We compare the low-lying eigenmodes of the $O(a)$ improved Wilson-Dirac operator on quenched and dynamical configurations and investigate methods of probing the topological properties of gauge configurations.

1. INTRODUCTION

The eigenmodes of the Dirac operator carry information about the topological content of the background gauge field. The Atiyah-Singer index theorem tells us that in a sector with topological charge $Q_t$ the Dirac operator has at least $|Q_t|$ zero modes. In the presence of light dynamical fermions, the light quark determinant should suppress topological sectors with large $|Q_t|$. As a consequence we expect a reduced value for the topological susceptibility

$$\chi_t = \frac{\langle Q_t^2 \rangle}{V}.$$  

The topological charge is still difficult to determine on the lattice. Various methods have been explored, usually gluonic methods. The latter, however, typically require smoothing procedures to reduce the ultraviolet fluctuations which do not take the fermionic part of the action, i.e. the presence of dynamical fermions, into account.

Fermionic methods, on the other hand, used to probe topological properties are expected to be sensitive to the effects caused by explicitly breaking chiral symmetry when using Wilson fermions. These effects are potentially large on coarse lattices, even when one reduces these effects using $O(a)$ Symanzik improved fermions.

![Figure 1. Eigenvalues of the massless improved Wilson-Dirac operator calculated on $O(150)$ quenched configurations (left) and dynamical (right) configurations.](image)

2. THE NON-HERMITIAN CASE

The $O(a)$ improved Wilson-Dirac operator

$$M(\kappa) = 1 - \kappa \left[ H + \frac{i}{2} c_{SW} F_{\mu\nu}^a \sigma_{\mu\nu} \right],$$  

is a non-hermitian operator. To calculate its eigenvalues $\lambda_i$ we used the Arnoldi algorithm \cite{Arnoldi:1951}. To speed up convergence and to maximize the
number of real modes found by the algorithm, we transformed $M$ using a “least-squares” polynomial of degree 40. This polynomial was tuned to make the eigenvalues with small real part lying in a band along the real axis fastest to converge. For our purpose it is crucial to increase the number of calculated small real modes. We calculated $O(100)$ eigenvalues per configuration.

The index theorem establishes a relation between the difference in the number of real modes with negative ($n^+$) and positive ($n^-$) chirality $\omega_i = (v_i, 5\gamma v_i)$ and the topological charge:

$$Q_t = n^+ - n^- .$$  \hfill (3)

However, on the lattice this relation can only hold approximately (if at all) in case of Wilson fermions. We can assume this approximation only to be reasonable if the spectrum of the Wilson-Dirac operator allows us to distinguish between physical modes and unphysical modes due to the doublers. To check for this precondition we considered the density of real modes. If our massless Dirac operator would be continuum-like we would expect the density of real modes being sharply peaked around zero and becoming small for larger modes. Comparing the densities at two different values of the gauge field coupling $\beta$ in the quenched case we indeed see a more continuum-like behaviour for larger $\beta$. In case of the dynamical configurations we expect small eigenvalues to be suppressed and therefore the density of real modes around zero to be smaller. This is indeed what we find, see Figs. 2, 3 and 4. We now consider $\chi_t$ using $Q_t$ from Eq. (3) taking all real modes $\lambda_i < \lambda$ into account. For larger values of the cut-off $\lambda$ one would expect $\chi_t$ to become constant. In the quenched case $\chi_t$ shows a reasonable plateau, which seems to improve for larger $\beta$. The same analysis done on an ensemble of dynamical configurations does not show any plateau.

3. THE HERMITIAN CASE

The Wilson-Dirac operator becomes Hermitian when multiplied by $\gamma_5$. To calculate $O(100)$ smallest eigenvalues of $\gamma_5M(\kappa)$ we used again the Arnoldi algorithm with a Chebychev polynomial to speed up convergence of the eigenvalues with smallest modulus. Note that there is no trivial relation between the eigenvalues of $\gamma_5M(\kappa)$ for different values of $\kappa$, while the eigenvalues of $M(\kappa)$ for different $\kappa$ are related by a trivial scale and shift operation. We used $\kappa = 0.1338$ and 0.1353 for $\beta = 6.0$ and $(\beta, \kappa_{\text{sea}}) = (5.2, 0.1355)$, respectively, which corresponds to $m_{PS}/m_V = 0.6 - 0.7$.  

\begin{figure}
\centering
\includegraphics[width=0.45\textwidth]{figure2.png}
\caption{The density of real modes (upper plot) and the topological susceptibility as a function of the largest real mode taken into account (lower plot) calculated on quenched configurations at $\beta = 5.9$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.45\textwidth]{figure3.png}
\caption{Same as in Fig. 2 for quenched configurations at $\beta = 6.0$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.45\textwidth]{figure4.png}
\caption{Same as in Fig. 2 for dynamical configurations at $(\beta, \kappa_{\text{sea}}) = (5.2, 0.1355)$.}
\end{figure}
Table 1
Simulation parameters. We used lattices of size $V = 16^3 \times 32$. The scale was set using $r_0 = 0.5\text{fm}$.

| $\beta$  | $\kappa_{\text{sea}}$, $m_{\text{PS,sea}}$ [MeV] | $a$ [fm] | #conf |
|---------|---------------------------------------------|---------|-------|
| 5.9     | quenched                                   | 0.112   | $O(100)$ |
| 6.0     | quenched                                   | 0.093   | $O(150)$ |
| 5.2     | 0.13550, 578(6)                             | 0.099   | $O(150)$ |

The eigenvalues of the Hermitian operator can be used to calculate the topological charge $Q_t$. Based on a chiral Ward identity one can define

$$Q_t = \lim_{m_q \to 0} m_q \text{Tr} \left[ \gamma_5 M(m_q)^{-1} \right].$$

If the largest of the $N$ calculated smallest eigenvalues is large enough, we expect

$$\text{Tr} \left[ \gamma_5 M(m_q)^{-1} \right] \approx \sum_{i=1}^{N} \frac{1}{\lambda_i},$$

to be a good approximation. For an extrapolation to $N = \dim(\gamma_5 M)$ we performed a fit using the ansatz $c_0 + c_{-1} |\lambda|^{-1}$.

While the theoretical basis for the definition of the topological charge using Eq. (4) is much better, there are a number of disadvantages and open problems. For instance, calculations have to be performed at quark masses which are heavy enough to avoid effects from “exceptional configurations”. Therefore, a chiral extrapolation is in principle required, but has not been done here. Initial results using smaller (and coarser) lattices show a very mild quark mass dependence.

Finally, renormalization is a problem which has not yet been addressed. In Fig. 5 we compare the results for $\chi_t$ calculated from the eigenvalues of $M$ and $\gamma_5 M$. For quenched configurations at $\beta = 6.0$ both methods give consistent results, indicating that the neglected renormalization factor is close to one. In the same figure we compare our results for $\chi_t$ with those obtained in [5] using a gluonic method. The results agree surprisingly well.

4. CONCLUSIONS

We calculated the eigenvalues of the Wilson-Dirac operator $M$ and the Hermitian operator $\gamma_5 M$ on dynamical and quenched gauge configurations at similar lattice spacings. We found clear signs for a suppression of small eigenmodes on configurations with dynamical fermions. We determined the topological susceptibility using two different definitions of the topological charge based on the index theorem and a chiral Ward identity. We found good agreement for quenched gauge configurations at $\beta = 6.0$ between the two methods. Our results indicate that the index theorem cannot hold on coarse lattices since the topological charge turns out to be ill-defined. Finally, we found a surprisingly good agreement between results for the topological susceptibility determined with gluonic and fermionic methods.

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