Difference Macdonald-Mehta conjecture

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Introduction

We formulate and check a difference counterpart of the Macdonald-Mehta conjecture and its generalization for the Macdonald polynomials. It solves the last open problem from the fundamental paper [M1] and, moreover, gives the formulas for the Fourier transforms of the polynomials multiplied by the Gaussian. We also introduce the reproducing kernel of the difference Fourier transform, which is an important step towards the difference Harish-Chandra theory.

Mehta suggested a formula for the integral of the $\prod_{1 \leq i < j \leq n} (x_i - x_j)^{2k}$ with respect to the Gaussian measure. Macdonald extended it from $A_{n-1}$ to other root systems and verified his conjecture for classical ones by means of Selberg’s integrals [M1]. It was established by Opdam in [O1] in full generality using the shift operators.

The integral is an important normalization constant for a $k$-deformation of the Hankel transform introduced by Dunkl [D]. The generalized Bessel functions [O3] multiplied by the Gaussian are eigenfunctions of this transform. The eigenvalues are given in terms of this constant. See [D,J] for detail. The Hankel transform is a rational degeneration of the Fourier transform in the Harish-Chandra theory of spherical functions when the symmetric space $G/K$ is replaced by its tangent space $T_e(G/K)$ with the adjoint action of $G$ (see [H]).

The harmonic analysis for $G/K$ is much more complicated than that in the rational case. The reproducing kernel of the Fourier transform is not symmetric, the Gaussian is not Fourier-invariant, and so on. The zonal spherical functions for dominant weights are (trigonometric) polynomials and have no counterparts in the rational theory. They play a great role in mathematics and physics. When $k = 1$ they are the characters of finite dimensional representations of $G$. Unfortunately the Fourier transform is not very helpful for the spherical polynomials, but for the characters.

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It is the same for the $k$-deformations, called the Jack - Heckman-Opdam polynomials. They have remarkable combinatorial properties (Macdonald, Stanley, Hanlon and others) and many applications. A variant of the Mehta-Macdonald conjecture in the trigonometric differential setup is the celebrated Macdonald constant term conjecture [M1]. It was proved by Opdam for all root systems [O1].

A difference generalization of the Harish-Chandra theory was started in [C3,C4]. It fuses together the Fourier transform and the Gaussian. From this viewpoint, it is similar to the rational case. Moreover, the Fourier transform acts very well on the Macdonald $q,t$-polynomials [M2,M3,M4,C4], generalizing the spherical polynomials, which has no analogue in the differential theory. The Macdonald polynomials form a basis in the smallest spherical irreducible representation of the double affine Hecke algebra. All spherical representations were classified in [C1]. They are expected to have promising applications in harmonic analysis and combinatorics.

The Macdonald constant term conjecture, as well as the norm, evaluation, and duality conjectures, were justified in the $q,t$-case in [C2,C3,C4]. In this paper we complete the theory calculating the Fourier transforms of the Macdonald polynomials multiplied by the Gaussian. The Gaussian and its transform are proportional. The formula for the coefficient of proportionality, a difference counterpart of the Mehta integral, resembles that for Macdonald’s constant term. It is not surprising because both are established using similar methods. However the Gaussian measure is very different from that due to Macdonald.

In the paper we mainly consider polynomials and the pairing based on the constant term. The Jackson integrals appear in the last section (see also [C1]). All statements remain valid for the Jackson (discrete) pairing. As an application, we check that the kernel of the Fourier transform reproduces the Macdonald polynomials, which makes its definition self-consistent.

Actually different concepts of integration do not affect the main formulas up to minor renormalizations. The shift operators always work well. This holds even when $q$ is a root of unity [C2,C3] or $k$ is special negative (see [C1], [DS]). In these cases the Jackson integrals become finite sums, but it does not change the formulas too much (as long as they are meaningful).

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1. Main results

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form $(z, z')$ on $\mathbb{R}^n \ni z, z'$, $W$ the Weyl group generated by the the
reflections $s_\alpha$. We assume that $(\alpha, \alpha) = 2$ for long $\alpha$. Let us fix the set $R_+$ of positive roots ($R_- = -R_+$), the corresponding simple roots $\alpha_1, \ldots, \alpha_n$, and their dual counterparts $a_1, \ldots, a_n, a_i = \alpha_i^\vee$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. The dual fundamental weights $b_1, \ldots, b_n$ are determined from the relations $(b_i, \alpha_j) = \delta_i^j$ for the Kronecker delta. We will also introduce the dual root system $R^\vee = \{\alpha^\vee, \alpha \in R\}, R^\vee_+$, the lattices

$$A = \oplus_{i=1}^n \mathbb{Z} a_i \subset B = \oplus_{i=1}^n \mathbb{Z} b_i,$$

and $A_\pm, B_\pm$ for $\mathbb{Z}_\pm = \{m \in \mathbb{Z}, \pm m \geq 0\}$ instead of $\mathbb{Z}$. In the standard notations, $A = Q^\vee$, $B = P^\vee$ (see [B]).

Later on, $(\theta, \theta) = 2$ for the maximal root $\theta$,

$$\nu_\alpha = (\alpha, \alpha), \nu_i = \nu_{\alpha_i}, \nu_R = \{\nu_\alpha, \alpha \in R\},$$

$$\rho_\nu = (1/2) \sum_{\nu_\alpha = \nu} \alpha = (\nu/2) \sum_{\nu_i = \nu} b_i, \text{ for } \nu \in R_+,$$

$$r_\nu = \rho_\nu^\vee = (2/\nu) \rho_\nu = \sum_{\nu_i = \nu} b_i, \quad 2/\nu = 1, 2, 3.$$

We will mainly use $r_\nu$ and $r = \sum_\nu r_\nu$ in the paper. The theory depends on the parameters $q, t, \nu \in \nu_R$. It is convenient to set

$$q_\nu = q^{2/\nu}, \quad t_\nu = q_\nu^{k_\nu}, \quad q_\alpha = q_\nu, t_\alpha = t_\nu \text{ for } \nu = \nu_\alpha \text{ and } r_k = \sum_\nu k_\nu r_\nu.$$

Let us formally put

$$x_i = q_i^{b_i}, \quad x_b = q_b = \prod_{i=1}^n x_i^{l_i} \text{ for } b = \sum_{i=1}^n l_i b_i,$$

and introduce the algebra $\mathbb{C}(q, t)[x]$ of polynomials in terms of $x_i^{\pm 1}$ with the coefficients belonging to the field $\mathbb{C}(q, t)$ of rational functions.

The coefficient of $x^0 = 1$ (the constant term) will be denoted by $\langle \cdot \rangle$. The following product (the Macdonald truncated $\theta$-function) is a Laurent series in $x$ with coefficients in the algebra $\mathbb{C}[t][[q]]$ of formal (holomorphic) series in $q$ over polynomials in $t$:

$$\Delta = \prod_{\alpha \in R_+} \prod_{i=0}^\infty \frac{(1 - x_a q_i a)}{(1 - x_a q_i a^0)(1 - x_a^{-1} t_i q_i^0)}, \quad a = \alpha^\vee.$$

Here $(1 - (\cdot))^{-1}$ are replaced by $1 + (\cdot) + (\cdot)^2 + \ldots$. We note that $\Delta \in \mathbb{C}(q, t)[x]$ if $t_\nu = q_\nu^{k_\nu}$ for $k_\nu \in \mathbb{Z}_+.$

By the Gaussians $\tilde{\gamma}^{\pm 1}$ we mean

$$\tilde{\gamma} = \sum_{b \in B} q^{-b(b)}/2 x_b, \quad \tilde{\gamma}^{-1} = \sum_{b \in B} q^{b(b)}/2 x_b.$$

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The multiplication by $\tilde{\gamma}^{-1}$ preserves the space of Laurent series with coefficients from $\mathbb{C}[t][[q]]$.

**Macdonald-Mehta Theorem 1.1.**

$$\langle \tilde{\gamma}^{-1}\Delta \rangle = |W| \prod_{\alpha \in R_+} \prod_{j=0}^\infty \left( \frac{1 - q_\alpha^{(r,k,\alpha)+j}}{1 - t_\alpha q_\alpha^{(r,k,\alpha)+j}} \right).$$

Here $t$ is either a formal parameter or the right hand side is to be understood as the corresponding limit when some $k_\nu \in \mathbb{Z}_+$.

The monomial symmetric functions $m_b = \sum_{c \in W(b)} x_c$ for $b \in B_-$ form a basis of the space $\mathbb{C}[x]^W$ of all $W$-invariant polynomials. We introduce the Macdonald polynomials $p_b(x)$, $b \in B_-$, by means of the conditions

$$p_b - m_b \in \oplus_c \mathbb{C}(q,t)m_c, \quad \langle p_b m_c \Delta \rangle = 0 \quad \text{for} \quad c > b,$$

where $c \in B_-$, $c > b$ means that $c - b \in A_+, c \neq b$.

They can be determined by the Gram - Schmidt process (see [M2,M3]) and form a basis in $\mathbb{C}(q,t)[x]^W$. As it was established by Macdonald, they are pairwise orthogonal for the pairing $\langle f(x)g(x^{-1})\Delta \rangle$. We consider $t$ as a formal parameter. If it is a number then it is necessary to avoid certain negative $k$.

**Theorem 1.2.** Given $b,c \in B_-$ and the corresponding Macdonald polynomials $p_b, p_c$,

$$\langle p_b p_c \tilde{\gamma}^{-1}\Delta \rangle = q^{(b,b)/2+(c,c)/2-(b+c,r_k)}p_c(q^{b-r_k})p_b(q^{r_k}) \langle \tilde{\gamma}^{-1}\Delta \rangle$$

$$= q^{(b,b)/2+(c,c)/2-(c,r_k)}p_c(q^{b-r_k}) |W| \prod_{\alpha \in R_+} \prod_{j=-(\alpha,b)}^\infty \left( \frac{1 - q_\alpha^{(r,k,\alpha)+j}}{1 - t_\alpha q_\alpha^{(r,k,\alpha)+j}} \right),$$

where $x_c(q^b) \overset{\text{def}}{=} q^{(b,c)}$.

In the last formula, we used the Macdonald evaluation conjecture proved in [C3]:

$$p_b(q^{r_k}) = q^{(r_k,b)} \prod_{\alpha \in R_+} \prod_{j=1}^{-\langle\alpha,b\rangle} \left( \frac{1 - t_\alpha q_\alpha^{(r,k,\alpha)+j-1}}{1 - q_\alpha^{(r,k,\alpha)+j-1}} \right).$$

To make the products meaningful we expand the coefficients of the polynomials $p_b$ in terms of $q$. They are from $\mathbb{C}[t][[q]]$.

The formula is an important particular case of the difference Fourier-Plancherel theorem. We will extend it to the non-symmetric Macdonald polynomials. More general results will be considered in the next paper.

There is a straightforward passage to non-reduced root systems. One may also take $x_\alpha$ in place of $x_a$, changing $q_\alpha$ by $q$, choosing $c$ from the lattice $P$, and
and substituting \((r_k, \alpha) \to (p_k, \alpha')\), \((b, r_k) \to (b, p_k)\) in (1.3), (1.5) and (1.7), (1.8). The Gaussian remains the same (for \(B\)).

A rational-differential counterpart of (1.7) was verified by Dunkl and de Jeu (see [D], Theorem 3.2). Theorem 1.2 is the cornerstone of the theory of the Fourier transform (cf. [J], Lemma 4.11). Generally speaking, the latter is the map \(f(x) \to \hat{f}(\lambda) = \int p_\lambda(x)f(x)\Delta\), where \(p_\lambda\) is a \(W\)-invariant eigenfunction of the generalized Macdonald operators for a proper choice of the integration and \(\lambda\). For \(\hat{f} = 0\), (1.7) determines the Fourier transform and its inverse in the space of \(W\)-symmetric Laurent polynomials multiplied by the Gaussian. This space is identified with a subspace of all functions in \(\lambda = q^{b-r_k}, b \in B\).

Both statements are new. In the case of rank one, (1.5) resembles the so-called quintuple product identity and the formulas from [AW] (the \(BC_1\) case). It is likely to be related to the known one-dimensional identities.

The definition of the Jackson integral for \(W\)-symmetric \(f\) is as follows: \(\langle f \rangle_\xi = \sum_{a \in B} f(q^{\xi_+ a})\). The integrals are formal series in terms of \(q\) or functions of \(q, t, \xi\) when \(|q| < 1\). We choose \(\gamma(q^2) = q^{(z, z)/2}\) and

\[
\Delta^\circ = \prod_{a \in R_+} \prod_{i=1}^{\infty} \frac{(1 - x_d t_{\alpha}^{-1} q_{\alpha}^i)(1 - x_a t_{\alpha}^{-1} q_{\alpha}^i)}{(1 - x_d q_{\alpha}^i)(1 - x_a q_{\alpha}^i)}.
\]

For instance, \(\langle \gamma \rangle_\xi = \sum_{a \in B} q^{(\xi_+ a_+ a)/2} = \gamma^{-1}(q^{\xi})q^{(\xi_+ \xi)/2}\). We assume that \(\Delta^\circ(q^{\xi+b})\) is well-defined, so \((\alpha, \xi) \not\in \mathbb{Z}\) for all \(\alpha \in R_+\).

**Theorem 1.3.** Given \(b, c \in B_-\) and the corresponding Macdonald polynomials \(p_b, p_c\),

\[
\langle p_b(x)p_c(x^{-1}) \gamma \Delta^\circ \rangle_\xi = q^{-(b, b)/2-(c, c)/2+(b+c, r_k)} p_c(q^{b-r_k}) p_b(q^{r_k}) \langle \gamma \Delta^\circ \rangle_\xi,
\]

\[
\langle \gamma \Delta^\circ \rangle_\xi = \langle \gamma \rangle_\xi \prod_{a \in R_+} \prod_{j=1}^{\infty} \frac{(1 - t_{\alpha}^{-1} q_{\alpha}^{-r_k a} a_j)}{1 - q_{\alpha}^{-r_k a} a_j}.
\]

In this formulas \(t\) is arbitrary provided the existence of \(p_b, p_c\). The right hand side of (1.11) is considered as the corresponding limit if \(k_\nu \in \mathbb{Z}_+ \setminus \{0\}\).

**2. Affine Weyl groups**

The vectors \(\tilde{\alpha} = [\alpha, k] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}\) for \(\alpha \in R, k \in \mathbb{Z}\) form the affine root system \(R_\alpha \supset R\) (\(z \in \mathbb{R}\) are identified with \([z, 0]\)). We add \(\alpha_0 \equiv [-\theta, 1]\) to the simple roots for the maximal root \(\theta \in R\). The corresponding set \(R_\alpha^+\) of positive roots coincides with \(R_+ \cup \{[\alpha, k], \alpha \in R, k > 0\}\).

We denote the Dynkin diagram and its affine completion with \(\{\alpha_j, 0 \leq j \leq n\}\) as the vertices by \(\Gamma\) and \(\Gamma^\alpha\). The set of the indices of the images of \(\alpha_0\) by
all the automorphisms of $\Gamma^a$ will be denoted by $O$ ( $O = \{0\}$ for $E_8,F_4,G_2$).
Let $O^* = r \in O, r \neq 0$. The elements $b_r$ for $r \in O^*$ are the so-called minuscule weights $((b_r, \alpha) \leq 1$ for $\alpha \in R_+)$.
Given $\tilde{\alpha} = [\alpha,k] \in R^a$, $b \in B$, let
\[
(2.1) \quad s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z,\alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z,\zeta - (z,b)]
\]
for $\tilde{z} = [z,\zeta] \in R^{a+1}$.

The affine Weyl group $W^a$ is generated by all $s_{\tilde{\alpha}}$ (we write $W^a = < s_{\tilde{\alpha}}, \tilde{\alpha} \in R^a_+ >$). One can take the simple reflections $s_j = s_{\alpha_j}, 0 \leq j \leq n$, as its generators and introduce the corresponding notion of the length. This group is the semi-direct product $W \ltimes A'$ of its subgroups $W = < s_{\alpha}, \alpha \in R_+ >$ and $A' = \{a', a \in A\}$, where
\[
(2.2) \quad a' = s_{\alpha}s_{[\alpha,1]} = s_{[-\alpha,1]}s_{\alpha} \quad \text{for} \quad \alpha = \alpha^\vee, \quad \alpha \in R.
\]

The extended Weyl group $W^b$ generated by $W$ and $B'$ (instead of $A'$) is isomorphic to $W \ltimes B'$:
\[
(2.3) \quad (wb')(\tilde{z},\zeta) = [w(z),\zeta - (z,b)] \quad \text{for} \quad w \in W, b \in B.
\]

Later on $b$ and $b'$ will not be distinguished.

Given $b \in B$, the decomposition $b = \pi_b \omega_b, \omega_b \in W$ can be uniquely determined from the condition: $\omega_b(b) = b_- \in B_-$ where the length $l(\omega_b)$ of $\omega$ in terms of $\{s_1, \ldots, s_n\}$ is the smallest possible. For instance, let $\pi_r = \pi_{b_r}, r \in O$.
They leave $\Gamma^a$ invariant and form a group denoted by $\Pi$, which is isomorphic to $B/A$ by the natural projection $\{b_r \rightarrow \pi_r\}$. As to $\{\omega_r\}$, they preserve the set $\{-\theta,\alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (\omega_r)^{-1}(-\theta)$ distinguish the indices $r \in O^*$. Moreover (see e.g. [C2]):
\[
(2.4) \quad W^b = \Pi \ltimes W^a, \quad \text{where} \quad \pi_r s_i \pi_r^{-1} = s_j \quad \text{if} \quad \pi_r(\alpha_i) = \alpha_j, \quad 0 \leq j \leq n.
\]

We extend the length to $W^b$. Given $r \in O^*$, $\tilde{w} \in W^a$, and a reduced decomposition $\tilde{w} = s_{j_1} \cdots s_{j_2} s_{j_1}$ with respect to $\{s_j, 0 \leq j \leq n\}$, we call $l = l(\tilde{w})$ the length of $\tilde{w} = \pi_r \tilde{w} \in W^b$. Similarly, $l_\nu(\tilde{w})$ is the number of $s_j$ with $\nu_j = \nu$.

Let us introduce a partial ordering on $B$. Here and further $b_-$ is the unique elements from $B_-$ which belong to the orbit $W(b)$. Namely, $b_-=\omega_b(b)$. So the equality $c_-=b_-$ means that $b, c$ belong to the same orbit. Set
\[
(2.5) \quad b \leq c, c \geq b \quad \text{for} \quad b, c \in B \quad \text{if} \quad c - b \in A_+,
\]
\[
(2.6) \quad b \preceq c, c \succeq b \quad \text{if} \quad b_- < c_- \quad \text{or} \quad b_- = c_- \text{ and } b \leq c.
\]
We use $\prec, \succ, \prec,<,\succ$ respectively if $b \neq c$.

3. Difference operators
We put \( m = 2 \) for \( D_{2k} \) and \( C_{2k+1} \), \( m = 1 \) for \( C_{2k}, B_k \), otherwise \( m = |\Pi| \). We use the parameters \( q, t_\nu = q^{k_\nu}, q_\nu = q^{2/\nu} (\nu \in \nu_R) \) and the variables \( x_1, \ldots, x_n \) are from Section 1,
\[
t_{[\alpha, k]} = t_\alpha = t_{\nu_\alpha}, \ t_j = t_{\alpha_j}, \ \text{where} \ [\alpha, k] \in R^a, \ 0 \leq j \leq n,
\]
\[
(3.1) \quad x_\tilde{b} = \prod_{i=1}^{n} x_i^{l_i} q^{k_i} \ \text{if} \ \tilde{b} = [b, k],
\]
for \( b = \sum_{i=1}^{n} l_i b_i \in B, \ k \in \frac{1}{m} \mathbb{Z}. \)

We will also consider polynomials in \( q^{\pm 1/m} \) and \( \{i_{\nu}^{\pm 1/2}\} \), using the notation \( C[q^{\pm 1/m}, t^{\pm 1/2}] \). The elements \( \hat{\omega} \in W^b \) act in \( C[q^{\pm 1/m}][x] \) by the formulas:
\[
(3.2) \quad \hat{\omega}(x_\tilde{b}) = x_{\hat{\omega}(\tilde{b})}.
\]
In particular:
\[
(3.3) \quad \pi_r(x_\tilde{b}) = x_{\omega_r^{-1}(b)} q^{(b_r, b)} \ \text{for} \ \alpha_r^* = \pi_r^{-1}(\alpha_0), \ r \in O^*.
\]

The **Demazure-Lusztig operators** (see [KL, KK, C2])
\[
(3.4) \quad T_j = t_j^{1/2} s_j + (t_j^{1/2} - t_j^{-1/2})(x_{\alpha_j} - 1)^{-1}(s_j - 1), \ 0 \leq j \leq n.
\]
preserve \( C[q^{\pm 1/m}, t^{\pm 1/2}][x] \). We note that only \( T_0 \) involves \( q \):
\[
(3.5) \quad T_0 = t_0^{1/2} s_0 + (t_0^{1/2} - t_0^{-1/2})(qX_{\hat{\theta}}^{-1} - 1)^{-1}(s_0 - 1),
\]
where \( s_0(X_i) = X_i X_{\hat{\theta}}^{-1} q^{(b_i, \hat{\theta})} \).

Given \( \hat{\omega} \in W^a, \ r \in O, \) the product
\[
(3.6) \quad T_{\pi_r \hat{\omega}} \overset{def}{=} \pi_r \prod_{k=1}^{l} T_{i_k}, \ \text{where} \ \hat{\omega} = \prod_{k=1}^{l} s_{i_k}, \ l = l(\hat{\omega}),
\]
does not depend on the choice of the reduced decomposition of \( \hat{\omega} \) (because \( \{T\} \) satisfy the same ‘braid’ relations as \( \{s\} \) do). Moreover,
\[
(3.7) \quad T_{\hat{\nu}} T_{\hat{\omega}} = T_{\hat{\nu} \hat{\omega}} \ \text{whenever} \ l(\hat{\nu} \hat{\omega}) = l(\hat{\nu}) + l(\hat{\omega}) \ \text{for} \ \hat{\nu}, \hat{\omega} \in W^b.
\]

In particular, we arrive at the pairwise commutative operators (see [C2]):
\[
(3.8) \quad Y_b = \prod_{i=1}^{n} Y_i^{b_i} \ \text{if} \ b = \sum_{i=1}^{n} k_i b_i \in B, \ \text{where} \ Y_i \overset{def}{=} T_{b_i},
\]
satisfying the relations
\[
(3.9) \quad T_i^{-1} Y_b T_i^{-1} = Y_b Y_{\alpha_i}^{-1} \ \text{if} \ (b, \alpha_i) = 1,
\]
\[
T_i Y_b = Y_b T_i \ \text{if} \ (b, \alpha_i) = 0, \ 1 \leq i \leq n.
\]
4. Macdonald polynomials

Recall that \( \langle f \rangle \) is the constant term of \( f \). We will switch from \( \Delta \) to

\[
\mu = \mu^{(k)} = \prod_{a \in R^+} \prod_{i=0}^{\infty} \frac{(1 - x_a q_i^a)(1 - x_a^{-1} q_i^{a+1})}{(1 - x_a t_{a, q_i}^a)(1 - x_a^{-1} t_{a, q_i}^{a+1})}, \quad a = \alpha^\vee.
\]

It is considered as a Laurent series with the coefficients in \( \mathbb{C}[t][[q]] \).

Let \( \mu_1 \overset{\text{def}}{=} \mu / \langle \mu \rangle \), where the formula for the constant term of \( \mu \) is as follows (see [C2]):

\[
\langle \mu \rangle = \prod_{a \in R^+} \prod_{i=1}^{\infty} \frac{(1 - q_i^{(r_k, \alpha)+1})^2}{(1 - t_{a, q_i}^{(r_k, \alpha)+1})(1 - t_{a, q_i}^{(r_k, \alpha)+1})}.
\]

It is a Laurent series with coefficients in \( \mathbb{C}(q, t) \), and \( \mu_1^* = \mu_1 \) with respect to the involution

\[
x_b^* = x_{-b}, \quad t^* = t^{-1}, \quad q^* = q^{-1}.
\]

Setting

\[
\langle f, g \rangle_1 = \langle \mu_1 f, g^* \rangle_1 = \langle g, f \rangle_1^* \quad \text{for} \quad f, g \in \mathbb{C}(q, t)[x],
\]

we introduce the non-symmetric Macdonald polynomials \( e_b(x) = e_b^{(k)}, \quad b \in B \), by means of the conditions

\[
e_b - x_b \in \perp_{c > b} \mathbb{C} x_c, \quad \langle e_b, x_c \rangle_1 = 0 \quad \text{for} \quad B_- \ni c > b.
\]

They are well-defined because the pairing is non-degenerate and form a basis in \( \mathbb{C}(q, t)[x] \).

Replacing here \( \mu_1 \) by \( \mu \) and involving the norm-formulas from [C4] (the Main Theorem), we establish that the expansions of the coefficients of \( e_b \) in terms of \( q \) belong to \( \mathbb{C}[t^{\pm 1}][[q]] \). Applying \( \ast \), these coefficients also belong to \( \mathbb{C}[t^{\pm 1}][(q^{-1})] \) when expanded in \( q^{-1} \). The same holds for \( e_b^* \).

This definition is due to Macdonald (for \( t_{\nu} = q_{\nu}^k, \quad k \in \mathbb{Z}_+ \)), who extended Opdam’s non-symmetric polynomials introduced in the degenerate (differential) case in [O2]. The general case was considered in [C4]. Another approach is based on the \( \mathcal{Y} \)-operators (see [M4],[C4]):

**Proposition 4.1.** The polynomials \( \{e_b, b \in B\} \) are eigenvectors of the operators \( \{L_f \overset{\text{def}}{=} f(Y_1, \cdots, Y_n), f \in \mathbb{C}[x]\} \):

\[
L_f(e_b) = f(q^{-b \#})e_b, \quad \text{where} \quad b_{\#} \overset{\text{def}}{=} b - \omega_b^{-1}(r_k),
\]

\[
x_a(q^{b \#}) = q^{(a, b)} \prod_{\nu} t_{\nu}^{-\omega_b^{-1}(r_{\nu})}(t_{\nu}), \omega_b \text{ is from Sec. 2}.
\]
Similarly, the symmetric Macdonald polynomials $p_b = p_b^{(k)}$ from (4.4) are eigenfunctions of the $W$-invariant operators $L_f = f(Y_1, \cdots, Y_n)$ for $f \in C(q,t)[x]^W$:

\[
L_f(p_b) = f(q^{-b+r_k})p_b, \ b \in B_-. \tag{4.7}
\]

They are connected with $\{e\}$ as follows (see [M4,C4] and [O2] in the differential case):

\[
p_b = \mathcal{P}_b^t e_b, \ b = b_ - \in B_-, \tag{4.8}
\]

where $w_c \overset{\text{def}}{=} \omega_c^{-1}w_0$ for the longest $w_0 \in W$.

Following [C2-C3], let us fix a subset $v \in \nu_R$ and introduce the shift operator by the formula:

\[
\mathcal{G}_v = \mathcal{G}_v^{(k)} = (\mathcal{X}_v)^{-1}\mathcal{Y}_v, \tag{4.9}
\]

\[
\mathcal{X}_v = \prod_{\nu \in v} (t a x a)^{1/2} - (t a x a)^{-1/2}, \quad \mathcal{Y}_v = \prod_{\nu \in v} (t a Y a^{-1})^{1/2} - (t a Y a^{-1})^{-1/2}. \]

Here $a = \alpha^\vee, a \in R_+,$ $\mathcal{X}_v = \mathcal{X}_v^{(k)}$ and $\mathcal{Y}_v = \mathcal{Y}_v^{(k)}$ belong to $C[t^{\pm 1/2}][x]$ and $C[t^{\pm 1/2}][Y]$ respectively.

**Proposition 4.2.** The operators $\mathcal{G}_v$ are $W$-invariant and preserve the space $C[q^{1/m}, t^{\pm 1/2}][x]^W$. If $t_{\nu} = 1$ for $\nu \not\in v$ then

\[
\mathcal{G}_v^{(k)}(p_b^{(k)}) = g_v^{(k)}(b)p_b^{k+v} \quad \text{for} \quad g_v^{(k)}(b) = \prod_{\alpha \in R_+, \nu \in v} (q_{a}^{-r_b+\nu})^{1/2} - (t a q_{a}^{(b-r_b, \alpha)})^{1/2}, \tag{4.10}
\]

where $r_v = \sum_{\nu \in v} r_{\nu}, \ k + v = \{k_\nu + 1, k_{\nu'}\}$ for $v \not\in \nu'$. $p_c = 0$ for $c \not\in B_-$. 

5. Fourier transforms

Proofs of the following theorems are based on the analysis of the automorphisms of the double affine Hecke algebras. The technique is similar to that from [C1-C4] and will be exposed in more detail in the next paper.

We will mainly use the renormalized Macdonald polynomials:

\[
\epsilon_b = \epsilon_b/e_b(q^{-r_k}) = q^{-(r_k,b_ -)} \prod_{[\alpha, j] \in \Lambda_b} \left( \frac{1 - q_{a}^{(r_k, \alpha)+j}}{1 - t a q_{a}^{(r_k, \alpha)+j}} \right) \epsilon_b, \tag{5.1}
\]

\[
\Lambda_b = \{[\alpha, j], \ \alpha \in R_+, 0 < j < - (a, b_ -) \text{ if } (a, b) > 0, \ 0 < j < - (a, b_ -) \text{ if } (a, b) < 0\}, \ b_ - = \omega_b(b). \tag{5.2}
\]
Here we applied the Main Theorem from [C4]. This normalization is very convenient in the difference harmonic analysis. For instance, the duality relations are especially simple: \( \epsilon_b(q^\gamma) = \epsilon_c(q^{\#}) \).

**Theorem 5.1.** Given \( b,c \in B \) and the corresponding renormalized polynomials \( \epsilon_b, \epsilon_c \),

\[
\begin{align*}
(5.3) \quad \langle \epsilon_b \epsilon_c \tilde{\gamma}^{-1} \mu \rangle &= q^{(b_c, b_c)/2 + (c_c, c_c)/2 - (r_k, r_k)} \epsilon_c(q^{\#}) \langle \tilde{\gamma}^{-1} \mu \rangle, \\
(5.4) \quad \langle \epsilon_b \epsilon_c \tilde{\gamma}^{-1} \mu \rangle &= q^{(b_c, b_c)/2 + (c_c, c_c)/2 - (r_k, r_k)} \epsilon_c(q^{\#}) \langle \tilde{\gamma}^{-1} \mu \rangle, \\
(5.5) \quad \langle \epsilon_b \epsilon_c \tilde{\gamma}^{-1} \mu \rangle &= q^{-(b_c, b_c)/2 - (c_c, c_c)/2 + (r_k, r_k)} \epsilon_c(q^{\#}) \langle \tilde{\gamma}^{-1} \mu \rangle.
\end{align*}
\]

The first two formulas make sense when we expand \( \epsilon_b, \epsilon_c, \epsilon_k^* \) in terms of \( q \). The coefficients of \( \epsilon, \epsilon^* \) belong to \( C[t^{\pm 1/2}[[q]] \). In the last formula, \( \mu^* \) is expanded in terms of powers of \( q^{-1} \), as well as \( \epsilon_b, \epsilon_c^* \). Thus (5.5) results from (5.4) and the duality. The product \( \mu \tilde{\gamma}^{-1} \) generalizes the (radial) Gaussian measure in the theory of Lie groups and symmetric spaces. Actually (5.5) is a formula for the Fourier transform of \( \epsilon_c \tilde{\gamma}^{-1} \) (see [C3]).

The following theorem is equivalent to Theorem 5.1 although it does not involve \( c \) and any scalar products. We will use the conjugation ‘*’ : \( q^* = q^{-1}, \ t^* = t^{-1}, \ x^*_b = x_b \).

**Theorem 5.2.** Given \( b \in B \),

\[
\begin{align*}
(5.6) \quad \epsilon_b(Y_1^{-1}, \ldots, Y_n^{-1})(\tilde{\gamma}) &= q^{(b, b)/2 - (r_k, r_k)/2} \epsilon_b \tilde{\gamma}, \\
(5.7) \quad \epsilon_b^*(Y_1, \ldots, Y_n)(\tilde{\gamma}) &= q^{(b, b)/2 - (r_k, r_k)/2} \epsilon_b^* \tilde{\gamma}, \\
(5.8) \quad \epsilon_b(Y_1, \ldots, Y_n)(\tilde{\gamma}^{-1}) &= q^{-(b, b)/2 + (r_k, r_k)/2} \epsilon_b \tilde{\gamma}^{-1}.
\end{align*}
\]

The first formula holds for any normalizations of \( \epsilon_b \). Moreover, since the coefficient of proportionality \( q^{(b, b)/2 - (r_k, r_k)/2} \) is the same for all \( b \) from the same \( W \)-orbit, it can be applied to linear combinations of \( \epsilon_c, c \in W(b) \). For instance, will use it for the symmetric Macdonald polynomials or for the \( t \)-antisymmetric ones (the next section).

Actually we do not need a presentation of \( \tilde{\gamma}^{\pm 1} \) as a Laurent series in this theorem. One may take \( \gamma^{\pm 1} = q^{\pm \sum_{i=1}^n b_i \alpha_i/2} \) or any \( W \)-invariant solution of the following system of difference equations:

\[
\begin{align*}
(5.9) \quad b_j(\gamma) &= q^{(1/2) \sum_{i=1}^n (b_i - (b_j, b_i)) (\alpha_i, -\delta_j^i)} = \\
&= \gamma q^{\pm b_j+b_j/2} = q^{(b, b)/2} x_j^{-1} \gamma, \\
&= b_j(\gamma^{-1}) = q^{-(b, b)/2} x_j^{-1} \gamma^{-1} \quad \text{for} \ 1 \leq j \leq n.
\end{align*}
\]
The formula (5.8) for the symmetric polynomials was verified in [C3] via the roots of unity. See (4.19) and the end of the proof of Corollary 5.4 (use that \( p_b = p_b \) for the symmetric polynomials). The same method works well in the non-symmetric case and for other identities. A more natural proof will appear in the next paper. To make the formulas complete we need to know the coefficient of proportionality:

**Theorem 5.3.**

\[
\langle \tilde{\gamma}^{-1} \mu \rangle = \prod_{a \in R_+} \prod_{j=1}^{\infty} \left( \frac{1 - q_a^{(r_k, a)+j}}{1 - t_a q_a^{(r_k, a)+j}} \right).
\]

When \( k_\nu^0 \in \mathbb{Z}_- \setminus \{0\} \), by the right hand side we mean the limit as \( k_\nu \to k_\nu^0 \).

There is one more reformulation of Theorem 5.1. Let us introduce another set of variables \( \lambda_i, 1 \leq i \leq n \), and the corresponding \( \tilde{\gamma}_\lambda^{\pm 1} \). We will also use the operators \( Y_b^\lambda, T_b^\lambda \) acting with respect to \( \lambda \) by the same formulas as for \( x \). The coefficients of the following Laurent series in terms of \( \{x_i, \lambda_j\} \) are well-defined:

\[
(5.11) \quad q^{(r_k, r_k)/2} \mathcal{E}_q(x, \lambda)\tilde{\gamma}_x^{\pm 1}\tilde{\gamma}_\lambda^{\pm 1} = \sum_{b \in B} q^{(b_y, b_y)/2-(r_k, r_k)/2} \frac{\epsilon_b(x)}{\epsilon_b^2} \frac{\epsilon_b(\lambda)}{\epsilon_b^2},
\]

\[
(5.12) \quad q^{-(r_k, r_k)/2} \mathcal{E}_{-1}(x, \lambda)\tilde{\gamma}_x \tilde{\gamma}_\lambda = \sum_{b \in B} q^{-(b_y, b_y)/2+(r_k, r_k)/2} \frac{\epsilon_b(x)}{\epsilon_b} \frac{\epsilon_b(\lambda)}{\epsilon_b}.
\]

They belong respectively to \( \mathbb{C}[t^{\pm 1/2}][[q^{1/m}]] \) and \( \mathbb{C}[t^{\pm 1/2}][[q^{-1/m}]] \). The latter series is obviously symmetric: \( \mathcal{E}_{-1}(x, \lambda) = \mathcal{E}_{q^{-1}}(\lambda, x) \). It holds for the first one too, but requires some proving (use (5.3) instead of (5.4)).

**Theorem 5.4.** The series \( \mathcal{E}_q(x, \lambda) \) is symmetric:

\[
(5.13) \quad q^{(r_k, r_k)/2} \mathcal{E}_q(x, \lambda)\tilde{\gamma}_x \tilde{\gamma}_\lambda = \sum_{b \in B} q^{(b_y, b_y)/2-(r_k, r_k)/2} \frac{\epsilon_b(x)}{\epsilon_b} \frac{\epsilon_b(\lambda)}{\epsilon_b}.
\]

Moreover,

\[
(5.14) \quad Y_b^x(\mathcal{E}(x, \lambda)) = (\lambda_b)^{-1} \mathcal{E}(x, \lambda), \quad Y_b^\lambda(\mathcal{E}(x, \lambda)) = (x_b)^{-1} \mathcal{E}(x, \lambda), \\
T_i^x(\mathcal{E}(x, \lambda)) = T_i^\lambda(\mathcal{E}(x, \lambda)), \quad 1 \leq i \leq n, \quad \text{for} \quad \mathcal{E} = \mathcal{E}_{q^{\pm 1}}.
\]

We note that the right hand side of (5.11) is a holomorphic function for all \( x_i \neq 0 \neq \lambda_j \) and generic \( t \) provided that \( |q| < 1 \). Similarly, (5.12) is holomorphic for non-zero \( x_i, \lambda_j \) when \( |q| > 1 \). It is checked by means of the recurrence relations (the Pieri rules) from [C4], Theorem 5.4. Dividing the Gaussians out, we see that \( \mathcal{E}(x, \lambda) \) are meromorphic with the poles coming
from the zeros of $\tilde{\gamma}^{\pm 1}$. The latter set (of divisors) is related to the Macdonald identities (product formulas). These functions are also invariant with respect to the intertwiners from Section 5, [C1]. It readily results from the definition and the theorem. The functions $E(x, \lambda)$ generalize the reproducing kernel of the Dunkl transform (which is also symmetric) and that from the Harish-Chandra theory (which is not). The passage to the $p$-polynomials is a straightforward $t$-symmetrization.*

6. Mehta integral

Following [C2,C3] we will use the shift operator to verify Theorem 5.3 in a way similar to that in the differential case [O1].

First we take $k_\nu \in \mathbb{Z}_+$ and replace $\mu$, which is a trigonometric polynomial in this case, by a proportional one:

$$\tilde{\mu} = \mu^{(k)} \overset{\text{def}}{=} \prod_{\alpha \in R_+} ((q_\alpha^{k_\alpha - 1} x_\alpha)^{1/2} - (q_\alpha^{k_\alpha - 1} x_\alpha)^{-1/2})$$

$$\cdots ((x_\alpha)^{1/2} - (x_\alpha)^{-1/2}) \cdots ((q_\alpha^{-k_\alpha} x_\alpha)^{1/2} - (q_\alpha^{-k_\alpha} x_\alpha)^{-1/2}).$$

Then $\tilde{\mu}^* = \tilde{\mu}$ and

$$\langle p_b \tilde{\mu} \tilde{\gamma}^{-1} \rangle = \langle (p_b \tilde{\mu} \tilde{\gamma})^* \rangle.$$

Let

$$\tilde{\mu} = \mu^{(k)}, \tilde{\mu}' = \mu^{(k+v)}, b' = b - r_v,$$

$$p = p^{(k)}_b, p' = p^{(k+v)}_{b - r_v} = (g^{(k)}(b))^{-1} g^{(k)}(p).$$

The notations are from Proposition 4.2.

**Key Lemma 6.1.**

$$(6.1) \quad \langle (p')^* \tilde{\mu}' \tilde{\gamma} \rangle = q^{(r_k, r_k)/2 - (b - r_k, b - r_k)/2} (d_{k+v}/d_k)$$

$$\times \prod_{\alpha \in R_+} \left( \frac{q_\alpha^{(b - r_k, \alpha)/2 + k_\alpha} - q_\alpha^{(r_k - b, \alpha)/2}}{(p')^* \tilde{\mu} \tilde{\gamma}} \right),$$

$$d_k = |W(r_k)|^{-1} \prod_{\alpha \in R_+, k_\alpha \neq 0} \left( \frac{q_\alpha^{((r_k, \alpha) + k_\alpha)/2} - q_\alpha^{-(r_k, \alpha) + k_\alpha)/2}}{q_\alpha^{(r_k, \alpha)/2} - q_\alpha^{-(r_k, \alpha)/2}} \right).$$

* In a recent paper by Baker, Forrester ([q-alg/9701039, 30 Jan 1997]), a certain generalization of the Dunkl kernel was suggested (formula (5.1)) in the case of $GL_n$. It is a certain sum in terms of the non-symmetric polynomials resembling our (5.13). However it is not $x \leftrightarrow \lambda$ symmetric (according to the authors), and there is no counterpart of the important formula $Y_\lambda^\lambda(\mathcal{E}(x, \lambda)) = (x_\lambda)^{-1} \mathcal{E}(x, \lambda)$ in the paper. The meaning (existence, convergence) of the infinite sums is not discussed by the authors. In our approach we separate the contribution of the Gaussians. After this the analysis is mainly due to [C4].
Proof. One has:

\[
\langle (p')^* \bar{\mu'} \bar{\gamma} \rangle = (d_{k+v}/d_k)\langle (\mathcal{X}_v^{(k)})^2 (p')^* \bar{\mu'} \bar{\gamma} \rangle \\
= (d_{k+v}/d_k)\langle (-1)^{\kappa_v} (\mathcal{X}_v^{(k)} \bar{\gamma}^2) (\mathcal{X}_v^{(k)} p')^* \bar{\mu} \rangle,
\]

where \(\kappa_v\) is the number of roots \(\alpha > 0\) with \(\nu_\alpha \in v\). See Lemma 4.4 and formula (5.7) from [C2]. We will treat the factors \(\mathcal{X}_v^{(k)}\) in two different ways.

First,

\[
\mathcal{X}_v^{(k)} \bar{\gamma} = q^{(r_k, r_k)/2 - (b - r_k, b - r_k)/2} \mathcal{Y}_v^{(k)}(\bar{\gamma})
\]

thanks to formula (5.6) applied to \(\mathcal{X}_v^{(k)}\), that is a linear combination of \(e_c^{(k)}\) for \(c \in W(r_v)\). Indeed, \(\mathcal{Y}_v^{(k)}(m_c)\) is proportional to \(\mathcal{X}_v^{(k)}\) (or zero) for all \(B_- \ni c \geq -r_v\). Applying any \(Y_a\) to \(e_b\) we get linear combinations of the \(e\)-polynomials from the same orbit.

Second,

\[
\mathcal{X}_v^{(k)} p' = (g^k(b))^{-1} \mathcal{Y}_v^{(k)}(p).
\]

Substituting, let us combine two \(\mathcal{Y}\) together using the unitarity of \(\{Y\}\) with respect to the pairing \(\langle fg^* \bar{\mu} \rangle\):

\[
\langle (p')^* \bar{\mu'} e \rangle = (d_{k+v}/d_k)\langle \bar{\gamma} (\{\mathcal{Y}_v^{(k)}\}^2 (p))^* \bar{\mu} \rangle.
\]

Because the constant term is \(W\)-invariant, we may multiply \(\mathcal{Y}^2\) on the left by \((\text{const}) P_v^{(k)}\) for the \(t\)-symmetrization from (4.8) with the idempotent normalization. As to \(\mathcal{Y}^2\), it can be replaced by \((-1)^{\kappa_v} \mathcal{Y}^2\) for

\[
\tilde{\mathcal{Y}}_v^{(k)} = \prod_{\alpha \in R_+, r_\alpha \in v} ((t_\alpha Y_\alpha)^{1/2} - (t_\alpha Y_\alpha)^{-1/2}), \ a = a^\vee
\]

(see [C2], formula (5.10)). Now we use formula (4.7). Collecting all the factors together we get the required.

The remaining part of the calculation is based on the following chain of the shift operators, that will be applied to \(p^{(0)}_{-r_k} = m_{-r_k}\) one after another:

\[
G_v^{(k-e)} G_v^{(k-2e)} \ldots G_v^{(k-s e)} G_v^{(k-se-v)} \ldots G_v^{(0)}
\]

where \(k_v = s + d, k_{v_2} = s, v = \{v_\nu\}, v_{v_1} = 1, v_{v_2} = 0,\) the set \(\{1, 1\}\) is denoted by \(e\). We assume that \(d \geq 0\) choosing the components properly. So \(k - se = dv\).

The product

\[
q^{(r_k, r_k - e)/2 - (r_k, r_k)/2} (d_k/d_{k-e}) \ldots q^{-(r_v, r_v)/2} (d_v/d_0)
\]
is equal to $q^{-(r_k,r_k)/2}d_k$. Taking into consideration that $p_{-r_k}(1) = |W(r_k)|$, we come to the final formula for $\langle \tilde{\mu}^{(k)} \tilde{\zeta} \rangle$. Conjugating,

\begin{equation}
(6.6) \quad \langle \gamma^{-1}\tilde{\mu}^{(k)} \rangle = q^{(r_k,r_k)} \prod_{\alpha \in R_+} \prod_{1 \leq j \leq k_\alpha} \left( q_\alpha^{(r_k,\alpha)/2+1/2} - q_\alpha^{-(r_k,\alpha)/2+1/2-j} \right).
\end{equation}

Multiplying by $\mu/\tilde{\mu}$, we verify (5.10) for $k_\alpha \in \mathbb{Z}_+$.

If $k$ are arbitrary we can still use the key lemma. It gives that the ratio $\phi(t)$ of the left and the right hand sides of (5.10) is a periodic function of $t_\nu$ with respect to the shift $t_\nu \to t_\nu q_\nu$. We may replace the constant term by a contour integral in $x$ for $|t_\alpha q_\alpha| < |x_a| < |t_\alpha^{-1}|$, $a = \alpha \vee \alpha > 0$, provided that $|t_\nu|, |q_\nu| < 1$. If $t$ has only one component, then $\phi(t)$ is analytic and $q$-periodic for such $q, t$ and has to be 1. Otherwise, $t = (t_{\nu_1}, t_{\nu_2})$ and we conclude that $\phi(t)$ depends only on $\tau = t_{\nu_1}/t_{\nu_2}^{1/2}$. Taking $t_{\nu_2} = 1$ and applying the shift operator for $v = \nu_1$, we see that $\phi(\tau) = \phi(q\tau)$ and $\phi = 1$ (cf. [O1]). The calculation is completed.

7. Jackson pairing

The formulas from the previous sections can be generalized for Jackson integrals taken instead of the constant term. We fix $\xi \in \mathbb{C}^n$ and set $\langle f \rangle_\xi \overset{def}{=} |W|^{-1} \sum_{w \in W, b \in \mathcal{B}} f(q^{w(\xi)+b})$ following [C1]. Here $f$ is a Laurent polynomial or any function well-defined at $\{q^{w(\xi)+b}\}$ provided the convergence. We assume that $|q| < 1$. One may also operate in formal series in terms of non-negative powers of $q$ considering $q^{(\xi,b)}$ as independent parameters.

Recall that $x_a(q^\xi) = q^{(a,\xi)}$, $\gamma(q^2) = q^{(z,z)/2}$. For instance,

\begin{equation}
\langle \gamma \rangle_\xi = \sum_{\alpha \in \mathcal{B}} q^{(\xi+a,\xi+a)/2} = \gamma^{-1}(q^\xi)q^{(\xi,\xi)/2}.
\end{equation}

Applying the shift operators for $k_\nu \in \mathbb{Z}_+$ we readily establish that

\begin{align}
\langle \gamma \mu^o \rangle_\xi &= |W|^{-1} q^{(r_k,r_k)/2} \langle \gamma p_{r_k} \rangle_\xi \\
&\times \prod_{\alpha \in R_+} \prod_{1 \leq j \leq k_\alpha} \left( 1 - q_\alpha^{-(r_k,\alpha)/2-j} \right), \quad \text{for} \quad \mu^o \overset{def}{=} \prod_{\alpha \in R_+} \prod_{0 \leq j \leq k_\alpha - 1} \left( 1 - q_\alpha^{-j} X_\alpha^{-1} - q_\alpha^{-j+1} X_\alpha \right).
\end{align}

We use the same Lemma 6.1.

The analytic continuation to any $t$ is

\begin{equation}
\mu^o = \prod_{\alpha \in R_+} \prod_{i=0}^{\infty} \frac{(1 - x_a^{-1} t_a^{-1} q_\alpha^i)(1 - x_a^{-1} t_a^{-1} q_\alpha^{i+1})}{(1 - x_a q_\alpha^i)(1 - x_a^{-1} q_\alpha^{i+1})}.
\end{equation}
In fact it is $\mu^{-1}(x,t^{-1})$. We assume in the next theorem that $\mu^o(q^{w(\xi)+b})$ is well-defined, i.e. $(\alpha, \xi) \not\in \mathbb{Z}$ for all $\alpha \in R_+$.

**Theorem 7.1.** Given $b, c \in B$ and the corresponding polynomials $\epsilon_b, \epsilon_c$,

(7.3) \[ \langle \epsilon_b \epsilon_c^* \gamma \mu^o \rangle_\xi = q^{-\frac{(b_\# \cdot b_\#)}{2} - \frac{(c_\# \cdot c_\#)}{2}} \epsilon_c(q^{b_\#}) \epsilon_c(q^{b_\#})^{-1} \langle \gamma \mu^o \rangle_\xi, \]

(7.4) \[ \langle \gamma \mu^o \rangle_\xi = |W|^{-1} \langle \gamma \rangle_\xi \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \left( 1 - t_\alpha^{-1} q_\alpha^{-(r_\alpha, \alpha) + j} \right). \]

In this formulas $t$ is arbitrary provided the existence of $\epsilon_b, \epsilon_c$. The right hand side of (7.4) is replaced by the limit when $k_\nu \in \mathbb{Z}_+$. We use the values at these points and the shift operators to deduce (7.4) from (7.1) (see the end of the previous section). There is one more similar formula corresponding to the conjugation of (5.3). These formulas can be extended even to the special cases when $\mu^o(q^{w(\xi)+b})$ has singularities, for instance, to spherical representations of double affine Hecke algebras from [C1]. One has to renormalize $\mu$ as follows.

Let us switch from $\mu^o$ to $\tilde{\mu} = \frac{\mu^o}{\mu^o(q^{-r_k})}$. See [C1], Proposition 4.2 and [M5]. Setting $\Lambda(bw) = R^a_+ \cap (bw)^{-1}(R^a_+)$,

(7.5) \[ \tilde{\mu}(q^{w(\xi)+b}) = \tilde{\mu}(q^{w(\xi)+b})^{-1} \prod_{[\alpha, j] \in \Lambda(bw)} \left( \frac{t_\alpha^{-1/2} - q_\alpha^{(\alpha, \xi)/2}}{t_\alpha^{-1/2} - q_\alpha^{(\alpha, \xi)/2}} \right), \]

where the conjugation of $\xi$ is trivial: $(q^{(\xi, a)})^{-1} = q^{-(\xi, a)}$. We note that (7.5) is the same for $\mu$ instead of $\mu^o$.

Let $\xi = -r_k$ for generic $k$. Then $\tilde{\mu}(q^{w(\xi)+b})$ is always well-defined and non-zero only for $\pi_b = b_\alpha b_\beta^{-1}$ [C1]. The previous theorem in this case reads as follows:

**Theorem 7.2.** For $\xi = -r_k$, $b, c \in B$,

(7.6) \[ \langle \epsilon_b \epsilon_c^* \gamma \tilde{\mu} \rangle_\xi = q^{-\frac{(b_\# \cdot b_\#)}{2} - \frac{(c_\# \cdot c_\#)}{2}} \epsilon_c(q^{b_\#}) \epsilon_c(q^{b_\#})^{-1} \langle \gamma \tilde{\mu} \rangle_\xi, \]

(7.7) \[ \langle \gamma \tilde{\mu} \rangle_\xi = |W|^{-1} \langle \gamma \rangle_\xi \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left( 1 - t_\alpha^{-1} q_\alpha^{-(r_\alpha, \alpha) + j} \right). \]

**Corollary 7.3.** For the function $\mathcal{E}_q$ from (5.13), arbitrary $b, c \in B$, and the constant $\langle \gamma \rangle_{r_k} = \gamma(q^{r_k})\gamma^{-1}(q^{r_k})$,

(7.8) \[ \mathcal{E}_q(q^{b_\#}, q^{c_\#}) \langle \gamma \rangle_{r_k}^2 = \epsilon_c(q^{b_\#}) |W| \langle \gamma \tilde{\mu} \rangle_{-r_k}, \]

(7.9) \[ \mathcal{E}_q(x, q^{c_\#}) \langle \gamma \rangle_{r_k} = \epsilon_c(x) \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left( 1 - t_\alpha^{-1} q_\alpha^{-(r_\alpha, \alpha) + j} \right). \]
Proof. One has:

\[ \mathcal{E}_q(q^{b\#}, q^{c\#})\tilde{\gamma}^{-1}(q^{b\#})\tilde{\gamma}^{-1}(q^{c\#}) = \]

\[ \sum_{a \in B} q^{(a\# a\#)/2-(r_k, r_k)} \epsilon_a(q^{b\#}) \epsilon_a^*(q^{c\#}) \]

\[ \langle \epsilon_a, \epsilon_a \rangle_{1} = \]

\[ \sum_{a \in B} q^{(a\# a\#)/2-(r_k, r_k)} \epsilon_b(q^{a\#}) \epsilon_c^*(q^{a\#}) \tilde{\epsilon}_\lambda(q^{a\#}) = \]

\[ q^{-(r_k, r_k)}|w| \langle \epsilon_b, \epsilon_c^* \tilde{\gamma}_\mu \rangle_{x,-r_k} = \]

\[ q^{-(r_k, r_k)} \tilde{\gamma}_{x, \lambda x \epsilon_c^* b} \gamma \tilde{\gamma}_{x, \lambda x \epsilon_c^* b} = \]

\[ q^{-(r_k, r_k)} \tilde{\gamma}_{x, \lambda x \epsilon_c^* b} \gamma \tilde{\gamma}_{x, \lambda x \epsilon_c^* b} \]

(7.10)

Here we used the duality and the formula \( \tilde{\mu}(q^{b\#}) = \langle \epsilon_b, \epsilon_b \rangle_{1}^{-1} \) from Theorem 5.6 [C1]. The function \( q^{(b\# a\#)/2} \tilde{\gamma}^{-1}(q^{b\#}) \) does not depend on \( b \in B \), so we get the constant \( \langle \gamma \rangle_{x, r_k} \).

The second formula (7.13) is an analytic continuation of the first. The ratio \( \psi = \mathcal{E}_q(x, q^{c\#})/\epsilon_c(x) \) is \( B \)-periodic and with poles in the \( B \)-periodic set of zeros of \( \tilde{\gamma}^{-1}(x) \). Therefore \( \psi \) is analytic for all \( x_i \neq 0 \) and must be a constant. □

Thus \( \mathcal{E}_q \) can be introduced more conceptually as a unique extension of the non-symmetric polynomials in the class of meromorphic functions of non-zero \( \{x_i, \lambda_j\} \) with the poles in the zeros of the Gaussians \( \tilde{\gamma}^{-1}(x) \), \( \tilde{\gamma}^{-1}(\lambda) \). It must satisfy (5.14) and be invariant with respect to the action of the intertwining operators.

Let us give a symmetric version of the above results (see Section 1). The reproducing kernel is given by the formula in terms of symmetric Macdonald’s polynomials:

\[ q^{(r_k, r_k)/2} \mathcal{P}(x, \lambda) \tilde{\gamma}^{-1}_x \tilde{\gamma}^{-1}_\lambda = \sum_{b \in B_{-}} q^{(b, b)/2-(r_k, b)} \frac{p_b(x)}{p_b(x)} \frac{p_b(\lambda^{-1})}{p_b(x)} \frac{\Delta}{\lambda^{-1}}. \]

(7.11)

The right hand side is analytic for non-zero \( x, \lambda \) when \( |q| < 1 \). The function \( \mathcal{P} \) is \( x \leftrightarrow \lambda \) symmetric and satisfies the equations

\[ f(Y_1, \ldots, Y_n)(\mathcal{P}(x, \lambda)) = f(\lambda^{-1})\mathcal{P}(x, \lambda) \text{ for } f \in \mathbb{C}[x]^W. \]

(7.12)

It extends \( \{p_c, c \in B_{-}\} \) with the same coefficient of proportionality as in

(7.13):

\[ \mathcal{P}(x, q^{c-r_k}) \langle \gamma \rangle_{r_k} = p_c(x)(p_c(q^{r_k}))^{-1} \prod_{\alpha \in R_k} \prod_{j=1}^{\infty} \left( \frac{1 - q_\alpha^{(r_k, \alpha)+j}}{1 - q_\alpha^{-(r_k, \alpha)+j}} \right). \]

In fact it is a corollary of Theorem 1.3 for \( \xi = -r_k \).

We note that the calculation of the constant \( \langle \gamma \mu^0 \rangle_{\xi} \) from Theorem 7.1 and that for \( \tilde{\mu} \) instead of \( \mu^0 \) is directly related to the Aomoto conjecture [A,Ito] proved in [M5]. Actually it gives another way to check this conjecture.
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