

\textbf{Abstract.} Let $F/\mathbb{Q}_p$ be a finite extension. This paper is about continuous admissible $p$-adic Banach space representations $\Pi$ of $G = GL_n(F)$ and their restriction to $H = SL_n(F)$. We first show that the restriction of any such absolutely irreducible $G$-representation decomposes as a finite direct sum of irreducible $H$-representations. Then we consider the restriction to $SL_2(\mathbb{Q}_p)$ of certain $p$-adic unitary Banach space representations of $GL_2(\mathbb{Q}_p)$, relying on the work of Colmez and Colmez-Dospinescu-Paškūnas. It is shown that if $\Pi$ is associated to an irreducible trianguline de Rham representation with distinct Hodge-Tate weights, then $\Pi|_H$ decomposes non-trivially if and only if this is true for the subrepresentation $(\Pi^{\text{alg}})|_H$ of locally algebraic vectors.

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1. Introduction

Let $F/{\mathbb{Q}}_p$ be a finite extension. $L$-packets of smooth representations of $H = \text{SL}_n(F)$ consist of the irreducible constituents $\pi_1, \ldots, \pi_r$ of the restriction of a smooth irreducible representation $\pi$ of $G = \text{GL}_n(F)$ to $H$, cf. [18, 2]. In the emerging (local) $p$-adic Langlands program (cf. [9, 12, 7, 6], to name just a few) the role of smooth representations is mainly played by continuous representations on $p$-adic Banach spaces. It is thus a natural problem to study the restriction $\Pi|_H$ of an admissible irreducible $p$-adic Banach space representation $\Pi$ of $G$. A first result which uses the general theory of Schneider-Teitelbaum [25] and some elementary arguments is

**Proposition 1.1.1.** (2.1.3, 2.1.4) Suppose the admissible Banach space representation $\Pi$ of $G$ has a central character.

Then the following statements hold.

(i) $\Pi|_H$ decomposes as a finite direct sum of closed $H$-stable subspaces $\Pi_1, \ldots, \Pi_r$, each of which is an irreducible admissible representation of $H$.

(ii) If the subspace $\Pi_{\text{alg}}$ of locally algebraic vectors is non-zero, and if $\Pi_1, \ldots, \Pi_r$ are as in (i), then

$$ (\Pi_{\text{alg}})|_H = (\Pi_1)_{\text{alg}} \oplus \cdots \oplus (\Pi_r)_{\text{alg}}. $$

We point out that the $H$-representations $(\Pi_i)_{\text{alg}}$ are not necessarily irreducible, even if the $G$-representation $\Pi_{\text{alg}}$ is irreducible (cf. 1.1.3 below).

In sections 3 and 4 we consider the case when $G = \text{GL}_2(\mathbb{Q}_p)$ and $H = \text{SL}_2(\mathbb{Q}_p)$. Theorems of Paškūnas [23, 1.1], Colmez-Dospinescu-Paškūnas [12, 1.4], and Abdellatif [1, 0.1, 0.7] imply that $\Pi|_{\text{SL}_2(\mathbb{Q}_p)}$ can have at most two irreducible constituents, cf. 4.1.1.

Our next aim is to clarify the relation between the irreducible constituents of $\Pi|_{\text{SL}_2(\mathbb{Q}_p)}$ and that of $\Pi_{\text{alg}}|_{\text{SL}_2(\mathbb{Q}_p)}$, provided the latter space is non-zero. Denote by $\mathcal{G}_{\mathbb{Q}_p}$ the absolute Galois group of $\mathbb{Q}_p$. We then have

1 Though smooth representations do retain their significance, cf. below.

2 This is the case when $\Pi$ is absolutely irreducible, by [14, 1.1].
Theorem 1.1.2. ([4.2.5]) Let $\psi$ be an absolutely irreducible 2-dimensional trianguline de Rham representation of $G_{Q_p}$ with distinct Hodge-Tate weights. Let $\Pi = \Pi(\psi)$ be the absolutely irreducible admissible Banach space representation of $GL_2(Q_p)$ associated to $\psi$ via Colmez’ $p$-adic Langlands correspondence.

Then $\Pi|_{SL_2(Q_p)}$ is decomposable if and only if $(\Pi^{\text{alg}})|_{SL_2(Q_p)}$ is decomposable. In this case, $\Pi|_{SL_2(Q_p)}$ has two irreducible non-isomorphic constituents $\Pi_1$ and $\Pi_2$. Furthermore, the representations $(\Pi_1)^{\text{alg}}$ and $(\Pi_2)^{\text{alg}}$ are the irreducible constituents of $(\Pi^{\text{alg}})|_{SL_2(Q_p)}$, and they are not isomorphic. □

We sketch the proof here. It is known that $\Pi^{\text{alg}} = \Pi^{\text{alg}} \otimes \pi$ with a finite-dimensional irreducible algebraic representation $\Pi^{\text{alg}}$ and a smooth representation $\pi$. If $\Pi^{\text{alg}}$ is decomposable, then $\pi$ must be a principal series representation whose restriction to $SL_2(Q_p)$ decomposes. This happens only in a very special case, and in that case one can show that the restriction to $SL_2(Q_p)$ of the locally analytic representation $\Pi^{\text{an}}$ decomposes too. This finishes the proof because the universal completion of $\Pi^{\text{an}}$ is $\Pi$, cf. [11, 0.2].

Remark 1.1.3. There are absolutely irreducible 2-dimensional de Rham representations $\psi$ which are not trianguline and for which $\Pi^{\text{alg}}|_H$ has four irreducible constituents, and therefore more irreducible constituents than $\Pi|_H$ (cf. [4.2.7] for more details). We are currently trying to determine the irreducible constituents of $\Pi|_H$ in this case.

In the case of smooth representations the cardinality of $L$-packets for $SL_n(F)$ can be computed via component groups of centralizers of projective Weil group representations. We consider the same problem in the context of 1.1.2.

Proposition 1.1.4. ([4.3.10]) Let $E/Q_p$ be a finite extension, and let $\psi : G_{Q_p} \to GL_2(E)$ be an absolutely irreducible trianguline de Rham representation with distinct Hodge-Tate weights. Denote by $\overline{\psi} : G_{Q_p} \to PGL_2(E)$ the corresponding projective representation. Let $S_{\overline{\psi}}$ be the centralizer in $PGL_2(E)$ of the image of $\overline{\psi}$. Then $S_{\overline{\psi}}$ is finite and its cardinality is equal to the number of irreducible constituents of $\Pi(\psi)|_{SL_2(Q_p)}$ (which is one or two). □

In section 5 we consider how restricting Banach space representations from $GL_n(F)$ to $SL_n(F)$ relates to the theory developed by C. Breuil and P. Schneider in [6, sec. 6]. There
they consider a connected split reductive group $G$, and they associate to a pair $(\xi, \zeta)$, consisting of a dominant algebraic character $\xi$ of a maximal torus $T \subset G$ and an $E$-valued point $\zeta$ of the dual torus $T'$, a Banach space representation $B_{\xi,\zeta}$ of $G(F)$. This representation they conjecture to be non-zero if the pair $(\xi, \zeta)$ is admissible in the sense that there are crystalline Galois representations $\gamma_{\nu,b}$ into the dual group $G'$ which have ”Hodge-Tate weights” given by $\xi$ and such that the semisimple part of the Frobenius action is given by $\zeta$, cf. [6, 6.3] for a precise statement. In the case of $GL_n$ and $SL_n$, the non-vanishing of $B_{\xi,\zeta}$ for $GL_n(F)$ should imply the non-vanishing of $B_{\xi,\zeta}$ for $SL_n(F)$, and this is what we show in 5.1.2.

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1.3. Notation. Let $\mathbb{Q}_p \subseteq F \subseteq E$ be a sequence of finite extensions, with the rings of integers $\mathbb{Z}_p \subseteq \mathcal{O}_F \subseteq \mathcal{O}_E$. We fix a uniformizer $\varpi_F$ of $F$ and denote $v_F : F^* \rightarrow \mathbb{Z}$ the normalized valuation (i.e., $v_F(\varpi_F) = 1$). Let $q = p^f$ be the cardinality of the residue field of $F$, and denote by $| \cdot | = | \cdot |_F$ the absolute value specified by $|x| = q^{-v_F(x)}$. From Section 3 on we assume that $E$ contains a square root of $q$.

All representations on Banach spaces are tacitly assumed to be continuous. Such a representation is called unitary if the group action is norm-preserving. If $V$ is an $E$-Banach space representation of $G$, we denote by $V^{an}$ the subspace of locally analytic vectors of $V$. Similarly, $V^{sm}$ is the subspace of smooth vectors and $V^{lalg}$ is the subspace of locally algebraic vectors of $V$.

Given a parabolic subgroup $P$ of $G$, with Levi decomposition $P = MU$, we will use several types of parabolic induction: $i_{G,M}(\cdot)$ denotes the smooth normalized induction, $\text{ind}_{G}^{P}(\cdot)$ the smooth non-normalized induction, and $\text{Ind}_{G}^{P}(\cdot)^{an}$ the locally analytic non-normalized induction.

The absolute Galois group of $\mathbb{Q}_p$ will be denoted by $G_{\mathbb{Q}_p}$.

\[3\text{which we take here in the introduction to be over } F = \mathbb{Q}_p, \text{ for simplicity}\]
2. Restrictions of representations of $p$-adic Lie groups

2.1. Restrictions of admissible Banach space representations. Let $G$ be a $p$-adic Lie group [29 sec. 13], and let $K \subset G$ be an open compact subgroup. Define

$$O_E[[K]] = \lim_{\leftarrow} O_E[K/N]$$

and

$$E[[K]] = E \otimes_{O_E} O_E[[K]],$$

where $N$ runs through all open normal subgroups of $K$. These are both noetherian rings, cf. [25, beginning sec. 3]. An $E$-Banach space representation $\Pi$ of $K$ is called admissible if the continuous dual $\Pi' = \text{Hom}_{E}^{\text{cont}}(\Pi, E)$ is a finitely generated $E[[K]]$-module, cf. [25, 3.4].

In the duality theory of [25] (building on Schikhof’s duality [24]) it is important that $\Pi'$ is equipped with the weak topology (topology of pointwise convergence), often indicated by a subscript $s$, i.e., $\Pi'_s$, though we will suppress the subscript in the following.

An $E$-Banach space representation of $G$ is called admissible if it is admissible as a representation of every compact open subgroup $K$ of $G$. Admissibility can be tested on a single compact open subgroup.

If $\Pi$ is an admissible $E$-Banach space representation of $G$, then the map $\Pi_0 \mapsto \Pi_0'$ is a bijection between the set of $G$-invariant closed vector subspaces $\Pi_0 \subset \Pi$ and the set of $G$-stable $E[[K]]$-quotient modules of $\Pi'$, cf. [25, 3.5].

**Proposition 2.1.1.** Let $G$ be a $p$-adic Lie group and $H$ an open normal subgroup of $G$ of finite index. Let $\Pi$ be an irreducible admissible $E$-Banach space representation of $G$. Then there are closed $H$-stable subspaces $\Pi_1, \ldots, \Pi_r$ of $\Pi$ such that each $\Pi_i$ is an irreducible admissible representation of $H$, and the canonical map $\Pi_1 \oplus \ldots \oplus \Pi_r \to \Pi$ is a topological isomorphism, i.e., we have an isomorphism

$$\Pi|_H = \Pi_1 \oplus \ldots \oplus \Pi_r$$

of $E$-Banach space representations of $H$. For each $i \in \{1, \ldots, r\}$ and each $g \in G$ one has $g.\Pi_i = \Pi_j$ for some $j \in \{1, \ldots, r\}$, and $G$ acts transitively on the set $\{\Pi_1, \ldots, \Pi_r\}$.

**Proof.** Choose a compact open subgroup $K \subset H$. As $\Pi$ is assumed to be irreducible, $\Pi \neq 0$, and hence $\Pi' \neq 0$, by [25, 3.5]. Therefore, $\Pi'$ contains proper $H$-stable $E[[K]]$-submodules (e.g., the zero submodule). Let $\mathcal{M}$ be the set of proper $E[[K]]$-submodules $M \subset \Pi'$ which are $H$-stable. $\mathcal{M}$ is not empty, by what we have just observed. As $\Pi$ is assumed to be admissible,
the continuous dual space \( \Pi' \) is a finitely generated \( E[[K]] \)-module. Given any ascending ascending chain \( M_0 \subset M_1 \subset \ldots \) in \( \mathcal{M} \) its union is again a proper \( H \)-stable submodule, and thus an upper bound in \( \mathcal{M} \). Zorn’s Lemma implies that \( \Pi' \) contains a maximal \( H \)-invariant \( E[[K]] \)-submodule \( M \). The quotient \( \Pi'/M \) then corresponds to a topologically irreducible closed \( H \)-subrepresentation \( \Pi_1 \) of \( \Pi|_H \), namely \( \Pi_1 = M^\perp := \{ v \in \Pi \mid \forall \ell \in M : \ell(v) = 0 \} \). Because \( \Pi'/M \) is finitely generated as \( E[[K]] \)-module, \( \Pi_1 \) is an admissible representation of \( H \).

For each \( g \in G \), \( g.\Pi_1 \) is an \( H \)-invariant closed vector subspace of \( \Pi \). It is irreducible as \( H \)-representation because if \( \Pi_0 \subsetneq g.\Pi_1 \) is a proper \( H \)-invariant closed subspace, then \( g^{-1}.\Pi_0 \subset \Pi_1 \) is a subspace of the same type, which must hence be zero. Let \( \{g_1, g_2, \ldots, g_k\} \) be a set of coset representatives of \( G/H \). Then

\[
\sum_{i=1}^k (g_i).\Pi_1
\]

is a \( G \)-invariant subspace of \( \Pi \). It is closed because it corresponds to the \( G \)-stable \( E[[K]] \)-submodule \( \bigcap_{i=1}^k (g_i).M \). Hence, it is equal to \( \Pi \), because \( \Pi \) is irreducible. We select a minimal subset \( \{g_{i_1}, g_{i_2}, \ldots, g_{i_r}\} \) of \( \{g_1, g_2, \ldots, g_k\} \) such that \( \sum_{j=1}^r (g_{i_j})\Pi_1 = \Pi \). We claim that this sum is direct. Suppose that

\[
(g_{i_l}).\Pi_1 \cap \sum_{j \neq l} (g_{i_j}).\Pi_1 \neq 0.
\]

Then, as \( \sum_{j \neq l} (g_{i_j}).\Pi_1 \) is closed (by the same argument as above) and \( H \)-stable, and because \( (g_{i_l}).\Pi_1 \) is topologically irreducible, the left-hand side of (2.1.2) must be \( (g_{i_l}).\Pi_1 \). This implies \( \sum_{j \neq l} (g_{i_j}).\Pi_1 = \Pi \), contradicting the minimality. The map \( \bigoplus_{j=1}^r (g_{i_j}).\Pi_1 \to \Pi \) is thus a continuous bijection, and hence a topological isomorphism [32, 8.7].

**Corollary 2.1.3.** Let \( \Pi \) be an irreducible admissible \( E \)-Banach space representation of the group \( GL_n(F) \). Assume that \( \Pi \) has a central character. Then there are closed \( H \)-stable subspaces \( \Pi_1, \ldots, \Pi_r \) of \( \Pi \), each of which is an irreducible admissible \( E \)-Banach representation of \( SL_n(F) \), and such that the canonical map \( \Pi_1 \oplus \ldots \oplus \Pi_r \to \Pi \) is an isomorphism of topological vector spaces. The group \( GL_n(F) \) acts transitively on the set \( \{\Pi_1, \ldots, \Pi_r\} \) as in 2.1.1.

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4This is the case when \( \Pi \) is absolutely irreducible, by [13, 1.1].
Proof. Let $Z$ be the center of $GL_n(F)$ and $H = ZSL_n(F)$. Then $H$ is an open normal subgroup of $GL_n(F)$ of finite index. The statement now follows from [2.1.1]. □

**Proposition 2.1.4.** Let $\Pi$ be an irreducible admissible $E$-Banach space representation of $GL_n(F)$ which has a central character. Suppose that the subspace of locally algebraic vectors $\Pi^{\text{alg}}$ is dense in $\Pi$. Write $\Pi = \Pi_1 \oplus \ldots \oplus \Pi_r$ as in [2.1.3]. Then for each $i$, the set $(\Pi_i)^{\text{alg}}$ is dense in $\Pi_i$, hence non-zero, and

$$(\Pi^{\text{alg}})|_H = (\Pi_1)^{\text{alg}} \oplus \ldots \oplus (\Pi_r)^{\text{alg}}.$$ 

**Proof.** Consider $v \in V^{\text{alg}}$ and write it as

$$v = v_1 + \ldots + v_k \quad \text{with} \quad v_i \in \Pi_i.$$ 

Because the projection map $\Pi \to \Pi_i$, $i = 1, \ldots, r$, is continuous and $H$-equivariant, it follows that $v_i \in (\Pi_i)^{\text{alg}}$, for all $i$. This shows the inclusion "$\subseteq$". Because we assume $\Pi^{\text{alg}}$ to be non-zero, the central character must be locally algebraic. But then any vector in $\Pi_i^{\text{alg}}$ is locally algebraic for $ZH$ (cf. proof of [2.1.3]), and thus locally algebraic for $G$, which shows "$\supseteq$". □

**2.2. Universal completions of restrictions of locally analytic representations.** Let $G$ be a $p$-adic Lie group. We recall the notion of a universal unitary Banach space completion as introduced by M. Emerton in [15, Def. 1.1]. Let $V$ be a locally convex $E$-vector space equipped with a continuous action of $G$, and let $U$ be an $E$-Banach space equipped with a unitary $G$-action. A continuous $G$-homomorphism $\alpha : V \to U$ is said to **realize** $U$ as a **universal unitary completion** of $V$, if any continuous $G$-homomorphism $\phi : V \to W$, where $W$ is an $E$-Banach space equipped with a unitary $G$-action, factors uniquely through a continuous $G$-homomorphism $\psi : U \to W$. Here are some easy facts, cf. [15, sec. 1].

**Remarks 2.2.1.** (i) If it exists, the universal unitary completion $U$, together with the map $\alpha : V \to U$, is unique up to unique isomorphism. We will henceforth denote it by $\hat{V}^u$. The map $V \to \hat{V}^u$, which we will call the **canonical map**, will be denoted by $\alpha_V$ or $\alpha$.

(ii) The map $\alpha_V : V \to \hat{V}^u$, if it exists, has dense image.
(iii) The universal unitary completion, if it exists, is functorial. That is, if $\beta : V \to W$ is a continuous $G$-homomorphism of locally convex $E$-vector spaces, and if $\hat{V}^u$ and $\hat{W}^u$ both exist, then there is a unique continuous $G$-homomorphism $\hat{\beta} : \hat{V}^u \to \hat{W}^u$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\beta} & W \\
\downarrow{\alpha_V} & & \downarrow{\alpha_W} \\
\hat{V}^u & \xrightarrow{\hat{\beta}} & \hat{W}^u
\end{array}
$$

commutes. And if $\gamma : P \to V$ is another continuous $G$-homomorphism from a locally convex $E$-vector space $P$ to $V$, and if $\hat{P}^u$ exists, then $\hat{\beta} \circ \hat{\gamma} = \hat{\beta} \circ \gamma : \hat{P}^u \to \hat{W}^u$.

**Proposition 2.2.2.** Let $G$ be a $p$-adic Lie group, and $V$ be a continuous representation of $G$ on a locally convex $E$-vector space. Suppose the universal unitary completion $\alpha : V \to \hat{V}^u$ exists. Suppose $V = V_1 \oplus V_2$ with closed $G$-invariant subspaces $V_1$ and $V_2$. Let $\iota_i : V_i \to V$ be the inclusion, and let $\text{pr}_i : V \to V_i$ be the corresponding projection ($i = 1, 2$).

(i) Let $\overline{\alpha(V_1)}$ be the closure of $\alpha(V_1)$ in $\hat{V}^u$, and denote by $\alpha_1 : V_1 \to \overline{\alpha(V_1)}$ the restriction of $\alpha$ to $V_1$. Then $\alpha_1 : V_1 \to \overline{\alpha(V_1)}$ realizes $\alpha(V_1)$ as the universal unitary Banach space completion of $V_1$.

(ii) Let $\hat{V}_1^u$ and $\hat{V}_2^u$ be the universal unitary completions (which exist by (i)). Then the maps $\hat{\iota}_1 + \hat{\iota}_2 : \hat{V}_1^u \oplus \hat{V}_2^u \to \hat{V}^u$ and $\hat{\text{pr}}_1 \oplus \hat{\text{pr}}_2 : \hat{V}^u \to \hat{V}_1^u \oplus \hat{V}_2^u$ are mutually inverse topological isomorphisms.

**Proof.** (i) Let $\phi : V_1 \to U$ be a continuous $G$-homomorphism to an $E$-Banach space $U$. Let $\overline{\iota_1 : \alpha(V_1)} \to \hat{V}^u$ be the inclusion and define $\tilde{\phi} = \phi \circ \text{pr}_1 : V \to U$. Because $V \xrightarrow{\alpha} \hat{V}^u$ is a universal unitary completion of $V$, there is a unique continuous $G$-equivariant map $\tilde{\psi} : \hat{V}^u \to U$ satisfying $\tilde{\psi} \circ \alpha = \tilde{\phi}$. Set $\psi = \tilde{\psi} \circ \overline{\iota_1 : \alpha(V_1)} \to U$. We then obtain $\psi \circ \alpha_1 = \tilde{\psi} \circ \overline{\iota_1 \circ \alpha_1} = \tilde{\psi} \circ \alpha \circ \iota_1 = \tilde{\phi} \circ \iota_1 = \phi \circ \text{pr}_1 \circ \iota_1 = \phi$. The situation is summarized in the following diagram
That $\psi$ is unique follows from the fact that $\text{im}(\iota_1)$ is dense in $\overline{\alpha(V_1)}$.

(ii) For $i = 1, 2$ we have $\text{pr}_i \circ \iota_i = \text{id}_{V_i}$, and so $\widehat{\text{pr}_i} \circ \widehat{\iota_i} = \widehat{\text{pr}_i \circ \iota_i} = \text{id}_{\widehat{V}_i} = \text{id}_{\widehat{V}_i^u}$. On the other hand, if $i \neq j$, then $\text{pr}_i \circ \iota_j = 0$, and thus $\widehat{\text{pr}_i} \circ \widehat{\iota_j} = \widehat{\text{pr}_i \circ \iota_j} = 0$. This implies $\widehat{\text{pr}_1} \circ \widehat{\iota_1} + \widehat{\text{pr}_2} \circ \widehat{\iota_2} = \text{id}_{\widehat{V}_1} \oplus \text{id}_{\widehat{V}_2}$. Using part (i) we identify $\widehat{V}_i^u$ with $\overline{\alpha(V_i)}$ and denote the canonical map $V_i \to \widehat{V}_i^u$ by $\alpha_i$. Because the formation of the universal unitary completion is functorial, we have the commutative diagrams

$$
\begin{array}{ccc}
V_i & \xrightarrow{\alpha_i} & \widehat{V}_i^u \\
\downarrow \iota_i & & \downarrow \widehat{\iota_i} \\
V & \xrightarrow{\alpha} & \widehat{V}^u
\end{array}
$$

i.e., $\alpha \circ \iota_i = \widehat{\iota_i} \circ \alpha_i$ and $\alpha_i \circ \text{pr}_i = \widehat{\text{pr}_i} \circ \alpha$. And therefore

$$
(\widehat{\iota_1} + \widehat{\iota_2}) \circ (\widehat{\text{pr}_1} \oplus \widehat{\text{pr}_2}) \circ \alpha = (\widehat{\iota_1} + \widehat{\iota_2}) \circ (\widehat{\text{pr}_1} \circ \alpha \oplus \widehat{\text{pr}_2} \circ \alpha) = (\widehat{\iota_1} + \widehat{\iota_2}) \circ (\alpha_1 \circ \text{pr}_1 \oplus \alpha_2 \circ \text{pr}_2)
$$

$$
= (\widehat{\iota_1} \circ \alpha_1 \circ \text{pr}_i) + (\widehat{\iota_2} \circ \alpha_2 \circ \text{pr}_2) = (\alpha \circ \iota_1 \circ \text{pr}_1) + (\alpha \circ \iota_2 \circ \text{pr}_2) = \alpha 
$$

because $\iota_1 \circ \text{pr}_1 + \iota_2 \circ \text{pr}_2 = \text{id}_V$. Applying the universal property of $(\widehat{V}^u, \alpha)$ to $\phi = \alpha : V \to \widehat{V}^u$ shows that $(\widehat{\iota_1} + \widehat{\iota_2}) \circ (\widehat{\text{pr}_1} \oplus \widehat{\text{pr}_2}) = \text{id}_{\widehat{V}_u}$.

\[\square\]

**Proposition 2.2.3.** Let $G$ be a $p$-adic Lie group, and let $H \subset G$ be an open subgroup of finite index. Let $V = (V, \rho_V)$ be a continuous representation of $G$ on a locally convex $E$-vector space. Suppose the universal unitary completion $\widehat{V}^u = (\widehat{V}^u, \alpha : V \to \widehat{V}^u)$ exists. Then
the universal unitary Banach space completion \( \hat{V}_H^u \) of the restriction \( V|_H \) of \( V \) to \( H \) exists and is equal to \( (\hat{V}^u)|_H \).

Proof. Let \( \phi : V|_H \to W = (W, \| \cdot \|_W, \rho_W) \) be a continuous \( H \)-homomorphism to a unitary \( E \)-Banach space representation. Let \( \text{Ind}^G_H(W) \) be the \( E \)-vector space of all maps \( f : G \to W \) such that \( f(hg) = \rho_W(h).f(g) \). The group \( G \) acts on \( \text{Ind}^G_H(W) \) by \( (g.f)(g') = f(g'g) \). We equip \( \text{Ind}^G_H(W) \) with the maximum norm \( \| f \| = \max \{ \| f(g) \|_W \mid g \in G \} \), which is well-defined, as \( |G : H| < \infty \) and \( H \) acts norm-preserving on \( W \). It is straightforward to see that in this way \( \text{Ind}^G_H(W) \) is a continuous unitary Banach space representation of \( G \). Define the following maps

\[
\begin{align*}
\text{pr} &: \text{Ind}^G_H(W) \to W, \quad \text{pr}(f) = f(1), \\
\tilde{\phi} &: V \to \text{Ind}^G_H(W), \quad \tilde{\phi}(v)(g) = \phi(\rho_V(g).v).
\end{align*}
\]

Then \( \text{pr} \) is a continuous \( H \)-homomorphism, and \( \tilde{\phi} \) is a continuous \( G \)-homomorphism:

\[
\tilde{\phi}(g.v)(g') = \phi(\rho_V(g'g).v)) = \tilde{\phi}(v)(g'g) = (g.\tilde{\phi}(v))(g').
\]

Moreover, \( \text{pr} \circ \tilde{\phi} = \phi \). Consider the diagram

![Diagram](image)

As \( \tilde{\phi} \) is a continuous \( G \)-homomorphism in the unitary representation \( \text{Ind}^G_H(W) \), there exists a continuous \( G \)-homomorphism \( \tilde{\psi} : \hat{V}^u \to \text{Ind}^G_H(W) \) such that \( \tilde{\phi} = \tilde{\psi} \circ \alpha \). Define \( \psi = \text{pr} \circ \tilde{\psi} \), which is clearly a continuous \( H \)-homomorphism. Then \( \psi \circ \alpha = \text{pr} \circ \tilde{\psi} \circ \alpha = \text{pr} \circ \tilde{\phi} = \phi \). That \( \psi \) is unique follows from the fact, recalled above, that the image of \( \alpha \) is dense in \( \hat{V}^u \). \( \square \)

3. Restricting locally analytic representations from \( GL_2(F) \) to \( SL_2(F) \)

In this section, \( G = GL_2(F) \) and \( H = SL_2(F) \). Denote by \( T \) the diagonal torus in \( G \) and by \( P \subset G \) the group of upper triangular matrices. From now on, we assume that \( E \) contains
a square root of $q$, which we henceforth fix once and for all, and denote by $q^{1/2}$. Accordingly, for any integer $h$ we write $\lfloor x \rfloor^{h/2}$ for $(q^{1/2})^{-h \cdot \nu_F(x)}$.

3.1. **Smooth principal series.** In this subsection, we review some well-known results about reducibility of principal series representations of $G = GL_2(F)$ and $H = SL_2(F)$. Given a smooth character $\chi : T \to E^\times$, we consider it as a character of $P$ via the canonical homomorphism $P \to T$. Then we denote by $i_{G,T}(\chi)$ the normalized smooth parabolic induction. It consists of all locally constant functions $f : G \to E$ with the property that $f(pg) = \delta_P(p)^{\frac{1}{2}} \chi(p)f(g)$ for all $p \in P$ and $g \in G$, where

$$\delta_P \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \frac{|a|}{|d|}$$

is the modulus character on $P$. Let $T_H = T \cap H$ be the torus of diagonal matrices in $H$, which is isomorphic to $F^\times$ via the map $a \mapsto \text{diag}(a,a^{-1})$. Given a character $\chi$ of $T_H$ we define the normalized smooth parabolically induced representation $i_{H,T_H}(\chi)$ in exactly the same way as above. Since $\delta_H^{\frac{1}{2}}(\text{diag}(a,a^{-1})) = |a|$, we see that $i_{H,T_H}(\chi)$ does not depend on the choice of the square root of $q$ that we fixed in the beginning.

**Proposition 3.1.1.** Let $\chi$ be a smooth character of $T_H$. If the smooth normalized induced representation $i_{H,T_H}(\chi)$ is reducible, then either $\chi(x) = \delta_P^{\pm \frac{1}{2}}$ or $\chi^2 = 1$, $\chi \neq 1$. More specifically:

(i) The representation $i_{H,T_H}(\delta_P^{\frac{1}{2}})$ fits in the following exact sequence

$$0 \to \text{St} \to i_{H,T_H}(\delta_P^{\frac{1}{2}}) \to 1 \to 0,$$

where St is the Steinberg representation and $1$ is the one-dimensional trivial representation. The above sequence does not split. The representation $i_{H,T_H}(\delta_P^{-\frac{1}{2}})$ is the smooth contragredient of $i_{H,T_H}(\delta_P^{\frac{1}{2}})$ and it fits in the following exact sequence

$$0 \to 1 \to i_{H,T_H}(\delta_P^{-\frac{1}{2}}) \to \text{St} \to 0.$$

(ii) If $\chi$ is a nontrivial quadratic character, then $i_{H,T_H}(\chi)$ is a direct sum of two inequivalent absolutely irreducible components.
3.2. Locally analytic principal series of $T$.

Proof. The proposition follows from the discussion in [19] ch. 2, §3-5. Although the representations considered in [19] are over the complex numbers, the arguments, in particular [19] statement 3, p. 164] applies also to $E$-valued principal series. It is assumed in [19] that the residue characteristic is odd, but the properties of these representations (in particular, their reducibility) also hold for residue characteristic two (see also [19] ch. 2, §8]). That the irreducible constituents in assertion (iii) are inequivalent is [31] 1.2].

Proposition 3.1.2. Let $\chi_1 \otimes \chi_2$ be a smooth character of $T$.

(i) The representation $i_{G,T}(\chi_1 \otimes \chi_2)$ is reducible if and only if $\chi_1\chi_2^{-1} = | \cdot |$, then there is a smooth character $\delta$ such that $(\chi_1, \chi_2) = (\delta | \cdot |^{\frac{1}{2}}, \delta | \cdot |^{-\frac{1}{2}})$ and $i_{G,T}(\chi_1 \otimes \chi_2)$ has $\delta \circ \det$ as a quotient and $(\delta \circ \det) \otimes \text{St}$ as a subrepresentation. If $\chi_1\chi_2^{-1} = | \cdot |^{-1}$, then there is a smooth character $\delta$ such that $(\chi_1, \chi_2) = (\delta | \cdot |^{-\frac{1}{2}}, \delta | \cdot |^{\frac{1}{2}})$ and $i_{G,T}(\chi_1 \otimes \chi_2)$ has $(\delta \circ \det) \otimes \text{St}$ as a quotient and $\delta \circ \det$ as a subrepresentation.

(ii) If $i_{G,T}(\chi_1 \otimes \chi_2)$ is irreducible, then $i_{G,T}(\chi_1 \otimes \chi_2) \cong i_{G,T}(\chi_2 \otimes \chi_1)$.

(iii) Suppose $i_{G,T}(\chi_1 \otimes \chi_2)$ is irreducible. Then its restriction to $H$ is reducible if and only if $(\chi_1 \otimes \chi_2)|_{T_H}$ is a nontrivial quadratic character.

Proof. (i) follows from [33] Proposition 1.11 (a)]. (ii) follows from [8] Theorem 6.3.11]. For (iii), note that $i_{G,T}(\chi_1 \otimes \chi_2)|_H = i_{H,T_H}((\chi_1 \otimes \chi_2)|_{T_H})$ and apply Proposition 3.1.1. □

As quadratic characters on $F^\times$ will play a significant role later on, let us recall how to describe them. Any such character is determined by its kernel, which is a subgroup of $F^\times$ of index two. By Local Class Field Theory, these subgroups are precisely the norm groups for the quadratic extensions of $F$, which in turn correspond to the non-trivial elements in $F^\times/(F^\times)^2$. For $\theta \in F^\times \setminus (F^\times)^2$, we define the quadratic character $\text{sgn}_\theta$ on $F^\times$ by

$$\text{sgn}_\theta(x) = \begin{cases} 1 & \text{if } x \text{ is in the image of } \text{Norm}_{F(\sqrt{\theta})/F}, \\ -1 & \text{otherwise.} \end{cases}$$

3.2. Locally analytic principal series of $GL_2(F)$ and $SL_2(F)$. Let $\chi : T \to E^\times$ be a locally algebraic $F$-analytic character. We will abuse notation and also write $\chi$ for the restriction $\chi|_{T_H}$ of $\chi$ to $T_H$. The groups of algebraic characters (resp. cocharacters) of $T$ and $T_H$ will be denoted, as usual, by $X^*(T)$ (resp. $X_*(T)$) and $X^*(T_H)$ (resp. $X_*(T_H)$), and we let $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ be the canonical pairing. The derivative of algebraic characters
$d : X^*(T) \to \text{Lie}(T)' = \text{Hom}_F(\text{Lie}(T), F)$ induces an isomorphism $X^*(T)_F = X^*(T) \otimes \mathbb{Z} F \cong \text{Lie}(T)'$, and we extend the pairing $\langle \cdot, \cdot \rangle$ linearly to a perfect pairing $\text{Lie}(T)' \times X_*(T)_F \cong X^*(T)_F \times X_*(T)_F \to F$, where $X_*(T)_F = X_*(T) \otimes \mathbb{Z} F$.

**Preliminaries on the functors $F^G_p$.** We are going to describe the locally analytic principal series $\text{Ind}^G_P(\chi)^{an}$ and $\text{Ind}^H_{P_H}(\chi)^{an}$ in terms of the functors $F^G_p$ and $F^H_{P_H}$ introduced in [22]. We will be working in the category $\mathcal{O}_{\text{alg}}$ (for $g = \mathfrak{gl}_2$ and $h = \mathfrak{sl}_2$, resp.), cf. [22, 2.6]. In the following we describe the necessary facts only for $G$ and $g = \mathfrak{gl}_2$, but everything holds also for $H$ and $h$, with the obvious modifications. Given $\lambda \in X^*(T)$ we have the associated one-dimensional Lie algebra representation $d\lambda : \text{Lie}(T) \to \mathfrak{gl}_1(F) = F \hookrightarrow E$ which we denote by $E_{d\lambda}$. The canonical map $P \to T$ induces the map $p = \text{Lie}(P) \to \text{Lie}(T)$ on Lie algebras, and the representation $E_{d\lambda}$ lifts to a representation of $p$. The latter gives rise to the Verma module

$$M(d\lambda) = U(g) \otimes_{U(p)} E_{d\lambda}.$$  

Applying the functor $F^G_p$ to $M(-d\lambda)$ gives a locally $F$-analytic principal series representation

$$F^G_p(M(-d\lambda)) = \text{Ind}^G_P(\lambda)^{an},$$

cf. [22, 5.10]. Note that on the left we have the Verma module attached to the character $-d\lambda$. This property is related to the fact that the functor $F^G_p(-, -)$ is contravariant in the first argument, cf. below. More generally, we can also treat locally algebraic characters as follows. Write the locally algebraic character $\chi$ as $\chi = \chi^{\text{alg}} \cdot \chi^{\text{sm}}$ with a uniquely determined algebraic character $\chi^{\text{alg}}$ and a smooth character $\chi^{\text{sm}}$. Denote by $\text{Rep}_E^{\infty, \text{adm}}(T)$ the category of smooth admissible representations of $T$ on $E$-vector spaces. The functor $F^G_p$ has an extension to a bi-functor on the product category $\mathcal{O}_{\text{alg}} \times \text{Rep}_E^{\infty, \text{adm}}(T)$, cf. [22, 4.7], and we have

(3.2.1)  \[ F^G_p\left( M(-d\chi^{\text{alg}}), \chi^{\text{sm}} \right) = \text{Ind}^G_P(\chi^{\text{alg}} \cdot \chi^{\text{sm}})^{an} = \text{Ind}^G_P(\chi)^{an}. \]

Define characters $\varepsilon_1, \Delta_T \in X^*(T)$ by $\varepsilon_1(\text{diag}(a, d)) = a$ and $\Delta_T(\text{diag}(a, d)) = ad$. Then $\{\varepsilon_1, \Delta_T\}$ is a $\mathbb{Z}$-basis of $X^*(T)$. One has $d(\varepsilon_1^2 \Delta_T^m) = nd\varepsilon_1 + md\Delta_T$, and there is the following relation for the Verma module

(3.2.2)  \[ M(d(\varepsilon_1^2 \Delta_T^m)) = M(nd\varepsilon_1 + md\Delta_T) \cong (d\Delta_T^m) \otimes_E M(nd\varepsilon_1) = (m \text{Tr}_g) \otimes_E M(nd\varepsilon_1), \]
where $\Delta_G$ is the determinant character on $G$, and $\text{Tr}_g$ is the trace, the derived representation of $\Delta_G$. It is well known that if $n < 0$ then $M(nd_{\varepsilon_1})$ is irreducible, whereas for $n \geq 0$ one has an exact sequence

\begin{equation}
0 \rightarrow M(nd_{\varepsilon_1} - (n + 1)d\alpha) \rightarrow M(nd_{\varepsilon_1}) \rightarrow L(n_{\varepsilon_1}) \rightarrow 0,
\end{equation}

where $L(nd_{\varepsilon_1}) \cong \text{Sym}^n(E^2)$ is the irreducible representation of $\mathfrak{g}$ with highest weight $nd_{\varepsilon_1}$, and $\alpha = \varepsilon_T^2/\Delta_T$. Define the cocharacter $\alpha^\vee : F \times T$ by $\alpha^\vee(t) = \text{diag}(t, t^{-1})$.

**Proposition 3.2.4.** Let $\chi = \chi^{\text{alg}} \cdot \chi^{\text{sm}}$ be a locally $F$-analytic character as above. Put $n = \langle -d\chi^{\text{alg}}, \alpha^\vee \rangle \in \mathbb{Z}$.

(i) Suppose $n < 0$. Then $\text{Ind}_G^P(\chi)^{\text{an}}$ (resp. $\text{Ind}_H^P(\chi)^{\text{an}}$) is a topologically irreducible $G$-representation (resp. $H$-representation) and has no non-zero locally algebraic vectors.

(ii) If $n \geq 0$, then there is an exact sequence of locally analytic $G$-representations

\begin{equation}
0 \rightarrow \Delta_G^{\alpha^{-1}n - m} \otimes \text{Sym}^n(E^2) \otimes \text{ind}_P^G(\chi^{\text{sm}}) \rightarrow \text{Ind}_P^G(\chi)^{\text{an}} \rightarrow \text{Ind}_P^G(\chi\alpha^{n+1})^{\text{an}} \rightarrow 0
\end{equation}

with continuous $G$-equivariant maps. Here, $m \in \mathbb{Z}$ is such that $\chi^{\text{alg}} = \varepsilon_T^{-n} \Delta_T^{-m}$, and $\text{ind}_P^G(\chi^{\text{sm}})$ denotes the smooth non-normalized induction.\footnote{See the notations at the end of the introduction.} The representation on the right is topologically irreducible, and the representation on the left is topologically irreducible if and only if $\text{ind}_P^G(\chi^{\text{sm}})$ is irreducible.

(iii) If $n \geq 0$, then there is an exact sequence of locally analytic $H$-representations

\begin{equation}
0 \rightarrow \text{Sym}^n(E^2) \otimes \text{ind}_H^P(\chi^{\text{sm}}) \rightarrow \text{Ind}_H^P(\chi)^{\text{an}} \rightarrow \text{Ind}_H^P(\chi\alpha^{n+1})^{\text{an}} \rightarrow 0
\end{equation}

with continuous $H$-equivariant maps. The representation on the right is topologically irreducible, and the representation on the left is topologically irreducible if and only if $\text{ind}_H^P(\chi^{\text{sm}})$ is irreducible.

(iv) The $G$-representation $\text{Ind}_P^G(\chi)^{\text{an}}$ has non-zero locally algebraic vectors if and only if $n \geq 0$. When $n \geq 0$, then the space of locally algebraic vectors $(\text{Ind}_P^G(\chi)^{\text{an}})^{\text{alg}} \subset \text{Ind}_P^G(\chi)^{\text{an}}$
is equal to the kernel of the canonical map \( \text{Ind}_P^G(\chi)^{an} \to \text{Ind}_P^G(\chi \alpha^{n+1})^{an} \) in (3.2.3). In particular,

\[
(3.2.7) \quad \left( \text{Ind}_P^G(\chi)^{an} \right)^{\text{alg}} \cong \Delta_G^{n-m} \otimes \text{Sym}^n(E^2) \otimes \text{ind}_P^G(\chi^{sm}) .
\]

(v) The \( H \)-representation \( \text{Ind}_{P_H}^H(\chi)^{an} \) has non-zero locally algebraic vectors if and only if \( n \geq 0 \). When \( n \geq 0 \), then the space of locally algebraic vectors \( \left( \text{Ind}_{P_H}^H(\chi)^{an} \right)^{\text{alg}} \subset \text{Ind}_{P_H}^H(\chi)^{an} \) is equal to the kernel of the canonical map \( \text{Ind}_{P_H}^H(\chi)^{an} \to \text{Ind}_{P_H}^H(\chi \alpha^{n+1})^{an} \) in (3.2.6). In particular,

\[
(3.2.8) \quad \left( \text{Ind}_{P_H}^H(\chi)^{an} \right)^{\text{alg}} \cong \text{Sym}^n(E^2) \otimes \text{ind}_{P_H}^H(\chi^{sm}) .
\]

Proof. Since \( n = (-d\chi^{\text{alg}}, \alpha^\vee) \), we can write \( \chi^{\text{alg}} = \varepsilon_1^{-n} \Delta_T^{-m} \) for some \( m \in \mathbb{Z} \).

(i) By (3.2.1) we have \( \text{Ind}_P^G(\chi)^{an} = \mathcal{F}_P^G \left( M(-d\chi^{\text{alg}}), \chi^{sm} \right) \). By (3.2.2) we have \( M(-d\chi^{\text{alg}}) = (m \text{Tr}_g) \otimes M(nd\varepsilon_1) \), and since \( M(nd\varepsilon_1) \) is irreducible if \( n < 0 \), the same is true for \( M(-d\chi^{\text{alg}}) \). By [22, 5.8], the representation \( \text{Ind}_P^G(\chi)^{an} \) is topologically irreducible. If the representation \( \text{Ind}_P^G(\chi)^{an} \) has a non-zero locally algebraic vector, then, by the discussion in [16, 4.2], there is a finite-dimensional irreducible algebraic representation \( W \) of \( G \) over \( E \) such that the \( G \)-subrepresentation \( V_{W-\text{alg}} \) of \( W \)-locally algebraic vectors is non-zero. We use here the notation introduced by Emerton in [16, 4.2.2]. By [16, 4.2.10] the subspace \( V_{W-\text{alg}} \) is closed in \( V \). Therefore, if \( \text{Ind}_P^G(\chi)^{an} \) has a non-zero locally algebraic vector, it has a non-zero closed \( G \)-subrepresentation which consists only of locally algebraic vectors. It is easy to see that not all vectors in \( \text{Ind}_P^G(\chi)^{an} \) are locally algebraic. Since \( \text{Ind}_P^G(\chi)^{an} \) is topologically irreducible if \( n < 0 \), its subspace of locally algebraic vectors must be zero in this case. The same arguments also apply to the representation \( \text{Ind}_{P_H}^H(\chi)^{an} = \mathcal{F}_{P_H}^H \left( M(-d\chi^{\text{alg}}), \chi^{sm} \right) \).

(ii) Now we assume \( n \geq 0 \). Tensoring the exact sequence (3.2.3) with the one-dimensional representation \( m \text{Tr}_g \) gives the exact sequence

\[
0 \to (m \text{Tr}_g) \otimes M(nd\varepsilon_1 - (n+1)d\alpha) \to (m \text{Tr}_g) \otimes M(nd\varepsilon_1) \to (m \text{Tr}_g) \otimes L(n\varepsilon_1) \to 0 ,
\]
which, using \(3.2.2\), can be rewritten as

\[
0 \rightarrow M(-d\chi^\text{alg} - (n + 1)d\alpha) \rightarrow M(-d\chi^\text{alg}) \rightarrow (m\operatorname{Tr}_\theta) \otimes L(n\varepsilon_1) \rightarrow 0.
\]

Applying the (contravariant) functor \(\mathcal{F}_P^G(\cdot, \chi^\text{sm})\) to this exact sequence, and using \(3.2.1\) we obtain the exact sequence

\[
0 \rightarrow \mathcal{F}_P^G((m\operatorname{Tr}_\theta) \otimes L(n\varepsilon_1), \chi^\text{sm}) \rightarrow \operatorname{Ind}_P^G(\chi)^\text{an} \rightarrow \operatorname{Ind}_P^G(\chi\alpha^{n+1})^\text{an} \rightarrow 0.
\]

The representation on the right is topologically irreducible because \(-d\chi^\text{alg}\alpha^{n+1}, \alpha^\vee\) = \(n - 2(n + 1) = -n - 2 < 0\), cf (i). \((m\operatorname{Tr}_\theta) \otimes L(n\varepsilon_1)\) is a finite-dimensional \(g\)-module which lifts uniquely to the algebraic representation \(\Delta_G^m \otimes \operatorname{Sym}^n(E^2)\) of \(G\). In this case, the formula in \([22, 4.9]\) shows that

\[
\mathcal{F}_P^G((m\operatorname{Tr}_\theta) \otimes L(n\varepsilon_1), \chi^\text{sm}) \cong \mathcal{F}_G^G((m\operatorname{Tr}_\theta) \otimes L(n\varepsilon_1), \operatorname{ind}_P^G(\chi^\text{sm}))
= \left(\Delta_G^m \otimes \operatorname{Sym}^n(E^2)\right)^!' \otimes \operatorname{ind}_P^G(\chi^\text{sm})
\]

where \(\left(\Delta_G^m \otimes \operatorname{Sym}^n(E^2)\right)^!'\) is the dual representation to \(\Delta_G^m \otimes \operatorname{Sym}^n(E^2)\). Looking at the central character, this dual representation is easily seen to be isomorphic to \(\Delta_G^{-n-m} \otimes \operatorname{Sym}^n(E^2)\).

The representation \(\Delta_G^{-n-m} \otimes \operatorname{Sym}^n(E^2) \otimes \operatorname{ind}_P^G(\chi^\text{sm})\) is irreducible if and only if the representation \(\operatorname{ind}_P^G(\chi^\text{sm})\) is irreducible, by \([27, \text{appendix, Thm. 1}], [16, 4.2.8]\).

(iii) All arguments in (ii) also apply to the case of the group \(H\).

(iv) If \(n \geq 0\), then \(3.2.3\) shows that \(\operatorname{Ind}_P^G(\chi)^\text{an}\) has a non-zero closed subspace of locally algebraic vectors, because \(\Delta_G^{-n-m} \otimes \operatorname{Sym}^n(E^2) \otimes \operatorname{ind}_P^G(\chi^\text{sm})\) is a locally algebraic representation. This subspace must be equal to the whole space of locally algebraic vectors, because otherwise the quotient

\[
\operatorname{Ind}_P^G(\chi)^\text{an} / \left(\Delta_G^{-n-m} \otimes \operatorname{Sym}^n(E^2) \otimes \operatorname{ind}_P^G(\chi^\text{sm})\right) \cong \operatorname{Ind}_P^G(\chi\alpha^{n+1})^\text{an}
\]

would have itself non-zero locally algebraic vectors, and would thus not be irreducible, which would contradict (i).

(v) All arguments in (iv) also apply to the case of the group \(H\).

\(\square\)
Remark 3.2.9. We keep the notation of the previous proposition. When $n \geq 0$ the inclusion $M(-d\chi^{\text{alg}} - (n + 1)d\alpha) \hookrightarrow M(-d\chi^{\text{alg}})$ induces the surjective continuous $G$-homomorphism

\begin{equation}
\text{Ind}^G_P(\chi)^{\text{an}} \longrightarrow \text{Ind}^G_P(\chi\alpha^{n+1})^{\text{an}}
\end{equation}

which appears on the right in 3.2.5. This map can be described by a $G$-equivariant differential operator. More precisely, one can identify $\text{Ind}^G_P(\chi)^{\text{an}}$ (resp. $\text{Ind}^G_P(\chi\alpha^{n+1})^{\text{an}}$) with the space of locally analytic functions on $F$ which satisfy some regularity condition at infinity which depends on $\chi$ (resp. $\chi\alpha^{n+1}$), cf. [20, 3.2.1]. With this description, the map 3.2.10 become the derivation map $f \mapsto \frac{d^{n+1}}{d x^{n+1}} f$ does not have a continuous linear section on the level of topological vector spaces. This implies that the map $f \mapsto \frac{d^{n+1}}{d x^{n+1}} f$ considered here does not have a continuous section. In particular, the sequences 3.2.5 and 3.2.6 do not allow continuous $G$-equivariant, resp. $H$-equivariant, splittings. □

In the following we are particularly interested in the case when the $G$-representation

$$
\left( \text{Ind}^G_P(\chi)^{\text{an}} \right)^{\text{alg}} \cong \Delta_G^{-n-m} \otimes \text{Sym}^n(E^2) \otimes \text{ind}^G_P(\chi^{\text{sm}}),
$$

is irreducible, but the restriction to $H$ of this locally algebraic representation, namely

$$\text{Sym}^n(E^2) \otimes \text{ind}^H_P(\chi^{\text{sm}}),$$

is reducible. By [27, appendix, Thm. 1], [16, 4.2.8], this happens if and only if the smooth representation $\text{ind}^G_P(\chi^{\text{sm}})$ is irreducible, but its restriction to $H$, namely $\text{ind}^H_P(\chi^{\text{sm}})$, is reducible. Denote by $e(E/F)$ the ramification index of $E/F$, and let $\varpi_E$ be a uniformizer of $E$.

**Proposition 3.2.11.** Let $\chi : T \to E^\times$ be a locally algebraic character. Write $\chi(\text{diag}(a, d)) = \chi_1(a)\chi_2(d)$, and $\chi_i^{\text{alg}}(x) = x^{-c_i}$ with $c_i \in \mathbb{Z}$ ($i = 1, 2$).

(i) Then $\left( \text{Ind}^G_P(\chi)^{\text{an}} \right)^{\text{alg}}$ is an irreducible $G$-representation and is reducible as a representation of $H$, if and only if the following conditions hold

(a) $c_1 \geq c_2$,

(b) there is a nontrivial quadratic character $\text{sgn}_g$ on $F^\times$, and a smooth character $\tau$ of $F^\times$, such that $\chi_1(x) = x^{-c_1}|x|^{\frac{1}{2}}\text{sgn}_g(x)\tau(x)$ and $\chi_2(x) = x^{-c_2}|x|^{-\frac{1}{2}}\tau(x)$.
Moreover, $\text{Ind}_P^G(\chi)^{an}$ has unitary central character if and only if $\varpi^{-c_1-c_2} \tau(\varpi)^2 \in \mathcal{O}_E^\times$. In that case, the integer $e(E/F)(c_1+c_2)$ is even.

(ii) Given integers $c_1 \geq c_2$ such that $h := e(E/F)(c_1+c_2)$ is even, then, for any non-trivial quadratic character $\text{sgn}_\theta$ of $F^\times$ and any smooth unitary character $\tau_{\text{unit}} : F^\times \to \mathcal{O}_E^\times$, if we define

$$\chi_1(x) = x^{-c_1}|x|^{\frac{1}{2}}\varpi_E^{\frac{h}{2}v_F(x)}\text{sgn}_\theta(x)\tau_{\text{unit}}(x) \quad \text{and} \quad \chi_2(x) = x^{-c_2}|x|^{-\frac{1}{2}}\varpi_E^{\frac{h}{2}v_F(x)}\tau_{\text{unit}}(x),$$

then $\text{Ind}_P^G(\chi)^{an}$ has unitary central character, and $\left(\text{Ind}_P^G(\chi)^{an}\right)^{\text{alg}}$ is irreducible as a representation of $G$ and is reducible as $H$-representation.

(iii) If $\text{Ind}_P^G(\chi)^{an}$ has unitary central character, and $\left(\text{Ind}_P^G(\chi)^{an}\right)^{\text{alg}}$ is an irreducible $G$-representation and is reducible as $H$-representation, then $\chi_1$ and $\chi_2$ are of the form in (ii). In that case, the restriction of $\left(\text{Ind}_P^G(\chi)^{an}\right)^{\text{alg}}$ to $H$ splits as a direct sum of two inequivalent absolutely irreducible $H$-representations.

Proof. (i) Write $\chi_i(x) = x^{-c_i}\chi_i^{\text{sm}}(x), i = 1, 2,$ where $\chi_i^{\text{sm}}$, is a smooth character. We have $n := \langle -d\chi^\text{alg}, \alpha^\vee \rangle = c_1 - c_2$. According to 3.2.4, $c_1 - c_2 \geq 0$ is necessary and sufficient for $\text{Ind}_P^G(\chi)^{an}$ to have non-zero algebraic vectors. In that case, by [27, appendix, Thm. 1], [16, 4.2.8], the representation $\left(\text{Ind}_P^G(\chi)^{an}\right)^{\text{alg}}$ is irreducible as $G$-representation but reducible as $H$-representation, if and only if

$$\text{ind}_P^G(\chi^{\text{sm}}) = i_{G,T}(\delta_{\mathcal{O}_F^\times}^{-\frac{1}{2}}\chi^{\text{sm}}) = i_{G,T}(|^{-\frac{1}{2}}\chi_1^{\text{sm}} \otimes |^{-\frac{1}{2}}\chi_2^{\text{sm}})$$

is irreducible as $G$-representation but reducible as $H$-representation. By 3.1.1 and 3.1.2, this happens if and only if $|^{-\frac{1}{2}}\chi_1^{\text{sm}}(|^{-\frac{1}{2}}\chi_2^{\text{sm}})^{-1}$ is a non-trivial quadratic character $\text{sgn}_\theta$ of $F^\times$. If we now define $\tau = |^{-\frac{1}{2}}\chi_2^{\text{sm}}$, then we see that condition (b) holds. Moreover, the central character is unitary if and only if $(\chi_1\chi_2)(\varpi) = \varpi^{-c_1-c_2}\text{sgn}_\theta(\varpi)\tau(\varpi)^2 \in \mathcal{O}_E^\times$.

(ii) This is an easy consequence of part (i).

(iii) The first assertion follows from (i) and (ii). The second assertion follows from 3.2.4 (v) and 3.1.2 (ii).
Remark 3.2.12. We see from part (ii) of 3.2.11 that, if \( e(E/F) \) is even, we may choose any integers \( c_1 \geq c_2 \), and define characters \( \chi_1, \chi_2 \) by the formulas in part (ii), so as to obtain a locally analytic principal series representation \( \text{Ind}_P^G(\chi)^\text{an} \) with unitary central character, and which has the property as in 3.2.11 (i). Of course, we may always enlarge the coefficient field \( E \) so as to have even ramification index over \( F \). Note also that if \( q^\frac{1}{2} \notin F \) (e.g., \( F = \mathbb{Q}_p \)), then \( e(E/F) \) is necessarily even, since we assume that \( E \) contains a square root of \( q \). □

3.2.13. Choose integers \( c_1 \geq c_2 \), a non-trivial quadratic character \( \text{sgn}_\theta \) of \( F^\times \), a smooth character \( \tau \) of \( F^\times \), and define the following characters of \( T \):

\[
(3.2.14) \quad \eta = \left( (\cdot)^{-c_1} \otimes (\cdot)^{-c_2} \right) \cdot \frac{1}{p} \cdot (\tau \cdot \text{sgn}_\theta \otimes \tau), \quad \mu = \left( (\cdot)^{-c_1} \otimes (\cdot)^{-c_2} \right) \cdot \frac{3}{p} \cdot (\tau \otimes \tau \cdot \text{sgn}_\theta).
\]

Then \( \mu = (\text{sgn}_\theta \circ \Delta_T)\eta \). Write \( \eta = \eta^\text{alg} \cdot \eta^\text{sm} \) and \( \mu = \mu^\text{alg} \cdot \mu^\text{sm} \). By 3.2.7 the representation \( U_\eta = \left( \text{Ind}_P^G(\eta)^\text{an} \right)^\text{alg} \) can be written as \( U_\eta = U_\eta^\text{alg} \otimes U_\eta^\text{sm}, \) where \( U_\eta^\text{sm} = \text{ind}_P^G(\eta^\text{sm}) \) is a smooth \( G \)-representation and \( U_\eta^\text{alg} \) is algebraic, and both are absolutely irreducible. Note that \( \eta^\text{alg} = \mu^\text{alg} \), and that irreducibility of \( U_\eta^\text{sm} = \iota_G(\tau \cdot \text{sgn}_\theta \otimes \tau) \) implies \( U_\eta^\text{sm} \cong U_\mu^\text{sm} \), cf. 3.1.2. It follows

\[
U_\eta = U_\eta^\text{alg} \otimes_E U_\eta^\text{sm} \cong U_\mu^\text{alg} \otimes_E U_\mu^\text{sm} = U_\mu,
\]

and because those representations are absolutely irreducible \( G \)-representations, any such isomorphism is unique up to a non-zero scalar multiple [14, 1.1]. We fix an isomorphism \( \iota : U_\eta \xrightarrow{\sim} U_\mu \) and define the amalgamated sum

\[
W := \text{Ind}_P^G(\eta)^\text{an} \oplus_{U_\eta} \text{Ind}_P^G(\mu)^\text{an}
\]

to be the quotient of \( \text{Ind}_P^G(\eta)^\text{an} \oplus \text{Ind}_P^G(\mu)^\text{an} \) by the subspace \( \{(v, -\iota(v)) \mid v \in U_\eta\} \). The restriction of this representation to \( H = SL_2(F) \) is described in the following proposition.

Proposition 3.2.15. Suppose \( \eta \) and \( \mu \) are the characters of \( T \) defined by (3.2.14), and put \( \eta' = \eta \alpha^{c_1-c_2+1} \). Then the restriction of \( W = \text{Ind}_P^G(\eta)^\text{an} \oplus_{U_\eta} \text{Ind}_P^G(\mu)^\text{an} \) to \( H \) decomposes as \( W|_H = W_1 \oplus W_2 \), where for \( i = 1, 2 \), \( W_i \) fits in the following exact sequence of \( H \)-modules

\[
0 \to W_i^\text{alg} \to W_i \to \text{Ind}_{P_H}^H(\eta')^\text{an} \to 0.
\]

The \( H \)-modules \( W_i^\text{alg} \) and \( \text{Ind}_{P_H}^H(\eta')^\text{an} \) are irreducible and \( W_1^\text{alg} \neq W_2^\text{alg} \). Moreover, for any \( g \in G \) such that \( \text{sgn}_\theta(\det(g)) = -1 \) we have \( g \cdot W_1 = W_2 \) and \( g \cdot W_2 = W_1 \).
Proof. We want to describe the amalgamated sum \( W = \text{Ind}^G_P(\eta)^{an} \oplus U_\eta \text{Ind}^G_P(\mu)^{an} \) explicitly. To this end we make use of the description of locally analytic principal series representations given in [20, 3.2.1]. Define locally analytic characters on \( F^\times \) as follows:

\[
\eta_1(z) = z^{-c_1}|z|^\frac{1}{2}\tau(z)\text{sgn}_\theta(z), \quad \eta_2(z) = z^{c_1-c_2}|z|^{-1}\text{sgn}_\theta(z), \\
\mu_1(z) = z^{-c_1}|z|^\frac{1}{2}\tau(z), \quad \mu_2(z) = z^{c_1-c_2}|z|^{-1}\text{sgn}_\theta(z),
\]

so that

\[
(3.2.16) \quad \mu_1 = \eta_1 \cdot \text{sgn}_\theta, \quad \mu_2 = \eta_2, \\
\eta(\text{diag}(a,d)) = \eta_1(ad)\eta_2(d), \quad \mu(\text{diag}(a,d)) = \mu_1(ad)\mu_2(d).
\]

Then the space underlying both \( \text{Ind}^G_P(\eta)^{an} \) and \( \text{Ind}^G_P(\mu)^{an} \) can be identified with the space of locally analytic functions \( f \) on \( F \) which have the property that the function

\[
F^\times \to E, \quad z \mapsto \eta_2(z)f\left(\frac{1}{z}\right) = \mu_2(z)f\left(\frac{1}{z}\right)
\]

extends to a locally analytic function on all of \( F \). We henceforth denote this space by \( V \).

The group action on \( V \), considered as the underlying vector space of \( \text{Ind}^G_P(\eta)^{an} \), is such that

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ maps } f \text{ to } (g_{\cdot \eta}f)(z) = \eta_1(\text{det}(g))\eta_2(bz+d)f\left(\frac{az+c}{bz+d}\right)
\]

We write \( V_\eta \) when we equip \( V \) with this group action. Similarly, when we consider the group action on \( V \) as defined by \( \text{Ind}^G_P(\mu)^{an} \), we find that

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ maps } f \text{ to } (g_{\cdot \mu}f)(z) = \mu_1(\text{det}(g))\mu_2(bz+d)f\left(\frac{az+c}{bz+d}\right)
\]

We write \( V_\mu \) when we equip \( V \) with this latter group action. Put \( K = \ker(\text{sgn}_\theta \circ \text{det}) \subset G \).

It follows from 3.2.16 that

\[
(3.2.17) \quad g_{\cdot \eta}f = \begin{cases} g_{\cdot \mu}f, & \text{if } g \in K, \\ -g_{\cdot \mu}f, & \text{if } g \in G \setminus K. \end{cases}
\]

Let \( U = V^{\text{alg}} \) be the subspace of locally algebraic vectors. This space is the same for either group action. We write \( U_\eta \) (resp. \( U_\mu \)) when we consider \( U \) as a subrepresentation of \( V_\eta \) (resp. \( V_\mu \)). By 3.2.7, we have \( U_\eta = U_\eta^{\text{alg}} \otimes U_\eta^{\text{sm}} \) with \( U_\eta^{\text{sm}} \simeq i_{G,T}(\tau \cdot \text{sgn}_\theta \otimes \tau) \), and
thus \( U_{\eta}^{\text{sm}}|_H \simeq i_{H,T}(\text{sgn}_\theta \otimes 1) \). By \(3.1.1\) (ii) the smooth representation \( U_{\eta}^{\text{sm}} \) splits into two absolutely irreducible inequivalent representations \( (U_{\eta}^{\text{sm}})_1 \) and \( (U_{\eta}^{\text{sm}})_2 \) of \( H \). Because \( U_{\eta}^{\text{alg}} \) is absolutely irreducible, the representation \( U_{\eta}|_H \) splits as \( U_{\eta,1} \oplus U_{\eta,2} \) with absolutely irreducible and inequivalent locally algebraic \( H \)-representations \( U_{\eta,i} = U_{\eta}^{\text{alg}} \otimes (U_{\eta}^{\text{sm}})_i, i = 1, 2 \), by \[16\] 4.2.8]. Moreover, \( U_{\mu}^{\text{sm}} \simeq i_{G,T}(\tau \otimes \tau \cdot \text{sgn}_\theta) \) and \( U_{\eta}^{\text{alg}} = U_{\mu}^{\text{alg}} \) (since \( \eta^{\text{alg}} = \mu^{\text{alg}} \)). Therefore, by \(3.1.2\) there is an intertwining operator of \( G \)-representations \( \iota: U_{\eta} \rightarrow U_{\mu} \). It follows from \(3.2.17\) that the map \( \iota \) is, on the underlying vector space, not given by a scalar multiplication.

**Claim.** The spaces \( U_{\eta,1} \) and \( U_{\eta,2} \) are both \( \mathcal{K} \)-invariant, and for \( g \in G \setminus \mathcal{K} \) we have \( g_{\eta}(U_{\eta,1}) = U_{\eta,2} \) and \( g_{\eta}(U_{\eta,2}) = U_{\eta,1} \).

**Proof.** If \( U_{\eta,1} \) would not be \( \mathcal{K} \)-invariant, then the smallest \( \mathcal{K} \)-invariant subrepresentation \( U' \) of \( U_{\eta} \) which contains \( U_{\eta,1} \) would necessarily also contain non-zero vectors in \( U_{\eta,2} \). But since \( U_{\eta,2} \) is irreducible as an \( H \)-representation, we must then have \( U_{\eta,2} \subset U' \), and therefore \( U' = U_{\eta} \). It thus follows that if \( U_{\eta,1} \) would not be \( \mathcal{K} \)-invariant, then \( U_{\eta} \) would have to be irreducible as \( \mathcal{K} \)-representation. The same argument applies after base change to any finite extension of \( E \), and \( U_{\eta} \) would thus be absolutely irreducible as \( \mathcal{K} \)-representation. By \(3.2.17\) the representations \( U_{\eta}|_{\mathcal{K}} \) and \( U_{\mu}|_{\mathcal{K}} \) are the same (the underlying vector space and the action of \( \mathcal{K} \) are identical), and any \( \mathcal{K} \)-intertwiner between them would have to be given by scalar multiplication, by \[14\] 1.1], if indeed \( U_{\eta}|_{\mathcal{K}} \) (and hence \( U_{\mu}|_{\mathcal{K}} \)) would be irreducible. But since the \( G \)-intertwiner \( \iota \) is not given by scalar multiplication, we can deduce that \( U_{\eta} \) (and hence \( U_{\mu} \)) is reducible as \( \mathcal{K} \)-representation, and \( U_{\eta,1} \) is therefore \( \mathcal{K} \)-invariant. Of course, the same argument applies in the same way to \( U_{\eta,2} \). The second assertion follows from the fact that \( \mathcal{K} \) is normal in \( G \) and that \( U_{\eta} \) is irreducible as \( G \)-representation. \(\square\)

We are now going to describe a \( G \)-intertwiner \( U_{\eta} \rightarrow U_{\mu} \) explicitly. To this end we set \( U_1 = U_{\mu,1} = U_{\eta,1} \) and \( U_2 = U_{\mu,2} = U_{\eta,2} \), where we consider \( U_{\mu,1} \) and \( U_{\mu,2} \) as subspaces of the \( G \)-representation \( U_{\mu} \). Define

\[
\varphi : U_{\eta} \rightarrow U_{\mu} , \quad \varphi(u_1 + u_2) = u_1 - u_2 ,
\]

with \( u_1 \in U_{\eta,1} \) and \( u_2 \in U_{\eta,2} \). Then for \( g \in \mathcal{K} \) we have

\[
\varphi(g_{\eta}(u_1 + u_2)) = \varphi(g_{\eta}u_1 + g_{\eta}u_2) = g_{\eta}u_1 - g_{\eta}u_2 = g_{\mu}u_1 - g_{\mu}u_2 = g_{\mu}(\varphi(u_1 + u_2)) ,
\]
and for $g \in G \setminus K$ we have
\[
\varphi(g \cdot \eta(u_1 + u_2)) = \varphi(g \cdot \eta u_2 + g \cdot \eta u_1) = g \cdot \eta u_2 - g \cdot \eta u_1 = g \cdot \mu \varphi(u_1 + u_2),
\]
because $g \cdot \eta u_2 \in U_{\eta,1}$ and $g \cdot \eta u_1 \in U_{\eta,2}$. Hence, $\varphi$ is a $G$-intertwining operator $U_\eta \rightarrow U_\mu$.

Therefore, $\varphi$ is a non-zero scalar multiple of $\iota$, and we may thus assume that our previously chosen $\iota$ is equal to $\varphi$. If we thus embed $U_\eta$ into $V_\mu$ via $\varphi$, then we can describe the amalgamated sum $W$ as $(V_\eta \oplus V_\mu)/U_0$, where
\[
U_0 = \{(u_1 + u_2, -u_1 + u_2) \mid u_1 \in U_1, u_2 \in U_2\}.
\]

We use here again the fact that the vector spaces underlying $V_\eta$ and $V_\mu$ are identical. Define
\[
\tilde{W}_1 = \{(v, -v) \mid v \in V\} \subset V_\eta \oplus V_\mu, \quad \tilde{W}_2 = \{(v, v) \mid v \in V\} \subset V_\eta \oplus V_\mu,
\]
\[
\tilde{U}_1 = \{(u_1, -u_1) \mid u_1 \in U_1\} \subset \tilde{W}_1, \quad \tilde{U}_2 = \{(u_2, u_2) \mid u_2 \in U_2\} \subset \tilde{W}_2.
\]
Set $W_1 = \text{im}(\tilde{W}_1 \rightarrow W)$ and $W_2 = \text{im}(\tilde{W}_2 \rightarrow W)$. It is straightforward to check that $W = W_1 \oplus W_2$ as $E$-vector spaces. Moreover, because the $K$-actions on $V_\eta$ and $V_\mu$ are identical, $W_1$ and $W_2$ are $K$-stable, hence $H$-stable, and we have $W|_H = W_1 \oplus W_2$ as $H$-representations. Moreover, it follows from \cite{3.2.17} that $g(W_1) = W_2$ and $g(W_2) = W_1$.

It remains to analyze $W_i$, $i = 1, 2$, as an $H$-representation. As the $H$-actions on $V_\eta$ and $V_\mu$ are identical, we now drop the subscripts $\eta$ and $\mu$. Note that $\ker(\tilde{W}_1 \rightarrow W_1) = \tilde{U}_1$ and $\ker(\tilde{W}_2 \rightarrow W_2) = \tilde{U}_2$, so that $W_i \cong \tilde{W}_i/\tilde{U}_i \cong V/U_i$. Choose $i \in \{1, 2\}$ and let $j$ be the other integer in $\{1, 2\}$. The inclusion $U \hookrightarrow V$ induces an embedding $U_i \hookrightarrow W_j$, and we obtain an exact sequence
\[
0 \rightarrow U_i \rightarrow W_j \rightarrow W_j/U_i \cong V/(U_1 + U_2) = V/V^{\text{alg}} \cong \text{Ind}_{P_H}^H(\eta') \rightarrow 0.
\]
We clearly have $U_i \subset W_j^{\text{alg}}$, and because $\text{Ind}_{P_H}^H(\eta')$ is topologically irreducible, its subspace of locally algebraic vectors is zero, cf. \cite{3.2.24}(i). Therefore $U_i = W_j^{\text{alg}}$. That $U_1$ and $U_2$ (and hence $W_1^{\text{alg}}$ and $W_2^{\text{alg}}$) are inequivalent as $H$-representations has already been noted above.

4. Restricting Banach space representations of $GL_2(\mathbb{Q}_p)$ to $SL_2(\mathbb{Q}_p)$

In this section we will have throughout $F = \mathbb{Q}_p$, $G = GL_2(\mathbb{Q}_p)$ and $H = SL_2(\mathbb{Q}_p)$.
4.1. At most two irreducible constituents.

Proposition 4.1.1 Let $\Pi$ be an absolutely irreducible admissible unitary $p$-adic Banach space representation of $G$. Then $\Pi|_{SL_2(\mathbb{Q}_p)}$ decomposes into at most two irreducible components.

Proof. Put $\overline{\Pi} = \Pi \leq 1 \otimes \mathcal{O}_E k_E$, where $\Pi \leq 1 = \{v \in \Pi \mid \|v\| \leq 1\}$ and $k_E$ is the residue field of $E$. This is a smooth $G$-representation. By [12, 1.4], after possibly replacing $E$ by an unramified quadratic extension, there are two possibilities for $\overline{\Pi}$, namely

(i) $\overline{\Pi}$ is an absolutely irreducible supersingular representation.

(ii) The semisimplification $\overline{\Pi}^ss$ of $\overline{\Pi}$ embeds into

$$\pi\{\chi_1, \chi_2\} := \left( \text{Ind}_P^G(\chi_1 \otimes \chi_2 \omega^{-1}) \right)^{ss} \oplus \left( \text{Ind}_P^G(\chi_2 \otimes \chi_1 \omega^{-1}) \right)^{ss},$$

where $\chi_1$ and $\chi_2$ are smooth characters $\mathbb{Q}_p^\times \to k_E^\times$, and $\omega : \mathbb{Q}_p^\times \to k_E^\times$ is the reduction of the cyclotomic character.

It is a result of R. Abdellatif that in case (i) $\overline{\Pi}|_H$ decomposes into two irreducible representations, cf. [11 Théorème 0.7]. In particular, $\Pi|_H$ cannot have more than two irreducible components.

Now suppose we are in case (ii). We consider the list given in [12, 2.14] which provides an explicit description of the decomposition of $\pi\{\chi_1, \chi_2\}$ into irreducible constituents. $\pi\{\chi_1, \chi_2\}$ is isomorphic to one (and only one) of the following:

1. $\text{ind}_P^G(\chi_1 \otimes \chi_2 \omega^{-1}) \oplus \text{ind}_P^G(\chi_2 \otimes \chi_1 \omega^{-1})$, if $\chi_1 \chi_2^{-1} \neq 1$, $\omega^{\pm 1}$;

2. $\text{ind}_P^G(\chi \otimes \chi \omega^{-1})^{\otimes 2}$, if $\chi_1 = \chi_2 = \chi$ and $p \geq 3$;

3. $\left( 1 \oplus \text{St} \oplus \text{ind}_P^G(\omega \otimes \omega^{-1}) \right) \otimes \chi \circ \text{det}$, if $\chi_1 \chi_2^{-1} = \omega^{\pm 1}$ and $p \geq 5$;

4. $\left( 1 \oplus \text{St} \oplus \omega \circ \text{det} \oplus \text{St} \otimes \omega \circ \text{det} \right) \otimes \chi \circ \text{det}$, if $\chi_1 \chi_2^{-1} = \omega^{\pm 1}$ and $p = 3$;

5. $\left( 1 \oplus \text{St} \right)^{\otimes 2} \otimes \chi \circ \text{det}$, if $\chi_1 = \chi_2$ and $p = 2$.

Here, St denotes the smooth Steinberg representation in characteristic $p$ which is irreducible by [11, 0.1]. The restriction of $\text{ind}_P^G(\chi_1 \otimes \chi_2 \omega^{-1})$ to $H$ is just $\text{ind}_P^H(\chi_1 \otimes \chi_2 \omega^{-1})$.

---

6We learned this fact from a comment by Matthew Emerton.
by [1, 0.1], this representation is irreducible if and only if $\chi_1 \neq \chi_2 \omega^{-1}$. Thus we see that in case (1) the restriction is of length two. The same is true in case (2) because the $\omega$ is not trivial when $p \geq 3$. In case (3), since $\omega^2 \neq 1$ for $p \geq 5$, the representation $\pi\{\chi_1, \chi_2\}|_H$ has two infinite-dimensional constituents, and the same is true in cases (4) and (5), because the Steinberg representation $St$ of $H$ is irreducible.

Write $\Pi|_H = \Pi_1 \oplus \ldots \oplus \Pi_r$, with irreducible $H$-representations $\Pi_i$. By [2, 1.3] the irreducible representations $\Pi_i$ are permuted by the action of $G$, and they must hence be all infinite-dimensional. Therefore, the representation $\left(\prod^s\right)|_H$ must have at least $r$ infinite-dimensional irreducible constituents. By what we have just seen, $r$ can then be at most two. □

4.2. The case of trianguline de Rham representations.

4.2.1. Locally algebraic vectors and Weil-Deligne representations. Let $\psi : \mathcal{G}_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(E)$ be an absolutely irreducible representation. We denote by $\Pi(\psi)$ the unitary Banach space representation of $GL_2(\mathbb{Q}_p)$ attached to $\psi$ by Colmez’ $p$-adic Langlands correspondence, cf. [9, 0.17]. The space of locally algebraic vectors $\Pi(\psi)|_{\text{lalg}}$ is non-zero if and only if $\psi$ is de Rham with distinct Hodge-Tate weights $a < b$ [9, Theorem 0.20]. In this case, by [9, VI.6.50], $\Pi(\psi)|_{\text{lalg}}$ decomposes as

$$\Pi(\psi)|_{\text{lalg}} = \Pi(\psi)|_{\text{alg}} \otimes \pi(\psi),$$

where $\Pi(\psi)|_{\text{alg}} \cong \text{det}^a \otimes_E \text{Sym}^{b-a-1}(E^2)$ is absolutely irreducible, and $\pi(\psi)$ is a smooth representation which is obtained from the associated $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-module $D_{\text{pst}}(\psi)$ as follows. $D_{\text{pst}}(\psi)$ is equipped with an $E$-linear action $\rho$ of $\mathcal{G}_{\mathbb{Q}_p}$ whose restriction to the inertia subgroup has finite image. Furthermore, $D_{\text{pst}}(\psi)$ is equipped with a bijective $E$-linear Frobenius $\varphi$ which commutes with that Galois action, and a nilpotent $E$-linear endomorphism $N$ satisfying $\varphi N \varphi^{-1} = pN$. One obtains a Weil group representation by letting $\gamma \in W_{\mathbb{Q}_p}$ act on $D_{\text{pst}}(\psi)$ as $\varphi^{-\deg(\gamma)} \rho(\gamma)$, where $\deg(\gamma) \in \mathbb{Z}$ is such that $\gamma$ acts on the residue field of $\overline{\mathbb{Q}}_p$ as $x \mapsto x^{p^{\deg(\gamma)}}$. The nilpotent endomorphism $N$ gives rise to a Weil-Deligne representation.

Let $WD(\psi)$ be the $F$-semisimplification of this Weil-Deligne representation, in the sense of [13, 8.5]. Then, up to a twist by a smooth character, $\pi(\psi)$ is equal to the smooth representation of $GL_2(\mathbb{Q}_p)$ associated to $WD(\psi)$ by the local Langlands correspondence,
except if the latter representation would be 1-dimensional, in which case \( \pi(\psi) \) is the unique principal series representation which surjects onto this character, cf. [9, before VI.6.50] for details.

4.2.3. The trianguline variety. We summarize some information about the trianguline variety from [10, 0.3]. We denote by \( \mathcal{F}(E) \) the set of continuous characters \( \delta : \mathbb{Q}_p^\times \to E^\times \). For \( \delta \in \mathcal{F}(E) \) let \( w(\delta) = w'(1) \) be its weight, i.e., its derivative at 1 (equivalently, \( w(\delta) = \frac{\log \delta(u)}{\log u} \), where \( u \in \mathbb{Z}_p^\times \) is not a root of unity).

Set \( \Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p) \) and let \( \mathcal{I} \) be the set of triples \( s = (\delta_1, \delta_2, \mathcal{L}) \), where \( \delta_1, \delta_2 \in \mathcal{F}(E) \) and \( \mathcal{L} \in \text{Proj}(\text{Ext}^1(\mathcal{A}(\delta_2), \mathcal{A}(\delta_1))) \). The latter set will be identified with \( \mathbb{P}^0(E) = \{ \infty \} \) if \( \dim(\text{Ext}^1(\mathcal{A}(\delta_2), \mathcal{A}(\delta_1))) = 1 \) and with \( \mathbb{P}^1(E) \) if \( \dim(\text{Ext}^1(\mathcal{A}(\delta_2), \mathcal{A}(\delta_1))) = 2 \). Let \( \Delta(s) \) be the \((\varphi, \Gamma)\)-module associated to \( s \):

\[
0 \rightarrow \mathcal{A}(\delta_1) \rightarrow \Delta(s) \rightarrow \mathcal{A}(\delta_2) \rightarrow 0.
\]

Set \( u(s) = v_p(\delta_1(p)) \) and \( w(s) = w(\delta_1) - w(\delta_2) \), and define

\[
\mathcal{I}_* = \{ s \in \mathcal{I} \mid v_p(\delta_1(p)) + v_p(\delta_2(p)) = 0 \text{ and } u(s) > 0 \},
\]

\[
\mathcal{I}_*^{\text{vg}} = \{ s \in \mathcal{I}_* \mid w(s) \notin \mathbb{Z}_{\geq 1} \},
\]

\[
\mathcal{I}_*^{\text{cris}} = \{ s \in \mathcal{I}_* \mid w(s) \in \mathbb{Z}_{\geq 1} , u(s) < w(s) , \mathcal{L} = \infty \},
\]

\[
\mathcal{I}_*^{\text{st}} = \{ s \in \mathcal{I}_* \mid w(s) \in \mathbb{Z}_{\geq 1} , u(s) < w(s) , \mathcal{L} \neq \infty \},
\]

\[
\mathcal{I}_*^{\text{ord}} = \{ s \in \mathcal{I}_* \mid w(s) \in \mathbb{Z}_{\geq 1} , u(s) = w(s) \}.
\]

Assume \( s \in \mathcal{I}_* \). Then \( \Delta(s) \) is étale if and only if \( s \in \mathcal{I}_*^{\text{vg}} \sqcup \mathcal{I}_*^{\text{cris}} \sqcup \mathcal{I}_*^{\text{st}} \sqcup \mathcal{I}_*^{\text{ord}} \), and hence corresponds to a Galois representation \( \psi(s) \) in this case. \( \psi(s) \) is irreducible if and only if \( s \in \mathcal{I}_*^{\text{vg}} \sqcup \mathcal{I}_*^{\text{cris}} \sqcup \mathcal{I}_*^{\text{st}} \). If \( \psi(s) \) is an irreducible Hodge-Tate representation, we have the following:

- \( \psi(s) \) is de Rham if and only if \( s \in \mathcal{I}_*^{\text{cris}} \sqcup \mathcal{I}_*^{\text{st}} \),
- \( \psi(s) \) is crystalline (i.e., becomes crystalline over an abelian extension of \( \mathbb{Q}_p \)) if and only if \( s \in \mathcal{I}_*^{\text{cris}} \).

\( \pi(\psi) \) is denoted \( \text{LL}(\text{WD}(\psi)) \) in loc. cit.
• \( \psi(s) \) is the twist of a semistable non-crystalline representation by a character of finite order if and only if \( s \in \mathcal{S}^\text{st} \),

cf. [10, 8.5]. Finally, every 2-dimensional absolutely irreducible trianguline representation of \( \mathcal{G}_{\overline{Q}_p} \) is of the form \( \psi(s) \) for some \( s \in \mathcal{S}^\text{ng} \cup \mathcal{S}^\text{cris} \cup \mathcal{S}^\text{st} \).

4.2.4. Notation for principal series representations. We denote by \( x \in \widehat{T}(E) \) the character \( x \mapsto x \) induced by the inclusion \( \mathbb{Q}_p \subseteq E \). Set \( \chi_{\text{cyc}} = x|_F \). For \( \delta_1, \delta_2 \in \widehat{T}(E) \) define

\[
B^\text{an}(\delta_1, \delta_2) = \text{Ind}^G_P(\delta_2 \otimes \delta_1 \chi_{\text{cyc}}^{-1})^\text{an},
\]

the locally analytic principal series representation, cf. [10, 0.4, after Thm. 8.6]. As introduced at the beginning of Section 3, \( p^{\frac{1}{2}} \) is the fixed square root in our coefficient field \( E \), and \( |x|_E = (p^{\frac{1}{2}})^{-v_p(x)} \) for \( x \in \mathbb{Q}_p^\times \). In particular, the ramification index \( e(E/F) = e(E/\mathbb{Q}_p) \) is even, and the condition that the integer \( h \) in 3.2.11 is even holds.

Theorem 4.2.5. Let \( \psi : \mathcal{G}_{\overline{Q}_p} \to GL_2(E) \) be an absolutely irreducible trianguline representation which is de Rham with distinct Hodge-Tate weights. Denote by \( \Pi = \Pi(\psi) \) the corresponding absolutely irreducible unitary admissible Banach space representation of \( G = GL_2(\mathbb{Q}_p) \), and let \( \Pi^{\text{alg}} \) be the subspace of locally algebraic vectors of \( \Pi \). Then the following assertions are equivalent:

1. \( \Pi|_H \) is reducible.
2. \( \Pi|_H \) is decomposable.
3. \( (\Pi^{\text{alg}})|_H \) is decomposable.

If one (equivalently all) of the above cases occurs, then both \( \Pi|_H \) and \( (\Pi^{\text{alg}})|_H \) have two absolutely irreducible inequivalent constituents.

Proof. ”(1) \( \Rightarrow \) (2)”. This implication follows from 2.1.3

”(2) \( \Rightarrow \) (3)” This follows from 2.1.4

”(3) \( \Rightarrow \) (1)”. Let us now suppose that \( \Pi^{\text{alg}} \) is decomposable. As we recalled in 4.2.2

\( \Pi^{\text{alg}} = \Pi^{\text{alg}} \otimes_E \pi \), with an (absolutely) irreducible finite-dimensional algebraic representation \( \Pi^{\text{alg}} \) and a smooth representation \( \pi \). This shows that \( \Pi^{\text{alg}} = (\Pi^{\text{alg}})|_{\Pi^{\text{alg}} - \Pi^{\text{alg}}} \), where the latter notation is the one of [10 4.2.2]. We equip \( \text{Hom}_E(\Pi^{\text{alg}}, \Pi^{\text{alg}}) \) with the usual \( G \)-action,
By 3.2.11, there are integers\( c \) such that \( \lambda \in \text{Hom}_E(\Pi^{\text{alg}}, \Pi^{\text{alg}}) \) and \( w \in \Pi^{\text{alg}} \). We denote by \( \text{Hom}_E(\Pi^{\text{alg}}, \Pi^{\text{alg}})^\text{sm} \) the set of smooth vectors for this \( G \)-action. Then the map

\[
(4.2.6) \quad \pi \rightarrow \text{Hom}_E(\Pi^{\text{alg}}, \Pi^{\text{alg}})^\text{sm}, \quad v \mapsto [w \mapsto w \otimes v],
\]

is an isomorphism of \( G \)-representations, cf. [16] before 4.2.4. Now suppose that \( \Pi^{\text{alg}|_H} = (\Pi^{\text{alg}})_1 \oplus (\Pi^{\text{alg}})_2 \) with non-zero \( H \)-representations \((\Pi^{\text{alg}})_1\) and \((\Pi^{\text{alg}})_2\). We then have also

\[
(\Pi^{\text{alg}})_i \Pi^{\text{alg}} \text{-representation},
\]

for \( i = 1, 2 \). By [16] 4.2.4, there is a smooth (non-zero) \( H \)-representation \( \pi_i \) such that \( (\Pi^{\text{alg}})_i = \Pi^{\text{alg}} \otimes_E \pi_i \) (as \( H \)-representations). The isomorphism \( 4.2.6 \) then shows that \( \pi|_H \cong \pi_1 \oplus \pi_2 \) is decomposable.

Denote by \( \text{LL}(D^{\text{pst}}(\psi)) \) the smooth \( G \)-representation associated to the filtered module \( D^{\text{pst}}(\psi) \), as defined in [9] VI, sec. 11. By [9] 0.21, we have \( \pi \cong \text{LL}(D^{\text{pst}}(\psi)) \). A quick glance at the list in [9] VI, sec. 11 shows that if \( \pi|_H \) is decomposable, then \( \pi \) must be an irreducible principal series representation. By [16] 4.2.8, \( \Pi^{\text{alg}} = \Pi^{\text{alg}} \otimes_E \pi \) is irreducible. Moreover, \( \psi \) is necessarily crystabelline and thus of the form \( \psi(s) \) with \( s = (\delta_1, \delta_2, \infty) \in \mathcal{S}_*^{\text{cris}} \). By [10] 4.6.1 we have

\[
\Pi^{\text{alg}} \cong B^{\text{alg}}(\delta_1, \delta_2) = \left( \text{Ind}(\delta_2 \otimes \delta_1 \chi_{\text{cyc}}^{-1}) \right)^{\text{alg}}.
\]

By 3.2.11 there are integers \( c_1 \geq c_2 \), a smooth character \( \tau \) of \( \mathbb{Q}^*_p \), and a non-trivial quadratic character \( \text{sgn}_\theta \) such that

\[
\delta_2(x) = x^{-c_1}|x|^s \text{sgn}_\theta(x) \tau(x), \quad \delta_1(x) \chi_{\text{cyc}}^{-1} = x^{-c_2}|x|^{-\frac{1}{2}} \tau(x).
\]

We then have \( w(s) = c_1 - c_2 + 1 > 0 \). Following [10] 4.6.1 we set \( \delta' = x^{w(s)} \delta_2 \) and \( \delta' = x^{-w(s)} \delta_1 \), and get thus

\[
\delta'_2(x) = x^{-c_1}|x|^s \tau(x), \quad \delta'_1(x) \chi_{\text{cyc}}^{-1} = x^{-c_2}|x|^{-\frac{1}{2}} \text{sgn}_\theta(x) \tau(x).
\]

By [10] 8.97, there is an exact sequence of \( G \)-representations

\[
0 \rightarrow \Pi^{\text{alg}} \rightarrow B^{\text{an}}(\delta_1, \delta_2) \oplus B^{\text{an}}(\delta'_1, \delta'_2) \rightarrow \Pi^{\text{an}} \rightarrow 0,
\]

where \( \Pi^{\text{alg}} \cong B^{\text{alg}}(\delta_1, \delta_2) \cong B^{\text{alg}}(\delta'_1, \delta'_2) \) is embedded diagonally in the representation in the middle, and \( \Pi^{\text{an}} \subset \Pi \) is the subspace of locally analytic vectors, equipped with its intrinsic
We are thus in the situation considered in 3.2.13 and by 3.2.15 we can infer that $\Pi_{an, H} = (\Pi_{an, 1} \oplus (\Pi_{an, 2})$ with two irreducible closed $H$-subrepresentations of $\Pi_{an}$. By [11, 0.2] the map $\Pi_{an} \to \Pi$ realizes $\Pi$ as the universal completion of $\Pi_{an}$. By 2.2.3 and 2.2.2 we conclude that 

$$\Pi_{H} = \hat{\Pi}_{an, H}$$

is decomposable. This completes the proof that assertion (3) implies assertion (1).

Now suppose that assertions (1)-(3) hold. As we have seen, if $\Pi_{lalg, H} = \Pi_{alg, H} \otimes \pi_{H}$ is decomposable, then so is $\pi_{H}$. Because $\psi$ is assumed to be trianguline and de Rham with distinct Hodge-Tate weights, $\pi$ is necessarily an irreducible principal series representation, and $\pi_{H}$ decomposes as the direct sum of two absolutely irreducible inequivalent constituents, cf. 3.1.1. On the other hand $\Pi_{H}$ must have at least two irreducible constituents, but it cannot have more, by 4.1.1. Those must be inequivalent by 2.1.4, since $\Pi_{lalg, H}$ has two inequivalent constituents. □

Remark 4.2.7. The supercuspidal case. Here we briefly consider the case when the smooth factor $\pi(\psi)$ of $\Pi(\psi)_{lalg}$ is supercuspidal. There are supercuspidal representations $\pi$ of $GL_{2}(\mathbb{Q}_{p})$ whose restriction to $SL_{2}(\mathbb{Q}_{p})$ have four components, cf. [30, sec. 12]. Suppose $\psi$ is such that $\pi(\psi)$ is a supercuspidal with this property. Then $\Pi(\psi)_{lalg, H}$ has four irreducible constituents, but we know from Proposition 4.1.1 that $\Pi_{H}$ has at most two irreducible constituents. Therefore, the analogue of Theorem 4.2.5 does not hold for such $\psi$.

Remark 4.2.8. The case when $\Pi_{lalg, H}$ is reducible but indecomposable. Let $\psi$ be as in 4.2.5. If $\Pi(\psi)_{lalg, H} = \Pi(\psi)_{lalg, H} \otimes \pi(\psi)_{H}$ is reducible but indecomposable, then $\pi(\psi)_{H}$ is reducible but indecomposable. By 3.1.1 and 3.1.2 this implies that $\pi(\psi)$, and hence $\Pi(\psi)_{lalg}$ are already reducible. This can indeed happen. For example, if $s = (x^{w} | \cdot, x^{-w}, \infty)$, with an integer $w \geq 2$. Then $s \in \mathcal{S}^{\text{cris}}$, and $\psi(s)$ is absolutely irreducible, crystalline, and has Hodge-Tate weights $w$ and $-w$, cf. [10, 8.5]. Moreover, $\delta_{s} := x_{\psi}^{-1} \delta_{1} \delta_{2}^{-1} = x^{-1+2w}$. In that case, the smooth representation $\pi(\psi)$ is the unique non-split extension of $\mathbf{1}$ by $\text{St}$ (cf. [10, 8.9]), and is thus reducible, but not decomposable.

4.3. Projective 2-dimensional Galois representations. Let $pr : GL_{2}(E) \to PGL_{2}(E)$ be the canonical projection. If $\psi : \mathcal{G}_{\mathbb{Q}_{p}} \to GL_{2}(E)$ is a continuous Galois representation,
then we denote by

$$\overline{\psi} = \text{pr} \circ \psi : G_{\mathbb{Q}_p} \to PGL_2(E)$$

the induced projective representation.

**Proposition 4.3.1.** Let $\psi : G_{\mathbb{Q}_p} \to GL_2(E)$ be an absolutely irreducible trianguline de Rham representation with distinct Hodge-Tate weights.

(i) The centralizer $S_\overline{\phi}$ in $PGL_2(E)$ of the image of $\overline{\psi}$ has one or two elements. The latter case occurs if and only if $\psi$ is equivalent to $\overline{\vartheta \psi}$ for some quadratic character $\vartheta \neq 1$.

(ii) Denote by $\phi$ the Weil group representation on $\text{WD}(\psi)^9$ associated to $\psi$, cf. 4.2.1, and set $\overline{\phi} = \text{pr} \circ \phi$. Then

$$S_{\psi} \cong S_{\overline{\phi}} / S_{\overline{\phi}}^0,$$

where $S_{\overline{\phi}}$ denotes the centralizer of the image of $\overline{\phi}$ in $PGL_2(E)$ (considered as an algebraic group) and $S_{\overline{\phi}}^0$ its identity component.

The proof of 4.3.1 will be given in section 4.3.9 below. The strategy is to determine the centralizer using the filtered modules attached to $\psi$. Note that we can twist $\psi$ by a power of the cyclotomic character so that its Hodge-Tate weights are 0 and $k - 1$, where $k \geq 2$, and this is what we will assume for the remainder of this subsection.

**4.3.2. Filtered modules in the crystabelline case.** Here we assume that $\psi$ is crystabelline, and we consider the filtered $(\varphi, G_{\mathbb{Q}_p})$-module $D_{\text{cris}}(\psi)$ as defined in [3, after 2.4.2]9. Let $\alpha, \beta : \mathbb{Q}_p^\times \to E^\times$ be locally constant characters such that

$$-(k - 1) < \text{val}(\alpha(p)) \leq \text{val}(\beta(p)) < 0 \quad \text{and} \quad \text{val}(\alpha(p)) + \text{val}(\beta(p)) = -(k - 1)$$

and which are trivial on $1 + p^n \mathbb{Z}_p$ for some $n \geq 1$. As in [3, 2.4.4], we define on $D(\alpha, \beta) = E \cdot e_\alpha \oplus E \cdot e_\beta$ the structure of a filtered $(\varphi, G_{\mathbb{Q}_p})$-module:

If $\alpha \neq \beta$, then:

$$\begin{cases}
\varphi(e_\alpha) = \alpha(p)e_\alpha \\
\varphi(e_\beta) = \beta(p)e_\beta
\end{cases} \quad \text{and if } g \in \Gamma, \text{ then:}
\begin{cases}
g(e_\alpha) = \alpha(\varepsilon(g))e_\alpha \\
g(e_\beta) = \beta(\varepsilon(g))e_\beta
\end{cases}$$

\[\text{8i.e., forgetting the monodromy operator } N\]

\[\text{9We use the notation } D_{\text{cris}}(\psi), \text{ as in [3, even though } \psi \text{ may not be crystalline, cf. [3 after 2.4.2].}\]
and

$$\text{Fil}^i(E_n \otimes_E D(\alpha, \beta)) = \begin{cases} 
E_n \otimes_E D(\alpha, \beta) & \text{if } i \leq -(k-1) \\
E_n \cdot (e_\alpha + G(\beta \alpha^{-1}) \cdot e_\beta) & \text{if } -(k-2) \leq i \leq 0 \\
0 & \text{if } i \geq 1.
\end{cases}$$

Here, $E_n = E \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n})$, $\varepsilon : \mathcal{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the cyclotomic character, and $G(\beta \alpha^{-1})$ is the Gauss sum [3, sec. 1.2].

If $\alpha = \beta$, then:

$$\begin{align*}
\varphi(e_\alpha) &= \alpha(p)e_\alpha \\
\varphi(e_\beta) &= \alpha(p)(e_\beta - e_\alpha)
\end{align*}$$

and if $g \in \Gamma$, then:

$$\begin{align*}
g(e_\alpha) &= \alpha(\varepsilon(g))e_\alpha \\
g(e_\beta) &= \alpha(\varepsilon(g))e_\beta
\end{align*}$$

and

$$\text{Fil}^i(E_n \otimes_E D(\alpha, \beta)) = \begin{cases} 
E_n \otimes_E D(\alpha, \beta) & \text{if } i \leq -(k-1) \\
E_n \cdot e_\beta & \text{if } -(k-2) \leq i \leq 0 \\
0 & \text{if } i \geq 1.
\end{cases}$$

The functor $D_{\text{cris}}$ is an equivalence between the category of $E$-linear potentially crystalline representations of $\mathcal{G}_{\mathbb{Q}_p}$ and the category of admissible filtered $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-modules. We give the following result of Colmez as stated in [3, 2.4.5]:

**Proposition 4.3.3.** If $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ is an absolutely irreducible crystabelline representation with Hodge-Tate weights 0 and $k-1$, where $k \geq 2$, then there exist characters $\alpha$ and $\beta$ as above such that $D_{\text{cris}}(\psi) = D(\alpha, \beta)$.

Conversely, if $\alpha$ and $\beta$ are such characters, then there exists an absolutely irreducible crystabelline representation $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ such that $D_{\text{cris}}(\psi) = D(\alpha, \beta)$.

**Lemma 4.3.4.** Suppose $D(\alpha, \beta)$ is equivalent to $\vartheta \otimes D(\alpha, \beta)$, for some character $\vartheta$ of $\mathbb{Q}_p^\times$. Then $\beta = \vartheta \alpha$ and $\vartheta^2 = 1$. Conversely, if $\beta = \vartheta \alpha$ with a non-trivial quadratic character $\vartheta$, then $D(\alpha, \beta)$ is equivalent to $\vartheta \otimes D(\alpha, \beta)$. The set of equivalences $D(\alpha, \beta) \cong \vartheta \otimes D(\alpha, \beta)$, if non-empty, is a torsor under $E^\times$. 
Proof. Suppose \( D(\alpha, \beta) \cong \vartheta \otimes D(\alpha, \beta) \), and the equivalence is given with respect to the basis \((e_\alpha, e_\beta)\) by \(y \in GL_2(E)\). First, we consider the case \( \alpha \neq \beta \). Then, for all \( t \in \mathbb{Q}_p \),

\[
y \left( \begin{array}{cc} \alpha(t) & 0 \\ 0 & \beta(t) \end{array} \right) y^{-1} = \left( \begin{array}{cc} \vartheta(t)\alpha(t) & 0 \\ 0 & \vartheta(t)\beta(t) \end{array} \right)
\]

and, since \( y \) respects filtration,

\[
y \left( \begin{array}{c} 1 \\ G(\beta\alpha^{-1}) \end{array} \right) = c \left( \begin{array}{c} 1 \\ G(\beta\alpha^{-1}) \end{array} \right)
\]

for some \( c \in E^\times \). Because \( \alpha \neq \beta \), (4.3.5) implies that \( y \) must be either a diagonal matrix or an anti-diagonal matrix (i.e., the entries on the diagonal vanish). If \( y \) is a diagonal matrix, then equation (4.3.5) gives \( \vartheta = 1 \) and equation (4.3.6) implies that \( y \) is a scalar matrix.

Now suppose that \( y \) is an anti-diagonal matrix. Equation (4.3.5) becomes

\[
y \left( \begin{array}{cc} \alpha(t) & 0 \\ 0 & \beta(t) \end{array} \right) y^{-1} = \left( \begin{array}{cc} \beta(t) & 0 \\ 0 & \alpha(t) \end{array} \right) = \left( \begin{array}{cc} \vartheta(t)\alpha(t) & 0 \\ 0 & \vartheta(t)\beta(t) \end{array} \right)
\]

It follows \( \beta = \vartheta\alpha, \alpha = \vartheta\beta \), and hence \( \vartheta^2 = 1 \). Finally, equation (4.3.6) implies that \( y \) is a scalar multiple of the matrix \( y_0 = \left( \begin{array}{cc} 0 & G(\beta\alpha^{-1})^{-1} \\ G(\beta\alpha^{-1}) & 0 \end{array} \right) \). Conversely, if \( \beta = \vartheta\alpha \) with a non-trivial quadratic character \( \vartheta \), then the matrix \( y_0 \) defines an equivalence \( D(\alpha, \beta) \cong \vartheta \otimes D(\alpha, \beta) \).

If \( \alpha = \beta \), then

\[
y \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) y^{-1} = \vartheta(p) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \quad \text{and} \quad y \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = c \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\]

imply that \( y \) is a scalar matrix, and hence \( \vartheta = 1 \). The last statement follows from our observation that \( y \) is a scalar matrix or unique up to a scalar matrix. \( \square \)

4.3.7. Filtered modules in the semistable case. Now assume that \( \psi \) is as in 4.3.1 but not crystalline. As we recalled in 4.2.3 \( \psi \) is then a twist of a semistable non-crystalline representation by a character of finite order. Without loss of generality we assume in the
following that $\psi$ is semistable. By [5, 1.3.5], the filtered $(\phi, N)$-module $D(k, \mathcal{L})$ of $\psi$ is given as follows:

\[
\begin{align*}
N(e_1) &= e_2, \\
N(e_2) &= 0,
\end{align*}
\]

\[
\begin{align*}
\varphi(e_1) &= p^{-k/2}e_1, \\
\varphi(e_2) &= p^{-k/2+1}e_2,
\end{align*}
\]

and

\[
\text{Fil}^i D(k, \mathcal{L}) = \begin{cases} 
D(k, \mathcal{L}) & \text{if } i \leq -(k - 1), \\
E(e_1 + \mathcal{L} \cdot e_2) & \text{if } -(k - 2) \leq i \leq 0, \\
0 & \text{if } i \geq 1.
\end{cases}
\]

Lemma 4.3.8. If $D(k, \mathcal{L})$ is equivalent to $\vartheta \otimes D(k, \mathcal{L})$ for some character $\vartheta$ of $\mathbb{Q}_p^\times$, then $\vartheta = 1$. In that case the group of auto-equivalences of $D(k, \mathcal{L}) \otimes_E \overline{E}$ is $\overline{E}^\times$.

Proof. Suppose the equivalence is given with respect to the basis $(e_1, e_2)$ by $y \in GL_2(E)$. By writing out what it means that $y$ is an equivalence of filtered $(\phi, N)$-modules, we find that $y$ is a scalar matrix and $\vartheta = 1$, cf. the proof of 4.3.4.

4.3.9. Proof of Proposition 4.3.1 (i) We consider $\overline{y} \in S_{\overline{y}} \subset PGL_2(\overline{E})$. Let $y$ be an element of $GL_2(E)$ such that $\text{pr}(y) = \overline{y}$. Then for any $\sigma \in \mathcal{G}_{\mathbb{Q}_p}$ we have

\[
y\psi(\sigma)y^{-1} = y_{\vartheta}(\sigma)\psi(\sigma)
\]

for some $\vartheta_{\vartheta}(\sigma) \in \overline{E}^\times$. This gives us a character $\vartheta_{\vartheta} : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \overline{E}^\times$, and we see that $\psi$ and $\vartheta \otimes \psi$ are equivalent. We distinguish two cases.

(1) Suppose first that for all nontrivial characters $\vartheta$ one has $\psi \not\cong \vartheta \otimes \psi$. Then $\vartheta_{\vartheta} = 1$ and $y$ commutes with all $\psi(\sigma)$. Because $\psi$ is assumed to be absolutely irreducible, $y$ is therefore a scalar. Hence $\overline{y} = 1$ and $S_{\overline{y}}$ is trivial.

(2) Suppose there is a non-trivial character $\vartheta$ of $\mathbb{Q}_p^\times$ such that $\psi \cong \vartheta \otimes \psi$. Then the filtered $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-modules $D_{\text{pst}}(\psi)$ and $D_{\text{pst}}(\vartheta \otimes \psi)$ are equivalent. By 4.3.4, 4.3.8, and 4.3.3, this means that $\vartheta^2 = 1$, $\psi$ is crystabelline, $D_{\text{cris}}(\psi) = D(\alpha, \beta)$, $\beta = \vartheta \alpha$, and there exists an intertwiner $D(\alpha, \beta) \xrightarrow{\cong} \vartheta \otimes D(\alpha, \beta)$ whose matrix w.r.t. to the basis $(e_\alpha, e_\beta)$ is the dual.
matrix \( y_0 \) appearing in the proof of 4.3.4. Because any two such intertwiners are the same up to a scalar multiple, the centralizer group \( S_{\psi} \) has two elements, the nontrivial given by the image of \( y_0 \) in \( PGL_2(E) \).

(ii) Consider \( \overline{y} \in S_{\phi} \subset PGL_2(E) \). As above we have then \( y\phi(\sigma)y^{-1} = \vartheta_y(\sigma)\phi(\sigma) \) for all \( \sigma \) in the Weil group \( W_{Q_p} \). Again we distinguish the two cases:

1. Suppose first that for all nontrivial characters \( \vartheta \) one has \( \psi \not\cong \vartheta \otimes \psi \). Then one of the following cases occurs:
   - \( \psi \) is crystabelline with \( \text{D}_{\text{cris}}(\psi) = D(\alpha, \beta) \) with \( \alpha \neq \beta \) and \( \beta \alpha^{-1} \) not a quadratic character. Equation 4.3.5 with \( \vartheta = 1 \), then shows that \( y \) must be a diagonal matrix, and we find \( S_{\phi} \cong (E^\times \times E^\times)/E^\times \cong E^\times \), and \( S_{\phi}/S_{\phi}^\circ \) is hence trivial.
   - \( \psi \) is crystabelline with \( \text{D}_{\text{cris}}(\psi) = D(\alpha, \alpha) \), or \( \psi \) is semistable (up to twist by a character of finite order, cf. 4.2.3). In both cases the projective \( F \)-semisimple Weil group representation is trivial, and \( S_{\phi} = PGL_2(E) \), i.e., \( S_{\phi}/S_{\phi}^\circ \) is trivial.

2. Now suppose that \( \psi \cong \vartheta \otimes \psi \) with a non-trivial character \( \vartheta \). Then \( \psi \) is crystabelline with \( \text{D}_{\text{cris}}(\psi) = D(\alpha, \beta) \), with \( \beta = \vartheta \alpha \) and a non-trivial quadratic character \( \vartheta \). In that case the Weil group representation is diagonal and the matrix \( y \) can be diagonal as well as anti-diagonal, cf. 4.3.5. Hence \( S_{\phi}/S_{\phi}^\circ \) has two elements. \( \square \)

**Proposition 4.3.10.** Let \( \psi \) be as in 4.3.1. Then the number of irreducible constituents of \( \Pi(\psi)|_H \) is equal to the cardinality of the centralizer \( S_{\phi} \) of the corresponding projective Galois representation (which is either one or two).

**Proof.** By 4.2.5 \( \Pi(\psi)|_H \) is decomposable if and only if \( \pi(\psi)|_H \) is decomposable. By 3.1.1 and 3.1.2 this happens if and only if \( \pi(\psi) \) is an irreducible \( G \)-representation whose restriction splits into two irreducible (non-isomorphic) components. By the local Langlands correspondence for smooth representations for \( SL_2(Q_p) \), this is equivalent to \( |S_{\phi}/S_{\phi}^\circ| = 2 \) (with the notation of 4.3.1). By 4.3.1 this is equivalent to \( |S_{\phi}| = 2 \). \( \square \)
5. Crystalline representations and the universal Banach space representation

In this section we go back to $G = GL_n(F)$ and $H = SL_n(F)$, and we want to consider what the theory of Breuil and Schneider in [6] may predict about restrictions to $H$ of unitary Banach space representations of $G$. To this end, we recall the setting considered in [6].

Let $G$ be an $F$-split connected reductive algebraic group defined over $F$ and $G = G(F)$. Let $\mathbf{T}$ be a maximal $F$-split torus in $G$ and $\mathbf{T} = (\text{Res}_{F/\mathbb{Q}_p} \mathbf{T})_E$. Let $(\rho, V_\rho)$ be a $\mathbb{Q}_p$-rational representation of $G$ on a finite dimensional $E$-vector space $V_\rho$, and let $\tilde{\xi} \in X^* (\mathbf{T})$ be the highest weight of $\rho$. Fix a good maximal compact subgroup $K \subset G$, and set $\rho_K = \rho|_K$. The corresponding Satake-Hecke algebra $\mathcal{H}(G, \rho_K)$ is the convolution algebra over $E$ of all compactly supported functions $\psi : G \to \text{End}_E(V_\rho)$ satisfying

$$\psi(k_1 g k_2) = \rho(k_1) \circ \psi(g) \circ \rho(k_2)$$

for any $k_1, k_2 \in K, g \in G$. The choice of a $K$-invariant norm on $V_\rho$ leads to a $G$-invariant norm on $\mathcal{H}(G, \rho_K)$, as explained in [6, sec. 2]. Denote by $\mathcal{B}(G, \rho_K)$ the completion of $\mathcal{H}(G, \rho_K)$ with respect to this norm. Similarly, the norm on $V_\rho$ gives rise to the supremum norm on the compact induction $c\text{-ind}_K^G(\rho_K)$, and we denote by $B_K^G(\rho_K)$ the completion of this $G$-representation. Let $\mathbf{T}^\vee$ be the $E$-torus dual to $\mathbf{T}$ and $\zeta \in \mathbf{T}^\vee (E)$. Denote by $\omega_\zeta$ the $E$-valued character of $\mathcal{H}(G, \rho_K)$ corresponding to $\zeta$ by the normalized Satake isomorphism [6 after 2.5], and by $E_\zeta$ the corresponding one dimensional $\mathcal{H}(G, \rho_K)$-module. Define

$$H_{\xi,\zeta}(G) = E_\zeta \otimes \mathcal{H}(G, \rho_K) \ c\text{-ind}_K^G(\rho_K).$$

Denote by $\mathbf{T}_{\xi,\text{norm}}^\vee$ the affinoid subdomain in (the rigid analytic space associated to) $\mathbf{T}^\vee$, as defined in [6 before 2.6]. For $\zeta \in \mathbf{T}_{\xi,\text{norm}}^\vee$ set

$$B_{\xi,\zeta}(G) = E_\zeta \hat{\otimes} \mathcal{B}(G, \rho_K) B_K^G(\rho_K).$$

**Conjecture 5.1.1.** [6, 6.1] The Banach space $B_{\xi,\zeta}(G)$ is non-zero for all $\zeta \in \mathbf{T}_{\xi,\text{norm}}^\vee$.

For $G = GL_n(F)$, conjecture 5.1.1 is known to be true in many cases by [7, sec. 1.2]. Here we study the spaces $H_{\xi,\zeta}$ and $B_{\xi,\zeta}$ for the groups $G = GL_n(F)$ and $H = SL_n(F)$. Roughly speaking, a non-zero (admissible) quotient of $B_{\xi,\zeta}(G)$ should correspond to a crystalline
Let \( \psi \) whose Hodge-Tate weights are given by \( \xi \) and the Frobenius is determined by \( \zeta \). Then we can consider the corresponding projective Galois representation \( \overline{\psi} \), which should conjecturally correspond to some (admissible) quotient of \( B_{\overline{\xi\zeta}}(H) \), and the latter Banach space is therefore non-zero. However, the implication \( B_{\xi\zeta}(G) \neq 0 \Rightarrow B_{\overline{\xi\zeta}}(H) \neq 0 \) can be shown without assuming the existence of \( p \)-adic Langlands correspondences.

**Proposition 5.1.2.** Let \( G = \text{GL}_n/F \), \( H = \text{SL}_n/F \), \( G = G(F) \), \( H = H(F) \), \( T \subset G \) the diagonal torus, \( T_H = T \cap H \), \( \xi \in X^*(T) \), and \( \zeta \in T^\vee(E) \). Let \( \overline{\xi} \in X^*(\overline{T}) \) be the restriction of \( \xi \) and \( \overline{\zeta} \in (T_H)^\vee(E) \) be induced by \( \zeta \). Assume \( \zeta \in T_{\xi,\text{norm}}^\vee \). Define the representations \( B_{\xi\zeta}(G) \) and \( B_{\overline{\xi\zeta}}(H) \) with respect to the maximal compact subgroups \( K \subset G \) and \( K_H = K \cap H \), as above.

Then \( \overline{\zeta} \in (T_H)^\vee_{\xi,\text{norm}} \), and if \( B_{\xi\zeta}(G) \) is non-zero, \( B_{\overline{\xi\zeta}}(H) \) is non-zero too.

**Proof.** We will first consider the situation when \( \xi = 1 \). Then \( \mathcal{H}(G, K) := \mathcal{H}(G, 1_K) \) is the Hecke algebra of compactly supported \( K \)-biinvariant functions \( \mu : G \to E \), with convolution product. Let \( \Lambda_G = T/(T \cap K) \) and \( \Lambda_H = T_H/(T_H \cap K) \), where \( T = T(F) \) and \( T_H = T_H(F) \).

Because the Weyl groups \( W(G, T) \) and \( W(H, T_H) \) are canonically isomorphic, we can and will identify them and just write \( W \) for it. Using the embedding \( \Lambda_H \to \Lambda_G \) and the Satake isomorphism

\[
S : \mathcal{H}(G, K) \to E[\Lambda_G]^W,
\]

cf. [6] after 2.2], we obtain an embedding \( \mathcal{H}(H, H \cap K) \to \mathcal{H}(G, K) \).

The compactly induced space \( \text{c-ind}^{G}_{K}(1_K) \) is equal to \( C_c(G/K, E) \). The spaces \( \mathcal{H}(G, K) \) and \( C_c(G/K, E) \) are equipped with the supremum norm. The \( G \)-action on \( C_c(G/K, E) \) is norm-preserving, and hence continuous. Note that \( C_c(G/K, E) \) is generated as an \( G \)-module by the characteristic function \( c_K \) on \( K \).

Given a function \( f \in C_c(H/(H \cap K), E) \), we can extend it to a function \( f_G \) of \( G \) by

\[
f_G(g) = \begin{cases} f(h), & \text{if } g = hK \\ 0, & \text{otherwise.} \end{cases}
\]

The map \( \iota : C_c(H/(H \cap K), E) \to C_c(G/K, E) \) given by \( \iota(f) = f_G \) is continuous and \( H \)-equivariant, and \( \iota(c_{H \cap K}) = c_K \). Denote by \( \gamma_1 \) the corresponding continuous map \( \gamma_1 : \)
$H_{1,\zeta}(H) \to H_{1,\zeta}(G)$. Then $\gamma_1$ must be non-zero, because $H_{1,\zeta}(G)$ is generated as an $G$-module by the image of $c_K$. We remark that [4, 2.4] tells us that the space of $K$-invariant vectors in $H_{1,\zeta}(G)$ is one dimensional.

As explained in [26, sec. 5], there is a $G$-equivariant isomorphism between $H_{\xi,\zeta}(G)$ and $\hom{G}{H_{\xi,\zeta}(H)}{H_{\xi,\zeta}(G)}$ tensoring $\gamma_1$ with the identity map on $V_\rho$, we obtain an $H$-homomorphism $\gamma : H_{\xi,\zeta}(H) \to H_{\xi,\zeta}(G)$.

The Banach space $B_{\xi,\zeta}(G)$ is the Hausdorff completion of $H_{\xi,\zeta}(G)$. Assume $B_{\xi,\zeta}(G) \neq 0$. We have the following commutative diagram:

\[
\begin{array}{ccc}
\hom{H_{\xi,\zeta}(H)}{H_{\xi,\zeta}(G)} & \gamma & \hom{B_{\xi,\zeta}(H)}{B_{\xi,\zeta}(G)} \\
\alpha \downarrow & \downarrow \beta & \\
H_{\xi,\zeta}(H) & \hom{\hat{\gamma}} & B_{\xi,\zeta}(G).
\end{array}
\]

The existence of $\hat{\gamma}$ follows from the universal property of the Hausdorff completion $B_{\xi,\zeta}(H)$ of $H_{\xi,\zeta}(H)$ [28, 7.5].

We claim that $\beta \circ \gamma \neq 0$. To prove the claim, we first consider $\gamma_1 : H_{1,\zeta}(H) \to H_{1,\zeta}(G)$. Since the $H$-representation $H_{1,\zeta}(H)$ is generated by its $K$-fixed vectors, it follows that $\gamma_1(H_{1,\zeta}(H))$ contains the one-dimensional subspace of $K$-fixed vectors of $H_{1,\zeta}(G)$. Then $\beta \circ \gamma_1(H_{1,\zeta}(H))$ must be non-zero, because $H_{1,\zeta}(G)$ is generated as a $G$-representation by $K$-fixed vectors. A similar reasoning implies $\beta \circ \gamma \neq 0$.

Finally, $\hat{\gamma} \circ \alpha(H_{\xi,\zeta}(H)) = \beta \circ \gamma(H_{\xi,\zeta}(H)) \neq 0$ implies $B_{\xi,\zeta}(H) \neq 0$. □

Let us go back to the case when $G = \text{GL}_2(\mathbb{Q}_p)$ and $H = \text{SL}_2(\mathbb{Q}_p)$. Let $\psi : \mathcal{G}_{\mathbb{Q}_p} \to \text{GL}_2(E)$ be an absolutely irreducible crystalline representation with distinct Hodge-Tate weights. By [26, sec. 5, ex. 1], we have $\Pi(\psi) = B_{\xi,\zeta}(G)$ and $\Pi(\psi)_{\text{alg}} = H_{\xi,\zeta}(G)$, where $\xi$ and $\zeta$ are determined by the Hodge-Tate weights and Frobenius eigenvalues. The representation $B_{\xi,\zeta}(G)$ is irreducible. If $B_{\xi,\zeta}(G)|_H$ is also irreducible, then $B_{\xi,\zeta}(G)|_H \cong B_{\xi,\zeta}(H)$.

Assume $B_{\xi,\zeta}(G)|_H$ is reducible. Then $B_{\xi,\zeta}(G)|_H \neq B_{\xi,\zeta}(H)$. The representation $\hat{\gamma}(B_{\xi,\zeta}(H))$ is the irreducible component of $B_{\xi,\zeta}(G)|_H$ containing the $K$-fixed vectors (tensored with $V_\rho$). The other component can be obtained using a maximal compact subgroup $K' \subset G$ such that $K \cap H$ and $K' \cap H$ are not conjugate in $H$. 
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