Pebbling in Semi-2-Trees

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Abstract

Graph pebbling is a network model for transporting discrete resources that are consumed in transit. Deciding whether a given configuration on a particular graph can reach a specified target is \( \text{NP} \)-complete, even for diameter two graphs, and deciding whether the pebbling number has a prescribed upper bound is \( \Pi_2^\text{P} \)-complete. Recently we proved that the pebbling number of a split graph can be computed in polynomial time. This paper advances the program of finding other polynomial classes, moving away from the large tree width, small diameter case (such as split graphs) to small tree width, large diameter, continuing an investigation on the important subfamily of chordal graphs called \( k \)-trees. In particular, we provide a formula, that can be calculated in polynomial time, for the pebbling number of any semi-2-tree, falling shy of the result for the full class of 2-trees.

**Key words.** pebbling number, \( k \)-trees, \( k \)-paths, Class 0, complexity

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1 Introduction

The fundamental question in graph pebbling is whether a given supply (configuration) of discrete pebbles on the vertices of a connected graph can satisfy a particular set of demands on the vertices. The operation of pebble movement across an edge \{u, v\} is called a pebbling step: while two pebbles cross the edge, only one arrives at the opposite end, as the other is consumed. We write \((u, v)\) to denote a pebbling step from \(u\) to \(v\).

The most studied scenario involves the demand of one pebble on a single root vertex \(r\). Satisfying this demand is often referred to as reaching or solving \(r\), and configurations are consequently called either \(r\)-solvable or \(r\)-unsolvable.

The size \(|C|\) of a configuration \(C : V \rightarrow \mathbb{N} = \{0, 1, \ldots\}\) is its total number of pebbles \(\sum_{v \in V} C(v)\). The pebbling number \(\pi(G) = \max_{r \in V} \pi(G, r)\), where \(\pi(G, r)\) is defined to be the minimum number \(s\) so that every configuration of size at least \(s\) is \(r\)-solvable. Simple sharp lower bounds like \(\pi(G) \geq n\) and \(\pi(G) \geq 2^{\text{diam}(G)}\) are easily derived. Graphs satisfying \(\pi(G) = n\) are called Class 0 and are a topic of much interest. Recent chapters in [12] and [11] include variations on the theme such as \(k\)-pebbling, fractional pebbling, optimal pebbling, cover pebbling, and pebbling thresholds, as well as applications to combinatorial number theory, combinatorial group theory, and \(p\)-adic diophantine equations, and also contain important open problems in the field.

Computing the pebbling number is difficult in general. The problem of deciding if a given configuration on a graph can reach a particular vertex was shown in [13] and [15] to be \(\text{NP}\)-complete, even for diameter two graphs ([9]) or planar graphs ([14]). Interestingly, the problem was shown in [14] to be in \(\text{P}\) for graphs that are both planar and diameter two, as well as for outerplanar graphs (which include 2-trees). The problem of deciding whether a graph \(G\) has pebbling number at most \(k\) was shown in [15] to be \(\Pi_2^p\)-complete.

In contrast, the pebbling number is known for many graphs. For example, in [16] the pebbling number of a diameter 2 graph \(G\) was determined to be \(n\) or \(n + 1\). Moreover, [8] and [4] characterized those graphs having \(\pi(G) = n + 1\), and it was shown in [10] that
one can recognize such graphs in quartic time, improving on the order $n^3 m$ algorithm of [3]. Beginning a program to study for which graphs their pebbling number can be computed in polynomial time, the authors of [1] produced a formula for the family of split graphs that involves several cases. For a given graph, finding to which case it belongs takes $O(n^{1.41})$ time. The authors also conjectured that the pebbling number of a chordal graph of bounded diameter can be computed in polynomial time.

In opposition to the small diameter, large tree width case of split graphs, we turn here to chordal graphs with large diameter and small tree width. In this paper we study 2-paths, the sub-class of 2-trees whose graphs have exactly two simplicial vertices, as well as what we call semi-2-trees, the sub-class of 2-trees, each of whose blocks are 2-paths, and prove an exact formula that can be computed in linear time.

2 Preliminary Definitions and Results

In order to simplify notation, for a subgraph $H \subset G$ or subset $H \subset V(G)$ we write $C(H)$ to denote $\sum_{v \in V(H)} C(v)$. We use $C_H$ for the restriction of $C$ to $H$.

A simplicial vertex in a graph is a vertex whose neighbors form a complete graph. It is $k$-simplicial if it also has degree $k$. A $k$-tree is a graph $G$ that is either a complete graph of size $k$ or has a $k$-simplicial vertex $v$ for which $G - v$ is a $k$-tree. A $k$-path is a $k$-tree with exactly two simplicial vertices. A semi-2-tree is a graph in which each of its blocks is a 2-path, with each of its cut-vertices being simplicial in all of its blocks. For the purpose of our work we derive a new characterization of 2-paths that facilitates the analysis of its pebbling number.

Let $P = x_0, x_1, \ldots, x_{d-1}, x_d$ be a shortest $rs$-path between two vertices $r = x_0$ and $s = x_d$ of $G$, where $d = \text{dist}(r, s) = \text{diam}(G)$. For $1 \leq i \leq d - 1$, an $x_{i-1}x_{i+1}$-fan (centered on $x_i$) is a subgraph $F$ of $G$ consisting of the subpath $x_{i-1}, x_i, x_{i+1}$ of $P$ and a

\footnote{One can find the definition of tree-width in [5], but it is not necessary for this paper.}
path \( Q = x_{i-1}, v_{i,1}, \ldots, v_{i,k_i}, x_{i+1} \) with \( k_i \geq 1 \) such that \( x_i \) is adjacent to every vertex of \( Q \). We call \( F' \) the set \( \{v_{i,1}, \ldots, v_{i,k_i}\} \).

Let \( F_i \) be an \( x_{i-1}x_{i+1} \)-fan and \( F_{i+1} \) be an \( x_i x_{i+2} \)-fan, centered on \( x_i \) and on \( x_{i+1} \), respectively. We say that \( F_i \) and \( F_{i+1} \) are opposite-sided if \( F'_i \cap F'_{i+1} = \emptyset \); and that they are same-sided when \( F'_i \cap F'_{i+1} = \{v_{i,k_i}\} \) and \( v_{i,k_i} = v_{i+1,1} \).

The graph \( G \) is an overlapping fan graph if the following three conditions are satisfied:

- for every \( 1 \leq i \leq d - 1 \), there is a subgraph \( F_i \) which is an \( x_{i-1}x_{i+1} \)-fan centered on \( x_i \),
- for every \( 1 \leq i \leq d - 2 \), \( F_i \) and \( F_{i+1} \) are either opposite-sided or same-sided, and
- \( G \) is the union of the subgraphs \( F_i \) for \( 1 \leq i \leq d - 1 \).

If we agree in calling \( F_1 \) an upper fan, then all further fans of an overlapping fan graph can be classified into upper or lower (opposite-sided from upper) — see Figure 1.

Notice that, in general, the description of a graph as an overlapping fan graph, may be done using different paths \( P \) (see the examples in the center and right of Figure 1). The path \( P \) used to describe \( G \) as an overlapping fan graph is called the spine of \( G \).

In an overlapping fan graph, \( |F'_i \cap F'_{i+3}| = 0 \); while \( |F'_{i-1} \cap F'_{i+1}| \leq 1 \), with equality if and only if \( k_i = 1 \). Notice that we can always choose the spine \( P \) so that \( |F'_{i-1} \cap F'_{i+1}| = 0 \) by swapping the names of vertices \( x_i \) and \( v_{i,1} \), changing the fans \( F_{i-1}, F_i, \) and \( F_{i+1} \) from being same-sided to \( F_i \) being opposite-sided from \( F_{i-1} \) and \( F_{i+1} \). Such a choice of path \( P \) is called pleasant (see Figure 1).

For an internal vertex \( x_i \) of the spine of an overlapping fan graph \( G \), we let \( A_{x_i} \) be the set of vertices of \( F'_i \) that are in no other fan of \( G \). If \( A_{x_i} = \emptyset \) then \( k_i = 1 \) and \( v_{i,1} \in F'_{i-1} \) or \( F'_{i+1} \); or \( k_i = 2 \) and \( v_{i,1} \in F'_{i-1} \) and \( v_{i,2} \in F'_{i+1} \). In the former let \( e_{x_i} \) be the edge \( x_{i-1}v_{i,1} \) or \( v_{i,1}x_{i+1} \) respectively, and in the latter let \( e_{x_i} = \{v_{i,1}, v_{i,2}\} \). The following fact will be used in Section 5.2.
Claim 1 If $A_{x_i}$ is empty (non empty) then $G - e_{x_i}$ $(G - A_{x_i})$ is the union of two overlapping fan graphs each one with $x_i$ as simplicial vertex and no other vertex in common.

A 2-path of diameter 1 is just a path on two vertices. In this case, its spine is the graph itself. For larger diameter we have the following lemma.

Lemma 2 A graph $G$ of $\text{diam}(G) \geq 2$ is a 2-path if and only if it is an overlapping fan graph.

Proof. An overlapping fan graph is certainly a 2-path.

Let $G$ be a 2-path with simplicial vertices $r$ and $s$ and diameter at least 2. The 2-path on 4 vertices is a fan, and hence an overlapping fan graph, so we assume that $G$ has at least 5 vertices. Let $G' = G - s$, with simplicial vertices $r$ and $s'$. Since $G'$ is a 2-path, by induction it is also an overlapping fan graph.

If $\text{diam}(G) > \text{diam}(G')$ then the inclusion of $s$ creates a new fan centered on $s'$. Otherwise, the inclusion of $s$ extends the last fan of $G'$. In both cases, then, $G$ is an overlapping fan graph. \qed
Recall that if $S$ is a set of vertices of $G$ then $G - S$ denote the subgraph of $G$ induced by $V(G) - S$. In an analogous way, if $F$ is a subgraph, we let $G - F$ denote the subgraph of $G$ induced by $V(G) - V(F)$.

With respect to pebbling configurations, we define an empty vertex (or zero) to be a vertex with no pebbles on it. A big vertex has at least two pebbles on it; of course, in an $r$-unsolvable configuration, every path from a big vertex to the root $r$ must contain at least one zero. A huge vertex $v$ has at least $2^{\text{dist}(v, r)}$ pebbles on it; of course, no $r$-unsolvable configuration has a huge vertex. The cost of a pebbling solution $\sigma$ is the number of pebbles lost during the pebbling steps of $\sigma$, plus one for the pebble that reaches $r$ — we denote this by $\text{cost}(\sigma)$. A cheap $r$-solution is an $r$-solution of cost at most $2^{\text{ecc}(r)}$, where $\text{ecc}(r) = \text{ecc}_G(r)$ is the eccentricity of $r$ in $G$.

The $t$-pebbling number $\pi_t(G)$ is the minimum number $s$ so that every configuration of size $s$ is $t$-fold solvable (i.e., can place $t$ pebbles on any root). The $t$-pebbling number is related to the fractional pebbling number, which measures the limiting average cost of repeated solutions; i.e. $\lim_{t \to \infty} \pi_t(G)/t$. It is also used as a powerful inductive tool for computing the pebbling number. The following theorem was proven in [10].

**Theorem 3** [10] If $G$ is a graph of diameter 2 then $\pi_t(G) \leq \pi(G) + 4t - 4$.

In what follows we outline the key lemmas and ideas of our proof of the pebbling number for semi-2-trees. In Section 3 we introduce the Cheap Lemma, a powerful mechanism used in tandem with $t$-pebbling techniques. Section 4 is devoted to 2-paths, which form the base step of our induction argument for semi-2-trees in Section 5. We finish with various remarks for further progress in Section 6.

### 3 The Cheap Lemma

We begin by introducing the Cheap Lemma, which we believe is a useful tool of independent interest. First we develop a general framework for some key ideas.
Fix a root $r$ in a graph $G$. We say that a pebbling step from $u$ to $v$ is greedy if $\text{dist}(v, r) < \text{dist}(u, r)$. Furthermore, an $r$-solution $\sigma$ is greedy if each of its pebbling steps is greedy, and a configuration $C$ is greedy if it has a greedy $r$-solution. Finally, $G$ is greedy if every configuration of size at least $\pi(G, r)$ is greedy. (If $r$ needs to be specified, we’ll use the term $r$-greedy.)

Given $\sigma$, let $G_\sigma$ denote the subgraph of edges of $G$ that are traversed by the pebbling steps of $\sigma$, oriented by the direction of travel (bi-directed edges are allowed). We say that $G_\sigma$ is acyclic if it contains no directed cycle. The $r$-solution $\sigma$ is called minimal if no subset of its pebbling steps solves $r$; it is minimum if no $r$-solution uses fewer steps. A well-known lemma of great use is the No-Cycle Lemma of [17].

**Lemma 4 (No-Cycle Lemma)** If $\sigma$ is a minimal $r$-solution of a configuration on $G$ then $G_\sigma$ is acyclic.

Because of the No-Cycle Lemma, we see that every tree is greedy. In particular, if $T$ is a breadth-first-search spanning tree of $G$, rooted at $r$, then $T$ is an example of an $r$-greedy spanning subgraph of $G$ preserving distances to $r$. Hence any configuration of size at least $\pi(T, r)$ on $G$ has a greedy solution. Indeed, more can be said. Our main point will be that minimal greedy solutions are cheap, which we will show by using weight functions. We say that a configuration is cheap if it has a cheap solution.

**Lemma 5 (Cheap Lemma)** Given a graph $G$ with root $r$, let $G^*$ be an $r$-greedy spanning subgraph of $G$ preserving distances to $r$. Then any configuration on $G$ of size at least $\pi(G^*, r)$ is cheap.

**Proof.** For a vertex $v$ define the weight function $w(v) = 2^{-\text{dist}(v, r)}$; let the weight of a configuration $C$ be $w(C) = \sum_v C(v)w(v)$. Note that the configuration with a single pebble on $r$ has weight 1.
Suppose that $C$ is a configuration on $G$ of size at least $\pi(G^*, r)$. Let $\sigma$ be a minimal greedy $r$-solution from $C$. Denote by $C_\sigma$ the configuration on $G^*$ using only the pebbles of $C$ that are used by $\sigma$. Then $\text{cost}(\sigma) = |C_\sigma|$.

For any configuration $C'$, let $C''$ be a configuration that results from making one greedy pebbling step. Then $w(C') = w(C'')$. Applied iteratively to $C_\sigma$, this means that $w(C_\sigma) = 1$.

Now $w(C_\sigma) = \sum_v C_\sigma(v)w(v) \geq \sum_v C_\sigma(v)2^{-\text{ecc}(r)}$, and so $\text{cost}(\sigma) = |C_\sigma| = \sum_v C_\sigma(v) \leq 2^{\text{ecc}(r)}w(C_\sigma) = 2^{\text{ecc}(r)}$. □

The pebbling number for a rooted tree $(T, r)$ was first derived in [7], using the notion of its maximum $r$-path partition $P$. One can compute such a thing iteratively as follows. Beginning with $F = T$, $W = \{r\}$, and $P = \emptyset$, we choose a longest path $P$ in $F$ having one endpoint in $W$. Then we add $P$ to $P$, add its vertices to $W$, remove its edges from $F$, and repeat.

**Theorem 6** [7] Let $P = \{P_1, \ldots, P_k\}$ be a maximum $r$-path partition of a rooted tree $(T, r)$, with each $P_i$ having length (number of edges) $a_i$. (By construction, $a_i \geq a_{i+1}$ for $1 \leq i < k$.) Then $\pi_t(T, r) = (t2^{a_1} - 1) + \sum_{i=2}^{k}(2^{a_i} - 1) + 1 = t2^{a_1} + \sum_{i=2}^{k}2^{a_i} - k + 1$.

The pebbling number $\pi_t(T)$ is given by choosing $r$ to be a leaf of a longest path of $T$. We say that a configuration $C$ is $t$-extremal for a rooted tree $(T, r)$ if the following holds. Let $P = \{P_1, \ldots, P_k\}$ be a maximum $r$-path partition of $(T, r)$ with each $P_i$ having leaf endpoint $v_i$. Then $C(v_1) = t2^{a_1} - 1$, $C(v_i) = 2^{a_i} - 1$ for $2 \leq i \leq k$, and $C(v) = 0$ otherwise. The proof of the lower bound in Theorem 6 involves showing (by induction) that such a configuration is not $t$-fold $r$-solvable.

For a 2-path $G$ with simplicial root $r$, we denote by $T^*(G, r)$ any spanning tree of $G$, rooted at $r$, that includes the spine of $G$ and all fan vertices as leaves, each one adjacent to its neighbor in the spine closest to $r$. Notice that $T^*(G, r)$ is an $r$-greedy spanning subgraph of $G$ preserving distances to $r$. 

For a 2-path $G$ with simplicial vertex $r$, root eccentricity $d$, and with $n$ vertices, we define the functions $p_t(G, r) = t2^d + n - 2d$ (suppressing $t$ when $t = 1$) and $q(G, r) = 2^d + n - d - 1$. Note that $p(G, r) < q(G, r) < p_2(G, r)$ when $1 < d$.

**Corollary 7** Let $G$ be a 2-path with simplicial vertex $r$ and diameter $d$. If $C$ is a configuration of size at least $q(G, r) + (t - 1)2^d$ then $C$ has $t$ distinct cheap $r$-solutions.

**Proof.** For $t = 1$ this follows from the Cheap Lemma 5 and Theorem 6 because for $T = T^*(G, r)$ we have $\pi(T^*, r) = q(G, r)$. The general statement follows by induction on $t$. □

The following two lemmas about pebbling in trees will be used in Section 5.2.

**Lemma 8** Let $T$ be a tree with diameter $d = \text{diam}(T)$, $r^*$ and $r$ be vertices with $\text{ecc}(r) < \text{ecc}(r^*) = d$. Let $P^*$ be a path $v_0v_1\cdots v_d$ with $v_0 = r^*$ and $v_d = s^*$, labeled so that $\text{dist}(r, s^*) \leq \text{dist}(r, r^*) = \text{ecc}(r)$. Denote by $P$ the path from $r$ to $r^*$, and set $P^* \cap P = v_0\cdots v_h$. Define $h = d - h'$. Then $\pi_t(T, r) \leq \pi_t(T, r^*) - t(2^d - 2^{\text{ecc}(r)}) + 2h - 1 \leq \pi_t(T, r^*) - 2^d - 2^{d-2}$.

**Proof.** Let $P^*$ be a maximum path partition of $T$ with root $r^*$. Define $P_0^* = P^*$, $P_1^*$, \ldots, $P_k^*$ to be the sequence of paths of $P^*$ that are used sequentially while traveling from $r^*$ to $r$ in $P$, and set $d_i^* = \text{length}(P_i^*)$ for each $0 \leq i \leq k$ (so $d_0^* = d$). Next define $P'_i = P \cap P_i^*$, with $h'_i = \text{length}(P'_i)$ and $h_i = d_i^* - h'_i$ (so $h'_0 = h'$ and $h_0 = h$). Notice that $\text{ecc}(r) = \sum_{i=0}^{k} h'_i$ and $h \leq d/2$.

Denote by $\mathcal{P}$ the maximum path partition of $T$ with $r$ as root. We will use the following facts in the calculations below.

- The longest path in $\mathcal{P}$ is $P$.

- In the component of the tree $T - P$ that contains the path $P_i^* = P_i^* - P'_i$, the longest path is $\hat{P}_i$. 


From these it follows that each $\hat{P}_i \in \mathcal{P}$ and, subsequently, that $\mathcal{P}^* - \{ P_0^*, \ldots, P_k^* \} = \mathcal{P} - \{ P, \hat{P}_0, \ldots, \hat{P}_k \}$. Now, by converting $\mathcal{P}^*$ to $\mathcal{P}$, we find that
\[
\pi_t(T, r) = \pi_t(T, r^*) - \left( t2^d - 1 + \sum_{i=1}^{k} (2^{d_i} - 1) \right) + \left( t2^{ecc(r)} - 1 + \sum_{i=0}^{k} (2^{h_i} - 1) \right)
\]
\[
\leq \pi_t(T, r^*) + t2^{ecc(r)} - \left( t2^d - 2^{h_0} \right) - \left[ \sum_{i=1}^{k} (2^{d_i} - 2^{h_i}) \right] - 1
\]
\[
\leq \pi_t(T, r^*) - t2^d - 2^{ecc(r)} + 2^{h_0} - 1
\]
\[
\leq \pi_t(T, r^*) - t2^{d-1} + 2^{\lfloor d/2 \rfloor} - 1
\]
\[
\leq \pi_t(T, r^*) - 2^{d-2} .
\]

\[\square\]

Lemma 9 Let $e = xy$ be a non pendant edge of a tree $T$ and assume that $ecc(x) \geq ecc(y)$. If $T'$ is the tree obtained by subdividing the edge $e$ with a new vertex $r$, then $\pi_t(T', r) = \pi_t(T, x) + 2^a$, where $a$ is the eccentricity of $x$ in the connected component of $T - y$ that contains $x$ (thus, $a + ecc(x) \leq diam(T)$).

Proof. Define $x'$ to be the vertex having $dist_T(x, x') = ecc_T(x)$, and denote the $xx'$-path by $P$. Because $ecc_T(y) \leq ecc_T(x)$ we know that $y \in P$. Let $x''$ be a vertex having $dist_{T-y}(x, x'') = a$, with $xx''$-path $Q$, and note that $a < ecc_T(x)$.

Now observe that $dist_{T'}(r, x') = dist_T(x, x')$ (witnessed by the $rx'$ path $P'$) and $dist_{T'}(r, x'') = dist_T(x, x'') + 1$ (witnessed by the $rx''$ path $Q'$). This means that the only changes from the maximum path partition of $T$ with root $x$ to the maximum path partition of $T'$ with root $r$ are that the longest path $P$ from $x$ in $T$ becomes the longest path $P'$ from $r$ in $T'$, and the longest path $Q$ from $x$ in $T - y$ becomes the longest path $Q'$ from $r$ in $T' - y$. Hence we have $\pi_t(T, x) = t2^{ecc_T(x)} + 2^a + F(x)$, for some $F(x)$, and $\pi_t(T', r) = t2^{ecc_{T'}(r)} + 2^{a+1} + F(x) = \pi_t(T, x) + 2^{a+1} - 2^a$. \[\square\]
4 2-Paths

In this section we calculate a $\pi_t(G,r)$ for $r$ a simplicial vertex of a 2-path $G$.

4.1 The Lower Bound

We now present some general removal techniques for finding lower bounds that may also be of independent interest. For a vertex $v$, define its open neighborhood $N(v)$ to be the set of vertices adjacent to $v$, and its closed neighborhood $N[v] = N(v) \cup \{v\}$. Also, for a set of vertices $A$ write $N(A) = \cup_{v \in A} N(v)$. Along the lines of the definition of twin vertices, for a non-root vertex $y$ we say that $y$ is a junior sibling of $x$ (or, more simply, junior to $x$) if $N(y) \subseteq N[x]$, and that $y$ is a junior if it is junior to some vertex $x$.

Lemma 10 (Junior Removal Lemma) Given the rooted graph $(G,r)$ with configuration $C$, suppose that $y$ is a junior with $C(y) = 0$. Then $C$ is $t$-fold $r$-solvable if and only if $C$ restricted to $G - y$ is $t$-fold $r$-solvable in $G - y$.

Proof. Sufficiency is obvious, so we only prove necessity. Suppose that $\sigma$ is an $r$-solution from $C$ that uses $y$. Let $y$ be junior to some vertex $x$. Construct $\sigma'$ from $\sigma$ by replacing every pebbling step $(u,y)$ with $(u,x)$ and every pebbling step $(y,v)$ with $(x,v)$. Then $\sigma'$ $t$-fold solves $r$ as well. \hfill \Box

We say that a set of vertices $W$ is a wart if it is a component of $G - X$ for some clique cutset $X$, where by clique we mean complete subgraph.

Lemma 11 (Wart Removal Lemma) Given the rooted graph $(G,r)$ with configuration $C$, suppose that $W$ is a wart of $G$ not containing $r$ and that $C(w) \leq 1$ for every $w \in W$. Then $C$ is $t$-fold $r$-solvable if and only if $C$ restricted to $G - W$ is $t$-fold $r$-solvable in $G - W$. 

Proof. Sufficiency is obvious, so we only prove necessity. We show that no minimum \( r \)-solution from \( C \) uses \( W \).

Suppose instead that \( \sigma \) is a minimum \( r \)-solution that uses \( W \). Let \( X \) be a clique cutset that witnesses the wart \( W \), and let \( u \) be a vertex of \( X \) having a pebbling step into \( W \). Because \( \sigma \) is minimum, there is a vertex \( v \in X \) that receives a pebble from \( W \) and that is different from \( u \). By replacing those two pebbling steps by the single step from \( u \) to \( v \) we find an \( r \)-solution with fewer steps, a contradiction. \( \square \)

Let \( G \) be a 2-path with simplicial root \( r \), pleasant path \( P \), and configuration \( C \). For a given \( t \) we say that \( C \) is \( t \)-extremal for \( r \) (simply, extremal if \( t = 1 \)) if there is an \( I \)-saturating matching \( M \) from the internal spine vertices \( I = \{ x_1, \ldots, x_{d-1} \} \) to the fan vertices \( \{ v_{i,j} \} \) such that \( C(x_d) = t2^d - 1 \), \( C(r) = 0 \), \( C(M) = 0 \), and \( C(v) = 1 \) otherwise. Notice that \( |C| = p_t(G, r) - 1 \).

If a configuration \( C \) on \( G \) is \( t \)-fold \( r \)-solvable if and only if \( C_H \) is \( t \)-fold \( r \)-solvable on the subgraph \( H \subset G \), then we say that \( G \) \( t \)-fold \( r \)-reduces to \( H \) for \( C \). If \( C, t \) and \( r \) are clear from the context we just write reduces.

Lemma 12 (Extremal Lemma) If \( C \) is \( t \)-extremal for the simplicial root \( r \) of a 2-path \( G \) then \( C \) is not \( t \)-fold \( r \)-solvable. Moreover, by using Lemmas 10 and 11 (repeatedly removing juniors and warts) \( G \) reduces to its spine, the path \( P_d \), where \( d = \text{diam}(G) \).

Proof. We use induction on \( d \). The result is trivial for \( d = 1 \). For \( d > 1 \) we suppose that \( C \) is \( t \)-fold \( r \)-solvable and let \( \sigma \) be a \( t \)-fold \( r \)-solution. Write \( y_i = v_{i,j_i} \) for the neighbor of \( x_i \) in \( M \) and let \( \ell \) be the smallest index \( i \) such that \( y_i \) is a junior. This exists because if \( y_i \) is not a junior then either \( y \in F_{i-1} \cap F_i \) or \( y \in F_i \cap F_{i+1} \) (it is a fan intersection), and there are more fans than fan intersections. Set \( y = y_\ell \) and \( x = x_\ell \). Then \( y \) is junior to \( x \) and so, by Lemma 10, \( C \) is \( t \)-fold \( r \)-solvable in \( G - y \).

Furthermore, let \( j^+ \) be the maximum \( j \) such that \( v_{\ell,j} \in F_\ell - F_{\ell+1} \). If \( j_\ell + 1 \leq j^+ \) then \( \{ v_{\ell,j_\ell+1} \} \) is a wart in \( G - y \), and so Lemma 11 says that we can remove it. Once we
do, \({v_{\ell,j\ell+2}}\) becomes a wart, and so on, until all the vertices \(v_{\ell,j}\) with \(j_\ell < j \leq j^+\) have been removed. Then the graph \(G_{\ell+1} = \bigcup_{i > \ell} F_i\) is a 2-path, with the restriction, \(C_{\ell+1}\), of \(C\) to \(G_{\ell+1}\) being \(2^t\)-extremal for \(x_\ell\). By induction, \(C_{\ell+1}\) is not \(2^t\)-fold \(x_\ell\)-solvable and \(G_{\ell+1}\) can be reduced to the path \(P_{d-\ell}\).

Similarly, let \(j^-\) be the minimum \(j\) such that \(v_{\ell,j}\) is a wart for \(j^- \leq j < j_\ell\). If \(j_\ell - 1 \geq j^-\) then the warts \(\{v_{\ell,j}\}\) for \(j^- \leq j < j_\ell\) can be successively removed, leaving the 2-path \(G^\ell = \bigcup_{i \leq \ell} F_i\). Since \(C\) is \(t\)-fold \(r\)-solvable and \(x_\ell\) is a cut-vertex of \(G - y\), all the pebbles of \(G_{\ell+1}\) used by \(\sigma\) must pass through \(x_\ell\). But because \(C_{\ell+1}\) is not \(2^t\)-fold \(x_\ell\)-solvable, the most number of pebbles that can reach \(x_\ell\) is \(2^t - 1\). After placing as many pebbles as possible on \(x_\ell\) from \(G_{\ell+1}\), the resulting configuration \(C^\ell\) is a subconfiguration of a configuration \(\hat{C}^\ell\) that is \(t\)-extremal for \(r\) on \(G^\ell\). By induction, \(\hat{C}^\ell\) is not \(t\)-fold \(r\)-solvable, a contradiction. Also, \(G^\ell\) can be reduced to the path \(P_{\ell}\), which reduces \(G\) to the path \(P_d\). \(\square\)

**Corollary 13** If \(r\) is a simplicial vertex of a 2-path \(G\) then \(\pi_t(G,r) \geq p_t(G,r)\). \(\square\)

### 4.2 The Upper Bound

We first note that a diameter two 2-path \(G\) is Class 0. Indeed, the following lemma is a corollary of the Class 0 characterization for diameter two graphs from [8] that shows that \(\pi(G) = n\) in this case and the \(t\)-pebbling bound of [10] that states \(\pi_t(G) \leq \pi(G) + 4t - 4\) for all diameter two graphs. Equality comes from Corollary 13. The diameter one case is from [10] also.

**Lemma 14** [10] If \(G\) is a 2-path on \(n\) vertices with diameter \(d \leq 2\) then \(\pi_t(G) = t2^d + n - 2d\).

**Theorem 15** Let \(G\) be a 2-path on \(n\) vertices with simplicial root vertex \(r\) having eccentricity \(d\), and configuration \(C\). If \(|C| \geq p(G,r)\) then \(C\) is \(r\)-solvable.
Proof. When \( d \leq 2 \), the result is taken care of by Lemma 14. So we will assume that \( d > 2 \) and use induction. Suppose that \( |C| = p(G, r) \) and let \( P = r, x_1, \ldots, x_{d-1}, s \) be a pleasant shortest \( rs \)-path between the two simplicial vertices of \( G \). Write \( x_0 = r \) and \( x_d = s \) and label \( G \) by its fan graph labeling, so that \( V(F_i) = \{x_{i-1}, x_i, x_{i+1}, v_{i,1}, \ldots, v_{i,k_i}\} \) and \( Q_i \) is the path \( x_{i-1}, v_{i,1}, \ldots, v_{i,k_i}, x_{i+1} \). Let \( G' \) be the restriction of \( G \) to the \( n' \) vertices of \( \cup_{i \geq 2} V(F_i) \), with \( C' \) denoting the restriction of \( C \) to \( G' \). We further use the abbreviations \( C_1 = C(F_1) \) and \( n_1 = |V(F_1)| \). Notice that \( \text{diam}(G') = d - 1 \), so that the Theorem holds for \( G' \). Define \( \phi = 1 \) (0) if \( F_2 \) is same-sided (opposite-sided) as \( F_1 \).

If \( C(x_1) \geq 1, C(x_2) \geq 2, \) or \( C(v_{1,j}) \geq 2 \) for some \( j \) (either \( \phi = 1 \) and \( j = k_1 \) or not), then we can place a pebble on \( x_1 \). If \( |C'| - (1, 2, 2, 0) \geq p(G', x_1) \), where the coordinates correspond, in order, to the four cases above (first two cases plus two sub-cases of the third case), then we can place another pebble on \( x_1 \), and then one on \( r \). Otherwise, \( |C'| - (1, 2, 2, 0) \leq p(G', x_1) - 1 \). That is, \( |C'| \leq [2^{d-1} + n' - 2(d - 1)] + (0, 1, 1, -1) \). Thus \( |C_1| \geq |C| - |C'| + (1, 2, 2, 0) \geq 2^{d-1} + (n_1 - 2 - \phi) - 2 + (1, 1, 1, 1) = n_1 + (2^{d-1} - 3 - \phi) \geq n_1 \), which means by Lemma 14 that we can solve \( r \).

On the other hand, if \( C(x_1) = 0, C(x_2) \leq 1, \) and \( C(v_{1,j}) \leq 1 \) for all \( j \), then \( C(\{r, v_{1,1}, \ldots, v_{1,k_1-1}\}) \leq k_1 - 1 \). Here we define \( \theta \) to be the number of zeros in \( \{v_{1,1}, \ldots, v_{1,k_1}, x_2\} \), so that \( |C_1| = n_1 - 2 - \theta \), and set \( \theta' \) to be the number of those zeros other than \( x_2 \) (i.e. \( \theta - \theta' = 1 - C(x_2) \)). Now we have

\[
|C'| \geq |C| - |C_1| + C(x_2) \\
= (2^d + n - 2d) - (n_1 - 2 - \theta) + C(x_2) \\
= (2)2^{d-1} + (n' - 2 - \phi) - 2d + 2 + \theta + C(x_2) \\
= [(2)2^d + n' - 2d'] + [C(x_2) + \theta - 2 - \phi] \\
= p_2(G', x_1) + [\theta' - 1 - \phi].
\]
If $\theta' - 1 - \phi \geq 0$ then $|C'| \geq p_2(G', x_1) > q(G', x_1)$, which means, by Corollary 7, that we can place one pebble on $x_1$ cheaply. Because the remaining configuration (after solving $x_1$ cheaply) has at least $p_2(G', x_1) - 2^d = p(G', x_1)$ pebbles, induction places a second pebble on $x_1$. Then we move one to $r$.

Otherwise, we have $\theta' - 1 - \phi < 0$, which means that $\theta' \leq \phi$. If $\theta' = 0$, that is $C(v_{1,j}) = 1$ for all $j$, then we will show that it is possible to place two pebbles on $x_2$, from which we solve $r$ by moving pebbles from $x_2$ along $Q_1$. Indeed, this is so if $C(x_2) = 1$ and $Q_2$ has a big vertex, or if $Q_2$ contains either a vertex with four pebbles or two big vertices, so we assume otherwise. In this case, we have $|C((F_1 \cup F_2) - G'')| \leq |V((F_1 \cup F_2) - G'')|$, where $G''$ is the restriction of $G$ to the $n''$ vertices of $\cup_{i \geq 3} V(F_i)$. For the restriction $C''$ of $C$ to $G''$, this implies that

$$|C''| = |C| - |C((F_1 \cup F_2) - G'')|$$

$$\geq 2^d + n'' - 2d$$

$$= [(2)2^{d-2} + n'' - 2(d - 2)] + [2^{d-1} - 4]$$

$$\geq p_2(G'', x_2) ,$$

since $d \geq 3$. As before, since $p_2(G'', x_2) > q(G'', x_2)$ and $p_2(G'', x_2) - 2^{d''} = p(G'', x_2)$, we can place one pebble on $x_2$ cheaply, followed by a second pebble on $x_2$.

We are left now with the final case (since $\theta' \leq \phi \leq 1$) in which $\theta' = 1$ (exactly one $v_{1,j}$ is empty), which means that $\phi = 1$ ($F_1$ and $F_2$ are same-sided, so that $v_{1,k_1} = v_{2,1}$).

If $v_{1,k_1}$ is not empty then $k_1 \geq 2$, and so

$$|C'| = |C| - (k_1 - 2)$$

$$= (2)2^{d-1} + (n - k_1) - 2(d - 1)$$

$$= p_2(G', x_1)$$

$$> q(G', x_1) .$$
As above, this means, by Corollary 7 and induction, that we can place two pebbles on \( x_1 \), and hence one on \( r \).

If instead \( v^1 \) is empty then set \( \hat{G} = G' - x_1 \) and \( \hat{C} = C(\hat{G}) \), so that

\[
|\hat{C}| = |C| - (k_1 - 1)
\]
\[
= (2)2^{d-1} + (n - k_1 - 1) - 2(d - 1)
\]
\[
= p_2(\hat{G}, v_{1,k_1}) .
\]

Again, this means that we can place two pebbles on \( v^1, k_1 \) and hence one on \( r \) (via \( Q_1 \)).

This completes the proof. \( \square \)

**Corollary 16** If \( r \) is a simplicial vertex of a 2-path \( G \) then \( \pi_t(G,r) = p_t(G,r) \).

**Proof.** The lower bound was stated in Corollary 13. The upper bound for \( t = 1 \) follows from Theorem 15. If \( t > 1 \), then for any configuration \( C \) of size \( p_t(G,r) = p_2(G,r) + (t - 2)2^d > q(G,r) + (t - 2)2^d \), we can place \( t - 1 \) pebbles on \( r \), each cheaply, by Corollary 7. The remaining configuration has at least \( p_t(G,r) - (t - 1)2^d = p(G,r) \) pebbles, from which we can place the \( t \)th pebble on \( r \) by Theorem 15. \( \square \)

## 5 Pebbling number of Semi-2-Trees

We define the **skeleton** \( T \) of a semi-2-tree \( G \) to be the union of the spines of its blocks; it is a geodesic tree spanning all of the simplicial vertices of \( G \). Let \( e(T) \) denote the number of edges of \( T \), \( b(G) \) denote the number of blocks of \( G \), and for a simplicial vertex or cut-vertex \( r \) and positive integer \( t \) define \( p_t(G,r) = \pi_t(T,r) + (n - 1) + b(G) - 2e(T) \) (suppressing \( t \) when \( t = 1 \)). Notice that this matches the corresponding formula for 2-paths because \( b = 1 \) and \( T \) is a path. In addition, we have \( p_t(G,r) = p_t(G,r) + 2^{\text{ecc}(r)} \) because of Theorem 6. We also define \( q(G,r) = \pi(T,r) + n - e(T) - 1 \); note that
\( q(G, r) = \pi(T^*, r) \), where \( T^* \) is a spanning tree of \( G \), rooted at \( r \), that contains its skeleton and all its fan vertices as leaves, each one adjacent to its neighbor in the skeleton closest to \( r \). Notice that \( T^* \) is an \( r \)-greedy spanning tree of \( G \) preserving distances to \( r \).

### 5.1 Simplicial or Cut-vertex Roots

We begin with another consequence of the Cheap Lemma, generalizing Corollary 7. The proof is similar and is left to the reader.

**Corollary 17** Let \( r \) be a simplicial vertex or cut-vertex with eccentricity \( d \) of a semi-2-tree \( G \). If \( C \) is a configuration of size at least \( q(G, r) + (t - 1)2^d \) then \( C \) has \( t \) distinct cheap \( r \)-solutions. □

For a tree \( T \) with maximum \( r \)-path partition \( \mathcal{P} = \{P_1, \ldots, P_k\} \), each \( P_i \) having length \( a_i \) (sorted so that \( a_i \geq a_{i+1} \)), let \( C_T \) be its \( t \)-extremal configuration for \( r \).

For a semi-2-tree \( G \), call a vertex of the skeleton \( T \) *internal* if it is not a simplicial vertex or cut-vertex, and let \( M \) be any \( I \)-saturating matching from the internal vertices \( I \) to the fan vertices of \( G \). For a simplicial or cut vertex \( r \) of \( G \), define the configuration \( C \) by \( C(T) = C_T, C(M) = 0, \) and \( C(v) = 1 \) otherwise — such a configuration we call \( t \)-extremal for \( r \). Note that \( |C| = p_t(G, r) - 1 \).

As in the proof of the Extremal Lemma 12, we can use the Removal Lemmas 10 and 11 to prove that \( G \) reduces to \( T \) for \( C \) and obtain the following more general extremal lemma, which we leave to the reader.

**Lemma 18** If \( C \) is \( t \)-extremal for the simplicial or cut-vertex root \( r \) of a semi-2-tree \( G \) then \( C \) is not \( t \)-fold \( r \)-solvable. Moreover, by using Lemmas 10 and 11 \( G \) can be reduced to its skeleton \( T \). □

Now we state and prove the solvability theorem in this case.
Theorem 19 Let $G$ be a semi-2-tree on $n$ vertices with simplicial or cut-vertex $r$ and configuration $C$. If $|C| \geq p(G, r)$ then $C$ is $r$-solvable.

Proof.

We use induction on $n$ with base case $\text{ecc}(r) = 1$, which is handled by Theorem 3. So we assume that $\text{ecc}(r) > 1$. We may also assume that $C(r) = 0$. We consider two cases.

1. $r$ is a cut-vertex.

Let $H_1, \ldots, H_k$ be the components of $G - r$, with $G_i$ induced by $V(H_i) \cup \{r\}$; then each $G_i$ is a semi-2-tree, so that the theorem holds for them by induction. Let $C_i$ and $T_i$ be the restrictions of $C$ and $T$ to $G_i$, with $n_i$ and $b_i$ counting the number of vertices and blocks of $G_i$. If some $|C_i| \geq p(G_i, r)$ then $C_i$ solves $r$, so we assume not. Then

$$|C| = \sum_i |C_i|$$
$$\leq \sum_i [p(G_i, r) - 1]$$
$$= \sum_i [\pi(T_i, r) + (n_i - 1) + b_i - 2e(T_i) - 1]$$
$$= \pi(T, r) + (n - 1) + b(G) - 2e(T) - k$$
$$< p(G, r),$$

a contradiction. Hence some $C_i$ solves $r$.

2. $r$ is a simplicial vertex.

Let $H$ be the block of $G$ containing $r$, with $r'$ the other simplicial vertex of $H$. If $|C(H)| \geq p(H, r)$ then we solve $r$ directly on $H$. Otherwise, we assume that $|C(H)| = p(H, r) - s$ for some $s > 0$. Recall that $p(H, r) = 2^{d_H} + n_H - 2d_H$, where $n_H = |H|$ and $d_H = \text{ecc}_H(r)$. Let $G'$ be the subgraph of $G$ induced by
Suppose that $s \leq 2^{d_H}$. Then
\[
|C'| = |C| - |C(H)| \\
= p(G, r) - p(H, r) + s \\
= \left[\pi(T, r) + (n - 1) + b(G) - 2e(T)\right] - \left[2^{d_H} + n_H - 2d_H\right] + s \\
= \left[\pi_s(T', r') + (n' - 1) + b' - 2e(T')\right] + \left[s - 2^{d_H} + 2 - s2^d\right] \\
= p_s(G', r') + (2^d - 1)(2^{d_H} - s) \\
\geq p_s(G', r') ,
\]
which means that we can place $s$ pebbles on $r'$, so that now there are $p(H, r)$ pebbles in $H$, enough to solve $r$.

Suppose that $s \geq 2^{d_H}$; i.e. $|C(H)| \leq n_H - 2d_H$. Then
\[
|C'| = |C| - |C(H)| \\
\geq \left[\pi(T, r) + (n - 1) + b(G) - 2e(T)\right] - \left[n_H - 2d_H\right] \\
= \left[\pi_{2^{d_H}}(T', r') + (n' - 1) + b' - 2e(T')\right] \\
\geq p_{2^{d_H}}(G', r') ,
\]
which means that we can place $2^{d_H}$ pebbles on $r'$, enough to solve $r$ on $T$.

\[\square\]

**Corollary 20** If $r$ is a simplicial vertex or cut-vertex of a semi-2-tree $G$ then $\pi_t(G, r) = p_t(G, r)$.

**Proof.** As in the proof of Corollary 16 \[\square\]
Theorem 21 If \( r \) is a simplicial vertex or cut-vertex of a semi-2-tree \( G \) and \( r^* \) is a simplicial vertex with \( \text{ecc}(r^*) = \text{diam}(G) \) then \( \pi_t(G, r) \leq \pi_t(G, r^*) \).

Proof. Let \( T \) be a skeleton of \( G \). Because the only term in \( p_t(G, r) = \pi_t(T, r) + (n - 1) + b(G) - 2e(T) \) that depends on \( r \) is \( \pi_t(T, r) \), it follows that \( \pi_t(G, r) \) is maximized precisely where \( \pi_t(T, r) \) is maximized, which is well-known ([7]) to be at \( r^* \).

\[ \square \]

5.2 Other Roots

We begin with two more removal lemmas of general use.

Lemma 22 (Edge Removal Lemma) Let \( r \) be a vertex of a connected graph \( G \) and suppose \( e \) is an edge between two neighbors of \( r \). Then \( \pi(G, r) = \pi(G - e, r) \).

Proof. Given any configuration on \( V(G) = V(G - e) \), every minimal \( r \)-solution in one graph is a minimal solution in the other.

\[ \square \]

Lemma 23 Let \( r \) be a cut-vertex of a graph \( G \), and denote the connected components of \( G - r \) by \( H_1, \ldots, H_k \). For each \( i \) define the graph \( G_i \) induced by \( H_i \cup \{ r \} \). Then \( \pi(G, r) = 1 + \sum_i (\pi(G_i, r) - 1) \).

Proof. The lower bound follows from the union of the individual maximum-sized \( r \)-unsolvable configurations on \( H_i \). The upper bound follows from the pigeonhole principle.

\[ \square \]

Lemma 24 (Neighbor Removal Lemma) Let \( r \) be a vertex of a connected graph \( G \). Suppose that \( A \subseteq N(r) \) such that \( N(A) \subseteq N[r] \). Let \( \{H_1, \ldots, H_k\} \) be the connected components of \( (G - r) - A \) and denote by \( G_i \) the subgraph of \( G \) induced by \( V(H_i) \cup \{ r \} \). Then \( \pi(G, r) = 1 + |A| + \sum_i (\pi(G_i, r) - 1) = |A| + \pi(G - A, r) \).
Figure 2: A non-semi-2-tree $G$ for which $G - x$ is a semi-2-tree. The configuration $C(r, x, y, z) = (0, 1, 0, 3)$ is extremal for $r$.

**Proof.** We can remove the edges incident with $A$ by Lemma 22. Then each $v \in A$ is its own component of $G - r$. The result follows from Lemma 23. □

Under the conditions of Lemma 24 if each $(G_i, r)$ is a rooted semi-2-tree, then we say that a configuration $C$ on $G$ is extremal for $r$ if $C(x) = 1$ for every $x \in A$ and each $C_{G_i}$ is extremal for $r$ on $G_i$.

A small example of a non-semi-2-tree to which Lemma 24 applies is shown in Figure 2. This idea is used later in the proof of Corollary 32.

A simple consequence (using the Cheap Lemma and induction) of Lemma 22 is the following.

**Corollary 25** Let $G$ be a semi-2-tree with skeleton $T$, and suppose that $r$ is a vertex of $T$ that is not a simplicial or cut vertex of $G$. Let $A_r$ be the set of vertices of the fan centered on $r$ that are in no other fan of $G$. If $A_r$ is empty and $e_r$ is as defined on Claim 7, then $\pi_t(G, r) = \pi_t(G - e_r, r)$ for all $t \geq 1$. □

Notice that the previous corollary allows one to calculate the pebbling number for $r$. In fact, by Claim 1 $G - e_r$ is a semi-2-tree with $r$ a simplicial or cut vertex, then we use Corollary 20 to calculate $\pi(G, r) = \pi(G - e_r, r)$.

Anagously, a consequence (using the Cheap Lemma and induction) of Lemma 24 is the following.
Corollary 26  Let $G$ be a semi-2-tree with skeleton $T$, and suppose that $r$ is a vertex of $T$ that is not a simplicial or cut vertex of $G$. Let $A_r$ be the set of vertices of the fan centered on $r$ that are in no other fan of $G$. If $A_r$ is non empty then $\pi_t(G, r) = \pi_t(G-A_r, r) + |A_r|$ for all $t \geq 1$.

Notice that the previous corollary allows one to calculate the pebbling number for $r$. In fact, by Claim 1, $G - A_r$ is a semi-2-tree with $r$ a simplicial or cut vertex, then we use Corollary 20 to calculate $\pi(G, r) = \pi(G - A_r, r)$.

Theorem 27  Let $G$ be a semi-2-tree with skeleton $T$. Suppose that $r$ is a vertex of $T$ that is not a simplicial or cut vertex of $G$, and let $r^*$ be a simplicial vertex of $G$ with $\text{ecc}(r^*) = \text{diam}(G)$. Then $\pi_t(G, r) < \pi_t(T, r^*)$.

Proof.  Let $A_r$ be the set of vertices of the fan centered on $r$ that are in no other fan of $G$. First assume $A_r = \emptyset$ and let $e_r$ be as in Corollary 25. Notice that the skeleton of $G - e_r$ is the same $T$, while $G - e_r$ has one block less than $G$, thus (using Corollary 25 and Corollary 20)

\[
\begin{align*}
\pi_t(G, r) & = \pi_t(G - e_r, r) \\
& = p_t(G - e_r, r) \\
& = \pi_t(T, r) + (n(G - e_r) - 1) + b(G - e_r) - 2e(T) \\
& \\n& = \pi_t(T, r) + (n(G) - 1) + (b(G) - 1) - 2e(T) \\
& \\n& \leq \pi_t(T, r^*) + (n(G) - 1) + b(G) - 2e(T) - 1 \\
& = \pi_t(G, r^*) - 1 \\
& < \pi_t(G, r^*) .
\end{align*}
\]

Analogously, if $A_r \neq \emptyset$ then (using Corollary 26)
\[ \pi_t(G, r) = \pi_t(G - A_r, r) + |A_r| \]
\[ = p_t(G - A_r, r) + |A_r| \]
\[ = \pi_t(T, r) + (n(G) - A_r) - 1 + b(G - A_r) - 2e(T) + |A_r| \]
\[ = \pi_t(T, r) + (n(G) - 1) + b(G) - 2e(T) - 1 \]
\[ \leq \pi_t(G, r^*) - 1 \]
\[ < \pi_t(G, r^*) . \]

□

Another consequence (again using the Cheap Lemma and induction) of Lemma 22 is the following. We say that a vertex \( r \) is not in any skeleton of \( G \) when for every skeleton \( T \) of \( G \), \( r \) is a fan vertex, i.e. \( r \in V(G - T) \).

**Corollary 28** Let \( r \) be a vertex of a semi-2-tree \( G \) that is not in any skeleton of \( G \). If the root \( r \) is in two fans of \( G \), centered on \( x \) and \( y \), with edge \( e = xy \), then \( \pi_t(G, r) = \pi_t(G - e, r) \).

□

Since \( G - e \) is a semi-2-tree with cut-vertex root \( r \), the value of \( \pi_t(G - e, r) \) is computed by Corollary 20.

**Theorem 29** Let \( r \) be a vertex of a semi-2-tree \( G \) that is not in any skeleton of \( G \). Suppose that the root \( r \) is in two fans of \( G \), centered on \( x \) and \( y \), with edge \( e = xy \), labeled so that \( \text{ecc}(x) \geq \text{ecc}(y) \), and let \( r^* \) be a simplicial vertex of \( G \) with \( \text{ecc}(r^*) = \text{diam}(G) \). Then \( \pi_t(G, r) < \pi_t(G, r^*) \).

**Proof.** We note that \( r \) is a cut vertex of \( G - e \), and so \( b(G - e) = b(G) + 1 \). Also, the skeleton \( T' \) of \( G - e \) has one more edge than does \( T \); in fact, \( T' \) can be seen as the tree
obtained from $T$ by subdividing the edge $e$ with the vertex $r$. Furthermore, $r \not\in N(r^*)$, which means that $\text{ecc}_{T'}(r) \leq \text{diam}(T') - 2$. As in Lemma 9, define $a = \text{ecc}_{T-y}(x)$, and set $d = \text{diam}(T)$. Then, by Corollaries 20 and 28, and Lemmas 8 and 9, we have

$$
\pi_t(G, r) = \pi_t(G - e, r)
= \pi_t(T', r) + (n(G - e) - 1) + b(G - e) - 2e(T')
= \pi_t(T', r) + (n(G) - 1) + b(G) + 1 - 2e(T) - 2
= \pi_t(T, x) + 2^a + (n(G) - 1) + b(G) + 1 - 2e(T) - 2
\leq \pi_t(T, x) + 2^{d-\text{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1.
$$

If $\text{ecc}(x) \geq 3$ then we write

$$
\pi_t(G, r) \leq \pi_t(T, x) + 2^{d-\text{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1
\leq \pi_t(T, r^*) - (2^{d-2} - 2^{d-3}) + (n(G) - 1) + b(G) - 2e(T) - 1
< \pi_t(G, r^*).
$$

Otherwise we have $\text{ecc}(x) \leq 2$ and so, with $h'$ defined as in Lemma 8, we find that

$$
\pi_t(G, r) \leq \pi_t(T, x) + 2^{d-\text{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1
\leq \pi_t(T, r^*) - t(2^d - 2^{\text{ecc}(x)}) + 2^{h'} - 1 + 2^{d-\text{ecc}(x)} + (n(G) - 1) + b(G) - 2e(T) - 1
= \pi_t(G, r^*) - t(2^d - 2^{\text{ecc}(x)}) + 2^{h'} - 1 + 2^{d-\text{ecc}(x)}
\leq \pi_t(G, r^*) - (2^d - 2^{d-2} - 2^{[d/2]} + 1)
< \pi_t(G, r^*),
$$

since $d \geq 3$. □

We pause to develop some notation that will be used in Corollary 32. Suppose that $r$ is a fan vertex of $G$, in a unique fan $F$ centered on $x$. Denote by $H_1$ and $H_2$ the two
components of $G - \{r, x\}$, and by $G_i$ the subgraph of $G$ induced by $V(H_i) \cup \{r, x\}$. Let $V_i$ be the vertices of $F_i \cap H_i$ that are not in any other fan. Define $G'_i = G_i - V_i$. Finally, let the subscripts be labeled either so that $V_2$ is empty or so that neither $V_1$ nor $V_2$ is empty and $\text{ecc}_{G_1}(r) \geq \text{ecc}_{G_2}(r)$.

We note that $G_i$ is a semi-2-tree except in the case that the block of $G_i$ containing $r, x$ and their unique common neighbor $y$ is a $K_3$ (as in Figure 2), because $K_3$ is not a 2-path. Observe that this happens if and only if $V_i = \emptyset$ and $y$ is a cut vertex of $G$, and that in such a case $G_i - x$ is a semi-2-tree. Moreover, by the Neighbor Removal Lemma with $A = \{x\}$, $\pi(G_i, r) = \pi(G_i - x, r) + 1$.

**Claim 30** Let $G$ be a semi-2-tree and suppose that $r$ is a fan vertex of $G$, in a unique fan $F$ centered on $x$. Define $G_i$ ($i \in \{1, 2\}$) as above, having $n_i$ vertices. Define $T_i(r)$ to be the skeleton of $G_i$ when $G_i$ is a semi-2-tree and of $G_i - x$ when $G_i$ is not a semi-2-tree. Define $T_i(x)$ to be the skeleton of $G_i - V_i - r$. Then for each $v \in \{r, x\}$ we have $\pi(G_i, v) = \pi(T_i(v), v) + (n_i - 1) + b(G_i) - 2e(T_i(v))$.

**Proof.** For $v = r$ the result is true by Corollary 20 when $G_i$ is a semi-2-tree, because $r$ is simplicial. If $G_i$ is not a semi-2-tree then $|V_i| = 0$ and $\{r, x, y\}$ is a $K_3$ block, where $y$ is the common neighbor of $r$ and $x$. In this case $G_i - x$ is a semi-2-tree, and so $\pi(G_i, r) = \pi(G_i - x, r) + 1$ by Lemma 24. This equals $\pi(T_i(r), r) + ((n_i - 1) - 1) + b(G_i - x) - 2e(T_i(r)) + 1 = \pi(T_i(r), r) + (n_i - 1) + b(G_i) - 2e(T_i(r))$.
For $v = x$ we have that $x$ is a simplicial vertex of the semi-2-tree $G_i - V_i - r$, and so by Lemma 24 we obtain that $\pi(G_i, x) = \pi(G_i - V_i - r, x) + |V_i| + 1 = \pi(T_i(x), x) + (n_i - |V_i| - 1) - 1) + b(G_i - V_i - r) - 2e(T_i(x)) + |V_i| + 1 = \pi(T_i(x), x) + (n_i - 1) + b(G_i - 2e(T_i(x)))$.

\[\Box\]

**Claim 31** Under the same hypotheses as in Claim 30 we have $\pi(G_1, r) + \pi(G_2, x) \geq \pi(G_1, x) + \pi(G_2, r)$.

**Proof.** Because of the cancellation of common terms, we have

\[
[\pi(G_1, r) + \pi(G_2, x)] - [\pi(G_1, x) + \pi(G_2, r)]
= [\pi(T_1(r), r) + \pi(T_2(x), x)] - [\pi(T_1(x), x) + \pi(T_2(r), r)]
= [\pi(T_1(r), r) - \pi(T_1(x), x)] - [\pi(T_2(r), r) - \pi(T_2(x), x)].
\]

Because of the cancellation of common branches, this equals

\[
[2^{ecc_{G_1}(r)} - 2^{ecc_{G_1}(x)}] - [2^{ecc_{G_2}(r)} - 2^{ecc_{G_2}(x)}].
\] (1)

We note that $ecc_{G_i}(x) \leq ecc_{G_i}(r) \leq ecc_{G_i}(x) + 1$ for each $i$, with $ecc_{G_i}(x) = ecc_{G_i}(r)$ precisely when $V_i = \emptyset$. Thus, the choice of labeling ensures that (1) is non-negative. \[\Box\]

**Corollary 32** Let $(G, r)$ be a rooted semi-2-tree with $r$ not in any skeleton of $G$. If $r$ is in a unique fan, centered on $x$, then (using the notation defined above) $\pi(G, r) = \pi(G_1, r) + \pi(G_2, x) - 2$.

**Proof.** The lower bound is argued as follows. Let $C_1$ be an extremal configuration for $r$ on $G_1$, $C_2$ be an extremal configuration for $x$ on $G_2 - r$ (which is defined by using $A = V_2$ in the Neighbor Removal Lemma 24), and define the configuration $C = C_1 + C_2$.

Now $|C_1| = \pi(G_1, r) - 1$ and $|C_2| = \pi(G_2 - r, x) - 1 = \pi(G_2, x) - 2$ (by Lemma 24 with $A = \{r\}$), and thus $|C| = |C_1| + |C_2| = \pi(G_1, r) + \pi(G_2, x) - 3$. Furthermore, we claim
that $C$ is $r$-unsolvable. Indeed, $C_1$ cannot solve $r$ by itself and cannot receive another pebble from $C_2$ through $x$, and $C_2$ (without its pebble already on $r$) cannot solve $r$ by itself (any step to $r$ can be replaced by a step to $x$, which would be a contradiction).

For the upper bound, assume that $|C| = \pi(G_1, r) + \pi(G_2, x) - 2$. Let $i \in \{1, 2\}$ and $j = 3 - i$. Define $C_i$ to be the restriction of $C$ to $G_i$. If $|C_i| \geq \pi(G_i, r)$ then $C_i$ can solve $r$, so we assume otherwise. Then $|C_j| = |C| - |C_i| + C(x) \geq [\pi(G_1, r) + \pi(G_2, x) - 2] - [\pi(G_i, r) - 1] + C(x) \geq \pi(G_j, x) - 1 + C(x)$. Indeed, this follows trivially for $j = 2$, and from Claim 31 for $j = 1$. If $C(x) \geq 2$ then we can move a pebble to $r$. If $C(x) = 1$ then, since we may assume that $C(r) = 0$, we have $|C(G_j - r - x)| \geq \pi(G_j, x) - 1 = \pi(G_j - r, x)$ by Lemma 23, and so we can move a second pebble to $x$ and then one to $r$. Hence we will assume that $C(x) = 0$.

If $|C(V_i)| \geq |V_i| + 2$ then $V_i$ either has a huge vertex or two big vertices, in which case it can solve $r$ through $x$, or it has a big vertex with a path of all ones to $r$, which also solves $r$. Hence we assume that each $|C(V_i)| \leq |V_i| + 1$. Thus we have that

$$|C(G'_i)| = |C_i| - |C(V_i)|$$

$$\geq \pi(G_i, x) - |V_i| - 2$$

$$= \pi(G'_i, x) - 1 ,$$

for each $i$.

Also, if some $|C(G'_j)| \geq \pi(G'_j, x)$ then we could place a pebble on $x$. This implies that $|C(V_j)| \leq |V_j|$ since a big vertex in $V_j$ could place a second pebble on $x$, and then one on $r$. Then we would have

$$|C(G'_j)| = |C_j| - |C(V_j)|$$

$$\geq \pi(G_j, x) - |V_j| - 1$$

$$= \pi(G'_j, x) ,$$

so that we could place a second pebble on $x$ and solve $r$. Thus we must have $|C(G'_i)| = \pi(G'_i, x) - 1$ for each $i$.  

28
Finally we see that

\[ |V_1| + |V_2| + 2 \geq |C(F)| \]

\[ = |C| - |C(G'_1)| - |C(G'_2)| \]

\[ = [\pi(G_1, r) - \pi(G'_1, x)] + [\pi(G_2, x) - \pi(G'_2, x)] \]

\[ = [\pi(G_1, r) - \pi(G_1, x)] + [\pi(G_1, x) - \pi(G'_1, x)] + [|V_2| + 1] \]

\[ = [2^{\text{ecc}_{G_1}(r)} - 2^{\text{ecc}_{G_1}(x)}] + [|V_1| + 1] + [|V_2| + 1], \]

which means that \(2^{\text{ecc}_{G_1}(x)} \geq 2^{\text{ecc}_{G_1}(r)}\), and hence \(2^{\text{ecc}_{G_1}(x)} = 2^{\text{ecc}_{G_1}(r)}\). That is, \(V_1 = \emptyset\), which implies by our labeling that \(V_2 = \emptyset\). Define \(x^-\) and \(x^+\) to be the common neighbors of \(r\) and \(x\). Then in the skeleton of \(G\) we can replace the path \(x^-xx^+\) by the path \(x^-rx^+\) to obtain a new skeleton containing \(r\), which is a contradiction, completing the proof. \(\square\)

Notice that the previous corollary allows one to calculate the pebbling number for \(r\). In fact, one can use Corollaries 20 and 26 to calculate \(\pi(G_1, r)\) and \(\pi(G_2, x)\), respectively.

As with Corollary 7, the following is a simple consequence of Lemma 5.

**Corollary 33** Let \((G, r)\) be a rooted semi-2-tree with \(r\) not in any skeleton of \(G\). If \(C\) is a configuration of size at least \(\pi(G, r) + (t - 1)2^{\text{ecc}(r)}\) then \(C\) has \(t\) distinct cheap \(r\)-solutions. \(\square\)

Similarly, Corollaries 28, 32, and 33 yield the following result.

**Corollary 34** Let \((G, r)\) be a rooted semi-2-tree with \(r\) not in any skeleton of \(G\). Then \(\pi_t(G, r) = \pi(G, r) + (t - 1)2^{\text{ecc}(r)}\). \(\square\)

**Theorem 35** Let \((G, r)\) be a rooted semi-2-tree with \(r\) not in any skeleton of \(G\), and let \(r^*\) be a simplicial vertex of \(G\) with \(\text{ecc}(r^*) = \text{diam}(G)\). Then \(\pi_t(G, r) \leq \pi_t(G, r^*)\), with equality if and only if \(\text{ecc}(r) = \text{diam}(G)\).
Proof. We prove that $\pi(G, r) \leq \pi(G, r^*)$; then $\pi_t(G, r) = \pi(G, r) + (t - 1)2^{\text{ecc}(r)} \leq \pi(G, r^*) + (t - 1)2^{\text{ecc}(r^*)} = \pi_t(G, r^*)$ will follow.

First we analyze the case in which $\text{ecc}(r) = \text{ecc}(r^*)$. Define $x$ to be the center of the fan containing $r$. Then we can suppose that $x$ is in first (longest) path $P^*$ in the maximum path partition of $T$ with root $r^*$. If $s^*$ is the other endpoint of $P^*$ then $x$ is adjacent to $s^*$. Hence $\text{ecc}_{G_2}(x) = 1$ and so $\pi(G_2, x) = n(G_2) = |V_2| + 3$. Thus

$$\pi(G, r) = \pi(G_1, r) + \pi(G_2, x) - 2$$
$$= \pi(G_1, r) + |V_2| + 3 - 2$$
$$= \pi(G_1, r) + |V_2| + 1$$

Also,

$$\pi(G, r^*) = \pi(T, r^*) + (n(G) - 1) + b(G) - 2e(T)$$
$$= \pi(T_1, r) + (n(G_1) + |V_2| + 1 - 1) + b(G_1) - 2e(T_1)$$
$$= \pi(G_1, r) + |V_2| + 1.$$
\[
\pi(G, r) = \pi(G_1, r) + \pi(G_2, x) - 2
\]
\[
= \pi(T_1, r) + (n(G_1) - 1) + b(G_1) - 2e(T_1)
+ \pi(T_2, x) + (n(G_2) - 1) + b(G_2) - 2e(T_2) - 2
= \pi(T_1, r) + \pi(T_2, x) + (n(G) - 1) + b(G) - 2e(T) - 2\epsilon.
\]

Analogous to the proof of Lemma 8, let \(P^*\) be a path \(v_0 v_1 \cdots v_d\) with \(v_0 = r^*\) and \(v_d = s^*\), labeled so that \(\text{dist}(x, s^*) \leq \text{dist}(x, r^*) = \text{ecc}(x)\). Denote by \(P\) the path from \(r^*\) to \(x\), and set \(P^* \cap P = v_0 \cdots v_{h'}\). Define \(P^*\) to be a maximum path partition of \(T\) with root \(r^*\). Define \(P^*_0 = P^*, P^*_1, \ldots, P^*_k\) to be the sequence of paths of \(P^*\) that are used sequentially in \(P\), and set \(d^*_i = \text{length}(P^*_i)\) for each \(0 \leq i \leq k\) (so \(d^*_0 = d\)). Next define \(P'_i = P \cap P^*_i\), with \(h'_i = \text{length}(P'_i)\) and \(\overline{h}_i = d^*_i - h'_i\) (so \(h'_0 = h'\) and \(\overline{h}_0 = \overline{h}\)). Note that \(1 \leq h'_i \leq d^*_i \leq d/2\).

Suppose that \(k > 0\). Then, since \(\text{ecc}(r) < \text{ecc}(r^*)\), we have

\[
\pi(T_1, r) + \pi(T_2, x) \leq \pi(T_1, r^*) - 2^{d-2} + \pi(T_2, x)
= \pi(T, r^*) + (2^{h'_k+1} - 1) + (2^{\overline{h}_k} - 1) - (2^{d^*_k} - 1) - 2^{d-2}
\leq \pi(T, r^*) + 1 - 2^{d-2}
< \pi(T, r^*),
\]

because \(d \geq 3\) and \(2^{a+1} + 2^b - 2^{a+b} \leq 2\) for all \(a, b \geq 1\).
Suppose instead that \( k = 0 \). Define \( Q \) to be the longest path in a maximum path partition of \( T_1 \) (choosing the partition to contain \( P \), if possible). If \( Q = P \) then we have

\[
\pi(T_1, r) + \pi(T_2, x) \leq \pi(T_1, r^*) - 2^{(h_0'+1)-2} + \pi(T_2, x)
\]

\[
= \pi(T, r^*) - (2^d - 1) + (2^{h_0'+1} - 1) + (2^{h_0} - 1) - 2^{h_0-1}
\]

\[
< \pi(T, r^*) - 2^d - 2^{h_0'+1} + 2^{h_0} - 2^{h_0-1}
\]

\[
< \pi(T, r^*) - 2^d + 2^{h_0}
\]

\[
\leq \pi(T, r^*) - 2^d + 2^{d-1}
\]

\[
< \pi(T, r^*) .
\]

Otherwise, when \( Q \neq P \) we define \( Q_0 = Q \cap P \) and \( Q_1 = Q - Q_0 \), having lengths \( q_0 \) and \( q_1 \), respectively, and set \( \hat{h}_0 = h_0' - q_0 \). Here we will use that \( q_0 + q_1 < d \), \( q_1 > \hat{h}_0 \), and \( \bar{h}_0 \leq d/2 \leq q_0 + \hat{h}_0 \leq q_0 + q_1 \). Hence

\[
\pi(T_1, r) + \pi(T_2, x) \leq \pi(T_1, r^*) - 2^{q_0+q_1-2} + \pi(T_2, x)
\]

\[
= \pi(T, r^*) - (2^d - 1) - (2^{q_1} - 1) + (2^{q_0+q_1} - 1) + (2^{h_0'+1} - 1)
\]

\[
+ (2^{h_0} - 1) - 2^{q_0+q_1-2}
\]

\[
< \pi(T, r^*) - 2^d - 2^{q_1} + 2^{q_0+q_1} + 2^{h_0+1} + 2^{q_0+q_1} - 2^{q_0+q_1-2}
\]

\[
< \pi(T, r^*) - (2^d - 2^{q_0+q_1+1}) - (2^{q_1} - 2^{h_0+1})
\]

\[
\leq \pi(T, r^*) .
\]

In all cases, then, we see that

\[
\pi(G, r) = \pi(T_1, r) + \pi(T_2, x) + (n(G) - 1) + b(G) - 2e(T) - 2\epsilon .
\]

\[
< \pi(T, r^*) + (n(G) - 1) + b(G) - 2e(T)
\]

\[
= \pi(G, r^*) ,
\]

and the result follows. \( \square \)
Theorem 36 If $G$ is a semi-2-tree then $\pi_t(G) = \pi_t(G, r^*)$, where $r^*$ is a simplicial vertex with $\text{ecc}(r^*) = \text{diam}(G)$.

Proof. Use Theorems 21, 27, 29, and 35.

Theorem 37 If $G$ is a semi-2-tree then $\pi_t(G)$ can be computed in linear time.

Proof. A breadth-first search from any simplicial vertex finds $r^*$, a simplicial vertex with $\text{ecc}(r^*) = \text{diam}(G)$. Indeed, this is true for trees, and the result extends to semi-2-trees as follows. Let $T$ be the skeleton of $G$ and let $A$ be a breadth-first search algorithm on $G$. Then $A$ is also a breadth-first search algorithm on $T$ and so finds a simplicial vertex $r$ with $\text{ecc}_T(r) = \text{diam}(T)$. Because $T$ is a geodesic tree spanning all of the simplicial vertices of $G$, we have $\text{ecc}_G(r) = \text{ecc}_T(r)$ and $\text{diam}(G) = \text{diam}(T)$, and so $r^* = r$.

At this point, we do not yet know $T$. However, we realize that $T$ can be constructed during $A$ because it is a geodesic tree spanning all of the simplicial vertices of $G$. Once we have $T$ we can remove its cut-vertices $S$ (those having degree bigger than 2) to reveal $b$, which equals the number of components of $T - S$.

Then $\pi_t(T, r)$ can be computed in linear time, according to Theorem 3 of [17].

6 Remarks

The obvious pressing question is how to extend this work to 2-trees. The pyramid is the graph on 6 vertices formed by adjoining a 2-simplicial vertex onto each of the three sides of a triangle. The pyramid is the key structure that forms the basis in the Class 0 characterization of diameter two graphs found in [8] and is what causes the extra 1 in their pebbling numbers — the configuration with 3 pebbles at two of the simplicial vertices cannot reach the third. The pyramid is also the smallest example of a 2-tree.
that is not a semi-2-tree, and it hints at the complexity that can ensue in a more general 2-tree.

Another natural question in the direction of this research program regards other simple examples of chordal graphs, such as interval graphs. It would seem that tackling \( k \)-paths is a necessary investigation toward approaching interval graphs. One interesting thing about the 2-path pebbling number is that both of the standard lower bounds of \( n(G) \) and \( 2^{\text{diam}(G)} \) for general graphs \( G \) appear in its formula. This is encouraging in light of the manner in which the size of \( k \) can determine which of those two terms is dominant.

It appears that parameters such as pathwidth and treewidth may figure prominently in the determination of pebbling numbers of general graphs. Other authors have made similar remarks, for example in [6]. Thus considering these classes of graphs seems the most productive direction of research.

Our final thought points to the many lemmas developed in this paper that should be of very general use, including the Cheap Lemma (5) and the four Removal lemmas: Junior (10), Wart (11), Edge (22), and Neighbor (24). We anticipate their ability to simplify the analysis of many future problems.

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