On the uncertainty relations for angular observables and beyond them

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Abstract

For angular observables pairs (angular momentum-angle and number-phase) the adequate reference element of normality is not the Robertson-Schrödinger uncertainty relation but a Schwarz formula regarding the quantum fluctuations. Beyond such a fact the traditional interpretation of the uncertainty relations appears as an unjustified doctrine.

Key words: Uncertainty relations, angular observables, traditional interpretation, quantum measurements

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1 Introduction

The problem of uncertainty relations (UR) for angular variables regards the anomalies of the pairs $L_z - \varphi$ (angular momentum - azimuthal angle) and $N - \phi$ (number - phase) in respect with the predominant conceptions about quantum mechanics (QM). It was brought forward as a burning question four decades ago [1,2,3] and since then it is known as subject of many debates (see the review works [4,5,6,7,8,9] and references. But in spite of such a history the mentioned problem still remains [9,10,11] as an open question without an agreement of opinions, because the existing approaches are dissimilar both quantitatively and qualitatively. Then the searches for new and deeper approaches of the same problem and related facts are topical. One of the searched approaches can be obtained by investigating the identity and viability of normality reference element comparatively with which the mentioned

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anomalies are evaluated. Such an investigation together with some connected
questions define the goal of the present paper. We shall show that the Robert-
son Schrödinger UR (RSUR), currently assumed as the normality reference
element, is inadequate in respect with the angular observables. Also we find
that the respective observables can be described without problems by the
usual procedures of QM. Then a simple relation of Schwartz type appears in
the posture of a true reference element of normality. The mentioned rebutment
of RSUR suggests directly a reconsideration of the traditional interpretation of
UR (TIUR). Here we shall present such a reconsideration by developing some
incipient ideas from our works [12,13,14,15,16]. Therethrough we find that
TIUR is an unjustified doctrine while UR require a natural reinterpretation.
So UR are found to be not crucial physical formulas but simple fluctuations
relations with natural analogues in non-quantum physics. Such findings give
a justification, from another perspective, of the observation that [17]: "there
is nothing especially quantum mechanical about the ...’coordinate momentum
UR’... per se”. Also the new regard about UR induce the idea that the
natural description of quantum measurements has to be separated from the
objectives of usual QM.

2 The RSUR as an incorrect reference element

Currently (see [1,2,3,4,5,6,7,8,9] and quoted publications) one assumes that
for $L_z - \varphi$ and $N - \phi$ pairs the alluded reference element of normality is given
by the RSUR

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|$$

(1)

Here $\Delta A$ and $\Delta B$ denote the standard deviations of observables $A$ and $B$
while $[\hat{A}, \hat{B}]$ signifies the commutator of $\hat{A}$ and $\hat{B}$ respectively $\langle ... \rangle$ represent
the mean value. According to the usual procedures of QM the observables
$L_z - \varphi$ should be described by the conjugated operators

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \quad \text{and} \quad \hat{\varphi} = \varphi.$$  

(2)

respectively by the commutation relation

$$[\hat{L}_z, \hat{\varphi}] = -i\hbar$$

(3)

So for $L_z - \varphi$ pair the RSUR (2.1) requires directly the relation

$$\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2}$$

(4)
The anomaly of $L_z - \varphi$ pair appears as the incorrectness of the relation (1) in respect with the usually quoted physical situations. In fact the alluded situations regard exclusively the restricted class of circular rotations (CR). Such CR are specific for a particle on a circle, a 2-D rigid rotator and non-degenerate spatial rotations. One finds examples of systems with spatial rotations in cases of a particle on a sphere, of a 3-D rigid rotator and an electron in a hydrogen like atom. The respective rotations are considered as non-degenerate if all the implied quantum numbers have unique values.

In all the cases of CR the part of the wave function important for $L_z - \varphi$ pair has the form

$$\psi_m(\varphi) = (2\pi)^{-1/2}e^{im\varphi}$$

with

$$\varphi \in [0, 2\pi), \quad \psi(2\pi) := \lim_{\varphi \to 2\pi} \psi(\varphi)$$

and a single value for the integer number $m$. Then in the respective cases one obtains the expressions

$$\Delta L_z = 0, \quad \Delta \varphi = \frac{\pi}{\sqrt{3}}$$

But these expressions are incompatible with the relation (4)

In order to avoid the mentioned incompatibility many publications promoted the conception that for the $L_z - \varphi$ pair the usual procedures of QM do not work correctly. Consequently the idea that the formula (4) must be prohibited and replaced by adjusted $\Delta L_z - \Delta \varphi$ relations resembling with RSUR (1) was accredited. So, along the years, a lot of such adjusted relations were promoted. In the main the respective relations are expressible in one of the following forms

$$\Delta L_z \cdot \Delta f(\varphi) \geq \hbar \langle g(\varphi) \rangle$$

$$\left(\Delta L_z\right)^2 + h^2 \langle \Delta u(\varphi) \rangle^2 \geq \hbar^2 \langle v(\varphi) \rangle^2$$

$$\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2} \left| 1 - 2\pi |\psi(2\pi)|^2 \right|$$

In (7)-(9) $f(\varphi)$, $g(\varphi)$, $u(\varphi)$ and $v(\varphi)$ denote various adjusting functions of $\varphi$ introduced by means of some circumstantial considerations.
A minute examination of the facts shows that, in essence, the set of the relations (7)-(9) is affected by the following shortcomings (Shc):

- **Shc.1**: None of the respective relations is agreed unanimously as a correct \( \Delta L_z - \Delta \varphi \) relation able to replace the formula (4).
- **Shc.2**: Mathematically the alluded relations are not mutually equivalent.
- **Shc.3**: The relations (7)-(8) do not have rational supports in the usual formalism of QM (that however works very well in a huge number of applications).
- **Shc.4**: The considerations appealed in the promotion of relations (7)-(8) are not based on natural physical arguments.

*Observation*: We do not associate the formula (9) with Shc.3-4 because it is justifiable within the usual framework of QM (see below the relations (22) and (31)).

In the context of the above discussions another fact connected with the relation (4) is of interest. In spite of the known idea that it must be prohibited the respective relation appears as valid in some nontrivial physical situations regarding the non-circular rotations (NCR) By NRC we refer to the quantum torsion pendulum (QTP) respectively to the degenerate spatial rotations (of the above mentioned systems). A rotation (motion) is degenerate if the energy of the system is well precised while the non-energetic quantum numbers take all the permitted values.

From the class of NCR let us firstly refer to the case of a QTP which [14,15] is nothing but a harmonic oscillator characterized by the Hamiltonian

\[
\hat{H} = -\frac{\hbar^2}{2I} \frac{\partial^2}{\partial \varphi^2} + \frac{I \omega^2}{2} \varphi^2
\]

with \( \varphi \in (-\infty, +\infty) \), \( I \) = moment of inertia and \( \omega \) = angular frequency. In the state with the energy \( E_n = \hbar \omega \left( n + \frac{1}{2} \right) \) the QTP is described by the wave function

\[
\psi_n(\varphi) = \psi_n(\xi) \propto \exp \left\{ -\frac{\xi^2}{2} \right\} \mathcal{H}_n(\xi), \quad \xi = \varphi \sqrt{\frac{I \omega}{\hbar}}
\]

(10)

Here: \( n = 0, 1, 2, ... \) = vibrational quantum number, \( \mathcal{H}_n(\xi) \) = Hermite polynomials of \( \xi \). For observables \( L_z \) and \( \varphi \) described by operators (2) by means of (10) one obtains

\[
\Delta L_z = \sqrt{\hbar I \omega \left( n + \frac{1}{2} \right)}, \quad \Delta \varphi = \sqrt{\frac{\hbar}{I \omega} \left( n + \frac{1}{2} \right)}
\]

(11)

With these expressions one finds that for QTP the pair \( L_z - \varphi \) satisfy the prohibited formula (4).
From the same class of NRC now let us refer to the cases of degenerate spatial rotations regarding a particle on a sphere or a 3D rigid rotator. In such cases the energy $E = \hbar^2 l(l + 1)/2I$ and orbital number $l$ have well-precised values while the magnetic number $m$ takes the values $-l, -l+1, ..., -1, 0, 1, ..., l-1, l$. The corresponding wave functions for the considered cases has the form:

$$\psi_l(\varphi) = \sum_{m=-l}^{l} c_m Y_{lm}(\vartheta, \varphi)$$

where $Y_{lm}(\vartheta, \varphi) =$ spherical functions and $c_m =$ complex coefficients which satisfy the condition $\sum_{m=-l}^{l} |c_m|^2 = 1$.

With $L_z$ and $\varphi$ described by the operators (2) and using (12) one obtains

$$(\Delta L_z)^2 = \sum_{m=-l}^{l} m^2 |c_m|^2 \hbar^2 - \left[ \sum_{m=-l}^{l} c_m \hbar m \right]^2$$

$$(\Delta \varphi)^2 = \sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m^* c_{m'} (Y_{lm}, \varphi^2 Y_{lm'} -$$

$$- \left[ \sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m^* c_{m'} (Y_{lm}, \varphi Y_{lm'}) \right]^2$$

where $(f, g)$ denotes the scalar product of the functions $f$ and $g$.

With (13) and (14) one finds that in the cases described by (12) it is possible that the prohibited formula (4) to be verified. The respective possibility is conditioned by the concrete values of the coefficients $c_m$.

Now let us refer to the pair $N - \phi$ (number-phase) which [4,5,6,7,8,9] was also found as showing an anomaly in respect with the same reference element given by RSUR (1). The mentioned pair refers to a quantum oscillator described by a wave function $\psi_n$ like (10) (with $\xi = x\sqrt{\frac{2M\omega}{\hbar}}$ in the case of a rectilinear oscillator of mass $M$ and Cartesian coordinate $x$). The corresponding operators $\hat{N}$ and $\hat{\phi}$ are introduced (according to a Dirac’s idea) by the relations

$$\hat{a} = e^{i\phi}\sqrt{\hat{N}}, \quad \hat{a}^+ = \sqrt{\hat{N}}e^{-i\phi}$$

where $\hat{a}$ and $\hat{a}^+$ are the known ladder (annihilation and creation) operators. From (15) it follows directly the commutation formula

$$[\hat{N}, \hat{\phi}] = i$$
Then RSUR (1) should imply the relation
\[ \Delta N \cdot \Delta \phi \geq \frac{1}{2} \]  
(17)

On the other hand, because in the considered case \( \phi \in [0,2\pi) \) and \( \psi_n \) is an eigenfunction of \( \hat{N} \), one obtains
\[ \Delta N = 0, \quad \Delta \phi \leq 2\pi \]  
(18)

But such values for \( \Delta N \) and \( \Delta \phi \) are in evident discordance with (17) and so they reveal the anomaly of the \( N - \phi \) pair in respect with RSUR (1) as reference element. We add here the observation that, in fact, the pair \( N - \phi \) is in a situation completely similar with that of the pair \( L_z - \varphi \) in CR cases. The respective similarity can be pointed out as follows. If the wave functions are taken in the \( \phi \) - representation from (16) it results that the operators \( \hat{N} \) and \( \hat{\phi} \) have the expressions
\[ \hat{N} = i \frac{\partial}{\partial \phi}, \quad \hat{\phi} = \phi. \]  
(19)

Then the Schrödinger equation for the oscillator take the form
\[ \hbar \omega \left( i \frac{d}{d\phi} + \frac{1}{2} \right) \psi = E\psi \]  
(20)

By considering \( \psi_n(2\pi) = \psi_n(0) \) and \( E \geq 0 \) from (20) one obtains
\[ \psi_n = \frac{1}{\sqrt{2\pi}} e^{-in\phi} \]  
(21)

respectively \( E_n = \hbar \omega (n + \frac{1}{2}) \) and \( n = 0,1,2,... \). So the couples of relations (3)/(16) and (5)/(21) attest the announced mathematical similarity between the pairs \( N - \phi \) and \( L_z - \varphi \).

The alluded circular similarity is evidenced also by the various adjusted \( \Delta N - \Delta \phi \) formulas proposed in the literature (see the works [4,5,6,7,8,9] and references) in order to replace (17) and to avoid the anomaly of \( N - \phi \) pair in respect with RSUR (1). In their essence the respective formulas are completely analogous with the relations (7)-(9) for \( L_z - \varphi \) pair. Moreover, it is easy to see, that the alluded \( \Delta N - \Delta \phi \) formulas are affected by shortcomings which are similar with the above mentioned Shc.1-4. Here it is important to remark that the above mentioned shortcomings, for both pairs \( L_z - \varphi \) and \( N - \phi \), have an unavoidable character. This means that RSUR (1) taken as normality reference element implies inevitable anomalies for the respective pairs in the CR cases. On the other hand, as it was shown above, RSUR (1) can not offer a basis for an unitar approach of all cases (of CR and NCR type) regarding the \( L_z - \varphi \) pair. Then one can conclude that, in fact for angular observables
$L_z - \phi$ and $N - \phi$, the RSUR (1) is not adequate for the role of reference element. In the next section we shall show that in usual QM one can find an adequate candidate for such an role.

3 The true reference element

Now we shall investigate the mentioned inadequacy of RSUR (1) by searching its true origin and validity conditions. Such a search can be done as follows with the aid of some elements/notations from usual QM.

We consider a quantum system in a state described by the wave function $\psi$. The observables $A_j (j = 1, 2, .., r)$ of the system are associated with the operators $\hat{A}_j$. If $(f,g)$ denote the scalar product of two functions $f$ and $g$, for two observables $A_1 = A$ and $A_2 = B$ one can write the following Schwarz relation

\[
\langle \delta \hat{A}_j \delta \hat{B}_k \rangle = \langle \delta \hat{A}_j \delta \hat{B}_k \rangle^* - i \frac{1}{2} \langle [\hat{A}, \hat{B}] \rangle
\]

This Schwarz formula is generally valid for any wave function $\psi$ and any observables $A$ and $B$. It implies the less general relation which is RSUR (1) only when the operators $\hat{A} = \hat{A}_1$ and $\hat{B} = \hat{A}_2$ satisfy the conditions:

\[
(\hat{A}_j \psi, \hat{A}_k \psi) = (\psi, \hat{A}_j \hat{A}_k \psi) \quad (j = 1, 2; k = 1, 2)
\]

Indeed when (24) are satisfied one can write

\[
\Delta A \cdot \Delta B \geq \frac{1}{2} | \langle [\hat{A}, \hat{B}] \rangle | \quad (26)
\]

The above presented considerations justify the following observations:

- (i) The Schwarz formula (23) is aboriginal in respect with the RSUR (26). Moreover it is (23) always valid, independently if the conditions (24) are satisfied or no.
(ii) The RSUR (26)/(1) is valid only in the circumstances strictly delimited by the conditions (24) and it is false in all other situations.

The noted observations suggest to investigate if the previously discussed behavior of RSUR in respect with the the pairs $L_z - \varphi$ and $N - \phi$ can be correlated with the conditions (24). For such an investigation the following facts are of direct interest. In the cases described by the wave functions (5) and (21) for $L_z - \varphi$ and $N - \phi$ one finds respectively

$$\left(\hat{L}_z \psi_m, \hat{\varphi}\psi_m\right) = (\psi_m, \hat{L}_z \hat{\varphi}\psi_m) - i\hbar$$

(27)

$$\left(\hat{N}\psi_n, \hat{\phi}\psi_n\right) = (\psi_n, \hat{N}\hat{\phi}\psi_n) + i$$

(28)

For the pair $L_z - \varphi$ in the cases associated with the wave functions (10) respectively (12) one obtains:

$$\left(\hat{L}_z \psi_n, \hat{\varphi}\psi_n\right) = (\psi_n, \hat{L}_z \hat{\varphi}\psi_n)$$

(29)

$$\left(\hat{L}_z \psi_l, \hat{\varphi}\psi_l\right) = (\psi_l, \hat{L}_z \hat{\varphi}\psi_l) + i\hbar \left\{1 + 2 \text{ Im} \left[\sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m^* c_m \phi(Y_{lm}, \hat{\varphi}Y_{lm'})\right]\right\}$$

(30)

Relations (27)-(30) justify the following remarks. RSUR (26)/(1) is essentially inapplicable for the pairs $L_z - \varphi$ and $N - \phi$ in the cases of CR described by (5) and (21). In respect with (10) the RSUR (26)/(1) is always applicable. In the situations associated with (12) the applicability of RSUR (26)/(1) to the $L_z - \varphi$ pair depends on the values of the second term from right side of (30). It is important that in all cases regarding the pairs $L_z - \varphi$ and $N - \phi$ the Schwarz formula (23) remains valid. In the above noted situations when (24) are not satisfied the respective formula degenerate into the trivial equality $0 = 0$.

Here is the place to mention also the fact that, for any wave function $\psi(\varphi)$ with $\varphi \in [0, 2\pi]$ and $\psi(2\pi) = \psi(0)$, the relation:

$$|\langle \delta \hat{L}_z \psi, \delta \hat{\varphi}\psi \rangle| \geq \frac{\hbar}{2} |1 - 2\pi |\psi(2\pi)||$$

(31)

is generally true. This result shows that the adjusted relation (9) is only a secondary piece derivable from the general Schwarz formula (23).

The facts pointed out in this section prove that, in respect with the pairs $L_z - \varphi$ and $N - \phi$ the RSUR (1) is not adequate for the role of normality reference element. It also results that for such a role the Schwarz formula (23)
is adequate without problems. The respective formula is valid for any state (wave function) and for all pairs of observables (particularly for \( L_z - \varphi \) and \( N - \phi \)). So one finds that in reality the usual procedures of QM work well and without anomalies in all situations of interest for physics.

Previous findings show that for \( L_z \) and \( \varphi \) the relations (2) - (3) are always viable. So in a natural framework of QM it is not necessary to replace the respective relations with some substitutions (like [18]: \( \hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} + \alpha \), or [6]: \( [\hat{L}_z, \hat{\varphi}] = -i\hbar + \delta \) (\( \delta = \text{Dirac's function} \)).

**Observation:** The deadlock of RSUR in respect with the pairs \( L_z - \varphi \) and \( N - \phi \) is directly connected with the conditions (24). Then it is strange that in almost all the QM literature the respective conditions are ignored. The reason seems to be related with the fact that in Dirac’s \( \langle \text{bra} \rangle \) and \( \langle \text{ket} \rangle \) notations (which dominate in the nowadays publications) the terms from the both sides of (24) have a unique notation - namely \( \langle \psi | \hat{A}_j \hat{A}_k | \psi \rangle \). Such a uniqueness in notations can induce the confusion (unjustified supposition) that the conditions (24) are always satisfied. It is interesting to note that systematic investigations about the confusions/surprises generated by the Dirac’s notations were started only recently [19]. Probably that further efforts on the line of such investigations will bring a new light on the conditions (24) as well as on some other QM questions.

### 4 Beyond the problem of angular observables

In the previous sections we did a reevaluation of the RSUR (1) in its role of normality reference element for angular observables. But, as it is known, the respective role is a piece of the TIUR (traditional interpretation of uncertainty relations) which is still largely present in the nowadays conceptions about QM. Then a re-examination of the TIUR global validity becomes of a direct interest.

The alluded re-examination requires firstly a brief presentation of the TIUR doctrine. In the main the respective doctrine is connected with the preoccupation for giving an unique and generic interpretation for the RSUR

\[
\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|
\]  

(32)

and for the thought-experimental (te) relations

\[
\Delta_{te} A \cdot \Delta_{te} B \geq \hbar
\]  

(33)

The relations (32) were introduced through the mathematical formalism of QM. On the other hand the relations (33), regarding the te- uncertain-
ties $\Delta_{te}A$ and $\Delta_{te}B$, were proposed by means of some so called thought (or mental) experiments. A bibliography of the significant publications on the history of debates about the relations (32) - (33) can be found in the works [7,20,21,22,23,24,25]. Related to the interpretation of the alluded relations it was promoted a whole doctrine known as TIUR (traditional interpretation of uncertainty relations). Due to the respective doctrine the relations (32)-(33) have a large popularity, they being frequently regarded as crucial physical formulas or [25] even as expression of "the most important principle of the twentieth century physics". But, as a strange aspect, in its partisan literature TIUR is often presented so fragmentary and esoteric that it seems to be rather a dim conception but not a well-delimited doctrine. However, in spite of such an aspect, from the mentioned literature one can infer that in fact TIUR is reducible to the following set of main assertions (Ass.):

- **Ass.1**: The quantities $\Delta A$ and $\Delta_{te}A$ from the relations (32) and (33) have similar significance of measurement uncertainty for the quantum observable $A$. Consequently the respective relations have also the same generic significance and regard the simultaneous measurements of observables $A$ and $B$.

- **Ass.2**: For an observable $A$ considered alone the the quantity $\Delta A$ can be indefinitely small (even null).

- **Ass.3**: For two observables $A$ and $B$, considered in simultaneous measurements, the quantities $\Delta A$ and $\Delta B$ are interrelated through RSUR (32) considered as reference formula. So in the non-commutative cases ($\{\hat{A}, \hat{B}\} \neq 0$) the respective quantities cannot be reduced concomitantly because their product $\Delta A \cdot \Delta B$ is lower limited by a non-null term which depends prevalently on $\hbar$. On the other hand in the commutative cases ($\{\hat{A}, \hat{B}\} = 0$) $\Delta A$ and $\Delta B$ are mutually independent, they being allowed to take simultaneously indefinitely small (even null) values.

- **Ass.4**: The relations (32) and (33) are typically QM formulas and they, as well as the Planck’s constant $\hbar$, have not analogues in classical (non-quantum) physics.

Now it is clearly that the announced re-examination of the global validity of TIUR can be materialized by scrutinizing the correctness of the assertions Ass.1-4. on the line of such scrutiny we note the following remarks (Rem.):

- **Rem.1**: First of all we note that the $te$-relations (33) are improper as a reference element for a supposed solid doctrine like TIUR. This because the respective relations have only a transitory character due to the fact that they were founded on old resolution criteria (introduced by Abbe and Rayleigh - see [26]). But in in modern experimental physics [27,28,29,30,31,32,33,34] some super-resolution techniques that overstep the respective criteria are known. Then it is possible to imagine some super-resolution-thought-experiments...
(srte ) which instead of (33) can promote the srte-relations like:

$$\Delta_{srte} A \cdot \Delta_{srte} B < \hbar$$

(34)

for the srte-uncertainties $\Delta_{srte} A$ and $\Delta_{srte} B$. The alluded possibility invalidate the assertion Ass.1 and incriminates TIUR in connection with one of its main points.

- Rem.2: From Rem.1 it results directly that for the debates about TIUR only the RSUR (32) remains of interest. But, as it was pointed out in the previous sections, the RSUR (32) is only a secondary relation, derivable in well-precised conditions from the primary Schwarz formula (23). Then it results that in fact TIUR is confronted with a formula which is not consonant with its assertion Ass.3.

- Rem.3: Now let us refer to the term ”uncertainty” used by TIUR for quantities like $\Delta A$ from (32). We think that the respective term is groundless because of the following facts. As it is defined in the mathematical framework of QM the quantity $\Delta A$ signifies a probabilistic estimator (standard deviation) of the observable $A$ regarded as a random variable. The mentioned framework deals with theoretical concepts and models about the intrinsic (inner) properties of the considered system but not with the elements referring to the (possible) measurements performed on the respective system. Consequently, for a physical system, $\Delta A$ refers to the intrinsic characteristics, reflected in the fluctuations (deviations from the mean value) of the observable $A$. Moreover, as the expressions (6) and (11) suggest, for a system in a given state $\Delta A$ has a well defined value, connected with the corresponding wave function. The respective value cannot be related with the modifiable evaluations (e.g. by independent or interdependent reductions) assumed by Ass. 2-3.

- Rem.4: The alluded modifiable evaluations can be associated with the measurements errors/uncertainties, due to the possible changes of the accuracy for the measuring devices and procedures. But, as a general rule, such changes regard all the characteristics of a random observable $A$ - i.e the mean value $\langle A \rangle$ and fluctuation estimators (like $\Delta A$). Moreover such evaluations refer to all the random observables of both quantum and classical type, without differences of principle. Also, according to the real practice of experimental physics, one can state that for avoiding the damages (misconceptions) the descriptions of measurements must not pertain to QM or to other chapters of actual theoretical physics. Such a statement is consonant with the thinking that [35] : "in fact the word ('measurement') has had such a damaging effect on the discussions, that ... it should be banned altogether in quantum mechanics". In the spirit of the mentioned statement and thinking the QM, as well as the whole theoretical physics, must be concerned only with the (conceptual and mathematical) models of the intrinsic properties for physical systems. But such a concern disagrees with the TIUR’s assertions Ass. 1-4.
• Rem.5: As we have shown in sections 2 and 3 the angular observables $L - \phi$ and $N - \phi$ imply situations which are in discordance with Ass.3. Surprisingly, similar situations are encountered even for commutable observables. Such is the case of Cartesian coordinates $x$ and $y$ regarding a microparticle in a bi-dimensional potential well with inclined walls in respect with the $x - y$ axes. In the respective case [14] the product $\Delta x \cdot \Delta y$ is a non-null quantity, with precisely defined values for $\Delta x$ and $\Delta y$. So one finds another example which disaccords with Ass.3.

• Rem.6: As it is known TIUR promoted the idea that two observables $A$ and $B$ to be denoted with the terms "compatible" respectively "incompatible" subsequently of the fact that their operators are commutable ([Â, ˆB] = 0) or not ([Â, ˆB] ≠ 0). The mentioned terms are directly connected with the suppositions of TIUR about the lower limit of the product $\Delta A \cdot \Delta B$. But it is easy to see that the facts presented in the Rem.5 prove the desuetude of the mentioned idea. Particularly the respective idea becomes self-contradictory for the pairs of observables $L$ and $\phi$ respectively $x$ and $y$ which ought to be both "compatible" and "incompatible".

• Rem.7: The quantities $\Delta A$ and $\Delta B$ from RSUR (32)/(1) are second order probabilistic estimators, evaluated for the same moment of time. Consequently RSUR is a simple uni-temporal probabilistic formula. But the respective formula is generalizable in form of some extended relations referring also to the second order probabilistic estimators. So one obtains [14,16] bi-temporal, many-observables respectively quantum-macroscopic relations. For the mentioned extended relations TIUR has to give an interpretation concordant with its own essence, if it is a well-grounded doctrine. But to find such an interpretation on natural ways (i.e. without esoteric and/or non-physical considerations) seems to be a difficult (even impossible) task. In this sense it is significant to remind the lack of success connected with the above alluded quantum-macroscopic relations. In order to adjust the respective relations to the TIUR’s assertions it was resorted to the so called "macroscopic-operators" (see [36] and references). But in fact [14,16] the mentioned resort does not ensure for TIUR the avoidance of the involved shortcomings. Moreover the respective "macroscopic-operators" are only fictitious concepts without any real applicability in physics. It is also interesting to observe that, in the last decades, the problem of the "macroscopic-operators" and related relations is eschewed in the literature regarding the UR.

• Rem.8: In classical physics for observables with random character an non-trivial interest can present also higher order estimators (correlations) [37,38]. This fact suggests that in the case of quantum observables, additionally to the second order estimators (like $\Delta A = (\delta \hat{A}\psi, \delta \hat{A}\psi)^{1/2}$ and $(\delta \hat{A}\psi, \delta \hat{B}\psi)$ from (23)) can be used also the higher order correlations such as $(\delta \hat{A})^r \psi, (\delta \hat{B})^s \psi))$ with $r + s \geq 3$. Then, naturally for the respective correlations TIUR has to give an interpretation incorporable in his own doctrine. But it seems
to be less probable (or even excluded) that such an interpretation can be promoted through credible arguments.

- **Rem.9**: In contradiction with **Ass.4** in non-quantum physics [14,39,40] there are really some classical formulas that are completely similar with the quantum relations (32) and (23). The alluded formulas have the form

\[
\Delta_{cf} A \cdot \Delta_{cf} B \geq \left| \langle \delta A \cdot \delta B \rangle_{cf} \right|
\]  

Here the standard deviations \(\Delta_{cf} A\) and \(\Delta_{cf} B\) respectively the correlation \(\langle \delta A \cdot \delta B \rangle_{cf}\) refer to the classical fluctuations (\(cf\)) of the macroscopic observables \(A\) and \(B\) considered as random variables. Note that in classical conception the fluctuations and consequently the relations (35) regard the intrinsic (own) properties of the macroscopic systems but not the aspects of the measurements performed on the respective systems. Then the relations (35) and (32) (or (23)) reveal a classical-quantum similarity. This fact suggests that the quantum quantities \(\Delta A\) and \(\Delta B\) from (32) or (23) describe intrinsic properties (fluctuations) of the quantum observables \(A\) and \(B\) but not the uncertainties regarding the respective observables.

- **Rem.10**: The size of the quantities \(\Delta A\) and \(\Delta_{cf} B\) from (32) and (35) disclose the level of stochasticity (randomness) for the referred observables and systems. On the other hand the concrete expressions of \(\Delta A\) and \(\Delta_{cf} B\) appear [41] as products between \(\hbar\) respectively \(k\) (Boltzmann’s constant) and quantities which do not contain \(\hbar\) respectively \(k\). So \(\hbar\) and \(k\) have similar roles of generic indicators for stochasticity. But such a similarity disagrees with the TIUR’s assertion **Ass.4**.

- **Rem.11**: The TIUR’s assertion **Ass.3** roused many debates regarding the pair \(t - E\) (time - energy) (see [7,42] and references). The alluded debates tried to subordinate the description of the pair \(t - E\) to the idea that within QM the RSUR (32) is an capital reference element. But as we have shown above the respective idea is unjustifiable. Then the direct conclusion is that the mentioned subordination is a groundless and unnatural requirement. Such a conclusion can be completed with some considerations related both with the above presented discussions and with the recent notification [42] that in QM time has a threefold role. In the spirit of the mentioned notification the time can be regarded respectively as an external, intrinsic or observational entity. The external time \(t_{ext}\) [42] "is identified as the parameter entering in the Schrödinger equation and measured by an external, detached laboratory clock". The intrinsic time \(t_{int}\) refers to the own properties of the quantum objects themselves (such are the spreading of a wave packet, the decay of an unstable state or the temporal evolution of a quantum observable \(A\)). In a certain contrast with [42] we think that the observational time \(t_{obs}\) must be principally associated with the performances of the measuring devices and procedures (e.g. with the resolution time of a device or with the duration of a measuring procedure). With such a view about time the announced considerations can be formulated as follows.
The external time \( t_{\text{ext}} \) is a deterministic (non-random or dispersion-free) variable without fluctuations. Consequently \( t_{\text{ext}} \) should not be endowed with an operator nor associated with RSUR (32). In fact, even such an endowment leave RSUR (32) inapplicable for \( t_{\text{ext}} \). Indeed, if the operators \( t_{\text{ext}} = t_{\text{ext}} \cdot \) and \( \hat{E} = i\hbar \frac{\partial}{\partial t_{\text{ext}}} \) (often promoted in publications) are used, one obtains:

\[
(\hat{E}\psi, \hat{t}_{\text{ext}}\psi) = (\psi, \hat{E}\hat{t}_{\text{ext}}\psi) - i\hbar
\]

With this relation one finds a violation of the conditions (24) and, consequently, a proof of inadequacy for RSUR (32)/(25) in respect with the operators \( \hat{t}_{\text{ext}} \) and \( \hat{E} \). However, independently of (36), for the respective operators a relation of (23)-type is true i.e.:

\[
\Delta E \cdot \Delta t_{\text{ext}} \geq \left| \left( \delta \hat{E}\psi, \delta \hat{t}_{\text{ext}}\psi \right) \right|
\]

But this relation reduces itself to the trivial equality \( 0 = 0 \) (because \( \langle t_{\text{ext}} \rangle = t_{\text{ext}}, \delta \hat{t} = 0 \) and \( \Delta t_{\text{ext}} = 0 \)). Such an equality signifies that, in fact, in the QM framework \( t_{\text{ext}} \) is a deterministic variable in the mentioned sense. In the same framework the energy is a random quantity described by the Hamiltonian operator \( \hat{H} \) (which can be substituted by \( \hat{E} = i\hbar \frac{\partial}{\partial t_{\text{ext}}} \) due to the Schrodinger equation).

For the role of intrinsic time \( t_{\text{int}} \) a lot of diverse quantities were promoted (see [42] and references). They describe various temporal characteristics of quantum systems and each of them is associated with a corresponding time-energy UR. In the main the respective relations are manipulated so as to take the generic form:

\[
\tau \cdot \epsilon_w \geq \frac{\hbar}{2}
\]

where \( \tau \) denote a characteristic time interval while \( \epsilon_w \) signify an energetic width. By means of the relation (38) one expects to harmonize the pair time-energy with TIUR and especially with RSUR (32) regarded as a capital physical formula. But, on the one hand, from a mathematical and/or physical perspective, the relations (38) are not assimilable with RSUR (32). On the other hand, as it was proved above, in fact TIUR is an unjustified doctrine and RSUR (32) is not at all a capital formula. Then it results that the expectations connected with the relations (38) have not a viable object and the respective relations appear as formulas without a major significance.

The observational time \( t_{\text{obs}} \) is dependent outstandingly on the modifications (choices or changes) of the measurement characteristics for the same measured system in a given state. On the other hand for the respective system QM associates entities (wave function, mean values and
standard deviations of observables) which are independent on the mentioned modifications. Such connections with the measurements characteristics suggest that the observational times must be described not in the framework of QM but inside of a scientific approach which is distinct and additional in respect with QM. In our opinion [43] an approach of the mentioned kind is required also by the general description of the measurements for quantum systems.

5 Conclusions

The above presented discussions can be summarized through the following concluding remarks (C.Rem.):

- **C.Rem.1**: The RSUR (1)/(26)/(32) is an incorrect reference element in respect with the pairs of quantum angular observables \( L_z - \varphi \) and \( N - \phi \).
- **C.Rem.2**: In fact the RSUR proves itself to be only a secondary formula valid in well delimited conditions.
- **C.Rem.3**: If instead of RSUR one appeals to the primary Schwarz formula (23) the description of the pairs \( L_z - \varphi \) and \( N - \phi \) can be integrated without problems in the framework of usual QM. So in the QM framework the true reference element of normality is the respective formula.
- **C.Rem.4**: Because the RSUR plays also the role of reference element for TIUR doctrine, the mentioned reevaluation of RSUR requires directly a minute reconsideration of the respective doctrine. Of course that such a reconsideration enlarges the observation that [44] UR "are probably the most controverted formulae in the whole of the theoretical physics".
- **C.Rem.5**: Through the alluded reconsideration one finds that in fact TIUR is nothing but an unjustified doctrine. Such a finding consolidate the statement [45]: "the idea that there are defects in the foundations of orthodox quantum theory is unquestionable present in the conscience of many physicists".
- **C.Rem.6**: According to the above findings the RSUR loses its aureola of crucial formula. In fact RSUR as well as its primary source (23)(Schwarz formula) must be legitimated as simple QM relations. They refer to the quantum fluctuations (regarded as intrinsic properties of quantum systems) and have natural analogues in classical physics. The mentioned legitimation confirms, from a more general perspective, the observation [17] that there is nothing especially quantum mechanical about UR per se.
- **C.Rem.7**: Because of the assertions Ass.1 -3 the above argued reevaluation of TIUR doctrine requires new and natural approaches regarding the description of quantum measurements. We opine [43] that such approaches must be done within a theoretical frame which is distinct and additional in respect with the usual QM.
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List of abbreviations

Ass. = assertion
$cf = \text{ classical fluctuation}$
CR = circular rotations
C.Rem = concluding remark
$ext = \text{ external}$
int = intrinsic
NCR = non-circular rotations
obs = observational
QM = quantum mechanics
QTP = quantum torsion pendulum
Rem. = remark
RSUR = Robertson Schrödinger uncertainty relation
Shc. = shortcoming
srte = super-resolution-thought-experimental
te = thought-experimental
TIUR = traditional interpretation of uncertainty relations
UR = uncertainty relation(s)
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