The book under review is a research monograph on the homogenization for the motion of suspensions. Starting from a microscopic model describing the dynamics of mixtures of fluid with fine solid particles, the aim is the rigorous derivation of macroscopic models using multi-scale techniques. The homogenization process is performed under the assumption of an *a priori* known evolution of the particles, and different scenarios are considered in terms of the size of the particles, the distance between neighboring particles, and the interaction between different particles, leading to different macroscopic models. For this several multi-scale techniques are developed which are quite important for homogenization of fluid problems in perforated (non-periodic) domains including small moving rigid particles. While there exists a wide literature on phenomenological approaches for the derivation of macroscopic models for suspension based on physical observations, starting with the pioneering work [5], results on the rigorous mathematical justifications from first principles seem to be rare. Here, the book offers significant contributions by using homogenization theory; see the seminal work [2] for an introduction.

Let us briefly explain the idea of homogenization. Starting from a microscopic model, including for example physical and biochemical processes on the micro-scale as well as the heterogeneity of the medium, the aim in the homogenization theory is the derivation of a macroscopic (effective) model, which is a good approximation (in a suitable mathematical sense) of the micro-model. The heterogeneity of the microscopic problem can be described by a small parameter $\epsilon$. From a mathematical point of view, the microscopic solutions, depending on $\epsilon$, converge in a suitable sense to
the solution of the macroscopic problem. We sketch this procedure for a model problem of diffusion or conductivity with a periodic coefficient: Let $\Omega \subset \mathbb{R}^n$ for $n \geq 1$ be open and bounded. For $0 < \epsilon \ll 1$ we consider the problem

$$-\nabla \cdot \left( a \left( \frac{x}{\epsilon} \right) \nabla u_\epsilon \right) = f \quad \text{in } \Omega,$$

$$u_\epsilon = 0 \quad \text{on } \partial \Omega,$$

with $f \in L^2(\Omega)$ and $a \in L^\infty_{\text{per}}(\mathbb{R}^n)^{n \times n}$ is symmetric, positive-definite, and $Y$-periodic, where $Y = (0, 1)^n$ is the unit cube in $\mathbb{R}^n$. It is easy to check by the Lax-Milgram lemma that (1) admits a unique weak solution $u_\epsilon \in H^1_0(\Omega)$, and the sequence $(u_\epsilon)_{\epsilon > 0}$ is bounded in $H^1(\Omega)$. However, even for such simple models the numerical treatment is quite challenging. Starting with the pioneering works [2] and [9], the homogenization theory deals with the derivation of macroscopic models for $\epsilon \to 0$ with effective/homogenized coefficients, the solution of which gives an approximation of the microscopic solution. In fact, for the solutions $u_\epsilon$ of (1) it can be shown that there exists $u \in H^1_0(\Omega)$ such that $u_\epsilon$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to $u$, and $u$ is the unique weak solution of the macroscopic problem

$$-\nabla \cdot (a^* \nabla u) = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$ 

Here the homogenized coefficient $a^* \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, defined by $(i, j = 1, \ldots, n)$

$$a^*_{ij} = \int_Y a(y)(\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j) dy,$$

with the unit vectors $e_i$, and $w_i \in H^1_{\text{per}}(Y)/\mathbb{R}$ (the space of functions in $H^1_{\text{loc}}(\mathbb{R}^n)$ which are $Y$-periodic and have mean value zero on $Y$) are the unique weak solutions of the cell problems

$$-\nabla_y \cdot \left( a \left( \nabla_y w_i + e_i \right) \right) = 0 \quad \text{in } Y,$$

$$w_i \text{ is } Y\text{-periodic and } \int_Y w_i dy = 0.$$

There is a huge literature on different methods in the homogenization theory dealing with the derivation of macroscopic models, see for example Tartar’s method of oscillating test-functions [10], the two-scale convergence [1, 8], or the unfolding method [4]. Another important approach for homogenization, especially for nonlinear problems, using variational methods is the $\Gamma$-convergence, see [3, 7]. This method is closely related to the method used in the book under review. We illustrate the methodology with the above example: The problem (1) is equivalent to the minimization of the functional

$$I_\epsilon(v) := \int_\Omega a \left( \frac{x}{\epsilon} \right) \nabla v \cdot \nabla v - f v dx.$$
over the space $H^1_0(\Omega)$. Let us define the functional $I_0 : H^1_0(\Omega) \to \mathbb{R}$ by

$$I_0(v) := \int_\Omega g_{\text{hom}}(\nabla v) - f v \, dx,$$

where for $\xi \in \mathbb{R}^n$ the function $g_{\text{hom}}$ is given by

$$g_{\text{hom}}(\xi) := \inf \left\{ \int_Y a(y)(\xi + \nabla \phi) \cdot (\xi + \nabla \phi) \, dy : \phi \in H^1_{\text{per}}(Y) \right\}.$$

We extend the functionals $I_\epsilon$ and $I_0$ to the space $L^2(\Omega)$ by setting $I_\epsilon (v) = I_0 (v) = \infty$ for $v \in L^2(\Omega) \setminus H^1_0(\Omega)$. Now, based on the methods of $\Gamma$-convergence, see [3], it is possible to show the following results ($\Gamma$-convergence with respect to the $L^2(\Omega)$-norm)

(i) (Lower bound) For every sequence $(v_\epsilon)_{\epsilon > 0}$ with $v_\epsilon \to v$ in $L^2(\Omega)$ it holds that

$$\liminf_{\epsilon \to 0} I_\epsilon (v_\epsilon) \geq I_0 (v).$$

(ii) (Recovery sequence) For every $v \in L^2(\Omega)$ there exists a sequence $v_\epsilon \in L^2(\Omega)$ with $v_\epsilon \to v$ in $L^2(\Omega)$ such that

$$\lim_{\epsilon \to 0} I_\epsilon (v_\epsilon) = I_0 (v).$$

An elementary calculation shows that $g_{\text{hom}}(\xi) = a^* \xi \cdot \xi$. In fact, the minimization problem in $g_{\text{hom}}$ for $\xi = e_i$ is equivalent to the cell problem (3). Hence, $I_0$ is the associated energy for the macroscopic problem (2). A crucial point in the homogenization theory is the identification and characterization of the homogenized coefficient $a^*$ respectively the homogenized integrand $g_{\text{hom}}$, which also constitutes an important aspect in the book of Khruslov.

Let us describe in more detail the underlying microscopic model in the book under review: Let $\Omega \subset \mathbb{R}^3$ be a fixed and bounded domain, filled with a viscous incompressible fluid with $N_\epsilon = O\left(\frac{\Omega}{\epsilon^3}\right)$ suspended small particles. Here, the parameter $0 < \epsilon \ll 1$ describes the diameter and the distance between nearest neighboring particles. More precisely, $\epsilon$ is defined as the mean distance between the centers of mass of the $i$-th particle of the nearest particle in the initial state. The diameter $d^i_\epsilon$ of a particle depends on an additional parameter $\alpha \in [1, 3]$ such that $d^i_\epsilon = O(\epsilon^\alpha)$.

The domain occupied by the $i$-th particle for $i \in \{1, \ldots, N_\epsilon\}$ at time $t \in [0, T]$ is denoted by $Q^i_\epsilon(t)$, and the domain occupied by the fluid at time $t$ is given by

$$\Omega_\epsilon(t) = \Omega \setminus \bigcup_{i \in \{1, \ldots, N_\epsilon\}} Q^i_\epsilon(t).$$

Now, the non-cylindrical domain of the $i$-th particle and the time-varying fluid domain are defined by

$$Q^i_{\epsilon T} := \bigcup_{t \in [0, T]} Q^i_\epsilon(t) \times \{t\}, \quad \Omega_{\epsilon T} := (\Omega \times [0, T]) \setminus \bigcup_{i \in \{1, \ldots, N_\epsilon\}} Q^i_{\epsilon T}.$$
The evolution of the fluid is described by the incompressible Navier-Stokes equations
\[
\rho_f \left( \partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon \right) - \mu \nabla^2 u_\epsilon + \nabla p_\epsilon = \rho_f f_\epsilon(x, t) \quad \text{in } \Omega_\epsilon T, \quad (4a)
\]
\[
\nabla \cdot u_\epsilon = 0 \quad \text{in } \Omega_\epsilon T. \quad (4b)
\]

On the boundaries of the moving particles a stick condition is assumed:
\[
u(x, t) = u_i^t(t) + \omega_i^t(t) \times (x - x_i^t(t)) \quad \text{on } \partial Q_i^\epsilon T, \quad (4c)
\]
with the center of mass \(x_i^t\) of the \(i\)-th particle, the velocity vector of the mass center \(u_i^t\), and the instantaneous angular velocity \(\omega_i^t\). The motion of the \(i\)-th particle is described by a system of ordinary differential equations depending on external forces and stresses induced by the fluid. The system is closed with suitable initial and boundary conditions on \(\partial \Omega\), which are not the main focus of the monograph. For existence of a generalized solution for the microscopic problem for fixed \(\epsilon\) the author refers to [6], where a similar problem is considered. However, existence can only be expected locally in time for a small time interval \((0, T_\epsilon)\), until the first collision of particles or with the boundary \(\partial \Omega\), and \(T_\epsilon\) may vanish for \(\epsilon \to 0\).

The numerical treatment of the model (4a)–(4c) leads to high challenges, due to the different scalings like the size of the domain \(\Omega\) and the small diameter, and simulations in appropriate time scales are almost impossible. A possibility to overcome this problem is the derivation of so called macroscopic models for \(\epsilon \to 0\) with homogenized (effective) coefficients, the solution of which approximates the solution of the microscopic model for fixed \(\epsilon\). The main part of the monograph is the rigorous derivation of such macro models for various relations between the given data, for example \(\epsilon, d_\epsilon,\) and \(f_\epsilon\). However, the treatment of the full time-dependent problem is rather complicated and perhaps impossible to solve. In fact, due to possible collisions (which can be expected in practical applications even after short times), the maximal existence time \(T_\epsilon\) for a solution in general tends to zero for \(\epsilon \to 0\). Furthermore, in contrast to most results in homogenization theory, the evolution of the fluid domain \(\Omega_\epsilon(t)\) is an unknown of the system and leads to great challenges. Hence, the author uses the so called method of particle fixation to simplify the model. Thereby it is assumed that the evolution of the fluid domain and the particles is known, i.e., the position of the particles, and therefore the varying fluid domain \(\Omega_\epsilon(t)\), is known for any time \(t \in [0, T]\). Of course, this is a quite strong simplification, but at the moment there seems to be no rigorous mathematical framework for a homogenization of the full model (4a)–(4c) with the unknown evolution of \(\Omega_\epsilon(t)\). The method of fixing particle positions leads to a system of stationary Stokes equations with suitable boundary conditions on the surface of the particles. This is obtained by introducing a new force (assuming existence and regularity for the microscopic solution) \(\tilde{f}_\epsilon\) including the material derivative from the Navier-Stokes equations. The asymptotic analysis for these stationary Stokes systems for \(\epsilon \to 0\) and the derivation of a macroscopic model is a main part of the book, and two geometrical settings for the particles are considered:

- The frozen particle mode for \(d_\epsilon^i = O(\epsilon^\alpha)\) with \(\alpha \in [1, 3)\). Here, asymptotically the velocities of the particles coincide with the mean velocity of the fluid and the
mixture can be considered as a single-phase medium. The microscopic model can be formulated as a variational problem: For fixed \( t \in [0, T] \) define the space
\[
\mathcal{J}_\varepsilon(\Omega) := \{ v_\varepsilon \in H_0^1(\Omega)^3 : \nabla \cdot v_\varepsilon = 0, \quad v_\varepsilon(x) = a_i^\varepsilon + b_i^\varepsilon \times (x - x_i^\varepsilon(t)) \text{ on } Q_i^\varepsilon(t) \text{ for some } a_i^\varepsilon, b_i^\varepsilon \in \mathbb{R}^3 \}.
\]
Then, we are looking for a solution \( u_\varepsilon \in \mathcal{J}_\varepsilon(\Omega) \) of the minimization problem
\[
\min_{v_\varepsilon \in \mathcal{J}_\varepsilon(\Omega)} \int_\Omega \mu |e[v_\varepsilon]|^2 + \tilde{f}_\varepsilon \cdot v_\varepsilon \, dx + \sum_{i=1}^{N_\varepsilon} M_i^\varepsilon \cdot b_i^\varepsilon, \quad (5)
\]
with the symmetric gradient \( e[v_\varepsilon] = \frac{1}{2} (\nabla v_\varepsilon + \nabla v_\varepsilon^T) \) and external moments \( M_i^\varepsilon \).

- The filtering particle mode for \( d_\varepsilon^i = O(\varepsilon^3) \) and \( \rho_s^\varepsilon = O(\varepsilon^{-6}) \), when particle velocities differ significantly from the mean fluid velocity. In this critical case \( \alpha = 3 \) asymptotically the suspension behaves like a multiphase flow with interacting fluids. In this case, it is assumed that beside the centers of mass \( x_i^\varepsilon(t) \) of the \( i \)-th particle at time \( t \), also its velocity \( u_i^\varepsilon(t) \) and angular velocity \( \omega_i^\varepsilon(t) \) is a priori known.

An additional problem considered in the frozen particle mode is the motion of electrified or magnetized particles in a very strong magnetic or electric field, as well as particles interacting with each other through forces (e.g., electrostatic, elastic, Van der Waals) in both cases (filter and frozen particle mode). The case \( \alpha > 3 \) is not considered in detail, as the particles then have almost no influence on the fluid.

The principal idea for the derivation of the macroscopic model in the limit \( \varepsilon \to 0 \) for the problem (5), respectively the problem for the filtering particle mode, is to use a direct approach from the calculus of variations. This could also be stated in the formal language of \( \Gamma \)-convergence, see [7], but the author makes no relation to this method. Perhaps, a formulation in a more abstract framework offers additional opportunities for the treatment of other related problems. Let us exemplarily consider in more detail the homogenization of the problem (5), since the other cases are treated in a similar way. The aim is to show that, under suitable assumptions on the data and the perforated domain \( \Omega_\varepsilon(t) \) for fixed \( t \), the sequence \( u_\varepsilon \) of minimizers of (5), which we denote by \( \Phi_\varepsilon \), converges to a divergence free limit function \( u \), which minimizes the functional
\[
\Phi[v] := \int_\Omega a(x)e[v] : e[v] - \tilde{f} \cdot v \, dx,
\]
where \( : \) denotes the Frobenius inner product on \( \mathbb{R}^{3 \times 3} \), \( \tilde{f} \) is the weak limit of \( \tilde{f}_\varepsilon \), and the fourth order tensor \( a = (a_{n,p,q,r})^{3}_{n,p,q,r=1} \) is a homogenized limit tensor, the characterization of which is an essential part of the book. The proof consists of two main steps:
(A) For a given smooth divergence free vector function $w$ a sequence of test functions $w_{\epsilon h} \in J_{\epsilon}(\Omega)$ (depending on an additional parameter $h$) is constructed with

$$\lim_{h \to 0} \limsup_{\epsilon \to 0} \Phi_{\epsilon}[w_{\epsilon h}] \leq \Phi[w].$$

(B) The limit function $u \in W^{1,2}_{0}(\Omega)^{3}$ of the minimizers $u_{\epsilon}$ fulfills

$$\liminf_{\epsilon \to 0} \Phi_{\epsilon}[u_{\epsilon}] \geq \Phi[u].$$

The construction of the approximating sequence in (A) (some kind of recovery sequence in the sense of $\Gamma$-convergence) is a quite technical procedure, to guarantee that the sequence belongs to $J_{\epsilon}(\Omega)$. The techniques for the construction of such test-functions is a strong benefit of the book and provides important methods for problems in the homogenization theory, especially in continuum mechanics. For the treatment of (B) a so called mean tensor of suspension viscosity

$$a(\epsilon, h, \xi) = (a(\epsilon, h, \xi)_{npqr})^{3}_{n,p,q,r=1} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$$

is introduced via a minimization problem over small cubes $K(\xi, h)$ with center $\xi$ and side length $h$, such that $\epsilon \ll h \ll 1$. Hence, the cubes are small compared to the whole domain $\Omega$, but large compared to the size of the particles and include many of them. In other words, with the parameter $h$ an additional mesoscale is introduced, and the mean tensor of suspension $a(\epsilon, h, \xi)$ gives a mesoscopic characteristic of the mixture. Now, under the strong assumption that $\lim_{h \to 0} \lim_{\epsilon \to 0} a(\epsilon, h, x) = a(x)$ for all $x \in \Omega$ and $a$ is continuous, it is possible to establish (B).

Altogether, the minimizer of $\Phi$ gives an approximation of the minimizer of $\Phi_{\epsilon}$ for every fixed time $t \in [0, T]$, if the evolution of the particles is known. Hence, the homogenized system describes the motion of a carrier fluid perturbed by particles moving in it. In this sense, the macroscopic system is not closed because it does not capture the influence of the fluid on the particles, and it includes unknown coefficients (the limit tensor $a$) which are depending on the dynamics of the particles and their properties. The closure of the homogenized system is the second main topic of the book. In a first step, the suspension viscosity tensor $a$ is derived for some specific cases:

- The particles are distributed locally periodic, leading to an expression of $a$ in terms of solutions of cell problems.
- Particles with an axisymmetric shape and arbitrary position, but diameters much less than the distances between neighboring particles ($= \text{low concentration}$).
- Formula for the mean value of the viscosity tensor $a$ for random distribution of diameters and orientations of particles.

Finally, the homogenized system is closed by the derivation of evolution equations for the mean orientation vector of

- axisymmetric particles in the frozen particle mode,
- spherical particles in filtration mode.
For this, the forces and moments acting on a single particle moving in an arbitrary linear unperturbed fluid flow (later the macroscopic fluid flow) are calculated. Combining all these steps, a closed macroscopic model with homogenized coefficients is obtained. Since the closed system is not obtained directly by homogenization from the microscopic model, existence of a weak solution is not guaranteed. This is obtained by some additional assumptions and simplifications on the homogenized coefficients.

An additional subject of the book is the treatment of “complex fluids”, where the particles are interacting with each other through different forces, in both cases, the frozen and the filtering particle mode. For the homogenization a reduced (linearized) model of (4a)–(4c) with some additional interaction forces between the particles is considered, where especially the microscopic domain is not evolving. In the frozen particle mode, the macroscopic system corresponds to an incompressible viscoelastic medium model including homogenized coefficients, whereas in the filtering particle mode it is shown for spherical particles, that the homogenized system is a two-phase model which can be interpreted as a generalized Brinkman law.

In summary, the monograph deals with an important problem for applications and is also highly interesting from a mathematical point of view. Starting from a general microscopic model, which takes into account the whole complexity of the system, especially the unknown dynamics of the particles, closed macroscopic models are derived by using the method of fixing particles. This seems to be necessary to overcome the difficulties arising due to the evolution of the domain. Nevertheless, the results are still highly nontrivial and state of the art. The monograph is appropriate for readers who are experts in the fields of homogenization and partial differential equations, especially in fluid dynamics. The proofs include methods from the existence and regularity theory for Stokes and Navier-Stokes equations (also for fluids with moving rigid bodies), variational methods, operator and spectral theory, as well as functional analysis. There is no introduction to these methods and the proofs including existing theory from the literature are quite short without many details, which makes it hard to read also for graduated students.

It would be highly interesting to include the results in more general frameworks, like the $\Gamma$-convergence, the method of two-scale convergence [1], or the periodic unfolding method [4]. The latter two are developed for locally periodic or stochastic problems. However, for some of the results in the book these methods could be applicable, for example under the assumption of the existence of a transformation to a periodic domain, which should be closely related to the assumption that the suspension velocity tensor $a(\epsilon, h, x)$ converges to a continuous tensor $a$. A combination of such results could be beneficial for other applications in this field, for example considering elastic particles instead of rigid ones, leading to a fluid structure interaction between fluid and deformable solids, where it is necessary to take into account elasticity equations.

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