LONELY POINTS REVISITED

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Abstract. In our previous paper, Lonely points, we introduced the notion of a lonely point, due to P. Simon. A point \( p \in X \) is lonely if it is a limit point of a countable dense-in-itself set, not a limit point a countable discrete set and all countable sets whose limit point it is, form a filter. We use the space \( G_\omega \) from a paper of A. Dow, A.V. Gubbi and A. Szymański (2) to construct lonely points in \( \omega^* \). This answers the question of P. Simon posed in our paper Lonely points (12).

1. Introduction

1.1. Definition. A topological type in a space \( X \) is a subset \( T \subseteq X \) which is invariant under homeomorphisms.

An example of a topological type are the discrete points in a space \( X \). Another more interesting type is given in the following definition. The first part is due to W. Rudin (10) the second to K. Kunen (8).

1.2. Definition (Rudin, Kunen). A point \( x \in X \) is a P-point if the countable intersection of neighbourhoods of \( x \) is again a neighbourhood of \( x \). It is a weak P-point if it is not a limit point of a countable subset of \( X \).

Clearly any isolated point is a P-point, and a P-point is a weak P-point. However none of the implications can be reversed.

If a space contains two distinct topological types, then it is not homogeneous.

The motivation for finding topological types in \( \omega^* \) was given by the following surprising result of Z. Frolík (5, 4):

1.3. Theorem (Frolík). \( \omega^* \) is not homogeneous.

His proof used a clever combinatorial argument but it gave no intrinsically topological reason for the non-homogeneity of \( \omega^* \). This motivated the question whether one can find a “topologically defined” topological type — an “honest” proof of nonhomogeneity. Under CH, this was answered already by W. Rudin in (10) where he proved that P-points exist in \( \omega^* \). However in ZFC the question remained open for some twenty years.

In his seminal paper (8), K. Kunen proved in ZFC that \( \omega^* \) contains a weak P-point:

1.4. Theorem (Kunen). \( \omega^* \) contains a weak P-point.

Since it obviously contains non weak P-points, this is an “honest” proof of non-homogeneity. In (9), J. van Mill had exploited the techniques of K. Kunen to prove, in ZFC, the existence of sixteen distinct topological types in \( \omega^* \)!

One of the types he introduced is given in the following theorem:

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1.5. **Theorem** (van Mill). There is a point \( p \in \omega^* \) which is a limit point of a countable discrete set and the countable sets whose limit point it is form a filter.

**Proof.** (Idea) Use Kunen’s result to construct a weak P-point \( p \in \omega^* \subseteq \beta\omega \). Now use a theorem of P. Simon (see theorem 2.5) to embed \( \beta\omega \) into \( \omega^* \) as a weak P-set. Then the image of \( p \) via the embedding will be as required, since \( p \) clearly has the property in \( \beta\omega \) and the embedding does not destroy it since the image of \( \beta\omega \) is a weak P-set. □

This motivated P. Simon to define the following notion, which we have called a lonely point in [12]. We want essentially the same type of point as in the above theorem only replacing the countable discrete set whose limit point it is by a crowded set:

1.6. **Definition.** A point \( p \in X \) is a lonely point provided:

(i) \( p \) is \( \omega \)-discretely untouchable, i.e. not a limit point of a countable discrete set,

(ii) \( p \) is a limit point of a countable crowded (i.e. without isolated points) set

and

(iii) the countable sets whose limit point \( p \) is form a filter.

In the paper we were able to show that lonely points exist in some open dense subspace of \( \omega^* \). Here we prove that they actually exist in \( \omega^* \):

1.7. **Theorem.** \( \omega^* \) contains a lonely point.

The idea is to construct a countable, perfectly disconnected space \( X \) with an \( \aleph_0 \)-bounded remainder and then embed it as a weak P-set into \( \omega^* \). Any point of \( X \) will then be a lonely point of \( \beta X \) and, since \( \beta X \) will be a weak P-set in \( \omega^* \), also a lonely point of \( \omega^* \).

2. **Basic definitions and theorems**

2.1. **Definition** (Kunen). \( F \subseteq X \) is a weak P-set of \( X \) if any countable \( D \subseteq X \) disjoint from \( F \) has closure disjoint from \( F \).

2.2. **Observation.** If \( F \subseteq X \) is a weak P-set of \( X \) and \( x \in F \) is a lonely point of \( F \) then it is also a lonely point of \( X \).

2.3. **Definition.** A space \( X \) is extremally disconnected (or ED for short) if the closure of any open set is open.

The following is standard, see e.g. [3]:

2.4. **Theorem.** If \( X \) is ED then so is \( \beta X \)

We shall also need the following theorem of P. Simon (see [11]):

2.5. **Theorem** (Simon). The Čech-Stone compactification of any \( T_3 \) ED space of weight \( \leq 2^{\aleph_0} \) can be embedded into \( \omega^* \) as a closed weak P-set.

3. **Irresolvable spaces**

In this section, unless otherwise stated, we assume all spaces to be crowded (i.e. without isolated points). The following definitions were introduced in [1]:

3.1. **Definition** (van Douwen). A crowded space \( X \) is perfectly disconnected if no point of \( X \) is a limit point of two disjoint subsets of \( X \). It is irresolvable, if it contains no disjoint dense sets. It is open-hereditarily-irresolvable (OHI for short), provided each open subspace is irresolvable. A crowded space is maximal regular if each finer topology either contains an isolated point or is not regular.
Irresolvable spaces were constructed by E. Hewitt \cite{Hewitt} and independently by M. Katětov \cite{Katetov}. They were extensively studied in \cite{vanDouwen} where the following theorems may be found:

3.2. **Theorem** (\cite{vanDouwen}1.7,1.11). Maximal regular spaces are zerodimensional, ED and OHI.

3.3. **Theorem** (\cite{vanDouwen}1.4,1.6). If $A, B$ are disjoint crowded subspaces of a maximal regular space, then $\overline{A}$ and $\overline{B}$ are disjoint.

3.4. **Theorem** (\cite{vanDouwen}2.2). If $X$ is ED and OHI and each nowhere dense subset of $X$ is closed then $X$ is perfectly disconnected.

The following theorem is not explicitly stated in van Douwen’s paper, but its proof is essentially given in his Lemma 3.2 and Example 3.3.

3.5. **Theorem** (van Douwen). Any countable maximal regular space $X$ contains an open perfectly disconnected subspace.

**Proof.** For each $Z \subseteq X$ let

$$A_Z = \{x \in Z : x \text{ is a limit point of a relatively discrete subset of } Z\}$$

**Claim** $A_Z \neq Z$ for each open subset $Z$ of $X$.

Assume otherwise. Enumerate $Z$ as $\langle x_n : n < \omega \rangle$. By induction construct pairwise disjoint, relatively discrete sets $\langle D_n : n < \omega \rangle$ such that:

(i) $\bigcup_{i<n} D_i \subseteq D_n$ for all $n < \omega$ and
(ii) $x_n \in \overline{D_n}$ for $n < \omega$.

This will lead to a contradiction with the irresolvability of $Z$ (by theorem 3.2, $X$ is OHI, so $Z$ is irresolvable) since $\bigcup_{n<\omega} D_{2n}$ and $\bigcup_{n<\omega} D_{2n+1}$ would then be disjoint dense subsets of $Z$. To see that the construction can be carried out let $D_0 = \{x_0\}$ and assume we have constructed $D_i$ for $i \leq n$. Let $Y = D_n \cup Z \setminus \overline{D_n}$. Since $D_n$ is relatively discrete, $Y$ is open. Since $Z$ is regular and $D_n$ is countable and relatively discrete, there is a pairwise disjoint collection of open sets $\langle U_x : x \in D_n \rangle$ such that $x \in U_x \subseteq Y$. Since we assumed $A_Z = Z$ we can choose for each $x \in D_n$ a relatively discrete set $D_x$ such that $D_x \subseteq U_x$ and $x \in \overline{D_x} \setminus D_x$. Let $D_{n+1}' = \bigcup_{x \in D_n} D_x$. If $x_{n+1}$ is a limit point of $D_n'$ let $D_{n+1} = D_{n+1}'$ otherwise let $D_{n+1} = D_{n+1}' \cup \{x_{n+1}\}$. Then $D_{n+1}$ is as required.

**Claim** $\text{int } A_X = \emptyset$.

For any clopen $U$, $A_X \cap U = A_U$. Since $X$ is regular and countable, it is zerodimensional. Suppose $U$ is clopen and $U \subseteq A_X$. By the previous claim $U \setminus A_U \neq \emptyset$ but then $U \setminus A_X \neq \emptyset$ a contradiction.

**Claim** $A_X$ is nowhere dense.

Take any open $U \subseteq X$. Then $U \setminus A_X$ is dense in $U$, since $\text{int } A_X = \emptyset$. Since $X$ is OHI (by theorem 3.2), $U$ is irresolvable so $A_X$ cannot be dense in $U$ so $U \subseteq \overline{A_X}$. Thus $\text{int } A_X = \emptyset$.

**Claim** If $A \subseteq X$ is nowhere dense then there is a discrete $D \subseteq A$ dense in $A$.

Let $D = \{x \in A : x$ is isolated in $A\}$. Since $X$ is regular and countable $D$ is relatively discrete. Since $A$ is nowhere dense, $D$ is discrete. Let $E = A \setminus \overline{D}$. Then $E$ has no isolated points. Also $X \setminus E$ has no isolated points. By theorem 3.3 $E$ must be open which contradicts that $A$ is nowhere dense.

Let

$$\vartheta = \{x \in X : x \text{ is not a limit point of a nowhere dense subset of } X\}$$

By the previous claim (and by the fact that each discrete subset of $X$ is nowhere dense)
\[ \vartheta = \{ x \in X : x \text{ is not a limit point of a discrete set} \} \]

Then \( X \setminus \vartheta \subseteq A_X \) so \( X \setminus \vartheta \) is nowhere dense, so \( \text{int } \vartheta \) is nonempty. We finally show that \( \text{int } \vartheta \) is perfectly disconnected. By the definition of \( \vartheta \) any nowhere dense subset of \( \text{int } \vartheta \) is closed. Now it remains to apply theorem 3.4 remembering that by theorem 3.3 \( \text{int } \vartheta \) is ED and OHI (any open subspace of a maximal regular space is maximal regular). \hfill \square

4. Proof of the main theorem

The following definition and theorem is taken from \[2\]:

4.1. Definition. Let \( p \in \omega^* \) be a weak P-point. The space \( G_\omega \) is the space \( \omega^{< \omega} \) of all finite sequences of natural numbers with \( G \subseteq \omega^{< \omega} \) being open precisely when for each \( \sigma \in G \) the set \( \{ n : \sigma \upharpoonright n \in G \} \) is in \( p \).

4.2. Theorem (Dow, Gubbi, Szymanski). The remainder of \( G_\omega \) is \( \aleph_0 \)-bounded. Moreover \( G_\omega \) is a \( T_2 \), zerodimensional, ED space.

Notice that if a space \( X \) has an \( \aleph_0 \)-bounded remainder, any finer topology also has an \( \aleph_0 \)-bounded remainder:

4.3. Proposition. If \((X, \tau)^*\) is a zerodimensional \( \aleph_0 \)-bounded space and \( \sigma \supseteq \tau \) is also zerodimensional, then \((X, \sigma)^*\) is \( \aleph_0 \)-bounded.

Proof. Note that any \( p \in (X, \tau)^* \) corresponds to a closed subset of \((X, \sigma)^*\) (denote it \([p]\)). Now given \( \{q_n : n < \omega\} \subseteq (X, \sigma)^* \) we can find \( \{p_n : n < \omega\} \subseteq (X, \tau)^* \) such that \( \{q_n : n < \omega\} \subseteq \bigcup \{[p_n] : n < \omega\} \). Since \((X, \tau)^*\) is \( \aleph_0 \)-bounded, \( \{p_n : n < \omega\} \cap X = \emptyset \) so also \( \{q_n : n < \omega\} \cap X = \emptyset \) which implies that \((X, \sigma)^*\) is \( \aleph_0 \)-bounded. \hfill \square

4.4. Theorem. There is a countable, ED, perfectly disconnected space \( X \) with an \( \aleph_0 \)-bounded remainder.

Proof. Take the space \( G_\omega \) from theorem 4.2, and refine the topology to a maximal regular topology. Then, by the previous proposition, this space still has an \( \aleph_0 \)-bounded remainder and so does its open perfectly disconnected subspace given by theorem 3.5. Let \( X \) be this subspace. \hfill \square

4.5. Theorem. \( \omega^* \) contains a lonely point.

Proof. Let \( X \) be the space from the previous theorem. Since it is crowded perfectly disconnected, each of its points is a lonely point of \( X \). Since its remainder is \( \aleph_0 \)-bounded, each of its points is also a lonely point of \( \beta X \). Since it is ED, \( \beta X \) is also ED and since it is countable, \( \beta X \) has weight at most \( 2^{8\omega} \). Hence, by theorem 2.5 \( \beta X \) can be embedded as a weak P-set into \( \omega^* \) and each point of \( X \) will be a lonely point of \( \omega^* \) (by observation 2.3). \hfill \square

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