Extension of the interior connection of a nonholonomic manifold with a Finsler metric

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Abstract

The notions of the interior and truncated connections of a nonholonomic manifold are introduced. A class of extended truncated connections is distinguished. For the case of a contact space with a Finsler metric, it is shown that there exists a unique extended truncated connection that satisfies additional properties. The curvature tensor of the obtained connection in the case of a sub-Riemannian space coincides with the Wagner curvature tensor that was constructed by Wagner for the case of an arbitrary nonholonomic manifold of codimension one endowed with an interior affine connection.

Introduction

A nonholonomic manifold is a smooth distribution on a smooth manifold. This distribution is in general not integrable. There are many works devoted to the theory of nonholonomic systems and nonholonomic geometry, among them there are a lot of works of prominent geometers of the first half of the twentieth century. The foundations of nonholonomic geometry were laid in the classical works on nonholonomic mechanics of physicists and mathematicians like Hertz, Hölder, Chaplygin, Appell, Korteweg and others. In the works of Gibbs and Caratheodory on the foundations of thermodynamics there appears for the first time the contact nonholonomic structure, which is the simplest nonholonomic structure. The nonholonomic theory is strongly related with the theory of distributions that in particular includes the theory
of Pfaffian systems. An important role here plays the works of É. Cartan, who developed the formalism of exterior differential forms and distributions. In the twenties of the last century, after the works of Levi-Civita and H. Weyl, where the Riemannian and affine connections were defined and the relation of geometry and mechanics was discovered, there appeared the understanding of what the nonholonomic mechanics should use for finding new geometric structures. The origin of such interplay was placed by Vranceanu and Synge. In USSR the nonholonomic theory was actively developed by V.F Kagan. In 1937 he proposed the following topic for the N.I. Lobachevskii prize: "to establish the foundations of the doctrine for the theory of nonholonomic spaces ... the applications to mechanics, physics and the theory of integration of Pfaffian equations are desirable". The most serious achievements on the nonholonomic geometry and its applications in mechanics belong to Vranceanu, Synge, Schouten and V.V. Wagner. The Romanian mathematician Vranceanu was the first who precisely formulated the notion of the nonholonomic structure on a Riemannian manifold as well as its relations to the dynamics of nonholonomic systems. Synge studied the problem of the stability of the inertial motion of nonholonomic mechanical systems and by this he anticipated the notion of the curvature of a nonholonomic manifold. Schouten defined the parallel transport of some vectors along some vector fields, afterwards this was called the truncated connection. He also defined the curvature tensor of this connection. The next step in the definition of the curvature tensor of a nonholonomic manifold belongs to V.V. Wagner, who constructed a curvature tensor expanding the Schouten tensor \cite{3}. The Wagner curvature tensor is zero if and only if the Schouten-Vranceanu connection is flat.

At present the increasing interest to the nonholonomic geometry is mostly related to the active usage of nonholonomic systems (sub-Riemannian spaces) in the control theory. The problems of the control theory frequently require the generalization of the Riemannian metric to the Finslerian one \cite{2}.

\section{Gradient coordinate system on a nonholonomic manifold}

In this paper $X$ denotes a connected $C^\infty$-manifold of dimension $n = 2m + 1$, $m \geq 2$. All objects on $X$ are assumed to be smooth.
A nonholonomic manifold $D$ is a smooth not involutive distribution of codimension one on $X$. Assume that there exists a one-dimensional distribution $D^\perp$ such that

\[ TX = D \oplus D^\perp. \]  

Following Wagner, we call $D^\perp$ a closing of the nonholonomic manifold $D$.

A vector field on $X$ is called admissible if it is tangent to the distribution $D$. A 1-form on $X$ is called admissible if it is zero on the closing $D^\perp$. At last, an admissible tensor field on $D$ is a linear combination of tensor products of the admissible vector fields and 1-forms. Denote by $f_{pq}^p(D)$ the module of admissible tensor fields of type $(p, q)$ on $D$.

A coordinate chart $K(x^\alpha)$ $(\alpha, \beta, \gamma = 1, \ldots, n; a, b, c = 1, \ldots, 2m)$ on the manifold $X$ is called adapted to the nonholonomic manifold $D$ if $\partial_n = \frac{\partial}{\partial x^n} \in f_{0}^0(D^\perp)$. It is not hard to check that any two adapted coordinate charts are related by a transformation of the form

\[ x^a = x^a(x'^a), \quad x^n = x^n(x'^a, x'^n). \]

Let $P : TX \to D$ be the projector defined by the decomposition (1) and let $K(x^a)$ be an adapted coordinate chart. Then the vector fields

\[ P(\partial_a) = \vec{e}_a = \partial_a - \Gamma_a^n \partial_n \]

are linearly independent and they generate the distribution $D$, i.e. $D = \text{span}(\vec{e}_a)$. Thus we have on the manifold $X$ the nonholonomic basis field $(\vec{e}_a, \vec{e}_n)$ and the corresponding cobasis field $(dx^a, \Theta^n = dx^n + \Gamma^n_a dx^a)$. The vector fields $\vec{e}_a$ defined in the nonholonomic manifold linear coordinates called by Wagner gradient coordinates.

The gradient coordinates satisfy the property that the transformation matrix in the formula $\vec{e}_a = A_a^a' \vec{e}_a'$ coincides with the matrix $\frac{\partial x'^a}{\partial x^a}$. This property implies in particular the following proposition.

**Proposition 1** If $t$ is an admissible tensor field of type $(p, q)$, then the object $\tilde{t}$ with the components $\partial_n t_{a_1 \cdots a_p}^{b_1 \cdots b_q}$ is an admissible tensor field of the same type.

**Proof.** Let $t$ be an admissible tensor field of type $(p, q)$, than in adapted coordinates it holds

\[ t = t_{b_1 \cdots b_q}^{a_1 \cdots a_p} \vec{e}_{a_1} \otimes \cdots \otimes \vec{e}_{a_p} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_q}. \]
The transformation (2) implies
\[ t^a_{b_1'...b_q'} = A^a_{a_1'} A^b_{b_1'} \cdots A^b_{b_q'} t^a_{b_1'...b_q}. \]

Equalities (2) show that the functions \( \frac{\partial x'}{\partial x_a} \) do not depend on the coordinate \( x^n \). This proves the proposition. \( \square \)

2 \quad \textbf{Interior and extended connections on a nonholonomic manifold}

Developing the geometry of nonholonomic manifolds, V.V. Wagner, introduces the notion of the interior geometry of a nonholonomic manifold \( D \) as the collection of the properties of the objects defined on \( D \) that depend only on \( D \) and its clothing \[3\). The parallel transport in a nonholonomic manifold is defined by a connection \( \nabla \) that in the terminology of V.V. Wagner is called interior. In some works in addition to the interior connections one considers also connections for which the parallel transport of vectors from \( D \) along arbitrary curves in \( X \) is defined. Such connections are called connections in the vector bundle \( D \) defined be the nonholonomic manifold. Defining in \[4\] the curvature of the interior connection \( \nabla \), V.V. Wagner constructs in a special way a connection in the vector bundle \( D \). The curvature tensor of this connection was called afterwards the Wagner curvature tensor. In this paper we interpret the Wagner curvature tensor as the tensor of the nonholonomicity of a smooth distribution and then, using this interpretation, we define an analog of the Wagner curvature tensor for nonholonomic manifolds with Finsler metrics \[1\].

Suppose that on \( X \) a contact structure \( \lambda \) is given, i.e. \( \lambda \) is a 1-form such that the rank of the 2-form \( \omega = d\lambda \) equals \( 2m \), and there exists a decomposition
\[ TX = D \oplus D^\perp, \]
where \( D = \ker \lambda \) and \( D^\perp = \ker \omega \). We call the nonholonomic manifold \( D \) a contact space.

We say that on the nonholonomic manifold \( D \) an interior connection is given, if the distribution \( \tilde{D} = \pi_*^{-1}(D) \), where \( \pi : D \to X \) is the natural projection, is decomposed into the direct sum
\[ \tilde{D} = HD \oplus VD, \]
(3)
where $VD$ is an vertical distribution on the total space $D$. Thus the interior connection is uniquely defined by the object $G^a_b(x^a, x^n)$ such that $HD = \text{span}(\vec{\varepsilon}_a)$, where
\[
\vec{\varepsilon}_a = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{a+b}.
\] (4)
The equality (4) implies the equalities of the form
\[
[\vec{\varepsilon}_a, \vec{\varepsilon}_b] = \omega_{ba} \partial_n + 2(\vec{e}_{[b}G^c_{a]} - G^d_{[a}G^c_{b]},d),
\] (5)
\[
[\vec{\varepsilon}_a, \partial_n] = \partial_n \Gamma^n_a \partial_n + (\partial_n G^b_a) \partial_{n+b}.
\] (6)
In (5) a dot denotes the derivative with respect to the coordinate $s$ on a fibre. Denote by $\mathcal{T}_{p,q}(D)$ the set of all admissible tensors of type $(p, q)$. By definition, an admissible Finslerian tensor field of type $(p, q)$ is a morphism $t : V \rightarrow \mathcal{T}_{p,q}(D)$ such that $t(z) \in \mathcal{T}_{p,q}(D \setminus \{0\})$ is the bundle of non-zero vectors from the distribution $D$.

The partial case, when the interior connection is given by a linear connection $(G^a_b(x^a, x^{n+a})) = \Gamma^a_{bc}(x^a)x^{n+a}$, the objects
\[
P^b_a = \partial_a G^b_a, \quad K^c_{ab} = 2(\vec{e}_{[b}G^c_{a]} - G^d_{[a}G^c_{b]},d)
\] were called by V.V. Wagner the first and the second Schouten curvature tensors, respectively. We use these terms for the object defined above.

Let $\vec{u}$ be the vector field defined by the conditions $\vec{u} \in f_1(D\perp)$ and $\lambda(\vec{u}) = 1$. If we assume that an adapted coordinate chart satisfies the additional condition $\partial_n = \vec{u}$, then the transformation (2) takes the following more simple form:
\[
x^a = x^a(x^a'), \quad x^n = x^{n'} + \text{const.}
\] (7)
In what follows we will consider only adapted coordinate charts related by the transformation (2). It is not hard to check the equality
\[
[\vec{e}_a, \vec{e}_b] = \omega_{ba} \partial_n.
\] (8)

The distribution $D$ may be considered as the total space of the vector bundle $\mu = (X, D, \pi)$, where $\pi : D \rightarrow X$ is the natural projection.

Any adapted coordinate chart $k(x^a)$ on the manifold $X$ defines the coordinate chart $\widehat{k}(x^a, x^{n+a})$ on the manifold $\widehat{D}$, where $x^{n+a}$ are the coordinates of the vector $\vec{v} \in D$ with respect to the basis $(\vec{e}_a)$. Thus $D$ is a smooth manifold of dimension $4m + 1$. 
A coordinate chart on the manifold $D$ defines the nonholonomic frame field $(\varepsilon_a, \partial_n, \partial_{n+a})$, where $\varepsilon_a$ are defined by the equality (4). Thus besides the distribution $HD$, which defines the exterior connection on the nonholonomic manifold $D$, we get the well defined distribution

$$\overline{HD} = HD \oplus \text{span}(\partial_n) \quad (9)$$
on the whole manifold $D$. This distribution defines an infinitesimal connection on $D$ as on the vector bundle. This means that we deal with a connection that is traditionally called a truncated connection. We call this connection an extension of the interior connection of the nonholonomic manifold. Any other extension of the interior connection is defined by a vector field that has the following coordinate form:

$$\tilde{u} = \partial_n - G_n^a \partial_{n+a},$$

where the object $G_n^a$ is an example of an admissible Finslerian vector field.

### 3 Contact space with a Finsler structure

Suppose now that on the manifold $D$ is defined a function $L(x^n, x^{n+a})$ that satisfies the following conditions:

1) $L$ is smooth at least on $D \setminus \{0\}$;

2) $L$ is homogeneous of degree 1 with respect to the coordinates of an admissible vector, i.e.

$$L(x^n, \lambda x^{n+a}) = \lambda L(x^n, x^{n+a}), \quad \lambda > 0; \quad (10)$$

3) $L(x^n, x^{n+a})$ is positive if not all $x^{n+a}$ are zero simultaneously;

4) the quadric form

$$L_{a\ b}\xi^a \xi^b = \frac{\partial^2 L^2}{\partial x^{n+a} \partial x^{n+b}}$$

is positive definite.

We call the triple $(X, D, F)$, where $F = L^2$, a contact sub-Finslerian manifold.

The following takes place.
**Theorem 1** On any contact sub-Finslerian manifold \( (X, D, F) \) there exists a unique truncated metric connection such that

\[
G^a_{b\cdot c} = G^a_{c\cdot b}.
\] (11)

**Proof.** Let \( \vec{v} \) be an admissible vector field such that

\[
\nabla_a v^b = \vec{e}_a v^b + G^b_a(x^\alpha, v^c) = 0,
\] (12)

\[
\nabla_n v^b = \partial_n v^b + G^b_n(x^\alpha, v^c) = 0.
\] (13)

We need to prove that the objects \( G^b_a \) and \( G^b_n \) are uniquely defined by the metrizability condition and by the condition (11). The metrizability condition implies that the function \( f \) is constant along any integrable curve of the vector field \( \vec{v} \), hence

\[
dF(x^\alpha, x^{n+a}) = \vec{e}_a F dx^a + \partial_n F \Theta^n = 0,
\]

where \( x^{n+a} = v^a \). This equality together with (12) and (13) implies the following:

\[
\vec{e}_a F - G^c_a F_c = 0,
\] (14)

\[
\vec{e}_n F - G^c_n F_c = 0.
\] (15)

Differentiating the equality (14) with respect to \( x^{n+b} \), we get

\[
\vec{e}_a F_b - G^c_{a\cdot b} F_c - G^c_a F_{c\cdot b} = 0.
\]

Contracting this with \( x^{n+a} \), we obtain

\[
x^{n+a} \vec{e}_a F_b - x^{n+a} G^c_{a\cdot b} F_c - x^{n+a} G^c_a F_{c\cdot b} = 0.
\] (16)

The homogeneity of the coefficients \( G^c_a \) and the condition (11) imply the equality

\[
x^{n+a} G^c_{a\cdot b} = G^c_b.
\]

Thus (16) takes the form

\[
x^{n+a} \vec{e}_a F_b - x^{n+a} G^c_a F_{c\cdot b} - G^c_b F_c = 0.
\] (17)

On the other hand, from (14) it follows that

\[
\vec{e}_a F = G^c_a F_c.
\]
Consequently, (17) can be rewritten in the form
\[ x^{n+a}G^c_aF_{c.b} = x^{n+a}\tilde{e}^a_bF - \tilde{e}_bF. \] (18)

Define the following admissible tensor field:
\[ g_{ab} = \frac{1}{2}F_{ab}. \]

Then (18) implies
\[ x^{n+a}g^c_a = \frac{1}{2}(g^{bc}(x^{n+a}\tilde{e}_aF - \tilde{e}_bF)). \] (19)

Differentiating this equality by \( x^{n+d} \), we get
\[ G^c_d + x^{n+a}G^c_{ad} = \frac{1}{2}(g^{bc}(x^{n+a}\tilde{e}_aF - \tilde{e}_bF))_d, \]
or
\[ G^c_d = \frac{1}{4}(g^{bc}(x^{n+a}\tilde{e}_aF - \tilde{e}_bF))_d. \] (19)

We may find now the components of the field \( G^c_n \) using (15).

Alternating the second derivative of an admissible vector field and using (8), (5), (17), we get
\[ K^c_{ab}(x^a, v^d) + 2\omega_{ab}\partial_nv^c = 0. \] (20)

From this it follows that
\[ \partial_nv^d + \omega^{ba}K^d_{ab} = 0. \] (21)

Comparing (21) with (13), we get
\[ G^d_n(x^\alpha, v^c) = \omega^{ba}(x^\alpha)K^d_{ab}(x^\alpha, v^c). \] (22)

Thus we conclude that if a metric connection satisfying (11) exists, then its coefficients are uniquely defined by the equalities (19) and (22). On the other hand, defining the coefficients of the connection using (19) and (22), we obtain a metric torsion free connection. The theorem is proved. □

As it is known, the curvature tensor of the constructed metric connection on the vector bundle is the tensor of the nonholonomicity of the corresponding
horizontal distribution. In order to write it down, one finds the Lie brackets of the vector fields generating the horizontal distribution:

\[
[\varepsilon_a, \varepsilon_b] = \omega_{ba} \bar{U} + R_{ab}^c \partial_{n+c},
\]

\[
[\varepsilon_a, \bar{U}] = \partial_n \Gamma_n^a \bar{U} + R_{na}^c \partial_{n+c},
\]

where

\[R_{ab}^c = K_{ab}^c + \omega_{ba} \omega^{ij} K_{ij}^c, \quad (23)\]

\[R_{na}^c = P_{na}^c + \tilde{\nabla}_a G_n^c. \quad (24)\]

The constructed tensor satisfies the usual properties of the Berwald curvature tensor.

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