First order field redefinition of relativistic hydrodynamic theory gives causal propagating mode

Sukanya Mitra

Department of Nuclear and Atomic Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

In this work, a first order relativistic dissipative hydrodynamic theory has been developed that redefines all the concerned hydrodynamic fields in order to include the out of equilibrium system dissipation from the medium. The resulting dispersion relation from the theory is observed to produce a propagating causal mode that was previously absent without the field redefinition. The condition for causality has been explicitly determined for linear perturbations around a hydrostatic state. Finally, the causality has been tested with the field correction coefficients estimated from momentum dependent relaxation time approximation (MDRTA) in relativistic transport equation.

The journey of relativistic dissipative hydrodynamic theory can be traced back from the relativistic extension of the Navier-Stokes (N-S) formalism introduced by Landau-Lifshitz (LL) [1] and Eckart [2]. These theories are popularly known as first order theories because of the presence of first order gradient corrections in the equilibrium deviations. The problem occurs with these theories when they exhibit superluminal speed of signal propagation causing severe causality violation, which further associates instabilities within the system posing major concerns to the practical applicability of the theory. To rescue the situation, second order hyperbolic theories have been introduced like the one formulated by Israel-Stewart (IS) [3] to recently developed DNMR [4] and resummed BRSSS theory [5,6]. These theories promote the dissipative fluxes as the fundamental dynamical variables and give rise to relaxation type evolution equations. Proven to be free from causality and stability related issues at least for linear perturbations around equilibrium [3,5], they have been used for a wide range of hydrodynamic numerical simulations.

Recently, attempts have been made [10–14] to establish a causal and stable hydrodynamic theory without incorporating extra dynamical degrees of freedom other than the fundamental ones such as temperature, hydrodynamic velocity and charge chemical potential. In other words, they derive first order theories that prohibit superluminal signal propagation as well as retain stability criteria besides other requirements like non-negative entropy production, that are essential for an acceptable hydrodynamic theory. The basic idea is to define the out of equilibrium thermodynamic variables in a general frame other than specified either by Landau-Lifshitz or Eckart, through their postulated constitutive relations. Motivated from these studies, in this work, a first order theory has been developed, where the out of equilibrium thermodynamic fields are not uniquely defined and are subjected to include dissipative effects from the medium.

In the usual N-S theory, the resulting dispersion relation from the conservation equations, at the limit of large wave number (k) gives rise to non propagating modes with k² dependence [6,7], identical to that of diffusion process which is acausal with an infinite propagation speed. This behaviour of the non propagating modes are considered to be the origin of acausality in N-S theory. However, the macroscopic thermodynamic quantities such as energy density and particle number density in these equations are usually set to their equilibrium values even in the dissipative medium by imposing certain matching or fitting conditions. In the current work, a first order relativistic dissipative hydrodynamic theory that includes the out of equilibrium contributions in thermodynamic fields and recently been derived from relativistic transport equation by gradient expansion technique [15], has been employed to derive the dispersion equations and the associated modes. The observation of a causal propagating mode in large k limit is the main focus of this article.

The basic problem is to estimate the first order out of equilibrium correction of the thermodynamic fields needed to define the particle four-flow Np and energy-momentum tensor Tµν. Here, relativistic transport equation serves the purpose by providing the first order correction in single particle distribution function via gradient expansion technique. The first order Chapman-Enskog (CE) gives the following equation [16],

\[ p^\mu \partial_\mu f^{(0)}(x,p) = -\mathcal{L}[\phi] , \]

with \( \mathcal{L}[\phi] \) as the linearized collision term over the out of equilibrium first order distribution deviation \( \phi \), \( f = f^{(0)} + f^{(1)}(1 \pm f^{(0)}) \phi \) with \( f^{(0)} = (\exp(\frac{\mu p}{T} - \frac{1}{2}) \mp 1)^{-1} \) for Bosons and Fermions respectively) as the equilibrium distribution) as follows,

\[ \mathcal{L}[\phi] = \int d\Gamma_{p_1} d\Gamma_{p_2} d\Gamma_{p'_1} f^{(0)}(1 \pm f^{(0)})(1 \pm f^{(0)}_1) \]
\[ \{ \phi + \phi_1 - \phi' - \phi'_1 \} W(p'p_1 | pp_1) , \]

\[ d\Gamma_p = \frac{d^3p}{(2\pi)^3} \] is the phase space factor and \( W \) is the microscopic interaction rate. \( T, \mu \) and \( \omega' \) are respectively the equilibrium temperature, chemical potential and hydrodynamic four velocity of the system. The left hand
The bracket quantities are defined as, \([\phi, \phi] = \int d\Gamma_p \phi \mathcal{L}[\phi]\) which are always non-negative. The homogeneous solutions are fully arbitrary and the field corrections in \((\ref{14})\) due to them is attributed solely to the hydrodynamic frame choice. In certain situations the frame is so chosen that the homogeneous part exactly cancels the interaction part giving rise to field correction zero such that the field can be identified with its equilibrium value even in dissipative medium. In current analysis, the field corrections will be retained to generate the frequency modes.

However, these field corrections are not independent but constrained to give the dissipative flux of same tensorial rank. The coefficients are shown to follow,

\[
c_\Omega - c_A \frac{\partial P_0}{\partial \rho_0} \rho_0 - c_T \left(\frac{\partial \rho_0}{\partial \rho_0}\right)^2 \tau_\rho = -\zeta, \tag{13}
\]

such that the field corrections add up to produce dissipative fluxes as,

\[
\delta P - \left(\frac{\partial \rho_0}{\partial \rho_0}\right)\rho_0 \delta \epsilon - \left(\frac{\partial \rho_0}{\partial \rho_0}\right)\tau_\rho \delta \rho = \Pi, \quad W^\alpha - \hat{h} T V^\mu = q^\alpha. \tag{15}
\]

Here, \(\Pi = -T^2 \int d\Gamma_p \phi \hat{Q} = -\zeta (\partial : u)\) and \(q^\alpha = T^2 \int d\Gamma_p \hat{p}^{(\alpha)}(\tau_p - \hat{h}) = -\hat{h} \nabla^\alpha \hat{p}\) are respectively the first order bulk viscous and diffusion flux. The coefficient of bulk viscosity (\(\zeta\)) and thermal conductivity (\(\lambda\)) in this theory are respectively given by,

\[
\zeta = T^2 \int d\Gamma_p \hat{Q} A, \quad \lambda = -\frac{T}{3} \int d\Gamma_p \hat{p}, \nabla \hat{p}, (\tau_p - \hat{h}) B. \tag{16}
\]

Now since, \(\hat{Q} f^{(0)}(1 \pm f^{(0)}) = \frac{1}{T} \mathcal{L}[A]\) and \(\hat{p}^{(\mu)}(\frac{T}{h} - 1) f^{(0)}(1 \pm f^{(0)}) = \frac{1}{T} \mathcal{L}[B^\mu]\), then by virtue of self adjoint property of collision integral \(\int d\Gamma_p \psi \mathcal{L}[\phi] = \int d\Gamma_p \phi \mathcal{L}[\psi]\) with \(\psi = \psi(x, p^\mu)\), \(\zeta\) and \(\lambda\) do not include the homogeneous solutions and purely depends upon interactions. Eq.\((13)\) and \((14)\) reveal that these combinations are frame invariant as suggested by \((12)\) which retain only the interaction part of the field corrections through the transport coefficients associated with dissipative fluxes. Detailed discussion for any order of gradient expansion will be available in \((13)\). Including field corrections the expressions for particle four-flow and energy-momentum tensor are respectively given as follows,

\[
N^\mu = (\rho_0 + \delta \rho) u^\mu + V^\mu, \tag{17}
\]

\[
T^{\mu\nu} = (\epsilon_0 + \delta \epsilon) u^\mu u^\nu - (P_0 + \delta P) \delta^{\mu\nu} + (W^\mu u^\nu + W^\nu u^\mu) + \pi^{\mu\nu}. \tag{18}
\]
\( \pi^{\mu\nu} = \Delta^{\mu\nu} \alpha \delta T_{\alpha\beta} = 2\eta\sigma^{\mu\nu} \) with \( \eta \) as the shear viscous coefficient.

To analyze the modes, small perturbations are considered around a hydrostatic equilibrium state of the fluid with a background that is in local rest frame such as,

\[
e = \epsilon_0 + \delta\epsilon(t, x) , \quad \rho = \rho_0 + \delta\rho(t, x) , \quad u^\mu = (1, \tilde{0}) + \delta u^\mu(t, x) .
\] (19)

In linear approximation, the velocity perturbation has only spatial components \( \delta u^\mu = (0, \delta u^x, \delta u^y, \delta u^z) \), since \( u_0^\mu \delta u_\mu = 0 \) to retain the normalization condition. It is convenient to express these fluctuations in their plane wave solutions via a Fourier transformation \( \delta\psi(t, x) \rightarrow e^{i(\omega t - kx)} \delta\psi(\omega, k) \), with wave 4-vector \( k^\mu = (\omega, k, 0, 0) \).

Following this prescription, the conservation equations \( \partial_\mu N^\mu = 0 \) and \( \partial_\mu T^{\mu\nu} = 0 \), over the (17) and (18) give the dispersion relation. Following the convention of [3], retaining the component \( \delta u^\mu \) parallel to \( k^\mu \), the dispersion relation for longitudinal or sound mode is obtained as the following,

\[
\omega^3(1 + Ak^2) - iB(\omega^2 k^2 - \omega(Ck^2 + Dk^4)) + iEk^4 = 0 ,
\] (20)

with,

\[
A = \tilde{h}c_\Sigma(c_\Lambda - \tilde{c}_T) , \\
B = (4\eta/3 + \zeta + \lambda T)/(\epsilon_0 + P_0) , \\
C = c_s^2 , \\
D = (4\eta/3 + \zeta)\lambda T/(\epsilon_0 + P_0)^2 \\
+ \tilde{h}(c_\Lambda - \tilde{c}_T)\left(\frac{\partial P_0}{\partial \epsilon_0} c_\Sigma + \frac{1}{\hbar} \frac{1}{T} \left(\frac{\partial P_0}{\partial \rho_0} c_0 \tilde{c}_T\right)\right) ,
\] (24)

\[
E = c_s^2\lambda T/(\epsilon_0 + P_0) .
\] (25)

The used notations read, \( \tilde{c}_\Lambda = \epsilon_0/(\epsilon_0 + P_0) \), \( \tilde{c}_\Sigma = c_\Sigma/(\epsilon_0 + P_0) \), \( \tilde{c}_T = c_T/\rho_0 \), \( \tilde{c}_0 = c_0/\rho_0 \) and \( c_s^2 = (\partial P_0/\partial \epsilon_0) \rho_0 + \frac{1}{\hbar} (\partial P_0/\partial \rho_0) c_0 \) is the velocity of sound squared [18]. The coefficients \( B, C, E \) and first part of \( D \) being the function of transport coefficients associated with dissipative fluxes (which are independent of hydrodynamic field corrections), they will be present in the usual N-S theory as well, i.e. without field redefinition in out of equilibrium scenario. However, as mentioned earlier, in most of the studies \( c_\Lambda \) and \( c_T \) are set to zero employing certain frame choice in order to keep the energy density and particle number density at their equilibrium values even in dissipative medium. In such cases \( A \) and the second part of \( D \) in Eq. (20) vanish. In such situations, propagating modes appears only at small \( k \) values with a propagation speed of usual sound velocity \( c_s \). The problem occurs at large \( k \) limit where the propagating modes are changed to non-propagating modes with \( k^2 \) dependence which makes the theory acausal [1]. Here the dispersion relation (20) is analyzed in presence of all the field corrections. At small \( k \) values the dispersion relation gives,

\[
\omega_{1,2} = \frac{i}{2} \left( B - \frac{E}{C} \right) k^2 \pm kc_s , \quad \omega_3 = \frac{iE}{C} k^2 ,
\] (26)

which is identical to the usual N-S theory without field redefinition. It is the large \( k \) limit that gives interesting results here. At large \( k \) the dispersion relation renders,

\[
\omega_{1,2} = \frac{i}{2} \left( \frac{B - E}{A} \right) k^2 \pm k\sqrt{\frac{D}{A}} , \quad \omega_3 = \frac{iE}{D} ,
\] (27)

where positive values of \( D/A \) give two propagating modes via the real part of frequency \( \omega \). Eq. (21) shows that in absence of field redefinition \( A \) vanishes and consequently (20) produces only non-propagating modes at large \( k \) limit. These propagating modes of Eq. (27) with a linear dependence of wave number \( k \) is the key finding of the current article.

Following the argument of [1], the propagation speed of the fluid is characterized by the group velocity of propagating modes. Hence, the real parts of the mode frequencies determine the causality situation of the theory. Although there is not much improvement with field redefinition at small \( k \) limit (which was any way causal according to its propagating mode, only the \( k^2 \) momentum behaviour of the non propagating mode was resembling that of the diffusion processes, where the propagation speed is infinite), it is the asymptotic value of \( \omega(k \rightarrow \infty) \) that determines whether the theory as a whole is causal or not [8]. Eq. (27) structure wise looks remarkably similar to the large wave number behaviour of causal second order theory [6] bearing two propagating modes (with \( D/A > 0 \)) and one non-propagating mode independent of \( k \). If \( D/A \) becomes negative all three modes turn out to be non-propagating. It is to be noted, that at large \( k \) limit, the non-propagating modes are either independent of \( k \) or have a weak \( k \) dependence than N-S modes (\( \sim k^2 \)). This is a signature that in the present case for large \( k \) limit, the non-propagating modes do not violate causality. Following the general thumb rule to investigate causality, that at \( k \rightarrow \infty \) the propagating mode can not grow faster than the linear dependence of \( k \), the propagating mode of (27) provides the possibility to preserve causality. Since propagation speed of the fluid is characterized by the group velocity of the propagating mode, in order to analyze the causality of the mode, here the asymptotic value of group velocity \( v_g \) has been defined as follows,

\[
v_g = \lim_{k \rightarrow \infty} \left| \frac{\partial \text{Re}(\omega)}{\partial k} \right| = \sqrt{\frac{D}{A}} .
\] (28)

In order to be subluminal, the theory must satisfy \( D/A < 1 \) along with \( D/A > 0 \). Eq. (21) and (24) show that \( A \) and \( D \) explicitly depend upon the field correction coefficients. So it can be derived that in order to preserve causality of the propagating mode, the coefficients must satisfy the following relation,

\[
\frac{(4\eta/3 + \zeta)\lambda T/(\epsilon_0 + P_0)^2}{\tilde{h}c_\Sigma(c_\Lambda - c_T)} < \left( 1 - \left( \frac{\partial P_0}{\partial \epsilon_0} \rho_0 + \frac{1}{\hbar} \frac{1}{T} \left( \frac{\partial P_0}{\partial \rho_0} \right) c_0 \tilde{c}_T \right) \tilde{c}_0 \right) .
\] (29)
Eq. (29) is the asymptotic causality condition of the theory.

The next job is to test the theory with a microscopic model that can explicitly determine the field correction coefficients from Eq. (39). For the same I propose here solving relativistic transport equation (1) in momentum dependent relaxation time approximation (MDRTA). In recent literature, a number of studies have been explored related to the application of MDRTA [19, 22] in solving the relativistic transport equation to extract the macroscopic thermodynamic quantities thereafter. The idea is just to replace $L_0[\phi]$ in Eq. (1) with the help of relaxation time $\tau_R$ of single particle distribution function as follows,

$$\tilde{p}^\mu \partial_\mu f = -\frac{\tau_p}{\tau_R} f^{(0)}(1 \pm f^{(0)}) \phi \ , \quad \tau_R(x, p) = \tau_R^0(x) \tau_p^n,$$

(30)

where the momentum dependence of $\tau_R$ is expressed as a power law of the scaled particle energy $\tau_p$ in comoving frame, with $\tau_R^0$ as the momentum independent part and $n$ as the exponent specifying the power of the scaled energy. In [19, 22] the interaction part of the out of equilibrium field corrections have been estimated using the MDRTA technique from the relativistic transport equation. Here, the first order field correction coefficients are listed below,

$$\frac{c_a}{\tau_R} = T^2 \left[ \frac{z^2}{3} a_{n+1} + \left\{ \left( \frac{\partial P_0}{\partial \varepsilon_0} \right)_{\rho_0} - \frac{1}{3} \right\} a_{n+3} \right. \left. + \frac{1}{3T} \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\varepsilon_0} a_{n+2} \right],$$

(31)

$$\frac{c_\Gamma}{\tau_R} = T \left[ \frac{z^2}{3} a_n + \left\{ \left( \frac{\partial P_0}{\partial \varepsilon_0} \right)_{\rho_0} - \frac{1}{3} \right\} a_{n+2} + \frac{1}{3T} \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\varepsilon_0} a_{n+1} \right].$$

(32)

$$\frac{c_\Omega}{\tau_R} = T^2 \left[ \frac{z^2}{9} a_{n+1} + \left\{ \left( \frac{\partial P_0}{\partial \varepsilon_0} \right)_{\rho_0} - \frac{1}{3} \right\} a_{n+3} \right. \left. + \frac{1}{3T} \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\varepsilon_0} a_{n+2} - \frac{z^2}{9} a_{n-1} \right. \left. - \frac{z^2}{3} \left( \left( \frac{\partial P_0}{\partial \varepsilon_0} \right)_{\rho_0} - \frac{1}{3} \right) a_{n+1} - \frac{z^2}{3T} \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\varepsilon_0} a_n \right],$$

(33)

$$\frac{c_\Sigma}{\tau_R} = T^2 \left[ \frac{1}{6} b_{n+1} - b_n \right],$$

(34)

$$\frac{c_\Xi}{\tau_R} = T \left[ \frac{1}{6} b_n - b_{n-1} \right].$$

(35)

The conservation of particle four-flow and energy-momentum tensor along with the non-negativity of entropy production rate have been confirmed with in the theory.

Since, the homogeneous part in the field correction in Eq. (39) can be chosen arbitrarily, first let us examine Eq. (28) only with the dissipative correction provided by the transport equation itself (Eq. (31, 35)). In Fig. (1) the group velocity of the propagating mode from Eq. (28) has been plotted as a function of the exponent $n$ (which provides a quantitative measure of momentum transfer for the underlying microscopic interaction processes).

Let us now analyze the scenario in different hydrodynamic frames. Frame choice being a vector condition that defines the out of equilibrium velocity flow, Landau frame is defined by $W^\mu = 0$ such that $T^{\mu\nu}u_\nu = (\varepsilon_0 + \delta \varepsilon) u^\mu$ and Eckart frame is defined by $V^\mu = 0$ such that $N^\mu = (\rho_0 + \delta \rho) u^\mu$. Conventionally, in both the frames $\delta \varepsilon$ and $\delta \rho$ are both set to zero, but in a number of recent studies [24, 22] it has been shown that in presence of dissipation $\varepsilon$ and $\rho$ can have extended matching conditions for both Landau and Eckart frames. In such situation, setting $\delta \varepsilon$ and $\delta \rho$ from interaction corrections only (from Eq. (31) and (32) respectively), Landau frame has $\delta \Sigma = 0$ and $\delta \Xi = \lambda T/(\varepsilon_0 + P_0)$ and Eckart frame has $\delta \Sigma = 0$ and $\delta \Xi = -\lambda T/(\varepsilon_0 + P_0)$. Clearly, Landau frame giving rise to $A = 0$ can not have a propagating mode. Eckart frame on the other hand can produce a propagating mode with asymptotic causality condition $(4n/3 + z)/z_0 + P_0_0/(c_T - c_\lambda) < 1 - (\partial P_0/\partial \rho_0)$, as long as energy density and particle number density corrections are non zero.

In this work the generation of a propagating mode in the first order relativistic hydrodynamic theory has been reported when the concerned thermodynamic fields are redefined in order to include the out of equilibrium dissipation corrections. Asymptotic causality condition (Eq. (29)) shows that certain constrained values of field correction coefficients can produce subluminal propagation velocity $v_g$ for the mode, making it causal. Here one thing needs to be mentioned. First order field redefinition estimated in the above mention gradient expansion technique does not improve the shear channel modes. Shear
channel is still constituted of the non-propagating mode 
\[ \omega = \frac{\text{i}\eta}{(\epsilon_0 + P_0)k^2}, \]
as in the usual N-S theory. Hence, calling the theory as a whole causal is a debatable issue. The author only reports the generation of a propagating mode with first order field redefinition which was otherwise absent in the usual first order theory. But following the general argument that the causality of a theory is determined by the behaviour of its propagating modes, i.e. the real parts of the frequencies \[7\], the generation of this causal propagating mode is certainly an improvement over the conventional first order relativistic theory.

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