Asymptotic Behavior of Manakov Solitons: Effects of Potential Wells and Humps

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Abstract

We consider the asymptotic behavior of the soliton solutions of Manakov’s system perturbed by external potentials. It has already been established that its multisoliton interactions in the adiabatic approximation can be modeled by the Complex Toda chain (CTC). The fact that the CTC is a completely integrable system, enables us to determine the asymptotic behavior of the multisoliton trains. In the present study we accent on the 3-soliton initial configurations perturbed by sech-like external potentials and compare the numerical predictions of the Manakov system and the perturbed CTC in different regimes. The results of conducted analysis show that the perturbed CTC can reliably predict the long-time evolution of the Manakov system.

1 Introduction

The Gross-Pitaevski (GP) equation and its multicomponent generalizations are important tools for analyzing and studying the dynamics of the Bose-Einstein condensates (BEC), see the monographs [34, 21, 25] and the numerous references therein among which we mention [36, 21, 25, 31, 32, 33, 30, 25, 24, 28, 29]. In the 3-dimensional case these equations can be analyzed solely by numerical methods. If we assume that BEC is quasi-one-dimensional then the GP equations mentioned above may be reduced to the nonlinear Schrödinger equation (NLSE) perturbed by the external potential $V(x)$

$$ \begin{align*}
    iu_t + \frac{1}{2}u_{xx} + |u|^2u(x,t) &= V(x)u(x,t),
\end{align*} $$

or to its vector generalizations (VNLSE)

$$ \begin{align*}
    i\bar{u}_t + \frac{1}{2}\bar{u}_{xx} + (\bar{u}^\dagger \bar{u})\bar{u}(x,t) &= V(x)\bar{u}(x,t). \tag{2}
\end{align*} $$

The Manakov model (MM) [31] is a two-component VNLSE with $V(x) = 0$ (for more details see [11, 20]).
The analytical approach to the $N$-soliton interactions was proposed by Zakharov and Shabat \cite{16,33} for the scalar NLSE, for a vector NLSE see \cite{27}. They treated the case of the exact $N$-soliton solution where all solitons had different velocities. They calculated the asymptotics of the $N$-soliton solution for $t \to \pm \infty$ and proved that both asymptotics are sums of $N$ one-soliton solutions with the same sets of amplitudes and velocities. The effects of the interaction were shifts in the relative center of masses and phases of the solitons. The same approach, however, is not applicable to the MM, because the asymptotics of the soliton solution for $t \to \pm \infty$ do not commute.

The present paper is an extension of \cite{10,16,17} where the main result is that the $N$-soliton interactions in the adiabatic approximation for the Manakov model can also be modeled by the CTC \cite{13,38,12,19}. More specifically, here we continue our analysis of the effects of external potentials on the soliton interactions. While in \cite{10,16} we studied the effects of periodic, harmonic and anharmonic potentials, here we consider potential wells and humps of the form:

$$V(x) = \sum_{s} c_{s} V_{s}(x, y_{s}), \quad V_{s}(x, y_{s}) = \frac{1}{\cosh^{2}(2\nu_{0} x - y_{s})}.$$  \hspace{1cm} (3)

If $c_{s}$ is negative (resp. positive) $V_{s}(x)$ is a well (resp. hump) with width 1.7 at half-height/depth. Adjusting one or more terms in \cite{33} with different $c_{s}$ and $y_{s}$ we can describe wells and/or humps with different widths/depths and positions.

In the present paper we in fact prove the hypothesis in \cite{6} and extend the results in \cite{6,7,9,29,38,15,10,8,14} concerning the model of soliton interactions of vector NLSE (2) in adiabatic approximation.

The corresponding vector $N$-soliton train is a solution of (2) determined by the initial condition:

$$\vec{u}(x, t = 0) = \sum_{k=1}^{N} \vec{u}_{k}(x, t = 0), \quad \vec{u}_{k}(x, t) = u_{k}(x, t) \vec{n}_{k}, \quad u_{k}(x, t) = \frac{2\nu_{k} e^{i\phi_{k}}}{\cosh(z_{k})}$$  \hspace{1cm} (4)

with

$$z_{k} = 2\nu_{k}(x - \xi_{k}(t)), \quad \xi_{k}(t) = 2\mu_{k} t + \xi_{k,0},$$

$$\phi_{k} = \frac{\mu_{k}}{\nu_{k}} z_{k} + \delta_{k}(t), \quad \delta_{k}(t) = 2(\mu_{k}^{2} + \nu_{k}^{2}) t + \delta_{k,0}.$$  \hspace{1cm} (5)

where the $s$-component polarization vector $\vec{n}_{k} = \left( n_{k,1} e^{i\beta_{k,1}}, n_{k,2} e^{i\beta_{k,2}}, \ldots, n_{k,s} e^{i\beta_{k,s}} \right)^{T}$ is normalized by the conditions

$$\langle \vec{n}_{k}^{\dagger}, \vec{n}_{k} \rangle = \sum_{p=1}^{s} n_{k,p}^{2} = 1, \quad \sum_{p=1}^{s} \beta_{k,s} = 0.$$  \hspace{1cm} (6)

The adiabatic approximation holds true if the soliton parameters satisfy \cite{24}:

$$|\nu_{k} - \nu_{0}| \ll \nu_{0}, \quad |\mu_{k} - \mu_{0}| \ll \mu_{0}, \quad |\nu_{k} - \nu_{0}| |\xi_{k+1,0} - \xi_{k,0}| \gg 1,$$  \hspace{1cm} (7)

for all $k$, where $\nu_{0} = \frac{1}{N} \sum_{k=1}^{N} \nu_{k}$, and $\mu_{0} = \frac{1}{N} \sum_{k=1}^{N} \mu_{k}$ are the average amplitude and velocity, respectively. In fact we have two different scales:

$$|\nu_{k} - \nu_{0}| \simeq \varepsilon_{0}^{1/2}, \quad |\mu_{k} - \mu_{0}| \simeq \varepsilon_{0}^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_{0}^{-1}.$$  

We remind that the basic idea of the adiabatic approximation is to derive a dynamical system for the soliton parameters which would describe their interaction. This idea was
initiated by Karpman and Solov’ev [24] and modified by Anderson and Lisak [2]. Later this idea was generalized to $N$-soliton interactions [18, 13, 12, 19] and the corresponding dynamical system for the $4N$-soliton parameters was identified as a $N$-site complex Toda chain (CTC).

The fact that the CTC, (just like its real counterpart – the Toda chain) is completely integrable gives additional possibilities. A detailed comparative analysis between the solutions of the RTC and CTC [11] shows that the CTC allows for a variety of asymptotic regimes, see Section 3 below. More precisely, knowing the initial soliton parameters one can effectively predict the asymptotic regime of the soliton train. Another possible use of the same fact is, that one can describe the sets of soliton parameters responsible for each of the asymptotic regimes. Another important advantage of the adiabatic approach consists in the fact, that one may consider the effects of various perturbations on the soliton interactions [24, 18].

The next step was to extend this approach to treat the soliton interactions of the Manakov solitons. More precisely, using the method of Anderson and Lisak [2] we derive a generalized version of the Complex Toda Chain (CTC) (see Eqs. (16), (17) below) as a model describing the behavior of the $N$-soliton trains of of the VNLSE (2) [6, 7, 8, 10, 17, 14]. This generalized CTC includes dependence on the polarization vectors $\vec{n}_k$. It allows us to analyze how the changes of the polarization vectors influence the soliton interactions. Besides, the generalized CTC is also integrable with the consequence that one can predict the asymptotic regimes of the Manakov solitons and can describe the sets of soliton parameters that are responsible for each of the asymptotic regimes. Of course, just like for the scalar case, one can also analyze the effects of the various perturbations on the soliton interactions.

In Section 2 we outline how the variational approach developed in [2] can be used to derive the perturbed CTC (PCTC) model [10] for sech-type external potentials. We also remind the reader about the asymptotic regimes of the soliton trains predicted by the CTC [13, 12]. In Section 3 we briefly treat the $N$-soliton interactions of the MM without external potential. Obviously in order to determine the $N$-soliton train for the MM, along with the usual sets of solitons parameters $\nu_k, \mu_k, \xi_k$ and $\delta_k$ we need also the set of polarization vectors $\vec{n}_s$. In Section 4 we derive the effects of the external potentials on the soliton interactions. This is a perturbed form of the CTC (PCTC) for generic potentials of the form (3). Section 5 is dedicated to the comparison between the numeric solutions of the perturbed VNLSE (2) with the predictions of the PCTC model. To this end we solve the VNLSE numerically by using an implicit scheme of Crank-Nicolson type in complex arithmetic. The concept of the internal iterations is applied (see [5]) in order to ensure the implementation of the conservation laws on difference level within the round-off error of the calculations [40, 41, 42]. The solutions of the relevant PCTC have been obtained using Maple. Knowing the numeric solution $\vec{u}$ of the perturbed VNLSE we calculate he maxima of $(\vec{u}^\dagger, \vec{u})$, compare them with the (numeric solutions) for $\xi_k(t)$ of the PCTC and plot the predicted by both models trajectories for each of the solitons. Thus we are able to analyze the effects of the external potentials on the soliton interactions. Finally, Sections 6 and 7 contain discussion and conclusions.

2 Preliminaries

Here we briefly remind the derivation of the CTC as a model describing the $N$-soliton interactions VNLSS systems using the variational approach [2].
2.1 Derivation of the CTC as a model for the soliton interaction of perturbed VNLS systems

The perturbed vector NLSE (2) allows Hamiltonian formulations with the Poisson brackets
\[
\{ \vec{u}_j(x,t), \vec{u}_k^*(y,t) \} = \delta_{jk}\delta(x-y)
\]  
and the Hamiltonian
\[
H = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} (\vec{u}_x, \vec{u}_x) - \frac{1}{2} (\vec{u}^\dagger, \vec{u}^\dagger)^2 + V(x)(\vec{u}^\dagger, \vec{u}) \right].
\]

It also admits a Lagrangian of the form:
\[
L = \int_{-\infty}^{\infty} dt \left[ \frac{i}{2} (\vec{u}_t, \vec{u}_t) - \langle \vec{u}_t, \vec{u} \rangle - H \right].
\]

In what follows we will analyze the large time behavior of the \( N \)-soliton train determined as the solution of the VNLS by the initial condition (4), (5).

The idea of the variational approach of [2] is to insert the anzatz (4) into the Lagrangian, perform the integration over \( x \) and retain only terms of the orders of \( \varepsilon_0^{-1/2} \) and \( \varepsilon_0 \). The first obvious observation is that only the nearest neighbors solitons will contribute such terms and
\[
L = \sum_{k=1}^{N} L_k + \sum_{k=1}^{N} \sum_{n=k+1}^{N} \tilde{L}_{kn}.
\]

Here \( L_k \) correspond to the terms involving only the \( k \)-th soliton (see [10]):
\[
L_k = 4\nu_k \left( \frac{i}{2} (\vec{n}_{k,t}, \vec{n}_{k,t}) - \langle \vec{n}_{k,t}, \vec{n}_k \rangle \right) + 2\mu_k \frac{d\xi_k}{dt} - \frac{d\delta_k}{dt} - 2\mu_k^2 + 2\nu_k^2 + \int_{x=-\infty}^{\infty} V(x)(\vec{n}^\dagger_k, \vec{n}_k).
\]

Finally the terms describing soliton-soliton interactions are given by:
\[
L_{kn} = 16\nu_0^3 e^{-\Delta_{kn}} (R_{kn} + R_{kn}^*) + \mathcal{O}(\varepsilon^{3/2}),
\]
\[
R_{kn} = e^{i(\tilde{\delta}_n - \delta_k)} \langle \vec{n}_k, \vec{n}_n \rangle, \quad \tilde{\delta}_k = \delta_k - 2\mu_0 \xi_k, \quad \Delta_{kn} = 2s_{kn}\nu_0(\xi_k - \xi_n),
\]

where \( s_{k,k+1} = 1 \) and \( s_{k,k-1} = -1 \).

The next step is to consider \( L \) (11) as a Lagrangian of the dynamical system, describing the motion of the \( N \)-soliton train and providing the equations of motion for the \((2s+2)N\) \((6N\) for the Manakov case) soliton parameters.

Let us first consider the unperturbed case, \( i.e., V(x) = 0 \). Deriving the dynamical system we get terms of three different orders of magnitude: (i) terms of order \( \Delta_{kn}^2 \exp(-\Delta_{kn}) \); (ii) terms of order \( \Delta_{kn} \exp(-\Delta_{kn}) \) and (iii) terms of order \( \exp(-\Delta_{kn}) \). However the terms of types (i) and (ii) are multiplied by factors that are of the order of \( \exp(-\Delta_{kn}) \) due to the evolution equations for the soliton parameters. Finally, we arrive at the following set of dynamical equations for the soliton parameters:
\[
\frac{d\xi_k}{dt} = 2\mu_k, \quad \frac{d\delta_k}{dt} = 2\mu_k^2 + 2\nu_k^2, \quad \frac{d\nu_k}{dt} = 8\nu_0^3 \sum_n e^{-\Delta_{kn}} i (R_{kn} - R_{kn}^*), \quad \frac{d\mu_k}{dt} = -8\nu_0^3 \sum_n e^{-\Delta_{kn}} (R_{kn} + R_{kn}^*).
\]
In addition we obtain also a system of equations for the evolution of the polarization vectors:

\[
\frac{d\tilde{n}_k}{dt} = 4\nu_0^2 i \sum_{n=k\pm 1} e^{-\Delta_k} \left[ e^{i(\delta_n - \delta_k)} \tilde{n}_n - R_{kn} \tilde{n}_k + e^{i(\delta_n - \delta_k)} \tilde{n}_n + R_{kn}^* \tilde{n}_k \right] + C_k \tilde{n}_k.
\] (15)

where the constants \( C_k \) are fixed up by the constraints on the polarization vectors. Indeed, from \( \langle \tilde{n}_k \dagger, \tilde{n}_k \rangle = 1 \) for all \( t \) one finds that \( C_k + C_k^* = 0 \), i.e., the constants \( C_k \) are purely imaginary. Let us now assume that \( C_k = i\theta_k \). Then from eqs. (10) and (15) we find that \( \beta_k, s \) become time-dependent and up to terms of the order of \( \epsilon \) evolve linearly with time:

\( \beta_k, s(t) = \beta_k, s(0) + \dot{\theta}_k t \). But such evolution is compatible with the second normalization condition in (6) only if \( \theta_k = 0 \); therefore \( C_k = 0 \). Thus, from Eqs. (14) we get:

\[
\frac{d(\mu_k + i\nu_k)}{dt} = 4\nu_0 \left[ \langle \tilde{n}_k, \tilde{n}_{k-1} \rangle e^{q_k - q_{k-1}} - \langle \tilde{n}_{k+1}, \tilde{n}_k \rangle e^{q_{k+1} - q_k} \right],
\] (16)

where

\[
q_k = -2\nu_0 \xi_k + k \ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0 \xi_k),
\]

\[
\nu_0 = \frac{1}{N} \sum_{s=1}^N \nu_s, \quad \mu_0 = \frac{1}{N} \sum_{s=1}^N \mu_s, \quad \delta_0 = \frac{1}{N} \sum_{s=1}^N \delta_s.
\] (17)

Besides, from (14) and (17) there follows (see [18]):

\[
\frac{dq_k}{dt} = -4\nu_0(\mu_k + i\nu_k).
\] (18)

and

\[
\frac{d^2 q_k}{dt^2} = 16\nu_0^2 \left[ \langle \tilde{n}_{k+1}, \tilde{n}_k \rangle e^{q_{k+1} - q_k} - \langle \tilde{n}_k, \tilde{n}_{k-1} \rangle e^{q_k - q_{k-1}} \right],
\] (19)

which proves the statement in [6]. Eq. (19), combined with the system of equations for the polarization vectors (15) provides the proper generalization of the CTC model for the MNLS.

The equations for the polarization vectors are nonlinear. So the whole system of equations for \( q_k \) and \( \tilde{n}_k \) seems to be rather complicated and nonintegrable even for the unperturbed MNLS. However, all terms in the right hand sides of the evolution equations for \( \tilde{n}_k \) are of the order of \( \epsilon \). This allows us to neglect the evolution of \( \tilde{n}_k \) and to approximate them with their initial values. As a result we obtain that the \( N \)-soliton interactions for the VNLSE in the adiabatic approximation are modeled by the CTC, see Section 3.

It is easy to see, that if all \( \langle \tilde{n}_{k+1} \dagger, \tilde{n}_k \rangle \) = const \( \neq 0 \) then the CTC (19) is a completely integrable dynamical system, just like the real Toda chain.

Note also that the CTC models the soliton interactions for the VNLSE with any number of components. The effect of the polarization vectors on the interaction comes into CTC only through the scalar products \( m_{0s} = \langle \tilde{n}_{k+1}, \tilde{n}_k \rangle \). It is well known, that a gauge transformation \( \tilde{u} \rightarrow g_0 \tilde{u} \) with any constant unitary matrix \( g_0 \) leaves the VNLSE, Eq. (2) invariant. Such transformation will change all polarization vectors simultaneously \( \tilde{n}_k \rightarrow g_0 \tilde{n}_k \) but preserves their scalar products, and so will not influence the soliton interaction. Obviously, our CTC model is invariant under such transformations. Due to the above arguments our choice of the initial values of \( \tilde{n}_{k0} \) can be changed into \( g_0 \tilde{n}_{k0} \) with no effect on the interaction. That is why we specify only the scalar products \( m_{k0} \) for our runs, which we have chosen to be real.

2.2 The effects of the sech-like potentials on CTC

Now we consider the effects of the external potentials of the form [3]. To this end we have to calculate the integrals in the right hand side of eq. (12) and see how they would change the right hand sides of eqs. (14).

\[\text{5}\]
As it is clear from above, we have to replace \( L_k \) in Eq. (12) by
\[
L_{k,\text{pert}} = L_k - 2\nu_k \int_{-\infty}^{\infty} dx \frac{V(x)}{\cosh^2(z_k)}.
\]
while \( L_k \) remains unchanged. Thus we obtain the following PCTC system:
\[
\frac{d\lambda_k}{dt} = -4\nu_0 \left( e^{\nu_{k+1} - \nu_k} (\bar{\eta}_{k+1} - \bar{\eta}_k) - e^{\nu_{k-1} - \nu_k} (\bar{\eta}_{k-1} - \bar{\eta}_k) \right) + M_k + iN_k,
\]
\[
\frac{dq_k}{dt} = -4\nu_0 \lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k,
\]
\[
\frac{d\bar{n}_k}{dt} = O(\epsilon),
\]
where \( \lambda_k = \mu_k + i\nu_k, X_k = 2\mu_k \Xi_k + D_k \) and
\[
N_k = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dz_k}{\cosh z_k} \text{Im} \left( V(y_k)u_ke^{-i\phi_k} \right),
\]
\[
M_k = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dz_k \sinh z_k}{\cosh^2 z_k} \text{Re} \left( V(y_k)u_ke^{-i\phi_k} \right),
\]
\[
\Xi_k = -\frac{1}{4\nu_k} \int_{-\infty}^{\infty} \frac{dz_k \sinh z_k}{\cosh z_k} \text{Im} \left( V(y_k)u_ke^{-i\phi_k} \right),
\]
\[
D_k = \frac{1}{2\nu_k} \int_{-\infty}^{\infty} \frac{dz_k (1 - z_k \tanh z_k)}{\cosh z_k} \text{Re} \left( V(y_k)u_ke^{-i\phi_k} \right),
\]
where \( y_k = z_k/(2\nu_0) + \xi_k \).

As a result for our specific choice of \( V(x) \) we get:
\[
M_k = \sum_s 2c_s\nu_k P(\Delta_{k,s}), \quad N_k = 0, \quad \Xi_k = 0, \quad D_k = \sum_s c_s R(\Delta_{k,s}), \quad (22)
\]
where \( \Delta_{k,s} = 2\nu_0 \xi_k - y_s \) and the integrals describing the interaction of the solitons with the potential (see Figure 1(left panel))
\[
P(\Delta) = \frac{\Delta + 2\Delta \cosh^2(\Delta) - 3 \sinh(\Delta) \cosh(\Delta)}{\sinh^4(\Delta)},
\]
\[
R(\Delta) = \frac{6\Delta \sinh(\Delta) \cosh(\Delta) - (2\Delta^2 + 3) \sinh^2(\Delta) - 3\Delta^2}{2 \sinh^4(\Delta)}.
\]

The details of deriving the integrals are given in the Appendix.

The corrections to \( N_k \) and \( P_k \), coming from the terms linear in \( u \) depend only on the parameters of the \( k \)-th soliton; i.e., they are ‘local’ in \( k \).

### 3 CTC and the Asymptotic Regimes of N-soliton Trains

The fact that the \( N \)-soliton trains for the scalar NLSE are modeled by an integrable model – CTC allowed one to to predict their asymptotic behavior. The method to do so was based on the exact integrability of the CTC [13] and on its Lax representation.

Here we shall show, that similar results hold true also for the CTC (14) modeling the soliton trains of the MM. Indeed, following Moser [32] we introduce the Lax pair
\[
\dot{L} = [B, L],
\]
where
\[
L = \sum_{k=1}^{N} (b_k E_{kk} + a_k (E_{k,k+1} + E_{k-1,k})), \quad (25)
\]
\[
B = \sum_{k=1}^{N} a_k ((E_{k,k+1} - E_{k-1,k}).
\]
Here the matrices \((E_{kn})_{pq} = \delta_{kp}\delta_{nq}\), and \(E_{kn} = 0\) whenever one of the indices becomes 0 or \(N + 1\); the other notations in (24) are as follows:

\[
a_k = \frac{1}{2} \sqrt{\langle \vec{n}_{k+1}, \vec{n}_k \rangle} e^{(q_{k+1} - q_k)/2}, \quad b_k = \frac{1}{2} (\mu_k + i\nu_k).
\]

One can check that the compatibility condition eq. (24) with \(L\) and \(B\) as in (25) is equivalent to the unperturbed CTC (14).

The first consequence of the Lax representation is that the CTC has \(N\) complex-valued integrals of motion provided by the eigenvalues of \(L\) which we denote by \(\zeta_k = \kappa_k + i\eta_k\), \(k = 1, \ldots, N\). Indeed the Lax equation means that the evolution of \(L\) is isospectral, i.e., \(d\zeta_k/dt = 0\).

Another important consequence from the results of Moser [32] is that for the real Toda chain one can write down explicitly its solutions in terms of the scattering data, which consist of \(\{\zeta_k, r_k\}_{k=1}^N\) where \(r_k\) are the first components of the properly normalized eigenvectors of \(L_0\) [32, 39]. For the real Toda chain both \(\zeta_k = \kappa_k\) and \(r_k\) are real; besides all \(\zeta_k\) are different. Next Moser calculated the asymptotics of these solutions for \(t \to \pm\infty\) and showed that \(\kappa_k\) determine the asymptotic velocities of the particles.

The formulae derived by Moser can easily be extended to the complex case [11]. The important difference is that all important ingredients such as eigenvalues \(\zeta_k\) and first components of the eigenvectors of \(L\) normalized to 1 now become complex valued. In addition, the important asset of \(L\) for the RTC, namely that all eigenvalues are real and different, is also lost. However the asymptotics of the solutions for \(t \to \pm\infty\) can be calculated with the result:

\[
q_k(t) = -2\nu_0\zeta_k t - B_k + O(e^{-Dt}), \quad (27)
\]

where \(D\) in (27) is some real positive constant which is estimated by the minimal difference between the asymptotic velocities. Equating the real parts in eq. (27) we obtain:

\[
\lim_{t \to \infty} (\xi_k + 2\kappa_k t) = \text{const} \quad (28)
\]

which means that the real parts \(\kappa_k\) of the eigenvalues of \(L\) determine the asymptotic velocities for the CTC. This fact will be used to classify the regimes of asymptotic behavior.

Let us also point out the important differences between RTC and CTC, namely:

D1) While for RTC \(q_k, r_k\) and \(\zeta_k\) are all real, for CTC they generically take complex values, e.g. \(\zeta_k = \kappa_k + i\eta_k\);

D2) While for RTC \(\zeta_k \neq \zeta_j\) for \(k \neq j\), for CTC no such restriction holds.

As a consequence we find that the only possible asymptotic behavior in the RTC is the asymptotically free motion of the solitons. For CTC it is \(\kappa_k\) that determines the asymptotic velocity of the \(k\)-th soliton. For simplicity and without loss of generality we assume that: \(\text{tr} L_0 = 0\); \(\zeta_k \neq \zeta_j\) for \(k \neq j\); and \(\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_N\). Then we have:

AFR The asymptotically free regime takes place if \(\kappa_k \neq \kappa_j\) for \(k \neq j\), i.e., the asymptotic velocities are all different. Then we have asymptotically separating, free solitons, see also [18, 12];

BSR The bound state regime takes place for \(\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0\), i.e., all \(N\) solitons move with the same mean asymptotic velocity, and form a “bound state”. The key question now will be the nature of the internal motions in such a bound state: is it quasi-equidistant or not?
a variety of intermediate situations, or mixed asymptotic regimes happen when one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.

Obviously the regimes (ii) and (iii), as well as the degenerate and singular cases, which we do not consider here have no analogies in the RTC and physically are qualitatively different from i).

The perturbed CTC taking into account the effects of the sech-like potentials to the best of our knowledge is not integrable and does not allow Lax representation. Therefore we are applying numeric methods to solve it.

4 Comparison Between the PCTC Model and Manakov Soliton Interactions

In this Section we will compare how well the PCTC model derived above predicts the soliton interactions as solutions of the MM with the external potentials of kind (3). Before that let us remind the well known result (see [13, 18, 12]) that the 3-soliton systems allow for three types of dynamical regimes for large times, namely

AFR) asymptotically free regime when all 3 solitons move away with different velocities. This regime takes place if the initial amplitudes are given by eq. (31) with [12]:

$$\Delta \nu < \nu_{\text{cr}} = 2 \sqrt{2} \cos(\theta_1 - \theta_2) \nu_0 \exp(-\nu_0 r_0)$$

and the phases are as in (32a). For our configuration with $r_0 = 8$ we have $\nu_{\text{cr}} = 0.0246$. Such asymptotic regime is shown on the left panel of Figure 2.

MAR) mixed asymptotic regime, when two of the solitons form bound state and the third soliton goes away from them with different velocity; Such regime takes place if the amplitudes are chosen as in (29) and the phases are as in (32ab), see the left panel of Figure 3.

BSR) bound state regime when all solitons move asymptotically with the same velocity. Such regime takes place for amplitudes with $\Delta \nu > \nu_{\text{cr}}$ and the phases are as in (32ab). Such asymptotic regime is shown on the left panel of Figure 4.

It is natural to analyze separately all three regimes and to see what would be the effect of the external wells/humps on them. In particular, one can determine for which positions and intensities of the external potentials the solitons will undergo from one asymptotic regime to another.

Remark 1 The CTC and its perturbative version PCTC use the adiabatic approximation. If we assume that the distance between the solitons is $r_0 = 8$, then the adiabatic parameter $\epsilon \simeq 0.01$, so one can expect that the CTC model will hold true up to times of the order of $1/\epsilon \simeq 100$. In figure 2 we see an excellent match between the MM and CTC up to times of the order of 300; after that the two models diverge.

Since the PCTC model is not integrable we will solve it numerically to find the predicted solitons trajectories $\xi_k(t)$. Besides we will solve numerically the MM with the initial condition (4) and extract the trajectories of $\max(|u_1|^2 + |u_2|^2)$, where $\bar{u} \equiv (u_1, u_2)$. 

Figure 1: Graphs of $P$ and $R$ functions: for a single sech-potential centered at the origin – in cyan and red colors; and for the superposed potential at the neighboring panel – in green and brown colors. (left); Single sech-potential in black color vs. superposed external potential $V(x) = \sum_{s=0}^{32} c_s \text{sech}^2(x - x_s)$, $c_s = -10^{-1}$, $x_s = -16 + sh$, $h = 1$, $s = 0, ..., 32$ – in blue color. The superposed potential forms a well. (right).

On the right panel of Figure 1 we plot samples of potential well with width 40 composed by 33 wells with depth $c_s = -0.1$ distributed uniformly between abscisas $-16$ and $16$ and distance between them $h = 1$.

Evidently each Manakov soliton solution is parameterized by 6 parameters and four of them are the usual velocity, position, amplitude and phase. Two more parameters fix up the polarization vector. Having in mind the big parametric phenomenology of the solutions we fix the velocities, positions and polarization vectors and vary the initial amplitudes and phases in order to ensure one or another asymptotic regime [12]. Even with only three solitons configuration but with 13 to 33 potential wells/humps we have a large variety of combinations.

Potential wells, especially when broad enough attract the solitons and may be used to stabilize in a bound state. Potential humps repel the solitons; choosing their positions appropriately one can either split a soliton bound state into free solitons or force free solitons into bound state.

In what follows we compare the PCTC models with the numeric solutions of the corresponding (perturbed) MM. In doing this, to have better base for comparison we keep fixed some of the the initial parameters of the soliton trains. The other parameters may vary from run to run; their particular values will be specified in the captions of the figures. To avoid any confusion we mark the PCTC solutions by dashed lines, and the numeric solutions of the MM and the perturbed MM by solid lines. Also, we plot the centers of solitons and track their trajectories. Since the PCTC are derived in the framework of the adiabatic approximation, they are expected to be adequate only up to times of the order of $\epsilon^{-1}$. So, if the distance between neighboring solitons is 8 units, then $\epsilon \simeq 10^{-2}$ and one might expect that the PCTC would be valid up to $t \simeq 100$. Rather surprisingly, see we find that the models work well until $t \simeq 1000$ or even longer. We also assume that $\xi_k < \xi_{k+1}$, $k = 1, 2$.

Below we provide several examples that illustrate our points. More specifically we set:

**IC-1** The initial velocities $\mu_{k,0} = 0$, $k = 1, 2, 3$;
Figure 2: AFR: Free potential behavior corresponding to real parts of eigenvalues of the Lax pair $\text{Re} \zeta_1 = -0.0116$, $\text{Re} \zeta_2 = 0$, $\text{Re} \zeta_3 = 0.0116$ (left panel); External potential well $V(x) = \sum_{s=0}^{32} c_s \text{sech}^2(x - x_s)$, $c_s = -10^{-1}$, $x_s = -16 + s$, $s = 0, ..., 32$. The shaded area denotes the external potential at a half level (right panel).

**IC-2** The initial positions $\xi_{1,0} = -8, \xi_{2,0} = 0, \xi_{2,0} = 8$;

**IC-3** Each of the initial polarization vectors $\vec{n}_{k,0}$ will be parameterized by its polarization angle $\theta_{k,0}$ and a phase $\gamma_{k,0}$ as follows:

$$
\vec{n}_{k,0} = \begin{pmatrix} e^{i\gamma_{k,0}} \cos(\theta_{k,0}) \\ e^{-i\gamma_{k,0}} \sin(\theta_{k,0}) \end{pmatrix}.
$$

(30)

Generically the scalar products $(\vec{n}_{k+1}^\dagger, \vec{n}_k)$ are complex-valued. For simplicity here we assume that $\gamma_{k,0} = 0$ for $k = 1, 2, 3$ so that $(\vec{n}_{k+1}^\dagger, \vec{n}_k) = \cos(\theta_{k+1,0} - \theta_{k,0})$ and $\theta_{k,0} = (4 - k)\pi/10$. Thus all scalar products just mentioned equal to $\cos(\pi/10) \approx 0.951$;

**IC-4** The initial amplitudes

$$
\nu_{1,0} = \nu_0 + \Delta \nu, \quad \nu_{2,0} = \nu_0 = 0.5, \quad \nu_{3,0} = \nu_0 - \Delta \nu;
$$

(31)

**IC-5** We use two types of initial phases configurations:

a) $\delta_{1,0} = 0, \quad \delta_{2,0} = \pi, \quad \delta_{3,0} = 0, \quad \Delta \nu = 0.01.$

If $\Delta \nu < 0.02526$ – asymptotically free behavior; (32a)

b) $\delta_{1,0} = \delta_{2,0} = \delta_{3,0} = 0, \quad \Delta \nu = 0.02.$

Bound state behavior for every $\Delta \nu > 0.$ (32b)
Figure 3: BSR: Free potential behavior corresponding to real parts of eigenvalues of the Lax pair \( \text{Re} \zeta_1 = \text{Re} \zeta_2 = \text{Re} \zeta_3 = 0 \) (left); External potential hump \( V(x) = \sum_{s=0}^{12} c_s \text{sech}^2(x - x_s) \), \( c_s = 10^{-2}, x_s = -10 + sh, h = 5/3, s = 0, ..., 12 \). The shaded area denotes the external potential at a half level (right).

5 Results and Discussion

On the next figures we show some examples of 3-soliton systems. On the figures we plot the trajectories predicted by the PCTC (green dashed lines) with the MM (red solid lines). In order to ensure the adiabaticity condition we assume that initially the distance between the neighboring solitons \( r_0 = 8 \). The first example (Figure 2) clearly demonstrates the role of the external well on the stability of the asymptotically free 3-soliton configuration. The potential (shaded strip) does not allow the lateral solitons to leave the well and they start to oscillate. So, the new regime is bound state.

On the next Figure 3 the potential free regime is bound state. The influence of potential hump of width 24 and amplitude \( c_s = 10^{-2} \) leads to fast violation of this regime and transition to asymptotically free behavior of the lateral solitons.

On the Figure 4 is demonstrated the influence of external potential on the third possible regime – mixed asymptotic regime. In potential free configuration we have two bound stated solitons and one freely propagating. The adding of an external potential as superposed wells with amplitude (depth) \( c_s = -10^{-2} \) leads to a bound state behavior of all the three solitons.

On these figures the solutions obtained by VNSE are plotted by red solid lines while those obtained by CTC model by dashed green line. The comparison of the numerical predictions of the both models is fully satisfactory.
Figure 4: MAR: Free potential behavior corresponding to real parts of eigenvalues of the Lax pair $\text{Re}\zeta_1 = \text{Re}\zeta_2 = -0.00321$, $\text{Re}\zeta_3 = 0.00642$ (left); External potential well $V(x) = \sum_{s=0}^{32} c_s \text{sech}^2(x - x_s)$, $c_s = -10^{-2}$, $x_s = -16 + sh$, $h = 1$, $s = 0, ..., 32$. The shaded area denotes the external potential at a half level (right).

6 Conclusion

We have analyzed the effects of the external potential wells and humps on the VNLSE soliton interactions using the PCTC model. The comparison with the predictions of the more general VNLSE model \[\begin{align*}
i\vec{u}_t + \frac{1}{2} \vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u} + \alpha \vec{U}(x,t) &= 0,
\end{align*}\] (33)
where $\vec{U} = (|u_2|^2 u_1, |u_1|^2 u_2)^T$ and the cross-modulation magnitude $\alpha$ is an excellent validation of the consistency and applicability of PCTC. The superposition of big number of wells/humps obviously complexify the motion of the soliton envelopes and can cause a transition from asymptotically free and mixed asymptotic regime to a bound state regime and vice versa. In particular, the latter means that the external potentials can be used to control the soliton motion in a given direction and therefore to achieve a predicted motion of the optical pulse. A general feature of the conducted experiments is that the predictions of both models match very well for a very long-time evolution. This means that PCTC is reliable model for predicting the evolution of the multisoliton solutions of Manakov model in adiabatic approximation.

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A List of integrals

We outline the method of calculating the integrals [17] needed to work out external well-type potentials as in Eq. (3). We start with the well known integral [37]:

\[
K(a, \Delta) \equiv \int_{-\infty}^{\infty} \frac{dz \ e^{iaz}}{2 \cosh z \cosh(z + \Delta)} = \frac{\pi}{\sinh \Delta} \frac{\sin(a\Delta/2)}{\sinh(\pi a/2)} e^{-ia\Delta/2}.
\]

Using the identity \(2 \cosh(z - \Delta/2) \cosh(z + \Delta/2) = \cosh(2z) + \cosh(\Delta)\) and applying the substitution \(z \to z - \Delta/2\) and we get

\[
e^{ia\Delta/2}K(a, \Delta) = \int_{-\infty}^{\infty} \frac{dze^{iaz}}{\cosh(2z) + \cosh(\Delta)} = \frac{\pi \sin(a\Delta/2)}{\sinh(\Delta) \sinh(\pi a/2)}.
\]

Next we differentiate with respect to \(\Delta\) and divide by \(\sinh \Delta\):

\[
\int_{-\infty}^{\infty} \frac{dze^{iaz}}{(\cosh(2z) + \cosh(\Delta))^2} = -\frac{1}{\sinh(\Delta)} \frac{\partial}{\partial \Delta} \left( \frac{\pi \sin(a\Delta/2)}{\sinh(\Delta) \sinh(\pi a/2)} \right),
\]

\[
\int_{-\infty}^{\infty} \frac{dze^{iaz}}{(\cosh(2z) + \cosh(\Delta))^3} = \frac{1}{\sinh(\Delta)} \frac{\partial}{\partial \Delta} \left( \frac{1}{\sinh(\Delta)} \frac{\partial}{\partial \Delta} \left( \frac{\pi \sin(a\Delta/2)}{\sinh(\Delta) \sinh(\pi a/2)} \right) \right).
\]

Thus we have:

\[
N(a, \Delta) = -\frac{2}{\sinh(\Delta)} e^{-ia\Delta/2} \frac{\partial}{\partial \Delta} \left( e^{ia\Delta/2}K(a, \Delta) \right), \quad P(\Delta) = -\frac{1}{2i} e^{ia\Delta} \frac{\partial N(a, \Delta)}{\partial a} \bigg|_{a=0},
\]

\[
R(\Delta) = N(0, \Delta) - \frac{1}{2i} \frac{\partial}{\partial a} \left( e^{ia\Delta} \frac{\partial N(a, \Delta)}{\partial \Delta} \right) \bigg|_{a=0}.
\]

If we have generic (complex-valued) potential \(V(x)\) we will have to calculate the integrals:

\[
N(\Delta) = \int_{-\infty}^{\infty} \frac{dz}{2 \cosh^2(z) \cosh^2(z - \Delta)} = \frac{2\Delta \cosh(\Delta) - 2 \sinh(\Delta)}{\sinh^3(\Delta)}
\]

\[
Q(\Delta) = \int_{-\infty}^{\infty} \frac{dzz}{2 \cosh^2(z) \cosh^2(z - \Delta)} = \frac{2\Delta \sinh(\Delta) - 2\Delta^2 \cosh(\Delta)}{\sinh^3(\Delta)}
\]

while \(P(\Delta)\) and \(R(\Delta)\) are given in eq. [23].

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