Abstract
My previous calculations of the Lamb shift in muonic hydrogen are reviewed and compared with other work. In addition, numerical results for muonic deuterium are presented.

Introduction
The energy levels of muonic atoms are very sensitive to effects of quantum electrodynamics (QED), nuclear structure, and recoil, since the muon is about 206 times heavier than the electron [1]. A number of theoretical analyses of the Lamb shift (the 2p-2s transition) in light muonic atoms have been published [2, 3, 4, 5, 15, 7, 9, 16], most recently in view of a proposed measurement of the Lamb shift in muonic hydrogen [8]. The present paper repeats the independent recalculation of some of the most important effects [3] and extends the numerical calculations to the case of muonic deuterium, including effects that were not considered previously [10]. Muonic deuterium is in many ways similar to muonic hydrogen, but there are some differences. In addition to the different mass the deuteron has spin 1 and both magnetic and quadrupole moments.

Vacuum Polarization
The most important QED effect for muonic atoms is the virtual production and annihilation of a single $e^+e^-$ pair. It has as a consequence an effective interaction of order $\alpha Z \alpha$ which is usually called the Uehling potential [17, 18]. This interaction describes the most important modification of Coulomb’s law. Numerically it is so important that it should not be treated using perturbation theory; instead the Uehling potential should be added to the nuclear electrostatic potential before solving the Dirac equation. However, a perturbative treatment is also useful in the case of very light atoms, such as hydrogen.
However, unlike some other authors, we prefer to use relativistic (Dirac) wave functions to describe the muonic orbit. Since these contributions have been extensively discussed in the literature [1, 2, 3, 4] (among others), there is no need to go into detail here. The results, calculated as the expectation value of the Uehling potential using point-Coulomb Dirac wave functions with reduced mass are, for muonic deuterium:

|        | point nucleus | $R_d = 2.139\text{ fm}$ |
|--------|---------------|--------------------------|
| Uehling| $2p_{1/2} - 2s_{1/2}$ | $2p_{3/2} - 2s_{1/2}$ | $2p_{1/2} - 2s_{1/2}$ | $2p_{3/2} - 2s_{1/2}$ |
|        | 227.6577      | 227.6635                 | 227.5985                 | 227.6043                 |
| Kaellen-Sabry | 1.66622          | 1.66626                 | 1.66577                 | 1.66582                 |

The effect of finite proton size calculated here can be parametrized as $-0.0129 \langle r^2 \rangle$. However higher iterations can change these results. Up to now, these have not been calculated well for muonic deuterium, as far as I know.

Corresponding numbers for muonic hydrogen, calculated as the expectation value of the Uehling potential using point-Coulomb Dirac wave functions with reduced mass are:

|        | point nucleus | $R_p = 0.875\text{ fm}$ |
|--------|---------------|--------------------------|
| Uehling| $2p_{1/2} - 2s_{1/2}$ | $2p_{3/2} - 2s_{1/2}$ | $2p_{1/2} - 2s_{1/2}$ | $2p_{3/2} - 2s_{1/2}$ |
|        | 205.0282      | 205.0332                 | 205.0199                 | 205.0250                 |
| Kaellen-Sabry | 1.50814          | 1.50818                 | 1.50807                 | 1.50811                 |

The effect of finite proton size calculated here can be parametrized as $-0.0109 \langle r^2 \rangle$. However higher iterations can change these results. The contribution due to two and three iterations have been calculated by [4] and [23], respectively, giving a total of 0.151 meV. An additional higher iteration including finite size and vacuum polarization is given in ref. [4] (equations(66) and (67)) and ref. [2] (equations(264) and (268)). These amount to $-0.0164 \langle r^2 \rangle$. The best way to calculate this would be an accurate numerical solution of the Dirac equation in the combined Coulomb-plus Uehling potential.

The mixed muon-electron vacuum polarization correction ([21, 2]) is 0.00007 meV for hydrogen and 0.00008 meV for deuterium.

The Wichmann-Kroll contribution was calculated using the parametrization for the potential given in [1]. The result obtained for hydrogen is -0.00103 meV, consistent with that given in [2]. For deuterium, the contribution is -0.00111 meV.

The equivalent potential for the virtual Delbrück effect was recomputed from the Fourier transform given in [22] and [1]. The resulting potential was checked by reproducing previously calculated results for the 2s-2p transition in muonic Helium, and the 3d-2p transitions in muonic Mg and Si. The result for hydrogen is $+(0.00135 \pm 0.00015)$ meV, and for deuterium it is $+(0.00147 \pm 0.00016)$ meV. As in the case of muonic helium, this contribution very nearly cancels the Wichmann-Kroll contribution. The contribution corresponding to three photons to the muon and one to the proton should be analogous to the light by light contribution to the muon anomalous moment; to my knowledge, the corresponding contribution to the muon form factor has never been calculated. It will be comparable to the other light by light contributions. This graph was included in contributions to the muon’s anomalous magnetic moment; the contribution to the muon form factor is one of the most significant unknown corrections.
The sixth order vacuum polarization corrections to the Lamb shift in muonic hydrogen have been calculated by Kinoshita and Nio [23]. Their result for the 2p-2s transition (in hydrogen) is

\[ \Delta E^{(6)} = 0.120045 \cdot (\alpha Z)^2 \cdot m_r \left( \frac{\alpha}{\pi} \right)^3 \approx 0.00761 \text{ meV} \]

and 0.00804 meV for muonic deuterium.

However, I should remark that the contributions from figures 1 and 2 of Ref. [23] were checked by direct integration. Although the results agreed perfectly for the case of hydrogen, there were small, but significant discrepancies for the case of deuterium. (hydrogen: Fig. 1 contributes 0.000396 meV and Fig. 2 contributes 0.002931 meV; deuterium: direct integration gave 0.000472 meV and 0.003364 meV, respectively, while the work of ref. [23] indicates values 0.000419 meV and 0.003906 meV, respectively). This indicates that, at least for these two graphs, integration over momentum transfer involves more than a single reduced mass factor.

The hadronic vacuum polarization contribution has been estimated by a number of authors [27, 28, 2]. It amounts to about 0.012 meV in hydrogen and 0.013 meV in deuterium. One point that should not be forgotten about the hadronic VP correction is the fact that the sum rule or dispersion relation that everyone (including myself) used does not take into account the fact that the proton (nucleus) can in principle interact strongly with the hadrons in the virtual hadron loop. It is irrelevant for the anomalous magnetic moment but probably not for muonic atoms. An estimation of this effect appears to be extremely difficult, and could easily change the correction by up to 50%. Eides et al. [2] point out that the graph related to hadronic vacuum polarization can also contribute to the measured value of the nuclear charge distribution (and polarizability). It is not easy to determine where the contribution should be assigned. This may also be true for the so-called "proton self-energy" [5, 2], which involves some of the same graphs as are present in the calculation of radiative corrections to electron scattering.

Finite nuclear size and nuclear polarization
The main contribution due to finite nuclear size has been given analytically to order \((\alpha Z)^6\) by Friar [24]. The main result is

\[
\Delta E_{ns} = -2\alpha Z \left( \frac{\alpha Z m_r}{n} \right)^3 \cdot \left[ \langle r^2 \rangle - \frac{\alpha Z m_r}{2} \langle r^3 \rangle_{(2)} + (\alpha Z)^2 (F_{RL} + m_r^2 F_{NR}) \right]
\]

where \(\langle r^2 \rangle\) is the mean square radius of the proton. For muonic hydrogen, the coefficient of \(\langle r^2 \rangle\) is 5.1975 (meV fm\(^{-2}\)), giving an energy shift (for the leading term) of (3.979\pm0.076) meV if the proton rms radius is 0.875 fm. Other values of the proton radius that have been reported recently in the literature are 0.880 fm [25] and (0.895\pm0.018 fm) [26]. The second term in Eq.(1) contributes -0.0232 meV for a dipole form factor and -0.0212 meV for a Gaussian form factor. The parameters were fitted to the proton rms radius. This can be written as -0.0347\(\langle r^2 \rangle^{3/2}\) or -0.0317\(\langle r^2 \rangle^{3/2}\). This differs slightly from the value given by Pachucki [5]. The model dependence introduces an uncertainty about \(\pm0.002\) meV. The remaining terms contribute 0.00046 meV. This estimate includes all of the terms given in [24], while other authors [4] give only some of them. Clearly the
neglected terms are not negligible. There is also a contribution of \(-3 \cdot 10^{-6}\) meV to the binding energy of the 2p_{1/2}-level, and a recoil correction of 0.013 meV to the binding energy of the 2s-level.

Pachucki [5] has estimated a correction similar to the second term (proportional to \(\langle r^3 \rangle_{(2)}\)) in Eq. (1). Since the logarithmic terms in the two-photon correction without finite size (see below) also seem to be suspect, this correction requires further investigation. In particular, the parametrization of the form factors used in any calculation should reproduce the correct proton radius.

For muonic deuterium, the main contribution amounts to 
\[-6.0732 \langle r^2 \rangle = - (27.787 \pm 0.078)\) meV. Depending on the model, the term proportional to \(\langle r^3 \rangle_{(2)}\) gives a contribution of 0.382 meV or 0.417 meV.

As mentioned previously, the finite-size contributions to vacuum polarization in muonic hydrogen can be parametrized as 
\[-0.0109 \langle r^2 \rangle - 0.0164 \langle r^2 \rangle^2 - 0.0209(6)\) meV if the proton radius is 0.875 fm. For deuterium, only the contribution corresponding to the first term of the sum \((-0.0129 \langle r^2 \rangle\) has been calculated.

The contribution due to nuclear polarization (in hydrogen) has been calculated by Rosenfelder [29] to be 0.017 \pm 0.004 meV, and by Pachuki [5] to be 0.012 \pm 0.002 meV. Other calculations [30, 31] give intermediate values (0.013 meV and 0.016 meV, respectively). The value appearing in table 2 is an average of the three most recent values, with the largest quoted uncertainty, which is probably underestimated.

### Relativistic Recoil

As is well-known, the center-of-mass motion can be separated exactly from the relative motion only in the nonrelativistic limit. Relativistic corrections have been studied by many authors, and will not be reviewed here. The relativistic recoil corrections summarized in [1] include the effect of finite nuclear size to leading order in \(m_\mu/m_N\) properly.

Up to now this method has been used to treat recoil corrections to vacuum polarization only in the context of extensive numerical calculations that include the Uehling potential in the complete potential, as described in [1]. They can be included explicitly, as a perturbation correction to point-Coulomb values. Recall that (to leading order in \(1/m_N\),

the energy levels are given by

\[ E = E_r - \frac{B_0^2}{2m_N} + \frac{1}{2m_N} \langle h(r) + 2B_0 P_1(r) \rangle \tag{2} \]

where \(E_r\) is the energy level calculated using the reduced mass and \(B_0\) is the unperturbed binding energy. Also

\[ h(r) = -P_1(r)(P_1(r) + \frac{1}{r}Q_2(r)) - \frac{1}{3r}Q_2(r)[P_1(r) + Q_4(r)/r^3] \tag{3} \]
Here

\[ P_1(r) = 4\pi\alpha Z \int_r^\infty r' \rho(r') \, dr' = -V(r) - rV'(r) \quad (4) \]

\[ Q_2(r) = 4\pi\alpha Z \int_0^r r^2 \rho(r') \, dr' = r^2 V'(r) \]

\[ Q_4(r) = 4\pi\alpha Z \int_0^r r^4 \rho(r') \, dr' \]

An effective charge density \( \rho_{VP} \) for vacuum polarization can be derived from the Fourier transform of the Uehling potential. Recall that (for a point nucleus)

\[ V_{\text{Uehl}}(r) = -\frac{\alpha Z}{r} \chi_1(2m_e r) \]

\[ = -\left(\frac{\alpha Z}{r}\right)^2 \chi_1(2m_e) \int_1^\infty dz \frac{(z^2 - 1)^{1/2}}{z^2} \left(1 + \frac{1}{2z^2}\right) \left(\frac{2}{\pi} \int_0^\infty \frac{q^2 \cdot j_0(q r)}{q^2 + 4m_e^2 z^2} dq\right) \]

where \( \chi_n(x) \) is defined in [1]. In momentum space, the Fourier transform of \( \nabla^2 V \) is obtained by multiplying the Fourier transform of \( V \) by \( -q^2 \). Note that using the normalizations of [1, 7], one has

\[ \nabla^2 V = -4\pi\alpha Z \rho \]

where \( \rho \) is the charge density. One then obtains

\[ 4\pi \rho_{VP}(r) = \frac{2\alpha}{3\pi} \int_1^\infty d\frac{x^2 - 1}{x^2} \left(1 + \frac{1}{2x^2}\right) \left(\frac{2}{\pi} \int_0^\infty \frac{q^4 \cdot j_0(q r)}{q^2 + 4m_e^2 z^2} dq\right) \]

where \( U_2(q) \) is defined in [2].

Keeping only the Coulomb and Uehling potentials, one finds

\[ P_1(r) = -\frac{2\alpha}{3\pi} (2m_e) \chi_0(2m_e r) \]

\[ Q_2(r) = \alpha Z \left(1 + \frac{2\alpha}{3\pi} \left[\chi_1(2m_e r) + (2m_e r) \chi_0(2m_e r)\right]\right) \]

\[ Q_4(r) = \frac{2\alpha}{3\pi} \int_1^\infty d\frac{z^2 - 1}{z^2} \left(1 + \frac{1}{2z^2}\right) \left(\frac{2}{\pi} \int_0^\infty \frac{q^4 \cdot j_0(q r)}{q^2 + 4m_e^2 z^2} dq\right) \]

\[ \cdot \left(\frac{2}{\pi} \int_0^\infty \frac{1}{q^2 + 4m_e^2 z^2} \frac{(6qr - (qr)^3) \cos(qr) + (3(qr)^2 - 6) \sin(qr)}{q} dq\right) \]

Details of the calculations for the case of vacuum polarization are given in Appendix 1 and in Ref.[3]. Corrections due to finite nuclear size can be included when a model for the charge distribution is given. This done by Friar [24] (and confirmed independently for two different model charge distributions); the contribution due to finite nuclear size to the recoil correction for the binding energy of the 2s-level is -0.013 meV. The factor \( 1/m_n \) is replaced by \( 1/(m_\mu + m_N) \), also consistent with the calculations presented in [24].

Combining the expectation values given in Appendix 1 according to equations 2 and 3, one finds a contribution to the 2p-2s transition of -0.00419 meV (hydrogen) and -0.00479 meV.
To obtain the full relativistic and recoil corrections, one must add the difference between the expectation values of the Uehling potential calculated with relativistic and nonrelativistic wave functions, giving a total correction of 0.0166 meV for muonic hydrogen. This is in quite good agreement with the correction of 0.0169 meV calculated by Veitia and Pachucki [33]. The treatment presented here has the advantage of avoiding second order perturbation theory. For deuterium, one obtains a total correction of 0.0179 meV.

The review by Eides et al. [2] gives a better version of the two photon recoil (Eq. 136) than was available for the review by Borie and G. Rinker [1]. Evaluating this expression for muonic hydrogen gives a contribution of -0.04497 meV to the 2p-2s transition in hydrogen and -0.02656 meV in deuterium. Higher order radiative recoil corrections give an additional contribution (in hydrogen) of -0.0096 meV [2]. However, some of the contributions to the expressions given in [2] involve logarithms of the mass ratio $m_{\mu}/m_N$. Logarithms can only arise in integrations in the region from $m_{\mu}$ to $m_N$; in this region the effect of the nuclear form factor should not be neglected. Pachucki [4] has estimated a finite size correction to this of about 0.02 meV, which seems to be similar to the term proportional to $\langle r^3 \rangle$ given in Eq. (11) as calculated in the external field approximation by Friar [24]. This two-photon correction requires further investigation. In particular, the parametrization of the form factors used in any calculation should reproduce the correct proton radius. Also the relationship among the different contributions needs to be specified more clearly.

An additional recoil correction for states with $\ell \neq 0$ has been given by [34] (see also [2]). It is

$$\Delta E_{n,\ell,j} = \frac{(\alpha Z)^4 \cdot m_{\mu}^3}{2n^3 m_N^2} (1 - \delta_{\ell 0}) \left( \frac{1}{\kappa(2\ell + 1)} \right)$$

When evaluated for the 2p-states of muonic hydrogen, one finds a contribution to the 2p-2s transition energy of 0.0575 meV for the 2p$_{1/2}$ state and -0.0287 meV for the 2p$_{3/2}$ state in hydrogen (0.0168 meV for the 2p$_{1/2}$ state and -0.0084 meV for the 2p$_{3/2}$ state in deuterium).

A final point about recoil corrections is that in the case of light muonic atoms, the mass ratio $m_{\mu}/m_N$ is considerably larger than the usual perturbation expansion parameter $\alpha Z$. Contributions of higher order in the mass ratio could be significant.

### Muon Lamb Shift

For the calculation of muon self-energy and vacuum polarization, the lowest order (one-loop approximation) contribution is well-known, at least in perturbation theory. Including also muon vacuum polarization (0.0168 meV) and an extra term of order $(Z\alpha)^5$ as given in [2], which contributes -0.00443 meV, one finds a contribution of -0.66788 meV for the $2s_{1/2} - 2p_{1/2}$ transition and -0.65031 meV for the $2s_{1/2} - 2p_{3/2}$ transition. For deuterium, the corresponding contributions are given by -0.77462 meV for the $2s_{1/2} - 2p_{1/2}$ transition and -0.75512 meV for the $2s_{1/2} - 2p_{3/2}$ transition. The second order calculation in deuterium includes muonic vacuum polarization (0.01968 meV); the extra term of order $(Z\alpha)^5$ as given in [2], contributes -0.00518 meV.
These results, and the higher order corrections \[1, 21\] can be summarized as

| Transition          | \(2p_{1/2} - 2s_{1/2}\) | \(2p_{3/2} - 2s_{1/2}\) |
|---------------------|--------------------------|--------------------------|
| Hydrogen            |                           |                           |
| second order        | -0.66788                 | -0.65031                 |
| higher orders       | -0.00172                 | -0.00165                 |
| Total               | -0.66960                 | -0.65196                 |
| Deuterium           |                           |                           |
| second order        | -0.774616                | -0.755125                |
| higher orders       | -0.002001                | -0.001926                |
| Total               | -0.776617                | -0.757051                |

Table 1: Contributions to the muon Lamb shift \(E(2p_{1/2}) - E(2s_{1/2})\) in muonic hydrogen and deuterium, in meV.

For hydrogen, Pachuki \[4\] has estimated an additional contribution of -0.005 meV for a contribution corresponding to a vacuum polarization insert in the external photon.

The higher order contributions can be written in the form

\[
\Delta E_{LS} = \frac{1}{m^2_{\mu}} \cdot \langle \nabla^2 V \rangle \left[ m^2_{\mu} F_1'(0) + \frac{a_\mu}{2} \right] + \frac{a_\mu}{2} m^2_{\mu} \langle \frac{2}{r} \frac{dV}{dr} \vec{L} \cdot \vec{\sigma}_\mu \rangle
\]

where \(F_2(0) = a_\mu\); the higher order contributions (fourth and sixth) can be taken from the well-known theory of the muon’s anomalous magnetic moment:

\[
F_2(0) = a_\mu = \frac{\alpha}{2\pi} + 0.7658(\alpha/\pi)^2 + 24.05(\alpha/\pi)^3.
\]

The fourth order contribution to \(F_1'(0)\) is

\[
0.46994(\alpha/\pi)^2 + 2.21656(\alpha/\pi)^2 = 2.68650(\alpha/\pi)^2 \quad \text{[1]}
\]

The sixth order contributions to \(F_1'(0)\) that involve electron vacuum polarization loops (especially the light-by-light graph) might contribute at an experimentally significant level, but have not been calculated.
Summary of contributions for muonic hydrogen

Using the fundamental constants from the CODATA 2002 (11) one finds the transition energies in meV in table 2. Here the main vacuum polarization contributions are given for a point nucleus, using the Dirac equation with reduced mass. Some uncertainties have been increased from the values given by the authors, as discussed in the text.

The finite size corrections for hydrogen up to order \((\alpha Z)^5\) can be parametrized as

\[
5.1975\langle r^2 \rangle + 0.0109\langle r^2 \rangle + 0.0164\langle r^2 \rangle + 0.0347\langle r^3 \rangle.
\]

The various contributions are discussed in the text.

| Contribution                  | Value (meV) | Uncertainty (meV) |
|-------------------------------|-------------|-------------------|
| Uehling                       | 205.0282    |                   |
| Källen-Sabry                  | 1.5081      |                   |
| Wichmann-Kroll                | -0.00103    |                   |
| virt. Delbrücke               | 0.00135     | 0.00015           |
| mixed mu-e VP                 | 0.00007     |                   |
| hadronic VP                   | 0.011       | 0.002             |
| sixth order [23]              | 0.00761     |                   |
| recoil [2] (eq136)            | -0.04497    |                   |
| recoil, higher order [2]      | -0.0096     |                   |
| recoil, finite size [24]      | 0.013       | 0.001             |
| recoil correction to VP [1]   | -0.0042     |                   |
| additional recoil [34]        | 0.0575      |                   |
| muon Lamb shift               |             |                   |
| second order                  | -0.66788    |                   |
| fourth order                  | -0.00169    |                   |
| nuclear size \((R_p=0.875\text{ fm})\) | 0.007 fm  |                   |
| main correction [24]          | -3.979      | 0.076             |
| order \((\alpha Z)^5\) [24]  | 0.0232      | 0.002             |
| order \((\alpha Z)^6\) [24]  | -0.0005     |                   |
| correction to VP              | -0.0083     |                   |
| polarization                  | 0.015       | 0.004             |
| Other (not checked)           |             |                   |
| VP iterations [4]             | 0.151       |                   |
| VP insertion in self energy [4]| -0.005     |                   |
| additional size for VP [2]    | -0.0128     |                   |

Table 2: Contributions to the muonic hydrogen Lamb shift (the \(2s_{1/2} - 2p_{1/2}\) transition). The proton radius is taken from [11].

Summary of contributions for muonic deuterium

For deuterium, one finds the transition energies in meV in table 3. Also here the main vacuum polarization contributions are given for a point nucleus, using the Dirac equation with reduced mass. The finite size corrections for deuterium up to order \((\alpha Z)^5\) can be
parametrized as
6.0732\langle r^2 \rangle + 0.0129\langle r^2 \rangle + 0.0409\langle r^3 \rangle_{(2)}, although not all contributions to the effect of finite size on the vacuum polarization correction are included.

| Contribution                              | Value (meV) | Uncertainty (meV) |
|-------------------------------------------|-------------|-------------------|
| Uehling                                   | 227.6577    |                   |
| Källén-Sabry                              | 1.6662      |                   |
| Wichmann-Kroll                            | -0.00111    |                   |
| virt. Delbrück                            | 0.00147     | 0.00016           |
| mixed mu-e VP                             | 0.00008     |                   |
| hadronic VP                               | 0.013       | 0.002             |
| sixth order                               | 0.00804     |                   |
| recoil [2] (eq136)                        | -0.02656    |                   |
| recoil, higher order [2]                  | ?           |                   |
| recoil, finite size [24]                  | 0.019       | 0.003             |
| recoil correction to VP [1]               | -0.0048     |                   |
| additional recoil [34]                    | 0.0168      |                   |
| muon Lamb shift                           |             |                   |
| second order                              | -0.77462    |                   |
| fourth order                              | -0.00200    |                   |
| nuclear size \((R_d=2.139\text{fm})\)     |             | 0.003\text{fm}   |
| main correction [24]                      | -27.787     | 0.078             |
| order \((\alpha Z)^5\) [24]              | 0.0400      | 0.018             |
| order \((\alpha Z)^6\) [24]              | -0.0045     |                   |
| correction to VP                          | -0.0592     |                   |
| polarization                              | ?           |                   |
| Other (not checked)                       |             |                   |
| VP iterations [4]                         | ?           |                   |
| VP insertion in self energy [4]           | ?           |                   |
| additional size for VP [2]                | ?           |                   |

Table 3: Contributions to the muonic deuterium Lamb shift. The deuteron radius is taken from [11].

Fine structure of the 2p state
The fine structure of the 2p states can be calculated by using the relativistic Dirac energies, computing the vacuum polarization contributions with Dirac wave functions, and including the effect of the anomalous magnetic moment in the muon Lamb shift. An additional recoil correction (Eq(38)) also has to be included. The results are summarized in table 4. One should also include the \(B^2/2M_N\)-type correction to the fine structure. (see [2], Eq(38)). This is tiny \((5.7\cdot10^{-6}\text{meV in hydrogen})\) and is not included in the table. Friar [24] has given expressions for the energy shifts of the 2p-states due to finite nuclear
|                | Hydrogen | Deuterium |
|----------------|----------|-----------|
| Dirac          | 8.41564  | 8.86430   |
| Uehling(VP)    | 0.0050   | 0.00575   |
| Källen-Sabry   | 0.00004  | 0.00005   |
| anomalous moment $a_\mu$ | 0.01757  | 0.01491   |
| higher orders  | 0.00007  | 0.00007   |
| Recoil (Eq. 4) | -0.0862  | -0.0252   |
| Total Fine Structure | 8.352    | 8.864     |

**Table 4:** Contributions to the fine structure ($E(2p_{3/2}) - E(2p_{1/2})$) of the 2p-state in muonic hydrogen and deuterium, in meV.

These were calculated and found to give a negligible contribution ($3.1 \times 10^{-6}$ meV) to the fine structure of the 2p-state in hydrogen.

**Hyperfine structure**

The Breit equation \[34, 2, 7\] contributions to the fine- and hyperfine interactions for general potentials and arbitrary spins were given by Metzner and Pilkuhn \[36\]. Here a version applicable to the case of muonic atoms ($Z_1 = -1, s_1 = 1/2, m_1 = m_\mu, \kappa_1 = a_\mu, Z_2 = Z$) is given.

$$V_{L,s_1} = \frac{1}{2m_\mu} \frac{1}{r} \frac{dV}{dr} \left[ \frac{1}{s_1 m_r} - \frac{1}{m_\mu} \right] \vec{L} \cdot \vec{s}_1$$

(7)

This can be rearranged to give the well-known form for spin 1/2 particles with an anomalous magnetic moment, namely

$$-\frac{1}{r} \frac{dV}{dr} \cdot \frac{1 + a_\mu + (a_\mu + 1/2)m_N/m_\mu}{m_N m_\mu} \vec{L} \cdot \vec{\sigma}_\mu$$

Note that

$$\frac{1}{m_N m_\mu} + \frac{1}{2m_\mu^2} = \frac{1}{2m_r^2} - \frac{1}{2m_N^2}$$

so that the terms not involving $a_\mu$ in the spin-orbit contribution are really the Dirac fine structure plus the Barker-Glover correction (Eq. 6).

Also

$$V_{L,s_2} = \frac{1}{2m_2} \frac{1}{r} \frac{dV}{dr} \left[ \frac{1 + \kappa_2/Z}{s_2 m_r} - \frac{1}{m_2} \right] \vec{L} \cdot \vec{s}_2$$

Usually one writes

$$\frac{Z + \kappa_2}{m_2} = \frac{\mu_2}{m_p}$$

where $\mu_2$ is the magnetic moment of the nucleus in units of nuclear magnetons ($\mu_N = e/2m_p$).

A value of $\mu_d = 0.85744 \mu_N = 0.307012 \mu_p$ corresponds to $\kappa_d = 0.714$. 

$$V_{s_1,s_2} = \frac{2(1 + a_\mu)\mu_2}{2s_2 m_\mu m_2} \left[ \frac{1}{r} \frac{dV}{dr} (3\vec{s}_1 \cdot \hat{r} \vec{s}_2 \cdot \hat{r} - \vec{s}_1 \cdot \vec{s}_2) - \frac{2}{3} \nabla^2 V \vec{s}_1 \cdot \vec{s}_2 \right]$$
$$V_Q = -\alpha Q \frac{1}{r} \frac{dV}{dr} \left[3\hat{s}_2 \cdot \hat{r}\hat{s}_2 \cdot \hat{r} - \hat{s}_2 \cdot \hat{s}_2\right]$$

with Q in units of $1/m^2$. The quadrupole moment of the deuteron is taken to be $Q = 0.2860(15) \text{ fm}^2$ \cite{12,13,14}. In other units, one finds $Q = 25.84/m^2 = 7.345 \times 10^{-6} \text{ MeV}^{-2}$.

Note that $V_{L,s_1}$ describes the fine structure, while the hyperfine structure is described (in perturbation theory) by the expectation values of $V_{L,s_2}$, $V_{s_1,s_2}$, and $V_Q$ (where applicable).

The Uehling potential has to be included in the potential $V(r)$. For states with $\ell > 0$ in light atoms, and neglecting the effect of finite nuclear size, we may take

$$\frac{1}{r} \frac{dV}{dr} = \frac{\alpha Z}{r^3} \cdot \left[1 + 2\alpha \frac{1}{3\pi} \int_1^\infty \frac{(z^2 - 1)^{1/2}}{z^2} \cdot \left(1 + \frac{1}{2z^2}\right) \cdot (1 + 2m_e rz) \cdot e^{-2m_e rz} \, dz\right]$$

which is obtained from the Uehling potential \cite{17,18} by differentiation. Then, assuming that it is sufficient to use nonrelativistic point Coulomb wave functions for the 2p state, one finds

$$\langle \frac{1}{r^3} \rangle_{2p} \rightarrow \langle \frac{1}{r^3} \rangle_{2p} \cdot (1 + \varepsilon_{2p})$$

where

$$\varepsilon_{2p} = 2\alpha \frac{1}{3\pi} \int_1^\infty \frac{(z^2 - 1)^{1/2}}{z^2} \cdot \left(1 + \frac{1}{2z^2}\right) \cdot \left(\frac{1}{(1 + az)^2} + \frac{2az}{(1 + az)^3}\right) \, dz$$

with $a = 2m_e/(\alpha Z m_r)$. For hydrogen, $\varepsilon_{2p} = 0.000365$, and for deuterium $\varepsilon_{2p} = 0.000391$.

### Hyperfine structure of the 2p state in muonic hydrogen

The hyperfine structure of muonic hydrogen is calculated in the same way as was done in earlier work \cite{7,16}, but with improved accuracy. Most of the formalism and results are similar to those given by \cite{4} and \cite{35}.

The hyperfine structure of the 2p-state is given by \cite{4,35} ($F$ is the total angular momentum of the state)

$$\frac{1}{4m\mu m_N} \left\langle \frac{1}{r} \frac{dV}{dr} \right\rangle_{2p} \cdot (1 + \kappa) \left[2(1 + x_p)\delta_{jj'}(F(F+1) - 11/4) + 6\hat{j}\hat{j'}(C_{F1}(1 + a_\mu) - 2(1 + x)) \left\{ \begin{array}{ccc} \ell & F & 1 \\ \frac{1}{2} & \frac{1}{2} & j \end{array} \right\} \left\{ \begin{array}{ccc} \ell & F & 1 \\ \frac{1}{2} & \frac{1}{2} & j' \end{array} \right\} \right\}$$

(10)

where $\hat{j} = \sqrt{2j + 1}$, the 6-j symbols are defined in \cite{37}, and $C_{F1} = \delta_{F1} - 2\delta_{F0} - (1/5)\delta_{F2}$.

Also

$$x_p = \frac{m\mu(1 + 2\kappa_p)}{2m_p(1 + \kappa_p)} = 0.09245$$

represents a recoil correction due to Thomas precession \cite{7,34,35}. The correction due to vacuum polarization (Eq. (21)) should be applied to the HFS shifts of the 2p-states.
As has been known for a long time \cite{16, 35}, the states with total angular momentum \( F = 1 \) are a superposition of the states with \( j = 1/2 \) and \( j = 3/2 \). Let the fine structure splitting be denoted by
\[
\delta = E_{2p3/2} - E_{2p1/2} = 8.352 \text{ meV},
\]
and \( \beta' = \beta_p \cdot (1 + \varepsilon_{2p}) \). The matrix elements for the hyperfine structure of the 2p-state are then given by

\[
\begin{array}{ccc}
  j & j' & \text{Energy} \\
  1/2 & 1/2 & \left( \beta'/8 \right) \left( 2 + x_p + a_\mu \right) \left[ -\delta_{F:0} + 1/3 \delta_{F:1} \right] \\
  3/2 & 3/2 & \delta + \left( \beta'/4 \right) \left( 4 + 5x_d - a_\mu \right) \left[ -1/12 \delta_{F:1} + 1/20 \delta_{F:2} \right] \\
  3/2 & 1/2 & \left( \beta'/24 \right) \left( 1 + 2x_p - a_\mu \right) \left[ \sqrt{2} \delta_{F:1} \right]
\end{array}
\]

Then for the 2p-level with \( j = j' = 1/2 \) and \( F = 0 \), the energy shift is given by
\[
- \left( \beta'/8 \right) \left( 2 + x_p + a_\mu \right) = -5.971 \text{ meV},
\]
and for the 2p-level with \( j = j' = 3/2 \) and \( F = 2 \), the energy shift is given by
\[
\delta + \left( \beta'/80 \right) \left( 4 + 5x_d - a_\mu \right) = 9.6243 \text{ meV}.
\]

For the 2p-levels with \( F = 1 \) the corresponding matrix has to be diagonalized. The resulting numerical values for the eigenvalues are \( (\Delta \pm R)/2 = 1.846 \text{ meV} \) and \( 6.376 \text{ meV} \), where
\[
\Delta = \delta - \beta' \left( x_p - a_\mu \right)/16
\]
\[
R^2 = \left[ \delta - \beta' \left( 1 + 7x_p/8 + a_\mu/8 \right) / 6 \right]^2 + \left( \beta' \right)^2 \left( 1 + 2x_p - a_\mu \right)^2 / 288
\]

**Hyperfine structure of the 2p-state in muonic deuterium**

For the 2p state, the matrix elements of the magnetic hyperfine structure have been given by Brodsky and Parsons \cite{35}. For hydrogen they are the same as those calculated above. Here the Uehling potential will be included in the expectation value of
\[
\left\langle \frac{1}{r} \frac{dV}{dr} \right\rangle
\]
as discussed above.

Let
\[
\beta_D = \frac{16(1 + \kappa_d)}{m_\mu m_d} \frac{\alpha}{(\alpha Z m_r/n)^3} \ell(\ell + 1)(2\ell + 1) = \frac{(1 + \kappa_d)}{6m_\mu m_d} (\alpha Z m_r)^3 = 4.0906 \text{ meV}
\]
(for a point Coulomb potential)

The matrix elements for the magnetic hyperfine structure are then given by

\[
\begin{array}{ccc}
  j & j' & \text{Energy} \\
  1/2 & 1/2 & \left( \beta_D/6 \right) \left( 2 + x_d + a_\mu \right) \left[ -\delta_{F:1/2} + 1/2 \delta_{F:3/2} \right] \\
  3/2 & 3/2 & \delta + \left( \beta_D/4 \right) \left( 4 + 5x_d - a_\mu \right) \left[ -1/6 \delta_{F:1/2} - 1/15 \delta_{F:3/2} + 1/10 \delta_{F:5/2} \right] \\
  3/2 & 1/2 & \left( \beta_D/48 \right) \left( 1 + 2x_d - a_\mu \right) \left[ \sqrt{2} \delta_{F:1/2} - \sqrt{5} \delta_{F:3/2} \right]
\end{array}
\]
where $x_d = (m^2_\mu/m_d m_r)(\kappa_d/(1 + \kappa_d)) = 0.0248$.

For the evaluation of the contributions of the quadrupole HFS, let

$$\epsilon_Q = \alpha Q \left\langle \frac{1}{r} \frac{dV}{dr} \right\rangle$$

For a point Coulomb potential, and the 2p-state, $\epsilon_Q = \alpha Q(Z\alpha m_r)^3/24 = 0.43243$ meV. The quadrupole interaction results in energy shifts of

| $j$  | $j'$ | Energy                                      |
|------|------|---------------------------------------------|
| 1/2  | 1/2  | 0                                           |
| 3/2  | 3/2  | $\epsilon_Q[\delta_{F,1/2} - 4/5 \delta_{F,3/2} + 1/5 \delta_{F,5/2}]$ |
| 3/2  | 1/2  | $\epsilon_Q[\sqrt{2}\delta_{F,1/2} - 1/\sqrt{5}\delta_{F,3/2}]$               |

As mentioned before, the Uehling potential has to be included in the potential $V(r)$. For states with $\ell > 0$ in light atoms, this can be taken into account by multiplying $\beta_D$ and $\epsilon_Q$ by $(1 + \varepsilon_{2p}^2)$ where $\varepsilon_{2p}$ is given by Eq. (9). With a numerical value of $\varepsilon_{2p} = 0.000391$ for muonic deuterium, the value of $\epsilon_Q$ is increased to 0.43440 meV and the value of $\beta_D$ is increased to $\beta_D' = 4.0922$ meV.

Then for the 2p-level with $j = j' = 3/2$ and $F = 5/2$, the energy shift is given by

$$\delta + \epsilon_Q/5 + (\beta_D'/40)(4 + 5x_d - a_\mu) = 9.373 \text{ meV}$$

For the 2p-levels with $F = 1/2$ and $F = 3/2$, the corresponding matrices have to be diagonalized. The resulting numerical values for the eigenvalues are, for $F = 1/2$, -1.3834 meV and 8.5974 meV; for $F = 3/2$ they are 0.6856 meV and 8.2410 meV.

**Hyperfine structure of the 2s-state:**

The expectation value of $V_{s_1 s_2}$ in an ns state with $j = 1/2$ is

$$\Delta E_{ns} = \frac{2\mu_2\alpha(\alpha Z)^3 m_r^3}{3n^3 m_\mu m_2 s_2} \cdot (1 + a_\mu)[F(F + 1) - s_2(s_2 + 1) - 3/4]$$

When $s_2 = 1/2$, and $\mu_2/m_p = (1 + \kappa_2)/m_2$, this reproduces the well-known result for muonic hydrogen:

$$\Delta E_{ns} = \frac{8(\alpha Z)^4 m_r^3}{3n^3 m_\mu m_2} \cdot (1 + \kappa_2) \cdot (1 + a_\mu) = (8/n^3)\beta_p \cdot (1 + a_\mu) = (8/n^3) \times 22.8332 \text{ meV}$$

(see, for example [2], Eq. (271,277)). The numerical value was calculated for hydrogen. For deuterium, with $s_2 = 1$, the corresponding hyperfine splitting is

$$\Delta E_{ns} = \frac{2(\alpha Z)^4 m_r^3}{3n^3 m_\mu m_2} \cdot (1 + \kappa_d) \cdot (1 + a_\mu)[F(F + 1) - 11/4] = (8/n^3) \times 2.04766 \text{ meV} \times [F(F + 1) - 11/4]$$

for a total splitting of 6.14298 meV in muonic deuterium. This is in reasonably good agreement with the result given by Carboni [10].
As was shown in [7, 2], the energy shift of the 2s-state in muonic hydrogen is given by:

\[
\Delta E_{2s} = \beta \cdot (1 + a_\mu) \cdot (1 + \varepsilon_{VP} + \varepsilon_{\text{vertex}} + \varepsilon_{\text{Breit}} + \varepsilon_{FS,rec}) \cdot \frac{[\delta F_1 - 3\delta F_0]}{4}
\]

(11)

The corrections due to QED effects, nuclear size and recoil are analogous for muonic deuterium.

The QED corrections have been discussed by Borie [3, 7, 16] (see also [38]), and are given in Appendix 2.

The correction due to finite size and recoil have been given in [4] as -0.145 meV, while a value of -0.152 meV is given in [42]. Ref. [4] also gives a correction as calculated by Zemach ([40]) equal to -0.183 meV, but claims that this correction does not treat recoil properly. The Zemach correction is equal to

\[
\varepsilon_{Zem} = -2\alpha Z m_r \langle r \rangle_{(2)}
\]

where \( \langle r \rangle_{(2)} \) is given in [7, 24, 41]. Using the value \( \langle r \rangle_{(2)} = 1.086 \pm 0.012 \text{fm} \) from [41], gives \( \varepsilon_{Zem} = -0.00702 \), and a contribution of of -0.1742 meV to the hyperfine splitting of the 2s state. Including this, but not other recoil corrections to the hyperfine structure of the 2s-state gives a total splitting of 22.7806 meV. Additional higher order corrections calculated in Ref. [42] amount to a total of -0.0003 meV and are not included here.

It would be very desirable to understand the reasons for the discrepancy between references [4] and [42] in the calculations of this effect. Also, since the Zemach radius seems to be sensitive to details of the electric and magnetic charge distributions [41], evaluations performed with a dipole-type form factor may not be good enough. This point requires further investigation.

For muonic deuterium, the coefficient of \( \langle r \rangle_{(2)} \) is -0.007398 fm\(^{-1}\), giving, with \( \langle r \rangle_{(2)} = 2.593 \pm 0.016 \text{fm} \) from [41], \( \varepsilon_{Zem} = -0.01918 \pm 0.00012 \).

The total hyperfine splitting of the 2s-state of muonic deuterium, including all corrections, is

\[
\Delta E_{2s} = \frac{3}{2} \beta_D \cdot (1 + a_\mu) \cdot (1 + \varepsilon_{VP} + \varepsilon_{\text{vertex}} + \varepsilon_{\text{Breit}} + \varepsilon_{FS,rec}) = 6.0582 \text{ meV}
\]
Table 5: Fine- and hyperfine contributions to the Lamb shift in muonic hydrogen.

| Transition | Energy shift in meV |
|------------|---------------------|
| $^1p_{1/2} - ^1s_{1/2}$ | 11.114 |
| $^3p_{1/2} - ^1s_{1/2}$ | 18.931 |
| $^3p_{3/2} - ^1s_{1/2}$ | 23.461 |
| $^1p_{1/2} - ^3s_{1/2}$ | -11.666 |
| $^3p_{1/2} - ^3s_{1/2}$ | -3.849 |
| $^3p_{3/2} - ^3s_{1/2}$ | 0.681 |
| $^5p_{3/2} - ^3s_{1/2}$ | 3.929 |

Table 6: Fine- and hyperfine contributions to the Lamb shift in muonic deuterium.

| Transition | Energy shift in meV |
|------------|---------------------|
| $^2p_{1/2} - ^2s_{1/2}$ | 2.655 |
| $^2p_{3/2} - ^2s_{1/2}$ | 12.636 |
| $^4p_{1/2} - ^2s_{1/2}$ | 4.724 |
| $^4p_{3/2} - ^2s_{1/2}$ | 12.280 |
| $^2p_{1/2} - ^4s_{1/2}$ | -3.403 |
| $^2p_{3/2} - ^4s_{1/2}$ | 6.578 |
| $^4p_{1/2} - ^4s_{1/2}$ | -1.334 |
| $^6p_{3/2} - ^4s_{1/2}$ | 6.222 |
| $^6p_{3/2} - ^4s_{1/2}$ | 7.354 |

Tables 5 and 6 give the contributions to the transition energies due to fine and hyperfine structure.

Summary of contributions and Conclusions
The most important contributions to the Lamb shift in muonic hydrogen, including hyperfine structure, have been independently recalculated. A new calculation of some terms that were omitted in the most recent literature, such as the virtual Delbrück effect [22] and an alternative calculation of the relativistic recoil correction have been presented.

Numerically the results given in table 2 add up to a total correction of 
\[ (206.032(6) - 5.225 \langle r^2 \rangle + 0.0347 \langle r^2 \rangle^{3/2}) \text{ meV} = 202.055 \pm 0.12 \text{ meV}. \] (for the value of the proton radius from [11]). As is well known, most of the uncertainty arises from the uncertainty in the proton radius.

Numerical results were also given for muonic deuterium. The total correction is 
\[ (228.573(6) - 6.086 \langle r^2 \rangle + 0.0409 \langle r^2 \rangle^{3/2}) \text{ meV} = 200.767 \pm 0.09 \text{ meV}. \] The complete dependence on the deuteron radius is uncertain since contributions from iteration of the potential are not included. Also, some other contributions are not included, as indicated in table 3.
Acknowledgments
The author wishes to thank M. Eides, E.-O. Le Bigot and F. Kottmann for extensive email correspondence regarding this work.

References
[1] E. Borie, G.A. Rinker, Rev. Mod. Phys. 54, 67 (1982)
[2] M.I. Eides, H. Grotch, V.A. Selyuto, Physic Reports, 342, 63-261 (2001)
[3] E. Borie, Phys. Rev. A71, 032508 (2005)
[4] K. Pachucki, Phys. Rev. A53, 2092 (1996)
[5] K. Pachucki, Phys. Rev. A60, 3593 (1999)
[6] E. Borie, Z. Phys. A275, 347 (1975)
[7] E. Borie, Z. Phys. A278, 127 (1976)
[8] F. Kottmann et al., Hyperf. Int. 138, 55 (2001)
[9] A. di Giacomo, Nucl. Phys. B11, 411 (1969)
[10] G. Carboni, Lett. Nuov. Cim. 7, 160 (1973)
[11] P.J. Mohr, B.N. Taylor, Rev. Mod. Phys. 77, 1 (2005)
[12] J.L. Friar, Can. J. Phys. 80, 1337 (2002)
[13] R.V. Reid, M.L. Vaida, Phys. Rev. Lett. 29, 494 (1972); (E) 34, 1064 (1975)
[14] D.M. Bishop, L.M. Cheung, Phys. Rev. A20, 381 (1979)
[15] E. Borie, G.A. Rinker, Phys. Rev. A18, 324 (1978)
[16] E. Borie, Z. Phys. A297, 17 (1980)
[17] E.A. Uehling, Phys. Rev. 48, 55 (1935)
[18] R. Serber, Phys. Rev. 48, 49 (1935)
[19] G. Källen, A. Sabry, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 29, #17 (1955)
[20] E.H. Wichmann, N.M. Kroll, Phys. Rev. 101, 843 (1956)
[21] E. Borie, Helv. Phys. Acta 48, 671 (1975)
[22] E. Borie, Nucl. Phys. A267, 485 (1976)
[23] T. Kinoshita, M. Nio, Phys. Rev. Lett. 82, 3240 (1999)
[24] J.L. Friar, Annals of Physics, 122, 151 (1979)
[25] R. Rosenfelder, Phys. Lett. B479, 381 (2000)
[26] I. Sick, Phys. Lett. B576, 62 (2003)
[27] E. Borie, Z. Phys. A302, 187 (1981)
[28] J.L. Friar, J. Martorell, D.W.L. Sprung, Phys. Rev. A59, 4061 (1999)
[29] R. Rosenfelder, Phys. Lett. B463, 317 (1999)
[30] S.A. Srartsev, et al., Phys. Atom Nucl. 1233 (1976)
[31] R.N. Faustov, A.P. Martynenko, AIP Conf. Proc. 564, 277 (2001)
[32] R. Barbieri, M. Caffo, E. Remiddi, Lett. Nuov. Cim. 7, 60 (1973)
[33] A. Veitia, K. Pachucki, Phys. Rev. A69, 042501 (2004)
[34] E.H. Barker, N.M. Glover, Phys. Rev. 99, 317 (1955)
[35] S.J. Brodsky, R.G. Parsons, Phys. Rev. 163, 134 (1967)
[36] R. Metzner, H. Pilkuhn, Z. Phys. A286, 147 (1978)
[37] A.R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press, 1960
[38] S.J. Brodsky, G.W. Erickson, Phys. Rev. 148, 26 (1966)
[39] M.M. Sternheim, Phys. Rev. 138, B430 (1965)
[40] A.C. Zemach, Phys. Rev. 104, 1771 (1956)
[41] J.L. Friar, I. Sick, Phys. Lett. B579, 285 (2004)
[42] A.P. Martynenko, Preprint SSU-HEP-04/09
[43] A.I. Akhiezer, V.B. Berestetskii, *Quantum electrodynamics*, Wiley Interscience, New York, 1965.
Appendix 1: Details of the Relativistic Recoil Calculation

As mentioned above, the energy levels of muonic atoms are given, to leading order in $1/m_N$ by

$$E = E_r - \frac{B_0^2}{2m_N} + \frac{1}{2m_N}\langle h(r) + 2B_0 P_1(r) \rangle$$

where $E_r$ is the energy level calculated using the reduced mass and $B_0$ is the unperturbed binding energy. Also

$$h(r) = -P_1(r)(P_1(r) + \frac{1}{r}Q_2(r)) - \frac{1}{3r}Q_2(r)[P_1(r) + Q_4(r)/r^3]$$

where $P_1$, $Q_2$, and $Q_4$ are defined in Eq. (4).

Keeping only the Coulomb and Uehling potentials, one finds

$$P_1(r) = -\alpha Z \frac{2\alpha}{3\pi} (2m_e) \chi_0(2m_e r)$$

$$Q_2(r) = \alpha Z \left( 1 + \frac{2\alpha}{3\pi} [\chi_1(2m_e r) + (2m_e r)\chi_0(2m_e r)] \right)$$

$$Q_4(r) = \alpha Z \frac{2\alpha}{3\pi} \int_1^\infty \frac{dz}{z^2} \left( \frac{(z^2 - 1)^{1/2}}{z} \left( 1 + \frac{1}{2z^2} \right) \right) \cdot \left( \frac{2}{\pi} \right) \int_0^\infty \frac{dq}{q^2 + 4m_e^2 z^2} \left( 6q^2 - (qr)^3 \right) \cos(qr) + (3(qr)^2 - 6) \sin(qr) \right) dq$$

where $\chi_n(x)$ is defined in [1].

Since vacuum polarization is assumed to be a relatively small correction to the Coulomb potential, it will be sufficient to approximate $Q_2(r)$ by $\alpha Z/r$. After some algebra, one can reduce the expectation values to single integrals:

$$\langle P_1(r) \rangle = 2m_e \alpha Z \frac{2\alpha}{3\pi} \int_1^\infty \frac{(z^2 - 1)^{1/2}}{z} \cdot \left( 1 + \frac{1}{2z^2} \right) \cdot \left( \frac{(az)^2 - az + 1}{(1 + az)^5} \delta_{\ell_0} + \frac{1}{(1 + az)^5} \delta_{\ell_1} \right) dz$$

When multiplied by $-2B_0/(m_\mu + m_N)$ this results in a shift of -0.00015 meV for the 2s-state and of -0.00001 meV for the 2p-state. For muonic deuterium, the corresponding numbers are -0.000176 meV and -0.000030 meV, respectively.

$$\langle \frac{\alpha Z}{r} P_1(r) \rangle = -\langle \alpha Z \rangle^3 m_\mu m_e \frac{2\alpha}{3\pi} \int_1^\infty \frac{(z^2 - 1)^{1/2}}{z} \cdot \left( 1 + \frac{1}{2z^2} \right) \cdot \left( \frac{2(az)^2 + 1}{2(1 + az)^3} \delta_{\ell_0} + \frac{1}{2(1 + az)^3} \delta_{\ell_1} \right) dz$$

When multiplied by $1/(m_\mu + m_N)$ this results in a shift of 0.00489 meV for the 2s-state and of 0.000017 meV for the 2p-state of muonic hydrogen. For muonic deuterium, the corresponding numbers are 0.005543 meV and 0.000206 meV, respectively.

These expectation values also appear when vacuum polarization is included in the Breit equation.
Finally,

\[ \langle \frac{\alpha Z}{3r} Q_4(r) \rangle = -\frac{(\alpha Z)^4 m_e^2 2\alpha}{3\pi} \int_1^\infty \frac{(z^2 - 1)^{1/2}}{z^2} \cdot \left( 1 + \frac{1}{2z^2} \right) \cdot \left[ -\frac{6}{az} \left( \frac{2 + az}{1 + az} - \frac{2}{az} \ln(1 + az) \right) + \frac{3(az)^2 + 2az - 1}{(1 + az)^3} + \frac{3 + az}{4(1 + az)^4} \right] \delta_{\ell 0} + \frac{1 - 3az - 2(az)^2}{4(1 + az)^4} \delta_{\ell 1} \right] dz \]

When multiplied by \(1/(m_\mu + m_N)\) this results in a shift of 0.002475 meV for the 2s-state and of 0.000238 meV for the 2p-state. For muonic deuterium, the corresponding numbers are 0.002753 meV and 0.000281 meV, respectively.

Combining these expectation values according to equations \ref{eq:2} and \ref{eq:3} one finds a contribution to the 2p-2s transition of -0.00419 meV (hydrogen) and -0.00479 meV (deuterium). To obtain the full relativistic and recoil corrections, one must add the difference between the expectation values of the Uehling potential calculated with relativistic and nonrelativistic wave functions, giving a total correction of 0.0166 meV for muonic hydrogen. This is in quite good agreement with the correction of .0169 meV calculated by Veitia and Pachucki \cite{33}. The treatment presented here has the advantage of avoiding second order perturbation theory. For deuterium, one obtains a total correction of 0.0179 meV.

**Appendix 2: Details of Corrections to the Hyperfine Structure of the 2s-state of Muonic Hydrogen and Deuterium**

The expectation value of \(V_{s_1s_2}\) in an ns state with \(j = 1/2\) is

\[ \Delta E_{ns} = \frac{2\mu_2 \alpha (\alpha Z m_e)^3}{3n^3m_\mu m_2s_2} \cdot (1 + a_\mu)[F(F + 1) - s_2(s_2 + 1) - 3/4] \]

As was shown in \cite{7,2}, the energy shift of the 2s-state has to be multiplied by:

\[ 1 + \epsilon_{VP} + \epsilon_{vertex} + \epsilon_{Breit} + \epsilon_{FS,rec} \]

Here (\cite{38})

\[ \epsilon_{vertex} = \frac{2\alpha (\alpha Z)}{3} \left( \ln(2) - \frac{13}{4} \right) = -1.36 \cdot 10^{-4} \]

and (\cite{2}, Eq. (277))

\[ \epsilon_{Breit} = \frac{17(\alpha Z)^2}{8} = 1.13 \cdot 10^{-4} \]

The vacuum polarization correction has two contributions. One of these is a result of a modification of the magnetic interaction between the muon and the nucleus and is given by (see \cite{16})

\[ \epsilon_{VP1} = \frac{4\alpha}{3\pi^2} \int_0^\infty r^2 dr \left( \frac{R_{ns}(r)}{R_{ns}(0)} \right)^2 \int_0^\infty q^4 j_0(qr) G_M(q) dq \]

\[ \int_1^\infty \frac{(z^2 - 1)^{1/2}}{z^2} \cdot \left( 1 + \frac{1}{2z^2} \right) \cdot \frac{dz}{4m_e^2[z^2 + (q/2m_e)^2]} \]
One can do two of the integrals analytically and obtains for the 2s-state (with \( a = 2m_e/(\alpha Z m_r) \) and \( \sinh(\phi) = q/(2m_e) = K/a \)

\[
\varepsilon_{VP1} = \frac{4\alpha}{3\pi^2} \int_0^\infty \frac{K^2}{(1+K^2)^2} F(\phi) G_M(\alpha Z m_r K) \, dK \left[ 2 - \frac{7}{(1+K^2)} + \frac{6}{(1+K^2)^2} \right] \tag{13}
\]

where \( F(\phi) \) is known from the Fourier transform of the Uehling potential (given as \( U_2(q) \) in Ref. [1]) and is given by

\[
F(\phi) = \frac{1}{3} + (\coth^2(\phi) - 3) \cdot [1 + \phi \cdot \coth(\phi)] = \frac{3\pi}{\alpha} U_2(q)
\]

with \( \sinh(\phi) = q/2m_e \).

The other contribution, as discussed by [38, 39] arises from the fact that the lower energy hyperfine state, being more tightly bound, has a higher probability of being in a region where vacuum polarization is large. This results in an additional energy shift of

\[
2 \int V_{Ueh}(r) \psi_{2s}(r) \delta_M \psi_{2s}(r) d^3r
\]

Following Ref. [38] with \( y = (\alpha Z m_r/2) \cdot r \), one has

\[
\delta_M \psi_{2s}(r) = 2m_e \Delta \nu_F \psi_{2s}(0) \left( \frac{2}{\alpha Z m_r} \right)^2 \exp(-y) \left[ (1-y)(\ln(2y) + \gamma) + \frac{13y - 3 - 2y^2}{4} \cdot \frac{1}{4y} \right]
\]

(\( \gamma \) is Euler’s constant), and

\[
\psi_{2s}(r) = \psi_{2s}(0)(1-y) \exp(-y)
\]

One finds after a lengthy integration

\[
\varepsilon_{VP2} = \frac{16\alpha}{3\pi^2} \int_0^\infty \frac{dK}{1+K^2} G_E(\alpha Z m_r K) F(\phi) \left\{ \frac{1}{2} - \frac{17}{(1+K^2)^2} + \frac{41}{(1+K^2)^3} - \frac{24}{(1+K^2)^4} \right. \]

\[+ \frac{\ln(1+K^2)}{1+K^2} \left[ 2 - \frac{7}{1+K^2} + \frac{6}{(1+K^2)^2} \right] \]

\[+ \frac{\tan^{-1}(K)}{K} \left[ 1 - \frac{19}{2(1+K^2)} + \frac{20}{1+K^2} \right] - \frac{12}{(1+K^2)^3} \right\} \tag{14}
\]

Sternheim[39] denotes the two contributions by \( \delta_M \) and \( \delta_E \), respectively. An alternative expression, obtained by assuming a point nucleus, using Eq.(131) from [1] for the Uehling potential, and doing the integrations in a different order, is

\[
\varepsilon_{VP2} = \frac{16\alpha}{3\pi} \int_0^\infty \frac{(z^2 - 1)^{1/2}}{z^2} \cdot \left( 1 + \frac{1}{2z^2} \right) \cdot \frac{1}{(1+az)^2} \left[ \frac{az}{2} - \frac{1}{1+az} + \frac{23}{8(1+az)^2} - \frac{3}{2(1+az)^3} \right. \]

\[+ \ln(1+az) \cdot \left( 1 - \frac{2}{1+az} + \frac{3}{2(1+az)^2} \right) \] \[dz \tag{15}
\]
with $a = 2m_e/(\alpha Z m_{\text{red}})$. Both methods give the same result.

In the case of ordinary hydrogen, each of these contributes $3\alpha^2/8 = 1.997 \cdot 10^{-5}$. The accuracy of the numerical integration was checked by reproducing these results. One can thus expect that muonic vacuum polarization will contribute $3\alpha^2/4 \simeq 4 \cdot 10^{-5}$, as in the case of normal hydrogen. This amounts to an energy shift of 0.0009 meV in muonic hydrogen and 0.0002 meV in muonic deuterium. Contributions due to the weak interaction or hadronic vacuum polarization should be even smaller. For muonic hydrogen, one obtains $\varepsilon_{VP1} = 0.00211$ and $\varepsilon_{VP2} = 0.00325$ for a point nucleus. Including the effect of the proton size (with $G_E(q) = G_M(q)$ as a dipole form factor) reduces these numbers to 0.00206 and 0.00321, respectively. For the case of muonic deuterium, the corresponding numbers are $\varepsilon_{VP1} = 0.00218$ (0.00207) and $\varepsilon_{VP2} = 0.00337$ (0.00326), respectively. The contribution to the hyperfine splitting of the 2s-state of hydrogen is then 0.0470 meV + 0.0733 meV = 0.1203 meV (0.1212 meV if muonic vacuum polarization is included). The combined Breit and vertex corrections reduce this value to 0.1207 meV. (0.1226 meV if the proton form factors are not taken into account).

The contribution to the hyperfine structure from the two loop diagrams [19] can be calculated by replacing $U_2(\alpha Z m_r K) = (\alpha/3\pi)F(\phi)$ by $U_4(\alpha Z m_r K)$ (as given in [1, 6]) in equations 13 and 14. The resulting contributions are $1.64 \cdot 10^{-5}$ and $2.46 \cdot 10^{-5}$ (for deuterium $1.69 \cdot 10^{-5}$ and $2.54 \cdot 10^{-5}$), respectively, giving a total shift of 0.0009 meV in muonic hydrogen and 0.0002 meV in muonic deuterium.