The most general structure of graphs with hamiltonian or hamiltonian connected square

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Abstract

On the basis of recent results on hamiltonicity, [4], and hamiltonian connectedness, [8], in the square of a 2-block, we determine the most general block-cutvertex structure a graph \( G \) may have in order to guarantee that \( G^2 \) is hamiltonian, hamiltonian connected, respectively. Such an approach was already developed in [9] for hamiltonian total graphs.

Keywords: hamiltonian cycle, hamiltonian path, block-cutvertex graph, square of a graph

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1 Introduction and Preliminary Discussion

As for standard terminology and other terminology used in this paper, we refer to the book by Bondy and Murty, [2], and to the papers quoted in the references. Let \( G \) be a connected graph. A 2-block is a 2-connected graph or a block of \( G \) containing more than two vertices. The square of a graph \( G \),

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denoted $G^2$, is the graph obtained from $G$ by joining any two nonadjacent vertices which have a common neighbor, by an edge.

It was shown in 1970 and published in 1974 that the square of every 2-block contains a hamiltonian cycle, [6]. Key in proving this was the existence of EPS-graphs $S$ in connected bridgeless graphs $G$, where $S$ is the edge-disjoint union of a not necessarily connected eulerian subgraph $E$ and a linear forest $P$, and $S$ is connected and spans $G$, [5]. In subsequent papers [7], [9] the existence of various types of EPS-graphs was established. Their relevance was based on the fact that the total graph $T(G)$ of any connected graph $G$ other than $K_1$ is hamiltonian if and only if $G$ has an EPS-graph, [9]. This and the theory of EPS-graphs led to a description of the most general block-cutvertex graph $bc(G)$ of a graph $G$ may have such that $T(G)$ is hamiltonian and if $bc(G)$ does not have the corresponding structure, then exchanging certain 2-blocks in $G$ with some special 2-blocks yields a graph $G^*$ such that $bc(G)$ and $bc(G^*)$ are isomorphic but $T(G^*)$ is not hamiltonian, [9]. In dealing with hamiltonian cycles and hamiltonian paths by methods developed up to that point, it was shown in [7] that in the square of graphs hamiltonicity and vertex-pancyclicity are equivalent concepts, and so are hamiltonian connectedness and panconnectedness. In this context Theorem 3 stated below was established as a tool needed to prove the equivalences just mentioned.

However, in the course of time much shorter proofs of Fleischner’s Theorem were developed [10], [13]; the same applies to Theorem 3 below, [12]. More recently, an algorithm yielding a hamiltonian cycle in the square of a 2-block in linear time, was developed, [1]. The methods developed in these much shorter proofs (including the algorithm just mentioned) do not seem to yield short proofs of Theorems 1 and 2 below, [4], [8]. These latter theorems are, on the other hand, instrumental in proving the central results of this paper, i.e., Theorems 1 and 2 and related algorithms.

Let $bc(G)$ denote the block-cutvertex graph of $G$. Blocks corresponding to leaves of $bc(G)$ are called endblocks, otherwise innerblocks. Note that a block in a graph $G$ is either a 2-block or a bridge of $G$. For each cutvertex $i$ of $G$, let $k_i$ be the number of 2-blocks of $G$ which include vertex $i$ and let $bn(i)$ be the number of nontrivial bridges of $G$ which are incident with vertex $i$. In what follows a bridge is called nontrivial if it is not incident to a leaf.

Let $H$ be a subgraph of the graph $G$. We define $G - H := G - E(H) - \{v \in V(H) : d_H(v) = d_G(v)\}$. 

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In Theorem 1, we introduce an array $m_i(B)$ of numbers with an entry for each pair consisting of a cutvertex $i$ and a 2-block $B$ of $G$. We may think of this number $m_i(B)$ as the number of edges of $B$ incident with $i$ which are possibly contained in a hamiltonian cycle in $G^2$.

Statement of Theorem 1 describes the most general block-cutvertex structure a graph $G$ may have in order to guarantee that $G^2$ is hamiltonian using parameters $m_i(B)$ as in [9].

**Theorem 1.** Let $G$ be a connected graph with at least three vertices. Let the 2-blocks of $G$ be labelled $B_1, B_2, \ldots, B_n$. Let the cutvertices of $G$ be labelled $1, 2, \ldots, s$. Suppose there is a labelling $m_i(B_t)$ for each $i \in \{1, 2, \ldots, s\}$ and each $t \in \{1, 2, \ldots, n\}$ such that the following conditions are fulfilled.

1) $0 \leq m_i(B_t) \leq 2$ for all $i$ and all 2-blocks $B_t$;
2) for 2-block $B_t$, $m_i(B_t) = 0$ if and only if cutvertex $i$ is not in $V(B_t)$;
3) for 2-block $B_t$, $m_i(B_t) \geq bn(i)$, if cutvertex $i \in V(B_t)$;
4) $bn(i) \leq 2$ for all $i \in \{1, 2, \ldots, s\}$;
5) $\sum_{i=1}^{s} m_i(B_t) \leq 4$ for each 2-block $B_t$ of $G$ and, if $m_i(B_t) = 2$ for some $i$, then $\sum_{i=1}^{s} m_i(B_t) \leq 3$; and
6) $\sum_{t=1}^{n} m_i(B_t) \geq 2k_i + bn(i) - 2$ for each $i \in \{1, 2, \ldots, s\}$.

Then $G^2$ is hamiltonian.

Moreover, if the labelling $m_i(B_t)$ satisfying conditions 1), 2) and 3) is given and at least one of conditions 4), 5), 6) is violated by some $G$, then there exists a class of graphs $G'$ with non-hamiltonian square but $bc(G')$ and $bc(G)$ are isomorphic.

Also, we obtain a similar result for hamiltonian connectedness (Theorem 2). Quite surprisingly, its formulation is much simpler than that of Theorem 1.

**Theorem 2.** Let $G$ be a connected graph such that the following conditions are fulfilled:

1) there is no nontrivial bridge of $G$;
2) every block contains at most 2 cutvertices.

Then $G^2$ is hamiltonian connected.

Moreover,

· if a graph $G$ contains a nontrivial bridge, then $G^2$ is not hamiltonian connected;
if $G$ contains a block containing more than 2 cutvertices, then there is a graph $G'$ such that $bc(G)$ and $bc(G')$ are isomorphic but $(G')^2$ is not hamiltonian connected.

A fundamental result regarding hamiltonicity in the square of a 2-block is the following theorem.

**Theorem 3.** [7] Suppose $v$ and $w$ are two arbitrarily chosen vertices of a 2-block $G$. Then $G^2$ contains a hamiltonian cycle $C$ such that the edges of $C$ incident to $v$ are in $G$ and at least one of the edges of $C$ incident to $w$ is in $G$. Furthermore, if $v$ and $w$ are adjacent in $G$, then these are three different edges.

The hamiltonian theme in the square of a 2-block has been recently re-visited ([3], [4], [8]), yielding the following results which are essential for this paper.

A graph $G$ is said to have the $H_k$ property if for any given vertices $x_1, ..., x_k$ there is a hamiltonian cycle in $G^2$ containing distinct edges $x_1y_1, ..., x_ky_k$ of $G$.

**Theorem 4.** [4] Given a 2-block $G$ on at least 4 vertices, then $G$ has the $H_4$ property, and there are 2-blocks of arbitrary order greater than 4 without the $H_5$ property.

By a $uv$-path we mean a path from $u$ to $v$ in $G$. If a $uv$-path is hamiltonian, we call it a $uv$-hamiltonian path. Let $A = \{x_1, x_2, ..., x_k\}$ be a set of $k \geq 3$ distinct vertices in $G$. An $x_1x_2$-hamiltonian path in $G^2$ which contains $k-2$ distinct edges $x_iy_i \in E(G), i = 3, ..., k$, is said to be $F_k$. A graph $G$ is said to have the $F_k$ property if, for any set $A = \{x_1, x_2, ..., x_k\} \subseteq V(G)$, there is an $F_k x_1x_2$-hamiltonian path in $G^2$.

**Theorem 5.** [8] Every 2-block on at least 4 vertices has the $F_4$ property.

A graph $G$ is said to have the strong $F_3$ property if, for any set of 3 vertices $\{x_1, x_2, x_3\}$ in $G$, there is an $x_1x_2$-hamiltonian path in $G^2$ containing distinct edges $x_3z_3, x_iz_i \in E(G)$ for a given $i \in \{1, 2\}$. Such an $x_1x_2$-hamiltonian path in $G^2$ is called a strong $F_3 x_1x_2$-hamiltonian path.

**Theorem 6.** [8] Every 2-block has the strong $F_3$ property.

**Theorem 7.** [8] Let $G$ be a 2-connected graph and let $x, y$ be two vertices in $G$. Then $G^2$ has an $xy$-hamiltonian path $P(x, y)$ such that

(i) $xz \in E(G) \cap E(P(x, y))$ for some $z \in V(G)$, and

(ii) either $yw \in E(G) \cap E(P(x, y))$ for some $w \in V(G)$, or else $P(x, y)$ contains an edge $uv$ for some vertices $u, v \in N(y)$. 


2 Proofs and algorithms

PROOF OF THEOREM 1

Proof. Set $P_0 = G - \bigcup_{t=1}^n B_t$. Then every component of $P_0$ is a tree. Since by 4) $bn(i) \leq 2$ every component of $P_0$ is even a caterpillar.

For every caterpillar $T$ of $P_0$ except $T = K_2$ we have the following observation which can be proved easily.

Observation: Let $T$ be a caterpillar with at least three vertices and $P = x_1x_2...x_m$ be some longest path in $T$. Then $T^2$ contains a hamiltonian cycle containing edges $x_1x_2, x_{m-1}x_m$ and different edges $u_jv_j$, where $u_j, v_j \in N_G(x_j)$ for $j = 2, 3, ..., m - 1$.

See Figure 1 for illustration in which for $x_3$ we have $u_3 = x_2$ and $v_3 = x_4$.

Figure 1: Hamiltonian cycle in a caterpillar for $m = 7$ (bold edges)

Every 2-block $B_t$ contains a hamiltonian cycle in $(B_t)^2$ which is one of two types depending on labellings $m_i(B_t)$:

Let $m_i(B_t) \neq 2$ for every $i = 1, 2, ..., s$. If $B_t \cong C_3$, then we set $C_t = B_t$. Otherwise for at most 4 cutvertices $a, b, c, d$ it holds that $m_j(B_t) = 1$ for $j = a, b, c, d$ by condition 5). By Theorem 3 $(B_t)^2$ has a hamiltonian cycle $C_t$ containing 4 different edges $aa', bb', cc', dd'$ of $B_t$.

If $m_i(B_t) = 2$ for some $i \in \{1, 2, ..., s\}$, then at most one cutvertex $a$ has $m_a(B_t) = 1$ by condition 5). By Theorem 3 $(B_t)^2$ has a hamiltonian cycle $C_t$ containing 3 different edges $ii', ii'', aa'$ of $B_t$.

The union of hamiltonian cycles $C_t$ in $(B_t)^2$, for $t = 1, 2, ..., n$, hamiltonian cycles in the square of each caterpillar (nontrivial component of $P_0$) and trivial components of $P_0$ is a connected spanning subgraph $S$ of $G^2$. 5
We construct a hamiltonian cycle $C$ in $G^2$ from $S$ repeating step by step the following procedure for every cutvertex $i$ of $G$ with $m_i(B) \geq 1$ for some 2-block $B$.

If $i$ does not exist, then $n = 0$ and $G = P_0$ is a caterpillar. Hence $S$ is a hamiltonian cycle in $G^2$. Otherwise we join all hamiltonian cycles from $S$ containing $i$ together with trivial components of $P_0$ containing $i$ to one cycle in the following way.

First assume that $b_n(i) = 0$.

By condition 6) we have $\sum_{t=1}^n m_i(B_t) \geq 2k_i - 2$. Without loss of generality for $k_i > 1$ we may assume that $m_i(B_1) \geq 1$, $m_i(B_2) \geq 1$ and $m_i(B_3) = \cdots = m_i(B_k) = 2$, where $m_i(B_t)$ corresponds to the number of edges of $B_t$ incident to $i$ in $C_i$. If $k_i = 1$, then by condition 2) we have $m_i(B_1) \geq 1$.

We find a cycle $C'$ on $\cup_{r=1}^{k_i} V(C_r) \cup L$, where $L$ is the set of all leaves incident to $i$, by appropriately replacing edges of $C_t \cap B_i$, $r = 1, 2, \ldots, k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) with edges of $G^2$ joining vertices in different $C_r$ adjacent to $i$ and leaves adjacent to $i$. Note that we preserve properties given by $m_j(B_i)$ for all $j \neq i$.

Now assume that $b_n(i) = 1$.

By condition 6) we have $\sum_{t=1}^n m_i(B_t) \geq 2k_i + 1 - 2 = 2k_i - 1$. Without loss of generality we may assume that $m_i(B_1) \geq 1$ and $m_i(B_2) = m_i(B_3) = \cdots = m_i(B_k) = 2$, where $m_i(B_t)$ corresponds to the number of edges of $B_t$ incident to $i$ in $C_i$. Let $T$ be the component of $P_0$ containing $i$.

If $T = K_2 = i'i^*$, where $i'$ is also a cutvertex of $G$ with $m_{i'}(B) \geq 1$ ($T$ is a trivial component of $P_0$), then we find a cycle $C'$ on $\cup_{r=1}^{k_i} V(C_r) \cup V(T)$ containing the edge $ii'$ by appropriately replacing edges of $C_t \cap B_i$, $r = 1, 2, \ldots, k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) with edges of $G^2$ joining $i'$ and vertices in different $C_r$ adjacent to $i$. Also here we preserve properties given by $m_j(B_i)$ for all $j \neq i$.

If $T$ is a nontrivial component of $P_0$, then $T^2$ contains a hamiltonian cycle $C_T$ containing end-edges of any fixed longest path $P$ in $T$ (we choose end-edges containing cutvertices of $G$ with $m_i(B_t) \geq 1$) - see Observation above. Again we find a cycle $C'$ on $\cup_{r=1}^{k_i} V(C_r) \cup V(C_T)$ by appropriately replacing edges of $C_t \cap B_i$, $r = 1, 2, \ldots, k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) and the end-edge $ii^*$ of $P$ with edges of $G^2$ joining $i^*$ and vertices in different $C_r$ adjacent to $i$. Again we preserve properties given by $m_j(B_i)$ for all $j \neq i$ and by $C_T$. 

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Finally assume that $bn(i) = 2$.

By condition 6) we have $\sum_{t=1}^{n} m_i(B_t) \geq 2k_i + 2 - 2 = 2k_i$. It follows necessarily that $m_i(B_1) = m_i(B_2) = \cdots = m_i(B_{k_i}) = 2$, where $m_i(B_t)$ corresponds to the number of edges of $B_t$ incident to $i$ in $C_t$.

Let $T$ be the nontrivial component of $P_0$ containing $i$. Note that $i$ is not an endvertex of $T$ because of $bn(i) = 2$. Then $T^2$ contains a hamiltonian cycle $C_T$ containing end-edges of any fixed longest path in $T$ (we choose end-edges containing cutvertices of $G$ with $m_i(B_t) \geq 1$) and an edge $u_iv_i$ of $G^2$ where $u_i, v_i \in N_G(i)$ (see Observation above). We find a cycle $C^i$ on $\bigcup_{r=1}^{k_i} V(C_r) \cup V(C_T)$ by appropriately replacing edges of $C_r \cap B_r$, $r = 1, 2, \ldots, k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) and the edge $u_iv_i$ of $P$ with edges of $G^2$ joining $u_i, v_i$ and vertices in different $C_r$ adjacent to $i$ if $k_i > 1$.

If, however, $k_i = 1$, then $u_i$ and $v_i$ are joined to the neighbors of $C_r \cap B_r$ in $N_G(i)$. Also here we preserve properties given by $m_j(B_t)$ for all $j \neq i$ and by $C_T$.

Now we choose next cutvertex $i$ with $m_i(B) \geq 1$ for some 2-block $B$ successively and we use all cycles formed in the previous steps instead of previously formed cycles. Note that we preserve all properties given by $m_j(B)$ for all $j \neq i$ in every case. We stop with the hamiltonian cycle in $G^2$ as required.

Now assume that there is no labelling satisfying conditions 1) - 6), that is, the labelling $m_i(B_t)$ satisfying conditions 1), 2) and 3) is given and at least one of conditions 4), 5), 6) is violated. We show that there exists a class of graphs $G'$ with non-hamiltonian square but $bc(G')$ and $bc(G)$ are isomorphic.

**Condition 4) does not hold.**

Hence $bn(i) \geq 3$ for at least one $i \in \{1, 2, \ldots, s\}$. Clearly this is a class of graphs $G'$ such that the square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a hamiltonian cycle in the square, then the degree of the cutvertex $i$ is at least 3, a contradiction), e.g. see the graph in Figure 2a, where $H_1$ is an arbitrary connected graphs, $H_2$, $H_3$, $H_4$ are arbitrary connected graphs with at least one edge each and $bn(i) = 3$. Note that conditions 5) and 6) may hold.
Condition 5) does not hold.

Hence $\sum_{i=1}^{s} m_i(B) \geq 5$ for some 2-block $B$ and $m_i(B) < 2$ for all $i$ or $\sum_{j=1}^{s} m_j(B) \geq 4$ for some 2-block $B$ and $m_i(B) = 2$ for some $i \in \{1, 2, ... , s\}$.

First suppose that $k = \sum_{i=1}^{s} m_i(B) \geq 5$ for some 2-block $B$ of $G$ and $m_i(B) < 2$ for all $i$. Clearly $B$ has exactly $k$ cutvertices by condition 2). Then we exchange $B$ with $K_{2,k}$ where $k$ 2-valent vertices are cutvertices of $G$ and all other blocks with arbitrary blocks to get a class of graphs $G'$ such that $bc(G')$ and $bc(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a hamiltonian cycle in the square, then the degree of at least one of the two $k$-valent vertices of $K_{2,k}$ is at least 3, a contradiction), e.g. see the graph in Figure 2(b), where $k = 5$ and $H_1, ..., H_5$ are arbitrary connected graphs with at least one edge each. Note that conditions 4), 6) and the second part of condition 5) may hold.

Now suppose that $\sum_{j=1}^{s} m_j(B) \geq 4$ for some 2-block $B$ and $m_i(B) = 2$ for some $i$. If $B$ contains at least 5 cutvertices of $G$, then we continue similarly as above. If $B$ contains $k$ cutvertices of $G$ where $2 \leq k \leq 4$, then without loss of generality we may assume that we tried to set the labelling $m_i(B_t)$ satisfying firstly condition 5) and subsequently condition 6). Hence $\text{bn}(i) \geq 2$ and $\text{bn}(j) \geq 2$ where $j$ is the second cutvertex of $G$ in $B$ if $k = 2$, otherwise we find a labelling $m_i(B_t)$ satisfying condition 5), a contradiction (see Algorithm 1 cases e) and f) below).

For $k = 3, 4$ we exchange $B$ with a cycle $C_k$ to get a class of graphs $G'$ such that $bc(G')$ and $bc(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a
hamiltonian cycle in the square, then the degree of the cutvertex $i$ is at least 3, a contradiction), e.g. see the graph in Figure 2 c), where $k = 3$ and $H_1, ..., H_4$ are arbitrary connected graphs with at least one edge each. Note that conditions 4), 6) and the first part of condition 5) may hold.

For $k = 2$, we exchange $B$ with $K_{2,3}$, where two of the three 2-valent vertices are $i$ and $j$, to get a class of graphs $G'$ such that $bc(G')$ and $bc(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (it is not possible to find a hamiltonian cycle in the square containing the third 2-valent vertex different from $i$, $j$, a contradiction), e.g. see the graph in Figure 2 d), where $H_1, ..., H_4$ are arbitrary connected graphs with at least one edge each. Note that conditions 4), 6) and the first part of condition 5) may hold.

**Condition 6) does not hold.**

Hence $\sum_{t=1}^{n} m_i(B_t) < 2k_i + bn(i) - 2$ for some $i$ and consequently $m_i(B_t) = 1$ for at least $3 - bn(i)$ 2-blocks containing $i$. Note that, clearly, $bn(i) < 2$ with respect to condition 3).

Let $r$ be the number of 2-blocks with $m_i(B_t) = 1$. Each of these 2-blocks contains either exactly 2 cutvertices of $G$ or at least 3 cutvertices of $G$. Note that for 2-blocks containing only cutvertex $i$ we have $m_i(B_t) = 2$ (see Algorithm 1 case d) below). We exchange every 2-block containing exactly 2 cutvertices of $G$ with a cycle $C_3$ and every 2-block containing $k$ cutvertices of $G$, $k \geq 3$, with a cycle $C_k$. In the first case note that we assume without loss of generality that there is no labelling such that we switch values 1 and 2 for both cutvertices of this 2-block to get a permissible labelling (again see Algorithm 1 case e) below).

Since $r \geq 3 - bn(i)$, by the exchanging 2-block mentioned above we get a class of graphs $G'$ such that $bc(G')$ and $bc(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a hamiltonian cycle in the square, then the degree of the cutvertex $i$ is at least 3, a contradiction), e.g. see graphs in Figure 2 e1) and e2). For the graph in Figure 2 e1) it holds that $r = 3 - bn(i) = 3 - 1 = 2$, the 2-block $B_1$ has exactly 2 cutvertices of $G$, the 2-block $B_2$ has $k = 3$ cutvertices of $G$ (and hence $B_1$, $B_2$ are isomorphic to $C_3$) and $H_1, ..., H_5$ are arbitrary connected graphs with at least one edge. For the graph in Figure 2 e2) it holds that $r = 3 - bn(i) = 3 - 0 = 3$, the 2-block $B_1$ has exactly 2 cutvertices of $G$, the 2-block $B_2$ has $k = 3$ cutvertices of $G$, the 2-block $B_3$ has $k = 4$ cutvertices of $G$ (hence $B_1$, $B_2$ are isomorphic to $C_3$ and $B_3$ is isomorphic
to $C_4$) and $H_1, \ldots, H_7$ are arbitrary connected graphs with at least one edge.
Note that conditions 4) and 5) may hold.

This finishes the proof of Theorem 1.

If there is a graph $G$ such that every labelling $m_i(B_t)$ violates at least one of the conditions 4) - 6) of Theorem 1 then there is a graph $G'$ with $bc(G') = bc(G)$ such that $(G')^2$ is not hamiltonian as it has been shown in the proof of Theorem 1. On the other hand, if we are able to construct a labelling $m_i(B_t)$ satisfying conditions 1) - 6) using the following algorithm, then $G^2$ is hamiltonian as it has been shown in the proof of Theorem 1.

**ALGORITHM 1:**

Set $P_0 = G - \cup_{t=1}^n B_t$. If any component of $P_0$ is not a caterpillar, then $bn(i) \geq 3$ for some $i \in \{1, 2, \ldots, s\}$ contradicting condition 4) in Theorem 1 and $G^2$ is not hamiltonian (e.g. see Figure 2 a)). STOP.

If $G = P_0$, then $G$ is a caterpillar, $n = 0$ and $G^2$ is hamiltonian (see Observation in the proof of Theorem 1) and $m_i(B_t)$ is not defined ($n = 0$). STOP.

If $G$ is a 2-block, $G^2$ is hamiltonian by Theorem 3 and $m_i(B_t)$ is not defined ($s = 0$ and $n = 1$). STOP.

We set $G_0 = G - P_0$ and $m_i(B_t) = 0$ if $i \notin V(B_t)$ for $i \in \{1, 2, \ldots, s\}$ and $t \in \{1, 2, \ldots, n\}$.

START

We choose a 2-block $B$ containing at most 1 cutvertex of $G_0$. Note that $B$ is either a component of $G_0$ or an endblock of some component of $G_0$. If such endblock does not exist, we choose 2-block $B$ as a component of $G_0 - H$ or an endblock of $G_0 - H$ where $H$ is the union of all 2-blocks for which the labelling $m_i(B_t)$ is already set. Let $c_1, c_2, \ldots, c_k$ be all cutvertices of $G$ contained in $B$, $k \geq 1$.

a) If $k \geq 5$, then by condition 2) $m_{c_i}(B) \geq 1$ for $i = 1, 2, \ldots, k$. Hence condition 5) in Theorem 1 does not hold and $G^2$ may not be hamiltonian (e.g. see Figure 2 b)). STOP.

b) If $k \geq 3$ and $bn(c_i) = 2$ for some $i \in \{1, 2, \ldots, k\}$, then by condition 3) $m_{c_i}(B) = 2$ and by 2) $m_{c_j}(B) \geq 1$ for $j = 1, 2, \ldots, k$. Hence condition 5) in Theorem 1 does not hold and $G^2$ may not be hamiltonian (e.g. see Figure 2 c)). STOP.
c) If \( k = 2 \) and \( \text{bn}(c_1) = \text{bn}(c_2) = 2 \), then by condition 3) \( m_{c_1}(B) = 2 \) and \( m_{c_2}(B) = 2 \). Hence condition 5) in Theorem 4 does not hold and \( G^2 \) may not be hamiltonian (e.g. see Figure 2d)). STOP.

d) If \( k = 1 \), then we set \( m_{c_1}(B) = 2 \) (we maximize values \( m_i(B_t) \) with respect to condition 6) in Theorem 1). Note that, if the labelling \( m_i(B_t) \) is set for all 2-blocks incident with \( c_1 \), then condition 6) holds for cutvertex \( c_1 \) with respect to the choice of \( B \).

If the labelling \( m_i(B_t) \) is set for all 2-blocks of \( G \), then the labelling \( m_i(B_t) \) satisfies the conditions of Theorem 1 and \( G^2 \) is hamiltonian. STOP.

Otherwise we go to START.

e) If \( k = 2 \) and \( \text{bn}(c_i) \leq 1 \) for \( i \in \{1, 2\} \), then we set \( m_{c_1}(B) \) and \( m_{c_2}(B) \) in the following way (without loss of generality \( i = 1 \)).

Let \( \text{bn}(c_2) = 2 \). Then we set \( m_{c_1}(B) = 1 \) and \( m_{c_2}(B) = 2 \) with respect to conditions 2), 3) and 5).

Let \( \text{bn}(c_2) \leq 1 \). Then for at least one of \( c_1, c_2 \) it holds that \( m_{c_j}(B_t) \) for \( j \in \{1, 2\} \) is set for all 2-blocks \( B_t \) except \( B \) with respect to the choice of \( B \) (again without loss of generality \( j = 1 \)). We set \( m_{c_1}(B) = 1 \) and we verify condition 6) for \( c_1 \). If it holds, then we set \( m_{c_2}(B) = 2 \) (again we maximize values \( m_i(B_t) \) with respect to condition 6)). If condition 6) for \( c_1 \) does not hold for \( m_{c_1}(B) = 1 \), then we set \( m_{c_1}(B) = 2 \) and \( m_{c_2}(B) = 1 \).

Now in both cases we verify condition 6) for \( c_1 \) and \( c_2 \) if the labelling \( m_{c_1}(B_t) \) and \( m_{c_2}(B_t) \) is set for all 2-blocks \( B_t \).

If condition 6) does not hold in at least one case, then \( G^2 \) may not be hamiltonian (e.g. see Figure 2e1)). STOP.

Hence suppose that condition 6) holds for \( c_1, c_2 \) if \( m_{c_1}(B_t), m_{c_2}(B_t) \) is set for all \( B_t \), respectively.

If the labelling \( m_i(B_t) \) is set for all 2-blocks, then the labelling \( m_i(B_t) \) satisfies the conditions of Theorem 1 and \( G^2 \) is hamiltonian. STOP.

Otherwise we go to START.
f) If \( k \in \{3, 4\} \) and \( bn(c_i) \leq 1 \), then we set \( m_{c_i}(B) = 1 \) for \( i = 1, 2, ..., k \). We verify condition 6) for all \( c_i \) if the labelling \( m_{c_i}(B_i) \) is set for all 2-blocks \( B_i \).

If condition 6) does not hold in at least one case, then \( G^2 \) may not be hamiltonian (e.g. see Figure 2e2)). STOP.

Hence suppose that condition 6) holds for all \( c_i, i = 1, 2, ..., k \), for which \( m_{c_i}(B_i) \) is set for all \( B_i \).

If the labelling \( m_i(B_i) \) is set for all 2-blocks, then the labelling \( m_i(B_i) \) satisfies the conditions of Theorem 1 and \( G^2 \) is hamiltonian. STOP.

Otherwise we go to START.

PROOF OF THEOREM 2

Proof. Let \( x, y \in V(G) \). First we prove that there exists an \( xy \)-hamiltonian path \( P \) in \( G^2 \) if there is no nontrivial bridge of \( G \) and every block contains at most 2 cutvertices.

(A) Suppose that \( x \) and \( y \) are in the same block \( B \) of \( G \). We proceed by induction on \( n \), where \( n \) is the number of blocks of \( G, n \geq 1 \).

For \( n = 1 \), clearly \( G = B \). If \( B = K_2 = xy \), then \( G \) is also the \( xy \)-hamiltonian path in \( G^2 \) as required. If \( B \) is a 2-block, then by Theorem 6, \( G^2 = B^2 \) contains an \( xy \)-hamiltonian path \( P \) as required.

Now suppose that the statement of Theorem 2 is true for every graph with \( n \) blocks and \( G \) is a graph with \( n + 1 \) blocks, \( n \geq 1 \). We distinguish 2 cases.

- \( B \) has exactly one cutvertex \( c \).

Without loss of generality we assume that \( x \neq c \). If \( B \) is a 2-block, then by Theorem 3, \( B^2 \) contains an \( xy \)-hamiltonian path \( P_B \) containing an edge \( cy' \) where \( y' \) is a neighbor of \( c \) in \( B \). Note that \( y' = x \) or \( c = y \) is possible. If \( B = K_2 \), then \( B = xy = y'c \) and \( P_B = xy \) is an \( xc \)-hamiltonian path in \( B^2 \). By the induction hypothesis \((G - B)^2 \) contains a \( cc' \)-hamiltonian path \( P_G \) where \( c' \) is a neighbor of \( c \) in \( G - B \). Then \( P = P_B \cup P_G - cy' + y'c' \) is an \( xy \)-hamiltonian path in \( G^2 \) as required.
• \( B \) has two cutvertices \( c_1, c_2 \).

We denote by \( G_1, G_2 \) the two components of \( G - B \) such that \( c_i \in V(G_i) \) and let \( c'_i \) be a neighbor of \( c_i \) in \( G_i, i = 1, 2 \). By the induction hypothesis \((G_i)^2\) contains a \( c_ic'_i\)-hamiltonian path \( P_{G_i}, i = 1, 2 \).

a) \( c_i \notin \{x, y\} \) (\( x \) and \( y \) are not cutvertices).

By Theorem 5, \( B^2 \) contains an \( xy\)-hamiltonian path \( P_B \) containing the edges \( c_iz_i \) where \( z_i \) is a neighbor of \( c_i \) in \( B, i = 1, 2 \). Note that \( z_i \in \{x, y\} \) is possible.

b) Up to symmetry \( c_1 = x \) and \( c_2 \neq y \) (either \( x \) or \( y \) is a cutvertex of \( G \)).

By Theorem 6, \( B^2 \) contains an \( xy\)-hamiltonian path \( P_B \) containing the edges \( c_iz_i \) where \( z_i \) is a neighbor of \( c_i \) in \( B, i = 1, 2 \). Note that \( z_1 = c_2 \) or \( z_2 = y \) is possible.

c) \( c_1 = x \) and \( c_2 = y \) (similarly \( c_1 = y \) and \( c_2 = x \)).

By Theorem 7, \( B^2 \) contains an \( xy\)-hamiltonian path \( P_B \) containing either the edges \( c_iz_i \) where \( z_i \) is a neighbor of \( c_i \) in \( B, i = 1, 2 \), or the edges \( c_1z_1, uv \) where \( z_1 \) is a neighbor of \( c_1 \) in \( B \) and \( u, v \) are neighbors of \( c_2 \) in \( B \).

In all cases except the case c), if \( uv \) is the edge of \( P_B \),

\[
P = P_{G_1} \cup P_B \cup P_{G_2} - \{c_1z_1, c_2z_2\} \cup \{c'_1z_1, c'_2z_2\}
\]

is an \( xy\)-hamiltonian path in \( G^2 \) as required.

It remains to find an \( xy\)-hamiltonian path in \( G^2 \) if \( uv \) is the edge of \( P_B \).

If \( G_2 = K_2 = c_2c'_2 \), then

\[
P = P_{G_1} \cup P_B - \{c_1z_1, uv, c_2c'_2\} \cup \{c'_1z_1, c'_2u, c'_2v\}
\]

is an \( xy\)-hamiltonian path in \( G^2 \) as required.

If \( G_2 \neq K_2 \), then we prove that \((G_2)^2\) contains a hamiltonian cycle \( C \) containing edges \( c_2u_2, c_2v_2 \) of \( G_2 \). Let \( B_1, B_2, ..., B_k \) be all 2-blocks of \( G_2 \) containing \( c_2 \). By Theorem 3 for \( i = 1, 2, ..., k \), \((B_i)^2\) contains a hamiltonian cycle \( C'_{i} \) containing three different edges \( c_2u'_{2i}, c_2v'_{2i}, y_{i}y'_{i} \) of \( B_i \) where \( y_i \) is the second cutvertex of \( G_2 \) in \( B_i \) if it exists.
If \( y_i \) exists, then we denote by \( H_i \), a component of \( G_2 - (B_i - y_i) \) containing \( y_i \). By the induction hypothesis \((H_i)^2\) contains a \( y_i d_i \)-hamiltonian path \( P_i \) where \( d_i \) is a neighbor of \( y_i \) in \( H_i \). Then we set \( C_i = C_i' \cup P_i - y_i y_i' + y_i' d_i \). If \( y_i \) does not exist, then we set \( C_i = C_i' \).

Let \( T \) be the set of all leaves of \( G_2 \) adjacent to \( c_2 \). Then we find a cycle \( C \) on \( \cup_{i=1}^{k} V(C_i) \cup T \) by appropriately replacing edges \( c_2 u_2', c_2 v_2' \) with edges of \( G^2 \) joining \( u_2', v_2' \) in different \( C_i \) and leaves adjacent to \( c_2 \) (similarly as in the proof of Theorem \( \text{II} \)) such that we preserve two edges \( (c_2 u_2', c_2 v_2') \) or \( (c_2 l_1, c_2 l_2) \) where \( l_1, l_2 \) are two leaves of \( G_2 \) adjacent to \( c_2 \) as \( c_2 u_2, c_2 v_2 \).

Now

\[
P = P_{G_1} \cup P_B \cup C - \{c_1 z_1, uv, c_2 u_2, c_2 v_2\} \cup \{c_1' z_1, u_2 u, v_2 v\}
\]

is an \( xy \)-hamiltonian path in \( G^2 \) as required.

(B) Suppose that \( x \) and \( y \) are in different blocks of \( G \).

Let \( P_G \) be any \( xy \)-path in \( G \) and \( c \in V(P_G) \setminus \{x, y\} \) be a cutvertex of \( G \). Let \( K \) be the component of \( G - c \) containing \( x \), \( G_y = G - V(K) \) and \( G_x = G - G_y \). Clearly \( G_x \cup G_y = G \) and \( G_x \cap G_y = c \). If \( G_x, G_y \) are isomorphic to \( K_2 \), then we set \( P_x = G_x, P_y = G_y \), respectively. If \( G_x, G_y \) are 2-blocks, then \( (G_x)^2, (G_y)^2 \) contains an \( xc \)-hamiltonian path \( P_x \), a \( cy \)-hamiltonian path \( P_y \) by Theorem \( \text{I} \) respectively. We proceed by induction on \( n \), where \( n \) is the number of blocks of \( G \), \( n \geq 2 \).

First assume that \( G \) has exactly 2 blocks. Hence \( G_x, G_y \) are isomorphic to \( K_2 \) or 2-blocks and \( P = P_x \cup P_y \) is an \( xy \)-hamiltonian path in \( G^2 \) as required.

Now suppose that the statement of Theorem \( \text{II} \) is true for every graph with \( n \) blocks and \( G \) is a graph with \( n + 1 \) blocks, \( n \geq 2 \). If \( G_x, G_y \) is not a block, then by the induction hypothesis \( (G_x)^2, (G_y)^2 \) contains an \( xc \)-hamiltonian path \( P_x \), a \( cy \)-hamiltonian path \( P_y \), respectively. Then \( P = P_x \cup P_y \) is an \( xy \)-hamiltonian path in \( G^2 \) as required.

Now it remains to prove that if there is a nontrivial bridge of \( G \), then \( G^2 \) is not hamiltonian connected and if \( G \) contains a block containing more than 2 cutvertices, then there is a graph \( G' \) such that \( bc(G) \) and \( bc(G') \) are isomorphic but \( (G')^2 \) is not hamiltonian connected.
Figure 3: Graphs without $xy$-hamiltonian path in the square

Clearly, if there exists a nontrivial bridge $xy$ in $G$, then there is no $xy$-hamiltonian path in $G^2$ and $G^2$ is not hamiltonian connected.

Finally assume that $G$ contains a block $B$ containing $r$ cutvertices, where $r > 2$. Then we exchange $B$ with a cycle $C_r$ and all other blocks with arbitrary blocks to get a class of graphs $G'$ such that $bc(G')$ and $bc(G)$ are isomorphic. Clearly the square of every such graph $G'$ does not contain a hamiltonian path between arbitrary two cutvertices of $G'$ in $C_r$ and hence $(G')^2$ is not hamiltonian connected, e.g. with Figure 3, where $r = 3$ and $H_1, H_2, H_3$ are arbitrary connected graphs with at least one edge.

Similarly as for Theorem 1 we state the following algorithm to verify conditions of Theorem 2.

ALGORITHM 2:

Let $G' = G - S$ where $S$ is the set of all endblocks of $G$. Let $cvn_G(B)$ be the number of cutvertices of $G$ in $B$.

START

Find an endblock $B$ of $G'$.

• If $B$ is a bridge of $G'$, then $B$ is a nontrivial bridge of $G$ and $G^2$ is not hamiltonian connected. STOP.

• Let $B$ be a 2-block.

  − If $cvn_G(B) > 2$, then $G^2$ may not be hamiltonian connected (e.g. see Figure 3). STOP.
  − If $cvn_G(B) \leq 2$, then $G' := G' - B$.
    • If $G' = \emptyset$, then $G^2$ is hamiltonian connected. STOP.
    • If $G' \neq \emptyset$, then go to START.
In both algorithms in this paper, determining blocks and especially endblocks and bridges, cutvertices, block-cutvertex graphs, and the parameters $bn(i)$, $cvn_G(B)$ can be determined in polynomial time.

As a consequence, polynomial running time in Algorithm 2 is guaranteed. For, determining (potentially) not being Hamiltonian connected, can be determined instantly once a nontrivial bridge, a block with more than 2 cutvertices has been found. And deleting an endblock reduces the size of $G'$ linearly.

Now consider the running time of Algorithm 1. The first decision to be made is whether $P_0$ is a forest of caterpillars – this can be done in linear time. After that, at every step 'one chooses a 2-block $B$ as a component of $G_0 - H$ or an endblock of $G_0 - H$ where $H$ is the union of all 2-blocks for which the labelling $m_i(B_t)$ is already set'. Clearly, identifying such $B$ can be done in linear time. The same applies to working through the cases for defining the various values of $m_i(B)$.

Summarizing, it follows that both algorithms run in polynomial time. We note however, that these algorithms can only decide the existence or potential non-existence of hamiltonian cycles or hamiltonian paths in the square of graphs under consideration; they do not construct any such cycle or path.

3 Conclusion

The main results of this paper are Theorem 1 and Theorem 2. As we mention in Introduction Fleischner in [7] proved that in the square of graphs hamiltonicity and vertex-pancyclicity are equivalent concepts, and so are hamiltonian connectedness and panconnectedness. Hence we proved in fact that for graphs satisfying assumptions of Theorem 1 Theorem 2 the square of these graphs is vertex-pancyclic, panconnected, respectively.

As an easy corollary of Theorem 2 we get the following result.

Corollary 8. Let $G$ be a block-chain. Then $G^2$ is panconnected if and only if every innerblock of $G$ is a 2-block.

Moreover Corollary 8 is also the answer to Problem 1 stated by Chia et al. in [11] that for a graph $G$ with only two cutvertices it is true that $G^2$ is panconnected if and only if the unique block containing the two cutvertices is not the complete graph on two vertices.

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References

[1] S. Alstrup, A. Georgakopoulos, E. Rotenberg, C. Thomassen; A Hamiltonian cycle in the square of a 2-connected graph in linear time; Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, 1645–1649, SIAM, Philadelphia, PA, 2018.

[2] J.A. Bondy, U.S.R. Murty; Graph Theory, Graduate Texts in Mathematics 244; Springer, New York 2008.

[3] G. L. Chia, J. Ekstein, H. Fleischner; Revisiting the Hamiltonian Theme in the Square of a Block: The Case of DT-graphs; Journal of Combinatorics 9 (2018), no.1, 119–161.

[4] J. Ekstein, H. Fleischner; A Best Possible Result for the Square of a 2-Block to be Hamiltonian; Discrete Mathematics 344 (1) (2021), 112158.

[5] H. Fleischner; On Spanning Subgraphs of a Connected Bridgeless Graph and Their Application to DT-Graphs; Journal of Combinatorial Theory 16, No. 1 (1974), 17-28.

[6] H. Fleischner; The Square of Every Two-Connected Graph is Hamiltonian; Journal of Combinatorial Theory 16, No. 1 (1974), 29-34.

[7] H. Fleischner; In the square of graphs, Hamiltonicity and pancyclicity, hamiltonian connectedness and panconnectedness are equivalent concepts; Monatsh. Math. 82 (1976), 125–149.

[8] H. Fleischner, G. L. Chia; Revisiting the Hamiltonian Theme in the Square of a Block: The General Case; Journal of Combinatorics 10 (2019), no.1, 163–201.
[9] H. Fleischner, A.M. Hobbs; Hamiltonian total graphs; Mathematische
Nachrichten 68 (1975), 59-82.

[10] A. Georgakopoulos; A Short Proof of Fleischner’s Theorem; Discrete
Mathematics 309 (2009), no. 23-24, 6632-6634.

[11] G. L. Chia, S.-H. Ong, L. Y. Tan; On graphs whose square have strong
hamiltonian properties; Discrete Mathematics 309 (13) (2009), 4608-4613.

[12] J. Müttel, D. Rautenbach; A short proof of the versatile version
of Fleischner’s theorem; Discrete Mathematics 313 (2013), no. 19,
1929–1933.

[13] S. Říha; A New Proof of the Theorem by Fleischner; Journal of Combinatorial Theory Series B 52 (1991) 117-123.