Resarch on Calculation Method of Pile Horizontal Response under Combined Loads

Yongzhi Li\textsuperscript{1,*}, Wei Huang\textsuperscript{1}, Fuquan Ji\textsuperscript{1}, Minghui Yang\textsuperscript{2}

\textsuperscript{1} CCC Second Harbour Engineering Company LTD., Wuhan, Hubei, 430040, China
\textsuperscript{2} Geotechnical Engineering Institute, Hunan University, Changsha, Hunan, 410082, China
\textsuperscript{*} Corresponding author’s e-mail: liyongzhi2@cccltd.cn

Abstract. The main function of the pile foundation is to bear the vertical load, but in actual engineering, the pile foundation usually bears combined loads which contain vertical load, lateral load and bending moment. Under combined loads, due to the P-Δ effect, the calculation of the bearing capacity and deformation of the pile foundation is very complicated. The small deformation superposition principle recommended by the current pile foundation codes fails to consider the resistance of the soil around the pile under the combined loads, and it is difficult to accurately evaluate the influence of P-Δ effect on the bearing capacity and deformation of pile foundation. In view of this, this paper firstly starts from the potential energy equation of the pile-soil system and deduces the calculation formula for the horizontal displacement of the pile considering the P-Δ effect, and it is verified in conjunction with related calculation examples. Then the factors affecting the P-Δ effect are discussed in detail, and the results show that the larger the load ratio is, the more obvious the P-Δ effect will be; when the pile top is fixed under the same load condition, the P-Δ effect will be significantly weakened.

1. Introduction

In recent years, with the rapid development of economy and society, China has made remarkable achievements in the field of high-rise buildings, ports, bridges and other infrastructure constructions. Because of its high bearing capacity and small settlement deformation, pile has become the main foundation form of such high-rise buildings. The main function of pile is to bear the vertical load from the superstructure, but in practical engineering, the pile is rarely only subjected to the vertical load, usually also to bear the horizontal load and eccentric moment load, that is, to bear the combined loads. For example, the pile of high-rise building will not only bear the vertical load caused by the self weight of the structure, but also bear the wind load, horizontal seismic action and so on. In the bridge, port, offshore drilling platform and other projects, the pile will also bear the horizontal load from the lateral wind, earth pressure, wave force and ship collision pressure. The mechanical behavior of pile under combined loads is much more complex than that under single load. On the one hand, there will be large deflection deformation in the pile due to horizontal load and eccentric moment load; on the other hand, due to the deflection deformation of pile, the vertical load will produce an additional moment, which forms P - Δ effect, which aggravates the deflection deformation of pile. In addition, the pile-soil interaction under combined loads is also very complex. Therefore, more and more attention has been paid to the mechanical analysis of pile under combined loads, and it has become one of the hot spots in the research field.
It is very complicated to consider both horizontal and vertical loads when determining the bearing capacity and deformation of pile. In order to solve this problem, the calculation method given in the relevant code for pile is as follows: firstly, the bearing capacity and deformation of pile under single load are considered; secondly, the bearing capacity and deformation of pile under combined loads are calculated according to the superposition principle of small deformation, and the section bending moment is corrected by the eccentricity increasing factor when checking the section strength, without considering the influence of horizontal and vertical load interaction on the bearing capacity of pile. It is difficult to evaluate the effect of P-Δ effect on the internal force and deformation of pile accurately, because this method fails to consider the exertion characteristics of soil resistance around pile under combined loads. In view of this, based on the theory of energy method in elastic mechanics, this paper puts forward the energy calculation method of horizontal response of pile under combined loads by introducing the additional work term of vertical load, and then deeply discusses the variation law of P-Δ effect. These research results can make up for the shortcomings of traditional design methods to a certain extent, and provide some reference for related engineering design.

2. Model description

2.1. Pile soil system model
The plane diagram of pile-soil system is shown in Fig. 1. The total length of pile is L, in which the length of free section is $L_0$ and the radius of pile body is $R$. Because the pile should be able to work stably under the maximum load, most of the soil around the pile is in the stage of elastic deformation. Therefore, it is assumed that the soil around the pile is an isotropic elastic body, and its elastic modulus and Poisson's ratio are $E_s$ and $\mu_s$ respectively. At the same time, in order to establish the governing equation of the pile-soil system, the virtual pile section is introduced at the bottom of the pile. The column coordinate system is adopted for the pile-soil system, as shown in Fig. 1.

![Figure 1. Calculation diagram of pile soil system](image)

2.2. Displacement model of soil around pile
Based on the research of Sun [2] and Seo [3], the displacement of soil around pile under combined loads can be expressed by the separation function related to $r$, $\theta$ and $z$ in cylindrical coordinate system:
Where, $w(z)$ is horizontal displacement of pile; $f(r)$ is attenuation function of horizontal displacement of soil around pile; $v(z)$ is vertical displacement of pile; $y(r)$ is attenuation function of vertical displacement of soil around pile.

### 2.3. Governing equation of pile soil system

Based on the theory of elastic mechanics and formula (1), the potential energy equation of pile-soil system can be expressed as:

$$\Pi = U_{pile} + U_{soil} + U_{external\ force} + U_{p-\Delta\ effect}$$

Where, $\sigma_{kl}$ is the stress component of soil, and its distribution is shown in Fig. 2. The calculation formula is:

$$\sigma_{kl} = 2G_{s} \varepsilon_{kl} + \lambda_{s} \varepsilon_{kl} \delta_{kl}$$

Where, $G_{s}$ and $\lambda_{s}$ are shear modulus and lame constant respectively. $\varepsilon_{kl}$ is the strain component of soil, which can be derived from formula (1):
The total potential energy $\Pi$ of pile-soil system is a functional containing four unknown functions $w(z)$, $v(z)$, $f(r)$ and $y(r)$. According to the principle of minimum potential energy, the variation of $\Pi$ should be equal to 0, that is $\delta \Pi = 0$. It can be sorted into the following forms by partial integral and merging similar items.

\[
Aw w' + Bw w' + Cv v' + Df f' + Ey y' = 0
\] (5)

Since $w$, $w'$, $v$, and $y$ are unknown in advance, their variational $\delta w$, $\delta (w')$, $\delta v$, $\delta f$ and $\delta y$ are not 0, that is, if and only if $A(w)$, $B(w)$, $C(v)$, $D(f)$, $E(y)$ are all equal to 0, then formula (5) holds.

2.4. Analysis of horizontal displacement of pile

It is assumed that the axial force $N(z)$ of the pile is linearly distributed along the depth [4], i.e

\[
N(z) = \left\{ \begin{array}{ll}
N_0 + f_0 z, & 0 \leq z \leq L_0 \\
N_0 + f_0 L_0 + f_2, & L_0 \leq z \leq L 
\end{array} \right.
\] (6)

Where, $f_0 = \gamma A_p$, $f = \gamma A_p - \pi u / 2$, $\gamma$ is the weight of pile, $A_p$ is the cross-sectional area of pile, $u$ is the circumference of pile section, and $\tau$ is the ultimate friction resistance of pile side.

From $A(w) = 0$, the differential control equation of horizontal displacement of pile can be obtained:

\[
\frac{d^4 w^{(0)}}{dz^4} + f_0 (z) \frac{d^2 w^{(0)}}{dz^2} + \alpha_0 \frac{dw^{(0)}}{dz} = 0, \quad 0 \leq z \leq L_0
\]

\[
\frac{d^4 w}{dz^4} + f(z) \frac{d^2 w}{dz^2} + \alpha^3 \frac{dw}{dz} + kw = 0, \quad L_0 \leq z \leq L
\] (7)

Where, $t$ and $k$ are the parameters to describe the resistance of soil around the pile.

\[
f_0 (z) = \lambda_0^2 + \lambda_0 z
\]

\[
f(z) = \alpha^3 (z - L_0) + \lambda^2 - 2t
\]

\[
t = \frac{\pi G_s}{E_p I_p} \int_0^\infty \phi^2 r dr, \quad \text{for equation (7)}
\]

\[
t = \frac{\pi G_s}{E_p I_p} \left( \int_0^\infty \phi^2 r dr + \frac{R^2}{2} \right), \quad \text{for parameters } \beta
\]

\[
k = \frac{\pi (\lambda_0 + 3G_s)}{E_p I_p} \left( \int_0^\infty \frac{d\phi}{dr} \right)^2 dr
\]

\[
\lambda_0^2 = \frac{P_0}{E_p I_p} ; \quad \lambda^2 = \frac{P_0 + f_0 L_0}{E_p I_p} ; \quad \alpha_0 = \frac{f_0}{E_p I_p} ; \quad \alpha^3 = \frac{f}{E_p I_p}.
\]
The boundary conditions of equation (7) can be obtained from the compatible conditions of the pile-top and pile-bottom boundary:

\[ z = 0, \]
\[ z = L_0, \]
\[ z = L, \]

Where,

\[ Q = \frac{Q_0}{E_p I_p}; \]
\[ M_0 = \frac{M_0}{E_p I_p}; \]
\[ \beta = \sqrt{\frac{k}{E_p I_p}}; \]
\[ \lambda^2 = \frac{P_0 + f_0 L_0 + f (L - L_0)}{E_p I_p}. \]

2.5. Analysis of horizontal displacement of soil around pile

The differential control equation of \( f(r) \) can be obtained from \( D(f) = 0 \):

\[ \frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d \phi}{dr} - \left( \frac{\gamma}{R} \right)^2 \phi = 0 \]

Where,
\[ \gamma = R \left( \frac{2G_0 \int_0^\infty (z')^2 dz}{(A_0 + 3G_z) \int_0^\infty z^2 dz} \right)^{1/2} \]

\(f(r)\) satisfies the boundary conditions:
\[
\phi(r) = \begin{cases} 
1, r = R \\
0, r = \infty 
\end{cases} \tag{12}
\]

Equation (11) is a zero-order Bessel equation, which satisfies the boundary conditions (12) and the analytical solution can be expressed as:
\[
\phi(r) = \frac{K_0(\gamma r / R)}{K_0(\gamma)} \tag{13}
\]

Where, \(K_0()\) is the second kind of modified zero order Bessel function.

3. Solution of the governing equations

3.1. Solution of governing equation based on finite difference method

For the differential control equation (7), it contains fourth derivative, and the second derivative coefficients \(f_0(z)\) and \(f(z)\) are all related to \(z\), and the analytical solution is not easy to obtain. Therefore, the finite difference method is used to solve it. The mesh division of pile is shown in Figure 3. The free section of pile is divided into \(n_0\) sections, embedded section is divided into \(n\) sections, and the distance between adjacent two nodes is \(H = L_0 / n_0 = (L-L_0) / n\). In order to use the center difference method, the virtual node numbers -2 and -1 are taken at the top of the pile, and the virtual node numbers \(n+n_0+1\) and \(n+n_0+2\) are taken at the bottom of the pile.

\[ K \cdot W = F \tag{14} \]

Where,
The Gauss-Seidel iteration of equation (14) can be used to obtain the horizontal displacement of the pile.

3.2. Solver
It can be seen from the above analysis that the solution of the horizontal displacement of the pile and the horizontal displacement of the soil around the pile are interdependent and must be solved at the same time. Therefore, the iterative calculation method is adopted, and the specific calculation process is shown in Figure 4.

4. Method verification

4.1. Example 1
Firstly, considering the pile only subjected to horizontal load and bending moment, the analysis results of this paper are compared with the field test results of McClelland and Focht [5]. In the above field test, the length of the pile is \( L = 23\) m, the diameter of the pile is \( r = 0.305\) m, the pile top bears horizontal load \( Q_0 = 300\) kN, bending moment \( M_0 = 265\) kN·m. The elastic modulus of pile and elastic parameters of soil around pile are calculated by Randolph (1981) [6]. The results are as follows: \( E_p = 68.42\) GPa, \( \lambda = 0.3, \) \( G_s = 0.82 \times 10^3 \) kN/m² (in this paper, the average value is taken as the representative value of soil shear modulus). Figure 5 shows the horizontal displacement distribution curve of pile calculated by different methods. The method in this paper is in good agreement with the field measured value and Basu method.
4.2. Example 2
In order to further verify the rationality of this method, the analysis results are compared with the model test results in reference [8] and the theoretical calculation results in reference [9]. In reference [8], the model is aluminum pipe pile, and the parameters are as follows: pile length $L = 0.5m$, pile outer diameter $D = 19mm$, wall thickness $\delta = 1.5mm$, pile elastic modulus $E_p = 70GPa$; the soil around the pile is soft clay, and the relevant parameters $w_L = 42\%$, $w_p = 24\%$, $c_u = 28kPa$. Since the elastic parameters of soil around pile are not given in reference [8], the empirical formula $E_s = 250 \sim 400 c_u$ proposed by Poulos and Davis (1980) [10] is used to estimate the elastic modulus of soil. Considering Poisson's ratio has little influence on the calculation, the Poisson's ratio of soft clay is taken as 0.49. Figure 6 shows the distribution diagram of pile bending moment obtained by different methods. The method in this paper is in good agreement with the method in reference [9], and the variation trend of pile bending moment is also in good agreement with model test. Because the soil around the pile is assumed to be elastic and the actual soil is elastic-plastic, the theoretical calculation value is slightly less than the measured value of the model test.

5. Parameter analysis
Referring to the actual project, the following pile and soil parameters are taken for analysis: pile length $L = 60m$, free section length $L_0 = 20m$, pile diameter $d = 1.8m$, elastic modulus of pile $E_p = 18GPa$; elastic modulus of soil around pile $E_s = 15MPa$, Poisson's ratio $\nu_s = 0.25$.

5.1. Influence of load ratio
In order to study the influence of load ratio ($P/Q$) on $P - \Delta$ effect, the displacement and bending moment of pile when $Q = 200kN$, 400kN and 600kN and $P/Q = 0$, 2, 5, 10 and 20 are calculated respectively, as shown in Fig. 7 ~ 9. It can be seen from the figure analysis that when the horizontal load is 200kN, the maximum horizontal displacement of pile at $P = 20Q$ only increases by 13.4% compared with that under pure horizontal load ($P = 0$), which indicates that the $P - \Delta$ effect is less affected at this time. When the horizontal load is 400kN, the influence of $P - \Delta$ effect begins to become more obvious. The maximum horizontal displacement of pile at $P = 20Q$ increases by 31.1% compared with that under pure horizontal load. When the horizontal load reaches 600kN, the $P - \Delta$ effect is very significant, and the maximum horizontal displacement of pile body at $P = 20Q$ increases by 55.3% compared with that under pure horizontal load. The variation law of bending moment of pile is similar to that of horizontal displacement of pile.
Figure 7. Displacement and bending moment diagram of pile when $Q = 200kN$

Figure 8. Displacement and bending moment diagram of pile when $Q = 400kN$

Figure 9. Displacement and bending moment diagram of pile when $Q = 600kN$
In order to more clearly reflect the variation law of $P - \Delta$ effect with load ratio, Fig. 10 shows the variation curves of maximum displacement and maximum bending moment of pile under different load ratios. The results show that when the horizontal load is small (less than 200kN), the $P - \Delta$ effect is not obvious; when the horizontal load increases to a certain value (more than 400kN), the $P - \Delta$ effect is more significant, and its influence on the horizontal response of the pile cannot be ignored, which is consistent with the above conclusion. In addition, comprehensive analysis of figures 8-11 also shows that when the load ratio is less than 2, the effect of $P - \Delta$ increase is small and can be ignored; when the load ratio is greater than 5, the effect of $P - \Delta$ increase is very significant and cannot be ignored. For the pile subjected to combined loads in practical engineering, the vertical load is usually much greater than the horizontal load, so the $P - \Delta$ effect should be considered in the design of pile.

![Figure 10. Variation curves of maximum horizontal displacement and maximum bending moment of pile under different load ratios](image)

**5.2. Influence of pile top boundary**

In order to study the influence of pile top boundary conditions, the pile top boundary is set as embedded, and the horizontal displacement and bending moment of pile are calculated when $Q = 800$ kN, 1000kN and 1200kN, respectively. Figure 11 shows the displacement and bending moment of pile when $Q = 800$ kN. It can be seen from the figure that when the pile top is embedded, the negative bending moment at the pile top is the maximum bending moment of pile, and the horizontal displacement of pile under the same load is far less than that when the pile top is free. Compared with the case of $P = 0$, when $P = 20Q$, the maximum horizontal displacement of pile increases by 15.7%, and the maximum bending moment of pile increases by 13.4%, which indicates that the influence of $P - \Delta$ effect is small. Fig. 12 shows the curves of the maximum displacement and bending moment of the pile with the load ratio under different horizontal loads. When the pile top is fixed, the $P - \Delta$ effect increases with the increase of the load ratio, but the increase rate is far less than that when the pile top is free. When $Q = 1200$ kN and $P = 20Q$, the maximum displacement of pile increases by 25.7% compared with that under pure horizontal load, while for the free pile top, the maximum displacement increases by 55.3% under the same load ratio. It can be seen that the influence of pile top boundary conditions on the horizontal response of pile is very significant, and the $P - \Delta$ effect can be significantly weakened by pile top fixation, so as to improve the horizontal bearing capacity of pile.
6. Conclusion

Based on the principle of minimum potential energy and variational method, the calculation method of horizontal response of pile under combined loads is derived. The P - Δ effect of pile deformation under combined loads is deeply analyzed. The conclusions are as follows:

(1) Based on the energy method, the differential equations of the horizontal displacement of the pile and the soil around the pile are derived, and the difference solution of the horizontal displacement of the pile is given. Combined with the existing theoretical method and model test, two examples are compared and analyzed. The results show that the errors are within a reasonable range, which proves the rationality of the method in this paper.

(2) The factors influencing P - Δ effect are analyzed. The results show that: when the horizontal load is small and the load ratio \( P/Q \) is less than 2, the P - Δ effect is small and can be ignored; when the horizontal load is large and the load ratio \( P/Q \) is greater than 5, the P - Δ effect is very significant and can not be ignored; when the pile top is embedded, there will not be angular displacement on the pile top, and the horizontal displacement of the pile will be significantly reduced, and the P - Δ effect will be greatly weakened.
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