AFFINE MANIFOLDS AND SOLVABLE GROUPS

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Let $M$ be a compact affine manifold. Thus $M$ has a distinguished atlas whose coordinate changes are locally in $\text{Aff}(E)$, the group of affine automorphisms of Euclidean $n$-space $E$. Assume $M$ is connected and without boundary.

The universal covering $\tilde{M}$ of $M$ has an affine immersion $D: \tilde{M} \rightarrow E$ which is unique up to composition with elements of $\text{Aff}(E)$. Corresponding to $D$ there is a homomorphism $\alpha: \pi \rightarrow \text{Aff}(E)$, where $\pi$ is the group of deck transformations of $\tilde{M}$, such that $D$ is equivariant for $\alpha$. Set $\alpha(\pi) = \Gamma$. Let $L: \text{Aff}(E) \rightarrow \text{GL}(E)$ be the natural map.

**Theorem 1.** If $\Gamma$ is nilpotent the following are equivalent:

(a) $M$ is complete, i.e. $D: \tilde{M} \rightarrow E$ is bijective;
(b) $D$ is surjective;
(c) no proper affine subspace of $E$ is invariant under $\Gamma$;
(d) $L(\Gamma)$ is unipotent;
(e) $M$ has parallel volume, i.e. $L(\Gamma) \subset \text{SL}(E)$;
(f) $M$ is affinely isomorphic to $\Gamma \backslash G$ where $G$ is a connected Lie group with a left-invariant affine structure and $\Gamma \subset G$ is a discrete subgroup;
(g) each de Rham cohomology class of $M$ is represented by a differential form whose components in affine charts are polynomials.

For abelian $\Gamma$ the equivalence of (a), (d), and (e) is due to J. Smillie. We conjecture that (a), (b), (e), and (g) are equivalent even without nilpotence (if $M$ is orientable). In general (a) $\Rightarrow$ (c) and (e) $\Rightarrow$ (c); but (c) $\not\Rightarrow$ (a) even for $\Gamma$ solvable and $M$ three-dimensional.

**Theorem 2.** The following are equivalent:

(i) $M$ is finitely covered by a complete affine nilmanifold $M_1$ (i.e. conditions (a) through (g) of Theorem 1 hold for $M_1$);
(ii) all eigenvalues of elements of $L(\Gamma)$ have norm 1;
(iii) $M$ has a Riemannian metric whose coefficients in affine charts are polynomials.

L. Auslander has conjectured that if $M$ is complete then $\pi = \Gamma = \pi_1(M)$ is virtually solvable (i.e. contains a solvable subgroup of finite index); see [M] for discussion. This conjecture is true in dimension three (see [FG]).
THEOREM 3. If \( \pi \) is virtually solvable and \( M \) is complete then (e), (f), (g) of Theorem 1 hold. If \( \alpha: \pi \to \Gamma \) factors through a virtually polycyclic group of rank \( \leq \dim M \) and \( M \) has parallel volume, then \( M \) is complete. In particular if \( M \) is finitely covered by a manifold homeomorphic to a solvmanifold then parallel volume is equivalent to completeness.

We briefly indicate the proof of Theorem 1.

(a) \( \Rightarrow \) (c). This holds for any compact complete \( M \). If \( F \subset E \) is a \( \Gamma \)-invariant affine subspace then both \( E/\Gamma \) and \( F/\Gamma \) are Eilenberg-Mac Lane spaces of type \( K(\pi, 1) \). Since they are compact manifolds their dimensions are equal; thus \( F = E \).

(e) \( \Rightarrow \) (c). This holds for all compact \( M \). The linear holonomy \( \rho = L \circ \alpha: \pi \to GL(E) \) determines a \( \pi \)-module \( E_\rho \). Let \( u: \pi \to E \) send \( g \in \pi \) into the translational part of \( \alpha(g) \). Then \( u \) is a crossed homomorphism whose cohomology class \( c_M \in H^1(\pi; E_\rho) \) depends only on \( M \). The \( n \)-th exterior power \( \Lambda^n c_M \) comes from \( H^1(\pi; \Lambda^n F) \).

From now on assume \( \Gamma \) is nilpotent.

(c) \( \Rightarrow \) (d). Let \( E_U \subset E \) be the maximal unipotent submodule. Then \( H^0(\pi; E/E_U) = 0 \), and nilpotent implies \( H^1(\pi; E/E_U) = 0 \) (Hirsch [H]). This means some coset of \( E_U \) is \( \Gamma \)-invariant.

(b) \( \Rightarrow \) (d). Suppose \( E_U \neq E \). Some coset of \( E_U \) is \( \Gamma \)-invariant; we may assume \( E_U \) is \( \Gamma \)-invariant. There is a unique \( L(\Gamma) \)-invariant splitting \( E = E_U \oplus F \). Let \( M_1 = p(D^{-1}E_U) \) where \( p: \tilde{M} \to M \) is the projection. Then \( M_1 \) is a compact affine manifold with unipotent holonomy, hence complete. Let \( Y \) be the vector field on \( \tilde{M} \) which is \( D \)-related to the vector field \( (x, y) \mapsto (0, y) \) on \( E_U \oplus F \). Then \( Y \) covers a vector field on \( M \), so \( Y \) is completely integrable. Every component of \( p^{-1}M_1 \) is a repellor for \( Y \). One uses these facts to prove that \( M \) is complete; but this implies (c), and hence (d).

(d) \( \Rightarrow \) (a). When \( L(\Gamma) \) is unipotent there is a flag \( E = E_n \supset \cdots \supset E_0 = \{0\} \) of \( L(\Gamma) \)-invariant linear subspaces with \( L(\Gamma) \) acting trivially on each \( E_i/E_{i-1} \).

There are nested foliations \( Z_2, \ldots, Z_0 \) on \( M \) covered by foliations \( \tilde{Z}_i \) on \( \tilde{M} \) such that \( D \) relates \( \tilde{Z}_i \) to the linear foliation \( E_i \) of \( E \) whose leaves are cosets of \( E_i \).

For each \( i \) there is a closed 1-form \( \omega_i \) on \( \tilde{Z}_i \) which vanishes on \( \tilde{Z}_{i-1} \) related by \( D \) to a constant 1-form on \( E \) vanishing on \( E_{i-1} \). There are completely integrable vector fields \( X_i \) in \( \tilde{Z}_i \) with \( \langle X_i, \omega_i \rangle = 1 \). Given any \( p \in \tilde{M}, x \in E \) one shows that the trajectory of \( X_n \) through \( p \) meets a point \( p_1 \) such that \( D(p_1) \) is the leaf of \( E_{n-1} \) through \( x \).

The trajectory of \( X_{n-1} \) through \( p_1 \) stays in a leaf of \( \tilde{Z}_{n-1} \) and eventually meets a \( p_2 \) such that \( D(p_2) \) is the leaf of \( E_{n-2} \) through \( x \), etc. In this
way one proves that $D(\tilde{M})$ contains a path from $D(p)$ to $x$. Hence $D$ is surjective. Injectivity is proved similarly.

(e) $\Rightarrow$ (d). If $E_U \neq E$ let $F \subset E$ be a complementary submodule to $E_U$. One shows that some element of $L(\Gamma)$ expands $F$, contradicting parallel volume.

(a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (e) are obvious.

(a) $\Rightarrow$ (f). $G$ is the algebraic hull of $\Gamma$ in $\text{Aff}(E)$.

(f) $\Rightarrow$ (g). By Nomizu's theorem [N] the cohomology of $M$ is represented by invariant forms on $G$; these turn out to be polynomial.

(g) $\Rightarrow$ (e). If $L(\Gamma)$ is not unipotent then one proves there is no polynomial volume form.

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