MATRICIAL BRIDGES FOR "MATRIX ALGEBRAS CONVERGE TO THE SPHERE"

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Abstract. In the high-energy quantum-physics literature one finds statements such as “matrix algebras converge to the sphere”. Earlier I provided a general setting for understanding such statements, in which the matrix algebras are viewed as quantum metric spaces, and convergence is with respect to a quantum Gromov-Hausdorff-type distance. In the present paper, as preparation of discussing similar statements for convergence of “vector bundles” over matrix algebras to vector bundles over spaces, we introduce and study suitable matrix-norms for matrix algebras and spaces. Very recently Latréomolière introduced an improved quantum Gromov-Hausdorff-type distance between quantum metric spaces. We use it throughout this paper. To facilitate the calculations we introduce and develop a general notion of “bridges with conditional expectations”.

1. Introduction

In several earlier papers [11, 12, 14] I showed how to give a precise meaning to statements in the literature of high-energy physics and string theory of the kind “matrix algebras converge to the sphere”. (See the references to the quantum physics literature given in [11, 13, 15, 4, 5, 2, 1].) I did this by introducing and developing a concept of “compact quantum metric spaces”, and a corresponding quantum Gromov-Hausdorff-type distance between them. The compact quantum spaces are unital C*-algebras, and the metric data is given by putting on the algebras seminorms that play the role of the usual Lipschitz seminorms on the algebras of continuous functions on ordinary compact metric spaces. The natural setting for “matrix algebras converge to the sphere” is that of coadjoint orbits of compact semi-simple Lie groups.

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But physicists need much more than just the algebras. They need vector bundles, gauge fields, Dirac operators, etc. So I now seek to give precise meaning to statements in the physics literature of the kind “here are the vector bundles over the matrix algebras that correspond to the monopole bundles on the sphere”. (See [13] for many references.) In [13] I studied convergence of ordinary vector bundles on ordinary compact metric spaces for ordinary Gromov-Hausdorff distance. From that study it became clear that one needed Lipschitz-type seminorms on all the matrix algebras over the underlying algebras, with these seminorms coherent in the sense that they form a “matrix seminorm” (defined below). The purpose of this paper is to define and develop the properties of such matrix seminorms for the setting of coadjoint orbits, and especially to study how these matrix seminorms mesh with quantum Gromov-Hausdorff distance.

Very recently Latrémolière introduced an improved version of quantum Gromov-Hausdorff distance [8] that he calls “propinquity”. We show that propinquity works very well for our setting of coadjoint orbits, and so propinquity is the form of quantum Gromov-Hausdorff distance that we use in this paper. Latrémolière defines his propinquity in terms of an improved version of the “bridges” that I had used in my earlier papers. For our matrix seminorms we need corresponding “matricial bridges”, and we show how to construct natural ones for the setting of coadjoint orbits.

It is crucial to obtain good upper bounds for the lengths of the bridges that we construct. In the matricial setting the calculations become somewhat complicated. In order to ease the calculations we introduce a notion of “bridges with conditional expectations”, and develop their general theory, including the matricial case, and including bounds for their lengths in the matricial case.

The main theorem of this paper, Theorem 6.10, states in a quantitative way that for the case of coadjoint orbits the lengths of the matricial bridges go to 0 as the size of the matrix algebras goes to infinity.

We also discuss a closely related class of examples coming from [12], for which we construct bridges between different matrix algebras associated to a given coadjoint orbit. This provides further motivation for our definitions and theory of bridges with conditional expectation.

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2. The first basic class of examples

In this section we describe the first of the two basic classes of examples underlying this paper. It consists of the main class of examples studied in the papers [11, 14]. We begin by describing the common setting for the two basic classes of examples.

Let $G$ be a compact group (perhaps even finite, at first). Let $U$ be an irreducible unitary representation of $G$ on a (finite-dimensional) Hilbert space $\mathcal{H}$. Let $B = L(\mathcal{H})$ denote the $C^*$-algebra of all linear operators on $\mathcal{H}$ (a “full matrix algebra”, with its operator norm). There is a natural action, $\alpha$, of $G$ on $B$ by conjugation by $U$, that is, $\alpha_x(T) = U_xTU_x^*$ for $x \in G$ and $T \in B$. Because $U$ is irreducible, the action $\alpha$ is “ergodic”, in the sense that the only $\alpha$-invariant elements of $B$ are the scalar multiples of the identity operator.

Let $P$ be a rank-one projection in $B(\mathcal{H})$ (traditionally specified by giving a non-zero vector in its range). For any $T \in B$ we define its Berezin covariant symbol [11], $\sigma_T$, with respect to $P$, by

$$\sigma_T(x) = \text{tr}(T\alpha_x(P)),$$

where $\text{tr}$ denotes the usual (un-normalized) trace on $B$. (When the $\alpha_x(P)$’s are viewed as giving states on $B$ via tr, they form a family of “coherent states” [11] if a few additional conditions are satisfied.) Let $H$ denote the stability subgroup of $P$ for $\alpha$. Then it is evident that $\sigma_T$ can be viewed as a (continuous) function on $G/H$. We let $\lambda$ denote the action of $G$ on $G/H$, and so on $\mathcal{A} = C(G/H)$, by left-translation. If we note that $\text{tr}$ is $\alpha$-invariant, then it is easily seen that $\sigma$ is a unital, positive, norm-nonincreasing, $\alpha$-$\lambda$-equivariant map from $B$ into $\mathcal{A}$.

Fix a continuous length function, $\ell$, on $G$ (so $G$ must be metrizable). Thus $\ell$ is non-negative, $\ell(x) = 0$ iff $x = e_G$ (the identity element of $G$), $\ell(x^{-1}) = \ell(x)$, and $\ell(xy) \leq \ell(x) + \ell(y)$. We also require that $\ell(xyx^{-1}) = \ell(y)$ for all $x, y \in G$. Then in terms of $\alpha$ and $\ell$ we can define a seminorm, $L^B$, on $B$ by the formula

$$L^B(T) = \sup\{\|\alpha_x(T) - T\|/\ell(x) : x \in G \quad \text{and} \quad x \neq e_G\}.$$
Then \((B, L_B)\) is an example of a compact C*-metric-space, as defined in definition 4.1 of [14], and in particular \(L_B\) satisfied the conditions given there for being a “Lip-norm”.

Of course, from \(\lambda\) and \(\ell\) we also obtain a seminorm, \(L_A\), on \(A\) by the evident analog of formula 2.1 except that we must permit \(L_A\) to take the value \(\infty\). It is shown in proposition 2.2 of [10] that the set of functions for which \(L_A\) is finite (the Lipschitz functions) is a dense *-subalgebra of \(A\). Also, \(L^A\) is the restriction to \(A\) of the seminorm on \(C(G)\) that we get from \(\ell\) when we view \(C(G/H)\) as a subalgebra of \(C(G)\), as we will often do when convenient. From \(L^A\) we can use equation 2.2 below to recover the usual quotient metric [16] on \(G/H\) coming from the metric on \(G\) determined by \(\ell\). One can check easily that \(L^A\) in turn comes from this quotient metric. Thus \((A, L^A)\) is the compact C*-metric-space associated to this ordinary compact metric space. Then for any bridge from \(A\) to \(B\) we can use \(L^A\) and \(L_B\) to define the length of the bridge in the way given by Latrémolière, which we will describe soon below.

For any two unital C*-algebras \(A\) and \(B\) a bridge from \(A\) to \(B\) in the sense of Latrémolière [8] is a quadruple \(\{D, \pi_A, \pi_B, \omega\}\) for which \(D\) is a unital C*-algebra, \(\pi_A\) and \(\pi_B\) are unital injective homomorphisms of \(A\) and \(B\) into \(D\), and \(\omega\) is a self-adjoint element of \(D\) such that 1 is an element of the spectrum of \(\omega\) and \(\|\omega\| = 1\). Actually, Latrémolière only requires a looser but more complicated condition on \(\omega\), but the above condition will be appropriate for our examples. Following Latrémolière we will call \(\omega\) the “pivot” for the bridge. We will often omit mentioning the injections \(\pi_A\) and \(\pi_B\) when it is clear what they are from the context, and accordingly we will often write as though \(A\) and \(B\) are unital subalgebras of \(D\).

For our first class of examples, in which \(A\) and \(B\) are as described in the paragraphs above, we take \(D\) to be the C*-algebra

\[ D = A \otimes B = C(G/H, B). \]

We take \(\pi_A\) to be the injection of \(A\) into \(D\) defined by

\[ \pi_A(a) = a \otimes 1_B \]

for all \(a \in A\), where \(1_B\) is the identity element of \(B\). The injection \(\pi_B\) is defined similarly. From the many calculations done in [11, 14] it is not surprising that we define the pivot \(\omega\) to be the function in \(C(G/H, B)\) defined by

\[ \omega(x) = \alpha_x(P) \]
for all $x \in G/H$. We notice that $\omega$ is actually a projection in $\mathcal{D}$, and so it satisfies the requirements for being a pivot. We will denote the bridge $\{\mathcal{D}, \omega\}$ by $\Pi$. 

For any bridge between two unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$ and any choice of seminorms $L^\mathcal{A}$ and $L^\mathcal{B}$ on $\mathcal{A}$ and $\mathcal{B}$, Latrémolière defines the “length” of the bridge in terms of these seminorms. For this he initially puts relatively weak requirements on the seminorms, but for the purposes of the matricial bridges that we will define later, we need somewhat different weak requirements. To begin with, Latrémolière only requires his seminorms, say $L^\mathcal{A}$ on a unital C*-algebra $\mathcal{A}$, to be defined on the subspace of self-adjoint elements of the algebra, but we need $\mathcal{A}$ to be defined on all of $\mathcal{A}$. To somewhat compensate for this we require that $L^\mathcal{A}$ be a *-seminorm. As with Latrémolière, our $L^\mathcal{A}$ is permitted to take value $+\infty$. Latrémolière also requires the subspace on which $L^\mathcal{A}$ takes finite values to be dense in the algebra. We do not really need this here, but for us there would be no harm in assuming it, and all interesting examples probably will satisfy this. Finally, Latrémolière requires that the null space of $L^\mathcal{A}$ (i.e where it takes value $0$) be exactly $C_1\mathcal{A}$. We must loosen this to simply requiring that $L^\mathcal{A}(1_{\mathcal{A}}) = 0$, but permitting $L^\mathcal{A}$ to also take value $0$ on elements not in $C_1\mathcal{A}$. We think of such seminorms as “semi-Lipschitz seminorms”. To summarize all of this we make:

**Definition 2.1.** By a *slip-norm* on a unital C*-algebra $\mathcal{A}$ we mean a *-seminorm, $L$, on $\mathcal{A}$ that is permitted to take the value $+\infty$, and is such that $L(1_{\mathcal{A}}) = 0$.

Because of these weak requirements on $L^\mathcal{A}$, various quantities in this paper may be $+\infty$, but most interesting examples will satisfy stronger requirements that will result in various quantities being finite. Latrémolière defines the length of a bridge by first defining its “reach” and its “height”. We apply his definitions to slip-norms.

**Definition 2.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras and let $\Pi = \{\mathcal{D}, \omega\}$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$. Let $L^\mathcal{A}$ and $L^\mathcal{B}$ be slip-norms on $\mathcal{A}$ and $\mathcal{B}$. Set

$$L^\mathcal{A}_1 = \{a \in \mathcal{A} : a = a^* \text{ and } L^\mathcal{A}(a) \leq 1\},$$

and similarly for $L^\mathcal{B}_1$. (We can view these as subsets of $\mathcal{D}$.) Then the *reach* of $\Pi$ is given by:

$$\text{reach}(\Pi) = \text{Haus}_\mathcal{D}\{L^\mathcal{A}_1 \omega \cup \omega L^\mathcal{B}_1\},$$

where $\text{Haus}_\mathcal{D}$ denotes the Hausdorff distance with respect to the norm of $\mathcal{D}$, and where the product defining $L^\mathcal{A}_1 \omega$ and $\omega L^\mathcal{B}_1$ is that of $\mathcal{D}$. 
Latréomilie`re shows just before definition 3.14 of [8] that, under conditions that include the case in which \((\mathcal{A}, L^A)\) and \((\mathcal{B}, L^B)\) are C*-metric spaces, the reach of \(\Pi\) is finite.

To define the height of \(\Pi\) we need to consider the state space, \(S(\mathcal{A})\), of \(\mathcal{A}\), and similarly for \(\mathcal{B}\) and \(\mathcal{D}\). Even more, we set
\[
S_1(\omega) = \{ \phi \in S(\mathcal{D}) : \phi(\omega) = 1 \},
\]
the “level-1 set of \(\omega\)”. The elements of \(S_1(\omega)\) are “definite” on \(\omega\) in the sense [7] that for any \(d \in \mathcal{D}\) we have
\[
\phi(d\omega) = \phi(d) = \phi(\omega d).
\]

Let \(\rho_A\) denote the metric on \(S(\mathcal{A})\) determined by \(L^A\) by the formula
\[
(2.2) \quad \rho_A(\mu, \nu) = \sup \{|\mu(a) - \nu(a)| : L^A(a) \leq 1\}.
\]
(Without further conditions on \(L_A\) we must permit \(\rho_A\) to take the value \(+\infty\). Also, it is not hard to see that the supremum can be taken equally well just over \(L^1_A\).) Define \(\rho_B\) on \(S(\mathcal{B})\) similarly.

**Notation 2.3.** We denote by \(S_1^A(\omega)\) the restriction of the elements of \(S_1(\omega)\) to \(\mathcal{A}\). We define \(S_1^B(\omega)\) similarly.

**Definition 2.4.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be unital C*-algebras and let \(\Pi = \{\mathcal{D}, \omega\}\) be a bridge from \(\mathcal{A}\) to \(\mathcal{B}\). Let \(L^A\) and \(L^B\) be slip-norms on \(\mathcal{A}\) and \(\mathcal{B}\).

The **height** of the bridge \(\Pi\) is given by
\[
\text{height}(\Pi) = \max \{ \text{Haus}_{\rho_A}(S_1^A(\omega), S(\mathcal{A})), \text{Haus}_{\rho_B}(S_1^B(\omega), S(\mathcal{B})) \},
\]
where the Hausdorff distances are with respect to the indicated metrics (and value \(+\infty\) is allowed). The length of \(\Pi\) is then defined by
\[
\text{length}(\Pi) = \max \{ \text{reach}(\Pi), \text{height}(\Pi) \}.
\]

In Section 6 we will show how to obtain a useful upper bound on the length of \(\Pi\) for our first class of examples.

### 3. The Second Basic Class of Examples

Our second basic class of examples has the same starting point as the first class, consisting of \(G, \mathcal{H}, U\) and \(P\) as before, with \(\mathcal{B} = L(\mathcal{H})\). But now we will also have a second irreducible representation. The more concrete class of examples motivating this situation, but for which we will not need the details, is that in [11, 12, 14] in which \(G\) is a compact semi-simple Lie group, \(\lambda\) is a positive integral weight, and our two representations of \(G\) are the representations with highest weights \(m\lambda\) and \(n\lambda\) for positive integers \(m, n\), \(m \neq n\). Furthermore, the projections \(P\) are required to be those along highest weight vectors.
The key feature of this situation that we do need to remember here is that the stability subgroups $H$ for the two projections coincide.

Accordingly, for our slightly more general situation, we will denote our two representations by $(\mathcal{H}^m, U^m)$ and $(\mathcal{H}^n, U^n)$, where now $m$ and $n$ are just labels. Our two C*-algebras will be $\mathcal{B}^m = \mathcal{L}(\mathcal{H}^m)$ and $\mathcal{B}^n = \mathcal{L}(\mathcal{H}^n)$. We will denote the action of $G$ on these two algebras just by $\alpha$, since the context should always make clear which algebra is being acted on. The corresponding projections will be $P^m$ and $P^n$. The crucial assumption that we make is that the stability subgroups of these two projections coincide. We will denote this common stability subgroup by $H$ as before.

We construct a bridge from $\mathcal{B}^m$ to $\mathcal{B}^n$ as follows. We let $\mathcal{A} = C(G/H)$ as in our first class of examples, and we define $\mathcal{D}$ by

$$\mathcal{D} = \mathcal{B}^m \otimes \mathcal{A} \otimes \mathcal{B}^n = C(G/H, \mathcal{B}^m \otimes \mathcal{B}^n).$$

We view $\mathcal{B}^m$ as a subalgebra of $\mathcal{D}$ by sending $b \in \mathcal{B}^m$ to $b \otimes 1_\mathcal{A} \otimes 1_{\mathcal{B}^n}$, and similarly for $\mathcal{B}^n$. From the many calculations done in [12] it is not surprising that we define the pivot, $\omega$, to be the function in $C(G/H, \mathcal{B}^m \otimes \mathcal{B}^n)$ defined by

$$\omega(x) = \alpha_x(P^m) \otimes \alpha_x(P^n).$$

We let $L^m$ be the Lip-norm defined on $\mathcal{B}^m$ determined by the action $\alpha$ and the length function $\ell$ as in Section 2, and similarly for $L^n$ on $\mathcal{B}^n$. In terms of these Lip-norms the length of any bridge from $\mathcal{B}^m$ to $\mathcal{B}^n$ is defined. Thus the length of the bridge described above is defined. In Section 7 we will see how to obtain useful upper bounds on the length of this bridge.

4. Bridgess with conditional expectations

We will now seek a somewhat general framework for obtaining useful estimates for the lengths of bridges such as those of our two basic classes of examples. To discover this framework we will explore some properties of our two basic classes of examples. We will summarize what we find at the end of this section.

On $G/H$ there is a unique probability measure that is invariant under left translation by elements of $G$. We denote the corresponding linear functional on $\mathcal{A} = C(G/H)$ by $\tau_\mathcal{A}$, and sometimes refer to it as the canonical tracial state on $\mathcal{A}$. On $\mathcal{B} = \mathcal{L}(\mathcal{H})$ there is a unique tracial state, which we denote by $\tau_\mathcal{B}$. These combine to form a tracial state, $\tau_\mathcal{D} = \tau_\mathcal{A} \otimes \tau_\mathcal{B}$ on $\mathcal{D} = \mathcal{A} \otimes \mathcal{B}$. Similarly, we have the unique tracial states $\tau_m$ and $\tau_n$ on $\mathcal{B}^m$ and $\mathcal{B}^n$, which combine with $\tau_\mathcal{A}$ to give a tracial state on $\mathcal{D} = \mathcal{B}^m \otimes \mathcal{A} \otimes \mathcal{B}^n$. 
For $\mathcal{D} = \mathcal{A} \otimes \mathcal{B}$, the tracial state $\tau_\mathcal{B}$ determines a conditional expectation, $E^\mathcal{A}$, from $\mathcal{D}$ onto its subalgebra $\mathcal{A}$, defined on elementary tensors by
\[
E^\mathcal{A}(a \otimes b) = a \tau_\mathcal{B}(b)
\]
for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$. (This is an example of a “slice map” as discussed in [3], where conditional expectations are also discussed.) This conditional expectation has the property that for any $d \in \mathcal{D}$ we have
\[
\tau_\mathcal{A}(E^\mathcal{A}(d)) = \tau_\mathcal{D}(d),
\]
and it is the unique conditional expectation with this property. (See corollary II.6.10.8 of [3].) In the same way the tracial state $\tau_\mathcal{A}$ determines a canonical conditional expectation, $E^\mathcal{B}$ from $\mathcal{D}$ onto its subalgebra $\mathcal{B}$.

For the case in which $\mathcal{D} = \mathcal{B}^m \otimes \mathcal{A} \otimes \mathcal{B}^n$, the tracial state $\tau_\mathcal{A} \otimes \tau_n$ on $\mathcal{A} \otimes \mathcal{B}^n$ determines a canonical conditional expectation, $E^m$, from $\mathcal{D}$ onto $\mathcal{B}^m$ in the same way as above, and the tracial state $\tau_m \otimes \tau_\mathcal{A}$ determines a canonical conditional expectation, $E^n$, from $\mathcal{D}$ onto $\mathcal{B}^n$.

These conditional expectations relate well to the pivots of the bridges. For the case in which $\mathcal{D} = \mathcal{A} \otimes \mathcal{B}$ we find that for any $F \in \mathcal{D} = C(G/H, \mathcal{B})$ we have
\[
E^\mathcal{A}(F \omega)(x) = \tau_\mathcal{B}(F(x) \alpha_x(P)).
\]
In particular, for any $T \in \mathcal{B}$ we have
\[
E^\mathcal{A}(T \omega)(x) = \tau_\mathcal{B}(T \alpha_x(P))
\]
for all $x \in G/H$. Aside from the fact that we are here using the normalized trace instead of the standard trace on the matrix algebra $\mathcal{B}$, the right-hand side is exactly the definition of the Berezin covariant symbol of $T$ that plays such an important role in [11, 14] (beginning in section 1 of [11]), and that is denoted there by $\sigma_T$. This indicates that for general $\mathcal{A}$ and $\mathcal{B}$ a map $b \mapsto E^\mathcal{A}(b \omega)$ might be of importance to us. For our specific first basic class of examples we note the following favorable properties:

1. Self-adjointness i.e. $E^\mathcal{A}(F^* \omega) = (E^\mathcal{A}(\omega F))^*$ for all $F \in \mathcal{D}$.
2. $E^\mathcal{A}(F \omega) = E^\mathcal{A}(\omega F)$ for all $F \in \mathcal{D}$.
3. Positivity, i.e. if $F \geq 0$ then $E^\mathcal{A}(F \omega) \geq 0$.
4. $E^\mathcal{A}(1_D \omega) = r^{-1}1_A$ where $\mathcal{B}$ is an $r \times r$ matrix algebra.

However, if we consider $E^\mathcal{B}$ instead $E^\mathcal{A}$, then for any $F \in \mathcal{D}$ we have
\[
E^\mathcal{B}(F \omega) = \int_{G/H} F(x) \alpha_x(P) \, dx,
\]
and we see that in general properties 1-3 above fail, although property 4 still holds, with the same constant \( r \). But if we restrict \( F \) to be any \( f \in A \), we see that properties 1-3 again hold. Even more, the expression

\[
\int_{G/H} f(x) \alpha_x(P) \, dx
\]

is, except for normalization of the trace, the formula involved in the Berezin contravariant symbol that in [11] is denoted by \( \hat{\sigma} \).

For our second class of examples, in which \( D = B^m \otimes A \otimes B^n \), we find that for \( F \in D = C(G/H, B^m \otimes B^n) \) we have

\[
E^m(F\omega) = \int_{G/H} (\iota_A \otimes \tau_n)(F(x)(\alpha_x(P^m) \otimes \alpha_x(P^n))) \, dx.
\]

Again we see that properties 1-3 above are not in general satisfied. But if we restrict \( F \) to be any \( T \in B^n \) then the above formula becomes

\[
\int_{G/H} \alpha_x(P^m) \tau_n(T \alpha_x(P^n)) \, dx,
\]

which up to normalization of the trace is exactly the second displayed formula in section 3 of [12]. It is not difficult to see that properties 1-3 above are again satisfied under this restriction.

We remark that it is easily seen that the maps \( T \mapsto E^m(T\omega) \) from \( B^n \) to \( B^m \) and \( S \mapsto E^n(S\omega) \) from \( B^m \) to \( B^n \) are each other’s adjoints when they are viewed as being between the Hilbert spaces \( L^2(B^m, \tau_m) \) and \( L^2(B^n, \tau_n) \). A similar statement hold for our first basic class of examples.

With these observations in mind, we begin to formulate a somewhat general framework. As before, we assume that we have two unital C*-algebras \( A \) and \( B \), and a bridge \( \Pi = (D, \omega) \) from \( A \) to \( B \). We now require that we are given conditional expectations \( E_A \) and \( E_B \) from \( D \) onto its subalgebras \( A \) and \( B \). (We do not require that they be associated to any tracial states.) We require that they relate well to \( \omega \). To begin with, we will just require that \( \omega \geq 0 \) so that \( \omega^{1/2} \) exists. Then the map

\[
D \mapsto E_A(\omega^{1/2}D\omega^{1/2})
\]

from \( D \) to \( A \) is positive.

Once we have slip-norms \( L^A \) and \( L^B \) on \( A \) and \( B \), we need to require that the conditional expectations are compatible with these slip-norms. To begin with, we require that if \( L^B(b) = 0 \) for some \( b \in B \) then \( L^A(E_A(\omega^{1/2}b\omega^{1/2})) = 0 \). But one of the conditions on a Lip-norm is that it takes value 0 exactly on the scalar multiples of the identity element, and the case of Lip-norms is important to us. For Lip-norms
we see that the above requirement implies that \( E^A(\omega) \in \mathbb{C}1_A \), and so 
\( E^A(\omega) = r_\omega 1_A \) for some positive real number \( r_\omega \). We require the same of \( E^B \) with the same real number, so that we require that 
\[ E^A(\omega) = r_\omega 1_A = E^B(\omega). \]

We then define a map, \( \Phi^A \), from \( D \) to \( A \) by 
\[ \Phi^A(d) = r_\omega^{-1} E^A(\omega^{1/2} d \omega^{1/2}). \]

In a similar way we define \( \Phi^B \) from \( D \) to \( B \). We see that \( \Phi^A \) and \( \Phi^B \) are unital positive maps, and so are of norm 1 (as seen by composing them with states). Then the main compatibility requirement that we need is that for all \( b \in B \) we have 
\[ L^A(\Phi^A(b)) \leq L^B(b), \]
and similarly for \( A \) and \( B \) reversed. Notice that this implies that if \( b \in L^1_B \) then \( \Phi^A(b) \in L^1_A \).

We now show how to obtain an upper bound for the reach of the bridge \( \Pi \) when the above requirements are satisfied. Let \( b \in L^1_B \) be given. As an approximation to \( \omega b \) by an element of the form \( a \omega \) for some \( a \in L^1_A \) we take \( a = \Phi^A(b) \). It is indeed in \( L^1_A \) by the requirements made just above. This prompts us to set 
(4.1) \[ \gamma^B = \sup \{ \| \Phi^A(b) \omega - \omega b \|_D : b \in L^1_B \}, \]
and we see that \( \omega b \) is then in the \( \gamma^B \)-neighborhood of \( L^1_A \omega \). Note that without further assumptions on \( L^B \) we could have \( \gamma^B = +\infty \). Interchanging the roles of \( A \) and \( B \), we define \( \gamma^A \) similarly. We then see that 
\[ \text{reach}(\Pi) \leq \max \{ \gamma^A, \gamma^B \}. \]

We will explain in Sections 6 and 7 why this upper bound is useful in the context of \([11, 12, 14]\).

We now consider the height of \( \Pi \). For this we need to consider \( S_1(\omega) \) as defined in Section 2. Let \( \mu \in S(A) \). Because \( \Phi^A \) is positive and unital, its composition with \( \mu \) is in \( S(D) \). When we evaluate this composition at \( \omega \) to see if it is in \( S_1(\omega) \), we obtain \( \mu(\omega^{-1} E^A(\omega^2)) \), and we need this to equal 1. Because \( \mu(\omega^{-1} E^A(\omega^2)) = 1 \), it follows that we need \( \mu(\omega^{-1} E^A(\omega - \omega^2)) = 0 \). If this is to hold for all \( \mu \in S(A) \), we must have \( E^A(\omega - \omega^2) = 0 \). If \( E^A \) is a faithful conditional expectation, as is true for our basic examples, then because \( \omega \geq \omega^2 \) it follows that \( \omega^2 = \omega \) so that \( \omega \) is a projection, as is also true for our basic examples. These arguments are reversible, and so it is easy to see that if \( \omega \) is a projection, then for every \( \mu \in S(A) \) we obtain an element, \( \phi_\mu \), of \( S_1(\omega) \), defined by 
\[ \phi_\mu(d) = \mu(\omega^{-1} E^A(\omega d \omega)) = \mu(\Phi^A(d)). \]
This provides us with a substantial collection of elements of $S_1(\omega)$.

Consequently, since to estimate the height of $\Pi$ we need to estimate the distance from each $\mu \in S(A)$ to $S_1^A(\omega)$, we can hope that $\phi_\mu$ restricted to $A$ is relatively close to $\mu$. Accordingly, for any $a \in A$ we compute

$$|\mu(a) - \phi_\mu(a)| = |\mu(a) - \mu(\Phi^A(a))| \leq \|a - \Phi^A(a)\|.$$ 

Set

$$\delta^A = \sup\{\|a - \Phi^A(a)\| : a \in L^1_A\}.$$ 

Then we see that

$$\rho_{L^A}(\mu, \phi_\mu|_A) \leq \delta^A.$$

We define $\delta^B$ in the same way, and obtain the corresponding estimate for the distances from elements of $S(B)$ to the restriction of $S_1(\omega)$ to $B$. In this way we see that

$$\text{height}(\Pi) \leq \max\{\delta^A, \delta^B\}.$$ 

(Notice that $\delta^A$ involves what $\Phi^A$ does on $A$, whereas $\gamma^A$ involves what $\Phi^B$ does on $A$.)

While this bound is natural within this context, it turns out not to be so useful for our two basic classes of example. In Proposition 4.6 below we will give a different bound that does turn out to be useful for our basic examples. But perhaps other examples will arise for which the above bound is useful.

We now summarize the main points discussed in this section.

**Definition 4.1.** Let $A$ and $B$ be unital C*-algebras and let $\Pi = (D, \omega)$ be a bridge from $A$ to $B$. We say that $\Pi$ is a bridge with conditional expectations if conditional expectations $E^A$ and $E^B$ from $D$ onto $A$ and $B$ are specified, satisfying the following properties:

1. The conditional expectations are faithful.
2. The pivot $\omega$ is a projection.
3. There is a constant, $r_\omega$, such that

$$E^A(\omega) = r_\omega 1_D = E^B(\omega).$$

For such a bridge with conditional expectations we define $\Phi^A$ on $D$ by

$$\Phi^A(d) = r_\omega^{-1}E^A(\omega d \omega).$$

We define $\Phi^B$ similarly, with the roles of $A$ and $B$ reversed. We will often write $\Pi = (D, \omega, E^A, E^B)$ for a bridge with conditional expectations.
I should mention here that at present I do not see how the class of examples considered by Latrémolière that involves non-commutative tori [9] fits into the setting of bridges with conditional expectations, though I have not studied this matter carefully. It would certainly be interesting to understand this better. I also do not see how the general case of ordinary compact metric spaces, as discussed in theorem 6.6 of [8], fits into the setting of bridges with conditional expectations.

**Definition 4.2.** With notation as above, let $L^A$ and $L^B$ be slip-norms on $A$ and $B$. We say that a bridge with conditional expectations $\Pi = (D, \omega, E^A, E^B)$ is admissible for $L^A$ and $L^B$ if

$$L^A(\Phi^A(b)) \leq L^B(b)$$

for all $b \in B$, and

$$L^B(\Phi^B(a)) \leq L^A(a)$$

for all $a \in A$.

We define the reach, height and length of a bridge with conditional expectations $(D, \omega, E^A, E^B)$ to be those of the bridge $(D, \omega)$.

From the earlier discussion we obtain:

**Theorem 4.3.** Let $L^A$ and $L^B$ be slip-norms on unital $C^*$-algebras $A$ and $B$, and let $\Pi = (D, \omega, E^A, E^B)$ be a bridge with conditional expectations from $A$ to $B$ that is admissible for $L^A$ and $L^B$. Then

$$\text{reach}(\Pi) \leq \max\{\gamma^A, \gamma^B\},$$

where

$$\gamma^A = \sup\{\|a\omega - \omega \Phi^B(a)\|_D : a \in L^1_A\},$$

and similarly for $\gamma^B$, while

$$\text{height}(\Pi) \leq \max\{\delta^A, \delta^B\},$$

where

$$\delta^A = \sup\{\|a - \Phi^A(a)\| : a \in L^1_A\}$$

and similarly for $\delta^B$. Consequently

$$\text{length}(\Pi) \leq \max\{\gamma^A, \gamma^B, \delta^A, \delta^B\}.$$

(Consequently the propinquity between $(A, L^A)$ and $(B, L^B)$, as defined in [8], is no greater than the right-hand side above.)

We could axiomitize the above situation in terms of just $\Phi^A$ and $\Phi^B$, without requiring that they come from conditional expectations, but at present I do not know of examples for which this would be useful. It would not suffice to require that $\Phi^A$ and $\Phi^B$ just be positive (and
unital) because for the matricial case discussed in the next section they would need to be completely positive.

The following result is very pertinent to our first class of basic examples.

**Proposition 4.4.** With notation as above, suppose that our bridge $\Pi$ has the quite special property that $\omega$ commutes with every element of $\mathcal{A}$, or at least that $E^A(\omega a) = E^A(a \omega)$ for all $a \in \mathcal{A}$. Then $\Phi^A(a) = a$ for all $a \in \mathcal{A}$. Consequently $\delta^A = 0$, and the restriction of $S_1(\omega)$ to $\mathcal{A}$ is all of $S(\mathcal{A})$.

**Proof.** This depends on the conditional expectation property of $E^A$. For $a \in \mathcal{A}$ we have

$$\Phi^A(a) = r^{-1}_\omega E^A(a \omega) = a r^{-1}_\omega E^A(\omega) = a.$$

The following steps might not initially seem useful, but in Sections 6 and 7 we will see in connection with our basic examples that they are quite useful. Our notation is as above. Let $\nu \in S(\mathcal{B})$. Then as seen above, $\nu \circ \Phi^B \in S_1(\omega)$, and so its restriction to $\mathcal{A}$ is in $S(\mathcal{A})$. But then $\nu \circ \Phi^B \circ \Phi^A \in S_1(\omega)$. Let us denote it by $\psi_\nu$. Then the restriction of $\psi_\nu$ to $\mathcal{B}$ can be used as an approximation to $\nu$ by an element of $S_1(\omega)$. Now for any $b \in \mathcal{B}$ we have

$$|\nu(b) - \psi_\nu(b)| = |\nu(b) - (\nu \circ \Phi^B \circ \Phi^A)(b)| \leq \|b - \Phi^B(\Phi^A(b))\|.$$

**Notation 4.5.** In terms of the above notation we set

$$\hat{\delta}^B = \sup\{\|b - \Phi^B(\Phi^A(b))\| : b \in \mathcal{L}_B^1\}.$$

We note that $L^B(\Phi^B(\Phi^A(b))) \leq L^B(b)$ because of the admissibility requirements of Definition 4.2. It follows that

$$\rho_{L^B}(\nu, \psi_\mu) \leq \hat{\delta}^B.$$

We define $\hat{\delta}^A$ in the same way, and obtain the corresponding estimate for the distances from elements of $S(\mathcal{A})$ to the restriction of $S_1(\omega)$ to $\mathcal{A}$. In this way we obtain:

**Proposition 4.6.** For notation as above,

$$\text{height}(\Pi) \leq \max\{\min\{\delta^A, \hat{\delta}^A\}, \min\{\delta^B, \hat{\delta}^B\}\}.$$

We will see in Section 6 that for our first class of basic examples, $\Phi^B \circ \Phi^A$ is exactly a term that plays an important role in [11, 14]. It is essentially an “anti-Berezin-transform”.
5. The corresponding matricial bridges

Fix a positive integer $q$. We let $M_q$ denote the C*-algebra of $q \times q$ matrices with complex entries. For any C*-algebra $\mathcal{A}$ we let $M_q(\mathcal{A})$ denote the C*-algebra of $q \times q$ matrices with entries in $\mathcal{A}$. We often identify it in the evident way with the C*-algebra $M_q \otimes \mathcal{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras, and let $\Pi = (D, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$. Then $M_q(\mathcal{A})$ can be viewed as a subalgebra of $M_q(D)$, as can $M_q(\mathcal{B})$. Let $\omega_q = 1_q \otimes \omega$, where $1_q$ is the identity element of $M_q$, so $\omega_q$ can be viewed as the diagonal matrix in $M_q(D)$ with $\omega$ in each diagonal entry. Then it is easily seen that $\Pi^q = (M_q(D), \omega_q)$ is a bridge from $M_q(\mathcal{A})$ to $M_q(\mathcal{B})$.

In order to measure the length of $\Pi^q$ we need slip-norms $L^A_q$ and $L^B_q$ on $M_q(\mathcal{A})$ and $M_q(\mathcal{B})$. It is reasonable to want these slip-norms to be coherent in some sense as $q$ varies. The discussion that we will give just after Theorem 6.8 suggests that the coherence requirement be that the sequences $\{L^A_q\}$ and $\{L^B_q\}$ form “matrix slipnorms”. To explain what this means, for any positive integers $m$ and $n$ we let $M_{mn}$ denote the linear space of $m \times n$ matrices with complex entries, equipped with the norm obtained by viewing such matrices as operators from the Hilbert space $C^n$ to the Hilbert space $C^m$. We then note that for any $A \in M_n(\mathcal{A})$, any $\alpha \in M_{mn}$, and any $\beta \in M_{nm}$ the usual matrix product $\alpha A \beta$ is in $M_m(\mathcal{A})$. The following definition, for the case of Lip-norms, is given in definition 5.1 of [18] (and see also [17, 19, 14, 6]).

**Definition 5.1.** A sequence $\{L^A_n\}$ is a matrix slip-norm for $\mathcal{A}$ if $L^A_n$ is a $*$-seminorm (with value $+\infty$ permitted) on $M_n(\mathcal{A})$ for each integer $n \geq 1$, and this family of seminorms has the following properties:

1. For any $A \in M_n(\mathcal{A})$, any $\alpha \in M_{mn}$, and any $\beta \in M_{nm}$, we have

   $$L^A_m(\alpha A \beta) \leq \|\alpha\| L^A_n(A) \|\beta\|.$$

2. For any $A \in M_m(\mathcal{A})$ and any $C \in M_n(\mathcal{A})$ we have

   $$L^A_{m+n} \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = \max(L^A_m(A), L^A_n(C)).$$

3. $L^A_1$ is a slip-norm.

We remark that the properties above imply that for $n \geq 2$ the null-space of $L^A_n$ contains all of $M_n$, not just the scalar multiples of the identity. This is why our definition of slip-norms does not require that the null-space is exactly the scalar multiples of the identity.

Now let $\Pi = (D, \omega, E^A, E^B)$ be a bridge with conditional expectations. For any integer $q \geq 1$ set $E_q^A = \iota_q \otimes E^A$, where $\iota_q$ is the identity
map from $M_q$ onto itself. Define $E_q^B$ similarly. Then it is easily seen that $E_q^A$ and $E_q^B$ are faithful conditional expectations from $M_q(D)$ onto its subalgebras $M_q(A)$ and $M_q(B)$ respectively. Furthermore, $E_q^A(\omega_q)$ is the diagonal matrix each diagonal entry of which is $E^A(\omega) = r_\omega 1_D$, and from this we see that $E_q^A(\omega_q) = r_\omega 1_{M_q(A)}$. Thus $r_\omega = r_\omega$. It is also clear that $\omega_q$ is a projection. Putting this all together, we obtain:

**Proposition 5.2.** Let $\Pi = (D, \omega, E^A, E^B)$ be a bridge with conditional expectations from $A$ to $B$. Then

$$\Pi^q = (M_q(D), \omega_q, E_q^A, E_q^B)$$

is a bridge with conditional expectations from $M_q(A)$ to $M_q(B)$. It has the same constant $r_\omega$ as does $\Pi$.

We can then set $\Phi^A_q = \iota_q \otimes \Phi^A$, and similarly for $\Phi^B_q$. Because $\Pi^q$ has the same constant $r_\omega$ as does $\Pi$, we see that for any $D \in M_q(D)$ we have

$$\Phi^A_q(D) = r_\omega^{-1} E_q^A(\omega_q D \omega_q).$$

Suppose now that $A$ and $B$ have matrix slip-norms $\{L^A_n\}$ and $\{L^B_n\}$. We remark that a matrix slip-norm $\{L^A_n\}$ is in general not at all determined by $L^A_1$. Thus a bridge that is admissible for $L^A_1$ as in Definition 4.2 need not relate well to the seminorms $L^A_n$ for higher $n$.

**Definition 5.3.** With notation as above, let $\{L^A_n\}$ and $\{L^B_n\}$ be matrix slip-norms on $A$ and $B$. We say that a bridge with conditional expectations $\Pi = (D, \omega, E^A, E^B)$ is admissible for $\{L^A_n\}$ and $\{L^B_n\}$ if for all integers $n \geq 1$ the bridge $\Pi^q$ is admissible for $L^A_n$ and $L^B_n$, that is, for all integers $n \geq 1$ we have

$$L^A_n(\Phi^A_q(B)) \leq L^B_n(B)$$

for all $B \in M_n(B)$, where

$$\Phi^A_q(B) = r_\omega^{-1} E_q^A(\omega_q B \omega_q),$$

and similarly with the roles of $A$ and $B$ reversed.

We assume now that $\Pi = (D, \omega, E^A, E^B)$ is admissible for $\{L^A_n\}$ and $\{L^B_n\}$. Since for a fixed integer $q$ the bridge $\Pi^q$ is admissible for $L^A_q$ and $L^B_q$, the length of $\Pi^q$ is defined. We now show how to obtain an upper bound for the length of $\Pi^q$ in terms of the data used in the previous section to get an upper bound on the length of $\Pi$.

We consider first the reach of $\Pi^q$. Set, much as earlier,

$$L_{q}^A = \{A \in M_q(A) : A = A^* \text{ and } L_q^A(A) \leq 1\},$$
and similarly for $L_{\mathcal{B}}^{1q}$. Then the reach of $\Pi^q$ is defined to be

$$\text{Haus}_{M_q(\mathcal{D})}\{L_{\mathcal{A}}^{1q}\omega_q, \omega_q L_{\mathcal{B}}^{1q}\}.$$

Suppose that $B \in L_{\mathcal{B}}^{1q}$. Then $\Phi_q^A(B) \in L_{\mathcal{A}}^{1q}$ by the admissibility requirement. So we want to bound

$$\|\Phi_q^A(B)\omega_q - \omega_q B\|_{M_q(\mathcal{D})}.$$

I don’t see any better way to bound this in terms of the data used in Theorem 4.3 for $\Pi$ than by using an entry-wise estimate as done in the third paragraph before lemma 14.2 of [14]). We use the fact that for a $q \times q$ matrix $C = [c_{jk}]$ with entries in a C*-algebra we have $\|C\| \leq q \max_{jk}\{\|c_{jk}\|\}$ (as is seen by expressing $C$ as the sum of the $q$ matrices whose only non-zero entries are the entries $c_{jk}$ for which $j - k$ is a given constant mod $q$). In this way, for $B \in M_q(\mathcal{B})$ with $B = [b_{jk}]$ we find that the last displayed term above is

$$\leq q \max_{jk}\{\|\Phi^A(b_{jk})\omega - \omega b_{jk}\|\}.$$

The small difficulty is that the $b_{jk}$’s need not be self-adjoint. But for any $b \in \mathcal{B}$, if we denote its real and imaginary parts by $b_r$ and $b_i$, then because $L^B$ is a $*$-seminorm it follows that $L^B(b_r) \leq L^B(b)$ and similarly for $b_i$. Consequently

$$\|\Phi^A(b)\omega - \omega b\| \leq \|\Phi^A(b_r)\omega - \omega b_r\| + \|\Phi^A(b_i)\omega - \omega b_i\| \leq \gamma^B L^B(b_r) + \gamma^B L^B(b_i) \leq 2\gamma^B L^B(b).$$

Thus the term displayed just before is

$$\leq 2q\gamma^B L^B(b_{jk}).$$

But $\{L^B_n\}$ is a matrix slip-norm, and by the first property of such seminorms given in Definition 5.1 we have

$$\max\{L^B(b_{jk})\} \leq L^B_q(B).$$

Thus for $B \in L_{\mathcal{B}}^{1q}$ we see that

$$\|\Phi_q^A(B)\omega_q - \omega_q B\|_{M_q(\mathcal{D})} \leq 2q\gamma^B,$$

so that $\omega_q B$ is in the $2q\gamma^B$-neighborhood of $L_{\mathcal{A}}^{1q}\omega_q$. In the same way $A\omega_q$ is in the $2q\gamma^A$-neighborhood of $\omega_q L_{\mathcal{B}}^{1q}$ for every $A \in L_{\mathcal{A}}^{1q}$. We find in this way that

$$\text{reach}(\Pi^q) \leq 2q \max\{\gamma^A, \gamma^B\}.$$

We now consider the height of $\Pi^q$. We argue much as in the discussion of height before Definition 4.4. For any $\mu \in S(M_q(\mathcal{A}))$ its
composition with $\Phi_q^A$ is an element, $\phi_\mu$, of $S_1(\omega_q)$, specifically defined by
\[
\phi_\mu(D) = \mu(\Phi_q^A(D)) = \mu(\tau^{-1}_\omega E_q(\omega_q D \omega_q)).
\]
We take $\phi_\mu|_{M_q(A)}$ as an approximation to $\mu$, and estimate the distance between these elements of $S(M_q(A))$. For $A \in L_{1q}^1$ we calculate
\[
|\mu(A) - \phi_\mu(A)| = |\mu(A) - \mu(\Phi_q^A(A))| \leq \|A - \Phi_q^A(A)\|.
\]
Again I don’t see any better way to bound this in terms of the data used in Theorem 4.3 for $\Pi$ than by using an entry-wise estimates. For $A \in M_q(A)$ with $A = [a_{jk}]$ we find (by using arguments as above to deal with the fact that the $a_{jk}$’s need not be self-adjoint) that the last displayed term above is
\[
\leq q \max_{jk} \{|a_{jk} - \Phi^A(a_{jk})|\} \leq 2q \delta^A \max \{L^A(a_{jk})\}.
\]
But again $\{L^A_q\}$ is a matrix slip-norm, and so by the first property of such seminorms given in Definition 5.1 we have
\[
\max \{L^A_q(a_{jk})\} \leq L_q^A(A).
\]
Since our assumption is that $A \in L_{1q}^1$, we see in this way that
\[
\rho_{L_q^A}(\mu, \phi_\mu|_{A}) \leq 2q \delta^A.
\]
Thus $S(M_q(A))$ is in the $2q \delta^A$-neighborhood of the restriction to $M_q(A)$ of $S_1(\omega_q)$.

We find in the same way that $S(M_q(B))$ is in the $2q \delta^B$-neighborhood of the restriction to $M_q(B)$ of $S_1(\omega_q)$. Consequently,
\[
\text{height}(\Pi^q) \leq 2q \max \{\delta^A, \delta^B\}.
\]
We can instead use $\hat{\delta}^A$ in the way done in Proposition 4.6. Using reasoning much like that used above, we find that for any $B \in M_q(B)$ we have:
\[
\|B - \Phi_q^B(\Phi_q^A(B))\| \leq q \max_{jk} \{|b_{jk} - \Phi^B(\Phi^A(b_{jk}))|\}
\]
\[
\leq 2q \hat{\delta}^B \max \{L^B(b_{jk})\} \leq 2q \hat{\delta}^B L_q^B(B)
\]
Consequently we see that
\[
\text{height}(\Pi^q) \leq 2q \max \{\hat{\delta}^A, \hat{\delta}^B\}.
\]
We summarize what we have found by:
Theorem 5.4. Let \( \{L^n_A\} \) and \( \{L^n_B\} \) be matrix slip-norms on unital \( C^* \)-algebras \( A \) and \( B \), and let \( \Pi = (D, \omega, E^A, E^B) \) be a bridge with conditional expectations from \( A \) to \( B \) that is admissible for \( \{L^n_A\} \) and \( \{L^n_B\} \). For any fixed positive integer \( q \) let \( \Pi^q \) be the corresponding bridge with conditional expectations from \( M_q(A) \) to \( M_q(B) \). Then
\[
\text{reach}(\Pi^q) \leq 2q \max\{\gamma^A, \gamma^B\},
\]
where as before
\[
\gamma^A = \sup\{\|a\omega - \omega \Phi^B(a)\|_D : a \in L^1_A\},
\]
and similarly for \( \gamma^B \); while
\[
\text{height}(\Pi^q) \leq 2q \max\{\min\{\delta^A, \hat{\delta}^A\}, \min\{\delta^B, \hat{\delta}^B\}\},
\]
where as before
\[
\delta^A = \sup\{\|\Phi^A(a) - a\| : a \in L^1_A\}
\]
and
\[
\hat{\delta}^A = \sup\{\|a - \Phi^A(\Phi^B(a))\| : a \in L^1_A\};
\]
and similarly for \( \delta^B \) and \( \hat{\delta}^B \). Consequently
\[
\text{length}(\Pi^q) \leq 2q \max\{\gamma^A, \gamma^B, \min\{\delta^A, \hat{\delta}^A\}, \min\{\delta^B, \hat{\delta}^B\}\}.
\]

6. The application to the first class of basic examples

We now apply the above general considerations to our first class of basic examples, described in Section 2. Thus we have \( G, H, B, A, D \) and \( \omega \) as defined there, as well as \( L^A \) and \( L^B \) given by equation (2.1). We proceed to obtain an upper bound for the length of the bridge \( \Pi = (D, \omega) \), where \( D = C(G/H, B) \) and \( \omega(x) = \alpha_x(P) \). We begin by considering its reach.

As seen in Section 4, for any \( F \in D = C(G/H, B) \) we have
\[
E^A(\omega F \omega)(x) = \tau_B(F(x)\alpha_x(P)).
\]
From this it is easily seen that \( r^{-1}_\omega \) is the dimension of \( H \), and so \( r^{-1}_\omega \tau_B \) is the usual unnormalized trace on \( B \), which we now denote by \( \text{tr}_B \). In particular, for any \( T \in B \) we have
\[
\Phi^A(T)(x) = r^{-1}_\omega E^A(\omega T \omega)(x) = \text{tr}_B(T\alpha_x(P))
\]
for all \( x \in G/H \). But this is exactly the covariant Berezin symbol of \( T \) (for this general context) as defined early in section 1 of [11] and denoted there by \( \sigma_T \). It is natural to put on \( D \) the action \( \lambda \otimes \alpha \) of \( G \). One then easily checks that \( \Phi^A \) is equivariant for \( \lambda \otimes \alpha \) and \( \lambda \). From this it is easy to verify that
\[
L^A(\Phi^A(T)) = L^A(\sigma_T) \leq L^B(T)
\]
for all \( T \in \mathcal{B} \), which is exactly the content of proposition 1.1 of [11]. Thus that part of admissibility is satisfied.

Now
\[
(\Phi^A(T)\omega - \omega T)(x) = \alpha_x(P)(\sigma_T(x)1_\mathcal{B} - T).
\]
Consequently
\[
\|\Phi^A(T)\omega - \omega T\|_D = \sup\{\|\alpha_x(P)(\sigma_T(x)1_\mathcal{B} - T)\|_B : x \in G/H\}.
\]
As discussed in the text before proposition 8.2 of [14], by equivariance this is
\[
= \sup\{\|P(\sigma_{\alpha_x(T)}1_\mathcal{B} - \alpha_x(T))\|_B : x \in G\}.
\]
Then because \( \alpha_x \) is isometric on \( \mathcal{B} \) for \( L^1_\mathcal{B} \) (as well as for the norm), we find for our present example that \( \gamma^B \), as defined in equation (4.1), is given by
\[
\gamma^B = \sup\{\|\Phi^A(T)\omega - \omega T\|_D : T \in L^1_\mathcal{B}\} \tag{6.2}
\]
\[
= \sup\{\|P(\text{tr}(PT)1_\mathcal{B} - T)\|_B : T \in L^1_\mathcal{B}\}.
\]
This last term is exactly the definition of \( \gamma^B \) given in proposition 8.2 of [14].

We next consider \( \gamma^A \). For any \( f \in \mathcal{A} \) we have
\[
(6.3) \quad \Phi^B(f) = r_A^{-1}E^B(f) = d_\mathcal{H} \int_{G/H} f(x)\alpha_x(P) \, dx.
\]
where \( d_\mathcal{H} = \dim(\mathcal{H}) \). But this is exactly the formula used for the Berezin contravariant symbol, as indicated in Section 4. Early in section 2 of [11] this \( \Phi^B \) is denoted by \( \tilde{\sigma}_f \), that is,
\[
(6.4) \quad \Phi^B(f) = \tilde{\sigma}_f
\]
for the present class of examples. One easily checks that \( \Phi^B \) is equivariant for \( \lambda \otimes \alpha \) and \( \alpha \). From this it is easy to verify that
\[
L^B(\Phi^B(f)) = L^B(\tilde{\sigma}_f) \leq L^A(f)
\]
for all \( f \in \mathcal{A} \), as shown in section 2 of [11]. Thus we obtain:

**Proposition 6.1.** The bridge with conditional expectations \( \Pi = (\mathcal{D}, \omega, E^A, E^B) \) is admissible for \( L^A \) and \( L^B \).

Much as in the statement of proposition 8.1 of [14] set
\[
(6.5) \quad \tilde{\gamma}^A = d_\mathcal{H} \int_{G/H} \rho_G(e, y)\|P\alpha_y(P)\| \, dy.
\]
In the proof of proposition 8.1 of [14] (given just before the statement of the proposition, and where \( \tilde{\gamma}^A \) is denoted just by \( \gamma^A \)) it is shown, with different notation, that
\[
(6.6) \quad \|f \omega - \omega \tilde{\sigma}_f\| \leq \tilde{\gamma}^A L^A(f).
\]
Thus if \( f \in L^1_A \) then
\[
\|f \omega - \omega \Phi^B(f)\| \leq \check{\gamma}^A.
\]
It follows that \( \gamma^A \leq \check{\gamma}^A \). We have thus obtained:

**Proposition 6.2.** For the present class of examples, with notation as above, we have

\[
\text{reach}(\Pi) \leq \max\{\gamma^A, \gamma^B\} \leq \max\{\check{\gamma}^A, \check{\gamma}^B\}
\]

where \( \check{\gamma}^A \) is defined in equation 6.3 and \( \gamma^B \) is defined in equation 6.2 (and 4.1) above.

Even more, for the case mentioned at the beginning of Section 3 in which \( G \) is a compact semisimple Lie group and \( \lambda \) is a positive integral weight, for each positive integer \( m \) let \((H^m, U^m)\) be the irreducible representation of \( G \) with highest weight \( m\lambda \). Then let \( B^m = L(H^m) \) with action \( \alpha \) of \( G \), and let \( P^m \) be the projection on the highest weight vector in \( H^m \). All the \( P^m \)'s will have the same \( \alpha \)-stability group, \( H \). As before, we let \( A = C(G/H) \). Then for each \( m \) we can construct as in Section 4 the bridge with conditional expectations, \( \Pi_m = (D_m, \omega^m, E^A_m, E^B_m) \). From a fixed length function \( \ell \) on \( G \) we will obtain Lip-norms \( \{L^B_m\} \) which together with \( \{L^A\} \) give meaning to the lengths of the bridges \( \Pi_m \). In turn the constants \( \gamma^A_m, \check{\gamma}^A_m, \gamma^B_m, \delta^A_m, \delta^B_m \) will be defined.

Now it follows from the discussion of \( \check{\gamma}^A \) above that \( \gamma^A_m \leq \check{\gamma}^A_m \) for each \( m \). But section 10 of [14] gives a proof that the sequence \( \check{\gamma}^A_m \) converges to 0 as \( m \) goes to \( \infty \). It follows that \( \gamma^A_m \) converges to 0 as \( m \) goes to \( \infty \). Then section 12 of [14] gives a proof that the sequence \( \gamma^B_m \) converges to 0 as \( m \) goes to \( \infty \). Putting together these results for \( \gamma^A_m \) and \( \gamma^B_m \), we obtain:

**Proposition 6.3.** The reach of the bridge \( \Pi_m \) goes to 0 as \( m \) goes to \( \infty \).

We now consider the height of \( \Pi \). For \( \delta^A \) something quite special happens. It is easily seen that \( A = C(G/H) \) is the center of \( D = A \otimes B \), and so all elements of \( A \) commute with \( \omega \). Thus we can apply Proposition 4.4 to conclude that \( \delta^A = 0 \).

In order to deal with \( B \) we use \( \check{\delta}^B \) of Notation 4.5 and the discussion surrounding it. For any \( T \in B \) we have
\[
\Phi^B(\Phi^A(T)) = r_{\omega}^{-1}E^B(\omega(r_{\omega}^{-1}E^A(\omega T\omega))\omega)
\]
\[
= d_H \int \alpha_x(P)(d_H \check{\tau}_B(\alpha_x(P)T\alpha_x(P))\alpha_x(P))dx
\]
\[
= d_H \int \alpha_x(P)(tr_B(\alpha_x(P)T))dx = \check{\sigma}(\sigma_T).
\]
where for the last term we use notation from [11, 14]. The term $\tilde{\sigma}(\sigma_T)$ plays an important role there. See theorem 6.1 of [11] and theorem 11.5 of [14]. The $\hat{\delta}^B$ of our Notation 4.5 is for the present class of examples exactly the $\delta^B$ of notation 8.4 of [14]. For use in the next section we here denote it by $\tilde{\delta}^B$, that is:

**Notation 6.4.** For $G$, $A = C(G/H)$, $B = \mathcal{L}(\mathcal{H})$, $\sigma$, $\tilde{\sigma}$, etc as above, we set

$$\tilde{\delta}^B = \sup\|T - \tilde{\sigma}(\sigma_T)\| : T \in \mathcal{L}^1_B.$$  

When we combine this with Propositions 4.6 and 6.2 we obtain:

**Theorem 6.5.** For the present class of examples, with notation as above, we have

$$\text{height}(\Pi) \leq \tilde{\delta}^B.$$  

Consequently

$$\text{length}(\Pi) \leq \max\{\gamma^A, \gamma^B, \min\{\delta^B, \tilde{\delta}^B\} \leq \max\{\tilde{\gamma}^A, \gamma^B, \min\{\delta^B, \tilde{\delta}^B\}\}.$$  

We will indicate in Section 8 Latrémolière’s definition of his propinquity between compact quantum metric spaces, but it is always no larger than the length of any bridge between the two spaces. He denotes his propinquity simply by $\Lambda$, but we will denote it here by “Prpq”.

Consequently, from the above theorem we obtain:

**Corollary 6.6.** With notation as above,

$$\text{Prpq}((A, L^A), (B, L^B)) \leq \max\{\tilde{\gamma}^A, \gamma^B, \min\{\delta^B, \tilde{\delta}^B\}\}.$$  

For the case of highest weight representations discussed just above, theorem 11.5 of [14] gives a proof that the sequence $\tilde{\delta}^B_m$ (in our notation) converges to 0 as $m$ goes to $\infty$. It follows from the above proposition that:

**Proposition 6.7.** The height of the bridge $\Pi_m$ goes to 0 as $m$ goes to $\infty$.

Combining this with Proposition 6.3 we obtain:

**Theorem 6.8.** The length of the bridge $\Pi_m$ goes to 0 as $m$ goes to $\infty$. Consequently $\text{Prpq}((A, L^A), (B^m, L^{B^m}))$ goes to 0 as $m$ goes to $\infty$.

We now treat the matricial case, beginning with the general situation in which $G$ is some compact group. We must first specify our matrix slip-norms. This is essentially done in example 3.2 of [18] and section 14 of [14]. As discussed in Section 2 we have the actions $\lambda$ and $\alpha$ on $A = C(G/H)$ and $B = \mathcal{B}(\mathcal{H})$ respectively. For any $n$ let $\lambda^n$ and $\alpha^n$ be the corresponding actions $\iota_n \otimes \lambda$ and $\iota_n \otimes \alpha$ on $M_n \otimes A = M_n(A)$
and $M_n \otimes B = M_n(B)$. We then use the length function $\ell$ and formula 2.1 to define seminorms $L^A_n$ and $L^B_n$ on $M_n(A)$ and $M_n(B)$. It is easily verified that $\{L^A_n\}$ and $\{L^B_n\}$ are matrix slip-norms. Notice that here $L^A_1 = L^A$ and $L^B_1 = L^B$ are actually lipnorms, and so, by property 1 of Definition 5.1, for each $n$ the null-spaces of $L^A_n$ and $L^B_n$ are exactly $M_n$.

Now fix $q$ and take $n = q$. From our bridge with conditional expectations $\Pi = (D, \omega, E^A, E^B)$ we define the bridge with conditional expectations $\Pi^q = (M_q(D), \omega_q, E^A_q, E^B_q)$ between $M_q(A)$ and $M_q(B)$ in the way done in Proposition 5.2. We then define $\Phi^A_q = \iota_q \otimes \Phi^A$, and similarly for $\Phi^B_q$ as done right after Proposition 5.2.

Because $\lambda$ and $\alpha$ and $\Phi^A$ and $\Phi^B$ act entry-wise on $M_q(D)$, and because $\Phi^A$ and $\Phi^B$ are equivariant for $\lambda \otimes \alpha$ and $\lambda$ and for $\lambda \otimes \alpha$ and $\alpha$ respectively, it is easily seen that $\Pi^q$ is admissible for $\{L^A_q\}$ and $\{L^B_q\}$.

We are thus in position to apply Theorem 5.4. From it and Theorem 6.5 we conclude that:

**Theorem 6.9.** With notation as above, we have

$$\text{reach}(\Pi^q) \leq 2q \max\{\gamma^A_q, \gamma^B_q\} \leq 2q \max\{\tilde{\gamma}^A_q, \tilde{\gamma}^B_q\},$$

where $\tilde{\gamma}^A_q$ is defined by formula 6.5. Furthermore

$$\text{height}(\Pi^q) \leq 2q \min\{\delta^B_q, \tilde{\delta}^B_q\},$$

where $\tilde{\delta}^B_q$ is defined in Notation 6.4. Thus

$$\text{length}(\Pi^q) \leq 2q \max\{\gamma^A_q, \gamma^B_q, \min\{\delta^B_q, \tilde{\delta}^B_q\}\}.$$

We remark that we could improve slightly on the above theorem by using a calculation given in section 14 of [14] in the middle of the discussion there of Wu’s results. Let $F \in M_q(A)$ be given, with $F = \{f_{jk}\}$, and set

$$t_{jk} = \check{\sigma}_{f_{jk}} = d_H \int f_{jk}(y)\alpha_y(P)dy,$$

and let $T = \{t_{jk}\}$. Then

$$(F \omega_q - \omega_q T)(x) = \{\alpha_x(P)(f_{jk}(x) - d_H \int f_{jk}(y)\alpha_y(P)dy)\} = \{d_H \int (f_{jk}(x) - f_{jk}(y))\alpha_x(P)\alpha_y(P)dy\}.$$
by $x$, in the way done shortly before proposition 8.1 of [14], it suffices to consider
\[
\sup \{ \| d_H \int (f_{jk}(e) - f_{jk}(y)) P \alpha_y(P) dy \| \}
\leq d_H \int \| F(e) - F(y) \| \| P \alpha_y(P) \| dy
\leq L^A_q(F) d_H \int \rho_{G/H}(e, y) \| P \alpha_y(P) \| dy = L^A_q(F) \tilde{\gamma}^A.
\]

In this way we see that $\gamma^A_q \leq \tilde{\gamma}^A$, with no factor of $2q$ needed.

We can apply Theorem 6.9 to the situation considered before Proposition 6.3 in which $G$ is a compact semisimple Lie group, $\lambda$ is a positive integral weight, and $(H^m, U^m)$ is the irreducible representation of $G$ with highest weight $m\lambda$ for each positive $m$, with $B^m = \mathcal{L}(H^m)$. We can then form the bridge with conditional expectations, $\Pi_m = (D_m, \omega, E^A_m, E^B_m)$ that is discussed there. For any positive integer $q$ we then have the matricial version involving $M_q(A), M_q(B_m),$ and the corresponding bridge $\Pi^q_m$. On applying Theorem 6.9 together with the results mentioned above about the convergence of the quantities $\tilde{\gamma}^A_m, \gamma^B_m,$ and $\tilde{\delta}^B_m$ to 0, we obtain one of the two main theorems of this paper:

**Theorem 6.10.** With notation as above, we have
\[
\text{length}(\Pi^q_m) \leq 2q \max \{ \tilde{\gamma}^A_m, \gamma^B_m, \min \{ \delta^B_m, \tilde{\delta}^B_m \} \}
\]
where $\tilde{\gamma}^A_m$ is defined as in formula (6.4) and where $\tilde{\delta}^B_m$ is defined as in Notation 6.4. Consequently $\text{length}(\Pi^q_m)$ converges to 0 as $m$ goes to $\infty$, for each fixed $q$.

We remark that because of the factor $q$ in the right-hand side of the above bound for $\text{length}(\Pi^q_m)$, we do not obtain convergence to 0 that is uniform in $q$. I do not have a counter-example to the convergence being uniform in $q$, but it seems to me very possible that the convergence will not be uniform.

7. The application to the second class of basic examples

We now apply our general considerations to our second basic class of examples, described in Section 6. We use the notation of that section. We also use much of the notation of Section 6, but now we have two representations, $(H^m, U^m)$ and $(H^n, U^n)$ (where for the moment $m$ and $n$ are just labels). We have corresponding $C^*$-algebras $B^m$ and $B^n$, and projections $P^m$ and $P^n$. 
We let $L^m$ be the Lip-norm defined on $B^m$ determined by the action $\alpha$ and the length function $\ell$ as in equation 2.1, and similarly for $L^n$ on $B^n$. In terms of these Lip-norms the length of any bridge from $B^m$ to $B^n$ is defined.

As in Section 3 we consider the bridge $\Pi = (D, \omega)$ for which

$$D = \mathcal{B}^m \otimes A \otimes \mathcal{B}^n = C(G/H, \mathcal{B}^m \otimes \mathcal{B}^n),$$

and the pivot, $\omega$, in $C(G/H, \mathcal{B}^m \otimes \mathcal{B}^n)$, is defined by

$$\omega(x) = \alpha_x(P^m) \otimes \alpha_x(P^n).$$

We view $\mathcal{B}^n$ as a subalgebra of $D$ by sending $T \in \mathcal{B}^m$ to $T \otimes 1_A \otimes 1_{\mathcal{B}^n}$, and similarly for $\mathcal{B}^m$.

As seen in Section 4, the tracial state $\tau_A \otimes \tau_n$ on $A \otimes \mathcal{B}^n$ determines a canonical conditional expectation, $E^m$, from $D$ onto $\mathcal{B}^m$, and the tracial state $\tau_m \otimes \tau_A$ determines a canonical conditional expectation, $E^n$, from $D$ onto $\mathcal{B}^n$. We find that for any $F \in D$ we have

$$E^m(F) = \int_{G/H} (\iota_m \otimes \tau_n)(F(x)) \, dx,$$

and similarly for $E^n$, where here $\iota_m$ is the identity map from $\mathcal{B}^m$ to itself. From this it is easily seen that

$$r^{-1}_\omega = d_m d_n$$

where $d_m$ is the dimension of $\mathcal{H}^m$ and similarly for $d_n$. Thus $\Pi = \{D, \omega, E^m, E^n\}$ is a bridge with conditional expectations.

Then

$$\Phi^m(F)(x) = r^{-1}_\omega E^m(\omega F \omega)(x) = d_m d_n E^m(\omega F \omega)(x).$$

But, if we set $\alpha_x(P^m \otimes P^n) = \alpha_x(P^m) \otimes \alpha_x(P^n)$, we have

$$E^m(\omega F \omega)(x) = \int (\iota_m \otimes \tau_n)(\alpha_x(P^m \otimes P^n)F(x)\alpha_x(P^m \otimes P^n)) \, dx.$$ 

In particular, for any $T \in \mathcal{B}^n$ we have

$$E^m(\omega T \omega)(x) = \int \alpha_x(P^m)\tau_n(T \alpha_x(P^n)) \, dx,$$

and so since $d_n \tau_{\mathcal{B}^n}$ is the usual unnormalized trace $\text{tr}_n$ on $\mathcal{B}^n$, we have

$$\Phi^m(T) = d_m \int \alpha_x(P^m)\text{tr}_n(T \alpha_x(P^n)) \, dx.$$

This is essentially the formula obtained in Section 4, and is exactly the second displayed formula in section 3 of [12]. Even more, with
notation as in Section 6, especially the $\Phi^A$ of equation (6.1), except for our different $B^m$ and $B^n$ etc, we see that we can write

$$(7.1) \quad \Phi^m(T) = \Phi^m(\Phi^A(T)) = \tilde{\sigma}^m(\sigma_T^m).$$

We have a similar equation for $\Phi^n(T)$, and we see that we depend on the context to make clear on which of the two algebras $B^m$ and $B^n$ we consider $\Phi^A$ to be defined.

As in the proof of Proposition 6.1, we can use the fact that $E^m$ and $E^n$ are equivariant, where the action of $G$ on $D$ is given by $\alpha \otimes \lambda \otimes \alpha$, to obtain:

**Proposition 7.1.** The bridge with conditional expectations $\Pi$ is admissible for $L^m$ and $L^n$.

The formula (7.1) suggests the following steps for obtaining a bound on the reach of $\Pi$ in terms of the data of the previous section. Let $S \in B^m$, $f \in C(G/H)$, and $T \in B^n$. Then, for the norm of $B^m \otimes B^n$ and for any $x \in G/H$, we have

$$|| (S\omega - \omega T)(x) || = || (S\alpha_x(P^m)) \otimes \alpha_x(P^n) - \alpha_x(P^n) \otimes \alpha_x(P^m) || T ||$$

$$\leq || (S\alpha_x(P^m)) \otimes \alpha_x(P^n) - f(x) (\alpha_x(P^m) \otimes \alpha_x(P^n)) ||$$

$$+ || f(x) (\alpha_x(P^m) \otimes \alpha_x(P^n)) - \alpha_x(P^n) \otimes \alpha_x(P^m) || T ||$$

$$(7.2) \quad = || S\alpha_x(P^n) - f(x) \alpha_x(P^n) || + || f(x) \alpha_x(P^n) - \alpha_x(P^n) || T ||.$$

Notice that the last two norms are in $B^m$ and $B^n$ respectively.

We will also use the $\Phi^B$ of equation (6.3), but now, to distinguish it from the $\Phi^m$ above, we indicate that it is defined on $A$ (and maps to $B^m$) by writing $\Phi^B_A$.

For fixed $T \in B^m$ let us set $f(x) = \Phi^A(T) = \text{tr}_n(\alpha_x(P^n)T)$, and then let us set $S = \Phi^B_A(f) = d_m \int f(x) \alpha_x(P^m)$. Thus $S = \tilde{\sigma}^m_f$ by equation (6.4). When we substitute these into the inequality (7.2), we obtain

$$|| (\Phi^B_A(f) \omega - \omega T)(x) ||$$

$$\leq || (\Phi^B_A(f)(x) - f(x) \alpha_x(P^n)) || + || \alpha_x(P^n)(f(x) - T) ||$$

In view of the definition of $f$, we recognize that the supremum over $x \in G/H$ of the second term on the right of the inequality sign is the kind of term involved in the supremum in the right-hand side of equality (6.2). Consequently that second term above is no greater than $\gamma^B L^n(T)$. To indicate that this comes from equality (6.2) we write $\gamma^B_A$ instead of just $\gamma^B$. Because $\Phi^B_A(f) = \tilde{\sigma}^m_f$, we also recognize that the supremum over $x \in G/H$ of the first term above on the right of the inequality sign is exactly (after taking adjoints to get $P^m$ on the correct side) the left hand side of inequality (6.6), where the $\omega$
there is that of Section \[6\]. Consequently that term is no greater than \(\bar{\gamma}_m^A L^A(f)\), where the subscript \(m\) on \(\bar{\gamma}_m^A\) indicates that \(P_m^m\) should be used in equation (6.5). But from the admissibility in Proposition 6.1 involving \(\Phi^B_n = \Phi^B_m\) and \(\Phi^A\) we have

\[
L^m(S) = L^m(\Phi^B_m(f)) \leq L^A(f) \leq L^n(T).
\]

Notice that it follows that if \(T \in L^1_B\), then \(S \in L^1_B\). Anyway, on taking the supremum over \(x \in G/H\), we obtain

\[
\|\Phi^B_n(f)\omega - \omega T\| \leq (\bar{\gamma}_m^A + \gamma^B_n) L^n(T).
\]

We see in this way that the distance from \(\omega_T\) to \(L^1_B\) is no bigger than \(\bar{\gamma}_m^A + \gamma^B_n\).

The role of \(\mathcal{A}\) in Theorem 4.3 is here being played by \(B^m\). So to reduce confusion we will here write \(\gamma^m_n\) for the \(\gamma^A\) of Theorem 4.3, showing also the dependence on \(n\). Thus by definition

\[
\gamma^m_n = \sup \{\|T\omega - \omega \Phi^n(T)\| : T \in L^1_B}\}.
\]

We define \(\gamma^n_m\) similarly. Then in terms of this notation, what we have found above is that

\[
\gamma^m_n \leq \bar{\gamma}_m^A + \gamma^B_n
\]

We now indicate the dependence of \(\Pi\) on \(m\) and \(n\) by writing \(\Pi_{m,n}\). The situation just above is essentially symmetric in \(m\) and \(n\), and so, on combining this with the first inequality of Theorem 4.3, we obtain:

**Proposition 7.2.** With notation as above, we have

\[
\text{reach}(\Pi_{m,n}) \leq \max\{\gamma^m_n, \gamma^n_m\} \leq \max\{\bar{\gamma}_m^A + \gamma^B_n, \gamma^n_m + \gamma^B_n\}
\]

As mentioned in the previous section, section 10 of [14] gives a proof that the sequence \(\bar{\gamma}_m^A\) converges to 0 as \(m\) goes to \(\infty\), while section 12 of [14] gives a proof that the sequence \(\gamma^B_n\) converges to 0 as \(m\) goes to \(\infty\). We thus see that we obtain:

**Proposition 7.3.** The reach of the bridge \(\Pi_{m,n}\) goes to 0 as \(m\) and \(n\) go to \(\infty\) simultaneously.

We now obtain an upper bound for the height of \(\Pi_{m,n}\). For this we will again use Proposition 4.6. We calculate as follows, using equation (7.1). For \(T \in B^n\) we have

\[
\Phi^n(\Phi^m(T)) = \Phi^n(\tilde{\sigma}^m(\sigma^n_T)) = \tilde{\sigma}^n(\sigma^m(\tilde{\sigma}^m(\sigma^n_T)))
\]

Thus

\[
\|T - \Phi^n(\Phi^m(T))\| \leq \|T - \tilde{\sigma}^n(\sigma^n_T)\| + \|\tilde{\sigma}^n(\sigma^n_T) - \tilde{\sigma}^n(\sigma^m(\tilde{\sigma}^m(\sigma^n_T)))\|
\]
\[ \leq \delta_A^{B^n} L^{B^n}(T) + \|\sigma^n_T - \sigma^m(\tilde{\sigma}^m(\sigma^n_T))\|, \]

where the first term of the last line comes from Notation 6.4 and we write \( \tilde{\delta}_A^{B^n} \) for the \( \tilde{\delta}_B^{B^n} \) there. But \( \sigma^n_T \) is just an element of \( \mathcal{A} \), and in inequality 11.2 of [14] it is shown that for any \( f \in \mathcal{A} \) we have

\[ \|f - \sigma^m(\tilde{\sigma}^m(f))\| \leq \tilde{\delta}_m^A L^A(f), \]

where \( \tilde{\delta}_m^A \) is defined in equation 11.1 of [14] by

\[ (7.4) \quad \tilde{\delta}_m^A = \int_{G/H} \rho_{G/H}(e,x) d_m \text{tr}(P_m^m \alpha_x(P^m)) \ dx. \]

(In equation 11.1 of [14] \( \tilde{\delta}_m^A \) is denoted just by \( \delta_m^A \). Also, \( \sigma^m \circ \tilde{\sigma}^m \) is, within our setting, the usual Berezin transform.) Thus we see that the second term of the last line of inequality (7.3) is no bigger than \( \tilde{\delta}_m^A L^A(\sigma^n_T) \). But \( L^A(\sigma^n_T) \leq L^{B^n}(T) \). From all of this we see that if \( T \in L^1_B^{B^n} \) then

\[ \|T - \Phi^m(\Phi^m(T))\| \leq \tilde{\delta}_m^{B^n} + \tilde{\delta}_m^A. \]

Again, the role of \( B \) in Notation 4.5 is being played here by \( B^n \), and so to reduce confusion we will here write \( \tilde{\delta}_m^n \) for the \( \tilde{\delta}_B^n \) of Notation 4.5. We then see that for our present class of examples, that depend on \( m \) and \( n \), we have

\[ \tilde{\delta}_m^n \leq \tilde{\delta}_m^{B^n} + \tilde{\delta}_m^A. \]

The situation is essentially symmetric in \( m \) and \( n \), and so, combining this with Propositions 4.6 and 7.2, we obtain:

**Theorem 7.4.** With notation as above, we have

\[ \text{height}(\Pi_{m,n}) \leq \max\{\tilde{\delta}_m^n, \tilde{\delta}_n^m\} \leq \max\{\tilde{\delta}_m^{B^n} + \tilde{\delta}_m^A, \tilde{\delta}_n^{B^n} + \tilde{\delta}_n^A\}. \]

Consequently

\[ \text{length}(\Pi_{m,n}) \leq \max\{\gamma_m^A + \gamma_n^{B^n}, \tilde{\gamma}_n^A + \gamma_m^{B^n}, \tilde{\delta}_n^{B^n} + \tilde{\delta}_m^A, \tilde{\delta}_m^{B^n} + \tilde{\delta}_n^A\}. \]

As mentioned in the previous section, theorem 11.5 of [14] gives a proof that the sequence \( \tilde{\delta}_m^{B^n} \) (in our notation) converges to 0 as \( m \) goes to \( \infty \), while theorem 3.4 of [11] shows that the sequence \( \delta_m^A \) (where it was denoted by \( \gamma_m \)) converges to 0 as \( m \) goes to \( \infty \). Thus when we combine this with Proposition 7.3 we obtain:

**Theorem 7.5.** The height of the bridge \( \Pi_{m,n} \) goes to 0 as \( m \) and \( n \) go to \( \infty \) simultaneously. Consequently the length of the bridge \( \Pi_{m,n} \) goes to 0 as \( m \) and \( n \) go to \( \infty \) simultaneously, and thus \( \text{Prpq}((B^n, L^n), (B^n, L^n)) \) goes to 0 as \( m \) and \( n \) go to \( \infty \) simultaneously.
We now consider the matricial case. For any natural number \( q \) we apply the constructions of Section 5 to obtain the bridge with conditional expectations

\[
\Pi_{m,n}^q = (M_q(\mathcal{D}), \omega_q, E^m_q, E^n_q)
\]

from \( M_q(\mathcal{B}^m) \) to \( M_q(\mathcal{B}^n) \). From this we then obtain the corresponding maps \( \Phi_m^q \) and \( \Phi_n^q \).

We have the actions \( \alpha^q \) of \( G \) on \( M_q(\mathcal{B}^m) \) and \( M_q(\mathcal{B}^n) \), much as discussed after Theorem 6.8. From these actions and the length function \( \ell \) we obtain the slip-norms \( L^m_q \) and \( L^n_q \). As \( q \) varies, these result in matrix slip-norms. One shows that \( \Pi_{m,n}^q \) is admissible for \( L^m_q \) and \( L^n_q \) by arguing in much the same way as done after formula (7.1).

We are thus in a position to apply Theorem 5.4, as well as the convergence to 0 indicated above for the various constants, to obtain the second main theorem of this paper:

**Theorem 7.6.** With notation as above, we have

\[
\text{length}(\Pi_{m,n}^q) \leq 2q \max\{\tilde{\gamma}_m^A + \gamma^B_m, \tilde{\gamma}_n^A + \gamma^B_n, \tilde{\delta}_m^A + \tilde{\gamma}_m^B, \tilde{\delta}_n^A + \tilde{\gamma}_n^B\}
\]

where \( \tilde{\gamma}_m^A \) is defined as in formula (6.5) while \( \gamma^B_m \) is the \( \gamma^B_m \) of equation (6.2), and where \( \tilde{\delta}_m^A = \tilde{\delta}_m^B \) is defined in Notation 6.4 while \( \tilde{\delta}_m^A \) is defined by equation (7.4), and similarly for \( n \). Consequently \( \text{length}(\Pi_{m,n}^q) \) converges to 0 as \( m \) and \( n \) go to \( \infty \) simultaneously, for each fixed \( q \).

8. Treks

Latréomiliez defines his propinquity in terms of “treks”. We will not give here the precise definition (for which see definition 3.20 of [8]), but the notion is quite intuitive. A trek is a finite “path” of bridges, so that the “range” of the first bridge should be the “domain” of the second, etc. The length of a trek is the sum of the lengths of the bridges in it. The propinquity between two quantum compact metric spaces is the infimum of the lengths of all the treks between them. Latréomiliez shows in [8] that propinquity is a metric on the collection of isometric isomorphism classes of quantum compact metric spaces. Notably, he proves the striking fact that if the propinquity between two quantum compact metric spaces is 0 then they are isometrically isomorphic.

There is an evident trek associated with our second class of examples. In this section we will briefly examine this trek. Let the notation be as in the early parts of the previous section. Thus we have \( \mathcal{A} = C(G/H) \), and the operator algebras \( \mathcal{B}^m \) and \( \mathcal{B}^n \). In Section 5 we have the bridge \( \Pi_m = (\mathcal{A} \otimes \mathcal{B}^m, \omega_m) \) from \( \mathcal{A} \) to \( \mathcal{B}^m \), and the corresponding bridge \( \Pi_n \) from \( \mathcal{A} \) to \( \mathcal{B}^n \). But by reversing the roles of \( \mathcal{A} \) and \( \mathcal{B}^m \) we obtain a bridge from \( \mathcal{B}^m \) to \( \mathcal{A} \). We do this by still viewing \( \mathcal{A} \) and \( \mathcal{B}^m \) as
subalgebras of $D_m = C(G/H, B^m)$, but we now let $A$ act on the right of $D_m$ and we let $B^m$ act on the left. We will denote this bridge by $D_m^{-1}$, which is consistent with the notation of Latrémolière at the beginning of the proof of proposition 4.7 of \[8\]. Of course $D_m^{-1}$ has the “same” conditional expectations $E^A$ and $E^{B_m}$ as those of $\Pi_m$. We will write $E^A$ as $E^A_m$ to distinguish it from the $E^A$ from $D_n$, which we will denote by $E^A_n$. Then $D_m^{-1}$ is a bridge with conditional expectations, which is easily seen to be admissible for $L^{B_m}$ and $L^A$. It is then easily seen that

$$\text{length}(D_m^{-1}) = \text{length}(D_m).$$

The pair $\Gamma_{m,n} = (D_m^{-1}, D_n)$ then forms a trek from $B^m$ to $B^n$, and

$$\text{length}(\Gamma_{m,n}) = \text{length}(D_m^{-1}) + \text{length}(D_n).$$

From Theorem 6.5 it follows that

$$\text{length}(\Gamma_{m,n}) \leq \max\{\tilde{\gamma}_m^A, \gamma^B_m, \min\{\delta^{B_m}, \tilde{\delta}^{B_m}\}\} + \max\{\tilde{\gamma}_n^A, \gamma^B_n, \min\{\delta^{B_n}, \tilde{\delta}^{B_n}\}\}.$$ 

Note that $\tilde{\delta}_m^A$ and $\tilde{\delta}_n^A$ do not appear in the above expression, in contrast to their appearance in the estimate in Theorem 7.4 for length($\Pi_{m,n}$). This opens the possibility that in some cases length($\Gamma_{m,n}$) gives a smaller bound for $\Prpq(B^m, B^n)$ than does length($\Pi_{m,n}$), and, even more, that this might give examples for which the lengths of certain multi-bridge treks are strictly smaller that the lengths of any single-bridge treks. But I have not tried to determine if this happens for the examples in this paper.

We can view the situation slightly differently as follows. Although Latrémolière does not mention it, it is natural to define the reach of a trek as the sum of the reaches of the bridges it contains, and similarly for the height of a trek. One could then give a new definition of the length of a trek as simply the max of its reach and height. This definition is no bigger that the original definition, and might be smaller. I have not examined how this might affect the arguments in \[8\], but I imagine that the effect would not be very significant. Anyway, for the above examples we see from Proposition 6.2 that we would have

$$\text{reach}(\Gamma_{m,n}) \leq \max\{\tilde{\gamma}_m^A, \gamma^B_m\} + \max\{\tilde{\gamma}_n^A, \gamma^B_n\},$$

so that the bound for reach($\Pi_{m,n}$) given in Proposition 7.2 is no bigger than that above for reach($\Gamma_{m,n}$). But from Theorem 6.5 we see that

$$\text{height}(\Gamma_{m,n}) \leq \tilde{\delta}^{B_m} + \tilde{\delta}^{B_n}$$

(where $\tilde{\delta}_m^A = \tilde{\delta}^{B_m}$), and this can clearly be less than the right-most bound for height($\Pi_{m,n}$) given in Theorem 7.4.
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