MULTI-PERIOD MEAN-VARIANCE PORTFOLIO SELECTION WITH FIXED AND PROPORTIONAL TRANSACTION COSTS

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Abstract. Portfolio selection problem is one of the core research fields in modern financial economics. Considering the transaction costs in multi-period investments makes portfolio selection problems hard to solve. In this paper, the multi-period mean-variance portfolio selection problems with fixed and proportional transaction costs are investigated. By introducing the Lagrange multiplier and using the dynamic programming approach, the indirect utility function is defined for solving the portfolio selection problem constructed in this paper. The optimal strategies and the boundaries of the no-transaction region are obtained in the explicit form. And the efficient frontier for the original portfolio selection problems is also given. Numerical result shows that the method provided in this paper works well.

1. Introduction. Portfolio selection problem is one of the most studied topics in financial economics. The problem is concerned with allocating the investors' wealth among different assets or asset classes properly. It is rather difficult to determine the optimal portfolio, which depends on the objective of the investors. A fundamental answer to the problem in single period setting was given by Markowitz who introduced the mean-variance model [16]. In the single period setting, the problem is well understood and can be easily solved by using the mean-variance model or other static models, see [14]. But in the multi-period setting, the problem is more complex than in single period setting. The explicit solution for the multi-period portfolio selection problem was proposed by Samuelson [19] and Merton [17] under some assumptions, such as the short sale is allowed; investment opportunities are constant; the market is complete and there are no transaction costs.

It is well known that an investor who ignores the transaction costs would end up bankrupt. Several works have been made of the effect of the transaction costs in the multi-period setting, see for example [5-22]. Kamin [13] introduced the transaction cost into the dynamic portfolio selection model, and found that the investor's behavior is systematically different from the one's without transaction costs. After that, Constantinides [8] extended Kamin's model to the HARA utility function. In [9], Davis and Norman studied the optimal consumption and investment decision with the transaction costs for an investor, and gave an algorithm for solving the free

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boundary problem. All these solutions above usually deal with the case in an infinite time horizon. But it is more realistic to analyze a finite terminal time. In this case, Gennotte and Jung [11] developed a numerical approximate values of the boundaries. Akian et al. [1] considered the \( n \) risky assets situation, and gave the viscosity solution. In [5], Boyle and Lin extended Gennotte and Jung’s work, and illustrated the solution procedure in which the returns on the risky asset follow a multiplicative binomial process. Framstad et al. [10] showed that the solution in a jump diffusion market has the same form as in the pure diffusion case. Jang [12] investigated an optimal portfolio selection problem with transaction costs when an illiquid asset pays cash dividends and there are constraints on the illiquid asset holding, and provided the closed form solutions for the problem. In [3], the robust optimization approaches were employed to solve a multi-period portfolio selection problem with transaction costs, and the comparative results with single period model were given. Yu et al. [23] proposed a new portfolio selection model with maximum absolute deviation in multi-period settings, and an analytical optimal strategy was obtained. Yao [21] constructed a multi-period mean-variance asset liability management problem and introduced the Lagrange duality method for solving this problem to obtain the closed form expressions for the investment strategies. In [20], they modeled the problem as a mixed-integer quadratic programme with transaction costs, and computational results were given for the test problem with 1317 assets.

In this paper, an explicit closed form solution to the finite horizon mean-variance portfolio selection problem is provided, when there are fixed and proportional transaction costs. A procedure to derive the boundaries of the no-transaction region and efficient frontier is also given.

This paper is organized as follows. In section 2, we set up a multi-period model and convert it to the dynamic programming problem. Section 3 describes the explicit form of the optimal strategies and the boundaries of the no-transaction region. Section 4 gives the efficient frontier and the main steps to solve the original dynamic programming problem. Section 5 presents some numerical examples. Some conclusions are given in the last section.

2. The model. Consider a financial market with two trading securities: one riskless asset and one risky asset at each time. Investors can make decision in this market for their sequential investment at \( T + 1 \) trading times over a finite planning period, indexed as \( t = 0, 1, \ldots, T \). Denote \( P^0_t \) the price of the risk-less asset and \( P_t \) the prices of the risky assets at time \( t \). Then, for \( t = 0, 1, \ldots, T - 1 \),

\[
\begin{align*}
r^0 &= \frac{P^0_{t+1}}{P^0_t}, & r_t &= \frac{P_{t+1}}{P_t},
\end{align*}
\]

are the total return on risk-less asset and risky assets, respectively. Thus, \( r^0 \) is a constant and \( r_t \) is a random variable.

Assume that an investor holds a portfolio with \( x^0_0 \geq 0 \) dollars of the risk-less asset and \( x_0 \geq 0 \) dollars of the risky asset at the initial time \( t = 0 \). At each trading time \( t = 0, 1, \ldots, T - 1 \), the investor may make his investment decision to maximize his expected utility of terminal wealth.

Let \( x^0_t \) be the dollar amounts of the risk-less asset and \( x_t \) be dollar amounts of the risky asset in the portfolio at time \( t \) before trading. Let \( \theta \) be the proportional transaction cost for buying or selling the risky asset, and \( F \) be the fixed transaction cost. We use \( u_t \) to denote investment decision at time \( t \). \( u_t \) is the amount of the
risky asset traded, \( u_t \geq 0 \) for buying and \( u_t \leq 0 \) for selling. Then, the following relationships can be built:

\[
y^0_t = x^0_t - u_t - \theta \cdot |u_t| - FH(u_t),
\]

(2)

\[
y_t = x_t + u_t,
\]

(3)

here \( y^0_t \geq 0 \) is the dollar amounts of the risk-less asset, \( y_t \geq 0 \) is the dollar amounts of the risky asset at time \( t \) after trading, and \( H(\cdot) \) is an impulse function, i.e.,

\[
H(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}
\]

Thus, at time \( t + 1 \) the portfolio amounts before trading can be written as:

\[
x^0_{t+1} = y^0_t r^0_t = (x^0_t - u_t - \theta \cdot |u_t| - FH(u_t)) r^0_t,
\]

(4)

\[
x_{t+1} = y_t r_t = (x_t + u_t) r_t.
\]

(5)

Equation (4)-(5) describe feasible investment decisions.

The investor’s objective is to minimize the variance of the terminal wealth

\[
\text{Var } W(T) = \text{E}[W(T) - EW(T)]^2 = \text{E}[W(T) - K]^2,
\]

for a prescribed target mean wealth \( EW(T) = K \), where \( K \geq W(0)e^T \). Then, the investor’s objective can be described as the following mean-variance (MV) portfolio selection problem:

\[
\min E[W(T) - K]^2
\]

subject to

\[
\begin{cases}
EW(T) = K, \\
(x^0_t, x_t, u_t) \text{ satisfy (4) and (5),} \\
t = 0, 1, \ldots, T - 1,
\end{cases}
\]

(6)

for the given initial portfolio \((x^0_0, x_0)\), where \( W(T) = x^0_T + x_T \).

It is clear that problem (6) is a dynamic quadratic convex optimization problem. To find the optimal strategy corresponding to the constraint \( EW(T) = K \), we introduce a Lagrange multiplier \( 2\lambda \in R \). Consequently, the new cost function is obtained as follows:

\[
\hat{J}(\lambda) = E[(W(T) - K)^2 + 2\lambda(W(T) - K)] = E[W(T) - (K - \lambda)]^2 - \lambda^2.
\]

Setting \( \gamma = K - \lambda \), problem (6) can be solved via the following optimal problem:

\[
\min \hat{J}(\gamma) = E[W(T) - \gamma]^2 - (K - \gamma)^2
\]

subject to

\[
\begin{cases}
(x^0_t, x_t, u_t) \text{ satisfy (4) and (5),} \\
t = 0, 1, \ldots, T - 1,
\end{cases}
\]

(7)

Noting that, the link between problem (6) and (7) is provided by the Lagrange duality theorem (see e.g. Luenberger [15])

\[
\min \text{Var } W(T) = \max_{\lambda \in R} \min_{\gamma \in R} \hat{J}(\lambda) = \max_{\gamma \in R} \min_{\lambda \in R} \hat{J}(\lambda).
\]

(8)

For a fixed constant \( \gamma \), problem (7) is clearly equivalent to

\[
\min E[x^0_T + x_T - \gamma]^2
\]

subject to

\[
\begin{cases}
(x^0_t, x_t, u_t) \text{ satisfy (4) and (5),} \\
t = 0, 1, \ldots, T - 1,
\end{cases}
\]

(9)
Problem (6) can be solved by a dynamic programming technique. In [5], it was assumed that the terminal utility function $U$ must be a concave, homogeneous differentiable function with some degree, say $\alpha$. Thus, the transformations are given as follows:

$$\bar{x}_T^0 = x_T^0 - \gamma,$$

$$\bar{x}_t^0 = x_t^0 - \gamma \cdot \left(\frac{1}{p_0}\right)^{T-t}, \ t = 0, 1, \cdots, T - 1. \tag{10}$$

Therefore, problem (9) is equivalent to

$$\min \ E[U(\bar{x}_T^0, x_T^0)] = E[(\bar{x}_T^0 + x_T^0)^2]$$

subject to

$$\begin{cases}
\bar{x}_{t+1}^0 = (x_t^0 - u_t - \theta \cdot |u_t| - FH(u_t))r_0^0 \\
x_{t+1} = (x_t + u_t)r_t \\
t = 0, 1, \cdots, T - 1.
\end{cases} \tag{11}$$

To apply dynamic programming, we define the indirect utility function $V_t$, $t = 0, 1, \cdots, T$, as follows:

$$V_t(\bar{x}_t^0, x_t) = \begin{cases} U(\bar{x}_T^0, x_T^0), & t = T, \\
\min_{u_t} E_t V_{t+1}(\bar{x}_{t+1}^0, x_{t+1}), & t = 0, 1, \cdots, T - 1,
\end{cases} \tag{12}$$

where $E_t$ denotes the expectation over $r_t$ conditional on $\bar{x}_t^0$ and $x_t$.

3. Optimal strategy. In this section, the procedures for solving problem (11) are developed. The derivation is based on the main theorem of [5].

Let

$$\phi_t(u_t, \bar{x}_t^0, x_t) = E_t V_{t+1}(\bar{x}_{t+1}^0, x_{t+1}). \tag{13}$$

The definition of the no-transaction region is given below. In this region, the expected value will not be increased by buying or selling the risky asset for any portfolio.

**Definition 3.1.** When the set of portfolio $\Phi_t$ satisfies

$$\Phi_t = \{(\bar{x}_t^0, x_t) | \phi_t(u_t, \bar{x}_t^0, x_t) \geq \phi_t(0, \bar{x}_t^0, x_t), \ for \ all \ u_t\}, \tag{14}$$

$\Phi_t$ is called the no-transaction region at time $t$.

According to the main theorem of [5], if $0 < a_t \leq b_t < \infty$, the no-transaction region can be written as

$$\Phi_t = \left\{ (\bar{x}_t^0, x_t) \ | \ a_t \leq \frac{x_t}{\bar{x}_t} \leq b_t \right\},$$

where

$$a_t = \min \left\{ x_t \ | \ \frac{\partial^+ \phi_t(0, 1, x_t)}{\partial u_t} = 0, \ x_t \geq 0 \right\},$$

$$b_t = \max \left\{ x_t \ | \ \frac{\partial^- \phi_t(0, 1, x_t)}{\partial u_t} = 0, \ x_t \geq 0 \right\},$$

and $\partial^+ \phi_t/\partial u_t$, $\partial^- \phi_t/\partial u_t$ denote the right and left derivatives of $\phi_t$, respectively. Then, the optimal policies are characterized in Theorem 3.2, when a portfolio is outside the no-transaction region.
Theorem 3.2. If \( a_t > \frac{x_t}{\pi_t} \), then
\[
\min_{u_t} \phi_t(u_t, \bar{x}_t^0, x_t) = \phi_t(u_t^+, \bar{x}_t^0, x_t) = \phi_t(0, y_t^0, y_t^+),
\]  
where
\[
\begin{align*}
  u_t^+ &= \frac{(\bar{x}_t^0 - F)a_t - x_t}{1 + (1 + \theta)a_t}, \\
y_t^0 &= \bar{x}_t^0 - (1 + \theta)u_t^+ - F, \\
y_t^+ &= x_t + u_t^+,
\end{align*}
\]  
with \((y_t^0, y_t^+) \in \Phi_t \) and \( \frac{y_t^+}{y_t^-} = a_t \).

If \( b_t < \frac{\bar{x}_t^0}{\pi_t} \), then
\[
\min_{u_t} \phi_t(u_t, \bar{x}_t^0, x_t) = \phi_t(u_t^-, \bar{x}_t^0, x_t) = \phi_t(0, y_t^0, y_t^-),
\]  
where
\[
\begin{align*}
  u_t^- &= \frac{(\bar{x}_t^0 - F)b_t - x_t}{1 + (1 - \theta)b_t}, \\
y_t^0 &= \bar{x}_t^0 - (1 - \theta)u_t^- - F, \\
y_t^- &= x_t + u_t^-,
\end{align*}
\]  
with \((y_t^0, y_t^-) \in \Phi_t \) and \( \frac{y_t^-}{y_t^-} = b_t \).

Proof. Case 1. \( a_t > \frac{x_t}{\pi_t} \).

If \( a_t < \infty \) since \( u_t^+ > 0 \),
\[
\frac{\partial \phi_t(u_t^+, \bar{x}_t^0, x_t)}{\partial u_t} = -r^0(1 + \theta)E_t \frac{\partial V_{t+1}((\bar{x}_t^0 - (1 + \theta)u_t^+ - FH(u_t^+))r^0, (x_t + u_t^+)r_t)}{\partial x_t} + E_t r_t \frac{\partial V_{t+1}((\bar{x}_t^0 - (1 + \theta)u_t^- - FH(u_t^-))r^0, (x_t + u_t^-)r_t)}{\partial x_t} + E_t r_t \frac{\partial \phi_t(0, a_t r_t)}{\partial x_t} = (\bar{x}_t^0 - (1 + \theta)u_t^+ - F) \cdot \frac{\partial \phi_t(0, 1, a_t)}{\partial u_t}.
\]

As we mention above that \( \frac{\partial \phi_t(0, 1, a_t)}{\partial u_t} = 0 \), thus
\[
\frac{\partial \phi_t(u_t^+, \bar{x}_t^0, x_t)}{\partial u_t} = 0.
\]

It is shown that \( u_t^+ \) is a minimum point.

If \( a_t = \infty \). As \( \phi_t \) is non-increasing, \( u_t^+ \) is the right end point of its domain, thus \( u_t^+ \) is the minimum point.

Case 2. \( b_t < \frac{\bar{x}_t^0}{\pi_t} \).

Similarly, if \( b_t < \infty \), then
\[
\frac{\partial \phi_t(u_t^-, \bar{x}_t^0, x_t)}{\partial u_t} = 0.
\]
Definition 3.3. The first, the definition of the indirect utility function is introduced. Therefore, \( u_t \) is the minimum point. If \( b_t = \infty \). As \( \phi_t \) is non-increasing, \( u_t^- \) is the right end point of its domain, thus \( u_t^- \) is the minimum point.}

We now present how to calculate \( a_t, b_t \) and the value function \( V_t(x_0, x_t) \). At first, the definition of the indirect utility function is introduced.

**Definition 3.3.** \( J \) is a piece-wise linear utility function with respect to the function \( U \), if there is a sequence of increasing numbers \( q_j, j = 1, \ldots, s \), and non-negative constants \( \alpha_{ij} \) and \( \beta_{ij} \) with respect to the underlying probability space \( \{ w_i; i = 1, \ldots, I \} \) such that

\[
J(x^0, x) = \sum_{i=1}^{l} U(\alpha_{ij} x^0, \beta_{ij} x) \Pr(w_i),
\]

for \( q_j \leq \frac{x}{x^0} \leq q_{j+1} \).

Assume that the value function \( V_t(x^0, x_t) \) is a piecewise linear utility function with respect to \( U \). Let \( \{ r_i^l; l = 1, \ldots, L \} \) be all possible outcomes for \( r_t \), and \( \{ w_i \} \) be all possible outcomes from \( \{ r^1_t, r^2_t, \ldots, r_T \} \), where \( \{ w_i \} \) represents the set of all future paths of the underlying tree structure starting at a node at time \( t + 1 \). Then, starting at time \( t \), all the paths of the underlying tree structure can be written as \( \{(r^l_t, w_i)\} \). We now calculate \( J_t \) recursively starting at \( t = T \).

At \( t = T \),

\[
V_T(x^0_T, x_T) = U(x^0_T, x_T) = (\bar{x}_T + x_T)^2.
\]

Suppose that

\[
V_{t+1}(x^0_{t+1}, x_{t+1}) = \sum_{i=1}^{l} U(\alpha_{ij} \bar{x}^0_{t+1}, \beta_{ij} x_{t+1}) \Pr(w_i)
\]

\[
= \sum_{i=1}^{l} (\alpha_{ij} x_{t+1}^0 + \beta_{ij} x_{t+1})^2 \Pr(w_i),
\]

\[
q_j \leq \frac{x_{t+1}}{\bar{x}_{t+1}} \leq q_{j+1}, \ j = 0, 1, \cdots, s.
\]

\[
q_0 = 0, \ q_{s+1} = \infty.
\]
Then
\[
\phi_t(0, \bar{x}_t^0, x_t) = E_t V_{t+1}(\bar{x}_t^0, x_t r_t) = \sum_{l=1}^{L} \sum_{i=1}^{I} U(a_{ij} r_0^0 \bar{x}_t^0, \beta_{ij} r_t^l x_t) \Pr \{(r_t^l, w_t)\} \]
\[
= \sum_{l=1}^{L} \sum_{i=1}^{I} (a_{ij} r_0^0 \bar{x}_t^0 + \beta_{ij} r_t^l x_t)^2 \Pr \{(r_t^l, w_t)\}. \tag{23}
\]

Let \( \tilde{\alpha}_{ij} = a_{ij} r_0^0 \), \( \tilde{\beta}_{ij} = \beta_{ij} r_t^l \), when \( u_t \geq 0 \),
\[
\phi_t(u_t, \bar{x}_t^0, x_t) = \sum_{l=1}^{L} \sum_{i=1}^{I} (\tilde{\alpha}_{ij} [x_t^0 - (1 + \theta) u_t - FH(u_t)] + \tilde{\beta}_{ij} [x_t + u_t]) \Pr \{(r_t^l, w_t)\}; \tag{24}
\]
and when \( u_t \leq 0 \),
\[
\phi_t(u_t, \bar{x}_t^0, x_t) = \sum_{l=1}^{L} \sum_{i=1}^{I} (\tilde{\alpha}_{ij} [x_t^0 - (1 - \theta) u_t - FH(u_t)] + \tilde{\beta}_{ij} [x_t + u_t]) \Pr \{(r_t^l, w_t)\}. \tag{25}
\]

Therefore, \( a_t \) is a solution of one of the equations
\[
\sum_{l=1}^{L} \sum_{i=1}^{I} (\tilde{\alpha}_{ij} + \tilde{\beta}_{ij} a_t) [\tilde{\beta}_{ij} - (1 + \theta) \tilde{\alpha}_{ij}] \Pr \{(r_t^l, w_t)\} = 0, \tag{26}
\]
when \( \tilde{q}_j \leq a_t < \tilde{q}_{j+1}, \ j = 0, 1, 2, \ldots \),

and \( b_t \) is a solution of one of the equations
\[
\sum_{l=1}^{L} \sum_{i=1}^{I} (\tilde{\alpha}_{ij} + \tilde{\beta}_{ij} b_t) [\tilde{\beta}_{ij} - (1 - \theta) \tilde{\alpha}_{ij}] \Pr \{(r_t^l, w_t)\} = 0, \tag{27}
\]
when \( \tilde{q}_j \leq b_t < \tilde{q}_{j+1}, \ j = 0, 1, 2, \ldots \).

The indirect utility function can be calculated as follows: rearrange all \((r_0^0/r_t^l)q_j\), \( l = 1, 2, \ldots, L, \ j = 0, 1, \ldots \), from smallest to largest and relabel them as \( \tilde{q}_h \) in order of magnitude.

For \( m = 1, \ldots, I \),
\[
\tilde{\alpha}_{mh} = \alpha_{mj} r_0^0, \quad \tilde{\beta}_{mh} = \beta_{mj} r_t^1,
\]
where \((r_0^0/r_t^1)q_j \leq \tilde{q}_h \leq \tilde{q}_{h+1} \leq (r_0^0/r_t^1)q_{j+1}; \)
for \( m = I+1, \ldots, 2I \),
\[
\tilde{\alpha}_{mh} = \alpha_{m-I,j} r_0^0, \quad \tilde{\beta}_{mh} = \beta_{m-I,j} r_t^2,
\]
where \((r_0^0/r_t^2)q_j \leq \tilde{q}_h \leq \tilde{q}_{h+1} \leq (r_0^0/r_t^2)q_{j+1}; \)
\ldots ;
for \( m = (L-1)I+1, \ldots, LI \),
\[
\tilde{\alpha}_{mh} = \alpha_{m-(L-1)L,j} r_0^0, \quad \tilde{\beta}_{mh} = \beta_{m-(L-1)L,j} r_t^L,
\]
where \((r_0^0/r_t^L)q_j \leq \tilde{q}_h \leq \tilde{q}_{h+1} \leq (r_0^0/r_t^L)q_{j+1}. \)
Then, change \( m \) and \( h \) back to \( i \) and \( j \) to avoid too much notation,

\[
\phi_t(0, \bar{x}_t^0, x_t) = \sum_{l=1}^{L} \sum_{i=1}^{I} (\tilde{\alpha}_{ij} \bar{x}_t^0 + \tilde{\beta}_{ij} x_t)^2 \Pr\{(r^l_t, w_t)\},
\]

\[
\tilde{q}_j \leq \frac{x_t}{x_t^0} \leq \tilde{q}_{j+1}.
\]

From Theorem 3.1, we have

\[
V_t(\bar{x}_t^0, x_t) = \begin{cases} 
\phi_t(0, \bar{y}_t^0, y_t^+), & \frac{\bar{x}_t}{x_t^0} < a_t, \\
\phi_t(0, \bar{x}_t^0, x_t), & a_t \leq \frac{\bar{x}_t}{x_t^0} \leq b_t, \\
\phi_t(0, \bar{y}_t^0, y_t^-), & \frac{\bar{x}_t}{x_t^0} > b_t.
\end{cases}
\]

Assume that

\[
\tilde{q}_{j_1} < a_t \leq \cdots \leq \tilde{q}_{j_2} \leq b_t < \cdots.
\]

Define

\[
\tilde{q}_0 = 0, \tilde{q}_1 = a_t, \tilde{q}_2 = \tilde{q}_{j_1}, \ldots, \tilde{q}_{j_2-j_1+2} = b_t, \tilde{q}_{j_2-j_1+3} = \infty,
\]

and

\[
\tilde{\alpha}_0 = \frac{\tilde{\alpha}_{i,j_1} + a_t \tilde{\beta}_{i,j_1}}{1 + (1 + \theta) a_t}, \quad \tilde{\beta}_0 = (1 + \theta) \tilde{\alpha}_0;
\]

\[
\tilde{\alpha}_{ij} = \tilde{\alpha}_{i,j_1+j-1}, \quad \tilde{\beta}_{ij} = \tilde{\beta}_{i,j_1+j-1}, \quad j = 1, 2, \ldots, j_2 - j_1 + 1;
\]

\[
\tilde{\alpha}_{i,j_2-j_1+2} = \frac{\tilde{\alpha}_{i,j_2} + b_t \tilde{\beta}_{i,j_2}}{1 + (1 - \theta) b_t}, \quad \tilde{\beta}_{i,j_2-j_1+2} = (1 - \theta) \tilde{\alpha}_{i,j_2-j_1+2}.
\]

Hence,

\[
V_t(\bar{x}_t^0, x_t) = \sum_{l=1}^{L} \sum_{i=1}^{I} (\tilde{\alpha}_{ij} \bar{x}_t^0 + \tilde{\beta}_{ij} x_t)^2 \Pr\{(r^l_t, w_t)\},
\]

\[
\tilde{q}_j \leq \frac{x_t}{x_t^0} \leq \tilde{q}_{j+1}, \quad j = 0, 1, \cdots.
\]

4. **Efficient frontier.** In this section, the relation between the variance and the expected value of the wealth for the problem (6) is obtained. According to the above derivations, the explicit solution of the value function is given as follows:

\[
V_t(x_t^0, x_t) = \sum_{l=1}^{L} \sum_{i=1}^{I} \left\{ \tilde{\alpha}_{ij} \left[ x_t^0 - \gamma \cdot \left( \frac{1}{\alpha^0} \right)^{T-t} \right] + \tilde{\beta}_{ij} x_t \right\}^2 \Pr\{(r^l_t, w_t)\},
\]

\[
\tilde{q}_j \leq \frac{x_t}{x_t^0 - \gamma \cdot \left( \frac{1}{\alpha^0} \right)^{T-t}} \leq \tilde{q}_{j+1}, \quad j = 0, 1, \cdots.
\]

Based on (8), there exists

\[
\min \Var W(T) = \max_{\lambda \in R} \min_{\gamma \in R} J(\lambda) = \max_{\gamma \in R} J_{\gamma}(x_t^0, x_t) - (K - \gamma)^2.
\]

So, its maximum attains at point \( \gamma^* \) which satisfies

\[
\sum_{l=1}^{L} \sum_{i=1}^{I} \left\{ \tilde{\alpha}_{ij} \left[ x_t^0 - \gamma^* \cdot \left( \frac{1}{\alpha^0} \right)^{T-t} \right] + \tilde{\beta}_{ij} x_t \right\} \tilde{\alpha}_{ij} \left( \frac{1}{\alpha^0} \right)^{T-t} \Pr\{(r^l_t, w_t)\} + (K - \gamma^*) = 0,
\]
\[ \gamma^* = \frac{K - \sum_{l=1}^{L} \sum_{i=1}^{I} (\bar{\alpha}_{ij} x^0_t + \bar{\beta}_{ij} x_t) \bar{\alpha}_{ij} \left( \frac{1}{r_0} \right)^{T-t} \Pr\{r^l_i, w_i\}}{1 - \sum_{l=1}^{L} \sum_{i=1}^{I} \left[ \bar{\alpha}_{ij} \left( \frac{1}{r_0} \right)^{T-t} \right]^2 \Pr\{r^l_i, w_i\}}. \]

For a given \((x^0_0, x_0)\) at time \(t = 0\), we have
\[ \gamma^* = \frac{K - \sum_{l=1}^{L} \sum_{i=1}^{I} (\bar{\alpha}_{ij} x^0_0 + \bar{\beta}_{ij} x_0) \bar{\alpha}_{ij} \left( \frac{1}{r_0} \right)^{T} \Pr\{r^l_0, w_i\}}{1 - \sum_{l=1}^{L} \sum_{i=1}^{I} \left[ \bar{\alpha}_{ij} \left( \frac{1}{r_0} \right)^{T} \right]^2 \Pr\{r^l_0, w_i\}}. \] (32)

Thus,
\[ \text{Var } W(T) = V_0(x^0_0, x_0, \gamma^*) - (K - \gamma^*)^2. \] (33)

Substituting (32) into (33), the efficient frontier is obtained.

Based on the above discussion, the problem (6) can be solved by the following procedures:

First, choose the proper distribution of the \(r_t\) and construct the scenario tree to determine the scenario’s paths.

Secondly, use (26) and (27) to calculate the boundaries of the no-transaction region.

Finally, determine whether the portfolio lies in the no-transaction region. If it is, the portfolio is the optimal strategy; if not, take the optimal strategies as (16) and (18) shown.

5. Numerical example. The model and the solution procedure will be illustrated in this section by the numerical examples. We assume \(T = 4\) and consider the case in which the rate of return for the risky asset in each period is dependent of \(t\) and has only two states \(u\) and \(d\). The parametrization for \(u\) and \(d\) in [5] is used in this section, i.e., \(u = e^{\sigma \sqrt{h}}, d = e^{-\sigma \sqrt{h}}, r^0 = e^{\delta h}\) that \(\sigma = 1, h = 0.25, \delta = 0.05\). Thus the return of risk-less asset is \(r^0 = 1.0126\) and the scenario data for return of the risky asset is shown in Table 1.

| i'th Scenario | \(r_0\) | \(r_1\) | \(r_2\) | \(r_3\) |
|---------------|--------|--------|--------|--------|
| 1             | 1.0205 | 1.6825 | 2.7739 | 4.5733 |
| 2             | 1.0205 | 1.6825 | 2.7739 | 1.6824 |
| 3             | 1.0205 | 1.6825 | 1.0204 | 1.6823 |
| 4             | 1.0205 | 1.6825 | 1.0204 | 0.6189 |
| 5             | 1.0205 | 0.6189 | 1.0204 | 1.6823 |
| 6             | 1.0205 | 0.6189 | 1.0204 | 0.6189 |
| 7             | 1.0205 | 0.6189 | 0.3754 | 0.6189 |
| 8             | 1.0205 | 0.6189 | 0.3754 | 0.2277 |

We can calculate the boundaries of no-transaction region recursively backwards from the last period by using the procedures in the last section. Table 2 shows the values, when \(\theta = 0.01\) and \(\theta = 0.1\) respectively. As shown in table 2, if an investor who holds the initial proportion of the risky asset is 0.05 and the initial
proportion of the risk-less asset is 0.95, then he should buy the risky asset to reach the boundary $a_0 = 0.55226$. From table 2, it is also shown that the transaction costs have an impact on the no-transaction region. It is clear that when the transaction cost increases from 0.01 to 0.1, the distance between $a_t$ and $b_t$ becomes larger, which means that the larger the transaction cost the wider the no-transaction region becomes. For any investor, when the transaction cost increases, they will more willingly to hold the risk-less asset than risky asset.

| $\theta$ | $a_0$ | $b_0$ | $a_1$ | $b_1$ | $a_2$ | $b_2$ | $a_3$ | $b_3$ |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.01     | 0.55226| 0.79526| 0.31089| 0.43902| 0.17999| 0.28314| 0.055457| 0.15572|
| 0.1      | 0.59226| 0.79226| 0.23633| 0.43716| 0.16516| 0.29668| 0.054259| 0.1569  |

To interpret the efficiency with and without transaction costs, assume that the initial wealth is 1000 dollars, of which initial proportion of risky asset is 10% and that of risk-less asset is 90%. Optimal investment decisions for problem (6) are shown in Table 3 and Table 4. For each entry in these tables, the first number $\gamma_t$ represents $x_t - x_{t-1}$. According to Theorem 2, it should be compared with $a_t$ and $b_t$ in Table 2, then we can determine how to calculate $u_t$. The second number stand for the amount of risky asset one should buy or sell, while the third and forth numbers give the amount of risk-less asset and risky asset one should hold, respectively. Comparing the results in table 3 with the one in table 4, we can see that the investors facing the higher transaction costs will behave more risk aversion.

Table 5 gives the optimal investment strategies in each period with different transaction costs. It shows that the investors will change his strategies according to his forecasting of the rate of the risky asset’s return. Take the first scenario as an example, the scenario shows the up-up tendency while the proportion of the risky asset increase. It means that when the investors in the bull market they are more willingly to invest on the risky asset to gain more returns, which is closer to real investment practice. For the other tendencies, the results obtained in table 5 also have the same performance shown in the real investment activities. Anyway, the above results show the efficiency of the method we proposed in this paper.

Figure 1 depicted the efficient frontiers with different proportional transaction costs. The figure clearly shows that the mean-variance efficient frontier declines as the transaction costs increase, which means that the risk aversion investor will be more cautious in trading risky asset when transaction costs exist.
Figure 1. Efficient frontier with different proportional transaction costs.

Table 3. Numerical results for $\theta = 0.01$

| $\theta$ | 0   | 1   | 2   | 3   | 4   |
|----------|-----|-----|-----|-----|-----|
| $\gamma_t$ | 0.1853 | 0.5566 | 0.7295 | 0.7756 | -   |
| $u_t$ | 125.3336 | -35.3831 | -157.5504 | -326.8636 | -   |
| $x^0_t$ | 900 | 778.0951 | 815.0674 | 981.4946 | 1316.5 |
| $x_t$ | 100 | 229.9529 | 327.3637 | 471.0451 | 659.3853 |
| $\gamma_t$ | 0.1853 | 0.5566 | 0.7295 | 0.7756 | -   |
| $u_t$ | 125.3336 | -35.3831 | -157.5504 | -326.8636 | -   |
| $x^0_t$ | 900 | 778.0951 | 815.0674 | 981.4946 | 1316.5 |
| $x_t$ | 100 | 229.9529 | 325.0010 | 471.0451 | 242.5710 |
| $\gamma_t$ | 0.1853 | 0.5566 | 0.7295 | 0.7756 | -   |
| $u_t$ | 125.3336 | -35.3831 | -157.5504 | -66.7803 | -   |
| $x^0_t$ | 900 | 778.0951 | 815.0674 | 981.4946 | 1055.7 |
| $x_t$ | 100 | 229.9529 | 325.0010 | 170.8666 | 64.4190 |
| $\gamma_t$ | 0.1853 | 0.5566 | 0.7295 | 0.7756 | -   |
| $u_t$ | 125.3336 | -35.3831 | -157.5504 | -66.7803 | -   |
| $x^0_t$ | 900 | 778.0951 | 815.0674 | 981.4946 | 1055.7 |
| $x_t$ | 100 | 229.9529 | 325.0010 | 122.8758 | 47.6862 |
| $\gamma_t$ | 0.1853 | 0.5566 | 0.7295 | 0.7756 | -   |
| $u_t$ | 125.3336 | -35.3831 | 0 | -45.8259 | -   |
| $x^0_t$ | 900 | 778.0951 | 815.0674 | 825.3372 | 879.9343 |
| $x_t$ | 100 | 229.9529 | 120.4192 | 122.8758 | 129.6210 |
| $\gamma_t$ | 0.1853 | 0.5566 | 0.7295 | 0.7756 | -   |
| $u_t$ | 125.3336 | -35.3831 | 0 | -45.8259 | -   |
| $x^0_t$ | 900 | 778.0951 | 815.0674 | 825.3372 | 879.9343 |
| $x_t$ | 100 | 229.9529 | 120.4192 | 122.8758 | 47.6862 |
| $\gamma_t$ | 0.1853 | 0.5566 | 0.7295 | 0.7756 | -   |
| $u_t$ | 125.3336 | -35.3831 | 0 | -45.8259 | -   |
| $x^0_t$ | 900 | 778.0951 | 815.0674 | 825.3372 | 835.7364 |
| $x_t$ | 100 | 229.9529 | 120.4192 | 45.2054 | 10.2933 |
Table 4. Numerical results for $\theta = 0.1$

|   | 0     | 1     | 2     | 3     | 4     |
|---|-------|-------|-------|-------|-------|
| 1 | $\gamma_t$ | 0.1853 | 0.5724 | 0.7295 | 0.7756 |
|   | $u_t$   | 125.3336 | -39.9796 | -156.9790 | -316.9547 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 955.8802 | 1251.7 |
|   | $x_t$   | 100 | 229.9529 | 319.6301 | 451.1779 | 613.8430 |
| 2 | $\gamma_t$ | 0.1853 | 0.5724 | 0.7295 | 0.7756 |
|   | $u_t$   | 125.3336 | -39.9796 | -156.9790 | -316.9547 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 955.8802 | 1251.7 |
|   | $x_t$   | 100 | 229.9529 | 319.6301 | 451.1779 | 225.8171 |
| 3 | $\gamma_t$ | 0.1853 | 0.5724 | 0.7295 | 0.7756 |
|   | $u_t$   | 125.3336 | -39.9796 | -156.9790 | -66.8041 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 955.8802 | 1023.7 |
|   | $x_t$   | 100 | 229.9529 | 319.6301 | 165.9692 | 166.8254 |
| 4 | $\gamma_t$ | 0.1853 | 0.5724 | 0.7295 | 0.2853 |
|   | $u_t$   | 125.3336 | -39.9796 | -156.9790 | -66.8041 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 955.8802 | 1023.7 |
|   | $x_t$   | 100 | 229.9529 | 319.6301 | 165.9692 | 61.3733 |
| 5 | $\gamma_t$ | 0.1853 | 0.5724 | 0.2683 | 0.2704 |
|   | $u_t$   | 125.3336 | -39.9796 | 0 | -45.3094 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 817.8820 | 864.4166 |
|   | $x_t$   | 100 | 229.9529 | 117.5745 | 119.9730 | 125.6066 |
| 6 | $\gamma_t$ | 0.1853 | 0.5724 | 0.2683 | 0.2704 |
|   | $u_t$   | 125.3336 | -39.9796 | 0 | -45.3094 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 817.8820 | 864.4166 |
|   | $x_t$   | 100 | 229.9529 | 117.5745 | 119.9730 | 46.2093 |
| 7 | $\gamma_t$ | 0.1853 | 0.5724 | 0.2683 | 0.0995 |
|   | $u_t$   | 125.3336 | -39.9796 | 0 | 0 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 817.8820 | 828.1873 |
|   | $x_t$   | 100 | 229.9529 | 117.5745 | 119.9730 | 27.3167 |
| 8 | $\gamma_t$ | 0.1853 | 0.5724 | 0.2683 | 0.0995 |
|   | $u_t$   | 125.3336 | -39.9796 | 0 | 0 |
|   | $x_t^0$ | 900 | 766.6729 | 807.7049 | 817.8820 | 828.1873 |
|   | $x_t$   | 100 | 229.9529 | 117.5745 | 44.1375 | 10.0501 |
Table 5. Proportion of riskless asset and risky asset

| Scenario | Time | $\theta = 0.001$ | $\theta = 0.01$ |
|----------|------|-----------------|-----------------|
|          | $x_t^0$ | $x_t$ | $x_t^0$ | $x_t$ |
| 1        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.769269 | 0.230731 | 0.769269 | 0.230731 |
|          | 0.716473 | 0.283527 | 0.716473 | 0.283527 |
|          | 0.679347 | 0.320653 | 0.679347 | 0.320653 |
|          | 0.670957 | 0.329043 | 0.670957 | 0.329043 |
| 2        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.718183 | 0.228117 | 0.769269 | 0.230731 |
|          | 0.714929 | 0.285071 | 0.716473 | 0.283527 |
|          | 0.851725 | 0.148275 | 0.852058 | 0.147942 |
|          | 0.857732 | 0.142268 | 0.859872 | 0.140128 |
| 3        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.771883 | 0.228117 | 0.769269 | 0.230731 |
|          | 0.714929 | 0.285071 | 0.716473 | 0.283527 |
|          | 0.851725 | 0.148275 | 0.852058 | 0.147942 |
|          | 0.857732 | 0.142268 | 0.859872 | 0.140128 |
| 4        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.771883 | 0.228117 | 0.769269 | 0.230731 |
|          | 0.871276 | 0.128274 | 0.872931 | 0.127069 |
|          | 0.870413 | 0.129587 | 0.872077 | 0.127923 |
|          | 0.948593 | 0.051407 | 0.949255 | 0.050745 |
| 5        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.771883 | 0.228117 | 0.769269 | 0.230731 |
|          | 0.871276 | 0.128274 | 0.872931 | 0.127069 |
|          | 0.870413 | 0.129587 | 0.872077 | 0.127923 |
|          | 0.948593 | 0.051407 | 0.949255 | 0.050745 |
| 6        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.771883 | 0.228117 | 0.769269 | 0.230731 |
|          | 0.871276 | 0.128274 | 0.872931 | 0.127069 |
|          | 0.948593 | 0.051407 | 0.949255 | 0.050745 |
| 7        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.771883 | 0.228117 | 0.769269 | 0.230731 |
|          | 0.871276 | 0.128274 | 0.872931 | 0.127069 |
|          | 0.948593 | 0.051407 | 0.949255 | 0.050745 |
| 8        | 0.900000 | 0.100000 | 0.900000 | 0.100000 |
|          | 0.771883 | 0.228117 | 0.769269 | 0.230731 |
|          | 0.871276 | 0.128274 | 0.872931 | 0.127069 |
|          | 0.948593 | 0.051407 | 0.949255 | 0.050745 |
6. Conclusion. Multi-period mean-variance portfolio selection problems with fixed and proportional transaction costs are investigated. We have developed the analytical expressions for the indirect utility function and the boundaries of the no-transaction region, and given the optimal strategies and efficient frontiers of the investment problem. The numerical example indicates the efficiency of the method.

The large-scale problems is common in the practical applications but it is difficult to calculate. Therefore, it is necessary to develop an efficient algorithm to solve the large-scale problems. This is left for future research.

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