Abstract

We develop general techniques and present an approach to solve the problem of constructing a maximal Banach ideal $(\mathfrak{A}, A)$ which does not satisfy a transfer of the norm estimation in the principle of local reflexivity to its norm $A$. This approach leads us to the investigation of product operator ideals containing $L^2$ (the collection of all Hilbertian operators) as a factor. Using the local properties of such operator ideals – which are typical examples of ideals with property (I) and property (S) –, trace duality and an extension of suitable finite rank operators even enable us to show that $L^\infty$ cannot be totally accessible – answering an open question of Defant and Floret.

Key words and phrases: Accessibility, Banach spaces, cotype 2, finite rank operators, Hilbert space factorization, Grothendieck’s Theorem, operator ideals, principle of local reflexivity, tensor norms

1991 AMS Mathematics Subject Classification: primary 46M05, 47D50; secondary 47A80.

1 Introduction

The aim of the present paper is to present a thorough investigation of operator ideals $(\mathfrak{A}, A)$ in relation to a transfer of the norm estimation in the classical principle of local reflexivity to their ideal (quasi–)norm $A$. In particular, we are interested in constructing examples of maximal Banach ideals which do not satisfy such a transfer. Due to the local nature of this principle of local reflexivity for operator ideals (called $\mathfrak{A} – LRP$) – which had been introduced and discussed in [18] and [19] – and the local nature of maximal Banach ideals, local versions of injectivity (right–accessibility) resp. surjectivity (left–accessibility) of suitable operator ideals and factorizations through operators with finite dimensional range even imply interesting relations between operators with infinite dimensional range. After extending
finite rank operators in certain quasi–Banach ideals \( \mathfrak{A} \), the \( \mathfrak{A} – LRP \) and the calculation of conjugate ideal norms then allow us to neglect the structure of the range space, so that we may leave the finite dimensional case. We will give sufficient conditions on \( \mathfrak{A} \) to guarantee that each finite rank operator \( L \) has a finite rank–extension \( \tilde{L} \) so that \( A(\tilde{L}) \leq (1 + \epsilon) \cdot A(L) \) – for given \( \epsilon > 0 \). Consequently, we are lead to the problem under which circumstances a finite rank operator \( L \in \mathfrak{A} \circ \mathfrak{B} \) has a factorization \( L = AB \) so that \( A(A) \cdot B(B) \leq (1 + \epsilon) \cdot A \circ B(L) \) and \( A \) resp. \( B \) has finite dimensional range. Operator ideals \( \mathfrak{A} \circ \mathfrak{B} \) with such a property \( (I) \) resp. property \( (S) \) had been introduced in \([14]\) to prepare a detailed investigation of trace ideals.

After introducing the necessary framework which also includes a full description of the technical concept of ultrastability, we recall the definition of the \( \mathfrak{A} – LRP \) and its first consequences. Not only in view of looking for a counterexample of a maximal Banach ideal \( (\mathfrak{A}_0, \mathfrak{A}_0) \) which does not satisfy the \( \mathfrak{A}_0 – LRP \), we will see that the property \( (I) \) of \( \mathfrak{A}^* \circ \mathfrak{L}_\infty \) plays a fundamental part in this paper; it even enables us to show that \( \mathfrak{L}_\infty \) is not totally accessible – answering a question of Defant and Floret (see Theorem 4.1)! We finish the paper with further applications, where we also consider linear operators acting between Banach spaces with cotype 2 which do not have the approximation property (such as Pisier’s space \( P \)). We apply the machinery of section 3 to product operator ideals which contain the operator ideal \( (\mathfrak{L}_2, \mathfrak{L}_2) \) as a factor and reveal surprising relations between the principle of local reflexivity for the maximal hull of such operator ideals and the existence of an ideal–norm on these product ideals.

2 The framework

In this section, we introduce the basic notation and terminology which we will use throughout this paper. We only deal with Banach spaces and most of our notations and definitions concerning Banach spaces and operator ideals are standard. We refer the reader to the monographs \([1, 7, 22]\) for the necessary background in operator ideal theory and the related terminology. Infinite dimensional Banach spaces over the field \( \mathbb{K} \in \{ R, \mathbb{C} \} \) are denoted throughout by \( W, X, Y \) and \( Z \) in contrast to the letters \( E, F \) and \( G \) which are used for finite dimensional Banach spaces only. The space of all operators (continuous linear maps) from \( X \) to \( Y \) is denoted by \( \mathfrak{L}(X, Y) \), and for the identity operator on \( X \), we write \( \text{Id}_X \). The collection of all finite rank (resp. approximable) operators from \( X \) to \( Y \) is denoted by \( \mathfrak{F}(X, Y) \) (resp. \( \mathfrak{T}(X, Y) \)), and \( \mathfrak{E}(X, Y) \) indicates the collection of all operators, acting between finite dimensional Banach spaces \( X \) and \( Y \) (elementary operators). The dual of a Banach space \( X \) is denoted by \( X' \), and \( X'' \) denotes its bidual \( (X')' \). If \( T \in \mathfrak{L}(X, Y) \) is an operator, we indicate that it is a metric injection by writing \( T : X \xrightarrow{1} Y \), and if it is a metric surjection, we write \( T : X \rightarrow Y \). If \( X \) is a Banach space, \( E \) a finite dimensional subspace of \( X \) and \( K \) a finite codimensional subspace of \( X \), then \( B_X := \{ x \in X : \|x\| \leq 1 \} \) denotes the closed unit ball, \( J_X^E : E \xrightarrow{1} X \) the canonical metric injection and \( Q_X^K : X \rightarrow X/K \) the canonical metric surjection. Finally, \( T' \in \mathfrak{L}(Y', X') \) denotes the dual operator of \( T \in \mathfrak{L}(X, Y) \).

If \( (\mathfrak{A}, \mathfrak{A}) \) and \( (\mathfrak{B}, \mathfrak{B}) \) are given quasi–Banach ideals, we will use throughout the shorter notation \( (\mathfrak{A}, \mathfrak{A}) \) for the dual ideal and the abbreviation \( \mathfrak{A} \cong \mathfrak{B} \) for the isometric equality \( (\mathfrak{A}, \mathfrak{A}) = (\mathfrak{B}, \mathfrak{B}) \). We write \( \mathfrak{A} \subseteq \mathfrak{B} \) if, regardless of the Banach spaces \( X \) and \( Y \), we have \( \mathfrak{A}(X, Y) \subseteq \mathfrak{B}(X, Y) \). If \( X_0 \) is a fixed Banach space, we write \( \mathfrak{A}(X_0, \cdot) \subseteq \mathfrak{B}(X_0, \cdot) \) (resp.
The metric inclusion \( (\mathfrak{A}, \mathfrak{A}) \subseteq (\mathfrak{B}, \mathfrak{B}) \) if, regardless of the Banach space \( Z \) we have \( \mathfrak{A}(X_0, Z) \subseteq \mathfrak{B}(X_0, Z) \) (resp. \( \mathfrak{A}(Z, X_0) \subseteq \mathfrak{B}(Z, X_0) \)). The metric inclusion \( (\mathfrak{A}, \mathfrak{A}) \subseteq (\mathfrak{B}, \mathfrak{B}) \) is often shortened by \( \mathfrak{A} \subseteq \mathfrak{B} \). If \( \mathbf{B}(T) \leq \mathbf{A}(T) \) for all finite rank (resp. elementary) operators \( T \in \mathfrak{F} \) (resp. \( T \in \mathfrak{E} \)), we sometimes use the abbreviation \( \mathfrak{A} \mathfrak{F} \subseteq \mathfrak{B} \) (resp. \( \mathfrak{A} \mathfrak{E} \subseteq \mathfrak{B} \)).

First we recall the basic notions of Grothendieck’s metric theory of tensor products (cf., e.g., [6], [8], [10], [16]), which together with Pietsch’s theory of operator ideals spans the mathematical frame of this paper. A **tensor norm** \( \alpha \) is a mapping which assigns to each pair \((X, Y)\) of Banach spaces a norm \( \alpha(;X,Y) \) on the algebraic tensor product \( X \otimes Y \) (shorthand: \( X \otimes_\alpha Y \) and \( X \tilde{\otimes}_\alpha Y \) for the completion) so that

- \( \varepsilon \leq \alpha \leq \pi \)
- \( \alpha \) satisfies the metric mapping property: If \( S \in \mathbb{L}(X, Z) \) and \( T \in \mathbb{L}(Y, W) \), then \( \| S \otimes T : X \otimes_\alpha Y \rightarrow Z \otimes_\alpha W \| \leq \| S \| \cdot \| T \| \).

Wellknown examples are the injective tensor norm \( \varepsilon \), which is the smallest one, and the projective tensor norm \( \pi \), which is the largest one. For other important examples we refer to [6], [8], or [16]. Each tensor norm \( \alpha \) can be extended in two natural ways. For this, denote for given Banach spaces \( X \) and \( Y \)

\[
\text{FIN}(X) := \{ E \subseteq X \mid E \in \text{FIN} \} \quad \text{and} \quad \text{COFIN}(X) := \{ L \subseteq X \mid X/L \in \text{FIN} \},
\]

where FIN stands for the class of all finite dimensional Banach spaces. Let \( z \in X \otimes Y \). Then the **finite hull** \( \bar{\alpha} \) is given by

\[
\bar{\alpha} (z; X, Y) := \inf \{ \alpha(z; E, F) \mid E \in \text{FIN}(X), \; F \in \text{FIN}(Y), \; z \in E \otimes F \},
\]

and the **cofinite hull** \( \tilde{\alpha} \) of \( \alpha \) is given by

\[
\tilde{\alpha} (z; X, Y) := \sup \{ \alpha(Q_K^X \otimes Q_L^Y(z); X/K, Y/L) \mid K \in \text{COFIN}(X), \; L \in \text{COFIN}(Y) \}. \]

\( \alpha \) is called **finitely generated** if \( \alpha = \bar{\alpha} \), **cofinitely generated** if \( \alpha = \tilde{\alpha} \) (it is always true that \( \bar{\alpha} \leq \alpha \leq \tilde{\alpha} \)). \( \alpha \) is called **right–accessible** if \( \bar{\alpha}(z; E, Y) = \bar{\alpha}(z; E, Y) \) for all \( (E, Y) \in \text{FIN} \times \text{BAN} \), **left–accessible** if \( \tilde{\alpha}(z; X, F) = \tilde{\alpha}(z; X, F) \) for all \( (X, F) \in \text{BAN} \times \text{FIN} \), and **accessible** if it is right–accessible and left–accessible. \( \alpha \) is called **totally accessible** if \( \bar{\alpha} = \tilde{\alpha} \). The injective norm \( \varepsilon \) is totally accessible, the projective norm \( \pi \) is accessible – but not totally accessible, and Pisier’s construction implies the existence of a (finitely generated) tensor norm which is neither left– nor right–accessible (see [1], 31.6).

There exists a powerful one–to–one correspondence between finitely generated tensor norms and maximal Banach ideals which links thinking in terms of operators with ”tensorial” thinking and which allows to transfer notions in the ”tensor language” to the ”operator language” and conversely. We refer the reader to [6] and [18] for detailed informations concerning this subject. Let \( X, Y \) be Banach spaces and \( z = \sum_{i=1}^n x'_i \otimes y_i \) be an Element in \( X' \otimes Y \). Then \( T_z(x) := \sum_{i=1}^n \langle x, x'_i \rangle y_i \) defines a finite rank operator \( T_z \in \mathfrak{F}(X, Y) \) which is independent of the representation of \( z \) in \( X' \otimes Y \). Let \( \alpha \) be a finitely generated tensor norm and \( (\mathfrak{A}, \mathfrak{A}) \) be a maximal Banach ideal. \( \alpha \) and \( (\mathfrak{A}, \mathfrak{A}) \) are said to be **associated**, notation:

\[
(\mathfrak{A}, \mathfrak{A}) \sim \alpha \quad \text{(shorthand: } \mathfrak{A} \sim \alpha, \text{ resp. } \alpha \sim \mathfrak{A})
\]
if for all $E, F \in \text{FIN}$

$$\mathfrak{A}(E, F) = E' \otimes_{\alpha} F$$

holds isometrically: $\mathfrak{A}(T_z) = \alpha(z; E', F)$.

Since we will use them throughout in this paper, let us recall the important notions of the conjugate operator ideal (cf. [8], [14] and [19]) and the adjoint operator ideal (all details can be found in the standard references [1] and [22]). Let $(\mathfrak{A}, A)$ be a quasi–Banach ideal.

- Let $\mathfrak{A}^\Delta(X, Y)$ be the set of all $T \in \mathcal{L}(X, Y)$ which satisfy

$$\mathfrak{A}^\Delta(T) := \sup\{| tr(TL) | \mid L \in \mathcal{F}(Y, X), A(L) \leq 1\} < \infty.$$  

Then a Banach ideal $(\mathfrak{A}^\Delta, A^\Delta)$ is obtained (here, $tr(\cdot)$ denotes the usual trace for finite rank operators). It is called the conjugate ideal of $(\mathfrak{A}, A)$.

- Let $\mathfrak{A}^{\ast}(X, Y)$ be the set of all $T \in \mathcal{L}(X, Y)$ which satisfy

$$\mathfrak{A}^{\ast}(T) := \sup\{| tr(TJ_{E}^{X}S_{F}^{Y}) | \mid E \in \text{FIN}(X), K \in \text{COFIN}(Y), A(S) \leq 1\} < \infty.$$  

Then a Banach ideal $(\mathfrak{A}^{\ast}, A^{\ast})$ is obtained. It is called the adjoint operator ideal of $(\mathfrak{A}, A)$.

By definition, it immediately follows that $\mathfrak{A}^{\ast} \subseteq \mathfrak{A}^{\ast}$. Another easy, yet important observation is the following: let $(\mathfrak{A}, A)$ be a quasi–Banach ideal and $(\mathfrak{B}, B)$ be a quasi–Banach ideal. If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{B}^{\ast} \subseteq \mathfrak{A}^{\ast}$, and $\mathfrak{A} \subseteq \mathfrak{B}$ implies the inclusion $\mathfrak{A}^{\Delta} \subseteq \mathfrak{A}^{\Delta}$. In particular, it follows that $\mathfrak{A}^{\Delta^{\ast}} \subseteq \mathfrak{A}^{\ast}$ and $(\mathfrak{A}^{\Delta})^{\ast} \subseteq \mathfrak{A}^{\ast}$.

In addition to the maximal Banach ideal $\mathcal{L} || \cdot || \sim \varepsilon$ we mainly will be concerned with the maximal Banach ideals $\mathcal{L} \sim \pi$ (integral operators), $(\mathcal{Q}, L) \sim \omega_2$ (Hilbertian operators), $(\mathcal{Q}, L) \sim \omega_2$ (2–dominated operators), $(\mathcal{Q}, L) \sim g_\pi$ (absolutely $p$–summing operators), $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $(\mathcal{L}, \mathcal{L}) \sim g_\pi$ (where $p$, $q$, and $w_1$ are also considered the maximal Banach ideals $\mathcal{Q}(c_0, C_0) \sim c_0$ (cotype 2 operators) and $(\mathfrak{A}, \mathfrak{A}) \sim \alpha_p$ (Pisier’s counterexample of a maximal Banach ideal which is neither right– nor left–accessible (cf. [3], 31.6)).

What about the regularity of conjugate ideals? We do not treat this problem in its whole generality in this paper. In the next section, we will include additional methods and tools which are of local nature, like accessibility or the principle of local reflexivity for operator ideals to prove the regularity of conjugate operator ideals of type $\mathfrak{A}^{\Delta_{\pi}}$ (cf. Proposition 3.2). However, if $(\mathfrak{A}, A)$ is a maximal Banach ideal, then $\mathfrak{A}^{\Delta}$ is regular, since:

**Proposition 2.1** Let $(\mathfrak{A}, A)$ be a quasi–Banach ideal. If $\mathfrak{A} \subseteq \mathfrak{A}^{dd}$, then $\mathfrak{A}^{\Delta}$ is regular.

**Proof:** Let $X, Y$ be arbitrary Banach spaces, $T \in \mathfrak{A}^{\Delta_{\pi}}(X, Y)$ and $L \in \mathfrak{F}(Y, X)$. Choose $A \in \mathfrak{F}(Y'', X)$ so that $L'' = j_X A$ (if $L = T_z$ with $z = \sum_{i=1}^{n} y_i \otimes x_i \in Y' \otimes X$, then $A = T_w$ where $w := \sum_{i=1}^{n} j_Y y_i \otimes x_i$). Since $\mathfrak{A} \subseteq \mathfrak{A}^{dd}$ in particular is regular, we have

$$|tr(TL)| = |tr(T''j_X A)| = |tr(j_Y TA)| \leq A^\Delta(j_Y T) \cdot A(A) \leq A^{\Delta_{\pi}}(T) \cdot A(L),$$
and the claim follows. ■

Given quasi–Banach ideals \((\mathfrak{A}, \mathfrak{A})\) and \((\mathfrak{B}, \mathfrak{B})\), let \((\mathfrak{A} \circ \mathfrak{B}, \mathfrak{A} \circ \mathfrak{B})\) be the corresponding product ideal and \((\mathfrak{A} \circ \mathfrak{B}^{-1}, \mathfrak{A} \circ \mathfrak{B}^{-1})\) (resp. \((\mathfrak{B}^{-1} \circ \mathfrak{B}, \mathfrak{A} \circ \mathfrak{B}^{-1})\)) the corresponding ”right–quotient” (resp. ”left–quotient”). We write \((\mathfrak{A}^{inj}, \mathfrak{A}^{inj})\), to denote the injective hull of \((\mathfrak{A}, \mathfrak{A})\), the unique smallest injective quasi–Banach ideal which contains \((\mathfrak{A}, \mathfrak{A})\), and \((\mathfrak{A}^{sur}, \mathfrak{A}^{sur})\), the surjective hull of \((\mathfrak{A}, \mathfrak{A})\), is the unique smallest surjective quasi–Banach ideal which contains \((\mathfrak{A}, \mathfrak{A})\). Of particular importance are the quotients \(\mathfrak{A}^{\Delta} := \mathfrak{J} \circ \mathfrak{A}^{-1}\) and \(\mathfrak{A}^{\Delta} := \mathfrak{A}^{-1} \circ \mathfrak{J}\) and their relations to \(\mathfrak{A}^{\Delta}\) and \(\mathfrak{A}^{*}\), treated in detail in [18] and [21]. Very useful will be the following statement which represents the injective hull (resp. the surjective hull) of a conjugate operator ideal as a quotient (cf. Corollary 3.4):

**Proposition 2.2** Let \((\mathfrak{A}, \mathfrak{A})\) be an arbitrary quasi–Banach ideal. Then

\[
(\mathfrak{A}^{\Delta})^{inj} \overset{1}{=} \mathfrak{P}_1 \circ \mathfrak{A}^{-1}
\]

and

\[
(\mathfrak{A}^{\Delta})^{sur} \overset{1}{=} \mathfrak{A}^{-1} \circ \mathfrak{P}_d
\]

**Proof:** It is sufficient to prove the statement only for the injective hull. Since

\[
(\mathfrak{A}^{\Delta})^{inj} \circ \mathfrak{A} \subseteq (\mathfrak{A}^{\Delta} \circ \mathfrak{A})^{inj} \subseteq \mathfrak{J}^{inj} \subseteq \mathfrak{P}_1,
\]

it follows that \((\mathfrak{A}^{\Delta})^{inj} \overset{1}{=} \mathfrak{P}_1 \circ \mathfrak{A}^{-1}\). To see the other inclusion, note that

\[
\mathfrak{A}^{\Delta}(\cdot, Y_0) \overset{1}{=} \mathfrak{J} \circ \mathfrak{A}^{-1}(\cdot, Y_0)
\]

holds for every Banach space \(Y_0\) of which the dual has the metric approximation property (this follows by an direct application of [22], Lemma 10.2.6.) Hence,

\[
\mathfrak{P}_1 \circ \mathfrak{A}^{-1} \overset{1}{=} \mathfrak{J}^{inj} \circ \mathfrak{A}^{-1} \subseteq (\mathfrak{J} \circ \mathfrak{A}^{-1})^{inj} \overset{1}{=} (\mathfrak{A}^{\Delta})^{inj},
\]

and the proof is finished. ■

A deeper investigation of relations between the Banach ideals \((\mathfrak{A}^{\Delta}, \mathfrak{A}^{\Delta})\) and \((\mathfrak{A}^{*}, \mathfrak{A}^{*})\) needs the help of an important local property, known as accessibility, which can be viewed as a local version of injectivity and surjectivity. All necessary details about accessibility and its applications can be found in [7], [19], [20] and [21]. So let us recall:

- A quasi–Banach ideal \((\mathfrak{A}, \mathfrak{A})\) is called right–accessible, if for all \((E, Y) \in \text{FIN} \times \text{BAN}\), operators \(T \in \mathfrak{L}(E, Y)\) and \(\varepsilon > 0\) there are \(F \in \text{FIN}(Y)\) and \(S \in \mathfrak{L}(E, F)\) so that \(T = J_{F, S}^Y S\) and \(A(S) \leq (1 + \varepsilon)A(T)\).

- \((\mathfrak{A}, \mathfrak{A})\) is called left–accessible, if for all \((X, F) \in \text{BAN} \times \text{FIN}\), operators \(T \in \mathfrak{L}(X, F)\) and \(\varepsilon > 0\) there are \(L \in \text{COFIN}(X)\) and \(S \in \mathfrak{L}(X/L, F)\) so that \(T = SQ_L^X\) and \(A(S) \leq (1 + \varepsilon)A(T)\).

- A left–accessible and right–accessible quasi–Banach ideal is called accessible.
• \((\mathfrak{A}, \mathfrak{A})\) is totally accessible, if for every finite rank operator \(T \in \mathcal{F}(X, Y)\) acting between Banach spaces \(X, Y\) and \(\varepsilon > 0\) there are \((L, F) \in \text{COFIN}(X) \times \text{FIN}(Y)\) and \(S \in \mathcal{L}(X/L, F)\) so that \(T = J_F^Y S Q_L^X\) and \(A(S) \leq (1 + \varepsilon)A(T)\).

Due to the existence of Banach spaces without the approximation property, we will see now that conjugate hulls are not "big enough" to contain such spaces. To this end, consider an arbitrary Banach ideal \((\mathfrak{A}, \mathfrak{A})\), and let \(X\) be a Banach space so that \(\text{Id}_X \in \mathfrak{A}^\Delta\) (i.e., \(X \in \text{space}(\mathfrak{A}^\Delta)\)). Since \((\mathfrak{M}, \mathfrak{N})\), the collection of all nuclear operators, is the smallest Banach ideal, it follows that \(\text{Id}_X \in \mathfrak{M}^\Delta\) and \(\mathfrak{N}^\Delta(\text{Id}_X) \leq \mathfrak{A}^\Delta(\text{Id}_X)\). Hence, if \(T \in \mathcal{L}(X, X)\) is an arbitrary linear operator, it follows that \(T = T \text{Id}_X \in \mathfrak{M}^\Delta(X, X)\) and \(\mathfrak{N}^\Delta(T) \leq \|T\| \cdot \mathfrak{A}^\Delta(\text{Id}_X)\). But this implies that

\[
\mathcal{L}(X, X) = \mathfrak{N}^\Delta(X, X).
\]

If \(\mathfrak{A}\) contains the class \(\mathfrak{I}\) of all integral operators (e.g., if \(\mathfrak{A}\) is maximal or if \(\mathfrak{A}\) is a conjugate of a quasi–Banach ideal), similar considerations lead to

\[
\mathcal{L}(X, X) = \mathfrak{I}^\Delta(X, X),
\]

and [14, Proposition 2.2.] now imply the following

**Remark 2.1** Let \((\mathfrak{A}, \mathfrak{A})\) be an arbitrary quasi–Banach ideal, and let \(X\) be a Banach space so that \(X \in \text{space}(\mathfrak{A}^\Delta)\). If \(\mathfrak{A}\) is normed, then \(X\) has the approximation property. If \(\mathfrak{I} \subseteq \mathfrak{A}\), then \(X\) has the bounded approximation property.

**Corollary 2.1** Let \((\mathfrak{A}, \mathfrak{A})\) be an arbitrary quasi–Banach ideal so that there exists a Banach space in \(\text{space}(\mathfrak{A}^{**})\) without the bounded approximation property, then \(\mathfrak{A}^\Delta\) (– in particular \(\mathfrak{A}^*\)) cannot be totally accessible.

**Proof:** Let \(X\) be a Banach space without the bounded approximation property so that \(X \in \text{space}(\mathfrak{A}^{**})\). Assume, \(\mathfrak{A}^\Delta\) is totally accessible, then

\[
\mathfrak{A}^{**} \subseteq \mathfrak{A}^* \subseteq \mathfrak{A}^{\Delta \Delta}
\]

Since \(\mathfrak{I} \subseteq \mathfrak{L}^\Delta \subseteq \mathfrak{A}^\Delta\), the previous Remark leads to a contradiction. 

Since ultrastable operator ideals play an important part in this paper, we completely recall the definition of an ultrastable operator ideal and its construction (cf. [4], [6], [7], [12], [13], and [22]): Let \(I\) be a non–empty set and \(U\) be an ultrafilter in \(I\). If \((X_i)_{i \in I}\) is a family of Banach spaces, consider in the Banach space

\[
\mathcal{L}_\infty(X_i; I) := \left\{ x = (x_i)_{i \in I} \in \prod_{i \in I} X_i \mid \|x\|_\infty := \sup_{i \in I} \|x_i\| < \infty \right\}
\]

the closed subspace \(N_U(X_i; I) := \{(x_i)_{i \in I} \in \mathcal{L}_\infty(X_i; I) \mid \lim_{U} \|x_i\| = 0 \}\). The ultraproduct of the family \((X_i)_{i \in I}\) with respect to the ultrafilter \(U\) is defined to be the Banach space

\[
\left( \prod_{i \in I} X_i \right)_U := \mathcal{L}_\infty(X_i; I)/N_U(X_i; I)
\]

\footnote{Proposition 21.6 in [8] is a special case of this Corollary.}
equipped with the canonical quotient norm. The elements of $(\prod_{i \in I} X_i)_U$ are denoted by $(x_i)_U$ (whenever $(x_i)_{i \in I} \in l_\infty(X_i; I)$), and the construction implies that $\|(x_i)_U\| = \lim_U \|x_i\|$. If $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ are two families of Banach spaces and $T_i \in \mathcal{L}(X_i, Y_i)$ with $c := \sup_{i \in I} \|T_i\| < \infty$, then

$$T^I(x_i) := (T_i x_i)$$

defines an operator $T^I : l_\infty(X_i; I) \to l_\infty(Y_i; I)$ with $\|T^I\| \leq c$ and which maps $N_U(X_i; I)$ into $N_U(Y_i; I)$. Consequently, there exists a linear operator $T : (\prod_{i \in I} X_i)_U \to (\prod_{i \in I} Y_i)_U$ so that $T(x_i)_U = (T_i x_i)_U$. $T$ is called the ultraproduct $(T_i)_U$ of the operators $T_i$. It satisfies $\|(T_i)_U\| = \lim_U \|T_i\|$.

Let $(\mathfrak{A}, \mathcal{A})$ be an arbitrary quasi–Banach ideal. $\mathfrak{A}$ is called ultrastable if for every ultrafilter $U$ on $I$ and every $\mathcal{A}$–bounded family of operators $T_i \in \mathfrak{A}(X_i, Y_i)$

$$(T_i)_U \in \mathfrak{A}((\prod_{i \in I} X_i)_U, (\prod_{i \in I} Y_i)_U) \quad \text{and} \quad \mathcal{A}((T_i)_U) \leq \lim_U \mathcal{A}(T_i).$$

The key part of ultrastable operator ideals is given by the following relation (see [22], Theorem 8.8.6.):

**Theorem 2.1 (Pietsch)** *Let $(\mathfrak{A}, \mathcal{A})$ be an ultrastable quasi–Banach ideal. Then

$$(\mathfrak{A}, \mathcal{A})^{\max} = (\mathfrak{A}, \mathcal{A})^{\reg}.$$*

Although $\mathfrak{A}^{\min}$ (resp. $(\mathfrak{A}^\Delta)^{dd}$) is always accessible, Pisier’s counterexample shows the existence of maximal Banach ideals which neither are left nor right–accessible. However, accessibility conditions of a quasi–Banach ideal at least can be transmitted to its regular hull:

**Proposition 2.3** *Let $(\mathfrak{A}, \mathcal{A})$ be an arbitrary quasi–Banach ideal. If $\mathfrak{A}$ is right–accessible (resp. totally–accessible), then the regular hull $\mathfrak{A}^{\reg}$ is also right–accessible (resp. totally–accessible).*

**Proof:** Let $\epsilon > 0$, $X$, $Y$ be Banach spaces and $T \in \mathfrak{B}(X, Y)$ an arbitrary finite rank operator. Assume that $\mathfrak{A}$ is totally accessible or that $X \in \text{FIN}$ and $\mathfrak{A}$ is right–accessible. In both cases, there exists a finite dimensional Banach space $F \in \text{FIN}(Y^m)$ and an operator $S \in \mathcal{L}(X, F)$, so that $j_Y T = J_F^{Y^m} S$ and

$$\mathcal{A}(S) < (1 + \epsilon) \cdot \mathcal{A}(j_Y T) = (1 + \epsilon) \cdot \mathcal{A}^{\reg}(T).$$

Due to the classical principle of local reflexivity for linear operators, there exists an operator $W \in \mathcal{L}(F, Y)$ so that $\|W\| < 1 + \epsilon$ and $j_Y W z = J_F^{Y^m} z$ for all $z \in F$ which satisfy $J_F^{Y^m} z \in j_Y(Y)$. Let $x \in X$ and put $z := Sx$. Then $J_F^{Y^m} z = j_Y Tx \in j_Y(Y)$, which therefore implies that $j_Y W S x = J_F^{Y^m} z = j_Y Tx$. Now, factor $W$ canonically through a finite dimensional subspace $G$ of $Y$ so that $W = J_G^Y U$ and $\|U\| < 1 + \epsilon$. Consequently, $T = W S = J_G^Y (US)$, and

$$\mathcal{A}^{\reg}(US) < (1 + \epsilon)^2 \cdot \mathcal{A}^{\reg}(T).$$
Hence, $\mathfrak{A}^{reg}$ is right–accessible (in each of the both cases). In the case of $\mathfrak{A}$ being totally accessible, the operator $S$ even can be chosen as $S = S_0 Q^X_K$, where $K \in COFIN(X)$ and $S_0 \in \mathfrak{L}(X \setminus K, F)$ so that

$$\mathcal{A}(S_0) < (1 + \epsilon) \cdot \mathcal{A}^{reg}(T),$$

and the proof is finished.

Let us finish this section with a short Remark concerning ultrastability versus accessibility. To this end, let $(\mathfrak{A}, \mathfrak{A})$ be a maximal Banach ideal. Then $\mathfrak{A}$ is right–accessible (resp. totally accessible) if and only if $\mathfrak{A}^* \overset{1}{=} \mathfrak{I} \circ \mathfrak{A}^{-1}$ (resp. $\mathfrak{A}^* \overset{1}{=} \mathfrak{A}^{\Delta}$) (cf. [21]). A straightforward calculation therefore leads to the following structurally interesting

**Remark 2.2** Let $(\mathfrak{A}, \mathfrak{A})$ be a maximal Banach ideal. Then the following statements are equivalent:

(i) $\mathfrak{A}$ is right–accessible (resp. totally accessible)

(ii) $\mathfrak{I} \circ \mathfrak{A}^{-1}$ (resp. $\mathfrak{A}^\Delta$) is ultrastable.

### 3 Extension of finite rank operators and the principle of local reflexivity for operator ideals

Let $(\mathfrak{A}, \mathfrak{A})$ be a maximal Banach ideal. Then, $\mathfrak{A}^\Delta$ always is right–accessible (cf. [21]). The natural question whether $\mathfrak{A}^\Delta$ is left–accessible is still open$^2$ and leads to interesting and non–trivial results concerning the local structure of $\mathfrak{A}^\Delta$. Deeper investigations of the left–accessibility of $\mathfrak{A}^\Delta$ namely lead to a link with a principle of local reflexivity for operator ideals (a detailed discussion can be found in [13] and [19]) which allows a transmission of the operator norm estimation in the classical principle of local reflexivity to the ideal norm $\mathfrak{A}$. So let us recall the

**Definition 3.1** Let $E$ and $Y$ be Banach spaces, $E$ finite dimensional, $F \in FIN(Y')$ and $T \in \mathfrak{L}(E, Y'')$. Let $(\mathfrak{A}, \mathfrak{A})$ be a quasi–Banach ideal and $\epsilon > 0$. We say that the principle of $\mathfrak{A}$–local reflexivity (short: $\mathfrak{A} – LRP$) is satisfied, if there exists an operator $S \in \mathfrak{L}(E, Y)$ so that

(1) $\mathcal{A}(S) \leq (1 + \epsilon) \cdot \mathcal{A}^{**}(T)$

(2) $\langle Sx, y' \rangle = \langle y', Tx \rangle$ for all $(x, y') \in E \times F$

(3) $j_Y Sx = Tx$ for all $x \in T^{-1}(j_Y(Y))$.

Although both, the quasi–Banach ideal $\mathfrak{A}$ and the 1–Banach ideal $\mathfrak{A}^{**}$ are involved, the asymmetry can be justified by the following statement which holds for arbitrary quasi–Banach ideals (see [19]):

$^2$For minimal Banach ideals $(\mathfrak{A}, \mathfrak{A})$, there exist counterexamples: The conjugate of $\mathfrak{A}^{P\min}$ neither is right–accessible nor left–accessible (cf. [19]).
Theorem 3.1 Let \((\mathfrak{A}, A)\) be a quasi–Banach ideal. Then the following statements are equivalent:

(i) \(A^\Delta\) is left–accessible

(ii) \(A^{**}(E, Y'') \cong A(E, Y)'\) for all \((E, Y) \in \text{FIN} \times \text{BAN}\)

(iii) The \(\mathfrak{A}–\text{LRP}\) holds.

One reason which leads to extreme persistent difficulties concerning the verification of the \(\mathfrak{A}–\text{LRP}\) for a given maximal Banach ideal \(A\), is the behaviour of the bidual \((A^\Delta)^{dd}\): although we know that in general \((A^\Delta)^{dd}\) is accessible (see [18] and [19]) and that \((A^\Delta)^{dd} \subseteq A^\Delta\), we do not know whether \(A^\Delta(X, Y)\) and \((A^\Delta)^{dd}(X, Y)\) coincide isometrically for all Banach spaces \(X\) and \(Y\). If we allow in addition the approximation property of \(X\) or \(Y\), then we may state the following

Lemma 3.1 Let \((\mathfrak{A}, A)\) be an arbitrary maximal Banach ideal and \(X, Y\) be arbitrary Banach spaces. Then

\[
\mathfrak{A}^d(X, Y) \overset{1}{\supseteq} A^d(X, Y)
\]

holds in each of the following two cases:

(i) \(X'\) has the metric approximation property

(ii) \(Y'\) has the metric approximation property and the \(\mathfrak{A}^d–\text{LRP}\) is satisfied.

Proof: Only the inclusion \(\subseteq\) is not trivial. So, let \(T \in \mathfrak{A}^d(X, Y)\) be given. First, we consider the case (i). Due to Proposition 2.3 of [14], it follows that in general

\[
\mathfrak{A}^d \lesssim (\mathfrak{A}^d)^{-1} \circ J \equiv (\mathfrak{A}^d)^d
\]

so that \(T' \in \mathfrak{A}^d(Y', X')\), and \(A^d(T') \lesssim A^d(T)\). Since \(X'\) has the metric approximation property we even obtain that \(T' = Id_{X'}T' \in J^\Delta \circ A^{-1}(Y', X') \lesssim A^\Delta(Y', X')\), and case (i) is finished.

To prove case (ii), we have to proceed in a total different way. Let \(L \in \mathfrak{F}(X', Y')\) be an arbitrary finite rank operator and \(\epsilon > 0\). Since \(Y'\) has the metric approximation property, there exists a finite rank operator \(A \in \mathfrak{F}(Y', Y'')\) so that \(L = AL\) and \(\|A\| \leq 1 + \epsilon\). Thanks to canonical factorization, we can find a finite dimensional space \(G\) and operators \(A_1 \in \mathcal{L}(Y', G''), A_2 \in \mathcal{L}(G'', Y')\) so that \(A = A_2A_1\), \(\|A_2\| \leq 1\) and \(\|A_1\| \leq 1 + \epsilon\). Now, look carefully at the composition of the two operators \(A_1L \in \mathfrak{F}(X', G'')\) and \(T'A_2 \in \mathcal{L}(G'', X')\). Using exactly the same considerations as in [22, E.3.2.], the assumed \(\mathfrak{A}^d–\text{LRP}\) implies the existence of an operator \(\Lambda \in \mathcal{L}(G', X)\) so that

\[
\mathfrak{A}^d(\Lambda) \leq (1 + \epsilon) \cdot \mathfrak{A}^d(A_1L') = (1 + \epsilon) \cdot A(A_1L)
\]

\footnote{We only have to substitute the operator norm through the ideal norm \(A^d\).}
and $A_1LT'A_2 = \Lambda'T'A_2$. Since $G$ is finite dimensional, we may represent $A_2$ as the dual of a finite rank operator $B_2 \in \mathfrak{F}(Y, G')$, and consequently it follows

$$
| tr(T'L) | \leq | tr(A_1LT'A_2) | \leq | tr(\Lambda'T'A_2) | \leq | tr(T\Lambda B_2) | \\
\leq A^{d\Delta}(T) \cdot A^d(\Lambda) \\
\leq (1 + \epsilon)^2 \cdot A^{d\Delta}(T) \cdot A(L).
$$

Hence, $T' \in \mathfrak{A}^{\Delta}(Y'', X')$, and $A^{\Delta}(T') \leq A^{d\Delta}(T)$, and case (ii) also is proved. ■

A straightforward dualization of the previous Lemma implies a result which we will use later again:

**Corollary 3.1** Let $(\mathfrak{A}, A)$ be an arbitrary maximal Banach ideal and $X, Y$ be arbitrary Banach spaces. Then

$$
\mathfrak{A}^{\Delta}(X, Y) \supseteq (\mathfrak{A}^{d})^{dd}(X, Y)
$$

holds in each of the following two cases:

(i) $X''$ has the metric approximation property and the $\mathfrak{A}^{d} - LRP$ is satisfied

(ii) $Y''$ has the metric approximation property and the $\mathfrak{A} - LRP$ is satisfied.

Using the considerations of the previous section, we obtain a closer approach to the $\mathfrak{A} - LRP$ in the following sense:

**Theorem 3.2** Let $(\mathfrak{A}, A)$ be an arbitrary Banach ideal. If $\mathfrak{A}$ is right–accessible and ultrastable, then the $\mathfrak{A} - LRP$ is satisfied.

**Proof:** Let $\epsilon > 0$. Let $E$ and $Y$ be Banach spaces, $E$ finite dimensional, $F \in \text{FIN}(Y')$ and $T \in \mathcal{L}(E, Y'')$. Since the right–accessibility of $\mathfrak{A}$ implies the right–accessibility of $\mathfrak{A}^{reg}$, there exists $G \in \text{FIN}(Y'')$ and an operator $B \in \mathcal{L}(E, G)$ so that $A(B) = A^{reg}(B) \leq (1 + \epsilon) \cdot A^{reg}(T)$ and $T = J_G^{y''}B$. The (classical) principle of local reflexivity, applied to the operator $J_G^{y''}$, implies the existence of a further operator $\Lambda \in \mathcal{L}(G, Y)$ so that $\|\Lambda\| \leq 1 + \epsilon$ and $\langle \Lambda z, y' \rangle = \langle y', J_G^{y''}z \rangle$ for all $(z, y') \in G \times F$. Hence, $S := \Lambda B \in \mathcal{L}(E, Y)$,

$$
A(S) \leq (1 + \epsilon)^2 \cdot A^{reg}(T),
$$

and $\langle Sx, y' \rangle = \langle y', J_G^{y''}(Bx) \rangle = \langle y', Tx \rangle$ for all $(x, y') \in E \times F$. Now, the assumption further implies that $A^{reg}(T) = A^{max}(T) = A^{**}(T)$, and the proof is finished. ■

Consequently, every right–accessible and maximal Banach ideal $(\mathfrak{A}, A)$ satisfies the $\mathfrak{A} - LRP$. Is the converse implication also true? Does the $\mathfrak{A}^{**} - LRP$ even imply the right–accessibility of $\mathfrak{A}^{**}$? A partial answer – involving Hilbert spaces – is given in Corollary 4.2. For minimal operator ideals, the previous considerations immediately lead to the following fact which we want to state separately:

**Corollary 3.2** Let $(\mathfrak{A}, A)$ be an arbitrary Banach ideal. If $\mathfrak{A}^{min}$ is ultrastable, then the $\mathfrak{A} - LRP$ is satisfied.
PROOF: Since $\mathfrak{A}$ is a Banach ideal, $\mathfrak{A}^{\text{min}}$ is also a Banach ideal (cf. [4], 9.2. and [5], 22.2.), and $\mathfrak{A}^{\text{min}}$ always is (right–)accessible. Therefore, the assumed ultrastability of $\mathfrak{A}^{\text{min}}$, implies the validity of the $\mathfrak{A}^{\text{min}} - LRP$ and in particular the validity of the $\mathfrak{A} - LRP.$ ■

Although operator ideals which are both, minimal and ultrastable, seem to be quite strange objects, there exist examples, such as the quasi–Banach ideal $(\mathfrak{N}_{(r,p,q)}, \mathfrak{N}_{(r,p,q)})$ (the collection of all $(r,p,q)$–nuclear operators). If $0 < r < \infty$, $1 \leq p, q \leq \infty$ and $1 + 1/r > 1/p + 1/q$, then $\mathfrak{N}_{(r,p,q)}$ is ultrastable and minimal (see [24], 18.1.4. and 18.1.9.).

Pisier’s counterexample of the maximal Banach ideal $(\mathfrak{A}_p, \mathfrak{A}_p)$ which neither is left–accessible nor right–accessible (cf. [3], 31.6) implies that in particular $(\mathfrak{A}_p^*, \mathfrak{A}_p^*)$ neither is left–accessible nor right–accessible. Thinking at $\mathfrak{A}_p^{1/2} \subseteq \mathfrak{A}_p$, this leads to the natural and even more tough question whether the $\mathfrak{A}_p - LRP$ is true or false. However, due to Corollary 2.1, we already know that $\mathfrak{A}_p^{1/2}$ cannot be totally accessible. Is it even true that $(\mathfrak{A}_p^{1/2})^{\text{inj}} \subseteq \mathfrak{A}_p$ is not totally accessible? If this is the case, the $\mathfrak{A}_p - LRP$ will be false. Unfortunately, we will later recognize that, in addition, $\mathfrak{A}_p^{1/2}$ cannot be injective. What about the left accessibility of $\mathfrak{A}_p^{1/2}$? Although we do not investigate the local structure of $\mathfrak{A}_p$ in this paper, we want to show a way how to construct other counterexamples. A first step towards an construction of such a candidate $(\mathfrak{A}, \mathfrak{A})$ is given by the following factorization property for finite rank operators which had been introduced by Jarchow and Ott in their paper [14]. It not only turns out to be a useful tool in constructing such a counterexample; later, we will also use this factorization property to show that $\mathfrak{L}_\infty$ is not totally accessible – answering an open question of Defant and Floret (see [3], 21.12)! So let us recall the definition of this factorization property and its implications:

**Definition 3.2 (Jarchow/Ott)** Let $(\mathfrak{A}, \mathfrak{A})$ and $(\mathfrak{B}, \mathfrak{B})$ be arbitrary quasi–Banach ideals. Let $L \in \mathcal{F}(X,Y)$ an arbitrary finite rank operator between two Banach spaces $X$ and $Y$. Given $\epsilon > 0$, we can find a Banach space $Z$ and operators $A \in \mathfrak{A}(Z,Y)$, $B \in \mathfrak{B}(X,Z)$ so that $L = AB$ and

$$A(A) \cdot B(B) \leq (1 + \epsilon) \cdot A \circ B(L).$$

(i) If the operator $A$ is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).

(ii) If the operator $B$ is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the property (S).

Important examples are the following (see [14], Lemma 2.4.):

- If $\mathfrak{B}$ is injective, or if $\mathfrak{A}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).

- If $\mathfrak{A}$ is surjective, or if $\mathfrak{B}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (S).

Since $\mathfrak{L}_2 \circ \mathfrak{A}$ is injective for every quasi–Banach ideal $(\mathfrak{A}, \mathfrak{A})$ (see [21], Lemma 5.1.), $\mathfrak{B} \circ \mathfrak{L}_2 \circ \mathfrak{A}$ therefore has the property (I) as well as the property (S), for all quasi–Banach ideals $(\mathfrak{B}, \mathfrak{B})$. Such ideals are exactly those which contain $\mathfrak{L}_2$ as factor – in the sense of [14].

The next statement will be also useful for our further investigatons (see [14], 2.5.):

**Proposition 3.1** Let $(\mathfrak{A}, \mathfrak{A})$ and $(\mathfrak{B}, \mathfrak{B})$ be arbitrary quasi–Banach ideals. Then

(i) $(\mathfrak{A} \circ \mathfrak{B})^\Delta \subseteq \mathfrak{B}^{-1} \circ \mathfrak{A}^\Delta$, if $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).
(ii) $(\mathcal{A} \circ \mathcal{B})^\Delta \subseteq \mathcal{B}^\Delta \circ \mathcal{A}^{-1}$, if $\mathcal{A} \circ \mathcal{B}$ has the property (S).

In both cases (i) and (ii), the inclusion $\subseteq$ holds in general – without any assumption on the ideals $\mathcal{A}$ and $\mathcal{B}$.

Later, we will recognize the particular importance of operator ideals of type $\mathcal{A}^* \circ \mathcal{L}_\infty$ which in addition have the property (I). First, let us note an implication of this factorization property which gives us a further insight into the local structure of conjugate operator ideals:

**Corollary 3.3** Let $(\mathcal{A}, \mathcal{A})$ be a quasi–Banach ideal so that $\mathcal{A}^\Delta$ is injective or surjective, then space$(\mathcal{A})$ cannot contain a Banach space without the approximation property.

**Proof:** Again, a proof for the injective case is enough. So, assume that the statement is false. Choose a Banach space $X \in$ space$(\mathcal{A})$ without the approximation property. Since $\mathcal{A}^\Delta$ is injective, it follows that $\mathcal{L} \circ \mathcal{A}^\Delta$ has the property (I), so that

$$Id_X \in \mathcal{A} \subseteq (\mathcal{A}^\Delta)^{-1} \circ \mathcal{J} \subseteq (\mathcal{L} \circ \mathcal{A}^\Delta)^\Delta \subseteq \mathcal{A}^\Delta,$$

which is a contradiction. $\blacksquare$

Next, we will see how the property (I) influences the structure of operator ideals of type $\mathcal{A}^{inj} \subseteq \mathcal{A}^*$ and their conjugates. To this end, first note that for all Banach spaces $X, Y$ and $X \to Z$, every operator $T \in (\mathcal{A}^{inj})^*(X, Y) \subseteq \mathcal{A}^*(X, Y)$ satisfies the following extension property: Given $\epsilon > 0$, there exists an operator $\tilde{T} \in \mathcal{A}^*(Z, Y'')$ so that $j_YT = \tilde{T}J_Z^{\epsilon}$ and $\mathcal{A}^*(\tilde{T}) \leq (1 + \epsilon) \cdot \mathcal{A}^*(T)$ (see [21], Satz 7.14). In particular, such an extension holds for all finite rank operators. However, we then cannot be sure that $\tilde{T}$ is also as a finite rank operator. Here, property (I) comes into play – in the following sense:

**Theorem 3.3** Let $(\mathcal{A}, \mathcal{A})$ be a maximal Banach ideal so that $\mathcal{A}^* \circ \mathcal{L}_\infty$ has the property (I). Let $\epsilon > 0$, $X$ and $Y$ be arbitrary Banach spaces and $L \in \mathcal{F}(Y, X)$. Let $Z$ be a Banach space which contains $Y$ isometrically. Then there exists a finite rank operator $V \in \mathcal{F}(Z, X'')$ so that $j_XL = VJ_Z^\epsilon$ and

$$\mathcal{A}^*(V) \leq (1 + \epsilon) \cdot (\mathcal{A}^{inj})^*(L).$$

If in addition, the $\mathcal{A}^* - LRP$ is satisfied, then $V$ even can be chosen to be a finite rank operator with range in $X$ and $L = VJ_Z^\epsilon$.

**Proof:** Let $L \in \mathcal{F}(Y, X)$ be an arbitrary finite rank operator between arbitrarily given Banach spaces $X$ and $Y$, and set $(\mathcal{B}, \mathcal{B}) := (\mathcal{A}^{inj}, \mathcal{A}^{inj})$. Let $\epsilon > 0$. Since $\mathcal{B}^* \subseteq (\mathcal{A}^* \circ \mathcal{L}_\infty)^{reg}$ (cf. [21]), there exist a Banach space $W$ and operators $A \in \mathcal{A}^*(W, X'')$, $B \in \mathcal{L}_\infty(Y, W)$ so that $j_XL = AB$ and

$$\mathcal{A}^*(A) \cdot \mathcal{L}_\infty(B) \leq (1 + \epsilon) \cdot \mathcal{B}^*(L).$$

Due to the assumed property (I) of $\mathcal{A}^* \circ \mathcal{L}_\infty$, we even may assume that $A$ is a finite rank operator. Further, we also may choose a Borel–Radon measure $\mu$ and operators $R \in \mathcal{L}(L_\infty(\mu), W'')$, $S \in \mathcal{L}(Y, L_\infty(\mu))$ so that $j_WB = RS$ and

$$\|R\| \cdot \|S\| \leq (1 + \epsilon) \cdot \mathcal{L}_\infty(B)$$

12
extended to an operator $\tilde{S} \in \mathcal{L}(Z, L_\infty(\mu))$ so that $S = \tilde{S}J_Y^Z$ and $\|\tilde{S}\| = \|S\|$. If we also take into account that $Id_{X''} = j'_{X''}j_{X''}$, then we obtain the following factorization of $j_X L$:

$$
j_X L = j'_{X'}(j_{X''}j_X L) = j'_{X'}(A''R\tilde{S}J_Y^Z).
$$

Therefore, $V := j'_{X'}, A''R\tilde{S} \in \mathfrak{F}(Z, X'')$ is the desired finite rank operator, and the factorization further shows that

$$
A^*(V) \leq (1 + \epsilon)^2 \cdot B^*(L),
$$

and the first part of our Theorem is proven.

Now let us assume that in addition the $\mathfrak{A}^* - LRP$ is satisfied. Since $Y$ embeds isometrically into $Y^\infty = l_\infty(B_Y)$, the previous considerations (in particular) imply the existence of a finite rank operator $V \in \mathfrak{F}(Y^\infty, X'')$, so that $j_X L = VJ_Y$ and $A^*(V) \leq (1 + \epsilon) \cdot B^*(L)$. Due to the metric approximation property of the dual of $Y^\infty$, we can find a finite dimensional subspace $F$ in $Y^\infty$ and an operator $B \in \mathcal{L}(Y^\infty, F)$ so that $\|B\| \leq 1 + \epsilon$ and $V = WB$ where

$$
W := VJ_Y^Z \in \mathcal{L}(F, X'').
$$

Due to the assumed $\mathfrak{A}^* - LRP$, we even can find an operator $W_0 \in \mathcal{L}(F, X)$ so that

$$
A^*(W_0) \leq (1 + \epsilon) \cdot A^*(W) \leq (1 + \epsilon)^2 \cdot B^*(L)
$$

and

$$
W x = j_X W_0 x \quad \text{for all } x \in W^{-1}(j_X(X)).
$$

Since for every $y \in Y$, $x = BJ_Y y \in F$ and $W x = WB J_Y y = V J_Y y = j_X L y \in j_X(X)$, it therefore follows that

$$
j_X L y = j_X W_0 B J_Y y \quad \text{for all } y \in Y.
$$

Hence, $L = W_0 BJ_Y$ and $A^*(W_0 B) \leq (1 + \epsilon)^3 \cdot (A^{inj})^*(L)$. Since $Y^\infty$ has the metric extension property, we can factorize $J_Y$ as $J_Y = J J_Y^Z$ so that $J \in \mathcal{L}(Z, Y^\infty)$, $\|J\| = 1$, and $V_0 := W_0 B J \in \mathcal{L}(Z, X)$ is our desired finite rank operator.

Let $(\mathfrak{A}, A)$ be a Banach ideal and $(\mathfrak{A}^{inj}, A^{inj})$ its injective hull. Thinking carefully about the previous statement, one might guess a strong relationship between the conjugate of $(\mathfrak{A}^{inj})^*$ and the injective hull of $\mathfrak{A}^\Delta$ involving the $\mathfrak{A}^* - LRP$ and further accessibility conditions. Indeed, we will show that such interesting relations exist and that they even support the search for a counterexample of a maximal Banach ideal $\mathfrak{A}_0$ which does not satisfy the $\mathfrak{A}_0 - LRP$. So, let us start with a deeper investigation of the Banach ideal $\mathfrak{A}^{inj\Delta}$.

**Proposition 3.2** Let $(\mathfrak{A}, A)$ be a 1–Banach ideal so that the $\mathfrak{A}^* - LRP$ is valid. Then

$$
\mathfrak{A}^{\Delta inj} \cong \mathfrak{A}^{min inj} \cong \mathfrak{A}^{inj \min} \cong \mathfrak{A}^{inj \Delta}.
$$

In particular, $\mathfrak{A}^{inj \Delta}$ is totally accessible and $(\mathfrak{A}^{inj*})^{\Delta \Delta}$ regular.
Proof: Let \( \epsilon > 0 \), \( X \) and \( Y \) be arbitrary Banach spaces and \( T \in \mathcal{F}(X,Y) \) an arbitrary finite rank operator. Due to the assumed \( \mathfrak{A}^* - LRP \), \( \mathfrak{A}^{\Delta} \) is left–accessible, and it follows the total accessibility of its injective hull \( \mathfrak{A}, B) := (\mathfrak{A}^{\Delta inj}, \mathfrak{A}^{* inj}) \). Hence, there exist \( F \in FIN(Y), K \in COFIN(X) \) and an elementary operator \( S \in \mathcal{L}(X/K,F) \) so that \( T = J_K^* SQ_K^X \) and \( B(S) < (1 + \epsilon) \cdot B(T) \). Since \( A \) is an ideal–norm, we obtain (cf. [22, 8.7.13., 9.2.2. and 9.3.1.])

\[
\mathfrak{A}^{inj}(S) = \mathfrak{A}^{inj\ast}(S) = \mathfrak{A}^{\ast\ast\ast inj}(S) = B(S),
\]

so that

\[
\mathfrak{A}^{inj\ast\Delta}(T) \leq \mathfrak{A}^{inj\ast\min}(T) \leq \mathfrak{A}^{inj\ast}(S) \leq B(S) \leq (1 + \epsilon) \cdot B(T) = (1 + \epsilon) \cdot \mathfrak{A}^{\ast\Delta inj}(T).
\]

Since further

\[
\mathfrak{A}^{*\infty}(Y^\infty, X) \supseteq \mathfrak{A}^{inj\ast}(Y^{\infty}, X),
\]

(cf. [8], 20.12.), conjugation leads to the remaining inclusion \( \mathfrak{A}^{inj\ast\Delta} \subseteq \mathfrak{A}^{\ast\Delta inj} \), which implies the equality

\[
\mathfrak{A}^{inj\ast\Delta}(T) = \mathfrak{A}^{inj\ast\min}(T) = \mathfrak{A}^{\ast\Delta inj}(T)
\]

for all \( T \in \mathfrak{F} \), so that in particular \( \mathfrak{A}^{inj\ast\Delta} \supseteq \mathfrak{A}^{inj\ast\min} \supseteq \mathfrak{B} \) is totally accessible. Since in general the inclusions \( \mathfrak{A}^{inj\ast\min} \supseteq \mathfrak{A}^{inj\ast\Delta} \) (cf. [3], 25.11.) and \( \mathfrak{A}^{inj\ast\min} \subseteq \mathfrak{A}^{\ast\Delta} \) (cf. [20]) are satisfied, the proof is finished.

Due to the existence of Banach spaces without the metric approximation property, we have \( \mathcal{L}^{\Delta inj} \supseteq \mathfrak{I}^{\Delta inj} \supseteq \mathfrak{I} \not\supseteq \mathfrak{A}^{\Delta} \supseteq \mathcal{L}^{\ast\Delta} \) and \( \mathfrak{A} \supseteq \mathfrak{A}^{\Delta inj} \supseteq \mathcal{L}^{inj\ast\Delta} \not\supseteq \mathcal{L}^{inj\ast\min} \supseteq \mathcal{F} \), so that in general we cannot transfer the previous Proposition to operators with infinite dimensional range. What about quasi–Banach ideals which are not normed? As the proof shows, the assumption \( p = 1 \) is essential. But we even can say more: The statement is false if we only assume the case \( 0 < p < 1 \)!

To see this, consider the injective \( \frac{1}{2} \)–Banach ideal \( \mathfrak{A} := \mathfrak{P}_2 \diamond \mathfrak{P}_2 \supseteq \mathfrak{L}_2 \diamond \mathfrak{M} \) (cf. [14], [21], [23]). Being a trace ideal, \( \mathfrak{A} \) cannot be normed. Since the self–adjoint Banach ideal \( \mathfrak{P}_2 \) is accessible, the quotient formula implies that \( \mathfrak{A}^{\ast\Delta} \supseteq \mathfrak{A}^{\ast\Delta} \supseteq \mathfrak{A}_2 \supseteq \mathfrak{P}_2 \diamond \mathfrak{P}_2 \supseteq \mathfrak{L} \) (cf. [6], 25.7.). In particular the \( \mathfrak{A}^{\ast} - LRP \) is valid, and we obtain \( \mathfrak{A}^{inj\ast\Delta} \supseteq \mathfrak{A}^{\ast\Delta} \supseteq \mathfrak{L} \supseteq \mathfrak{I} \). On the other hand \( \mathfrak{A}^{\ast\Delta inj} \supseteq \mathfrak{I}^{inj} \supseteq \mathfrak{P}_1 \). The assumption \( \mathfrak{I} \supseteq \mathfrak{P}_1 \) would imply the (global) equality \( \mathfrak{I}^{\Delta} \supseteq \mathfrak{P}_1^{\Delta} \supseteq \mathcal{L} \) which is a contradiction\(^4\) since \( \mathcal{L} \supseteq \mathfrak{L}_2 \supseteq \not\supseteq \mathfrak{I}^{\Delta} \).

However, there exists an additional sufficient condition which allows the extension of the previous Proposition to operators between infinite dimensional Banach spaces, namely the property (I) of the product ideal \( \mathfrak{A}^{\ast} \circ \mathcal{L}_\infty \):

**Theorem 3.4** Let \( \mathfrak{A}, A) \) be a maximal Banach ideal ideal so that the \( \mathfrak{A}^{\ast} - LRP \) is satisfied. Then

\[
\mathfrak{A}^{\ast\Delta inj} \supseteq (\mathfrak{A}^{\ast\Delta inj})^{dd}
\]

\(^4\)Note, that these considerations even show that \( \mathfrak{A}^{inj\ast\Delta} \not\supseteq \mathfrak{A}^{\ast\ast\ast inj} \).
If in addition, \( \mathfrak{A}^* \circ \mathfrak{L}_\infty \) has the property (I), then
\[
(\mathfrak{A}^{\ast\inj})^{dd} \subseteq \mathfrak{A}^{\ast\inj} \subseteq \mathfrak{A}_1 \circ (\mathfrak{A}^*)^{-1} \subseteq \mathfrak{A}^{\inj\ast\Delta} \subseteq (\mathfrak{A}^{\inj\ast\Delta})^{dd} \tag{**}
\]

**Proof:** First, let the \( \mathfrak{A}^* - \text{LRP} \) be satisfied. Let \( T \in \mathfrak{A}^{\ast\inj}(X, Y) \) be given and \( X, Y \) be arbitrary Banach spaces. Due to Corollary 3.1 and the assumed validity of the \( \mathfrak{A}^* - \text{LRP} \), it follows that \( J_Y T'' = (J_Y T)' \in \mathfrak{A}^{\Delta}(X'', (Y''\infty))'' \) and
\[
\mathfrak{A}^{\ast\Delta}(J_Y T'') \subseteq \mathfrak{A}^{\ast\Delta}(J_Y T) = (\mathfrak{A}^{\ast\Delta})^{\inj}(T).
\]
Since \( J_Y'' : Y'' \rightarrow (Y''\infty)'' \) is an isometric embedding (cf. \([22]\), B.3.9.), the metric extension property of \( (Y''\infty)'' \) implies the existence of an operator \( \overline{J} = \mathfrak{L}((Y''\infty), (Y''\infty)) \) so that \( J_Y'' = \overline{J} J_Y'' \) and \( \|J_Y''\| = 1 \). Hence, \( T'' \in (\mathfrak{A}^{\ast\Delta})^{\inj}(X'', Y'') \) and
\[
(\mathfrak{A}^{\ast\Delta})^{\inj}(T'') \subseteq (\mathfrak{A}^{\ast\Delta})^{\inj}(T),
\]
which implies the inclusion (\( * \)). To prove (\( ** \)), note, that the second isometric identity already has been proven in this paper (see Proposition 2.2). Recalling that always
\[
\mathfrak{A}^{\inj\ast\Delta} \subseteq (\mathfrak{A}^{\ast\Delta})^{\inj},
\]
we only have to prove the inclusion
\[
(\mathfrak{A}^{\ast\inj})^{dd} \subseteq \mathfrak{A}^{\inj\ast\Delta}
\]
– given the property (I) of \( \mathfrak{A}^* \circ \mathfrak{L}_\infty \). To this end, let \( T \in (\mathfrak{A}^{\ast\inj})^{dd}(X, Y) \) be given, with arbitrarily chosen Banach spaces \( X \) and \( Y \), and put \( (\mathfrak{B}, \mathfrak{B}) := (\mathfrak{A}^{\inj}, \mathfrak{A}^{\inj}) \). Since \( \mathfrak{B}^{\ast\Delta} \) is regular (see Proposition 2.1), we only have to show that \( j_Y T \in \mathfrak{B}^{\ast\Delta}(X, Y'') \) and
\[
\mathfrak{B}^{\ast\Delta}(j_Y T) \subseteq (\mathfrak{A}^{\ast\Delta})^{\inj}(T'').
\]
So, let \( L \in \mathfrak{L}(Y'', X) \) be an arbitrary finite rank operator – considered as an element of \( \mathfrak{B}^{\ast}(Y'', X) \). Due to the assumed property (I) of \( \mathfrak{A}^* \circ \mathfrak{L}_\infty \), Theorem 3.3 shows us the existence of a finite rank operator \( V \in \mathfrak{L}((Y'')\infty, X'') \) so that \( j_X L = V J_Y'' \) and
\[
\mathfrak{A}^*(V) \leq (1 + \epsilon) \cdot \mathfrak{B}^*(L).
\]
Hence,
\[
\begin{align*}
| \text{tr}(j_Y T L) | &= | \text{tr}(T'' j_X L) | = | \text{tr}(T'' V J_Y'') | = | \text{tr}(J_Y'' T'' V) | \\
&\leq \mathfrak{A}^{\ast\Delta}(J_Y'' T'') \cdot \mathfrak{A}^*(V) \\
&\leq (1 + \epsilon) \cdot (\mathfrak{A}^{\ast\Delta})^{\inj}(T'') \cdot \mathfrak{B}^*(L),
\end{align*}
\]
which implies that \( j_Y T \in \mathfrak{B}^{\ast\Delta}(X, Y'') \) and \( \mathfrak{B}^{\ast\Delta}(j_Y T) \subseteq (\mathfrak{A}^{\ast\Delta})^{\inj}(T'') \). Summing up all the previous steps in our proof, we have shown that
\[
\mathfrak{A}^{\ast\inj} \subseteq \mathfrak{A}^{\inj\ast\Delta} \subseteq (\mathfrak{A}^{\ast\inj})^{dd}
\]
which obviously implies (\( ** \)), and the proof is finished. \( \blacksquare \)
Corollary 3.4 Let \((\mathfrak{A}, A)\) be a maximal and left–accessible Banach ideal so that \(\mathfrak{A}^* \circ \mathcal{L}_\infty\) has the property (I). Then both, \(\mathfrak{A}^{inj}\) and \((\mathfrak{A}^{inj})^*\) are totally accessible.

**Proof:** Since \(\mathfrak{A}\) is left–accessible, Proposition 2.2 and the previous statement imply that

\[
A^{inj} \subseteq P^{-1} (A^* - 1) \subseteq (A^* \Delta)^{inj} \subseteq \mathfrak{A}^{inj},
\]

and it follows that

\[
\mathfrak{B}^* \subseteq \mathfrak{B}^A
\]

where we have put \(\mathfrak{B} := (\mathfrak{A}^{inj})^*\). Hence, [21, Theorem 3.1.] finishes the proof.\[\Box\]

As the careful reader might guess, Theorem 3.4 and Corollary 3.4 imply a lot of interesting consequences. Let us note the most important one:

**Theorem 3.5** Let \((\mathfrak{A}, A)\) be a maximal Banach ideal so that \(\mathfrak{A}^* \circ \mathcal{L}_\infty\) has the property (I). If space(\(\mathfrak{A}\)) contains a Banach space \(X_0\) so that \(X_0\) has the bounded approximation property but \(X_0''\) has not, then the \(A^* - LRP\) cannot be satisfied.

**Proof:** Assume, that the statement is false and hence the \(A^* - LRP\) is satisfied. Since \(X_0\) has the bounded approximation property, \(Id_{X_0} \in \mathcal{I}^\Delta(X_0, X_0)\) and \(c := \mathcal{I}^\Delta(Id_{X_0}) < \infty\). By definition of \(\mathcal{I}^\Delta\) and of the adjoint \(A^*\), one immediately derives the inclusion

\[
\mathfrak{A} \circ \mathcal{I}^\Delta \circ A^* \subseteq \mathcal{I},
\]

so that in particular

\[
\mathfrak{A}(X_0, X_0) \subseteq \mathcal{I} \circ (A^*)^{-1}(X_0, X_0) \subseteq P^{-1} (A^* - 1) \subseteq (A^* \Delta)^{inj}(X_0, X_0)
\]

and

\[
A^{\Delta^{inj}}(Id_{X_0}) \leq c \cdot A(Id_{X_0}).
\]

Hence, due to the assumed property (I) of \(A^* \circ \mathcal{L}_\infty\), Theorem 3.4 implies that even \(X_0'' \in \text{space}(\mathfrak{A}^{inj^*\Delta})\) and

\[
A^{inj^*\Delta}(Id_{X_0''}) \leq c \cdot A(Id_{X_0}).
\]

But this would imply that \(X_0'' \in \text{space}(\mathcal{I}^\Delta)\), leading to the conclusion that \(X_0''\) would have the bounded approximation property – with constant \(c \cdot A(Id_{X_0})\), which is a contradiction.\[\Box\]

Now, the reader may ask for explicite examples for such maximal Banach ideals. To this end, note again that \(A^* \circ \mathcal{L}_\infty\) has the property (I), if \(A^*\) contains \(\mathcal{L}_2\) as a factor. Since \(A^*\) is a Banach ideal, we therefore have to look for maximal Banach ideals of type \(\mathfrak{B} \circ \mathcal{L}_2 \circ \mathcal{C}\). A first investigation of geometrical properties of such product ideals was given in [21]. Unfortunately, we cannot present explicite sufficient criteria which show the existence of (an equivalent) ideal norm on product ideals. It seems to be much more easier to show that a certain product ideal cannot be a normed one by using arguments which involve trace ideals and the ideal of nuclear operators (the smallest Banach ideal). However, let us turn to the following section.
4 Applications

Among other things, we will see in this section how deep the properties (I) and (S) reflect the local structure of operator ideals. A first example considers the question of Defant and Floret (see [6], 21.12) whether $L_\infty$ is totally accessible or not. We are able to show that $L_\infty$ is not totally accessible, and the idea of the proof is the following: Assuming the opposite, leads to the property (I) for a suitable class of quasi–Banach ideals of type $A^* \circ L_\infty$. On the other hand, there exists a well known left–accessible candidate $A$ so that $(A^{\text{inj}})^*$ is not totally accessible – a contradiction to Corollary 3.4. To prepare the steps carefully, we first state a fact which is of its own interest:

**Lemma 4.1** Let $(A, A)$ and $(B, B)$ be arbitrary quasi–Banach ideals so that

(i) $A \circ B$ has the property (S)

(ii) $B$ is totally accessible.

Then $A \circ B$ is left–accessible and has the property (I).

**Proof:** Let $X$, $Y$ be arbitrary Banach spaces and $L \in \mathfrak{F}(X, Y)$ an arbitrary finite rank operator. Given $\epsilon > 0$, there exists a Banach space $Z$ and operators $A \in A(Z, Y)$, $B \in B(X, Z)$ so that $L = AB$ and

$$A(A) \cdot B(B) \leq (1 + \epsilon) \cdot A \circ B(L).$$

Due to the property (S) of $A \circ B$, we may assume that $B$ is of finite rank. Hence, since $B$ is totally accessible, there exist $K \in \text{COFIN}(X)$, $E \in \text{FIN}(Z)$ and an operator $\Gamma \in \mathfrak{L}(X / K, E)$ so that $B = J^K_E \Gamma Q^X_E$ and

$$B(\Gamma) \leq (1 + \epsilon) \cdot B(B).$$

Therefore, $L = A_0 \Gamma Q^X_E$ where $A_0 := AJ^K_E \in \mathfrak{F}(E, Y)$ and

$$A \circ B(A_0 \Gamma) \leq A(A_0) \cdot B(\Gamma) \leq (1 + \epsilon)^2 \cdot A \circ B(L),$$

and the claim follows.

Obviously, similar arguments allow a transfer of property (S) to property (I), and we obtain the ”(I)–version”:

**Lemma 4.2** Let $(A, A)$ and $(B, B)$ be arbitrary quasi–Banach ideals so that

(i) $A \circ B$ has the property (I)

(ii) $A$ is totally accessible.

Then $A \circ B$ has the property (S) and is right–accessible.

Associating these results with Proposition 2.3, we immediately obtain (with the help of a factor diagram) a quite useful result:

---

\[5\] Notice, that Proposition 21.4. in [8] is a Corollary of this result.

17
Proposition 4.1 Let \((\mathfrak{A}, \mathfrak{A})\) and \((\mathfrak{B}, \mathfrak{B})\) be arbitrary quasi–Banach ideals so that one of the following properties hold:

1. \(\mathfrak{A} \circ \mathfrak{B}\) has the property (I), \(\mathfrak{A}\) is totally accessible and \(\mathfrak{B}\) is left–accessible
2. \(\mathfrak{A} \circ \mathfrak{B}\) has the property (S), \(\mathfrak{A}\) is right–accessible and \(\mathfrak{B}\) is totally accessible,

then \(\mathfrak{A} \circ \mathfrak{B}\) has the property (I) as well as the property (S), and \((\mathfrak{A} \circ \mathfrak{B})^{\text{reg}}\) is totally accessible.

Now, we are well prepared to investigate the total accessibility of \(\mathcal{L}\): 

Theorem 4.1 The maximal Banach ideals \(\mathcal{L}_\infty \sim g_\infty\) and \(\mathcal{L}_1 \sim w_1\) are not totally accessible.

PROOF: Since \(\mathcal{L}_1 = \mathcal{L}_\infty^d\), we only have to prove the claim for \(\mathcal{L}_\infty\). Assume the opposite. Consider the maximal Banach ideal \(\mathfrak{A} := \mathcal{L}_1^\perp = (\mathfrak{P}_1^d)^*\). Since

\[(\mathfrak{A}^*)^\text{sur} \supset (\mathfrak{P}_1^d)^\text{sur} \supset (\mathfrak{P}_1^\text{inj})^d = \mathfrak{P}_1^d = \mathfrak{A}^*\]

it follows that \(\mathfrak{A}^*\) is surjective, so that \(\mathfrak{A}^* \circ \mathcal{L}_\infty\) has the property (S). Due to Lemma 4.1, the assumed total accessibility of \(\mathcal{L}_\infty\) even leads to the property (I) of \(\mathfrak{A}^* \circ \mathcal{L}_\infty\), and Corollary 3.4 implies that \((\mathfrak{A}^\text{inj})^* \supset (\mathfrak{P}_1^\text{inj})^*\) is totally accessible. On the other hand, [3, Corollary 21.6.2] tells us that the adjoint of \(\mathfrak{P}_1^\text{inj}\) cannot be totally accessible (because of the existence of subspaces of \(l_1\) without the approximation property), and we obtain a contradiction.■

Corollary 4.1 \(\mathcal{L}_\infty \circ \mathcal{L}_\infty\) neither has property (I) nor property (S) and is not regular. In particular, \(\mathcal{L}_\infty^1 \neq \mathcal{L}_\infty \circ \mathcal{L}_\infty\).

PROOF: First, assume that \(\mathcal{L}_\infty \circ \mathcal{L}_\infty\) has property (I). Then, Proposition 3.1 implies that

\[\mathcal{L}_\infty^{-1} \circ \mathcal{L}_\infty^\Delta = (\mathcal{L}_\infty \circ \mathcal{L}_\infty)^\Delta\]

Since \(\mathfrak{P}_1\) is right–accessible, it follows that

\[\mathcal{L}_\infty \circ \mathfrak{P}_1 = \mathfrak{P}_1 \circ \mathfrak{P}_1 \subseteq \mathfrak{I} \subseteq \mathcal{L}_\infty^\Delta \subseteq \mathfrak{L}_\infty^\Delta,\]

and hence \(\mathfrak{P}_1 \subseteq (\mathcal{L}_\infty \circ \mathfrak{P}_1)^\Delta\). But this inclusion further implies that

\[\mathfrak{P}_1^d \subseteq (\mathcal{L}_\infty \circ \mathfrak{P}_1)^d \subseteq ((\mathcal{L}_\infty \circ \mathcal{L}_\infty)^\text{reg})^d \subseteq \mathcal{L}_\infty^\text{reg} \subseteq \mathfrak{L}_1^\Delta,\]

(since \(\mathcal{L}_\infty^\Delta = (\mathcal{L}_\infty \circ \mathcal{L}_\infty)^\text{reg}\) and we obtain the contradiction \(\mathfrak{I}^\Delta \neq \mathfrak{L}_1^\Delta\) (since \(\mathfrak{I}\) is not totally accessible). Using similar arguments, the assumption of property (S) of \(\mathcal{L}_\infty \circ \mathcal{L}_\infty\) also leads to a contradiction. Obviously, \(\mathcal{L} \circ \mathcal{L}_\infty \neq \mathcal{L}_\infty\) has the property (S), and the proof is finished.■

Given two ultrastable quasi–Banach ideals \((\mathfrak{A}, \mathfrak{A})\) and \((\mathfrak{B}, \mathfrak{B})\), we know that the product ideal \(\mathfrak{B} \circ \mathfrak{A}\) is also ultrastable (see [3]). If \(\mathfrak{B} \circ \mathfrak{A}\) is normed, Theorem 2.1 further implies that

\[(\mathfrak{B} \circ \mathfrak{A})^{**} \supset (\mathfrak{B} \circ \mathfrak{A})^{\text{max}} \supset (\mathfrak{B} \circ \mathfrak{A})^{\text{reg}},\]

and, if in addition \((\mathfrak{A}, \mathfrak{A})\) is injective, then [2, Corollary 2.1.] even implies that

\[(\mathfrak{B} \circ \mathfrak{A})^{**} \supset (\mathfrak{B} \circ \mathfrak{A})^{\text{reg}} \supset \mathfrak{B}^{\text{reg}} \circ \mathfrak{A}.\]

Bearing this situation in mind, the \(\mathfrak{B} – LRP\) leads to a further surprising result which has rich consequences:
Lemma 4.3  Let $(\mathfrak{A}, \mathfrak{A})$ and $(\mathfrak{B}, \mathfrak{B})$ be quasi–Banach ideals, so that $\mathfrak{L}_2$ is a left–factor of $\mathfrak{A}$ and the $\mathfrak{B} – LRP$ is satisfied. Then,

$$\mathfrak{B} \circ \mathfrak{A} \cong (\mathfrak{B} \circ \mathfrak{A})^{reg} \subseteq \mathfrak{B}^{reg} \circ \mathfrak{A}.$$  

**Proof:** Let $X, Y$ be arbitrary Banach spaces, $\epsilon > 0$ and $L \in \mathfrak{B}(X, Y)$ be an arbitrary finite rank operator. Let $(\mathfrak{C}, \mathfrak{C})$ be a quasi–Banach ideal so that $\mathfrak{A} \cong L_2 \circ \mathfrak{C}$. Due to the property (S) of the product ideal $\mathfrak{B} \circ \mathfrak{A}$ and the property (I) of (the injective product) $L_2 \circ \mathfrak{C}$, we may write the finite rank operator $j_Y L$ as the composition

$$j_Y L = BAC,$$

where $C \in \mathfrak{C}(X, U)$, $\Lambda \in \mathfrak{B}(U, V)$, $B \in \mathfrak{B}(V, Y'')$ and $U, V$ are Banach spaces so that

$$\mathfrak{B}(B) \cdot L_2(\Lambda) \cdot C(C) < (1 + \epsilon)^2 \cdot (\mathfrak{B} \circ \mathfrak{A})^{reg}(L).$$

Since $\mathfrak{L}_2$ is totally accessible, there exists a finite dimensional subspace $F$ of $V$ and an operator $\Lambda_0 \in \mathfrak{L}(U, F)$ so that $\Lambda = J_F^* \Lambda_0$ and $L_2(\Lambda_0) < (1 + \epsilon) \cdot L_2(\Lambda)$. Hence, we now may apply the assumed $\mathfrak{B} – LRP$ to the operator $BJ_F' \in \mathfrak{L}(F, Y'')$, and it therefore follows the existence of an operator $B_0 \in \mathfrak{L}(F, Y)$ so that

$$\mathfrak{B}(B_0) \leq (1 + \epsilon) \cdot \mathfrak{B}(B)$$

and

$$BJ_F' v = j_Y B_0 v \text{ for all } v \in (BJ_F')^{-1}(j_Y(Y)).$$

Let $x \in X$ be given. Then $v = \Lambda_0 Cx \in (BJ_F')^{-1}(j_Y(Y))$, which implies that

$$j_Y L x = BACx = BJ_F' v = j_Y B_0 v = j_Y B_0 \Lambda_0 C x.$$

Since $x \in X$ was chosen arbitrary, we therefore obtain that $L = B_0 \Lambda_0 C$ and

$$\mathfrak{B} \circ \mathfrak{A}(L) = \mathfrak{B} \circ L_2 \circ C(L) \leq \mathfrak{B}(B_0) \cdot L_2(\Lambda_0) \cdot C(C) \leq (1 + \epsilon)^2 \cdot \mathfrak{B} \circ \mathfrak{A}(L),$$

and the proof is finished. $\blacksquare$

For completion, let us note the following fact:

**Proposition 4.2** Let $(\mathfrak{A}, \mathfrak{A})$ be an arbitrary quasi–Banach ideal. Then, $\mathfrak{A}^* \circ \mathfrak{L}_2 \circ \mathfrak{A}$ cannot be a $1-$ Banach ideal.

**Proof:** Put $\mathfrak{B} := \mathfrak{L}_2 \circ \mathfrak{A}$. Then both, $\mathfrak{B}$ and $\mathfrak{B} \circ \mathfrak{A}$ are injective (see [21]). Assume, that the statement is false. The previous considerations imply that the regular hull $(\mathfrak{A}^* \circ \mathfrak{B} \circ \mathfrak{A})^{reg}$ coincides isometrically with $\mathfrak{A}^* \circ \mathfrak{B} \circ \mathfrak{A}$. Using exactly the same technique as presented in the proof of Proposition 8.3. in [2] (a factorization through Banach spaces of type $l_2^2(Z_1, Z_2)$ consisting of elements $(z_1, z_2) \in Z_1 \times Z_2$ so that $\| (z_1, z_2) \|_{\infty} := \max(\|z_1\|, \|z_2\|) < \infty$) shows, that the assumed existence of an ideal norm on $\mathfrak{A}^* \circ \mathfrak{B}$ even implies the existence of an ideal norm on the smaller ideal $\mathfrak{A}^* \circ \mathfrak{B} \circ \mathfrak{A}$. Since $\mathfrak{A}$, the ideal of the nuclear operators is the
smallest Banach ideal, the right–accessibility of $\mathfrak{B}$ therefore implies that $\mathfrak{N} \subseteq \mathfrak{A}^* \circ \mathfrak{B} \circ \mathfrak{F} \subseteq \mathfrak{B}^* \circ \mathfrak{B} \circ \mathfrak{F} \subseteq \mathfrak{N}$, and we obtain that

$$\mathfrak{N} \subseteq \mathfrak{A}^* \circ \mathfrak{B} \circ \mathfrak{F} \subseteq (\mathfrak{A}^* \circ \mathfrak{B} \circ \mathfrak{F})^{\text{reg}} \subseteq \mathfrak{N}_{\text{reg}} \subseteq \mathfrak{N}^d$$

– a contradiction (cf. [3, Proposition 16.8.]).

Next, we again turn our attention to candidates $(\mathfrak{A}, \mathfrak{L})$ which do not satisfy the $\mathfrak{A} - \mathfrak{L}$ holds. Although it seems, that property (I) and property (S) do not play the fundamental part in the next statement, the proof opens the reader’s eyes:

**Theorem 4.2** Let $(\mathfrak{A}, \mathfrak{L})$ be a maximal Banach ideal so that the injective hull of $(\mathfrak{A} \circ \mathfrak{L}_2)^{\text{max}}$ is not totally accessible, then the $(\mathfrak{A}^{\text{inj}})^* - \mathfrak{L}^\text{RP}$ (and hence the $\mathfrak{A}^* - \mathfrak{L}^\text{RP}$) cannot be satisfied.

**Proof:** Assume that the $(\mathfrak{A}^{\text{inj}})^* - \mathfrak{L}^\text{RP}$ holds. Then $(\mathfrak{A}^{\text{inj}})^* \mathfrak{A}$ is totally accessible – due to Proposition 3.2. Since $\mathfrak{L}_2$ is injective, $\mathfrak{A}_0 := (\mathfrak{A}^{\text{inj}})^* \circ \mathfrak{L}_2$ has the property (I), and the total accessibility of $(\mathfrak{A}^{\text{inj}})^* \mathfrak{A}$ together with the left–accessibility of $\mathfrak{L}_2$ further implies that $\mathfrak{A}_0$ is totally accessible (Proposition 4.1). Since $\mathfrak{L}_2$ is a factor of $\mathfrak{A}_0$, $\mathfrak{L}_\infty \circ \mathfrak{A}_0$ has the property (S), and Proposition 4.1 even implies that $(\mathfrak{L}_\infty \circ \mathfrak{A}_0)^{\text{reg}}$ is totally accessible. Because of the metric approximation property of Hilbert spaces and spaces of type $\mathfrak{L}_\infty$, $(\mathfrak{L}_\infty \circ \mathfrak{A}_0)^{\text{reg}} = (\mathfrak{L}_\infty \circ \mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2)^{\text{reg}}$, so that in particular $\mathfrak{L}_\infty \circ \mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2$ is totally accessible. Therefore, we obtain the total accessibility of the injective hull of $\mathfrak{L}_\infty \circ \mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2$ which (by definition) equals isometrically $(\mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2)^{\text{inj}} = (\mathfrak{A} \circ \mathfrak{L}_2)^{\text{inj}}$. Hence, $((\mathfrak{A} \circ \mathfrak{L}_2)^{\text{max}})^{\text{inj}} = ((\mathfrak{A} \circ \mathfrak{L}_2)^{\text{reg}})^{\text{inj}} = (\mathfrak{A} \circ \mathfrak{L}_2)^{\text{inj}}$ must be totally accessible.

**Corollary 4.2** Let $(\mathfrak{B}, \mathfrak{B})$ be a maximal Banach ideal so that $\mathfrak{A} := \mathfrak{B} \circ \mathfrak{L}_2$ is normed. Then, $\mathfrak{A}$ is a maximal Banach ideal, and the following statements are equivalent:

(i) The $(\mathfrak{A}^{\text{inj}})^* - \mathfrak{L}^\text{RP}$ is satisfied

(ii) $\mathfrak{A}^{\text{inj}}$ is totally accessible.

**Proof:** Only the inclusion (i)$\implies$(ii) is not trivial. Since

$$\mathfrak{A}^* = \mathfrak{A}^{\text{max}} \supseteq \mathfrak{B} \circ \mathfrak{L}_2 \supseteq \mathfrak{B} \circ \mathfrak{L}_2 \supseteq \mathfrak{A},$$

$\mathfrak{A}$ is a maximal Banach ideal, and we therefore may apply Theorem 4.2 to $\mathfrak{A}$. Since $\mathfrak{L}_2 \supseteq \mathfrak{L}_2 \circ \mathfrak{L}_2$, assumption (i) therefore implies that $\mathfrak{A}^{\text{inj}} = (\mathfrak{A} \circ \mathfrak{L}_2)^{\text{inj}} \supseteq ((\mathfrak{A} \circ \mathfrak{L}_2)^{\text{max}})^{\text{inj}}$ is totally accessible, and the Corollary is proven.

The careful reader now may (and should) ask whether there exist operator ideals $\mathfrak{A}$ so that $\mathfrak{A} \circ \mathfrak{L}_2$ is not injective. Indeed, we will show that this is the case – in contrast to ideals of type $\mathfrak{L}_2 \circ \mathfrak{A}$ which always are injective (see [21], Lemma 5.1). To this end, we need the help of some “exotic” Banach spaces: G.T. spaces. Recall that a Banach space $X$ is called a G.T. space (a space which satisfies Grothendieck’s Theorem) if

$$\mathfrak{L}(X, \mathfrak{L}_2) = \mathfrak{P}_1(X, \mathfrak{L}_2)$$

Details and further informations about these Banach spaces are listed in [1] and [24]. We now will work with the famous Pisier space $\mathfrak{P}$ which is a G.T. space without the approximation
property (see [24], Theorem 10.6.). By the \( L_p \) – Local Technique Lemma for Operator Ideals (see [3], 23.1.), it follows that \( L_2(P, \cdot) \subseteq \mathcal{P}_1(P, \cdot) \) which is equivalent to \( Id_P \in L_2^{-1} \circ \mathcal{P}_1 \triangleq (L_\infty \circ L_2)^* \) (since \( L_2 \) is totally accessible). Assuming now that \( L_\infty \circ L_2 \) is injective, would imply the contradiction \( Id_P \in (L_\infty \circ L_2)^* \frac{1}{L_2} ((L_\infty \circ L_2)^{inj})^* \frac{1}{L_2} \frac{1}{L_2^\Delta} \subseteq \mathcal{I}^\Delta \). Hence, \( L_\infty \circ L_2 \) is not injective. But even more holds:

**Proposition 4.3** Let \((\mathfrak{A}, A)\) be a maximal Banach ideal which contains \( L_2 \) as a factor. Then \( L_\infty \circ \mathfrak{A} \) is not injective.

**Proof:** Put \( \mathfrak{B}_0 := (L_\infty \circ L_2)^* \). Then \( \mathfrak{B}_0 \subseteq (L_\infty \circ \mathfrak{A})^* \). Assuming the injectivity of \( L_\infty \circ \mathfrak{A} \) leads to the inclusion \( \mathfrak{B}_0 \subseteq (\mathfrak{A}^{inj})^* \), and we obtain

\[
\mathfrak{B}_0 \circ \mathfrak{A} \subseteq (\mathfrak{A}^{inj})^* \circ \mathfrak{A} \subseteq (\mathfrak{A}^{inj})^* \circ \mathfrak{A}^{inj} \subseteq \mathcal{I}
\]

(since \( \mathfrak{A}^{inj} \) always is right–accessible), which implies that \( Id_P \in \mathfrak{B}_0 \subseteq \mathcal{I} \circ \mathfrak{A}^{-1} \subseteq \mathcal{P}_1 \circ \mathfrak{A}^{-1} \triangleq (\mathfrak{A}^{\Delta})^{inj} \). Since \( L_2 \) is a factor of \( \mathfrak{A} \), \( \mathfrak{A} \circ L_\infty \) has the property (I), and the proof of Theorem 3.4 implies that \( Id_P \in \mathfrak{B}_0 \frac{1}{L_2} \mathfrak{B}_0^{dd} \subseteq ((\mathfrak{A}^{\Delta})^{inj})^{dd} \subseteq (\mathfrak{A}^{inj})^{\Delta} \subseteq \mathcal{I} \) which is a contradiction. \( \blacksquare \)

To round off these interesting considerations, we next prove a quotient version of Grothendieck’s Theorem:

**Proposition 4.4** Let \( \mathfrak{B}_0 := (L_\infty \circ L_2)^* \). Then \( L_1 \not\subseteq \mathfrak{B}_0 \), and \( \text{space}(\mathfrak{B}_0) \) contains Banach spaces without the approximation property. Moreover,

\[
L_2 \frac{1}{L_2} \mathfrak{B}_0^{-1}.
\]

**Proof:** The inclusion \( \frac{1}{L_2} \) already has been shown. To see the other inclusion, note that \( \mathfrak{D}_2 \triangleq L_2^{-1} \subseteq \mathfrak{B}_0 \). Since \( L_2 \) is injective, it therefore follows that \( \mathfrak{P}_1 \circ \mathfrak{B}_0^{-1} \triangleq (\mathfrak{B}_0^{\Delta})^{inj} \subseteq (\mathfrak{D}_2^{\Delta})^{inj} \triangleq L_2. \)

The situation completely changes, if we permute the factors \( L_\infty \) and \( L_2 \) in the product ideal \( L_\infty \circ L_2 \), since:

\[
(L_2 \circ L_\infty)^* \frac{1}{L_2} \frac{1}{L_2} \mathfrak{P}_1 \circ \mathfrak{L}_2^{-1} \mathfrak{D}_2^{inj} \subseteq \mathfrak{P}_2. \quad (*)
\]

If \((\mathfrak{B}, B)\) is a quasi–Banach ideal so that \( \mathfrak{B} \subseteq \mathfrak{D}_2 \), then we already know that \( \mathfrak{B} \circ L_2 \) is a trace ideal and therefore cannot admit an (equivalent) ideal–norm (see [14], 3.7.). What can we say if we only assume the existence of one (suitable) Banach space \( X_0 \) so that \( \mathfrak{B}(\cdot, X_0) \subseteq \mathfrak{D}_2(\cdot, X_0) \)? In this case, the existence of an ideal–norm on the product ideal \( \mathfrak{B} \circ L_2 \) a priori cannot be excluded, and we will see that the property (I) implies a surprising connection between the principle of local reflexivity for operator ideals and the existence of such an ideal–norm. To prepare the right instruments, we need the following statement:

**Lemma 4.4** Let \((\mathfrak{B}, B)\) be an arbitrary ultrastable quasi–Banach ideal so that \( \mathfrak{B} \circ L_2 \) is normed. If the \((\mathfrak{B} \circ L_2)^{**} \triangleq LRP \) is satisfied, then \((\mathfrak{B} \circ L_2)^{**} \circ L_\infty \) has the property (I) as well the property (S).

---

6 Note, that we do not assume the regularity of \((\mathfrak{B}, B)\) in Lemma 4.4!
Proof: Put $\mathfrak{A} := (\mathfrak{B} \circ \mathfrak{L}_2)^\ast$. Due to the assumptions on $\mathfrak{B}$ and the product ideal $\mathfrak{B} \circ \mathfrak{L}_2$, it follows that

$$\mathfrak{A}^\ast \overset{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{\ast\ast} \overset{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{\max} \overset{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{reg}$$

is a maximal Banach ideal which even implies that

$$\mathfrak{A}^\ast \overset{1}{=} ((\mathfrak{B} \circ \mathfrak{L}_2)^{reg})^{dd} \overset{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{dd}.$$ 

Let $X, Y$ be arbitrary Banach spaces and $T \in \mathfrak{F}(X,Y)$ an arbitrary finite rank operator. Given $\epsilon > 0$, the definition of $\mathfrak{L}_2$ implies the existence of a Borel–Radon measure $\mu$, a Banach space $Z$ and operators $S_1 \in \mathcal{L}(X, L_\infty(\mu))$, $S_2 \in \mathcal{L}(L_\infty(\mu), Z''')$, $R \in \mathfrak{A}^\ast(Z,Y) \overset{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{dd}(Z,Y)$ so that $j_T T = R''S_2S_1$ and

$$\mathfrak{B} \circ \mathfrak{L}_2(R''S_2) \cdot \|S_1\| \leq \mathfrak{B} \circ \mathfrak{L}_2(R'') \cdot \|S_2\| \cdot \|S_1\| < (1 + \epsilon)^2 \cdot \mathfrak{A}^\ast \circ L_\infty(T).$$

Since $\mathfrak{L}_2$ is a factor of the product ideal $\mathfrak{B} \circ \mathfrak{L}_2$, we may copy the proof of [14, Lemma 2.4], which even allows us to substitute the operator $R''S_2$ through a finite rank operator $V \in \mathfrak{F}(L_\infty(\mu), Y'')$ so that $j_T T = VS_1$ and

$$\mathfrak{B} \circ \mathfrak{L}_2(V) \cdot \|S_1\| < (1 + \epsilon)^3 \cdot \mathfrak{A}^\ast \circ L_\infty(T).$$

Due to the metric approximation property of $L_\infty(\mu)'$, we then obtain a finite dimensional subspace $F$ of $L_\infty(\mu)$ and operators $B \in \mathcal{L}(L_\infty(\mu), F)$ and $W \in \mathcal{L}(F, Y'')$ so that $V = WB$, $\|B\| \leq 1 + \epsilon$ and $\mathfrak{B} \circ \mathfrak{L}_2(W) \leq \mathfrak{B} \circ \mathfrak{L}_2(V)$. Now, we proceed as in the proof of the second part of Theorem 3.3, and the assumed $\mathfrak{A}^\ast - LRP$ even implies the existence of an operator $W_0 \in \mathcal{L}(F,Y)$ so that $T = W_0(BS_1)$ and

$$\mathfrak{A}^\ast(W_0) \cdot L_\infty(BS_1) \leq (1 + \epsilon) \cdot \mathfrak{A}^\ast(W_0) \cdot \|S_1\| \leq (1 + \epsilon)^2 \cdot \mathfrak{A}^\ast(W) \cdot \|S_1\| \leq (1 + \epsilon)^2 \cdot \mathfrak{B} \circ \mathfrak{L}_2(W) \cdot \|S_1\| \leq (1 + \epsilon)^5 \cdot \mathfrak{A}^\ast \circ L_\infty(T),$$

and we have obtained the properties (I) and (S) of $\mathfrak{A}^\ast \circ L_\infty.\blacklozenge$

Theorem 4.3 Let $(\mathfrak{B}, \mathfrak{B})$ be an ultrastable quasi–Banach ideal so that $\mathfrak{B} \circ \mathfrak{L}_2$ is normed. Let $X_0$ be a Banach space so that $X_0$ has the bounded approximation property but $X_0''$ has not. If

$$\mathfrak{B}(\cdot, X_0) \subseteq \mathfrak{D}_2(\cdot, X_0), \quad (**)$$

then the $(\mathfrak{B} \circ \mathfrak{L}_2)^{**} - LRP$ is not satisfied.

Proof: Put $\mathfrak{A} := (\mathfrak{B} \circ \mathfrak{L}_2)^\ast$. Assume that the $\mathfrak{A}^\ast - LRP$ is satisfied. Thanks to the previous Lemma, even $\mathfrak{A}^\ast \circ L_\infty$ has the property (I). Conjugating the inclusion (**) the total accessibility of $\mathfrak{D}_2$ leads to the inclusion

$$\mathfrak{L}_2(X_0, \cdot) \overset{1}{=} \mathfrak{D}_2^\Delta(X_0, \cdot) \subseteq \mathfrak{A}(X_0, \cdot) \overset{1}{=} \mathfrak{B}(X_0, \cdot),$$

and the quotient formula ([E], 25.7) therefore implies that $Id_{X_0} \in \mathfrak{L}_2^{-1} \circ \mathfrak{B}^\ast(X_0, X_0) \overset{1}{=} \mathfrak{A}(X_0, X_0)$, and Theorem 3.5 implies the bounded approximation property of $X_0''$ which is a contradiction.$\blacksquare$

Even the case $\mathfrak{B} \subseteq \mathfrak{D}_2^{inj}$ implies the same situation -- yet requiring a different proof:
Theorem 4.4 Let $(\mathfrak{B}, \mathcal{B})$ be an ultrastable quasi–Banach ideal so that $\mathfrak{B} \subseteq D_2^{\text{inj}}$. If $\mathcal{B} \circ \mathcal{L}_2$ is a 1–Banach ideal, then the $(\mathfrak{B} \circ \mathcal{L}_2)^{**} - LRP$ cannot be satisfied.

**Proof:** As before, put $\mathfrak{A} := (\mathfrak{B} \circ \mathcal{L}_2)^*$. Assume the validity of the $\mathfrak{A}^* - LRP$. Then, $\mathfrak{A}^* \circ \mathcal{L}_\infty$ has the property (I), and Theorem 3.4 therefore implies

$$(\mathfrak{A}^* \Delta)^{\text{inj}} \overset{1}{\subseteq} (\mathfrak{A}^{\text{inj}})^* \Delta.$$

On the other hand, since $\mathfrak{B} \subseteq D_2^{\text{inj}}$, (*) implies

$$(\mathfrak{B} \circ \mathcal{L}_2)^{\text{reg}} \subseteq \mathfrak{P}_1,$$

and it follows

$$\mathcal{L}_\infty \overset{1}{\subseteq} \mathfrak{P}_1^\Delta \subseteq ((\mathfrak{B} \circ \mathcal{L}_2)^{\text{reg}})^\Delta \overset{1}{\subseteq} \mathfrak{A}^\Delta.$$

Hence, $\mathcal{L}_\infty \overset{1}{\subseteq} \mathfrak{P}_1^\Delta \subseteq ((\mathfrak{B} \circ \mathcal{L}_2)^{\text{reg}})^\Delta \overset{1}{\subseteq} \mathfrak{A}^\Delta$, and we obtain a contradiction $\blacksquare$.

Permuting the factors $\mathfrak{B}$ and $\mathcal{L}_2$, we again obtain a different situation which even shows us a beautyful application of the principle of local reflexivity for operator ideals to the geometry of Banach spaces. Let $(\mathfrak{B}, \mathcal{B})$ be an ultrastable quasi–Banach ideal so that $\mathcal{L}_2 \circ \mathfrak{B}$ is normed. Then, we already know that in this case $\mathcal{L}_2 \circ \mathfrak{B}$ is an injective (hence, right–accessible) and even maximal Banach ideal, so that the $\mathcal{L}_2 \circ \mathfrak{B} - LRP$ automatically is satisfied. Since $\mathcal{L}_2 \circ \mathfrak{B} \circ \mathcal{L}_\infty$ has the property (I), Theorem 3.5 implies that every Banach space $X \in \text{space}(\mathfrak{B}^* \circ \mathcal{L}_2^{-1})$ with the bounded approximation property even must have a bidual $X''$ with the bounded approximation property! In other words:

**Theorem 4.5** Let $(\mathfrak{B}, \mathcal{B})$ be an ultrastable quasi–Banach ideal so that $\mathcal{L}_2 \circ \mathfrak{B}$ is normed. Let $X$ be a Banach space with the bounded approximation property. If

$$\mathfrak{B}(X, \cdot) \subseteq \mathcal{D}_2(X, \cdot),$$

then even $X''$ has the bounded approximation property.

To end up this section, we turn to another application of the property (I), involving Banach spaces of cotype 2. Using a deep result of Pisier, we only have to implement some of our own techniques at the right place, to prove the next result$^7$:

**Theorem 4.6** Let $(\mathfrak{A}, \mathcal{A})$ be a maximal and left–accessible Banach ideal so that $\mathfrak{A}^* \circ \mathcal{L}_\infty$ has the property (I). Let $X$ and $Y$ be Banach spaces so that both $X'$ and $Y$ have cotype 2. Then

$$\mathfrak{A}^{\text{inj}}(X, Y) \subseteq \mathcal{L}_2(X, Y),$$

and

$$\mathcal{L}_2(T) \leq (2C_2(X') \cdot C_2(Y))^{3/2} \cdot \mathfrak{A}^{\text{inj}}(T)$$

$^7$Note that it is not necessary to assume that $X$ resp. $Y$ has the Gordon–Lewis property (cf. $^7$, Theorem 17.12).
for all operators $T \in \mathfrak{A}^{inj}(X, Y)$.

**Proof:** Let $X$ and $Y$ be as above and put $C := (2C_2(X') \cdot C_2(Y))^\frac{1}{2}$. Then, [24, Theorem 4.9.] tells us, that any finite rank operator $L \in \mathfrak{S}(Y, X)$ satisfies

$$N(L) \leq C \cdot D_2(L).$$

Hence,

$$\mathfrak{N}^\Delta(X, Y) \subseteq D_2^\Delta(X, Y) \subseteq \mathfrak{L}_2(X, Y),$$

and

$$L_2(T) \leq C \cdot N^\Delta(T)$$

for all operators $T \in \mathfrak{N}^\Delta(X, Y)$. Since $\mathfrak{N} \subseteq (\mathfrak{A}^{inj})^*$, we therefore obtain

$$(\mathfrak{A}^{inj})^\Delta(X, Y) \subseteq \mathfrak{L}_2(X, Y),$$

and

$$L_2(T) \leq C \cdot (\mathfrak{A}^{inj})^\Delta(T)$$

for all operators $T \in (\mathfrak{A}^{inj})^\Delta(X, Y)$. Given our assumptions on $\mathfrak{A}$, Corollary 3.4 reveals that $(\mathfrak{A}^{inj})^*$ is totally accessible, and the claim follows. ■

5 Concluding remarks and open questions

Summing up our previous investigations, we recognize deep and still surprising relations between (the validity of) the principle of local reflexivity for operator ideals, the existence of a norm on product operator ideals of type $\mathfrak{B} \circ \mathfrak{L}_2$ and the extension of finite rank operators with respect to a suitable operator ideal norm. The basic objects, connecting these different aspects, are Jarchow/Otts' product operator ideals with property (I) and property (S). In the widest sense, a product $\mathfrak{A} \circ \mathfrak{B}$ has the property (I), if

$$(\mathfrak{A} \circ \mathfrak{B}) \cap \mathfrak{F} = (\mathfrak{A} \cap \mathfrak{F}) \circ \mathfrak{B}$$

and the property (S), if

$$(\mathfrak{A} \circ \mathfrak{B}) \cap \mathfrak{F} = \mathfrak{A} \circ (\mathfrak{B} \cap \mathfrak{F}),$$

so that each finite rank operator in $\mathfrak{A} \circ \mathfrak{B}$ is the composition of two operators, one of which is of finite rank. Since each operator ideal which contains $\mathfrak{L}_2$ as a factor, has both, the property (I) and the property (S), Hilbert space factorization is a fundamental key.

However, we do not know whether Corollary 4.2 holds for all maximal Banach ideals. If this is the case, the $\mathfrak{A}^{inj} - LRP$ will be false.

Is the property (I) of $\mathfrak{C}_2 \circ \mathfrak{L}_\infty$ satisfied? If this is the case, then the injective Banach ideal $\mathfrak{C}_2$ will be not left–accessible (due to Corollary 2.1 and Corollary 3.4) – answering another open question of Defant and Floret (see [4, 21.2., p. 277]).

We still do not know criteria which are sufficient for the existence of an ideal–norm on a given product of quasi–Banach ideals. It seems to be much easier to give arguments which imply the non–existence of such an ideal norm (using trace ideals). In particular, we would like to know whether $\mathfrak{T}_2 \circ \mathfrak{D}_2$ is a 1–Banach ideal ($\mathfrak{T}_2$, $\mathfrak{D}_2$ denotes the collection of all type 2 operators).
References

[1] R. Alencar, *Multilinear mappings of nuclear and integral type*, Proc. Amer. Math. Soc. 44 (1985), 33 - 38.

[2] A. Braunß, *Multi-ideals with special properties*, Preprint, Potsdam, Pädagogische Hochschule "Karl-Liebknecht" (1987).

[3] B. Carl, A. Defant, and M. S. Ramanujan, *On tensor stable operator ideals*, Michigan Math. J. 36 (1989), 63 - 75.

[4] D. Dacunha–Castelle and J. L. Krivine, *Applications des ultraproducts à l'étude des espaces et des algèbres de Banach*, Studia Math. 41 (1972), 315–334.

[5] A. Defant, *Produkte von Tensornormen*, Habilitationsschrift, Oldenburg 1986.

[6] A. Defant and K. Floret, *Tensor norms and operator ideals*, North - Holland Amsterdam, London, New York, Tokio 1993.

[7] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge University Press 1995.

[8] J. E. Gilbert and T. Leih, *Factorization, tensor products and bilinear forms in Banach space theory*, Notes in Banach spaces, pp. 182 - 305, Univ. of Texas Press, Austin, 1980.

[9] Y. Gordon, D. R. Lewis, and J. R. Retherford, *Banach ideals of operators with applications*, J. Funct. Analysis 14 (1973), 85 - 129.

[10] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo 8 (1956), 1 - 79.

[11] J. Harksen, *Tensornormtopologien*, Dissertation, Kiel 1979.

[12] S. Heinrich, *Ultraproducts in Banach space theory*, J. reine angew. Math. 313 (1980), 72–104.

[13] H. Jarchow, *Locally convex spaces*, Teubner 1981.

[14] H. Jarchow and R. Ott, *On trace ideals*, Math. Nachr. 108 (1982), 23 - 37.

[15] K. D. Kürsten, *s–Zahlen und Ultraprodukte von Operatoren in Banachräumen*, Dissertation, Leipzig, 1976.

[16] H. P. Lotz, *Grothendieck ideals of operators in Banach spaces*, Lecture notes, Univ. Illinois, Urbana, 1973.

[17] J. Lindenstrauss and H. P. Rosenthal, *The L_p-spaces*, Israel J. Math. 7 (1969), 325-349.

[18] F. Oertel, *Konjugierte Operatorenideale und das Λ-lokale Reflexivitätsprinzip*, Dissertation, Kaiserslautern 1990.

[19] F. Oertel, *Operator ideals and the principle of local reflexivity*, Acta Universitatis Carolinae - Mathematica et Physica 33, No. 2 (1992), 115 - 120.
[20] F. Oertel, *Composition of operator ideals and their regular hulls*; Acta Universitatis Carolinae - Mathematica et Physica 36, No. 2 (1995), 69 - 72.

[21] F. Oertel, *Local properties of accessible injective operator ideals*; Czech. Math. Journal, 48 (123) (1998), 119-133.

[22] A. Pietsch, *Operator ideals*, North - Holland Amsterdam, London, New York, Tokio 1980.

[23] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge studies in advanced mathematics 13 (1987).

[24] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*; CBMS Regional Conf. Series 60, Amer. Math. Soc. 1986.