The modified Poynting theorem and the concept of mutual energy

Shuang-ren Zhao
*Imrecons Inc, Toronto, Canada

Kevin Yang, Kang Yang, and Xingang Yang
*Imrecons Inc, Toronto, Canada

Xintie Yang
Northwestern Polytechnical university, Xi’an, China
Abstract

The goal of this article is to derive the reciprocity theorem, mutual energy theorem from Poynting theorem instead of from Maxwell equation. In this way the reciprocity theorem will become the energy theorem. In order to realize this purpose the followings have been done. The Poynting theorem is generalized to the modified Poynting theorem. In the modified Poynting theorem the electromagnetic field is superimposition of different electromagnetic fields including the field of retarded potential and advanced potential, electric/magnetic mirrored field, time-reversed field, time-offset field, space-offset field. The media epsilon (permittivity) and mu (permeability) can also be different in the different fields. The concept of mutual energy is introduced which is the difference between the total energy and self-energy. First we try to derive the mutual energy theorem from complex Poynting theorem, it is failed. As a side effect we obtained the mixed mutual energy theorem. We applied the average process to derive the mutual energy theorem from Poynting theorem. This is derivation is not strictly. Then we derive the mutual energy from Fourier domain, instead of obtained the mutual energy theorem from time-domain. We obtain the time-reversed mutual energy theorem. A time-reverse transform needed to further derive the mutual energy theorem. The time-reverse transform contains some information from Maxwell equation, hence the derivation is not a purely derivation from Poynting theorem. Then we derive the mutual energy theorem in time-domain. Using the modified Poynting theorem with the concept of the mutual energy. The instantaneous modified mutual energy theorem is derived. Applying time-offset transform and time integral to the instantaneous modified mutual energy theorem, the time-correlation modified mutual energy theorem is obtained. Assume there are two electromagnetic fields one is retarded potential and one is advanced potential, the convolution reciprocity theorem can be derived. Corresponding to the modified time-correlation mutual energy theorem and the time-convolution reciprocity theorem in Fourier domain, there is the modified mutual energy theorem and the Lorentz reciprocity theorem. Hence all mutual energy theorem and the reciprocity theorems are put in one frame of the concept of the mutual energy. The inner product is introduced for two different electromagnetic fields in both time domain and Fourier domain. The concept of inner product of electromagnetic fields simplifies the theory of the wave expansion. The concept of reaction is re-explained as the mutual energy of two fields with retarded potential and advanced potential.
I. INTRODUCTION

In electromagnetic field theory, the Poynting theorem[1] is energy conservation theorem. The Lorentz reciprocity theorem[2–5]

\[
\int_S (E_1(\omega) \times H_2(\omega) - E_2(\omega) \times H_1(\omega)) \, dt \, dS
\]

\[
= \int_V (J_1(\omega) \cdot E_2(\omega) - J_2(\omega) \cdot E_1(\omega) - K_1(\omega) \cdot H_2(\omega) + K_2(\omega) \cdot H_1(\omega)) \, dV = 0 \tag{1}
\]

is close related to Poynting theorem. The two theorems look similar, J.R. Carson call the reciprocity theorem as “Reciprocal energy theorem” in the reference[4]. But until now the two theorems are two different theorems derived from Maxwell equations respectively.

Many efforts try to reveal the relationship between the reciprocity theorem and Poynting theorem have been done. V. H. Rumsey proposed the concept of the reaction[6] in 1954 which is related to Lorentz reciprocity theorem. But what is the concept of the “reaction” in behind scene?. W.J. Welch has derived a reciprocity theorem[7] in 1960, which is in the following,

\[
- \int_S \int_{t=-\infty}^{\infty} (E_1(t) \times H_2(t) + E_2(t) \times H_1(t)) \, dt \, dS
\]

\[
= \int_V \int_{t=-\infty}^{\infty} (J_1(t) \cdot E_2(t) + K_1(t) \cdot H_2(t) + J_2(t) \cdot E_1(t) + K_2(t) \cdot H_1(t)) \, dt \, dV \tag{2}
\]

In the formula all variables with subscript “1” is belong to retarded potential and all variables with subscript “2” belong to advanced potential. Welch has derived another reciprocity theorem in 1961,

\[
\int_S \int_{t=-\infty}^{\infty} (E_1(t) \times H_2(t) - E_2(t) \times H_1(t)) \, dt \, dS
\]

\[
= \int_V \int_{t=-\infty}^{\infty} (J_1(t) \cdot E_2(t) - K_1(t) \cdot H_2(t) - J_2(t) \cdot E_1(t) + K_2(t) \cdot H_1(t)) \, dt \, dV \tag{3}
\]
Rumsey has derived a another reciprocity theorem in 1963 \cite{9} from his reaction concept, which is the following,

\[
\int_{V} (J_{1}(\omega) \cdot E_{2}^{*}(\omega) + J_{2}^{*}(\omega) \cdot E_{1}(\omega) + K_{1}(\omega) \cdot H_{2}^{*}(\omega) + K_{2}^{*}(\omega) \cdot H_{1}(\omega)) \, dV = 0 \quad (4)
\]

S. N. Samaddar suggest a reciprocity theorem and applied it to solve wave expansion problems in 1964 \cite{10} in plasma media,

\[
\nabla \cdot (E_{l} \times \hat{H}_{l'} + \hat{E}_{l'} \times H_{l}) + \dot{v}_{el'} \cdot p_{el} + v_{el} \hat{p}_{el'} + \dot{v}_{l'} \cdot p_{l} + v_{l} \hat{p}_{l} + + \dot{v}_{nl'} \cdot p_{nl} + v_{in} \hat{p}_{nl'} = 0 \quad (5)
\]

The corresponding time-domain theory of the reciprocity theorem \cite{2–5} is the time-convolution reciprocity theorem \cite{11, 12} derived by G. Goubau in 1960 and B. Ru-shao Cheo in 1965 which is

\[
\int_{V} \int_{-\infty}^{\infty} (J_{1}(\tau - t) \cdot E_{2}(t) - J_{2}(t) \cdot E_{1}(\tau - t) - K_{1}(\tau - t) \cdot H_{2}(t) + K_{2}(t) \cdot H_{1}(\tau - t)) \, dt \, dV = 0 \quad (6)
\]

The further development of the time-convolution reciprocity theorem can be found \cite{13, 14}. J. A. Kong offers the details of conjugate transform and the concept of the modified reciprocity theorem \cite{15, 16} in 1972. In the modified reciprocity theorem the \( \epsilon \) (permittivity) and \( \mu \) (permeability) of two electromagnetic fields appeared in the reciprocity theorem are allowed to be different. Norbert N. Bojarski has further developed the Welch’s reciprocity theorem in 1983 \cite{17}. Shuang-Ren Zhao proposed the mutual energy theorem and modified mutual energy theorem in May of 1987 \cite{18}, which is

\[
- \int_{S} (E_{1}(\omega) \times H_{2}^{*}(\omega) + E_{2}^{*}(\omega) \times H_{1}(\omega)) \cdot \hat{n} \, dS
\]

\[
= \int_{V} (J_{1}(\omega) \cdot E_{2}^{*}(\omega) + J_{2}^{*}(\omega) \cdot E_{1}(\omega) + K_{1}(\omega) \cdot H_{2}^{*}(\omega) + K_{2}^{*}(\omega) \cdot H_{1}(\omega)) \, dV \quad (7)
\]

The derivation of the mutual energy theorem is based on Lorenz reciprocity theorem \cite{2–5} and the conjugate transform \cite{16}. The mutual energy theorem is defined in Fourier domain or complex domain and is further developed in the reference \cite{19, 20}. Compare to the Rumsey’s formula in the mutual energy theorem there is an item of the surface integral.
which does not vanish. The surface integral has been applied to define an inner product of electromagnetic fields on the surface and hence to solve the wave expansion problems. Welch’s reciprocity theorem\cite{7, 17} is further developed by A. T. de Hoop in December 1987 to become the so called time-correlation reciprocity theorem\cite{21} which is

\[
-\int_{S} \int_{t=-\infty}^{\infty} (E_1(t+\tau) \times H_2(t) + E_2(t) \times H_1(t+\tau)) \, dt \, d\hat{S}
\]

\[
= \int_{V} \int_{t=-\infty}^{\infty} (J_1(t+\tau) \cdot E_2(t) + K_1(t+\tau) \cdot H_2(t) + J_2(t) \cdot E_1(t+\tau) + K_2(t) \cdot H_1(t+\tau)) \, dt \, dV
\] (8)

de Hoop’s time-correlation reciprocity theorem can be seen as the mutual energy theorem\cite{18} in time-domain instead of in the Fourier domain. Welch’s reciprocity theorem is a special situation of the time-correlation reciprocity theorem where the time variable $\tau = 0$. By the way in the theorem of de Hoop\cite{21} the surface integral appeared but has been thought that it will vanish on the infinite sphere. Baun has a book\cite{22} in 1995 which systematically introduced reciprocity theorems. The reference\cite{23, 24} solved the wave expansion problem directly from Maxwell equation which is close related to the mutual energy theorem.

Later the mutual energy theorem has been rediscovered and has been referred as the second reciprocity theorem\cite{25} in 2009. In this second reciprocity theorem the surface integral was also thought to be vanish on the infinite big sphere.

Application of reciprocity theorem and mutual energy theorem can also be found in the examples\cite{26–29}. There are a few reference discussed the relationship between Poynting theorem and reciprocity theorem\cite{30–32}. However they only discussed them together and did not offer the direct relationship between the two theorems. The reference\cite{15} discussed the reciprocity theorem in bi-anisotropic media.

There is a concept “mixed Poynting vector”\cite{36, 37} which is close related to the concept mutual energy and Poynting vector.

It is worth to notice that in the reference\cite{18–20}, the concept of “mutual energy” was not well defined and the related concept “total energy” and “self energy” were also not defined. The so called mutual energy theorem is only derived from the modified reciprocity theorem instead of Poynting theorem, hence the concept of mutual energy is still not widely acceptable.
II. THE CONTRIBUTION OF THIS ARTICLE

The goal of this article is to convince the reader that the mutual energy theorem is a real energy. To achieve this goal the mutual energy theorem have to be derived from Poynting theorem instead of from Maxwell equation directly. In this article a few new concept is defined or generalized. Among them there is “modified”, “time-reversed transform”, “mutual energy”, “self energy”, “total energy”. The concept of reaction is reexplained. The mutual energy theorem is re-derived from Poynting theorem. The derivation include different versions and the history is introduced. A few new theorem is obtained from the derivation, the following gives the details.

A. The modified Maxwell equation

The concept of “modified” is borrowed from the modified reciprocity theorem, where two electromagnetic field put in different media with different $\varepsilon$ and $\mu$ can be superimposed. We found for this situation the Maxwell equation is still established. In this kind of media, the Maxwell equation is referred as modified Maxwell equation.

B. The modified Poynting theorem

The Poynting theorem is generalized to the modified Poynting theorem. The concept of “modified” is borrowed from the modified reciprocity theorem[16] as above modified Maxwell equation. This idea has also been used in the mutual energy theorem which is modified mutual energy[18,20].

C. Time-reversed transform

We knew that after a magnetic mirror transform the electromagnetic field $(E, H)$ is still electromagnetic field. A electromagnetic field after a time-reversed transform is not a electromagnetic field any more. That means after the time-reversed transform it does not satisfy the Maxwell equation. However we modified the time-reversed transform through introducing the negative media with negative $\varepsilon$, $\mu$, this new time-reversed transform is given in the following,
\[ [E_r(t), H_r(t), J_r(t), K_r(t), \epsilon_r(t), \mu_r(t)] \]
\[ \equiv \tau[E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] \]
\[ = [E(-t), H(-t), J(-t), K(-t), -\epsilon(-t), -\mu(-t)] \]  

A electromagnetic fields after the above time-reverse transform is still electromagnetic fields. This time reverse transform is one of the import tool to derive mutual energy theorem from Poynting theorem in Fourier domain.

D. The difference the substitution and the replace of a transform is noticed

There are many transforms: magnetic mirror transform, electric mirror transform, time-reversed transform. Time-offset transform. There are two process for the above transform, one is substitution of the transform, the other is replace of a transform. Replace and substitution are two different process, in the history many mistakes were made since the confusion with this two processes. In this article we try to clarify the difference of the replacement and substitution of a transform.

E. Introduced the instantaneous mutual energy theorem

In this article the concept of mutual energy is defined as the difference between the total energy and the self energy. The instantaneous mutual energy theorem is derived from Poynting theorem with the concept of mutual energy. The instantaneous mutual energy theorem is following,

\[ -\nabla \cdot (E_1 \times H_2 + E_2 \times H_1) \]

\[ = J_1 \cdot E_2 + J_2 \cdot E_1 + K_1 \cdot H_2 + K_2 \cdot H_1 + E_1 \cdot \partial D_2 + E_2 \cdot \partial D_1 + H_1 \cdot \partial B_2 + H_2 \cdot \partial B_1 \]  

F. Derived the time-reversed mutual energy theorem in Fourier domain

Time-reversed mutual energy theorem is introduced from Fourier domain through instantaneous mutual energy theorem, and hence from the Poynting theorem. The reversed
The mutual energy theorem in Fourier domain is shown in the following,

\[- \int_{S} (E_{1}(\omega) \times H_{2}(\omega) + E_{2}(\omega) \times H_{1}(\omega)) \, \hat{n} dS \]

\[= \int_{V} (J_{1}(\omega) \cdot E_{2}(\omega) + J_{2}(\omega) \cdot E_{1}(\omega) + K_{1}(\omega) \cdot H_{2}(\omega) + K_{2}(\omega) \cdot H_{1}(\omega)) \, dV \quad (11)\]

The time-reversed mutual energy theorem in time domain is

\[- \int_{S} \int_{t=-\infty}^{\infty} (E_{1}(\tau - t) \times H_{2}(t) + E_{2}(t) \times H_{1}(\tau - t)) \, \hat{n} dS \]

\[= \int_{V} \int_{t=-\infty}^{\infty} (J_{1}(\tau - t) \cdot E_{2}(t) + J_{2}(t) \cdot E_{1}(\tau - t) + K_{1}(\tau - t) \cdot H_{2}(t) + K_{2}(t) \cdot H_{1}(\tau - t)) \, dV \quad (12)\]

**G. Derived the mutual energy theorem in Fourier domain**

The mutual energy theorem can be derived from reversed mutual energy theorem with time-reversed transform. The mutual energy theorem is shown in the following,

\[- \int_{S} (E_{1}(\omega) \times H_{2}^{*}(\omega) + E_{2}^{*}(\omega) \times H_{1}(\omega)) \cdot \hat{n} dS \]

\[= \int_{V} (J_{1}(\omega) \cdot E_{2}^{*}(\omega) + J_{2}^{*}(\omega) \cdot E_{1}(\omega) + K_{1}(\omega) \cdot H_{2}^{*}(\omega) + K_{2}^{*}(\omega) \cdot H_{1}(\omega)) \, dV \quad (13)\]

The corresponding in the time domain which is time-correlation mutual energy theorem or time-correlation reciprocity theorem[21].

\[- \int_{S} \int_{t=-\infty}^{\infty} (E_{1}(t + \tau) \times H_{2}(t) + E_{2}(t) \times H_{1}(t + \tau)) \, dt \, \hat{n} dS \]

\[= \int_{V} \int_{t=-\infty}^{\infty} (J_{1}(t + \tau) \cdot E_{2}(t) + K_{1}(t + \tau) \cdot H_{2}(t) + J_{2}(t) \cdot E_{1}(t + \tau) + K_{2}(t) \cdot H_{1}(t + \tau)) \, dt \, dV \quad (14)\]
H. Derived the time-reversed reciprocity theorems from the mutual energy theorems in Fourier domain

The time-reversed reciprocity theorem is shown as following,

\[
\int_S \left( (E_1(\omega) \times H_2^*(\omega) - E_2(\omega) \times H_1^*(\omega)) \right) dt \cdot \hat{n} ds
\]

\[
= \int_V \left( J_1(\omega) \cdot E_2^*(\omega) - J_2(\omega) \cdot E_1^*(\omega) - K_1(\omega) \cdot H_2^*(\omega) + K_2(\omega) \cdot H_1(\omega) \right) dV = 0 \quad (15)
\]

In time-domain, the corresponding theorem is time-correlation reciprocity theorem is derived which is in the following,

\[
\int_S \int_{t=-\infty}^{\infty} \left( (E_1(t) \times H_2(t+\tau) - E_2(t+\tau) \times H_1(t)) \right) dt \cdot \hat{n} ds
\]

\[
= \int_V \int_{t=-\infty}^{\infty} \left( J_1(t) \cdot E_2(t+\tau) - K_1(t) \cdot H_2(t+\tau) - J_2(t+\tau) \cdot E_1(t) + K_2(t+\tau) \cdot H_1(t) \right) dV \quad (16)
\]

I. Re-derived the Lorenz reciprocity theorem from the mutual energy theorem

The Lorenz reciprocity theorem is shown as a special situation of the mutual energy theorem. In the special situation where two electromagnetic fields are different, one is the field of retarded potential and the other is the field of advanced potential. The concept of the reaction is re-explained as the mutual energy (or power) of the two electromagnetic fields where one is the field of retarded potential and the other is the field of advanced potential.

J. Re-derived time-correlation mutual energy theorem from Poynting theorem

In the above we have derived the mutual energy theorem in Fourier domain. The derivation is that first derive the time-reversed mutual energy theorem and then through a time-reverse transform we obtained the mutual energy theorem. We did not very satisfy with this process of derivation. Since it is not a pure derivation from Poynting theorem. The time-reversed transform need the Maxwell equation to prove. Hence we actually derived the
mutual energy theorem form Poynting theorem plus Maxwell equation. A purely derivation from Poynting theorem should not use the transform for example mirrored transform or time-reversed transform. Hence we seek another way to prove the mutual energy theorem and avoid the time-reversed transform. We have proved that the time-correlation mutual energy theorem from Poynting theorem. And then considered a Fourier transform for the time-correlation mutual energy theorem, we obtained the mutual energy theorem. This way we have purely derived the mutual energy theorem form the Poynting theorem.

K. Introduced the mixed mutual energy theorem

We have try to derive the mutual energy theorem from complex Poynting theorem. But we have failed. Instead to obtained the mutual energy theorem we obtained the following mixed mutual energy theorem.

\[
- \int_{S}(E_1 \times H_2^* + E_2 \times H_1^*) \cdot \hat{n}dS
\]

\[
= \int_{V}(E_1 \cdot J_2^* + E_2 \cdot J_1^* + H_1^* \cdot K_2 + H_2^* \cdot K_1)dV
\]

\[
+ j\omega \int_{V}(H_1^* \cdot \mu_2 H_2 + H_2^* \cdot \mu_1 H_1 - E_1 \cdot \epsilon_2^* E_2^* - E_2 \cdot \epsilon_1^* E_1^*)dV
\]

(17)

or corresponding mixed time-correlation mutual energy theorem,

\[
- \int_{S} \int_{t=-\infty}^{\infty}(E_1(t + \tau) \times H_2^*(t) + E_2(t + \tau) \times H_1^*(\tau))dt \cdot \hat{n}dS
\]

\[
= \int_{V} \int_{t=-\infty}^{\infty}(E_1(t + \tau) \cdot J_2^*(t) + E_2(t + \tau) \cdot J_1^*(t) + H_1^*(t) \cdot K_2(t + \tau) + H_2^*(t) \cdot K_1(t + \tau)) dt dV
\]

\[
+ \partial_\tau \int_{V} \int_{t=-\infty}^{\infty}(H_1^*(t) \cdot (\mu_2 \ast H_2)(t + \tau) + H_2^*(t) \cdot (\mu_1 \ast H_1)(t + \tau)
\]

\[
- E_1(t + \tau) \cdot (\epsilon_2^* \ast E_2^*)(t) - E_2(t + \tau) \cdot (\epsilon_1^* \ast E_1^*)(t)) dt dV
\]

(18)

The mixed mutual energy theorem is related with the concept of mixed Poynting vector\[36, 37\]. We do not clear perhaps the mixed mutual energy has some usage in the future.
L. Introduced the inner product to the mutual energy theorem

This author has introduced the inner product in Fourier domain[18–20]. In this article this idea is generalized to time domain. In time domain the inner product is defined as following

$$(\zeta_1, \zeta_2)_\tau = \int_S \int_{t=\infty}^t (E_1(t+\tau) \times H_2(t) + E_2(t) \times H_1(t+\tau)) \, dt \, dS$$

The author has shown that $(\zeta_1, \zeta_2)_\tau$ is not a good inner product, but $(\zeta_1, \zeta_2)_{\tau=0}$ is a good inner product. The inner product is applied to the wave expansion problem. The normal function expansion method can be applied to electromagnetic field expansions.

M. Re-explained the concept of the reaction

Many confused concept about transform is clarified. There are time-reversed transform, mirror transform, time offset transform. In the above transform there are two different process in derivation of new theory one is substitution and the other is replacement. Mistake is often caused by confusing the replacement as substitution. The concept of causal field, advanced potential, retard potential, offset field, transmitting filed, receiving field is clarified too. The concept reaction is re-explained as the mutual energy of two fields one is retarded potential and the other one is advanced potential.

N. Complementary theorems

Chen-To Tai has derived the complementary reciprocity theorem[38]. We have obtained 4 theorems, 2 mutual energy theorem and 2 reciprocity theorem. We apply the electromagnetic field swapping transform

$$\zeta_s = s\zeta = [ZH, \frac{1}{Z}E, -\frac{1}{Z}K, -ZJ, -\frac{1}{Z^2}\mu, -Z^2\epsilon]$$

4 corresponding complementary theorems are obtained. Among them one is the Chen-To tai’s complementary reciprocity theorem.
III. MODIFIED MAXWELL EQUATION

A. The Maxwell equation and the modified Maxwell equation

There are two kinds of electromagnetic fields, one is transmitting field and the other one is receiving field. Assume $\xi = [E, H]$ is radiated from the source $\rho = [J, K]$. $\rho$ is inside the volume $V$, the boundary of the volume is $S = \partial V$. The example of this kind of electromagnetic field is the electromagnetic field radiated from an antenna. $\xi = [E, H]$ is the retarded potential. Another kind of field is the field received by the sink of $\rho = [J, K]$. The example of this kind of electromagnetic field is the electromagnetic field received by an antenna which is advanced potential.

Assume $\zeta = [E, H, J, K, \epsilon, \mu]$ is a electromagnetic system, where $\xi = [E, H]$ are electric field intensity and magnetic field intensity. $\rho = [J, K]$ are electric current distribution and magnetic current distribution. $\epsilon, \mu$ are permittivity and permeability, we assume $\zeta$ satisfies the Maxwell equation,

$$\nabla \times H = J + \partial D$$
$$\nabla \times E = -K - \partial B$$

Where $\partial = \partial_t = \frac{\partial}{\partial t}$, $t$ is time. $\nabla$ is gradient operator to the space variable $x = [x_1, x_2, x_3]$ is the rectangle coordinates, “×” is vector cross product operator. “$\nabla \times$” is “curl” operator. $\nabla \cdot$ is divergence operator Here $D$ is electric displacement field intensity; $B$ is magnetic induction field intensity. And

$$D(t) = \int_{\tau=-\infty}^{\infty} \epsilon(t - \tau)E(\tau)\,d\tau$$
$$B(t) = \int_{\tau=-\infty}^{\infty} \mu(t - \tau)H(\tau)\,d\tau$$

If there is only one media $\epsilon, \mu$, the electromagnetic field can also be written as $\zeta = [E, H, J, K, D, B]$. In general we assume $\epsilon$ and $\mu$ are tensors

$$\epsilon = [\epsilon_{ij}], \quad \mu = [\mu_{ij}] \quad i = j = 1, 2, 3$$

(23)
If there are $N$ electromagnetic fields $\zeta_i = [E_i, H_i, J_i, K_i, D_i, B_i]$, $i = 1, 2, \ldots, N$. Assume $\zeta_1$, $\zeta_2$ ... $\zeta_N$ satisfy the above Maxwell equation. Assume the superimposing electromagnetic field is

$$\zeta = \zeta_1 + \zeta_2 \cdots \zeta_N$$  \hfill (24)

There is the relationship,

$$D(t) = \int_{\tau=-\infty}^{\infty} \epsilon(t-\tau)(E_1(\tau) + E_2(\tau) + \cdots + E_N(\tau)) d\tau$$  \hfill (25)

$$B(t) = \int_{\tau=-\infty}^{\infty} \epsilon(t-\tau)(H_1(\tau) + H_2(\tau) + \cdots + H_N(\tau)) d\tau$$  \hfill (26)

### B. The modified Maxwell equation

Where $\zeta = [E, H, J, K, D, B]$. In the space with media (permittivity and permeability) $\epsilon, \mu$, normally $D$ and $B$ should satisfy the above formula. For the above formula the $D(t)$ and $B(t)$ are not linear. $D(t)$ and $B(t)$ are only linear when all fields $\zeta_1 + \zeta_2 \cdots \zeta_N$ has same media, i.e.

$$\epsilon_{i=1,\ldots,N} = \epsilon, \quad \mu_{i=1,\ldots,N} = \mu \quad \hfill (27)$$

The above relationship can be loosen by defining that the relation from $D$ to $E$ and $B$ to $H$ are linear, which is.

$$D(t) = D_1(t) + D_2(t) + \cdots + D_N(t)$$  \hfill (28)

$$B(t) = B_1(t) + B_2(t) + \cdots + B_N(t)$$  \hfill (29)

where

$$D_i(t) = \int_{\tau=-\infty}^{\infty} \epsilon_i(t-\tau) E_i(\tau) d\tau \quad i = 1, \cdots, N$$  \hfill (30)

$$B_i(t) = \int_{\tau=-\infty}^{\infty} \mu_i(t-\tau) E_i(\tau) d\tau \quad i = 1, \cdots, N$$  \hfill (31)
Hence there is

\[ D(t) = \int_{-\infty}^{\infty} (\epsilon_1(t - \tau)E_1(\tau) + \epsilon_2(t - \tau)E_2(\tau) + \cdots + \epsilon_N(t - \tau)E_N(\tau))d\tau \quad (32) \]

\[ B(t) = \int_{-\infty}^{\infty} (\mu_1(t - \tau)H_1(\tau) + \mu_2(t - \tau)H_2(\tau) + \cdots + \mu_N(t - \tau)H_N(\tau))d\tau \quad (33) \]

This modification can also be found in reference[14]. It can be proven that if \( \zeta_1, \zeta_2, \cdots, \zeta_N \) satisfy the Maxwell equation, with different media Eq(30,31), the superimposing electromagnetic field \( \zeta = \zeta_1 + \zeta_2 + \cdots + \zeta_N \) will also satisfies Maxwell equation Eq.(19,20) with the above loosen relation Eq.(32,33). In the following we will combine the media equation to Maxwell equation. In case the field \( \zeta \) satisfies the Maxwell equation with the media Eq.(25,26), it is will be referred satisfying the Maxwell equation. In case the field satisfies the Maxwell equation with the media condition Eq.(32,33) it is referred as the modified Maxwell equation. The concept of “modified” is borrowed from the modified reciprocity theorem[15, 16].

IV. THE TRANSFORM OF ELECTROMAGNETIC FIELD

A. Time reverse transform

Assume \( r \) is time reversed transform[33], \( \zeta = [E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] \) is electromagnetic system and satisfies Maxwell Equation. \( [E_r(t), H_r(t), J_r(t), K_r(t)] \) is the transformed electromagnetic system. \( \zeta_r = r\zeta \), or

\[ [E_r(t), H_r(t), J_r(t), K_r(t), D_r(t), B_r(t)] \equiv r[E(t), H(t), J(t), K(t), D(t), B(t)] \]

\[ = [E(-t), H(-t), J(-t), K(-t), D(-t), B(-t)] \]

(34)

It can be proved that if the electromagnetic field \( \zeta \) satisfied Maxwell equation, the time-reversed electromagnetic field \( \zeta_r \) is also satisfies the the time reversed Maxwell equation:

\[ \nabla \times H = J - \partial D \quad (35) \]

\[ \nabla \times E = -K + \partial B \quad (36) \]
Proof: If the electromagnetic field $\zeta$ is normal field (satisfies Maxwell equation), If $\zeta$ is time reversed field then $r\zeta$ is normal field. $\zeta_r = r\zeta$ will be time-reversed satisfies time-reversed Maxwell equation. There is $\zeta = r\zeta_r$ or

$$[E(t), H(t), J(t), K(t), D(t), B(t)] \equiv r[E_r(t), H_r(t), J_r(t), K_r(t), D_r(t), B_r(t)]$$

$$= [E_r(-t), H_r(-t), J_r(-t), K_r(-t), D_r(-t), B_r(-t)] \quad (37)$$

Substituting this to the Maxwell equation, there is

$$\nabla \times H_r(-t) = J_r(-t) + \partial_t D_r(-t) \quad (38)$$

$$\nabla \times E_r(-t) = -K_r(-t) - \partial_t B_r(-t) \quad (39)$$

Assume $-t = \tau$, $\partial_t = -\partial_\tau$

$$\nabla \times H_r(\tau) = J_r(\tau) - \partial_\tau D_r(\tau) \quad (40)$$

$$\nabla \times E_r(\tau) = -K_r(\tau) + \partial_\tau B_r(\tau) \quad (41)$$

Hence $\zeta_r$ satisfies the time reversed Maxwell equation. Proof finish.

The above time reversed transform has been applied to produce a few reciprocity theorems[10][17]. It is worth to say, even the time-reverse transformed field does not satisfy the Maxwell equation, but if we put the minus sign insider the media, i.e., define

$$\epsilon_r(-t) = -\epsilon(-t), \quad \mu_r(-t) = -\mu(-t) \quad (42)$$

or

$$[E_r(t), H_r(t), J_r(t), K_r(t), \epsilon_r(t), \mu_r(t)] \equiv r[E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)]$$

$$= [E(-t), H(-t), J(-t), K(-t), -\epsilon(-t), -\mu(-t)] \quad (43)$$

$$\quad \equiv \quad (44)$$

We can prove that $\zeta_r = [E_r(t), H_r(t), J_r(t), K_r(t), \epsilon_r(t), \mu_r(t)]$ satisfies the Maxwell equation as following

$$\nabla \times H_r(\tau) = J_r(\tau) + \partial_\tau (\epsilon_r(t) * E_r(\tau)) \quad (45)$$

$$\nabla \times E_r(\tau) = -K_r(\tau) - \partial_\tau (\mu_r(t) * H_r(\tau)) \quad (46)$$

Hence whether or not the time-reverse transformed field satisfies Maxwell equation, is depending how the media after the transform is defined. In the following only the Eq(44) will be referred as time reversed transform. A electromagnetic field after time-reverse transform is still magnetic field.
B. Magnetic mirror transform

Assume $h$ is magnetic mirror transform, $\zeta = [E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)]$ is electromagnetic field and satisfies Maxwell Equation. $[E_h, H_h, J_h, K_h, \epsilon_h, \mu_h]$ is the transformed electromagnetic system. $\zeta_h = h\zeta$, or

$$[E_h(t), H_h(t), J_h(t), K_h(t), \epsilon_h(t), \mu_h(t)] = h[E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)]$$

$$= [E(-t), -H(-t), -J(-t), K(-t), \epsilon(-t), \mu(-t)] \quad (47)$$

$\zeta_h = h\zeta$, it can be easily proven that $\zeta = h\zeta_h$, i.e.,

$$[E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] = h[E_h(t), H_h(t), J_h(t), K_h(t), \epsilon_h(t), \mu_h(t)]$$

$$= [E_h(-t), -H_h(-t), -J_h(-t), K_h(-t), \epsilon_h(-t), \mu_h(-t)] \quad (48)$$

It can be proved that if the electromagnetic field $\zeta$ satisfied Maxwell equation, the magnetic mirror transformed field $\zeta_h$ also satisfies the Maxwell equation.

Proof: Substitute Eq. (48) to the Maxwell equation Eq. (19-20)

$$\nabla \times (-1)H_h(-t) = (-1)J_h(-t) + \partial D_h(-t) \quad (49)$$

$$\nabla \times E_h(-t) = -K_h - \partial(-1)B(-t) \quad (50)$$

or

$$\nabla \times (-1)H_h(-t) = (-1)J_h(-t) + (-1)\partial_{-t}D_h(-t) \quad (51)$$

$$\nabla \times E_h(-t) = -K_h - (-1)\partial_{-t}(-1)B(-t) \quad (52)$$

or

$$\nabla \times H_h(-t) = J_h(-t) + \partial_{-t}D_h(-t) \quad (53)$$

$$\nabla \times E_h(-t) = -K_h - \partial_{-t}B(-t) \quad (54)$$

substitute $\tau = -t$, there is

$$\nabla \times H_h(\tau) = J_h(\tau) + \partial_{\tau}D_h(\tau) \quad (55)$$
\[ \nabla \times E_h(\tau) = -K_h - \partial_\tau B_h(\tau) \] (56)

Hence \( \zeta_h \) satisfied the Maxwell equation. Proof finish.

Hence an electromagnetic fields after the magnetic mirror transform, it is stall electromagnetic field satisfying Maxwell equation.

\[ \zeta \in \mathbb{C} \]

C. Electric mirror transform

Assume \( e \) is electric reversed transform, \( \zeta = [E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] \) is electromagnetic system and satisfies Maxwell Equation. \( [E_e, H_e, J_e, K_e, \epsilon_e, \mu_e] \) is the transformed electromagnetic system. \( \zeta_e = e\zeta \), or

\[ [E_e(t), H_e(t), J_e(t), K_e(t), \epsilon_e(t), \mu_e(t)] \equiv e[E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] \]

\[ = [-E(-t), H(-t), J(-t), -K(-t), \epsilon(-t), \mu(-t)] \] (57)

It can be proven that \( \zeta_e = e\zeta \) satisfies Maxwell equation.

Proof: \( \zeta = e\zeta_e \), substitute this to Maxwell equation Eq(19,20). There is

\[ \nabla \times H_e(-t) = J_e(-t) + \partial(-1)D_e(-t) \] (58)

\[ \nabla \times (-1)E_e(-t) = -(1)K_e - \partial B_e(-t) \] (59)

or considering \( \partial_t = (-1)\partial_{-t} \)

\[ \nabla \times H_e(-t) = J_e(-t) + (-1)\partial_{-t}(-1)D_e(-t) \] (60)

\[ \nabla \times (-1)E_e(-t) = -(1)K_e - (-1)\partial_{-t}B_e(-t) \] (61)

or

\[ \nabla \times H_e(-t) = J_e(-t) + \partial_{-t}D_e(-t) \] (62)

\[ \nabla \times E_e(-t) = -K_e - \partial_{-t}B(-t) \] (63)

or considering \( -t = \tau \)

\[ \nabla \times H_e(\tau) = J_e(\tau) + \partial_\tau D_e(\tau) \] (64)
\[ \nabla \times E_e(\tau) = -K_e - \partial_\tau B_e(\tau) \]  

(65)

Hence \( \zeta_e \) is also satisfies the Maxwell equation too. Proof finish.

D. Conjugate transform corresponding to time-reversed transform

Considering a real function \( f(t) \),

\[ f(t) = F^{-1}\{F(\omega)\} \equiv \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t) \, dt \]

Hence

\[ f(-t) = \int_{-\infty}^{\infty} F(\omega) \exp(-j\omega t) \, dt \]

\[ = \{ \int_{-\infty}^{\infty} F^*(\omega) \exp(j\omega t) \, dt \}^* \]

Considering \( f(-t) \) is a real function

\[ f(-t) = f^*(-t) \]

\[ = \int_{-\infty}^{\infty} F^*(\omega) \exp(j\omega t) \, dt \]

\[ = F^{-1}\{F^*(\omega)\} \]

In the time-reversed transform considering \( f(-t) \to F^*(\omega) \), we can obtained the corresponding time-reversed transform in Fourier domain.

Assume \( r \) is time reversed transform [33], \( \zeta = [E(\omega), H(\omega), J(\omega), K(\omega), \epsilon(\omega), \mu(\omega)] \) is electromagnetic system and satisfies Maxwell Equation. \( [E_r, H_r, J_r, K_r, \epsilon_r, \mu_r] \) is the transformed electromagnetic system. \( \zeta_r = r\zeta \), or

\[ [E_r(\omega), H_r(\omega), J_r(\omega), K_r(\omega), \epsilon_r(\omega), \mu_r(\omega)] \equiv r[E(\omega), H(\omega), J(\omega), K(\omega), \epsilon(\omega), \mu(\omega)] \]

\[ = [E^*(\omega), H^*(\omega), J^*(\omega), K^*(\omega), -\epsilon^*(\omega), -\mu^*(\omega)] \]  

(66)

or

\[ \zeta_r \equiv r\zeta \]
There is

\[ \zeta = r \zeta_r \]

or

\[ [E(\omega), H(\omega), J(\omega), K(\omega), \epsilon(\omega), \mu(\omega)] \equiv r[E_r(\omega), H_r(\omega), J_r(\omega), K_r(\omega), \epsilon_r(\omega), \mu_r(\omega)] \]

\[ = [E^*_r(\omega), H^*_r(\omega), J^*_r(\omega), K^*_r(\omega), -\epsilon^*_r(\omega), -\mu^*_r(\omega)] \] (67)

Assume \( \zeta = [E(\omega), H(\omega), J(\omega), D(\omega), B(\omega)] \) satisfies Maxwell Equation Eq.(Eq(19,20).).

\[ \nabla \times H^*_r = J^* + (j\omega)(-\epsilon^*_r)E^*_r \] (68)

\[ \nabla \times E^*_r = -K^*_r - (j\omega)(-\mu^*_r)H^*_r \] (69)

or

\[ \nabla \times H_r = J_r + j\omega \epsilon_r E_r \] (70)

\[ \nabla \times E_r = -K_r - j\omega \mu_r H_r \] (71)

Hence \([E_r, H_r]\) is the solution of Maxwell equation with the current \([J_r, K_r]\) and media \([\epsilon_r, \mu_r]\)

E. The conjugate transform corresponding magnetic mirror transform

Considering \( F\{f(-t)\} = f^*(\omega) \) which \( F\{\bullet\} \) is Fourier transform, tn Fourier domain (or complex space), the magnetic mirrored transform become conjugate transform. Assume \( h \) is magnetic mirrored transform, \( \zeta = [E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] \) is electromagnetic system and satisfies Maxwell Equation. \( \zeta_h(t) = [E_h(t), H_h(t), J_h(t), K_h(t), \epsilon_h(t), \mu_h(t)] \) is the mirror transformed electromagnetic field, The corresponding conjugate transform is defined as following \( \zeta_h(\omega) = h\zeta(\omega), \) or

\[ [E_h(\omega), H_h(\omega), J_h(\omega), K_h(\omega), \epsilon_h(\omega), \mu_h(\omega)] \equiv h[E(\omega), H(\omega), J(\omega), K(\omega), \epsilon(\omega), \mu(\omega)] \]

\[ = [E^*(\omega), -H^*(\omega), -J^*(\omega), K^*(\omega), \epsilon^*(\omega), \mu^*(\omega)] \] (72)

Since \( \zeta_h = h\zeta \), it can be easily proven that there is \( \zeta = h\zeta_h, \) or
\[ [E(\omega), H(\omega), J(\omega), K(\omega), \epsilon(\omega), \mu(\omega)] = h[E_h(\omega), H_h(\omega), J_h(\omega), K_h(\omega), \epsilon_h(\omega), \mu_h(\omega)] \]
\[ = [E^*_h(\omega), -H^*_h(\omega), -J^*_h(\omega), K^*_h(\omega), \epsilon^*_h(\omega), \mu^*_h(\omega)] \quad (73) \]

If \( \zeta(\omega) \) is electromagnetic field, after conjugate transform, \( h\zeta \) is also electromagnetic field.

**F. The difference between the replacement and the substitution of a transform**

Assume there is a formula \( f(\ldots, \zeta) \) which contents the field \( \zeta = [J, K, E, H, \epsilon, \mu] \).

\[ f(\ldots, \zeta) = f(\ldots, [J, K, E, H, \epsilon, \mu]) \quad (74) \]

Here \( f(\ldots, \zeta) \) can be any known electromagnetic formula derived from Maxwell equation for example the reciprocity theorem.

There is difference between the replacement and substitution of a transform. Assume here \( [J, K] \) are the source. Assume \( \zeta_h \) is the magnetic mirror transformed filed, so that \( \zeta_h = [J_h, K_h, E_h, H_h, \epsilon_h, \mu_h] = h\zeta \). And hence there is \( \zeta = h\zeta_h \). \( h \) is mirror transform Hence \( \zeta = h\zeta_h = [-J_h(-t), K_h(-t), -H_h(-t), E(-t), \epsilon_h(-t), \mu_h(-t)] \). we can substitute \( \zeta = h\zeta_h \) to above formula, which is

\[ f(\ldots, \zeta) = f(\ldots, [-J_h(-t), K_h(-t), E_h(-t), -H_h(-t), \epsilon_h(-t), \mu_h(-t)]) \quad (75) \]

In this situation the formula does not change. However if we replace the \( \zeta \) using \( \zeta_h \), which is

\[ f(\ldots, \zeta_h) = f(\ldots, [J_h, K_h, E_h, H_h, \epsilon_h, \mu_h]) \quad (76) \]

The formula is changed. Since clearly that \( \zeta \) is not \( \zeta_h \). Substituting \( \zeta_h = h\zeta \) to the above formula, there is

\[ f(\ldots, \zeta_h) = f(\ldots, [-J(-t), K(-t), E(-t), -H(-t), \epsilon(-t), \mu(-t)]) \quad (77) \]

is a different formula compare to,

\[ f(\ldots, \zeta) = f(\ldots, [J, K, E, H, \epsilon, \mu]) = \]

21
Hence if \( f(\ldots, \zeta) = 0 \) we can not guarantee that \( f(\ldots, \zeta_h) = 0 \).

Substitution will not change the original formula, but the replacement will change the original formula. Using replacement actually derive a new formula which is the dual of the original formula. It must be very careful to the replacement and substitution of the transform which are two different manipulations.

G. Time offset transform

Assume \( T \) is time offset transform \( \zeta = [E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] \) is electromagnetic system and satisfies Maxwell Equation. \( \zeta_T = [E_T, H_T, J_T, K_T, \epsilon_T, \mu_T] \) is the transformed electromagnetic system.

\[
\zeta_T \equiv [E_T(t), H_T(t), J_T(t), K_T(t), \epsilon_T(t), \mu_T(t)]
\]

\[
\equiv T[E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)]
\]

\[
= [E(t + T), H(t + T), J(t + T), K(t + T), \epsilon(t + T), \mu(t + T)]
\]

(79)

It can be proved that if the electromagnetic field \( \zeta \) satisfies the Maxwell equation, then the time-offset electromagnetic field \( \zeta_T \) is also satisfies the Maxwell equation.

H. Space offset transform

Assume \( T \) is time offset transform \( \zeta = [E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)] \) is electromagnetic system and satisfies Maxwell Equation. \( \zeta_X = [E_X, H_X, J_X, K_X, \epsilon_X, \mu_X] \) is the transformed electromagnetic system. Where \( X \) is stand for space transform. \( X \) is also a value of space offset \( X = [X_1, X_2, X_3] \). Assume \( x = [x_1, x_2, x_3] \).

\[
\zeta_X \equiv [E_X(t, x), H_X(t, x), J_X(t, x), K_X(t, x), \epsilon_X(t, x), \mu_X(t, x)]
\]

\[
\equiv X[E(t, x), H(t, x), J(t, x), K(t, x), \epsilon(t, x), \mu(t, x)]
\]

\[
= [E(t, x + X), H(t, x + X), J(t, x + X), K(t, x + X), \epsilon(t, x + X), \mu(t, x + X)]
\]

(80)
It can be proved that if the electromagnetic field $\zeta$ satisfies the Maxwell equation, then the space-offset electromagnetic field $\zeta_X$ is also satisfies the Maxwell equation.

The idea of time offset and space offset has been used in reference [17].

I. Transform by swapping electric field and magnetic field

Assume $s$ is a swap transform which can swap electric field and magnetic field, $\zeta = [E, H, J, K, \epsilon, \mu]$

$$\zeta_s = s\zeta = [aH, bE, cK, dJ, e\mu, f\epsilon]$$

and this transform should be able to reverse, that means

$$\zeta = s\zeta_s$$

$$\zeta = s\zeta_s = [aH_s, bE_s, cK_s, dJ_s, e\mu_s, f\epsilon_s]$$

Where $a, b, c, d, e, f$ are constant to be found. Assume $\zeta = [E, H, J, K, \epsilon, \mu]$ satisfy Maxwell equation

$$\nabla \times H = J + j\omega \epsilon E$$

$$\nabla \times E = -K - j\omega \mu H$$

We can substitute $\zeta$ to the above Maxwell equation, $\zeta_s = [E_s, H_s, J_s, K_s, \epsilon_s, \mu_s]$ should also satisfy the Maxwell equation

$$\nabla \times H_s = J_s + j\omega \epsilon_s E_s$$

$$\nabla \times E_s = -K_s - j\omega \mu_s H_s$$

After substituting we have

$$\nabla \times (bE_s) = cK_s + j\omega e\mu_s aH_s$$

$$\nabla \times (aH_s) = -dJ_s - j\omega f \epsilon bE_s$$

or

$$\nabla \times E_s = \frac{c}{b} K_s + \frac{ea}{b} j\omega \mu_s H_s$$
\[ \nabla \times H_s = \frac{d}{a} J_s - \frac{fb}{a} j\omega \epsilon_s E_s \]

We found that the constant should following

\[
\frac{c}{b} = -1 \\
 \frac{-d}{a} = 1 \\
 \frac{ea}{b} = -1 \\
 \frac{-fb}{a} = 1
\]

Or

\[
c = -b \\
d = -a \\
e = a \\
f = b
\]

What about \(a\) and \(b\)? Since this transform is not only satisfy the Maxwell equation, there has unit dimension problem. After the transform it should has same unit dimension.

\[ E_s = aH \]
\[ H = bE_s \]
\[ E_s = abE_s \]

or

\[ ab = 1 \]

Now if \(a\) is known all other constant can be found. \(a\) actually can be any value, however considering

\[ \epsilon_0 E^2, \quad \mu_0 H^2 \]

have the same unit dimension of energy, hence the following has the same unit dimension. Here \(\epsilon_0\) and \(\mu_0\) is values in empty space.

\[ \sqrt{\epsilon_0 E}, \quad \sqrt{\mu_0 H} \]
or have same unit dimension.

\[ E, \quad ZH \]

where

\[ Z = \sqrt{\frac{\mu_0}{\epsilon_0}} \]

Hence we can just take

\[ a = Z \]

From above we can find if we take

\[ b = \frac{1}{a} = \frac{1}{Z} \]

Further we have

\[ c = -\frac{1}{Z} \]
\[ d = -Z \]
\[ e = -\frac{1}{Z^2} \]
\[ f = -Z^2 \]

or

\[ \zeta_s = s\zeta = [ZH, \frac{1}{Z} E, -\frac{1}{Z} K, -ZJ, -\frac{1}{Z^2} \mu, -Z^2 \epsilon] \]

substitute \( \zeta_s \) to

\[ \nabla \times H_s = J_s + j\omega \epsilon_s E_s \]
\[ \nabla \times E_s = -K_s - j\omega \mu_s H_s \]

we have

\[ \nabla \times \left( \frac{1}{Z} E \right) = \left( -\frac{1}{Z} K \right) + j\omega \left( -\frac{1}{Z^2} \mu \right) (ZH) \]
\[ \nabla \times (ZH) = -(-ZJ) - j\omega (-Z^2 \epsilon) \left( \frac{1}{Z} E \right) \]

or

\[ \nabla \times E = -K - j\omega \mu H \]
\[ \nabla \times H = J + j\omega \epsilon E \]

That is the Maxwell equation.
V. ADVANCED POTENTIAL AND RETARDED POTENTIAL

A. Mirrored transform for $A(t), \phi(t), \varrho(t)$

Assume $A(t), \phi(t)$ are vector potential and scale potential and is defined as following

$$E = -\nabla \phi - \partial A$$

(81)

$$B = \nabla \times A$$

(82)

Assume the magnetic field $\zeta = [E(t), H(t), J(t), K(t), \epsilon(t), \mu(t)]$. Assume the magnetic current is assumed as $K = 0$. Assume $\varrho$ is electric charge distribution which is related current distribution through continue equation

$$\nabla \cdot J + \partial \varrho = 0$$

(83)

Hence, $J$ is the only source of the potential $[A(t), \phi(t)]$. Now let us find out the magnetic mirror transformed potential and electric charge distribution $[A(t), \phi(t), \varrho(t)]$, i.e. $[A_h(t), \phi_h(t), \varrho_h(t)] = h[A(t), \phi(t), \varrho(t)]$. Here $h$ is magnetic mirror transform.

Assume the magnetic mirror transform for $\phi(t)$ and $\varrho(t)$ is same as to $E(t)$, The mirror transform of $A(t)$ is same as $H$ or $B$.

$$[A_h(t), \phi_h(t), J_h(t), K_h(t), \varrho_h(t), t, \partial_t] \equiv h[A(t), \phi(t), J(t), K(t), \varrho(t)]$$

$$= [-A(-t), \phi(-t), -J(-t), K(-t), \varrho(-t)]$$

(84)

It can be proven that the calculated $E_h(t) = -\nabla \phi_h - \partial A_h$, $H_h(t) = \nabla \times A_h$ satisfies the magnetic mirrored transform Eq.(87). Here $h$ is magnetic mirror transform. We need to prove that

1) $$E_h(t) = E(-t)$$

(85)

2) $$H_h(t) = -H(-t)$$

(86)

3) $$\nabla \cdot J_h + \partial \varrho_h = 0$$

(87)
Hence, guarantees the magnetic transformed field \([E_h(t), H_h(t)]\) satisfies the Maxwell equation Eq.(19,20). It also guarantees the current continue function still satisfies.

Proof: After the magnetic mirror transform, the time \(t\) is changed to \(-t\) and the \(\partial_t\) change to \(-\partial_{-t} = -\partial_\tau\), here \(-\tau = -t\) that is For example

1) 
\[
E_h(t) = -\nabla \phi_h - \partial_t A_h = -\nabla \phi(-t) - \partial_t(-A(-t)) = -\nabla \phi(-t) - \partial_{-t}(A(-t)) = (\nabla \phi(\tau) - \partial_\tau A(\tau))|_{\tau=-t} = E(\tau)|_{\tau=-t} = E(-t)
\]

2) 
\[
B_h = \nabla \times A_h = \nabla \times ((-1)A(-t)) = -\nabla \times A(-t) = -B(-t)
\]

Hence 
\[
B_h = -B(-t) = -(\mu(\tau) * H(\tau))|_{\tau=-t} = -\mu(-t) * H(-t) = \mu_h * (-H(-t))
\]

considering 
\[
B_h = \mu_h * H_h
\]

Hence we have 
\[
H_h = -H(-t)
\]

3) 
\[
\nabla \cdot J_h + \partial_\tau g_h = \nabla \cdot (-J(-t)) + \partial_{-t} g(-t) = \nabla \cdot (-J(-t)) - \partial_{-t} g(-t) = -(\nabla \cdot J(-t) + \partial_{-t} g(-t)) = -(\nabla \cdot J(\tau) + \partial_\tau g(\tau)) = 0
\]

In the above proof \(\tau = -t\) has been used. Proof finish.
B. Advanced potential and retarded potential

In empty space the retarded potential is widely accept which are following,

\[ E^r = -\nabla \phi^r - \partial A^r \]  \hspace{1cm} (94)

\[ B^r = \nabla \times A^r \]  \hspace{1cm} (95)

\[ \phi^r(x, t) = \frac{1}{4\pi \epsilon_0} \int_V \frac{\phi(x', t^r)}{|x - x'|} dV \]  \hspace{1cm} (96)

\[ A^r(x, t) = \frac{\mu_0}{4\pi} \int_V \frac{J(x', t^r)}{|x - x'|} dV \]  \hspace{1cm} (97)

\[ t^r = t - \frac{|x - x'|}{c} \]  \hspace{1cm} (98)

Where \( c \) is the speed of light wave in empty space. Here \( K = 0 \) is also assumed.

There is corresponding advanced potential, where

\[ E^a = -\nabla \phi^a - \partial A^a \]  \hspace{1cm} (99)

\[ B^a = \nabla \times A^a \]  \hspace{1cm} (100)

\[ \phi^a(x, t) = \frac{1}{4\pi \epsilon_0} \int_V \frac{\phi(x', t^a)}{|x - x'|} dV \]  \hspace{1cm} (101)

\[ A^a(x, t) = \frac{\mu_0}{4\pi} \int_V \frac{J(x', t^a)}{|x - x'|} dV \]  \hspace{1cm} (102)

\[ t^a = t + \frac{|x - x'|}{c} \]  \hspace{1cm} (103)

The above electromagnetic field \( \zeta^r = [E^r, H^r, J^r, K^r, \epsilon^r, \mu^r] \) is the corresponding field of the retarded potential. \( \zeta^a = [E^a, H^a, J^a, K^a, \epsilon^a, \mu^a] \) is the corresponding field of the advanced potential. In the above we can assume that \( J^r = J^a = J, \epsilon^r = \epsilon^a = \epsilon, \mu^r = \mu^a = \mu \). That means for the electric current \( J \) and media \( \epsilon, \mu \), the superscript \( r \) and \( a \) can be dropped.

If the electric current or magnetic current \( \rho = [J, K] \) sending electromagnetic wave out, \( \rho^r = [J^r, K^r] \) is the source. If the electric current or magnetic current \( \rho^a = [J^a, K^a] \) receiving electromagnetic wave, \( \rho^a = [J^a, K^a] \) is the sink. The retarded potential is the electromagnetic field \( \xi^r = [E^r, H^r] \) transmitting from the source \( \rho^r = [J^r, K^r] \). The advanced potential
is the electromagnetic field $\xi = [E^a, H^a]$ received by the sink $\rho^a = [J^a, K^a]$. Advanced potential and retarded potential are all normal electromagnetic field which satisfies the Maxwell equation. In next subsection we will shown that the field of advanced potential is magnetic mirror transformed transformed field. It is not the time reversed field.

Here the mirror transformed field and time reversed field are different. Mirror transformed fields still satisfy the Maxwell equation and hence is a normal electromagnetic field. But the time reversed field does not satisfy the Maxwell equation, it satisfies the time-reversed Maxwell equation Eq.(40 41) which is very close to Maxwell equation (It is noticed if the minus sign is put to the media, the field after the time-reversed transform is still satisfies the Maxwell equation). Time reversed transform can be apply to derive some reciprocity theorems[10][17].

C. Obtain advanced potential from mirrored transform

The retarded potential is widely accept. Advanced potential is not widely accept. However magnetic mirror transformed field is accept widely, since mirror transformed field satisfies the Maxwell equation and also easy to explained as the reflect field on a magnetic super conductor mirror. In this subsection we will derive the advanced potential from magnetic mirror transform and retarded potential.

Assume

$$\zeta_1(t) = [E_1(t), H_1(t), J_1(t), K_1(t), \epsilon_1(t), \mu_1(t)]$$

$$\zeta_2(t) = [E_2(t), H_2(t), J_2(t), K_2(t), \epsilon_2(t), \mu_2(t)]$$

are both retarded potentials. Assume there is only electron current hence $K_1 = 0$ and $K_2 = 0$. Assume

$$[J_2(t), K_2(t), \epsilon_2(t), \mu_2(t), \varrho_2(t)] = [-J_1(-t), K_1(-t), \epsilon_1(-t), \mu_1(-t), \varrho_1(-t)]$$

corresponding to the source $\rho_2$ is the retarded potential, and there is

$$\phi_2(x, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\varrho_2(x', t - \frac{|x-x'|}{c})}{|x-x'|} dV$$

$$A_2(x, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{J_2(x', t - \frac{|x-x'|}{c})}{|x-x'|} dV$$

29
and the corresponding fields are

\[ E_2 = -\nabla \phi_2 - \partial A_2 \]  

\[ B_2 = \nabla \times A_2 \]

Assume

\[ \zeta_3 = h \zeta_2 \]

\( \zeta_3 \) is magnetic mirrored field of the retarded potential \( \zeta_2 \). Considering

\[ A_3(x, t) = h A_2(x, t) = -A_2(x, -t) \]  

\[ \phi_3(x, t) = h \phi_2(x, t) = \phi_2(x, -t) \]

Considering Eq. (107), there is

\[ A_3(x, t) = (-) \frac{1}{4\pi \epsilon_0} \int_V \frac{J_2(x', -t - \frac{|x-x'|}{c})}{|x-x'|} dV \]  

\[ \phi_3(x, t) = \frac{1}{4\pi \epsilon_0} \int_V \frac{\rho_2(x', -t - \frac{|x-x'|}{c})}{|x-x'|} dV \]

Considering Eq. (106), there is

\[ A_3(x, t) = \frac{1}{4\pi \epsilon_0} \int_V \frac{J_1(x', t + \frac{|x-x'|}{c})}{|x-x'|} dV \]  

\[ \phi_3(x, t) = \frac{1}{4\pi \epsilon_0} \int_V \frac{\rho_1(x', t + \frac{|x-x'|}{c})}{|x-x'|} dV \]

The field can be obtained

\[ E_3 = -\nabla \phi_3 - \partial A_3 \]  

\[ B_3 = \nabla \times A_3 \]

This way It is proven that the magnetic mirror transformed field

\[ \zeta_3(t) = [E_3(t), H_3(t), J_3(t), K_3(t), \epsilon_3(t), \mu_3(t)] \]

from a field of the retarded potential

\[ \zeta_2(t) = [E_2(t), H_2(t), J_2(t), K_2(t), \epsilon_2(t), \mu_2(t)] \]
is just the field of the advanced potential of the sink \( \rho_1 = [J_1(t), 0] \).

The field of the advanced potential is obtained by using a magnetic mirror transform. Hence the advanced potential should be acceptable same as the field obtained form a magnetic mirror transform. In the following the transmitting field and retarded potential are field that send out from the source \( \rho = [J, K] \). Receiving potential and advanced potential are field receiving by the sink \( \rho = [J, K] \). Electric and magnetic current can receive and transmit the field or do both in the same time, i.e. \( \rho = [J, K] \) can be sink or source.

VI. THE MODIFIED POYNTING THEOREM

A. The superimposing electromagnetic field

The superimposing electromagnetic field is considered which contains the following electromagnetic field compounds,

1. retarded potential which send from the source.
2. advanced potential which is received by the sink.
3. time reversed field of the retarded potential and advanced potential.
4. mirror transformed field of the retarded potential and advanced potential.
5. time-offset field of the above fields.
6. space-offset field of the above fields.
7. transmitting field, which is sent from the antenna.
8. receiving field, which receiving from antenna and reflected by the antenna

The Maxwell equation Eq. (19, 20) satisfies. This is also referred as the modified Poynting theorem. The simple example is shown in the following. Assume \( \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6 \) are all the field of the retarded potential,

\[
\zeta = \zeta_1 + h\zeta_2 + e\zeta_3 + T\zeta_4 + X\zeta_5 + r\zeta_6
\]  

(122)

is superimposed field. Where \( h \) and \( e \) are magnetic and electric mirror transforms. \( h\zeta_2 \) and \( e\zeta_3 \) are receiving fields or advanced potential. \( T \) is time offset transform, \( X \) is space offset transform. \( T\zeta_4 \) is time-offset field, \( X\zeta_5 \) is spatial offset fields. \( r\zeta_6 \) is time reversed transform.
B. The Poynting theorem

The Poynting theorem can be proved as following from Maxwell equation Eq.(19,20),

\[- \nabla \cdot (E \times H) = J \cdot E + K \cdot H + E \cdot \partial D + H \cdot \partial B \tag{123}\]

Where “·” is vector point product. The superimposed field Eq.(122) satisfies the Maxwell equation, if the media satisfies Eq.(25,26), i.e., there is only one media, \([\epsilon, \mu]\) we say that the Poynting theorem establishes.

C. The modified Poynting theorem

The Eq.(123) is also the modified Poynting theorem if we consider the media Eq.(32,33). The derivation of the modified Poynting theorem can be done with Maxwell equation and the modified media Eq.(32,33). The derivation of the modified Poynting theorem is exactly same as derivation Poynting theorem form Maxwell equation. The only different is that the media have been generalized to Eq.(32,33) from Eq.(25,26). If Eq.(25,26) is applied, the word “modified” can be dropped, it become Poynting theorem. The concept of “modified” is borrowed from the modified reciprocity theorem[16]. We extended this idea to the Maxwell equation and also Poynting theorem.

D. The Poynting theorem in Fourier space

Considering

\[f(t) = F^{-1}(f(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega) \exp(j\omega t) \, d\omega\]

\[
\partial f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega) \partial \exp(j\omega t) \, d\omega
\]

\[= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega)(j\omega) \exp(j\omega t) \, d\omega\]

Hence there is

\[F\{\partial f(t)\} = (j\omega)f(\omega)\]
or there is the transform

\[ \partial \rightarrow (j\omega) \]

from time-domain to Frequency domain. And the Maxwell equation in Fourier space becomes,

\[ \nabla \times H = J + (j\omega)D \tag{124} \]
\[ \nabla \times E = -K - (j\omega)B \tag{125} \]

In frequency domain the Poynting theorem Eq.(123) can be written as

\[ -\nabla \cdot (E \times H) = J \cdot E + K \cdot H + E \cdot j\omega D + H \cdot j\omega B \tag{126} \]

E. The complex Poynting theorem

Considering \( \partial_t \rightarrow j\omega \) in the Maxwell equation Eq.(19,20), there is the Maxwell equation in the Fourier domain,

\[ \nabla \times H = J + (j\omega)\varepsilon E \tag{127} \]
\[ -\nabla \times E = K + j\omega\mu H \tag{128} \]

take complex conjugate to Eq.\((127)\), there is

\[ \nabla \times H^* = J^* + (j\omega)^*\varepsilon^* E^* \tag{129} \]

point product a variable \( H^* \) and \( E \) to Eq.(128,129), there are

\[ -H^* \cdot \nabla \times E = H^* \cdot K + j\omega H^* \cdot \mu H \tag{130} \]
\[ E \cdot \nabla \times H^* = E \cdot J^* + (j\omega)^* E \cdot \varepsilon^* E^* \tag{131} \]

Add them together

\[ -H^* \cdot \nabla \times E + E \cdot \nabla \times H^* = H^* \cdot K + j\omega H^* \cdot \mu H + E \cdot J^* + (j\omega)^* E \cdot \varepsilon^* E^* \tag{132} \]
Poynting theorem in Fourier domain can be written as

\[- \nabla \cdot (E \times H^*) = E \cdot J^* + H^* \cdot K + j\omega (H^* \cdot \mu H - E \cdot \epsilon^* E^*)\]  \hspace{1cm} (133)

**VII. FAIL TO DERIVE THE MUTUAL ENERGY THEOREM FROM COMPLEX POYNTING THEOREM**

Our first try is to obtain mutual energy theorem from complex Poynting theorem. This try is failed, instead of obtaining the mutual energy theorem we obtained the mixed mutual energy theorem.

Assume \( \zeta = \zeta_1 + \zeta_2 \), the above formula can be rewritten as

\[- \nabla \cdot ((E_1 + E_2) \times (H_1^* + H_2^*)) \]

\[= (E_1 + E_2) \cdot (J_1^* + J_2^*) + (H_1^* + H_2^*) \cdot (K_1 + K_2) \]

\[+ j\omega ((H_1^* + H_2^*) \cdot (\mu_1 H_1 + \mu_2 H_2) - (E_1 + E_2) \cdot (\epsilon_1^* E_1^* + \epsilon_2^* E_2^*))\]  \hspace{1cm} (134)

take out all self energy compounds, it becomes

\[- \nabla \cdot (E_1 \times H_2^* + E_2 \times H_1^*) \]

\[= E_1 \cdot J_2^* + E_2 \cdot J_1^* + H_1^* \cdot K_2 + H_2^* \cdot K_1 \]

\[+ j\omega (H_1^* \cdot \mu_2 H_2 + H_2^* \cdot \mu_1 H_1 - E_1 \cdot \epsilon_2^* E_2^* - E_2 \cdot \epsilon_1^* E_1^*)\]  \hspace{1cm} (135)

or in integral form,

\[- \int_S (E_1 \times H_2^* + E_2 \times H_1^*) \cdot \hat{n} dS \]

\[= \int_V (E_1 \cdot J_2^* + E_2 \cdot J_1^* + H_1^* \cdot K_2 + H_2^* \cdot K_1) dV \]

\[+ j\omega \int_V (H_1^* \cdot \mu_2 H_2 + H_2^* \cdot \mu_1 H_1 - E_1 \cdot \epsilon_2^* E_2^* - E_2 \cdot \epsilon_1^* E_1^*) dV\]  \hspace{1cm} (136)

This is referred as mixed mutual energy theorem. It is related to the concept of mixed Poynting vector. It is corresponding to Poynting theorem in the complex form. The inverse
Fourier transform of above formula is

\[
-\int_S \int_{t=-\infty}^{\infty} (E_1(t+\tau) \times H_2^*(t) + E_2(t+\tau) \times H_1^*(\tau)) dt \cdot \hat{n} dS
\]

\[
= \int_V \int_{t=-\infty}^{\infty} (E_1(t+\tau) \cdot J_2^*(t) + \epsilon_2^* \cdot E_2^*(t) - E_2(t+\tau) \cdot (\epsilon_1^* \cdot E_1^*(t))) dt dV
\]

\[
+ \partial_\tau \int_V \int_{t=-\infty}^{\infty} (H_1^*(t) \cdot (\mu_2 \ast H_2)(t+\tau) + H_2^*(t) \cdot (\mu_1 \ast H_1)(t+\tau)) dt dV
\]

\[
= \int_S \int_{t=-\infty}^{\infty} (E_1(t+\tau) \times H_2(t) + E_2(t+\tau) \times H_1(\tau)) \cdot \hat{n} dS
\]

\[
= \int_V \int_{t=-\infty}^{\infty} (E_1(t+\tau) \cdot J_2(t) + E_2(t+\tau) \cdot J_1(t) + H_1(t) \cdot K_2(t+\tau) + H_2(t) \cdot K_1(t+\tau)) dV
\]

\[
+ \partial_\tau \int_V \int_{t=-\infty}^{\infty} (H_1(t) \cdot (\mu_2 \ast H_2)(t+\tau) + H_2(t) \cdot (\mu_1 \ast H_1)(t+\tau)) dt dV
\]

\[
- E_1(t+\tau) \cdot (\epsilon_2^* \ast E_2^*) - E_2(t+\tau) \cdot (\epsilon_1^* \ast E_1^*) dt dV
\]

In the above formula \((f \ast g)(t)\) means convolution of the function \(f(t)\) and \(g(t)\). Considering \(E(t), H(t), \epsilon(t), \mu(t)\) are all real variables, the above formula can be rewritten as following,

\[
- \int_S \int_{t=-\infty}^{\infty} (E_1(t+\tau) \times H_2(t) + E_2(t+\tau) \times H_1(\tau)) dt \cdot \hat{n} dS
\]

\[
+ \partial_\tau \int_V \int_{t=-\infty}^{\infty} (H_1(t) \cdot (\mu_2 \ast H_2)(t+\tau) + H_2(t) \cdot (\mu_1 \ast H_1)(t+\tau)) dt dV
\]

\[
- \int_V \int_{t=-\infty}^{\infty} (E_1(t+\tau) \cdot (\epsilon_2 \ast E_2^*) - E_2(t+\tau) \cdot (\epsilon_1 \ast E_1^*)) dt dV
\]

This can be seen as mixed mutual energy theorem in time domain, it also can be referred as mixed time-correlation reciprocity theorem. It can be prove that the the real part of mixed mutual energy theorem Eq.(136) is same as the real part of the mutual energy theorem Eq.(257). Mixed mutual energy theorem is related to the concept of mixed poynting vector [36, 37].

**VIII. DERIVATION OF MUTUAL ENERGY THEOREM BY AVERAGE**

We define the mutual energy of a electromagnetic field system as the difference between the total energy and the self energy. We derive mutual energy from the idea that subtract the self energy from the total energy, the rest is the mutual energy. There is energy conservation theorem which is Poynting theorem which guarantee the total energy and self energy
are conservation. Hence the mutual energy should also be conserved. The mutual energy is conserved is referred as mutual energy theorem. The following gives the detail of the derivation of the mutual energy theorem

A. Spatial-temporal mutual energy theorem

Assume

\[ E = E_1 + E_2 \quad (139) \]
\[ D = D_1 + D_2 \quad (140) \]
\[ H = H_1 + H_2 \quad (141) \]
\[ B = B_1 + B_2 \quad (142) \]

\[ J = J_1 + J_2 \quad (143) \]

Assume \((E_1, H_1)\) is produced from \(J_1\) and assume \((E_2, H_2)\) is produced from \(J_2\). It is clear we have the Poynting theorem as following,

\[ - \nabla \cdot (E_1 \times H_1) = J_1 \cdot E_1 + E_1 \cdot \partial D_1 + H_1 \cdot \partial B_2 \quad (144) \]

\[ - \nabla \cdot (E_2 \times H_2) = J_2 \cdot E_2 + E_2 \cdot \partial D_2 + H \cdot \partial B_2 \quad (145) \]

Substitute Eq.\((139,140,141,142)\) to Eq.\((123)\) and subtract Eq.\((144)\) and Eq.\((145)\) from Eq.\((123)\), we obtain,

\[ - \nabla \cdot (E_1 \times H_2 + E_2 \times H_1) = J_1 \cdot E_2 + J_2 \cdot E_1 + E_1 \cdot \partial D_2 + E_2 \cdot \partial D_1 + H_1 \cdot \partial B_2 + H_2 \cdot \partial B_1 \quad (146) \]

Actually we can call Eq.\((123)\) total part of Poynting theorem. The Eq.\((144)(145)\) is self part of Poynting theorem. Eq.\((146)\) is mutual part of Poynting theorem. The corresponding integral form of the above formula is

\[ - \int_S (E_1 \times H_2 + E_2 \times H_1) \cdot \hat{n} dS = \int_V \int (J_1 \cdot E_2 + J_2 \cdot E_1 + E_1 \cdot \partial D_2 + E_2 \cdot \partial D_1 + H_1 \cdot \partial B_2 + H_2 \cdot \partial B_1) dV \quad (147) \]
Where \( \hat{n} \) is norm vector of the the surface \( S \). The above Eq.(146,147) are referred as spatial-temporal mutual energy theorem in this article. In the derivation of the above Eq.(146,147) we have considered the media,

\[
D_1 = \epsilon E_1 \quad \quad \quad (148) \\
D_2 = \epsilon E_2 \quad \quad \quad (149) \\
B_1 = \mu H_1 \quad \quad \quad (150) \\
B_2 = \mu H_2 \quad \quad \quad (151)
\]

**B. Modified spatial-temporal mutual energy theorem**

If we assume that

\[
D_1 = \epsilon_1 E_1 \quad \quad \quad (152) \\
D_2 = \epsilon_2 E_2 \quad \quad \quad (153) \\
B_1 = \mu_2 H_1 \quad \quad \quad (154) \\
B_2 = \mu_2 H_2 \quad \quad \quad (155)
\]

Eq.(146,147) are referred as modified spatial-temporal mutual energy theorem. Modified spatial-temporal mutual energy theorem can be derived directly from modified Poynting theorem. We can see in the derivation if only consider \( E, H, D, B, J \). The medium \( \epsilon_1, \mu_1 \) and \( \epsilon_2, \mu_2 \) did not appear in the proving process. Hence using the exactly same proving process we can obtain the modified spatial-temporal mutual energy theorem.

It is remarkable that we did not care whether or not the modified spatial-temporal mutual energy theorem is a true physical theorem, we can consider it as a mathematical theorem. In real situation there is only

\[
\epsilon_1 = \epsilon_2 = \epsilon \quad \quad \quad (156) \\
\mu_1 = \mu_2 = \mu \quad \quad \quad (157)
\]

Modified mutual energy theorem can be used to simplify some calculation of electromagnetic fields which will be shown in the following.
C. Mutual energy theorem in complex space

Considering \( f = \text{Re}\{f_0 \exp(j \omega t)\} \) \( g = \text{Re}\{g_0 \exp(j \omega t)\} \),

\[
f g = \text{Re}\{f_0 \exp(j \omega t)\} \text{Re}\{g_0 \exp(j \omega t)\}
\]

\[
= \frac{1}{2}(f_0 \exp(j \omega t) + f_0^*(\exp(j \omega t))^*)
\cdot \frac{1}{2}(g_0 \exp(j \omega t) + g_0^*(\exp(j \omega t))^*)
\]

\[
= \frac{1}{4}(f_0 g_0 \exp(j 2 \omega t) + f_0^* g_0^*(\exp(j 2 \omega t))^* + f_0 g_0^* + f_0^* g_0)
\]

\[
= \frac{1}{2}(\text{Re}\{f_0 g_0 \exp(j 2 \omega t)\} + \text{Re}\{f_0 g_0^*\}) \quad (158)
\]

Considering

\[
< f_0 g_0 \exp(j 2 \omega t) > = 0 \quad (159)
\]

Where \(< \bullet >\) means average of some variable with time. Hence we have

\[
< f, g > = \frac{1}{2}\text{Re}\{f_0 g_0^*\}
\]

\[
= \frac{1}{2}\text{Re}\{f_0 \exp(j \omega t) g_0^*(\exp(j \omega t))^*\}
\]

\[
= \frac{1}{2}\text{Re}\{f g^*\} \quad (160)
\]

Hence we have the following transform after the average if do not consider the constant \( \frac{1}{2} \),

\[
f g \implies f g^* \quad (161)
\]

or \( \implies \) means take an average

\[
f g \implies f^* g \quad (162)
\]

According to this, we have

\[
E_1 \times H_2 \implies E_1 \times H_2^* \quad (163)
\]

\[
E_2 \times H_1 \implies E_2^* \times H_1 \quad (164)
\]

\[
J_1 \times E_2 \implies J_1 \times E_2^* \quad (165)
\]
\[ J_2 \times E_1 \Rightarrow J_2^* \times E_1 \] (166)

In the above substitution we always put the * to the all variable with subscript 2. considering,

\[
\partial f = \partial \text{Re}\{f_0 \exp(j\omega t)\} \\
= \partial \frac{1}{2}(f_0 \exp(j\omega t) + f_0^*(\exp(j\omega t))^*) \\
= \frac{1}{2}(j\omega f_0 \exp(j\omega t) + f_0^*(-j\omega)(\exp(j\omega t))^*) \\
= \frac{1}{2} \text{Re}\{(j\omega f_0 \exp(j\omega t)\} \\
= \frac{1}{2} \text{Re}\{(j\omega f\} \\
\] (167)

Hence we have the following transform

\[ \partial \Rightarrow j\omega \] (168)

We obtain,

\[
E_1 \cdot \partial D_2 \Rightarrow E_1 \cdot (\partial \epsilon E_2)^* = E_1 \cdot (j\omega \epsilon E_2)^* = (j\omega)^* E_1 \cdot \epsilon^* E_2^* = -j\omega E_1 \cdot \epsilon^* E_2^* \] (169)

and

\[
E_2 \cdot \partial D_1 \Rightarrow E_2^* \cdot (j\omega \epsilon E_1) = j\omega E_2^* \cdot \epsilon E_1 \] (170)

similarly we have

\[
H_1 \cdot \partial B_2 \Rightarrow -j\omega H_1 \cdot \mu^* H_2^* \] (171)

\[
H_2 \cdot \partial B_1 \Rightarrow j\omega H_2^* \cdot \mu B_1 \] (172)

Substitute the above formula to Eq.(146) we have

\[
-\nabla \cdot (E_1 \times H_2^* + E_2^* \times H_1) = \\
J_1 \cdot E_2^* + J_2^* \cdot E_1 - j\omega E_1 \cdot \epsilon^* E_2^* + j\omega E_2^* \cdot \epsilon E_1 - j\omega H_1 \cdot \mu^* H_2^* + j\omega H_2^* \cdot \mu B_1 \] (173)

Considering

\[
a \cdot Cb = a_i C_{ij} b_j = C_{ij} a_i b_j = C_{ji}^T a_i \cdot b_j = b \cdot C^T a \] (174)

Hence we have

\[
E_1 \cdot \epsilon^* E_2^* = E_2^* \cdot (\epsilon^*)^T E_1 = E_2^* \cdot \epsilon^\dagger E_1 \] (175)
Where $A^\dagger = (A^*)^T$. $\dagger$ means matrix conjugate. $T$ means matrix transpose. Hence we have

$$E_1\epsilon^* E_2^* - E_2^* \epsilon E_1 = E_2^* \epsilon^\dagger E_1 - E_2^* \epsilon E_1 = E_2^* (\epsilon^\dagger - \epsilon) E_1$$

$$H_1 \mu^* H_2^* - H_2^* \mu H_1 = H_2^* (\mu^\dagger - \mu) H_1$$

Hence Eq. (173) can be rewritten as following,

$$- \nabla \cdot (E_1 \times H_2^* + E_2^* \times H_1) = J_1 \cdot E_2^* + J_2^* \cdot E_1 - j\omega E_2^* \cdot (\epsilon^\dagger - \epsilon) E_1 - j\omega H_2^* \cdot (\mu^\dagger - \mu) H_1$$

(178)

The corresponding integral form is

$$- \int \int_S (E_1 \times H_2^* + E_2^* \times H_1) \cdot \hat{n} dS = \int \int_V (J_1 \cdot E_2^* + J_2^* \cdot E_1 - j\omega E_2^* \cdot (\epsilon^\dagger - \epsilon) E_1 - j\omega H_2^* \cdot (\mu^\dagger - \mu) H_1) dV$$

(179)

D. mutual energy theorem in complex space with lossless medium

If the medium is lossless, there are following condition,

$$\epsilon^\dagger = \epsilon$$

$$\mu^\dagger = \mu$$

Hence there is

$$j\omega E_2^* \cdot (\epsilon^\dagger - \epsilon) E_1 + j\omega H_2^* \cdot (\mu^\dagger - \mu) H_1 = 0$$

(182)

Hence the complex mutual energy theorem can be simplified to the following form,

$$- \nabla \cdot (E_1 \times H_2^* + E_2^* \times H_1) = J_1 \cdot E_2^* + J_2^* \cdot E_1$$

(183)

or if the integral form,

$$- \int \int_S (E_1 \times H_2^* + E_2^* \times H_1) \cdot \hat{n} dS = \int \int_V (J_1 \cdot E_2^* + J_2^* \cdot E_1) dV$$

(184)

This is referred as simplified form of the mutual energy theorem. Or for simplification, just mutual energy theorem [18, 20]. It has been referred as the second reciprocity theorem [25], generalized reciprocity theorem, adjoint reciprocity theorem, lossless reciprocity theorem.
E. Modified mutual energy theorem in complex space

The above are referred as the complex mutual energy theorem. The corresponding modified complex mutual energy theorem is

\[-\nabla \cdot (E_1 \times H_2^* + E_2^* \times H_1) \cdot \hat{n} dS = J_1 \cdot E_2^* + J_2^* \cdot E_1 - j\omega E_2^* \cdot (\epsilon_2^\dagger - \epsilon_1) E_1 - j\omega H_2^* \cdot (\mu_2^\dagger - \mu_1) H_1 \]  

(185)

\[-\int_S (E_1 \times H_2^* + E_2^* \times H_1) \cdot \hat{n} dS = \int \int \int (J_1 \cdot E_2^* + J_2^* \cdot E_1 - j\omega E_2^* \cdot (\epsilon_2^\dagger - \epsilon_1) E_1 - j\omega H_2^* \cdot (\mu_2^\dagger - \mu_1) H_1) dV \]  

(186)

F. Modified mutual energy theorem in complex space with loss medium

If the lossless condition does not satisfy. The mutual energy formula can not be written as the above simplified form. However we always can choose

\[\epsilon_2^\dagger = \epsilon_1 \]  

(187)

\[\mu_2^\dagger = \mu_1 \]  

(188)

So that the Eq.(183,184) are still true in the meaning of modified mutual energy theorem. We must keep in mind that Eq.(183,184) are both the mutual energy theorem and the modified mutual energy theorem depending whether or not the medium i.e. the Eq.(156,157) satisfy or not. Here usually one of media is in the real space for example \(\epsilon_1, \mu_1\), the other \(\epsilon_2, \mu_2\) are in virtual space and it is effective only with mathematical meaning. In this situation we can free to choose their values. The simplified form of the modified mutual energy theorem can be found in reference[18–20] in which is just called as modified mutual energy theorem.

It is worth to see that the above (modified) mutual energy theorem is derived from modified Poynting theorem, time-offset, time-reversed, space-offset.

IX. DERIVE THE THEOREMS IN FOURIER DOMAIN DIRECTLY

In the last section we have derived the mutual energy theorem from complex Poynting theorem using the process of average. However that derivation is not strictly. Since actually we have only proved the real part of the theorem. The mutual energy theorem has image
part. We have not prove the image part of mutual energy theorem. In this section we will solve this problem.

A. The modified reversed mutual energy theorem

Assume \( \zeta_1 = [E_1, H_1, J_1, K_1, D_1, B_1] \), \( \zeta_2 = [E_2, H_2, J_2, K_2, D_2, B_2] \), \( \zeta = \zeta_1 + \zeta_2 \), there is the total energy formula,

\[
-\nabla \cdot ((E_1 + E_2) \times (H_1 + H_2))
\]

\[
= (J_1 + J_2) \cdot (E_1 + E_2) + (K_1 + K_2) \cdot (H_1 + H_2) + (E_1 + E_2) \cdot j\omega (D_1 + D_2) + (H_1 + H_2) \cdot j\omega (B_1 + B_2)
\]

(189)

And the self energy formula,

\[
-\nabla \cdot (E_1 \times H_1) = J_1 \cdot E_1 + K_1 \cdot H_1 + E_1 \cdot j\omega D_1 + H_1 \cdot j\omega B_1
\]

(190)

\[
-\nabla \cdot (E_2 \times H_2) = J_2 \cdot E_2 + K_2 \cdot H_2 + E_2 \cdot j\omega D_2 + H_2 \cdot j\omega B_2
\]

(191)

Subtract the above two self energy formulas form the total energy formula, we obtain

\[
-\nabla \cdot (E_1 \times H_2 + E_2 \times H_1)
\]

\[
= J_1 \cdot E_2 + J_2 \cdot E_1 + K_1 \cdot H_2 + K_2 \cdot H_1 + j\omega (E_1 \cdot D_2 + E_2 \cdot D_1 + H_1 \cdot B_2 + H_2 \cdot B_1)
\]

(192)

Among the above formula we notice that

\[
E_1 \cdot D_2 + E_2 \cdot D_1 = E_1 \cdot \epsilon_2 E_2 + E_2 \cdot \epsilon_1 E_1
\]

\[
= E_1 \cdot \epsilon_2 E_2 + E_1 \epsilon^T_1 E_2
\]

\[
= E_1 \cdot (\epsilon_2 + \epsilon^T_1) E_2
\]

(193)

and

\[
H_1 \cdot B_2 + H_2 \cdot B_1 = H_1 \cdot (\mu_2 + \mu^T_1) H_2
\]

(194)
Hence if we choose \( \varepsilon_2(\omega) \) and \( \mu_2(\omega) \) satisfy

\[
\varepsilon_2(\omega) + \varepsilon_1^T(\omega) = 0, \quad \mu_2(\omega) + \mu_1^T(\omega) = 0
\]

(195)

There is

\[
-\nabla \cdot (E_1(\omega) \times H_2(\omega) + E_2(\omega) \times H_1(\omega))
\]

\[
= J_1(\omega) \cdot E_2(\omega) + E_1(\omega) \cdot J_2(\omega) + K_1(\omega) \cdot H_2(\omega) + H_1(\omega) \cdot K_2(\omega)
\]

(196)

the integral form can be written as

\[
-\int_S (E_1(\omega) \times H_2(\omega) + E_2(\omega) \times H_1(\omega)) \, \hat{n} dS
\]

\[
= \int_V (J_1(\omega) \cdot E_2(\omega) + E_1(\omega) \cdot J_2(\omega) + K_1(\omega) \cdot H_2(\omega) + H_1(\omega) \cdot K_2(\omega)) \, dV
\]

(197)

The above two formula can be referred as the modified reversed mutual theorem. Define

\[
(\xi_1, \xi_2)_{r\omega} = \int_S (E_1(\omega) \times H_2(\omega) + E_2(\omega) \times H_1(\omega)) \, \hat{n} dS
\]

(198)

\[
(\rho_1, \xi_2)_{r\omega} = \int_V (J_1(\omega) \cdot E_2(\omega) + K_1(\omega) \cdot H_2(\omega)) \, dV
\]

(199)

\[
(\xi_1, \rho_2)_{r\omega} = \int_V (E_1(\omega) \cdot J_2(\omega) + H_1(\omega) \cdot K_2(\omega)) \, dV
\]

(200)

Subscript “\( r \)” is used to express this inner product has not use any complex conjugate symbols. We have the reversed mutual energy theorem as following,

\[
(\xi_1, \xi_2)_{r\omega} + (\rho_1, \xi_2)_{r\omega} + (\xi_1, \rho_2)_{r\omega} = 0
\]

(201)

\[
\varepsilon_2(\omega) + \varepsilon_1^T(\omega) = 0, \quad \mu_2(\omega) + \mu_1^T(\omega) = 0
\]

(202)

In the history, the closed work related Eq.(197) can be found in reference.\[^{[10]}\] or see Eq.(5).

In case \( \epsilon_1 = \epsilon_2 = \epsilon, \mu_1 = \mu_2 = \mu \), the word “modified” can be dropped. It becomes the reverse modified mutual energy theorem. The media condition is changed to

\[
\epsilon(\omega) = -\epsilon^T(\omega), \quad \mu(\omega) = -\mu^T(\omega)
\]

(203)
That means the media is anti-symmetric. We do not clear whether or not this kind media exist in the nature, however if it exist, in this anti-symmetric media, the above reversed mutual energy theorem is established. Here we call it “reversed” is because this media is anti-symmetric. It is also used to distinguish it with the theorem will be discussed in following subsection.

B. The modified mutual energy theorem

Considering a conjugate transform for the time-reverse transform to all the variable with subscript “2” to the formula Eq.(197)

\[
[E(\omega), H(\omega), J(\omega), K(\omega), \epsilon(\omega), \mu(\omega)] \equiv [E_r(\omega), H_r(\omega), J_r(\omega), K_r(\omega), \epsilon_r(\omega), \mu_r(\omega)]
\]

\[
= [E^*_r(\omega), H^*_r(\omega), J^*_r(\omega), K^*_r(\omega), -\epsilon^*_r(\omega), -\mu^*_r(\omega)]
\] (204)

There is

\[-\nabla \cdot (E_1(\omega) \times H^*_2(\omega) + E^*_2(\omega) \times H_1(\omega))
\]

\[= J_1(\omega) \cdot E^*_2(\omega) + E_1(\omega) \cdot J^*_2(\omega) + K_1(\omega) \cdot H^*_2(\omega) + H_1(\omega) \cdot K^*_2(\omega)
\] (205)

or

\[-\int_S (E_1(\omega) \times H^*_2(\omega) + E^*_2(\omega) \times H_1(\omega)) \hat{n}dS
\]

\[= \int_V (J_1(\omega) \cdot E^*_2(\omega) + E_1(\omega) \cdot J^*_2(\omega) + K_1(\omega) \cdot H^*_2(\omega) + H_1(\omega) \cdot K^*_2(\omega)) dV
\] (206)

The above can be referred as mutual theorem, or

\[(\xi_1, \xi_2)_\omega + (\rho_1, \xi_2)_\omega + (\xi_1, \rho_2)_\omega = 0
\] (207)

where

\[(\xi_1, \xi_2)_\omega = \int_S (E_1(\omega) \times H^*_2(\omega) + E^*_2(\omega) \times H_1(\omega)) \hat{n}dS
\] (208)
\begin{align}
\rho_1, \xi_2) = \int_V (J_1(\omega) \cdot E^*_2(\omega) + K_1(\omega) \cdot H^*_2(\omega))dV \\
(\xi_1, \rho_2) = \int_V (E_1(\omega) \cdot J^*_2(\omega) + H_1(\omega) \cdot K^*_2(\omega))dV
\end{align}

The media equation becomes,

\begin{align}
- \epsilon^*_2(\omega) + \epsilon^T_1(\omega) = 0, \\
- \mu^*_2(\omega) + \mu^T_1(\omega) = 0
\end{align}

or

\begin{align}
\epsilon_2(\omega) = \epsilon^T_1(\omega), \\
\mu_2(\omega) = \mu^T_1(\omega)
\end{align}

If \( \epsilon_1 = \epsilon_2 \) and \( \mu_1 = \mu_2 \), the word “modified” can be dropped, the above theorem becomes modified mutual energy theorem. For the modified mutual energy theorem the media satisfy

\begin{align}
\epsilon(\omega) = \epsilon^T(\omega), \\
\mu(\omega) = \mu^T(\omega)
\end{align}

which is lossless media. Hence mutual energy theorem is established in lossless media. The above mutual energy theorem has been derived by this author in reference [18–20].

In case \( \zeta_1 \) and \( \zeta_2 \) one is retarded potential, and the other is advanced potential, the surface integral \( (\xi_1, \xi_2) = 0 \), or

\[ \int_S (E_1(\omega) \times H_2^*(\omega) + E_2^*(\omega) \times H_1(\omega)) \hat{n}dS = 0 \]

the proof can been seen from the Appendix 3. In this case the mutual energy current will not go to the outside of the the surface.

If there are \( N \) electromagnetic fields, the modified mutual energy theorem is

\[ \sum_{i \leq j \leq N} \left( (\xi_i, \xi_j) + (\rho_i, \xi_j) + (\xi_i, \rho_j) \right) = 0 \]

C. The modified Lorenz reciprocity theorem

Considering a conjugate transform corresponding to magnetic mirror transform Eq. (72) to the above modified mutual energy theorem to all variable with subscript “2”, there is

\[-\nabla \cdot (E_1(\omega) \times (-1)H_2(\omega) + E_2(\omega) \times H_1(\omega)) \]
\[ J_1(\omega) \cdot E_2(\omega) - E_1(\omega) \cdot J_2(\omega) - K_1(\omega) \cdot H_2(\omega) + H_1(\omega) \cdot K_2(\omega) \quad (215) \]

\[-\epsilon_2(\omega) + \epsilon_1^T(\omega) = 0, \quad -\mu_2(\omega) + \mu_1^T(\omega) = 0 \quad (216)\]

This is modified Lorenz reciprocity theorems. In case \( \epsilon_2(\omega) = \epsilon_1(\omega) = \epsilon \mu_2(\omega) = \mu_1(\omega) = \mu \), hence

\[ \epsilon^T(\omega) = \epsilon(\omega), \quad \mu^T(\omega) = \mu(\omega) \]

The “modified” can be dropped. It becomes Lorenz reciprocity theorem. Lorenz reciprocity theorem is established in symmetric media.

\[ \int_S (E_1(\omega) \times H_2(\omega) - E_2(\omega) \times H_1(\omega)) \, dt \, \hat{n} dS = 0 \quad (217) \]

In case \( \zeta_1 \) and \( \zeta_2 \) one is retarded potential and one is advanced potential, we knew from last subsection the surface integral \( (\zeta_1, \zeta_2)_\omega = 0 \). Assume \( \zeta_1 \) is retarded potential and \( \zeta_2 \) is advanced potential, since we have did a magnetic mirror transform to \( \zeta_2 \), after the transform \( \zeta_2 \) become retarded potential. Hence there is if \( \zeta_1 \) and \( \zeta_2 \) are all retarded potential there is the surface integral for reciprocity theorem

\[ \int_S (E_1(\omega) \times H_2(\omega) - E_2(\omega) \times H_1(\omega)) \, dt \, \hat{n} dS = 0 \quad (218) \]

**D. The modified reversed reciprocity theorem**

After a reverse conjugate transform for subscript “2”. The above formula can be written as

\[ \epsilon_2^*(\omega) + \epsilon_1^T(\omega) = 0, \quad \mu_2^*(\omega) + \mu_1^T(\omega) = 0 \quad (219) \]

and
\[-\nabla \cdot (-E_1(\omega) \times H_2^*(\omega) + E_2^*(\omega) \times H_1(\omega))\]
\[= J_1(\omega) \cdot E_2^*(\omega) - J_2^*(\omega) \cdot E_1(\omega) - K_1(\omega) \cdot H_2^*(\omega) + K_2^*(\omega) \cdot H_1(\omega) \quad (220)\]

The corresponding integral form is

\[-(E_1(\omega) \times H_2^*(\omega) - E_2^*(\omega) \times H_1(\omega)) \hat{n} dS\]
\[= \int_V (J_1(\omega) \cdot E_2^*(\omega) - J_2^*(\omega) \cdot E_1(\omega) - K_1(\omega) \cdot H_2^*(\omega) + K_2^*(\omega) \cdot H_1(\omega)) dV \quad (221)\]

This can be referred as the reversed reciprocity theorem. In case \(\epsilon_2(\omega) = \epsilon_1(\omega) = \epsilon \mu_2(\omega) = \mu_1(\omega) = \mu\), the word “modified” can be dropped, it becomes reversed reciprocity theorem. The media condition becomes

\[\epsilon(\omega) = -\epsilon^\dagger(\omega), \quad \mu(\omega) = -\mu^\dagger(\omega) \quad (222)\]

This kind media can be referred as anti-lossless media. Hence the reversed reciprocity theorem is established in anti-lossless media.

E. The surface integral in the mutual energy theorem

If both \(\zeta_1\) and \(\zeta_2\) are both transmitting fields or the retarded potential, in general

\[(\xi_1, \xi_2)_\tau \neq 0 \quad (223)\]

Since, that means \((\xi_1, \xi_2)_\tau\) are mutual energy current go through the surface. For example if \(\zeta_1 = \zeta_2 = \zeta\) then

\[(\xi, \xi)_\tau = \int_S \int_{t=-\infty}^{\infty} (E(t + \tau) \times H(t) + E(t) \times H(t + \tau)) \hat{n} dS \quad (224)\]

and

\[F\{(\xi, \xi)_\tau\} = \int_S (E(\omega) \times H^*(\omega) + E^*(\omega) \times H(\omega)) \hat{n} dS\]
\[= 2 \int_S Re\{E(\omega) \times H^*(\omega)\} \hat{n} dS \quad (225)\]
$E(\omega) \times H^*(\omega)$ is the Fourier domain Poynting vector, $\int_S Re\{E(\omega) \times H^*(\omega)\} \hat{n}dS$ is the power flow out the surface which is not vanish in general. It is only vanish if the surface $S$ is super conductor or magnetic super conductor wall.

Hence there is

$$\langle \xi, \xi \rangle_{\omega} \neq 0$$

and hence, after an inverse Fourier transform, there is

$$\langle \xi, \xi \rangle_{\tau} = F^{-1}\{\langle \xi, \xi \rangle_{\omega} \neq 0 \}$$

in general. If both of them $\zeta_1$, $\zeta_2$ are the field of retarded potential, the mutual energy current will have the same direction from inner side to the outside of the surface $S$, the surface integral is not vanish in general.

In other hand if one of them is the field of retarded potential and the other is the field of advanced potential. For example $\rho_1 = [J_1, K_1]$ is the source and $\rho_2 = [J_2, K_2]$ is sink. $\rho_1$ and $\rho_2$ are inside the surface $S$. In this case, $\xi_1$ is retarded potential. $\xi_2$ is advanced potential, there is

$$\langle \xi_1, \xi_2 \rangle_{\tau} = 0$$

The proof can been seen in Appendix 3 of the reference[22]. In the proof where the Sommerfeld’s radiation condition has been applied.

X. MUTUAL ENERGY THEOREMS IN TIME DOMAIN

In the section we has derived the mutual energy theorem directly from Fourier domain. The derivation has first derived the time-reversed mutual energy theorem from Poynting theorem. The time reversed transform is applied to derive the mutual energy theorem from time reversed mutual energy theorem. However time reversed transform need Maxwell equation to prove. This means actually we did not derive the mutual energy theorem from Poynting theorem but through time reversed transform, the Maxwell equation is involved. Hence we did not purely derived the mutual energy theorem from Poynting theorem.

In this article the mutual energy of a electromagnetic field system is defined as the difference between the total energy (power) and the self energy (power). The mutual energy
is obtained from the idea that subtract the self energy from the total energy, the rest is the mutual energy. There is energy conservation theorem which is Poynting theorem. The Poynting theorem guarantees that the total energy and self energy are conservative. Hence the mutual energy should be also conservative. The mutual energy is conservative is referred as mutual energy theorem. The following offers the detail of the derivation of the mutual energy theorem.

A. The instantaneous-time mutual energy theorem

Assume \( \zeta_1 = [E_1, H_1, J_1, K_1, D_1, B_1] \) and \( \zeta_2 = [E_2, H_2, J_2, K_2, D_2, B_2] \) are electromagnetic fields, which can be retarded potential, advanced potential, time-offset or space-offset. Let \( \zeta = \zeta_1 + \zeta_2 \) be superimposing electromagnetic field. Assume that \( \zeta_1, \zeta_2 \) satisfy Maxwell equation Eq.(19, 20) and Eq.(32, 33). Hence \( \zeta_1, \zeta_2 \) satisfy the modified Poynting theorem Eq.(123), that means,

\[
- \nabla \cdot (E_1 \times H_1) = J_1 \cdot E_1 + K_1 \cdot H_1 + E_1 \cdot \partial D_1 + H_1 \cdot \partial B_1
\]

(229)

\[
- \nabla \cdot (E_2 \times H_2) = J_2 \cdot E_2 + K_2 \cdot H_2 + E_2 \cdot \partial D_2 + H_2 \cdot \partial B_2
\]

(230)

Then the superimposing electromagnetic field also satisfies the modified Poynting theorem Eq.(123),

\[
- \nabla \cdot ((E_1 + E_2) \times (H_1 + H_2))
= (J_1 + J_2) \cdot (E_1 + E_2) + (K_1 + K_2) \cdot (H_1 + H_2) + (E_1 + E_2) \cdot \partial (D_1 + D_2) + (H_1 + H_2) \cdot \partial (B_1 + B_2)
\]

(231)

Eq.(231) tell us the total energy should be conservative. Eq.(229, 230) tell us the self energy is conservative. Subtract the self energy from the total energy we can obtained the mutual energy. The mutual energy should be also conservative. Subtract Eq.(229, 230) from Eq.(231), there is,

\[
- \nabla \cdot (E_1 \times H_2 + E_2 \times H_1)
\]

\[
= J_1 \cdot E_2 + J_2 \cdot E_1 + K_1 \cdot H_2 + K_2 \cdot H_1 + E_1 \cdot \partial D_2 + E_2 \cdot \partial D_1 + H_1 \cdot \partial B_2 + H_2 \cdot \partial B_1
\]

(232)
The corresponding integral form of the above formula is

\[- \int_S (E_1 \times H_2 + E_2 \times H_1) \cdot \hat{n} dS\]

\[= \int_V (J_1 \cdot E_2 + J_2 \cdot E_1 + K_1 \cdot H_2 + K_2 \cdot H_1 + E_1 \cdot \partial D_2 + E_2 \cdot \partial D_1 + H_1 \cdot \partial B_2 + H_2 \cdot \partial B_1) dV \quad (233)\]

\(V\) is volume, \(S\) is the boundary surface of \(V\). \(S = \partial V\). Where \(\hat{n}\) is norm vector of the surface \(S\). The above Eq.\((232, 233)\) are referred as the mutual energy theorem. Same as Poynting theorem, if the medial Eq.\((25, 26)\), the above formula Eq.\((233)\) is referred as the mutual energy theorem. If the medial Eq.\((32, 33)\) is satisfied, Eq.\((233)\) is referred as modified mutual energy theorem.

Eq.\((233)\) is too long. Inner product will be defined to shorten the formula.

**B. Inner product of two electromagnetic systems in spatial-temporal domain**

Assume \(\xi_i = [E_i(t), H_i(t)], \eta_i = [D_i(t), B_i(t)], \rho_i = [J_i(t), K_i(t)], i = 1, 2\). A inner product on the surface can be defined as following

\[(\xi_1(t), \xi_2(t)) = \int_S (E_1(t) \times H_2(t) + E_2(t) \times H_1(t)) \hat{n} dS \quad (234)\]

In the same way we can also define

\[(\rho_1(t), \xi_2(t)) = \int_V (J_1(t) \cdot E_2(t) + K_1(t) \cdot H_2(t)) dV \quad (235)\]

\[(\xi_1(t), \partial \eta_2(t)) = \int_V (E_1 \cdot \partial D_2 + H_1 \cdot \partial B_2) dV \quad (236)\]

The mutual energy theorem can be rewritten as following,

\[- (\xi_1, \xi_2) = (\rho_1, \xi_2) + (\xi_1, \rho_2) + (\xi_1, \partial \eta_2) + (\partial \eta_1, \xi_2) \quad (237)\]

or

\[(\xi_1, \xi_2) + (\rho_1, \xi_2) + (\xi_1, \rho_2) + (\xi_1, \partial \eta_2) + (\partial \eta_1, \xi_2) = 0 \quad (238)\]

The above formula tell us the summation of the mutual energy current flow out the surface \(S: (\xi_1, \xi_2)\), the mutual energy loss contributed from the source \(\rho_1, \rho_2\): \((\rho_1, \xi_2) + (\xi_1, \rho_2)\) and
the mutual energy loss in the space: \((\xi_1, \partial \eta_2) + (\partial \eta_1, \xi_2)\) are zero. The above formula is instantaneous mutual energy theorem.

In case the superimposition electromagnetic field contains \(N\) electromagnetic fields, the above instantaneous mutual energy theorem can be written as

\[
\sum_{i<j, j \leq N} ((\xi_i, \xi_j) + (\rho_i, \xi_j) + (\xi_i, \rho_j) + (\xi_i, \partial \eta_j) + (\partial \eta_i, \xi_j)) = 0 \tag{239}
\]

C. The modified time-correlated mutual energy theorem

Considering if we use \(\zeta_1\tau = \tau \zeta_1\) to replace \(\zeta_1\), since after the time-offset transform, \(\zeta_1\tau\) still satisfies the Maxwell equation, and hence the satisfies the modified Poynting theorem and the modified mutual energy theorem, hence there is

\[
(\xi_1(t + \tau), \xi_2(t)) + (\rho_1(t + \tau), \xi_2(t)) + (\xi_1(t + \tau), \rho_2(t))
\]

\[
+ (\xi_1(t + \tau), \partial_t \eta_2(t)) + (\partial_{t+\tau} \eta_1(t + \tau), \xi_2(t)) = 0 \tag{240}
\]

taking the integral to the above formula to the variable \(t\), there is

\[
\int_{-\infty}^{\infty} ((\xi_1(t + \tau), \xi_2(t)) + (\rho_1(t + \tau), \xi_2(t)) + (\xi(t + \tau), \rho_2(t))
\]

\[
+ (\xi_1(t + \tau), \partial_t \eta_2(t)) + (\partial_{t+\tau} \eta_1(t + \tau), \xi_2(t))) \, dt = 0 \tag{241}
\]

In the Appendix 1 it is proven that if the media satisfies

\[
\epsilon_1^\dagger(\omega) = \epsilon_2(\omega), \quad \mu_1^\dagger(\omega) = \mu_2(\omega) \tag{242}
\]

or after a inverse Fourier transform \(F^{-1}\{\bullet\} = \int_{-\infty}^{\infty} \exp(j\omega t) \bullet \, dt\) the media satisfies,

\[
\epsilon_1^T(-t) = \epsilon_2(t), \quad \mu_1^T(-t) = \mu_2(t)
\]

The last 2 items of Eq.(241) disappear i.e.,

\[
\int_{-\infty}^{\infty} ((\xi_1(t + \tau), \partial_t \eta_2(t)) + (\partial_{t+\tau} \eta_1(t + \tau), \xi_2(t))) \, dt = 0 \tag{243}
\]

Hence there is
\[
\int_{-\infty}^{\infty} (\xi_1(t + \tau), \xi_2(t)) + (\rho_1(t + \tau), \xi_2(t)) + (\xi_1(t + \tau), \rho_2(t)) \, dt = 0 \tag{244}
\]

This is referred as the modified time-correlation mutual energy theorem. In case \( \epsilon_2(\omega) = \epsilon_1(\omega) = \epsilon(\omega) \), there is
\[
\epsilon^\dag(\omega) = \epsilon(\omega), \quad \mu^\dag(\omega) = \mu(\omega) \tag{245}
\]
or after an inverse Fourier transform
\[
\epsilon^T(-t) = \epsilon(t) \quad \mu^T(-t) = \mu(t) \tag{246}
\]
This means the media must be symmetry with time \( t \).

Define new inner product in spatial-temporal space
\[
(\xi_1, \xi_2)_{\tau} = \int_{t=\infty}^{-\infty} ((\xi_1(t + \tau), \xi_2^*(t)) \, dt \tag{247}
\]
\[
(\rho_1, \xi_2)_{\tau} = \int_{t=\infty}^{-\infty} (\rho_1(t + \tau), \xi_2^*(t)) \, dt \tag{248}
\]
\[
(\xi_1, \rho_2)_{\tau} = \int_{t=\infty}^{-\infty} (\xi_1(t), \rho_2^*(t + \tau)) \, dt \tag{249}
\]
The modified time-correlation mutual energy theorem can be written as,
\[
(\xi_1, \xi_2)_{\tau} + (\rho_1, \xi_2)_{\tau} + (\xi_1, \rho_2)_{\tau} = 0 \tag{250}
\]
Perhaps you will argue the field variable \( \zeta_1(t), \zeta_2(t) \) are real variable, why the above define the inner product here with complex conjugate symbol “\(*\)” inside? The reason is in following subsection we need to make a Fourier transform of above formula that also need this conjugate symbol.

In case the media satisfies Eq.(25, 26), the modified time-correlation mutual energy theorem becomes the time-correlation mutual energy theorem. The word of “modified” is dropped. It also be referred as time-correlation reciprocity theorem in reference[21]. The above modified time-correlation mutual energy theorem can be written as following,
\[
\int_S \int_{t=-\infty}^{\infty} (E_1(t + \tau) \times H_2^*(t) + E_2^*(t) \times H_1(t + \tau)) \, dt \, d\sigma \tag{251}
\]
\[ + \int_V \int_{t=-\infty}^{\infty} \left( J_1(t+\tau) \cdot E_2^*(t) + K_1(t+\tau) \cdot H_2^*(t) + \rho_2(t) + E_1(t+\tau) \cdot \rho_2^*(t) + H_1(t+\tau) \cdot K_2^*(t) \right) dt \, dV = 0 \]  
\tag{251}

In the above formula the conjugate symbol "\( ^* \)" can be dropped since \( \zeta_1(t) \) and \( \zeta_2(t) \) is a real variable, however we put "\( ^* \)" there the formula is still correct. It will used in next subsection.

If there are \( N \) electromagnetic fields, the modified time-correlation mutual energy theorem is

\[ \sum_{i=1}^{i<j} \sum_{j=1}^{j\leq N} \left( (\xi_i, \xi_j)_\tau + (\rho_i, \xi_j)_\tau + (\xi_i, \rho_j)_\tau \right) = 0 \]  
\tag{252}

XI. MUTUAL ENERGY THEOREMS IN FOURIER DOMAIN

A. The modified complex mutual energy theorem

Considering \( F\{\bullet\} = \int_{t=-\infty}^{\infty} \exp(-j\omega t) \cdot \bullet \, dt \) is Fourier transform, we have

\[ (\xi_1, \xi_2)_\omega \equiv F\{(\xi_1, \xi_2)_\tau\} \]

\[ = \int_S \left( E_1(\omega) \times H_2^*(\omega) + E_2^*(\omega) \times H_1(\omega) \right) \cdot \hat{n} \, dS \]  
\tag{253}

Please see the Appendix 1 for definition of \( F\{\bullet\} \) and the details of calculation.

\[ (\rho_1, \xi_2)_\omega \equiv F\{(\rho_1, \xi_2)_\tau\} \]

\[ = \int_V \left( J_1(\omega) \cdot E_2^*(\omega) + K_1(\omega) \cdot H_2^*(\omega) \right) \, dV \]  
\tag{254}

\[ (\xi_1, \rho_2)_\omega \equiv F\{(\xi_1, \rho_2)_\tau\} \]

\[ = \int_V \left( E_1(\omega) \cdot J_2^*(\omega) + H_1(\omega) \cdot K_2^*(\omega) \right) \, dV \]  
\tag{255}

the corresponding to frequency theorem is

\[ (\xi_1, \xi_2)_\omega + (\rho_1, \xi_2)_\omega + (\xi_1, \rho_2)_\omega = 0 \]  
\tag{256}
or
\[
\int_S (E_1(\omega) \times H_2^*(\omega) + E_2^*(\omega) \times H_1(\omega)) \cdot \hat{n} dS
\]
\[+
\int_V (J_1(\omega) \cdot E_2^*(\omega) + E_1(\omega) \cdot J_2^*(\omega) + K_1(\omega) \cdot H_2^*(\omega) + H_1(\omega) K_2^*(\omega)) dV
\]
\[= 0 \quad (257)
\]
Where the media have to meet the condition,
\[
\epsilon_1^\dagger(\omega) = \epsilon_2(\omega), \quad \mu_1^\dagger(\omega) = \mu_2(\omega)
\]
This formula has been referred the modified complex mutual energy theorem [18–20] by this author. In the reference [18–20], This author has obtained the modified complex mutual energy theorem from modified reciprocity theorem through a conjugate transform. Conjugate transform is the magnetic mirror transform in Fourier domain, see sub-section 2.5. In this article, the above complex mutual energy theorem is re-obtained through the modified Poynting theorem and the concept of mutual energy.

If the \(\zeta_1\) and \(\zeta_2\) are in the same media
\[
\epsilon(\omega) = \epsilon_1(\omega) = \epsilon_2(\omega) \quad \mu(\omega) = \mu_1(\omega) = \mu_2(\omega)
\]
There is
\[
\epsilon^\dagger(\omega) = \epsilon(\omega) \quad \mu^\dagger(\omega) = \mu(\omega)
\]
This is referred as lossless media. In lossless media the corresponding theorem is referred as the complex mutual energy theorem. The word “modified” can be dropped, if \(\epsilon_1(\omega) = \epsilon_2(\omega)\) and \(\mu_1(\omega) = \mu_2(\omega)\). The complex mutual energy theorem has been rediscovered later and is referred as the second reciprocity theorem [25].

If there are \(N\) electromagnetic fields, the corresponding mutual energy theorem,
\[
\sum_{i<j, i,j \leq N}^{i<j, j \leq N} ((\xi_i, \xi_j)\omega + (\rho_i, \xi_j)\omega + (\xi_i, \rho_j)\omega) = 0 \quad (261)
\]

\[B. \quad \text{The surface integral in the mutual energy theorem}\]

If both \(\zeta_1\) and \(\zeta_2\) are both retarded potential, in general
\[
(\xi_1, \xi_2)_r \neq 0 \quad (262)
\]
Since, that means \((\xi_1, \xi_2)_\tau\) are mutual energy current go through the surface. For example if \(\zeta_1 = \zeta_2 = \zeta\) then

\[
(\xi, \xi)_\tau = \int_S \int_{t=-\infty}^{\infty} (E(t + \tau) \times H(t) + E(t) \times H(t + \tau)) \hat{n}dS \tag{263}
\]

and

\[
F\{(\xi, \xi)_\tau\} = \int_S (E(\omega) \times H^*(\omega) + E^*(\omega) \times H(\omega)) \hat{n}dS = 2 \int_S \text{Re}\{E(\omega) \times H^*(\omega)\} \hat{n}dS \tag{264}
\]

\(E(\omega) \times H^*(\omega)\) is the Fourier domain Poynting vector, \(\int_S \text{Re}\{E(\omega) \times H^*(\omega)\} \hat{n}dS\) is the power flow out the surface which is not vanish in general. It is only vanish if the surface \(S\) is super conductor or magnetic super conductor wall.

Hence there is

\[
(\xi, \xi)_\omega \neq 0 \tag{265}
\]

and hence, after a inverse Fourier transform, there is

\[
(\xi, \xi)_\tau = F^{-1}\{(\xi, \xi)_\omega\} \neq 0 \tag{266}
\]

in general. If both of them \(\zeta_1, \zeta_2\) are the field of retarded potential, the mutual energy current will have the same direction from inner side to the outside of the surface \(S\), the surface integral is not vanish in general.

In other hand if one of them is the field of retarded potential and the other is the field of advanced potential. For example \(\rho_1 = [J_1, K_1]\) is the source and \(\rho_2 = [J_2, K_2]\) is sink. \(\rho_1\) and \(\rho_2\) are inside the surface \(S\). In this case, \(\xi_1\) is retarded potential. \(\xi_2\) is advanced potential, there is

\[
(\xi_1, \xi_2)_\tau = 0 \tag{267}
\]

The proof can been seen in Appendix 3. In the proof where the Sommerfeld’s radiation condition has been applied.
XII. RECIPROCITY THEOREMS

In this section we assume the mutual energy theorem is known but the reciprocity theorem is unknown. We derive the reciprocity theorem from the mutual energy theorem. This way we show that the reciprocity theorem actually is a sub-theorem of mutual energy theorem. Actually is it is a special situation of the mutual energy theorem.

A. Time convolution reciprocity theorem

In above mutual energy theorems, \( \zeta_1 \) and \( \zeta_2 \) can be retarded potential or advanced potential or even the combination of retarded potential and advanced potential. In a special situation where \( \zeta_1 \) and \( \zeta_2 \) one is retarded potential and the other one is advanced potential. Assume \( \zeta_1 \) is the retarded potential, \( \zeta_2 \) is the advanced potential. From the above case we have know in mutual energy theorem the surface integral vanish on the infinite sphere \( S \).

\[
(\xi_1, \xi_2)_{\tau} = 0
\]

or

\[
\int_{\mathcal{S}} \int_{t=-\infty}^{\infty} (E_1(t+\tau) \times H_2(t) + E_2(t) \times H_1(t+\tau)) \, dt \, \hat{n} \, dS = 0
\]

(269)

Hence according the time correlation mutual energy theorem Eq(250), there is

\[
(\rho_1, \xi_2)_{\tau} + (\xi_1, \rho_2)_{\tau} = 0
\]

(270)

or

\[
\int_{\mathcal{V}} \int_{-\infty}^{\infty} (J_1(t+\tau) \cdot E_2(t) + E_1(t+\tau) \cdot J_2(t) + K_1(t+\tau) \cdot H_2(t) + H_1(t+\tau) \cdot K_2(t)) \, dt \, dV = 0
\]

(271)

Since we know \( \zeta_2 \) is advanced potential, hence \( \zeta_{h2} = h\zeta_2 \) become retarded potential. Here \( h \) is magnetic mirror transform defined in Eq.(17). Hence \( \zeta_2 = h\zeta_{h2} \), or

\[
[E_2(t), H_2(t), J_2(t), K_2(t), \epsilon_2(t), \mu_2(t)]
\]

\[
= [E_{h2}(-t), -H_{h2}(-t), -J_{h2}(-t), K_{h2}(-t), \epsilon_{h2}(-t), \mu_{h2}(-t)]
\]

(272)
or substitute this to the above formula Eq.(269,271) there is

\[ \int_{S}^{\infty} \int_{t=-\infty}^{\infty} (E_1(t + \tau) \times (-1)H_2(-t) + E_2(-t) \times H_1(t + \tau)) \, dt \, d\mathbf{n} \, dS = 0 \quad (273) \]

and

\[ \int_{V}^{\infty} \int_{t'=-\infty}^{\infty} (J_1(t+\tau) \cdot E_2(-t) + (-1)J_2(-t) \cdot E_1(t+\tau) + K_1(t+\tau) \cdot (-1)H_2(-t) + K_2(-t) \cdot H_1(t+\tau)) \, dt \, dV = 0 \quad (274) \]

or in the above integral substitute \( t' = -t \)

\[ \int_{S}^{\infty} \int_{t'=-\infty}^{\infty} (E_1(-t' + \tau) \times (-1)H_2(t') + E_2(t') \times H_1(-t' + \tau)) \, dt' \, d\mathbf{n} \, dS = 0 \quad (275) \]

and

\[ \int_{V}^{\infty} \int_{t'=-\infty}^{\infty} (J_1(-t' + \tau) \cdot E_2(t') + (-1)J_2(t') \cdot E_1(-t' + \tau) + K_1(-t' + \tau) \cdot (-1)H_2(t) + K_2(t) \cdot H_1(-t' + \tau)) \, dt' \, dV = 0 \quad (276) \]

using \( t \) to replace \( t' \), there is

\[ \int_{S}^{\infty} \int_{t'=-\infty}^{\infty} (-E_1(\tau - t) \times H_2(t) + E_2(t) \times H_1(\tau - t)) \, dt \, d\mathbf{n} \, dS = 0 \quad (277) \]

and

\[ \int_{V}^{\infty} \int_{t'=-\infty}^{\infty} (J_1(\tau - t) \cdot E_2(t) - J_2(t) \cdot E_1(\tau - t) - K_1(\tau - t) \cdot H_2(t) + K_2(t) \cdot H_1(\tau - t)) \, dt \, dV = 0 \quad (278) \]

The media formula

\[ \epsilon_1^T(-t) = \epsilon_2(t), \quad \mu_1^T(-t) = \mu_2(t) \quad (279) \]
after the substitution Eq.(272) become

\[
\varepsilon_T^1(-t) = \varepsilon_h^2(-t), \quad \mu_T^1(-t) = \mu_h^2(-t)
\] (280)

or

\[
\varepsilon_T^1(t) = \varepsilon_h^2(t), \quad \mu_T^1(t) = \mu_h^2(t)
\] (281)

Considering \( \zeta_h^2 \) is retarded potential. We can take the subscript “\( h \)” and keep in mind that \( \zeta_2 \) is the retarded potential, there is,

\[
\int_S \int_t^{\infty} (-E_1(\tau - t) \times H_2(t) + E_2(t) \times H_1(\tau - t)) dt \, \hat{n}dS = 0
\] (282)

and

\[
\int_V^{\infty} \int_{-\infty}^{\infty} (J_1(\tau - t) \cdot E_2(t) - E_1(\tau - t) \cdot J_2(t) - K_1(\tau - t) \cdot H_2(t) + H_1(\tau - t) \cdot K_2(t) dt \, dV = 0
\] (283)

In the above formula the item of the surface integral is zero is only correct for this case where \( \zeta_1 \) and \( \zeta_2 \) are all retarded potential (or all are advanced potential). In general case the surface integral are not zero. Hence in the above formula the surface integral is still put there. The media should satisfy

\[
\varepsilon_T^1(t) = \varepsilon_2(t), \quad \mu_T^1(t) = \mu_2(t)
\] (284)

The above last second formula can be rewritten as

\[
\int_V^{\infty} \int_{-\infty}^{\infty} (J_1(\tau - t) \cdot E_2(\tau) - K_1(\tau - t) \cdot H_2(\tau)) dt \, dV
\]

\[
= \int_V^{\infty} \int_{-\infty}^{\infty} (J_2(\tau) \cdot E_1(\tau - t) - K_2(\tau) \cdot H_1(\tau - t)) dt \, dV
\] (285)

In the above formula we keep in mind that both \( \zeta_1 \) \( \zeta_2 \) are all retarded potential. This is the modified convolution reciprocity theorem. In the modified convolution reciprocity theorem, the media can be arbitrary, it does not need to be symmetry. However if the media is symmetry, We can choose \( \varepsilon_1 = \varepsilon_2 = \varepsilon, \mu_1 = \mu_2 = \mu \),

\[
\varepsilon_T^T(t) = \varepsilon(t), \quad \mu_T^T(t) = \mu(t)
\] (286)

58
In this situation, “modified” can be dropped. So the modified convolution reciprocity becomes the convolution reciprocity theorem.

**B. Lorenz reciprocity theorem**

Assume \( \zeta_1 \) and \( \zeta_2 \) are retarded potential. Considering the Fourier transform of the above time-convolution reciprocity theorem, there is

\[
\int_{S} (-E_1(\omega) \times H_2(\omega) + E_2(\omega) \times H_1(\omega)) \, dt \, \hat{n} dS +
\]

\[
\int_{V} (J_1(\omega) \cdot E_2(\omega) - E_1(\omega) \cdot J_2(\omega) - K_1(\omega) \cdot H_2(\omega) + H_1(\omega) \cdot K_2(\omega)) \, dV = 0 \tag{287}
\]

in case, \( \zeta_1 \) and \( \zeta_2 \) are retarded potential, there is

\[
\int_{S} (-E_1(\omega) \times H_2(\omega) + E_2(\omega) \times H_1(\omega)) \, dt \, \hat{n} dS = 0 \tag{288}
\]

The above last second formula can be rewritten as

\[
\int_{V} (J_1(\omega) \cdot E_2(\omega) - K_1(\omega) \cdot H_2(\omega)) \, dV
\]

\[
= \int_{V} (J_2(\omega) \cdot E_1(\omega) - H_1(\omega) \cdot K_2(\omega)) \, dV \tag{289}
\]

After the Fourier transform the media condition become,

\[
\epsilon^T_1(\omega) = \epsilon_2(\omega), \quad \mu^T_1(\omega) = \mu_2(\omega) \tag{290}
\]

This is the modified reciprocity theorem\(^{16}\). In case choose \( \epsilon_1 = \epsilon_2 = \epsilon, \mu_1 = \mu_2 = \mu \), and hence have,

\[
\epsilon^T(\omega) = \epsilon(\omega), \quad \mu^T(\omega) = \mu(\omega) \tag{291}
\]

The “modified” can be dropped, it become the reciprocity theorem, or the Lorentz reciprocity theorem\(^{6}\). Lorenz reciprocity theorem can be obtained also thorough conjugate transform from the Fourier domain mutual energy theorem.
According to the above discussion the Lorentz reciprocity theorem, and time-convolution reciprocity theorem are a special case of the mutual energy theorem where the two electromagnetic fields one is the retarded potential and the other one is the advanced potential.

In the Lorenz reciprocity theorem and convolution reciprocity theorem the two fields are both retarded potentials, even originally one is retarded potential and another is advanced potential in the time-correlation mutual energy theorem or complex mutual energy theorem. The concept reaction\[9\] is a special mutual energy (power) where two fields are in opposite, one is retarded potential the other one is the advanced potential.

C. The relationship of Poynting theorem, mutual energy theorem and reciprocity theorem

Even the modified complex mutual energy theorem and modified Lorenz reciprocity theorem can be derived from each other through a conjugate transform, the modified time-correlation mutual energy theorem and time-convolution reciprocity theorem can be derived through a magnetic mirrored transform, they are still independent theorems. The reason is that the mirror transform and conjugate transform are not mathematical equation, it is physical equation which contains some information coming from the Maxwell equation. When conjugate transform or mirror transform is applied in a derivation, it is same as the Maxwell equation is used again.

If we drop out the word “modified”, The complex mutual energy theorem and the Lorenz reciprocity theorem are thoroughly different theorems, the complex mutual energy theorem is established in losseless media and the Lorenz reciprocity theorem is established in symmetry media. They are suitable in different situation and are different theorems.

However even the word “modified” is dropped, the time-correlation mutual energy theorem and the complex mutual energy theorem can be derived from Poynting theorem. This can be proved exactly following the derivation of this article but do not use the word “modified”. Hence time-correlation mutual energy theorem and complex mutual energy theorem are really a sub-theorem of Poynting theorem. From this point of view, time-correlation mutual energy theorem and complex mutual energy theorem (with and without “modified”) are much closer related to the Poynting theorem than the convolution reciprocity theorem and the Lorenz reciprocity theorem.
Time-correlation mutual energy theorem and complex mutual energy theorem can be easily extended to the situation there is $N$ fields. In principle, the reciprocity theorem can do the same, however if it is done, there will be too many minus and positive sign in the extended theorem which will confuse all of us.

XIII. THE APPLICATION OF MUTUAL ENERGY THEOREM

A. Inner product

The 3 inner product ($\xi_1, \xi_2$), ($\xi_1, \xi_2)_{r=0}$, ($\xi_1, \xi_2)_{\omega}$ have been defined in Eq. (234, 247, and 253), it should be remarkable these are not just a notation for simplification. These 3 inner products are the real inner products. It can be proved that these inner products satisfy the inner product standard 3 conditions as following, if the electromagnetic fields $\zeta_1$ and $\zeta_2$ are all retarded potential, there are

1. Positive-definiteness:

$$ (\xi, \xi) \geq 0 \quad (\xi, \xi) = 0 \text{ iff } \xi = 0 $$

(292)

2. Conjugate symmetry:

$$ (\xi_1, \xi_2) = (\xi_2, \xi_1)^* $$

(293)

or if ($\xi_2, \xi_1$) is real,

$$ (\xi_1, \xi_2) = (\xi_2, \xi_1) $$

(294)

3. Linearity:

$$ (a \xi_1 + b \xi_2, \xi_3) = a(\xi_1, \xi_3) + b(\xi_2, \xi_3) $$

(295)

Where $a$ and $b$ are any constant. Here ($\xi_1, \xi_2$) represent all the 3 inner products ($\xi_1, \xi_2$), ($\xi_1, \xi_2)_{r=0}$ and ($\xi_1, \xi_2)_{\omega}$.

With the inner product, the norm can be defined as

$$ ||\xi|| = \sqrt{(\xi, \xi)} $$

(296)

Using the inner product, the mutual energy theorem in the Fourier domain and in time domain has nearly same formula, the only difference is the subscript of $\omega$ or $\tau$. 

61
It is worth to notice that \((\xi_1, \xi_2)_\tau\) does not satisfy the above standard inner product conditions.

\[
(\xi_1, \xi_2)_\tau = \int_S \int_{t=-\infty}^{\infty} (E_1(t + \tau) \times H'_2(t) + E'_2(t) \times H_1(t + \tau)) \, dt \, \hat{n} \, dS
\]

\[
= \int_S \int_{t=-\infty}^{\infty} (E'_1(t') \times H'_2(t' - \tau) + E_2(t') \times H'_1(t')) \, dt \, \hat{n} \, dS
\]

\[
= \int_S \int_{t=-\infty}^{\infty} (E'_2(t' - \tau) \times H_1(t') + E_1(t') \times H'_2(t' - \tau)) \, dt \, \hat{n} \, dS
\]

\[
= (\int_S \int_{t=-\infty}^{\infty} (E_2(t' - \tau) \times H'_1(t') + E'_1(t') \times H_2(t' - \tau)) \, dt \, \hat{n} \, dS)^*
\]

\[
= (\xi_2, \xi_1)_{-\tau}^*
\]

(297)

However if let \(\tau = 0\), the above formula means that

\[
(\xi_1, \xi_2)_{\tau=0} = (\xi_2, \xi_1)_{\tau=0}^*
\]

(298)

Hence \((\xi_1, \xi_2)_{\tau=0}\) is a good inner product.

**B. Applied the mutual energy theorem to the wave expansion**

Assume \(\zeta = (E, H, J, K, \epsilon, \mu)\) is a field of retarded potential. \(\zeta\) is in spatial-temporal domain or in Fourier domain. Choose that \(\zeta_i = (E_i, H_i, J_i, K_i, \epsilon_i, \mu_i)\) as also retarded potential. \(\zeta_i\) is at the same domain as \(\zeta\). \(i = 0, 1, \cdots \infty\). It can be taken that \(\epsilon_i(\omega) = \epsilon_0(\omega) = \epsilon^\dagger(\omega)\), \(\mu_i(\omega) = \mu_0(\omega) = \mu^\dagger(\omega)\) in complex space, or time space \(\epsilon_i(\tau) = \epsilon_0(\tau) = \epsilon^T(-\tau)\), \(\mu_i(\tau) = \mu_0(\tau) = \mu^T(-\tau)\). Here \(\epsilon_0\) and \(\mu_0\) are not the permittivity and permeability in empty space, instead, they are permittivity and permeability of the electromagnetic field \(\zeta_i\) when \(i = 0\). Hence there is \(\zeta_i = [E_i, H_i, J_i, H_i, \epsilon_0, \mu_0]\). In the following the inner product \((\xi, \xi_i)\) also means ether in time domain which is \((\xi, \xi_i)_{\tau=0}\) or in Fourier domain which is \((\xi, \xi_i)_{\omega}\). It can be in complex space or in spatial-temporal space. If there is any method the electromagnetic field \(\xi\) can be written as a expansion form

\[
\xi = \sum_{i=0}^{\infty} a_i \xi_i
\]

(299)
Where $\xi = [E(x, \tau), H(x, \tau)]$ or $\xi = [E(x, \omega), H(x, \omega)]$, here $x = [x_1, x_2, x_3]$ is a space variable which is often does not write out. $x$ can be express according other coordinates for example spherical coordinates. $a_i$ is expansion coefficients which need to be found in the following.

$$\xi_i(x, \omega) = R_l(r)Y_{nm}(\theta, \phi)$$ (300)

$(r, \theta, \phi)$ are spherical coordinates, $Y_{nm}(\theta, \phi)$ is a orthogonal function on the for $\theta$ and $\phi$ variable. $R_n(r)$ is a orthogonal variable alone the variable $r$, and $(r, \theta, \phi)$ is the spherical coordinates. The index $i = [l, m, n]$.

$$\xi_i(x, t) = R_l(r)Y_{nm}(\theta, \phi)\Phi_k(t)$$ (301)

where $\Phi_k(t)$ is the Fourier series,

$$\Phi_k(t) = \exp(j\frac{2k\pi t}{P})$$ (302)

The Fourier series is expansion in the region $-\frac{P}{2} \leq t \leq \frac{P}{2}$. In the numerical calculation for the time variable a fixed number is used to replace $-\infty < t < \infty$. The index $i = [l, m, n, k]$.

Assume $\xi_i$ is with the property of normalized orthogonality,

$$(\xi_i, \xi_j) = \delta_{ij}$$ (303)

where $(\xi_i, \xi_j)$ means $(\xi_i, \xi_j)_{\tau=0}$ or $(\xi_i, \xi_j)_{\omega}$ and

$$\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}$$ (304)

considering Eq.(299) with (303), there is

$$(\xi, \xi_i) = a_i$$ (305)

where from the modified mutual energy theorem Eq.(250 or 256) we know that

$$(\xi, \xi_i) = -(\rho, \xi_i) - (\xi, \rho_i)$$ (306)

or the expansion can be written as

$$\xi = -\sum_{i=0}^{\infty}((\rho, \xi_i) + (\xi, \rho_i))\xi_i$$ (307)
In general if both $\xi$ and $\xi_i$ are both retarded potential or both are advanced potential the inner product does not disappear. Hence the electromagnetic field $\xi$ can be expanded as $\xi_i$. It is worth to see the above expansion can be done in Fourier domain and also in time domain. The medial $\epsilon(\omega), \mu(\omega)$ can be arbitrary. $\epsilon(\omega), \mu(\omega)$ do not need to be lossless, because $\epsilon_0, \mu_0$ can always be chosen so to satisfy Eq. (258) even with loss media $\epsilon(\omega), \mu(\omega)$.

The spherical wave expansion and plane wave expansion in Fourier domain can be found in reference [18, 20]. Where the modified mutual energy theorem is applied in Fourier domain. In this article the expansion method has been extended to the time domain. Similar discussions about the wave expansion can be found also in reference [23, 24].

C. One example of mutual energy theorem

Assume there are electromagnetic field systems $\zeta_1, \zeta_2$ and $\zeta_3$ are known. Assume, $\zeta_1, \zeta_2$ are retarded potential, $\zeta_3$ is an advanced potential. Please find out the mutual energy current radiate to the outside of the infinite sphere $S$.

Solution: the all mutual energy current radiate to the out of the surface $S$ is

$$ \sum_{i<j, j=1}^{i=1} (\xi_i, \xi_j) $$

Since $\zeta_3$ is advanced potential, and $\zeta_1$ and $\zeta_2$ are retarded potential, there is

$$ (\xi_1, \xi_3) = 0, \quad (\xi_2, \xi_3) = 0 $$

(308)

The mutual energy current radiate to the outside of the surface $S$ is

$$ \sum_{i=1, j=1}^{i<j, j=3} (\xi_i, \xi_j) = (\xi_1, \xi_2) + (\xi_1, \xi_3) + (\xi_2, \xi_3) $$

$$ = (\xi_1, \xi_2) = -((\rho_1, \xi_2) + (\xi_1, \rho_2)) $$

(309)

In the last step the mutual energy theorem for $\zeta_1$ and $\zeta_2$ has been applied. $\zeta_1$ and $\zeta_2$ are retarded potential, $(\xi_1, \xi_2) \neq 0$. Finished.

In this case the result is only the mutual energy current of $\zeta_1$ and $\zeta_2$ which have the contribution to the energy current going to the outside of the surface $S$.

XIV. COMPLEMENTARY THEOREMS

Chen-To Tai has derived the complementary reciprocity theorem [38]. We have obtained 4 theorems 2 mutual energy theorem and 2 reciprocity theorem. We apply the electromagnetic
field swapping transform

\[ \zeta_s = s\zeta = [ZH, \frac{1}{Z}E, -\frac{1}{Z}K, -ZJ, -\frac{1}{Z^2}\mu, -Z^2\epsilon] \]

to the 4 theorem, 4 complementary theorems can be obtained, among them one is the Chen-
to Ta’s complementary theorem. Consider swapping transform for the following 4 theorems

A. Corresponding to Lorenz reciprocity theorem

\[
\int \left( (E_1(\omega) \times H_2(\omega) - E_2(\omega) \times H_1(\omega)) \right) dt \hat{ndS}
\]

\[
= \int \left( J_1(\omega) \cdot E_2(\omega) - E_1(\omega) \cdot J_2(\omega) - K_1(\omega) \cdot H_2(\omega) + H_1(\omega) \cdot K_2(\omega) \right) dV \quad (310)
\]

\[
\epsilon_1(\omega) = \epsilon_2^T(\omega), \quad \mu_1(\omega) = \mu_2^T(\omega)
\]

\[
\zeta_s = s\zeta = [ZH, \frac{1}{Z}E, -\frac{1}{Z}K, -ZJ, -\frac{1}{Z^2}\mu, -Z^2\epsilon] \]

The corresponding theorem is

\[
\int \left( (E_1(\omega) \times \frac{1}{Z}E_2(\omega) - ZH_2(\omega) \times H_1(\omega)) \right) dt \hat{ndS}
\]

\[
= \int \left( J_1(\omega) \cdot ZH_2(\omega) - E_1(\omega) \cdot \left( -\frac{1}{Z}K_2(\omega) \right) - K_1(\omega) \cdot \frac{1}{Z}E_2(\omega) + H_1(\omega) \cdot \left( -ZJ_2(\omega) \right) \right) dV \quad (312)
\]

or

\[
\int \left( (E_1(\omega) \times E_2(\omega) - Z^2H_2(\omega) \times H_1(\omega)) \right) dt \hat{ndS}
\]

\[
= \int \left( Z^2J_1(\omega) \cdot H_2(\omega) + E_1(\omega) \cdot K_2(\omega) - K_1(\omega) \cdot E_2(\omega) - Z^2H_1(\omega) \cdot J_2(\omega) \right) dV \quad (313)
\]

\[
\epsilon_1(\omega) = -\frac{1}{Z^2}\mu_2^T(\omega), \quad \mu_1(\omega) = -Z^2\epsilon_2^T(\omega)
\]
This is complementary reciprocity of Chen-To Ta. In the above derivation we have applied the concept of “modified”, if take the “modified” away, there is,

\[ \epsilon(\omega) = -\frac{1}{Z^2} \mu^T(\omega), \quad \mu(\omega) = -Z^2 \epsilon^T(\omega) \]  

(315)

This media can not be realized in empty space.

B. Corresponding to reverse reciprocity theorem

\[ \int S (E_1(\omega) \times H_2^*(\omega) - E_2^*(\omega) \times H_1(\omega)) \hat{n} dS \]

\[ = \int_V (J_1(\omega) \cdot E_2^*(\omega) - J_2^*(\omega) \cdot E_1(\omega) - K_1(\omega) \cdot H_2^*(\omega) + K_2^*(\omega) \cdot H_1(\omega)) dV \]  

(316)

\[ \epsilon_1(\omega) = -\epsilon_2^\dagger(\omega), \quad \mu_1(\omega) = -\mu_2^\dagger(\omega) \]  

(317)

\[ \int S (E_1(\omega) \times \frac{1}{Z} E_2^*(\omega) - Z H_2^*(\omega) \times H_1(\omega)) \hat{n} dS \]

\[ = \int_V (J_1(\omega) \cdot Z H_2^*(\omega) + \frac{1}{Z} K_2^*(\omega) \cdot E_1(\omega) - K_1(\omega) \cdot \frac{1}{Z} E_2^*(\omega) - Z J_2^*(\omega) \cdot H_1(\omega)) dV \]  

(318)

\[ \int S (E_1(\omega) \times E_2^*(\omega) - Z^2 H_2^*(\omega) \times H_1(\omega)) \hat{n} dS \]

\[ = \int_V (Z^2 J_1(\omega) \cdot H_2^*(\omega) + K_2^*(\omega) \cdot E_1(\omega) - K_1(\omega) \cdot E_2^*(\omega) - Z^2 J_2^*(\omega) \cdot H_1(\omega)) dV \]  

(319)

\[ \epsilon_1(\omega) = \frac{1}{Z^2} \mu_2^\dagger(\omega), \quad \mu_1(\omega) = Z^2 \epsilon_2^\dagger(\omega) \]  

(320)

if the word “modified” is taken away, there is

\[ \epsilon(\omega) = \frac{1}{Z^2} \mu^\dagger(\omega), \quad \mu(\omega) = Z^2 \epsilon^\dagger(\omega) \]  

(321)

This media can be realized in empty space.
C. Corresponding to reverse mutual energy theorem

\[- \int_S (E_1(\omega) \times H_2(\omega) + E_2(\omega) \times H_1(\omega)) \, \hat{n} \, dS \]

\[= \int_V (J_1(\omega) \cdot E_2(\omega) + E_1(\omega) \cdot J_2(\omega) + K_1(\omega) \cdot H_2(\omega) + H_1(\omega) \cdot K_2(\omega)) \, dV \quad (322)\]

\[\epsilon_1(\omega) = -\epsilon_2^T(\omega), \quad \mu_1(\omega) = -\mu_2^T(\omega) \quad (323)\]

\[- \int_S (E_1(\omega) \times \left( \frac{1}{Z} E_2(\omega) \right) + Z H_2(\omega) \times H_1(\omega)) \, \hat{n} \, dS \]

\[= \int_V (J_1(\omega) \cdot Z H_2(\omega) + E_1(\omega) \cdot (-\frac{1}{Z} K_2(\omega)) + K_1(\omega) \cdot \frac{1}{Z} E_2(\omega) + H_1(\omega) \cdot (-Z J_2(\omega)) \, dV \quad (324)\]

after the transform it becomes,

\[- \int_S (E_1(\omega) \times E_2(\omega) + Z^2 H_2(\omega) \times H_1(\omega)) \, \hat{n} \, dS \]

\[= \int_V (Z^2 J_1(\omega) \cdot H_2(\omega) - E_1(\omega) \cdot K_2(\omega) + K_1(\omega) \cdot E_2(\omega) - Z^2 H_1(\omega) \cdot J_2(\omega)) \, dV \quad (325)\]

\[\epsilon_1(\omega) = \frac{1}{Z^2} \mu_2^T(\omega), \quad \mu_1(\omega) = Z^2 \epsilon_2^T(\omega) \quad (326)\]

if the “modified” is taken away, there is

\[\epsilon(\omega) = \frac{1}{Z^2} \mu^T(\omega), \quad \mu(\omega) = Z^2 \epsilon^T(\omega) \quad (327)\]

This kind of media can be realized in empty space.

D. Corresponding to mutual energy theorem

\[- \int_S (E_1(\omega) \times H_2^*(\omega) + E_2^*(\omega) \times H_1(\omega)) \, \hat{n} \, dS \]

\[= \int_V (J_1(\omega) \cdot E_2^*(\omega) + E_1(\omega) \cdot J_2^*(\omega) + K_1(\omega) \cdot H_2^*(\omega) + H_1(\omega) \cdot K_2^*(\omega)) \, dV \quad (328)\]
\[ \epsilon_2(\omega) = \epsilon_1^\dagger(\omega), \quad \mu_2(\omega) = \mu_1^\dagger(\omega) \] (329)

after transform, it becomes

\[ \zeta_s = s\zeta = [ZH, \frac{1}{Z}E, \frac{1}{Z}K, -ZJ, -\frac{1}{Z^2}\mu, -Z^2\epsilon] \]

\[-\int_S (E_1(\omega) \times \frac{1}{Z}E_2^*(\omega) + ZH_2^*(\omega) \times H_1(\omega)) \hat{n}dS \]

\[= \int_V (J_1(\omega) \cdot ZH_2^*(\omega) + E_1(\omega) \cdot (-\frac{1}{Z})K_2^*(\omega) + K_1(\omega) \cdot \frac{1}{Z}E_2^*(\omega) + H_1(\omega) \cdot (-Z)J_2^*(\omega)) dV \] (330)

or

\[-\int_S (E_1(\omega) \times E_2^*(\omega) + Z^2H_2^*(\omega) \times H_1(\omega)) \hat{n}dS \]

\[= \int_V (Z^2J_1(\omega) \cdot H_2^*(\omega) - E_1(\omega) \cdot K_2^*(\omega) + K_1(\omega) \cdot E_2^*(\omega) - Z^2H_1(\omega) \cdot J_2^*(\omega)) dV \] (331)

\[\epsilon_1(\omega) = -\frac{1}{Z^2}\mu_2^\dagger(\omega), \quad \mu_1(\omega) = -Z^2\epsilon_2^\dagger \] (332)

If the "modified" is taken away, there is

\[\epsilon(\omega) = -\frac{1}{Z^2}\mu^\dagger(\omega), \quad \mu(\omega) = -Z^2\epsilon^\dagger \] (333)

This media can not realized in empty space.

If inverse Fourier transform is made, we can obtained the 4 corresponding time domain mutual energy or reciprocity theorems.

**XV. CONCLUSION**

The modified Poynting theorem is introduced, so it functions for the superimposition of the electromagnetic fields which includes retarded potential, mirrored field of the retarded potential which is a field of advanced potential and time-offset electromagnetic fields. Each fields can also be in a different media.

The concept of the mutual energy is introduced, which is the difference between the total energy and self-energy. Using the concept of the mutual energy a few mutual energy
theorems are derived from the modified Poynting theorem. The mutual energy theorems introduced in this article includes,

1) The instantaneous-time mutual energy theorem.
2) The time-reversed mutual energy theorem.
3) Mixed mutual energy theorem.
4) The time-correlation reciprocity theorem is re-derived from the above instantaneous-time mutual energy theorem. Hence it can be referred as time-correlation mutual energy theorem too.

5) The mutual energy theorem in Fourier domain is re-derived from Poynting theorem in Fourier domain and time domain.

6) The Lorenz reciprocity theorem is re-derived as a special case of the mutual energy theorem, where the electromagnetic fields are opposite, one is retarded potential and the other is mirrored field of retarded potential which is advanced potential. The concept of the reaction is also explained as a special mutual energy where two field one is retarded potential one is advanced potential.

7) We also extended the mutual energy theorem to the case there are \( N \) electromagnetic fields instead of only 2. Since the use of inner product, the formula for \( N \) electromagnetic fields is very simple and easy to understand.

8) The 3 additional complementary theorems are derived.

This article has built a bridge between the Poynting theorem and the reciprocity theorem. This bridge is mutual energy theorem.

**Appendix 1**

Assume \( f(t) \) and \( g(t) \) is real function, and \( f(\omega) = F\{f(t)\}, g(\omega) = F\{g(t)\} \)

\[
F\{ \int_{t=-\infty}^{\infty} f(t+\tau) g(t) \, dt \} = f(\omega)(g(\omega))^* \tag{334}
\]

**Appendix 2**

Prove the formula
\[
\int_{t=-\infty}^{\infty} (\xi_2(t), \partial_{t+\tau} \eta_1(t + \tau)) + (\xi_1(t + \tau), \partial_t \eta_2(t)) dt = 0 \quad (335)
\]

In case there is
\[
\epsilon_2^\dagger(\omega) - \epsilon_1(\omega) = 0, \quad \mu_2^\dagger(\omega) - \mu_1(\omega) = 0 \quad (336)
\]

Do the Fourier transform \(F\{\bullet\}\) to the above formula,

\[
F\{ \int_{t=-\infty}^{\infty} (\xi_2(t), \partial_{t+\tau} \eta_1(t + \tau)) + (\xi_1(t + \tau), \partial_t \eta_2(t)) dt \} = F\{ \int_{V} \int_{t=-\infty}^{\infty} (E_2(t) \cdot \partial_{t+\tau} D_1(t + \tau) + E_1(t + \tau) \cdot \partial D_2(t) + H_2(t) \cdot \partial_{t+\tau} B_1(t + \tau) + H_1(t + \tau) \cdot \partial B_2(t)) dt dV \} = 0 \quad (337)
\]

Let \(U = \partial_t D_1(t)\)

\[
F\{ \int_{t=-\infty}^{\infty} (E_2(t) \cdot \partial_{t+\tau} D_1(t + \tau)) dt \} = F\{ \int_{t=-\infty}^{\infty} (E_2(t) \cdot U(t + \tau)) dt \} = F\{ \partial_\tau U \} = E_2^\ast(\omega) U(\omega) \quad (338)
\]

Where

\[
U(\omega) = F\{U\} = F\{\partial_t D_1(t)\} = \partial_t \int_{\tau=-\infty}^{\infty} \epsilon_1(t - \tau) E_1(\tau) d\tau = -j\omega \epsilon_1(\omega) E_1(\omega) \quad (339)
\]

or

\[
U^*(\omega) = (-j\omega)^* \epsilon_2^*(\omega) E_2^*(\omega) \quad (340)
\]

Hence the Eq.(337) becomes

\[
= \int_{V} (E_2^*(\omega) \cdot (-j\omega) \epsilon_1(\omega) E_1(\omega) + E_1(\omega) \cdot (-j\omega)^* \epsilon_2^*(\omega) E_2^*(\omega) + H_2^*(\omega) \cdot (-j\omega) \mu_1(\omega) H_1(\omega) + H_1(\omega) \cdot (-j\omega)^* \epsilon_2^*(\omega) H_2^*(\omega)) dV = (-j\omega) \int_{V} (E_2^*(\omega)(\epsilon_1(\omega) - \epsilon_2^*(\omega)) E_1(\omega)) dV
\]
+(-j\omega)\int_{V}(H_{2}^*(\omega)(\mu_{1}(\omega) - \mu_{2}^{\dagger}(\omega))H_{1}(\omega)dV = 0

Where we have considered that

\begin{align*}
E_{1}\epsilon_{2}^{*}E_{2}^{*} & = E_{2}^{*}(\epsilon_{2}^{*})^{T}E_{1} = E_{2}^{\dagger}\epsilon_{2}E_{1} \quad (341) \\
E_{1}\mu_{2}^{*}E_{2}^{*} & = E_{2}^{*}(\mu_{2}^{*})^{T}E_{1} = E_{2}^{\dagger}\mu_{2}E_{1} \quad (342)
\end{align*}

The last step Eq.(336) has been considered. Hence we have Eq.(335),

\begin{align*}
\epsilon_{2}^{\dagger} & = \epsilon_{1} \quad (343) \\
\mu_{2}^{\dagger} & = \mu_{1} \quad (344)
\end{align*}

**Appendix 3**

Prove if \(\xi_{1}\) is retarded potential and \(\xi_{2}\) is advanced potential and the integral satisfies that,

\[(\xi_{1}, \xi_{2}) = 0 \quad (345)\]

We can assume \(\xi_{2} = h\xi_{2o}\), here \(h\) is magnetic mirror transform. \(\xi_{2o}\) is the corresponding field of \(\xi_{2}\). \(\xi_{2o}\) is retarded potential,

\[(\xi_{1}, \xi_{2}) = \int_{s}(E_{1}(t) \times H_{2}(t) + E_{2}(t) \times H_{1}(t)) \hat{n}dS \quad (346)\]

In order to the above formula, the Fourier transform is applied,

\[f(\tau) \equiv \int_{s}(E_{1}(t) \times H_{2o}(\tau - t) - E_{2o}(\tau - t) \times H_{1}(t)) \hat{n}dS \quad (347)\]
\[ F\{f(\tau)\} = \int_S (E_1(\omega) \times H_{2o}(\omega) - E_{2o}(\omega) \times H_1(\omega)) \hat{n} dS \]  

(348)

Where \( F\{\cdot\} \) is Fourier transform. In the big sphere. Assume \( r \to \infty \). Considering the Silver-Muller radiation condition or Sommerfeld’s radiation condition,

\[ \lim_{r \to \infty} r(H \times \hat{n} - E) = 0 \]  

(349)

In \( r \to \infty \), \( \hat{n} \) can be calculated,

\[ \hat{n} = \frac{E_1(\omega) \times H_1(\omega)}{||E_1(\omega) \times H_1(\omega)||} \]  

(350)

and

\[ \hat{n} = \frac{E_{2o}(\omega) \times H_{2o}(\omega)}{||E_{2o}(\omega) \times H_{2o}(\omega)||} \]  

(351)

or at \( r \to \infty \) there is

\[ E_1(\omega) = E_1(\omega) \times \hat{n} \]  

(352)

\[ E_{2o}(\omega) = E_{2o}(\omega) \times \hat{n} \]  

(353)

\[ F\{f(\tau)\} = \int_S (E_1(\omega) \times (E_{2o}(\omega) \times \hat{n}) - E_{2o}(\omega) \times (E_1(\omega) \times \hat{n})) \hat{n} dS \]

\[ = \int_S (E_1(\omega) \cdot E_{2o}(\omega)) \hat{n} - (E_{2o}(\omega) \cdot E_1(\omega)) \hat{n} \cdot \hat{n} dS \]

\[ = 0 \]  

(354)

In the above we have considered

\[ a \times (b \times c) = (a \cdot b)c - (a \cdot c)b \]  

(355)

That means that

\[ F\{f(\tau)\} = 0 \]  

(356)

Hence

\[ f(\tau) = 0 \]  

(357)
i.e.

\[ f(\tau) = \int_{S} (E_1(t) \times H_{2o}(\tau - t) - E_{2o}(\tau - t) \times H_1(t)) \hat{n}dS = 0 \]  

(358)

Hence

\[ \int_{S} (E_1(t) \times H_{2o}(\tau - t) - E_{2o}(\tau - t) \times H_1(t)) \hat{n}dS)_{\tau=0} = 0 \]  

(359)

or

\[ (\int_{S} (E_1(t) \times H_{2o}(-t) - E_{2o}(-t) \times H_1(t)) \hat{n}dS = 0 \]  

(360)

or

\[ (\xi_1, \xi_2) \]

\[ = \int_{S} (E_1(t) \times H_{2o}(-t) - E_{2o}(-t) \times H_1(t)) \hat{n}dS = 0 \]  

(361)

---

[1] J. H. Poynting (1884). "On the Transfer of Energy in the Electromagnetic Field". Philosophical Transactions of the Royal Society of London 175: 343–361. doi:10.1098/rstl.1884.0016

[2] H. A. Lorentz, "Het theorema van Poynting over de energie in het electromagnetisch veld en een paar algemene stellingen over de voortplanting van licht," Verhandelingen en bijdragen uitgegeven door de Afdeeling Natuurkunde, Koninklijke Nederlandse Akademie van Wetenschappen te Amsterdam, vol. 4, pp. 176-187, 1895-1896.

[3] J.R. Carson. Reciprocal theorems in Radio communication. Proc. IRE 17. 952(1929)

[4] J.R. Carson. The Reciprocal energy theorem. Bell Syst. Tech. Jour., 9 525, (1930)

[5] STUART-BALLANTINE, RECIPROCITY IN ELECTROMAGNETIC, MECHANICAL, ACOUSTICAL, AND INTERCONNECTED SYSTEMS, proceedings of the Institute of Radio Engineers Volume 17, Number 6 June, 1929

[6] Rumsey, V. H., Reaction concept in electromagnetic theory, Phys. Rev., vol 94 Jun 1954, pp 1483-1491.

[7] Welch, W. J., Reciprocity theorems for electromagnetic fields whose time dependence is arbitrary, IRE trans. On Antennas and Propagation, vol AP-8, Jan 1960, PP68-73.

[8] Welch, W. J., Comment on “Reciprocity Theorems for Electromagnetic Fields Whose Time Dependence Is Arbitrary”, IRE Transactions on antennas and propagation January 1961 p114.
[9] Rumsey, V. H. A Short Way of Solving Advanced Problems in electromagnetic Fields and Other Linear Systems, IEEE Transactions on antennas and Propagation, Jan 1963, pp 73-86.

[10] S. N. Samaddar, Orthogonality properties of modes in a compressible partially ionized plasma, Applied Scientific Research, Section B February 1964, Volume 11, Issue 1-2, pp 84-102.

[11] Goubau, G., A reciprocity theorem for non-periodic fields, IRE Trans. on Antennas and Propagation, vol AP-8, May 1960, pp 339-342

[12] B. Ru-shao, Cheo, A Reciprocity Theorem for electromagnetic fields with general time dependence. Antennas and Propagation, 1965 IEE Volume:13 Issue:2

[13] Adrianus T. de Hoop, Reciprocity, Causality, and Huygens’ principle in electromagnetic wave theory. Huygens’ Principle, 1690-1990: Theory and Applications, H. Blok, H.A. Ferwerda, H.K. Kuiken (editors). © 1992 Elsevier Science Publishers B. V. All rights reserved.

[14] Anders Karlsson, Constitutive relations, dissipation and Gerhard Kristensson, reciprocity for the Maxwell equations in the time domain, Editor: Gerhard Kristensson © Anders Karlsson and Gerhard Kristensson, Lund, August 4, 1999.

[15] Jin Au Kong, Theorems of Bianisotropic Media, Theorems of Bianisotropic Media, Proceeding of IEEE, vol, 60, No. 9, September 1972.

[16] J. A. Kong, Electromagnetic Wave Theory, EMW Publishing, 1016 pg, 2008 (Previous editions by Wiley-Interscience: 1975, 1986 and 1990 and EMW Publishing: 1998, 2000 and 2005)

[17] Norbert N. Bojarskbi, Generalized reaction principles and reciprocity theorems for the wave equations, and the relationship between the time-advanced and time-retarded fields, J. Acoust. Soc. Am 74(1), July 1983

[18] Zhao S. R., The Application of ‘Mutual Energy Theorem’ in Expansion of Radiation Fields in Spherical Waves, published on ACTA Electronica Sinica, Vol. 15, No. 3 May 1987. P. R. of China, P88

[19] Zhao S.R., The Simplification of Formulas of Electromagnetic Fields by Using ‘Mutual Energy Formula’ published on Journal of Electronics, Vol. 11 and No. 1 Jan. 1989. P. R. of China, P73 Simplification

[20] Zhao S.R., The Application of ‘Mutual Energy Formula’ in Expansion of Plane Waves published on Journal of Electronics, Vol. 11 and No. 2. March 1989. P. R. of China, P204 Mutual Energy Formula

[21] Adrianus T. de Hoop, Time-domain reciprocity theorems for electromagnetic fields in disper-
sive media Radio Science, volume 22, number 7, pages 1171-1178, December 1987

[22] Book of “Electromagnetic Symmetry” edited by Carl E. Baum, Haralambos N. Kritikos, 1995

[23] D. Marcuse Coupled-mode Theory for Anisotropic optical waveguides, The Bell System technical Journal Vol. 54, No. 6 July-August 1975.

[24] A. A. Barybin , MODAL EXPANSIONS AND ORTHOGONAL COMPLEMENTS IN THE THEORY OF COMPLEX MEDIA WAVEGUIDE EXCITATION BY EXTERNAL SOURCES FOR ISOTROPIC, ANISOTROPIC, AND BIANISOTROPIC MEDIA Progress In Electromagnetics Research, PIER 19, 241–300, 1998

[25] I.V. P and Yu.K. Sirenko, The Lost “Second Lorentz Theorem” in the Phasor Domain, Telecommunications and Radio Engineering, 68(7):555-560 (2009)

[26] Steven G. Johnson, Peter Bienstman, M. A. Skorobogatiy, Mihai Ibanescu, Elefterios Lidorikis, and J. D. Joannopoulos, Adiabatic theorem and continuous coupled-mode theory for efficient taper transitions in photonic crystals, PHYSICAL REVIEW E 66, 066608 (2002)

[27] Anatoly A. Barybin, Excitation theory for space-dispersive active media waveguides, J. Phys. D: Appl. Phys. 32 (1999) 2014–2028. Printed in the UK

[28] T. D. Carozzi, J. E. S. Bergman and R. L. Karlesson. Complex Poynting Theorem as a Conservation Law, 2005, IEEE Trans. Antennas Propag.

[29] Jiuping Chen, Douglas W. Oldenburg, and Eldad Haber, Reciprocity in electromagnetics: Application to marine magnetometric resistivity Physics of the Earth and Planetary Interiors Volume 150, Issues 1–3, 16 May 2005, Pages 45–61

[30] SHUN-LIEN CHUANG, A Coupled-Mode Theory for multiwaveguide Systems Satisfying the Reciprocity Theorem and Power Conservation, JOURNAL OF LIGHTWAVE TECHNOLOGY, VOL. LT-5, NO. 1, JANUARY 1987.

[31] Burak Polat, On Poynting’s Theorem and Reciprocity Relations for Discontinuous Fields, Electronics Engineering Department, Faculty of Engineering and Architecture Uludag University, Gorukle, Bursa, TR-1 6059 Turkey, E-mail: burakpolat@uludag.edu.tr

[32] Wen Cho Chew, A New Look at Reciprocity and Energy Conservation Theorems in Electromagnetics, IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, VOL. 56, NO. 4, APRIL 2008.

[33] Altman, C., Suchy, K., Reciprocity, Spatial Mapping and Time Reversal in Electromagnetics, ISBN 978-94-007-1530-1
[34] Zhou Dongfang, Zhou Yonghua, Reciprocity and Unitarity of non-loss linear networks in zero state, Journal of electronics Vol.13, No.2 Mar., 1991

[35] Steven H. Schot, Eigty years of Sommerfeld’s Radiation Condition, Historia mathematica 19 (1992), 385-401

[36] Fragstein, Conrad von, The History of the Mixed Poynting Vector, Veröffentlichung in: Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft Band 39, 1987, S.25-29

[37] Angus Macleod, The Mixed Poynting Vector, http://www.svc.org/DigitalLibrary/documents/2014_Summer_AMacleod.pdf

[38] Chen-To Tai, Life Fellow, IEEE, Complementary Reciprocity Theorems in Electromagnetic Theory, IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, VOL. 40, NO. 6, JUNE 1992 675