Deducibility of Identicals, Reflection Principle and Synthetic Connectives

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Abstract

Sambin et al (2000) introduced Basic Logic as an uniform framework for various logics. At the same time, they also introduced the principle of reflection as a criterion for being a connective of Basic Logic. We make explicit a relationship between Hacking’s deducibility of identicals condition (Hacking, 1979) and the principle of reflection by proving the equivalence between them. Moreover, despite Sambin et al.’s conjecture that only six connectives satisfy the principle of reflection, we show the following; a logical connective satisfies the principle of reflection if and only if the connective is Girard’s synthetic connective.

1 Introduction

We consider the deducibility of identicals condition in this paper. The deducibility of identical condition (abbreviated DoI) is introduced by Hacking (1979) as a deducibility condition along with transitivity (i.e. admissibility of the cut rule) and dilution (i.e. admissibility of the weakening rule). Importance of transitivity condition is clear from the example of Prior’s tonk connective (Belnap, 1962). On the contrary, such persuasive example is not known in the case of DoI condition. Necessity of DoI condition has not fully been clarified. While the transitivity condition is well discussed, few studies have done on DoI condition. Our aim in this paper is to provide a technical basis for further discussion on the DoI condition. We explain a relationship between DoI and another criterion on a logical connective. The criterion is Sambin et al.’s principle of reflection (reflection principle). Sambin et al. introduced Basic Logic as an uniform framework for various logics. At the same time, they introduced the principle of reflection, which connectives of Basic Logic should satisfy. Logical connectives in Basic Logic are obtained by solving definitional equations which connect logical connectives with meta-linguistic links between assertions. It is considered that a connective $C$ satisfies Sambin et al.’s reflection principle if and only if $C$ satisfies DoI and the main
step of the cut elimination (Schroeder-Heister, 2013). However, the equivalence had not been fully spelled out. We prove it in our setting. To consider a relationship between the DoI and the reflection principle, we consider the reflection principle in a different way. Sambin et al. first consider a definitional equation and then construct inference rules (Sambin et al., 2000, p.983). Conversely, given arbitrary inference rules, we construct a definitional equation from them, and we solve the equation according to their procedure. Sambin et al. did not explain the case that a definitional equation is unsolvable, that is, their procedure fails. We make explicit when definitional equations are unsolvable. We consider \( n \)-ary connectives instead of binary connective in order to consider which logical connectives satisfy the reflection principle and which does not.

2 Preliminaries

2.1 Deducibility of Identicals

We briefly explain the deducibility of identicals, the principle of reflection and synthetic connectives in this chapter. The deducibility of identicals condition is formalized in (Naibo and Petrolo, 2015, p.147) as follows.

We must be able to show that logical constants are uniquely identified by their inference rules: given a \( n \)-ary operator \( C \), for all \( A_1,\ldots,A_n \), the sequent \( C(A_1,\ldots,A_n) \vdash C(A_1,\ldots,A_n) \) has to be derivable using imperatively at least one of the rules of \( C \) and, when needed, the reflexivity axiom rule in order to close the derivation. No other rules are admitted.

Satisfiability of this condition is checked by the following procedure:

Start by applying the left (resp. right) rule(s) of the operator undFer analysis, and immediately conclude by applying its right (resp. left) rule(s) (Naibo and Petrolo, 2015, p.147).

Hence, a procedure which decides satisfiability of deducibility of identicals condition amounts to proof search. DoI condition is strictly weaker than the uniqueness condition (Naibo and Petrolo, 2015, pp.153-154). The uniqueness condition is that for two logical connectives \( C, C' \) which have the same inference rules, \( C \vdash C' \) and \( C' \vdash C \) are deducible (Belnap, 1962). DoI condition corresponds to the eta-expansion under the presence of structural rules. However, these do not correspond in the absence of structural rules; the tensor connective satisfies DoI condition and does not satisfy the eta-expansion.
2.2 Principle of Reflection

The principle of reflection says that a logical connective is reflection of metalinguistic link between assertions at the level of object language (Sambin et al., 2000, p.980). According to Sambin et al. (2000), “A logical constant obeys to the principle of reflection if it is characterized semantically by an equation binding it with a metalinguistic link between assertions, and if its synthetic inference rules are obtained by solving the equation” (p.279).

We explain the principle of reflection for the cases of the multiplicative conjunction and the additive conjunction. The cases of disjunctions similarly hold by symmetrically interchanging left and right sides of sequents. A definitional equation of the tensor connective is as follows; For all $\Delta$, $A \otimes B \vdash \Delta$ if and only if $A, B \vdash \Delta$.

The ‘if’ direction corresponds to the following formation rule.

$$\frac{A, B \vdash \Delta}{A \otimes B \vdash \Delta} \text{- formation}$$

The ‘only if’ direction corresponds to the following rule.

$$\frac{A \otimes B \vdash \Delta}{A, B \vdash \Delta} \text{ implicit } \otimes \text{- reflection}$$

This implicit reflection rule contains the connective at issue in the premise. Hence, we construct the equivalent rule which does not contain the connective in the premise. We first substitute $A \otimes B$ into the context $\Delta$ and obtain the identity. Secondly, we replace $A$ and $B$ in $A, B \vdash A \otimes B$ with $\Gamma_1$ and $\Gamma_2$ respectively by the cut rule.

$$\frac{\Gamma \vdash A \quad A, B \vdash A \otimes B}{\Gamma, B \vdash A \otimes B} \text{ Cut} \quad \frac{\Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B} \text{ Cut}$$

Finally, we obtain the following rule.

$$\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B} \text{ explicit } \otimes \text{- reflection}$$

In the case of the additive conjunction, we solve the following definitional equation; $\Gamma \vdash A \& B$ if and only if $\Gamma \vdash A$ and $\Gamma \vdash B$.

The ‘if’ direction corresponds to the following formation rule.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \text{ &- formation}$$
The ‘only if’ direction corresponds to the following rules.

\[ \frac{\Gamma \vdash A \& B}{\Gamma \vdash A} \quad \text{implicit \&-reflection 1} \]
\[ \frac{\Gamma \vdash A \& B}{\Gamma \vdash B} \quad \text{implicit \&-reflection 2} \]

We substitute \( A \& B \) into \( \Gamma \) and obtain two sequents \( A \& B \vdash A \) and \( A \& B \vdash B \). We replace \( A \) and \( B \) with a context \( \Delta \) by the cut rule. Finally, we obtain the following rules.

\[ \frac{A \vdash \Delta}{A \& B \vdash \Delta} \quad \text{explicit \&-reflection 1} \]
\[ \frac{B \vdash \Delta}{A \& B \vdash \Delta} \quad \text{explicit \&-reflection 2} \]

**Remark 1.** We do not deal with the implication rules in this paper because it needs nested sequents. We exclude the implication rules (and constants) in the following our results.

### 2.3 Synthetic Connective

We assume basic background knowledges of linear logic in this paper. The synthetic connectives are explained by polarity theory of linear logic (Andreoli, 1992). The connectives and inference rules of the multiplicative additive linear logic MALL are classified into two groups. The tensor \( \otimes \) and the plus \( \oplus \) have the positive polarity and the with \( \& \) and the par \( \otimes \) have the negative. An polarity of a inference rule is defined by a polarity of its principal formula. Inference rules of synthetic connectives are obtained by decomposing the sequent which only contain a formula \( A \) (where \( A \) is composed of connectives (and meta-variables) which have the same polarity).

![Figure 1: An example of synthetic connective](image)

If we define inference rules of the synthetic connectives by the rules of binary connectives as in (Girard, 1999), the synthetic connectives are conceptually dependent on another connectives. Hence, we directly define rules of the synthetic connectives.
Definition 1. A logical connective $C$ is a synthetic connective if its inference rules are an instance of either scheme in Figure 2 or scheme in Figure 3 (where $m \in \mathbb{N}$, $A_{pq} \in \{A_1, \ldots, A_n\}$, $1 \leq p, g \leq m$, $1 \leq q \leq k_r$).

\[
\begin{align*}
\Gamma \vdash A_{11}, \ldots, A_{1k_1} & \quad \ldots \quad \Gamma \vdash A_{m1}, \ldots, A_{1km} & \text{C-right rule} \\
& \quad \Gamma \vdash C(A_1, \ldots, A_n) \\
& \quad \Delta \vdash A_{i1}, \ldots, A_{ik_i} & \text{C-left rule} \\
& \quad \Delta \vdash C(A_1, \ldots, A_m) & \quad \Delta \vdash C(A_1, \ldots, A_m) \\
\end{align*}
\]

Figure 2: Inference rule scheme I of synthetic connective

\[
\begin{align*}
& \quad \Delta \vdash A_{11}, \ldots, A_{1k_1} \quad \ldots \quad \Delta \vdash A_{m1}, \ldots, A_{1km} & \text{C-right rule} \\
& \quad \Delta \vdash C(A_1, \ldots, A_n) \quad \Delta \vdash C(A_1, \ldots, A_n) \\
& \quad \Gamma \vdash A_{i1}, \ldots, A_{ik_i} & \text{C-left rule} \\
\end{align*}
\]

Figure 3: Inference rule scheme II of synthetic connective

Synthetic connectives in (Girard, 2001) have the property that its active formulas appear on reverse side in the premise and conclusion in order to take turns to appear a left rule and right rule. Synthetic connectives in (Girard, 2001) are composed of negation and binary connectives. In order to consider the reflection principle for the synthetic connectives in this paper, we exclude the negation from the constituents of synthetic connectives. Hence, the active formulas appear on the same side of sequents. Synthetic connectives in (Girard, 1999) satisfy this property \(^1\).

\(^1\) Girard explained that a new connective is not defined by any combination of binary connective and only the connectives which has the same polarity define a connective, and its reason is that DoI does not hold in the case that connectives have different polarities (Girard, 1999, p.271).
3 Equivalence between two criteria

Sambin et al.'s deducibility of identicals condition can be regarded as a procedure for obtaining left or right rules from the rules of the other side. Sambin et al. considered definitional equations by considering all of combinations of the meta-implication ‘yields’ and the meta-conjunction ‘and’. Then, they make correspondence each solvable combination of these with logical constant. In an opposite way, we make correspondence given logical connectives with its definitional equation. After that, we apply the procedure of the reflection principle to the definitional equations. However, our method has the following problem. Assume that the left rule of the tensor connective is given. If we consider the definitional equation obtained from the tensor’s left rule, the equation is as follows; \( \Gamma_1, \Gamma_2 \vdash A \otimes B \) if and only if \( \Gamma_1 \vdash A \) and \( \Gamma_2 \vdash B \). This equation is unsolvable. On the contrary, if we consider the equation obtained from the tensor’s right rule, its definitional equation is solvable. The problem that which rule we start from cannot be ignored. Sambin et al. first construct a definitional equation and then obtain a formation rule and implicit reflection rules from it. We first consider formation rule(s). We regard a definitional equation as what assumption(s) and a conclusion of an inference rule are combined by “if and only if”.

A set of formation rules may not be singleton. In such case, we postulate that assumptions in several rules are combined by “and” in a definitional equation. The class of the definitional equations obtained in this way are larger than those in (Sambin et al., 2000). For example, the left rules of the additive conjunction produces the equation; \( A \vdash \Delta \) and \( B \vdash \Delta \) if and only if \( A \& B \vdash \Delta \). Such equations are not permitted in (Sambin et al., 2000). However, the set of solvable definitional equations in our definition is identical with those of Sambin et al. because we apply the same procedure as Sambin et al’s. Definitional equations only in our terminology is unsolvable (in our terminology). Two definitional equations are obtainable from the left rule and right rule of a connective. Sambin et al. deal with only solvable cases. We consider the question: when are definitional equations unsolvable. From this consideration, we make explicit forms of inference rules which make definitional equations solvable. From our proof of the equivalence between DoI and the reflection principle, DoI may be characterized by success of unification between an assumption of one rule (formation rule) and a conclusion of the other rule (explicit reflection rule).

In the following, we assume that contexts in a side which have active formulas of a synthetic connective are empty. It is the visibility condition in Basic Logic. According to (Sambin et al, 2000), “a rule satisfies visibility if it operates on a formula (or two formulae) only if it is (they are) the only formula(e), either in the antecedent or in the succedent of a sequent (p.981)”.

We denote a set of contexts contained in premisses (resp. conclusions) of
formation rules obtained from a definitional equation $E$ by $Cxt_{Prem}(E)$. (resp. $Cxt_{Concl}(E)$)

**Remark 2.** In this paper, a logical connective denote a pair of sets of right rules and left rules such that the main step of the cut elimination holds between them. Sambin et al. stated that the cut elimination theorem is needed to justify the validity of the cut rule (Sambin et al., 2000, pp.993-994). From that, it may be appropriate to restrict our consideration to pairs of rules which admit the main step of the cut elimination.

**Definition 2.** Let $\mathcal{C}$ be a logical connective, $F$ be a formation rule obtained from $\mathcal{C}$ and $E$ be the definitional equation obtained from $F$. $E$ is solvable if the main step of the cut elimination holds between the explicit reflection rule(s) and $F$.

Satisfiability of the main step of the cut elimination and the solvability of one of definitional equation of a logical connective are not the same. If we consider the definitional equation obtained from the tensor’s right rule, it is unsolvable. On the other hand, the tensor connective satisfies the main step of the cut elimination.

**Definition 3.** Let $\mathcal{C}$ be a logical connective and $F$ be a formation rule obtained from $\mathcal{C}$. $F$ is context changing if $Cxt_{Prem}(F) \neq Cxt_{Concl}(F)$ holds. The definitional equation obtained from $F$ is context changing if $F$ is context changing.

**Lemma 1.** Let $\mathcal{C}$ be a logical connective and $E$ be its one of definitional equation. If $E$ is context changing, then $E$ is unsolvable.

**Proof.** We assume that $E$ is context changing. Let $F$ be one of formation rules of $E$. $F$ is written as follows (where $A_{ij} \in \{A_1, \ldots, A_n\}$, $1 \leq i, l \leq m$, $1 \leq j, j' \leq k_l$, $\bigcup \Gamma' \subset \bigcup \Gamma$).

\[
\begin{align*}
\Gamma_1 \vdash A_{11}, \ldots, A_{1l_1} & \quad \cdots \quad \Gamma_m \vdash A_{m1}, \ldots, A_{1k_m} \\
\Gamma_1', \ldots, \Gamma_k' & \vdash \mathcal{C}(A_1, \ldots, A_n)
\end{align*}
\]

The implicit reflection rule has the following form.

\[
\begin{align*}
\Gamma_1', \ldots, \Gamma_k' & \vdash \mathcal{C}(A_1, \ldots, A_n) \\
\Gamma_i & \vdash \Delta
\end{align*}
\]

The explicit reflection rule has the following form.

\[
\begin{align*}
A_{ii'} & \vdash \Delta \\
\mathcal{C}(A_1, \ldots, A_n) & \vdash \Delta
\end{align*}
\]

Then, we obtain $\Gamma_1', \ldots, \Gamma_k' \vdash \Delta$ by the cut on $\mathcal{C}(A_1, \ldots, A_n)$ and $\Gamma_i \vdash \Delta$ by the cut on $A_{ii'}$. Therefore, the main step of the cut elimination does not hold. Hence, $E$ is unsolvable. 

\[\square\]
The following lemma was pointed out by (Schroeder-Heister, 2012, p.498).

**Lemma 2.** Let $\mathcal{C}$ be a logical connective and $E$ be a definitional equation of $\mathcal{C}$. If $E$ has several formation rules, then $E$ is unsolvable.

*Proof.* If $E$ is context changing, then it is unsolvable by Lemma 1. Hence, we assume that contexts of formation rules are the same in its premise and conclusion. Let formation rules of $E$ be $F_1, \ldots, F_m$ (where $m$ is a natural number). We assume that $F_i$ ($1 \leq i \leq m$) has the following form (where $m \in \mathbb{N}$, $A_{pq}^i \in \{A_1, \ldots, A_n\}$, $1 \leq p, g \leq m$, $1 \leq q \leq k_g$).

$$
\begin{align*}
\Gamma \vdash A_{11}^i, \ldots, A_{1k_1}^i & \quad \ldots \quad \Gamma \vdash A_{m1}^i, \ldots, A_{1k_m}^i \\
\Gamma \vdash \mathcal{C}(A_1, \ldots, A_n)
\end{align*}
$$

We assume that another formation rule $F_j$ ($j \neq i$ and $1 \leq j \leq m$) has the following form (where $A_{pq}^j \in \{A_1, \ldots, A_n\}$, $1 \leq p, g \leq m$, $1 \leq q \leq k_g$).

$$
\begin{align*}
\Gamma \vdash A_{11}^j, \ldots, A_{1l_1}^j & \quad \ldots \quad \Gamma \vdash A_{m1}^j, \ldots, A_{1l_m}^j \\
\Gamma \vdash \mathcal{C}(A_1, \ldots, A_n)
\end{align*}
$$

An assumption of the explicit reflection rule obtained from $F_i$ is $A_{ss'}^i \vdash \Delta$ ($1 \leq s, g \leq m$, $1 \leq s' \leq k_g$), and an assumption of the explicit reflection rule obtained from $F_j$ is $A_{tt'}^j \vdash \Delta$ ($1 \leq t, g \leq m$, $1 \leq t' \leq k_g$). There is a pair of different formulas in $A_{ss'}^i$ and $A_{tt'}^j$. Otherwise, $F_i = F_j$ holds and it contradicts our assumption. The main step of the cut elimination between a pair of different formulas in $A_{ss'}^i$ and $A_{tt'}^j$ does not hold. Hence, $E$ is unsolvable. \(\square\)

**Theorem 1.** A logical connective $\mathcal{C}$ satisfies the deducibility of identicals condition if and only if $\mathcal{C}$ satisfies the reflection principle.

*Proof.* (if-direction) We assume that $\mathcal{C}$ satisfies the reflection principle. We apply explicit reflection rule(s) to axiom(s). Let $S_0$ be the resulting sequent(s). One side of the assumption of explicit reflection rule(s) contains one propositional variable and the other side contains context variable(s). We trivialize the context variable and obtain an axiom. Hence, the application of the explicit reflection rule(s) is possible. Next, we consider a formation rule. A formation rule contains the logical connective in one side and a context in the other side. We replace this context with the formula $\mathcal{C}(A_1, \ldots, A_n)$. Let $S_1$ be the sequent(s) in assumptions of this instance. Matching between one side of sequent in the assumption of a formation rule and one in conclusion of an explicit reflection rule succeeds because of the construction of explicit reflection rules. Hence, the matching between $S_0$ and $S_1$ succeeds and the deducibility of identicals condition holds.
By definition of DoI, the sequent $\Gamma \vdash \mathcal{C}(A_1, \ldots, A_n)$ is obtainable by applying (several) left (resp. right) rule(s) and then a right (resp. left) rule. We regard the rule(s) we first apply as the explicit reflection rule and the rules we secondly apply as the formation rule. We replace contexts in a definitional equation with the formula $\mathcal{C}(A_1, \ldots, A_n)$. From one direction of a definitional equation, we obtain a sequent $S$ because the identity $\mathcal{C}(A_1, \ldots, A_n) \vdash \mathcal{C}(A_1, \ldots, A_n)$ holds. We can replace active subformulas of $\mathcal{C}(A_1, \ldots, A_n)$ in $S$ with contexts by the cut. Hence, the reflection principle holds.

**Corollary 1.** A pair of sets of inference rules $(L, R)$ satisfies the main step of cut elimination and the deducibility of identicals condition if and only if it satisfies the reflection principle.

We give a characterization of synthetic connectives.

**Theorem 2.** A logical connective $\mathcal{C}$ (its arity is an arbitrary natural number $n$) satisfies the reflection principle if and only if $\mathcal{C}$ is a synthetic connective.

**Proof.** We assume that $\mathcal{C}$ appears in the succedent of a sequent. We also assume that $\mathcal{C}$ satisfies the reflection principle. By Lemma 2, the set of the formation rules of $\mathcal{C}$ is singleton. By Lemma 1, the contexts in the assumption and conclusion are the same. Hence, one of the rules of $\mathcal{C}$ has the following form (where $m \in \mathbb{N}$, $A_{ij} \in \{A_1, \ldots, A_n\}$, $1 \leq i, l \leq m$, $1 \leq j, l' \leq k_l$).

\[
\begin{array}{c}
\Gamma \vdash A_{11}, \ldots, A_{1k_1} \\
\vdots \\
\Gamma \vdash A_{m1}, \ldots, A_{1k_m} \\
\Gamma \vdash \mathcal{C}(A_1, \ldots, A_n)
\end{array}
\]

Hence, $\mathcal{C}$ is a synthetic connective. The reverse direction follows by easy calculation.

Sambin et al. conjectured that only six definitional equations, which are two conjunctions, two disjunctions and two implications, are solvable (Sambin et al., 2000, p.985). From our result, Theorem 2 in addition to six equations, the definitional equations which are obtained from synthetic connectives are solvable. Of course, if we consider that synthetic connectives are reducible to the conjunctions and disjunctions, Sambin et al.’s conjecture is true.

### 4 Conclusion

We proved the equivalence between the reflection principle and the deducibility of identical condition in the absence of the implications. Moreover, we proved that a logical connective satisfy the reflection principle if and only if the connective is
a synthetic connective. The property that a set of formation rules is singleton is essential to satisfy the reflection principle.

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