Static cylindrically symmetric dyonic wormholes
in 6-dimensional Kaluza–Klein theory: Exact solutions

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We study cylindrically symmetric Abelian wormholes (WhC) in (4 + n)-dimensional Kaluza–Klein theory. It is shown that static, four-dimensional, cylindrically symmetric solutions in (4 + n)-dimensional Kaluza–Klein theory with maximal Abelian isometry group \( U(1)^n \) of the internal space with diagonal internal metric can be obtained, as in the case of a supersymmetric static black hole \( 1 \), only if the isometry group of the internal space is broken down to the \( U(1)_e \times U(1)_m \) gauge group; they correspond to dyonic configurations with one electric \( (Q_e) \) and one magnetic \( (Q_m) \) charge that are related either to the same \( U(1)_e \) or \( U(1)_m \) gauge field or to different factors of the \( U(1)_e \times U(1)_m \) gauge group of the effective 6-dimensional Kaluza–Klein theory. We find new exact solutions of the 6-dimensional Kaluza–Klein theory with two Abelian gauge fields, dilaton and scalar fields, associated with the internal metric. We obtain new types of cylindrically symmetric wormholes supported by the radial and longitudinal electric and magnetic fields.

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I. INTRODUCTION

As a starting point we recall the comment in a paper \( 2 \) where a class of static spherically symmetric solutions in (4 + n)-dimensional Kaluza–Klein theory with Abelian isometry was studied: "We assumed that the internal isometry group \( G \) is Abelian. In this case, different supersymmetric static spherical solutions spontaneously break \( G \) down to different \( U(1)_E \times U(1)_M \) factors as the vacuum configurations. We suspect that the same thing will happen for axially symmetric stationary configurations, but it remains to be proven". This article is motivated by the desire to find out to what extent the numerous results (e.g. \( 3 \)–\( 6 \), and others) obtained for (spherically symmetric) black holes in higher-dimensional unified theories, related to supergravity and string theory, can be transferred to axially symmetric wormholes. We started with a cylindrically symmetric wormhole; the first task was derivation from the higher-dimensional theory of an effective 4-dimensional theory of pure gravity, including "external gravitons" – the space-time metric, and "internal gravitons" – the scalar and gauge fields associated with the extra dimensions, as well as finding the exact solutions of the resulting theory. This problem is solved in this paper. The study of geodesic structure, singularity structure and thermal properties of the wormhole solutions we found, as well as inclusion of fermions in the theory and finding connections between Kaluza-Klein wormhole solutions and string theory, would be the next step in the research of cylindrically symmetric wormholes within supersymmetric unified field theories.

By wormholes one usually means topological features in the form of “handles” (throats) connecting different regions of the space–time continuum.

The typically-discussed wormhole models are endowed with spherical symmetry (see, for example, the survey \( 7 \)). It has been shown \( 8 \) that spherically symmetric wormholes can exist only in the presence of so-called “exotic matter”, which refers to a variety of field configurations that have, for example, negative energy density and negative pressure.

A cylindrically symmetric space-time has a preferred direction – the axis of (axial) symmetry. Examples of axially symmetric systems include cosmic strings, among others. We consider a static cylindrically symmetric space-time metric \( 9 \)

\[
ds^2 = e^{2\gamma(u)}dt^2 - e^{2\omega(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2,
\]

(1)

where \( u \) is an arbitrary cylindrical radial coordinate, \( z \in (-\infty, +\infty) \) is the longitudinal coordinate, and \( \phi \in [0, 2\pi] \) is the angular coordinate. The “circle radius” \( R(u) := e^{\beta(u)} \) is non-negative and tends to \(+\infty\).
when \( u \to \pm \infty \). Space-time (1) has one timelike Killing vector \( \xi_1 = \partial_t \) and two spacelike Killing vectors \( \xi_2 = \partial_\vartheta, \xi_3 = \partial_\varphi \), which define the axial symmetry. By a suitable choice of coordinate \( u \) the equality
\[
\omega(u) = \beta(u) + \gamma(u) + \xi(u)
\]
can be satisfied (in what follows we assume that this equality holds), and from (1) we have
\[
ds^2 = e^{2\gamma(u)}dt^2 - e^{2[\beta(u) + \gamma(u) + \xi(u)]}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\vartheta^2.
\]

Following K. Bronnikov and J. Lemos [9] we make the definitions:

**Definition 1** We say that the metric (1) describes a cylindrically symmetric wormhole if the circle radius \( R(u) \) has an absolute minimum \( R(u_0) > 0 \) at some point \( u = u_0 \) and for all possible values of \( u \) the metric functions \( \omega(u), \beta(u), \gamma(u), \xi(u) \) in (1) are smooth and finite.

**Definition 2** The throat of a cylindrically symmetric wormhole with metric (1) is a cylindrical hypersurface defined by the equation
\[
u = u_0.
\]

It has been shown in [9] that the existence of the static, cylindrically symmetric wormholes does not require violation of the weak or null energy conditions near the throat, and the cylindrically symmetric geometry of wormhole configurations can be generated by less exotic sources compared to the case of spherical symmetry. The exact solutions of Einstein’s theory of gravity with scalar, spinor, and electromagnetic fields describing cylindrically symmetric wormholes with metric (2) have been obtained in [9], and all the solutions are not asymptotically flat. It has been proved that in the absence of material fields that violate the weak or null energy conditions, i.e. in the case of everywhere nonnegative energy density of matter, flat asymptotic behavior on both sides of a cylindrically symmetric wormhole is impossible [9].

In this article we discuss static, cylindrically symmetric space-times (2) within \((4+n)\)-dimensional Kaluza–Klein theory with Abelian isometry, and our results confirm the validity of the above “no–go” statement.

The paper is organized as follows. In section II we discuss dimensional reduction of \((4+n)\)-dimensional gravity with a 4-dimensional space-time metric (2) and gauge and scalar fields that are compatible with cylindrical symmetry. We show that the isometry group \( U(1)_n \) of the internal space with diagonal metric is broken down to the \( U(1)_c \times U(1)_m \) gauge group and obtain the constraints on charges in the case of compactification on a 2-torus. In section III we derive and integrate equations of motion for cylindrically symmetric configurations with two Abelian gauge fields, dilaton and scalar fields. We find the exact solutions describing cylindrically symmetric wormholes in the six-dimensional effective Kaluza–Klein theory with radial electric and magnetic fields. In section IV the exact solutions describing cylindrically symmetric 6-d Kaluza–Klein wormholes with the longitudinal electric and magnetic fields are found. Conclusions are given in Section V.

In this paper we use units in which the speed of the light in vacuum \( c \) and gravitational constant \( G \) are taken to be one: \( c = G = 1 \).

**II. DIMENSIONAL REDUCTION OF (4+N)-D GRAVITY WITH DIAGONAL INTERNAL METRIC**

The effective 4-dimensional Kaluza-Klein theory is obtained from \((4+n)\)-dimensional pure gravity with the Einstein-Hilbert action (1):
\[
S_{4+n} = \frac{1}{16\pi G_{4+n}} \int \sqrt{-g^{(4+n)}} R^{(4+n)} d^{4+n}x,
\]
by compactifying the \( n \) extra spatial coordinates on a compact manifold. Here \( G_{4+n} \) is the gravitational constant in \( 4+n \) dimensions, \( R^{(4+n)} \) is the Ricci scalar and \( g^{(4+n)} \) is the determinant of a metric \( g_{AB} \) of the unified space \( M^{4+n} \).

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1 Upper-case letters \( A, B, \ldots \) are used for the indices of the coordinates \( x^0, \ldots , x^{3+n} \) in a \((4+n)\)-dimensional space. The Greek letters \( \mu, \nu, \ldots \) denote the indices of coordinates \( x^0, x^1, x^2, x^3 \) of the four-dimensional space-time and the lower-case letters \( a, b, \ldots \) are used for the coordinates \( x^4, \ldots , x^{4+n} \) of an internal space.
The dimensional reduction of the \((4+n)\)-d gravity is achieved by requiring right invariance of the metric \(g_{AB}\) under the action of an isometry group \(G_n\) with Killing vector fields \(X_a (a = 1, 2, ..., n)\) and structure constants \(f^c_{ab}\); \(L_{X_a} g_{AB} = 0\), \([X_a, X_b] = f^c_{ac} X_c\). \((4+n)\)-d is considered as a principal fiber bundle \(P(M^4, G_n)\) with the four-dimensional space-time \(M^4\) as the base manifold and \(G_n\) as the structure group. In a local direct-product coordinate basis \((\partial_\mu, \partial_n)\) the metric \(g_{AB}\) is written as

\[
\begin{pmatrix}
\left(\epsilon^{\sigma/\rho_{ab}} A^\rho_{a\mu} \epsilon^{\sigma/\rho_{bc}} A^\sigma_{b\nu} + 2\epsilon^{\sigma/\rho_{ab}} A^\rho_{a\sigma} A^\mu_{b\nu} \right) \\
\left(\epsilon^{\sigma/\rho_{ac}} A^\rho_{a\mu} \epsilon^{\sigma/\rho_{bd}} A^\mu_{b\nu} + 2\epsilon^{\sigma/\rho_{ac}} A^\rho_{a\sigma} A^\mu_{b\nu} \right)
\end{pmatrix},
\]

where \(\sigma\) is dilaton field and \(A_1^\mu, ..., A_n^\mu\) are gauge potentials, \(\alpha = \sqrt{1+2/n}\) is the coupling constant of the dilaton to the gauge fields, \(\rho_{ab}\) is the unimodular part of the internal metric: \(\det(\rho_{ab}) = 1\). The dependence of \(g_{AB}\) on internal space coordinates is determined by its right invariance under the action of \(G_n\) \([2, 11]\):

\[
\partial_\alpha g_{\mu\nu} = 0, \quad \partial_\alpha A^\mu_b = -f^c_{ab} A^b_c, \quad \partial_\alpha \rho_{bc} = f^d_{ac} \rho_{bd} + f^d_{ad} \rho_{bc}.
\]

When the isometry group \(G_n\) is unimodular, the \((4+n)\)-d Lagrangian density of the Einstein-Hilbert action \([3]\) becomes explicitly independent of internal coordinates. The dimensional reduction is obtained by integrating over the internal space and, up to a total divergence, the Lagrangian of the effective four-dimensional Kaluza-Klein theory is given by \([11]\)

\[
L = \frac{1}{16\pi\sqrt{-g}} [R + e^{-\alpha\sigma} \tilde{R} - \frac{1}{4} e^{\alpha\sigma} \rho_{ab} F^a_{\mu\nu} F^b_{\mu\nu} + \frac{1}{2} \partial_{\mu} \sigma \partial^{\mu} \sigma - \frac{1}{2} e^{\alpha\sigma} \rho_{ab} \sigma^d D_\mu \rho_{ac} D^\mu \rho_{bd} + \lambda (\det(\rho_{ab}) - 1)],
\]

where \(g = \det(g_{\mu\nu})\), \(\tilde{R}\) is the Ricci scalar of the unimodular part \(\rho_{ab}\) of the internal metric, \(F^a_{\mu\nu} = \partial_a A^\mu_{b\nu} - \partial_b A^\mu_{a\nu} - g f^c_{ab} A^c_{b\nu} A^\mu_{a\nu}\) is the field strength of \(A^\mu_b\), \(\tilde{g}\) is the gauge coupling constant, \(D_\mu \rho_{ab} = \partial_\mu \rho_{ab} - f^c_{ad} A^c_{d\nu} \rho_{bd}\) is the corresponding gauge covariant derivative, \(\rho_{ab} = \rho_{ab} + \rho_{bc}\), and \(\lambda\) is a Lagrange multiplier; the 4-d gravitational constant has been set equal to 1. When \(G_n\) is Abelian isometry group (compactification on an \(n\)-torus \(T^n\)), all structure constants \(f^c_{ab}\) vanish, the gauge covariant derivatives in \([3]\) become partial derivatives, and the Ricci scalar \(\tilde{R}\) vanishes.

We use the diagonal internal metric ansatz \([2, 10]\):

\[
(\rho_{ab}) = \text{diag} \left( \rho_1, ..., \rho_{n-1}, \prod_{h=1}^{n-1} \rho_h^{-1} \right),
\]

and the static cylindrically symmetric ansatzes for the 4-d space-time metric and for the gauge and scalar fields associated with the internal metric. We choose the space-time metric \(g_{\mu\nu}\) of the form \([2]\). The ansatzes for electric and magnetic fields, compatible with cylindrical symmetry, are obtained by using the Yang–Mills equations \(\nabla_\mu (e^{\alpha\sigma} \rho_{ab} F^{a\mu\nu}) = 0\) derived from the Lagrangian \([11]\) and are of the following three forms:

\[
F^a_{\nu u} = \frac{Q^a_{\sigma} e^{2\gamma(u)}}{e^{\alpha\sigma(u)} \rho_{ab}(u)} \equiv \tilde{E}^a(u) e^{2\gamma(u)}, \quad F^a_{\nu \phi} = Q^a_{\phi}, \quad \text{for electric and magnetic radial fields},
\]

\[
F^a_{\nu z} = Q^a_z, \quad F^a_{\nu \phi} = \frac{Q^a_{m} e^{2\gamma(u)}}{e^{\alpha\sigma(u)} \rho_{ab}(u)} \equiv \tilde{H}^a(u) e^{2\beta(u)} \quad \text{for electric and magnetic longitudinal fields},
\]

\[
F^a_{\nu r} = Q^a_r, \quad F^a_{\nu \phi} = \frac{Q^a_{z} e^{2\gamma(u)}}{e^{\alpha\sigma(u)} \rho_{ab}(u)} \equiv \tilde{B}^a(u) e^{2\xi(u)} \quad \text{for electric and magnetic azimuthal fields},
\]

where \(a = 4, ..., n + 3\), the constant \(Q^a_{m}\) is the magnetic charge and the constant \(Q^a_{z}\) is the electric charge of the configuration, the other components of the strength fields \(F^a_{\mu\nu}\) are equal to zero (see arguments after \([13]\) and \([14]\)).

The allowed charge configurations are restricted by the special choice of internal metric \([5]\). Really, the Euler–Lagrange equations for \(\rho_{ab}(u)\), derived from the Lagrangian \([11]\), read:

\[
e^{\alpha\sigma(u) + 2\gamma(u)} [Q^a_{m} Q^b_{m} - \tilde{E}^a(u) \tilde{E}^b(u)] + \lambda \rho_{ab}(u) = \frac{d^2}{du^2} \rho_{ab}(u) \quad \text{for the radial fields},
\]
\[ e^{\alpha(u)}H^a(u)H^b(u) - Q^a_cQ^b_c + \lambda \rho^{ab}(u) = \frac{d^2}{du^2} \rho^{ab}(u) \quad \text{for the longitudinal fields}, \]

\[ e^{\alpha(u) + 2\xi(u)}[B^a(u)B^b(u) - Q^a_cQ^b_c] + \lambda \rho^{ab}(u) = \frac{d^2}{du^2} \rho^{ab}(u) \quad \text{for the azimuthal fields}. \]

It follows from this that for the diagonal metric the following constraints have to be satisfied:

\[ Q^a_cQ^b_c - e^{2\alpha} \rho_a \rho_b Q^a_m Q^b_m = 0 \quad \text{for the radial fields}, \]

\[ Q^a_cQ^b_c - e^{2\alpha} \rho_a \rho_b Q^a_m Q^b_m = 0 \quad \text{for the longitudinal and azimuthal fields}, \]

whence for radial fields (i) \( Q^a_cQ^b_c = Q^a_m Q^b_m = 0 \) if \( a \neq b \), or, generally, (ii) \( Q^a_cQ^b_c - \kappa_a \kappa_b Q^a_m Q^b_m = 0 \) if \( a \neq b \), with \( \kappa_a \equiv e^{\alpha} \rho_a = \text{const} \). The latter case would imply the equation of motion for \( e^{\alpha} \rho_a \) with \( Q^a_c = Q^a_m = 0 \). Thus, the constraint (ii) reduces to the subset of constraints (i) which imply that the same (electric or magnetic) type of charge can appear in at most one gauge field. The same result is valid for longitudinal and azimuthal electric and magnetic fields. Consequently, the internal isometry group \( U(1)^n \) is broken down to at most \( U(1) \times U(1) \), with only one electric and one magnetic charge. Without loss of generality, we associate two \( U(1) \) factors of the internal isometry group \( U(1) \times U(1) \) with the \((n - 1)\)-th and the \( n \)-th internal dimensions, i.e., with gauge fields \( A^{(1)}_\mu \) and \( A^{(n)}_\mu \). When the first \((n - 2)\) gauge fields are turned off the first \((n - 2)\) components of the diagonal internal metric become constant: \( e^{2\alpha} \rho_a = \text{const}, \quad a = 1, \ldots, n - 2 \). As a result the solutions of \((4 + n)\)-dimensional Kaluza–Klein theory are those of effective six-dimensional Kaluza–Klein theory with action:

\[
S_4 = \frac{1}{16\pi} \int \sqrt{-g} [R - 2(\nabla \psi)^2 - 4(\nabla \chi)^2 - e^{2\sqrt{\psi + \chi}} K^{\mu \nu} K_{\mu \nu} - e^{2\sqrt{\psi - \chi}} F^{\mu \nu} F_{\mu \nu}] d^4x,
\]

which is obtained from (4) by using the field redefinition: \( \chi_a \equiv 1/\sqrt{2} \ln \rho_a + 2\sigma/(na) = \text{const}, \quad a = 1, \ldots, n - 2, \chi_{n-1} \equiv 1/\sqrt{2} \ln \rho_{n-1} + (2 - n)\sigma/(na) \equiv \chi, \chi_n = -\chi, \psi \equiv \sqrt{2\alpha} \sigma, F^{\mu \nu -1} \equiv 2K_{\mu \nu}, F^{\mu \nu} \equiv 2F_{\mu \nu} \) (cf. [2], see also [6, 10]).

Note that the above result can also be obtained by solving the Killing spinor equations for supersymmetric Kaluza–Klein configurations with cylindrical symmetry (the proof will be given in a separate paper).

### III. EXACT SOLUTIONS OF EINSTEIN–YANG–MILLS–DILATON EQUATIONS WITH RADIAL ELECTRIC AND MAGNETIC FIELDS

In this section we will obtain the Einstein equations, Yang–Mills equations for the gauge fields \( A_\mu, B_\nu \), and the equations for the dilaton \( \psi \) and scalar field \( \chi \). We will find exact solutions of the Einstein–Yang–Mills–dilaton (EYMD) equations in the case of the radial electric and magnetic fields.

We consider the four-dimensional action (6) of 6-d Kaluza–Klein theory with

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad K_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,
\]

where \( A_\mu, B_\nu \) are the two Abelian gauge fields, and \( \psi \) (dilaton) and \( \chi \) are two scalar fields. The non-zero components of the Ricci tensor \( R^a_b \) and the Ricci scalar \( R \) of the metric (2) are:

\[
R^t_t = e^{-2(\beta + \gamma + \xi)} \gamma'',
R^u_u = e^{-2(\beta + \gamma + \xi)} [\gamma'' + \xi'' + \beta'' - 2(\beta' \xi' + \xi' \gamma' + \gamma' \beta')],
R^z_z = e^{-2(\beta + \gamma + \xi)} \xi'',
R^\phi_\phi = e^{-2(\beta + \gamma + \xi)} \beta'',
R = 2e^{-2(\beta + \gamma + \xi)} (\gamma'' + \xi'' + \beta'' - \beta' \xi' - \xi' \gamma' - \gamma' \beta'),
\]
where the prime denotes differentiation with respect to \( u \). The Euler–Lagrange equations for the scalar fields \( \psi \) and \( \chi \), derived from the action \( (8) \), are given by:

\[
\Box \psi = (1/\sqrt{2}) \left( e^{2\sqrt{2}(\psi+\chi)} K_{\mu\nu} K^{\mu\nu} + e^{2\sqrt{2}(\psi-\chi)} F_{\mu\nu} F^{\mu\nu} \right),
\]

\[
\Box \chi = (1/(2\sqrt{2})) \left( e^{2\sqrt{2}(\psi+\chi)} K_{\mu\nu} K^{\mu\nu} - e^{2\sqrt{2}(\psi-\chi)} F_{\mu\nu} F^{\mu\nu} \right).
\]

\( (\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \). Variation in the fields \( A_\nu \), \( B_\nu \) gives the Yang–Mills equations:

\[
\nabla_\nu F^{\mu\nu} = 2\sqrt{2} e^{2\sqrt{2}(\psi-\chi)} F^{\mu\nu} \nabla_\nu (\chi - \psi), \quad \nabla_\nu K^{\mu\nu} = -2\sqrt{2} e^{2\sqrt{2}(\psi+\chi)} K^{\mu\nu} \nabla_\nu (\psi + \chi),
\]

where \( \nabla_\nu \) is covariant derivative with respect to \( x^\nu \).

Finally, varying the metric \( g_{\mu\nu} \), we obtain the Einstein equations

\[
R^\mu_\nu = \tilde{T}^\mu_\nu, \quad \tilde{T}^\mu_\nu := T^\mu_\nu - (1/2)\delta^\mu_\nu T \quad (T \equiv T^\chi_\chi)
\]

with the energy-momentum tensor

\[
T_{\mu\nu} = 2\nabla_\mu \psi \nabla_\nu \psi - g_{\mu\nu} (\nabla \psi)^2 + 4\nabla_\mu \chi \nabla_\nu \chi - 2g_{\mu\nu} (\nabla \chi)^2 + e^{2\sqrt{2}(\psi+\chi)} (2K_{\mu\nu} K^{\nu\tau} - (1/2)g_{\mu\nu} K^{\nu\tau}) + e^{2\sqrt{2}(\psi-\chi)} (2F_{\mu\tau} F^{\nu\tau} - (1/2)g_{\mu\nu} F^{\nu\tau}).
\]

We expand the gauge fields \( A_\mu \) and \( B_\mu \) into temporal and spatial components: \( A_\mu = (A_t, \vec{A}), \quad B_\mu = (B_t, \vec{B}) \), where \( \vec{A} = (A_u, A_z, A_{\phi}) \), \( \vec{B} = (B_u, B_z, B_{\phi}) \), and consider the ansatz for the form of the next static Abelian gauge fields, dilaton and scalar field:

\[
A_\mu = (A_t(u), 0, 0, 0), \quad B_\mu = (0, 0, 0, \text{const} \cdot z), \quad \psi = \psi(u), \quad \chi = \chi(u).
\]

We introduce the three-dimensional vector fields:

\[
\vec{E}_A := -\vec{\partial} A_t + \vec{\partial} \vec{A}, \quad \vec{H}_A := [\vec{\partial}, \vec{A}], \quad \vec{E}_B := -\vec{\partial} B_t + \frac{\partial \vec{B}}{\partial t}, \quad \vec{H}_B := [\vec{\partial}, \vec{B}],
\]

where \( \vec{\partial} = (\partial_u, \partial_z, \partial_{\phi}) \), \( ([\vec{\partial}, \vec{A}]^h) = e^{-(\gamma+2\lambda \chi+2\beta \phi)} e^{h k l} \partial_k A_l \) with totally antisymmetric \( e^{h k l \phi} = 1 \), and \( h, k, l \) running over \( u, z, \phi \); components are defined similarly for \( [\vec{\partial}, \vec{B}] \). By an analogy with the electromagnetic field, \( \vec{E}_A, \vec{E}_B \) are called electric fields and \( \vec{H}_A, \vec{H}_B \) are called magnetic fields. Since from \( (12) \) only the components \( E^u_A, H^u_B \) are non-zero, we call the fields \( A_u, B_u \) radial.

Using \( (8) \) and \( (12) \) we find the solutions of the Yang–Mills equations \( (8) \), describing the radial electric and magnetic fields:

\[
K^{\pm \phi} = Q_m e^{-2(\beta + \xi)}, \quad K^{t \phi} = K^{t u} = K^{u z} = K^{u \phi} = 0, \quad Q_m = \text{const};
\]

\[
F^{u t} = Q_e e^{-2(\beta + \gamma + \xi + \sqrt{2}(\psi - \chi))}, \quad F^{t \phi} = F^{t z} = F^{u z} = F^{u \phi} = F^{z \phi} = 0, \quad Q_e = \text{const}.
\]

The non-zero components of the tensor \( \tilde{T}^\mu_\nu \) are:

\[
\tilde{T}^\t z = \tilde{T}^z_t = Q_e^2 e^{-2(\beta + \gamma + \xi + \sqrt{2}(\psi - \chi))} + Q_m^2 e^{-2(\beta + \xi + \sqrt{2}(\psi + \chi))},
\]

\[
\tilde{T}^u = -2 e^{-2(\gamma + \xi + \beta)} (\psi^2 + 2 \chi^2) - Q_e^2 e^{-2(\beta + \xi + \sqrt{2}(\psi - \chi))} - Q_m^2 e^{-2(\beta + \xi - \sqrt{2}(\psi + \chi))}.
\]

Equations \( (8) \) give:

\[
\chi'' = -(1/\sqrt{2}) \left( Q_e^2 e^{2(\gamma - \sqrt{2}(\psi - \chi))} + Q_m^2 e^{2(\gamma + \sqrt{2}(\psi + \chi))} \right),
\]

\[
\psi'' = \sqrt{2} \left( Q_e^2 e^{2(\gamma - \sqrt{2}(\psi - \chi))} - Q_m^2 e^{2(\gamma + \sqrt{2}(\psi + \chi))} \right).
\]

The Einstein equations \( R^\mu_\nu = \tilde{T}^\mu_\nu \) \( (10) \) reduce to the next four equations:

\[
\gamma'' = -Q_e^2 e^{2(\gamma - \sqrt{2}(\psi - \chi))} - Q_m^2 e^{2(\gamma + \sqrt{2}(\psi + \chi))}
\]

\( (R^t_t = \tilde{T}^t_t) \),

\( (16) \).
\[ \xi'' = Q_c^2 e^{2(\gamma - \sqrt{2}(\psi - \chi))} + Q_m^2 e^{2(\gamma + \sqrt{2}(\psi + \chi))} \quad (R_z^2 = \tilde{T}_z^2), \quad (17) \]

\[ \beta'' = Q_c^2 e^{2(\gamma - \sqrt{2}(\psi - \chi))} + Q_m^2 e^{2(\gamma + \sqrt{2}(\psi + \chi))} \quad (R_\phi^0 = \tilde{T}_\phi^0), \quad (18) \]

\[ \gamma'' + \xi'' + \beta'' - 2(\beta' + \xi' + \gamma') = -2\psi'^2 - 4\chi'^2 - Q_c^2 e^{2(\gamma - \sqrt{2}(\psi - \chi))} - Q_m^2 e^{2(\gamma + \sqrt{2}(\psi + \chi))} \quad (R_u^0 = \tilde{T}_u^0). \quad (19) \]

The other six Einstein equations are satisfied identically. Substituting (16)–(18) into (19), we obtain

\[ \gamma' + \xi' + \beta' = \psi'^2 + 2\chi'^2 + Q_c^2 e^{2(\gamma - \sqrt{2}(\psi - \chi))} + Q_m^2 e^{2(\gamma + \sqrt{2}(\psi + \chi))}. \quad (20) \]

We notice that the equation obtained by differentiating (20) is satisfied identically as consequence of (14)–(18), which imposes restrictions on the initial data for this system. Integrating equations (14), (16) we see that \( \gamma'' = \sqrt{2}\gamma'' \). Setting the constants of integration equal to zero, we obtain

\[ \gamma = \sqrt{2}\chi. \quad (21) \]

In addition, from (16)–(18) we get: \( \beta = -\gamma + au + \beta_0, \quad \xi = -\gamma + bu + \xi_0 \), where \( a, d, \beta_0, \xi_0 \) are the constants of integration. By scale transformations of coordinates \( z \) and \( \phi \) the additive constants \( \beta_0, \xi_0 \) can be made to vanish, so we have

\[ \beta = -\gamma + au, \quad \xi = -\gamma + bu. \quad (22) \]

From (15), (16) and (21) it follows that

\[ \psi'' = \sqrt{2} \left( Q_c^2 e^{2(\gamma - \sqrt{2}\psi)} - Q_m^2 e^{2(\gamma + \sqrt{2}\psi)} \right), \quad (23) \]

supplemented by the condition (20), which in view of (21) and (22) takes the form

\[ -2\gamma'^2 + ab = \psi'^2 + Q_m^2 e^{2(\gamma + \sqrt{2}\psi)} + Q_c^2 e^{2(\gamma - \sqrt{2}\psi)}. \quad (24) \]

The system (23) is equivalent to the following system:

\[ \eta'' = -2Q_c^2 e^{4\eta}, \quad \zeta'' = -2Q_m^2 e^{4\zeta}, \quad (25) \]

where

\[ \eta = \gamma - (1/\sqrt{2})\psi, \quad \zeta = \gamma + (1/\sqrt{2})\psi. \quad (26) \]

Consider all possible cases: 1) \( Q_cQ_m \neq 0, 2) Q_c \neq 0, Q_m = 0, 3) Q_c = 0, Q_m \neq 0, \) and 4) \( Q_c = Q_m = 0. \)

Solving the system (26) in the case \( Q_cQ_m \neq 0 \) and taking into account that, by (20), \( \gamma = (\zeta + \eta)/2, \)

\[ \psi = (\zeta - \eta)/\sqrt{2}, \]

we find

\[ \gamma(u) = -(1/4) \ln \left( \frac{4|q_cq_m| \cosh[h_e(u - u_e)] \cosh[h_m(u - u_m)]}{\cosh[h_e(u - u_m)] \cosh[h_m(u - u_e)]} \right), \quad (27) \]

\[ \psi(u) = -(1/(2\sqrt{2})) \ln \left( \frac{|q_m/q_c| \cosh[h_m(u - u_m)]}{\cosh[h_e(u - u_e)]} \right), \quad (28) \]

where \( h_e \neq 0, h_m \neq 0, u_e, u_m \) are constants of integration, \( Q_e/h_e \equiv q_e \) and \( Q_m/h_m \equiv q_m. \) Without loss of generality we can assume that \( h_e > 0, h_m > 0. \) From (22) we obtain:

\[ \beta(u) = (1/4) \ln \left( \frac{4|q_cq_m| \cosh[h_e(u - u_e)] \cosh[h_m(u - u_m)]}{\cosh[h_e(u - u_m)]} \right) + au, \quad (29) \]

\[ \xi(u) = (1/4) \ln \left( \frac{4|q_cq_m| \cosh[h_e(u - u_e)] \cosh[h_m(u - u_m)]}{\cosh[h_e(u - u_m)]} \right) + bu. \quad (30) \]

Substituting (27)–(30) into (20), we have

\[ 4ab = h_e^2 + h_m^2. \]
From (29) we derive

$$\beta'(u) = (1/4) (h_e \tanh[h_e(u - u_c)] + h_m \tanh[h_m(u - u_m)]) + a.$$  (31)

Due to the monotonicity of $\tanh$ the derivative $\beta'(u)$ can vanish at no more than one point. If

$$|a| < (h_e + h_m)/4$$  (32)

then there exists a (unique) point $u_0 \in \mathbb{R}$ for which $\beta'(u_0) = 0$. If the condition (32) is not satisfied, then there is no point at which the derivative (31) vanishes.

Note that because of the transcendence of the equation $\beta'(u) = 0$ the value $u_0$ can be found analytically only for some particular values of the parameters $h_e, h_m$.

Since, by (15), the second derivative of $\beta(u)$ is positive, taking into account the condition (32), the function $\beta(u)$ has an absolute minimum at $u = u_0$. In accordance with definition 1, metric (2) with functions (27), (29), (30) together with condition (32) describes a family of cylindrically symmetric wormholes characterized by an electric charge $Q_e$, a magnetic charge $Q_m$ and parameters $h_e, h_m, a$ and $b$. Note that this solution is not asymptotically flat.

We put $r_e = \exp(h_e u_c)$, $r_m = \exp(h_m u_m)$ and introduce a new “radial” coordinate $r := \exp(u) \in [0, +\infty)$. With these new coordinates the metric (2) with the functions (27), (29), (30) takes the form

$$ds^2 = \kappa \Omega \left( \frac{dt^2}{\kappa^2 \Omega^2} - r^{2(a + b - 1)} dr^2 - r^{2b} dz^2 - r^{2a} d\phi^2 \right),$$  (33)

where

$$\kappa = \sqrt{|q_e q_m|}, \quad \Omega = \sqrt{[(r/r_e)^{h_e} + (r/r_e)^{-h_e}][(r/r_m)^{h_m} + (r/r_m)^{-h_m}]}, \quad 4ab = h_e^2 + h_m^2$$

and $h_e h_m Q_e Q_m \neq 0$. If $4|a| < h_e + h_m$ then (33) is the metric of the wormhole with the throat radius $r_0$ defined by the equation

$$(4a + h_e + h_m)(r_0/r_e)^{2h_e}(r_0/r_m)^{2h_m} + (4a - h_e + h_m)(r_0/r_m)^{2h_m} + (4a + h_e - h_m)(r_0/r_e)^{2h_e} + 1 = 0.$$  (34)

The wormhole (33) with $4|a| < h_e + h_m$ which we denote by WhCR$^{e;m}$ is generated by the following Abelian gauge fields and scalar fields:

$$A_\mu = \left( - (h_e/(4q_e)) \right) \left( [(r/r_e)^{h_e} - (r/r_e)^{-h_e}] / [(r/r_e)^{h_e} + (r/r_e)^{-h_e}] \right), \quad B_\mu = (0, 0, 0, -h_m q_m z),$$

$$F^{\mu\nu} = (h_e/q_e) \left( [(r/r_e)^{h_e} + (r/r_e)^{-h_e}]^{-2} \delta_\nu^\mu \delta_\nu^\mu \right), \quad K_{\mu\nu} = h_m q_m \delta_\mu^\nu \delta_\mu^\nu, \quad \psi = (1/(2\sqrt{2})) \ln \left( [q_e q_m] \left[ [(r/r_e)^{h_e} + (r/r_e)^{-h_e}] / [(r/r_m)^{h_m} + (r/r_m)^{-h_m}] \right] \right),$$

$$\chi = -(1/(4\sqrt{2})) \ln \left( [q_e q_m] \left[ [(r/r_e)^{h_e} + (r/r_e)^{-h_e}] / [(r/r_m)^{h_m} + (r/r_m)^{-h_m}] \right] \right).$$  (34)

In the second case when $Q_e \neq 0$ and $Q_m = 0$ we obtain:

$$\gamma = \sqrt{2} \chi = -(1/4) \ln D(u) + cu, \quad \beta = -\gamma + au, \quad \xi = -\gamma + bu, \quad \psi = (1/(2\sqrt{2})) \ln D(u) + \sqrt{2} cu,$$

where $D(u) \equiv 2 |q_e| \cosh[h_e(u - u_c)]$ and, by (21), $4(ab - 4c^2) = h_e^2$. We have

$$\beta'(u) = (h_e/4) \tanh[h_e(u - u_c)] + a - c.$$  (34)

The single point at which the derivative of $\beta(u)$ vanishes, exists only if $4|a - c| < h_e$, and it is given by

$$u_0 = u_e - (1/h_e) \arctanh[4(a - c)/h_e].$$

The hypersurface $u = u_0$ defines a throat of the cylindrically symmetric wormhole, which we denote by WhCR$^e$. Its metric in coordinates $t, r, z, \phi$ is

$$ds^2 = k_e A_e \left( \frac{r_{2c} dr^2}{k_e A_e} - r^{2(a + b - c - 1)} dt^2 - r^{2b} dz^2 - r^{2a - c} d\phi^2 \right) \quad (a, b, c = \text{const}),$$  (35)
where
\[ k_e = \sqrt{|q_e|}, \quad \Lambda_e = \sqrt{(r/r_e)^{h_e} + (r/r_e)^{-h_e}}, \quad 4ab = h_e^2 + 16c^2 \quad (h_e, q_e = \text{const}, h_e q_e \neq 0). \] (36)

The throat radius of the WhCR is
\[ r_0 = r_e \left( \frac{h_e - 4(a - c)}{h_e + 4(a - c)} \right)^{1/(2h_e)}, \quad h_e > 4|a - c|, \] (37)
and the wormhole is generated by the following single Abelian gauge field \( A_\mu \), dilaton \( \psi \) and scalar field \( \chi \):
\[ A_\mu = \left( -\frac{h_e}{4q_e} \right) \left( \frac{(r/r_e)^{h_e} - (r/r_e)^{-h_e}}{\left[(r/r_e)^{h_e} + (r/r_e)^{-h_e}\right]} \right)^{-2} \left( 0, 0, 0 \right), \]
\[ F^{\mu\nu} = \left( \frac{h_e}{q_e} \right) \left[ \frac{(r/r_e)^{h_e} + (r/r_e)^{-h_e}}{\left[(r/r_e)^{h_e} + (r/r_e)^{-h_e}\right]} \right]^2 \delta^\mu_\nu, \]
\[ \psi = (1/(2\sqrt{2})) \ln \left( |q_e|^4[(r/r_e)^{h_e} + (r/r_e)^{-h_e}] \right), \]
\[ \chi = -(1/(4\sqrt{2})) \ln \left( |q_e|^4[(r/r_e)^{h_e} + (r/r_e)^{-h_e}] \right). \] (38)

The wormhole WhCR corresponding to the third case \( Q_e = 0, Q_m \neq 0 \) is defined by the metric
\[ ds^2 = k_m \Lambda_m \left( \frac{r^{2c}dt^2}{k_m^2 \Lambda_m^2} - r^{2(a + b - c - 1)}dr^2 - r^{2(b - c)}dz^2 - r^{2(a - c)}d\phi^2 \right) \quad (a, b, c = \text{const}), \] (39)
where
\[ k_m = \sqrt{|q_m|}, \quad \Lambda_m = \sqrt{(r/r_m)^{h_m} + (r/r_m)^{-h_m}}, \quad 4ab = h_m^2 + 16c^2 \quad (h_m, q_m = \text{const}, h_m q_m \neq 0), \] (40)
and by the formula
\[ B_\mu = (0, 0, 0, -h_m q_m), \quad K_{\mu\nu} = h_m q_m \delta_\mu^\lambda \delta_\nu^\lambda, \]
\[ \psi = -(1/(2\sqrt{2})) \ln \left( |q_m|^4[(r/r_m)^{h_m} + (r/r_m)^{-h_m}] \right), \]
\[ \chi = -(1/(4\sqrt{2})) \ln \left( |q_m|^4[(r/r_m)^{h_m} + (r/r_m)^{-h_m}] \right), \]
\[ r_0 = r_m \left( \frac{h_m - 4(a - c)}{h_m + 4(a - c)} \right)^{1/(2h_m)}, \quad h_m > 4|a - c|. \] (41)

Finally, in the case \( Q_e = Q_m = 0 \) we have non-wormhole exact solutions of the Einstein-dilaton-scalar field equations:
\[ ds^2 = r^{2c}dt^2 - r^{2(a+b+c-1)}dr^2 - r^{2b}dz^2 - r^{2a}d\phi^2 \quad (a, b, c = \text{const}, \ r \in (0, +\infty)), \] (42)
\[ \psi = k_1 \ln r + k_0, \quad \chi = s_1 \ln r + s_0, \quad ab + ac + bc = k_1^2 + 2s_1^2 \quad (k_0, k_1, s_0, s_1 = \text{const}). \]

Let's consider the question of the stability of the solutions. Introducing the variables \( \gamma_1 := \gamma', \psi_1 := \psi' \), we rewrite the system in the form:
\[ \begin{align*}
\gamma' & = \gamma_1, \\
\gamma'_1 & = Q_e^2 e^{2(2\gamma - \sqrt{2}\psi)} + Q_m^2 e^{2(2\gamma + \sqrt{2}\psi)}, \\
\psi' & = \psi_1, \\
\psi'_1 & = \sqrt{2} \left( Q_e^2 e^{2(2\gamma - \sqrt{2}\psi)} - Q_m^2 e^{2(2\gamma + \sqrt{2}\psi)} \right). 
\end{align*} \] (43)

Given that the equilibrium point of a system: \( d\vec{x}/d\tau = \vec{f}(\vec{x}(\tau)) \), where \( \vec{x}, \vec{f} \) are vector strings, is the point at which the right-hand side vanishes: \( \vec{f}(\vec{x}) = 0 \) [16], we find that the equilibrium points of the system [13] are defined by the equations: \( \gamma_1 = \psi_1 = Q_e^2 e^{2(2\gamma - \sqrt{2}\psi)} = Q_m^2 e^{2(2\gamma + \sqrt{2}\psi)} = 0 \). We can see from here that, if at least one of the charges \( Q_e \) or \( Q_m \) is nonzero, then the equilibrium points are absent, indicating the stability of our wormhole solutions.
IV. EXACT SOLUTIONS OF EINSTEIN–YANG–MILLS–DILATON EQUATIONS WITH LONGITUDINAL ELECTRIC AND MAGNETIC FIELDS

In this section we write and integrate the Einstein–Yang–Mills equations and dilaton and scalar field equations for the following ansatz:

\[ A_\mu = (\text{const}, z, 0, 0), \quad B_\mu = (0, 0, 0, B_\phi(u)), \quad \psi = \psi(u), \quad \chi = \chi(u). \]  

(44)

Of the components \( B_\mu \) only \( E_\chi \) and \( H_\tilde{\beta} \) are non-zero, so one can speak of longitudinal electric and magnetic fields. Using (7) and solving the Yang–Mills equations (9) we have

\[ K^{u\phi} = Q_m e^{-2[\beta+\gamma+\xi+\sqrt{2}(\psi+\chi)]}, \quad K^{u\phi} = K^{t\alpha} = K^{\alpha \alpha} = K^{u\phi} = 0, \quad Q_m = \text{const}; \]

(44)

\[ F^{zt} = Q_e e^{-2(\gamma+\xi)}, \quad F^{t\phi} = F^{uz} = F^{\mu t} = F^{u\phi} = F^{z\phi} = 0, \quad Q_e = \text{const}. \]

(44)

The equations (8) give:

\[ \chi'' = -(1/\sqrt{2}) \left( Q^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q^4 e^{2[\beta-\sqrt{2}(\psi+\chi)]} \right), \]

(45)

\[ \psi'' = \sqrt{2} \left( Q^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} - Q^4 e^{2[\beta-\sqrt{2}(\psi+\chi)]} \right). \]

(46)

The non-trivial Einstein equations (10) are

\[ -\gamma'' = Q^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}, \]

(47)

\[ -\xi'' = Q^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}, \]

(48)

\[ \beta'' = Q^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}, \]

(49)

\[ \gamma'' + \xi'' + \beta'' - 2(\beta'\xi' + \xi'\beta') = -2\psi'' - 4\chi'' + Q^2 e^{2\beta} e^{-2\sqrt{2}(\psi+\chi)} + Q^2 e^{2\beta} e^{2\sqrt{2}(\psi-\chi)}. \]

(50)

From (17)–(50) we get

\[ \gamma' + \xi' + \beta' = \psi'^2 + 2\chi'^2 - Q^2 e^{2\beta} e^{-2\sqrt{2}(\psi+\chi)} - Q^2 e^{2\beta} e^{2\sqrt{2}(\psi-\chi)}. \]

(51)

By virtue of (15)–(19) the differential consequence of the last equation is satisfied identically, which imposes restrictions on the initial data for the system. From (15), (19) it follows \( \beta'' = -\sqrt{2}\chi'' \). We put

\[ \beta = -\sqrt{2}\chi \]

(52)

and, from (17)–(19),

\[ \gamma = -\beta + au, \quad \xi = -\beta + bu \quad (a, b = \text{const}), \]

(53)

where the additive integration constants are eliminated by rescaling of \( t, z \). The system (15)–(19) reduces to the form:

\[ \beta'' = Q^2 e^{2(2\beta-\sqrt{2}\psi)} + Q^2 e^{2(2\beta-\sqrt{2}\psi)}, \quad (1/\sqrt{2})\psi'' = Q^2 e^{2(2\beta-\sqrt{2}\psi)} - Q^2 e^{2(2\beta-\sqrt{2}\psi)}. \]

(54)
From (51) we have
\[ Q^2 e^{2(\beta + \sqrt{2}\psi)} + Q_m^2 e^{2(\beta - \sqrt{2}\psi)} + ab = \psi'^2 + 2\beta'^2. \] (55)

In variables \( \bar{\eta} = \beta + (1/\sqrt{2})\psi, \bar{\zeta} = \beta - (1/\sqrt{2})\psi \) (54) takes the form
\[ \bar{\eta}'' = 2Q^2 e^{4\bar{\eta}}, \quad \bar{\zeta}'' = 2Q_m^2 e^{4\bar{\zeta}}. \] (56)

Assume \( Q, Q_m \neq 0 \). From (56) it follows
\[ \bar{\eta}'' - Q^2 e^{4\bar{\eta}} = (\varepsilon/4)h_e^2, \quad \bar{\zeta}'' - Q_m^2 e^{4\bar{\zeta}} = (\bar{\varepsilon}/4)h_m^2, \]
where \( h_e, h_m \) are positive integration constants and \( \varepsilon, \bar{\varepsilon} \) take values 0, ±1. Considering all possible cases we obtain the solutions: \( \bar{\eta}_k = -\ln \Lambda_{k|e}, \bar{\zeta}_k = -\ln \Lambda_{k|m} \) where \( k = 1, 2, 3 \) and
\[ \Lambda_{1|e} = \sqrt{2} \sqrt{|Q_e(u - u_e)|}, \quad \Lambda_{2|e} = \sqrt{2} \sqrt{|g_e \sinh[h_e(u - u_e)]|}, \quad \Lambda_{3|e} = \sqrt{2} \sqrt{|g_e \sin[h_e(u - u_e)]|}, \]
\[ \Lambda_{1|m} = \sqrt{2} \sqrt{|Q_m(u - u_m)|}, \quad \Lambda_{2|m} = \sqrt{2} \sqrt{|g_m \sin[h_m(u - u_m)]|}, \quad \Lambda_{3|m} = \sqrt{2} \sqrt{|g_m \sin[h_m(u - u_m)]|}, \] (57)

\( (g_e \equiv Q_e/h_e, g_m \equiv Q_m/h_m, h_e > 0, h_m > 0) \). Substituting these solutions in formulas \( \bar{\eta} = (\bar{\eta} + \bar{\zeta})/2, \psi = (\bar{\eta} - \bar{\zeta})/\sqrt{2} \) and using (55), (57), we obtain in the case \( Q, Q_m \neq 0 \) nine types \((k|e; j|m)\) of exact solutions of the EYMD equations with two indices \( k, j \) running over 1, 2, 3:
\[ ds^2_{k|e;j|m} = \Lambda_{k|e} \Lambda_{j|m} (e^{2au} du^2 - e^{2(a+b)u} du^2 - e^{2bu} dz^2) - \frac{d\phi^2}{\Lambda_{k|e} \Lambda_{j|m}}, \quad 4ab = \varepsilon_k h_e^2 + \bar{\varepsilon}_j h_m^2, \]
\[ \psi_{k|e;j|m} = (1/\sqrt{2}) \ln(\Lambda_{j|m}/\Lambda_{k|e}), \quad \chi_{k|e;j|m} = (1/(2\sqrt{2})) \ln(\Lambda_{k|e} \Lambda_{j|m}), \]
\[ A_\mu = (-h_e g_e z, 0, 0, 0), \quad B^{1|m}_\phi = (0, 0, 0, B^{1|m}_\phi), \] (58)

here \( \varepsilon_1 = \bar{\varepsilon}_1 = 0, \varepsilon_2 = \bar{\varepsilon}_2 = 1, \varepsilon_3 = \bar{\varepsilon}_3 = -1 \) and
\[ B^{1|m}_\phi = -(1/(4Q_m))(u - u_m), \quad B^{2|m}_\phi = -(1/(4Q_m)) \coth[h_m(u - u_m)], \]
\[ B^{3|m}_\phi = -(1/(4Q_m)) \cot[h_m(u - u_m)]. \] (59)

In the same way we obtain three types \((k|e), k = 1, 2, 3, \) of exact solutions of the EYMD equations in the case \( Q_e \neq 0, Q_m = 0 \):
\[ ds^2_{k|e} = e^{cu} \Lambda_{k|e} (e^{2au} du^2 - e^{2(a+b)u} du^2 - e^{2bu} dz^2) - \frac{e^{cu} d\phi^2}{\Lambda_{k|e}}, \quad 4(ab - c^2) = \varepsilon_k h_e^2, \]
\[ \psi_{k|e} = (1/\sqrt{2}) \left[(cu - \ln \Lambda_{k|e}) \right], \quad \chi_{k|e} = (1/(2\sqrt{2})(\ln \Lambda_{k|e} + cu), \]
\[ A_\mu = (-h_e g_e z, 0, 0, 0), \quad B_\mu = 0, \quad \varepsilon_1 = 0, \quad \varepsilon_2 = 1, \quad \varepsilon_3 = -1, \] (60)

and three types \((k|m), k = 1, 2, 3, \) of exact solutions of the EYMD equations in the case \( Q_e = 0, Q_m \neq 0 \):
\[ ds^2_{k|m} = e^{cu} \Lambda_{k|m} (e^{2au} du^2 - e^{2(a+b)u} du^2 - e^{2bu} dz^2) - \frac{e^{cu} d\phi^2}{\Lambda_{k|m}}, \quad 4(ab - c^2) = \bar{\varepsilon}_k h_m^2, \]
\[ \psi_{k|m} = (1/\sqrt{2}) \left[(\ln \Lambda_{k|m} - cu) \right], \quad \chi_{k|m} = (1/(2\sqrt{2})(\ln \Lambda_{k|m} + cu), \]
\[ A_\mu = 0, \quad B_\mu = (0, 0, 0, B^{k|m}_\phi), \quad \bar{\varepsilon}_1 = 0, \quad \bar{\varepsilon}_2 = 1, \quad \bar{\varepsilon}_3 = -1, \] (61)

where \( \Lambda_{k|e}, \Lambda_{k|m}, B^{k|m}_\phi \) are defined by (57) and (59), \( c = \text{const}, k = 1, 2, 3. \)

In the case \( Q_e = Q_m = 0 \) we again obtain non-wormhole exact solutions (42) of the Einstein-dilaton-scalar field equations.
Arguing similarly to the previous case, it is easy to see that all the above solutions are Jacobi stable. We now turn to a discussion of the wormhole properties of the exact solutions found in this section. We require that the radicand of $\Lambda_k$, $k = 1, 2, 3$, be nonzero. The sets $U_{k|m}$ of the roots of the equations $\Lambda_k = 0$ are $U_{1|e} = \{u_e\}$, $U_{3|e} = \{u_e + \pi N/h_e \mid N \in \mathbb{Z}\}$, and the sets $U_{k|m}$ of the roots of the equations $\Lambda_k = 0$ are $U_{1|m} = U_{2|m} = \{u_m\}$, $U_{3|m} = \{u_m + \pi N/h_m \mid N \in \mathbb{Z}\}$. For $Q_{k} Q_{m} \neq 0$ the domains of the functions $\ln[(\Lambda_k \Lambda_j)^{-1/2}] = \beta(u)$ and $\psi_{k|m}$ are $D_{k|j|m} = \mathbb{R} \cup (U_{k|e} \cup U_{j|m})$. Using (57) we find for the types $(j|e; 3|m)$, $j = 1, 2, 3$:

$$
\beta'(u) = \begin{cases}
-\left(\frac{1}{4}\right)((u - u_e)^{-1} + h_e \cot h_e (u - u_m)) & \text{for type } (1|e; 3|m), \\
-\left(\frac{1}{4}\right)((u - u_e)^{-1} + h_m \cot h_m (u - u_m)) & \text{for type } (2|e; 3|m), \\
-\left(\frac{1}{4}\right)(h_e \cot h_e (u - u_e) + h_m \cot h_m (u - u_m)) & \text{for type } (3|e; 3|m) \\
\end{cases}
\quad (h_e > 0, h_m > 0).
$$

We put $V_{n}^m = (u_m + \pi N/h_m, u_m + \pi (N + 1)/h_m)$ for $N \in \mathbb{Z}$. Suppose for $j = 1, 2$ we have $u_e \in V_{N_0}^m$ for some $N_0 \in \mathbb{Z}$. Then the domain $D_{j|e; 3|m}$ of $\beta(u)$ is the set of the intervals $V_{n}^m \equiv (u_m + \pi N_0/h_m, u_m)$, $V_{n}^m \equiv (u_e, u_m + \pi (N_0 + 1)/h_m)$ and $V_{n}^m$ for $N \neq N_0$, $D_{j|e; 3|m} = V_{n}^m \cup V_{n}^m \cup N \neq N_0$ $V_{n}^m$. At the boundaries of each of the above intervals the function $\beta(u)$ and the circle radius $R(u) = e^{\beta(u)}$ tend to $+\infty$. It follows from (62) that $\beta''(u) > 0$ in the domain of $\beta(u)$, therefore, $\beta'(u)$ grows monotonically in each interval of $D_{j|e; 3|m}$. Since the monotone continuous function $\beta'(u)$ tends to $-\infty$ at the left boundary of each interval of $D_{j|e; 3|m}$ and tends to $+\infty$ at the right boundary of the interval, there exists a single point for each interval: $u_0 - \in V_{n}^m, u_0 + \in V_{n}^m, u_m \in V_{n}^m, N \neq N_0$, at which $\beta'(u)$ vanishes and $\beta(u)$ has a minimum. According to the definitions 1 and 2 there is a throat in each interval of $D_{j|e; 3|m}$, and the space–time “consists” of an infinite countable number of “universes” with $u \in V_{n}^m \cap N \neq N_0$ or $u \in V_{n}^m$ or $u \in V_{n}^m$; each universe has a throat. Similarly, when $u_e = u_m + \pi N/h_m$ for some $N \in \mathbb{Z}$ we obtain an infinite countable number of universes with $u \in V_{n}^m$, $N \in \mathbb{Z}$. We denote these wormhole solutions by WhCL$^{|j|e; 3|m}$, $j = 1, 2$. The solutions WhCL$^{|j|e; 3|m}$, $j = 1, 2$, are similarly defined.

We put $V_{N}^{n} = (u_e + \pi N/h_e, u_e + \pi (N + 1)/h_e)$, $N \in \mathbb{Z}$. The domain $D_{j|e; 3|m} = \bigcup_{N_i, N_2} \mathbb{Z}(V_{N_1}^m \cap V_{N_2}^m)$ of the function $\beta(u)$ for the solution of type $(3|e; 3|m)$ is the infinite set of intervals $(u^K, u^K)$, $K \in \mathbb{Z}$, where $u^K, u^K \in U_{3|e} \cup U_{3|m}$, $u^K < u^K$ and the interval $(u^K, u^K)$ does not contain any element of $U_{3|e} \cup U_{3|m}$. It follows from (62) that $\beta(u)$ and the circle radius $R(u) = e^{\beta(u)}$ tend to $+\infty$ at the boundaries of each interval $(u^K, u^K)$. As the monotonically increasing function $\beta'(u)$ tends to $-\infty$ when $u \rightarrow u^K + 0$ and tends to $+\infty$ when $u \rightarrow u^K - 0$, in each interval $(u^K, u^K)$ there exists a single point $u^K$ at which $\beta'(u)$ vanishes and $\beta(u)$ has a minimum. As in the previous case the space–time splits into multiple universes with $u \in (u^K, u^K)$, $K \in \mathbb{Z}$; each universe has a throat defined by the equation $u = u^K$. This wormhole solution is denoted by WhCL$^{|j|e; 3|m}$.

Consider the solutions of the remaining types $(f|e; h|m)$, $f, h = 1, 2$. We have

$$
\beta'(u) = \begin{cases}
-\left(\frac{1}{4}\right)((u - u_e)^{-1} + (u - u_m)^{-1}) & \text{for type } (1|e; 1|m), \\
-\left(\frac{1}{4}\right)((u - u_e)^{-1} + h_e \coth h_e (u - u_m)) & \text{for type } (1|e; 2|m), \\
-\left(\frac{1}{4}\right)(h_e \coth h_e (u - u_e) + (u - u_m)^{-1}) & \text{for type } (2|e; 1|m), \\
-\left(\frac{1}{4}\right)(h_e \coth h_e (u - u_e) + h_m \coth h_m (u - u_m)) & \text{for type } (2|e; 2|m). \\
\end{cases}
$$

If $u_e = u_m$ then the domain of $\beta(u)$ is $(-\infty, u_e) \cup (u_e, +\infty)$. As the first derivative of the smooth function $\beta(u)$ does not vanish in the domain, it has no minimum and the solutions of the types $(f|e; h|m)$, $f, h = 1, 2$, with $u_e = u_m$ are non-wormholes. In the case $u_e \neq u_m$ we have $D_{f|e; h|m} = (-\infty, u_j) \cup (u_j, u_f) \cup (u_f, +\infty)$ where $u_j = \min\{u_e, u_m\}$ and $u_f = \max\{u_e, u_m\}$. The first derivative of the monotonically increasing function $\beta'(u)$ at $(-\infty, +\infty)$ vanishes at a single point $u_0 \in (u_j, u_f)$, and the hypersurface $u = u_0$ defines a throat of the wormhole denoted by WhCL$^{|f|e; h|m}$, $u_e \neq u_m$, $f, h = 1, 2$. Equation $\beta'(u) = 0$ can be solved analytically only for the wormhole WhCL$^{|j|e; 3|m}$ with the throat $u_0 = (u_e + u_m)/2$.

In the case $Q_{m} \neq 0, Q_{e} = 0$ the domains $D_{j|m}$ of the function $\beta(u)$ for the solutions of the types $(j|m)$, $j = 1, 2, 3$, are $D_{1|m} = D_{2|m} = (-\infty, u_m) \cup (u_m, +\infty)$ and $D_{3|m} = \mathbb{R} \cup \mathbb{Z} V_{n}^m$. The first derivative of $\beta(u)$ is

$$
\beta'(u) = \begin{cases}
-\left(\frac{1}{4}\right)((u - u_m)^{-1} + 2c), & \text{for type } (1|m), \\
-\left(\frac{1}{4}\right)(h_m \coth h_m (u - u_m) + 2c), & \text{for type } (2|m), \\
-\left(\frac{1}{4}\right)(h_m \coth h_m (u - u_m) + 2c), & \text{for type } (3|m). \\
\end{cases}
\quad (63)
$$

For type $(1|m)$ the throat $u = u_0 \equiv u_m - (2c)^{-1}$ exists when $c > 0$, $-\infty < u_0 < u_m$ or $c < 0$, $u_m < u_0 < +\infty$; the solution with $c = 0$ is a non-wormhole. Solutions of the type $(2|m)$ are wormholes with
the throat \( u = u_0 \equiv u_m - \text{arcoth}(2c/h_m) \) only if \( 2c/h_m > 1, -\infty < u < u_m \) or \( 2c/h_m < -1, u_m < u < +\infty \); we have no wormholes when \( |2c/h_m| \leq 1 \). In the case (3|m) we obtain an infinite countable number of the “universes” \( (61) \) with \( u \in V_{N \in \mathbb{Z}} \), each contains a throat \( u = u_0 \equiv u_m + h_m^{-1}(\pi N - \text{arccot}(2c/h_m)) \). We denote the above wormhole solutions by WhCL\(^{k|m}, k = 1, 2, 3 \). The wormhole solutions WhCL\(^{k|e} \) (60) are similarly defined.

The purely dilatonic exact solutions (42) are non-wormholes. So the wormhole solutions do not exist if both electric and magnetic charges are equal to zero.

Formulas (58) in the case \( Q_m \neq 0 \), (60) in the case \( Q_e \neq 0, Q_m = 0 \) and (61) in the case \( Q_m \neq 0, Q_e = 0 \) define new cylindrically symmetric exact solutions of EYMD equations (38–40) with longitudinal electric and magnetic fields as sources; these solutions determine wormholes of the types WhCL\(^{|e;j|m}, WhCL\(^{k|e} \) and WhCL\(^{k|m}, j, k = 1, 2, 3 \). All of them are not asymptotically flat.

V. CONCLUSION

We studied static axially (cylindrically) symmetric solutions in \( 4 + n \)-dimensional Kaluza–Klein theory with \( n \) gauge fields in the case that the internal symmetry group is the maximal Abelian isometry group \( U(1)^n \). It was shown that, as in the case of spherical symmetry [6], the consistency of the Euler–Lagrange equations for pure Einstein–Poincaré gravity in \( 4 + n \) dimensions with a diagonal internal metric and a cylindrically symmetric ansatz for gauge, dilaton and internal scalar fields imposes constraints on the possible charge configurations of the solutions. Such solutions can exist only for configurations with no more than one nonzero electric \( (Q_e) \) and one nonzero magnetic \( (Q_m) \) charge. In fact, \( 4 + n \)-dimensional Kaluza–Klein theory is reduced to an effective six-dimensional Kaluza–Klein theory (9) with two Abelian gauge fields \( A_\mu, B_\mu \), dilaton field \( \psi \) and scalar field \( \chi \). We solved the Einstein–Yang–Mills–Dilaton equations (8–10) derived from the action (3) for the cylindrically symmetric space–time (2) and found the exact solutions of these equations in the cases of (cylindrically symmetric) radial and longitudinal magnetic and electric fields.

Following the definitions 1 and 2 (K. Bronnikov and J. Lemos [9]) we investigated the wormhole properties of the obtained solutions. In the case of radial fields we found three types of static cylindrically symmetric dilatonic wormholes: dyonic WhCR\(^{e;m} \) defined by (33–34), WhCR\(^{e} \) (35–38) with nonzero electric charge and WhCR\(^{m} \) (39), (40), (41) with nonzero magnetic charge. Nine types of dyonic wormholes WhCL\(^{|e;j|m}, k, j = 1, 2, 3 \), determined by (58) are found in the case of longitudinal gauge fields as well as the wormholes WhCL\(^{j|e} \) (60) with nonzero electric charge and the wormhole WhCL\(^{j|m} \) (61) with nonzero magnetic charge. All the solutions we found are Jacobi stable.

The purely dilatonic exact solutions (42) are non-wormholes. So wormhole solutions do not exist if both electric and magnetic charges are equal to zero. Hence, the wormhole properties of these solutions are generated only by electric and/or magnetic charges of static Abelian gauge fields. All obtained wormhole solutions are asymptotically nonflat; this confirms the “no–go” statement [9] about nonflat asymptotic behavior of a cylindrically symmetric wormhole in the case of everywhere nonnegative energy density of matter, i.e. in the absence of ghost fields.

The detailed study of the structure of the new wormhole solutions obtained in this paper will be a task for the next paper.

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