RELATIVISTIC RIGID PARTICLES: CLASSICAL TACHYONS AND QUANTUM ANOMALIES

JAN GOVAERTS†

Department of Mathematical Sciences
University of Durham, Durham DH1 3LE, UK

Abstract

Causal rigid particles whose action includes an arbitrary dependence on the world-line extrinsic curvature are considered. General classes of solutions are constructed, including causal tachyonic ones. The Hamiltonian formulation is developed in detail except for one degenerate situation for which only partial results are given and requiring a separate analysis. However, for otherwise generic rigid particles, the precise specification of Hamiltonian gauge symmetries is obtained with in particular the identification of the Teichmüller and modular spaces for these systems. Finally, canonical quantisation of the generic case is performed paying special attention to the phase space restriction due to causal propagation. A mixed Lorentz-gravitational anomaly is found in the commutator of Lorentz boosts with world-line reparametrisations. The subspace of gauge invariant physical states is therefore not invariant under Lorentz transformations. Consequences for rigid strings and membranes are also discussed.

† Address from 1st October 1992:
Institut de Physique Nucléaire, Université Catholique de Louvain,
B-1348 Louvain-la-Neuve (Belgium)
1. Introduction

Some time ago, motivated by different physical considerations, Polyakov [1,2] proposed a modification of the ordinary Nambu-Goto string action by including a dependence on the world-sheet extrinsic curvature. In spite of the great deal of activity that followed [3], a complete and exact understanding of these systems is still lacking, especially at the quantum level. Only partial results and educated guesses obtained through semi-classical approximation schemes to classical solutions are available. It is generally believed [4], though not demonstrated explicitly, that higher derivative terms due to extrinsic curvature contributions would render quantum unitarity impossible through physical states either of negative norm or of energy unbounded below. Indeed, a semi-classical analysis [5] indicates instabilities of the latter type for specific solutions.

This situation has led some authors [6-22] to consider the same class of actions in the simpler case of relativistic particles. Such an investigation is interesting not only in its own right, but it also has some relevance to the string case in as far as particles may be viewed as collapsed strings. However, the information available in the literature concerning these so called rigid particles is confusing and self-contradictory. It is thus appropriate to analyse these systems again paying greater attention to specific issues, in particular those of canonical quantisation not properly addressed so far.

The class of rigid particle systems considered here is described by the general action

\[ S[x^\mu] = -\mu c \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\dot{x}^2(\tau)} F(\kappa^2 K^2(\tau)), \]  

(1.1)

where the extrinsic curvature vector is given by

\[ K^\mu = \frac{(\dot{x}\ddot{x}^\mu - \dot{x}^2 \ddot{x}^\mu)}{(\dot{x}^2)^2}, \]  

(1.2)

so that

\[ K^2 = \frac{\dot{x}^2 \ddot{x}^2 - (\dot{x}\ddot{x})^2}{(\dot{x}^2)^3}. \]  

(1.3)

Our notations and conventions are given in Appendix A. As is also explained there, (1.1) provides the next simplest generalisation of the ordinary action for a relativistic scalar particle corresponding to the choice \( F(x) = 1 \). Indeed, (1.1) involves not only the velocities \( \dot{x}^\mu(\tau) \) but also the accelerations \( \ddot{x}^\mu(\tau) \) of the particle. By considering a dependence on higher \( j \)-torsions (see Appendix A), still higher order derivatives of \( x^\mu \) could be included in a consistent and systematic way.

Within the context of (1.1), two main cases have been analysed [9,16] corresponding to the choices

\[ F(x) = \alpha_0 \sqrt{x} + \beta_0, \]  

(1.4)

and

\[ F(x) = \alpha_0 x + \beta_0. \]  

(1.5)
In the present work, arbitrary choices for $F(x)$ are considered. However, (1.4) turns out [9] to define a distinguished case, referred to as the “degenerate case”. In contradistinction, all other choices define the “generic case”. Note that due to our insistence on considering (strictly) time-like velocities only, the quantity $\kappa^2 K^2$ is positive (see Appendix A). Any function $F(x)$ defined for positive arguments is a priori allowed in (1.1). The only restriction we shall assume here is that $F(x)$ is not constant.

First, in sect.2, the classical system is considered in some detail. Noether (or Ward) identities [23] following from the spacetime and world-line symmetries of (1.1) are given, and generic classes of solutions are presented. In particular, causal but nevertheless tachyonic solutions of constant extrinsic curvature are found to exist always, thus generalising the observation made in Ref.[16] for $F(x)$ given by (1.4) and (1.5). In sect.3, we turn to the Hamiltonian formulation. The system of constraints is analysed and the local gauge invariances associated to first-class constraints are identified [15] – including the associated Teichmüller and modular spaces – for all choices of $F(x)$ except for the degenerate case (1.4) for which only partial results are given, leaving the complete analysis of that case to subsequent work. Sect.4 addresses the issue of canonical quantisation in the generic case, i.e. for all choices of $F(x)$ different from (1.4). Due to the restriction on phase space following from causal propagation, first a certain change of variables is required whereby manifest spacetime Poincaré covariance – still a symmetry of course – is lost. Even though the algebras of Poincaré and gauge transformations are easily seen to be preserved at the quantum level, the quantised system turns out to have no physically acceptable interpretation, certainly in the context of models for particle physics. Gauge invariant physical states cannot be defined in a manner which is at the same time consistent from the spacetime point of view. Indeed, there appears [24] a quantum anomaly in the commutator of Lorentz boosts with world-line reparametrisations. Consequently, being gauge invariant becomes a frame dependent property and in fact only the mass but not the spin of physical states can be defined in a consistent manner. This result is derived so far only in the generic case. In the degenerate case (1.4), the analysis of the same issues still needs to be developed and is therefore left for future work. However, one expects that the same conclusion concerning anomalies would extend further to the degenerate case, and probably also to actions including a dependence on $j$-torsions of higher order still, such as the extrinsic torsion. Finally, in sect.5, further discussion and comments are presented, including some consequences of our results concerning rigid strings. Additional results secondary to the main arguments are described in two appendices.

2. Classical Solutions

Even though the action (1.1) is a higher order one, there is actually no difficulty in applying the usual variational principle in order to derive the associated Euler-Lagrange equations of motion as well as the Noether identities and conserved quantities following from the spacetime Poincaré and world-line local reparametrisation invariances of the system. In particular, and as is typical, the equations of motion are precisely the conservation equations for the total energy-momentum of the system. Note also that these
equations of motion are of fourth order in $\tau$-derivatives, whose general solution thus requires $(4D)$ integration constants. Here, we shall take for these integration constants the initial ($\tau = \tau_i$) and final ($\tau = \tau_f$) values of the coordinates $x^\mu(\tau)$ and velocities $\dot{x}^\mu(\tau)$. Of course, these integration constants will also have to obey some constraints following from local reparametrisation invariance.

However, having in mind canonical quantisation, it would rather be more convenient to have an action involving velocities only. Such a redefinition of the action is readily achieved by introducing additional degrees of freedom and associated Lagrange multipliers whereby (1.1) is re-expressed as

$$S[x^\mu, q^\mu, \lambda^\mu] = \int_{\tau_i}^{\tau_f} d\tau \ L(\dot{x}^\mu, q^\mu, \dot{q}^\mu, \lambda^\mu) ,$$  \hspace{1cm} (2.1)

with

$$L = -\mu c \sqrt{-q^2} \ F(\kappa^2 q^2 - (q\dot{q})^2) - \mu c \ \lambda^\mu (q^\mu - \dot{x}^\mu) .$$  \hspace{1cm} (2.2)

Here, $q^\mu(\tau)$ are new degrees of freedom – corresponding to the velocities $\dot{x}^\mu(\tau)$ – with a dimension of length, and $\lambda^\mu(\tau)$ are dimensionless Lagrange multipliers for the constraints $q^\mu = \dot{x}^\mu$.

The action (2.1) is thus our definition for the rigid particles under consideration. As the reader is invited to verify, it should be clear that (1.1) and (2.1) indeed describe the same classical physical system for any given choice of $F(x)$. Corresponding to our previous remarks, the only restrictions are that $q^\mu(\tau)$ is (strictly) time-like ($q^2(\tau) < 0$) and that $F(x)$ is not constant ($F'(x) \neq 0$).

### 2.1 Symmetries and Noether identities

By construction, (2.1) possesses different symmetries. Let us consider them in turn. Spacetime Poincaré transformations act as

$$x'^\mu = \Lambda^\mu_\nu \ x^\nu + a^\mu, \quad q'^\mu = \Lambda^\mu_\nu \ q^\nu, \quad \lambda'^\mu = \Lambda^\mu_\nu \ \lambda^\nu ,$$  \hspace{1cm} (2.3)

with $\Lambda^\mu_\nu$ being a Lorentz transformation and $a^\mu$ a constant spacetime translation. Correspondingly, the conserved energy-momentum $P_\mu$ and angular-momentum $M_{\mu\nu}$ are given by

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \mu c \ \lambda^\mu ,$$  \hspace{1cm} (2.4)

and

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} ,$$  \hspace{1cm} (2.5)

with

$$L_{\mu\nu} = P_\mu x_\nu - P_\nu x_\mu , \quad S_{\mu\nu} = \frac{\partial L}{\partial \dot{q}^\mu q_\nu} - \frac{\partial L}{\partial q^\mu} q_\nu .$$  \hspace{1cm} (2.6)
Note that the Lagrange multiplier $\lambda$ is essentially the energy-momentum of the system. On the other hand, $L_{\mu\nu}$ corresponds to the covariant orbital angular-momentum, so that $S_{\mu\nu}$ is to be interpreted as some internal spin. Such an interpretation is indeed consistent, as confirmed by later results. Moreover, we have

$$S^{\mu\nu} = 2\mu c k \sqrt{\kappa^2 K^2} F'(\kappa^2 K^2) K_{(2)}^{\mu\nu},$$

with $K_{(2)}^{\mu\nu}$ given in (A.8) (and the constraint $q^{\mu} = \dot{x}^{\mu}$ is to be understood of course). Thus, the internal spin is a direct measure of any extrinsic curvature in the world-line. In particular for the degenerate case (1.4), the invariant $S^{\mu\nu} S_{\mu\nu}/2$ takes the constant value $- (\mu c k \alpha_0)^2$ irrespectively of the equations of motion. Finally, associated to the invariance under (2.3), we have the Noether identities [23]

$$\frac{d}{d\tau} P_{\mu} = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^{\mu}},$$

(2.8)

$$\frac{d}{d\tau} M_{\mu\nu} = \frac{3}{\alpha} \left[ (\frac{d}{d\tau} \frac{\partial L}{\partial \dot{z}_{\alpha}^{\mu}} - \frac{\partial L}{\partial z_{\alpha}^{\mu}}) z_{\alpha\nu} - (\mu \leftrightarrow \nu) \right],$$

(2.9)

where we set $(z_1^{\mu}, z_2^{\mu}, z_3^{\mu}) = (x^{\mu}, q^{\mu}, \lambda^{\mu})$ for convenience.

The action (2.1) is also invariant under reparametrisations ($\tau \rightarrow \tilde{\tau} = \tilde{\tau}(\tau)$) which preserve or reverse the orientation of the world-line and leave the interval $[\tau_i, \tau_f]$ invariant (namely, $\tau_i$ and $\tau_f$ are invariant (resp. interchanged) under orientation preserving (resp. reversing) reparametrisations). These transformations are defined by

$$\tilde{x}^{\mu}(\tilde{\tau}) = x^{\mu}(\tau), \quad \tilde{q}^{\mu}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}} q^{\mu}(\tau), \quad \tilde{\lambda}^{\mu}(\tilde{\tau}) = \text{sign}(\frac{d\tau}{d\tilde{\tau}}) \lambda^{\mu}(\tau).$$

(2.10)

In particular, for infinitesimal reparametrisations $\tilde{\tau} = \tau - \eta(\tau)$ with $\eta(\tau_i) = 0 = \eta(\tau_f)$, we have

$$\delta_\eta x^{\mu} = \eta \dot{x}^{\mu}, \quad \delta_\eta q^{\mu} = \frac{d}{d\tau} (\eta q^{\mu}), \quad \delta_\eta \lambda^{\mu} = \eta \dot{\lambda}^{\mu}.$$

(2.11)

The associated generator is the canonical Hamiltonian

$$H_0 = \sum_{\alpha=1}^{3} \dot{z}_\alpha^{\mu} \frac{\partial L}{\partial \dot{z}_\alpha^{\mu}} - L,$$

(2.12)

and the corresponding Noether identities – Noether’s second theorem [23] – are

$$q^{\mu} \frac{\partial L}{\partial \dot{q}^{\mu}} = 0,$$

(2.13)

$$H_0 = q^{\mu} \left[ \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^{\mu}} - \frac{\partial L}{\partial q^{\mu}} \right],$$

(2.14)
\[
\frac{d}{d\tau} H_0 = \sum_{\alpha=1}^{3} \dot{z}_\alpha \left[ \frac{d}{d\tau} \frac{\partial L}{\partial \dot{z}_\alpha} - \frac{\partial L}{\partial z_\alpha} \right].
\]  
(2.15)

Note that for classical solutions, not only is \( H_0 \) a conserved quantity – Noether’s first theorem (2.15) – but it actually then also vanishes as shown in (2.14) expressing the invariance of solutions under local world-line reparametrisations. On the other hand, from the relation
\[
\frac{\partial L}{\partial \dot{q}^\mu} = -2\mu c \kappa \frac{F'(\kappa^2 K^2)}{\sqrt{-q^2}} \kappa K^\mu,
\]  
(2.16)

(where the relation \( q^\mu = \dot{x}^\mu \) is again understood), it is clear that the identity (2.13) is equivalent to the relation \( nK = 0 \) in (A.6) following from the definition of the extrinsic curvature vector \( K^\mu \) as the variation of the normalised tangent vector \( n^\mu \).

### 2.2 Equations of motion

The equations of motion following from (2.1) are readily obtained. Variations in \( x^\mu \) lead to
\[
\dot{\lambda}^\mu = 0,
\]  
(2.17)

thus expressing the conservation of the energy-momentum \( P^\mu \). The equation for \( x^\mu \) is
\[
\dot{x}^\mu = q^\mu,
\]  
(2.18)

which is solved by
\[
x^\mu(\tau) = x^\mu_i + \int_{\tau_i}^{\tau} d\tau' q^\mu(\tau') .
\]  
(2.19)

Finally, the equation for \( q^\mu \) reduces to
\[
\frac{d}{d\tau} \left[ \kappa^2 \sqrt{-q^2} F'(\kappa^2 K^2) \frac{\partial K^2}{\partial q^\mu} \right] = \kappa^2 \sqrt{-q^2} F'(\kappa^2 K^2) \frac{\partial K^2}{\partial q^\mu} - \frac{q^\mu}{\sqrt{-q^2}} F(\kappa^2 K^2) + \lambda^\mu,
\]  
(2.20)

with \( K^2 \) being of course given by
\[
K^\mu = \frac{(q^\mu q^\mu - q^2 \dot{q}^\mu)}{(q^2)^2}, \quad K^2 = \frac{q^2 \dot{q}^2 - (q^\mu)^2}{(q^2)^3}.
\]  
(2.21)

Clearly, these equations are solved by specifying the boundary values \((x^\mu_i, q^\mu_i)\) and \((x^\mu_f, q^\mu_f)\) of \((x^\mu(\tau), q^\mu(\tau))\) at \( \tau = \tau_i \) and \( \tau = \tau_f \) respectively. Of course, due to local reparametrisation invariance, these boundary conditions will have to obey a certain set of constraints. The value for \( \lambda^\mu \) is determined through (2.20) and (2.19) since we must have
\[
\int_{\tau_i}^{\tau_f} d\tau \ q^\mu(\tau) = x^\mu_f - x^\mu_i = \Delta x^\mu.
\]  
(2.22)
Before considering solutions to these equations, let us present some of the identities that follow from them. First, given the variables

\[ Q^\mu = \frac{\partial L}{\partial \dot{q}_\mu} = -2\mu c \kappa \frac{F'(\kappa^2 K^2)}{\sqrt{-q^2}} \kappa K^\mu , \quad (2.23) \]

we clearly have the identity

\[ qQ = 0 , \quad (2.24) \]

equivalent to the orthogonality condition \( nK = 0 \) in (A.6) and the Noether identity (2.13). On the other hand, by projection of the equation of motion for \( q^\mu \) on \( \lambda^\mu \), we also obtain

\[ qP + \mu c \sqrt{-q^2} \left[ F(\kappa^2 K^2) - 2\kappa^2 K^2 F'(\kappa^2 K^2) \right] = 0 . \quad (2.25) \]

In the degenerate case (1.4), this last relation reduces to

\[ qP + \mu c \beta_0 \sqrt{-q^2} = 0 , \quad (2.26) \]

whereas we then also have the further identities

\[ q^2 Q^2 + (\alpha_0 \mu c \kappa)^2 = 0 , \quad PQ = 0 , \quad P^2 + (\mu c \beta_0)^2 = \alpha_0 \beta_0 (\mu c)^2 \sqrt{\kappa^2 K^2} . \quad (2.27) \]

These additional constraints are indicative of the distinguished rôles played [9] by the degenerate case. In particular, since \( P^\mu \) is conserved under time evolution, the last equality shows that in the degenerate case all classical solutions are [16] of constant extrinsic curvature. Moreover, the same relation also establishes that a necessary condition for the existence of classical solutions in the degenerate case is \( \beta_0 \neq 0 \). Indeed, \( \beta_0 = 0 \) would imply \( P^2 = 0 \), but such an identity is incompatible with the other constraints \( qP = 0, \ PQ = 0 \) and \( q^2 Q^2 + (\alpha_0 \mu c \kappa)^2 = 0 \) when only configurations with \( q^2 < 0 \) are considered.

In order to solve the equations of motion, it is most convenient to consider a proper-time gauge fixing condition with

\[ q^2(\tau) = -k^2 , \quad k \neq 0 , \quad (2.28) \]

where \( k \) is some real constant with the dimension of length. Note that the sign of \( k \) is not specified by (2.28). This ambiguity is related [25,23] to the fact that the condition (2.28) only fixes the gauge freedom under local reparametrisations but not under global \( \mathbb{Z}_2 \) modular transformations corresponding to orientation reversing world-line reparametrisations. Given the gauge (2.28), the extrinsic curvature \( K^2 \) is simply

\[ K^2 = \frac{\dot{q}^2}{k^4} , \quad (2.29) \]

implying that the equation for \( q^\mu \) now reads

\[ \frac{2\kappa^2}{|k|^3} F'(\kappa^2 \frac{\dot{q}^2}{k^4}) \ddot{q}^\mu + \frac{4\kappa^4}{|k|^7} (\dot{q} \ddot{q}) F''(\kappa^2 \frac{\dot{q}^2}{k^4}) \ddot{q}^\mu = \frac{1}{|k|} \left[ 4\kappa^2 \frac{\dot{q}^2}{k^4} F'(\kappa^2 \frac{\dot{q}^2}{k^4}) - F(\kappa^2 \frac{\dot{q}^2}{k^4}) \right] q^\mu + \lambda^\mu , \quad (2.30) \]
2.3 Straight trajectories

In the proper-time gauge any straight trajectory corresponds to

\[ x^\mu(\tau) = x_i^\mu + \frac{\Delta x^\mu}{\Delta \tau} (\tau - \tau_i), \quad q^\mu(\tau) = \frac{\Delta x^\mu}{\Delta \tau}, \quad (2.31) \]

with \( \Delta \tau = \tau_f - \tau_i \) and \( \Delta x^\mu = x_f^\mu - x_i^\mu \). Thus, such solutions may exist only if the boundary conditions are such that

\[ q_f^\mu = q_i^\mu = \frac{\Delta x^\mu}{\Delta \tau}, \quad (\Delta x)^2 < 0, \quad (2.32) \]

with the parameter \( k \) then given by \(|k|\Delta \tau = \sqrt{-(\Delta x)^2}\). This is not sufficient however. In addition, the choice for \( F(x) \) must also be such that the quantity equal to \( \lambda^\mu \) in (2.30) is finite and non vanishing for \( q^\mu(\tau) \) given in (2.31), in which case this quantity takes a value of the form

\[ A \frac{\Delta x^\mu}{\sqrt{-(\Delta x)^2}}, \quad (2.33) \]

corresponding to the value of \( \lambda^\mu \), with \( A \) thus a dimensionless non vanishing constant. The spacetime conserved quantities are then

\[ P^\mu = \frac{\mu c A}{\sqrt{-(\Delta x)^2}} \Delta x^\mu, \quad L^{\mu\nu} = \frac{\mu c A}{\sqrt{-(\Delta x)^2}} \left[ x_j^\mu x_i^\nu - x_i^\mu x_j^\nu \right], \quad S^{\mu\nu} = 0. \quad (2.34) \]

This shows that straight trajectories have indeed no extrinsic curvature, thus also no internal spin, and that the invariant mass of such solutions is simply \( \mu |A| \). Note that the sign of \( A \) is related to whether we are describing a particle as opposed to its particle (with the particle corresponding to solutions with positive (resp. negative) energy propagating forward (resp. backward) in time).

Therefore, provided \( F(x) \) is chosen appropriately so that \( A \) in (2.33) is finite and non vanishing, rigid particles always have straight trajectories as particular classical solutions. These are precisely all classical solutions for the ordinary scalar particle corresponding to \( F(x) = 1 \). However, rigid particles possess far more solutions. Nevertheless, as long as there would exist regimes where extrinsic curvature effects are small, rigid particles could be regarded as a viable generalisation of the ordinary scalar particle, with even the intriguing possibility that internal spin would follow from extrinsic curvature effects. As will become clear later on, such a suggestion is unfortunately not tenable at the quantum level, not even for integer spin.

2.4 Solutions of constant curvature

Obviously, it is difficult to completely solve (2.30) given an arbitrary function \( F(x) \). Nevertheless, a quite general class of solutions can be obtained when restricting to trajectories of constant curvature \( K^2 \). In fact, this class of configurations actually provides the
complete solution in the degenerate case, as was pointed out above. Note that constant values for \( K^2 \) are independent of the world-line parametrisation (see Appendix A) so that we may indeed work in the proper-time gauge (2.28) without loss of generality. Thus, given an arbitrary choice for \( F(x) \), consider configurations such that

\[
K^2 = a^2, \quad \dot{q}^2 = a^2 k^4 ,
\]

with \( a \) being a positive constant with the dimension of \((\text{length})^{-1}\). The equation (2.30) then reduces to

\[
\ddot{q}^\mu = \frac{k^2}{2\kappa^2} \left[ 4a^2 \kappa^2 - \frac{F(a^2 \kappa^2)}{F'(a^2 \kappa^2)} \right] q^\mu + \frac{|k|^3}{2\kappa^2 F'(a^2 \kappa^2)} \lambda^\mu .
\]

Three cases must therefore be considered related to the sign of the coefficient of \( q^\mu \) which depends on the choice for \( F(x) \) and the value of \( a \). Defining

\[
\alpha^2 = 4a^2 \kappa^2 - \frac{F(a^2 \kappa^2)}{F'(a^2 \kappa^2)} ,
\]

we shall refer to these cases as being parabolic, elliptic or hyperbolic corresponding respectively to whether \( \alpha^2 \) is vanishing, negative or positive. Note that \( \alpha^2 \) is independent of \( a^2 \) only if \( F(x) = \alpha_0 (x - \beta_0)^{1/4} \), in which case \( \alpha^2 = 4\beta_0 \).

Solving (2.36) is straightforward enough though tedious due to the constraints (2.28) and (2.35). The general solution is constructed as follows. Given a choice \( F(x) \) and a value \( a \) for the extrinsic curvature, introduce the quantities

\[
\beta = \frac{\alpha}{\kappa \sqrt{2}} \sqrt{-q^2_i} , \quad \gamma = \frac{1}{2} \beta \Delta \tau ,
\]

where \( \alpha \) is a square root of \( \alpha^2 \) in (2.37) and \( q^\mu_i \) is the initial boundary value for \( q^\mu(\tau) \), i.e. the initial velocity of the particle. Note that \( \alpha, \beta \) and \( \gamma \) are pure imaginary in the elliptic case and real in the hyperbolic case. Associated to this choice, a solution of constant curvature then exists provided we have \( F(x) \) and \( a \) such that

\[
\frac{F(a^2 \kappa^2)}{F'(a^2 \kappa^2)} \geq 2a^2 \kappa^2 ,
\]

and boundary conditions obeying the following constraints

\[
q^2_f = q^2_i < 0 , \quad \Delta q \Delta x = 0 ,
\]

\[
(\Delta q)^2 = a^2 (\Delta \tau)^2 (q^2_i)^2 \left( \frac{\cosh 2\gamma - 1}{2\gamma^2} \right) > 0 ,
\]

\[
\frac{(\Delta x)^2}{(\Delta \tau)^2} = q_i^2 - \frac{1}{4} (\Delta q)^2 \left( \frac{\cosh 2\gamma + 1}{\sinh \gamma} \right)^2 - 2 \left( \frac{\tanh \gamma}{\cosh 2\gamma - 1} \right) ,
\]

\[
\frac{\Delta x}{\Delta \tau} = q_i^2 - \frac{1}{4} (\Delta q)^2 \left( \frac{\cosh 2\gamma + 1}{\sinh \gamma} \right)^2 - 2 \left( \frac{\tanh \gamma}{\cosh 2\gamma - 1} \right) .
\]
The solution is then given as
\[ x^\mu(\tau) = x_i^\mu - \tilde{\lambda}^\mu(\tau - \tau_i) + \frac{1}{\beta} (q_f^\mu + \tilde{\lambda}^\mu) \sinh \beta(\tau - \tau_i) + \]
\[ + \frac{1}{\beta} [(q_f^\mu + \tilde{\lambda}^\mu) - (q_i^\mu + \tilde{\lambda}^\mu) \cosh 2\gamma] \frac{\cosh \beta(\tau - \tau_i) - 1}{\sinh 2\gamma}, \tag{2.41} \]

where
\[ \tilde{\lambda}^\mu = \left[ \frac{1}{2}(q_f^\mu + q_i^\mu) \left( \frac{\tanh \gamma}{\gamma} - \frac{\Delta x^\mu}{\Delta \tau} \right) \right] \left[ 1 - \left( \frac{\tanh \gamma}{\gamma} \right) \right]^{-1} = \sqrt{-q_i^2} \mu c \alpha F'(a^2 \kappa^2) P^\mu. \tag{2.42} \]

In particular, the invariant mass of such a solution is
\[ M^2 = -\frac{1}{c^2}P^2 = \mu^2 F''(a^2 \kappa^2) \alpha^2 (\alpha^2 - 2a^2 \kappa^2). \tag{2.43} \]

Obviously, some comments are in order. First of all, the expressions above are valid only when \((\sinh 2\gamma \neq 0)\) and \((\tanh \gamma \neq \gamma)\) whenever \(\gamma \neq 0\). However, a situation with \(\gamma \neq 0\) and \((\sinh 2\gamma = 0)\) or \((\tanh \gamma = \gamma)\) can only occur in the elliptic case, and the apparent singularities in the expressions above are only a reflection of the fact that some of the integration constants of the then periodic solutions are left undetermined. Such a situation is analogous to that [23] for the ordinary harmonic oscillator when the time interval happens to coincide with an integer multiple of the half-period. Similarly here, no additional physical understanding is to be gained by solving the equations whenever \((\tanh \gamma = \gamma)\) or \((\sinh 2\gamma = 0)\) with \(\gamma \neq 0\). In any case, these singular situations may always be avoided by slightly changing the value for \(\Delta \tau = \tau_f - \tau_i\).

The expressions above also define the solution when \((\alpha = 0 = \gamma)\) through the appropriate limit in that variable. Correspondingly, we then have the constraints on the integration constants
\[ q_f^2 = q_i^2 < 0, \quad \Delta q \Delta x = 0, \]
\[ (\Delta q)^2 = a^2(\Delta \tau)^2(2q_i^2)^2 > 0, \]
\[ (\Delta x)^2 \frac{(\Delta \tau)^2}{(\Delta \tau)^2} = q_i^2 - \frac{1}{12}(\Delta q)^2, \]
\[ q_i \Delta x \Delta \tau = q_i^2 - \frac{1}{6}(\Delta q)^2, \tag{2.44} \]

while the solution reads
\[ x^\mu(\tau) = x_i^\mu + q_i^\mu(\tau - \tau_i) + \left[ 3 \frac{\Delta x^\mu}{\Delta \tau} - (q_f^\mu + 2q_i^\mu) \right] \frac{(\tau - \tau_i)^2}{\Delta \tau} + \]
\[ + \left[ (q_f^\mu + q_i^\mu) - 2 \frac{\Delta x^\mu}{\Delta \tau} \right] \frac{(\tau - \tau_i)^3}{(\Delta \tau)^2}, \tag{2.45} \]

with
\[ P^\mu = \frac{24 \mu c a^2 F'(a^2 \kappa^2)}{(\Delta \tau)^2(-q_i^2)^{3/2}} \left[ \frac{q_f^\mu + q_i^\mu}{2} - \frac{\Delta x^\mu}{\Delta \tau} \right]. \tag{2.46} \]
Obviously, the condition (2.39) is always satisfied for these parabolic solutions, and their invariant mass vanishes identically.

Therefore, solutions of constant curvature always exist for boundary conditions obeying (2.40) or (2.44), whatever the choice for \( F(x) \) and extrinsic curvature \( a \) obeying (2.39). In Appendix B, it is shown how the set of solutions to (2.40) and (2.44) is indeed non empty and may completely be specified up to arbitrary Poincaré transformations. In particular, the condition (2.39) is necessary for the existence of solutions to the constraints (2.40). For example in the degenerate case, (2.39) requires that the parameters \( \alpha_0 \) and \( \beta_0 \) are of the same sign, in agreement with Ref.[16] (that reference however, does not establish the existence of solutions to (2.40)). In addition, it may be shown from (2.40) and (2.44) that in all cases \((\Delta x)^2\) is strictly negative, corresponding to a causal observation of the particle. This property is consistent with the conditions \( \Delta x \Delta q = 0 \) and \((\Delta q)^2 > 0\). Nevertheless, in spite of this causality, the above solutions are all tachyonic in the hyperbolic case! Their energy-momentum lies inside the light-cone only in the elliptic case, and on the light-cone in the parabolic case. For example in the degenerate case again, all solutions of curvature \((\kappa a > \beta_0/\alpha_0 > 0)\) are tachyonic. This completes the discussion of solutions of constant extrinsic curvature for an arbitrary choice of \( F(x) \). Solutions of non constant curvature are more difficult to come by, unless a specific choice is made for \( F(x) \) (see for example Ref.[16] in the case (1.5)).

3. The Hamiltonian Description

Corresponding to the action (2.1), the conjugate momenta are simply

\[
\begin{align*}
P_\mu &= \frac{\partial L}{\partial \dot{x}^\mu} = \mu c \lambda_\mu , \\
Q_\mu &= \frac{\partial L}{\partial \dot{q}^\mu} = -2\mu c \kappa \frac{F'(\kappa^2 K^2)}{\sqrt{-q^2}} \kappa K_\mu , \\
\pi_\mu &= \frac{\partial L}{\partial \lambda_\mu} = 0 .
\end{align*}
\]

(3.1)

The Poisson bracket structure on the associated phase space is thus the ordinary one, namely

\[
\{x^\mu, P_\nu\} = \delta^\mu_\nu , \quad \{q^\mu, Q_\nu\} = \delta^\mu_\nu , \quad \{\lambda^\mu, \pi_\nu\} = \delta^\mu_\nu .
\]

(3.2)

Clearly, due to its local gauge invariance and the presence of Lagrange multipliers, the Hamiltonian description of the system is subject to constraints [23]. The following are primary constraints

\[
\begin{align*}
\chi_1^\mu &= \pi^\mu = 0 , \\
\chi_2^\mu &= P_\mu - \mu c \lambda^\mu , \\
\phi_1 &= q^\mu Q_\mu = 0 .
\end{align*}
\]

(3.3)

There may exist further primary constraints however. This issue is easily settled by considering the total number of zero modes of the Hessian of the Lagrange function (2.2). An
explicit calculation shows that for a generic function $F(x)$, the constraints (3.3) are the full set of primary constraints. It is only in the degenerate case that an additional primary constraint arises [9], corresponding to (compare with (2.27))

$$\chi_3 = q^2 Q^2 + (\alpha_0 \mu c \kappa)^2 = 0. \tag{3.4}$$

Consequently, the Hamiltonian analysis requires a separate treatment only for the degenerate case. All other choices of $F(x)$ may be studied together which is done in the next section. The meaning of the constraints above is clear. The constraints $\chi_1^\mu = 0$ and $\chi_2^\mu = 0$ appear since $\lambda^\mu$ are Lagrange multipliers, actually also measuring the total energy-momentum of the particle. The constraint $\phi_1 = 0$ corresponds to the Noether identity (2.13) and is thus equivalent to the relation $nK = 0$ in (A.6). Therefore, this constraint will always appear whatever the dependence on all extrinsic $j$-torsions ($j = 1, 2, \ldots, D-1$) (see Appendix A) in the most general case. The constraint $\phi_1 = 0$ is not particular to our restriction of a dependence on the 1-torsion or extrinsic curvature only. Finally, the constraint $\chi_3 = 0$ is a direct representation of the fact that for the degenerate case the combination $xF'^2(x)$ is constant (see (2.23)).

3.1 The generic case

In the non degenerate case, the analysis of constraints shows that there is only one secondary constraint which actually corresponds to the canonical Hamiltonian. This secondary constraint is (compare with (2.25))

$$\phi_2 = qP + \mu c \sqrt{-q^2} \Phi(q^2 Q^2), \tag{3.5}$$

with the function $\Phi$ defined by

$$\Phi(q^2 Q^2) = F(x_0) - 2x_0 F'(x_0), \tag{3.6}$$

and $x_0$ being a solution to the equation

$$x_0 F'^2(x_0) = \frac{-q^2 Q^2}{(2 \mu c k)^2} > 0. \tag{3.7}$$

Therefore, in order to define the Hamiltonian description of generic rigid particles, the function $F(x)$ must also be such that given any $y > 0$ there always exists a unique $x > 0$ for which $xF'^2(x) = y$. This condition puts some restriction on the class of acceptable functions $F(x)$ in (2.2), which is assumed to be met in our analysis. However, one may also take the point of view that the Hamiltonian formulation is not necessarily directly related to the Lagrangian one in (2.2), in which case only $\Phi(q^2 Q^2)$ needs to be given and may be assumed to be any arbitrary non constant function ($\Phi(q^2 Q^2)$ constant indeed corresponds to the degenerate rigid particle). The first-class Hamiltonian $-H_*$ in the notation of Ref.[23] – including only the primary first-class constraints is obtained as

$$H_* = \phi_2 + u_1 \phi_1, \tag{3.8}$$
with \( u_1 \) being an arbitrary Lagrange multiplier for the primary constraint \( \phi_1 \).

It turns out that \( \chi_1^\mu \) and \( \chi_2^\mu \) are second-class constraints, while \( \phi_1 \) and \( \phi_2 \) are first-class ones. Solving for \( \chi_1^\mu \) and \( \chi_2^\mu \) through the associated Dirac brackets [23] is straightforward enough. As a result, phase space reduces to the variables \((x^\mu, \varphi)\) fundamental (Poisson-Dirac) brackets in (3.3), leaving only the first-class constraints \( \phi_1 \) and \( \phi_2 \) with the algebra

\[
\{ \phi_1, \phi_2 \} = -\phi_2 .
\]

Hence, the total Hamiltonian generating time evolution in \( \tau \) is simply

\[
H_T = \lambda_1 \phi_1 + \lambda_2 \phi_2 ,
\]

with \( \lambda_1 \) and \( \lambda_2 \) being arbitrary Lagrange multipliers. The above description thus provides the Hamiltonian definition of generic rigid particles. The associated first-order action is

\[
S[x^\mu, P^\mu; q^\mu, Q^\mu; \lambda_1, \lambda_2] = \int_{\tau_i}^{\tau_f} d\tau \left[ \dot{x}^\mu P_\mu + \dot{q}^\mu Q_\mu - \lambda_1 \phi_1 - \lambda_2 \phi_2 \right] .
\]

The generators of spacetime Poincaré transformations are obviously \( P^\mu \) and \( M^{\mu \nu} \) with

\[
M^{\mu \nu} = L^{\mu \nu} + S^{\mu \nu} ,
\]

\[
L^{\mu \nu} = P^\mu x^\nu - P^\nu x^\mu ,
\]

\[
S^{\mu \nu} = Q^\nu q^\mu - Q^\mu q^\nu .
\]

Indeed, the Poisson bracket algebra of \( P^\mu \) and \( M^{\mu \nu} \) is isomorphic to the Poincaré algebra. In addition, \( L^{\mu \nu} \) and \( S^{\mu \nu} \) separately define the Lorentz algebra, \( S^{\mu \nu} \) commutes with \( P^\mu \) and \( L^{\mu \nu} \), whereas \( L^{\mu \nu} \) and \( P^\mu \) also define the Poincaré algebra. The identification of \( S^{\mu \nu} \) with internal spin commuting with the orbital angular-momentum is thus consistent.

Given (3.10), it is possible to write down the Hamiltonian equations of motion (see (3.17) below). Using the equation for \( \dot{q}^\mu \) to express \( Q^\mu \) in terms of the remaining degrees of freedom – this requires \( \lambda_2 \) to be nowhere vanishing (at least in the interval \([\tau_i, \tau_f]\)), a condition which we therefore assume to be met throughout –, the action (3.11) reduces to

\[
S[x^\mu, P^\mu; q^\mu; \lambda_1, \lambda_2] = -\mu c \int_{\tau_i}^{\tau_f} d\tau \left[ \lambda_2 \sqrt{-q^2} F(\kappa^2 \frac{(\dot{q}^\mu - \lambda_1 q^\mu)^2}{\lambda_2^2 (q^2)^2}) + \frac{1}{\mu c} P_\mu (\lambda_2 q^\mu - \dot{x}^\mu) \right] .
\]

Clearly, any dependence on \( \lambda_2 \) may be absorbed into a rescaling of \( q^\mu \). The resulting action is then still as in (3.13), with \( \lambda_2 \) then set equal to 1, \( \lambda_1 \) replaced by \( \lambda_3 = \lambda_1 + \lambda_2 / \lambda_2 \) and \( F(x) \) multiplied by \( \text{sign} \lambda_2 \). Obviously, the same remark applies to the Hamiltonian formulation before the Lagrangian reduction of \( Q^\mu \) is performed. Simply \( q^\mu \) and \( Q^\mu \) are rescaled as \( \tilde{q}^\mu = \lambda_2 q^\mu \) and \( \tilde{Q}^\mu = Q^\mu / \lambda_2 \) – a transformation which preserves the canonical brackets (3.2) –, \( \lambda_2 \) is set to 1, \( \lambda_1 \) is shifted into \( \lambda_3 = \lambda_1 + \lambda_2 / \lambda_2 \) and \( F(x) \) is multiplied by \( \text{sign} \lambda_2 \). As will become clear shortly when considering the local Hamiltonian gauge invariances of (3.11), \( \lambda_2 q^\mu, Q^\mu / \lambda_2 \) and \( \lambda_3 \) are indeed combinations of degrees of freedom invariant under transformations generated by \( \phi_1 \).
To conclude as to the equivalence of (3.13) with (2.1) up to the sign of $\lambda_2$, it is still necessary to solve for $\lambda_1$ using the constraint $\phi_1 = 0$, namely

$$\lambda_1 = \frac{q\dot{q}}{q^2}.$$  \hfill (3.14)

Upon rescaling of $q^\mu$ by $\lambda_2$, this amounts to the relation

$$\lambda_3 = \lambda_1 + \frac{\dot{\lambda}_2}{\lambda_2} = \frac{\dot{q}\dot{q}}{q^2}, \quad \tilde{q}^\mu = \lambda_2 q^\mu.$$  \hfill (3.15)

Note that this expression for $\lambda_3$ coincides with the value for $\dot{e}/e$ with $e$ being the induced world-line einbein $\kappa|e| = \sqrt{-\dot{q}^2}$. The combination $\lambda_3 = \lambda_1 + \lambda_2/\lambda_2$ is indeed related to an intrinsic world-line einbein, as will soon become clear.

It is well known [23] that first-class constraints generate local (Hamiltonian) gauge symmetries through Poisson brackets. Considering the infinitesimal generator

$$\phi_\epsilon = \epsilon_1 \phi_1 + \epsilon_2 \phi_2,$$  \hfill (3.16)

with $\epsilon_1(\tau)$ and $\epsilon_2(\tau)$ arbitrary infinitesimal functions, the associated transformations are

$$\delta_\epsilon x^\mu = \epsilon_2 q^\mu,$$

$$\delta_\epsilon P^\mu = 0,$$

$$\delta_\epsilon q^\mu = \epsilon_1 q^\mu + \epsilon_2 \frac{q^2 \sqrt{-q^2}}{2\mu c k^2 F'(x_0)} Q^\mu,$$

$$\delta_\epsilon Q^\mu = -\epsilon_1 Q^\mu + \epsilon_2 \left[-P^\mu + \mu c \frac{q^\mu}{\sqrt{-q^2}} \Phi(q^2 Q^2) - \frac{q^2 \sqrt{-q^2}}{2\mu c k^2 F'(x_0)} q^\mu \right],$$

$$\delta_\epsilon \lambda_1 = \dot{\epsilon}_1,$$

$$\delta_\epsilon \lambda_2 = \dot{\epsilon}_2 + \lambda_1 \epsilon_2 - \lambda_2 \epsilon_1.$$  \hfill (3.17)

(Note that the Hamiltonian equations of motion for $(\dot{x}^\mu, \dot{P}^\mu, \dot{q}^\mu, \dot{Q}^\mu)$ are obtained from the r.h.s. expressions in $(\delta_\epsilon x^\mu, \delta_\epsilon P^\mu, \delta_\epsilon q^\mu, \delta_\epsilon Q^\mu)$ through the substitutions $\epsilon_1 = \lambda_1$ and $\epsilon_2 = \lambda_2$). Correspondingly, the variation of the first-order action (3.11) is simply

$$\delta_\epsilon S = \int_{\tau_i}^{\tau_f} d\tau \left\{ \epsilon_2 \left( \frac{\partial \phi_2}{\partial Q^\mu} Q_\mu - \mu c \sqrt{-q^2} \Phi(q^2 Q^2) \right) \right\}.$$  \hfill (3.18)

This expression is independent of $\epsilon_1$. Therefore, when requiring the action to be exactly invariant – as opposed to a possible surface term – only $\epsilon_2(\tau)$ is restricted by the boundary conditions

$$\epsilon_2(\tau_i) = 0, \quad \epsilon_2(\tau_f) = 0,$$  \hfill (3.19)

whereas $\epsilon_1(\tau)$ remains totally arbitrary. This is a first indication that the (Hamiltonian) generator of world-line reparametrisations must involve the constraint $\phi_2.$
Hence, transformations generated by \( \phi_1 \) alone are not related to world-line diffeomorphisms. Actually, these transformations are easily integrated for finite ones leading to

\[
\begin{align*}
x^{\mu}(\tau) &= x^{\mu}(\tau), \quad P^{\mu}(\tau) = P^{\mu}(\tau), \\
q^{\mu}(\tau) &= \left[1 + h(\tau)\right] q^{\mu}(\tau), \quad Q^{\mu}(\tau) = \frac{1}{1 + h(\tau)} Q^{\mu}(\tau), \\
\lambda_1'(\tau) &= \lambda_1(\tau) + \frac{\dot{h}(\tau)}{1 + h(\tau)}, \quad \lambda_2' = \frac{1}{1 + h(\tau)} \lambda_2(\tau), \\
\lambda_3'(\tau) &= \lambda_3(\tau),
\end{align*}
\tag{3.20}
\]

where \( h(\tau) \) is an arbitrary function with the only restriction that \( (1 + h(\tau)) \) must be strictly positive. The local gauge symmetry generated by \( \phi_1 \) – the constraint expressing the orthogonality of the tangent and extrinsic curvature vectors \( n^\mu \) and \( K^\mu \) – thus induces a rescaling of \( q^\mu \), \( Q^\mu \) and \( \lambda_2 \) and a shift of \( \lambda_1 \) such that \( \tilde{q}^\mu = \lambda_2 q^\mu \), \( \tilde{Q}^\mu = Q^\mu / \lambda_2 \) and \( \lambda_3 = \lambda_1 + \lambda_2 / \lambda_2 \) are invariant. The existence of this symmetry thus explains why \( \lambda_2 \) scales out when using the variables \((\tilde{q}^\mu, \tilde{Q}^\mu, \lambda_3)\). Since \( \lambda_2 \) is assumed to be nowhere vanishing, note that there always exists a \( h \)-transformation (3.20) with \( (h(\tau) = |\lambda_2(\tau)| - 1) \) such that

\[
\begin{align*}
q^{\mu}(\tau) &= \sigma \tilde{q}^\mu(\tau), \quad Q^{\mu}(\tau) = \sigma \tilde{Q}^\mu(\tau), \\
\lambda_1'(\tau) &= \lambda_3'(\tau) = \lambda_3(\tau) = \lambda_1(\tau) + \frac{\dot{\lambda}_2(\tau)}{\lambda_2(\tau)}, \quad \lambda_2'(\tau) = \sigma,
\end{align*}
\tag{3.21}
\]

where \( \sigma = \text{sign}(\lambda_2(\tau)) = \pm 1 \). Hence, without loss of information concerning all possible gauge inequivalent configurations of the system, we may always assume that \( \lambda_2(\tau) = \pm 1 \).

To discover how transformations also involving \( \phi_2 \) generate the remaining local gauge invariance of the system – namely local world-line reparametrisations – it is useful to consider the variation under (3.16) and (3.17) of the invariant combinations \( \tilde{q}^\mu \), \( \tilde{Q}^\mu \) and \( \lambda_3 \). This suggests that infinitesimal (Hamiltonian) world-line reparametrisations are generated by the combination

\[
\phi^{(R)}_\epsilon = \left[ \frac{d}{d\tau} \left( \frac{\epsilon_2}{\lambda_2} \right) + \lambda_1 \frac{\epsilon_2}{\lambda_2} \right] \phi_1 + \epsilon_2 \phi_2.
\tag{3.22}
\]

Indeed, with the identification

\[
\epsilon_2(\tau) = \lambda_2(\tau) \eta(\tau),
\tag{3.23}
\]

and using the Lagrangian reduction of \( Q^\mu \), \( q^\mu \) and \( P^\mu \) following from the Hamiltonian equations of motion, it is easily seen that the transformations \( \delta_\epsilon x^{\mu} \) and \( \delta_\epsilon q^{\mu} \) (and \( \delta_\epsilon P^{\mu} \)) induced by \( \phi^{(R)}_\epsilon \) in (3.22) agree with the infinitesimal (Lagrangian) reparametrisations in (2.11). Also, note that the total Hamiltonian (3.10) which generates evolution in \( \tau \) coincides with the generator (3.22) of reparametrisations in \( \tau \) when choosing \( \epsilon_2(\tau) = \lambda_2(\tau) \), which also corresponds to \( \eta(\tau) = 1 \) in (3.23).

Actually, having identified the (Hamiltonian) generator of local world-line diffeomorphisms, it becomes possible to integrate the associated transformations to finite ones, at
least for the Lagrange multipliers $\lambda_1$ and $\lambda_2$. For this purpose, first note that $\lambda_3$ varies under $\phi^{(R)}_\epsilon$ as

$$\delta_\epsilon \lambda_3 = \frac{d}{d\tau} [\dot{\eta} + \eta \lambda_3] , \quad (3.24)$$

where $\eta(\tau) = \epsilon_2(\tau)/\lambda_2(\tau)$. Let us introduce the quantity

$$e(\tau) = e_i \exp \int_{\tau_i}^{\tau} d\tau' \lambda_3(\tau') = e_i \frac{|\lambda_2(\tau)|}{|\lambda_2(\tau_i)|} \exp \int_{\tau_i}^{\tau} d\tau' \lambda_1(\tau') , \quad (3.25)$$

with $e_i = e(\tau_i)$ being an arbitrary integration constant extraneous to the system. Assuming that $e_i$ transforms under $\phi^{(R)}_\epsilon$ as

$$\delta_\epsilon e_i = \dot{\eta}(\tau_i) e_i + \eta(\tau_i) \dot{e}(\tau_i) , \quad (3.26)$$

(imposing the boundary condition (3.19) would imply $\eta(\tau_i) = 0$ ) the variation of $e(\tau)$ under $\phi^{(R)}_\epsilon$ is simply

$$\delta_\epsilon e(\tau) = \frac{d}{d\tau} \left[ \eta(\tau) e(\eta) \right] . \quad (3.27)$$

(This result is of course consistent with (3.26) and the fact that $e_i = e(\tau_i)$). Hence, $e(\tau)$ is identified with an intrinsic world-line einbein which however, couples to the system only through the combination

$$\lambda_1 + \frac{\dot{\lambda}_2}{\lambda_2} = \lambda_3 = \frac{\dot{e}}{e} . \quad (3.28)$$

Using this fact and the infinitesimal variations of $\lambda_1$ and $\lambda_2$ under $\phi^{(R)}_\epsilon$, it is straightforward to obtain their transformations for finite world-line reparametrisations as

$$\lambda'_1(\tau) = \dot{f}(\tau) \lambda_1(f(\tau)) + \frac{\dot{f}(\tau)}{f(\tau)} ,$$

$$\lambda'_2(\tau) = \lambda_2(f(\tau)) , \quad (3.29)$$

$$\lambda'_3(\tau) = \dot{f}(\tau) \lambda_3(f(\tau)) + \frac{\dot{f}(\tau)}{f(\tau)} ,$$

with $f(\tau)$ being an arbitrary function such that $f(\tau_i) = \tau_i$ and $f(\tau_f) = \tau_f$, corresponding to the world-line diffeomorphism. Infinitesimal variations are then obtained with $f(\tau) = \tau + \epsilon_2(\tau)/\lambda_2(\tau)$.

The algebra of constraints (3.9) being closed, the notion of Teichmüller space is applicable. Having obtained the explicit form for finite gauge transformations of Lagrange multipliers, the issue of gauge fixing of the system through gauge fixing in Teichmüller space may be addressed (for a general discussion of these points, see Ref.[23]). Teichmüller space – namely [25,23] the quotient of the space of Lagrange multipliers by the group of local (Hamiltonian) gauge transformations generated by all first-class constraints – is
very simple for the present system. In fact, it reduces to only two points with the simple representation

\[ \lambda_1(\tau) = 0 , \quad \lambda_2(\tau) = 1 \quad \text{or} \quad \lambda_2(\tau) = -1 . \] (3.30)

Indeed, under gauge transformations generated by \( \phi_1 \) and \( \phi_2 \), any configuration for \((\lambda_1, \lambda_2)\) – with \( \lambda_2 \) nowhere vanishing in \([\tau_i, \tau_f]\) – is always related to one of the two configurations (3.30). On the one hand, as was remarked in (3.21), \( \lambda_2(\tau) \) may always be set equal to \( \text{sign} \lambda_2 = \pm 1 \) through some \( h \)-transformation. Note that the configurations \( \lambda_2(\tau) = \pm 1 \) are invariant under world-line diffeomorphisms in (3.29). On the other hand, any configuration of \( \lambda_3(\tau) \) may always be set to zero through some world-line reparametrisation. As is well known, given an intrinsic einbein \( e(\tau) \), there always exists a local world-line diffeomorphism such that \( e(\tau) \) is transformed into the constant configuration

\[ \frac{1}{\Delta \tau} \int_{\tau_i}^{\tau_f} d\tau \ e(\tau) , \] (3.31)

corresponding to the total world-line intrinsic “length” of the interval \([\tau_i, \tau_f]\). Using this transformation and the correspondence (3.28), it is clear that any \( \lambda_3 \) configuration is gauge equivalent to \( \lambda_3 = 0 \). Hence finally, combining the two classes of local gauge transformations, any configuration of \((\lambda_1(\tau), \lambda_2(\tau))\) is gauge equivalent to one of the configurations in (3.30): Teichmüller space consists of only two points.

This conclusion also enables us to consider the issue [23] of a complete and global, hence admissible gauge fixing of the system through gauge fixing in Teichmüller space. Indeed, once the configuration (3.30) is reached, there are no further non trivial local gauge transformations possible leaving (3.30) invariant, as is easily seen from (3.20) and (3.29). Hence, any specific choice for \((\lambda_1(\tau), \lambda_2(\tau))\) defines a complete gauge fixing of the system, with the sign of \( \lambda_2(\tau) \) determining which of the two Teichmüller points is selected. Such a gauge fixing however, is not global and thus not admissible in Teichmüller space since only one of its two elements is singled out. Nevertheless, a gauge fixing leading to a unique specification for \((\lambda_1(\tau), \lambda_2(\tau))\) is global and thus admissible for the system itself. This follows by considering the issue of modular invariance, namely transformations under orientation reversing world-line diffeomorphisms. As is the case [25,23] for the ordinary relativistic particle, the Hamiltonian description of rigid particles is not modular invariant, even though the original Lagrangian (2.1) possesses this symmetry. Modular transformations act on phase space as

\[ x^\mu(\tau) \to x^\mu(\tau) , \quad P^\mu(\tau) \to -P^\mu(\tau) , \]
\[ q^\mu(\tau) \to -q^\mu(\tau) , \quad Q^\mu(\tau) \to Q^\mu(\tau) , \]
\[ \lambda_1(\tau) \to \lambda_1(\tau) , \quad \lambda_2(\tau) \to -\lambda_2(\tau) . \] (3.32)

In particular, note that in the same way [25,23] as for the ordinary particle, the modular group \( \mathbb{Z}_2 \) acts by exchanging the particle with its antiparticle, since the sign of the energy \( P^0 \) is then reversed. Therefore, modular invariance of the original Lagrangian (2.1) is enforced in the Hamiltonian description by requiring that \( \lambda_2(\tau) \) be, say, positive, in order to describe a rigid particle as opposed to its antiparticle – the latter identification being
also correlated to the sign of $F(x)$. Hence, by specifying $(\lambda_1(\tau), \lambda_2(\tau))$ uniquely, with $\lambda_2(\tau)$ say positive, a complete and global, hence admissible gauge fixing of the system is effected. Such a procedure singles out one of the two Teichmüller points, corresponding to all Lagrange multiplier configurations which are gauge equivalent to

$$\lambda_1(\tau) = 0 , \quad \lambda_2(\tau) = +1 . \quad (3.33)$$

Note that this choice precisely corresponds to the proper-time gauge (2.28) used in solving the equations of motion. Moreover, the configuration (3.33) also defines modular space – namely [25,23] the quotient of Teichmüller space by the modular group – which indeed reduces to a single point for the present system due to the action (3.32) of the modular group on Teichmüller space.

### 3.2 The degenerate case

The primary constraints in the degenerate case were already given in (3.3) and (3.4). The analysis of constraints reveals two further constraints, namely

$$\phi_2 = qP + \mu c \beta_0 \sqrt{-q^2} , \quad \chi_4 = PQ , \quad (3.34)$$

while the first-class Hamiltonian $H_*$ including only first-class primary constraints is obtained as

$$H_* = \phi + u_1 \phi_1 , \quad (3.35)$$

where $u_1$ is an arbitrary Lagrange multiplier for the primary constraint $\phi_1$ and $\phi$ is defined by

$$\phi = \phi_2 + \frac{1}{2(\mu c \beta_0)} \frac{[P^2 + (\mu c \beta_0)^2]}{(\alpha_0 \mu c \kappa)^2} \sqrt{-q^2} \chi_3 . \quad (3.36)$$

As in the generic case, the constraints $\chi^\mu_1$ and $\chi^\mu_2$ turn out to be second-class ones which are easily reduced by using Dirac brackets. Doing so leaves only the conjugate phase space degrees of freedom $(x^\mu, P_\mu; q^\mu, Q_\mu)$ with the brackets (3.2). The remaining constraints are then (compare with (2.26) and (2.27))

$$\phi_1 = qQ = 0 , \quad \phi_2 = qP + \mu c \beta_0 \sqrt{-q^2} = 0 , \quad (3.37)$$

$$\chi_3 = q^2 Q^2 + (\alpha_0 \mu c \kappa)^2 = 0 , \quad \chi_4 = PQ = 0 ,$$

with the brackets

$$\{\phi_1, \phi_2\} = -\phi_2 , \quad \{\phi_1, \chi_3\} = 0 , \quad \{\phi_1, \chi_4\} = \chi_4 ,$$

$$\{\phi_2, \chi_3\} = 2q^2 \chi_4 + 2\mu c \beta_0 \sqrt{-q^2} \phi_1 , \quad \{\phi_2, \chi_4\} = [P^2 + (\mu c \beta_0)^2] - \frac{\mu c \beta_0}{\sqrt{-q^2}} \phi_2 , \quad (3.38)$$

$$\{\chi_3, \chi_4\} = 2Q^2 qP .$$
From these results, it follows that the four constraints must separate into two first-class and two second-class constraints. Indeed, the number of second-class constraints must be even and $\chi_4$ is clearly second-class whereas $\phi_1$ is obviously first-class. Given (3.38), one easily finds that the two first-class constraints are precisely $\phi_1$ and $\phi$ given above, with the algebra

$$\{\phi_1, \phi\} = -\phi.$$  

(3.39)

Of course, that these are the two first-class constraints of the system is to be expected since the first-class Hamiltonian $H_*$ in (3.35) involves them both. The two remaining second-class constraints are then

$$\chi = \alpha_2 \phi_2 + \alpha_3 \chi_3, \quad \chi_4 = PQ,$$  

(3.40)

where $\alpha_2$ and $\alpha_3$ are arbitrary functions on phase space such that

$$\alpha_2 \left[ P^2 + (\mu c \beta_0)^2 \right] - 2(\mu c \beta_0) \frac{(\alpha_0 \mu c \kappa)^2}{\sqrt{-q^2}} \alpha_3 \neq 0.$$  

(3.41)

The total Hamiltonian of the system is therefore

$$H = \lambda_1 \phi_1 + \lambda_2 \phi,$$  

(3.42)

with $\lambda_1$ and $\lambda_2$ being arbitrary Lagrange multipliers for the two independent first-class constraints. As in the generic case, the constraint $\phi_1$ generates local Hamiltonian gauge symmetries corresponding to local rescalings of $q^\mu, Q^\mu$ and $\lambda_2$ and local shifts of $\lambda_1$, as given in (3.20). On the other hand, the algebra (3.39) shows that $\phi$ is now the generator of local world-line reparametrisations as opposed to $\phi_2$ in the generic case. All the considerations that applied to $\phi_2$ and its generated symmetries in the generic case should now apply essentially to $\phi$ in the degenerate case. However, the complete analysis of local gauge invariances associated to the two first-class constraints $\phi_1$ and $\phi$ along the lines of the previous section in the generic case – including the discussion of Teichmüller and modular spaces and gauge fixing – is not developed here and is left for future work. The reason is that the remaining two second-class constraints need to be included as well, and the present author has not been able so far to do this is any satisfactory way. On the one hand, introducing Dirac brackets associated to $\chi$ and $\chi_4$ in (3.40) is possible but far from being elegant. Not only are the corresponding expressions rather involved, rendering any possible geometrical or physical insight at least very difficult if not impossible, but the reduced Hamiltonian formulation is then no longer manifestly Poincaré covariant – a symmetry which one would like to maintain as much as is possible. On the other, there exist other possible approaches which in principle preserve manifest Poincaré covariance by extending the formulation of the system in one or another way – either [26] by introducing further degrees of freedom of opposite Grassmann parities or [27] by viewing the second-class constraints as resulting from the gauge fixing of some other system which encompasses the present one and possesses first-class constraints only (these two points of view are probably related). However, it has not been possible so far to complete either of these two programs for degenerate rigid particles.
For these reasons, a detailed and complete Hamiltonian analysis of degenerate rigid particles is left for subsequent work. Such a discussion is also necessary in order to address in a satisfactory and complete way the canonical quantisation of these systems and in particular their quantum physical spectrum. Therefore, the remainder of this paper will be concerned with the canonical quantisation of generic rigid particles only.

4. Canonical Quantisation of Generic Rigid Particles

Naively, canonical quantisation of generic rigid particles would proceed from their Hamiltonian formulation discussed in sect.3.1. Heisenberg commutation relations for the fundamental degrees of freedom would simply follow from the Poisson brackets (3.2) and in the associated representation space of quantum states – necessarily equivalent to a wave-function representation – physical states would be identified as being those states annihilated by two quantum operators in direct correspondence with the first-class constraints $\phi_1$ and $\phi_2$ for some consistent choice of normal ordering of composite operators. However, this approach – the one so far always adopted for rigid particles [8-10,22,28] – overlooks one important feature concerning the degrees of freedom $q^\mu$, namely the fact that this sector of phase space is restricted by the requirement ($-q^2 > 0$) or in other words that $q^\mu$ must lie inside the light-cone. Hence, in the same way that the canonical quantisation of the nonrelativistic particle moving freely on the positive real axis needs some specification [29], we must – if we are to avoid using the more abstract methods of geometric quantisation [30,29] – first find a (canonical) transformation for the restricted degrees of freedom $q^\mu$ and $Q^\mu$ such that the new set of variables is unrestricted and preferably, is also equipped with a canonical symplectic structure. Then, in terms of the transformed degrees of freedom, the above quantisation program may be applied. Such a set of transformed degrees of freedom indeed exists [24] for rigid particles.

4.1 The unrestricted phase space map

Consider the following definitions

$$ y^0 = \eta \sqrt{-q^2} , \quad R^0 = \frac{-q^0 Q^0 + \vec{q} \cdot \vec{Q}}{\sqrt{-q^2}} , \quad (4.1a) $$

$$ y^i = \eta \frac{q^i}{\sqrt{-q^2}} , \quad R^i = \eta \sqrt{-q^2} [Q^i - \frac{q^i}{q^0} Q^0] , \quad (4.1b) $$

where $\eta$ is the sign of $q^0$ and $(i = 1, 2, \cdots, D - 1)$ are space indices. The inverse relations are

$$ q^0 = y^0 \sqrt{1 + \vec{y}^2} , \quad Q^0 = \sqrt{1 + \vec{y}^2} \left[ -R^0 + \frac{\vec{y} \cdot \vec{R}}{y^0} \right] , \quad (4.2a) $$

$$ q^i = y^0 y^i , \quad Q^i = \frac{R^i}{y^0} + y^i \left[ -R^0 + \frac{\vec{y} \cdot \vec{R}}{y^0} \right] \quad (4.2b) $$

19
In geometrical terms, \( y^0 \) measures the invariant length of the vector \( q^\mu \) with a sign related to whether \( q^\mu \) lies in the forward or in the backward light-cone, while the remaining variables \( y^i \) are in fact the parameters \( \Lambda^0_i \) of the Lorentz boost in the direction \( \vec{q} \) mapping the vector \( q^\mu = (q^0, \vec{q}) \) into the vector \( (y^0, \vec{0}) \). The variables \( R^0 \) and \( R^i \) are then obtained as degrees of freedom conjugate to \( y^0 \) and \( y^i \) respectively. Namely, the Poisson brackets

\[
\{q^\mu, Q^\nu\} = \eta^{\mu\nu},
\]

and the following canonical brackets

\[
\{y^0, R^0\} = 1, \quad \{y^i, R^j\} = \delta^{ij},
\]

are mapped into one another under the transformations (4.1) and (4.2).

Clearly, the canonically conjugate degrees of freedom \( (y^0, R^0; y^i, R^i) \) are no longer restricted as are the original ones \( (q^\mu, Q^\mu) \), thereby achieving the required properties. However, the price to pay is a loss of manifest Lorentz covariance. Spacetime translations generated by \( P^\mu \) and space rotations generated by \( M^{ij} = L^{ij} + S^{ij} \) are still manifest symmetries in the transformed representation of the system, but this is no longer the case for Lorentz boost generators \( M^{0i} = I^{0i} + S^{0i} \). Indeed, while the expressions (3.12) for \( L^{\mu\nu} \) are not affected by the redefinitions (4.2), those for the spin tensor become

\[
S^{0i} = -R^i \sqrt{1 + \vec{y}^2}, \quad S^{ij} = R^i y^j - R^j y^i.
\]

Nevertheless, it is a straightforward calculation to check that with the brackets (4.4), the full Poincaré algebra is still obtained for the generators \( P^\mu \) and \( M^{\mu\nu} \) expressed in terms of the transformed variables (4.1), thereby establishing the consistency of this alternative Hamiltonian description of generic rigid particles (the redefinitions (4.1) are of course also applicable in the degenerate case but only the complete Hamiltonian analysis in that case would confirm whether these redefinitions are also appropriate for degenerate rigid particles). In the generic case, the first-class constraints (3.3) and (3.5) and the associated Hamiltonian (3.10) are then given by

\[
\phi_1 = y^0 R^0,
\]

and

\[
\phi_2 = y^0 [\vec{y} \vec{R} - P^0 \sqrt{1 + \vec{y}^2}] + \eta \mu c y^0 \Phi \left( (y^0 R^0)^2 - (\vec{y} \vec{R})^2 - \vec{R}^2 \right),
\]

with

\[
q^2 Q^2 = (y^0 R^0)^2 - (\vec{y} \vec{R})^2 - \vec{R}^2.
\]

From these expressions and the brackets (4.4), the gauge algebra (3.9) is obviously also recovered. The present Hamiltonian formulation of generic rigid particles is thus the one appropriate for their canonical quantisation.
4.2 A mixed Lorentz-gravitational anomaly

Quantised generic rigid particles are thus specified by the Heisenberg commutation relations
\[ [x^\mu, P^\nu] = i\hbar\eta^{\mu\nu}, \quad [y^0, R^0] = i\hbar, \quad [y^i, R^j] = i\hbar\delta^{ij}, \quad (4.9) \]
and an abstract representation space of this algebra equipped with an inner product for which these operators are all hermitian and self-adjoint. Representations of this algebra are unitarily equivalent [23] to wave-function ones either in position or in momentum space for each pair of conjugate degrees of freedom. This determines the space of quantum states for such systems, each of these states being therefore of positive norm.

Turning to the ordering problem, let us first consider the situation for the Poincaré generators. Clearly, \( P^\mu \) does not require an ordering prescription. For \( L^{\mu\nu} \) and \( S^{\mu\nu} \) we choose
\[ L^{\mu\nu} = P^\mu x^\nu - P^\nu x^\mu, \quad (4.10) \]
and
\[ S^{0i} = -\frac{1}{2} \left[ R^i \sqrt{1 + \vec{y}^2} + \sqrt{1 + \vec{y}^2} \ R^j \right], \quad S^{ij} = R^i y^j - R^j y^i, \quad (4.11) \]
in order that these operators be hermitian and self-adjoint. Obviously, \( L^{\mu\nu} \) and \( P^\mu \) generate the Poincaré algebra. On the other hand, while it is clear that \( S^{ij} \) generates the algebra of rotations in space, it is not difficult to check that with the choice of ordering in (4.11) the operators \( S^{ij} \) and \( S^{0i} \) in fact obey the whole Lorentz algebra. Thus, the total angular-momentum \( (M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}) \) and energy-momentum \( P^\mu \) operators generate the whole Poincaré algebra, thereby establishing that this algebra is anomaly free and that in spite of the loss of manifest spacetime covariance, quantum states of rigid particles indeed span a linear representation space for spacetime translations and Lorentz transformations. However, since wave-function representations of the Heisenberg algebra are single-valued, the space of states of quantised generic rigid particles can only support integer spin representations of the Lorentz group (though erroneous [9,10,28], claims for half-integer spin have indeed been made [10,14] in the literature).

Let us now turn to the ordering problem for the first-class constraints \( \phi_1 \) and \( \phi_2 \), a necessary prerequisite in order to define physical, i.e. gauge invariant states of quantum rigid particles. Again, in order to have hermitian and self-adjoint operators, we must choose for the quantum constraints
\[ \phi_1 = \frac{1}{2} \left[ y^0 R^0 + R^0 y^0 \right], \quad (4.12a) \]
and
\[ \phi_2 = y^0 \left[ \vec{y} \cdot \vec{R} - R^0 \sqrt{1 + \vec{y}^2} \right] + \frac{1}{2} \eta \mu c \left[ y^0 \Phi ( y^0 R^0 )^2 : - : ( y^0 \cdot \vec{R} )^2 : - \vec{R}^2 \right] + \Phi ( ( y^0 R^0 )^2 : - : ( \vec{y} \cdot \vec{R} )^2 : - \vec{R}^2 : - y^0 ) y^0 \], \quad (4.12b) \]
where \( ( y^0 R^0 )^2 : \) and \( ( \vec{y} \cdot \vec{R} )^2 : \) stand for normal ordered expressions of the corresponding operators to be specified presently. By considering all possible orderings for the products
in these operators, one concludes that all possible choices always reduce to expressions of the following form

\[ (y^0 \dot{R}^0)^2 = R^0 y^0 \dot{y}^0 + i\hbar A_1 y^0 \dot{R}^0 + \hbar^2 A_2, \quad (4.13a) \]

\[ (\vec{y} \cdot \vec{R})^2 = R^i y^i \dot{R}^i + \hbar B_1 y^i \dot{y}^i + \hbar^2 B_2, \quad (4.13b) \]

where \( A_1, A_2, B_1 \) and \( B_2 \) are undetermined free complex coefficients. Requiring that these operators be also hermitian and self-adjoint only leads to the restrictions

\[ A_1^* = -A_1, \quad A_2^* = A_2 - A_1, \quad (4.14a) \]
\[ B_1^* = -B_1, \quad B_2^* = B_2 - (D - 1)B_1. \quad (4.14b) \]

With these definitions, it is now possible to determine the commutation relations for the quantum gauge algebra. One easily finds

\[ [\phi_1, \phi_2] = -i\hbar \phi_2. \quad (4.15) \]

Comparison with the classical bracket (3.9) shows that the gauge algebra is indeed anomaly free. Therefore, both local world-line reparametrisations and the local rescalings generated by \( \phi_1 \) are symmetries of quantised generic rigid particles. From that point of view, it is thus meaningful to define their quantum physical states \( |\psi> \) as being the solutions to the conditions

\[ \phi_1 |\psi> = 0, \quad \phi_2 |\psi> = 0, \quad (4.16) \]

thereby ensuring invariance of these states under all local gauge symmetries including local world-line reparametrisations.

However, this definition must also be consistent with the other symmetries of the system. Namely, the generators of gauge symmetries must commute with those of spacetime Poincaré transformations, as do the corresponding classical brackets. Otherwise, physical states solving (4.16) cannot define linear representations of the Poincaré group. In other words, a state physical in a given reference frame would no longer be physical in some other frame! Nor would it be possible to define consistently the mass or the spin, or both these quantities for physical states!

Clearly, this type of problem does not arise for the gauge generator \( \phi_1 \) since

\[ [L^{\mu\nu}, \phi_1] = 0, \quad [S^{\mu\nu}, \phi_1] = 0, \quad [M^{\mu\nu}, \phi_1] = 0, \quad (4.17) \]
\[ [P^\mu, \phi_1] = 0. \]

Moreover, we also have for the generator of world-line reparametrisations

\[ [P^\mu, \phi_2] = 0. \quad (4.18) \]

Therefore, at least the energy-momentum \( P^\mu \) hence also the mass \( (M^2 = -P^2/c^2) \) of quantum physical states are well defined observables for generic rigid particles. To analyse
the situation for the remaining commutators $[M^{\mu\nu}, \phi_2]$, it is useful to decompose $\phi_2$ in (4.12b) as $\phi_2 = \chi_1 + \chi_2$ with

$$\chi_1 = y^0 \left[ \vec{y} \cdot \vec{P} - P^0 \sqrt{1 + \vec{y}^2} \right]. \quad (4.19)$$

A simple calculation then finds that

$$\begin{align*}
[L^0, \chi_1] &= i\hbar (P^0 y^0 y^i - P^i y^0 \sqrt{1 + \vec{y}^2}) = -[S^0, \chi_1], \quad (4.20a) \\
[L^j, \chi_1] &= i\hbar (P^j y^0 y^i - P^j y^0 y^j) = -[S^{ij}, \chi_1], \quad (4.20b)
\end{align*}$$

leading to

$$\begin{align*}
[M^{\mu\nu}, \chi_1] &= 0, \quad (4.21)
\end{align*}$$

and

$$\begin{align*}
[M^{\mu\nu}, \phi_2] &= [M^{\mu\nu}, \chi_1]. \quad (4.22)
\end{align*}$$

Moreover, since $L^{\mu\nu}$ clearly also commutes with $\chi_2$, only the commutators of $S^{\mu\nu}$ with $\chi_2$ are left to be computed. In fact, since both $y^0$ and $R^0$ commute with $S^{\mu\nu}$, the crucial commutators to be determined are those of $S^{\mu\nu}$ with $\left( : (y^0 R^0)^2 : - : (\vec{y} \cdot \vec{R})^2 : -\vec{R}^2 \right)$. Using the normal ordered expressions (4.13), a direct calculation shows that

$$\begin{align*}
[S^{ij}, : (y^0 R^0)^2 : - : (\vec{y} \cdot \vec{R})^2 : -\vec{R}^2] &= 0, \quad (4.23)
\end{align*}$$

so that finally

$$\begin{align*}
[M^{ij}, \phi_2] &= 0. \quad (4.24)
\end{align*}$$

This result is indeed to be expected owing to the manifest rotation covariance of the quantisation procedure.

On the other hand, the commutator with Lorentz boost generators gives

$$\begin{align*}
[S^{0i}, : (y^0 R^0)^2 : - : (\vec{y} \cdot \vec{R})^2 : -\vec{R}^2] &= \frac{1}{2} i\hbar \frac{y^i}{(1 + \vec{y}^2)^{3/2}} + \\
&+ \frac{1}{2} \hbar^2 B_1 \left[ R^i \sqrt{1 + \vec{y}^2} + \frac{1}{\sqrt{1 + \vec{y}^2}} \vec{R}^i \right]. \quad (4.25)
\end{align*}$$

Hence, we certainly have for any choice of $F(\kappa^2 K^2)$ in the generic case

$$\begin{align*}
[M^{0i}, \phi_2] &\neq 0. \quad (4.26)
\end{align*}$$

This result thus represents [24] a mixed Lorentz-gravitational anomaly in the commutator of world-line reparametrisations and Lorentz boosts. Generally, this anomaly is of order $\hbar^2$ unless an ordering for $\phi_2$ corresponding to $B_1 = 0$ in (4.13b) happens to be chosen, in which case the anomaly is of order $\hbar^3$. Therefore, given any ordering for the generator of local world-line reparametrisations, physical states (4.16) do not transform covariantly under Lorentz boosts! The subspace of physical states (4.16) is not closed under the action
of Lorentz generators, even though these generators act covariantly on the entire space of states and the gauge algebra is anomaly free.

5. Conclusions

By paying closer attention to some issues not always properly addressed in previous works, this paper considered classical and quantum causal rigid particles for any possible dependence of their action on the world-line extrinsic curvature. At the classical level, general classes of solutions of constant extrinsic curvature were constructed, extending results of Refs.[9,16] to any curvature dependence. These solutions always include tachyonic ones even though the corresponding world-line trajectories always lie inside the local light-cone in agreement with spacetime causality. Conditions for the existence of straight trajectory solutions, i.e. the solutions of ordinary relativistic scalar particles, were also discussed, implying some restriction on the extrinsic curvature dependence.

The Hamiltonian formulation of these systems was also reconsidered. Except for one degenerate situation [9] which requires a separate analysis still to be completed – this degenerate case represents rigid particles whose classical trajectories are all [16] of constant curvature – the identification [15] of local Hamiltonian gauge invariances associated to all first-class constraints was described in detail. In particular, Teichmüller and modular spaces for the generic case were shown to reduce to only two and one points respectively. Consequently, the complete, global and thus admissible gauge fixing [23] of the Hamiltonian description of generic rigid particles was shown to be possible, thereby demonstrating the absence of Gribov problems of any kind for these systems. The degenerate case is distinguished by additional second-class constraints rendering a manifestly Poincaré covariant analysis more difficult. However, partial results in that case were presented as well.

Canonical quantisation of generic rigid particles was then considered using the associated Hamiltonian formulation. However, due to the restriction of causal propagation inside the light-cone, i.e. strictly time-like velocities, a certain sector of phase space turns out to be restricted to the interior of the light-cone. Therefore, in order to quantise the system without having recourse to the methods of geometric quantisation, first a certain map to an unrestricted set of canonically conjugate phase space degrees of freedom is required. This specific issue, which so far has never properly been addressed in the literature, is actually essential for a correct quantisation of rigid particles. As a consequence of the unrestricted phase space map, Poincaré covariance is no longer a manifest symmetry of the formalism. Nevertheless, it turns out that both the spacetime Poincaré algebra and the local gauge algebra are each anomaly free even at the quantum level. Thus, the full quantum space of states transforms covariantly under Poincaré transformations and physical quantum states may be defined as being all those states annihilated by all quantum gauge generators. In fact, the complete space of states only supports integer spin representations of the Lorentz group.

However, due to the necessary operator ordering of composite quantum operators as are the gauge generators, a quantum anomaly was found [24] for the commutator of Lorentz boosts and world-line reparametrisations. Consequently, Lorentz boosts map out-
side of the subspace of physical states even though these transformations act covariantly on the complete space of states. A quantum state physical in one reference frame is not necessarily physical is some other frame! In fact, the only Poincaré invariant quantum observable which is well defined for physical states is their mass. The notion of spin has no meaning for quantum physical states due to this mixed Lorentz-gravitational anomaly. Therefore, quantum rigid particles cannot be considered as being consistent models for particle physics! Their physical states cannot be defined in a way which is compatible with the requirements of local world-line reparametrisation invariance and spacetime Poincaré covariance both at the same time. The present quantum anomaly is quite similar to the usual mixed triangular anomaly in four dimensions for two gravitons and one U(1) gauge boson [31], Poincaré invariance playing from the world-line point of view the rôle of an internal global symmetry for rigid particles.

Strictly speaking, this conclusion applies so far only for an arbitrary dependence on the extrinsic curvature not including the degenerate case (1.4), whose Hamiltonian description and thus canonical quantisation still remains to be analysed. However, the same type of anomaly would presumably be obtained in the degenerate case as well. Most probably, the same conclusion would also extend further to theories of particles whose actions include a dependence on other possible \( j \)-torsions (see Appendix A). If this expectation were indeed proved to be correct, the only consistent quantum model for point-particles using as fundamental degrees of freedom spacetime coordinates \( x^\mu \) only – as well as the associated second-quantised field theories – would simply be the ordinary action for scalar particles, namely (1.1) with \( F(x) = 1 \).

If these conclusions are to be any guide, one may also like to argue that the quantum anomaly for rigid particles is the strongest indication yet as to the probable inconsistency of quantised rigid strings and membranes usually [4] expected on the grounds of higher derivative couplings leading to physical states either of negative norm or of energy unbounded below. Indeed, rigid strings and membranes possess collapsed configurations corresponding to rigid particles. Since quantised rigid particles are not consistent, quantised rigid strings and membranes cannot be consistent either. Note that quantum inconsistency of rigid particles is not related either to negative-norm physical states nor to energy unbounded below but actually follows from a quantum anomaly. Strictly speaking, if this type of reasoning is justified, the conclusion applies so far only to those rigid strings and membranes whose collapsed configurations are not degenerate rigid particles.

Specifically, consider for example the dimensional reduction [32,12] of a rigid string in a spacetime of \((D + 1)\) dimensions whose action depends on the world-sheet extrinsic curvature through some dimensionless function \( G(x) \),

\[
S[\phi^M] = -\frac{\mu c}{\kappa} \int d\tau d\sigma \sqrt{-g} G(\kappa^2 \triangle \phi^M \Delta \phi_M) .
\] (5.1)

Here, \( \phi^M(M = 0, 1, \cdots, D) \) are the string coordinates, \( g_{\alpha\beta} = \eta_{MN} \partial_\alpha \phi^M \partial_\beta \phi^N \) is the induced world-sheet metric (\( \eta_{MN} \) is the Minkowski metric in \((D + 1)\) dimensions), \( \triangle \) is the Laplacian

\[
\triangle = \frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} g^{\alpha\beta} \partial_\beta ,
\] (5.2)
and as usual $\xi^{\alpha=0} = \tau$ and $\xi^{\alpha=1} = \sigma$ are dimensionless world-sheet coordinates with $\alpha, \beta = 0, 1$. When identifying [32,12] one of the space coordinates $\phi^M$ with $\sigma$ and assuming that the remaining string coordinates $\phi^M \sim x^\mu$ are independent of $\sigma$, (5.2) reduces to (1.1) with (the integral here is over the finite range of $\sigma$)

$$F(x) = G(x) \int d\sigma .$$

(5.3)

Thus for instance, Polyakov’s rigid strings [1,12] correspond to the choice

$$F(x) = \alpha_0 x + \beta_0 .$$

(5.4)

Since this function does not define the degenerate case (1.4), we must conclude from the analysis of this paper that Polyakov’s rigid strings cannot be consistent fundamental quantum theories. Of course, this does not necessarily exclude the possible relevance of rigid string and membrane theories – whose actions include higher derivative couplings characterizing the extrinsic geometry of these objects as embedded manifolds – as effective theories for a semi-classical approximation to the dynamics of specific solutions possessing some extended structure in more fundamental theories.

Acknowledgements

The help of Anna Sioras in checking some of the expressions for classical solutions is acknowledged. This work was supported through a Senior Research Assistant position funded by the S.E.R.C.
Appendix A

Consider a point particle propagating freely in a Minkowski spacetime of $D$ dimensions with metric $\eta_{\mu\nu} = \text{diag}(-++\ldots+) \ (\mu, \nu = 0, 1, \cdots, D-1)$. The spacetime embedding of the particle world-line is specified by $D$ coordinates $x^\mu(\tau)$ transforming as vectors under space-time Poincaré symmetries and functions of the world-line parameter $\tau$. Correspondingly, the induced world-line metric is simply

$$\gamma(\tau) = -\dot{x}^2(\tau) \ ,$$

(A.1)

where as usual a dot stands for a derivative with respect to $\tau$. For obvious physical reasons, the entire discussion is restricted to (classical) time-like trajectories $x^\mu(\tau)$, namely trajectories for which $\gamma(\tau)$ is strictly positive corresponding to strictly time-like velocities (configurations with $\dot{x}^2 = 0$ are excluded as they correspond to a degenerate world-line metric $\gamma(\tau)$).

Given the proper-time parametrisation implicitly defined by

$$ds = \gamma^{1/2} \ d\tau \ ,$$

(A.2)

consider now the normalised tangent vector

$$n^\mu = \frac{dx^\mu}{ds} = \gamma^{-1/2} \dot{x}^\mu \ ,$$

(A.3)

with the time-like value

$$n^2(\tau) = -1 \ ,$$

(A.4)

following from the restriction $\dot{x}^2 < 0$. Any variation in $n^\mu(\tau)$ corresponds to some extrinsic curvature of the embedded trajectory $x^\mu(\tau)$, namely

$$K^\mu = \frac{dn^\mu}{ds} = \gamma^{-1/2} \frac{d}{d\tau} \left[ \gamma^{-1/2} \dot{x}^\mu \right] .$$

(A.5)

It readily follows from (A.4) that we have

$$nK = 0 \ , \quad K^2 \geq 0 \quad \text{(A.6)}$$

(note that if $\dot{x}^{\mu}(\tau)$ were space-like rather than time-like as assumed here, the sign of $K^2$ would remain underdetermined, depending on the configuration $x^\mu(\tau)$).

Whenever $K^\mu$ is non vanishing, (A.6) shows that the vectors $n^\mu$ and $K^\mu$ are linearly independent. They thus define a plane: the “osculating plane”. Any variation in this plane corresponds to some extrinsic torsion of the embedded trajectory $x^\mu(\tau)$. Clearly, this extrinsic torsion is non vanishing whenever the vectors $(n^\mu, K^\mu)$ and $(dn^\mu/ds = K^\mu, dK^\mu/ds)$ are linearly independent. An orthogonal decomposition of $dK^\mu/ds$ with respect to $(n^\mu, K^\mu)$.
then leads to the following definition of extrinsic torsion (\(\kappa\) is some arbitrary physical positive constant with a dimension of length)

\[
T^\mu = \frac{dK^\mu}{ds} - \left(\frac{d}{ds} \ln \sqrt{\kappa^2 K^2}\right) K^\mu - K^2 n^\mu .
\] (A.7)

Equivalently, consider the normalised two-form characterizing the linear independence of \(n^\mu\) and \(K^\mu\) when \(K^\mu \neq 0\)

\[
K^{\mu\nu}_{(2)} = n^\mu \frac{K^\nu}{\sqrt{K^2}} - n^\nu \frac{K^\mu}{\sqrt{K^2}} .
\] (A.8)

Any variation in \(K^{\mu\nu}\) corresponds to some extrinsic torsion. Indeed, one has

\[
T^{\mu\nu}_{(2)} = \frac{dK^{\mu\nu}_{(2)}}{ds} = \frac{1}{\sqrt{K^2}} \left[ n^\mu T^\nu - n^\nu T^\mu \right] ,
\] (A.9)

with \(T^\mu\) given in (A.7). From these definitions, it follows again that

\[
nT = 0 , \quad KT = 0 , \quad T^2 \geq 0 .
\] (A.10)

Clearly, the type of considerations above generalises easily, leading to the definition of higher order quantities further characterizing the extrinsic geometry of the embedded world-line \(x^\mu(\tau)\). For example, when \(K^\mu\) and \(T^\mu\) are non vanishing, (A.6) and (A.10) establish the linear independence of the vectors \((n^\mu, K^\mu, T^\mu)\) which thus span an “osculating 3-volume”. Any variation in this 3-volume represents some further extrinsic “3-torsion” of the embedded world-line. Obviously, in a spacetime of \(D\) dimensions, there only exist \(D\) such vector quantities, each corresponding to some extrinsic “\(j\)-torsion” \((j = 0, 1, \cdots, D-1)\) characterizing the extrinsic geometry of the spacetime trajectory \(x^\mu(\tau)\), with the 0-torsion, 1-torsion and 2-torsion being the tangent, curvature and torsion vectors \(n^\mu, K^\mu\) and \(T^\mu\) respectively. In terms of the coordinates \(x^\mu(\tau)\), the extrinsic \(j\)-torsion \(T^\mu_{(j)}\) involves derivatives with respect to \(\tau\) of order \((j + 1)\) and less.

By construction, all \(j\)-torsions \(T^\mu_{(j)}\) are manifestly covariant vectors for spacetime Poincaré transformations, and invariant quantities under local, \(i.e.\) orientation preserving world-line reparametrisations. For global, \(i.e.\) orientation reversing world-line reparametrisations, all \((2j)\)-torsions \((j = 0, 1, \cdots, [(D + 1)/2] - 1; [x] denotes the integer part of \(x\)) change sign whereas all \((2j + 1)\)-torsions \((j = 0, 1, \cdots, [D/2] - 1)\) are invariant. However, since by definition all \(j\)-torsions are mutually orthogonal, the only possible independent Poincaré invariants are simply the quantities

\[
T^2_{(j)} = \eta_{\mu\nu} T^\mu_{(j)} T^\nu_{(j)} , \quad j = 0, 1, \cdots, D-1 ,
\] (A.11)

with

\[
T^2_{(0)} = n^2 = -1 , \quad T^2_{(j)} \geq 0 , \quad j = 1, 2, \cdots, D-1 .
\] (A.12)
Clearly, all \( T^2_{(j)} \) are also invariant under local and global world-line reparametrisations.

Hence, the most general manifestly Poincaré and world-line reparametrisation invariant action that can be constructed using only the world-line scalar “fields” \( x^\mu(\tau) \) is of the form \((\tau_i < \tau_f)\)

\[
S[x^\mu] = -\mu c \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\dot{x}^2} \mathcal{F}(\kappa^2 T^2_{(j)})
\]

(A.13)

(actually for \( D = 2 \), a 2-cocycle term proportional to \( \epsilon_{\mu\nu} x^\mu \dot{x}^\nu \) could be added [33] to (A.13). Such a term however, breaks spacetime parity and changes sign under global world-line reparametrisations). It is understood that the coordinates \( x^\mu(\tau) \) have a dimension of length, with in particular \( x^0(\tau) = ct(\tau) - c \) being the speed of light and \( t(\tau) \) the physical time. Thus, \( \mu \) and \( \kappa \) in (A.13) are fundamental positive physical constants characteristic of the system, with a dimension of mass and length respectively. Finally, \( \mathcal{F} \) is some specific dimensionless function of the \( j \)-torsion invariants \( T^2_{(j)}(j = 1, 2, \ldots, D - 1) \). Clearly, \( \mu \) sets the mass scale of the system and \( \kappa \) the intrinsic length scale in the world-line. For example, the ordinary case of the relativistic scalar particle of mass \( m \) corresponds to the choices \( \mathcal{F} = \pm 1 \) and \( \mu = m \) – with the sign of \( \mathcal{F} \) distinguishing between the particle and its antiparticle.

Appendix B

In this appendix, we show that there do indeed exist solutions to the constraints (2.40) and (2.44) for the boundary conditions \( (x^\mu_i, x^\mu_f, q^\mu_i, q^\mu_f) \). Taking advantage of Poincaré invariance at the classical level, any solution to (2.40) is equivalent, under a spacetime translation and a Lorentz transformation, to the following choice of boundary conditions \( (\tau_i < \tau_f) \)

\[
x^\mu_i = (0, 0, 0, \vec{0}) , \quad \frac{\Delta x^\mu}{\Delta \tau} = (\delta, 0, 0, \vec{0}) ,
\]

\[
q^\mu_i = (q^0_i, -\frac{1}{2}\Delta q^1_i, q^\mu_i=2_i, \vec{0}) , \quad q^\mu_f = (q^0_i, \frac{1}{2}\Delta q^1_i, q^\mu_i=2_i, \vec{0}) ,
\]

(B.1)

where

\[
\delta = \epsilon_1 |k| \left[ 1 + \left( \frac{ak\Delta \tau}{2} \right)^2 \frac{(\cosh 2\gamma + 1)\left(\frac{\tanh \gamma}{\gamma}\right)^2 - 2}{2\gamma^2} \right]^{1/2} ,
\]

\[
q^0_i = \frac{k^2}{\delta} \left[ 1 + \left( \frac{ak\Delta \tau}{2} \right)^2 \frac{(\cosh 2\gamma + 1)\left(\frac{\tanh \gamma}{\gamma}\right) - 2}{2\gamma^2} \right] ,
\]

\[
\frac{1}{2}\Delta q^1 = \epsilon_2 k \left( \frac{ak\Delta \tau}{2} \right) \sqrt{\cosh 2\gamma - 1} \frac{1}{2\gamma^2} ,
\]

\[
q^\mu_i=2_i = \epsilon_1 \epsilon_3 \frac{k^2}{\delta a\kappa\sqrt{2}} \left( \frac{ak\Delta \tau}{2} \right)^2 \left[ \frac{(\cosh 2\gamma + 1)\left(\frac{\tanh \gamma}{\gamma} - 1\right)^2}{2\gamma^4} \frac{F(a^2\kappa^2)}{F'(a^2\kappa^2) - 2a^2\kappa^2} \right]^{1/2} .
\]

(B.2)
Here, $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, $\epsilon_3 = \pm 1$ are arbitrary sign factors – corresponding to arbitrary spacetime reflections – and $k$ is such that $\eta_{\mu\nu} q^\mu_i q^\nu_i = - k^2$ (see (2.28)). The remainder of the notation is defined in sect.2.4.

In the limit $\gamma = 0$ corresponding to the parabolic case, the expressions in (B.2) reduce to

$$
\delta = \epsilon_1 |k|[1 + \frac{1}{12} (ak\Delta\tau)^2]^{1/2},
$$

$$
q^0_i = \frac{k^2}{\delta} \left[ 1 + \frac{1}{6} (ak\Delta\tau)^2 \right],
$$

$$
\frac{1}{2} \Delta q^1 = \epsilon_2 k(\frac{ak\Delta\tau}{2}),
$$

$$
q^\mu = 2 \epsilon_3 \frac{k^2}{\delta} \frac{1}{12} (ak\Delta\tau)^2,
$$

or equivalently

$$
q^0_i = \epsilon_1 |\delta|(1 + \eta), \quad \frac{1}{2} \Delta q^1 = \epsilon_2 |\delta| \sqrt{3\eta}, \quad q^\mu = 2 \epsilon_3 |\delta| \eta,
$$

(B.4)

where

$$
\eta = \frac{\frac{1}{12} (ak\Delta\tau)^2}{1 + \frac{1}{12} (ak\Delta\tau)^2}, \quad 0 < \eta < 1.
$$

(B.5)

In this latter parametrisation, we then have

$$
|k| = |\delta| \sqrt{1 - \eta}, \quad a = \frac{2 \sqrt{3\eta}}{\Delta\tau |\delta| (1 - \eta)}.
$$

(B.6)

Therefore, up to arbitrary Poincaré transformations, any solution to (2.40) and (2.44) is parametrised by the choice of the value $a$ for the extrinsic curvature and the value for $\eta_{\mu\nu} q^\mu_i q^\nu_i = - k^2$. In other words, given any initial time-like velocity $q^\mu_i$ and an extrinsic curvature value $a$, a solution of constant extrinsic curvature may always be found for any choice of function $F(x)$ such that (2.39) is obeyed.
REFERENCES

[1] A. M. Polyakov, Nucl. Phys. B268 (1986) 406.
[2] H. Kleinert, Phys. Lett. B174 (1986) 335.
[3] For a recent review, see
  G. German, Mod. Phys. Lett. A6 (1991) 1815.
[4] J. Polchinski and Z. Yang, “High Temperature Partition Function of the Rigid String”,
  Texas/Rochester preprint UTTG-08-92, UR-1254, ER-40685-706.
[5] E. Braaten and C. K. Zachos, Phys. Rev. D34 (1987) 1512.
[6] R. D. Pisarski, Phys. Rev. D34 (1986) 670.
[7] C. Battle, J. Gomis and N. Roman-Roy, J. Phys. A21 (1988) 2693.
[8] V. V. Nesterenko, J. Phys. A22 (1989) 1673; Theor. Math. Phys. 86 (1991) 168; Mod.
  Phys. Lett. A6 (1991) 719.
[9] M. S. Plyushchay, Mod. Phys. Lett. A3 (1988) 1299; Int. J. Mod. Phys. A4 (1989)
  3851; Phys. Lett. B253 (1991) 50.
[10] M. S. Plyushchay, Mod. Phys. Lett. A4 (1989) 837; ibid A4 (1989) 2747; Phys. Lett.
  B235 (1990) 47; ibid B236 (1990) 291; ibid B243 (1990) 383; ibid B248 (1990) 107;
  ibid B248 (1990) 299; ibid B262 (1991) 71; ibid B280 (1992) 232; Nucl. Phys. B362
  (1991) 54.
[11] A. Dhar, Phys. Lett. B214 (1988) 75.
[12] J. Grundberg, J. Isberg, U. Lindström and H. Nordström, Phys. Lett. B231 (1989)
  61.
[13] J. P. Gauntlett, K. Itoh and P. K. Townsend, Phys. Lett. B238 (1990) 65;
  J. P. Gauntlett and C. F. Yastremiz, Class. Quantum Grav. 7 (1990) 2089;
  J. P. Gauntlett, Phys. Lett. B272 (1991) 25.
[14] M. Pavšič, Phys. Lett. B205 (1988) 231; ibid B221 (1989) 264.
[15] M. Huq, P. I. Obiakor and S. Singh, Int. J. Mod. Phys. A5 (1990) 4301.
[16] H. Arodz, A. Sitarz and P. Wegrzyn, Acta Phys. Polonica B20 (1989) 921.
[17] T. Dereli, D. H. Harley, M. Önder and R. W. Tucker, Phys. Lett. B252 (1990) 601.
[18] D. Zoller, Phys. Rev. Lett. 65 (1990) 2236.
[19] G. Fiorentini, M. Gasperini and G. Scapetta, Mod. Phys. Lett. A6 (1991) 2033.
[20] A. M. Polyakov, Mod. Phys. Lett. A3 (1988) 325.
[21] S. Iso, C. Itoi and H. Mukaida, Phys. Lett. B236 (1990) 287; Nucl. Phys. B346 (1990)
  293.
[22] Yu. A. Kuznetsov and M. S. Plyushchay, “The Model of the Relativistic Particle with
  Curvature and Torsion”, Protvino preprint IHEP 91-162 (October 1991), and reference
  therein.
[23] For a recent review, see
  J. Govaerts, Hamiltonian Quantisation and Constrained Dynamics, Lecture Notes in
  Mathematical and Theoretical Physics 4 (Leuven University Press, Leuven, 1991).
[24] J. Govaerts, “A Quantum Anomaly for Rigid Particles”, Durham preprint DTP-92/37 (July 1992), hepth/9207068.

[25] J. Govaerts, Int. J. Mod. Phys. A4 (1991) 173; ibid A4 (1991) 4487.

[26] I. A. Batalin and E. S. Fradkin, Phys. Lett. B180 (1986) 157; Nucl. Phys. B279 (1987) 514;
I. A. Batalin, E. S. Fradkin and T. E. Fradkina, Nucl. Phys. B314 (1989) 158.

[27] K. Harada and H. Mukaido, Z. Phys. C48 (1990) 151.

[28] J. Isberg, U. Lindström and H. Nordström, Mod. Phys. Lett. A5 (1990) 2491.

[29] C. J. Isham, in Relativity, Groups and Topology II, Les Houches 1983, eds. B. S. DeWitt and R. Stora (North Holland, Amsterdam, 1984), p. 1162.

[30] See for example
N. M. J. Woodhouse, Geometric Quantisation (Oxford University Press, Oxford, 1980).

[31] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269.

[32] M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, Phys. Lett. B191 (1987) 70.

[33] D. R. Grigore, “A Derivation of the Nambu-Goto Action from Invariance Principles”,
preprint CERN-TH.6101/91 (May 1991).