A WEAK TO STRONG TYPE $T_1$ THEOREM FOR GENERAL SMOOTH CALDERÓN-ZYGMUND OPERATORS WITH DOUBLING WEIGHTS, II

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Abstract. We consider the weak to strong type problem for two weight norm inequalities for Calderón-Zygmund operators with doubling weights. We show that if a Calderón-Zygmund operator $T$ is weak type $(2, 2)$ with doubling weights, then it is strong type $(2, 2)$ if and only if the dual cube testing condition for $T^*$ holds, alternatively if and only if the dual cancellation condition of Stein holds. This continues the weighted theory begun in [Saw6].

The testing condition can be taken with respect to either cubes or balls, and more generally, this is extended to a weak form of $Tb$ theorem.

Finally, we show that for all pairs of locally finite positive Borel measures, and all Stein elliptic Calderón-Zygmund operators $T$, the weak type $(2, 2)$ inequalities for $T$ and its associated maximal truncations operator $T_0$ are equivalent. Thus the characterization of weak type for $T_0$ in [LaSaUr1] applies to $T$ as well.

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1. Introduction

This paper is a sequel to the second author’s paper \[\text{[Saw0]},\] and continues to be dedicated to the memory of Professor Elias M. Stein.

In this paper we attack the problem of characterizing the two weight norm inequality for Calderón-Zygmund operators in two separate steps. First, we characterize the weak type problem for general weights in terms of a testing type condition. Second, we solve the weak to strong type problem for doubling weights in terms of the standard cube testing condition.

**Problem 1.** Given a pair of locally finite positive Borel measures on \(\mathbb{R}^n\), and an \(\alpha\)-fractional Calderón-Zygmund operator \(T^{\alpha}\) that is weak type \((2, 2)\), i.e.
\[
\|T^{\alpha}_\sigma f\|_{L^2(\omega)} \leq \text{weak } M(\sigma, \omega) \|f\|_{L^2(\sigma)},
\]
characterize when \(T^{\alpha}_\sigma\) is strong type \((2, 2)\), i.e.
\[
\|T^{\alpha}_\sigma f\|_{L^2(\omega)} \leq M(\sigma, \omega) \|f\|_{L^2(\sigma)}.
\]

One rationale for considering the two step approach used here is the notorious difficulty of the general \(T1\) conjecture, namely that an \(\alpha\)-fractional Calderón-Zygmund operator \(T^{\alpha}\) is strong type if and only if both the testing condition and its dual formulation hold, as well as an extension of the classical \(A_2\) condition of Muckenhoupt. In the two step approach used here, we are concerned with two logically easier problems, that of characterizing weak type boundedness, followed by characterizing the passage from weak type to strong type. It is notable that there is no known counterexample to the general \(T1\) conjecture, yet the conjecture has only been shown to hold for the Hilbert transform (see the two part paper \[\text{[LaSaShUr3], [Lac]}\] and certain perturbations (see \[\text{[SaShUr10]}\]).

A corollary of our main theorem below is that when \(T^{\alpha}\) is a Stein elliptic Calderón-Zygmund operator on \(\mathbb{R}^n\), and when \(\sigma\) and \(\omega\) are both doubling measures, then given that \(T^{\alpha}\) is weak type \((2, 2)\), it is strong type \((2, 2)\) if and only if the cube testing constant \(\Xi_{T^{\alpha}}(\sigma, \omega)\) for \(T^{\alpha}\) is finite, where
\[
\Xi_{T^{\alpha}}(\sigma, \omega) \equiv \sup_{Q \in \mathcal{P}^n} \frac{1}{|Q}|Q \int_Q |T^{\alpha}_\sigma 1_Q|^2 \omega.
\]

Under the above hypotheses of doubling measures, we can also replace the cube testing constant \(\Xi_{T^{\alpha}}(\sigma, \omega)\) with the cancellation constant of Stein \(\mathcal{A}_{K^{\alpha}}(\sigma, \omega)\), which is defined as the best constant \(C(\sigma, \omega)\) in the inequality,
\[
\int_{|x-x_0|<\varepsilon} \left| \int_{|x-y|<\varepsilon} K^{\alpha}(x, y) \, d\sigma(y) \right|^2 \omega(x) \leq C(\sigma, \omega) \int_{|x_0-y|<\varepsilon} d\sigma(y),
\]
for all \(0 < \varepsilon < N\) and \(x_0 \in \mathbb{R}^n\).

The cancellation constant \(\mathcal{A}_{K^{\alpha}}(\sigma, \omega)\) has the historical form of bounding, in an average sense, integrals of the kernel over annuli, and arose originally in the setting of Lebesgue measure, in the form of the \(T1\) theorem presented by E. Stein in \[\text{[Ste2}, \text{Theorem 4, page 306]}\). Similar suprema and inequalities define the dual constants \(\Xi_{T^{\alpha}}^*\) and \(\mathcal{A}_{K^{\alpha}}^*\), in which the measures \(\sigma\) and \(\omega\) are interchanged and \(K^{\alpha}(x, y)\) is replaced by \(K^{\alpha*}(x, y) = K^{\alpha}(y, x)\). It should also be noted that (1.3) is not simply the testing condition for a truncation of \(T\) over a ball.

Define the classical Muckenhoupt constant \(A^\alpha_{2}(\sigma, \omega)\) by
\[
A^\alpha_{2}(\sigma, \omega) \equiv \sup_{Q \in \mathcal{P}^n} \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}},
\]
and the indicator / cube testing constant \(\Xi^{\text{ind}}_{T^{\alpha}}(\sigma, \omega)\) by
\[
\Xi^{\text{ind}}_{T^{\alpha}}(\sigma, \omega) \equiv \sup_{Q \in \mathcal{P}^n} \left| \frac{1}{|Q|_{\sigma}} \sup_{E \subset Q} \left| \int_Q (T^{\alpha}_{\sigma} 1_E)^2 \omega \right| \right|^\frac{1}{2}.
\]

\[\text{[see e.g. [Hyt2}, \text{[Lac], [LaSaShUr3], [LaWi], [NTV4], [SaShUr7], [SaShUr12], [Vol] and many more of the references.}\]
Our main theorem is this, which improves upon the corresponding theorem in [Saw6] by eliminating one of the indicator / cube testing conditions, as well as a comparability hypothesis on the pair of measures \((\sigma, \omega)\). As in [Saw6], one can also replace the testing constants with the cancellation constants of Stein.

**Theorem 2.** Suppose \(\sigma, \omega\) are doubling measures on \(\mathbb{R}^n\), and that \(T^\alpha\) is a smooth Stein elliptic Calderón-Zygmund operator on \(\mathbb{R}^n\). Then (1.3) holds if and only if the testing constants in (1.3) and (1.6) are finite, and furthermore, we have the equivalences,

\[
\mathcal{R}_{T^\alpha}(\sigma, \omega) \approx \mathcal{F}_{T^\alpha}(\sigma, \omega) + \mathcal{K}^{\text{ind}}_{T^\alpha}(\sigma, \omega) \approx \mathcal{S}_{T^\alpha}(\sigma, \omega) + \mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega) \approx \mathcal{K}^{\ast}_{T^\alpha}(\sigma, \omega) + \mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega) ,
\]

and the corresponding equivalences with \(T^\alpha\) and \(T^{\ast, \alpha}\) and their constants interchanged. Here \(\mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega)\) denotes the weak type norm of \(T^\alpha\).

**Remark 3.** It is easy to see that the second line in (1.7) follows from the first line in (1.7) using

\[
\mathcal{R}_{T^\alpha}(\sigma, \omega) \geq \text{weak } \mathcal{R}_{T^\alpha}(\sigma, \omega) \geq \mathcal{K}^{\text{ind}}_{T^\alpha}(\sigma, \omega) .
\]

To see this last display, recall that the dual space of the Banach space \(L^{2, 1}(\mu)\) is \(L^{2, \infty}(\mu)\) for any \(\sigma\)-finite nonatomic measure, see e.g. [Gra] Theorem 1.4.17 (v) page 52. Thus

\[
\mathcal{N}^{\text{rest}*}_{T^\alpha}(\omega, \sigma) \equiv \sup_{\|g\|_{L^2, 1}(\omega) \leq 1} \|T^\alpha_{\omega} g\|_{L^2(\sigma)} = \sup_{\|g\|_{L^2, 1}(\omega) \leq 1} \sup_{\|f\|_{L^2(\sigma)} \leq 1} \left| \int (T^\alpha_{\omega} f) g \, d\sigma \right| = \sup_{\|f\|_{L^2(\sigma)} \leq 1} \|T^\alpha_{\omega} f\|_{L^2(\omega)} = \text{weak } \mathcal{R}_{T^\alpha}(\sigma, \omega) ,
\]

and then by [StWa] Theorem 3.13 page 195] we have

\[
\mathcal{R}_{T^\alpha}(\sigma, \omega) \geq \text{weak } \mathcal{R}_{T^\alpha}(\sigma, \omega) = \mathcal{N}^{\text{rest}*}_{T^\alpha}(\omega, \sigma) \approx \sup_{E} \sqrt{\frac{\int |T^\alpha_{\omega} 1_E|^2 \, d\sigma}{|E|}} \approx \mathcal{K}^{\text{ind}}_{T^\alpha}(\omega, \sigma) .
\]

We thus have the following solution to a case of the weak to strong type problem.

**Corollary 4.** Suppose \(\sigma, \omega\) are doubling measures, and \(T^\alpha\) is a Stein elliptic Calderón-Zygmund operator on \(\mathbb{R}^n\) of weak type \((2, 2)\) with respect to \((\sigma, \omega)\). Then

\[
T^\alpha \text{ is strong type } (2, 2) \iff \mathcal{R}_{T^\alpha}(\sigma, \omega) \text{ is finite} \iff \mathcal{K}^{\ast}_{T^\alpha}(\sigma, \omega) \text{ is finite},
\]

and moreover,

\[
\mathcal{R}_{T^\alpha}(\sigma, \omega) \approx \mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega) + \mathcal{S}_{T^\alpha}(\sigma, \omega) \approx \mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega) + \mathcal{K}^{\ast}_{T^\alpha}(\sigma, \omega) .
\]

### 1.1. Weak type inequalities.

The above corollary begs the question of characterizing the weak type norm \(\mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega)\) in terms of a testing condition, which is the first step in our two step program. In order to shed light on this question, we recall the maximal truncation operator \(T^\alpha\) associated to \(T^\alpha\) defined by \(T^\alpha_{\omega} f(x) \equiv \sup_{\epsilon > 0} \|T_{\epsilon, \omega} f(x)\|\). In the final section of the paper we show that for a Stein elliptic operator \(T\), the weak type inequalities for \(T\) and \(T^\alpha\) are equivalent.

**Theorem 5.** Suppose \(T^\alpha\) is an \(\alpha\)-fractional Calderón-Zygmund operator in \(\mathbb{R}^n\) with kernel satisfying just (9.7) below, and \(\sigma\) and \(\omega\) are locally finite positive Borel measures in \(\mathbb{R}^n\). Then we have

\[
\mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega) \leq \mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega) \leq \mathcal{N}^{\text{weak}}_{T^\alpha}(\sigma, \omega) + \sqrt{A_2^{\alpha}(\sigma, \omega)} ,
\]

where \(A_2^{\alpha}(\sigma, \omega)\) is the weak type norm for \(A_2^{\alpha}(\sigma, \omega)\).
Remark 7. We showed above that the weak type $(2,2)$ norms of $T^\alpha$ and $T^\alpha_\sharp$ respectively. If in addition $T^\alpha$ is Stein elliptic, then the Muckenhoupt constant $\sqrt{A_2^\alpha(\sigma,\omega)}$ can be dropped from the right hand side of (1.8).

Then from the characterization of weak type for maximal truncations in [LaSaUr1] Theorem 1.8 (1), we obtain the corollary that

$$\mathcal{N}_T^{\text{weak}}(\sigma,\omega) \approx \mathcal{N}_T^{\text{weak}}(\sigma,\omega) \approx \mathcal{T}_{T^\alpha}^{\text{flat}}(\sigma,\omega),$$

where the flat testing constant $\mathcal{T}_{T^\alpha}^{\text{flat}}(\sigma,\omega)$ (introduced in [LaSaUr1]) is the best constant in the inequality,

$$\left| \int_Q T^\alpha_{\flat,\sigma} (1_Q f)(x) \, d\sigma(x) \right| \leq \mathcal{T}_{T^\alpha}^{\text{flat}}(\sigma,\omega) \|f\|_{L^2(\sigma)} \|1_Q\|_{L^2(\omega)}.$$

Note that if $T^\alpha$ were linear, then the inequality in (1.10) would be equivalent to the dual cube testing condition,

$$\int_Q T^\alpha_{\flat,\sigma} (1_Q f)(x) \, d\sigma(x) \lesssim |Q|_\omega.$$

In general, we can only apply duality to linearizations $L$ of $T^\alpha$ (see [LaSaUr1] for definitions), and (1.10) is then equivalent to

$$\int_Q L_{\omega} (1_Q f)(x) \, d\sigma(x) \leq C \mathcal{T}_{T^\alpha}^{\text{flat}}(\sigma,\omega) |Q|_\omega,$$

taken uniformly over all linearizations $L$ of $T^\alpha$.

Note also that in the weak type characterization (1.9), the conditions required of the kernel of $T$ are very weak, namely just (9.1) below, which consists of the size condition together with a Dini smoothness condition in the first variable of the kernel. In conclusion we obtain the following theorem.

**Theorem 6.** Suppose $\sigma$ and $\omega$ are doubling measures on $\mathbb{R}^n$, and that $T^\alpha$ is a smooth Stein elliptic Calderón-Zygmund operator on $\mathbb{R}^n$. Then,

$$\mathcal{N}_T^{\alpha}(\sigma,\omega) \approx \mathcal{T}_{T^\alpha}(\sigma,\omega) + \mathcal{T}_{T^\alpha}^{\text{flat}}(\sigma,\omega).$$

**Remark 7.** We showed above that

$$\mathcal{T}_{T^\alpha}^{\text{ind}}(\omega,\sigma) \lesssim \mathcal{N}_T^{\text{weak}}(\sigma,\omega),$$

and so we have

$$\mathcal{T}_{T^\alpha}^{\text{ind}}(\omega,\sigma) \lesssim \mathcal{T}_{T^\alpha}^{\text{flat}}(\sigma,\omega).$$

Thus Theorem 6 follows from (1.11) and Theorem 5. However, we do not know how to obtain (1.11) without going through the weak type norm $\mathcal{N}_T^{\text{weak}}(\sigma,\omega)$ and using Theorem 5 and [LaSaUr1] Theorem 1.8 (1)).

**Remark 8.** The arguments below will show that for doubling measures $\sigma$ and $\omega$, and any smooth fractional Calderón-Zygmund operator $T^\alpha$ on $\mathbb{R}^n$, not necessarily Stein elliptic, we have the inequalities,

$$\mathcal{N}_T^{\alpha}(\sigma,\omega) \lesssim \mathcal{T}_{T^\alpha}(\sigma,\omega) + \mathcal{T}_{T^\alpha}^{\text{ind}}(\omega,\sigma) + \sqrt{A_2^\alpha(\sigma,\omega)},$$

$$\mathcal{N}_T^{\text{weak}}(\sigma,\omega) \lesssim \mathcal{T}_{T^\alpha}^{\text{flat}}(\sigma,\omega) + \sqrt{A_2^\alpha(\sigma,\omega)},$$

where (2.1) is assumed in the first line, and (9.1) in the second line.

2. Preliminaries

Denote by $\mathcal{P}^n$ the collection of cubes in $\mathbb{R}^n$ having sides parallel to the coordinate axes; all cubes mentioned in this paper will be elements of $\mathcal{P}^n$. A positive locally finite Borel measure $\mu$ on $\mathbb{R}^n$ is said to satisfy the doubling condition if $|2Q|_\mu \leq C_{\text{doub}} |Q|_\mu$ for all cubes $Q \in \mathcal{P}^n$. It is well known (see e.g. the introduction in [SaUr]) that doubling implies reverse doubling, which means that there exists a positive constant $\theta_\mu^{\text{rev}}$, called a reverse doubling exponent, such that

$$\sup_{Q \in \mathcal{P}^n} \frac{|sQ|_\mu}{|Q|_\mu} \leq s^{\theta_\mu^{\text{rev}}},$$

for all sufficiently small $s > 0$. 

2.1. Standard fractional singular integrals and the norm inequality. Let \( 0 \leq \alpha < n \) and \( \kappa \in \mathbb{N} \). We define a standard \((\kappa + \delta)\)-smooth \(\alpha\)-fractional Calderón-Zygmund kernel \( K^\alpha(x, y) \) to be a function \( K^\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) satisfying the following fractional size and smoothness conditions: for \( x \neq y \), and with \( \nabla_1 \) and \( \nabla_2 \) denoting gradient in the first and second variables respectively,

\[
\nabla_1^j K^\alpha(x, y) \leq C_{CZ} |x - y|^{\alpha - j - \kappa}, \quad 0 \leq j \leq \kappa,
\]

\[
|\nabla_2^\kappa K^\alpha(x, y) - \nabla_2^\kappa K^\alpha(x', y)| \leq C_{CZ} \left( \frac{|x - x'|}{|x - y|} \right)^\delta |x - y|^{\alpha - \kappa - \kappa}, \quad |x - x'| \leq \frac{1}{2},
\]

and where the same inequalities hold for the adjoint kernel \( K^{\alpha,*}(x, y) \equiv K^\alpha(y, x) \), in which \( x \) and \( y \) are interchanged, and where \( \nabla_1 \) is replaced by \( \nabla_2 \).

2.1.1. Ellipticity of kernels. Following [Ste] (39) on page 210, we say that an \(\alpha\)-fractional Calderón-Zygmund kernel \( K^\alpha \) is \emph{elliptic in the sense of Stein} if there is a unit vector \( u_0 \in \mathbb{R}^n \) and a constant \( c > 0 \) such that

\[
|K^\alpha(x, x + tu_0)| \geq c |t|^\alpha, \quad \text{for all } t \in \mathbb{R}.
\]

**Remark 9.** The functions and kernels in the Calderón-Zygmund operators considered here, are assumed to be complex-valued. However, it should be noted that for all of the sufficiency proofs in this paper, one may assume without loss of generality that the functions and kernels are real-valued. It is only in (2.2) that both real and imaginary parts might be needed.

2.1.2. Defining the norm inequality. We follow the approach in [SaShUr9] (see page 314). So we suppose that \( K^\alpha \) is a standard \((\kappa + \delta)\)-smooth \(\alpha\)-fractional Calderón-Zygmund kernel, and we introduce a family \( \{\eta_{\delta, R}^\alpha\}_{0 < \delta < R < \infty} \) of nonnegative functions on \([0, \infty)\) so that the truncated kernels \( K_{\delta, R}^\alpha(x, y) = \eta_{\delta, R}^\alpha(|x - y|) K^\alpha(x, y) \) are bounded with compact support for fixed \( x \) or \( y \), and uniformly satisfy (2.1).

Then the truncated operators

\[
T_{\sigma, \delta, R}^\alpha f(x) \equiv \int_{\mathbb{R}^n} K_{\delta, R}^\alpha(x, y) f(y) d\sigma(y), \quad x \in \mathbb{R}^n,
\]

are pointwise well-defined when \( f \) is bounded with compact support, and we will refer to the pair \( (K^\alpha, \{\eta_{\delta, R}^\alpha\}_{0 < \delta < R < \infty}) \) as an \(\alpha\)-fractional singular integral operator, which we typically denote by \( T^\alpha \), suppressing the dependence on the truncations.

**Definition 10.** We say that an \(\alpha\)-fractional singular integral operator \( T^\alpha = \left( K^\alpha, \left\{ \eta_{\delta, R}^\alpha \right\}_{0 < \delta < R < \infty} \right) \) satisfies the norm inequality

\[
\|T^\alpha f\|_{L^2(\omega)} \leq \mathcal{N}_{T^\alpha} \|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma),
\]

provided

\[
\|T_{\sigma, \delta, R}^\alpha f\|_{L^2(\omega)} \leq \mathcal{N}_{T^\alpha} (\sigma, \omega) \|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma), \quad 0 < \delta < R < \infty.
\]

**Independence of Truncations:** In the presence of the classical Muckenhoupt condition \( A^\alpha_2 \), the norm inequality (2.3) is independent of the choice of truncations used, including nonsmooth truncations as well - see [LaSaShUr3]. However, in dealing with the Energy Lemma (12) below, where \( \kappa \)th order Taylor approximations are made on the truncated kernels, it is necessary to use sufficiently smooth truncations. Similar comments apply to the Cube Testing conditions.

2.2. \(\kappa\)-cube testing conditions. In this subsection we describe a variety of testing conditions that arise in the course of our proof, but which do not appear in the statement of our main Theorem 2 where only the classical testing condition in (1.3) is used.

The \(\kappa\)-cube testing conditions associated with an \(\alpha\)-fractional singular integral operator \( T^\alpha \) are given by

\[
\left( \mathcal{T}_{T^\alpha, \sigma, \omega}^{(\gamma)} \right)^2 = \sup_{Q \in \mathcal{P}^n, 0 \leq |\beta| < \kappa} \frac{1}{|Q|_\omega} \int_Q \left| T^\alpha \left( 1_{Q^m} \right) \right|^2 \omega < \infty,
\]

\[
\left( \mathcal{T}_{(T^\alpha)^*, \omega, \sigma}^{(\gamma)} \right)^2 = \sup_{Q \in \mathcal{P}^n, 0 \leq |\beta| < \kappa} \frac{1}{|Q|_\sigma} \int_Q \left| (T^\alpha)^* \left( 1_{Q^m} \right) \right|^2 \sigma < \infty,
\]
where \((T^{\alpha,x})_Q = (T^\alpha)_Q\), with \(m^\beta_Q(x) = \left(\frac{x-c_Q}{\ell(Q)}\right)^\beta\) for any cube \(Q\) and multiindex \(\beta\), where \(c_Q\) is the center of the cube \(Q\), and where we interpret the right hand sides as holding uniformly over all sufficiently smooth truncations of \(T^\alpha\). Equivalently, in the presence of \(A^2_2\), we can take a single suitable truncation, see Independence of Truncations in Subsubsection 2.1.2.

We also use the triple \(\kappa\)-cube testing conditions in which the integrals are over the triple \(3Q\) of \(Q\):

\[
\left(\mathcal{T}^\kappa_{P,Q}\left(\sigma, \omega\right)\right)^2 \equiv \sup_{Q \subset I_{\ell} \in \mathcal{P}} \max_{0 \leq |\beta| < \kappa} \frac{1}{|Q|} \int_{3Q} \left|T^\alpha \left(1_{Q} m^\beta_{Q}\right)\right|^2 \omega < \infty,
\]

\[
\left(\mathcal{T}^\kappa_{R,Q}\left(\omega, \sigma\right)\right)^2 \equiv \sup_{Q \subset I_{\ell} \in \mathcal{P}} \max_{0 \leq |\beta| < \kappa} \frac{1}{|Q|} \int_{3Q} \left|T^{\alpha,x} \left(1_{Q} m^\beta_{Q}\right)\right|^2 \sigma < \infty.
\]

The following lemma from [Saw6, Subsection 4.1 on pages 12-13, especially Remark 15] was the point of departure for freeing the theory from reliance on energy conditions when the measures are doubling.

**Lemma 11 ([Saw6]).** If \(\sigma\) is a doubling measure, then for sufficiently large \(\kappa\) depending on the doubling constant of \(\sigma\), we have

\[
P^\kappa_{\sigma}(Q, \sigma) \approx \frac{|Q|_\sigma}{|Q|^\frac{1}{\kappa}} \text{ and } P^\kappa_{\sigma}(Q, \sigma)^2 |Q|_\sigma \leq C A^2_{2}(\sigma, \omega) |Q|_\sigma.
\]

### 2.3. Weighted Alpert bases for \(L^2(\mu)\) and \(L^\infty\) control of projections.

We now recall the construction of weighted Alpert wavelets in [RaSaWi], and refer also to [AlSaUr] for the correction of a small oversight in [RaSaWi]. Let \(\mu\) be a locally finite positive Borel measure on \(\mathbb{R}^n\), and fix \(\kappa \in \mathbb{N}\). For each cube \(Q\), denote by \(L^2_{Q,\kappa}(\mu)\) the finite dimensional subspace of \(L^2(\mu)\) that consists of linear combinations of the indicators of the children \(\mathcal{E}(Q)\) of \(Q\) multiplied by polynomials of degree less than \(\kappa\), and such that the linear combinations have vanishing \(\mu\)-moments on the cube \(Q\) up to order \(\kappa - 1\):

\[
L^2_{Q,\kappa}(\mu) \equiv \left\{ f = \sum_{Q' \in \mathcal{E}(Q)} 1_{Q'} p_{Q'x^\beta} : \int_Q f(x) x^\beta d\mu(x) = 0, \text{ for } 0 \leq |\beta| < \kappa \right\},
\]

where \(p_{Q'x^\beta} = \sum_{|\beta| \leq \kappa} a_{Q'} x^\beta\) is a polynomial in \(\mathbb{R}^n\) of degree less than \(\kappa\). Here \(x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}\).

Let \(d_{Q,\kappa} \equiv \dim L^2_{Q,\kappa}(\mu)\) be the dimension of the finite dimensional linear space \(L^2_{Q,\kappa}(\mu)\).

Consider an arbitrary dyadic grid \(\mathcal{D}\). For \(Q \in \mathcal{D}\), let \(\Delta^\mu_{Q,\kappa}\) denote orthogonal projection onto the finite dimensional subspace \(L^2_{Q,\kappa}(\mu)\), and let \(\mathbb{E}^\mu_{Q,\kappa}\) denote orthogonal projection onto the finite dimensional subspace

\[
\mathcal{P}^\kappa_{Q,\kappa}(\mu) = \text{Span}\{1_Q x^\beta : 0 \leq |\beta| < \kappa\}.
\]

For a doubling measure \(\mu\), it is proved in [RaSaWi], that we have the orthonormal decompositions

\[
f = \sum_{Q \in \mathcal{D}} \Delta^\mu_{Q,\kappa} f, \quad f \in L^2_{\mathbb{R}^n}(\mu), \quad \text{ where } \left\langle \Delta^\mu_{I,\kappa} f, \Delta^\mu_{Q,\kappa} f \right\rangle = 0 \text{ for } P \neq Q,
\]

where convergence holds both in \(L^2_{\mathbb{R}^n}(\mu)\) norm and pointwise \(\mu\)-almost everywhere, the telescoping identities

\[
1_Q \sum_{I : Q' \subset I \subset P} \Delta^\mu_{I,\kappa} = \mathbb{E}^\mu_{Q,\kappa} - 1_Q \mathbb{E}^\mu_{P,\kappa} \quad \text{for } P, Q \in \mathcal{D} \text{ with } Q \subsetneq P,
\]

and the moment vanishing conditions

\[
\int_{\mathbb{R}^n} \Delta^\mu_{Q,\kappa} f(x) x^\beta d\mu(x) = 0, \quad \text{ for } Q \in \mathcal{D}, \beta \in \mathbb{Z}^n_+, \quad 0 \leq |\beta| < \kappa.
\]

We have the bound for the Alpert projections \(\mathbb{E}^\mu_{I,\kappa}\) ([Saw6] see (4.7) on page 14]):

\[
\left\| \mathbb{E}^\mu_{I,\kappa} f \right\|_{L^2(\mu)} \lesssim E^\mu_I |f| \lesssim \frac{1}{|I|_\mu} \int_I |f|^2 d\mu, \quad \text{ for all } f \in L^2_{\mathbb{R}^n}(\mu).
\]
In terms of the Alpert coefficient vectors \( \hat{f}(I) \equiv \left\{ \langle f, h_{I,n}^\alpha \rangle \right\}_{a \in \Gamma_{I,n,\kappa}} \) for an orthonormal basis \( \left\{ h_{I,n}^\alpha \right\}_{a \in \Gamma_{I,n,\kappa}} \) of \( L^2_I(\mu) \) where \( \Gamma_{I,n,\kappa} \) is a convenient finite index set of size \( d_{Q,\kappa} \), we thus have

\[
\left| \hat{f}(I) \right| = \| \Delta_{I,n} f \|_{L^2(\sigma)} \leq \| \Delta_{I,n} f \|_{L^\infty(a)} \sqrt{|I|_{\sigma}} \leq C \| \Delta_{I,n} f \|_{L^2(\sigma)} = C \left| \hat{f}(I) \right|.
\]

2.4. The Pivotal Lemma. For \( 0 \leq \alpha < n \), let \( \mathcal{P}^\alpha(J,\mu) \equiv \mathcal{P}^\alpha(J,\mu) \) denote the standard Poisson integral, where \( \mathcal{P}^\alpha(J,\mu) \) is as defined in (2.10). The following extension of the ‘energy lemma’ is due to Rahm, Sawyer and Wick [RaSaWi] in the case the polynomial \( R(x) \) is constant, and this case is proved in detail in [Saw6] Lemmas 28 and 29 on pages 27-30. Note the crucial assumption below is that \( \Psi_{J} \) has at least \( 2\kappa - 1 \) vanishing moments while the polynomial \( R(x) \) has degree at most \( \kappa - 1 \).

**Lemma 12 (Pivotal Lemma).** Fix \( \kappa \geq 1 \). Let \( J \) be a cube in \( \mathcal{D} \), and let \( \Psi_{J} \) be an \( L^2(\omega) \) function supported in \( J \) with vanishing \( \omega \)-means of all orders less than \( 2\kappa \). Let \( R(x) \) be a polynomial of degree less than \( \kappa \) that satisfies \( \sup_{x \in J} |R(x)| \leq 1 \). Let \( \sigma \) be a positive measure supported in \( \mathbb{R}^n \setminus \gamma J \) with \( \gamma > 1 \), and let \( T^\alpha \) be a standard \( \alpha \)-fractional singular integral operator with \( 0 \leq \alpha < n \). Then we have the ‘pivotal’ bound

\[
\left| \left( R T^\alpha(\varphi) \right)_{\Psi_{J}} \right|_{L^2(\omega)} \lesssim C_{\gamma} \mathcal{P}^\alpha(J,\nu) \sqrt{|J|_\omega} \| \Psi_{J} \|_{L^2(\omega)} \lesssim C_{\gamma} \sqrt{A^\alpha_{\gamma}(\sigma,\omega)} \sqrt{|J|_\nu} \| \Psi_{J} \|_{L^2(\omega)},
\]

for any function \( \varphi \) with \( |\varphi| \leq 1 \).

To obtain Lemma 12 in the case of a general polynomial \( R(x) \) of degree at most \( \kappa \), note that we cannot simply replace the Taylor expansion of the function \( x \to K(x,y) \) that arises in the proof in [Saw6] Lemmas 28 and 29 on pages 27-30, by the Taylor expansion of the function \( x \to R(x)K(x,y) \), since this latter kernel no longer satisfies Calderón-Zygmund estimates in the \( x \) variable. For example, if all of the derivatives in \( \partial_x^\alpha [R(x)K(x,y)] \) hit the polynomial \( R(x) = x^\beta \), then we get

\[
|\partial_x^\beta R(x)| \sim (|x-y|^\alpha)^{\alpha-n},
\]

which can be much larger than the Calderón-Zygmund bound \( C_{\text{CZ}} |x-y|^{\alpha-n-|\beta|} \) for \( |x-y| > 1 \).

On the other hand, the function \( R(x) \Psi_{J}(x) \) is supported in \( J \) and has at least \( (2\kappa - 1) - (\kappa - 1) = \kappa \) vanishing moments since the degree of \( R(x) \) is at most \( \kappa - 1 \), and there are at least \( 2\kappa - 1 \) vanishing moments in \( \Psi_{J}(x) \). Indeed,

\[
\int R(x) \Psi_{J}(x) x^\beta dx = \int \Psi_{J}(x) [R(x) x^\beta] dx = 0,
\]

since \( R(x) x^\beta \) is a polynomial of degree less than \( 2\kappa \). Thus the argument in [Saw6] applies to prove the Pivotal Lemma 12.

We also recall from [Saw6] Lemma 33] the following Poisson estimate, that is a straightforward extension of the case of \( m = 1 \) due to Nazarov, Treil and Volberg in [NTV4]. This lemma is the key to exploiting the crucial reduction to good cubes \( J \) that we use below, see [NTV4] and [NTV].

**Lemma 13.** Fix \( m \geq 1 \). Suppose that \( J \subset I \subset K \) and that \( \text{dist}(J,\partial I) > 2\sqrt{n} \ell(J)^\gamma \ell(I)^{1-\gamma} \). Then

\[
\mathcal{P}^\alpha_{m}(J,\nu 1_{K \setminus J}) \lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{m-\varepsilon(n+m-\alpha)} \mathcal{P}^\alpha_{m}(I,\sigma 1_{K \setminus J}).
\]

3. The Calderón-Zygmund corona decomposition

To set the stage for control of the stopping form below in the absence of the energy condition, we construct the Calderón-Zygmund corona decomposition for a function \( f \in L^2(\sigma) \) that is supported in a dyadic cube \( F_0^0 \). Fix \( \Gamma > 1 \) and define \( \mathcal{G}_0 = \{ F_0^0 \} \) to consist of the single cube \( F_0^0 \), and define the first generation \( \mathcal{G}_1 = \{ F_1^0 \}_k \) of CZ stopping children of \( F_1^0 \) to be the maximal dyadic subcubes \( I \) of \( F_0^0 \) satisfying

\[
E_I^\gamma |f| \geq \Gamma E_{F_1^0}^\gamma |f|.
\]

Then define the second generation \( \mathcal{G}_2 = \{ F_2^0 \}_k \) of CZ stopping children of \( F_1^0 \) to be the maximal dyadic subcubes \( I \) of some \( F_2^1 \in \mathcal{G}_1 \) satisfying

\[
E_I^\gamma |f| \geq \Gamma E_{F_2^1}^\gamma |f|.
\]
Continue by recursion to define $G_n$ for all $n \geq 0$, and then set
\[ F = \bigcup_{n=0}^{\infty} G_n = \{ F^n_k : n \geq 0, k \geq 1 \} \]
to be the set of all CZ $\kappa$-pivotal stopping intervals in $F^1_1$ obtained in this way.

The $\sigma$-Carleson condition for $F$ follows as usual from the first step,
\[ \sum_{F' \in \mathcal{F}(F)} |F'|_{\sigma} \leq \frac{1}{n} \sum_{F' \in \mathcal{F}(F)} \left\{ \mathbb{P}_n(F', 1_F \sigma)^2 |F'|_{\omega} + \frac{1}{E^F_{F'}} \int_{F'} |f| \, d\sigma \right\} \leq \frac{1}{n} (A^2 \sigma, \omega + 1) |F|_{\sigma}. \]
Moreover, if we define
\[ \alpha_F(F) \equiv \sup_{F' \in \mathcal{F}(F) : F' \subset F} E^F_{F'} |f|, \]
then in each corona
\[ C_F \equiv \{ I \in D : I \subset F \text{ and } I \nsubseteq F' \text{ for any } F' \in \mathcal{F} \text{ with } F' \nsubseteq F \}, \]
we have, from the definition of the stopping times, the average control
\[ \sum_{F' \subseteq F} |F'|_{\sigma} \leq C_0 |F|_{\sigma} \text{ for all } F \in \mathcal{F}; \text{ and } \sum_{F \in \mathcal{F}} \alpha_F(F)^2 |F|_{\sigma} \leq C_0^2 \|f\|_{L^2(\sigma)}^2. \]

Define the two corona projections
\[ P^\sigma_{C_F} \equiv \sum_{I \in C_F} \Delta^\sigma_{I, \kappa}, \text{ and } P^{\sigma_{\text{shift}}}_{C_F} \equiv \sum_{J \in C^{\sigma_{\text{shift}}}_{F}} \Delta^\sigma_{J, \kappa}, \]
where
\[ C^{\sigma_{\text{shift}}}_{F} \equiv [C_F \setminus N^3_{D}(F)] \cup \bigcup_{F' \in \mathcal{F}(F)} \{ N^3_{D}(F') \setminus N^3_{D}(F) \}; \]
and note that $f = \sum_{F \in \mathcal{F}} P^\sigma_{C_F} f$. Thus the corona $C^{\sigma_{\text{shift}}}_{F}$ has the top $\tau$ levels from $C_F$, removed and includes the first $\tau$ levels from each of its $\mathcal{F}$-children, except if they have already been removed.

**Remark 14.** The shifted coronas are pairwise disjoint, since if $F'' \subset F \in \mathcal{F}$ and $J \in C^{\sigma_{\text{shift}}}_{F''}$, then $J \in C^{\sigma_{\text{shift}}}_{F'}$, then either $F'' \subset F$ or $F \subset F''$. Thus it suffices to assume $F'' \nsubseteq F$ and derive a contradiction. But then $J \subset F''$ and by $[3.4]$ and the assumption that $J \in C^{\sigma_{\text{shift}}}_{F''}$, we have $J \in N^3_{D}(F'') \setminus N^3_{D}(F)$ for some $F' \in \mathcal{F}(F)$. Thus $J \subset F'' \subset F' \subset F$, and the assumption that $J \in C^{\sigma_{\text{shift}}}_{F''}$ implies that $J \notin N^3_{D}(F'')$, contradicting $J \in N^3_{D}(F')$ with $F'' \subset F'$.

The main result we need from [Saw6] regarding these coronas is the Intertwining Proposition.

**Proposition 15** (The Intertwining Proposition [Saw6] see Subsection 6.4). Suppose that $F$ satisfies both
\[ \sum_{F' \in \mathcal{F}(F) : F' \subset F} |F'|_{\sigma} \leq C_0 |F|_{\sigma} \text{ for all } F \in \mathcal{F}; \text{ and } \sum_{F \in \mathcal{F}} \alpha_F(F)^2 |F|_{\sigma} \leq C_0^2 \|f\|_{L^2(\sigma)}^2, \]
where $\alpha_F(F)$ is as in $[3.1]$, and that
\[ \|\Delta^\sigma_{I, \kappa} f\|_{L^\infty(\sigma)} \leq C \alpha_F(F), \quad f \in L^2(\sigma), \quad I \in C_F. \]
Then for good functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, and with $\kappa \geq 1$, we have
\[ \sum_{F \in \mathcal{F}} \sum_{1 \equiv F} \langle T^\sigma_{\alpha} \Delta^\sigma_{I, \kappa} f, P^{\sigma_{\text{shift}}}_{C_F} g \rangle \omega \lesssim \left( \sqrt{A^2_2 + \mathbb{T}^\alpha_F} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \]
3.1. Parallel corona decompositions. In this subsection, we recall certain material on parallel corona decompositions from [Saw6]. Strictly speaking, we will not use any of this material in our paper, but it is included here as it motivates a key construction used later on for the ‘above’ stopping form. Let \((C_0,A,\alpha_A)\) constitute stopping data for \(f \in L^2(\sigma)\), and let \((C_0,B,\alpha_B)\) constitute stopping data for \(g \in L^2(\omega)\) as in the previous subsection. We now organize the bilinear form,\
\[
\langle T^\sigma_\omega f, g \rangle_\omega = \left\langle T^\sigma_\omega \left( \sum_{I \in D} \Delta^\sigma_{I,\kappa_1} f \right), \left( \sum_{J \in D} \Delta^\omega_{J,\kappa_2} g \right) \right\rangle_\omega = \sum_{I \in D \text{ and } J \in D} \left\langle T^\sigma_\omega \left( \Delta^\sigma_{I,\kappa_1} f \right), \left( \Delta^\omega_{J,\kappa_2} g \right) \right\rangle_\omega,
\]
as a sum over the families of Calderón-Zygmund stopping cubes \(A\) and \(B\), and then decompose this sum, by the Parallel Corona decomposition, in which the ‘diagonal cut’ in the bilinear form is made according to the relative positions of intersecting coronas, rather than the traditional way of making the ‘diagonal cut’ according to relative side lengths of cubes. See e.g. [Saw6] for more on the parallel corona decomposition.

We have
\[
\langle T^\sigma_\omega f, g \rangle_\omega = \sum_{(A,B) \in A \times B} \left\langle T^\sigma_\omega \left( P^\sigma_{C_A(A)} f \right), P^\omega_{C_B(B)} g \right\rangle_\omega = \sum_{(A,B) \in \text{Near}(A \times B)} + \sum_{(A,B) \in \text{Disjoint}(A \times B)} + \sum_{(A,B) \in \text{Far}(A \times B)} \langle T^\sigma_\omega \left( P^\sigma_{C_A(A)} f \right), P^\omega_{C_B(B)} g \rangle_\omega
\]
\[
\equiv \text{Near}(f,g) + \text{Disjoint}(f,g) + \text{Far}(f,g).
\]
Here \(\text{Near}(A \times B)\) is the set of pairs \((A,B) \in A \times B\) such that one of \(A, B\) is contained in the other, and there is no \(A_1 \in A\) with \(B \subseteq A_1 \subseteq A\), nor is there \(B_1 \in B\) with \(A \subseteq B_1 \subseteq B\). The set \(\text{Disjoint}(A \times B)\) is the set of pairs \((A,B) \in A \times B\) such that \(A \cap B = \emptyset\). The set \(\text{Far}(A \times B)\) is the complement of \(\text{Near}(A \times B) \cup \text{Disjoint}(A \times B)\) in \(A \times B\):
\[
\text{Far}(A \times B) = (A \times B) \setminus \{\text{Near}(A \times B) \cup \text{Disjoint}(A \times B)\}.
\]
Note that if \((A,B) \in \text{Far}(A \times B)\), then either \(B \subset A'\) for some \(A' \in C_A(A)\), or \(A \subset B'\) for some \(B' \in C_B(B)\).

We further decompose the near form \(\text{Near}(f,g)\) into
\[
\text{Near}(f,g) = \sum_{B \subseteq A} \left\langle T^\sigma_\omega \left( P^\sigma_{C_A(A)} f \right), P^\omega_{C_B(B)} g \right\rangle_\omega = \text{Near}_{\text{below}}(f,g) + \text{Near}_{\text{above}}(f,g).
\]
The \(\text{Near}_{\text{below}}(f,g)\) form can be controlled by the Indicator/Cube Testing condition \([\text{LT6}]\) if we define projections
\[
Q^\omega_A g = \sum_{B \in B_1: (A,B) \in \text{Near}(A \times B)} P^\omega_{C_B(B)} g,
\]
and observe that, while the Alpert support of \(Q^\omega_A\) need not be contained in the corona \(C_A(A)\), these projections are nevertheless mutually orthogonal in the index \(A \in A\), since for \((A,B) \in \text{Near}(A \times B)\) there is no \(A_1 \in A\) with \(B \subseteq A_1 \subseteq A\). Indeed,
\[
\langle \text{Near}_{\text{below}}(f,g) \rangle = \sum_{A \in A} \left\langle T^\sigma_\omega P^\sigma_{C_A(A)} f, Q^\omega_A g \right\rangle_\omega \leq \sum_{A \in A} \left\| T^\sigma_\omega P^\sigma_{C_A(A)} f \right\|_{L^2(\omega)} \left\| Q^\omega_A g \right\|_{L^2(\omega)} \leq \sum_{A \in A} \alpha_A(A) \left( \sum_{A \in A} \left\| Q^\omega_A g \right\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}} \leq \sum_{A \in A} \alpha_A(A) \sqrt{|A|} \left\| Q^\omega_A g \right\|_{L^2(\omega)} \leq \sum_{A \in A} \alpha_A(A) \sqrt{|A|} \left\| f \right\|_{L^2(\sigma)} \left\| g \right\|_{L^2(\omega)},
\]
by quasi-orthogonality and the fact that the projections $Q_A^T$ are mutually orthogonal in the index $A \in \mathcal{A}$.

4. Reduction of the proof to local forms

The proof of Theorem 2 will require significant new arguments beyond those in [Saw6]. In particular, we will use the shifted corona decomposition as in [LaSaShUr3] and [SaShUr7], instead of the parallel corona decomposition used in [Saw6], and construct a paraproduct - stopping - neighbour decomposition of NTV type for a local Alpert form, but complicated by the fact that singular integrals do not in general commute with multiplication by polynomials.

To prove Theorem 2 we begin by proving the bilinear form bound,

$$|⟨T^nf, g⟩_ω| \lesssim \left( \sqrt{A^2_2(σ, ω)} + Ξ^r(σ, ω) + Ξ^{r, ε}(σ, ω) \right) \left\| f \right\|_{L^2(σ)} \left\| g \right\|_{L^2(ω)},$$

for functions $f \in L^2(σ)$ and $g \in L^2(ω)$. Following the weighted Haar expansions of Nazarov, Treil and Volberg, we write $f$ and $g$ in weighted Alpert wavelet expansions,

$$⟨T^nf, g⟩_ω = \left( T^σ_σ \left( \sum_{I \in \mathcal{D}} \Delta^n_{I; ρ, 1} f \right), \left( \sum_{J \in \mathcal{D}} \Delta^n_{J; ρ, 2} g \right) \right)_ω = \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} ⟨T^σ_σ (△^n_I; ρ, 1) f, (△^n_J; ρ, 2) g⟩_ω.$$

Then the sum is further decomposed by first Cube Size Splitting, then using the Shifted Corona Decomposition, according to the Canonical Splitting. We assume the reader is reasonably familiar with the notation and arguments in the first eight sections of [SaShUr7]. The $n$-dimensional decompositions used in [SaShUr7] are in spirit the same as the one-dimensional decompositions in [LaSaShUr3], as well as the $n$-dimensional decompositions in [LaWi], but differ in significant details.

A fundamental result of Nazarov, Treil and Volberg [NTV] is that all the cubes $I$ and $J$ appearing in the bilinear form above may be assumed to be $(r, ε) - good$, where a dyadic interval $K$ is $(r, ε) - good$, or simply good, if for every dyadic supercube $L$ of $K$, it is the case that either $K$ has side length at least $2^{1-r}$ times that of $L$, or $K \subseteq (r, ε) L$. We say that a dyadic cube $K$ is $(r, ε)-deeply embedded in a dyadic cube $L$, or simply $r$-deeply embedded in $L$, which we write as $K \subseteq_{r, ε} L$, when $K \subseteq L$ and both

$$\ell (K) \leq 2^{-r} \ell (L),$$

$$\text{dist} \left( K, \bigcup_{L' \in \mathcal{T}_L} \partial L' \right) \geq 2\ell (K)^\varepsilon \ell (L)^{1-\varepsilon}.$$

Here is a brief schematic diagram derived from [SaShUr7] summarizing the shifted corona decompositions that we will follow using Alpert wavelet expansions below. We first introduce parameters as in [SaShUr7]. We will choose $ε > 0$ sufficiently small later in the argument, and then $\varepsilon$ must be chosen sufficiently large depending on $ε$ in order to reduce matters to $(r, ε) - good$ functions by the NTV argument in [NTV].

Definition 16. The parameters $τ$ and $ρ$ are fixed to satisfy

$$τ > r \text{ and } ρ > \tau + r,$$

where $r$ is the goodness parameter already fixed.

Here is a brief diagram highlighting the various decompositions of the bilinear form $⟨T^nf, g⟩_ω$. We will treat the below form $B_{<ρ, r} (f, g)$ in detail under the assumption that $κ_2 \geq 2κ_1$, and then turn to the above form $B_{>ρ, r} (f, g)$, which is handled in the same way except for the diagonal form, which requires the above indicator / cube testing condition due to the asymmetry in the assumption $κ_2 \geq 2κ_1$.

\(^2\) See also [SaShUr10] Subsection 3.1 for a treatment using finite collections of grids, in which case the conditional probability arguments are elementary.
4.1. Cube Size Splitting. The NTV Cube Size Splitting of the inner product \( \langle T_\sigma^a f, g \rangle_\omega \) given in (4.1) splits the pairs of cubes \((I, J)\) in a simultaneous Alpert decomposition of \(f\) and \(g\) into four groups determined by relative position, and is given by

\[
\langle T_\sigma^a f, g \rangle_\omega = \sum_{I, J \in D} \langle T_\sigma^a (\Delta_{I,R} f), (\Delta_{J,R} g) \rangle_\omega + \sum_{I, J \in D} \langle T_\sigma^a (\Delta_{I,R} f), (\Delta_{J,R}^\omega g) \rangle_\omega + \sum_{I, J \in D} \langle T_\sigma^a (\Delta_{I,R}^\omega f), (\Delta_{J,R}^\omega g) \rangle_\omega
\]

where the absolute values are placed inside the sum. We have the following bound for the sublinear intersection and comparable forms from \cite{Saw6} see Lemma 31], which in turn followed the NTV arguments for Haar wavelets in \cite{SaShUr7} see the proof of Lemma 7.1] (see also \cite{LaSaShUr3}),

\[
|B_\cap| (f, g) + |B_\setminus| (f, g) \leq C \left( \sum_{T_\sigma \in D} T_\sigma^a + \sum_{T_\sigma \in D} WBP_{T_\sigma}^{(k_1, k_2)} (\sigma, \omega) + \frac{1}{(Q')_{\text{norm}} (\sigma)} \left| \int_{Q'} T_\sigma^a (1_Q f) \, gd\omega \right| \right),
\]

where if \( \Omega \) is the set of all dyadic grids,

\[
WBP_{T_\sigma}^{(k_1, k_2)} (\sigma, \omega) = \sup_{\Omega} \sup_{Q, Q' \in D} \frac{1}{(Q')_{\text{norm}} (\sigma)} \left| \int_{Q'} T_\sigma^a (1_Q f) \, gd\omega \right| < \infty
\]

is a weak boundedness constant introduced in \cite{Saw6}. This constant will be removed in the final section below using the following bound proved in \cite{Saw6} see (6.25) in Subsection 6.7 and note that only triple testing is needed there by choosing \( \ell (Q') \leq \ell (Q) \) (using duality and \( T_\sigma^{a,*} \) if needed)],

\[
WBP_{T_\sigma}^{(k_1, k_2)} (\sigma, \omega) \leq C_{k_1, k_2} \left( \sum_{T_\sigma \in D} T_\sigma^a (\sigma, \omega) + \sum_{T_\sigma \in D} T_\sigma^a (\omega, \sigma) \right).
\]

Since the below and above forms \( B_{\cap, \setminus} (f, g), B_{\cap, \setminus} (f, g) \) are symmetric, matters are reduced to proving

\[
|B_{\cap, \setminus} (f, g)| \lesssim \left( \sum_{T_\sigma \in D} T_\sigma^{a_1} + T_\sigma^{a_2}, + \sqrt{A_2} \right) \| f \|_{L^2 (\sigma)} \| g \|_{L^2 (\omega)} .
\]
4.2. Shifted Corona Decomposition. For this we recall the *Shifted Corona Decomposition*, as opposed to the *parallel* corona decomposition used in [Saw00], associated with the Calderón-Zygmund $\kappa$-pivotal stopping cubes $\mathcal{F}$ introduced above. But first we must invoke standard arguments, using the $\kappa$-cube testing conditions (2.5), to permit us to assume that $f$ and $g$ are supported in a finite union of dyadic cubes $F_0$ on which they have vanishing moments of order less than $\kappa$.

4.2.1. The initial reduction using testing. For this construction, we will follow the treatment as given in [SaShUr12]. We first restrict $f$ and $g$ to be supported in a large common cube $Q_\infty$. Then we cover $Q_\infty$ with $2^n$ pairwise disjoint cubes $I_\infty \in \mathcal{D}$ with $\ell(I_\infty) = \ell(Q_\infty)$. We now claim we can reduce matters to consideration of the $2^{2n}$ forms

$$\sum_{I \in \mathcal{D}} \sum_{I \subset J \subset I_\infty} \int (T^\alpha_{\sigma} \triangle^\eta_{I_\infty} f) \triangle^\omega_{J_\infty} g d\omega,$$

as both $I_\infty$ and $J_\infty$ range over the dyadic cubes as above. First we note that when $I_\infty$ and $J_\infty$ are distinct, the corresponding form is included in the sum $\mathcal{B}c(f, g) + \mathcal{Bf}(f, g)$, and hence controlled. Thus it remains to consider the forms with $I_\infty = J_\infty$ and use the cubes $I_\infty$ as the starting cubes in our corona construction below. Indeed, we have from (2.7) that

$$f = \sum_{I \in \mathcal{D}} \sum_{I \subset J \subset I_\infty} \triangle^\eta_{J_\infty} f + \mathcal{E}_{I_\infty}^{\eta} f \quad \text{and} \quad g = \sum_{J \in \mathcal{D}} \sum_{J \subset I_\infty} \triangle^\omega_{I_\infty} g + \mathcal{E}_{I_\infty}^{\omega} g,$$

which can then be used to write the bilinear form $\int (T_{\sigma} f) g d\omega$ as a sum of the forms

(4.6)

$$\int (T_{\sigma} f) g d\omega = \sum_{I_\infty} \left\{ \int (T^\alpha_{\sigma} \triangle^\eta_{I_\infty} f) \triangle^\omega_{I_\infty} g d\omega + \sum_{J \in \mathcal{D}} \sum_{J \subset I_\infty} \left( \int (T^\alpha_{\sigma} \triangle^\eta_{J_\infty} f) \mathcal{E}_{I_\infty}^{\omega} g d\omega + \int (T^\alpha_{\sigma} \mathcal{E}_{I_\infty}^{\eta} f) \triangle^\omega_{I_\infty} g d\omega \right) \right\},$$

taken over the $2^n$ cubes $I_\infty$ above.

The second, third and fourth sums in (4.6) can be controlled by the $\kappa$-testing conditions (2.5), e.g. using Cauchy-Schwarz,

(4.7)

$$\left| \sum_{J \in \mathcal{D}} \int (T^\alpha_{\sigma} \triangle^\eta_{J_\infty} f) \mathcal{E}_{I_\infty}^{\omega} g d\omega \right| \leq \left\| \sum_{I \in \mathcal{D}} \sum_{I \subset J \subset I_\infty} \triangle^\eta_{J_\infty} f \right\|_{L^2(\sigma)} \left\| \mathcal{E}_{I_\infty}^{\omega} g \right\|_{L^\infty} \left\| 1_{I_\infty} T^\alpha_{\sigma} \frac{\mathcal{E}_{I_\infty}^{\eta} g}{\| \mathcal{E}_{I_\infty}^{\eta} g \|_{L^\infty}} \right\|_{L^2(\sigma)} \lesssim \| f \|_{L^2(\sigma)} \sqrt{\mathcal{E}_{I_\infty}^{\eta} \frac{\mathcal{E}_{I_\infty}^{\omega}}{T^\alpha_{\sigma}} \sqrt{\| g \|_{L^\infty}} \leq T^\eta_{\mathcal{F}} \| f \|_{L^2(\sigma)} \| g \|_{L^2(\sigma)},$$

and similarly for the third and fourth sum.

4.2.2. The shifted corona. Recall the shifted corona $C^\tau_{\mathcal{F}}$ defined in (3.4). A simple but important property is the fact that the $\tau$-shifted coronas $C^\tau_{\mathcal{F}}$ have overlap bounded by $\tau$:

(4.8)

$$\sum_{F \in \mathcal{F}} 1_{C^\tau_{\mathcal{F}}} (J) \leq \tau, \quad J \in \mathcal{D}.$$

It is convenient, for use in the canonical splitting below, to introduce the following shorthand notation for $F, G \in \mathcal{F}$:

$$\left\langle T^\alpha_{\sigma} (P^\alpha_{\mathcal{F}} f), P^\alpha_{\mathcal{G}} g \right\rangle_{\omega} \equiv \sum_{I \in \mathcal{F} \text{ and } J \in C_\omega^\tau} \left\langle T^\alpha_{\sigma} \left( \triangle^\eta_{I_\infty} f \right), \left( \triangle^\omega_{J_\infty} g \right) \right\rangle_\omega.$$
4.3. Canonical Splitting. We then proceed with the Canonical Splitting of $B_{C,\rho,\varepsilon}(f,g)$ in (4.3) as in [SaShUr7], but with Alpert wavelets in place of Haar wavelets,

$$B_{C,\rho,\varepsilon}(f,g) = \sum_{T \in \mathcal{D}} \left\langle T_{\sigma}(P_{C,\rho,\varepsilon}^f)P_{C,\rho,\varepsilon}^{\omega - \text{shift}}g \right\rangle_{\omega} + \sum_{T \in \mathcal{D}} \left\langle T_{\sigma}(P_{G,\rho,\varepsilon}^f)P_{G,\rho,\varepsilon}^{\omega - \text{shift}}g \right\rangle_{\omega}$$

$$+ \sum_{T \in \mathcal{D}} \left\langle T_{\sigma}(P_{C,\rho,\varepsilon}^f)P_{C,\rho,\varepsilon}^{\omega - \text{shift}}g \right\rangle_{\omega} + \sum_{T \in \mathcal{D}} \left\langle T_{\sigma}(P_{G,\rho,\varepsilon}^f)P_{G,\rho,\varepsilon}^{\omega - \text{shift}}g \right\rangle_{\omega}$$

$$= T_{C,\rho,\varepsilon,\text{diagonal}}(f,g) + T_{C,\rho,\varepsilon,\text{above}}(f,g) + T_{C,\rho,\varepsilon,\text{disjoint}}(f,g).$$

The final two forms $T_{C,\rho,\varepsilon,\text{above}}(f,g)$ and $T_{C,\rho,\varepsilon,\text{disjoint}}(f,g)$ each vanish just as in [SaShUr7], since there are no pairs $(I,J) \in \mathcal{C}_F \times \mathcal{C}_G$ such that both $(i) J \subset I$ and $(ii)$ either $F \subset G$ or $G \cap F = \emptyset$. The below far below form $T_{C,\rho,\varepsilon,\text{far below}}(f,g)$ is then further split into two forms $T_{1,\rho,\varepsilon,\text{far below}}(f,g)$ and $T_{2,\rho,\varepsilon,\text{far below}}(f,g)$ as in [SaShUr7],

$$T_{C,\rho,\varepsilon,\text{far below}}(f,g) = \sum_{T \in \mathcal{D}} \sum_{F \in \mathcal{D}} \left\langle T_{\sigma}^\alpha((\Delta_{I,\rho,\varepsilon}^\alpha f), (\Delta_{J,\rho,\varepsilon}^\alpha g)) \right\rangle_{\omega}$$

$$= \sum_{F \in \mathcal{D}} \sum_{G \in \mathcal{D}} \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} \left\langle T_{\sigma}^\alpha((\Delta_{I,\rho,\varepsilon}^\alpha f), (\Delta_{J,\rho,\varepsilon}^\alpha g)) \right\rangle_{\omega}$$

Remark 17. For the remainder of the proof of Theorem 2, one should keep in mind that if $T_{\alpha}$ is a Stein elliptic Calderón-Zygmund operator on $\mathbb{R}^n$, then $A_{\alpha}^2(\sigma,\omega) \lesssim \mathcal{R}_{T_{\alpha}}(\sigma,\omega)$.

The second form $T_{2,\rho,\varepsilon,\text{far below}}(f,g)$ is easily seen to satisfy

$$|T_{2,\rho,\varepsilon,\text{far below}}(f,g)| \lesssim C \left(T_{T_{\alpha}}^{\mathcal{S}_1} + \mathcal{T}_{T_{\alpha}}^{\mathcal{S}_2} + \mathcal{W}_{BP_{T_{\alpha}}}^{(\mathcal{S}_1,\mathcal{S}_2)}(\sigma,\omega) + \sqrt{A_{\alpha}^2}\right) \|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},$$

just as for the analogous inequality in [SaShUr7] for Haar wavelets, by (4.3). To control the first and main form $T_{1,\rho,\varepsilon,\text{far below}}(f,g)$, we use the $\kappa$-pivotal Intertwining Proposition 1 recalled from [Saw6] in the earlier section on preliminaries. This proposition then immediately gives the bound

$$|T_{1,\rho,\varepsilon,\text{far below}}(f,g)| \lesssim \left(T_{T_{\alpha}}^{\mathcal{S}_1} + \sqrt{A_{\alpha}^2}\right) \|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

To handle the below diagonal form $T_{C,\rho,\varepsilon,\text{diagonal}}(f,g)$, we decompose according to the stopping times $F$,

$$T_{C,\rho,\varepsilon,\text{diagonal}}(f,g) = \sum_{F \in \mathcal{D}} T_{C,\rho,\varepsilon,\text{Far}}(f,g),$$

and it is enough, using Cauchy-Schwarz and quasiorthogonality [SaShUr7] in $f$, together with orthogonality in both $f$ and $g$, to prove the following bound involving the usual cube testing constant,

$$|T_{C,\rho,\varepsilon,\text{Far}}(f,g)| \lesssim \left(T_{T_{\alpha}}^{\mathcal{S}_1} + \sqrt{A_{\alpha}^2}\right) \left(\alpha_{\mathcal{F}}(F)\|F\|_\sigma + \|P_{C,F}^\alpha f\|_{L^2(\sigma)}\right) \|P_{C,F}^{\omega - \text{shift}}g\|_{L^2(\omega)}.$$

Indeed, this then gives the estimate,

$$|T_{C,\rho,\varepsilon,\text{diagonal}}(f,g)| \lesssim \left(T_{T_{\alpha}}^{\mathcal{S}_1} + \sqrt{A_{\alpha}^2}\right) \|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

It is important for this below estimate that we choose $\kappa_2 \geq 2\kappa_1$ and $\kappa_1$ sufficiently large.

On the other hand, when we turn to bounding the above diagonal form $T_{\text{diagonal}}(f,g)$, we will not have $\kappa_1 \geq 2\kappa_2$, and we will have to argue differently in order to use the dual indicator / cube testing constant $\mathcal{T}_{T_{\alpha}}^{\mathcal{S}_1}(\omega,\sigma)$,

$$|T_{\text{diagonal}}(f,g)| \lesssim \left(\mathcal{T}_{T_{\alpha}}^{\mathcal{S}_1}(\omega,\sigma) + \sqrt{A_{\alpha}^2}\right) \left(\alpha_{\mathcal{F}}(F)\|F\|_\sigma + \|P_{C,F}^\alpha f\|_{L^2(\sigma)}\right) \|P_{C,F}^{\omega - \text{shift}}g\|_{L^2(\omega)}.$$

Thus at this point we have reduced the proof of Theorem 2 to
(1) proving \([4.12]\),
(2) proving \([4.14]\).
(3) and controlling the triple polynomial testing condition \([4.14]\) by the usual cube testing condition and the classical Muckenhoupt condition \([1.3]\).

In the next section we address the first issue by proving the inequality \([4.12]\) for the below diagonal forms \(T_{\Delta,\rho,e}^{\perp} (f,g)\), and in the subsequent section we prove \([4.14]\) for the above diagonal form \(T_{\Delta,\rho,e}^{\perp} (f,g)\). In the final section, we address the second issue and complete the proof of Theorem 2 by drawing together all of the estimates.

5. Below diagonal form and the NTV reach for Alpert wavelets

In this section we give the main new argument of this paper. It will be convenient to denote our fractional singular integral operators by \(T_\lambda\), \(0 \leq \lambda < n\), instead of \(T_\alpha\), as \(\alpha\) will denote a multi-index in \(\mathbb{Z}_+^n\). But first, we note that for a doubling measure \(\mu\), a cube \(I\) and a polynomial \(P\), we have \(\|P1_I\|_{L^\infty(\mu)} = \sup_{x \in I} |P(x)|\); in particular, \(\|P1_I\|_{L^\infty(\rho)} = \|P1_I\|_{L^\infty(\omega)} = \|P1_I\|_{L^\infty}\).

We will adapt the classical reach of NTV using Haar wavelet projections \(\Delta_I^\sigma\), namely the ingenious ‘thinking outside the box’ idea of the paraproduct / stopping / neighbour decomposition of Nazarov, Treil and Volberg \([NTV4]\). Since we are using weighted Alpert wavelet projections \(\Delta_{\ell,\tau}\) instead, the projection \(\mathbb{E}_{\ell,\tau}^\rho \Delta_{\ell,\tau}^\rho\) onto the child \(I' \in \mathcal{C}_\tau(I)\) equals \(M_{\ell,\tau}1_{I'\tau}\), where \(M = M_{\ell,\tau}\) is a polynomial of degree less than \(\kappa\) restricted to \(I'\), as opposed to a constant in the Haar case, and hence no longer commutes in general with the operator \(T_\lambda\). As mentioned in the introduction, this results in a new commutator form to be bounded, and complicates bounding the remaining forms as well.

We will treat the below diagonal forms \(T_{\Delta,\rho,e}^{\perp} (f,g)\) for \(F \in \mathcal{F}\) in detail in this section, and turn to the analogous above diagonal forms \(T_{\Delta,\rho,e}^{\perp,\perp} (f,g)\) for \(G \in \mathcal{G}\) in the next section. We have from \([4.11]\), that \(T_{\Delta,\rho,e}^{\perp,\perp} (f,g)\) equals

\[
\sum_{I \in \mathcal{F}} \sum_{J \subset I} \langle T_\sigma^\alpha (1_{I_J} \Delta_{I_J}^\tau f) , \Delta_{J_K}^\omega g \rangle_\omega + \sum_{I \in \mathcal{F}} \sum_{J \subset I} \langle T_\sigma^\alpha (1_{I_J} \Delta_{I_J}^\tau f) , \Delta_{J_K}^\omega g \rangle_\omega
\]

where we write \(\kappa = (\kappa_1, \kappa_2)\), and we further decompose the below home form using

\[
M_{I_J} = M_{I_J,\omega} = 1_{I_J} \Delta_{I_J}^\tau f = \mathbb{E}_{I_J,\tau}^\rho \Delta_{I_J}^\tau f = \mathbb{E}_{I_J,\tau}^\rho \Delta_{I_J}^\tau f \mathcal{P}_{\mathcal{C}_\tau} f,
\]

where \(\mathcal{P}_{\mathcal{C}_\tau} f \equiv \sum_{I \in \mathcal{C}_\tau} \Delta_{I_K}^\tau f\), to obtain

\[
T_{\Delta,\rho,e}^{\perp,\perp} (f,g) = \sum_{I \in \mathcal{F}} \sum_{J \subset I} \langle M_{I_J} T_\sigma^\alpha 1_{F_J} \Delta_{J_K}^\tau g \rangle_\omega + \sum_{I \in \mathcal{F}} \sum_{J \subset I} \langle M_{I_J} T_\sigma^\alpha 1_{F_J} \Delta_{J_K}^\tau g \rangle_\omega
\]

Altogether then we have the weighted Alpert version of the NTV paraproduct decomposition:\[3\]

\[
T_{\Delta,\rho,e}^{\perp,\perp} (f,g) = T_{\perp,\perp}^{\text{paraproduct}} (f,g) + T_{\perp,\perp}^{\text{stop}} (f,g) + T_{\perp,\perp}^{\text{commutator}} (f,g) + T_{\perp,\perp}^{\text{neighbour}} (f,g).
\]

In fact, we will see that all forms above, except for the paraproduct, are absolutely convergent with respect to the double sum over the cubes \(I, J\).

\[3\]In \([\text{Saw6}]\) see the end of Section 10 on Concluding Remarks] it was remarked that one cannot extend a nonconstant polynomial, normalized to a cube \(Q\), to a supercube \(F\) without destroying the normalization in general. This obstacle to the paraproduct decomposition of NTV is overcome here by controlling the commutator form.
5.1. The below paraproduct form. First pigeonhole the sum over pairs $I$ and $J$ according to which child $I' \in C_D(I)$ contains $J$ to get
\[
\mathbb{T}^\square_{\text{paraproduct}}^\square_{\text{paraproduct}}(f, g) = \sum_{I \in C_F} \sum_{I' \in C_{D}(I)} \sum_{J \in \mathcal{C}^{\eta \text{-shift}}_{\mathcal{D}}(I)} \langle M_{I', \kappa_1}^\square T_1^\square F, \Delta_{J, \kappa_2}^\omega \rangle.
\]
This form $\mathbb{T}^\square_{\text{paraproduct}}(f, g)$ can be handled as usual, using the telescoping property (2.8) to sum the restrictions to a cube $J \in C_F^{\eta \text{-shift}}$ of the polynomials $M_{I', \kappa_1}$ over the relevant cubes $I$, to obtain a restricted polynomial $1_J P_{I', \kappa_1}$ that is controlled by $\alpha_F(F)$, and then passing the polynomial $M_{J, \kappa_1}$ over to $\Delta_{J, \kappa_2}^\omega$. More precisely, for each $J \in C_F^{\eta \text{-shift}}$, let $I_j^\square$ denote the smallest $K \in C_F$ such that $J \subset_{\rho, \varepsilon} K$ (and as a consequence $J \subset_{\rho, \varepsilon} I$ for all $I \supset I_j^\square$), and let $I_j^\square$ denote the $D$-child of $I_j^\square$ that contains $J$. Then we further consider the two possibilities where $I_j^\square$ is in $C_F$ or not. We have
\[
\sum_{I \in C_F: I_j^\square \subset I} 1_J M_{I, \kappa_1} = 1_J \sum_{I \in C_F: I_j^\square \subset I} M_{I, \kappa_1} = 1_J \left( E^\square_{I_j^\square, \kappa_1} f - E^\square_{I_j^\square, \kappa_1} f \right) = 1_J P_{J, \kappa_1}
\]
and we set $P_{J, \kappa_1} \equiv 0$ if $I_j^\square \notin C_F$ and $I_j^\square \in C_F$. We now claim that
\[
\|1_J P_{J, \kappa_1}\|_{L^\infty(\sigma)} \leq \left\| E^\square_{I_j^\square, \kappa_1} f \right\|_{L^\infty(\sigma)} + \left\| E^\square_{I_j^\square, \kappa_1} f \right\|_{L^\infty(\sigma)} \leq \alpha_F(F).
\]
Indeed, we note that
\[
\left\| E^\square_{I_j^\square, \kappa_1} f \right\|_{L^\infty(\sigma)} \leq E_F f \lesssim \alpha_F(F).
\]
And as for $\left\| E^\square_{I_j^\square, \kappa_1} f \right\|_{L^\infty(\sigma)}$, there are two cases: if $I_j^\square \in C_F$, then
\[
\left\| E^\square_{I_j^\square, \kappa_1} f \right\|_{L^\infty(\sigma)} \leq E_F f \lesssim \alpha_F(F)
\]
by (2.10) and the definition of the stopping time, and if $I_j^\square \in F$, then because $\sigma$ is doubling and $\pi_D F' \in C_F$, we get
\[
\left\| E^\square_{I_j^\square, \kappa_1} f \right\|_{L^\infty(\sigma)} \lesssim E^\square_{F'} f \lesssim \alpha_F(F).
\]
Thus
\[
\left| \mathbb{T}^\square_{\text{paraproduct}}(f, g) \right| = \left| \sum_{J \in C_F^{\eta \text{-shift}}} \left( 1_J \sum_{I \in C_F: I_j^\square \subset I} M_{I, \kappa_1} \right) T_1^\square F, \Delta_{J, \kappa_2}^\omega \right| \lesssim \sum_{J \in C_F^{\eta \text{-shift}}} \left( 1_J P_{J, \kappa_1} T_1^\square F, \Delta_{J, \kappa_2}^\omega \right) \lesssim \left| T_1^\square F \right|_{L^2(\omega)} \alpha_F(F) \left| \sum_{J \in C_F^{\eta \text{-shift}}} \frac{P_{J, \kappa_1}}{\alpha_F(F)} \Delta_{J, \kappa_2}^\omega \right|_{L^2(\omega)}.
\]
Now we will use an almost orthogonality argument that exploits the fact that for $J'$ small compared to $J$, and $\kappa_1 \leq \kappa_2$, the function $M_{J', \kappa_1} \Delta_{J', \kappa_2}^\omega g$ has vanishing $\omega$-means up to order $\kappa_2 - \kappa_1 + 1$, and the polynomial $1_J P_{J, \kappa_1} \Delta_{J, \kappa_2}^\omega g$ is relatively smooth at the scale of $J'$, together with the fact that the polynomials
$R_{J;\kappa_l} \equiv \frac{P_{J;\kappa_l}}{\alpha_{J;\kappa_l}}$ of degree at most $\kappa_l - 1$, have $L^\infty$ norm uniformly bounded by \((13)\), to show that

\[
(5.4) \quad \left\| \sum_{J \in C_F^{\tau-\text{shift}}} R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g \right\|_{L^2(\omega)}^2 = \sum_{J \in C_F^{\tau-\text{shift}}} \| R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g \|_{L^2(\omega)}^2 + \sum_{J, J' \in C_F^{\tau-\text{shift}} \iff J \neq J'} \int (R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g)(R_{J';\kappa_l} \Delta^\omega_{J';\kappa_l} g) \, d\omega \\
\lesssim \sum_{J \in C_F^{\tau-\text{shift}}} \| R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g \|_{L^2(\omega)}^2 \lesssim \sum_{J \in C_F^{\tau-\text{shift}}} \| \Delta^\omega_{J;\kappa_l} g \|_{L^2(\omega)}^2 \quad \iff \left\| P_{C_F^{\tau-\text{shift}}} g \right\|_{L^2(\omega)}^2.
\]

Indeed, if $J'$ is small compared to $J$, and $J' \subset J', J \subset J$, we have

\[
\left| \int (R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g)(R_{J';\kappa_l} \Delta^\omega_{J';\kappa_l} g) \, d\omega \right| = \| R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g \|_{L^\infty(\omega)} \| R_{J';\kappa_l} \Delta^\omega_{J';\kappa_l} g \|_{L^\infty(\omega)} \int \left( \frac{R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g}{\| R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g \|_{L^\infty(\omega)}} \right) \left( \frac{R_{J';\kappa_l} \Delta^\omega_{J';\kappa_l} g}{\| R_{J';\kappa_l} \Delta^\omega_{J';\kappa_l} g \|_{L^\infty(\omega)}} \right) \, d\omega \lesssim \| R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g \|_{L^\infty(\omega)} \| R_{J';\kappa_l} \Delta^\omega_{J';\kappa_l} g \|_{L^\infty(\omega)} \| \Delta^\omega_{J,\kappa_l} g \|_{L^2(\omega)} \| \Delta^\omega_{J',\kappa_l} g \|_{L^2(\omega)},
\]

by \((2.11)\), i.e.

\[
\| R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g \|_{L^\infty(\omega)} \sqrt{|J'|_\omega} \lesssim \| \Delta^\omega_{J',\kappa_l} g \|_{L^2(\omega)} \quad \iff \quad \sqrt{|J'|_\omega} \lesssim \| \Delta^\omega_{J',\kappa_l} g \|_{L^2(\omega)}.
\]

Thus

\[
\sum_{J, J' \in C_F^{\tau-\text{shift}} \iff J \neq J'} \left| \int (R_{J;\kappa_l} \Delta^\omega_{J;\kappa_l} g)(R_{J';\kappa_l} \Delta^\omega_{J';\kappa_l} g) \, d\omega \right| \lesssim \sum_{J, J' \in C_F^{\tau-\text{shift}} \iff J \neq J'} \sqrt{|J'|_\omega} \| \Delta^\omega_{J',\kappa_l} g \|_{L^2(\omega)} \| \Delta^\omega_{J;\kappa_l} g \|_{L^2(\omega)}
\]

\[
= \sum_{m=1}^{\infty} 2^{-m} \sum_{J, J' \in C_F^{\tau-\text{shift}} \iff J \neq J'} \sqrt{|J'|_\omega} \| \Delta^\omega_{J',\kappa_l} g \|_{L^2(\omega)} \| \Delta^\omega_{J;\kappa_l} g \|_{L^2(\omega)}
\]

\[
\leq \sum_{m=1}^{\infty} 2^{-m} \left( \sum_{J \in C_F^{\tau-\text{shift}}} \| \Delta^\omega_{J;\kappa_l} g \|_{L^2(\omega)}^2 \right)^{1/2} \left( \sum_{J \in C_F^{\tau-\text{shift}}} \frac{|J'|_\omega}{|J|_\omega} \| \Delta^\omega_{J';\kappa_l} g \|_{L^2(\omega)}^2 \right)^{1/2} \lesssim \sum_{J \in C_F^{\tau-\text{shift}}} \| \Delta^\omega_{J;\kappa_l} g \|_{L^2(\omega)}^2.
\]

Altogether we have shown

\[
(5.5) \quad \left| T_{\text{paraproduct}}^{C_\rho;\tau,F}(f, g) \right| \leq \mathcal{T}_{p_\alpha} \mathcal{A}_F(f) \| F \|_{L^2(\omega)} \| P_{C_F^{\tau-\text{shift}}} g \|_{L^2(\omega)}
\]

as required by \((1.12)\).

5.2. The below commutator form. We show that below commutator form converges absolutely, in the sense that

\[
(5.6) \quad \left| T_{\text{commutator}}^{C_\rho;\tau,F}(f, g) \right| \leq \sum_{I \in C_F \iff J \in C_F^{\tau-\text{shift}} \iff J \neq I} \left| \int \left( T^\lambda_{\sigma, M_{I;\kappa}} \right) 1_I \cdot \Delta^\omega_{J;\kappa_l} g \right| \lesssim \sqrt{A_2} \| P_{C_F^T} f \|_{L^2(\sigma)} \| P_{C_F^{\tau-\text{shift}}} g \|_{L^2(\omega)}.
\]
If \( T = H \) is the Hilbert transform on the real line, and if \( P_\ell (x) = x^\ell \) with \( 1 \leq \ell \leq \kappa \), then by the moment vanishing properties of Alpert projections, we get that \( H \) commutes with polynomials \( P \) of degree at most \( \kappa \) when acting on a function \( f \) with vanishing \( \sigma \)-means up to order \( \kappa - 1 \), i.e.

\[
\langle (H_\sigma P - PH_\sigma) f, g \rangle_\omega = \langle 0, g \rangle_\omega = 0.
\]

By duality we also have

\[
\langle (H_\sigma P - PH_\sigma) f, g \rangle_\omega = \langle f, (H_\omega P - PH_\omega) g \rangle_\omega = \langle f, 0 \rangle_\omega = 0,
\]

if \( g \) has vanishing \( \omega \)-means up to order \( \kappa - 1 \). This motivates the following argument.

**Notation 18.** We will take \( \kappa_2 \geq \kappa_1 \) throughout this subsection, and for convenience we write \( \kappa = \kappa_1 \). Then \( \Delta_{j,\kappa_2}^\omega g (x) \) has vanishing \( \omega \)-means up to order \( \kappa_2 - \kappa_1 + 1 \geq 1 \). At many points in the arguments below we will simply use that \( \Delta_{j,\kappa_2}^\omega g (x) \) has 1 vanishing \( \omega \)-moment, and continue with this single vanishing \( \omega \)-moment in subsequent estimates, without further reference to the fact that the estimates could be improved for \( \kappa_2 \) strictly larger than \( \kappa = \kappa_1 \) (as these improved estimates are not needed for the commutator form).

Fix \( \kappa \geq 1 \). Assume that \( K^\lambda \) is a general standard \( \lambda \)-fractional kernel in \( \mathbb{R}^n \), and \( T^\lambda \) is the associated Calderón-Zygmund operator, and define

\[
P_{\alpha,a,I'} (x) = \left( \frac{x - a}{\ell (I')} \right)^\alpha = \left( \frac{x_1 - a_1}{\ell (I')} \right)^{\alpha_1} \cdots \left( \frac{x_\ell - a_\ell}{\ell (I')} \right)^{\alpha_\ell},
\]

where \( 1 \leq |\alpha| \leq \kappa - 1 \) (since when \( |\alpha| = 0 \), \( P_{\alpha,a,I'} \) commutes with \( T^\lambda \)) and \( I' \in \mathcal{E}_D (I), I \in \mathcal{E}_F \).

We consider the renormalization \( Q_{I',\kappa}^\mu \) of the polynomial \( M_{I',\kappa}^{\mu} \) introduced earlier, given by

\[
Q_{I',\kappa}^\mu = \frac{1}{|I'|^{\mu}} |I'| \Delta_{I',\kappa}^\mu f = \frac{1}{|I'|^{\mu}} M_{I',\kappa}^{\mu}.
\]

For \( c_j \in J \subset I' \), write

\[
Q_{I',\kappa}^\mu (y) = \sum_{|\alpha| < \kappa} b_\alpha \left( \frac{y - c_j}{\ell (I')} \right)^\alpha = \sum_{|\alpha| < \kappa} b_\alpha P_{\alpha,c_j,I'} (y).
\]

By rescaling to the unit cube and invoking the fact that any two norms on a finite dimensional vector space are equivalent, followed by then noting that from (2.11) we get \( \| Q_{I',\kappa}^\mu \|_\infty \approx \frac{1}{\sqrt{|I'|_\sigma}} \), then we have

\[
(5.7) \quad \sum_{|\alpha| < \kappa} |b_\alpha| \approx \| Q_{I',\kappa}^\mu \|_\infty \approx \frac{1}{\sqrt{|I'|_\sigma}}.
\]

We then bound

\[
\left| \langle [M_{I',\kappa}^{\mu}, T_\sigma^\lambda] 1_{I'}, \Delta_{j,\kappa}^\omega g \rangle_\omega \right| \leq \left| \frac{\hat{f} (I)}{|I'|_\sigma} \right| \left| \langle Q_{I',\kappa}^\mu 1_{I'}, \Delta_{j,\kappa}^\omega g \rangle_\omega \right| \leq \sum_{|\alpha| < \kappa} |b_\alpha| \left| \langle [P_{\alpha,c_j,I'}, T_\sigma^\lambda] 1_{I'}, \Delta_{j,\kappa}^\omega g \rangle_\omega \right|
\]

\[
\lesssim \frac{\left| \frac{\hat{f} (I)}{|I'|_\sigma} \right| \max_{|\alpha| < \kappa} \left| \langle [P_{\alpha,c_j,I'}, T_\sigma^\lambda] 1_{I'}, \Delta_{j,\kappa}^\omega g \rangle_\omega \right|},
\]

so we turn to estimating

\[
\left| \langle [P_{\alpha,c_j,I'}, T_\sigma^\lambda] 1_{I'}, \Delta_{j,\kappa}^\omega g \rangle_\omega \right|
\]

uniformly in \( \alpha \).

Taking \( J \subset I' \), we begin by writing

\[
(5.8) \quad \left| \langle [P_{\alpha,a,I'}, T_\sigma^\lambda] 1_{I'}, \Delta_{j,\kappa}^\omega g \rangle_\omega \right| = \left| \langle [P_{\alpha,a,I'}, T_\sigma^\lambda] 1_{I'} (x) \Delta_{j,\kappa}^\omega g (x) d\omega (x) \right|
\]

\[
= \left| \int [P_{\alpha,a,I'}, T_\sigma^\lambda] 1_{I' \setminus J} (x) \Delta_{j,\kappa}^\omega g (x) d\omega (x) + \int [P_{\alpha,a,I'}, T_\sigma^\lambda] 1_{J} (x) \Delta_{j,\kappa}^\omega g (x) d\omega (x) \right|
\]

\[
\equiv \text{Int}^\lambda (J) + \text{Int}^\lambda (J),
\]

where we are suppressing the dependence on both \( \alpha \) and \( \kappa \).
Let us address the first term. We use the known identity

\[ x^\alpha - y^\alpha = \sum_{k=1}^{n} (x_k - y_k) \sum_{|\beta| + |\gamma| = |\alpha| - 1} c_{\alpha,\beta,\gamma} x^\beta y^\gamma, \]

to write the pointwise equality

\[
1_{I'}(x) \left[ P_{\alpha,\alpha',I'}, T_{\lambda}^x \right] 1_{I'}(x) = 1_{I'}(x) \int K^\lambda (x - y) \left\{ P_{\alpha,\alpha',I'}(x) - P_{\alpha,\alpha',I'}(y) \right\} 1_{I'}(y) \, d\sigma(y)
= 1_{I'}(x) \int K^\lambda (x - y) \left\{ \sum_{k=1}^{n} \frac{(x_k - y_k)}{\ell(I')} \sum_{|\beta| + |\gamma| = |\alpha| - 1} c_{\alpha,\beta,\gamma} \left( \frac{x - a}{\ell(I')} \right)^\beta \left( \frac{y - a}{\ell(I')} \right)^\gamma \right\} 1_{I'}(y) \, d\sigma(y)
= \sum_{k=1}^{n} \sum_{|\beta| + |\gamma| = |\alpha| - 1} c_{\alpha,\beta,\gamma} 1_{I'}(x) \left[ \Phi^\lambda_k (x - y) \left\{ \left( \frac{y - a}{\ell(I')} \right)^\gamma \right\} 1_{I'}(y) \, d\sigma(y) \right] \left( \frac{x - a}{\ell(I')} \right)^\beta,
\]

where \( \Phi^\lambda_k (x - y) = K^\lambda (x - y) \left( \frac{x_k - y_k}{\ell(I')} \right) \).

Integrating the above against \( \Delta^\omega_{J,\kappa_2} g \) and then pulling out the double sum \( \sum_{k=1}^{n} \sum_{|\beta| + |\gamma| = |\alpha| - 1} \) lets us write

\[
\text{Int}^\lambda_{k,\beta,\gamma}(J) = \sum_{k=1}^{n} \sum_{|\beta| + |\gamma| = |\alpha| - 1} c_{\alpha,\beta,\gamma} \text{Int}^\lambda_{k,\beta,\gamma}(J),
\]

where with the choice \( a = c_J \) the center of \( J \), we define

\[
(5.9) \quad \text{Int}^\lambda_{k,\beta,\gamma}(J) = \int_J \int_{I' \setminus 2J} \Phi^\lambda_k (x - y) \left( \frac{y - c_J}{\ell(I')} \right)^\gamma \, d\sigma(y) \left( \frac{x - c_J}{\ell(I')} \right)^\beta \Delta^\omega_{J,\kappa_2} g(x) \, d\omega(x)
= \int_{I' \setminus 2J} \left\{ \int_J \Phi^\lambda_k (x - y) \left( \frac{x - c_J}{\ell(I')} \right)^\beta \Delta^\omega_{J,\kappa_2} g(x) \, d\omega(x) \right\} \left( \frac{y - c_J}{\ell(I')} \right)^\gamma \, d\sigma(y).
\]

Taking

\[ h(x) = \left( \frac{x - c_J}{\ell(I')} \right)^\beta \Delta^\omega_{J,\kappa_2} g(x), \]

which has support in \( J \) and at \( \kappa - |\beta| + 1 \) vanishing moments, by Taylor’s formula and the Calderon-Zygmund estimates for \( K^\lambda \), we have the inner-most integral has absolute value

\[
\left| \int \Phi^\lambda_k (x - y) h(x) \, d\omega(x) \right| = \left| \int \frac{1}{(\kappa - |\beta|)!} ((x - c_J) \cdot \nabla)^{|\beta|} \Phi^\lambda_k (y) \, h(x) \, d\omega(x) \right|
\leq \| h \|_{L^1(\omega)} \frac{\ell(J)^{\kappa - |\beta|}}{[\ell(J) + \text{dist}(y,J)]^{\kappa - |\beta| + n - \lambda - 1} \ell(I')} \leq \frac{\ell(J)^{|\beta|}}{[\ell(J) + \text{dist}(y,J)]^{\kappa - |\beta| + n - \lambda - 1} \ell(I')} \sqrt{|J|_\omega \| g \| (J)},
\]

where in the last estimate we used the estimate

\[
\| h \|_{L^1(\omega)} = \int_J \left| \left( \frac{x - c_J}{\ell(I')} \right)^\beta \Delta^\omega_{J,\kappa_2} g(x) \right| \, d\omega(x) \leq \left( \frac{\ell(J)}{\ell(I')} \right)^{|\beta|} \| \Delta^\omega_{J,\kappa_2}g \|_{L^1(\omega)} \leq \left( \frac{\ell(J)}{\ell(I')} \right)^{|\beta|} \sqrt{|J|_\omega \| g \| (J)}.
\]

Thus (5.9) yields

\[
\left| \text{Int}^\lambda_{k,\beta,\gamma}(J) \right| \leq \int_{I' \setminus 2J} \left| \int \Phi^\lambda_k (x - y) \left( \frac{x - c_J}{\ell(I')} \right)^\beta \Delta^\omega_{J,\kappa_2} g(x) \, d\omega(x) \right| \left( \frac{y - c_J}{\ell(I')} \right)^\gamma \, d\sigma(y)
\leq \left( \frac{\ell(J)}{\ell(I')} \right)^{|\beta|} \frac{\ell(J)^{\kappa - |\beta|}}{[\ell(J) + \text{dist}(y,J)]^{\kappa - |\beta| + n - \lambda - 1} \ell(I')} \sqrt{|J|_\omega \| g \| (J)} \left( \frac{\ell(J) + \text{dist}(y,J)}{\ell(I')} \right)^{|\gamma|} \, d\sigma(y)
= \left( \frac{\ell(J)}{\ell(I')} \right)^{|\alpha| - 1} \sqrt{|J|_\omega \| g \| (J)} \left\{ \int_{I' \setminus 2J} \left( \frac{\ell(J)}{\ell(J) + \text{dist}(y,J)} \right)^{\kappa - |\beta| + 1} \frac{1}{[\ell(J) + \text{dist}(y,J)]^{n - \lambda - 1} \ell(I')} \, d\sigma(y) \right\}.
\]
Now we fix \( t \in \mathbb{N} \), and estimate the sum of \(|\text{Int}^{\lambda, \delta}(J)|\) over those \( J \subset I' \) with \( \ell(J) = 2^{-t} \ell(I') \) by splitting the integration in \( y \) according to the size of \( \ell(J) + \text{dist}(y, J) \), to obtain the following bound:

\[
\sum_{\substack{J \subset I' \colon \\
\ell(J) = 2^{-t} \ell(I')}} \left| \text{Int}^{\lambda, \delta}(J) \right| 
\lesssim 2^{-t|\alpha|-1} \sum_{\substack{J \subset I' \colon \\
\ell(J) = 2^{-t} \ell(I')}} \sqrt{|J|_\omega} \left| \hat{g}(J) \right| \left\{ \int_{I \setminus 2J} \left( \frac{\ell(J)}{\ell(J) + \text{dist}(y, J)} \right)^{\kappa - |\alpha| + 1} \frac{d\sigma(y)}{(2^s \ell(J))^{n-\lambda-1} \ell(I')} \right\}
\lesssim 2^{-t|\alpha|-1} \sum_{J \subset I' \colon \ell(J) = 2^{-t} \ell(I')} \sqrt{|J|_\omega} \left| \hat{g}(J) \right| \left\{ \frac{\ell(I)}{(2^s \ell(J))^{n-\lambda-1} \ell(I')} \right\}
\lesssim 2^{-t|\alpha|} \sum_{J \subset I' \colon \ell(J) = 2^{-t} \ell(I')} \sqrt{|J|_\omega} \left| \hat{g}(J) \right| \sum_{s=1}^{t} (2^{-s})^{\kappa - |\alpha| + 1} 2^{-s(n-\lambda-1)} \frac{2^s |J|_\sigma}{\ell(J)^{n-\lambda}},
\]

which, upon pigeonholing the sum in \( J \) according to membership in the grandchildren of \( I \) at depth \( t - s \), gives:

\[
\sum_{J \in \mathcal{C}^{(t-s)}_D(I')} \left| \text{Int}^{\lambda, \delta}(J) \right| \lesssim 2^{-t|\alpha|} \sum_{J \in \mathcal{C}^{(t-s)}_D(I')} \sqrt{|J|_\omega} \left| \hat{g}(J) \right| \sum_{s=1}^{t} (2^{-s})^{\kappa - |\alpha| + 1} 2^{-s(n-\lambda-1)} \frac{2^s |J|_\sigma}{\ell(J)^{n-\lambda}}
\]

\[
= 2^{-t|\alpha|} \sum_{s=1}^{t} (2^{-s})^{\kappa - |\alpha|} \sum_{K \in \mathcal{C}^{(t-s)}_D(I')} \sqrt{|K|_\omega} \left| \hat{g}(J) \right| \frac{2^s |J|_\sigma}{\ell(K)^{n-\lambda}}
\]

\[
\lesssim 2^{-t|\alpha|} \sum_{s=1}^{t} (2^{-s})^{\kappa - |\alpha|} \sum_{K \in \mathcal{C}^{(t-s)}_D(I')} \left( \frac{3K|_\sigma}{\ell(K)^{n-\lambda}} \right) \sqrt{|K|_\omega} \sum_{J \in \mathcal{C}^{(t-s)}_D(K)} \left| \hat{g}(J) \right|^2
\]

\[
\lesssim 2^{-t|\alpha|} \sum_{s=1}^{t} (2^{-s})^{\kappa - |\alpha|} \sum_{K \in \mathcal{C}^{(t-s)}_D(I')} \left( \frac{3K|_\sigma}{\ell(K)^{n-\lambda}} \right) \sqrt{|K|_\omega} \sum_{J \in \mathcal{C}^{(t-s)}_D(K)} \left| \hat{g}(J) \right|^2,
\]

where we used the \( A_2^\lambda \) condition and doubling for \( \sigma \) in the last inequality. Thus we have

\[
\sum_{J \in \mathcal{C}^{(t-s)}_D(I')} \left| \text{Int}^{\lambda, \delta}(J) \right| \lesssim 2^{-t|\alpha|} \sqrt{A_2^\lambda} \sum_{s=1}^{t} (2^{-s})^{\kappa - |\alpha|} \sqrt{|I'|_\sigma} \sum_{J \in \mathcal{C}^{(t-s)}_D(I')} \left| \hat{g}(J) \right|^2
\]

\[
\lesssim 2^{-t} \sqrt{A_2^\lambda} \sqrt{|I'|_\sigma} \sum_{J \in \mathcal{C}^{(t-s)}_D(I')} \left| \hat{g}(J) \right|^2,
\]

since \( 1 \leq |\alpha| \leq \kappa - 1 \).

We now claim the same estimate holds for the sum of \(|\text{Int}^{\lambda, \delta}(J)|\) over \( J \subset I' \) with \( \ell(J) = 2^{-t} \ell(I') \). We write

\[
\text{Int}^{\lambda, \delta}(J) = \sum_{k=1}^{n} \sum_{|\beta|+|\gamma|=|\alpha|-1} c_{\alpha, \beta, \gamma} \text{Int}^{\lambda, \delta}_{k, \beta, \gamma}(J),
\]
and estimate
\[ \left| \text{Int}^{\lambda,b}_{\nu,\eta}(J) \right| \lesssim \left| \int J \left( \int_{2J} \Phi^\nu_{\kappa}(x-y) \left( \frac{y-c_J}{\ell(I')} \right)^\gamma \, d\sigma(y) \right) \frac{x-c_J}{\ell(I')} \Delta^\omega_{\nu,\kappa_2} g(x) \, d\omega(x) \right| \]
\[ \leq \int J \left( \int_{2J} \frac{1}{\ell(I')} |y-c_J|^{\gamma} \, d\sigma(y) \right) \frac{|x-c_J|}{\ell(I')} \left| \Delta^\omega_{\nu,\kappa_2} g(x) \right| \, d\omega(x) \]
\[ \lesssim \left( \frac{\ell(J)}{\ell(I')} \right)^{|\gamma|+|\beta|} \frac{1}{\ell(I')} \left| \int J \int_{2J} \frac{d\sigma(y) \, d\omega(x)}{|x-y|^{n-\lambda-1}} \lesssim \sqrt{\frac{A^2}{\ell(J)} \frac{\ell(I')}{\ell(I')} \left| \int J \right|_{\sigma} \left| J \right|_{\omega},} \right. \]
where in the last inequality we used that \(|\beta| + |\gamma| = |\alpha| \geq 1\) and the estimate
\[ \int J \int_{2J} \frac{d\sigma(y) \, d\omega(x)}{|x-y|^{n-\lambda-1}} \lesssim \sqrt{\frac{A^2}{\ell(J)} \frac{\ell(I')}{\ell(I')} \left| \int J \right|_{\sigma} \left| J \right|_{\omega}.} \]
Indeed, in order to estimate the double integral using the \(A^2\) condition we cover the band \(|x-y| \leq C2^{-m} \ell(J)\) by a collection of cubes \(Q(\pm m, C2^{-m} \ell(J)) \times Q(\pm m, C2^{-m} \ell(J))\) in \(CJ \times CJ\) with centers \((z_m, z_m)\) and bounded overlap. Then we have
\[ \int J \int_{2J} \frac{d\sigma(y) \, d\omega(x)}{|x-y|^{n-\lambda-1}} \lesssim \sum_{m=0}^{\infty} \sum_{z_m \in Q(\pm m, C2^{-m} \ell(J))} \frac{\ell(J)}{\ell(I')} \left| Q(\pm m, C2^{-m} \ell(J)) \right|_{\sigma} \left| Q(\pm m, C2^{-m} \ell(J)) \right|_{\omega} \]
\[ \lesssim \sum_{m=0}^{\infty} \frac{1}{\ell(J)} \sum_{z_m \in Q(\pm m, C2^{-m} \ell(J))} 2^{m(n-\lambda-1)} \right| Q(\pm m, C2^{-m} \ell(J)) \right|_{\sigma} \left| Q(\pm m, C2^{-m} \ell(J)) \right|_{\omega} \]
\[ \lesssim \sqrt{\frac{A^2}{\ell(J)} \sum_{m=0}^{\infty} 2^{m(n-\lambda-1)} 2^{-m(n-\lambda)} \left| CJ \right|_{\sigma} \left| CJ \right|_{\omega} \lesssim \sqrt{\frac{A^2}{\ell(J)} \frac{\ell(I')}{\ell(I')} \left| J \right|_{\sigma} \left| J \right|_{\omega}.} \]
Now
\[ \sum_{J \in E_D^{(i)}(I')} \left| \text{Int}^{\lambda,b}_{\nu,\eta}(J) \right| \lesssim \sum_{J \in E_D^{(i)}(I')} \sqrt{\frac{A^2}{\ell(J)} \frac{\ell(I')}{\ell(I')} \left| \bar{g}(J) \right| \left| J \right|_{\sigma} \leq 2^{-t} \sqrt{2^{-t} \left| I' \right|_{\sigma} \left| \sum_{J \in E_D^{(i)}(I')} \left| \bar{g}(J) \right|^2 \right|},} \]
and so altogether we have
\[ \sum_{J \in E_D^{(i)}(I')} \left| \left[ \left[ P_{a,c_I,I'}, T_{\lambda_{\sigma}} \right] \right] 1_{I'}, \Delta^\omega_{\nu,\kappa} g \right|_{\omega} = \sum_{J \in E_D^{(i)}(I')} \left| \text{Int}^{\lambda}_{\nu}(J) \right| \]
\[ \leq \sum_{J \in E_D^{(i)}(I')} \left| \text{Int}^{\lambda,b}_{\nu}(J) \right| + \sum_{J \in E_D^{(i)}(I')} \left| \text{Int}^{\lambda,b}_{\nu}(J) \right| \lesssim 2^{-t} \sqrt{2^{-t} \left| I' \right|_{\sigma} \left| \sum_{J \in E_D^{(i)}(I')} \left| \bar{g}(J) \right|^2 \right|}. \]
Finally then we obtain from this and (5.7),
\[ \sum_{J \in E_D^{(i)}(I')} \left| \left[ \left[ Q_{\nu,\kappa, I'}, T_{\lambda_{\sigma}} \right] \right] 1_{I'}, \Delta^\omega_{\nu,\kappa} g \right|_{\omega} = \sum_{J \in E_D^{(i)}(I')} \left| \sum_{|\alpha| \leq \kappa-1} b_{\alpha} \left( \left[ \left[ P_{a,c_I,I'}, T_{\lambda_{\sigma}} \right] \right] 1_{I'}, \Delta^\omega_{\nu,\kappa} g \right) \right|_{\omega} \]
\[ \lesssim 2^{-t} \sqrt{2^{-t} \left| \sum_{J \in E_D^{(i)}(I')} \left| \bar{g}(J) \right|^2 \right|}. \]
Now using $M_{I';\kappa} = |\hat{f}(I)| Q_{I';\kappa}$, and applying the above estimates with $I' = I_I$, we can sum over $t$ and $I \in \mathcal{C}_F$ to obtain the following estimate for the below commutator form,

$$
\left| T_{\text{commutator}}^{C_{F},F}(f, g) \right| \lesssim \sum_{I \in \mathcal{C}_F} \left| \hat{f}(I) \right| \left| \left\langle T_{\sigma}^\lambda, Q_{I;\kappa} \right| \mathbf{1}_{I_I}, \Delta_{J;\kappa}^\omega g \rangle \right|_{\omega}
$$

$$
\lesssim \sum_{t = r}^{\infty} \sum_{I \in \mathcal{C}_F} 2^{-t} \sqrt{A_2^2} \left| \hat{f}(I) \right| \left| \sum_{J \in \mathcal{C}_F^{(I)}} |\hat{g}(J)|^2 \right| \lesssim \sqrt{A_2^2} \sum_{t = r}^{\infty} 2^{-t} \left| \hat{f}(I) \right|^2 \sum_{I \in \mathcal{C}_F} \sum_{J \in \mathcal{C}_F^{(I)}} |\hat{g}(J)|^2
$$

$$
\lesssim \sqrt{A_2^2} \sum_{t = r}^{\infty} 2^{-T} \left\| P_{C_{F},F}^\sigma f \right\|_{L^2(\sigma)} \left\| P_{C_{F},F}^\omega \Delta_{J;\kappa}^\omega g \right\|_{L^2(\omega)} \lesssim \sqrt{A_2^2} \left\| P_{C_{F},F}^\sigma f \right\|_{L^2(\sigma)} \left\| P_{C_{F},F}^\omega \Delta_{J;\kappa}^\omega g \right\|_{L^2(\omega)}.
$$

5.3. The below neighbour form. We show the neighbour form converges absolutely, in the sense that the neighbour form is bounded above by the sublinear form

$$
\left| T_{\text{neighbour}}^{C_{F},F}(f, g) \right| \equiv \sum_{I \in \mathcal{C}_F} \sum_{J \in \mathcal{C}_F^{(I)}} \left| \left\langle T_{\sigma}^\alpha (\theta(I_I)) \Delta_{J;\kappa}^\omega f, \Delta_{J;\kappa}^\omega g \right\rangle \right| \lesssim \sqrt{A_2^2} \left\| P_{C_{F},F}^\sigma f \right\|_{L^2(\sigma)} \left\| P_{C_{F},F}^\omega \Delta_{J;\kappa}^\omega g \right\|_{L^2(\omega)}.
$$

We revert to using $\kappa_1$ and $\kappa_2$ in treating the below neighbour form, and we also revert to using $\alpha$ instead of $\lambda$ as was done in the previous subsection. In the neighbour form we obtain the required bound by taking absolute values inside the sum, and then arguing as in the case of Haar wavelets in [SaShUr7], end of Subsection 8.4. We begin with $M_{I';\kappa} = \mathbf{1}_{I'} \Delta_{J;\kappa}^\omega f$ as in (5.1) to obtain

$$
\left| T_{\text{neighbour}}^{C_{F},F}(f, g) \right| = \sum_{I \in \mathcal{C}_F} \sum_{J \in \mathcal{C}_F^{(I)}} \left| \left\langle T_{\sigma}^\alpha (\theta(I_I)) \Delta_{J;\kappa}^\omega f, \Delta_{J;\kappa}^\omega g \right\rangle \right| \lesssim \sum_{I \in \mathcal{C}_F} \sum_{J \in \mathcal{C}_F^{(I)}} \left| \left\langle T_{\sigma}^\alpha (\theta(I_I)) \Delta_{J;\kappa}^\omega f, \Delta_{J;\kappa}^\omega g \right\rangle \right|_{\omega}
$$

Using the pivotal bound (2.12) on the inner product with $\nu = \|M_{I';\kappa}\|_{L^\infty(\sigma)} \mathbf{1}_{I'} d\sigma$, and then estimating by the usual Poisson kernel,

$$
\left| \left\langle T_{\sigma}^\alpha (M_{I';\kappa} \mathbf{1}_{I'}), \Delta_{J;\kappa}^\omega g \right\rangle \right| \lesssim P_{\kappa_2}^\sigma (J, \|M_{I';\kappa}\|_{L^\infty(\sigma)} \mathbf{1}_{I'} d\sigma) \sqrt{|J|} \left\| \Delta_{J;\kappa}^\omega g \right\|_{L^2(\omega)}
$$

$$
\lesssim \|M_{I';\kappa}\|_{L^\infty(\sigma)} P_{\kappa_2}^\sigma (J, \mathbf{1}_{I'} d\sigma) \sqrt{|J|} \left\| \Delta_{J;\kappa}^\omega g \right\|_{L^2(\omega)}.
$$
and the estimate $\|M_{I',\rho_1}\|_{L^\infty(\sigma)} \approx \frac{1}{|I'|} |\hat{f}(I)|$ from (2.11), to obtain

$$\left|T_{\text{neighbour}}^{C,\rho,\sigma,F} (f, g) \lesssim \sum_{I \in C_F} \sum_{J \in C_{I',\sigma}^{\rho-\text{shift}}} \sum_{I_0 \neq I_0} \frac{|\hat{f}(I)|}{|I_0|} P_{\kappa_2}^\alpha (J, 1_{I_0}) \sqrt{|I_0|} \|\Delta_{J; I_0} g\|_{L^2(\omega)} \right|$$

$$= \sum_{s=r} \sum_{I \in C_F} \sum_{I_0 \neq I_0} \sum_{J \in C_{I',\sigma}^{\rho-\text{shift}}} \sum_{J \subseteq I_0} \frac{|\hat{f}(I)|}{|I_0|} P_{\kappa_2}^\alpha (J, 1_{I_0}) \sqrt{|I_0|} \|\Delta_{J; I_0} g\|_{L^2(\omega)}$$

$$\lesssim \sum_{s=r} \sum_{I \in C_F} \sum_{I_0 \neq I_0} \sum_{J \in C_{I',\sigma}^{\rho-\text{shift}}} \sum_{J \subseteq I_0} \frac{|\hat{f}(I)|}{|I_0|} \left\{ (2^{-s})^{1-\varepsilon(n+1-\lambda)} P_{\kappa_2}^\alpha (I_0, 1_{I_0}) \right\} \sqrt{|I_0|} \|\Delta_{J; I_0} g\|_{L^2(\omega)}$$

By Lemma 11 and the Muckenhoupt condition, this is

$$\lesssim \sqrt{A_2^\alpha (\sigma, \omega)} \sum_{s=r} \sum_{I \in C_F} \sum_{I_0 \neq I_0} \sum_{J \in C_{I',\sigma}^{\rho-\text{shift}}} \sum_{J \subseteq I_0} \frac{|\hat{f}(I)|}{|I_0|} \left\{ (2^{-s})^{1-\varepsilon(n+1-\lambda)} \right\} \|\Delta_{J; I_0} g\|_{L^2(\omega)}$$

and by Cauchy-Schwarz, this is at most

$$\sqrt{A_2^\alpha (\sigma, \omega)} \sum_{s=r} (2^{-s})^{1-\varepsilon(n+1-\lambda)} \left( \sum_{I \in C_F} \sum_{I_0 \neq I_0} \sum_{J \subseteq I_0} \frac{|\hat{f}(I)|}{|I_0|} \left\{ (2^{-s})^{1-\varepsilon(n+1-\lambda)} \right\} \|\Delta_{J; I_0} g\|_{L^2(\omega)} \right)^{\frac{1}{2}}$$

$$\times \left( \sum_{I \in C_F} \sum_{I_0 \neq I_0} \sum_{J \subseteq I_0} \|\Delta_{J; I_0} g\|_{L^2(\omega)} \right)^{\frac{1}{2}}.$$

Now we note that

$$\sum_{I \in C_F} \sum_{I_0 \neq I_0} \sum_{J \subseteq I_0} \|\Delta_{J; I_0} g\|_{L^2(\omega)}^2 \lesssim \|P_{C_F}^\sigma g\|_{L^2(\omega)}^2,$$

and

$$\sum_{J \in C_{I',\sigma}^{\rho-\text{shift}}} \frac{|J|_\omega}{|I|_\omega} \lesssim 1,$$

and use $\|P_{C_F}^\sigma f\|_{L^2(\omega)}^2 = \sum_{I \in C_F} |\hat{f}(I)|^2$, to obtain

$$\left|T_{\text{neighbour}}^{C,\rho,\sigma,F} (f, g) \lesssim \sqrt{A_2^\alpha} \sum_{s=r} (2^{-s})^{1-\varepsilon(n+1-\lambda)} \|P_{C_F}^\sigma\|_{L^2(\sigma)} \|P_{C_{I',\sigma}^{\rho-\text{shift}}} g\|_{L^2(\omega)} \lesssim \sqrt{A_2^\alpha} \|P_{C_F}^\sigma\|_{L^2(\sigma)} \|P_{C_{I',\sigma}^{\rho-\text{shift}}} g\|_{L^2(\omega)}.$$

5.4. The below stopping form. We also show the below stopping form converges absolutely, in the sense that the below stopping form is bounded by the sublinear form

\[ T_{\text{stop}}^{C_{\rho,\varepsilon,F}} (f, g) \leq \sum_{I \in C_{\rho,\varepsilon}} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{\rho,\varepsilon}^\text{shift}} \left| \left\langle M_{I',\kappa_1} T_{\sigma}^\omega 1_{F \setminus I'} \Delta_{J,\kappa_2}^\omega g \right\rangle \omega \right| \lesssim \sqrt{A_2} \left\| P_{C_{\rho,\varepsilon}}^\sigma \right\|_{L^2(\omega)} \left\| P_{C_{\rho,\varepsilon}^\text{shift}}^\omega \right\|_{L^2(\omega)}, \]

using nearly the same proof as for (5.10), except that here we need to use the extra vanishing moments of \( \Delta_{J,\kappa_2}^\omega \) to penetrate past the polynomial \( M_{I',\kappa_1} \) of degree \( \kappa_1 \) and reach the kernel \( T_{\sigma}^\omega \).

To bound the below stopping form \( T_{\text{stop}}^{C_{\rho,\varepsilon,F}} (f, g) \), we will use the \( \kappa_1 \)-pivotal condition together with a variant of the Haar stopping form argument due to Nazarov, Treil and Volberg [NTV4]. Most importantly we will assume in this subsection that

\[ \kappa_2 \geq 2\kappa_1. \]

Recall that

\[ M_{I',\kappa} \equiv \left| \hat{f}(I) \right| Q_{I',\kappa} = E_{I',\kappa}^\sigma f - 1_{I'} E_{I',\kappa}^\sigma f. \]

We begin the proof by pigeonholing the ratio of side lengths of \( I \) and \( J \) in the stopping form:

\[ T_{\text{stop}}^{C_{\rho,\varepsilon,F}} (f, g) = \sum_{I \in C_{\rho,\varepsilon}} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{\rho,\varepsilon}^\text{shift}} \left| \hat{f}(I) \right| \left| \left\langle Q_{I',\kappa} T_{\sigma}^\omega 1_{F \setminus I'} \Delta_{J,\kappa_2}^\omega g \right\rangle \omega \right| \]

\[ = \sum_{s=0}^{\infty} \sum_{I \in C_{\rho,\varepsilon}} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{\rho,\varepsilon}^\text{shift}} \left| \hat{f}(I) \right| \left| \left\langle Q_{I',\kappa} T_{\sigma}^\omega 1_{F \setminus I'} \Delta_{J,\kappa_2}^\omega g \right\rangle \omega \right| \equiv \sum_{s=0}^{\infty} T_{\text{stop},s}^{C_{\rho,\varepsilon,F}} (f, g). \]

By (2.12) with \( R = \frac{Q_{I',\kappa}}{\|Q_{I',\kappa}\|_\infty} \), and then using \( \|Q_{I',\kappa}\|_\infty \lesssim \frac{1}{|I'|^{\kappa_1}} \), the Poisson inequality (2.13), and the crucial assumption (5.12), we obtain the last display is at most a constant times

\[ \sum_{I \in C_{\rho,\varepsilon}} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{\rho,\varepsilon}^\text{shift} \text{and } \ell(I) = 2^{-s} \ell(I)} \left| \hat{f}(I) \right| \left| \left\langle Q_{I',\kappa} T_{\sigma}^\omega 1_{F \setminus I'} \Delta_{J,\kappa_2}^\omega g \right\rangle \omega \right| \leq \sum_{s=0}^{\infty} \left| T_{\text{stop},s}^{C_{\rho,\varepsilon,F}} (f, g) \right|. \]

By Lemma [11] and the Muckenhoupt condition, this is

\[ \lesssim \sqrt{A_2} (\sigma, \omega) \sum_{I \in C_{\rho,\varepsilon}} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{\rho,\varepsilon}^\text{shift} \text{and } \ell(I) = 2^{-s} \ell(I)} \left| \hat{f}(I) \right| \left| \left\langle Q_{I',\kappa} T_{\sigma}^\omega 1_{F \setminus I'} \Delta_{J,\kappa_2}^\omega g \right\rangle \omega \right| \]

\[ \lesssim (2^{-s})^{\kappa_1 - \varepsilon(n+\kappa_1-\lambda)} \sqrt{A_2} (\sigma, \omega) \sum_{I \in C_{\rho,\varepsilon}} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{\rho,\varepsilon}^\text{shift} \text{and } \ell(I) = 2^{-s} \ell(I)} \left| \hat{f}(I) \right|^2 \left| \frac{J}{|I|} \right| \left| \left\langle Q_{I',\kappa} T_{\sigma}^\omega 1_{F \setminus I'} \Delta_{J,\kappa_2}^\omega g \right\rangle \omega \right|^2. \]


Now we note that

$$\sum_{I \in C_F} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_F^{I'-\text{shift}}} \| \Delta^\omega_{f, \kappa_2} g \|_{L^2(\omega)}^2 \lesssim \| P^\omega_{C_F^{I'-\text{shift}}} g \|_{L^2(\omega)}^2,$$

and use $\| P^\omega_{C_F^{I'-\text{shift}}} f \|_{L^2(\sigma)}^2 \equiv \sum_{I \in C_F} \| \hat{f}(I) \|_2^2$, and

$$\sum_{J \in C_F^{I'-\text{shift}}} \sum_{(j) = 2^{-r} t(I)} \| \eta_j \|_{T_{\omega}^{I-I'}} \leq 1,$$

to obtain

$$\bigg| T^C_{\mathrm{stop},s} (f, g) \bigg| \lesssim (2^{-s})^{\kappa_1 - \epsilon(n + \kappa_1 - \alpha)} \sqrt{A^2_2 (\sigma, \omega)} \bigg\| P^\omega_{C_F^{I'-\text{shift}}} f \bigg\|_{L^2(\sigma)} \bigg\| P^\omega_{C_F^{I'-\text{shift}}} g \bigg\|_{L^2(\omega)}.$$

Finally, then sum in $s$ to obtain

$$\bigg| T^C_{\mathrm{stop},s} (f, g) \bigg| \leq \sqrt{A^2_2 (\sigma, \omega)} \sum_{s=0}^{\infty} \bigg| T^C_{\mathrm{stop},s} (f, g) \bigg| \lesssim \sqrt{A^2_2 (\sigma, \omega)} \bigg\| P^\omega_{C_F^{I'-\text{shift}}} f \bigg\|_{L^2(\sigma)} \bigg\| P^\omega_{C_F^{I'-\text{shift}}} g \bigg\|_{L^2(\omega)},$$

if we take $0 < \epsilon < \frac{\kappa_1}{n + \kappa_1 - \alpha}$.

6. Above diagonal form and the parallel corona

In order to treat the above form $B_{\Delta^\omega_{f, \kappa_2}} (f, g)$, we again start with the Canonical Splitting of $B_{\Delta^\omega_{f, \kappa_2}} (f, g)$ in $\mathbb{R}^n$ just as we did for the below form $B_{\Delta^\omega_{f, \kappa_2}} (f, g)$,

$$B_{\Delta^\omega_{f, \kappa_2}} (f, g) = \sum_{G \in \mathcal{G}} \bigg( T_{\sigma} \left( P^G_{C_G} f \right), P^G_{C_G^{\text{shift}}} g \bigg)_{\omega} + \sum_{F, G \in \mathcal{G}} \bigg( T_{\sigma} \left( P^G_{C_F} f \right), P^G_{C_F^{\text{shift}}} g \bigg)_{\omega},$$

where the underlying Alpert projections defining $P^\sigma_{C_F^{I'-\text{shift}}}$ and $P^\omega_{C_F^{I'-\text{shift}}}$ are $\Delta^\sigma_{f, \kappa_1} f$ and $\Delta^\omega_{f, \kappa_2} f$ respectively with $\kappa_2 \geq 2\kappa_1$ and $\kappa_1 \geq \kappa$. Since the treatment of the forms $T^C_{\mathrm{far},s} (f, g)$, $T^C_{\mathrm{far},s} (f, g)$ and $T^C_{\mathrm{disjoint}} (f, g)$ in the case of the below form $B_{\Delta^\omega_{f, \kappa_2}} (f, g)$ did not use the crucial assumption $\kappa_1 = 2\kappa_2$, which we do not have because of (5.12). Instead, we must manipulate the above diagonal form $T^C_{\mathrm{diagonal}} (f, g)$ by adding 'pieces' of the below form $B_{\Delta^\omega_{f, \kappa_2}} (f, g)$ to create a parallel corona like decomposition to which we can apply the indicator / cube testing condition.

Here is a brief schematic diagram of the decompositions used for the above diagonal form $T^C_{\mathrm{diagonal}} (f, g)$,

$\begin{array}{c}
T^C_{\mathrm{diagonal}} (f, g) \\
\downarrow \\
T^C_{\mathrm{parallel}} \quad (f, g) \\
\downarrow \\
T^C_{\mathrm{alt}} \quad (f, g) \\
\downarrow \\
T^C_{\mathrm{diff}} \quad (f, g)
\end{array}$

The decoupled parallel above form $T^C_{\mathrm{parallel}} (f, g)$ will be controlled by the dual indicator / cube testing condition. The diagonal below form $T^C_{\mathrm{diagonal}} (f, g)$ has already been bounded in the previous section, and the difference diagonal below form $T^C_{\mathrm{diff}} (f, g)$ will be handled here in a virtually identical way. Finally, the error form Error1 $(f, g)$ will be controlled by sublinear variants of the comparable and disjoint forms.
6.1. Bounding the above diagonal form. We write,

\[ T_{\chi,\pi,\nu}^{\tau}(f, g) = \sum_{G \in \mathcal{G}} \langle T^\alpha_{\chi} (P_{G_{\delta}} f), P_{C_{G_{\delta}-shift}}^\omega g \rangle_{\omega} \]

where we now have \( \kappa_1 \leq \frac{1}{2} \kappa_2 < \kappa_2 \), which prevents us from bounding the stopping form if we were to use the NTV reach. Instead, for each \( G \in \mathcal{G} \), we will add the missing inner products \( \langle T^\alpha_{\chi} (\Delta_{J_{\kappa_2}} g), \Delta_{I_{\kappa_1}} f \rangle_{\sigma} \) to \( T_{\chi,\pi,\nu}^{\tau}(f, g) \) that are needed to result in the decoupled parallel form,

\[ T_{\chi,\pi,\nu,parallel}^{\tau}(f, g) = \sum_{G, J \in \mathcal{G}} \sum_{I \in G_{\delta}^{-shift}} \left( \sum_{J \in \mathcal{G}} \sum_{I \in G_{\delta}^{-shift}} \langle T^\alpha_{\chi} (\Delta_{J_{\kappa_2}} g), \Delta_{I_{\kappa_1}} f \rangle_{\sigma} \right) , \]

However, we compute

\[ T_{\chi,\pi,\nu,parallel}^{\tau}(f, g) = T_{\chi,\pi,\nu}^{\tau}(f, g) - \sum_{G \in \mathcal{G}} \sum_{J \in \mathcal{G}} \sum_{I \in G_{\delta}^{-shift}} \langle T^\alpha_{\chi} (\Delta_{J_{\kappa_2}} g), \Delta_{I_{\kappa_1}} f \rangle_{\sigma} \]

where the sublinear form

\[ |Error(f, g)| = \sum_{G \in \mathcal{G}} \sum_{J \in \mathcal{G}} \sum_{I \in G_{\delta}^{-shift}} \left| \langle T^\alpha_{\chi} (\Delta_{J_{\kappa_2}} g), \Delta_{I_{\kappa_1}} f \rangle_{\sigma} \right| \]

is controlled by the sublinear comparable form \( |B_{\chi}(f, g)| \) with absolute values inside, together with the sublinear disjoint form \( |B_{\chi}(f, g)| \) with absolute values inside.

The alternate below diagonal form

\[ T_{\chi,\pi,\nu,alt}^{\tau}(f, g) = \sum_{G \in \mathcal{G}} \sum_{J \in \mathcal{G}} \sum_{I \in G_{\delta}^{-shift}} \langle T^\alpha_{\chi} (\Delta_{J_{\kappa_2}} g), \Delta_{I_{\kappa_1}} f \rangle_{\omega} \]

was essentially treated in the previous section on the below form by breaking it up using the NTV reach plus error terms. To see this we write

\[ C_{G}^{-shift} = \bigcup \left( \bigcup_{G \in \mathcal{G}} (G \cap C_{G}^{-shift}) \cup \bigcup_{G \in \mathcal{G}} (G \cap C_{G}^{-shift}) \right) \]

where \( \pi_{\mathcal{G}} F \) is the smallest cube \( G \in \mathcal{G} \) containing \( F \). Then fixing a pair \( (F, J) \) with \( F \in \mathcal{F} \) and \( J \in C_{F}^{-shift} \), and noting Remark 13 the set of cubes \( I \) arising in the sum for \( T_{\chi,\pi,\nu,alt}^{\tau}(f, g) \) are precisely those \( I \in C_{G}^{-shift} \) satisfying \( J \subset \mathcal{J} \), i.e.

\[ \pi^{(\rho)}_{\mathcal{G}} J \subset I \subset \pi^{(\tau)}_{\mathcal{G}} J, \]
where $\pi^{[\tau]}_J$ is the $\tau$-grandchild of $\pi_G J$ that contains $J$, or more precisely, is the unique cube $K$ in $D$ such that $J \subset K$ and $\pi^{(\tau)}_G K = \pi_G J$. Using Remark 14 we have

$$T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g) = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^\tau} \sum_{I \in \mathcal{D}_{\rho,\varepsilon}^J} \langle T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega,$$

where if $K \supset L$ then $\sum_{I \in \mathcal{I}[K,L]} \Delta_{I,\varepsilon}^1 f = 0$.

The form $T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g)$ matches the form,

$$T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g) = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^\tau} \langle T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega = \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{D}_{\rho,\varepsilon}^J} \langle T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega,$$

considered in the previous section, with the only exception that the sum in $I$ stops at $\pi^{[\tau]}_G J$ in $T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g)$, instead of at $F$ as it does in $T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g)$.

This suggests we decompose the form $T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g)$ as

$$(6.1) \quad T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g) = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^\tau} \langle T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega$$

$$= \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^\tau} \langle T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega$$

$$- \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^\tau} \langle T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega$$

$$+ \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^\tau} \langle T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega$$

$$= T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g) - T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g) + T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g).$$

We now claim that the bottom difference form $T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g)$ can be treated using the NTV reach in the same way that the form $T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g)$ was treated using the crucial assumption (5.12). Indeed, by the absolute convergence of the commutator, stopping and neighbor forms given by (5.6), (5.11) and (5.10), it suffices to bound the resulting paraproduct form for $T_{\text{c,alt}}^{C_{\rho,\varepsilon}}(f,g)$ given by

$$\sum_{I \in \mathcal{D}_{\rho,\varepsilon}^J} \sum_{J \in \mathcal{C}_F^\tau} \langle M_{IJ \varepsilon} T_\sigma^\alpha(\Delta_{I,\varepsilon}^1 f), \Delta_{J,\varepsilon}^2 g \rangle_\omega.$$
Because the sum in $I$ is along a tower of dyadic intervals, then as in \[5.2\], we may write

$$
\sum_{I \in \bigcup_{\pi\tau G} J \subseteq F} 1_I M_{I, \tau G} \leq 1_I P_{I, \tau G},
$$

for some polynomial $P_{I, \tau G}$ of degree strictly less than $\kappa_1$. Then noting that $\pi^{[r]}_G J$ is either in $C_F$, or is in the corona associated to an $F$-descendant of $F$ a bounded number of generations below, then one can check that \([5.3]\) still holds. Then the rest of the proof the paraproduct term follows verbatim.

We now further decompose the top difference form as a sum of a `plug' form and a `hole' form,

\[
T_{\text{top}, \text{diagonal}}^{C_{\rho, \varepsilon}} (f, g) = \sum_{F \in F} \sum_{J \subseteq C_F} \left< T_\sigma^{\alpha} \left( 1_F \sum_{I \in \bigcup_{\pi\tau G} J \subseteq F} \Delta_{I, \tau G} f \right), \Delta_{J, \tau G} g \right>.
\]

\[
+ \sum_{F \in F} \sum_{J \subseteq C_F} \left< T_\sigma^{\alpha} \left( 1_{\mathbb{R}^n \setminus F} \sum_{I \in \bigcup_{\pi\tau G} J \subseteq F} \Delta_{I, \tau G} f \right), \Delta_{J, \tau G} g \right>,
\]

where the `hole' form is handled in the same way as the below stopping form was handled above, since the $\omega$-vanishing moments up to order less than $\kappa_2$ of the projection $\Delta_{J, \tau G}$ are here applied directly to the kernel of $T_\sigma^{\alpha}$. Indeed, we now use,

\[
P_{\alpha, \kappa_1} (J, 1_{\mathbb{R}^n \setminus F}) \lesssim \left( \frac{\ell(J)}{\ell(F)} \right)^\varepsilon P_{\alpha, \kappa_1} (F, 1_{\mathbb{R}^n \setminus F}) \lesssim \left( \frac{\ell(J)}{\ell(F)} \right)^\varepsilon |F|^{1 - \frac{\varepsilon}{\kappa_1}},
\]

together with the telescoping identity,

\[
\left| \sum_{I \in \bigcup_{\pi\tau G} J \subseteq F} \Delta_{I, \tau G} f \right| = \left| E_{\pi\tau G} f - E^{\tau\pi G}_{\tau G} f \right| \lesssim E|f|,
\]
to obtain,

\[
\left| T_{\mathcal{F}_x}^{\Delta_{\sigma}} \right|_{\text{diff top hole diagonal}} (f, g) \lesssim \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^{-\text{shift}}} \left| T_{\sigma}^{\Delta_{J,k_1} f} \left( \mathbf{1}_{\mathcal{F}_x} \right) , \Delta_{J,k_2}^\omega g \right|_{\omega}
\]

\[
\lesssim \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^{-\text{shift}}} \mathbf{1}_{\mathcal{F}_x} (J, \mathbf{1}_{\mathcal{F}_x} (E_F^\sigma |f|) \sigma) \left[ |J|_{\omega} \right] \left| \Delta_{J,k_2}^\omega g \right|_{L^2(\omega)}
\]

\[
\lesssim \sum_{F \in \mathcal{F}} \left( E_F^\sigma |f| \right) \left[ F \sigma \right] \left| J \right| \left( \frac{\ell(F)}{\ell(J)} \right)^{\kappa_1 - \varepsilon (\kappa_1 + n - \alpha)} \left| J \right| \left| \Delta_{J,k_2}^\omega g \right|_{L^2(\omega)}
\]

\[
\lesssim \sum_{F \in \mathcal{F}} \left( E_F^\sigma |f| \right) \left[ F \sigma \right] \left| \Delta_{J,k_2}^\omega g \right|_{L^2(\omega)}^2
\]

So we finally turn to analyzing the ‘plug form’ above. We again use the telescoping identity

\[
\mathbf{1}_{\mathcal{F}_x} \sum_{I \in \left[ F \sigma \right]} \Delta_{I,k_1}^\alpha f = E_{\mathcal{F}_x}^{\sigma} f - \mathbf{1}_{\mathcal{F}_x} E_{\mathcal{F}_x}^{\sigma \mid J \mid k_1 f}, \quad \text{for } F \subseteq \pi_0^{-1} J
\]

to write the top difference plug form as,

\[
T_{\mathcal{F}_x}^{\Delta_{\sigma}} \left( f, g \right) = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^{-\text{shift}}} \left| T_{\sigma}^{\Delta_{J,k_1} f} \left( \mathbf{1}_{\mathcal{F}_x} \right) , \Delta_{J,k_2}^\omega g \right|_{\omega}
\]

The first form above is controlled by the local testing condition,

\[
\left| T_{\mathcal{F}_x}^{\Delta_{\sigma}} \left( f, g \right) \right|_{\text{diagonal}} \lesssim \sum_{F \in \mathcal{F}} \left| T_{\sigma}^{\Delta_{J,k_1} f} \left( \mathbf{1}_{\mathcal{F}_x} \right) , \Delta_{J,k_2}^\omega g \right|_{\omega}
\]

\[
\lesssim \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_F^{-\text{shift}}} \left( E_F^\sigma |f| \right) \left[ F \sigma \right] \left| \Delta_{J,k_2}^\omega g \right|_{L^2(\omega)}
\]

\[
\lesssim \sum_{F \in \mathcal{F}} \left( E_F^\sigma |f| \right) \left[ F \sigma \right] \left| \Delta_{J,k_2}^\omega g \right|_{L^2(\omega)}^2
\]
To control the second form above we note that if $F \subseteq \pi_{g}^{[r]} J$, then we must have $\pi_{g} J = \pi_{g} F$ and so,

$$T_{\text{diagonal}}^{C_{\rho, \varepsilon}, \text{diff top:2}}(f, g) = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_{F}^{C_{\rho, \varepsilon}}} \left\langle T_{\sigma}^{\alpha} \left(1_{F} E_{\pi_{g}^{[r]}}^{\sigma} f, \Delta_{J, \kappa_{2}}^{\omega} g \right) \right\rangle_{\omega}$$

$$= \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_{F}^{C_{\rho, \varepsilon}}} \left\langle T_{\sigma}^{\alpha} \left(1_{F} E_{\pi_{g}^{[r]}}^{\sigma} f, \Delta_{J, \kappa_{2}}^{\omega} g \right) \right\rangle_{\omega}$$

$$= \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_{F}^{C_{\rho, \varepsilon}}} \left\langle T_{\sigma}^{\alpha} \left(1_{F} E_{\pi_{g}^{[r]}}^{\sigma} f, \Delta_{J, \kappa_{2}}^{\omega} g \right) \right\rangle_{\omega}$$

whose modulus is at most,

$$\sum_{F \in \mathcal{F}} \left\| T_{\sigma}^{\alpha} \left(1_{F} E_{\pi_{g}^{[r]}}^{\sigma} f, \Delta_{J, \kappa_{2}}^{\omega} g \right) \right\|_{L^{2}(\omega)} \leq \mathcal{T}_{T \sigma}^{\alpha} (\sigma, \omega) \sum_{F \in \mathcal{F}} \left\| 1_{F} E_{\pi_{g}^{[r]}}^{\sigma} f \right\|_{\infty} \left\| 1_{F} \right\|_{L^{2}(\sigma)} \sum_{J \in \mathcal{C}_{F}^{C_{\rho, \varepsilon}}} \left\| \Delta_{J, \kappa_{2}}^{\omega} g \right\|_{L^{2}(\omega)}$$

and we can now continue with Cauchy-Schwarz in $F'$, noting that the projections

$$\left\{ \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{C}_{F}^{C_{\rho, \varepsilon}}} \left\| \Delta_{J, \kappa_{2}}^{\omega} g \right\|_{L^{2}(\omega)} \right\}_{F' \in \mathcal{F}}$$

have pairwise disjoint Alpert supports, and that

$$\left\| 1_{F} E_{\pi_{g}^{[r]}}^{\sigma} f \right\|_{\infty} \left\| 1_{F} \right\|_{L^{2}(\sigma)} \lesssim \left( E_{F} f, \pi_{g}^{[r]} f \right)$$

by the construction of the Calderón-Zygmund corona in which averages at the tops of coronas increase. Thus we obtain that

$$\left\| T_{\text{diagonal}}^{C_{\rho, \varepsilon}, \text{diff top:2}}(f, g) \right\|_{L^{2}(\omega)} \lesssim \mathcal{T}_{T \sigma}^{\alpha} (\sigma, \omega) \left\| f \right\|_{L^{2}(\sigma)} \left\| g \right\|_{L^{2}(\omega)}.$$

Finally then we conclude that the alternate below diagonal form $\mathcal{T}_{\text{diagonal}}^{C_{\rho, \varepsilon}, \text{alt}}(f, g)$ is controlled by

$$\left\| T_{\text{diagonal}}^{C_{\rho, \varepsilon}, \text{alt}}(f, g) \right\| \lesssim \left( \mathcal{T}_{T \sigma}^{\alpha} + \mathcal{T}_{T \sigma}^{\alpha} + \mathcal{	ext{WBP}}_{T \sigma}^{\alpha, \kappa_{2}} + \sqrt{A_{2}^{\alpha}} \right) \left\| f \right\|_{L^{2}(\sigma)} \left\| g \right\|_{L^{2}(\omega)}.$$

6.1.1. The parallel form. Thus it remains to bound the decoupled parallel form $T_{\text{diagonal}}^{C_{\rho, \varepsilon}, \text{parallel}, G}(f, g)$, which we do here using the dual indicator / cube testing condition, defined by

$$\mathcal{T}_{T \sigma}^{\text{ind}} (\omega, \sigma) \equiv \sup_{Q} \left( \frac{1}{|Q|_{\sigma}} \sup_{E \subset Q} \int_{Q} |T_{\omega}^{\sigma}(\chi_{E})|^{p} \omega \right)^{\frac{1}{p}}.$$
For this we recall the following elementary observation from \cite{LaSaUr1}. For any linear operators $T$, which we may assume have real-valued kernel by Remark 9, we have with $g_{h,Q} = T^*(\chi_Q h \omega)$,

\begin{equation}
(6.2) \quad \sup_{|g| \leq 1} \frac{1}{|Q|} \int_{Q} |T(\chi_Q g \sigma)|^p \omega = \sup_{|g| \leq 1} \sup_{\|h\|_{L^p(\omega)} \leq 1} \left| \frac{1}{|Q|} \int_{Q} T(\chi_Q g \sigma) h \omega \right| \leq \sup_{|h|_{L^p(\omega)} \leq 1} \left| \frac{1}{|Q|} \int_{Q} T^*(\chi_Q h \omega) g \sigma \right| = \sup_{|h|_{L^p(\omega)} \leq 1} \frac{1}{|Q|} \int_{Q} T^*(\chi_Q h \omega) g_{h,Q} \sigma \leq \sup_{|h|_{L^p(\omega)} \leq 1} \left| \frac{1}{|Q|} \int_{Q} T(\chi_Q g_{h,Q} \sigma) h \omega \right| \leq \sup_{|g| \leq 1} \left| \frac{1}{|Q|} \int_{Q} T(\chi_Q g \sigma) h \omega \right|,
\end{equation}

upon noting that both of the functions $g$ and $h$ appearing in the suprema are real-valued, and hence we can write $g_{h,Q} = 1_A - 1_B$.

Then we have,

\[
\left| 1_{T \cap \sigma \text{ parallel}, G}(f,g) \right| = \left| \int_{T} T_{G, \sigma}^* P_{\mathcal{C}_G}^\sigma g \, P_{\mathcal{C}_G}^\sigma \right|_{L^2(\sigma)} \leq \|T_{\alpha} \mathcal{C}_G^\sigma g\|_{L^2(\sigma)} \leq 2 \|T_{\alpha} \mathcal{C}_G^\sigma g\| \leq 2 \|T_{\alpha} \mathcal{C}_G^\sigma g\|_{L^2(\sigma)},
\]

and collecting all of the estimates above, we conclude that

\begin{equation}
(6.3) \quad \left| 1_{T \cap \sigma \text{ diagonal}, G}(f,g) \right| \leq \sum_{G \in \mathcal{G}} \left| 1_{T \cap \sigma \text{ diagonal}, G}(f,g) \right| \leq \sum_{G \in \mathcal{G}} \left| 1_{T \cap \sigma \text{ diagonal}, G}(f,g) \right| \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right) \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right) \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right) \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right) \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right) \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right) \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right) \leq \left( \sum_{G \in \mathcal{G}} \left| 1_{\mathcal{C}_G^\sigma}(f,g) \right| \right),
\end{equation}

since the collection of Alpert projections $\{P_{\mathcal{C}_G^\sigma} \}_{G \in \mathcal{G}}$ have pairwise disjoint Alpert support.

7. Conclusion of the Proofs

Collecting all the estimates proved above, namely (4.4), (4.7), (4.9), (4.10), (4.11), (4.13), (5.5), (5.6), (6.11), and (7.10), and the relevant corresponding dual estimates including (6.3) for the dual diagonal form, we obtain that for any dyadic cube $D$, and any admissible truncation of $T_\alpha$,

\[
\left| (T_\sigma P_{\text{good}} f, P_{\text{good}} g) \right| \leq C \left( T_{\mathcal{G}}^{(k)}(f,\omega) + T_{\mathcal{G}}^{(k)}(g,\omega) + \sqrt{A_2} (\sigma,\omega) + \varepsilon_3 \mathcal{N}_{T_\alpha}(\sigma,\omega) \right) \times \|P_{\text{good}} f\|_{L^2(\sigma)} \|P_{\text{good}} g\|_{L^2(\omega)}.
\]

Thus for any admissible truncation of $T_\alpha$ we obtain from \cite{borel-ntv} Sections 7.4 - 7.7], followed by the previous display,

\begin{equation}
(7.1) \quad \mathcal{N}_{T_\alpha}(\sigma,\omega) \leq C \sup_{D} \sup_{f \in L^2(\sigma) \text{ and } g \in L^2(\omega)} \left| \langle T_\sigma P_{\text{good}} f, P_{\text{good}} g \rangle \right| \leq C \left( T_{\mathcal{G}}^{(k)}(f,\omega) + T_{\mathcal{G}}^{(k)}(g,\omega) + \sqrt{A_2} (\sigma,\omega) + \varepsilon_3 \mathcal{N}_{T_\alpha}(\sigma,\omega) \right). + C \varepsilon_3 \mathcal{N}_{T_\alpha}(\sigma,\omega).
\end{equation}

Our next task is to use the doubling hypothesis to replace the triple $\kappa$-testing constants by the usual cube testing constants, and we start with a lemma.
Lemma 19. Let \( m \geq 1 \). Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \), with \( \sigma \) doubling. If \( T^\alpha \) is a bounded operator from \( L^2(\sigma) \) to \( L^2(\omega) \), then for every grandchild \( I' \in \mathcal{E}_D^{(m)}(I) \), and each \( 0 < \varepsilon_1 < 1 \), there is a positive constant \( C_{m, \varepsilon_1} \) such that

\[
\sqrt{\int_{3I' \setminus I'} |T^\alpha_{\sigma} 1_I'|^2 \, d\omega} \leq \left\{ C_{m, \varepsilon_1} \sqrt{A^\alpha_2 (\sigma, \omega)} + \varepsilon_1 \mathfrak{N}_{T^\alpha} (\sigma, \omega) \right\} \sqrt{|I'|_\sigma}.
\]

Proof. Given \( 0 < \delta < 1 \), we split the left hand side as usual,

\[
\sqrt{\int_{3I' \setminus I'} |T^\alpha_{\sigma} 1_I'|^2 \, d\omega} \leq \sqrt{\int_{3I' \setminus (1-\delta) I'} |T^\alpha_{\sigma} 1_{(1-\delta) I'}|^2 \, d\omega} + \sqrt{\int_{3I' \setminus (1-\delta) I'} |T^\alpha_{\sigma} 1_{(1-\delta) I'}|^2 \, d\omega}
\]

\[\equiv A + B.\]

Using the halo estimate in [Saw6, Lemma 24 on page 24], we obtain

\[B \leq \mathfrak{N}_{T^\alpha} (\sigma, \omega) \sqrt{|I' \setminus (1-\delta) I'|_\sigma} \leq \sqrt{\frac{C}{\ln \frac{1}{\delta}} \mathfrak{N}_{T^\alpha} (\sigma, \omega)} \sqrt{|I'|_\sigma}.
\]

For term \( A \) we have

\[
A \leq \sqrt{\int_{3I' \setminus I'} \left| \int_{(1-\delta) I'} [\delta \ell(I')]^{\alpha-n} \, d\sigma \right|^2 \, d\omega} = \sqrt{\left( 3 \cdot 2^m \delta \right)^{2(\alpha-n)} |(1-\delta) I'|_\sigma |3I' \setminus I'|_\omega |(1-\delta) I'|_\sigma}
\]

\[\leq \sqrt{\left( 2^m \delta \right)^{2(\alpha-n)} |3I'|_\sigma |3I'|_\omega |(1-\delta) I'|_\sigma} \leq (2^m \delta)^{\alpha-n} A^\alpha_2 (\sigma, \omega) \sqrt{|I'|_\sigma}.
\]

Now we choose \( 0 < \delta < 1 \) so that \( \sqrt{\frac{C}{\ln \frac{1}{\delta}}} \leq \varepsilon_1 \).

Recall that the triple \( \kappa \)-cube testing conditions use the \( Q \)-normalized monomials \( m^\beta_Q (x) \equiv 1_Q (x) \left( \frac{x - c}{\ell(x)} \right)^{\beta} \), for which we have \( \|m^\beta_Q\|_{L^\infty} \approx 1 \).

Theorem 20. Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \), with \( \sigma \) doubling, and let \( \kappa \in \mathbb{N} \). If \( T^\alpha \) is a bounded operator from \( L^2(\sigma) \) to \( L^2(\omega) \), then for every \( 0 < \varepsilon_2 < 1 \), there is a positive constant \( C(\kappa, \varepsilon_2) \) such that

\[
\mathfrak{T}\mathfrak{R}_{T^\alpha}^{(\kappa)} (\sigma, \omega) \leq C(\kappa, \varepsilon_2) \left[ \mathfrak{T}\mathfrak{R}_{T^\alpha} (\sigma, \omega) + A^\alpha_2 (\sigma, \omega) \right] + \varepsilon_2 N_{T^\alpha} (\sigma, \omega), \quad \kappa \geq 1,
\]

and where the constants \( C(\kappa, \varepsilon_2) \) depend only on \( \kappa \) and \( \varepsilon \), and not on the operator norm \( N_{T^\alpha} (\sigma, \omega) \).

Proof. Fix a dyadic cube \( I \). If \( P \) is an \( I \)-normalized polynomial of degree less than \( \kappa \) on the cube \( I \), i.e. \( \|P\|_{L^\infty} \approx 1 \), then we can approximate \( P \) by a step function

\[
S \equiv \sum_{I' \in \mathcal{E}_D^{(m)}(I)} a_{I'} 1_{I'}
\]

satisfying

\[\|S - 1_{I} P\|_{L^\infty(\sigma)} < \frac{\varepsilon_2}{2} \].
provided we take \( m \geq 1 \) sufficiently large depending on \( n \) and \( \kappa \), but independent of the cube \( I \). Then using
the above lemma with \( C2^\frac{\alpha}{3} \varepsilon_1 \leq \frac{C}{\varepsilon} \), and the estimate \( |a_P| \lesssim \|P\|_{L^\infty} \leq 1 \), we have

\[
\sqrt{\int_{3I} |T_\sigma^1 1_I|^2 \, d\omega} \leq \sqrt{\int_{3I} \sum_{I' \in \mathcal{C}_D^{(m)}(I)} a_{I'} T_\sigma^1 1_{I'}} \, d\omega + \sqrt{\int_{3I} |T_\sigma^1 [(\mathcal{S} - P) 1_I]|^2 \, d\omega}
\]

\[
\leq C \sum_{I' \in \mathcal{C}_D^{(m)}(I)} \sqrt{\int_{3I'} |a_{I'} T_\sigma^1 1_{I'}|^2 \, d\omega} + C \sum_{I' \in \mathcal{C}_D^{(m)}(Q)} \sqrt{\int_{I'} |a_{I'} T_\sigma^1 1_{I'}|^2 \, d\omega} + \frac{\varepsilon_2}{2} \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \sqrt{|I|_\sigma}
\]

\[
\leq C \sum_{I' \in \mathcal{C}_D^{(m)}(Q)} \left\{ C_{m, \varepsilon_1} \|A_2^{\varepsilon_1} (\sigma, \omega) + \varepsilon_1 \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) + \mathfrak{M}_{T_\sigma^0} \right\} \sqrt{|I|_\sigma} + \left( C2^\frac{\alpha}{3} \varepsilon_1 + \frac{\varepsilon_2}{2} \right) \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \sqrt{|I|_\sigma}.
\]

Combining this with (7.1) we obtain

\[
\mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \leq C \left( \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) + \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) + \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \right) + C (\varepsilon_2 + \varepsilon_3) \mathfrak{M}_{T_\sigma^0} (\sigma, \omega).
\]

Since \( \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) < \infty \) for each truncation, we may absorb the final summand on the right into the left hand side provided \( C (\varepsilon_2 + \varepsilon_3) < \frac{1}{2} \), to obtain

\[
\mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \lesssim \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) + \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) + \mathfrak{M}_{T_\sigma^0} (\sigma, \omega).
\]

Now we eliminate the Muckenhoupt constant from the right hand side using the following lemma.

**Lemma 21.** If \( K^\alpha \) is elliptic in the sense of Stein, then

\[
\sqrt{A_2^{\varepsilon_1} (\sigma, \omega)} \lesssim \mathfrak{M}_{T_\sigma^0} (\sigma, \omega).
\]

**Proof.** The argument in [Ste2] see the proof of Proposition 7 page 210] shows that if \( T^\alpha \) satisfies (2.2), there
is \( \varepsilon > 0 \), depending only on the constant \( C_{C_{\mathbf{Z}}} \) in (2.1), such that

\[
\frac{|Q|_{\sigma} |Q'|_{\sigma}}{\ell (Q)^{2(\alpha - n)}} \lesssim \mathfrak{M}_{T_\sigma^0} (\sigma, \omega)^2,
\]

when \( \text{dist} (Q, Q') \approx \ell (Q) = \ell (Q') \), and \( \frac{x - y}{|x - y|} - u_0 \ll \varepsilon \) whenever \( x \in Q \) and \( y \in Q' \). Indeed, given such
cubes \( Q \) and \( Q' \) we may assume they are subcubes of a cube \( I \) with \( \ell (I) \gtrsim \ell (Q) \), and then we have

\[
\mathfrak{M}_{T_\sigma^0} (\sigma, \omega)^2 \gtrsim \sup_{E \subseteq I} \left( \frac{\int_{3I} |T_\sigma^1 1_E|^2 \, d\omega}{|I|_{\sigma}} \right) \gtrsim \left( \frac{|Q|_{\sigma} |Q'|_{\sigma}}{\ell (I)^{2(\alpha - n)}} \right)^2 \gtrsim \frac{|Q|_{\sigma} |Q'|_{\sigma}}{\ell (Q)^{2(\alpha - n)}} \gtrsim \frac{|Q|_{\sigma} |Q'|_{\sigma}}{\ell (Q)^{2(\alpha - n)}},
\]

since \( \sigma \) is doubling. The lemma follows upon taking the supremum over cubes \( Q \). \( \Box \)

Since the norm constant obviously bounds the two testing constants, we have proved

\[
\mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \approx \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) + \mathfrak{M}_{T_\sigma^0} (\sigma, \omega),
\]

and the remaining equivalences in Theorem [2] that do not involve cancellation conditions \( \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \) and \( \mathfrak{M}_{T_\sigma^0} (\sigma, \omega) \) follow by symmetry. By [Saw6], the testing conditions are equivalent to the cancellation conditions when the measures are doubling.
8. A $T_{\text{translate}}$ theorem for doubling measures

The arguments used above rely crucially on the nested property of dyadic cubes, and break down completely even for balls. Here we instead use a fairly elementary argument to show that testing over indicators of cubes $1_Q$ in the doubling theorem above can be replaced by testing over indicators of balls $1_B$. In fact, we can replace cubes or balls by translates and dilates of any fixed bounded set having positive Lebesgue measure. The theorem below extends this to translates of fixed bounded functions $b_k, b_k^*$ with integral 1 at each length scale $2^k$, which we refer to as a “$T_{\text{translate}}$ theorem” because at any given scale, we test a function “$b_k$” and all of its translates at that scale.

More precisely, suppose that for each $t \in (0, \infty)$, we are given a bounded complex-valued function $b_t$ on $\mathbb{R}^n$ satisfying

1. $\operatorname{Supp} b_t \subset B(0,t)$,
2. $|b_t(x)| \leq \frac{C}{|Q|}$,
3. $\int b_t(x) \, dx = 1$.

**Definition 22.** Set $B = \{b_Q\}_{Q \in \mathcal{P}^n}$, where $b_Q(x) = |Q| b_{t(Q)}(x - c_Q)$ is a translation and normalization of $b_{t(Q)}$ (so that it satisfies estimates similar to $1_Q$) where the functions $b_t$ are as above. For $\sigma$ and $\omega$ locally finite positive Borel measures on $\mathbb{R}^n$, define the $B$-testing constant and the $\delta$-full $B$-testing constant for the operator $T^\sigma$ by

$$
\mathfrak{T}_{T^\sigma}^B(\sigma, \omega) \equiv \sup_Q \frac{\int_Q |T^\sigma b_Q(x)|^2 \, d\omega(x)}{\int_Q |b_Q(y)|^2 \, d\sigma(y)},
$$

$$
\mathfrak{F}_\delta \mathfrak{T}_{T^\sigma}^B(\sigma, \omega) \equiv \sup_Q \frac{\int_Q |T^\sigma b_Q(x)|^2 \, d\omega(x)}{\int_Q |b_Q(y)|^2 \, d\sigma(y)}.
$$

If we take $b_t(x) = \frac{1}{|Q|} 1_{B(0,t)}(x)$, then the $B$-testing constant is simply the ball-testing constant. Fix $0 < \delta < 1$. Given a cube $Q \in \mathcal{P}$, define the $\delta$-convolution

$$
b_{Q,\delta}(y) \equiv 1_Q \ast b_{t(Q)}(y) = \int_{\mathbb{R}^n} 1_Q(y - z) b_{t(Q)}(z) \, dz
$$

$$
= \int_Q b_{t(Q)}(y - z) \, dz = \int_Q \tau_z b_{t(Q)}(y) \, dz, \quad x \in \mathbb{R}^n.
$$

Now we compute

$$
\sqrt{\int_Q |T^\sigma 1_Q(x)|^2 \, d\omega(x)} \leq \sqrt{\int_Q |T^\sigma b_Q,\delta(x)|^2 \, d\omega(x)} + \sqrt{\int_Q |T^\sigma (1_Q - b_Q,\delta)(x)|^2 \, d\omega(x)}
$$

$$
\leq \sqrt{\int_Q |T^\sigma b_Q,\delta(x)|^2 \, d\omega(x) + \mathfrak{m}_{T^\sigma} \int_Q \left| (1_Q - b_Q,\delta) \right|^2 \, d\sigma(x)},
$$

where

$$
\sqrt{\int_Q |T^\sigma b_Q,\delta(x)|^2 \, d\omega(x)} = |Q| \sqrt{\int_Q T^\sigma \left( \int_Q \tau_z b_{t(Q)} \, dz \right)(x) \, d\omega(x)}
$$

$$
\leq \int \mathfrak{m}_{T^\sigma} \frac{\mathfrak{T}_{T^\sigma}^B(\sigma,\delta)}{|Q|} \sqrt{\int_Q \left| (1_Q - b_Q,\delta) \right|^2 \, d\sigma(y) \, dy}
$$

$$
\leq \mathfrak{m}_{T^\sigma} \mathfrak{T}_{T^\sigma}^B \left| \frac{C |Q|}{|Q|} \sqrt{\int_Q \left| (1_Q - b_Q,\delta) \right|^2 \, d\sigma(y)} \right| 
\leq \mathfrak{T}_{T^\sigma}^B \left| \frac{C |Q|}{|Q|} \sqrt{\int_Q \left| (1_Q - b_Q,\delta) \right|^2 \, d\sigma(y)} \right|.
$$
The only property of cubes used in Theorem 23, was that their boundaries are not charged

Thus altogether we obtain

and now using \( \sqrt{A_2} \) to control \( \tilde{\mathcal{B}}_\delta \), we have shown that a T1 theorem for \( T^\alpha \) over cubes follows from a Tb theorem where the functions \( b_Q \) are translates of the fixed function \( b_{\ell(Q)} \). In particular we can take \( b_t = \frac{1}{\omega} 1_{B(t, t)}(x) \) to get ball-testing implies cube-testing.

More precisely, we have the following theorem.

**Theorem 23.** Let \( \sigma \) and \( \omega \) be doubling measures on \( \mathbb{R}^n \). Then with \( T^\alpha \) and \( B \) as above we have both

\[
\tilde{\mathcal{B}}_\delta \tilde{T}_\sigma^B (\sigma, \omega) \leq C_\delta \tilde{T}_\sigma^B (\sigma, \omega) + C_\delta \sqrt{A_2^2} (\sigma, \omega) + C \sqrt{1 \ln \frac{1}{\delta}} \mathcal{N}_{T^\alpha} (\sigma, \omega),
\]

\[
\tilde{T}_\sigma^B (\sigma, \omega) \leq C_\delta \tilde{\mathcal{B}}_\delta \tilde{T}_\sigma^B (\sigma, \omega) + C \sqrt{1 \ln \frac{1}{\delta}} \mathcal{N}_{T^\alpha} (\sigma, \omega).
\]

Altogether then we have

\[
\mathcal{N}_{T^\alpha} (\sigma, \omega) \leq C \left( \tilde{T}_\sigma^B + \tilde{T}_{\sigma}^{\text{ind,B}} (T^\alpha) + \sqrt{A_2^2} \right),
\]

and hence

\[
\mathcal{N}_{T^\alpha} (\sigma, \omega) \leq C \left( \tilde{T}_\sigma^B + \tilde{T}_{\sigma}^{\text{ind,B}} (T^\alpha) + \sqrt{A_2^2} \right),
\]

where

\[
\tilde{T}_{\sigma}^{\text{ind,B}} (T^\alpha) \equiv \sup_B \int B \sup_{E \subset B} \int E \left| T_{\omega}^\alpha 1_E \right|^2 \sigma,
\]

and the supremum is taken over all balls \( B \).

**Proof.** The bound for \( \tilde{\mathcal{B}}_\delta \tilde{T}_\sigma^B (\sigma, \omega) \) in the first line of (8.2) is in Theorem 20 and the bound for \( \tilde{T}_\sigma^B (\sigma, \omega) \) in the second line of (8.2) is in (8.1). The reader can easily check that the indicator / cube constant \( \tilde{T}_{\sigma}^{\text{ind,B}} (T^\alpha) \) is comparable to the indicator / ball constant \( \tilde{T}_\sigma^B (\sigma, \omega) \), because the measures are doubling. An absorption argument proves the final display.

**Remark 24.** The only property of cubes used in Theorem 23 was that their boundaries are not charged by doubling measures (the specific decay \( \sqrt{1 \ln \frac{1}{\delta}} \) of the halo can be replaced by any decay to zero). Thus we can replace cube testing \( \tilde{T}_\sigma^B (\sigma, \omega) \) in Theorem 23 by ball testing, or any other ‘shape testing’ in which the boundary of the ‘shape’ is not charged by doubling measures.

In fact one can extend Theorem 2 to testing over balls instead of testing over cubes. Due to the above arguments, it remains only to verify that the ball testing conditions are equivalent to the cancellation conditions over spherical annuli, which follows by mimicking the arguments in Saw6, see Subsection 9.2, with spherical annuli replacing cubical annuli.

9. A characterization of weak type inequalities

The passage from weak type to strong type above required the recent technology of NTV good cubes [NTV], weighted Alpert wavelets [RaSaWi] and corona decompositions [NTV4]. On the other hand, the characterization of weak type inequalities derived here uses the classical machinery of Whitney decompositions, maximum principles and good \( \lambda \) inequalities for maximal singular integrals; see e.g. [LaSaUr1] and [Ste2]. Theorem 3 links the two approaches.
Remark 25. The reader can easily check that all the results in this section extend to weighted $L^p$ spaces for $1 < p < \infty$ as well.

9.1. Maximal singular integrals. We begin by adapting an argument in [Ste2, Chapter I, subsection 7.3, pages 34-36] in order to control $T_\sigma^\alpha$ by weighted operators and $T^\alpha$.

1) For any locally finite positive Borel measure $\mu$ we define the centered maximal operator $M_\mu$ acting on a finite positive Borel measure $\nu$ by

$$M_\mu \nu (x) = \sup_{B \in B^n: \, c_B = x} \frac{|B|_\nu}{|B|_\mu}.$$

where $B^n$ is the collection of all balls $B$ in $\mathbb{R}^n$, and $c_B$ is the center of $B$.

2) For $0 \leq \alpha < n$, we define $M^\alpha$ on a finite positive Borel measure $\nu$ by

$$M^\alpha \nu (x) = \sup_{B \in B^n: \, c_B = x} \frac{1}{|B|^{1-\frac{\alpha}{n}}_\mu} \int_B \nu.$$

3) Given any $0 < r < \infty$, we define $M_{\mu, r}$ acting on a function $f \in L^1_{\text{loc}}(\mu)$ by

$$M_{\mu, r} f (x) = \sup_{B \in B^n: \, c_B = x} \left( \frac{1}{|B|_\mu} \int_B |f|_r^r \, d\mu \right)^\frac{1}{r}.$$

Lemma 26. Suppose $\sigma$ and $\omega$ are locally finite positive Borel measures on $\mathbb{R}^n$, and that $T^\alpha$ is a Calderón-Zygmund operator on $\mathbb{R}^n$ with kernel satisfying just (9.1). Then for any $r > 0$,

$$T^\alpha_{\omega, \sigma} f (x) \leq A_{\alpha, n, r} \left\{ M_\omega (|T^\alpha_\sigma f| (x)) + M_\omega (|f|^2 \sigma) (x) + M^\alpha (|f| \sigma) (x) \right\}, \quad x \in \mathbb{R}^n.$$

Proof. Fix $z \in \mathbb{R}^n$ and $\varepsilon > 0$. Write

$$f = f 1_{B(z, \varepsilon)} + f 1_{\mathbb{R}^n \backslash B(z, \varepsilon)} = f_1 + f_2,$$

so that

$$T^\alpha_{\varepsilon, \sigma} f_1 (z) = T^\alpha_{\varepsilon, \sigma} f_2 (z) = T^\alpha_\sigma f_2 (z).$$

We now claim that

$$|T^\alpha_\sigma f_2 (z) - T^\alpha_{\varepsilon, \sigma} f_2 (x)| \leq A' M^\alpha (|f| \sigma) (z), \quad \text{for } |x - z| < \frac{\varepsilon}{c}.$$

Indeed, we have

$$|T^\alpha_\sigma f_2 (z) - T^\alpha_{\varepsilon, \sigma} f_2 (x)| \leq \int_{|y - z| \geq \varepsilon} |K^\alpha (x, y) - K^\alpha (z, y)| \, |f(y)| \, d\sigma (y)$$

$$\leq \sum_{k=0}^{\infty} \int_{2^{k+1} \varepsilon \leq |y - z| \leq 2^k \varepsilon} |K^\alpha (x, y) - K^\alpha (z, y)| \, |f(y)| \, d\sigma (y)$$

$$\leq \sum_{k=0}^{\infty} \eta \left( \frac{1}{2^k c} \right) \frac{1}{|B(z, 2^k |\varepsilon|)|^{1 - \frac{\alpha}{n}}} \int_{B(z, 2^k \varepsilon)} |f(y)| \, d\sigma (y)$$

$$\leq c \sum_{k=0}^{\infty} \eta \left( \frac{1}{2^k c} \right) M^\alpha (|f| \sigma) (z) \leq A' M^\alpha (|f| \sigma) (z).$$

Thus we conclude that

$$|T^\alpha_{\varepsilon, \sigma} f (z)| \leq |T^\alpha_\sigma f (x)| + |T^\alpha_{\varepsilon, \sigma} f_1 (x)| + A' M^\alpha (|f| \sigma) (z), \quad \text{for } x \in B \left( z, \frac{\varepsilon}{c} \right).$$
Now we note that
\[
\left\{ x \in B \left( z, \frac{\varepsilon}{c} \right) : |T_{\sigma}^\alpha f (x) | > \lambda \right\}_\omega \leq \lambda^{-r} \int_{B \left( z, \frac{\varepsilon}{c} \right)} |T_{\sigma}^\alpha f (x) |^r \, d\omega (x)
\]
\[
\leq \lambda^{-r} \left| B \left( z, \frac{\varepsilon}{c} \right) \right|_\omega M_\omega |T_{\sigma}^\alpha f |^r (z) \leq \frac{1}{4} \left| B \left( z, \frac{\varepsilon}{c} \right) \right|_\omega ,
\]
provided
\[
\lambda \geq 4^\frac{1}{r} ( M_\omega |T_{\sigma}^\alpha f |^r ) (z)^\frac{1}{r} .
\]
Moreover we have
\[
\left\{ x \in B \left( z, \frac{\varepsilon}{c} \right) : |T_{\sigma}^\alpha f_1 (x) | > \lambda \right\}_\omega \leq \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) \frac{1}{\lambda^2} \int |f_1 |^2 \, d\sigma
\]
\[
= \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) \frac{1}{\lambda^2} \int_{B \left( z, \frac{\varepsilon}{c} \right)} |f |^2 \, d\sigma \leq \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) \frac{1}{\lambda^2} \left| B \left( z, \frac{\varepsilon}{c} \right) \right|_\omega \mathcal{M}_\omega \left( |f|^2 \sigma \right) (z) \leq \frac{1}{4} \left| B \left( z, \frac{\varepsilon}{c} \right) \right|_\omega ,
\]
provided
\[
\lambda \geq 2 \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) \sqrt{\mathcal{M}_\omega \left( |f|^2 \sigma \right) (z)} .
\]
Altogether, if
\[
\lambda = \max \left\{ 4^\frac{1}{r} ( M_\omega |T_{\sigma}^\alpha f |^r ) (z)^\frac{1}{r} , 2 \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) \sqrt{\mathcal{M}_\omega \left( |f|^2 \sigma \right) (z)} \right\} ,
\]
then there exists \( x \in B \left( z, \frac{\varepsilon}{c} \right) \) such that both \( |T_{\sigma}^\alpha f (x) | \) and \( |T_{\sigma}^\alpha f_1 (x) | \) are at most \( \lambda \), and it follows that
\[
|T_{\sigma}^\alpha f (z) | \leq |T_{\sigma}^\alpha f (x) | + |T_{\sigma}^\alpha f_1 (x) | + A' \mathcal{M}_\omega \left( |f|^2 \sigma \right) (z) \leq 2 \lambda + A' \mathcal{M}_\omega \left( |f|^2 \sigma \right) (z)
\]
\[
\leq 2 \cdot 4^\frac{1}{r} ( M_\omega |T_{\sigma}^\alpha f |^r ) (z)^\frac{1}{r} + 2 \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) A' \sqrt{\mathcal{M}_\omega \left( |f|^2 \sigma \right) (z)} + A' \mathcal{M}_\omega \left( |f|^2 \sigma \right) (z) .
\]
Taking the supremum in \( \varepsilon > 0 \) now completes the proof of Lemma 26. \( \square \)

Now we can quickly prove Theorem 5 using the well known properties that \( M_{\mu, r} \) is bounded on both \( L^p (\mu) \) and \( L^{p, \infty} (\mu) \) for \( 1 \leq r < p \leq \infty \) (use the Besicovitch covering lemma for strong type, and then interpolation for weak type), and in addition, that \( \mathcal{M}_\mu \) is `weak type on measures \( \nu ' \), i.e.
\[
\{|x \in \mathbb{R}^n : \mathcal{M}_\mu \nu (x) > \lambda \}|_\mu \leq \frac{1}{\lambda} \int_{\mathbb{R}^n} d\nu .
\]
See e.g. [Ste2], 8.17 on page 44.

**Proof of Theorem 5.** Recall that we are assuming only the conditions in (9.1) on the kernel of \( T \). The first inequality in (18) follows from [LaSuUr] Theorem 1.8 (3)]. To prove the second inequality in (18), we use Lemma 26 to write,
\[
\left\{ T_{\sigma}^\alpha f \geq \lambda \right\}_\omega \leq \left\{ ( M_\omega |T_{\sigma}^\alpha f |^r (x) )^{\frac{1}{r}} > \frac{1}{3 \mathcal{A}_{\alpha, n, r} } \lambda \right\}_\omega
\]
\[
+ \left\{ \mathcal{M}_\omega \left( |f|^2 \sigma \right) (z) > \frac{1}{3 \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) \mathcal{A}_{\alpha, n, r} } \lambda \right\}_\omega
\]
\[
+ \left\{ \mathcal{M}_\omega \left( |f|^2 \sigma \right) (x) > \frac{1}{3 \mathcal{A}_{\alpha, n, r} } \lambda \right\}_\omega = I ( \lambda ) + II ( \lambda ) + III ( \lambda ) ,
\]
where
\[
\lambda \sqrt{I ( \lambda )} = \lambda \sqrt{\left\{ M_{\omega, r} ( T_{\sigma}^\alpha f ) (x) > \frac{1}{3 \mathcal{A}_{\alpha, n, r} } \lambda \right\}_\omega \leq \| M_{\omega, r} ( T_{\sigma}^\alpha f ) \|_{L^2, \infty (\omega)}
\]
\[
\leq \| T_{\sigma}^\alpha f \|_{L^2, \infty (\omega)} \leq \mathfrak{M}_{\mathfrak{T}_\alpha} ( \sigma, \omega ) \| f \|_{L^2 (\sigma)} .
\]
Using the Besicovitch covering lemma, we can write,
\[
\left\{ z : \sqrt{M_\omega \left( |f|^2 \sigma \right)}(z) > \frac{1}{\|\mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega)\| A_{\alpha,n,r}} \right\} = \bigcup_{i=1}^\infty B_i
\]
where the collection of balls \{B_i\}_{i=1}^\infty has bounded overlap \beta, and each \(B_i\) satisfies
\[
\sqrt{\frac{1}{|B_i|_\omega} \int_{B_i} |f|^2 d\sigma} > \frac{1}{\|\mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega)\| A_{\alpha,n,r}}
\]
Then
\[
\lambda \sqrt{I(\lambda)} = \lambda \left\{ \left\{ z : M_\omega \left( |f|^2 \sigma \right)(z) > \frac{1}{\|\mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega)\| A_{\alpha,n,r}} \right\} \right\} \lesssim \sum_{i \in \mathbb{N}} \lambda^2 |B_i|_\omega
\]
\[
\leq \sum_{i \in \mathbb{N}} 9A_{\alpha,n,r}^2 \mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega)^2 \left( \frac{1}{|B_i|_\omega} \int_{B_i} |f|^2 d\sigma \right) |B_i|_\omega
\]
\[
\leq 3A_{\alpha,n,r} \mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega) \beta \int_{\mathbb{R}^n} |f|^2 d\sigma = 3A_{\alpha,n,r} \beta \mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega) \|f\|_{L^2(\omega)}.
\]
Finally we use the Besicovitch covering lemma once more to write,
\[
\left\{ M_\alpha \left( |f| \sigma \right)(x) > \frac{1}{3} \lambda \right\} = \bigcup_{i=1}^\infty B_i
\]
where the collection of balls \{B_i\}_{i=1}^\infty has bounded overlap \beta and each \(B_i\) satisfies
\[
\frac{1}{|B_i|^{1-\frac{1}{\mathbb{N}}} \|f\|_{L^2(\omega)}} \int_{B_i} |f| d\sigma > \frac{1}{3} \lambda.
\]
Then
\[
\lambda \sqrt{III(\lambda)} = \lambda \left\{ \left\{ M_\alpha \left( |f| \sigma \right)(x) > \frac{1}{3} \lambda \right\} \right\} = \sum_{i \in \mathbb{N}} \lambda^2 |B_i|_\omega
\]
\[
\leq \sum_{i \in \mathbb{N}} 9 \left( \frac{1}{|B_i|^{1-\frac{1}{\mathbb{N}}} \|f\|_{L^2(\omega)}} \int_{B_i} |f| d\sigma \right)^2 |B_i|_\omega \leq 3 \sum_{i \in \mathbb{N}} \frac{|B_i|_\omega |B_i|_\sigma}{|B_i|^{2(1-\frac{1}{\mathbb{N}}} \int_{B_i} |f|^2 d\sigma} \leq 3A_{\alpha}^2 (\sigma, \omega) \|f\|_{L^2(\sigma)}.
\]
Altogether we have,
\[
\mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega) = \sup_{f \in L^2(\omega)} \frac{\|T^\omega_{\sigma, \sigma} f\|_{L^2(\omega)}}{\|f\|_{L^2(\omega)}} = \sup_{\|f\|_{L^2(\omega)}} \frac{\sqrt{\lambda}}{\|f\|_{L^2(\omega)}} \left\{ T^\omega_{\sigma, \sigma} f > \lambda \right\}
\]
\[
\lesssim \sup_{f \in L^2(\omega)} \frac{\sqrt{\lambda} \sqrt{I(\lambda)}}{\|f\|_{L^2(\omega)}} + \sup_{f \in L^2(\omega)} \frac{\sqrt{\lambda} \sqrt{II(\lambda)}}{\|f\|_{L^2(\omega)}} + \sup_{f \in L^2(\omega)} \frac{\sqrt{\lambda} \sqrt{III(\lambda)}}{\|f\|_{L^2(\omega)}}
\]
\[
\lesssim \mathbf{1}_{T^\omega}^{\text{weak}}(\sigma, \omega) + A_\omega^2 (\sigma, \omega),
\]
which completes the proof of the second inequality in (1.8). \(\square\)

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