bar{\theta}(z) M(s, z) \frac{dxdy}{y^2}
\end{equation}
\begin{equation}
= \frac{1}{2\sqrt{k}} \int_0^\infty \int_{\Gamma_0(4k) \setminus H} f(z) \bar{\theta}(z) \sum_{h_1, h_2} \exp \left( - \frac{4\pi t^2}{y} \left| \frac{h_1}{4\sqrt{k}} + \sqrt{k}h_2 \right|^2 \right) t^{s+1} dt / t.
\end{equation}

Since $\int_{\Gamma_0(4k) \setminus H} f(z) \bar{\theta}(z) \frac{dxdy}{y^2} = 0$, then
\begin{equation}
\Omega(s) = \frac{1}{2\sqrt{k}} \int_0^\infty \int_{\Gamma_0(4k) \setminus H} f(z) \bar{\theta}(z) \sum_{h_1, h_2} \exp \left( - \frac{4\pi t^2}{y} \left| \frac{h_1}{4\sqrt{k}} + \sqrt{k}h_2 \right|^2 \right) t^{s+1} dt / t.
\end{equation}

where $\sum_{h_1, h_2}'$ is the sum over integers $h_1, h_2 \in \mathbb{Z}$ excluding $h_1 = h_2 = 0$. Next, we change the variable to $\tau := \frac{2\sqrt{\pi} |h_1|}{4\sqrt{k}} + \sqrt{k}|h_2|$. Then $t = \frac{\tau \sqrt{\pi}}{2\sqrt{\pi} |h_1| + \sqrt{k}|h_2|}$ and $d\tau = dt / t$. Therefore,
\begin{equation}
\int_0^\infty \sum_{h_1, h_2}' \exp \left( - \frac{4\pi t^2}{y} \left| \frac{h_1}{4\sqrt{k}} + \sqrt{k}h_2 \right|^2 \right) t^{s+1} dt / t
\end{equation}
\begin{equation}
= \left( \int_0^\infty \exp(-\tau^2) \tau^{s+1} \frac{d\tau}{\tau} \right) \sum_{h_1, h_2}' \left( \frac{y}{\pi} \frac{2\sqrt{\pi} |h_1| + \sqrt{k}|h_2|}{4\sqrt{k}} \right)^{s+1}
\end{equation}

\begin{equation}
= 2^s k^{-\frac{s+1}{2}} \pi^{-\frac{s+1}{2}} \Gamma\left( \frac{s+1}{2} \right) \sum_{h_1, h_2}' \left( \frac{y}{|h_1 + 4h_2 k|^2} \right)^{s+1}.
\end{equation}

We define
\begin{equation}
E(s, z) := \sum_{h_1, h_2}' \left( \frac{y}{|h_1 + 4h_2 k|^2} \right)^s,
\end{equation}

Therefore,
\begin{equation}
\Omega(s) = k^{s/2} 2^{s-1} \Gamma\left( \frac{s+1}{2} \right) \pi^{-\frac{s+1}{2}} \int_{\Gamma_0(4k) \setminus H} f(z) \bar{\theta}(z) E\left( \frac{s+1}{2}, z \right) \frac{dxdy}{y^2}.
\end{equation}

This completes the proof of the lemma. \hfill \blacksquare

Let
\begin{equation}
I(s) := \int_{\Gamma_0(4k) \setminus H} f(z) \bar{\theta}(z) E\left( \frac{s+1}{2}, z \right) \frac{dxdy}{y^2}.
\end{equation}

Hence,
\begin{equation}
\Omega(s) = k^{s/2} 2^{s-1} \Gamma\left( \frac{s+1}{2} \right) \pi^{-\frac{s+1}{2}} I(s).
\end{equation}
Next, we give an explicit formula for $I(s)$ in terms of the Fourier coefficients of $f$. We begin by writing $E(s, z)$ as a linear combination of Eisenstein series associated to the cusps of $\Gamma_0(4k)$. Then by unfolding method we write the integral $I(s)$ as a Dirichlet series with coefficients associated to the Fourier coefficients of weight $1/2$ modular form $f(z)$. First we parametrize the cusps of $\Gamma_0(4k)$. We cite [KY17, Proposition 3.1].

**Proposition 5.2.** [KY17, Proposition 3.1.] Every cusp of $\Gamma_0(N)$ is equivalent to one of the form $1/w$ with $1 \leq w \leq N$. Two cusps of the form $1/w$ and $1/v$ with $1 \leq v, w \leq N$ are equivalent to each other if and only if
\[(v, N) = (w, N), \quad \text{and} \quad \frac{v}{(v, N)} \equiv \frac{w}{(w, N)} \left( \mod \left((w, N), \frac{N}{(w, N)} \right) \right).
\]
A cusp of the form $p/q$ is equivalent to one of the form $1/w$ with $w \equiv p'q \left( \mod N \right)$ where $p' \equiv p \left( \mod (q, N) \right)$ and $(p', N) = 1$. In particular, the cusp at $\infty$ is associated to $w = N$.

For each cusp $a \in \mathbb{Q} \cup \{\infty\}$ of a finite covolume discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})$, we call $\sigma_a \in SL_2(\mathbb{R})$ a scaling matrix for cusp $a$ if
\[\sigma_a \infty = a \]
\[\sigma_a^{-1} \Gamma_a \sigma_a = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}, \]
where $\Gamma_a$ is the centralizer of the cusp $a$. Note that scaling matrices are not unique. If $\sigma_a$ is a scaling matrix for $a$ so does $\sigma_a \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$. We use [KY17, Proposition 3.3.], where the authors give a representative for scaling matrix $\sigma_{1/w}$ of each cusp $1/w$ of $\Gamma_0(N)$.

**Proposition 5.3.** [KY17, Proposition 3.3.] Let $1/w$ be a cusp of $\Gamma = \Gamma_0(N)$, and set
\[N = (N, w)N' \quad w = (N, w)w' = (N', w)w'' \quad N' = (N', w)N'' \]
The stabilizer of $1/w$ is given as
\[\Gamma_{1/w} = \left\{ \pm \begin{bmatrix} 1 - w''N't & N''t \\ -w'w''Nt & 1 + w''N't \end{bmatrix} : t \in \mathbb{Z} \right\}, \]
and one may choose the scaling matrix as
\[\sigma_{1/w} = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{bmatrix}.\]

For each cusp $1/w$ of $\Gamma_0(4k)$, we write $E_{1/w,4k}(s, z)$ for the Eisenstein series associated to the cusp $1/w$
\[E_{1/w,4k}(s, z) := \sum_{\gamma \in \Gamma_{1/w} \setminus \Gamma_0(4k)} \text{im}(\sigma_{1/w}^{-1}\gamma z)^s.\]
By the spectral theory of $\Gamma_0(4k) \setminus H$, the continuous spectrum of the Laplacian operator on $\Gamma_0(4k) \setminus H$ is spanned by the Eisenstein series associated to the cusps of
$\Gamma_0(4k)$. In the following lemma, we write $E(s, z)$ that is defined in equation (5.19) as a linear combination of $E_{1/w, 4k}(s, z)$.

**Lemma 5.4.** Let $E(s, z)$ and $E_{1/w, 4k}(s, z)$ be the Eisenstein series as above. Then

$$E(s, z) = \sum_{1/w \in \text{cusps of } \Gamma_0(4k)} \phi_1/w(s) E_{1/w}(s, z),$$

where $\phi_1/w(s) := 2\zeta(2s)\left(\frac{N''/w}{N''}ight)^s$ with $N''$ and $N''_w$ defined in Proposition 5.3.

**Proof.** We note that the Eisenstein series $E_{1/w}(s, z)$ is zero asymptotically at every cusp except the cusp $1/w$ that is

$$\lim_{\text{Im}(z) \to \infty} E_{1/w}(s, \sigma_1/w z) = y^s,$$

for $\Re(s) > 1/2$. Hence, the asymptotic of $E(s, z)$ at cusp $1/w$ gives the coefficient of the associated Eisenstein series $E_{1/w}(s, z)$ in the basis of $\{E_{1/w}(s, z), w \in \text{cusps of } \Gamma_0(4k)\}$ for the continuous spectrum of $\Gamma_0(4k)$. Next, we give the asymptotic of $E(s, z)$ at cusp $1/w$. By definition (5.19) we have

$$E(s, z) = \sum_{h_1, h_2 \in \mathbb{Z}} \frac{y^s}{|4kh_1 z + h_2|^2}.$$

We use the scaling matrix

$$\sigma_1/w = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{bmatrix},$$

that is given in Proposition 5.3 in order to compute the asymptotic of $E(s, z)$ at cusp $1/w$. We have

$$E(s, \sigma_1/w z) = \sum_{h_1, h_2 \in \mathbb{Z}} \frac{\text{Im}(\sigma_1/w z)^s}{|4kh_1 \sigma_1/w z + h_2|^2} = \sum_{h_1, h_2 \in \mathbb{Z}} \frac{N''/w y^s}{|wN''/w z + 1|^2 |4kh_1 \frac{N''/w z}{wN''/w z + 1} + h_2|^2} = \sum_{h_1, h_2 \in \mathbb{Z}} \frac{N''/w y^s}{|4kh_1 N''/w z + h_2(wN''/w z + 1)|^2} = \zeta(2s) \sum_{\gcd(h_1, h_2) = 1} \frac{N''/w y^s}{|4kh_1 \frac{N''/w z}{wN''/w z + 1} + h_2|^2}.$$

We note that as $\text{Im}(z) \to \infty$ then all the terms in the above sum goes to zero except $h_1$ and $h_2$ such that the coefficient of $z$ in the denominator is zero, that is

$$4kh_1 N''/w + h_2 w N''/w = 0.$$

Since $\gcd(h_1, h_2) = 1$ then $h_2 = \pm\frac{4k}{\gcd(w, 4k)} = N'_w$ by the notation of the Proposition 5.3. Therefore,

$$\lim_{\text{Im}(z) \to \infty} E(s, \sigma_1/w z) = 2\zeta(2s) \frac{N''/w y^s}{N''/w^2}.$$
As a corollary,
\begin{equation}
E(s, z) = \sum_{1/w \in \text{cusp of } \Gamma_0(4k)} 2\zeta(2s) \left( \frac{N''}{N'} \right)^s E_w(s, z).
\end{equation}

This completes the proof of our lemma.

\[\square\]

5.1.1. Fourier expansion of the Jacobi function at every cusp of \(\Gamma_0(4k)\): In this section we give the Fourier expansion of the classical Jacobi theta series at each cusp of \(\Gamma_0(4k)\). We note that the Fourier expansion of the Jacobi theta series at \(\infty\) is
\begin{equation}
\theta(z) := y^{1/4} \sum_{n \in \mathbb{Z}} e(n^2 z).
\end{equation}
\(\theta(z)\) is a weight 1/2 modular form invariant by \(\Gamma_0(4)\) that has 3 inequivalent cusps \(\infty, 0\) and \(1/2\). Hence, it suffices to give the Fourier expansion of \(\theta(z)\) at 1/2 and 0. We use the following scaling matrices for \(\Gamma_0(4)\). We let
\[
\tau_0 := \begin{bmatrix} 0 & -1/2 \\
2 & 0 \end{bmatrix}, \quad \tau_{1/2} := \begin{bmatrix} 1 & -1/2 \\
2 & 0 \end{bmatrix},
\]
where \(\tau_0\) and \(\tau_{1/2}\) are scaling matrices for cusps 0 and 1/2 of \(\Gamma_0(4)\). The Fourier expansion of \(\theta(z)\) at cusp 0 is given by expanding \(\theta|_{\tau_0}\) that is
\[
\theta|_{\tau_0} := e^{i\pi/4} \left( \frac{z}{|z|} \right)^{-1/2} \theta(-1/4z)
\]
at \(\infty\). We use the following formula from [KS93, equation (2.4)]
\begin{equation}
\theta(z)|_{\tau_0} = \theta(z).
\end{equation}
Next, we give the Fourier expansion of \(\theta(z)\) at cusp 1/2. We have
\[
\theta(\tau_{1/2}z) = \Im(\tau_{1/2}z)^{1/4} \sum_{n \in \mathbb{Z}} e(n^2(\tau_{1/2}z))
\]
\[
= \frac{y^{1/4}}{|2z|^{1/2}} \sum_{n \in \mathbb{Z}} e(n^2(1/2 - 1/(4z)))
\]
\[
= \frac{y^{1/4}}{|2z|^{1/2}} \sum_{n \in \mathbb{Z}} (-1)^n e(-n^2/(4z))
\]
\[
= \frac{y^{1/4}}{|2z|^{1/2}} \left(2 \sum_{n \text{ even}} e(-n^2/(4z)) - \sum_{n \in \mathbb{Z}} e(-n^2/4z)\right)
\]
\[
= \frac{y^{1/4}}{|2z|^{1/2}} \left(2 \sum_{n \in \mathbb{Z}} e(-n^2/z) - \sum_{n \in \mathbb{Z}} e(-n^2/4z)\right)
\]
\[
= \sqrt{2} \theta(-1/z) - \theta(-1/4z).
\]

We use the transformation formula of \(\theta(z)\) under \(\gamma_2 := \begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix}\); see [KS93, Page 202]
\begin{equation}
\theta(-1/z) = i^{-1/2} \left( \frac{z}{|z|} \right)^{1/2} \theta(z) + \theta(z + 1/2)
\end{equation}
By equations 5.31 and 5.32 we have

\[
\theta(\tau_{1/2} z) = e^{-\pi/4 \left( \frac{z}{|z|} \right)^{1/2}} \left( \theta(z) + \theta(z + 1/2) - \theta(z) \right)
\]

(5.33)

\[= e^{-\pi/4 \left( \frac{z}{|z|} \right)^{1/2}} \theta(z + 1/2)\]

We note that \(\theta_{\sigma_{1/w}}\) is invariant under \(\Gamma_{\infty}\). Hence, we have

\[
\theta_{\sigma_{1/w}}(z) := y^{1/4} \sum_{n \in \mathbb{Z}} b_{\theta,1/w}(n) e(nz),
\]

(5.34)

where \(b_{\theta,1/w}(n)\) is the \(n\)th Fourier coefficient of \(\theta(z)\) at cusp \(1/w\) associated to scaling matrices \(\sigma_{1/w}\). In the following lemma, we give the Fourier coefficients of \(\theta(z)\).

**Lemma 5.5.** Let \(\theta(z) = y^{1/4} \sum_{n \in \mathbb{Z}} e(n^2 z)\) and \(\sigma_{1/w}\) be the scaling matrices introduced above. Then \(\theta(z)\) has the following Fourier coefficients for each cusp \(1/w\) of \(\Gamma_0(4k)\). If \(w \equiv 0 \mod 4\) then

\[
\theta_{\sigma_{1/w}} = \theta(N_{1/w}' z),
\]

(5.35)

\[
|b_{\theta,1/w}(n)| := \begin{cases} 
(N_{1/w}'^2)^{1/4} & \text{if } n = m^2 N_{1/w}' \text{ for some } m \in \mathbb{Z} \\
0 & \text{Otherwise.}
\end{cases}
\]

If \(w \equiv \pm 1 \mod 4\) then \(N_{1/w}' = 4\alpha\) and

\[
\theta_{\sigma_{1/w}}(z) = \theta(\alpha z \pm 1/4),
\]

(5.36)

\[
|b_{\theta,1/w}(n)| := \begin{cases} 
\alpha^{1/4} & \text{if } n = m^2 \alpha \text{ for some } m \in \mathbb{Z} \\
0 & \text{Otherwise.}
\end{cases}
\]

Finally if \(w \equiv 2 \mod 4\)

\[
\theta_{1/w}(z) = \theta(N_{1/w}' z),
\]

(5.37)

\[
|b_{\theta,1/w}(n)| := \begin{cases} 
(N_{1/w}'^2)^{1/4} & \text{if } n = m^2 N_{1/w}' \text{ for some } m \in \mathbb{Z} \\
0 & \text{Otherwise.}
\end{cases}
\]

**Proof.** We note that \(\theta(z)\) is invariant under \(\Gamma_0(4)\) and \(\Gamma_0(4)\) has 3 cusps \(\{0, 1/2, \infty\}\). If \(w \equiv 0 \mod 4\) then the cusp \(1/w\) is equivalent to \(\infty\) in \(\Gamma_0(4)\) and the Fourier expansion of \(\theta_{\sigma_{1/w}}\) is given by the following identity

\[
\theta_{\sigma_{1/w}}(z) = \theta(N_{1/w}' z).
\]

If \(w = 4\alpha + 2\) then \(1/w\) is equivalent to \(1/2\) in \(\Gamma_0(4)\) and we have

\[
\sigma_{1/w} = \begin{bmatrix} 1 & 0 \\ 4\alpha & 1 \end{bmatrix} \tau_{1/2} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N_{1/w}'^2} & 0 \\ 0 & 1/\sqrt{N_{1/w}'^2} \end{bmatrix}.
\]

(5.38)

By the above decomposition and equation 5.33 we have

\[
\theta_{1/w}(z) = \theta(N_{1/w}' z).
\]

If \(w = 4\alpha + 1\) then \(1/w\) is equivalent to \(0\) in \(\Gamma_0(4)\) and we have

\[
\sigma_{1/w} = \begin{bmatrix} 1 & 1 \\ 4\alpha & 4\alpha + 1 \end{bmatrix} \tau_0 \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N_{1/w}'^2/4} & 0 \\ 0 & 1/\sqrt{N_{1/w}'^2/4} \end{bmatrix}.
\]

(5.39)
By the above decomposition and equation (5.31) we have
\[
\theta_{1/w}(z) = \theta(N''z/4 + 1/4).
\]
Finally if \( w = 4\alpha + 3 \) then \( 1/w \) is equivalent to 0 in \( \Gamma_0(4) \) and we have
\[
\sigma_{1/w} = \begin{bmatrix}
-1 & 1 \\
-4(\alpha + 1) & 4\alpha + 3
\end{bmatrix}
\begin{bmatrix}
1 & -1/4 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\sqrt{N''}/4 & 0 \\
0 & 1/\sqrt{N''}/4
\end{bmatrix}
\]
By the above decomposition and equation (5.31) we have
\[
\theta_{1/w}(z) = \theta(N''z/4 - 1/4).
\]
This completes the proof of our lemma.

Note that \( f_{\sigma_{1/w}} \) is invariant under \( \Gamma_\infty = \{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \} \). So, we can write the Fourier expansion of \( f_{\sigma_{1/w}} \) at \( \infty \) and obtain
\[
f_{\sigma_{1/w}} := \sum_{n \neq 0} b_{f,1/w}(n)W_{1/4\text{sgn}(n),ir}(4\pi|n|y)e(nz).
\]
Next, we apply Hardy’s method in order to give the trivial bound on \( b_{f,1/w}(n) \) the \( n \)th Fourier coefficient of \( f \) at cusp \( 1/w \). This method was implemented by Matthes for real analytic cusp forms \([\text{Mat92}, \text{Page 157}]\).

**Lemma 5.6.** Let \( f \) be a weight \( 1/2 \) modular form defined on \( \Gamma_0(4k) \) with Laplacian eigenvalue \( 1/4 + r^2 \) and \( |f|_2 = 1 \). Then we have
\[
|b_{f,1/w}(m)| \ll r^{(1-1/4,\text{sgn}(m))}e^{\pi r N^{1/2}_{1/w}}(1 + O(|r|^{-1})),
\]
Proof. Let
\[
\Lambda_{y_0} := \{ z = x + iy : |x| < 1/2 \text{ and } y \geq y_0 \}.
\]
For each \( z \in H \), we denote the number of elements of the orbit of \( z \) by the discrete group \( \sigma_{1/w}^{-1}\Gamma_0(4k)\sigma_{1/w} \) that lies inside \( \Lambda_{y_0} \) by
\[
N(z,1/w,y_0).
\]
For each cusp \( 1/w \) of \( \Gamma_0(4k) \), let
\[
c_{1/w} := \min \{ c > 0 : \begin{bmatrix} * & * \\
c & * \end{bmatrix} \in \sigma_{1/w}^{-1}\Gamma_0(4k)\sigma_{1/w} \}.
\]
By definition of \( \sigma_{1/w} \) in Proposition 5.3 it is easy to check that \( c_{1/w} \in 1/N''_{1/w}\mathbb{Z} \).
Hence
\[
c_{1/w} \geq 1/N''_{1/w}.
\]
By \([\text{Iwa02b}, \text{Lemma 2.10}]\), we have the following upper bound on \( N(z,1/w,y_0) \)
\[
N(z,1/w,y_0) \leq 1 + \frac{10}{c_{1/w}y_0}
\]
\[
\leq 1 + \frac{10N''_{1/w}}{y_0}.
\]
By inequality 5.48 and $|f|^2 = 1$, we have

$$
\int_{\Lambda_{y_0}} |f(\sigma_{1/w})|^2 d\mu(z) = \int_{\sigma_{1/w}^{-1} \mathcal{Y}_0(4k) \sigma_{1/w} \setminus H} N(z, 1/w, y_0) |f(\sigma_{1/w})|^2 d\mu(z) \\
\leq \left(1 + \frac{10N''}{y_0}\right).
$$

(5.49)

Next, for each $m \in \mathbb{Z}$, we give an upper bound on $|b_{f,1/w}(m)|$, the $m$th Fourier coefficient of $f$ at cusp $1/w$ defined in equation 5.44

$$
\int_{\Lambda_{y_0}} |f(\sigma_{1/w})|^2 d\mu(z) = \sum_{n \neq 0} |b_{f,1/w}(n)|^2 \int_{y_0}^{\infty} |W_{1/4\text{sgn}(n),ir}(4\pi |n| y)|^2 dy/y^2 \\
= \sum_{n \neq 0} |b_{f,1/w}(n)|^2 4\pi |n| \int_{4\pi |n||y_0}}^{\infty} |W_{1/4\text{sgn}(n),ir}(u)|^2 du/u^2 \\
\geq |b_{f,1/w}(m)|^2 4\pi |m| \int_{4\pi |m||y_0}}^{\infty} |W_{1/4\text{sgn}(m),ir}(u)|^2 du/u^2
$$

We take $y_0 := (4\pi |m|)^{-1}$ then by inequalities 5.49 and 5.50 we have

$$
|b_{f,1/w}(m)|^2 \int_{1}^{\infty} |W_{1/4\text{sgn}(m),ir}(u)|^2 du/u^2 \ll N''_{1/w}.
$$

(5.51)

For $t \to \infty$ and bounded $y$, we have

$$
W_{\text{sgn}(m)1/4,ir}(y) = \frac{\Gamma(-2ir)}{\Gamma(1/2 - \mu - \text{sgn}(m)1/4)} y^{1/2 + ir} + \frac{\Gamma(2ir)}{\Gamma(1/2 + 2ir - \text{sgn}(m)1/4)} y^{1/2 - ir} + O(t^{-1})
$$

(5.52)

By Stirling formula, we have

$$
\Gamma(x + iy) = \sqrt{2\pi} y^{x-1/2} e^{-\pi |y|/2} (1 + O(|y|^{-1})), \quad x \text{ bounded},
$$

(5.53)

By using the above asymptotic formula, equation 5.52 and 5.51, we have

$$
|b_{f,1/w}(m)|^2 \ll r^{1/4} \text{sgn}(m) e^{\pi r} N''_{1/w}(1 + O(|r|^{-1})),
$$

(5.54)

with an absolute constant. This completes the proof of our lemma.

Finally, we compute the integral $I(s)$ defined in equation 5.20. By Lemma 5.4 and unfolding method we simplify the right hand side.

**Lemma 5.7.** We have

$$
I(s) = \psi(s) \sum_{n \geq 1} \frac{\rho(n)}{n^{s-1/2}}.
$$

(5.55)

where

$$
\rho(n) := \frac{1}{\sqrt{2}} \sum_{w \text{ odd}} \frac{N''}{N^{3/2}} b_{f,1/w}(\frac{2n}{N_w})^2 N''_{w}/4 + \sum_{w \text{ even}} \frac{N''}{N^{3/2}} b_{f,1/w}(\frac{n}{N_w})^2 N''_{w},
$$

(5.56)

and

$$
\psi(s) := 2\zeta(s+1)(4\pi)^{-(s/2-1/4)} \frac{\Gamma(s/2 + 1/4 + ir)\Gamma(s/2 + 1/4 - ir)}{\Gamma(s+1/4)}.
$$

(5.57)
Proof.

(5.58)

\[ I(s) = \int_{\Gamma_0(4k) \setminus \mathbb{H}} f(z) \tilde{\theta}(z) E_1 \left( \frac{s+1}{2}, z \right) d\mu(z) \]

\[ = \sum \phi_{1/w}(s+1/2) \int_{\Gamma_0(4k) \setminus \mathbb{H}} f(z) \tilde{\theta}(z) E_1 \left( \frac{s+1}{2}, z \right) d\mu(z) \]

\[ = \sum \phi_{1/w}(s+1/2) \int_{\Gamma_1 \setminus \mathbb{H}} f(z) \tilde{\theta}(z) \sum \text{Im}(\sigma_{1/w} \gamma z)^{s+1} d\mu(z) \]

\[ = \sum \phi_{1/w}(s+1/2) \int_{\Gamma_1 \setminus \mathbb{H}} f(\sigma_{1/w} z) \tilde{\theta}(\sigma_{1/w} z) \sigma_{1/w} y^{s+1} d\mu(z) \]

By Lemma 5.5 and 5.6, we write \( I(s) \) as a Dirichlet series

(5.59)

\[ I(s) = \sum \phi_{1/w}(s+1/2) \sum_{n>0} b_{f,1/w}(n) \tilde{b}_{\theta,1/w}(n) \int_0^\infty W_{1/4,ir}(4\pi|n|y) \exp(-2\pi ny)y^{s/2-1/4} dy/y \]

\[ = \sum \phi_{1/w}(s+1/2) \sum_{n>0} b_{f,1/w}(n) \tilde{b}_{\theta,1/w}(n) \frac{\int_0^\infty W_{1/4,ir}(4\pi u) \exp(-2\pi u u^{s/2-1/4} du/u}{\Gamma(\frac{s+1}{2})} \]

\[ = \psi(s) \sum \frac{\rho(n)}{n^{s-1/2}}, \]

where

(5.60)

\[ \rho(n) := \frac{1}{\sqrt{2}} \sum_{w \text{ odd}} \frac{N''}{Nw^{1/2}} b_{f,1/w}((2n/Nw)^2 N''/4) + \sum_{w \text{ even}} \frac{N''}{Nw^{1/2}} b_{f,1/w}((n/Nw)^2 N''), \]

and

(5.61)

\[ \psi(s) := 2\zeta(s+1)(4\pi)^{(s/2-1/4)} \Gamma(s/2+1/4+ir) \Gamma(s/2+1/4-ir) \Gamma(\frac{s+1}{2}). \]

This completes the proof of the lemma.
Corollary 5.8. By the above formulas and equation 5.5, we have

\[
\Omega(s) = 2\zeta(s + 1)(4\pi)^{-s/2} \Gamma(s/2 + 1/4 + ir) \Gamma(s/2 + 1/4 - ir) k^{s/2} 2^{-s-1} \pi^{-s} \sum_{n \geq 1} \frac{\rho(n)}{ns^{1/2}}.
\]

\[
= \sqrt{2\pi}^{-s-1/4} \zeta(s + 1) \Gamma(s/2 + 1/4 + ir) \Gamma(s/2 + 1/4 - ir) k^{s/2} \sum_{n \geq 1} \frac{\rho(n)}{ns^{1/2}}.
\]

5.2. Bounding the \(L^2\) norm of the theta transfer. Let \(\varphi(g)\) be the theta transfer of the weight \(1/2\) modular form \(f\) on \(\Gamma_0(4k) \backslash H\) with \(\Delta_{1/2}\) eigenvalue \(1/4 + r^2\) and \(|f|_2 = 1\). Recall that

\[
\varphi(g) := \int_{\Gamma_0(4k) \backslash H} f(x + iy) \Theta(x + iy, g) \frac{dx dy}{y^2}.
\]

In the following theorem, we give an upper bound on \(|\varphi|_2\), the \(L^2\) norm of \(\varphi\).

Theorem 5.9. Let \(f\), \(\varphi\) and \(r\) be as above. Then \(\varphi\) can be realized as a Maass form of weight \(0\) on \(\Gamma_0(k) \backslash H\). Moreover

\[
|\varphi|_2 \ll \cosh(-\pi r/2) k^{17+\epsilon} r^9,
\]

where the constant in \(\ll\) is absolute.

Proof. Recall that \(\Theta(z, g)\) is \(\Gamma\) invariant from the left and \(G_{x_0}\) invariant from the right in \(g\) variable where \(x_0 := \begin{bmatrix} \sqrt{\frac{x}{n}} & \sqrt{\frac{y}{n}} \\ \sqrt{\frac{x}{n}} & 0 \end{bmatrix}\). By Theorem 3.6, \(\varphi(g)\) is a Maass form of weight \(0\) on \(\Gamma \backslash V_m\) by

\[
\varphi(v) := \varphi(g_v)
\]

where \(v \in V_m\) and \(g_v \in SO(q)\) is an element such that \(g_v x_0 = v\). Define the involution \(\tau : G \to G\) by

\[
\tau(g) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} g \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]

By definition of theta series at 3.34 it is easy to check that

\[
\Theta(z, g) = \Theta(z, \tau(g)).
\]

As a result \(\varphi(g) = \varphi(\tau(g))\) and this means that \(\varphi\) is an even Maass form on \(\Gamma \backslash V_m\). We identify the orthogonal group \(G := SO(q)\) where \(q(x, y, z) = z^2 - 4kxy\) with \(SL_2(\mathbb{R})\) so that the discrete subgroup \(\Gamma = G(\mathbb{Z})\) is identified with \(\Gamma'\) a discrete subgroup of \(SL_2(\mathbb{R})\) that contains the congruence subgroup \(\Gamma_0(k) := \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } k|c\}\). As a result \(\varphi(g)\) is also identified with an even Maass form \(u(z)\) of weight \(0\) on \(\Gamma' \backslash H\). More precisely, \(PSL_2(\mathbb{R})\) acts on the space of binary quadratic forms \(V := \{ax^2 + bxy + cy^2 : a, b, c \in \mathbb{R}\}\) by linear change of variables and it preserves the discriminant of the binary quadratic forms

\[
F(x, y, \begin{bmatrix} a & b \\ c & d \end{bmatrix}) = F(ax + cy, bx + dy).
\]
This identifies $\text{PSL}_2(\mathbb{R})$ with $SO(q_0)$ where $q_0(x, y, z) = z^2 - 4xy$ through the map

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow g_\gamma = \begin{bmatrix} a^2 & b^2 & ab \\ c^2 & d^2 & cd \\ 2ac & 2bd & ad + bc \end{bmatrix}.$$

As a result $\text{PSL}_2(\mathbb{Z})$ is isomorphic to the integral points of $SO(q_0)(\mathbb{Z})$. Let

$$C := \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$C' = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} C = \begin{bmatrix} 0 & -2k & 0 \\ -2k & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We note that if $g \in SO(q_0)$ then $C^{-1} g C \in SO(q)$. This identifies $\text{PSL}_2(\mathbb{R})$ with $SO(q)$

$$\gamma \in \text{PSL}_2(\mathbb{R}) \rightarrow g_\gamma \in SO(q_0) \rightarrow C^{-1} g_\gamma C \in SO(q).$$

By the above isomorphism the lattice $\Gamma \subset SO(q)$ is identified with $\Gamma' \subset \text{PSL}_2(\mathbb{R})$, where

$$\Gamma' := \{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : C^{-1} g_\gamma C \in \Gamma = SO(q)(\mathbb{Z}) \}.$$

It is easy to check that the congruence subgroup $\Gamma_0(k) \subset \Gamma'$. Moreover, let $V_m := \{(x, y, z) : z^2 - 4kyz = m\}$ and $U_m := \{(x, y, z) : z^2 - 4yz = m\}$ and identify them by the linear transformation $C$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_m \rightarrow C v = \begin{bmatrix} x \\ ky \\ z \end{bmatrix} \in U_m.$$

Then $SO(q)$ and $SO(q_0)$ acts on $V_m$ and $U_m$ respectively and their action commute with $C$. Let $(a_1, a_2, a_3) = a \in U_m$ with $m < 0$ then the quadratic equation $a_1 x^2 + a_3 x + a_2$ has a unique root in the upper half plane that we denote by $z_a$

$$z_a := -a_3 + i\sqrt{|m|} \over 2a_1.$$

We define the following map from $V_m$ to the upper half plane $H$

$$a := \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V_m \rightarrow C a = \begin{bmatrix} a_1 \\ k a_2 \\ a_3 \end{bmatrix} \in U_m \rightarrow z_a = -a_3 + i\sqrt{|m|} \over 2a_1 \in H.$$

This map defines an equivariant map between $H$ with the action of $\text{PSL}_2(\mathbb{R})$ and $V_m$ with the action of $SO(q)$. As a result, we can realize $\varphi(g)$ as an even Maass form $u(z)$ with Laplacian eigenvalue $1/4 + (2r)^2$ on the congruence curve $\Gamma_0(k) \setminus H$

$$u(z_a) := \varphi(a),$$

where $a \in V_m$ and $z_a \in H$. Next, we relate the coefficients of $\Omega(s)$ defined in (5.3) to the Fourier coefficients of $u(z)$ at the cups $\infty$ of $\Gamma_0(k)$. Recall that

$$\Omega(s) := \int_0^\infty \varphi(g_t) t^s dt,$$
where
\[ g_t = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G. \]

By equation (5.67), \( z_0 = i\sqrt{D} \). Moreover, by isomorphism (5.65)
\[ g_t = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{t}^{-1} \end{bmatrix} \in SL_2(\mathbb{R}). \]

Hence, \( \varphi(g_t) = u(it\sqrt{D}) \) and as a result
\[ \Omega(s) = \int_{t=0}^{\infty} u(it\sqrt{k}) t^s \frac{dt}{t}. \]

\( u(z) \) is an even Maass form with eigenvalue \( 1/4 + (2r)^2 \) on \( \Gamma_0(k) \), we write the Fourier expansion of \( u(z) \) at \( \infty \) and obtain

\[ u(x + iy) = 2 \sum_{n=1}^{\infty} a_\varphi(n) n^{-1/2} \cos(2\pi nx) W_{0,2ir}(4\pi ny), \]

where \( W_{0,2ir}(y) \) is the usual Whittaker function which is normalized so that
\[ W_{\beta,\mu}(y) \approx e^{-y/2} y^\beta \text{ as } y \to \infty. \]

By Ramanujan conjecture, we expect \( |a_\varphi(n)| \leq n^r \). By using the above expansion, we have

\[ \Omega(s) = 2 \int_{t=0}^{\infty} \sum_{n=1}^{\infty} a_\varphi(n) n^{-1/2} W_{2ir}(4\pi nt\sqrt{D}) t^s \frac{dt}{t}, \]

\[ = 2k^{-s/2} \pi^{-s} \sum_{n=0}^{\infty} \frac{a_\varphi(n)}{n^{s+1/2}} \Gamma\left(\frac{s+1/2+2ir}{2}\right) \Gamma\left(\frac{s+1/2-2ir}{2}\right) \sum_{n=1}^{\infty} \frac{a_\varphi(n)}{n^{s+1/2}}. \]

where we used

\[ \int_{0}^{\infty} W_{2ir}(4u) u^s \frac{du}{u} = \frac{\pi^{-1/2}}{2} \Gamma\left(\frac{s+1/2+2ir}{2}\right) \Gamma\left(\frac{s+1/2-2ir}{2}\right), \]

from [GR15].

By the equations (5.62) and (5.70), we obtain

\[ \sqrt{2\pi}^{-s-1/4} \zeta(s+1) \Gamma(s/2 + 1/4 + ir) \Gamma(s/2 + 1/4 - ir) k^{s/2} \sum_{n \geq 1} \frac{\rho(n)}{n^{s-1/2}} \]

\[ = k^{-s/2} \pi^{-s-1/2} \Gamma\left(\frac{s+1/2+2ir}{2}\right) \Gamma\left(\frac{s+1/2-2ir}{2}\right) \sum_{n=1}^{\infty} \frac{a_\varphi(n)}{n^{s+1/2}}. \]

Hence,

\[ \sqrt{2\pi}^{1/4} k^s \zeta(s+1) \sum_{n \geq 1} \frac{\rho(n)}{n^{s-1/2}} = \sum_{n=1}^{\infty} \frac{a_\varphi(n)}{n^{s+1/2}} \]
Therefore,
\[
(5.74) \quad a_\varphi(n) = n^{1/2} \sqrt{2\pi}^{1/4} \sum_{l \equiv k} l^{-1} m^{1/2} \rho(m),
\]
where
\[
\rho(m) := \frac{1}{\sqrt{2}} \sum_{w \text{ odd}} N'' \frac{b(f, 1/w)}{N''_{1/2}} \left( \frac{2m}{N_w} \right)^2 N''_{1/2} + \sum_{w \text{ even}} N'' \frac{b(f, 1/w)}{N''_{1/2}} \left( \frac{m}{N_w} \right)^2 N''.
\]
By Lemma 5.8 and Proposition 5.3, we have
\[
(5.75) \quad \rho(m) = \frac{1}{\sqrt{2}} \sum_{w \text{ odd}} N'' \frac{b(f, 1/w)}{N''_{1/2}} \left( \frac{2m}{N_w} \right)^2 N''_{1/2} + \sum_{w \text{ even}} N'' \frac{b(f, 1/w)}{N''_{1/2}} \left( \frac{m}{N_w} \right)^2 N'' \ll \frac{r^{5/8} e^{\pi r/2}}{\text{Cusp of } \Gamma_0(4k)} N'' \frac{1}{N''^{1/2}} \ll \frac{r^{5/8} e^{\pi r/2} k^\epsilon}{k^\epsilon}.
\]
Therefore,
\[
(5.76) \quad |a_\varphi(n)| \ll n^{1/2} \sum_{l \equiv k} l^{-1} m^{1/2} \rho(m) \ll n^{1/2} (kn)^{1/2+\epsilon} \max_{1 \leq m \leq kn} |\rho(m)| \ll n^{1+\epsilon} k^{1/2+\epsilon} r^{5/8} e^{\pi r/2}.
\]
Recall that \( \varphi \) is a Maass form of weight 0 on the congruence group \( \Gamma_0(k) \). We use [Iwa02a, Page 110, equation (8.17)].
\[
(5.77) \quad \sum_{|n| \leq X} |\nu_\varphi(n)|^2 = 8|SL_2(\mathbb{Z}) : \Gamma_0(k)|^{-1} X \nu_\varphi^2 + O(kr X^{7/8} |\varphi|^2).
\]
where \( \nu_\varphi(n) = \left( \frac{4\pi}{\cosh 2\pi r} \right)^{1/2} a_\varphi(n) \). We have
\[
[SL_2(\mathbb{Z}) : \Gamma_0(k)] = k \prod_{p \mid k} \left( 1 + 1/p \right) \leq k \log(k).
\]
Let \( (k^2 r)^{8+\epsilon} < X \) then the main term \( 8|SL_2(\mathbb{Z}) : \Gamma_0(k)|^{-1} X |\varphi|^2 \) dominates the error term \( O(kr X^{7/8} |\varphi|^2) \) and we obtain
\[
(5.78) \quad |\varphi|^2 \leq \frac{k^{1+\epsilon} X}{X} \sum_{|n| \leq X} |\nu_\varphi(n)|^2.
\]
By inequality \( 5.76 \), we have
\[
(5.79) \quad |\nu_\varphi(n)|^2 = \left( \frac{4\pi}{\cosh 2\pi r} \right)^{1/2} |a_\varphi(n)|^2 \ll \cosh(-\pi r) n^{2+\epsilon} k^{1/2+\epsilon} r^{5/4}.
\]
We apply the above inequality in \( 5.78 \) and obtain
\[
(5.80) \quad |\varphi|^2 \ll \cosh(-\pi r) k^{2+\epsilon} r^{5/4} \frac{1}{X} \sum_{1 \leq n \leq X} n^{2+\epsilon} \ll \cosh(-\pi r) k^{2+\epsilon} r^{5/4} X^{2+\epsilon}.
\]
By choosing $X = (k^2r)^{8+\epsilon}$, we deduce that

$$|\varphi|_2^2 \ll \cosh(-\pi r)k^{34+\epsilon}r^{18}.$$ 

This completes the proof of our lemma.

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