Multiple-scales analysis of cosmological perturbations in brane-worlds

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We present a new approximation method for solving the equations of motion for cosmological tensor perturbations in a Randall–Sundrum brane-world model of the type with one brane in a five-dimensional anti-de Sitter spacetime. This method avoids the problem of coordinate singularities inherent in some methods. At leading order, the zero-mode solution replicates the evolution of perturbations in a four-dimensional Friedmann–Robertson–Walker universe in the absence of any tensor component to the matter perturbation on the brane. At next order, there is a mode-mixing effect, although, importantly, the zero-mode does not source any other modes.

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I. INTRODUCTION

The concept of brane-worlds is not a very new one\textsuperscript{1}, but has risen in popularity in the last five years. Central to this high level of interest has been the observation that the experimental bounds on the inverse square behaviour of the gravitational force are surprisingly large, with the current bound being about $10^{-4}$ m\textsuperscript{2}; by contrast, the equivalent bound on the Coulomb law is about $10^{-18}$ m. In a seminal paper, Arkani-Hamed, Dimopoulos and Dvali\textsuperscript{3} observed that, if the Standard Model interactions were confined to a four-dimensional subspace, the length-scale of extra dimensions could be made as large as these bounds on Newton’s law. A further revelation came from the work of Randall and Sundrum\textsuperscript{4, 5}, who demonstrated that the extra dimensions could be made infinite in extent if one removed the assumption of a product geometry.

We will focus on the single-brane Randall–Sundrum (RS) model\textsuperscript{5}. The cosmological background solution for this model was found shortly after the model was suggested\textsuperscript{6}; however, the question of how perturbations evolve in such a background is still unresolved. Some progress has been made, for example, it is possible to analyse the evolution of perturbations perturbations during a primordial de Sitter phase\textsuperscript{7, 8} of the brane-world assuming an initial vacuum state. Our interest here is in the a more general cosmological background dominated by either radiation or matter.

Several approaches have been proposed to study this problem. One is to try to construct an effective theory in four dimensions and study perturbations in that. This is particularly useful for brane-world models with a finitely sized extra dimension since it is then not possible to excite Kaluza–Klein states below a certain energy; however, it is less well motivated in the case of the single-brane model. In this method one ignores most of the higher-dimensional effects by reducing the problem to a scalar-tensor theory, so even where it is an accurate approximation it can never offer a “smoking gun” for brane-world models by predicting a uniquely higher-dimensional effect. Other approaches have been to try to solve the full five-dimensional problem, either numerically\textsuperscript{9} or analytically, using some approximation scheme. Typically, these methods fall into two categories: brane-based and bulk. The brane-based methods use a Gaussian normal (GN) coordinate system, constructed from a congruence of geodesics normal to the brane, where the brane is chosen to lie at a fixed value of one of the coordinates. These GN coordinates make the boundary conditions at the brane simple at the expense of rendering the equation of motion for the perturbations complicated. The GN coordinates also suffer from a coordinate horizon, meaning that they do not cover the whole spacetime. In the case of a brane with the induced metric of de Sitter space, the coordinate singularity coincides with the horizon, and, as mentioned above, it is possible to solve the problem\textsuperscript{10, 11}.

In the bulk-based approach one typically chooses a coordinate system that makes manifest the symmetries of the bulk, resulting in a simple equation of motion for the perturbations; the penalty for this is that the brane is not at a fixed value of one of the coordinates and, consequently, applying the boundary condition is complicated. In the special case of a brane with a Minkowski induced metric, the brane is at a fixed value of one of the coordinates and the perturbation equations can be solved; indeed, this was done by Randall and Sundrum did in their original paper\textsuperscript{12}. One major advantage of the bulk-based approach is that it does not have any coordinate singularities. The relation between the GN and bulk coordinates is presented in Appendix A.

Several methods for solving this problem have been suggested in the literature, all of which have some limitations. One possibility is to make an educated guess about the behaviour of the bulk effects\textsuperscript{13} and treat these effects as a source term in the usual equations for the evolution of perturbations in four dimensions. The obvious drawback is...
that the guess might not be educated enough; furthermore, the perturbations on the brane can source the bulk effects, meaning that a simple ansatz is unlikely to give a realistic answer. To go beyond this, many authors have suggested methods of approximating the problem, for example, the near-brane approximation or the gradient expansion method. The main limitation with both of these approaches is that they rely on the GN coordinates, which have a coordinate horizon.

In this article, we present a new approximate method for solving the evolution equations for tensor perturbations. There are two stages to this: a change of coordinates to a hybrid system — a bulk-based system modified to make the boundary condition tractable within a well-motivated approximation scheme — and an adaptation of the method of multiple scales to find approximate solutions to the equations of motion. Intuitively, this approximation works by assuming that the boundary condition evolves slowly compared to the time-scale of the solutions. We will comment at the end on the consequences of the solutions we obtain and on how we hope to extend this method to work for scalar perturbations.

II. THE EQUATIONS OF MOTION

We will start from the Poincaré coordinates, as advocated in Ref. [12], where the metric is manifestly conformally related to Minkowski space. The line element has the form

$$ds^2 = \frac{l^2}{Z^2} \eta_{\alpha \beta} dx^\alpha dx^\beta = \frac{l^2}{Z^2} \left( -dT^2 + \delta_{ij} dx^i dx^j + dZ^2 \right),$$

where $l$ is the AdS length-scale. We have assumed spatial flatness of the background cosmology here. This coordinate system makes the five-dimensional linearized Einstein equations simple, the price being that the brane has locus given by $Z = l/a$, rather than being at a fixed value of one of the coordinates, making the boundary condition much more difficult to impose. If we write the perturbed line element as

$$ds^2 = \frac{l^2}{Z^2} \left( \eta_{\alpha \beta} + h_{\alpha \beta} \right) dx^\alpha dx^\beta,$$

and choose the transverse-traceless (TT) gauge, following Ref. [15], where $h_{\alpha \beta}$ satisfies $h_{\alpha 2} = 0$, $\eta^{\alpha \beta} h_{\alpha \beta} = 0$ and $\nabla_\alpha h^{\alpha \beta} = \ddot{g}^{\gamma \mu} \ddot{g}^{\nu \beta} \nabla_\mu h^\nu = 0$, with $\ddot{g}$ and $\nabla$ representing, respectively, the projected metric and derivative on the brane. This setup has the advantage that the resulting perturbed Einstein equations exactly soluble. After making a Fourier transform $x^i \to k^i$ in the spatial directions parallel to the brane, the equation is simply

$$\frac{\partial^2 h}{\partial T^2} - \frac{\partial^2 h}{\partial Z^2} + \frac{3 \partial h}{\partial Z} + k^2 h = 0,$$

where the indices on $h$ have been dropped for convenience. This equation can be solved by separation of variables to give mode functions $h_m = \exp \left( \pm i \sqrt{k^2 + m^2 T} \right) Z^2 B_2 (mZ)$, where $B_2$ is a Bessel function of order 2. There will be five different polarizations of the graviton in five-dimensions. However, in this gauge the position of the brane is not on the same locus as for the background but is displaced, an effect which has been dubbed “brane-bending” in the literature. This can be written as $\Delta n^\mu$, that is, a displacement of $\Delta$ from the background brane position along the direction of the unit normal $n$, which is given by

$$n^\mu = \frac{1}{a} \left( \varepsilon, 0, 0, -\sqrt{1 + \varepsilon^2} \right), \quad \varepsilon = \frac{l}{a^2} \frac{da}{dt} = \frac{l}{a} \frac{dt}{dr} = lH,$$

where $a$ is the scale factor on the brane, and $\tau$ and $t$ are, respectively, conformal and proper time on the brane. Combined with the five polarizations of the gravitational perturbations in the bulk, this gives six degrees of freedom, corresponding to the scalar, vector, and tensor degrees of freedom in four-dimensional cosmology. Since the perturbation of the position of the brane is a scalar degree of freedom, we can ignore this effect when studying tensor perturbations to linear order. This approach should also apply to the bulk perturbations which give rise to vector modes on the brane, although we will not discuss these here.

The boundary condition at the brane is given by the Israel junction condition [16], relating the jump in the normal derivative of the metric to the matter supported on the brane. (With reflection symmetry imposed between the two sides of the brane, the jump is equal to twice the value on one side.) For tensor perturbations this is simply

$$(n.\nabla) h_{ij}^T \big|_{\text{brane}} = -\kappa \Sigma_{ij}^T,$$
where the r.h.s. is the tensor part of the perturbation of the brane stress-energy-momentum tensor. For most of the history of the universe, there is no matter source for the tensor modes so the r.h.s. of (5) will reduce to zero. Only when the CMB photons develop a quadrupole moment in the late universe is this assumption no longer valid.

In summary, the problem we must solve for the evolution of the tensor perturbations is

\[
\frac{\partial^2 h}{\partial T^2} - \frac{\partial^2 h}{\partial Z^2} + 3 \frac{\partial h}{\partial Z} + k^2 h = 0, \quad \text{subject to} \quad \sqrt{1 + \varepsilon^2} \frac{\partial h}{\partial Z} \bigg|_{Z=t/a} = \varepsilon \frac{\partial h}{\partial T} \bigg|_{Z=t/a},
\]

which, although seemingly innocuous, is extremely difficult to do exactly. We will try to find approximate solutions based on the observation that there are two independent dimensionless parameters in the problem: \( \varepsilon = lH \) and \( p = kl \), which are, respectively, the Hubble parameter and the frequency of the perturbation measured in terms of the AdS length-scale. The presence of two scales means that the solutions should have time dependence on two different scales—much like an amplitude- or frequency-modulated radio broadcast, which is a signal of a high frequency with its amplitude or frequency changing on a much longer time-scale. The method of multiple scales [13, 14] is specifically designed to find approximate solutions in precisely these situations.

III. NEW COORDINATE SYSTEM

We will be constructing an approximation scheme valid when \( \varepsilon = lH \) is small. If one naively applies the method of multiple scales as described in section IV, one obtains an incorrect answer. This is because, to get the correct evolution at zeroth order, it is necessary for the boundary condition to have no first-order term. In order to make the method work, we devise a new set of coordinates designed to remove the first-order term in the boundary condition.

From the change of coordinates in the appendix, we have that, on the brane,

\[
d\tau = \sqrt{1 + \varepsilon^2} dT,
\]

where \( \tau \) is conformal time for an observer on the brane moving with the Hubble flow. We can then evaluate

\[
d\alpha = \frac{d\tau}{dT} \frac{dt}{d\tau} = a^2 \varepsilon \sqrt{1 + \varepsilon^2}.
\]

If we also assume that the Friedmann law \( H^2 = \frac{8\pi G}{3} \frac{\rho}{3} \) holds to sufficient accuracy, that is, we are late enough in the history of the universe for the \( \rho^2 \) correction to be ignorable, then we can compute that

\[
\frac{dH}{dt} = -4\pi G(\rho + p) = -\frac{3}{2} \Gamma H^2,
\]

for a cosmological fluid with polytropic index \( \Gamma \) satisfying \( \rho + p = \Gamma \rho \). This gives

\[
\frac{d\varepsilon}{dT} = -\frac{3\Gamma}{2l} a \varepsilon^2 \sqrt{1 + \varepsilon^3}.
\]

If we make the change of coordinates

\[
\xi = \frac{Z}{l}, \quad \eta = \frac{T}{l} + \frac{a \varepsilon Z^2}{2l^2},
\]

the equation of motion becomes

\[
\frac{\partial^2 h}{\partial \eta^2} - \frac{\partial^2 h}{\partial \xi^2} + 3 \frac{\partial h}{\partial \xi} + p^2 h = 2\varepsilon a \left\{ \xi \frac{\partial^2 h}{\partial \xi^2} - \frac{\partial h}{\partial \xi} \right\} + \mathcal{O}(\varepsilon^2),
\]

where \( p = kl \) is the dimensionless frequency of the mode. The boundary condition is simply

\[
\frac{\partial h}{\partial \xi} \bigg|_{\xi=1/a} = \mathcal{O}(\varepsilon^3).
\]

Note that the brane is still at \( \xi = 1/a \) in this new coordinate system.
IV. MULTIPLE-SCALES ANALYSIS

For the purpose of multiple-scales analysis [13, 14], we introduce two new time variables: “fast” time $f$, and “slow” time $s$, given by

$$ f = \eta, \quad s = \frac{1}{a}. \tag{14} $$

Note that these are dimensionless (as is $\xi$). The variables $f$ and $s$ are both, of course, functions of $\eta$ so the definition is a technical construct to allow us simply to handle the fact that the boundary condition changes with time on a scale related to $s$. Let us list our various time coordinates for clarity:

- $T$ is the time coordinate in the static coordinates,
- $\eta$ is the time coordinate in the new coordinates,
- $\tau$ is conformal time on the brane,
- $t$ is proper time on the brane,
- $f$ is the “fast” time coordinate,
- $s$ is the “slow” time coordinate.

We assume that $\varepsilon$ is small, which is valid except in the very early universe. In the usual application of the method of multiple scale [13, 14] the small parameter $\varepsilon$ is a constant; here it is a function of $T$. We then write derivatives w.r.t. $\eta$ in terms of the fast and slow time derivatives as

$$ \frac{\partial}{\partial \eta} = \frac{\partial}{\partial f} - \varepsilon \frac{\partial}{\partial s} + \mathcal{O}(\varepsilon^2), \tag{15} $$

$$ \frac{\partial^2}{\partial \eta^2} = \frac{\partial^2}{\partial f^2} - 2\varepsilon \frac{\partial^2}{\partial f \partial s} + \mathcal{O}(\varepsilon^2). \tag{16} $$

Expressed in terms of these fast and slow variables, the equation of motion is

$$ \frac{\partial^2 h}{\partial f^2} - \frac{\partial^2 h}{\partial \xi^2} + 3 \frac{\partial h}{\partial \xi} + p^2 h = 2\varepsilon \left( \frac{\partial^2 h}{\partial \eta \partial s} - a \frac{\partial h}{\partial f} + a \xi \frac{\partial^2 h}{\partial \xi \partial f} \right) + \mathcal{O}(\varepsilon^2), \tag{17} $$

subject to the boundary condition $\mathcal{B}^3$ at $\xi = s$.

V. SOLUTIONS

We will use the separation of variables technique, where one tries to find a solution of the form

$$ h = \int \Psi_m(f,s,\varepsilon) \Phi_m(\xi, s, \varepsilon) \, dm. \tag{18} $$

We will write $\Psi_m(f,s,\varepsilon)$ and $\Phi_m(\xi, s, \varepsilon)$ as series in $\varepsilon$:

$$ \Psi_m(f,s,\varepsilon) \approx \Psi_m^{(0)}(f,s) + \varepsilon \Psi_m^{(1)}(\eta, s) + \ldots, \quad \Phi_m(\xi, s, \varepsilon) \approx \Phi_m^{(0)}(\xi, s) + \varepsilon \Phi_m^{(1)}(\xi, s) + \ldots, \tag{19} $$

which we will require to be asymptotic as $\varepsilon \to 0$, and we will require that the $\Phi_m(\xi, s)$ satisfy the boundary condition $\mathcal{B}^3$ individually.

A. Zeroth-order solution

We will construct the solution order by order in $\varepsilon$. The equation of motion to zeroth order is

$$ \frac{1}{\Psi^{(0)}} \frac{\partial^2 \Psi^{(0)}}{\partial f^2} + p^2 = \frac{1}{\Phi^{(0)}} \frac{\partial^2 \Phi^{(0)}}{\partial \xi^2} - \frac{3}{\xi} \frac{1}{\Phi^{(0)}} \frac{\partial \Phi}{\partial \xi}, \tag{20} $$

which is separable with solutions

$$ \Psi_m^{(0)}(f,s) = \sum_{\pm} A_m^{\pm}(s) \exp \left( \pm i \omega_m f \right), \quad \text{where} \quad \omega_m = \sqrt{p^2 + m^2}, \tag{21} $$

$$ \Phi_m^{(0)}(\xi, s) = \frac{\pi}{2} m \xi^2 \left( Y_1(ms) J_2(m\xi) - J_1(ms) Y_2(m\xi) \right), \tag{22} $$
with \( \Phi_m^{(0)} \) being chosen to satisfy the boundary condition \( \xi \) to zeroth order:

\[
\frac{\partial \Phi^{(0)}}{\partial \xi} \bigg|_{\xi= s} = 0. \tag{23}
\]

The zero-mode solution \( \Phi_0^{(0)}(\xi, s) = s \) is the limit of \( \Phi_0^{(0)} \) as \( m \to 0 \). Of course, one has the freedom to redistribute the \( s \) dependence between \( \Phi \) and \( \Psi \) so as to leave the product unchanged. We can use the relation derived from the Wronskian of Bessel functions, given in the Appendix B, to evaluate \( \Phi^0(\xi, s) \) on the brane as

\[
\Phi^0_m(s, s) = s = \frac{1}{a(\tau)}. \tag{24}
\]

Note that what would be constants, fixed by the initial conditions, in a standard solution of \( \Phi^0 \) are now functions of the slow time variable, \( s \), and it is these functions which encode the crucial evolution of the amplitude.

### B. First-order solution

The function \( A_m(s) \) in expression \( \Phi^0 \) is undetermined by the zeroth-order calculation; in order to calculated it, we must consider the first order equation and construct the so-called seularity condition which forces the solution to asymptotic. The order \( \varepsilon \) terms in the equation of motion \( \Phi^0 \) give us

\[
\int d m \Phi^0_m \left( \frac{\partial^2 \Psi^{(1)}_m}{\partial f^2} + \omega^2_m \Psi^{(1)}_m \right) - \int d m \Psi^{(0)}_m \left( \frac{\partial^2 \Phi^{(1)}_m}{\partial z^2} - \frac{3}{z} \frac{\partial \Phi^{(1)}_m}{\partial z} + m^2 \Phi^{(1)}_m \right) = 2 \int d m \left( \frac{\partial \Psi^{(0)}_m}{\partial f} \frac{\partial \Phi^{(1)}_m}{\partial s} + \frac{\partial \Psi^{(0)}_m}{\partial \xi} \frac{\partial \Phi^{(0)}_m}{\partial s} - \frac{1}{s} \frac{\partial \Psi^{(0)}_m}{\partial f} \frac{\partial \Phi^{(0)}_m}{\partial s} + \frac{s \partial \Phi^{(0)}_m}{\partial f} \frac{\partial \Phi^{(0)}_m}{\partial \xi} \right), \tag{25}
\]

and the boundary condition to order \( \varepsilon \) is the same as at zeroth order

\[
\frac{\partial \Phi^{(1)}_m}{\partial \xi} \bigg|_{\xi= s} = 0. \tag{26}
\]

We still have some freedom in constructing the solution, which we shall use to impose

\[
\frac{\partial^2 \Phi^{(1)}_m}{\partial \xi^2} - \frac{3}{\xi} \frac{\partial \Phi^{(1)}_m}{\partial \xi} + m^2 \Phi^{(1)}_m = 0, \tag{27}
\]

so that \( \Phi^{(1)}_m \) satisfies the same equation as \( \Phi^{(0)}_m \). Given that the boundary condition for \( \Phi^{(1)}_m \) is also the same as \( \Phi^{(0)}_m \), it is actually be possible to impose the stronger condition \( \Phi^{(1)}_m \equiv 0 \) without any loss of generality. We can then use the orthogonality relation,

\[
\int_s^\infty \frac{d \xi}{\xi^3} \Phi^{(0)}_m \Phi^{(0)}_n = \frac{\pi^2}{4} m \left( (J_1(ms))^2 + (Y_1(ms))^2 \right) \delta(m - n), \tag{28}
\]

to write the equation of motion as an equation for \( \Phi^{(1)}_m \), giving

\[
\frac{1}{2} \left[ \frac{\partial^2 \Psi^{(1)}_m}{\partial f^2} + \omega^2_m \Psi^{(1)}_m \right] = \frac{\partial^2 \Psi^{(0)}_m}{\partial f \partial s} - \frac{1}{s} \frac{\partial \Psi^{(0)}_m}{\partial f} + \frac{4}{m \pi^2} \left( J_1(ms)^2 + Y_1(ms)^2 \right)^{-1} \times

\left[ \int d n \frac{\partial \Psi^{(0)}_n}{\partial f} \right] \int d \xi \frac{\Phi^{(0)}_m \partial \Psi^{(0)}_n}{\partial s} + \frac{1}{s} \int d n \frac{\partial \Psi^{(0)}_m}{\partial f} \int d \xi \frac{\Phi^{(0)}_m \partial \Phi^{(0)}_n}{\partial \xi}. \tag{29}
\]

The calculation of the various integrals involving Bessel functions is explained in Appendix B. We can write the equation of motion for \( \Psi^{(1)}_m \) as

\[
\frac{1}{2} \left[ \frac{\partial^2 \Psi^{(1)}_m}{\partial f^2} + \omega^2_m \Psi^{(1)}_m \right] = \frac{\partial^2 \Psi^{(0)}_m}{\partial f \partial s} + \frac{d}{ds} \log \left( (J_1(ms))^2 + (Y_1(ms))^2 \right) \frac{\partial \Psi^{(0)}_m}{\partial f} + \int d n \mathcal{M}_{mn} \frac{\partial \Psi^{(0)}_m}{\partial f}, \tag{30}
\]
for \( m \neq 0 \), and
\[
\frac{1}{2} \left[ \frac{\partial^2 \Phi^{(1)}_m}{\partial f^2} + \omega_m^2 \Phi^{(1)}_m \right] = \frac{\partial^2 \Phi^{(0)}_m}{\partial f \partial s} + \int \! dm \, M_{mn} \frac{\partial \Phi^{(0)}_n}{\partial f}, \tag{31}
\]
for the zero-mode. The trace-free matrix \( M_{mn} \) is defined as
\[
M_{mn} = \frac{4}{\pi^2 m} \left( J_1(m \eta)^2 + Y_1(m \eta) \right)^{-1} \left\{ \int \! \frac{d\xi}{\xi^3} \Phi^{(0)}_n \frac{\partial \Phi^{(0)}_m}{\partial s} + \frac{1}{s} \int \! \frac{d\xi}{\xi^2} \Phi^{(0)}_n \frac{\partial \Phi^{(0)}_m}{\partial \xi} \right\}, \tag{32}
\]
for \( m \neq n \) and is calculated in Appendix B; these terms do not contribute to the secularity condition.

Substituting in the solution (21) for \( \Phi^{(0)}_0 \), we get terms on the r.h.s. of (31) for the form
\[
\sum_{\pm} \pm ipe^{\pm ipf} \frac{dA^{\pm}_m}{ds} , \tag{33}
\]
This is a forcing term in equation (30) which, as a function of the fast time \( f \), has the same frequency as the natural frequency of the l.h.s. and so will cause a resonance. If \( \varepsilon \) decays as \( 1/\eta \) or slower, this is forbidden since the \( h^{(1)} \) term would not be of asymptotic order. So we can impose the secularity condition that \( A^{\pm}_m(s) \) must satisfy
\[
\frac{dA^{\pm}_m}{ds} = 0 \quad \Rightarrow \quad A^{\pm}_m = C^{\pm}_m, \tag{34}
\]
where the \( C^{\pm}_m \) are constants independent of \( s \).

For the massive modes \((m \neq 0)\), the requirement that (30) be free of forcing terms gives the secularity condition
\[
\frac{d}{ds} \left[ A^{\pm}_m(s) \sqrt{\left( J_1(m \eta)^2 + Y_1(m \eta) \right)} \right] = 0 \quad \Rightarrow \quad A^{\pm}_m(s) = C^{\pm}_m \left( J_1(m \eta)^2 + Y_1(m \eta) \right)^{-1/2}, \tag{35}
\]
where, again, the \( C^{\pm}_m \) are constants independent of \( s \). We will choose \( C^{\pm}_m \propto m \) so that the limit as \( m \to 0 \) is a constant. With the secularity condition imposed, equation of motion for \( \Phi^{(1)}_m \) is then
\[
\frac{\partial^2 \Phi^{(1)}_m}{\partial f^2} + \omega_m^2 \Phi^{(1)}_m = 2i \int \! dm \, M_{mn} A^{\pm}_m \omega_n e^{\pm i\omega_n f}, \tag{36}
\]
which will have solution
\[
\Phi^{(1)}_m = B^{\pm}_m(s) e^{\pm i\omega_n f} + 2i \sum_{\pm} \int \! dm \, \frac{A^{\pm}_m \omega_n}{\omega_n - \omega_n} M_{mn} e^{\pm i\omega_n f}. \tag{37}
\]
In order to determine \( B_m(s) \), one would have to consider its secularity condition, coming from the second-order equations.

**VI. MODE-MIXING AND THE ZERO-MODE**

A great advantage of the method of multiple scales is that the leading-order solution includes the slow-time variation derived from the secularity condition so is usually a very good approximation. The modes do not influence each other at leading order, and our calculation gives the zero-mode as
\[
h_0(\eta, \xi) = \frac{1}{a} e^{\pm ip\eta}, \tag{38}
\]
and the massive modes as
\[
h_m(\eta, \xi) = \frac{\pi m^2 \xi^2 \left( Y_1(m/a) J_2(m \xi) - J_1(m/a) Y_2(m \xi) \right)}{2 \sqrt{(J_1(m/a))^2 + (Y_1(m/a))^2}} e^{\pm i\omega_m \eta}, \tag{39}
\]
which takes the value
\[ h_m(\eta, 1/a) = \frac{m/a}{\sqrt{(J_1(m/a))^2 + (Y_1(m/a))^2}} e^{\pm i \omega_m \eta}, \]  
(40)
on the brane.

By re-writing the equation of motion in a different way, we can see more directly that the zero-mode to leading order gives the same answer as four-dimensional cosmology. By treating one of the order \( \varepsilon \) terms in the equation of motion as a zeroth order term, one gets
\[ \frac{\partial^2 h}{\partial f^2} + 2\varepsilon \frac{\partial h}{\partial f} + \frac{\partial^2 h}{\partial \xi^2} + \frac{3}{\xi} \frac{\partial h}{\partial \xi} + p^2 h = 2\varepsilon \left( \frac{\partial^2 h}{\partial f \partial s} + a \frac{\partial^2 h}{\partial \xi \partial f} \right) + O(\varepsilon^2). \]  
(41)

Setting the r.h.s. to zero for the zeroth-order calculation and performing the separation of variables in the same way as before gives the time evolution equation
\[ \frac{\partial^2 \Psi}{\partial f^2} + \frac{2}{a} \frac{\partial \Psi}{\partial f} + k^2 \Psi = 0 \]  
(42)
for the time component of the zero-mode, which is exactly the same as the equation in four dimensions. Not surprisingly, the secularity condition at first order reveals no slow-time dependence for the zero-mode. So, by slightly dishonest accountancy with the \( \varepsilon \) terms, one can make the time dependence of the zero-mode look exactly like the four-dimensional mode.

At first order in \( \varepsilon \), there is a modification in the phase due to the \( \varepsilon \)-dependent term in (11). Evaluated on the brane, we have
\[ \eta = \frac{\tau}{l} + \frac{\varepsilon}{2a} + O(\varepsilon^2), \]  
(43)
so the zeroth-order part of the solution on the brane can we written
\[ \frac{m/a}{\sqrt{(J_1(m/a))^2 + (Y_1(m/a))^2}} \exp \left\{ \pm i \omega_m \left( \frac{\tau}{l} + \frac{\varepsilon}{2l} \right) \right\} \]  
(44)
for the massive mode, or
\[ \frac{1}{a} \exp \left\{ \pm ik \left( \frac{\tau}{l} + \frac{\varepsilon}{2a} \right) \right\} \]  
(45)
for the zero-mode. This is in addition to the contribution of the \( \Psi^{(1)} \Phi^{(0)} \) term.

The first-order equation in \( \varepsilon \) has an integral term on the r.h.s., which describes how a mode is sourced by the other modes. The matrix \( M_{mn} \), which is a function of \( s \), is calculated in Appendix B. Of particular note is that the zero-mode cannot source any other modes, that is, \( M_{0n} = 0 \). Massive modes, however, can decay into other modes. Again, this is an effect which occurs at first order in \( \varepsilon \). The effects of mode-mixing would be very small later in the history of the universe so one would expect all the interesting signatures coming from mode-mixing to arise when \( \varepsilon \) is not too small, that is, when the Hubble radius is only one or two orders of magnitude larger than the AdS length-scale. Of course, our approximation scheme breaks down as \( \varepsilon \) approaches unity.

From studies of the evolution of perturbations during a de Sitter era in the history of the universe we know that production of massive modes is heavily suppressed. So it is reasonable to take the zero-mode as an approximation for the spectrum of perturbations after the end of inflation. If the AdS length-scale, \( l \), is sufficiently small that our approximation method is valid at the end of inflation, then our analysis shows that no massive perturbations will be sourced during the subsequent evolution, except possibly at \( O(\varepsilon^2) \).

\section{VII. DISCUSSION AND CONCLUSIONS}

The approximation method we have used has several advantages over the alternatives. Firstly, our coordinates are related to the Poincaré coordinates advocated by by a non-singular transformation and so, unlike GN coordinates do not have a coordinate singularity. This means that the answers we get are valid right up to the horizon, allowing
one meaningfully to impose initial conditions on a Cauchy surface or a boundary condition of no incoming radiation at the Cauchy horizon. Secondly, we have not had to specify the evolution of the scale factor, \(a\), so the method can be applied quite generally in different eras of the universe and at transitions between such eras; by contrast, when using a GN coordinate system the time evolution has to be specified in order to find approximate solutions and, even then, the equations are only tractable in certain cases. Finally, it should be simple to extend the method to other bulk geometries where the perturbation spectrum is known.

In particular, it is worth comparing our method to the gradient expansion method. The gradient expansion method works by expanding the solution in the bulk in orders of the derivatives normal to the brane. The small parameter is exactly the same one as used here. The two methods are quite similar, the main difference being that the method presented here uses a coordinate system free from coordinate horizons. Our multiple-scales method can also allow for the presence of very massive modes, although intuition tells us not expect such modes to be important at low energies.

An important case is when the initial state for our calculation is the zero-mode solution, because this is what one expects from a de Sitter phase in the history of the universe. Our analysis shows that the evolution of the solution will be the same as for perturbations in a four-dimensional FRW universe and that the massive modes are not sourced. This is somewhat obvious from physical intuition if these modes are viewed as Kaluza–Klein particles because one would expect particles to decay into less massive states, not more massive ones. The picture would be very different if there were sources on the brane: massive modes could be sourced by the brane-based matter. For simplicity, and because it is a reasonable approximation to make for the tensor part of the spectrum, we have assumed that there is no matter source on the brane to source the bulk metric perturbations. If one were to consider scalar perturbations, the brane sources could not be neglected in the same way.

Our calculation applies strictly to the tensor component of the metric perturbations on the brane since they are gauge invariant, but it should be simple to apply it to the vector modes. Tensor modes are gauge-invariant, but for vector modes there is the added complication of gauge choice; the gauge we have used here is different from the gauges usually used to study four-dimensional cosmological perturbations, for example, the synchronous gauge. It should also be possible to extend this method to allow for a tensor matter source, such as the photon quadrupole which develops after last scattering. In addition, we hope, in future work, to include the effect of perturbing the brane position, so as to apply the method to study scalar perturbations.

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APPENDIX A: COORDINATE CHANGE

We consider the coordinate change between Gaussian Normal (GN) coordinates and the conformally flat bulk coordinates. The metrics are given respectively by

\[
\begin{align*}
\text{d}s^2 &= -N^2 \text{d}t^2 + \text{d}\zeta^2 + A^2 \delta_{ij} \text{d}x^i \text{d}x^j = \frac{l^2}{Z^2} \left( - \text{d}T^2 + \text{d}Z^2 + \delta_{ij} \text{d}x^i \text{d}x^j \right), \\
0 &= -\frac{\partial T}{\partial \tau} \frac{\partial T}{\partial \zeta} + \frac{\partial Z}{\partial \tau} \frac{\partial Z}{\partial \zeta}, \\
1 &= -\frac{l^2}{Z^2} \left( \frac{\partial T}{\partial \zeta} \right)^2 + \frac{l^2}{Z^2} \left( \frac{\partial Z}{\partial \zeta} \right)^2.
\end{align*}
\]  

(A1)

We use a trick to simplify the transformation by noting that \( -\text{d}T \) is a Killing vector, so that its inner product with the normal vector \( \partial / \partial \zeta \) is independent of \( \zeta \) along the geodesics in the \( \zeta \) direction, and hence we can write

\[
\frac{\partial T}{\partial \zeta} = E(\tau). 
\]  

(A2)

The usual formula for coordinate change gives

\[
\begin{align*}
-N^2 &= -\frac{l^2}{Z^2} \left( \frac{\partial T}{\partial \tau} \right)^2 + \frac{l^2}{Z^2} \left( \frac{\partial Z}{\partial \tau} \right)^2, \\
0 &= -\frac{\partial T}{\partial \tau} \frac{\partial T}{\partial \zeta} + \frac{\partial Z}{\partial \tau} \frac{\partial Z}{\partial \zeta}, \\
1 &= -\frac{l^2}{Z^2} \left( \frac{\partial T}{\partial \zeta} \right)^2 + \frac{l^2}{Z^2} \left( \frac{\partial Z}{\partial \zeta} \right)^2.
\end{align*}
\]  

(A3)
Solving these simultaneously, we find

\[
\left( \frac{\partial Z}{\partial \xi} \right)^2 = N^2 E^2. \tag{A6}
\]

On the brane, \( N = A = a(\tau) \) and \( Z = l/a \) so

\[
\frac{\partial Z}{\partial \tau} \bigg|_{\xi=0} = -\frac{l}{a^2} \frac{da}{d\tau} = -\varepsilon. \tag{A7}
\]

Thus, \( E(\tau) = -\varepsilon/a(\tau) \), where the sign can be determined by considering the components of the normal vector in the \( T, Z \) coordinates. So, the Jacobian matrix and its inverse are

\[
\begin{pmatrix}
\frac{\partial T}{\partial \xi} & \frac{\partial Z}{\partial \xi} \\
\frac{\partial T}{\partial \zeta} & \frac{\partial Z}{\partial \zeta}
\end{pmatrix}
= \begin{pmatrix}
N \sqrt{\frac{x^2}{a^2} + \frac{z^2}{a^2}} & -\frac{z N}{a} \\
-\frac{z}{a} & \sqrt{\frac{x^2}{a^2} + \frac{z^2}{a^2}}
\end{pmatrix},
\]

\begin{align}
\frac{\partial \tau}{\partial \xi} & \quad \frac{\partial \xi}{\partial \tau} \\
\frac{\partial \tau}{\partial \zeta} & \quad \frac{\partial \zeta}{\partial \tau}
\end{align}
= \begin{pmatrix}
\frac{4}{N} \sqrt{1 + \frac{z^2 A^2}{a^2}} & \frac{\varepsilon A^2}{a} \\
-\frac{\varepsilon A}{a N} & A \sqrt{1 + \frac{z^2 A^2}{a^2}}
\end{pmatrix}. \tag{A8}

APPENDIX B: INTEGRALS OF PRODUCTS OF BESSEL FUNCTIONS AND ORTHOGONALITY RELATIONS

1. Standard results

There is a standard formula for Bessel functions (see, for example, Ref. [17]) which tells us that

\[
\int x B_\nu(mx) \tilde{B}_\nu(nx) \, dx = \frac{x}{m^2 - n^2} \left( m B_{\nu+1}(mx) \tilde{B}_\nu(nx) - n B_\nu(mx) \tilde{B}_{\nu+1}(nx) \right), \tag{B1}
\]

where \( B_\nu \) and \( \tilde{B}_\nu \) are any two cylinder functions of order \( \nu \) and \( m \neq n \). We will apply this to two specific cylinder function defined by

\[
C^{(m)}(x) = m Y_1(ms) J_\nu(x) - m J_1(ms) Y_\nu(x), \tag{B2}
\]

\[
D^{(m)}(x) = Y_2(ms) J_\nu(x) - J_2(ms) Y_\nu(x), \tag{B3}
\]

with \( J \) and \( Y \) being Bessel functions of the first and second kinds respectively. Cylinder functions satisfy the recurrence relations [17]

\[
z B_{\nu-1}(z) + z B_{\nu+1}(z) = 2 \nu B_\nu(z) \quad \text{and} \quad B_{\nu-1}(z) - B_{\nu+1}(z) = 2 \frac{d B_\nu(z)}{dx}, \tag{B4}
\]

from which it can be seen that

\[
\frac{\partial}{\partial s} C^{(m)}(sx) = \frac{1}{s} C^{(m)}(sx) - m^2 D^{(m)}(sx), \tag{B5}
\]

where the l.h.s. is also a cylinder function of order \( \nu \). The Wronskian of Bessel functions of the first and second kinds is [17]

\[
W[J_\nu, Y_\nu](z) = J'_\nu(z) Y_\nu(z) - J_\nu(z) Y'_\nu(z) = Y_{\nu-1}(z) J_\nu(z) - J_{\nu-1}(z) Y_\nu(z) = \frac{2}{\pi z}, \tag{B6}
\]

which, in conjunction with the recurrence relations, allows us to evaluate the following

\[
C^{(m)}_2(ms) = \frac{2}{\pi s}, \quad \frac{\partial}{\partial s} C^{(m)}_2(ms) = \frac{2}{\pi s},
\]

\[
C^{(m)}_3(ms) = \frac{2}{\pi ms^2}, \quad \frac{\partial}{\partial s} C^{(m)}_3(ms) = \frac{2}{\pi ms}, \quad \frac{2m}{\pi s}, \tag{B7}
\]

which will be of use to use later on.

We will also make use of the asymptotic expansions for large argument [17]

\[
J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{2} \nu - \frac{\pi}{4} \right), \quad Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\pi}{2} \nu - \frac{\pi}{4} \right), \tag{B8}
\]
the integral \[ \int x^{-p+1} B_p(x) \, dx = -x^{-p+1} B_{p-1}(x), \quad \text{(B9)} \]

and the orthogonality relation \[ \int_0^\infty x J_{\nu}(mx) J_{\nu}(nx) = \frac{1}{m} \delta(m - n), \quad \text{(B10)} \]

when \( \nu \) is a whole number.

2. Orthogonality relation

From \( \text{B11} \) it is clear that, for \( m \neq n \)

\[ \int_{x_0}^\infty x C^{(m)}_\nu(mx) C^{(n)}_\nu(nx) \, dx = 0. \quad \text{(B11)} \]

because the derivatives of the \( C^{(m)} \) functions with respect to \( z \) vanish on the boundary \( z = s \). To determine the coefficient of the delta function in the orthogonality relation, we note that the integral is dominated by the behaviour near infinity, so we can consider instead the leading order asymptotic expression

\[ C^{(m)}_2(mx) \sim \sqrt{\frac{2m}{\pi x}} (Y_1(ms)^2 + J_1(ms)^2)^{1/2} \sin (mx + \text{phase angle}), \quad \text{(B12)} \]

valid when \( x \) is large. So the constant of proportionality can be deduced to be

\[ \int_{x_0}^\infty x C^{(m)}_\nu(mx) C^{(n)}_\nu(nx) \, dx = m (Y_1(ml)^2 + J_1(ml)^2) \delta(m - n), \quad \text{(B13)} \]

by comparison to the equivalent asymptotic expression for eq. \( \text{B10} \).

This argument breaks down in the case of the zero-mode, which must be evaluated separately. The inner product of the zero-mode with itself is simply

\[ \int_s^\infty \frac{d\xi}{\xi^3} \Phi_0^{(0)}(0) \Phi_0^{(0)}(0) = \int_s^\infty \frac{d\xi}{\xi^3} \xi^2 = \frac{1}{2}, \quad \text{(B14)} \]

while \( \text{B15} \) gives us

\[ \int_s^\infty \frac{d\xi}{\xi^3} \Phi_0^{(m)}(0) \Phi_0^{(0)}(0) = 0. \quad \text{(B15)} \]

3. Resonant terms

The integrals in the first order equation of motion \( \text{B15} \) which need to be calculated to obtain the secularity condition are

\[ \int_s^\infty \frac{d\xi}{\xi^3} \Phi_0^{(0)}(0) \frac{\partial \Phi_0^{(0)}}{\partial s}, \quad \text{(B16)} \]

\[ \int_s^\infty \frac{d\xi}{\xi^2} \Phi_0^{(m)}(0) \frac{\partial \Phi_0^{(0)}}{\partial s}, \quad \text{(B17)} \]

To evaluate these when \( m = 0 \) is simple, because \( \Phi_0^{(0)} = 0 \), and we can easily see that

\[ \int_s^\infty \frac{d\xi}{\xi^3} \Phi_0^{(0)}(0) \frac{\partial \Phi_0^{(0)}}{\partial s} = \frac{1}{2s}, \quad \int_s^\infty \frac{d\xi}{\xi^2} \Phi_0^{(0)}(0) \frac{\partial \Phi_0^{(0)}}{\partial \xi} = 0. \quad \text{(B18)} \]
To evaluate \(B16\) when \(m \neq 0\), we can differentiate the orthogonality relation \(B28\) w.r.t. \(s\) to get
\[
\int_s^\infty \frac{d\xi}{\xi^3} \left( \Phi_m^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial s} + \frac{\partial \Phi_n^{(0)}}{\partial s} \frac{\partial \Phi_m^{(0)}}{\partial \xi} \right) = \frac{\pi^2}{4} m \frac{d}{ds} \left( (J_1(ms))^2 + (Y_1(ms))^2 \right) \delta(m - n) + \frac{1}{s^3} \Phi_m^{(0)}(s) \Phi_n^{(0)}(s, s). \tag{B19}
\]
and note that, for \(m = n\), the last term is ignorable compared to the delta function term. Evaluating \(B17\) when \(m \neq 0\), can be done integrating by parts to get
\[
\int_s^\infty \frac{d\xi}{\xi^2} \Phi_m^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial \xi} = \int_s^\infty \frac{d\xi}{\xi^3} \Phi_m^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial s} + \frac{1}{2} \left[ \Phi_n^{(0)} \frac{\partial \Phi_m^{(0)}}{\partial s} \right]_s^\infty,
\tag{B20}
\]
which is just given by the orthogonality relations since, again, the second term is insignificant compared to the first.

4. Mode-mixing terms

The mode-mixing, that is, the effect of modes as source terms for other modes, arises from the terms involving the following integrals
\[
\int_s^\infty \frac{d\xi}{\xi^3} \Phi_m^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial s},
\tag{B21}
\]
\[
\int_s^\infty \frac{d\xi}{\xi^2} \Phi_m^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial \xi},
\tag{B22}
\]
when \(m \neq n\). These integrals are simple to calculate when either \(m = 0\) or \(n = 0\), using \(\Phi_0^{(0)} = s\). For \(B21\) we calculate that
\[
\int_s^\infty \frac{d\xi}{\xi^3} \Phi_m^{(0)} \frac{\partial \Phi_0^{(0)}}{\partial s} = 0, \quad \int_s^\infty \frac{d\xi}{\xi^3} \Phi_0^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial s} = \frac{1}{s}, \tag{B23}
\]
while \(B22\) is evaluated to be
\[
\int_s^\infty \frac{d\xi}{\xi^2} \Phi_m^{(0)} \frac{\partial \Phi_0^{(0)}}{\partial \xi} = 0, \quad \int_s^\infty \frac{d\xi}{\xi^2} \Phi_0^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial \xi} = \frac{1}{n} C_2^{(n)} (ns) = \frac{2}{\pi ns}, \tag{B24}
\]
by applying the relation \(B39\).

When \(m\) and \(n\) are both non-zero, we can evaluate \(B21\) using the formula \(B1\) to give
\[
\int_s^\infty \frac{d\xi}{\xi^3} \Phi_m^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial s} = \frac{\pi^2}{4} \left[ \frac{\xi}{m^2 - n^2} \left( mC_3^{(m)}(m\xi) \frac{\partial}{\partial s} C_2^{(n)}(n\xi) - nC_2^{(m)}(m\xi) \frac{\partial}{\partial s} C_3^{(n)}(n\xi) \right) \right]_s^\infty. \tag{B25}
\]
The recurrence formulæ \(B24\) allow us to write
\[
C_\mu^{(m)}(m\xi) = \left[ \frac{4}{s} Y_2(ms) - mY_3(ms) \right] J_\mu(m\xi) - \left[ \frac{4}{s} J_2(ms) - mJ_3(ms) \right] Y_\mu(m\xi), \tag{B26}
\]
\[
\frac{\partial}{\partial s} C_\nu^{(n)}(n\xi) = \left[ \left( \frac{4}{s^2} - n^2 \right) Y_2(ns) - \frac{n}{s} Y_3(ns) \right] J_\nu(n\xi) - \left[ \left( \frac{4}{s^2} - n^2 \right) J_2(ns) - \frac{n}{s} J_3(ns) \right] Y_\nu(n\xi), \tag{B27}
\]
whence, we see that
\[
\int_s^\infty \frac{d\xi}{\xi^3} \Phi_m^{(0)} \frac{\partial \Phi_n^{(0)}}{\partial s} = -\frac{4}{(m + n)ms^2} \frac{mn}{(m^2 - n^2)s}. \tag{B28}
\]
To calculate \(B22\) we first note that
\[
\frac{\partial}{\partial \xi} \left( \xi^2 J_2(n\xi) \right) = n\xi^2 J_1(n\xi) = \frac{\xi}{n} \frac{\partial}{\partial n} \left( n^2 J_2(n\xi) \right) \tag{B29}
\]
and similarly for $Y_2$. Now

$$\frac{\partial \Phi^{(0)}_m}{\partial \xi} = \frac{\pi}{2n} \left[ Y_1(ns) \frac{\partial}{\partial \xi} \left( \xi^2 J_2(n\xi) \right) - J_1(ns) \frac{\partial}{\partial \xi} \left( \xi^2 Y_2(n\xi) \right) \right],$$

$$\frac{\partial \Phi^{(0)}_n}{\partial n} = \frac{\pi}{2n} \left[ Y_1(ns) \frac{\partial}{\partial n} \left( n^2 J_2(n\xi) \right) - J_1(ns) \frac{\partial}{\partial n} \left( n^2 Y_2(n\xi) \right) \right],$$

and

$$\frac{\partial \Phi^{(0)}_m}{\partial \xi} = \frac{\pi}{2n} \left[ Y_1(ns) \frac{\partial}{\partial \xi} \left( n^2 J_2(n\xi) \right) - J_1(ns) \frac{\partial}{\partial \xi} \left( n^2 Y_2(n\xi) \right) \right]$$

$$\frac{\partial \Phi^{(0)}_n}{\partial n} = \frac{\pi}{2n} \left[ Y_1(ns) \frac{\partial}{\partial n} \left( n^2 J_2(n\xi) \right) - J_1(ns) \frac{\partial}{\partial n} \left( n^2 Y_2(n\xi) \right) \right],$$

so that

$$\frac{\partial \Phi^{(0)}_m}{\partial \xi} = \frac{\pi}{2n} \left[ Y_1(ns) \frac{\partial}{\partial \xi} \left( n^2 J_2(n\xi) \right) - J_1(ns) \frac{\partial}{\partial \xi} \left( n^2 Y_2(n\xi) \right) \right]$$

$$\frac{\partial \Phi^{(0)}_n}{\partial n} = \frac{\pi}{2n} \left[ Y_1(ns) \frac{\partial}{\partial n} \left( n^2 J_2(n\xi) \right) - J_1(ns) \frac{\partial}{\partial n} \left( n^2 Y_2(n\xi) \right) \right].$$

and we can write

$$\int_0^\infty \frac{d\xi}{\xi^2} \frac{\partial \Phi^{(0)}_m}{\partial \xi} = \frac{n}{\xi} \frac{\partial}{\partial n} \int_0^\infty \frac{d\xi}{\xi} \Phi^{(0)}_m \Phi^{(0)}_n + \frac{\pi}{2n} \int_0^\infty \frac{d\xi}{\xi} \Phi^{(0)}_m D_2^{(n)}(n\xi).$$

The first integral on the r.h.s. is zero when $m \neq n$; the second integral is

$$\frac{\pi^2}{4} n^2 s \int_0^\infty \xi C_2^{(m)}(m\xi) D_2^{(n)}(n\xi) d\xi = \frac{\pi^2}{4} \int_0^\infty \xi C_2^{(m)}(m\xi) C_2^{(n)}(n\xi) d\xi - \frac{\pi^2}{4} s \int_0^\infty \xi C_2^{(m)}(m\xi) \frac{\partial}{\partial s} C_2^{(n)}(n\xi) d\xi$$

the first term of which is zero for $m \neq n$ by the orthogonality relation, the second having already been calculated above.

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