Implications from analyticity constraints used in a Landshoff-Donnachie fit

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**Abstract**

Landshoff and Donnachie [hep/ph 0509240, (2005)] parametrize the energy behavior of $pp$ and $p\bar{p}$ scattering cross sections with five parameters, using:

\[
\begin{align*}
\sigma^+ &= 56.08s^{-0.4525} + 21.70s^{0.0808} \quad \text{for } pp, \\
\sigma^- &= 98.39s^{-0.4525} + 21.70s^{0.0808} \quad \text{for } p\bar{p}.
\end{align*}
\]

(1) (2)

Using the 4 analyticity constraints of Block and Halzen [M. M. Block and F. Halzen, Phys. Rev. D 72, 036006 (2005)], we simultaneously fit the Landshoff-Donnachie form to the same “sieved” set of $pp$ and $p\bar{p}$ cross section and $\rho$ data that Block and Halzen used for a very good fit to a $\ln^2 s$ parametrization. We show that the satisfaction of the analyticity constraints will require complicated modifications of the Landshoff-Donnachie parametrization for lower energies, greatly altering its inherent appeal of simplicity and universality.
Landshoff and Donnachie[1,2,3] parameterize the scattering cross section with five parameters, using:

\[ \sigma^+ = 56.08 s^{-0.4525} + 21.70 s^{0.0808} \quad \text{for } pp, \]
\[ \sigma^- = 98.39 s^{-0.4525} + 21.70 s^{0.0808} \quad \text{for } p\bar{p}, \]

where \( s \) is in GeV\(^2\). Using the high energy limit \( s \to 2m\nu \), where \( \nu \) is the laboratory energy and \( m \) is the proton mass, we rewrite these equations, as well as \( \rho \), the ratio of the real part to the imaginary part of the forward scattering amplitude, more generally as

\[ \sigma^\pm = A \left( \frac{\nu}{m} \right)^{\alpha - 1} + B \left( \frac{\nu}{m} \right)^{\beta - 1} \pm D \left( \frac{\nu}{m} \right)^{\alpha - 1}, \]
\[ \frac{d\sigma^\pm}{d(\nu/m)} = A(\alpha - 1) \left( \frac{\nu}{m} \right)^{\alpha - 2} + B(\beta - 1) \left( \frac{\nu}{m} \right)^{\beta - 2} \pm D(\alpha - 1) \left( \frac{\nu}{m} \right)^{\alpha - 2}, \]
\[ \rho^\pm = \frac{1}{\sigma^\pm} \left\{ -A \cot \left( \frac{\pi \alpha}{2} \right) \left( \frac{\nu}{m} \right)^{\alpha - 1} - B \cot \left( \frac{\pi \beta}{2} \right) \left( \frac{\nu}{m} \right)^{\beta - 1} + 4\pi \frac{\nu}{\nu} f_+(0) \right\} \pm D \tan \left( \frac{\pi \alpha}{2} \right) \left( \frac{\nu}{m} \right)^{\alpha - 1}, \]

where \( m \) is the proton mass, the upper sign is for \( pp \) and the lower for \( p\bar{p} \) scattering. We have used the analyticity properties of real analytic amplitudes to write \( \rho^\pm \) in Eq. (7). The 6 real parameters which are needed are: 3 Regge coefficients, \( A, B \) and \( D \) in mb, 2 Regge powers, \( \alpha \) and \( \beta \), which are dimensionless and \( f_+(0) \). The real constant \( f_+(0) \) introduced in Eq. (7) is the subtraction constant at \( \nu = 0 \) needed for a singly-subtracted dispersion relation[4][5].

Using Eq. (3) and Eq. (4), along with Eq. (5), we find \( \alpha = 0.5475 \) and \( \beta = 1.0808 \), with \( A = 59.8 \text{ mb} \), \( B = 22.71 \) and \( D = -16.38 \text{ mb} \), where the energy variable is now \( \nu/m \), instead of \( s \).

Let us now consider a transition energy \( \nu_0 \), defined as an energy slightly higher than the energy where the resonances average out, i.e., an energy where the cross sections already have a smooth behavior (a useful choice for \( pp \) and \( p\bar{p} \) reactions is \( \nu_0 = 7.59 \text{ GeV} \), corresponding to the c.m. (center-of-mass) energy \( \sqrt{s_0} = 4 \text{ GeV} \). At the transition energy \( \nu_0 \), it is convenient to define the 4 analyticity conditions[4,5]

\[ \sigma_{av} = \frac{\sigma^+ (\nu_0) + \sigma^- (\nu_0)}{2} = A(\nu_0/m)^{\alpha - 1} + B(\nu_0/m)^{\beta - 1}, \]
\[ \Delta \sigma = \frac{\sigma^+ (\nu_0) - \sigma^- (\nu_0)}{2} = D(\nu_0/m)^{\alpha - 1}, \]
\[ m_{av} = \frac{1}{2} \left[ \frac{d\sigma^+}{d(\nu/m)} + \frac{d\sigma^-}{d(\nu/m)} \right]_{\nu = \nu_0} = A(\alpha - 1)(\nu_0/m)^{\alpha - 2} + B(\beta - 1)(\nu_0/m)^{\beta - 2}, \]
\[ \Delta m = \frac{1}{2} \left[ \frac{d\sigma^+}{d(\nu/m)} - \frac{d\sigma^-}{d(\nu/m)} \right]_{\nu = \nu_0} = D(\alpha - 1)(\nu_0/m)^{\alpha - 2}. \]

Using these definitions of the experimental quantities \( \sigma_{av}, \Delta \sigma, m_{av} \) and \( \Delta m \), we now write the four analyticity constraints at energy \( \nu_0 \), using Eq. (11) and Eq. (12)—see references [4].
and $\alpha = 1 + \frac{\Delta m}{\Delta \sigma} (\nu_0/m)$, 

$$\alpha = 1 + \frac{\Delta m}{\Delta \sigma} (\nu_0/m),$$

$$D = \Delta \sigma (\nu_0/m)^{1-\alpha},$$

$$\beta(A) = 1 + \frac{m_{av}(\nu_0/m) - A(\alpha - 1)(\nu_0/m)^{\alpha-1}}{\sigma_{av} - A(\nu_0/m)^{\alpha-1}},$$

$$B(A) = \sigma_{av}(\nu_0/m)^{1-\beta} - A(\nu_0/m)^{\alpha-\beta}. $$

These analyticity consistency conditions utilize the two experimental cross sections and their first derivatives at the transition energy $\nu_0$, where we join on to the asymptotic fit. We have chosen $\nu_0$ as the (low) energy just after which resonance behavior finishes. At $\sqrt{\mathcal{s}_0} = 4$ GeV (corresponding to $\nu_0 = 7.59$ GeV), Block and Halzen found that

$$\sigma^+ (\nu_0) = 40.18 \text{ mb}, \quad \sigma^- (\nu_0) = 56.99 \text{ mb},$$

$$\left. \frac{d\sigma^+}{d(\nu/m)} \right|_{\nu = \nu_0} = -0.2305 \text{ mb}, \quad \left. \frac{d\sigma^-}{d(\nu/m)} \right|_{\nu = \nu_0} = -1.4456 \text{ mb},$$

using a local fit in the neighborhood of $\nu_0$.

These values yield the 4 constraints required by analyticity, i.e.,

$$\sigma_{av}(\nu_0) = 48.59 \text{ mb}, \quad \Delta \sigma(\nu_0) = -8.405 \text{ mb},$$

$$m_{av}(\nu_0) = -0.8381 \text{ mb}, \quad \Delta m(\nu_0) = 1.215 \text{ mb}. $$

Using the numerical values in Eq. (18) and Eq. (19) for the odd amplitude, along with Eq. (12) and Eq. (13), we note that the odd amplitude is completely specified. This is true even before we make a fit to the high energy data. The two odd analyticity conditions constrain the odd parameters to be

$$D = -28.56 \text{ mb}, \quad D_{LD} = -16.38 \text{ mb},$$

$$\alpha = 0.4150, \quad \alpha_{LD} = 0.545. $$

where we have contrasted these value with the values found by Landshoff and Donnachie, $D_{LD}$ and $\alpha_{LD}$, which are clearly incompatible.

Next, we use the two constraint equations, Eq. (14) and Eq. (15), along with Eq. (20) and Eq. (21), together with the even amplitude portions of Eq. (18) and Eq. (19), to simultaneously fit a “sieved” data set of high energy cross sections and $p$-values for $pp$ and $p\bar{p}$ with energies above $\sqrt{\mathcal{s}} = 6$ GeV, derived from the archives of the Particle Data Group. The “sieve” algorithm which was used to find this data set is fully described in ref. [9].

This same data set has already been successfully used to make an excellent $\ln^2 \mathcal{s}$ fit of the type used in Eq. (24), using the same analyticity constraints as we use here. It should be further noted that a $\ln \mathcal{s}$ fit, i.e., setting the coefficient $c_2 = 0$ in Eq. (24), again using the identical analyticity constraints of Eq. (18) and Eq. (19) as well as the same “sieved” data set, was conclusively ruled out.

After employing the 4 constraints, the number of fit parameters has been reduced from 6 to 2, i.e., the two free parameters $A$ and $f_+ (0)$. It should be noted that the subtraction
constant $f_{c}(0)$ only enters into $\rho^\pm$-values and not into cross section determinations $\sigma^\pm$. In essence, the cross section fit is a one-parameter fit, $A$.

The results of the fit are given in Table 1. When the cross sections using the parameters of Table 1 are rewritten in terms of $s$, for direct comparison with the Landshoff-Donnachie cross sections of Eq. (3) and Eq. (4), we find that our analyticity-constrained cross sections are:

\begin{align}
\sigma^+ &= 23.97s^{-0.5850} + 33.02s^{0.0255} \quad \text{for } pp, \\
\sigma^- &= 109.1s^{-0.4525} + 33.02s^{0.0255} \quad \text{for } p\bar{p}.
\end{align}

Clearly, our Eq. (22) and Eq. (23) are in sharp disagreement with the Landshoff-Donnachie cross sections given in Eq. (3) and Eq. (4). This is graphically seen in Figures 1 and 2, which plot $\sigma$ and $\rho$, respectively, against the c.m. energy $\sqrt{s}$, where we see that the fits are very much below all of the high energy points. The renormalized $\chi^2$ per degree of freedom is 21.45, for 185 degrees of freedom, yielding the incredibly large value of 3576.24 for the total $\chi^2$.

Thus, there is essentially zero probability that a fit of the Landshoff-Donnachie type—of the form given in Eq. (3) and Eq. (4)—is a good representation of the high energy data ($\sqrt{s} \geq 6 \text{ GeV}$). Certainly, at the very high energy end, their functional form violates unitarity. We now see that it does not have the proper shape to satisfy analyticity at the lower energy end. Clearly, the form requires substantial ad hoc modifications to join on to the low energy constraints. Thus its primary virtue—its simplicity of form—requires serious modification. We bring to the reader’s attention that a $\ln^2 s$ fit of the form

\[ \sigma^\pm(\nu) = c_0 + c_1 \ln(\nu/m) + c_2 \ln^2(\nu/m) + \beta \nu/m - 5 \pm \delta(\nu/m)^{\alpha-1}, \]

was carried out on the same “sieved” sample of $\sigma^\pm$ and $\rho^\pm$ in ref. [5, 6], using the same 4 analyticity constraints, where it gave a renormalized $\chi^2$ per degree of freedom of 1.095 for 184 degrees of freedom, an excellent fit. Further, this $\ln^2 s$ type fit was shown to be independent of the choice of transition energy $s_0$ [5, 6], for $4 \leq s_0 \leq 6 \text{ GeV}$.

In conclusion, a functional form of the type

\begin{align}
\sigma(pp) &= A's^{\alpha-1} + B's^{\beta-1}, \\
\sigma(p\bar{p}) &= C's^{\alpha-1} + B's^{\beta-1},
\end{align}

with $\beta \sim 1.1$, although conceptually very simple, cannot be used for fitting high energy scattering for energies $\sqrt{s} > 6 \text{ GeV}$, since it can not satisfy the 4 analyticity requirements of Equations (18) and (19). In addition, the term $s^{\beta-1}$ violates unitarity at the highest energies. Thus, this simple type of parametrization—which was widely used because of its inherent simplicity—is effectively excluded, since it now requires substantial modification to its low energy behavior. We suspect that the earlier success of the Landshoff-Donnachie model reflected its validity in only a very limited energy region. In contrast, the $\ln^2 s$ parametrization of Block and Halzen [5] gives an excellent fit, satisfying unitarity in a natural way, as well as satisfying the 4 analyticity constraints.

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Table 1: The fitted results for a 2-parameter fit of the Landshoff-Donnachie type, \((\sigma^\pm = A(\nu/m)^{\alpha-1} + B(\nu/m)^{\beta-1} \pm D(\nu/m)^{\gamma-1}, \) where \(\nu\) is the laboratory projectile energy and \(m\) is the proton mass), simultaneously to both the total cross sections and \(\rho\)-values for \(pp\) and \(p\bar{p}\) scattering, for c.m. energies \(\sqrt{s} \geq 6\) GeV, using the sieved data set of Ref. [5, 6]. The renormalized \(\chi^2\) per degree of freedom, taking into account the effects of the \(\Delta \chi^2 = 6\) cut [9], is given in the row labeled \(R \times \chi^2\)/d.f. The errors in the fitted parameters have been multiplied by the appropriate \(r_{\chi^2}\).

| Parameters       | Even Amplitude | Odd Amplitude |
|------------------|----------------|---------------|
| \(A\) (mb)       | 44.65 \(\pm 0.0031\) |               |
| \(B\) (mb)       | 33.60           |               |
| \(\beta\)        | 1.0255          |               |
| \(f_+(0)\) (mb GeV) | 2.51 \(\pm 0.57\) |               |
| \(D\) (mb)       |                 | \(-28.56\)    |
| \(\alpha\)       | 0.415           |               |
| \(\chi^2_{\text{min}}\) |               | 3576.2        |
| \(R \times \chi^2_{\text{min}}\)/d.f. |               | 3968.1        |
| d.f.             | 185             |               |
| \(R \times \chi^2_{\text{min}}/\text{d.f.}\) |               | 21.45         |
Figure 1: The fitted total cross sections $\sigma_{pp}$ and $\sigma_{\bar{p}p}$ in mb, vs. $\sqrt{s}$, in GeV, using the 4 constraints of Equations (18) and (19). The circles are the sieved data\cite{5,9} for $\bar{p}p$ scattering and the squares are the sieved data\cite{5,9} for $pp$ scattering for c.m. energies $\sqrt{s} \geq 6$ GeV. The solid curve ($\bar{p}p$) and the dotted curve ($\bar{p}p$) are the $\chi^2$ fits from Table 1.

Figure 2: The fitted $\rho$-values, $\rho_{pp}$ and $\rho_{\bar{p}p}$, vs. $\sqrt{s}$, in GeV, using the 4 constraints of Equations (18) and (19). The circles are the sieved data\cite{5,9} for $\bar{p}p$ scattering and the squares are the sieved data\cite{5,9} for $pp$ scattering for c.m. energies $\sqrt{s} \geq 6$ GeV. The solid curve ($\bar{p}p$) and the dotted curve ($\bar{p}p$) are the $\chi^2$ fits from Table 1.