INVERSE-CLOSEDNESS OF SUBALGEBRAS OF INTEGRAL OPERATORS WITH ALMOST PERIODIC KERNELS

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Abstract. The integral operator of the form

\[(Nu)(x) = \sum_{k=1}^{\infty} e^{i(\omega_k, x)} \int_{\mathbb{R}^c} n_k(x - y) u(y) \, dy\]

acting in \(L_p(\mathbb{R}^c), 1 \leq p \leq \infty,\) is considered. It is assumed that \(\omega_k \in \mathbb{R}^c,\)
\(n_k \in L_1(\mathbb{R}^c),\) and

\[\sum_{k=1}^{\infty} \|n_k\|_{L_1} < \infty.\]

We prove that if the operator \(1 + N\) is invertible, then \((1 + N)^{-1} = 1 + M,\)
where \(M\) is an integral operator possessing the analogous representation.

Introduction

This paper is devoted to almost periodic linear operators, i.e., operators with almost periodic ‘coefficients’ (we interpret the kernel of an integral operator as a kind of ‘coefficients’). Different properties of such operators were investigated in [1, 2, 7, 8, 15, 16, 17, 21, 39, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 56] and other works. Equations and operators with almost periodic ‘coefficients’ often arise in applications for the following reason. It is known [40, p. 8] that if the flow generated by a differential equation is equicontinuous and the image of a solution is relatively compact, then the solution is an almost periodic function. Therefore the linearization of the equation along such a solution gives an equation with almost periodic ‘coefficients’.

It is almost evident that the inverse of an almost periodic operator is also almost periodic (Theorem 2.1). In this paper, we mainly discuss the structure of inverses of integral operators with almost periodic kernels. Our main result (Theorem 7.2) states that if the Fourier series of the kernel of an integral operator converges absolutely, then the kernel of the inverse operator possesses the same property. As an auxiliary result, we prove that the inverse of an integral operator with an almost periodic kernel is also an integral operator with an almost periodic kernel (Theorem 6.3).

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It is natural to formulate and discuss these problems using the language of full [13, ch. 1, § 1.4] or inverse closed [27, p. 183] subalgebras. The history of full subalgebras have its origin in Wiener’s theorem on absolutely convergent Fourier series [57]. For further results related to inverse closed classes, see [3, 4, 5, 6, 10, 11, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 38, 41, 47, 54, 55] and references therein.

The paper is organized as follows. In Section 1, we recall and specify the notation and terminology. In Section 2 we recall the definitions of an almost periodic function and an almost periodic operator. In Sections 3, 4, and 5 we describe the auxiliary facts which we use in the proof: the inverse closedness of the general algebra of almost periodic operators having an absolutely convergent Fourier series (Theorem 3.16), some technical results concerning integral operators acting in \( L_\infty \), and the inverse closedness of the algebra of integral operators whose kernels vary continuously in \( L_1 \)-norm (Theorem 5.6). In Sections 6 we prove the inverse closedness of the algebra of integral operators with almost periodic kernels (Theorem 6.3); the proof essentially uses Theorem 5.6. Finally, in Section 7, we prove the inverse closedness of the algebra of integral operators with almost periodic kernels whose Fourier series converges absolutely (Theorem 7.2); the proof is based on Theorems 3.16 and 6.3.

1. General notation and terminology

Let \( X \) and \( Y \) be complex Banach spaces. We denote by \( \mathcal{B}(X,Y) \) the space of all bounded linear operators acting from \( X \) to \( Y \). If \( X = Y \) we use the brief notation \( \mathcal{B}(X) \). We denote by \( 1 \in \mathcal{B}(X) \) the identity operator.

As usual, \( \mathbb{Z} \) is the set of all integers and \( \mathbb{N} \) is the set of all positive integers. Let \( c \in \mathbb{N} \). The linear space \( \mathbb{R}^c \) is considered with the Euclidean norm \( | \cdot | \) and the associated inner product \( \langle \cdot , \cdot \rangle \).

Let \( E \) be a complex Banach space with the norm \( | \cdot | \); in Sections 2 and 3 we assume that \( E \) is arbitrary, but in subsequent sections we assume that \( E \) is finite dimensional and Hilbert. We denote by \( \mathcal{L}_p = \mathcal{L}_p(\mathbb{R}^c, E) \), \( 1 \leq p < \infty \), the space of all measurable functions \( u : \mathbb{R}^c \to E \) bounded in the semi-norm

\[
\| u \| = \| u \|_{L_p} = \left( \int_{\mathbb{R}^c} |u(x)|^p \, dx \right)^{1/p},
\]

and we denote by \( \mathcal{L}_\infty = \mathcal{L}_\infty(\mathbb{R}^c, E) \) the space of all measurable essentially bounded functions \( u : \mathbb{R}^c \to E \) with the semi-norm

\[
\| u \| = \| u \|_{L_\infty} = \text{ess sup} |u(x)|.
\]

Finally, we denote by \( L_p = L_p(\mathbb{R}^c) = \mathcal{L}_p(\mathbb{R}^c, E) \), \( 1 \leq p \leq \infty \), the Banach space of all classes of functions \( u \in \mathcal{L}_p \) with the identification almost everywhere. For more details, see [14]. Usually they do not distinguish the spaces \( \mathcal{L}_p \) and \( L_p \).

For any \( h \in \mathbb{R}^c \), we define the shift operator

\[
(S_h u)(x) = u(x-h),
\]

and for any \( \omega \in \mathbb{R}^c \), we define the oscillation (modulation) operator

\[
(\Psi_\omega u)(x) = e^{i\langle \omega , x \rangle} u(x).
\]
Proposition 1.1. For all $h, \omega \in \mathbb{R}^c$,
\[ S_h \Psi_\omega = e^{-i\langle \omega, h \rangle} \Psi_\omega S_h. \]

Proof. The proof is by direct calculations. \(\square\)

All algebras \([13]\) are considered over the field of complex numbers. A complete normed algebra is called \textit{Banach}. We denote the unit of an algebra by the symbol $1$. If an algebra has a unit, it is called \textit{unital}.

A subset $R$ of an algebra $B$ is called a \textit{subalgebra} if $R$ is stable under the algebraic operations (addition, scalar multiplication, and multiplication), i.e. $A + B, \lambda A, AB \in R$ for all $A, B \in R$ and $\lambda \in \mathbb{C}$. If the unit $1$ of an algebra $B$ belongs to its subalgebra $R$, then $R$ is called a \textit{unital subalgebra}. Any non-unital subalgebra $R$ of an algebra $B$ can be extended to a unital subalgebra, which we denote by $\tilde{R}$.

A unital subalgebra $R$ of a unital algebra $B$ is called \textit{full} \([13], \text{ch. 1, \S 1.4}\) or \textit{inverse closed} \([27], \text{p. 183}\) if every $B \in R$ that is invertible in $B$ is also invertible in $R$. This definition is equivalent to the following one: for any $B \in R$, the existence of $B^{-1} \in B$ such that $BB^{-1} = B^{-1}B = 1$ implies that $B^{-1} \in R$.

2. Almost periodic functions and operators

A subset of a Banach space is called \textit{relatively compact} if its closure is compact with respect to the norm topology.

Let $X$ be a Banach space, and $T$ be a topological space. We denote by $C(T, X)$ the Banach space of all bounded continuous functions $u : T \to X$ with the norm
\[ \|u\|_C = \sup_{x \in \mathbb{R}^c} \|u(x)\|. \]

The main example is the space $C = C(\mathbb{R}^c, X)$. Clearly, the operators $S_h$, $h \in \mathbb{R}^c$, defined by formula (1.1) act in $C$ and $\|S_h\| = 1$. A function $x \in C(\mathbb{R}^c, X)$ is called \textit{almost periodic} \([1, 18, 40, 45]\) if the set $\{S_h x : h \in \mathbb{R}^c\} \subseteq C$ is relatively compact. We denote by $C_{AP} = C_{AP}(\mathbb{R}^c, X)$ the subset of $C$ consisting of all almost periodic functions. Clearly, $C_{AP}$ is a closed subspace of $C$.

An operator $T \in \mathcal{B}(L_p(\mathbb{R}^c))$, $1 \leq p \leq \infty$, is called \([16, 52]\) \textit{almost periodic} if the family
\[ T\{h\} = S_h TS_{-h}, \quad h \in \mathbb{R}^c, \quad (2.1) \]
where $S_h$ is defined by formula (1.1), continuously (in the norm) depends on $h$ and is relatively compact in $\mathcal{B}(L_p(\mathbb{R}^c))$. We denote the set of all almost periodic operators $T \in \mathcal{B}(L_p(\mathbb{R}^c))$ by $\mathcal{B}_{AP}(L_p) = \mathcal{B}_{AP}(L_p(\mathbb{R}^c))$.

Theorem 2.1 (see, e.g., \([36, \text{Theorem 6.5.2}]\)). The set $\mathcal{B}_{AP}(L_p)$ is a closed full subalgebra of the algebra $\mathcal{B}(L_p)$.

Proof. The proof immediately follows from the definition of an almost periodic operator. \(\square\)
3. Almost periodic operators with absolutely convergent Fourier series

In this Section we assume that $E$ is an arbitrary Banach space. We call an operator $A \in B(L_p)$ shift invariant if

$$AS_h = S_h A, \quad h \in \mathbb{R}^c.$$  

We denote by $A(L_p)$ the set of all shift invariant operators $A \in B(L_p)$.

**Example 3.1.** (a) The integral convolution operator

$$(Gu)(x) = \int_{\mathbb{R}^c} g(x-y)u(y)\, dy,$$

where $g \in L_1(\mathbb{R}^c, B(E))$, belongs to $A(L_p)$, $1 \leq p \leq \infty$, see, e.g., [36, Corollary 4.4.11]. (b) More generally, the convolution operator

$$(Tu)(x) = \int_{\mathbb{R}^c} d\mu(x-y)u(y),$$

with a bounded operator-valued measure $\mu$ on $\mathbb{R}^c$ belongs to $A(L_p)$, $1 \leq p \leq \infty$, see [36, Theorem 4.4.4] for more details. (c) Let the space $E$ be Hilbert, $\mathcal{F} : L_2(\mathbb{R}^c) \to L_2(\mathbb{R}^c)$ be the Fourier transform, and $\xi \in L_\infty(\mathbb{R}^c, B(E))$. Then the operator $u \mapsto \mathcal{F}^{-1}(\xi \cdot \mathcal{F} u)$ belongs to $A(L_2)$.

**Proposition 3.2.** The set $A(L_p)$ is a full closed subalgebra of $B(L_p)$, $1 \leq p \leq \infty$.

**Proof.** The proof is evident. 

**Proposition 3.3.** Let $A \in A(L_p)$, $1 \leq p \leq \infty$. Then $\Psi_{-\omega} A \Psi_{\omega} \in A(L_p)$ for any $\omega \in \mathbb{R}^c$.

**Proof.** The proof follows from Proposition 1.1. 

We denote by $\mathcal{B}_{APW}(L_p)$, $1 \leq p \leq \infty$, the set of all operators of the form

$$K = \sum_{\omega \in \mathbb{R}^c} \Psi_{\omega} A_{\omega},$$  

where $A_{\omega} \in A(L_p)$ and at most a countable number of operators $A_{\omega}$ are nonzero, with

$$\sum_{\omega \in \mathbb{R}^c} \|A_{\omega}\| < \infty.$$  

We call an operator $K \in \mathcal{B}_{APW}(L_p)$ an almost periodic operator with absolutely convergent Fourier series.

**Proposition 3.4.** The set $\mathcal{B}_{APW}(L_p)$ is a subalgebra of $B(L_p)$, $1 \leq p \leq \infty$.

**Proof.** The proof follows from Propositions 3.3 and 3.2. 

For $K \in \mathcal{B}_{APW}(L_p)$, in accordance with (2.1), we set

$$K\{h\} = S_h KS_{-h}, \quad h \in \mathbb{R}^c.$$  

(3.2)
**Proposition 3.5.** Let an operator \( K \in B_{APW}(L_p) \) has the form (3.1). Then
\[
K \{ h \} = \sum_{\omega \in \mathbb{R}^c} e^{-i(\omega, h)} \Psi_\omega A_\omega, \quad h \in \mathbb{R}^c.
\]

*Proof.* The proof follows from Proposition 1.1. \( \Box \)

We denote by \( \mathbb{U} \) the multiplicative group \( \{ z \in \mathbb{C} : |z| = 1 \} \). A function \( \varkappa : \mathbb{R}^c \to \mathbb{U} \) is called \([13, \text{ch. II, } \S 1.1], [31, 22.15] \) a **character** of the group \( \mathbb{R}^c \) if
\[
\varkappa(x + y) = \varkappa(x)\varkappa(y), \quad x, y \in \mathbb{R}^c,
\]
\[
|\varkappa(x)| = 1, \quad x \in \mathbb{R}^c.
\]
We denote by \( \mathfrak{X}_b = \mathfrak{X}_b(\mathbb{R}^c) \) the set of all **characters** and call elements of \( \mathfrak{X}_b \) (discontinuous) **characters** of \( \mathbb{R}^c \). Sometimes we will denote the action of a character \( \varkappa \in \mathfrak{X}_b \) on \( x \in \mathbb{R}^c \) by the symbol \( \langle x, \varkappa \rangle \).

We denote by \( \mathfrak{X} = \mathfrak{X}(\mathbb{R}^c) \) the set of all **continuous** (with respect to the usual topology on \( \mathbb{R}^c \)) **characters**. We endow \( \mathfrak{X}_b = \mathfrak{X}_b(\mathbb{R}^c) \) with the topology of pointwise convergence. And we endow \( \mathfrak{X} = \mathfrak{X}(\mathbb{R}^c) \) with the topology of uniform convergence on compact sets. Clearly, the topology on \( \mathfrak{X}_b = \mathfrak{X}_b(\mathbb{R}^c) \) can also be interpreted as a topology of uniform convergent on compact sets provided one considers the group \( \mathbb{R}^c \) with the discrete topology.

We define the sum of elements of \( \mathfrak{X}_b \) in the pointwise sense:
\[
(\varkappa_1 + \varkappa_2)(x) = \varkappa_1(x)\varkappa_2(x).
\]

**Proposition 3.6 (\([31, 23.2]\)).** The set \( \mathfrak{X}_b = \mathfrak{X}_b(\mathbb{R}^c) \) of all characters of \( \mathbb{R}^c \) is an abelian group, and \( \mathfrak{X} = \mathfrak{X}(\mathbb{R}^c) \) is its subgroup.

Clearly, the zero element of the groups \( \mathfrak{X}_b \) and \( \mathfrak{X} \) is the function \( 0(x) \equiv 1 \).

**Proposition 3.7 (\([31, 23.27f]\)).** The group \( \mathfrak{X}(\mathbb{R}^c) \) is topologically isomorphic to the group \( \mathbb{R}^c \). Namely, \( \mathfrak{X} = \mathfrak{X}(\mathbb{R}^c) \) consists of the functions \( \chi = \chi_\omega : \mathbb{R}^c \to \mathbb{U} \) of the form
\[
\chi(x) = \chi_\omega(x) = e^{i(x, \omega)},
\]
where \( \omega \) runs over \( \mathbb{R}^c \), and the mapping \( J : \omega \mapsto \chi_\omega \) is an isomorphism.

**Corollary 3.8.** The function \( K \{ \cdot \} : \mathbb{R}^c \to B(L_p) \) defined by rule (3.3) is equivalent to the function \( K \{ \cdot \} : \mathfrak{X}(\mathbb{R}^c) \to B(L_p) \) defined by the formula
\[
K \{ \chi \} = \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \chi \rangle \Psi_\omega A_\omega, \quad \chi \in \mathfrak{X}(\mathbb{R}^c).
\]

More precisely, the ‘equivalence’ means that the diagram
\[
\begin{array}{ccc}
\mathbb{R}^c & \xrightarrow{J} & \mathfrak{X}(\mathbb{R}^c) \\
\kappa \{ \cdot \} \downarrow & & \kappa \{ \cdot \}
\end{array}
\]
\[
\begin{array}{ccc}
B(L_p) & \xrightarrow{\kappa \{ \cdot \}} & B(L_p)
\end{array}
\]

is commutative (see Proposition 3.7 for the definition of \( J \)).

*Proof.* The proof follows from Proposition 3.7. \( \Box \)
We denote by $\mathbb{R}_d^c$ the group $\mathbb{R}^c$ considered with the discrete topology.

**Proposition 3.9.** The group $X_b(\mathbb{R}^c)$ is compact.

*Proof.* Clearly, the set $X_b = X_b(\mathbb{R}^c)$ can be interpreted as the set of all continuous characters of $\mathbb{R}_d^c$. Now the statement follows from [31, Theorem 23.17].

For a more detailed structure of $X_b = X_b(\mathbb{R}^c)$, see, e.g., [36, Example 4.2.1e].

**Proposition 3.10 ([31, 26.15]).** The subgroup $X(\mathbb{R}^c) \simeq \mathbb{R}^c$ is dense in the group $X_b(\mathbb{R}^c)$ (in the topology of $X_b(\mathbb{R}^c)$).

In accordance with Proposition 3.7, we identify the initial group $\mathbb{R}^c$ with $X(\mathbb{R}^c)$; and according to Proposition 3.10 we extend the group $\mathbb{R}^c \simeq X(\mathbb{R}^c)$ to the group $X_b(\mathbb{R}^c)$; in the latter case we denote the extension $X_b(\mathbb{R}^c)$ by $\mathbb{R}_b^c$. We denote by $C(\mathbb{R}_b^c, X)$ the Banach space of all continuous functions $u : \mathbb{R}_b^c \to X$ with the norm $\|u\| = \max_{t \in \mathbb{R}_b^c} \|u(t)\|$.

**Theorem 3.11 ([18, 16.2.1], [45, p. 7]).** Let $X$ be a Banach space. For a function $u \in C(\mathbb{R}^c, X)$, the following assumptions are equivalent.

(a) $u \in C_{AP}(\mathbb{R}^c, X)$.
(b) The function $u$ can be approximated in the norm of $C(\mathbb{R}^c, X)$ by functions of the form

$$p(t) = \sum_{k=1}^{m} \langle x, \chi_k \rangle u_k,$$

where $\chi_k \in X(\mathbb{R}^c)$ and $u_k \in X$.
(c) The function $u$ possesses an extension to a function $u \in C(\mathbb{R}_b^c, X)$. By Proposition 3.10, this extension is unique.

In order not to complicate our notation, we use the same symbol for an initial function defined on $\mathbb{R}^c$ and for its extension to $\mathbb{R}_b^c$. We will always make it clear which function we mean by mentioning its domain.

**Corollary 3.12.** Let $T \in B_{AP}(L_p(\mathbb{R}^c))$. Then family (2.1) possesses a unique extension to a function $T\{\cdot\} \in C(\mathbb{R}_b^c, B(L_p(\mathbb{R}^c)))$.

*Proof.* The proof follows from Theorem 3.11.

A positive measure on $X_b = \mathbb{R}_b^c$ is a bounded linear functional $\mu$ on $C(X_b, \mathbb{C})$ that is non-negative on non-negative functions $u \in C(X_b, \mathbb{C})$. A Haar measure on the group $X_b = \mathbb{R}_b^c$ is a non-zero shift invariant positive measure $\mu$, i.e., $\mu(u) = \mu(S_x u)$ for all $u \in C(X_b, \mathbb{C})$ and $x \in X_b$; here $(S_x u)(x) = u(x - x)$, $x \in X_b$.

**Proposition 3.13 ([31, Theorem 15.5]).** The Haar measure exists and is determined uniquely, up to a constant factor.

We normalize the Haar measure on $X_b = \mathbb{R}_b^c$ so that $\int_{X_b} d\nu = 1$.

**Lemma 3.14** (see, e.g., [36, Lemma 5.2.3]). For $h \in \mathbb{R}_d^c$ one has

$$\int_{X_b} \langle h, \nu \rangle d\nu = \begin{cases} 1 & \text{if } h = 0, \\ 0 & \text{if } h \neq 0. \end{cases}$$
We consider the Banach algebra \( l_1 = l_1(\mathbb{R}^c, B(L_p)) \). The algebra \( l_1 \) consists of all families \( T = \{ T_\omega \in B(L_p) : \omega \in \mathbb{R}^c \} \) such that the norm \( \| T \| = \sum_{\omega \in \mathbb{R}^c} \| T_\omega \| \) is finite (it is assumed that at most a countable number of operators \( T_\omega \) are nonzero), with the point-wise linear operations. The multiplication in \( l_1 \) is defined as the convolution: the family \( \mathcal{R} = T \ast S \), where \( \mathcal{T} = \{ T_\omega \in B(L_p) : \omega \in \mathbb{R}^c \} \) and \( S = \{ S_\omega \in B(L_p) : \omega \in \mathbb{R}^c \} \), consists of the elements

\[
R_\omega = \sum_{\nu \in \mathbb{R}^c} T_{\omega - \nu} S_{\nu}, \quad \omega \in \mathbb{R}^c.
\]

We will use the following variant of the Bochner–Phillips theorem [12].

**Lemma 3.15 ([36, Corollary 4.5.2(g)])**. An element \( \{ T_\omega \in B(L_p) : h \in \mathbb{R}^c \} \in l_1 \) is invertible in the algebra \( l_1 \) if and only if the operators \( T\{ \kappa \} = \sum_{\omega \in \mathbb{R}^c} \langle \omega, \kappa \rangle T_\omega \) are invertible for all \( \kappa \in \mathbb{R}^c \).

**Theorem 3.16**. The subalgebra \( B_{APW}(L_p) \), \( 1 \leq p \leq \infty \), is full in the algebra \( B(L_p) \). More precisely, if an operator \( K \in B_{APW}(L_p) \) is invertible, then \( K^{-1} = \sum_{\omega \in \mathbb{R}^c} \Psi_{-\omega} B_\omega \), where \( B_\omega \in A(L_p) \), with \( \sum_{\omega \in \mathbb{R}^c} \| B_\omega \| < \infty \), and \( \Psi \) is invertible for all \( \kappa \in \mathbb{X}_h(\mathbb{R}^c) \).

Theorem 3.16 can be obtained as a special case of [3, Theorem 3.2], see also [4] and [36, Theorem 5.2.5]. Nevertheless, we give here a detailed proof for the completeness of the exposition.

**Proof.** Let an invertible operator \( K \in B_{APW}(L_p) \) be represented in the form (3.1).

For all \( \kappa \in \mathbb{X}_h(\mathbb{R}^c) \), we consider the operator

\[
K\{ \kappa \} = \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa \rangle \Psi_\omega A_\omega.
\]  

It is clear that function (3.5) is an extension from \( \mathbb{X}(\mathbb{R}^c) \) to \( \mathbb{X}_h(\mathbb{R}^c) \) of the function \( K\{ \cdot \} \) defined by (3.4). Moreover, since the series in (3.5) is absolutely convergent, function (3.5) is continuous with respect to the topology of \( \mathbb{X}_h(\mathbb{R}^c) \). By Proposition 3.10, it coincides with the extension of (3.4) by continuity.

Since the operator \( K \) is invertible in the algebra \( B(L_p) \), from (3.2) and Corollary 3.8 it is evident that the operators \( K\{ \chi \}, \chi \in \mathbb{X}(\mathbb{R}^c) \), are also invertible and \( \|(K\{ \chi \})^{-1}\| = \|K^{-1}\| \). Because \( \mathbb{X}(\mathbb{R}^c) \) is dense in \( \mathbb{X}_h(\mathbb{R}^c) \) (see Proposition 3.10), the operators \( K\{ \kappa \} \) are invertible for all \( \kappa \in \mathbb{X}_h(\mathbb{R}^c) \) as well, with \( \|(K\{ \kappa \})^{-1}\| = \|K^{-1}\| \).

We briefly denote the operators \( \Psi_\omega A_\omega \) from (3.1) by \( T_\omega \) and consider the family \( \{ T_\omega \in B(L_p) : h \in \mathbb{R}^c \} \in l_1(\mathbb{R}^c, B(L_p)) \) associated with the operator \( K \). As we have seen above, the operators \( K\{ \kappa \}, \kappa \in \mathbb{X}_h(\mathbb{R}^c) \), are invertible. Hence, by Lemma 3.15, the family \( \{ T_\omega \} \) is invertible in the algebra \( l_1(\mathbb{R}^c, B(L_p)) \); we denote its inverse by \( \{ R_\omega \} \in l_1(\mathbb{R}^c, B(L_p)) \). Thus,

\[
\sum_{\nu \in \mathbb{R}^c} T_{\omega - \nu} R_\nu = \sum_{\nu \in \mathbb{R}^c} R_{\omega - \nu} T_\nu = \begin{cases} 
1 & \text{if } \omega = 0, \\
0 & \text{otherwise}.
\end{cases}
\]
We show that the function \( \kappa \mapsto \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa \rangle R_\omega \) is the point-wise inverse of the function \( \kappa \mapsto K\{\kappa\} \). In other words,

\[
(K\{\kappa\})^{-1} = \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa \rangle R_\omega
\]  

(3.6)

for some \( R_\omega \in \mathcal{B}(L_p) \), with \( \sum_{\omega \in \mathbb{R}^c} \|R_\omega\| < \infty \). Indeed,

\[
\left( \sum_{\mu \in \mathbb{R}^c} \langle -\mu, \kappa \rangle T_\mu \right) \left( \sum_{\nu \in \mathbb{R}^c} \langle -\nu, \kappa \rangle R_\nu \right) = \sum_{\mu \in \mathbb{R}^c} \sum_{\nu \in \mathbb{R}^c} \langle -\mu - \nu, \kappa \rangle T_\mu R_\nu
\]

\[
= \sum_{\nu \in \mathbb{R}^c} \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa \rangle T_{\omega - \nu} R_\nu
\]

\[
= \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa \rangle \sum_{\nu \in \mathbb{R}^c} T_{\omega - \nu} R_\nu
\]

\[
= \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa \rangle \begin{cases} 1 & \text{if } \omega = 0, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
= 1.
\]

The equality \( \left( \sum_{\nu \in \mathbb{R}^c} \langle -\nu, \kappa \rangle R_\nu \right) \left( \sum_{\mu \in \mathbb{R}^c} \langle -\mu, \kappa \rangle T_\mu \right) = 1 \) is established in a similar way. In particular, substituting in these formulae \( \kappa = 0 \) we obtain

\[
K^{-1} = (K\{0\})^{-1} = \sum_{\omega \in \mathbb{R}^c} R_\omega.
\]  

(3.7)

To complete the proof, we show that \( R_\omega \) has the form \( R_\omega = \Psi_{-\omega} B_\omega \), where \( B_\omega \in \mathcal{A}(L_p) \).

It is straightforward to verify that

\[
K\{\kappa + \chi_h\} = S_h K\{\kappa\} S_h^{-1}, \quad h \in \mathbb{R}^c, \, \kappa \in \mathcal{X}_b.
\]

From this identity, it follows that for an arbitrary \( \omega_0 \)

\[
\langle \omega_0, \kappa \rangle K\{\kappa + \chi_h\} = \langle \omega_0, \kappa \rangle S_h K\{\kappa\} S_h^{-1}, \quad \kappa \in \mathcal{X}_b.
\]

We invert both the left and right sides:

\[
\langle -\omega_0, \kappa \rangle (K\{\kappa + \chi_h\})^{-1} = \langle -\omega_0, \kappa \rangle S_h (K\{\kappa\})^{-1} S_h^{-1}, \quad \kappa \in \mathcal{X}_b.
\]

Substituting representation (3.6) is this formula, we obtain

\[
\sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa + \chi_h \rangle \langle -\omega_0, \kappa \rangle R_\omega = \langle -\omega_0, \kappa \rangle S_h \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \kappa \rangle R_\omega S_h^{-1}, \quad \omega \in \mathbb{R}^c, \, \kappa \in \mathcal{X}_b.
\]

Or

\[
\sum_{\omega \in \mathbb{R}^c} \langle -\omega, \chi_h \rangle \langle -\omega - \omega_0, \kappa \rangle R_\omega = S_h \left( \sum_{\omega \in \mathbb{R}^c} \langle -\omega - \omega_0, \kappa \rangle R_\omega \right) S_h^{-1}, \quad \omega \in \mathbb{R}^c, \, \kappa \in \mathcal{X}_b.
\]
Next, we integrate the both sides with respect to \( \kappa \in X_b \) (using the Haar measure and taking into account Lemma 3.14):

\[
\int_{X_b} \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \chi h \rangle \langle -\omega - \omega_0, \kappa \rangle R_{\omega} \, d\kappa
\]

\[
= \int_{X_b} S_h \left( \sum_{\omega \in \mathbb{R}^c} \langle -\omega - \omega_0, \kappa \rangle R_{\omega} \right) S_h^{-1} \, d\kappa, \quad \omega \in \mathbb{R}^c,
\]

or

\[
\langle -\omega_0, \chi h \rangle R_{-\omega_0} = S_h R_{-\omega_0} S_h^{-1},
\]

or

\[
R_{-\omega_0} = e^{i(\omega_0, h)} S_h R_{-\omega_0} S_h^{-1},
\]

or

\[
R_{\omega_0} = e^{-i(\omega_0, h)} S_h R_{\omega_0} S_h^{-1},
\]

for all \( \omega_0 \in \mathbb{R}^c \). By Proposition 1.1, the last equality implies that

\[
\Psi_{\omega_0} R_{\omega_0} = S_h \Psi_{\omega_0} R_{\omega_0} S_h^{-1},
\]

which means that \( B_{\omega_0} = \Psi_{\omega_0} R_{\omega_0} \in A(L_p) \). \( \square \)

**Remark 1.** It is useful to notice that a one-sided invertibility of an almost periodic operator often implies its two-sided invertibility [21, 33, 35, 42, 43, 49, 52, 53].

### 4. Integral operators in \( L_\infty \)

In this and subsequent sections we assume that \( E \) is a finite-dimensional Hilbert space.

We denote by \( N_b = N_b(\mathbb{R}^c, E) \) the set of all measurable functions \( n : \mathbb{R}^c \times \mathbb{R}^c \to B(E) \) such that for almost all \( x \in \mathbb{R}^c \) the function \( n(x, \cdot) \) (is defined almost everywhere and) belongs to \( L_1(\mathbb{R}^c, B(E)) \) and

\[
||n||_{N_b} = \text{ess sup}_{x \in \mathbb{R}^c} ||n(x, \cdot)||_{L_1} < \infty.
\]

We introduce in \( N_b(\mathbb{R}^c, E) \) the identification almost everywhere. After that \( N_b(\mathbb{R}^c, E) \) becomes a normed space with the natural linear operations and the norm \( ||\cdot||_{N_b} \).

**Proposition 4.1.** The function \( n : \mathbb{R}^c \times \mathbb{R}^c \to B(E) \) is measurable if and only if the function \( n_1(x, y) = n(x, x - y) \) is measurable.

**Proof.** The proof is a word for word repetition of that of [36, Lemma 4.1.5]. \( \square \)

For any \( n \in N_b(\mathbb{R}^c, E) \), we denote by \( \bar{n} \) the function that assigns to each \( x \in \mathbb{R}^c \) the function \( \bar{n}(x) : \mathbb{R}^c \to B(E) \) defined by the rule

\[
\bar{n}(x)(y) = n(x, x - y); \quad (4.1)
\]

and for any \( n \in N_b(\mathbb{R}^c, E) \), we denote by \( \tilde{n} \) the function that assigns to each \( x \in \mathbb{R}^c \) the function \( \tilde{n}(x) : \mathbb{R}^c \to B(E) \) defined by the rule

\[
\tilde{n}(x)(y) = n(x, y). \quad (4.2)
\]
Proposition 4.2. The normed space $N_b(\mathbb{R}^c, E)$ (with the identification almost everywhere) is isometrically isomorphic to $L_\infty(\mathbb{R}^c, L_1(\mathbb{R}^c, B(E)))$.

Proof. Let $n \in N_b(\mathbb{R}^c, E)$. By the Fubini theorem [14], the restriction of $n$ to the set $K \times \mathbb{R}^c$, where $K \subset \mathbb{R}^c$ is an arbitrary compact set, is summable. Therefore the restriction $\tilde{n}_K$ of $\tilde{n}$ to $K$ is a function of the class $L_1$. It is clear that actually $\tilde{n} \in L_\infty$ and $\|\tilde{n}\|_{L_\infty} = \|n\|_{N_b}$.

Conversely, let $\tilde{n} \in L_\infty(\mathbb{R}^c, L_1(\mathbb{R}^c, B(E)))$. Let $K \subset \mathbb{R}^c$ be compact. Clearly, the restriction $\tilde{n}_K$ of $\tilde{n}$ to $K$ belongs to $L_\infty(K, L_1(\mathbb{R}^c, B(E)))$. Further, the space $L_\infty(K, L_1(\mathbb{R}^c, B(E)))$ is included into $L_1(K, L_1(\mathbb{R}^c, B(E)))$. It is known (see, e.g., [36, Corollary 1.5.9]) that the space $L_1(K, L_1(\mathbb{R}^c, B(E)))$ is naturally isometrically isomorphic to $L_1(K \times \mathbb{R}^c, B(E))$. Thus, we obtain a measurable function $n_K : K \times \mathbb{R}^c \to B(E)$. For almost all $x \in \mathbb{R}^c$ the function $n_K(x, \cdot)$ (is defined almost everywhere and) belongs to $L_1(\mathbb{R}^c, B(E))$ and

$$\text{ess sup}_{x \in \mathbb{R}^c} \|n_K(x, \cdot)\|_{L_1} \leq \|\tilde{n}\|_{L_\infty}.$$ 

Clearly, if $K \subseteq K_1$ are compact sets, the restriction of $n_{K_1}$ to $K \times \mathbb{R}^c$ coincides almost everywhere with $n_k$. \hfill \Box

Proposition 4.3. For any $n \in N_b(\mathbb{R}^c, E)$, the operator

$$ (Nu)(x) = \int_{\mathbb{R}^c} n(x, x - y) u(y) \, dy $$

acts in $L_\infty(\mathbb{R}^c, E)$. More precisely, for any $u \in L_\infty(\mathbb{R}^c, E)$ the function $y \mapsto n(x, x - y) u(y)$ is integrable for almost all $x$, and the function $Nu$ belongs to $L_\infty(\mathbb{R}^c, E)$; if $u_1$ and $u_2$ coincide almost everywhere, then $Nu_1$ and $Nu_2$ also coincide almost everywhere. Besides,

$$ c\|n\|_{N_b} \leq \|N : L_\infty \to L_\infty\| \leq \text{ess sup}_{x \in \mathbb{R}^c} \|n(x, \cdot)\|_{L_1}, $$

where $c$ is independent of $n$.

Proof. Let $K \subset \mathbb{R}^c$ be compact. Then the function $(x, y) \mapsto n(x, x - y) u(y)$, $x \in K$, $y \in \mathbb{R}^c$, belongs to $L_1(K \times \mathbb{R}^c, B(E))$. Therefore, by the Fubini theorem, the function $y \mapsto n(x, x - y) u(y)$ is integrable for almost all $x \in K$. The estimate

$$ \text{ess sup}_{x \in \mathbb{R}^c} \left\| \int_{\mathbb{R}^c} n(x, x - y) u(y) \, dy \right\| \leq \text{ess sup}_{x \in \mathbb{R}^c} \|n(x, \cdot)\|_{L_1} $$

is evident.

The proof of the first part of estimate (4.4) repeats the proof of [38, Proposition 3.7]. \hfill \Box

Proposition 4.4. Let $n \in N_b$, and the operator $N$ be defined by formula (4.3). Then for any $h \in \mathbb{R}^c$ the operator $S_h NS_{-h}$ is defined by the formula

$$ (S_h NS_{-h} u)(x) = \int_{\mathbb{R}^c} n(x - h, x - y) u(y) \, dy. $$

Thus, the operator $S_h NS_{-h}$ is generated by the kernel $(x, y) \mapsto n(x - h, y)$. \hfill \Box
Proof. Indeed, we have

\[
(\mathcal{N}S_h u)(x) = \int_{\mathbb{R}^c} n(x, x - y) u(y + h) \, dy,
\]

\[
(S_h \mathcal{N}S_{-h} u)(x) = \int_{\mathbb{R}^c} n(x - h, x - h - y) u(y + h) \, dy
\]

\[
= \int_{\mathbb{R}^c} n(x - h, x - y) u(y) \, dy. \quad \square
\]

We denote by \(\mathcal{N}_b(L_{\infty})\) the set of all operators induced by kernels \(n \in \mathcal{N}_b(\mathbb{R}^c, E)\).

Corollary 4.5. The spaces \(L_{\infty}(\mathbb{R}^c, L_1(\mathbb{R}^c, B(E)))\) and \(\mathcal{N}_b(L_{\infty})\) are topologically isomorphic. Moreover, the natural correspondence \(U : \tilde{n} \mapsto N\) maps \(S_h \tilde{n}\) to \(N\{h\}\) for all \(h \in \mathbb{R}^c\). Thus, a function \(\tilde{n} \in L_{\infty}(\mathbb{R}^c, L_1(\mathbb{R}^c, B(E)))\) is almost periodic if and only if the associated operator \(N \in \mathcal{N}_b(L_{\infty})\) is almost periodic.

Proof. By Propositions 4.2 and 4.3, the correspondence \(U : \tilde{n} \mapsto N\) is a topological isomorphism from \(L_{\infty}(\mathbb{R}^c, L_1(\mathbb{R}^c, B(E)))\) onto \(\mathcal{N}_b(L_{\infty})\). From Proposition 4.3 it follows that the operator \(S_h \mathcal{N}S_{-h}\) is generated by the function \(S_h \tilde{n}, h \in \mathbb{R}^c\). Thus, the two functions \(h \mapsto S_h \tilde{n}\) and \(h \mapsto N\{h\}\) are connected by the isomorphism \(U : \tilde{n} \mapsto N\), i.e., \(U : S_h \tilde{n} \mapsto N\{h\}\) for all \(h \in \mathbb{R}^c\). Since \(U\) is a topological isomorphism, the images of these two functions are relatively compact simultaneously. Thus, \(\tilde{n}\) and \(N\) are almost periodic simultaneously as well. \(\square\)

Corollary 4.6. Let an operator \(N \in \mathcal{N}_b\) be shift invariant, i.e.,

\[NS_h = S_h N, \quad h \in \mathbb{R}^c.\]

Then \(N\) is an operator of convolution with a function \(g \in L_1(\mathbb{R}^c, B(E))\).

Proof. By Propositions 4.3 and 4.4, it is enough to prove that if a function \(\tilde{n} \in L_{\infty}(\mathbb{R}^c, L_1(\mathbb{R}^c, B(E)))\) possesses the equalities (almost everywhere)

\[S_h \tilde{n} = \tilde{n} ,\]

for all \(h \in \mathbb{R}^c\), then \(\tilde{n}\) is a constant function almost everywhere.

We consider the function

\[v(t, h) = \tilde{n}(t) - \tilde{n}(t - h), \quad t, h \in \mathbb{R}^c.\]

It is measurable, see, e.g., [36, Lemma 4.1.5]. By assumption, for any \(h \in \mathbb{R}^c\) the function \(t \mapsto v(t, h)\) equals zero almost everywhere. Hence, by the Fubini theorem, the function \(v\) equals zero almost everywhere. Therefore for almost all \(t \in \mathbb{R}^c\) the function \(h \mapsto v(t, h)\) equals zero almost everywhere. Let us take such a point \(t\). It follows that \(\tilde{n}(t - h) = \tilde{n}(t)\) for almost all \(h\), which was to be proved. \(\square\)

Proposition 4.7. The set \(\mathcal{N}_b(L_{\infty})\) is a subalgebra of the algebra \(B(L_{\infty})\).
Proof. Let \( n, m \in \mathbb{N}_b \). Let \( K \subset \mathbb{R}^c \) be a compact set. For almost all \( x \in K \) we have

\[
(NM u)(x) = \int_{\mathbb{R}^c} n(x, x - y) (M u)(y) \, dy
= \int_{\mathbb{R}^c} n(x, x - y) \int_{\mathbb{R}^c} m(y, y - z) u(z) \, dz \, dy.
\]

The function \( (x, y, z) \mapsto n(x, x - y) m(y, y - z) u(z) \) is measurable and majorized by an absolutely integrable function on \( K \times \mathbb{R}^c \times \mathbb{R}^c \); therefore, by the Fubini theorem, the order of integration may be changed. Thus,

\[
(NM u)(x) = \int_{\mathbb{R}^c} \left( \int_{\mathbb{R}^c} n(x, x - y) m(y, y - z) \, dy \right) u(z) \, dz.
\]

(4.5)

Again by the Fubini theorem, the function

\[
k(x, z) = \int_{\mathbb{R}^c} n(x, x - y) m(y, y - z) \, dy
\]

defined by the internal integral in (4.5) exists for almost all \( (x, z) \in K \times \mathbb{R}^c \); moreover, it is a summable function. Finally, for almost all \( x \), we have

\[
\int_{\mathbb{R}^c} \|k(x, z)\| \, dz \leq \int_{\mathbb{R}^c} \left( \int_{\mathbb{R}^c} \|n(x, x - y)\| \|m(y, y - z)\| \, dy \right) \, dz
= \int_{\mathbb{R}^c} \|n(x, x - y)\| \left( \int_{\mathbb{R}^c} \|m(y, y - z)\| \, dz \right) \, dy
\leq \|n\|_{\mathbb{N}_b} \|m\|_{\mathbb{N}_b},
\]

which means that \( k \in \mathbb{N}_b \). From (4.5) it follows that the operator \( NM \) is induced by the kernel \( k \).

5. Integral operators with \( L_1 \)-continuously varying kernels

We denote by \( \mathbb{N}_1 = \mathbb{N}_1(\mathbb{R}^c, E) \) the set of all measurable functions \( n : \mathbb{R}^c \times \mathbb{R}^c \to \mathcal{B}(E) \) satisfying the assumption: there exists a function \( \beta \in \mathcal{L}_1(\mathbb{R}^c, \mathbb{R}) \) such that

\[
\|n(x, y)\| \leq \beta(y)
\]

(5.1)

for almost all \( (x, y) \in \mathbb{R}^c \times \mathbb{R}^c \). For convenience (without loss of generality), we assume that \( \beta \) is defined everywhere. Clearly, \( \mathbb{N}_1 \subset \mathbb{N}_b \). Kernels of the class \( \mathbb{N}_1 \) and operators induced by them were considered in [10, 9, 20, 24, 36, 37, 38].

Proposition 5.1 ([36, Proposition 5.4.3]). For any \( n \in \mathbb{N}_1(\mathbb{R}^c, \mathcal{B}(E)) \), the operator

\[
(N u)(x) = \int_{\mathbb{R}^c} n(x, x - y) u(y) \, dy
\]

acts in \( L_p(\mathbb{R}^c, \mathcal{B}(E)) \) for all \( 1 \leq p \leq \infty \). More precisely, for any \( u \in \mathcal{L}_p(\mathbb{R}^c, E) \) the function \( y \mapsto n(x, x - y) u(y) \) is integrable for almost all \( x \), and the function \( Nu \) belongs to \( L_p(\mathbb{R}^c, \mathcal{B}(E)) \); if \( u_1 \) and \( u_2 \) coincide almost everywhere, then \( Nu_1 \) and \( Nu_2 \) also coincide almost everywhere. Besides,

\[
\|N : L_p \to L_p\| \leq \|\beta\|_{\mathcal{L}_1}.
\]

(5.3)
We denote by \( N_1 = N_1(L_p) \), \( 1 \leq p \leq \infty \), the set of all operators \( N \in B(L_p) \) of the form (5.2) generated by \( n \in N_1(R^c, E) \).

**Proposition 5.2 ([38, Proposition 3.3]).** If two functions \( n, n_1 \in N_1(R^c, E) \) coincide almost everywhere on \( R^c \times R^c \), then they induce the same operator (5.2).

**Theorem 5.3 ([36, Theorem 5.4.7]).** The subalgebra \( \bar{N}_1(L_p) \), \( 1 \leq p \leq \infty \), is full in the algebra \( B(L_p) \) i.e., if the operator \( 1 + N \), where \( N \in \bar{N}_1(L_p) \), is invertible, then \( (1 + N)^{-1} = 1 + M \), where \( M \in N_1(L_p) \).

**Corollary 5.4 ([36, Corollary 5.4.8]).** Let \( n \in N_1 \), and the operator \( N \) be defined by (5.2). If the operator \( 1 + N \) is invertible in \( L_p \) for some \( 1 \leq p \leq \infty \), then it is invertible in \( L_p \) for all \( 1 \leq p \leq \infty \). Moreover, the kernel \( m \) of the operator \( M \), where \( (1 + N)^{-1} = 1 + M \), does not depend on \( p \).

We denote by \( C N_1 = C N_1(R^c, E) \) the class of kernels \( n \in N_1 \) such that the function \( n \) can be redefined on a set of measure zero so that it becomes defined everywhere, estimate (5.1) holds for all \( x \) and \( y \), and the associated function \( x \mapsto \tilde{n}(x) \) becomes continuous in the norm of \( L_1(R^c, B(E)) \). Unless otherwise stated, we will always assume that such an override has already been performed. We note that in this case, integral (5.2) exists for all \( x \in R^c \) provided \( u \in L_\infty \).

**Proposition 5.5.** For any \( n \in N_1(R^c, E) \) the functions \( \tilde{n} : R^c \to L_1(R^c, B(E)) \) and \( \tilde{n} : R^c \to L_1(R^c, B(E)) \) are continuous in the norm of \( L_1(R^c, B(E)) \) simultaneously. Thus, a kernel \( n \in N_1 \) belongs to \( C N_1 = C N_1(R^c, E) \) if and only if the kernel \( n \in N_1 \) can be redefined on a set of measure zero so that it becomes defined everywhere, estimate (5.1) holds for all \( x \) and \( y \), and the associated function \( x \mapsto \tilde{n}(x) \) is continuous in the norm of \( L_1(R^c, B(E)) \).

**Proof.** The proof follows from the fact that for any function \( g \in L_1(R^c, B(E)) \) the family \( S_hg \), where \( (S_hg)(x) = g(x - h) \), depends on \( h \in R^c \) continuously in the norm of \( L_1 \).

We denote by \( C N_1(L_p) \), \( 1 \leq p \leq \infty \), the set of all operators \( N \in B(L_p) \) induced by kernels \( n \in C N_1 \) in accordance with formula (5.2).

**Theorem 5.6 ([38, Theorem 5.3]).** The subalgebra \( \bar{C N}_1(L_p) \), \( 1 \leq p \leq \infty \), is full in the algebra \( B(L_p) \) i.e., if the operator \( 1 + N \), where \( N \in \bar{C N}_1(L_p) \), is invertible, then \( (1 + N)^{-1} = 1 + M \), where \( M \in \bar{C N}_1(L_p) \).

6. Integral operators with almost periodic kernels

We denote by \( C N_{1, AP} = C N_{1, AP}(R^c, E) \) the class of kernels \( n \in C N_1 \) such that the function \( n \) can be redefined on a set of measure zero so that it becomes defined everywhere, estimate (5.1) holds for all \( x \) and \( y \), and the associated function \( x \mapsto \tilde{n}(x) \) becomes almost periodic in the norm of \( L_1(R^c, B(E)) \) i.e., \( \tilde{n} \in C_{AP}(R^c, L_1(R^c, B(E))) \).

**Proposition 6.1.** For any \( n \in C N_{1, AP}(R^c, E) \), the continuous function \( \tilde{n} : R^c \to L_1(R^c, B(E)) \) possesses a unique extension to a function \( \tilde{n} \in C(R^c, L_1(R^c, B(E))) \).
Proof. The proof follows from Proposition 3.11. \hfill \Box

We denote by $\mathbf{CN}_{1,AP}(L_p), \ 1 \leq p \leq \infty$, the set of all operators $N \in B(L_p)$ induced by kernels $n \in \mathbf{CN}_{1,AP}$.

**Proposition 6.2.** The set $\mathbf{CN}_{1,AP}(L_p)$ is a subalgebra of the algebra $B(L_p), \ 1 \leq p \leq \infty$.

Proof. The closedness with respect to addition and scalar multiplication is evident. We prove the closedness with respect to multiplication. Let $M, N \in \mathbf{CN}_{1,AP}(L_p)$. From Theorem 5.3 it follows that $M, N \in \mathbf{CN}_{1}(L_p)$. It remains to prove that the operator $K = NM$ is induced by an almost periodic function $x \mapsto \tilde{k}(x)$.

Let $p = \infty$. By Corollary 4.5, the operators $N, M \in B(L_{\infty})$ are almost periodic. Consequently, by Theorem 2.1, the operator $K = NM \in B(L_{\infty})$ is almost periodic as well. Therefore, again by Corollary 4.5, the associated function $\tilde{k}$ is almost periodic. \hfill \Box

**Theorem 6.3.** The subalgebra $\mathbf{CN}_{1,AP}(L_p), \ 1 \leq p \leq \infty$, is full in the algebra $B(L_p)$, i.e., if the operator $1 + N$, where $N \in \mathbf{CN}_{1,AP}(L_p)$, is invertible, then $(1 + N)^{-1} = 1 + M$, where $M \in \mathbf{CN}_{1,AP}(L_p)$.

Proof. Since $\mathbf{CN}_{1,AP} \subseteq \mathbf{CN}_1$, by Theorem 5.6 the operator $(1 + N)^{-1}$ can be represented in the form $1 + M$, where

$$ (Mu)(x) = \int_{\mathbb{R}^c} m(x, x - y) u(y) \, dy $$

and $m \in \mathbf{CN}_1$. By the definition of the class $\mathbf{CN}_1$, we can assume without loss of generality that $m$ is defined everywhere, estimate of the kind (5.1) holds for all $x$ and $y$, and the associated function $x \mapsto \tilde{m}(x)$ is continuous in the norm of $L_1(\mathbb{R}^c, B(\mathbb{E}))$.

Since $\mathbf{CN}_1 \subseteq \mathbf{N}_1$, by Corollary 5.4 the operator $1 + N$ is invertible in $L_p$ for all $p$ and the inverse operator is defined by the formula $1 + M$, where the operator $M$ is generated by the same kernel $m \in \mathbf{N}_1$ for all $p$ as well. So, we consider the operators $1 + N$ and $1 + M$ as acting in $L_{\infty}$.

We recall [19, I.6.14] that a subset $K$ of a Banach space is called *totally bounded* if for every $\varepsilon > 0$ it is possible to cover $K$ by a finite number of balls $B(k_i, \varepsilon) = \{ x \in K : \|x - k_i\| < \varepsilon \}, \ i = 1, \ldots, n$, with centers $k_i \in K$. It is well known [19, I.6.15] that a subset of a Banach space is relatively compact if and only if it is totally bounded.

By Theorem 2.1, the operator $M : L_{\infty} \rightarrow L_{\infty}$ is almost periodic, i.e., the family

$$ M\{h\} = S_hMS_{-h}, \quad h \in \mathbb{R}^c, $$

is relatively compact or, equivalently, totally bounded. Let, for a given $\varepsilon > 0$, the set of balls

$$ B(M\{h_i\}, \varepsilon) = \{ M\{h\} : \|M\{h\} - M\{h_i\}\| < \varepsilon \}, \quad i = 1, \ldots, n, $$

forms a finite covering of the set $M\{h\}, \ h \in \mathbb{R}^c$. 
By Proposition 4.4, the operator $M\{h\} = S_h MS_{-h}$ is induced by the function $S_h \tilde{m}$. It follows from Proposition 4.3 that the balls

$$B(S_h \tilde{m}, C\varepsilon) = \{ S_h \tilde{m} : \| S_h \tilde{m} - S_h \tilde{m} \|_{L_1} < C\varepsilon \}, \quad i = 1, \ldots, n,$$

(where $C = 1/c$) form a finite covering of the set $\{ S_h \tilde{m} : h \in \mathbb{R}^c \}$. Indeed,

$$\| S_h \tilde{m} - S_h \tilde{m} \|_{C(\mathbb{R}^c, L_1(\mathbb{R}^c, \mathcal{B}(E)))} = \text{ess sup}_{x \in \mathbb{R}^c} \| \tilde{m}(x - h) - \tilde{m}(x - h_i) \|_{L_1}$$

$$\leq C \| S_h MS_{-h} - S_h MS_{-h_i} : L_\infty \to L_\infty \| = C \| M\{h\} - M\{h_i\} : L_\infty \to L_\infty \|.$$

Since $C$ is a constant, the set $S_h \tilde{m}, h \in \mathbb{R}^c$, is totally bounded and, consequently, relatively compact. Thus, $m \in \text{CN}_1\text{, AP}$. □

7. Almost periodic integral operators with absolutely convergent Fourier series

We denote by $\text{CN}_{1,APW} = \text{CN}_{1,APW}(\mathbb{R}^c, E)$ the class of kernels $n : \mathbb{R}^c \times \mathbb{R}^c \to \mathcal{B}(E)$ that can be represented in the form

$$n(x,y) = \sum_{k=1}^{\infty} e^{i\langle \omega_k, x \rangle} n_k(y), \quad (7.1)$$

where $\omega_k \in \mathbb{R}^c$ and $n_k \in L_1(\mathbb{R}^c, \mathcal{B}(E))$, with

$$\sum_{k=1}^{\infty} \| n_k \|_{L_1} < \infty.$$

Clearly,

$$\text{CN}_{1,APW} \subset \text{CN}_{1,AP} \subset \text{CN}_1.$$

We note that for kernel (7.1) we have

$$\tilde{n}(x) = \sum_{k=1}^{\infty} e^{i\langle \omega_k, x \rangle} n_k,$$

$$(Nu)(x) = \sum_{k=1}^{\infty} e^{i\langle \omega_k, x \rangle} \int_{\mathbb{R}^c} n_k(x - y) u(y) \, dy,$$

$$(N\{h\}u)(x) = (S_h NS_{-h}u)(x) = \sum_{k=1}^{\infty} e^{i\langle \omega_k, x - h \rangle} \int_{\mathbb{R}^c} n_k(x - y) u(y) \, dy.$$

We denote by $\text{CN}_{1,APW}(L_p), 1 \leq p \leq \infty$, the set of all operators $N \in \mathcal{B}(L_p)$ induced by kernels $n \in \text{CN}_{1,APW}$. We say that an operator $N \in \text{CN}_{1,APW}(L_p)$ has an almost periodic kernel with an absolutely convergent Fourier series.

For any $g \in L_1(\mathbb{R}^c, \mathcal{B}(E))$, we define the convolution operator

$$\widehat{(G_g u)}(x) = \int_{\mathbb{R}^c} g(x - y) u(y) \, dy \quad (7.2)$$
With this notation the operator $N$ induced by kernel (7.1) can be represented in the form
\[ N = \sum_{k=1}^{\infty} \Psi_{\omega_k} G_{n_k}. \]  
(7.3)

For brevity, instead of (7.3) we will use the representation (cf. (3.1))
\[ N = \sum_{\omega \in \mathbb{R}^c} \Psi_{\omega} G_{\omega} \]  
(7.4)

for operators $N \in \mathbf{CN}_{1, APW}(L_p)$, where $G_{\omega}$ are operators of convolution with functions $g_{\omega} \in L_1(\mathbb{R}^c, \mathbf{B}(\mathbb{E}))$ and $\sum_{\omega \in \mathbb{R}^c} \| g_{\omega} \|_{L_1} < \infty$. We assume that only a countable number of operators $G_{\omega}$ in (7.4) are nonzero. Clearly, $\mathbf{CN}_{1, APW}(L_p) \subset \mathbf{B}_{APW}(L_p)$. In particular, we have (see Proposition 3.5)
\[ N\{h\} = \sum_{\omega \in \mathbb{R}^c} e^{-i(\omega, h)} \Psi_{\omega} G_{\omega}, \quad h \in \mathbb{R}^c. \]

**Proposition 7.1.** The set $\mathbf{CN}_{1, APW}(L_p)$ is a subalgebra of the algebra $\mathbf{B}(L_p)$, $1 \leq p \leq \infty$.

**Proof.** The closedness with respect to addition and scalar multiplication is evident. We prove the closedness with respect to multiplication. Let operators $N, M \in \mathbf{B}(L_p)$ be represented in the form
\[ N = \sum_{\omega \in \mathbb{R}^c} \Psi_{\omega} G_{\omega} \quad \text{and} \quad M = \sum_{\nu \in \mathbb{R}^c} \Psi_{\nu} H_{\nu}, \]

where $G_{\omega}$ and $H_{\nu}$ are operators of convolution with functions $g_{\omega}, h_{\nu} \in L_1(\mathbb{R}^c, \mathbf{B}(\mathbb{E}))$ respectively, with $\sum_{\omega \in \mathbb{R}^c} \| g_{\omega} \|_{L_1} < \infty$ and $\sum_{\nu \in \mathbb{R}^c} \| h_{\nu} \|_{L_1} < \infty$. We have
\[ NM = \sum_{\omega \in \mathbb{R}^c} \sum_{\nu \in \mathbb{R}^c} \Psi_{\omega} G_{\omega} \Psi_{\nu} H_{\nu} = \sum_{\omega \in \mathbb{R}^c} \sum_{\nu \in \mathbb{R}^c} \Psi_{\omega} \Psi_{\nu} \left( \Psi_{\nu} G_{\omega} \Psi_{\nu} \right) H_{\nu}. \]

It remains to observe that $\Psi_{-\nu} G_{\omega} \Psi_{\nu}$ is an operator of convolution with the function $x \mapsto g_{\omega}(x)e^{-i(\nu, x)}$ (evidently, the $L_1$-norm of this function is equal to the $L_1$-norm of the function $g_{\omega}$). Indeed,
\[
\left( \Psi_{-\nu} G_{\omega} \Psi_{\nu} \right)(x) = e^{-i(\nu, x)} \int_{\mathbb{R}^c} g_{\omega}(x-y)e^{i(\nu, y)} u(y) dy \\
= \int_{\mathbb{R}^c} (g_{\omega}(x-y)e^{-i(\nu, x-y)}) u(y) dy. \quad \Box
\]

**Theorem 7.2.** The subalgebra $\mathbf{CN}_{1, APW}(L_p)$, $1 \leq p \leq \infty$, is full in the algebra $\mathbf{B}(L_p)$, i.e., if the operator $1 + N$, where $N \in \mathbf{CN}_{1, APW}(L_p)$, is invertible, then $(1 + N)^{-1} = 1 + M$, where $M \in \mathbf{CN}_{1, APW}(L_p)$.

**Proof.** By Theorems 6.3 and 5.6, the operator $M$ has the form
\[ (Mu)(x) = \int_{\mathbb{R}^c} m(x, x - y) u(y) dy, \]

with the same $m \in \mathbf{CN}_{1, AP}$ for all $1 \leq p \leq \infty$. So, it remains to prove that $m \in \mathbf{CN}_{1, APW}$. 

We suppose that \( p = \infty \).

We extend the function \( \tilde{m} \in C_{AP}(\mathbb{R}^c, L_1(\mathbb{R}^c, B(\mathbb{E}))) \) to the function \( \tilde{m} \in C(\mathbb{R}^c, L_1(\mathbb{R}^c, B(\mathbb{E}))) \) in accordance with Corollary 6.1. By Proposition 3.10, this extension can be interpreted as the extension by continuity. Clearly, for any \( \omega \in \mathbb{R}^c \) there exists the integral

\[
m_\omega = \int_{\mathbb{R}^c} \langle \omega, \chi \rangle \tilde{m}(\chi) \, d\chi
\]

with respect to the Haar measure.

We consider the family

\[
M\{h\} = S_h MS_{-h} : L_\infty \to L_\infty, \quad h \in \mathbb{R}^c.
\]

By Theorem 2.1, the operator \( M : L_\infty \to L_\infty \) is almost periodic, i.e., \( M\{\cdot\} \in C_{AP}(\mathbb{R}^c, B(L_\infty)) \). Therefore, by Corollary 3.12, it has a unique extension to \( M\{\cdot\} \in C(\mathbb{R}^c, B(L_\infty)) \). Again by Proposition 3.10, this extension can be interpreted as the extension by continuity.

By Corollary 4.5, the two functions \( h \mapsto S_h \tilde{m} \) and \( h \mapsto M\{h\} \) are connected by the isomorphism \( U : \tilde{m} \mapsto M \), i.e., \( U : S_h \tilde{m} \mapsto M\{h\} \) for all \( h \). Therefore their extensions on \( \mathbb{R}^c \) by continuity are connected by \( U \) as well. Consequently, the integrals

\[
\int_{\mathbb{R}^c} \langle \omega, \chi \rangle S_\chi \tilde{m} \, d\chi \quad \text{and} \quad \int_{\mathbb{R}^c} \langle \omega, \chi \rangle M\{\chi\} \, d\chi
\]

are also connected by the same isomorphism \( U : \tilde{m} \mapsto M \); here \( \chi \mapsto S_\chi \tilde{m} \) means the extension by continuity of the function \( h \mapsto S_h \tilde{m} \) from \( \mathbb{R}^c \) to \( \mathbb{R}^c \).

From Theorem 3.16 it follows that the operator \( (1 + N)^{-1}\{\chi\} \) has the form

\[
(1 + N)^{-1}\{\chi\} = \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \chi \rangle \Psi_{-\omega} B_\omega,
\]

where \( B_\omega \in A(L_\infty) \) and \( \sum_{\omega \in \mathbb{R}^c} \|B_\omega\| < \infty \). Consequently,

\[
M\{\chi\} = \left[ 1 - (1 + N)^{-1} \right]\{\chi\} = 1 - \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \chi \rangle \Psi_{-\omega} B_\omega.
\]

By Lemma 3.14, for \( \omega_0 \in \mathbb{R}^c \), we have

\[
\int_{\mathbb{R}^c} \langle \omega_0, \chi \rangle \left( 1 - \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \chi \rangle \Psi_{-\omega} B_\omega \right) \, d\chi = \begin{cases} -\Psi_{-\omega_0} B_{\omega_0} & \text{for } \omega_0 \neq 0, \\ 1 - B_0 & \text{for } \omega_0 = 0. \end{cases} \tag{7.5}
\]

On the other hand, by the above, the operator

\[
\int_{\mathbb{R}^c} \langle \omega_0, \chi \rangle \left( 1 - \sum_{\omega \in \mathbb{R}^c} \langle -\omega, \chi \rangle \Psi_{-\omega} B_\omega \right) \, d\chi
\]

corresponds to the function (here the integrand \( \chi \mapsto S_\chi \tilde{m} \) takes values in the space \( C(\mathbb{R}^c, L_1(\mathbb{R}^c, B(\mathbb{E}))) \))

\[
\int_{\mathbb{R}^c} \langle \omega_0, \chi \rangle S_\chi \tilde{m} \, d\chi,
\]
where \( \varphi \mapsto S_\varphi \tilde{m} \) means the extension by continuity of the function \( h \mapsto S_h \tilde{m} \) from \( \mathbb{R}^c \) to \( \mathbb{R}^c_\partial \). Therefore operator (7.5) is the integral operator \( M_{\omega_0} \) generated by a function \( \tilde{m}_{\omega_0} \in C(\mathbb{R}^c_\partial, L_1(\mathbb{R}^c, B(E))) \).

Let \( \omega_0 \neq 0 \). Then \( M_{\omega_0} = -\Psi_{-\omega_0} B_{\omega_0} \). Hence, the operator \( -\Psi_{\omega_0} M_{\omega_0} = B_{\omega_0} \) belongs to \( A(L_\infty) \). Clearly, \( \Psi_{\omega_0} M_{\omega_0} \) is an integral operator. Since it is shift invariant, by Corollary 4.6 it is an operator of convolution with a function of the class \( L_1 \). The case \( \omega_0 = 0 \) is considered in a similar way. \( \square \)

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