VERTEX OPERATOR ALGEBRAS, EXTENDED $E_8$ DIAGRAM, AND MCKAY’S OBSERVATION ON THE MONSTER SIMPLE GROUP

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Abstract. We study McKay’s observation on the Monster simple group, which relates the 2A-involutions of the Monster simple group to the extended $E_8$ diagram, using the theory of vertex operator algebras (VOAs). We first consider the sublattices $L$ of the $E_8$ lattice obtained by removing one node from the extended $E_8$ diagram at each time. We then construct a certain coset (or commutant) subalgebra $U$ associated with $L$ in the lattice VOA $V \sqrt{2}E_8$. There are two natural conformal vectors of central charge $1/2$ in $U$ such that their inner product is exactly the value predicted by Conway [1]. The Griess algebra of $U$ coincides with the algebra described in [1, Table 3]. There is a canonical automorphism of $U$ of order $|E_8/L|$. Such an automorphism can be extended to the Leech lattice VOA $V_\Lambda$ and it is in fact a product of two Miyamoto involutions. In the sequel [12] to this article we shall develop the representation theory of $U$. It is expected that if $U$ is actually contained in the Moonshine VOA $V^\natural$, the product of two Miyamoto involutions is in the desired conjugacy class of the Monster simple group.

1. Introduction

The Moonshine vertex operator algebra $V^\natural$ constructed by Frenkel-Lepowsky-Meurman [7] is one of the most important examples of vertex operator algebras (VOAs). Its full automorphism group is the Monster simple group. The weight 2 subspace $V_2^\natural$ of $V^\natural$ has a structure of commutative non-associative algebra which coincides with the 196884-dimensional algebra investigated by Griess [9] in his construction of the Monster simple group (see also Conway[11]). The structure of this algebra, which is called the Monstrous Griess algebra, has been studied by group theorists. It is well known [11] that each 2A-involution $\phi$ of the Monster simple group uniquely defines an idempotent $e_\phi$ called an axis in the Monstrous Griess algebra. Moreover, the inner product $\langle e_\phi, e_\psi \rangle$ of any two axes $e_\phi$ and $e_\psi$ is uniquely determined by the conjugacy class of the product $\phi\psi$ of 2A-involutions. Actually, 2A-involutions of the Monster simple group satisfy a 6-transposition property, that is, $|\phi\psi| \leq 6$ for any two 2A-involutions $\phi$ and $\psi$. In addition, the conjugacy class of $\phi\psi$ is one of $1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $4B$, $2B$, or $3C$.

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John McKay [14] observed that there is an interesting correspondence with the extended $E_8$ diagram. Namely, one can assign $1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$, and $3C$ to the nodes of the extended $E_8$ diagram as follows (cf. Conway [1], Glauberman and Norton [8]):

$$1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$$

where the numerical labels are equal to the multiplicities of the corresponding simple roots in the highest root and the numbers behind the labels denote the inner product $\langle 2e_\phi, 2e_\psi \rangle$ of $2e_\phi$ and $2e_\psi$.

On the other hand, from the point of view of VOAs, Miyamoto [15, 17] showed that an axis is essentially a half of a conformal vector $e$ of central charge $1/2$ which generates a Virasoro VOA $\text{Vir}(e) \simeq L(1/2, 0)$ inside the Moonshine VOA $V^\natural$. Moreover, an involutive automorphism $\tau_e$ can be defined by

$$\tau_e = \begin{cases} 
1 & \text{on } W_0 \oplus W_{1/2}, \\
-1 & \text{on } W_{1/16},
\end{cases}$$

where $W_\alpha$ denotes the sum of all irreducible $\text{Vir}(e)$-modules isomorphic to $L(1/2, \alpha)$ inside $V^\natural$. In fact, $\tau_e$ is always of class $2A$ for any conformal vector $e$ of central charge $1/2$ in $V^\natural$.

In this article, we try to give an interpretation of the McKay diagram (1.1) using the theory of VOAs. We first observe that there is a conformal vector $\hat{e}$ of central charge $1/2$ in the lattice VOA $V_{\sqrt{2}E_8}$ which is fixed by the action of the Weyl group of type $E_8$. Let $\Phi$ be the root system corresponding to the Dynkin diagram obtained by removing one node from the extended $E_8$ diagram and $L = L(\Phi)$ the root lattice associated with $\Phi$. Then the Weyl group $W(\Phi)$ of $\Phi$ and the quotient group $E_8/L$ both act naturally on $V_{\sqrt{2}E_8}$ and their actions commute with each other. The action of the quotient group $E_8/L$ can be extended to the Leech lattice VOA $V_\Lambda$.

The main idea is to construct certain vertex operator subalgebras $U$ of the lattice VOA $V_{\sqrt{2}E_8}$ corresponding to the nine nodes of the McKay diagram. In each case, $U$ is constructed as a coset (or commutant) subalgebra of $V_{\sqrt{2}E_8}$ associated with $\Phi$. In fact, $U$ is chosen so that the Weyl group $W(\Phi)$ acts trivially on it. We show that in each of the nine cases $U$ always contains $\hat{e}$ and another conformal vector $\hat{f}$ of central charge $1/2$ such that the inner product $\langle \hat{e}, \hat{f} \rangle$ is exactly the value listed in the McKay diagram. Both of $\hat{e}$ and $\hat{f}$ are fixed by the Weyl group $W(\Phi)$. Thus the Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$ commute with the action of $W(\Phi)$. Furthermore, the quotient group $E_8/L$ naturally induces some automorphism of $U$ of order $n = |E_8/L|$, which is identical with the numerical label of the corresponding node in the McKay diagram. Such an automorphism can be extended to the Leech lattice VOA $V_\Lambda$ and it is in fact a product $\tau_{\hat{e}}\tau_{\hat{f}}$ of two Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$. 
In the sequel to this article we shall study the properties of the coset subalgebra $U$ in detail. Except the $4A$ case, $U$ always contains a set of mutually orthogonal conformal vectors such that their sum is the Virasoro element of $U$ and the central charge of those conformal vectors are all coming from the unitary series

$$c = c_m = 1 - \frac{6}{(m+2)(m+3)}, \quad m = 1, 2, 3, \ldots.$$ 

Such a conformal vector generates a Virasoro VOA isomorphic to $L(c_m, 0)$ inside $U$. The structure of $U$ as a module for a tensor product of those Virasoro VOAs is determined.

In the $4A$ case, $U$ is isomorphic to the fixed point subalgebra $V_N^+$ of $\theta$ for some rank two lattice $\mathcal{N}$, where $\theta$ is an automorphism of $V_N$ induced from the $-1$ isometry of the lattice $\mathcal{N}$.

The VOA $U$ is generated by $\hat{e}$ and $\hat{f}$. As a consequence we know that every element of $U$ is fixed by the Weyl group $W(\Phi)$. The weight 1 subspace $U_1$ of $U$ is 0. The Griess algebra $U_2$ of $U$ is also generated by $\hat{e}$ and $\hat{f}$ and it has the same structure as the algebra studied in Conway Table 3]. The automorphism group of $U$ is a dihedral group of order $2n$ except for the cases for $1A$, $2A$, and $2B$. It is a trivial group in the $1A$ case, a symmetric group of degree 3 in the $2A$ case, and of order 2 in the $2B$ case. Furthermore, we shall discuss the rationality of $U$ and the classification of irreducible modules. The product $\tau_\epsilon \tau_f$ of two Miyamoto involutions should be in the desired conjugacy class of the Monster simple group, provided that the Moonshine VOA $V^\natural$ contains a subalgebra isomorphic to $U$.

Further mysteries concerning the McKay diagram can be found in Glauberman and Norton. Among other things, some relation between the Weyl group $W(\Phi)$ and the centralizer of a certain subgroup generated by two 2A-involutions and one 2B-involution in the Monster simple group was discussed. That every element of $U$ is fixed by $W(\Phi)$ seems quite suggestive.

Let us recall some terminology (cf. [7]). A VOA is a $\mathbb{Z}$-graded vector space $V = \oplus_{n \in \mathbb{Z}} V_n$ with a linear map $Y(\cdot, z) : V \to (\text{End } V)[[z, z^{-1}]]$ and two distinguished vectors; the vacuum vector $1 \in V_0$ and the Virasoro element $\omega \in V_2$ which satisfy certain conditions. For any $v \in V$, $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$ is called a vertex operator and $v_n \in \text{End } V$ a component operator. Each homogeneous subspace $V_n$ is the eigenspace for the operator $L(0) = \omega_1$ with eigenvalue $n$. The eigenvalue for $L(0)$ is called a weight. Suppose $V = \oplus_{n=0}^\infty V_n$ with $V_0 = C1$ and $V_1 = 0$. For $u, v \in V_2$, one can define a product $u \cdot v$ by $u_1 v$ and an inner product $\langle u, v \rangle$ by $u_3 v = \langle u, v \rangle 1$. The inner product is invariant, that is, $\langle u_1 v, w \rangle = \langle v, u_1 w \rangle$ for $u, v, w \in V_2$ (cf. [7, Section 8.9]). With the product and the inner product $V_2$ becomes an algebra, which is called the Griess algebra of $V$.

The organization of the article is as follows. In Section 2 we review some notation for lattice VOAs from [4] and certain conformal vectors in the lattice VOA $V_{\sqrt{7}R}$ given by [5], where $R$ is a root lattice of type $A$, $D$, or $E$. Moreover, we study some highest weight vectors in irreducible modules of $V_{\sqrt{7}R}$ with respect to those conformal vectors. In Section 3 we consider the sublattice $L$ of $E_8$ and define the coset subalgebra $U$ and two conformal vectors $\hat{e}$ and $\hat{f}$ of central charge $1/2$. We calculate the inner product $\langle \hat{e}, \hat{f} \rangle$ and verify that it is identical with the value given in the McKay diagram. A canonical automorphism $\sigma$ of order $n = |E_8/L|$ induced by the quotient group $E_8/L$ is also discussed. Then in
Section 4 we consider an embedding of an orthogonal sum \( \sqrt{2}E_8 \) of three copies of \( \sqrt{2}E_8 \) into the Leech lattice \( \Lambda \) and show that the product \( \tau_\ell \tau_j \) of two Miyamoto involutions \( \tau_\ell \) and \( \tau_j \) is of order \( n \) as an automorphism of \( V_\Lambda \). Finally, in Section 5 we give an explicit correspondence between the Griess algebra \( U_2 \) of \( U \) and the algebra in Conway [1] Table 3].

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2. Conformal vectors in lattice VOAs

In this section, we review the construction of certain conformal vectors in the lattice VOA \( V_{\sqrt{2}R} \) from [5], where \( R \) is a root lattice of type \( A_n, D_n, \) or \( E_8 \). The notation for lattice VOAs here is standard (cf. [2]). Let \( N \) be a positive definite even lattice with inner product \( \langle \cdot, \cdot \rangle \). Then the VOA \( V_N \) associated with \( N \) is defined to be \( M(1) \otimes \mathbb{C}\{N\} \). More precisely, let \( \mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} N \) be an abelian Lie algebra and \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \) its affine Lie algebra. Then \( M(1) = \mathbb{C}[\alpha(n) \mid \alpha \in \mathfrak{h}, n < 0] \cdot 1 \) is the unique irreducible \( \mathfrak{h} \)-module such that \( \alpha(n) \cdot 1 = 0 \) for \( \alpha \in \mathfrak{h}, n \geq 0 \) and \( K = 1 \), where \( \alpha(n) = \alpha \otimes t^n \). Moreover, \( \mathbb{C}\{N\} \) denotes a twisted group algebra of the additive group \( N \). In the case for \( N = \sqrt{2}R \), the twisted group algebra \( \mathbb{C}\{\sqrt{2}R\} \) is isomorphic to the ordinary group algebra \( \mathbb{C}[\sqrt{2}R] \) since \( \sqrt{2}R \) is a doubly even lattice. The standard basis of \( \mathbb{C}[\sqrt{2}R] \) is denoted by \( e^{\sqrt{2}a}, \alpha \in R \).

Then the vacuum vector \( 1 = 1 \otimes e^0 \).

Let \( \Phi \) be the root system of \( R \) and \( \Phi^+ \) and \( \Phi^- \) the set of all positive roots and negative roots, respectively. Then \( \Phi = \Phi^+ \cup \Phi^- = \Phi^+ \cup (-\Phi^+) \). The Virasoro element \( \omega \) of \( V_{\sqrt{2}R} \) is given by

\[
\omega = \omega(\Phi) = \frac{1}{2h} \sum_{\alpha \in \Phi^+} \alpha(-1)^2 \cdot 1,
\]

where \( h \) is the Coxeter number of \( \Phi \). Now define

\[
s = s(\Phi) = \frac{1}{2(h+2)} \sum_{\alpha \in \Phi^+} \left( \alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}) \right),
\]

(2.1)

\[
\tilde{\omega} = \tilde{\omega}(\Phi) = \omega - s.
\]

It is shown in [3] that \( \tilde{\omega} \) and \( s \) are mutually orthogonal conformal vectors, that is, \( \tilde{\omega}_1 \tilde{\omega} = 2\tilde{\omega}, s_1 s = 2s \), and \( \tilde{\omega}_1 s = 0 \). The central charge of \( \tilde{\omega} \) is \( 2n/(n+3) \) if \( R \) is of type \( A_n, 1 \) if \( R \) is of type \( D_n \) and \( 6/7, 7/10 \) and 1/2 if \( R \) is of type \( E_6, E_7 \) and \( E_8 \), respectively.

Let \( W(\Phi) \) be the Weyl group of \( \Phi \). Any element \( g \in W(\Phi) \) induces an automorphism of the lattice \( R \) and hence it defines an automorphism of the VOA \( V_{\sqrt{2}R} \) by

\[
g(u \otimes e^{\sqrt{2}\alpha}) = gu \otimes e^{\sqrt{2}ga} \quad \text{for} \quad u \otimes e^{\sqrt{2}\alpha} \in M(1) \otimes e^{\sqrt{2}\alpha} \subset V_{\sqrt{2}R}.
\]

Note that both \( s \) and \( \tilde{\omega} \) are fixed by the Weyl group \( W(\Phi) \).

We shall study certain highest weight vectors with respect to the subalgebra \( \text{Vir}(s) \otimes \text{Vir}(\tilde{\omega}) \), where \( \text{Vir}(s) \) and \( \text{Vir}(\tilde{\omega}) \) denote the Virasoro VOAs generated by the conformal vectors \( s \) and \( \tilde{\omega} \), respectively.

Let \( R^* = \{ \alpha \in \mathbb{Q} \otimes \mathbb{Z} R \mid \langle \alpha, R \rangle \subset \mathbb{Z} \} \) be the dual lattice of \( R \).
Lemma 2.1. Let $R$ be a root lattice of type $A$, $D$, or $E$ and $\gamma + R$ a coset of $R$ in $R^*$. Let $k = \min\{\langle \alpha, \alpha \rangle | \alpha \in \gamma + R\}$. For any $\eta \in \gamma + R$ with $\langle \eta, \eta \rangle = k$, we define

$$X_\eta = \{(\alpha, \beta) \in R \times (\gamma + R) \mid \langle \alpha, \alpha \rangle = 2, \langle \beta, \beta \rangle = k \text{ and } \alpha + \beta = \eta\}.$$ 

Then $|X_\eta| = kh$, where $h$ is the Coxeter number of $R$.

Proof. The proof is just by direct verification. We only discuss the case for $R = A_n$. The other cases can be proved similarly.

Let $R = A_n$. Then the Coxeter number $h$ is $n + 1$ and the roots of $A_n$ are given by the vectors in the form $\pm(1, -1, 0^{n-1}) \in \mathbb{R}^{n+1}$, that is, the vectors whose one entry is $\pm 1$, another entry is $\mp 1$, and the remaining $n - 1$ entries are 0. Let $\mu = \frac{1}{n+1}(1, \ldots, 1, -n)$. Then $\mu + R$ is a generator of the group $R^*/R$. Denote $\gamma = j\mu$ for $j = 0, \ldots, n$. Then $k = \min\{\langle \alpha, \alpha \rangle | \alpha \in \gamma + R\} = j \frac{(n+1-j)}{n+1}$, and the elements of square norm $k$ in $\gamma + R$ are of the form $\frac{1}{n+1}(j^{n+1-j}, (-n-1+j)^j)$.

Now it is easy to see that $|X_\eta| = (n+1-j)j = kh$ for any $\eta$ with $\langle \eta, \eta \rangle = k$. \hfill \Box

Proposition 2.2. Let $\gamma + R$ be a coset of $R$ in $R^*$ and $k = \min\{\langle \alpha, \alpha \rangle | \alpha \in \gamma + R\}$. Define

$$v = \sum_{\alpha \in \gamma + R, \langle \alpha, \alpha \rangle = k} e^{\sqrt{2}a} \in V_{\sqrt{2}(\gamma + R)}.$$ 

Then $v$ is a highest weight vector of highest weight $(0, k)$ in $V_{\sqrt{2}(\gamma + R)}$ with respect to $\text{Vir}(s) \otimes \text{Vir}(\tilde{\omega})$, that is, $s_j v = \tilde{\omega}_j v = 0$ for all $j \geq 2$, $s_1 v = 0$, and $\tilde{\omega}_1 v = kv$.

Proof. Since $k$ is the minimum weight of $V_{\sqrt{2}(\gamma + R)}$, it is clear that $s_j v = \tilde{\omega}_j v = 0$ for all $j \geq 2$. Since $\omega_1 v = kv$, it suffices to show that $s_1 v = 0$. By the definition (2.1) of $s$ and the above lemma, we have

$$s_1 v = \frac{1}{2(h+2)} \sum_{\alpha \in \Phi^+} \left( \alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}a} + e^{-\sqrt{2}a}) \right)_1 v,$$

$$= \left( \frac{h}{h+2} \omega - \frac{1}{h+2} \sum_{\alpha \in \Phi^+} (e^{\sqrt{2}a} + e^{-\sqrt{2}a}) \right)_1 v,$$

$$= \frac{hk}{h+2} v - \frac{hk}{h+2} v = 0.$$ 

Hence the assertion holds. \hfill \Box
3. Extended $E_8$ diagram and sublattices of the root lattice $E_8$

In this section, we consider certain sublattices of the root lattice $E_8$ by using the extended $E_8$ diagram

$$\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7$$

(3.1)

where $\alpha_1, \alpha_2, \ldots, \alpha_8$ are the simple roots of $E_8$ and

$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0.$$  

(3.2)

Thus $\langle \alpha_i, \alpha_i \rangle = 2$, $0 \leq i \leq 8$. Moreover, for $i \neq j$, $\langle \alpha_i, \alpha_j \rangle = -1$ if the nodes $\alpha_i$ and $\alpha_j$ are connected by an edge and $\langle \alpha_i, \alpha_j \rangle = 0$ otherwise. Note that $-\alpha_0$ is the highest root.

For any $i = 0, 1, \ldots, 8$, let $L(i)$ be the sublattice generated by $\alpha_i, 0 \leq j \leq 8, j \neq i$. Then $L(i)$ is a rank 8 sublattice of $E_8$. In fact, $L(i)$ is the lattice associated with the Dynkin diagram obtained by removing the corresponding node $\alpha_i$ from the extended $E_8$ diagram (3.1). Note that the index $|E_8/L(i)|$ is equal to $n_i$, where $n_i$ is the coefficient of $\alpha_i$ in the left hand side of (3.2). Actually, we have

$$L(0) \cong E_8, \quad L(1) \cong A_1 \oplus E_7, \quad L(2) \cong A_2 \oplus E_6,$$

$$L(3) \cong A_3 \oplus D_5, \quad L(4) \cong A_4 \oplus A_4, \quad L(5) \cong A_5 \oplus A_2 \oplus A_1,$$

$$L(6) \cong A_7 \oplus A_1, \quad L(7) \cong D_8, \quad L(8) \cong A_8.$$  

(3.3)

**Remark 3.1.** If $n_i$ is not a prime, there is an intermediate sublattice as follows.

$$A_3 \oplus D_5 \subset D_8 \subset E_8,$$

$$A_5 \oplus A_2 \oplus A_1 \subset A_2 \oplus E_6 \subset E_8, \quad A_5 \oplus A_2 \oplus A_1 \subset A_1 \oplus E_7 \subset E_8,$$

$$A_7 \oplus A_1 \subset A_1 \oplus E_7 \subset E_8.$$

There are corresponding power maps between conjugacy classes of the Monster simple group, namely,

$$(4A)^2 = 2B, \quad (6A)^2 = 3A, \quad (6A)^3 = 2A, \quad (4B)^2 = 2A,$$

where $(mX)^k = nY$ means that the $k$-th power $g^k$ of an element $g$ in the conjugacy class $mX$ is in the conjugacy class $nY$ (cf. [2]).

3.1. **Coset subalgebras of the lattice VOA $V_{\sqrt{2}E_8}$.** We shall construct some VOAs $U$ corresponding to the nine nodes of the McKay diagram (1.1). In each case, we show that the VOA $U$ contains some conformal vectors of central charge 1/2 and the inner products among these conformal vectors are the same as the numbers given in the McKay diagram.

Let us explain the details of our construction. First, we fix $i \in \{0, 1, \ldots, 8\}$ and denote $L(i)$ by $L$. In each case, $|E_8/L| = n_i$ and $\alpha_i + L$ is a generator of the quotient group $E_8/L$. Hence we have

$$E_8 = L \cup (\alpha_i + L) \cup (2\alpha_i + L) \cup \cdots \cup ((n_i - 1)\alpha_i + L).$$  

(3.4)
Then the lattice VOA $V_{E_8}$ can be decomposed as
\[ V_{E_8} = V_{L} \oplus V_{L_0} \oplus V_{L_1} \oplus \cdots \oplus V_{L_{n-1}}, \]
where $V_{L_j} = V_{L_j} \oplus V_{L_{j+1}} \oplus \cdots \oplus V_{L_{n-1}}$, $j = 0, 1, \ldots, n-1$, are irreducible modules of $V_{L}$ (cf. [4]).

The quotient group $E_8/L$ induces an automorphism $\sigma$ of $V_{E_8}$ such that
\[ \sigma(u) = \xi^j u \quad \text{for any} \quad u \in V_{E_8/L}, \quad \xi = e^{2\pi \sqrt{-1}/n_i}, \]
where $\xi$ is a primitive $n_i$-th root of unity. More precisely, let
\[
\mathbf{a} = \begin{cases} 
\alpha_1 & \text{if } i = 0, \\
-\frac{1}{i+1}(\alpha_0 + 2\alpha_1 + \cdots + i\alpha_{i-1}) & \text{if } 1 \leq i \leq 5, \\
-\frac{1}{8}(\alpha_0 + 2\alpha_1 + \cdots + 6\alpha_5 + 7\alpha_8) & \text{if } i = 6, \\
\frac{1}{7}(\alpha_6 + \alpha_8) & \text{if } i = 7, \\
-\frac{1}{8}(\alpha_0 + 2\alpha_1 + \cdots + 8\alpha_7) & \text{if } i = 8.
\end{cases}
\]
Then $\langle \mathbf{a}, \alpha \rangle \in \mathbb{Z}$ for $0 \leq j \leq 8$ with $j \neq i$ and $\langle \mathbf{a}, \alpha \rangle \equiv -1/n_i \pmod{\mathbb{Z}}$. The automorphism $\sigma : V_{E_8} \rightarrow V_{E_8}$ is in fact defined by
\[ \sigma = e^{-\pi \sqrt{-1}\beta(0)} \quad \text{with} \quad \beta = \sqrt{2}\mathbf{a}. \]
For $u \in M(1) \otimes e^{\alpha} \subset V_{E_8}$, we have $\sigma(u) = e^{-\pi \sqrt{-1}(\beta, \alpha)} u$. Note that $\mathbf{a} + R$ is a generator of the quotient group $R^*/R$ for the cases $i \neq 0, 7$, where $R$ is an indecomposable component of the lattice $L$ of type $A$ and $R^*$ is the dual lattice of $R$.

For any lattice VOA $V_N$ associated with a positive definite even lattice $N$, there is a natural involution $\theta$ induced by the isometry $\alpha \rightarrow -\alpha$ for $\alpha \in N$. If $N = \sqrt{2}E_8$, which is doubly even, we may define $\theta : V_{E_8} \rightarrow V_{E_8}$ by
\[ \alpha(-n) \rightarrow -\alpha(-n) \quad \text{and} \quad e^{\alpha} \rightarrow e^{-\alpha} \]
for $\alpha \in \sqrt{2}E_8$ (cf. [7]). Then $\theta \sigma \theta = \sigma^{-1}$ and the group generated by $\theta$ and $\sigma$ is a dihedral group of order $2n_i$.

Let $R_1, \ldots, R_l$ be the indecomposable components of the lattice $L$ and $\Phi_1, \ldots, \Phi_l$ the corresponding root systems of $R_1, \ldots, R_l$ (cf. [5,3]). Then $L = R_1 \oplus \cdots \oplus R_l$ and
\[ V_{L_0} \cong V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_l}, \]
(see [6] for tensor products of VOAs). By [2,4], one obtains $2l$ mutually orthogonal conformal vectors
\[ s_k = s(\Phi_k), \quad \tilde{\omega}_k = \tilde{\omega}(\Phi_k), \quad k = 1, \ldots, l \]
such that the Virasoro element $\omega$ of $V_{L_0}$, which is also the Virasoro element of $V_{E_8}$, can be written as a sum of these conformal vectors
\[ \omega = s^1 + \cdots + s^l + \tilde{\omega}^1 + \cdots + \tilde{\omega}^l. \]

Now we define $U$ to be a coset (or commutant) subalgebra
\[ U = \{ v \in V_{E_8} \mid (s_k)^{1}v = 0 \text{ for all } k = 1, \ldots, l \}. \]
Note that $U$ is a VOA with the Virasoro element $\omega' = \tilde{\omega}^1 + \cdots + \tilde{\omega}^l$ and the automorphism $\sigma$ defined by (3.5) induces an automorphism of order $n_i$ on $U$. By abuse of notation, we denote it by $\sigma$ also.
Remark 3.2. In [11], it is shown that \( \{ v \in V_{\sqrt{2}A_n} \mid s(A_n) v = 0 \} \) is isomorphic to a parafermion algebra \( W_{n+1}(2n/(n+3)) \) of central charge \( 2n/(n+3) \). Thus, if \( L \) has some indecomposable component of type \( A_n \), then \( U \) contains some subalgebra isomorphic to a parafermion algebra. It is well known [18] that the parafermion algebra \( W_{n+1}(2n/(n+3)) \) possesses a certain \( \mathbb{Z}_{n+1} \) symmetry in the fusion rules among its irreducible modules. The automorphism \( \sigma \) is in fact related to such a symmetry. More details about the relation between coset subalgebra \( U \) and the parafermion algebra \( W_{n+1}(2n/(n+3)) \) can be found in [12].

3.2. Conformal vectors of central charge 1/2. Next, we shall study some conformal vectors in \( V_{\sqrt{2}E_8} \). We shall also show that the coset subalgebra \( U \) always contains some conformal vectors of central charge 1/2. Moreover, the inner products among these conformal vectors will be discussed.

Recall that the lattice \( \sqrt{2}E_8 \) can be constructed by using the \([8,4,4]\) Hamming code \( H_8 \) and the Construction A (cf. [3]). That means
\[
\sqrt{2}E_8 = \left\{ (a_1, \ldots, a_8) \in \mathbb{Z}^8 \mid (a_1, \ldots, a_8) \in H_8 \mod 2 \right\}. \tag{3.11}
\]

We denote the vectors \( (0,0,0,0,0,0,0,0) \) and \( (1,1,1,1,1,1,1,1) \) by \( 0 \) and \( 1 \), respectively. For any \( \gamma \in H_8 \), we define
\[
X^0_\gamma = \sum_{\alpha \equiv \gamma \mod 2 \atop \langle \alpha, \alpha \rangle = 4} (-1)^{(\alpha,0)/2} e^\alpha = \sum_{\alpha \equiv \gamma \mod 2 \atop \langle \alpha, \alpha \rangle = 4} e^\alpha,
\]
and for any binary word \( \delta \in \mathbb{Z}_2^8 \), we define
\[
\hat{e}_\delta^\epsilon = \frac{1}{16} \hat{\omega} + \frac{1}{32} \sum_{\gamma \in H_8} (-1)^{\langle \delta, \gamma \rangle} X^\epsilon_\gamma, \quad \epsilon = 0, 1,
\]
where \( \omega \) is the Virasoro element of the VOA \( V_{\sqrt{2}E_8} \). Note that \( X^1_\gamma = 0 \) for any \( \epsilon = 0, 1 \) and that \( \hat{e}_\delta^\epsilon = \hat{e}_\eta^\epsilon \) if and only if \( \eta \in \delta + H_8 \)

Lemma 3.3. For any \( \epsilon = 0, 1 \) and \( \delta \in \mathbb{Z}_2^8 \), \( \hat{e}_\delta^\epsilon \) is a conformal vector of central charge 1/2. The inner product among them are as follows.
\[
\langle \hat{e}_\delta^\epsilon, \hat{e}_\eta^\epsilon \rangle = \begin{cases} 0 & \text{if } \delta + \eta \text{ is even} \\ 1/32 & \text{if } \delta + \eta \text{ is odd} \end{cases}
\]
for any \( \eta \notin \delta + H_8 \), and
\[
\langle \hat{e}_\delta^0, \hat{e}_\eta^1 \rangle = 0
\]
for any \( \delta, \eta \in \mathbb{Z}_2^8 \).
Proof. We have
\[(X^{e}_\gamma)_{1}(X^{e}_\zeta) = 4X^{e}_{\gamma+\zeta}\] if \(|\gamma+\zeta| = 4\),
\[(X^{e}_0)_{1}(X^{e}_0) = \sum_{\alpha \equiv \gamma \mod 2 \atop \langle \alpha,\alpha \rangle = 4} \frac{1}{2}(-1)^2 \cdot 1.\]

Moreover, for any \(\gamma \in H_8\) with \(|\gamma| = 4\),
\[(X^{e}_\gamma)_{1}(X^{e}_\gamma) + (X^{e}_{1+\gamma})_{1}(X^{e}_{1+\gamma}) = \sum_{\alpha \equiv \gamma \mod 2 \atop \langle \alpha,\alpha \rangle = 4} \frac{1}{2}(-1)^2 \cdot 1 + \sum_{\alpha \equiv 1+\gamma \mod 2 \atop \langle \alpha,\alpha \rangle = 4} \frac{1}{2}(-1)^2 \cdot 1 + 8X^{e}_0.\]

Note also that
\[\sum_{\gamma \in H_8} \sum_{\beta \in \Phi(E_8)} \frac{1}{2}(-1)^2 \cdot 1 = \sum_{\beta \in \Phi^+(E_8)} \beta(-1)^2 \cdot 1 = 2 \sum_{\beta \in \Phi^+(E_8)} \beta(-1)^2 \cdot 1.\]

In addition, we have
\[\langle X^{e}_\gamma, X^{e}_\zeta \rangle = \begin{cases} 16 & \text{if } \gamma = \zeta \text{ and } \langle \gamma, \gamma \rangle \neq 8, \\ 0 & \text{otherwise}, \end{cases}\]
\[\langle X^{0}_\gamma, X^{1}_\zeta \rangle = \begin{cases} -16 & \text{if } \gamma = \zeta = 0, \\ 0 & \text{otherwise}. \end{cases}\]

Then since \(\omega_1\omega = 2\omega\) and \(\langle \omega, \omega \rangle = 4\), it follows that
\[\begin{aligned}(\hat{e}^{e}_\delta, \hat{e}^{e}_\delta) &= \left(\frac{1}{16}\omega + \frac{1}{32} \sum_{\gamma \in H_8} (-1)^{\langle \delta, \gamma \rangle} X^{e}_\gamma \right)_{1} \left(\frac{1}{16}\omega + \frac{1}{32} \sum_{\gamma \in H_8} (-1)^{\langle \delta, \gamma \rangle} X^{e}_\gamma \right) \\
&= \frac{1}{2^8} \times 2\omega + 2 \times \frac{1}{16} \times \frac{1}{32} \times 2 \sum_{\gamma \in H_8} (-1)^{\langle \delta, \gamma \rangle} X^{e}_\gamma \\
&\quad + \frac{1}{2^{10}} \sum_{\beta \in \Phi^+(E_8)} \sum_{\gamma \in H_8} 2\beta(-1)^2 \cdot 1 + 56 \sum_{\gamma \in H_8} (-1)^{\langle \delta, \gamma \rangle} X^{e}_\gamma \\
&= \frac{1}{8}\omega + \frac{1}{16} \sum_{\gamma \in H_8} (-1)^{\langle \delta, \gamma \rangle} X^{e}_\gamma = 2\hat{e}^{e}_\delta,\end{aligned}\]
and
\[\langle \hat{e}^{e}_\delta, \hat{e}^{e}_\delta \rangle = \frac{1}{2^8} \times 4 + \frac{1}{2^{10}} \times 240 = \frac{1}{4}.\]

Hence \(\hat{e}^{e}_\delta\) is a conformal vector of central charge \(1/2\).

For any \(\eta \notin \delta + H_8\), we calculate that
\[\langle \hat{e}^{e}_\delta, \hat{e}^{e}_\eta \rangle = \frac{1}{2^8} \times 4 + \frac{1}{2^{10}} \sum_{\gamma \in H_8} (-1)^{\langle \delta+\eta, \gamma \rangle} \langle X^{e}_\gamma, X^{e}_\gamma \rangle \]
\[= \begin{cases} \frac{1}{64} + \frac{1}{2^9} \times 16 \times (7 - 8) = 0 & \text{if } \delta + \eta \text{ is even,} \\
\frac{1}{64} + \frac{1}{2^9} \times 16 \times (8 - 7) = \frac{1}{32} & \text{if } \delta + \eta \text{ is odd}. \end{cases}\]
Note that there are exactly eight elements in $H_8$ which are orthogonal to $\delta + \eta$. Note also that $\delta + \eta$ is orthogonal to $(1, 1, 1, 1, 1, 1, 1, 1)$ if and only if $\delta + \eta$ is even.

Finally, for any $\delta, \eta \in \mathbb{Z}_2^8$ we obtain
\[
\langle \hat{e}_\delta^0, \hat{e}_\eta^1 \rangle = \frac{1}{2^8} \times 4 - \frac{1}{2^{10}} \times 16 = 0.
\]
\[
\square
\]

In Miyamoto [16], certain conformal vectors of central charge $1/2$ are constructed inside the Hamming code VOA. Our construction of $\hat{e}_\delta^0$ is essentially a lattice analogue of Miyamoto's construction. In fact, take $\lambda_j = (0, \ldots, 2, \ldots, 0) \in \mathbb{Z}^8$ to be the element in $\sqrt{2}E_8$ such that the $j$-th entry is 2 and all other entries are zero. Then we have a set of 16 mutually orthogonal conformal vectors of central charge $1/2$ given by
\[
\omega_{\lambda_j}^\pm = \frac{1}{16} \lambda_j (-1)^2 \cdot 1 \pm \frac{1}{4} (e^{\lambda_j} + e^{-\lambda_j}), \quad j = 1, 2, \ldots, 8.
\]

A set of mutually orthogonal conformal vectors of central charge $1/2$ whose sum is equal to the Virasoro element in a VOA is called a Virasoro frame. Thus, $\{\omega_{\lambda_j}^\pm | 1 \leq j \leq 8\}$ is a Virasoro frame of $V_{\sqrt{2}E_8}$. With respect to this Virasoro frame, the lattice VOA $V_{\sqrt{2}E_8}$ is a code VOA (cf. [16]). Let $V_{\sqrt{2}E_8}^+$ be the fixed point subalgebra of $V_{\sqrt{2}E_8}$ under the automorphism $\theta$ (cf. (3.8)). Then $\omega_{\lambda_j}^\pm \in V_{\sqrt{2}E_8}^+$ and $V_{\sqrt{2}E_8}^+$ is isomorphic to a code VOA $M_D$, where $D$ is the second order Reed-Müller code $RM(4, 2)$ of length 16. Note that $\dim \ RM(4, 2) = 11$ and the dual code of $RM(4, 2)$ is the first order Reed-Müller code $RM(4, 1)$ with the generating matrix
\[
\left(\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}\right)
\]

Let $H^+$ and $H^-$ be the subcodes of $D$ whose supports are contained in the positions corresponding to $\{\omega_{\lambda_j}^+ | 1 \leq j \leq 8\}$ and $\{\omega_{\lambda_j}^- | 1 \leq j \leq 8\}$, respectively. Then $H^+$ and $H^-$ are both isomorphic to the $[8, 4, 4]$ Hamming code $H_8$. The conformal vectors $\hat{e}_0^0$ and $\hat{e}_0^1$ are actually the conformal vectors $s_8$ constructed by Miyamoto [16] using the Hamming code VOAs $M_{H^+}$ and $M_{H^-}$, respectively.

**Proposition 3.4.** The set $\{\hat{e}_0^0, \hat{e}_0^1 | \delta, \zeta \in \mathbb{Z}_2^8 / H_8, \delta, \zeta \text{ are even}\}$ is a Virasoro frame of $V_{\sqrt{2}E_8}^+$. Moreover, $V_{\sqrt{2}E_8}^+ \cong M_{RM(4, 2)}$ with respect to this frame, where $M_{RM(4, 2)}$ denotes the code VOA associated with the second order Reed-Müller code $RM(4, 2)$.

**Proof.** The first assertion follows from Lemma 3.3. As mentioned above, we know that $V_{\sqrt{2}E_8}^+ \cong M_D$ with respect to the frame $\{\omega_{\lambda_j}^+ | 1 \leq j \leq 8\}$, where $D \cong RM(4, 2)$. It contains a subalgebra isomorphic to $M_{H^+} \otimes M_{H^-}$. For convenience, we arrange the positions of $\{\omega_{\lambda_j}^+\}$ so that the support supp $H^+$ of $H^+$ is $(1^8, 0^8)$ and the support supp $H^-$ of $H^-$ is $(0^8, 1^8)$. Let $\{\beta_0, \beta_1, \ldots, \beta_7\}$ with $\beta_0 = 0$ be a complete set of coset representatives of
Let
\[ V_+^{\sqrt{2E_8}} \cong M_{H^+ \oplus H^-} \oplus \bigoplus_{i=1}^{7} M_{\beta_i(H^+ \oplus H^-)}. \]

By a result of Miyamoto [16], \( M_{H^+ \oplus H^-} \) is still isomorphic to the code VOA \( M_{H^+ \oplus H^-} \) associated with \( H^+ \oplus H^- \) with respect to the frame \( (e^0_x, e^1_x) \) \( \delta, \zeta \in \mathbb{Z}_2^8 / H_8, \delta, \zeta \) (are even). Moreover, we know that \( (1^8, 0^8) \) and \( (0^8, 1^8) \) are contained in the dual code of \( D \). Thus \( \langle (1^8, 0^8), \beta_i \rangle = \langle (0^8, 1^8), \beta_i \rangle = 0 \) for all \( i \). Let \( \beta^+ \) and \( \beta^- \) be such that \( \text{supp}\beta^+ \subset \text{supp}\beta^- \subset \text{supp}H^- \), and \( \beta_i = \beta^+ + \beta^- \). Then \( M_{\beta_i(H^+ \oplus H^-)} \cong M_{\beta^+ H^+} \otimes M_{\beta^- H^-} \) and both of \( M_{\beta^+ H^+} \) and \( M_{\beta^- H^-} \) are of integral weight. Hence, by [16], \( M_{\beta_i(H^+ \oplus H^-)} \) is again isomorphic to \( M_{\beta_i(H^+ \oplus H^-)} \) with respect to the frame \( (e^0_x, e^1_x) \) \( \delta, \zeta \in \mathbb{Z}_2^8 / H_8, \delta, \zeta \) (are even) and thus we still have \( V_+^{\sqrt{2E_8}} \cong M_D \).

Now let
\[
\hat{e} = e^0_x = \frac{1}{16} \omega + \frac{1}{32} \sum_{\alpha \in \Phi(H^+)} (e^{\sqrt{2} \alpha} + e^{-\sqrt{2} \alpha}),
\]
\[
\hat{f} = \sigma \hat{e},
\]
where \( \sigma \) is the automorphism defined by (3.5). These conformal vectors of central charge \( 1/2 \) play an important role for the rest of the paper.

Let \( \Phi \) be the root system of \( L = L(i) \). Let \( H_j = \{ \alpha \in j\alpha_i + L \mid \langle \alpha, \alpha \rangle = 2 \} \) be the set of all roots in the coset \( j\alpha_i + L \) for \( j = 1, \ldots, n_i - 1 \). Then
\[
\Phi(E_8) = \Phi \cup \bigcup_{j=1}^{n_i-1} H_j.
\]

We introduce weight 2 elements \( X^j \), namely,
\[
X^j = \sum_{\alpha \in H_j} e^{\sqrt{2} \alpha}, \quad j = 1, \ldots, n_i - 1.
\]

Then
\[
\hat{e} = \frac{1}{16} \omega + \frac{1}{32} \left( \sum_{\alpha \in \Phi} e^{\sqrt{2} \alpha} + \sum_{j=1}^{n_i-1} X^j \right),
\]
\[
\hat{f} = \frac{1}{16} \omega + \frac{1}{32} \left( \sum_{\alpha \in \Phi} e^{\sqrt{2} \alpha} + \sum_{j=1}^{n_i-1} \xi^j X^j \right),
\]

where \( \xi = e^{2\pi \sqrt{-1}/n_i} \) is a primitive \( n_i \)-th root of unity.

**Lemma 3.5.** (1) \( X^j \in U, \ j = 1, \ldots, n_i - 1. \)
(2) \( \hat{e}, \hat{f} \in U. \)

**Proof.** Let \( s^k \) be defined as in (3.9). Then by a similar argument as in the proof of Proposition 2.2, we can verify that \( (s^k)_1 X^j = 0 \) and \( (s^k)_1 \hat{e} = 0 \) for \( k = 1, \ldots, l \). Thus \( X^j, \hat{e} \in U \) by the definition (3.10) of \( U \). Since \( \sigma \) leaves \( U \) invariant, we also have \( \hat{f} \in U. \)
Remark 3.6. The Weyl group \( W(E_8) \) of the root system of type \( E_8 \) acts naturally on the lattice VOA \( V_{\sqrt{2}E_8} \) and \( \hat{e} \) is the only conformal vector among \( \hat{e}_0^0, \hat{e}_1^1 \) which is fixed by \( W(E_8) \). The conformal vector \( \hat{f} \) is fixed by the Weyl group \( W(\Phi) = W(\Phi_1) \times \cdots \times W(\Phi_l) \) of the root system \( \Phi = \Phi_1 \oplus \cdots \oplus \Phi_l \) of \( L = L(i) \). The conformal vector \( \hat{e} \) is also fixed by the automorphism \( \theta \) (cf. (3.8)). However, \( \hat{f} \) is not fixed by \( \theta \) in general.

Theorem 3.7. Let \( \hat{e}, \hat{f} \) be defined as in (3.12). Then

\[
\langle \hat{e}, \hat{f} \rangle = \begin{cases} 
1/4 & \text{if } i = 0, \\
1/32 & \text{if } i = 1, \\
13/2^{10} & \text{if } i = 2, \\
1/2^7 & \text{if } i = 3, \\
3/2^9 & \text{if } i = 4, \\
5/2^{10} & \text{if } i = 5, \\
1/2^8 & \text{if } i = 6, \\
0 & \text{if } i = 7, \\
1/2^8 & \text{if } i = 8.
\end{cases}
\]

(3.15)

In other words, the values of \( \langle \hat{e}, \hat{f} \rangle \) are exactly the values given in McKay’s diagram (1.1).

Proof. By (3.14), we can easily obtain that

\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}} \left( |\Phi| + \sum_{j=1}^{n-1} \xi^j |H_j| \right),
\]

where \( H_j = \{ \alpha \in j\alpha_i + L \mid \langle \alpha, \alpha \rangle = 2 \} \).

If \( i = 0 \), then \( n_0 = 1 \) and \( |\Phi| = 240 \). Hence

\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{240}{2^{10}} = \frac{1}{4}.
\]

If \( i = 1 \), then \( n_1 = 2 \), \( |\Phi| = |\Phi(A_1)| + |\Phi(E_7)| = 128 \), and \( |H_1| = 112 \). Hence

\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}} (128 - 112) = \frac{1}{32}.
\]

If \( i = 2 \), then \( n_2 = 3 \), \( |\Phi| = |\Phi(A_2)| + |\Phi(E_8)| = 78 \), and \( |H_1| = |H_2| = 81 \). Hence

\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}} (78 - 81) = \frac{13}{210}.
\]

If \( i = 3 \), then \( n_3 = 4 \), \( |\Phi| = |\Phi(A_3)| + |\Phi(D_5)| = 52 \), \( |H_1| = |H_3| = 64 \), and \( |H_2| = 60 \). Hence

\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}} (52 - 60) = \frac{1}{27}.
\]

If \( i = 4 \), then \( n_4 = 5 \), \( |\Phi| = |\Phi(A_4)| + |\Phi(A_4)| = 40 \), and \( |H_1| = |H_2| = |H_3| = |H_4| = 50 \). Hence

\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}} (40 - 50) = \frac{3}{29}.
\]
If \( i = 5 \), then \( n_5 = 6 \), \(|\Phi| = |\Phi(A_1)| + |\Phi(A_2)| + |\Phi(A_5)| = 38\), \( |H_1| = |H_5| = 36\), \( |H_2| = |H_4| = 45 \), and \( |H_3| = 40 \). Hence
\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{26} + \frac{1}{2^{10}} (38 + 36 - 45 - 40) = \frac{5}{2^{10}}.
\]

If \( i = 6 \), then \( n_6 = 4 \), \(|\Phi| = |\Phi(A_1)| + |\Phi(A_7)| = 58\), \( |H_1| = |H_3| = 56 \), and \( |H_2| = 70 \). Hence
\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{26} + \frac{1}{2^{10}} (58 - 70) = \frac{1}{2^{8}}.
\]

If \( i = 7 \), then \( n_7 = 2 \), \(|\Phi| = |\Phi(D_8)| = 112\), and \( |H_1| = 128 \). Hence
\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{26} + \frac{1}{2^{10}} (112 - 128) = 0.
\]

If \( i = 8 \), then \( n_8 = 3 \), \(|\Phi| = |\Phi(A_8)| = 72\), and \( |H_1| = |H_2| = 84 \). Hence
\[
\langle \hat{e}, \hat{f} \rangle = \frac{1}{26} + \frac{1}{2^{10}} (72 - 84) = \frac{1}{2^{8}}.
\]

Thus we have proved the theorem. \( \square \)

Remark 3.8. The same result still holds if we replace \( \hat{e} \) by \( \hat{e}_\delta \) and \( \hat{f} = \sigma \hat{e} \) by \( \sigma \hat{e}_\delta \) for any \( \epsilon = 0, 1 \) and \( \delta \in \mathbb{Z}_2^8 \).

4. Miyamoto’s \( \tau \)-involutions and the canonical automorphism \( \sigma \)

Let \( V \) be a VOA. If \( V \) contains a conformal vector \( w \) of central charge \( 1/2 \) such that the subalgebra \( \text{Vir}(w) \) generated by \( w \) is isomorphic to the Virasoro VOA \( L(1/2, 0) \), then an automorphism \( \tau_w \) of \( V \) with \( (\tau_w)^2 = 1 \) can be defined. Indeed, \( V \) is a direct sum of irreducible \( \text{Vir}(w) \)-modules. Denote by \( W_\delta \) the sum of all irreducible direct summands which are isomorphic to \( L(1/2, \delta) \). Denote by \( W_\delta \) the sum of all irreducible direct summands which are isomorphic to \( L(1/2, \delta) \). Then \( \tau_w \) is defined to be 1 on \( W_0 \oplus W_1/2 \) and \(-1 \) on \( W_{1/16} \) (cf. [15, 17]). Thus \( \tau_w \) is the identity if \( V \) has no irreducible direct summand isomorphic to \( L(1/2, 1/16) \). We call \( \tau_w \) the Miyamoto involution or the \( \tau \)-involution associated with \( w \).

In this section, we shall study the relationship between the canonical automorphism \( \sigma \) and the Miyamoto involutions \( \tau_\hat{e}, \tau_{\hat{e}} \), \ldots, and \( \tau_{\sigma^{n_1-1} \hat{e}} \). Let us recall two conformal vectors \( \hat{e} \) and \( \hat{f} \) of central charge \( 1/2 \) defined by (3.12) and two automorphisms \( \sigma \) and \( \theta \) introduced in Subsection 3.1.

Lemma 4.1. As automorphisms of \( V_{\sqrt{2} E_8} \), \( \tau_\hat{e} = \theta \).

Proof. By Proposition 3.4 we know that \( \{ \hat{e}_{\delta}, \hat{e}_\zeta | \delta, \zeta \in \mathbb{Z}_2^8/H_8, \delta, \zeta \text{ even} \} \) is a Virasoro frame of \( V_{\sqrt{2} E_8}^+ \) and with respect to this frame, \( V_{\sqrt{2} E_8}^+ \) is a code VOA isomorphic to \( M_{\text{RM}(4,2)} \). Therefore, \( \tau_{\hat{e}}|_{V_{\sqrt{2} E_8}^+} = \text{id} \). On the other hand,
\[
\hat{e}_1 \gamma(-1) \cdot 1 = \frac{1}{16} \gamma(-1) \cdot 1 + \frac{1}{32} \sum_{\alpha \in \Phi^+(E_8)} (e^{\sqrt{2} \alpha} + e^{-\sqrt{2} \alpha})_1 \gamma(-1) \cdot 1
\]
\[
= \frac{1}{16} \gamma(-1) \cdot 1
\]
for any $\gamma \in \sqrt{2}E_8$. By the definition of $\tau_\hat{e}$, this implies that $\tau_\hat{e}(\gamma(-1) \cdot 1) = -\gamma(-1) \cdot 1$. Then $\tau_\hat{e}|_{\sqrt{2}E_8} = -\text{id}$, since $V_{\sqrt{2}E_8}$ is an irreducible $V_{\sqrt{2}E_8}$-module. Hence $\tau_\hat{e} = \theta$ as automorphisms of $V_{\sqrt{2}E_8}$. □

**Theorem 4.2.** As automorphisms of $V_{\sqrt{2}E_8}$, $\tau_\hat{e}\tau_f = (\sigma^{-1})^2 = e^{2\pi \sqrt{-1}\theta(0)}$ and thus $|\tau_\hat{e}\tau_f| = n_i$ if $n_i$ is odd and $|\tau_\hat{e}\tau_f| = n_i/2$ if $n_i$ is even.

**Proof.** Since $\hat{f} = \sigma \hat{e}$, we have $\tau_f = \sigma \tau_\hat{e}\sigma^{-1}$. By (3.5) and the preceding lemma, we also have $\tau_\hat{e}\sigma\tau_\hat{e} = \sigma\theta\theta = \sigma^{-1}$. Hence the assertion holds by (3.7). □

Next, we shall extend $\tau_\hat{e}, \tau_f,$ and $\sigma$ to the Leech lattice VOA $V_\Lambda$. According to the presentation (3.11) of $\sqrt{2}E_8$, the dual lattice $L$ of $\sqrt{2}E_8$ is given by

$$L = \{(a_1, \ldots, a_8) \in \frac{1}{2}\mathbb{Z}^8 \mid 2(a_1, \ldots, a_8) \in H_8 \mod 2\}.$$

Note that $|L/\sqrt{2}E_8| = 2^8$. Note also that

$$V_L = S(h^-) \otimes \mathbb{C}\{L\} \cong \bigoplus_{\alpha + \sqrt{2}E_8 \in L/\sqrt{2}E_8} V_{\alpha + \sqrt{2}E_8}$$

as a module of $V_{\sqrt{2}E_8}$.

For any coset $\alpha + \sqrt{2}E_8$ of $\sqrt{2}E_8$ in $L$, one can always find a coset representative $\alpha$ whose square norm is minimum in the coset such that $\alpha$ is in one of the following forms.

$$(0)^8, \quad (1,0)^7, \quad (1^2,0^6), \quad ((1/2)^4,0^4),$$

$$(1/2)^3,-1/2,0^4), \quad ((1/2)^2,(-1/2)^2,0^4), \quad ((1/2)^4,1,0^3) \quad ((1/2)^3,-1/2,1,0^3), \quad ((1/2)^5), \quad ((1/2)^7,,-1/2), \quad ((1/2)^6,(-1/2)^2).$$

The square norm $\langle \alpha, \alpha \rangle$ of such $\alpha$ is 0, 1, or 2. Moreover, if $\langle \alpha, \alpha \rangle = 2$, then $\alpha$ can be written as a sum $\alpha = a + b$, where $a, b \in L$ are in the above forms with $\langle a, a \rangle = \langle b, b \rangle = 1$ and $\langle a, b \rangle = 0$. In particular, the minimal weight of the irreducible module $V_{\alpha + \sqrt{2}E_8}$ is either 1/2 or 1 for $\alpha \notin \sqrt{2}E_8$.

Now $\sigma = e^{-\pi \sqrt{-1}\theta(0)}$ (cf. (3.7)) acts on $V_L$ as an automorphism of order $2n_i$. The $\tau$-involution $\tau_\hat{e}$ also acts on $V_L$. In fact, $V_{\alpha + \sqrt{2}E_8}$ is $\tau_\hat{e}$-invariant for any coset $\alpha + \sqrt{2}E_8$ of $\sqrt{2}E_8$ in $L$.

**Lemma 4.3.** For any $x \in L$ with $\langle x, x \rangle = 1$, $\tau_\hat{e}(e^x) = -e^{-x}$.

**Proof.** If $\langle \gamma, \gamma \rangle = 4$ and $\langle \gamma + x, \gamma + x \rangle = 1$ for some $\gamma \in \sqrt{2}E_8$, then $\langle \gamma, x \rangle = -2$ and $\gamma + x = -x$. Thus, by the definition of $\hat{e}$ it follows that

$$\hat{e}_1e^x = \frac{1}{16}e^{1/2}e^x + \frac{1}{32}e^{-x} \quad \text{and} \quad \hat{e}_1e^{-x} = \frac{1}{16}\left(\frac{1}{2}e^{-x}\right) + \frac{1}{32}e^x.$$

Therefore, $\hat{e}_1(e^x + e^{-x}) = \frac{1}{6}(e^x + e^{-x})$ and $\hat{e}_1(e^x - e^{-x}) = 0$. Hence $\tau_\hat{e}(e^x + e^{-x}) = -(e^x + e^{-x})$ and $\tau_\hat{e}(e^x - e^{-x}) = e^x - e^{-x}$ by the definition of $\tau_\hat{e}$, and so $\tau_\hat{e}(e^x) = -e^{-x}$. □

**Lemma 4.4.** Let $\alpha + \sqrt{2}E_8$ be a coset of $\sqrt{2}E_8$ in $L$. Then for any $u \in V_{\alpha + \sqrt{2}E_8}$, $\tau_\hat{e}\sigma\tau_\hat{e}(u) = \sigma^{-1}(u)$. 
Proof. We have $V_{\alpha + \sqrt{2}E_8} = \text{span}_{C}\{v_ne^\alpha \mid v \in V_{\sqrt{2}E_8}, \; n \in \mathbb{Z}\}$, since $V_{\alpha + \sqrt{2}E_8}$ is an irreducible $V_{\sqrt{2}E_8}$-module. If $\langle \alpha, \alpha \rangle = 1$, then we know that $\tau_e(e^\alpha) = -e^{-\alpha}$ by Lemma 4.3. Thus $\tau_e \sigma \tau_e(e^\alpha) = \sigma^{-1}(e^\alpha)$ and so

$$\tau_e \sigma \tau_e(v_ne^\alpha) = (\tau_e \sigma \tau_e(v))_n (\tau_e \sigma \tau_e(e^\alpha)) = \sigma^{-1}(v)_n \sigma^{-1}(e^\alpha) = \sigma^{-1}(v_ne^\alpha)$$

for any $v \in V_{\sqrt{2}E_8}$ by Lemma 4.4.

If $\langle \alpha, \alpha \rangle = 2$, then $\alpha = a + b$ for some vectors $a, b$ in the forms of (4.1) with $\langle a, a \rangle = \langle b, b \rangle = 1$ and $\langle a, b \rangle = 0$. In this case, $e^\alpha = (e^a)_1 e^b$ and we still have $\tau_e \sigma \tau_e(e^\alpha) = \sigma^{-1}(e^\alpha)$. Thus for any $v \in V_{\sqrt{2}E_8}$,

$$\tau_e \sigma \tau_e(v_ne^\alpha) = \sigma^{-1}(v)_n \sigma^{-1}(e^\alpha) = \sigma^{-1}(v_ne^\alpha)$$

as required. \qed

As a consequence, we have the following proposition.

**Proposition 4.5.** For any $u \in V_L$, $\tau_e \sigma \tau_e(u) = \sigma^{-1}(u)$. Hence $\tau_e \tau_f = (\sigma^{-1})^2 = e^{2\pi \sqrt{-1} \beta(0)}$ as automorphisms of $V_L$.

Now we discuss the situation in the Leech lattice VOA $V_\Lambda$. First let us recall the following theorem [5, Theorem 4.1] (see also [10, 13]).

**Theorem 4.6.** For any even unimodular lattice $N$ of rank 24, there is at least one (in general many) isometric embedding of $\sqrt{2}N$ into the Leech lattice $\Lambda$.

It is well known (cf. [10]) that the Leech lattice $\Lambda$ can be constructed by “Construction A” for $\mathbb{Z}_4$-codes of length 24. In fact,

$$\Lambda = A_4(C) = \frac{1}{2}\{x \in \mathbb{Z}^{24} \mid x \equiv c \mod 4 \quad \text{for some} \; c \in C\}$$
for some type II self-dual $\mathbb{Z}_4$-code $C$ of length 24. By [10], $C$ can be taken to be the $\mathbb{Z}_4$-code having the generating matrix (4.2).

$$
\begin{pmatrix}
2222 & 0000 & 0000 & 0000 & 0000 & 0000 \\
0222 & 2200 & 0000 & 0000 & 0000 & 0000 \\
0000 & 0022 & 2020 & 0000 & 0000 & 0000 \\
0000 & 0000 & 0202 & 2020 & 0000 & 0000 \\
0000 & 0000 & 0000 & 0202 & 2002 & 0000 \\
2020 & 2020 & 0000 & 0000 & 0000 & 0000 \\
0000 & 0220 & 2200 & 0000 & 0000 & 0000 \\
0000 & 0000 & 0000 & 0202 & 2002 & 0000 \\
0000 & 0000 & 0000 & 0022 & 2020 & 0000 \\
2000 & 2000 & 2000 & 0000 & 0000 & 0000 \\
3012 & 1010 & 1001 & 1001 & 1100 & 1100 \\
3201 & 1001 & 1100 & 1100 & 1010 & 1010
\end{pmatrix}
$$

For any $\mathbb{Z}_4$-code $C$ of length $n$, one can obtain a binary code

$$
B(C) = \{(b_1, \dots, b_n) \in \mathbb{Z}_4^n \mid (2b_1, \dots, 2b_n) \in C\},
$$

where $2b_j$ should be considered as $0 \in \mathbb{Z}_4$ if $b_j = 0 \in \mathbb{Z}_2$ and $2 \in \mathbb{Z}_4$ if $b_j = 1 \in \mathbb{Z}_2$. Moreover, the lattice

$$
L_{B(C)} = \{x \in \mathbb{Z}^n \mid x \in B(C) \mod 2\}
$$

is a sublattice of $A_4(C)$. In the case for $C = C$, the binary code $B(C)$ contains a subcode isomorphic to $H_8 \oplus H_8 \oplus H_8$. Thus by (3.11), we have an explicit embedding of $\sqrt{2}E_8^3$ into the Leech lattice $\Lambda$.

Now let $\sqrt{2}E_8^3 \to \Lambda$ be any embedding of $\sqrt{2}E_8^3$ into the Leech lattice $\Lambda \subset \mathcal{L}^3$. Let $\tilde{\beta} = \sqrt{2}(a, 0, 0) \in \mathcal{L}^3$, where $a$ is defined as in (3.6). Define $\tilde{\sigma} : (V_{\Lambda})^{33} \to (V_{\Lambda})^{33}$ by

$$
\tilde{\sigma} = \sigma \otimes 1 \otimes 1 = e^{-\pi \sqrt{-1} \tilde{\beta}(0)}.
$$

Then $\tilde{\sigma}$ is an automorphism of $V_{\Lambda}$. Moreover, the following theorem holds.

**Theorem 4.7.** Let $\tilde{\beta}$ and $\tilde{\sigma}$ be defined as above. Then as automorphisms of $V_{\Lambda}$, $\tau_i \tau_j = (\tilde{\sigma}^{-1})^2 = e^{2\pi \sqrt{-1} \tilde{\beta}(0)}$ and $|\tau_i \tau_j| = n_i$ for any $i = 0, 1, \ldots, 8$.

5. **Correspondence with Conway’s axes.**

Recall the elements $\tilde{\omega}^k$ and $X^j$ defined by (3.9) and (3.13). It turns out that the Griess algebra $U_2$ of $U$ is generated by $\hat{e}$ and $\hat{f}$ and is of dimension $l + n_i - 1$ with basis $\tilde{\omega}^k, 1 \leq k \leq l$ and $X^j, 1 \leq j \leq n_i - 1$ (see [12] for details). We can verify that the Griess algebra $U_2$ coincides with the algebra described in Conway [1 Table 3]. In [1], it is shown that for each $2A$-involution of the Monster simple group, there is a unique idempotent in
the Monstrous Griess algebra $V_2^*$ corresponding to the involution. Such an idempotent is called an axis. By Miyamoto [13], an axis is exactly half of a conformal vector of central charge 1/2. Note that the product $t \ast t'$ and the inner product $(t, t')$ of two axes $t, t'$ in $\mathfrak{H}$ are equal to $t \cdot t' = t_1 t'$ and $(t, t')/2$, respectively in our notation. Let $t_n$ be as in $\mathfrak{H}$. We denote $t, u, v,$ and $w$ of $\mathfrak{H}$ by $t_{2A}, u_{3A}, v_{4A},$ and $w_{5A}$, respectively.

In each of the nine cases, we obtain an isomorphism of our Griess algebra $\mathfrak{H}$ to Conway’s algebra generated by two axes through the following correspondence between our conformal vectors and Conway’s axes.

1A case. $\hat{e} \leftrightarrow \frac{1}{32} t_0.$

2A case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad j = 0, 1, \quad \tilde{\omega}^1 \leftrightarrow \frac{1}{45} t_{2A}.$

3A case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad j = 0, 1, 2, \quad \tilde{\omega}^1 \leftrightarrow \frac{1}{45} u_{3A}.$

4A case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad 0 \leq j \leq 3, \quad \tilde{\omega}^1 \leftrightarrow \frac{1}{96} v_{4A}.$

5A case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad 0 \leq j \leq 4, \quad \tilde{\omega}^1 - \tilde{\omega}^2 \leftrightarrow -\frac{1}{30\sqrt{5}} w_{5A}.$

6A case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad 0 \leq j \leq 5, \quad \tilde{\omega}^2 \leftrightarrow \frac{1}{45} t_{2A}, \quad \tilde{\omega}^1 \leftrightarrow \frac{1}{45} u_{3A}.$

4B case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad 0 \leq j \leq 3, \quad \tilde{\omega}^1 \leftrightarrow \frac{1}{32} t_{2A}.$

2B case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad j = 0, 1.$

3C case. $\sigma_1^j \hat{e} \leftrightarrow \frac{1}{32} t_j, \quad j = 0, 1, 2.$

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