Do Absolutely Irreducible Group Actions Have Odd Dimensional Fixed Point Spaces?

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Abstract

In his volume \cite{5} on “Symmetry Breaking for Compact Lie Groups” Mike Field quotes a private communication by Jorge Ize claiming that any bifurcation problem with absolutely irreducible group action would lead to bifurcation of steady states. The proof should come from the fact that any absolutely irreducible representation possesses an odd dimensional fixed point space.

In this paper we show that there are many examples of groups which have absolutely irreducible representations but no odd dimensional fixed point space. This observation may be relevant also for some degree theoretic considerations concerning equivariant bifurcation. Moreover we show that our examples give rise to some interesting Hamiltonian dynamics and we show that despite some complications we can go a long way towards doing explicit computations and providing complete proofs. For some of the invariant theory needed we will depend on some computer aided computations. The work presented here greatly benefited from the computer algebra program GAP\cite{6}, which is an indispensable aid for doing the required group theory computations.
1 Introduction

Symmetric systems may have solutions with less symmetry than the original problem. In bifurcation theory one can have the situation that branching of fully symmetric solutions leads to less symmetric solutions. Here the notion of a solution refers to steady states of some dynamical problem. Questions on symmetries of solutions come up in pattern formation and other applied problems. The general problem can be stated as follows: let $X$ be a state space with a group action, let us say by a compact Lie group $G$, let $P$ be a parameter space and suppose that $F: X \times P \rightarrow X$ is the right hand side of a differential equation which is equivariant with respect to this action, i.e.

$$F(gx, p) = gF(x, p)$$

for all $g \in G$, $p \in P$. If $x_0$ for given $p_0$ is a steady state solution which is fully symmetric, then we have

$$F(gx_0, p_0) = 0$$

for all $g \in G$. Symmetry breaking occurs if for $p$ near $p_0$ we find steady state solutions $x(p)$ with $gx(p) \neq x(p)$ for at least one $g \in G$. Steady state bifurcation with symmetry has a long history, see for example Vanderbauwhede [18], Satttinger, [17], Golubitsky et al. [7, 8], Chossat and Lauterbach [3]. One of the main results is the so called Equivariant Branching Lemma. It addresses the situation when $X$ is a real Banach space and the branching comes from a change of stability of the steady state solution where at criticality the linearisation has a kernel which is an absolutely irreducible representation of $G$.

Let us briefly recall these notions. If $X$ is a Banach space, $F: X \times P \rightarrow X$ is sufficiently smooth and equivariant. Let $F(x_0, p_0) = 0$ and assume that $K = D_xF(x_0, p_0)$ is the kernel of the linearisation of $F$ at this point. If $D_xF(x_0, p_0)$ is an isomorphism of $X$, then locally near $p_0$ the solution manifold can be parameterized over $p$. If we assume the parameter space to be one-dimensional, then the solution manifold is locally a one dimensional manifold. If we assume, that at $p_0$ the operator $D_xF(x_0, p_0)$ is not an isomorphism, and moreover if we assume, that dim ker $D_xF(x_0, p_0) > 0$, then the kernel $K = D_xF(x_0, p_0)$ is invariant under the group action. It is a generic property that $K$ is an absolutely irreducible representation, i.e. that a linear map commuting with the group action is a multiple of the identity. In such a
situation the eigenvalue 0 can be prolonged to neighboring parameter values and the group action will remain the same [3]. Therefore the multiplicity of the critical eigenvalue will not change and we have to look at solutions in the kernel $K$. The geometry of the group action helps to overcome some of the problems of the higher multiplicity of the eigenvalue.

**Theorem 1.1 (Sattinger, Vanderbauwhede, Cicogna) [8, 10]** If $H < G$ is an isotropy subgroup with $\dim \text{Fix}(H) = 1$, then the Hopf condition

$$\sigma'(p) \neq 0$$

implies bifurcation of a branch of steady states with isotropy $H$.

Here we assume that the space of parameters is one-dimensional and $\sigma(p)$ is the curve of eigenvalues prolonging the critical eigenvalue. It is an open question whether the loss of stability through an absolutely irreducible kernel always leads to a bifurcation of a branch of steady states. In the abstract we find a strategy to prove such a result. It relies on a slight generalization of the Equivariant Branching Lemma.

**Theorem 1.2 [3]** Given a compact Lie group $G$ an $G$-equivariant bifurcation problem

$$F(x, \lambda) = 0$$

with an absolutely irreducible $G$-action on the kernel

$$K = \ker D_x F(0, 0)$$

of the linearisation. If $H < G$ is an isotropy subgroup for which $\dim \text{Fix}(H)$ is odd, then the Hopf condition

$$\sigma'(p) \neq 0$$

implies bifurcation of a branch of steady states with isotropy at least $H$.

Here as before we write $\sigma$ for the prolongation of the critical eigenvalue.

Then, if we can show that each group with an absolutely irreducible group action has an isotropy subgroup with an odd dimensional kernel, then a general result on bifurcation of equilibria in the presence of absolutely irreducible group actions would follow from the above statement. As far as we are aware, this property holds for all group actions previously considered.
in the context of equivariant bifurcation theory. In particular, it is true for all absolutely irreducible group actions on $\mathbb{R}^2$ and $\mathbb{R}^3$. In dimension 2 the groups acting absolutely irreducible are $D_m$, $m \geq 3$ and $O(2)$. These groups all contain reflections which have a one-dimensional fixed point space. In $\mathbb{R}^3$, the relevant groups are $O(3)$ and some of its subgroups, and all of these groups contain a rotation with a one-dimensional fixed point space.

In this paper, we will show that this strategy cannot be successful, by providing infinite series of finite groups each of which acts absolutely irreducibly on $\mathbb{R}^4$, for which the only non-trivial isotropy subgroups have two dimensional fixed point subspaces. In section 2 the powerful and compact quaternion notation for actions on $\mathbb{R}^4$ is introduced. The subgroups of interest and our main results are given in section 3. The equivariant vector fields and their Hamiltonian structure are discussed in sections 4 and 5 followed by the proofs of the main results in section 6. Some computational results are given in section 7.

2 The group $SO(4)$, its subgroups and quaternion notation

A classification of subgroups of $SO(4)$ and $O(4)$ goes back to Goursat [9]; some data relevant for bifurcation theory has been given by Becker and Krämer [1]. Here we use the classification of subgroups as it is presented by Conway and Smith [4]. In this recent (and very nice) book the quaternions are used to give a geometric way to describe the subgroups of $SO(3)$, $O(3)$, $SO(4)$ and some others. This quaternion notation provides a much more compact and elegant description of $SO(4)$ than the use of $4 \times 4$ matrices. Although we will use the form described by [4], it is worth noting that an equivalent notation was described by Felix Klein [12, 13], who in turn attributes the key result to Cayley [2].

We will denote the set of unit quaternions by $Q$. The set of pairs of such quaternions forms a six-dimensional group, called the spinor group and denoted by $\text{Spin}_4$. We get a map

$$\text{Spin}_4 \to SO(4) : (l, r) \mapsto [l, r] = \{x \mapsto \bar{l}xr\} \quad (1)$$
where a vector in $\mathbb{R}^4$ is identified with a quaternion via
\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \iff x_1 + i x_2 + j x_3 + k x_4.
\]

It is obvious that
\[
[-l, -r] = [l, r]
\]
and [4] show that this is the only way that injectivity fails, so the map is two-to-one.

In a similar way we can obtain a map $\text{Spin}_4 \to \mathbf{O}(4) \setminus \mathbf{SO}(4)$ by
\[
\text{Spin}_4 \to \mathbf{O}(4) \setminus \mathbf{SO}(4) : (l, r) \mapsto \star[l, r] = \{ x \mapsto \bar{l} \bar{x} r \},
\]
but this form will not be used in this paper. Using the map (1), composition of elements of $\mathbf{SO}(4)$ can be written in a natural way,
\[
[l_1, r_1][l_2, r_2] = [l_1 l_2, r_1 r_2],
\]
since $\bar{l}_1 \bar{l}_2 = \bar{l}_2 \bar{l}_1$ (recall that for any unit quaternion $q$, $\bar{q} = q^{-1}$). A number of other properties can easily be obtained. If $l = r$, then the real ($x_1$) axis is preserved, so this special case represents an element of $\mathbf{SO}(3)$ acting on $(x_2, x_3, x_4)$.

Elements in $\mathbf{SO}(4)$ are either single rotations, that fix all points on a two-dimensional plane, or double rotations that fix only the origin. These two types of rotation can be easily be distinguished in the quaternion notation (see Lemma 6.5).

Using the map (1), Conway and Smith [4] use the classification of subgroups of $\mathbf{O}(3)$ to present a complete list of subgroups of $\mathbf{SO}(4)$.

3 Series of Groups

In a search for examples of groups with an absolutely irreducible action on a finite dimensional space, where all the isotropy subgroups have even dimensional fixed point spaces, we came across three groups of order 48 having this property. Further investigation showed that these formed part of three infinite series of such groups.
We are interested in three series of groups $G_j(m), j = 1, 2, 3$ and $m \geq 3$ an odd integer. We follow the notation in [4], where each group is related to two subgroups of $O(3)$. The groups can also be defined by a (non-minimal) set of generators in the quaternion notation described above (see [4], Tables 4.1 and 4.2). Let us write

\[
G_1(m) = \pm \frac{1}{2}[D_{2m} \times D_8] \\
G_2(m) = \pm \frac{1}{4}[D_{4m} \times D_8] \\
G_3(m) = \pm [D_{2m} \times D_4].
\]

The orders of these groups are (see [4]): $|G_1(m)| = \frac{1}{2} \cdot 2 \cdot 2m \cdot 8 = 16m$, $|G_2(m)| = 2 \cdot \frac{1}{4} \cdot 8 \cdot 4m = 16m$ and $|G_3(m)| = 2 \cdot 2m \cdot 4 = 16m$. So we get group orders 48, 80, 112, ..., all of which have the form $16 + 32 \cdot \ell, \ell \in \mathbb{N}$ and $m = 2\ell + 1$. Observe that in the notation of [4] the group $D_{2n}$ has $2n$ elements. Table 1 translates this notation for small values of $m$ into the SmallGroupLibrary notation of GAP [6]. For some of the computations this program and its library are extremely useful (some computational results are given in section 7).

| $m$ | $G_1(m)$ | $G_2(m)$ | $G_3(m)$ |
|-----|----------|----------|----------|
| 3   | [48:17]  | [48:15]  | [48:41]  |
| 5   | [80:17]  | [80:15]  | [80:42]  |
| 7   | [112:16]| [112:14]| [112:34]|
| 9   | [144:18]| [144:16]| [144:44]|
| 11  | [176:16]| [176:14]| [176:34]|
| 13  | [208:17]| [208:15]| [208:42]|

Table 1: The SmallGroupLibrary names for our groups for small values of $m$.

The main results concerning these groups are collected in the following theorems.

**Theorem 3.1** 1. Given any two groups within the same series $G_j(m)$ and $G_j(m')$ then, if $m$ divides $m'$, we have

$$G_j(m) \subset G_j(m').$$
2. The closure of the union of all groups within one family is a compact, one-dimensional Lie-Group \( G_j \). The groups \( G_1 \) and \( G_2 \) are isomorphic.

3. The infinitesimal generator of \( G_j \) is given by
   \[
   L_j = [i, 0].
   \]

4. Each of the groups \( G_j(m) \), \( j = 1, 2, 3 \) contains a unique index 2 subgroup \( F_j(m) \) which commutes with
   \[
   J = \begin{pmatrix}
   0 & 1 & 0 & 0 \\
   -1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 \\
   0 & 0 & -1 & 0
   \end{pmatrix} = [i, 1].
   \]

5. The closure of the unions of all \( F_j(m) \) are again subgroups of \( SO(4) \), denoted by \( F_j \), which are compact one dimensional Lie Groups which are contained in \( G_j \) and which commute with \( J \).

6. The elements in \( G_j(m) \setminus F_j(m) \), and in \( G_j \setminus F_j \) anti-commute with \( J \).

The following theorem describes the actions of these groups on \( \mathbb{R}^4 \).

**Theorem 3.2**

1. The natural actions \( \rho \) of \( G_j(m) \) on \( \mathbb{R}^4 \) for \( j = 1, 2, 3 \) and \( m \geq 3 \) are absolutely irreducible.

2. If \( m \geq 3 \) is odd then corresponding to the natural representation \( \rho \) of \( G_j(m) \) on \( \mathbb{R}^4 \) there exists at least one nontrivial isotropy subgroup. All isotropy subgroups are of order 2 and the corresponding fixed point space is 2-dimensional.

3. In each of the groups \( G_j(m) \) we have precisely \( j \) isotropy types.

4. The normalizer of the isotropy subspaces acts on the fixed point subspaces in the following way:
   
   (a) \( j = 1 \): The normalizer is isomorphic to \( D_2 \) and it acts on \( \text{Fix}(Z_2) \) as \( Z_2 \), namely as a rotation by \( \pi \).
   
   (b) \( j = 2 \): here we have two isotropy subgroups: in one case the normalizer is isomorphic to \( D_2 \) and it acts as in the previous case. The other normalizer is isomorphic to \( Z_{2m} \times Z_2 \) and it acts on \( \text{Fix}(Z_2) \) as a rotation by \( \frac{\pi}{m} \).
(c) \( j = 3 \): in this case we have three isotropy types; each one is represented by a group isomorphic to \( \mathbb{Z}_2 \). In each case the normalizer is isomorphic to \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) and the normalizer acts as \( \mathbb{Z}_4 \).

4 Flows

In this section we want to look at the set of \( G_j(m) \)-equivariant vector fields on \( \mathbb{R}^4 \) and we show that generically we find that loss of stability leads to bifurcating equilibria. In order to determine the fine structure of the equivariant maps we need to know the number of equivariant polynomial maps in a given dimension. Computing the Poincare series gives this information. The dimension of the space of equivariant polynomials can be computed directly using a formula given in Sattinger [17]. The computations needed for such a detailed study are given in Section 8. Tables 2, 3 and 4 show the number of invariant polynomials and equivariant polynomial maps respectively in the various degrees for the groups \( G_j(m), F_j(m) \) for \( j = 1, 2, 3 \) and \( m \geq 3, m \) odd. Looking at the lowest order nontrivial \( G_j(m) \)-equivariant polynomial maps, we observe that we expect in each case three independent

| \( m \) | \( G_1(m) \) | \( e3 \) | \( i4 \) | \( i6 \) | \( i8 \) | \( F_1(m) \) | \( e1 \) | \( i2 \) | \( i3 \) | \( i4 \) | \( i6 \) | \( i8 \) |
|------|--------|------|------|------|------|--------|------|------|------|------|------|------|
| 3    | [48,17]| 3    | 2    | 4    | 9    | [24,11]| 2    | 1    | 6    | 3    | 6    | 15   |
| 5    | [80,17]| 3    | 2    | 3    | 5    | [40,11]| 2    | 1    | 6    | 3    | 4    | 7    |
| 7    | [112,16]| 3   | 2    | 3    | 5    | [56,10]| 2    | 1    | 6    | 3    | 4    | 7    |
| 9    | [144,18]| 3   | 2    | 3    | 5    | [72,11]| 2    | 1    | 6    | 3    | 4    | 7    |
| 11   | [176,16]| 3   | 2    | 3    | 5    | [88,10]| 2    | 1    | 6    | 3    | 4    | 7    |
| 13   | [208,17]| 3   | 2    | 3    | 5    | [104,11]| 2    | 1    | 6    | 3    | 4    | 7    |
| 15   | [240,78]| 3   | 2    | 3    | 5    | [120,33]| 2    | 1    | 6    | 3    | 4    | 7    |
| 17   | [272,17]| 3   | 2    | 3    | 5    | [136,11]| 2    | 1    | 6    | 3    | 4    | 7    |
| 19   | [304,16]| 3   | 2    | 3    | 5    | [152,10]| 2    | 1    | 6    | 3    | 4    | 7    |
| 21   | [336,103]| 3   | 2    | 3    | 5    | [168,41]| 2    | 1    | 6    | 3    | 4    | 7    |

Table 2: The information on the invariants/equivariants for the groups in \( G_1(m) \). Here \( e \) stands for equivariants, \( i \) for invariants and the number behind these letters for the degree of the polynomial map. The number in the table gives the dimension of the space of equivariants/invariants in the given degrees. Observe, here and in the following tables the groups in the left column act absolutely irreducibly and hence we always have \( e1 = i2 = 1 \).
maps of order three. In the cases $j = 1, 2$ we expect two of these maps to be variational. This is because $i4 = 2$ and the gradient of an invariant of degree 4 is an equivariant of degree 3 that is variational. In the case $j = 3$ all of the equivariants are variational. For fixed $j$ the numbers $e_3, i_4, e_5$ (which are not displayed in the tables) are monotonically decreasing in $m$. So if we find 3 independent equivariant maps of order 3, which are equivariant for all $m$, we see that these three maps are the ones to be looked at. Moreover they are equivariant with respect to $G_j$, which is a compact Lie group. In order to discuss specifics for each group, one has to look at higher order equivariants. However we expect, that generically the bifurcation scenario will be decided at the cubic level. In the case $j = 3$, the equivariant maps up to order 3 are variational, so restricting to third order we will have bifurcation to equilibria. We collect the results in the following two theorems, which will proved in Section 6.

**Theorem 4.1** For each $j = 1, 2, 3$ and each $m \geq 3$, $m$ odd, there are precisely three linearly independent cubic equivariant maps. For $j = 1, 2$ two of these maps are gradients of invariant polynomials, the third one is a Hamiltonian vectorfield. In the case $j = 3$ all three vector fields are gradients of invariant polynomials.
Table 4: The information on the invariants/equivariants for the groups in $G_3(m)$. Here $e$ stands for equivariants, $i$ for invariants and the number after these letters for the degree of the polynomial map. The number in the table gives the dimension of the space of equivariants/invariants in the given degrees.

Theorem 4.2  The third order polynomial equations lead to bifurcation to one or more circles of equilibria. At least one of these circles intersects one fixed point spaces in discrete points. Each of these points is a regular point, so if we restrict the map to the complement of a ball around zero in the nontrivial fixed point spaces, we find at least one point which persists under perturbation with higher order terms.

In a short form we have shown:

Theorem 4.3  Generically, i.e. for an open and dense set in the set of $C^\infty$-equivariant vector fields we have bifurcation of nontrivial equilibria.

In this sense the Ize conjecture holds for the groups under consideration.

## 5 Hamiltonian structure

In this section we want to describe a Hamiltonian structure which we have in all the groups discussed here, but has not been well studied. Whether the generic behaviour for this type of group is different from the usual one is not clear, it could well be that there are new phenomena. Let us briefly
describe the situation of equivariant Hamiltonians. On $\mathbb{R}^{2n}$ we look at linear operators $J$ with $J^2 = -I$. We call a vectorfield $v : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ Hamiltonian, if there exists a function $H : \mathbb{R}^{2n} \to \mathbb{R}$ with

$$v = J \nabla H.$$

The vectorfield $v$ is equivariant with respect to a group $G$ if one of two conditions hold:

1. $H$ is an invariant for $G$, and $J$ commutes with $g \in G$. Then $\nabla H$ is equivariant, and $v(gx) = J \nabla H(gx) = Jg \nabla H(x) = gJ \nabla H(x) = gv(x)$. Observe that $J$ is not a multiple of the identity and commutes with $g \in G$, therefore the action of $G$ is not absolutely irreducible.

2. In this case we require an index 2 subgroup $F$ of $G$ and $H$ is an invariant of $F$, $J$ commutes with $F$ and for $g \in G \setminus F$ we have $H(gx) = -H(x)$ and $g^{-1}Jg = -J$. Then, obviously $v$ is equivariant. In such a case $G$ can act absolutely irreducibly.

In all our examples we have a pair $(G, F)$ of index 2 subgroups and we have functions $H$ and operators $J$ which are invariant under $F$ and anti-commute with elements in $G \setminus F$. The details can be found in the next section.

6 Proofs

[4] give a set of generators for these groups. For elements $a, b, c, \ldots$ of a group we write

$$\langle a, b, c, \ldots \rangle$$

for the smallest subgroup containing these elements. Following [4] we define elements using the short notation

$$e_s = e^{i\tau s}.$$

With this notation the groups are given by (see [4], Tables 4.1 and 4.2)

$$G_1(m) = \langle [e_m, 1], [1, i], [1, j], [j, e_4] \rangle$$
$$G_2(m) = \langle [e_m, 1], [1, i], [e_{2m}, j], [j, e_4] \rangle$$
$$G_3(m) = \langle [e_m, 1], [1, i], [j, 1], [1, j] \rangle.$$
Observe that these sets of generators do not form minimal sets of generators. A first observation is the following: if we multiply the first two elements in \( Q \times Q \), then the product generates the same group as the two elements. For this it suffices to prove, that the element \([e_m, i]\) is in the group generated by \([e_m, i]\). Since
\[
[e_m, i]^4 = [e_m^4, 1]
\]
and \(m\) and 4 are relatively prime, \(e_m\) and \(e_m^4\) generate the same group in \(S^1\).

**Lemma 6.1** The element \([e_m, i]\) generates a group \(H(m)\) of order 4m.

**Proof.** The order of the group generated by this element is obviously a multiple of \(m\). So we have
\[
[e_m, i]^m = [−1, ±i] \neq [1, 1], \quad [e_m, i]^{2m} = [1, (−1)^m] \neq [1, 1], \quad \text{and} \quad [e_m, i]^{4m} = [1, 1].
\]

6.1 Proof of Theorem 3.1.

In this section we give the proof of the six parts of Theorem 3.1 on the structure of the groups.

1. If \(m\) divides \(m'\), so \(m' = pm\) for some integer \(p\), then
\[
[e_m, 1]^p = [e^{p\pi i/m}, 1] = [e^{\pi i/m}, 1] = [e_m, 1]
\]
so \(⟨[e_m, 1]⟩ \subset ⟨[e_m', 1]⟩\) and hence \(G_1(m) \subset G_1(m')\) and \(G_3(m) \subset G_3(m')\). Similarly, considering the element \([e_{2m}, j] \in G_2(m)\), \([e_{2m'}, j]^p = [e_{2m}, j^p]\), but we know that \(p\) is odd (since \(m'\) is odd), so this is \(±[e_{2m}, j] \in G_2(m)\), so \(⟨[e_{2m}, j]⟩ \subset ⟨[e_{2m'}, j]⟩\) and hence \(G_2(m) \subset G_2(m')\).

2. The closure of the union of all the groups \(G_1(m)\) is
\[
G_1 = ⟨[e^{iθ}, 1], [1, i], [1, j], [j, e_4]⟩
\]
where \( \theta \in [0, 2\pi] \). This is a compact one-dimensional Lie group. For the group \( G_2 \), we also have elements of the form \([e^{i\phi}, j]\), but these are already included in \( G_1 \), so \( G_2 \subset G_1 \). But also \( G_1 \subset G_2 \), since \([1, j] \in G_2\), so \( G_1 = G_2 \). By the same argument the closure of \( G_3(m) \) is a compact Lie group, but this is not the same group as \( G_1 \).

3. Note that \( H(m) \) is in the intersection of all \( G_j(m) \). The closure of the union over all groups \( H(m) \) is a one-parameter group and hence isomorphic to \( S^1 \) with the generator of its Lie algebra given by \([i, 0]\). The group generated by \( H(m) \) and the remaining generators of \( G_j(m) \) produce a extension of finite index, therefore the connected component of this group is isomorphic to \( S^1 \).

4. We define

\[
F_1(m) = \langle [e_m, i], [1, j] \rangle \\
F_2(m) = \langle [e_m, i], [e_{2m}, j] \rangle \\
F_3(m) = F_1(m).
\]

Let us write

\[ F(m) = ([e_m, i]^2) = ([e_{m/2}, -1]). \]

Clearly \( F(m) \subset F_j(m) \). The generator of \( F(m) \) commutes with all elements in \( F_j(m) \) and hence \( F(m) \) is contained in the center of \( F_j(m) \) for \( j = 1, 2, 3 \). \( F(m) \) contains \( 2m \) elements, including minus the identity. Since \([e_m, i]\) does not commute with the second generator of \( F_j(m) \), \( F(m) \) is the center of \( F_j(m) \). Now the square of the second generator of \( F_j(m) \) is in both cases in \( F(m) \), since for \( F_1(m) \) the square of the second generator is \(-1 \in F(m)\) and for \( F_2(m) \) the square of the second generator is \([e_m, -1] = \pm[e_{m/2}, -1]^{(m+1)/2} \in F(m)\). Therefore \( F_j(m) \) is an index 4 extension of \( F(m) \) and hence it has \( 8m \) elements. Therefore \( F_j(m) \) is an index 2 subgroup of \( G_j(m) \).

5. It is obvious that each element of \( F_j(m) \) commutes with \([i, 1]\). Therefore \( F_j(m) \) commutes with \( J \) which is the map induced by \([i, 1]\).

6. The elements of \( G_j(m) \) which are not in \( F_j(m) \) anti-commute with \([i, 1]\). Note that \( F(m) \) being the center of \( F_j(m) \) is normal in \( G_j(m) \) and \(|G_j(m)/F(m)| = 8\).
Before we enter the proof of Theorem 3.2, we state some useful little lemmas which should be well known, but we could not find a reference.

**Lemma 6.2** If a Lie Group $G$ acts on a real space $V$ and the condition

$$(\forall A \in O(V) \ : \ Ag = gA) \Rightarrow A = \pm \mathbb{I}$$

is true, then the action is absolutely irreducible.

**Proof.** First observe that the action is irreducible: assume $U \subset V$ is a $G$-invariant subspace with an orthogonal complement $W$. Then the orthogonal projection onto $U$ along $Q$ and vice versa commute with $G$. Especially the operator $I$ which acts as $\mathbb{I}$ on $U$ and as $-\mathbb{I}$ on $W$ commutes with $G$. But this operator is in $O(V)$ and therefore $I = \pm \mathbb{I}$ and one of these spaces is $\{0\}$ and the other equal to $V$.

Now the set of commuting matrices forms a division algebra and if the action is not absolutely irreducible it contains a subspace isomorphic to $\mathbb{C}$. Let $J$ be the operator corresponding to $i$, then $J$ is skew and $J^T J = \mathbb{I}$. Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, $\beta \neq 0$ with $\alpha^2 + \beta^2 = 1$. Then $\alpha \mathbb{I} + \beta J$ commutes with $G$ and it is orthogonal, since

$$(\alpha \mathbb{I} + \beta J)^T (\alpha \mathbb{I} + \beta J) = \alpha^2 \mathbb{I} + \alpha \beta (J^T + J) + \beta^2 J^T J = (\alpha^2 + \beta^2) \mathbb{I} = \mathbb{I}.$$ 

Therefore $\alpha \mathbb{I} + \beta J$ is a multiply of the identity and therefore we have a contradiction.

**Lemma 6.3** Elements $g \in SO(4)$ with $g^2 = \mathbb{I}$ and which are not equal to $\pm \mathbb{I}$ have a two-dimensional fixed point space.

**Proof.** All eigenvalues $\lambda$ of $g$ satisfy $\lambda^2 = 1$ and hence they are equal to $\pm 1$. $\det g = 1$ implies that the number of eigenvalues equal to $-1$ is even, and so this number is $0, 2, 4$. The cases $0, 4$ are excluded by our assumption $g \neq \pm \mathbb{I}$.

**Lemma 6.4** Let $a, b$ be two unitary quaternions, such that $[a, b]$ has order two and $[a, b] \neq \pm \mathbb{I}$. Then $[a, b]$ fixes the two elements $1 + a^{-1}b, a + b$. These two elements span a two dimensional subspace.
Proof. \([a, b]^2 = 1\) means \([a^2, b^2] = [1, 1]\) or \([-1, -1]\). In either case, \(a^2 = b^2\).

We observe that
\[
[a, b](1 + a^{-1}b) = a^{-1}b + a^{-2}b^2 = a^{-1}b + 1.
\]

In the same fashion we get
\[
[a, b](a + b) = b + a^{-1}b^2 = b + a.
\]

This leads to a two-dimensional space except in the case
\[(a + b) = r(1 + a^{-1}b)\]
for some real number \(r \in \mathbb{R}\). Since the nonzero quaternions form a multiplicative group, we have \(r = a\). Therefore \(a \in \mathbb{R}\). Since \([a, b]^2 = \mathbb{I}\), we have \(a^2 = \pm 1\) and so \(a = \pm 1\) and \(a^2 = 1\). Since \(b^2 = a^2 = 1, b = \pm 1\). Then we have
\[
[a, b] = [\pm 1, \pm 1] = \pm \mathbb{I}
\]
which contradicts our assumption. 

Lemma 6.5 An element \([l, r], l, r \in Q\) fixes an element \(p\) if and only if \(l\) and \(r\) are conjugate.

Proof. If \([l, r]p = p\) then
\[
l^{-1}pr = p
\]
or
\[
r = p^{-1}lp.
\]
So \(l\) and \(r\) are conjugate. Note that \(p\) is mapped to \(p\) and \(lp\) is mapped to \(lp\), so we have a two-dimensional fixed-point space (except in the case \(l = 1\), but in that case \([l, r]\) is the identity).

This lemma is useful to determine whether elements have fixed point subspaces. It also distinguishes between the single rotations and double rotations in \(\textbf{SO}(4)\). To make use of this we need the following observation.

Lemma 6.6 Two elements in \(Q\) are conjugate if and only if they have the same real part.
Proof. This follows from the work of Janowskà and Opfer [11]. They prove that two quaternions are conjugate, if they have the same length and same real parts.

It is also useful to have an explicit form of the space fixed by a single rotation.

Lemma 6.7 Let \([l, r]\) be a single rotation, with \(\text{Re}(l) = \text{Re}(r)\). Then \([l, r]\) fixes the space spanned by the two vectors \(l - \bar{r}\) and \(1 - \bar{l}\bar{r}\).

Proof. \([l, r]\) maps \(l - \bar{r}\) to \(\bar{l}(l - \bar{r})r = r - \bar{l}\). But if \(\text{Re}(l) = \text{Re}(r)\), then \(l + \bar{l} = r + \bar{r}\), so \(l - \bar{r} = r - l\). Similarly, \(1 - \bar{l}\bar{r}\) is mapped to \(\bar{l}(1 - \bar{l}\bar{r})r = \bar{l}r - \bar{l}^2\) but since \(l + \bar{l} = r + \bar{r}\), we have \(1 + \bar{l}^2 = \bar{l}r + \bar{l}\bar{r}\) so this mapping is also the identity.

For example, consider the element \([i, j]\), which is of order 2. The real part of both quaternions is zero so this is a single rotation. The fixed vectors are \(i + j\) and \(1 - k\).

6.2 Proof of Theorem 3.2

1. To prove that the action of \(G_j(m)\) on \(\mathbb{R}^4\) is absolutely irreducible, we use Lemma 6.2. Each of our groups \(G_j(m)\) contains the element \([1, i]\). If \([l, r] \in Q \times Q\) commutes with \([1, i]\) then \(r = q_1 + q_2i\). Now we also have an element of the form \([*, j]\) in the group, and \(r \cdot j = j \cdot r\) implies \(q_2 = 0\) showing that the right element \(r\) is real.

Now we prove a similar statement for \(l\). Each group contains \([e_m, 1] = [\cos(\pi/m) + i\sin(\pi/m), 1]\). Since \(m \geq 3\), the property \(e_m \cdot l = l \cdot e_m\) implies that \(l = p_1 + ip_2\). Now we also have in each group an element of the form \([j, *]\), implying \(p_2 = 0\). Therefore the only commuting elements are of the form \([p_1, q_1]\) with \(p_1, q_1 \in \mathbb{R}\). Since \(l\) and \(r\) are unit quaternions, we conclude \(p_1, q_1 = \pm 1\). Therefore all the commuting elements in \(O(4)\) are of the form \(\pm 1\) and we deduce absolute irreducibility from Lemma 6.2.

To prove our main results concerning the isotropy subgroups of \(G_j(m)\), we begin with some general observations which are relevant for all three groups; the second part of the proof will address each group separately.

Note first that none of the nontrivial elements of the group \(H(m) = \langle [e_m, i] \rangle\) of order \(4m\) fixes any point. Using Lemma 6.5 \([e_m, i]^r\) can only fix a subspace if \(\text{Re}(e_m^r) = \text{Re}(i^r)\), which implies that \(\cos(\pi r/m) = \text{Re}(i^r)\).
If \( r \) is odd this equation cannot be satisfied since the right-hand side is zero but the left-hand side is not zero, because \( m \) is odd. If \( r \) is even the condition can only be satisfied if \( r \) is a multiple of \( m \). If \( r = 2m \) then the left-hand side is 1 and the right-hand side is \(-1\). For \( r = 4m \) the equation is satisfied but that element is the identity. For any isotropy subgroup we conclude that it intersects \( H(m) \) only at the trivial element.

The remaining argument for parts 2, 3 and 4 of the theorem is different for the three groups in question and we discuss each case separately.

(a) The case \( G_1(m) \):

2. We consider the subgroup \( F_1(m) \) generated by \( H(m) \) and the element \([1, j] \). It was shown in the proof of Theorem \( 3.1 \) that \( F_1(m) \) is of order \( 8m \). Hence \( H(m) \) is a normal subgroup of \( F_1(m) \) and therefore any isotropy subgroup in \( F_1(m) \) has order 2 and the nontrivial element is in the coset \( H(m) \cdot [1, j] \). The elements in this coset have the form

\([e_m^r, i^r j]\).

None of these elements has order 2, because \( i^r j = \pm j \) or \( \pm k \) which when squared gives \(-1\), but \( e_m^r \neq -1 \), and therefore, none of these elements fixes any nontrivial \( x \). So, isotropy subgroups are subgroups of \( G_1(m) \) and intersect \( F_1(m) \) only at the identity. Hence all isotropy subgroups have order 2, and by Lemma \( 6.3 \) have a two-dimensional fixed point space. The nontrivial element lies in the nontrivial coset of \( F_1(m) \). So it has the form

\([e_m^r j, i^r e_4] \) or \([e_m^r j, i^r je_4] \).

Since the square of the second component of the first element is never \( \pm 1 \) the first element is never of order 2. So, we concentrate on the second element. We begin with two simple remarks:

\[ j1j = -1, jij = i, \]

and therefore for \( z \in C \) we find

\[ jzj = -\bar{z}. \]

From this we conclude for \( |z| = 1 \) that \( zjzj = -1 \). In particular, for \( p \in \mathbb{N} \) we get

\[ e_pje_pj = -1. \]
The square of the second element has the form

\[ [e_m^r j e_m^r j, i^r j e_4 i^r j e_4] = [-1, i^r j i^r e_4 j e_4] = [-1, j e_4 j e_4] = [-1, -1] = [1, 1]. \]

Therefore this coset of \( F_1(m) \) consists of 4\( m \) elements of order 2. By Lemma 6.3 each of these elements generates an isotropy subgroup isomorphic to \( Z_2 \) with a two-dimensional fixed point space. The fixed point spaces for these elements are given by Lemma 6.4 with \( a = e_m^r j, b = i^r j e_4 \) we get the fixed point space is spanned by

\[ e_m^r j + i^r j e_4, \quad 1 + (e_m^r j)^{-1} i^r j e_4. \]

3. From

\[ [e_m, i][j, je_4] = [e_m j, i je_4] = [je_m, -j i e_4] = [je_m, je_4 i] = [j, je_4][e_m, i]^{-1} \]

we get

\[ [e_m, i]^2[j, je_4] = [e_m, i][j, je_4][e_m, i]^{-1}. \]

This shows that at least 2\( m \) of these 4\( m \) elements are conjugate under \( F(m) \). Consider

\[ [1, -j][j, je_4][1, j] = [j, -j j e_4 j] = [j, e_4 j] \]

which is another conjugate element of order 2. Observe that for \( q = 1 \) or \( q = 3 \)

\[ [e_m^q m j, i^q m j e_4][j, e_4 j] = [-j^2, i^q m j e_4 e_4 j] = [1, i^q m j j] = [1, i^{q+1} m] = [1, 1] \]

where \( q = 3 \) if \( m = 1 \) mod 4 and \( q = 1 \) if \( m = 3 \) mod 4. This proves that all elements of order 2 in this coset are conjugate and hence there is precisely one isotropy type with two dimensional fixed point space.

4. Let \( S_1(m) \) be a representative of this isotropy type. Now we have 4\( m \) objects (either the subgroups or their invariant planes) that are permuted by the group \( G_1(m) \), so by the orbit-stabilizer theorem, the stabilizer of any of these objects must be of order 4. The stabilizer is also the normalizer of \( S_1(m) \), that is, the largest subgroup of \( G_1(m) \) in which \( S_1(m) \) is normal. Clearly the stabilizer includes \(-\mathbb{I}\), so the stabilizer is isomorphic to \( D_2 \) and acts on \( \text{Fix}(S_1(m)) \) as minus the identity, i.e. as a rotation through \( \pi \).
(b) The case $G_2(m)$:
Here we follow the lines of the previous argument. None of the elements in $H(m)$ fixes anything. $H(m)$ is an index 2 subgroup of $F_2(m)$ and therefore every isotropy subgroup of $F_2(m)$ is of order 2 and intersects $H(m)$ in the trivial element. Again we look for order 2 elements in $F_2(m)$. The nontrivial coset of $H(m)$ in $F_2(m)$ is the coset of $[e_{2m}, j]$, the general element in this coset therefore is given by

$$[e^r_m, i^r] [e_{2m}, j], \quad r = 0, \ldots, 4m - 1.$$ 

Squaring these elements gives us

$$[e_{2m}^{2r} e_{2m}, i^r j^r]^2 = [e_{2m}^{4r+2}, i^r j^r j^r] = [e_{2m}^{4r+2}, -1].$$

So we are interested in those $r$ with

$$e_{2m}^{4r+2} = -1$$

or

$$4r + 2 = 2m \mod 4m \iff 2r + 1 = m \mod 2m.$$ 

We get four solutions in the set $0 \leq r \leq 4m - 1$: writing $m = 2\tau + 1$ then (obviously) the solutions $r$ have the form $r = \tau \mod m$.

$$r_j = \tau + qm, \quad \text{for } q = 0, 1, 2, 3.$$ 

From Lemma 6.3 it follows that the dimension of the corresponding fixed point space is two. Depending on the parities of $q$ and $\tau$ the exponent $\tau + qm$ can be even or odd: for each parity of $\tau$ there are two parities of $q$ leading to an odd exponent and also two parities leading to an even exponent. In the odd case the element takes the form

$$[i, \pm k],$$

in the even case the form

$$[i, \pm j].$$

In any case the elements

$$[i, k] \text{ and } [i, -k]$$

and the elements

$$[i, j] \text{ and } [i, -j]$$
are conjugate in $F_2(m)$ (the conjugating element is $[1, i]$). The elements $[i, j]$ and $[i, k]$ are not conjugate (as one can easily check) in $F_2(m)$, so we get two isotropy types in $F_2(m)$. However these two isotropy subgroups in $F_2(m)$ are conjugate within $G_2(m)$, since $[j, i e_4][i, j][-j, -i e_4] = [i, k]$. Hence we have so far only found one isotropy type in $G_2(m)$.

Now we have to look at the full group. $F_2(m)$ is an index 2 subgroup of $G_2(m)$ and therefore any isotropy subgroup of $G_2(m)$ intersects $F_2(m)$ in the trivial group or one of the isotropy subgroups of order 2 in $F_2(m)$. Therefore we have to look for elements of order 2 and 4 in the coset of $F_2(m)$. This coset consists of $8m$ elements $f[j, e_4]$ where $f \in F_2(m)$. So we get this coset as a union of two sets

$$\left\{ [e^r_m, i^r] [j, e_4] \mid 0 \leq r < 4m \right\}$$

and

$$\left\{ [e^r_m e_2 e_4, i^r j] [j, e_4] \mid 0 \leq r < 4m \right\}.$$

So the first class of elements has the form

$$[e^r_m j, i^r e_4]$$

and the squares are of the form

$$[e^r_m j e^r_m j, i^{2r} i] = [-1, \pm i] \neq [1, 1].$$

Squaring again gives

$$[1, -1] \neq [1, 1]$$

and so there are no elements of order 2 or 4 within this class, therefore none of these elements belongs to any isotropy subgroup.

The second class of elements has the form

$$[e^{2r+1}_m j, i^r j e_4].$$

The squares have the form

$$[e^{2r+1}_m j e^{2r+1}_m j, i^r j e_4 i^r j e_4] = [-1, (-1)^r i^r j e_4 j e_4 i^r].$$

This gives

$$[-1, (-1)^r (-1)(i^r)^2] = [-1, -1] = [1, 1].$$
None of these elements of order 2 can be conjugate to any of the four in
$F_2(m)$ we had found before, since $F_2(m)$ is a normal subgroup of $G_2(m)$.
To show that they are conjugate to each other, consider
\[ [e_m, i][e_{2m, j}, je_4][e_m, -i] = [e_m, i]^2[e_{2m, j}, je_4]. \]
As before this type of conjugation gives us two classes
\[ [e_m, i]^{2r}[e_{2m, j}, j][e_4] \]
and
\[ [e_m, i]^{2r+1}[e_{2m, j}, j][e_4]. \]
Conjugation with $[e_{2m, j}]$ shows that the elements in these two classes are
all conjugate, since
\[ [e_{2m, j}, j][e_{2m, j}, j][e_m, -i] = [e_{2m, j}, e_4j] = [e_m, i][e_{2m, j}, j][e_4]. \]
So in total we find in $G_2(m)$ two classes of order 2 subgroups. By Lemma 6.3
the fixed point space is two dimensional and is given according to Lemma 6.4.

$G_2(m)$ acts as a permutation group on these elements. The first class has
length 4, and so 4$m$ elements fix each of the groups isomorphic to $Z_2$, i.e.
the normalizer of each subgroup is of order 4$m$. Consider the representative
$\langle [i, j] \rangle$ of this class. Since $[e_{2m, j}]$ and $-1$ commute with $[i, j]$, and $[e_{2m, j}]$ is
of order 2$m$, these two elements generate a group $Z_{2m} \times Z_2$ of order 4$m$
which must be the normalizer. The action of the normalizer on the two-dimensional
space is the normalizer quotient which acts as $Z_{2m}$, a rotation through an
angle $\pi/m$.

In the other class of isotropy subgroups we have 4$m$ elements, so each one is
fixed by 4 elements and the normalizer acts as $D_2/Z_2 = Z_2$, as in the case
of $G_1(m)$.

(c) The case $G_3(m)$:
The argument here is very similar to that for $G_1(m)$. It has already been
shown that there are no isotropy subgroups in $F_3(m) = F_1(m) = \langle [e_m, i], [1, j] \rangle$,
which is an index 2 subgroup of $G_3(m)$. Therefore isotropy subgroups of
$G_3(m)$ can only be of order 2 and must be generated by an element in the
nontrivial coset of $F_3(m)$ in $G_3(m)$, $F_3(m)[j, 1]$. We find 6$m$ elements of
order 2:
\[ [e_{m, j}, \varepsilon], \text{ where } r \in \{0, \ldots, 2m - 1\} \text{ and } \varepsilon \in \{i, j, k\}. \]
Each of these elements is clearly of order 2 and hence generates a subgroup isomorphic to \(Z_2\). The second entry determines the conjugacy class, since there is no possible conjugating element in \(G_3(m)\) that could alter the second entry. In fact any two of those elements with the same second element are conjugate. So the subgroups of order two come in three conjugacy classes, each of the subgroups has a two-dimensional fixed point space and the length of each conjugacy class under \(G_3(m)\) is \(2m\). Each element is fixed by \(8\) elements, the normalizer of the group \(\Sigma = \langle [e_m^r j, \varepsilon] \rangle\) has the form
definition

\[
N_{G_3(m)}(\Sigma) = \{[1, 1], [1, -1], [e_m^r, \varepsilon], [1, \varepsilon], [e_m^r j, 1], [-1, \varepsilon], [j, -1], [-j, \varepsilon]\}.
\]

Therefore it consists of a group of order 8 with two generators, isomorphic to \(Z_4 \times Z_2\). The quotient is a cyclic group of order 4. This concludes the proof.

**Proof of Theorem 4.1.** We begin to investigate the structure of the invariant polynomials. We will not give a complete description of all invariants, but just enough to prove the bifurcation results. The results and the arguments are slightly different for the three cases, so we discuss them partially separately. Since the groups \(G_j(m)\) operate absolutely irreducibly there is no linear invariant, this means we look only for invariants which are at least quadratic. Let us write

\[
I_2(x) = \sum_{\nu=1}^{4} x_{\nu}^2.
\]

This is clearly a quadratic invariant for all groups in question. Now we restrict our attention to the groups \(F_j(m), j = 1, 2\). We recall the generating elements:

\[
F_1(m) = \langle [e_m, 1], [1, i], [1, j] \rangle, \quad F_2(m) = \langle [e_m, 1], [1, i], [e_m^r, j] \rangle.
\]

In both cases we have as one of the generating elements the element \([e_m, 1]\). In order to describe its invariant functions we introduce complex notation via

\[
z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4 \text{ with } x = x_1 + ix_2 + jx_3 + kx_4 = z_1 + z_2j.
\]

Then \(I_2 = |z_1|^2 + |z_2|^2\). In order to describe further invariants let us look at the action of \([e_m, 1]\) on the complex variables \(z_1, z_2\). By

\[
[e_m, 1]x = e_m x = e_m (z_1 + z_2j) = e_m z_1 + e_m z_2j.
\]
This means the first generator sends the pair \((z_1, z_2)\) to \((e^{-i\pi m} z_1, e^{-i\pi m} z_2)\). The second generator maps
\[(z_1, z_2) \mapsto (iz_1, -iz_2)\. For the third one we obtain in the case \(F_1(m)\)
\[[1, j](z_1, z_2) = (-z_2, z_1)\]and in the case \(F_2(m)\) the third generator maps
\[[e_{2m}, j](z_1, z_2) = -(\bar{e}_{2m} z_2, \bar{e}_{2m} z_1)\. We look at monomials in the form \(z_1^{k_1} z_2^{k_2} \bar{z}_1^{\ell_1} \bar{z}_2^{\ell_2}\). So for the action of the first element we simply get
\[e^{-\frac{2\pi i}{m} (k_1 + k_2 - \ell_1 - \ell_2)} \cdot z_1^{k_1} z_2^{k_2} \bar{z}_1^{\ell_1} \bar{z}_2^{\ell_2}\]In order that this is invariant under the action of the first element and which has an order at most 4 we have
\[k_1 + k_2 = \ell_1 + \ell_2\]This implies that functions of
\[|z_1|^2, |z_2|^2, z_1 \bar{z}_2, \bar{z}_1 z_2\]are invariant under this particular action. The second element multiplies the last two expressions with \(-1\), so we should look at the squares of these elements of products of two sign changing functions, i.e. we look at functions of
\[|z_1|^2, |z_2|^2, z_1^2 z_2^2, \bar{z}_1^2 \bar{z}_2^2\]Since the third element, in the case of \(F_1(m)\), basically interchanges \(z_1, z_2\) the invariants for \(F_1(m)\) have to be symmetric in \(z_1, z_2\). Therefore invariants for \(F_1(m)\) have to be functions of
\[I_2 = |z_1|^2 + |z_2|^2, I_{4,1} = |z_1|^2 |z_2|^2, I_{4,2} = z_1^2 z_2^2 + z_1^2 \bar{z}_2^2, I_6 = (|z_1|^2 - |z_2|^2) i (z_1^2 \bar{z}_2^2 - z_1^2 z_2^2)\]In the case of \(F_2(m)\) we find it leaves the same functions invariant, observe that the extra factors multiply to 1.
With this information we construct fourth and six order invariants for \( F_j(m) \). Of course \( I_2^2 \) is a fourth order invariant for \( F_j(m) \), \( j = 1, 2, 3 \). Let us consider

\[
I_{4,1}(x_1, \ldots, x_4) = |z_1|^2|z_2|^2 = (x_1^2 + x_2^2)(x_3^2 + x_4^2).
\]

It is (obviously) invariant under \( F_j(m) \). Let

\[
I_{4,2} = z_1^2z_2^2 + z_1^2z_2^2.
\]

Let us write

\[
I_6 = (|z_1|^2 - |z_2|^2)i(z_1^2z_2^2 - z_1^2z_2^2).
\]

This is invariant under \( F_j(m) \). We now write down the invariants up to order 6 (for \( m \) sufficiently large, for small \( m \) there might be additional invariants): \( I_2 \) is the unique quadratic invariant, \( I_2^2, I_{4,1}, I_{4,2} \) are the quartic invariants, and

\[
I_2^3, I_{4,1} \cdot I_2, I_{4,2} \cdot I_2, I_6
\]

are sextic invariants. In a similar way we can construct 7 invariants of order 8 and 9 invariants of order 10.

1. **The case \( G_1(m) \)**

   We provide the explicit form of the generating elements outside \( F_1(m) \) in complex notation. We get

   \[
   [j, e_4](z_1 + z_2j) = -\bar{e}_4\bar{z}_1j + e_4\bar{z}_2.
   \]

   We can write this as \((z_1, z_2) \mapsto (e_4\bar{z}_2, -\bar{e}_4\bar{z}_1)\). Obviously \( I_2, I_2^2, I_{4,1}, I_2^3, I_{4,1}I_2 \) are invariant. The function \( I_{4,2} = z_1^2z_2^2 + z_1^2z_2^2 \) is mapped by the above substitution to \(-I_{4,2} \) (observe \( e_4^* = -1 \)). Then \( I_{4,2} \) is invariant under this element and hence under \( G_1(m) \). This substitution applied to \( I_6 \) changes the signs of both factors and hence \( I_6 \) is invariant.

2. Since the invariants we have constructed are the same for \( F_1(m) \) and \( F_2(m) \) they have to be the same for the groups \( G_1(m) \) and \( G_2(m) \), because the extension is defined with the same element.

3. **The case \( G_3(m) \).** In this case \( F_3(m) = F_1(m) \) and therefore we just have to look at the remaining generating element \([j, 1] \) which acts as

   \[
   [j, 1](z_1 + z_2j) = \bar{j}z_1 + j\bar{z}_2j = -\bar{z}_1j + \bar{z}_2.
   \]
Therefore we get the substitution \((z_1, z_2) \mapsto (\bar{z}_2, -\bar{z}_1)\). Applying this to the invariants \(I_2, I_2^3, I_{4,1}, I_{4,2}^3, I_2, I_{4,1}\) we easily see that they are invariant as well. Note however, that \(I_{4,2}\) is invariant under this element as well, so we find three independent quartic polynomial invariants in this case.

If we look at the gradients of \(I_2^2, I_{4,1}, I_{4,2}\) they give rise to three independent equivariant maps. In the case of \(G_1(m)\) and \(G_2(M)\) the first two are gradient vector fields, the third gives rise a Hamiltonian field as we discussed before.

In the case \(G_3(m)\) we get three equivariant gradients, so up to cubic level a \(G_3(m)\)-equivariant vector fields is a gradient.

Observe that our proof so far shows only that there are at least three equivariant vector fields for the given groups. However, the character formula for the number of equivariant fields in Section 8 yields three equivariant cubic fields (compare Tables 2, 3 and 4). Since for a given degree the number of equivariant vector fields is as a function of \(m\) non increasing, we conclude that we have precisely three cubic equivariants for all \(j = 1, 2, 3\) and all odd \(m \geq 3\).

**Proof of Theorem 4.2.** We use the standard theory of complex structures (see e.g. Range [15], Chapter III, Section II) to derive the real vectorfield from the complex form of the invariants. We differentiate with respect to \(\bar{z}_s, s = 1, 2\) and write the resulting differential equation in complex form as

\[
\begin{align*}
\dot{z}_1 &= \lambda z_1 + c_1 z_1(|z_1|^2 + |z_2|^2) + c_2 \bar{z}_1 |z_2|^2 + c_3 i \bar{z}_1 z_2^2 \quad (4) \\
\dot{z}_2 &= \lambda z_2 + c_2 z_2(|z_1|^2 + |z_2|^2) + c_2 \bar{z}_2 |z_1|^2 + c_3 i z_1^2 \bar{z}_2. \quad (5)
\end{align*}
\]

For the last equivariant map we observe in real coordinates we take \(J \nabla I_{4,2}\). We get \(\nabla I_{4,2}\) in the complex form by computing the gradient with respect \(\partial/\partial \bar{z}_j\) and multiplying with \(J\). The last operation is obtained by multiplying acting with \([i, 1]\) on the equation. This is the same as multiplying the \(\bar{z}\)-gradient with \(i\). Observe that in the case \(G_3(m)\) we have to take the real gradient of this invariant and obtain up to order 3 a fully gradient map.

Let us add one more remark: the number of cubic equivariant fields is non increasing function in \(m\). From the character formula we find, that for \(m = 3\) we have three equivariant maps. So the vectorfield given here is the cubic truncation in each case.

In a similar way we can construct the four sextic terms from \(I_2^3, I_{4,1} I_2, I_6\) and a Hamiltonian field from \(I_{4,2} I_2\) (respectively non-Hamiltonian for \(G_3(m)\)). This
listing is not complete, the character formula predicts 9 sextic equivariant maps. Due to our method to require invariance for all \( m \), we have constructed the invariants and equivariants for the Lie groups \( G_j \). Therefore equilibria occur in circles, therefore they are not hyperbolic. Of course this means that they could be destroyed by adding higher order terms. Next we are looking at the fixed point subspaces. Since these spaces are different for the three groups, we discuss them one by one.

1. The case \( G_1(m) \).

There is one isotropy type, a representative of this type is given by the nontrivial element of order 2: \( Z_2 = \langle [j, je_4] \rangle \), where \( \langle \cdot \rangle \) denotes the group generated by the listed elements. Its fixed point space can be explicitly computed by Lemma [6.4]. We get

\[
\left\{ \alpha (1 + e_4) + \beta (1 + \bar{e}_4) j \left| \alpha, \beta \in \mathbb{R} \right. \right\} = \left\{ \alpha e_8 + \beta \bar{e}_8 j \left| \alpha, \beta \in \mathbb{R} \right. \right\}.
\]

2. The case \( G_2(m) \).

From the computation which we have given in the proof of Theorem [3.2] we can read off, two representatives for the two classes of isotropy subgroups. We have

\[
\Sigma_1 = \langle [i, j] \rangle \subset F_2(m)
\]

and

\[
\Sigma_2 = \langle [e_2 j, je_4] \rangle.
\]

The fixed point spaces are given

\[
\text{Fix}(\Sigma_1) = \left\{ \alpha (1 - ij) + \beta (i + j) \left| \alpha, \beta \in \mathbb{R} \right. \right\}
\]

and

\[
\text{Fix}(\Sigma_2) = \left\{ \alpha (1 + \bar{e}_2 e_4) + \beta (e_2 m + \bar{e}_4) j \left| \alpha, \beta \in \mathbb{R} \right. \right\}.
\]

3. The case \( G_3(m) \).

We had seen that we have three conjugacy classes of groups of order 2, in each class we look at one representative, i.e. we look at

\[
\Sigma_1 = \langle [j, i] \rangle, \Sigma_2 = \langle [j, j] \rangle, \Sigma_3 = \langle [j, k] \rangle
\]

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with fixed point spaces given by Lemma 6.4 as

\[
\begin{align*}
\text{Fix}(\Sigma_1) &= \left\{ \alpha(1 - ji) + \beta(i + j) \mid \alpha, \beta \in \mathbb{R} \right\}, \\
\text{Fix}(\Sigma_2) &= \left\{ \alpha + \beta j \mid \alpha, \beta \in \mathbb{R} \right\}, \\
\text{Fix}(\Sigma_3) &= \left\{ \alpha(1 - jk) + \beta(j + k) \mid \alpha, \beta \in \mathbb{R} \right\} \\
&= \left\{ \alpha(1 - i) + \beta(1 + i)j \mid \alpha, \beta \in \mathbb{R} \right\}.
\end{align*}
\]

The next observation comes from the equivariance with respect to the Lie groups \(G_j\). As a consequence the equations are equivariant with respect to the action

\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto e^{i\phi} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.
\]

The only fixed point of this group action is the origin and therefore all equilibria are not isolated and therefore perturbations with higher order terms might destroy all equilibria. Now we look at restriction of the equation to the various fixed point spaces. We do it again case by case.

1. \(G_1(m)\)

There is only one isotropy type with a nontrivial fixed point space. The fixed point space is \(\alpha e_8 + \beta \bar{e}_8 j\) and the equation for equilibria reads (up to cubic order)

\[
\begin{align*}
0 &= \lambda \alpha e_8 + c_1 \alpha e_8 (\alpha^2 + \beta^2) + c_2 \alpha e_8 \beta^2 + c_3 i \bar{e}_8 e_8^2 \alpha \beta^2, \\
0 &= \lambda \beta \bar{e}_8 + c_1 \beta \bar{e}_8 (\alpha^2 + \beta^2) + c_2 \beta \bar{e}_8 \alpha^2 + c_3 i \bar{e}_8 e_8 \beta \alpha^2.
\end{align*}
\]

This gives the equation

\[
\begin{align*}
0 &= \alpha (\lambda + c_1 (\alpha^2 + \beta^2) + (c_2 + c_3) \beta^2) e_8, \\
0 &= \beta (\lambda + c_1 (\alpha^2 + \beta^2) + (c_2 - c_3) \beta \alpha^2) \bar{e}_8.
\end{align*}
\]

We can look for solutions \((\lambda, \alpha, \beta) \in \mathbb{R}^3\) with \(\alpha \beta \neq 0\). However then we get sign conditions on the coefficients \(c_1, c_2, c_3\). As a consequence we would not prove generic bifurcation results. Let us look for solutions of the form \((\lambda, 0, \beta)\) with \(\beta \neq 0\). With the ansatz \(\alpha = 0\) the first equation is identically satisfied, the second equation then leads to

\[
0 = \lambda \beta + c_1 \beta^3.
\]
which gives
\[ \beta^2 = -\frac{\lambda}{c_1}. \]
where we assume \( 0 < |c_1| < \infty \) and we choose \( \lambda \) such that the term on the right hand side is positive. With the additional assumptions \( c_2 + c_3 \neq c_1 \) we obtain the linearisation of the map along the given branch as

\[
\begin{pmatrix}
\lambda - \frac{c_2 + c_3}{c_1} \lambda & 0 \\
0 & -2\lambda
\end{pmatrix},
\]
which is regular and yields the persistence of this branch under higher order perturbation.

2. \( G_2(m) \)
   In this case we use the second fixed point space and we obtain precisely the same system as in the case before, therefore we obtain the same result.

3. \( G_3(m) \)
   Here we use the third fixed point space, again, we get the real equation and therefore the same result

\[ \square \]

**Proof of Theorem 4.3.** In each case we have constructed a branch in a generic third order equation which is stable under higher order perturbation and hence the result follows. \[ \square \]

7 GAP Computations

7.1 General remarks

In this paper we have shown that there are three infinite series of groups of orders \( 48+32\mu, \mu \in \mathbb{N} \), which act absolutely irreducibly on \( \mathbb{R}^4 \) and which have no odd-dimensional fixed point space. In this section we collect together some data obtained with the computational group theory package GAP [6], using the groups in the Small Group Library. For any of these groups, it is possible to obtain its character table and hence determine whether it acts irreducibly on \( \mathbb{R}^4 \). The subgroup lattice is obtained and then character formulas are
used to determine the isotropy subgroups and the dimension of their fixed point spaces (see [14] for further details).

This approach was applied to actions on $\mathbb{R}^4$ and also to actions on $\mathbb{R}^N$, for $4 < N \leq 20$ with $N$ even. The results lead us to the following conjectures: For dimensions $N = 0 \text{ mod } 4$, there are infinitely many groups acting absolutely irreducibly on $\mathbb{R}^N$ that have no isotropy subgroups with odd-dimensional fixed point spaces. But for dimensions $N = 2 \text{ mod } 4$, there are no such groups.

We have checked most of the groups of order up to 1000, however in the dimensions 4, 8 we did not look at the groups of order 512, due to the sheer number of such groups: there are 10494213 groups of order 512. Even if this number of groups can be checked with a computer, there are 49487365422 groups of order 1024 and this number of groups is certainly out of reach for present day computers.

### 7.2 The case $0 \text{ mod } 4$

The following tables gives the GAP numbers for finite groups of orders up to 1000 (and some cases higher) which act absolutely irreducibly and have only even dimensional fixed point spaces. Some of these groups have several inequivalent representations in these dimensions with the same properties. However we do not provide this information. The tables are based on computations on different computers using the computer algebra package GAP.

#### 7.2.1 $\mathbb{R}^4$

The results of the GAP computations for actions on $\mathbb{R}^4$ are summarised in Tables 5 and 6 which list the Small Group Library number of groups that act on $\mathbb{R}^4$ and have no isotropy subgroup with an odd-dimensional fixed point space. Note that this list contains the groups from the series $G_1(m)$, $G_2(m)$, $G_3(m)$ (compare Table 1), but also many other groups. Most of the groups in this table belong to the two-parameter families

$$\pm\frac{1}{2}[D_{2m} \times \mathcal{D}_{4n}], \quad \pm\frac{1}{4}[D_{4m} \times \mathcal{D}_{4n}], \quad \pm[D_{2m} \times D_{2n}],$$

in the notation of [4]. These families include $G_1(m)$, $G_2(m)$, $G_3(m)$ in the case $n = 2$. The remaining groups belong to four of the one-parameter
families in Table 4.1 of [4],

$$\pm [O \times D_{2n}] \text{ for } n = 5, 7 (480:969, 672:1053),$$

$$\pm \frac{1}{2} [O \times D_{2n}] \text{ for } n = 5, 7, 11, 13, 17 (240:106, 336:119, 528:91, 624:135, 816:102),$$

$$\pm \frac{1}{6} [O \times D_{6n}] \text{ for } n = 3, 9, 15 (144:32, 432:38, 720:106),$$

$$\pm [T \times D_{2n}] \text{ for } n = 3, 5, 7, 9, 11, 13, 15, 17$$

(144:127, 240:108, 336:131, 432:262, 528:93, 624:147, 720:544, 816:104).

In most of these cases the isotropy subgroups are isomorphic to \(Z_2\). But in some cases, for example the group 144:127, there is an isotropy subgroup isomorphic to \(Z_3\). All the groups which we have discussed here are subgroups of \(SO(4)\). It is easy to see, that groups with elements in \(O(4) \setminus SO(4)\) do have odd dimensional fixed point spaces. If \(g\) is such an element, its determinant is \(-1\), so \(-1\) is an eigenvalue of multiplicity 1 or 3. In the second case, 1 is an eigenvalue of multiplicity one. In the first case, 1 is an eigenvalue of multiplicity 1 or 3. So in any case the group generated by \(g\) has an odd dimensional fixed point space.

7.2.2 \(\mathbb{R}^8\)

Table 7 is a similar list for groups acting absolutely irreducibly on \(\mathbb{R}^8\). The case of groups of order 512 has not been checked. In the case of groups of order 768 we have the complete answer, the list presented here is only part of the list we have obtained so far.

7.2.3 \(\mathbb{R}^{12}\)

Table 8 is a list of groups acting on \(\mathbb{R}^{12}\), with the same property.

7.2.4 \(\mathbb{R}^{16}\)

A similar list for groups acting on \(\mathbb{R}^{16}\) is given in Table 9. The gap numbers of all groups of order 768 have been determined, but the number is too large to present all of them here.
| Order | Groups       | Order | Groups       |
|-------|--------------|-------|--------------|
| 48    | 15 17 41     | 352   | 32 34 114    |
| 80    | 15 17 42     | 360   | 9 10 13     |
| 96    | 35 33 126    | 368   | 14 16 34    |
| 112   | 14 16 34     | 384   | 183 185 1959|
| 120   | 10 12 13     | 400   | 16 18 42    |
| 144   | 16 18 32 44 127 | 408 | 9 11 12 |
| 160   | 35 33 140    | 416   | 33 35 140   |
| 168   | 14 16 17     | 432   | 16 18 38 50 262 |
| 176   | 14 16 34     | 440   | 19 21 22    |
| 192   | 78 80 478    | 448   | 76 78 453   |
| 208   | 15 17 42     | 456   | 14 16 17    |
| 224   | 32 34 114    | 464   | 15 17 42    |
| 240   | 14 15 19 21 76 78 106 108 130 134 182 | 480 | 14 15 19 21 186 188 347 351 884 969 |
| 264   | 7 9 10       | 496   | 14 16 34    |
| 272   | 15 17 43     | 504   | 14 15 18    |
| 280   | 9 11 12      | 520   | 13 15 16    |
| 288   | 33 35 129    | 528   | 12 13 17 19 74 76 91 |
| 304   | 14 16 34     | 544   | 33 35 141   |
| 312   | 17 19 20     | 552   | 7 9 10      |
| 320   | 77 79 546    |       |            |
| 336   | 30 31 35 37 101 103 119 131 142 146 201 | 560 | 14 15 19 21 75 77 114 118 163 |

Table 5: Small Group Library numbers of groups acting on $\mathbb{R}^4$ that have no odd-dimensional isotropy subgroups. The groups of order 512 have not been checked.
| Order | Groups       | Order | Groups       |
|-------|--------------|-------|--------------|
| 576   | 78 80 481    | 792   | 7 8 11       |
| 592   | 15 17 42     | 800   | 33 35 140    |
| 600   | 9 10 13      | 816   | 14 15 19 21 76 78 102 |
| 608   | 32 34 114    | 104   | 126 130 178  |
| 616   | 7 9 10       | 832   | 77 79 546    |
| 624   | 33 34 38 40 104 106 | 840   | 58 60 61 72 74 75 79 81 82 |
|       | 135 147 172 176 231 | 848   | 15 17 42     |
| 640   | 180 182 2258 | 864   | 34 36 135    |
| 656   | 15 17 43     | 880   | 40 41 45 47 101 103 154 158 203 |
| 672   | 55 56 60 62 247 249 415 419 984 1053 | 888   | 17 19 20     |
|       | 896 179 181 1903 |       |               |
| 680   | 13 15 16     | 912   | 30 31 35 37 101 103 118 130 141 |
| 688   | 14 16 34     | 920   | 9 11 12      |
| 696   | 9 11 12      | 928   | 33 35 140    |
| 704   | 76 78 451    | 720   | 12 13 18 22 77 79 106 |
|       |               | 127   | 134 182 544  |
| 728   | 9 11 12      | 936   | 17 18 21     |
| 736   | 32 34 114    | 944   | 14 16 34     |
| 744   | 14 16 17     | 952   | 9 11 12      |
| 752   | 14 16 34     | 960   | 14 15 19 21 535 537 1029 1033 |
| 760   | 9 11 12      | 976   | 15 17 42     |
| 784   | 15 17 34     | 984   | 9 11 12      |
|       |               | 992   | 32 34 114    |

Table 6: Small Group Library numbers of groups acting on $\mathbb{R}^4$ that have no odd-dimensional isotropy subgroups (continued).
| Order | Groups                                      |
|-------|--------------------------------------------|
| 160   | 82 85 208                                  |
| 192   | 36 308 310 312 313 758 761 762 804 990 1333 1336 1337 1394 1396 1484 1527 |
| 240   | 96 99 101                                   |
| 288   | 382 383 433 572 573 582 583 586 587 589 593 596 597 598 937 964 966 968 |
| 320   | 35 242 244 245 247 266 376 378 380 381 826 829 830 872 1078 1079 1122 1123 1446 1450 1507 1509 1598 1625 |
| 384   | 114 117 126 128 346 349 1744 1748 1748 1834 1836 1842 1844 1847 1847 1848 3570 3668 3670 3673 3674 4093 4095 4099 4099 4698 4700 5713 5715 12858 12874 12878 13519 13522 14612 14615 14616 14616 16583 16592 16596 19786 |
| 416   | 82 85 207                                  |
| 448   | 34 283 285 287 288 733 736 737 779 1227 1230 1231 1288 1290 1382 |
| 480   | 227 228 233 234 249 250 553 – 557 563 567 568 571 572 574 576 – 580 582 588 591 592 595 – 597 599 959 964 966 970 973 990 992 1007 1100 1102 1103 1105 1108 1109 1111 |
| 544   | 83 86 216                                  |
| 560   | 89 92 94                                   |
| 576   | 37 311 313 315 316 762 765 766 808 1065 1415 1758 1761 1762 1819 1821 1906 1907 1927 1938 1940 1946 1947 1949 2078 2080 2090 2099 2905 2906 2915 2916 2919 2920 2922 2926 2929 2930 2931 3390 3522 3525 3528 3540 4982 4983 5025 5026 5028 5263 5297 5547 5599 5601 5669 5670 5713 5714 6643 6644 6647 6649 6653 6979 7201 7205 8273 8317 8330 8332 8338 8470 8507 8526 8571 |
| 624   | 125 128 130                                 |
| 640   | 111 114 123 125 343 346 760 763 765 766 829 832 835 839 2043 2047 2133 2135 2141 2143 2146 2147 3869 3967 3969 3972 3973 4392 4394 4398 4997 4999 6218 6219 6219 6722 6912 6916 6929 14095 14111 14115 15852 15854 15853 14756 14759 15849 17820 17829 17833 19519 21193 |
| 672   | 621 – 625 631 635 636 640 642 644 645 – 648 650 656 659 660 663 664 – 667 1053 1054 1057 1153 1155 1156 1158 1161 1162 1164 |
| 704   | 34 281 283 285 286 731 734 735 777 1224 1227 1228 1285 1287 1376 |
| 720   | 96 98 101 450 452 457 459 460 462 475 476 481 482 490 495 496 501 502 504 509 510 513 517 520 521 523 524 |
| 768   | 57401 57403 57429 80778 82966 82967 82970 83806 83807 83820 83821 89833 89899 90043 90249 90250 90252 90255 90259 90262 90263 90264 |

Table 7: As Table 5 but for actions on $\mathbb{R}^8$. 33
| Order | Groups   |
|-------|----------|
| 336   | 18 20 128 134 |
| 432   | 83 85 153 155 161 163 245 248 261 269 295 369 371 |
| 504   | 69 71 72 |
| 576   | 184 1399 1401 1990 2010 4976 4988 5060 5061 5065 5067 5531 5533 5584 5586 8307 8310 8464 8483 8499 8510 |
| 624   | 19 21 144 150 |
| 672   | 36 38 337 337 |
| 720   | 415 418 |
| 840   | 18 20 21 |
| 864   | 220 222 428 430 436 438 662 822 1157 1195 1206 1532 1535 2195 2222 2271 2487 |
| 912   | 18 20 127 133 |
| 936   | 76 78 79 |
| 960   | 789 809 5713 5714 5769 5770 5774 5776 6329 6331 6382 6384 10946 10949 11098 11117 11133 11144 |
| 1008  | 224 225 229 231 286 288 288 522 524 532 536 664 682 882 881 |
| 1080  | 92 93 96 99 100 103 145 147 149 282 |
| 1152  | 153939 153941 153959 153960 153963 153969 154100 154102 154147 154149 154375 154498 154503 154506 154507 154509 154590 154592 154596 154598 154690 154961 154964 154970 155100 155102 155172 155191 155194 155356 155358 155809 155823 156072 156207 156208 156214 156214 157025 157340 157585 157644 |

Table 8: As Table 5 but for actions on $R^{12}$.

| Order | Groups   |
|-------|----------|
| 576   | 5153 8369 |
| 640   | 653 657 660 915 917 6009 6012 6014 6016 6948 6950 6951 7102 19529 19534 19535 19641 19644 21497 |
| 768   | 57413 57415 79717 79730 80103 80107 80108 80111 80408 80412 80557 1045835 1045841 1045859 1045863 1045866 |
| 960   | 6130 6131 6133 6134 6135 6136 6137 6144 6148 6149 6152 6153 6156 6158 6159 6160 6161 6166 6167 6169 6172 6173 6176 6177 10895 10897 10900 11035 11037 11040 11043 11046 11048 11049 |

Table 9: As Table 5 but for actions on $R^{16}$.
| Order | Groups   |
|-------|---------|
| 880   | 18 20 121 |
| 1320  | 17 19 20 |
| 1760  | 36 38 329 |

Table 10: As Table 5 but for actions on $\mathbb{R}^{20}$.

7.2.5 $\mathbb{R}^{20}$

Here we present the list for the groups acting on $\mathbb{R}^{20}$.

7.3 The case $2 \mod 4$

We have already remarked that in the case of $\mathbb{R}^2$ there is no absolutely irreducible representation without an odd dimensional fixed point space. We have checked all groups up to the following orders in the various dimensions and have not found any groups which act absolutely irreducibly and have no odd dimensional fixed point spaces.

| dimension | Order |
|-----------|-------|
| 2         | $\infty$ |
| 6         | 1013  |
| 10        | 999   |
| 14        | 1007  |
| 18        | 1151  |

At this point we mention a recent result by Ruan [16]: in dimension 6 all solvable groups which act absolutely irreducibly have an odd dimensional fixed point space.

8 Characters and Invariant Theory

In Sattinger [17] we find a formula which allows to compute the vector space dimensions of the space of invariant polynomials for a group action of a given degree. Consider the action of a compact Lie group $G$ on some finite dimensional space $V$. We write $C^\infty_G(V)$ for the $G$ invariant smooth functions. It is well known that they form an algebra which is finitely generated by invariant polynomials. Therefore we are interested in the space of homogeneous
invariant polynomials of some given degree \( d \). The dimension of this space is denoted by \( c_d \) and in a similar way we write \( C_d \) for the dimension of the space of homogeneous equivariant polynomial maps for \( V \to V \). Sattinger [17] defines the quantities
\[
\chi(d)(g) = \sum_{j=1}^{d} \frac{\chi^{i_1}(g)\chi^{i_2}(g^2)\chi^{i_3}(g^3) \cdots \chi^{i_d}(g^d)}{1^{i_1}i_1!2^{i_2}i_2! \cdots d^{i_d}i_d!}
\]
and obtains the following representations for \( c_d, C_d \):
\[
c_d = \int_G \chi(d)(g) \, dg \tag{6}
\]
and
\[
C_d = \int_G \chi(d)(g)\chi(g) \, ds. \tag{7}
\]
We obtain for (following Sattinger [17])
\[
\chi(2) = \frac{1}{2} \left( \chi(g^2) + \chi^2(g) \right).
\]
For the next values we derive the following expressions (using that
\[
i_1 + 2i_2 + 3i_3 = 3
\]
leads to the choices \((3, 0, 0), (1, 1, 0), (0, 0, 1)\) for \((i_1, i_2, i_3)\) and
\[
i_1 + 2i_2 + 3i_3 + 4i_4 = 4
\]
to \((4, 0, 0, 0), (1, 0, 1, 0), (2, 1, 0, 0), (0, 2, 0, 0), (0, 0, 0, 1)\) for \((i_1, i_2, i_3, i_4)\))
\[
\chi(3) = \frac{1}{3!} \chi^3(g) + \frac{1}{2} \chi(g)\chi(g^2) + \frac{1}{3} \chi(g^3).
\]
and
\[
\chi(4) = \frac{1}{4!} \chi^4(g) + \frac{1}{3} \chi(g)\chi(g^3) + \frac{1}{4} \chi^2(g)\chi(g^2) + \frac{1}{2!} \chi^2(g^2) + \frac{1}{4} \chi(g^4).
\]
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