A stochastic flow arising in the study of local times

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Abstract

A stochastic flow of homeomorphisms of \( \mathbb{R} \) previously studied by Bass and Burdzy \cite{2} and Hu and Warren \cite{4} is shown to arise in the study of the local times of Brownian motion. This leads to a new proof of the Ray-Knight theorems for the flow via the classical Ray-Knight theorems for Brownian motion.

1 Introduction

Let \( \beta_1 \) and \( \beta_2 \) be fixed real constants. Suppose \( B \) is a Brownian motion on the real line issuing from 0. Associated with the equation

\[
X_t(x) = x + B_t + \beta_1 \int_0^t ds1(X_s(x) \leq 0) + \beta_2 \int_0^t ds1(X_s(x) > 0),
\]

is a stochastic flow. There exists a random flow of homeomorphisms of the real line \( (X_t; t \geq 0) \) so that \( t \mapsto X_t(x) \) satisfies the above equation for all \( x \in \mathbb{R} \). For each \( t \) the map \( x \mapsto X_t(x) \) is increasing and differentiable. We denote the derivative by \( DX_t(x) \). It has been shown by Bass and Burdzy, \cite{2}, and Hu and Warren, \cite{4}, that for certain stopping times \( T \) the process \( x \mapsto DX_T(x) \) is a diffusion process. Such a result was called a Ray-Knight theorem for the flow.

Let \( (W_t; t \geq 0) \) be a real-valued Brownian motion starting from \( W_0 = \xi > 0 \). Let \( T_0 \) be the stopping time

\[
T_0 = \inf\{ t \geq 0 : W_t = 0 \}.
\]

Let \( (l(t,x); t \geq 0, x \in \mathbb{R}) \) be the family of bi-continuous local times of \( W \). The celebrated Ray-Knight theorem states that the process \( x \mapsto l(T_0, x) \) is a diffusion. The problem studied by Warren and Yor \cite{7} (actually a slight variant of it) was that of obtaining a description of \( (W_t; 0 \leq t \leq T_0) \) after conditioning on the values of the entire family of random variables \( (l(T_0, x); x \in \mathbb{R}) \). A related problem was treated by Aldous, \cite{1}. The method of \cite{7} was to write \( W \) as a random transformation of a process \( W \), christened the Burglar process, which is independent of \( (l(T_0, x); x \in \mathbb{R}) \). It was shown, using a skew product linking squared Bessel and Jacobi diffusions, that Ray-Knight type theorems hold for \( W \) itself. The principal result of this paper is the following theorem which establishes a close connection between the this problem and a stochastic flow of the Bass-Burdy type. Let \( l(t,x) \) be the local times of a real-valued Brownian motion \( W \) as above. Define

\[
\lambda(t,x) = l(T_0, x) - l(t, x), \quad \text{for } x \in \mathbb{R}, \quad t \in [0, T_0].
\]

Let \( M = \sup\{W_t : t \in [0,T_0]\} \) and \( T_M = \inf\{t : W_t = M\} \). Now define a family of random changes of scale via:

\[
\phi_t(x) = \int_x^{W_t} \frac{dz}{\lambda(t,z)} \quad \text{for } x \in (0, M), \quad t \in [0, T_M].
\]

Each map \( \phi_t \) is a homeomorphism between \( (0, M) \) and \( \mathbb{R} \) and satisfies \( \phi_t(W_t) = 0 \). Next define a random clock via:

\[
A_t = \int_0^t \frac{ds}{\lambda(s, W_s)^2} \quad \text{for } 0 \leq t < T_M.
\]

We will see below that \( A_{T_M} = \infty \), almost surely.

**Theorem 1.** The family of maps \( (X_u; 0 \leq u < \infty) \) defined via

\[
X_{A_t}(x) = \phi_t \circ \phi_0^{-1}(x) \quad \text{for } x \in \mathbb{R}, \quad 0 \leq t < T_M,
\]

is distributed as the Bass-Burdy flow associated with \cite{1} with \( \beta_1 = 0 \) and \( \beta_2 = 1 \). Moreover it is independent of \( (l(T_0, x); x \in \mathbb{R}) \).
2 From the Burglar to a flow

In this section we prove Theorem 1. According to this result, the flow \((X_u; 0 \leq u < \infty)\) constructed from the Brownian motion \(W\) is independent of the local times \((l(T_0, x); x \in \mathbb{R})\). Because of this, Theorem 1 is intimately related to the problem considered in [7] of describing the process \(W\) conditional on knowing these local times. One standard approach for studying this type of problem is the method of enlargement of filtrations. We assume that the Brownian motion \((W_t; t \geq 0)\) is carried by a filtered probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). Let \(\mathcal{L} = \sigma(l(T_0, x); x \in \mathbb{R})\) and introduce the enlarged filtration \((\mathcal{F}^*_t; t \geq 0)\) where \(\mathcal{F}^*_t = \mathcal{F}_t \vee \mathcal{L}\). Typically we would seek to show that a martingale in the original filtration is a semimartingale with respect to the enlarged filtration, and to obtain some expression for its canonical decomposition as such. However in this case there is no reason to believe that \(W\) is a semimartingale with respect to \(\mathcal{F}^*_t\); Jacod’s absolute continuity criteria, [3], does not hold. Thus another approach is required to generate examples of martingales and semi-martingales relative to \(\mathcal{F}^*_t\).

For \(x \in \mathbb{R}\) we denote by \(T_x\) the first time \(W\) reaches the level \(x\). We adopt the usual convention that \(\exp(-\infty) = 0\).

Proposition 2. Let \(\kappa > 2\) be a constant, and let \(a, \xi\) and \(c\) be constants satisfying \(W_0 = \xi\) and \(0 < a < \xi < c\). The process

\[
M^*_t = \exp\left\{\frac{-\kappa^2}{2} \int_a^{W_t} \frac{dy}{\lambda(t, y)} - \frac{\kappa(\kappa - 2)}{2} \int_a^c \frac{dy}{\lambda(t, y)}\right\}, \quad t \leq T_a \wedge T_c,
\]

\[
M^*_t = M^*_{T_a \wedge T_c}, \quad t > T_a \wedge T_c,
\]

is an \(\mathcal{F}^*_t\)-martingale.

Admitting this proposition for the moment, let us see how it leads to a proof the main result.

First note:

Lemma 1.

\[
\int_a^c \frac{dz}{\lambda(t, z)} - \int_a^c \frac{dz}{\lambda(0, z)} = \int_0^t \mathbf{1}(a \leq W_s \leq c) dA_s
\]

almost surely on the event \(\{c < M, t < T_M\}\).

Proof. Notice that on the event \(\{c < M, t < T_M\}\) we have, almost surely, \(\lambda(s, z)\) being bounded away from 0 for all \(z \in [a, c]\) and \(s \in [0, t]\). Then the result is a consequence of applying the occupation time formula, see Exercise (1.15), Chapter VI of Revuz and Yor, [6], to obtain:

\[
\int_a^c dz \int_0^t \frac{d\lambda(s, z)}{\lambda(s, z)^2} = \int_0^t \mathbf{1}(a \leq W_s \leq c) \frac{ds}{\lambda(s, W_s)^2}.
\]

\(\square\)

We can use this lemma to verify that \(A_{T_M^-} = \infty\) almost surely. For we see that on \(\{c < M\}\),

\[
A_{T_M^-} \geq \int_a^c \frac{dz}{\lambda(T_M, z)} - \int_a^c \frac{dz}{\lambda(0, z)}.
\]

The distributions of \(\lambda(T_M, \cdot)\) and \(\lambda(0, \cdot)\) can be described using the classical Ray-Knight theorems for Brownian motion, as in Lemma 4 and then it follows from asymptotics of squared Bessel processes, to be found in for instance in [6], that almost surely,

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\log(1/\epsilon)} \int_a^{M^- \epsilon} \frac{dz}{\lambda(T_M, z)} = \infty \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \frac{1}{\log(1/\epsilon)} \int_a^c \frac{dz}{\lambda(0, z)} = 1/2.
\]

We define a process \((B_t; t \geq 0)\) on the sample space \(\Omega\) which carries \(W\) via

\[
B_{A_t} = \int_a^c \frac{dz}{\lambda(0, z)} - \int_a^{W_t} \frac{dz}{\lambda(t, z)} \quad \text{on} \quad \{t < T_c\} \cap \{c < M\}.
\]

Note that this equation defines \(B\) consistently as \(c\) varies, and that \(B\) is defined for arbitrarily large times by letting \(c \uparrow M\) so that \(T_c \uparrow T_M\) and \(A_{T_c} \uparrow \infty\).
Let \( X \) be defined as in Theorem 1. Suppose that \( 0 < x < M \) and \( t < T_M \). Then for \( c \in (x, M) \) satisfying \( t < T_c \),

\[
X_{A_t}(\phi_0(x)) - \phi_0(x) = \int_x^{W_t} \frac{dz}{\lambda(t, z)} - \int_x^{W_0} \frac{dz}{\lambda(0, z)}
\]

\[
= \int_x^c \frac{dz}{\lambda(t, z)} - \int_x^c \frac{dz}{\lambda(0, z)} + \int_x^c \frac{dz}{\lambda(0, z)}
\]

\[
= B_{A_t} + \int_x^t \frac{dz}{\lambda(t, z)} - \int_x^c \frac{dz}{\lambda(0, z)}
\]

\[
= B_{A_t} + \int_0^t 1(X_{A_t}(\phi_0(x)) \geq 0) dA_u.
\]

Here on the last line we have used Lemma 1 together with the fact that for \( s < T_c \),

\[
1(x \leq W_s \leq c) = 1(x \leq W_s) = 1(X_{A_t}(\phi_0(x)) \geq 0).
\]

Time-changing equation (9), and putting \( y = \phi_0(x) \), we see that \( X_t(y) \) solves equation (11) driven by the process \( B \).

We must next show that \( B \) is a Brownian motion with respect to a suitable filtration. Using Lemma 1 we can re-write the martingale \( M^*_t \) as,

\[
\frac{M^*_t}{M^*_0} = \exp\{-\kappa B_{A_t} - \frac{1}{2}\kappa^2 A_t\} \quad \text{on } \{t < T_a \wedge T_c\} \cap \{c < M\}.
\]

By letting \( a \) decrease to 0 and \( c \) increase to \( M \) we see that \( \exp\{-\kappa B_{A_t} - \frac{1}{2}\kappa^2 A_t\} \) is \( \mathcal{F}_t \)-local martingale on the stochastic interval \([0, T_M]\). Note here that \( T_M \) is an \( \mathcal{F}_t \)-stopping time. Introduce that filtration \( (\mathcal{G}_u; u \geq 0) \) via

\[
\mathcal{G}_u = \mathcal{F}_{a_u}, \quad u \in [0, \infty),
\]

where \((a_u; u \geq 0)\) is the inverse of \((A_t; t \geq 0)\). The local martingale property is preserved by the time-change and we are able to conclude that \( B \) is a Brownian motion by virtue of the next lemma.

**Lemma 2.** If \((B_t; t \geq 0)\) is a continuous \( \mathcal{G}_t \)-adapted process such that

\[
\exp\{-\kappa B_t - \frac{1}{2}\kappa^2 t\} \quad \text{is a } \mathcal{G}_t \text{-local martingale}
\]

for at least two distinct and non-zero values of \( \kappa \), then \((B_t; t \geq 0)\) is a \( \mathcal{G}_t \)-Brownian motion.

**Proof.** By considering logarithms it is clear that \( B_t \) is a \( \mathcal{G}_t \) semimartingale. Let its canonical decomposition be \( B_t = B_0 + N_t + A_t \) where \( A_t \) is a finite variation process and \( N_t \) is a \( \mathcal{G}_t \)-local martingale. Applying Itô’s formula to \( Z_{t}^{(c)} = \exp\{-\kappa B_t - \frac{1}{2}\kappa^2 t\} \), we obtain

\[
Z_{t}^{(c)} = Z_0^{(c)} - \kappa \int_0^t Z_s^{(c)} dB_s + \kappa^2/2 \int_0^t Z_s^{(c)} (d[N]_s - ds).
\]

Using this we see that the finite variation part of the local martingale \( \frac{1}{2} \int_0^t dZ_s^{(c)} / Z_s^{(c)} \), which must be identically zero, is given by

\[
-A_t + \frac{\kappa}{2} (d[N]_t - ds).
\]

From this we conclude that \( A_t \) is identically zero, that is to say \( B_t \) is a local martingale itself, and that \([B]_t = [N]_t = t\). Thus \( B \) is a \( \mathcal{G}_t \)-Brownian motion by virtue of Itô’s characterization.

Finally to complete the proof of Theorem 1 we must note that the Brownian motion \( B \) is independent of the \( \sigma \)-algebra \( \mathcal{G}_0 \) which contains \( \mathcal{L} \), and that since the stochastic differential equation (11) is exact by Zvonkin’s observation, the flow \( X \) is measurable with respect to \( B \), and hence also independent of \( \mathcal{L} \).

We turn now to proving Proposition 2. The principle tool used in constructing \( \mathcal{F}_t \)-martingales is a certain Markov property. Assume that the sample space \( \Omega \) is the canonical space \( \mathcal{C}([0, \infty), \mathbb{R}) \) and \((W_t; t \geq 0)\) realized as the co-ordinate process on \( \Omega \). Then we can introduce the family \((\theta_t; t \geq 0)\) of shift operators:

\[
W_t \circ \theta_s = W_{t+s}, \quad s, t \geq 0.
\]
For \( x \in \mathbb{R} \) let \( \mathbb{P}^x \) be the probability measures on \( \Omega \) under which the co-ordinate process \( W \) is a Brownian motion starting from \( x \). Then let \( \mathbb{P}^{x,\ell} \) for \( \ell \in L \) denote a version of the regular conditional probability for \( \mathbb{P}^x \) given \( l(T_0, \cdot) = \ell(\cdot) \), where \( L \) is some suitable space of local time profiles.

**Proposition 3.** The process \( (W_t, \lambda(t, \cdot); 0 \leq t < T_0) \) is Markovian relative to the filtration \( (\mathcal{F}_t^\nu; t \geq 0) \):

\[
\mathbb{E}^x \left[ F \circ \theta_t | \mathcal{F}_t^\nu \right] = \mathbb{E}^{W_t, \lambda(t, \cdot)} (F), \quad \text{a.s. on } \{ t < T_0 \},
\]
for all \( x > 0 \), and non-negative, measurable \( F \) on \( \Omega \).

**Proof.** Let \( \nu \) be a finite measure supported in \((0, \infty)\). Observe that, on \( \{ t < T_0 \} \),

\[
\mathcal{E}_\nu(l(T_0, \cdot)) = \mathcal{E}_\nu((\lambda(t, \cdot))_\nu(\ell(t, \cdot)),
\]
where \( \mathcal{E}_\nu(l) \) denotes

\[
\exp \left\{ -\frac{1}{2} \int_0^\infty \nu(dy)l(y) \right\}.
\]

If we define \( F' \) to be \( F\mathcal{E}_\nu(l(T_0, \cdot)) \) then

\[
F' \circ \theta_t \mathcal{E}_\nu(l(t, \cdot)) = F \circ \theta_t \mathcal{E}_\nu(l(T_0, \cdot)) \quad \text{on } \{ t < T_0 \}.
\]

Also put \( G' = \mathbb{P}^{W_0, l(T_0, \cdot)}(F)\mathcal{E}_\nu(l(T_0, \cdot)) \). Then \( \mathbb{E}^y[G'] = \mathbb{E}^y[F'] \), for all \( y > 0 \), and

\[
G' \circ \theta_t = \mathbb{P}^{W_t, \lambda(t, \cdot)}(F)\mathcal{E}_\nu(\lambda(t, \cdot)) \quad \text{on } \{ t < T_0 \}.
\]

Using these various observations and two applications of the Markov property of Brownian motion we have, for each \( A \in \mathcal{F}_t \)

\[
\mathbb{E}^x \left[ F \circ \theta_t 1_A(1(t < T_0)) \mathcal{E}_\nu(l(T_0, \cdot)) \right] = \mathbb{E}^x \left[ F' \circ \theta_t 1_A(1(t < T_0)) \mathcal{E}_\nu(l(t, \cdot)) \right]
\]

\[
= \mathbb{E}^x \left[ \mathbb{E}^{W_t} \left[ F' \right] 1_A(1(t < T_0)) \mathcal{E}_\nu(l(t, \cdot)) \right]
\]

\[
= \mathbb{E}^x \left[ \mathbb{E}^{W_t} \left[ G' \right] 1_A(1(t < T_0)) \mathcal{E}_\nu(l(t, \cdot)) \right]
\]

\[
= \mathbb{E}^x \left[ G' \circ \theta_t 1_A(1(t < T_0)) \mathcal{E}_\nu(l(t, \cdot)) \right]
\]

\[
= \mathbb{E}^x \left[ \mathbb{P}^{W_t, \lambda(t, \cdot)}(F) 1_A(1(t < T_0)) \mathcal{E}_\nu(l(T_0, \cdot)) \right].
\]

Since, the random variables \( 1_A \) and \( \mathcal{E}_\nu(l(T_0, \cdot)) \) generate \( \mathcal{F}_t^\nu \) as \( A \) and \( \nu \) vary we are done. \( \Box \)

Consider \( F \) given by

\[
F = 1_{(T_a < T_\xi)} \exp \left\{ \frac{\kappa}{2} \int_a^\xi \frac{dy}{\lambda(T_a, y)} \right\} + 1_{(T_\xi < T_a)} \exp \left\{ \frac{\kappa(\kappa - 2)}{2} \int_a^\xi \frac{dy}{\lambda(T_a, y)} \right\},
\]

where \( 0 < a < \xi < c \), recalling \( W_0 = \xi \). Using the Markov property just proved, the martingale \( M_t^\nu \) is constructed as

\[
M_t^\nu = \mathbb{E}^\xi \left[ F | \mathcal{F}_t^\nu \right] = \mathbb{E}^\xi \left[ F \circ \theta_t 1_{(t < T_\xi \wedge T_a)} + F 1_{(t > T_\xi \wedge T_a)} | \mathcal{F}_t^\nu \right]
\]

\[
= \mathbb{P}^{W_t, \lambda(t, \cdot)}(F) 1_{(t < T_\xi \wedge T_a)} + F 1_{(t > T_\xi \wedge T_a)},
\]

and to complete the proof of Proposition 2 we will show that

\[
\mathbb{E}^b[F|\mathcal{L}] = G(b) \quad \text{a.s.,}
\]

for all \( b \in (a, c) \), where \( G = G(b) \) is given by

\[
G = \exp \left\{ \frac{\kappa^2}{2} \int_a^b \frac{dy}{l(T_0, y)} - \frac{\kappa(\kappa - 2)}{2} \int_b^c \frac{dy}{l(T_0, y)} \right\}.
\]

As a first step in establishing (15), let us verify the integrated version \( \mathbb{E}^b[F] = \mathbb{E}^b[G] \). According to the Ray-Knight theorems the local time process \( (l(T_0, y); y \geq 0) \) is distributed, under \( \mathbb{P}^b \), as a inhomogeneous diffusion: starting from zero it behaves as a squared Bessel process of dimension 2 for
Finally the $\delta$ give us $I$

Proof. Distributed as the value at time $t$

The joint law of $Z$ if $l(0, a) = u, l(0, b) = v$ and $l(0, c) = w$, we have $(l(0, y); a \leq y \leq b)$ and $(l(0, y); b \leq y \leq c)$ are independent and distributed as squared Bessel bridges of dimension 2 and dimension 0 respectively.

For any dimension $d \geq 0$ we will denote the density of the transition density of the $\delta$-dimensional squared Bessel process by $q^\delta_t(x, y)$, for $x, y, t > 0$. Note that if $\delta = 0$ then the transition kernel also has an atom at zero of size $q^0_t(x, 0)$. It is an outcome of the absolute continuity relations between the squared Bessel processes that:

$$
E \left[ \exp \left\{ -\kappa^2/2 \int_0^t ds/Z_s \right\} \mid Z_0 = u, Z_t = v \right] = \left( \frac{u}{v} \right)^{\kappa/2} \frac{q^{2\kappa+2}_t(u, v)}{q^\delta_t(u, v)}
$$

if $Z$ is a squared Bessel with dimension 2; while

$$
E \left[ \exp \left\{ -\kappa(\kappa - 2)/2 \int_0^t ds/Z_s \right\} \mid Z_0 = v, Z_t = w \right] = \left( \frac{u}{w} \right)^{\kappa/2} \frac{q^{2\kappa}(v, w)}{q^\delta_t(v, w)}
$$

if $Z$ has dimension 0. Combining these results gives

$$
E^b \left[ \exp \left\{ -\kappa^2/2 \int_a^b dy/l(T_0, y) - \kappa(\kappa - 2)/2 \int_b^c dy/l(T_0, y) \right\} \mid l(T_0, a) = u, l(T_0, b) = v, l(T_0, c) = w \right] = \left( \frac{u}{w} \right)^{\kappa/2} \frac{q^{2\kappa+2}_t(u, v)q^\delta_{b-c}(v, w)}{q^\delta_{b-a}(u, v)q^\delta_{c-b}(v, w)}. \quad (17)
$$

The joint law of $l(T_0, a), l(T_0, b)$ and $l(T_0, c)$ (restricted to $w > 0$) is

$$
q^\delta_{b-a}(0, u)q^\delta_{b-c}(u, v)q^0_{c-b}(v, w) \, du \, dv \, dw.
$$

So integrating out, noting that there is no contribution to the expectation from the event $\{l(T_0, c) = 0\}$,

$$
E^b \left[ \exp \left\{ -\kappa^2/2 \int_a^b dy/l(T_0, y) - \kappa(\kappa - 2)/2 \int_b^c dy/l(T_0, y) \right\} \right] = \int_0^\infty du \int_0^\infty dv \int_0^\infty dw \left( \frac{u}{w} \right)^{\kappa/2} q^\delta_{b-a}(0, u)q^{2\kappa+2}_{b-c}(u, v)q^{2\kappa}_c(v, w). \quad (18)
$$

Lemma 3. For any $u, w, s, t > 0$, and any $\delta \geq 0$,

$$
\int_0^\infty dv \, q^\delta_{s+2}(u, v)q^\delta_t(v, w) = \frac{s}{s+t} q^{\delta+2}_{s+t}(u, w) + \frac{t}{s+t} q^\delta_{s+t}(u, w).
$$

Proof. The identity can be rewritten as an identity-in-law. Let $Z^\delta_t(u)$ denote a random variable distributed as the value at time $t$ of a $\delta$-dimensional squared Bessel process starting from $u$. Then the claim is that

$$
Z^\delta_t(Z^\delta_{s+2}(u)) \overset{law}{=} 1_{(t=0)}Z^\delta_{s+t}(u) + 1_{(t=1)}Z^\delta_{s+t}(u)
$$

where $I$ is a suitable Bernoulli random variable independent of other variables. Assume this holds for $\delta = 0$, then several applications of the additive property,

$$
Z^\delta_{i+t}(x + y) \overset{law}{=} Z^\delta_i(x) + Z^\delta_t(y)
$$

give us

$$
Z^\delta_t(Z^\delta_{s+2}(u)) \overset{law}{=} Z^\delta_t(Z^\delta_s(u) + Z^\delta_0(0)) \overset{law}{=} Z^\delta_t(Z^\delta_s(u)) + Z^\delta_t(0) \overset{law}{=} 1_{(t=0)}Z^\delta_{s+t}(u) + 1_{(t=1)}Z^\delta_{s+t}(u).
$$

Finally the $\delta = 0$ case is easily explained by combining the Ray-Knight theorems with hitting probabilities for Brownian motion. □
Applying this lemma to the expression at \( \text{18} \) gives
\[
\mathbb{E}^b[G] = \frac{b-a}{c-a} \mathbb{E}^c[G_+] + \frac{c-b}{c-a} \mathbb{E}^a[G_-],
\]  
(19)
where
\[
G_+ = \exp \left\{ -\frac{\kappa^2}{2} \int_a^c \frac{dy}{l(T_0, y)} \right\},
\]
(20)
\[
G_- = \exp \left\{ -\frac{\kappa(\kappa - 2)}{2} \int_a^c \frac{dy}{l(T_0, y)} \right\}.
\]
(21)

On the other hand, applying the strong Markov property of Brownian motion at \( T_a \land T_c \) and using \( \mathbb{P}^b(T_a < T_c) = c - b/c - a \) we obtain
\[
\mathbb{E}^b[F] = \frac{b-a}{c-a} \mathbb{E}^c[G_+] + \frac{c-b}{c-a} \mathbb{E}^a[G_-].
\]
(22)

Thus, comparing this with the preceding \( \text{19} \), we have obtained
\[
\mathbb{E}^b[F] = \mathbb{E}^b[G] \quad \text{for all } a \leq b \leq c.
\]
(23)

Next we introduce a family of transformations. Suppose that \( h : [0, \infty) \rightarrow [0, \infty) \) is increasing with strictly positive derivative, and satisfies \( h(0) = 0, h(\infty) = \infty \). If \( W \) is path beginning at \( W_0 > 0 \) let \( T_h W = W_h \) denote the transformed path satisfying
\[
h(W_u^h) = W_{H_u}, \quad \text{for } H_u < T_0,
\]
(24)
and \( W_u^h = W_{H_u} \) for \( H_u \geq T_0 \), where \( u \mapsto H_u \) is the inverse of the increasing process
\[
t \mapsto \int_0^{t \land T_0} \frac{ds}{(h' \circ h^{-1}(W_s))^2} + (t - T_0)^+.
\]
Let \( \nu \) be a positive, finite measure supported on \((0, \infty)\). The Sturm-Liouville equation
\[
\phi'' = \phi\nu
\]
admits a unique, strictly positive, decreasing solution \( \phi(0) = 1 \). We shall denote this solution by \( \Phi_\nu \). It has the following probabilistic characterization:
\[
\Phi_\nu(x) = \mathbb{E}^x[E_\nu(l(T_0, \cdot))] \quad \text{for } x \geq 0.
\]
(25)
Define the probability measure \( \mathbb{P}^{b, \nu} \) via
\[
\mathbb{P}^{b, \nu} = \frac{1}{\Phi_\nu(b)} \mathcal{E}_\nu(l(T_0, \cdot)) \cdot \mathbb{P}^b.
\]

**Lemma 4.** Let \( \nu \) be a positive, finite measure supported on \((0, \infty)\). Take \( h \) to be given by
\[
h(y) = \int_0^y \frac{dx}{\Phi_\nu(x)^2}.
\]
Then if \( W \) has law \( \mathbb{P}^{h(b)} \), the transformed process \( T_h W \) has law \( \mathbb{P}^{b, \nu} \).

**Proof.** Under \( \mathbb{P}^b \) the density
\[
\frac{1}{\Phi_\nu(b)} \exp \left\{ -\frac{1}{2} \int_0^\infty \nu(dy) l(t \land T_0, y) \right\}
\]
is the terminal value of the exponential martingale
\[
\frac{\Phi_\nu(W_{t \land T_0})}{\Phi_\nu(b)} \exp \left\{ -\frac{1}{2} \int_0^\infty \nu(dy) l(t \land T_0, y) \right\} =
\exp \left\{ \ln \Phi_\nu(W_{t \land T_0}) - \ln \Phi_\nu(W_0) - \frac{1}{2} \int_0^\infty \nu(dy) l(t \land T_0, y) \right\} =
\exp \left\{ \frac{1}{2} \int_0^{t \land T_0} \frac{\Phi_\nu'(W_s)}{\Phi_\nu(W_s)} dW_s - \frac{1}{2} \int_0^{t \land T_0} \Phi_\nu(W_s) \frac{\Phi_\nu'(W_s)^2}{\Phi_\nu(W_s)} ds \right\}.
\]
Whence, by Girsanov’s formula, under $\mathbb{P}^{h,\nu}$,

$$W_t - \int_0^{t\wedge T_0} \frac{\Phi'_\nu(W_s)}{\Phi_\nu(W_s)} \, ds$$

is a martingale. Now we just need to notice that the scale function corresponding to this drift $b(\cdot) = \Phi'_\nu(\cdot)/\Phi_\nu(\cdot)$ is proportional to

$$\int_0^y dx \exp \left\{ - \int_0^x 2b(z) \, dz \right\} = \int_0^y \frac{dx}{\Phi_\nu(x)^2}.$$

Standard scale/speed measure arguments complete the proof. \(\square\)

**Lemma 5.** If the occupation measure of the path $W$ admits local times $l(t, y)$ then the transformed path $T^h W$ admits as local times $l^h(\cdot, \cdot)$ given by

$$l^h(t, y) = \frac{1}{h'(y)} l(H_t, h(y)), \quad \text{for } 0 \leq H_t \leq T_0.$$

In particular $T^h_0$, the time at which the path $W^h$ first reaches zero, satisfies $H_{T^h_0} = T_0$ and

$$l^h(T^h_0, y) = \frac{1}{h'(y)} l(T_0, h(y)).$$

**Proof.** For any test function $f$, and $t$ satisfying $H_t \leq T_0$,

$$\int_t^T f(W^h_s) \, ds = \int_t^T f \circ h^{-1}(W^h_s) \, ds = \int_0^{H_t} f \circ h^{-1}(W_s) \, ds \left( \frac{h'(W_s)}{h'(W_s)} \right)^2 ds = \int_0^{\infty} \frac{f \circ h^{-1}(x)}{(h'(h^{-1}(x)))^2} l(H_t, x) \, dx = \int_0^{\infty} \frac{f(y)}{h'(y)} l(H_t, h(y)) \, dy.$$

Treating $T_h$ as an almost everywhere defined application from $\Omega$ to $\Omega$, we have with obvious notation,

$$F^h = F \circ T_h = \begin{cases} 1(T^h_c < T^h_a) \exp \left\{ \frac{-\kappa^2}{2} \int_a^c \frac{dy}{\lambda^h(T^h_c, y)} \right\} + 1(T^h_a < T^h_c) \exp \left\{ -\frac{\kappa(\kappa - 2)}{2} \int_a^c \frac{dy}{\lambda^h(T^h_c, y)} \right\} = \\
1((T^h_a < T^h(a)) \exp \left\{ \frac{-\kappa^2}{2} \int_a^{h(b)} \frac{dy}{\lambda^h(T^h_c, y)} \right\} + 1(T^h(a) < T^h(c)) \exp \left\{ -\frac{\kappa(\kappa - 2)}{2} \int_a^{h(b)} \frac{dy}{\lambda^h(T^h_c, y)} \right\} \right. \end{cases}.$$ (26)

Similarly we have

$$G^h = G \circ T_h = \exp \left\{ \frac{-\kappa^2}{2} \int_a^{h(b)} \frac{dy}{l^h(T^h_0, y)} - \frac{\kappa(\kappa - 2)}{2} \int_a^{h(b)} \frac{dy}{l^h(T^h_0, y)} \right\} = \exp \left\{ -\frac{\kappa^2}{2} \int_a^{h(b)} \frac{dy}{l(T^h_0, y)} - \frac{\kappa(\kappa - 2)}{2} \int_a^{h(b)} \frac{dy}{l(T^h_0, y)} \right\}. \quad (27)$$

Since the arguments leading to (26) hold equally if we replace throughout $a$ by $h(a)$, $b$ by $h(b)$ and $c$ by $h(c)$, we deduce that

$$\mathbb{E}^{h(b)}[F^h] = \mathbb{E}^{h(b)}[G^h]. \quad (28)$$

Now suppose that $h$ and $\nu$ are associated as in Lemma 4. Then we have, by virtue of (28),

$$\frac{1}{\Phi(b)}\mathbb{E}^{b,\nu}[FE_{\nu}(l(T_0, \cdot))] = \mathbb{E}^{b,\nu}[F] = \mathbb{E}^{h(b)}[F^h] = \mathbb{E}^{h(b)}[G^h] = \mathbb{E}^{b,\nu}[G] = \frac{1}{\Phi(b)}\mathbb{E}^{b}[G \mathcal{E}_{\nu}(l(T_0, \cdot))]. \quad (29)$$

The measure $\nu$ being arbitrary this proves that $\mathbb{E}^h[F \mid \mathcal{L}] = G$ and the proof of Theorem 4 is complete.
3 Ray-Knight theorems

Suppose the Bass-Burdzy flow $X$ is constructed from a Brownian motion $W$ as in Theorem 1. Then

$$DX_{X_t}(\cdot) = \frac{\phi_t \circ \phi_0^{-1}(\cdot)}{\phi_t \circ \phi_0^{-1}(\cdot)} = \frac{\lambda(0, \phi_0^{-1}(\cdot))}{\lambda(t, \phi_0^{-1}(\cdot))}. \quad (30)$$

This allows us to convert statements concerning the distribution of the derivative of the flow into statements concerning the local times of $W$ and vice-versa. In this section we illustrate this by giving a proof of one of the Ray-Knight theorems for the flow by means of verifying the corresponding statement regarding Brownian local times.

The following result is part of Theorem 1.1 of [4], together with equation (1.18) there. Notice that, for each $\xi \in \mathbb{R}$, the process $t \mapsto DX_t(\xi)$ is constant except when $t \mapsto X_t(\xi)$ visits zero. In the case $\beta_2 = 1$ and $\beta_1 = 0$ then $t \mapsto X_t(\xi)$ is transient in the sense $X_t(\xi) \to \infty$ as $t \to \infty$ and so $t \mapsto DX_t(\xi)$ is eventually constant. It is thus meaningful to consider the limit $DX_\infty(\xi)$.

**Theorem 4.** Suppose that $(X_t; t \geq 0)$ is the Bass-Burdzy flow with $\beta_1 = 0$ and $\beta_2 = 1$. Then $1/DX_\infty(0)$ is uniformly distributed on $[0, 1]$. Conditionally on $\{1/DX_\infty(0) = y\}$ the processes $(Y^+_x; x \geq 0)$ and $(Y^-_x; x \geq 0)$ given by

$$Y^+_x = \frac{1}{DX_\infty(x)} \quad \text{and} \quad Y^-_x = \frac{1}{DX_\infty(-x)},$$

are independent diffusions on $[0, 1]$, starting from $y$, with infinitesimal generators

$$2y(1-y)\frac{d^2}{dy^2} + 2(1-y)\frac{d}{dy} \quad \text{and} \quad 2y(1-y)\frac{d^2}{dy^2} + (2(1-y) - 2y)\frac{d}{dy},$$

respectively.

It should be noted that versions of this result hold for other values of $\beta_1$ and $\beta_2$, but by a combination of scaling and Girsanov transformations they may be deduced from the special case stated here.

The diffusions appearing in the above description of $DX_\infty$ are sometimes called Jacobi diffusions. A Jacobi diffusion with dimensions $d_1$ and $d_2$ has generator

$$2y(1-y)\frac{d^2}{dy^2} + (d_1(1-y) - d_2y)\frac{d}{dy}$$

The behaviour at the boundaries 0 and 1 is determined from the corresponding dimension, $d_1$ for the boundary at 0 and $d_2$ for the boundary at 1, according to the following rule. If $d = 0$ the boundary is absorbing, if $0 < d < 2$ then the boundary is instantaneously reflecting, while if $d \geq 2$ then the boundary point is an inaccessible entrance point.

Next we give a description of Brownian local times corresponding to Theorem 4. Recall that the Brownian motion $W$ starts from $W_0 = \xi > 0$, that $M = \sup\{W_t; t \in [0, T_0]\}$, and $T_M$ is the almost surely unique time such that $W_{T_M} = M$. Now as was noted earlier $A_{T_M} = \infty$. So if we take $t = T_M$ in equation (30) we obtain

$$Y^-\left(\int_{\xi}^{x} \frac{dz}{l(T_0, z)}\right) = \frac{l(T_0, x) - l(T_M, x)}{l(T_0, x)} \quad \text{for} \quad \xi \leq x \leq M, \quad (31)$$

$$Y^+\left(\int_{x}^{\xi} \frac{dz}{l(T_0, z)}\right) = \frac{l(T_0, x) - l(T_M, x)}{l(T_0, x)} \quad \text{for} \quad 0 < x \leq \xi, \quad (32)$$

where $Y^-$ and $Y^+$ are defined as in Theorem 4. Thus Theorem 4 is equivalent to the following proposition.

**Proposition 5.** Suppose that processes $Y^-$ and $Y^+$ are determined from the local times of a Brownian motion via equations (31) and (32). Then $Y^-_0 = Y^+_0$ is uniformly distributed on $[0, 1]$ and conditionally on $Y^-_0 = Y^+_0 = y$ the processes $Y^-$ and $Y^+$ are independent Jacobi diffusions with dimensions $d^+_1 = d^-_2 = 2$ and $d^+_2 = 2, d^-_1 = 0$ respectively.
The principal tool used in proving this proposition is the skew-product involving Jacobi processes and squared Bessel processes, see [7]. Suppose that \((Z(t); t \geq 0)\) and \((Z'(t); t \geq 0)\) are independent squared Bessel processes with dimensions \(d\) and \(d'\) respectively and that \(d + d' > 0\). Let \(Z^{+}(t) = Z(t) + Z'(t)\) and suppose \(Z^{+}(0) > 0\). Define \(Y\) via,

\[
Y \left( \int_0^t ds/Z^{+}(s) \right) = \frac{Z(t)}{Z^{+}(t)}. \tag{33}
\]

Then \(Y\) is a Jacobi process with dimensions \(d\) and \(d'\) independent of \(Z^{+}\) which is a squared Bessel process of dimension \(d + d'\). In fact, as was remarked in [7], the skew product holds also in the case that \(Z\) and \(Z'\) are replaced by \(Z\) and \(Z'\) which are obtained from \(Z\) and \(Z'\) by

\[
\tilde{Z}_t = \frac{1}{u'(t)} Z_{u(t)} \quad \text{and} \quad \tilde{Z}'_t = \frac{1}{u'(t)} Z'_{u(t)} \tag{34}
\]

where \(u\) is a strictly increasing, continuously differentiable function satisfying \(u(0) = 0\). We will make use of this in the case \(u(t) = t/(h - t)\), in which case \(Z\) and \(Z'\) are independent squared Bessel bridges leading to \(0\) over the interval \([0, h]\).

We may now proceed with the proof of Proposition 5. We decompose the path \((W_t; 0 \leq t \leq T_0)\) about its maximum: let

\[
W_t^{(1)} = W_t \quad \text{for} \quad 0 \leq t \leq T_M \tag{35}
\]

\[
W_t^{(2)} = W_{T_0 - t} \quad \text{for} \quad 0 \leq t \leq T_0 - T_M. \tag{36}
\]

Then conditionally on \(M = m \in (\xi, \infty)\) the path segments \((W_t^{(1)}; 0 \leq t \leq T_M)\) and \((W_t^{(2)}; 0 \leq t \leq T_0 - T_M)\) are independent Bessel processes of dimension three run until first hitting the level \(m\), started from \(\xi\) and \(0\) respectively. Using this decomposition we obtain the description of the local times of \(W\) in the following lemma, and then Proposition 5 and hence Theorem 4 follow by virtue of the skew product.

We use the notation \(Q_{d,h}^{x,y}\) to denote the law of a squared Bessel bridge of dimension \(d\) leading from \(x\) to \(y\) over the interval \([0, h]\).

**Lemma 6.** Conditionally on \(M = m\), the local time processes \(l(T_M, \cdot)\) and \(l(T_0, \cdot) - l(T_M, \cdot)\) are distributed as follows.

\((l(T_M, x), 0 \leq x \leq m)\) and \((l(T_0, x) - l(T_M, x), 0 \leq x \leq m)\) are independent.

\(l(T_M, \xi)\) and \(l(T_0, \xi) - l(T_M, \xi)\) each have the exponential distribution with mean \(2\xi(m - \xi)/m\).

Conditionally on \(l(T_M, \xi) = z\) the processes \((l(T_M, \xi - x), 0 \leq x \leq \xi)\) and \((l(T_M, \xi + x), 0 \leq x \leq m - \xi)\) are independent and are distributed as \(Q_{z,0}^{0,\xi}\) and \(Q_{z,0}^{2,m-\xi}\) respectively.

Conditionally on \(l(T_0, \xi) - l(T_M, \xi) = z\) the processes \((l(T_0, \xi - x) - l(T_M, \xi - x), 0 \leq x \leq \xi)\) and \((l(T_0, \xi + x) - l(T_M, \xi + x), 0 \leq x \leq m - \xi)\) are independent and are distributed as \(Q_{z,0}^{2,\xi}\) and \(Q_{z,0}^{2,m-\xi}\) respectively.

**Proof.** \((l(T_M, x), 0 \leq x \leq m)\) are the local times of the path segment \(W^{(1)}\) while \((l(T_0, x) - l(T_M, x), 0 \leq x \leq m)\) are the local times of \(W^{(2)}\). Thus their independence follows from that of \(W^{(1)}\) and \(W^{(2)}\).

Next it is well known (see [8] for instance) that the local times of a Bessel three process starting from zero taken at its hitting time of a level \(m\) are distributed as \(Q_{0,0}^{2,m}\). So the distribution of \(l(T_0, \xi) - l(T_M, \xi)\) is obtained as that of this bridge at time \(\xi\):

\[
\frac{q^2_{\xi}(0, z) q_{m-\xi}^2(z, 0)}{q^2_{m}(0, 0)} = \frac{m}{2\xi(m - \xi)} \exp \left\{ -\frac{zm}{2\xi(m - \xi)} \right\},
\]

where \(q^2_{\xi}(x, y)\) are the transition densities of a squared Bessel process of dimension \(d\). Moreover, by standard Markovian properties of bridges, a process distributed as \(Q_{0,0}^{2,m}\) can be constructed by conditioning on its value at time \(\xi\) being \(z\), and placing “back to back” two independent bridges.
distributed as $\mathbb{Q}_{z,0}^{2,m-\xi}$ and $\mathbb{Q}_{z,0}^{2,\xi}$. This proves that the distribution of $(l(T_0, x) - l(T_M, x); 0 \leq x \leq m)$ is as asserted.

Turn now to the process $(l(T_M, m - x); 0 \leq x \leq m)$ which is distributed of the local times of a Bessel three process starting from $\xi$ and taken at its hitting time of a level $m$. This is obtained by conditioning the inhomogeneous diffusion which starts from 0, evolves a squared Bessel process of dimension two until time $m - \xi$, and then evolves as a squared Bessel process of dimension zero, to have been absorbed at zero by time $m$. To see this, just note that according to the Ray-Knight theorems, Brownian motion started from $\xi$ and taken at its hitting time of $m$ has local times distributed as the unconditioned diffusion, and that the conditioning corresponds exactly to conditioning the path of the Brownian motion not to have reached zero before hitting level $m$. Thus the distribution of $l(T_M, \xi)$ is given by

$$\frac{m}{\xi} q_{m - \xi}^2(0, z) q_{m}^0(z, 0) = \frac{m}{2\xi(m - \xi)} \exp\left\{-\frac{zm}{2\xi(m - \xi)}\right\}.$$  

Finally, once again, by standard Markovian arguments we observe that this conditioned, inhomogeneous diffusion, can be constructed by conditioning on its value at time $\xi$ being $z$ and placing “back to back” two independent bridges distributed as $\mathbb{Q}_{z,0}^{0,\xi}$ and $\mathbb{Q}_{z,0}^{2,m-\xi}$.

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**References**

[1] D. Aldous, *Brownian excursion conditioned on its local time*. Elect. Comm. in Probab. 3, 79-90, (1998).

[2] Bass, R.F. and Burdzy, K., *Stochastic bifurcation models*, Annals of Probability, 27, 50-108, (1999).

[3] C. Donati-Martin and M. Yor, *Some Brownian functionals and their laws*. Annals of Probability, 25:3, 1011-1058, (1997).

[4] Hu, Y. and Warren, J., *Ray-Knight theorems for a stochastic flow*, Stochastic processes and its applications, 86, 287-305, (2000).

[5] Jacod, J., *Grossissement initial, hypothèse (H’), et théorème de Girsanov*. In Grossissements de filtrations: exemples et applications, Lecture notes in Mathematics, 1118. Springer, (1985).

[6] D.Revuz and M.Yor, *Continuous martingales and Brownian motion*, Springer, (1998).

[7] Warren, J. and Yor, M., *The Brownian burglar: conditioning Brownian motion by its local time process*, Seminaire de Probabilites 32, 328-342, Lecture notes in Mathematics, 1709, Springer, (1998).