ON OPEN SCATTERING CHANNELS
FOR MANIFOLDS WITH ENDS

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Abstract. In the framework of time-dependent geometric scattering theory, we study
the existence and completeness of the wave operators for perturbations of the Riemann-
ian metric for the Laplacian on a complete manifold of dimension \( n \). The smallness
condition for the perturbation is expressed (intrinsically and coordinate free) in purely
geometric terms using the harmonic radius; therefore, the size of the perturbation can
be controlled in terms of local bounds on the injectivity radius and the Ricci-curvature.
As an application of these ideas we obtain a stability result for the scattering matrix
with respect to perturbations of the Riemannian metric. This stability result implies
that a scattering channel which interacts with other channels preserves this property
under small perturbations.

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1. Introduction

The first fundamental problem in multi-channel scattering theory is to establish the
existence and the (asymptotic) completeness of the wave operators. These questions
are currently quite well understood in various situations including the case of the \( N \)-
body problem in quantum mechanics, multi-channel scattering in perturbed acoustic
and electromagnetic wave guides, and scattering on manifolds with ends; cf., e.g., [Y10],
[DG97], [W91], and the literature discussed at the end of this introduction. Roughly
speaking, asymptotic completeness in multi-channel scattering means the following: as
time goes to \( \pm \infty \), any scattering state decays into a number of states living in subsystmes
(channels); these subsystems then evolve according to a simpler reference dynamics, like
clusters of particles in the quantum mechanical case, radiation and guided modes for
perturbed wave guides, and components that travel into the various ends of a manifold.
However, given an initial state belonging to a particular channel (as time goes to \( -\infty \)),
asymptotic completeness does not tell us into which channels our state will decay as time
goes to \( +\infty \), or, put differently, which subsystems will actually be non-zero. We are
therefore led to ask which channels are \textit{open} to an initial state belonging to a particular channel in the past \((t \to -\infty)\). Clearly, one would expect that two scattering channels will be open to each other unless a particular obstruction prevents the decay from one into the other; put differently, two channels should be open to each other in some generic sense. It appears, though, that there are no general methods in mathematical multi-channel scattering which would allow to prove such a result.

1.1. \textbf{Open scattering channels.} As a first step in the analysis of this issue, the present paper studies the interaction of the channels in geometric scattering theory where the dynamics is given by the Laplacian on a complete \(n\)-dimensional Riemannian manifold with a finite number of ends. Any geometric end gives rise to a scattering channel provided the corresponding decoupled part of the Laplacian has a non-zero absolutely continuous part. One of our main results (cf. Theorem 5.1 and Corollary 5.3) roughly says the following: Suppose that the \(i\)-th scattering channel is open to the \(k\)-th channel in the sense that the channel scattering operator \(S_{ik}\) for these channels satisfies

\[ S_{ik} \neq 0; \tag{1.1} \]

then the same property holds for small perturbations of the metric. In other words, we derive a \textit{stability theorem} for property (1.1). The smallness of the perturbation is expressed in geometric terms that involve the \textit{harmonic radius} \(r_M(x)\) at a point \(x \in M\) of a Riemannian manifold \(M = (M,g)\), defined as in the work of Anderson and Cheeger [AC92]. According to [AC92] and [HH98], \(r_M(x)\) depends only on (local) lower bounds for the radius of injectivity and the Ricci curvature. Note that we do not need to require any particular structure of the unperturbed manifolds or its ends. As explained at the end of Section 5, the property (1.1) is symmetric in \(k\) and \(i\), i.e., \(S_{ik} \neq 0 \iff S_{ki} \neq 0\). Some comments on the notion of openness of scattering channels can be found at the end of Section 5 in Remark 5.7.

1.2. \textbf{An intrinsic trace class perturbation result.} In preparation for the above analysis we derive a rather general theorem which establishes existence and completeness of the wave operators for a pair of Laplacians on a manifold \(M\) with two different metrics. This result generalizes Theorem 0.1 of Müller and Salomonson [MS07] in several directions and may be of independent interest.

Although the definitions are somewhat involved, let us try and give a description of our basic construction:

For \(x \in M\) and quasi-isometric metrics \(g_1\) and \(g_2\) (see Definition 3.1), let \(A(x)\) be the endomorphism from \(T^*_x M\) to itself defined by

\[ g_2(x)(\xi, \zeta) = g_1(x)(A(x)\xi, \zeta), \quad \xi, \zeta \in T^*_x M. \]

We denote the (positive) eigenvalues of \(A(x)\) by \(\alpha_k(x), k = 1, \ldots, n\). Then our distance function is defined as

\[ \tilde{d}(g_1, g_2)(x) := 2 \sinh\left(\frac{n}{4} \cdot \max_k |\ln \alpha_k(x)|\right) = \max_k |\alpha_k(x)^{n/4} - \alpha_k(x)^{-n/4}|. \]

Note that \(g_1\) and \(g_2\) are quasi-isometric if and only if

\[ \tilde{d}_\infty(g_1, g_2) := \sup_{x \in M} \tilde{d}(g_1, g_2)(x) < \infty. \]

(The tilde \(\tilde{\cdot}\) here and below indicates that the distance functions \(\tilde{d}, \tilde{d}_\infty\) etc. only satisfy a weaker version of the triangle inequality, see Appendix A.) Of equal importance is a distance function in form of a weighted integral,

\[ \tilde{d}_1(g_1, g_2) := \int_M \tilde{d}(g_1, g_2) \cdot r_0^{-(n+2)} \cdot (1 + g_{g_2-g_1}) \, \text{dvol}_{g_1}, \tag{1.2} \]
where \( \tilde{d}(g_1, g_2) \) is the pointwise distance introduced above, \( r_0 : M \rightarrow (0, 1] \) is a continuous function (in practice \( r_0(x) \) is a common lower bound for the harmonic radii of \( g_1 \) and \( g_2 \) at the point \( x \in M \)), and \( \varrho_{g_2, g_1} \) is the density of \( \text{d} \text{vol}_{g_2} \) with respect to \( \text{d} \text{vol}_{g_1} \). It is a key element of our analysis that the trace-class condition for relative scattering theory with respect to the metrics \( g_1 \) and \( g_2 \) is satisfied provided \( \tilde{d}_1(g_1, g_2) \) is finite. By a basic result of Birman and Belopol’skii (\cite{RS79} Thm. XI.13 or Theorem \ref{thm2.4} below), the trace class condition then implies existence and completeness of the wave operators. Passing from our weighted integral condition to the existence and completeness of the wave operators is almost immediate, and thus the finiteness of \( \tilde{d}_1 \) is a very natural, intrinsic condition. Also note that we express our perturbations in terms of quadratic forms to keep the assumptions minimal.

In the above construction, we use harmonic coordinates in conjunction with elliptic regularity theory in \( L_p \) as in \cite{AC92} to obtain estimates for the Green’s function and its first order derivatives. These estimates are then employed to verify the trace class condition that is required in the Birman-Belopol’skii theorem. A similar approach can be found in \cite{WS84} in the case of higher-order operators in domains with infinite boundary.

Let us note as an aside that the distance \( \tilde{d}_1^\ast(g_1, g_2) \) of eqn. \ref{eqn5.1}, which is defined as \( \tilde{d}_1(g_1, g_2) \) above but without the factor \( 1 + \varrho_{g_2, g_1} \), can be computed more or less explicitly in some particularly simple cases. In fact, in Remark \ref{rem6.2} we consider the case of two quasi-isometric Riemannian metrics \( g_1 \) and \( g_2 \) on \( M = \mathbb{R} \times S^{n-1} \) of the form \( g_1 = ds^2 + r_i(s)^2 dg_{S^{n-1}}, \) for \( i = 1, 2, \) where the functions \( r_i \) have to satisfy some natural conditions. Here one obtains

\[
\tilde{d}_1^\ast(g_1, g_2) = \omega_{n-1} \int_{-\infty}^{\infty} \left( \frac{r_2}{r_1} \right)^{n/2} - \left( \frac{r_1}{r_2} \right)^{n/2} \frac{1}{(\min\{1, r_1, r_2\})^{n+2}} \, ds,
\]

where \( \omega_{n-1} \) denotes the volume of the \((n-1)\)-sphere, and \( \tilde{d}_1^\ast(g_1, g_2) \leq \tilde{d}_1(g_1, g_2) \leq c\tilde{d}_1^\ast(g_1, g_2) \) for some constant \( c \geq 1 \) depending only on the quasi-isometric distance \( \tilde{d}_\infty(g_1, g_2). \)

1.3. Structure of the paper. This paper is organized as follows. Section \ref{sec2} introduces some basic definitions concerning scattering in a two-Hilbert space setting, Sobolev spaces on Riemannian manifolds, and the harmonic radius according to \cite{AC92}.

Section \ref{sec3} presents a first main result, Theorem \ref{thm3.7}, which establishes the existence and completeness of the wave operators for the Laplacian on a Riemannian manifold with respect to perturbations of the metric tensor. The trace class condition required in the Birman-Belopol’skii theorem can be verified under fairly general and simple conditions that depend on (local) lower bounds for the Ricci curvature and the injectivity radius given in eqn. \ref{eqn2.9}. Note that we do not need any assumptions on the derivatives of the curvature tensor nor do we need to control the derivative of the relative perturbation.

In Section \ref{sec4} we introduce a class of Riemannian manifolds with ends where we discuss the Laplacian \( H \) and a decoupled version \( H_{\text{dec}} \). It is well known that, under mild conditions, the wave operators for the pair \( (H, H_{\text{dec}}) \) exist and are complete (cf., e.g., Carron \cite{Ca02}). This allows us to define the scattering operator and the scattering matrix for the pair \( (H, H_{\text{dec}}) \) in a natural way.

In Section \ref{sec5} we arrive at our second main result, Theorem \ref{thm5.1} which establishes strong continuity of the scattering operator under perturbations of the metric that are small at infinity. More precisely, we allow for perturbed metrics \( g \) which are quasi-isometric to the given metric \( g_0 \) and enjoy roughly the same bounds for the injectivity radius and the Ricci curvature as \( g_0 \). Furthermore, the perturbation has to satisfy a trace class condition on each end, similar to the condition required in Theorem \ref{thm3.7}. As a direct consequence, we then obtain a continuity result for the scattering matrix that implies,
in particular, that scattering channels which are open for the metric \( g_0 \) will also be open for metrics \( g \) that are close to \( g_0 \) in the sense explained above.

Some simple examples are discussed in Section 6. We restrict our attention to manifolds \((M, g)\) with two ends and \( M = \mathbb{R} \times S^{n-1} \). We first give examples for Theorem 3.7 where neither the unperturbed nor the perturbed metric enjoy rotational symmetry (however, a surface of revolution is used for the sake of comparison to obtain a lower bound for the injectivity radius). Two examples illustrating Corollary 5.3 have a surface of revolution as unperturbed manifold while the perturbed manifolds may be more general. For simplicity, we restrict our attention to the case where one end is a horn while the other end is asymptotically Euclidean. We show by standard techniques that condition (1.1) holds for the rotationally symmetric case and determine a class of admissible perturbations of the metric for which property (L.1) is preserved. Obtaining suitable lower bounds for the radius of injectivity is a cumbersome obstacle and we have been happy to use a comparison theorem of [MS07] which, however, requires a global bound on the sectional curvature. These issues are discussed in Appendix D.

The paper comes with three more Appendices. Appendix A contains a coordinate free way of measuring the distance between two Riemannian metrics \( g \) and \( g_0 \) on a manifold. Appendix B gives some details on how to apply elliptic regularity theory ([GT83]) to the Laplacian in harmonic coordinates; here we mostly follow [AC92]. Appendix C provides some (actually rather standard) material concerning scattering on the line for \(-\frac{d^2}{dx^2}\) and \(-\frac{d^2}{dx^2} + w\) for short-range potentials \( w \).

1.4. Notes on the literature. There exists an extensive literature dealing with the major issues in Riemannian scattering, most notably the existence and completeness of the wave operators, absence of singular continuous spectrum, absence of embedded eigenvalues, counting of resonances, and the construction of a reference dynamics from the geometry. A part of this work was done from a starting point in mathematical physics, while other groups rather originate in differential geometry, like the school of R. Melrose who introduced the important and fruitful concept of scattering manifolds. Closest to our work is the recent paper by Müller and Salomonsen [MS07] mentioned above; Theorem 0.1 of [MS07] yields existence and completeness of the wave operators for perturbations of the curvature tensor. This appears to be the first coordinate-free perturbative result. Our Theorem 3.7 is stronger than their Theorem 0.1; cf. Remark 3.9 for a detailed comparison.

More detailed information on the scattering operator can be obtained if one assumes that the ends possess some additional structure, often expressed in terms of a coordinate system which is globally defined on each end. Then advanced analytical tools from micro-local analysis, pseudo-differential operators or Fourier integral operators can be used to gain rather precise information on the behavior of wave packets and the scattering matrix. Let us highlight some of these developments. A vast body of work has been devoted to scattering manifolds and the scattering on hyperbolic ends (the survey by Perry [Pe07] lists some 170 references). From its inception, the study of scattering manifolds in the sense of Melrose has produced a constant stream of papers, devoted to various aspects and issues. We have to restrict ourselves to a small selection which nonetheless, as we hope, displays some of the variety and depth of what has been achieved by various groups in two decades: Melrose [Me95], Datchev [Da09], Guillopé and Zworski [GZ97], Hassell and Wunsch [HW05], [HW08], Ito and Nakamura [IN10], [IN12], Ito and Skibsted [ISk13b], Mazzeo and Vasy [RV05], Melrose, Sá Barreto and Vasy [MBV13], Sá Barreto [SB05], The papers [GZ97] and Wunsch and Zworski [WZ00] deal with the counting of resonances on scattering manifolds. Early on, the Mourre-method has been applied by Froese and Hislop [FH89] and others to exclude singular continuous spectrum, while Donnelly [Do99], Kumura [Ku10] study asymptotically Euclidean manifolds where
they exclude singular continuous spectrum and embedded eigenvalues as well; cf. also the recent paper by Ito and Skibsted [ISk13]. The decay of solutions of the Schrödinger equation on asymptotically conical ends has recently been studied by Schlag, Soffer and Staubach [SSSI0A, SSSI0B]. In many instances progress is achieved via the (non-perturbative) construction of “natural” dynamics on the ends, starting from the classical geodesic flow; cf., e.g., Herbst and Skibsted [HSk04] and the most recent work of Ito and Skibsted [ISk13b] or Ito and Nakamura [IN10], and the references therein. From our point of view, some of the non-perturbative methods provide natural comparison dynamics which can be used as a reference for perturbed systems. This point of view will be taken up again in Remark 4.8. We emphasize that neither these remarks nor the list of references are in any way complete.

2. Preliminaries

In this section, we introduce basic notation and definitions concerning scattering in two Hilbert spaces and Laplacians on manifolds; this material is fairly standard. We then discuss harmonic coordinates and the harmonic radius in the sense of Anderson and Cheeger [AC92].

2.1. Some basic notation. Let $U \subset \mathbb{R}^d$ be open. The vector space of infinitely differentiable functions $\varphi: U \rightarrow \mathbb{C}$ with compact support in $U$ is denoted as $C^\infty_c(U)$. For $0 < \alpha \leq 1$ and $k \in \mathbb{N}_0$, we denote as $C^{k,\alpha}(U)$ the space of functions $f: U \rightarrow \mathbb{C}$ that are $k$-times continuously differentiable with all partial derivatives of order $k$ being locally Hölder-continuous, and similarly, we denote as $C^{k,\alpha}(\overline{U})$ the subspace of $C^{k,\alpha}(U)$ with all derivatives of order $k$ being uniformly Hölder-continuous functions, as defined in [GT83, Sec. 4.1]. In the special case $U = \mathbb{R}^d$, we will also need the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions.

For $1 \leq p < \infty$, the space of (equivalence classes of) Borel-measurable functions $f: U \rightarrow \mathbb{C}$ with $\int_U |f(x)|^p \, dx < \infty$, equipped with the usual norm, is denoted as $L_p(U)$; $L_\infty(U)$ is the Banach space of (equivalence classes of essentially) bounded Borel-functions $f: U \rightarrow \mathbb{C}$ with the usual norm. For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, we let $W^k_p(U)$ denote the Sobolev space of all functions $f \in L_p(U)$ with distributional derivatives up to order $k$ in $L_p(U)$, equipped with the canonical norm as in [GT83, Sec. 7.5]. In the special case $p = 2$ we write $H^k(U) := W^k_2(U)$, a Hilbert space. The local spaces $W^k_{p,\text{loc}}(U)$, $H^k_{\text{loc}}(U)$ are defined accordingly. We will also need the subspaces $W^k_p(U)$ and $H^k(U)$ obtained as the closure of $C^\infty_c(U)$ in the respective norms.

For linear operators $T$ acting in a Hilbert space $\mathcal{H}$ we denote by $\text{Dom} \, T$, $\text{Ran} \, T$, and $\ker T$ the domain, the range, and the kernel of $T$, respectively.

2.2. Scattering in two Hilbert spaces. Let $H_1$ and $H_2$ be self-adjoint operators acting in separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $J$ be a bounded operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. We define the wave operators

$$W_{\pm}(H_2, H_1, J) = \lim_{t \to \pm \infty} e^{itH_2} e^{-itH_1} P_{\text{ac}}(H_1),$$

provided that the strong limits exist, with $P_{\text{ac}}(H_1)$ denoting the projection onto the subspace of absolute continuity of $H_1$. We say that the wave operators $W_{\pm}(H_2, H_1, J)$ are complete if $(\ker W_{\pm}(H_2, H_1, J))^\perp = \mathcal{H}_{\text{ac}}(H_1)$ and

$$\text{Ran} \, W_{\pm}(H_2, H_1, J) = \mathcal{H}_{\text{ac}}(H_2),$$

where $\mathcal{H}_{\text{ac}}(H_i)$ denotes the subspace of absolute continuity of $H_i$; note that since we only assume that $J$ is bounded, $\text{Ran} \, W_{\pm}(H_2, H_1, J)$ is not necessarily closed.

A bounded linear operator $T: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ ($T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$) is said to be trace class if $(T^*T)^{1/2}$ is trace class in $\mathcal{H}_1$; the corresponding space of trace class operators is denoted
as $\mathcal{B}_1(\mathcal{H}_1, \mathcal{H}_2)$. Equivalently, $T$ is trace class if $T$ can be factorized as $T = T_2T_1$ with $T_1: \mathcal{H}_1 \rightarrow \mathcal{H}_0$ being Hilbert-Schmidt operators into a third Hilbert space $\mathcal{H}_0$. The space of such Hilbert-Schmidt operators is denoted by $\mathcal{B}_2(\mathcal{H}_1, \mathcal{H}_0)$.

For further basic definitions and results in two-Hilbert space scattering, we refer to [RS79, K67]. Our main result will be based on the Birman-Belopol’skii theorem as given in [RS79, Thm. XI.13]:

**Theorem 2.1.** For $j = 1, 2$, let $H_j$ be a self-adjoint and semi-bounded operator in a Hilbert space $\mathcal{H}_j$ with associated quadratic form $\mathcal{h}_j$ and spectral projectors $E_0(H_j)$. Suppose that $I \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that

(a) $I$ has a two-sided bounded inverse;
(b) we have $E_1(H_2)(H_2I - IH_1)E_2(H_1) \in \mathcal{B}_1(\mathcal{H}_1, \mathcal{H}_2)$ for any bounded interval $I$;
(c) the operator $(I^*I - 1)E_2(H_1)$ is compact for any bounded interval $I \subset \mathbb{R}$;
(d) $I(\text{Dom } \mathcal{h}_1) = \text{Dom } \mathcal{h}_2$.

Then the wave operators $W_{\pm}(H_2, H_1, I)$ exist, are complete, and partially isometric with initial space $\mathcal{H}_{ac}(H_1)$ and final space $\mathcal{H}_{ac}(H_2)$.

### 2.3. Manifolds and their Laplacians.

Let $M$ be a smooth, complete, oriented, connected manifold of dimension $n \geq 2$; typically, $M$ will not be compact.

We will define various objects such as Hilbert spaces or norms intrinsically without referring to an atlas, in order to make the definitions as natural and general as possible.

Let $T^*M$ denote the cotangent bundle on $M$. A Riemannian metric $g$ on $M$ is a smooth family of positive definite sesquilinear forms on $T^*M$. The corresponding Riemannian manifold will be written as $\mathcal{M} = (M, g)$. As explained in Remark 2.6 we may weaken the regularity assumption on the metric.

We assume—chiefly for simplicity—that $M$ has no boundary, and that $(M, g)$ is complete. Our results easily extend to many cases where the manifold has a boundary. In this case, we have to specify suitable boundary conditions, such as Dirichlet or Neumann, to obtain a self-adjoint realization of the Laplacian.

The metric $g$ naturally induces a volume measure on $M$, denoted by $d\text{vol}_g$. The corresponding Hilbert space of square integrable (equivalence classes of) functions on $M$ is denoted by $L_2(\mathcal{M}) = L_2(M, g)$ with inner product $\langle u, v \rangle := \int_M u\overline{v} \, d\text{vol}_g$ and corresponding norm

$$\|u\|_{L_2(\mathcal{M})}^2 = \|u\|^2_{L_2(\mathcal{M})} = \int_M |u|^2 \, d\text{vol}_g.$$  

Similarly, we denote by $L_2(T^*\mathcal{M})$ the square integrable sections in the Riemannian cotangent bundle $T^*\mathcal{M} = (T^*M, g)$ with norm

$$\|\omega\|^2_{L_2(T^*\mathcal{M})} = \|\omega\|^2_{L_2(T^*\mathcal{M})} = \int_M |\omega|^2_g \, d\text{vol}_g,$$

where $|\omega|^2_g = g(\omega, \omega)$ depends on the metric $g$. We assume that the fibres $T^*_xM$ are complex vector spaces, and that $g_x$ is a sesquilinear form on $T^*_xM$, linear in the first and anti-linear in the second argument. In a coordinate chart $\Phi: B \rightarrow U$ with $B \subset M$ and $U \subset \mathbb{R}^n$ open and $(x^1, \ldots, x^n) = \Phi(x)$, $x \in B$, we have the local expression

$$|\omega|^2_g = \sum_{i,j=1}^n g^{ij} \omega_i \overline{\omega}_j$$

where $\omega = \sum_i \omega_i dx^i$ and $g^{ij} = g(dx^i, dx^j)$. On the tangent bundle $T\mathcal{M}$, the metric components are given by the inverse matrix $(g_{ij})$ with $g_{ij} = g(\partial_i, \partial_j)$. Moreover, $d\text{vol}_g = \sqrt{\det g} \, dx^1 \cdot \cdot \cdot dx^n$ is the volume measure in the coordinate chart, where $\det g$ is the determinant of the matrix $(g_{ij})$. 
Let $du$ be the exterior derivative of $u$, a section in the cotangent bundle. Let $C^\infty_c(M)$ be the space of smooth functions with compact support. We denote by $H^1(\mathcal{M})$ the closure of $C^\infty_c(M)$ with respect to the norm

$$
\|u\|^2_{H^1(\mathcal{M})} := \|u\|^2_{L^2(\mathcal{M})} + \|du\|^2_{L^2(T^*\mathcal{M})}.
$$

We define the operator $D: H^1(\mathcal{M}) \to L^2(T^*\mathcal{M})$ by $Du := du$. Note that by definition of $H^1(\mathcal{M})$, $D$ is a closed operator. Similarly, the quadratic form $\mathfrak{d}$ given by $\mathfrak{d}(u) := \|du\|^2_{L^2(T^*\mathcal{M})}$ and $\text{Dom} \mathfrak{d} := H^1(\mathcal{M})$ is closed. We denote the corresponding sesquilinear form obtained via the polarization identity by the same symbol $\mathfrak{d}$. By the first representation theorem (cf. [K66, Thm. VI.2.1]), there exists a self-adjoint, non-negative operator $\Delta = \Delta_{\mathcal{M}}$, the Laplacian of the Riemannian manifold $\mathcal{M}$, satisfying

$$
\mathfrak{d}(u,v) = \langle \Delta u, v \rangle
$$

for all $u \in \text{Dom} \Delta$ and $v \in \text{Dom} \mathfrak{d}$. We also write $\Delta = \Delta_{\mathcal{M}} = \Delta_g$ in order to stress the dependence on the metric $g$. Note that we define the Laplacian as a non-negative operator—instead of a non-positive operator—as is often the case in the mathematical physics literature.

2.4. Harmonic radius. We denote by $B(x, r) = B_{\mathcal{M}}(x, r)$ the open geodesic ball of radius $r$ around $x$ in the Riemannian manifold $\mathcal{M} = (M, g)$. A central role in our analysis is played by the (local) harmonic radius of $\mathcal{M}$. Roughly speaking, for any point $x \in M$, the harmonic radius $r_{\mathcal{M}}(x)$ at $x$ is the largest radius (less than the injectivity radius at $x$) with the property that there exists a system of harmonic coordinates in $B_{\mathcal{M}}(x, r_{\mathcal{M}}(x))$. This means that there is an open set $U \subset \mathbb{R}^n$ and a diffeomorphism $\Phi: B_{\mathcal{M}}(x, r_{\mathcal{M}}(x)) \to U$ such that each component function $\Phi^i$ satisfies $\Delta_g \Phi^i = 0$ in $B_{\mathcal{M}}(x, r_{\mathcal{M}}(x))$. The actual definition, given below, is a bit more technical and provides several rather precise estimates. We mostly follow the work of Anderson and Cheeger [AC92]; cf. also [DK81, HB96, HH98, HB99]. We first define the harmonic radius at $x \in M$ as in [AC92]:

**Definition 2.2.** For $p \in (n, \infty)$ and $Q \in (1, \infty)$, the $W^1_p$-harmonic radius of $\mathcal{M}$ at $x \in M$ is the largest number $r_{\mathcal{M}}(x) = r_{\mathcal{M}}(x;p, Q)$ such that there is a system of harmonic coordinates in $B_{\mathcal{M}}(x, r_{\mathcal{M}}(x))$ with the property that the metric tensor $(g_{ij})$ in these coordinates satisfies

$$
Q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq Q(\delta_{ij}),
$$

as bilinear forms, and

$$
r_{\mathcal{M}}(x)^{1-n/p}\|\partial_k g_{ij}\|_{L_p(U)} \leq Q - 1,
$$

where $U \subset \mathbb{R}^n$ is the domain of the coordinates.

We are mainly interested in $Q \in (1, 2]$ close to 1; the actual choice of $Q$ will only become important in Appendix B (see the text before eqn. (B.10)) when we transfer estimates in elliptic regularity theory from Euclidean space to the manifold). At the same time, we will fix $p := n + 1$. Our choice of $Q$ in Appendix B is not mandatory; other $Q$’s would lead to different constants, however.

Note that [AC92] gives control of the $g_{ij}$ in $\alpha$-Hölder-norm with exponent $\alpha := 1 - n/p$. A fine point in the way how [AC92] use elliptic regularity theory in conjunction with harmonic coordinates concerns the modulus of continuity of the $g_{ij}$ which enters the $L^p$-estimate of [CT83, Thm. 9.11]. By [CT83, Thm. 7.19], the above eqn. (2.4) implies the following Hölder estimate: there exists a constant $C = C(n, \alpha) > 0$ such that for any ball $B$ satisfying $\overline{B} \subset U$ we have

$$
|g_{ij}(y) - g_{ij}(y')| \leq C(Q - 1)r_{\mathcal{M}}(x)^{-\alpha}|y - y'|^\alpha, \quad y, y' \in \overline{B},
$$

where $\overline{B}$ is the closure of $B$. This estimate is crucial for our analysis.
with $\alpha = 1 - n/p$. In particular, $|g_{ij}(y) - g_{ij}(y')| \leq C(2Q)^\alpha(2 - 1)$, for $y, y' \in \overline{B} \subset U$. By eqn. (2.3) we also have the elementary estimate $1/Q \leq |g_{ij}(y)| \leq Q$, for all $i, j$ and all $y \in U$.

We need a local lower bound on the harmonic radius in terms of local lower bounds on the injectivity radius $\text{inj}_{\mathcal{M}}$ and the Ricci curvature $\text{Ric}_{\mathcal{M}}$. Denote by $\text{Ric}_{\mathcal{M}}^- : M \rightarrow \mathbb{R}$ the (pointwise) lowest eigenvalue of $\text{Ric}_{\mathcal{M}}$ viewed as an endomorphism on $T^*M$.

For $\delta > 0$ and a continuous function $f : M \rightarrow \mathbb{R}$ denote by
\[
(\inf_\delta f)(x) := \inf_{y \in B_{\mathcal{M}}(x, \delta)} f(y)
\]
the $\delta$-homogenized lower bound of $f$. Note that if $f$ is Lipschitz-continuous with Lipschitz constant $L > 0$ on $B_{\mathcal{M}}(x, \delta)$ then
\[
f(x) - L\delta \leq \inf_\delta f(x) \leq f(x).
\]

As in [AC92], we denote by
\[
\iota_{\mathcal{M}}(x) := \sup_{\delta > 0} \min \{\delta, \inf_\delta \text{inj}_{\mathcal{M}}(x)\}
\]
the largest radius $\delta$ for which the injectivity radius at $y \in B(x, \delta)$ is bounded from below by $\delta$; we call $\iota_{\mathcal{M}}(x)$ the homogenized injectivity radius. An important technical ingredient in the proof of Thm. 0.3 in Anderson and Cheeger [AC92, p. 271] consists in the observation that
\[
\iota_{\mathcal{M}}(y) = \frac{1}{2} \text{dist}(y, \partial B(x, r))
\]
for $y \in B(x, r)$ and $r = \text{inj}_{\mathcal{M}}(x)$, where $\mathcal{B} = (B(x, r), g)$. This fact justifies the complicated definition of $\iota_{\mathcal{M}}(x)$. In particular, for $y = x$, we have $\iota_{\mathcal{M}}(x) = r/2$. We use a slight generalisation of this fact in the following form that $r \leq \text{inj}_{\mathcal{M}}(x)$ implies $\iota_{\mathcal{M}}(x) \geq r/2$.

This inequality can be easily seen as follows: $r \leq \text{inj}_{\mathcal{M}}(x)$ implies that the exponential map is well-defined in $B(y, r/2)$ for any $y \in B(x, r/2)$, hence $\iota_{\mathcal{M}}(x) \geq r/2$ with $\mathcal{B}' = (B(y, r/2), g)$. The domain monotonicity of the homogenized injectivity radius implies
\[
r/2 \leq \iota_{\mathcal{M}}(x) \leq \iota_{\mathcal{M}}(x).
\]

The harmonic radius is a purely geometric quantity with a lower bound depending only on lower bounds for the injectivity radius and the Ricci curvature (cf. [AC92, Thm. 0.3]):

**Proposition 2.3.** Let $\mathcal{M} = (M, g)$ be a smooth Riemannian manifold and let $r_0 : M \rightarrow (0, 1]$ be a continuous function such that the homogenized injectivity radius and Ricci curvature satisfy the lower bounds
\[
\iota_{\mathcal{M}}(x) \geq r_0(x) \quad \text{and} \quad \inf_{r_0(x)} \text{Ric}_{\mathcal{M}}^- \geq -\frac{1}{r_0(x)^2}
\]
for all $x \in M$. Then there exists a constant $c = c(n, p, Q) > 0$ such that the $W^p_1$-harmonic radius of $\mathcal{M}$ at $x \in M$ satisfies the lower bound
\[
r_{\mathcal{M}}(x) \geq cr_0(x),
\]
for all $x \in M$.

**Proof.** For $x \in M$ given, we apply Thm. 0.3 of [AC92] to the specific choice of $\mathcal{B} = (B, g)$ with $B := B(x, r_0(x))$ (which now replaces the manifold $M$ of [AC92]). The constant $\lambda$ of Thm. 0.3 is then replaced by $1/r_0(x)$; notice that $\text{Ric}_{\mathcal{M}}^-(y) \geq -1/r_0(x)^2$ for all $y \in B$ by the very definition of $\inf_\delta$, for $\delta = r_0(x)$. Applying their theorem, we obtain that (with constants $c_1$ and $c_2$ depending only on $Q$, $n$ and $p$)
\[
r_{\mathcal{M}}(x) \geq r_{\mathcal{M}}(x) \geq \min\{c_1\lambda^{-1}, c_2\iota_{\mathcal{M}}(x)\} \geq \min\{c_1r_0(x), c_2r_0(x)/2\} = cr_0(x)
\]
with $c := \min\{c_1, c_2/2\}$; here we have used (2.8) with $r = r_0(x)$. \qed
As an alternative to the above proof of Proposition 2.3 one could just follow the proof of Theorem 0.3 in [AC92]. Indeed, Thm. 0.3 in [AC92] is of a purely local nature (as noted by the authors), and the first step in their proof of Thm. 0.3 consists in a reduction to geodesic balls.

Note that Theorem 0.3 in [AC92] is purely local (as noted by the authors). Indeed, the first step in their proof of Theorem 0.3 consists in a reduction to geodesic balls.

Without loss of generality we may assume in the sequel, as we have already done in Proposition 2.3, that the function \( r_0 \), serving as lower bound for the injectivity radius and the Ricci curvature, is bounded from above by 1; this assumption is convenient in the proofs of Appendix [B].

For further reference, we use the following notation:

**Definition 2.4.** For a continuous positive function \( r_0 : M \to (0, 1] \), we denote by \( \text{Met}_{r_0}(M) \) the set of smooth metrics \( g \) on \( M \) that satisfy the lower bounds (2.9).

Let us mention a particularly simple situation where the homogenized injectivity radius and Ricci curvature can be estimated from below using a pointwise lower bound on the injectivity radius and Ricci curvature itself.

**Proposition 2.5.** Assume that

\[
\text{Ric}^r_{(M,g)}(x) \geq -\frac{1}{\beta(x)^2} \quad \text{and} \quad \operatorname{inj}_{(M,g)}(x) \geq r(x),
\]

for all \( x \in M \), where \( \beta, r : M \to (0, 1] \) are \( C^1 \)-functions enjoying the following properties: \( r \) has bounded derivative, and \( \beta \) satisfies an estimate \( |\beta'(x)| \leq C/\beta(x)^3 \) for all \( x \in M \), for some constant \( C \geq 0 \).

Then the lower bound (2.10) on the homogenized injectivity radius and the homogenized Ricci curvature holds with

\[
r_0(x) := \min \left\{1, \frac{r(x)}{1 + \|r'\|_\infty^2}, \frac{\beta(x)}{\sqrt{1 + 2C}} \right\},
\]

i.e., \( g \in \text{Met}_{r_0}(M) \). In particular, if \( \beta \) and \( r \) are constant, then \( r_0(x) = \min\{1, r, \beta\} \) can be chosen as a constant function.

**Proof.** Note first that \( f \geq g \) implies \( \inf_\delta f \geq \inf_\delta g \). Applying (2.9) to the function \( -\beta^{-2} \) with \( \delta \in (0, 1] \) we obtain

\[
\inf_\delta \left( -\frac{1}{\beta^2} \right)(x) \geq -\frac{1}{\beta(x)^2} - L\delta \geq -\frac{1}{\beta(x)^2} - 2C \geq -\frac{1 + 2C}{\beta(x)^2}
\]

with \( L = \|\beta^{-2}\|_\infty \leq 2C \) and similarly, \( \psi_{(M,g)}(x) \geq \min\{\delta, r(x) - \|r'\|_\infty \delta\} \). The latter expression is greater than or equal to \( \delta \) iff \( r(x) - \|r'\|_\infty \delta \geq \delta \). This inequality yields the inequality \( r(x)(1 + \|r'\|_\infty)^{-1} \geq r_0(x) \) on \( r_0(x) = \delta \). \( \square \)

**Remark 2.6.** We may weaken the regularity assumptions on the metric \( g \) as follows:

It is sufficient to assume that \( g \in C^1_{loc} \subset W^1_{p,loc} \), i.e., we assume that there is a covering with (for simplicity, smooth) charts, such that the metric tensor \( (g_{ij}) \) in each of these charts is of class \( C^1_{loc} \). In this case, the Ricci curvature is still defined and \( \text{Ric}(M,g) \in C^0_{loc} \) (cf. the paper [AKK+03] for a detailed discussion of related ideas and results).

More precisely, we can argue as follows:

(a) The Hölder regularity \( g \in C^1_{loc} \) allows us to apply the results of deTurck and Kazdan [DK81], notably their Lemma 1.2 which states that, for any point \( x \in M \), there exist harmonic coordinate charts of class \( C^2_{loc} \) near \( x \) and that, moreover, all harmonic charts near \( x \) have this regularity. In particular, the metric tensor \( g_{ij} \) in any harmonic chart has regularity \( C^1_{loc} \).
As before, Theorem 0.3 of [AC92] yields a lower bound for the \( W^1_{p,\text{loc}} \)-harmonic radius at \( x \in M \) as in Proposition 2.3 (Note that [AC92] seem to consider smooth \( g_{ij} \), but a simple approximation argument allows to establish the result of Theorem 0.3 in [AC92] under the weaker assumption \( g_{ij} \in W^1_{p,\text{loc}} \).

(c) By (m), the metric tensor \( g_{ij} \) is locally Lipschitz and, by (b), the estimates (2.3)–(2.4) hold. This quality of the \( g_{ij} \) is required for an application of elliptic regularity theory in \( L_p \) to the Laplacian, expressed in harmonic coordinates, cf. [AC92] and Appendix B. Note that the Laplacian, written in harmonic coordinates, has no first-order terms (cf. [DK81], [AC92]), an important simplification.

3. Existence and completeness of the wave operators

We are now going to derive a criterion for the existence and completeness of the wave operators for the Laplacian of two (non-compact) manifolds that are close to one another in a suitable sense. It is our aim to find conditions that only involve geometric quantities and do not assume a particular structure of the unperturbed situation. This problem has recently been studied in [MS07] where the (relative) smallness of the perturbation at the ends of the manifold is expressed in terms of bounds on the curvature tensor and its derivatives. Here we propose an approach which, in several respects, is even closer to the geometry and, hopefully, even simpler. To this end, we advocate the use of the (local) harmonic radius (cf. Section 2.4) as a basic geometric quantity which can be used to express conditions on the perturbed metric that translate into trace-class conditions for the difference of resolvents (or, more precisely, the difference of suitable powers of resolvents). Note that the manifolds considered in this section are more general than what we discuss later on where we will restrict our attention to manifolds with ends.

Suppose that \( M \) is an \( n \)-dimensional, smooth, oriented manifold with two metrics \( g_1, g_2 \) such that \( \mathcal{M}_k := (M, g_k) \) is a complete \( n \)-dimensional Riemannian manifold, for \( k = 1, 2 \). Let us first compare the corresponding norms defined with respect to \( g_1 \) and \( g_2 \). In Appendix A we define a quasi-distance

\[
\tilde{d}_\infty(g_1, g_2) := \sup_{x \in M} \tilde{d}(g_1, g_2)(x),
\]

where

\[
\tilde{d}(g_1, g_2)(x) := 2 \sinh \left( \frac{n}{4} \cdot \max_k |\ln \alpha_k(x)| \right) = \max_k |\alpha_k(x)^{n/4} - \alpha_k(x)^{-n/4}|,
\]

and where \( \alpha_k(x) \) is the \( k \)-th eigenvalue of the positive definite endomorphism \( A(x) \in \mathcal{B}(T^*_x M) \) given by

\[
g_2(x)(\xi, \zeta) = g_1(x)(A(x)\xi, \zeta), \quad \xi, \zeta \in T^*_x M.
\]

Let us give a simple example. If \( g_2 = e^{2\mu} g_1 \) for some (bounded) function \( \mu : M \to \mathbb{R} \), then \( g_2 \) is a conformal perturbation of \( g_1 \). In this case, \( A = e^{2\mu} \), \( \tilde{d}_\infty(g_1, g_2) = 2 \sinh(n|\mu|_\infty/2) \) and \( d_\infty(g_1, g_2) = 2|\mu|_\infty \) (cf. Appendix A for a definition of \( d_\infty(g_1, g_2) \)), i.e., \( \tilde{d}(g_1, g_2) \) measures the distortion rate of the conformal factor.

The following definition is standard (cf., e.g., [MS07]).

**Definition 3.1.** We say that the metrics \( g_1, g_2 \) are quasi-isometric if there exists a constant \( \eta > 0 \) such that

\[
\eta g_1(x)(\xi, \xi) \leq g_2(x)(\xi, \xi) \leq \eta^{-1} g_1(x)(\xi, \xi)
\]

for all \( \xi \in T^*_x M \) and \( x \in M \).

Note that \( g_1, g_2 \) are quasi-isometric if and only if

\[
\tilde{d}_\infty(g_1, g_2) < \infty
\]
(see Remark A.2 in Appendix A).

The following equivalence of norms follows immediately from the definition and (A.4)–(A.5):

**Proposition 3.2.** If the metrics \( g_1, g_2 \) on \( M \) are quasi-isometric, i.e., if the quasi-distance fulfills \( \tilde{d}_\infty = \tilde{d}_\infty(g_1, g_2) < \infty \), then

\[
(1 + \tilde{d}_\infty)^{-1} \| u \|_{\tilde{F}_1} \leq \| u \|_{\tilde{F}_2} \leq (1 + \tilde{d}_\infty) \| u \|_{\tilde{F}_1},
\]

\[
(1 + \tilde{d}_\infty)^{-1} \| \omega \|_{\tilde{F}_1} \leq \| \omega \|_{\tilde{F}_2} \leq (1 + \tilde{d}_\infty) \| \omega \|_{\tilde{F}_1},
\]

\[
(1 + \tilde{d}_\infty)^{-1} \| u \|_{\tilde{F}_1} \leq | u |_{\tilde{F}_2} \leq (1 + \tilde{d}_\infty) \| u \|_{\tilde{F}_1},
\]

for \( u \) resp. \( \omega \) in the appropriate spaces. Here, \( \mathcal{H}_k := L_2(M, g_k) \), \( \mathcal{H}_k^\delta := L_2(T^*M, g_k) \) and \( \mathcal{H}_k^\delta := H^1(M, g_k) \). In particular, the spaces for \( k = 1 \) and \( k = 2 \) are identical as vector spaces and have equivalent norms.

Let us assume for the rest of this section that \( \tilde{d}_\infty(g_1, g_2) < \infty \), i.e., that \( g_1, g_2 \) are quasi-isometric (cf. Remark A.2).

We let \( I : L_2(\mathcal{M}_1) \to L_2(\mathcal{M}_2), I f_1 := f_1 \), denote the natural identification operator; its adjoint, \( I^* \), is given by \( I^* g = \varrho g \), where \( \varrho = \varrho_{g_2, g_1} \) is the density of \( d \text{vol}_{g_2} \) with respect to \( d \text{vol}_{g_1} \), i.e.,

\[
d \text{vol}_{g_2} = \varrho_{g_2, g_1} d \text{vol}_{g_1}.
\]

Note that \( \varrho = (\det A)^{-1/2} \). Finally, we let \( H_k \) denote the (self-adjoint, non-negative) Laplacian operator \( \Delta_{\mathcal{H}_k} \) acting in \( L_2(\mathcal{H}_k) \); for simplicity of notation, we write \( R_k := (H_k + 1)^{-1} \) for the resolvents.

The aim of the following is to find conditions which will allow us to show that, for sufficiently large \( m \in \mathbb{N} \), the operators

\[
V := R_m^2 (H_2 I - IH_1) R_1^m
\]

(3.4) can be written as a sum of products of Hilbert-Schmidt operators; this will be achieved in Lemma 3.3 and Proposition 3.5. We begin with some technicalities. Let

\[
S := 2 \sinh \frac{1}{2} \ln \varrho = \varrho^{1/2} - \varrho^{-1/2} : M \to \mathbb{R},
\]

\[
\hat{S} := 2 \sinh \frac{1}{2} \ln (\varrho A) = (\varrho A)^{1/2} - (\varrho A)^{-1/2} : M \to \mathcal{B}(T^*M),
\]

where \( \mathcal{B}(T^*M) \) denotes the vector bundle of (fiberwise) endomorphisms of \( T^*M \). We denote the corresponding multiplication operators on \( \mathcal{H}_k = L_2(M, g_k) \) (resp. \( \mathcal{H}_k^\delta = L_2(T^*M, g_k) \)) by \( S_k \) (resp. \( \hat{S}_k \)). (Strictly speaking, \( \hat{S}(x) \) acts as an endomorphism on the fiber \( T^*_x M \), but we call \( \hat{S}_k \) also a multiplication operator.) We use the (pointwise) polar decomposition \( S = |S|(\text{sgn } S) \) and \( \hat{S} = |\hat{S}|(\text{sgn } \hat{S}) \), where \( |S|(x) = |S(x)| \geq 0 \) and \( |\text{sgn } S(x)| = 1 \), and where \( |\hat{S}|(x) \) is a non-negative endomorphism and \( \text{sgn } \hat{S}(x) \) is unitary on \( T^*_x M \).

The following pointwise estimates will be used in Proposition 3.5.

**Lemma 3.3.** We have the pointwise estimate

\[
0 \leq |S|, |\hat{S}|_{\mathcal{B}(T^*\mathcal{M})} \leq \tilde{d}(g_1, g_2),
\]

where \( |\cdot|_{\mathcal{B}(T^*\mathcal{M})} \) denotes the pointwise operator norm in \( \mathcal{B}(T^*\mathcal{M}) \).

Proof. Let us prove the estimate for \( \hat{S} \), the estimate \( 0 \leq |S| \leq \tilde{d}(g_1, g_2) \) can be seen in the same way. We have

\[
|\hat{S}|_{\mathcal{B}(T^*\mathcal{M})} = \left| (\varrho A)^{1/2} - (\varrho A)^{-1/2} \right|_{\mathcal{B}(T^*\mathcal{M})} = 2 \sinh \frac{1}{2} \ln (\varrho A)\right|_{\mathcal{B}(T^*\mathcal{M})}.
\]
Moreover, the $i$-th eigenvalue of $\ln(gA)$ is given by

$$-\sum_{k=1}^{n} \frac{\ln \alpha_k}{2} + \ln \alpha_i.$$

Let $k_0$ be such that $|\ln \alpha_{k_0}| = \max_k |\ln \alpha_k|$. Then

$$\left| -\sum_{k=1}^{n} \frac{\ln \alpha_k}{2} + \ln \alpha_i \right| \leq \frac{n}{2} |\ln \alpha_{k_0}|.$$

Therefore, we have

$$|\tilde{S}|_{\mathcal{H}(T^*\mathcal{M})} \leq 2 \sinh \frac{n}{4} |\ln \alpha_{k_0}| = \tilde{d}(g_1, g_2). \quad \square$$

Denote by $D_k$ the exterior derivative viewed as a closed operator from $\mathcal{H}_k$ into $\tilde{\mathcal{H}}_k$ with domain $\text{Dom} D_k = \mathcal{H}_k^1 = H^1(M, g_k)$. We also set

$$U: \mathcal{H}_1 \longrightarrow \mathcal{H}_2, \quad U u := (\text{sgn } S) g^{-1/2} u,$$

$$\hat{U}: \mathcal{H}_1 \longrightarrow \tilde{\mathcal{H}}_2, \quad \hat{U} \omega := (\text{sgn } \hat{S})(gA)^{-1/2} \omega.$$

It is easily seen that $U$ and $\hat{U}$ are unitary. We now define $V$ as a quadratic form and provide a decomposition of $V$ which involves two terms, each of them a product containing factors of operators $B^{(m)}_k$ or $\hat{B}^{(m)}_k$, as defined below. We will show in a second step that these factors are Hilbert-Schmidt operators, provided $m$ is large enough.

**Lemma 3.4.** For any $m \in \mathbb{N}$ the operator $V$ in eqn. (3.4) can be written as

$$V = (B^{(m)}_2)^* \hat{U} \hat{B}^{(m)}_1 - (B^{(m)}_2)^* U B^{(m-1)} \hat{1} R_1,$$

where

$$B^{(m)}_k := |S_k|^{1/2} R^m_k: \mathcal{H}_k \longrightarrow \mathcal{H}_k \quad \text{ and } \quad \hat{B}^{(m)}_k := |\hat{S}_k|^{1/2} D_k R^m_k: \mathcal{H}_k \longrightarrow \tilde{\mathcal{H}}_k.$$

**Proof.** Let us first introduce a second identification operator $I': \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ given by $I' f_2 = f_2$ for $f_2 \in \mathcal{H}_2$; note that $I' = I^{-1}$. Letting $\mathfrak{h}_k(u) = \|D_k u\|_{\mathcal{H}_k}^2$ denote the quadratic forms of the operators $H_k$, Proposition 5.2 implies $I(\text{Dom } \mathfrak{h}_1) = \text{Dom } \mathfrak{h}_2$. We now consider $f_k \in \mathcal{H}_k$ and write $h_k := R^m_k f_k$. With $I^* h_2 = \mathfrak{g} h_2$ we then compute

$$\langle V f_1, f_2 \rangle_{\mathcal{H}_2} = \langle \mathfrak{h}_2 \mathbf{1} h_1, h_2 \rangle - \langle \langle I - (I')^* \rangle H_1 h_1, h_2 \rangle_{\mathcal{H}_2} = \int_M \langle (1 - g^{-1} A^{-1}) d h_1, d h_2 \rangle_{g_2} - \langle 1 - g^{-1} \rangle (H_1 h_1, \mathfrak{g} h_2) \rangle_{\mathcal{H}_2} \quad \text{and the desired factorization follows.} \quad \square$$

The advantage of using the identification operator $I'$ instead of $I^*$ in the quadratic form is to avoid a condition on $d \mathfrak{g}$.

We define another “distance” between two metrics $g_1, g_2$ on the $n$-dimensional manifold $M$: for a given continuous function $r_0: M \longrightarrow (0, 1]$ we let

$$\tilde{d}_1(g_1, g_2) := \int_M \tilde{d}(g_1, g_2)(x) \cdot r_0(x)^{(n+2)} \cdot (1 + \varrho_{g_2, g_1}(x)) \text{ d vol}_{g_1}(x) \quad (3.5)$$

denote the weighted $L_1$-quasi-distance of $g_1$ and $g_2$. The factor $1 + \varrho_{g_2, g_1}$ is introduced in order to make $\tilde{d}_1$ symmetric. We will be interested in situations where $\tilde{d}_1(g_1, g_2)$ is finite, and we will show in Remark 5.6 that $\tilde{d}_1$ is actually a quasi-distance (see Definition A.1) on a suitable subset.

We next discuss a Hilbert-Schmidt property of the operators $B^{(m)}_k$ and $\hat{B}^{(m)}_k$, for $m$ large:
Proposition 3.5. Let \( r_0 : M \to (0, 1] \) be a continuous function. Suppose we are given two quasi-isometric metrics \( g_1, g_2 \) on \( M \) (i.e., \( \tilde{d}_\infty(g_1, g_2) < \infty \)) which have \( r_0 \) as a common lower bound for the harmonic radii, i.e., \( r_{(M,g_k)}(x) \geq r_0(x) \) for all \( x \in M \) and \( k = 1,2 \).

If \( \tilde{d}_1(g_1, g_2) < \infty \) then for any \( m \in \mathbb{N} \) with \( m \geq [n/4] + 2 \), the operators \( B_k^{(m)} \) and \( \tilde{B}_k^{(m)} \), defined in Lemma 3.4, are Hilbert-Schmidt. Furthermore, their Hilbert-Schmidt norms satisfy the estimate

\[
\| B_k^{(m)} \|_{\mathcal{B}^2}^2, \quad \| \tilde{B}_k^{(m)} \|_{\mathcal{B}^2}^2 \leq C \tilde{d}_1(g_1, g_2), \quad k = 1,2,
\]

where \( C \) depends only on \( m, n, p \) and \( Q \).

Proof. By the Riesz Representation Theorem and Theorem [3.1] it is easy to see that the resolvents \( R_k^m \) are integral operators with (measurable) kernels \( G_k^{(m)}(x,y) \) satisfying

\[
\int_M |G_k^{(m)}(x,y)|^2 \, d \text{vol}_{g_k}(y) \leq C (\min \{1, r_{(M,g_k)}(x)\})^{-n} \leq C r_0(x)^{-n},
\]

where \( C \) depends only on \( m, n, p, \) and \( Q \). We similarly obtain for the kernels \( d_x G_k^{(m)}(x,y) \) of the \( D_k R_k^m \) that

\[
\int_M |d_x G_k^{(m)}(x,y)|^2 \, d \text{vol}_{g_k}(y) \leq C (\min \{1, r_{(M,g_k)}(x)\})^{-n-2} \leq C r_0(x)^{-n-2}.
\]

The Hilbert-Schmidt norm of \( B_k^{(m)} \) is given by

\[
\| B_k^{(m)} \|_{\mathcal{B}^2}^2 = \int_{M \times M} |S_k(x)||G_k^{(m)}(x,y)|^2 \, d \text{vol}_{g_k}(y) \, d \text{vol}_{g_k}(x),
\]

hence with the previous estimate on the kernel and the pointwise estimate \( |S| \leq \tilde{d}(g_1, g_2) \) from Lemma [3.3] we obtain

\[
\| B_1^{(m)} \|_{\mathcal{B}^2}^2 \leq C \int_M \tilde{d}(g_1, g_2)(x)r_0(x)^{-n} \, d \text{vol}_{g_1}(x) \leq C \tilde{d}_1(g_1, g_2),
\]

\[
\| B_2^{(m)} \|_{\mathcal{B}^2}^2 \leq C \int_M \tilde{d}(g_1, g_2)(x)r_0(x)^{-n} \, d \text{vol}_{g_1}(x) \leq C \tilde{d}_1(g_1, g_2).
\]

The estimate on \( \| \tilde{B}_k^{(m)} \|_{\mathcal{B}^2}^2 \) follows in a similar fashion. \( \square \)

Combining the last two propositions leads to the following result:

Corollary 3.6. Under the assumptions of the preceding Proposition 3.5 the operator \( V := R_2^m(H_2 I - IH_1)R_1^m \) is trace class. Furthermore, \( \| V \|_{\mathcal{B}_1} \), the trace norm of \( V \), satisfies the estimate

\[
\| V \|_{\mathcal{B}_1} \leq 2C \tilde{d}_1(g_1, g_2),
\]

where the constant \( C \) depends only on \( m, n, p, \) and \( Q \).

We are now ready for the main theorem of this section. Recall that \( \text{Met}_{r_0}(M) \) consists of the set of metrics \( g \) on \( M \) that satisfy the lower bounds (2.9) for the injectivity radius and the Ricci-curvature in terms of a continuous function \( r_0 : M \to (0, 1] \). Also recall the definition of the wave operators \( W_\pm(H_2, H_1, I) \) in Section 2.2.

Theorem 3.7. Suppose we are given a smooth manifold \( M \) and a continuous function \( r_0 : M \to (0, 1] \). Let \( g_1, g_2 \in \text{Met}_{r_0}(M) \) denote two quasi-isometric Riemannian metrics on \( M \). Furthermore, we assume that the difference between \( g_1 \) and \( g_2 \) satisfies the \( r_0 \)-dependent weighted integral condition \( \tilde{d}_1(g_1, g_2) < \infty \), with \( \tilde{d}_1 \) as in (3.5).

Then the wave operators \( W_\pm(H_2, H_1, I) \) exist and are complete. Furthermore, the \( W_\pm(H_2, H_1, I) \) are partial isometries with initial space \( \mathcal{H}_{ac}(H_1) \) and final space \( \mathcal{H}_{ac}(H_2) \).
Proof. From Proposition 2.3 we obtain the lower bound \( r(M,g_1)(x) \geq cr_0(x) \) on the harmonic radii. We are now going to check the assumptions of the Birman-Belopol’skii theorem as given in Theorem 2.1 or [RS79, Thm. XI.13]: That \( I \) is bounded and has a bounded inverse is nothing but the equivalence of the \( L_2 \)-norms on \( L_2(M,g) \), cf. Proposition 3.2. Note that \( I^{-1} \) is simply the identification \( I^{-1}h = h \), for \( h \in L_2(M) \).

For the trace class condition, let \( m \in \mathbb{N} \) satisfy \( m \geq [n/4]+2 \) and let \( E_1(H) \) denote the spectral projection of \( H \) associated with a bounded interval \( I \subset \mathbb{R} \). As \( E_1(H)(H+1)^m \) is bounded, it follows from Corollary 3.6 that \( E_1(H)(H^2I-IH_1)E_1(H) \) is trace class.

Moreover, \( V_1 := (I^*-I)R_1^m = (q-1)R_1^m = T_1B_1^{(m)} \), where \( T_1 = q^{1/2} \text{sgn} S |S|^{1/2} \) is a bounded multiplication operator. From Proposition 3.5 we see, using again \( \tilde{d}_1(g_1,g_2) < \infty \), that \( B_1^{(m)} \) (and therefore \( V_1 \)) is Hilbert-Schmidt. In particular, \( (I^*-I)E_1(H_1) \) is compact.

We also have \( I(\text{Dom} \ h_1) = \text{Dom} \ h_2 \) (by Proposition 3.2), where \( \text{Dom} \ h_k \) is the quadratic form domain of \( H_k \). Note that we have used the quasi-isometry of \( g_1 \) and \( g_2 \) here. The desired results now follow from Theorem 2.1.

Remark 3.8. In typical applications to manifolds with ends it is evident that \( q(x) \rightarrow 1 \) “at infinity” in the sense that for each \( \varepsilon > 0 \) there is a compact subset \( M_\varepsilon \subset M \) such that \( |1-q(x)| < \varepsilon \) for all \( x \notin M \setminus M_\varepsilon \). The compactness of \( (I^*-I)(H_1+1)^{-m} \) is then immediate by local compactness of the Laplacian.

Remark 3.9. Let us comment on the result by Müller and Salomonsen [MS07], mentioned in the introduction: We first note that the assumptions of Theorem 0.1 of [MS07] already imply the assumptions of our Theorem 3.7, namely, \( g_1 \sim_\beta g_2 \) in the sense of [MS07] implies that the metrics are quasi-isometric (this follows from [MS07, Lem. 1.7]) and that \( \tilde{d}(g_1,g_2)(x) \leq C_1 |g_1(x) - g_2(x)|_{g_1(x)} \leq C_2 \beta(x) \). Here, \( \beta \) is a function called of moderate decay (actually, \( \beta \) is a function of \( d_{g_i}(x,p) \) for some reference point \( p \in M \)), and in particular bounded.

Müller and Salomonsen require a weighted bound on a modified injectivity radius which implies that \( \tilde{d}_1(g_1,g_2) < \infty \): From condition (iii) in their Thm. 0.1, it follows that the injectivity radius \( \text{inj}(M,g_1)(x) \) is bounded from below by \( r_0(x) := C_3 \beta(x)^{2a/(3n(n+2))} \) for some \( a \leq 1 \). Since the curvature is assumed to be bounded in Thm. 0.1 of [MS07], our condition on the Ricci curvature in (2.9) is automatically fulfilled. Moreover, by (ii) in Thm. 0.1, one has \( \beta^{b/3} \in L_1(M,g_1) \) with \( b = 2 - a \geq 1 \). The latter condition, together with \( d(g_1,g_2)(x) \leq C_2 \beta(x) \), implies that

\[
\tilde{d}_1(g_1,g_2) = \int_M \tilde{d}(g_1,g_2)(x)r_0(x)^{-2a/(3n+2)(1+\text{sgn} g_2,g_1(x))} dx \leq C_4 \int_M \beta(x)^{1-2a/(3n+2)} dx.
\]

Since \( 1-2a/(3n) = 1-4/(3n) + 2b/3n \geq b/3 \) for \( n \geq 2 \), and since \( \beta \) is bounded, it follows from \( \beta^{b/3} \in L_1(M,g_2) \) that \( \tilde{d}_1(g_1,g_2) < \infty \).

Let us also note that [MS07] require more regularity on the deviation of the metric \( g_2 \) from \( g_1 \). In fact, \( g_1 \sim_\beta g_2 \) means that the derivatives up to order 2 have to be close to each other with respect to a weight function \( \beta \); furthermore, the manifolds \( (M,g_i) \) are supposed to have curvature bounded up to order 2. Our assumptions here are weaker in the sense that there are only relative conditions on the metrics, i.e., the metrics \( g_1 \) and \( g_2 \) only have to be quasi-isometric and that \( \tilde{d}_1(g_1,g_2) < \infty \). No global boundedness assumption on the curvature of \( (M,g_i) \) has to be made, and no condition on the derivatives of \( g_1 \) and \( g_2 \).

4. Manifolds with ends

The general setup presented in this section is fairly standard and has been used in a similar way by many authors; cf. Remark 4.5 below. Let \( M \) be a smooth, orientable,
connected $n$-dimensional manifold and let $g$ be a metric on $M$ such that the Riemannian manifold $\mathcal{M} = (M, g)$ is complete. Our manifolds with ends are characterized by geometric and spectral assumptions. We first describe the geometry:

**Assumption 4.1.** We assume that $M$ can be decomposed into $\ell + 1$ open submanifolds $M_k$, $k = 0, \ldots, \ell$, where $M_0$ is compact and $M_k$, $k = 1, \ldots, \ell$, are non-compact. More precisely, we assume that the boundaries $\Sigma_k := \partial M_k = \overline{M}_k \cap \overline{M}_0$, $k = 1, \ldots, \ell$, are pairwise disjoint, smooth and compact manifolds of dimension $n - 1$; in particular,

$$M = \overline{M}_0 \cup M_1 \cup \cdots \cup M_\ell.$$

We denote the corresponding Riemannian manifolds by $\mathcal{M}_k = (M_k, g)$, $k = 0, \ldots, \ell$. In addition, we denote the boundaries, now considered as Riemannian manifolds, by $\partial \mathcal{M}_k = \mathcal{I}_k := (\Sigma_k, \nu^* g)$, $k = 1, \ldots, \ell$, where $\nu : \Sigma \hookrightarrow M$ denotes the natural embedding and $\nu^* g$ the induced metric.

**Remark 4.2.** If we allow metrics $g$ of class $C^{1, \alpha}_0(M)$, then the induced metric $\nu^* g$ is of class $C^{1, \alpha}(\partial M_k)$ which is sufficient for our purposes, especially for the proof of Proposition 4.6; cf. also [AKK+03] for a related discussion of regularity properties.

We next turn to the spectral assumptions. Let $H$ denote the Laplacian of $\mathcal{M}$ and let $H_k$ denote the Laplacian of $\mathcal{M}_k$, $k = 0, \ldots, \ell$, where $H_0$ satisfies Dirichlet boundary conditions on $\Sigma := \Sigma_1 \cup \cdots \cup \Sigma_\ell = \partial M_0$, while the $H_k$ satisfy Dirichlet boundary conditions on $\Sigma_k$. We can define $H_0$ as the self-adjoint operator associated with the quadratic form $h_0$ (cf. [K66, Thm. VI.2.1]), where $h_0$ is given by $h_0(u) := \|du\|_{L^2(\mathcal{M}_0)}^2$ and $\text{Dom } h_0 := H^1(\mathcal{M}_0)$. Moreover, the Sobolev space $H^1(\mathcal{M}_0)$ is the completion of $C_0^\infty(M_0)$ (functions with support away from $\partial M_0$) in the norm of $H^1(\mathcal{M}_0)$. The operators $H_k$, $k = 1, \ldots, \ell$, are defined in a similar way.

We will need the decoupled Laplacians $H_{\text{dec}} = \bigoplus_{k=0}^\ell H_k$ and $H'_{\text{dec}} = \bigoplus_{k=1}^\ell H_k$. Finally, let $I : L_2(\mathcal{M}) \to L_2(\mathcal{M})$ denote the natural embedding for $k = 0, \ldots, \ell$, and, similarly, let $I$ denote the natural embedding of $\bigoplus_{k=1}^\ell L_2(\mathcal{M}_k)$ into $L_2(\mathcal{M})$. The Laplacian $H_0$ of $\mathcal{M}_0$ has compact resolvent and thus purely discrete spectrum.

**Assumption 4.3.** In addition to Assumption 4.1 we assume that each of the decoupled Laplacians $H_k$, $k = 1, \ldots, \ell$, has a (non-trivial) absolutely continuous part.

**Remark 4.4.** Given Assumption 4.1 this is an assumption on the metric $g$. If Assumption 4.3 is satisfied, each of the ends of $\mathcal{M}$ will constitute a scattering channel. Assumption 4.3 is made chiefly for simplicity of notation later on. Indeed, it would be easy to adapt our results to the case where some of the ends do not participate in the scattering at all; e.g., an (infinite) horn may have purely discrete spectrum if it shrinks fast enough (cf. [Br89]). This point will be discussed further in Section 6.

**Remark 4.5.** As mentioned above, this setup can be considered standard. In fact, in many papers it is even assumed right from the beginning that each end of the manifold $\mathcal{M}$ is given as a warped product on $N \times (0, \infty)$ where $N$ is a compact manifold (cf., e.g., [DBHS92] for an early, and [IN10] for a recent reference). In particular, many examples are constructed using simple coordinate systems of this type, and our examples in Section 6 are no exception to that.

Let us also note that the concept of a scattering manifold in the sense of Melrose fits into our general scheme. Indeed, Melrose removes from a compact, smooth manifold a finite number of open sets $U_k$ (with $\partial U_k$ smooth and $\overline{U}_k$ pairwise disjoint, $k = 1, \ldots, \ell$), requiring the metric to become singular near $\partial U_k$ in a specific way. Here we may consider
open sets $V_k \supset U_k$, again with $\partial V_k$ smooth and $\nabla_k$ pairwise disjoint, to model the ends on $M_k := V_k \setminus U_k$; of course, the metric has to be singular near $\partial U_k$ in a certain sense.

We next discuss the decoupling by Dirichlet boundary conditions on the submanifold $\Sigma$. Similar decoupling arguments have been used (for the smooth case) in Birman [B62, B63], Weder [W84], Yafaev [Y92], Hempel and Weder [HW93], and Carron [Ca02], to name just a few. There are different approaches to check that a trace class condition is satisfied; for example, Carron [Ca02] uses techniques from pseudo-differential operators from Appendix B that are indispensable for our analysis anyway.

**Proposition 4.6.** Under the above Assumption $\mathbb{4.7}$ the wave operators

$$W_\pm(H, H_{\text{dec}}) = \lim_{t \to \pm \infty} e^{itH} e^{-itH_{\text{dec}}} P_{\text{ac}}(H_{\text{dec}})$$

and

$$W_\pm(H, H'_{\text{dec}}, I) = \lim_{t \to \pm \infty} e^{itH} I e^{-itH'_{\text{dec}}} P_{\text{ac}}(H'_{\text{dec}})$$

exist, are complete, and partially isometric. In particular,

$$\text{Ran} W_\pm(H, H_{\text{dec}}) = \mathcal{H}_{\text{ac}}(H),$$

where

$$\text{Ran} W_\pm(H, H_{\text{dec}}) = \text{Ran} W_\pm(H, H'_{\text{dec}}, I) = \bigoplus_{k=1}^{\ell} \text{Ran} W_\pm(H, H, I_k).$$

**Proof.** The proof is a modification of the proof of Theorem $\mathbb{5.7}$ and we only give a sketch. We first need some notation. For $k = 1, \ldots, \ell$, let $\mathcal{U}_k = (U_k, g)$ be a collar neighborhood of $\partial \mathcal{M}_k = \mathcal{J}_k$ in $\mathcal{M}_k$. Similarly, we let $\mathcal{U}_0$ denote a collar neighborhood of $\partial \mathcal{M}_0 = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_\ell$ in $\mathcal{M}_0$. Without loss of generality we may assume that $U_0, \ldots, U_\ell$ are relatively compact. For brevity, we denote the $(\ell + 1)$-tuple of these collar neighborhoods as $\mathcal{U}$ and we let $H^2(\mathcal{U})$ denote the $(\ell + 1)$-tuple of the second order Sobolev spaces $H^2(\mathcal{U}_k)$ with $k = 0, \ldots, \ell$. We define the boundary operators

$$\Gamma^1: H^1(\mathcal{M}) \to \mathcal{G}, \quad \Gamma^1 f := f|_{\Sigma},$$

$$\Gamma^2: H^2(\mathcal{U}) \to \mathcal{G}, \quad \Gamma^2 f := \partial_n f|_{\Sigma} + \partial_n f|_{\Sigma},$$

where $\mathcal{G} = L_2(\mathcal{J}) = \bigoplus_k L_2(\mathcal{J}_k)$, $\partial_n f = df \cdot n_\pm$, and where $n_\pm$ are unit vector fields on $\mathcal{U}_k$ resp. $\mathcal{U}_0$, normal to $\Sigma_k$ and pointing outwards of $\mathcal{M}_k$ resp. $\mathcal{M}_0$.

Defining $V$ via $\langle V \tilde{f}, \tilde{h} \rangle := \langle R^m_{\text{dec}} \tilde{f}, H R^m \tilde{h} \rangle - \langle H_{\text{dec}} R^m_{\text{dec}} \tilde{f}, R^m \tilde{h} \rangle$ for $\tilde{f}, \tilde{h} \in \mathcal{H} = L_2(\mathcal{M})$ and fixing $m \in \mathbb{N}$, $m \geq [n/4] + 2$, we then have

$$\langle V \tilde{f}, \tilde{h} \rangle = \langle f, H \tilde{h} \rangle - \langle H_{\text{dec}} f, \tilde{h} \rangle = \langle \Gamma^1 f, \Gamma^1 h \rangle_{\mathcal{G}} = \langle (\Gamma^2)^* \langle \Gamma^2 R^m_{\text{dec}} \tilde{f}, \tilde{h} \rangle$$

by Green’s formula, where $f = R^m_{\text{dec}} \tilde{f}$ and $h = R^m \tilde{h}$. We have to show that $A := \Gamma^2 R^m$ and $B := \Gamma^2 R^m_{\text{dec}}$ are Hilbert-Schmidt operators. Their integral kernels are given by $\Gamma^2 G^{(m)}(x, y)$ and $\Gamma^2 G^{(m)}_{\text{dec}}(x, y)$ respectively, for $x \in \Sigma$, $y \in M$, where $G^{(m)}$ resp. $G^{(m)}_{\text{dec}}$ denote the kernel of $R^m$ resp. $R^m_{\text{dec}}$. Let $C^1(\mathcal{M})$ denote $C^1(M)$ equipped with the usual Fréchet-topology generated by the semi-norms $p_K(u) := \sup_{x \in K} |u(x)| + |du(x)|_g$, with $K \subset M$ compact. Similarly, we let $C^1(\mathcal{M}_s)$ denote the space of functions $u \in C^1(\mathcal{M}_s)$ with the same family of semi-norms as above. The restrictions $\chi_{\mathcal{M}_s}$ define continuous embeddings of $C^1(\mathcal{M}_s)$ into $C^1(\mathcal{M}_s)$, for $s = 0, \ldots, \ell$. By Proposition $\mathbb{3.5}$, $R^m$ maps $\mathcal{H}$ continuously into $C^1(\mathcal{M})$, while $\chi_{\mathcal{M}_s} R^m_{\text{dec}}$ maps $\mathcal{H}$ continuously into $C^1(\mathcal{M}_s)$. As in the
proof of Proposition \[3.5\] it follows by the Riesz Theorem that there is a constant \( \tilde{C} > 0 \) such that for all \( x \in \Sigma \)

\[
\int_M |\Gamma_x G(m)(x,y)|^2 \, d\text{vol}_g(y) \leq \tilde{C}, \quad \int_M |\Gamma_x G'(m)(x,y)|^2 \, d\text{vol}_g(y) \leq \tilde{C}.
\]

Since \( \Sigma \) is compact, and \( A \) and \( B \) are Hilbert-Schmidt and the result follows. \hfill \Box

We now discuss incoming and outgoing states. It follows from \[1.3\] and \[4.3\] that for each state \( f \in \mathcal{H}_\text{ac}(H) \) there are vectors \( g_{k,\pm} \in \mathcal{H}_\text{ac}(H_k) \), \( k = 1, \ldots, \ell \), such that

\[
\left\| e^{\pm i t H} f - \bigoplus_{k=1}^\ell I_k e^{\pm i t H_k} g_{k,\pm} \right\| \to 0, \quad t \to \pm \infty.
\]

(4.5)

We call \( \text{Ran} W_+(H, H_k, I_k) \) the \textit{outgoing} (for \(+\)) and the \textit{incoming} subspace (for \(-\)) of the \( k \)-th scattering channel. In this sense, scattering on \( \mathcal{M} \) can be understood as an interaction of \( \ell \) scattering channels. If \( K \subset M \) is compact and \( f \in \mathcal{H}_\text{ac}(H) \), then \( \chi_K e^{itH} f \to 0 \), as \( t \to \pm \infty \), by the usual arguments.

In many cases (and in particular in concrete examples), one would like to describe the long-time behavior of the system by suitable \textit{asymptotes}: for any given channel \( \mathcal{M}_k \), \( k \in \{1, \ldots, \ell\} \), there may exist a (simple) \textit{comparison dynamics} given by an operator \( h_k \) acting in a Hilbert space \( \mathcal{H}_k \), which can be used to describe the asymptotic evolution of \( e^{-itH_k} f \) as \( t \to \pm \infty \), for \( f \) in the absolutely continuous subspace of \( H_k \). More precisely, suppose there is a bounded operator \( j_k : \mathcal{H}_k \to L_2(\mathcal{M}_k) \) such that the wave operators

\[
W_\pm(H_k, h_k, j_k) := \text{s-lim}_{t \to \pm \infty} e^{itH_k} j_k e^{-it h_k} P_{\text{ac}}(h_k)
\]

exist, are partially isometric, and complete, so that, in particular,

\[
\text{Ran} W_\pm(H_k, h_k, j_k) = \mathcal{H}_\text{ac}(H_k).
\]

If we assume, for the moment, that such reference operators \( h_k \) exist for all \( k = 1, \ldots, \ell \), the chain rule implies that

\[
W_\pm(H, \bigoplus_{k=1}^\ell h_k, I_j) = W_\pm(H, H'_\text{dec}, I) \circ W_\pm(H'_\text{dec}, \bigoplus_{k=1}^\ell h_k, j)
\]

\[
= W_\pm(H, H'_\text{dec}, I) \circ \left( \bigoplus_{k=1}^\ell W_\pm(H_k, h_k, j_k) \right),
\]

where \( j \) is the direct sum of the \( j_k \). In view of \[4.4\] we then see that

\[
\mathcal{H}_\text{ac}(H) = \bigoplus_{k=1}^\ell \text{Ran} W_\pm(H, h_k, I_k j_k).
\]

(4.7)

Note that the definition of the incoming/outgoing subspaces is independent of the choice of the comparison dynamics given by the pairs \( (\mathcal{H}_k, h_k) \).

Standard examples for ends include (asymptotically) Euclidean ends where we may take the (flat) Laplacian on \( \mathbb{R}^n \) to define the comparison dynamics, half-cylinders, cusps or horns, funnels, etc. In many of these examples the notion of “incoming/outgoing” corresponds to the geometric notion of coming in from infinity or going out to infinity; cf. Eq. \[4.5\]. Some of these examples will be discussed in more detail in Section \[6\].

Abstracting from the above situation, we use the following terminology:

**Definition 4.7.** Let \( H \) and \( \mathcal{M} \) satisfy Assumptions \[4.1\] and \[4.3\]. Suppose we are given Hilbert spaces \( \mathcal{H}_k \) and self-adjoint operators \( h_k \) acting in \( \mathcal{H}_k \), for \( k = 1, \ldots, \ell \). We say that \( h := \bigoplus_k h_k \) is a \textit{reference operator with \ell channels} for the Laplacian \( H \) on \( \mathcal{M} \), if there are bounded operators \( J_k : \mathcal{H}_k \to L_2(\mathcal{M}_k) \) such that the following holds:
If we define the identification operator $J \colon \mathcal{H} = \bigoplus_k \mathcal{H}_k \to L_2(\mathcal{M})$ by $Jf := \sum_k Jkf_k$ then the wave operators $W_\pm(H, h, J)$ exist and are complete, and they are partially isometric with initial space $\bigoplus_k \mathcal{H}_0(h_k)$ and final space $\mathcal{H}_0(h)$. We then define the associated scattering operator by

$$S = S(H, h, J) := W_+^*(H, h, J) \circ W_-(H, h, J) : \mathcal{H} \to \mathcal{H}. \quad (4.8)$$

**Remark 4.8.** Note that we always have the trivial choice $h_k = H_k$, $\mathcal{H}_k = L_2(\mathcal{M}_k)$ which leads to the scattering operator $S(H, H_0')$ with $H_0'$ and $I$ as in the beginning of this section. While the scattering operators $S(H, h, J)$ depend on the choice of the reference operators, the chain rule implies that they are all unitarily equivalent to $S(H, I', I)$. In many instances a “natural” choice of a reference dynamics can be derived from the geometry of the ends, cf. e.g. the recent work of Ito and Skibsted [ISk13b] and the literature quoted there.

In order to study the interaction of the channels, we introduce the scattering matrix $(S_{ik})_{i,k=1,\ldots,\ell}$ where

$$S_{ik} = W_+^*(H, h_i, J_i) \circ W_-(H, h_k, J_k) : \mathcal{H}_k \to \mathcal{H}_i;$$

in particular, $Sf = \left(\sum_{k=1}^{\ell} S_{ik}f_k\right)_{i=1,\ldots,\ell}$ for all $f \in \mathcal{H}$. Clearly, $S_{ik} \neq 0$ for a given pair of indices $(i, k)$ is equivalent to the fact that some incoming states which for $t \to -\infty$ asymptotically lie in the $k$-th end have a non-zero asymptotic part in the $i$-th end, as $t \to +\infty$. Put differently, $S_{ik} \neq 0$ is equivalent to having non-zero transmission from the $k$-th into the $i$-th channel or end, i.e., the $k$-th scattering channel is open to the $i$-th channel.

Note that $S_{ik} \neq 0$ is equivalent to $\text{Ran} W_-(H, h_k, J_k) \cap \text{Ran} W_+(H, h_i, J_i) \neq \{0\}$. Furthermore, $\text{Ran} W_-(H, h_k, J_k) = \text{Ran} W_+(H, h_k, J_k)$ implies that $S_{ik} = 0$ for all $i \neq k$. Conversely, suppose that $S_{ik} \neq 0$ for some $i \neq k$, then, necessarily, $\text{Ran} W_-(H, h_k, J_k) \neq \text{Ran} W_+(H, h_k, J_k)$.

The property $S_{ik} \neq 0$ is symmetric in the indices $i$ and $k$, as we will show now: Denoting by $T$ the operator of complex conjugation, $T\varphi(x) = \varphi(x)$ for complex-valued functions $\varphi$ ($T$ is the operator of time reversal in Quantum Mechanics), we clearly have

$$Te^{-itH} = e^{itH}T, \quad Te^{-ith_k} = e^{ith_k}T,$$

and it follows that

$$TW_\pm(H, h_k, J_k) = W_\pm(H, h_k, J_k)T, \quad TW_\pm^*(H, h_k, J_k) = W_\pm^*(H, h_k, J_k)T.$$

Then

$$TS_{ik} = TW_\pm^*(H, h_i, J_i) \circ W_-(H, h_k, J_k) = W_\pm^*(H, h_i, J_i) \circ W_+(H, h_k, J_k)T = S_{ ik}^* T;$$

as a consequence, $S_{ik} \neq 0 \iff S_{ki}^* \neq 0 \iff S_{ki} \neq 0$. We are thus justified in saying that the $i$-th and the $k$-th scattering channel are open to one another.

In the next section, we will show that the property $S_{ik} \neq 0$ is stable under small perturbations of the metric.

5. **Non-zero transmission under perturbation of the metric**

In this section, we prove strong continuity of the scattering matrix $S$ with respect to perturbations of the metric. This immediately implies the result mentioned at the end of Section 4 on the stability of the openness of scattering channels.

Let $M$ be as in Assumption 4.1 and let $g_0$ be a metric on $M$ satisfying Assumption 4.3. we consider $g_0$ as being fixed. Let $r_0 > 0$ be a function satisfying (2.9) for $\mathcal{M}_0 = (M, g_0)$, and let $H_0$ denote the Laplacian of $\mathcal{M}_0$. Let $h_0 = \bigoplus_{k=1}^{\ell} h_k$ denote a reference operator for $H_0$ with $\ell$ scattering channels as in Definition 4.7; in particular, there is a Hilbert
space $\mathcal{H}_0$ and a bounded operator $J: \mathcal{H}_0 \to L_2(\mathcal{M})$ such that the wave operators $W_{\pm}(H_0, h_0, J)$ exist and are complete. We may then define the associated scattering operator $S(H_0, h_0, J)$ as in Eq. (1.8).

We first describe the set of admissible metrics that are close to the metric $g_0$. Recall the definitions of $\tilde{d}_\infty$ and $\tilde{d}_1$ in Section 3 equ. (3.1), (3.2), and (3.3). For $\gamma > 0$ and $\varepsilon > 0$ we set

\[
\text{Met}_{\gamma, \varepsilon}(g_0) := \{ g \in \text{Met}_{\gamma, \varepsilon}(M) \mid \tilde{d}_\infty(g_0, g) \leq \gamma, \quad \tilde{d}_1(g_0, g) \leq \varepsilon \},
\]

i.e., $\text{Met}_{\gamma, \varepsilon}(M, g_0)$ is the set of smooth metrics $g$ enjoying the following properties:

(i) The homogenized Ricci curvature and the homogenized injectivity radius of $g$ are controlled locally from below by the function $r_0$, cf. (2.9).

(ii) The metric $g$ is quasi-isometric to $g_0$ ($\tilde{d}_\infty(g_0, g) \leq \gamma < \infty$, cf. Remark A.2).

(iii) The weighted $L_1$-quasi-distance $\tilde{d}_1(g_0, g)$ is smaller than or equal to $\varepsilon$. The quasi-distance is defined in (3.5) with respect to the weight function $r_0$.

We comment on the structure of the space $\text{Met}_{\gamma, \varepsilon}(M, g_0)$ of admissible metrics later on in Remark 5.6.

Let us now present a stability result for the scattering operator under perturbation of the metric:

**Theorem 5.1.** Let $g_0$, $H_0$, $h_0$ and $S(H_0, h_0, J)$ as above. For $g_\varepsilon \in \text{Met}_{\gamma, \varepsilon}(M, g_0)$, denote by $H_\varepsilon$ the Laplacian of $\mathcal{M}_\varepsilon = (M, g_\varepsilon)$ and by $I_\varepsilon: L_2(\mathcal{M}_0) \to L_2(\mathcal{M}_\varepsilon)$ the natural identification. Then $h_0$ is also a reference operator for $H_\varepsilon$, and the associated scattering operators $S(H_\varepsilon, h_0, I_\varepsilon J)$ converge strongly to $S(H_0, h_0, J)$, as $\varepsilon \to 0$.

**Remark 5.2.** In particular, it follows that, for $\varepsilon > 0$ small, all $\ell$ channels constitute a scattering channel for $H_\varepsilon$. Moreover, if the $k$-th and the $i$-th channels are open to each other for $H$ they will also be open to each other for $H_\varepsilon$. Conversely, suppose there are $k_0, i_0 \in \{1, \ldots, \ell\}$ such that the $k_0$-th and the $i_0$-th channels are not open to each other for a sequence $(H_{\varepsilon_j})_{j \in \mathbb{N}}$ where $\varepsilon_j \to 0$ as $j \to \infty$. Then, the $k_0$-th and the $i_0$-th channels are also not open to each other for $H$.

**Proof of Theorem 5.1.** It follows from Theorem 3.7 that the wave operators $W_{\pm}(H_\varepsilon, H_0, I_\varepsilon)$ exist, are complete and partially isometric. The same is true for the wave operators $W_{\pm}(H_0, h_0, J)$ by Definition 4.7, with $J: \mathcal{H}_0 \to L_2(\mathcal{M}_0)$ as above. By the chain rule, we have existence and completeness of the wave operators

\[
W_{\pm}(H_\varepsilon, h_0, I_\varepsilon J) = W_{\pm}(H_\varepsilon, H_0, I_\varepsilon) \circ W_{\pm}(H_0, h_0, J).
\]

In particular, $h_0$ is also a reference operator for $H_\varepsilon$, and the associated scattering operator is given by

\[
S_\varepsilon := S(H_\varepsilon, h_0, I_\varepsilon J) = (W_+(H_\varepsilon, h_0, I_\varepsilon J))^* \circ W_-(H_\varepsilon, h_0, I_\varepsilon J).
\]

For $u, v \in \mathcal{H}_{ac}(h_0)$, we then have

\[
\langle S_\varepsilon u, v \rangle_{\mathcal{H}_0} = \langle W_-(H_\varepsilon, H_0, I_\varepsilon) \cdot W_-(H_0, h_0, J) u, W_+(H_\varepsilon, H_0, I_\varepsilon) \cdot W_+(H_0, h_0, J) v \rangle_{L_2(\mathcal{M}_\varepsilon)} = \langle I_\varepsilon^* \cdot W_-(H_\varepsilon, H_0, I_\varepsilon) \cdot W_-(H_0, h_0, J) u, I_\varepsilon^* \cdot W_+(H_\varepsilon, H_0, I_\varepsilon) \cdot W_+(H_0, h_0, J) v \rangle_{L_2(\mathcal{M}_0)}.
\]

Using Lemma 5.3 below, we find that

\[
\langle S_\varepsilon u, v \rangle_{\mathcal{H}_0} \to \langle P_{ac}(H_\varepsilon) \circ W_-(H_0, h_0, J) u, P_{ac}(H_\varepsilon) \circ W_+(H_0, h_0, J) v \rangle_{L_2(\mathcal{M}_0)} = \langle S_0 u, v \rangle_{\mathcal{H}_0}
\]

as $\varepsilon \to 0$, where $S_0 = S(H_0, h_0, J)$. The strong convergence follows from the fact that the operators $S(H_\varepsilon)$ are unitary.

\footnote{For less regular metrics, cf. Remark 2.6}
As we observed in Remark 5.2, Theorem 5.1 immediately gives the following stability result for the scattering matrix:

**Corollary 5.3.** Let $M$, $g_0$ and $H_0$ as above and suppose that for a given pair of indices $i, k \in \{1, \ldots, \ell \}$ we have

$$S_{ik}(H_0, h_0, J) \neq 0. \quad (5.2)$$

Then for any $\gamma > 0$ fixed, there exists $\varepsilon_0 > 0$ such that $S_{ik}(H, h_0, I_\varepsilon J) \neq 0$ for all metrics $g_\varepsilon \in \text{Met}_r\langle M, g_0, \gamma, \varepsilon \rangle$ and all $0 < \varepsilon \leq \varepsilon_0$.

**Remark 5.4.** (a) As discussed in Remark 4.8, the property $S_{ik}(H_0, h_0, J) \neq 0$ is independent of the choice of the reference operator $h_0$.

(b) It is not easy to establish property (5.2) in concrete situations. We will give simple examples with $\ell = 2$ in Section 4 where we exploit rotational symmetry.

The following lemma has been used in the proof of Theorem 5.1. We consider here a family of metrics $(g_\varepsilon)_\varepsilon$ converging to a metric $g_0$ in the sense of (5.1), so that, in particular, $\hat{d}_1(g_\varepsilon, g_0) \leq \varepsilon$.

**Lemma 5.5.** With the assumptions and notation of Theorem 5.1, let $H_\varepsilon$ be the Laplacian of $\mathcal{M}_\varepsilon = (M, g_\varepsilon)$ associated with a metric $g_\varepsilon \in \text{Met}_r\langle M, g_0, \gamma, \varepsilon \rangle$, for $0 < \varepsilon < \varepsilon_0$. Then

$$\lim_{\varepsilon \to 0} \langle I_\varepsilon \rangle W_\varepsilon(H_\varepsilon, H_0, I_\varepsilon) = P_{ac}(H_0). \quad (5.3)$$

Proof. For $\varepsilon \geq 0$ set $R_\varepsilon := (H_\varepsilon + 1)^{-1}$, and let $m := [n/4] + 2$. Defining $V_\varepsilon$ in analogy with $V$ in (3.4) by

$$V_\varepsilon := R_\varepsilon^m (H_\varepsilon I_\varepsilon - I_\varepsilon H_0) R_0^m,$$

it follows from our assumptions and Corollary 3.9 that $\|V_\varepsilon\|_{L^1} \to 0$ as $\varepsilon \to 0$. This implies (cf. the Corollary following Theorem X.7 in [RS79]) that

$$\|W_\varepsilon(H_\varepsilon, H_0, I_\varepsilon) R_\varepsilon^m \varphi - R_\varepsilon^m I_\varepsilon R_\varepsilon^m P_{ac}(H_0) \varphi\| \leq 16\pi \|V_\varepsilon\|_{L^1} \cdot \|\varphi\|^2 \cdot \|I_\varepsilon\| \to 0 \quad (5.4)$$

as $\varepsilon \to 0$, for all $\varphi \in \mathcal{M}(H_0)$ that satisfy $\|\varphi\| < \infty$; the notation is as in [RS79]. Note that, by assumption and Proposition 3.2, $\|I_\varepsilon\| \leq 1 + \tilde{d}_1(g_0, g_\varepsilon) \leq 1 + \gamma$ independently of $\varepsilon$. By the intertwining relations [RS79],

$$W_\varepsilon(H_\varepsilon, H_0, I_\varepsilon) R_\varepsilon^m = W_\varepsilon(H_\varepsilon, H_0, R_\varepsilon^m I_\varepsilon R_\varepsilon^m). \quad (5.5)$$

Furthermore, the arguments used in Section 3 yield

$$\|R_\varepsilon^m I_\varepsilon R_0^m - I_\varepsilon R_0^m\| \to 0, \quad \varepsilon \to 0; \quad (5.6)$$

a proof of Eq. (5.6) will be given below. We conclude from (5.4)–(5.6) that for all $\varphi$ with $\|\varphi\| < \infty$

$$\|W_\varepsilon(H_\varepsilon, H_0, I_\varepsilon) R_\varepsilon^m \varphi - I_\varepsilon P_{ac}(H_0) R_\varepsilon^m \varphi\| \to 0, \quad \varepsilon \to 0.$$ 

The set of vectors $\{ R_\varepsilon^m \varphi \mid \varphi \in L^2(\mathcal{M}_\varepsilon), \|\varphi\| < \infty \}$ is dense and (5.3) follows.

It remains to prove (5.6). Here we first note that, by a standard expansion,

$$(R_\varepsilon^m I_\varepsilon - I_\varepsilon R_0^m) R_0^m = - \sum_{j=1}^m R_\varepsilon^j (H_\varepsilon I_\varepsilon - I_\varepsilon H_0) R_0^{2m-j-1};$$

therefore, it is clearly enough to show that $\|R_\varepsilon(H_\varepsilon I_\varepsilon - I_\varepsilon H_0) R_0^{m+1}\| \to 0$, as $\varepsilon \to 0$. By a simple variant of Lemma 5.4 we have

$$R_\varepsilon(H_\varepsilon I_\varepsilon - I_\varepsilon H_0) R_0^{m+1} = (\hat{B}_\varepsilon^{(1)})^* \hat{U} \hat{B}_\varepsilon^{(m+1)} - (B_\varepsilon^{(1)})^* U B_\varepsilon^{(m)} R_0 H_0,$$

where $B_\varepsilon^{(m)} = |S_\varepsilon|^{1/2} R_\varepsilon^m$ and $\hat{B}_\varepsilon^{(m+1)} = |\hat{S}_\varepsilon|^{1/2} D_\varepsilon R_\varepsilon^{m+1}$ (see Section 3). From Proposition 3.3 we now conclude

$$\|R_\varepsilon(H_\varepsilon I_\varepsilon - I_\varepsilon H_0) R_0^{m+1}\|_{L^2} \leq 2(\gamma C\varepsilon)^{1/2} \to 0$$
as $\varepsilon \to 0$, using the norm bound $\|B^{(1)}_\varepsilon\|_2^2 \leq \|S_\varepsilon\|_\infty \leq \gamma$ (see Lemma 3.3) and similarly for $\tilde{B}^{(1)}_\varepsilon$. Moreover, the convergence in Hilbert-Schmidt norm implies the convergence in operator norm and \ref{5.6} follows. \hfill $\square$

To conclude this section, let us comment on the metric structure of the spaces of metrics used so far.

Remark 5.6. (a) On $\text{Met}(M, g_0, \gamma) := \{ g \in \text{Met}(M) \mid \tilde{d}_\infty(g_0, g) \leq \gamma \}$, the distance function $\tilde{d}_\infty$ is a quasi-distance with constant $\tau = 1 + \gamma/2$ (cf. Definition A.1 and what is said before that definition). A quasi-distance induces a unique topology and a uniform structure (cf. the comment after Definition A.1).

(b) Moreover, $\tilde{d}_1$ is a quasi-distance on the subspace of the metrics $g \in \text{Met}(M, g_0, \gamma)$ for which $\tilde{d}_1(g_0, g)$ is finite. Indeed, straightforward calculations, using \ref{A.4} for the estimates on $\rho$ and the fact that $\tilde{d}$ is a quasi-distance with constant $\tau = 1 + \gamma/2$, show that

$$\tilde{d}_1(g_1, g_2) \leq \tau(1 + \gamma)^4(\tilde{d}_1(g_1, g_2) + \tilde{d}_1(g_2, g_3)).$$

On $\text{Met}(M, g_0, \gamma)$, we can also work with the quasi-distance

$$\tilde{d}_1^*(g_1, g_2) := \int_M \tilde{d}(g_1, g_2) r_0^{-\eta(n+2)} \, d\text{vol}_{g_0}$$

(5.7)

(with constant $\tau$) which is equivalent to $\tilde{d}_1$. We prefer to work with $\tilde{d}_1$ since this is the quasi-distance appearing naturally in the estimates of Proposition 3.5 and Corollary 3.6.

The set $\text{Met}(M, g_0, \gamma, \varepsilon) = \{ g \in \text{Met}(M, g_0, \gamma) \mid \tilde{d}_1(g_0, g) \leq \varepsilon \}$ is now the closed $\varepsilon$-ball with respect to the quasi-distance $\tilde{d}_1$, and the set $\text{Met}^{\varepsilon}(M, g_0, \gamma, \varepsilon)$ defined above is the intersection of this $\varepsilon$-ball with the space $\text{Met}^{\varepsilon}(M)$ of metrics fulfilling the (local) lower bounds \ref{2.5} on the homogenized Ricci curvature and injectivity radius.

(c) On the $\varepsilon$-ball $\text{Met}_\varepsilon(M, g_0, \gamma, \varepsilon)$, the wave operators $W_{\pm}(\Delta_{(M,g_0)}, \Delta_{(M,g_1)}, I)$ exist and are complete for any two metrics $g_1, g_2$ in this ball due to Theorem 3.7. Theorem 5.1 can be restated as saying that the map $g \mapsto S(\Delta_{(M,g)}, h_0, IJ)$ (with $I : L_2(M, g_0) \rightarrow L_2(M, g), \, I f = f$), associating to a metric the scattering operator of $\Delta_{(M,g)}$ and a reference operator $h_0$ (see Definition A.7), is continuous with respect to the quasi-distance $\tilde{d}_1$ and the topology of strong convergence of operators in $\mathcal{B}(\mathcal{H}_0)$.

Moreover, Corollary 3.3 can be restated by saying that the set of metrics $g$ such that the $k$-th and the $i$-th channel of the scattering operator $S(\Delta_{(M,g)}, h_0, IJ)$ are open to each other, is a neighborhood of $g_0$ (i.e., it contains a ball of radius $\varepsilon_0$ around $g_0$) in the quasi-distance space $(\text{Met}^{\varepsilon}(M, g_0, \gamma), \tilde{d}_1)$.

It would be interesting to analyze in more detail the structure of $\text{Met}(M, g_0, \gamma, \varepsilon)$ and $\text{Met}_\varepsilon(M, g_0, \gamma, \varepsilon)$.

We conclude this section with some remarks of a more general nature concerning the question of openness of scattering channels.

Remark 5.7. (a) One might conjecture that an end can only be closed if the (decoupled) Laplacian of this end has no absolutely continuous spectrum. In particular, this would mean that there are no geometric or topological obstructions that might prevent sending wave packets from the $i$-th channel into any of the other channels. We are not aware of any counter-examples.

(b) Another conjecture would say that—as long as the ends have some absolutely continuous spectrum—condition \ref{1.1} holds in a generic sense. Our work provides a step in this direction since we show that the set of metrics enjoying property \ref{1.1} for all pairs $(i, k)$ is open in the sense of Remark 5.6 (c); whether this set is also dense is a rather difficult question.
(c) In this paper we consider the openness of the scattering channels in a global sense, i.e., we ask—without imposing any restrictions on the energy of the wave packets—whether the scattering channels are open. From a physical point of view, the following question of local openness would also be of great interest. Suppose we restrict our attention to a compact set with respect to energy, i.e., we ask—without imposing any restrictions on the energy of the wave packets—which wave packets are open for wave packets with energy in a compact set, $K$, are no longer open for such wave packets if the metric is perturbed in such a way that the set $K$ and the absolutely continuous spectrum of the “receiving channel” become disjoint.

(d) Strong distortions of the metric on one of the ends may destroy the a.c. spectrum there and then this end (with the new metric) would be closed for scattering. It is easy to construct examples of the type discussed in Section 6 below where the radial function $s \mapsto r(s)$ is distorted in such a way that the corresponding potential $w$ in eqn. (6.3) changes from short range to a potential that tends to $+$ as $s \to \infty$; consider, e.g., $r(s) := e^{-s^2}$ for the distorted radial function.

It is a much harder question to ask whether a channel can close while staying within a given class $\text{Met}(M, g_0, \gamma, \varepsilon_0)$, maybe with some large $\varepsilon_0$; note that for metrics from this class the a.c. spectrum is stable.

6. Examples

In this section, we present some examples where our main results, Theorem 3.7 and Corollary 5.3, can be applied. We only consider manifolds with two ends; dealing with more than two ends would require additional efforts.

6.1. Surfaces of revolutions and warped products. As a preparation, we first recall and establish some facts on manifolds which are symmetric with respect to rotation around some axis (surfaces of revolution for $n = 2$). Such manifolds are special cases of warped products [O’N83]. In some of our examples such manifolds are used to find a suitable function $r_0$ and in other examples we consider perturbations of manifolds of this type.

A particularly simple case is given by the following situation: Let $\mathcal{M}_0$ denote an $n$-dimensional manifold in $\mathbb{R}^{n+1}$ which is symmetric with respect to rotation around the $x_{n+1}$-axis and homeomorphic to $\mathbb{R} \times S^{n-1}$. Writing $\bar{x} = (x_1, \ldots, x_n)$, we introduce coordinates $(s, \xi)$, $\xi = |\bar{x}|^{-1}\bar{x} \in S^{n-1}$ for $\bar{x} \neq 0$, where $s$ is arc-length along any line $\xi = \text{const.}$ Defining $r(s) = |\bar{x}|$, we assume that $r$ is a positive function of class $C^2$. In these coordinates, the Riemannian manifold $\mathcal{M}_0 = (M, g_0)$ is given by $M = \mathbb{R} \times S^{n-1}$ and $g_0 = ds^2 + r(s)^2g_{S^{n-1}}$, where $g_{S^{n-1}}$ denotes the standard metric on $S^{n-1}$. More generally, we can start with a metric $g_0$ and define the Riemannian manifold $(M, g_0)$ abstractly, without referring to the ambient space $\mathbb{R}^{n+1}$ (this is necessary, e.g., if $r(s)$ grows fast). Some of the results in this section remain true if we replace $S^{n-1}$ by any compact Riemannian manifold $Y$; for simplicity, we only treat the case $Y = S^{n-1}$ here.

As is well known, there is a unitary operator $U: L_2(\mathcal{M}) \to L_2(\mathbb{R}, L_2(S^{n-1}))$ with the property that

$$H = U^* \tilde{H} U, \quad \tilde{H} = \bigoplus_{m=0}^{\infty} \tilde{H}_m,$$

where

$$\tilde{H}_m = -\frac{d^2}{ds^2} + w(r) + \frac{\lambda_m}{r^2}, \quad m \in \mathbb{N}_0,$$
a self-adjoint operator in $L_2(\mathbb{R}, E_m)$, and

$$w(r) = \left(\frac{n-1}{2}\right)^2 q + \left(\frac{n-1}{2}\right)^2 q^2, \quad q = \frac{\hat{r}}{r};$$

(6.3)

here, $E_m$ denotes the eigenspace associated with $\lambda_m$, the $m$-th eigenvalue of the Laplacian on the sphere $S^{n-1}$. We have $\dim E_0 = 1$ and $\lambda_0 = 0$ with constant eigenfunction. The Ricci curvature (viewed as a symmetric tensor on $T^*M$) is given by

$$\text{Ric} = -(n-1)(\dot{q} + q^2) ds^2 + \left(\frac{n-1}{r^2} - (\dot{q} + (n-1)q^2)\right) r^2 g_{S^{n-1}}.$$ \(\text{Similarly, we obtain for the sectional curvature}

$$K(\partial_s, \partial_{\xi_i}) = -\frac{\dot{j}}{r} \quad \text{and} \quad K(\partial_{\xi_j}, \partial_{\xi_k}) = \frac{1-\dot{j}^2}{r^2}$$

for $i \neq j$ (provided $n \geq 3$), where $\{\partial_{\xi_j}\}_j$ is a basis of $T_xS^{n-1}$; cf., e.g., [O'N83, p. 209ff].

In particular, if $n = 2$, we have

$$\text{Ric} = -\dot{q} g^2 = -\frac{\dot{j}}{r} g^2 \quad \text{and} \quad K(\partial_s, \partial_{\xi}) = -\frac{\dot{j}}{r}.$$ \(6.2. \text{Reference operators on the two ends.} \text{We decouple } \mathcal{M} = (M, g) \text{ into just two pieces by a surface } \Sigma \subset M \text{ which corresponds to } s = 0. \text{ With the same unitary operator } U \text{ as in } [6.1], \text{ the decoupled operators } H_{\text{dec}} \text{ and } H_{\text{dec,m}} \text{ satisfy}

$$H_{\text{dec}} = U^* H_{\text{dec}} U, \quad H_{\text{dec}} = \bigoplus_{m=0}^{\infty} \tilde{H}_{\text{dec},m},$$

with

$$\tilde{H}_{\text{dec},m} = \tilde{H}_{-,m} \oplus \tilde{H}_{+,m};$$

here $\tilde{H}_{-,m}$ and $\tilde{H}_{+,m}$ are, respectively, the operator [6.2] in $L_2(\mathbb{R}, E_m)$ and in $L_2(\mathbb{R}, E_m)$ with Dirichlet boundary condition at $s = 0$, and $\mathbb{R}_- := (-\infty, 0)$ and $\mathbb{R}_+ := (0, \infty)$. \text{We then have:}

(a) Applying a celebrated result of Deift and Killip [DK99] to our situation, we see that $\tilde{H}_{+,m}$ (respectively, $\tilde{H}_{-,m}$) has absolutely continuous (a.c.) spectrum $[0, \infty)$, provided the potential $w + \lambda_m/r^2$ is square integrable over $(1, \infty)$ (respectively, over $(-\infty, -1)$).

(b) If $r(s) \to 0$ as $s \to \pm\infty$, the decoupled operators $H_{\pm,m}$ have purely discrete spectrum for $m \geq 1$ (while $H_{\pm,0}$ still may have a non-trivial a.c. part). Br"{u}ning [Br89] provided conditions for Laplacians on cusps that guarantee purely discrete spectrum.

(c) We will mostly work with the following assumption on $w$: we say that $w$ is short range on $\mathbb{R}$ if there exists a constant $\alpha > 1$ such that

$$|w(s)| \leq C(1 + |s|)^{-\alpha}. \quad (6.4)$$

Under this assumption, the Enss method yields existence and completeness of the wave operators for the pair $(h_0, h_0 + w)$ and the absence of singular continuous spectrum for $h_0 + w$, with $h_0$ denoting the unique self-adjoint realization of $-d^2/ds^2$ in $L_2(\mathbb{R})$ with domain $H^2(\mathbb{R})$; cf., e.g., [E78, S79]. Analogous results hold for the operators $H_{\pm,m}$, $m \geq 1$, provided $w + \lambda_m/r^2$ is short-range.

The cases where $r$ is of the form $r(s) = s^\beta$, for some $\beta \in \mathbb{R}$ and $s \geq 1$, are particularly simple. Let

$$r(s) := \tau s^\beta, \quad s \geq 1,$$

for some $\beta \in \mathbb{R}$ and $\tau > 0$. In this case,

$$q(s) = \frac{\beta}{s^\beta}, \quad w(s) = \left(\frac{n-1}{2}\beta - 1\right)^{\frac{n-1}{2}} \frac{1}{s^\beta}.$$
and the potentials $w$ are short-range. Let us describe the geometry of such an end in more detail in the case $n = 2$. The Ricci curvature is then given by

$$\text{Ric} = - (\beta - 1) \beta \frac{1}{s^2} \cdot g.$$ 

In particular, the curvature tends to zero as $s \to \infty$.

If $\beta > 1$, the end is large and negatively curved. If $\beta = 1$, we have a Euclidean (flat) end for $\tau = 1$ and a cone for $0 < \tau < 1$. If $0 < \beta < 1$, we have a positively curved parabolic-type end. If $\beta = 0$, we have a flat cylinder of radius $\tau > 0$, and if $\beta < 0$, we have a negatively curved shrinking horn.

On the real line, one may combine different asymptotics of the above type for $\beta > 1$, the end is large and negatively curved. If $0 < \beta < 1$, the end is large and negatively curved. If $\beta = 0$, the end is large and negatively curved. If $\beta < 0$, the end is large and negatively curved.

Example 6.1 (Existence and completeness of wave operators). Suppose we are given a manifold $\mathcal{M} = (M, g_0)$ with a warped product metric $g_0 = ds^2 + r(s)^2 dg_{S^{n-1}}$ satisfying the following conditions: $r \in C^2(\mathbb{R})$, the functions $\dot{r}/r$ and, for $n \geq 3$, $(1 + \dot{r}^2)/r^2$, are bounded, and there exists a constant $m \geq 1$ such that

$$\frac{1}{m} r(s_0) \leq r(s) \leq m r(s_0), \quad \forall s \in [s_0 - 2, s_0 + 2], \quad (6.5)$$

for all $s_0 \in \mathbb{R}$. Notice that $\mathcal{M}_0$ has bounded sectional curvature. It follows from Lemma D.2 and Remark D.5 that $\iota_{\mathcal{M}_0}(s)$, the (homogenized) injectivity radius of $\mathcal{M}_0$ at points $(s, y) \in M$, is bounded below by $r_0(s) := c_0 \min\{r(s), 1\}$, where $c_0 > 0$ is a suitable constant; without loss of generality, we may assume $c_0 \leq 1$.

Now consider two metrics $g_1$ and $g_2$ on $M$ with the following properties:

(i) $g_1$ and $g_2$ are quasi-isometric to $g_0$.

(ii) $\mathcal{M}_k = (M, g_k)$ have bounded sectional curvature.

(iii) We assume that

$$\tilde{d}_1^2(g_1, g_2) = \int_M \tilde{d}(g_1, g_2)(x) r_0(x)^{-(n+2)} \, d\text{vol}_{g_0}(x) < \infty, \quad (6.6)$$

with the function $r_0$ obtained above, i.e., we assume that the quasi-distance $\tilde{d}_1^2(g_1, g_2)$ as defined in eqn. is finite. Note that this quasi-distance is equivalent to the original quasi-distance $\tilde{d}_1$ defined in eqn. (3.2), cf. Remark 5.6 (ii).

Then the wave operators for the Laplacians $H_1$ and $H_2$, associated with the metrics $g_1$ and $g_2$, exist and are complete.

Indeed, it follows from \[1, 3\] and \[MS07, Prop. 2.1\] (cf. also Proposition D.1) that the injectivity radius for $\mathcal{M}_{1,2}$ is bounded from below by $cr_0(x)$ for some positive constant $c$. By (6.5), a similar lower bound holds for the homogenized injectivity radius of
Remark 6.2. Suppose we are given two quasi-isometric metrics $g_1$ and $g_2$ on $M = \mathbb{R} \times S^{n-1}$ of the form $g_i = ds^2 + r_i(s)^2 \, ds_{S^{n-1}}$, for $i = 1, 2$, where the functions $r_i$ satisfy the same conditions as $r_0$ in Example 6.1. In particular, the manifolds $\mathcal{M}_i = (M, g_i)$ have bounded sectional curvature and their (homogenized) injectivity radius is bounded below by $cr_i(s)$ for some constant $c > 0$. We may thus work with $r_0(s) := \min\{1, r_1(s), r_2(s)\}$ in eqn. (3.5); note that the function $r_0$ defined here may be different from the function $r_0$ of Example 6.1.

It is easy to see that the matrix $A$ from eqn. (5.3) has one eigenvalue 1 while the other $n-1$ eigenvalues are equal to $(r_2/r_1)^2$ so that

$$\tilde{d}(g_1, g_2) = \left| (\frac{r_2}{r_1})^{n/2} - (\frac{r_1}{r_2})^{n/2} \right|.$$ 

We can now compute $\tilde{d}_1^1(g_1, g_2)$ (as in eqn. (5.7)) as

$$\tilde{d}_1^1(g_1, g_2) = \omega_{n-1} \int_{-\infty}^{\infty} \left| (\frac{r_2}{r_1})^{n/2} - (\frac{r_1}{r_2})^{n/2} \right| \frac{1}{(\min\{1, r_1, r_2\})^{n+2}} \, ds;$$

since $g_1$ and $g_2$ are quasi-isometric, there is a constant $c_1 \geq 1$ depending only on the quasi-isometric distance $\tilde{d}_\infty(g_1, g_2)$ such that $\tilde{d}_1^1(g_1, g_2) \leq \tilde{d}_1^1(g_1, g_2) \leq c_1 \tilde{d}_1^1(g_1, g_2)$.

We next discuss examples without a global curvature bound:

Example 6.3. (Existence and completeness of wave operators without a global curvature bound). We start with a manifold $\mathcal{M}_0$ with a warped product metric obtained from a function $r \in C^2(\mathbb{R})$ satisfying condition (6.5). We define the function $\kappa_0: \mathbb{R} \to \mathbb{R}_+$ as in eqn. (6.5).

Given $r$, we then consider metrics $g_1, g_2$ as in Proposition D.3. In particular, $g_1$ and $g_2$ have to be quasi-isometric to $g_0 = ds^2 + r(s)^2 \, ds_{S^{n-1}}$ and there is a (more complicated) condition on the sectional curvature of $\mathcal{M}_1 := (M, g_1)$ and $\mathcal{M}_2 := (M, g_2)$ expressed in terms of functions $\kappa_1$ and $\kappa_2$ on the real line, defined as in assumption (ii) of Proposition D.3. By this proposition and Remark D.5, the (homogenized) injectivity radius and then also the (homogenized) harmonic radius of $\mathcal{M}_1$ and $\mathcal{M}_2$ at points $(s, y) \in M$ are bounded below by

$$r_0(s) := C \min\{r(s), \min\{ (\sqrt{\kappa_i(s)})^{-1} \mid i = 0, 1, 2 \} \},$$

for some positive constant $C$. Proceeding now as in the above example, we find that the wave operators for the Laplacians $H_1$ and $H_2$ exist and are complete if $g_1$ and $g_2$ satisfy condition (6.6).

It may be of interest to note that lower bounds for the radius of injectivity are the main limitation and difficulty in the application of our theorems to concrete examples.

6.4. Non-trivial scattering for warped products. Here we provide examples of functions $r$ on the real line which yield scattering channels that are open to each other. We use the notation from the beginning of this section concerning $H_{\pm m}$ etc. We give the details in the case of the scattering channels, $\tilde{H}_{\pm,0}$, with zero angular momentum. The case of scattering channels with non-zero zero angular momentum follows in the same way.

We require that the potential $w$, defined in eqn. (6.3), is short-range, i.e., that it satisfies (6.4). Note that the ends can be flat, horns, etc., as long as $w$ is short-range.
We first introduce some notation that we need (see Appendix C). Denote by $h_0$ the unique self-adjoint realization of $-d^2/ds^2$ in $\mathcal{H} := L_2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. We define $h := h_0 + w$, a self-adjoint operator in $\mathcal{H}$, with domain $\text{Dom } h = H^2(\mathbb{R})$. Let $h_{\pm,0}$ be the self-adjoint realizations of $-d^2/ds^2$ in $L_2(\mathbb{R}_\pm)$ with Dirichlet boundary condition at zero, i.e., with domain $\text{Dom } h_{\pm,0} = H^1(\mathbb{R}_\pm) \cap H^2(\mathbb{R}_\pm)$. We denote by $t_\pm$ the natural embeddings of $L_2(\mathbb{R}_\pm)$ into $L_2(\mathbb{R})$ (extension by zero). We then consider the wave operators

$$\Lambda_+^+ := \text{s-lim}_{t \to \pm \infty} e^{it} J_+ e^{-it}, \quad \text{and} \quad \Lambda_-^- := \text{s-lim}_{t \to \pm \infty} e^{it} J_- e^{-it}.$$ 

We define a mapping $j_0$ which associates with a function $u \in L_2(\mathbb{R})$ the function $j_0 u \in L_2(\mathbb{R}, E_0)$, defined by

$$(j_0 u)(s, y) = \frac{1}{\sqrt{\omega_{n-1}}} u(s),$$

where $\omega_{n-1} := \text{vol}_{n-1}(\mathbb{S}^{n-1})$, and define $J := U^* j_0$, an isometry from $L_2(\mathcal{M})$ into $L_2(\mathcal{M})$.

A natural choice of a reference operator for the ends are the operators $h_{\pm,0}$. We then define the wave operators for the right end as

$$W_+^+ := \text{s-lim}_{t \to \pm \infty} e^{it} J_+ e^{-it} h_{+}, \quad \text{and} \quad W_-^- := \text{s-lim}_{t \to \pm \infty} e^{it} J_- e^{-it} h_{-}.$$ 

By the chain rule, we have

$$W_+^+ = W_+^+ (H, h, J) \circ \Lambda_+^+ \quad \text{and} \quad W_-^- = W_-^- (H, h, J) \circ \Lambda_-^-.$$ 

Furthermore, as $UJ = UU^* j_0 = j_0$,

$$W_+^+ (H, h, J) = U^* W_+^+ (\tilde{H}, h, j_0) = U^* W_+^+ (\tilde{H}_0, h, j_0) = U^* j_0 = J,$$ 

since $e^{it \tilde{H}} j_0 u = j_0 e^{it} u$. By (6.9) and (6.10), we have

$$W_+^+ = J \circ \Lambda_+^+ \quad \text{and} \quad W_-^- = J \circ \Lambda_-^-.$$ 

Since $J$ is an isometry, it follows from Lemma C.3 and (6.11) that

$$\text{Ran } W_+^+ \neq \text{Ran } W_-^- \quad \text{and} \quad \text{Ran } W_+^+ \neq \text{Ran } W_-^-,$$ 

which shows that the left and the right scattering channels are open to each other, i.e.,

$$S_{+-} := (W_+^+)^* W_-^- \neq 0.$$ 

Let us give an example of a high-velocity asymptotic state which comes in from the left end for large negative times, and which travels to the right end as time tends to infinity, with a reflected part that is very small if the velocity is large. As in Appendix C we define

$$\varphi^+_v(s) := e^{iv s} \varphi_0, \quad \text{with } \mathcal{F} \varphi_0 \in C^\infty_c(\mathbb{R}),$$

where $\mathcal{F}$ denotes the Fourier transform. Since

$$\left(\mathcal{F} \varphi^+_v\right)(k) = \left(\mathcal{F} \varphi_0\right)(k - v),$$

this state has large velocity if $v > 0$ is taken large enough. We set

$$\varphi^+_{v,0} := \varphi^+_v(s) - \varphi^+_v(-s), \quad s > 0,$$

then by (6.11) and (C.17)–(C.18) we have

$$S_{+-} \varphi^+_{v,0} = (\Lambda_+^+)^* \varphi^+_{v,0} = \varphi^+_v + \mathcal{O}(\frac{1}{v}),$$

and $S_{+-} \varphi^+_{v,0} = (\Lambda_+^+)^* \varphi^+_{v,0} \neq 0$ for $v$ large enough.
6.5. Perturbations of warped products: Open scattering channels (Applications of Corollary 5.3). We now consider perturbations of the rotationally symmetric situation discussed in the preceding subsection.

Example 6.4 (Open scattering channels). For simplicity, let \( n = 2 \) and let us assume that the metric \( g_0 \) is a warped product obtained from a function \( r: \mathbb{R} \to \mathbb{R}_+ \) of class \( C^2 \) satisfying \( r(s) = |s| \) for \( s \leq -1 \) and that \( r(s) = \tau s^\beta \), for \( s \geq 1 \), for some \( \beta < 0 \) and \( \tau > 0 \). We let \( \mathcal{M}_0 = (M, g_0) \), a manifold of bounded sectional curvature. Let \( K \geq 0 \) denote an upper bound for the curvature of \( \mathcal{M}_0 \). By the above discussion, the scattering channels for the Laplacian of \( \mathcal{M}_0 \) are open. As in Example 6.1, we find that the (homogenized) injectivity radius of \( \mathcal{M}_0 \) has a lower bound of the form \( c_0 \min \{ 1, r(s) \} \) for \( s \in \mathbb{R} \), with a constant \( c_0 > 0 \); cf. Lemma [D.2]. Since the Ricci curvature is of order \( s^{-2} \) as \( s \to \infty \), and as it is equal to zero for \( s \leq -1 \), we may choose \( r_0(s) = c_0 \) for \( s \leq -1 \) and \( r_0(s) = \tau s^\beta \) for \( s \geq 1 \) in \([0, \infty)\).

For some \( \gamma > 0 \) and \( K \) as above, let us consider the class \( \text{Met}_{r_0}(M, g_0, \gamma, K, \varepsilon) \) consisting of all metrics \( g \in \text{Met}_{r_0}(M, g_0, \gamma, \varepsilon) \) with sectional curvature bounded by \( K \). Since any \( g \in \text{Met}_{r_0}(M, g_0, \gamma, K, \varepsilon) \) is quasi-isometric to \( g_0 \) (with relative constants depending only on \( \gamma \)), Proposition 2.1 of [MS07] (cf. also Proposition D.1) implies that the injectivity radius of \( g \) is bounded below by \( c r_0 \), for some constant \( c > 0 \); it is easy to see that a similar estimate then holds for the homogenized injectivity radius as well. As in Example 6.4, the wave operators for the pair \( H \) and \( H_0 \) exist and are complete. From Corollary 5.3 we infer that there exists \( \varepsilon_0 > 0 \) such that the scattering channels for the Laplacian of \( \mathcal{M} = (M, g) \) are open for any metric \( g \in \text{Met}_{r_0}(M, g_0, \gamma, K, \varepsilon) \), provided \( \varepsilon < \varepsilon_0 \).

Here is, finally, an example for Corollary 5.3 without a global bound on the curvature:

Example 6.5 (Open scattering channels without global curvature bound). Let \( n \in \mathbb{N} \), \( n \geq 2 \). For \( r: \mathbb{R} \to \mathbb{R}_+ \), \( r \in C^2 \) satisfying (8.5), let \( g_0 := ds^2 + r(s)^2 g_{S^{n-1}} \) and assume that the scattering channels are open. Define \( \kappa_0: \mathbb{R} \to \mathbb{R}_+ \) as in eqn. (12.5). Let \( \kappa: \mathbb{R} \to \mathbb{R}_+ \) be continuous and satisfy \( \kappa(s) \geq \kappa_0(s) \) and let \( \text{Met}_{r_0}(M, g_0, \gamma, K, \varepsilon) \) denote the set of all metrics \( g \in \text{Met}_{r_0}(M, g_0, \gamma, \varepsilon) \) which satisfy the curvature condition \( \kappa \) of Proposition D.3. By this proposition and Remark D.5, the (homogenized) harmonic radius of any \( \mathcal{M} = (M, g) \) with \( g \in \text{Met}_{r_0}(M, g_0, \gamma, K, \varepsilon) \) at points \((s, y) \in M \) is bounded below by

\[
r_0(s) := C \min \left\{ \frac{1}{\sqrt{\kappa(s)}}, r(s) \right\}.
\]

By Corollary 5.3 there exists \( \varepsilon_0 > 0 \) such that the scattering channels of any \( \mathcal{M} = (M, g) \) with \( g \in \text{Met}_{r_0}(M, g_0, \gamma, K, \varepsilon) \) are open, provided \( 0 < \varepsilon < \varepsilon_0 \).

Appendix A. Pointwise distance functions on the set of metrics

Let us introduce two pointwise distance functions on the set of metrics \( \text{Met}(M) \) on a manifold \( M \). We use the terminology “distance (function)” for what is usually called “metric” in the sense of metric spaces. Let \( V \) be an \( n \)-dimensional \( \mathbb{C} \)-vector space and denote by \( \text{Sesq}_+(V) \) the set of all positive definite sesquilinear forms on \( V \). Given \( g_1, g_2 \in \text{Sesq}_+(V) \), we define a positive definite endomorphism \( A = A_{g_2, g_1} \) on \( V \) via

\[
g_2(\xi, \zeta) = g_1(A\xi, \zeta) \tag{A.1}
\]

for all \( \xi, \zeta \in V \), the relative distortion of \( g_2 \) with respect to \( g_1 \). The distance of \( g_1 \) to \( g_2 \) is defined as

\[
d(g_1, g_2) := \max_k |\ln \alpha_k|,
\]
where \(\alpha_1, \ldots, \alpha_n\) denote the \(n\) positive eigenvalues of \(A\); hence \(d(g_1, g_2)\) equals the operator norm of \(\ln A\). Moreover, setting \(\tilde{g}_1, \tilde{g}_2 := \ln A_{g_2, g_1}\), the set \(\text{Sesq}_+(V)\) becomes an affine space with associated vector space \(\mathcal{B}(V)\) (endomorphisms on \(V\)). Moreover, we have \(d(g_1, g_2) = |\ln A_{g_2, g_1}|_{\mathcal{B}(V)}\), which shows that \(d\) is indeed a distance function on \(\text{Sesq}_+(V)\).

When dealing with Riemannian metrics (and especially with our trace class estimate in Section 3), it will be convenient to work with the following modified distance function, namely,

\[
\tilde{d}(g_1, g_2) := 2 \sinh \left( \frac{\mu}{4} \cdot d(g_1, g_2) \right).
\]

Note that \(\tilde{d}\) is symmetric and definite (i.e., \(\tilde{d}^*(g_1, g_2) = 0\) implies \(g_1 = g_2\)), but does not fulfill the triangle inequality. Instead, using the addition theorem

\[
\sinh(u + v) = \sinh u \sqrt{\sinh^2 v + 1} + \sinh v \sqrt{\sinh^2 u + 1}
\]

and the triangle inequality for \(d\), one can see that

\[
\tilde{d}(g_1, g_3) \leq \mu(\tilde{d}(g_1, g_2), \tilde{d}(g_2, g_3))
\]

with \(\mu(a, b) = a \sqrt{(b/2)^2 + 1} + b \sqrt{(a/2)^2 + 1}\). Since \(\mu(a, b) \leq a + b + ab =: \bar{\mu}(a, b)\), we can also use \(\bar{\mu}\) instead of \(\mu\). Replacing \(d\) with the equivalent (uniformly) bounded distance

\[
d^*(g_1, g_2) := \min\{d(g_1, g_2), \delta\}
\]

for some fixed \(\delta > 0\), we see that

\[
\tilde{d}^*(g_1, g_3) \leq \tilde{d}^*(g_1, g_2) + \tilde{d}^*(g_2, g_3) + \tilde{d}^*(g_1, g_2) \tilde{d}^*(g_2, g_3)
\]

\[
\leq \left( 1 + \sinh \frac{n\delta}{4} \right) \tilde{d}^*(g_1, g_2) + \tilde{d}^*(g_2, g_3),
\]

where \(\tilde{d}^*\) is defined as \(\tilde{d}\) in (A.2) but with \(d^*\) instead of \(d\), i.e., the triangle inequality is fulfilled up to a factor \(\tau = 1 + \sinh(n\delta/4)\). Note that if \(\tilde{d}\) is bounded by \(\gamma\), then we can choose \(\tau = 1 + \gamma/2\).

**Definition A.1.** A function \(\tilde{d}^* \geq 0\) is called quasi-distance if it is symmetric (\(\tilde{d}^*(g_1, g_2) = \tilde{d}^*(g_2, g_1)\)), definite (\(\tilde{g}^*(g_1, g_2) = 0\) implies \(g_1 = g_2\)), and fulfills the weak triangle inequality

\[
\tilde{d}^*(g_1, g_3) \leq \tau(\tilde{d}^*(g_1, g_2) + \tilde{d}^*(g_1, g_3))
\]

for some factor \(\tau \geq 1\).

Usually, a quasi-distance is called a quasi-metric, but we prefer the terminology “distance” in order not to interfere with the word “metric” for the points of the space, being Riemannian metrics. Sometimes, such distance functions are also called semi-metrics. For more details on such spaces we refer to [Hn01, Sec. 14] (see also [X09] for a more recent list of references). Let us just mention Proposition 14.5 from [Hn01] stating that a power of a quasi-distance \(d^*\) is equivalent to a metric, i.e., for any \(\varepsilon \in (0, \varepsilon_0)\) there is a metric \(\tilde{d}_\varepsilon\) and a constant \(C = C(\varepsilon, \tau) > 0\) such that

\[
C^{-1}(\tilde{d}^*(g_1, g_2))^{\varepsilon} \leq \tilde{d}_\varepsilon(g_1, g_2) \leq C(\tilde{d}^*(g_1, g_2))^{\varepsilon}
\]

for all \(g_1, g_2\), where \(\varepsilon_0 = \ln 2/(2 \ln \tau)\). In particular, a quasi-distance uniquely determines a topology and a uniform structure independent of \(\varepsilon\). Hence, the notions of convergence and completeness are well-defined on a space with a quasi-distance.

Let us now pass to metrics \(g_1, g_2\) on a manifold \(M\), i.e., to sections in the bundle \(\text{Met}(M) := \text{Sesq}_+T^*M\). (For the following considerations, no smoothness assumptions on \(g_1, g_2\) are needed.) In particular, \(g_1, g_2\) induce a section \(A = A_{g_2, g_1}\) into the bundle of positive definite endomorphisms on \(T^*M\) applying (A.1) pointwise. Denote by
the uniform (quasi-)distance. Note that \( d_\alpha \) and \( \tilde{\kappa} \) have the same estimate holds with \( \tilde{\kappa} \). It is easy to see that Remark A.2. \( \eta > 0 \)

where the last statement means that there exists \( \eta > 0 \) such that

\[
\eta^{-1} g_2(x) \leq g_1(x) \leq \eta g_1(x)
\]

for all \( x \in M \) in the sense of quadratic forms (one may choose \( \eta = \exp d_\infty(g_1, g_2) \)).

Recall that the notion “quasi-isometric” was already defined in Definition 3.1 in Section 3.

For a further analysis of the topological (or uniform) structure of \( \text{Met}(M) \), we refer to [Ei07].

**Appendix B. Pointwise bounds for \( u(x) \) and \( du(x) \)**

Let \( \mathcal{M} = (M, g) \) denote a complete Riemannian manifold satisfying the assumptions of Section 2.3. The metric \( g \) is assumed to be smooth (or to have the “minimal” regularity of Remark 2.6). Let \( H \) denote the self-adjoint and non-negative extension of the Laplacian of \( \mathcal{M} \). We provide pointwise bounds for \( u \) and \( du \) where \( u = (H + 1)^{-m} f, f \in L_2(\mathcal{M}), \) and \( m \) is sufficiently large. For \( n < p < \infty \) and \( 1 < Q \leq 2 \) let \( r(x) = r(\mathcal{M}, x, p, Q) \) denote the harmonic radius at \( x \in M \) as in Section 2.4. We then have the following theorem.

**Theorem B.1.** Let \( H \) and \( r(x) \) be as above, let \( f \in L_2(\mathcal{M}), \) and let \( u := (H + 1)^{-m} f, \) where \( m \in \mathbb{N}, m \geq [n/4] + 2. \) Then \( u \in C^1(\mathcal{M}) \) and there exist constants \( C > 0, \) depending only on \( m, n, p, \) and \( Q, \) such that

\[
|u(x)| \leq C(\min\{1, r(x)\})^{-n/2} \|f\|_{L_2(\mathcal{M})} \tag{B.1a}
\]

and

\[
|du(x)|_g \leq C(\min\{1, r(x)\})^{-n/2-1} \|f\|_{L_2(\mathcal{M})}. \tag{B.1b}
\]

While the first estimate is well-known (cf., e.g., [CGT82]), the gradient estimate requires some additional work. The most crucial ingredient is an estimate in elliptic regularity theory (eqn. (0.10) in [AC92]) which we adapt to our situation. We will employ elliptic regularity theory in \( L_2 \) for equations in divergence form as well as elliptic regularity theory in \( L_p \) for strong solutions. The first one will allow us to show that weak solutions are in fact strong solutions in the sense of [GT83]; the actual estimates will then be obtained from Theorem 9.11 in [GT83].
Let us recall how the equation \((H + 1)u = f\) reads in local coordinates. Note first that
\[
|du|^2_g = \sum_{i,j=1}^n g^{ij} \partial_i u \partial_j \pi. \tag{B.2}
\]
Therefore, by the very definition of the operator \(H\) in Section 2.3, the equation \(Hu + u = f\) (in the weak sense) means that \(u\) belongs to \(H^1(M)\) and satisfies
\[
\sum_{i,j=1}^n \int g^{ij} \partial_i u \partial_j \varphi \sqrt{g} \,dx + \int u \varphi \sqrt{g} \,dx = \int f \varphi \sqrt{g} \,dx, \tag{B.3}
\]
for all \(\varphi \in C_c^\infty(U)\), where \(U\) is as in Section 2.4 and where \(\sqrt{g} = \sqrt{\det(g_{ij})}\). (Note that we do not distinguish in the notation between \(u\) and \(u \circ \Phi^{-1}\) if \(\Phi\) is a coordinate map.) Therefore, the weak form of the partial differential equation is formally given by
\[
-\sum_{i,j=1}^n \partial_j (\sqrt{g} g^{ij} \partial_i u) + \sqrt{g} u = \sqrt{g} f. \tag{B.4}
\]
We will see shortly that, under suitable assumptions, \(u\) belongs to \(H^2_{\text{loc}}(U)\) and is also a strong solution. In the special case of harmonic coordinates the first-order terms cancel, and we see that \(u\) then satisfies the partial differential equation
\[
-\sum_{i,j=1}^n g^{ij} \partial_i \partial_j u + u = f; \tag{B.5}
\]
see, e.g., [DK81]. Since, in harmonic coordinates, \(g\) is close to 1 in its coordinate patch (cf. eqn. (2.3)), the extra factors of \(\sqrt{g}\) in eqn. (B.4) pose no problem.

As a preparation, we recall some facts from (interior) elliptic regularity theory in a bounded domain \(\Omega \subset \mathbb{R}^n\) where we now use the symbols \(u\) and \(f\) in a different context. We begin with weak solutions \(u \in H^1_{\text{loc}}(\Omega)\) of an elliptic equation \(L_w u = f\),
\[
L_w u := -\sum_{i,j=1}^n \partial_j A_{ij} \partial_i u + \gamma u, \tag{B.6}
\]
where the coefficient matrix \((A_{ij})\) is uniformly positive definite with \((A_{ij}) \geq 1/2\), the \(A_{ij}\) are of class \(C^{0,1}(\Omega)\), and \(\gamma\) is bounded; finally, we assume \(f \in L^q(\Omega)\) for some \(q \geq 2\). From [GT83, Thm. 8.8] we then infer that \(u \in H^2_{\text{loc}}(\Omega)\) and that \(u\) is a strong solution of eqn. (B.6) in the sense of [GT83]. Furthermore, [GT83, Lemma 9.16] yields that \(u \in W^2_{q,\text{loc}}(\Omega)\). This type of regularity will be needed later on.

We next consider strong solutions \(u \in W^2_{q,\text{loc}}(\Omega)\) of an elliptic equation \(Lu = f\) with
\[
Lu := -\sum_{i,j=1}^n a_{ij} \partial_i \partial_j u + \gamma u \tag{B.7}
\]
where, as above, \(q \geq 2\), \(a_{ij} \geq 1/2\), \(a_{ij} \in C^{0,1}(\Omega)\), and \(\gamma\) is bounded. In view of the Sobolev Embedding Theorem, we define the exponent
\[
\sigma(q) := \begin{cases} 
\frac{qn}{n - 2q}, & 2q < n, \\
q + 1, & 2q = n, \\
\infty, & 2q > n,
\end{cases} \tag{B.8}
\]
for \(q \in [1, \infty]\). For \(a \in C^{0,\alpha} \overline{\Omega}\) (the space of uniformly Hölder-continuous functions), we denote the \(\alpha\)-Hölder-constant of \(a\) by \([a]_{0,\alpha}\). We then have the following lemma.
Lemma B.2. Let $\Omega := B_2 \subset \mathbb{R}^n$ and let $L$ as in (B.7) with $(a_{ij}) \geq 1/2$, $a_{ij} \in C^{0,1}(\Omega)$, and $\gamma$ bounded. Let $\alpha \in (0, 1]$ and let $\Lambda > 0$ be such that $\|a_{ij}\|_{\infty} \leq \Lambda$, $|a_{ij}|_{0,\alpha} \leq \Lambda$ and $|\gamma|_{\infty} \leq \Lambda$. Let $q \in [2, \infty)$ and let $u \in W^2_q(B_2)$. We then have:

(i) There exists a constant $C_1 \geq 0$, depending only on $n$, $\alpha$, and $\Lambda$, such that

$$\|u\|_{W^2_q(B_2)} \leq C_1 (\|Lu\|_{L_q(B_2)} + \|u\|_{L_q(B_2)}).$$

(ii) Let $q_1 := \sigma(q)$ as in (B.8). Then there exists a constant $C_2 \geq 0$, depending only on $n$, $\alpha$, and $\Lambda$, such that

$$\|u\|_{L_q(B_1)} \leq C_2 (\|Lu\|_{L_q(B_2)} + \|u\|_{L_q(B_2)}).$$

Remark B.3. (a) It is essential for later applications that the last term in eqns. (B.9) and (B.10) is an $L_2$-norm, as in eqn. (0.10) in [AC92].

(b) If $L$ would also contain first order terms $b_i \partial_i u$, it appears that we would need to require the coefficients $b_i$ to be bounded. In general, the first order terms of the Laplacian contain derivatives of the $g^{ij}$ and we would need an assumption like $\|g^{ij}\|_{W^{1,\infty}} \leq \Lambda$, while the harmonic coordinates only come with an estimate for $\|g^{ij}\|_{W^{1,p}}$.

Proof. The proof of Lemma B.2 combines elliptic regularity in $L_2$ with a simple bootstrap argument for which we fix a sequence of radii $2 > q_1 > q_2 > \cdots > 1$.

(i) Interior elliptic regularity in $L_2(B_2)$ as in [GT83 Thm. 9.11] gives us a constant $c_1$, depending only on $\Lambda$ and $q_1$ such that

$$\|u\|_{H^2(B_{q_1})} \leq c_1 (\|Lu\|_{L_2(B_2)} + \|u\|_{L_2(B_2)}).$$

(ii) Let $p_1 := \min\{q, \sigma(2)\}$. By the Sobolev Embedding Theorem, there is a constant $c_2$, depending only on $n, \alpha, \Lambda$, and $q$, such that

$$\|u\|_{L_{p_1}(B_{q_2})} \leq c_2 \|u\|_{H^2(B_{q_1})} \leq c_2 (\|Lu\|_{L_2(B_2)} + \|u\|_{L_2(B_2)}).$$

(iii) We apply [GT83 Thm. 9.11] in $L_{p_1}(B_{q_2})$ to obtain a constant $c_3$, depending only on $\Lambda, q_1, q_2, q_3,$ and $p_1$ such that

$$\|u\|_{W^2_{p_1}(B_{q_3})} \leq c_3 (\|Lu\|_{L_{p_1}(B_{q_2})} + \|u\|_{L_{p_1}(B_{q_2})}) \leq c_4 \|Lu\|_{L_{p_1}(B_{q_2})} + c_5 \|u\|_{L_{p_1}(B_{q_2})}.$$

If $p_1 = q$ the proof of the first inequality is finished. Otherwise we continue with another application of the Sobolev Embedding Theorem. The proof terminates after a finite number of steps (which depends only on $n$ and $q$). The second inequality follows by Sobolev.

By a simple scaling argument we now transfer the estimate (B.10) from $B_2$ to $B_{2r}$ where $0 < r \leq 1$.

Lemma B.4. Let $0 < r \leq 1$ and consider $\Omega := B_{2r}$. Let $L$ be as in (B.7) with $(a_{ij}) \geq 1/2$, $a_{ij} \in C^{0,1}(\Omega)$, and $\gamma$ bounded. Let $\Lambda > 0$ be such that $\|a_{ij}\|_{\infty} \leq \Lambda$, $|a_{ij}|_{0,\alpha} \leq \Lambda r^{-\alpha}$, and $|\gamma|_{\infty} \leq \Lambda$ on $B_{2r}$. Let $q \in [2, \infty)$, $q_1 := \sigma(q)$, and let $u \in W^2_q(B_{2r})$.

Then there exists a constant $C \geq 0$, depending only on $n, \alpha, \Lambda$, and $r$, such that

$$\|u\|_{L_{q_1}(B_r)} \leq Cr^2 \cdot r^{-n(1/q - 1/n)} \cdot \|Lu\|_{L_q(B_{2r})} + C \cdot r^{-n(1/2 - 1/n)} \cdot \|u\|_{L_2(B_{2r})}.$$  

(B.11)

Proof. Write $f := Lu \in L_q(B_{2r})$ and scale out (i.e., set $y := x/r$) to obtain

$$\tilde{u}(y) := u(ry), \quad y \in B_{2r}.$$

Similarly, write $\tilde{a}_{ij}(y) := a_{ij}(ry), \tilde{\gamma}(y) := \gamma(ry)$, and $\tilde{f}(y) := f(ry)$. Defining

$$\tilde{L} = -\sum \tilde{a}_{ij} \partial_{y_i} \partial_{y_j} + r^2 \tilde{\gamma},$$

we have

$$\tilde{L} \tilde{u} = \tilde{f},$$

and therefore

$$\|\tilde{u}\|_{L_{q_1}(B_1)} \leq C \cdot \|\tilde{f}\|_{L_q(B_{2r})} + C \cdot r^{-n(1/2 - 1/n)} \cdot \|\tilde{u}\|_{L_2(B_{2r})}.$$
the equation $Lu = f$ in $L_2(2r)$ is then equivalent with

$$\tilde{Lu} = r^2 \tilde{f}(y)$$

(B.12)

in $L_2(2)$. Applying Lemma [B.2] to eqn. [B.12] in $2$, we find that

$$\|\tilde{u}\|_{L_1(B_2)} \leq C r^2 \|\tilde{f}\|_{L_1(B_2)} + C \|\tilde{u}\|_{L_2(2)},$$

with a constant $C$ depending only on $n, \alpha$, and $\Lambda$; note that $[\tilde{a}_{ij}]_{0,\alpha} \leq r^n [a_{ij}]_{0,\alpha} \leq \Lambda$. Scaling back yields

$$r^{-n/q} \|u\|_{L_1(B_r)} \leq C r^{2r^{-n/q} \cdot \|f\|_{L_1(B_2r)} + r^{-n/2} \|u\|_{L_2(B_r)}$$

and the result follows. \hfill \Box

The estimate (B.11) is slightly more precise than what follows from the estimate (0.10) in [AC92] in the sense that the dependence on the (local) harmonic radius is made explicit in eqn. (B.11).

The proof of Theorem [B.3] will be based on an iteration of Lemma [B.4]. To illustrate the idea, let $f \in L_2(2r)$ and let $u_1, u_2$ satisfy $Lu_1 = f$ and $Lu_2 = u_1$ in $B_2$, in the sense of a strong solution; in particular, we assume $u_1, u_2 \in H_2^1(B_2r)$. By Sobolev, we then have $u_1 \in L_{q_1,loc}(B_2r)$, where $q_1 := \sigma(2)$ and, by Lemma 9.15 in [GT83], $u_2 \in W_2^{q_1,loc}(B_2r)$. Again, we assume $0 < r \leq 1$. Then, Lemma [B.4] again yields

$$\|u_1\|_{L_1(B_r)} \leq C \cdot r^{2r^{-n/(1-1/q_1)} \cdot \|f\|_{L_1(B_{2r})} + C \cdot r^{-n/(1-1/q_1)} \cdot \|u_1\|_{L_2(B_2r)}.$$  
(B.13)

Similarly, we see that $u_2 \in L_{q_2}(B_{r/2})$ with $q_2 := \sigma(q_1)$. Again, Lemma [B.4] gives

$$\|u_2\|_{L_2(1/2)} \leq C \cdot r^{-n/(1-1/q_2)} \cdot \|u_1\|_{L_1(B_{r/2})} + C \cdot r^{-n/(1-1/q_2)} \cdot \|u_2\|_{L_2(B_{r/2})}.$$  
(B.14)

Inserting (B.13) into (B.14) we obtain

$$\|u_2\|_{L_2(B_{r/2})} \leq C r^{-n/(1-1/q_2)} \cdot \left( C r^4 \|f\|_{L_1(B_{2r})} + C \|u_1\|_{L_2(B_{2r})} + \|u_2\|_{L_2(B_2r)} \right) \leq C r^{-n/2} \left( \|f\|_{L_1(B_{2r})} + \|u_1\|_{L_2(B_{2r})} + \|u_2\|_{L_2(B_2r)} \right);$$

(B.15)

note that the powers $r^{1/q_2}$ have dropped out. Clearly, analogous estimates hold for (finite) chains of equations where $Lu_{k+1} = u_k, \text{ for } k = 0, \ldots, m$.

Finally, we now return to manifolds $\mathcal{M}$ with Laplacian $H$. In view of Definition 2.2 we fix once and for all some $p \in (n, \infty)$ and let $\alpha := 1 - n/p$. We also fix some $1 < Q \leq 2$ close enough to 1 to ensure that any of the neighborhoods $U$ of Section 2.3 contains a Euclidean ball of radius $r_{\#}(x)/2$; here we use eqn. (2.3) and the usual formula for the length of curves in local coordinates. In the sequel, we will suppress the dependence on the constants $p, Q$ and $n$ in the notation.

Let $f \in L_2(\mathcal{M})$ and consider

$$u_k := (H + 1)^{-k} f, \quad k \in \mathbb{N}_0;$$

we then have $u_k \in \text{Dom } H$,

$$(H + 1)u_{k+1} = u_k, \quad k \in \mathbb{N}_0;$$

(B.16)

and

$$\|u_k\|_{L_2(\mathcal{M})} \leq \|f\|_{L_2(\mathcal{M})}, \quad k \in \mathbb{N}.$$ 

Let $r(x) = r_{\#}(x, p, Q)$ denote the harmonic radius at $x \in M$. Passing to harmonic coordinates in a geodesic ball $B = B_{\#}(x, r(x)) \subset M$ of radius $r(x)$ around $x$, the functions $u_k$ (or, more precisely, $u_k \circ \Phi^{-1}$ etc.) are weak solutions of the divergence form equation $-\partial_j \sqrt{g} g^{ij} \partial_i u_{k+1} + \sqrt{g} u_{k+1} = \sqrt{g} u_k$, but then, as explained above, they are also strong solutions of

$$- g^{ij} \partial_j u_{k+1} + u_{k+1} = u_k$$

(B.17)
in $\Phi(B) \subset \mathbb{R}^n$, for $k \in \mathbb{N}_0$; note that we may apply Theorem 8.8 of [GT83] to the weak equation since $\sqrt{g}g^{ij}$ is in $W^{1,\infty}_{loc}$ and thus locally Lipschitz.

**Proof of Theorem B.4.** The $g^{ij}$ satisfy the estimates (2.3)–(2.5) and we see that $\Lambda$ (defined as in Lemma B.3) depends only on $n$ and $p$. Applying Lemma B.4 successively to the equations (B.17), as indicated above, we obtain the estimate (B.1a), which, in fact, holds for $m \geq \lfloor n/4 \rfloor + 1$.

As for the gradient estimate, we let $k_0 := \lfloor n/4 \rfloor + 1$ and consider the equation $Lu_{k_0+1} + u_{k_0+1} = u_{k_0}$ where, by the above, $u_{k_0}$ and $u_{k_0+1}$ are locally bounded with estimates

$$|u_{k_0}(x')|, |u_{k_0+1}(x')| \leq C r(x)^{-n/2} \|f\|_{L^2(\#)}, \quad |x - x'| \leq 2^{-k_0} r(x).$$

Scaling out as in the proof of Lemma B.4, but now with a factor of $2^{k_0+3} r(x)$, we find that the scaled function $\tilde{u}_{k_0+1}$ satisfies an equation

$$\tilde{L} \tilde{u}_{k_0+1} = 4^{-k_0-3} r(x)^2 \tilde{u}_{k_0}$$

in $B_2 \subset \mathbb{R}^n$, where $\tilde{L} = -g^{ij} \partial_i \partial_j + 4^{-k_0-3} r(x)^2$.

Fix some $q \in (n, \infty)$, e.g., $q := n + 1$. As above, we have $\tilde{u}_{k_0+1} \in W^2_q,\text{loc}(B_2)$ and Lemma B.2 yields an estimate

$$\|\tilde{u}_{k_0+1}\|_{W^2_q(B_1)} \leq c \left( \|\tilde{u}_{k_0}\|_{W^1_q(B_2)} + \|\tilde{u}_{k_0+1}\|_{L^q(B_2)} \right) \leq c' r^{-n/2} \|f\|_{L^2(\#)},$$

where the constants $c, c'$ depend only on $n, p,$ and $Q$.

By the Sobolev Embedding Theorem, we now conclude that $\tilde{u}_{k_0+1} \in C^1(B_1)$ and

$$|\nabla \tilde{u}_{k_0+1}(x)| \leq C' \|\tilde{u}_{k_0+1}\|_{W^2_q(B_2)} \leq c'' r^{-n/2} \|f\|_{L^2(\#)}, \quad x \in B_1,$$

with $C'$ depending only on $n$, and $c'' := C' c'$.

Scaling back gives the estimate $|\nabla u_{k_0+1}(x)| \leq cr^{-n/2-1} \|f\|_{L^2(\#)}$, with a constant $c$ depending only on $n, p$, and $Q$. We conclude by combining (B.2) with the estimate (2.3). □

Since it fits well into the context of this appendix, we indicate here how to deal with (smooth) boundaries $\Sigma$ and some of the mapping properties of $R^m = (H + 1)^{-m}$ and of $R_{dec}^m = (H_{dec} + 1)^{-m}$, as required in the proof of Proposition 4.5. Recall from Section 4 that $\Sigma_s = \partial M_s$, for $s = 1, \ldots, \ell$, and $\Sigma = \bigcup_{s=1}^\ell \Sigma_s = \partial M_0$ where we now label by the index $s$ instead of $k$. The spaces $C^1(\mathcal{M})$ and $C^1(\mathcal{M}_s)$, for $s = 0, \ldots, \ell$, are as in Section 4.

**Proposition B.5.** For $m \in \mathbb{N}$, $m \geq \lfloor n/4 \rfloor + 2$ we have the following:

(a) $R^m$ is a bounded operator from $L^2_2(\mathcal{M})$ to $C^1(\mathcal{M})$;

(b) $R_{dec}^m$ is a bounded operator from $L^2_2(\mathcal{M})$ to $C^1(\mathcal{M}_0) \times \cdots \times C^1(\mathcal{M}_\ell)$. Furthermore, for all $f \in L^2_2(\mathcal{M})$ we have $(R_{dec}^m f | \Sigma = 0$.

(c) For any $K \subset M$ compact there exists a constant $C_K$ such that

$$\sup_{x \in K} \|u(x)\|_{L^2(\#)} + \|du(x)\|_{L^2(\#)} \leq C \|f\|_{L^2(\#)}, \quad (B.18)$$

for all $f \in L^2_2(\mathcal{M})$ and $u = R^m f$ or $u = R_{dec}^m f$.

One might say that this result is a routine consequence of elliptic regularity theory, and, indeed, its proof is very similar to the proof of Theorem B.1. Let again $r_0 : M \to (0, 1]$ denote the (continuous) function introduced in Proposition 2.3. Notice that, in Proposition B.5, we do not need to control the constants in our estimates as functions of $r_0(x)$, which simplifies the argument as compared to the proof of Theorem B.1 on the other hand, the presence of a boundary requires the use of appropriate tools from elliptic regularity theory.
Proof. For $K \subset M$ compact, there exists a constant $\varrho_0 > 0$ such that $r_0(x) \geq \varrho_0$ for all $x \in K$. The desired results in (a) and (c) for $R^m$ and $u = (H + 1)^{-m}f$ are now immediate from Theorem B.1 (but note that we could also use the simpler arguments given below).

We next consider $R^m_{\text{dec}}$. For any $x \in K$ there exists an open neighborhood $U_x$ which admits a system of harmonic coordinates $\Phi_x$ that map $U_x$ diffeomorphically to a Euclidean ball $B(0, r_x)$ of radius $r_x > 0$. By compactness, there exists a finite selection of points $x_1, \ldots, x_J \in K$ such that the union of $U_{x_1}, \ldots, U_{x_J}$ covers $K$. We may assume, in addition, that for any $j$ we either have $U_{x_j} \cap \Sigma = \emptyset$ or $x_j \in \Sigma$. We write $N_{s,j} := \Phi_{x_j}(M_s \cap U_{x_j}) \subset B(0, r_j)$, for $s = 0, \ldots, \ell$ and $j = 1, \ldots, J$.

Letting $r_j := r_{x_j}$ and $\Psi_j : B(0, r_j) \rightarrow M$ denote the inverse of $\Phi_{x_j}$, there exist radii $0 < r'_j < r_j$ with the property that the sets $\Psi_j(B(0, r'_j))$ also cover $K$. We therefore only have to produce the required bounds on Euclidean balls $B(0, r'_j)$.

As in eqn. (B.16), we write $u_0 := f$ and $u_k := (H_{\text{dec}} + 1)^{-k}f$ for $k \in \mathbb{N}$; we also let $u_{k,j} := u_k \circ \Psi_j$. We then have $\|u_k\| \leq \|f\|$ and $\|du_k\|_{L^2_r(T^*\mathcal{M})} \leq C$ for all $k$. Furthermore, the equation $(H_{\text{dec}} + 1)u_{k+1} = u_k$ in $L^2(\mathcal{M})$ implies that $u_{k+1}$ is a weak solution of the associated divergence form elliptic partial differential equation in local coordinates (cf. eqn. (B.1)) in the sets $N_{s,j}$, satisfying Dirichlet boundary conditions on $B(0, r_j) \cap \Phi_{x_j}(\partial M_s \cap U_{x_j})$, for $s = 0, \ldots, \ell$.

For simplicity of notation, let us assume that $r_j = 1$ and $r'_j = 1/2$. It will be convenient to introduce the radii $\varrho_\nu := 1/2 + 2^{-\nu-1}$, for $\nu \in \mathbb{N}_0$, so that $1/2 < \varrho_\nu+1 < \varrho_\nu < 1$ for all $\nu \in \mathbb{N}$. We may assume without loss of generality that the domains $N_{s,j,\nu} := N_{s,j} \cap B(0, \varrho_\nu)$ are Lipschitz so that the Sobolev Embedding Theorem in the form of [GT83, Eqn. (7.30)] can be applied to each of the $N_{s,j,\nu}$ (with a constant which may depend on $\nu$).

(i) Now, applying [GT83] Thm. 8.12] in a suitable domain with $C^2$-boundary yields $u_{k,j} \in H^2(N_{s,j,1})$ for $k \in \mathbb{N}$; furthermore, $u_{k,j}$ satisfies the estimate (8.25) of [GT83] with $\Omega := N_{s,j,1}$ and $\varphi = 0$. (If $x_i \in \Sigma$ the application of [GT83] Thm. 8.12] requires some care: we first pick a cut-off function $\psi \in C_c^\infty(B(0, 1))$ which is 1 on $B(0, \varrho_1)$ and plug $\psi u_{k,j}$ into the p.d.e. satisfied by $u_{k,j}$.) In addition, we may conclude that $u_{k,j}$ is a strong solution of the associated partial differential equation in $N_{s,j,1}$; cf. eqn. (B.17).

(ii) Since, by the first step, $u_{1,j} \in H^2(N_{s,j,1})$, the Sobolev Embedding Theorem yields $u_{1,j} \in L^q_{\varphi_i}(N_{s,j,1})$ with $q_i := \sigma(2)$, where $\sigma(\cdot)$ is defined in (B.8). Now [GT83] Theorem 9.13 and Lemma 9.16] imply that $u_{2,j} \in W^2_{q_i}(N_{s,j,2})$ together with an estimate as in Lemma B.2

$$\|u_{2,j}\|_{W^2_{q_i}(N_{s,j,2})} \leq C\left(\|(H_{\text{dec}} + 1)u_{2,j}\|_{L^2(N_{s,j,1})} + \|u_{2,j}\|_{L^2(N_{s,j,1})}\right)$$
$$\leq C\left(\|u_{1,j}\|_{L^2(N_{s,j,1})} + \|u_{2,j}\|_{L^2(N_{s,j,1})}\right) \leq C\|f\|_{L^2(\mathcal{M})}.$$

In a similar fashion we subsequently obtain an estimate of $\|u_{3,j}\|_{W^2_{q_i}(N_{s,j,3})}$ in terms of $\|f\|_{L^2(\mathcal{M})}$ etc.

(iii) Iterating the above steps $m$ times we arrive at $q_m > n/2$ with an estimate

$$\|u_{m+1,j}\|_{W^2_{q_m}(N_{s,j,m+1})} \leq C\|f\|_{L^2(\mathcal{M})}.$$

A variant of the Sobolev Embedding Theorem ([GT83] Thm. 7.26], which uses the Sobolev Extension Theorem) yields $u_{m+1,j} \in C^1(\overline{N_{s,j,m+2}})$ and a corresponding estimate estimate of $|u_{m+1,j}(x)|$ and $|\nabla u_{m+1,j}(x)|$ for $x \in N_{s,j,m+2}$. The estimate of $|u(x)|$ is now immediate, while the estimate of $|\nabla u(x)|$ follows as in the proof of Theorem B.1.
We state below the standard stationary-phase estimate (cf. [RS 79, Corollary to Thm. XI.14]):

We only give the proof for $\Theta$. We first prove the existence of the $\Theta$. Let $\varphi \in L^2_2(\mathbb{R})$ satisfy $\mathcal{F}\varphi \in C_c^\infty(\mathbb{R})$ with supp $\mathcal{F}\varphi = K$. Then, for any open set $U$ such that $K \subset U$ and for any $m \in \mathbb{N}$, there is a constant $C_m$ such that

$$|e^{-ith_0 \varphi}(x)| \leq C_m(1 + |x| + |t|)^{-m}, \quad \text{for all } x \in \mathbb{R} \text{ such that } x/(2t) \notin U. \quad (C.1)$$

Similar estimates hold for the $x$-derivatives of $e^{-ith_0 \varphi}$. Let $h_{t,0}$ be the self-adjoint realizations of $-d^2/dx^2$ in $L^2_2(\mathbb{R})$ with Dirichlet boundary condition at zero, i.e., with domain $\text{Dom}(h_{t,0}) = H^1(\mathbb{R}) \cap H^2(\mathbb{R})$. We denote by $\mathcal{T}_\pm$ the natural embeddings of $L^2_2(\mathbb{R})$ into $L^2_2(\mathbb{R})$ (extension by zero) and by

$$\mathcal{H}_\pm := \mathcal{F}^{-1}(\mathcal{T}_\pm L^2_2(\mathbb{R})) = \{ u \in \mathcal{H} | \hat{u}|_{\mathbb{R}^+} = 0 \}.$$ 

We define the wave operators

$$\Theta^+_\pm := \text{s-lim}_{t \to \pm \infty} e^{ith_0 t} e^{-ith_{t,0}} \quad \text{and} \quad \Theta^-_\pm := \text{s-lim}_{t \to \pm \infty} e^{ith_0 t} e^{-ith_{-t,0}},$$

provided that the strong limits exist.

**Lemma C.1.** The wave operators $\Theta^+_\pm$ and $\Theta^-_\pm$ exist, are isometric and we have

$$\text{Ran } \Theta^+_\pm = \mathcal{H}_\pm \quad \text{and} \quad \text{Ran } \Theta^-_\pm = \mathcal{H}_\mp.$$

**Proof.** We only give the proof for $\Theta^+_\pm$; the case of $\Theta^-_\pm$ is similar. We first prove the existence of the $\Theta^+_\pm$. Let $j \in C^2(\mathbb{R})$ satisfy $0 \leq j(x) \leq 1$, $j(x) = 0$ for $x < 0$ and $j(x) = 1$ for $x \geq 2$. We also denote by $j$ the bounded operator from $L^2_2(\mathbb{R})$ into $\mathcal{H}$ given by multiplication by $j$. As a first step, we replace $\mathcal{T}_\pm$ in the definition of $\Theta^+_\pm$ by $j$, using a well-known and simple argument; cf. [RS 79] p. 35 and problem 18: The function $1 - j$, defined on $\mathbb{R}^+$, is bounded and has compact support; hence the Rellich local compactness theorem implies that $(1 - j)(h_{+0} + 1)^{-1}$ is compact. Since $h_{+0}$ is absolutely continuous, we have s-lim$_{t \to \pm \infty} (1 - j)(h_{+0} + 1)^{-1} e^{-ith_{+0}} = 0$ and therefore

$$(1 - j)e^{-ith_{+0}} \varphi = (1 - j)(h_{+0} + 1)^{-1} e^{-ith_{+0}}(h_{+0} + 1)\varphi \to 0$$

in norm, for all $\varphi \in C_c^\infty(\mathbb{R})$. Therefore, it is enough to prove the existence of the wave operators

$$M_\pm := \text{s-lim}_{t \to \pm \infty} e^{ith_0 t} j e^{-ith_{+0}}.$$
By Duhamel’s formula, to prove the existence of \( M_+ \varphi_{\text{odd}} \) it is enough to show that
\[
\int_0^\infty \| e^{ith_0} (h_0 j - j h_{0,+}) e^{-ith_0} \varphi_{\text{odd}} \|_{L^2(\mathbb{R})} \, dt
= \int_0^\infty \left\| \left( j'' + 2j' \frac{d}{dx} \right) \left( (e^{-ith_0} \varphi)(x) - (e^{-ith_0} \varphi)(-x) \right) \right\|_{L^2(\mathbb{R})} \, dt < \infty,
\]
but this is immediate from (C.1). The case of \( M_\varphi \) follows in the same way. \( \Theta_\pm \) are isometric since \( e^{ith_0} \) and \( e^{-ith_0} \) are unitary.

Set \( \hat{\mathcal{D}}_\pm := \{ \varphi_\pm \in L_2(\mathbb{R}) \mid \mathcal{F} \varphi_\pm \in C_\infty^c(\mathbb{R}_\pm) \} \) and
\[
\varphi_{\pm, t}(x) := (e^{-ith_0} \varphi_\pm)(x) - (e^{-ith_0} \varphi_\pm)(-x), \quad \varphi_\pm \in \hat{\mathcal{D}}_\pm.
\]
Then, as above,
\[
\varphi_{\pm, t} = e^{-ith_0} \varphi_{\pm, 0}, \quad \text{where} \quad \varphi_{\pm, 0}(x) = \varphi_\pm(x) - \varphi_\pm(-x), \quad \varphi_\pm \in \hat{\mathcal{D}}_\pm.
\]
It follows from (C.1) that \( \lim_{t \to \pm \infty} \| e^{-ith_0} \varphi_\pm - t e^{-ith_0} \varphi_{\pm, 0} \|_{L^2(\mathbb{R})} = 0 \) and we see that
\[
\varphi_\pm = \Theta_\pm^+ \varphi_{\pm, 0}, \quad \forall \varphi_\pm \in \hat{\mathcal{D}}_\pm. \tag{C.2}
\]
Then, \( \hat{\mathcal{D}}_\pm \subset \text{Ran} \Theta_\pm^+ \), and as the \( \Theta_\pm^+ \) are isometric,
\[
\mathcal{H}_\pm = \overline{\hat{\mathcal{D}}_\pm \subset \text{Ran} \Theta_\pm^+}. \tag{C.3}
\]
We prove in the same way, using (C.1), that \( \hat{\mathcal{D}}_\pm \subset (\text{Ran} \Theta_\pm^+) \perp \), and then that
\[
\mathcal{H}_\pm \subset (\text{Ran} \Theta_\pm^+) \perp. \tag{C.4}
\]
Finally, as \( L_2(\mathbb{R}) = \mathcal{H}_- \oplus \mathcal{H}_+ \) and \( L_2(\mathbb{R}) = \text{Ran} \Theta_\pm^+ \oplus (\text{Ran} \Theta_\pm^+) \perp \), equations (C.3) and (C.4) imply that \( \mathcal{H}_\pm = \text{Ran} \Theta_\pm^+ \).

Let \( w \) be a real-valued, bounded function defined on \( \mathbb{R} \) satisfying
\[
|w(x)| \leq C(1 + |x|)^{-1-\beta}
\]
for some constants \( C \geq 0, \beta > 0 \), and let
\[
h := h_0 + w.
\]
The operator \( h \) is self-adjoint in \( \mathcal{H} \) with domain \( \text{Dom} h = H^2(\mathbb{R}) \). We consider the wave operators
\[
\Omega_\pm^+ := \text{s-lim}_{t \to \pm \infty} e^{ith} t e^{-ith_0}, \quad \text{and} \quad \Omega_\pm^- := \text{s-lim}_{t \to \pm \infty} e^{ith} t e^{-ith_0}.
\]

**Theorem C.2.** The wave operators \( \Omega_\pm^+, \Omega_\pm^- \) exist and are partially isometric. Moreover, their initial subspaces, respectively, \( \mathcal{H}_{\text{ini}, \pm}^+ \) and \( \mathcal{H}_{\text{ini}, \pm}^- \), are given by
\[
\mathcal{H}_{\text{ini}, \pm}^+ = \mathcal{H}_\pm^+ \quad \text{and} \quad \mathcal{H}_{\text{ini}, \pm}^- = \mathcal{H}_\pm^- \tag{C.5}
\]
Furthermore,
\[
\text{Ran} \Omega_\pm^+ \neq \text{Ran} \Omega_\pm^- \quad \text{and} \quad \text{Ran} \Omega_\pm^- \neq \text{Ran} \Omega_\pm^+.
\]

**Proof.** The existence of \( \Omega_\pm^+ \varphi \) and \( \Omega_\pm^- \varphi \) for \( \mathcal{F} \varphi \in C_\infty^c(\mathbb{R} \setminus \{0\}) \) follows upon replacing \( t_\pm \) by multiplication with a smooth cut-off function together with Duhamel’s formula and equation (C.1). We omit the details. This proves the existence of the wave operators in a dense set, and by continuity in \( \mathcal{H} \).

By (C.1), we have for \( \varphi_+ \in \hat{\mathcal{D}}_+, \)
\[
\| t_+ e^{-ith_0} \varphi_+ \| \to \begin{cases} \| \varphi_+ \|, & \text{as } t \to \infty, \\ 0, & \text{as } t \to -\infty, \end{cases}
\]
and similarly for $\varphi_+ \in \hat{\mathcal{D}}_+$. It follows that
\[
\|\Omega^\pm \varphi\| = \begin{cases} 
\|\varphi\|, & \text{if } \varphi \in \hat{\mathcal{D}}_\pm, \\
0, & \text{if } \varphi \in \hat{\mathcal{D}}_\mp.
\end{cases}
\] (C.6)

Since the subspaces $\hat{\mathcal{D}}_\pm$ are dense in $\mathcal{H}_\pm$, the first equality in (C.5) follows from (C.6). We prove the second equality in (C.5) in the same way.

We now show that $\text{Ran } \Omega^+_+ \neq \text{Ran } \Omega^+_+$. The intuition behind our proof is as follows. Consider an incoming state with large negative mean velocity and assume that this state is localized near $+\infty$ for large negative times. Such a state will be in the range of $\Omega^+_+$. As time increases it will propagate to the left, but as it has large velocity it will "go across" the potential $w$ and will travel to $-\infty$ as time goes to infinity (with only a small reflected part of the state travelling to $+\infty$). Since this state has a non-trivial component localized near $-\infty$ for large positive times, it cannot be in the range of $\Omega^+_-$.

Let us consider the following asymptotic state with high negative velocity,
\[
\varphi^- \vphantom{e} := e^{-ivx} \varphi_0, \quad \text{with } \mathcal{F} \varphi_0 \in C^\infty_c(\mathbb{R}).
\]
As
\[
(\mathcal{F} \varphi^-)(k) = (\mathcal{F} \varphi_0)(k + v),
\]
this state will have large negative velocity if $v > 0$ is taken large enough.

Let us introduce the wave operators
\[
\Omega^+_\pm := \text{s-lim}_{t \to \pm \infty} e^{it\theta} e^{-ith_0}.
\]
The existence of these wave operators follows from (C.1) and Duhamel's formula; they are also complete (cf. [E78, S79]). Defining $\psi^- := \Omega^- \varphi^-$, we have $\text{s-lim}_{t \to -\infty} e^{i\theta} e^{-ith_0} \varphi^-_0 = 0$, by (C.1), and then
\[
\psi^- := \text{s-lim}_{t \to -\infty} e^{it\theta} e^{-ith_0} \varphi^-_0 = \text{s-lim}_{t \to -\infty} e^{it\theta} e^{-ith_0} \varphi^-_0 = \Omega^- \varphi^-.
\] (C.7)

This proves that $\psi^- \in \text{Ran } \Omega^-_+$. It follows from Corollary 2.3 of [EW95] and the intertwining relation $e^{-ith\varphi} = \Omega^- e^{-ith_0}$ that
\[
\|e^{-ith} \psi^-_v - e^{-ith_0} \varphi^-_v\|_{L^2(\mathbb{R})} = \mathcal{O}(1/v),
\]
uniformly in $t \in \mathbb{R}$. Hence, by (C.1),
\[
\lim_{t \to \infty} \|\mathcal{L}_- e^{-ith} \psi^-_v\|_{L^2(\mathbb{R})} \geq \lim_{t \to \infty} \|\mathcal{L}_- e^{-ith_0} \varphi^-_v\|_{L^2(\mathbb{R})} - \mathcal{O}(1/v)
\]
\[
= \|\varphi^-_v\|_{L^2(\mathbb{R})} - \mathcal{O}(1/v) \geq 1/2 \|\varphi^-_v\|_{L^2(\mathbb{R})} > 0,
\]
for $v$ large enough, which proves that $\psi^- \notin \text{Ran } \Omega^-_+$.

The asymptotic state $\varphi^+_+$, defined as $\varphi^+_+(x) := e^{ixv} \varphi_0$, with $\mathcal{F} \varphi_0 \in C^\infty_c(\mathbb{R})$, has large positive velocity if $v > 0$ is large enough. We define $\psi^+_+ := \Omega^+ \varphi^+_+$ and prove as above that $\psi^+_+ = \Omega^- \varphi^+_+$, so that $\psi^+_+ \notin \text{Ran } \Omega^-_+$, which proves that $\text{Ran } \Omega^+_+ \neq \text{Ran } \Omega^-_+$. □

We need yet another set of wave operators. Let us denote by $\Lambda^+_\pm$ and $\Lambda^-_\pm$ the wave operators
\[
\Lambda^+_\pm := \text{s-lim}_{t \to \pm \infty} e^{it\theta} e^{-ith_0} \quad \text{and} \quad \Lambda^-_\pm := \text{s-lim}_{t \to \pm \infty} e^{it\theta} e^{-ith_0}.
\] (C.8)

**Lemma C.3.** The wave operators $\Lambda^+_\pm$ and $\Lambda^-_\pm$ exist, are isometric and, furthermore,
\[
\text{Ran } \Lambda^+_+ \neq \text{Ran } \Lambda^-_+, \quad \text{and} \quad \text{Ran } \Lambda^-_+ \neq \text{Ran } \Lambda^-_+.
\] (C.9)

**Proof.** By the chain rule
\[
\Lambda^+_+ = \Omega^+_+ \circ \Theta^+_+ \quad \text{and} \quad \Lambda^-_+ = \Omega^-_+ \circ \Theta^-_+.
\] (C.10)
Then, the lemma follows by Lemma [C.1] and Theorem [C.2]. □
Remark C.4. The above proofs yield explicit examples of asymptotic states that belong to Ran $\Lambda^\pm$ and do not belong to Ran $\Lambda^\mp$. In fact, it follows from (C.2) that
\[ \varphi^+_v = \Theta_+^\pm \varphi^+_{v,0}, \quad \text{where} \quad \varphi^+_{v,0} := \varphi^+_v(x) - \varphi^+_v(-x). \] (C.11)
We prove in the same way that
\[ \varphi^-_v = \Theta_-^\pm \varphi^-_{v,0}. \]
By (C.7), (C.10) and (C.11) we have
\[ \psi^-_v = \Omega^+ \Theta^- \varphi^-_{v,0} = \Lambda^- \varphi^-_{v,0}. \] (C.12)
Moreover as $\psi^-_v \notin \text{Ran} \Omega^+_v$ it follows from (C.10) that $\psi^-_v \notin \text{Ran} \Lambda^+_v$. We prove in the same way that
\[ \psi^+_v = \Omega^- \Theta^- \varphi^+_{v,0} = \Lambda^- \varphi^+_{v,0}. \] (C.13)
Moreover as $\psi^+_v \notin \text{Ran} \Omega^+_v$ it follows from (C.10) that $\psi^+_v \notin \text{Ran} \Lambda^+_v$.
By the definition of $\psi^+_v = \Omega^+ \varphi^+_v$, equation (C.13), and Corollary 2.3 of [EW95], we have
\[ \Lambda^- \varphi^+_{v,0} = \Omega^+ \varphi^+_v = \varphi^+_v + \Theta^+ \left( \frac{1}{v} \right). \] (C.14)
As in the proof of (C.7) we prove that
\[ \Omega^+_v \varphi^+_v = \Omega^+_v \varphi^+_v. \] (C.15)
Then, as above, it follows from (C.10), (C.11), (C.15) and Corollary 2.3 of [EW95] that
\[ \Lambda^+_v \varphi^+_{v,0} = \varphi^+_{v,0} + \Theta^+ \left( \frac{1}{v} \right). \] (C.16)
By (C.14) and (C.16) and since $\Lambda^+_v$ is isometric, we have
\[ (\Lambda^+_v)^* \Lambda^- \varphi^+_{v,0} = \varphi^+_{v,0} + \Theta^+ \left( \frac{1}{v} \right). \] (C.17)
Then
\[ (\Lambda^+_v)^* \Lambda^- \varphi^+_{v,0} \neq 0, \quad v \gg 1, \] (C.18)
proving that the $-$ and the $+$ channels are open to each other. Note that $\varphi^+_{v,0}$ is a high-velocity asymptotic state, coming in from the left for large negative times; as time increases, it travels to the right and is transmitted through the potential—with a reflected part that is very small for $v$ large—and it goes to $+\infty$ as time tends to $+\infty$.

Appendix D. Lower bounds for the injectivity radius

The main purpose of this final appendix is to acquaint the reader with a comparison result of M"uller and Salomonsen [MS07, Prop. 2.1] for the injectivity radius of two complete manifolds with bounded curvature. We use their result in a two-step process to deal with warped products (and perturbed warped products) where the curvature does not obey a global curvature bound, as happens for shrinking ends of dimension $n \geq 3$.

In some sense, we attempt to produce local versions of [MS07, Prop. 2.1]; note that the proof given in their paper uses some non-local arguments and completeness is an issue.

The basic idea is to extend finite sections of our manifolds to complete manifolds with two cylindrical ends and to obtain a lower bound for the injectivity radius of these extended manifolds by comparison with a straight cylinder.

We begin with the comparison theorem for the injectivity radius of Müller and Salomonsen [MS07]. Our statement displays a lower bound with explicit constants taken from their proof; cf. eqn. (D.2).
Lemma D.2. Let \( M \) denote a smooth \( n \)-dimensional manifold. Suppose that the Riemannian manifolds \( \mathcal{M}_0 := (M, g_0) \) and \( \mathcal{M}_1 := (M, g_1) \) are complete with quasi-isometric metrics \( g_0 \) and \( g_1 \), i.e.,

\[ \eta g_0 \leq g_1 \leq \eta^{-1} g_0, \tag{D.1} \]

for some constant \( \eta \in (0, 1] \). Furthermore, suppose that the sectional curvature of \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) is bounded (in absolute value) by some constant \( K \geq 0 \). Let \( \text{inj}_{\mathcal{M}_0}(x) \) and \( \text{inj}_{\mathcal{M}_1}(x) \) denote the injectivity radius of \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \), respectively, at the point \( x \in M \). We then have

\[ \text{inj}_{\mathcal{M}_1}(x) \geq \frac{1}{2} \min \left\{ \frac{\pi}{\eta \text{inj}_{\mathcal{M}_0}(x)}, \frac{\pi}{\eta} \right\}, \quad x \in M. \tag{D.2} \]

We next use the above comparison theorem to obtain lower bounds for the injectivity radius of manifolds \( \mathcal{M} = (M_+, g) \) where \( M_+ = \mathbb{R}_+ \times S^{n-1}, \mathbb{R}_+ = (0, \infty), \) and \( g \) is a warped product metric generated by a function \( r \). We require a local bound on the variation of \( r \), cf. eqn. (D.3). From this point on, we restrict our attention to the case \( n \geq 3 \). The corresponding results in the case \( n = 2 \) are obtained by dropping the term \( (1 + \dot{r}(t)^2)/r(t)^2 \) in the definition of \( \kappa \) in eqn. (D.5) below etc.

Lemma D.2. Let \( r : \mathbb{R}_+ \to \mathbb{R}_+, \ r \in C^2(\mathbb{R}_+), \) satisfy the condition

\[ \frac{1}{m} r(s_0) \leq r(s) \leq mr(s_0), \quad s \in [s_0 - 2, s_0 + 2], \tag{D.3} \]

for all \( s_0 > 2 \), where \( m \geq 1 \) is a constant. Let \( \mathcal{M}_+ = (M_+, ds^2 + r(s)^2 g_{S^{n-1}}) \) with \( g_{S^{n-1}} \) denoting the standard metric on \( S^{n-1} \).

Then \( \text{inj}_{\mathcal{M}_+}(s) \), the injectivity radius of \( \mathcal{M}_+ \) at the points \( (s, \omega) \in M_+ \), satisfies the lower bound

\[ \text{inj}_{\mathcal{M}_+}(s) \geq C_0 \min \left\{ \frac{1}{\sqrt{\kappa(s)}}, r(s) \right\}, \quad s > 2, \tag{D.4} \]

where \( C_0 > 0 \) is a constant that is independent of \( s \), and

\[ \kappa(s) := \max_{|s-t| \leq 2} \max \left\{ \frac{\dot{r}(t)}{r(t)}, \frac{1 + \dot{r}(t)^2}{r(t)^2}, 1 \right\}. \tag{D.5} \]

Proof. We fix a function \( \varphi \in C^\infty(-2, 2) \) satisfying \( 0 \leq \varphi \leq 1 \) and \( \varphi(x) = 1 \) for \(-1 \leq x \leq 1 \). Let \( c_\varphi := \max\{\|\varphi\|_\infty, \|\varphi''\|_\infty\} \). Furthermore, let \( \varphi_{s_0} = \varphi(\cdot - s_0) \) and define

\[ \varrho_{s_0}(s) := \varphi_{s_0}(s)r(s) + (1 - \varphi_{s_0}(s))r(s_0), \quad s \in \mathbb{R}, \tag{D.6} \]

for \( s_0 \geq 2 \); note that \( \varrho_{s_0} \) is defined on all of \( \mathbb{R} \). We have

\[ \frac{1}{m} r(s_0) \leq \varrho_{s_0}(s) \leq mr(s_0), \quad s \in \mathbb{R}. \tag{D.7} \]

Let \( \mathcal{M}_{s_0} = (M, g_{s_0}) \) with \( M := \mathbb{R} \times S^{n-1} \) and \( g_{s_0} = ds^2 + g_{s_0}^2 g_{S^{n-1}} \). Then \( \mathcal{M}_{s_0} \) is complete and, by (D.7), \( \mathcal{M}_{s_0} \) is quasi-isometric (with a constant \( \eta := 1/m \in (0, 1) \)) to \( \widetilde{\mathcal{M}_{s_0}} = (M, ds^2 + r(s_0)^2 g_{S^{n-1}}) \), a cylinder of constant radius \( r(s_0) \).

In order to obtain a curvature bound for \( \mathcal{M}_{s_0} \) we first note that \( |r(s) - r(s_0)| \leq r(s) \) if \( r(s) \geq r(s_0) \) while \( |r(s) - r(s_0)| \leq r(s_0) \leq mr(s) \) if \( r(s) < r(s_0) \); in both cases the estimate \( |r(s) - r(s_0)| \leq mr(s) \) is valid. For \( |s - s_0| \leq 2 \) the derivatives of the function \( \varrho_{s_0} \) satisfy

\[ |\varrho'_{s_0}(s)| \leq \|\varphi\|_\infty |r(s) - r(s_0)| + |\dot{r}(s)| \leq c_\varphi mr(s) + \sqrt{\kappa(s)} r(s) \leq c_1 \sqrt{\kappa(s_0)} r(s_0), \]

by definition of \( c_\varphi \) and \( \kappa \). Similarly, we have for \( |s - s_0| \leq 2 \)

\[
|\varrho''_{s_0}(s)| \leq \|\varphi''\|_\infty |r(s) - r(s_0)| + 2 \|\varphi''\|_\infty |\dot{r}(s)| + |\ddot{r}(s)| \\
\leq c_\varphi mr(s) + 2c_\varphi \sqrt{\kappa(s_0)} r(s) + \kappa(s_0) r(s) \leq c_2 \kappa(s_0) r(s_0).
\]
We now find for $|s - s_0| \leq 2$

$$\frac{|1 - \bar{\gamma}_{s_0}(s)|^2}{\bar{\gamma}_{s_0}(s)^2} \leq \frac{m^2}{r(s_0)^2} + c_1^2\kappa(s_0) \leq c_2\kappa(s_0), \quad \frac{|\bar{\gamma}_{s_0}(s)|}{\bar{\gamma}_{s_0}(s)} \leq c_3\kappa(s_0).$$

Thus the sectional curvature of $\mathcal{M}_{s_0}$ is bounded by $C_1\kappa(s_0)$ with some constant $C_1 \geq 0$ which is independent of $s_0 \geq 2$.

We may now apply Proposition D.1 to $\mathcal{M}_{s_0}$ and $\mathcal{M}_{s_0}$ with $\eta := 1/m$ and $K := C_1\kappa(s_0)$ to obtain

$$\operatorname{inj}_{\mathcal{M}_{s_0}} \geq C_0 \min\left\{ \frac{1}{\sqrt{\kappa(s_0)}}, r(s_0) \right\}.$$ Since the manifolds $\mathcal{M}_{s}$ and $\mathcal{M}_{s_0}$ have the same metric for $|s - s_0| \leq 1$ it is clear that $\operatorname{inj}_{\mathcal{M}_{s_0}}(s_0) \geq \min\{\operatorname{inj}_{\mathcal{M}_{s}}, 1\}$, and the desired result follows. □

The idea of proof used in obtaining Lemma D.2 can easily be generalized to perturbations of a warped product metric. For simplicity we work here with $M = \mathbb{R} \times S^{n-1}$. The functions $\varphi_{s_0} \in C^\infty(-2, 2)$ are as above.

Proposition D.3. Let $M = \mathbb{R} \times S^{n-1}$, let $r : \mathbb{R} \to \mathbb{R}_+$ be a $C^2$-function satisfying the estimate (D.3) for all $s_0 \in \mathbb{R}$. Also define $\kappa_0 = \kappa(s)$ as in eqn. (D.5), but now for all $s \in \mathbb{R}$. Let $g_0$ denote the metric $ds^2 + r(s)^2g_{S^{n-1}}$.

Let $g$ denote another metric on $M$ that satisfies the following conditions:

(i) There exists a constant $m \geq 1$ such that, for all $s_0 \in \mathbb{R}$,

$$\frac{1}{m}g_0(s_0) \leq g(s) \leq mg_0(s_0), \quad s \in [s_0 - 2, s_0 + 2]. \quad (D.8)$$

(ii) There exists a (continuous) function $\kappa$ on $\mathbb{R}$ such that the sectional curvature of $\mathcal{M}_{s_0} := (M, \varphi_{s_0}(s)g(s) + (1 - \varphi_{s_0}(s))g_0(s_0))$ is bounded (in absolute value) by $\kappa(s_0)$, for all $s_0 \in \mathbb{R}$.

Then $\operatorname{inj}_{\mathcal{M}}(s, \omega)$, the injectivity radius of $\mathcal{M} = (M, g)$ at $(s, \omega) \in M$, obeys the lower bound

$$\operatorname{inj}_{\mathcal{M}}(s, \omega) \geq C \min\left\{ \frac{1}{\sqrt{\kappa_0(s)}}, \frac{1}{\sqrt{\kappa(s)}}, r(s) \right\}, \quad s \in \mathbb{R}, \quad (D.9)$$

for some positive constant $C$.

Proof. For $s_0 \geq 2$ given, let us consider the (complete) manifold

$$\mathcal{M}_{0, s_0} := (M, \varphi_{s_0}(s)g_0(s) + (1 - \varphi_{s_0}(s))g_0(s_0)).$$

Note that the manifolds $\mathcal{M}_{0, s_0}$ and $\mathcal{M}_{s_0}$ are identical outside the interval $[s_0 - 2, s_0 + 2]$. Furthermore, both have a metric that is independent of $s$ outside of $[s_0 - 2, s_0 + 2]$. It is immediate from assumption (i) that $\mathcal{M}_{0, s_0}$ and $\mathcal{M}_{s_0}$ are quasi-isometric with a constant $\eta = 1/m$. As in the proof of Lemma D.2, one shows that the sectional curvature of $\mathcal{M}_{0, s_0}$ is bounded by $C_1\kappa_0(s_0)$, with a constant $C_1$ that is independent of $s_0 \in \mathbb{R}$. As for $\mathcal{M}_{s_0}$, assumption (i) says that the sectional curvature of $\mathcal{M}_{s_0}$ is bounded by $\kappa(s_0)$. By Lemma D.2 the injectivity radius of $\mathcal{M}_{0, s_0}(s)$ is bounded below by $C_0 \min\{(\kappa_0(s))^{-1}, r(s)\}$ and thus Proposition D.1 yields the estimate

$$\operatorname{inj}_{\mathcal{M}_{s_0}}(s_0, \omega) \geq \min\left\{ \frac{\eta^2\pi}{\kappa(s_0)}, \frac{\eta C_0}{\sqrt{\kappa_0(s_0)}}, \eta C_0 r(s_0) \right\}$$

and the desired estimate follows. □

Remark D.4. In concrete applications the required bound $\kappa$ on the sectional curvature can be obtained by direct calculation in terms of the metric (cf., e.g., [ON83], p. 204 ff.).
Remark D.5. It is clear from the assumptions of Lemma D.2 and Proposition D.3 (in particular, eqns. (D.3) and (D.8)) that the lower bounds (D.4) and (D.9) for the radius of injectivity also hold for the homogenized radius of injectivity $\iota_{\#}$ as defined in equation (2.7), possibly with a smaller positive constant. In both cases our assumptions imply that the Ricci curvature satisfies the lower bound required in Proposition 2.3 in the form $\text{Ric}_{\#}(s, \omega) \geq - (n-1) \kappa(s_0)$ for $|s - s_0| < 2$ since the Ricci curvature at a point $x$ is the sum of the sectional curvatures of any $n - 1$ orthogonal non-degenerate planes through $x$ ([O’N83, p. 88]). Therefore, Proposition 2.3 yields that the homogenized harmonic radius $\iota_{\#}$ obeys lower bounds analogous to the lower bounds for $\text{inj}_{\#}$; in other words, the function $r_0(x)$ of Proposition 2.3 can be read off from the right hand side of eqns. (D.4) or (D.9).

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