A GENERALIZATION OF POWERS-STØRMER INEQUALITY

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Abstract. Let $A$, $B$ be the positive semidefinite matrices. A matrix version of the famous Powers-Størmer's inequality

$$2 \text{Tr}(A^\alpha B^{1-\alpha}) \geq \text{Tr}(A + B - |A - B|), \quad 0 \leq \alpha \leq 1,$$

was proven by Audenaert et. al. We establish a comparison of eigenvalues for the matrices $A^\alpha B^{1-\alpha}$ and $A + B - |A - B|$, $0 \leq \alpha \leq 1$, subsuming the Powers-Størmer’s inequality. We also prove several related norm inequalities.

1. Introduction

Let $M_n$ denote the algebra of all $n \times n$ complex matrices. A Hermitian member $A$ of $M_n$ with all non-negative eigenvalues is known as positive semi-definite matrix, simply denoted by $A \geq 0$. We shall denote by $P_n$, the collection of all such matrices. For $A$, $B$ Hermitian in $M_n$, we employ the positive semi-definite ordering: $A \geq B$ if and only if $A - B \geq 0$. By $|A|$, we mean the positive square root of the matrix $A^*A$, i.e., $(A^*A)^{1/2}$. The Jordan decomposition of a Hermitian matrix $A$ is given by $A = A_+ - A_-$, where $A_+$ and $A_-$ are the members of the $P_n$ along with $A_+A_- = 0$ (see [3], page 99). We shall consider $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \geq 0$, the eigenvalues of $A \in P_n$, arranged in decreasing order and repeated according to their multiplicity. Similarly $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0$, denote the singular values (eigenvalues of $|A|$) of a matrix $A \in P_n$, arranged in decreasing order and repeated according to their multiplicity. By $||| . |||$, we mean any unitarily invariant norm, while $|| . ||$ denotes operator norm on $M_n$.

In 2007, Audenaert et. al. [1] solved a long standing open problem to identify the classical quantum Chernoff bound in the area of information theory. After the mathematical formulation of that problem, they proved a nontrivial and fundamental inequality relating to the trace distance to the quantum Chernoff bound. That became a key result to a solution of the problem and is stated as follows:

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Let $A, B$ be positive matrices and $0 \leq \alpha \leq 1$. Then

$$2Tr(A^\alpha B^{1-\alpha}) \geq Tr(A + B - |A-B|)$$

(1.1)

holds. A particular case $\alpha = 1/2$ in (1.1) is a well known Powers-Størmer’s inequality [7], which was proved in 1970. For such literature and detail of inequalities the reader may refer [6]. Subsequently in 2008, again Audenaert et. al. [2] worked on symmetric as well as with asymmetric quantum hypothesis testing. In [2] also, they proved some similar type of inequalities as that of (1.1) which played a key role in getting the optimal solution to the symmetric classical hypothesis test.

In 2011, Y. Ogata [5] generalised the Powers-Størmer inequality to von Neumann algebras. Recently several authors including D. Hoa et. al. [4] generalised this inequality on $C^*$-algebras using the technique of operator monotone functions on $[0, \infty)$.

We aim to prove the comparison of eigenvalues of $A + B - |A-B|$ and $2A^\alpha B^{1-\alpha}$, generalizing all the forms of Powers-Størmer’s inequality. We shall also prove several other associated norm inequalities.

2. Main Results

**Lemma 2.1.** Let $A, B \in P_n$ then there exist a matrix $S \in P_n$ satisfying

1. $S \preceq A$, $S \preceq B$
2. If $T \preceq A$, $T \preceq B$, is a fixed Hermitian matrix then $\lambda_i(T) \leq \lambda_i(S)$ for $1 \leq i \leq n$.

*Proof.* We shall first prove this result for either of $A$ or $B$ invertible. So assume $B$ is invertible i.e. $B$ is Hermitian and whose all the eigenvalues are positive. As is well-known that $B^{-1/2}AB^{-1/2} \in P_n$ and so unitarily diagonalizable. We assume that $B^{-1/2}AB^{-1/2} = U^*DU$ for some $U$ a unitary and $D$ a diagonal matrix with diagonal entries as $d_1 \geq d_2 \geq d_3 \cdots \geq d_n \geq 0$. Choose $S = B^{1/2}U^*D_1UB^{1/2}$, where $D_1$ is a diagonal matrix with diagonal entries as $t_1 \geq t_2 \geq t_3 \cdots \geq t_n \geq 0$, such that $t_i = \min\{d_i, 1\}$. This choice of $S$ satisfies

$$S = B^{1/2}U^*D_1UB^{1/2} \leq B^{1/2}U^*DUB^{1/2} = A,$$

$$S = B^{1/2}U^*D_1UB^{1/2} \leq B^{1/2}U^*IUB^{1/2} = B.$$ 

For (2), let $T \preceq A$ as well as $T \preceq B$ be a fixed Hermitian matrix, then by Weyl’s monotonicity principle we have $\lambda_i(T) \leq \lambda_i(A)$ and $\lambda_i(T) \leq \lambda_i(B)$ for all $i = 1, 2, \cdots, n$. If

$$\lambda_i(T) \leq \lambda_i(S) \text{ for } 1 \leq i \leq n,$$
the above construction of $S$ meets both the requirements.

If

$$\lambda_j(T) \geq \lambda_j(S) \text{ for some } 1 \leq j \leq n,$$

then we replace that particular $t_j$ with $\lambda_j(T)$ in $D_1$. Then, this choice of $S$ meets both the requirements.

The general case follows by using continuity argument. □

Now onwards, we shall denote $S$ by $\min\{A, B\}$.

Theorem 2.2. Let $A, B \in P_n$ then

$$\lambda_i(A + B - |A - B|) \leq 2\lambda_i(A^\alpha B^{1-\alpha})$$

for $0 \leq \alpha \leq 1$ and $1 \leq i \leq n$.

Proof. Let $T$ be any Hermitian matrix with Jordan decomposition $T_+ - T_-$. Then, $|T| = T_+ + T_-$, so $T - |T| = -2T_- \leq 0$. Using this fact for $A - B$, we can write,

$$A + B - |A - B| = 2(B - (A - B)_-) \leq 2B.$$  \hfill (2.2)

Replacing $B$ by $A$ in above inequality, we obtain

$$A + B - |A - B| = 2(A - (B - A)_-) \leq 2A.$$  \hfill (2.3)

Now, on using Lemma 2.1, we obtain

$$\lambda_i(A + B - |A - B|) \leq 2\lambda_i(\min\{A, B\} = S) \leq 2\lambda_i(S^{\alpha/2}B^{1-\alpha}S^{\alpha/2}), \text{ for } 1 \leq i \leq n.$$  

To complete the proof, it is enough to show

$$\lambda_i(S^{\alpha/2}B^{1-\alpha}S^{\alpha/2}) \leq \lambda_i(A^\alpha B^{1-\alpha}), \text{ for } 1 \leq i \leq n.$$  

Indeed,

$$2\lambda_i(S^{\alpha/2}B^{1-\alpha}S^{\alpha/2}) = 2\lambda_i(B^{(1-\alpha)/2}S^{\alpha}B^{(1-\alpha)/2}) \leq 2\lambda_i(B^{(1-\alpha)/2}A^\alpha B^{(1-\alpha)/2}) = 2\lambda_i(A^\alpha B^{1-\alpha}) \text{ for } 1 \leq i \leq n.$$  

□

Corollary 2.3. (Cf. [1, 2],Theorem 1,Theorem 2 ) Let $A, B \in P_n$ then for $0 \leq \alpha \leq 1$

$$0 \leq Tr(A + B - |A - B|) \leq 2Tr(A^\alpha B^{1-\alpha}).$$  \hfill (2.4)
Proof. Let $A - B = (A - B)_+ - (A - B)_-$ be the Jordan decomposition of $A - B$, then for $1 \leq i \leq n$,

$$\lambda_i(A - B)_- \leq \lambda_i(B),$$

(see Lemma IX.4.1 of [3]). The first inequality from the left side in (2.4) follows immediately from (2.2) and (2.5). The last inequality follows from Theorem 2.2. \qed

Corollary 2.4. Let $A, B \in P_n$ then for $0 \leq \alpha \leq 1$

(i) $|||(A + B - |A - B|)_+||| \leq 2|||A^\alpha B^{1-\alpha}|||

(ii) $|||(A + B - |A - B|)_-||| \leq 2|||A^\alpha B^{1-\alpha}|||.$

Proof. (i) As $A, B \in P_n$, hence, without loss of generality we assume

$$\lambda_1(A + B - |A - B|) \geq \lambda_2(A + B - |A - B|) \geq \cdots \geq \lambda_k(A + B - |A - B|) \geq 0$$
$$> \lambda_{k+1}(A + B - |A - B|) \geq \cdots \geq \lambda_n(A + B - |A - B|)$$

and

$$\lambda_1(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}) \geq \lambda_2(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}) \geq \cdots \geq \lambda_n(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}) \geq 0.$$ 

The matrix $A + B - |A - B|$ is Hermitian, so unitarily diagonalizable, i.e.,

$$A + B - |A - B| = W^*D_2W,$$

for $W$ a unitary matrix and $D_2$ a diagonal matrix given by

$$D_2 = diag(\lambda_1(A + B - |A - B|), \cdots, \lambda_n(A + B - |A - B|)).$$

Now, using Jordan decomposition of $A + B - |A - B|$, (see [3], page 99) provides that

$$(A + B - |A - B|)_+ = W^*D_{2+}W$$
and
$$(A + B - |A - B|)_- = W^*D_{2-}W,$$

where $D_{2+}$ and $D_{2-}$ are diagonal matrices in $P_n$, given by

$$D_{2+} = diag(\lambda_1(A + B - |A - B|), \cdots, \lambda_k(A + B - |A - B|), 0, \cdots 0)$$
and

$$D_{2-} = diag(0, \cdots, -\lambda_{k+1}(A + B - |A - B|), \cdots, -\lambda_n(A + B - |A - B|)).$$

By the above discussion, we clearly obtain

$$\lambda_i(A + B - |A - B|)_+ = \begin{cases} \lambda_i(A + B - |A - B|), & \text{for } i = 1, 2, \cdots, k \\ 0, & \text{for } i = k + 1, k + 2, \cdots, n, \end{cases}$$
and
\[
\lambda_i(A + B - |A - B|)_- = \begin{cases} 
0, & \text{for } i = 1, 2, \ldots, k \\
-\lambda_i(A + B - |A - B|), & \text{for } i = k + 1, k + 2, \ldots, n.
\end{cases}
\]

Now, using inequality (2.1) along with \(\lambda_i(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}) = \lambda_i(A^{\alpha}B^{1-\alpha})\), we obtain
\[
\lambda_i((A + B - |A - B|)_+) \leq 2\lambda_i(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}), \quad \text{for } i = 1, 2, \ldots, n,
\]
i.e.,
\[
s_i((A + B - |A - B|)_+) \leq 2s_i(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}), \quad \text{for } i = 1, 2, \ldots, n. \tag{2.6}
\]

On using Theorem IV.2.2 and then Proposition IX.1.1 of [3] in (2.6), we obtain
\[
|||(A + B - |A - B|)_+||| \leq 2|||A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}|||
\leq 2|||A^{\alpha}B^{1-\alpha}|||. \tag{2.7}
\]
This completes the proof of (i).

For a proof of (ii), use (2.2) and (2.5) to obtain,
\[
\lambda_i((A + B - |A - B|)_-) \leq 2\lambda_i(B). \tag{2.8}
\]
Now, replace \(B\) by \(A\) in (2.8), we obtain,
\[
\lambda_i((A + B - |A - B|)_-) \leq 2\lambda_i(A). \tag{2.9}
\]
Again, on using similar technique as in Theorem 2.2, we get the desired result. \(\square\)

The following corollary is an immediate consequence of Corollary 2.4.

**Corollary 2.5.** Let \(A, B \in P_n\) then for \(0 \leq \alpha \leq 1\)
\[
||A + B - |A - B|||| \leq 2||A^{\alpha}B^{1-\alpha}||. \tag{2.10}
\]

*Proof.* The operator norm for any Hermitian matrix \(T\) is given by
\[
||T|| = \max\{|T_+||, |T_-||\}.
\]
Using the above fact for the matrix \(A + B - |A - B|\) and Corollary 2.4 to obtain (2.10). \(\square\)

**Theorem 2.6.** Let \(A, B \in P_n\) then for \(0 \leq \alpha \leq 1\), some projection \(P\) and \(\beta \geq 0\),
\[
|||A + B - |A - B|||| \leq 2|||A^{\alpha}B^{1-\alpha} - \beta A^{\alpha/2}PA^{-\alpha/2}|||.
\]
\[
\tag{2.11}
\]
Proof. Let $X = \text{diag}(x_1, x_2, \ldots, x_n)$ and $T = \text{diag}(t_1, t_2, \ldots, t_n)$ be the matrices comprised of $x_i$'s and $t_i$'s as eigenvalues of $A + B - |A - B|$ and $2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}$ in decreasing order respectively. Using Theorem (2.2) on $X$ and $T$, we get

$$x_i \leq t_i \text{ for } i = 1, 2, \ldots, n.$$  

If $\beta = Tr(T) - Tr(X)$, then on using Corollary 2.3, we obtain $\beta \geq 0$. Consider

$$T_1 = 2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2} - \beta Q_n,$$

where $\sum_{i=1}^{n} t_i Q_i$ is the spectral decomposition of $2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}$. It is clear from the construction of $T_1$ that eigenvalues of $T_1$ are all same and in the same order as that of $2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}$ except the last one. So we may assume $(t_1, t_2, \ldots, t_{n-1}, \gamma_n)^t$ as a column vector of eigenvalues of $T_1$, satisfying

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} t_i \text{ for } k = 1, 2, 3, \ldots, n - 1,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-1} t_i + \gamma_n.$$

Finally, using Example II.3.5 in [3], we get

$$\sum_{i=1}^{k} |x_i| \leq \sum_{i=1}^{k} t_i \text{ for } k = 1, 2, \ldots, n - 1,$$

and

$$\sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n-1} t_i + |\gamma_n|.$$  

Equivalently,

$$\sum_{i=1}^{k} s_i(A + B - |A - B|) \leq \sum_{i=1}^{k} s_i(T_1) \text{ for } k = 1, 2, \ldots, n.$$  

Hence,

$$|||A + B - |A - B||| \leq |||2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2} - \beta Q_n|||$$

$$= |||A^{-\alpha/2}(2A^\alpha B^{1-\alpha}A^{\alpha/2} - \beta A^{\alpha/2}Q_n)|||$$

$$\leq |||2A^\alpha B^{1-\alpha} - \beta A^{\alpha/2}Q_n A^{-\alpha/2}|||,$$

using Theorem IV.2.2 of [3] for the first inequality and Proposition IX.1.1 of [3] for the second inequality. This completes the proof.
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