The semi-classical energy of open Nambu-Goto strings

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Abstract
We compute semi-classical corrections to the energy of rotating Nambu-Goto strings, using methods from quantum field theory on curved space-times. We find that the energy density diverges in a non-integrable way near the boundaries. Regularizing these divergences with boundary counterterms, we find the Regge intercept $a = 1 + \frac{D-2}{24}$ for $D$ dimensional target space.

1 Introduction

For several reasons, the Nambu-Goto string is an interesting model: It exhibits diffeomorphism invariance, making it a toy model for (quantum) gravity. It also provided motivation for the Polyakov string, which led to string theory as a candidate for a fundamental theory. Furthermore, it constitutes a phenomenological model for QCD vortex lines connecting quarks, i.e., for the description of hadrons.

It is well-known [1, 2] that in the covariant quantization of the open Nambu-Goto string, the intercept $a$ is a free parameter, only constrained by the fact that the theory is consistent only for $a \leq 1$ and $D \leq 25$ or $a = 1$ and $D = 26$. Furthermore, the ground state energies $E_{\ell_{1,2}}$ for a given angular momentum $\ell_{1,2} > 0$, say in the $1-2$ plane, lie on the Regge trajectory

$$E_{\ell_{1,2}}^2 = 2\pi \gamma (\ell_{1,2} - a),$$

with $\gamma$ the string tension.

Interestingly, $a$ can be fixed in light cone gauge quantization, as the requirement of the existence of a representation of the Lorentz group implies $a = 1$ and $D = 26$. But we tend to take these results with caution: On the classical level there is the problem that the light cone gauge can not be
achieved for general string configurations. As an example, consider light cone coordinates in the 1-direction, i.e., \( X^+ = \frac{1}{\sqrt{2}} (X^0 + X^1) \). It is easy to see that a world-sheet (or a patch of a world-sheet) that extends in the 0–1 plane can not be parameterized in light cone gauge.\(^1\) Also on the quantum level, light cone gauge is problematic, as discussed in [3]. By this choice of gauge the canonical pair \( \{ q^+, p^- \} \) of center-of-mass position and momentum operators is eliminated, making it possible to localize states arbitrarily well in \( q^+ \) and \( p^- \), which is not possible for the other canonical pairs. Hence, a preferred direction is manifestly singled out.

The result \( D = 26 \) and \( a = 1 \) is also found for the Polyakov string from the condition of the absence of the trace anomaly. However, one has to keep in mind that the Polyakov string is quantized in a vacuum in which the expectation values of the embedding coordinates \( X \) all vanish. Classically, this corresponds to an event, and it is a configuration at which the Nambu-Goto action is not continuously differentiable. Also the classical equivalence of the Polyakov and the Nambu-Goto action does not hold at such a configuration. Furthermore, in perturbation theory, the expansions around a degenerate and a non-degenerate configuration differ substantially. For instance, as an effective field theory (in the sense of perturbation theory around arbitrary non-degenerate classical solutions), the Nambu-Goto string is anomaly-free in any target space dimension [3].

As this seems to be largely unappreciated, let us sketch the very simple argument. Let \( X : \Sigma \to \mathbb{R}^D \) describe the embedding and split \( X \) into a classical solution \( \bar{X} \) and a perturbation \( \varphi \), i.e.,

\[
X = \bar{X} + \lambda \varphi, \tag{2}
\]

with some formal expansion parameter \( \lambda \) (later we identify it for convenience with \( \gamma^{-\frac{1}{2}} \)). The BRST transformation of \( \varphi \) is then

\[
s \varphi^a = e^\mu \partial_\mu X^a = e^\mu \partial_\mu \bar{X}^a + \lambda e^\mu \partial_\mu \varphi^a,
\]

with \( e^a \) the re-parametrization ghosts. Its free, i.e., \( O(\lambda^0) \), part is

\[
s_0 \varphi^a = e^\mu \partial_\mu \bar{X}^a. \tag{3}
\]

For a non-degenerate embedding \( \bar{X}, \partial_\mu \bar{X}^a \) has maximal rank. It follows that the fluctuations \( \varphi^a \) tangential to the embedding \( \bar{X} \) form trivial pairs with the ghosts \( e^\mu \) and drop out of the cohomology of \( s_0 \). Hence, \( s_0 \) has trivial cohomology at positive ghost number. By cohomological perturbation theory [4], this carries over to the cohomology of \( s \), and also to the cohomology of \( s \mod d \). Hence, there is a renormalization scheme in which no anomalies

\(^1\) Also for the rotating string solution [16] discussed below, the passage to light cone coordinates is not globally possible, as the corresponding Jacobians degenerate at one-dimensional submanifolds.
are present, in any order of perturbation theory. Note that it is crucial here that no derivatives of $c$ are present in (3). Furthermore, it is crucial that the classical embedding $\bar{X}$ is non-degenerate. This explains why this result does not hold in approaches in which the ground state corresponds to an event or a world-line. More details can be found in [3, Sect. 7].

For the reasons mentioned above, we think that an independent determination of the intercept is highly desirable. Similar calculations have appeared in [3, 5–8].

The starting point of our approach are classical rotating string solutions for the Nambu-Goto string. We then quantize the perturbations to these solutions at second order in the perturbation, obtaining a free quantum field living on the world-sheet. This is a curved manifold, and the equations of motion for the fluctuations only depend on the world-sheet geometric data, i.e., the induced metric and the second fundamental form. Hence, it seems natural, in line with the framework of [3], to use methods from quantum field theory on curved space-time [9, 10] for the renormalization of the free world-sheet Hamiltonian $H^0$. The crucial requirements are that the renormalization is performed in a local and covariant way, and that the renormalization conditions are fixed only once. The latter means that they are “the same” on all the classical solutions for the same bare parameters (in the present case, the only bare parameter is the string tension). We find that there is only one renormalization freedom in $H^0$, which amounts to an Einstein-Hilbert counterterm. Furthermore, the energy density is locally finite but diverges in a non-integrable fashion at the boundaries. In line with the usual treatment of such divergences [11], we regularize them by introducing geodesic curvature counterterms at the boundaries. The correspondence between the world-sheet Hamiltonian and the target space energy then gives corrections to the classical Regge trajectories.

Let us analyze this in a bit more detail. The classical target space energy and angular momentum for the string rotating in the $1-2$ plane are

$$\bar{E} = \gamma \pi R, \quad \bar{L}_{1,2} = \frac{1}{2} \gamma \pi R^2, \quad (4)$$

with $2R$ the string length in target space. In the parametrization that we are using, the world-sheet time $\tau$ is dimensionless, and so should be the world-sheet Hamiltonian $H$. Its free part $H^0$ does not contain any further parameters, the string tension $\gamma$ appearing in inverse powers in the interaction terms. By dimensional analysis, we must thus have

$$H = H^0 + O(R^{-1} \gamma^{-\frac{1}{2}}),$$

with $H^0$ independent of $R$ and $\gamma$. In our parametrization, the relation between the world-sheet Hamiltonian $H$, the quantum correction $E^q$ to the
target space energy $E$, and the quantum correction $L_{1,2}^q$ to the angular momentum $L_{1,2}$ is

$$E^q = \frac{1}{R}(H + L_{1,2}^q),$$

leading to

$$E^2 = (E + E^q)^2$$

$$= \gamma^2 \pi^2 R^2 + 2\gamma \pi (H + L_{1,2}^q) + O(R^{-2})$$

$$= 2\gamma \pi (L_{1,2} + H^0) + O(L_{1,2}^{-1}).$$

By comparison with (1), one can directly read off the intercept $a$ from the expectation value of $H^0$, i.e.,

$$a = -\langle H^0 \rangle.$$  

Such semi-classical approximations are generally believed to provide the correct sub-leading behavior for large angular momentum (a quantum mechanical example is discussed in Section 2). Or, put differently, agreement, at sub-leading order, of a semi-classical result and the corresponding result from a non-perturbative quantization of a given action may serve as a consistency check for the non-perturbative quantization.

The value for the intercept $a$ that we find with our methods is

$$a = 1 + \frac{D - 2}{24},$$

which is always larger than 1. In particular, the covariantly quantized string would be inconsistent for this value, for any dimension $D$. Different interpretations of this result are possible:

- Insisting that a non-perturbative quantization of a given action should reproduce the semi-classical results derived from that action, one would conclude that the usual non-perturbative quantum strings are not quantizations of the Nambu-Goto action, or at least that they do not properly describe the regime of long extended strings.

- Semi-classical analysis is not applicable. But as it is based on an anomaly-free effective theory, it is unclear what precisely should be responsible for the failure.

It is certainly interesting to investigate these issues further, also with a view to quantum gravity, which also has a well-defined effective theory [12,13].

Let us comment on the relation to other calculations of the intercept. In [5], the non-relativistic limit of the rotating string with masses at the

by renormalization. This would imply that the intercept is ambiguous. However, we find that such a term is not present.
ends was considered. The calculation of the energy proceeds via the series of eigenfrequencies. Now there are many different ways to regularize such a series, so without any physical input, one can get an arbitrary dependence of the energy on the angular momentum. This is exemplified by considering two, mathematically well-motivated, schemes, that lead to qualitatively different results. This constitutes a good example for the need for a physically motivated renormalization scheme in order to obtain unambiguous results. We think that our local renormalization scheme fulfills this criterion.

In [7], building on results in [6], the full relativistic problem was considered. This work is closest in spirit to our calculation, so we discuss the differences in some detail. The quantization of the fluctuations around the rotating string solution with masses at the ends there led to the intercept

$$a = \frac{D - 2}{24},$$

(9)

which would be consistent with the above mentioned results for $D = 26$. However, some comments are in order. First, for the fluctuations, Dirichlet boundary conditions are imposed. These are not the ones that one obtains with masses at the ends [14]. Second, the renormalization, in particular of the logarithmic divergences, is not manifestly local on the world-sheet. As discussed in Section 4, in a local renormalization scheme, the renormalization scale should always be considered w.r.t. the local metric. Third, for the corrections to the energy as a function of the classical angular momentum $\bar{L}$,

$$\bar{a} = \frac{1}{2} + \frac{D - 2}{24} \quad (10)$$

is obtained. The result (9) is then gotten upon the replacement

$$\bar{L} = \ell + \frac{1}{2}.$$  

(11)

While this so-called Langer modification is well known in semi-classical calculations, it applies to quantum mechanical problems in three spatial dimensions if no fluctuations perpendicular to the plane of rotation are allowed, as explained in Section 2. All these criteria are not fulfilled in the setting of [6,7], so the substitution (11) does not seem to be justified. Finally, let us remark that the additional term $\frac{1}{2}$ in (10) is due to the fact that a mode with frequency equal to the rotation frequency is absent from the spectrum.

In [8], the fluctuations around solutions to the massless Nambu-Goto string were quantized. The calculation of the intercept then proceeded by $\zeta$ function regularization of the series of eigenmodes [4] leading to (10). As before, the reason is the absence of a certain mode. A similar calculation is also performed for the Polyakov action, leading to the intercept (9).

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4The problem with such calculations was already discussed above.
There is also another effective string theory, defined by the Polchinski-Strominger (PS) action \[15\]. It is derived by fixing the parametrization to conformal gauge and introducing singular supplementary terms in order to preserve the conformal symmetry at the quantum level. Concretely, it is the full embedding \( X : \Sigma \to \mathbb{R}^D \), cf. \(2\), which is gauge fixed. In particular, this implies constraints on the parametrization of the classical embedding \( \bar{X} \).

Conceptually, this is quite different from our approach to effective string theory \[3\]: There, gauge conditions are only imposed on the fluctuations \( \varphi \), the parametrization of the classical solution \( \bar{X} \) being arbitrary. This corresponds to the standard treatment of Yang-Mills theories in background fields or of perturbative quantum gravity. As discussed above, there are no anomalies for any \( D \) and no further consistency requirements need to be imposed. The difference in the approach to gauge fixing may explain the seeming contradiction between the statements i) that effective string theory in non-critical dimension is consistent only with the PS term and ii) that the perturbatively quantized Nambu-Goto string is consistent in any dimension.\[5\]

Due to the unusual (from the point of view of quantum field theory in non-trivial backgrounds) approach to gauge fixing, it is not clear how the PS action can be perturbatively treated: Let \( \bar{X}(\tau, \sigma) \) be a non-degenerate solution fulfilling the gauge condition that the corresponding induced metric \( g_{\mu\nu} \) is conformal to the Minkowski metric in the coordinates \((\tau, \sigma)\). Considering perturbations \( \varphi \) as in \(2\), and requiring that the gauge condition still holds at first order in \( \varphi \), one obtains a linear constraint on \( \varphi \):

\[
\partial_\mu \bar{X}^a \partial_\nu \varphi_a + \partial_\mu \varphi^a \partial_\nu \bar{X}_a - g_{\mu\nu} g^{\lambda\rho} \partial_\lambda \bar{X}^a \partial_\rho \varphi_a = 0. \tag{12}
\]

This supplements the equation of motion \( \Box g \varphi^a = 0 \). But it is unclear how to implement the constraint \(12\) in practice in a quantum field theoretical calculation of the Casimir energy of \( \varphi \).

For the PS action, the intercept \( a = 1 \) was obtained in \[19\] for rotating open strings, independently of the dimension \( D \). However, the Casimir energy was obtained from the energy of angular momentum eigenstates in the quantized non-perturbative theory\[6\] with supplementary corrections due to the PS term. It therefore seems that the result can not be interpreted as a semi-classical value, in the sense that it is not obtained by perturbation theory around classical solutions\[7\].

As already mentioned, we find that the energy density diverges near the boundaries. This is due to the fact that also the scalar curvature of the

\[5\] We refer to \[16–18\] for further discussions of the relations between different effective string theories.

\[6\] I thank S. Hellerman for clarifying discussions on this work.

\[7\] Note that \(12\) is the linearization of the Virasoro constraint, which is quadratic in the fields. Thus, \(12\) differs qualitatively from the Virasoro constraint, and is in particular not implemented on the physical subspace of the Hilbert space of the non-perturbative theory.
classical rotating solutions diverges there. Even though the renormalization of these boundary divergences seems very natural, using only geometric (geodesic curvature) data, one may still have doubts about the validity of the semi-classical approximation in such a situation. It is thus re-assuring that the same value \(8\) has been found for the closed string \([20]\), where boundary effects are absent.

As mentioned above, both in \([7]\) and \([8]\), a certain mode, that one might naively expect to be present, is absent. In our terminology, introduced below, this is the planar \(n = 1\) mode, and it is also absent in our approach. In \([14]\) it was shown, in the context of the open string with masses at the ends, that this mode can be interpreted as a Nambu-Goldstone mode for the broken translation invariance in the plane of rotation. There is thus a geometric reason for its absence. This aspect will be further discussed in a forthcoming publication covering the quantization of the string with masses at the ends. In any case, it seems worthwhile to point out that this mode is also absent, more precisely represented by a null state, in the covariantly quantized open string for the critical intercept \(a = 1\).

The article is structured as follows: In the next section, we discuss, as a motivating example for our semi-classical calculation, the hydrogen atom. In Section 3 we discuss the fluctuations of classical rotating string solutions and their canonical quantization. In Section 4 the locally covariant renormalization of the world-sheet Hamiltonian is explained and the semi-classical value of the Regge intercept is calculated. Section 5 is devoted to the comparison of the excitation spectra of the semi-classical string and that of the covariantly quantized string. An appendix contains some calculations that were omitted in the main part.

2 A motivating example: The hydrogen atom

As a motivating example, we consider the hydrogen atom treated with our method of perturbation theory around a classical solution. It has a non-smooth potential, making it similar to the Nambu-Goto string. Another similarity is that there is no classical ground state, or more precisely, that the ground state is at a singularity of the Lagrangian. We will see that our semi-classical analysis produces the correct sub-leading behavior for large angular momentum.

Fixing one component \(L_3 > 0\) of the classical angular momentum and working in cylindrical coordinates, we obtain the Hamiltonian

\[
H = \frac{m}{2} \left( \dot{\rho}^2 + \dot{z}^2 \right) + \frac{L_3^2}{2m \rho^2} - \frac{e^2}{\varepsilon \rho}.
\]

*For the classical Nambu-Goto string, one may consider a particle at rest as the ground state, for which the induced metric degenerates. One is thus dealing with a configuration at which the Lagrangian is not smooth.
One minimum of this potential, corresponding to the classical solution around which we want to do perturbation theory, lies at

\[ \rho_0 = a_0 \frac{L_3^2}{\hbar^2}, \quad z_0 = 0, \]

with the Bohr radius \( a_0 = \frac{\varepsilon_0 \hbar^2}{m e^2} \). Considering perturbations

\[ \rho = \rho_0 + \delta \rho, \quad z = z_0 + \delta z, \]

and expanding the Hamiltonian to second order, we obtain

\[ H^0 = -\frac{1}{2ma_0^2 L_3^2} \frac{\hbar^4}{2} + \frac{m}{2} (\delta \dot{\rho}^2 + \delta \dot{z}^2) + \frac{1}{2ma_0^2 L_3^6} (\delta \rho^2 + \delta z^2) \]

Quantizing this system, we thus find that for a given \( L_3 \), the \( k \)th excited energy level is \( k + 1 \) times degenerate and given by

\[ E_{k,L_3} = -\frac{\hbar^2}{2ma_0^2 \left( \frac{h^2}{L_3^2} - \frac{h^3 2(k + 1)}{L_3^3} \right)}. \]

On the other hand, we know the correct energy of the \( k \)th excited state with a given angular momentum \( L_3 = h \ell_3 > 0 \): It is \( k + 1 \) times degenerate, has quantum number \( n = \ell_3 + k + 1 \), and hence energy

\[ E_{k,\ell_3} = -\frac{\hbar^2}{2ma_0^2} \left( \frac{1}{(\ell_3 + k + 1)^2} \right) \simeq -\frac{h^2}{2ma_0^2} \left( \frac{1}{\ell_3^2} - \frac{2(k + 1)}{\ell_3^3} \right) + O(\ell_3^{-4}). \]

Comparison with the above shows that our semi-classical approximation yields the correct degeneracy and the correct sub-leading term in a large \( \ell_3 \) expansion of the energy.

It should be noted that one could also fix the total angular momentum \( L^2 \), thus getting rid of the harmonic oscillator in the \( z \)-direction. One would then obtain the correct semi-classical behavior upon using the Langer modification \([11]\), which is well established in semi-classical approximations \([21, 22]\), and which can also be motivated by the expansion of \( L = \hbar \sqrt{\ell (\ell + 1)} \).

In our treatment of the Nambu-Goto string, we do not even have to fix the angular momentum component, as due to \([5]\), the quantum correction \( L_{1,2}^q \) to the angular momentum combines with the classical angular momentum \( \bar{L}_{1,2} \) in the calculation \([6]\) leading to the identification \([7]\) of world-sheet Hamiltonian and Regge intercept.

### 3 Perturbations of classical rotating strings

We recall the action

\[ S = -\gamma \int_{\Sigma} \sqrt{|g|} d^2 x \]
for the Nambu-Goto string. Here $\Sigma$ is the world-sheet and $g$ the induced metric (we work with the signature $(-, +)$). Denoting by $X : \Sigma \to \mathbb{R}^D$ the embedding, the equations of motion and boundary conditions can be written as

\[
\Box_g X = 0, \\
\sqrt{|g|} g^{1\mu} \partial_\mu X = 0,
\]

assuming that the components of the boundary $\partial \Sigma$ reside at $x^1 = \text{const.}$ We also recall the target space energy and angular momentum derived from the action:

\[
E = \int \frac{\delta S}{\delta \partial_0 X_0} d\sigma = -\gamma \int \sqrt{gg} g^{0\mu} \partial_\mu X^0 d\sigma,
\]

\[
L_{i,j} = \int \left[ \frac{\delta S}{\delta \partial_0 X_j} X_i - i \leftrightarrow j \right] d\sigma = \gamma \int \sqrt{gg} g^{0\mu} (X_j \partial_\mu X^i - X_i \partial_\mu X^j) d\sigma.
\]

We parameterize the rotating string solutions as

\[
\bar{X}(\tau, \sigma) = R(\tau, \cos \tau \cos \sigma, \sin \tau \cos \sigma, 0),
\]

where $\sigma \in (0, \pi)$. For simplicity, we here assumed that the target space-time is four dimensional. Adding further dimensions is straightforward. The induced metric on the world-sheet and the scalar curvature, in the coordinates introduced above, are

\[
g_{\mu\nu} = R^2 \sin^2 \sigma \eta_{\mu\nu},
\]

\[
\mathcal{R} = -\frac{2}{R^2 \sin^3 \sigma}.
\]

For later use, it is convenient to note that an imaginary boundary at $\sigma = s$ would have the geodesic curvature

\[
\kappa_s = \frac{|\cot s|}{R \sin s}.
\]

Energy and angular momentum of the above solution were given in (4).

Our goal is now to perform a (canonical) quantization of the fluctuations $\varphi$ around the classical background $\bar{X}$, i.e., we consider

\[
X = \bar{X} + \gamma^{-\frac{1}{2}} \varphi.
\]

At second order in $\varphi$, i.e., at $O(\gamma^0)$, the fluctuations parallel to the world-sheet drop out of the action [3], so that it is natural to parameterize the fluctuations as

\[
\varphi = f_s v_s + f_p v_p = f_s \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + f_p \begin{pmatrix} \cot \sigma \\ -\sin \tau / \sin \sigma \\ \cos \tau / \sin \sigma \\ 0 \end{pmatrix},
\]

\[
f_s \begin{pmatrix} \cot \sigma \\ -\sin \tau / \sin \sigma \\ \cos \tau / \sin \sigma \\ 0 \end{pmatrix}.
\]
Here the scalar component \(f_s\) describes the fluctuations in the direction perpendicular to the plane of rotation, and the planar component \(f_p\) describes the fluctuations in the plane of rotation (at least approximately for \(\sigma \sim \frac{\pi}{2}\)). The vectors \(v_s, v_p\) are orthonormal to each other and the world-sheet. At \(O(\gamma^0)\), we thus obtain the action

\[
S^0 = \frac{1}{2} \int \left( f_p'^2 - f_p'^2 - \frac{2}{\sin^2 \sigma} f_p'^2 + \frac{\dot{f}_s'^2 - \dot{f}_s'^2}{\sin^2 \sigma} \right) \, d\sigma d\tau. \tag{21}
\]

We note that we are implicitly working in transversal gauge here, cf. [23]. One may perform an explicit gauge fixing in transversal gauge by reintroducing parallel fluctuations together with anti-ghosts \(\bar{c}_\mu\) and Lagrange multipliers \(b_\mu\) and adding a gauge fixing term

\[
b_\mu \left( \partial_\mu \bar{X}_a \right) \varphi^a + i e^{\mu} c_\mu,
\]

which is BRST exact under the free BRST transformation given by (3) and \(s_0 \bar{c}_\mu = i b_\mu\) (with the action on the other fields vanishing). But obviously, the unphysical fields \(c_\mu, \bar{c}_\mu, b_\mu\) and the parallel fluctuations \((\partial_\mu \bar{X}_a \varphi^a)\) vanish by the equation of motion, so it is consistent to set them to zero. In particular there are then no anomalies. Furthermore, they also do not contribute to the energy density, which we compute later on.

Obviously, going to higher dimensional target space-time simply amounts to multiplying the number of scalar fields. Furthermore, it should be noted that the world-sheet is actually curved, cf. (18). This does not matter for the canonical quantization procedure described in this section, but will be important in the discussion of renormalization in the following one.

From the action (21), one obtains the equations of motion (where derivates w.r.t. \(\tau\) are denoted by dots and those w.r.t. \(\sigma\) by primes)

\[
\begin{align*}
\ddot{f}_s &= \Delta_s f_s := -f_s'', & \quad \text{(22)} \\
\ddot{f}_p &= \Delta_p f_p := -f_p'' + \frac{2}{\sin^2 \sigma} f_p'.
\end{align*}
\]

Furthermore, from (13), we obtain the boundary conditions

\[
\begin{align*}
0 &= f_s'(0) = f_s' (\pi), & \quad \text{(24)} \\
0 &= f_p'(0) = f_p'(\pi) = f_p'(0) = f_p'(\pi).
\end{align*}
\]

This is shown in Appendix A.

The operators \(\Delta_s, \Delta_p\) on \(L^2([0, \pi])\) defined in (22), (23), on the domain \(C^2([0, \pi])\) with boundary conditions (24), (25), are essentially self-adjoint, so they admit a unique self-adjoint extension\(^{10}\).

Defining \(N_N = \{n \in \mathbb{N} | n \geq 9\}\) the same action was obtained for different parameterizations of the fluctuations [7,8].

\(^{9}\)This is shown in Appendix A.

\(^{10}\)For \(\Delta_s\), this is clear. \(\Delta_p\) is obviously symmetric. It thus remains to show that the deficiency indices vanish. The generic solution to \(\Delta_p f = \pm i f\) is

\[
f(x) = C_1 \sqrt{\sin \sigma} P_{(\pm)}^{3/2}(\frac{\pi}{2}) (\cos \sigma) + C_2 \sqrt{\sin \sigma} Q_{(\pm)}^{3/2}(\frac{\pi}{2}) (\cos \sigma).
\]

It is easy to see that there are no normalizable solutions of this form.
\( N \), these have spectrum \( N_0, N_2 \), with normalized eigenvectors
\[
 f_{s,n} = \frac{\sqrt{2}}{\sqrt{\pi}} \cos n\sigma,
 \]
\[
 f_{p,n} = \frac{\sqrt{2}}{\sqrt{\pi(n^2-1)}} (n \cos n\sigma - \cot \sigma \sin n\sigma).
\]
The absence of the planar \( n = 1 \) mode was already noted in [6, 8]. The scalar zero mode corresponds to translations perpendicular to the plane of rotation. There is also an associated momentum. For the purposes of the calculation of the Regge intercept, we want to fix the spatial momentum, so we do not consider the zero modes in the following. The usual canonical quantization then yields quantum fields \( \phi_s, \phi_p \) with two-point functions
\[
 w_s(x; x') := \langle \Omega | \phi_s(x) \phi_s(x') | \Omega \rangle = \sum_{n \geq 1} \frac{1}{2\pi} f_{s,n}(\sigma) f_{s,n}(\sigma') e^{-in(\tau - \tau' - i\epsilon)},
\]
\[
 w_p(x; x') := \langle \Omega | \phi_p(x) \phi_p(x') | \Omega \rangle = \sum_{n \geq 2} \frac{1}{2\pi} f_{p,n}(\sigma) f_{p,n}(\sigma') e^{-in(\tau - \tau' - i\epsilon)}.
\]

The canonical quantization scheme in particular implies that the physical fluctuations are represented on a positive definite Fock space.

4 Renormalizing the world-sheet Hamiltonian

Before turning to the renormalization of the free world-sheet Hamiltonian \( H^0 \), let us first justify the relation (5) between the world-sheet Hamiltonian \( H \) and the quantum corrections \( \bar{E}^q \) and \( \bar{L}^q_{1,2} \) to the target space energy and the angular momentum. The latter are obtained by expanding (14) and (15) in \( \phi \) and neglecting the zeroth order parts \( \bar{E} \) and \( \bar{L}^{1,2}_1 \). The world-sheet Hamiltonian \( H \) generates (world-sheet) time translations on the world-sheet. The quantum target space energy \( E^q \), should generate time translations in target space. However, the background world-sheet is rotating in the 1-2 plane and so is the vector \( v_p \) parameterizing the fluctuations. The correct relation between \( H \) and \( E^q \) is thus (5), where the factor \( \frac{1}{R} \) corrects the different scaling of target space time \( X^0 \) and world-sheet time \( \tau \), and \( L^q_{1,2} \) corrects for the missing rotation. One can easily check (5) up to \( O(\phi^2) \), which is the relevant order for our purposes. The relation (5) was also obtained in [8], albeit for the Polyakov action.

The free Hamiltonian corresponding to the free action (21) is
\[
 H^0 = \frac{1}{2} \int_0^\pi \left( \dot{\phi}_p^2 + \dot{\phi}_p'^2 + \frac{2}{\sin^2 \sigma} \phi_p^2 + \phi_s'^2 + \phi_s^2 \right) d\sigma.
\]
In the treatment of this expression, let us first concentrate on the scalar sector. Formally, the vacuum expectation value is given by
\[
 \langle H^0_s \rangle = \frac{1}{2} \sum_{n \in \mathbb{N}_1} n.
\]
This sum is of course quadratically divergent. As long as one does not impose some conditions on the renormalization prescription, one can obtain any result. The renormalization prescription that we are going to employ is based on the framework of locally covariant field theory \cite{9}, where the renormalization is performed locally, by using the local geometric data. In that framework, the expectation value of Wick squares (possibly with derivatives) is determined as follows:

\[
\langle \Omega | (\nabla^\alpha \phi \nabla^\beta \phi)(x) | \Omega \rangle = \lim_{x' \to x} \nabla^\alpha \nabla^\beta \left( w(x; x') - h(x; x') \right)
\]

Here \(\alpha, \beta\) are multiindices, \(w\) is the two-point function in the state \(\Omega\), defined as on the l.h.s. of (26), (27), and \(h\) is a distribution which is covariantly constructed out of the geometric data, the Hadamard parametrix. Importantly, for physically reasonable states (ground states in particular), the difference \(w - h\) is smooth, so that the above coinciding point limit exists and is independent of the direction from which \(x'\) approaches \(x\). This method has been reliably used for the computation of Casimir energies and vacuum polarization, cf. \cite{10,24,25} for example.

For our purposes, it is advantageous to perform the limit of coinciding points from the time direction, i.e., we take \(x = (\tau, \sigma), x' = (\tau + t, \sigma)\), and \(t \to +0\). Performing the summation in (26), we find

\[
\frac{1}{2} (\partial_0 \partial'_0 + \partial_1 \partial'_1) w_s(x; x') = -\frac{1}{2\pi(t + i\varepsilon)^2} - \frac{1}{24\pi} + \mathcal{O}(t).
\]

For a minimally coupled scalar field with a variable mass \(m^2(x)\) in two dimensional space-time, the Hadamard parametrix is given by (see, e.g., \cite{26})

\[
h(x; x') = -\frac{1}{4\pi} \left( 1 + \frac{1}{2} m^2(x) \rho(x, x') + \mathcal{O}((x - x')^3) \right) \log \frac{\rho \varepsilon(x, x')}{\Lambda^2},
\]

where \(\rho\) is the Synge world function, i.e., \(\frac{1}{2}\) times the squared (signed) geodesic distance of \(x\) and \(x'\), cf. \cite{27}, and \(\Lambda\) is a length scale (the “renormalization scale”). For the local covariance, it is crucial that \(\Lambda\) is fixed and does not depend on any geometric data \cite{9}. Inside of the logarithm, the world function is equipped with an \(i\varepsilon\) prescription as follows:

\[
\rho \varepsilon(x, x') = \rho(x, x') + i\varepsilon(\tau - \tau').
\]

For the scalar part, the mass term is absent, and we obtain, using (18) and standard identities for the coinciding point limit of derivatives of \(\rho\), cf. \cite{27} \footnote{Here and in the following, \(\mathcal{O}(t)\) also includes terms of the form \(t \log t\).}

\[
\frac{1}{2} (\partial_0 \partial'_0 + \partial_1 \partial'_1) h_s = -\frac{1}{2\pi(t + i\varepsilon)^2} + \frac{1}{12\pi \sin^2 \sigma} + \mathcal{O}(t).
\]
For the scalar contribution to the energy density, we thus obtain

$$\langle H^0_s(\sigma) \rangle = -\frac{1}{24\pi} - \frac{1}{12\pi \sin^2\sigma}. \quad (28)$$

This is locally finite, but diverges in a non-integrable fashion at the boundaries.

Also the two-point function of the planar part can be computed explicitly. Evaluating the sums in (27), one obtains, cf. Appendix B,

$$\frac{1}{2}(\partial_0 \partial'_0 + \partial_1 \partial'_1 + \frac{2}{\sin^2\sigma})w_p(x;x') = -\frac{1}{2\pi} \left[ \frac{1}{(t+i\epsilon)^2} + \frac{1}{2\sin^2\sigma} \log \frac{(t+i\epsilon)^2}{4\sin^2\sigma} + \frac{3}{2\sin^2\sigma} + \frac{1}{12} \right] + \mathcal{O}(t). \quad (29)$$

For the parametrix, we note that given the metric (17), the mass square which is implicit in (21) is

$$m^2 = \frac{2}{R^2 \sin^4\sigma},$$

so that we obtain

$$\frac{1}{2} \left( \partial_0 \partial'_0 + \partial_1 \partial'_1 + \frac{2}{\sin^2\sigma} \right) h_p = -\frac{1}{2\pi} \left[ \frac{1}{(t+i\epsilon)^2} + \frac{1}{2\sin^2\sigma} \log \frac{(t+i\epsilon)^2 R^2 \sin^2\sigma}{\Lambda^2} + \frac{1}{12\sin^2\sigma} \right] + \mathcal{O}(t). \quad (30)$$

Hence, for the planar contribution to the energy density, we find

$$\langle H^0_p(\sigma) \rangle = -\frac{1}{24\pi} - \frac{1}{2\pi \sin^2\sigma} \log \frac{\Lambda}{2R \sin^2\sigma} - \frac{17}{24\pi \sin^2\sigma}. \quad (31)$$

In the last term, we have the same non-integrable divergence that we already found in (28). However, we see that both these terms can be absorbed in a change of the scale \(\Lambda\). Noting that \(\frac{1}{\sin^2\sigma} = -\frac{1}{2} \sqrt{|g|} R\), this corresponds to an Einstein-Hilbert counterterm.\(^{12}\) Our final expression for the local energy density in \(D\) dimensional target space is thus

$$\langle H^0(\sigma) \rangle = -\frac{D-2}{24\pi} - \frac{1}{2\pi \sin^2\sigma} \log \frac{\Lambda}{R \sin^2\sigma}. \quad (31)$$

The final expression (31) still contains a non-integrable singularity at the boundaries. We recall that near Dirichlet boundaries, the energy density of a massive scalar field in two space-time dimensions behaves as

$$\varepsilon \sim -\frac{m^2}{2\pi} \log \frac{\lambda}{md},$$

\(^{12}\)The most general redefinition of a parametrix that affects Wick powers with up to two derivatives is \(h \rightarrow h + c_0 + c_1 R + c_2 m^2 R\). This has no effect on the scalar contribution to the energy density and its effect on the planar contribution is exactly as above.
with $d$ the distance to the boundary, cf. [28] for example. In view of this and the divergence of $m^2$ near the boundary, a divergence as in the second term in [31] has to be expected. Such non-integrable divergences near boundaries are a well-known phenomenon [29], in particular in space-time dimensions larger than two. For the treatment of our singularity, we follow the approach proposed in [11], i.e., to introduce boundary counterterms. Concretely, one performs the integration of the energy density only up to a distance $d$ to the boundary and introduces a $d$-dependent local counterterm on this boundary. We denote by $s$ the value of $\sigma$ at which this shifted boundary resides. In the spirit of locally covariant field theory, a boundary counterterm may only depend on the boundary geometric data and the proper distance $d_s = 2R \sin^2 \frac{s}{2}$ to the boundary. More precisely, it should be of the form

$$\sqrt{|h^s|} p(d_s^{-1}, \log d_s / \Lambda_{bd}, \kappa_s, \mathcal{R}(s), m^2(s)),$$

with $h^s$ and $\kappa_s$ the induced metric and geodesic curvature on the boundary, $p$ a polynomial (which may also contain normal derivatives of $\mathcal{R}$ and $m^2$), and $\Lambda_{bd}$ a renormalization scale. We compute

$$\int_{s}^{\pi-s} \frac{1}{\sin^2 \sigma} \log \frac{\Lambda}{R \sin^2 \sigma} \, d\sigma = -4s + 2\pi + 2 \cot s \log \frac{\Lambda}{e^2 R \sin^2 s}.$$

The only way to cancel the divergence in the last term with a counterterm of the form (32) is to add the counterterm

$$-\frac{1}{\pi} \sqrt{|h^s|} \kappa_s \log \frac{e^2 d_s}{\Lambda},$$

cf. (19). It is important to note that the scale $\Lambda_{bd}$ in the logarithm is fixed by the renormalization scale $\Lambda$, so that there is no renormalization ambiguity (a change in $\Lambda$ would lead to a non-integrable divergence of the energy density, unless compensated by a change of $\Lambda_{bd}$). Similar (geodesic curvature) boundary counterterms for open strings were also used in [7, 19] for the calculation of the energy. Hence, for the renormalized total energy, we finally obtain

$$\langle H^0_{\text{ren}} \rangle = -\frac{D-2}{24} - 1,$$

which, by (7), yields the intercept (8).

Let us close by remarking that upon omitting the factor $\sin^2 \sigma$ in the logarithm in the planar parametrix (30)\footnote{Such a modification would single out a preferred parametrization of the world-sheet.}, one would obtain the value (10) for the intercept, as in [7] (before the application of the Langer modification (11)) and [8]. In this sense, it is the locally covariant renormalization that contributes another term $\frac{1}{2}$ to the intercept.
5 Degeneracies of excited states

For the semi-classical spectrum of excitations of the string with a fixed angular momentum component $L_{1,2}$, we found oscillators with frequency $n \geq 1$ for each of the $D-3$ directions perpendicular to the plane of rotation and oscillators with frequency $n \geq 2$ for excitations in the plane of rotation. The goal of this section is to compare with the spectrum of excitations of the covariantly quantized open string, i.e., to investigate the physical states that are eigenstates of the energy and of angular momentum $L_{1,2} = \ell$. A particular focus will be on the presence or absence of an $n = 1$ excitation in the plane of rotation.

In the covariantly quantized Nambu-Goto string, the state of minimal energy for a fixed angular momentum $\ell$ in the $1-2$ plane is given by (for simplicity, we fix $\pi \gamma = 1$)

$$|\ell\rangle = (\xi \cdot \alpha_{-1})^\ell |0, \sqrt{2(\ell - a)}\rangle.$$  

Here

$$\xi = \frac{1}{\sqrt{2}}(0, 1, i, 0, \ldots, 0)$$

and $|0, m\rangle$ stands for the ground state with vanishing spatial momentum and rest mass $m$. We recall the definitions, cf. [30],

$$L_m = \frac{1}{2} \sum_{n=\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \quad m \neq 0$$

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n,$$

$$J^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^{\nu} \alpha_n^{\mu} - \alpha_{-n}^{\mu} \alpha_n^{\nu}),$$

and the commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = m \delta_{m+n} \eta^{\mu\nu}.$$  \hspace{1cm} (33)

In order to avoid confusion with the Virasoro generators $L_m$, we here switch to the notation $J^{\mu\nu}$ for the angular momentum. We also omitted the center-of-mass contribution to $J^{\mu\nu}$. We note that $\alpha_0 = p$, the momentum operator. The commutation relations (33) imply

$$[L_m, \zeta \cdot \alpha_{-k}] = k \zeta \cdot \alpha_{m-k},$$

$$[J^{12}_{12}, \zeta \cdot \alpha_{-k}] = \tilde{\zeta} \cdot \alpha_{-k},$$

where

$$\tilde{\zeta} = (0, -i\zeta^2, i\zeta^1, 0, \ldots, 0).$$
With the last equation, one straightforwardly checks that $|\ell\rangle$ is an eigenstate of $J^{12}$ of eigenvalue $\ell$. Furthermore, one checks that the state $|\ell\rangle$ is physical, i.e., it fulfills the conditions

$$(L_m - \delta_{0m} a) |\ell\rangle = 0 \quad \forall m \geq 0.$$ 

Finally, (33) implies that $|\ell\rangle$ has positive norm.

Let us begin by considering the minimal excitations of $|\ell\rangle$, i.e., the physical states which are eigenstates of $J^{12}$ with eigenvalue $\ell$ and of $p^2$ with eigenvalue $2(\ell + 1 - a)$. It is easy to find $D-3$ linearly independent states:

$$\zeta \cdot \alpha_{-1}(\xi \cdot \alpha_{-1})^\ell |0, \sqrt{2(\ell + 1 - a)}\rangle.$$ 

Here $\zeta$ is an element of the subspace spanned by $e_{3} - e_{D-1}$. These correspond to the $D-3$ scalar excitations for $n=1$ of the semi-classical open rotating string. These states obviously have positive norm, so they count as proper physical excitations.

We can see the $D-3$ linearly independent operators $\zeta \cdot \alpha_{-1}$ as the creation operators for the oscillator of frequency $n=1$. As a slight complication, also the momentum needs to be shifted and when powers of these operators are applied, correction terms need to be added to ensure physicality. For example,

$$\left[(\zeta \cdot \alpha_{-1})^2 - \frac{\zeta^2}{1 - 2p} \left(\frac{1}{p^2}(p \cdot \alpha_{-1})^2 - p \cdot \alpha_{-2}\right)\right] (\xi \cdot \alpha_{-1})^\ell |0, \sqrt{2(\ell + 2 - a)}\rangle$$ 

is the state obtained by twice acting with $\zeta \cdot \alpha_{-1}$ and adding corrections to ensure physicality. Similarly, one may see the $D-3$ linearly independent operators $\zeta \cdot \alpha_{-n}$ as the creation operators for the oscillator with frequency $n$, up to correction terms. In the scalar sector, we thus have complete agreement of the spectra of the semi-classical and the covariantly quantized Nambu-Goto string.

The analog of the first excitation of the planar $n=2$ mode is given by

$$\left[\zeta \cdot \alpha_{-1}\xi \cdot \alpha_{-1} - \frac{\ell+1}{1 - 2p^2} \left(\frac{1}{p^2}(p \cdot \alpha_{-1})^2 - p \cdot \alpha_{-2}\right)\right] (\xi \cdot \alpha_{-1})^\ell |0, \sqrt{2(\ell + 2 - a)}\rangle.$$ 

This state has positive norm, at least in the range $a \leq 2$. Higher excitations of this mode are constructed by applying $\zeta \cdot \alpha_{-1}\xi \cdot \alpha_{-1}$ several times, and adding correction terms to ensure physicality. Similarly, excitations of the $n$th planar mode are obtained by acting with $\zeta \cdot \alpha_{-n+1}\xi \cdot \alpha_{-1}$ and applying correction terms. This exhausts the excitation spectrum of the semi-classical string.

However, there is also a state corresponding to a planar $n=1$ mode:

$$\left[\zeta \cdot \alpha_{-1}(\zeta \cdot \alpha_{-1})^{\ell - 1} - 2p^{-2}p \cdot \alpha_{-1}(\xi \cdot \alpha_{-1})^\ell\right] |0, \sqrt{2(\ell + 1 - a)}\rangle.$$ 

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It is straightforward to check that this is an eigenstate of $J^{12}$ of eigenvalue $\ell$ and of $L_0$ with eigenvalue $a$. Also the physicality conditions are fulfilled. However, one finds that this state has positive norm for $a < 1$, is null for $a = 1$, and has negative norm for $a > 1$. In the critical covariantly quantized string, i.e., with $a = 1$, this state would thus not correspond to a physical excitation. In this sense, the spectra of excitations in the semi-classical and the critical covariantly quantized string coincide, up to the global shift in energies due to the fact that $a > 1$ in the former case. In view of the fact that linearization around a non-degenerate solution changes the Virasoro constraint qualitatively, cf. footnote 7, this similarity of the two spectra seems surprising.

A The boundary conditions

The boundary is a submanifold of co-dimension $D - 1$, so in addition to the scalar and planar perturbations, also radial perturbations could be relevant there. To the r.h.s. of (20), we thus add $f_r v_r$ with $v_r = (0, \cos \tau, \sin \tau, 0)$.

To work out the implication of the boundary condition (13) on the perturbations $\phi$, we first determine the variation of the metric (the brackets denote symmetrization in $\mu, \nu$):

$$
\delta g_{\mu\nu} = 2\partial_\mu \bar{X}_a \partial_\nu \phi^a
= 2\partial_\mu \bar{X}_a \partial_\nu v_p^a f_p + 2\partial_\mu \bar{X}_a \partial_\nu v_r^a f_r + 2\partial_\mu \bar{X}_a v_p^a \partial_\nu f_r
= 2R \left( f_p - \frac{1}{2} \sin \sigma f'_r \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cos \sigma f_r \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \sin \sigma f'_r \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).
$$

Here we used that the vectors $v_s, v_p$ are orthogonal to the world-sheet, that $\partial_\nu v_s = 0$ and

$$
\partial_0 \bar{X} = R \cos \sigma \sin \sigma v_p + R \sin^2 \sigma e_0,
\partial_1 \bar{X} = -R \sin \sigma v_r,
\nu'_p = -\cot \sigma v_p - e_0,
$$

with $e_0$ the unit vector in time direction. This implies

$$
\delta \sqrt{|g|} = -R \cos \sigma f_r - R \sin \sigma f'_r,
\delta g^{\mu\nu} = \frac{2f_p - \sin \sigma f'_r}{R^3 \sin^3 \sigma} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{2\cos \sigma f_r}{R^3 \sin^3 \sigma} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{2f'_r}{R^3 \sin^3 \sigma} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

We thus obtain

$$
\delta \left[ \sqrt{|g|} g^{1\nu} \partial_\nu X \right] = \cot \sigma f_r v_r + f'_s v_s + \left( \cot \sigma f_p - \cos \sigma f'_r + f'_p \right) v_p + \left( f_p - \sin \sigma f'_r \right) e_0.
$$

Linear independence of $v_p, v_s, v_r, e_0$ implies that $f_r = f_p = f'_s = 0$ at the boundary. Furthermore, with l’Hôpital’s rule, we also obtain $f'_p = 0$. 

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The planar two-point function

To compute the l.h.s. of (29), we have to evaluate

$$\sum_{n=2}^{\infty} \frac{1}{4n} \left[ \left( n^2 + \frac{2}{\sin^2 \sigma} \right) f_{p,n}^2 + f_{p,n}'^2 \right] e^{in(t+i\epsilon)}$$

Straightforward manipulations simplify this to

$$\sum_{n=2}^{\infty} \frac{n}{2\pi(n^2-1)} \left[ n^2 + \cot^2 \sigma + \frac{2}{\sin^2 \sigma} \cos 2n\sigma - \frac{3}{n\sin^2 \sigma} \sin 2n\sigma \right. 
+ \frac{2}{n^2\sin^4 \sigma} - \frac{\sin^2 n\sigma}{n\sin^2 \sigma} \left] e^{in(t+i\epsilon)} \right.$$}

Using

$$\sum_{n=2}^{\infty} \frac{n^3}{n^2-1} e^{in(t+i\epsilon)} = -\frac{1}{(t+i\epsilon)^2} - \frac{11}{6} + O(t),$$
$$\sum_{n=2}^{\infty} \frac{n}{n^2-1} e^{in(t+i\epsilon)} = -\frac{1}{2} \log[-(t+i\epsilon)^2] - \frac{3}{4} + O(t),$$
$$\sum_{n=2}^{\infty} \frac{n \cos 2n\sigma}{n^2-1} e^{in(t+i\epsilon)} = -\frac{1}{4} - \frac{1}{4} \cos 2\sigma - \frac{1}{2} \cos 2\sigma \log[4\sin^2 \sigma] + O(t),$$
$$\sum_{n=2}^{\infty} \frac{\sin 2n\sigma}{n^2-1} e^{in(t+i\epsilon)} = \frac{1}{4} \sin 2\sigma - \frac{1}{2} \sin 2\sigma \log[4\sin^2 \sigma] + O(t),$$
$$\sum_{n=2}^{\infty} \frac{\sin^2 n\sigma}{n(n^2-1)} e^{in(t+i\epsilon)} = \frac{3}{4} \sin^2 \sigma - \frac{1}{2} \sin^2 \sigma \log[4\sin^2 \sigma] + O(t),$$

one obtains the r.h.s. of (29).

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