Effective temperature of self–similar time series: analytical and numerical developments

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Abstract

Within both slightly non–extensive statistics and related numerical model, a picture is elaborated to treat self–similar time series as a thermodynamic system. Thermodynamic–type characteristics relevant to temperature, pressure, entropy, internal and free energies are introduced and tested. Predictability conditions of time series analysis are discussed on the basis of Van der Waals model. Maximal magnitude for time interval and minimal resolution scale of the value under consideration are found and analyzed in details. The statistics developed is shown to be governed by effective temperature being exponential measure of the fractal dimension of the time series. Testing of the analytical consideration is based on numerical scheme of non–extensive random walk. A statistical scheme is introduces to present numerical model as a grand canonical ensemble for which entropy and internal energy are calculated as functions of particle number. Effective temperature is found numerically to show that its value is reduced to averaged energy per one degree of freedom.

Key words: Time series; non–extensive statistics; simulation
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1 Introduction

Time series analysis allows one to elaborate and verify macroscopic models of complex systems evolution on the basis of data analysis [1]. This analysis is known to be focused on numerical calculations of correlation sum for delay vectors that allow to find principle characteristics of the time series. Being traditionally a branch of the theory of statistics, time series analysis is based on the class of models of harmonic oscillator which are related to the simplest
case of the Gaussian random process [2]. But well known a real time series is relevant rather to the Lévy stable processes, than the Gaussian ones being very special case [3]. Because the former are invariant with respect to dilatation transformation [4], the problem is reduced to consideration of self–similar stochastic processes.

The simplest characteristic of a time series is known to be the Lyapunov exponent the largest of whose positive values yields predictability domain in the system behavior. A range of complexity in such behavior is determined by the Kolmogorov–Sinai entropy that equals to sum over positive magnitudes of whole set of the Lyapunov exponents\(^1\) and can be reduced to the usual Shannon value in information theory. With passage to nonlinear system the probability \(p_n\) of \(n\)-th scenario of the system behavior is transformed into power function \(p_n^q\) determined by index \(q \leq 1\), so that the Kolmogorov–Sinai entropy should be replaced by the Renyi one

\[
K_q \equiv \frac{1}{1 - q} \ln \sum_n p_n^q. \tag{1}
\]

Respectively, governing equation \(dp/d\epsilon = -\beta p^q\), \(\beta = \text{const} > 0\) describes probability variation with energy \(\epsilon = \epsilon_n\) to derive the Tsallis distribution [5]

\[
p(\epsilon) \propto [1 - (1 - q)\beta\epsilon]^{1-q}. \tag{2}
\]

In the limit \(q \to 1\), this distribution takes usual Boltzmannian form \(p(\epsilon) \propto \exp(-\beta\epsilon)\) falling down exponentially fast in contrary to the power asymptotic of the generalized exponent (2). Physically, such a behavior is caused by self–similarity of non–extensive system [6] — [9].

As a result, the problem appears to study self–similar time series that present processes corresponded to power–law distribution type of Zipf–Mandelbrot ones [10]. This work is devoted to both analytical and numerical considerations of such type time series as a thermodynamic system. It is worthwhile to stress that our approach is principally different off the pseudo–thermodynamic formalism developed for consideration of multifractal–type objects (see [11], [12] and references therein). Indeed, if within latter formalism the role of state parameters play related multifractal indices, we introduce a set of effective state parameters, whose meaning is of true thermodynamic type (so, an effective temperature is a measure of data scattering being related to the fractal dimension of the time series).

The paper consists of two main Sections 2 and 3, the first of which is devoted to analytical consideration and the second one — to numerical study. We start

\(^1\) We have in mind hyperbolic attractors.
analytical consideration with elaboration of a model which allows us to address a self–similar time series as slightly non-extensive thermodynamic system (see Subsection 2.1). Following Subsection 2.2 is based on calculations of both entropy and internal energy of the time series. As a result, thermodynamic–type characteristics such as temperature and entropy, volume and pressure, internal and free energies are introduced. Their testing for the model of ideal gas is shown to be basis for statistics of self–similar time series. Subsection 2.3 contains calculations of corrections to the ideal gas approach when external field and particle interaction are switched on. On the basis of Van der Waals model, we obtain the expressions for the specific heat and susceptibility as functions of the temperature. Subsection 2.4 is devoted to discussion of the physical meaning of the results obtained within the framework where the predictability of behavior of time series is mimicked as stability conditions of a non–ideal gas. We find maximal magnitudes for time interval and minimal resolution scale of the value under consideration. Subsection 2.5 shows that a temperature governing time series statistics is exponential measure of a self–similarity index related to fractal dimension of the time series.

Testing of the analytical consideration is based on numerical scheme of non–extensive random walk [13] whose stochastic equation and its solutions are treated in Subsection 3.1. As a result, we obtain non–trivial time series whose form is governed by a friction coefficient fixing related fractal dimension. In Subsection 3.2 we introduce a statistical scheme that allows us to consider modeled time series as a grand canonical ensemble for which we calculate entropy and internal energy as functions of particle number. In Subsection 3.3 we find numerically effective temperature and show that its value is reduced to averaged kinetic energy per one particle of ideal gas. In concluding Section 4 we confirm numerically exponential dependence of the effective temperature on the fractal dimension that was found analytically in Subsection 2.5. We discuss also multifractional and clustering properties of time series.

Finally, several equalities needed in quoting are placed in Appendices A, B.

2 Analytical study of self–similar time series

The principle peculiarity of time series is that it evolves during large enough but finite time interval $T < \infty$. On the other hand, the self–similarity means that a series relates to a fractal manifold characterized by the fractal dimension $D$. We show below that both properties pointed out are taken into account in natural way if we shall use non-extensive statistical mechanics where a simple combination of quantities $T$ and $D$ fixes the non-extensivity exponent $q$ (see Eqs.(26), (31) below).
2.1 Description of time series as slightly non-extensive statistical system

We start with consideration of \( d \)-dimensional time series \( x(t_i) \) related to the set \( \{x_i\} \) of consequent values \( x_i \equiv x(t_i) \) of principle variable \( x(t) \) taken at discrete time instant \( t_i \equiv i\tau \) that we obtain as result of dividing a whole time series length \( T \equiv N_0\tau \) by \( N_0 \) equal intervals \( \tau \). Following the ergodic hypothesis we shall imitate the time series \( x(t_i) \) by the set of generalized coordinates \( \{x_n\} \) supplemented by conjugated set \( \{v_n\} \) of velocities that show jumping rates of coordinates \( x_i \) with the time variation. It is principally important to take into account that mapping of the time series \( x(t_i) \) into the manifold \( \{x_n\} \) is not mutually single valued because different terms \( x(t_i) \) and \( x(t_j) \) may be equal (similarly, it appears at mapping of velocity time series \( v(t_i) \) into related manifold \( \{v_n\} \)). This peculiarity is displayed first of all in that the number \( N_0 \) of specimens of the time series is much more than the total number \( N \) of terms of related manifold. In the following, just the latter number \( N \) plays the role of the particle number of statistical system. Obviously, this system should be considered as grand canonical ensemble with variable number \( N_n \) of particles in a state \( n \).

In the simplest case of Markovian consequence, one has the velocity \( v_n \equiv (x_n - x_{n-1})/\tau \). For more complicated series with \( m \)-step memory, the velocity magnitude is defined as follows:

\[
   v^{(m)}_n \equiv \sqrt{\frac{1}{m} \sum_{l=1}^{m} \delta^2_l(m, n)}, \quad \delta_l(m, n) \equiv \frac{x_{(n-m)+l} - x_{(n-m)+(l-1)}}{\tau}.
\]

Along the line of the ergodic hypothesis, the paradigm of our approach is to address the time series as a physical system defined by an effective Hamiltonian \( \mathcal{H} = \mathcal{H}\{x_n, v_n\} \) on whose basis statistical characteristics of this series could be found. If one proposes that series terms \( x_n \) related to different \( n \) are not connected, the effective Hamiltonian is additive:

\[
   \mathcal{H} = \sum_{n=1}^{N} \varepsilon_n N_n, \quad \varepsilon_n \equiv \varepsilon(x_n, v_n).
\]

Physically, this means that the series under consideration is relevant to an ideal gas comprising of \( N \) identical particles with energy \( \varepsilon_n \) and number \( N_n \) in state \( n \). Further, we suppose different terms of time series to be statistically identical, so that effective particle energy does not depend on coordinate \( x_n \): \( \varepsilon(x_n, v_n) \rightarrow \varepsilon(v_n) \). Moreover, since this energy does not vary with inversion of the coordinate jumps \( x_n - x_{n-1} \), the function \( \varepsilon(v_n) \) should be even. We use the simplest square form.
\[ \varepsilon_n = \frac{1}{2} v_n^2, \quad (5) \]

which is reduced to the usual kinetic energy for a particle with mass 1. With switching on an external force \( F = \text{const} \), particle energy (5) becomes as follows:

\[ \varepsilon_n = \frac{1}{2} v_n^2 - Fx_n. \quad (6) \]

Finally, when time series has a microscopic memory, dimension of the delay vectors \( \delta_i(m, n) \) in the definition (3) needs taking \( m > 1 \). Moreover, if time series terms \( x_m, x_n \) with \( m \neq n \) are clustered, the Hamiltonian becomes relevant to a non–ideal gas with interaction \( w_{mn}, m \neq n \):

\[ \mathcal{H} = \frac{1}{2} \sum_{n=1}^{N} v_n^2 + \frac{1}{2} \sum_{m \neq n} w_{mn}. \quad (7) \]

To study a behavior of the time series as a whole one needs to fulfill summation over a set of states given by manifold \( \{x_n, v_n\} \) that is relevant to the system phase space. In so doing, it is convenient to pass to related integrations as following:

\[ \sum_{\{x_n, v_n\}} \Rightarrow \int \prod_{n=1}^{N} dx_n dv_n = N^{-1} \prod_{n=1}^{N} \int dy_n du_n. \quad (8) \]

Here, the factorial takes into account statistical identity of the time series terms, \( \Delta \) is effective Planck constant that determines a minimal volume of the phase space per a particle related to a term. The inverted factor

\[ \mathcal{N} \equiv N! \left( \frac{X^2}{\tau \Delta} \right)^{-dN} \approx \left[ \frac{e^{X^2 d}}{N (\tau \Delta)^d} \right]^{-N} \quad (9) \]

is caused by change of variables

\[ y_n \equiv \frac{x_n}{X}, \quad u_n \equiv \frac{\tau v_n}{X} \quad (10) \]

rescaled with respect to macroscopic length \( X \) being chosen to guarantee the conditions

\[ \int dy_n = 1, \quad \int du_n = 1. \quad (11) \]
According to Introduction self-similarity condition forces to use a statistics type of given by the Tsallis–type distribution (2). However, the latter is appeared to be inconvenient because the condition

$$\sum_n p_n^q \equiv \langle 1 \rangle_q \neq 1$$

(12)
takes place and definition of the internal energy reads:

$$E = \sum_n \frac{\varepsilon_n p_n^q}{\langle 1 \rangle_q}.$$  

(13)

To take into account these constraints an escort distribution was proposed to use [14]

$$P_n \equiv \frac{p_n^q}{\langle 1 \rangle_q}.$$  

(14)

In explicit form it reads as follows:

$$P_q\{y_n, u_n\} = \begin{cases} \frac{1}{Z} \left[ 1 - (1 - q)^{\mathcal{H}\{y_n, u_n\} - E} \right]^{\frac{1}{1-q}} \quad \text{at} \quad (1 - q)^{\mathcal{H}\{y_n, u_n\} - E} < 1, \\ 0 \quad \text{otherwise.} \end{cases}$$  

(15)

Here, the partition function is defined by the condition

$$Z \equiv N^{-1} \prod_{n=1}^N \iint \left[ 1 - (1 - q)^{\mathcal{H}\{y_n, u_n\} - E} \right]^{\frac{1}{1-q}} d\mathbf{y}_n d\mathbf{u}_n,$$  

(16)

where $0 < q < 1$ is a parameter of non-extensivity, $T_s$ is energy scale. Internal energy $E$ is determined by the equality

$$E \equiv N^{-1} \prod_{n=1}^N \iint \mathcal{H}\{y_n, u_n\} P_q\{y_n, u_n\} d\mathbf{y}_n d\mathbf{u}_n,$$  

(17)

and normalization parameter $\langle 1 \rangle_q$ is expressed by the partition function (16) in accordance with Eq.(88) in Appendix A.

To check the statistical scheme proposed let us address firstly trivial case of time series $x_n = \text{const}$. Here, the particle energy $\varepsilon$ is a constant as well, so
that the Hamiltonian is $H = N\varepsilon$. The partition function $Z = N^{-1}$ and the normaliza\tion parameter $\langle 1 \rangle_q = N^{-(1-q)}$ are given by inverted normalization factor (9), while the internal energy $E = N\varepsilon$ is reduced to the Hamiltonian. Then, the entropy $H = -a \ln N$, $a = \frac{1}{2}(1 - q)dN \neq 0$ obtained according to definition (86) given in Appendix A is reduced to zero if only the normalization factor takes the value $N = 1$. As a result, we find effective Planck constant:

$$\Delta = \left(\frac{e}{N}\right)^{\frac{1}{2}} \frac{X^2}{\tau}. \quad (18)$$

Our future consideration is stated on the assumption that the volume $V \equiv X^d$ of $d$-dimensional domain of the $x_n$ coordinate variation in dependence of the particle number $N$ is governed by Lévy-type law

$$X^d = x^d N^{\frac{1}{2}}. \quad (19)$$

Here, $x$ is a microscopic scale and a dynamic exponent $z$ is reduced to the fractal dimensionality $D$ of self–similar manifold [7]

$$z = D. \quad (20)$$

Then, one obtains the following scaling relation for the phase space volume per a term of the time series:

$$\Delta^d = e \left(\frac{x^2}{\tau}\right)^d N^{\frac{z}{D}-1}. \quad (21)$$

In the case of Gaussian scattering, when $D = 2$, the minimal volume $\Delta^d$ of the phase space does not depend on number $N$ of the time series terms. Such a condition approves our choice of the relation (19) for the whole volume $V \equiv X^d$ as function of the number $N$ of time series terms.

2.2 Non-extensive thermodynamics of time series as an ideal gas

Calculations of main thermodynamic quantities of non-extensive ideal gas arrives at the following expressions for the partition function (16) and the internal energy (17) (see [14] — [16])
\[ Z = \frac{V^N \gamma(q)}{N!} \left[ \frac{\theta \langle 1 \rangle_q}{1 - q} \right] \frac{dN}{1 + \left(1 - q \right) \frac{dN}{2}} \cdot \left[ 1 + \left(1 - q \right) \frac{E}{\langle 1 \rangle_q T_s} \right]^{\frac{1}{1 - q} + \frac{dN}{2}}, \]  
\[ E = \frac{dN}{2} V^N \gamma(q) T_s \left[ \frac{\theta \langle 1 \rangle_q}{1 - q} \right] \frac{dN}{1 + \left(1 - q \right) \frac{dN}{2}} \cdot \left[ 1 + \left(1 - q \right) \frac{E}{\langle 1 \rangle_q T_s} \right]^{\frac{1}{1 - q} + \frac{dN}{2}} Z^{-q}. \]  

Here, \( d \)-dimensional gas in volume \( V \equiv X^d \) is addressed and the notations are introduced:

\[ \theta \equiv \frac{2\pi T_s}{\Delta^2}, \quad \gamma(q) \equiv \frac{\Gamma \left( \frac{1}{1 - q} \right)}{\Gamma \left( \frac{1}{1 - q} + \frac{dN}{2} \right)}, \]  

(24)


\[ \langle 1 \rangle_q = \left\{ \frac{X^d N^{\gamma(q)}}{N!} \left[ \frac{\theta(1 + a)^{\frac{dN}{2}}}{1 - q} \right] \right\}^{\frac{1 - q}{1 - a}}, \]  

(25)

\[ a \equiv \frac{1}{2} \left(1 - q \right) dN. \]  

(26)

In the limits

\[ 1 - q \ll 2/d, \quad N \gg 1 \]  

(27)

when

\[ \gamma(q) \simeq \left[ \xi(1 - q) \right]^{\frac{dN}{2}} \left(1 + a\right)^{-\frac{1 + a}{1 - q}}, \]  

(28)

one obtains for the first of entropies (86):

\[ H \simeq \frac{Na}{2(1 - a)} \ln \left[ e^{2 + d} \theta^d \left( \frac{X^d}{N} \right)^2 \right]. \]  

(29)
With accounting scaling relation (19), this expression takes the usual form

\[ H = N \frac{D-1}{D} \ln \left( \frac{G}{N} \right), \quad G \equiv (2\pi eT_s)^{dD} \frac{dD}{2} \left( \frac{x}{\tau} \right)^{-dD} \]  

(30)

if the dynamic exponent is determined as

\[ z \equiv D = \frac{1}{1 - a}. \]  

(31)

Respectively, the internal energy (23) and the normalization parameter (25) read:

\[ E = \frac{dN}{2} \left( \frac{G}{N} \right)^{2a} T_s, \quad \langle 1 \rangle_q = \left( \frac{G}{N} \right)^{2a}; \quad a \equiv \frac{D-1}{D}. \]  

(32)

The physical temperature is defined as follows [15]

\[ T \equiv \langle 1 \rangle_q T_s = \left( \frac{G}{N} \right)^{2a} T_s \]  

(33)

where the last equality takes into account the second of relations (32). This definition guarantees the equipartition law

\[ E = CT, \quad C \equiv cN, \quad c \equiv \frac{d}{2} \]  

(34)

where the quantity

\[ C = \frac{\partial E}{\partial T} \]  

(35)

is the specific heat. It is easily to convince that equations (30) — (33) arrive at standard thermodynamic relation

\[ \frac{\partial H}{\partial E} = \frac{1}{T}; \]  

(36)

Above used treatment is addressed to a fixed value of the internal energy \( E \) [16]. In alternative case when the principle state parameter is the temperature \( T \), we should pass to the conjugate formalism [14]. Here, standard definition
\[ F \equiv E - TH \quad (37) \]

of the free energy arrives at the dependence

\[ F = -CT \ln \left( \frac{T}{eT_s} \right) . \quad (38) \]

Then, the thermodynamic identity

\[ \frac{\partial F}{\partial T} \equiv -H \quad (39) \]

yields the relation

\[ H = C \ln \left( \frac{T}{T_s} \right) \quad (40) \]

that plays a role of the heat equation of states. It arrives at the usual definition of the specific heat (cf. Eq. (35))

\[ C = T \frac{\partial H}{\partial T} . \quad (41) \]

Let us introduce now a specific entropy per unit time

\[ h \equiv (d\tau)^{-1} \frac{\partial H}{\partial N} = \tau^{-1}H_1 - r \quad (42) \]

to be determined by a minimal entropy

\[ H_1 = (D - 1) \left[ \ln \sqrt{2\pi eT_s} - \ln \left( \frac{x}{\tau} \right) \right] \quad (43) \]

and a redundancy

\[ r = \frac{a}{d\tau} \ln(eN) , \quad a \equiv \frac{D - 1}{D} . \quad (44) \]

Dependencies on the scale \( x \)

\[ h(x) = \text{const} - \frac{D - 1}{\tau} \ln \left( \frac{x}{\tau} \right) , \quad r(x) = \text{const} \quad (45) \]
notice that the system behaves in a stochastic manner [17].

Effective pressure is defined as

\[ p \equiv -\tau \frac{\partial h}{\partial x} \]  

(46)

to measure specific entropy variation with respect to the time series scale. Then, we arrive at a mechanic–type equation of states

\[ px = D - 1 \]  

(47)

being additional to the relation (40) of entropy to temperature. According to Eq. (47), definition of the pressure coefficient

\[ \kappa \equiv \frac{\partial p}{\partial (x^{-1})} \]  

(48)

shows that it is fixed by the dynamic exponent (31) being the fractal dimensionality of self–similar time series:

\[ \kappa = D - 1. \]  

(49)

It is worthwhile to note that effective pressure (46) is introduced as derivative of the specific entropy \( h \) with respect to the microscopic scale \( x \). That reduces a susceptibility of the time series to the pressure coefficient (48) inverted.

### 2.3 Corrections to the ideal gas approach

Let us focus now on effect of both external field and particle interaction whose switching on is expressed by equalities (6), (7). An external force \( \mathbf{F} = \text{const} \) causes the second term in Hamiltonian (6) to arrive at the factor

\[ Z_{ext} = Z_d \left[ \sinh \left( \frac{FX}{2T} \right) \right]^N \]  

(50)

in partition function (22). Here, \( Z_d \) is a factor depended on the dimension \( d \) only and we put \( q \to 1 \) due to the conditions (27). According to the definition (86) the factor (50) yields the entropy addition
where we suppress unessential term. With increasing homogeneous external field, the entropy $H_{ext}$ grows quadratically at $FX \ll T$ and linearly at $FX \gg T$.

According to [18], cluster expansion of the particle interaction in Hamiltonian (7) results in additional factor in partition function (22):

$$Z_{int} = 1 - \frac{N^2}{2V} \left( v + \frac{w}{T} \right);$$

$$w \equiv S_d \int_{\epsilon}^{\infty} w(x)x^{d-1}dx, \quad S_d \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)}; \quad v \equiv \frac{S_d}{d}\epsilon^d$$

where $\epsilon$ is effective radius of the particle core. Relevant entropy addition

$$H_{int} = -\frac{aN^2}{2V} \left( v + \frac{w}{T} \right)$$

decreases monotonically with growth of the particle volume $v$. Increase of the temperature $T$ causes the entropy decrease for attractive interaction $w < 0$ and its increase in the case of repelling one $(w > 0)$.

Entropy additions (51), (53) arrive at total value of the specific heat (41) in the following form:

$$C = \frac{d}{2}N - aN \left\{ \left[ \frac{FX}{2T} \coth \left( \frac{FX}{2T} \right) - 1 \right] - \frac{N w}{2VT} \right\}$$

where formula (34) are taken into account. On the other hand, making use of equalities (42), (46), (51), (53) arrives at the pressure addition $\delta p$ determined by the equality

$$\delta p \cdot x = -\frac{a}{d} \left[ \frac{FX}{2T} \coth \left( \frac{FX}{2T} \right) - 1 \right] + a \frac{N}{V} \left( v + \frac{w}{T} \right).$$

Then, with accounting (49) the pressure coefficient (48) takes the form

$$\kappa = \frac{a}{1-a} + \frac{a}{d} \left\{ 1 + \left[ \frac{FX}{\cosh \left( \frac{FX}{2T} \right)} \right]^2 \right\} + (1 + d)an \left( v + \frac{w}{T} \right)$$

(56)
where effective density
\[
n \equiv \frac{N}{V} = x^{-d} N^a
\] (57)
is introduced. In the limiting case of low temperatures \( T \ll FX/2 \), we obtain:
\[
C \simeq \frac{d}{2} N - \frac{aN}{2T} (FX - nw),
\] (58)
\[
\kappa \simeq \frac{a}{1-a} + \frac{a}{d} \left[ 1 + \left( \frac{FX}{T} \right)^2 \exp \left( -\frac{FX}{T} \right) \right] + (1 + d)an \left( v + \frac{w}{T} \right).\] (59)

Respectively, in opposite case \( T \gg FX/2 \), one has:
\[
C \simeq \frac{d}{2} N + \frac{aN}{2T} \left[ nw - \frac{(FX)^2}{6T} \right],
\] (60)
\[
\kappa \simeq \frac{a}{1-a} + \frac{a}{d} \left[ 1 + \frac{1}{4} \left( \frac{FX}{T} \right)^2 \right] + (1 + d)an \left( v + \frac{w}{T} \right).\] (61)

Thus, in the limits of both low and high temperatures, the influence of external field \( F \) reduces to hyperbolically decreasing addition \( (FX/T)^n, n = 1, 2 \) to the specific heat \( C \); accordingly, the pressure coefficient \( \kappa \) gets the constant \( a/d \) only. In a similar manner, the particle interaction affects hyperbolically on the specific heat, while the pressure coefficient takes the term being proportional to the particle volume \( v \).

2.4 Stability conditions

It might seem the analysis of a time series would appeared to be the subject of more much study if both the interval \( \tau \) and the scale \( x \) take infinitely decreasing magnitudes (respectively, the number \( N \) tends to infinitely large values). However, we will show now that external influence and term clustering arrive at predictability boundaries in behavior of relevant system that are represented as instability of modeling thermodynamic system. Our analysis is stated on stability conditions
\[
C > 0, \quad \kappa > 0
\] (62)
for the specific heat $C$ and the pressure coefficient $\kappa$ given by equalities (54) and (56).

In the simplest approach being modeled by the ideal gas in Subsection 2.2, we arrives at the natural restriction for the dynamic exponent:

$$z \equiv D > 1.$$  \hspace{1cm} (63)

It means non-extensive dependence of the system volume (19) on the particle number $N$. On the other hand, in accordance with relation (31) the magnitude of the principle index (26) is limited by condition

$$a < 1$$ \hspace{1cm} (64)

which restricts the rate of increasing effective density (57) with $N$.

As is seen from expressions (54), (58), (60), the specific heat falls down monotonically with temperature decrease. To analyze such a behavior quantitatively let us consider the case of low temperatures $T \ll FX/2$, when we can use the simplest dependence (58). Then, it is easily to see that at temperature less than critical magnitude

$$T_c = \frac{a}{d} \left( FN^{-\frac{1-a}{d}} - wx^{-d}N^a \right)$$ \hspace{1cm} (65)

the system becomes nonstable due to external force $F$ and interparticle attraction $w < 0$. If the number $N$ is much less than critical value $N_c$ defined as

$$N_c^{-\frac{1+d}{d}} = \frac{|w|}{F} x^{-(1+d)},$$ \hspace{1cm} (66)

main contribution gets interparticle attraction $w < 0$, whereas in opposite case $N \gg N_c$ — external force $F$. With passage to more complicated case of high temperatures $T \gg FX/2$, when one follows to use temperature dependence (60), the physical situation does not change qualitatively.

As show estimations (59), (61), the pressure coefficient becomes negative if microscopic scale $x^d$ is less than a critical magnitude

$$x_c^d = \frac{d(1+d)}{1+d} \left( \frac{|w|}{T} - v \right) N^a$$ \hspace{1cm} (67)

that is determined at attractive nature of interaction only ($w < 0$). If both parameters $w$ and $v$ takes non–zeroth magnitudes, the temperature values
are characterized by another minimal magnitude \( T_v = |w|/v \), lower which the microscopic scale \( x_c \) becomes principle. This temperature is less than critical value (65) if elementary volume takes magnitudes lower than value

\[
v_c \equiv \frac{d}{a} n^{-1} = \frac{d}{a} x^d N^{-a}
\]

where we take into account definition (57).

In the limits \( v \to 0, F \to 0 \), boundary magnitudes (65), (67) arrive at appearance of the critical value

\[
a_c = \frac{(1 + d)(1 + d^2)}{2} \left[ 1 - \sqrt{1 - \frac{4d^2}{(1 + d)(1 + d^2)^2}} \right].
\]

Relevant fractal dimension of the time series (31) grows monotonically with dimension of the time series \( d \) from the magnitude \( D_c = 1 + \sqrt{2} \simeq 2.414 \) at \( d = 1 \) to \( D_c \simeq d^2 \) at \( d \gg 1 \) (see Figure 1). As a result, the critical value (69) arrives at the upper boundary of the fractal dimensionality \( D_c \) being more than topologic magnitude \( D_t = 2d \) always \( (D_c > D_t) \). Thus, we can conclude the upper boundary \( D_c \) of application of our approach is appeared to be inessential at all.

Fig. 1. a. The dependence of the critical value (69) on the time series dimensionality. b. The same for related maximal value (31) of the fractal dimension of the time series (solid line) and topologic magnitude \( D_t = 2d \) (lower line). The physical domain is shaded.

2.5 Effective temperature of time series

We have considered above the simplest model which have allowed us to examine analytically a self–similar time series in standard statistical manner.
Characteristic peculiarity of related equalities is a scale invariance with respect to variation of the non-extensivity parameter $1 - q$ which is contained everywhere through the parameter (26) that is related to the dynamic exponent $z$ and the fractal dimension $D$ of the time series according to Eq. (31). This invariance is clear to be caused by self–similarity of the system under consideration. For a given time series, the value $D$ is fixed if it is addressed to a monofractal manifold and takes a closed set of magnitudes in the case of self–similar system relevant to a multifractal.

In real time series the property of self–similarity can be broken, so that a dependence on the non-extensivity parameter itself could appear to be very weak. However, above introduced set of pseudo–thermodynamic characteristics of time series is kept as applicable and accustomed thermodynamic relations (34) — (37), (39), (41), (46) and (48) can be applied to analysis of arbitrary time series.

Main progress in our consideration is that time series statistics is governed completely by the temperature (33). With accounting the second of equalities (30) and rescaling the temperature unit $T_s$ into $T_{sc} \equiv (2\pi e)^a T_s$ we derive to the expression

$$\frac{T}{T_{sc}} = \left[\left(\frac{\tau}{X}\right)^2 T_{sc}\right]^{\frac{1}{1-a}}. \quad (70)$$

Being independent of the number of terms $N$, the time series temperature shows exponential dependence on the index (26) located under critical magnitude (69). To establish a character of the power dependence on the ratio of the range $X$ of the principle variable to the time interval $\tau$, it is naturally to choice measure units of the temperature in the following manner:

$$T_{sc} \equiv e\left(\frac{X}{\tau}\right)^2, \quad T_s \equiv \frac{e^{1-a}}{(2\pi)^a} \left(\frac{X}{\tau}\right)^2. \quad (71)$$

Then, the expression (70) for the time series temperature takes the simplest form

$$T = \left(\frac{X}{\tau}\right)^2 e^D, \quad D \equiv \frac{1}{1-a} \quad (72)$$

according to which the value $T$ is the exponential measure of the fractal dimensionality $D$ of the self–similar time series.

According to consideration given in Subsection 2.4, time series subjected to external influence and term clustering is limited in predictability that is emerged
as instability of modeling thermodynamic system. On the basis of Van der Waals model, we have found minimal magnitudes (67), (68) for resolution scale $x_c$ of the value under consideration. Respectively, making use of the critical temperature (65) and the definition (72) shows that a time series is predictable if the microscopic time interval $\tau$ takes magnitudes less than a critical magnitude $\tau_c$. In the case $N \ll N_c$, where the boundary magnitude is determined by Eq. (66), we find

$$\tau_c = \left(\frac{a}{d}\right)^{-\frac{1}{2}} e^{\frac{a}{2} x^2} N^{\frac{1}{2} \frac{1}{a} - \frac{1}{2}}.$$  \hspace{1cm} (73)

In opposite case $N \gg N_c$, one obtains

$$\tau_c = \left(\frac{a}{dF}\right)^{-\frac{1}{2}} e^{\frac{a}{2} x^2} N^{\frac{1}{2} \frac{1}{a} - \frac{1}{2}}.$$  \hspace{1cm} (74)

It is easily to convince that the entropy (40) takes positive values within the stability domain $\tau < \tau_c$.

3 Numerical study of time series

In this Section, we are aimed to verify numerically the definitions based on expressions (3) — (5), (15) — (17), (26), (30) — (41). As above analysis has shown, one of peculiarities of the statistical system under consideration is that it is slightly non–extensive due to condition (27). Thus, we can put $q = 1$ in the following numerical consideration.

3.1 Simulations of time series

In our consideration, we shall follow the numerical scheme of non-extensive random walk [13] based on the discrete stochastic equation

$$x_{i+1} = \sqrt{\tau} \zeta_i + \left[(1 - \gamma \tau) + \sqrt{\tau} \xi_i\right] x_i.$$  \hspace{1cm} (75)

Here, discrete time $t_i = i \tau$ is fixed by integers $i = 0, 1, ..., N_0$ and minimal interval $\tau$; $\zeta_i$ and $\xi_i$ are additive and multiplicative stochastic sources normed with white–noise conditions $\langle \zeta_i \zeta_j \rangle = \langle \xi_i \xi_j \rangle = \delta_{ij}$; friction coefficient $\gamma$ determines parameter $\nu = \gamma/(1 + \gamma)$ which fixes, in accordance with stationary distribution, non–extensivity parameter $q \equiv (2 - \nu)^{-1} = (1 + \gamma)/(2 + \gamma)$. Making use of iteration procedure (75) arrives at stochastic time series, whose form
is shown in Figure 2 for total $T = 500$ and minimal $\tau = 0.01$ time intervals and different $\nu$ values. As is seen from these series, at friction coefficient $-1 < \gamma$,

![Diagram](image)

**Fig. 2.** The form of modeled time series at $T = 500$, $\tau = 0.01$ and different $\nu$. The magnitudes of fractal dimension $D$ are pointed out being determined in correspondence with $R/S$–method [20].

when the parameter $\nu$ is negative, time series are relevant explicitly to the Lévy flights. With passage to positive parameters $\gamma$, $\nu$, their growth arrives at gradual transformation of superdiffusion process into Brownian diffusion that
is related to the madnitudes $\gamma = \infty$, $\nu = 1$. As $\nu$ increase, the relevant fractal dimension grows from the value $D \geq 1$ at $\nu < 0$ to $D \leq 2$ at $\nu \leq 1$.

In Figure 3, we show velocity time dependencies $v(t)$ obtained from origin time series $x(t)$ according to definition (3) where the simplest non–clustering case $m = 1$ is taken. It is easily to see a velocity time dependence $v(t)$ has much more rugged form in comparison with related time series $x(t)$. As a result, fractal dimension for the velocity series $v(t)$ is always more than its value for the initial series $x(t)$ (see data in Figure 3). Moreover, it turned out that

![Figure 3](image-url)

Fig. 3. The form of velocity time dependencies $v(t)$ related to time series $x(t)$. Because $R/S$–method gives overestimated values $D_{R/S}$ near upper boundary $D = 2$ of fractal dimension, we apply also the values $D_{bc}$ determined within box–counting method [20].
\textit{R/S}–method and box–counting method (see [20]) give quite different values \(D_{R/S}\) and \(D_{bc}\) of fractal dimension. In our opinion, this difference which grows with passage from time series \(x(t)\) to velocity \(v(t)\) is caused by that related manifolds are more likely multifractals than fractals being corresponded with single value of the fractal dimension.

3.2 Statistical consideration of simulated time series

To study statistical properties of above time series we are stated on the ergodic hypothesis that supposes identity of averaging over both velocity time series \(v(t_i), i = 1, 2, ..., N_0\) and statistical ensemble \(\{v_n\}, n = 1, 2, ..., N\), which describes velocity scattering in phase space. We determine such an ensemble dividing maximum interval of the velocity variation domain into \(N \gg 1\) zones \(n = 1, 2, ..., N\), within of which velocities have mean value \(v_n\) and small variation \(\delta v_n\). Then, the probability \(p_n\) to hit the interval \([v_n - \delta v_n/2, v_n + \delta v_n/2]\) and the relevant probability density function \(\pi(v_n)\) are determined as follows:

\[
p_n \equiv \frac{\nu_n}{N_0}, \quad \pi(v_n) \equiv N_0^{-1} \frac{\nu_n}{\delta v_n} \tag{76}
\]

where \(\nu_n\) is a number of the time series specimens with mean value \(v_n\). According to Figure 4 the probability density function \(\pi(v)\) has the usual bell–shaped form centered at the velocity \(v = 0\). By this, non-monotonic variation of the \(\pi(v)\) form in dependence of the fractal dimension \(D\) attracts attention to oneself: in particular, the most broad velocity scattering (curve 5) corresponds to negative value of the parameter \(\nu\) to display fat tails.

Fig. 4. The probability density function of the velocity distribution at different fractal dimensions (curves 1 — 5 correspond to parameters \(\nu = 0.85; 0.50; 0.35; 0.99; -0.10; \) respectively, fractal dimensions are \(D = 1.81; 1.58; 1.45; 2.57; 1.45\).

form centered at the velocity \(v = 0\). By this, non-monotonic variation of the \(\pi(v)\) form in dependence of the fractal dimension \(D\) attracts attention to oneself: in particular, the most broad velocity scattering (curve 5) corresponds to negative value of the parameter \(\nu\) to display fat tails.
Because distribution (76) corresponds to grand canonical ensemble, related thermodynamic functions are determined by the $n$–state number

$$N_n \equiv Np_n = \frac{N}{N_0}\nu_n \quad (77)$$

and energy (5). Thus, the total energy of ideal gas\(^2\) reads:

$$E \equiv \sum_{n=1}^{N} \varepsilon_n N_n = \frac{N}{N_0} \sum_{n=1}^{N} \frac{v_n^2}{2}\nu_n. \quad (78)$$

The plots of dependencies $E(N)$ of energy (78) accompanied with related dependencies $H(N)$ of entropy (87) are shown in Figures 5 for different values of fractal dimensions $D$. It is principally important, the energy $E$ is directly proportional to the particle number $N$, while the entropy $H$ increases much more slowly with $N$. As for the velocity distribution $\pi(v)$, non–monotonic variations of both energy and entropy as functions of the fractal dimension take place, but main tendency is $E$ decrease and $H$ increase with $D$ growth.

**3.3 Calculations of effective temperatures of time series**

The peculiarity of the self–similar system under consideration is that it is finite to be characterized with the following effective temperatures:

$$\Theta^{-1} \equiv \frac{\partial H}{\partial E} \bigg|_D, \quad T^{-1} \equiv \frac{N\partial H}{\partial E} \bigg|_N. \quad (79)$$

Being determined at constant value of the fractal dimension $D$, the first of these is a function of the particle number $N$, whereas the second one depends\(^2\) Do not confuse with Hamiltonian (4).
on magnitude $D$ to be determined at fixed value $N$. As show simple calculations in Appendix B, above temperatures are connected by the following relation:

$$T = \frac{\Theta}{N} \left[ 1 - \frac{1}{(1-q)D^2} \left( \frac{H}{D-1} + \frac{D-1}{D^2} N \ln N \right) \right]^{-1}. \tag{80}$$

Taking into account definitions (26), (31) and (34) accompanied with Eq.(40) where $T_s$ rescaled by factor $N$, one obtains

$$T = \frac{\Theta}{N} \left\{ 1 - \left[ D \ln \left( \frac{NT}{T_s} \right) + \frac{2(D-1)^2}{d} \frac{\ln N}{D} \right]^{-1} \right\}^{-1}. \tag{81}$$

Making use of the data presented in Figure 5 gives for the first temperature (79) the dependence $\Theta(N)$ shown in the left panel of the Figure 6. It is seen this temperature takes very large values which increase directly proportionally with the $N$ number growth. Such a behavior displays an intermediate character of the quantity $\Theta$ that plays rather a role of effective energy being complementary to the energy (78). On the other hand, usage of the relation (81) arrives at the dependencies of the temperature $T(N)$ shown in the right panel of the Figure 6. It is seen with increase of the fractal dimension to values $D$ which are close to $D = 2$ the dependencies $T(N)$ approach to constant magnitudes. These dimensions relates to large values of the parameter $\nu \leq 1$ which is connected with non–extensivity index $q$ according to equality $q = (2 - \nu)^{-1}$. Thus, we find the temperature $T$ takes values which are non–dependent on the particle number $N$ in region $q \approx 1$ that is relevant to slightly non–extensive limit where above analytical consideration is valid. Due to such a behavior we can conclude the temperature $T$ has usual physical sense.

To confirm this conclusion we examine the equipartition law (34) rewritten with usage of the averaged kinetic energy (5):
\[ T = \langle \varepsilon \rangle \equiv \frac{\langle v^2 \rangle}{2}. \]  

With this aim we compare the values \( \langle \varepsilon \rangle \) and \( T \) at different magnitudes of the fractal dimension \( D \) in Figure 7. It is seen the difference between above values is not exceed 10\% if one does not take into account the points related to three smallest magnitudes of \( D \) where our analysis is not applicable.

![Fig. 7. Dependencies of the averaged kinetic energy \( \langle \varepsilon \rangle \) and the physical temperature \( T \) on the fractal dimension \( D \).](image)

4 Discussion

As shows above consideration, the quantity \( T \) defined with the second equation (79) plays the role of the physical temperature that determines the energy (82) per one particle of slightly non-extensive ideal gas. According to connection (72) this temperature should grow monotonically with increase of the fractal dimension, whereas Figure 7 shows non-monotonic dependence. As is seen from Figure 2, such a behavior is caused by strong falling down of the maximal variation \( X \) with \( D \) increase. To examine exponential relation in Eq.(72) we plot in Figure 8 the dependencies of the averaged special energy \( \langle \varepsilon \rangle \) and the temperature \( T \) on the fractal dimension \( D \) using the factor \((X/\tau)^2\) as the scale of these quantities. It is seen with such a scaling above dependencies take approximately exponential form. Thus, main result of our analytical consideration is conformed numerically.

It is worthwhile to pay attention to essential scattering of the numerical data presented in Figures 6 — 8. In our opinion, there are two main reasons of above scattering. Firstly, this is caused by multifractal nature of non-extensive time series generated according to recurrent procedure (75). To take into account this reason it needs naturally to consider a multifractional set whose fractal
Fig. 8. Graphical check-up of the dependence (72): left panel — for mean value of kinetic energy \( \langle \varepsilon \rangle \); right panel — for the physical temperature \( T \).

dimensions are distributed as a function of the parameter \( q = (2 - \nu)^{-1} \) in accordance with the constraint

\[
D = \left( 1 - \frac{dN}{2} \cdot \frac{1 - \nu}{2 - \nu} \right)^{-1}.
\] (83)

that follows from Eqs. (26), (31). The second reason of the pointed out scattering is clustering of time series that is displayed essentially at small values of the parameter \( \nu \) (see Figure 2). This reason is taken into consideration by means of introducing effective interaction as it is done in Subsection 2.3.

**Appendix A: Main statements of the non-extensive statistics**

Our approach is stated on using entropy definitions

\[
H = a \ln[\exp_q(H_q)], \quad H_q = \ln_q[\exp(H/a)]; \quad a \equiv \frac{1}{2}(1 - q)dN
\] (84)

that are alternations of the usual functions logarithm \( \ln(x) \) and exponential \( \exp(x) \) with corresponding Tsallis generalizations [5]

\[
\ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q}, \quad \exp_q(x) \equiv [1 + (1 - q)x]^{\frac{1}{1-q}}.
\] (85)

In explicit form, relations (84) appears as

\[
H \equiv a \ln Z, \quad H_q \equiv \frac{\langle 1 \rangle_q - 1}{1-q}.
\] (86)
Making use of the first equality (84) and the second formulae (85) and (86) shows that physically defined entropy $H$ is extensive value being reduced to the Renyi definition (1):

$$H = aK_q = \frac{Nd}{2} \ln \langle 1 \rangle_q.$$  

(87)

Comparison of this result with the first of equalities (86) arrives at the relation

$$\langle 1 \rangle_q = Z^{1-q}$$  

(88)

that ensures the normalization condition [5].

Appendix B: Relationship between effective temperatures corresponding to constant values of fractal dimension $D$ and particle number $N$

Within determinant representation, the first of definitions (79) reads as follows [19]

$$\Theta^{-1} = \left[ \begin{array}{c} \frac{\partial E}{\partial H} \bigg|_N - \frac{\partial E}{\partial N} \bigg|_H \cdot \frac{\partial D}{\partial H} \bigg|_N \cdot \left( \frac{\partial D}{\partial N} \bigg|_H \right)^{-1} \\ \frac{\partial D}{\partial N} \bigg|_H \end{array} \right].$$  

(89)

Making use of Eqs.(26), (31), (34) gives

$$\frac{\partial D}{\partial N} \bigg|_H = \frac{d}{2}(1-q)D^2, \quad \frac{\partial E}{\partial N} \bigg|_H = \frac{d}{2} T.$$  

(90)

Respectively, rewriting relation (30) in the form

$$S = N \left[ -\ln(gN) + \left( D \ln g + \frac{1}{D} \ln N \right) \right], \quad g \equiv G^{1/D} = (2\pi cT_s)^{\frac{d}{2}} \left( \frac{x}{T} \right)^{-d},$$

(91)

25
one finds
\[
\frac{\partial H}{\partial D} = N \ln g - \frac{N}{D^2} \ln N = N \left[ \frac{1}{D} \ln \left( \frac{G}{N} \right) + \frac{1}{D} \ln N - \frac{1}{D^2} \ln N \right]
\]
\[
= N \left[ -\frac{H}{N} + \ln \left( \frac{G}{N} \right) + \frac{D - 1}{D^2} \ln N \right] = \frac{H}{D - 1} + \frac{D - 1}{D^2} N \ln N.
\]

Inserting Eqs. (79), (90), (92) in last relation (89), one obtains the result (80).

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