DISTRIBUTED CONTAINMENT REFERENCE SIGNAL FOR NONHOLONOMIC PLANAR VEHICLES

A PREPRINT

Lixia Yan
The Seventh Research Division, School of Automation Science and Electrical Engineering
Beihang University
Beijing 100191, People’s Republic of China
yanlixia@buaa.edu.cn

December 8, 2021

ABSTRACT

Cooperative of multiple nonholonomic vehicles can be converted into tracking problems of a single-vehicle. The reference trajectory design within distributed features for each vehicle in the group is addressed in this note. The motivation is that nonholonomic vehicles cannot achieve asymptotical stabilization of non-feasible reference signals, and modifications about the virtual reference trajectory design are needed. Reduced-order design and time-varying technique, and some simple geometry tricks are applied to derive the dynamic reference trajectory.

Keywords: Distributed Containment · Nonlinear Observer · Reduced-order Design

1 Preliminaries

1.1 Notations and definitions

Throughout this paper, \(\mathbb{R}\) denotes the set of real numbers, \(\|\cdot\|\) represents the Euclidean norm, \(|\cdot|\) is the absolute value of a scalar, \(\text{diag}\{\cdot\}\) denotes the diagonal matrix formed by a vector, \(I_n\) is an \(n\)-dimensional identity matrix, \(1_n\) is an \(n\)-dimensional identity vector, \(0_n\) denotes an \(n\)-dimensional zero vector. For a given square matrix, \(\lambda(\cdot)\), \(\lambda_m(\cdot)\) and \(\lambda_M(\cdot)\) represent the eigenvalue, the smallest and largest eigenvalue, respectively.

1.2 Graph Theory

Using graph theory to model the interactions among followers and leaders [Wei et al., 2005], let \(G = (\mathcal{N}, \mathcal{E}, \mathcal{A})\) be the undirected graph with node set \(\mathcal{N} = F \cup R = \{1, ..., n, n + 1, ..., n + m\}\), edge set \(\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}\) and adjacency matrix \(\mathcal{A} = \{a_{ij}\} \in \mathbb{R}^{n \times n}\). A direct edge \(\{(j, i) : i \neq j\} \in \mathcal{E}\) means that node \(i\) has access to node \(j\). The entry \(a_{ij}\) is constant weight and defined as: \(a_{ij} = 1\), if \((j, i) \in \mathcal{E}; a_{ij} = 0\), otherwise. Self interaction is not allowed, i.e., \(a_{ii} = 0\).

The graph \(G\) is called undirected if matrix \(\mathcal{A}\) has symmetric weights, i.e., \(a_{ij} = a_{ji}, \forall i, j \in \mathcal{N}\). A path from node \(i\) to node \(j\) denotes an edge sequence \(\{(i, j_1), (j_1, j_2), ..., (j_{s}, j)\}\), where \(i, j_1, j_2, ..., j_{s}, j \in \mathcal{N}\). The Laplacian matrix \(\mathcal{L} = \{l_{ij}\} \in \mathbb{R}^{(n+m) \times (n+m)}\) is denoted as: \(l_{ij} = -a_{ij}\), if \(i \neq j\); \(l_{ij} = \sum_{j=1}^{n+m} a_{ij}\), if \(i = j\).

Partition Laplacian matrix \(\mathcal{L}\) into

\[
\mathcal{L} = \begin{bmatrix}
\mathcal{L}_1 & \mathcal{L}_2 \\
0_{m \times n} & 0_{m \times m}
\end{bmatrix}
\]

where \(\mathcal{L}_1 \in \mathbb{R}^{n \times n}\) and \(\mathcal{L}_2 \in \mathbb{R}^{n \times m}\). The interaction graph \(G\) is supposed to satisfy,

Assumption 1 The communication among follower hovercrafts are fixed, undirected and connected. For each individual follower, there exists one path from it to one leader at least.
Lemma 1 [Cao et al., 2012] Under Assumption 1 the matrix $L_1$ is positive definite, every entry of $-L_1^{-1}L_2$ is non-negative, and the sum of entries of each row of $-L_1^{-1}L_2$ equals to 1.

2 Problem Formulation

Consider $n$ nonholonomic planar vehicles, being viewed as followers with regards to the leaders introduced later. Let $F = \{1, \ldots n\}$ and define the pose as

$$
\eta_{F,i} = [p_{F,i}, \theta_{F,i}]^T, \forall i \in F,
$$

where $p_{F,i} = [x_{F,i}, y_{F,i}]^T$ and $\theta_{F,i}$ denote position and orientation in Cartesian coordinate system. The planar vehicles are supposed to satisfy a certain nonholonomic constraints, obstructing the Brockett’s necessary condition [Brockett, 1983] for the existence of smooth time-invariant stabilizer. Typical nonholonomic planar vehicles include wheeled mobile robots [Yan and Ma, 2020], underactuated surface vehicles [Ma, 2009] and underactuated hovercrafts [Yan et al., 2021].

Besides $n$ followers, consider $m$ virtual leaders, labeled as $n+1$ to $n+m$. At present, we assume that leaders have already achieved a fixed formation pattern and their motions are independent of followers. Define the following leaders’ poses,

$$
\eta_{L,j}(t) = \eta_c(t) + d_{L,j} \in \mathbb{R}^3, j \in \mathcal{R} = \{n+1, \ldots, n+m\},
$$

where $\eta_{L,j}(t) = [p_{L,j}(t), \theta_{L,j}(t)]^T, \eta_c(t) = [p_c(t), \theta_c(t)]^T$ and $d_{L,j} = [d_{L,jx}, d_{L,jy}]^T \in \mathbb{R}^3$ is a constant vector with $d_{L,j} = [d_{L,jx}, d_{L,jy}]^T$. Intuitively, $p_{L,j} = [x_{L,j}, y_{L,j}]^T$ and $\theta_{L,j}$ denote the position and orientation of the leader $j$, respectively.

Assumption 2 The $\eta_c(t)$ is third-differentiable with uniformly bounded time derivatives.

Assumption 3 The $m$ constant coordinates $d_{L,jp}, \forall j \in \mathcal{R}$ span a two-dimensional convex $m$-polygon hull that encloses the origin; and $0 \in [\min(d_{L,j0}), \max(d_{L,j0})], \forall j \in \mathcal{R}$.

Define two convex sets formed by leader coordinates $\mathcal{L}_p$ as

$$
\mathcal{L}_p = \left\{ [x, y] \in \mathbb{R}^2 | [x, y]^T = \sum_{j=n+1}^{n+m} b_{p,j}p_{L,j} \right\}, \mathcal{L}_\theta = \left\{ \theta \in \mathbb{R} | \theta = \sum_{j=n+1}^{n+m} b_{p,j}\theta_{L,j} \right\},
$$

where $b_{p,j}$ satisfy $b_{p,j} \in [0, 1]$ and $\sum_{j=n+1}^{n+m} b_{p,j} = 1$.

Define

$$
\eta_{F,ir} = [p_{F,ir}, \theta_{F,ir}]^T \in \mathbb{R}^3, \forall i \in F.
$$

where $p_{F,ir} = [x_{F,ir}, y_{F,ir}]^T$ denotes position and $\theta_{F,ir}$ is orientation.

Then, the problem is formally stated as: Under Assumptions 1-3, design a virtual reference signal for the holonomic vehicles and derive the conditions within which the nonholonomic vehicles can converge into the convex hull spanned by leaders.

3 Reference Design and Stability Analysis

Let us design

$$
\begin{align*}
\dot{\eta}_{F,ir} &= \varphi_{F,ir}, \\
\dot{\varphi}_{F,ir} &= \rho_{F,ir}, \\
\dot{\rho}_{F,ir} &= -g_1\varphi_{F,ir} - g_2\rho_{F,ir} - g_3\delta_{F,ir} - g_4 \frac{s_{F,ir}}{\sqrt{\|s_{F,ir}\|^2 + \gamma_1^2e^{-2\gamma_2t}}}
\end{align*}
$$

where $s_{F,ir}$
where \( \eta_{F,ir} \triangleq [p_{F,ir}^T, \theta_{F,ir}]^T \), \( g_1 > 0, g_2 > 0, g_3 > 0, g_4 > 0, \gamma_1 > 0, \gamma_2 > 0 \) and

\[
s_{F,ir} = \sum_{j=1}^{n} a_{ij} (\rho_{F,ir} - \rho_{F,jr}) + g_1 \sum_{j=1}^{n} a_{ij} (\varphi_{F,ir} - \varphi_{F,jr}) \\
+ g_2 \sum_{j=1}^{n} a_{ij} (\eta_{F,ir} - \eta_{F,jr}) + \sum_{j=n+1}^{n+m} a_{ij} (\rho_{F,ir} - \tilde{\eta}_{L,j}) \\
+ g_1 \sum_{j=n+1}^{n+m} a_{ij} (\varphi_{F,ir} - \tilde{\eta}_{L,j}) + g_2 \sum_{j=n+1}^{n+m} a_{ij} (\eta_{F,ir} - \tilde{\eta}_{L,j}),
\]

(7)

where \( \tilde{\eta}_{L,j} = \eta_c(t) + \mu \tilde{d}_{L,j} \) and \( 0 < \mu < 1 \).

**Theorem 1** Given Assumptions 2-1 and \( g_1 > 0, g_2 > 0, g_3 > 0, g_4 \geq n \tilde{\eta}_c, \tilde{\eta}_c = \sup_{t \geq 0} \| \tilde{\eta}_c + g_1 \tilde{\eta}_c + g_2 \tilde{\eta}_c \|, \gamma_1 > 0 \) and \( \gamma_2 > 0 \), then the \( \eta_{F,ir} = [p_{F,ir}^T, \theta_{F,ir}]^T, \forall i \in \mathcal{F} \) generated by (6) satisfies

\[
\lim_{t \to +\infty} \eta_{F,r} = - (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L,
\]

(8)

where

\[
\eta_{F,r} \triangleq [\eta_{F,1r}^T, ..., \eta_{F,nr}^T]^T \in \mathbb{R}^{3n}, \tilde{\eta}_L \triangleq [\tilde{\eta}_{n+1}^T, ..., \tilde{\eta}_{n+m}^T]^T \in \mathbb{R}^{3m}.
\]

**Proof.** Define three stacked vectors belonging to \( \mathbb{R}^{3n}, \)

\[
\varphi_{F,r} = [\varphi_{F,1r}^T, ..., \varphi_{F,nr}^T]^T, \rho_{F,r} = [\rho_{F,1r}^T, ..., \rho_{F,nr}^T]^T, s_{F,r} = [s_{F,1r}^T, ..., s_{F,nr}^T]^T.
\]

(9)

By (1) and (7), we obtain

\[
\begin{aligned}
\dot{\eta}_{F,r} &= \varphi_{F,r}, \\
\dot{\varphi}_{F,r} &= \rho_{F,r}, \\
\dot{\rho}_{F,r} &= -g_1 \varphi_{F,r} - g_2 \rho_{F,r} - g_3 s_{F,r} - g_4 \rho_{F,s_{F,r}},
\end{aligned}
\]

(10)

where \( P_F = \text{diag} \left[ \left( \frac{\mathbf{1}_1^T}{\| s_{F,1d} \|^2 + \sum_{i=2}^{n} \mathbf{1}_1^T}, ..., \frac{\mathbf{1}_1^T}{\| s_{F,nr} \|^2 + \sum_{i=2}^{n} \mathbf{1}_1^T} \right) \right] \in \mathbb{R}^{3n \times 3n} \). Then, arrange \( s_{F,r} \) as follow,

\[
s_{F,r} = (\mathcal{L}_1 \otimes I_3) (\rho_{F,r} + g_1 \varphi_{F,r} + g_2 \eta_{F,r}) + (\mathcal{L}_2 \otimes I_3) [\tilde{\eta}_L + g_1 \tilde{\eta}_L + g_2 \tilde{\eta}_L],
\]

(11)

where the facts \( \tilde{\eta}_L = \tilde{\eta}_L \) and \( \tilde{\eta}_L = \tilde{\eta}_L \) are used. Define

\[
\xi_{F} = \eta_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L,
\]

(12)

and

\[
S_F = \dot{\xi}_F + g_1 \dot{\xi}_F + g_2 \xi_F
\]

(13)

= \rho_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L + g_1 [\varphi_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L] + g_2 [\rho_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L].
\]

By (11) and (12), we get

\[
s_{F,r} = (\mathcal{L}_1 \otimes I_3) S_F.
\]

(14)

It is obvious that the convergence to zero of \( S_F \) can lead to the convergence to zero of \( \xi_{F} \), and hence, achieve \( \lim_{t \to +\infty} \eta_{F,r} = - (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L \). Differentiate \( S_F \) with respect to time \( t \) and yield

\[
\dot{S}_F = \dot{\tilde{\eta}}_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L + g_1 [\dot{\tilde{\eta}}_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L] + g_2 [\dot{\tilde{\eta}}_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L]
\]

(15)

\[
= -g_1 \varphi_{F,r} - g_2 \rho_{F,r} - g_3 s_{F,r} - g_4 \rho_{F,s_{F,r}} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L
\]

\[
+ g_1 [\varphi_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L] + g_2 [\rho_{F,r} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) \tilde{\eta}_L]
\]

\[
= -g_3 s_{F,r} - g_4 \rho_{F,s_{F,r}} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) (\tilde{\eta}_L + g_1 \tilde{\eta}_L + g_2 \tilde{\eta}_L)
\]

\[
= -g_3 (I_3 \otimes I_3) S_F - g_4 (I_3 \otimes I_3) S_F + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_3) (\tilde{\eta}_L + g_1 \tilde{\eta}_L + g_2 \tilde{\eta}_L).
\]

Using the property that \( \mathcal{L}_1 \) is positive definite, choose a Lyapunov candidate as,

\[
V_F = 0.5 S_F^T (I_3 \otimes I_3) S_F,
\]

(16)
which satisfies $0.5\lambda_{\min}(L_1)\|S_F\|^2 \leq V_1 \leq 0.5\lambda_{\max}(L_1)\|S_F\|^2$ and $\|S_F\|^2 \geq 2\frac{V_1}{\lambda_{\max}(L_1)}$. The time derivative of (16) on the trajectory of (15) is

$$
\dot{V}_1 = -g_3S_F^T(L_1 \otimes I_3)^2S_F - g_4P_T^F(L_1 \otimes I_3)^2S_F + S_F^T(L_1 \otimes I_3) (\tilde{\eta}_L + g_1\tilde{\eta}_L + g_2\tilde{\eta}_L)
$$

$$
= -g_3S_F^T(L_1 \otimes I_3)^2S_F - g_4P_T^FS_T^r (L_1 \otimes I_3) (\tilde{\eta}_L + g_1\tilde{\eta}_L + g_2\tilde{\eta}_L),
$$

where the equation $s_T^r = S_T^r(L_1 \otimes I_3)$ is utilized. It then by $-L_1\dot{L}_2 1_m = 1_n$ follows that

$$
s_T^r (L_1^{-1}L_2 \otimes I_3) (\tilde{\eta}_L + g_1\tilde{\eta}_L + g_2\tilde{\eta}_L) = s_T^r (L_1^{-1}L_2 \otimes I_3) [1_m \otimes (\tilde{\eta}_c + g_1\tilde{\eta}_c + g_2\tilde{\eta}_c)]
$$

$$
= s_T^r [1_m \otimes (\tilde{\eta}_c + g_1\tilde{\eta}_c + g_2\tilde{\eta}_c)]
$$

$$
\leq \eta \sum_{i=1}^{n} \|s_T^r\| \|\tilde{\eta}_c\|,
$$

where the fact $\eta^{(q)}(t), \forall j \in \mathcal{R}, q \in \mathbb{Z}_{\geq 1}$ is also applied. The combination of (17) and (18) leads to

$$
\dot{V}_1 \leq -g_3S_F^T(L_1 \otimes I_3)^2S_F - \sum_{i=1}^{n} \frac{g_3s_T^r s_F^r}{\|s_F^r\|^2 + \gamma_1 e^{-2\gamma_2 t}}\|n\tilde{\eta}_c\| + \sum_{i=1}^{n} \|s_T^r\| \|\tilde{\eta}_c\|
$$

$$
= -g_3S_F^T(L_1 \otimes I_3)^2S_F - \sum_{i=1}^{n} \left( \frac{g_4\|s_F^r\|^2}{\|s_F^r\|^2 + \gamma_1 e^{-2\gamma_2 t}} - \frac{\|s_T^r\|}{\|s_F^r\|^2 + \gamma_1 e^{-2\gamma_2 t}} \|n\tilde{\eta}_c\| \right).
$$

After some direct computations,

$$
\dot{V}_1 \leq -g_3\lambda_{\min}^2(L_1)\|S_F\|^2 - \sum_{i=1}^{n} \left( \frac{(g_4 - n\tilde{\eta}_c)\|s_F^r\|^2}{\|s_F^r\|^2 + \gamma_1 e^{-2\gamma_2 t}} - \frac{\|s_T^r\|}{\|s_F^r\|^2 + \gamma_1 e^{-2\gamma_2 t}} \|n\tilde{\eta}_c\| \right)
$$

$$
\leq -2\frac{g_3\lambda_{\min}^2(L_1)}{\lambda_{\max}(L_1)}V_1 + \sum_{i=1}^{n} \|s_T^r\| \|n\tilde{\eta}_c\| e^{-2\gamma_2 t}
$$

(20)

where the $g_4 \geq n\tilde{\eta}_c$ is used. Let $\lambda_1 = \frac{2g_3\lambda_{\min}^2(L_1)}{\lambda_{\max}(L_1)}$, convert the second inequality of (20) into

$$
\dot{V}_1 \leq -\lambda_1 V_1 + n^2\gamma_1 \tilde{\eta}_c e^{-2\gamma_2 t}.
$$

(21)

Integrating both sides of (21) by comparison principle, we yield

$$
V_1(t) \leq e^{-\lambda_1 t}V_1(0) + \int_0^t e^{-\lambda_1 t} n^2\gamma_1 \tilde{\eta}_c e^{-2\gamma_2 t} d\chi
$$

(22)

$$
= \begin{cases} 
  e^{-\lambda_1 t}V_1(0) + n^2\gamma_1 \tilde{\eta}_c e^{-2\gamma_2 t}, & \text{if } \lambda_1 = \gamma_2, \\
  e^{-\lambda_1 t}V_1(0) + n^2\gamma_1 \tilde{\eta}_c e^{-2\gamma_2 t}, & \text{if } \lambda_1 \neq \gamma_2,
\end{cases}
$$

and find out that $V_1$ converges to zero globally exponentially. Hence, $S_F$ is globally exponentially convergent. By (13), we obtain $\xi_F = -g_1\xi_F - g_2\xi_F + S_F$, which further implies that $\xi_F$ and $\xi_F$ are globally exponentially convergent to zero. Henceforth, we have

$$
\lim_{t \to +\infty} \eta_{F,r} = -\left( L_1^{-1}L_2 \otimes I_3 \right) \tilde{\eta}_L.
$$

(23)

Because the sum of each row of $-L_1^{-1}L_2$ equals 1, we obtain $\eta_{F,r} \to \tilde{C}_{L_2}$ and $\theta_{F,r} \to \tilde{C}_{L_0}$ as $t \to +\infty$. Additionally, by the above analysis and Assumption 2.2, it is direct to verify that the derivatives $\dot{\eta}_{F,r} = [[\dot{\eta}_{F,r}, \dot{\theta}_{F,r}]^T, \eta_{F,r} = [\dot{\eta}_{F,r}, \dot{\theta}_{F,r}]^T$ and $\dot{\eta}_{F,r} = [[\dot{\eta}_{F,r}, \dot{\theta}_{F,r}]^T$ are bounded for all $t \geq 0$. This completes the proof. □

According to Lemma 1, the Theorem 1 tells that the virtual reference signal propagated by (9) would converge into the convex set formed by vectors $\tilde{\eta}_{L,j} = \eta_j(t) + \mu d_{L,j}, \forall j \in \mathcal{R}$. Note that the constant $\mu$ scales down that original convex hulls. This scaling is an essential step for steering the hovercrafts converge into the original convex hulls spanned by leaders. Let $\tilde{C}_{L_2}$ and $\tilde{C}_{L_0}$ denote the convex hulls formed by $\tilde{\eta}_{L,j}, \forall j \in \mathcal{R}$, we have $\tilde{C}_{L_2} \subseteq C_{L_2}$ and $\tilde{C}_{L_0} \subseteq C_{L_0}$. Define $\tilde{C}_{L_2} = \mathbb{R}^2 - C_{L_2}$ and $\tilde{C}_{L_0} = \mathbb{R} - C_{L_0}$ and propose the following lemma.
We denote the edge connected by $E$. According to basic geometric knowledge, the distance between $E$. To prove the property 2, we calculate the straight-line equations of paired edges. The constant $\alpha$ is selected as the minimum distance between the parallel paired edges $E$. Suppose that the $C_L$ is composed of $m$ edges, labeled as $E_{L,n+1}, \ldots, E_{L,n+m}$ by connecting the coordinates $p_{L,i}, i \in R$ in counter-clockwise (or clockwise) circular order. Different from the edge for communication in graph theory, the edge $E_{L,i}, i \in R$ here refers to the line segment connected by position coordinates $p_{L,i} = [x_{L,i}, y_{L,i}]^T$ and $p_{L,j} = [x_{L,j}, y_{L,j}]^T$, where, without making any confusion, the subscript is chosen as: $j = i + 1$ if $i \neq n + m$; otherwise $j = n + 1, \forall i \in R$. By definition (3), the vertex coordinates of the edge $E_{L,i}$ are

$$p_{L,i} = [x_c(t) + d_{L,i}, y_c(t) + d_{L,j}]^T,$$

$$p_{L,j} = [x_c(t) + d_{L,j}, y_c(t) + d_{L,ix}]^T.$$  

(25)

Then, the scaled coordinates of (25), according to $\tilde{\eta}_{L,j}$ in (7), become

$$\tilde{p}_{L,i} = [x_c(t) + \mu d_{L,i}, y_c(t) + \mu d_{L,j}]^T,$$

$$\tilde{p}_{L,j} = [x_c(t) + \mu d_{L,j}, y_c(t) + \mu d_{L,ix}]^T.$$  

(26)

We denote the edge connected by $\tilde{p}_{L,i}$ and $\tilde{p}_{L,j}$ as $E_{L,i}$. The relationship between the paired edges $(E_{L,i}, E_{L,i})$, by (25) and (26), satisfies

$$\|p_{L,i} - p_{L,j}\| = \mu \|\tilde{p}_{L,i} - \tilde{p}_{L,j}\|,$$

(27)

which means that the length of edge $E_{L,i}$ is proportional to that of its scaled part $E_{L,i}$ at a ratio of $\mu$. This is true for all $m$ paired edges $(E_{L,i}, E_{L,i})$, $\forall i \in R$. Therefore, the coordinates $\tilde{p}_{L,i}, \forall i \in R$ span a sub-hull $\tilde{C}_{L}$ that is not only similar to $C_{L,j}$ but also is convex. According to Assumption 3 and the fact that $0 < \mu < 1$, any point in $\tilde{C}_{L,j}$ is also in $C_{L,j}$, namely, $\tilde{C}_{L,i} \subseteq C_{L,i}$. For $\tilde{C}_{L,i}$, we have $\tilde{C}_{L,i} = \{\theta \in R| \theta \in [\theta_{L,i}(t) + \mu \min\{d_{L,j}\}, \theta_{L,j}(t) + \mu \max\{d_{L,j}\}], \forall j \in R\}$. Thus, it is direct to prove that $\tilde{C}_{L,i} \subseteq C_{L,i}$.

To prove the property 2, we calculate the straight-line equations of paired edges $E_{L,i}$ and $E_{L,i}$ as follows,

$$(d_{L,j} - d_{L,j})x + (d_{L,ix} - d_{L,j})y + [x_c(t) + d_{L,j}][y_c(t) + d_{L,j}] - [x_c(t) + d_{L,ix}][y_c(t) + d_{L,j}] = 0;$$

(28a)

$$\mu (d_{L,j} - d_{L,j})x + \mu (d_{L,ix} - d_{L,j})y + [x_c(t) + \mu d_{L,j}][y_c(t) + \mu d_{L,j}] - [x_c(t) + \mu d_{L,ix}][y_c(t) + \mu d_{L,j}] = 0.$$  

(28b)

Obviously, the straight lines described by (28a) and (28b) are parallel; and so are the paired edges $E_{L,i}$ and $E_{L,i}$. According to basic geometric knowledge, the distance between $E_{L,i}$ and $E_{L,i}$ can be calculated as,

$$\text{dis}(E_{L,i}, E_{L,i}) = (1 - \mu) \frac{|d_{L,j}d_{L,j}d_{L,ix}d_{L,j}|}{\sqrt{(d_{L,j} - d_{L,j})^2 + (d_{L,ix} - d_{L,j})^2}},$$

(29)

where ‘dis’ means ‘the distance of’. Note that the Assumption 3 ensures that both the numerator and denominator of (29) are greater than zero. Therefore, $\text{dis}(E_{L,i}, E_{L,i}) > 0$ holds. Let us choose

$$\alpha_p = (1 - \mu) \min \left\{ \frac{|d_{L,j}d_{L,ix}d_{L,j}|}{\sqrt{(d_{L,j} - d_{L,j})^2 + (d_{L,ix} - d_{L,j})^2}}, \forall i, j \in R \right\}.$$  

(30)

The constant $\alpha_p$ is selected as the minimum distance between the parallel paired edges $E_{L,i}$ and $E_{L,i}, \forall i \in R$. Hence, the distance between any point in $\tilde{C}_{L,j}$ and the other one outside $C_{L,j}$ is greater than $\alpha_p$. Following the same routine above, we can calculate $\alpha_\theta$ as

$$\alpha_\theta = (1 - \mu) \min \{|d_{L,j}|, \forall j \in R\}.$$  

(31)

A graphical version of the above analysis refers to Fig. 2. The choice of $\alpha_p$ and $\alpha_\theta$ is to guarantee that the distance from any point outside the original convex hull to that in the scaled one is greater than the minimum distance between the ‘edges’ before and after scaling. This completes the proof. □

By the expressions of (30) and (31), both $\alpha_p$ and $\alpha_\theta$ can be set arbitrarily small by decreasing $\mu$, so that the scaled convex sub-hulls $\tilde{C}_{L,j}$ and $\tilde{C}_{L,j}$ can approximate the original ones with arbitrarily small difference. This implies that the control design for the vehicle in question ought to ensure that the ultimate bounds of pose tracking errors $p_{F,i} - p_{F,i}$ and $\theta_{F,i} - \theta_{F,i}$ be tunable and arbitrarily small.

5
References
Ren Wei, R. W. Beard, and E. M. Atkins. A survey of consensus problems in multi-agent coordination. In Proceedings of the American Control Conference, volume 3, pages 1859–1864, 2005.

Yongcan Cao, Wei Ren, and Magnus Egerstedt. Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks. Automatica, 48(8):1586–1597, 2012.

Roger W Brockett. Asymptotic stability and feedback stabilization. 1983.

Lixia Yan and Baoli Ma. Adaptive practical leader-following formation control of multiple nonholonomic wheeled mobile robots. International Journal of Robust and Nonlinear Control, 30(17):7216–7237, 2020.

Baoli Ma. Global $\kappa$-exponential asymptotic stabilization of underactuated surface vessels. Systems & Control Letters, 58(3):194–201, 2009. ISSN 0167-6911.

Lixia Yan, Baoli Ma, and Wenjing Xie. Robust practical tracking control of an underactuated hovercraft. Asian Journal of Control, 23(5):2201–2213, 2021.

Hassan K Khalil. Nonlinear Systems. Upper Saddle River. 2002.