On the Martingale Representation with Respect to the super-Brownian Filtration

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Abstract

We derive the explicit form of the martingale representation for square-integrable processes that are martingales with respect to the natural filtration of the super-Brownian motion. This is done by using a weak extension of the Dupire derivative for functionals of superprocesses.

Introduction

Dupire’s landmark work on the functional Itô-formula [Dupire, 2009] gave rise to a completely new approach to numerous questions in the field of stochastic calculus. In this paper, we use it to derive a martingale representation of functionals of the super-Brownian motion, a well-studied, infinite-dimensional, namely measure-valued, stochastic process.

Cont and Fournie (Cont and Fournie, 2010, Cont and Fournie, 2013, Cont, 2016) as well as Levental et al. (Levental et al., 2013) used different formalizations of Dupire’s idea to derive versions of a functional Itô-formula for real-valued semi-martingales using different derivatives. In addition, Cont used the functional Itô-formula to derive a martingale representation formula for Brownian martingales and extended the functional derivative to a weak derivative for square-integrable functionals (see Cont and Fournie, 2013, Cont, 2016).

In Mandler and Overbeck, 2021 the functional Itô-formula for $B(A, c)$-superprocesses is derived using derivatives extending the ones introduced by Cont and Fournie to functionals on the infinite-dimensional space of measures. In this paper, we extend the vertical derivative to a weak one and use it to derive
a martingale representation for functionals of the super-Brownian motion.

The martingale representation of functionals of superprocesses has been studied in [Overbeck, 1995], [Evans and Perkins, 1994] and [Evans and Perkins, 1995], where the authors proved the existence of a martingale representation and derive the explicit form for the richer class of so-called historical processes using their cluster representation, from which one can derive the representation for superprocesses as a projection. The approach presented in Section 2 is based on the ideas presented by Cont and Fournié in [Cont and Fournié, 2013] and [Cont, 2016], does not use the concept of historical processes and only considers the super-Brownian motion as a prototype of more general measure-valued diffusions.

1 The setting

Consider a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\), where \(\mathbb{P}\) is the law of the super-Brownian motion on \(\mathbb{R}^d\), \(d \geq 1\), starting at \(m \in M_F(\mathbb{R}^d)\), \(\mathcal{F}\) is the Borel-\(\sigma\)-field corresponding to \(\Omega = C([0,T], M_F(\mathbb{R}^d))\) and \((\mathcal{F}_t)_{t \in [0,T]}\) is the canonical filtration satisfying the usual conditions. Also, denote the Laplacian operator by \(\Delta\). From [Dawson, 1993], we know that the probability measure \(\mathbb{P}\) is a solution to the following martingale problem:

\[
\mathbb{P}(X(0) = m) = 1 \text{ and for all } \phi \in D\left(\frac{1}{2}\Delta\right) \text{ the process } \begin{align*}
M(t)(\phi) &= \langle X(t), \phi \rangle - \langle X(0), \phi \rangle - \int_0^t \langle X(s), \frac{1}{2}\Delta\phi \rangle ds, \quad t \in [0,T] \\
\text{is a } (\mathcal{F}_t)_t\text{-martingale } \mathbb{P}\text{-a.s. and has quadratic variation} \\
[M(\phi)]_t &= \int_0^t \langle X(s), \phi^2 \rangle ds \quad \forall t \in [0,T] \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]

As the super-Brownian motion is a \(B(A,c)\)-superprocess, the process \(M(\phi)\) in the martingale problem gives rise to a martingale measure in the sense of [Walsh, 1986] which we denote by \(M_X\). This martingale measure is defined on \(\Omega \times B[0,T] \times \mathcal{E}\) and plays a crucial role in the Itô-formulas presented in [Mandler and Overbeck, 2021] as well as the results introduced below.

Denote by \(S(\mathbb{R}^d)\) the Schwartz space on \(\mathbb{R}^d\). Functions in \(S(\mathbb{R}^d)\) are infinitely continuously differentiable. Thus \(S(\mathbb{R}^d) \subset D(\frac{1}{4}\Delta)\) and for any \(h \in S(\mathbb{R}^d)\) it holds \(\frac{1}{2}\Delta h \in D(\frac{1}{4}\Delta)\).

In addition, let \(\mathcal{S}\) be the space of simple functions, i.e. the space of functions that are linear combinations of functions \(f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}\) of form

\[
f(\omega, s, x) = X(\omega)1_{(a,b]}(s)1_A(x)
\]

with \(0 \leq a < b \leq t\), \(X\) bounded and \(\mathcal{F}_a\)-measurable and \(A \in \mathcal{B}(\mathbb{R}^d)\). Denote by \(\mathcal{P}\) the predictable \(\sigma\)-field on \(\Omega \times [0,T] \times \mathbb{R}^d\), i.e. the \(\sigma\)-field generated by \(\mathcal{S}\). If
a function is $\mathcal{P}$-measurable, we say it is predictable.

In Section 2 we consider functions $F$ of the path of the super-Brownian motion $X$. More precisely, we consider the stopped path $X_t$ of the process $X$. To formalize the notion of (stopped) paths, consider an arbitrary path $\omega \in D([0, T], M_F(\mathbb{R}^d))$, the space of right continuous functions with left limits. We equip $D([0, T], M_F(\mathbb{R}^d))$ with the metric $d$ given by

$$
d(\omega, \omega') = \sup_{u \in [0, T]} d_P(\omega(u), \omega'(u))$$

for $\omega, \omega' \in D([0, T], M_F(\mathbb{R}^d))$, where $d_P$ is the Prokhorov metric on $M_F(\mathbb{R}^d)$. For such a path, define the path stopped at time $t$ by $\omega_t(u) = \omega_t(u \wedge t)$. Using this, we define an equivalence relation on the space $[0, T] \times D([0, T], M_F(\mathbb{R}^d))$ by

$$(t, \omega) \sim (t', \omega') \iff t = t' \text{ and } \omega_t = \omega'_t.$$ 

This relation gives rise to the quotient space

$$\Lambda_T = \{(t, \omega_t) \mid (t, \omega) \in [0, T] \times D([0, T], M_F(\mathbb{R}^d))\}$$

$$= ([0, T] \times D([0, T], M_F(\mathbb{R}^d)))/\sim.$$ 

Next, define a metric $d_\infty$ on $\Lambda_T$ by

$$d_\infty((t, \omega), (t', \omega')) = \sup_{u \in [0, T]} d_P(\omega(u \wedge t), \omega'(u \wedge t')) + |t - t'|.$$ 

A functional $F : \Lambda_T \to \mathbb{R}$ is continuous with respect to $d_\infty$ if for all $(t, \omega) \in \Lambda_T$ and every $\varepsilon > 0$ there exists an $\eta > 0$ such that for all $(t', \omega') \in \Lambda_T$ with $d_\infty((t, \omega), (t', \omega')) < \eta$ we have

$$|F(t, \omega) - F(t', \omega')| < \varepsilon.$$ 

Further, a functional $F$ on $[0, T] \times D([0, T], M_F(\mathbb{R}^d))$ is called non-anticipative if it is a measurable map on the space of stopped paths, i.e. $F : \Lambda_T \to \mathbb{R}$. In other words, $F$ is non-anticipative if $F(t, \omega) = F(t, \omega_t)$ holds for all $\omega \in D((0, T], M_F(\mathbb{R}^d))$.

For continuous non-anticipative functionals, we can define the following derivative. For this, denote by $\delta_x$, $x \in \mathbb{R}^d$, the Dirac measure with unit mass at $x$.

**Definition 1.** A continuous non-anticipative functional $F : \Lambda_T \to \mathbb{R}$ is called vertically differentiable at $(t, \omega) \in \Lambda_T$ in direction $\delta_x$, $x \in \mathbb{R}^d$, if the limit

$$\mathcal{D}_x F(t, \omega) = \lim_{\varepsilon \to 0} \frac{F(t, \omega_t + \varepsilon \delta_x 1_{[t, T]}) - F(t, \omega_t)}{\varepsilon}$$

exists. If this is the case for all $(t, \omega) \in \Lambda_T$, we call $\mathcal{D}_x F$ the vertical derivative of $F$ in direction $\delta_x$. Higher order vertical derivatives are defined iteratively.
2 The Explicit Form of the Martingale Representation for the Super-Brownian Motion

Let $L^2(M_X)$ be the space of predictable functions $\phi : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that
\[ \|\phi\|^2_{L^2(M_X)} = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \phi^2(s, x) X(s)(dx)ds \right] < \infty, \]
$M^2$ be the space of square-integrable $(\mathcal{F}_t)_t$-martingales with initial value zero and with norm
\[ \|Y\|^2_{M^2} = \mathbb{E}[Y(T)^2]. \]
Further define the space $U$ which is the linear span of functions of form
\[ \phi_{\Gamma, a, h}(\omega, t, x) = \Gamma(\omega) \cdot h(x)1_{(a,T]}(t) \]
with $\Gamma$ bounded, $a \in [0, T)$ and $\mathcal{F}_a$-measurable and $h \in \mathcal{S}(\mathbb{R}^d)$. As functions in $U$ can be expressed as pointwise limits of functions in $\mathcal{S}$, these functions are predictable.

**Proposition 1.** The space $U$ is a dense subset of $L^2(M_X)$.

**Proof.** From the above, we already know that functions $\phi \in U$ are predictable. Hence to show that $U$ is indeed a subspace of $L^2(M_X)$, we have to prove that $\|\phi\|^2 < \infty$ holds for all $\phi \in U$.

It is enough to show that $\|\phi_{\Gamma, a, h}\|^2_{L^2(M_X)} < \infty$ for all functions $\phi_{\Gamma, a, h}$ generating $U$. Assuming that the bounds of $\Gamma^2$ and $h^2$ are given by $C_{\Gamma^2}$ and $C_{h^2}$, respectively, we get
\[
\|\phi_{\Gamma, a, h}\|^2_{L^2(M_X)} = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} (\Gamma h(x)1_{(a,T]}(s))^2 X(s)(dx)ds \right]
\leq C_{\Gamma^2} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} h^2(x)1_{(a,T]}(s) X(s)(dx)ds \right]
\leq C_{\Gamma^2} C_{h^2} \mathbb{E} \left[ \int_a^T X(s)(\mathbb{R}^d)ds \right]
\leq C_{\Gamma^2} C_{h^2} (T - a) \mathbb{E} \left[ \max_{t \in [a, T]} X(t)(\mathbb{R}^d) \right].
\]

As the total mass of $X$ is a critical Feller continuous state branching process, we know that
\[ \mathbb{E}[X(t)(\mathbb{R}^d)] = X(0)(\mathbb{R}^d) < \infty, \]
\[ \mathbb{E}[X(t)(\mathbb{R}^d)] = X(0)(\mathbb{R}^d) < \infty, \]
\[ \text{This space coincides with the space } P_M \text{ in } [\text{Walsh, 1986}]. \]
Therefore the stochastic integral with respect to the martingale measure $M_X$ is well defined for all $\phi \in L^2(M_X)$.
holds for all $t \in [0,T]$, which yields that $\phi_{t,a,b}$ has finite $\| \cdot \|_{L^2(M_X)}$-norm. Consequently $\phi \in L^2(M_X)$ and thus $U \subset L^2(M_X)$.

To obtain the density of $U$ in $L^2(M_X)$, note that $S$ is dense in $L^2(M_X)$, i.e. $\bar{S} = L^2(M_X)$ holds. Since the closures of $S$ and $U$ coincide, we obtain $\bar{U} = \bar{S} = L^2(M_X)$ and thus that $U$ is dense in $L^2(M_X)$.

In the following, we derive the martingale representation formula for all square-integrable $(\mathcal{F}_t)$-martingales. We do so by extending the notion of vertical derivatives to obtain an operator $\nabla_M$ on $M^2$ which plays a crucial role in the martingale representation. To get the general representation formula, we start by deriving the formula and defining the operator $\nabla_M$ for martingales in a subset of $M^2$. As the considered subspace is dense in $M^2$, we can then extend the operator as well as the martingale representation formula to all $Y \in M^2$ and thus obtain a martingale representation for all square-integrable $(\mathcal{F}_t)$-martingales.

**Definition 2.** A linear operator $\Pi$ mapping from its domain $D(\Pi)$ into a Hilbert space $H$ is called an extension of the linear operator $\tilde{\Pi} : D(\tilde{\Pi}) \to H$ if $D(\tilde{\Pi}) \subset D(\Pi)$ and $\tilde{\Pi}v = \Pi v$ for all $v \in D(\tilde{\Pi})$.

To construct the just mentioned subspace of $M^2$, we consider the stochastic integral $I_{M_X}(\phi)$ of a function $\phi \in L^2(M_X)$ with respect to the martingale measure $M_X$ corresponding to $X$. This allows us to define the set

$$I_{M_X}(U) = \{Y \mid Y(t) = \int_0^t \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx), \; \phi \in U, \; t \in [0,T]\}.$$ 

To prove that this is indeed a subset of $M^2$, we need parts of the proof of the following result.

**Proposition 2.** The mapping

$$I_{M_X} : \mathcal{L}^2(M_X) \rightarrow M^2$$

$$\phi \mapsto \int_0^t \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx)$$

is an isometry.

**Proof.** Let $Q_{M_X}$ be the covariation of $M_X$ and $\phi, \psi \in \mathcal{L}^2(M_X)$. Then, by adapting the proof of Theorem 2.5 in Walsh, 1986 to our setting, we get that

$$\int_0^t \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx) \int_0^t \int_{\mathbb{R}^d} \psi(s,x)M_X(ds,dx)$$

$$- \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s,x)\psi(s,y)Q_{M_X}(ds,dx,dy)$$
is a martingale for all \( \phi, \psi \in \mathcal{L}^2(M_X) \) and thus
\[
E \left[ \int_0^t \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx) \int_0^t \int_{\mathbb{R}^d} \psi(s,x)M_X(ds,dx) \right] = E \left[ \int_0^t \int_{\mathbb{R}^d} \phi(s,x)\psi(s,y)Q_{M_X}(ds,dx) \right].
\]

In the scenario we consider \( M_X \) is orthogonal. For orthogonal martingale measures, the covariation \( Q_{M_X} \) simplifies to \( Q(\{0,t\} \times B \times B) = \nu(\{0,t\} \times B) \) with \( \nu \) being a measure on \( \mathbb{R}^d \times [0,T] \). From Example 7.1.3 in [Dawson, 1993] we know that \( \nu \) has the following form if \( M_X \) is the martingale measure associated with the \( B(A,c) \)-superprocess:
\[
\nu(ds,dx) = cX(s)(dx)ds.
\]
As \( c = 1 \) in our setting, this yields
\[
E \left[ \int_0^t \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx) \int_0^t \int_{\mathbb{R}^d} \psi(s,x)M_X(ds,dx) \right] = E \left[ \int_0^t \int_{\mathbb{R}^d} \phi(s,x)\psi(s,x)X(s)(dx)ds \right].
\]
for all \( t \in [0,T] \) and therefore
\[
\|I_{M_X}(\phi)\|^2_{M^2} = E \left[ \left( \int_0^T \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx) \right)^2 \right] = E \left[ \int_0^T \int_{\mathbb{R}^d} \phi^2(s,x)X(s)(dx)ds \right] = \|\phi\|^2_{L^2(M_X)}.
\]

For any function in \( \phi = \phi_{\Gamma,a,h} \in U \) we have
\[
I_{M_X}(\phi)(t) = \int_0^t \int_{\mathbb{R}^d} \phi_{\Gamma,a,h}(s,x)M_X(ds,dx)
= \int_0^t \int_{\mathbb{R}^d} \Gamma \cdot h(x)1_{(a,T]}(s)M_X(ds,dx)
= \Gamma \cdot (M(t)(h) - M(a)(h))1_{t>a}
= \Gamma \cdot \left( \langle X(t), h \rangle - \langle X(a), h \rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2} \Delta h(y)X(s)(dy)ds \right)1_{t>a}.
\]
From the martingale problem defining the super-Brownian motion, we get that \((M(t)(h))_{t \in [0,T]}\) is a martingale as \( h \in \mathcal{S}(\mathbb{R}^d) \subset D(\frac{1}{2}\Delta) \) and thus \((I_{M_X}(\phi)(t))_{t \in [0,T]}\)
is also a martingale for any \( \phi \in U \). The process is also square-integrable as we have for all \( t \in [0,T] \)

\[
\mathbb{E}[|I_{M_X}(\phi)(t)|^2] = \mathbb{E}
\left[
\left(\int_0^t \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx)\right)^2
\right]
\]

\[
= \mathbb{E}
\left[
\int_0^t \int_{\mathbb{R}^d} \phi^2(s,x)X(s)(dx)ds
\right]
\]

\[
\leq \mathbb{E}
\left[
\int_0^T \int_{\mathbb{R}^d} \phi^2(s,x)X(s)(dx)ds
\right]
\]

\[
< \infty,
\]

since \( \phi \in \mathcal{L}^2(M_X) \). Hence \( I_{M_X}(\phi) \) is square-integrable and thus \( I_{M_X}(U) \subset \mathcal{M}^2 \).

Next, consider the function \( F \) of form

\[
F : \Lambda_T \rightarrow \mathbb{R}
\]

\[
(t,\omega) \rightarrow \Gamma(\omega)
\left(\langle \omega(t), h \rangle - \langle \omega(a), h \rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2} \Delta h(y)\omega(s)(dy)ds\right)_{t>a}
\]

Plugging \( X_t \) into \( F \) for \( \omega \) yields

\[
F(t,X_t) = \Gamma(X_t)
\left(\langle X_t(t), h \rangle - \langle X_t(a), h \rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2} \Delta h(x)X_t(s)(dx)ds\right)_{t>a}
\]

\[
= \Gamma(X_t)
\left(\langle X(t), h \rangle - \langle X(a), h \rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2} \Delta h(x)X(s)(dx)ds\right)_{t>a}
\]

and, as \( X_t(\omega) = \omega_t \) and \( X(t)(\omega) = \omega(t) \), we get

\[
F(t,X_t)(\omega) = \Gamma(\omega_a)
\left(\langle \omega(t), h \rangle - \langle \omega(a), h \rangle - \int_a^t \int_{\mathbb{R}^d} \langle \omega(s), \frac{1}{2} \Delta h(x)\omega(s)(dx)ds\right),
\]

from which we get, as \( \Gamma \) is \( \mathcal{F}_a \)-measurable, that \( F(t,X_t) = I_{M_X}(\phi_{T,a,h})(t) \).

As, in addition, for any path \( \omega \in C([0,T],\mathcal{M}_{\mathcal{P}}(E)) \), it holds

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}
\left(\langle (\omega + \epsilon \delta_x 1_{[t,T]})(t), h \rangle - \langle (\omega + \epsilon \delta_x 1_{[t,T]})(a), h \rangle
\right.

\[
- \int_a^t \int_{\mathbb{R}^d} \frac{1}{2} \Delta h(y)(\omega + \epsilon \delta_x 1_{[t,T]})(r)(dy)dr
\]

\[
- \langle \omega(t), h \rangle + \langle \omega(a), h \rangle + \int_a^t \int_{\mathbb{R}^d} \frac{1}{2} \Delta h(y)\omega(r)(dy)dr
\]

\[
= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}
\left(\epsilon h(x) - \epsilon \int_a^t \int_{\mathbb{R}^d} \frac{1}{2} \Delta h(y)1_{[t,T]}(r)\delta_x(dy)dr
\right)
\]

\[
= h(x)
\]
and, as $\Gamma$ is $\mathcal{F}_a$-measurable, for $t \in (a, T]$ it holds

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \Gamma(\omega + \epsilon \delta_x 1_{[t,T]}) - \Gamma(\omega) \right) = 0,$$

we obtain, for all $(t, \omega) \in \Lambda_T$,

$$\mathcal{D}_x F(t, \omega) = \Gamma(\omega) 1_{(a,T]}(t) h(x) = \phi_{\Gamma,a,h}(t, x).$$

Now, for a process $Y$ defined by

$$Y(t) = I_{M_X}(\phi_{\Gamma,a,h})(t) = F(t, X_t)$$

for $t \in [0, T]$, we can define an operator $\nabla_M$ of the form

$$\nabla_M : I_{M_X}(U) \to \mathcal{L}^2(M_X)$$

$$Y \mapsto \nabla_M Y,$$

(3)

where $\nabla_M Y$ is given by

$$\nabla_M Y : (\omega, t, x) \mapsto \nabla_M Y(\omega, t, x) := \mathcal{D}_x F(t, X_t(\omega)) = \mathcal{D}_x F(t, \theta)|_{\theta=X_t(\omega)}.$$

Further, by the definition of $Y$, we get the martingale representation

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} \nabla_M Y(s, x) M_X(ds, dx),$$

(4)

which is equal to

$$F(t, X_t) = \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_x F(s, X_s) M_X(ds, dx).$$

Thus, we derived a martingale representation formula for martingales in the subspace $I_{M_X}(U)$ of $\mathcal{M}^2$ using the operator $\nabla_M$ defined by (3) on $I_{M_X}(U)$.

In [Evans and Perkins, 1994] the authors proved that, if $Y$ belongs to $\mathcal{M}^2$, there exists a unique $\rho \in \mathcal{L}^2(M_X)$ such that

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} \rho(s, x) M_X(ds, dx) \quad \forall t \geq 0$$

(5)

holds $\mathbb{P}$-a.s.. Consequently, the representation in (4) is unique. Further, this yields that the mapping $I_{M_X}$ is bijective. These two properties allow us to extend $\nabla_M$ to all processes $Y \in \mathcal{M}^2$ by using the following result.

**Proposition 3.** The space $\{\nabla_M Y \mid Y \in I_{M_X}(U)\}$ is dense in $\mathcal{L}^2(M_X)$ and the space $I_{M_X}(U)$ is dense in $\mathcal{M}^2$.

**Proof.** As $U$ is dense in $\mathcal{L}^2(M_X)$ (see Proposition 1),

$$U = \{\nabla_M Y \mid Y \in I_{M_X}(U)\} \subset \mathcal{L}^2(M_X)$$

yields that $\{\nabla_M Y \mid Y \in I_{M_X}(U)\}$ is dense in $\mathcal{L}^2(M_X)$. Further, as $I_{M_X}$ is a bijective isometry, we get the density of $I_{M_X}(U)$ in $\mathcal{M}^2$. 

$\square$
This density result allows to prove the following proposition that can be interpreted as an integration by parts formula on [0, T] \times \mathbb{R}^d \times \Omega.

**Proposition 4.** If \( Y \in I_{M^X}(U) \), then \( \nabla_M Y \) is the unique element in \( L^2(M^X) \) such that

\[
\mathbb{E}[Y(T)Z(T)] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) \nabla_M Z(s,x) X(s)(dx) ds \right]
\]

holds for all \( Z \in I_{M^X}(U) \).

*Proof.* From (2) and (4) we get that, for every \( Y, Z \in I_{M^X}(U) \),

\[
\mathbb{E}[Y(T)Z(T)] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) M^X(ds,dx) \int_0^T \int_{\mathbb{R}^d} \nabla_M Z(s,x) M^X(ds,dx) \right]
\]

\[
= \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) \nabla_M Z(s,x) X(s)(dx) ds \right]
\]

holds.

To prove the uniqueness, let \( \psi \in L^2(M^X) \) be another process such that for \( Z \in I_{M^X}(U) \) we have

\[
\mathbb{E}[Y(T)Z(T)] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \psi(s,x) \nabla_M Z(s,x) X(s)(dx) ds \right].
\]

Then, by subtraction

\[
0 = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} (\psi(s,x) - \nabla_M Y(s,x)) \nabla_M Z(s,x) X(s)(dx) ds \right]
\]

for all \( Z \in I_{M^X}(U) \), which yields the uniqueness of \( \nabla_M Y \) in \( L^2(M^X) \) as \( \{ \nabla_M Z \mid Z \in I_{M^X}(U) \} \) is dense in \( L^2(M^X) \).

The interpretation as an integration by parts formula becomes clear when we consider the following alternative form of equation (6), which holds for all \( \phi \in L^2(M^X) \):

\[
\mathbb{E}[Y(T) \int_0^T \int_{\mathbb{R}^d} \phi(s,x) M^X(ds,dx)] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) \phi(s,x) X(s)(dx) ds \right].
\]

Now, we have all the necessary results on hand to introduce the extension of \( \nabla_M \) to \( M^2 \).
Theorem 1. The operator defined in (3) can be extended to an operator
\[ \nabla_M : \mathcal{M}^2 \to \mathcal{L}^2(M_X) \]
\[ Y \mapsto \nabla Y. \]

This operator is a bijection and the unique continuous extension of the operator defined in (3) given by the following: For a given \( Y \in \mathcal{M}^2 \), \( \nabla_M Y \) is the unique element in \( \mathcal{L}^2(M_X) \) such that
\[
E[Y(T)Z(T)] = E \left[ \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) \nabla_M Z(s,x)X(s)(dx)ds \right] \tag{7}
\]
holds for all \( Z \in I_{M_X}(U) \).

Proof. As \( \nabla_M : I_{M_X}(U) \to \mathcal{L}^2(M_X) \) is a bounded linear operator, \( \mathcal{L}^2(M_X) \) is a Hilbert space and \( I_{M_X}(U) \) is dense in \( \mathcal{M}^2 \), the the BLT theorem (see e.g. Theorem 5.19 in Hunter and Nachtergaele, 2001) yields the existence of a unique continuous extension \( \nabla_M : \mathcal{M}^2 \to \mathcal{L}^2(M_X) \). To prove that (7) defines the extension, we have to prove that its restriction to \( I_{M_X}(U) \) is equal to the initial operator. As we immediately get this from Proposition 4, the unique continuous extension is given by (7).

For \( Y \in \mathcal{M}^2 \), there exists a unique \( \rho \in \mathcal{L}^2(M_X) \) such that (5) holds. This \( \rho \) satisfies (7) as we get from combining (5) with (4) and (2) that
\[
E[Y(T)Z(T)] = E \left[ \int_0^T \int_{\mathbb{R}^d} \rho(s,x)M_X(ds,dx)Z(T) \right] = E \left[ \int_0^T \int_{\mathbb{R}^d} \rho(s,x)\nabla_M Z(s,x)X(s)(dx)ds \right]
\]
holds for all \( Z \in I_{M_X}(U) \). The uniqueness of the integrand in (5) then yields that \( \rho \) and \( \nabla_M Y \) coincide in \( \mathcal{L}^2(M_X) \).

Next, assume \( Y, Y' \in \mathcal{M}^2 \) with \( \nabla_M Y = \nabla_M Y' \) for \( \nabla_M Y, \nabla_M Y' \in \mathcal{L}^2(M_X) \). Then, for all \( Z \in I_{M_X}(U) \),
\[
0 = E \left[ \int_0^T \int_{\mathbb{R}^d} (\nabla_M Y(s,x) - \nabla_M Y'(s,x))\nabla_M Z(s,x)X(s)(dx)ds \right] = E \left[ \left( \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x)M_X(ds,dx)ds - \int_0^T \int_{\mathbb{R}^d} \nabla_M Y'(s,x)M_X(ds,dx)ds \right) \nabla_M Z(T) \right] = E[(Y(T) - Y'(T))Z(T)],
\]
which implies that \( Y = Y' \) in \( \mathcal{M}^2 \) as \( I_{M_X}(U) \) is dense in \( \mathcal{M}^2 \). Therefore, the operator \( \nabla_M \) is injective. Further, as for every \( \phi \in \mathcal{L}^2(M_X) \) there exists the process
\[
Y = \int_0^T \int_{\mathbb{R}^d} \phi(s,x)M_X(ds,dx) \in \mathcal{M}^2
\]
for which $\nabla_M Y = \phi$ holds, the operator is also surjective and thus bijective.

The extension of $\nabla_M$ from $I_{M^X}(U)$ to $M^2$ now immediately yields the following martingale representation for square-integrable martingales.

**Theorem 2.** For any square-integrable $(\mathcal{F}_t)_{t}$-martingale $Y$ and every $t \in [0, T]$ it holds

$$Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d} \nabla_M Y(s, x)M_X(ds, dx) \quad \mathbb{P}\text{-a.s.}.$$  

**Proof.** First, assume $Y \in M^2$. Then there exists a unique $\rho \in \mathcal{L}^2(M^X)$ such that (5) holds and from the proof of Theorem 1 we get that this $\rho$ satisfies (7). Therefore, by the uniqueness of $\rho$ and the definition of $\nabla_M Y$, we have $\rho = \nabla_M Y$ in $\mathcal{L}^2(M^X)$ and thus

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} \nabla_M Y(s, x)M_X(ds, dx)$$

holds $\mathbb{P}\text{-a.s.}.$

Now, let $Y$ be a square-integrable $(\mathcal{F}_t)_{t}$-integral. Then $\hat{Y} = Y - Y(0) \in M^2$ and thus we can apply the above to $\hat{Y}$. Adding $Y(0)$ to both sides of the resulting representation for $\hat{Y}$ yields the desired martingale representation formula for $Y$.

We conclude this paper with the two following properties of the extended operator $\nabla_M$ defined on $M^2$, which are worth mentioning.

**Lemma 1.** The operator $\nabla_M$ defined on $M^2$ is an isometry and the adjoint operator of $I_{M^X}$, the stochastic integral with respect to the martingale measure $M^X$.

**Proof.** Let $Y \in M^2$. Then we get, using the same arguments as in Proposition 2

$$\|\nabla_M Y\|_{\mathcal{L}^2(M^X)}^2 = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} (\nabla_M Y(s, x))^2 X(s)(dx)ds \right]$$

$$= \mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s, x)M_X(ds, dx) \right)^2 \right]$$

$$= \left\| \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s, x)M_X(ds, dx) \right\|_{\mathcal{M}^2}^2$$

$$= \|Y\|_{\mathcal{M}^2}^2.$$
Further, let $\phi \in \mathcal{L}^2(M_X)$. As

$$\langle I_{M_X}(\phi), Y \rangle_{M^2} = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \phi(s, x)M_X(ds, dx) Y(T) \right]$$

$$= \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \phi(s, x)M_X(ds, dx) \int_0^T \nabla_M Y(s, x)M_X(ds, dx) \right]$$

$$= \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \phi(s, x)\nabla_M Y(s, x)X(s)(dx)ds \right]$$

$$= \langle \phi, \nabla_M Y \rangle_{\mathcal{L}^2(M_X)}$$

holds, we get that $\nabla_M$ is the adjoint operator of $I_{M_X}$. 

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