On the distribution of \( k \)-free numbers and \( r \)-tuples of \( k \)-free numbers. A survey

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Abstract: This paper presents a brief survey of the current state of research the distribution of \( k \)-free numbers and \( r \)-tuples of \( k \)-free numbers. We state the main problems in the field, sketch their history and the basic machinery used to study them.

Keywords: \( k \)-free numbers, Consecutive \( k \)-free numbers, Asymptotic formula.

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1 Notations

Let \( k \) and \( n \) be integers and \( k \geq 2 \). We say that \( n \) is \( k \)-free if there is no prime \( p \) such that \( p^k | n \).

By convention, a 2-free integer is called square-free and a 3-free integer is called cube-free. We denote by \( \mu_k \) the characteristic function of the \( k \)-free numbers, i.e.,

\[
\mu_k(n) = \begin{cases} 
1, & \text{if } n \text{ is a } k\text{-free number,} \\
0, & \text{otherwise.} 
\end{cases}
\]

As usual, \( \varphi(n) \) is Euler’s function, \( \zeta(n) \) is Riemann’s zeta function, \( \mu(n) \) is the Möbius’ function and \( \tau(n) \) denotes the number of positive divisors of \( n \). Let \( X \) be a sufficiently large positive number. By \( \varepsilon \) we denote an arbitrary small positive number, not necessarily the same in different occurrences. Moreover \( \lfloor t \rfloor \) and \( \{ t \} \) denote the integer part and the fractional part of \( t \), respectively. The letter \( p \) will always denote a prime number.
2 Introduction

The investigation of the $k$-freeness of the numbers is important and plays a significant role in the contemporary analytic number theory. Many Diophantine equations are solved with square-free numbers. For example, in 2014, Dudek [17] proved that every integer greater than two may be written as the sum of a prime and a square-free number. This paper presents a brief survey of the current state of the distribution of $k$-free numbers and $r$-tuples of $k$-free numbers. We state the main problems in the field, sketch their history and the basic machinery used to study them.

3 On the distribution of $k$-free numbers

3.1 $k$-free numbers of arbitrary type

It is well known that the density of $k$-free integers is $1/\zeta(k)$, and an elementary sieve shows

$$\sum_{n \leq X} \mu_k(n) = \frac{X}{\zeta(k)} + O \left( X^{1/k} \right).$$

No better exponent is known for the remainder term. In the case $k = 2$ assuming RH the exponent $1/2$ has been refined several times [1, 2, 49] and currently to $17/54 = 0.31$ by [38]. It is expected that

$$\sum_{n \leq X} \mu_2(n) = \frac{6}{\pi^2} X + O \left( X^{1/4+\varepsilon(1)} \right).$$

3.2 $k$-free numbers of the form $[nc]$

3.2.1 Square-free numbers of the form $[nc]$

In 1978, Rieger [57] showed that for any fixed $1 < c < 3/2$ the asymptotic formula

$$\sum_{n \leq X} \mu_2([nc]) = \frac{6}{\pi^2} X + O \left( X^{2c+1+\varepsilon_0} \right)$$

holds.

From the above formula it follows that for any fixed $1 < c < 3/2$ there exist infinitely many square-free numbers of the form $[nc]$.

Subsequently Cao and Zhai [9] using estimation of multiple exponential sums with monomials ([8, Theorem 7]), estimation of three-dimensional exponential sums with monomials ([58, Theorem 3]) and Heath-Brown’s identity [24], improved the result of Rieger by proving the following:

**Theorem 1** ([9]). For any fixed $1 < c < 149/87$, $\gamma = e^{-1}$ and $0 < \varepsilon < (149\gamma - 87)/400$ the asymptotic formulas

$$\sum_{n \leq X} \mu_2([nc]) = \frac{6}{\pi^2} X + O \left( X^{1-\varepsilon} \right),$$

$$\sum_{p \leq X} \mu_2([pc]) = \frac{6}{\pi^2} \int_2^X \frac{dt}{\ln t} + O \left( X e^{-c_0 \sqrt{\ln \ln X}} \right),$$
\[ \sum_{n \leq X, \mu_2(n) = 1} \mu_2([n^c]) = \frac{36}{\pi^4} X + \mathcal{O}(X^{1-\varepsilon}) \]

hold. Here \( c_0 > 0 \) is an absolute constant.

Their earlier result [7] covers the narrower range \( 1 < c < 61/36 \).

### 3.2.2 Cube-free numbers of the form \([n^c]\)

In 2017, Zhang and Li [67] using the method of Cao and Zhai [9] showed that for any fixed \( 1 < c < 11/6 \) there exist infinitely many cube-free numbers of the form \([n^c]\). More precisely, they proved the following:

**Theorem 2 ([67]).** For any fixed \( 1 < c < 11/6 \), \( \gamma = c^{-1} \) and \( 0 < \varepsilon < 10^{-10} \) the asymptotic formulas

\[
\sum_{n \leq X} \mu_3([n^c]) = \frac{X}{\zeta(3)} + \mathcal{O}(X^{1-\varepsilon}) ,
\]

\[
\sum_{p \leq X} \mu_3([p^c]) = \frac{1}{\zeta(3)} \int_2^X \frac{dt}{\ln t} + \mathcal{O}\left( X e^{-c_0 \sqrt{\log X}} \right) ,
\]

\[
\sum_{n \leq X, \mu_3(n) = 1} \mu_3([n^c]) = \frac{X}{\zeta^2(3)} + \mathcal{O}(X^{1-\varepsilon})
\]

hold. Here \( c_0 > 0 \) is an absolute constant.

### 3.3 Square-free numbers of the form \([\alpha n]\)

In 2008, Güloğlu and Nevans [22] showed that there exist infinitely many square-free numbers of the form \([\alpha n]\), where \( \alpha > 1 \) is an irrational number of finite type. More precisely, they proved that the asymptotic formula

\[
\sum_{n \leq N} \mu_2([\alpha n]) = \frac{6}{\pi^2} N + \mathcal{O}\left( \frac{N \log \log N}{\log N} \right)
\]

holds.

Subsequently in 2013, Victorovich [64] showed that there exist infinitely many square-free numbers of the form \([\alpha n]\), where \( \alpha \) is irrational number with bounded partial quotient or algebraic number. More precisely, he proved the following:

**Theorem 3 ([64]).** For each \( A > 0 \) the asymptotic formula

\[
\sum_{n \leq N} \mu_2([\alpha n]) = \frac{6}{\pi^2} N + \mathcal{O}\left( AN^{\frac{2}{3}} \log^5 N \right)
\]

holds. Here \( A = A(N) = \max_{1 \leq m \leq N^2} \tau(m) \).
3.4 Square-free numbers of the form $p - 1$

Further in 2013, Victorovich [64] showed that there exist infinitely many square-free numbers of the form $p - 1$, where $p$ is prime. More precisely he proved the following:

**Theorem 4 ([64]).** For each $A > 0$ the asymptotic formula

$$\sum_{p \leq N} \mu_2(p - 1) = \prod_p \left(1 - \frac{1}{p(p - 1)}\right) \int_2^N \frac{dt}{\ln t} + \mathcal{O}\left(\frac{N}{\log^A N}\right)$$

holds.

3.5 $k$-free values of polynomials

3.5.1 $k$-free values of polynomial of arbitrary type

Let $k$ and $n$ be integers and $k \geq 2$. Consider the irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree $d$. Assume that for every prime $p$ there is at least one integer $n_p$ for which $p^k \nmid f(n_p)$. It is conjectured that the set $f(\mathbb{Z}) = \{ f(n), n \in \mathbb{Z} \}$ contains infinitely many $k$-free values. The first result in this direction belongs to Ricci [56], who proved the following:

**Theorem 5 ([56]).** Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial of degree $d$. Then for $k \geq d$ the asymptotic formula

$$N_{f,k}(x) \sim C(f,k)x \quad (x \to \infty)$$

holds. Here

$$N_{f,k}(x) = \# \{ n \leq x : f(n) \text{ is } k\text{-free} \},$$

$$C(f,k) = \prod_p \left(1 - \frac{\rho_f(p^k)}{p^k}\right)$$

and

$$\rho_f(n) = \# \{ a (\mod n) : n \mid f(a) \}.$$  

Further progress was made by Erdős [18] who proved the conjecture in the case $k = d - 1$ for $d \geq 3$. Later, Hooley [34] derived the asymptotic formula (1) for each such $k$. Using an alternative approach, Nair [50] established (1) for

$$k \geq \sqrt{2d^2 + 1} - \frac{d + 1}{2}.$$  

In 2006, Heath-Brown [27] showed how the determinant method could be applied to the problem, and demonstrated that the asymptotic formula (1) remained valid for

$$k \geq \frac{3d + 2}{4}.$$  

The idea behind Heath-Brown’s approach is to translate the problem into one that involves counting suitably constrained integral points on a certain affine surface.
In 2011 Browning [6] proved that the asymptotic formula (1) holds for
\[ k \geq \frac{3d + 1}{4}, \]
and \( d \geq 3. \)

There is a related question concerning \( k \)-free values of polynomial \( f \) at prime arguments. Such results can be found in [6, 30–33, 35, 39, 42, 50–52, 54, 63].

### 3.5.2 Square-free values of the form \( n^2 + 1 \)

It was shown in 1931 by Estermann [19] that there exist infinitely many square-free numbers of the form \( n^2 + 1 \). More precisely, he proved the following:

**Theorem 6 ([19]).** For \( x \geq 2 \) the asymptotic formula
\[ \mathcal{N}(x) = c_0 x + \mathcal{O} \left( X^{2/3} \log x \right) \]
holds. Here
\[ \mathcal{N}(x) = \# \{ n \leq x : n^2 + 1 \text{ is square-free} \} \]
and
\[ c_0 = \prod_{p \equiv 1 \pmod{4}} \left( 1 - \frac{2}{p^2} \right). \]

In 2012, Heath-Brown [29] improved the remainder term in the theorem of Estermann with \( \mathcal{O} \left( X^{7/12+\epsilon} \right) \). In order to obtain this result Heath-Brown used a variant of the determinant method, developed in his papers [26, 28].

### 3.5.3 \( k \)-free values of the form \( x^d + c \)

In 2013, Heath-Brown [30] investigated \( k \)-freeness of the polynomials of type \( x^d + c \). He proved the following:

**Theorem 7 ([30]).** Let \( f(x) = x^d + c \in \mathbb{Z}[x] \) be an irreducible polynomial, and suppose that \( k \geq (5d + 3)/9 \). Then, there is a constant \( \delta(d) \) such that
\[ N_{f,k}(x) = C(f, k)x + \mathcal{O} \left( X^{1-\delta(d)} \right) \]
holds. Here
\[ N_{f,k}(x) = \# \{ n \leq x : f(n) \text{ is } k\text{-free} \}, \]
\[ C(f, k) = \prod_p \left( 1 - \frac{\rho_f(p^k)}{p^k} \right) \]
and
\[ \rho_f(n) = \# \{ a \pmod{n} : n \mid f(a) \}. \]
The implied constant may depend on \( f \) and \( k \).
3.6  \( k \)-free values of multivariable polynomials

3.6.1 \( k \)-free values of multivariable polynomials of arbitrary type

Let \( n \geq 1, d \geq 2 \) be two integers. Consider the power-free values of the multivariable polynomial \( F(x_1, \ldots, x_n) \) with integer coefficients and degree \( d \). Denote

\[
N_{F,k}(B) = \#\{(x_1, \ldots, x_n) \in \mathbb{Z}^n : |x_i| \leq B \text{ for } i = 1, \ldots, n, F(x_1, \ldots, x_n) \text{ is } k\text{-free}\}
\]

Most of the work has been done for binary forms. The asymptotic formula for \( N_{F,k}(B) \) for binary forms \( F \) was established for: \( k \geq (d-1)/2 \) by Greaves [21], \( k > (2\sqrt{2}-1)d/4 \) by Filaseta [20], \( k > 7d/16 \) by Browning [6] and \( k > 7d/18 \) by Xiao [65]. For other results concerning power-free values of polynomials in two variables we refer to [36, 37] and [60].

In 2018, Lapkova and Xiao [41] derived an asymptotic formula for \( N_{F,k}(B) \).

**Theorem 8 ([41, Theorem 1]).** Let \( k \geq 2 \) be a positive integer and let \( F \) be a polynomial with integer coefficients and degree \( d \geq 2 \), in \( n \) variables, such that for all primes \( p \), there exists an integer \( n \)-tuple \( (m_1, \ldots, m_n) \) such that \( p^k \nmid F(m_1, \ldots, m_n) \). Then, there exists a positive number \( C_{F,k} \) such that the asymptotic relation

\[
N_{F,k}(B) \sim C_{F,k}B^n
\]

holds whenever \( k \geq (3d+1)/4 \).

Here the constant term is given by the limit of an absolutely convergent infinite product

\[
C_{F,k} = \prod_p \left(1 - \frac{\rho_F(p^k)}{p^{kn}}\right)
\]

and

\[
\rho_F(m) = \#\{(m_1, \ldots, m_n) \in (\mathbb{Z}/m\mathbb{Z})^n : m \mid F(m_1, \ldots, m_n)\}.
\]

Lapkova and Xiao [41] also proved a similar result when the inputs are restricted to be primes (see [41, Theorem 2]).

For another result concerning \( k \)-free values of multivariable polynomials we refer to [3,4,53] and [66].

3.6.2 Square-free values of the form \( x^2 + y^2 + 1 \)

Using the properties of the Gauss sum and A. Weil’s estimate for the Kloosterman sum in 2010 Tolev [61] showed that there exist infinitely many square-free numbers of the form \( x^2 + y^2 + 1 \). More precisely he proved the following:

**Theorem 9 ([61]).** The asymptotic formula

\[
\sum_{1 \leq x, y \leq H} \mu_2(x^2 + y^2 + 1) = cH^2 + \mathcal{O}\left(H^{\frac{4}{5} + \varepsilon}\right),
\]

holds. Here
\[ c = \prod_p \left( 1 - \frac{\lambda(p^2)}{p^4} \right) \]

and
\[ \lambda(q) = \sum_{1 \leq x, y \leq q} x^2 + y^2 + 1 \equiv 0 \quad (q) \]

3.6.3 \textit{k-free values of the form } t_1 \cdots t_r - 1

Let \( k, r \geq 2 \) be two integers. Let \( N_{k,r}(x) \) denotes the number of the \( k \)-free values of the \( r \) variables polynomial \( t_1 \cdots t_r - 1 \) over \([1, x]^r \cap \mathbb{Z}^r\). In 2011 P. Le Boudec [43] proved an asymptotic formula for \( N_{k,r}(x) \).

\textbf{Theorem 10 ([43]).} Let \( \varepsilon > 0 \) be fixed. As \( x \to \infty \), if \( \delta_{k,r} \leq 1 \) we have the estimate
\[ N_{k,r}(x) = c_{k,r} x^r + \mathcal{O}(x^{r-\delta_{k,r}+\varepsilon}) \],

where
\[ c_{k,r} = \prod_p \left( 1 - \frac{1}{p^k} \right) \left( 1 - \frac{1}{p} \right)^{r-1} \],

and if \( 1 < \delta_{k,r} \leq 2 \) we have the estimate
\[ N_{k,r}(x) = c_{k,r} x^r - \theta_{k,r}^{(1)}(x) x^{r-1} + \mathcal{O}(x^{r-\delta_{k,r}+\varepsilon}) \],

where
\[ \theta_{k,r}^{(1)}(x) = r \sum_{d=1}^{\infty} \frac{\mu(d)}{\varphi(d^k)} \left( \frac{\varphi(d)}{d} \right)^{r-1} \sum_{m|d} \mu(m) \left\{ \frac{x}{m} \right\} \],

and finally, if \( \delta_{k,r} > 2 \) we have the estimate
\[ N_{k,r}(x) = c_{k,r} x^r - \theta_{k,r}^{(1)}(x) x^{r-1} + \theta_{k,r}^{(2)}(x) x^{r-2} + \mathcal{O}(x^{r-\delta_{k,r}+\varepsilon}) \],

where
\[ \theta_{k,r}^{(2)}(x) = \frac{r(r-1)}{2} \sum_{d=1}^{\infty} \frac{\mu(d)}{\varphi(d^k)} \left( \frac{\varphi(d)}{d} \right)^{r-2} \left( \sum_{m|d} \mu(m) \left\{ \frac{x}{m} \right\} \right)^2 \].

3.6.4 \textit{k-free values of the form } xy^k + C

In 2012, Lapkova [40] considered the polynomial \( f(x, y) = xy^k + C \) for \( k \geq 2 \) and any nonzero integer constant \( C \). She derived an asymptotic formula for the \( k \)-free values of \( f(xy) \) when \( x, y \leq H \).

\textbf{Theorem 11 ([40, Theorem 1]).} Let \( f(x, y) = xy^k + C \in \mathbb{Z}[x, y] \) for \( k \geq 2 \) and \( C \neq 0 \). Then, for some real \( \delta = \delta(k, f) > 0 \), we have
\[ S(H) = c_f H^2 + \mathcal{O}(H^{2-\delta}) \]
holds. Here
\[ S(H) = \#\{1 \leq x, y \leq H : f(x,y) \text{ is } k\text{-free}\}, \]
\[ c_f = \prod_p \left( 1 - \frac{\rho(p^k)}{p^{2k}} \right) \]
and
\[ \rho(m) = \#\{(\mu, \nu) \in (\mathbb{Z}/m\mathbb{Z})^2 : m \mid f(\mu, \nu)\}. \]

Lapkova [40] also proved a similar result for the \( k\)-free values of \( f(p,q) \) when \( p,q \leq H \) are primes (see [40, Theorem 2]).

4 On the distribution of \( r \)-tuples of \( k \)-free numbers

4.1 Pairs of \( k \)-free numbers of arbitrary type

The problem for the pairs of \( k \)-free numbers arises in 1932 when Carlitz [10] proved the following:

Theorem 12 ([10]). The asymptotic formula
\[ \sum_{n \leq X} \mu_k(n)\mu_k(n+1) = \prod_p \left( 1 - \frac{2}{p^k} \right) X + O \left( X^{\frac{2}{p^k} + \varepsilon} \right) \]
holds.

Further we find the result of Mirsky:

Theorem 13 ([47]). The asymptotic formula
\[ \sum_{n \leq X} \mu_k(n)\mu_k(n+h) = \prod_p \left( 1 - \frac{2}{p^k} \right) \prod_{p^k \mid h} \left( \frac{p^k-1}{p^k-2} \right) X + O \left( X^{\frac{2}{p^k} + \varepsilon} \right) \]
holds.

Subsequently Mirsky [48] improved his result:

Theorem 14 ([48]). The asymptotic formula
\[ \sum_{n \leq X} \mu_k(n)\mu_k(n+h) = \prod_p \left( 1 - \frac{2}{p^k} \right) \prod_{p^k \mid h} \left( \frac{p^k-1}{p^k-2} \right) X + O \left( X^{\frac{2}{p^k} \log X^{\frac{p^k}{h^2}}} \right) \]
holds.

Further Meng [46] improved the result of Mirsky as follows

Theorem 15 ([46]). The asymptotic formula
\[ \sum_{n \leq X} \mu_k(n)\mu_k(n+h) = \prod_p \left( 1 - \frac{2}{p^k} \right) \prod_{p^k \mid h} \left( \frac{p^k-1}{p^k-2} \right) X + O \left( X^{\frac{2}{p^k}} \right) \]
holds.
In 1984, Heath-Brown [25] improved the result of Meng for \( k = 2, h = 1 \).

**Theorem 16 ([25]).** The asymptotic formula

\[
\sum_{n \leq X} \mu_2(n)\mu_2(n+1) = \prod_p \left( 1 - \frac{2}{p^2} \right) X + O \left( X^{\frac{7}{11}} (\log X)^{\frac{7}{11}} \right)
\]

holds.

Finally, Reuss [55] using a generalization of the approximate determinant method proved the best result.

**Theorem 17 ([55]).** The asymptotic formula

\[
\sum_{n \leq X} \mu_k(n)\mu_k(n+h) = \prod_p \left( 1 - \frac{2}{p^k} \right) \prod_{p^j|h} \left( \frac{p^k - 1}{p^k - 2} \right) X + O \left( X^{\omega(k)+\epsilon} \right)
\]

holds. Here

\[
\omega(k) = \begin{cases} 
\frac{26+\sqrt{433}}{81} & \text{if } k = 2, \\
\frac{169}{144} & \text{for } k \geq 3.
\end{cases}
\]

For intermediate results concerning the distribution of the pairs of \( k \)-free numbers of arbitrary type we refer to Brandes [5] and Dietmann and Marmon [11].

### 4.2 \( r \)-tuples of \( k \)-free numbers of arbitrary type

In 2014, Reuss [55], using a generalization of the approximate determinant method gave an asymptotic formula for \( r \)-tuples of \( k \)-free integers.

**Theorem 18 ([55]).** Let \( k \geq 2, r \geq 2 \) and \( l_i(x) = a_ix + b_i \in \mathbb{Z}[x] \) for \( i = 1, \ldots, r \) such that \( a_i b_j - a_j b_i \neq 0 \) and \( a_i \neq 0 \) for all \( i, j \) with \( 1 \leq i, j \leq r \) and \( i \neq j \). Then define

\[
\rho(p) = \#\{n(\mod p^k) : p^k | l_i(n) \text{ for some } i\},
\]

and let

\[
c = \prod_p \left( 1 - \frac{\rho(p)}{p^k} \right).
\]

If \( N(x) \) is the number of integers \( n \leq x \) such that \( l_1(n), \ldots, l_r(n) \) are all \( k \)-free. Then for any \( \epsilon > 0 \) and any sufficiently large \( x \) we have that

\[
N(x) = cx + O_{\epsilon} \left( x^{\frac{1}{11} + \epsilon} \right).
\]

It should be pointed that the implied constant in Theorem 18 depends on the choice of the \( l_i \) and that the best reminder term up to now for \( k = 2 \) was \( \mathcal{O}(x^{7/11 + \epsilon}) \) (See Tsang [62]). Tsang’s proof uses a form of the Rosser–Iwaniec sieve and the version of Theorem 17 due to Heath-Brown. It should be noted that even though Tsang’s error term is weaker than Reuss’s, his implied constants are uniform in \( r \) and \( \max_i ||l_i|| \).

For other results on \( r \)-tuples of \( k \)-free numbers of arbitrary type we refer the reader to [?].
### 4.3 $r$-tuples of $k$-free numbers of the form $p + \alpha_1, \cdots, p + \alpha_s$

In 2016, Hablizel [23] using the circle method evaluated the behavior of limit-periodic functions on primes on average.

As an application he showed that for arbitrary $\alpha_i \in \mathbb{N}_0$, $r_i \in \mathbb{N}_{>1}$ and $s \in \mathbb{N}$ the asymptotic formula

$$\sum_{p \leq x} \mu_{r_1}(p + \alpha_1) \cdots \mu_{r_s}(p + \alpha_s) = \prod_p \left( 1 - \frac{D^*(p)}{\varphi(p^{r_s})} \right) \frac{x}{\log x} + o \left( \frac{x}{\log x} \right),$$

holds. Here $D^*(p)$ is a computable function of the prime $p$, depending on the choice of the numbers $\alpha_i$ and $r_i$.

A weaker result related to this was obtained by Dimitrov in [15].

### 4.4 Consecutive $k$-free numbers of the form $[n^c], [n^c] + 1$

#### 4.4.1 Consecutive square-free numbers of the form $[n^c], [n^c] + 1$

In 2018, Dimitrov [13] using the method of Cao and Zhai [8] showed that for any fixed $1 < c < 22/13$ there exist infinitely many consecutive square-free numbers of the form $[n^c], [n^c] + 1$. More precisely, he proved the following:

**Theorem 19 ([13]).** Let $1 < c < 22/13$, $\gamma = c^{-1}$ and $0 < \varepsilon < (22\gamma - 13)/5(14 - \gamma)$ is a sufficiently small constant. Then

$$\sum_{n \leq X} \mu_2([n^c]) \mu_2([n^c] + 1) = \prod_p \left( 1 - \frac{2}{p^2} \right) X + O \left( X^{1-\varepsilon^2/2} \right),$$

$$\sum_{p \leq X} \mu_2([p^c]) \mu_2([p^c] + 1) = \prod_p \left( 1 - \frac{2}{p^2} \right) \int_2^X \frac{dt}{\ln t} + O \left( X e^{-c_0 \sqrt{\log X}} \right),$$

$$\sum_{n \leq X \atop \mu_2(n) = 1} \mu_2([n^c]) \mu_2([n^c] + 1) = 6 \pi^2 \prod_p \left( 1 - \frac{2}{p^2} \right) X + O \left( X^{1-\varepsilon^2/2} \right),$$

where $c_0 > 0$ is an absolute constant.

His earlier result [12] covers the narrower range $1 < c < 7/6$.

#### 4.4.2 Consecutive cube-free numbers of the form $[n^c], [n^c] + 1$

In 2018, Dimitrov [14] using the method of Zhang and Li [67] showed that for any fixed $1 < c < 31/17$ there exist infinitely many consecutive cube-free numbers of the form $[n^c], [n^c] + 1$. More precisely, he proved:
Theorem 20 ([14]). Let $1 < c < 31/17$, $\gamma = c^{-1}$ and $0 < \varepsilon < \min\{(31\gamma - 17)/(9 - 9\gamma), 10^{-10}\}$ is a sufficiently small constant. Then

$$\sum_{n \leq X} \mu_3([n^c])\mu_3([n^c] + 1) = \prod_p \left(1 - \frac{2}{p^3}\right) X + O\left(X^{1 - \varepsilon^2/2}\right),$$

$$\sum_{p \leq X} \mu_3([p^c])\mu_3([p^c] + 1) = \prod_p \left(1 - \frac{2}{p^3}\right) \int_2^X \frac{dt}{\ln t} + O\left(Xe^{-c_0\sqrt{\log X}}\right),$$

$$\sum_{n \leq X, \mu_3(n) = 1} \mu_3([n^c])\mu_3([n^c] + 1) = \frac{1}{\zeta(3)} \prod_p \left(1 - \frac{2}{p^3}\right) X + O\left(X^{1 - \varepsilon^2/2}\right),$$

where $c_0 > 0$ is an absolute constant.

4.5 Consecutive square-free numbers of the form $x^2 + y^2 + 1, x^2 + y^2 + 2$

Recently Dimitrov [16] using the method of Tolev [61] showed that there exist infinitely many consecutive square-free numbers of the form $x^2 + y^2 + 1, x^2 + y^2 + 2$. He also gave an asymptotic formula for the number of pairs of positive integers $x, y \leq H$ such that $x^2 + y^2 + 1, x^2 + y^2 + 2$ are square-free.

Theorem 21 ([16]). The asymptotic formula

$$\sum_{1 \leq x, y \leq H} \mu_2(x^2 + y^2 + 1) \mu_2(x^2 + y^2 + 2) = \sigma H^2 + O\left(H^{8/5 + \varepsilon}\right)$$

holds. Here

$$\sigma = \prod_p \left(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^4}\right)$$

and

$$\lambda(q_1, q_2) = \sum_{1 \leq x, y \leq q_1, q_2 \atop x^2 + y^2 + 2 \equiv 0 (q_1)} 1.$$}

4.6 On the distribution of consecutive square-free primitive roots modulo $p$

Let $p$ be an odd prime. For any integer $n$ with $(n, p) = 1$, the smallest positive integer $f$ such that $n^f \equiv 1 \pmod{p}$ is called the exponent of $n$ modulo $p$. If the exponent of $n$ modulo $p$ is $p - 1$, then $n$ is called a primitive root mod $p$.

Let $A(n)$ be the characteristic function of the square-free primitive roots modulo $p$. In 2015 Liu and Dong [45] investigated the distribution of consecutive square-free primitive roots modulo $p$ as follows.
Theorem 22 ([45]). Let $p$ be an odd prime, and let $A(n)$ be the characteristic function of the square-free primitive roots modulo $p$. Then we have

$$\sum_{n \leq x} A(n)A(n+1) = \frac{x\varphi^2(p-1)p(p-2)}{(p-1)^2} \frac{p-2}{p^2-2} \prod_{p_i} \left(1 - \frac{2}{p_i^2}\right)$$

$$+ \mathcal{O}\left(4^{\omega(p-1)}p^{-1/2}(\log p)x + 4^{\omega(p-1)}p^{1/4}(\log p)^{1/2}x^{1/2}\log x\right),$$

where the $\mathcal{O}$-constant is absolute and $\omega(q)$ denotes the number of the distinct prime factors of $q$.

For results concerning the distribution of positive square-free primitive roots modulo $p$ not exceeding $x$ we refer to Liu and Zhang [44] and Shapiro [59].

References

[1] Axer, A. (1911). Über einige Grenzwert sätze, S.-B. Math.-Natur. Kl. Akad. Wiss. Wien 120 (2a), 1253–1298.

[2] Baker, R. C. & Pintz, J. (1985). The distribution of squarefree numbers, Acta Arith., 46, 73–79.

[3] Bhargava, M. (2014). The geometric sieve and the density of squarefree values of invariant polynomials, arXiv:1402.0031 [math.NT].

[4] Bhargava, M., Shankar, A., & Wang, X. (2016). Squarefree values of polynomial discriminants I, arXiv:1611.09806 [math.NT].

[5] Brandes, J. (2013). Twins of $s$-free numbers, arXiv:1307.2066v1 [math.NT].

[6] Browning, T. D. (2011). Power-free values of polynomials, Arch. Math., 96, 139–150.

[7] Cao, X., & Zhai, W. (1998). The distribution of square-free numbers of the form $[n^c]$, Journal de Théorie des Nombres de Bordeaux, 10, 287–299.

[8] Cao, X., & Zhai, W. (2000). Multiple exponential sums with monomials, Acta Arith., 92, 195–213.

[9] Cao, X., & Zhai, W. (2008). The distribution of square-free numbers of the form $[n^c]$, II, Acta Math. Sinica (Chin. Ser.), 51, 1187–1194.

[10] Carlitz, L. (1932). On a problem in additive arithmetic II, Quart. J. Math., 3, 273–290.

[11] Dietmann, R. & Marmon, O. (2014). The density of twins of $k$-free numbers, arXiv:1307.2481v2 [math.NT].

[12] Dimitrov, S. I. (2018). Consecutive square-free numbers of a special form, arXiv:1702.03983v3 [math.NT].
[13] Dimitrov, S. I. (2018). Consecutive square-free numbers of the form \([n^c], [n^c] + 1\), *JP Journal of Algebra, Number Theory and Applications*, 40 (6), 945–956.

[14] Dimitrov, S. I. (2018). Consecutive cube-free numbers of the form \([n^c], [n^c] + 1\), *Appl. Math. in Eng. and Econ.–44th. Int. Conf., AIP Conf. Proc.*, 2048, 050004.

[15] Dimitrov, S. I. (2018). Consecutive square-free numbers of the form \(p + 1, p + 2\), *Far East Journal of Mathematical Sciences*, 107(2), 449–456.

[16] Dimitrov, S. I. (2019). On the number of pairs of positive integers \(x, y \leq H\) such that \(x^2 + y^2 + 1, x^2 + y^2 + 2\) are square-free, *arXiv:1901.04838v1 [math.NT]* 5 Jan 2019.

[17] Dudek, A. (2014). On the sum of a prime and a square-free number, *arXiv:1410.7459v1 [math.NT]*.

[18] Erdős, P. (1953). Arithmetical properties of polynomials, *J. London Math. Soc.*, 28, 416–425.

[19] Estermann, T. (1931). Einige Sätze über quadratfeie Zahlen, *Math. Ann.*, 105, 653–662.

[20] Filaseta, M. (1994). Powerfree values of binary forms, *Journal of Number Theory*, 49, 250–268.

[21] Greaves, G. (1992). Power-free values of binary forms, *Q. J. Math*, 43(1), 45–65.

[22] Güloğlu, A. M., & Nevans, C. W. (2008). Sums of multiplicative functions over a Beatty sequence, *Bull. Austral. Math. Soc.*, 78, 327–334.

[23] Hablizel, M. (2016). The asymptotic behavior of limit-periodic functions on primes and an application to \(k\)-free numbers, *arXiv:1609.08183v1 [math.NT]* 26 Sep 2016.

[24] Heath-Brown, D. R. (1982). Prime numbers in short intervals and a generalized Vaughan identity, *Canad. J. Math.*, 34, 1365–1377.

[25] Heath-Brown, D. R. (1984) The square-sieve and consecutive square-free numbers, *Math. Ann.*, 266, 251–259.

[26] Heath-Brown, D. R. (2002). The density of rational points on curves and surfaces, *Ann. of Math.* 155 (2), 553–595.

[27] Heath-Brown, D. R. (2006). Counting rational points on algebraic varieties, *Lecture Notes in Math.*, 1891, 51–95.

[28] Heath-Brown, D. R. (2009). Sums and differences of three \(k\)-th powers, *J. Number Theory*, 129, 1579–1594.

[29] Heath-Brown, D. R. (2012), Square-free values of \(n^2 + 1\), *Acta Arith.*, 155, 1–13.
[30] Heath-Brown, D. R. (2013). Power-free values of polynomials, *Quart. J. Math.*, 64, 177–188.

[31] Helfgott, H. (2007). Power-free values, large deviations and integer points on irrational curves, *J. Theorie des Nombres de Bordeaux*, 19, 433–472.

[32] Helfgott, H. (2008). Power-free values, repulsion between points, differing beliefs and the existence of error, *CRM Proceedings and Lecture Notes*, 46, 81–88.

[33] Helfgott, H. (2014). Square-free values of $f(p)$, $f$ cubic, *Acta Math.*, 213, 107–135.

[34] Hooley, C. (1967). On the power-free values of polynomials, *Mathematika*, 14, 21–26.

[35] Hooley, C. (1977). On the power-free numbers and polynomials II, *J. Reine Angew. Math.*, 295, 1–21.

[36] Hooley, C. (2009). On the power-free values of polynomials in two variables, *Analytic number theory*, 235–266.

[37] Hooley, C. (2009). On the power-free values of polynomials in two variables: II, *Journal of Number Theory*, 129, 1443–1455.

[38] Jia, C. H. (1993). The distribution of squarefree numbers, *Sci. China Ser. A*, 36, 154–169.

[39] Lando, G. Square-free values of polynomials evaluated at primes over a function field, *arXiv:1409.7633v3 [math.NT]*.

[40] Lapkova, K. (2012). On the $k$-free values of the polynomial $xy^k + C$, *Acta Math. Hung.*, 149, 190–207.

[41] Lapkova, K. & Xiao, S. (2018). Density of power-free values of polynomials, *arXiv:1801.04481v1 [math.NT]*.

[42] Lee, J. & Murty, M. R. (2007). Dirichlet series and hyperelliptic curves, *Forum Math.*, 19 (4), 677–705.

[43] Le Boudec, P. (2012). Power-free values of the polynomial $t_1 \cdots t_r - 1$, *Bull. Aust. Math. Soc.*, 85 (1), 154–163.

[44] Liu, H. & Zhang, W. (2005). On the squarefree and squarefull numbers, *J. Math. Kyoto Univ.*, 45, 247–255.

[45] Liu, H. & Dong, H. (2015). On the distribution of consecutive square-free primitive roots modulo $p$, *Czechoslovak Mathematical Journal*, 65, 555–564.

[46] Meng, Z. (2006). Some new results on $k$-free numbers, *Journal of Number Theory*, 121, 45–66.
[47] Mirsky, L. (1947). Note on an asymptotic formula connected with $r$-free integers, *Quart. J. Math. Oxford*, 18, 178–182.

[48] Mirsky, L. (1949). On the frequency of pairs of square-free numbers with a given difference, *Bull. Amer. Math. Soc.*, 55, 936–939.

[49] Montgomery, H. L. & Vaughan, R. C. (1981). The distribution of squarefree numbers, Recent progress in analytic number theory, 1, (Durham, 1979), 247–256, Academic Press, London-New York.

[50] Nair, M. (1976). Power free values of polynomials, Mathematika, 23, 159–183.

[51] Nair, M. (1979). Power free values of polynomials II, *Proc. London Math. Soc.*, 38, 353–368.

[52] Pasten, H. (2014). The ABC conjecture, arithmetic progressions of primes and square-free values of polynomials at prime arguments, *Int. J. Number Theory* DOI: 10.1142/S1793042115500396.

[53] Poonen, B. (2003). Squarefree values of multivariable polynomials, *Duke Math. J.*, 118 (2), 353–373.

[54] Reuss, T. (2013). Power-free values of polynomials, *arXiv:1307.2802v1 [math.NT]*.

[55] Reuss, T. (2014). Pairs of k-free numbers, consecutive square-full numbers, *arXiv: 1212.3150v2 [math.NT]*.

[56] Ricci, G. (1933). Riecenche aritmetiche sui polynomials, *Rend. Circ. Mat. Palermo*, 57, 433–475.

[57] Rieger, G. J. (1978) Remark on a paper of Stux concerning square-free numbers in non-linear sequences, *Pacific J. Math.*, 78, 241–242.

[58] Robert, O., & Sargos, P. (2006). Three-dimemsional exponential sums with monomials, *J. Reine Angew. Math.*, 591, 1–20.

[59] Shapiro, H. N. (1983). *Introduction to the Theory of Numbers*, Pure and Applied Mathematics. Wiley-Interscience Publication, John Wiley & Sons, New York.

[60] Stewart, C., & Xiao, S. (2018). On the representations of $k$-free integers by binary forms, *arXiv:1612.00487v2 [math.NT]*.

[61] Tolev, D. I. (2012) On the number of pairs of positive integers $x, y \leq H$ such that $x^2 + y^2 + 1$ is squarefree, *Monatsh. Math.*, 165, 557–567.

[62] Tsang, K.-M. (1985). The distribution of $r$-tuples of square-free numbers, *Mathematika*, 32, 265–275.
[63] Uchiyama, S. (1972). On the power-free values of a polynomial, *Tensor (N.S.)*, 24, 43–48.

[64] Victorovich, G. D. (2013). *On additive property of arithmetic functions*, Thesis, Moscow State University, (in Russian).

[65] Xiao, S. (2017). Power-free values of binary forms and the global determinant method, *Int. Math. Res. Notices*, 16, 5078–5135.

[66] Xiao, S. (2018). Square-free values of decomposable forms, *Canadian Journal of Mathematics*, 70, 1390–1415.

[67] Zhang, M. & Li, J. (2017). On the distribution of cube-free numbers with the form \([n^3]\), *arXiv:1702.00165v1[math.NT]*.