Gamma matrices, Majorana fermions, and
discrete symmetries in Minkowski and
Euclidean signature

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Abstract
I describe the interplay between Minkowski and Euclidean signature gamma matrices, Majorana fermions, and discrete and continuous symmetries in all spacetime dimensions.

1 Introduction

Textbook discussions of the discrete $C$, $T$ symmetries of the Dirac equation tend to feel unsatisfactory because they make use of representation-specific properties of the gamma matrices and other basis-dependent operations such as complex conjugation and transposition. Complex conjugation of the components of a vector is basis-dependent operation because the vector can have real components in one basis and imaginary components in another, and when a matrix acts on a vector space $V$ its transpose acts on the dual space $V^*$. In the absence of some preferred metric there is no natural identification of a vector space with its dual and equating a spinor to a transposed spinor as we do in charge conjugation is necessarily unnatural. Other linear-algebra aspects of the Dirac equation — the dimensions of the irreducible
gamma-matrix representations, Lorentz transformations, and the existence of the Weyl spinors in even dimensions — can be derived purely from the Clifford algebra obeyed by the gamma matrices without using these operations. A particular consequence of the basis-choice dependence is that Majorana fermions are usually defined by ugly equations containing non-covariant-looking products of gamma matrices that vary from book to book

It is also widely asserted that there is no such thing as a Majorana fermion in four Euclidean dimensions. This last is a great pity because we would like to study Majorana fermions using heat-kernel regularized path integrals or by lattice-theory computations, and these tools are only available in Euclidean signature.

The problem with this assertion is that many authors identify the existence of Majorana fermions with the existence of real gamma matrices and so, when the metric has \( p \) plus signs and \( q \) negative signs, they need to delve into the reality properties of representations of the Clifford algebras \( \text{Cl}(p,q) \). These properties depend intricately on \( p - q \mod 8 \) \cite{2,3} and, in particular because

\[
\text{Cl}(p,q) \not\cong \text{Cl}(q,p),
\]

they suggest that the Bjorken and Drell “West Coast” or “mostly minus” metric \( g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \) is necessarily different from the “East Coast,” mostly plus, metric \( g_{\mu\nu} = \text{diag}(-1,+1,+1,+1) \) that is preferred by general relativists. Surely nothing physical can depend on our choice of metric convention?

The representation theory of Clifford algebras defined over the field \( \mathbb{R} \) contains much beautiful mathematics, but for physics applications it rather misses the point. We compute in Euclidean signature not because we are interested in how physics would look were our world Euclidean, but rather because it is a useful tool for studying our Minkowski-signature universe. Osterwalder and Schrader \cite{4} showed that when the Euclidean-signature \( n \)-point functions satisfy the condition of reflection positivity then one can obtain the Minkowski-signature \( n \)-point functions as analytic continuations of the Euclidean ones by taking the external momenta \( p \) into the Minkowski region where particles are on-shell when \( p^2 = -m^2 \). The *physical* Majorana condition in Euclidean signature does not need real gamma matrices, but rather that the reconstructed \( n \)-point functions be those of Minkowski-signature Majorana particles \cite{5}. The physical Majorana condition then depends only
the symmetry properties of certain matrices \( C_{\alpha\beta} \) and \( T_{\alpha\beta} \) which are indifferent to the ± signs in the metric [6].

A particular advantage of exploring the properties of the \( C \) and \( T \) matrices is that they make manifest an eightfold period in dimension that also appears in the classification of random matrices [7], Cartan symmetric spaces, and in the classification of topological insulators and superconductors [8, 9]. This “eightfold way” is an aspect of of Bott periodicity [11, 12] whose connection with the eightfold periodicity of gamma matrix properties was pointed out by Hořava [13] and Kitaev [14].

Our aim is to use the \( C \) and \( T \) matrices to reconcile the problems stated above in as straightforward a manner as possible, and to show how one can consistently switch between the field-operator language of Minkowski space and the Grassmann-variable path integral language in Euclidean signature. Our overall philosophy is similar that of Wetterich [5] although we stress different aspects of the mapping.

To achieve our goal we review in section 2 the construction of gamma matrices in \( d \) spacetime dimensions and arbitrary signature and in doing so uncover the properties of the \( C \) and \( T \) matrices that relate the gamma matrices to their transpose. In section 3 we will define the operation of charge conjugation in a basis and signature independent manner, and define three distinctly different types of Majorana fermions. We then show how the dimensions in which the three types occur — \( d = 2, 3, 4 \) (mod 8) for Majorana, \( d = 8, 9, 10 \) (mod 8) for the necessarily massless pseudo-Majorana and \( d = 5, 6, 7 \) (mod 8) for symplectic Majorana — are the same in both Minkowski and Euclidean signature. We show how the period 8 pattern governing both existence of the Majorana classes and their possible flavour and gauge symmetries is related to the eightfold version of Bott periodicity. Section 5 will then discuss how parity and time-reversal symmetries operate in different dimensions. Three appendices discuss how integrals over anticommuting variables differ between Dirac and Majorana fermions (determinants versus Pfaffians) and also use the example of condensed matter systems to show that both charge-conjugation and time-reversal act antilinearly on the one-particle space, but in the induced action on the many-particle Fock space charge conjugation acts linearly while time-reversal remains antilinear.
2 Gamma matrices

We begin with the construction and properties of the Dirac gamma matrices in various dimensions and space-time signatures. This is well-known material and we will use the notation and strategy from Hitoshi Murayama’s online lecture notes \[6\].

2.1 \( \mathcal{C} \) and \( \mathcal{T} \) matrices: eightfold periodicity

In Euclidean \( d = 2k \) dimensions we can construct a matrix representation of the generators \( \gamma^i \) of the Clifford algebra

\[
\gamma^i \gamma^j + \gamma^j \gamma^i = 2 \delta^{ij}
\]  

(1)

by using fermion annihilation and creation operators \( \hat{a}_n, \hat{a}^\dagger_n \), \( n = 1, \ldots, k \), that obey

\[
\{\hat{a}_n, \hat{a}_m\} = 0 = \{\hat{a}^\dagger_n, \hat{a}^\dagger_m\}, \quad \{\hat{a}_n, \hat{a}^\dagger_m\} = \delta_{nm},
\]  

(2)

and by setting

\[
\gamma^{2n-1} = \hat{a}_n + \hat{a}^\dagger_n, \\
\gamma^{2n} = i(\hat{a}_n^\dagger - \hat{a}_n).
\]  

(3)

When they act on the Fock space built on a vacuum vector \( |0\rangle \) such that \( \hat{a}_n |0\rangle = 0 \), \( n = 1, \ldots, k \), the \( \gamma^i \) are represented by a set of \( 2^k \)-by-\( 2^k \) Hermitian matrices that are symmetric for odd \( i \) and antisymmetric for even \( i \).

For odd dimension \( d = 2k + 1 \) we append an extra gamma matrix

\[
\gamma^{2k+1} = (-i)^k \gamma^1 \ldots \gamma^{2k},
\]

which is equal to \( (-1)^{\sum \hat{a}_n^\dagger \hat{a}_n} \) and, in this basis, is diagonal and so symmetric.

This construction displays a clear even-odd periodicity in the dimension \( d \) because of the special treatment of \( \gamma^{2k+1} \), but it also contains a rather less obvious period-eight property. To reveal this hidden structure we define \[6\]

\[
C_1 = \prod_{i \text{ odd}} \gamma^i, \quad C_2 = \prod_{i \text{ even}} \gamma^i,
\]  

(4)

and use them to construct matrices \( \mathcal{C} \) and \( \mathcal{T} \) such that

\[
\mathcal{C} \gamma^i \mathcal{C}^{-1} = -(\gamma^i)^T, \\
\mathcal{T} \gamma^i \mathcal{T}^{-1} = + (\gamma^i)^T.
\]  

(5)
We find that

\[ k=0, \text{mod } 4: C = C_1 \text{ symmetric, } T = C_2 \text{ symmetric.} \]

Both commute with \( \gamma^{2k+1} \).

\[ k=1, \text{mod } 4: C = C_2 \text{ antisymmetric, } T = C_1 \text{ symmetric.} \]

Both anticommute with \( \gamma^{2k+1} \).

\[ k=2, \text{mod } 4: C = C_1 \text{ antisymmetric, } T = C_2 \text{ antisymmetric.} \]

Both commute with \( \gamma^{2k+1} \).

\[ k=3, \text{mod } 4: C = C_2 \text{ symmetric, } T = C_1 \text{ antisymmetric.} \]

Both anticommute with \( \gamma^{2k+1} \).

Under a change of basis \( \gamma^\mu \rightarrow A \gamma^\mu A^{-1} \) the matrices \( C \) and \( T \) will no longer given by the explicit product expressions \( C_1 \) and \( C_2 \), but instead transform as

\[ C \rightarrow A^T C A, \quad T \rightarrow A^T T A. \]  

(6)

The symmetry or antisymmetry of \( C, T \) is unchanged, and is thus a basis-independent property.

Another way to think of this symmetry is by making use of the transpose of their defining transformations to see that \( C^{-1} C^T \) and \( T^{-1} T^T \) commute with all \( \gamma^\mu \). As our Fock-space gamma representation is clearly irreducible, Schur's lemma tells us that both \( C, T \) are proportional to their transpose so

\[ C^T = \lambda C \]  

(7)

with \( \lambda \) basis independent. Then, transposing again,

\[ C = \lambda C^T \Rightarrow C = \lambda^2 C \]  

(8)

showing that \( \lambda = \pm 1 \). Similarly \( T^T = \pm T \) with a basis independent sign.

If we restrict to transformations in which \( A \) is unitary, the Euclidean \( \gamma^\mu \) remain Hermitian and a similar argument shows that \( C^T C \) is proportional to the identity. As \( C^T C \) is a positive operator the factor of proportionality is real and positive. Consequently \( C \) (and \( T \)) can be scaled by real numbers so as to be unitary. We will assume that we have done this.

If we regard a gamma matrix with elements \( \gamma^{\alpha \beta} \) as representing a linear map from \( V \rightarrow V \), where \( V \) is the spinor representation space, then its transpose \( \gamma^T \) with matrix elements \( (\gamma^T)^{\alpha}_{\beta} \) represents a linear map from \( V^* \rightarrow V^* \) where \( V^* \) is the dual space of \( V \). We can think of the matrices \( C^\alpha_{\beta} \) and \( T^\alpha_{\beta} \), and their inverses \( (C^{-1})^\alpha_{\beta} \) and \( (T^{-1})^\alpha_{\beta} \), as “metrics” on spinor-space.
that allow us to raise and lower the spinor indices on the Fermi fields $\psi^\alpha$ and $\bar{\psi}_\beta$ and so identify $V$ with $V^*$. The existence of $C_{\alpha\beta}$ and $T_{\alpha\beta}$ thus resolves one of the unnaturalness issues raised in the introduction. It is then clear that a matrix product such as $C\gamma^i$ is well formed and covariant because an upstairs index on the gamma matrix is contracted with a downstairs index on $C$. A matrix product such as $\gamma^iC$ is not well-formed as we are contracting a pair of downstairs indices. These observations can serve as a useful check on manipulations.

In a similar vein one sometimes sees assertions such as "$T^2 = I$" (implicitly in Bjorken and Drell [10] eq. (15.134) for example) but the manner in which $T$ and $C$ transform under $\gamma^i \rightarrow A\gamma^i A^{-1}$ shows that there is no basis-independent notion of a product of $T$ or $C$ matrices with themselves. Such formulæ are not operator identities therefore, and can only hold in specific bases. There is no such problem with $C^{-1}CT = \pm I$, etc.

If the matrices $\gamma^i$ constitute a representation of the Clifford algebra, so do $\pm(\gamma^i)^T$. In $d = 2k$ dimensions the Dirac representation is unique, so these three representations must be equivalent. Consequently, even if we did not have the explicit construction given above, the existence of $T, C$ is guaranteed. In odd dimensions, however, there are two inequivalent representations of the Clifford algebra because we can replace $\gamma^{2k+1} = (-i)^k\gamma_1\cdots\gamma_{2k}$ by minus this expression and still satisfy the Clifford algebra. The representation with the minus sign is inequivalent to that with the plus sign because $\gamma_1\gamma_2\cdots\gamma_{2k+1} = \pm I$ and the sign cannot be changed by any transformation $\gamma^\mu \rightarrow U^{-1}\gamma^\mu U$. The existence of $T, C$ matrices that transpose all $2k + 1$ $\gamma^i$ is no therefore longer assured. We have to ask whether conjugation by the even-dimensional $T, C$ continues to correctly transpose the extra gamma matrix. Table 1 summarizes the outcome of this examination.

When $T$ is symmetric we can find a unitary matrix $U$ with which to transform $T \rightarrow T' = U^T T U$ to a basis in which $T' = I$ (see appendix B for a proof of this). In this basis all the Euclidean gamma matrices are symmetric, still Hermitian, and therefore all real. When $C$ is symmetric we can find a basis in which $C = I$ and all the Euclidean gamma matrices are antisymmetric, still Hermitian, and therefore purely imaginary.

In other signatures the $\delta^{\mu\nu}$ in the defining equation (11) will be replaced by $\text{diag}(\pm 1, \pm 1, \ldots, \pm 1)$ and for each minus sign the corresponding $\gamma^\mu$ must be multiplied by $\pm i$ in order to satisfy the new Clifford algebra. The reality properties of the gamma matrices are obviously changed by this. One advantage of focussing on the $C$ and $T$ matrices is that their existence and
symmetry properties are indifferent to the metric signature.

2.2 Numbering convention for Minkowski-signature matrices

We labelled our Euclidean gamma matrices as $\gamma^1, \gamma^2, \ldots, \gamma^{2k}$ with $\gamma^{2k+1}$ being a product of the $2k$ lower-numbered matrices. In contemporary physics usage four-dimensional Minkowski-signature gamma matrices are universally numbered as $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ with $\gamma^0$ associated with $x^0 = t$. For historical reasons their product still called $\gamma^5$ — although there is no $\gamma^4$. In Minkowski signature $\gamma^0$ and $\gamma^5$ have special roles and renaming either $\gamma^0 \rightarrow \gamma^1$ or $\gamma^5 \rightarrow \gamma^4$ to close the “$\gamma^4$ gap” is likely to generate more fog than light. It seems simplest to keep the chirality operator as $\Gamma^5 \equiv \gamma^{2k+1}$, and when “$\gamma^0$” appears in the familiar definition $\bar{\psi} = \psi^\dagger \gamma^0$ it should be born mind that the Minkowski signature “$\gamma^0$” corresponds to the Euclidean $\gamma^4$.

3 Charge conjugation and Majorana fermions

3.1 Charge conjugation

In a Euclidean-signature path integral the Fermi fields $\bar{\psi}$ and $\psi$ are unrelated row and column vectors of Grassmann variables (see Appendix A). Nonetheless it is useful to define the Euclidean-signature charge-conjugate fields $\psi^c$ and $\bar{\psi}^c$ so as to be consistent with the Minkowski-signature operator language in which $\bar{\psi}$ and $\psi$ are related by $\bar{\psi} = \psi^\dagger \gamma^0$. We arrange for this by defining

\[
\psi^c = C^{-1} \bar{\psi}^T,
\]

\[
\bar{\psi} = \psi^\dagger \gamma^0
\]
\[ \bar{\psi}^c = -\psi^T C. \] \hspace{1cm} (9)

To obtain the motivating Minkowski version we recall that in any signature we can use exactly the same \( T \) and \( C \) matrices as in Euclidean signature — the insertion of factors of \( i \) in some of the \( \gamma^\mu \) does not affect the formula for their transposition and no \( i \)'s need be inserted in \( T \) and \( C \).

Consider first the mostly-minus “West-Coast” Minkowski metric \((+,−,−,\ldots)\) in which \( \gamma^0 \) is Hermitian and obeys \((\gamma^0)^2 = 1\). Then with \( \bar{\psi}^T = (\psi^\dagger \gamma^0)^T \), and writing \( \psi^\ast \) for the quantum Hilbert-space adjoint of \( \psi \) without the column \( \rightarrow \) row operation implicit in \( \psi^\dagger \), we have

\[ \psi^c = C^{-1} \bar{\psi}^T = C^{-1} (\gamma^0)^T \psi^\ast \Rightarrow (\psi^c)^\dagger = \psi^T (\gamma^0)^T C \] \hspace{1cm} (10)

because \( C \) remains unitary in Minkowski space. We define

\[ \bar{\psi}^c \equiv (\psi^c)^\dagger \gamma^0 \]
\[ = \psi^T (\gamma^0)^T C \gamma^0 \]
\[ = -\psi^T C \gamma^0 C^{-1} C \gamma^0 \]
\[ = -\psi^T C. \] \hspace{1cm} (11)

In the mostly-plus “East-Coast” Minkowski metric \((-+,+,\ldots\ldots)\), in which \( \gamma^0 \) is skew Hermitian and obeys \((\gamma^0)^2 = -1\), we have \((\psi^c)^\dagger = -\psi^T (\gamma^0)^T C\) and

\[ \bar{\psi}^c \equiv (\psi^c)^\dagger \gamma^0 \]
\[ = -\psi^T (\gamma^0)^T C \gamma^0 \]
\[ = \psi^T C \gamma^0 C^{-1} C \gamma^0 \]
\[ = -\psi^T C. \gamma^0 \gamma^0 \]
\[ = -\psi^T C. \] \hspace{1cm} (12)

In both signatures, therefore, \( \bar{\psi}^c = -\psi^T C. \)

From these results, and with anticommuting Grassmann \( \psi \)'s, we find that

\[ \bar{\psi}^c \gamma^\mu \psi^c = [-\psi^T C] \gamma^\mu [C^{-1} \bar{\psi}^T] = \psi^T (\gamma^\mu)^T \bar{\psi}^T = -\bar{\psi} \gamma^\mu \psi, \] \hspace{1cm} (13)

so the number current changes sign. The spin-current density transforms as

\[ \bar{\psi}^c \gamma^0 [\gamma^i, \gamma^j] \psi^c = -\bar{\psi} \gamma^0 [\gamma^j, \gamma^i] \psi = \bar{\psi} \gamma^0 [\gamma^j, \gamma^i] \psi, \quad (i,j \neq 0), \] \hspace{1cm} (14)
and is left unchanged. Similarly
\[
\bar{\psi}^{c}\psi^{c} = -\psi^{T}\bar{\psi}^{T} = \bar{\psi}\psi.
\] (15)

In Euclidean signature, and using the anticommuting property of the Grassmann fields, the action for Dirac fermions minimally-coupled to a skew-Hermitian vector gauge field \( A_{\mu} \) has the property
\[
S = \int d^{n}x \bar{\psi}\gamma^{i}(\partial_{i} + A_{i}) + m|\psi = \int d^{n}x \bar{\psi}^{c}\gamma^{i}(\partial_{i} - A_{i}^{T}) + m|\psi^{c}.
\] (16)

The \(-A^{T}\) are the Lie algebra representation-valued fields in the the conjugate representation to that of \( A \), and so \( \psi^{c} \) has the opposite gauge-field “charge” to \( \psi \).

### 3.2 Minkowski-signature Majorana Fermions

We have defined
\[
\psi^{c} = C^{-1}\bar{\psi}^{T} = C^{-1}(\gamma^{0})^{T}\psi^{*}
\] (17)

so, with \( C^{T} = \lambda C \) we find (in both mostly-plus and mostly-minus metrics)
\[
(\psi^{c})^{c} = C^{-1}(\gamma^{0})^{T}(C^{-1}(\gamma^{0})^{T}\psi^{*})^{*}
= C^{-1}(\gamma^{0})^{T}C^{T}(\gamma^{0})^{\dagger}\psi
= \lambda C^{-1}(\gamma^{0})^{T}C(\gamma^{0})^{\dagger}\psi
= -\lambda\gamma^{0}(\gamma^{0})^{\dagger}\psi
= -\lambda\psi.
\] (18)

We can therefore consistently impose the Minkowski Majorana condition that \( \psi^{c} = \psi \) only if \( \lambda = -1 \) so \( C \) is antisymmetric: i.e. in 2, 3, 4 (mod 8) dimensions.

The equal-time anti-commutator of an operator-valued Majorana field can be taken to be
\[
\{\psi^{\alpha}(x),\psi^{\beta}(x')\}_{t = t'} = [\gamma^{0}C^{-1}]^{\alpha\beta}\delta^{d-1}(x - x')
\] (19)

where \( \gamma^{0}C^{-1} = -C^{-1}(\gamma^{0})^{T} \) is symmetric when \( C \) is antisymmetric.

We can regard the map \( C : \psi \mapsto \psi^{c} = C^{-1}γ^{0}T\psi^{*} \) as an antilinear map \( C : V \to V \) where \( V \) is the gamma-matrix representation space. If \( C^{2} = \text{id} \),

\footnote{Charge conjugation is antilinear only when acting on the field components. It is a linear map when acting on the states in the many-body Hilbert space. See Appendix C.}
this is real structure on the complex $V$ space. Vectors that are left fixed by $C$ are regarded “real” because there is a basis in which their components are real — even even though these components will be complex in other bases.

The antilinear map $C$ commutes with the gamma matrices only in the mostly plus East Coast metric. With this metric choice, and in the basis in which the Majorana spinor components are real, the gamma matrices become purely real and so preserve the reality condition. In the the West-Coast-metric Majorana representation the gamma’s are purely imaginary and we have to remove a factor of $i$ to get matrices that commute with the antilinear $C$. This does not matter though, because it is the Dirac equation that must preserve the reality of of the spinor solutions, and in the West Coast Minkowski metric the Dirac equation is

$$(-i\gamma^i\partial_i + m)\psi = 0 \quad \text{(West Coast).}$$  \hspace{1cm} (20)

This version of the equation puts the necessary factor of $i$ with the $\gamma$’s, while on the East Coast the Dirac equation reads

$$(\gamma^i\partial_i + m)\psi = 0, \quad \text{(East Coast),}$$  \hspace{1cm} (21)

where there is no factor of $i$.

To verify that $C$ commutes with the $\gamma^i$ in the East Coast metric we begin by observing that $(\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0$ in both conventions. Then

$$C^{-1}(\gamma^0)^T(\gamma^i\psi)^* = C^{-1}(\gamma^0)^T(\gamma^i\gamma^0)^T\psi^*$$
$$= C^{-1}(\gamma^0)^T(\gamma^0\gamma^i\gamma^0)^T\psi^*$$
$$= C^{-1}(\gamma^0)^T(\gamma^0)^T(\gamma^i)^T(\gamma^0)^T\psi^*$$
$$= C^{-1}(\gamma^0)^TCC^{-1}(\gamma^0)^TCC^{-1}(\gamma^i)^TCC^{-1}(\gamma^0)^T\psi^*$$
$$= (-\gamma^0)(-\gamma^0)(-\gamma^i)C^{-1}(\gamma^0)^T\psi^*$$
$$= -(\gamma^0)^2\gamma^iC^{-1}(\gamma^0)^T\psi^*.$$  \hspace{1cm} (22)

Thus $C\gamma^i = \gamma^iC$, or equivalently

$$C^{-1}(\gamma^0)^T(\gamma^i\psi)^* = C^{-1}(\gamma^0)^T\psi^*$$  \hspace{1cm} (23)

holds only if $(\gamma^0)^2 = -1.$
3.3 Minkowski-signature pseudo-Majorana fermions

We can define an alternative “charge conjugation” operation

\[
\psi^\tau = T^{-1}\bar{\psi}^T, \\
\bar{\psi}^\tau = \psi^T T.
\]  

(24)

This operation reverses the current, again leaves the spin unchanged, but flips the sign of \(\bar{\psi}\psi\). Almost identical algebra to the conventional charge conjugation case shows that the condition \(\psi^\tau = \psi\) is consistent only when \(T\) is symmetric, hence in \(d=8,9,10\) (mod 8). Fermions such that \(\psi^\tau = \psi\) are said by some authors \([15, 16, 17, 18]\) to be pseudo-Majorana\(^2\).

Repeating the algebra for the \(C\) conjugation, but with \(C\) replaced by \(T\), gives an extra minus sign. Consequently in the mostly-minus West-Coast metric the gamma matrices of a pseudo-Majorana representation can be chosen to be real, while in a Majorana representation they are pure imaginary. It is the other way around in the mostly-plus East-Coast metric.

Because this “conjugation” flips \(\bar{\psi}\psi\), these pseudo-Majorana fermions are necessarily massless. Indeed the absence of the mass term is necessary for the real gamma matrices in the West Coast pseudo-Majorana representation and the pure imaginary gamma matrices in the East Coast pseudo-Majorana representation to avoid conflict with their appropriate Dirac equation.

3.4 Euclidean-signature Majorana fermions

We now explore to what extent the constraints on the Minkowski signature space-time dimensions in which Majorana and pseudo-Majorana fermions exist are compatible with Euclidean-signature Grassmann-variable path integration.

Assume that any gauge fields in the skew-Hermitian Euclidean-signature Dirac operator

\[
\mathcal{D} = \gamma^\mu (\partial_\mu + A_\mu) 
\]  

(25)

are in real representations so that \(A_\mu\) is a real matrix and \(A_\mu = -A^{T}_\mu\). Then, if we have an eigenfunction such that

\[
\mathcal{D} u_n = i\lambda_n u_n,
\]  

(26)

\(^2\)José Figueroa-O’Farril \([3]\) also uses the term pseudo-Majorana spinors, (“a nebulous concept best kept undisturbed”) but by this I believe he means the purely imaginary gamma matrices of the West-Coast Majorana representation.
complex conjugation gives
\[ \mathcal{D}^* u^* = -i \lambda_n u_n^*. \]  
(27)

This can be written as
\[ \mathcal{C} \mathcal{D} \mathcal{C}^{-1} u_n^* = i \lambda_n u_n^*, \]  
(28)
or
\[ \mathcal{D} \mathcal{C}^{-1} u_n^* = i \lambda_n \mathcal{C}^{-1} u_n^*. \]  
(29)

Thus \( u_n \) and \( \mathcal{C}^{-1} u_n^* \) are both eigenfunctions of \( \mathcal{D} \) with the same eigenvalue. They will be orthogonal, and therefore linearly independent, when \( \mathcal{C} \) is antisymmetric — something that happens in \( d = 2, 3, 4 \) (mod 8) Euclidean dimensions. These are precisely the dimensions in which Minkowski space Majorana spinors can occur. This suggests that we can take the Euclidean Majorana-Dirac action to be
\[ S[\psi] = \frac{1}{2} \int d^d x \psi^T \mathcal{C}(\mathcal{D} + m) \psi. \]  
(30)

The combination \( \psi^T \mathcal{C} \) is called by Peter van Nieuwenhuizen the “Majorana adjoint.”

Expanding the fields out as
\[ \psi(x) = \sum_n [\xi_n u_n(x) + \eta_n (\mathcal{C}^{-1} u_n^*)(x)], \]
\[ \psi^T(x) = \sum_n [\xi_n u_n^T(x) - \eta_n (u_n(x) \mathcal{C}^{-1})], \]  
(31)

where \( \xi_n \) and \( \eta_n \) are Grassmann variables we find that
\[ S = \frac{1}{2} \int d^d x \sum_{i,j} (\xi_i u_i^T - \eta_i u_i^\dagger \mathcal{C}^{-1})(\xi_j u_j + \eta_j \mathcal{C}^{-1} u_j^*) \] 
\[ + \frac{1}{2} \int d^d x \sum_{i,j} \left\{ \xi_i \eta_j (u_i^T u_j^*) + \xi_j \eta_i (u_i^\dagger u_j) \right\} \]  
\[ = \sum_i \xi_i \eta_i (i \lambda_i + m). \]  
(32)

\(^3\)We do not really need the complex conjugation. If we do not demand that the \( \mathcal{A} \) in \( \gamma^\mu \rightarrow A \gamma^\mu A^{-1} \) be unitary our Euclidean gamma-matrices are no longer Hermitian but we can still obtain the second eigenfunction for a given \( \lambda \) as \( \mathcal{C}^{-1} u^\beta v_\beta \) where \( v_\beta \) is a left eigenvector of \( \mathcal{D} \); i.e. one that obeys \( v^\dagger \mathcal{D} = i \lambda v \) where \( v^\dagger \mathcal{D} \equiv (-\partial_\mu v)\gamma^\mu \). See appendix A eq 125 et seq.
As the Grassmann integration uses only one copy of the doubly degenerate eigenvalue, we obtain a square-root of the full Dirac determinant \( \text{Det}(\not{D} + m) \).

We anticipate that the resulting partition function is the Pfaffian (see Appendix [A.3])

\[
Z = \text{Pf}[\mathcal{C}(\not{D} + m)]
\]

of the skew-symmetric bilinear kernel \( \mathcal{C}(\not{D} + m) \).

To confirm that the kernel is skew symmetric we take a transpose and find

\[
\begin{align*}
[C(\gamma^\mu \partial_\mu + m)]^T &= \left(\partial^T_\mu (\gamma^\mu) + m\right)\mathcal{C}^T \\
&= \left[(-\partial_\mu)(-\mathcal{C}\gamma^\mu\mathcal{C}^{-1}) + m\right](-\mathcal{C}) \\
&= -\mathcal{C}(\gamma^\mu \partial_\mu + m).
\end{align*}
\]

(34)

Note that \( \partial^T = -\partial \) because its \( x \)-basis matrix element is \( \delta'(x - x') \) is skew. The skew symmetry continues to hold for \( \not{D} \) in curved space and with gauge fields in real representations.

One further step is needed to confirm that

\[
\text{Pf}[\mathcal{C}(\not{D} + m)] = \prod_n (i\lambda_n + m).
\]

(35)

We know from Appendix [A.5] that under a change of basis a Pfaffian transforms as \( \text{Pf}[B^TQB] = \text{Pf}[Q]\text{det}[B] \). The mode expansion

\[
\psi^\alpha(x) = \sum_n [\xi_n u^{\alpha n}(x) + \eta_n (\mathcal{C}^{-1})^{\alpha\beta} u^{*\beta n}(x)],
\]

(36)

is a such linear change of variables in the Grassmann integral. It corresponds to a matrix \( B \) with indices \( B_{\alpha n;\beta x} \), where the range of \( n \) is doubled to include both the labels on \( u_n \) and on \( \mathcal{C}^{-1}u_n^* \). To show that diagonalizing \( \not{D} \) has not altered the value of the Pfaffian we need to show that this matrix is unimodular — or at least does not depend on the background fields. However

\[
\int d^4x \frac{1}{2} \psi^T(x)\mathcal{C}\psi(x) = \sum_n \xi_n \eta_n = \frac{1}{2} \sum_n (\xi_n \eta_n - \eta_n \xi_n)
\]

(37)

holds for any set of eigenmodes \( u_n \). In other words

\[
B^T (\mathcal{C} \otimes \mathbb{I}) B = \mathcal{C} \otimes \mathbb{I}
\]

(38)
where \([C \otimes I]_{\alpha n; \beta m} = C_{\alpha \beta} \delta_{nm}\) and \([C \otimes \tilde{I}])_{\alpha x; \beta x'} = C_{\alpha \beta} \delta_d(x - x')\). The matrices \(B\) from different background fields can differ only by left factors that preserve the symplectic form \((C \otimes I)\). Such symplectic matrices are automatically unimodular.

We could redefine the mode expansion as in [19]

\[
\psi^\alpha(x) = \sum_n [\xi_n u^\alpha_n(x) + \eta_n e^{i\theta_n}(\xi^{-1})^{\alpha\beta} u^\beta_n(x)],
\]

where the \(e^{i\theta_n}\) are arbitrary phases. This modification replaces \((i\lambda_n + m) \rightarrow e^{i\theta_n}(i\lambda_n + m)\) in the eigenvalue product and apparently alters the Pfaffian. The \(B\) matrix is no longer unimodular, though, and its phase cancels the product of the phase factors in the modified eigenvalue product leaving \(\text{Pf}[C(\partial + m)]\) unaffected. We therefore disagree with the claim in [19] that the phase of the Pfaffian is ill-defined.

3.4.1 4-d Dirac vs. Majorana

As a further illustration of the need to be careful when performing linear transformations on the Pfaffian integrands we consider the pulling apart a Dirac field into two Majorana fields. In four dimensions we can do this by relabelling its Weyl fermion components as

\[
\psi^\alpha = \begin{bmatrix} \psi_R^\alpha \\ \psi_L^\alpha \end{bmatrix} = \begin{bmatrix} \chi_{2R}^\alpha \\ \chi_{1L}^\alpha \end{bmatrix}, \quad \bar{\psi}^\alpha = \begin{bmatrix} \bar{\psi}_L^\alpha, \bar{\psi}_R^\alpha \end{bmatrix} = \begin{bmatrix} -\chi_{1R}^\alpha, -\chi_{2L}^\alpha \end{bmatrix} C.
\]

If all gauge fields are in real representations we can rewrite the kinetic part of the action density

\[
\mathcal{L} = [\bar{\psi}_L, \bar{\psi}_R] \begin{bmatrix} 0 & \mathcal{D}_L \\ \mathcal{D}_R & 0 \end{bmatrix} [\psi_R^\alpha, \psi_L^\alpha]
\]

as

\[
\mathcal{L} = -\frac{1}{2} \left\{ [\chi_{1R}^T, \chi_{2L}^T] C \begin{bmatrix} 0 & \mathcal{D}_L \\ \mathcal{D}_R & 0 \end{bmatrix} [\chi_{2R}^{1T}, \chi_{1L}^{1T}] + [\chi_{2R}^{1T}, \chi_{1L}^{1T}] C \begin{bmatrix} 0 & \mathcal{D}_L \\ \mathcal{D}_R & 0 \end{bmatrix} [\chi_{1R}^T, \chi_{2L}^T] \right\}
\]

\footnote{If we decompose a Dirac spinor into its Weyl-fermion chiral parts \(\psi = [\psi_R, \psi_L]^T\) then in the Minkowski operator language we have \(\bar{\psi} = [\psi_R^\dagger, \psi_L^\dagger]\). It is a common convention to label the entries in \(\psi\) so that \(\psi_R^\dagger \rightarrow \psi_R\) and \(\psi_L^\dagger \rightarrow \psi_L\) making \(\psi = [\psi_L, \psi_R]\).}
\[ \chi^T_{1R}, \chi^T_{1L}, \chi^T_{2R}, \chi^T_{2L} \] 
\[ \chi_{1L}, \chi_{1R}, \chi_{2L}, \chi_{2R} \] 
\[ \bar{\chi}_{1R}, \bar{\chi}_{1L}, \bar{\chi}_{2R}, \bar{\chi}_{2L} \]
\[ \bar{\chi} = -\chi^TC \]

Here \( \bar{\chi} = [\bar{\chi}_{L}, \bar{\chi}_{R}] = [-\chi^T_{R}, -\chi^T_{L}]C \) and both \( \chi_1 \) and \( \chi_2 \) are Majorana because \( \bar{\chi} = -\chi^T C \). The averaging in the first line comes from antisymmetry of \( C \bar{D} \).

A Dirac mass term

\[ m\bar{\psi}\psi = m[\bar{\psi}_L, \bar{\psi}_R] \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \]

becomes

\[ \frac{m}{2} \{ \bar{\chi}_1 \chi_2 + \bar{\chi}_2 \chi_1 \} = \frac{m}{2} \bar{\chi} \sigma_1 \chi \]

where the \( \sigma_1 \) acts on the “flavour” indices 1,2. We can do a flavour diagonalization by

\[ \chi \rightarrow e^{i\pi \sigma_1 \gamma^5/4} \chi = \frac{1}{\sqrt{2}}(1 + i\sigma_1 \gamma^5)\chi \]
\[ \bar{\chi} \rightarrow \bar{\chi} e^{i\pi \sigma_1 \gamma^5/4} = \bar{\chi} \frac{1}{\sqrt{2}}(1 + i\sigma_1 \gamma^5) \]

that takes

\[ \bar{\chi} \sigma_1 \chi \rightarrow \bar{\chi} \sigma_1 e^{i\pi \sigma_1 \gamma^5/2} \chi = \bar{\chi} i(\sigma_1)^2 \gamma^5 \chi \]

to get

\[ \frac{m}{2} \{ \bar{\chi}_1 (i\gamma^5)\chi_1 + \bar{\chi}_2 (i\gamma^5)\chi_2 \} \]

The transformation is unimodular for both eigenvalues of \( \gamma^5 \), and so it seems as if the path integral outputs the product of two Pfaffians (and therefore a determinant) of fields with an \( im\gamma^5 \) chiral mass. As every eigenvalue occurs twice this means that product of the two identical Pfaffians, which should reproduce the Dirac determinant, has become

\[ |m|^{n_+ + n_-} e^{i(\pi/2)(n_+ - n_-)} \prod \frac{m^2 + \lambda_n^2}{n} \]

Here \( n_+ \) and \( n_- \) are the number of zero modes with plus or minus \( \Gamma^5 \) chiralities. This determinant has apparently acquired a factor of \( (-1) \) for each pair of zero modes when compared to the original Dirac determinant, which does not contain the factor \( e^{i(\pi/2)(n_+ - n_-)} \). However, for the Grassman integral to
give the product of the two Pfaffians, we need reorder the $d[\chi]$ measure factors to get all the $d\chi_2$’s to the right of the $d\chi_1$’s. For each mode number $n$ we have

$$
d\bar{\psi}_n,Ld\psi_n,Rd\bar{\psi}_n,Rd\psi_n,L = \det[C]^{-1}d\chi_{n,1R}d\chi_{n,2R}d\chi_{n,2L}d\chi_{n,1L},
$$

where the order of the factors in the measure on the LHS is mandated so that we get $\det(\not{\slashed{D}} + m)$. The factors of $\det[C]^{-1}$, one for each mode $n$, cancel the “metric” factor $\det[C] = \det[C \otimes I]$ that always occurs when we use eigenvalues of a linear operator such as $\not{\slashed{D}} + m$ to compute the Pfaffian of an associated skew symmetric matrix such as $C(\not{\slashed{D}} + m)$ (Appendix A.3). If all modes are present the remaining factors on the RHS can be rearranged to get the Pfaffian without changing the sign. If there is a zero mode then (taking into account that each zero mode occurs twice) the factors of $(-1)$ that arise from each exchange of pairs of $d\chi$’s cancel the extra sign from the $i\gamma^5$ mass. This resolves the apparent paradox that motivated the paper 19.

3.5 Euclidean-signature pseudo-Majorana fermions

If

$$
\not{\slashed{D}} u_n = i\lambda_n u_n
$$

then

$$
\not{\slashed{D}}^* u_n^* = -i\lambda_n u_n^*
$$

or, assuming that any gauge fields are in real representations,

$$
\not{\slashed{D}} \mathcal{T}^{-1} u_n^* = -i\lambda_n \mathcal{T}^{-1} u_n^*.
$$

If $\lambda_n \neq 0$ we will have $u_n$ and $\mathcal{T}^{-1} u_n^*$ orthogonal because they have different eigenvalues. Let us choose $u_n$ to be the positive-$\lambda_n$ eigenfunctions and consider the cases $d=8, 9, 10$ (mod 8) in which $\mathcal{T}$ is symmetric and we can consistently impose the Minkowski signature pseudo-Majorana condition and take the action to be

$$
S = \frac{1}{2} \int d^d x \psi^T \mathcal{T} \not{\slashed{D}} \psi.
$$

Note that were $\mathcal{T}$ skew-symmetric, then

$$
\frac{1}{2} \int d^d x \psi^T \mathcal{T} \not{\slashed{D}} \psi = 0.
$$
Consequently Euclidean-signature pseudo-Majoranas are only available in the same dimensions as Minkowski-signature pseudo-Majoranas.

If there are no zero modes we can expand

\[
\psi(x) = \sum_n [\xi_n u_n(x) + \eta_n (T^{-1} u_n^*(x))]
\]

\[
\psi^T(x) = \sum_n [\xi_n u_n^T(x) + \eta_n (u_n^T(x)T^{-1})]
\]

and find that

\[
\frac{1}{2} \int d^d x \psi^T \mathcal{T} \psi = \sum_{\lambda_n > 0} (i\lambda_n) \eta_n \xi_n.
\]

As in the ordinary Majorana case, the partition function is the product of only half the eigenvalues, so again we get a square-root of the full Dirac determinant which we expect to identify with the Pfaffian Pf[\mathcal{T} \mathcal{D}] of the skew-symmetric kernel \(\mathcal{T} \mathcal{D}\).

What is different from the ordinary Majorana case is that we cannot add a mass term by taking

\[
S = \frac{1}{2} \int d^d x \psi^T (\mathcal{D} + m) \psi
\]

because \(\psi^T \mathcal{T} \psi = 0\) by the symmetry of \(\mathcal{T}\). Consequently Euclidean pseudo-Majorana fermions are necessarily massless — just as are their Minkowski brethren.

As second consequence of \(\psi^T \mathcal{T} \psi = 0\) is that establishing that the essential unimodularity of the diagonalizing matrices requires a slightly different tactic. If we replace the anticommuting \(\xi_n\) and \(\eta_n\) by commuting variables \(X_n\) and \(Y_n\) and define

\[
\phi_\alpha(x) = \sum_n [X_n u_\alpha n(x) + Y_n (T^{-1} u_\beta^* n(x))]
\]

\[
\phi^T_\alpha(x) = \sum_n [X_n u^T_\alpha n(x) + \eta_n (u^T_\beta n(x) T^{-1})].
\]

Then

\[
\int d^d x \phi^T(x) \mathcal{T} \phi(x) = \sum_n (X_n Y_n + Y_n X_n),
\]
independently of particular form of the \( u_n \) eigenfunctions. All diagonalizing transformations \( B \) therefore preserve the same (non-positive definite) symmetric form and can differ only by factors drawn from some orthogonal group. Orthogonal matrices obey \( \det[B]^2 = 1 \), so, unlike the ordinary Majorana case where the \( B \) matrices differ by symplectic (and therefore unimodular) matrix factors, here we have the possibility \( \det[B] = -1 \) and the Pfaffian changing sign. Indeed we have already seen an inherent sign ambiguity because we arbitrarily assigned the positive eigenvalue \( \lambda_n \) to \( u_n \) and \( \xi_n \), rather than to \( T^{-1}u_n^* \) and \( \eta_n \). This ambiguity is a potential source of global anomalies: if we smoothly interpolate between two gauge-equivalent background fields (which necessarily have the same set of \( \lambda_n \)), and an odd number of \( \lambda_n \) change sign during the interpolation then the partition function changes sign and the theory inconsistent [20].

### 3.6 Majorana-Weyl fermions

In \( 2 \ (\text{mod} \ 8) \) dimensions \( \Gamma^5 \equiv \gamma^{8k+3} \) obeys \( C\Gamma^5C^{-1} = -(\Gamma^5)^T \) and therefore \( C\mathcal{T}\Gamma^5 = (\Gamma^5)^T C\mathcal{T} \). We can thus decompose

\[
\psi^T C(\mathcal{D} + m)\psi = \psi^T_R C(\mathcal{D})\psi_R + \psi^T_L C(\mathcal{D})\psi_L + m(\psi^T_R C\psi_L + \psi^T_L C\psi_R). \tag{60}
\]

If \( m = 0 \), we may retain only one of the the right or left fields, in which case we have a Euclidean Majorana-Weyl fermion.

### 3.7 Rokhlin’s theorem

Recall that in \( d = 4 \ (\text{mod} \ 8) \) the \( \mathcal{T} \) matrix is antisymmetric and obeys \( \mathcal{T}\Gamma^5\mathcal{T}^{-1} = (\Gamma^5)^T = (\Gamma^5)^* \), where \( \Gamma^5 \equiv \gamma^{8k+5} \). Assume that no gauge fields are present (i.e. gravity only), then similar algebra to the previous section shows that if \( u_n \) obeys

\[
\mathcal{D} u_n = i\lambda_n u_n, \tag{61}
\]

then

\[
\mathcal{D} \Gamma^5 u_n^* = -i\lambda_n \Gamma^5 u_n^*. \tag{62}
\]

In particular, if \( \lambda_n = 0 \) then \( \Gamma^{-1}u_n^* \) is also a zero mode, is orthogonal to \( u_n \), and has the same \( \gamma^{8k+5} \) eigenvalue. It follows that chiral zero modes come in pairs, and so the Dirac index

\[
n_+ - n_- = \int_M \hat{A}(R) \tag{63}
\]
is an even integer. In dimension 4 this result implies Rokhlin’s theorem that the signature of a 4-dimensional spin manifold is divisible by 16. This is because when \( d = 4 \)

\[
\hat{A}_1 = -\frac{1}{24}p_1, \quad \text{Dirac Index},
\]

\[
L_1 = +\frac{1}{3}p_1, \quad \text{Signature},
\]

where

\[
p_1 = -\frac{1}{(2\pi)^2} \text{tr} \left\{ \frac{1}{2} R \wedge R \right\},
\]

is the four-form Pontryagin class. We see that the \( \hat{A} \)-genus whose integral gives \( n_+ - n_- \) evaluates to minus one-eighth of the signature.

Observe that iterating the antilinear map \( u_n \to T^{-1}u_n^* \) twice gives

\[
u_n \to T^{-1}[T^{-1}u_n^*]^* = -u_n,
\]

where we have used \( T^* = [T^{-1}]^T = -T^{-1} \). Thus our map gives rise to a quaternionic structure — i.e. an antilinear map that squares to minus the identity — on the zero mode space. This is how Rokhlin’s theorem is explained in the mathematics literature.

We could, of course, have deduced the doubling of the zero modes from the Majorana doubling given by the antisymmetric \( C \). This also gives rise to a quaternionic structure. Indeed in \( d = 4 \pmod{8} \) we have

\[
C = \Gamma^5 T,
\]

so we can take

\[
Q_\pm \overset{\text{def}}{=} \frac{1}{2} (1 \pm \Gamma^5) T
\]

to be pair of independent quaternionic structures, one for each of the \( \Gamma^5 \to \pm 1 \) subspaces of chiral zero-modes.

### 3.8 Symplectic Majorana fermions

In \( d = 5, 6, 7 \pmod{8} \) Euclidean dimensions neither Majorana nor pseudo-Majorana fermion actions can be constructed. There is however the option
of symplectic Majorana fermions. To obtain these we start from a pair of fermions $\psi_1$, $\psi_2$. In $d = 7 \ (\text{mod} \ 8)$ where $C$ is symmetric we can set

$$S = \frac{1}{2} \int \psi_a^T C e^{ab}(\mathbb{D} + m)\psi_b d^{8k+7}x. \quad (68)$$

In $d = 5 \ (\text{mod} \ 8)$ where $\mathcal{T}$ is antisymmetric we can take

$$S = \frac{1}{2} \int \psi_a^T \mathcal{T} e^{ab}\mathbb{D}\psi_b d^{8k+5}x. \quad (69)$$

In $d = 6 \ (\text{mod} \ 8)$ we can use either of $C$ or $\mathcal{T} = C\Gamma^5$, and if the mass vanishes we can have symplectic Majorana-Weyl fermions.

In all three cases the Minkowski-signature Majorana constraint is

$$\psi_a = \epsilon_{ab}(C \text{ or } \mathcal{T})^{-1}(\gamma^0)^T \psi_b^*. \quad (70)$$

4 Multiplets, continuous symmetries, and Bott periodicity

We have so far considered Majorana fields one at a time. Let us add a “flavour” index taking values $n = 1, \ldots, N$ to the $\psi$’s and consider the possible symmetry groups that the resulting $N$-tuplets can possess.

4.1 Massless fields

For massless fields, the resulting symmetry groups are displayed in table 2. Not all of these groups are immediately obvious. Consider for example $d = 4 \ (\text{mod} \ 8)$. The Euclidean Majorana action is invariant under separate chiral transformation $\psi_R \to U\psi_R$, $\psi_L = V\psi_L$ provided the $N$-by-$N$ matrices $U$, $V$, acting on the new indices obey $U^TV = \mathbb{I}_N$. In Euclidean signature there is no obvious reason to impose any reality conditions on $U$ and $V$, but when we desire the invariance to extend to a symmetry in Minkowski signature where $\psi = C^{-1}(\gamma^0)^T \psi^*$ preserving this relation requires that $V = U^*$. This condition makes $U^TU = \mathbb{I}_N$ and hence $U \in U(N)$.

In $d = 5, 6, 7$ preserving the kinetic term requires $\psi \to V\psi$ with $V$ a $2N$-by-2$N$ matrix obeying $V^T\Omega V = \Omega$. Here $\Omega$ is block diagonal with $N$ copies
Table 2: The Minkowski signature flavour groups $G$ that preserve the action of massless Majorana fermions in $d$ spacetime dimensions. In the table $\text{Sp}(N) \equiv \text{Sp}(2N, \mathbb{C}) \cap \text{U}(2N)$ is the unitary symplectic group. The subgroups $H$ are those that survive the addition of suitable mass terms.

| $d \pmod{8}$ | $\mathcal{T}$ | $\mathcal{C}$ | $G$ | $H$ |
|-------------|--------------|-------------|----|-----|
| 0           | S            | S           | $\text{U}(N)$ | $\text{Sp}(N/2)$ |
| 1           | S            |             | $\text{O}(N)$ | $\text{U}(N/2)$ |
| 2           | S            | A           | $\text{O}(N) \times \text{O}(N)$ | $\text{O}(N)$ |
| 3           | A            |             | $\text{O}(N)$ | $\text{O}(N/2) \times \text{O}(N/2)$ |
| 4           | A            | A           | $\text{U}(N)$ | $\text{O}(N)$ |
| 5           | A            |             | $\text{Sp}(N)$ | $\text{U}(N)$ |
| 6           | A            | S           | $\text{Sp}(N) \times \text{Sp}(N)$ | $\text{Sp}(N)$ |
| 7           | S            |             | $\text{Sp}(N)$ | $\text{Sp}(N/2) \times \text{Sp}(N/2)$ |

on the diagonal of the two-by-two matrix with entries $\epsilon_{ab}$. This condition requires $V$ to be in $\text{Sp}(2N, \mathbb{C})$, and, as $\Omega^{-1} = -\Omega$, can also be written as

$$-\Omega V \Omega = (V^T)^{-1}. \quad (71)$$

The Minkowski-signature reality conditions are of the form $\psi = \Omega \psi^*$ (ignoring the spinor indices) and preserving them requires $-\Omega V \Omega = V^*$. Consistency with equation (71) now needs $(V^T)^{-1} = V^*$, meaning that $V$ is a unitary matrix. $V$ is therefore an element of the unitary symplectic group $\text{Sp}(N) \equiv \text{Sp}(2N, \mathbb{C}) \cap \text{U}(2N)$.

The other cases are more straightforward except that in $2, 6 \pmod{8}$ dimensions the products $\text{O}(N) \times \text{O}(N)$ and $\text{Sp}(N) \times \text{Sp}(N)$ arise as a consequence of the existence of Majorana-Weyl fermions in those dimensions.

We can elevate the global flavour symmetries to local symmetries in which case the flavour indices become “colour” indices that couple to gauge fields in the fundamental representation of the corresponding group. It may seem unlikely that a four-dimensional Majorana field can couple to a $\text{U}(N)$ gauge field but, because the left- and right-handed fermions transform under conjugate representations, the induced coupling is to an *axial* gauge field. Unlike the vector current $\bar{\psi} C \lambda_\alpha \gamma^\mu \psi$, the axial current $\bar{\psi} C \lambda_\alpha \gamma^5 \gamma^\mu \psi$ is not identically zero. An example of such a Majorana $U(1)$ axial gauge-field coupling appears in models of topological superconductors [21, 22, 23].

21
Table 3: The Cartan classification of the symmetric spaces arising from quotienting each of the flavour symmetry groups by its successor starting from \(d = 0 \pmod{8}\). The Bott periodicity theorem asserts that, for sufficiently large \(N\), the homotopy group \(\pi_k(G/H)\) is isomorphic to \(\pi_0(G/H)\) of the symmetric space \(k\) rows \((\pmod{8})\) lower down the table.

What is most interesting about the pattern of groups in table 2 is that, starting arbitrarily from \(d = 3 \pmod{8}\) and with suitable \(d\)-dependent choices of \(N\), we have a natural sequence of group embeddings

\[
\ldots \circlearrowleft O(16N) \supset U(8N) \supset \text{Sp}(4N) \supset \text{Sp}(2N) \times \text{Sp}(2N) \supset \text{Sp}(2N) \\
\supset U(2N) \supset O(2N) \supset O(N) \times O(N) \supset O(N) \ldots
\]

of each group in its predecessor. Starting from \(d = 3\) is not essential. The sequence can be extended to the left, and to the right when \(N\) is a suitably large power of two. The pattern repeats with period eight, and the successive quotients of the groups are the reductive symmetric spaces that appear in Bott’s periodicity theorem for stable homotopy groups. This theorem is displayed in table 3. The same pattern appears, and for the same reason — the trading of spinor indices for flavour indices — in the sequence of “R” symmetries that appear during the reduction of \(N = 1\) supersymmetry algebras to larger \(N\) algebras in lower dimensions [24, 25].

### 4.2 Mass terms and symmetry breaking

In \(4 \pmod{8}\) dimensions the addition of a \(m\psi^T \mathcal{C} \psi\) mass term to the Majorana action breaks the chiral flavour symmetry from \(U(N)\) down to \(O(N)\). This
is because preserving such a mass term requires the matrices $U$ and $V = U^*$ of the previous section to be equal. Note that $O(N)$ is the group in the Bott cycle that *precedes* the $U(N)$ in row 4 of table 2. This backstepping is part of a general pattern.

To illustrate this pattern consider some possible symmetry breaking mass terms:

$$
\psi^T T M \psi, \quad M \text{ skew-symmetric in } d = 0, 1 \mod 8 \text{ dimensions,}
$$
$$
\psi^T C M \psi, \quad M \text{ symmetric in } d = 3, 4 \mod 8 \text{ dimensions}
$$
$$
\psi^T T \Omega M \psi, \quad \Omega M \text{ symmetric in } d = 5 \mod 8 \text{ dimensions,}
$$
$$
\psi^T C \Omega M \psi, \quad \Omega M \text{ skew-symmetric in } d = 7 \mod 8 \text{ dimensions.}
$$

For a moment let us assume the maximally symmetric situation where $M = \text{diag}(m, \ldots m)$ in $d = 3, 4 \mod 8$ and $M = \Omega \text{diag}(m, \ldots m)$ in $d = 5 \mod 8$ dimensions.

In $d = 5 \mod 8$ dimensions preserving the mass term $\psi^T T \text{diag}(m, \ldots m) \psi$ requires the unitary symplectic matrix $V$ to obey $V^T V = I$ *i.e.* to be an orthogonal matrix. Consequently the $\text{Sp}(N) \simeq \text{Sp}(2N, \mathbb{C}) \cap U(2N)$ flavour group is broken to $\text{Sp}(2N, \mathbb{R}) \cap O(2N) \simeq U(N)$.

Similarly, in $d = 0 \mod 8$ dimensions $M$ must be skew, and if non-degenerate, $N$ must be an even number. Preserving the skew symmetric form $\psi^T T M \psi$ requires a symplectic group. The symmetry is therefore reduced from $U(2N)$ to its subgroup $\text{Sp}(2N, \mathbb{C}) \cap U(2N) \simeq \text{Sp}(N)$. In both these cases the preserved sub group $H \subseteq G$ is again the preceding one the Bott cycle. In $d = 3, 7 \mod 8$ we have an *option* of taking $N_+$ of the of diagonal elements of the symmetric matrix to be $-m$ and the remaining $N_- = N - N_+$ to be $+m$, so so in $d=3 \mod 8$ we can take $O(N) \to O(N_+) \times O(N_-)$, and similarly the $d = 7 \mod 8$ symplectic case.

If the original symmetry group is of the form $G_{\text{gauge}} \otimes G_{\text{flavour}}$ and the mass terms arise from spontaneous breaking of the flavour symmetry $G_{\text{flavour}} \to H_{\text{flavour}}$ by the gauge interactions, the orbit of equivalent vacua is the appropriate symmetric space $G_{\text{flavour}}/H_{\text{flavour}}$.

---

5The first line of this list shows that prohibition of a mass for pseudo-Majorana fermions can be evaded when there is more than one of them.
5 Discrete symmetries

5.1 Intrinsic parity of Dirac and Majorana fermions

In even space-time dimensions parity is defined by \( P : (t, x) \mapsto (t, -x) \). In the mostly-minus metric \( P \) is implemented on spinor-valued fields as

\[
P : \psi(t, x) \mapsto \psi^p(t, x) = \eta \gamma^0 \psi(t, -x),
\]

where the phase \( \eta \) is the particle’s intrinsic parity. We usually take \( \eta = \pm 1 \) so that that \( P^2 = \text{id} \). However if \( P \) is to be compatible with the Majorana condition \( \psi^c = \psi \) and if we require \( (\psi^c)^p = (\psi^p)^c \) then we must have same parity transformation rule for \( \psi \) and \( \psi^c \). Let us see what this requires.

Using \( (\psi^p(t, x))^* = \eta^*(\gamma^0)^* \psi^*(t, -x) \) we have

\[
\eta \gamma^0 [C^{-1} (\gamma^0)^T \psi^*(t, -x)] = C^{-1} (\gamma^0)^T \eta^* (\gamma^0)^* \psi^*(t, -x)
\]

which reduces to

\[
\eta C \gamma^0 C^{-1} (\gamma^0)^T = \eta^* (\gamma^0)^T (\gamma^0)^* \tag{74}
\]

or

\[
-\eta (\gamma^0)^T = \eta^* (\gamma^0)^* \tag{75}
\]

Since \( \gamma^0 \) is Hermitian in the mostly minus metric we see that \( \eta^* = -\eta \), and so for a Majorana particle we must have \( \eta = \pm i \) and so \( P^2 = -1 \).

If we have the freedom to allow

\[
(\psi^c)^p(t, x) = \eta^c \gamma^0 \psi^c(t, -x) \tag{76}
\]

then the same algebra shows that \( \eta^c = -\eta^* \). For particles that are distinct from their antiparticles we are therefore allowed to have \( \eta \) to be \( \pm 1 \), but then the parity of an antiparticle is minus that of the particle.

In the mostly-plus metric parity is usually implemented by\(^6\)

\[
P : \psi(t, x) \mapsto \psi^p(t, x) = i \eta \gamma^0 \psi(t, -x) \tag{77}
\]

The reason for the extra factor of \( i \) is that when \( \eta = 1 \) we again want \( P^2 = \text{id} \), and the extra \( i \) compensates for \( (\gamma^0)^2 = -1 \). For Majorana fields we still find that \( \eta^* = -\eta \).

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\(^6\)Steven Weinberg’s *The Quantum Theory of Fields* \( [26] \) is the only text that I know that uses the mostly plus convention, and he has this “\( i \)” factor. Mark Srednicki’s *Quantum Field Theory* \( [27] \) claims to use the mostly-plus convention, but he defines his Clifford algebra by \( \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \delta^{\mu \nu} \), so his are the mostly-minus gamma matrices.
5.2 R symmetry

In odd space-time dimension $d = 2k + 1$, the standard parity operation $(t, x) \mapsto (t, -x)$ is an $SO(2k)$ rotation, so “parity” is instead defined as the inversion of an odd number of the spatial coordinates. In the case that we flip only one direction Witten calls it R symmetry \[20\] Let us define $R$ to invert $x^1$ so

$$R : (t, x_1, x_2, \ldots x_{2k}) \mapsto (t, -x_1, x_2, \ldots x_{2k}) \equiv (t, \tilde{x}). \quad (78)$$

In Euclidean signature the natural way to flip the sign of $\gamma^1$ only is by the using the Clifford algebra “twisted map” reflection in the plane perpendicular to the $x^1$ axis:

$$R : \gamma^\mu \mapsto (-\gamma^1)\gamma^\mu\gamma^1 = \tilde{\gamma}^\mu \quad (79)$$

To get

$$R : \bar{\psi}(x)\gamma^\mu\psi(x) \mapsto \bar{\psi}(\tilde{x})\tilde{\gamma}^\mu\psi(\tilde{x}) \quad (80)$$

we must therefore set

$$R : \psi(x) \mapsto \gamma^1\psi(\tilde{x})$$
$$R : \bar{\psi}(\tilde{x}) \mapsto \bar{\psi}(\tilde{x})(-\gamma^1) \quad (81)$$

In Euclidean signature the $R : u_n(x) \mapsto \gamma^1u_n(\tilde{x})$ anticommutates with $\partial$ and so changes the sign of the corresponding eigenvalue $\lambda_n$.

In Minkowski space consider first the mostly plus metric in which $(\gamma^1)^2 = 1$ and $\gamma^1$ is Hermitian. When

$$\psi(t, x) \mapsto \eta\gamma^1\psi(t, \tilde{x}) \quad (82)$$

we have

$$\bar{\psi}(t, x) \mapsto \eta\gamma^1\psi(t, \tilde{x}) = \eta^*\psi^\dagger(t, \tilde{x})(\gamma^1)^1\gamma^0 = \eta^*\psi(t, \tilde{x})(-\gamma^1) \quad (83)$$

and so we have

$$R : \bar{\psi}\gamma^\mu\psi \mapsto \bar{\psi}\tilde{\gamma}^\mu\psi. \quad (84)$$

In the mostly-minus metric when $R$ acts on the Fermi fields as

$$\psi(t, x) \mapsto \eta\gamma^1\psi(t, \tilde{x}) \quad (85)$$

then

$$\bar{\psi}(t, x) \mapsto \eta\gamma^1\psi(t, \tilde{x}) = \eta^*\psi^\dagger(t, \tilde{x})(\gamma^1)^1\gamma^0 = \eta^*\psi(t, \tilde{x})\gamma^1. \quad (86)$$

25
This appears to differ from the Euclidian $R$, but $(\gamma^1)^2 = -1$, so we still have

$$R : \bar{\psi} \gamma^\mu \psi \mapsto \bar{\psi} \tilde{\gamma}^\mu \psi.$$  \hfill (87)

As $\partial_x \mapsto \partial_{\bar{x}}$ both signature versions of $R$ leave the kinetic part of the Dirac action invariant. However

$$R : \bar{\psi} \psi \mapsto -\bar{\psi} \psi.$$  \hfill (88)

so the mass term is not invariant. In even space-time dimensions we can undo the flip with a $\Gamma^5$ and so obtain the usual parity operation which leaves $m$ fixed. This option is not available in odd space-time dimensions, where a mass is unavoidably parity-violating.

Requiring that reflection commutes with charge conjugation leads to $\eta_c = -\eta^*$. To see this compare

$$[\psi^c(t, x)]^r = \eta_c \gamma^1 [C^{-1} \gamma^{0T} \psi^* (t, \bar{x})]$$  \hfill (89)

with

$$[\psi^r(t, x)]^c = C^{-1} \gamma^{0T} [\eta^* \gamma^1 \psi^* (t, \bar{x})]$$

$$= -CC \gamma^0 C^{-1} \gamma^1 \psi^* (t, \bar{x})$$

$$= -\eta^* \gamma^0 C^{-1} \gamma^1 \psi^* (t, \bar{x})$$

$$= +\eta^* \gamma^0 C^{-1} \gamma^{1T} \psi^* (t, \bar{x})$$

$$= -\eta^* \gamma^0 C^{-1} C \gamma^1 C^{-1} \psi^* (t, \bar{x})$$

$$= -\eta^* \gamma^0 \gamma^1 C^{-1} \psi^* (t, \bar{x})$$

$$= +\eta^* \gamma^0 \gamma^1 C^{-1} \psi^* (t, \bar{x})$$

$$= -\eta^* \gamma^1 [C^{-1} \gamma^{0T} \psi^* (t, \bar{x})].$$  \hfill (90)

### 5.3 Time reversal

At first sight time reversal should simply be an $R$ map applied to $x^0 \equiv t$ rather than $x^1$. However such an $R$ reverses the direction of particle trajectories in time and so converts particles into antiparticles. The conventional particle-physics (Wigner) time reversal operation does not charge-conjugate and so $T$ is defined by composing an $R$ with a compensating charge conjugation operation. There is still a problem: time reversal does not play nicely with the passage from Euclidean to Minkowski signature because, as in non-relativistic
quantum mechanics, time reversal must be implemented on the many-particle Hilbert space by an antiunitary operator $\mathcal{J}$.

An operator $\Omega$ is said to be antiunitary with respect to a conjugate-symmetric sesquilinear inner product $\langle \cdot, \cdot \rangle$ if
\[
\langle \Omega a, \Omega b \rangle = \langle a, b \rangle^* = \langle b, a \rangle.
\] (91)

Consider the vector
\[
X = \Omega(\alpha a + \beta b) - \alpha^*(\Omega a) - \beta^*(\Omega b).
\] (92)
Using the definition of antiunitarity and the antilinearity of $\langle \cdot, \cdot \rangle$ in its first slot and linearity in the second, we can expand out $\|X\|^2 = \langle X, X \rangle$ and find that it is zero. For a positive definite inner product a vanishing norm implies that $X = 0$, and so for such a product we have
\[
\Omega(\alpha a + \beta b) = \alpha^*(\Omega a) + \beta^*(\Omega b).
\] (93)
Thus an antiunitary operator acting on a positive-definite Hilbert space is necessarily antilinear.

One consequence of the antilinearity is that there is no way to define an adjoint $\Omega^\dagger$. The standard definition $\langle \Omega^\dagger a, b \rangle = \langle a, \Omega b \rangle$ leads to
\[
\langle b, a \rangle = \langle \Omega a, \Omega b \rangle = \langle \Omega^\dagger \Omega a, b \rangle
\] (94)
and a contradiction: the leftmost expression is antilinear in $b$ while the rightmost is linear in $b$. A similar issue leads to
\[
\langle \langle a | \Omega \rangle | b \rangle \neq \langle a | \Omega \rangle | b \rangle \quad (95)
\]
and so makes Dirac notation “matrix elements” $\langle a | \Omega | b \rangle$ ambiguous. It also prevents us from defining a left action of $\Omega$ on bra vectors $\langle a \rangle$. Instead we have the useful identity
\[
\langle b, \Omega a \rangle = \langle a, \Omega^{-1} b \rangle.
\] (96)

Another useful result is that if $A$ is a linear operator then so is $\Omega^{-1} A \Omega$, and we can compute its adjoint as follows
\[
\langle b, (\Omega^{-1} A \Omega) a \rangle = \langle \Omega b, A \Omega a \rangle^* = \langle A^\dagger \Omega b, \Omega a \rangle^* = \langle (\Omega^{-1} A^\dagger) b, a \rangle
\] (97)

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7Some sources—for example the Wikipedia article on antiunitary operators—define an “$\Omega^\dagger$” by equating it to $\Omega^{-1}$. I think that this notation is dangerously confusing.
so
\[(\Omega^{-1}A\Omega)^\dagger = \Omega^{-1}A^\dagger \Omega.\] (98)

In the mostly minus Minkowski metric the time reversal operator \(\mathcal{J}\) is usually taken to acts on Dirac field operators as
\[
\mathcal{J}^{-1}\psi(x, t)\mathcal{J} = \eta_T T\psi(x, -t)
\]
\[
\mathcal{J}^{-1}\bar{\psi}(x, t)\mathcal{J} = \eta_T^* \bar{\psi}(x, -t)T^{-1},
\] (99)

where \(\eta_T\) is a phase. Despite the antilinearity of \(\mathcal{J}\) the field operator is not Hermitian-conjugated: time reversal changes the sign of momentum and the spin, but does not change particle to antiparticle. This, however, is the action on the field operator. The action of \(\mathcal{J}\) on wavefunctions does involve complex conjugation, and will be described later.

We can decompose the action of \(\mathcal{J}\) into a composition of \(T = CR\) followed by complex conjugation:

\[
\psi(x, t) \xrightarrow{R} \gamma^0 \psi(x, -t)
\]
\[
\gamma^0 \psi(x, -t) \xrightarrow{C} \eta_T^{-1}(\gamma^0)^T(\gamma^0)^* \psi(x, -t) = \eta_T^{-1} \psi^*(x, -t)
\]
\[
\eta_T^{-1} \psi^*(x, -t) \xrightarrow{\gamma^0} \eta_T^{-1} \psi^*(x, -t) = \lambda \eta_T^* T \psi(x, -t),
\] (100)

where \(T^T = \lambda T\). We have elected to use the \(T\) version of charge conjugation rather than the \(C\) version because a \(T\) conjugation inverts \(\bar{\psi}\bar{\psi}\) and so undoes the \(\bar{\psi}\bar{\psi}\) inversion due to the \(R\). As a result
\[
\bar{\psi}(x, t)\psi(x, t) \rightarrow \mathcal{J}^{-1}\bar{\psi}(x, t)\psi(x, t)\mathcal{J} = \mathcal{J}\bar{\psi}(x, t)\mathcal{J}^{-1}\mathcal{J}^{-1}\psi(x, t)\mathcal{J} = \bar{\psi}(x, -t)\psi(x, -t).
\] (101)

A \(\Gamma^5\) chiral mass term does change sign.

The transformation of \(\bar{\psi}\) follows that of \(\psi\) via
\[
(\mathcal{J}^{-1}A\mathcal{J})^* = \mathcal{J}^{-1}A^*\mathcal{J}.
\] (102)

We use \(*\) instead of \(\dagger\) to indicate that the Hermitian adjoint in the quantum-state Hilbert space does not transpose column-matrix spinors to row-matrix spinors. Then
\[
\mathcal{J}^{-1}\psi(x, t)\mathcal{J} = \eta_T T\psi(x, -t) \Rightarrow \mathcal{J}^{-1}\psi^*(x, t)\mathcal{J} = \eta_T^* T^* \psi^*(x, -t).
\] (103)
Transposing and using antilinearity

\[
\mathcal{J}^{-1}\psi^\dagger(x, t)\gamma^0\mathcal{J} = \eta_T^* \psi^\dagger(x, -t)\mathcal{T}^\dagger(\gamma^0)^* \\
= \eta_T^* \psi(x, -t)\gamma^0\mathcal{T}^{-1}(\gamma^0)^* \\
= \eta_T^* \psi(x, -t)\mathcal{T}^{-1}\mathcal{T}\gamma^0\mathcal{T}^{-1}(\gamma^0)^* \\
= \eta_T^* \psi(x, -t)\mathcal{T}^{-1}(\gamma^0)^{\dagger}\mathcal{T}(\gamma^0)^* \\
= \eta_T^* \psi(x, -t)\mathcal{T}^{-1}. \tag{104}
\]

The time reversal of the current is

\[
\mathcal{J}^{-1}\bar{\psi}(x, t)\gamma^\mu\psi(x, t)\mathcal{J} = \mathcal{J}^{-1}\bar{\psi}(x, t)\mathcal{J}\mathcal{J}^{-1}\gamma^\mu\psi(x, t)\mathcal{J} \\
= \mathcal{J}^{-1}\bar{\psi}(x, t)\mathcal{J}(\gamma^\mu)^*\mathcal{J}^{-1}\psi(x, t)\mathcal{J} \\
= \bar{\psi}(x, -t)\mathcal{T}^{-1}(\gamma^\mu)^*\mathcal{T}\psi(x, -t) \\
= \bar{\psi}(x, -t)\mathcal{T}^{-1}(\gamma^\mu)^{\dagger}\mathcal{T}\psi(x, -t) \\
= \pm\bar{\psi}(x, -t)\mathcal{T}^{-1}(\gamma^\mu)^{\dagger}\mathcal{T}\psi(x, -t) \\
= \pm\bar{\psi}(x, -t)\gamma^\mu\psi(x, -t). \tag{105}
\]

Here \(\gamma^\mu = \pm\gamma^\mu\) so we have + for the Hermitian \(\gamma^0\), so the charge is not altered, and -1 for the antiHermitian \(\gamma^\mu\) which changes the sign of the the spatial current.

If we act twice we find

\[
\mathcal{J}^{-2}\psi(x, t)\mathcal{J}^2 = |\eta_T|^2\mathcal{T}\mathcal{T}^*\psi(x, t). \tag{106}
\]

Now \(\mathcal{T}^{-1} = \mathcal{T}^\dagger = (\mathcal{T}^T)^* = \lambda\mathcal{T}^*\) so

\[
\mathcal{J}^{-2}\psi(x, t)\mathcal{J}^2 = \lambda\psi(x, t). \tag{107}
\]

Thus conjugating by \(\mathcal{J}\) twice gives a -1 in 4, 5, 6 (mod 8) dimensions in which \(\mathcal{T}\) is antisymmetric. As the vacuum is left fixed

\[
\mathcal{J}|0\rangle = |0\rangle \tag{108}
\]

and the field operators change the fermion number by \(\pm 1\), the -1 means that when \(\mathcal{J}\) acts on the many-particle Hilbert space we have \(\mathcal{J}^2 = (-1)^F\) id where \(F\) is the fermion number. We get a +1, and hence \(\mathcal{J}^2 = \text{id}\), in 0, 1, 2 (mod 8) dimensions in which \(\mathcal{T}\) is symmetric.
In 3 and 7 (mod 8) dimensions the $T$ matrix does not exist. We can however use the $C$ charge-conjugation operation to define an alternative time reversal

$$\bar{\psi}(\mathbf{x}, t) \bar{T} = \eta T C \psi(\mathbf{x}, -t), \quad \bar{\psi}(\mathbf{x}, t) \bar{T} = -\eta^* \bar{\psi}(\mathbf{x}, -t) C^{-1},$$

at the expense of flipping the the sign of $\bar{\psi} \psi$. Thus a mass term is necessarily time-reversal-symmetry violating in 3 and 7 (mod 8) dimensions. Acting twice, this time reversal gives a $(-1)^F$ in 3 (mod 8) dimensions and a plus sign in 7 (mod 8).

5.3.1 $T$ and anomaly inflow

Consider a chiral (Weyl) fermion in space-time dimension $d = 2k$ and interacting with an abelian gauge field $A_\mu$. This is an anomalous theory in which the anomaly can be be accounted for by current flowing into the $d = 2k$ dimensional surface from the $D = 2k + 1$ bulk at a rate [28]

$$J^{2k+1} = \frac{1}{(2\pi)^k k!} \text{sgn}(M) \epsilon^{2k+1,i_1,\ldots,i_{2k}} F_{i_1i_2} \cdots F_{i_{2k-1}i_{2k}}. \quad (110)$$

Here $M$ is a large Dirac mass in the $2k + 1$ dimensional theory. If we reverse time, the direction of this flow should reverse. How does this reversal relate to our discussion so far?

In Minkowski signature time reversal acts on the components of the gauge field as

$$A_0 \mapsto A_0, \quad \mathbf{A} \mapsto -\mathbf{A}. \quad (111)$$

From this we see that the “electric field”

$$F_{0i} = \partial_0 A_i - \partial_i A_0 \quad (112)$$

is unaffected by time reversal, but all other (magnetic) $F^{ij}$ change sign. Consequently the gauge-field $2k$-form $F^k$ in equation (110) is time-reversal invariant in $2k + 1 = 3$ and 7 (mod 8) bulk dimensions and changes sign in 1 and 5 dimensions (mod 8). The reversal of $J^{2k+1}$ is therefore accounted for by the mass $M$ changing sign in 3 and 7 (mod 8) and by the $F^k$ factor changing sign in the other odd dimensions.

A change in sign of the $2k + 1$ bulk theory Dirac mass should also cause a change in the $\Gamma^5$ chirality of the surface-trapped $2k$-dimensional fermions.
An inspection of the table shows that it is precisely in 2 and 6 dimensions that $\Gamma^5$ anticommutes with both $C$ and $T$, and so the change in chirality is consistent with the

$$\psi(t, x) \mapsto (C \text{ or } T)\psi(-t, x)$$  \hspace{1cm} (113)

time reversal transformation.

5.3.2 The action of $T$ on wavefunctions

The $c$-number spinor wavefunction corresponding to a single particle state $|\phi\rangle$ is

$$\phi(x, t) = \langle 0|\psi(x, t)|\phi\rangle.$$  \hspace{1cm} (114)

The wavefunction of the time reversed state is then

$$\langle 0|(\psi(x, t)\mathcal{J}|\phi\rangle) = \langle 0|(\mathcal{J}^{-2}\psi(x, t)\mathcal{J}|\phi\rangle)$$

$$= \langle 0|\mathcal{J}^{-1}(\mathcal{J}^{-1}\psi(x, t)\mathcal{J}|\phi\rangle)$$

$$= \langle 0|(\mathcal{J}^{-1}\psi(x, t)\mathcal{J}|\phi\rangle)^*$$

$$= \langle 0|\mathcal{J}^{-1}\psi(x, -t)|\phi\rangle)^*$$

$$= \eta^*_T T^*(\langle 0|\psi(x, -t)|\phi\rangle)^*$$

$$= \lambda\eta^*_T T^{-1}\phi^*(x, -t).$$  \hspace{1cm} (115)

In the first line we used $\mathcal{J}^2 = (\lambda)^F$ with fermion number $F = 0$, and in passing from the second to third line we used

$$\langle 0|\mathcal{J}^{-1}|v\rangle \equiv \langle 0|(\mathcal{J}^{-1}|v\rangle) = \langle v|\langle \mathcal{J}|0\rangle) = \langle v|0\rangle = \langle 0|v\rangle^*. \hspace{1cm} (116)$$

The result is that, in contrast with the transformation of the operator, the single-particle wavefunction is complex-conjugated.

6 Conclusion

We have shown that for each class of Majorana fermions (original, pseudo and symplectic) there is a Euclidean-signature Grassmann action integral that exists in precisely the same dimensions as the corresponding Minkowski-signature Majorana fermion fields. We have described how the pattern of dimensions in which these classes exist, and the possible continuous symmetries that they might possess, is controlled by the same eightfold Bott periodicity that appears in many areas of mathematics and physics. We have
also related the manner in which the discrete \( C, P, T \) manifest themselves in both signatures to this periodicity.

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Appendices

A Berezin Integrals

The path integral for fermions requires a formal integration over Grassmann-valued fields. Felix Berezin’s recipe for this process is purely algebraic but is called “integration” because its output mirrors, up to signs, the result of the corresponding analytic operation on real and complex variables. The general Grassmann/Berezin integral requires the sophisticated mathematics of sheaf theory [29], but we require only “Gaussian” integrals, and these are relatively straightforward.

A.1 Finite number of variables

If $\bar{\psi}_\alpha$ and $\psi^\beta$, $\alpha, \beta = 1, \ldots, N$ are a set of anticommuting Grassmann variables, we define their Berezin integral by setting

$$\int [d\bar{\psi}d\psi] \bar{\psi}_1\psi^1 \cdots \bar{\psi}_N\psi^N \equiv \int \left[ \prod_{\alpha=1}^{N} d\bar{\psi}_\alpha d\psi^\alpha \right] \bar{\psi}_1\psi^1 \cdots \bar{\psi}_N\psi^N = 1. \quad (117)$$

To obtain a non-zero answer all $2N$ anticommuting variable must be present in the integrand, and in an even permutation of increasing numerical order to get a $+1$. Each interchange of adjacent variables gives a factor of $-1$.

Under linear changes of variables

$$\psi^\alpha \rightarrow \psi'^\alpha = A^\alpha_\beta \psi^\beta$$
$$\bar{\psi}_\alpha \rightarrow \bar{\psi}'_\alpha = \bar{\psi}_\beta B^\beta_\alpha$$

we have

$$d[\psi] \rightarrow [d\psi'] = \det[A]^{-1}d[\psi],$$
$$d[\bar{\psi}] \rightarrow d[\bar{\psi}'] = d[\bar{\psi}] \det[B]^{-1} \quad (119)$$

in which the jacobian factors are the inverse of the commuting variable version.

If $L$, with entries $L^\alpha_\beta$, is an $N$-by-$N$ matrix representing a linear map $L : V \rightarrow V$, we expand the exponential function in the first line below and
use the definition to get
\[ Z(L) = \int [d\bar{\psi}d\psi] \exp\{\bar{\psi}_\alpha L^\alpha_\beta \psi^\beta\}, \]
\[ = \frac{1}{N!} \epsilon_{\alpha_1\ldots\alpha_N} \epsilon^{\beta_1\ldots\beta_N} L^{\alpha_1}_{\ \beta_1} \cdots L^{\alpha_N}_{\ \beta_N} \]
\[ = \det [L]. \]  

(120)

The integral for the two-variable correlator or propagator
\[ \langle \bar{\psi}_\rho \psi^\sigma \rangle \overset{\text{def}}{=} \frac{1}{Z(L)} \int [d\bar{\psi}][d\psi] \bar{\psi}_\rho \psi^\sigma \exp\{\bar{\psi}_\alpha L^\alpha_\beta \psi^\beta\}, \]
\[ = [L^{-1}]^\sigma_\rho \]  

(121)

follows because the explicit \( \bar{\psi}_\rho \psi^\sigma \) factor forces the omission of the term containing \( L^\sigma_\rho \) in the expansion of the exponential, and from the formula for the inverse of a matrix
\[ L^{-1} = \frac{1}{\det[L]} \text{Adj}[L], \]  

(122)

where \( \text{Adj}[L] \) is the adjugate matrix \( i.e. \) the transposed matrix of the cofactors. We can check the sign and index placement by observing that the claimed expression gives
\[ \langle \bar{\psi}_\alpha L^\alpha_\beta \psi^\beta \rangle = L^\alpha_\beta [L^{-1}]^\beta_\alpha = \text{tr} \{\mathbb{I}_N\} = N, \]  

(123)

which is correct because inserting an explicit factor of \( \bar{\psi}_\alpha L^\alpha_\beta \psi^\beta \) into the integral means that we need to expand the exponential only to order \( N - 1 \) to get all the \( \psi \)’s, and hence we get \( N/(N - 1)! = N \times \) the integral without the explicit factor.

Linear maps are naturally associated with eigenvectors and eigenvalues. When \( L \) is diagonalizable \( i.e. \) possesses sufficient eigenvectors \( u_n \) to form a basis \( \) the determinant is the product of the eigenvalues
\[ \det[L] = \prod_{n=1}^{N} \lambda_n. \]  

(124)

We can extract formula from the integral by diagonalizing \( L \to A^{-1}LA = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and then by setting \( B = A^{-1} \) in the change of variables formulæ given above.
Again when $L : V \rightarrow V$ is diagonalizable, we can express the inverse $[L^{-1}]_{\alpha \beta}$ in terms of the left and right eigenvectors

\begin{align*}
Lu_n &= \lambda_n u_n, \\
v^T_n L &= \lambda_n v^T_n, \\
\text{(125)}
\end{align*}

where $u_n \in V$, $v^T_n \in V^*$. We have not distinguished between the left and right eigenvalues because the two sets of eigenvalues coincide: they are determined by the same characteristic equation. For a non-hermitian matrix a right (left) eigenvector with eigenvalue $\lambda_n$ is no longer orthogonal to all right (left) eigenvectors with a different eigenvalue $\lambda_m$, but it remains true that a left eigenvector with eigenvalue $\lambda_n$ is orthogonal to every right eigenvector with a different eigenvalue $\lambda_m$. This is because

\begin{equation}
\lambda_n v^T_n u_m = (v^T_n L)u_m = v^T_n (Lu_m) = \lambda_m v^T_n u_m, \
\text{(126)}
\end{equation}

so

\begin{equation}
0 = (\lambda_n - \lambda_m)v^T_n u_m. \
\text{(127)}
\end{equation}

We may choose the phases and normalization of the two eigenfunction sets so that

\begin{equation}
v^T_n u_m \equiv v_n^{\alpha}, u_m^\alpha = \delta_{mn}, \
\text{(128)}
\end{equation}

and, as the eigenvectors are being assumed complete, we have

\begin{equation}
\sum_n u_n^{\alpha} v_n^\beta = \delta^\alpha_\beta. \
\text{(129)}
\end{equation}

The two sets of eigenvectors $u_n^\alpha$ and $v^T_n$ then compose mutually dual bases for $V$ and $V^*$ respectively, and

\begin{equation}
[L^{-1}]_{\alpha \beta} = \sum_n \frac{1}{\lambda_n} u_n^\alpha v_n^\beta. \
\text{(130)}
\end{equation}

We now naturally expand

\begin{align*}
\psi^\alpha &= \sum_n \chi_n u_n^\alpha, \\
\bar{\psi}_\beta &= \sum_n \bar{\chi}_n v_n^\beta, \\
\text{(131)}
\end{align*}

where $\chi_n, \bar{\chi}_n$ are Grassmann variables.
A.2 Continuous fields

Now consider how these finite integrals work in the continuum where we have an infinite set of Grassmann fields $\psi(x)$, one Grassmann variable for each point $x$, and similarly $\bar{\psi}(x)$. We will assume that we have chosen our gamma matrices to be Hermitian, in which case the $v_n$ left eigenvectors of the previous section coincide with the Hermitian conjugate $u_n^\dagger$ of the right eigenvectors $u_n$.

To avoid dealing with continuous spectra, we will restrict the discussion to a closed (compact without boundary) $d$-dimensional spin manifold on which the skew-adjoint Dirac operator

$$\mathcal{D} = \gamma^a D_a = \gamma^a e^\mu_a \left( \partial_\mu + \frac{1}{2} \sigma^{bc} \omega_{bc\mu} \right)$$

possesses a complete orthonormal set of c-number spinor eigenfunctions $u_n(x)$ labeled by $n \in \mathbb{Z}$, and with the properties

$$\mathcal{D} u_n = i \lambda_n u_n, \quad \int d^d x \sqrt{g} u_n^\dagger(x) u_m(x) = \delta_{mn}, \quad \sum_n u_n(x) u_n^\dagger(x') = \mathbb{I} \delta^d(x-x').$$

Here the the $\lambda_n$ are real, $\mathbb{I}$ is the identity matrix in spinor space, and the distribution $\delta^d(x-x')$ obeys

$$\int d^d x \sqrt{g} \delta^d(x-y) = 1.$$

In Euclidean signature there is no preferred $\gamma^0$ and therefore no inherent need to distinguish between $\psi(x)$ and $\bar{\psi}(x)$, but when we use the eigenmodes to expand out the Grassmann-valued Fermi fields it is convenient to write

$$\psi(x) = \sum_n u_n(x) \chi_n,$$

$$\bar{\psi}(x) = \sum_n u_n^\dagger(x) \bar{\chi}_n.$$

As before, the Grassmann variables $\bar{\chi}_n$ and $\chi_n$ are independent, and not related by any notion of complex conjugation, but when $(\ldots)$ is applied to an expression containing $\psi(x)$ we understand that it not only transposes and complex conjugates matrices and the spinor functions $u_n(x)$ but it also changes any $\chi_n$'s into $\bar{\chi}_n$'s.
The Euclidean action functional for the Dirac field can therefore be taken as

$$S[\psi, \bar{\psi}] = \int d^d x \sqrt{g} \left\{ \frac{1}{2} \left( \bar{\psi}(\slashed{D}\psi) - \overline{(\slashed{D}\bar{\psi})} \psi \right) + m\psi^\dagger \psi \right\}. \quad (136)$$

Equivalently

$$S[\psi, \bar{\psi}] = \int d^d x \sqrt{g} \left\{ \frac{1}{2} \left( \bar{\psi}\gamma^a (D_a \psi) - (D_a \bar{\psi})\gamma^a \psi \right) + m\psi^\dagger \psi \right\}, \quad (137)$$

where the covariant derivative $D_a$ acting on conjugate spinors $\psi^\dagger$ or $\bar{\psi}$ is

$$D_a \bar{\psi} = e^\mu_a \bar{\psi} \left( \overleftarrow{\partial}_\mu - \frac{1}{2} \sigma_{bc} \omega_{bc\mu} \right) \quad (138)$$

with $\bar{\psi} \overleftarrow{\partial}_\mu = \partial_\mu \bar{\psi}$. The second form has the advantage of treating $\psi$ and $\bar{\psi}$ symmetrically.

On inserting the eigenfunction expansions and using the eigenfunction orthonormality to evaluate the space-time integrals, the Euclidean action functional becomes diagonal

$$S[\psi, \bar{\psi}] = \int d^d x \sqrt{g} \left\{ \frac{1}{2} \left( \bar{\psi}(\slashed{D}\psi) - \overline{(\slashed{D}\bar{\psi})} \psi \right) + m\psi^\dagger \psi \right\} = \sum_n (i\lambda_n + m) \bar{\chi}_n \chi_n. \quad (139)$$

The vacuum-amplitude partition function is now formally given by the Berezin integral

$$Z = \int d[\bar{\psi}] d[\psi] \exp\{ S[\psi, \psi^\dagger] \} = \prod_n d[\bar{\chi}_n] d[\chi_n] \exp\left\{ \sum_n (i\lambda_n + m) \bar{\chi}_n \chi_n \right\} = \prod_n (i\lambda_n + m) = \text{Det}(\slashed{D} + m). \quad (140)$$

Here $\text{Det}(\slashed{D} + m)$ is the Matthews-Salam functional determinant. The infinite product over the eigenvalues usually needs some form of regularization.
In the path integral the Berezinian version of the jacobian determinants involved in the change of integration measure from $d[\bar{\psi}(x)]d[\psi(x)]$ to $d[\bar{\chi}_n]d[\chi_n]$ cancel one another just as they do in the finite case. We are, in effect, performing a unitary similarity transformation

$$(\mathcal{D} + m) = U^\dagger \operatorname{diag}(i\lambda_n + m)U$$

in which $\operatorname{Det}(U) = [\operatorname{Det}(U^\dagger)]^{-1}$. This formal cancellation is not affected by some of the $\lambda_n$ being zero.

### A.3 Majorana fermions: determinants vs. Pfaffians

For Majorana fermions we require an integral containing a skew symmetric matrix $Q_{ij}$ representing a skew bilinear (symplectic) form $Q : V \times V \to \mathbb{C}$. As the matrix $Q$ is equipped with two lower indices, we no longer need distinguish between $\psi^\alpha$ and $\bar{\psi}_\alpha$. For a $2N$-by-$2N$ matrix we have $\psi^\alpha$, $\alpha = 1, \ldots, 2N$, and the defining integral becomes

$$\int [d\psi] \psi^1 \cdots \psi^{2N} = 1. \quad (141)$$

Again all $\psi^\alpha$ must be present and in numerical order to get +1. Thus

$$\int [d\psi] \psi^{\alpha_1} \cdots \psi^{\alpha_{2N}} = \epsilon^{\alpha_1 \cdots \alpha_{2N}}. \quad (142)$$

Using this definition we evaluate

$$Z(Q) = \int [d\psi] \exp \left\{ \frac{1}{2} \psi^\alpha Q_{\alpha\beta} \psi^\beta \right\}$$

$$= \frac{1}{2^N N!} \epsilon^{\alpha_1 \cdots \alpha_{2N}} Q_{\alpha_1 \alpha_2} \cdots Q_{\alpha_{2N-1} \alpha_{2N}}$$

$$= \operatorname{Pf} [Q]. \quad (143)$$

The last two lines serve to define the Pfaffian of the skew symmetric matrix $Q$. The two-variable correlator is now

$$\langle \psi^\rho \psi^\sigma \rangle = \frac{1}{Z(Q)} \int [d\psi] \psi^\rho \psi^\sigma \exp \left\{ \frac{1}{2} \psi^\alpha Q_{\alpha\beta} \psi^\beta \right\}$$

$$= [Q^{-1}]^{\rho\sigma}. \quad (144)$$
Again we check the sign and index placement by computing
\[
\frac{1}{2} \langle \psi^\rho Q_{\rho\sigma} \psi^\sigma \rangle = \frac{1}{2} Q_{\rho\sigma} [Q^{-1}]^{\sigma\rho} = \frac{1}{2} \text{tr} \{I_{2N}\} = N. \tag{145}
\]

Regarding \( Q \) simply as a numerical matrix there is a well-known identity
\[
(Pf Q)^2 = \det [Q], \tag{146}
\]
which implies that the Pfaffian of a matrix is a square-root of its determinant. We need to interpret this statement with care. A linear map \( L \) and a symplectic form \( Q \) are rather different mathematical objects. A linear map \( L : V \to V \) possesses eigenvalues and eigenvectors while a bilinear form \( Q : V \times V \to \mathbb{C} \) does not. The placement of the indices on their entries indicates that their matrix representatives respond differently to a change of basis in the vector space \( V \):
\[
L \to B^{-1}LB, \quad \text{(Similarity transformation)},
\]
\[
Q \to B^TQB, \quad \text{(Congruence transformation)}. \tag{147}
\]

From
\[
\det [B^TQB] = \det [Q] \det [B]^2 \tag{148}
\]
we see that a bilinear form does not possess a basis-independent determinant. We will show later that
\[
Pf[B^TQB] = Pf[Q] \det[B], \tag{149}
\]
so a skew bilinear form does not possess a basis-independent Pfaffian.

To convert a linear map into a bilinear form, or vice-versa, we need to have some sort of “metric” to lower or raise the first index on \( L^{\alpha\beta} \) or \( Q_{\alpha\beta} \). For relativistic Majorana fermions in spacetime dimensions 2, 3, 4 (mod 8) this role is played by the antisymmetric charge-conjugation matrix \( C_{\alpha\beta} \) and its inverse \( [C^{-1}]^{\alpha\beta} \). For pseudo-Majorana fermions in spacetime dimensions 0, 1, 2 (mod 8) the role is taken by the symmetric time-reversal matrix \( T_{\alpha\beta} \) and its inverse \( [T^{-1}]^{\alpha\beta} \). For symplectic Majorana’s we need to include an \( \epsilon^{a\beta} \) acting on the labels \( a, b = 1, 2 \) distinguishing the fermion pair.

The cases are different. In the finite dimensional version of the first we are given a \( 2N \)-by-\( 2N \) non-degenerate skew-symmetric matrix \( C \) with entries \( C_{ij} \) together with a self-adjoint linear operator represented by a \( 2N \)-by-\( 2N \)
hermitian matrix $L$ with entries $L_{ij}$, such that their product $Q_{ij} = C_{ik}L^k_j$ is skew symmetric. The simplest case is
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix}
= 
\begin{bmatrix}
0 & \lambda \\
-\lambda & 0
\end{bmatrix}.
\] 
(150)

We wish to evaluate $\text{Pf}[Q] = \text{Pf}[CL]$ in terms of the eigenvalues of $L$. Each eigenvalue occurs twice and, after some algebra that we will display later, we find that we need only one of the pair in the result
\[
\text{Pf}[Q] = \text{Pf}[C] \prod_{n=1}^{N} \lambda_n.
\] 
(151)

In the second case $C$ is replaced by a $2N$-by-$2N$ non-degenerate symmetric matrix $T$ with entries $T_{ij}$ such that $Q_{ij} = T_{ik}L^k_j$ is skew symmetric. The simplest example is
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0 & \lambda \\
\lambda & 0
\end{bmatrix}
= 
\begin{bmatrix}
0 & \lambda \\
-\lambda & 0
\end{bmatrix}.
\] 
(152)

We can again evaluate $\text{Pf}[Q] = \text{Pf}[TL]$ in terms of the eigenvalues of $L$, but the result is more complicated. The eigenvalues occur in $\pm \lambda_n$ pairs, and if we arbitrarily select $\lambda_n$ to the positive eigenvalue, we find
\[
\text{Pf}[TL] = \pm \sqrt{(-1)^N \text{det}[T]} \prod_{n=1}^{N} (-\lambda_n).
\] 
(153)

It is not possible to decide what sign to take for the $\pm$ without more information. In the simplest example above, we need the minus sign if $\lambda$ is positive and the plus sign if $\lambda$ is negative.

The source of the difference between the $C$ case and the $T$ case is that after reducing $C$ to a standard symplectic form, the matrices $B$ in the subsequent normal-form reduction
\[
Q \rightarrow B^TQB = \bigoplus_{n=1}^{N} \begin{bmatrix}
0 & \lambda_n \\
-\lambda_n & 0
\end{bmatrix}
\] 
(154)

belong to $\text{Sp}(2N)$ and symplectic matrices are automatically unimodular. After reducing $T$ to a standard metric the matrices $B$ lie in $\text{SO}(N,N)$ and such orthogonal matrices can have either $\pm 1$ as their determinant. The proofs of these Pfaffian formulæ are given below.
A.4 Pfaffian to Determinant

We can always rewrite the linear operator “action” $\bar{\psi} L \psi$ as

$$\bar{\psi} L \psi = \frac{1}{2} \bar{\psi} \left[ \begin{array}{cc} 0 & L \\ -L^T & 0 \end{array} \right] \psi.$$  \hspace{1cm} (155)

When $L$ is $N$-by-$N$ we can now compute the Pfaffian of the skew symmetric matrix and so find that

$$\text{Pf} \left[ \begin{array}{cc} 0 & L \\ -L^T & 0 \end{array} \right] = (-1)^{N(N-1)/2} \det[L],$$  \hspace{1cm} (156)

The sign comes from the need to rearrange the $d\psi$ and $d\bar{\psi}$ so as to put all the $d\bar{\psi}$’s before the $d\psi$’s instead of in adjacent pairs. The dependence on $N$ makes this rewriting less useful in infinite dimensions.

A.5 Proofs of some Pfaffian formulæ

$\text{Pf}[B^TQB] = \det[B] \text{Pf}[Q]$: Start from the definition of the Pfaffian

$$\text{Pf}[Q] = \frac{1}{2^N N!} \epsilon^{j_1 \ldots j_{2N}} Q_{j_1 j_2} \cdots Q_{j_{2N-1} j_{2N}}$$  \hspace{1cm} (157)

and recall that

$$\epsilon^{j_1 \ldots j_{2N}} \det[B] = \epsilon^{i_1 \ldots i_{2N}} B_{j_1 i_1} \cdots B_{j_{2N} i_{2N}}.$$  \hspace{1cm} (158)

Thus

$$\text{Pf} [B^TQB] = \frac{1}{2^N N!} \epsilon^{j_1 \ldots j_{2N}} [B^TQB]_{j_1 j_2} \cdots [B^TQB]_{j_{2N} i_{2N}}$$
$$= \frac{1}{2^N N!} \epsilon^{j_1 \ldots j_{2N}} B_{j_1 i_1} Q_{j_1 j_2} B_{j_2 i_2} \cdots B_{j_{2N} i_{2N}} Q_{j_{2N} i_{2N}}$$
$$= \frac{1}{2^N N!} \epsilon^{j_1 \ldots j_{2N}} \det[B] Q_{j_1 j_2} \cdots Q_{j_{2N} i_{2N}}$$
$$= \det[B] \text{Pf} [Q].$$  \hspace{1cm} (159)

$(\text{Pf} Q)^2 = \det[Q]$: Given a $2N$-by-$2N$ non-degenerate skew-symmetric matrix $Q$ with entries in a field, we can repeatedly complete squares to find a linear map $B$ that reduces $Q$ to the canonical form $[34]$

$$Q = B^T JB, \quad J = \bigoplus_1^N \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \hspace{1cm} (160)$$
Taking the Pfaffian of this equation we get
\[ \text{Pf}[Q] = \det[B] \text{Pf}[J] = \det[B]. \quad (161) \]

Taking the determinant gives
\[ \det[Q] = \det[(B^T J B)] = \det[B]^2 = (\text{Pf}[Q])^2. \quad \blacksquare \quad (162) \]

\[ \text{Pf}[CL] = \text{Pf}[C] \prod_{n=1}^{N} \lambda_n: \] Here \( C \) is skew symmetric and \( L \) is a \( 2N \)-by-\( 2N \) hermitian matrix such that \( Q_{ij} = C_{ik} L^k_i \) is skew symmetric, observe that the hermiticity of \( L \) implies that \( L^* = L^T \), and hence
\[ Lu_n = \lambda u_n \quad \Rightarrow \quad LC^{-1} u_n^* = \lambda C^{-1} u_n^*. \]
The skew symmetry of \( C^{-1} \) guarantees that \( u_n \) and \( C^{-1} u_n^* \) are mutually orthogonal
\[ u_n^T (C^{-1} u_n^*) = u_n^T C^{-1} u_n^* = 0, \]
so each eigenvalue of \( L \) is therefore doubly degenerate and we can assume that \( u_n \) and \( C^{-1} u_n^* \), \( n = 1, \ldots, N \), together constitute a complete orthonormal set. Let us introduce vectors \( \tilde{X} = (\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_n, \tilde{y}_n) \) and \( X = (x_1, y_1, \ldots, x_n, y_n) \)

Using the orthonormality we have
\[
\tilde{X}^T B^T (CL) B X = \sum_n (\tilde{x}_n u_n + \tilde{y}_n C^{-1} u_n^*)^T C D (x_n u_n + y_n C^{-1} u_n^*) \\
= \sum_n (\tilde{x}_n u_n^T - \tilde{y}_n u_n^T C^{-1}) C D (x_n u_n + y_n C^{-1} u_n^*) \\
= \sum_n \lambda_n (\tilde{x}_n y_n - \tilde{y}_n x_n) \\
= \tilde{X}^T \Lambda X.
\]

Here
\[ \Lambda = \bigoplus_{n=1}^{N} \begin{bmatrix} 0, & \lambda_n \\ -\lambda_n, & 0 \end{bmatrix}, \]
and \( B \) is the \( 2N \)-by-\( 2N \) matrix
\[ B = [u_1, C^{-1} u_1^*, \ldots, u_n, C^{-1} u_n^*]. \]

We have reduced \( CL \) to a canonical form, and taking the Pfaffian we have
\[ \text{Pf}[B^T (CL) B] = \text{Pf}[CL] \det[B] = \prod_n \lambda_n. \]
We need to find an expression for $\det[B]$. To do this replace $L$ by $I_{2N}$ while keeping the $u_n$ unchanged. This results in

$$B^T C B = J = \bigoplus_{n=1}^{N} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

But $\text{Pf}[J] = 1$ so $\det[B] = \text{Pf}[C]^{-1}$. The end result is that

$$\text{Pf}[CL] = \text{Pf}[C] \prod_n \lambda_n. \quad \blacksquare$$

$\text{Pf}[TL] = \pm \sqrt{(-1)^N \det[T]} \prod_{n=1}^{N} (-\lambda_n)$: We are given 2$N$-by-2$N$ non-degenerate symmetric matrix $T$ with entries $T_{ij} = T_{ik} L_{ki}$ is skew symmetric. An example to bear in mind is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}.$$

To evaluate $\text{Pf}[Q] = \text{Pf}[TL]$ in terms of the eigenvalues of $L$ we use a similar strategy as before. In this case the hermiticity of $L$ gives us

$$Lu_n = \lambda_n u_n \quad \Rightarrow \quad LT^{-1} u_n^* = -\lambda T^{-1} u_n^*,$$

and if $\lambda_n$ is non-zero $T^{-1} u_n^*$ is orthogonal to $u_n$ because they have different eigenvalues. The non-zero-mode eigenvectors of $L$ therefore come in opposite eigenvalue pairs. If there are no zero modes the $u_n$ and $T^{-1} u_n^*$, $n = 1, \ldots, N$, together constitute a complete orthonormal set. We will take $\lambda_n$ to be the positive eigenvalue. Again set $\tilde{X} = (\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_n, \tilde{y}_n)$ and $X = (x_1, y_1, \ldots, x_n, y_n)$ and use the orthonormality and symmetry of $T^{-1}$ to conclude that

$$\tilde{X}^T B^T (T)L B X = \sum_n (\tilde{x}_n u_n + \tilde{y}_n T^{-1} u_n^*)^T T L (x_n u_n + y_n T^{-1} u_n^*)$$

$$= \sum_n (\tilde{x}_n u_n^T + \tilde{y}_n u_n^T T^{-1}) T L (x_n u_n + y_n T^{-1} u_n^*)$$

$$= \sum_n (-\lambda_n) (\tilde{x}_n y_n - \tilde{y}_n x_n)$$

$$= \tilde{X}^T (-\Lambda) X$$

45
where

\[ \Lambda = \bigoplus_{n=1}^{N} \begin{bmatrix} 0, & \lambda_n \\ -\lambda_n & 0 \end{bmatrix}, \]

and \( B \) is the \( 2N \)-by-\( 2N \) matrix

\[ B = [u_1, T^{-1}u_1^*, \ldots, u_n, T^{-1}u_n^*] \]

Thus

\[ \text{Pf}[B^T(TL)B] = \text{Pf}[TL] \text{det}[B] = \prod_n (-\lambda_n). \]

In this case, however, we cannot take the Pfaffian after replacing \( L \) by \( I \) because \( T \) is not skew symmetric. We can still replace \( L \rightarrow I \) and find that

\[ \tilde{X}B^T T B X = \sum_n (\tilde{x}_n y_n + \tilde{y}_n x_n) = \tilde{X}G X \]

where

\[ G = \bigoplus_{n=1}^{N} \begin{bmatrix} 0, & 1 \\ 1 & 0 \end{bmatrix}. \]

We can now take the determinant to conclude that

\[ \text{det}[B^T T B] = \text{det}[G] = (-1)^N \]

and so \( \text{det}[B]^2 \text{det}T = (-1)^N \). Hence

\[ \text{Pf}[TL] = \pm \sqrt{(-1)^N \text{det}[T]} \prod_n (-\lambda_n), \]

where the \( \pm \) sign comes from the need to take \( \sqrt{\text{det}[B]^2} \). The uncertainty as to which root to take is inevitable. We arbitrarily assigned \( u_n \) to the positive eigenvalue rather than to the negative. If we make the opposite choice for some eigenvalue pair, the product formula must change sign whilst \( \text{Pf}[Q] \) itself is indifferent to our choice. ■

**B  Canonical forms for complex matrices**

Here are some lesser known, but useful reductions of matrices with complex entries:
If $M$ is an $n$-by-$n$ complex symmetric matrix, then there exists a unitary matrix $\Omega$ such that

$$\Omega^T M \Omega = \text{diag}(m_1, \ldots, m_n)$$

(163)

where the numbers $m_i$ are real and non-negative. This result is useful for diagonalizing symmetric $C$ and $T$ matrices and also for Majorana-mass matrices.

**Proof:** the matrix $N = M^\dagger M$ is Hermitian and non-negative, so there is a unitary matrix $V$ such that $V^\dagger NV$ is diagonal with non-negative real entries. Thus $C = V^T MV$ is complex symmetric with $C^\dagger C \equiv V^\dagger NV$ real. Writing $C = X + iY$ with $X$ and $Y$ real symmetric matrices, we have $C^\dagger C = X^2 + Y^2 + i[X,Y]$. As this expression is real, the commutator must vanish. Because $X$ and $Y$ commute, there is a real orthogonal matrix $W$ such that both $WXW^T$ and $WYW^T$ are simultaneously diagonal. Set $U = WV^T$ then $U$ is unitary and the matrix $UMU^T$ is complex diagonal. By post-multiplying $U$ by another diagonal unitary matrix, the diagonal entries can be made to be real and non-negative. Since their squares are the eigenvalues of $M^\dagger M$, they coincide with the singular values of $M$. ■

If $A$ is a complex skew-symmetric matrix, one can use the same strategy to show there exists a unitary matrix $\Omega$ such that

$$\Omega^T A \Omega = \bigoplus_i \begin{bmatrix} 0 & \lambda_i \\ \lambda_i & 0 \end{bmatrix} \oplus \text{diag}(0, \ldots, 0),$$

and the $\lambda_i$ are the positive square-roots of the eigenvalues $\lambda_i^2$ of $A^\dagger A$. We can use this to show that the Pfaffian of a skew matrix $Q$ is the product of the square roots of the eigenvalues of $Q^\dagger Q$, but only up to a phase equal to $\det[\Omega]$.

## C  The $C$ and $T$ operations in Condensed Matter Systems

The eightfold Bott periodicity we have uncovered in the Dirac equation manifests itself in the various discrete symmetries of non-relativistic systems. For a review see [9]. This setting is useful for explaining why charge conjugations is a unitary linear map on the many-particle Hilbert space despite involving a complex conjugation operation.
Condensed matter physics is usually formulated in Hamiltonian language. and we will restrict ourselves to non-interacting Hamiltonians built from a set of fermion annihilation and creation operators $\Psi_\alpha$ and $\Psi^\dagger_\alpha$ that obey
\[
\{\Psi_\alpha, \Psi_\beta\} = 0, \quad \{\Psi_\alpha, \Psi^\dagger_\beta\} = \delta_{\alpha\beta}.
\] (164)
We will need to distinguish between a vacuum state $|\text{empty}\rangle$ such that $\Psi_\alpha |\text{empty}\rangle = 0$ for all $\alpha$, and a ground state $|\text{gnd}\rangle$ in which all negative energy states are occupied.

### C.1 C or particle-hole symmetry

Suppose that we have a non-interacting many-fermion Hamiltonian
\[
\hat{H} = \Psi^\dagger_\alpha H_{\alpha\beta} \Psi_\beta
\] (165)
where the $N$-by-$N$ one-particle Hamiltonian matrix $H_{\alpha\beta}$ is traceless and obeys
\[
CH^*C^{-1} = -H
\] (166)
for some unitary matrix $C$. The Bogoliubov-de-Gennes Hamiltonian for superconducting systems has this property. Now
\[
Hu_n = \lambda_n u_n \quad \Rightarrow \quad HCu^*_n = -\lambda_n Cu^*_n,
\] (167)
so, when $\lambda$ is non zero, the single-particle eigenfunctions come in opposite-eigenvalue pairs. In the absence of zero energy states the ground state $|\text{gnd}\rangle$ has all negative-energy states occupied and is non-degenerate.

We define the action of a unitary particle-hole operator $C$ on the many-body Fock space by $C|\text{empty}\rangle = |\text{empty}\rangle$ and
\[
C\Psi_\beta C^{-1} = \Psi^\dagger_\alpha C_{\alpha\beta}, \quad C\Psi^\dagger_\beta C^{-1} = C^\dagger_{\beta\alpha} \Psi_\alpha.
\] (168)
When $C$ acts on the Hamiltonian we have
\[
C\hat{H}C^{-1} = C\Psi^\dagger_\alpha H_{\alpha\beta} \Psi_\beta C^{-1} = C\Psi^\dagger_\alpha C^{-1} H_{\alpha\beta} C\Psi_\beta C^{-1} = C\Psi^\dagger_\alpha C^{-1} H_{\alpha\beta} C\Psi_\beta C^{-1} = C^\dagger_{\alpha\rho} \Psi_\rho H_{\alpha\beta} C^\dagger_{\beta\sigma} \Psi_\sigma
\] = $-\Psi^\dagger_\alpha C_{\sigma\beta} H_{\alpha\beta} C^\dagger_{\alpha\rho} \Psi_\rho$

48
Thus the one-particle transformation on \( H \) leaves the many-particle Hamiltonian invariant.

We used \( C^* = C^T \) and the tracelessness (line 5 \( \rightarrow \) 6) and hermiticity of \( H \) in the above manipulations. More importantly, and despite the appearance of the complex conjugation \( \ast \) in \( H^* = -C^{-1}HC \), the many-body operator \( \mathcal{C} \) must act on the Fock space linearly:

\[
\mathcal{C}(\lambda \lvert \psi_1 \rangle + \mu \lvert \psi_2 \rangle) = \lambda \mathcal{C} \lvert \psi_1 \rangle + \mu \mathcal{C} \lvert \psi_2 \rangle.
\]

The linearity is required in the step

\[
\mathcal{C} H_{\alpha\beta} \mathcal{C}^{-1} = H_{\alpha\beta}.
\]

If we write

\[
\Psi_\alpha = \sum_n u_{n\alpha} \hat{a}_n
\]

with \( n > 0 \) corresponding to positive energy and \( n < 0 \) to negative, then in the absence of zero energy states the ground state \( \lvert \text{gnd} \rangle \) is specified up to phase by

\[
\hat{a}_n \lvert \text{gnd} \rangle = \hat{a}_n^\dagger \lvert \text{gnd} \rangle = 0, \quad n > 0.
\]

Let \( C^T = \lambda C, \lambda = \pm 1 \). From this we deduce that \( \mathcal{C} a_n \mathcal{C}^{-1} = \lambda a_n^\dagger \) and hence that \( \mathcal{C} \lvert \text{gnd} \rangle = \lvert \text{gnd} \rangle \). We also have that

\[
\mathcal{C}(\Psi^\dagger_\beta \Psi_\beta - N/2) \mathcal{C}^{-1} = \Psi^\dagger_\alpha \Psi_\alpha - N/2 = -(\Psi^\dagger_\alpha \Psi_\alpha - N/2),
\]

so the sign of the normal-ordered charge operator

\[
\hat{Q} = \frac{1}{2}(\Psi^\dagger_\beta \Psi_\beta - \Psi^\dagger_\beta \Psi_\beta) = \Psi^\dagger_\beta \Psi_\beta - N/2
\]

is reversed.
C.2 T and time-reversal

Again consider a non-interacting many-fermion Hamiltonian
\[ \hat{H} = \Psi_\alpha^\dagger H_{\alpha\beta} \Psi_\beta, \] (176)
but now assume that the one-particle Hamiltonian matrix \( H_{\alpha\beta} \) obeys
\[ TH^*T^{-1} = +H \] (177)
for some unitary matrix \( T \). This condition tells us that if \( u_n(x, t) \) obeys
\[ \left( i \frac{\partial}{\partial t} - H \right) u_n(x, t) = 0 \] (178)
then \( Tu_n^*(x, -t) \) obeys the same equation:
\[ \left( i \frac{\partial}{\partial t} - H \right) Tu_n^*(x, -t) = 0. \] (179)

We define the action of an anti-unitary time reversal operator \( \mathcal{T} \) on the many-body Fock space by
\[ \mathcal{T} \Psi_\beta^\dagger \mathcal{T}^{-1} = T_\beta^\dagger \Psi_\alpha, \quad \mathcal{T} \Psi_\beta^\dagger \mathcal{T}^{-1} = \Psi_\alpha^\dagger T_{\alpha\beta}. \] (180)
Then \( \mathcal{T} \Psi_\beta^\dagger \mathcal{T}^{-1} = \Psi_\beta^\dagger \mathcal{T}^{-1} \) so the charge is unchanged, and
\[ \mathcal{T} \hat{H} \mathcal{T}^{-1} = \mathcal{T} \Psi_\alpha^\dagger H_{\alpha\beta} \Psi_\beta \mathcal{T}^{-1} \]
\[ = \mathcal{T} \Psi_\alpha^\dagger \mathcal{T}^{-1} \mathcal{T} H_{\alpha\beta} \mathcal{T}^{-1} \mathcal{T} \Psi_\beta \mathcal{T}^{-1} \]
\[ = \mathcal{T} \Psi_\alpha^\dagger \mathcal{T}^{-1} H_{\alpha\beta}^* \mathcal{T} \Psi_\alpha \mathcal{T}^{-1} \]
\[ = \Psi_\mu T_{\rho\mu} H_{\alpha\beta}^* T_{\beta\sigma} \Psi_\sigma \]
\[ = \Psi_\mu H_{\rho\sigma} \Psi_\sigma, \]
\[ = \hat{H}. \] (181)

Again the transformation of \( H \) leaves the many-particle Hamiltonian invariant. We required \( \mathcal{T} \) to be an anti-linear map because we need
\[ \mathcal{T} H_{\alpha\beta} \mathcal{T}^{-1} = H_{\alpha\beta}^*. \] (182)