Entire one-periodic maximal surfaces

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Abstract. In the present paper we study two-dimensional maximal surfaces with harmonic level-sets. As a corollary we obtain a new class of one-periodic maximal surfaces.

1. Introduction

Let $\mathbb{R}^{n+1}$ be $(n + 1)$-dimensional Minkowski space with the standard metric

$$\langle \chi', \chi'' \rangle = x_1' \cdot x_1'' + \ldots + x_n' \cdot x_n'' - t' \cdot t'',$$

where $\chi = (x_1, \ldots, x_n, t)$ is a point in $\mathbb{R}^{n+1}$.

Let $M$ be a surface in $\mathbb{R}^{n+1}$ given by $t = f(x_1, \ldots, x_n)$, and $f$ be a function in a domain $\Omega \subset \mathbb{R}^n \equiv \{ \chi : t = 0 \}$. We shall assume that $f$ is $C^2$-smooth everywhere in $\Omega$ except for a set $A \subset \Omega$ consisting of isolated points only.

A surface $M$ is space-like if the induced from $\mathbb{R}^{n+1}$ metric is the positive definite metric. This is equivalent to the following inequality

$$|\nabla f|^2 = \sum_{i=1}^n f_{x_i}^2(x) < 1, \quad \forall x \in \Omega \setminus A,$$ (1)

where the lower index denotes a partial derivative with respect to the corresponding variable: $f_{x_i} = \frac{\partial f}{\partial x_i}$, and $|\nabla f|^2 = \sum_{i=1}^n f_{x_i}^2$.

A space-like surface $M$ is called maximal if the following equality holds

$$\sum_{i,j=1}^n f_{ij}'' \left( \delta_{ij}(1 - |\nabla f|^2) + f_{x_i} f_{x_j} \right) = 0$$ (2)

everywhere in $\Omega \setminus A$. Then it is well-known that (2) is equivalent to the vanishing of the mean curvature of $M$ (with respect to its embedding in the Minkowski space $\mathbb{R}^{n+1}$).

Cheng and Yau in [2] proved that for every entire maximal surface $M$ (i.e. the surface to be defined over the whole $\mathbb{R}^n$) satisfies the Bernstein

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property. In other words, an entire solution \( f(x) \) to (1)-(2) is always an affine function. On the other hand, the study of almost-entire solutions to (1)-(2), i.e. solutions that they are of \( C^2 \) outside of a non-empty discrete set \( A \), is of great interest. Really, there are a lot of connections between this theory and the modern physics (see [1]).

The first break-through in this direction was due to Ecker [3], who established that the rotationally symmetric maximal surfaces (‘maximal catenoids’)

\[
\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}, \quad t = c \int_0^{\|x\|} \left( c^2 + \lambda^{2(n-1)} \right)^{-\frac{1}{2}} d\lambda, \tag{3}
\]

are only almost-entire solutions with \( A \) consisting of the origin. Moreover, in the same paper it was proved that every isolated singular point of a maximal (not necessarily entire) surface behaves as a light cone. Namely, let \( a \in \mathbb{R}^n \) be an isolated singular point of a solution \( f \) which is defined at a neighborhood of \( a \) then \( f \) can be extended continuously at \( a \) and

\[
f(x) - f(a) = \pm \|x - a\| + o(\|x - a\|).
\]

An important characteristic of a singular point is the flux of the solution which is defined as

\[
\mu_f(a) = \int_{C_a} \frac{\langle \nabla f, \nu \rangle}{\sqrt{1 - |\nabla f|^2}}, \tag{4}
\]

where the integral is taken over a closed surface \( C_a \subset \mathbb{R}^n \) which encloses \( a \) and contains no singular points, and \( \nu \) denotes the outward unit normal to \( C_a \). It follows from (2) that integral (4) is independent on a choice of \( C_a \). It was shown in [4] that \( \mu_f(a) \) is a Lorentzian invariant of \( M \). It is not hard to prove that a point \( a \in \Omega \) is an essential singularity of the solution \( f \) if and only if \( \mu_f(a) \neq 0 \) [5].

In recent papers [6], [7] the asymptotic behaviour and the existence questions for solutions to (1)-(2) were studied. In particularly, it was shown in [6] that under some natural geometrical assumptions on the finite set \( A \) there exists a unique almost-entire solution \( f \) with the prescribed fluxes \( \mu_f(a_i), a_i \in A \), where \( A \) is the singular set of the solution.

On the other hand, there are no explicit examples except for the mentioned above ”maximal catenoids” (3) even in the two-dimensional case. In this paper we construct a one-parametric family of periodic almost-entire maximal surfaces whose singular set is discrete (consists of isolated points) and located on the fixed line. These examples are based on the following general assertion which completely characterizes all maximal surfaces with harmonic level-sets.

**Theorem 1.** Let \( f(x, y) \) satisfy (1)-(2) and let \( f(x, y) = F(\varphi(x, y)) \), where \( \varphi(x, y) \) is a harmonic function. Then \( \varphi(x, y) \) is a real part of the holomorphic function \( h(w) = \int \frac{dw}{g(w)} \), where \( g(w) \) is one of the following

(i) \( g(w) = aw + c \);
(ii) \( g(w) = ae^{bw} \);
(iii) \( g(w) = a \sin(bw + c) \).

Here \( w = x + iy \) and \( a^2, b^2 \in \mathbb{R}, c \in \mathbb{C} \).
The cases (i) and (ii) lead us to the well-known examples: the plane, the rotational surface, the helicoid and Sherk’s maximal surfaces. In the case (iii) the surface is space-like only if \(a, b \in \mathbb{R}\), moreover in that case it has infinitely many isolated singularity points. The explicit expression and analysis we shall give in the remained part of the paper.

In fact, the examples of one-periodic maximal surfaces constructed in this paper are a small part of a bigger family of double- and one-periodic maximal two-dimensional surfaces which we treat in the forthcoming paper \([8]\).

2. Preliminaries

Let \(u(x, y)\) be a \(C^2\)-function such that \(|\nabla u(x, y)| < 1\) and

\[
(1 - u'^2_y)u''_{xx} + 2u'_{x}u'_y u''_{xy} + (1 - u'^2_x)u''_{yy} = 0.
\]

Then the graph of \(u(x, y)\) is a maximal surface in \(\mathbb{R}^3\).

Now we consider the function \(u(x, y)\) such that \(u(x, y) = F(\varphi(x, y))\), where \(F = F(\eta)\) and \(\varphi(x, y)\) are some twice-differentiable functions. Then (5) can be brought to the following form:

\[
A(x, y)F''''_{\eta\eta} + B(x, y)F'_{\eta} + C(x, y)F^{3}_{\eta} = 0,
\]

where

\[
A(x, y) = \varphi'^2_{x} + \varphi'^2_{y}, \quad B(x, y) = \varphi''_{xx} + \varphi''_{yy}, \quad C(x, y) = -\varphi'^2_{x}\varphi''_{yy} + 2\varphi'_{x}\varphi'_{y}\varphi''_{xy} - \varphi'^2_{y}\varphi''_{xx}.
\]

**Lemma 1.** Let \(f(w), g(w)\) be holomorphic functions, \(w = x + iy\). Then the following identities take place

\[
\frac{\partial}{\partial x} \text{Re}(f\bar{g}) = \text{Re}(f'\bar{g} + f\bar{g}'); \\
\frac{\partial}{\partial y} \text{Re}(f\bar{g}) = -\text{Im}(f'\bar{g} - f\bar{g}').
\]

*Here \(\bar{g}\) denotes the conjugate to \(g\) function.*

**Proof.** The Cauchi-Riemann conditions imply

\[
\frac{\partial}{\partial x} \text{Re } f = \text{Re } f', \quad \frac{\partial}{\partial y} \text{Re } f = -\text{Im } f', \\
\frac{\partial}{\partial x} \text{Im } f = \text{Im } f', \quad \frac{\partial}{\partial y} \text{Im } f = \text{Re } f'.
\]

We prove the validity of the first equality only:

\[
\frac{\partial}{\partial x} \text{Re}(f\bar{g}) = \frac{\partial}{\partial x} (\text{Re } f \text{ Re } g + \text{Im } f \text{ Im } g) = \text{Re } f' \text{ Re } g + \text{Re } f \text{ Re } g' \\
+ \text{Im } f' \text{ Im } g + \text{Im } f \text{ Im } g' = \text{Re } (f'\bar{g} + f\bar{g}').
\]

The second equality can be proved by the same way, hence the lemma is proved completely.

Let \(\varphi(x, y) = \text{Re } h(w), w = x + iy\), where \(h(w) \in \mathcal{H}(D)\) is a holomorphic function in the domain \(D\). In order to find the coefficients \(A, B\) and \(C\) of equation (6), from (8) we notice

\[
\varphi'_{x} = \frac{\partial}{\partial x} \text{Re } h = \text{Re } h', \quad \varphi''_{xx} = \frac{\partial}{\partial x} \text{Re } h' = \text{Re } h''.
\]
\[
\varphi_y' = \frac{\partial}{\partial y} \text{Re } h = -\text{Im } h', \quad \varphi'' = \frac{\partial}{\partial y} \text{Re } h' = -\text{Im } h'', \quad \varphi''_{yy} = -\text{Re } h''.
\]

Then
\[
A(x, y) = \varphi_x^2 + \varphi_y^2 = |h'(w)|^2,
\]
\[
B(x, y) = \varphi_{xx} + \varphi_{yy} = 0,
\]
\[
C(x, y) = \text{Re}(h''\bar{h}^2)
\]
and the equation (6) becomes
\[
|h'|^2 F'' + \text{Re}(h''\bar{h}^2)F' = 0.
\]

Setting
\[
g(w) \equiv \frac{1}{h'(w)}
\]
we find that
\[
\frac{F''(\varphi)}{F'^3(\varphi)} = \frac{\text{Re } g'}{|g|^2}.
\]

**Lemma 2.** The term \(\frac{1}{|g|^2} \text{Re } g'\) in the equation (10) depends only on \(\varphi(x, y) = \text{Re } h(w)\) if and only if
\[
 gg'' - g'^2 = c,
\]
where \(c\) is a real constant.

**Proof.** Let \(\psi(x, y) = \text{Re}(h'\bar{h}g') \equiv \frac{1}{|g|^2} \text{Re } g'.\) We show that the condition of the functional dependence \(\frac{\partial}{\partial(x, y)}(\varphi(x, y), \psi(x, y)) = 0\) is equivalent to \(gg'' - g'^2 = c, c \in \mathbb{R}\). Indeed, by virtue of (8) \(\varphi_x' = \text{Re } h', \varphi_y' = -\text{Im } h'\) and (7), we have
\[
\psi_x' = 2 \text{Re}(h''\bar{h}') \text{Re } g' + \text{Re}(h''\bar{h}g''),
\]
\[
\psi_y' = -2 \text{Im}(h''\bar{h}') \text{Re } g' - \text{Im}(h''\bar{h}g'').
\]
Then
\[
0 = \frac{\partial}{\partial(x, y)}(\varphi, \psi) = \varphi_x'\psi_y' - \varphi_y'\psi_x' = -(2 \text{Im}(h''\bar{h}') \text{Re } g' + \text{Im}(h''\bar{h}g'') + \text{Re}(h''\bar{h}g'')) \cdot \text{Re } h' =
\]
\[
= -2 \text{Im}(h''\bar{h}^2) \text{Re } g' - \text{Im}(g''h'h''^2),
\]
and, finally,
\[
-2|h'|^4 \text{Im} \left( \frac{h''}{h''^2} \right) \text{Re } g' - |h'|^4 \text{Im} \left( g' \frac{1}{h''} \right) = 0.
\]
Simplifying the last expression yields
\[
0 = -2 \text{Im } g' \text{Re } g' + \text{Im}(g''g) = \text{Im}(g''g - g'^2),
\]
The latter identity holds in a non-empty domain \(D\), hence by the uniqueness theorem for analytic functions, there exists a real constant \(c\) such that \(gg'' - g'^2 = c\). The lemma is proved. \(\Box\)
3. The construction of examples

Now we consider the differential equation (11) with a holomorphic in some domain $D$ function $g(w)$. One can easily show that the set of solutions of this equation makes up the following functional family: (a) $g(w) = aw + c$; (b) $g(w) = ae^{bw}$; (c) $g(w) = a\sin(bw + c)$, $a^2, b^2 \in \mathbb{R}$, $c \in \mathbb{C}$.

The cases (a)-(b) lead us to the classic examples of the maximal surfaces such as the plane, the rotational surface, the helicoid and Sherk’s maximal surfaces.

Now we consider the last case, when $g(w) = \sin w$. Here we have $h'(w) = \frac{1}{g(w)}$ and, hence,

$$h(w) = \frac{1}{2} \ln \frac{\cos w - 1}{\cos w + 1} + \text{const.}$$

Without loss of generality, we may assume that the constant in the right hand side of the last equality is identically zero. Then

$$\varphi(x, y) = \text{Re} h(w) = \frac{1}{2} \ln \left| \frac{\cos w - 1}{\cos w + 1} \right| = \frac{1}{2} \ln \frac{\cosh y - \cos x}{\cosh y + \cos x},$$

and

$$\frac{1}{|g|^2} \text{Re} g' = \frac{\text{Re} \cos w}{|\sin w|^2}.$$

On the other hand,

$$2 \sinh 2 \varphi(x, y) = \left| \frac{\cos w - 1}{\cos w + 1} \right| - \left| \frac{\cos w + 1}{\cos w - 1} \right| = -4 \frac{\text{Re} \cos w}{|\sin w|^2} = -4 \frac{1}{|g|^2} \text{Re} g'.$$

Then the equation (10) takes the form

$$F''(\eta) + \frac{1}{2} F'_{\eta}^3 \sinh 2\eta = 0.$$  

By solving the ordinary differential equation we arrive at

$$\frac{1}{F'_{\eta}^2} = \frac{1}{2} \cosh 2\eta + \frac{k}{2}, \quad k \equiv \text{const},$$

and

$$F'_{\eta}(\eta) = \frac{1}{\sqrt{\frac{1}{2} \cosh 2\eta + \frac{k}{2}}} = \frac{1}{\sqrt{\frac{1}{4}(e^{2\eta} + e^{-2\eta}) + \frac{k}{2}}} = \frac{2 e^{\eta}}{\sqrt{e^{2\eta} + 2ke^{2\eta} + 1}}.$$

To find the admissible values of the parameter $k$ which correspond to the space-like examples, we check when the inequality $|\nabla F(\varphi(x, y))| < 1$ holds. For this purpose we write

$$|\nabla F(\varphi(w))| = |F'(\varphi)||\nabla \varphi| = |F'(\varphi)||h'(w)|$$ (13)

and by using a new variable $\gamma = \frac{\cos x}{\cosh y}$, we obtain

$$|h'(w)| = \frac{1}{|\sin w|} = \frac{1}{\sqrt{\cosh^2 y - \cos^2 x}} = \frac{1}{\cosh y \sqrt{1 - \gamma^2}}.$$  

\footnote{The general case (c) is reduced to this equation by a suitable isometry and homothety}
On the other hand, using the exact form of \( \phi \) given above, we find

\[
|F'(\phi)| = \sqrt{\frac{2(1 - \gamma^2)}{(1 + k) - (k - 1)\gamma^2}}.
\]

Substituting the above expression in (13) yields

\[
|\nabla F(\phi(w))| = \frac{1}{\cosh y} \sqrt{\frac{2}{(1 + k) - (k - 1)\gamma^2}}.
\]

Taking into account, that \( \cosh y \) and \( \gamma = \cos x/\cosh x \) may change by independent manner, we obtain that the space-likeness condition takes place only when \( k > 1 \).

We denote \( \xi = e^\eta \) and assume without loss of generality that \( F(0) = 0 \). Then

\[
F(\eta) = 2 \int_1^{e^\eta} \frac{d\xi}{\sqrt{\xi^4 + 2k\xi^2 + 1}} = \sqrt{2} \int_0^{\tanh \eta} \frac{dt}{\sqrt{1 - t^2} \sqrt{(1 + k) - (k - 1)t^2}} =
\]

\[
= \frac{\sqrt{2}}{1 + k} \int_0^{\tanh \eta} \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - \alpha^2 t^2}},
\]

where \( \eta = \frac{1 - \xi^2}{1 + \xi^2} \) and \( \alpha^2 = \frac{k - 1}{k + 1} \). Let us introduce \( \alpha'^2 = 1 - \alpha^2 = \frac{2}{k + 1} \). Then

\[
F(\eta) = \alpha' \int_0^{\tanh \eta} \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - \alpha'^2 t^2}}
\]

i.e. by means of the Jacobi elliptic sinus, we find

\[
\text{sn} \left( \frac{F(\eta)}{\alpha'}; \alpha \right) = \tanh \eta
\]

Figure 1. One-periodic surface, \( \alpha = 0.6 \)
Thus, we have the solution 
\[ z = F(\varphi(x, y)) \]
given by
\[ \text{sn} \left( \frac{z}{\alpha'}; \alpha \right) = \tanh \varphi(x, y) = -\frac{\cos x}{\cosh y}. \]

The above can be we can summarized as follows.

**Theorem 2.** Let \( \alpha \in (0; 1) \) and \( \alpha' = \sqrt{1 - \alpha^2} \). Then the surface \( M(\alpha) \) given implicitly by
\[ \text{sn} \left( \frac{z}{\alpha'}; \alpha \right) = \cos x \cosh y, \]
is a maximal surface in \( \mathbb{R}^3 \). Moreover, this surface is a graph of a real analytic function everywhere except for the set consisting of singular points
\[ A_k = (\pi k; 0), \quad k \in \mathbb{Z}. \]

We observe that for different values of \( \alpha \in (0, 1) \), the surfaces \( M(\alpha) \) are Lorentz non-isometric. One can also see that \( M(\alpha) \) is located in the parallel slab \(|z| \leq K(\alpha)\alpha'\), where \( K(\alpha) \) is the complete elliptic integral of the first kind (the least positive solution of equation \( \text{sn}(K(\alpha), \alpha) = 1 \)). Moreover, one can show that the flux \( \mu(A_k) \) at the singular point \( A_k = (\pi k, 0) \) is equal to
\[ \mu(A_k) = 4 \int_0^{\pi/2} \frac{\alpha' dt}{\sqrt{1 - \alpha'^2 \cos^2 t}} = 4\alpha' K(\alpha'). \]

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