Source number estimation using dual eigenspace projection

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(Received 20 August 2012, Accepted for publication 26 November 2012)

Abstract: A new tensorial model is herein established for a special multidimensional array of vector sensors composed of a pressure part and orthogonal velocity parts, under which a multi-linear algebra based source number estimation algorithm is proposed. The estimation projects the covariance tensor into the signal subspace. Owing to the orthogonality between the newly defined tensor signal subspace and tensor noise subspace, it is easy to differentiate the contribution of the signal and noise by using the decision value in a compact tensorial fashion, which is the magnitude of projections from twofold mode signal subspaces. To reduce computer burden, meanwhile, the real-valued preprocessing method is applied to the covariance tensor for simplicity. Simulation and experiment results exhibit the superiority of the proposed algorithm over the conventional ones based on the matrix method.

Keywords: Covariance tensor, Dual projection, Real-valued preprocessing

PACS number: 43.30.Wi, 43.60.Gk [doi:10.1250/ast.34.271]

1. INTRODUCTION

Multidimensional subspace-based estimation problems are encountered in a variety of signal processing applications [1–3] including radar, sonar, communications, and medical imaging. Since the measurement data is multidimensional, current approaches require stacking the dimensions into one highly structured matrix. A tensor-based subspace estimate through higher order singular value decomposition (HOSVD) [4] is a better estimate of the desired signal subspace than the multilinear algebra approach reported in the literature. In this paper, we show how the model order of the multidimensional data, i.e., the source number, can be estimated by the foundation method described in the paper [4].

State-of-the-art model-order estimation techniques include information theoretic criteria such as Akaike’s Information theoretic Criterion (AIC) [5], the Minimum Description Length (MDL) criterion [6] and the Gerschgorin disks method [7], etc. These classical model-order selection methods often fail when the number of available snapshots is small and SNR is low. To deal with such cases, several improved algorithms were put forward by experts which still exploit the two-dimensional structure of the matrix without determining the characteristics of the multidimensional data. In this paper, we propose a three-order tensor model of a vector sensor array and derive the source number by dual projection onto the signal eigenspace that is proved by the simulation and experiment results.

In this paper, we suggest an alternative tensor model (termed the multiway model) to accommodate the multidimensional structure of the output of a vector sensor array. Using the multiway model, we redefined the array manifold and the signal subspace from a tensorial point of view. Thus the covariance tensor can be projected onto the signal subspace in the twofold mode in a multilinear sense, which is herein referred to as dual-mode projection. This dual projection can be viewed as a tensorial extension of the well-known subspace projection employed in the conventional subspace decomposition methods. We then propose to perform a twofold mode projection to utilize such dual mode orthogonality and ultimately find the desired source number according to the projection extrema.

The remainder of this paper is organized as follows. In Sect. 2, we introduce the tensor, matrix notations and important tensor operations. In Sect. 3, the three dimension tensor output of the vector sensor array are presented, then the HOSVD based signal eigenspace estimation technique is described, and the covariance tensor is projected onto the signal eigenspace from the twofold mode. The detailed algorithm is given in Sect. 4. In Sect. 5, real-valued preprocessing is extended to the complex tensor to lower the complexity of computing. Finally, the benefits of the
new source number estimation algorithm are demonstrated through numerical simulations and experiments in Sects. 6 and 7, respectively, before the conclusions are drawn in Sect. 8.

2. MATRIX AND TENSOR NOTATION

In order to facilitate the distinction between scalars, matrices, and tensors, the following notation is used: scalars are denoted as italic letters (a, b, ..., A, B, ...), column vectors as lowercase boldface letters (a, b, ...), matrices as boldface capital letters (A, B, ...), and tensors are written as boldface calligraphic letters (\( \mathcal{A}, \mathcal{B}, ... \)). Lower order parts are consistently named: the \((i,j)\)-element of matrix \( A \), is denoted as \( a_{ij} \) and the \((i,j,k)\)-element of a three-order tensor \( \mathcal{A} \) as \( a_{ijk} \).

The tensor operations we use are consistent with [8]. The \( n \)-mode product of tensor \( \mathcal{A} \in \mathbb{C}^{l_1 \times l_2 \times \cdots \times l_k} \) and matrix \( U \in \mathbb{C}^{l_1 \times I} \) along the \( n \)-th mode is denoted as \( \mathcal{A} \times_n U \). It is obtained by multiplying all \( n \)-mode \( \mathcal{A} \) vectors from the left-hand side by matrix \( U \).

The outer product of tensor \( \mathcal{A} \in \mathbb{C}^{l_1 \times l_2 \times \cdots \times l_k} \) and tensor \( \mathcal{B} \in \mathbb{C}^{l_1 \times l_2 \times \cdots \times l_k} \) is defined by

\[
(\mathcal{A} \circ \mathcal{B})_{i_1i_2...i_nj_1j_2...j_0} \triangleq a_{i_1i_2...i_n}b_{j_1j_2...j_0}
\]

for all values of indices.

The \( n \)-mode matrix unfolding of a given tensor \( \mathcal{A} \in \mathbb{C}^{l_1 \times l_2 \times \cdots \times l_k} \) is \( [\mathcal{A}]_{(n)} \in \mathbb{C}^{l_1l_2...l_n \times l_{n+1}...l_k} \). The unfolding operation involved with the construction of \([\mathcal{A}]_{(1)}, [\mathcal{A}]_{(2)}, [\mathcal{A}]_{(3)}\) is visualized in Fig. 1, for the three-order case.

The higher order SVD of tensor \( \mathcal{A} \in \mathbb{C}^{l_1 \times l_2 \times \cdots \times l_k} \) is given by

\[
\mathcal{A} = \delta \times_1 U_1 \times_2 U_2 \cdots \times_N U_N,
\]

where \( \delta \in \mathbb{C}^{l_1 \times l_2 \times \cdots \times l_k} \) is the core tensor that satisfies the all-orthogonal condition [9] and \( U_n \in \mathbb{C}^{l_n \times l_k} \) are the unitary matrices of \( n \)-mode singular vectors.

The concatenation of two tensors \( \mathcal{A} \) and \( \mathcal{B} \) along the \( n \)-th mode is denoted by \([\mathcal{A} \cup_n \mathcal{B}]\).

3. DATA MODEL

Consider that there are \( K \) narrow-band, far-field and spatially distinct signals impinging upon a vector sensor array composed of \( M \) identical vector sensors spaced at the same interval along a line. Under the above assumption, the corresponding array manifold concerning the \( k \)-th impinging wave can be written as

\[
a_k \triangleq a(\theta_k) \otimes u(\theta_k),
\]

where \( a(\theta_k) = [1, \exp(-j2\pi d \sin(\theta_k)/\lambda), \ldots, \exp(-j(M - 1)2\pi d \sin(\theta_k)/\lambda)]^T \) and \( u(\theta_k) = [\cos \theta_k \sin \theta_k] \). \( \otimes \) denotes the Kronecker product. With these notations, the \( l \)-th temporal samples collected at the output of the vector sensor array can be organized as a \( 3M \times K \) matrix:

\[
X(l) = A \cdot S(l)^T + N(l).
\]

A 3rd-order tensor model of the vector sensor array with \( L \) snapshots is

\[
\mathcal{X} = \mathcal{A} \otimes_3 S^T + \mathcal{N},
\]

where \( \mathcal{X} \in \mathbb{C}^{M \times 3 \times L} \), \( \mathcal{A} \in \mathbb{C}^{M \times 3 \times K} \), and \( \mathcal{N} \in \mathbb{C}^{M \times 3 \times L} \) satisfy

\[
\mathcal{X}_{i,j,l} = X_{(i-1)j+l}
\]

\[
\mathcal{A}_{i,j,l} = A_{(i-1)j+l}
\]

\[
\mathcal{N}_{i,j,l} = N_{(i-1)j+l}.
\]

According to [10], 3rd-order manifold tensor \( \mathcal{A} \) can be decomposed as

\[
\mathcal{A} = \mathcal{T}_{3,K} \times_1 A_{11} \times_2 A_{12},
\]

where \( \mathcal{T}_{3,K} \in \mathbb{C}^{K \times K \times K} \) is the identity tensor and contains nothing about signal information, except for the source number \( K \). However, the factor matrices \( A_{11} \) and \( A_{12} \) extract valid signal eigenspace information in the twofold mode efficiently, so output tensor \( \mathcal{X} \) can be expressed in the following way:

\[
\mathcal{X} = \mathcal{T}_{3,K} \times_1 A_{11} \times_2 A_{12} \times_3 S^T + \mathcal{N}.
\]

Meanwhile, the HOSVD of the measurement tensor \( \mathcal{X} \) yields

\[
\mathcal{X} = \mathcal{C} \times_1 U_1 \times_2 U_2 \times_3 U_3,
\]

where \( \mathcal{C} \in \mathbb{C}^{M \times 3 \times L} \) is the core tensor, \( U_1 \in \mathbb{C}^{M \times M} \) is the left singular vectors of \( [\mathcal{X}]_{(1)} \), \( U_2 \in \mathbb{C}^{3 \times 3} \) is the left singular vectors of \( [\mathcal{X}]_{(2)} \), and \( U_3 \in \mathbb{C}^{L \times L} \) is the left singular vectors of \( [\mathcal{X}]_{(3)} \).
In the absence of noise, $\mathcal{X}$ can be expressed in terms of "economy size" HOSVD as

$$\mathcal{X} = \mathcal{C} \times_1 U_1^r \times_2 U_2^r \times_3 U_3^r,$$

where $\mathcal{C} \in \mathbb{C}^{\min(K,M) \times \min(K,L) \times \min(K,L)}$. $U_1^r \in \mathbb{C}^{M \times K}$ are the dominant left singular vectors of $[\mathcal{X}]_{k1}$, $U_2^r \in \mathbb{C}^{L \times K}$ are the dominant left singular vectors of $[\mathcal{X}]_{k2}$, and $U_3^r \in \mathbb{C}^{L \times K}$ are the dominant left singular vectors of $[\mathcal{X}]_{k3}$. Under the assumption that $M > K$, there exist two nonsingular $K \times K$ complex matrices $T_1$ and $T_2$ such that

$$A_{11} = U_1^r T_1,$$
$$A_{22} = U_2^r T_2.$$  

In fact, the source number is unknown before HOSVD of the tensor output, so we can solve the above problem in this way:  

$$\mathcal{X} = T_{1,1} \times_1 U_{1,1} \times_2 U_{2,2} \times_3 U_{3,3},$$

where core tensor $T_{1,1}$ = 1 and the factor matrices are $U_{1,1} \in \mathbb{C}^{M \times 1}$, $U_{2,2} \in \mathbb{C}^{L \times 1}$, and $U_{3,3} \in \mathbb{C}^{L \times 1}$. However, the signal eigenspace information is still contained in matrices $U_{1,1}$ and $U_{2,2}$.

Assuming that signals and noise are uncorrelated, and the noise is i.i.d. centered, complex Gaussian, and spatially white, we define a covariance tensor estimate of the received data:

$$\mathcal{R}_{XX} \triangleq \frac{1}{L} \sum_{l=1}^{L} \mathcal{X}(\cdot,*;l) \circ \mathcal{X}^*(\cdot,*;l),$$

where $\mathcal{R}_{XX} \in \mathbb{C}^{M \times 3 \times M \times 3}$, $\circ$ is the tensor outer product and superscript $\dagger$ stands for the conjugate of a tensor.

The signal eigenspace spans the same space as the vector sensor array manifold, the corresponding value of projection should be numerically large when the covariance tensor is projected onto the signal subspace, whereas noise eigenspace is orthogonal to the signal eigenspace, so the corresponding value of projection is theoretically zero. Indeed, the value is relatively small. Via HOSVD of the tensor model of the vector sensor array, there exist two vectors, i.e., $U_{1,1}$ and $U_{2,2}$, onto which the covariance tensor can be projected, thereby distinguishing signals from noise with more accuracy. This is the theoretic foundation of our work. More details of the algorithm will be shown in the following.

4. ALGORITHM

We denote the projectors of the twofold-mode signal subspaces as $\mathcal{R}_{XX} \times_1 U_{1,1}$ and $\mathcal{R}_{XX} \times_2 U_{2,2}$. Then, according to Eq. (16), the twofold mode projector can be expressed as

$$P = \mathcal{R}_{XX} \times_1 U_{1,1} \times_2 U_{2,2},$$

where $P \in \mathbb{C}^{M \times 3}$ and the $m$th row of $P$ is $P_m = [p_{m1}, p_{m2}, p_{m3}], m = 1, 2, \ldots, M$.

The average norm of every element in the $m$th row is

$$\bar{P}_m = \frac{1}{3}(|p_{m1}| + |p_{m2}| + |p_{m3}|),$$

where $\bar{P}_m$ is the source number estimation criteria. By rearranging $\bar{P}_m$ ($m = 1, 2, \ldots, M$) in descending order of the norm value of $\bar{P}_m$, they are divided into two groups comprising the signals with values greater than 1 and the noise with values less than 1. According to the above description, if $\bar{P}_m$ is the signal projection, its value is not close to zero; otherwise its value is less than 1. Then the ratio of $\beta_m = |\bar{P}_m|^2/|\bar{P}_{m+1}|$ is calculated; the source number equals to the value of $m$ while $\beta_m$ reaches its maximum value.

5. REAL-VALUED APPROACH FOR SOURCE NUMBER ESTIMATION

For the aim of reducing the computer burden, real-valued pre-processing [11] is applied before the algorithm. Define the sparse matrix $Q_M$ as

$$Q_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & i_n \\ J_n & -i_n \end{bmatrix},$$
$$Q_{2n+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & 0 & i_n \\ 0 & \sqrt{2} & 0 \\ J_n & 0 & -i_n \end{bmatrix},$$

where $I_n$ is the $n \times n$ identity matrix and $J_n$ is an $n \times n$ exchange matrix having ones on its antidiagonal and zeros elsewhere.

The forward-backward average version of the measurement tensor $\mathcal{X}$ is computed through

$$\mathcal{Y} = [\mathcal{X} \oplus \mathcal{X}^* \times_1 J_M \times_2 J_N \times_3 I_L],$$

where $\oplus$ denotes concatenation of two tensors along the $n$-mode.

The set of tensor $\mathcal{Y}$ can be mapped onto the set of real-valued tensors using the transformation

$$Z = \mathcal{Y} \times_1 Q^H \times_2 Q^H \times_3 Q^H,$$

where $Z \in \mathbb{C}^{M \times 3 \times 2L}$ and the covariance tensor of $Z$ is

$$\mathcal{R}_{ZZ} = \frac{1}{2L} \sum_{l=1}^{2L} Z(\cdot,*;l) \circ Z^*(\cdot,*;l).$$

From Eq. (22), the accuracy of the covariance tensor is enhanced. The latter steps are the same as those of the complex form and will be omitted. After the transformation, the complexity of the algorithm can be reduced to $1/5$ that of the complex form. Taking the spending of trans-
formation also into consideration, in total, our algorithm has only $3/5$ the burden of the complex form.

6. COMPUTER SIMULATION

In this section, we compare the estimation performance of the proposed method with that of MDL using the information theoretic criterion and Gerschgorin disks method. Assume that a uniform vector sensor array with elements at intervals of half the wavelength possesses 10 vector hydrophones.

In a single simulation, if the source number estimation equals the actual source number, then we should judge it to be a correct differentiation. The correct estimation probability is the ratio of the number of the correct differentiations to the total number of simulations.

To enable comparison with the conventional methods, it is appropriate to introduce MDL and Ger-disks algorithms primarily.

\[
MDC(n) = L(M - n) \ln \frac{1}{M - n} \sum_{i=n+1}^{M} \lambda_i \ln \left( \prod_{i=n+1}^{M} \lambda_i \right) + \frac{1}{2} n(2M - n) \ln L \tag{24}
\]

\[
GDE(n) = r_n - \frac{D(L)}{M - 1} \sum_{i=1}^{M-1} r_i \tag{25}
\]

Here, \( L \) is the number of snapshots, \( M \) is the number of receivers in the array, \( \lambda_i \) is the eigenvalue of the corresponding covariance matrix of output data. \( r_n \) is the radius of the Gerschgorin disk and can be calculated as shown in [8]. The source number is that when Eq. (24) reaches the minimum value or Eq. (25) is less than zero.

In the first test, we compare these algorithms under correct estimation probability verse distance conditions. There are two independent sinusoids with equal power impinging on the array from the directions of $40^\circ$ and $100^\circ$. The number of snapshots is 300, and distances between sources and the vector array vary from 25 m to 500 m. The Monte Carlo simulation results can be seen in Fig. 2. The proposed method’s performance greatly exceeds those of the other two because of the multiway data reception.

In the third test, we examine the performance of correct estimation probability verse correlative coefficient of colored noise. The spatial correlative colored noise model is taken from [12]:

\[
R_{m,n} = \beta |m-n| \exp[j2\pi\phi(m - n)], \tag{26}
\]

where \( \beta \in [0,1] \) is the spatial correlative coefficient between adjacent vector sensors. Stochastic coefficient \( \phi \) determines the peak value position of the noise spectrum density.

The SNR is 10 dB and other conditions are the same as in test 2. In Fig. 4 although there is a close relationship between spatial noise, the proposed method’s correct estimation probability is higher than those of the other two methods.

In the last test, we compare the three methods’ performances in terms of correct estimation probability verse $\Delta \theta$. Assume that SNR is 5 dB, the number of
snapshots is 100, and there are two far-field signals impinging onto the array. One signal’s incident angle is $\theta_1 = 40^\circ$, while the other signal’s angle varies from $41^\circ$ to $50^\circ$ at intervals of $1^\circ$. From Fig. 5, we can see that the proposed method and Ger-disk has almost always the same performance; they are both better than the MDL method. When $\Delta \theta \geq 5^\circ$, the correct estimation probability is approximately 1.

7. EXPERIMENT

Lake experiments were conducted in Hubei Province, China, where the lake depth is 15–20 m. A 5-element ULA with 0.5 m interelement spacing was used as the receiving array. Two sources were placed at distances far from the array. During the experiments, the directions of arrival from the two sources were $40^\circ$ and $100^\circ$ constantly for two different sinusoids at frequencies of 1 kHz and 1.5 kHz while the sample frequency was 10 kHz throughout the experiments. Meanwhile, there was no other interferes. Here we set five different distances between the source and the array: 25 m, 50 m, 100 m, 125 m and 200 m.

It can be seen from Tables 1 and 2 that when the distance is less than 100 m, the correct estimation probability of the three methods are 100% because of high SNR. However, when the distance is over 100 m, the Ger-disk method’s result is underestimated and the threshold is not known and must be set artificially. The proposed method’s performance is still superior to those of MDL and Ger-disk.

8. CONCLUSION

We have proposed a new method for estimating the source number through dual projection of the norm onto the signal eigenspace, although real-valued preprocessing is necessary to lower the complexity. Simulation and experimental results show that the new method is efficient when dealing with, for example, the multiway vector sensor array.

ACKNOWLEDGMENT

This work is supported by the Doctorate Creation Foundation of Naval University of Engineering (No. 201110).

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