Nonlinear Schrödinger lattices
II: Persistence and stability of discrete vortices

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Abstract

We study discrete vortices in the anti-continuum limit of the discrete two-dimensional non-linear Schrödinger (NLS) equations. The discrete vortices in the anti-continuum limit represent a finite set of excited nodes on a closed discrete contour with a non-zero topological charge. Using the Lyapunov–Schmidt reductions, we find sufficient conditions for continuation and termination of the discrete vortices for a small coupling constant in the discrete NLS lattice. An example of a closed discrete contour is considered that includes the vortex cell (also known as the off-site vortex). We classify the symmetric and asymmetric discrete vortices that bifurcate from the anti-continuum limit. We predict analytically and confirm numerically the number of unstable eigenvalues associated with various families of such discrete vortices.

1 Introduction

Following the first paper of this series \cite{1}, we address discrete systems and differential-difference equations, which have become topics of increasing physical and mathematical importance. The variety of physical applications where such models are relevant, and their significant differences from the mathematical theory of partial differential equations, contribute to recent interest in these topics. The applicability of such models extends to areas as diverse as nonlinear optics, atomic and soft condensed-matter physics, as well as biophysics: specific details and references can be found in our first paper \cite{1} as well as in reviews \cite{2,3,4,5,6,7}.

The second paper is devoted to existence and stability of coherent structures in two-dimensional lattices, which include both discrete solitons \cite{6,7} and discrete vortices \cite{8}. These two-dimensional coherent structures have emerged recently in studies of photorefractive crystals in nonlinear optics \cite{9,10} and droplets of optical lattices in Bose-Einstein condensates \cite{11,12}. A significant boost to this subject was given by the experimental realization of two-dimensional photonic crystal lattices with periodic potentials based on the ideas of \cite{9}. As a result, discrete solitons were observed in \cite{13,14}, while more complex structures such as dipoles, soliton trains and vector solitons were observed in \cite{15,16,17}.

1.1 Introduction

We study discrete vortices in the anti-continuum limit of the discrete two-dimensional non-linear Schrödinger (NLS) equations. The discrete vortices in the anti-continuum limit represent a finite set of excited nodes on a closed discrete contour with a non-zero topological charge. Using the Lyapunov–Schmidt reductions, we find sufficient conditions for continuation and termination of the discrete vortices for a small coupling constant in the discrete NLS lattice. An example of a closed discrete contour is considered that includes the vortex cell (also known as the off-site vortex). We classify the symmetric and asymmetric discrete vortices that bifurcate from the anti-continuum limit. We predict analytically and confirm numerically the number of unstable eigenvalues associated with various families of such discrete vortices.
Most recently, observations of discrete vortices were reported by two independent groups \[18, 19\] where the fundamental vortices with topological charge one were experimentally created and detected in photorefractive crystals. Two main examples of charge-one discrete vortices include a vortex cross (an on-site centered vortex) and a vortex cell (an off-site centered vortex). These structures were also recently predicted in a continuous two-dimensional model with the periodic potential \[20\].

These discoveries have stimulated further theoretical work and numerical computations. Thus, while in \[8\], discrete vortices of charge two were shown to be unstable, recently in \[21\] discrete vortices of charge three were found to be stable. Based on the theoretical predictions of \[21\], further experiments on localized structures were undertaken to unveil other interesting structures, such as the discrete soliton necklace which is more globally stable compared to the charge three vortex \[22\]. The discrete vortices have been also extended to three-dimensional discrete models \[23\]. Furthermore, asymmetric vortices have been recently predicted in the two-dimensional lattices in \[24\].

The above activity clearly signals the importance and experimental relevance of discrete solitons and vortices in two-dimensional discrete lattices. However, most of the above mentioned works are predominantly of experimental or numerical nature, while the mathematical theory of existence and stability of discrete localized structures has not been developed to a similar extent. The aim of the present paper is to develop a categorization of discrete solitons and vortices in the discrete two-dimensional nonlinear Schrödinger (NLS) equations. We start from a well-understood limit (the so-called anti-continuum case of zero coupling between the lattice nodes) and examining persistence of the limiting solutions for small coupling by means of the Lyapunov-Schmidt theory. This method allows us to discuss persistence and stability of the localized structures by analyzing finite-dimensional linear eigenvalue problems. The theoretical predictions agree well with full numerical computations of the discrete two-dimensional NLS equation.

Our main results are summarized for the simplest localized structures in Table 1. These results corroborate and extend the previously reported experimental and numerical findings. We quantify the stability of the charge-one vortex in accordance to \[8, 18, 19, 20\], the instability of the charge-two vortex in accordance to \[8, 21\] and the stability of the charge-three vortex in accordance to \[21\]. We further demonstrate the instability of all asymmetric vortices proposed in \[24\]. Furthermore, our results can be used to extract the spectral stability of the dipole mode considered in \[15\] and of the soliton necklace of \[22\].

| contour | vortex of charge L | # of unstable eigenvalues | # of stable eigenvalues | # of continuation parameters |
|---------|--------------------|--------------------------|-------------------------|-----------------------------|
| \(M = 1\) | symmetric \(L = 1\) | none | two pairs | two parameters |
| \(M = 2\) | symmetric \(L = 1\) | six complex one real | none | one parameter |
| \(M = 2\) | symmetric \(L = 2\) | one real | five pairs | two parameters |
| \(M = 2\) | symmetric \(L = 3\) | none | seven pairs | one parameter |
| \(M = 2\) | asymmetric \(L = 1\) | six real | one pair | one parameter |
| \(M = 2\) | asymmetric \(L = 2\) | three real | three pairs | two parameters |
| \(M = 2\) | asymmetric \(L = 3\) | one real | six pairs | one parameter |
Table 1: The numbers of small unstable, stable and zero eigenvalues, associated to vortices of the discrete two-dimensional NLS equation with small coupling constant.

The paper is structured as follows. Abstract results on existence of discrete solitons and vortices are derived in Section 2. Persistence of localized modes for a particular square discrete contour is considered in Section 3. Stability of the persistent solutions is addressed in Section 4. Analytical results are compared to numerical computations in Section 5. Section 6 concludes the paper with a summary of our results and discussions of interesting directions for future study.

2 Existence of discrete vortices

We consider the discrete nonlinear Schrödinger (NLS) equation in two space dimensions [5]:

\[ i\dot{u}_{n,m} + \epsilon (u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}) + |u_{n,m}|^2u_{n,m} = 0, \]

where \( u_{n,m}(t) : \mathbb{R}_+ \rightarrow \mathbb{C}, (n, m) \in \mathbb{Z}^2 \), and \( \epsilon > 0 \) is the inverse squared step size of the discrete two-dimensional NLS lattice. Time-periodic localized modes of the discrete NLS equation (2.1) take the form:

\[ u_{n,m}(t) = \phi_{n,m}e^{i(\mu - 4\epsilon)t + i\theta_0}, \quad \phi_{n,m} \in \mathbb{C}, \quad (n, m) \in \mathbb{Z}^2, \]

where \( \theta_0 \in \mathbb{R} \) and \( \mu \in \mathbb{R} \) are parameters. Since localized modes in the focusing NLS lattice (2.1) with \( \epsilon > 0 \) may exist only for \( \mu > 4\epsilon \) [4] and the parameter \( \mu \) is scaled out by the scaling transformation,

\[ \phi_{n,m} = \sqrt{\mu}\hat{\phi}_{n,m}, \quad \epsilon = \mu\hat{\epsilon}, \]

the parameter \( \mu > 0 \) will henceforth be set to \( \mu = 1 \). In this case, the complex-valued \( \phi_{n,m} \) solve the nonlinear difference equations on \((n, m)\in\mathbb{Z}^2\):

\[ (1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon (\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}). \]

As \( \epsilon = 0 \), the localized modes of the difference equations (2.4) are given by the limiting solution:

\[ \phi_{n,m}^{(0)} = \begin{cases} e^{i\theta_{n,m}}, & (n, m) \in S, \\ 0, & (n, m) \in \mathbb{Z}^2\setminus S, \end{cases} \]

where \( S \) is a finite set of nodes on the lattice \((n, m)\in\mathbb{Z}^2\) and \( \theta_{n,m} \) are parameters for \((n, m)\in S\). Since \( \theta_0 \) is arbitrary in the ansatz (2.2), we can set \( \theta_{n_0,m_0} = 0 \) for a particular node \((n_0, m_0)\in S\). Using this convention, we define two special types of localized modes, called discrete solitons and vortices.

**Definition 2.1** The localized solution of the difference equations (2.4) with \( \epsilon > 0 \), that has all real-valued amplitudes \( \phi_{n,m}, (n, m) \in \mathbb{Z}^2 \) and satisfies the limit (2.5) with all \( \theta_{n,m} = \{0, \pi\}, (n, m) \in S \), is called a discrete soliton.

**Definition 2.2** Let \( S \) be a simply-connected discrete contour on the plane \((n, m)\in\mathbb{Z}^2\). The localized solution of the difference equations (2.4) with \( \epsilon > 0 \), that has complex-valued \( \phi_{n,m}, (n, m) \in \mathbb{Z}^2 \) and satisfies the limit (2.5) with \( \theta_{n,m} \in [0, 2\pi], (n, m) \in S \), is called a discrete vortex.
Let the introduction of some relevant notations.

Abstract results on existence of such continuations are formulated and proved below, after vortices is based on the Implicit Function Theorem and Lyapunov–Schmidt Reduction Theorem \cite{26, 27}.

\( \dim(S) = 4M \). According to Definition \ref{definition2} the contour \( S_M \) for a fixed \( M \) could support symmetric and asymmetric vortices with some charge \( L \). An example of the simplest vortices for \( M = 1 \) is the symmetric charge-one vortex cell \( (\theta_{1,1} = 0, \theta_{2,1} = \frac{\pi}{2}, \theta_{2,2} = \pi, \theta_{1,2} = \frac{3\pi}{2}) \) \cite{26, 20} and an asymmetric charge-one vortex \( (\theta_{1,1} = 0, \theta_{2,1} = \theta, \theta_{2,2} = \pi, \theta_{1,2} = \pi + \theta) \), where \( \theta \neq \{0, \frac{\pi}{2}, \pi\} \) \cite{24}.

It follows from the general method \cite{2, 25} that the discrete solitons of the two-dimensional NLS lattice \cite{24} (see Definition \ref{definition2}) can be continued to the domain \( 0 < \epsilon < \epsilon_0 \) for some \( \epsilon_0 > 0 \). It is more complicated to find a configuration of \( \theta_{n,m} \) for \( (n, m) \in S \) that satisfies (i) \( \phi_{n,m} \in \mathbb{R} \), \( (n, m) \in \mathbb{Z}^2 \) and (ii) \( \lim_{\epsilon \rightarrow 0} \phi_{n,m} = \phi_{n,m}^{(0)} \), where \( \phi_{n,m}^{(0)} \) is given by \cite{20} with \( \theta_{n,m} = \{0, \pi\} \), \( (n, m) \in S \). The solution \( \phi(\epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \).

**Proposition 2.4** There exists a unique (discrete soliton) solution of the difference equations \cite{24} in the domain \( \epsilon \in \mathcal{O}(0) \) that satisfies (i) \( \phi_{n,m} \in \mathbb{R} \), \( (n, m) \in \mathbb{Z}^2 \) and (ii) \( \lim_{\epsilon \rightarrow 0} \phi_{n,m} = \phi_{n,m}^{(0)} \), where \( \phi_{n,m}^{(0)} \) is given by \cite{20} with \( \theta_{n,m} = \{0, \pi\} \), \( (n, m) \in S \). The solution \( \phi(\epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \).

**Proof.** Assume that \( \phi_{n,m} \in \mathbb{R} \) for all \( (n, m) \in \mathbb{Z}^2 \). The difference equations \cite{24} are rewrittten as zeros of the nonlinear vector-valued function:

\[
\begin{aligned}
f_{n,m}(\phi, \epsilon) &= (1 - \phi_{n,m}^2)\phi_{n,m} - \epsilon (\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}) = 0. \\
\end{aligned}
\]

(2.8)

The mapping \( f : \Omega \times \mathcal{O}(0) \rightarrow \Omega \) is \( C^1 \) on \( \phi \in \Omega \) and has a bounded continuous Fréchet derivative, given by:

\[
\mathcal{L}_{n,m} = (1 - 3\phi_{n,m}^2) - \epsilon (s_{1,0} + s_{-1,0} + s_{0,1} + s_{0,-1}),
\]

(2.9)

where \( s_{n',m'} \) is the shift operator, such that \( s_{n',m'} u_{n,m} = u_{n+n',m+m'} \). It is obvious that

\[
f(\phi^{(0)}, 0) = 0, \quad \ker(\mathcal{L}^{(0)}) = \emptyset,
\]

(2.10)
where \( \phi^{(0)} \) is the discrete soliton of Definition 2.4 and \( \mathcal{L}^{(0)} \) is the operator \( \mathcal{L} \) computed at \( \phi = \phi^{(0)} \) and \( \epsilon = 0 \). It follows from Lemma 2.2 that \( \mathcal{L}^{(0)} : \Omega \to \Omega \) has a bounded inverse. By the Implicit Function Theorem [27, Appendix 1], there exists a local \( C^1 \) mapping \( \phi : \mathcal{O}(0) \to \Omega \), such that \( \phi(\epsilon) \) is continuous in \( \epsilon \in \Omega(0) \) and \( \phi^{(0)} = \phi(0) \). Moreover, since \( f(\phi, \epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \), then \( \phi(\epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \). \( \square \)

**Remark 2.5** Proposition 2.4 does not exclude a possibility of continuation of the limiting solution (2.4) with \( \theta_{n,m} = \{0, \pi\} \) for all \((n,m) \in S\) to the complex-valued solution \( \phi(\epsilon) \) in \( \epsilon \in \mathcal{O}(0) \).

**Proposition 2.6** There exists a vector-valued function \( g : T \times \mathcal{O}(0) \to \mathbb{R}^N \), such that the limiting solution (2.5) is continued to the domain \( \epsilon \in \mathcal{O}(0) \) if and only if \( \theta \in T \) is a root of \( g(\theta, \epsilon) = 0 \) in \( \epsilon \in \mathcal{O}(0) \). Moreover, the function \( g(\theta, \epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \) and \( g(\theta, 0) = 0 \) for any \( \theta \in T \).

**Proof.** When \( \phi_{n,m} \in \mathbb{C} \) for some \((n,m) \in \mathbb{Z}^2\), the difference equations (2.4) are complemented by the complex conjugate equations in the abstract form:

\[
\begin{align*}
\mathcal{H}(\phi, \overline{\phi}, \epsilon) &= 0, \\
\mathcal{F}(\phi, \overline{\phi}, \epsilon) &= 0.
\end{align*}
\]  

(2.11)

Taking the Fréchet derivative of \( f(\phi, \overline{\phi}, \epsilon) \) with respect to \( \phi \) and \( \overline{\phi} \), we compute the linearization operator \( \mathcal{H} \) for the difference equations (2.4):

\[
\mathcal{H}_{n,m} = \begin{pmatrix}
1 - 2|\phi_{n,m}|^2 & -\phi_{n,m}^2 \\
-\overline{\phi}_{n,m}^2 & 1 - 2|\overline{\phi}_{n,m}|^2
\end{pmatrix} - \epsilon \begin{pmatrix}
s_{1,0} + s_{-1,0} + s_{0,1} + s_{0,-1} & 1 \\
0 & 1
\end{pmatrix},
\]  

(2.12)

Let \( \mathcal{H}^{(0)} = \mathcal{H}(\phi^{(0)}, 0) \). It is clear that \( \mathcal{H}^{(0)} : \Omega \times \Omega \to \Omega \times \Omega \) is a self-adjoint Fredholm operator of index zero with \( \dim \ker(\mathcal{H}^{(0)}) = N \). Moreover, eigenvectors of \( \ker(\mathcal{H}^{(0)}) \) re-normalize the parameters \( \theta_{n,m} \) for \((n,m) \in S\) in the limiting solution (2.4). By the Lyapunov Reduction Theorem [27, Chapter 7.1], there exists a decomposition \( \Omega = \ker(\mathcal{H}^{(0)}) \oplus \omega \), such that \( g(\theta, \epsilon) \) is defined in terms of the projections to \( \ker(\mathcal{H}^{(0)}) \). Let \( \{e_{n,m}\}_{(n,m) \in S} \) be a set of \( N \) linearly independent eigenvectors in the kernel of \( \mathcal{H}^{(0)} \). It follows from the representation,

\[
\mathcal{H}_{n,m}^{(0)} = -\begin{pmatrix}
1 & e^{2i\theta_{n,m}} \\
e^{-2i\theta_{n,m}} & 1
\end{pmatrix}, \quad (n,m) \in S,
\]  

(2.13)

that each eigenvector \( e_{n,m} \) in the set \( \{e_{n,m}\}_{(n,m) \in S} \) has the only non-zero element \((e^{i\theta_{n,m}}, -e^{-i\theta_{n,m}})^T\) at the \((n,m)\)-th position of \( u \in \Omega \). By projections of the nonlinear equations (2.11) to \( \ker(\mathcal{H}^{(0)}) \), we derive an implicit representation for the functions \( g(\theta, \epsilon) \):

\[
2i g_{n,m}(\theta, \epsilon) = \begin{align*}
(1 - |\phi_{n,m}|^2)(e^{-i\theta_{n,m}} \phi_{n,m} - e^{i\theta_{n,m}} \overline{\phi}_{n,m}) \\
- \epsilon e^{-i\theta_{n,m}} (\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}) \\
+ \epsilon e^{i\theta_{n,m}} (\overline{\phi}_{n+1,m} + \overline{\phi}_{n-1,m} + \overline{\phi}_{n,m+1} + \overline{\phi}_{n,m-1}),
\end{align*}
\]  

(2.14)

for \((n,m) \in S\), where the factor \((2i)\) is introduced for convenient notations. Let \( \phi_{n,m} = e^{i\theta_{n,m}} u_{n,m} \) for \((n,m) \in S\) and \( \phi_{n,m} = u_{n,m} \) for \((n,m) \in \mathbb{Z}^2\setminus S\). Since eigenvectors of \( \ker(\mathcal{H}^{(0)}) \) are excluded from the solution \( \phi \) in \( \omega \subset \Omega \), we have \( u_{n,m} \in \mathbb{R} \) for \((n,m) \in S\), such that

\[
-2i g_{n,m}(\theta, \epsilon) = \begin{align*}
\epsilon e^{-i\theta_{n,m}} (\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}) \\
- \epsilon e^{i\theta_{n,m}} (\overline{\phi}_{n+1,m} + \overline{\phi}_{n-1,m} + \overline{\phi}_{n,m+1} + \overline{\phi}_{n,m-1})
\end{align*}
\]  

(2.15)
and \( g(\theta, 0) = 0 \) for any \( \theta \in T \). Since \( f(\phi, \dot{\phi}, \epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \), then \( g(\theta, \epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \) \cite{20} Appendix 3].

**Corollary 2.7** The function \( g(\theta, \epsilon) \) can be expanded into convergent Taylor series in \( \mathcal{O}(0) \):

\[
g(\theta, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k g^{(k)}(\theta), \quad g^{(k)}(\theta) = \frac{1}{k!} \partial_{\epsilon}^k g(\theta, 0). \tag{2.16}
\]

If the root \( \theta(\epsilon) \) of \( g(\theta, \epsilon) = 0 \) is analytic in \( \epsilon \in \mathcal{O}(0) \), then the solution \( \phi(\epsilon) \) is analytic in \( \epsilon \in \mathcal{O}(0) \), such that

\[
\phi(\epsilon) = \phi^{(0)} + \sum_{k=1}^{\infty} \epsilon^k \phi^{(k)}, \tag{2.17}
\]

where \( \phi^{(0)} \) is given by \( g^{(0)} \).

**Lemma 2.8** Let \( \theta(\epsilon) \) be a root of \( g(\theta, \epsilon) = 0 \) in \( \epsilon \in \mathcal{O}(0) \). An arbitrary shift \( \theta(\epsilon) + \theta_0 p_0 \), where \( \theta_0 \in \mathbb{R} \) and \( p_0 = (1, 1, ..., 1)^T \), gives a one-parameter family of roots of \( g(\theta, \epsilon) = 0 \) for the same \( \epsilon \).

**Proof.** The statement follows from the symmetry of the difference equations \( \text{[2.14]} \) with respect to gauge transformation \( \text{[2.17]} \) Chapter 7.3].

**Proposition 2.9** Let \( \theta_* \) be the root of \( g^{(1)}(\theta) = 0 \) and \( M_1 \) be the Jacobian matrix of \( g^{(1)}(\theta) \) at \( \theta = \theta_* \). If the matrix \( M_1 \) has a simple zero eigenvalue, there exists a unique (modulo gauge transformation) analytic continuation of the limiting solution \( \text{[2.13]} \) to the domain \( \epsilon \in \mathcal{O}(0) \).

**Proof.** By Lemma \( \text{[2.8]} \) the matrix \( M_1 \) has always a non-empty kernel with the eigenvector \( p_0 = (1, 1, ..., 1) \) due to gauge transformation. Let \( X_0 \) be the constrained subspace of \( \mathbb{C}^N \):

\[
X_0 = \{ u \in \mathbb{C}^N : (p_0, u) = 0 \}. \tag{2.18}
\]

If the matrix \( M_1 \) is non-singular in the subspace \( X_0 \), then there exists a unique (modulo the shift) analytic continuation of the root \( \theta_* \) in \( \epsilon \in \mathcal{O}(0) \) by the Implicit Function Theorem, applied to the nonlinear equation \( g(\theta, \epsilon) = 0 \) \cite{20} Appendix 1].

**Proposition 2.10** Let \( \theta_* \) be a \( (1 + d) \)-parameter solution of \( g^{(1)}(\theta) = 0 \) and \( M_1 \) have a zero eigenvalue of multiplicity \( (1 + d) \), where \( 1 \leq d \leq N - 1 \). The limiting solution \( g^{(1)}(\theta) = 0 \) can be continued in the domain \( \epsilon \in \mathcal{O}(0) \) only if \( g^{(2)}(\theta_*) \) is orthogonal to \( \ker(M_1) \).

**Proof.** Let \( p_0 \) and \( \{ p_i \}_{i=1}^d \) be eigenvectors of \( \ker(M_1) \). We define the constrained subspace of \( X_0 \):

\[
X_d = \{ u \in X_0 : (p_l, u) = 0, \ l = 1, ..., d \}. \tag{2.19}
\]

If \( g^{(2)}(\theta_*) \not\in X_d \), the Lyapunov-Schmidt Reduction Theorem in finite dimensions \( \text{[2.17]} \) Chapter 1.3 shows that the solution \( \theta_* \) cannot be continued in \( \epsilon \in \mathcal{O}(0) \).}

**Proposition 2.11** gives an abstract formulation of the continuation problem for the limiting solution \( \text{[2.14]} \) for \( \epsilon \neq 0 \). Proposition \( \text{[2.10]} \) gives a sufficient condition of existence and uniqueness (up to gauge invariance) of such continuations. Proposition \( \text{[2.10]} \) gives a sufficient condition for termination of
multi-parameter solutions. Particular applications of Propositions 2.6, 2.9 and 2.10 are limited by the complexity of the set $S$ in the limiting solution (2.5), since computations of the vector-valued function $g^{(1)}(\theta)$, $g^{(2)}(\theta)$, and the Jacobian matrix $M_1$ could be technically involved. We apply the abstract results of Propositions 2.6, 2.9 and 2.10 to the simply-connected discrete contour $S_M$, defined in (2.6).

3 Persistence of discrete vortices

We consider discrete solitons and vortices on the contour $S_M$ defined by (2.6). Let the set $\theta_j$ correspond to the ordered contour $S_M$, starting at $\theta_1 = \theta_{1,1}$, $\theta_2 = \theta_{2,1}$ and ending at $\theta_N = \theta_{1,2}$, where $N = 4M$. In what follows, we use the periodic boundary conditions for $\theta_j$ on the circle, such that $\theta_0 = \theta_N$, $\theta_1 = \theta_{N+1}$, and so on.

The discrete vortex has the charge $L$ if the phase difference changes on $2\pi L$ along the discrete contour $S_M$. By gauge transformation, we can always set $\theta_1 = 0$ for convenience. We will also choose $\theta_2 = \theta$ with $0 \leq \theta \leq \pi$ for convenience, which corresponds to discrete vortices with $L \geq 0$.

3.1 Solutions of the first-order reductions

Substituting the limiting solution $\phi^{(0)}_{n,m}$ in the bifurcation function (2.15), we find that $g^{(1)}(\theta)$ in the Taylor series (2.16) is non-zero for the contour $S_M$ and it takes the form:

$$g^{(1)}_j(\theta) = \sin(\theta_j - \theta_{j+1}) + \sin(\theta_j - \theta_{j-1}), \quad 1 \leq j \leq N. \quad (3.1)$$

The bifurcation equations $g^{(1)}(\theta) = 0$ are rewritten as a system of $N$ nonlinear equations for $N$ parameters $\theta_1, \theta_2, \ldots, \theta_N$ as follows:

$$\sin(\theta_2 - \theta_1) = \sin(\theta_3 - \theta_2) = \ldots = \sin(\theta_N - \theta_{N-1}) = \sin(\theta_1 - \theta_N). \quad (3.2)$$

We classify all solutions of the bifurcation equations (3.2) and give explicit examples for $M = 1$ and $M = 2$.

**Proposition 3.1** Let $a_j = \cos(\theta_{j+1} - \theta_j)$ for $1 \leq j \leq N$, such that $\theta_1 = 0$, $\theta_2 = \theta$, and $\theta_{N+1} = 2\pi L$, where $N = 4M$, $0 \leq \theta \leq \pi$ and $L$ is the vortex charge. All solutions of the bifurcation equations (3.2) reduce to the four families:

(i) discrete solitons with $\theta = \{0, \pi\}$ and

$$\theta_j = \{0, \pi\}, \quad 1 \leq j \leq N, \quad (3.3)$$

such that the set $\{a_j\}_{j=1}^N$ includes $l$ coefficients $a_j = 1$ and $N - l$ coefficients $a_j = -1$, where $0 \leq l \leq N$.

(ii) symmetric vortices of charge $L$ with $\theta = \frac{\pi L}{2M}$, where $1 \leq L \leq 2M - 1$, and

$$\theta_j = \frac{\pi L(j-1)}{2M}, \quad 1 \leq j \leq N, \quad (3.4)$$

such that all $N$ coefficients are the same: $a_j = a = \cos\left(\frac{\pi L}{2M}\right)$.
(iii) one-parameter families of asymmetric vortices of charge \( L = M \) with \( 0 < \theta < \pi \) and

\[
\theta_{j+1} - \theta_j = \left\{ \begin{array}{ll}
\theta & \text{mod}(2\pi), \\
\pi - \theta & \end{array} \right. \quad 2 \leq j \leq N,
\]

such that the set \( \{a_j\}_{j=1}^N \) includes \( 2M \) coefficients \( a_j = \cos \theta \) and \( 2M \) coefficients \( a_j = -\cos \theta \).

(iv) zero-parameter asymmetric vortices of charge \( L \neq M \) and

\[
\theta = \theta_* = \frac{\pi}{2} \left( \frac{n + 2L - 4M}{n - 2M} \right), \quad 1 \leq n \leq N - 1, \ n \neq 2M,
\]

such that the set \( \{a_j\}_{j=1}^N \) includes \( n \) coefficients \( a_j = \cos \theta_* \) and \( N - n \) coefficients \( a_j = -\cos \theta_* \) and the family (iv) does not reduce to any of the families (i)–(iii).

**Proof.** All solutions of the bifurcation equations (3.2) are given by the binary choice (3.5) in the two roots of the sine-function on \( \theta \in [0, 2\pi] \), where the first choice gives \( a_j = \cos \theta \) and the second choice gives \( a_j = -\cos \theta \). Let us assume that there are totally \( n \) first choices and \( N - n \) second choices, where \( 1 \leq n \leq N \). Then, we have

\[
\theta_{N+1} = n\theta + (N-n)(\pi - \theta) = (2n - N)\theta + (N - n)\pi = 2\pi L,
\]

where \( L \) is the integer charge of the discrete vortex. There are only two solutions of the above equation. When \( \theta \) is arbitrary parameter, we have \( n = \frac{N}{2} \) and \( L = M \), which gives the one-parameter family (iii). When \( \theta = \theta_* \) is fixed, we have

\[
\theta_* = \frac{\pi}{2} \left( \frac{n + 2L - 4M}{n - 2M} \right)
\]

When \( n = N - 2L \), we have the family (i) with \( N - 2L \) phases \( \theta_j = 0 \) and \( 2L \) phases \( \theta_j = \pi \). Since discrete solitons do not have topological charge, the parameter \( L \) could be half-integer: \( L = (N - l)/2 \), where \( 0 \leq l \leq N \). When \( n = 4M \), we have the family (ii) for any \( 1 \leq L \leq 2M - 1 \). Other choices of \( n \), which are irreducible to the families (i)–(iii), produce the family (iv).

**Remark 3.2** The one-parameter family (iii) connects special solutions of the families (i) and (ii). When \( \theta = 0 \) and \( \theta = \pi \), the family (iii) reduces to the family (i) with \( l = 2M \). When \( \theta = \frac{\pi}{2} \), the family (iii) reduces to the family (ii) with \( L = M \). We shall call the corresponding solutions of family (i) as the super-symmetric soliton and of family (ii) as the super-symmetric vortex.

**Remark 3.3** There exist \( N_1 = 2^{N-1} \) solutions of family (i), \( N_2 = 2M - 1 \) solutions of family (ii), and \( N_3 \) solutions of family (iii), where

\[
N_3 = 2^{N-1} - \sum_{k=0}^{2M-1} \frac{N!}{k!(N-k)!}.
\]

The number \( N_4 \) of solutions of family (iv) can not be computed in general. We consider such solutions only in the explicit examples of \( M = 1 \) and \( M = 2 \).

**Example** \( M = 1 \) and \( N = 4 \): There are \( N_1 = 8 \) solutions of family (i), \( N_2 = 1 \) solution of family (ii), \( N_3 = 3 \) solutions of family (iii), and no solutions of family (iv). The three one-parameter
asymmetric vortices are given explicitly by

\[(a) \, \theta_1 = 0, \, \theta_2 = \theta, \, \theta_3 = \pi, \, \theta_4 = \pi + \theta \]  
\[(b) \, \theta_1 = 0, \, \theta_2 = \theta, \, \theta_3 = 2\theta, \, \theta_4 = \pi + \theta \]  
\[(c) \, \theta_1 = 0, \, \theta_2 = \theta, \, \theta_3 = \pi, \, \theta_4 = 2\pi - \theta. \]  

(3.8)

(3.9)

(3.10)

**Example** \(M = 2\) and \(N = 8\): There are \(N_1 = 128\) solutions of family (i), \(N_2 = 3\) solutions of family (ii), \(N_3 = 35\) solutions of family (iii), and \(N_4 = 14\) solutions of family (iv). The three symmetric vortices have topological charge \(L = 1\) (\(\theta = \frac{\pi}{2}\)), \(L = 2\) (\(\theta = \frac{\pi}{2}\)), and \(L = 3\) (\(\theta = \frac{3\pi}{4}\)).

The one-parameter asymmetric vortices include 35 combinations of 4 upper choices and 4 lower choices in (3.10), starting with the following three solutions:

\[(a) \, \theta_1 = 0, \, \theta_2 = \theta, \, \theta_3 = 2\theta, \, \theta_4 = 3\theta, \, \theta_5 = 4\theta, \, \theta_6 = \pi + 3\theta, \, \theta_7 = 2\pi + 2\theta, \, \theta_8 = 3\pi + \theta, \]

\[(b) \, \theta_1 = 0, \, \theta_2 = \theta, \, \theta_3 = 2\theta, \, \theta_4 = 3\theta, \, \theta_5 = \pi + 2\theta, \, \theta_6 = \pi + 3\theta, \, \theta_7 = 2\pi + 2\theta, \, \theta_8 = 3\pi + \theta, \]

\[(c) \, \theta_1 = 0, \, \theta_2 = \theta, \, \theta_3 = 2\theta, \, \theta_4 = 3\theta, \, \theta_5 = \pi + 2\theta, \, \theta_6 = 2\pi + \theta, \, \theta_7 = 2\pi + 2\theta, \, \theta_8 = 3\pi + \theta, \]

and so on. The zero-parameter asymmetric vortices include 7 combinations of vortices with \(L = 1\) for seven phase differences \(\frac{\pi}{6}\) and one phase difference \(\frac{\pi}{6}\), and 7 combinations of vortices with \(L = 3\) for one phase difference \(\frac{\pi}{6}\) and seven phase differences \(\frac{\pi}{6}\).

### 3.2 Continuation of solutions of the first-order reductions

We compute the Jacobian matrix \(\mathcal{M}_1\) from the bifurcation function \(g^{(1)}(\theta)\), given in (3.11):

\[(\mathcal{M}_1)_{i,j} = \begin{cases} 
\cos(\theta_{j+1} - \theta_j) + \cos(\theta_{j-1} - \theta_j), & i = j, \\
-\cos(\theta_j - \theta_i), & i = j \pm 1 \\
0, & |i-j| \geq 2
\end{cases} \quad (3.11)\]

subject to the periodic boundary conditions. The matrix \(\mathcal{M}_1\) is defined by the coefficients \(a_j = \cos(\theta_{j+1} - \theta_j)\) for \(1 \leq j \leq N\). It has the same structure as that in the perturbation theory of continuous multi-pulse solitons in coupled NLS equations [28]. Three technical results establish location of eigenvalues of the matrix \(\mathcal{M}_1\).

**Lemma 3.4** Let \(n_0, \, z_0, \, \text{and} \, p_0\) be the numbers of negative, zero and positive terms of \(a_j = \cos(\theta_{j+1} - \theta_j)\), \(1 \leq j \leq N\), such that \(n_0 + z_0 + p_0 = N\). Let \(n(\mathcal{M}_1), \, z(\mathcal{M}_1), \, \text{and} \, p(\mathcal{M}_1)\) be the numbers of negative, zero and positive eigenvalues of the matrix \(\mathcal{M}_1\), defined by (3.11). Assume that \(z_0 = 0\) and denote:

\[A_1 = \sum_{i=1}^{N} \prod_{j \neq i} a_j = \left( \prod_{i=1}^{N} a_i \right) \left( \sum_{i=1}^{N} \frac{1}{a_i} \right). \quad (3.12)\]

If \(A_1 \neq 0\), then, \(z(\mathcal{M}_1) = 1, \, \text{and} \, \text{either} \, n(\mathcal{M}_1) = n_0 - 1, \, p(\mathcal{M}_1) = p_0 \) or \(n(\mathcal{M}_1) = n_0, \, p(\mathcal{M}_1) = p_0 - 1\). Moreover, \(n(\mathcal{M}_1)\) is even if \(A_1 > 0\) and is odd if \(A_1 < 0\). If \(A_1 = 0\), then \(z(\mathcal{M}_1) \geq 2\).

**Proof.** The first statement follows from Appendix A of [28]. Let the determinant equation be \(D(\lambda) = \det(\mathcal{M}_1 - \lambda I) = 0\). By induction arguments in [28, 29], it can be found that \(D(0) = 0\) and \(D'(0) = -NA_1\). On the other hand, \(D'(0) = -\lambda_1 \lambda_2 \cdots \lambda_{N-1}, \, \text{where} \, \lambda_N = 0\) (which exists always with the eigenvector \(p_0 = (1, 1, ..., 1)^T\), see Proposition [29]). Then, it is clear that \((-1)^n(M_1) = \text{sign}(A_1)\). When \(A_1 = 0\), at least one more eigenvalue is zero, such that \(z(\mathcal{M}_1) \geq 2\).
Lemma 3.5 Let all coefficients \( a_j = \cos(\theta_{j+1} - \theta_{j}) \), \( 1 \leq j \leq N \) be the same: \( a_j = a \). Eigenvalues of the matrix \( \mathcal{M}_1 \) are computed explicitly as follows:

\[
\lambda_n = 4a \sin^2 \frac{\pi n}{N}, \quad 1 \leq n \leq N. \tag{3.13}
\]

**Proof.** When \( a_j = a \), \( 1 \leq j \leq N \), the eigenvalue problem for the matrix \( \mathcal{M}_1 \) takes the form of the linear difference equation with constant coefficients:

\[
a(2x_j - x_{j+1} - x_{j-1}) = \lambda x_j, \quad x_0 = x_N, \ x_1 = x_{N+1}, \tag{3.14}
\]

The discrete Fourier mode \( x_j = \exp \left( \frac{2 \pi i j}{N} \right) \) for \( 1 \leq j, n \leq N \) results in the solution \( 3.13 \).

Lemma 3.6 Let all coefficients \( a_j = \cos(\theta_{j+1} - \theta_{j}) \), \( 1 \leq j \leq N \) alternate the sign: \( a_j = (-1)^j a \), where \( N = 4M \). Eigenvalues of the matrix \( \mathcal{M}_1 \) are computed explicitly as follows:

\[
\lambda_n = -\lambda_{n+2M} = 2a \sin \frac{\pi n}{2M}, \quad 1 \leq n \leq 2M, \tag{3.15}
\]

such that \( n(\mathcal{M}_1) = 2M - 1 \), \( z(\mathcal{M}_1) = 2 \), and \( p(\mathcal{M}_1) = 2M - 1 \). These numbers do not change if the set \( \{a_j\}_{j=1}^{N}\ ) is obtained from the sign-alternating set by permutations.

**Proof.** When \( a_j = (-1)^j a \), \( 1 \leq j \leq 4M \), the eigenvalue problem for the matrix \( \mathcal{M}_1 \) takes the form of a coupled system of linear difference equation with constant coefficients:

\[
a(y_j - y_{j-1}) = \lambda x_j, \quad a(x_j - x_{j+1}) = \lambda y_j, \quad 1 \leq j \leq 2M, \tag{3.16}
\]

subject to the periodic boundary conditions: \( x_1 = x_{2M+1} \) and \( y_0 = y_{2M} \). The discrete Fourier mode \( x_j = x_0 \exp \left( \frac{2 \pi i j}{2M} \right) \) and \( y_j = y_0 \exp \left( \frac{2 \pi i j}{2M} \right) \) for \( 1 \leq j, n \leq 2M \) results in the solution \( 3.15 \). In this case, we have \( n(\mathcal{M}_1) = 2M - 1 \), \( z(\mathcal{M}_1) = 2 \), and \( p(\mathcal{M}_1) = 2M - 1 \), such that \( D(0) = D'(0) = 0 \) in the determinant equation \( D(\lambda) = \det(\mathcal{M}_1 - \lambda I) \). In order to prove that \( z(\mathcal{M}_1) = 2 \) remains invariant with respect to permutations of the sign-alternating set \( \{a_j\}_{j=1}^{N} \), we find from Mathematica that

\[
D''(0) = \left( \prod_{i=1}^{N} a_i \right) \left( \alpha_N \left( \sum_{i=1}^{N-1} \frac{1}{a_i a_{i+1}} + \frac{1}{a_1 a_N} \right) + \beta_N \left( \sum_{i=1}^{N-2} \sum_{i=1}^{N} \frac{1}{a_i a_{i+1}} - \frac{1}{a_1 a_N} \right) \right), \tag{3.17}
\]

where \( 0 < \alpha_N < \beta_N \) are numerical coefficients. Let \( A_* \) denote the sign-alternating set \( \{a_j\}_{j=1}^{N} \), such that \( a_j = (-1)^j a \), and \( A \) denote a set obtained from \( A_* \) by permutations. It is clear that

\[
\left( \sum_{i=1}^{N-1} \frac{1}{a_i a_{i+1}} + \frac{1}{a_1 a_N} \right)_{A_*} \leq \left( \sum_{i=1}^{N-1} \frac{1}{a_i a_{i+1}} + \frac{1}{a_1 a_N} \right)_{A},
\]

and

\[
\left( \sum_{i=1}^{N-1} \sum_{l=i+1}^{N} \frac{1}{a_i a_l} \right)_{A_*} = \left( \sum_{i=1}^{N-1} \sum_{l=i+1}^{N} \frac{1}{a_i a_l} \right)_{A}.
\]

Therefore, the expression in brackets in \( 3.17 \) can be estimated as follows:

\[
(\alpha_N - \beta_N) \left( \sum_{i=1}^{N-1} \frac{1}{a_i a_{i+1}} + \frac{1}{a_1 a_N} \right)_{A_*} + \beta_N \left( \sum_{i=1}^{N-1} \sum_{l=i+1}^{N} \frac{1}{a_i a_l} \right)_{A_*} \leq
\]

\[
(\alpha_N - \beta_N) \left( \sum_{i=1}^{N-1} \frac{1}{a_i a_{i+1}} + \frac{1}{a_1 a_N} \right)_{A_*} + \beta_N \left( \sum_{i=1}^{N-1} \sum_{l=i+1}^{N} \frac{1}{a_i a_l} \right)_{A_*} < 0,
\]

where the last inequality follows from the fact that \( D''(0) < 0 \) for \( A_* \). Therefore, \( z(M_1) = 2 \) for \( A \). Combining it with estimates from Appendix A in \( 23 \), we have \( n(M_1) = p(M_1) = 2M - 1 \)
Using Lemmas 3.4 and 3.9 we classify the continuation of solutions of the first-order reductions, which are described in the families (i)–(iv) of Proposition 3.1.

For family (i), excluding the case of super-symmetric solitons (see Remark 2.8), the numbers of positive and negative signs of $a_j$ are different, such that the conditions $z_0 = 0$ and $A_1 \neq 0$ are satisfied in Lemma 3.4 and hence $z(M_1) = 1$. By Proposition 2.9 the family (i) has a unique continuation to discrete solitons (see Definition 2.1). Continuations described in Remark 2.8 are only possible for super-symmetric solitons.

For family (ii), all coefficients $a_j$ are the same: $a_j = a = \cos \left( \frac{\pi j}{N} \right)$, $1 \leq j \leq N$. By Lemma 3.5 there is always a zero eigenvalue ($\lambda_N = 0$), while remaining $(N - 1)$ eigenvalues are all positive for $a > 0$ (when $1 \leq L \leq M - 1$), negative for $a < 0$ (when $M + 1 \leq L \leq 2M - 1$), and zero for $a = 0$ (when $L = M$). By Proposition 2.9 the family (ii) has a unique continuation to symmetric vortices with charge $L$, where $1 \leq L \leq 2M - 1$ and $L \neq M$ (see Definitions 2.2 and 2.3).

For family (iii), there are $2M$ coefficients $a_j = \cos \theta$ and $2M$ coefficients $a_j = -\cos \theta$, which are non-zero for $\theta \neq \frac{\pi}{2}$. By Lemma 3.6 we have $n(M_1) = 2M - 1$, $z(M_1) = 2$, and $p(M_1) = 2M - 1$. The additional zero eigenvalue is related to the derivative of the family of the asymmetric discrete vortices 3.3 with respect to the parameter $\theta$. Therefore, continuations of family (iii) of asymmetric vortices, including the particular cases of super-symmetric solitons of family (i) and super-symmetric vortices of family (ii), must be considered beyond the first-order reductions.

For family (iv), since $n \neq 2M$, the conditions $z_0 = 0$ and $A_1 \neq 0$ are satisfied in Lemma 3.4 and hence $z(M_1) = 1$. By Proposition 2.9 the family (iv) has a unique continuation to asymmetric vortices for $\epsilon \neq 0$.

### 3.3 Continuation of solutions to the second-order reductions

Results of the first-order reductions are insufficient to conclude persistence of the asymmetric vortices of family (iii), including the super-symmetric soliton of family (i) and the super-symmetric vortex of family (ii). Therefore, we continue the bifurcation function $g(\theta, \epsilon)$ to the second order of $\epsilon$ in the Taylor series (3.17). It follows from (3.17) that the first-order correction of the Taylor series (3.17) satisfies the inhomogeneous problem:

\[
(1 - 2[\phi_{n,m}^{(0)}]^2)\phi_{n,m}^{(1)} - \phi_{n,m}^{(0)}\phi_{n,m}^{(1)} = \phi_{n+1,m}^{(0)} + \phi_{n-1,m}^{(0)} + \phi_{n,m+1}^{(0)} + \phi_{n,m-1}^{(0)}. \tag{3.18}
\]

We define solution of the inhomogeneous problem (3.18) in $\omega \subset \Omega$, such that the homogeneous solutions in ker($\mathcal{H}^{(0)}$) are removed from the solution $\phi^{(1)}$. This is equivalent to the constraint: $\phi_{n,m} = u_{n,m}e^{i\theta_{n,m}}$, $u_{n,m} \in \mathbb{R}$ for all $(n, m) \in S_M$. We develop computations for three distinct cases: $M = 1$, $M = 2$ and $M \geq 3$. This separation is due to the special structure of the discrete contours $S_M$.

**Case $M = 1$:** The inhomogeneous problem (3.18) has a unique solution $\phi^{(1)} \in \omega \subset \Omega$:

\[
\phi_{n,m}^{(1)} = -\frac{1}{2} \left[ \cos(\theta_{j-1} - \theta_j) + \cos(\theta_{j+1} - \theta_j) \right] e^{i\theta_j}, \tag{3.19}
\]

where the index $j$ enumerates the node $(n, m)$ on the contour $S_M$,

\[
\phi_{n,m}^{(1)} = e^{i\theta_j}, \tag{3.20}
\]
where the node \((n, m)\) is adjacent to the \(j\)-th node on the contour \(S_M\), while \(\phi_{n,m}^{(1)}\) is empty for all remaining nodes. By substituting the first-order correction term \(\phi_{n,m}^{(1)}\) into the bifurcation function \(g_{n,m}^{(1)}\), we find the correction term \(g_{j}^{(2)}(\theta)\) in the Taylor series \(g_j(\theta)\):

\[
g_j^{(2)}(\theta) = \frac{1}{2} \sin(\theta_{j+1} - \theta_j) [\cos(\theta_j - \theta_{j+1}) + \cos(\theta_{j+2} - \theta_{j+1})] \\
+ \frac{1}{2} \sin(\theta_{j-1} - \theta_j) [\cos(\theta_j - \theta_{j-1}) + \cos(\theta_{j-2} - \theta_{j-1})], \quad 1 \leq j \leq N. \tag{3.21}
\]

We compute the vector \(g_2 = g_j^{(2)}(\theta)\) at the asymmetric vortex solutions \(3.8\)–\(3.10\):

(a) \(g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\), (b) \(g_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}\) \(\sin \theta \cos \theta\), (c) \(g_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}\) \(\sin \theta \cos \theta\).

The kernel of \(M_1\) is two-dimensional with the eigenvectors \(p_0\) and \(p_1\). The second eigenvector \(p_1\) is related to derivatives of the solutions \(3.8\)–\(3.10\) in \(\theta\):

(a) \(p_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\), (b) \(p_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}\), (c) \(p_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\).

The Fredholm alternative \((p_1, g_2) = 0\) is satisfied for the solution (a) but fails for the solutions (b) and (c), unless \(\theta = \{0, \frac{\pi}{2}, \pi\}\). The latter cases are included in the definitions of super-symmetric discrete solitons and vortices (see Remark \(3.2\)). By Proposition \(2.10\) the solutions (b) and (c) can not be continued in \(\epsilon \neq 0\), while the solution (a) can be continued up to the second-order reductions.

**Case** \(M = 2\): The solution \(\phi^{(1)} \in \omega \subset \Omega\) of the inhomogeneous problem \(3.18\) is given by \(3.19\) and \(3.20\), except for the center node \((2,2)\), where

\[
\phi_{2,2}^{(1)} = e^{i\theta_2} + e^{i\theta_4} + e^{i\theta_6} + e^{i\theta_8}. \tag{3.22}
\]

The correction term \(g_j^{(2)}(\theta)\) is given by \(3.21\) but the even entries are modified as follows:

\[
g_j^{(2)}(\theta) \rightarrow g_j^{(2)}(\theta) + \sin(\theta_j - \theta_{j-2}) + \sin(\theta_j - \theta_{j+2}) + \sin(\theta_j - \theta_{j+4}), \quad j = 2, 4, 6, 8. \tag{3.23}
\]

The vector \(g_2 = g^{(2)}(\theta)\) can be computed for each of 35 one-parameter asymmetric vortex solutions, starting with the first three solutions:

(a) \(g_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ 0 \end{pmatrix}\) \(\sin \theta \cos \theta\), (b) \(g_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}\) \(\sin \theta \cos \theta\), (c) \(g_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\) \(\sin \theta \cos \theta\).
The second eigenvector $p_1$ of the kernel of $M_1$ is related to derivatives of the family in $\theta$, e.g.

\[
\begin{align*}
(a) \ p_1 &= \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \\ 2 \\ 1 \end{pmatrix}, & (b) \ p_1 &= \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{pmatrix}, & (c) \ p_1 &= \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{pmatrix}.
\end{align*}
\]

The Fredholm alternative condition $(p_1, g_2) = 0$ fails for all solutions of family (iii) but one, excluding the special values $\theta = \{0, \frac{\pi}{2}, \pi \}$. The only solution of family (iii), where $g_2 = 0$, is characterized by the alternating signs of coefficients $a_j = \cos(\theta_{j+1} - \theta_j)$ for $1 \leq j \leq N$.

**Case $M \geq 3$:** The solution $\phi^{(1)}(\omega \subset \Omega)$ of the inhomogeneous problem given by (3.18) is given by (3.19) and (3.20), except for the four interior corner nodes $(2, 2), (M, 2), (M, M)$, and $(2, M)$, where

\[
\phi^{(1)}_{n,m} = e^{i\theta_{j} - 1} + e^{i\theta_{j} + 1}, \quad j = 1, M + 1, 2M + 1, 3M + 1. \tag{3.24}
\]

The correction term $g^{(2)}(\theta)$ is given by (3.24), except for the adjacent entries to the four corner nodes on the contour $S_M$: $(1, 1), (1, M + 1), (M + 1, M + 1)$, and $(M + 1, 1)$, which are modified by:

\[
\begin{align*}
g^{(2)}_j(\theta) &\to g^{(2)}_j(\theta) + \sin(\theta_j - \theta_{j-2}), \quad j = 2, M + 2, 2M + 2, 3M + 2, \\
g^{(2)}_j(\theta) &\to g^{(2)}_j(\theta) + \sin(\theta_j - \theta_{j+2}), \quad j = M, 2M, 3M, 4M. \tag{3.25}
\end{align*}
\]

Again, there is only one solution of family (iii), where $g_2 = 0$, which is characterized by the alternating signs of coefficients $a_j = \cos(\theta_{j+1} - \theta_j)$ for $1 \leq j \leq N$. All other solutions of family (iii) do not satisfy the Fredholm alternative condition $(p_1, g_2) = 0$.

By using results of these computations, we classify continuations of solutions of the super-symmetric solitons of family (i) and asymmetric vortices of family (iii). Let $M_2$ be the Jacobian matrix computed from the bifurcation function $g^{(2)}(\theta)$, given in (3.24), (3.26) and (3.25). Since $(p_1, g_2) \neq 0$ for $\theta \neq \{0, \frac{\pi}{2}, \pi \}$, except for the case of sign-alternating set $\{a_j\}_{j=1}^N$ with $a_j = (-1)^j a$, it follows from regular perturbation theory that $(p_1, M_2 p_1) \neq 0$. Therefore, the second zero eigenvalue of $M_1$ bifurcates off zero for the matrix $M_1 + \epsilon M_2$. By Proposition (2.40) (which needs to be modified for the Jacobian matrix $M_1 + \epsilon M_2$), the super-symmetric solutions of family (i), which are different from sign-alternating sets $a_j = (-1)^j a$, are uniquely continued to discrete solitons (see Definition (2.41)).

By Proposition (2.10) all asymmetric vortices of family (iii), except for the sign-alternating set $a_j = \cos(\theta_{j+1} - \theta_j) = (-1)^{j+1} \cos \theta$, $1 \leq j \leq N$, can not be continued to $\epsilon \neq 0$. The only solution which can be continued up to the second-order reductions has the explicit form:

\[
\theta_{4j-3} = 2\pi (j - 1), \quad \theta_{4j-2} = \theta_{4j-3} + \theta, \quad \theta_{4j-1} = \theta_{4j-3} + \pi, \quad \theta_{4j} = \theta_{4j-3} + \pi + \theta, \tag{3.26}
\]

where $1 \leq j \leq M$ and $0 \leq \theta \leq \pi$. This solution includes two particular cases of super-symmetric solitons of family (i) for $\theta = 0$ and $\theta = \pi$ and super-symmetric vortices of family (ii) for $\theta = \frac{\pi}{2}$.

Continuation of the solution (3.26) must be considered beyond the second-order reductions.
3.4 Jacobian matrix of the second-order reductions

The Jacobian matrix $\mathcal{M}_1$ of the first-order reductions is empty for super-symmetric vortices of family (ii) with $L = M$. In order to study stability of super-symmetric vortices, we need to compute the Jacobian matrix $\mathcal{M}_2$ from the second-order bifurcation function $g^{(2)}(\theta)$, given in (3.21), (3.23), and (3.25). These computations are developed separately for three cases $M = 1$, $M = 2$, and $M \geq 3$.

**Case $M = 1$:** Non-zero elements of $\mathcal{M}_2$ are given by:

$$
(M_2)_{i,j} = \begin{cases} 
+1, & i = j, \\
-\frac{1}{2}, & i = j \pm 2 \\
0, & |i - j| \neq 0,2
\end{cases} \quad (3.27)
$$

or explicitly for $N = 4$:

$$
\mathcal{M}_2 = \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
\end{pmatrix}.
$$

(3.28)

The matrix $\mathcal{M}_2$ has four eigenvalues: $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = \lambda_4 = 0$. The two eigenvectors for the zero eigenvalue are $p_3 = (1, 0, 1, 0)^T$ and $p_4 = (0, 1, 0, 1)^T$. The eigenvector $p_4$ corresponds to the derivative of the asymmetric vortex (3.8) with respect to parameter $\theta$, while the eigenvector $p_0 = p_3 + p_4$ corresponds to the shift due to gauge transformation.

**Case $M = 2$:** The Jacobian matrix $\mathcal{M}_2$ is given in (3.27) except for the even entries which are modified as follows:

$$
(M_2)_{i,j} \rightarrow (M_2)_{i,j} + \begin{cases} 
-1, & i = j, \\
+1, & i = j \pm 2 \\
-1, & i = j \pm 4
\end{cases} \quad (j = 2, 4, 6, 8).
$$

(3.29)

The matrix $\mathcal{M}_2$ for $N = 8$ takes the explicit form:

$$
\mathcal{M}_2 = \begin{pmatrix}
1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & -1 \\
0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2} & 0 \\
0 & -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\
0 & \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} & 0 & 0
\end{pmatrix}.
$$

(3.30)

The eigenvalue problem for $\mathcal{M}_2$ decouples into two linear difference equations with constant coefficients:

$$
2x_j - x_{j+1} - x_{j-1} = 2\lambda x_j, \quad j = 1, 2, 3, 4
$$

$$
-2y_{j+2} + y_{j+1} + y_{j-1} = 2\lambda y_j, \quad j = 1, 2, 3, 4,
$$

subject to the periodic boundary conditions for $x_j$ and $y_j$. By the discrete Fourier transform, see the proof of Lemma 3.6, the first problem has eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 1$, and $\lambda_4 = 0$, while the
The eigenvalue problem for $M$ or $S$ related by the Toeplitz transformation. As a result, the spectra of these two matrices are identical.

The second problem has eigenvalues: $\lambda_5 = 1$, $\lambda_6 = -2$, $\lambda_7 = 1$, and $\lambda_8 = 0$. The two eigenvectors for the zero eigenvalue are $p_4 = (1,0,1,0,1,0,0,0)^T$ and $p_8 = (0,1,0,1,0,1,0,0)^T$, where the eigenvector $p_8$ corresponds to the derivative of the asymmetric vortex (3.26) with respect to parameter $\theta$ and the eigenvector $p_0 = p_4 + p_8$ corresponds to the shift due to gauge transformation.

**Case $M \geq 3$:** The Jacobian matrix $M_2$ is given in (3.27), except for the adjacent entries to the four corner nodes on the contours $S_M$: $(1,1)$, $(1, M + 1)$, $(M + 1, M + 1)$, and $(M + 1, 1)$, which are modified by

$$(M_2)_{i,j} \rightarrow (M_2)_{i,j} + \begin{cases} 
-1, & i = j = 2, M, M + 2, 2M, 2M + 2, 3M, 3M + 2, 4M, \\
+1, & i = j = 2, M, 2M, 3M, 4M \\
+1, & i = j + 2 = 2, M + 2, 2M, 2M + 2, 3M + 2 
\end{cases} \quad (3.31)$$

The matrix $M_2$ for $N = 12$ takes the explicit form:

$$M_2 = \begin{pmatrix}
 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
 -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\
 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}. \quad (3.32)$$

The eigenvalue problem for $M_2$ decouples into eigenvalue problems for two 6-by-6 matrix, which are related by the Toeplitz transformation. As a result, the spectra of these two matrices are identical with the eigenvalues, obtained with the use of MATLAB:

$$\lambda_1 = \lambda_7 = -0.780776, \quad \lambda_2 = \lambda_8 = -0.5, \quad \lambda_3 = \lambda_9 = 0, \quad \lambda_4 = \lambda_{10} = 0.5, \quad \lambda_5 = \lambda_{11} = 1.28078, \quad \lambda_6 = \lambda_{12} = 1.5.$$ 

We confirm that the matrix $M_2$ has exactly two zero eigenvalues, one of which is related to the derivative of the asymmetric vortex (3.26) in $\theta$ and the other one is related to the shift due to gauge transformation.

Computations of the matrix $M_2$ for super-symmetric vortices of family (ii) confirm the results of the second-order reductions for asymmetric vortices of family (iii). Although all $N_3$ solutions of family (iii) reduce to the super-symmetric vortex of family (ii) in the first-order reductions, it is the only family (3.26) that survives in the second-order reductions, such that the super-symmetric vortex of family (ii) with $L = M$ and $\theta = \frac{\pi}{2}$ can be deformed and continued up to the second-order reductions to the asymmetric vortex (3.20).

All individual results on persistence of localized modes on the discrete contour $S_M$ are summarized as follows.
Proposition 3.7 Consider the discrete soliton and vortices of the nonlinear equations (2.4) that bifurcate from the limiting solution \( \phi^{(0)}_{n,m} \) in (2.4) on the discrete contour \( S_M \) in (2.4). There exists a unique (modulo gauge transformation) continuation to the domain \( \epsilon \in \mathcal{O}(0) \) of discrete solitons of family (i) in (3.3), except for the case \( l = 2M \) and \( a_j = (-1)^j a \), of symmetric vortices of family (ii) in (3.4), except for the case \( L = M \), and of zero-parameter asymmetric vortices of family (iv) in (3.6). Asymmetric vortices of family (iii) in (3.4) cannot be continued to the domain \( \epsilon \in \mathcal{O}(0) \), except for the only solution (3.26).

Hypothesis 3.8 There exists a one-parameter (modulo gauge transformation) continuation to the domain \( \epsilon \in \mathcal{O}(0) \) of the family of asymmetric vortices (3.4) with \( 0 \leq \theta \leq \pi \), which includes the one-parameter continuation of the two exceptions of Proposition 3.7.

We will not be proving Hypothesis 3.8 beyond the second-order reductions. Instead, we use Proposition 3.7 to study stability of persistent localized modes of the discrete NLS equation (2.4).

4 Stability of discrete vortices

The spectral stability of discrete vortices is studied with the standard linearization:

\[
\begin{align*}
    u_{n,m}(t) &= e^{i(1-4\epsilon)t+i\theta_0} \left( \phi_{n,m} + a_{n,m}e^{\lambda t} + b_{n,m}e^{\bar{\lambda}t} \right), \quad (n, m) \in \mathbb{Z}^2, \\
\end{align*}
\]

where \( \lambda \in \mathbb{C} \) and \( (a_{n,m}, b_{n,m}) \in \mathbb{C}^2 \) solve the linear eigenvalue problem on \( (n, m) \in \mathbb{Z}^2 \):

\[
\begin{align*}
    (1 - 2|\phi_{n,m}|^2) a_{n,m} - \phi_{n,m}^2 b_{n,m} - \epsilon (a_{n+1,m} - a_{n-1,m} + a_{n,m+1} - a_{n,m-1}) &= i\lambda a_{n,m}, \\
    -\phi_{n,m}^2 a_{n,m} + (1 - 2|\phi_{n,m}|^2) b_{n,m} - \epsilon (b_{n+1,m} - b_{n-1,m} + b_{n,m+1} - b_{n,m-1}) &= -i\lambda b_{n,m}.
\end{align*}
\]

The stability problem (4.2) can be formulated in the matrix-vector form:

\[
\mathcal{H}\psi = i\lambda \sigma \psi,
\]

where \( \psi \in \Omega \times \Omega \) consists of 2-blocks of \( (a_{n,m}, b_{n,m})^T \), \( \mathcal{H} \) is defined by the linearization operator (4.2), and \( \sigma \) consists of 2-by-2 blocks of

\[
\begin{pmatrix}
    1 & 0 \\
    0 & -1
\end{pmatrix}
\]

The discrete vortex is called spectrally unstable if there exists \( \lambda \) and \( \psi \in \Omega \times \Omega \) in the problem (4.2), such that \( \text{Re}(\lambda) > 0 \). Otherwise, the discrete vortex is called weakly spectrally stable. By using the Taylor series (2.6), the linearized operator \( \mathcal{H} \) is expanded as follows:

\[
\mathcal{H} = \mathcal{H}^{(0)} + \epsilon \mathcal{H}^{(1)} + \epsilon^2 \mathcal{H}^{(2)} + O(\epsilon^3),
\]

where \( \mathcal{H}^{(0)} \) is defined in (2.6), while the first-order and second-order corrections are given by

\[
\mathcal{H}^{(1)}_{n,m} = -2 \begin{pmatrix}
    \phi_{n,m}^{(0)} \phi_{n,m}^{(1)} + \phi_{n,m}^{(0)} \bar{\phi}_{n,m}^{(1)} \\
    \bar{\phi}_{n,m}^{(0)} \phi_{n,m}^{(1)} + \phi_{n,m}^{(0)} \bar{\phi}_{n,m}^{(1)}
\end{pmatrix} - (\delta_{+1,0} + \delta_{-1,0} + \delta_{0,+1} + \delta_{0,-1}) \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}
\]

and

\[
\mathcal{H}^{(2)}_{n,m} = -2 \begin{pmatrix}
    \phi_{n,m}^{(0)} \phi_{n,m}^{(2)} + \phi_{n,m}^{(0)} \phi_{n,m}^{(2)} \\
    \phi_{n,m}^{(0)} \phi_{n,m}^{(2)} + \phi_{n,m}^{(0)} \phi_{n,m}^{(2)}
\end{pmatrix} - \begin{pmatrix}
    2|\phi_{n,m}^{(1)}|^2 & \phi_{n,m}^{(1)} \\
    \phi_{n,m}^{(1)} & 2|\phi_{n,m}^{(1)}|^2
\end{pmatrix}.
\]
It is clear from the explicit form (2.13) that the spectrum of $\mathcal{H}(0)\varphi = \gamma \varphi$ has exactly $N$ negative eigenvalues $\gamma = -2$, $N$ zero eigenvalues $\gamma = 0$, and infinitely many positive eigenvalues $\gamma = +1$. The negative and zero eigenvalues $\gamma = -2$ and $\gamma = 0$ map to $N$ double zero eigenvalues $\lambda = 0$ in the eigenvalue problem $\sigma\mathcal{H}(0)\psi = i\lambda\psi$. The positive eigenvalues $\gamma = +1$ map to the infinitely many eigenvalues $\lambda = \pm i$. Since zero eigenvalues of $\sigma\mathcal{H}(0)$ are isolated from the rest of the spectrum of $\sigma\mathcal{H}(0)$, their splitting can be studied through regular perturbation theory \[30\]. On the other hand, if the discrete vortex solutions $\phi_{n,m}$ for $(n,m) \in \mathbb{Z}^2$ decays sufficiently fast as $|n| + |m| \to \infty$, the continuous spectral bands of $\sigma\mathcal{H}$ are located on the imaginary axis of $\lambda$ near the points $\lambda = \pm i$, similarly to the case $\phi_{n,m} = 0$ for $(n,m) \in \mathbb{Z}^2$ \[31\]. Therefore, the infinite-dimensional part of the spectrum does not produce any unstable eigenvalues $\text{Re}(\lambda) > 0$ in the stability problem (4.2) with small $\epsilon \in \mathcal{O}(0)$. We shall consider how zero eigenvalues of $\mathcal{H}(0)$ and $\sigma\mathcal{H}(0)$ split as $\epsilon \neq 0$ for solutions of the nonlinear equations (2.11), which are categorized by Propositions 3.1 and 3.4.

### 4.1 Splitting of zero eigenvalues in the first-order reductions

The splitting of zero eigenvalues of $\mathcal{H}$ is related to the Lyapunov–Schmidt reductions of the nonlinear equations (2.11). We show that the same matrix $\mathcal{M}_1$, which gives the Jacobian of the bifurcation functions $g^{(1)}(\theta)$, defines also small eigenvalues of $\mathcal{H}$ that bifurcate from zero eigenvalues of $\mathcal{H}(0)$ in the first-order reductions.

**Lemma 4.1** Let the Jacobian matrix $\mathcal{M}_1$ have eigenvalues $\mu_j^{(1)}$, $1 \leq j \leq N$. The eigenvalue problem $\mathcal{H}\varphi = \gamma \varphi$ has $N$ eigenvalues $\gamma_j$ in $\epsilon \in \mathcal{O}(0)$, such that

$$\lim_{\epsilon \to 0} \frac{\gamma_j}{\epsilon} = \mu_j^{(1)}, \quad 1 \leq j \leq N. \quad (4.4)$$

**Proof.** We assume that there exists an analytical solution $\phi(\epsilon)$ of the nonlinear equations (2.11). The Taylor series of $\phi(\epsilon)$ is defined by (2.17). By taking the derivative in $\epsilon$, we rewrite the problem (2.11) in the form:

$$\mathcal{H}_p\psi(\epsilon) + \epsilon\mathcal{H}_s\psi(\epsilon) + \mathcal{H}_s\phi(\epsilon) = 0, \quad \psi(\epsilon) = \phi'(\epsilon), \quad (4.5)$$

where the linearization operator (2.12) is represented as $\mathcal{H} = \mathcal{H}_p + \epsilon\mathcal{H}_s$. Using the series (2.17) and (4.3), we have the linear inhomogeneous equation:

$$\mathcal{H}^{(0)}\phi^{(1)} + \mathcal{H}_s\phi^{(0)} = 0. \quad (4.6)$$

Let $e_j(\theta)$, $j = 1, \ldots, N$ be eigenvectors of the kernel of $\mathcal{H}^{(0)}$. Each eigenvector $e_j(\theta)$ contains the only non-zero block $i(e^{i\theta_j}, -e^{-i\theta_j})^T$ at the $j$-th position, which corresponds to the node $(n,m)$ on the contour $S_M$. It is clear that the eigenvectors are orthogonal as follows:

$$(e_i(\theta), e_j(\theta)) = 2\delta_{i,j}, \quad 1 \leq i, j \leq N. \quad (4.7)$$

Let $\hat{e}_j(\theta)$, $j = 1, \ldots, N$ be generalized eigenvectors, such that each eigenvector $\hat{e}_j(\theta)$ contains the only non-zero block $(e^{i\theta_j}, e^{-i\theta_j})^T$ at the $j$-th position. Direct computations show that

$$\mathcal{H}^{(0)}\hat{e}_j(\theta) = 2i\sigma e_j(\theta), \quad 1 \leq j \leq N. \quad (4.8)$$
The limiting solution can be represented as follows:

$$\phi^{(0)}(\theta) = \sum_{j=1}^{N} \hat{e}_j(\theta).$$

By comparing the inhomogeneous equation (4.5) with the definition of the bifurcation function \( g(\theta) \) and its Taylor series (2.16), we have the correspondence:

$$g_j^{(1)}(\theta) = \frac{1}{2} \left( e_j(\theta), H_s \phi^{(0)}(\theta) \right).$$

Consider a perturbation to a fixed point of \( g^{(1)}(\theta^*_s) = 0 \) in the form \( \theta = \theta^*_s + \epsilon c \), where \( c = (c_1, c_2, ..., c_N)^T \in \mathbb{R}^N \). It is clear that

$$\phi^{(0)}(\theta) = \phi^{(0)}(\theta^*_s) + \epsilon \sum_{i=1}^{N} c_i e_i + O(\epsilon^2), \quad e_j(\theta) = e_j - \epsilon c_j \hat{e}_j + O(\epsilon^2),$$

where \( \phi^{(0)} = \phi^{(0)}(\theta^*_s) \), \( e_j = e_j(\theta^*_s) \), and \( \hat{e}_j = \hat{e}_j(\theta^*_s) \). By expanding the bifurcation function \( g^{(1)}(\theta) \) near \( \theta = \theta^*_s \), we define the Jacobian matrix \( M_1 \):

$$g_j^{(1)}(\theta) = g_j^{(1)} + \epsilon (M_1 c)_j + O(\epsilon^2),$$

where

$$(M_1 c)_j = \frac{1}{2} \sum_{i=1}^{n} (e_j, H_s e_i) c_i - \frac{1}{2} c_j \sum_{i=1}^{N} (\hat{e}_j, H_s \hat{e}_i). \quad (4.9)$$

On the other hand, the regular perturbation series for small eigenvalues of the problem \( H\varphi = \gamma \varphi \) are defined as follows:

$$\varphi = \varphi^{(0)} + \epsilon \varphi^{(1)} + \epsilon^2 \varphi^{(2)} + O(\epsilon^3), \quad \gamma = \epsilon \gamma_1 + \epsilon^2 \gamma_2 + O(\epsilon^3), \quad (4.10)$$

where \( \varphi^{(0)} = \sum_{j=1}^{N} c_j e_j \), according to the kernel of \( H^{(0)} \). The first-order correction term \( \varphi^{(1)} \) satisfies the inhomogeneous equation:

$$H^{(0)} \varphi^{(1)} + H^{(1)} \varphi^{(0)} = \gamma_1 \varphi^{(0)}. \quad (4.11)$$

Projection to the kernel of \( H^{(0)} \) gives the eigenvalue problem for \( \gamma_1 \):

$$\frac{1}{2} \sum_{i=1}^{N} (e_j, H^{(1)} e_i) c_i = \gamma_1 c_j. \quad (4.12)$$

By direct computations:

$$- \frac{1}{2} \sum_{i=1}^{N} (\hat{e}_j, H_s \hat{e}_i) = \cos(\theta_j - \theta_{j+1}) + \cos(\theta_j - \theta_{j-1}) = \sum_{i=1}^{N} \left( e_j, H^{(1)} e_i \right),$$

such that the limiting relation follows from (4.9) and (4.12) with \( H^{(1)} = H_p^{(1)} + H_s \). \hfill \blacksquare

We apply results of Lemma 4.1 to the solutions of the first-order reductions, which are described in families (i)–(iv) of Propositions 3.1 and 3.7. The numbers of negative, zero and positive eigenvalues of \( M_1 \) are denoted as \( n(M_1) \), \( z(M_1) \) and \( p(M_1) \) respectively. These numbers determine the
numbers of small negative and positive eigenvalues of $\mathcal{H}$ for small positive $\epsilon$. They are predicted from Lemmas 3.4, 3.5 and 3.6.

For family (i), we compute the parameter $A_1$ in Lemma 3.4 as $A_1 = (-1)^{N-l}(2l-N)$, where $l$ is defined in Proposition 3.1. By Lemma 3.4 we have $n(M_1) = N - l - 1$, $z(M_1) = 1$, and $p(M_1) = l$ for $0 \leq l \leq 2M - 1$ and $n(M_1) = N - l$, $z(M_1) = 1$, and $p(M_1) = l - 1$ for $2M + 1 \leq l \leq 4M$. In the case of super-symmetric vortices for $l = 2M$, by Lemma 3.6, we have $n(M_1) = 2M - 1$, $z(M_1) = 2$, and $p(M_1) = 2M - 1$.

For family (ii), by Lemma 3.4 we have $n(M_1) = 0$, $z(M_1) = 1$, and $p(M_1) = N - 1$ for $1 \leq L \leq M - 1$ and $n(M_1) = N - 1$, $z(M_1) = 1$, and $p(M_1) = 0$ for $M + 1 \leq L \leq 2M - 1$, where $L$ is defined in Proposition 3.1. The case of super-symmetric vortices $L = 2M$ can only be studied in the second-order reductions, since $M_1 = 0$.

The family (iii) is represented by the only one-parameter solution 3.26 for each $L = M$. By Lemma 3.6, we have $n(M_1) = 2M - 1$, $z(M_1) = 2$, and $p(M_1) = 2M - 1$ for $0 \leq \theta \leq \pi$, excluding the case of super-symmetric vortices $\theta = \pi$ but including the case of super-symmetric solitons $\theta = 0$ and $\theta = \pi$.

The family (iv) is characterized by the value of $\cos \theta_\ast \neq 0$, $L \neq M$, and $1 \leq n \leq N - 1$, $n \neq 2M$, specified in Proposition 3.1. The parameter $A_1$ in Lemma 3.4 is computed as $A_1 = (-1)^{N-n}(\cos \theta_\ast)^{N-1}(2n-N)$, such that $z(M_1) = 1$ in all cases. In the case $\cos \theta_\ast > 0$, by Lemma 3.4 we have $n(M_1) = N - n - 1$ and $p(M_1) = n$ for $1 \leq n \leq 2M - 1$ and $n(M_1) = N - n$ and $p(M_1) = n - 1$ for $2M + 1 \leq n \leq N - 1$. In the opposite case of $\cos \theta_\ast < 0$, we have $n(M_1) = n$ and $p(M_1) = N - n - 1$ for $1 \leq n \leq 2M - 1$ and $n(M_1) = n - 1$ and $p(M_1) = N - n$ for $2M + 1 \leq n \leq N - 1$.

The splitting of zero eigenvalue of $\mathcal{H}$ is related to splitting of zero eigenvalues of $\sigma \mathcal{H}$ in the stability problem 4.2.

**Lemma 4.2** Let the Jacobian matrix $\mathcal{M}_1$ have eigenvalues $\mu^{(1)}_j$, $1 \leq j \leq N$. The eigenvalue problem $\mathcal{H}\psi = i\lambda \sigma \psi$ has $N$ pairs of eigenvalues $\lambda_j$ in $\epsilon \in O(0)$, such that

$$\lim_{\epsilon \to 0} \frac{\lambda^2_j}{\epsilon} = 2\mu^{(1)}_j, \quad 1 \leq j \leq N.$$  \hspace{1cm} (4.13)

**Proof.** The regular perturbation series for small eigenvalues of $\sigma \mathcal{H}$ are defined as follows:

$$\psi = \psi^{(0)} + \sqrt{\epsilon}\psi^{(1)} + \epsilon\psi^{(2)} + \epsilon\sqrt{\epsilon}\psi^{(3)} + O(\epsilon^2), \quad \lambda = \sqrt{\epsilon}\lambda_1 + \epsilon\lambda_2 + \epsilon\sqrt{\epsilon}\lambda_3 + O(\epsilon^2),$$  \hspace{1cm} (4.14)

where, due to the relations 4.7 and 4.8, we have:

$$\psi^{(0)} = \sum_{j=1}^{N} c_j e_j, \quad \psi^{(1)} = \frac{\lambda_1}{2} \sum_{j=1}^{N} c_j \hat{e}_j,$$  \hspace{1cm} (4.15)

according to the kernel and generalized kernel of $\sigma \mathcal{H}^{(0)}$. The second-order correction term $\psi^{(2)}$ satisfies the inhomogeneous equation:

$$\mathcal{H}^{(0)}\psi^{(2)} + \mathcal{H}^{(1)}\psi^{(0)} = i\lambda_1 \sigma \psi^{(1)} + i\lambda_2 \sigma \psi^{(0)}.$$  \hspace{1cm} (4.16)

Projection to the kernel of $\mathcal{H}^{(0)}$ gives the eigenvalue problem for $\lambda_1$:

$$\mathcal{M}_1 \epsilon = \frac{\lambda^2_1}{2} \epsilon,$$  \hspace{1cm} (4.17)
where \( c = (c_1, c_2, ..., c_N)^T \) and the matrix \( M_1 \) is the same as in the eigenvalue problem \( 4.12 \). As a result, the relation \( 4.13 \) is proved.

The numbers of negative, zero and positive eigenvalues of \( M_1 \), denoted as \( n(M_1) \), \( z(M_1) \) and \( p(M_1) \), are computed above. Let \( r_1, z_1, \) and \( i_1 \) be the numbers of pairs of real, zero and imaginary eigenvalues of the reduced eigenvalue problem \( 4.17 \). We consider stability of families (i)–(iv), described in Propositions 3.1 and 3.7.

For family (i), we have the inhomogeneous equation:

\[ H(M) \psi = \lambda \psi. \]

For family (ii), we have the stability problem \( 4.13 \). For super-symmetric vortices of family (ii), when \( n = 2 \) and \( a_j \neq (-1)^j a \), the additional zero eigenvalue splits at the second-order reductions, which leads to an additional non-zero eigenvalues of the stability problem \( 4.12 \). For super-symmetric vortices of family (ii), all non-zero eigenvalues occur only in the second-order reductions. Finally, for symmetric vortices of family (ii), multiple real non-zero eigenvalues of the first-order reductions, according to the roots of \( \sin^2 \frac{\pi N}{2} \) in the explicit solution \( 4.18 \), split into the complex domain in the second-order reductions. These questions are studied next in the reverse order.

### 4.2 Splitting of non-zero eigenvalues in the second-order reductions

We continue the regular perturbation series \( 4.14 \) to the second-order reductions. By using the explicit first-order correction term \( 4.19 \), we compute the explicit solution:

\[
\psi^{(2)} = \frac{\lambda_2}{2} \sum_{j=1}^{N} c_j \hat{e}_j + \frac{1}{2} \sum_{j=1}^{N} (\sin(\theta_{j+1} - \theta_j) c_{j+1} + \sin(\theta_{j-1} - \theta_j) c_{j-1}) \hat{e}_j + \sum_{j=1}^{N} c_j (e_{j+1} + e_{j-1}), \quad (4.18)
\]

where the vectors \( e_{j+1} \) are obtained from \( e_j \) by shifts of non-zero elements of \( e_j \) from the node \((n, m) \in S_M \) to the adjacent nodes \((n, m) \in \mathbb{Z}^2 \setminus S_M \). The third-order correction term \( \psi^{(3)} \) satisfies the inhomogeneous equation:

\[
\mathcal{H}(0) \psi^{(3)} + \mathcal{H}(1) \psi^{(1)} + i\lambda_1 \sigma \psi^{(2)} + i\lambda_2 \sigma \psi^{(1)} = +i\lambda_3 \sigma \psi^{(0)}.
\]

Projection to the kernel of \( \mathcal{H}(0) \) gives the extended eigenvalue problem for \( \lambda_1 \) and \( \lambda_2 \):

\[
M_1 c = \frac{\lambda^2}{2} c + \sqrt{\epsilon} (\lambda_1 \lambda_2 c + \lambda_1 \mathcal{L}_1 c),
\]

\( (4.20) \)
where the matrix $L_1$ is defined by

$$
(L_1)_{i,j} = \begin{cases} 
\sin(\theta_j - \theta_i), & i = j \pm 1, \\
0, & |i - j| \neq 1 
\end{cases} \quad (4.21)
$$

subject to the periodic boundary conditions. Let $\gamma_j$ be an eigenvalue of the symmetric matrix $M_1$ with the linearly independent eigenvector $c_j$. Then,

$$
\lambda_1 = \pm \sqrt{2\gamma_j}, \quad \lambda_2 = -\frac{(c_j, L_1 c_j)}{(c_j, c_j)}. \quad (4.22)
$$

Since the matrix $L_1$ is skew-symmetric, the second-order correction term $\lambda_2$ is purely imaginary, unless $(c_j, L_1 c_j) = 0$. For discrete solitons of family (i), we have $\sin(\theta_{j+1} - \theta_j) = 0$, such that $L_1 = 0$ and $\lambda_2 = 0$.

For symmetric vortices of family (ii) with $L \neq M$, the matrix $M_1$ has double eigenvalues, according to the roots of $\sin^2 \frac{\pi n}{N}$ in the explicit solution (3.13). Using the same discrete Fourier transform as in the proof of Lemma 3.5, one can find the values of $\lambda_1$ and $\lambda_2$ in this case.

**Lemma 4.3** Let all coefficients $a_j = \cos(\theta_{j+1} - \theta_j)$ and $b_j = \sin(\theta_{j+1} - \theta_j)$, $1 \leq j \leq N$ be the same: $a_j = a$ and $b_j = b$. Eigenvalues of the reduced problem (4.20) are given explicitly:

$$
\lambda_1 = \pm \sqrt{8a} \sin \frac{\pi n}{N}, \quad \lambda_2 = -2ib \sin \frac{2\pi n}{N}, \quad 1 \leq n \leq N. \quad (4.23)
$$

According to Lemma 4.3, all double roots of $\lambda_1$ for $n \neq \frac{N}{2}$ and $n \neq N$ split along the imaginary axis in $\lambda_2$. When $a > 0$, the splitting occurs in the transverse directions to the real values of $\lambda_1$. When $a < 0$, the splitting occurs in the longitudinal directions to the imaginary values of $\lambda_1$. The simple roots at $n = \frac{N}{2}$ and $n = N$ are not affected, since $\lambda_2 = 0$ for $n = \frac{N}{2}$ and $n = N$.

For asymmetric vortices of family (iii), it follows from the explicit solution (3.15) that $\lambda_2 = 0$ for all roots, so that the splitting of eigenvalues does not occur in the second-order reductions.

For asymmetric vortices of family (iv), the value of $\lambda_2$ can not be computed in general.

### 4.3 Splitting of zero eigenvalues in the second-order reductions

We extend results of the regular perturbation series (4.10) and (4.14) to the case $M_1 = 0$, which occurs for super-symmetric vortices of family (ii) with charge $L = M$. It follows from the problem (4.11) with $M_1 = 0$ that $\gamma_1 = 0$ and the first-order correction term $\varphi^{(1)}$ has the explicit form:

$$
\varphi^{(1)} = \frac{1}{2} \sum_{j=1}^{N} (c_{j+1} - c_{j-1}) \hat{e}_j + \sum_{j=1}^{N} c_j (e_{j+1} + e_{j-1}), \quad (4.24)
$$

where the vectors $e_{j\pm 1}$ are the same as in the formula (4.18). The second-order correction term $\varphi^{(2)}$ satisfies the inhomogeneous equation:

$$
H^{(0)} \varphi^{(2)} + H^{(1)} \varphi^{(1)} + H^{(2)} \varphi^{(0)} = \gamma_2 \varphi^{(0)}. \quad (4.25)
$$

Projection to the kernel of $H^{(0)}$ gives the eigenvalue problem for $\gamma_2$:

$$
\frac{1}{2} \left( e_j, H^{(1)} \varphi^{(1)} \right) + \frac{1}{2} \sum_{i=1}^{N} \left( e_j, H^{(2)} e_i \right) c_i = \gamma_2 c_j. \quad (4.26)
$$

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By direct computations in the three separate cases, one can show that the matrix on the left-hand-side of the eigenvalue problem (4.26) is nothing but the matrix $M_2$, which is the Jacobian of the nonlinear function $g(2)(\theta)$, computed for the super-symmetric vortex of family (ii). Let the number of negative, zero and positive eigenvalues of $M_2$ be denoted as $n(M_2)$, $z(M_2)$ and $p(M_2)$ respectively. For super-symmetric vortices of family (ii), we have $n(M_2) = 0$, $z(M_2) = 2$ and $p(M_2) = 2$ for $M = 1$; $n(M_2) = 1$, $z(M_2) = 2$ and $p(M_2) = 5$ for $M = 2$; and $n(M_2) = 4$, $z(M_2) = 2$ and $p(M_2) = 6$ for $M = 3$.

Splitting of zero eigenvalues of $\sigma H$ is studied with the regular perturbation series (4.14). When $M_1 = 0$, it follows from the problem (4.16) that $\lambda_1 = 0$, such that the regular perturbation series (4.14) can be re-ordered as follows:

$$
\psi = \psi^{(0)} + \epsilon \psi^{(1)} + \epsilon^2 \psi^{(2)} + O(\epsilon^3), \quad \lambda = \epsilon \lambda_1 + \epsilon^2 \lambda_2 + O(\epsilon^3),
$$

(4.27)

where

$$
\psi^{(0)} = \sum_{j=1}^{N} c_j e_j, \quad \psi^{(1)} = \varphi^{(1)} + \frac{\lambda_1}{2} \sum_{j=1}^{N} c_j \hat{e}_j,
$$

(4.28)

and $\varphi^{(1)}$ is given by (4.24). The second-order correction term $\psi^{(2)}$ is found from the inhomogeneous equation:

$$
H^{(0)} \psi^{(2)} + H^{(1)} \psi^{(1)} + H^{(2)} \psi^{(0)} = i \lambda_1 \sigma \psi^{(1)} + i \lambda_2 \sigma \psi^{(0)}.
$$

(4.29)

Projection to the kernel of $H^{(0)}$ gives the eigenvalue problem for $\lambda_1$:

$$
M_2 c = \lambda_1 L_2 c + \frac{\lambda_1^2}{2} c,
$$

(4.30)

where $c = (c_1, c_2, ..., c_N)^T$, the matrix $M_2$ is the same as in the eigenvalue problem (4.26), and the matrix $L_2$ follows from the matrix $L_1$ in the form (4.21) with $\sin(\theta_{j+1} - \theta_j) = 1$, or explicitly:

$$
(L_2)_{i,j} = \begin{cases} +1, & i = j - 1, \\ -1, & i = j + 1 \\ 0, & |i - j| \neq 1 \end{cases}
$$

(4.31)

subject to the periodic boundary conditions. Since $M_2$ is symmetric and $L_2$ is skew-symmetric, the eigenvalues of the problem (4.30) occur in pairs $(\lambda_1, -\lambda_1)$. Computations of eigenvalues of the reduced eigenvalue problem (4.30) are reported in the three distinct cases: $M = 1$, $M = 2$ and $M \geq 3$.

**Case $M = 1$:** The reduced eigenvalue problem (4.30) takes the form of the difference equation with constant coefficients:

$$
-c_{j+2} + 2c_j - c_{j-2} = \lambda_1^2 c_j + 2\lambda_1 (c_{j+1} - c_{j-1}), \quad j = 1, 2, 3, 4,
$$

subject to the periodic boundary conditions. By the discrete Fourier transform, the difference equation reduces to the characteristic equation:

$$
\left(\lambda_1 + 2i \sin \frac{\pi n}{2}\right)^2 = 0, \quad n = 1, 2, 3, 4.
$$

The reduced eigenvalue problem (4.30) has two eigenvalues of algebraic multiplicity two at $\lambda_1 = -2i$ and $\lambda_1 = 2i$ and zero eigenvalue of algebraic multiplicity four.
Case $M = 2$: The reduced eigenvalue problem (4.30) takes the form of the system of difference equations with constant coefficients:

$$\begin{align*}
-x_{j+1} + 2x_j - x_{j-1} &= \lambda_1^2 x_j + 2\lambda_1 (y_j - y_{j-1}), & j = 1, 2, 3, 4, \\
y_{j+1} - 2y_{j+2} + y_{j-1} &= \lambda_1^2 y_j + 2\lambda_1 (x_{j+1} - x_j), & j = 1, 2, 3, 4,
\end{align*}$$

where $x_j = c_{2j-1}$ and $y_j = c_{2j}$ subject to the periodic boundary conditions. The characteristic equation for the linear system takes the explicit form:

$$\lambda^4 - 2\lambda_1^2 \left(1 - (-1)^n - 8\sin^2 \frac{\pi n}{4}\right) + 8\sin^2 \frac{\pi n}{4} \left(1 - (-1)^n - 2\sin^2 \frac{\pi n}{4}\right) = 0, \quad n = 1, 2, 3, 4.$$ 

The reduced eigenvalue problem (4.30) has three eigenvalues of algebraic multiplicity four at $\lambda_1 = -\sqrt{2}i$, $\lambda_1 = \sqrt{2}i$, and $\lambda_1 = 0$, and four simple eigenvalues at $\lambda_1 = \pm i\sqrt{80+8}$ and $\lambda_1 = \pm \sqrt{80-8}$.

Case $M \geq 3$: Eigenvalues of the reduced eigenvalue problem (4.30) with $M = 3$ are computed numerically by using Mathematica. The results are as follows:

$$\begin{align*}
\lambda_{1,2} &= \pm 3.68497i, \quad \lambda_{3,4} = \lambda_{5,6} = \pm 3.20804i, \quad \lambda_{7,8} = \pm 2.25068i, \quad \lambda_{9,10} = \lambda_{11,12} = \pm i, \\
\lambda_{13,14} = \lambda_{15,16} &= \pm 0.53991, \quad \lambda_{17,18,19,20} = \pm 0.63426 \pm 0.282851i, \quad \lambda_{21,22,23,24} = 0.
\end{align*}$$

Using these computations of eigenvalues, we summarize that the second-order reduced eigenvalue problem (4.30) has no unstable eigenvalues $\lambda$ when $L = M = 1$; a simple unstable (positive) eigenvalue when $L = M = 2$; two unstable real and two unstable complex eigenvalues when $L = M = 3$. We note that destabilization of the super-symmetric vortex with $M = 2$ occurs due to the center node $(2, 2)$, which couples the four even-numbered nodes of the contour $S_2$ in the second-order reductions. Due to this coupling, there exists a simple negative eigenvalue of the Jacobian matrix $M_2$ and a simple positive eigenvalue in the reduced eigenvalue problem (4.30). Similarly, destabilization of the super-symmetric vortex with $M = 3$ occurs due to the coupling of eight nodes of the contour $S_3$ with four interior corner points $(2, 2)$, $(2, M)$, $(M, M)$, and $(M, 2)$, which result in the four negative eigenvalues of the matrix $M_2$ and four unstable eigenvalues in the reduced eigenvalue problem (4.30).

We note that if the matrix $M_2$ would be defined by the formula (3.24) for any $M \geq 1$ (i.e. if all nodes inside the contour $S_M$ would be removed by drilling a hole), the eigenvalues of $M_2$ would be all positive and the eigenvalues of the reduced problem (4.30) would be all purely imaginary, similarly to the case $M = 1$.

### 4.4 Additional splitting in the second-order reduction

For super-symmetric solitons of family (i), when $l = 2M$ but $a_j \neq (-1)^ja$, the Jacobian matrix $M_1$ has two zero eigenvalues with eigenvectors $p_0$ and $p_1$, but the Jacobian matrix $M_1 + \epsilon M_2$ has only one zero eigenvalue with eigenvector $p_0$. Therefore, the splitting of zero eigenvalue occurs in the second-order reduction. Extending the regular perturbation series (4.10) to the next order, we find that $\gamma_1 = 0$ for $c = p_1$, and

$$\gamma_2 = \frac{\langle p_1, M_2 p_1 \rangle}{\langle p_1, p_1 \rangle}.$$
Extending the regular perturbation series (4.14) to the second order, we have find that $\lambda_1^2 = 0$ for $c = p_1, L_1 = 0$, and

$$\lambda_2^2 = 2 \left( \frac{p_1, M_2 p_1}{p_1, p_1} \right) = 2 \gamma_2.$$ 

Thus, the splitting of zero eigenvalue in the second-order reduction is the same as the splitting of zero eigenvalues in the first-order reductions. A positive eigenvalue $\gamma_2$ results in a pair of real eigenvalues $\lambda_2$, while a negative eigenvalue $\gamma_2$ results in a pair of purely imaginary eigenvalues $\lambda_2$.

All individual results on stability of localized modes on the discrete contour $S_M$ are summarized as follows.

**Proposition 4.4** Consider the stability problem (4.2) in the domain $\epsilon \in O(0)$, associated to the families of discrete solitons and vortices of Propositions 3.1 and 3.7. The following solutions are spectrally stable in the domain $\epsilon \in O(0)$: discrete solitons of family (i) with $l = 0$; symmetric vortices of family (ii) with the charge $M + 1 \leq L \leq 2M - 1$; and symmetric vortex of family (ii) with the charge $L = M = 1$. All other solutions have at least one unstable eigenvalue with $\text{Re}(\lambda) > 0$.

We note that stability of discrete solitons of family (i) with $l = 0$ is equivalent to stability of discrete solitons in the one-dimensional NLS lattice, which is proved in the first paper [1]. When $l = 0$, the limiting solution consists of alternating up and down pulses along the contour $S_M$, similar to Theorem 3.6 in [1].

We also note that the purely imaginary eigenvalue of $\lambda$ have negative Krein signature, such that the Hamiltonian Hopf bifurcations may occur for larger values of $\epsilon$ beyond the neighborhood $O(0)$. The number of eigenvalues of negative Krein signature is related to the closure relation for the number of negative eigenvalues of the linearized Hamiltonian $\mathcal{H}$ (see [1]). We can see from computations of families (i)-(iv) that the closure relation is satisfied in all cases, except for the super-symmetric vortices of family (ii) with $L = M$. Indeed, in the case $L = M = 1$, the Hamiltonian $\mathcal{H}$ has four negative eigenvalues for $\epsilon \in O(0)$, while it has two pairs of purely imaginary eigenvalues of negative Krein signature and two pairs of zero eigenvalues, which exceeds the allowed negative index of $\mathcal{H}$. Derivation of a modified closure relation for vortex solutions of the discrete NLS equations is beyond the scope of this manuscript.

5 Numerical Results

We perform direct numerical simulations of the discrete NLS equation (2.1) in order to examine stability of the discrete vortices in the simplest cases $M = 1$ and $M = 2$. The results are shown on Figures 1-6 (see also summary in Table 1).

In computations of solutions of the problems (2.4) and (4.2), we will use an equivalent renormalization of the problem with parameter:

$$\varepsilon = \frac{\epsilon}{1 - 4\epsilon}.$$ 

This renormalization is equivalent to keeping the diagonal term $-4\epsilon \phi_{n,m}$ in the right-hand-side of the difference equations (2.4). As a result, the discrete solitons and vortices exist typically in the
semi-infinite domain \( \varepsilon > 0 \). The anti-continuous limit is not affected by the renormalization since \( \varepsilon \approx \varepsilon \) for small \( \varepsilon \).

On the figures, the top left panel shows the profile of the vortex solution for a specific value of \( \varepsilon \) by means of contour plots of the real (top left), imaginary (top right) modulus (bottom left) and phase (bottom right) two-dimensional profiles. The top right panel shows the complex plane \( \lambda = \lambda_r + i\lambda_i \) for the linear eigenvalue problem (4.2) for the same value of \( \varepsilon \). The bottom panel shows the dependence of small eigenvalues as a function of \( \varepsilon \), obtained via continuation methods from the anti-continuum limit of \( \varepsilon = 0 \). The solid lines represent numerical results, while the dashed lines show results of the first-order and second-order reductions.

Figure 1 shows results for the super-symmetric vortex of charge \( L = 1 \) on the contour \( S_M \) with \( M = 1 \). In the second-order reduction, the stability spectrum of the vortex solution has a zero eigenvalue of algebraic multiplicity 4 and two pairs of imaginary eigenvalues \( \lambda \approx \pm 2\varepsilon i \). These pairs split along the imaginary axis beyond the second-order reductions and undertake Hamiltonian Hopf bifurcations for larger values of \( \varepsilon \) upon collision with the continuous spectrum (only the first bifurcation at \( \varepsilon \approx 0.38 \) is shown on Fig. 1). It follows from Fig. 1 that the zero eigenvalue splits along the imaginary axis for larger values of \( \varepsilon \), beyond the second-order reductions.

Figure 2 shows results for the symmetric vortex of charge \( L = 1 \) on the contour \( S_M \) with \( M = 2 \). There are three double and one simple real unstable eigenvalues in the first-order reductions, but all double eigenvalues split into the complex plane in the second-order reductions. The asymptotic result (4.23) for eigenvalues \( \lambda = \sqrt{2} \varepsilon \lambda_1 \) with \( N = 8, a = \cos(\pi/4) \) and \( b = \sin(\pi/4) \) are shown on Fig. 2 in perfect agreement with numerical results.

Figure 3 shows results for the super-symmetric vortex with \( L = M = 2 \). The second-order reductions have a pair of simple real eigenvalues \( \lambda \approx \pm \varepsilon \sqrt{80 - 8} \), a pair of simple imaginary eigenvalues \( \lambda \approx \pm \varepsilon \sqrt{80 + 8} \), zero eigenvalue of algebraic multiplicity 4 and a pair of imaginary eigenvalues of algebraic multiplicity 4 at \( \lambda \approx \pm \varepsilon \sqrt{2} \). The bottom right panel of Fig. 3 shows splitting of multiple imaginary eigenvalues beyond the second-order reductions and also subsequent Hamiltonian–Hopf bifurcations for larger values of \( \varepsilon \).

Figure 4 shows results for the symmetric vortex with \( L = 3 \) and \( M = 2 \). The first-order reductions predict three pairs of double imaginary eigenvalues, a pair of simple imaginary eigenvalues and a double zero eigenvalue. The double eigenvalues split in the second-order reductions along the imaginary axis, given by (4.23) with \( N = 8, a = \cos(3\pi/4) \) and \( b = \sin(3\pi/4) \). The seven pairs of imaginary eigenvalues lead to a cascade of seven Hamiltonian Hopf bifurcations for larger values of \( \varepsilon \) due to their collisions with the continuous spectrum. The first Hamiltonian–Hopf bifurcation when the symmetric vortex becomes unstable occurs for \( \varepsilon \approx 0.096 \).

Zero-parameter asymmetric vortices of family (iv) on the contour \( S_M \) with \( M = 2 \) are shown in Fig. 5 for \( L = 1 \) and in Fig. 6 for \( L = 3 \). In the case of Fig. 5 all the phase differences between adjacent sites in the contour are \( \pi/6 \), except for the last one which is \( 5\pi/6 \), completing a phase trip of \( 2\pi \) for a vortex of topological charge \( L = 1 \). Eigenvalues of the matrix \( M_1 \) in the first-order reductions can be computed numerically as follows: \( \gamma_1 = -1.154, \gamma_2 = 0, \gamma_3 = 0.507, \gamma_4 = 0.784, \gamma_5 = 1.732, \gamma_6 = 2.252, \gamma_7 = 2.957 \) and \( \gamma_8 = 3.314 \). As a result, the corresponding eigenvalues \( \lambda \approx \pm \sqrt{2\varepsilon} \) yield one pair of imaginary eigenvalues and six pairs of real eigenvalues, in agreement with our numerical results. The bottom panel of Fig. 5 shows that two pairs of real eigenvalues
collide for $\varepsilon \approx 0.047$ and $\varepsilon \approx 0.057$ and lead to two quartets of eigenvalues.

In the case of Fig. 6, all the phase differences in the contour are $5\pi/6$, except for the last one which is $\pi/6$, resulting in a vortex of topological charge $L = 3$. Eigenvalues of the matrix $M_1$ are found numerically as follows: $\gamma_1 = -3.314$, $\gamma_2 = -2.957$, $\gamma_3 = -2.252$, $\gamma_4 = -1.732$, $\gamma_5 = -0.784$, $\gamma_6 = -0.507$, $\gamma_7 = 0$, and $\gamma_8 = 1.154$. Consequently, this solution has six pairs of imaginary eigenvalues and one pair of real eigenvalues. The first Hamiltonian–Hopf bifurcation occurs for $\varepsilon \approx 0.086$.

We note that numerical results for asymmetric vortices of family (iii) are not shown. Both analytical and numerical analysis of such solutions are delicate problems which are open for future studies. However, all such solutions (if they persist) are unstable in the first-order reductions.

6 Conclusions

In this series of two papers, we have developed the mathematical analysis of discrete soliton and vortex solutions of the discrete NLS equations close to the anti-continuum limit. These solutions are relevant to recent experimental and numerical studies in the context of nonlinear optics, photonic crystal lattices, soft condensed-matter physics, and Bose-Einstein condensates.

In the present paper, we have examined the persistence of discrete vortices starting from the anti-continuum limit of uncoupled oscillators and continuing towards a finite coupling constant $\epsilon$. We have found persistent families of such solutions that include symmetric and asymmetric vortices. We have ruled out other non-persistent solutions by means of the Lyapunov-Schmidt method. We have subsequently categorized the persisting solutions to discrete solitons, symmetric vortices and one-parameter and zero-parameter asymmetric vortices. For persistent solutions, we have derived the leading-order asymptotic approximations for small unstable and neutrally stable eigenvalues of the stability problem, up to the first-order and second-order corrections.

We have applied the results to particular computations of discrete vortices of topological charge $L = 1$, $L = 2$, and $L = 3$ on the discrete square contours $S_M$ with $M = 1$ and $M = 2$. Besides particular computations collected in Table 1 and Figs. 1–6, these results offer a road map on stability predictions for larger contours, as well as predictions on how to stabilize the discrete vortices. For example, super-symmetric vortices of charge $L = M \geq 2$ can be stabilized by excluding the inner nodes inside the discrete contour $S_M$.

There are remaining open problems for future analytical work. First, it would be nice to extend this analysis to three-dimensional structures, such as the discrete solitons, vortices, or vortex “cubes” (see [23]). Second, other types of contours can be studied in the two-dimensional lattice, such as the diagonal square contours which would include the “vortex cross” or the octagon (see [22]). Persistence of one-parameter asymmetric vortices including the super-symmetric vortices has not yet been proved in the present paper and it leaves space for future work. Finally, the closure relation for negative indices of linearized Hamiltonian associated with the discrete vortex solutions must be derived and applied independently of the current studies. While conceptually, the methodologies and techniques presented here can be adapted to the problems mentioned above, actual computations of the higher-order Lyapunov–Schmidt reductions become technically involved.
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Figure 1: The (super-symmetric) vortex cell with $L = M = 1$. The top left panel shows the profile of the solution for $\varepsilon = 0.6$. The subplots show the real (top left), imaginary (top right), modulus (bottom left) and phase (bottom right) fields. The top right panel shows the spectral plane $(\lambda_r, \lambda_i)$ of the linear eigenvalue problem (4.2). The bottom panel shows the small eigenvalues versus $\varepsilon$ (the top subplot shows the imaginary part, while the bottom shows the real part). The solid lines show the numerical results, while the dashed lines show the results of the second-order reductions.
Figure 2: The same features as in the previous figure are shown for the symmetric vortex with $L = 1$ and $M = 2$ for $\varepsilon = 0.1$. 
Figure 3: The super-symmetric vortex with $L = M = 2$ for $\varepsilon = 0.1$. The bottom right panel is an extension of the bottom left panel to larger values of $\varepsilon$. Remarkable agreement of the theoretical predictions (dashed lines) with the numerical results (solid lines) can be observed for small values of $\varepsilon$. 
Figure 4: The same features as in the previous figure are shown for the symmetric vortex with $L = 3$ and $M = 2$ for $\varepsilon = 1$. 
Figure 5: Same as Figure 1 but for the asymmetric vortex with $L = 1$ and $M = 2$ for $\varepsilon = 0.1$. 
Figure 6: Same as Figure 1 but for the asymmetric vortex with $L = 3$ and $M = 2$ for $\varepsilon = 0.1$. 