Exploring Increasing-Chord Paths and Trees *

Yeganeh Bahoo1, Stephane Durocher1, Sahar Mehrpour2, and Debajyoti Mondal3

1Department of Computer Science, University of Manitoba, Winnipeg, Canada, 
\{bahoo,durocher\}@cs.umanitoba.ca

2School of Computing, University of Utah, Utah (UT), USA, 
mehrpour@cs.utah.edu

3Cheriton School of Computer Science, University of Waterloo, Canada, 
dmondal@uwaterloo.ca

February 28, 2017

Abstract

A straight-line drawing $\Gamma$ of a graph $G = (V, E)$ is a drawing of $G$ in the Euclidean plane, where every vertex in $G$ is mapped to a distinct point, and every edge in $G$ is mapped to a straight line segment between their endpoints. A path $P$ in $\Gamma$ is called increasing-chord if for every four points (not necessarily vertices) $a, b, c, d$ on $P$ in this order, the Euclidean distance between $b, c$ is at most the Euclidean distance between $a, d$. A spanning tree $T$ rooted at some vertex $r$ in $\Gamma$ is called increasing-chord if $T$ contains an increasing-chord path from $r$ to every vertex in $T$. In this paper we prove that given a vertex $r$ in a straight-line drawing $\Gamma$, it is NP-complete to determine whether $\Gamma$ contains an increasing-chord spanning tree rooted at $r$. We conjecture that finding an increasing-chord path between a pair of vertices in $\Gamma$, which is an intriguing open problem posed by Alamdari et al., is also NP-complete, and show a (non-polynomial) reduction from the 3-SAT problem.

1 Introduction

In 1995, Icking et al. [6] introduced the concept of self-approaching curve. A curve is called self-approaching if for any three points $a, b$ and $c$ on the curve in this order, $|bc| \leq |ac|$, where $|xy|$ denotes the Euclidean distance between $x$ and $y$. A curve is called increasing-chord if it is self-approaching in both directions. In this paper we examine increasing-chord paths in the context of planar graph drawing. A straight-line drawing $\Gamma$ of a graph $G$ in $\mathbb{R}^2$ maps every vertex of $G$ to a distinct point, and every edge of $G$ to a straight line segment. A path $P$ in $\Gamma$ is called an increasing-chord path if for every four points (not necessarily vertices) $a, b, c, d$ on $P$ in this order, the inequality $|bc| \leq |ad|$ holds. $\Gamma$ is called an increasing-chord drawing if there exists an increasing-chord path between every pair of vertices in $\Gamma$.

The study of increasing-chord drawings was motivated by greedy routing in geometric networks, where given two vertices $s$ and $t$, the goal is to send a message from $s$ to $t$ using some greedy strategy, i.e., at each step, the next vertex in the route is selected greedily as a function of the positions of the neighbors of the current vertex $u$ relative to the positions of $u$, $s$, and $t$ [10]. A polygonal path $u_1, u_2, \ldots, u_k$ is called a greedy path if for every $i$, where $0 < i < k$, the inequality $|u_iu_k| > |u_{i+1}u_k|$ holds. If a straight-line drawing is greedy, i.e., there exists a greedy path between every pair of vertices, then it is straightforward to route the message between any pair of vertices by following a greedy path. For example, we can repeatedly forward the message to some node which is closer to the destination than the current vertex. A disadvantage

\footnote{Work of S. Durocher and D. Mondal is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).}
of a greedy drawing, however, is that the dilation, i.e., the ratio of the graph distance to the Euclidean distance between a pair of vertices, may be unbounded. Increasing-chord drawings were introduced to address this problem, where the dilation of increasing-chord drawings can be at most $2\pi/3 \leq 2.094$.

Alamdari et al. [1] examined the problem of recognizing increasing-chord drawings, and the problem of constructing such a drawing on a given set of points. They showed that it is NP-hard to recognize increasing-chord drawings in $\mathbb{R}^3$, and asked whether it is also NP-hard in $\mathbb{R}^2$. They also proved that for every set of $n$ points $P$ in $\mathbb{R}^2$, one can construct an increasing-chord drawing $\Gamma$ with $O(n)$ vertices and edges, where $P$ is a subset of the vertices of $\Gamma$. In this case, $\Gamma$ is called a Steiner network of $P$, and the vertices of $\Gamma$ that do not belong to $P$ are called Steiner points. Dehkordi et al. [3] proved that if $P$ is a convex point set, then one can construct an increasing-chord network with $O(n \log n)$ edges, and without introducing any Steiner point; the resulting network is non-planar in general. Mastakas and Symvonis [8] improved the $O(n \log n)$ upper bound on edges to $O(n)$ with at most one Steiner point. Nöllenburg et al. [9] examined the problem of computing increasing-chord drawings of given graphs. They proved that every planar triangulation admits an increasing-chord drawing on the Euclidean plane, and also showed example of graphs that do not admit increasing-chord drawing.

Another related concept is angle monotonicity. A path $P = (v_0, v_1, \ldots, v_k)$ is called angle-monotone with width $\gamma < 180^\circ$, if there exists some angle $\beta$ such that the vector of every edge $(v_{i-1}, v_i)$, where $1 \leq i \leq k$, lies within a closed wedge of angle $\gamma$ between $\beta - (\gamma/2)$ and $\beta + (\gamma/2)$, e.g., see Figure 1(a). A straight-line drawing is angle-monotone with width $\gamma$ if every pair of vertices in the drawing is connected by such an angle-monotone path. Any angle-monotone path of width at most $90^\circ$ is an increasing-chord path, but not angle-monotone with width $90^\circ$. Recently, Bonichon et al. [2] showed how to find angle-monotone paths of width $0 \leq \gamma < 180^\circ$ in a straight-line drawing in polynomial time, and explored local routing strategies on angle-monotone graphs. Since every increasing-chord path is monotone [4], and hence angle-monotone for some $\gamma \in [0, 180^\circ)$, it may initially appear that one may be able to find an increasing chord path between a pair of vertices (if exists) by searching angle-monotone paths for all possible values of $\gamma$. However, there can be exponential number of angle-monotone paths with the same width, where only a few of them may be increasing-chord, e.g., see Figure 1(b). Therefore, Bonichon et al.’s [2] algorithm for deciding angle-monotone paths of width $\Gamma$ does not settle the complexity of finding increasing-chord paths. Consequently, Alamdari et al.’s [1] question of recognizing increasing-chord drawings remains open.

Since geometric routing algorithms are often based on spanning trees, it is natural to seek efficient algorithms that for any given vertex $v$ in the network, can compute an increasing-chord spanning tree rooted at $v$ (if exists). Besides, a polynomial-time algorithm for finding increasing-chord trees is very likely to contain some interesting strategies to find increasing-chord paths. In this paper, we prove that the problem of computing an increasing-chord rooted spanning tree is NP-complete. We conjecture that the problem of computing an increasing-chord path between a given pair of vertices is also NP-complete, and describe a (non-polynomial) reduction from the 3-SAT problem. Recall that Alamdari et al. [1] proved the problem to be hard in $\mathbb{R}^3$. Although their hardness proof is based on a reduction from 3-SAT, the proof relies heavily on the freedom of rotating gadgets in space, which is completely different from our approach.

## 2 Technical Background

In this section we introduce some preliminary definitions and notation.

Given a straight line segment $l$, the slab of $l$ is an infinite region lying between a pair of parallel straight lines that are perpendicular to $l$, and pass through the endpoints of $l$. Let $\Gamma$ be a straight-line drawing, and let $P$ be a path in $\Gamma$. Then the slabs of $P$ are the slabs of the line segments of $P$. We denote by $\Psi(P)$ the arrangement of the slabs of $P$. Figure 1(c) illustrates a path $P$, where the slabs of $P$ are shown in shaded regions. Let $A$ be an arrangement of a set of infinite straight lines such that no line in $A$ is vertical. Then the upper envelope of $A$ is a polygonal chain $U(A)$ such that each point of $U(A)$ belongs to some straight line of $A$.

1
Figure 1: (a) An angle-monotone path with width $\gamma$, which is also an increasing-chord path. (b) Any path between $a$ to $b$ is an angle-monotone path of width $135^\circ$. The only increasing-chord path between $a$ and $b$ is shown in bold. (c) Illustration for $\Psi(P)$, where the upper envelope is shown in dashed line. (d) An increasing-chord extension of $a,b,\ldots,p$ is shown in bold.

and they are visible from the point $(0, +\infty)$. The upper envelope of a set of slabs is the upper envelope of the arrangement of lines corresponding to slab boundaries, as shown in dashed line in Figure 1(c).

Let $t$ be a vertex in $\Gamma$ and let $Q = (a,b,\ldots,p)$ be an increasing-chord path in $\Gamma$. A path $Q' = (a,b,\ldots,p,\ldots,t)$ in $\Gamma$ is called an increasing-chord extension of $Q$ if $Q'$ is also an increasing-chord path, e.g., see Figure 1(d). The following property can be derived from the definition of an increasing-chord path.

**Observation 1 (Icking et al. [7])** A polygonal path $P$ is increasing-chord if and only if for each point $v$ on the path, the line perpendicular to $P$ at $v$ does not properly intersect $P$ except at $v$.

A straightforward consequence of Observation 1 is that every polygonal chain which is both $x$- and $y$-monotone, is an increasing-chord path. We will use Observation 1 throughout the paper to verify whether a path is increasing-chord. Let $v$ be a point in $\mathbb{R}^2$. By the quadrants of $v$ we refer to the four regions determined by the vertical and horizontal lines through $v$.

### 3 Finding Increasing-Chord Rooted Spanning Trees

In this section we prove that given a vertex $r$ in a straight-line drawing $\Gamma$, it is NP-complete to find an increasing-chord spanning tree of $\Gamma$ rooted at $r$. Here is a formal definition of the increasing-chord rooted spanning tree (IC-Tree) problem:

**Problem:** Increasing-Chord Spanning Tree (IC-Tree)

**Instance:** A straight-line drawing $\Gamma$ in $\mathbb{R}^2$, and a vertex $r$ in $\Gamma$.

**Question:** Determine whether $\Gamma$ contains a tree $T$ rooted at $r$ such that for each vertex $v(\neq r)$ in $\Gamma$, $T$ contains an increasing-chord path between $r$ and $v$.

We will reduce the NP-complete problem 3-SAT [5] to IC-Tree. Let $I = (X,C)$ be an instance of 3-SAT, where $X$ and $C$ are the set of literals and clauses. We construct a straight-line drawing $\Gamma$ and choose a vertex $r$ in $\Gamma$ such that $\Gamma$ contains an increasing-chord spanning tree rooted at $r$ if and only if $I$ admits a satisfying truth assignment. Here we briefly review the construction of $\Gamma$, and give an outline of the hardness proof. We describe the details in the subsequent sections.

Assume that $\alpha = |X|$, and $\beta = |C|$. Let $l_h$ be the line determined by the $X$-axis. $\Gamma$ will contain $O(\beta)$ points above $l_h$, one point $t$ on $l_h$, and $4\alpha + 2$ points below $l_h$, as shown in
Their negations, and two other points, i.e., \(\text{peak}_1\) and \(\text{peak}_2\), correspond to the clauses \(c_1\) and \(c_2\), respectively. The slabs of the edges of \(H\) that determine the upper envelope are shown in gray straight lines. Each literal and its negation correspond to a pair of adjacent line segments on the upper envelope of the slabs. See Figure 6 in Appendix A for a better illustration.

Figures 2(a)–(b). Each clause \(c \in C\) with \(j\) literals, will correspond to a set of \(j + 1\) points above \(l_h\), and we will refer to the point with the highest \(y\)-coordinate among these \(j + 1\) points as the peak \(t_c\) of \(c\). Among the points below \(l_h\), there are \(4\alpha\) points that correspond to the literals and their negations, and two other points, i.e., \(s\) and \(r\). In the reduction, the point \(t\) and the points below \(l_h\) altogether will help to set the truth assignments of the literals.

We will first create a straight-line drawing \(H\) such that every increasing-chord path between \(r\) and \(t_c\), where \(c \in C\), passes through \(s\) and \(t\). Consequently, any increasing-chord tree \(T\) rooted at \(r\) (not necessarily spanning), which spans the points \(t_c\), must contain an increasing-chord path \(P = (r, s, \ldots, t)\). We will use this path to set the truth values of the literals.

The edges below \(l_h\) will create a set of thin slabs, and the upper envelope of these slabs will determine a convex chain \(W\) above \(l_h\). Each line segment on \(W\) will correspond to a distinct literal, as shown in Figure 2(b). The points that correspond to the clauses will be positioned below these segments, and hence some of these points will be ‘inaccessible’ depending on the choice of the path \(P\). In brief, these clause-points will ensure that for any clause \(c \in C\), there exists an increasing-chord extension of \(P\) from \(t\) to \(t_c\) if and only if \(c\) is satisfied by the truth assignment determined by \(P\).

From the above discussion we can observe that \(I\) admits a satisfying truth assignment if and only if there exists an increasing-chord tree \(T\) in \(H\) that connects the peaks to \(r\). But \(H\) may still contain some vertices that do not belong to this tree. Therefore, we construct the final drawing \(\Gamma\) by adding some new paths to \(H\), which will allow us to reach these remaining vertices from \(r\) using increasing-chord paths. Furthermore, we will add these new paths in such a way that the tree becomes a spanning tree, without affecting the above reduction technique. In the subsequent sections, we describe the construction in details.

**Construction of \(H\):** We first construct an arrangement \(A\) of \(2\alpha\) straight line segments. The endpoints of the \(i\)th line segment \(L_i\), where \(1 \leq i \leq 2\alpha\), are \((0, i)\) and \((2\alpha - i + 1, 0)\). We now extend each \(L_i\) downward by scaling its length by a factor of \((2\alpha + 1)\), as shown in Figure 3(a). Later, the literal \(x_j\), where \(1 \leq j \leq \alpha\), and its negation will be represented using the lines \(L_{2j-1}\) and \(L_{2j}\).

Let \(l_v\) be a vertical line segment with endpoints \((2\alpha + 1, 2\alpha)\) and \((2\alpha + 1, -5\alpha^2)\). Since the slope of a line in \(A\) is at least \(1/(2\alpha)\), each \(L_i\) intersects \(l_v\). Note that the coordinates of the endpoints of \(L_i\) are of size \(O(\alpha^2)\). Therefore, we can represent the intersection point of a pair of line segments of \(A\) in polynomial space using the corresponding endpoints. Similarly, since the coordinates of the endpoints of \(l_v\) are of polynomial size, we can represent the intersection point of \(L_i\) and \(l_v\) using the endpoints of \(L_i\) and \(l_v\).

Let \(U(A)\) be the upper envelope of \(A\). By construction, every line segment appears on \(U(A)\) in the order of the literals, i.e., the first two segments (from right) of \(U(A)\) correspond to \(x_1\) and
The next two segments correspond to \( x_2 \) and \( \overline{x_2} \), and so on.

**Literal Gadgets:** Let \( l_b \) be a line that coincides with the \( x \)-axis. We denote the intersection point of \( l_b \) and \( l_v \) by \( t \), and the endpoint \( (2\alpha + 1, -5\alpha^2) \) of \( l_v \) by \( s \). We now create the points that correspond to the literals and their negations. Recall that \( L_{2j-1} \) and \( L_{2j} \) correspond to the literal \( x_j \) and its negation \( \overline{x_j} \), respectively. Denote the intersection point of \( L_{2j-1} \) and \( l_b \) by \( p_{x_j} \), and the intersection point of \( L_{2j} \) and \( l_b \) by \( p_{\overline{x_j}} \), e.g., see Figure 3(b). For each \( p_{x_j} \) (\( p_{\overline{x_j}} \)), we create a new point \( p'_{x_j} \) (\( p'_{\overline{x_j}} \)) such that the straight line segment \( p_{x_j}p'_{x_j} \) (\( p_{\overline{x_j}}p'_{\overline{x_j}} \)) is perpendicular to \( L_{2j-1} \) (\( L_{2j} \)), as shown using the dotted (dashed) line in Figure 3(b). Without loss of generality we may assume that all the points \( p'_{x_j} \) and \( p'_{\overline{x_j}} \) lie on a vertical line \( l'_v \), where \( l'_v \) lies \( \varepsilon \) distance away to the left of \( l_v \). The value of \( \varepsilon \) would be determined later. In the following we use the points \( p_{x_j} \), \( p_{\overline{x_j}} \), \( p'_{x_j} \), and \( p'_{\overline{x_j}} \) to create some polygonal paths from \( s \) to \( t \).

For each \( j \) from 1 to \( \alpha \), we draw the straight line segments \( p_{x_j}p'_{x_j} \) and \( p_{\overline{x_j}}p'_{\overline{x_j}} \). Then for each \( k \), where \( 1 < k \leq \alpha \), we make \( p_{x_k} \) and \( p_{\overline{x_k}} \) adjacent to both \( p'_{x_{k-1}} \) and \( p'_{\overline{x_{k-1}}} \), as illustrated in Figure 3(c). We then add the edges from \( s \) to \( p'_{x_1} \) and \( p'_{\overline{x_1}} \), and finally, from \( t \) to \( p_{x_1} \) and \( p_{\overline{x_1}} \). For each \( x_j \) (\( \overline{x_j} \)), we refer to the segment \( p_{x_j}p'_{x_j} \) (\( p_{\overline{x_j}}p'_{\overline{x_j}} \)) as the needle of \( x_j \) (\( \overline{x_j} \)). Figure 3(c) illustrates the needles in bold. Denote the resulting drawing by \( H_b \).

Recall that \( l'_v \) is \( \varepsilon \) distance away to the left of \( l_v \). We choose \( \varepsilon \) sufficiently small such that for each needle, its slab does not intersect any other needle in \( H_b \), e.g., see Figure 3(d). Note that the upper envelope of the slabs of all the straight line segments of \( H_b \) coincides with \( U(A) \). Since the distance between any pair of points that we created on \( l_v \) is at least \( 1/\alpha \) units, it suffices to choose \( \varepsilon = 1/\alpha^4 \). Note that the points \( p'_{x_j} \) and \( p'_{\overline{x_j}} \) can be represented in polynomial space using the endpoints of \( l'_v \) and the endpoints of the segments \( L_{2j-1} \) and \( L_{2j} \). The following lemma states that every increasing-chord path \( P \) between \( s \) and \( t \) must pass through exactly one point in \( \{ p_{x_j}, p_{\overline{x_j}} \} \), and vice versa. The proof of the lemma is included in Appendix A.

**Lemma 1** Every increasing-chord path \( P \) that starts at \( s \) and ends at \( t \) must pass through exactly one point among \( p_{x_j} \) and \( p_{\overline{x_j}} \), where \( 1 \leq j \leq \alpha \), and vice versa.
We now complete the construction of $H$, e.g., at position $(0, -\alpha^5)$, such that the slab of the straight line segment $rs$ does not intersect $H_b$ (except at $s$), and similarly, the slabs of the line segments of $H_b$ do not intersect $rs$. Furthermore, the slab of $rs$ does not intersect any segment $L_j$, and vice versa. We then add the point $r$ and the segment $rs$ to $H_b$. Let $P$ be an increasing-chord path from $r$ to $t$. Recall that $\Psi(P)$ denotes the arrangement of the slabs of $P$. Observe that the upper envelope $U(P)$ of $\Psi(P)$ is determined by the needles in $P$, e.g., see Figure 2(b). For each $x_j$, $P$ passes through exactly one point among $p_{x_j}$ and $p_{\overrightarrow{t_j}}$. Therefore, for each literal $x_j$, either the slab of $x_j$, or the slab of $\overrightarrow{t_j}$ appears on $U(P)$. Later, if $P$ passes through point $p_{x_j}$ ($p_{\overrightarrow{t_j}}$), then we will set $x_j$ to false (true). Since $P$ is an increasing-chord path, it cannot pass through both $p_{x_j}$ and $p_{\overrightarrow{t_j}}$ simultaneously. Therefore, all the truth values will be set consistently.

**Clause Gadgets:** We now complete the construction of $H$ by adding clause gadgets to $H_b$. For each clause $c_i$, where $1 \leq i \leq \beta$, we first create the peak point $t_{c_i}$ at position $(0, 2\alpha + i)$. For each literal $x_j$, let $\lambda_{x_j}$ be the interval of $L_{2j-1}$ that appears on the upper envelope of $A$. Similarly, let $\lambda_{x_j}$ be the interval of $L_{2j}$ on the upper envelope of $A$. For each $c_i$, we construct a point $q_{x_i, c_i} = (\overrightarrow{t_i} \cap \lambda_{x_j} (\overrightarrow{t_i}))$ inside the cell of $A$ immediately below $\lambda_{x_j} (\overrightarrow{t_i})$. We will refer to these points as the literal-points of $c_i$. Figure 4(a) depicts these points in black squares; see Figure 8 in Appendix B for a better illustration.

For each literal $x$ of $c_i$, we create a path $(t, x, t_c)$. Figure 4(a) illustrates an example, where $c_1 = (\overrightarrow{t_1} \cap x_3)$. Note that there does not exist any edge connecting $t$ to $t_{c_i}$. In the reduction we will see that if at least one of the literals of $c_i$ is true, then we can take the corresponding path to connect $t_c$ to $t$. Let the resulting drawing be $H$. In the following we add some other edges to $H$ to construct the final drawing $\Gamma$, which will be used in the hardness reduction.

**Construction of $\Gamma$:** Let $q$ be a literal-point in $H$. We now add an increasing-chord path $P' = (r, a, q)$ to $H$ in such a way that $P'$ cannot be extended to any larger increasing-chord path in $H$. We place the point $a$ at the intersection point of the horizontal line through $q$ and the vertical line through $r$, as shown in Figure 4(b). We refer to the point $a$ as the anchor of $q$. By the construction of $H$, all the neighbors of $q$ that have a higher $y$-coordinate than $q$ lie in the top-left quadrant of $q$, as illustrated by the dashed rectangle in Figure 4(b). Let $q'$ be the first neighbor in the top-left quadrant of $q$ in counter clockwise order. Since $\angle a q q' < 90^\circ$, $P'$ cannot be extended to any larger increasing-chord path $(r, a, q, w)$ in $H$, where the $y$-coordinate of $w$ is higher than $q$. On the other hand, every clause-point $w$ in $H$ with $y$-coordinate smaller than $q$ intersects the slab of $ra$. Therefore, $P'$ cannot be extended to any larger increasing-chord path.

For every literal-point $q$ in $H$, we add such an increasing-chord path from $t$ to $q$. To avoid edge overlaps, one can perturb the anchors such that the new paths remain increasing-chord and non-extensible to any larger increasing-chord paths. Let the resulting drawing be $\Gamma$. In the following section, we prove that $\Gamma$ admits an increasing-chord rooted spanning tree if and only if $I$ admits a satisfying truth assignment.

**Reduction:** First assume that $I$ admits a satisfying truth assignment. We now construct an increasing-chord spanning tree $T$ rooted at $r$. We first choose a path $P$ from $r$ to $t$ such that it passes through either $p_{x_j}$ or $p_{\overrightarrow{t_j}}$, i.e., if $x_j$ is true (false), then we route the path through $p_{\overrightarrow{t_j}}$ ($p_{x_j}$). Figure 4(c) illustrates such a path $P$ in a thick black line, where $x_{a-n} = true$ and $x_{a-n-1} = false$.

Observe that only $2\alpha$ points remain below $h_b$, two points per literal, that do not belong to $P$. We connect these points in a $y$-monotone polygonal path $Q$ starting at $s$, as illustrated in a thin black line in Figure 4(c). Note that $Q$ corresponds to a truth value assignment, which is opposite to the truth values determined by $P$. Therefore, by Lemma 4, $Q$ is also an increasing-chord path. Consequently, the point $t$ and the points that lie below $h_b$ are now connected to $t$ through increasing-chord paths.

The tree $T$ now consists of the paths $P$ and $Q$, and thus does not span the vertices that lie above $h_b$. We now add more paths to $T$ to span the points above $h_b$. Since every clause $c_i$ is satisfied, we can choose a path $P'$ from $t$ to $t_c$ that passes through a literal point whose
corresponding literal \( x \in c \) is true. Since the literal-points corresponding to true literal lie above the slabs of \( P \), the path \( P' \) determines an increasing-chord extension of \( P \). Therefore, all the peaks and some literal-points above \( l_h \) are now connected to \( r \) via increasing-chord paths.

For each remaining literal-point \( q \), we add \( q \) to \( T \) via the increasing-chord path through its anchor. There are still some anchors that are not connected to \( r \), i.e., the anchors whose corresponding literal-points are already connected to \( r \) via an increasing-chord extension of \( P \).

We connect each anchor \( a \) to \( r \) via the straight line segment \( ar \).

We now assume that \( \Gamma \) contains an increasing-chord rooted spanning tree \( T \), and show how to find a satisfying truth assignment for \( I \). Since \( T \) is rooted at \( r \), and the peaks are not reachable via anchors, \( T \) must contain an increasing-chord path \( P = (r, s, \ldots, t) \) that for each literal \( x_j \), passes through exactly one point among \( px_j \) and \( px'_j \). If \( P \) passes through \( px_j \ (px'_j) \), then we set \( x_j \) to false (true). Observe that passing through a literal \( x_j \) or its negation selects a corresponding needle segment \( px_j p_{x_j} \) or \( px'_j p_{x'_j} \). Recall that the interval \( \lambda_{x_j} (\lambda_{x'_j}) \), which corresponds to \( px_j p_{x_j} (px'_j p_{x'_j}) \), lies above the literal point \( q_{x_j,c_i} (q_{x'_j,c_i}) \), e.g., see Figure 4(a).

Therefore, if the above truth assignment does not satisfy some clause \( c \), then there cannot be any increasing-chord extension of \( P \) that connects \( t \) to \( t_c \). Therefore, \( T \) would not be a spanning tree.

Since determining whether a straight-line drawing of a tree is an increasing-chord drawing is polynomial-time solvable \([\mathbb{P}]\), the problem IC-TREE is NP-complete. Hence we obtain the following theorem.

**Theorem 1** Given a vertex \( r \) in a straight-line drawing \( \Gamma \), it is NP-complete to decide whether \( \Gamma \) admits an increasing-chord spanning tree rooted at \( r \).
4 Finding Increasing-Chord Paths

In this section we examine the problem of finding an increasing-chord path (IC-PATH) between a pair of vertices in a given straight-line drawing. Given an instance $I = (X, C)$ of 3-SAT, we compute a straight-line drawing $D$ and a pair of vertices $t, t'$ in $D$ such that $I$ admits a straight-line drawing if and only if there exists an increasing-chord path between $t$ and $t'$. Unfortunately, we could not prove the coordinates of $D$ to be polynomial, and hence the reduction is not a polynomial-time reduction.

Here we briefly describe the idea of the reduction. For each clause $c_i$, where $1 \leq i \leq \beta$, $D$ will contain a straight line drawing $D_{i-1}$, e.g., see Figure 3(a). We will refer to the bottommost (topmost) point of $D_{i-1}$ as $t_{c_{i-1}} (t_{c_{i+1}})$. We will choose $t_{c_0}$ and $t_{c_{\beta}}$ as the points $t$ and $t'$, respectively, and show that $I$ admits a satisfying truth assignment if and only if there exists an increasing-chord path $P$ from $t$ to $t'$ that passes through every $t_{c_i}$. For every $i$, the subpath $P_{i-1}$ of $P$ between $t_{c_{i-1}}$ and $t_{c_i}$ will correspond to a set of truth values for all the literals in $X$. The most involved part is to show that the truth values determined by $P_{i-1}$ and $P_i$ are consistent. This consistency will be ensured by the construction of $D$, i.e., any increasing-chord path $P'$ from $t_{c_{i-1}}$ to $t_{c_i}$ in $D_{i-1}$ will determine a set of slabs, which will force a unique increasing-chord path in $D_i$ between $t_{c_i}$ and $t_{c_{i+1}}$ with the same truth values as determined by $P'$.

Construction of $D$: For each clause $c_i$, we construct a straight line drawing $D_{i-1}$, where the construction of $D_{i-1}$ depends on an arrangement of lines $A^{i-1}$. The construction of $A^i$ is the same as the construction of arrangement $A$, which we described in Section 3. Figure 3(c) illustrates $A^0$ in dotted lines. For each literal $x_j$, where $1 \leq j \leq \alpha$, there exists an interval $\lambda^{0}_{x_j}$ of $L_{2j-1}$ on the upper envelope of $A^0$. Similarly, for each $x_j$, there exists an interval $\lambda^{+}_{x_j}$ on the upper envelope of $A^0$.

We now describe the construction of $D_0$. Choose $t_{c_0}$ ($t_{c_1}$) to be the bottommost (topmost) point of $\lambda^{0}_{x_1}$ ($\lambda^{+}_{x_1}$). We then slightly shrink the intervals $\lambda^{0}_{x_1}$ and $\lambda^{+}_{x_1}$ such that $t_{c_0}$ and $t_{c_1}$ no longer belong to these segments. If $c_1$ contains $\kappa$ literals, then there are $2^\kappa - 1$ distinct truth assignment for its variables to satisfy $c_1$. For each satisfying truth assignment $\sigma$, where $1 \leq k \leq 2^\kappa - 1$, we construct a set of vertices and edges in $D_0$, as follows. For each $x_j$ ($x_j'$), we construct a point $q^{+}_{x_j}$ ($q^{-}_{x_j}$) at the midpoint of $\lambda^{0}_{x_j}$ ($\lambda^{+}_{x_j}$) and its negation. We will explain the reason later, when we construct $D_1$. Finally, for every literal $x \in c_1$, if the $x$ is true in $\sigma$, then we remove the edges incident to $q^{+}_{x_j}$. We may assume that for each $x_j$, the points $q^{+}_{x_j}$ lie adjacent to the same location. At the end of the construction, one may perturb them to remove vertex overlaps.

By Observation 4, any $y$-monotone path $P'$ between $t_{c_0}$ and $t_{c_1}$ must be an increasing-chord path. Furthermore, by construction, for each $x_j$, $P'$ must pass through one of the two points $q^{+}_{x_j}$, $q^{-}_{x_j}$. If $P'$ passes through $q^{+}_{x_j}$, then we set $x_j$ to true. Otherwise, $P'$ must pass through $q^{-}_{x_j}$, and we set $x_j$ to false.

In the following we replace each $q$-point by a small segment. The slabs of these segments will determine $A^1$. Consider an upward ray $r^1$ with positive slope starting at the $q$-point on $\lambda^{x_1}$, e.g., see Figure 6(c). Since all the edges that are currently in $D_0$ have negative slopes, we can choose a sufficiently large positive slope for $r^1$ and a point $a^1$ on $r^1$ such that all the slabs of $D_0$ lie below $a^1$. We now find a point $b^1$ above $a^1$ on $r^1$ with sufficiently large $y$-coordinate such that the slab of $t_{c_1}$ $b^1$ does not intersect the edges in $D_0$. Let $l_{b^1}$ be the line determined by $b^1$. For each $x_j$ (except for $x_1$) and $x_j'$, we now construct the line $l_{x_j}$ and $l_{x_j}'$ that pass through their corresponding $q$-points and intersect $r$ above $b$. The lines $l_{x_j}$ and $l_{x_j}'$ determine the arrangement $A^1$. Observe that one can construct these lines in the decreasing order of the $x$-coordinates of their $q$-points, and ensure that for each $l_{x_j}$ ($l_{x_j}'$), there exists an interval $\lambda^{0}_{x_j}$ ($\lambda^{+}_{x_j}$) on the upper envelop of $A^1$.

For each $l_{x_j}$ ($l_{x_j}'$), we draw a small perpendicular segment $s^{0}_{x_j}$ ($s^{+}_{x_j}$) at the $q$-point and to the
\[ t = (t_c^0)_{x_1} \lambda_0 \]

\[ \lambda_0 \]

\[ x_1 \]

\[ \lambda_0 \]

\[ x_2 \]

\[ \lambda_0 \]

\[ x_3 \]

\[ \lambda_0 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]

\[ \lambda_1 \]

\[ x_2 \]

\[ \lambda_1 \]

\[ x_1 \]

\[ \lambda_1 \]

\[ x_3 \]
left of \( q \), e.g., see Figure 5(d). We will refer to these segments as the \( s \)-segments. We choose the length of the \( s \)-segments small enough such that the slabs of these segments still behave as lines of \( A^1 \). For each \( s \)-segment, if there exists a segment \( wq \) above the \( q \)-point of \( s \), then we delete the segment, and make \( w \) incident to the other endpoint of \( s \). Since the slopes of the \( s \)-segments are negative, it is straightforward to verify that any \( y \)-monotone path between \( t_{c_0} \) and \( t_{c_1} \) will be an increasing-chord path.

This completes the construction of \( D_0 \) and \( A^1 \). The construction for the subsequent drawing \( D_i \) depends on \( A_i \), where \( 1 \leq i < \beta \), and the arrangement \( A^{i+1} \) is determined by \( D_i \). Although the construction of \( D_i \) from \( A^i \) is similar to the construction of \( D_0 \) from \( A^0 \), we need \( D_i \) to satisfy some further conditions, as follows.

(A) In \( A^i \), \( \lambda_{ij}^{-1} \) corresponds to \( \lambda_{ij}^{-1} \), and \( \lambda_{ij}^{+1} \) corresponds to \( \lambda_{ij}^{-1} \). Therefore, the \( q \)-vertices and edges of \( D_i \) must be constructed accordingly. As a consequence, if an increasing-chord path \( P' \) between \( t_{c_{i-1}} \) and \( t_{e_i} \) passes through some \( s_{ij}^{-1} (s_{ij}^{-1}) \) in \( D_{i-1} \), then any increasing-chord extension of \( P' \) to \( t_{c_{i+1}} \) must pass through \( s_{ij}^{+1} (s_{ij}^{-1}) \) in \( D_i \).

(B) While constructing \( D_i \), we must ensure that the slabs of the segments in \( D_i \) do not intersect the segments in \( D_0, \ldots, D_{i-1} \). We now describe how to construct such a drawing \( D_i \), e.g., see Figure 6 in Appendix B. Without loss of generality assume that \( i \) is odd. The construction when \( i \) is even is symmetric. Let \( \Delta_{i-1} \) be the largest \( x \)-coordinate among all the vertices in \( D_0, \ldots, D_{i-1} \).

Recall that the drawing of \( D_i \) depends on \( A_i \), and we construct \( A_i \) starting with an upward ray \( r^i \) and choosing a point \( b_i \) on \( r^i \). We choose a positive slope for \( r^i \), which is larger than all the positive slopes determined by the slabs of \( D_0, \ldots, D_{i-1} \). We then choose \( b_i \) with a sufficiently large \( y \)-coordinate such that the \( x \)-coordinate of \( b_i \) is larger than \( \Delta_{i-1} \). It is now straightforward to choose the lines of \( A_i \) such that their intersection points are close to \( b_i \), and have \( x \)-coordinates larger than \( \Delta_{i-1} \). Since the segments of \( D_i \) will have positive slopes, their slabs cannot intersect the segments of \( D_0, \ldots, D_{i-1} \).

**Reduction:** Any increasing-chord path \( P \) from \( t \) to \( t' \) contains the points \( t_{c_i} \). We set a literal \( x_j \) true or false depending on whether \( P \) passes through \( s_{ij}^0 \) or \( s_{ij}^0 \). By Condition (A), if \( P \) passes through \( s_{ij}^{-1} (s_{ij}^{-1}) \), then it must pass through \( s_{ij}^{+1} (s_{ij}^{+1}) \). Hence the truth values in all the clauses are set consistently. By construction of \( D \), any increasing-chord path between \( t_{c_{i-1}} \) to \( t_{e_i} \) determines a satisfying truth assignment for \( c_i \). Hence the truth assignment satisfies all the clauses in \( C \).

On the other hand, if \( I \) admits a satisfying truth assignment, then for each clause \( c_i \), we choose the corresponding increasing-chord path \( P_i \) between \( t_{c_{i-1}} \) and \( t_{e_i} \). Let \( P \) be the union of all \( P_i \). By construction of \( D \), the slabs of \( P \) do not intersect \( P \) except at \( P_i \). Hence, \( P \) is the required increasing-chord path from \( t \) to \( t' \).

The following theorem summarizes the results of this section.

**Theorem 2** There exists a (non-polynomial) reduction from 3-SAT to IC-Path.

5 Conclusion

We proved that the decision problem of whether there exists an increasing-chord rooted spanning tree is NP-complete. The straight-line drawing \( \Gamma \) that we constructed in our hardness proof is non-planar. An interesting question is whether the problem remains NP-complete under the planarity constraint; a potential attempt could be replacing the edge intersections by dummy vertices.

We also gave a (non-polynomial) reduction from 3-SAT to the problem of computing an increasing-chord path between a pair of vertices in a given straight-line drawing. The drawings we used in the reduction may use coordinates of exponential size. Consequently, the question of recognizing increasing-chord drawings [11] remains open, and it would be interesting to examine whether our reduction can be carried out using polynomial-size drawings.
References

[1] Soroush Alamdari, Timothy M. Chan, Elyot Grant, Anna Lubiw, and Vinayak Pathak. Self-approaching graphs. In Proceedings of the 20th International Symposium on Graph Drawing (GD), volume 7704 of LNCS, pages 260–271. Springer, 2013.

[2] Nicolas Bonichon, Prosenjit Bose, Paz Carmi, Irina Kostitsyna, Anna Lubiw, and Sander Verdonschot. Gabriel triangulations and angle-monotone graphs: Local routing and recognition. In Proceedings of the 24th International Symposium on Graph Drawing and Network Visualization (GD), volume 9801 of LNCS, pages 519–531. Springer, 2016.

[3] Hooman Reisi Dehkordi, Fabrizio Frati, and Joachim Gudmundsson. Increasing-chord graphs on point sets. Journal of Graph Algorithms and Applications, 19(2):761–778, 2015.

[4] Stefan Felsner, Alexander Igamberdiev, Philipp Kindermann, Boris Klemz, Tamara Mehedlidze, and Manfred Scheucher. Strongly monotone drawings of planar graphs. In Proceedings of the 32nd International Symposium on Computational Geometry (SoCG), volume 51 of LIPIcs, pages 37:1–37:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

[5] Michael Randolph Garey and David Stier Johnson. Computers and Intractability. Freeman, San Francisco, 1979.

[6] Christian Icking and Rolf Klein. Searching for the kernel of a polygon - A competitive strategy. In Proceedings of the Eleventh Annual Symposium on Computational Geometry (SoCG), pages 258–266. ACM, 1995.

[7] Christian Icking, Rolf Klein, and Elmar Langetepe. Self-approaching curves. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 125, pages 441–453. Cambridge Univ Press, 1999.

[8] Konstantinos Mastakas and Antonios Symvonis. On the construction of increasing-chord graphs on convex point sets. In Proceedings of the 6th International Conference on Information, Intelligence, Systems and Applications (IISA), pages 1–6. IEEE, 2015.

[9] Martin Nöllenburg, Roman Prutkin, and Ignaz Rutter. On self-approaching and increasing-chord drawings of 3-connected planar graphs. In Proceedings of the 22nd International Symposium on Graph Drawing (GD), volume 8871 of LNCS, pages 476–487. Springer, 2014.

[10] Ananth Rao, Christos H. Papadimitriou, Scott Shenker, and Ion Stoica. Geographic routing without location information. In Proceedings of the 9th Annual International Conference on Mobile Computing and Networking (Mobicom), pages 96–108. ACM, 2003.

[11] G. Rote. Curves with increasing chords. Mathematical Proceedings of the Cambridge Philosophical Society, 115:1–12, 1994.
Appendix A

Figure 6: Illustration for the hardness proof using a schematic representation of $\Gamma$. The points that correspond to $c_1$ and $c_2$ are connected in paths of black, and lightgray, respectively. The slabs of the edges of $H$ that determine the upper envelope are shown in lightgray straight lines. Each literal and its negation correspond to a pair of adjacent line segments on the upper envelope of the slabs.

Lemma 1 Every increasing-chord path $P$ that starts at $s$ and ends at $t$ must pass through exactly one point among $p_{x_j}$ and $p_{\overline{x}_j}$, where $1 \leq j \leq \alpha$, and vice versa.

Proof: By Observation 1, $P$ must be $y$-monotone. Consequently, for each $j$, the edge on the $(2j)$th position on $P$ is a needle, which corresponds to either $p_{x_j}p'_{x_j}$ or $p_{\overline{x}_j}p'_{\overline{x}_j}$. Therefore, it is straightforward to observe that $P$ passes through exactly one point among $p_{x_j}$ and $p_{\overline{x}_j}$.

Now consider a path $P$ that starts at $s$, ends at $t$, and for each $j$, passes through exactly one point among $p_{x_j}$ and $p_{\overline{x}_j}$. By construction, $P$ must be $y$-monotone. We now show that $P$ is an increasing-chord path. Note that it suffices to show that for every straight-line segment $\ell$ on $P$, the slab of $\ell$ does not properly intersect $P$ except at $\ell$. By Observation 1, it will follow that $P$ is an increasing-chord path.

For every interior edge $e$ on $P$, which is not a needle, $e$ corresponds to some segment $\ell \in \{p_{x_j}p'_{x_j}, p_{x_j}p'_{\overline{x}_j}, p_{\overline{x}_j}p'_{x_j}, p_{\overline{x}_j}p'_{\overline{x}_j}\}$, for some $1 < j \leq \alpha$. By construction, in each of these four cases, the needles incident to $\ell$ lie either on the boundary or entirely outside of the the slab of $\ell$, and hence the slab does not properly intersect $P$ except at $\ell$. Figures 7(a)–(b) illustrate the scenario when $\ell \in \{p_{x_j}p'_{\overline{x}_j}, p_{x_j}p'_{\overline{x}_{j-1}}\}$.

Let $(s,a)$ and $(b,t)$ be the edges on $P$ incident to $s$ and $t$, respectively. By construction, these edges behave in the same way, i.e., all the needles on $P$ are above the slab of $(s,a)$ and below the slab of $(b,t)$. Consequently, the slab of $(s,a)$ (resp., $(b,t)$) does not properly intersect $P$ except at $(s,a)$ (resp., $(b,t)$).
For every interior edge $e$ on $P$, which is a needle, $e$ corresponds to some segment $\ell \in \{p_x, p'_x, p_{x-1}, p'_{x-1}\}$. By construction, the needles following (resp., preceding) $\ell$ on $P$ are above (resp., below) the slab of $\ell$. Consequently, the slab does not properly intersect $P$ except at $\ell$. Figures 7(c)–(d) illustrate these scenarios.

\[\square\]
Appendix B

Figure 8: Construction of clause gadgets, where $c_1 = (\pi_1 \lor x_3)$.
Figure 9: (a) A schematic representation of $\mathcal{D}$. The upward slabs of each $\mathcal{D}_i$ are illustrated in gray. (b) Construction of $\mathcal{D}_i$. The slabs of $\mathcal{D}_j$, where $j$ is odd, are shown in gray. Since the downward slabs of the edges of $\mathcal{D}_i$ have positive slopes, and since the vertices of $\mathcal{D}_i$ have larger $x$-coordinates than $\Delta_{i-1}$, the slabs of $\mathcal{D}_i$ do not intersect $\mathcal{D}_0, \ldots, \mathcal{D}_{i-1}$. 