Bi-conformal vector fields and their applications to the characterization of conformally separable pseudo-Riemannian manifolds

New criteria for the existence of conformally flat foliations in pseudo-Riemannian manifolds

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Abstract: In this paper a thorough study of the normal form and the first integrability conditions arising from bi-conformal vector fields is presented. These new symmetry transformations were introduced in Class. Quantum Grav. 21, 2153-2177 and some of their basic properties were addressed there. Bi-conformal vector fields are defined on a pseudo-Riemannian manifold through the differential conditions $\mathcal{L}_\xi P_{ab} = \phi P_{ab}$ and $\mathcal{L}_\xi \Pi_{ab} = \chi \Pi_{ab}$ where $P_{ab}$ and $\Pi_{ab}$ are orthogonal and complementary projectors with respect to the metric tensor $g_{ab}$. One of the main results of our study is the discovery of a new geometric characterization of conformally separable spaces with conformally flat leaf metrics similar to the vanishing of the Weyl tensor for conformally flat metrics. This geometric characterization seems to carry over to any pseudo-Riemannian manifold admitting conformally flat foliations which would open the door to the systematic searching of these type of foliations in a given pseudo-Riemannian metric. Other relevant aspects such as the existence of invariant tensors under the finite groups generated by these transformations are also addressed.

Key words. Differential Geometry, Symmetry transformations, Differential invariants

1. Introduction

The research of symmetry transformations in Differential Geometry and General Relativity has been an important subject during the years. Here by symmetries we mean a group of transformations of a given pseudo-Riemannian manifold complying with certain geometric property. By far the most studied symmetries are isometries and conformal transformations which are defined through the conditions

\[
\mathcal{L}_\xi g_{ab} = 0, \quad \mathcal{L}_\xi g_{ab} = 2\phi g_{ab},
\]  

(1.1)
where $g_{ab}$ is the metric tensor of the manifold, $\xi$ is the \textit{infinitesimal generator} of the transformation and $\phi$ is a function which we will call \textit{gauge} of the symmetry (this terminology was first employed in [4] and it will be explained later). Infinitesimal generators of these symmetries are known as Killing vectors and conformal Killing vectors. As is very simple to check they are a Lie algebra with respect to the Lie bracket of vector fields and the transformations generated by these vector fields give rise to subgroups of the diffeomorphism group.

Important questions are the possible dimensions of these Lie algebras and the geometric characterizations of spaces admitting the symmetry. The general answer to these questions can in principle be obtained by solving the differential conditions written above although for general enough cases the explicit evaluation of such solutions gets too complex and other methods are required. Notwithstanding these difficulties we can obtain easily from the differential conditions the cases in which the Lie algebras are finite dimensional, the greatest dimension of these finite Lie algebras and geometric characterizations of the spaces admitting these Lie algebras as solutions. This is done by finding the \textit{normal form} of the above equations (if such form exists) and the complete integrability conditions coming from this set of equations. In this way we deduce that isometries are always finite dimensional whereas conformal motions are finite dimensional iff the space dimension is greater or equal than three. The spaces in which the greatest dimension is achieved are constant curvature and conformally flat spaces respectively and as is very well known they are characterized by the geometric conditions

$$R_{abcd}^a = \frac{R}{n(n - 1)}(\delta^a_d g_{bd} - \delta^a_b g_{dc}), \quad C_{abcd}^a = 0, \quad n > 3,$$

where $n$ is the dimension of the manifold, $R_{abcd}^a$ is the curvature tensor, $R$ the scalar curvature and $C_{abcd}^a$ the Weyl tensor$^1$.

The procedure followed for isometries and conformal motions is carried over to other symmetries such as linear and affine collineations and conformal collineations (see [13,9] for a very good account of this). However, little research has been done for symmetries different from these mostly because the cases under consideration were infinite dimensional \textit{generically}. This means that it is not possible to obtain a normal set of equations out of the differential conditions (see section 4) which greatly complicates matters.

In cite [7] we put forward a new symmetry transformation for general pseudo-Riemannian manifolds. Infinitesimal generators of these symmetries (bi-conformal vector fields) fulfill the differential conditions

$$\mathcal{L}_\xi P_{ab} = \phi P_{ab}, \quad \mathcal{L}_\xi \Pi_{ab} = \chi \Pi_{ab}, \quad (1.2)$$

where $P_{ab}$ and $\Pi_{ab}$ are orthogonal and complementary projectors with respect to the metric tensor $g_{ab}$ and $\phi, \chi$ are the gauges of the symmetry. These are functions which, as happened in the conformal case, depend on the vector field $\xi$ so a solution of (1.2) is formed by $\xi$ itself and the gauges $\phi$ and $\chi$ (we will usually omit the dependence on $\xi$ in the gauges). The finite transformations generated by bi-conformal vector fields are called bi-conformal transformations. In a sense, these symmetries can be regarded as conformal transformations with respect to both $P_{ab}$ and $\Pi_{ab}$ so we can expect that

$^1$ In the case of dimension three the Weyl tensor is replaced by a three rank tensor called the Cotton-York tensor.
some properties of these will resemble those of conformal transformations. In [7] it was shown that bi-conformal vector fields comprise a Lie algebra under the Lie bracket and that this algebra is finite dimensional if none of the projectors has algebraic rank one or two being the greatest dimension

\[ N = \frac{1}{2} p(p + 1) + \frac{1}{2} (n - p)(n - p + 1), \]

with \( p \) the algebraic rank of one of the projectors. We provided also explicit examples in which this dimension is achieved, namely bi-conformally flat spaces which in local coordinates \( x = \{ x^1, \ldots, x^n \} \) look like \((\alpha, \beta = 1, \ldots p, A, B = p + 1, \ldots n)\)

\[ ds^2 = \Xi_1(x)\eta_{\alpha\beta}dx^\alpha dx^\beta + \Xi_2(x)\eta_{AB}dx^A dx^B, \quad \Xi_1, \Xi_2 \in C^3, \]

where \( \eta_{\alpha\beta}, \eta_{AB} \) are flat metrics depending only on the coordinates \( x^\alpha \) and \( x^A \) respectively. These spaces play the same role for bi-conformal vector fields as conformally flat spaces or spaces of constant curvature do for the classical symmetries. An open question was the finding of a geometric characterization for bi-conformally flat spaces similar to those of the spaces of constant curvature or conformally flat spaces. In the scheme developed in [7] this sort of characterization could not be extracted due to the complexity of the calculations and it had to be postponed. This matter and other important properties of bi-conformal vector fields are dealt with in the present work.

In this paper we perform the full calculation of the normal form and the first integrability conditions for the equations (1.2). In our previous work we were able to get the normal form for these equations but it turned out to be rather messy and relevant geometric information could not be obtained. This was so because all these calculations were done using the covariant derivatives arising from the metric connection which is not adapted to the calculations. Here we show that the definition of a new symmetric connection greatly simplifies the calculations making it possible to get a simpler form for the normal system and to work out its integrability conditions. In particular a geometric characterization for bi-conformally flat spaces in terms of certain tensors \( T_{a b c d} \) and \( T_{a b c} \) constructed from the projectors \( P_{a b} \) and \( \Pi_{a b} \) is derived. We prove the remarkable result that these both tensors are zero if and only if the space is bi-conformally flat being \( P_{a b} \) and \( \Pi_{a b} \) the conformally flat pieces in which the metric is split up (here neither of the projectors can have algebraic rank three). Along the way we obtain other interesting results such as the geometric characterization of conformally separable pseudo-Riemannian manifolds (in this case this comes through the vanishing of \( T_{a b c} \)) and the existence of geometric invariants similar to the Riemann tensor or the Weyl tensor for isometries or conformal vector fields. We can also give geometric conditions under which a conformally separable pseudo-Riemannian manifold admits conformally flat leaf metrics (see definitions 7.1 and 7.2 for explanations about this terminology). These conditions take the form

\[ P^a P^b P^c P^d T^{r}_{sqt} = 0. \]

In fact these conditions appear to be true in the case of general pseudo-Riemannian manifolds admitting conformally flat slices as we show by means of explicit examples. This is very interesting because it would enable us to systematically search foliations by conformally flat hypersurfaces in general pseudo-Riemannian manifolds (and even decide whether these foliations exist or not).

The outline of the paper is as follows: section 2 introduces the basic notation and definitions. In section 3 we define a new symmetric connection (bi-conformal connection)
and set its main properties. Section 4 presents the calculation of the normal form associated to (1.2) and the calculation of the maximum dimension of any finite dimensional Lie algebra of bi-conformal vector fields is carried out. The first integrability conditions of the equations comprising the normal form are worked out in section 5 and in these calculations a family of invariant tensors under bi-conformal transformations comes up naturally. The complete integrability conditions are the subject of section 6 and they lead to the geometric characterizations of conformally separable and bi-conformally flat pseudo-Riemannian manifolds explained above. These characterizations are spelt out in detail in section 7 where we provide a full account of all the possible cases and we point out that some of them correspond to the complete integrability conditions. Finally in section 8 we show explicit examples of how to use the geometric characterizations found in the paper and we hint how these conditions may be extended to metrics which are not conformally separable.

2. Bi-conformal vector fields and bi-conformal transformations

Let us start by setting our notation and conventions for the paper (they are though rather general and obvious so the paper should present no difficulties in this point). We work on a differentiable manifold $V$ in which a $C^\infty$ metric $g_{ab}$ of arbitrary signature has been defined (pseudo-Riemannian manifold). Vectors and vector fields are denoted with arrowed characters $u, \vec{v}$ (we leave to the context the distinction between each of these entities) when expressed in coordinate-free notation whereas 1-forms are written in bold characters $u$. Sometimes this same notation will be employed for other higher rank objects such as contravariant and covariant tensors. Indexes of tensors are represented by lowercase Latin characters $a, b, \ldots$ and the metric $g_{ab}$ or its inverse $g^{ab}$ are used to respectively raise or lower indexes. Rounded and square brackets are used for symmetrization and antisymmetrization respectively and whenever a group of indexes is enclosed between strokes they are excluded from the symmetrization or antisymmetrization operation. Partial derivatives with respect to local coordinates are $\partial_a \equiv \partial/\partial x^a$.

The metric connection associated to $g_{ab}$ is $\gamma^a_{bc}$ (Ricci rotation coefficients) reserving the symbol $\Gamma^a_{bc}$ only for the Christoffel symbols, namely, the connection components in a natural basis. The covariant derivative and the Riemann tensor constructed from this connection are denoted by $\nabla$ and $R^a_{bcd}$ respectively being our convention for the Riemann tensor

$$R^a_{bcd} \equiv \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^r_{cb} \Gamma^a_{rd} - \Gamma^r_{cd} \Gamma^a_{rb}.$$  

Under this convention Ricci identity becomes

$$\nabla_b \nabla_c u^a - \nabla_c \nabla_b u^a = R^a_{rbc} u^r, \quad \nabla_b \nabla_c u_a - \nabla_c \nabla_b u_a = -R^r_{abc} u_r.$$  

All the above relations are still valid for a non-metric symmetric connection. The push-forward and pull-back of elements of the tensor bundle $T^r_s(V)$ (bundle of $r$-covariant $s$-contravariant tensors) or its sections under a diffeomorphism $\Phi : V \rightarrow V$ are represented by the standard notation $\Phi_* \tilde{T}$ and $\Phi^* \tilde{T}$ respectively. We shall use $^* \Phi^* T$ if $T$ is a tensor with contravariant and covariant indexes.

The set of smooth vector fields of the manifold $V$ is denoted by $\mathfrak{X}(V)$. This is an infinite dimensional Lie algebra which is sometimes regarded as the Lie algebra of the group of diffeomorphisms of the manifold $V$. Finally the Lie derivative operator with respect to a vector field $\xi$ is $\mathcal{L}_\xi$. 

One of the main subjects of this paper is the study of bi-conformal vector fields whose definition given in [7] we recall here.

**Definition 2.1.** A $C^2$ vector field $\xi$ on $V$ is said to be a bi-conformal vector field if it fulfills the condition

$$\mathcal{L}_\xi P_{ab} = \phi P_{ab}, \quad \mathcal{L}_\xi \Pi_{ab} = \chi \Pi_{ab}, \quad (2.1)$$

for some functions $\phi, \chi \in C^2(V)$.

$P_{ab}$ and $\Pi_{ab}$ are elements of the tensor bundle $T^0_2(V)$ such that at each point $x \in V$ they form a pair of orthogonal and complementary projectors with respect to the metric tensor $g_{ab}|_x$ which leads to $P_{ab} + \Pi_{ab} = g_{ab}$, $P_{ap}P^p_b = P_{ab}$, $\Pi_{ap}\Pi^p_b = \Pi_{ab}$ and $P_{ap}\Pi^p_b = 0$. Equation (2.1) can be re-written in a number of equivalent ways if we use the square root of the metric tensor $S_{ab}$ defined in terms of the projectors $P_{ab}$ and $\Pi_{ab}$ [7]

$$S_{ab} \equiv P_{ab} - \Pi_{ab}, \iff P_{ab} = \frac{1}{2}(g_{ab} + S_{ab}), \quad \Pi_{ab} = \frac{1}{2}(g_{ab} - S_{ab}), \implies S_{ap}S^p_b = g_{ab}, \quad (2.2)$$

getting

$$\mathcal{L}_\xi g_{ab} = \alpha g_{ab} + \beta S_{ab}, \quad \mathcal{L}_\xi S_{ab} = \alpha S_{ab} + \beta g_{ab}, \quad \alpha = \frac{1}{2}(\phi + \chi), \quad \beta = \frac{1}{2}(\phi - \chi). \quad (2.3)$$

From equation (2.2) we deduce that both projectors are fixed by the square root $S_{ab}$ so we may use the latter instead of the projectors when working with a given set of bi-conformal vector fields. Following [7] the set of bi-conformal vector fields possessing $S_{ab}$ as the associated square root will be denoted by $\mathcal{G}(S)$. In this paper only expressions involving $P_{ab}$ and $\Pi_{ab}$ will be used in our calculations.

The next set of relations come straight away from (2.1)

$$\mathcal{L}_\xi P^a_b = \mathcal{L}_\xi \Pi^a_b = 0, \quad \mathcal{L}_\xi P^{ab} = -\phi P^{ab}, \quad \mathcal{L}_\xi \Pi^{ab} = -\chi \Pi^{ab}. \quad (2.4)$$

A very important property of $\mathcal{G}(S)$ is that it forms a Lie subalgebra of $\mathfrak{X}(V)$ (proposition 5.2 of [7]) which can be finite or infinite dimensional. Conditions upon the tensor $S_{ab}$ (or equivalently the projectors) for this Lie algebra to be finite dimensional were given in [7] and they will be re-derived in section 4 in a more efficient way. Observe that the functions $\alpha$ and $\beta$ appearing in definition 2.1 (or $\phi$ and $\chi$) do depend on the bi-conformal vector field (this dependence can be dropped if we work with a single bi-conformal vector field but it should be added when working with Lie algebras of bi-conformal vector fields). In the latter case $\alpha$ and $\beta$ are called gauge functions (see [4] for an explanation of this terminology).

The finite counterpart of bi-conformal vector fields (bi-conformal transformations) was also presented in section 4 of [7] but for the sake of completeness we repeat again the definitions here referring the interested reader to that paper for further details.

**Definition 2.2.** Under the assumptions of definition 2.1 a diffeomorphism $\Phi : V \to V$ is called a bi-conformal transformation if the conditions $\Phi^* P_{ab} = \alpha_1 P_{ab}$, $\Phi^* \Pi_{ab} = \alpha_2 \Pi_{ab}$ are satisfied for some functions $\alpha_1, \alpha_2 \in C^0(V)$.

In this paper the majority of our calculations will only involve bi-conformal vector fields and we will draw our results from infinitesimal equations instead of expressions arising from bi-conformal relations.
3. The bi-conformal connection

As we have commented in the introduction the ordinary metric connection $\gamma^a_{bc}$ defined in the standard way from the metric is not suitable to study the normal form and the integrability conditions coming from the differential condition (2.1) as they result in rather cumbersome expressions. In order to proceed further in our study we are going to show next that the definition of a new symmetric connection greatly simplifies the normal form calculated in [7] and what is more, it allows us to work out thoroughly the complete integrability conditions arising from this normal form.

To start with we recall some identities satisfied by any bi-conformal vector field $\xi$ which were obtained in [7]. These identities are in fact linear combinations of the first covariant derivative of (2.1) and we also indicate briefly how they are obtained as this information will be needed later. Using equation (A.4) we easily obtain the Lie derivative of the metric connection $\gamma^a_{bc}$ ($\phi_b \equiv \partial_b \phi, \chi_b \equiv \partial_b \chi$)

$$\mathcal{L}_{\xi} \gamma^a_{bc} = \frac{1}{2} (\phi_b P^a_{\ c} + \phi_c P^a_{\ b} - \phi^a P_{bc} + \chi_b \Pi^a_{\ c} + \chi_c \Pi^a_{\ b} - \chi^a \Pi_{bc} + (\phi - \chi) M^a_{bc}),$$  \quad (3.1)

where the tensor $M_{abc}$ is defined by

$$M_{abc} \equiv \nabla_b P_{ac} + \nabla_c P_{ab} - \nabla_a P_{bc}. \quad (3.2)$$

The Lie derivative of $M_{abc}$ can be worked out by means of (A.2) getting

$$\mathcal{L}_{\xi} M_{abc} = \phi M_{abc} + (\chi - \phi) P_{ap} M^p_{\ bc} - P_{bc} \Pi_{ap} \phi^p + \Pi_{cb} P_{ap} \chi^p =$$

$$= \chi M_{abc} + (\phi - \chi) \Pi_{ap} M^p_{\ bc} - P_{bc} \Pi_{ap} \phi^p + \Pi_{cb} P_{ap} \chi^p, \quad (3.3)$$

from which, projecting down with $P_{bc}$ and $\Pi_{bc}$, we deduce

$$\mathcal{L}_{\xi} E_a = -p \Pi_{ap} \phi^p, \quad \mathcal{L}_{\xi} W_a = (p - n) P_{ap} \chi^p, \quad (3.4)$$

with the definitions

$$E_a \equiv M_{acb} P^{cb}, \quad W_a \equiv -M_{acb} \Pi^{cb}, \quad p = P^a_{a}. \quad (3.5)$$

The following algebraic properties of the tensors $E_a$ and $W_a$ are useful

$$\Pi_{ac} E^c = E_a, \quad P_{ac} W^c = W_a, \quad 0 = P^{ab} E_b = \Pi^{ab} W_b. \quad (3.6)$$

Now, we plug (3.4) into (3.3) yielding

$$\mathcal{L}_{\xi} \left( M_{abc} - \frac{1}{p} E_a P_{bc} + \frac{1}{n - p} W_a \Pi_{cb} \right) = \phi \left( \Pi_{ap} M^p_{\ cb} - \frac{E_a P_{bc}}{p} \right) +$$

$$+ \chi \left( P_{ap} M^p_{\ bc} + \frac{\Pi_{cb} W_a}{n - p} \right). \quad (3.6)$$

This equation can be written in a more compact form

$$\mathcal{L}_{\xi} T_{abc} = (\phi \Pi_{ar} + \chi P_{ar}) T^r_{\ bc} = \phi B_{abc} + \chi A_{abc}, \quad (3.7)$$
where the definitions of the tensors $T_{abc}, A_{abc}, B_{abc}$ are

$$T_{abc} \equiv M_{abc} + \frac{1}{n-p} W_a \Pi_{bc} - \frac{1}{p} E_a P_{bc}, \quad (3.8)$$

$$A_{abc} \equiv P_a^d T_{dbc} = P_a^d M_{dbc} + \frac{1}{n-p} W_a \Pi_{cb}, \quad (3.9)$$

$$B_{abc} \equiv \Pi_a^d T_{dbc} = \Pi_a^d M_{dbc} - \frac{1}{p} E_a P_{cb}. \quad (3.10)$$

Using equation (3.7) we can calculate the Lie derivatives of $A_{bc}^a$ and $B_{bc}^a$

$$\mathcal{L}_\xi A_{bc}^a = (\chi - \phi) A_{bc}^a, \quad \mathcal{L}_\xi B_{bc}^a = (\phi - \chi) B_{bc}^a, \quad (3.11)$$

a relation which shall be used later. See [7] for further properties of these tensors.

Let us now use all this information to write the Lie derivative of the connection in a convenient way. Note that in equation (3.3) $\phi_a$ and $\chi_a$ appear projected with $\Pi_a^b$ and $\Pi^a_b$ respectively suggesting that it could be interesting to write any derivative of $\phi$ and $\chi$ decomposed in longitudinal and transverse parts

$$\phi_a^a \equiv \Pi_{ab}\phi^b, \quad \tilde{\phi}_a \equiv P_{ab}\phi^b, \quad \chi_a^a \equiv \Pi_{ab}\chi^b, \quad \bar{\chi}_a \equiv \Pi_{ab}\chi^b. \quad (3.12)$$

If we perform this decomposition in equation (3.1) and replace the longitudinal terms $\phi_a^a$ and $\chi_a^a$ by means of (3.4) we get the relation

$$\mathcal{L}_\xi \left( \gamma_{bc}^a + \frac{1}{2p} (E_b P_c^a + E_c P_b^a - P_{bc} E^a) + \frac{1}{2(n-p)} (W_b \Pi^a_c + W_c \Pi^a_b - W^a \Pi_{cb}) \right) =$$

$$= \frac{1}{2} (\tilde{\phi}_b P_c^a + \tilde{\phi}_c P_b^a) + \bar{\chi}_b \Pi^a_c + \bar{\chi}_c \Pi^a_b - \tilde{\phi}_a T_{bc}^a, \quad (3.13)$$

but from (3.11), (3.9) and (3.10) we easily deduce

$$\mathcal{L}_\xi (A_{bc}^a - B_{bc}^a) = (\chi - \phi) T_{bc}^a,$$

hence equation (3.13) becomes, after some simplifications

$$2 \mathcal{L}_\xi \left( \gamma_{bc}^a + \frac{1}{2p} (E_b P_c^a + E_c P_b^a) + \frac{1}{2(n-p)} (W_b \Pi^a_c + W_c \Pi^a_b) + \frac{1}{2} (P_a^p - \Pi^a_p) M_{cb}^p \right) =$$

$$= \tilde{\phi}_b P_c^a + \tilde{\phi}_c P_b^a - \tilde{\phi}_a P_{bc} + \bar{\chi}_b \Pi^a_c + \bar{\chi}_c \Pi^a_b - \tilde{\phi}_a T_{bc}^a. \quad (3.14)$$

The geometric object inside the Lie derivative, denoted by $\gamma_{bc}^a$, is the sum of the metric connection $\gamma_{bc}^a$ plus the rank-3 tensor

$$L_{bc}^a \equiv \frac{1}{2p} (E_b P_a^c + E_c P_b^a) + \frac{1}{2(n-p)} (W_b \Pi^a_c + W_c \Pi^a_b) + \frac{1}{2} (P_a^p - \Pi^a_p) M_{bc}^p, \quad (3.15)$$

so it is clear that it represents a new linear connection. As we will see during our calculations this linear connection is fully adapted to the calculations involving bi-conformal vector fields and it will be extensively used in this paper.

**Definition 3.1 (bi-conformal connection).** The connection whose components are given by $\gamma_{bc}^a$ is called bi-conformal connection. The covariant derivative constructed from the bi-conformal connection shall be denoted by $\nabla$ and the curvature tensor constructed from it by $R_{abcd}^a$. 
Since $L^a_{bc}$ is symmetric in the indexes $bc$, we see that the bi-conformal connection is symmetric so it has no torsion and all the identities involving only the covariant derivative $\nabla$ or the curvature $\bar{R}^a_{bcd}$ remain the same as for the case of a metric connection. However, this connection does not in general stem from a metric tensor as we will see later in explicit examples. This means that certain symmetries of the curvature tensor of a metric connection are absent for $\bar{R}^a_{bcd}$. We recall that for a symmetric connection the Riemann tensor is only antisymmetric in the last pair of indexes. Bianchi identities however, remain the same as for the case of a metric connection.

It is not very difficult to derive an identity relating the curvature tensor calculated from the bi-conformal connection and the curvature tensor associated to the connection $\gamma^a_{bc}$

$$\bar{R}^a_{bcd} = R^a_{bcd} + 2\nabla_{[c}L^a_{d]b} + 2L^a_{[c}L^r_{d]b}.$$  \hspace{1cm} (3.16)

being this a thoroughly general identity for two symmetric connections $\gamma^a_{bc}$ and $\gamma^a_{bc}$ differing in a tensor $L^a_{bc}$ ([11], p. 141).

The relation between the covariant derivatives $\nabla$ and $\nabla$ acting on any tensor $X^{a_1\ldots a_r}_{b_1\ldots b_q}$ is

$$\nabla_a X^{a_1\ldots a_r}_{b_1\ldots b_q} = \nabla_a X^{a_1\ldots a_r}_{b_1\ldots b_q} + \sum_{s=1}^r L^a_{ac} X^{a_1\ldots a_{s-1}a_{s+1}\ldots a_r}_{b_1\ldots b_q} -$$

$$- \sum_{s=1}^q L^c_{ab_s} X^{a_1\ldots a_r}_{b_1\ldots b_{s-1}c_{b_{s+1}}\ldots b_q},$$  \hspace{1cm} (3.17)

which again has general validity for two symmetric connections whose difference is a tensor $L^a_{bc}$ [5]. As a first application of this identity we may compare the covariant derivatives of the tensor $L^a_{bc}$ which leads us to the identity

$$\nabla_{[a}L^b_{c]} = \nabla_{[a}L^b_{c]} + 2L^b_{[a}L^r_{c]d},$$

from which we can rewrite (3.16) in terms of $\nabla$

$$\bar{R}^a_{bcd} = R^a_{bcd} - 2\nabla_{[c}L^a_{d]b} + 2L^a_{[c}L^r_{d]b}.$$  \hspace{1cm} (3.18)

Of course this last equation could have been obtained from (3.16) by means of the replacements $L^a_{bc} \rightarrow -L^a_{bc}$ and $\nabla_a \rightarrow -\nabla_a$.

**Example 3.1.** To realize the importance of bi-conformal connection in future calculations, let us calculate its components for a *conformally separable* pseudo-Riemannian manifold (see definition 7.2) given in local coordinates $x \equiv \{x^1, \ldots, x^n\}$ by

$$ds^2 = \Xi_1(x)G_{\alpha\beta}(x^k)dx^\alpha dx^\beta + \Xi_2(x)G_{AB}(x^C)dx^A dx^B.$$  \hspace{1cm} (3.19)

Here Greek indexes range from 1 to $p$ and uppercase Latin indexes from $p+1$ to $n$ so the metric tensors $G_{\alpha\beta}$ and $G_{AB}$ are of rank $p$ and $n-p$ respectively. The non-zero Christoffel symbols for this metric are

$$\Gamma^\gamma_{\beta\gamma} = \frac{1}{2\Xi_1}G^{\alpha\beta}(\partial_\beta(\Xi_1G_{\alpha\rho}) + \partial_\gamma(\Xi_1G_{\rho\beta}) - \partial_\rho(\Xi_1G_{\beta\gamma})), $$

$$\Gamma^A_{BC} = \frac{1}{2\Xi_1}G^{AD}(\partial_B(\Xi_1G_{CD}) + \partial_C(\Xi_1G_{DB}) - \partial_D(\Xi_1G_{BC})), $$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2\Xi_1}S^\alpha_\beta \partial_\gamma \Xi_1, \text{ } \Gamma^A_{B\alpha} = \frac{1}{2\Xi_2}S^A_B \partial_\alpha \Xi_2.$$


from which the only nonvanishing components of $M_{abc}, E_a, W_a$ are
\[
M_{\alpha AB} = \partial_\alpha (\Xi_2 G_{AB}), \quad M_{\alpha \beta} = -\partial_\alpha (\Xi_1 G_{\alpha \beta}),
\]
\[
E_A = -\partial_A \log |\det(\Xi_\alpha^n G_{AB})|, \quad W_a = -\partial_a \log |\det(\Xi_\alpha^n G_{AB})|.
\]
(3.20)

Therefore we get for the components of the bi-conformal connection
\[
\Gamma_{\beta \phi}^\alpha = \frac{1}{2 \Xi_1} (\delta^\alpha_\beta \partial_\phi \Xi_1 + \delta^\alpha_\phi \partial_\beta \Xi_1 - G^{\alpha \phi} G_{\beta \phi} \partial_\rho \Xi_1) + \Gamma_{\beta \phi}^\alpha (G),
\]
\[
\Gamma_{BC}^A = \frac{1}{2 \Xi_2} (\delta^A_B \partial_C \Xi_2 + \delta^A_C \partial_B \Xi_2 - G^{AB} G_{BC} \partial_R \Xi_2) + \Gamma_{BC}^A (G)
\]
\[
\Gamma_{\beta C}^\alpha = \Gamma_{\beta C}^A = 0,
\]
(3.21)

where $\Gamma_{\beta \phi}^\alpha (G)$ and $\Gamma_{BC}^A (G)$ are the Christoffel symbols of the metrics $G_{\alpha \beta}$ and $G_{AB}$ respectively. From the above formulae we deduce that the bi-conformal connection is fully adapted to conformally separable pseudo-Riemannian manifolds because its components clearly split in two parts being each of them the Christoffel symbols of the metrics $G_{\alpha \beta}, G_{AB}$ plus terms involving the derivatives of the factors $\Xi_1$ and $\Xi_2$. We will take advantage of this property in section 7 where we will provide invariant characterizations of conformally separable and bi-conformally flat pseudo-Riemannian manifolds.

We calculate next the covariant derivative with respect to the bi-conformal connection of a number of tensors.

**Proposition 3.1.** The following covariant identities hold true
\[
\nabla_a P_{bc} = \nabla_a P_{bc} - \frac{1}{p} E_a P_{bc} - \frac{1}{2p} (E_b P_{ac} + E_c P_{ab}) - \frac{1}{2} (P_{cp} M_{ab}^p + P_{bp} M_{ac}^p),
\]
(3.22)

\[
2 \nabla_a P_{bc} = 2 \nabla_a P_{bc} + P_{bq} P_{cr} M_{qra} - P_{bq} P_{cr} M_{qra} - P_{bq} M_{ac}^q + \frac{1}{n-p} W_c P_{bc} - \frac{1}{p} E_c P_{bc},
\]
(3.23)

\[
\nabla_a P_{bc} = \nabla_a P_{bc} + \frac{1}{p} E_a P_{bc} + \frac{1}{2(n-p)} (W_c P_{bc} + W^b P_{ca} - \frac{1}{2} (M_{ar} P_{rb} + M_{cr} P_{rb}),
\]
(3.24)

and all the identities formed with the replacements $P_{ab} \rightarrow \Pi_{ab}$. $p \rightarrow n-p$.

**Proof.** All these identities are proven by means of (3.17) and the use of properties (3.5).

Using the above properties we can get more interesting identities to be used later on.
\[
\nabla_a P^{ab} = \nabla_a \Pi^{ab} = \nabla_a P_{ab} = \nabla_a \Pi^a_b = 0,
\]
(3.25)

\[
P_{bc} \nabla_a P_{bc} = -E_a, \quad P_{bc} \nabla_a P_{bc} = E_a,
\]
(3.26)

\[
\Pi_{bc} \nabla_a \Pi_{bc} = -W_a, \quad \Pi_{bc} \nabla_a \Pi_{bc} = W_a.
\]
(3.27)

\[
P^d_r \nabla_b \Pi^a_d = \Pi^{d}_r \nabla_b P^a_r = P^d_r \nabla_a P_{rb} = \Pi^{d}_r \nabla_a \Pi_{rb} = 0.
\]
(3.28)

Note that index raising and lowering do not commute with $\nabla$ so we must be very careful when we raise or lower indexes in tensor expressions involving $\nabla$.\]
4. Normal form and dimension of maximal Lie algebras of bi-conformal fields

We turn now our attention to the calculation of the full normal form coming from the differential conditions (2.1). A detailed explanation of the general procedure and relevance of this calculation for a general symmetry can be found in [6, 7] (see also [13] for the calculation in the cases of the most studied symmetries in General Relativity such as isometries and conformal motions). Before starting the calculation and for the sake of completeness let us give a very brief sketch of the whole procedure. We must differentiate the condition (2.1) a number of times in such a way that we get enough equations to isolate the derivatives of certain variables (system variables) in terms of themselves (this is achieved typically by means of the resolution of a linear system of equations). The so obtained derivatives give rise to the normal form associated to our symmetry. Along the differentiating process one may obtain equations whose linear combinations no longer contain derivatives of the system variables (constraints). Examples of such constraints in our case are the differential conditions themselves and (3.4) (actually these are the only constraints as we will show in §4.2).

As we will see it is possible to meet cases in which the normal form cannot be achieved. This means that one cannot obtain enough equations to isolate all the derivatives obtained through the derivation process. The main implication of this is that the Lie algebra of vector fields fulfilling the starting differential condition is infinite dimensional as opposed to the case in which there is such a normal form. Therefore the calculation of the normal form allows us to tell apart the cases with an infinite dimension as opposed to the case in which there is such a normal form. The so obtained equations give rise to the normal form associated to our symmetry. Along the differentiating process one may obtain equations whose linear combinations no longer contain derivatives of the system variables (constraints). Examples of such constraints in our case are the differential conditions themselves and (3.4) (actually these are the only constraints as we will show in §4.2).

We start now our calculation with the substitution of (3.14) into (A.1) which yields

\[ \nabla_b \Psi_c^a + \xi^d \bar{R}^a_{cd} = \frac{1}{2} \left( \phi_b P^a_c + \phi_c P^a_b - \phi^a P_{cb} + \bar{\chi}_b \Pi^a_c + \bar{\chi}_c \Pi^a_b - \bar{\chi}^a \Pi_{cb} \right), \Psi^a_c = \nabla_c \xi^a. \]

Next we replace in (A.3) the Lie derivatives of the bi-conformal connection by their expressions given by (3.14) getting

\[ \mathcal{L}_{\xi} \bar{R}^a_{cd} = \nabla_c \bar{\phi}_b P^a_c + P^d_c \nabla_b \bar{\phi}_d - P^d_c \bar{\chi}_b \Pi^a_d + \Pi^d_c \bar{\chi}_d \bar{\phi}_c - \phi^a \nabla_c \bar{\phi}_b - \bar{\phi}^a \nabla_c \bar{\chi}_b + \bar{\chi}_b \nabla_c \bar{\phi}_b - \bar{\chi}_c \nabla_c \bar{\chi}_b. \]

The game is now to isolate from this expression \( \nabla_a \bar{\phi}_b \) and \( \nabla_a \bar{\chi}_b \) (these rank-2 tensors are not symmetric in general). The forthcoming calculations split in two groups which are dual under the interchange \( P_{ab} \leftrightarrow \Pi_{ab} \), \( p \leftrightarrow n - p \) (only the calculations for \( \nabla_a \bar{\phi}_b \) are performed). Multiplying (4.2) by \( P^a_r \), we obtain

\[ \mathcal{L}_{\xi} \left( P^d_r \bar{R}^r_{cad} \right) = P^d_c \nabla_c \bar{\phi}_b + P^d_c \bar{\chi}_b \Pi^a_d - P^d_p \bar{\chi}_b \Pi^a_d + P^d_r \Pi_{rb} \nabla_c \bar{\phi}_b + \bar{\chi}_b \Pi_{rb} \nabla_c \bar{\phi}_b - \bar{\chi}_c \Pi_{rb} \nabla_c \bar{\phi}_b. \]

Contraction of the indexes \( d-c \) in the above expression yields

\[ \nabla_a \bar{\phi}_b = \nabla_b \bar{\phi}_a + \frac{2}{p} \mathcal{L}_{\xi} \left( P^d_r \bar{R}^r_{dac} \right), \]
while the contraction of indexes $d - a$ and use of identities (3.25)-(3.28) entails

$$2\mathcal{L}_\xi (P^d_r \tilde{R}^r_{cd}) = P^d_c \tilde{\nabla} \phi + P^d_b \tilde{\nabla} P_{bc} - \tilde{\nabla}_a \phi^a P_{bc} - P \tilde{\nabla} \phi.$$  

Equations (4.4) and (4.5) can now be combined in a single expression which is

$$2\mathcal{L}_\xi \left[ P^d_r \tilde{R}^r_{cd} - \frac{1}{p} (P^d_c P^q_r \tilde{R}^q_{rdb} + P^d_b P^q_r \tilde{R}^q_{rdb} - P^d_c \tilde{R}^q_{rbc}) \right] = (2 - p) \tilde{\nabla} \phi - \tilde{\nabla} \phi^a P_{bc} - \phi_d (\tilde{\nabla}_b \phi^b + \tilde{\nabla}_c \phi^c + P^{r} \tilde{\nabla}_r P_{bc}).$$  

Multiplying here with $P^{cd}$ leads, after a little bit of algebra, to

$$\tilde{\nabla} \phi^a = \frac{1}{1 - p} (\mathcal{L}_\xi \phi^0 + \phi^0), \quad p \neq 1, \quad \tilde{R}^0 = P^d_r \tilde{R}^r_{cd} P^{cd}, \quad \tilde{\nabla} \phi^a = \frac{1}{1 - p} (\mathcal{L}_\xi \phi^0 + \phi^0).$$  

On the other hand (4.6) can be further simplified if we take into account the identity

$$\tilde{\nabla}_b P^d_c + \tilde{\nabla}_c P^d_b - P^d_r \tilde{\nabla}_r P_{bc} = \Pi^d_{abc} \tilde{\nabla}_r P_{bc} + \frac{1}{2} (\Pi^{dq} \Pi^r_{c} M_{qr} + \Pi^{dq} \Pi^r_{b} M_{qr} c) + \frac{1}{2 (n - p)} (W \Pi^d_c \Pi^d_c + W \Pi^d_c),$$

which is easily derived by writing all the covariant derivatives with respect to the bi-conformal connection of the projectors in terms of ordinary covariant derivatives (proposition 3.1). So plugging (4.7) into (4.6) and using previous identity we get

$$(2 - p) \tilde{\nabla} \phi = \mathcal{L}_{\xi} L^0_{bc} + 2 \tilde{\nabla} \phi^a P_{bc},$$

where

$$L^0_{bc} = 2 \left[ P^d_r \tilde{R}^r_{cd} - \frac{1}{p} (P^d_c P^q_r \tilde{R}^q_{rdb} + P^d_b P^q_r \tilde{R}^q_{rdb} - P^d_c \tilde{R}^q_{rbc}) \right] + \frac{\tilde{R}^0}{1 - p} P_{bc}. \quad \text{(4.9)}$$

The duals of (4.8) and (4.9) are

$$(2 - n + p) \tilde{\nabla} \phi = \mathcal{L}_{\xi} L^1_{bc} + 2 \tilde{\nabla} \phi^a P_{bc},$$

and

$$L^1_{bc} = 2 \left[ \Pi^d_r \tilde{R}^r_{cd} - \frac{1}{n - p} (\Pi^d_c \Pi^q_r \tilde{R}^q_{rdb} + \Pi^d_b \Pi^q_r \tilde{R}^q_{rdb} - \Pi^r_q \tilde{R}^q_{rbc}) \right] + \frac{\tilde{R}^1}{1 - n + p} \Pi_{bc}, \quad \tilde{R}^1 \equiv \Pi^d_r \tilde{R}^r_{cd} \Pi^{cd}. \quad \text{(4.11)}$$

To complete the normal form we need now the derivatives of $\phi^a$ and $\chi^a$ which are obtained through the differentiation of (3.4) (identity (A.2) must be used to get these derivatives)

$$-p \tilde{\nabla} \phi^a = \mathcal{L}_{\xi} (\tilde{\nabla}_b E_a) + \frac{1}{2} (\tilde{\nabla}_b E_a + \tilde{\nabla}_a E_b - (\tilde{\nabla} \phi^r) E_r) \Pi_{ab} \quad \text{(4.12)}$$

and

$$(p - n - 1) \tilde{\nabla}_b \phi^a = \mathcal{L}_{\xi} (\tilde{\nabla}_b W_a) + \frac{1}{2} (\tilde{\nabla}_b W_a + \tilde{\nabla}_a W_b - (\tilde{\nabla} \phi^r) P_{ab}), \quad \text{(4.13)}$$
4.1. Normal form of the differential conditions. The above calculations give us the sought normal form for the differential conditions (2.1) being these gathered in the following set of equations

\[(a) \ \nabla_a \phi = \phi_a + \phi_a^*; \ \nabla_a \chi = \bar{\chi}_a + \chi_a^*,\]
\[(b) \ \nabla_b \phi_a^* = -\frac{1}{p} \left[ \mathcal{L}_{\bar{\xi}} (\nabla_b E_a) + \frac{1}{2} (\bar{\chi}_b E_a + \bar{\chi}_a E_b - (\bar{\chi}^r E_r) P_{ab}) \right],\]
\[(c) \ \nabla_b \chi_a^* = \frac{1}{p - n} \left[ \mathcal{L}_{\bar{\xi}} (\nabla_b W_a) + \frac{1}{2} (\bar{\phi}_b W_a + \bar{\phi}_a W_b - (\bar{\phi}^r W_r) P_{ab}) \right],\]
\[(d) \ \nabla_b \bar{\phi}_c = \frac{1}{2 - p} \left[ \mathcal{L}_{\bar{\xi}} L_{bc}^0 + 2 \bar{\phi}^r \nabla_r P_{bc} \right], \quad (4.14)\]
\[(e) \ \nabla_b \bar{\chi}_c = \frac{1}{2 - n + p} \left[ \mathcal{L}_{\bar{\xi}} L_{bc}^1 + 2 \bar{\chi}^r \nabla_r P_{bc} \right],\]
\[(f) \ \nabla_b \xi^a = \bar{\psi}_b^a,\]
\[(g) \ \bar{\nabla}_b \bar{\psi}_c^a = \frac{1}{2} (\bar{\phi}_b P_{c}^a + \bar{\phi}_c P_{b}^a - \bar{\phi}^a P_{cb} + \bar{\chi}_b \Pi_{c}^a + \bar{\chi}_c \Pi_{b}^a - \bar{\chi}^a \Pi_{cb} - \xi^d \bar{R}_{cda}).\]

A first glance at these equations reveals us that this normal form does not always exist. To be precise if either \(p = 2\) or \(p = n - 2\) the derivatives \(\nabla_a \phi_b\) and \(\nabla_a \bar{\chi}_b\) cannot be isolated and the system cannot be “closed” (in fact these derivatives cannot be isolated even if we perform further derivatives of any of the above equations). We must also remember at this point that the tensors \(L_{ab}^0\) and \(L_{ab}^1\) are well-defined unless \(p = 1\) and \(p = n - 1\) respectively (see equations (4.9) and (4.11)). Therefore we have proven the following theorem (see proposition 6.2 of [7])

**Theorem 4.1.** The only cases in which the Lie algebra \(\mathcal{G}(S)\) can be infinite dimensional occur if and only if \(p = 1, p = 2, p = n - 1, p = n - 2\).

The result of this theorem is intuitively clear if we realize that bi-conformal vector fields are somehow conformal motions for the projectors \(P_{ab}\) and \(\Pi_{ab}\). Therefore if any of them projects onto one or two dimensional vector spaces the associated Lie algebras may turn out to be infinite dimensional as we have just found.

Equations (4.14) can fail to exist in the form written above if some of the derivatives involved vanish. This happens for instance if part of the gauge functions are constants or their second covariant derivatives with respect to the bi-conformal connection are zero. In all this work we will assume the conditions for all the derivatives involved in (4.14) to exist leaving the study of any other cases for a forthcoming publication. Under these hypotheses bi-conformal vector fields turn out to be smooth vector fields.

**Proposition 4.1.** Let \(\bar{\xi}\) be a bi-conformal vector field at least \(C^2\) in a neighbourhood \(U_x\) of a point \(x\) belonging to a manifold \(V\) with a \(C^\infty\) metric tensor. If \(\nabla_b \phi_a, \nabla_b \bar{\chi}_a\) are not identically zero on \(U_x\) then \(\bar{\xi} \in C^\infty(U_x)\).

**Proof.** To prove this result it is enough to show that the covariant derivatives of \(\bar{\xi}\) with respect to the bi-conformal connection exist at any order. The first and second derivatives of \(\bar{\xi}\) are equations (4.14)-f and (4.14)-g and higher derivatives are calculated from this last equation. When we derive (4.14)-g we need \(\nabla_b \phi_c\) and \(\nabla_b \bar{\chi}_c\) which by assumption are not zero on \(U_x\) and these are obtained through equations (4.14)-d and
(4.14)-e in which only derivatives of \( \tilde{\xi}, \tilde{\phi}_a \) and \( \tilde{\chi}_a \) of order less or equal than one appear. This makes clear that no other equation of (4.14) but the ones mentioned so far are involved in the calculation of the derivatives of \( \xi \) and so we can obtain them in as high order as we wish.  

Related to this is the following result.

**Proposition 4.2.** Under the hypotheses of previous proposition if a bi-conformal vector field \( \xi \) is such that \( \xi^a|_x = 0, \nabla_b \xi^a|_x = 0, \nabla_c \nabla_b \xi^a|_x = 0 \) then \( \tilde{\xi} \equiv 0 \) in a neighbourhood of \( x \).

**Proof.** Evaluation of the last equation of (4.14) at \( x \) entails

\[
(\tilde{\phi}_b P^a_c + \tilde{\phi}_c P^a_b - \tilde{\phi}^a P_{cb} + \tilde{\chi}_b \Pi^a_c + \tilde{\chi}_c \Pi^a_b - \tilde{\chi}^a \Pi_{cb})|_x = 0,
\]

from which, projecting down with \( \Pi^{ab} \) and \( \Pi^{ab} \), we deduce that \( \tilde{\phi}_a|_x = \tilde{\chi}_a|_x = 0 \). Now let \( \gamma(t) \) be a smooth curve on \( V \) such that \( \gamma(0) = x \) with \( \gamma(t) \) lying in a coordinate neighbourhood of \( x \) for all \( t \) in the interval \( (-\epsilon, \epsilon) \). If we denote by \( \tilde{\gamma}^a(t) \) the tangent vector to this curve we may define the derivatives

\[
\frac{D\tilde{\phi}_a}{dt} \equiv \tilde{\gamma}^r \nabla_r \tilde{\phi}_a, \quad \frac{D\tilde{\chi}_a}{dt} \equiv \tilde{\gamma}^r \nabla_r \tilde{\chi}_a, \quad \frac{D\tilde{\xi}^a}{dt} \equiv \tilde{\gamma}^r \nabla_r \tilde{\xi}^a, \quad \frac{D\tilde{\psi}_a^c}{dt} \equiv \tilde{\gamma}^r \nabla_r \tilde{\psi}_a^c,
\]

where all quantities are evaluated on \( \gamma(t) \). Contracting (4.14)-d-(4.14)-g with \( \tilde{\gamma}^b \) we can transform these equations into a first order ODE system in the variables \( \tilde{\phi}_a(\gamma(t)), \tilde{\chi}_a(\gamma(t)), \xi^a(\gamma(t)) \) and \( \tilde{\psi}_a^b(\gamma(t)) \). From the above all these variables vanish at \( t = 0 \) so according to the standard theorem of uniqueness for ODE systems the variables are identically zero along the curve \( \gamma(t) \) (and in particular \( \tilde{\xi}|_{\gamma(t)} = 0 \)). As \( \gamma(t) \) was chosen arbitrarily we conclude that \( \tilde{\xi} \equiv 0 \) in a whole neighbourhood of \( x \).  

From the calculations performed in this proof and in the proof of proposition 4.1 we deduce as a simple corollary that \( \nabla^{(m)} \xi^a|_x = 0 \) for all \( m \in \mathbb{N} \) if it holds for \( m = 1, 2 \).

### 4.2. Constraints.

From the above calculation of the normal form, the system variables are read off at once. These are the variables appearing under derivation in the left hand side of (4.14). However, they are not algebraically independent because in the calculation process of (4.14) some of the equations involved do not contain derivatives of the variables at all (system constraints). The most evident case of these constraints are the differential conditions (2.1) themselves. If we review the whole procedure followed to get (4.14) we deduce that the other set of constraints between the system variables is (3.4) so (4.14) must be complemented with

\[
(I) \quad \mathcal{L}_\xi P_{ab} = \phi P_{ab}, \quad (II) \quad \mathcal{L}_\xi \Pi_{ab} = \chi \Pi_{ab}.
\]

In order to clarify that these two sets of equations are truly the constraints associated to (4.14) we must show that they arise as a linear combination of some of the higher covariant derivatives of (2.1) employed to get the normal form (either with respect to \( \nabla \) or \( \nabla \)). Such higher derivatives are (3.3), (4.1) which are a linear combination of the
first derivative of (2.1) (they are constructed through the Lie derivatives of the connections \(\gamma^a_{bc}\) and \(\bar{\gamma}^a_{bc}\)) and (A.3) which is a linear combination of the first derivative of the bi-conformal connection and hence the second derivative of (2.1) (see equations (A.3), (A.4) of appendix A). In the case of (3.3) only its projections with \(P^{ab}\) and \(\Pi^{ab}\) (equation 3.4) really matter to work out the normal form and these are (4.15)-II. As for the other derivatives they do not give rise to any more equations with no derivatives of the system variables so the above equations are the only constraints we must care about.

A first application of all the above calculations comes in the following result, already proven in [7] using a normal form system written in terms of different variables.

**Theorem 4.2.** If the Lie algebra \(G(S)\) is finite dimensional then its dimension is bounded from above by \(N = p(p + 1)/2 + (n - p)(n - p + 1)/2\).

**Proof.** To prove this theorem we must state what the upper bound to the maximum number of integration constants for the system (4.14) is. As is very well known from the theory of normal systems of P.D.E’s (see e. g. [6]) such number is the number of system variables minus the number of linearly independent constraints. The following table summarises these numbers for our system.

| system variables | Constraints |
|------------------|-------------|
| \(\phi, \chi\)   | eq. (4.15)-I |
| \(\phi^a, \phi_a\) | eq. (4.15)-II |
| \(\chi^a, \chi_a\) |
| \(\xi^a\)        |
| \(\psi^b, \psi_a\) |

We have written explicitly the system variables and the total number for each of them. The constraints are also indicated together with how many linearly independent equations each constraint amounts to. This last part is not evident as opposed to the counting for the system variables so the rest of the proof is devoted to show that the numbers given in table 4.1 for the constraint equations are indeed correct.

**Equations (4.15)-I.** First of all, we expand the Lie derivatives of these equations

\[
\xi^c \nabla_c P_{ab} + \psi^c_p (\delta^{a}_p P_{bc} + \delta^{b}_p P_{ac}) = \phi P_{ab}, \quad \xi^c \nabla_c \Pi_{ab} + \psi^c_p (\delta^{a}_p \Pi_{bc} + \delta^{b}_p \Pi_{ac}) = \chi \Pi_{ab},
\]

where the standard definition of the Lie derivative of a tensor \(P_{ab}\) has been applied

\[
\mathcal{L}_\xi P_{ab} \equiv \xi^c \partial_c P_{ab} + \partial_a \xi^c P_{cb} + \partial_b \xi^c P_{ac},
\]

(observe that the general formula of the Lie derivative with respect to a vector in terms of its covariant derivatives still holds under an symmetric connection). We define the new indexes \(A, B, B'\) in such a way that

\[
P_A \equiv P_{ab}, \quad \Psi_B \equiv \nabla_p \xi^q, \quad \xi^{B'} = \xi^c,
\]

so capital indexes group together certain combinations of small indexes (explicitly \(A = \{a, b\}, B = \{p, q\}\) and \(B' = c\). The ranges of the new indexes are \(A = 1, \ldots, n(n + 1)/2\), \(B = 1, \ldots, n^2\), \(B' = 1, \ldots, n\) (note that the above expressions are symmetric in
Using these new labels we can write in matrix notation the homogeneous system posed by these constraints (we only concentrate in the first of (4.16))

\[
(M^B_A (\bar{\nabla} P)_{AB'} P_A) \begin{pmatrix} \psi_{B'} \\ \xi^{B'} \\ -\phi \end{pmatrix} = 0,
\]

(4.18)

\((A=\text{row index}, B, B'=\text{column indexes})\) where the explicit expressions of the matrices read

\[
(\bar{\nabla} P)_{AB'} = \bar{\nabla}_C P_{ab}, \quad M^B_A = \delta^b_ac + \delta^c_a b.
\]

The number of linearly independent equations is just the rank of the matrix system of (4.18). In principle the rank of this matrix will depend on the projector \(P_{ab}\) and its covariant derivative, meaning this that it will depend on the geometry of the manifold. However, since we are interested in spaces with a maximum number of bi-conformal vector fields it is enough to find the least rank of the above matrix for all possible projectors \(P_{ab}\). We start first studying the rank of \(M^B_A\) whose nonvanishing components occur in the following cases (no summation over the repeated indexes)

\[
p = a, c = b \Rightarrow \begin{cases} M^Q_Q = \delta^a_a P_{ab}, & a \neq b, \\
M^Q_Q = 2P_{aa}, & a = b. 
\end{cases}
\]

\[
p = b, c = a \Rightarrow M^Q_Q = \delta^b_b P_{aa}, & a \neq b,
\]

where we have assumed that we are working in the common (orthonormal) basis of eigenvectors of \(P_{ab}\) and \(\Pi^a_b\) so

\[
P^a_b = \text{diag}(1 \ldots 1 0 \ldots 0), \quad \Pi^a_b = \text{diag}(0 \ldots 0 1 \ldots 1).
\]

Hence we only need to count how many components of the type \(M^Q_Q\) are different from zero because by construction these elements give rise to linearly independent rows of the matrix \(M^B_A\) (the elements \(M^Q_Q\) are in the same row of the matrix \(M^B_A\) and they do not increase its rank). The sought number can be obtained from the following diagram gathering into blocks the \(A\) indexes of the rows containing non-zero elements (we express each index in terms of tensor indexes following the notation \(A = (a, b)\))

| Block 1 = \{(1, 1), (1, 2) \ldots (1, p)\} | Block 1 + 1 = \{(p + 1, 1) \ldots (p + 1, p)\} |
| Block 2 = \{(2, 2), (2, 3) \ldots (2, p)\} | Block 1 + 2 = \{(p + 2, 1) \ldots (p + 2, p)\} |
| Block 3 = \{(3, 3), (3, 4) \ldots (3, p)\} | Block 1 + 3 = \{(p + 3, 1) \ldots (p + 3, p)\} |
| \[\vdots\] | \[\vdots\] |
| Block 1 + \(p\) = \{(p, p)\} | Block 1 + \(p + n\) = \{(n, 1) \ldots (n, p)\}, |

from which the rank of \(M^B_A\) is

\[
1 + \ldots + p + p(n - p) = \frac{1}{2}p(p + 1) + p(n - p).
\]
Notice that this rank only depend on algebraic properties of the projector $P_{ab}$ and not on its actual form at some concrete space. Addition of the matrices $\bar{\nabla} A B$ and $P_{A} A$ only increase the rank of the matrix of the homogeneous system (4.18) and so we do not need to take them into account. The total number of constraints posed by (4.15)-I is then the rank of $M_{A} A$ plus the rank of the matrix $N_{B} A$ constructed replacing $P_{ab}$ by $\Pi_{ab}$

$$\text{rank}(M) + \text{rank}(N) = \frac{1}{2} p(p + 1) + p(n - p) + \frac{1}{2} (n - p)(n - p + 1) + p(n - p) = \frac{1}{2} n(n + 1) + p(n - p).$$

**Equations (4.15)-II.** In order to perform the analysis of these constraints it is enough to realize that the 1-forms $\phi_{a}^{*}$ and $\chi_{a}^{*}$ appearing in the right hand side of each equation are invariant under the projectors $\Pi_{ab}^{a}$ and $P_{ab}^{a}$ respectively. Therefore each constraint contains at least $n - p$ and $p$ linearly independent equations being $n$ the total sum of them.

The upper bound $N$ is then

$$N = 2 + n + n + n^2 - (n + \frac{1}{2} n(n + 1) + p(n - p)) = \frac{1}{2} (p + 1)(p + 2) + \frac{1}{2} (n - p + 1)(n - p + 2).$$

□

**Remark 4.1.** This proof does not guarantee the existence of a Lie algebra $G(S)$ in which the dimension $N$ is attained. However in section 7 we will exhibit explicit examples of $p$ pseudo-Riemannian spaces possessing $N$ linearly independent bi-conformal vector fields deriving also a geometric characterization of these spaces.

**5. First integrability conditions**

Once the normal form (4.14) has been obtained we must study next its integrability conditions. These are geometric conditions arising from the consistency between the covariant derivatives commutation rules and the expressions for these derivatives given by (4.14). In this paper we are only interested in the first integrability conditions so we will concentrate next in compatibility conditions obtained from the commutation of two covariant derivatives (Ricci identity)

$$\bar{\nabla}_{a} \bar{\nabla}_{b} \Xi^{a_1\ldots a_r}_{b_1\ldots b_s} - \bar{\nabla}_{b} \bar{\nabla}_{a} \Xi^{a_1\ldots a_r}_{b_1\ldots b_s} = \sum_{q=1}^{r} R^{t}_{a b} \Xi^{a_1\ldots a_{q-1} t a_{q+1}\ldots a_r}_{b_1\ldots b_s} - \sum_{q=1}^{s} R^{t}_{b q a} \Xi^{b_1\ldots b_{q-1} t b_{q+1}\ldots b_s}_{a_1\ldots a_r},$$

where we must replace the tensor $\Xi^{a_1\ldots a_r}_{b_1\ldots b_s}$ by the system variables and apply (4.14) to work out the covariant derivatives. Each one of the equations derived in this fashion only involves system variables and is called integrability condition. These integrability conditions can be further covariantly differentiated yielding integrability conditions of higher degree. The constraint equations (4.15) also give rise to integrability conditions when differentiated in the obvious way.
We are going next to work out the full set of first integrability conditions of the normal form (4.14) and the constraints (4.15). Part of these calculations were already performed in [7] but we will repeat everything in order to make the work presented in this section self-contained. The order followed to work out the integrability conditions of each equation of (4.14) will not coincide with the order followed in their presentation.

The simplest integrability condition is that of equation (4.14)−f

\[ \nabla_c \nabla_b \xi^a - \nabla_b \nabla_c \xi^a = R^a_{\ rcb} \xi^r = \nabla_c \Psi^a_{\ b} - \nabla_b \Psi^a_{\ c}, \]

being this expression an identity as is easily checked by replacing the covariant derivatives of \( \Psi^a_{\ b} \). Next we tackle the integrability conditions of (4.14)−d and (4.14)−e. To see this just recall that (4.2) was obtained through identity (A.1) which can be rewritten as

\[ (4.14)− \]

\[ \text{which in fact are } \]

\[ \text{given by (4.2) with all the covariant derivatives of } \bar{\phi}_a \text{ and } \bar{\chi}_a \text{ replaced by their values } \]

\[ \text{(4.14)−d and (4.14)−e. To see this just recall that (4.2) was obtained through identity (A.3) which can be rewritten as } \]

\[ \nabla_a \nabla_b \Psi^d - \nabla_b \nabla_a \Psi^d = \frac{1}{2} \xi^{-1} R^d_{\ cab} - \Psi^r_{\ a} \bar{R}^d_{\ crb} + \Psi^r_{\ b} \bar{R}^d_{\ era} - \xi^r \nabla_r \bar{R}^d_{\ cab}, \]

once the Lie derivatives of the connection (identity (A.1) have been substituted. Clearly the direct substitution in (4.2) results in a rather long expression so it is better to perform the replacements separately in (4.3) and its counterpart with the interchanges \( P_{ab} \leftrightarrow \Pi_{ab} \), etc. Adding the two equations so obtained yields after lengthy algebra

\[ \frac{1}{2} \xi^{-1} T^d_{\ cab} = \frac{\bar{\phi}_a}{2 - p} (P^d_{[b} A^r_{a]c} + P^d_{a} \bar{\gamma}^{r\ q} P_{a]c}) + \frac{\bar{\chi}_r}{2 - (n - p)} (\Pi^d_{[b} A^r_{a]c} + \Pi^d_{a} \bar{\gamma}^{r\ q} P_{a]c}) + \]

\[ + \bar{\phi}_a \nabla_a \Psi^d_{\ b} + \bar{\chi}^d \nabla_d \Pi_{a]c} + \chi^d \nabla_d \Pi_{[a]c} + \bar{\chi}_{[a} \nabla_a \Pi^d_{b]}, \]

(5.1)

where by convenience we introduce the tensors

\[ A^d_{bc} = 2 P^{dr} \nabla_r P^d_{bc}, \quad \bar{A}^d_{bc} = 2 \Pi^{dr} \nabla_r \Pi^d_{bc}, \]

\[ T^d_{bc} = 2 P^{sr} P^d_{[b} P^q_{c]} + (2 - p) \nabla_b P^d_{a]c} + T^d_{bc} = 2 \Pi^{sr} \Pi^{cq} \nabla_r \Pi^d_{qb} + (2 - n + p) \nabla_b \Pi^d_{sc}, \]

and the tensor \( T^d_{cab} \) is given by

\[ T^d_{cab} = 2 \bar{R}^d_{cab} - \frac{2}{2 - p} (P^d_{[b} L^0_{a]c} + P^d_{a} \bar{L}^0_{[b]c} + P_{c[a} \bar{L}^0_{b]q}) P^{qd} + \]

\[ - \frac{2}{2 - n + p} (\Pi^d_{[b} L^1_{[a]c} + \Pi^d_{a} L^1_{[b]c} + \Pi_{c[a} \bar{L}^1_{b]q}) P^{qd}. \]

(5.2)

This tensor will play an important role as will be seen later.

An interesting invariance property of some of the above tensors needed in future calculations is

\[ \xi^{-1} A^d_{bc} = 0, \quad \xi^{-1} \bar{A}^d_{bc} = 0 \]

(5.3)

which are easily obtained from (5.11).

Next we address the first integrability conditions of (4.14)−b and (4.14)−c. Again the calculations are tedious but straightforward (formula (A.2) is used to commute the
Lie derivative and the covariant derivative when differentiating both equations and the covariant derivatives of \( \tilde{\phi}_e \) and \( \tilde{\chi}_b \) are calculated through (4.14)-d and (4.14)-e
\[
E_d \mathcal{L}_\xi \left( \Pi^d_T \right) \left|_{abc} \right) = \tilde{\chi}_a \nabla_c E_b + \tilde{\chi}_b \nabla_c E_a - \tilde{\chi}^r \nabla^r_e \left( \Pi^{d}_{b]a} E_r \right) + \\
\left( P^r_a \tilde{\phi}_e + \tilde{\phi}_a P^r_e - \tilde{\phi}^* P^r_a + \Pi^r_a \tilde{\chi}_e + \tilde{\chi}_a \Pi^r_e - \tilde{\chi}^r \Pi^{a}[c] \nabla_b E_r \right) + \\
\left( \frac{1}{2} + \frac{n}{p} \right) \tilde{\phi}_a E_r \bar{T}_{ab}^{qr} \left| \right. \Pi_{c]} \right) a,
\]
\[
W_d \mathcal{L}_\xi \left( \Pi^d_T \right) \left|_{abc} \right) = \tilde{\phi}_a \nabla_c W_b + \tilde{\phi}_b \nabla_c W_a - \tilde{\phi}^* \nabla^c_e \left( P^b_{a]} \right) W_r + \\
\left( \Pi^r_a \tilde{\chi}_e + \tilde{\chi}_a \Pi^r_e - \tilde{\chi}^r \Pi^{a}[c] + P^r_a \tilde{\phi}_e + \tilde{\phi}_a P^r_e - \tilde{\phi}^* P^r_a \nabla_b W_r + \\
\left( \frac{1}{2} - \frac{n}{p} \right) \tilde{\phi}_a W_r \bar{T}_{ab}^{qr} \left| \right. \Pi_{c]} \right) a.
\]

The first integrability conditions of (4.14)-a are easier to handle
\[
0 = \nabla_a \nabla_b \phi - \nabla_b \nabla_a \phi = \nabla_a \tilde{\phi}_b - \nabla_b \tilde{\phi}_a + \nabla_a \tilde{\phi}_b^* - \nabla_b \tilde{\phi}_a^*
\]
\[
0 = \nabla_a \nabla_b \chi - \nabla_b \nabla_a \chi = \nabla_a \tilde{\chi}_b - \nabla_b \tilde{\chi}_a + \nabla_a \tilde{\chi}_b^* - \nabla_b \tilde{\chi}_a^*,
\]

from which we readily obtain by means of (4.14)-b, (4.14)-c, (4.14)-d, (4.14)-e
\[
\mathcal{L}_\xi \left( \frac{1}{2} - \frac{n}{p} \right) L^0_{[ab]} - \frac{1}{n - p} \nabla_e L^0_{[ab]} = 0,
\]
\[
\mathcal{L}_\xi \left( \frac{1}{2} - \frac{n}{p} \right) L^1_{[ab]} - \frac{1}{n - p} \nabla_e L^1_{[ab]} = 0.
\]

These are in principle independent integrability conditions. However, several calculations in explicit examples always have shown that the tensors inside the rounded brackets vanish identically so we believe that (5.6) and (5.7) are in fact identities holding for any vector field \( \xi \). The explicit proof of this has not been completed yet.

Finally only the first integrability conditions of (4.14)-d and (4.14)-e are left. The calculation method is similar to the above cases and the result is (equation (5.3) is used along the way)
\[
\frac{2 - \frac{n}{p} \bar{T}_{ee}^{qr}}{2} = 2 \mathcal{L}_\xi \left( \nabla_e L^0_{b]} + \frac{1}{2 - \frac{n}{p}} A^d_{c}[b L^0_{d]} \right) + \nabla_e L^0_{b]} \Pi^q_e + \\
\bar{\chi}_c \Pi^q_e L^1_{b]} - \bar{\chi}^q \Pi^q_e L^0_{b]} + \frac{2}{2 - \frac{n}{p}} \bar{T}_{ee}^{qr} \left( A^d_{c}[b A^r_{c]} + (2 - \frac{n}{p}) \nabla_e A^r_{b]} \right) c,
\]
\[
\frac{2 - \frac{n}{p} \bar{T}_{ee}^{qr}}{2} = 2 \mathcal{L}_\xi \left( \nabla_e L^1_{b]} + \frac{1}{2 - \frac{n}{p}} A^d_{c}[b L^1_{d]} \right) + \nabla_e L^1_{b]} P^q_e + \\
\bar{\phi}_c P^q_e L^1_{b]} - \bar{\phi}^q P^c_e L^1_{b]} + \frac{2}{2 - \frac{n}{p}} \bar{T}_{ee}^{qr} \left( A^d_{c}[b A^r_{c]} + (2 - \frac{n}{p}) \nabla_e A^r_{b]} \right) c.
\]

The integrability conditions of the constraints (4.15) result from their covariant derivative. We only need to take care of (4.15)-I because the differentiation of (4.15)-II results in (4.14)-b and (4.14)-c which are part of the normal form. Differentiation of these equations with respect to \( \nabla \) yield after some algebra
\[
\mathcal{L}_\xi \nabla_e P_{ab} = \phi \nabla_e P_{ab} + \phi^* P_{ab}, \quad \mathcal{L}_\xi \nabla_e \Pi_{ab} = \chi \nabla_e \Pi_{ab} + \chi^* \Pi_{ab}.
\]
For completeness we provide also these equations with the \( ab \) indexes raised
\[
\mathcal{L}_\xi \Lambda^a P^b_a = -\phi^a \varphi^b_a \mathcal{L}_\xi \Lambda^a P^b_a, \quad \mathcal{L}_\xi \nabla_c \Pi^a b = -\chi^a \Pi^a b - \chi \nabla_c \Pi^a b, \tag{5.11}
\]
and the invariance laws
\[
\mathcal{L}_\xi \nabla_c P^a b = 0, \quad \mathcal{L}_\xi \nabla_c \Pi^a b = 0. \tag{5.12}
\]
These equations close the whole suite of first integrability conditions. We will obtain geometric information from these conditions in the next section but before doing that let us show that likewise other symmetry transformations studied in Differential Geometry we can find geometric invariants associated to bi-conformal vector fields.

### 5.1. Geometric invariants

Equations (5.1)-(5.12) look rather more complicated than the first integrability conditions of other symmetries such as isometries or conformal motions. There is though a common point between all of them and it is the existence of geometric invariants, namely, tensors whose Lie derivative with respect to any vector field generating the symmetry under study vanishes. Here we face a richer set of invariants than in the isometric or conformal case as we are able to construct continuous multi-parameter families of invariants.

**Theorem 5.1.** The six-parameter family of rank-3 tensors given by
\[
W^d_{abc}(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3) = \lambda_1 A_d^{bc} + \lambda_2 \nabla_c P^d_a + \lambda_3 \nabla_b P^d_a + \mu_2 \nabla_c \Pi^d_{ba} + \mu_3 \nabla_b \Pi^d_a,
\]
is a geometric invariant.

**Proof.** This is straightforward from (5.3) and (5.12). \( \square \)

These invariants are not the only ones which can be found for bi-conformal vector fields.

**Theorem 5.2.** Let \( \| C^d_{abc} \) and \( \perp C^d_{abc} \) be the tensors constructed from \( T^d_{abc} \) by the following definitions
\[
\| C^d_{abc} \equiv P^d_a P^e_b P^q_c T^r_{stq}, \quad \perp C^d_{abc} \equiv \Pi^d_a \Pi^q_{ba} T^r_{stq}. \tag{5.13}
\]

Then, for any bi-conformal vector field \( \xi \) we have the family of invariance rules
\[
\mathcal{L}_\xi (\lambda_1 \| C^d_{abc} + \lambda_2 \perp C^d_{abc}) = 0, \quad \forall \lambda_1, \lambda_2 \in \mathbb{R}. \tag{5.14}
\]

**Proof.** It is enough to show that the Lie derivative of \( \perp C^d_{abc} \) and \( \| C^d_{abc} \) vanishes. The proof of this result is accomplished by projecting equations (5.1) in all its indexes with the projectors \( P_{ab} \) and \( \Pi_{ab} \) respectively and realising that the right hand side is zero. In fact all the terms of the right hand side of (5.1) are zero when full projected in all their indexes with either \( P_{ab} \) or \( \Pi_{ab} \). To see this we note that the simple result
\[
P^d_a P^e_b P^q_c \nabla_r P^s_a = 0, \quad \Pi^d_a \Pi^q_{ba} \nabla_r \Pi^s_a = 0,
\]
entails
\[
P^d_a P^e_b P^q_c \nabla_r P^s_a = 0, \quad \Pi^d_a \Pi^q_{ba} \nabla_r \Pi^s_a = 0,
\]
as can be explicitly checked from identities (3.22)-(3.24) (this result is still true even if we raise or lower any index of the projectors). From this last equation it is now easy to prove the asserted claim for all of the right hand side terms of (5.1). \( \square \)
Therefore the diffeomorphisms generated by bi-conformal vector fields keep invariant all the afore-mentioned families of tensors. Some of these results have immediate geometric consequences. If \( \{ \varphi_s \}_{s \in \mathbb{R}} \) is a one-parameter group of bi-conformal transformations then the invariance laws of the tensors \( {\| C}_{abcd}^\alpha \) and \( {\perp C}_{abcd}^\alpha \) are equivalent to (we use index-free notation)

\[
(\ast \varphi_s^\ast C)(g, P) = {\| C}(g, P), \quad (\ast \varphi_s^\ast C)(g, \Pi) = {\perp C}(g, \Pi),
\]

where by \( {\| C}(g, P) \) and \( {\perp C}(g, \Pi) \) we mean the tensors \( {\| C} \) and \( {\perp C} \) calculated from the metric \( g \) and the orthogonal complementary projectors \( P, \Pi \). This equation is still true for any bi-conformal transformation \( \Phi \) even if it does not belong to any one-parameter group of bi-conformal transformations (see appendix B). On the other hand the formula

\[
(\ast \Phi^\ast \| C)(g, P) = \| C(\Phi^\ast g, \Phi^\ast P), \quad (\ast \Phi^\ast \perp C)(g, \Pi) = \perp C(\Phi^\ast g, \Phi^\ast \Pi),
\]

holds for any diffeomorphism \( \Phi : V \to V \). Therefore combination of (5.15) and (5.16) yields the geometric property

\[
\| C(\alpha_1 P + \alpha_2 \Pi, \alpha_1 P) = \| C(g, P), \quad \perp C(\alpha_1 P + \alpha_2 \Pi, \alpha_2 \Pi) = \perp C(g, \Pi),
\]

which is true for any bi-conformal relation \( \Phi : V \to V \) with \( \Phi^\ast P = \alpha_1 P, \Phi^\ast \Pi = \alpha_2 \Pi \). Moreover this is true for any pair of positive functions \( \alpha_1, \alpha_2 \) even if they do not arise from any bi-conformal transformation (see appendix B for a proof of this). Properties (5.15)-(5.17) still hold if we replace \( {\| C}, \ast {\perp C} \) by any member of the family of tensors introduced in theorems 5.1 and 5.2.

**Theorem 5.3.** If \( P \) and \( \Pi \) are orthogonal complementary projectors with respect to a flat pseudo-Riemannian metric \( \eta \) then for any \( \alpha_1, \alpha_2 \in C^2 \)

\[
\| C(\alpha_1 P + \alpha_2 \Pi, \alpha_1 P) = 0, \quad \perp C(\alpha_1 P + \alpha_2 \Pi, \alpha_2 \Pi) = 0.
\]

**Proof.** This follows straight away from (5.17) because trivially \( \| C(\eta, P) = \perp C(\eta, \Pi) = 0 \) for any pair of orthogonal complementary projectors \( P, \Pi \). \( \square \)

We will prove in section 7 that the converse of this theorem is also true, that is to say if both \( \| C(g, P) \) and \( \perp C(g, \Pi) \) vanish for a certain metric \( g = P + \Pi \) then \( P \) and \( \Pi \) are conformal to orthogonal and complementary projectors with respect to a certain flat metric. Moreover we suspect (see example 8.2) that the vanishing of \( \| C(g, P) \)

\( \perp C(g, \Pi) \)

alone is a necessary and sufficient condition for the existence of conformally flat foliations on \( (V, g_{ab}) \) being \( P_{ab} \) \( (\Pi_{ab}) \) the metric tensor of the hypersurfaces forming these foliations. The rigorous proof of this statement is under investigation.

**6. Complete integrability**

Our next task is to decide when first integrability conditions presented in previous section become a set of identities for every election of the independent variables of the normal form (4.14). Under such circumstances the Lie algebra of bi-conformal vector fields attains its greatest dimension \( N \) and so if we are able to find spaces \( (V, g) \) in which this happens we will have proven that the bound \( N \) is actually reached (see [6]).
The first problem which we come across to is that not all the variables appearing in (4.14) are independent as there are constraints which must be taken into account. However, some of the variables of (4.14) are not involved in the constraint equations (4.15) and this fact will allow us to find necessary and sufficient geometric conditions for the whole set of first integrability conditions to become identities. The variables which are not constrained by (4.15) are $\bar{\phi}_a$ and $\bar{\chi}_a$ so we will separate out in the integrability conditions all the contributions involving these variables and derive the geometric implications. We will perform next this procedure step by step for each of the integrability conditions obtained before (actually we will not need to analyse all the conditions as some of them will turn into identities if the geometric conditions entailed by others are imposed). The full calculation is rather cumbersome and only its main excerpts will be shown so the reader interested in the geometric conditions characterizing the complete integrability should jump straight to theorem 6.1.

Equation (5.1). This equation can be rewritten as

$$\xi^d T^r_{\phantom{r}d} = \bar{\phi}_r M^r_{\phantom{r}c} + \bar{\chi}_r N^r_{\phantom{r}c},$$

where

$$M^r_{\phantom{r}c} = \frac{2}{2 - p} P^r_{\phantom{r}s}(A^r_{\phantom{r}c} P^d_{\phantom{d}b} + \Upsilon^{sd}_{\phantom{sd}b} P^c_{\phantom{c}a}) + 2 P^r_{\phantom{r}[b} \nabla^c_{\phantom{c}a} P^{d}_{\phantom{d}c} + 2 P^r_{\phantom{r}c} \nabla^d_{\phantom{d}a} P^{c}_{\phantom{c}b} + 2 P^{dr}_{\phantom{dr}c} \nabla^c_{\phantom{c}[b} P^{a}_{\phantom{a}c},$$

$$N^r_{\phantom{r}c} = \frac{2}{2 - n + p} \Pi^r_{\phantom{r}s}(\bar{A}^r_{\phantom{r}c} \Pi^d_{\phantom{d}b} + \bar{\Upsilon}^{sd}_{\phantom{sd}b} \Pi^c_{\phantom{c}a}) + 2 \Pi^r_{\phantom{r}[b} \bar{\nabla}^c_{\phantom{c}a} \Pi^{d}_{\phantom{d}c} + 2 \Pi^r_{\phantom{r}c} \bar{\nabla}^d_{\phantom{d}a} \Pi^{c}_{\phantom{c}b} + 2 \Pi^{dr}_{\phantom{dr}c} \bar{\nabla}^c_{\phantom{c}[b} \Pi^{a}_{\phantom{a}c}. $$

As (6.1) must be true for every $\bar{\phi}_a$, $\bar{\chi}_a$ we have

$$P^r_{\phantom{r}s} M^{sd}_{\phantom{sd}c} = M^{rd}_{\phantom{rd}c} = 0, \quad \Pi^r_{\phantom{r}s} N^{sd}_{\phantom{sd}c} = N^{rd}_{\phantom{rd}c} = 0. $$

Let us study these geometric conditions (it is enough to concentrate on the first condition because the other is dual and comes through the usual replacements). Contracting the indexes $d$ with $b$ in this equation we get

$$p P^{dr}_{\phantom{dr}c} \nabla^c_{\phantom{c}[b} P^{a}_{\phantom{a}c} - P^{rq}_{\phantom{rq}c} (P^r_{\phantom{r}a} \nabla^q_{\phantom{q}c} P^d_{\phantom{d}a} + P^c_{\phantom{c}a} \nabla^q_{\phantom{q}a} P^{d}_{\phantom{d}c}) + P^{as}_{\phantom{as}c} \nabla^s_{\phantom{s}a} P^{ac} = 0. $$

Projecting this in the $a$-index with $P^a_{\phantom{a}s}$ gives the condition

$$P^a_{\phantom{a}s} P^{rq}_{\phantom{rq}c} \nabla^q_{\phantom{q}c} P^d_{\phantom{d}a} = 0,$$

which put back in (6.5) yields

$$P^{rs}_{\phantom{rs}} \nabla^s_{\phantom{s}a} P^{ac} = 0 \Rightarrow A^r_{\phantom{r}ac} = 0, \quad \Upsilon^{sc}_{\phantom{sc}b} = (2 - p) \nabla^c_{\phantom{c}b} P^{sc}. $$

Now using this result we contract $r$ with $b$ in the first of (6.4)

$$(p - 1) \nabla^d_{\phantom{d}a} P^{d}_{\phantom{d}c} - P^s_{\phantom{s}a} \nabla^s_{\phantom{s}c} P^d_{\phantom{d}c} - P^s_{\phantom{s}c} \nabla^d_{\phantom{d}a} P^s_{\phantom{s}a} + P^s_{\phantom{s}c} \nabla^d_{\phantom{d}a} P^s_{\phantom{s}a} = 0,$$

from which we get

$$P^s_{\phantom{s}a} P^s_{\phantom{s}c} \nabla^q_{\phantom{q}c} P^{d}_{\phantom{d}a} = 0.$$
Combining this with (6.8) we readily obtain
\[ \bar{\nabla}_a P^d_c = 0. \] (6.10)

Finally multiplying by \( P^{ac} \) in the first of (6.4) gives
\[ P^r_b ( - P^a_b \bar{\nabla}_a P^{sd} + p \bar{\nabla}_b P^{sd} ) - P^{dr} E_b = 0. \] (6.11)

Projection of this equation in the index \( b \) leads to
\[ P^a_b P^r_s \bar{\nabla}_a P^{sd} = 0, \]
which combined with (6.11) implies
\[ \bar{\nabla}_b P^{rd} = \frac{1}{p} E_b P^{rd}. \] (6.12)

This last condition can be written with the indexes of the projector lowered if we use (6.10)
\[ \bar{\nabla}_b P_{cd} = - \frac{1}{p} E_b P_{dc}. \] (6.13)

The dual conditions for the projectors \( \Pi^{ab}, \Pi_{ab} \) coming from the vanishing of (6.3) are
\[ \bar{\nabla}_c \Pi^{ab} = \frac{1}{n - p} W_c \Pi^{ab}, \bar{\nabla}_c \Pi^a_b = 0, \bar{\nabla}_c \Pi_{ab} = - \frac{1}{p} W_c \Pi_{ab}. \] (6.14)

Conversely if equations (6.10), (6.12), (6.13) and (6.14) are assumed a simple calculation tells us that the tensors defined by (6.2) and (6.3) vanish. In fact all the above geometric conditions can be combined in a single simpler expression.

**Proposition 6.1.** The following assertion is true
\[ T_{abc} = 0 \iff \bar{\nabla}_a P_{bc} = - \frac{1}{p} E_a P_{bc}, \bar{\nabla}_a \Pi_{bc} = - \frac{1}{n - p} W_a \Pi_{bc}. \] (6.15)

**Proof.** First of all, it is convenient to rewrite the condition \( T_{abc} = 0 \) in an appropriate form. From (3.8) we have
\[ M_{abc} = \frac{1}{p} E_a P_{bc} - \frac{1}{n - p} W_a \Pi_{bc}. \] (6.16)

Using the definition \( M_{abc} = \bar{\nabla}_b P_{ac} + \bar{\nabla}_c P_{ab} - \bar{\nabla}_a P_{bc} \) we can isolate \( \bar{\nabla}_b P_{ac} \) getting
\[ \bar{\nabla}_b P_{ac} = \frac{1}{2p} (E_a P_{bc} + E_c P_{ab}) - \frac{1}{2(n - p)} (W_a \Pi_{bc} + W_c \Pi_{ba}). \] (6.17)

Each (6.16) and (6.17) are equivalent to \( T_{abc} = 0 \). Next we show the equivalence of \( T_{abc} = 0 \) to the conditions (6.15). Expanding \( \bar{\nabla}_a P_{bc} \) by means of (3.22) and use of the conditions upon the covariant derivatives yields
\[ \bar{\nabla}_b P_{ac} = \frac{1}{2p} (E_a P_{bc} + E_c P_{ab}) + \frac{1}{2} (P_{cp} M^p_{ba} + P_{ap} M^p_{bc}) \] (6.18)
\[ \bar{\nabla}_b \Pi_{ac} = \frac{1}{2(n - p)} (W_a \Pi_{bc} + W_c \Pi_{ba}) - \frac{1}{2} (\Pi_{cp} M^p_{ba} + \Pi_{ap} M^p_{bc}) \]. (6.19)
which are equivalent to (write $M_{abc}$ in terms of $\nabla_a P_{bc}$ and $\nabla_a \Pi_{bc}$ respectively)

$$P_{cp}M_{ab}^p = -\frac{1}{n-p}W_c \Pi_{ab}, \quad \Pi_{cp}M_{ab}^p = \frac{1}{p}E_c P_{ab}, \quad (6.20)$$

whose addition leads to $T_{abc} = 0$. Conversely, suppose that $T_{abc} = 0$. Then inserting (6.16) and (6.17) into (3.22) gives us the condition on $\nabla_a P_{bc}$ at once. The calculation for $\nabla_a \Pi_{bc}$ is similar using the identities written in terms of $\Pi_{ab}$.

**Proposition 6.2.**

$$T_{abc} = 0 \implies \nabla_c P^{ab} = 0.$$

**Proof.** To prove this we use identity (3.23) and replace $M_{abc}$ by the expression found in (6.16)

$$\nabla_a P^c = \nabla_a P_c - \frac{1}{2p} (E_b P_{ac} + E_c P_{ab}) + \frac{1}{2(n-p)} (W_c \Pi_b^a + W_b \Pi_c^a), \quad (6.21)$$

which vanishes by (6.17). $\Box$

**Remark 6.1.** Note that in view of the above result the condition $T_{abc} = 0$ entails

$$\nabla_c P^{ab} = \frac{1}{p} E_c P^{ab}, \quad \nabla_c \Pi^{ab} = \frac{1}{n-p} W_c \Pi^{ab}.$$

Therefore all the conditions coming from (6.2) and (6.3) are summarized in $T_{abc} = 0$. The geometric significance of this condition will be investigated in section 7.

Some of the first integrability conditions achieve a great simplification if $T_{abc}$ vanishes. For instance (5.9)-(5.12) become zero identically under this condition as is obvious from propositions 6.1 and 6.2 so we do not need to care about these integrability conditions any more. Equation (6.1) acquires the invariance law

$$\mathcal{L}_\xi T_{abc}^d = 0. \quad (6.22)$$

Other simplifications will be shown in the forthcoming analysis.

**Equations (5.8) and (5.9).** If $T_{abc} = 0$ then $\Lambda^a_{bc} = 0$, $\bar{\Lambda}^a_{bc} = 0$ so these equations take the form

$$-\bar{\phi}_q P^q T_{r c e b} = 2 \mathcal{L}_\xi \nabla_e [e \mathcal{L}_b^0]+ \bar{\chi}_r E_{r c e b}, \quad (6.23)$$

$$-\bar{\chi}_q \Pi^q T_{r c e b} = 2 \mathcal{L}_\xi \nabla_e [e \mathcal{L}_b^1]+ \bar{\phi}_r F_{r c e b}, \quad (6.24)$$

where

$$E_{r c e b} = \frac{2}{2-p} (\Pi^r_{\epsilon c} \Pi_{\epsilon b q} + \Pi^r_{\epsilon b} \Pi^q_{\epsilon c} - \Pi^r_{\epsilon q} \Pi_{\epsilon b c}^q), \quad (6.25)$$

$$F_{r c e b} = \frac{2}{2-n-p} (P^r_{\epsilon c} P^q_{\epsilon b q} + P^r_{\epsilon b} P^q_{\epsilon c} - P^r_{\epsilon q} P_{\epsilon b c}^q). \quad (6.26)$$

From (6.23) and (6.24) we find the conditions of complete integrability

$$P^q T_{r c e b} = 0, \quad \Pi^q_0 T_{r c e b} = 0 \implies T_{r c e b}^q = 0,$$

so equation (6.22) is trivially fulfilled. To proceed further with the calculations we need a lemma.
Lemma 6.1. If \( \nabla_a P_c^b = 0 \) then
\[
L_{bq}^0 \Pi^q_c = 0, \quad L_{bq}^1 P_c^q = 0.
\]

Proof. These properties are proven through the Ricci identity applied to the tensors \( P_c^b, \Pi^q_c \) which take a remarkably simple form under our conditions (we only perform the calculations for the tensor \( P_c^b \))
\[
0 = \nabla_e \nabla_a P_c^b - \nabla_a \nabla_e P_c^b = \vec{R}^b_{cea} P^q_c - \vec{\Pi}^q_{cea} P^b_c,
\]
whence
\[
L_{bq}^0 \Pi^q_c = 2 P^d q_0 d_r q_0 r_0 d_s \Pi^r_c,
\]
The first term of this expression is zero according to (6.27) and the second one can be transformed by means of the first Bianchi identity into
\[
\Pi^q_c \Pi^r_q \vec{R}^q_{rds} = - \Pi^q_c \Pi^r_q (P^r_q \vec{R}^q_{ars} + P^r_q \vec{R}^q_{ards}) = - \Pi^q_c \Pi^r_q (P^r_q \vec{R}^q_{ars} + P^r_q \vec{R}^q_{ards}),
\]
which also vanishes. \( \square \)

Therefore conditions (6.23) and (6.24) are further simplified to
\[
\mathcal{L}_\xi \nabla_{[e} L^0_{b]c} = 0, \quad \mathcal{L}_\xi \nabla_{[e} L^1_{b]c} = 0.
\]

(6.28)

It is our next aim to show that indeed these two equations are identities if \( T_{cab}^d = 0 \).

Lemma 6.2. If \( p \neq 3 \), \( n - p \neq 3 \) and \( \nabla_a P_c^b = 0 \) then
\[
T_{cab}^d = 0 \implies \nabla_{[e} L^0_{b]c} = 0, \quad \nabla_{[e} L^1_{b]c} = 0
\]

Proof. To prove this we start from the identity
\[
(2 - p) [\nabla_e (P^e_a T_{cab}^q) + \nabla_b (P^d_{a [cd} T_{e]qb]}) - \nabla_a (P^d_{[cd} T_{e]qb]})] = 2 P^e_a \nabla_e L^0_{[ba]} - 2 P^e_a \nabla_{[b} L^0_{|a]} + 2 P^e_a \nabla_{[b} L^0_{|a]} + 2 P^d_{[cd} \nabla_{[e} L^0_{b]|a]} + 2 P^d_{[cd} \nabla_{[e} L^0_{b]q} + 2 P^d_{[cd} \nabla_{[e} P_{b]q} L^0_{a|q}.
\]

(6.29)

which is easily obtained from equation (5.2) and the second Bianchi identity for the tensor \( \vec{R}^d_{bed} \) if the condition \( \nabla_a P_c^b = 0 \) holds. In our case the left hand side of this identity vanishes so we only need to study the right hand side equated to zero. Contracting such equation with \( P^{ca} \) we get
\[
2(1 - p) P^{ca} \nabla_e L^0_{ba} - 2(1 - p) P^{da} \nabla_b L^0_{da} + 2 P^d_{b} P^{ac} \nabla_a L^0_{dc} - 2 P^e_b P^{ac} \nabla_e L^0_{ac} = 0,
\]
and a further contraction with \( P^{rd} \) yields
\[
P^{ca} \nabla_e L^0_{a|q} = P^{ca} \nabla_e P_{rd}.
\]

This last property implies that the last two terms of (6.29) are zero and it becomes
\[
2 P^e_a \nabla_e L^0_{ba} - 2 P^d_{b} \nabla_{[e} L^0_{d|a]} + 2 P^d_{b} \nabla_{[e} P^0_{d|a]} + 2 P^d_{b} \nabla_{[e} P^0_{q} L^0_{d|a]} = 0.
\]

(6.30)
If we multiply this last equation by $P^r_a P^s_b$ we obtain

$$S_{cba} - S_{bca} + S_{acb} - S_{cab} + (p - 2)(S_{bac} - S_{abc}) = 0,$$

where

$$S_{abc} \equiv P^r_a P^s_b \nabla_r L^0_{sc}. $$

Permuting indexes in (6.31) we get the equations

\[
\begin{align*}
S_{cba} - S_{bca} + S_{acb} - S_{cab} + (p - 2)(S_{bac} - S_{abc}) &= 0, \\
S_{abc} - S_{bac} + S_{acb} + (p - 2)(S_{bca} - S_{cba}) &= 0, \\
S_{cba} - S_{bca} + S_{acb} + (p - 2)(S_{bac} - S_{abc}) &= 0.
\end{align*}
\]

(6.32)

Setting the variables $x = S_{eba} - S_{bca}$, $y = S_{acb} - S_{cab}$, $z = S_{bac} - S_{abc}$ we deduce that previous equations form a homogeneous system in these variables whose matrix is

\[
\begin{pmatrix}
1 & 1 & p - 2 \\
2 - p & -1 & -1 \\
p - 2 & 1 & 1
\end{pmatrix}
\Rightarrow
\begin{vmatrix}
1 & 1 & p - 2 \\
2 - p & -1 & -1 \\
p - 2 & 1 & 1
\end{vmatrix} = -(p - 3)^2.
\]

So unless $p = 3$ ($p = 0$ makes no sense in the current context) we conclude that $x = y = z = 0$ and hence

$$P^r_a P^s_b \nabla_r L^0_{sc} = 0.$$ 

Application of this in the expression resulting of multiplying (6.30) by $P^r_b$ leads to

$$P^d_c \nabla_d L^0_{ab} - P^e_c \nabla_e L^0_{ab} + (p - 1)(P^r_b \nabla_r L^0_{ac} - P^r_b \nabla_r L^0_{bc}) = 0. $$

By setting $Q_{abc} \equiv P^d_c \nabla_d L^0_{ab} - P^e_c \nabla_e L^0_{ab}$ we can rewrite this as

$$Q_{abc} - (p - 1)Q_{abc} = 0, \Rightarrow Q_{abc} - (p - 1)Q_{abc} = 0,$$

which entails $Q_{abc} = 0$ (recall that $p \neq 1$ by definition of $L^0_{ab}$). This last property applied to (6.30) yields

$$\nabla_{[b} L^0_{ac]} = 0,$$

as desired. The result for $L^1_{ab}$ is proven in a similar way. $\square$

**Equations (5.6) and (5.7)** The analysis of these conditions is performed by means of the following result.

**Proposition 6.3.** If $T_{abc} = 0$ then

$$\frac{1}{2 - p} (L^0_{ab} - L^0_{ba}) - \frac{1}{p} (\nabla_a E_b - \nabla_b E_a) = 0,$$

$$\frac{1}{2 - n + p} (L^1_{ab} - L^1_{ba}) - \frac{1}{n - p} (\nabla_a W_b - \nabla_b W_a) = 0.$$
Proof. We only carry on the proof for the first identity as the calculations are similar for the second one. From (4.9) and applying the first Bianchi identity is easy to obtain

\[ L_0^{ab} - L_0^{ba} = \frac{2(2 - p)}{p} P_r^r R_{rab}. \]

(6.33)

On the other hand

\[ \bar{\nabla}_a E_b - \bar{\nabla}_b E_a = P^{qr} (\bar{\nabla}_a \bar{\nabla}_b P_{qr} - \bar{\nabla}_b \bar{\nabla}_a P_{qr}) - \bar{\nabla}_a P^{qr} \bar{\nabla}_b P_{qr} + \bar{\nabla}_b P^{qr} \bar{\nabla}_a P_{qr}. \]

(6.34)

If we impose now the condition \( T_{abc} = 0 \) then combination of proposition 6.1 and remark 6.1 entails

\[ \bar{\nabla}_a P^{qr} \bar{\nabla}_b P_{qr} = -\frac{1}{p} E_a E_b = \bar{\nabla}_b P^{qr} \bar{\nabla}_a P_{qr}, \]

so (6.34) becomes

\[ \bar{\nabla}_a E_b - \bar{\nabla}_b E_a = 2 P^{qr} \bar{R}_{qab}. \]

(6.35)

Combination of (6.33) and (6.35) leads to the desired result. \( \Box \)

Equations (5.4) and (5.5)

Proposition 6.4. If \( T_{cab}^d = 0 \) and \( T_{abc} = 0 \) then (5.4) and (5.5) are identities.

Proof. The left hand side of both equations vanishes trivially if \( T_{cab}^d = 0 \) so we just need to show that the right hand side vanishes as well. The characterization of the condition \( T_{abc} = 0 \) in terms of the covariant derivatives of the projectors entails

\[ T_{bc}^a = \frac{2 - p}{p} E_b P_{ac}, \quad T_{bc}^a = \frac{2 - n + p}{n - p} W_b \Pi^{ac}, \]

which means that the terms of (5.4) and (5.5) containing these tensors are zero. The property \( \bar{\nabla}_c P_b^a = \bar{\nabla}_c \Pi_{bc}^a = 0 \) can be used now to get rid of some terms and simplify others on these couple of equations getting

\[ 0 = \bar{\chi}_a \bar{\nabla}_c E_b + \bar{\chi}_b \bar{\nabla}_c E_a + \bar{\chi}_a \bar{\nabla}_b E_c + \bar{\chi}_c \bar{\nabla}_b E_a \]

\[ 0 = \bar{\phi}_a \bar{\nabla}_c W_b + \bar{\phi}_b \bar{\nabla}_c W_a + \bar{\phi}_a \bar{\nabla}_b W_c + \bar{\phi}_c \bar{\nabla}_b W_a, \]

which is obviously zero. \( \Box \)

All our calculations are thus summarized in the next result which is one of the most important of this paper.

Theorem 6.1 (Complete integrability conditions). The necessary and sufficient geometric conditions for a space \((V, g)\) to possess \( N \) linearly independent bi-conformal vector fields associated to the projectors \( P_{ab}, \Pi_{ab} \) not projecting on a subspace of dimension three are

\[ T_{abc} = 0, \quad T_{cab}^d = 0. \]
7. Geometric characterisation of conformally separable pseudo-Riemannian manifolds

Once we have found the mathematical characterization of the spaces admitting a maximum number of bi-conformal vector fields we must next settle if there is actually any space whose metric tensor complies with the conditions stated in theorem 6.1 or on the contrary there are no pseudo-Riemannian manifolds fulfilling such requirement. Indeed we will find that each geometric condition has a separate meaning related with the geometric characterization of certain separable pseudo-Riemannian manifolds. Hence the tensors $T_{abc}$ and $T^b_{bcd}$ bear a geometric interest on their own regardless to the existence or not of bi-conformal vector fields on the space $(V, g)$ where they are defined. Before addressing this issue we need some preliminary definitions.

**Definition 7.1.** The pseudo-Riemannian manifold $(V, g)$ is said to be separable at the point $q \in V$ if there exists a local coordinate chart $x \equiv \{x^1, \ldots, x^n\}$ based at $q$ in which the metric tensor takes the form

$$g_{ab}(x) = \begin{cases} 
g_{\alpha\beta}(x), & 1 \leq \alpha, \beta \leq p 
g_{AB}(x), & p + 1 \leq A, B \leq n 
0 & \text{otherwise.}
\end{cases} \quad (7.1)$$

$(V, g)$ is separable if it is so at every point $p \in V$. Any of the metric tensors $g_{\alpha\beta}$, $g_{AB}$ in which $g_{ab}$ is split shall be called leaf metric of the separation.

This same concept is already defined in [14] for the case of Riemannian manifolds (they call these manifolds locally product Riemannian manifolds)

Henceforth all our results will deal with separable pseudo-Riemannian manifolds but they have a clear extension to the case in which only local separability at a point holds. From now on when working with separable spaces written in the form of (7.1) we adopt the convention that Greek letters label indexes associated to one of the leaf metrics whereas uppercase Latin characters are used for the other one. The coordinate system of definition 7.1 is fully adapted to the decomposition of the metric tensor but in general we cannot expect this to be the case when dealing with arbitrary separable pseudo-Riemannian manifolds. Therefore it would be desirable to have a result characterizing separable pseudo-Riemannian manifolds in a coordinate-free way. Nonetheless no general result characterizing separable spaces in a coordinate-free way is known although there are already available results for particular cases of separable spaces. Before considering them we need to give a brief account of the most studied types of separable pseudo-Riemannian manifolds in the literature.

**Definition 7.2.** Let $x \equiv \{x^a\}$, $a = 1 \ldots n$ be the local coordinate system introduced in definition 7.1. A separable manifold can then be classified in terms of the form the line element $ds^2$ takes in these coordinates as

1. decomposable or reducible:
$$ds^2 = g_{\alpha\beta}(x^a)dx^\alpha dx^\beta + g_{AB}(x^C)dx^A dx^B,$$
2. semi-decomposable, semi-reducible or warped product:
$$ds^2 = g_{\alpha\beta}(x^a)dx^\alpha dx^\beta + \Xi(x^r)G_{AB}(x^C)dx^A dx^B, \quad \Xi(x^r) \text{ warping factor},$$
3. generalized decomposable or double warped:
$$ds^2 = \Xi_1(x^C)G_{\alpha\beta}(x^a)dx^\alpha dx^\beta + \Xi_2(x^r)G_{AB}(x^C)dx^A dx^B, \quad \Xi_1(x^C), \Xi_2(x^r) \text{ warping factors},$$
4. conformally reducible:
\[ ds^2 = \Xi(x^\alpha)(G_{\alpha\beta}(x^\xi)dx^\alpha dx^\beta + G_{AB}(x^C)dx^A dx^B). \]
5. conformally separable or double twisted:
\[ ds^2 = \Xi_1(x^\alpha)G_{\alpha\beta}(x^\xi)dx^\alpha dx^\beta + \Xi_2(x^\alpha)G_{AB}(x^C)dx^A dx^B. \]
6. bi-conformally flat:
\[ ds^2 = \Xi_1(x^\alpha)\eta_{\alpha\beta}(x^\xi)dx^\alpha dx^\beta + \Xi_2(x^\alpha)\eta_{AB}(x^C)dx^A dx^B. \]

There are here classic results in Differential Geometry characterizing invariantly some of the above types of separable spaces.

**Theorem 7.1.** A pseudo-Riemannian manifold is decomposable if and only if either of the conditions written below holds

1. The manifold possesses two orthogonal families of foliations by totally geodesic hypersurfaces.
2. There exists a symmetric and idempotent tensor \( P_{ab} \neq g_{ab} \) such that \( \nabla_a P_{bc} = 0 \).

In this case the tensor \( P_{ab} \) is one of the leaf metrics of the decomposition being \( g_{ab} \) the other one.

**Proof.** The original proof of the first point was given in [2] (see also p.186 of [5] and p. 420 of [14]). The second point was proven in [10] in the context of General Relativity ([14] also proves this result indirectly) but we provide its proof as it employs techniques needed later. Choose an orthonormal co-basis \( \{ \bar{\theta}^1, \ldots, \bar{\theta}^p \} \) adapted to \( P_{ab} \), that is to say, its \( p \) first elements are dual to a basis of the eigenspace of \( P_{ab} \) with nonvanishing eigenvalue (observe that under the assumptions of this theorem \( P_{ab} \) is an orthogonal projector). In terms of this basis \( P_{ab} \) takes the form (we use index-free notation and index label splitting as in definition 7.1)

\[ P = \sum_{\alpha=1}^p \epsilon_\alpha \bar{\theta}^\alpha \otimes \bar{\theta}^\alpha, \]

where \( \epsilon_\alpha = \pm 1 \) (the exact value for each index \( \alpha \) will depend on the signature of \( g_{ab} \)).

Now since
\[ \nabla_c \bar{\theta}^\alpha = -\gamma^\alpha_{\beta c} \bar{\theta}^\beta - \gamma^\alpha_{B c} \bar{\theta}^B, \]

we have
\[ 0 = \nabla_c P = -\sum_{\alpha=1}^p \epsilon_\alpha [\gamma^\alpha_{\beta c}(\bar{\theta}^\beta \otimes \bar{\theta}^\alpha + \bar{\theta}^\alpha \otimes \bar{\theta}^\beta) + \gamma^\alpha_{B c}(\bar{\theta}^B \otimes \bar{\theta}^\alpha + \bar{\theta}^\alpha \otimes \bar{\theta}^B)], \]

which entails \( \gamma^\alpha_{B c} = 0 \). Similarly using the property \( \nabla_c (g_{ab} - P_{ab}) = 0 \) we can show that \( \gamma^\alpha_{\beta c} = 0 \). These two conditions upon the connection coefficients imply by means of Frobenius theorem that the distributions spanned by \( \{ \bar{\theta}^1, \ldots, \bar{\theta}^p \} \) and \( \{ \bar{\theta}^{p+1}, \ldots, \bar{\theta}^n \} \) are both integrable. In the local coordinate system \( \{ x^1, \ldots, x^n \} \) adapted to the manifolds generated by these distributions (i.e. \( x^\alpha = c^\alpha, x^\beta = c^\beta \)) the line element takes the form
\[ ds^2 = g_{\alpha\beta}(x^\alpha)dx^\alpha dx^\beta + g_{AB}(x^\alpha)dx^A dx^B. \]
Moreover if we impose the conditions $\nabla_{c} P_{ab} = \nabla_{c} (g_{ab} - P_{ab}) = 0$ on this metric we get
\[
\partial_{A} g_{\alpha\beta} = 0, \quad \partial_{\alpha} g_{AB} = 0,
\]
which implies that the metric is decomposable. The converse is straightforward if we write the metric in the form of point 1 of definition 7.2 and realize that the tensor
\[
P_{ab} = \delta_{a}^{\alpha} \delta_{b}^{\beta} g_{\alpha\beta}
\]
has the required properties.

There are also special characterizations of four dimensional decomposable pseudo-Riemannian manifolds devised in the framework of General Relativity using Holonomy groups (see [8]).

The invariant characterization of conformally separable pseudo-Riemannian manifolds is covered by the next result (all the tensors appearing herein are understood in terms of their homonym counterparts of section 5).

**Theorem 7.2.** A pseudo-Riemannian manifold $(V, g)$ is conformally separable if and only if any of the following conditions are satisfied.

1. The manifold admits two orthogonal families of foliations by totally umbilical hypersurfaces. The family of first fundamental forms of each hypersurface gives rise to the leaf metrics of the decomposition of $g_{ab}$ in the obvious way.
2. There exists an orthogonal projector $P_{ab}$ such that the tensor $T_{abc}$ formed with $P_{ab}$ and its complementary $\Pi_{ab} = g_{ab} - P_{ab}$ is zero identically. In such case $P_{ab}$ and $\Pi_{ab}$ are the leaf metrics of the separation.

**Proof.** The first point was proven in [12] and the second point is a new result which comes naturally from the geometric tools of previous sections (only the latter result is shown in this proof). The proof of the only if implication is theorem 7.2 of [7] but for the sake of completeness we reproduce its main details here. Choose the coordinate system and notation of definition 7.1 and to keep generality assume that the manifold is only separable. In this coordinate system the leaf metrics $P_{ab}$ and $\Pi_{ab}$ look like
\[
P_{ab} = g_{\alpha\beta} \delta_{a}^{\alpha} \delta_{b}^{\beta}, \quad \Pi_{ab} = g_{AB} \delta_{a}^{A} \delta_{b}^{B},
\]
and the non-zero components of the Christoffel symbols are

\[
\Gamma_{\beta\gamma}^{\alpha} = \frac{1}{2} g_{\alpha\rho} (\partial_{\beta} g_{\gamma\rho} + \partial_{\gamma} g_{\rho\beta} - \partial_{\rho} g_{\beta\gamma}), \quad \Gamma_{\beta A}^{\alpha} = \frac{1}{2} g_{\alpha\rho} \partial_{A} g_{\beta\rho},
\]
\[
\Gamma_{BA}^{\alpha} = -\frac{1}{2} g^{\alpha\rho} \partial_{A} g_{B\rho}, \quad \Gamma_{B\alpha}^{A} = \frac{1}{2} g^{AB} \partial_{A} g_{B\alpha},
\]
\[
\Gamma_{A\alpha}^{\beta} = -\frac{1}{2} g^{AB} \partial_{D} g_{B\alpha}, \quad \Gamma_{B\alpha}^{\beta} = \frac{1}{2} g^{AB} (\partial_{B} g_{C\beta} + \partial_{C} g_{B\beta} - \partial_{\beta} g_{BC}),
\]
where
\[
g_{\alpha\beta} \delta_{a}^{\beta} = \delta_{a}^{\alpha}, \quad g^{AB} \delta_{DB} = \delta_{B}^{A}.
\]
The only nonvanishing components of $M_{abc}$, $E_{a}$, $W_{a}$ are thus
\[
M_{aAB} = \partial_{a} g_{AB}, \quad M_{A\alpha\beta} = - \partial_{A} g_{\alpha\beta},
\]
\[
E_{A} = - \partial_{A} \log |\det(g_{\alpha\beta})|, \quad W_{a} = - \partial_{a} \log |\det(g_{AB})|.
\]

\[
(7.2)
\]
from which we deduce that those of $T_{abc}$ are

$$T_{\alpha AB} = \partial_\alpha g_{AB} + \frac{1}{n-p} g_{AB} W_\alpha, \ T_{A\alpha\beta} = \partial_A g_{\alpha\beta} + \frac{1}{p} g_{\alpha\beta} E_A, \quad (7.3)$$

which are identically zero if the metric tensor represents a conformally separable space. To prove the converse we use the equivalent condition in terms of the covariant derivative with respect to the bi-conformal connection (proposition 6.1)

$$\bar{\nabla}_c P_{ab} = -\frac{1}{p} E_c P_{ab}, \quad \bar{\nabla}_c \Pi_{ab} = -\frac{1}{n-p} W_c \Pi_{ab}. \quad (7.4)$$

Proceeding along the same lines as in theorem 7.1 we can easily show that this condition entails that the space is separable (we only need to replace the components of the metric connection by those of the bi-conformal connection $\bar{\gamma}^{bc}_{\alpha}$). Thus we are left with the couple of equations (7.3) equalled zero. The general solution of the resulting PDE system is

$$g_{\alpha\beta} = G_{\alpha\beta}(x) e^{A_1(x^\gamma)}, \quad g_{AB} = G_{AB}(x^D) e^{A_2(x^\gamma)},$$

where $G_{\alpha\beta}, G_{AB}, A_1, A_2$ are arbitrary functions of their respective arguments with no restrictions other than $\det(G_{\alpha\beta}) \neq 0, \det(G_{AB}) \neq 0$. Comparing these expression with the fifth point of definition 7.2 the result follows. \(\square\)

This theorem clearly states the geometric relevance of $T_{abc}$ as a tool to characterize conformally separable pseudo-Riemannian manifolds. In fact the condition $T_{abc} = 0$ can be re-written in terms of the factors $\Xi_1, \Xi_2$ introduced in the definition of a conformally separable metric. To that end we use the equivalent condition (6.17) and replace the 1-forms $E_a, W_a$ by the values given by equation (7.2) which are

$$E_a = -p \Pi_a^\gamma \partial_\gamma \log|\Xi_1|, \quad W_a = (p-n) \Pi_a^\gamma \partial_\gamma \log|\Xi_2|,$$

whence

$$\nabla_b P_{ac} = P_{bc} u_a + P_{ab} u_c - P_{ar}^\gamma u_r g_{bc} - P_{ar}^\gamma u_r g_{ab}, \quad (7.4)$$

where

$$u_a = \frac{E_a}{2p} + \frac{W_a}{2(n-p)}.$$

If $\Xi_1 = \Xi_2 = \Xi$ then the factor $\Xi$ is invariantly determined by the relation

$$\frac{E_a}{p} + \frac{W_a}{n-p} = -\partial_a \log|\Xi|,$$

which means that the 1-form $u_a$ of (7.4) is exact. This characterization of conformally reducible pseudo-Riemannian manifolds was already proven in [1] by other means. Finally let us point out that studies of conformally reducible spaces in the particular case of dimension four have been performed in [3].

Bi-conformally flat spaces are a particular and interesting case of conformally separable pseudo-Riemannian manifolds so they comply with the condition written above. In this case we can even go further and refine the characterization in terms of the tensor $T_{abc}$ for these spaces.
Theorem 7.3. A conformally separable pseudo-Riemannian manifold with leaf metrics of rank greater than 3 is bi-conformally flat if and only if the tensor $T^a_{bcd}$ constructed from its leaf metrics is identically zero.

Proof. We choose the same coordinates and notation for our conformally separable space as in example 3.1. Using the components of the bi-conformal connection calculated there for this metric we deduce after some algebra

$$\bar{R}^\alpha_{\beta\gamma\phi} = R^\alpha_{\beta\gamma\phi}, \quad \bar{R}^A_{BCD} = R^A_{BCD}, \quad \bar{R}^\alpha_{\beta F\phi} = \partial_F \Gamma^\alpha_{\phi\beta}, \quad \bar{R}^A_{BF\phi} = -\partial_\phi \bar{R}^A_{FB}$$

$$L^0_{\alpha\beta} = 2R_{\alpha\beta}, \quad L^1_{AB} = 2R_{AB} + \frac{R^C}{1-n+p}g_{AB}, \quad L^0 = L^1 = 0,$$

$$L^0_{A\alpha} = \frac{2(2-p)}{p} \partial_A F^\alpha_{\rho}, \quad L^1_{\alpha A} = \frac{2(2-n+p)}{n-p} \partial_\alpha R^A_{AR}$$

(7.5)

where $R_{\alpha\beta}, R_{AB}, R^1_{\gamma}$ and $R^C_{\gamma}$ are the Ricci tensors and Ricci scalars of $R^\alpha_{\beta\gamma\phi}$ and $R^A_{BCD}$ respectively (observe that these curvature tensors are calculated from the leaf metrics and not from their conformal counterparts $G_{\alpha\beta}$ and $G_{AB}$). The tensor components not shown in (7.5) are equal to zero. From this and equations (5.2), (3.21) we get that the only nonvanishing components of the tensor $T^a_{bcd}$ are

$$T^\alpha_{\beta\gamma\phi} = 2C^\alpha_{\beta\gamma\phi}, \quad T^A_{BCD} = 2C^A_{BCD},$$

(7.6)

being $C^\alpha_{\beta\gamma\phi}$ and $C^A_{BCD}$ the Weyl tensors constructed from each leaf metric through the relations

$$C^\alpha_{\beta\gamma\phi} = R^\alpha_{\beta\gamma\phi} + \frac{1}{2-p} (g_{\beta\gamma} L^0_{\phi}, g^{\rho\alpha} + \delta^\alpha_{\phi} L^0_{\beta\gamma}),$$

$$C^A_{BCD} = R^A_{BCD} + \frac{1}{2-n+p} (g_{B[D} L^1_{C]E}, g^{EA} + \delta^A_{[C} L^1_{D]B}).$$

Hence $T^\alpha_{\beta\gamma\phi}$ and $T^A_{BCD}$ are both zero if and only if the leaf metrics $g_{\alpha\beta}$ and $g_{AB}$ are both conformally flat which proves the theorem. □

Remark 7.1. Note that the tensors $\bar{R}^\alpha_{\beta\gamma\phi}$ and $\bar{R}^A_{F\phi}$ do not vanish in general which means that the bi-conformal connection does not stem from a metric tensor in this case.

From the calculations performed above is clear that theorem 7.3 can be generalized to conformally separable spaces in which only one of the leaf metrics is conformally flat.

Theorem 7.4. Under the assumptions of theorem 7.3 a leaf metric is conformally flat if and only if the tensor $||C^a_{bcd}$ calculated from the leaf metric $P_{ab}$ is equal to zero.

Proof. From the proof of theorem 7.3 we deduce that for a pseudo-Riemannian conformally separable manifold the only non-vanishing components of $||C^a_{bcd}$ are

$$||C^\alpha_{\beta\gamma\phi} = T^\alpha_{\beta\gamma\phi} = 2C^\alpha_{\beta\gamma\phi},$$

so the vanishing of $||C^a_{bcd}$ implies that the Weyl tensor calculated from the corresponding leaf metric is zero as well. □

Remark 7.2. This last theorem can be restated as the converse of theorem 5.3.
In the case of any of the leaf metrics being of rank three is clear from the above that the corresponding tensor $\bar{\nabla}^a L^0_{bc} = 0$, which is zero as the Weyl tensor of any three dimensional pseudo-Riemannian metric vanishes identically. Hence the results presented so far cannot be used to characterize conformally separable pseudo-Riemannian manifolds with conformally flat leaf metrics. This lacking is remedied in the next theorem.

**Theorem 7.5.** A conformally separable pseudo-Riemannian manifold has a conformally flat leaf metric of rank three if and only if the condition

$$\bar{\nabla}^a L^0_{bc} = 0, \quad (7.7)$$

holds for the leaf metric tensor $P_{ab}$.

**Proof.** To show this we will rely on the notation and calculations performed in the proof of theorem 7.3. If the manifold is conformally separable then the only non-zero components of the tensor $\bar{\nabla}^a L^0_{bc}$ are

$$\bar{\nabla}^a L^0_{bc} = \nabla^a L^0_{bc}, \quad \bar{\nabla}^a L^0_{Be} = \nabla^a L^0_{Be}, \quad \bar{\nabla}^a L^0_{Ae} = \nabla^a L^0_{Ae}, \quad \bar{\nabla}^a L^0_{Bc} = \nabla^a L^0_{Bc},$$

where $\nabla^a$ is the covariant derivative compatible with $g_{\alpha \beta}$. Trivially

$$\bar{\nabla}^a L^0_{bc} = \nabla^a L^0_{bc}, \quad (7.8)$$

here the tensor $\bar{\nabla}^a L^0_{bc}$ is the Cotton-York tensor of the 3-metric $g_{\alpha \beta}$ which vanish if and only if such metric is conformally flat. Therefore to finish the proof of this theorem we must show that all the remaining components of $\bar{\nabla}^a L^0_{bc}$ are zero for any conformally separable metric $g_{ab}$. These are

$$\bar{\nabla}^a L^0_{bc} = \nabla^a L^0_{bc}, \quad 2\bar{\nabla}^a L^0_{bc} = \nabla^a L^0_{bc}.$$ 

Clearly the first expression is zero and the second one is worked out by replacing the connection coefficients by their expressions given in (3.21) and using of the identity

$$L^0_{\alpha \beta} = L^0_{\alpha \beta} + (2 - p)(2 \sigma_{\alpha \beta} + G_{\alpha \beta} \sigma^2),$$

$$\sigma = \frac{1}{2} \log |\Xi^1(x^a)|, \quad \sigma_{\alpha \beta} = \nabla^a \nabla^b \sigma - \partial_a \sigma \partial_b \sigma, \quad (\partial \sigma)^2 = G_{\alpha \beta} \partial_a \sigma \partial_b \sigma,$$

where $L^0_{\alpha \beta}$ is calculated using curvature tensors computed from $G_{\alpha \beta}$ and $\nabla$ is the connection compatible with this metric. The sought result comes after some simple algebraic manipulations. $\square$

Theorems 7.3, 7.4 and 7.5 supply an invariant geometric characterization of conformally separable pseudo-Riemannian manifolds with conformally flat leaf metrics. This means that the conditions imposed by theorem 6.1 are satisfied by a nontrivial set of pseudo-Riemannian manifolds, namely, that formed by bi-conformally flat spaces. These spaces can thus be characterized as those admitting a maximum number of bi-conformal vector fields. On the other hand it is straightforward to check (proposition 6.1 of [7]) that for these spaces any conformal Killing vector of the leaf metrics is a bi-conformal vector field of the whole metric $g_{ab}$. As the number of conformal Killing
vectors for each leaf metric is the biggest possible as well we get at once that for any bi-conformally flat space the total number of linearly independent bi-conformal vector fields is

\[ N = \tfrac{1}{2}p(p+1) + \tfrac{1}{2}(n-p)(n-p+1), \quad p, \ n-p \neq 2 \]

so the upper bound placed by theorem 4.2 is actually achieved. Summing up we obtain the following result

**Theorem 7.6.** A pseudo-Riemannian manifold possesses \( N \) bi-conformal vector fields \((P^a \neq 3, \Pi^a \neq n - 3)\) if and only if it is bi-conformally flat. \( \square \)

Bi-conformally flat spaces in which any of the leaf metrics has rank three still admit \( N \) linearly independent bi-conformal vector fields (in fact the complete integrability conditions are also satisfied for these spaces as a result of theorem 7.5). However, we do not know yet if there are spaces with \( N \) linearly independent bi-conformal vector fields with either of the projectors \( P_{ab} \) or \( \Pi_{ab} \) projecting on a 3-dimensional vector space other than bi-conformally flat spaces. This is so because in such case the complete integrability conditions (6.28) may in principle be fulfilled by other conformally separable spaces not necessarily with conformally flat leaf metrics. The true extent of these assertion and the complete characterization of spaces with \( N \) linearly independent bi-conformal vector fields under these circumstances will be placed elsewhere.

All in all the geometric conditions proven in theorems 7.2, 7.3, 7.4 and 7.5 provide a set of equations which can be used to search systematically for conformally flat foliations of a given pseudo-Riemannian manifold \((V, g)\). For if we set to \( P_{ab} \) the leaf metric of such foliation as we did before then the differential equations

\[ T_{abc} = 0, \quad ||C^{da}_{bcd} = 0, \quad (7.9) \]

could in principle allow us to find \( P_{ab} \) in a local coordinate system and hence the foliation. Of course the second condition can be replaced by \( \nabla[a L^b_{cde}] \) if we look for leaf metrics of rank three or \( T^a_{bde} = 0 \) if we wish to check whether the metric is bi-conformally flat or not. If such system does not admit any solution then previous results guarantee that our metric cannot be foliated by conformally flat hypersurfaces whose first fundamental forms are leaf metrics of a conformal separation.

A natural question is now the generalization of the above conditions to pseudo-Riemannian manifolds which are not conformally separable, i.e. there is a set of equations similar to (7.9) for cases with \( T_{abc} \neq 0 \)? A first thought could be that (7.9) or its generalizations for three dimensional cases with the condition \( T_{abc} = 0 \) removed would still be true for metrics not conformally separable. The results of the calculations performed in example 8.2 hint towards this direction.

### 8. Examples

**Example 8.1.** As our first example we consider the four dimensional pseudo-Riemannian manifold with metric given by

\[ ds^2 = (\Psi^2 \sin^2 \theta - \alpha^2)dt^2 + 2\Psi^2 \sin^2 \theta d\phi dt + B^2 (dr^2 + r^2 d\theta^2) + \Phi^2 d\phi^2, \]

where the coordinate ranges are \(-\infty < t < \infty, \ 0 < r < \infty, \ 0 < \theta < \pi, \ 0 < \theta < 2\pi\) and the functions \( \Psi, \ \alpha, \ B \) and \( \Phi \) only depend on the the coordinates \( r, \ \theta \). We will try to find out the conditions under which the metric is conformally separable with
the hypersurfaces \( t = \text{cons} \) as one of the leaf metrics. A simple calculation shows that the projector \( P^a_b \) coming from these hypersurfaces is (now and henceforth all the components omitted in an explicit tensor representation are understood to be zero)

\[
P^a_r = P^\theta_\theta = P^\phi_\phi = 1, \quad P^\phi_\ell = \frac{\Psi^2}{\phi^2},
\]

which entails

\[
P_{tt} = \frac{\Psi^4}{\phi^2} \sin^2 \theta, \quad P_{rr} = B^2, \quad P_{\theta \theta} = r^2 B^2, \quad P_{\phi \phi} = \Phi^2 \sin^2 \theta, \quad P_{t \phi} = \Psi^2 \sin^2 \theta.
\]

From here we can calculate the components of the tensor \( T_{abc} \) and set them equal to zero. After doing that we find the following independent conditions

\[
-\Psi \Phi_r + \Psi_r \Phi = 0, \quad -\Psi \Phi_\theta + \Psi_\theta \Phi = 0,
\]

\[
(\alpha^2 \Phi^2 + \Psi^3 (2 \Phi^2 - \Phi^2) \sin^2 \theta)(-\alpha \Phi^3 \Phi_r + \Phi_r (\Phi^2 - 2 \Psi^2)) = 0,
\]

\[
\Psi^3 \Phi_\theta \sin^2 \theta - \Psi \Phi^3 (\Psi \cos \theta + 2 \Psi \Phi \sin \theta) \sin \theta + \Phi^3 (-\alpha \Phi_x + \Psi \sin \theta (\Psi \cos \theta + \Psi \Phi \sin \theta)) = 0,
\]

which are fulfilled if and only if

\[
\Phi = \Psi = k \sin \theta, \quad \alpha = \pm k \sin^2 \theta, \quad k = \text{cons}.
\]

Under these conditions the metric takes the form

\[
ds^2 = k^2 \sin^4 \theta (2 dt d\phi + d\phi^2) + B^2 (dr^2 + r^2 d\theta^2),
\]

which is conformally separable as desired.

**Example 8.2.** In the foregoing results we have only concentrated on conformally separable pseudo-Riemannian manifolds but nothing was said about manifolds with conformally flat slices and not conformally separable. To illustrate this case let us consider the four dimensional pseudo-Riemannian manifold given in local coordinates \( \{x^1, x^2, x^3, x^4\} \) by

\[
ds^2 = \Phi(x)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + 2 \sum_{i=1}^{3} \beta_i(x)dx^i dx^4 + \Psi(x)(dx^4)^2, \quad (8.1)
\]

where \( x = \{x^1, x^2, x^3, x^4\} \) and \( \Phi(x), \beta_i(x), \Psi(x) \) are functions at least \( C^3 \) in an open domain. Clearly the above line element is the most general four dimensional metric admitting a conformally flat foliation by three dimensional Riemannian hypersurfaces. The nonzero components of the orthogonal projector \( P^a_b \) associated to this foliation are

\[
P_1^1 = P_2^2 = P_3^3 = 1, \quad P_4^i = \beta_i(x)/\Phi(x), \quad i = 1, 2, 3, \quad (8.2)
\]

from which we easily get

\[
P_{11} = P_{22} = P_{33} = \Phi(x), \quad P_{44} = \beta_i(x), \quad i = 1, 2, 3, \quad P_{44} = \sum_{i=1}^{3} \beta_i^2(x)/\Phi(x).
\]

Using this we can check condition (7.7) and see what is obtained. This is a rather long calculation which is easily performed with any of the computer algebra systems available today (the system used here was GRTensorII). The result is that the tensor \( \nabla_{[a} \nabla_{b]} c^0 \) does not vanish in this case although a calculation using (8.2) shows the important property

\[
P_a^a P_i^b P^q_c \nabla_{[r} L^0_{s]} q = 0. \quad (8.3)
\]
Theorem 8.1. A necessary condition that a four dimensional pseudo-Riemannian mani-
ifolds can be foliated by conformally flat Riemannian hypersurfaces with associated
orthogonal projector $P_{ab}$ is equation (8.3).

This result suggests that it may well be possible to generalize the conditions of theo-
rem 7.5 to metrics of arbitrary dimension which are not conformally separable replacing
these conditions by (8.3). This example can be generalized if we consider pseudo-
Riemannian manifolds of higher dimension foliated by hypersurfaces of dimension ar-
bitrary and not necessarily Riemannian. The condition which must be checked in this case is $\nabla^a_{b c d} = 0$ being this condition found to be true in all the examples tried.
Therefore it seems that the conditions of theorems 7.4 and 7.5 hold even though the
pseudo-Riemannian manifold is not conformally separable. We thus feel inclined to be-
lieve that these theorems are true for any pseudo-Riemannian manifold admitting con-
formally flat foliations and they can indeed be used to search systematically for such
foliations. The true extent of this assertion is under current research.

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A. Basic identities involving the Lie derivative

In this appendix we recall some properties of the Lie derivative needed in the main text.
Despite their basic character, they are hardly presented in basic Differential Geometry
textbooks and the author is only aware of [13,11] as the only references in which they
are studied.

Proposition A.1. For any symmetric connection $\bar{\nabla}$ defined in a differentiable manifold
$V$, any vector field $\bar{\xi}$ at least $C^2$ and a tensor field $T_{b_1...b_q}^{a_1...a_p} \in T^p_q(V)$ we have the
following identities

\[ \mathcal{L}_{\bar{\xi}} \bar{\gamma}_{b c}^a = \bar{\nabla}_b \bar{\nabla}_c \bar{\xi}^a + \xi^d \bar{R}_{cd b}^a, \]  
(A.1)
\[ \bar{\nabla}_e \mathcal{L}_{\bar{\xi}} T_{b_1...b_q}^{a_1...a_p} - \mathcal{L}_{\bar{\xi}} \bar{\nabla}_e T_{b_1...b_q}^{a_1...a_p} = - \sum_{j=1}^{s} (\mathcal{L}_{\bar{\xi}} \bar{\gamma}_{a j}^r b_{j} b_{j+1} b_{j+1}... b_{q}) T_{b_1...b_{j-1}}^{a_1...a_{j+1}} + \sum_{j=1}^{q} (\mathcal{L}_{\bar{\xi}} \bar{\gamma}_{j c b_j}^r) T_{b_1...b_{j-1}}^{a_1...a_{j+1}}... b_{q}, \]  
(A.2)
\[ \mathcal{L}_{\bar{\xi}} \bar{\mathcal{R}}_{c a b} = \bar{\nabla}_c (\mathcal{L}_{\bar{\xi}} \bar{\gamma}_{d a c b}) - \bar{\nabla}_d (\mathcal{L}_{\bar{\xi}} \bar{\gamma}_{a b c}), \]  
(A.3)

where $\bar{\gamma}_{a b}^r$ are the components of the connection $\bar{\nabla}$ and $\bar{R}_{b c d}^a$ its curvature. Furthermore if a
metric tensor $g_{a b}$ is set in $V$ and $\nabla$ is now the metric connection associated
to it then

\[ \mathcal{L}_{\bar{\xi}} \bar{\gamma}_{b c}^a = \frac{1}{2} \bar{\gamma}_{a}^{b c} \left[ \bar{\nabla}_b (\mathcal{L}_{\bar{\xi}} g_{c e}) + \bar{\nabla}_c (\mathcal{L}_{\bar{\xi}} g_{b e}) - \bar{\nabla}_e (\mathcal{L}_{\bar{\xi}} g_{b c}) \right]. \]  
(A.4)
B. Finite transformation rules

In this appendix we show explicitly that the invariance laws found in theorems 5.1 and 5.2 for one-parameter groups of bi-conformal transformations do indeed hold for any bi-conformal transformation. Let \( g_{ab} \) be a pseudo-Riemannian metric and \( P_{ab} \), \( \Pi_{ab} \) orthogonal and complementary projectors with respect to \( g_{ab} \). Define now the metric \( \tilde{g}_{ab} \) by

\[
\tilde{g}_{ab} = Z P_{ab} + X \Pi_{ab}, \quad \tilde{g}^{ab} = \frac{1}{Z} P_{ab} + \frac{1}{X} \Pi_{ab}, \quad \tilde{g}_{ap} \tilde{g}^{pb} = \delta_a^b,
\]

where \( Z \) and \( X \) are strictly positive \( C^2 \) functions on the manifold. We are going now to calculate the tensors \( \nabla_a P^b_c, A^a_{bc} \), \( ||C^a_{bcd}|| \) using the metric \( \tilde{g}_{ab} \) and the projectors \( Z P_{ab}, X \Pi_{ab} \) instead of \( g_{ab}, P_{ab}, \Pi_{ab} \). The result will be that these tensors are kept invariant as expected. Note that in principle the metric \( \tilde{g}_{ab} \) does not need to be the bi-conformal transformed of \( g_{ab} \) (actually our manifold may even not admit such transformations).

We will add a prime over the tensor objects calculated with \( \tilde{g}_{ab} \) and try to relate these objects to their unprimed counterparts. The basic equation is the relation between the curvature tensors

\[
\bar{\nabla}_a P^b_c = \nabla_a P^b_c, \quad 1 \frac{1}{Z} \bar{\nabla}_a (Z P_{bc}) = P^{ar} \bar{\nabla}_r (Z P_{bc}),
\]

so \( \nabla_a P^b_c \) and \( A^a_{bc} \) are kept invariant. To show the invariance of \( ||C^a_{bcd}|| \) takes more efforts as we need to find the relation between the curvature tensors \( R^a_{bcde} \) and \( \tilde{R}^a_{bcde} \). This is done by means of (3.18) getting

\[
\begin{align*}
\tilde{R}^a_{bcde} &= \tilde{R}^a_{bcde} + \nabla_\gamma [\tilde{Z}_d] P^a_d + P^a_d \nabla_\gamma \tilde{Z}_b - P^a_d \nabla_\gamma \tilde{Z}_c \tilde{P}^b_c + \nabla_\gamma [\tilde{X}_d] \tilde{A}^a_d + \nabla_\gamma [\tilde{X}_d] \tilde{A}^a_d \\
&- \tilde{P}_b \nabla_\gamma \tilde{X}^a_c + \tilde{Z}_b \nabla_\gamma \tilde{P}^a_c + \tilde{Z}_c \nabla_\gamma \tilde{P}^a_b - \tilde{Z}^a \nabla_\gamma \tilde{P}^b_c + \tilde{X}_d \nabla_\gamma \tilde{P}^a_d \\
&- \tilde{X}^a \nabla_\gamma \tilde{P}^b_c \tilde{Z}_d + \tilde{Z}_r \tilde{Z}_d \nabla_\gamma \tilde{P}^a_d [\tilde{P}_b [\tilde{Z}] + \tilde{Z}_r \tilde{Z}_d \nabla_\gamma \tilde{P}^a_d [\tilde{P}_b [\tilde{Z}] + \tilde{Z}_r \tilde{Z}_d \nabla_\gamma \tilde{P}^a_d [\tilde{P}_b [\tilde{Z}]] + \\
&+ \frac{1}{2} \tilde{P}^a_d \tilde{Z}_d \tilde{X}_b + \tilde{X}_b \tilde{X}_d \tilde{Z}^a \tilde{P}^b_c [\tilde{Z}_b + \tilde{Z}_r \tilde{Z}_d \tilde{P}^a_d [\tilde{P}_d [\tilde{Z}]] + \frac{1}{2} \tilde{P}^a_d \tilde{Z}_d \tilde{X}_b + \tilde{X}_b \tilde{X}_d \tilde{Z}^a \tilde{P}^b_c [\tilde{Z}_b + \tilde{Z}_r \tilde{Z}_d \tilde{P}^a_d [\tilde{P}_d [\tilde{Z}]]],
\end{align*}
\]

from which we get

\[
\begin{align*}
\tilde{T}^0_{ab} &= L^0_{ab} + (2 - p) \nabla_a \tilde{Z}_b - 2 \tilde{Z}_r \nabla_r P_{ab} + \frac{1}{2} (p - 2) \left( \tilde{Z}_a \tilde{Z}_b + \frac{1}{2} \tilde{Z}_r \tilde{Z}_r P_{bc} \right) \\
\tilde{T}^1_{ab} &= L^1_{ab} + (2 - 2 + n + p) \nabla_a \tilde{X}_b - 2 \tilde{X}_r \nabla_r \Pi_{ab} + \frac{1}{2} (p - 2) \left( \tilde{X}_a \tilde{X}_b + \frac{1}{2} \tilde{X}_r \tilde{X}_r \Pi_{bc} \right).
\end{align*}
\]

Plugging these expressions in the definitions of \( T^a_{bcde} \) and \( T^a_{bcde} \) and using the above relations involving \( \tilde{T}^a_{bcde} \) and \( \tilde{P}^a_{bcde} \) we find an equation similar to (5.1) but containing the difference \( T^a_{bcde} - T^a_{bcde} \) instead of the Lie derivative of \( T^a_{bcde} \) and the variables \( Z_a, X_a \) in place of \( \phi_a, \chi_a \). From this point it is already easy to conclude

\[
|| \tilde{T}^a_{bcde} = ||C^a_{bcd},
\]

as desired. Of course all this procedure can be repeated \textit{mutatis mutandis} to get similar invariance rules for the tensors \( A^a_{bc}, \nabla_a \Pi^b_c \) and \( C^a_{bcd} \).
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