ON THE SIZE OF THE LARGEST EMPTY BOX AMIDST A POINT SET

CHRISTOPH AISTLEITNER, AICKE HINRICHS, AND DANIEL RUDOLF

Abstract. The problem of finding the largest empty axis-parallel box amidst a point configuration is a classical problem in computational geometry. It is known that the volume of the largest empty box is of asymptotic order $1/n$ for $n \to \infty$ and fixed dimension $d$. However, it is natural to assume that the volume of the largest empty box increases as $d$ gets larger. In the present paper we prove that this actually is the case: for every set of $n$ points in $[0,1]^d$ there exists an empty box of volume at least $c_d n^{-1}$, where $c_d \to \infty$ as $d \to \infty$. More precisely, $c_d$ is at least of order roughly $\log d$.

1. Introduction

The problem of finding the largest empty axis-parallel rectangle amidst a point configuration in the unit square is a standard problem in computational geometry and computational complexity theory. Here the emphasis is on the word finding; that is, researchers are actually interested in the complexity of algorithms whose output is the largest empty rectangle. The problem has probably been introduced by Naamad, Lee and Hsu [8], and generalizes in a natural way to the multi-dimensional case, where one has to find the largest empty axis-parallel box amidst a point configuration in the $d$-dimensional unit cube. Given the prominence of the problem, it is quite surprising that very little is known on the size (or, more precisely, the area resp. $d$-dimensional volume) of the largest empty box. There are actually two problems, a “lower bound problem” and an “upper bound problem”: one asking for the minimal size of the largest empty box for any point configuration (a kind of “irregularities of distributions” problem), and one asking for the maximal size of the largest empty box for an optimal point configuration.

Our attention was drawn to this problem by the fact that questions asking for the size of the largest empty box have recently appeared in approximation theory in problems concerning the approximation of high-dimensional rank one tensors; see [1] and [10]. However, we believe that the problem is very natural and interesting in its own right. Our main interest was in the multi-dimensional case, and the main question was: If the dimension $d$ of the problem is increased, does the minimal size of the largest empty box (taking the minimum over all configurations of $n$ points) necessarily increase as $d$ increases? As it turns out, the answer is affirmative. However, we are still far from a complete quantitative solution of the problem.

The first author is supported by Schrödinger scholarship J-3311 and by projects I1751-N26 and F5507-N26 of the Austrian Science Fund (FWF). The third author was supported by the DFG Priority Program 1324 and the DFG Research Training Group 1523.
To state our results, let us fix the notation. Let \( d \geq 2 \) and let \([0,1]^d\) be the \( d \)-dimensional unit cube. For \( x, y \in [0,1]^d \) with \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) we write \( x \leq y \) if this inequality holds coordinate-wise. For \( x \leq y \) we write \([x,y)\) for the axis-parallel box \( \Pi_{i=1}^d[x_i,y_i)\), and define
\[
\mathcal{B} = \{ [x,y) : x, y \in [0,1]^d, \ x \leq y \}.
\]
For \( n \geq 1 \) let \( T \) be a set of points in \([0,1]^d\) of cardinality \(|T| = n\). The volume of the largest empty axis-parallel box, which we call the dispersion of \( T \), is then given by
\[
\text{disp}(T) = \sup_{B \in \mathcal{B}, B \cap T = \emptyset} \lambda(B),
\]
where \( \lambda(B) \) denotes the volume of \( B \). We are in particular interested in the minimal dispersion of point sets; thus we set
\[
\text{disp}^\ast(n,d) = \inf_{T \subset [0,1]^d, |T| = n} \text{disp}(T).
\]
It turns out that this minimal dispersion is of asymptotic order \( 1/n \) as a function of \( n \). Consequently, to capture the dependence of this quantity on the dimension \( d \) we define the number
\[
(1) \quad c_d = \liminf_{n \to \infty} n \ \text{disp}^\ast(n,d).
\]
Moreover, we define the inverse function of the minimal dispersion
\[
(2) \quad N(\varepsilon,d) = \min \{ n : \text{disp}^\ast(n,d) \leq \varepsilon \}, \quad \text{for } \varepsilon \in (0,1).
\]
In particular, this number is of interest in the applications in approximation theory mentioned above.

For \( \text{disp}^\ast(n,d) \) one trivially has the lower bound
\[
\text{disp}^\ast(n,d) \geq \frac{1}{n+1},
\]
which follows directly from splitting the unit cube into \( n + 1 \) parts and using the pigeon hole principle. On the other hand, the best published upper bound is
\[
(3) \quad \text{disp}^\ast(n,d) \leq \frac{1}{n} \left( 2^{d-1} \prod_{i=1}^{d-1} p_i \right),
\]
where \( p_i \) denotes the \( i \)-th prime. This upper bound is essentially due to Rote and Tichy [12], with a detailed proof given in [5]. Note that by the prime number theorem the product on the right-hand side of (4) grows super-exponentially in \( d \). After reading a draft version of our manuscript, Gerhard Larcher communicated to us a proof of the upper bound
\[
(4) \quad \text{disp}^\ast(n,d) \leq \frac{2^{7d+1}}{n}.
\]
With his permission, we include the argument here in the last section. This bound is better than (3) for \( d \geq 54 \).
The only non-trivial lower bound known to date is
\[ \text{disp}^*(n, d) \geq \frac{5}{4(n + 5)}, \]
due to Dumitrescu and Jiang [5]. Thus for the constant \( c_d \) from (1) we have
\[ c_d \in \left[ \frac{5}{4}, 2^{7d+1} \right], \quad d \geq 2. \]

It is a natural question to ask whether \( c_d \to \infty \) as \( d \to \infty \); that is, whether there is some kind of “irregularity of distributions”-type behavior in high dimensions.\(^1\) As we will prove below, the answer is affirmative.

Before we state our main results, we note that a lower bound of the minimal dispersion leads also to a lower bound of the minimal extremal discrepancy. Moreover, the point sets leading to the upper bounds (3) and (4) are well-known low discrepancy sequences. This provides a link of the problem investigated in this paper to the theory of uniform distribution modulo one and discrepancy theory, and to the theory of irregularities of distributions. For the first two subjects see for example [4, 7], for the latter see [2]. The theory of digital nets and low-discrepancy sequences, which is used to prove (4), is explained in detail in [3]. Finally, the theory of information-based complexity, where quantities such as the one in (2) are investigated, is described in [11].

The topic addressed in this paper has been recently taken up by Ullrich [13], who considered the dispersion on the \( d \)-dimensional unit torus rather than on the unit cube, proved lower bounds for this notion of dispersion and obtained results which have an interesting resemblance of known inverse of the discrepancy-type results.

2. Main result

In the statement of the following theorem, and in the sequel, \( \log_2 \) denotes the dyadic logarithm.

**Theorem 1.** For all positive integers \( d \) and \( n \) we have
\[ \text{disp}^*(n, d) \geq \frac{\log_2 d}{4(n + \log_2 d)}. \]

We can state this result also in terms of the inverse of the minimal dispersion.

**Corollary 1.** For \( \varepsilon \in (0, 1/4) \) and \( d \geq 1 \) we have
\[ N(\varepsilon, d) \geq (4\varepsilon)^{-1}(1 - 4\varepsilon) \log_2 d. \]

For the constant \( c_d \) we can directly deduce the following corollary from Theorem 1.

\(^1\)As we learned from one referee, this problem was also stated as *Open question 6* in the *Computational Geometry Column* of Dumitrescu and Jiang; see [6].
Corollary 2. We have

\[ c_d \geq \frac{\log_2 d}{4} \quad \text{for } d \geq 1. \]

In particular \( \lim_{d \to \infty} c_d = \infty. \)

Note that by Corollary 2 the constant \( c_d \) grows at least logarithmically as \( d \) increases; on the other hand, the upper bound for \( c_d \) in (5) grows super-exponentially. So there remains a large gap for the “correct” asymptotic order of \( c_d. \)

3. Proof and auxiliary lemmas

The result is trivial for \( d = 1 \). Thus we will always assume in the sequel that \( d \geq 2. \)

Lemma 1. Let positive integers \( \ell, n \) be given. Then

\[ \text{disp}^*(n, d) \geq \frac{(\ell + 1)\text{disp}^*(\ell, d)}{n + \ell + 1}. \]

Proof. The proof of the lemma uses an idea similar to the one in the proof of [5, Theorem 1]. First we note that there is an integer \( k \geq 0 \) such that \( n = (\ell + 1)k + r \) for \( r \in \{0, 1, \ldots, \ell\}. \) We will use the following two facts:

- For any decomposition of \([0, 1]^d\) into \( k + 1 \) boxes of equal volume, there is one box which contains at most \( \ell \) points.
- For an arbitrary box \( A \in \mathcal{B}, \) writing
  \[ \text{disp}^*(A, \ell, d) := \inf_{T \subseteq A, \ |T| = \ell} \sup_{B \subseteq \mathcal{B}, \ B \subseteq A, \ B \cap T = \emptyset} \lambda(B) \]
  we always have
  \[ \text{disp}^*(A, \ell, d) \geq \lambda(A)\text{disp}^*(\ell, d). \]

By the application of these facts we obtain

\[ \text{disp}^*(n, d) \geq \frac{\text{disp}^*(\ell, d)}{k + 1}, \]

which by \( n \geq (\ell + 1)k \) proves the assertion.

The next result tells us that if \( d \) is sufficiently large, then for a small number of points in \([0, 1]^d\) we can find a strong structure in the coordinates of these points.

Lemma 2. Let \( \ell \) be an positive integer and assume that \( d \geq 2^\ell - 1. \) Then

\[ \text{disp}^*(\ell, d) \geq 1/4. \]

Proof. Let \( T = \{t_1, \ldots, t_\ell\} \subset [0, 1]^d \) be an arbitrary point set. To denote the coordinates of a point \( t \in [0, 1]^d \) we use the notation \( t = (t(1), \ldots, t(d)). \) Set

\[ \chi(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq 1/2 \\
1 & \text{if } 1/2 < x \leq 1.
\end{cases} \]
Furthermore, for \( m \in \{1, \ldots, \ell\} \), define
\[
\tau_m = (\chi(t_m(1)), \chi(t_m(2)), \ldots, \chi(t_m(d))).
\]
If there exists an index \( i \in \{1, \ldots, d\} \) such that
\[
(7) \quad (\tau_1(i), \ldots, \tau_\ell(i)) = (0, \ldots, 0) \quad \text{or} \quad (\tau_1(i), \ldots, \tau_\ell(i)) = (1, \ldots, 1),
\]
then all elements of \( T \) are contained in one half of \([0, 1]^d\), and we have \( \text{disp}(T) \geq 1/2 \). On the other hand, if (7) fails then by \( d \geq 2\ell - 1 \) and by the pigeon hole principle there must exist two distinct indices \( i, j \in \{1, \ldots, d\} \) such that
\[
(\tau_1(i), \ldots, \tau_\ell(i)) = (\tau_1(j), \ldots, \tau_\ell(j)).
\]
We assume without loss of generality that \( i = 1 \) and \( j = 2 \). Then we have
\[
T \in \left( [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, 1)^{d-2} \right) \cup \left( \left[ \frac{1}{2}, 1 \right] \times \left[ \frac{1}{2}, 1 \right] \times [0, 1)^{d-2} \right).
\]
Let \( 0 < \varepsilon < \frac{1}{2} \), and let \( A \) denote the box
\[
A = \left[ 0, \frac{1}{2} - \varepsilon \right] \times \left[ \frac{1}{2} + \varepsilon, 1 \right] \times [0, 1)^{d-2}.
\]
Then
\[
T \cap A = \emptyset,
\]
and
\[
\lambda(A) = \left( \frac{1}{2} - \varepsilon \right)^2.
\]
Letting \( \varepsilon \to 0 \) we obtain the conclusion of the lemma. \( \square \)

Now we are able to prove Theorem 1.

**Proof of Theorem 1.** We set
\[
\ell = \lfloor \log_2 d \rfloor.
\]
Then we obviously have \( 2^\ell \leq d \), which allows us to apply Lemma 2. Thus by Lemma 1 and Lemma 2 we have
\[
\text{disp}^*(n, d) \geq \frac{(\ell + 1)\text{disp}^*(\ell, d)}{\ell + 1 + n}
\]
\[
\geq \frac{\ell + 1}{4(\ell + 1 + n)}
\]
\[
\geq \frac{\log_2 d}{4(n + \log_2 d)}.
\]
This proves the theorem. \( \square \)
4. Upper bound

We now present the argument for the upper bound (4) discussed in the introduction. This argument was shown to us by Gerhard Larcher, who generously allowed us to include it here.

A dyadic interval is an interval of the form \([a2^{-k}, (a + 1)2^{-k})\) for non-negative integers \(a, k\) with \(0 \leq a < 2^k\). A \((d\text{-dimensional})\) dyadic box is the Cartesian product of \(d\) dyadic intervals. Let \(t\) be a non-negative integer and let \(m\) be a positive integer with \(t \leq m\). A \((t, m, d)\)-net (in base 2) is a set \(T\) of \(2^m\) points in \([0, 1]^d\) such that each dyadic box of volume \(2^{d-m}\) contains exactly \(2^t\) points of \(T\). Since any interval \([x, y) \subset [0, 1)\) contains a dyadic interval of length at least \(4^{-1}(y - x)\), any box \(B \in \mathcal{B}\) contains a dyadic box of volume at least \(4^{-d}\lambda(B)\). It follows that, if \(T\) is a \((t, m, d)\)-net and \(4^{-d}\lambda(B) \geq 2^{t-m}\), then \(B\) contains a point of \(T\). Hence
\[
\text{disp}^*(2^m, d) \leq \text{disp}(T) \leq 2^{d-m+2d}.
\]
In [9], \((t, m, d)\)-nets with \(t \leq 5d\) are constructed for all dimensions \(d\) and \(m \geq t\). If now \(m\) is chosen such that \(2^m \leq n < 2^{m+1}\), we arrive at
\[
\text{disp}^*(n, d) \leq \text{disp}^*(2^m, d) \leq 2^{d-m} \leq \frac{2^{7d+1}}{n}.
\]
This shows (4) in the case \(n \geq 2^{5d}\). For \(n < 2^{5d}\) the same bound holds trivially.

References

[1] M. Bachmayr, W. Dahmen, R. DeVore, and L. Grasedyck. Approximation of high-dimensional rank one tensors. Constr. Approx., 39(2):385–395, 2014.
[2] J. Beck and W. W. L. Chen. Irregularities of distribution, volume 89 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1987.
[3] J. Dick and F. Pillichshammer. Digital nets and sequences. Cambridge University Press, Cambridge, 2010.
[4] M. Drmota and R. F. Tichy. Sequences, discrepancies and applications, volume 1651 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1997.
[5] A. Dumitrescu and M. Jiang. On the largest empty axis-parallel box amidst \(n\) points. Algorithmica, 66(2):225–248, 2013.
[6] A. Dumitrescu and M. Jiang. Computational geometry column 60. SIGACT News, 45(4):76–82, Dec. 2014.
[7] L. Kuipers and H. Niederreiter. Uniform distribution of sequences. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.
[8] A. Naamad, D. T. Lee, and W.-L. Hsu. On the maximum empty rectangle problem. Discrete Appl. Math., 8(3):267–277, 1984.
[9] H. Niederreiter and C. Xing. Low-discrepancy sequences and global function fields with many rational places. Finite Fields Appl., 2(3):241–273, 1996.
[10] E. Novak and D. Rudolf. Tractability of the approximation of high-dimensional rank one tensors. Constr. Approx., 43(1):1–13, 2016.
[11] E. Novak and H. Woźniakowski. Tractability of multivariate problems. Volume II: Standard information for functionals, volume 12 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2010.
[12] G. Rote and R. F. Tichy. Quasi-Monte Carlo methods and the dispersion of point sequences. Math. Comput. Modelling, 23(8-9):9–23, 1996.
[13] M. Ullrich. A lower bound for the dispersion on the torus. *Math. Comput. Simulation*, to appear. Available at http://arxiv.org/abs/1510.04617.

**Institute of Financial Mathematics and Applied Number Theory, University Linz**  
*E-mail address*: aistleitner@math.tugraz.at

**Institute of Analysis, University Linz**  
*E-mail address*: aicke.hinrichs@jku.at

**Institut für Mathematische Stochastik, Universität Göttingen**  
*E-mail address*: daniel.rudolf@uni-goettingen.de