Multiflavor and multiband observations of neutrinos from core collapse supernovae

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Abstract

It has been proposed that the gamma ray burst - supernova connection may manifest itself in a significant fraction of core collapse supernovae possessing mildly relativistic jets with wide opening angles that do not break out of the stellar envelope. Neutrinos would provide proof of the existence of these jets. In the present letter we calculate the event rate of \( \gtrsim 100 \) GeV neutrino-induced cascades in km\(^3\) detectors. We also calculate the event rate for \( \gtrsim 10 \) GeV neutrinos of all flavors with the DeepCore low energy extension of IceCube. The added event rate significantly improves the ability of km\(^3\) detectors to search for these gamma-ray dark bursts. For a core collapse supernova at 10 Mpc we find \(~4\) events expected in DeepCore and \(~6\) neutrino-induced cascades in IceCube/KM3Net. Observations at \( \gtrsim 10 \) GeV are mostly sensitive to the pion component of the neutrino production in the choked jet, while the \( \gtrsim 100 \) GeV depends on the kaon component. Finally we discuss extensions of the on-going optical follow-up programs by IceCube and Antares to include neutrinos of all flavors at \( \gtrsim 10 \) GeV and neutrino-induced cascades at \( \gtrsim 100 \) GeV energies.

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I. INTRODUCTION

Long duration gamma-ray bursts (GRBs) and core collapse supernovae are known to be correlated \([1, 2]\). MeV photons that give rise to GRBs are thought to be produced by accelerated electrons in internal shock of jets emitted by the GRB progenitor. Many details about GRB phenomenology remains uncertain. These GRB jets have very narrow opening angles and very large Lorentz boost factors (\( \Gamma \gtrsim 300 \)). GRBs are one of the plausible candidates sources for the highest energy cosmic rays \([3, 4]\) and as such would observable in \( \sim 100 \) TeV neutrinos by km\(^3\) detectors such as IceCube \([5]\) and KM3Net \([6]\).

Only a very small fraction of core collapse supernovae result in GRBs. It has been speculated that the rate of production of jets in core collapse supernovae may be significantly higher than the rate of GRBs, but that often these jets are choked within the supernova. Evidence for this hypothesis is the observed asymmetry in the explosion of core collapse supernovae \([7, 8]\). Very recently evidence for mildly relativistic jets (\( \Gamma \sim 1 \)) that did break from the progenitor has been presented for type Ic supernovae 2007gr and 2009bb \([9, 10]\). Supernovae with hidden jets, supernovae with mildly relativistic jets that manage to break out and long GRBs could form a continuum class of astronomical objects. Neutrinos would provide us with information about these hidden jets and early neutrino observations could also lead the way in finding and studying supernovae with mildly relativistic jets that do break out. A model has been proposed by Razzaque, Mészáros and Waxman (henceforth RMW) \([11]\). This model was extended by Ando and Beacom (henceforth AB) to include kaon production \([12]\). The RMW/AB spectrum is very soft leading to \( \gtrsim 100 \) GeV neutrino observations in IceCube/KM3Net. A model for neutrino production in reverse shocks for both choked and successful relativistic jets associated with SN type Ib has also been proposed \([13]\). While we did not consider this latter model in the present letter, the broad conclusions still apply.

A promising way to search for \( \gtrsim 100 \) GeV neutrinos from core collapse supernovae is the optical follow-up \([14]\). Neutrino multiplets in directional and time coincidence trigger observations by fast robotic telescopes. This method has the advantage of being able to dramatically reduce the intrinsic atmospheric neutrino background because the signal search window is only \( O(100s) \). A neutrino multiplet may not be significant on its own because the accidental rate due to atmospheric neutrino background is \( O(10/year) \), but the subsequent observation of the rising light-curve of a supernova in spatial and temporal coincidence with a neutrino multiplet would be highly significant. Optical follow-up programs are already in operation by IceCube \([15]\) and Antares \([16]\).

The objective of this letter is to calculate the event rate from core collapse supernovae as observed by \( \sim 10 \) GeV detectors like DeepCore \([17]\). We propose the optical follow-up programs to be enhanced so as to require at least one \( \gtrsim 100 \) GeV \( \nu_\mu \) (minimum requirement to achieve good sky localization) and several \( \gtrsim 10 \) GeV events. We also calculate the event rate on \( \gtrsim 100 \) GeV neutrino-induced cascades. A most interesting case is that of KM3Net, which promises good angular reconstruction of cascades. We propose the optical follow-up programs to be enhanced so as to require at least one \( \sim 100 \) GeV \( \nu_\mu \) and one \( \sim 100 \) GeV neutrino-induced cascade. The enhanced sensitivity of these two new modes of observation will allow IceCube/KM3Net to provide a more thorough
test of the RMW/AB model.

II. NEUTRINO PRODUCTION IN CHOKED JETS

The RMW/AB model supposes a choked jet with bulk Lorentz boost factor \( \Gamma_b \sim 3 \) and an opening angle \( \theta_j \sim 0.3 \). The kinetic energy of the jet is set to \( E_j = 3 \times 10^{31} \) erg in analogy to GRBs. Also similar to GRBs the time variability of the engine is set to \( t_v \sim 0.1 \) s. Neutrinos are produced in p-p interactions via both pions and kaons. The accelerated proton spectrum is assumed to be \( dN_p/dE = E^{-2} \). The density and energies involved are similar to those of atmospheric neutrinos in which muons do not decay, but instead are subject to radiative cooling. Thus in the case of neutrinos generated by pion decay, the neutrino flavor flux ratio \( \phi_{\nu_e}/\phi_{\nu_\mu}/\phi_{\nu_\tau} \) is 0:1:0. In the case of kaons there is a small flux of \( \nu_e \) due to \( K^0 \) decay, but the AB paper does not include neutral kaon contribution. We therefore also assume 0:1:0 as the flavor flux ratio. Taking into account vacuum oscillations the expected flavor flux ratio at Earth is 0.2:0.4:0.4 for both pions and kaons. Note that vacuum oscillations were not taken into account in previous calculations \[11, 12\], we include them here as they are critical to describe neutrino fluxes at Earth. All our numbers include vacuum oscillations except where explicitly noted.

The mesons product of p-p interactions have the same initial spectrum as the protons, but they are subject to hadronic and radiative cooling. This results in two break energies above which the meson spectrum is steeper. Neutrinos follow the energy spectrum of their parent mesons. The neutrino flux resulting from pion and kaon contributions can be described as a doubly broken power law:

\[
\frac{dN_{\nu}}{dE} = A \begin{cases} 
E^{-2} & E > E_{\nu}^{(1)} \\
E_{\nu}^{(1)} E^{-3} & E_{\nu}^{(1)} E < E_{\nu}^{(2)} \\
E_{\nu}^{(1)} E_{\nu}^{(2)} E^{-4} & E_{\nu}^{(2)} < E < E_{\nu}^{(max)} 
\end{cases}
\]

\( A \) is the flux normalization (for all flavors combined) such that for pions(kaons) \( dN_{\nu}/dE \) is \( 5 \times 10^{-2} \) GeV\(^{-1}\) cm\(^{-2}\) (5 \times 10^{-5} \) GeV\(^{-1}\) cm\(^{-2}\) at \( E_{\nu}^{(1)} \). The energies \( E_{\nu}^{(1)} \) and \( E_{\nu}^{(2)} \) are the energies above which hadronic and radiative cooling become relevant respectively. For pions (kaons) these energies are: \( E_{\nu}^{(1)} = 30 \) GeV (200 GeV) and \( E_{\nu}^{(2)} = 100 \) GeV (20 TeV). The values above correspond to a supernova at 10 Mpc with default choices of parameters. See RMW\[11\] and AB\[12\] for details.

III. EFFECTIVE AREAS AND EVENT RATES IN ICECUBE/KM3NET/DEEPCORE

To calculate the expected number of events \( N_{\text{obs}} \) given a neutrino flux \( dN_{\nu}/dE \) the neutrino effective area \( A_{\text{eff}} \) of a detector is needed:

\[
N_{\text{obs}} = \int dE A_{\text{eff}}(E) \frac{dN_{\nu}}{dE}
\]

The neutrino effective area of IceCube for muon neutrinos averaged over the northern hemisphere and assuming equal \( \nu \) and \( \bar{\nu} \) fluxes has been published by the IceCube collaboration \[18\] and it is reproduced in Fig.1. The neutrino effective area of a detector can be calculated from first principles and the procedure for doing so has been reported in many publications. In particular \[19\] and refs. therein have a careful discussion of this subject.

The calculation of neutrino effective area requires knowledge of the muon effective area, which in turn depends on detailed knowledge of the operation of the detector. In the literature it is common to assume that IceCube has an effective area of 1 km\(^2\) for muons above a threshold of 100 GeV. This approximation is a good one for the hard spectra, such as that expected from the sources of cosmic rays. In contrast this approximation is a bad one for soft fluxes, such as those from the RMW/AB model. For RMW/AB most of the events observed have energies that are close to the muon detection threshold and thus detailed knowledge of the detector functioning (and hence the muon effective area) are needed. As similar argument also applies to DeepCore and cascade events.

We have used ANIS \[20\] to simulate neutrinos in the vicinity of IceCube and DeepCore. We assume that IceCube is a cylinder of 1000 m of height and 564 m in radius. The center of this cylinder is placed 1950 m below the ice surface. DeepCore is simulated as a cylinder of 350 m of height and 125 m of radius at 2275 m of depth below the ice surface. The ANIS simulation takes into account the ice/rock boundary, Earth neutrino absorption (though this is a very small effect for the soft RMW/AB spectrum), neutral current regeneration, etc. Our simulations include neutrinos of all flavors. We have assumed flavor flux ratio as appropriate for the injected fluxes by either pions or kaons and we have taken into account neutrino oscillations. We have also assumed equal ratios of neutrino to anti-neutrino for a given flavor. Muon propagation is calculated with MMC \[21\]. We assume that events are detected for the following conditions: a) through-going muons or entering muons must have a minimum energy when entering the detector cylinder b) the total visible energy (cascades, muons, etc) of a contained event must be greater than the same threshold as through-going events. For IceCube we have chose 100 GeV as the threshold and for DeepCore 10 GeV. IceCube results also apply to KM3Net. We show our calculations for neutrino effective areas for IceCube and DeepCore in Fig. 1.
FIG. 1: The three panels show the neutrino effective areas for $\nu_e$ (left), $\nu_\mu$ (center) and $\nu_\tau$ (right). The effective areas shown are for IceCube (solid histogram - black online version) and DeepCore (short dashed histogram - blue online version) as calculated in this letter. Also shown (long dashed curve - red online version) is the effective area for $\nu_\mu$ published by the IceCube collaboration. The effective area has been averaged over the northern hemisphere ($\delta$: 0° to 90°). The effective area is also averaged over neutrinos and anti-neutrinos. The difference between IceCube’s effective area for $\nu_{\mu\nu_{\mu}}$ as calculated in this letter and as published by the IceCube collaboration is due to threshold effects. In this letter’s calculation we have simplistically assumed that all muons with $>100$ GeV energy that go through IceCube are detected. For details see section III.

Note good agreement between our calculations and IceCube published effective for $\gtrsim$TeV. As described above, for low energies, detailed simulation of the detector performance becomes important. For DeepCore our simulation is in good agreement with the results published by the IceCube collaboration [17]. The IceCube collaboration has not published an effective area for neutrino-induced cascades. Therefore our results concerning cascades will have the largest uncertainty.

Performing the integral on Eq. 2 using IceCube’s published neutrino effective area with the RMW/AB flux for a core collapse supernova at 10 Mpc confirms previous estimates of expected $\nu_\mu$ events [12]. Using IceCube’s published $\nu_\mu$ effective area we obtain an expectation of 11.2 events (28 if we don’t assume oscillations as AB did.) The comparison remains equally valid at high energies for which threshold effects are less relevant. Using our calculation of the $\nu_\mu$ effective area we obtain 3.3 events. Our discrepancy with AB and the IceCube published effective area can be attributed exclusively to threshold effects near 100 GeV.

The all-flavor expectation for DeepCore is 4 events. The neutrino-induced cascade signal expected in IceCube is 6.1 events. DeepCore events are broken down by flavor to 0.4 due to $\nu_e$, 3 are due to $\nu_\mu$ and 0.6 due to $\nu_\tau$. Cascade events break down to 2 due to $\nu_e$, 1.3 due to $\nu_\mu$ (via neutral current interactions) and 2.8 due to $\nu_\tau$.

Figure 2 shows the observed event spectra for all three neutrino flavors in DeepCore due to a reference supernova at 10 Mpc that follows the RMW/AB model. Figure 3 is similar but neutrino-induced cascades in a km$^3$ detector. It is clear that the pion component makes the most significant contribution to the observations in DeepCore. In contrast events expected in IceCube, both $\nu_\mu$ and cascades, are mostly due to kaons with the exception of energies near to the detector threshold.

IV. ATMOSPHERIC NEUTRINO BACKGROUND

In DeepCore 10$^5$ atmospheric neutrinos of all flavors over 2$\pi$ sr per year are expected [17]. IceCube has an atmospheric muon neutrino rate of 10$^5$ events per year over 2$\pi$ sr [5]. IceCube has a rate of 2 $\times$ 10$^4$ atmospheric neutrino-induced cascades per year over 4$\pi$ sr [22].

When searching for neutrino transients, background is significantly reduced by using a narrow time window and by pointing in a specific direction. Km$^3$ neutrino detectors have angular resolutions of $O(1^\circ)$ [30]. Due to the optical properties of ice, IceCube is not expected to be able to reconstruct the direction of cascades well, but it is anticipated that KM3Net will be able to reconstruct cascades with $5^\circ$ resolution [23]. It is also expected that DeepCore will reconstruct the direction of events. For charged current $\nu_\mu$ events above $\sim$10-20 GeV DeepCore will have a resolution of 10-15$^\circ$. For cascade-like events in DeepCore the angular resolution is expected to be $\sim$ 30$^\circ$ but the minimum energy at which this is feasible is still unknown. DeepCore will also have the ability to separate track events (from C.C. $\nu_\mu$ interactions) from cascades-like events for energies greater than 10 GeV [24]. In our calculations below we assume, perhaps pessimistically, an angular resolution of $\sim$ 30$^\circ$ for all flavors in DeepCore. The appropriate time search window is set by the size of the star: $R_\ast/c_{\text{light}} \sim$ 100s). In fact the radii of progenitors just prior to core collapse are known with poor detail. Candidate progenitor stars have been identified for about a dozen of core collapse supernovae, including SN1987A (type IIP, $R_\ast \sim 10^{11}$ cm) [22] and SN2008D (type Ib, $R_\ast \sim 10^{11}$ cm) [26]. But SN1987A is unusual and type II progenitors are more frequently red giants with $R_\ast$ as large as $\sim$ 10$^{13}$ cm.

The rate of accidental atmospheric neutrino multiplets with $N$ events of channel a (e.g. $>$100 GeV $\nu_\mu$) and $M$ (e.g DeepCore) events of channel b is:
FIG. 2: The panels show the expected signal above a given energy for DeepCore for $\nu_e$ (left), $\nu_\mu$ (center) and $\nu_\tau$ (right). The solid (black in the online version) histogram is the total expectation of the RMW/AB model, the long dashed histogram (blue in the online version) is the pion contribution and the short dashed (red in the online version) histogram is the kaon contribution.

FIG. 3: The panels show the expected signal above a given energy for cascades in IceCube/KM3Net for $\nu_e$ (left), $\nu_\mu$ (center) and $\nu_\tau$ (right). The solid line (black in the online version) is the total expectation of the RMW/AB model, the long dashed line (blue in the online version) is the pion contribution and the short dashed line (red in the online version) is the kaon contribution.

\[ R_{N,M} = R_a^N R_b^M \left( \frac{\Omega_a}{2\pi} \right)^{N-1} \left( \frac{\Omega_b}{2\pi} \right)^M \frac{\Delta T^{M+N-1}}{(N-1)!M!}, \tag{3} \]

where $R_a$ and $R_b$ are the atmospheric neutrino background rates for channels $a$ and $b$, $\Omega_a$ and $\Omega_b$ are the angular areas search areas and $\Delta T$ is the temporal search window. Note that eq. 3 is only valid for $N \geq 1$ and $M \geq 0$. In the case that the neutrino multiplet with a single species, it is necessary to set $M = 0$.

The rate of accidentals should be matched to at most, what is feasible to follow-up with robotic optical telescopes, about 50/year. The resulting angular uncertainty of the multiband or multiflavor neutrino multiplet must also match the field of view of robotic optical telescopes, about $2^\circ \times 2^\circ$. Using eq. 3 we find that optical follow-ups are possible for 2 or more $\gtrsim 100$ GeV $\nu_\mu$ (which has already been proposed), but also (with KM3Net) one $\gtrsim 100$ GeV $\nu_\mu$ and one $\gtrsim 100$ GeV cascade. Two $\gtrsim 100$ GeV cascades also produces an accidental rate that might be appropriate, but its follow up would require a very large field of view telescope. Coincidences of one $\gtrsim 100$ GeV $\nu_\mu$ and at least three DeepCore events also provides a good match to perform optical follow up. Tables II and III show the accidental false rate of neutrino multiplets for $\nu_\mu$, DeepCore events and cascades.

V. DISCUSSION

Following the RMW/AB model, we have calculated the event expectation in DeepCore and for cascades in km$^3$ detectors like IceCube and KM3Net. We find that for a reference core-collapse supernova at 10 Mpc $\sim 4$ events would be detected by DeepCore and $\sim 6$ cascades would be detected by km$^3$ detectors. These signals are strong enough to allow for the search of neutrinos in coincidence with a known supernova in the scale of 10 Mpc. In the case of neutrino only searches, the atmospheric neutrino background is higher than what we described in the text, as the time of the explosion can only be established with about 1 day precision [27].

A better alternative is to expand the already running optical follow-up programs. A single TeV $\nu_\mu$ event can provide accurate location in the sky, while a coincident observation with at least three 10 GeV events in DeepCore would result in an acceptable false accidental rate. One muon event and one cascade event at $\gtrsim 100$ GeV also has a very low accidental rate in KM3Net, because of its good angular resolution for cascades. A DeepCore-like component in KM3Net may have better angular resolution than in IceCube, but the level of improvement is limited by the kinematical direction difference between the neutrino and the outgoing muon or shower.

The $\gtrsim 10$ GeV observations provide another advantage. Because they are sensitive to the pion contribution and the $\gtrsim 100$ GeV neutrinos are sensitive to the
TABLE I: Expected coincident yearly rates of $N\nu_\mu$ events and M DeepCore events. We assume the $\nu_\mu$ events coincident in a circle of $1^\circ$ radius and $30^\circ$ for DeepCore events. We assume coincidence time window of 100 s.

| DC / $\nu_\mu$ | 0      | 1      | 2      | 3      |
|----------------|--------|--------|--------|--------|
| 0              | -      | -      | 4.8    | $1.1\times10^{-4}$ |
| 1              | -      | -      | 0.2    | $5.0\times10^{-6}$ |
| 2              | -      | 94     | $4.6\times10^{-3}$ | $1.1\times10^{-3}$ |
| 3              | 94     | 1.4    | $6.6\times10^{-5}$ | $1.6\times10^{-9}$ |
| 4              | 1.4    | $1.5\times10^{-2}$ | $7.2\times10^{-7}$ | $1.7\times10^{-11}$ |

TABLE II: Expected coincident yearly rates of $N\nu_\mu$ events and M cascade events. We assume the $\nu_\mu$ events coincident in a circle of $1^\circ$ radius and $5^\circ$ for cascade events. We assume coincidence time window of 100 s.

| casc / $\nu_\mu$ | 0      | 1      | 2      | 3      |
|------------------|--------|--------|--------|--------|
| 0                | -      | -      | 4.8    | $1.1\times10^{-4}$ |
| 1                | -      | -      | 5.8    | $1.4\times10^{-8}$ |
| 2                | 1.2    | $7.3\times10^{-3}$ | $3.5\times10^{-8}$ | $8.5\times10^{-13}$ |
| 3                | $7.3\times10^{-5}$ | $2.9\times10^{-8}$ | $1.4\times10^{-12}$ | $3.4\times10^{-17}$ |

kaon contribution, the multiband channel helps the IceCube/DeepCore combination to maintain sensitivity if actual supernovae deviate from the reference model.

Observations of $\gtrsim10$ GeV and $\gtrsim100$ GeV neutrinos in coincidence with core collapse supernovae would be very strong evidence for the existence of choked jets. This observations would help understand the correlation between long duration gamma ray bursts and core collapse supernovae. Finally these observations may provide an alternative way of detecting gamma-ray dark core collapse supernovae with mildly relativistic jets as those observed in SN2007gr and SN2009bb.

The expansion of the optical follow-up programs proposed here is promising in light of the core-collapse supernova rate within 10 Mpc: 1-2 core collapse SNe/yr [28, 29]. The expected opening angle of the choked jets implies a random aligning of one of the jets with the line of sight for 10-20%. This would result in a positive observation every 2.5 - 10 years within 10 Mpc if all core collapse supernovae have hidden jets. But this is a conservative estimation of the relevant supernova rate. The neutrino event rates calculated in this letter imply that observations are possible at distances beyond 10 Mpc and the supernova rate depends of the cube of the distance [31].

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for higher energy. But at 100 GeV - 1 TeV, the angular resolution is dominated by the directional difference between the parent $\nu_\mu$ and the daughter muon.

[31] At close distances the detailed distribution of local galaxies is important and does deviate somewhat from the continuous limit.