Milnor fibration and fibred links at infinity

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29 January 1999

Introduction

Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a polynomial function. By definition $c \in \mathbb{C}$ is a regular value at infinity if there exists a disc $D$ centred at $c$ and a compact set $C$ of $\mathbb{C}^2$ such that the map $f : f^{-1}(D) \setminus C \rightarrow D$ is a locally trivial fibration. There are only a finite number of critical (or irregular) values at infinity. For $c \in \mathbb{C}$ and a sufficiently large real number $R$, the link at infinity $K_c = f^{-1}(c) \cap S^3_R$ is well-defined.

In this paper we sketch the proof of the following theorem which gives a characterization of fibred multilinks at infinity.

**Theorem.** A multilink $K_0 = f^{-1}(0) \cap S^3_R$ is fibred if and only if all the values $c \neq 0$ are regular at infinity.

We first obtain theorem 1, a version of this theorem was proved by A. Némethi and A. Zaharia in [NZ] (with “semitame” as a hypothesis). Here we give a new proof using resolution of singularities at infinity. This method enables us to describe the fibre and the monodromy of the Milnor fibration in terms of combinatorial invariants of a resolution of $f$.

**Theorem 1.** If there is no critical value at infinity outside $c = 0$ then in the homotopy class of

$$\frac{f}{|f|} : S^3_R \setminus f^{-1}(0) \rightarrow S^1$$

there exists a fibration.

The value 0 may be regular or not. One may specify what kind of fibration it is; if $f$ is a reduced polynomial, then this is an open book decomposition, otherwise it is a multilink fibration of $K_0 = f^{-1}(0) \cap S^3_R$ (see paragraph 2). The weights of $K_0$ are given by the multiplicities of the factorial decomposition of $f$.

If 0 is a regular value at infinity and $c \neq 0$ is a critical value at infinity, W. Neumann and L. Rudolph proved in [NR] that the link $f^{-1}(0) \cap S^3_R$ is not fibred. In the following theorem we do not have any hypothesis on the value 0, in particular 0 can be a critical value at infinity.
**Theorem 2.** Suppose that $c \neq 0$ is a critical value at infinity for $f$, then the multilink $K_0 = f^{-1}(0) \cap S^3_R$ is not a fibred multilink.

We begin with definitions, the second part is devoted to the proof of theorem 1. We conclude with the proof of theorem 2.

## 1 Definitions

As in [EN], a **multilink** $L(m)$ $(m = (m_1, \ldots, m_k))$ is a link having each component $L_i$ weighted by the integer $m_i$.

The multilink $L(m)$ is a **fibred multilink** if there exists a differentiable fibration $\theta : S^3_R \setminus L \to S^1$ such that $m_i$ is the degree of the restriction of $\theta$ on a meridian of $L_i$. A fibre $\theta^{-1}(x)$ is a **Seifert surface** for the multilink. The link $K_0 = f^{-1}(0) \cap S^3_R$ is a multilink, the weights being naturally given by the multiplicities of the factorial decomposition of $f$.

A **fibred link** is a fibred multilink having all its components weighted by $+1$. Then $\theta$ is called an **open book decomposition**.

Next we give definitions and results about resolutions, see [LW]. Let $n$ be the degree of $f$ and $F$ be the corresponding homogeneous polynomial with the same degree. The map $\tilde{f} : \mathbb{C}P^2 \to \mathbb{C}P^1$, $\tilde{f}(x : y : z) = (F(x, y, z) : z^n)$ is not everywhere defined, nevertheless there exists a minimal composition of blowing-ups $\pi_w : \Sigma_w \to \mathbb{C}P^2$ such that $\tilde{f} \circ \pi_w$ extends to a well-defined morphism $\phi_w$ from $\Sigma_w$ to $\mathbb{C}P^1$. This is the **weak resolution**.

\[
\begin{array}{ccc}
\mathbb{C}^2 & \xrightarrow{f} & \mathbb{C}P^2 \\
\downarrow \quad & \quad & \downarrow \pi_w \\
\mathbb{C} & \xrightarrow{\phi_w} & \mathbb{C}P^1
\end{array}
\]

For an irreducible component $D$ of $\pi_w^{-1}(L_\infty)$ ($L_\infty$ is the line of $\mathbb{C}P^2$ having the equation $(z = 0)$), we distinguish three cases:

1. $\phi_w(D) = \infty$, we denote $D_\infty = \phi_w^{-1}(\infty)$.
2. $\phi_w(D) = \mathbb{C}P^1$, $D$ is a **dicritical component**, the restriction of $\phi_w$ to $D$ is a ramified covering, the **degree** of $D$ is the degree of this restriction. The divisor which contains all these components is the **dicritical divisor** $D_{dic}$.
3. $\phi_w(D) = c \in \mathbb{C}$, there is a finite number of such components, collected in $D_{crit} = D_{c_1} \cup \ldots \cup D_{c_g}$.

The irregular values at infinity for $f$ are the values $c_1, \ldots, c_g$ and the critical values of the map $\phi_w$ restricted to $D_{dic}$; moreover each divisor $D_{c_i}$ is a disjoint union of bamboos.

We now increase the number of blowing-ups of $\pi_w$ in a minimal way, in order to obtain $\pi_p : \Sigma_p \to \mathbb{C}P^2$ and $\phi_p = \tilde{f} \circ \pi_p : \Sigma_p \to \mathbb{C}P^1$ such that the fibre
\( \phi_t^{-1}(0) \) cuts the divisor \( D_{\text{dic}} \) transversally and is a normal crossing divisor. This is the \textit{partial resolution} for the value \( c = 0 \).

We continue with blowing-ups in order to obtain \( \pi_t, \Sigma_t, \phi_t \) such that each fibre of \( \phi_t \) cuts the divisor \( D_{\text{dic}} \) transversally and all the fibres of \( \phi_t \) are normal crossing divisors. This is the \textit{total resolution}.

For the total resolution the values \( c_1, \ldots, c_g' \) coming from the components \( D \) of the new \( D_{\text{crit}} \) with \( \phi_t(D) = c_i \) are the critical values at infinity.

## 2 Milnor fibration at infinity

Until the end of this section, we suppose that the only irregular value at infinity for \( f \) can be the value 0. Let \( \phi = \phi_t \) coming from the total resolution. In \( \Sigma_t \) the sphere \( \pi_t^{-1}(S^3_R) \) is diffeomorphic to the boundary \( S \) of a neighbourhood of \( \pi_t^{-1}(L_{\infty}) \) (see [D]).

Instead of studying \( f|f|^{-1}(0) \) we study \( \phi|\phi|^{-1}(0) \). Let \( \theta \) be the restriction of \( \phi|\phi|^{-1}(0) \). By changing the sphere \( \pi_t^{-1}(S^3_R) \) into \( S \) we only know that \( \theta \) is in the homotopy class of \( f|f|^{-1}(0) \).

As in [LMW] there is a correspondence between the irreducible components of \( \pi_t^{-1}(L_{\infty}) \) and a Waldhausen decomposition of \( S \setminus \phi^{-1}(0) \) into Seifert three-manifolds. We will prove that the restriction of \( \theta \) to the Seifert manifold \( \sigma(D) \) associated to any irreducible component \( D \) of \( \pi_t^{-1}(L_{\infty}) \) is a fibration. If \( D \subset D_{\infty} \cup D_0 \), the equations are similar to the local case; we thus have to look at what happens with the components of the dicritical divisor.

**Lemma 1.** The smooth points in \( \pi_t^{-1}(L_{\infty}) \) of each dicritical component with non-empty intersection with \( D_{\text{crit}} = D_0 \) is an annulus.

In other words the intersection of \( D_0 \) with each dicritical component is empty or reduced to a single point.

**Proof.** This is a consequence of the fact that above \( \mathbb{CP}^1 \setminus \{0, \infty\} \), \( \phi \) is a regular covering. \( \square \)

With similar arguments, one can prove:

**Lemma 2.** Each dicritical component \( D \) with \( D \cap D_{\text{crit}} = \emptyset \) is of degree 1.

### 2.1 Fibration on \( \sigma(D) \) for \( D \subset D_{\text{dic}} \)

Let \( D \) be a dicritical component and let \( U \) be the simple points of \( D \) in \( \pi_t^{-1}(L_{\infty}) \cup \phi^{-1}(0) \). By lemmas 1 and 2 we know that \( U \) is an annulus and \( \phi_U : U \rightarrow \mathbb{CP}^1 \setminus \{0, \infty\} \) is a regular covering of order \( d \).

Let \( u \in \mathbb{C}^* \) be a parametrisation of \( U \). For each point of \( U \) we choose local coordinates \( (u, v) \) such that \( \phi \) can be written \( \phi(u, v) = u \). We choose \( S \) so that \( S \) is locally given by \( (|v| = \varepsilon) \) where \( \varepsilon \) is a small positive real number.

With these facts one can calculate that the restriction of \( \theta \) to the Seifert component \( \sigma(D) \) associated to \( D \) is a fibration whose fibres consist of \( d \) annuli.
2.2 Fibration in a neighbourhood of a non-simple point

In a neighbourhood $V$ of a non-simple point, i.e. a point belonging to a dicritical component $D$ and another component $D' \in \pi^{-1}_i(L_\infty) \cup \phi^{-1}(0)$, $\phi$ is defined in appropriate local coordinates by $(u,v) \mapsto u^d$.

Let $T$ be the tubular neighbourhood of $D \cap V$ given by $(|v| \leq \varepsilon)$. $\theta_T$ defines a fibration whose fibres consist of $d$ annuli:

$$\theta^{-1}(e^{i\alpha}) \cap T = \{(u,v) \in T; |v| = \varepsilon, u \neq 0 \text{ and } u^d/|u|^d = e^{i\alpha}\}.$$

Moreover, this fibration is a multilink fibration, because on a torus $D$ of $\partial T \cap \partial T'$, $\theta$ is a fibration on $V$.

2.3 Fibration in a neighbourhood of the strict transform

Let $F$ be an irreducible component of $\phi^{-1}(0) \setminus D_0$ (which corresponds to the affine set $f^{-1}(0)$). $F$ can intersect $D_0$ or $D_{dic}$. If $F \cap D_0 \neq \emptyset$ then locally in a neighbourhood $V$, $\phi(u,v) = u^p v^q$ with $(v = 0)$ is an equation for $D_0$. The associated component of the link is $\phi^{-1}(0) \cap S \cap V$. Then $\theta|_V$ is a fibration whose fibres consist of $\gcd(p,q)$ annuli:

$$\theta^{-1}(e^{i\alpha}) \cap V = \{(u,v) \in V; |v| = \varepsilon, u \neq 0 \text{ and } u^p v^q/|u^p v^q| = e^{i\alpha}\}.$$ 

Moreover this fibration is a multilink fibration, because on a torus $D_3^2 \times S_1^1 \setminus \{0\}$, the fibre of the fibre at $v = cst$ is $p$ radii of the annulus $D_3^2 \setminus \{0\} \times v$. If $f$ is a reduced polynomial function then $p = 1$ and $\theta$ is an open book decomposition.

Similarly, $\theta$ is still locally a fibration if $F \cap D_{dic} \neq \emptyset$.

We now conclude by collecting and gluing previous results. $\phi/|\phi|$ is a fibration in a neighbourhood of $S \cap \phi^{-1}(0)$ and on all $V \cap S$ which cover $S \setminus \phi^{-1}(0)$, so $\phi/|\phi| : S \setminus \phi^{-1}(0) \rightarrow S^1$ is a fibration. Furthermore with the discussion above $\phi/|\phi|$ is an open book decomposition or a multilink fibration depending on $f$ being reduced or not.

3 Non-fibred multilinks

Under the hypotheses of theorem $2$ and without loss of generality we suppose that $\{c \alpha \text{ with } \lambda < 0\}$ does not contain critical values of $f$ at infinity. The surface $\mathcal{F} = (f/|f|)^{-1}(c/|c|) \cap S^3_R$ is a Seifert surface for the multilink $K_0 = f^{-1}(0) \cap S^3_R$. Moreover, for complex numbers $\omega$ with $0 \leq |\omega - c| \ll |c|$ the links $f^{-1}(\omega) \cap S^3_R$ do not cut $\mathcal{F}$.

We choose $\omega$ as a regular value at infinity. For the partial resolution $\phi = \phi_p$ at infinity for $f$ and the value $0$, there exists one dicritical component with a valency at least $3$ in $\pi_p^{-1}(L_\infty) \cup \phi^{-1}(0)$; let $D$ be a dicritical component where
$c$ is a critical value at infinity. If the intersection $\phi^{-1}(0) \cap D$ has more than two points or if there is a bamboo of $D_c$ that cuts $D$ then we can easily conclude. But no other case is possible because $\phi|_D$, with the critical values 0 and $c$, has more than two zeroes. So the manifold $\sigma(D)$ induces a Seifert manifold of the minimal decomposition of $S^3_R \setminus K_0$; by crossing this component, $f^{-1}(\omega)$ defines a virtual component of $M = S^3_R \setminus f^{-1}(0)$ (see [LMW]); that is to say a regular fibre of the minimal Waldhausen decomposition of the manifold $M$.

According to [EN] th. 11.2, since $F$ and a virtual component of $M$ have empty intersection, $K_0$ is not a fibred multilink, if we exclude the case where $M$ is $S^1 \times S^1 \times [0, 1]$. This case is studied in the following lemma.

**Lemma 3.** If the underlying link associated to $K_0 = f^{-1}(0) \cap S^3_R$ is the Hopf link then $c \neq 0$ is a regular value at infinity for $f$.

**Proof.** We suppose first that $f$ is a reduced polynomial function. Then $K_0$ is the Hopf link, and since $K_0$ is an iterated torus link around Neumann’s multilink $L$ [N] §2], this multilink can only be the trivial knot or the Hopf link.

**Case of $L$ being the trivial knot:** There is only one dicritical component. If $f$ is not a primitive polynomial (i.e. with connected generic fibre) then with the use of the Stein factorisation, let $h \in \mathbb{C}[t]$ and let $g \in \mathbb{C}[x, y]$ be a primitive polynomial with $f = h \circ g$. By the Abhyankar-Moh theorem (see [A]), there exists an algebraic automorphism $\Theta$ of $\mathbb{C}^2$ with $g \circ \Theta(x, y) = x$ and then $f \circ \Theta(x, y) = h(x)$.

Let $x_1, \ldots, x_n$ be the zeroes of $h$; $x_1 \times \mathbb{C}, \ldots, x_n \times \mathbb{C}$ are the solutions of $f \circ \Theta(x, y) = 0$. Therefore the link $K_0$ is a union of trivial knots with zero linking numbers, so $K_0$ is not the Hopf link.

**Case of $L$ being the Hopf link:** $K_0$ and the multilink $L$ are isotopic. On the one hand in the weak resolution for $f$, the restriction of $\phi = \phi_w$ to $D_{dic}$ cannot have the critical value 0 without a bamboo. If so, one component of $K_0$ would be a true iterated torus knot around a component of $L$, in contradiction with the hypothesis. On the other hand, each component of the multilink $L$ can be represented by a disc which crosses transversally the last component of each bamboo (start counting at the dicritical component). If there exists a bamboo for the value 0, the component $C$ of $\phi^{-1}(0) \setminus D_0$ with $C \cap D_0 \neq \emptyset$ must be irreducible, reduced and cross $D_0$ transversally at the last component; this configuration is excluded by lemma 8.24 of [MW]. So 0 is a regular value at infinity and since $K_0$ is isotopic to $L$, all the dicritical components have degree one and there is no value having a bamboo, so $c$ is a regular value at infinity for $f$.

If $f$ is not reduced, let $g$ be the reduced polynomial function associated to $f$. Then the link $g^{-1}(0) \cap S^3_R$ is the Hopf link and from the discussion above we know that 0 is a regular value at infinity for $g$. From the classification of regular algebraic annuli [N] §8], there exists an algebraic automorphism $\Theta$ of $\mathbb{C}^2$ with $\Theta(0, 0) = (0, 0)$ such that $g \circ \Theta(x, y) = xy + \lambda$, $\lambda \in \mathbb{C}$. So $f \circ \Theta(x, y) = (xy + \lambda)^t$ if $\lambda \neq 0$ and $f \circ \Theta(x, y) = x^py^q$ if $\lambda = 0$. In both cases, $c$ is a regular value at infinity for $f$. \qed
In conclusion, whether 0 is a regular value at infinity or not, the multilink $K_0 = f^{-1}(0) \cap S^3_R$ is not fibred when $c \neq 0$ is a critical value at infinity.

Acknowledgments

I thank Professor Françoise Michel for long discussions and for her many ideas.

References

[A] E. Artal-Bartolo, *Une démonstration géométrique du théorème d’Abhyankar-Moh*, J. reine angew. Math. 464 (1995), 97-108.

[D] A. Durfee, *Neighborhoods of algebraic sets*, Trans. Amer. Math. Soc. 276 (1983), 517-530.

[EN] D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Ann. of Math. Stud. 110, Princeton Univ. Press (1985).

[LMW] D.T. Lê, F. Michel and C. Weber, *Courbes polaires et topologie des courbes planes*, Ann. scient. Éc. Norm. Sup. 24 (1991), 141-169.

[LW] D.T. Lê and C. Weber, *A geometrical approach to the Jacobian conjecture*, Kodai Math. J. 17 (1994), 374-381.

[MW] F. Michel and C. Weber, *On the monodromies of a polynomial map from $\mathbb{C}^2$ to $\mathbb{C}$*, preprint (1998).

[NZ] A. Némethi and A. Zaharia, *Milnor fibration at infinity*, Indag. Math. N. S. 3 (1992), 323-335.

[N] W. Neumann, *Complex algebraic plane curves via their links at infinity*, Invent. Math. 98 (1989), 445-489.

[NR] W. Neumann and L. Rudolph, *Unfoldings in knot theory*, Math. Ann. 278 (1987), 409-439 and *Corrigendum: Unfoldings in knot theory*, Math. Ann. 282 (1988), 349-351.

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