1. Introduction

As an important branch of mathematics, matrix theory has been widely applied in the fields of mathematics and technology, such as optimization theory ([1]), differential equations ([2]), numerical analysis, operations ([3]) and quantum theory ([4]).

In this manuscript, let $\mathbb{C}^n$ be an $n$-dimensional complex vector space with the inner product $(\vec{x}, \vec{y}) = \vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i^* y_i$ for $\vec{x} = (x_1, \cdots, x_n)^T$, $\vec{y} = (y_1, \cdots, y_n)^T \in \mathbb{C}^n$, where the superscripts $\vec{x}^T$ and $\vec{y}^T$ denote the conjugated transpose of $\vec{x}$ and the matrix transpose, respectively. Let $M_n$ denote the whole set of $n \times n$ matrices with complex entries, and we call $\vec{x} \in \mathbb{C}^n$ the eigenvector of $A \in M_n$ when $A \vec{x} = \lambda \vec{x}$ (where $\lambda$ is called the eigenvalue of $A$). We denote $H_n$ the set of all Hermitian matrices. For any $A \in H_n$, we have $A = \sum_{i=1}^{n} \lambda_i P_i$, where $\lambda_i$ is the eigenvalue of $A$ and $\sum_{i=1}^{n} P_i = I$, $P_i P_j = 0$ ($i \neq j$); specially, when $\vec{x}^T A \vec{x} \geq 0$ for any $\vec{x} \in \mathbb{C}^n$, we denote $A \in H_n^+$ ($H_n^+$ is the set of $n \times n$ positive-definite Hermitian matrices whose eigenvalues are nonnegative). Let $f$ be a function with the domain $(0, +\infty)$; for any $A \in H_n^+$, the matrix function is defined as

$$f(A) = \sum_{i=1}^{n} f(\lambda_i) P_i.$$

On the basis of this definition, we have a formula relating the trace of matrix $A$ and the eigenvalue of $A$:

$$\text{Tr}[A] = \sum_{i=1}^{n} \lambda_i(A),$$

where $\lambda_i(A)$ is the eigenvalue of $A$. It is well known that $\text{Tr}[AB] = \text{Tr}[BA]$ for any $A, B \in H_n^+$. However, $\text{Tr}[e^{A+B}] = \text{Tr}[e^A e^B]$ only if $AB = BA$. Fortunately, in 1965, Thompson and Golden independently discovered an inequality called the Thompson–Golden theorem (refer to [5–7]):
\[ \text{Tr}[e^{A+B}] \leq \text{Tr}[e^A e^B]. \]

In general, the following limit holds (called the Lie–Trotter formula [8]):

\[ \lim_{p \to 0} \text{Tr}\left( \left( e^{\frac{pA}{p}} e^{\frac{pB}{p}} e^{\frac{pA}{p}} \right)^\frac{1}{p} \right) = \text{Tr}[e^{A+B}]. \]

Furthermore, the following inequality holds when \( p \geq 1 \):

\[ \text{Tr}\left( A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p} \right) = \text{Tr}\left( e^{\frac{\ln A}{p}} e^{\frac{\ln B}{p}} e^{\frac{\ln A}{p}} \right)^\frac{1}{p} \leq \text{Tr}\left( e^{\frac{\ln A}{p}} e^{\frac{\ln B}{p}} e^{\frac{\ln A}{p}} \right)^\frac{1}{p} = \text{Tr}\left( A^\frac{1}{p} B A^\frac{1}{p} \right)^\frac{1}{p}, \]

which is the Lieb–Thirring–Araki theorem ([9,10]). Since the function \( F(A) = \text{Tr} e^{B+\ln A} \) is a Fréchet differential function for any \( A \in H_m \), the concavity of \( F(A) \) implies the Thompson–Golden theorem. At the same time, one can also obtain the Thompson–Golden theorem by using the relationship \( \lambda_1(A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p}) \) and \( \lambda_1(A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p}) \) ([11]). It is known that

\[ \det(A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p}) = [\det(A^\frac{1}{p} B A^\frac{1}{p})]^\frac{1}{p}. \]

By using the matrix exterior algebra, we have

\[ \det(A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p}) = \det(A^\frac{1}{p} B A^\frac{1}{p}) \]

According to the convexity of \( \text{Tr}[\wedge_k e^A] \), Huang proved the following inequality ([12]):

\[ \text{Tr}[\wedge_k e^{A+B}] \leq \text{Tr}[\wedge_k (A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p})]. \]

With this motivation, we utilize the Stein–Hirschman operator interpolation inequality to show that \( \lambda_1(A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p}) \) is a monotone increasing function for any \( \alpha > 0 \). Then, we generalize the Lieb–Thirring–Araki theorem and provide a new proof of the Furuta theorem ([13]). The rest of the paper is organized as follows. In Section 2, some general definitions and important conclusions are introduced. In Section 3, a new proof of the monotonicity of \( \lambda_1(A^\frac{1}{p} B^\frac{1}{p} A^\frac{1}{p}) \) and some general results are offered.

2. Preliminary

In this section, we recall some notions and definitions from matrix analysis, and introduce some important results of the matrix-monotone function, which are used through the article (refer to [14–17]).

2.1. Tensor Product and Exterior Algebra

The tensor product, denoted by \( \otimes \), is also called the Kronecker product. It is a generalization of the outer product from vectors to matrices, so the tensor product of matrices is referred to as the outer product as well in some contexts. For an \( m \times n \) matrix \( A \) and a \( p \times q \) matrix \( B \), the tensor product of \( A \) and \( B \) is defined by

\[ A \otimes B = \begin{pmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & \ddots & \vdots \\ a_{m1} B & \cdots & a_{mn} B \end{pmatrix}, \]

where \( A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \).

The tensor product is different from matrix multiplication, and one of the differences is commutativity:

\[ (I \otimes B)(A \otimes I) = (A \otimes I)(I \otimes B) = A \otimes B. \]
From this relation, one can obtain
\[
AC \otimes BD = (AC \otimes I)(I \otimes BD) \\
= (A \otimes I)(C \otimes I)(I \otimes B)(I \otimes D) \\
= (A \otimes I)(I \otimes B)(C \otimes I)(I \otimes D) \\
= (A \otimes B)(C \otimes D).
\]

For convenience, we denote
\[
\otimes_k A = A \otimes A \otimes \cdots \otimes A.
\]

In addition to the tensor product, there is another common product named exterior algebra ([18]). Exterior algebra, denoted by \( \wedge \), is a binary operation for any \( A_{n \times n} \) that is
\[
(A_1 \wedge A_2 \wedge \cdots \wedge A_k)(\xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k})_{1 \leq i_1 < \cdots < i_k \leq n} = (A_1 \xi_{i_1} \wedge A_2 \xi_{i_2} \cdots \wedge A_k \xi_{i_k})_{1 \leq i_1 < \cdots < i_k \leq n},
\]
where \( \{ \xi_i \}_{i=1}^n \) is an orthogonal basis of \( \mathbb{C}^n \) and
\[
\xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k} = \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} (-1)^{\pi} \xi_{\pi(i_1)} \otimes \xi_{\pi(i_2)} \cdots \otimes \xi_{\pi(i_k)},
\]
where \( S_n \) is the family of all permutations on \( \{1, 2, \cdots, n\} \).

Let \( \bigwedge^k \mathbb{C}^n \) be the span of the \( \{ \xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k} \}_{1 \leq i_1 < \cdots < i_k \leq n} \); a simple calculation shows that
\[
\lambda_1(\bigwedge^k A) = \prod_{i=1}^k \lambda_i(A). \tag{1}
\]

2.2. Schur-Convex Function

Let \( \vec{x} = (x_1, \cdots, x_n)^\top, \vec{y} = (y_1, \cdots, y_n)^\top \in \mathbb{R}^n \) and denote
\[
x_{[1]} = \max_{i=1,2,\cdots,n} \{x_i\}, \cdots, x_{[n]} = \min_{i=1,2,\cdots,n} \{x_i\}, \quad y_{[1]} = \max_{i=1,2,\cdots,n} \{y_i\}, \cdots, y_{[n]} = \min_{i=1,2,\cdots,n} \{y_i\}.
\]

If \( \vec{x} \) and \( \vec{y} \) satisfy
\[
\begin{cases}
\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, & k \leq n - 1; \\
\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}, & k = n.
\end{cases}
\]
then \( \vec{x} \) is said to be majorized by \( \vec{y} \), denoted by \( \vec{x} \prec \vec{y} \). Meanwhile, we denote \( \vec{x} \prec_{w} \vec{y} \) if \( \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} \) for any \( k \leq n \).

Suppose \( f \) is a real-valued function defined on a set \( A \subseteq \mathbb{R}^n \); then, \( f \) is said to be a Schur-convex function on \( A \) if, for any \( \vec{x}, \vec{y} \in \mathbb{R}^n \) and \( \vec{x} \prec \vec{y} \), one obtains \( f(\vec{x}) \leq f(\vec{y}) \) ([11]).

If \( f \) is differentiable and defined on \( I^n \) (\( I \subset \mathbb{R} \) being an open interval), then the following lemma holds (refer to [11]).

Lemma 1. \( f \) is Schur-convex on \( I^n \subset \mathbb{R}^n \) if and only if
\[
(x_i - x_j)(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}) \geq 0 \text{ for all } \vec{x} \in I^n.
\]
2.3. The Matrix-Monotone Function

For a matrix $A \in H_n^+$, according to the spectral theorem ([19]), it can be decomposed as

$$A = P^* \Lambda_1 P,$$

where $P$ is the unitary matrix and $\Lambda_1 := \text{diag}\{\lambda_1, ..., \lambda_n\}$ is a diagonal matrix with eigenvalues as elements. When $\vec{x}^*(A - B)\vec{x} > 0$ for any $\vec{x} \in \mathbb{C}^n$, we denote $A \succeq B$.

Associated with a function $f(x)$ on $(0, +\infty)$, the matrix function $f(A)$ is defined as

$$f(A) = P^* f(\Lambda_1) P,$$

where $f(\Lambda_1) = \text{diag}\{f(\lambda_1), ..., f(\lambda_n)\}$. Then, the function $f$ is said to be matrix-monotonic if it satisfies

$$f(A) \succeq f(B) \text{ for all } A \succeq B > 0.$$

Since the matrix-monotone function is a special type of operator monotone function, we present the following general conclusion about the operator-monotone function, which can be found in [20,21].

Lemma 2. The following statements for a real-valued continuous function $f$ on $(0, +\infty)$ are equivalent:

(i) $f$ is operator-monotone;

(ii) $f$ admits an integral representation

$$f(\lambda) = \alpha + \beta \lambda + \int_{-\infty}^{0} (1 + \lambda t) (t - \lambda)^{-1} \, d\mu(t), \text{ for any } \lambda > 0,$$  \hspace{1cm} (2)

where $\alpha$ is a real number, $\beta$ is non-negative and $\mu$ is a finite positive measure on $(-\infty, 0)$.

3. The Main Results

For any $A, B \in M_n$, it is known $\ln AB$ is not equal to $\ln A + \ln B$ in general when $AB \neq BA$. Generally, $(A^{\frac{1}{2}} B A^{\frac{1}{2}})^n$ is not equal to $A^{\frac{n}{2}} B^n A^{\frac{n}{2}}$. Therefore, many people pay much attention to studying the relation between $(A^{\frac{1}{2}} B A^{\frac{1}{2}})^n$ and $A^{\frac{n}{2}} B^n A^{\frac{n}{2}}$. A famous result regarding the trace inequality is the Lieb–Thirring–Araki theorem:

$$\text{Tr}[(A^{\frac{1}{2}} B A^{\frac{1}{2}})^n] \leq \text{Tr}[(A^{\frac{n}{2}} B^n A^{\frac{n}{2}})],$$

where $\alpha \geq 1$. In the following, we further study the relation between $(A^{\frac{1}{2}} B A^{\frac{1}{2}})^n$ and $A^{\frac{n}{2}} B^n A^{\frac{n}{2}}$ and provide the main results.

Theorem 1. For any $0 < \alpha \leq \beta$ and $A, B \in H_n^+$, the following inequality holds

$$\lambda_1(A^{\frac{n}{2}} B^n A^{\frac{n}{2}})^{\frac{1}{n}} \leq \lambda_1(A^{\frac{\beta}{2}} B^\beta A^{\frac{\beta}{2}})^{\frac{1}{\beta}}.$$  \hspace{1cm} (3)

Proof. By using the Cauchy inequality, we have

$$|\vec{x}^* A B \vec{x}|^2 = |\langle \vec{x}, A B \vec{x} \rangle|^2 \leq (\vec{x}^* A^* A \vec{x}) (\vec{x}^* B^* B \vec{x}), \text{ } \vec{x} \in \mathbb{C}^n.$$

If we denote

$$\lambda_1(A) = \max\left\{ \frac{\langle \vec{x}, A \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} \right\} = \max_{\vec{x} \in \mathbb{C}^n, \langle \vec{x}, \vec{x} \rangle = 1} \{ \langle \vec{x}, A \vec{x} \rangle \}$$

$$= \max_{\vec{x} \in \mathbb{C}^n, \langle \vec{x}, \vec{x} \rangle = 1} \{ \langle \vec{x}, A \vec{x} \rangle \}$$
as the maximum eigenvalue of \( A \), then we can obtain

\[
\lambda_1(A^{\frac{1}{2}} B A^{\frac{1}{2}}) = \max \{ \langle x, A^{\frac{1}{2}} B A^{\frac{1}{2}} x \rangle \}^{\frac{1}{2}} \\
= \max \{ \langle A^{\frac{1}{2}} B A^{\frac{1}{2}} x, x \rangle \}^{\frac{1}{2}} \\
\leq \max \{ \langle ABx, A^{\frac{1}{2}} B A^{\frac{1}{2}} x \rangle \}^{\frac{1}{2}} \\
\leq \max \{ \langle ABx, ABx \rangle \}^{\frac{1}{2}} \max \{ \langle x, A^{\frac{1}{2}} B A^{\frac{1}{2}} \rangle \}^{\frac{1}{2}} \\
= \lambda_1(A B A^{\frac{1}{2}})^{\frac{1}{2}},
\]

Here, we use the fact that \( \lambda_1(A B) = \lambda_1(B A) \).

Through a simple deformation, we have \( \lambda_1(A^{\frac{k+1}{2} B^{k-1} A^{\frac{k+1}{2}}})^{\frac{1}{k+1}} \leq \lambda_1(A^{\frac{k}{2} B^{k}} A^{\frac{k}{2}}) \).

From this inequality, we have the expression

\[
\lambda_1(A^{\frac{k}{2} B^{k}} A^{\frac{k}{2}}) = \left[ \lambda_1(A^{\frac{k+1}{2} B^{k} A^{\frac{k+1}{2}}}) \right]^{\frac{1}{k+1}} \leq \left[ \lambda_1(A^{\frac{k+1}{2} B^{k-1} A^{\frac{k+1}{2}}}) \right]^{\frac{1}{k+1}} \leq \left[ \lambda_1(A^{\frac{k}{2} B^{k}} A^{\frac{k}{2}}) \right]^\frac{k}{k+1}.
\]

Furthermore,

\[
\left[ \lambda_1(A^{\frac{k}{2} B^{k}} A^{\frac{k}{2}}) \right]^\frac{1}{k+1} \leq \left[ \lambda_1(A^{\frac{k+1}{2} B^{k} A^{\frac{k+1}{2}}}) \right]^\frac{1}{k+1}.
\]

This implies

\[
\left[ \lambda_1(A^{\frac{m_1 n_2}{2} B^{m_1 n_2} A^{\frac{m_1 n_2}{2}}} \right)^\frac{1}{m_1 n_2} \leq \left[ \lambda_1(A^{\frac{m_2 n_1}{2} B^{m_2 n_1} A^{\frac{m_2 n_1}{2}}} \right)^\frac{1}{m_2 n_1},
\]

for any \( m_1, m_2, n_1, n_2 > 0 \) and \( m_1 n_2 \leq m_2 n_1 \). Let \( A = A^{\frac{1}{m_1 n_2}}, B = B^{\frac{1}{m_2 n_1}} \); we obtain

\[
\left[ \lambda_1(A^{\frac{m_1 n_1}{2} B^{m_1 n_1} A^{\frac{m_1 n_1}{2}}} \right)^\frac{m_1}{n_1} \leq \left[ \lambda_1(A^{\frac{m_2 n_2}{2} B^{m_2 n_2} A^{\frac{m_2 n_2}{2}}} \right)^\frac{m_2}{n_2}.
\]

Namely, for \( 0 < \alpha \leq \beta \), we obtain

\[
\lambda_1(A^{\frac{\alpha}{2} B^{\beta}} A^{\frac{\alpha}{2}}) \leq \lambda_1(A^{\frac{\beta}{2} B^{\beta}} A^{\frac{\beta}{2}}).
\]

This completes the proof of Theorem 1. \( \square \)

Although Theorem 1 has been obtained from the Cauchy inequality, the frequency of retractions improves the inequality. In the following, we obtain Theorem 1 by using operator interpolation. First, let us introduce the Stein–Hirschman operator interpolation inequality (12).

**Lemma 3.** Supposing \( G(z) \) to be an analytic family of linear operators of admissible growth defined in the strip \( 0 \leq \Re(z) \leq 1 \) and \( 1 \leq p_1, p_2, q_1, q_2 \leq \infty \), when \( \frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{q_1} = \frac{1}{q_2} \) (0 \( \leq \), \( t \leq 1 \)), \( \| G_{t}(f) \| q_1 \leq A_0(y) \| f \| p_1 \) and \( \| G_{t+iy}(f) \| q_2 \leq A_1(y) \| f \| p_2 \), we can obtain

\[
\ln \| G_t(f) \| q \leq \int_{R} \left( \omega(1-t,y) \ln A_0(y) + \omega(t,y) \ln A_1(y) \right) dy + \ln \| f \| p, \tag{4}
\]

where \( \ln \| A_0(y) \|, \ln \| A_1(y) \| \leq A e^{a|y|} (\alpha < \pi) \) and \( \omega(t,y) = \frac{\tan(\frac{\pi}{4})}{2 \tan^2(\frac{\pi}{4}) + \tan^2(\frac{\pi}{4}) \cos\theta(\frac{\pi}{4})}. \)
From Lemma 3, we can improve the result in Theorem 1 and obtain the following theorem.

**Theorem 2.** For any $A, B \in H^+_2$, the following inequality holds

$$
\lambda_1(A^*B^tA^{**})^\frac{1}{2} \leq \int_{\mathbb{R}} \left( \frac{\omega(t,y)}{t} \right) \lambda_1((A^*B^{1-iy}AB^{1+iy}A^{**}))^\frac{1}{2} \, dy \leq \lambda_1(A^*BA^{**}).
$$

(5)

**Proof.** Firstly, let $f$ be an analytic function in $\mathbb{C}$. When $A, B \in H^+_2$, let $G_z(A,B) = \lambda_1(A^*B^zA^{**})$; obviously, $G_z(A,B)$ is an analytic function for any $z$ in strip $0 \leq R(z) \leq 1$. Since

$$
| G_{iy}(A,B) | = | \lambda_1(A^*B^{iy}A^{**}) | \leq | \lambda_1((A^*B^{1-iy}A^{1+iy})A^*B^{iy}A^{**}) |^\frac{1}{2} = 1,
$$

and

$$
| G_{i+iy}(A,B) | = | \lambda_1(A^*B^{iy}1^{iy}A^{**}) | \leq | \lambda_1((A^*B^{1-iy}A^{1-iy})A^*B^{iy}A^{**}) |^\frac{1}{2} \leq | \lambda_1(B^iA^iBA^{**}) |,
$$

and the formula

$$
\int_{\mathbb{R}} \omega(1-t,y) \, dy = 1-t,
$$

then, for any $f \in L^1_1(\mathbb{C})$, we can obtain

$$
\ln \| G_t(f) \|_1 = \ln | \lambda_1(A^*B^tA^{**}) | + \ln \| f \|_1 \leq \int_{\mathbb{R}} \left( \omega(1-t,y) \ln A_0(y) + \omega(t,y) \ln A_1(y) \right) \, dy + \ln \| f \|_1 \leq \int_{\mathbb{R}} \left( \omega(1-t,y) \ln A_0(y) + \omega(t,y) \ln \lambda_1((A^*B^{1-iy}AB^{1+iy}A^{**}))^\frac{1}{2} \right) \, dy + \ln \| f \|_1 \leq \int_{\mathbb{R}} \left( \omega(1-t,y) \ln 1 + \omega(t,y) \ln | \lambda_1(A^*BA^{**}) | \right) \, dy + \ln \| f \|_1 \leq t \ln | \lambda_1(A^*BA^{**}) | + \ln \| f \|_1.
$$

This implies

$$
\lambda_1(A^*B^tA^{**})^\frac{1}{2} \leq \int_{\mathbb{R}} \left( \frac{\omega(t,y)}{t} \right) \lambda_1((A^*B^{1-iy}AB^{1+iy}A^{**}))^\frac{1}{2} \, dy \leq \lambda_1(A^*BA^{**})
$$

or

$$
\lambda_1(A^*B^tA^{**}) \leq \int_{\mathbb{R}} \left( \frac{\omega(t,y)}{t} \right) \lambda_1((A^*B^{1-iy}AB^{1+iy}A^{**})) \, dy \leq \lambda_1(A^*BA^{**}),
$$

for any $0 < t < 1$, and the first “$\leq$” is obtained by the Jensen inequality ([22]). This completes the proof of Theorem 2. □
Theorem 2 is very useful. On one hand, when $\alpha < \beta$, letting $t = \frac{\alpha}{\beta}$, we can obtain Theorem 1. On the other hand, using the matrix exterior algebra, we obtain

$$\prod_{i=1}^{k} \lambda_i(A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta})^\frac{1}{k} \leq \prod_{i=1}^{k} \lambda_i(A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha})^\frac{1}{k}.$$  

Furthermore, we can deduce the following inequality

$$\text{Tr}[\wedge_k(A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta})^\frac{1}{k}] \leq \text{Tr}[\wedge_k(A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha})^\frac{1}{k}],$$

whether it is true or not for any $k \leq n$, and this inequality can be regarded as a generalization of the Lieb–Thirring–Araki theorem.

3.1. Generalization of Lieb–Thirring–Araki Theorem

According to Theorem 1 and Formula (1), we can show that

$$\begin{cases} \sum_{i=1}^{k} \ln \lambda_i(A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta})^\frac{1}{k} \leq \sum_{i=1}^{k} \ln \lambda_i(A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha})^\frac{1}{k}, \ k \leq n - 1; \\ \sum_{i=1}^{n} \ln \lambda_i(A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta})^\frac{1}{k} = \sum_{i=1}^{n} \ln \lambda_i(A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha})^\frac{1}{k}. \end{cases}$$

Furthermore, we have

$$\left(\ln \lambda_1(A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta})^\frac{1}{k}, \ldots, \ln \lambda_n(A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta})^\frac{1}{k}\right) < \left(\ln \lambda_1(A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha})^\frac{1}{k}, \ldots, \ln \lambda_n(A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha})^\frac{1}{k}\right).$$

Let $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} e^{x_i}$; direct calculations show that

$$(x_j - x_i) \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) = (x_j - x_i)(e^{x_i} - e^{x_j}) \geq 0,$$

which implies $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} e^{x_i}$ is a Schur-convex function. Hence, we have the Thompson–Golden theorem

$$\text{Tr}[A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta}]^\frac{1}{k} = \sum_{i=1}^{n} \lambda_i(A^\frac{\alpha}{\beta} B^\beta A^\frac{\alpha}{\beta})^\frac{1}{k} \leq \sum_{i=1}^{n} \lambda_i(A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha})^\frac{1}{k} = \text{Tr}[A^\frac{\beta}{\alpha} B^\alpha A^\frac{\beta}{\alpha}]^\frac{1}{k},$$  

when $\alpha \leq \beta$. Specially, when $\beta = 1$ and $A = A^\frac{1}{\alpha}$, $B = B^\frac{1}{\beta}$, we have the Lieb–Thirring–Araki theorem

$$\text{Tr}[A^\frac{1}{\alpha} B A^\frac{1}{\alpha}]^\frac{1}{k} \leq \text{Tr}[A^\frac{1}{\beta} B A^\frac{1}{\beta}].$$  

We know that the Lieb–Thirring–Araki theorem can be obtained from the Schur-convex function $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} e^{x_i}$. Generally, we can prove the following conclusion.

**Theorem 3.** For any $x_i \geq 0$, the function

$$g(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} e^{\gamma x_{i_1} + e^{\gamma x_{i_2}} \cdots e^{\gamma x_{i_k}}}$$

is Schur-convex for any $\gamma > 0$.

**Proof.** Since

$$g(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} e^{\gamma x_{i_1} + e^{\gamma x_{i_2}} \cdots e^{\gamma x_{i_k}}},$$
we have
\[ \frac{\partial \gamma}{\partial x_i} = \gamma e^{x_i} \left( \sum_{1 \leq i_1 < \cdots < i_k \leq n, \hat{x}_i} e^{x_{i_1} + x_{i_2} + \cdots + x_{i_k}} \right), \]
where \( \hat{x}_i \) denotes the removal of \( x_i \).

This implies
\[ (x_i - x_j)(\frac{\partial \gamma}{\partial x_i} - \frac{\partial \gamma}{\partial x_j}) = \gamma(x_i - x_j)(e^{x_i} - e^{x_j}) \left( \sum_{1 \leq i_1 < \cdots < i_k \leq n, \hat{x}_i} e^{x_{i_1} + x_{i_2} + \cdots + x_{i_k}} \right) \geq 0. \]

This completes the proof of Theorem 3. \( \square \)

From Theorem 3, we can deduce the following inequality immediately.

**Corollary 1.** For any \( \alpha \geq 1 \) and \( A, B > 0 \), the following inequality holds
\[ \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} BA^{\frac{1}{2}})^{\alpha y} \right] \leq \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} B^{\alpha} A^{\frac{1}{2}})^{y} \right], \tag{8} \]

or
\[ \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} BA^{\frac{1}{2}})^{y} \right] \leq \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} B^{\alpha} A^{\frac{1}{2}})^{y} \right] \leq \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} B^{\alpha} A^{\frac{1}{2}})^{y} \right], \tag{9} \]
for any \( y > 0 \).

From (8), it can be seen that, when \( k = 1 \), Corollary 1 is just the Lieb–Thirring–Araki theorem. Especially, \( \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} BA^{\frac{1}{2}})^{y} \right] \leq \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} B^{\alpha} A^{\frac{1}{2}})^{y} \right] \) when \( y = 1 \) ([12]).

Using Theorem 1, we can obtain
\[ \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} B^{\alpha} A^{\frac{1}{2}})\right] \leq \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}} B^{\beta} A^{\frac{1}{2}})\right]. \tag{10} \]
for any \( 0 \leq \alpha \leq \beta \), and this is a generalization of the Thompson–Golden theorem. For some other generalizations of the Thompson–Golden theorem, see [8,23]. Moreover, since
\[ \lambda(A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}}) < \lambda[(A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}})^{y}], \tag{11} \]
where \( 0 < \alpha \leq 1 \) and \( r \geq 1 \), we can obtain the following corollary.

**Corollary 2.** For any \( r \geq 1 \) and \( A, B > 0 \), the following inequality holds:
\[ \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}})^{y} \right] \geq \text{Tr} \left[ \wedge_k (A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}})^{y} \right]. \tag{12} \]

**3.2. Applications in Matrix-Monotone Function**

In this subsection, we obtain some other corollaries from Theorem 1 associated with the matrix-monotone function. Since
\[ \lambda_1(A^{\frac{1}{2}} B^{\alpha} A^{\frac{1}{2}}) \leq \lambda_1(A^{\frac{1}{2}} B^{\beta} A^{\frac{1}{2}}), \]
we can obtain \( \lambda_1(A^{\frac{1}{2}} B A^{\frac{1}{2}}) \geq \lambda_1(A^{\frac{1}{2}} B^\alpha A^{\frac{1}{2}}) \) for any \( 0 < \beta \leq 1 \). While \( A \leq Id \) is equivalent to \( \lambda_1(A) \leq 1 \), using this fact, for \( A \leq B \), we have \( B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \leq I \). For \( 0 < \beta \leq 1 \), we have
\[ \lambda_1(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) \leq \lambda_1(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) \beta \leq 1. \]

Hence, we obtain the Löwner–Heinz Theorem ([4]).

**Corollary 3.** For \( 0 < \alpha \leq 1 \) and \( A \leq B, A^\alpha \leq B^\alpha \).
Unfortunately, when $\alpha > 1$, Corollary 3 is false. However, we can use Theorem 1 to obtain the Furuta theorem [13].

**Corollary 4.** Let $0 \leq B \leq A$; then

$$A^{\frac{p+2r}{r+2}} \geq (A' B^p A')^{\frac{1}{r}},$$

(13)

where $p, r \geq 0, q \geq 1$ and $q \geq \frac{p+2r}{1+2r}$.

**Proof.** When $0 \leq p \leq 1$, we know that $B^p \leq A^p$ from Corollary 3; this implies

$$(A' B^p A')^{\frac{1}{r}} \leq (A' A^p A')^{\frac{1}{r}} = A^{\frac{p+2r}{r+2}}.
$$

Hence, we suppose $p \geq 1$. Meanwhile, we know $\frac{p+2r}{1+2r} \leq q$, so we only consider $p+2r = q$. Firstly, when $0 \leq r \leq \frac{1}{2}$, we have

$$\lambda_1 \left[ B^{-\frac{1}{2}} A^{-r} (A' B^p A')^{\frac{1}{r}} A^{-r} B^{-\frac{1}{2}} \right]$$

$$= \lambda_1 \left[ B^{-\frac{1}{2}} B^p (B^{-\frac{1}{2}} A^{-2r} B^{-\frac{1}{2}})^{\frac{1}{r}} B^{-\frac{1}{2}} B B^{-\frac{1}{2}} \right]$$

$$= \lambda_1 \left[ \left( B^{\frac{p+2r}{2}} \right)^{\frac{r-1}{p+2r}} (B^{-\frac{1}{2}} A^{-2r} B^{-\frac{1}{2}})^{\frac{p-1}{p+2r}} \right]$$

$$= \lambda_1 \left[ (A^{-r} B^{2r} A^{-r})^{\frac{p-1}{p+2r}} \right]$$

$\leq 1.$

This implies

$$A^{-r} (A' B^p A')^{\frac{1}{r}} A^{-r} \leq B \leq A.$$

That is,

$$(A' B^p A')^{\frac{1}{r}} \leq A^{1+2r}.$$  

Secondly, let $A_1 = A^{1+2r}, B_1 = (A' B^p A')^{\frac{1}{r}}$; then

$$\left( A_1^{\frac{1}{q}} B_1^{\frac{1}{q}} A_1^{\frac{1}{q}} \right)^{\frac{q}{q-1}} \leq A_1^{1+2r}.$$  

That is, $(A^{2r+\frac{1}{2}} B^p A^{2r+\frac{1}{2}})^\frac{1}{q} \leq A^{2(1+2r)}$, where $q_1 = \frac{p+4r+1}{1+4r+1}$.

This implies

$$(A^q B^p A^q)^\frac{1}{q} \leq A^{1+2s},$$

where $s = 2r + \frac{1}{2} \in [\frac{1}{2}, \frac{3}{2}]$.

Repeating this process, we have finished the proof. \(\square\)

### 3.3. Some Other Applications

In this subsection, we obtain a corollary associated with the matrix determinant. We suppose $A, B \in H_n$ and $\lambda_n(e^{\frac{p}{2}B} e^{\frac{q}{2}}) \geq 1$; then, from Theorem 1, we have

$$\left( \ln \lambda_1(e^{\frac{p}{2}B} e^{\frac{q}{2}}), \ldots, \ln \lambda_n(A B^p A^q) \right) \leq \left( \ln \lambda_1(e^{\frac{p}{2}B} e^{\frac{q}{2}})^\frac{1}{r}, \ldots, \ln \lambda_n(e^{\frac{p}{2}B} e^{\frac{q}{2}})^\frac{1}{r} \right).$$
where $0 < \alpha \leq \beta$ and $\ln \lambda_i(e^{\frac{\lambda}{\alpha}A \frac{\beta}{\lambda} B}) \geq 0 \ (i = 1, 2 \cdots, n)$. Let
\[
d(x_1, \cdots, x_n) = \prod_{i=1}^{n} x_i \quad (x_i \geq 0).
\]

Then, a straightforward calculation indicates
\[
(x_i - x_j)(\frac{\partial d(x_1, \cdots, x_n)}{\partial x_i} - \frac{\partial d(x_1, \cdots, x_n)}{\partial x_j}) \leq 0.
\]

Hence, $d(x_1, \cdots, x_n)$ is a Schur-concave function and the following inequality holds \([8]\).

**Corollary 5.** Supposing $A, B \in H_n$ and $\lambda_\rho(e^{\frac{\lambda}{\alpha}A \frac{\beta}{\lambda} B}) \geq 1$,
\[
det(A + B) \geq \det(\ln(e^{\frac{\lambda}{\alpha}A \frac{\beta}{\lambda} B} \frac{\lambda}{\lambda} A)) \geq \det(\ln(e^{\frac{\beta}{\alpha}A \frac{\beta}{\beta} B} B)) \geq 0, \quad \lambda > 0, \quad \beta > 0.
\]

In fact, for any $A \in H_n$, we have
\[
det(A) = \text{Tr}[\wedge_n A].
\]

Hence, Corollary 5 can be generalized as the following corollary.

**Corollary 6.** Supposing $A, B \in H_n$ and $\lambda_\rho(e^{\frac{\lambda}{\alpha}A \frac{\beta}{\lambda} B}) \geq 1$,
\[
\text{Tr}[\wedge_k \ln(e^{\frac{\lambda}{\alpha}A \frac{\beta}{\lambda} B} \frac{\lambda}{\lambda} A)] \geq \text{Tr}[\wedge_k \ln(e^{\frac{\beta}{\alpha}A \frac{\beta}{\beta} B} B)]
\]
where $1 \leq k \leq n$ and $0 < \alpha \leq \beta$.

**Proof.** Since
\[
\left( \ln \lambda_1(A \frac{\beta}{\alpha} B \frac{\beta}{\beta} A), \cdots, \ln \lambda_\rho(A \frac{\beta}{\alpha} B \frac{\beta}{\beta} A) \right) < \left( \ln \lambda_1(A \frac{\beta}{\alpha} B \frac{\beta}{\beta} A), \cdots, \ln \lambda_\rho(A \frac{\beta}{\alpha} B \frac{\beta}{\beta} A) \right),
\]
we can finish the proof if we show that the function
\[
a(x_1, x_2 \cdots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}
\]
is Schur-concave for any $x_i \geq 0$. In fact, we have
\[
(x_i - x_j)(\frac{\partial a(x_1, \cdots, x_n)}{\partial x_i} - \frac{\partial a(x_1, \cdots, x_n)}{\partial x_j}) = (x_i - x_j)^2 \left( \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} - \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \right) \leq 0.
\]

Hence, we know
\[
\text{Tr}[\wedge_k \ln(e^{\frac{\lambda}{\alpha}A \frac{\beta}{\beta} B} \frac{\lambda}{\lambda} A)] \geq \text{Tr}[\wedge_k \ln(e^{\frac{\beta}{\alpha}A \frac{\beta}{\beta} B} B)].
\]

This completes the proof of Corollary 6. \(\square\)
4. Conclusions

In the paper, we discuss the relationship between $\lambda_1(A^{1/2}BA^{1/2})^a$ and $\lambda_1(A^{1/2}B^aA^{1/2})$ by using the Stein–Hirschman operator interpolation inequality. Through in-depth study, we obtain some eigenvalue inequalities such as the generalization Golden–Thompson theorem and Lieb–Thirring–Araki theorem. Moreover, the Furuta theorem is also shown by using the eigenvalue inequality. At last, we generalize an important determinant inequality by using the matrix exterior algebra.

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References

1. Treanta, S.; Arana-Jimenez, M.; Antczak, T. A necessary and sufficient condition on the equivalence between local and global optimal solutions in variational control problems. *Nonlinear Anal.* **2020**, *191*, 111640. [CrossRef]
2. Bellman, R. *Introduction to Matrix Analysis*; McGraw-Hill: New York, NY, USA, 1960.
3. Anderson, W.N., Jr. Shorted operators. *SIAM J. Appl. Math.* **2006**, *20*, 520–525. [CrossRef]
4. Carlen, E. Trace inequalities and quantum entropy: An introductory course. *Contemp. Math.* **2010**, *529*, 73–140. [CrossRef]
5. Golden, S. Lower bounds for Helmholtz function. *Phys. Rev.* **1965**, *137*, 1127–1128. [CrossRef]
6. Thompson, C.J. Inequality with applications in statistical mechanics. *J. Math. Phys.* **1965**, *6*, 1812–1813. [CrossRef]
7. Lenard, A. Generalization of the Golden-Thompson inequality *Zntiana Univ. Math.* **1971**, *21*, 457–467. [CrossRef]
8. Ando, T.; Hiai, F. Log majorization and complementary Golden-Thompson type inequalities. *Linear Algebra Appl.* **1994**, *197–198*, 113–131. [CrossRef]
9. Araki, H. An inequality of Lieb and Thirring. *Lett. Math. Phys.* **1990**, *19*, 167–170. [CrossRef]
10. Lieb, E.H.; Thirring, W. Inequalities for the moments of the eigenvalues of the Schrodinger Hamiltonian and their relation to Sobolev inequalities. In *Studies in Mathematical Physics*; Lieb, E.H., Simon, B., Wightman, A., Eds.; Princeton University Press: Princeton, NJ, USA, 1976; pp. 269–303.
11. Marshall, A.W.; Olkin, I.; Arnold, B.C. *Inequalities: Theory of Majorization and Its Applications*; Springer: New York, NY, USA, 2011.
12. Huang, D. Generalizing Lieb’s concavity theorem via operator interpolation. *Adv. Math.* **2020**, *369*, 107208. [CrossRef]
13. Furuta, T. $A \geq B \geq 0$ assures $(B^rA^sB^r)^{1}\geq B^{r(1\times,r)}$ for $r \geq 0, p > 0, q \geq 1$ with $\frac{1+p}{q} \geq p + 2r$. *Proc. Am. Math. Sot.* **1987**, *101*, 85–88.
14. Bhatia, R. *Matrix Analysis*; Springer: New York, NY, USA, 1997.
15. Bhatia, R. *Positive Definite Matrices*; Princeton University Press: Princeton, NJ, USA; Oxford, UK, 2007.
16. Hall, B. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*; Springer: New York, NY, USA, 2003.
17. Zhang, F. *Matrix Theory: Basic Results and Techniques*; Springer: New York, NY, USA, 1999.
18. Simon, B. *Trace Ideals and Their Applications*, 2nd ed.; Math. Surveys and Monographs. A.M.S.: Cambridge, UK, 2005.
19. Tropp, J.A. An introduction to matrix concentration inequalities. *Found. Trends Mach. Learn.* **2015**, *8*, 1–2. [CrossRef]
20. Davis, C. Notions generalizing convexity for functions defined on spaces of matrices. In *Proceedings of the Symposia in Pure Mathematics*; American Mathematical Society: Providence, RI, USA, 1963; Volume 7, pp. 187–201.
21. Donoguе, W. *Monotone Matrix Functions and Analytic Continuation*; Springer: New York, NY, USA, 1974.
22. Bellman, R. *Inequalities*; Springer Verlag: Berlin/Gottingen/Heidelberg, Germany, 1961.
23. Hansen, F. Golden-Thompson’s inequality for deformed exponentials. *J. Stat. Phys.* **2015**, *159*, 1300–1305. [CrossRef]