Dielectric Correction to the Chiral Magnetic Effect

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We derive an electric current density \( j_{em} \) in the presence of a magnetic field \( B \) and a chiral chemical potential \( \mu \). That is induced by \( B \) if the chiral chemical potential \( \mu \) is non-zero, which is parallel to each other. The axial current may be realistic in the cores of compact stellar objects or in some condensed matter systems. Generally speaking, the Eq. (1) can also arise from low-energy effective descriptions. That is, from symmetry reason, the anomalous current is to be expressed as \( j_{em} = -\langle \partial \phi \rangle \hat{F}^{\mu \nu} \) where \( \phi \) is a pseudo-scalar field. Under the magnetic field \( B^i = \hat{F}^{0i} \) the current is thus proportional to \( B \) and \( \partial \phi \) that translates into \( \mu_5 \) of Eq. (1). This gives a physical interpretation of the \( \mu_5 \) as a time derivative of the pseudo-scalar condensate. Equivalently one can also say that \( \mu_5 \) appears as a result of a time derivative on the pseudo-scalar meson enhancement through mixture of the \( \sigma \) and \( \eta_0 \) condensates.

It is tremendously important to verify Eq. (1) in order to quantify the CME in such a way that theoretical predictions can be compared to experimental measurements. At the same time the estimate of background (non-topological) effects is indispensable for experimental confirmation of the CME. The purpose of this Letter is to point out that the induced current itself must receive a non-anomalous correction from interaction effects even at the level of the mean-field quasiparticle description. The notable feature of the correction is that the coefficient in Eq. (1), which seems to be protected by anomaly, is modified by a factor. Existence of such correction is a novel insight in theory.

In this Letter we do not directly consider the interaction mediated by gauge bosons but instead make use of an effective form of the interaction in terms of fermionic degrees of freedom. It should be a reasonable approximation to utilize the current-current interaction,

\[
\mathcal{L}_V = -G_V\langle \bar{\psi} \gamma_\mu \psi \rangle \langle \bar{\psi} \gamma^\mu \psi \rangle ,
\]

as long as the typical interaction energy is lower than the gauge boson mass, which is the case for Fermi’s effective theory of the weak interactions. In QCD the one-gluon exchange can result in an interaction form of the color current in which the color generators are inserted. It is then possible to extract a specific form out from the Fierz transformation. Accordingly \( G_V \) should be of order \( g^2/M^2 \) where \( g \) is the gauge coupling constant and \( M \) is the gauge boson mass. [Gluons should be massive non-perturbatively though their mass is zero at the QCD Lagrangian level.] For the moment we treat \( G_V \) as just a parameter and will plug a concrete value in later discussions.

The vector interaction is special in QCD; it is invariant under chiral rotations and thus its presence is naturally anticipated. Introducing a mean-field \( j^\mu = \langle \bar{\psi} \gamma^\mu \psi \rangle \) the interaction (2) is decomposed into \( \mathcal{L}_V \to -G_V j^2 + 2G_V j^2 \bar{\psi} \gamma^3 \psi \), then, apart from quantum gauge fluctuations, we have the kinetic term; \( \mathcal{L}_{\text{kin}} = \bar{\psi} (i\gamma^\mu D^\mu - \frac{M}{2} \gamma^5 ) \psi \), where \( M \) is a mass which may be dynamically generated, but for the present purpose the microscopic origin of \( M \) is irrelevant. In writing the above we have introduced an effective (classical) gauge field in \( D^\mu = \partial^\mu - i\bar{\psi}A^\mu \) given by

\[
A^x = A^y = 0, \quad A^z = -2G_V j^z ,
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for which the coupling constant is chosen as unity. [This is a convenient choice for later generalization to QCD with quark flavors having different electric charges.] The grand potential (divided by the volume \( V \)) can be read from the zero-point oscillation energy in addition to the

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The Chiral Magnetic Effect (CME) is a general mechanism to induce an electric current \( j_{em} \) along the direction of an external magnetic field \( B \) for systems that have a non-vanishing chirality charge \( N_5 = N_R - N_L \). Using the chiral chemical potential \( \mu_5 \) to express the result of a time derivative on the strong interaction effects even at the level of the mean-field quasi-particle description.

This type of anomaly relation is quite generic and also discussed in various contexts. Particularly, in prior to the CME, an anomaly relation is quite generic and also discussed in various contexts [3–6]. This type of anomaly relation is quite generic and also discussed in various contexts. Particularly, in prior to the CME, an anomaly relation is quite generic and also discussed in various contexts [3–6].

\[
j_{em} = \frac{e^2 \mu_5 B}{2\pi^2},
\]

which originates from the quantum anomaly and thus is an exact relation insensitive to any infrared scales such as the particle mass, temperature, etc. This type of anomaly relation is quite generic and also discussed in various contexts. Particularly, in prior to the CME, a dual situation had attracted attention; it is the axial current \( j_5 \) which is induced by \( B \) if the quark chemical potential \( \mu_q \) is non-zero, which is parallel to each other under the replacement \( \mu_5 \leftrightarrow \mu_q \) and \( j_5 \leftrightarrow j_5 \). The resulting axial current may be realistic in the cores of compact stellar objects or in some condensed matter systems. Generally speaking, the Eq. (1) can also arise from low-energy effective descriptions. That is, from symmetry reason, the anomalous current is to be expressed as \( j_{em} = -\langle \partial \phi \rangle \hat{F}^{\mu \nu} \) where \( \phi \) is a pseudo-scalar field. Under the magnetic field \( B^i = \hat{F}^{0i} \) the current is thus proportional to \( B \) and \( \partial \phi \) that translates into \( \mu_5 \) of Eq. (1). This gives a physical interpretation of the \( \mu_5 \) as a time derivative of the pseudo-scalar condensate. Equivalently one can also say that \( \mu_5 \) appears as a result of a time derivative on the strong \( \theta \)-angle parameter which leads to pseudo-scalar meson enhancement through mixture of the \( \sigma \) and \( \eta_0 \) condensates.

It is tremendously important to verify Eq. (1) in order to quantify the CME in such a way that theoretical predictions can be compared to experimental measurements. At the same time the estimate of background (non-topological) effects is indispensable for experimental confirmation of the CME. The purpose of this Letter is to point out that the induced current itself must receive a non-anomalous correction from interaction effects even at the level of the mean-field quasiparticle description. The notable feature of the correction is that the coefficient in Eq. (1), which seems to be protected by anomaly, is modified by a factor. Existence of such correction is a novel insight in theory.

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as long as the typical interaction energy is lower than the gauge boson mass, which is the case for Fermi’s effective theory of the weak interactions. In QCD the one-gluon exchange can result in an interaction form of the color current in which the color generators are inserted. It is then possible to extract a specific form out from the Fierz transformation. Accordingly \( G_V \) should be of order \( g^2/M^2 \) where \( g \) is the gauge coupling constant and \( M \) is the gauge boson mass. [Gluons should be massive non-perturbatively though their mass is zero at the QCD Lagrangian level.] For the moment we treat \( G_V \) as just a parameter and will plug a concrete value in later discussions.

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for which the coupling constant is chosen as unity. [This is a convenient choice for later generalization to QCD with quark flavors having different electric charges.] The grand potential (divided by the volume \( V \)) can be read from the zero-point oscillation energy in addition to the
mean-field condensation energy, i.e.
\[ \Omega/V = G Vij^2 - |eB|/2\pi \sum_{s,k} \alpha_{sk} \int dp^2/2\pi \omega_s(p), \]
(4)
where \( s \) is the spin and \( k \) refers to the Landau level. The spin degeneracy factor is taken care of \( \alpha_{sk} \) defined as
\[ \alpha_{sk} = \begin{cases} \delta_{s,1} & \text{for } k = 0, \ eB > 0, \\ \delta_{s,-1} & \text{for } k = 0, \ eB < 0, \\ 1 & \text{for } k \neq 0. \end{cases} \]
(5)
Now the quasi-particle dispersion relations are derived from the eigenvalues of the Dirac operator which are \( \omega_s^2 = M^2 + |p|^2 + \text{sgn}(p^z)s\mu_5|^2 \), where \( |p|^2 = (p^x + A^x)^2 + 2|eB|k \) with \( k \) being a non-negative integer to label the Landau level.

The stationary condition for \( \Omega \) with respect to \( j \), i.e. \( \partial\Omega/\partial j = 0 \), yields a self-consistent condition to determine the current density,
\[ \frac{\partial\Omega}{\partial j} = 0 \rightarrow j = \frac{\partial(\Omega/V)}{\partial A}. \]
(6)
We note that the derivative with respect to \( \mathcal{A} \) acts only on the latter term of \( \partial\Omega/\partial A \) that explicitly depends on \( \mathcal{A} \).

The expression (4) looks like a standard one utilized in Ref. [2] but the essential difference is that the derivative should be evaluated at non-zero \( \mathcal{A} \) which is specified by (3) and thus \( j \) should be solved in a self-consistent manner.

By taking the derivative explicitly on Eq. (4) only the surface terms contribute to the current. Here we assume some regularization scheme and introduce a ultraviolet momentum scale \( \Lambda_k \) in such a way that \( \Lambda_k^2 + 2|eB|k = \Lambda^2 \). Then the surface terms are picked up as
\[ \frac{\partial(\Omega/V)}{\partial A^2} \bigg|_\mathcal{A} = \frac{|eB|}{4\pi^2} \left( (g_+ - g_-) + \sum_{k=1}^{\kappa_{\text{max}}} \sum_s (f_{s+} - f_{s-}) \right), \]
(7)
with the energies at \( p^z = \pm\Lambda_k \), namely,
\[ g_{\pm} = \sqrt{M^2 + \left( |\pm\Lambda_0 \pm A^z| + \text{sgn}(eB)\mu_5 \right)^2}, \]
\[ \approx \Lambda_0 \pm A^z \pm \text{sgn}(eB)\mu_5, \]
(8)
for the Landau zero-mode \( k = 0 \) and the second line is an approximation valid for sufficiently large \( \Lambda \). In the same way we have
\[ f_{s\pm} = \sqrt{M^2 + \left( |\pm\Lambda_k + A^z| + 2|eB|(k + s\mu_5) \right)^2}, \]
\[ \approx \Lambda_0 \pm \frac{\Lambda_k}{\Lambda} A^z, \]
(9)
for the Landau non-zero modes \( k > 0 \). From our definition of \( \Lambda_k \) and \( \Lambda \) it is obvious that \( \kappa_{\text{max}} = |\Lambda^2/(2|eB|)| \).

It should be noted that the first contribution in (7) involving \( g_{\pm} \) leads to the electric current (11) in the limit of vanishing \( \mathcal{A} \). The latter term involving \( f_{s\pm} \) is simply zero if \( \mathcal{A} \) is absent.

When \( \Lambda \) is sufficiently larger than \( |eB| \), the sum over \( k \) can be well approximated as \( \sum_k (\Lambda_k/\Lambda) \approx \Lambda^2/(3|eB|) \). Using this we reach,
\[ j^z = \frac{|eB|}{2\pi^2} \left( \text{sgn}(eB)\mu_5 - 2G_V\left( 1 + \frac{2\Lambda^2}{3|eB|} \right) j^z \right). \]
(10)
From the above one might be able to solve \( j \), though the divergent term still remains. The expression is not physically meaningful yet as it is. One needs to formulate the renormalization procedure that we address in what follows below.

Here it should be mentioned that our result (10) is quite analogous to what is discussed in Ref. [6]. The formulation may look different since no current-current interaction was introduced in Ref. [6] but familiar Nambu–Jona-Lasinio (NJL) type (scalar and pseudo-scalar) four-fermion interactions were considered with the Dyson-Schwinger equation which goes beyond the present mean-field analysis. Nevertheless our treatment suffices to grasp the essential point of Ref. [6]. That is, the interaction term like Eq. (2) would be always induced by mixing interactions from scalar and pseudo-scalar channels, which might be the case in the Dyson-Schwinger calculation.

For the purpose of renormalization let us consider the following susceptibility defined by
\[ C = \frac{\partial^2(\Omega/V)}{\partial A^2^2} \bigg|_\mathcal{A} = \frac{|eB|}{2\pi^2} \left( 1 + \frac{2\Lambda^2}{3|eB|} \right), \]
(11)
which is still divergent. Because this susceptibility can be interpreted as the gauge boson (screening) mass, it must become vanishing in the vacuum (i.e. for \( B = 0 \)) so that the gauge invariance is maintained. This imposes the following requirement: \( C \to 0 \) for \( B \to 0 \), which sets a natural condition for minimal subtraction. In the definition of \( C \) in Eq. (11) the second term is divergent as \( \Lambda^2/(3\pi^2) \), which must be subtracted through renormalization. Consequently the renormalized susceptibility should be given by the finite first term:
\[ C_R = \frac{|eB|}{2\pi^2}. \]
(12)
This result is consistent with the conclusion in Ref. [12] in which an independent derivation of the susceptibility (12) is given by means of the linear response formula with the current generation due to anomaly (11). Since the derivation in Ref. [12] is free from ultraviolet divergence, this confirms the validity of Eq. (12).

It is straightforward to solve \( j \) with respect to \( j^z \) with \( C \) replaced by \( C_R \). We finally find,
\[ j = \frac{1}{1 + 2G_V C_R} \frac{\mu_5 eB}{2\pi^2}, \]
(13)
We note that $\kappa$ deduced from Eq. (13) is a standard formula representing the screening effects by vacuum polarization [18].

As we pointed out, the current-current term like Eq. (2) should be generally present as a result of non-perturbative interactions, and thus Eq. (1) is no longer the exact answer in the fully interacting case. In this sense we would claim that the holographic calculations as in Ref. [19] may miss some back-reaction because the correction factor (13) has not been found there. This difference might explain subtleties on the holographic chiral magnetic conductivity [17], Eq. (13) as follows;

$$j_{em} = \kappa \cdot j_{em}(G_V = 0).$$

(14)

We note that $\kappa$ is zero if quark masses are degenerate; the system is then automatically electric-charge neutral. Hence, there is no correction appearing at all, and thus identically $\kappa = 1$. This result is intuitively understandable if quark masses are degenerate; the system is then automatically electric-charge neutral. Hence, there is no coupling between the baryon current $j$ and the electric current $j_{em}$ and thus no back-reaction from their entanglement. It is, however, non-trivial that the conclusion of no correction for the three-flavor case holds regardless of whether quark masses are degenerate or not.

After some calculations we find,

$$\kappa = \left(1 + \frac{12}{5\pi^2} G_V |eB| \right) \left(1 + \frac{3}{\pi^2} G_V |eB| \right)^{-1}.$$  

(19)

This result [19] behaves different qualitatively from Eq. (13). One can easily see that $\kappa$ asymptotically approaches a finite number $4/5 = 0.8$ in the limit of $G_V \to \infty$ and $\kappa$ never goes to zero. To show this clearly, we make a plot for the above $\kappa$ as a function of $G_V |eB|$ in Fig. 1. It is obvious from the figure that $\kappa$ slowly decays to the asymptotic value 0.8 and thus the dielectric correction is only a minor effect in contrast to the one-flavor situation.

**Three-flavor case:** It is interesting to think of the ideal case with three flavors. Because the chiral magnetic current has the anomaly origin and is independent of the quark masses, the three-flavor case might be realistic if $\mu_5$ is large enough. Let us imagine that the system has $s$ quarks with $q_s = (-1/3)e$ in addition to $u$ and $d$ quarks. Then the situation is totally changed again. In view of Eq. (14), $j$ is proportional to $\sum_f q_f j_f^l$ which becomes vanishing for the three-flavor case: $q_u + q_d + q_s = 0$. Because $j$ is zero, there is no correction appearing at all, and thus identically $\kappa = 1$. This result is intuitively understandable if quark masses are degenerate; the system is then automatically electric-charge neutral. Hence, there is no coupling between the baryon current $j$ and the electric current $j_{em}$ and thus no back-reaction from their entanglement. It is, however, non-trivial that the conclusion of no correction for the three-flavor case holds regardless of whether quark masses are degenerate or not.

Now let us plug concrete numbers in our final expression to see how large/small the correction is in specific examples. We first need to determine a value of $G_V$. From the discussions below [20] we can postulate $G_V \sim q^2/M_g^2$ and let us choose $g = 2$ ($\alpha_s \sim 0.3$) and $M_g = 0.8$ GeV (that is roughly a half of the glueball mass) here. Then we have a rough estimate as $G_V \approx 6.3$ GeV$^{-2}$. We can make it sure that this is a reasonable estimate from the empir-
ically adopted $G_V$ in the NJL model; $G_V = 0.2 \sim 0.5G_S$ where $G_S$ is the four-fermion coupling in the scalar and pseudo-scalar channel and fixed as $G_S \approx 9.2 \text{ GeV}^{-2}$ to reproduce the pion mass and decay constant [21]. This is not far from our estimate $G_V \approx 6.3 \text{ GeV}^{-2}$.

In the heavy-ion collision it would be convenient to express the magnetic field strength $|eb|$ in the unit of the pion mass squared $m_\pi^2$ from the UrQMD simulation [23]. If we use the value $|eb| = m_\pi^2$, then, we find $G_V|eb| = 0.11$. This is a small number and the dielectric constant stays close to unity for any case of flavor number. Therefore, fortunately, we can conclude that the dielectric correction from back-reaction is only minor and practically negligible for phenomenology.

Finally let us make a comment on a possible application of our result to the lattice-QCD simulation. The chiral magnetic effect has been investigated in the lattice-QCD simulation [23,24] with extremely strong magnetic fields. For example, in Ref. [23], the applied magnetic field can be as strong as $|eb| \sim \text{ GeV}^2$, which is of order hundred in the unit of $m_\pi^2$. If we use $G_V \approx 6.3 \text{ GeV}^{-2}$ then $G_V|eb| \sim 6.3$ for $|eb| \sim 1 \text{ GeV}^2$. According to our expressions the one-flavor system would lead to a substantial suppression factor $\kappa \sim 0.34$. Even in the two-flavor case the suppression is a sizable effect; $\kappa \sim 0.87$. Therefore, it should be possible to confirm the existence of such dielectric corrections as discussed here using the lattice-QCD simulation.

The lattice-QCD simulation opens an intriguing possibility that $G_V$ may be determined from $\kappa$. In fact the determination of $G_V$ provides us with very useful information on the QCD phase diagram. Especially it crucially depends on $G_V$ whether the chiral phase transition can become of first order at finite density and whether the QCD critical point can exist on the phase diagram. Once a finite baryon chemical potential is turned on, of course, the sign problem hinders the simulation. Nevertheless the two-color two-flavor simulation is still feasible even at finite density, which may give $G_V$ as a function of density, if the precise determination of chiral magnetic current is possible from the lattice data.

In summary we computed the back-reaction coming from the vector interaction, which should result in a dielectric correction on the chiral magnetic current even at the mean-field level. Our final expressions show that the qualitative behavior of the correction strongly depends on the relevant number of flavors in the system. The one-flavor case has a substantial suppression on the chiral magnetic current due to screening effects, while the two-flavor case has only a minor modification however strong the vector interaction is. There is no correction at all for the three-flavor case. It should be possible to quantify the correction in the lattice-QCD simulation, which in turn would give useful information on the strength of the effective vector interaction.

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