The Complexity of Some Combinatorial Puzzles

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Abstract

We show that the decision versions of the puzzles Knossos and The Hour-Glass are complete for NP.

1 Introduction

Two puzzles that repeatedly appeared in the Munich newspaper tz are Knossos and The Hour-Glass (“Das Stundenglas”) [Bul10a, Bul10b, Bul10c, Bul11], see also [Pre].

Knossos asks the reader to insert walls into a grid of cells representing the famous palace of Knossos. Every room is described by a number that indicates the length of its walls in terms of segments enclosing the room (we assume that the walls form a connected curve and thus rooms are not nested). Note that there is only one shape for rooms with walls of lengths 4 or 6, where the room with length 6 contains two cells and can appear in two orientations. For lengths 8 and more there are rooms of different shapes containing different numbers of cells.

As an example consider the initial grid in Figure 1. A possible solution is depicted in Figure 2.

The Hour-Glass consists of two triangular shapes containing numbers and a target sum. The goal is to find a path from the first row to the last row such that the numbers traversed add up to the target sum. Each step moves diagonally down the structure.

An example is shown in Figure 3 where a solution for the target sum 53 is indicated by bold-face numbers.

We generalize these puzzles in several ways:

1. The original puzzles are of a fixed size (a 10 × 10-grid for Knossos and a structure with 11 rows for The Hour-Glass), while we consider arbitrary sizes.
2. As usual for puzzles, in the original versions a solution always exists and the task is to find it. We consider here also instances that might not have a solution. The problem is to decide whether the puzzles have a solution.

3. In the original puzzles the numbers are bounded (10 for Knossos and 9 for The Hour-Glass), whereas we allow arbitrary numbers.

The first two generalizations are necessary in order to investigate a decision problem in the framework of asymptotic complexity. A discussion of the relationship between the challenge presented by a puzzle and the complexity of its generalization can be found in [Epp].

The last generalization is also essential for The Hour-Glass (under reasonable assumptions) and will be discussed separately in Section 3.

Figure 1: Initial grid for Knossos

Figure 2: Solution for Knossos
2 Results

Theorem 1 Generalized Knossos is complete in NP.

Proof. The problem is easily seen to be in NP. Given a grid, the positions of walls are guessed and marked. For every number, the reachable cells can be determined by a variation of a recursive method used in computer graphics for contour filling by connectivity, see [Pav82, Section 8.4]. With the help of this algorithm it can be checked that all cells belong to some room and that every room contains at most one number. By starting at one wall and counting the walls of the circumference it can be verified that the number of walls is consistent with the inscribed number. Since there is a constant upper bound on the number of times a cell is visited, the complexity is linear in the size of the grid.

For showing hardness we reduce the well-known NP-complete satisfiability problem for boolean formulas (SAT) to Generalized Knossos, see e.g. [Sip06, p. 280].

In order to represent the following constructions in a clear way, we replace the number 4 by the symbol ♦, indicating that the four positions surrounding the cell will necessarily contain walls in every solution. Such cells will serve as borders and fill space not involved in the construction.

A truth value will in general be represented by the position of a room with walls of length 6 containing two cells. Two neighboring cells of the cell with inscription 6 are empty and the position of the room will indicate the truth value of a variable.

The first task we will explain is how to propagate a truth value of a variable to a position on the grid where it can be processed. We describe this for a horizontal propagation. It should be clear how to generalize the construction of Figure 3 to other directions. In Figure 5 we describe how to change the direction of propagation.

Notice that depending on the distance between the origin of a propagated
value and its destination, it might arrive “out of phase” or we might have to invert the propagated value. We can achieve these goals by inserting a “phase-shift” construction that contains a room with 8 walls instead of 6. This is outlined in Figure 6.

Given a boolean formula, it is necessary to encode a truth-assignment in terms of distribution of walls. In Figure 7 we demonstrate how to do this for a variable \( x \) that occurs at most 6 times in literals of the formula to be encoded. Three of the copies are initially complemented but can be inverted by the “phase-shift”-construction presented above. It is obvious how to generalize the construction to larger numbers (see also Section 3).

We now outline, why the construction correctly assigns truth values to the copies of variable \( x \). Notice that not both of the cells \( A \) and \( B \) can be in a room including the central 12, since this would require at least 20 walls. At least one of the cells is not in the room with 26 walls. Thus either all cells above the cells \( x_1 \) through \( x_3 \) or \( \bar{x}_1 \) through \( \bar{x}_3 \) are in one room with 12 walls. This induces a consistent assignment of the copies of \( x \) (\( x_1, x_2, \) and \( x_3 \)) and of the copies of \( \bar{x} \) (\( \bar{x}_1, \bar{x}_2, \) and \( \bar{x}_3 \)).

We now know how to assign values to variables and how to propagate these
values. But the encoding on a grid dictates a planar layout, and we may have to introduce cross-overs into the construction. Figure 8 shows how to do this.

We first argue, that the value of variable $x$ is transferred correctly in the sense, that the upper 6 and the cell marked $x$ are in one room, if and only if the lower 6 is not in one room with the cell containing $x$.

If both rooms including the cells with 6 do not contain $x$, then the room including the 26 can have at most 14 walls and therefore this option is excluded. If on the other hand both rooms contain $x$, then the central three cells are isolated and the room including the 26 cannot have connected walls. This shows correctness for $x$.

Without loss of generality we now assume that the room with the upper 6 does not contain the cell marked $x$. If both rooms containing the left and right 6 also include the cells marked $y$, then the wall of the room including 26 has length 28. If both rooms do not include the cells marked $y$, then the length is 24. This shows that $y$ is transferred correctly.
Next we show how to compute the AND of values of two variables $x$ and $y$. We note that together with the option to invert values (see Figure 9) all boolean functions can be computed.

![Figure 9: AND](image)

The value of result variable $z$ depends on the conjunction of $x$ and $y$.

We also have to “erase” values if an odd number of values is required, since by the construction in Figure 7 an even number of values is generated. We do this with the help of the “slack” construction described in Figure 10.

![Figure 10: Slack construction](image)

Finally the resulting value of the boolean formula encoded has to be true. This can be guaranteed by closing the propagating construction of Figure 4.

**Theorem 2** Generalized Hour-Glass is complete in NP.

Proof. The problem is in NP by the following algorithm. A path from the first to the last row of the input is guessed and the binary numbers along the path are added. The sum is compared to the target and the input is accepted if the values are equal.

For hardness we reduce the NP-complete problem SUBSET-SUM to The Hour-Glass, see [Sip06, Theorem 7.56].

The input of SUBSET-SUM consists of a sequence $S = (x_1, \ldots, x_n)$ of positive integers and a positive integer $t$, where the integers are encoded in binary. There is a subset $S' \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S'} x_i = t$. 
Without loss of generality $n$ is even, since we can add 0 to a list of odd length without affecting possible sums.

For a sequence $S = (x_1, \ldots, x_n)$ and target $t$ we form a structure with $2n - 1$ rows and put $x_i$ into column $n - 1$ in row $2i - 1$. All other non-empty positions contain 0. As an example we include the structure for $n = 6$:

Suppose there is a $S' \subseteq \{1, \ldots, n\}$ solving the instance of SUBSET-SUM. We construct a path along cells adding up to $t$. If $1 \in S'$ we choose the number in column $n - 1$ in row 1 as the first number of the path, otherwise we choose the 0 in column $n + 1$. If the path has been constructed up to row $2i - 1$ for $i \geq 1$, the next cell is the 0 in column $n$. If the path has been constructed up to row $2i$, the next step is to $x_i$ in column $n - 1$ if $i \in S'$ and to the 0 in column $n + 1$ if $i \notin S'$. Since $\sum_{i \in S'} x_i = t$, the numbers along the path add up to $t$. If conversely a path adds up to $t$, we can construct a $S'$ by collecting all rows in which numbers in column $n - 1$ are on the path. 

\[\square\]

3 Bounds on Numbers

If in Knossos only 4 is allowed as a number of walls, then a solution exists if all cells contain this number. This can be checked quickly.

In the construction from Section 2 only the choice of a truth value requires a variable size of rooms. In order to bound the size of the rooms, we duplicate the structure for the choice and connect the structures by signals. If the structures for choice are restricted to four output values and the upper room (with size 26) is split into a chain of rooms of size 6, then the bound 10 can be satisfied. The bound 26 in the cross-over structure can likely be improved, but whether the construction is possible with bound 10 remains open.

In the case of The Hour-Glass any constant bound on the numbers in the structure admits a non-deterministic solution in logarithmic space. The $O(n)$ numbers along a path are added in $O(\log n)$ bits and compared to the target sum. Unless NP and NL coincide, the constant bound makes the problem easier.
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