Canonical measures and dynamical systems of Bergman kernels

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Abstract

In this article, we construct the canonical semipositive current or the canonical measure (= the potential of the canonical semipositive current) on a smooth projective variety with nonnegative Kodaira dimension in terms of a dynamical system of Bergman kernels. This current is considered to be a generalization of a Kähler-Einstein metric and coincides the one considered independently by J. Song and G. Tian ([S-T]). The major difference between [S-T] and the present article is that they found the canonical measure in terms of Kähler-Ricci flows, while I found the canonical measure in terms of dynamical systems of Bergman kernels. Hence the present approach can be viewed as a discrete version of a Kähler-Ricci flow.

The advantage of the dynamical construction is two folds. First, it enables us to deduce the logarithmic plurisubharmonic variation propery of the canonical measures on a projective family. Second, we can overcome the difficulty arising from the singularities of the solution of a Kähler-Ricci flow.

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1 Introduction

In \cite{T4}, I have constructed a canonical Kähler-Einstein current on a smooth projective variety of general type in terms of a dynamical system of Bergman kernels originated in \cite{T3}. This Kähler-Einstein current is the same one which has been studied in \cite{T0, Su}.

Since the same kind of dynamical systems has been defined on a smooth projective varieties of nonnegative Kodaira dimension (or even for a smooth projective variety with pseudoeffective canonical bundle) in \cite{T3}, it is natural to expect that the normalized limit of the dynamical system of Bergman kernels yields a substitute of a Kähler-Einstein metric for a smooth projective variety (of non general type) with nonnegative Kodaira dimension.

The purpose of this article is to prove that the limit satisfies the partial differential equation (see \eqref{eq:1.22}) similar to the Kähler-Einstein equation on the base of the Iitaka fibration (not on the original variety) and give a natural generalization of the notion of Kähler-Einstein volume form. We call the (normalized) limit the canonical measure. And we call the $-\text{Ric}$ of the canonical measure (in the sense of current) the canonical semipositive current.

There are two major differences between Kähler-Einstein metrics and the canonical semipositive currents.

First of all in general the canonical semipositive current is strictly positive not on the original variety but on the base space of the Iitaka fibration. In other words, the current is the pullback of a closed generically strictly positive current on the base space of the Iitaka fibration.

Secondary although the canonical semipositive current satisfies a similar partial differential equation as a Kähler-Einstein metric on the base space of the Iitaka fibration, the equation has an additional term coming from variation of Hodge structure on the Iitaka fibration.

The objective of this generalization is to study the deformation of projective varieties with nonnegative Kodaira dimension. Actually the dynamical construction of the canonical semipositive current yields the existence of a closed semipositive current on the family which restricts the canonical semipositive current on the general fibers (Theorem \ref{thm:1.1}). We discuss the applications of Theorem \ref{thm:1.1} in \cite{T7}. And we also note that there are similar constructions of canonical measures (canonical and supercanonical AZD’s) on smooth projective varieties with pseudoeffective canonical bundles (\cite{T5}).
After the completion of this work, I have noticed a paper of Song and Tian (S-T) which also constructed the canonical semipositive current from a different point of view. Actually first they have constructed the canonical semipositive current as the limit of the Kähler-Ricci flow in the case of semiample canonical bundle. Then they constructed the current which satisfies the same equation without using a Kähler-Ricci flow on a projective variety with nonnegative Kodaira dimension whose canonical bundle is not necessarily semiample. In this sense their construction is modeled after the case of semiample canonical bundles. But in their construction, the meaning of the canonical semipositive current is not clear (although it is apparently a generalization of a Kähler-Einstein metric).

The main contribution of this article is to give a dynamical construction of the canonical semipositive currents (or the canonical measure in (S-T)) and give the authenticity to the canonical semipositive current.

The advantage of the dynamical construction is that we can overcome the difficulty arising from the singularity of the currents. For example it seems to be difficult to deduce the plurisubharmonic variation property of the canonical measures on a projective family (Theorem 4.1) by direct calculation. On the other hand Theorem 4.1 is an immediate consequence of the dynamical construction by using the logarithmic plurisubharmonic variation properties of Bergman kernels.

Also difficulty arises to study a Kähler-Ricci flow, when we consider non minimal algebraic varieties. In this case the flow of the Kähler class associated with a Kähler-Ricci flow reaches the boundary of the Kähler cone in finite time. Hence in this case it is inevitable to deal with a singular Kähler Ricci flow. But the dynamical construction (Theorem 1.7) automatically produces the canonical semipositive current (or the canonical measure) as soon as the Kodaira dimension of the variety is nonnegative. One may consider Theorem 1.7 as a discretization of a Kähler-Ricci flow and it overcomes the difficulty arising from singularities.

The author would like to express his sincere thanks to the referee for pointing out several errors.

Notations

- For a real number $a$, $\lceil a \rceil$ denotes the minimal integer greater than or equal to $a$ and $\lfloor a \rfloor$ denotes the maximal integer smaller than or equal to $a$.
- Let $X$ be a projective variety and let $D$ be a Weil divisor on $X$. Let $D = \sum d_i D_i$ be the irreducible decomposition. We set

$$\lceil D \rceil := \sum \lceil d_i \rceil D_i, \lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i.$$  

(1.1)

- Let $L$ be a line bundle on a compact complex manifold $X$. A singular hermitian metric $h$ on $L$ is given by

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{loc}(X)$ is an arbitrary function on $X$. We call $\varphi$ a weight function of $h.$

\footnote{In [T6], I have used another discretization of a Kähler-Ricci flow.}
The curvature current $\Theta_h$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\Theta_h := \Theta_{h_0} + \partial \bar{\partial} \varphi,$$

where $\partial \bar{\partial} \varphi$ is taken in the sense of current. The $L^2$ sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\mathcal{L}^2(L, h)(U) := \{ \sigma \in \Gamma(U, \mathcal{O}_X(L)) \mid h(\sigma, \sigma) \in L^1_{\text{loc}}(U) \},$$

where $U$ runs over the open subsets of $X$. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L, h) = \mathcal{O}_X(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the multiplier ideal sheaf of $(L, h)$.

- For a closed positive $(1, 1)$ current $T$, $T_{abc}$ denotes the absolutely continuous part of $T$.
- A line bundle $L$ on a compact complex manifold $X$ is said to be pseudo-effective, if it admits a singular hermitian metric with semipositive curvature current. A singular hermitian line bundle $(L, h)$ is said to be pseudoeffective if the curvature current $\sqrt{-1} \Theta_h$ is semipositive. If $X$ is a smooth projective variety, this is equivalent to the fact that $c_1(L)$ is on the closure of the effective cone.
- Let $(X, D)$ be a pair of a normal variety and a $\mathbb{Q}$-divisor on $X$. Suppose that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \rightarrow X$ be a log resolution. Then we have the formula:

$$K_Y = f^*(K_X + D) + \sum a_i E_i,$$

where $E_i$ is a prime divisor and $a_i \in \mathbb{Q}$. The pair $(X, D)$ is said to be subKLT (resp. subLC, if $a_i > -1$ (resp. $a_i \geq -1$) holds for every $i$). $(X, D)$ is said to be KLT (resp. LC), if $(X, D)$ is subKLT (resp. subLC) and $D$ is effective.

### 1.1 Kähler-Einstein metrics

Let $X$ be a compact Kähler manifold with the Kähler form

$$\omega := \frac{\sqrt{-1}}{2} \sum_{i,j} g_{ij} dz^i \wedge d\bar{z}^j.$$

$(X, \omega)$ is said to be Kähler-Einstein, if there exists a constant $c$ such that

$$\text{Ric}_\omega = c\omega$$

holds, where $\text{Ric}_\omega$ denotes the Ricci form:

$$\text{Ric}_\omega := -\sqrt{-1} \partial \bar{\partial} \log \det(g_{ij})$$

and we call $\omega$ a Kähler-Einstein form on $X$. If a compact complex manifold $X$ admits a Kähler-Einstein form, then $c_1(X)$ is negative or 0 or positive. Conversely by the celebrated solution of Calabi’s conjecture ([X], [Y]), for a compact
Kähler manifold $X$, if $c_1(X)$ is negative, there exists a unique Kähler-Einstein form $\omega_E$ such that

\begin{equation}
-\text{Ric}_{\omega_E} = \omega_E
\end{equation}

and if $c_1(X)$ is 0 in $H^2(X, \mathbb{R})$, then in every Kähler class on $X$, there exists a unique Kähler-Einstein form $\omega_E$ such that

\begin{equation}
\text{Ric}_{\omega_E} = 0
\end{equation}

holds ([Y1]).

1.2 Kähler-Einstein currents

There are numerous applications of Kähler-Einstein metrics. But in general, a smooth complex projective variety does not admit a Kähler-Einstein metric, since the first Chern class is not definite in general. One way to overcome this defect is to consider Kähler-Einstein metrics allowing singularities.

In [T0], I have constructed a Kähler-Einstein current $\omega_E$ on smooth minimal algebraic variety $X$ of general type. More precisely there exists a unique closed semipositive current $\omega_E$ such that

1. There exists a nonempty Zariski open subset $U$ of $X$ such that $\omega_E$ is a $C^\infty$ Kähler form on $U$.

2. $-\text{Ric}_{\omega_E} = \omega_E$ holds on $U$.

3. $\omega_E$ is absolutely continuous on $X$.

Later K. Sugiyama proved that there exists a Kähler-Einstein current on the canonical model of general type ([Su]). Also I have constructed a Kähler-Einstein current on an arbitrary smooth projective variety of general type (without using the finite generation of canonical rings) in [T4]. Hence for smooth projective varieties of general type, we have a substitute of a Kähler-Einstein metric.

We note that the above Kähler-Einstein current $\omega_E$ on a projective variety $X$ of general type have the following properties: $h_E := n!(\omega_E^n)^{-1}$ is a singular hermitian metric on $K_X$ such that the curvature current $\sqrt{-1} \Theta_{h_E}$ is a closed semipositive current and

\begin{equation}
H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_E^m)) \simeq H^0(X, \mathcal{O}_X(mK_X))
\end{equation}

holds for every $m \geq 1$, i.e., $h_E$ is an AZD of $K_X$ (cf. Definition [T4] below). In other words, $h_E$ is a singular hermitian metric which extracts all the positivity of $K_X$.

But for smooth projective varieties of non general type, the above results do not say anything.
1.3 Iitaka fibration

The simplest way to squeeze out the positivity of canonical bundles is to use the pluricanonical systems.

Let $X$ be a smooth projective variety. The Kodaira dimension $\text{Kod}(X)$ is defined by

\[(1.8) \quad \text{Kod}(X) := \limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(mK_X))}{\log m}.\]

It is known that Kod$(X)$ is $-\infty$ or a nonnegative integer between 0 and $\dim X$.

Let $X$ be a smooth projective variety with Kod$(X) \geq 0$. Then for a sufficiently large $m > 0$, the complete linear system $|mK_X|$ gives a rational fibration (with connected fibers):

\[(1.9) \quad f : X \to \cdots \to Y.\]

We call $f : X \to \cdots \to Y$ the **Iitaka fibration** of $X$.

The Iitaka fibration is independent of the choice of the sufficiently large $m$ up to birational equivalence. In this sense the Iitaka fibration is unique. By taking a suitable modification, we may assume that $f$ is a morphism and $Y$ is smooth.

The Iitaka fibration $f : X \to Y$ satisfies the following properties:

1. For a general fiber $F$, Kod$(F) = 0$ holds.
2. $\dim Y = \text{Kod}(Y)$.

1.4 Analytic Zariski decompositions

Let $L$ be a pseudoeffective line bundle on a compact complex manifold $X$. To analyze the ring:

\[(1.10) \quad R(X, L) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mL)),\]

it is useful to introduce the notion of analytic Zariski decompositions.

**Definition 1.1** Let $M$ be a compact complex manifold and let $L$ be a holomorphic line bundle on $M$. A singular hermitian metric $h$ on $L$ is said to be an **analytic Zariski decomposition** (AZD in short), if the followings hold.

1. $\sqrt{-1} \Theta_h$ is a closed positive current.
2. For every $m \geq 0$, the natural inclusion:

\[(1.11) \quad H^0(M, \mathcal{O}_M(mL) \otimes I(h^m)) \to H^0(M, \mathcal{O}_M(mL))\]

is an isomorphism.

**Remark 1.2** If an AZD exists on a line bundle $L$ on a compact complex manifold $M$, $L$ is pseudoeffective by the condition 1 above.
It is known that for every pseudoeffective line bundle on a compact complex manifold, there exists an AZD on \( L \) (cf. \([T1, T2, D-P-S]\)). The advantage of the AZD is that we can handle pseudoeffective line bundle \( L \) on a compact complex manifold \( X \) as a singular hermitian line bundle with semipositive curvature current as long as we consider the ring \( R(X, L) \).

One may construct an AZD for a pseudoeffective line bundle on a compact complex manifold as follows. Let \( L \) be a pseudoeffective line bundle on a compact complex manifold \( X \). Let \( h_0 \) be a \( C^\infty \) hermitian metric on \( L \). We set

\[ h_{\min} := \inf \{ h \mid \text{a singular hermitian metric on } L, \ h \geq h_0, \sqrt{-1}\Theta_h \geq 0 \} \]

Then \( h_{\min} \) is an AZD on \( L \) with minimal singularities in the following sense.

**Definition 1.3** Let \( L \) be a pseudoeffective line bundle on a compact complex manifold \( X \). An AZD \( h \) on \( L \) is said to be a **AZD of minimal singularities**, if for any AZD \( h' \) on \( L \), there exists a positive constant \( C \) such that

\[ h \leq C \cdot h' \]

holds. \( \square \)

In general an AZD of a pseudoeffective line bundle \( L \) on a smooth projective variety is not necessarily of minimal singularities.

### 1.5 Requirement of the canonical semipositive current

Let \( f : X \rightarrow Y \) be the Iitaka fibration of a smooth projective variety \( X \) of nonnegative Kodaira dimension. In this article, we shall consider a canonical semipositive current, say \( \omega_X \) associated with the Iitaka fibration.

It is natural to require that \( \omega_X \) has the following properties:

1. \( \omega_X \) is unique and birationally invariant, i.e., if \( X \) is birational to \( X' \) and let \( \mu : X'' \rightarrow X \) and \( \mu' : X'' \rightarrow X' \) be modifications from a smooth projective variety \( X'' \). Then

\[ \omega_{X''} = \mu^*\omega_X + 2\pi E = (\mu')^*\omega_{X'} + 2\pi E' \]

hold, where \( E := K_{X''} - \mu^*K_X \) and \( E' := K_{X''} - (\mu')^*K_{X'} \) respectively.

2. There exists an AZD \( h_K \) of \( K_X \) such that \( \omega_X = \sqrt{-1}\Theta_{h_K} \).

3. There exists a closed semipositive current \( \omega_Y \) on \( Y \) such that \( \omega_X = f^*\omega_Y \).

4. There exists a nonempty Zariski open subset \( U \) of \( Y \) such that \( \omega_Y | U \) is a \( C^\infty \)-Kähler form.

### 1.6 Canonical bundle formula and the correction term for the Kähler-Einstein equations

To construct the AZD \( h_K \) above, we shall solve a partial differential equation. The equation is similar to the Kähler-Einstein equation: \(-\text{Ric}_{\omega_E} = \omega_E\), but there are two major differences:
(1) The equation is defined on $Y$ not on $X$.

(2) The equation has the additional term which comes from variation of Hodge structures.

For a graded ring $R := \oplus_{i=0}^{\infty} R_i$ and a positive integer $m$, we set

\[(1.15)\quad R^{(m)} := \oplus_{i=0}^{\infty} R_{mi}.\]

For a KLT pair $(M, D)$, we set

\[(1.16)\quad R(M, K_M + D) := \oplus_{m=0}^{\infty} \Gamma(M, \mathcal{O}_M(|m(K_M + D)|))\]

and

\[(1.17)\quad \text{Kod}(M, D) := \limsup_{m \to \infty} \frac{\log \dim \Gamma(M, \mathcal{O}_M(|m(K_M + D)|))}{\log m}.\]

**Theorem 1.4** ([F-M, p.183, Theorem 5.2]) Let $(X, \Delta)$ be a proper KLT pair with

\[(1.18)\quad \text{Kod}(X, K_X + \Delta) = n.\]

Then there exists a $n$-dimensional KLT pair $(Y', \Delta')$ with $\text{Kod}(Y', \Delta') = n$, two positive integers $e, e'$ such that

\[(1.19)\quad R(X, K_X + \Delta)^{(e)} \simeq R(Y', K_{Y'} + \Delta')^{(e')} .\]

Let us consider the case that $\Delta = 0$ in Theorem 1.4. Then the canonical ring $R(X, K_X)^{(e)}$ is (a subring of) the pullback of the log canonical ring of some KLT pair $(Y', \Delta')$ of log general type.

Let us explain the equation. Let $f : X \to Y$ be an Iitaka fibration such that $f_* \mathcal{O}_X(m!K_{X/Y})^{**}$ is locally free on $Y$ for some $m$ (hence for every sufficiently large $m$), where $^{**}$ denotes the double dual. Such $f : X \to Y$ exists by [F-M, p.169, Proposition 2.2]. The divisor $\Delta'$ is related to the $\mathbb{Q}$-line bundle

\[(1.20)\quad L_{X/Y} := \frac{1}{m_0!} f_* \mathcal{O}_X(m_0!K_{X/Y})^{**}\]

on $Y$ in terms of the canonical bundle formula (See [F-M] for detail. In [F-M Section 2]), where $m_0$ is a sufficiently large positive integer. We note that $L_{X/Y}$ is independent of a sufficiently large $m_0$ (cf. [F-M Section 2]). We call $L_{X/Y}$ the **Hodge $\mathbb{Q}$-line bundle** of $f : X \to Y$. $L_{X/Y}$ carries a natural singular hermitian metric $h_{L_{X/Y}}$ defined by

\[(1.21)\quad h_{L_{X/Y}}^{m_0!} (\sigma, \sigma) = \left( \int_{X_y} |\sigma|^{2} \right)^{m_0!},\]

where $y \in Y, X_y := f^{-1}(y)$ and $\sigma \in L_{X/Y,y}$. It is known that $h_{L_{X/Y}}$ has semipositive curvature in the sense of current ([Ka2], [Ka3, p.174, Theorem 1.1])
by using the variation of Hodge structures. Let $\Omega$ be a $C^\infty$ volume form on $Y$. We shall consider the following equation:

(1.22) \[-\text{Ric}_{\omega_Y} + \sqrt{-1} \Theta_{h_{L/X/Y}} = \omega_Y,\]

where

(1.23) \[\omega_Y = -\text{Ric} \Omega + \sqrt{-1} \Theta_{h_{L/X/Y}} + \sqrt{-1} \partial \bar{\partial} u,\]

and $u$ is the unknown function. Here the term $\sqrt{-1} \Theta_{h_{L/X/Y}}$ corresponds to the boundary divisor of the KLT pair $(Y', \Delta')$ in the canonical bundle formula ([F-M, p.183, Theorem 5.2]). The authenticity of the equation (1.22) can be verified by checking the fact that the dynamical system of Bergman kernels on $X$ as in [T4] yields the current on $Y$ which satisfies the equation (1.22) (cf. Theorem 1.7).

Actually I first constructed the current by using the dynamical system of Bergman kernels and then found the equation (1.22) inspired by the [F-M].

There are several difficulties to solve the equation (1.22). First of all we cannot expect that there exists a $C^\infty$-solution $\omega_Y$. In fact $\Theta_{h_{L/X/Y}}$ is not $C^\infty$ in general. And if $\Theta_{h_{L/X/Y}}$ is not $C^\infty$, (1.22) has no $C^\infty$ solution $\omega_Y$. Moreover $h_{L/X/Y}$ is not of algebraic singularities (cf. Definition 2.1) in general. Moreover even if $h_{L/X/Y}$ is $C^\infty$, $u$ is not $C^\infty$ in general. Secondary the solution $u$ is not unique. But if we require that $\Omega^{-1} \cdot h_{L/X/Y} \cdot e^u$ is an AZD of $K_Y + L_{X/Y}$, then the solution $u$ is actually unique and the resulting current $\omega_Y$ is nothing but the current constructed by the dynamical system of Bergman kernels (see Section 1.8 and Section 3 below).

1.7 Canonical Kähler currents

Now we shall state the existence of the canonical semipositive current on a smooth projective variety of nonnegative Kodaira dimension.

**Theorem 1.5** (cf. [S-T, Theorem B.2]) In the above notations, there exists a unique singular hermitian metric on $h_K$ on $K_Y + L_{X/Y}$ and a nonempty Zariski open subset $U$ in $Y$ such that

1. $h_K$ is an AZD of $K_Y + L_{X/Y}$,
2. $f^* h_K$ is an AZD of $K_X$, Here we have used the inclusion:

\[f^* \mathcal{O}_Y(m_0!(K_Y + L_{X/Y})) \hookrightarrow \mathcal{O}_X(m_0!K_X)\]

3. $h_K$ is $C^\infty$ on $U$,
4. $\omega_Y = \sqrt{-1} \Theta_{h_K}$ is a Kähler form on $U$,
5. $-\text{Ric}_{\omega_Y} + \sqrt{-1} \Theta_{L_{X/Y}} = \omega_Y$ holds on $U$. \qed
Although the proof of Theorem 1.5 is given in [S-T], I shall give an alternative proof in this paper for the completeness, since my original proof seems to be different from that in [S-T]. One can see the KLT version of the above theorem in [T0].

**Definition 1.6** The current $\omega_Y$ on $Y$ is said to be the canonical Kähler current of the Iitaka fibration $f : X \to Y$. Also $\omega_X := f^*\omega_Y$ is said to be the canonical semipositive current on $X$. We define the measure $d\mu_{\text{can}}$ on $X$ by

$$(1.24) \quad d\mu_{\text{can}} := \frac{1}{n!} f^*\omega_Y^n \cdot h_{L_{X/Y}}^{-1}$$

and is said to be the canonical measure, where $n$ denotes $\dim Y$. □

The existence of the canonical Kähler current is proven in terms of solving Monge-Ampère equations. The proof given here is similar to the one of [T4, Section 5.1, Theorem 5.1]. We shall give a proof in Section 2 (see also [S-T, Section 4]).

### 1.8 Dynamical construction of the canonical Kähler currents

The canonical Kähler current in Theorem 1.5 can be constructed as the limit of a dynamical system as in ([T4]).

Let $X$ be a smooth projective $n$-fold with $\text{Kod}(X) \geq 0$. And let

$$(1.25) \quad f : X \to Y$$

be the Iitaka fibration associated with the complete linear system $|m_0!K_X|$ for some sufficiently large positive integer $m_0$. By taking a suitable modifications, we shall assume the followings:

1. $Y$ is smooth and $f$ is a morphism.
2. $f_*\mathcal{O}_X(m_0!K_X/Y)^{**}$ is a line bundle on $Y$, where ** denotes the double dual.

We define the Hodge $\mathbb{Q}$-line bundle $L_{X/Y}$ by

$$(1.26) \quad L_{X/Y} := \frac{1}{m_0!} f_*\mathcal{O}_X(m_0!K_X/Y)^{**}.$$ 

Let $a$ be a positive integer such that $f_*\mathcal{O}_X(aK_X/Y) \neq 0$. Then we see that

$$(1.27) \quad H^0(X, \mathcal{O}_X(m_0!K_X)) \simeq H^0(Y, \mathcal{O}_Y(m_0(K_Y + L_{X/Y})))$$

holds for every $m \geq 0$. In particular $\text{Kod}(X) = \dim Y$ holds. Hence by (1.27), we see that $K_Y + L_{X/Y}$ is big. Taking $m_0$ sufficiently large, we may assume that the following conditions are satisfied:

1. There exists an effective Cartier divisor $M$ such that $A := m_0!(K_Y + L_{X/Y}) - M$ is very ample.
We continue this process. Suppose that we have constructed $K_h$ (1.30) where $n$ denotes dim $Y$. Then we set

$$K_{m+1} := \left\{ \begin{array}{ll}
K(Y, K_Y + m! (K_Y + L_{X/Y}), h_{m!}), & \text{if } a > 1 \\
K(Y, K_Y + L_{X/Y} + m! (K_Y + L_{X/Y}), h_{L_{X/Y} \cdot h_{m!}}), & \text{if } a = 1
\end{array} \right.$$

(1.28)

where for a singular hermitian line bundle $(F, h_F)$ $K(Y, K_Y + F, h_F)$ is (the diagonal part of) the Bergman kernel of $H^0(Y, \mathcal{O}_Y(K_Y + F) \otimes \mathcal{I}(h_F))$ with respect to the $L^2$-inner product:

$$(\sigma, \sigma') := (\sqrt{-1})^n \int_Y h_F \cdot \sigma \wedge \bar{\sigma'},$$

(1.29)

where $n$ denotes dim $Y$. Then we set

$$h_{m+1} := (K_{m+1})^{-1}.$$

(1.30)

We continue this process. Suppose that we have constructed $K_m$ and the singular hermitian metric $h_m$ on $[\frac{m}{a}]a(K_Y + L_{X/Y}) + (m - [\frac{m}{a}]a)K_Y. We define $K_{m+1}$ by

(1.31)

$$K_{m+1} := \left\{ \begin{array}{ll}
K(Y, (m+1)K_Y + [\frac{m+1}{a}]aL_{X/Y}, h_m), & \text{if } m+1 \neq 0 \mod a \\
K(Y, (m+1)(K_Y + L_{X/Y}), h_{L_{X/Y}}^0 \otimes h_m), & \text{if } m+1 \equiv 0 \mod a
\end{array} \right.$$

and

$$h_{m+1} := (K_{m+1})^{-1}.$$

(1.32)

Thus inductively we construct the sequences $\{h_m\}_{m \geq m_0}$ and $\{K_m\}_{m > m_0!}$. This inductive construction is essentially the same one originated by the author in [T3]. But since $L_{X/Y}$ is a $\mathbb{Q}$-line bundle, the above dynamical system is slightly more complicated than in [T3]. We shall call $(L_{X/Y}, h_{L_{X/Y}})$ the boundary of the dynamical system of the Bergman kernels.

The following theorem asserts that the above dynamical system yields the canonical Kähler current on $Y$.

---

2In [T6], we consider dynamical systems with more general (singular) boundary. Actually in that case we need to consider a family of dynamical systems which are related to a Ricci iteration. I believe that the dynamical systems in [T6] are better than the one here. But the dynamical system here is much simpler.
Theorem 1.7 Let $X$ be a smooth projective variety of nonnegative Kodaira dimension and let $f : X \to Y$ be the Iitaka fibration as above. Let $\{h_m\}_{m \geq m_0}$ be the sequence of hermitian metrics as above and let $n$ denote $\dim Y$. $\omega_Y$ is the canonical Kähler current on $Y$ as in Theorem 1.5. Then

\begin{equation}
(1.33) 
\liminf_{m \to \infty} \frac{1}{\sqrt{(m!)^n}} \cdot h_m
\end{equation}

is a singular hermitian metric on $K_Y + L_{X/Y}$ such that

\begin{equation}
(1.34) 
\omega_Y = (2\pi)^n \cdot \left( \frac{1}{n!} \omega_Y^{abc} \right)^{-1} \cdot h_{L_{X/Y}}
\end{equation}

holds almost everywhere on $Y$ and

\begin{equation}
(1.35) 
\omega_Y = \sqrt{-1} \Theta_{h_\infty}
\end{equation}

holds on $Y$. In particular $h_\infty$ (and hence $\omega_Y$) is unique and is independent of the choice of $A$ and $h_A$. \qed

Here it may be better to replace $h_\infty$ by its lower-semi-continuous envelope because of the following classical theorem.

Theorem 1.8 ([L, p.26, Theorem 5]) Let $\{u_\alpha\}_{\alpha \in A}$ be a family of plurisubharmonic function on a domain $\Omega$ in $\mathbb{C}^n$. Suppose that $\{u_\alpha\}_{\alpha \in A}$ is locally uniformly bounded from above. Then the upper-semi-continuous envelope of $\sup_{\alpha \in A} u_\alpha$ is again plurisubharmonic on $\Omega$. \qed

But anyway this adjustment occurs on a set of measure 0.

Remark 1.9 In the above dynamical system, we start from an ample divisor $A$ of the form $m_0!(K_Y + L_{X/Y}) - M$ with $M$ effective. But every ample divisor $A$ is written in this form, since

\begin{equation}
(1.36) 
H^0(Y, \mathcal{O}_Y(m!(K_Y + L_{X/Y}) - A)) \neq 0
\end{equation}

for every sufficiently large $m$ by the bigness of $K_Y + L_{X/Y}$. \qed

In the above construction the dynamical system as above is more complicated than in [T4] because $L_{X/Y}$ is a $\mathbb{Q}$-line bundle and not a genuine line bundle on $Y$. But since $h_{L_{X/Y}}$ is very close to be a smooth metric (see Section 2.2), we can handle the singularity of $h_{L_{X/Y}}$. In [T6], we handle more singular boundary than $(L_{X/Y}, h_{L_{X/Y}})$.

1.9 Dynamical system on $X$

Let $\{h_m\}_{m \geq m_0}$ be the dynamical system as in the last subsection. Since $\mathcal{O}_X (f^*(a(K_Y + L_{X/Y})))$ is a subsea of $\mathcal{O}_X (aK_X)$ by the definition of $L_{X/Y}$, if $a| m$, we may identify $f^* h_m$ as a singular hermitian metric on $aK_X$.

Moreover, if $a = 1$, then by the definition of the Hodge metric $h_{L_{X/Y}}$, we see that

\begin{equation}
(1.37) 
(f^* h_m)^{-1} = K(X, mK_X, f^* h_{m-1})
\end{equation}
holds. Hence in this case, we consider the dynamical system \( \{(f^*h_m)^{-1}\} \) as a dynamical system of Bergman kernels on \( X \).

If \( a > 1 \),
\[
(1.38) \quad (f^*h_{m+1})^{-1} = \begin{cases} 
K(X, (m+1)K_X - (\lfloor \frac{m+1}{a} \rfloor a)L_{X/Y}, f^*h_{L_{X/Y}}^{-1} \cdot f^*h_m) & \text{if } m + 1 \not\equiv 0 \mod a \\
K(X, (m+1)K_X, f^*(h_m \cdot h_{L_{X/Y}}^{a})) & \text{if } m + 1 \equiv 0 \mod a
\end{cases}
\]

where for a real number \( \lambda \), \( \{\lambda\} \) denotes the fractional part \( \lambda - \lfloor \lambda \rfloor \). Here we note that
\[
(1.39) \quad (m+1)K_X - \left( \frac{m+1}{a} \right) aL_{X/Y} = \left( \lfloor \frac{m+1}{a} \rfloor a \right) K_X + \left( \{ \frac{m+1}{a} \} a \right) f^*K_Y
\]
is a line bundle on \( X \).

In this way, we may translate the dynamical system of Bergman kernels on \( Y \) in the last subsection as the dynamical system of Bergman kernels on \( X \).

2 Construction of the canonical measure by solving Monge-Ampère equations

In this section we shall prove Theorem 1.5. Although the proof of Theorem 1.5 is given in [S-T] independently, we shall give an original proof here for the completeness. Also the proof in [S-T] is quite close to the proof of [T4, Theorem 5.1].

The present proof is different from that in [S-T], in the following points:

1. We consider a smoothing of the Hodge metric \( h_{L_{X/Y}} \) which need not be of algebraic singularities. Hence we need to consider the sequence of modified equations.

2. The \( C^0 \)-estimate of the solution depends on the estimate of the Hodge metric near the discriminant locus and the notion of the minimal AZD.

The techniques used here are quite standard and have been known for more than twenty years (cf. [Su, T0]). In this sense the proof of Theorem 1.5 is not essentially new. But as in [Su], we need to require the finite generation of canonical ring ([B-C-H-M]) to prove the \( C^2 \)-regularity of the metrics on a Zariski open subset.

2.1 Setup

Let \( X \) be a smooth projective \( n \)-fold with \( \text{Kod}(X) \geq 0 \). And let
\[
(2.1) \quad f : X - \cdots \to Y
\]
be the Iitaka fibration associated with the complete linear system \( |m_0!K_X| \). By taking a suitable modifications, we shall assume the followings.

\footnote{The original proof of [T4 Theorem 5.1] has a gap in the proof of \( C^2 \)-regularity. I have not checked the proof in [S-T] in full detail.}
(1) \( f \) is a morphism.
(2) \( Y \) is smooth.
(3) \( f_* \mathcal{O}_X(m_0!K_{X/Y})^{**} \) is a line bundle on \( Y \).
(4) The discriminant locus \( D \) of \( f \) is a divisor with normal crossings on \( Y \).

We define the \( \mathbb{Q} \)-line bundle \( L_{X/Y} \) on \( Y \) by

\[
L_{X/Y} := \frac{1}{m_0!} f_* \mathcal{O}_X(m_0!K_{X/Y})^{**}
\]

and let \( h_{L_{X/Y}} \) be the singular hermitian metric on \( L_{X/Y} \) defined by

\[
h_{L_{X/Y}}^{m_0!}(\sigma, \sigma) := \left( \int_{X_y} |\sigma|^2 \frac{\omega^{n}}{m_0!} \right)^{m_0!}
\]

It is clear that \( h_{L_{X/Y}} \) is smooth on

\[
Y^o := \{ y \in Y | f \text{ is smooth over } y \} = Y \setminus D
\]

and the singularity of \( h_{L_{X/Y}} \) around \( D \) is described in terms of variation of Hodge structures. And we see that

\[
H^0(Y, \mathcal{O}_Y(m!(K_Y + L_{X/Y}))) \hat{\otimes} \mathcal{I}(h_{L_{X/Y}}^{m_0!}) \simeq H^0(Y, \mathcal{O}_Y(m!(K_Y + L_{X/Y})))
\]

holds for every sufficiently large \( m \), i.e., the \( L^2 \)-condition with respect to the singular hermitian metric \( h_{L_{X/Y}} \) does not affect the global section of \( m!(K_Y + L_{X/Y}) \).

Let \( \Omega \) be a \( C^\infty \)-volume form on \( Y \). Let us consider the equation :

\[
-\text{Ric}_\omega + \sqrt{-1} \Theta_{h_{L_{X/Y}}} = \omega_Y
\]

on \( Y \), where

\[
\omega_Y = -\text{Ric} \Omega + \sqrt{-1} \Theta_{h_{L_{X/Y}}} + \sqrt{-1} \partial \bar{\partial} u
\]

for some unknown upper-semi-continuous function \( u \) bounded from above on \( Y \). Then the above equation is equivalent to

\[
\log \left( \frac{\omega + \sqrt{-1} \partial \bar{\partial} u}{\Omega} \right)^n = u,
\]

where \( n := \dim Y \) and

\[
\omega := -\text{Ric} \Omega + \sqrt{-1} \Theta_{h_{L_{X/Y}}}.
\]

We note that since \( \text{Kod}(X) = \dim Y, K_Y + (L_{X/Y}, h_{L_{X/Y}}) \) is big, i.e.,

\[
\limsup_{m \to \infty} m!^{-n} \dim H^0(Y, \mathcal{O}_Y(m!(K_Y + L_{X/Y}))) \hat{\otimes} \mathcal{I}(h_{L_{X/Y}}^{m_0!}) > 0
\]

holds, where \( n = \dim Y \). The main difficulty for solving the equation (2.6) is the fact that \( h_{L_{X/Y}} \) is not of algebraic singularities in the following sense.
2.2 Smoothing of the Hodge metric \( h_{X/Y} \)

We say that \( h \) is of algebraic singularities, if there exists a positive integer \( m_0 \), global holomorphic sections \( \sigma_0, \ldots, \sigma_N \) of \( m_0L_{X/Y} \) and a \( C^\infty \) function \( \phi \) such that

\[
(2.11) \quad h = e^\phi \cdot \left( \sum_{i=0}^N |\sigma_i|^2 \right)^{-\frac{1}{m_0}}
\]

holds. \( \square \)

There are two ways to treat the singularity of \( h_{L_{X/Y}} \) in (2.10). One way is to smooth out the singularities of \( h_{L_{X/Y}} \) and the other way is to consider the metric of Poincaré growth on the complement of the discriminant locus of \( f : X \to Y \).

The both methods depend on the analysis of the singularities of \( h_{L_{X/Y}} \) in terms of the theory of variation of Hodge structures.

2.2 Smoothing of the Hodge metric \( h_{L_{X/Y}} \)

The singular hermitian metric \( h_{L_{X/Y}} \) is generically \( C^\infty \), but need not be of algebraic singularities.

The singularity of \( h_{L_{X/Y}} \) can be described by using variation of Hodge structures. Let \( \sigma \) be the minimal positive integer such that \( f_*O_X(aK_{X/Y})^{**} \) is not 0. Then the \( \sigma \)-th root of local holomorphic section of \( f_*O_X(aK_{X/Y}) \) can be considered to be a family of canonical forms on the family of cyclic \( \sigma \)-covers of the fibers. In this way the Hodge metric can be described in terms of the theory of variation of Hodge structures (cf. [Sch]). Let us assume that the discriminant locus \( D \) of \( f : X \to Y \) is a divisor with normal crossings. As in [Ka1], the locally free extension of the Hodge bundle is controlled by the monodromy which is quasi-unipotent. And in this setting the local monodromy is abelian.

**Definition 2.1** Let \( h \) be a singular hermitian metric on a line bundle \( L_{X/Y} \). We say that \( h \) is of algebraic singularities, if there exists a positive integer \( m_0 \), global holomorphic sections \( \sigma_0, \ldots, \sigma_N \) of \( m_0L_{X/Y} \) and a \( C^\infty \) function \( \phi \) such that

\[
(2.11) \quad h = e^\phi \cdot \left( \sum_{i=0}^N |\sigma_i|^2 \right)^{-\frac{1}{m_0}}
\]

holds. \( \square \)

**Definition 2.2** Let \( (M, B) \) a pair of a complex manifold \( M \) and a divisor \( B \) with normal crossings. Let \( \omega_P \) be a Kähler form on \( M - B \). \( \omega_P \) is said to be of Poincaré growth, if for any polydisk \( \Delta^n := \{(z_1, \ldots, z_n); |z_i| < 1, 1 \leq i \leq n \} \) in \( M \) such that

\[
(2.12) \quad \Delta^n \cap B = \{(z_1, \ldots, z_n) \in \Delta^n|z_1 \cdots z_k = 0\},
\]

there exist locally bounded positive continuous functions \( a \leq b \) on \( \Delta^n \) such that

\[
(2.13) \quad a \left( \sum_{i=1}^k \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^2(\log |z_i|)^2} + \sum_{j=k+1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j \right) \leq \omega_P \leq b \left( \sum_{i=1}^k \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^2(\log |z_i|)^2} + \sum_{j=k+1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j \right)
\]

hold on \( \Delta^n \cap (M - B) \).

Let \( \Omega_P \) be a volume form on \( M - B \). \( \Omega_P \) is said to be of Poincaré growth, if for any polydisk \( \Delta^n \) in \( M \) such that

\[
(2.14) \quad \Delta^n \cap B = \{(z_1, \ldots, z_n) \in \Delta^n|z_1 \cdots z_k = 0\},
\]

there exists a locally bounded positive continuous function \( c(z) \) on \( \Delta^n \) such that

\[
(2.15) \quad \Omega_P = c(z) \cdot \left( \wedge_{i=1}^k \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^2(\log |z_i|)^2} \right) \wedge \left( \wedge_{j=k+1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j \right)
\]

holds on \( \Delta^n \cap (M - B) \). \( \square \)
Remark 2.3 Let $\Omega_P$ be a volume form of Poincaré growth on $(M, B)$ with $M$ compact. If for every polydisk $\Delta^n$ as in Definition 2.2 the function $c(z)$ above is $C^2$ on $\Delta^n$, then $-\text{Ric} \Omega_P$ is of Poincaré growth in the sense that there exists a positive constant $C$ such that

$$
(2.16) \quad -C \cdot \omega_P \leq -\text{Ric} \Omega_P \leq C \cdot \omega_P,
$$

where $\omega_P$ is a Kähler form of $M - B$ with Poincaré growth. This is standard and is easily verified by direct calculation. □

Then we have the following lemma.

Lemma 2.4 There exists a positive integer $m_0$ such that $h^{m_0!}_L X/Y$ is a singular hermitian metric on the line bundle $f^* \mathcal{O}_Y(m_0!K_{X/Y})^{**}$ such that with respect to a local holomorphic frame

$$
(2.17) \quad h_L X/Y = O(\log |\sigma_D|)^q)
$$

holds, where $\sigma_D$ is a local defining function of $D$ and $q$ is a positive integer. And the curvature $\sqrt{-1} \Theta h_L X/Y$ is dominated by a constant times a Kähler form $\omega_P$ with Poincaré growth on $Y \setminus D$. □

Remark 2.5 Besides (2.17), we see that $h_L X/Y$ is bounded from below by a smooth metric on $L_{X/Y}$, since $h_L X/Y$ has semipositive curvature in the sense of current ([Ka2], [Ka3, p.174, Theorem 1.1]). This fact is essentially due to the theory of variation of Hodge structures.

The estimate of $h_L X/Y$ in Lemma 2.4 follows from [Ka1] which uses the theory of variation of Hodge structures due to W. Schmidt ([Sch]). And the latter estimate of $\sqrt{-1} \Theta h_L X/Y$ follows from the fact that holomorphic sectional curvature in the horizontal direction of the period domain is dominated by a negative constant ([G]) and the Yau-Royden Schwarz lemma ([Y2, ?]).

In this sense $h_L X/Y$ is very close to a smooth metric. To smooth out $h_L X/Y$, we take a finite open covering $\mathcal{U} := \{U_\alpha\}$ of $Y$ such that every $U_\alpha$ is biholomorphic to the open unit ball in $\mathbb{C}^n$ with center $O$ via the coordinate $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$. Taking $\mathcal{U}$ properly, we may and do assume $z_\alpha$ is a holomorphic coordinate on a larger open subset $\hat{U}_\alpha$ which is biholomorphic to the open ball with radius 2 in $\mathbb{C}^n$ with center $O$ via $z_\alpha$. Let $h_0$ be a $C^\infty$ hermitian metric on the $\mathbb{Q}$ line bundle $L_{X/Y}$. We set

$$
(2.18) \quad \varphi := \log \frac{h_L X/Y}{h_0}.
$$

Let $\rho$ be a $C^\infty$ function on $\mathbb{C}^n$ such that $0 \leq \rho \leq 1$, supp $\rho$ is contained in the unit open ball in $\mathbb{C}^n$ with center $O$ and

$$
(2.19) \quad \int_{\mathbb{C}^n} \rho(z) d\mu(z) = 1,
$$

where $d\mu$ is the usual Lebesgue measure on $\mathbb{C}^n$. For every $0 < \delta < 1$ we set

$$
(2.20) \quad \rho_\delta(z) = \delta^{-2n} \rho(z/\delta).
$$
We shall take a mollification $\varphi_{\alpha,\delta}$ of $\varphi|U_\alpha$ using the convolution with the mollifier $\rho_\delta$ as

\[(2.21) \quad \varphi_{\alpha,\delta} := (\varphi|U_\alpha) \ast \rho_\delta\]

with respect to the coordinate $z_\alpha$. Since $z_\alpha$ is a holomorphic coordinate on $U_\alpha$, $\varphi_{\alpha,\delta}$ is a well defined $C^\infty$ function on $U_\alpha$ for every $0 < \delta < 1$. Then $\varphi_{\alpha,\delta}$ converges to $\varphi$ in $L^1$-topology on $U_\alpha$ and compact uniformly in $C^\infty$-topology on $U_\alpha \setminus D$ as $\delta \downarrow 0$.

Let $\{\phi_\alpha\}$ be a partition of unity subordinate to $U$. We set

\[(2.22) \quad h_{L_X/Y,\delta} := \exp(\sum_\alpha \phi_\alpha \cdot \varphi_{\alpha,\delta}) \cdot h_0.\]

Then $h_{L_X/Y,\delta}$ is a $C^\infty$-hermitian metric on $L_X/Y$ and there exists a positive constant $C$ independent of $\delta > 0$ such that

\[(2.23) \quad \sqrt{-\Theta} h_{L_X/Y,\delta} \leq C \cdot \omega_P\]

holds. In general $h_{L_X/Y,\delta}$ does not have semipositive curvature. But by the construction, there exists a continuous function $e(\delta)$ on $Y$ such that

\[(2.24) \quad \sqrt{-\Theta} h_{L_X/Y,\delta} \geq -e(\delta) \cdot \omega_P\]

and

\[(2.25) \quad \lim_{\delta \downarrow 0} e(\delta) = 0\]

holds uniformly on $Y$.

### 2.3 The construction of the canonical Kähler currents

In this subsection, we shall prove the existence of the current $\omega_Y$ satisfying (2.6) without assuming the finite generation of canonical ring ([B-C-H-M]). This result is slightly weaker than Theorem 1.5. But the same strategy works to construct canonical Kähler-Einstein currents on LC pairs (cf. [T6]). Hence the following theorem has independent interest.

**Theorem 2.6** In the notations in Section 2.1, there exists a closed positive current $\omega_Y$ on $Y$ such that

1. $\omega_Y$ represents $2\pi c_1(K_Y + L_{X/Y})$. 
2. $h_K := n! (\omega_{Y,abc}^n)^{-1} \cdot h_{L_X/Y}$ (n = dim $Y$) is an AZD of $K_Y + L_{X/Y}$, where $\omega_{Y,abc}$ denotes the absolutely continuous part of $\omega_Y$. 
3. $\omega_Y = \sqrt{-\Theta} h_K$ holds on $Y$. 
4. $-\text{Ric}_{\omega_Y} + \sqrt{-\Theta} h_{L_X/Y} = \omega_Y$ holds on $Y$ in the sense of current, where $\text{Ric}_{\omega_Y} := \sqrt{-\Theta} \partial \bar{\partial} \log \omega_{Y,abc}$. □
The proof of Theorem 2.6 depends on the monotonicity lemma (Lemma 2.7).

Let \( m_0 \) be a sufficiently large positive integer such that for every \( m \geq m_0 \), \( m!(K_Y + L_{X/Y}) \) is a Cartier divisor on \( Y \) and \(|m!(K_Y + L_{X/Y})|\) gives a birational embedding of \( Y \). Let \( \pi_m : Y_m \to Y \) be the resolution of Bs \(|m!(K_Y + L_{X/Y})|\) such that for every \( m > m_0 \)

\[
\pi_m : Y_m \to Y
\]
factors through \( \pi_{m-1} : Y_{m-1} \to Y \). Let

\[
\mu_m : Y_m \to Y_{m-1}
\]
be the natural morphism. Here we may and do take \( \mu_m : Y_m \to Y_{m-1} \) such that the exceptional divisor of \( \pi_m \) is contained in \( \pi_{m-1}^{-1}(V) \). This is certainly possible by the definition of \( V \). Hence we have an (possibly infinite) tower

\[
\cdots \mu_{m+2} Y_{m+1} \xrightarrow{\mu_{m+1}} Y_m \xrightarrow{\mu_m} Y_{m-1} \xrightarrow{\mu_{m-1}} \cdots
\]

In the following proof, we shall consider this tower. But by the recent result on finite generation of canonical ring ([B-C-H-M]) we may avoid to consider an infinite tower. Namely we just need to consider one sufficiently large \( m_0 \). This certainly simplify the proof. The reason why we do not use the finite generation of canonical ring is that it is not essential from the analytic point of view and one may extend the theory to the case of LC pairs (cf. [T6]). Let

\[
\pi_m^* |m!(K_Y + L_{X/Y})| = |P_m| + F_m
\]
be the decomposition of \( \pi_m^* |m!(K_Y + L_{X/Y})| \) into the free part \( |P_m| \) and the fixed component \( F_m \). Let \( V \) be the analytic subset of \( Y \) defined by:

\[
V := \{ y \in Y^* | y \in \cap_{m>0} \text{Bs}\{|m!(K_Y + L_{X/Y})|\} \text{ or } \Phi_{|m!(K_Y + L_{X/Y})|} \text{ is not an embedding around } y \text{ for } m >> m_0 \} \cup \{ \text{the discriminant locus of } f \}.
\]

By taking a suitable modification of \( Y \), we may and do assume that \( V \) is a divisor with normal crossings.

There exists an effective \( \mathbb{Q} \)-divisor \( E_m \) on \( Y_m \) respectively such that the followings hold for every \( m \geq m_0 \).

1. \( P_m - E_m \) is ample on \( Y_m \).
2. All the coefficients of \( E_m \) are less than 1, i.e., \( |E_m| = 0 \).
3. \( \text{Supp} \ E_m = \pi_m^{-1}(V) \).
4. \( ((m+1)!)^{-1}(P_{m+1} - E_{m+1}) - \mu_{m+1}^{-1}(m!)^{-1}(P_m - E_m) \) is effective.

The existence of such \( \{E_m\} \) follows from the definition of \( V \) and the trivial fact that for any composition of successive blowing up

\[
\varpi : \mathbb{P}^\nu \to \mathbb{P}^\nu
\]

of a projective space \( \mathbb{P}^\nu \) with smooth centers, there exists an effective \( \mathbb{Q} \)-divisor \( B \) supported on the exceptional divisors of \( \varpi \) such that \( \varpi^*(\mathcal{O}(1) - B) \) is ample.
Hence by taking a suitable successive blowing ups with smooth centres over $V$, if necessary, we may assume the existence of such effective $\mathbb{Q}$-divisors $\{E_m\}$ by considering the image of $Y_m$ by the morphism associated with $|P_m|$ from $Y_m$ into a projective space.

After taking such a sequence $\{E_m\}$, we replace $\{E_m\}$ by $\{2^{-m}E_m\}$. Then it has the same properties as above. And we shall denote $\{2^{-m}E_m\}$ again by $\{E_m\}$. Then instead of (4) we have

\begin{equation}
\text{(2.30) } \ (m+1)!^{-1}(P_{m+1} - E_{m+1}) - \mu_{m+1}(m!)^{-1}(P_m - E_m) \text{ is effective}
\end{equation}

and contains $\varepsilon_m(\pi_{m+1}^{-1}V)_{red}$ for some positive number $\varepsilon_m$.

Let $h_{(m)}$ be a $C^\infty$ hermitian metric on $\pi_m^*(m!)^{-1}(P_m - E_m)$ with strictly positive curvature. We note that by (2.30) $\{h_{(m)}\}$ is getting less singular as $m$ tends to infinity and $h_{(m+1)}$ is strictly less singular than $h_{(m)}$ along $V$ (if we consider the metrics as singular hermitian metrics on $K_Y + L_{X/Y}$). Then

\begin{equation}
\Omega_{m, \delta} := h_{(m)}^{-1} \cdot (\pi_m^* h_{L_{X/Y}, \delta})
\end{equation}

is considered as a degenerate volume form on $Y_m$, where $\{h_{L_{X/Y}, \delta}\}$ is the smoothing of the Hodge metric $h_{L_{X/Y}}$ as in Section 2.2. We note that $\Omega_{m, \delta}^{-1} \cdot (\pi_m^* h_{L_{X/Y}, \delta})^{-1} = h_{(m)}$ is a metric with algebraic singularities on $K_Y + L_{X/Y}$.

Now we shall consider the equation:

\begin{equation}
\ - \text{Ric} \omega_{m, \delta} + \sqrt{-1} \pi_m^* \Theta_{h_{L_{X/Y}, \delta}} + \sqrt{-1} \partial \bar{\partial} u_{m, \delta} = \Omega_{m, \delta} \cdot e^{u_{m, \delta}}.
\end{equation}

on $Y_m$. Then by the definition (2.31) of $\Omega_{m, \delta}$, (2.32) is equivalent to

\begin{equation}
(\sqrt{-1} \Theta_{h_{(m)}} + \sqrt{-1} \partial \bar{\partial} u_{m, \delta})^n = \Omega_{m, \delta} \cdot e^{u_{m, \delta}}.
\end{equation}

Then since $\sqrt{-1} \Theta_{h_{(m)}}$ is a $C^\infty$ Kähler form on $Y_m$ by [Y] p.387, Theorem 6, solving (2.32), we see that there exists a $u_{m, \delta} \in C^\infty(Y_m \setminus \pi_m^{-1}V)$ and the closed positive current:

\begin{equation}
\omega_{m, \delta} := - \text{Ric} \omega_{m, \delta} + \sqrt{-1} \pi_m^* \Theta_{h_{L_{X/Y}, \delta}} + \sqrt{-1} \partial \bar{\partial} u_{m, \delta}
\end{equation}

on $Y_m$ such that

1. $- \text{Ric} \omega_{m, \delta} + \sqrt{-1} \pi_m^* \Theta_{h_{L_{X/Y}, \delta}} = \omega_{m, \delta}$ holds on $Y_m - \text{Supp} E_m$,

2. The absolutely continuous part $\omega_{m, \delta, abc}$ of $\omega_{m, \delta}$ is closed and represents $2\pi(m!)^{-1}(P_m - E_m)$,

3. $(\pi_m)_* \omega_{m, \delta}$ represents the class $2\pi C_1(K_Y + L_{X/Y})$,

4. There exists a positive constant $C(m, \delta)$ such that

\begin{equation}
|u_{m, \delta}| \leq C(m, \delta)
\end{equation}

holds on $Y_m$. 

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Here we note that
\[(2.36) \quad -\text{Ric} \Omega_{m,\delta} + \sqrt{-1} \pi_m^* \Theta_{h_{X/Y,\delta}} = \sqrt{-1} \Theta_{h(m)}\]
holds, hence it is independent of \(\delta\). If we set
\[(2.37) \quad \omega_{(m)} := \sqrt{-1} \Theta_{h(m)},\]
then the equation is transcribed as:
\[(2.38) \quad \log \frac{\omega^n_{m,\delta}}{\Omega_{m,\delta}} = \log \frac{(\omega(m) + \sqrt{-1} \partial \bar{\partial} u_{m,\delta})^n}{\Omega_{m,\delta}} = u_{m,\delta}.
\]
Let us consider \(\{\omega^n_{m,\delta}\}\) as a sequence of volume forms on \(Y \setminus V\). And we shall identify \(\pi_m^* \omega_{m,\delta}\) with \(\omega_{m,\delta}\) on \(Y \setminus V\). Hereafter we shall identify \(Y \setminus V\) with a Zariski open subset of \(Y_m\) for every \(m\) and consider everything on \(Y \setminus V\) (if without fear of confusion). Then by the maximum principle we have the following monotonicity lemma.

**Lemma 2.7 (Monotonicity Lemma)**

\[(2.39) \quad \omega^n_{m,\delta} \leq \omega^n_{m+1,\delta}\]
holds on \(Y \setminus V\). \(\square\)

**Proof of Lemma 2.7** We note that by the construction the followings hold.

1. The absolutely continuous parts of \(\omega_{m,\delta}\) and \(\omega_{m+1,\delta}\) represent \(2\pi (m!)^{-1}(P_m - E_m)\) and \(2\pi ((m + 1)!)^{-1}(P_{m+1} - E_{m+1})\) respectively.
2. \(\mu_{m+1}^* ((m!)^{-1}(F_m + E_m)) - ((m + 1)!)^{-1}(F_{m+1} + E_{m+1})\) is effective and contains \(\varepsilon_m (\pi^{-1}_{m+1} V)\) for some positive number \(\varepsilon_m\).

We note that by the boundedness of \(u_{m,\delta}\) and \(u_{m+1,\delta}\) (cf. (2.35)) and the equation (2.38), we see that the asymptotics of \(\omega^n_{m,\delta}\) and \(\omega^n_{m+1,\delta}\) near \(V\) is the same as \(\Omega_{m,\delta}\) and \(\Omega_{m+1,\delta}\) respectively. Since by the condition (2) above, \(\Omega_{m,\delta}/\Omega_{m+1,\delta}\) tends to 0 toward \(V\), the function \(\phi_{m,\delta}\) defined by
\[(2.40) \quad \phi_{m,\delta} := \log \frac{\omega^n_{m,\delta}}{\omega^n_{m+1,\delta}}\]
tends to \(-\infty\) toward \(V\). Hence there exists a point \(p_m \in Y \setminus V\) where \(\phi_{m,\delta}\) takes its maximum. Then
\[(2.41) \quad \sqrt{-1} \partial \bar{\partial} \phi_{m,\delta}(p_m) \leq 0\]
holds. By the equation
\[(2.42) \quad -\text{Ric}_{\omega_{k,\delta}} + \sqrt{-1} \pi_k^* \Theta_{h_{X/Y,\delta}} = \omega_{k,\delta}, (k = m, m + 1)\]
(2.41) implies that
\[(2.43) \quad \omega_{m,\delta}(p_m) \leq \omega_{m+1,\delta}(p_m)\]
Since we consider \(\omega_{m,\delta}\) as a current, it seems to be more authentic to denote \(\omega^n_{m,\delta,abc}\) instead of \(\omega_{m,\delta}\). But we consider the equation (2.15) on \(Y \setminus V\).
holds. In particular \( \phi_{m, \delta}(p_m) \leq 0 \) holds. Hence by the definition of \( p_m \), this implies that \( \phi_{m, \delta} \leq 0 \) holds on \( Y \setminus V \). Hence \( \omega^n_{m, \delta} \leq \omega^n_{m+1, \delta} \) holds on \( Y \setminus V \). This completes the proof of Lemma 2.7.

Now we shall consider the uniform \( C^0 \)-estimate on every compact subset of \( Y \setminus V \). Let us fix a positive integer \( s \geq m_0 \) and let

\[ \omega(s) = \sqrt{-1} \Theta h(s) \]

be the Kähler form defined as (2.37). We shall use \( h(s) \) and \( \omega(s) \) as standards in the following estimate. For every \( m > s \), let \( v_{m, \delta} \) be a \( C^\infty \)-function on \( Y \setminus D \) defined by

\[ v_{m, \delta} := u_{m, \delta} + \log \frac{h(s)}{h(m)} \]

Then

\[ \omega_{m, \delta} = \omega(s) + \sqrt{-1} \partial \bar{\partial} v_{m, \delta} \]

and

\[ \log \left( \frac{\omega(s) + \sqrt{-1} \partial \bar{\partial} v_{m, \delta}}{\Omega_{s, \delta}} \right)^n = v_{m, \delta} \]

hold. By the condition (2.30), we see that for every \( m > s \), \( \log(h(s)/h(m)) \) tends to \(+\infty\) toward \( V \). And by the boundedness of \( u_{m, \delta} \) (cf. (2.35)), we have the estimate:

\[ -C(m, \delta) + \log \frac{h(s)}{h(m)} \leq v_{m, \delta} \leq C(m, \delta) + \log \frac{h(s)}{h(m)}, \]

where \( C(m, \delta) \) is the positive constant as in (2.35). Hence \( v_{m, \delta} \) tends to \(+\infty\) toward \( V \). This implies that there exists a point \( p_0 \in Y \setminus V \), where \( v_{m, \delta} \) takes its minimum. Now we note that by (2.48),

\[ \log \left( \frac{\omega(s) + \sqrt{-1} \partial \bar{\partial} v_{m, \delta}}{\omega(s)} \right)^n = \int_0^1 \Delta_{(s, m, \delta, t)} v_{m, \delta} \, dt = v_{m, \delta} - \log \frac{\omega^n_{s, \delta}}{\Omega_{s, \delta}} \]

hold, where \( \Delta_{(s, m, \delta, t)} \) denotes the trace of \( \sqrt{-1} \partial \bar{\partial} v_{m, \delta} \) with respect to the Kähler form: \( (1 - t)\omega_{m, \delta} + t\omega(s) \). Hence by the minimum principle,

\[ v_{m, \delta}(p_0) \geq \log \frac{\omega^n_{s, \delta}(p_0)}{\Omega_{s, \delta}(p_0)} \]

holds.

On the other hand, by (2.31) and the definition of \( h_{X/Y, \delta}(\text{cf. (??)}) \), \( \Omega_{s, \delta} \) tends to 0 toward \( V \). Hence there exists a positive constant \( C_-(s) \) independent of \( \delta \) such that

\[ \min_{y \in Y} \log \frac{\omega^n_{s, \delta}(y)}{\Omega_{s, \delta}(y)} \geq C_-(s) \]
holds. By (2.50) and (2.49), we see that

\[
(2.51) \quad v_{m,\delta}(y) \geq C_-(s)
\]

holds for every \( y \in Y \setminus V \). By the definition of \( v_{m,\delta} \) (cf. (2.44)), we have that

\[
(2.52) \quad u_{m,\delta} \geq \log \frac{h(m)}{h(s)} + C_-(s)
\]

holds. Replacing \( s \) by \( t > s \), for \( m > t > s \) we have

\[
(2.52) \quad u_{m,\delta} \geq \log \frac{h(m)}{h(t)} + C_-(t)
\]

and hence by (2.44), we obtain the following lemma.

**Lemma 2.8** There exists a positive constant \( C_- (t) \) depending only on \( t > s \) such that for every \( m > t \)

\[
(2.53) \quad v_{m,\delta} \geq \log \frac{h(s)}{h(t)} + C_-(t)
\]

holds. In particular \( v_{m,\delta} \) tends to infinity toward \( V \). \( \square \)

On the other hand, we obtain the upper estimate of \( u_{m,\delta} \) as follows. We may and do assume that \( V \) is a divisor with normal crossings. Let \( \Omega_P \) be a volume form on \( Y \setminus V \) with Poincaré growth, i.e., for every polydisk \( \Delta^n \) in \( Y \) such that

\[
(2.54) \quad \Delta^n \cap V = \{(z_1, \ldots, z_n) \in \Delta^n | z_1 \cdots z_k = 0\},
\]

\[
(2.55) \quad \Omega_P = c |dz_1 \wedge \cdots \wedge dz_n|^2 \prod_{i=1}^k |z_i|^{(\log |z_i|^2)^2},
\]

where \( c \) is a positive \( C^\infty \) function on \( \Delta^n \). Such a \( \Omega_P \) can be constructed easily by using a partition of unity. We set

\[
(2.56) \quad \tilde{u}_{m,\delta} := u_{m,\delta} + \log \frac{\Omega_{m,\delta}}{\Omega_P}
\]

By the condition (2.30) and the boundedness of \( u_{m,\delta} \), there exists a point \( p'_0 \) on \( Y \setminus V \) such that \( \tilde{u}_{m,\delta} \) takes its maximum at \( p'_0 \). Then since

\[
(2.57) \quad \log \frac{\omega_{m,\delta}}{\Omega_P} = u_{m,\delta} + \log \frac{\Omega_{m,\delta}}{\Omega_P} = \tilde{u}_{m,\delta}
\]

hold, at \( p'_0 \) we have that

\[
(2.58) \quad \sqrt{-1} \partial \bar{\partial} \log \frac{\omega_{m,\delta}}{\Omega_P}(p'_0) \leq 0
\]

holds. Hence we have the inequality:

\[
(2.59) \quad -\text{Ric}_{\omega_{m,\delta}} \leq (-\text{Ric}_P)(p'_0).
\]
By the equation:

$$-\text{Ric} \omega_{m,\delta} + \sqrt{-1} \Theta_{h_{L_{X/Y},\delta}} = \omega_{m,\delta},$$

we see that

$$\omega_{m,\delta}(p_0') \leq -\text{Ric} \Omega_P + \sqrt{-1} \Theta_{h_{L_{X/Y},\delta}}$$

holds. In particular,

$$\omega^n_{m,\delta}(p_0') \leq (-\text{Ric} \Omega_P + \sqrt{-1} \Theta_{h_{L_{X/Y},\delta}})^n(p_0')$$

holds. This implies that

$$\tilde{u}_{m,\delta} = u_{m,\delta} + \log \frac{\Omega_{m,\delta}}{\Omega_P} \leq \log \frac{(-\text{Ric} \Omega_P + \sqrt{-1} \Theta_{h_{L_{X/Y},\delta}})^n}{\Omega_P}(p_0')$$

holds on $Y$. By the construction of $h_{L_{X/Y},\delta}$, Lemma 2.4 and Remark 2.3, there exists a positive constant $C_+$ independent of $\delta$ such that

$$\frac{(-\text{Ric} \Omega_P + \sqrt{-1} \Theta_{h_{L_{X/Y},\delta}})^n}{\Omega_P} \leq \exp(C_+)$$

holds on $Y \setminus V$. Combining (2.63) and (2.64) we have that

$$u_{m,\delta} \leq C_+ - \log \frac{\Omega_{m,\delta}}{\Omega_P}$$

and

$$e^{u_{m,\delta}} \Omega_{m,\delta} \leq \exp(C_+) \cdot \Omega_P$$

hold on $Y$.

On the other hand by the definition (2.31) and the definition of the smoothing \{h_{L_{X/Y},\delta}\} (2.22), there exists a sequence of positive number \{\epsilon(\delta)\} such that

(1) $\lim_{\delta \to 0} \epsilon(\delta) = 0$ holds,

(2) For every $0 < \lambda < \delta$, the inequality:

$$\Omega_{m,\delta} \leq (1 + \epsilon(\delta))\Omega_{m,\lambda}$$

holds.

Then for $0 < \lambda < \delta$ by (2.38)

$$\log \frac{(\omega_{m}(t) + \sqrt{-1} \partial \bar{\partial} u_{m,\lambda})^n}{(\omega_{m}(t) + \sqrt{-1} \partial \bar{\partial} u_{m,\delta})^n} = \log \frac{\Omega_{m,\lambda}}{\Omega_{m,\delta}} + (u_{m,\lambda} - u_{m,\delta})$$

holds. We note that

$$\log \frac{(\omega_{m}(t) + \sqrt{-1} \partial \bar{\partial} u_{m,\lambda})^n}{(\omega_{m}(t) + \sqrt{-1} \partial \bar{\partial} u_{m,\delta})^n} = \int_0^1 \bar{\Delta}_{\lambda}(u_{m,\lambda} - u_{m,\delta}) \, dt,$$
where \( \tilde{\Delta} \) denotes the Laplacian with respect to \( (1-t)\tilde{\omega}_{m,\lambda} + t\tilde{\omega}_{m,\delta} \). Then by \((2.68),(2.69)\) and the maximum principle, we see that

\[
(2.70) \quad u_{m,\lambda} - u_{m,\delta} \leq -\min_Y \log \frac{\Omega_{m,\lambda}}{\Omega_{m,\delta}} < \log(1 + \epsilon(\delta))
\]

holds on \( Y \). This argument is not quite right, since \( u_{m,\lambda}, u_{m,\delta} \) are not \( C^2 \) on \( Y \) (although they are \( C^2 \) bounded).

To justify the argument we proceed as in the proof of [p.387, Theorem 6][Y1], i.e., we shall consider the perturbation of the equation \((2.38)\). By the construction of \( \Omega_{m,\delta} \) (cf. \((2.31)\)) the 0-locus of \( \Omega_{m,\delta} \) is the divisor \( (\pi^*_{m} V)_{red} \) with normal crossings on \( Y_m \). Let \( (\pi^*_{m} V)_{red} = \sum V^{(m)}_j \) be the irreducible decomposition. Let us write \( \Omega_{m,\delta} \) as

\[
(2.71) \quad \Omega_{m,\delta} = \left( \prod_j \| \tau_j \|^2 a_j \right) \cdot \tilde{\Omega}_{m,\delta},
\]

where \( \tilde{\Omega}_{m,\delta} \) is a nondegenerate \( C^\infty \) volume form on \( Y_m \) and \( \| \tau_j \| \) denotes the hermitian norm of a holomorphic section \( \tau_j \) of the line bundle \( \mathcal{O}_{Y_m}(V^{(m)}_j) \) with divisor \( V^{(m)}_j \) on \( Y_m \) with respect to a fixed \( C^\infty \) hermitian metric and \( \{a_j\} \) are positive rational numbers. We may and do take the factor \( \left( \prod_j \| \tau_j \|^2 a_j \right) \) can be taken independent of \( \delta \). Now for \( 0 < \epsilon << 1 \), we shall consider the perturbed equation:

\[
(2.72) \quad \log \frac{(\omega(m) + \sqrt{-1}\partial\bar{\partial} u_{m,\delta}(\epsilon))}{\Omega_{m,\delta}(\epsilon)} = u_{m,\delta}(\epsilon),
\]

where \( \Omega_{m,\delta}(\epsilon) \) is a \( C^\infty \) nondegenerate volume form on \( Y \) defined by

\[
(2.73) \quad \Omega_{m,\delta}(\epsilon) = \left( \prod_j (\| \tau_j \|^2 + \epsilon a_j) \right) \cdot \tilde{\Omega}_{m,\delta}.
\]

Then \((2.72)\) has a unique \( C^\infty \)-solution \( u_{m,\delta}(\epsilon) \) and

\[
(2.74) \quad u_{m,\delta} = \lim_{\epsilon \downarrow 0} u_{m,\delta}(\epsilon)
\]

holds as in [Y1, p.387, Theorem 6] in the \( C^2 \)-norm with respect to the Kähler form \( \omega(m) \) on \( Y_m \). Then replacing \( u_{m,\lambda}, u_{m,\delta} \) by \( u_{m,\lambda}(\epsilon), u_{m,\delta}(\epsilon) \) respectively and letting \( \epsilon \downarrow 0 \), we may justify \((2.70)\). Since \( \{u_{m,\delta}\} \) is almost monotone decreasing as \( \delta \downarrow 0 \) as \((2.70)\), we have that

\[
(2.75) \quad dV_m := \frac{1}{n!} \lim_{\delta \downarrow 0} e^{u_{m,\delta}} \Omega_{m,\delta}
\]

exists and by \((2.38)\), we have

\[
(2.76) \quad dV_m = \frac{1}{n!} \lim_{\delta \downarrow 0} \omega^n_{m,\delta,abc}
\]

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holds on $Y$. We note that 
\[
\int_Y \omega_{m,\delta,abc}^n = (m!)^{-n}(P_m - E_m)^n
\]
holds. By (2.66) and the Lebesgue’s bounded convergence theorem, we see that 
\[
(2.77) \frac{1}{(2\pi)^n} \int_Y dV_m = \frac{1}{m!(m!)^n}(P_m - E_m)^n
\]
holds. We set 
\[
(2.78) \omega_m := \lim_{\delta \downarrow 0} \omega_{m,\delta}
\]
in the sense of current and 
\[
(2.79) h_m := dV_m^{-1} \cdot h_{L_{X/Y}}.
\]
Then $h_m$ is a singular hermitian metric on $K_Y + L_{X/Y}$ with semipositive curvature in the sense of current by (2.32) and (2.38). But also we may consider $h_m$ as a singular hermitian metric $\tilde{h}_m$ on $(m!)^{-1}(P_m - E_m)$ by the natural inclusion:
\[
\mathcal{O}_Y(m! \pi_m^*(m!(P_m - E_m))) \hookrightarrow \mathcal{O}_Y(m! \pi_m^*(K_Y + L_{X/Y}))
\]
where $\ell$ is a sufficiently large positive integer such that $\ell! \pi_m^*(m!(P_m - E_m))$ is Cartier. Now we introduce the following notion.

**Definition 2.9** Let $M$ be a projective manifold of dimension $n$ and let $(L, h_L)$ be a pseudoeffective singular hermitian line bundle on $X$. We define the number $\mu(L, h_L)$ by 
\[
\mu(L, h_L) := n! \lim_{m \to \infty} m^{-n} h^L(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h_m^m))
\]
is called the volume of $(L, h_L)$. □

**Remark 2.10** This definition is easily generalized to the case of singular hermitian $\mathbb{Q}$-line bundles. □

Suppose that the Lelong number $\nu(\sqrt{-\Delta} \Theta_{h_m})$ of $\sqrt{-\Delta} \Theta_{h_m}$ satisfies the inequality: $\nu(\sqrt{-\Delta} \Theta_{h_m}, y_m) > c$ for some $y_m \in Y_m$ and a positive number $c$, then by the basic property of the Lelong number, we see that 
\[
(2.80) \mathcal{I}(\tilde{h}^\ell_{m,y_m}) \subseteq m_{y_m}^{[\ell!]}\]
holds for every sufficiently large $\ell$, where $m_{y_m}$ denotes the maximal ideal at $y_m$. Hence the strict inequality 
\[
(2.81) \mu((m!)^{-1}(P_m - E_m), \tilde{h}_m) < (m!)^{-n}(P_m - E_m)^n
\]
holds, where $\mu((m!)^{-1}(P_m - E_m), \tilde{h}_m)$ denotes the volume of $(m!)^{-1}(P_m - E_m), \tilde{h}_m)$ (cf. Definition 3.3 below). On the other hand 
\[
(2.82) \frac{1}{(2\pi)^n} \int_Y \omega_{m,abc}^n = \mu((m!)^{-1}(P_m - E_m), \tilde{h}_m)
\]
holds by (2.77) and a theorem of Boucksom ([Bo, Proposition 3.1]). By the equality (2.77), (2.82) contradicts (2.81). Hence \( \nu(\sqrt{-1}\Theta h_m) \) is identically 0 and hence \( h_m \) is an AZD of \((m!)^{-1}(P_m - E_m)\). By Lemma 2.7 we see that \( \{dv_m\} \) is monotone increasing in \( m \) on \( Y \), hence \( \{h_m\} \) is getting less singular as \( m \) tends to infinity as metrics on \( K_Y + L_{X/Y} \). Then by (2.66) and Lebesgue’s bounded convergence theorem, we have the following lemma.

**Lemma 2.11**

\[
dV_Y := \frac{1}{n!} \lim_{m \to \infty} dV_m
\]

exists as a degenerate volume form on \( Y \). And if we define the singular hermitian metric \( h_K \) on on \( K_Y + L_{X/Y} \) by

\[
h_K := dV_Y^{-1} \cdot h_{L_{X/Y}},
\]

then \( h_K \) is an AZD of \( K_Y + L_{X/Y} \).

By the construction of \( \omega_m \),

\[
-\text{Ric}_{\omega_m} + \sqrt{-1} \pi_m^* \Theta h_{L_{X/Y}} = \omega_m
\]

holds for every \( m \geq 1 \). Then by (2.85) and Lemma 2.11 if we set

\[
\omega_Y := \sqrt{-1} \Theta h_K,
\]

then \( \omega_Y := \lim_{m \to \infty} \omega_m \) holds and \( \omega_K \) is a closed positive current on \( Y \). Moreover

\[
\omega^n_Y = \lim_{m \to \infty} \omega^n_m
\]

holds on \( Y \setminus V \) by the definition. Hence \( \omega_Y \) satisfies equation:

\[
-\text{Ric}_{\omega_Y} + \sqrt{-1} \Theta h_{L_{X/Y}} = \omega_Y.
\]

This completes the proof of Theorem 2.6.

### 2.4 Regularity of the canonical Kähler current

Here we shall prove Theorem 1.5 by using the recent result on the finite generation of canonical ring ([B-C-H-M]). By Theorem 2.6, we only need to prove the \( C^\infty \)-regularity of \( h_K \) and \( \omega_Y \) on a nonempty Zariski open subset of \( Y \).

The proof here is more or less parallel to the existence of the singular Kähler-Einstein metrics in [Su, T4] and is based on [Y1] and the idea in [T0]. But since the Hodge metric \( h_{L_{X/Y}} \) is not of algebraic singularities, we need to consider the smoothing of the Hodge metric. This is the major difference. We continue to use the notations in Section 2.3 Let us start the proof of Theorem 1.5. By [B-C-H-M], we see that the canonical ring \( R(X, K_X) \) (cf. (1.10)) is finitely generated. Then by the definition of \( L_{X/Y} \), we see that \( R(Y, K_Y + L_{X/Y}) \) is finitely generated also. Hence this implies that the tower (2.28) above can be taken to be finite. Here we shall assume that (2.28) is a finite tower. Moreover taking \( n_0 \)
sufficiently large, we may assume that $\mu_m : Y_m \rightarrow Y_{m-1}$ (cf. (2.27)) is identity for every $m \geq m_0$ and

\begin{equation}
P = \frac{1}{m!}P_m
\end{equation}

is independent of $m$. Moreover we may and do assume that $Y_m = Y$ holds for all $m$. In this case, only $E_m$ varies and we may assume that $E_m$ is of the form

\begin{equation}
E_m = \frac{1}{2^m}E,
\end{equation}

where $E$ is a fixed effective $\mathbb{Q}$-divisor supported on $V$ such that $P - E$ is ample.

Next we shall fix a $C^\infty$ hermitian metric $h_{(m)}$ with strictly positive curvature on $(m!)^{-1}(P_m - E_m)$ as in Section 2.3. Let $h_P$ be a $C^\infty$ hermitian metric defined by the pull back of the Fubini-Study metric on the hyperplane bundle on $\mathbb{P}^N$ via the morphism $\Phi_{|P_m|} : Y \rightarrow \mathbb{P}^N$ and let $h_0$ be a $C^\infty$ hermitian metric on $P - E$ with strictly positive curvature on $Y$. And we shall take $h_{(m)}$ in the previous section as

\begin{equation}
h_{(m)} := h_P^{\frac{1}{m!}} \cdot h_0^{\frac{1}{m!}}.
\end{equation}

Let us fix $s \geq m_0$ as in the Section 2.3.

By (2.52) and (2.65), we see that there exists a positive constant $C'_0$ independent of $m$ and $\delta$ such that

\begin{equation}
\int_Y |u_{m, \delta}| \omega_{(s)}^n < C'_0
\end{equation}

holds. Then since

\begin{equation}
\omega_{(m)} + \sqrt{-1}\partial\bar{\partial}u_{m, \delta}
\end{equation}

is a closed positive current on $Y$ and $\omega_{(m)}$ is a $C^\infty$ Kähler form on $Y$, $u_{m, \delta}$ is an almost plurisubharmonic function on $Y$. By the sub-mean-value inequality for plurisubharmonic functions, we see that by (2.92) there exists a positive constant $C_0$ such that

\begin{equation}
u_{m, \delta} \leq C_0
\end{equation}

holds on $Y$.

By Lemma 2.8, (2.92) and (2.74), we have the following lemma.

**Lemma 2.12** Let $s < t < m$ be as above. Then there exists a positive constant $C_0$ independent of $m$ and $\delta$ such that for every $\delta > 0$,

\begin{equation}
-C(t) + \log \frac{h_{(s)}}{h_{(t)}} \leq v_{m, \delta} \leq C_0 + \log \frac{h_{(s)}}{h_{(m)}},
\end{equation}

hold on $Y \setminus V$, where $C(t)$ is the constant as in (2.55) in Lemma 2.8.

Now we shall estimate the $C^2$-norm of $\{v_{m, \delta}\}_{\delta > 0}$ on every compact subset of $Y \setminus V$. 

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Lemma 2.13 (\textit{[TY] p. 127, Lemma 2.2}) We set

\[ (2.96) \quad f := \log \frac{\omega_{m}^{n}}{\Omega}, \]

Let \( C \) be a positive number such that

\[ (2.97) \quad C + \inf_{\alpha \neq \beta} R_{\alpha\bar{\beta}\beta} > 1 \]

holds on \( Y \), where \( R_{\alpha\bar{\beta}\beta} \) denotes the bisectional curvature of \( \omega_{(s)} \).

Then

\[ (2.98) \quad e^{C v_{m,\delta}} \Delta_{m,\delta}(e^{-C v_{m,\delta}}(n + \Delta_{s} v_{m,\delta})) \geq (n + \Delta_{s} v_{m,\delta}) \]

\[ + \Delta_{s} \left( f + \log \frac{h_{(m)}}{h_{(s)}} \right) - (n + n^2 \inf_{\alpha \neq \beta} R_{\alpha\bar{\beta}\beta}) \]

\[ - C \cdot n(n + \Delta_{s} v_{m,\delta}) + (n + \Delta_{s} v_{m,\delta}) \]

\[ \geq \left( \frac{h_{(m)}}{h_{(s)}} \right)^{\frac{1}{n-1}} \exp \left( \frac{1}{n-1} \right) \cdot \exp \left( - \frac{1}{n-1} (v_{m,\delta} + f) \right) \]

holds, where \( \Delta_{s} \) denotes the Laplacian with respect to \( \omega_{(s)} \) (i.e., \( \Delta_{s} = \text{trace}_{\omega_{(s)}(\sqrt{1-I\bar{\partial}\partial})} \)) and \( \Delta_{m,\delta} \) denotes the Laplacian with respect to \( \omega_{m,\delta} \). \( \Box \)

We note that in Lemma 2.13, \( C > 0 \) does not depend on \( m \) and \( \delta \), but \( C \) depends on \( s \) (more precisely \( \inf_{\alpha \neq \beta} R_{\alpha\bar{\beta}\beta} \)).

Lemma 2.14 For any choice of \( C > 0 \) satisfying (2.97), \( m \) and \( \delta \), there exists a point \( y_{0} \in Y \setminus \bar{V} \) where \( e^{-C v_{m,\delta}(n + \Delta v_{m,\delta})} \) takes its maximum. \( \Box \)

Proof. By Lemma 2.8 above (cf. (2.95))

\[ v_{m,\delta} \geq \log \frac{h_{(s)}}{h_{(t)}} + C_{-}(t) \]

holds. We note that for every \( t > s \), \( h_{(s)}/h_{(t)} \) has pole of positive order along \( V \) by the condition (2.30). Hence by the lower estimate Lemma 2.8 \( e^{-C v_{m,\delta}} \) tends to 0 toward \( V \). More precisely

\[ (2.99) \quad e^{-C v_{m,\delta}} \leq \exp(-C_{-}(t)) \left( \frac{h_{(t)}}{h_{(s)}} \right)^{C} \]

holds on \( Y \setminus \bar{V} \). On the other hand, we have that \( \sqrt{1-I\bar{\partial}\partial v_{m,\delta}} \) is bounded with respect to the Kähler form \( \omega_{(m)} \) on \( Y \) as in \textit{[Y1] p.387, Theorem 6}], hence also with respect to \( \omega_{(s)} \), since both \( \omega_{(m)} \) and \( \omega_{(s)} \) are Kähler forms on \( Y \) by the assumption. Hence for any \( C > 0 \) satisfying (2.97), for every \( m \) and \( \delta \)

\[ (2.100) \quad e^{-C v_{m,\delta}(n + \Delta_{s} v_{m,\delta})} = O(1) \]

holds on \( Y \) and

\[ (2.101) \quad \lim_{y \to V} \left( e^{-C v_{m,\delta}(n + \Delta_{s} v_{m,\delta})} \right) (y) = 0 \]
holds. Hence for any \( C > 0 \) satisfying (2.97), for every \( m \) and \( \delta \) there exists a point \( y_0 \in Y \setminus V \) where \( e^{-Cv_{m,\delta}(y_0)}(n + \Delta v_{m,\delta}) \) takes its maximum. This completes the proof of Lemma 2.14.

Here we have used the fact that the Monge-Ampère equation (2.32) of \( u_{m,\delta} \) has algebraic singularities. Then by Lemma 2.13 we have the following lemma.

**Lemma 2.15** If we take \( C > 0 \) satisfying (2.97), then there exists a positive constant \( C_2 \) independent of \( m \) and \( \delta \) such that

\[
0 \leq e^{-Cv_{m,\delta}(y_0)}(n + \Delta v_{m,\delta})(y_0) \leq C_2
\]

holds.

**Proof.** By the maximal principle, we have

\[
(n + \Delta v_{m,\delta}(y_0)) + \Delta \left( f + \log \frac{h_{(m)}}{h_{(s)}} \right)(y_0) - (n + n^2 \inf_{\alpha \neq \beta} R_{\alpha\beta\bar{\alpha}\bar{\beta}} - C \cdot n)(n + \Delta v_{m,\delta})(y_0)
\]

\[
+ \frac{n}{n} \cdot \left( \frac{h_{(m)}}{h_{(s)}} \right)(y_0) \cdot \exp \left( - \frac{1}{n-1} v_{m,\delta} + f \right)(y_0) \leq 0
\]

holds. Then we see that there exists a positive constant \( C_3 \) independent of \( m \) and \( \delta \) such that

\[
n + \Delta v_{m,\delta}(y_0) \leq C_3 \left( 1 + \left| \Delta \left( f + \log \frac{h_{(m)}}{h_{(s)}} \right)(y_0) \right| \right)^n
\]

holds. Since \( f \) is \( C^\infty \) on \( Y \) and by the definition of \( h_{(m)} \) (cf. (2.91)), there exists a positive constant \( C_4 \) independent of \( m \) such that

\[
\left| \Delta \log \frac{h_{(m)}}{h_{(s)}} \right| < C_4
\]

holds on \( Y \). Hence by (2.104), we see that there exists a positive constant \( C_5 \) independent of \( m,\delta \) and \( y_0 \) such that

\[
n + \Delta v_{m,\delta}(y_0) \leq C_5
\]

holds. Next we shall consider the factor \( e^{-Cv_{m,\delta}(y_0)} \). We note that for every \( t > s \), \( h_{(s)}/h_{(t)} \) has pole of positive order along \( V \) by the condition (2.30). Hence by the lower estimate Lemma 2.8, \( e^{-Cv_{m,\delta}} \) tends to 0 toward \( V \). Then by the \( C^0 \)-estimate Lemma 2.8, we see that there exists a positive constant \( C_6 \) independent of \( m \) and \( \delta \) such that

\[
e^{-Cv_{m,\delta}} \left( 1 + \left| \Delta \left( f + \log \frac{h_{(m)}}{h_{(s)}} \right) \right| \right)^n \leq C_6
\]

holds on \( Y \setminus V \). Combining (2.104) and (2.107), by the definition of \( y_0 \), we complete the proof of Lemma 2.15.

---

5By the definition (2.31), \( \Omega_{m,\delta} \) has algebraic singularities.
Let us take $C > 0$ satisfying (2.97) as in Lemma 2.13. By Lemma 2.15 and the definition of $y_0$

\[(2.108) \quad e^{-C_{v_{m,\delta}}(n + \Delta_s v_{m,\delta})} \leq e^{-C_{v_{m,\delta}}(y_0)(n + \Delta_s v_{m,\delta}(y_0))} \leq C_2\]

hold. Hence by (2.108) we have the inequality:

\[(2.109) \quad 0 \leq n + \Delta_s v_{m,\delta} \leq \exp(C \cdot v_{m,\delta}) \cdot C_2.\]

Estimating $\exp(C \cdot v_{m,\delta})$ from above by Lemma 2.12, (2.109) implies that there exists a positive constant $C_7$ independent of $m$ and $\delta$ such that

\[(2.110) \quad n + \Delta_s v_{m,\delta} \leq C_7 (\frac{h(s)}{h(m)})^C \]

holds on $Y \setminus V$. Applying the general theory of fully nonlinear elliptic equations ([Tr]), to the equation (2.46), we get a uniform higher order estimate of $\{v_{m,\delta}\}_{\delta > 0}$ on every compact subset of $Y \setminus V$. Hence there exists a sequence $\{\delta_j\}$ with $\delta_j \downarrow 0$ as $j$ tends to infinity such that

\[(2.111) \quad \omega_m := \lim_{j \to \infty} \omega_{m,\delta_j}\]

exists in $C^\infty$-topology on every compact subset of $Y \setminus V$. By using the diagonal argument, we may take $\{\delta_j\}$ independent of $m$. Then

\[(2.112) \quad \omega_m := \omega(s) + \sqrt{-1} \partial \bar{\partial} v_m\]

satisfies the equation:

\[(2.113) \quad \log \frac{\omega_m^n}{\Omega_s} = v_m,\]

on $Y \setminus V$, where

\[(2.114) \quad \Omega_s := h^{-1}(s) \cdot h_{L_X/Y}\]

and hence

\[(2.115) \quad - \text{Ric}_{\omega_m} + \sqrt{-1} \Theta h_{L_X/Y} = \omega_m\]

holds.

Let $h_{min}$ be an AZD of $K_Y + L_{X/Y}$ with minimal singularities as in Section 1.4 (cf. Definition 1.3). We set

\[(2.116) \quad \Omega_{min} := h_{min}^{-1} \cdot h_{L_X/Y}.\]

Then we have the following uniform $C^0$-estimate for $\{v_m\}$.

**Lemma 2.16** There exists a positive constant $C$ such that for every $m > s$

\[(2.117) \quad v_m \leq C + \log \frac{\Omega_{min}}{\Omega_s}\]

holds on $Y$. □
Proof of Lemma 2.16  Since
\begin{equation}
- \operatorname{Ric}_{\omega_m} + \sqrt{-1} \Theta_{h_{L_X/Y}} = \omega_m
\end{equation}
holds, we see that
\begin{equation}
\omega_m^n = O(\Omega_{\min})
\end{equation}
holds by Definition 1.3. Hence by the uniform upper bound (2.6) and Lemma 2.7 we see that there exists a positive constant $C$ such that
\begin{equation}
v_m \leq C + \log \frac{\Omega_{\min}}{\Omega_s}
\end{equation}
holds for every $m > s$. □

Let $dV_Y$ be as in (2.83) in Lemma 2.11. We set
\begin{equation}
v := \log \frac{n! \cdot dV_Y}{\Omega_s}.
\end{equation}

Then by Lemmas 2.13 and 2.16
\begin{equation}
\omega_Y = \omega_{(s)} + \sqrt{-1} \partial \bar{\partial} v
\end{equation}
satisfies
\begin{equation}
\omega_Y^n = e^v \cdot \Omega_s.
\end{equation}

By Lemmas 2.13, 2.7 and 2.16, we see that taking a suitable subsequence \{m_k\}, if necessary, we may assume that
\begin{equation}
v = \lim_{k \to \infty} v_{m_k}
\end{equation}
holds in $C^\infty$-topology on every compact subset of $Y \setminus V$. Then by the above construction (cf. (2.115))
\begin{equation}
- \operatorname{Ric}_{\omega_Y} + \sqrt{-1} \Theta_{h_{L_X/Y}} = \omega_Y
\end{equation}
holds on $Y \setminus V$.

Let us define the singular hermitian metric on $K_Y + L_{X/Y}$ by
\begin{equation}
h_K := (dV_Y)^{-1} \cdot h_{L_{X/Y}} = n! \cdot (e^v \cdot \Omega_s)^{-1} \cdot h_{L_{X/Y}}.
\end{equation}

We shall check $h_K$ is an AZD of $K_Y + L_{X/Y}$. First it is clear that $\sqrt{-1} \Theta_{h_K}$ is a closed semipositive current by (2.125) and the $C^0$-estimate: Lemma 2.16. We note that all the coefficients of $E_m$ is less than 1 by the construction. Then by the construction every global holomorphic section of $m! \cdot \pi^*_m(K_Y + L_{X/Y})$ is $L^2$ integrable on $Y$ with respect to $h_{L_{X/Y}}^{m!} \cdot (\omega_m^n)^{-m!-1}$ by the Monge-Ampère equation of $u_m$ and the almost boundedness of $u_m$. Then by the monotonicity of $\{\omega_m^n\}$ (Lemma 2.7), we see that $h_K = (dV_Y)^{-1} h_{L_{X/Y}}$ is an AZD of $K_Y + L_{X/Y}$. This completes the proof of Theorem 1.5 except the uniqueness of the canonical semipositive current. The uniqueness is the direct consequence of Theorem 1.7.

□
2.5 Generalization to general adjoint line bundles

In the proof of Theorem 1.5, we have not used the property of the Hodge bundle $L_{X/Y}$ (cf. 1.20) except the Poincaré growth property of the curvature of the Hodge metric $h_{X/Y}$ (cf. (1.21)). Hence without changing the proof, we have the following variant of Theorem 1.5.

**Theorem 2.17** Let $Y$ be a smooth projective $n$-fold and let $(L, h_L)$ be a $\mathbb{Q}$-line bundle with $C^\infty$ metric $h_L$ with semipositive curvature. Suppose that $K_Y + L$ is big. Let $U$ be the Zariski open subset of $U$ defined by

$$(2.127) \quad U := \{ y \in Y \mid m!(K_Y + L) \text{ is very ample around } y \text{ for } m >> 1 \}.$$

Then there exist a closed positive current $\omega_Y$ on $Y$ such that

1. There exists a sequence of closed positive currents $\{\omega_m\}$ such that $\omega_m|U$ is $C^\infty$ and satisfies the equation

$$(2.128) \quad - \text{Ric}_{\omega_m} + \sqrt{-1} \Theta_{h_L} = \omega_m$$

holds on $U$ and

$$(2.129) \quad \omega_Y = \lim_{m \to \infty} \omega_m$$

in the sense of currents.

2. 

$$(2.130) \quad h_{\text{can}} := \left( \frac{1}{n!} \omega_{Y,abc}^n \right)^{-1} \cdot h_L$$

is an AZD of $K_Y + L$.

Moreover if the log canonical ring

$$(2.131) \quad R(Y, a(K_Y + L)) = \bigoplus_{m=0}^{\infty} \Gamma(Y, \mathcal{O}_Y(ma(K_Y + L)))$$

is finitely generated, then $\omega_Y$ is $C^\infty$ on the Zariski open subset $U$, where $a$ is the minimal positive integer such that $aL$ is a genuine line bundle. \qed

3 Proof of Theorem 1.7

Let $f : X \to Y$ be the Iitaka fibration and let $(L_{X/Y}, h_{L_{X/Y}})$ be the singular hermitian $\mathbb{Q}$-line bundle on $Y$ as in Theorem 1.5. Let $\omega_Y$ be the canonical Kähler current on $Y$ (cf. Definition 1.9). Then there exists a nonempty Zariski open subset $U$ of $Y$ such that $\omega_Y$ is a $C^\infty$ on $U$ and

$$(3.1) \quad - \text{Ric}_{\omega_Y} + \sqrt{-1} \Theta_{h_{L_{X/Y}}} = \omega_Y$$

constructed in Theorem 1.5.

Let $m_0$ be the sufficiently large positive integer, $M$ be a effective Cartier divisor such that $A := m_0!(K_Y + L_{X/Y}) - M$ is sufficiently ample as in Section
Let \( h_A \) be the \( C^\infty \)-hermitian metric on \( A \). Hereafter we shall consider \( h_A \) as a singular hermitian metric on \( m_0!(K_Y+L_{X/Y}) \) by identifying \( h_A \) with

\[
(3.2) \quad h_A/|\tau_M|^2,
\]

where \( \tau_M \) is a global holomorphic section of \( \mathcal{O}_Y(M) \) with divisor \( M \). Let \( \{K_m\}_{m \geq m_0} \) be the dynamical system of Bergman kernels as in Section 1.8. Let \( \{h_m\}_{m \geq m_0} \) be the corresponding dynamical system of singular hermitian metrics defined by

\[
(3.3) \quad h_m := K_m^{-1}
\]
as in Section 1.8.

Now we shall prove Theorem 1.7. Let \( dV_Y = \frac{1}{n!}\omega^n_{Y,abc} \) be the volume form associated with \( (Y, \omega_Y) \). This \( dV_Y \) is the same as the volume form defined as (2.83) by the proof of Theorem 1.5.

Lemma 3.1

\[
(3.4) \quad \limsup_{m \to \infty} h_{L_{X/Y}} \cdot \sqrt{(m!)^{-n}K_m} \geq (2\pi)^{-n}dV_Y
\]
holds on \( X \).

Proof of Lemma 3.1. First we shall assume that \( L_{X/Y} \) is a genuine line bundle on \( Y \) for simplicity. If \( L_{X/Y} \) is not a genuine line bundle, we tensorize \((aL_{X/Y}, h_{aL_{X/Y}})\) in every \( a \)-steps instead of tensorize \((L_{X/Y}, h_{L_{X/Y}})\) every step, where \( a \) is the least positive integer such that \( aL_{X/Y} \) is Cartier. But of course this is a minor technical difference. Hence we shall give a proof assuming that \( L_{X/Y} \) is Cartier. The general case is left to readers to avoid inessential complication.

Let us consider the (singular) hermitian line bundle \((K_Y+L_{X/Y}, dV_Y^{-1}, h_{L_{X/Y}})\) on \( Y \). Let \( U \) be a nonempty Zariski open subset of \( Y \) such that \( \omega_Y|U \) is a \( C^\infty \) Kähler form. Let \( p \in U \) be a point. Then by the equation (1.22), there exists a holomorphic normal coordinate \((U, z_1, \cdots, z_n)\) of \((Y, \omega_Y)\) around \( p \) and a local holomorphic frame \( e_{L_{X/Y}} \) of \( L_{X/Y} \) on \( U \) such that

\[
(3.5) \quad dV_Y^{-1} \cdot h_{L_{X/Y}} = \prod_{i=1}^n (1 - |z_i|^2) + O(||z||^3) \cdot 2^n \cdot |dz_1 \wedge \cdots \wedge dz_n|^{-2} \cdot |e_{L_{X/Y}}|^{-2}
\]
and \( h_{L_{X/Y}}(e_{L_{X/Y}}, e_{L_{X/Y}})(p) = 1 \). Suppose that

\[
(3.6) \quad C_{m-1} \cdot h_A^{-1} \cdot dV_Y^{m-\text{mul}^{-1}} \cdot h_{L_{X/Y}}^{(m-\text{mul}^{-1})} \leq K_{m-1}
\]
holds on \( Y \) for some positive constant \( C_{m-1} \). We note that

\[
(3.7) \quad K_m(y) = \sup\{ ||\sigma||^2(y); \sigma \in H^0(Y, \mathcal{O}_Y(m(K_Y+L_{X/Y}))), (\sqrt{-1})^n \int_X h_{m-1} \cdot \sigma \wedge \bar{\sigma} = 1 \}
\]

\[\text{Please do not confuse } h_m \text{ in Section 2}\]
holds for every $y \in Y$, by the extremal property of the Bergman kernel. We note that for the open unit disk $\Delta = \{ t \in \mathbb{C} \mid |t| < 1 \}$, 

\begin{equation}
\sqrt{-1} \int_{\Delta} (1 - |t|^2)^m dt \wedge \overline{d}t = \frac{2\pi}{m+1}
\end{equation}

holds. Then by Hörmander’s $L^2$-estimate of $\overline{\partial}$-operators, we see that there exists a positive constant $\lambda_m$ such that 

\begin{equation}
(\lambda_m \cdot (2\pi)^{-n} \cdot m^n \cdot C_{m-1} \cdot dV_Y^{m-m_0!} \leq h_{L_X/Y}^{m-m_0!} \cdot h_A \cdot K_m
\end{equation}

with

\begin{equation}
\lambda_m \geq 1 - \frac{C}{\sqrt{m}},
\end{equation}

where $C$ is a positive constant independent of $m$.

In fact this can be verified as follows. Let $y \in Y\setminus\text{Supp}(M \cup V)$ and let $(U, z_1, \ldots, z_n)$ be the normal coordinate as above. We may assume that $U$ is biholomorphic to the polydisk $\Delta^n(r)$ of radius $r$ with center $0$ in $\mathbb{C}^n$ for some $0 < r < 1$ via $(z_1, \ldots, z_n)$.

Taking $r < 1$ sufficiently small we may assume that there exists a $C^\infty$-function $\rho$ on $Y$ such that

1. $\rho$ is identically $1$ on $\Delta^n(r/3)$.
2. $0 \leq \rho \leq 1$.
3. $\text{Supp } \rho \subset U$.
4. $|d\rho| < 3/r$, where $\mid \mid$ denotes the pointwise norm with respect to $\omega_Y$.

We note that by the equation (3.5), the mass of $\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^m \otimes e_{L_X/Y}^m$ concentrates around the origin as $m$ tends to infinity. Hence by (3.8) we see that the $L^2$-norm

\begin{equation}
\| \rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^m \otimes e_{L_X/Y}^m \|
\end{equation}

of $\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^m \otimes e_{L_X/Y}^m$ with respect to $(dV_Y)^{-m} \cdot h_{L_X/Y}^m$ and $\omega_Y$ is asymptotically

\begin{equation}
\| \rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^m \otimes e_{L_X/Y}^m \|^2 \sim 2^{mn} \left( \frac{2\pi}{m} \right)^n
\end{equation}

as $m$ tends to infinity, where $\sim$ means that the ratio of the both sides converges to $1$ as $m$ tends to infinity. We set

\begin{equation}
\phi := n\rho \log \sum_{i=1}^n |z_i|^2.
\end{equation}

We may and do assume that $m$ is sufficiently large so that

\begin{equation}
(m - m_0! - 1) \cdot \omega_Y + \sqrt{-1} \Theta_{h_A} + \sqrt{-1} \Theta_{h_{L_X/Y}} + \sqrt{-1} \partial \overline{\partial} \phi > 0
\end{equation}

\footnote{This is well known. See for example, [Kō, p.46, Proposition 1.3.16].}
holds on $Y$. We note that $\bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) = e_L^{m_{X/Y}})$ vanishes on the polydisc of radius $r/3$ with center $p$ as above. Then by (3.12), the $L^2$-norm
\[ \| \bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m}) \|_{\phi} \]
of $\bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m})$ with respect to $e^{-\phi} \cdot h_A \cdot (dV_Y)^{-(m-m_0l-1)} \otimes h_{L_{X/Y}}^{m-m_0l}$ and $\omega_Y$ satisfies the inequality
\[ (3.15) \]
\[ \| \bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m}) \|_{\phi}^2 \leq C_0 \cdot \left( \frac{3}{r} \right)^{2n+2} \left( 1 - \frac{r}{4} \right)^m \frac{2^{2m}}{m^n} \]
for every $m$, where $C_0$ is a positive constant independent of $m$. By Hörmander's $L^2$-estimate applied to the adjoint line bundle of the hermitian line bundle:

\[ (3.16) \]
\[ ((m-1)(K_Y + L_{X/Y}) + L_{X/Y}, e^{-\phi} \cdot h_A \cdot dV_Y^{-(m-m_0l-1)} \otimes h_{L_{X/Y}}^{m-m_0l}), \]
we see that for every sufficiently large $m$, there exists a $C^\infty$ solution $u$ of the equation:

\[ (3.17) \]
\[ \bar{\partial}u = \bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m}) \]
such that

\[ (3.18) \]
\[ u(p) = 0 \]
and

\[ (3.19) \]
\[ \| u \|_{\phi}^2 \leq \frac{2}{m} \| \bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m}) \|_{\phi}^2 \]
hold, where $\| \|_{\phi}$'s denote the $L^2$ norms with respect to $e^{-\phi} \cdot h_A \cdot dV_Y^{-(m-m_0l-1)} \otimes h_{L_{X/Y}}^{m-m_0l}$ and $\omega_Y$ respectively. Then $\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m}$ is a holomorphic section of $(K_Y + L_{X/Y})$ such that

\[ (3.20) \]
\[ (\rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m})(p) = (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m}(p) \]
and

\[ (3.21) \]
\[ \| \rho \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes e_{L_{X/Y}}^{m} - u \| \leq 1 + C_0 \cdot \left( \frac{3}{r} \right)^{2n+2} \cdot \frac{2}{m} \left( 1 - \frac{r}{4} \right)^m \cdot 2^{mn} \cdot \left( \frac{2\pi}{m} \right)^n. \]

Hence by the assumption of the induction and the extremal property of Bergman kernels, this implies that there exists a positive constant $C$ independent of $m$ such that

\[ (3.22) \]
\[ K_m(p) \geq \left( 1 - \frac{C}{\sqrt{m}} \right) \cdot m^n \cdot (2\pi)^{-n} \cdot C \cdot (m_{m-1} \cdot (h_{A}^{-1} \cdot h_{L_{X/Y}}^{-(m-m_0l)} \cdot dV_Y^{-(m-m_0l)}) \]) \]
holds, since the point norm of $(dz_1 \wedge \cdots \wedge dz_n) \otimes e_{A}$ at $p$ (with respect to $h_A \cdot dV_Y^{-(m-m_0l)} \cdot h_{L_{X/Y}}^{(m-m_0l)})$ is asymptotically equal to $2^{mn}$. Then by induction
on \( m \), using (3.7) and (3.9), we see that there exist a positive constant \( C' \) and a positive integer \( m_1 > m_0 \) such that \( C/\sqrt{m_1} < 1 \) and for every \( m > m_1 \)

\[
(3.23) \quad K_m \geq C' \left( \prod_{k=m_1}^{m} \left( 1 - \frac{C}{\sqrt{k}} \right) \right) \cdot (m!)^{n} \cdot (2\pi)^{-m} \cdot h_A^{-1} \cdot h_{L_{X/Y}}^{-m} \cdot dV_{m-m_0!}
\]

holds at \( p \). Moving \( p \), this implies that

\[
(3.24) \quad \limsup_{m \to \infty} h_{L_{X/Y}} \cdot \sqrt{(m!)^{-n} K_m} \geq (2\pi)^{-n} dV_Y
\]

holds on \( Y \). \( \square \)

**Lemma 3.2**

\[
(3.25) \quad \int_Y h_{L_{X/Y}} \cdot \frac{K^\frac{1}{m}}{K_{m-1}^\frac{1}{m}} \leq \left( \prod_{k=m_0}^{m} (N_k + 1) \right)^{\frac{1}{m}} \cdot \left( \int_Y h_{L_{X/Y}} \cdot \sqrt{K_{m_0}} \cdot h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}} \right)^{\frac{m_0}{m}}
\]

holds, where \( N_k := \dim | k(K_Y + L_{X/Y}) | = \dim H^0(Y, O_Y(k(K_Y + L_{X/Y}))) - 1 \).

\( \square \)

**Proof.** First we note that the trivial equality:

\[
(3.26) \quad \int_Y h_{L_{X/Y}} \cdot \frac{K_m}{K_{m-1}} = N_m + 1
\]

holds by the definition of \( K_m \) and the equality \( h_{m-1} = 1/K_{m-1} \). Then by Hölder’s inequality, we have

\[
\int_Y h_{L_{X/Y}} \cdot K_{m}^\frac{1}{m} = \int_Y h_{L_{X/Y}} \cdot \frac{K_m}{h_{L_{X/Y}} \cdot K_{m-1}^\frac{1}{m}} \cdot h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}}
\]

\[
\leq \left( \int_Y h_{L_{X/Y}}^\frac{m_0}{m} \cdot \frac{K_m}{h_{L_{X/Y}} \cdot K_{m-1}^\frac{1}{m}} \cdot (h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}}) \right)^{\frac{1}{m}} \cdot \left( \int_Y h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}} \right)^{\frac{m_0}{m}}
\]

\[
= \left( \int_Y h_{L_{X/Y}} \cdot \frac{K_m}{K_{m-1}} \right)^{\frac{1}{m}} \cdot \left( \int_Y h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}} \right)^{\frac{m_0}{m}}
\]

\[
= (N_m + 1)^{\frac{m_0}{m}} \cdot \left( \int_Y h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}} \right)^{\frac{m_0}{m}}.
\]

Hence we obtain the inequality:

\[
(3.27) \quad \int_Y h_{L_{X/Y}} \cdot K_{m}^\frac{1}{m} \leq (N_m + 1)^{\frac{m_0}{m}} \cdot \left( \int_Y h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}} \right)^{\frac{m_0}{m}}.
\]

Continuing this process, by using

\[
(3.28) \quad \int_Y h_{L_{X/Y}} \cdot K_{m-1}^{-\frac{1}{m}} \leq (N_{m-1} + 1)^{\frac{m_0}{m}} \cdot \left( \int_Y h_{L_{X/Y}} \cdot K_{m-2}^{-\frac{1}{m}} \right)^{\frac{m_0}{m}}
\]

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we have that
\begin{equation}
(3.29)
\int_Y h_{L_X/Y} \cdot (K_m)^{\frac{1}{m}} \leq \{(N_m + 1) \cdot (N_{m-1} + 1)\} \cdot \left( \int_Y h_{L_X/Y} \cdot (K_{m-2})^{\frac{1}{m-2}} \right)^{\frac{m-2}{m}}
\end{equation}
holds. Continuing this process we obtain the lemma. \qed

To estimate the growth of \( \{N_m\}_{m \geq m_0!} \), we introduce the following notion.

**Definition 3.3** Let \( L \) be a line bundle on a compact complex manifold \( M \) of dimension \( n \). We define the volume \( \mu(M, L) \) of \( M \) with respect to \( L \) by
\begin{equation}
(3.30)
\mu(M, L) := n! \cdot \limsup_{m \to \infty} \frac{m^\frac{n}{n} \cdot \dim H^n(M, \mathcal{O}_M(mL))}{m^n + o(m^n)}
\end{equation}
We note that Definition 3.3 can be generalized to the case of \( \mathbb{Q} \)-line bundles in an obvious way. Using Lemma 3.2, we obtain the following lemma.

**Lemma 3.4**
\begin{equation}
(3.31)
\limsup_{m \to \infty} \frac{1}{(m!)^\frac{n}{m}} \int_Y h_{L_X/Y} \cdot (K_m)^{\frac{1}{m}} \leq \frac{\mu(Y, K_Y + L_{X/Y})}{n!}
\end{equation}
holds. \qed

**Proof**. By the definition of the volume \( \mu(Y, K_Y + L_{X/Y}) \),
\begin{equation}
(3.32)
N_m + 1 = \mu(Y, K_Y + L_{X/Y}) \cdot \frac{m^n}{n!} + o(m^n)
\end{equation}
holds. Then by Lemma 3.2, we see that
\begin{equation}
(3.33)
\limsup_{m \to \infty} \frac{1}{(m!)^\frac{n}{m}} \int_Y h_{L_X/Y} \cdot (K_m)^{\frac{1}{m}} \leq \frac{\mu(Y, K_Y + L_{X/Y})}{n!}
\end{equation}
holds. \qed

**Lemma 3.5**
\begin{equation}
(3.34)
\frac{1}{(2\pi)^n} \int_Y dV_Y = \frac{n!}{n!} \int_Y \left( \frac{1}{2\pi} \omega_{Y,abc} \right)^n = \frac{n!}{n!} \mu(Y, K_Y + L_{X/Y})
\end{equation}
holds. \qed

**Proof of Lemma 3.5** Let \( |P_m| \) be the free part of \( |\pi_m^* m!(K_Y + L_{X/Y})| \), where \( \pi_m \) is the resolution of \( Bs|m!(K_Y + L_{X/Y})| \) as in the last section (cf. (2.26)). By Fujita’s theorem ([F, p.1, Theorem]), we see that
\begin{equation}
(3.35)
\lim_{m \to \infty} (m!)^{-n} P_m^n = \mu(Y, K_Y + L_{X/Y})
\end{equation}
Then by (2.66), (2.82), Lemma 2.7 and (3.35), Lebesgue’s bounded convergence theorem implies that
\begin{equation}
(3.36)
\mu(Y, K_Y + L_{X/Y}) = \lim_{m \to \infty} \int_Y \left( \frac{1}{2\pi} \omega_{m,abc} \right)^n = \int_Y \left( \frac{1}{2\pi} \omega_{Y,abc} \right)^n
\end{equation}
hold. This implies the lemma.

We note that by Lemma 3.4 and the submeanvalue inequality for plurisubharmonic functions, \( \{ h_{L_{X/Y}} \cdot \sqrt{(m!)^{-n}K_m} \} \) is a family of uniformly bounded semi-positive \((n, n)\) forms on \( Y \). Then by Lebesgue’s bounded convergence theorem, we see that

\[
\limsup_{m \to \infty} \int_Y h_{L_{X/Y}} \cdot \sqrt{(m!)^{-n}K_m} = \int_Y \limsup_{m \to \infty} h_{L_{X/Y}} \cdot \sqrt{(m!)^{-n}K_m}
\]

holds. Combining Lemmas 3.1, 3.4 and 3.5, we have the equality:

\[
\limsup_{m \to \infty} \frac{1}{(m!)^n} h_{L_{X/Y}} \cdot K_m^n = (2\pi)^{-n} dV_Y,
\]

holds almost everywhere on \( Y \). Hence by the definition of \( h_{\infty} \) (cf. (1.33))

\[
h_{\infty} = \left( \limsup_{m \to \infty} \frac{1}{(m!)^n} K_m^n \right)^{-1} = (2\pi)^n \cdot dV_Y^{-1} \cdot h_{L_{X/Y}}
\]

hold almost everywhere on \( Y \). This implies the equality (1.34) in Theorem 1.7. Then by the equation (1.22) we have the equality (1.35) in Theorem 1.7:

\[
\omega_Y = \sqrt{-1} \Theta h_{\infty}.
\]

This completes the proof of Theorem 1.7 assuming that \( L_{X/Y} \) is Cartier. The proof of the general case can be obtained by entirely the same estimates. More precisely if \( L_{X/Y} \) is not a genuine line bundle, we may have small ripple in the \( L^2 \)-estimates in the proof of Lemma 3.1 since we tensorize \((L_{X/Y}^{\otimes a}, L_{X/Y}^{\otimes a})\) every \( a \) steps. But the ripple disappears when we take the normalized limit as is easily be seen.

For the uniqueness of the canonical measure, we have the following uniqueness.

**Corollary 3.6** \( d\mu_{can} \) is birationally invariant.

**Proof of Corollary 3.6** Let \( f : X \to Y \) be as above and let us consider the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\tilde{f} \downarrow & & \downarrow f \\
\tilde{Y} & \xrightarrow{\varpi} & Y
\end{array}
\]

where \( \pi : \tilde{X} \to X, \varpi : \tilde{Y} \to Y \) are modifications. Let \( \tilde{A} \) and \( A \) be ample line bundles on \( \tilde{Y} \) and \( Y \) respectively.

Then \( \varpi^* A \) is nef and big on \( \tilde{Y} \). Hence by Kodaira’s lemma, we have that there exists a positive integers \( a_1, a_2 \) such that

\[
a_1 \tilde{A} - \varpi^* A, a_2 \varpi^* A - \tilde{A}
\]

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are \( \mathbb{Q} \)-effective. Let \( d\mu_{\tilde{X},can}, d\mu_{X,can} \) be canonical measures on \( \tilde{X} \) and \( X \) respectively. Then since \( a_1 A - \varpi^* A \) is \( \mathbb{Q} \)-effective, by Theorem 1.7 and its proof, we see that

\[
d\mu_{\tilde{X},can} \geq \pi^* d\mu_{X,can}
\]

holds. In fact this can be verified as follows. Let us fix \( C^\infty \) hermitian metrics \( h_A \) and \( h_{\tilde{A}} \) on \( A \) and \( \tilde{A} \) respectively. Let \( b \) be a sufficiently large positive integer and let

\[
\tau \in H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(b(a_1 \tilde{A} - \varpi^* A)))
\]

be a nonzero section such that

\[
(h^{a_1}_{\tilde{A}} \cdot \varpi^* h^{-1}_{A})^b (\tau, \tau) \leq 1.
\]

Let \( \{ K_m \}_{m \geq 0}, \{ K_m \}_{m \geq 0} \) be the dynamical systems of Bergman kernels as in Theorem 1.7 starting from \( (ba_1 \tilde{A}, h^{ba_1}_{\tilde{A}}) \) and \( (bA, h^b_A) \) on \( \tilde{Y} \) and \( Y \) respectively. Then by the extremal property of Bergman kernels, we see that

\[
\tilde{K}_m \geq |\tau|^2 \cdot \varpi^* K_m
\]

holds. Hence by Theorem 1.7 we see that

\[
d\mu_{\tilde{X},can} \geq \pi^* d\mu_{X,can}
\]

holds.

Similarly since \( a_2 \varpi^* A - \tilde{A} \) is \( \mathbb{Q} \)-effective, we have the opposite inequality:

\[
d\mu_{\tilde{X},can} \leq \pi^* d\mu_{X,can}
\]

Hence we have that the equality

\[
d\mu_{\tilde{X},can} = \pi^* d\mu_{X,can}
\]

holds. This completes the proof of Corollary 3.6.

4 Relative version of Theorems 1.5 and 1.7

In this section we shall consider variation of the canonical measures on projective families. Our result is as follows.

Theorem 4.1 Let \( f : X \to S \) be a projective family such that \( X, S \) are smooth and \( f \) has connected fibers. Suppose that \( f_* \mathcal{O}_S(mK_{X/S}) \neq 0 \) for some \( m > 0 \). There exists a relative measure \( d\mu_{can,X/S} \) such that the singular hermitian metric \( h_{X/S} := d\mu_{can,X/S}^{-1} \) on \( K_{X/S} \) satisfies:

1. \( \omega_{X/S} := \sqrt{-1} \Theta_{h_{X/S}} \) is semipositive on \( X \).
2. For every smooth fiber \( X_s := f^{-1}(s), h_{X/S}|X_s \) is well defined and is an AZD of \( K_{X_s} \).
(3) There exists a set $T$ of measure 0 on $S$ such that for every $s \in S \setminus T$, $X_s$ is smooth and $\omega_{X/S}|_{X_s}$ is the canonical semipositive current on $X_s$ constructed as in Theorems 1.5 and 1.7.

**Remark 4.2** Even for $s \in S \setminus T$, $d\mu_{\text{can},X/S}|_{X_s}$ may not be precisely equal to the canonical measure $d\mu_{\text{can},s}$ on $X_s$ as a degenerate volume form on $X_s$. But as a measure $d\mu_{\text{can},X/S}|_{X_s} = d\mu_{\text{can},s}$ holds in exact sense.

We call $d\mu_{\text{can},X/S}|_{X_s}$ the relative canonical measure of $f : X \to S$. Combining the logarithmic plurisubharmonicity of Bergman kernels ([B3, B-P] and [T4, Theorem 3.4]), this theorem strengthens the following famous result due to Y. Kawamata.

**Theorem 4.3** ([Ka2, p.57, Theorem 1]) Let $f : X \to S$ be an algebraic fiber space. Suppose that $\dim S = 1$. Then for every positive integer $m$, $f_*\mathcal{O}_X(mK_{X/S})$ is a semipositive vector bundle on $Y$, in the sense that every quotient $\mathcal{Q}$ of $f_*\mathcal{O}_X(mK_{X/S})$, $\deg \mathcal{Q} \geq 0$ holds.

The main difference between Theorems 4.1 and 4.3 is that the semipositivity is on the total space in Theorem 4.1, while the semipositivity is on the direct image of the relative pluricanonical systems in Theorem 4.3. In [K6], we consider the relative log canonical bundle of a family of log canonical pairs. In the case of log canonical pairs, this difference becomes an essential one.

**Proof of Theorems 4.1** Since the assertion is local, we may assume that $S$ is the unit open polydisk in $\mathbb{C}^n$. Let $m_0$ be a sufficiently large positive integer and let

$$F_{m_0} := f_*\mathcal{O}_X(m_0! K_{X/S})$$

and (shrinking $S$, if necessary) let $\sigma_0, \ldots, \sigma_{N(m)}$ be a set of global generators of $F_{m_0}$ on $S$. We take the image $Y$ of the rational map

$$\Phi_{m_0} : X \to \mathbb{P}^{N(m)}_S.$$ 

If we take $m_0$ sufficiently large, taking modifications of $\hat{X}$ of $X$ and $\hat{Y}$ of $Y$ respectively, we have the relative Iitaka fibration

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{g} & \hat{Y} \\
\hat{f} & \downarrow & \cr & & \downarrow h \\
S & & \\
\end{array}$$

such that $\hat{X}$ and $\hat{Y}$ are smooth and $g_*\mathcal{O}_{\hat{X}}(m! K_{\hat{X}/\hat{Y}})^{**}$ is a line bundle on $\hat{Y}$. We define the $\mathbb{Q}$-line bundle $L_{X/Y}$ on $\hat{Y}$ by

$$L_{X/Y} = \frac{1}{m_0!} g_*\mathcal{O}_{\hat{X}}(m_0! K_{\hat{X}/\hat{Y}})^{**}$$

and let $a$ be the least positive integer such that $\hat{f}_*\mathcal{O}_{\hat{X}}(aK_{\hat{X}/\hat{Y}}) \neq 0$. Hereafter we shall replace $f : X \to S$ by $\hat{f} : \hat{X} \to S$ and replace $X$ and $Y$ by $\hat{X}$ and $\hat{Y}$.
\(Y\) respectively. This does not affect the proof of Theorem 4.1 by the birational invariance of canonical measures.

Let \(S^0\) be the locus of \(S\) such that \(f\) is smooth over \(S^0\). Let \(A\) be a sufficiently ample line bundle on \(Y\) and let \(h_A\) be a \(C^\infty\) metric on \(A\) with strictly positive curvature. Then as in Section 3 for every \(s \in S^0\), we define the dynamical system of Bergman kernels \(\{K_{m,s}\}_{m \geq m_0}\) as in Section 3, i.e.,

\[
K_{1,s} := \begin{cases} 
K(Y_s, K_{Y_s} + A|Y_s, h_A|Y_s), & \text{if } a > 1 \\
K(Y_s, K_{Y_s} + L_{X/Y}|Y_s + (K_{Y_s} + L_{X/Y}|Y_s), h_{L_{X/Y}} \cdot h_A|Y_s), & \text{if } a = 1 
\end{cases}
\]

and \(h_{1,s} = K_{1,s}^{-1}\). And we define \(\{K_{m,s}\}\) inductively as in Section 3 fiberwise. More precisely if we have already defined \(K_{m,s}\) and \(h_{m,s}\), we shall define \(K_{m+1,s}\) and \(h_{m+1,s}\) by

\[
K_{m+1,s} := \begin{cases} 
K(Y_s, (m+1)K_{Y_s} + \lfloor \frac{m+1}{a} \rfloor aL_{X/Y}|Y_s, h_{m,s}), & \text{if } m+1 \equiv 0 \mod a \\
K(Y_s, (m+1)(K_{Y_s} + L_{X/Y}|Y_s), (h_{L_{X/Y}}|Y_s)^a \otimes h_{m,s}), & \text{if } m+1 \equiv 0 \mod a 
\end{cases}
\]

and

\[
h_{m+1,s} := (K_{m+1,s})^{-1}.
\]

Now we shall define \(K_m\) by

\[
K_m|Y_s = K_{m,s} \quad (s \in S^0)
\]

and set

\[
K^*_m := \text{the upper-semi-continuous envelope of } K_m.
\]

We note that the Hodge metric \(h_{L_{X/Y}}\) on \(L_{X/Y}\) defined as in Section 3 has semipositive curvature in the sense of current on \(Y\) (not on every fiber \(Y_s\)) by (\text{[Ka2, Ka3, p.174, Theorem 1.1]}). Then by the plurisubharmonicity of the Bergman kernel (\text{[B3, B-P, T4, Theorem 3.4]}) of the adjoint line bundle of singular hermitian line bundle of semipositive curvature current, by induction on \(m\), we see that

\[
h_m := (K^*_m)^{-1}
\]

extends to a singular hermitian metric on

\[
mK_{Y/S} + \left(\left\lfloor \frac{m}{a} \right\rfloor aL_{X/Y}\right)\cdot L_{X/Y}
\]

on \(Y\) and the extended metric has semipositive curvature in the sense of current, i.e. \(\log K^*_m\) is plurisubharmonic on \(Y\) by Theorem 1.8. Then by Theorem 1.7

\[
K_\infty := \text{the upper semicontinuous envelope of } \limsup_{m \to \infty} \sqrt[m]{(m!)^{-a}K^*_m}
\]

exists as a nontrivial \(L_{X/Y}\)-valued relative volume form on \(Y\) and

\[
h_\infty := K_\infty^{-1}
\]
is a singular hermitian metric on $K_{Y/S} + L_{X/Y}$ with semipositive curvature current. We set

\[(4.14) \quad h_{X/S} := g^* h_{\infty}.\]

Then as before we may consider $h_{X/S}$ as a singular hermitian metric on $K_{X/S}$ with semipositive curvature current, i.e.,

\[(4.15) \quad \omega_{X/S} := \sqrt{-1} \Theta h_{X/S}\]

is semipositive on $X$. By Theorem 1.7 and the birational invariance of the canonical semipositive current (Corollary 3.6), there exists a subset $T$ of measure 0 on $S$ such that $S \setminus T$ is contained in $S^\circ$ and for every $s \in S \setminus T$, $\omega_{X/S}|X_s$ is the canonical semipositive current on $X_s$. Moreover for $s \in T \cap S^\circ$, we see that $h_{X/S}|X_s$ is an AZD of $K_{X_s}$ by the very definition of the upper-semi-continuous envelope. This completes the proof of Theorem 4.1.

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