Killing spinor initial data sets

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Abstract

A 3+1 decomposition of the twistor and valence-2 Killing spinor equation is made using the space spinor formalism. Conditions on initial data sets for the Einstein vacuum equations are given so that their developments contain solutions to the twistor and/or Killing equations. These lead to the notions of twistor and Killing spinor initial data. These notions are used to obtain a characterisation of initial data sets whose development are of Petrov type N or D.

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1 Introduction

In this work we study the kind of conditions to be imposed on an initial data set \((S, h_{ij}, K_{ij})\) of the Einstein vacuum field equations for its development \((\mathcal{M}, g_{\mu\nu})\) to be endowed with a Killing spinor. For an \(r\)-valence Killing spinor, it will be understood a totally symmetric spinor \(\kappa_{(AB...P)} = \kappa_{AB...P}\), of valence \(r\) such that

\[\nabla_{Q'}(Q\kappa_{AB...P}) = 0,\]
although a more general definition exists with an arbitrary number of sym-
metrised primed indices —see [17]. For reasons of physical interest to be elab-
orated below, our analysis will be concentrated on the cases with \( r = 1, 2 \), namely

\[
\nabla_{A'}(A B) = 0, \quad \nabla_{A'}(A B C) = 0.
\]

(1a)

(1b)

The interest in spinors satisfying equations (1a) or (1b) stems from their po-
tential use in the characterisation of initial data sets yielding developments
of a particular Petrov type. An ultimate goal of this analysis is to yield a
characterisation of initial data sets for the Schwarzschild or Kerr spacetimes.

Equation (1a) is usually called the *twistor equation*. It is well known that its
solvability imposes very strong restrictions on the curvature (Weyl tensor) of
the spacetime. One of these restrictions is

\[
\Psi_{ABCD} = 0,
\]

(2)

where \( \Psi_{ABCD} \) is the Weyl spinor —see section 2 for more information on
the conventions being used. The latter condition will occur if and only if the
spacetime is of Petrov type N —see [17]. Therefore if we are able to find
conditions on a vacuum initial data set ensuring the existence of a spinor \( \kappa_A \)
fulfilling the twistor equation on at least an open subset of the development,
then these conditions will also guarantee that such a subset is of Petrov type
N. These explicit conditions are presented in theorem 2.

Equation (1b) has more physical relevance, as it can be shown that all Petrov
type D vacuum solutions to the Einstein field equations —and in particular
the Schwarzschild and Kerr spacetimes— possess a valence-2 Killing spinor
[24] —see proposition 7 in the appendix. In any case, if the spacetime has a
valence-2 Killing spinor then it has to be algebraically special and must satisfy
the integrability condition

\[
\Psi_{ABCD} = 0.
\]

Further, if the valence-2 Killing spinor is non-null —i.e. its principal spinors
are not repeated— then it has to be of Petrov type D [11]. If the valence-2
Killing spinor is null, then it can be shown that the Weyl tensor has to be of
Petrov type N —see again proposition 7 in the appendix for a proof of this
statement. Hence if we could find conditions ensuring that a non-null valence-
2 Killing spinor exists in at least a neighbourhood of the initial data set then
we could conclude that in such a neighbourhood the spacetime is of type D.
This idea is the core of theorem 3.

Another interesting feature about vacuum spacetimes admitting a spinor \( \kappa_{AB} \)
solving (1b) is that the vector $\xi^\mu$, defined in terms of its associated spinor $\xi_{AA'}$ by

$$\xi_{AA'} \equiv \nabla^Q_{A'}\kappa_{AQ}$$

is a —possibly complex— Killing vector. Thus, one has

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 0.$$

This paper is organised as follows: basic conventions followed in this paper are given in section 2. In section 3 we review the space spinor formalism, which is essential to find the orthogonal decomposition of spinor expressions. Section 4 provides a result on wave equations which will be used extensively. Section 5 is devoted to the study of conditions —Killing initial data, KID— which ensure the existence of Killing vectors in the development of a vacuum initial data set. Although these conditions are well-known, they are usually derived and presented in a tensor form which is not suitable for this work. We carry out this derivation in spinor form, so that it also illustrates procedures to be used later. In section 6 we find the necessary and sufficient conditions which a vacuum initial data set must fulfil for the existence of a twistor in a neighbourhood of the data development. Similarly, in section 7 conditions are derived for the existence of a valence-2 Killing spinor in the data development. The main results of the paper are contained in theorem 2 where it is shown when the development of a vacuum initial data set has a Petrov type N region, and in theorem 3 which provides a similar result for Petrov type D.

In order to prove some of the results presented in this paper, long computations with spinors are needed. These calculations have been performed with the computer algebra system $xAct$ [14], which is a suite of MATHEMATICA packages tailored for tensor calculus. In particular, it can neatly handle all the essential rules of the spinor calculus.

It should be noted that several of our results could be reformulated in the language of Killing tensors and Killing-Yano tensors —see e.g. [17]. As it is often the case, we expect the tensorial expressions to be much more complicated that their spinorial counterparts. However, for some particular applications it may prove useful to have a tensorial expression.

2 Conventions

Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth Lorentzian manifold. Due to the systematic use of spinors in our discussion, the spacetime metric will be taken to have signature $(+, - , - , -)$. We shall follow the conventions of [16,17] for spinor calculus. Abstract indices are used throughout with Greek letters denoting spacetime
abstract indices and capital Latin letters—with or without dash—denoting abstract spin indices. The Riemann, Ricci and Weyl tensors will be denoted, respectively, by $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$, $C_{\mu\nu\alpha\beta}$. The volume element is $\eta_{\mu\nu\alpha\beta}$. We define the Hodge dual in the standard fashion. It will be assumed that $(M, g_{\mu\nu})$ is a globally hyperbolic spacetime. This implies the existence of a smooth time function $t : M \to \mathbb{R}$ foliating $M$—see [4]. Let $S_s = \{ p \in M \mid t(p) = s, s \in \mathbb{R} \}$ denote a leaf of the foliation of $M$ induced by $t$. Then $S_s$ is an embedded spacelike hypersurface on $M$, for all $s \in \mathbb{R}$. The timelike 1-form $dt$ is orthogonal to the leaves of the foliation. We can construct another 1-form $\tau_\mu$ such that $\tau_\mu \tau^\mu = 2$ —the reasons for this normalisation will become apparent later. From $\tau_\mu$ and $g_{\mu\nu}$ we introduce the tensor $h_{\mu\nu}$ by means of the definition

$$h_{\mu\nu} = -\frac{1}{2} \tau_\mu \tau_\nu + g_{\mu\nu}$$

The tensor $h_{\mu\nu}$ plays the role of the intrinsic metric (first fundamental form) of $S_s$, for all $s \in \mathbb{R}$. Note that this intrinsic metric is negative definite, due to the signature convention chosen for $g_{\mu\nu}$. Using $\tau_\mu$ and $h_{\mu\nu}$ one can define all the mathematical objects used in the standard 3+1 decomposition. Our notation for the spatial derivative is $D_\mu$ and our conventions for the second fundamental form $K_{\mu\nu}$ and the acceleration $A^\mu$ are

$$K_{\mu\nu} = -h_\mu^\alpha h_\nu^\beta \nabla_\alpha \tau_\beta,$$

$$A^\mu = -\frac{1}{2} n^\alpha \nabla_\alpha n^\mu.$$  

Any covariant spatial tensor $T$ corresponds to a unique tensor field defined on any of the leaves $S_s$ which is obtained by means of the pull-back $i^*$ where $i : S_s \to M$ is the inclusion embedding. The tensor field $i^*T$ is an element of the tensor algebra constructed by taking $S_s$ as the base manifold. Latin characters will be used as the abstract indices of this tensor algebra and therefore if $T_\mu$ is spatial then $T_j$ will denote its pull-back under $i$.

3 The space spinor formalism

It is of interest to find a spinor formulation of the standard 3+1 decomposition. It can be seen that one can construct spinors which transform under the group $SU(2)$. These spinors play the role of the “spatial elements” and shall be called space spinors or $SU(2)$ spinors. Accounts of the definition and main properties of space spinors can be found in [18,19]. Here we will review the basics of this formalism as it is an essential tool in this work.
Let $\tau_{AA'}$ be the spinor equivalent of $\tau^A_{\mu A'}$. Clearly, $\tau^A_{\mu A'}\tau^{BA'} = 2$ and

$$\tau^A_{\mu A'}\tau^{BA'} = \epsilon^{AB}.$$ 

Using this spinor we may define the spatial soldering form as

$$\sigma^{AB} \equiv \sigma_{AB}(A^A\tau^{B'A}).$$

The spatial soldering form $\sigma^{AB}$ should not be confused with the soldering form $\sigma^{AA'}$. They are in fact different objects despite having the same kernel letter. The algebraic properties of the spatial soldering form are

$$\sigma^{\mu AB}\sigma_{\nu CD} = \frac{1}{2}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}), \quad \tau^{\mu}_{\sigma^{AB}} = 0. \quad (4)$$

Inspecting these algebraic properties one can see that $\sigma^{\mu AB}$ can be used to define a spin structure associated to the metric $h_{\mu\nu}$. Spinors belonging to such spin structure are the space spinors mentioned above and it is possible to relate spatial tensors to space spinors and viceversa by means of the spatial soldering form. As an example of this let $W^j_i$ be a spatial tensor. Its space-spinor counterpart is given by $W_{AB}^{CD} = \sigma^{j}_{AB}\sigma^{CD}_{i}W^j_i$. Another important example is the spatial metric $h_{ij}$ whose space spinor equivalent can be obtained by using the second and third expressions of (4):

$$h_{ABCD} = \frac{1}{2}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}).$$

In order to avoid complicating the notation, the same Kernel letter will be used to denote a spatial tensor and its space spinor equivalent; the nature of the relevant object will be indicated in the text or in most cases it will be clear from the context.

An important issue is to find when a given $SL(2, \mathbb{C})$ spinor arises from a spatial tensor —and thus is a $SU(2)$ spinor. In order to answer this question we need to introduce the Hermitian conjugation operation. Given any spinor $Z_{A_1\cdots A_n}$ we define its Hermitian conjugate as

$$\bar{Z}_{A_1\cdots A_n} \equiv \tau_{A_1}^{A_1'} \cdots \tau_{A_n}^{A_n'} \bar{Z}_{A_1'\cdots A_n'},$$

where the overline denotes the standard complex conjugation. Any even rank spinor $W_{A_1B_1\cdots A_nB_n}$ stems from a real spatial tensor if and only if the following two conditions hold:

1 Here and in the sequel, the spinor associated to a tensor, say $\Phi_{\mu\nu}^A$, is the spinor $\Phi_{AB}^{A'B'} = \sigma^{\mu}_{AA'}\sigma_{\nu}^{BB'}\Phi_{\mu\nu}^A$, where $\sigma^{AA'}$ denotes the soldering form and $g_{\mu\nu} = \epsilon_{AB}\epsilon_{A'B'}\sigma_{\mu}^{AA'}\sigma_{\nu}^{BB'}$. 

5
\[
\begin{align*}
\hat{W}_{A_1B_1\cdots A_nB_n} &= (-1)^n W_{A_1B_1\cdots A_nB_n}, \\
W_{A_1B_1\cdots A_nB_n} &= W_{(A_1B_1)\cdots (A_nB_n)}. 
\end{align*}
\]

(5a)

Any spinor of even rank fulfilling condition (5a) will be said to be real. Therefore we deduce that there exists an equivalence between real and space spinors. If for a spinor \(W_{A_1B_1\cdots A_nB_n} = W_{(A_1B_1)\cdots (A_nB_n)}\) the condition

\[
\hat{W}_{A_1B_1\cdots A_nB_n} = (-1)^{n+1} W_{A_1B_1\cdots A_nB_n},
\]

holds, then the spinor is said to be imaginary.

As in the case of spatial tensors, space spinors can also be regarded as intrinsic objects in any of the Riemannian manifolds \(S_s\). This means that there is an isomorphism between the subalgebra of space spinors and a tensor algebra constructed from a vector bundle arising from a spin structure on \((S_s, h_{ij})\).

In order to avoid excessive notation we are not going to define new abstract indices for this vector bundle and instead we will simply add a tilde over the kernel letter of a space spinor whenever we wish to consider it as an element of the spin bundle constructed over \((S_s, h_{ij})\).

### 3.1 Spatial spin covariant derivatives

The spinorial covariant derivative can be decomposed as follows

\[
\nabla_{AA'} = \frac{1}{2} \tau_{AA'} \nabla - \tau_{A'C} \nabla_{AC},
\]

(6)

where

\[
\nabla \equiv \tau_{AA'} \nabla^{AA'},
\]

\[
\nabla_{AB} \equiv \tau^{A'}_{(A} \nabla_{B)A'} = \sigma_{AB} \nabla_{\mu}.
\]

The operator \(\nabla_{AB}\) is usually referred to as the Sen connection. Next we introduce the spinors

\[
\begin{align*}
K_{AB} &\equiv \tau_{B}^{A'} \nabla_{AA'}, \\
K_{ABCD} &\equiv \tau_{C'}^{D'} \nabla_{AB} \tau_{CC'}. 
\end{align*}
\]

These spinors satisfy the following algebraic properties

\[
\begin{align*}
K_{(AB)} &= K_{AB}, \quad \hat{K}_{AB} = -K_{AB}, \quad K_{(AB)(CD)} = K_{ABCD}, \\
\hat{K}_{ABCD} &= K_{ABCD}, \quad K_{FD} = -2A_{\mu} \sigma_{FD}^{\mu}, \\
K_{ABCD} &= -\sigma_{AB}^{\mu} \sigma_{CD}^{\nu} K_{\mu\nu}. 
\end{align*}
\]

(7a)

(7b)

(7c)
The spinor $K_{AB}$ is called the acceleration spinor while $K_{ABCD}$ corresponds to the second fundamental form of the leaves. Formulae (6) and (7a)-(7c) hold regardless to whether $\tau_\mu$ is hypersurface forming or not — if $\tau_\mu$ is not hypersurface forming then $K_{\mu\nu}$ fails to be symmetric. In the case of $\tau_\mu$ being integrable then we get the extra symmetry

$$K_{ABCD} = K_{CDAB},$$

which is a straightforward consequence of $K_{\mu\nu} = K_{\nu\mu}$. As we are working with an hypersurface forming $\tau_\mu$ we will take for granted condition (8) in what is to follow.

Many of our arguments will make use of a foliation for which $K_{AB} = 0$. From (7a)-(7c) it follows directly that $A_\mu = 0$, that is, $\tau_{AA'}$ is geodesic. Such a foliation can always be constructed in, at least, a neighbourhood of any spacelike hypersurface.

The operator $\nabla_{AB}$ corresponds to the operator $h_\mu^{\nu}\nabla_\nu$ which is not intrinsic to the hypersurfaces $S_s$ —the action of $\nabla_{AB}$ on a space spinor does not result in a space spinor. In order to obtain a differential operator which maps space spinors into space spinors let us start by defining

$$D\pi_A \equiv \nabla\pi_A - \frac{1}{2} K_B^A \pi_B, \quad D\pi_{A'} \equiv \nabla\pi_{A'} - \frac{1}{2} K_{A'}^B \pi_B,$$

$$D_{AB}\pi_C \equiv \nabla_{AB}\pi_C - \frac{1}{2} K_{ABC}^D \pi_D,$$

$$D_{AB}\pi_{C'} \equiv \nabla_{AB}\pi_{C'} + \frac{1}{2} K_{ABCD}\tau_{C'A'} \tau_D^C \pi_{A'}.$$

The operators $D$, $D_{AB}$ are extended to the full spinor algebra by requiring them to satisfy the Leibnitz rule. Important properties of these operators are

$$D\tau^A_{B'} = 0, \quad D_{AB}\tau^C_{C'} = 0, \quad D\epsilon_{AB} = 0, \quad D_{AB}\epsilon_{CD} = 0.$$

Using these properties and equation (5a) it is now a simple matter to check that the action of $D_{AB}$ on a space spinor is again a space spinor. Therefore whenever $\tau_{AA'}$ is surface forming, the operator $D_{AB}$ can be regarded as the spinorial counterpart of the spatial derivative $D_\mu$.

In the sequel, the following commutators will be used
\[ [\nabla, \nabla_{AB}]\alpha_C = -\frac{1}{2} K_{AB} \nabla \alpha_C - K_{(Q}^{(Q} \nabla_{B)Q} \alpha_C - K_{ABPQ} \nabla^{PQ} \alpha_C + \]
\[ + \hat{\square}_{AB} \alpha_C - \square_{AB} \alpha_C, \] (9a)
\[ [\nabla_{AB}, \nabla_{CD}]\alpha_E = \frac{1}{4} (\epsilon_{AC} \square_{BD} + \epsilon_{AD} \square_{BC} + \epsilon_{BC} \square_{AD} + \epsilon_{BD} \square_{AC}) \alpha_E \]
\[ + \frac{1}{4} \left( \epsilon_{AC} \hat{\square}_{BD} + \epsilon_{AD} \hat{\square}_{BC} + \epsilon_{BC} \hat{\square}_{AD} + \epsilon_{BD} \hat{\square}_{AC} \right) \alpha_E \]
\[ - \frac{1}{2} (K_{ABCQ} \nabla_{Q}^{D} + K_{ABDQ} \nabla_{Q}^{C} + K_{CDAQ} \nabla_{Q}^{B} + K_{CDBQ} \nabla_{Q}^{A}) \alpha_E, \] (9b)

where \( \square_{AB} = \nabla_{C'}(A \nabla_{B})^{C'} \), and \( \hat{\square}_{AB} = \tau_{A'A'} \tau_{B'B'} \square_{A'B'} \). The action of these operators on any spinor \( \alpha_C \) is
\[ \square_{AB} \alpha_C = \Psi_{ABCD} \alpha_D + \frac{\Lambda}{2} \epsilon_{C(\alpha_B),} \]
\[ \hat{\square}_{AB} \alpha_C = -\tau_{A'A'} \tau_{B'B'} \Phi_{FCA'B'} \alpha_F, \] (10)

with \( \Psi_{ABCD} \) the Weyl spinor, \( \Lambda \) the scalar curvature and \( \Phi_{ABA'B'} \) the Ricci spinor.

From (9a)-(9b) we can obtain similar commutation relations for the operators \( D \) and \( D_{AB} \). These are
\[ [D, D_{AB}]\alpha_C = -K_{ABDF} D^{DF} \alpha_C - \frac{1}{2} K_{AB} D \alpha_C + \square_{AB} \alpha_C + \hat{\square}_{AB} \alpha_C \]
\[ + \frac{\alpha^D}{4} \left( K_{AB} K_{CD} - K_{DF} K_{ABC}^{F} - K_{CF} K_{ABD}^{F} + 2K_{ABFH} K_{CD}^{FH} - 2D_{AB} K_{CD} + 2D K_{ABCD} \right), \] (11a)
\[ [D_{AB}, D_{CD}]\alpha_F = \frac{\alpha^H}{4} \left( -K_{AB} H_{L} K_{CD} + K_{CD}^{HL} K_{ABF} + 2D_{CD} K_{ABF}^{H} - 2D_{AB} K_{CD}^{H} \right) \]
\[ + \frac{1}{4} \left( \epsilon_{AC} \hat{\square}_{BD} + \epsilon_{AD} \hat{\square}_{BC} + \epsilon_{BC} \hat{\square}_{AD} + \epsilon_{BD} \hat{\square}_{AC} \right) \alpha_F \]
\[ + \frac{1}{4} \left( \epsilon_{AC} \square_{BD} + \epsilon_{AD} \square_{BC} + \epsilon_{BC} \square_{AD} + \epsilon_{BD} \square_{AC} \right) \alpha_F. \] (11b)

The commutation relations (9a)-(9b) and (11a)-(11b) can be generalised if we let the commutators act on spinors of higher rank. For instance if we choose the spinor \( \tau_{AA'} \) then we obtain the identities
\[ DK_{AFCD} = 2(E_{ACDF} + \hat{\Phi}_{ACDF}) - \frac{1}{2} K_{AF} K_{CD} - K_{AF}^{BH} K_{CDBH} \]
\[ - 2\Lambda(\epsilon_{AD} \epsilon_{CF} + \epsilon_{AC} \epsilon_{DF}) + D_{CD} K_{AF}, \] (12a)
\[ B_{ABCD} = -i D_{Q}^{(A} K_{|BC|D)Q}. \] (12b)
where we have introduced the electric and magnetic parts of the Weyl spinor \( \Psi_{ABCD} \) which are given, respectively, by

\[
E_{ABCD} = \frac{1}{2}(\hat{\Psi}_{ABCD} + \Psi_{ABCD}), \quad B_{ABCD} = \frac{i}{2}(\hat{\Psi}_{ABCD} - \Psi_{ABCD}).
\]

Note that both \( E_{ABCD} \) and \( B_{ABCD} \) are totally symmetric and real. They are related to the electric and magnetic parts of the Weyl tensor by the relations

\[
\sigma_{\mu}^{\, AB} \sigma_{\nu}^{\, CD} E_{ABCD} = E_{\mu\nu}, \quad \sigma_{\mu}^{\, AB} \sigma_{\nu}^{\, CD} B_{ABCD} = B_{\mu\nu},
\]

with the definitions

\[
E_{\mu\nu} \equiv \frac{1}{2} \tau^\alpha \tau^\beta C_{\mu\alpha\nu\beta}, \quad B_{\mu\nu} \equiv \frac{1}{2} \tau^\alpha \tau^\beta C^*_{\mu\alpha\nu\beta},
\]

where \( C^*_{\mu\alpha\nu\beta} = \frac{1}{2} \eta_{\nu\beta\lambda\pi} C_{\mu\alpha}^{\lambda\pi} \). It is a property of the space-spinorial formalism that symmetric trace-free spatial tensors are associated to totally symmetric space-spinors.

As discussed in \cite{23,10} the values of \( E_{\mu\nu} \) and \( B_{\mu\nu} \) on the initial hypersurface \( S \) can be calculated entirely from the values of \( h_{ij} \) and \( K_{ij} \). This is also true for their spinorial counterparts. In the case of the magnetic part this assertion is obvious from (12b). To obtain the corresponding expression of the electric part we start from the identity

\[
(D_{CD}D_{FH} - D_{FH}D_{CD}) \tilde{\zeta}_{AB} = \tilde{r}_{LPABCDFH} \tilde{\zeta}_{LP},
\]

where \( \tilde{\zeta}_{AB} \) is any symmetric spinor and \( \tilde{r}_{LPABCDFH} \) is a spinor which represents the spatial Riemann tensor \( \tilde{r}_{ijkl} \). The explicit expression for \( \tilde{r}_{LPABCDFH} \) can be obtained from the generalisation of (11b) to \( \tilde{\zeta}_{AB} \) and shall be omitted as it is somewhat long. As happens with the spinor representing the spacetime Riemann tensor, the spinor \( \tilde{r}_{LPABCDFH} \) can be decomposed in irreducible parts which are obtained by taking suitable traces. For instance, the totally symmetric part \( \tilde{r}_{BCFL} \) is defined by

\[
\tilde{r}_{BCFL} \equiv -\tilde{r}_{LA(BC|}^{\, A}D_{F)D},
\]

from which using the expression for \( \tilde{r}_{ABCDFHLP} \) deduced from (13) we get the desired expression for \( \tilde{E}_{ABCD} \):

\[
\tilde{E}_{ABCD} = -\tilde{E}_{ABCD} - \frac{1}{6} \tilde{\Omega}_{ABCD} \tilde{K} + \frac{1}{2} \tilde{\Omega}_{(AB}^{\, PQ} \tilde{\Omega}_{CD)PQ}, \quad \tilde{K} \equiv \tilde{K}^{PQ}_{PQ},
\]

\[
\tilde{\Omega}_{ABCD} \equiv \tilde{K}_{(ABCD)}
\]

For completeness we include also the expressions for the Hamiltonian and momentum constraints in the space spinor formalism. These are
\[ \tilde{r} + \frac{1}{6} \tilde{K}^2 - \frac{1}{4} \tilde{\Omega}_{ABCD} \tilde{\Omega}^{ABCD} = 0, \quad \tilde{r} \equiv \tilde{r}^{AC} C^D A^L D_L, \]

\[ D^{PQ} \tilde{K}_{PQAB} - \frac{1}{2} D_{AB} \tilde{K} = 0. \]

The Hamiltonian constraint is obtained by taking the suitable traces in the formula for \( \tilde{r}_{ABCD} \), while the momentum constraint is the nonvanishing trace of (12b).

4 Homogeneous second order hyperbolic systems

In this work we will make use of the following result which can be consulted for instance in [19] — p. 378, Proposition 3.2.

**Theorem 1** Let \( g_{\mu\nu} \) be a Lorentzian metric on a smooth manifold \( \mathcal{M} \) and define the differential operator \( \square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \) (D’Alembertian). Consider the second order hyperbolic system

\[ \square X + H(X, \nabla_\mu X) = 0, \tag{14} \]

where \( X = (X_0, \ldots, X_m) \) is a set of scalar functions on \( \mathcal{M} \) and \( H(X, \nabla_\mu X) \) is a smooth linear, homogeneous function of the components of \( X \) and their first covariant derivatives \( \nabla_\mu X \). Let \( S \subset \mathcal{M} \) be a spacelike hypersurface with respect to \( g_{\mu\nu} \) and assume that

\[ X|_S = f, \quad \nabla_{AA'} X|_S = g, \]

where \( f, g \) are smooth on \( S \). Then there exists a unique smooth solution of (14) in a neighbourhood of \( S \). In particular if \( f = 0, \ g = 0 \) then such unique solution is given by \( X = 0 \).

5 Killing initial data

In this section, we study the conditions under which the development of a vacuum initial data set will contain Killing vectors. The contents of this section follow the ideas and methods of [6,15], but in our case the results are cast in the spinorial language and the techniques used in the proofs will be needed later on. From this point on, \( (\mathcal{M}, g_{\mu\nu}) \) will be assumed to be a vacuum spacetime.

**Proposition 1** Let \( \eta_{AA'} \) be a spinorial field on \( \mathcal{M} \), \( S_0 \subset \mathcal{M} \) a Cauchy spacelike hypersurface and suppose that the following conditions hold
\[
(\nabla_{AA'}\eta_{BB'} + \nabla_{BB'}\eta_{AA'})|_{S_0} = 0, \tag{15a}
\]
\[
\nabla_{CC'}(\nabla_{AA'}\eta_{BB'} + \nabla_{BB'}\eta_{AA'})|_{S_0} = 0. \tag{15b}
\]

Assume further that
\[
\square \eta_{AA'} = 0,
\]
in an open set \(\mathcal{W}\) which contains \(S_0\). Then, there exists an open set \(\mathcal{U} \subset \mathcal{W}\) containing \(S_0\) such that the condition
\[
\nabla_{AA'}\eta_{BB'} + \nabla_{BB'}\eta_{AA'} = 0,
\]
holds on \(\mathcal{U}\).

**Proof.** Define
\[
S_{AA'BB'} \equiv \nabla_{AA'}\eta_{BB'} + \nabla_{BB'}\eta_{AA'}. \tag{16}
\]
A lengthy, but straightforward calculation renders
\[
\square S_{AA'BB'} = \nabla_{AA'}\square \eta_{BB'} + \nabla_{BB'}\square \eta_{AA'} +
\[+ 2\Psi_{AB}^{PQ} S_{PA'QB'} + 2\Psi^{P'Q'}_{A'B'} S_{AP'BQ'}, \tag{17}\]
\[
\square \eta_{BB'} = \nabla_{AA'} S_{BB'}^{AA'} - \frac{1}{2} \nabla_{BB'} S_{AA'}^{AA'}, \tag{18}\]
which can be regarded as a system of partial differential equations in the variables \(\eta_{AA'}, S_{AA'BB'}\). In the open set \(\mathcal{W}\) the system (17)-(18) becomes
\[
\square S_{AA'BB'} = 2\Psi_{AB}^{PQ} S_{PA'QB'} + 2\Psi^{P'Q'}_{A'B'} S_{AP'BQ'}, \tag{19a}\]
\[
\nabla_{AA'} S_{BB'}^{AA'} - \frac{1}{2} \nabla_{BB'} S_{AA'}^{AA'} = 0. \tag{19b}\]

This is to be supplemented by the initial conditions
\[
S_{AA'BB'}|_{S_0} = 0, \quad \nabla_{CC'} S_{AA'BB'}|_{S_0} = 0, \tag{20}\]
implied by (15a)-(15b). Theorem 1 tells us that the initial value problem posed by (19a) and the conditions (20) has the unique solution \(S_{AA'BB'} = 0\) in a neighbourhood \(\mathcal{U}\) containing \(S_0\). Trivially, this solution is consistent with (19b) and therefore the proof is complete. \(\square\)

**Remark 1** It is important to point out that although the result given by proposition 5 is local, a global version which guarantees the existence of a Killing vector on the whole of \(\mathcal{M}\) can be obtained if the spacetime and \(\eta_{AA'}|_{S_0}\) are suitably smooth. One needs to ensure that the Killing vector candidate \(\eta_{AA'}\) satisfying \(\square \eta_{AA'} = 0\) exists on the whole of \(\mathcal{M}\). An example of the conditions ensuring this can be found in [5].
Proposition [1] enables us to determine under which conditions a Killing vector will exist in a neighbourhood of $S_0$. In our framework it is important to express this result in terms of objects which are intrinsic to $S_0$ and the resulting conditions are called Killing initial data. In order to find the Killing initial data let us start by finding the orthogonal decomposition of the Killing equation

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 0, \quad (21)$$

with respect to $\tau_{AA'}$. First, we decompose $\xi_{AA'}$ by writing

$$\xi_{AA'} = \frac{1}{2} \tau_{AA'}\xi - \tau_{AQ}^A \xi_{AQ}.$$  

Contracting the Killing equation (21) in all possible manners with $\tau_{AA'}$ and symmetrising when necessary one obtains the following expressions:

$$\nabla\xi - K_{PQ}\xi^{PQ} = 0, \quad (22a)$$

$$\nabla_{BC}\xi + \nabla_{BC}\xi + \frac{1}{2} K_{BC}\xi - K_{P(B}\xi_{C)}^P - K_{BCPQ}\xi^{PQ} = 0, \quad (22b)$$

$$\nabla_{(AB}\xi_{CD)} + \frac{1}{2} K_{(ABCD)}\xi - 2 K_{(ABC}^Q \xi_{D)Q} = 0. \quad (22c)$$

where $\nabla_{AA'}$ has been decomposed in terms of $\nabla_{AB}$ and $\nabla$ —see (6). Equations (22a)- (22c) are equivalent to

$$D\xi_{CF} = K_{CFAL}\xi_{AL} - \frac{1}{2} K_{CF}\xi - D_{CF}\xi, \quad (23a)$$

$$D_{AB}\xi_{CD} + D_{CD}\xi_{AB} + \xi K_{ABCD} = 0. \quad (23b)$$

Condition (23b) is intrinsic to the leaves of $S_s$ and in particular to $S_0$. If we apply the operator $D$ to it one obtains

$$DD_{AB}\xi_{CD} + DD_{CD}\xi_{AB} + K_{ABCD}D\xi + \xi DK_{ABCD} = 0. \quad (24)$$

To see that this last expression is indeed intrinsic to $S_0$ we transform it by using the commutator $[11a]$ together with equations (22a), (23a) and (12a). This yields

$$D_{DL}D_{CF}\xi + D_{CF}D_{DL}\xi + K_{DL}^{AB}D_{AB}\xi_{CF} + K_{CF}^{AB}D_{AB}\xi_{DL} + \xi (2E_{CDLF} - K_{CF}^{AB}K_{DLAB}) - D_{DL}(\xi^{AB}K_{CFAB}) - D_{CF}(\xi^{AB}K_{DLAB}) + 4i \xi^A_{(C}B_{DFL)A} = 0, \quad (25)$$

where a foliation has been chosen for which the acceleration vanishes —as we discussed earlier this can always be done in a neighbourhood of $S_0$.  

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The aim of previous calculations is to show that equations (23b) and (25) will hold on $S_0$ whenever a Killing vector $\eta_{AA'}$ exists. The next proposition asserts that there is a converse to this property.

**Proposition 2 (Killing initial data)** Let $(\mathcal{S}, h_{ij}, K_{ij})$ be an initial data set for the vacuum Einstein field equations and let $(\mathcal{M}, g_{\mu\nu})$ be the data development. Assume further that on $\mathcal{S}$ there exist a scalar $\tilde{\xi}$ and a space spinor $\tilde{\xi}_{AB}$ satisfying the conditions (Killing initial data)

$$D_{AB}\tilde{\xi}_{CD} + D_{CD}\tilde{\xi}_{AB} + \tilde{\xi}\tilde{K}_{ABCD} = 0,$$

$$D_{DL}D_CF\tilde{\xi} + D_CF D_{DL}\tilde{\xi} + \tilde{K}_{DL}^{AB}D_{AB}\tilde{\xi}_{CF} + \tilde{K}_{CF}^{AB}D_{AB}\tilde{\xi}_{DL} - \tilde{\xi}(2\tilde{E}_{CDFL} - \tilde{K}_{CF}^{AB}\tilde{K}_{DLAB}) - D_{DL}(\tilde{\xi}_{AB}\tilde{K}_{CFAB}) - D_{CF}(\tilde{\xi}_{AB}\tilde{K}_{DLAB}) + 4i\tilde{\xi}^A\frac{\epsilon^C}{(C\tilde{B}_{DFL})_A} = 0.$$  

(26a)  

Then there exists a spinorial field $\dot{\xi}_{AA'}$ which is a Killing vector of $g_{\mu\nu}$ on a neighbourhood of $\mathcal{M}$.

**Proof.** Construct a foliation in the development $\mathcal{M}$ with leaves $\mathcal{S}_s$ such that $S_0$ is identified with $\mathcal{S}$ and $K_{AB} = 0$ in at least a neighbourhood of $S_0$. If $\tau_{AA'}$ is the normal to this foliation we define a spacetime spinor $\dot{\xi}_{AA'}$—the Killing vector candidate— by requiring that it satisfies the wave equation

$$\square\dot{\xi}_{AA'} = 0,$$

supplemented with the initial conditions

$$\dot{\xi}|_{S_0} = \tilde{\xi}, \quad \dot{\xi}_{AB}|_{S_0} = \tilde{\xi}_{AB},$$

$$D\dot{\xi}|_{S_0} = 0, \quad D\dot{\xi}_{AB}|_{S_0} = -D_{AB}\tilde{\xi} + \tilde{K}_{ABPQ}\tilde{\xi}^{PQ}. \quad (28b)$$

According to theorem 1 a solution for this initial value problem exists in at least a neighbourhood of $S_0$ if the spacetime and the initial data for $\xi_{AA'}$ are suitably smooth. Next, we define the spinor $\dot{S}_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'}$. By a procedure similar to the one followed in the calculation of the orthogonal decomposition of equation (21), we work out the orthogonal decomposition of $\dot{S}_{AA'BB'}$ with respect to the foliation $\mathcal{S}_s$ to obtain

$$\dot{S}_{CC'D'D'}\tau_F^{CC'}\tau_L^{D'D'} = \dot{\Sigma}_{DLCF} + \frac{1}{4}\epsilon_{CF}\epsilon_{DL}D\dot{\xi} + \frac{1}{2}\epsilon_{CF}\dot{G}_{DL} + \frac{1}{2}\epsilon_{DL}\dot{G}_{CF}. \quad (29)$$

with
\[ \dot{G}_{DL} \equiv -K_{DLAB} \dot{\xi}_{AB} + D_{DL} \dot{\xi} + D_{DL} \dot{\xi}, \]
\[ \dot{\Sigma}_{DLCF} \equiv D_{DL} \dot{\xi}_{CF} + D_{CF} \dot{\xi}_{DL} + \dot{\xi} K_{CFDL} \]

Equations (26a) and (28a)-(28b) entail
\[ \dot{\Sigma}_{ABCD} |_{S_0} = 0, \quad \dot{G}_{AB} |_{S_0} = 0, \]
and thus
\[ \dot{S}_{AA'BB'} |_{S_0} = \left( \nabla_{AA'} \dot{\xi}_{BB'} + \nabla_{BB'} \dot{\xi}_{AA'} \right) |_{S_0} = 0. \]

Next, it is noted that
\[ \nabla_{CC'} \dot{S}_{AA'BB'} = \frac{1}{2} \tau_{CC'} D \dot{S}_{AA'BB'} - \tau_{Q} C_{D} D_{C} Q \dot{S}_{AA'BB'} + F_{CC'} A \dot{S}_{AA'BB'}, \quad (30) \]
where \( F_{CC'} A \dot{S}_{AA'BB'} \) is linear in \( \dot{S}_{AA'BB'} \). Now, from \( \dot{S}_{AA'BB'} |_{S_0} = 0 \), it follows that \( D_{CD} \dot{S}_{AA'BB'} |_{S_0} = 0 \), and further that \( F_{CC'} A \dot{S}_{AA'BB'} |_{S_0} = 0 \). On the other hand, applying the operator \( D \) to equation (29) yields

\[ \frac{1}{4} D S_{AA'} A + \frac{1}{2} \tau_{AB'} A D S_{BB'} A - \frac{1}{2} D_{BP} S_{AA'} A \]
\[ \quad - \tau_{AB'} A D_{C} B S_{BB'} C + F_{BP} = 0, \]
where \( F_{BP} \) is linear in \( \dot{S}_{AA'BB'} \). If we replace here \( \dot{S}_{AA'BB'} \) by the expression given by (29) and evaluate the result on \( S_0 \) we deduce
\[ \frac{1}{2} D \dot{G}_{AB} |_{S_0} + \frac{1}{4} \epsilon_{AB} D^2 \dot{\xi} |_{S_0} = 0, \]
from which we conclude that \( D \dot{G}_{AB} |_{S_0} = 0 \) and \( D^2 \dot{\xi} |_{S_0} = 0 \). Combining this with (31) and (32) we obtain that \( DS_{AA'BB'} |_{S_0} = 0 \) and thus due to the above
\[ \nabla_{CC'} \hat{S}_{AA'B'B'}|_{S_0} = \nabla_{CC'}(\nabla_{AA'}\hat{\xi}_{BB'} + \nabla_{BB'}\hat{\xi}_{AA'})|_{S_0} = 0. \] (33)

Finally, proposition 1 now implies that \( \hat{S}_{AA'B'B'} \) vanishes in a neighbourhood of \( S_0 \) so that \( \hat{\eta}_{AA'} \) is a Killing vector in that neighbourhood. \( \Box \)

**Remark 2** From the previous discussion we deduce that equations (26a)–(26b) are necessary and sufficient conditions for the existence of a Killing vector in the data development. These conditions are the (spinorial version of the) *Killing initial data equations (KID equations)*—see e. g. [3,6] for a derivation of the tensorial version. These can be regarded as an overdetermined elliptic system for the *longitudinal* \( \hat{\xi} \) and *transverse* \( \hat{\xi}_{AB} \) parts of the Killing vector \( \xi_{AA'} \) at the initial hypersurface.

### 6 Twistor initial data

Consider now a spinor of valence-1 satisfying the twistor equation (1a), that is

\[ \nabla_{A'(A\kappa_B)} = 0. \]

The spinor \( \kappa_A \) is called a *valence-1 Killing spinor*. As mentioned in the introduction, a necessary condition for the solvability of equation (1a) is that the spacetime be of Petrov type N—this can be seen by differentiating (1a) and using the identities (10) to render condition (2).

If one is able to find conditions on a vacuum initial data set ensuring the existence of a valence-1 Killing spinor in the data development then one may conclude that the development must be of Petrov type N.

**Proposition 3** Let \( \kappa_A \) be a spinorial field on \( \mathcal{M} \) such that

\[ \nabla_{A'(A\kappa_B)}|_{S_0} = 0, \quad (\nabla_{EE'}\nabla_{A'(A\kappa_B)})|_{S_0} = 0, \] (34)

where \( S_0 \subset \mathcal{M} \) is a spacelike Cauchy hypersurface. Assume further that

\[ \Box \kappa_A = 0 \]

in an open set \( \mathcal{W} \) which contains \( S_0 \). Then, there exists an open set \( \mathcal{U} \subset \mathcal{W} \) containing \( S_0 \) such that the condition

\[ \nabla_{A'(A\kappa_B)} = 0, \]

holds on \( \mathcal{U} \).
Proof. Define
\[ H_{A'AB} \equiv 2\nabla_A(A\kappa_B). \tag{35} \]
A straight-forward calculation using the commutators of the covariant derivatives renders
\[ \Box H_{A'AB} = 2\nabla_A(\Box \kappa_B) + 2\Psi \nabla^{PQ} H_{A'PQ}, \tag{36a} \]
\[ \Box \kappa_A = \frac{2}{3} \nabla^{PP'} H_{P'PA}, \tag{36b} \]
which in \( \mathcal{W} \) becomes
\[ \Box H_{A'AB} = 2\Psi \nabla^{PQ} H_{A'PQ}, \quad \nabla^{PP'} H_{P'PA} = 0. \tag{37} \]
Now, (34) implies that
\[ H_{A'AB}|_{S_0} = 0, \quad \nabla_{EE'} H_{A'AB}|_{S_0} = 0. \tag{38} \]
Again, the initial value problem (37)-(38) has the unique solution \( H_{A'AB} = 0 \) in at least a neighbourhood \( \mathcal{U} \subset \mathcal{W} \) containing \( S_0 \). Hence \( \kappa_A \) is a solution of the twistor equation on \( \mathcal{U} \). \( \Box \)

Remark 3 As in the case of the proof of proposition 1 the local character of the last proposition can be improved if the spacetime and \( \kappa_A|_{S_0} \) are suitably smooth.

Next, we deduce intrinsic conditions on a vacuum initial data set for its development to have a valence-1 Killing spinor, and accordingly to be of Petrov type N. The procedure is rather similar to that followed in section 3 for the case of Killing vectors. Suppose that \( \kappa_A \) is a valence-1 Killing spinor. Then the orthogonal decomposition of the twistor condition with respect to the spinor \( \tau^{AA'} \) gives
\[ \nabla \kappa_B + \frac{2}{3} \nabla_B \kappa_Q = 0, \tag{39a} \]
\[ \epsilon_{AC} \nabla \kappa_B + \epsilon_{BC} \nabla \kappa_A + 2\nabla_{AC} \kappa_B + 2\nabla_{BC} \kappa_A = 0. \tag{39b} \]
The symmetrisation of the last equation yields
\[ \nabla_{(AB} \kappa_C) = 0. \tag{40} \]
It is also noted that the contraction of any two indices in (39b) renders equation (39a). Hence the whole content of the twistor equation is expressed in equations (39a) and (40). Next, we use the relation between the operators \( \nabla_{AB} \) and \( D_{AB} \) to rewrite equation (40) in the form
\[ D_{(AB} \kappa_C) - \frac{1}{2} K_{Q(ABC)} \kappa_Q = 0, \tag{41} \]
which is intrinsic to each of the integral surfaces $S_s$ of $\tau_{AA'}$. This equation is known in older accounts as the spatial twistor equation \[22,12\]. From equation (41) we obtain the condition

$$ DD_{(AB\kappa_C)} - \frac{1}{2} (\kappa^Q DK_Q(ABC) + K_Q(ABC) D\kappa^Q) = 0, $$

which, by using the commutator relation (11a) and equation (12a), can be transformed into

$$ 2D_{(AB,D_F)C}\kappa^C + \frac{1}{2} \kappa_{(F,D_AB)} K^{CD}_{\ CD} - \frac{1}{2} K^{CD}_{\ CD} D_{(BF\kappa_A)} - 3\Omega_{CD(BF} D^{CD \kappa_A)} + \frac{3}{2} \Omega_{(AF^{DH} \Omega_B)CDH} \kappa^C - 3(iB_{ABFC} + E_{ABFC}) \kappa^C $$

$$ + \frac{3}{4} K^{CD}_{\ CD} \kappa^C - \Omega_{ABFD} D^D \kappa^C = 0. $$

where we have defined $\Omega_{ABCD} \equiv K_{(ABCD)}$, the trace-free part of the second fundamental form.

Equations (40) and (42) are intrinsic to each of the leaves $S_s$ and therefore they are necessary conditions for the existence of a twistor in the development of a vacuum initial data set. The converse of this statement also holds and its proof is analogous to that of proposition 2.

**Proposition 4 (Twistor initial data)** Let $(S,h_{ij},K_{ij})$ be an initial data set for the vacuum Einstein field equations and let $(M,g_{\mu\nu})$ be its data development. Assume further that on $S$ there exists a spinor $\tilde{\kappa}_A$ satisfying the conditions (twistor initial data equations)

\[D_{(AB\tilde{\kappa}_C)} + \frac{1}{2} \tilde{\kappa}^Q_{(ABC)} \tilde{\kappa}_Q = 0,\] (43a)

\[2D_{(AB,D_F)C}\tilde{\kappa}^C + \frac{1}{2} \tilde{\kappa}_{(F,D_AB)} \tilde{K}^{CD}_{\ CD} - \frac{1}{2} \tilde{K}^{CD}_{\ CD} D_{(BF\tilde{\kappa}_A)} + \frac{3}{2} \Omega_{(AF^{DH} \tilde{O}_B)CDH} \tilde{\kappa}^C - 3(i\tilde{B}_{ABFC} + \tilde{E}_{ABFC}) \tilde{\kappa}^C $$

$$ + \frac{3}{4} \tilde{K}^{CD}_{\ CD} \tilde{\kappa}^C + \tilde{\Omega}_{ABFD} D^D \tilde{\kappa}^C = 0. \] (43b)

Then there exists a spinorial field $\tilde{\kappa}_A$ satisfying the twistor equation (1a) on an open subset of $S$.

**Proof.** Consider a foliation of the development $M$ with leaves $S_s$. Identify $S_0$ with $S$. In a neighbourhood of $S_0$ this foliation is chosen in such a way that $K_{AB} = 0$. We introduce a spacetime spinor $\check{\kappa}_A$ —the Killing spinor candidate— satisfying

$$ \square \check{\kappa}_A = 0, $$

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subject to the initial conditions

\[ \kappa_A|_{S_0} = \tilde{\kappa}_A, \quad \text{(44a)} \]

\[ D\kappa_A|_{S_0} = \frac{1}{6} K^{CF} C_F \kappa_A - \frac{2}{3} D_A C \kappa_C. \quad \text{(44b)} \]

Again, theorem 1 guarantees that the above initial value problem has a solution in at least a neighbourhood of \( S_0 \) if the spacetime and initial data are suitably smooth. Define \( \hat{H}_{AB} = 2 \nabla_A \kappa_B \). A computation similar to that carried out to obtain equations (39a)-(39b) gives

\[ \hat{H}_{AB} = 2 \Sigma_{ABD} + \epsilon_B D \hat{\kappa}_A + \frac{1}{2} \epsilon_{AB} \hat{G}_D, \quad \text{(45)} \]

with

\[ \Sigma_{ABD} \equiv \left( D_{(AB)} \kappa_D \right) - \frac{1}{2} K_{(AB)C} \kappa^C, \quad \hat{G}_A \equiv \left( D \kappa_A \right) - \frac{1}{6} K^{CF} C_F \kappa_A - \frac{2}{3} D_A C \kappa_C. \]

From (44a)-44b we deduce \( \hat{G}_A|_{S_0} = 0 \) and (43a) yields \( \Sigma_{ABC}|_{S_0} = 0 \). Therefore

\[ \hat{H}_{AB}|_{S_0} = 2 \nabla_A \kappa_B|_{S_0} = 0. \]

Next, we consider the relation

\[ \tau^C \nabla_{CCP} \hat{H}_{AB} = D_{CP} \hat{H}_{AB} + \frac{1}{2} \epsilon_{CP} D \hat{H}_{AB} + F_{ABCPA'}, \quad \text{(46)} \]

where \( F_{ABCPA'} \) is linear in \( \hat{H}_{AB} \). Since we have \( \hat{H}_{AB}|_{S_0} = 0 \) we deduce \( D_{CP} \hat{H}_{AB}|_{S_0} = 0 \). On the other hand applying the operator \( D \) to equation (45) we obtain

\[ \tau^A D \hat{H}_{AB} - 2 D \Sigma_{ABD} = \epsilon_B D \hat{G}_A + \frac{1}{2} \epsilon_{AB} D \hat{G}_D, \quad \text{(47)} \]

From condition (43b) we deduce

\[ D \Sigma_{ABD}|_{S_0} = D \left( D_{(AB)} \kappa_D \right) - \frac{1}{2} K_{(AB)C} \kappa^C \big|_{S_0} = 0, \]

with an argument similar to that of the calculation which led to (42). Also the condition \( \Box \kappa_{AB} = 0 \) implies

\[ \nabla^{PP'} \hat{H}_{P'PA} = 0, \]

as is clear from the proof of proposition 3. Using in this expression (46) and (47) we get
\[
\frac{3}{4} D\hat{G}_A + 2D_{PB}\hat{\Sigma}_A^{PB} + \frac{1}{2} D_{AP}\hat{G}^P - \\
- \frac{1}{4}\hat{G}^P K_A^B_{PB} + \hat{G}_A K^{PB}_{PB} - K_{APBC}\hat{\Sigma}^{PBC} = 0,
\]
from which we conclude \(D\hat{G}_{AB}|_{S_0} = 0\) and hence, via (47), \(D\hat{H}_{A'AB}|_{S_0} = 0\). Thus (46) yields
\[
\nabla_{CC'}\hat{H}_{A'AB}|_{S_0} = \nabla_{CC'}\nabla_{A'(A}\hat{k}_{B)}|_{S_0} = 0. \quad (48)
\]
Proposition 3 implies that \(\hat{H}_{A'AB} = 0\) in a neighbourhood of \(S_0\) and hence \(\hat{k}_{A}\) is a solution of the twistor equation (1a) in such a neighbourhood. \(\square\)

An application of the last result is the following

**Theorem 2 (Type N initial data)** Let \((S, h_{ij}, K_{ij})\) be a vacuum initial data set and suppose that there exists a spinor \(\tilde{\kappa}_A\) on \(S\) fulfilling (43a)-(43b). Then the development \((M, g_{\mu\nu})\) contains an open set \(W\) such that \((W, g_{\mu\nu})\) is of Petrov type N.

**Proof.** This is a direct consequence of equation (2) and of proposition 4. \(\square\)

7 **Valence-2 Killing spinor initial data**

Next, we explain how to construct vacuum initial data such that their development contains a valence-2 Killing spinor. As we did in the case of Killing vectors and twistors, we start by finding a hyperbolic system which will be used as the basis to study the propagation of the differential condition which guarantees the existence of valence-2 Killing spinors. In order to obtain such a system of propagation equations, one has to consider simultaneously the propagation of the Killing vector associated to the Killing spinor.

**Proposition 5** Let \(\kappa_{AB}\) be a spinor defined on \(M\) and such that on a spacelike Cauchy hypersurface \(S_0\) one has
\[
\nabla_{A'(A}\kappa_{BC)}|_{S_0} = 0, \quad (49a)
\]
\[
\nabla_{EE'}\nabla_{A'(A}\kappa_{BC)}|_{S_0} = 0, \quad (49b)
\]
\[
(\nabla_{AA'}\nabla_{B'}^{P}K_{BP} + \nabla_{BB'}\nabla_{A'}^{P}\kappa_{AP})|_{S_0} = 0, \quad (49c)
\]
\[
\nabla_{EE'}\left(\nabla_{AA'}\nabla_{B'}^{P}K_{BP} + \nabla_{BB'}\nabla_{A'}^{P}\kappa_{AP}\right)|_{S_0} = 0. \quad (49d)
\]
Assume further that the condition
\[
\square\kappa_{AB} + \Psi_{ABPQ}\kappa^{PQ} = 0
\]
holds on an open set $W$ containing $S_0$. Then there exists an open set $U \subset M$ containing $S_0$ such that

$$\nabla_{A'(A^K_{BC})} = 0,$$

on $U$.

**Proof.** Define

$$H_{A'ABC} \equiv 3 \nabla_{A'(A^K_{BC})}, \quad (50a)$$

$$\xi_{AA'} \equiv \nabla^D A^K_{DAA'}, \quad (50b)$$

$$S_{AA'BB'} \equiv \nabla_{AA'} \xi_{BB'} + \nabla_{BB'} \xi_{AA'}. \quad (50c)$$

As in previous sections, the general strategy will be to construct a hyperbolic system with $H_{A'ABC}$ as one of its unknowns. First of all, we need to find a relation between $S_{AA'BB'}$ and $H_{A'ABC}$. To that end, we replace in (50c) the spinor $\xi_{AA'}$ by its expression in terms of $\kappa_{AB}$. This gives

$$S_{CC'DD'} = -\nabla_{DD'} \nabla^A C^K_{CA} - \nabla_{CC'} \nabla^A D^K_{DA}. \quad (51)$$

Now, in this last expression we use the identity

$$2\nabla_{DD'} \nabla^A C^K_{CA} = -\epsilon_{CD} \epsilon_{C'D'} \nabla^B A^K_{A'B} \nabla^A'A^K_{AB} + 2\nabla_{C'(D')} \nabla^A_{A'K_{D'A}} + 2\epsilon_{CD} \nabla_{C'} A^K_{A'B} \nabla_{D'} A^K_{A'B} + \epsilon_{C'D'} \nabla_{C} A^K_{A'B} \nabla_{D} A^K_{A'B}. \quad (52)$$

After some lengthy algebra involving the commutation of the covariant derivatives and the grouping of some terms by means of $H_{A'ABC} = 3 \nabla_{A'(A^K_{BC})}$, we arrive at —see appendix B for further details about this calculation—

$$S_{CC'DD'} = -\frac{1}{2} \nabla^A C^K_{C'D'CD}. \quad (53)$$

which is kept for later use. A straightforward calculation using the decomposition of a spinor in terms of its totally symmetric part and symmetrised contractions yields

$$\nabla_{E'E'H_{A'ABC}} = \nabla_{E'(E'H_{ABC})A'} + \frac{1}{2} (\epsilon_{EA} S_{BE'C'A'} + \epsilon_{EB} S_{AE'C'A'} + \epsilon_{EC} S_{AE'B'A'}),$$

where equation (53) has been used. Now, using that

$$\nabla_{E'(E'H_{ABC})A'} = \frac{1}{4} \left( \nabla_{E'E'H_{ABC}A'} + \nabla_{E'A'H_{EBCA'}} + \nabla_{E'B'H_{EACA'}} + \nabla_{E'C'H_{EABA'}} \right),$$

we obtain
\(\Box H_{A'ABC} = \frac{1}{4} \Box H_{A'ABC} \)

\[
+ \frac{1}{4} \left( \nabla^{EE'} \nabla_{E'A} H_{A'ABC} + \nabla^{EE'} \nabla_{E'B} H_{A'EC} + \frac{1}{4} \nabla^{EE'} \nabla_{E'B} H_{A'EC} \right) \\
+ \frac{1}{2} \left( \nabla^{E' \mathcal{C}} S_{B'\mathcal{C}'A'} + \nabla^{E' \mathcal{C}} S_{A'\mathcal{C}'A'} + \nabla^{E' \mathcal{C}} S_{A'\mathcal{C}'B'} \right).
\]

(54)

Now, if we make use of the identity

\[
\nabla_{AC'} \nabla_{C'B} = \Box_{AB} + \frac{1}{2} \epsilon_{AB} \Box,
\]

(55)

and of expression (10), equation (54) reduces to

\[
\Box H_{A'ABC} = 4(\Psi_{(AB)\mathcal{P}Q} H_{C} H_{\mathcal{P}Q}A' + \nabla_{(A' S_{BC})Q'\mathcal{P}}).
\]

(56)

The latter hyperbolic equation for \(H_{A'ABC}\) has to be complemented with another hyperbolic equation for \(S_{AA'BB'}\) which we compute next. We calculate \(\nabla^{CA'} H_{A'ABC}\) from equation (50a) to obtain

\[
\nabla_{AC'} \nabla_{C'B} = \Box_{AB} - 2\nabla_{(A'} \nabla_{C)}\nabla_{B').
\]

We work out the last term of the right hand side with equations (55) and (10). The final result is

\[
\nabla^{CA'} H_{A'ABC} = \Box_{AB} + \Psi_{ABPQ} \kappa_{PQ} = 0.
\]

(57)

where, by hypothesis, the last equality only holds in the open set \(\mathcal{W}\). Also a direct calculation shows that in \(\mathcal{W}\)

\[
\Box S_{A'\mathcal{A}B'} = -\nabla_{AA'}(\Psi_{B}^{PQR} H_{B'PQR}) - \nabla_{BB'}(\Psi_{A}^{PQR} H_{A'PQR}) \\
+ 2\Psi_{AB}^{PQ} S_{AP'QB'} + 2\Psi_{A'B'}^{P'Q'} S_{A'P'B'Q'},
\]

(59)

which is the required hyperbolic equation for \(S_{A'\mathcal{A}B'}\). Now, we note that the system of hyperbolic partial differential equations formed by equations (56) and (59) falls within theorem 1 if we take \(H_{A'ABC}\) and \(S_{A'\mathcal{A}B'}\) as the unknowns. Conditions (49a)-(49d) imply that the initial data of such a hyperbolic system are

\[
H_{A'ABC}|_{S_{0}} = 0, \quad \nabla_{DD'} H_{A'ABC}|_{S_{0}} = 0, \\
S_{A'\mathcal{A}B'}|_{S_{0}} = 0, \quad \nabla_{DD'} S_{A'\mathcal{A}B'}|_{S_{0}} = 0,
\]

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and therefore we deduce that \( H_{A'ABC} = 0 \) and \( S_{AA'BB'} = 0 \) in a neighbourhood of \( S_0 \). Note that equations (53), (57) which are constraints of the dependent variables in the hyperbolic system are now trivially fulfilled.

**Remark 4** Again, if the spacetime and the restriction of \( \kappa_{AB} \) to \( S_0 \) are suitably smooth, then it is possible to extend the existence of the Killing spinor to the whole of \( \mathcal{M} \).

From the proof of this proposition we note the following

**Corollary 1** If \( \tilde{\xi}_{AA'} = \nabla^Q_A \tilde{\kappa}_{QA} \) is such that \( \tilde{\xi}_{AA'} = \xi_{AA'}, \ \nabla_{CC'} \tilde{\xi}_{AA'} = \nabla_{CC'} \xi_{AA'} \) on \( S_0 \) then \( \tilde{\xi}_{AA'} = \xi_{AA'} \) on \( \mathcal{M} \).

**Proof.** This follows directly from the wave equation (58).

Hence, if the Killing vector associated to the Killing vector is real on the initial hypersurface, then it is also real at later times. This corollary is useful to characterise a class of initial data sets for Petrov type D spacetimes which includes initial data sets for the Kerr spacetime — see section 8.

Next, we proceed to obtain necessary conditions for the development of a vacuum initial data set to admit valence-2 Killing spinors. This is accomplished in a similar fashion as in the previous sections. The orthogonal decomposition of the Killing spinor equation (11) renders the expressions

\[
\nabla \kappa_{BC} + \nabla (A \kappa_{C})_A = 0, \tag{60a}
\]

\[
\epsilon_{AD} \nabla \kappa_{BC} + \epsilon_{CD} \nabla \kappa_{AB} + \epsilon_{BD} \nabla \kappa_{AC} + 2(\nabla_{AD} \kappa_{BC} + \nabla_{CD} \kappa_{AB} + \nabla_{BD} \kappa_{AC}) = 0. \tag{60b}
\]

The \( \nabla \)-derivative in (60a) can be transformed into a \( D \)-derivative to yield

\[
D \kappa_{AC} = -D_{(C} \kappa_{A)B} - \frac{1}{4} K^B_{BF} \kappa_{BFC} - \frac{1}{2} K_{ABC} \kappa_{BF} - K_{(C} B \kappa_{A)B}. \tag{61}
\]

Hence equation (60b) is equivalent to

\[
\nabla_{(AB} \kappa_{CD)} = 0
\]

so if we transform in this expression the covariant derivative \( \nabla_{AB} \) into \( D_{AB} \) we obtain.

\[
D_{(AB} \kappa_{CD)} - K_{E(ABC)D} E = 0. \tag{62}
\]

Equations (61)–(62) are completely equivalent to (60a)–(60b). Equation (62) is intrinsic to the leaves \( S_s \) and hence it is a necessary condition for the existence of a valence-2 Killing spinor in the data development. Another necessary
condition is obtained from

\[ D(D_{(AB} \kappa_{CD)} - K_{E(ABC} \kappa_{D)}^E) = 0. \]

As in previous sections we transform this equation by means of the commutator \((11a)\), equation \((12a)\) and equation \((61)\). We choose a foliation with vanishing acceleration in at least a neighbourhood of \(S_0\) in order to perform these calculations. The resulting expression is

\[
\begin{align*}
D_{(AC} D_{B} \kappa_{D)}^F &+ \frac{1}{2} D_{(AB} \kappa_{CD)}^F \Omega_{CF} D_{D)}^H + \Omega_{DH(BF} D_{DF)}^H \kappa_{AC)}^D \\
- \frac{1}{2} \Omega_{H(ABC} D_{F)}^D \kappa_{D)}^H &- \frac{1}{2} \Omega_{H(ACF} D_{DH)}^D \kappa_{B)}^D + 2(i B_{D(BCF} + E_{D(BCF)} \kappa_{A)}^D \\
- \left( \frac{1}{3} K_{HL}^H \Omega_{D(ABC} + \Omega_{DHL(A} \Omega_{BC)}^H) \kappa_{B)}^D \right) &+ \frac{1}{2} \kappa_{D}^D \Omega_{DHL} \Omega_{BCF}^L \\
- \frac{1}{3} \kappa_{(AB} D_{CF)} \kappa_{D)}^H &= 0.
\end{align*}
\]

Another set of necessary conditions arises from the orthogonal decomposition of \(\xi_{FA'} = \nabla D A' \kappa_{DF}\). In the spirit of the space-spinor formalism we write again

\[
\xi_{AA'} = \frac{1}{2} \xi_{\tau AA'} - \tau^Q A' \xi_{AQ}.
\]

A direct calculation shows that

\[
\begin{align*}
\xi &= \xi_{AA'} \tau_{AA'} = \nabla^P Q K_{PQ} = D^P Q K_{PQ}, \\
\xi_{AB} &= \tau_{(A} C_{(B}^C C^Q - \frac{1}{2} \nabla \kappa_{AB} \\
&= - \frac{1}{2} K_{PQ} K_{AB}^P + \frac{3}{4} K_{PQ} \Omega_{ABPQ} + \frac{3}{2} D_{(A} K_{B)P} \quad (63a)
\end{align*}
\]

where in the last equation the propagation equation \((61)\) has been used to simplify.

As it is to be expected, one has the following result

**Proposition 6 (Valence-2 Killing spinor initial data)** Let \((S, h_{ij}, K_{ij})\) be an initial data set for the vacuum Einstein field equations such that there exists a space spinor \(\kappa_{AB}\) on \(S\) satisfying the equations
\begin{equation}
D_{(AB}\tilde{\kappa}_{CD)} - \tilde{K}_{E(ABC}\tilde{\kappa}_{D)}^E = 0, \tag{64}
\end{equation}

\begin{align*}
D_{(AC}D_B^D\tilde{\kappa}_{F)D} + & \frac{1}{2}D_{(AB}(\tilde{\kappa}^{DH}\tilde{\Omega}_{CF)}DH) + \tilde{\Omega}_{DH(BF}D^{DH}\tilde{\kappa}_{AC)}
- & \frac{1}{2}\tilde{\Omega}_{H(ABC}D^D\tilde{\kappa}^H_{D)} - \frac{1}{2}\tilde{\Omega}_{H(ACF}D^{DH}\tilde{\kappa}_{B)D} + 2(i\tilde{\Omega}_{BCF} + \tilde{E}_{D(CBF)}\tilde{\kappa}^D_{A})
- & \left(\frac{1}{3}\tilde{\kappa}^{HL}_{H L}\tilde{\Omega}_{D(ABC} + \tilde{\Omega}_{DHL}(\tilde{\Omega}^H_{BC})\right)\tilde{\kappa}^D_F + \frac{1}{2}\tilde{\kappa}^{DH}\tilde{\Omega}_{DHL}(\tilde{\Omega}^L_{BCF})
- & \frac{1}{3}\tilde{\kappa}(AB)D_{CF}\tilde{\kappa}^{DH}_{DH} = 0. \tag{65}
\end{align*}

In addition, assume that the space spinors \(\tilde{\xi}, \tilde{\xi}_{BF}\) defined by

\begin{align*}
\tilde{\xi} & \equiv D^{PQ}\tilde{\kappa}_{PQ}, \tag{66}
\tilde{\xi}_{BF} & \equiv -\frac{1}{2}\tilde{\kappa}^{DA}\tilde{\kappa}_{BF} + \frac{3}{4}\tilde{\kappa}^{DA}\tilde{\Omega}_{BFDA} + \frac{3}{2}D_{(F}\tilde{\kappa}_{B)D}, \tag{67}
\end{align*}

are such that they fulfil the conditions (26a)-(26b) of proposition \(\text{2}\). Then, there exists a spacetime spinor \(\hat{\kappa}_{AB}\) in a neighbourhood of the data development \(\mathcal{M}\) which is a valence-2 Killing spinor.

**Proof.** The proof of this result proceeds in a similar way as the proofs of propositions \(\text{2}\) and \(\text{4}\). We consider a foliation of the data development \(\mathcal{M}\) whose leaves are \(\mathcal{S}_s\). Identify \(\mathcal{S}_0\) with \(\mathcal{S}\). In a neighbourhood of \(\mathcal{S}_0\) the foliation is constructed in such a way that \(\tilde{K}_{AB} = 0\). We consider a Killing spinor candidate \(\hat{\kappa}_{AB}\) satisfying

\begin{align*}
\Box \hat{\kappa}_{AB} = -\Psi_{ABPQ}\hat{\kappa}^{PQ},
\end{align*}

with initial data on \(\mathcal{S}_0\) given by

\begin{align*}
\hat{\kappa}_{AB}|_{\mathcal{S}_0} & = \bar{\kappa}_{AB}, \tag{68a}
D\hat{\kappa}_{AC}|_{\mathcal{S}_0} & = -D_{(C}\hat{\kappa}_{B)A} + \frac{1}{2}\tilde{\kappa}^{BF}_{B F}\hat{\kappa}_{AC} - \frac{1}{2}\tilde{\kappa}_{ABC}^{\kappa BF}, \tag{68b}
\end{align*}

Again, theorem \(\text{1}\) ensures that this initial value problem has a solution in at least a neighbourhood of \(\mathcal{S}_0\) if the spacetime and the initial data for the Killing spinor are suitably smooth. Next we define \(\hat{H}_{A'ABC} \equiv 3\nabla_{A'(A}\hat{\kappa}_{BC)}\) and compute its orthogonal decomposition by a procedure similar to that followed to obtain the relations (61)-(62). This renders

\begin{equation}
\hat{H}_{A'ABC}A'F = 3\hat{\Sigma}_{ABCF} + \frac{1}{8}(\epsilon_{CF}\hat{G}_{AB} + \epsilon_{AB}\hat{G}_{CF}) + \frac{1}{4}\hat{e}_{BF}\hat{G}_{AC}, \tag{69}
\end{equation}

where
\[
\hat{G}_{AB} \equiv -K^{DH} D_H \hat{k}_{AB} + 2K_{ABDH} \hat{k}^{DH} + 4D_{(B} \hat{k}_{A)D} + 4D \hat{k}_{AB},
\]
\[
\hat{\Sigma}_{ABC} \equiv (D_{(AB} \hat{k}_{CF)} - K_{E(ABC)CF})^E.
\]

Clearly, conditions (68b) and (64) entail \(\hat{G}_{AB}|_{S_0} = 0\) and \(\hat{\Sigma}_{ABCD}|_{S_0} = 0\) respectively from which we get
\[
\hat{H}_{A'ABC}|_{S_0} = \nabla_{A'(A\hat{k}_{BC})}|_{S_0} = 0.
\]

In addition, we have
\[
\nabla_{AA'} \hat{H}_{B'BCD} = \frac{1}{2} \tau_{AA'} D \hat{H}_{B'BCD} - \tau_{AF} D_{AF} \hat{H}_{B'BCD} + F_{AA'B'BCD},
\]
(70)

with \(F_{AA'B'BCD}\) linear in \(\hat{H}_{B'BCD}\). Therefore, from the above \(D_{AB} \hat{H}_{A'CDF}|_{S_0} = 0\) and \(F_{AA'B'BCD}|_{S_0} = 0\). On the other hand, if we apply the operator \(D\) to equation (69) we obtain
\[
\tau_{A'F} D \hat{H}_{A'ABC} - 3D \hat{\Sigma}_{ABC} = \frac{1}{4} \tau_{BF} D \hat{G}_{AC} + \frac{1}{8} (\epsilon_{CF} D \hat{G}_{AB} + \epsilon_{AB} D \hat{G}_{CF}),
\]
(71)

Condition (65) entails
\[
D \hat{\Sigma}_{ABCD}|_{S_0} = D \left( D_{(AB} \hat{k}_{CF)} - K_{E(ABC)CF})^E \right)|_{S_0} = 0,
\]
as it is shown by a computation similar to the one which enabled us to obtain equation (63). Also the condition \(\square \hat{k}_{AB} + \Psi_{ABCD} \hat{k}^{CD} = 0\) implies —cfr. equation (57)—
\[
\nabla^{A'} \hat{H}_{A'ABC} = 0.
\]

We work out the orthogonal splitting of this condition by using (70) and (69) with the result
\[
\frac{1}{4} D \hat{G}_{AB} - \frac{1}{4} D_B \hat{C}_{AC} + \frac{1}{8} \epsilon_{AB} D^{CD} \hat{G}_{CD} + 3D^{CD} \hat{\Sigma}_{ABCD} +
\frac{1}{3} \hat{G}_{AB} \hat{k}^{CD} - \frac{1}{8} \hat{G}^{CD} \hat{Q}_{ABCD} - 3\hat{\Sigma}_{(B} \hat{C}^{DF} \hat{Q}_{A)CDF} = 0,
\]
from which we deduce that \(D \hat{G}_{AB}|_{S_0} = 0\). Thus (71) implies \(D \hat{H}_{A'ABC}|_{S_0} = 0\) and hence (70) yields
\[
\nabla_{AA'} \hat{H}_{D'ABC}|_{S_0} = \nabla_{AA'} \nabla_{D'(A\hat{k}_{BC})}|_{S_0} = 0.
\]

Now, let us define \(\hat{\xi}_{FA'} \equiv \nabla^{D} \hat{A}_{D} \hat{k}_{DF}\). As usual the orthogonal decomposition of this spinor is written in terms of \(\hat{\xi} \equiv \hat{\xi}_{AA'} \tau^{AA'}\), \(\hat{\xi}_{AB} \equiv \tau_{(A} \hat{\xi}_{B)C'}\). By a computation similar to that giving equations (63a)-(63b) and using (66)-(67) we conclude that \(\hat{\xi}|_{S_0} = \hat{\xi}, \hat{\xi}_{AB}|_{S_0} = \hat{\xi}_{AB}\). The hypothesis that \(\hat{\xi}, \hat{\xi}_{AB}\) fulfill
(26a)-(26b) and a reasoning similar to that used in the proof of proposition 2 enable us to prove
\[
(\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'})|_{S_0} = 0, \quad \nabla_{CC'}(\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'})|_{S_0} = 0.
\]

Proposition 5 now applies and therefore we conclude that \( k_{AB} \) is a valence-2 Killing spinor in a neighbourhood of \( S_0 \). \( \Box \)

### 7.1 Valence-2 Killing spinor development

If the scalar field \( \tilde{\xi} \) defined by equation (66) is nonzero on \( S \)—that is, if the Killing initial data associated to \( \tilde{k}_{AB} \) is transversal— one can make use of the notion of Killing development introduced in [2,3] to obtain a spacetime containing a valence-2 Killing spinor. Given \( (S, h_{ij}, K_{ij}) \) satisfying the Einstein (vacuum) constraints, let \( \mathcal{M} = \mathbb{R} \times S \) and define the metric
\[
\tilde{g} = \tilde{N}^2 du^2 + \tilde{h}_{ij}(dx^i + \tilde{Y}^i du)(dx^j + \tilde{Y}^j du),
\]  
(72)

where \( \tilde{N}(u, x) = \text{Re} \tilde{\xi}(x) \), if \( \text{Re} \tilde{\xi} \neq 0 \) on \( S \). Alternatively, if the imaginary part of \( \tilde{\xi} \) satisfies \( \text{Im} \tilde{\xi} \neq 0 \), then set \( \tilde{N}(u, x) = \text{Im} \tilde{\xi}(x) \). If \( \text{Re} \tilde{\xi} \neq 0 \), then the shift \( \tilde{Y}^i \) is constructed by setting
\[
\tilde{Y}^i(u, x) \equiv \sigma^i_{AB} \left( \tilde{\xi}^{AB}(x) + \tilde{\xi}^{-AB}(x) \right),
\]
with \( \tilde{\xi}^{AB} \) given by equation (67) and \( \tilde{\xi}^{AB} \) its Hermitian conjugate. If \( \tilde{N}(u, x) = \text{Im} \tilde{\xi}(x) \), then set
\[
\tilde{Y}^i(u, x) \equiv -i\sigma^i_{AB} \left( \tilde{\xi}^{AB}(x) - \tilde{\xi}^{-AB}(x) \right).
\]

If \( \tilde{\xi} \) and \( \tilde{\xi}^{AB} \) satisfy the KID equations (26a) and (26b), then using well known results about the formulation of General Relativity as a dynamical system, the metric (72) is a solution to the vacuum Einstein field equations and \( \partial_u \) is a Killing vector —see e.g. [2]. Now, if in addition conditions (64) and (65) of proposition 6 hold, then \( (\mathcal{M}, \tilde{g}_{\mu\nu}) \) has a valence-2 Killing spinor —the spinor \( \tilde{k}_{AB} \) is constructed from objects with vanishing Lie derivative along the flow defined by \( (\tilde{N}, \tilde{Y}^i) \). The spacetime \( (\mathcal{M}, \tilde{g}_{\mu\nu}) \) is then called valence-2 Killing spinor development.
8 Type D initial data sets

As discussed in the introduction, the rationale of studying conditions on a vacuum initial data set \((\mathcal{S}, h_{ij}, K_{ij})\) for the existence of Killing spinors in the development is to obtain results enabling us to decide its Petrov type. Spacetimes containing valence-2 Killing spinors are very special. Indeed, from theorem 7 discussed in appendix A, these spacetimes can only be of Petrov type N or D. The type N case can be excluded by requiring the nonexistence of solutions to the twistor initial data equations (43a) and (43b). Hence combining the results of the previous sections with theorem 7 one obtains the following

**Theorem 3** Let \((\mathcal{S}, h_{ij}, K_{ij})\) be a suitably smooth initial data set for the Einstein vacuum field equations. Suppose that there is on \(\mathcal{S}\) a valence-2 spinor \(\tilde{\kappa}_{AB}\) solving equations (64) and (65), and assume that

\[
\tilde{\xi} \equiv D^{PQ}\tilde{K}_{PQ}, \quad \tilde{\xi}_{BF} \equiv -\frac{1}{2}\tilde{K}^{DA}_{DA}\tilde{K}_{BF} + \frac{3}{4}\tilde{\kappa}^{DA}\tilde{\Omega}_{BFDA} + \frac{3}{2}\tilde{D}(F\tilde{\kappa}_{B})D, \]

satisfy the spinorial KID equations (26a) and (26b). If in addition, there is no spinor \(\tilde{\kappa}_{A}\) on \(\mathcal{S}\) satisfying the twistor initial data set equations (43a) and (43b), then there is at least a neighborhood of \(\mathcal{S}_0\) in the development of the initial data where the spacetime is strictly of Petrov type D.

It is well-known that a property of the Kerr spacetime is that the Killing vector constructed from the valence-2 Killing spinor is degenerate, in the sense that one can always choose the phase of the Killing spinor so that the real and imaginary parts of the (complex) Killing vector are proportional —see e.g. [17]. This however, does not suffice to characterise the Kerr solution —see e.g. [8]. Possibly one may require some assumption on the asymptotic flatness of the spacetime. In basis of this, and using corollary 1 one can see that necessary —but certainly not sufficient— conditions for an initial data set for the Einstein vacuum equations to be Kerr initial data are the conditions of theorem 3 together with

\[
\xi = \tilde{\xi}, \quad \xi_{AB} = -\tilde{\xi}_{AB}. \]

9 Conclusions

In this paper we have established conditions on a vacuum initial data set ensuring that a neighbourhood of the initial data hypersurface is either type N
or type D. The strategy behind has been to identify the circumstances under which the development of the initial data will be endowed with Killing spinors. An important point of this approach is that the characterisation is expressed in terms of differential equations rather than in terms of algebraic conditions on, say, the electric and magnetic parts of the Weyl tensor. Arguably, these conditions will be hard to verify in practise. However, their formulation in terms of a system of (possibly elliptic) overdetermined partial differential equations could make it possible the introduction of global arguments. The structure of at least a subset of the conditions that has been obtained here is similar to that of the KID equations. In [7] it has been shown that it is possible to construct certain geometric invariants for an initial data set that indicate whether its development is static or not. It is conceivable then, that an analogous construction for the conditions in theorem 3—or less ambitiously in propositions 4 or 6—could be implemented rendering geometric invariants by means of which it could be possible to decide whether a given initial data set will give rise to Petrov type N or D spacetime. These ideas will be investigated elsewhere.

As it has been mentioned in several places, our results on the existence of Killing spinors in the development of the initial data sets are local —i.e. they only ensure the existence of the spinors in a neighbourhood of the initial hypersurface. Global results, valid for the maximal globally hyperbolic development can be obtained if the spacetime and the initial data for the spinors are suitably smooth. It is of interest whether it is possible to relax these assumptions by using alternative arguments which do not require solving wave equations on the whole spacetime ($\mathcal{M}, g_{\mu\nu}$).

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A A rigidity result for spacetimes with valence-2 Killing spinors

The following result is used to give a characterisation of Petrov type D spacetimes.
Proposition 7  Any vacuum Petrov type D spacetime \((\mathcal{M}, g_{\mu \nu})\) admits a valence-2 Killing spinor. Conversely, if \(\kappa_{AB}\) is a valence-2 Killing spinor then the spacetime is either of Petrov type D —and \(\kappa_{AB}\) is non-degenerate, i.e. it has two different principal spinors— or the spacetime is of Petrov type N —and \(\kappa_{AB}\) is degenerate.

Proof. Asssume first that \((\mathcal{M}, g_{\mu \nu})\) is of type D. Then the Weyl spinor \(\Psi_{ABCD}\) has two principal spinors \(o_A, \iota_A\) in terms of which it takes the form

\[
\Psi_{ABCD} = 6\psi_0(o_A o_B \iota_C \iota_D), \quad o_A \iota_A = 1.
\]

Then it is known —see e.g. [24]— that the spinor \(\kappa_{AB}\) defined by

\[
\kappa_{AB} \equiv \psi^{-1/3} o_0(A o_B),
\]

is a Killing spinor. Conversely, if the spinor \(\kappa_{AB}\) is a Killing spinor on a spacetime \((\mathcal{M}, g_{\mu \nu})\) then we distinguish two separate cases.

Case A: the spinor \(\kappa_{AB}\) is non-degenerate. This means that \(\kappa_{AB} = 2\omega o_0(A o_B)\), with \(o_A \iota^A = 1\). This case was studied in [11] and it was shown by means of the GHP formalism that the only possible Petrov type for \(\Psi_{ABCD}\) is D.

Case B: the spinor \(\kappa_{AB}\) is degenerate. Therefore, \(\kappa_{AB}\) takes the form \(\kappa_{AB} = \omega o_A o_B\). Let \(\iota_A\) be any spinor such that \(o_A \iota^A = 1\) and regard \(\{o_A, \iota_A\}\) as the spin basis used in the Newman-Penrose formalism. Our conventions for the Newman-Penrose formalism follow [21]. Expanding the condition \(\nabla_{A'}(A \kappa_{BC}) = 0\) in the Newman-Penrose spin basis and simplifying the resulting conditions we obtain

\[
\begin{align*}
D \omega &= -2\omega (\epsilon + \rho), \quad \Delta \omega = -2\gamma \omega, \\
\delta \omega &= -2(\beta + \tau) \omega, \quad \bar{\delta} \omega = -2\alpha \omega \\
\sigma &= 0, \quad \kappa = 0.
\end{align*}
\]

We use this information in the Newman-Penrose commutation relations which are thus reduced to

\[
\begin{align*}
D \beta &= -\bar{\alpha} (\epsilon + \rho) + \bar{\Pi} (\epsilon + \rho) - \beta (\bar{\epsilon} + \rho - \bar{\rho}) + (\epsilon - \bar{\epsilon} + \bar{\rho}) \tau - D \tau + \delta \epsilon + \delta \rho, \\
D \alpha &= \alpha (-2\epsilon + \bar{\epsilon}) - \left(\bar{\beta} - \Pi\right) (\epsilon + \rho) + \bar{\delta} \epsilon + \delta \rho, \\
D \gamma &= -\bar{\gamma} (\epsilon + \rho) - \gamma (2\epsilon + \bar{\epsilon} + \rho) + \alpha \left(\bar{\Pi} + \tau\right) + (\beta + \tau) (\Pi + \bar{\tau}) + \Delta \epsilon + \Delta \rho, \\
\Delta \alpha &= \bar{\beta} \gamma + \alpha (\bar{\gamma} - \bar{\mu}) + \nu (\epsilon + \rho) - \lambda (\beta + \tau) - \gamma \tau + \bar{\delta} \gamma, \\
\Delta \beta &= \bar{\alpha} \gamma - \alpha \bar{\lambda} + \beta (2\gamma - \bar{\gamma} + \mu) + \bar{\nu} (\epsilon + \rho) - (\bar{\gamma} + \mu) \tau - \Delta \tau + \delta \gamma, \\
\delta \alpha &= (\mu - \bar{\mu}) (\epsilon + \rho) + \gamma (\rho - \bar{\rho}) + \alpha (\bar{\alpha} - 2\beta - \tau) + \bar{\beta} (\beta + \tau) + \delta \beta + \bar{\delta} \tau.
\end{align*}
\]
Finally we combine these conditions with the Newman-Penrose “field equations”. After some manipulations in the resulting set of equations we obtain the conditions
\[ \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \]
thus proving that \((\mathcal{M}, g_{\mu\nu})\) is of Petrov type N. \(\square\)

B Completion of the proof of proposition 5

We fill in the details of the calculations needed to prove proposition 5. First of all we transform the identity (52) by means of (55) and insert the result into (51) obtaining
\[ S_{CC'D'D'} = -\nabla_{D'}(\nabla^A_{C'}\kappa_{CA}) - \nabla_C(\nabla^A_{C'}\kappa_{DA}). \]
The covariant derivatives in this expression can be commuted using the spinor Ricci identity with the result
\[ S_{CC'D'D'} = -\nabla^A_{C'}\nabla_{D'}(\kappa_{CA}) - \nabla^A_{D'}\nabla_{C'}(\kappa_{DA}). \]
If in this equation we use the identity \(\nabla_{D'}(\kappa_{CA}) = (H_{D'CA} - \nabla_{A'D'}\kappa_{CD})/2\) we get
\[ S_{CC'D'D'} = -\frac{1}{2}\nabla^A_{C'}H_{D'CA} - \nabla_A(\nabla^A_{D'}\kappa_{CD}). \]
Finally, if we use the identity (55) and the fact that in a vacuum spacetime \(\Box_{A'B'}\kappa_{CD} = 0\) we end up with equation (53).

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