LOWER ORDER TERMS IN SZEGÖ THEOREMS ON ZOLL MANIFOLDS

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Abstract. We give an outline of the computation of the third order term in a generalization of the Strong Szegö Limit Theorem for a zeroth order pseudo-differential operator (PsDO) on a Zoll manifold of an arbitrary dimension, see [Gi2] for the detailed proof. This is a refinement of a result by V. Guillemin and K. Okikiolu who have computed the second order term in [GO2]. An important role in our proof is played by a certain combinatorial identity which generalizes the formula of G. A. Hunt and F. J. Dyson to an arbitrary natural power, see [Gi3]. This identity is a different form of the renowned Bochner–Spitzer combinatorial theorem which is related to the maximum of a random walk with i.i.d. steps on the real line.

A corollary of our main result is a fourth order Szegö type asymptotics for a zeroth order PsDO on the unit circle, which in matrix terms gives a fourth order asymptotic formula for the determinant of the truncated sum $P_n(T_1 + T_2 D)P_n$ of a Toeplitz matrix $T_1$ with the product of another Toeplitz matrix $T_2$ and a diagonal matrix $D$ of the form $\text{diag}(\cdots , \frac{1}{3} , \frac{1}{2} , 1 , 0 , 1 , \frac{1}{2} , \frac{1}{3} , \cdots )$. Here $P_n = \text{diag}(\cdots , 0 , 1 , \cdots , 1 , 0 , \cdots ) , \ (2n + 1)$ ones.

1. Introduction and main results

The main motivation for this work was to find an explicit formula for a “Szegö–regularized” determinant of a zeroth order pseudodifferential operator (PsDO) on a Zoll manifold introduced in [GO1, O2], see Remark 2.4. Our main result, Theorem 1.2, is valid for any dimension $d \in \mathbb{N}$. In the case $d = 2$, Theorem 1.2 gives such a formula.

In this paper we find a third order generalization of the Strong Szegö Limit Theorem (SSLT) for a zeroth order PsDO on the unit circle (Theorem 1.1), and on a Zoll manifold of an arbitrary dimension $d \in \mathbb{N}$ (Theorem 1.2). We give also an outline of the proof of Theorem 1.2 and derive Theorem 1.1 from the former. The detailed proof of Theorem 1.2 can be found in [Gi1, Chapter 1] and will be published in a forthcoming paper [Gi2]. In Section 3, we sketch a proof of the generalized Hunt–Dyson combinatorial formula (Theorem 3.1), see [Gi3] for the detailed proof. We give also a brief review of the related work.

Recall that $M$ is called a Zoll manifold if it is compact, closed and such that the geodesic flow on $M$ is simply periodic with period $2\pi$. The unit circle and
the standard sphere of any dimension are Zoll manifolds. A second order generalization of the SSLT for a Zoll manifold of any dimension has been obtained by V. Guillemin and K. Okikiolu [GO1, GO2], see also an important preceding work [O1] by K. Okikiolu for the case of the two- and three-dimensional sphere. The proofs in [O1, GO1, GO2] use a combinatorial identity due to G. A. Hunt and F. J. Dyson and proceed in the spirit of the combinatorial proof of the classical SSLT by M. Kac [K]. See also [GO3, O2] where the combinatorial approach and the Hunt–Dyson formula are used in a different setting to obtain a second order generalization of the SSLT for a manifold with the set of closed geodesics of measure zero in the unit cotangent bundle.

In the proof of Theorem 1.2 we use the method of [GO2]. A central role in our proof is played by a certain combinatorial identity which generalizes the Hunt–Dyson formula mentioned above to an arbitrary natural power. We call this identity the generalised Hunt–Dyson formula (gHD), see Theorem 3.1.

After having discovered and proved the formula gHD we realized that it is related to another combinatorial theorem, which has a long history. The mentioned theorem is a result due to H. F. Bohnenblust that appeared in an article by F. Spitzer on random walks [S, Theorem 2.2], and is now commonly known as the Bohnenblust–Spitzer theorem (BSt). The characteristic function of the maximum of a random walk with independent identically distributed steps has been computed in [S] with the help of the BSt. On the other hand, using the gHD, we can compute the moment of an arbitrary order of the mentioned maximum (note that the usual Hunt–Dyson formula allows one to compute only the expected value of the maximum, see [K]). This indicates that the gHD and the BSt should be essentially the same. And indeed this is the case: it turns out that the gHD can be derived from the BSt, and vice versa, see [Gi3].

We have found the formula gHD being unaware of the BSt. In [Gi1, Chapter 2], a proof of the gHD “from scratch” can be found. This together with a derivation of the BSt from the gHD in [Gi1, Gi3] gives a new proof of the BSt. There are known at least four other proofs: the original one [S, F], a proof by G. Baxter [B1, B2], by J. G. Wendel [We], and finally, a unifying approach of G.-C. Rota in the framework of (Glen) Baxter algebras [Ro, RoSm]. See Section 3 for more combinatorial details.

Let det denote the determinant of a finite rank operator. In Section 2, we give explicit asymptotic formulas for $\log \det P_n BP_n$, $n \to \infty$, where $P_n$ is the projection onto the first $n$ eigenspaces of the Laplace–Beltrami operator on a Zoll manifold, and $B$ is an arbitrary zeroth order PsDO, see Corollary 2.2 (for $d = 1$) and Corollary 2.3 ($d \geq 2$).

Let $S^1$ be the unit circle $\mathbb{R}/2\pi \mathbb{Z}$. Denote by $P_n$, $n \in \mathbb{N}$, the orthogonal projection from $L^2(S^1)$ to the subspace spanned by $\{e^{ikx} \mid |k| \leq n\}$. For a function $f \in L^1(S^1)$ denote its $k$th Fourier coefficient by

$$(1.1) \quad \hat{f}_k := \int_0^{2\pi} f(x) e^{-ikx} \frac{dx}{2\pi}, \quad k \in \mathbb{Z}. $$

Let $b(x)$ be a strictly positive function on $S^1$ such that $\sum_{k \in \mathbb{Z}} |k| |(\log b)_k|^2 < \infty$. Denote by $B$ the operator of multiplication by $b$ acting in $L^2(S^1)$. Recall that the matrix representation of the operator $B$ is the Toeplitz matrix $(\hat{b}_{j-k})_{j,k \in \mathbb{Z}}$. The
classical Strong Szegő Limit Theorem (SSLT) \[\text{Sz}\] states that

\[
\text{Tr} \log P_n BP_n = \text{Tr} P_n (\log B) P_n + \sum_{k=1}^{\infty} k (\log b)_k (\log b)_{-k} + o(1), \quad n \to \infty.
\]

Here \(\text{Tr} \log P_n BP_n = \log \det P_n BP_n\) and \(\text{Tr} P_n (\log B) P_n = (2n + 1) \int_0^{2\pi} \log b(x) \frac{dx}{2\pi}\).

It has been shown by H. Widom that the remainder is \(O(n^{-\infty})\) if \(b(x) \in C^\infty(S^1)\), see \[W1\].

Let \(M\) be a closed manifold of dimension \(d \in \mathbb{N}\). Let \(\Psi^m(M), m \in \mathbb{Z}\), denote the space of the classical PsDO of order \(m\) on \(M\). Recall that for a given \(G \in \Psi^m(M)\), its principal symbol \(\sigma_m(G)\) and subprincipal symbol \(\text{sub}(G)\) are well-defined on \(T^*M\). Let \(S^*M\) be the unit cotangent bundle and denote by \(dx d\xi\) the standard measure on \(S^*M\) divided by \((2\pi)^d\).

The simplest form of our result is for \(d = 1\) and the function \(\psi(u) = \log u\) which is analytic in a neighborhood of \(u = 1\).

**Theorem 1.1.** Let \(M = S^1\) and \(P_n\) be the projection on the linear span of \(\{e^{ikx}\}_{|k| \leq n}\).

Let \(B \in \Psi^0(M)\) and assume that \(\sigma_0(B)\) is strictly positive and a certain symbolic norm of \(I - B\) is sufficiently small. Then \(\log B \in \Psi^0(M)\) and the following holds (1.2)

\[
\text{Tr} \log P_n BP_n = \text{Tr} P_n (\log B) P_n + \frac{1}{2} \int_{S^*M} \sum_{k=1}^{\infty} k (\sigma_0(\log B))_k (\sigma_0(\log B))_{-k} dx d\xi
\]

\[
+ \frac{1}{n} \cdot \frac{1}{2} \int_{S^*M} \sum_{k=1}^{\infty} k (\sigma_0(\log B))_k (\text{sub}(\log B))_{-k} dx d\xi + O\left(\frac{1}{n^2}\right), \quad n \to \infty.
\]

In (1.2) the argument \((x, \xi) \in S^*M\) is omitted for brevity. By the Fourier coefficient we mean the following: for a fixed \((x, \xi) \in S^*M\) compute the Fourier coefficient in accordance with (1.1) along the unit circle being the closed geodesic starting at \((x, \xi)\).

Let us denote \(b_0 := \sigma_0(B)\) and \(b_{\text{sub}} := \text{sub}(B)\). Recall that for an analytic \(f\), \(\sigma_0(f(B)) = f(\sigma_0(B))\) and \(\text{sub}(f(B)) = f'(\sigma_0(B))\) sub\((B)\). Then (1.2) takes the form

(1.3)

\[
\text{Tr} \log P_n BP_n = \text{Tr} P_n (\log B) P_n + \frac{1}{2} \int_{S^*M} \sum_{k=1}^{\infty} k (\log b_0)_k (\log b_0)_{-k} dx d\xi
\]

\[
+ \frac{1}{n} \cdot \frac{1}{2} \int_{S^*M} \sum_{k=1}^{\infty} k (\log b_0)_k (b_{\text{sub}}/b_0)_{-k} dx d\xi + O\left(\frac{1}{n^2}\right), \quad n \to \infty.
\]

**Remark 1.1.** Theorem 1.1 is a particular case of Theorem 1.2 below, see Remark 1.4.

Let us fix some notations and then state the result for the higher dimensional case. Let \(M\) be a Zoll manifold of dimension \(d \in \mathbb{N}\). Let \(\Delta\) denote the Laplace–Beltrami operator on \(M\). It is known \[CD\] that there exists a constant \(\alpha \in \mathbb{R}\) such that the spectrum of \(\sqrt{-\Delta}\) lies in bands around the points \(k + \frac{d}{4}, k \in \mathbb{N}\). Moreover, in has been shown in \[CDV\] that there exists \(A_{-1} \in \Psi^{-1}(M)\) such that \([\Delta, A_{-1}] = 0\) and the spectrum of the operator

(1.4)

\[
A := \sqrt{-\Delta} - \frac{\alpha}{4} - A_{-1}
\]
is $\mathbb{N}$. Let $P_n$, $n \in \mathbb{N}$, denote the projection from $L^2(M)$ onto the subspace spanned by the eigenfunctions of $A$ corresponding to the eigenvalues $1, 2, \cdots, n$. Let $dx d\xi$ be the defined above measure on $S^*M := \{(x, \xi) : \sigma_1(A)(x, \xi) = 1\}$. Following [1], for a $B \in \Psi^0(M)$ and $t \in \mathbb{R}$ introduce the operator

$$B^t := e^{-itA}Be^{itA}.$$  

By Egorov’s theorem, $B^t \in \Psi^0(M)$, and also

$$\sigma_0(B^t)(x, \xi) = \sigma_0(B)(\Theta^t(x, \xi)),$$

where $\Theta^t$ stands for the shift by $t$ units along geodesic flow. Note that because $\text{sub}(A) = \text{const}$ the following also holds [1, Lemma 2.2]

$$\text{sub}(B^t)(x, \xi) = \text{sub}(B)(\Theta^t(x, \xi)).$$

Because $\text{spec}(A) = \mathbb{N}$, the operator $B^t$ is periodic in $t$ with period $2\pi$.

The result we are about to state, and also Theorem [1] above, gives information on the asymptotic behavior of

$$\text{Tr } \psi(P_nBP_n) - \text{Tr } P_n\psi(B)P_n, \quad n \to \infty,$$

for the analytic in a neighborhood of 1 function $\psi(u) = \log u$. In [2, 3] we obtain the corresponding statements for an arbitrary analytic function $\psi(u)$. These results involve certain maps $W, \Phi, \Upsilon$ whose action on an analytic function is a continuous function from $\mathbb{C}^j$ to $\mathbb{C}$ for some $j \in \mathbb{N}$. The map

$$W[\psi](y_1, y_2) := \frac{1}{2} \int_0^1 \int_{y_1}^1 \int_{y_2}^1 \frac{\psi'(u_1) - \psi'(u_2)}{u_1 - u_2} du_1 du_2, \quad y_1, y_2 > 0,$$

was obtained by A. Laptev, D. Robert and Yu. Safarov in [4] (here $j = 2$). Earlier, an equivalent to $W$ map has been obtained by H. Widom in [5, 6] in a two-term Szegö type asymptotics for integral operators with discontinuous symbols. It has been noticed by H. Widom, and also by the authors [1, 2], that for the function $\psi(u) = \log u$, the action $W[\log]$ takes a simple form

$$W[\log](y_1, y_2) = -\frac{1}{2} \log y_1 \log y_2.$$  

This explains the fact that the map $W$ is not required in the next theorem.

Now define for $y_1, y_2, y_3 < 1$,

$$\Phi[\log](y_1, y_2, y_3) = y_3 \int_0^{y_1} \int_0^{y_2} \bigg[ \frac{\log(1 - u_1)}{u_1 (u_1 - u_2)(u_1 - u_3)} - \frac{\log(1 - u_2)}{u_2 (u_1 - u_2)(u_2 - u_3)} + \frac{\log(1 - y_3)}{y_3 (u_1 - y_3)(u_2 - y_3)} \bigg] du_1 du_2.$$  

For the map $\Phi$, $j = 3$. The expression for $\Upsilon[\log]$ appearing in the statement of the next theorem is complicated and is not given here, see [2, (1.31)] and [3] for details. It depends only on the principal symbol $\sigma_0(B)$ and involves certain Poisson brackets of the type $\{\sigma_0(B^t), \sigma_0(B^s)\}$, $0 \leq t, s \leq 2\pi$.

For a function $f \in C^\infty(S^*M)$ introduce the $k$th Fourier coefficient along the closed geodesic of length $2\pi$ starting at a given point $(x, \xi)$

$$\hat{f}_k(x, \xi) := \int_0^{2\pi} e^{-ikt} f(\Theta^t(x, \xi)) \frac{dt}{2\pi}, \quad k \in \mathbb{Z}.$$  

For an arbitrary $B \in \Psi^0(M)$, let us write $b_0 := \sigma_0(B)$, $b_{\text{sub}} := \text{sub}(B)$, $b'_0(x, \xi) := b_0(\Theta^t(x, \xi))$, and omit the argument $(x, \xi) \in S^*M$ for brevity.
Theorem 1.2. Let $M$ be a Zoll manifold of dimension $d \in \mathbb{N}$. Let $A$ be defined by (1.4). Assume that $\sigma_1(A)(x, \xi) = \sigma_1(A)(x, -\xi)$ for all $(x, \xi) \in T^* M$. Let $B \in \Psi^0(M)$ be such that $\sigma_0(B)$ is strictly positive and a certain symbolic norm of $I - B$ is sufficiently small. Then $\log B \in \Psi^0(M)$ and the following holds

\begin{equation}
\Tr \log P_n BP_n = \Tr P_n (\log B) P_n
\end{equation}

\begin{align*}
&\quad + n^{-d} \cdot \frac{1}{2} \int_{S^r M} \sum_{k=1}^{\infty} k (\log b_0)_k (\log b_0)_{-k} dxd\xi \\
&\quad + n^{-d-2} \cdot \frac{1}{2} \int_{S^r M} \left( \sum_{k=1}^{\infty} k (\log b_0)_k (b_{sub}/b_0)_{-k} + (d - 1) \left[ \sum_{k=1}^{\infty} \left( k^2 + (1 + \alpha/2)k \right) (\log b_0)_k (\log b_0)_{-k} \\
&\quad + \sum_{k,l=1}^{\infty} k l \int_0^{2\pi} \int_0^{2\pi} e^{i k (t-r) + i l (s-r)} \Phi[\log](b_0, b_0) \frac{dt ds dr}{2\pi 2\pi 2\pi} \right] \\
&\quad + O(n^{-d-3}), \quad n \to \infty.
\end{align*}

Remark 1.2. The existence of a full expansion of the type (1.8) has been proven in [GO1]. Explicit formulas for the coefficients of the first first two terms in (1.8) (and also of the log $n$ term which is a part of $\Tr P_n (\log B) P_n$) have been obtained in [GO1, GO2].

Remark 1.3. The formula (1.7) for $\Phi[\log]$ has a structure similar to the coefficient in the third asymptotic term in a Szeg"{o} type expansion for convolution operators obtained by R. Roccaforte in [R]. A version of the Bohnenblust–Spitzer combinatorial theorem is used in the proof in [R]. A second order Szeg"{o} type expansion for convolution operators was established by H. Widom in [W4] with the help of the usual Hunt–Dyson combinatorial formula (1.4). A full asymptotic expansion for convolution operators has been found in [W4].

Remark 1.4. For $d = 1$, the square bracket in (1.8) disappears. Also $\Upsilon$ vanishes, because all the Poisson brackets vanish in this case (for each of the two cotangent directions the angle does not change and $\sigma_0(B)$ is homogeneous of degree 0 in $\xi$). Theorem 1.1 follows.

Remark 1.5. As we have noticed above, Theorem 1.1 and Theorem 1.2 give lower order corrections to (1.6) for $\psi(u) = \log u$, as $n \to \infty$. In the next section we find explicit formulas for the coefficients in the asymptotic expansion of $\Tr \log P_n BP_n$, as $n \to \infty$. These expansions are fourth order for $d = 1, 2, 3$ and third order for $d \geq 4$, as $n \to \infty$.

Proof of Theorem 1.2: an outline. We start by expanding $\psi(u) = \log u$ in a power series about the point 1. After that it suffices to prove (1.8) for $\psi(u) = u^m$ for all $m \in \mathbb{N}$.

Let us recall the method of [GO1]. Let $\pi_k$, $k \in \mathbb{N}$, be the projection on the $k$th eigenspace of the operator $A$ and set $\pi_k := 0$ for $k \leq 0$. Then $P_n = \sum_{k=1}^{n} \pi_k$ for
$n \in \mathbb{N}$, and we set $P_n := 0$, $n \leq 0$. Recall that because $M$ is a Zoll manifold, for an arbitrary $B \in \Psi^0(M)$, the operator $B^t$, $t \in \mathbb{R}$, defined by (1.5) is $2\pi$-periodic. Therefore one can introduce the Fourier expansion $B = \sum_{j \in \mathbb{Z}} B_j$ where $B_j \in \Psi^0(M)$, $j \in \mathbb{Z}$, is defined by

$$B_j = \frac{1}{2\pi} \int_0^{2\pi} e^{ijt} e^{-itA} B e^{itA} dt.$$ 

For $m \in \mathbb{N}$ and $j_1, \ldots, j_m \in \mathbb{Z}$ introduce the notation

$$M_m(j) := \min(0, j_1, j_1 + j_2, \ldots, j_1 + \cdots + j_m).$$

The following commutation relation is of central importance here

(1.9) $B_j P_n = P_{n+j} B_j$, $n \in \mathbb{N}$, $j \in \mathbb{Z}$.

Using (1.9) one moves all the projectors to the left in the expression

$$(P_n B P_n)^m = \sum_{j_1, \ldots, j_m} P_{n+j_1} P_{n+j_2} P_{n+j_3} \cdots P_{n+j_m} P_n$$

obtaining $P_n B^m P_n$ plus a correction. This implies for all $n \in \mathbb{N}$

(1.10) $\text{Tr} (P_n B P_n)^m - \text{Tr} P_n B^m P_n$

$$= - \sum_{j_1, \ldots, j_m=0}^{M_m(j)+1} \text{Tr} \left( (P_n - P_{n+j_1} \cdots P_{n+j_1+\cdots+j_m}) B_{j_1} \cdots B_{j_m} \right)$$

Next, it is important that for any $G \in \Psi^0(M)$, $M$ being a Zoll manifold, there exists a full asymptotic expansion for $\text{Tr}(\pi_k G)$, as $k \to \infty$, see Lemma 1.3 below. This result is due to Y. Colin de Verdière [CdV]. The coefficients in that expansion are certain Guillemin–Wodzicki residues. Recall that for any compact closed manifold $M$ of dimension $d \in \mathbb{N}$ the Guillemin–Wodzicki residue of a pseudodifferential operator $G \in \Psi^m(M)$ of order $m \in \mathbb{Z}$ is defined by

$$\text{Res}(G) := \int_{S^*M} \sigma_{-d}(G)(x,\xi) \, dx d\xi.$$ 

For an arbitrary $G \in \Psi^0(M)$ denote

$$R_l(G) := \text{Res}(A^{-d+l} G), \quad l = 0, 1, 2, \ldots.$$ 

Lemma 1.3. Let $M$ be a Zoll manifold of dimension $d \in \mathbb{N}$. Assume $G \in \Psi^0(M)$. Then for any $N = 0, 1, 2, \ldots$, there exists $C_N < \infty$ such that

(1.11) $\left| \text{Tr}(\pi_k G) - \sum_{l=0}^{N} k^{d-1-l} R_l(G) \right| \leq C_N k^{d-2-N}$, $k \in \mathbb{N}$.

See [CdV] and [GO2, Appendix] for the proof of (1.11).
Now assume that for some $K < \infty$ we have $B_k = 0$ for $|k| > K$ (this is not assumed in the full proof in [Gi2]). Then (1.10) and Lemma 1.3 imply (1.12)

$$\text{Tr}(P_n B P_n)^m - \text{Tr}P_nB^m P_n$$

$$= \sum_{j_1 + \cdots + j_m = 0} \left\{ n^{d-1} \cdot M_m(\mathcal{J}) \int_{S^*M} \sigma_0(B_{j_1} \cdots B_{j_m}) \, dx \, d\xi \right. \right.$$  

$$+ n^{d-2} \cdot \left[ \frac{d-1}{2} \left( [M_m(\mathcal{J})]^2 + (1 + \frac{\alpha}{2}) M_m(\mathcal{J}) \right) \right] \int_{S^*M} \sigma_0(B_{j_1} \cdots B_{j_m}) \, dx \, d\xi$$

$$\left. + M_m(\mathcal{J}) \int_{S^*M} \text{sub}(B_{j_1} \cdots B_{j_m}) \, dx \, d\xi \right\}$$

$$+ O(n^{d-3}), \quad n \to \infty.$$  

Now the symmetrization argument from [K, S, O1, GO2, GO3, O2] comes into play. Note that the domain of summation over $j_1, \cdots, j_m$ in (1.12) is symmetric with respect to the permutations of the $j$’s. Also the factors involving

$$\sigma_0(B_{j_1} \cdots B_{j_m}) = \sigma_0(B_{j_1}) \cdots \sigma_0(B_{j_m})$$

are symmetric. Therefore one can write

$$\sum_{j_1 + \cdots + j_m = 0} M_m(\mathcal{J}) \int_{S^*M} \sigma_0(B_{j_1} \cdots B_{j_m}) \, dx \, d\xi$$

$$= \sum_{j_1 + \cdots + j_m = 0} \left( \frac{1}{m!} \sum_{\tau \in S_m} M_m(\mathcal{J}_\tau) \right) \int_{S^*M} \sigma_0(B_{j_1} \cdots B_{j_m}) \, dx \, d\xi,$$

where $S_m$ is the set of all permutations of $j_1, \cdots, j_m$ and $\mathcal{J}_\tau := (j_{\tau_1}, \cdots, j_{\tau_m})$. After an application of the usual Hunt–Dyson combinatorial formula (see (3.4) in Section 3)

$$\sum_{\tau \in S_m} M_m(\mathcal{J}_\tau) = \sum_{\tau \in S_m} \sum_{k=1}^{m} \min(0, j_{\tau_1} + \cdots + j_{\tau_k}),$$

all summations but one in $\sum_{j_1 + \cdots + j_m = 0}$ become free and can be carried out.

Now we also see the two difficulties with the computation of the $n^{d-2}$ term. The first difficulty is in computing the minimum raised to power 2. Here one needs a generalization of the Hunt–Dyson formula, see Theorem 3.4. The second difficulty is that the factor

$$\int_{S^*M} \text{sub}(B_{j_1} \cdots B_{j_m}) \, dx \, d\xi$$

is generally speaking not symmetric with respect to the permutations $j_1, \cdots, j_m$, and the straightforward symmetrization as above fails. Indeed, recall that

$$\text{sub}(B_{j_1} \cdots B_{j_m}) = \sum_{k=1}^{m} \text{sub}(B_{j_k}) \prod_{p=1 \atop p \neq k}^{m} \sigma_0(B_{j_p})$$

$$+ \frac{1}{2i} \sum_{1 \leq k < l \leq m} \left\{ \sigma_0(B_{j_k}), \sigma_0(B_{j_l}) \right\} \prod_{p=1 \atop p \neq k, p \neq l}^{m} \sigma_0(B_{j_p}).$$

(1.13)

where $\{,\}$ stands for the Poisson bracket. We see that the first sum in (1.13) is symmetric, whereas the second one is generally speaking not (not even after the
integration over $S^*M$). However, each of the $m(m - 1)/2$ terms in the second sum on the right-hand side of \eqref{1.13} possesses a partial symmetry. More precisely, we are allowed to permute the principal symbols that do not enter the Poisson bracket. This fact can be utilized in a modification of the symmetrization procedure. The corresponding contribution is lengthy, this is where the functional $\Upsilon$ arises.

Remark 1.6. We would like to mention here that in the computation of $\Upsilon$ one really needs the information on the set of values $\{M_m(\tau)\}_{\tau \in S_m}$ counted with multiplicities, that is the Bohnenblust–Spitzer theorem (see Section 3 for its statement), and not just a formula for $\sum_{\tau \in S_m}[M_m(\tau)]^p$ for $p = 1, 2$.

2. Explicit asymptotic formulas for $\log \det P_n BP_n$, as $n \to \infty$.

Theorem 1.1 and 1.2 give an expression for $\text{Tr} \log P_n BP_n = \log \det P_n BP_n$ as a sum of $\text{Tr} P_n(\log B)P_n$ and the two lower order corrections, as $n \to \infty$. We would like to compute the coefficients in the asymptotic expansion of $\log \det P_n BP_n$, as $n \to \infty$.

An auxiliary asymptotic expansion for

\begin{equation}
\text{Tr} P_n GP_n = \sum_{k=1}^{n} \text{Tr}(\pi_k G), \quad n \to \infty,
\end{equation}

where $G \in \Psi^0(M)$, is given in Proposition 2.1 below. A corresponding result for the coefficients of $n^d$, $n^{d-1}$ and $\log n$ for $d \geq 2$ can be found in [GO2, after Lemma 0.2]. To prove Proposition 2.1, we sum over $k = 1, \ldots, n$ in \eqref{1.11}, see [Gi2] for details. The subtle point is the constant coefficient in \eqref{2.1}, which we need for dimension $d = 1, 2$. The terms of all orders in \eqref{1.11}, and also the possible rapidly decaying term, will contribute to it. Let us therefore for $d = 1, 2$ make an additional assumption

\begin{equation}
\sum_{l=0}^{\infty} |R_l(G)| < \infty
\end{equation}

and set, for all $k \in \mathbb{N}$,

\begin{equation}
\epsilon_k(G) := \text{Tr}(\pi_k G) - \sum_{l=0}^{+\infty} k^{d-1-l} R_l(G).
\end{equation}

If \eqref{2.2} holds, the series in \eqref{2.3} (and also in \eqref{2.4} below) is absolutely convergent, and also for any $N \in \mathbb{N}$ there exists $c_N < \infty$ such that $|\epsilon_k| \leq c_N k^{-N}$, $k \in \mathbb{N}$. Set

\begin{equation}
C(G) := \sum_{k=1}^{\infty} \epsilon_k(G).
\end{equation}

Note that in \eqref{2.4} and also in the proposition below there only appears the series $\sum_{l=0}^{\infty} R_l(G)$. However we need the absolute convergence \eqref{2.2} in the proof of the remainder estimate.

Let $\gamma$ denote the Euler constant and $\zeta$ the Riemann zeta function.

Proposition 2.1. Let $M$ be a Zoll manifold of dimension $d \in \mathbb{N}$. Let $P_n$ be as above and assume that $G \in \Psi^0(M)$. For $d = 1, 2$, assume in addition that \eqref{2.2}
holds, and let $C(G)$ be defined by (2.4). Then the following holds, as $n \to \infty$,

(i) for $d = 1$,

$$\operatorname{Tr} P_n GP_n = n \cdot R_0(G) + \log n \cdot R_1(G) + \left( C(G) + \gamma R_1(G) + \sum_{l=2}^{\infty} \zeta(l) R_l(G) \right) + \frac{1}{n} \left( \frac{1}{2} R_1(G) - R_2(G) \right) + O\left( \frac{1}{n^2} \right),$$

(ii) for $d = 2$,

$$\operatorname{Tr} P_n GP_n = n^2 \cdot \frac{1}{2} R_0(G) + n \cdot \left( \frac{1}{2} R_0(G) + R_1(G) \right) + \log n \cdot R_2(G) + \left( C(G) + \gamma R_2(G) + \sum_{l=2}^{\infty} \zeta(l) R_{l+1}(G) \right) + O\left( \frac{1}{n} \right),$$

(iii) for $d \geq 3$,

$$\operatorname{Tr} P_n GP_n = n^d \cdot \frac{1}{d} R_0(G) + n^{d-1} \cdot \left( \frac{1}{2} R_0(G) + \frac{1}{d-1} R_1(G) \right) + n^{d-2} \cdot \left( \frac{d-1}{12} R_0(G) + \frac{1}{2} R_1(G) + \frac{1}{d-2} R_2(G) \right) + \log n \cdot R_d(G) + O(n^{d-3}).$$

Remark 2.1. We see that $\operatorname{Tr} P_n (\log B) P_n$ in Theorem 1.1 and 1.2 ($G = \log B$) contributes to the leading asymptotic term of order $n^d$, and also to all lower order terms of order $n^j$, $j = d - 1, \cdots, 1, 0, -1, \cdots$, and to the logarithmic term $\log n$, as $n \to \infty$. In the classical SSLT the situation is much simpler: $\log B$ is just the Toeplitz matrix of the operator of multiplication by $\log b$, and so $\operatorname{Tr} P_n (\log B) P_n = (2n + 1)(\log b)_0$.

Now we are ready to state the two corollaries.

**Corollary 2.2.** Let $B \in \Psi^0(\mathbb{S}^1)$ have a strictly positive principal symbol and be such that a certain symbolic norm of $I - B$ is sufficiently small. Assume also that (2.2) holds. Then the following holds, as $n \to \infty$,

$$\log \det P_n BP_n = c_1 \cdot n + c_{\log} \cdot \log n + c_0 + c_{-1} \cdot \frac{1}{n} + O\left( \frac{1}{n^2} \right),$$

where the coefficients are the sums of the corresponding coefficients from Theorem 1.2 and Proposition 2.1(i).

Assume further that $\sigma_0(B)$ and $\sub(B)$ do not depend on the direction of $\xi$, that is $\sigma_0(B)(x, \xi) = b_0(x)$ and $\sub(B)(x, \xi) = b_{\sub}(x)|\xi|^{-1}$, for $(x, \xi) \in S^*\mathbb{S}^1$. Assume
also that $b_{-2} = 0$. Then the following holds, as $n \to \infty$,

\begin{equation}
\log \det P_n BP_n = n \cdot 2 \int_0^{2\pi} \log b_0(x) \frac{dx}{2\pi} + \log n \cdot 2 \int_0^{2\pi} \frac{b_{\text{sub}}(x)}{b_0(x)} \frac{dx}{2\pi} + \left( \sum_{k=1}^{\infty} k (\log b_0)_k (\log b_0)_{-k} + C(\log B) + \gamma R_1(\log B) + \sum_{l=2}^{\infty} \zeta(l) R_l(\log B) \right) + \frac{1}{n} \left( \sum_{k=1}^{\infty} k (\log b_0)_k (b_{\text{sub}}/b_0)_{-k} + \int_0^{2\pi} \left[ \frac{b_{\text{sub}}(x)}{b_0(x)} + \left( \frac{b_{\text{sub}}(x)}{b_0(x)} \right)^2 \right] \frac{dx}{2\pi} \right) + O\left( \frac{1}{n^2} \right),
\end{equation}

where $C(\log B)$ is given by (2.4).

Remark 2.2. In some simple cases, for instance for $b_{\text{sub}}(x) = \pm \frac{1}{2} b_0(x)$, the left-hand side in (2.5) can be computed explicitly. The coefficients of $n$, $\log n$, and $\frac{1}{n}$ on the right in (2.5) in these cases are as expected, see also Remark 2.7.

Corollary 2.3. Let $M$ be a Zoll manifold of dimension $d \geq 2$. Assume that $P_n$ and $A$ are as in Theorem 1.4. Let $B \in \Psi^0(M)$ have a strictly positive principal symbol and be such that the symbolic norm of $I - B$ is sufficiently small. For $d = 2$, assume in addition (2.2). Then the following holds, as $n \to \infty$,

\begin{equation}
\log \det P_n BP_n = C_d(0)^{d} \cdot n^d + C_{d-1}^{(d)} \cdot n^{d-1} + C_{d-2}^{(d)} \cdot n^{d-2} + C_{\log}^{(d)} \cdot \log n + O(n^{d-3}),
\end{equation}

where the coefficients are the sums of the corresponding coefficients from Theorem 1.4 and Proposition 2.1(ii) or (iii). If one counts the logarithmic term, this expansion is fourth order for $d = 2, 3$ and third order for $d \geq 4$.

Remark 2.3. The coefficients $C_d^{(d)}$, $C_{d-1}^{(d)}$, and also $C_{\log}^{(d)}$, $d \in \mathbb{N}$, have been found in [GO1, GO2].

Remark 2.4. The most interesting coefficient in (2.4) is the constant one, since one can think of $\exp C_0^{(d)}$ as of a regularized determinant of $B \in \Psi^0(M)$, see [GO1, GO2]. The sum

\[ \gamma R_d(\log B) + \sum_{l=2}^{\infty} \zeta(l) R_{l+d-1}(\log B) \]

will for all $d \in \mathbb{N}$ be a part of $C_0^{(d)}$. For $d = 1$, Corollary 2.2 gives a full expression for $C_0^{(1)}$. For $d = 2$, Corollary 2.3 gives a full expression for $C_0^{(2)}$, which is quite lengthy.

Remark 2.5. Let us compare the result of Corollary 2.3 with a generalization of SSLT to the case of $B$ being an operator of multiplication by a function $b(x)$ having discontinuities which is due to H. Widom and E. Basor. In this case $\log b(x)$ also has discontinuities, and so the series $\sum_{k \geq 2} |k| (\log b)_k^2$ diverges logarithmically. The following third order asymptotic formula holds for the operator of multiplication by a piecewise $C^2$ function $b(x)$

\begin{equation}
\log \det P_n BP_n = a_1 \cdot n + a_2 \cdot \log n + a_3 + o(1), \quad n \to \infty,
\end{equation}

where $a_1$, $a_2$, and $a_3$ are constants.
where $a_1$ as in (2.3), the coefficient $a_2$ has been computed by H. Widom in [W2], and the constant term $a_3$ has been found by E. Basor in [B]. Note that the matrix of $B$ in (2.7) is still Toeplitz, the logarithmic order of the subleading term being due to a slower decay of the Fourier coefficients of $b(x)$. In our case the matrix of the operator $B \in \Psi^0(S^1)$ is not Toeplitz (see Remark 2.7), and the logarithm comes from the contribution of sub$(B)$.

Remark 2.6. It would be interesting to find a compact formula for the constant term in (2.5). We mention that the constant $a_3$ in (2.7) found in [B] has a form similar to the one in (2.3). It contains a “finite” term and an infinite series of certain integrals multiplied by the values of the Riemann zeta function at the points 3, 5, ⋯. Interestingly, an “invariant” form of that series has been found in [W1]. It is written as a single integral involving the function

$$\Psi(x) := \frac{d}{dx} \log \Gamma(x).$$

This gives the hope that a similar formula can be found for the constant (2.5). We have done some computations trying to find the constant term in (2.3), and the function $\Psi(x)$ has been appearing there.

Remark 2.7. The matrix interpretation of Corollary 2.2 is as follows. Assume for simplicity that $B \in \Psi^0(S^1)$ is as in the second part of Corollary 2.2, that is $\sigma_0(B)(x, \xi) = b_0(x)$ and sub$(B)(x, \xi) = b_{sub}(x) |\xi|^{-1}$, for all $(x, \xi) \in S^* S^1$. Assume also that $b_{-2} = b_{-3} = ⋯ = 0$. Let $B_0$ and $B_{sub}$ be the operators of multiplication by $b_0$ and $b_{sub}$, respectively. Let $D$ be the linear operator in $L^2(S^1)$ such that

$$De^{ikx} = \begin{cases} \frac{1}{|\xi|} e^{ikx}, & |k| \geq 1 \\ 0, & k = 0 \end{cases}$$

Note that this is not a differential, but rather a smoothing operator of order $-1$. There is known a correspondence between the classical PsDO’s on the circle and their discrete counterparts, see [B] for details. By that correspondence, the zeroth order PsDO $B$ we started with, equals $B_0 + B_{sub}D$. Introduce two Toeplitz matrices, $\tilde{B}_0 := \{(b_{0})_{j,k}\}_{j,k \in \mathbb{Z}}$ and $\tilde{B}_{sub} := \{(b_{sub})_{j,k}\}_{j,k \in \mathbb{Z}}$. Set also $\tilde{D} := \text{diag}(\cdots , \frac{1}{2}, \frac{1}{2}, 1, 0, 1, \frac{1}{2}, \frac{1}{2}, \cdots )$. Then the matrix representation of $B_0 + B_{sub}D$ is $\tilde{B}_0 + \tilde{B}_{sub} \cdot \tilde{D}$. Finally, set $\tilde{P}_n = \text{diag}(\cdots , 0, 1, \cdots , 1, 0, \cdots )$ $(2n + 1$ ones). We see that Corollary 2.2 gives a fourth order asymptotics of the determinant of the truncated matrix $\tilde{P}_n \cdot (\tilde{B}_0 + \tilde{B}_{sub} \cdot \tilde{D}) \cdot \tilde{P}_n$.

Now we can reformulate the question of finding the constant term in (2.5) in purely matrix terms. Drop the hats and the dots for brevity. Let $C_1$ be a Toeplitz matrix that corresponds to the operator of multiplication by $b_{sub}/b_0$, and let the matrix $D$ be as above. Clearly, the matrices $C_1$ and $D$ do not commute. Assume that the matrix $\log(I - C_1D)$ is well-defined. The question is to compute the constant coefficient in $Tr \ P_n \log(I - C_1D)P_n$, or which is the same, the constant coefficient in

$$Tr \ P_n \log(I - D^{1/2}C_1D^{1/2})P_n, \quad n \to \infty.$$

As we have noticed in Remark 2.1, this question is trivial for a Toeplitz matrix $T$ in place of $D^{1/2}C_1D^{1/2}$.
3. Generalized Hunt–Dyson combinatorial formula

In this section we state the generalized Hunt–Dyson formula (gHD) and remind the reader the Bohnenblust–Spitzer theorem (BSt). After that we briefly explain how the gHD is derived from the BSt in [Gi3]. We refer to [Gi3, Section 4] for the first step of an independent proof of the gHD. This first step is a generalization of \(F. J. \) Dyson’s idea on which the proof of the usual Hunt–Dyson formula in [K] is based. It is also explained in [Gi3, Section 4] how to reprove the BSt starting with the gHD. See [Gi1, Chapter 2] for the details of the independent proof of the gHD.

We state the result for the maximum, for the corresponding result for the minimum one should replace the positive parts with the negative parts. For \(a \in \mathbb{R}\), \(n \in \mathbb{N}\) denote
\[
(a)_+ := \max(0, a), \quad (a)^n_+ := ((a)_+)^n.
\]
Fix any \(m \in \mathbb{N}\) and assume \(a_1, \ldots, a_m \in \mathbb{R}\). Let \(S_m\) be the set of all permutations \(\tau\) of the numbers \(1, \ldots, m\). For each \(\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in S_m\) denote
\[
a_\tau := (a_{\tau_1}, \cdots, a_{\tau_m}).
\]
Introduce the notation
\[
(3.1) \quad M_j(a_\tau) := \begin{cases} 
\max(0, a_{\tau_1}, a_{\tau_1} + a_{\tau_2}, \cdots, a_{\tau_1} + \cdots + a_{\tau_j}), & j = 1, \cdots, m, \\
0, & j = 0.
\end{cases}
\]
Fix any \(j = 1, \cdots, m\). For arbitrary \(k_1 \geq 1, \cdots, k_j \geq 1\), \(k_1 + \cdots + k_j = m\), we introduce the notation
\[
(3.2) \quad k_1(a_\tau) := a_{\tau_1} + \cdots + a_{\tau_{k_1}},
\]
\[
k_2(a_\tau) := a_{\tau_{k_1+1}} + \cdots + a_{\tau_{k_1+k_2}},
\]
\[
\vdots
\]
\[
k_j(a_\tau) := a_{\tau_{k_1+\cdots+k_{j-1}+1}} + \cdots + a_{\tau_{k_1+\cdots+k_{j-1}+k_j}}.
\]
Each of \(k_l(a_\tau), l = 1, \cdots, j\), is a sum of \(k_l\) permuted variables out of \(a_{\tau_1}, \cdots, a_{\tau_m}\), so that each of the permuted variables enters exactly one sum. Recall that \(\binom{n}{l_1, \cdots, l_j} := \frac{n!}{l_1! \cdots l_j!}\) denotes a multinomial coefficient, here \(n, j \in \mathbb{N}\), \(l_1, \cdots, l_j \in \mathbb{N} \cup \{0\}\), and \(l_1 + \cdots + l_j = n\). We are ready to state the generalized Hunt–Dyson formula (gHD).

**Theorem 3.1.** For an arbitrary power \(n \in \mathbb{N}\), an arbitrary number of variables \(m \in \mathbb{N}\), and for arbitrary \(a_1, \cdots, a_m \in \mathbb{R}\), the following holds
\[
(3.3) \quad \sum_{\tau \in S_m} \min(m, n) \left[ (M_m(a_\tau))^n - (M_{m-1}(a_\tau))^n \right]
= \sum_{\tau \in S_m} \sum_{j=1}^{\min(m, n)} \frac{1}{j!} \sum_{k_1, \cdots, k_j \geq 1} \binom{n}{l_1, \cdots, l_j} \frac{(k_1(a_\tau))^l_1}{k_1} \cdots \frac{(k_j(a_\tau))^l_j}{k_j}.
\]
Remark 3.1. In the case \( n = 1 \) we obtain \( j = 1, \ k_1 = m, \ l_1 = 1, \) and (3.3) becomes the usual Hunt–Dyson combinatorial formula (HD) [3, after (4.8)]

\[
\sum_{\tau \in S_m} \left[ M_m(a_{\tau}) - M_{m-1}(a_{\tau}) \right] = \sum_{\tau \in S_m} \frac{(a_{\tau_1} + \cdots + a_{\tau_m})_+}{m} = (m-1)! (a_1 + \cdots + a_m)_+.
\]

Recall now the statement of the Bohnenblust–Spitzer theorem (BST) [5, Theorem 2.2]. It asserts that for any \( m \in \mathbb{N} \) and arbitrary \( a_1, \cdots, a_m \in \mathbb{R} \), the set \( \{M_m(a_{\tau}) \}_{\tau \in S_m} \) contains the same numbers with the same multiplicities as the set of sums of positive parts of the sums of \( a_1, \cdots, a_m \), arranged according to the cyclic representations of all \( m! \) permutations. This becomes clear if we consider a simple example. Let us choose \( m = 3 \) and any \( a_1, a_2, a_3 \in \mathbb{R} \). The symmetric group \( S_3 \) consists of six permutations that can be written via the cyclic representations as

\[
S_3 = \{ (123), (132), (12)(3), (13)(2), (23)(1), (1)(2)(3) \}
\]

In this case the BST states that the set

\[
\{ \max(0, a_{\sigma_1}, a_{\sigma_2}, a_{\sigma_3}) \}_{\sigma \in S_3}
\]

contains the same numbers with the same multiplicities as the set

\[
\{(a_1 + a_2 + a_3)_+, (a_1 + a_3 + a_2)_+, (a_1 + a_3 + a_2)_+, (a_2 + a_3)_+, (a_1)_+, (a_2)_+, (a_3)_+ \}.
\]

Note that a certain maximum of zero and accumulating sums of the permuted variables does not need to equal the element of the set on the right-hand side corresponding to the cyclic representation of that permutation. The statement of the BST is merely that the whole multisets are identical.

Proof of Theorem 3.1. an outline. By the BST, for any \( m \in \mathbb{N} \) and arbitrary real \( a_1, \cdots, a_m \), the counterparts of the sets (3.3) and (3.6) contain the same numbers with the same multiplicities. Therefore for any function \( f : \mathbb{R}_+ \rightarrow \mathbb{C} \), the sum \( \sum_{\tau \in S_m} f(M_m(a_{\tau})) \) equals the sum of \( f \)'s values over the counterpart of (3.6).

It turns out that for the multinomial function \( f(t) = t^m \) for any \( n \in \mathbb{N} \), a further computation can be performed which leads to the gHD, see [Gi3, Section 2].

Without going into details let us just mention here that when we apply the multinomial formula in the sum over the analog of (3.6), some terms have zero powers. Recall that we want all factors to be present in the right-hand side of the gHD (3.3). However it turns out that the terms having at least one zero power can be summed together, and their sum gives exactly the sum of the \( n \)th power of the “previous” maximum \( M_{m-1}(a_{\sigma}) \) over \( S_m \).

At this step we use the principle of inclusion and exclusion and the Cauchy and Cayley identities from the theory of partitions. \( \square \)

Remark 3.2. The steps of the derivation of the gHD from the BST can be reversed. After that having started with the gHD, we can conclude that for any monomial \( f(t) \) its sum over (3.3) equals the sum over (3.6). Due to an additional linearity, this actually holds for an arbitrary polynomial \( f(t) \). Now using the polynomial interpolation we arrive at the BST, see [Gi3, Section 4].
Remark 3.3. The independent proof of the gHD in [Gi1, Chapter 2] proceeds by induction on the power \( n \in \mathbb{N} \). The base of induction is the usual HD \((n = 1)\). In the proof of the inductive step, the key cancellation of the highest power \( n \) of the maximum after taking a sum over \( S_m \) follows from a generalization of F. J. Dyson’s argument from the proof of the usual Hunt–Dyson formula in [K], see [Gi2, Section 4]. The proof of the inductive step is however quite technical.

Remark 3.4. In [RS], the authors rediscover a version of the usual Hunt–Dyson formula starting from the much more powerful BSt.

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