Bulk viscosity of strange quark matter: Urca versus non-leptonic processes

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A general formalism for calculating the bulk viscosity of strange quark matter is developed. Contrary to the common belief that the non-leptonic processes alone give the dominant contribution to the bulk viscosity, the inclusion of the Urca processes is shown to play an important role at intermediate densities when the characteristic r-mode oscillation frequencies are not too high. The interplay of non-leptonic and Urca processes is analyzed in detail.

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I. INTRODUCTION

Compact (neutron) stars provide a natural laboratory of matter under extreme conditions. In the central regions of such stars the baryon density of matter could reach values up to 10 times the nuclear saturation density (i.e., $10\rho_0$ where $\rho_0 \simeq 0.15$ fm$^{-3}$). At such high density matter is likely to be in a deconfined state in which quarks rather than hadrons are the natural dynamical degrees of freedom.\[1, 2, 3, 4, 5\]

One can argue that the ground state of deconfined quark matter is a color superconductor. (For reviews on color superconductivity, see Refs. [6, 7, 8, 9, 10, 11, 12, 13]). Many phases of color superconductivity are known that could possibly be realized in dense matter. It remains unclear, however, which of these describe the ground state of matter under the specific conditions in stars. This is because of theoretical uncertainties in treating the strongly coupled, non-perturbative dynamics in QCD at the baryon densities of relevance.

In order to clarify this one could use observational data from stars to narrow down the range of possibilities. For this program to work, one requires a detailed knowledge of the physical properties of various phases of matter that are likely to exist in stars. The transport properties are most suitable for developing sufficiently sensitive, as well as unique and verifiable predictions regarding observational signals from stars. In this paper, we study the bulk viscosity of the normal phase of three-flavor quark matter. This work extends our recent analysis of the bulk viscosity in two-flavor quark matter \[14\] by including strange quarks. This also augments existing studies of the topic \[15, 16, 17, 18, 19, 20, 21, 22\], in which the role of the Urca processes was not thoroughly investigated.

In general, the bulk viscosity is a measure of the kinetic energy dissipation during expansion and compression of a fluid. In compact stars, the density oscillations of interest have characteristic frequencies that are of the same order of magnitude as the stellar rotation frequencies. These are bound from below and from above, $1 \text{ s}^{-1} \lesssim \omega \lesssim 10^{3} \text{ s}^{-1}$. (For the fastest-spinning pulsar currently known, PSR J1748-2446ad, one has $\omega \approx 4.5 \times 10^{3} \text{ s}^{-1}$ corresponding to $\nu = 716$ Hz \[23\].) The most important microscopic processes that provide the energy dissipation on the corresponding time scales are weak processes. Under conditions in stars, in particular, the bulk viscosity of quark matter is determined by the combined effect of the flavor-changing weak processes diagrammatically shown in Fig. \[1\]. When an instantaneous departure from chemical equilibrium is induced by expansion/compression of matter, the weak processes try to restore the equilibrium state and, while doing this, reduce the oscillation energy.

It is commonly argued that the bulk viscosity in the normal phase of three-flavor quark matter is dominated by the non-leptonic weak processes $u + d \leftrightarrow u + s$ \[15, 16, 17, 18, 19, 20, 21, 22\]. These are shown diagrammatically in Figs. \[1a\] and \[1b\]. As for the Urca (semileptonic) processes, see Figs. \[1c\] and \[1f\], they have considerably lower rates which are suppressed parametrically by a factor of order $(T/\mu_e)^2$. This is in contrast to two-flavor (non-strange) quark matter, in which case the Urca processes $u + e^{-} \rightarrow d + \nu$ and $d \rightarrow u + e^{-} + \bar{\nu}$ [see Figs. \[1c\] and \[1f\)] are the only ones that contribute \[14\]. (For studies of the viscosity in various phases of dense nuclear matter, see Refs. \[24, 25, 26, 27, 28, 29, 30, 31, 32, 33\].)

In this paper, we study the bulk viscosity of three-flavor (strange) quark matter. One of the key issues that we address here is the interplay between the Urca and the non-leptonic processes. We shall show that, because a resonance-type phenomenon determines the bulk viscosity and because there is a subtle interference of the two weak processes, the simple argument about the dominance of the non-leptonic processes is not always justified.

This paper is organised as follows. In the next section, we develop the general formalism for calculating the bulk viscosity in three-flavor quark matter, paying special attention to the interplay of several different weak
In this section, we give a general derivation of the bulk viscosity resulting from the combined effect of all weak processes shown diagrammatically in Fig. 1. Following the same method as in Ref. [14], one can relate the bulk viscosity \( \zeta \) to the average energy-density dissipation in pulsating matter,

\[
\langle \dot{\mathcal{E}}_{\text{diss}} \rangle = -\frac{\zeta \omega^2}{2} \left( \frac{\delta n_0}{n} \right)^2.
\]  

(1)

In the derivation of this relation, we assumed that small oscillations of the quark matter density are described by \( \delta n = \delta n_0 \Re(e^{i\omega t}) \) where \( \delta n_0 \) and \( \omega \) are the magnitude and the frequency of the oscillations, respectively. The energy dissipation can be also expressed in terms of the induced pressure oscillations,

\[
\langle \dot{\mathcal{E}}_{\text{diss}} \rangle = \frac{n}{\tau} \int_0^\tau P\dot{V} \, dt
\]  

(2)

where \( V \equiv 1/n \) is the specific volume, and \( \tau \equiv 2\pi/\omega \) is the period of the oscillation.

The density oscillations drive strange quark matter out of \( \beta \) equilibrium. While the weak processes tend to restore equilibrium, they always lag behind. Thermal equilibrium, in contrast, is restored almost without any delay by very fast strong processes. The corresponding instantaneous quasi-equilibrium state is unambiguously characterised by the total baryon density \( n \), the lepton fraction \( X_e \), and the strangeness fraction \( X_s \),

\[
n = \frac{1}{3} (n_u + n_d + n_s),
\]  

(3a)

\[
X_e = \frac{n_e}{n},
\]  

(3b)

\[
X_s = \frac{n_s}{n},
\]  

(3c)

where \( n_u, n_d, \) and \( n_s \) are the number densities of up, down, and strange quarks, while \( n_e \) is the number density of electrons. Because of the charge neutrality constraint, the number densities satisfy the following relation:

\[
\frac{2}{3} n_u - \frac{1}{3} n_d - \frac{1}{3} n_s - n_e = 0.
\]  

(4)

(Note that there are no microscopic processes that would lead to deviations from this charge neutrality constraint.) Using this constraint together with the definitions in Eq. (3), one can express the separate densities and, in fact, any thermodynamic quantity of quasi-equilibrium quark matter in terms of \( n, X_e, \) and \( X_s \). For the number densities, for example, one finds

\[
n_e = X_e n, \quad n_u = (1 + X_e) n, \quad n_d = (2 - X_e - X_s) n, \quad n_s = X_s n.
\]  

(5a)

(5b)

(5c)

(5d)

The number densities can also be expressed in terms of the corresponding chemical potentials, \( \mu_i = n_i(\mu_i) \). In \( \beta \) equilibrium, the chemical potentials are related as follows: \( \mu_s = \mu_d = \mu_u + \mu_e \). In pulsating matter, on the other hand, the instantaneous departure from equilibrium is described by the following two independent parameters:

\[
\delta \mu_1 \equiv \mu_s - \mu_d = \delta \mu_s - \delta \mu_d, \quad \quad \delta \mu_2 \equiv \mu_s - \mu_u - \mu_e = \delta \mu_s - \delta \mu_u - \delta \mu_e,
\]  

(6a)

(6b)

where \( \delta \mu_i \) denotes the deviation of the chemical potential \( \mu_i \) from its value in \( \beta \) equilibrium. Note that the
third possible combination of the chemical potentials, i.e., $\delta \mu_3 \equiv \mu_d - \mu_u - \mu_e = \delta \mu_2 - \delta \mu_1$, is not independent.

In quasi-equilibrium, both quantities $\delta \mu_1$ and $\delta \mu_2$ can be expressed in terms of the variations of the three independent variables, $\delta n, \delta X_e$, and $\delta X_s$, defining

$$
\delta \mu_i = C_i \frac{\delta n}{n} + B_i \delta X_e + A_i \delta X_s, \quad (i = 1, 2) \tag{7}
$$

where, as follows from the definition, the coefficient functions are given by

$$
A_1 = n \left( \frac{\partial \mu_d}{\partial n} + \frac{\partial \mu_u}{\partial n} \right), \quad (8a)
$$

$$
B_1 = n \frac{\partial \mu_d}{\partial n}, \quad (8b)
$$

$$
C_1 = n_s \frac{\partial \mu_s}{\partial n} - n_d \frac{\partial \mu_d}{\partial n}, \quad (8c)
$$

$$
A_2 = n \frac{\partial \mu_s}{\partial n}, \quad (8d)
$$

$$
B_2 = -n \left( \frac{\partial \mu_u}{\partial n} + \frac{\partial \mu_e}{\partial n} \right), \quad (8e)
$$

$$
C_2 = n_s \frac{\partial \mu_s}{\partial n} - n_u \frac{\partial \mu_u}{\partial n} - n_c \frac{\partial \mu_e}{\partial n}. \quad (8f)
$$

When the quantity $\delta \mu_i$ is non-zero, the two weak processes in the $i$th row of Fig. [1] have slightly different rates. To leading order in $\delta \mu_i$, we could write

$$
\Gamma_{(a)} - \Gamma_{(b)} = -\lambda_1 \delta \mu_1, \quad (9a)
$$

$$
\Gamma_{(c)} - \Gamma_{(d)} = -\lambda_2 \delta \mu_2, \quad (9b)
$$

$$
\Gamma_{(e)} - \Gamma_{(f)} = -\lambda_3 (\delta \mu_2 - \delta \mu_1), \quad (9c)
$$

(Note that our conventions are such that the quantities $\lambda_i$ are non-negative.) The net effect of having different rates for each pair of the weak processes in Fig. [1] is a change of the electron and strangeness fractions in the system, i.e.,

$$
\frac{d\delta X_e}{dt} = \lambda_2 \delta \mu_2 + \lambda_3 (\delta \mu_2 - \delta \mu_1), \quad (10a)
$$

$$
\frac{d\delta X_s}{dt} = -\lambda_1 \delta \mu_1 - \lambda_2 \delta \mu_2. \quad (10b)
$$

According to these equations, the instantaneous values of the electron and strangeness fractions, i.e., $X_e \equiv X_e^{(0)} + \delta X_e$ and $X_s \equiv X_s^{(0)} + \delta X_s$, tend to approach their equilibrium values, $X_e^{(0)}$ and $X_s^{(0)}$, respectively. For example, a deficit of electrons (indicated by either $\delta \mu_2 > 0$ or $\delta \mu_2 - \delta \mu_1 > 0$, or both) causes their production, see Eq. (10a). Similarly, a surplus of strange quarks (indicated by either $\delta \mu_1 > 0$ or $\delta \mu_2 > 0$, or both) is to be removed through the weak processes, see Eq. (10b). Because of finite rates for the weak processes, however, the oscillations of $\delta X_e$ and $\delta X_s$ always lag behind the density oscillations. In order to see this explicitly, we substitute $\delta \mu_i$ from Eq. (7) into Eq. (10), and get the following set of equation for $\delta X_e$ and $\delta X_s$:

$$
\frac{n d(\delta X_e)}{dt} = [(\lambda_2 + \lambda_3) C_2 - \lambda_3 C_1] \frac{\delta n}{n} + [(\lambda_2 + \lambda_3) B_2 - \lambda_3 B_1] \delta X_e + [(\lambda_2 + \lambda_3) A_2 - \lambda_3 A_1] \delta X_s, \quad (11a)
$$

$$
\frac{n d(\delta X_s)}{dt} = -[\lambda_1 C_1 + \lambda_2 C_2] \frac{\delta n}{n} - [\lambda_1 B_1 + \lambda_2 B_2] \delta X_e - [\lambda_1 A_1 + \lambda_2 A_2] \delta X_s. \quad (11b)
$$

The periodic solution to this equation can be found most easily by making use of complex variables. Thus, by defining

$$
\delta X_e = \text{Re} \left( \delta X_{e,0} e^{i\omega t} \right), \quad (12a)
$$

$$
\delta X_s = \text{Re} \left( \delta X_{s,0} e^{i\omega t} \right), \quad (12b)
$$

we derive the following results for the complex magnitudes:

$$
\delta X_{e,0} = \frac{\delta n_0 d_1 + i\delta n_0 f_1}{g_1 + ig_2}, \quad (13a)
$$

$$
\delta X_{s,0} = \frac{\delta n_0 d_2 + i\delta n_0 f_2}{g_1 + ig_2}, \quad (13b)
$$

where

$$
d_1 = (\alpha_1 + \alpha_2 + \alpha_3) (A_1 C_2 - A_2 C_1), \quad (14a)
$$

$$
d_2 = \alpha_1 \alpha_2 (C_2 - C_1) + \alpha_1 \alpha_3 C_2, \quad (14b)
$$

$$
f_1 = (\alpha_1 + \alpha_2 + \alpha_3) (C_1 B_2 - C_2 B_1), \quad (14c)
$$

$$
f_2 = -\alpha_1 \alpha_3 C_2 - \alpha_2 \alpha_3 C_1, \quad (14d)
$$

$$
g_1 = -\alpha_1 \alpha_2 \alpha_3 + (\alpha_1 + \alpha_2 + \alpha_3) (B_1 A_2 - A_1 B_2), \quad (14e)
$$

$$
g_2 = \alpha_1 \alpha_2 (B_1 - B_2) + \alpha_1 \alpha_3 (A_2 - B_2) + \alpha_2 \alpha_3 A_1, (14f)
$$

with $\alpha_i \equiv n_{i}/\lambda_i \ (i = 1, 2)$. The pressure oscillations can be given in terms of the instantaneous values of $\delta n, \delta X_e$, and $\delta X_s$, by

$$
\delta P = \frac{\partial P}{\partial n} \delta n + n (C_1 - C_2) \delta X_e + n C_1 \delta X_s, \quad (15)
$$
where the $C_i$ are the same as in Eq. (3). In the derivation we took into account that $n_i = \partial P / \partial \mu_i$, and that the total pressure is given by the sum of the partial contributions of the quarks and electrons, $P = \sum_i P_i (\mu_i)$.

After taking into account the relation (15) together with the solution for $C$ where viscosity. In the limit of vanishing rates for the semi-leptonic contributions to the bulk viscosity, we took into account that $\lambda_i \to \infty$, let us consider the interplay of the non-leptonic and semi-leptonic contributions to the bulk viscosity. In Fig. 1. Because of an interference of the weak processes, each of these contributions depends on all three rates $\lambda_i$, $i = 1, 2, 3$. The separation occurs only in the high-frequency limit, i.e., $\alpha_i \equiv n \omega / \lambda_i \to \infty$. Indeed, in this case $\zeta_i \sim \frac{\lambda_i}{\omega^2} C_i$, (no sum over $i$), (21) where $C_3 \equiv C_2 - C_1$. Note that the formal criterion for this separation reads $\omega \gg \omega_{\text{sep}}$. An estimate for $\omega_{\text{sep}}$ is derived from the parametric dependence of the coefficient functions $A_i$ and $B_i$ on densities,

$$\omega_{\text{sep}} \sim \frac{\sqrt{\lambda_1 (\lambda_2 + \lambda_3)}}{n e^{2/3}}. \quad (22)$$

In order to understand the general features of the result in Eq. (20), let us consider the interplay of the non-leptonic and semi-leptonic contributions to the bulk viscosity. In the limit of vanishing rates for the semi-leptonic processes, i.e., $\lambda_2, \lambda_3 \to 0$, the bulk viscosity reduces to the following well-known result for strange quark matter [16, 17]:

$$\zeta_{\text{non}} \sim \frac{\lambda_1 C_1^2}{\omega^2 + (\lambda_1 A_1 / n)^2}. \quad (23)$$

In this limit, there are only non-leptonic processes left, and they induce the dissipation of the oscillation energy.

In order to better understand the interplay of different types of processes, it is instructive to consider also the limit of an infinitely large non-leptonic rate, $\lambda_1 \to \infty$, keeping the semi-leptonic rates $\lambda_2$ and $\lambda_3$ finite. In this case, the expression for the bulk viscosity is given by

$$\zeta_{\text{sep}} \sim \frac{(\lambda_2 + \lambda_3) (C_2 - C_1 A_2 / A_1)^2}{\omega^2 + [(\lambda_2 + \lambda_3) (B_2 - B_1 A_2 / A_1) / n]^2}. \quad (24)$$

It may appear puzzling that the two seemingly equivalent limits, namely $\lambda_1 \gg \lambda_2, \lambda_3$, lead to such very different results. The problem can be resolved by noting that
the result in Eq. (23) is reliable only if the following additional constraint is satisfied: \((\lambda_2 + \lambda_3) \ll n^{4/3} \omega^2 / \lambda_1\). [In deriving this constraint, we assumed that the coefficient functions \(A_i\) and \(B_i\) scale with the density as \(n^{1/3}\), which will turn out to be a reasonable approximation, see Eq. (27) below.] In contrast, the result in Eq. (24), on the other hand, should be used at sufficiently low frequencies to dampen the kinetic energy efficiently. The result in Eq. (23) is to be used when the frequency of the density pulsations is sufficiently high, \(\omega \gg \omega_0\), so that only the fast non-leptonic processes have a chance of the non-leptonic and semi-leptonic processes. Below we study this interference in some more detail.

III. BULK VISCOSITY IN NORMAL PHASE

In order to calculate the bulk viscosity in the normal phase of three-flavor quark matter, we need to determine the corresponding thermodynamic coefficients \(A_i\), \(B_i\) and \(C_i\) [see Eq. (8)] and calculate the difference of the rates of the three pairs of weak processes in Fig. 1. Let us start from the derivation of the coefficients \(A_i\), \(B_i\) and \(C_i\). For this purpose, we use of the following zero-temperature expressions for the number densities of quarks and electrons:

\[
n_f = \left( \frac{\mu_f^2 - m_f^2}{\pi^2} \right)^{3/2} - \frac{2 \alpha_s}{\pi^3} \mu_f \left( \mu_f^2 - m_f^2 \right) \left( 1 - \frac{3m_f^2}{\mu_f \sqrt{\mu_f^2 - m_f^2}} \ln \frac{\mu_f + \sqrt{\mu_f^2 - m_f^2}}{m_f} \right), \quad \text{for } f = u, d, s, \quad (26a)
\]

\[
n_e = \frac{1}{3 \pi^2} \mu_e^3. \quad (26b)
\]

Note that the expressions for quarks include the leading-order corrections due to strong interactions [34, 35, 36, 37]. By making use of these relations together with the definitions in Eq. (8), we derive the following approximate expressions for the coefficient functions:

\[
A_1 \simeq \frac{2 \pi^2 n}{3 \mu_d^2}, \quad (27a)
\]

\[
B_1 \simeq \frac{\pi^2 n}{3 \mu_s^2}, \quad (27b)
\]

\[
C_1 \simeq - \frac{m_d^2 - m_s^2}{3 \mu_d} - C_d' + C_s', \quad (27c)
\]

\[
A_2 \simeq \frac{\pi^2 n}{3 \mu_s^2}, \quad (27d)
\]

\[
B_2 \simeq - \frac{\pi^2 n}{3} \left( \frac{3}{\mu_s^2} + \frac{1}{\mu_u^2} \right), \quad (27e)
\]

\[
C_2 \simeq - \frac{m_d^2}{3 \mu_s} + \frac{m_u^2}{3 \mu_u} - C_s' + C_d'. \quad (27f)
\]

On the right-hand side of these expressions, we made use of the equilibrium relations satisfied by the chemical potentials, \(\mu_s = \mu_d = \mu_u + \mu_e\), and neglected higher-order mass corrections in the expressions for \(A_i\) and \(B_i\). In the expressions for \(C_i\), on the other hand, we kept the leading-order mass corrections, which also include the contributions linear in the strong coupling constant \(\alpha_s\),

\[
C_f' = - \frac{4 \alpha_s m_f^2}{3 \pi \mu_f} \left( \ln \frac{2 \mu_f}{m_f} - \frac{2}{3} \right). \quad (28)
\]

The reason for keeping the mass corrections is that the functions \(C_i\) vanish in the massless limit. In that limit, all \(\zeta_i\)’s are zero as appropriate for scale invariant theories [38].

In order to further proceed with the calculation of the bulk viscosity in the normal phase of three-flavor quark matter, we also need to know the rate difference of the three pairs of weak processes in Fig. 1. The rates of both the non-leptonic and semi-leptonic processes have been calculated in the literature, see Refs. [15, 16, 39, 40] and Refs. [11, 12], respectively. In the limit of three massless quarks, for example, the rates are

\[
\lambda_1 \simeq \frac{64}{5 \pi^2} G_F^2 \cos^2 \theta_C \sin^2 \theta_C m_u^2 T^2, \quad (29a)
\]

\[
\lambda_2 \simeq \frac{17}{40 \pi} G_F^2 \sin^2 \theta_C m_e T^2, \quad (29b)
\]

\[
\lambda_3 \simeq \frac{17}{15 \pi^2} G_F^2 \cos^2 \theta_C \alpha_s m_d T^4. \quad (29c)
\]

When the value of the strange quark mass is not small, the expressions for the rates are more complicated [39, 40].
and can only be calculated numerically. For the purposes of this paper it suffices to use the approximate expressions in Eq. (20). These provide a reasonable approximation for the range of strange quark masses considered below.

Now, by making use of the results for the rates (20) as well as for the coefficient functions in Eq. (27), we can straightforwardly calculate the bulk viscosity. In the calculation, we use the following two representative sets of model parameters:

| set A | set B |
|-------|-------|
| $n = 5\rho_0$ | $n = 10\rho_0$ |
| $m_s = 300 \text{ MeV}$ | $m_s = 140 \text{ MeV}$ |
| $\alpha_s = 0.2$ | $\alpha_s = 0.1$ |

In both cases the light quark masses are $m_u = 5 \text{ MeV}$ and $m_d = 9 \text{ MeV}$. Regarding the choice of parameters for sets A and B, several comments are in order. Set B is supposed to be characteristic for the conditions in the inner core of a quark star. The strong coupling constant $\alpha_s$ is small due to asymptotic freedom and the in-medium constituent strange quark mass $m_s$ assumes a value close to its current value on account of chiral symmetry restoration at large baryon densities. Set A applies to intermediate densities, such as occur in the outer core of a quark star. Here, the strong coupling constant $\alpha_s$ and the in-medium constituent strange quark mass $m_s$ assume larger values.

It can be shown that the electron chemical potential is driven by both the strange quark mass and the leading-order corrections due to strong interactions, see Eq. (20). Without $\alpha_s$ corrections, the electron chemical potential would be 9.9 and 50.5 MeV for strange quark masses of 140 MeV ($n = 10\rho_0$) and 300 MeV ($n = 5\rho_0$), respectively. The leading-order corrections due to strong interaction tend to reduce the values of $\mu_e$. A simple analysis shows that, in fact, the electron chemical potential could even formally change the sign when the value of $\alpha_s$ is sufficiently large (e.g., $\alpha_s \gtrsim 0.5$). This would mean that strange quark matter requires the presence of the positrons rather than electrons to stay neutral. While not forbidden, such a possibility should be accepted with great caution. Indeed, the leading-order $\alpha_s$ corrections may be unreliable in this regime.

The general algorithm for calculating the bulk viscosity is as follows. First, by assuming a fixed value of the baryon density of neutral quark matter $n$ (i.e., $n = 5\rho_0 \approx 0.75 \text{ fm}^{-3}$ or $n = 10\rho_0 \approx 1.5 \text{ fm}^{-3}$ for the two cases considered), we determine the chemical potentials of the quarks and electrons in $\beta$ equilibrium. For the two representative sets, the values of the chemical potentials as well as the coefficient functions $A_i$, $B_i$ and $C_i$ are given in Table I. These are used in the calculation of the rates of the weak processes. Putting everything together, the result for the bulk viscosity follows from Eq. (18).

The numerical results for the bulk viscosity as a function of the period of density oscillations $\tau$ are presented in Fig. 2 for the two cases: (i) $n = 10\rho_0$ and $m_s = 140 \text{ MeV}$ (upper panel), and (ii) $n = 5\rho_0$ and $m_s = 300 \text{ MeV}$ (lower panel). Different line types correspond to different values of the temperature: $T = 0.1 \text{ MeV}$ (solid lines), $T = 0.2 \text{ MeV}$ (dashed lines), $T = 0.4 \text{ MeV}$ (dotted lines), and $T = 0.8 \text{ MeV}$ (dashed-dotted lines). The thin lines show the high- and low-frequency approximations, defined in Eqs. (23) and (24), for each value of the temperature.

Note the double-step structure of the bulk viscosity as a function of $\tau$. The first step at small $\tau$ is the usual one. It corresponds to the low-frequency saturation (with $\zeta^{(\text{max})} \approx 1/\Lambda_1$) of the non-leptonic contribution to the bulk viscosity, see Eq. (23). The second step at about $\tau = 2\pi/\omega_0$ (marked by dots in the figures) is a qualitatively new feature. As should be clear from our dis-
discussion in the preceding section, its appearance is the consequence of the interplay between the non-leptonic and the semi-leptonic weak processes contributing to the bulk viscosity of strange quark matter. At sufficiently large \( \tau \) (i.e., low frequencies) the contribution of the semi-leptonic processes also saturates.

The increase of the bulk viscosity at low frequencies due to the contributions of the slower semi-leptonic processes is not unexpected. (For the same reason, weak processes are much more important for compact stars than strong processes, operating on typical QCD time scales of order \( 1 \, \text{fm}/c \).) However, the main observation here is that the increase of the bulk viscosity due to the interplay between the non-leptonic and semi-leptonic weak processes could already be visible at frequencies relevant for the physics of compact stars. Moreover, by comparing the results in the two panels of Fig. 2, we find that for the conditions corresponding to lower densities, the range of frequencies where the semi-leptonic processes contribute widens significantly.

The bulk viscosity as a function of the temperature is given in Fig. 3 for set B (upper panel) and for set A (lower panel). The additional increase of the bulk viscosity due to the semi-leptonic processes is seen as a “bump” at intermediate values of the temperature. The results in the two panels demonstrate once again that the relative role of the semi-leptonic processes increases with increasing period of density oscillations, provided the baryon density is sufficiently small (set A).

From Fig. 3 we find that the substantial modification of the bulk viscosity due to the Urca processes occurs at temperatures between about 0.1 MeV and 1 MeV. This temperature range is of relevance to young neutron stars and, therefore, should be studied in more detail.

Perhaps the best way to appreciate the relative role of the Urca processes is to study the ratio between the complete expression for the bulk viscosity [18] and the commonly used approximate form [23] that takes only the non-leptonic interactions into account. In the more interesting case of parameter set A, this ratio is shown in Fig. 4 as a function of temperature for four different values of the period of density oscillations: \( \frac{1}{\tau} = 1 \, \text{Hz} \) (solid line), \( \frac{1}{\tau} = 10 \, \text{Hz} \) (dashed line), \( \frac{1}{\tau} = 100 \, \text{Hz} \) (dotted line), \( \frac{1}{\tau} = 1000 \, \text{Hz} \) (dashed-dotted line). The plot shows how the ratio \( \zeta/\zeta_{\text{non}} \) changes when the period of oscillations \( \tau \) varies in the whole range from 1 ms to 1 s.

The results presented in Fig. 4 are the main results of this paper. They show that neglecting the Urca processes can result in underestimating the value of the bulk viscosity by an order of magnitude, at intermediate densities and sufficiently large in-medium strange quark masses.

We believe this finding might be of relevance for strange quark matter under conditions realized inside young neutron stars.

Before concluding this section, it is appropriate to note that the role of the semi-leptonic processes is negligible in the case of the smaller strange quark mass, \( m_s = 140 \, \text{MeV} \). This is seen from Fig. 3 (upper panel),

### Table I: Two sets of parameters used in the calculation of the bulk viscosity.

| \( \frac{n_s}{n_0} \) | \( m_s \) [MeV] | \( \mu_u \) [MeV] | \( \mu_d \) [MeV] | \( \mu_s = \mu_u \) [MeV] | \( A_1 \) [MeV] | \( A_2 \) [MeV] | \( B_1 \) [MeV] | \( B_2 \) [MeV] | \( C_1 \) [MeV] | \( C_2 \) [MeV] |
|-----------------|---------------|-----------------|-----------------|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 5               | 300           | 39.139          | 402.463         | 441.602         | 239.432     | 127.937     | 111.386     | -3.726 × 10^4 | -60.463     | -60.460     |
| 10              | 140           | 7.396           | 495.275         | 502.671         | 324.118     | 164.288     | 160.268     | -2.080 × 10^4 | -10.692     | -10.709     |
and this can also be confirmed by studying the ratio $\zeta/\zeta_{\text{non}}$. Its value deviates from 1 by at most 28%. This is in qualitative agreement with earlier studies [15, 16, 17, 18, 19, 20, 21, 22], neglecting the semi-leptonic processes.

IV. DISCUSSION

In this paper we have studied the subtle interplay between the Urca (semi-leptonic) and the non-leptonic weak processes in determining the bulk viscosity of neutral, $\beta$-equilibrated strange quark matter. In general, the contributions of the two types of weak processes are not separable. The exception is the high-frequency limit, $\omega \gg \omega_{\text{sep}}$, see Eq. (22), in which case the contributions naturally separate. Because of a much higher rate of the non-leptonic processes, they dominate in this high-frequency limit.

With decreasing frequency the role of the Urca processes gradually increases. The maximal mixing with the non-leptonic processes occurs at about the frequency $\omega_0$, see Eq. (25) for the definition. Depending on the specific choice of model parameters, typical values of $\omega_0$ are in the range from about 1 s$^{-1}$ to 10$^3$ s$^{-1}$, see also Fig. 2 for the corresponding values of the period of oscillations.

Our numerical results for the normal-conducting phase of strange quark matter demonstrate that the commonly used approximation, in which the Urca processes are completely neglected, could substantially underestimate the value of the bulk viscosity, see Figs. 2 and 3. By sweeping over a wide range of model parameters, we find that the role of the Urca processes is most important in the range of temperatures between about 0.1 MeV and 1 MeV. In the outer core of a quark star, i.e., when the density is not too large, the strange quark mass not too small, and/or the period of oscillations not too short, the inclusion of the Urca processes could lead to an increase of the viscosity by an order of magnitude, see Fig. 4.

The findings of this work could have important implications for the physics of young neutron stars with strange quark matter interiors and/or for pure strange stars that could potentially exist too. In connection to the r-modes instabilities driven by gravitational radiation, the increase of the bulk viscosity due to the Urca processes is likely to broaden the region of stellar stability in the temperature-angular-frequency plane. The fact that the largest change occurs for the temperature range from about 0.1 MeV to 1 MeV might be also very important. Indeed, at the lower end of this range, the dominant role in suppressing the r-modes is expected to pass from the bulk viscosity to the shear viscosity. If the bulk viscosity is substantially higher than previously estimated, it may dominate the dissipative dynamics to much lower temperatures.

In view of possible color superconductivity in strange quark matter, in the future one should also investigate the role of the Urca processes in color-superconducting phases with various types of spin-zero and spin-one Cooper pairing. Because of the gaps in the quasiparticle spectra, the interplay between the semi- and non-leptonic processes in superconductors is expected to become more complicated due to the suppression of the rates. Certain general features of the corresponding dissipative dynamics could be predicted even without a detailed study. One could say, for example, that (i) the suppression of the rates should lead to the suppression of the bulk viscosity in the high-frequency limit, $\omega \gg \omega_{\text{sep}}$, (ii) the border line itself, $\omega_{\text{sep}}$, should shift to a lower value, (iii) the rates of the Urca processes could possibly become even higher than the rates of the non-leptonic ones. The first two observations are simple consequences of our general results in Sec. II. The situation described in item (iii) could be realized when all quark quasiparticles are gapped, for example, in one of the versions of the color-spin-locked phase. At low temperatures, the rates of the semi- and non-leptonic processes are suppressed by exponential factors $\exp(-\phi/T)$ and $\exp(-2\phi/T)$, respectively. A more detailed discussion of spin-one color-superconducting phases of strange quark matter is subject of a subsequent paper [43].

Note added. While writing our paper, we learned that another study of the bulk viscosity of strange quark matter is done by H. Dong, N. Su, and Q. Wang [44].

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