Reduced Order Modeling of Large Power Grid Model with POD-DEIM

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Abstract. This paper addresses the issue of computational complexity of a large power system network, specifically the swing dynamics problem. Swing equation is nonlinear model which required mathematical model to be solved for simulating the swing dynamics. It has been seen that numerical computation becomes intractable for such models. This issue can be solved with model order reduction. Dynamics of interest is represented by a minimum size aims to reduce the computational time and memory requirement. Proper Orthogonal Decomposition technique is most often used to reduce computational efforts. However, it does not reduce the size of the nonlinear function. The discrete empirical interpolation method was proposed for POD to overcome the large size of nonlinear function by providing its discrete computations. The POD-DEIM approach is experimented on power grid network model to show significant reduction in computational cost with high degree of accuracy.

Keywords: Model order reduction, Discrete empirical interpolation method (DEIM), Proper orthogonal decomposition (POD), Power Grid Network.

1. Introduction
In power grid systems, a higher order nonlinear dynamics often arises for better description of its behavior which increase numerical complexity. Stability analysis of nonlinear power grid is an active research area as it has direct implications for planning, operation, and control of power systems [1, 2]. However, the large size of power grid models and highly nonlinear nature of the associated mathematical equations make the transient analysis computationally costly procedure. Model order reduction (MOR) strategies can be used to derive computationally efficient models while preserving the important dynamical properties of the system [3]. MOR of linear systems is a well-studied problem, but the reduction of nonlinear and parametric systems is still an open problem.

A handful of researchers have applied MOR strategies to reduce the size of large power systems, but most of these methods involve linearization of the power system model about an equilibrium point [4, 5]. Hence, the model order reduction is applicable only for small excursions about the point of linearization. The work of Parrilo[6] proposed the study of proper orthogonal decomposition (POD) for reducing the nonlinear model of a power network. However, POD
suffers from a major disadvantage in the evaluation of non-linear function in the reduced model as it has the same size as that of the original large scale model. The computational difficulties associated with nonlinear function were recognized early by the model order reduction (MOR) community, and a number of approaches were proposed to overcome these difficulties. These approaches include the missing point estimation (MPE) [7], best points method, empirical interpolation method (EIM) [8], and the gappy POD method [9, 10]. The MPE method computes the Galerkin projection over a restricted subset of the spatial domain and the gappy POD is used in the case of sparse measurements. The applications of these methods are closely related to the number of spatial grid points. For a small number of grid points, MPE fails to converge whereas, for the same small number of grid points, gappy POD, EIM and DEIM may converge [11].

EIM was proposed to avoid the complete evaluation of the nonlinear function and for the Jacobian matrix to work iteratively, whereas DEIM is a discrete variant of EIM and represents a well-received effort towards a solution to the nonlinear function. In this paper, a discrete empirical interpolation method (DEIM) [12] is proposed to be used in conjunction with POD for getting computationally efficient reduced model of the large power grid network.

This paper is organized as follows, after the introductory section, mathematical modeling of swing dynamics in section-II. Section-III gives a mathematical description of the MOR strategy used. In the next section, the proposed strategy is implemented on a 1000 bus power grid network. Followed by a conclusion in the last section.

2. Mathematical Modeling of Swing Dynamics

A mathematical model to describe the transient dynamics of an alternator is the well known swing equation. It involves a second order differential equation representing the alternator node or bus that originates from the rotor dynamics of the alternator and algebraic equation associated with load bus [13, 1]. The dynamical behavior of the \( i^{th} \) alternator is given by following equation,

\[
\frac{2H_i}{\omega} \frac{d^2 \delta_i}{dt^2} + D_i \frac{d \delta_i}{dt} = P_{m_i} - P_{e_i}, \text{ for } i = 1, \ldots, N
\]  

(1)

This is a swing equation which, describes the power flow of the \( i^{th} \) bus [14]. The parameter \( H \) is the inertia constant of the generator, \( \omega \) is the reference frequency of the system, \( D \) is damping factor, \( \delta \) is rotor angle, \( P_m \) is the mechanical power provided to the generator, and \( P_e \) is the power demanded on the generator by the network (including the power lost to damping). Under equilibrium, \( P_{m_0} = P_{e_0} \) and if system experience fault or fluctuation, the power demand \( P_e \) varies. In (1) any variation in the difference between the power demand and power supply is compensated by either increasing or decreasing the angular momentum of the rotor [15].

The power grid network voltage is given by \( V_i = |V_i|e^{j\delta_i} \), where \( j = \sqrt{-1} \). Complex admittance of \( i^{th} \) line connected to \( i^{th} \) alternator is expressed as \( y_i = g_{i\ell} + jb_{i\ell} \), with \( b_{i\ell} \) is the line susceptance and \( g_{i\ell} \) is the line conductance. It is assumed that voltage magnitude is kept constant and transmission lines are purely reactive i.e. \( g_{i\ell} = 0 \). The transient state of the power system is described by rotor dynamics of \( i^{th} \) alternator in the following form [1],

\[
m_i \ddot{\delta} + d_i \dot{\delta} = P_{m_i} - \sum_{\ell=1}^{N} |V_i||V_\ell|b_{i\ell} \sin(\delta_i - \delta_\ell)
\]  

(2)

\( i = 1, 2, \ldots \ldots, N \). Equation (2) has solution, \( \delta_i(t) \) and it describes the transient behavior of the power grid network [6]. All the nodes are considered as generator or alternator (or PV) nodes. Equation(2) can be formulated in matrix form [1] as follows:

\[
M \ddot{\delta} + D \dot{\delta} = P_m - P(\delta(t))
\]  

(3)

where, \( M \) and \( D \) are \( N \times N \) diagonal matrices, while \( \ddot{\delta}, \dot{\delta} \) and \( \delta \) are vectors in \( R^N \), \( N \) being the number of nodes in the grid. The representation \( M \ddot{\delta} \) is for the matrix-vector product. The
vector $P(\delta)$ is a nonlinear function of $\delta$. Equation (3) is the high fidelity model of the swing dynamics. Electrical power depends on $\delta$ as well as $\dot{\delta}$ that means it depends on relative angular velocity as compared with a synchronously rotating system.

The nonlinear power grid network model (3) is formulated in state space as below:

\[
\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} -M^{-1}D\delta + M^{-1}(P_m - P(\delta(t))) \end{bmatrix}
\] (4)

The computational requirement to solve (4) is expensive due to large size ($2N$) and nonlinearity of the model [16]. Therefore, requirement of MOR is raised. Numerical techniques of MOR are elaborated in the next sections to reduce the model size and simulation time.

3. Description of MOR

3.1. Proper Orthogonal Decomposition (POD)

The aim of POD is to obtain a compact system by projecting a high-dimensional system into a low-dimensional subspace while retaining the dominant features of state evolution dynamics. The low dimensional subspace is obtained from state snapshots in response to certain inputs to which the high-dimensional system, i.e. full order model (FOM) is subjected [17]. For given an input function or an initial condition applied to a dynamical system whose state-trajectory evolves in $\mathbb{R}^n$, the objective is to approximate it by a trajectory in $\mathbb{R}^r$ with $r < n$. The problem can be stated as: Given a set of snapshots $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let $\chi = \text{span}(x_1, x_2, \ldots, x_n) \subset \mathbb{R}^n$, and let $r = \text{rank}(\chi)$. Find an orthonormal basis vector set $(\phi_1, \phi_2, \ldots, \phi_k)$ whose span best approximates $\chi$ for given $k < r$. This can be re-written as a matrix approximation problem: Let $X = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^{n \times n}$, Find $\hat{X}$ with $\text{rank}(\hat{X}) = k$ such that the error $E = ||X - \hat{X}||_F^2$ is minimised.

The solution to this, $\hat{X}^*$, is given by the Schmidt-Mirsky result [17, p. 37]:

\[
\hat{X}^* = V_k \Sigma_k W_k^T
\] (5)

Where $X = V \Sigma W^T$ is the Singular value decomposition (SVD) of $X$ with the columns of $V$ and $W$ consisting of left and right singular vectors respectively, $\Sigma \in \mathbb{R}^{n \times n}$ having singular-values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ on the main-diagonal and other elements zero, $\Sigma_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbb{R}^{k \times k}$, $V_k \in \mathbb{R}^{n \times k}$, consists of the first $k$ columns of, $V \in \mathbb{R}^{n \times n}$ and $W_k^T$ consists of the top $k$ rows of $W^T$. The error in the approximation (5) is

\[
||X - \hat{X}^*||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2
\] (6)

Hence the optimal orthonormal basis of rank $k$ which can approximate the state-trajectory evolving in $\mathbb{R}^n$ can be written as:

\[
\{\phi_i\}_{i=1}^k = \{v_i\}_{i=1}^k
\] (7)

The approximation is optimal in the 2-induced norm as well with the error given by:

\[
||X - \hat{X}^*||_2 = \sigma_{k+1}
\] (8)

In reduced order modelling, POD is used in conjunction with the Galerkin or orthogonal projection. The orthogonal projection matrix $V_k$ for forming the ROM is the basis obtained from the SVD of the snapshot-ensemble $X$ and is given by:

\[
V_k = \{v_i\}_{i=1}^k
\] (9)
In particular, consider the following nonlinear system with input $u(t)$, state vector $x(t)$ and nonlinear function $f(x(t))$ where state-trajectory, $x(t)$ evolves in $R^n$:

$$\dot{x}(t) = f(x(t)) + Bu(t)$$
$$y(t) = Cx(t)$$

(10)

$C$ is output vector matrix. The reduced order system of order $k \ll n$ that approximates the original system from a subspace spanned by a reduced basis of dimension $k$ in $R^n$ is obtained by Galerkin projection. Using the projection matrix obtained in (9) and projecting the system in (10) onto $V_k$, the reduced system is of the form:

$$\dot{\tilde{x}}(t) = V_k^T f(V_k \tilde{x}(t)) + V_k^T Bu(t)$$

(11)

Here $\tilde{x}(t) \in R^k$ and $x(t) \approx V_k \tilde{x}(t)$. The system (11) is reduced by orthogonal projection in state vector only. However, the order of nonlinear function for $k - modes$ from orthogonal projection, $f(V_k \tilde{x}(t)) \in R^n$ is not reduced.

3.2. Discrete Empirical Interpolation Method (DEIM)

The Empirical interpolation method (EIM) is proposed by finite elements (FE) community and its improved method is DEIM. The discrete empirical interpolation method (DEIM) was introduced used to address the computational inefficiency that occurs in solving the reduced-order model derived above(11). Equation (11) has the nonlinear term

$$\tilde{N}(\tilde{x}(t)) = V_k^T f(V_k \tilde{x}(t))$$

(12)

$\tilde{N}(\tilde{x})$ has a computational complexity that depends on $n$, the dimension of the full-order system[18]. DEIM [12] is an effective way to solve this difficulty. The function in (11) is approximated by projecting it onto a subspace that approximates the space generated by the nonlinear function and is of dimension $m \ll n$. From [12], the nonlinear function in (11) can be approximated as follows:

$$f(V_k \tilde{x}(t)) \approx U_m(P^T U_m)^{-1} P^T f(V_k \tilde{x}(t))$$

(13)

Where the nonlinear function is first projected onto the subspace spanned by the columns of $U_m \in R^{n \times m}$ and then $m$ distinguished rows are selected by pre-multiplying the whole system with $P^T$, where $P = [e_{\rho_1}, \ldots, e_{\rho_m}] \in R^{n \times m}$ and $e_{\rho_i}$ is the $\rho_i$th column of $I_n \in R^{n \times n}$. The Projection basis $U_m = [u_{1}, \ldots, u_{m}]$ for the nonlinear function $f(V_k \tilde{x}(t))$ is constructed by applying POD on the nonlinear function snapshots $F = [f(x_1(t)), \ldots, f(x_n(t))]$ obtained from the original system (10). The interpolation indices $\rho_1, \ldots, \rho_m$ are selected from the basis $[u_{1}, \ldots, u_{m}]$ by the DEIM algorithm. Substituting equation (13) in equation (12), the nonlinear term can be written as:

$$\tilde{N}(\tilde{x}(t)) = V_k^T U_m(P^T U_m)^{-1} f(P^T V_k \tilde{x}(t))$$

(14)

Where $P^T f(V_k \tilde{x}(t)) \in R^m$ in (12) has been replaced by $f(P^T V_k \tilde{x}(t))$ in (14) if the function $f$ evaluates component-wise at $x(t)$ (see [12]). Further the part $\Psi = V_k^T U_m(P^T U_m)^{-1}$ can be precomputed. From (11), (12) and (14) the final reduced order model can be written as [19]:

$$\dot{\tilde{x}}(t) = \Psi f(P^T V_k \tilde{x}(t)) + V_k^T Bu(t)$$

(15)
Algorithm 1 Reduced model using DEIM Algorithm

1: Input $U_m = \{u_i\}_{i=1}^m$
2: $[|p|, \rho_1] = \max |u_1|$
3: $U_m = [u_1], \ P = [e_{\rho_1}], \ \rho = [\rho_1]$
4: for $l = 2$ to $m$ do
5: Solve $(P^T U_m)c = P^T u_l$ for $c$
6: Residual, $r = u_l - U_m c$
7: $[|p|, \rho_l] = \max |r|$
8: $U_m \leftarrow [U_m \ u_l], \ P \leftarrow [P \ e_{\rho_l}]$
9: end for
10: Pre-compute offline $\Psi = V_k^T U_m (P^T U_m)^{-1}$
11: Reduced model, $\dot{\tilde{x}}(t) = \Psi f(P^T V_k \tilde{x}(t)) + V_k^T Bu(t)$

Algorithm 2 Steps to obtain ROM with POD-DEIM

1: Run full simulation for time $t$ and collect snapshots of the state and nonlinear function
2: two Snapshots matrices are $X = [x^1 \ x^2 \ ... \ x^{ns}]$ and $F = [f(x_1(t)), \ ... \ , f(x_{ns}(t))]$.
3: Perform SVD of snapshots matrices to compute the truncated POD basis of the system are, $V_k$ for state and $U_m$ for nonlinear term.
4: Determine the index matrix $P$ and approximate the nonlinear term as (13).
5: Construct ROM as (15).

This model in (15) has reduced the complexity of the nonlinear function in $k – modes$ from the order $O(k^n)$ to $O(k)$.

The steps of ROM are given in algorithm-2.

The value of $\| (I - UU^T) f \|_2$ should be decreased and $\| (P^T U_m)^{-1} \|_2$ must be minimal for increasing the accuracy of approximation [20]. In our study, large power grid model is proposed with and without POD-DEIM techniques.

4. Model Experiments

The network grid studied is a ring grid having only the alternators with one reference node connected to all the alternators, as indicated in fig-1 [21]. At slack bus both voltage and $\delta$ are known and used to balance power deviation in the system while performing the load flow study [14]. The slack bus ideally has infinite power capacity, so that the $\delta = 0$. To make calculations easy, certain assumptions have been taken for the study of transient power grid network.

- The power system model is lossless.
- Length of transmission lines between two consecutive alternators is very small than the line connecting alternators with the infinite bus.
- The length of transmission lines between infinite bus and all the alternators are identical.
- All alternators are mutually connected with same-sized transmission lines.

These assumptions enable us to derive a mathematical model of the motion of alternators where $\delta_i$ is the angular position of the rotor with respect to the reference bus of the $i^{th}$ alternator.

\[
\frac{1}{P_r} P_{e_1}(\delta_1, .., \delta_n) = \frac{V V_{\text{ref}}}{X_{\text{ref}} b} \sin \delta_i + \frac{V^2}{X_{\text{inf}} b_{\text{int}}} \{\sin(\delta_i - \delta_{i-1}) + \sin(\delta_i - \delta_{i+1})\}
\]
where $P_r$ is rated active power in watts. Parameters $V, V_{ref}, X_{ref}$ and $X_{int}$ are constant per unit values and respectively terminal voltage of the alternator, voltage of reference bus, the impedance of AC transmission lines joining alternator with reference bus, and impedance of AC transmission lines joining alternator $i$ and $i + 1$ in the grid network[21]. The nonlinear term $P(\delta)$ with four assumptions can be represented in per unit values for $i^{th}$ bus as follows,

$$P_{ei} = b\sin\delta_i + b_{int} \left[ \sin(\delta_i - \delta_{i-1}) + \sin(\delta_i - \delta_{i+1}) \right]$$ (16)

where $b$ is the critical transmission parameter between the reference bus and the $i^{th}$ alternator, and $b_{int}$ is the critical transmission parameter between generator $i$ and $i + 1$. The constants $b$ and $b_{int}$ are inversely proportional to the lengths of associated AC transmission lines.

The swing equation formulated in the standard form as given in (10), where $x$ is rotor angle, $f(x)$ is given by (16) and $u$ is mechanical power input $u(t) = P_m$. The output is represented by an average value of rotor angle of alternators $y(t) = \delta_{av}$. The average value of $\delta$ is defined as follows,

$$\delta_{av} = \frac{1}{N} \sum \delta_i$$ (17)

The data used for the study are given in Table-1, which are in per unit values [21, 1]. To ensure steady state stability of the power grid, a damping factor is included. The reduced-order model size is $k = 36$ (POD modes) with $m = 41$ (DEIM modes). The model simulation time and

**Table 1. Data Table**

| Symbol | Description               | value     |
|--------|---------------------------|-----------|
| $m_i$  | mass of alternator        | 1.0 (pu)  |
| $d_i$  | damping of alternator     | 0.25 (pu) |
| $P_m$  | power demanded by alternator | 0.95 (pu) |
| $b$    | susceptance between alternator and reference bus | 1(pu) |
| $b_{int}$ | susceptance between consecutive alternators | 100(pu) |
| $N$    | number of alternators     | 1000      |
approximation error with proposed algorithm is investigated with perturbation of node. For this purpose, four distinct cases are possible as given below.

1. Single node perturbation in non-equilibrium condition
2. All nodes starting from under perturbation
3. All nodes starting from over-perturbation
4. All nodes starting from synchronously equilibrium condition

To validate the proposed approach only one test case considered in this work.

**Test case: Single node perturbation in non-equilibrium condition**

This case we have all the Alternators, $i$ starting from an equilibrium point $\delta_i = 1$, with $D = 0.25$ except node $-2$ was perturbed by nearly 0.12 i.e. $\delta_2 = 1.12$. Using algorithm-1 and with the steps as given in algorithm-2, the time taken to simulate ROM is much lesser as compared to the time required for the FOM, as enumerated in table-2 and quantitative error in table-3.

| Time (s) | FOM | POD | DEIM |
|----------|-----|-----|------|
| 195.51   | 145.74 | 14.38 |

Dynamical profile of model is given in fig-2 and error profile between FOM and ROM are shown in fig-3.

![Figure 2. Dynamical profile starting with single node perturbation $\delta = 1.12$ for all Alternators](image)

From this test case it is clearly demonstrate that there is a definite saving in simulation time in ROM as compared to the FOM. The source of errors are also estimated that result due to approximation of FOM to ROM. The error plot is showing a very small error magnitude.
Magnitude of errors are associated with various factors such as the number of modes, step size, etc. The dynamical behavior of the ROM mimics the FOM with very little error. The error magnitude is associated with the singular value truncation (ranges between $10^{-1}$ to $10^{-5}$) and number of modes to be discarded. Therefore, the accuracy of the MOR can be associated with the number of modes kept.

For model accuracy in ROM, we have also observed that the order for the FOM, as well as the normal CPU times are significantly reduced. As shown in respective test cases, the biggest contribution to the simulation acceleration goes to the DEIM whenever ROM using POD is not significantly increasing the computational speed. As a matter of fact, without DEIM, the projection of the FOM completely lacks its sparse nature. The advantage of having a lower-dimensional system in POD leads to a ROM which might be even still slower as the FOM. Moreover, an efficient evaluation of the nonlinear terms saves the substantial computational burden. DEIM helps in online computation saving subject to better selection of modes in the orthogonal and interpolatory projections. The selection of modes are pertaining to singular values. Therefore, the proposed strategy for nonlinear model gives reduced size of state as well as nonlinear function. Experimental test cases serve the purpose of developing a scheme, which shall be reliable and has less simulation time with insignificant error.

![Figure 3. Error profile for MOR techniques](image-url)

**Figure 3.** Error profile for MOR techniques

**Table 3.** Quantitative comparison of errors.

| Model | % error ($\delta_{av}$) |
|-------|------------------------|
| POD   | $9.47 \times 10^{-10}$ |
| DEIM  | $9.49 \times 10^{-10}$ |
5. Conclusion
The addressed model is simulated with and without MOR and compared in terms of model order and computational time. The POD-DEIM approach minimizes the simulation time and order of nonlinear power grid network. It will make model computation faster. This work verified the POD-DEIM approach for large power grid network operating under the distinct possible scenarios. The performance and control of the system with POD-DEIM technique present an opportunity to focus on a minimum size dominant part of the dynamics. The orthogonal projection along with interpolatory projection reduces both offline as well as online computational burden by selecting appropriate number of modes. It provides better reliability and monitoring of the employed model. As shown in the test case, the DEIM approach ensures the reduced size same as the reduced number POD modes and hence the computational cost decreases drastically with insignificant error and meets the standard of real time simulation.

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