Rough solutions of a Schrödinger - Benjamin - Ono system

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Abstract

The Cauchy problem for a coupled Schrödinger and Benjamin-Ono system is shown to be globally well-posed for a class of data without finite energy. The proof uses the I-method introduced by Colliander, Keel, Staffilani, Takaoka, and Tao.

0 Introduction

Consider the following weakly coupled dispersive system

\begin{align}
    i\partial_t u + \partial_x^2 u &= \alpha uv \\
    \partial_t v + \nu \partial_x |\partial_x| v &= \beta \partial_x (|u|^2)
\end{align}

with Cauchy data

\[ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \]

where \( x, t \in \mathbb{R}, \alpha, \beta, \nu \in \mathbb{R} \).

This system was introduced by Funakoshi and Oikawa to model the interaction of two fluids described by a short wave term \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \), which fulfills a Schrödinger type equation and a long wave term \( v : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), which fulfills a Benjamin-Ono type equation. Bekiranov, Ogawa and Ponce showed local well-posedness for data \( u_0 \in H^s(\mathbb{R}), \quad v_0 \in H^{s-\frac{1}{2}}(\mathbb{R}) \) and \( |\nu| \neq 1, \quad s \geq 0 \). Because the system satisfies three conservation laws (cf. below) it is not difficult to see that this solution exists globally if \( \nu > 0 \) and \( \frac{2}{\nu} < 0 \) in the case \( s \geq 1 \) (finite energy solutions).

In this paper we first show local well-posedness also in the case \( |\nu| = 1 \), if \( s > 0 \). Then we use the Fourier restriction norm method and especially the

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so-called I-method to show global well-posedness for data with infinite energy (and $\nu > 0$, $\frac{a}{B} < 0$), we assume only $s > \frac{1}{2}$. This method was introduced by Colliander, Keel, Staffilani, Takaoka, and Tao and successfully applied in various situations [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15], in most of the cases using a scaling invariance of the problem. Such an invariance is also very helpful in our case. One introduces for given $0 < s < 1$ and $N >> 1$ the Fourier multiplier $I_N f(\xi) := m_N(\xi) \hat{f}(\xi)$, where $m_N$ is a smooth, radially symmetric and nonincreasing function of $|\xi|$ and

$$m(\xi) := m_N(\xi) := \begin{cases} \frac{1}{(N/2)^{1-s}} & |\xi| \leq N \\ |\xi| & |\xi| \geq 2N \end{cases}$$

Then $I = I_N$ is a smoothing operator which maps $H^s(\mathbb{R})$ to $H^1(\mathbb{R})$ in the sense that

$$\|u\|_{H^s} \leq c \|Iu\|_{H^1} \leq c N^{1-s} \|u\|_{H^s}$$

and similarly

$$\|v\|_{H^{s-\frac{1}{2}}} \leq c \|Iv\|_{H^{s-\frac{1}{2}}} \leq c N^{1-s} \|v\|_{H^{s-\frac{1}{2}}}.$$ 

One then considers the conserved functionals $L$ and $E$ (cf. (14) and (15) below) replacing $u$ and $v$ by $I_N u$ and $I_N v$, so that they make sense for $u \in H^s$, $v \in H^{s-\frac{1}{2}}$, whereas they originally are only defined for $u \in H^1$, $v \in H^{\frac{3}{2}}$. These modified functionals are then shown to be almost conserved in the sense that their increment on a local existence interval is bounded by $cN^{-1}$. One can show that this is enough to make the continuation process by reapplying the local existence theorem uniform, provided $s$ is close enough to 1, namely $s > \frac{1}{2}$.

We use the following norms (for $s \in \mathbb{R}$, $-1 < b < 1$):

$$\|u\|_{X^{s,b}} := \|\langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s \hat{u}(\xi,\tau)\|_{L^2(\mathbb{R}^2)}$$

$$\|v\|_{Y^{s,b}} := \|\langle \tau + \nu \xi \rangle^b \langle \xi \rangle^s \hat{\upsilon}(\xi,\tau)\|_{L^2(\mathbb{R}^2)}$$

belonging to the Schrödinger and Benjamin - Ono equation, respectively. We also need the local in time norm $\|u\|_{X^{s,b}} := \inf \|\psi\|_{X^{s,b}} = \|\psi\|_{X^{s,b}}$ and similarly $\|v\|_{Y^{s,b}}$.

The standard facts about the Fourier restriction norm method which we use without further comments can be found in [1], Chapter 2. The Strichartz estimates for the homogeneous Schrödinger and Benjamin - Ono equation read

$$\|e^{i\theta\xi^2} u_0\|_{L^6_{xt}} \leq c \|u_0\|_{L^2_x}$$

and

$$\|e^{i\nu \partial_x |\partial_x| u_0\|_{L^6_{xt}} \leq c \|u_0\|_{L^2_x}$$

(cf. [14] and [11]), which immediately imply $\|u\|_{L^p_{xt}} \leq c \|u\|_{X^{0,b}}$ and $\|v\|_{L^p_{xt}} \leq c \|v\|_{Y^{0,b}}$ for $2 \leq p \leq 6$ and $b > \frac{1}{2}$. We also use the following bilinear Strichartz type estimate for the Schrödinger equation

$$\|D^{1/2}_x(u_1 \overline{u_2})\|_{L^2_{xt}} \leq c \|u_1\|_{X^{0,b}} \|u_2\|_{X^{0,b}}$$

(4)

(for a proof cf. e.g. [1], Lemma 3.2).

We denote by $a+$ and $a-$ a number slightly larger and smaller than $a$, respectively.
1 Local existence

Proposition 1.1 For $|\nu| = 1$ we have

\[ \|uv\|_{X^{s,0}} \leq c\|u\|_{X^{s,b}}\|v\|_{Y^{s,-\frac{1}{2},b}} \]

if $-\frac{1}{2} < a < 0 < \frac{1}{2} < b$ and $s > 1 - 2|a|$ ($\iff |a| > \frac{1-s}{2}$) (especially $s > 0$).

Proof: (along the lines of [1], Lemma 3.4)

Assume first $\nu = 1$. We have to prove the following estimate

\[ \int \int \int \frac{(\xi)^s g(\tau,\sigma,\xi - \eta)d\sigma d\eta d\xi}{(\eta)^{s-\frac{1}{2}}(\xi - \eta)^{a}(\tau + \xi^2)^{b}} \leq c\|g\|_{L^2}\|f\|_{L^2}\|\phi\|_{L^2} \]

(5)

We split $(\tau,\sigma,\xi,\eta) \in \mathbb{R}^4$ into several regions:

\[
\begin{align*}
A &= \{ |\eta| < 1 \} \\
B &= \{ \eta < 0, |\xi| \geq \frac{1}{2}|\eta|, |\eta| \geq 1 \} \\
C &= \{ \eta < 0, |\xi| < \frac{1}{2}|\eta|, |\eta| \geq 1 \} \\
D &= \{ \eta > 0, |\xi - \eta| \leq \frac{1}{2}|\eta|, |\eta| \geq 1 \} \\
E &= \{ \eta > 0, |\xi - \eta| > \frac{1}{2}|\eta|, |\eta| \geq 1 \}
\end{align*}
\]

Now in $E$ we have

\[ |\nu \eta| + \eta^2 - 2\xi \eta = 2|\eta|^2 - \xi \eta = 2|\eta||\eta - \xi| > |\eta|^2 \]

and thus

\[ |\sigma + \nu \eta|| + |\tau - \sigma + (\xi - \eta)^2| \geq |\nu \eta| + \eta^2 - 2\xi \eta > |\eta|^2. \]

According to which of the terms on the l.h.s. is dominant we split $E$ into 3 parts:

\[
\begin{align*}
E_1 &= E \cap \{ |\tau + \xi^2| \geq |\sigma + \nu \eta||, |\tau - \sigma + (\xi - \eta)^2|, |\tau + \xi^2| \geq \frac{1}{3}|\eta|^2 \} \\
E_2 &= E \cap \{ |\sigma + \nu \eta|| \geq |\tau + \xi^2|, |\tau - \sigma + (\xi - \eta)^2|, |\sigma + \nu \eta|| \geq \frac{1}{3}|\eta|^2 \} \\
E_3 &= E \cap \{ |\tau - \sigma + (\xi - \eta)^2| \geq |\tau + \xi^2|, |\sigma + \nu \eta||, |\tau - \sigma + (\xi - \eta)^2| \geq \frac{1}{3}|\eta|^2 \}
\end{align*}
\]

Define $R_1 = A \cup B \cup D \cup E_1$, $R_2 = C \cup E_2$, $R_3 = E_3$. In order to prove (5) in the region $R_1$ it is sufficient to show

\[
\left\| \frac{\langle \xi \rangle^s}{\langle \tau + \xi^2 \rangle^a} \left( \int \frac{\langle \eta \rangle \chi_{R_3} d\sigma d\eta}{\langle \eta \rangle^{2s} \langle \xi - \eta \rangle^{2s} \langle \sigma + \nu \eta|| \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \right)^{\frac{1}{2}} \right\|_{L^\infty(L^2)} < \infty
\]

(6)
Similarly, in order to prove (5) in $R_2$ we have to show
\[ \left\| \frac{\langle \eta \rangle^{\frac{1}{2}}}{\langle \eta \rangle^{*} (\sigma + \nu \eta |\eta|)|^{2b}} \left( \int \int \frac{\langle \xi \rangle^{2s} \chi_{R_2} d\tau d\xi}{(\xi - \eta)^{2s}(\tau + \xi^2)^{2a}(\tau - \sigma + (\xi - \eta)^2)^{2b}} \right)^{\frac{1}{2}} \right\|_{L_{\infty}^p(L_{\xi}^\infty)} < \infty \]

Finally, in order to prove (6) in $R_3$ we use the transformed region
\[ \tilde{R}_3 = \{ (\rho, \sigma, \zeta, \eta) \in \mathbb{R}^4 : |\rho| \geq 1, |\rho - \zeta^2| \geq \frac{1}{3}|\eta|^2 \}, \]
where $\rho := \sigma - \tau$, $\zeta := \eta - \xi$. We have to show
\[ \left\| \frac{1}{\langle \xi \rangle^{*} (\rho - \zeta^2)^b} \left( \int \int \frac{\langle \eta \rangle^{2s} \chi_{\tilde{R}_3} d\sigma d\eta}{(\eta - \zeta)^{2s-1}(\sigma + \nu \eta |\eta|)^{2b}(\sigma - \rho + (\eta - \zeta)^2)^{2a}} \right)^{\frac{1}{2}} \right\|_{L_{\infty}^p(L_{\xi}^\infty)} < \infty . \]

We start to prove (6). In the regions $A, B$ and $D$ we use the estimate $\langle \xi \rangle \leq \langle \eta \rangle$ so that it suffices to show
\[ \left\| \int \int \frac{\langle \eta \rangle d\sigma d\eta}{(\sigma + \nu \eta |\eta|)^{2b}(\tau - \sigma + (\xi - \eta)^2)^{2b}} \right\|_{L_{\infty}^p(L_{\xi}^\infty)} < \infty . \]

Performing the $\sigma$-integration we get by [1], Lemma 2.5 (2.11):
\[ \int \int \frac{\langle \eta \rangle d\sigma d\eta}{(\sigma + \nu \eta |\eta|)^{2b}(\tau - \sigma + (\xi - \eta)^2)^{2b}} \leq c \int \frac{\langle \eta \rangle d\eta}{(\tau + \xi^2 + \nu \eta |\eta| + \eta^2 - 2\xi \eta)^{2b}}. \]
This is trivially bounded in the region $A$, whereas in region $B$ we substitute $\tau + \xi^2 + \nu \eta |\eta| + \eta^2 - 2\xi \eta = \tau + \xi^2 + 2\eta^2 - 2\xi \eta =: \eta'$, so that $\frac{d\eta'}{d\eta} = -2\xi$, and we get the bound using $|\xi| \geq \frac{1}{2}|\eta|$ and $|\eta| \geq 1$:
\[ c \int \frac{\langle \eta \rangle d\eta}{\langle \eta' \rangle^{2b}} \leq c \int \frac{d\eta'}{\langle \eta' \rangle^{2b}} < \infty \]
for $b > \frac{1}{2}$.

In the region $D$ we have $\tau + \xi^2 + \nu \eta |\eta| + \eta^2 - 2\xi \eta = \tau + \xi^2 + 2\eta^2 - 2\xi \eta =: \eta'$, so that $|\frac{d\eta'}{d\eta}| = |4\eta - 2\xi| = |2(2\eta - \xi)| = 2|\eta + (\eta - \xi)| \geq 2(|\eta| - |\eta - \xi|) \geq |\eta|$, because $|\xi - \eta| \leq \frac{1}{2}|\eta|$. Thus we get the bound
\[ c \int \frac{\langle \eta \rangle d\eta}{\langle \eta' \rangle^{2b}} < \infty \]
for $b > \frac{1}{2}$ and $|\eta| \geq 1$.

It remains to prove (6) in the region $E_1$. First we consider the case $0 < s < \frac{1}{2}$. This implies $|a| > \frac{1}{2} b > \frac{1}{4}$. We use again the estimate $\langle \xi \rangle \leq \langle \eta \rangle$ so that it suffices to show
\[ \left\| \frac{1}{(\tau + \xi^2)^{|a|}} \left( \int \int \frac{\langle \eta \rangle d\sigma d\eta}{(\sigma + \nu \eta |\eta|)^{2b}(\tau - \sigma + (\xi - \eta)^2)^{2b}} \right)^{\frac{1}{2}} \right\|_{L_{\infty}^p(L_{\xi}^\infty)} < \infty . \]

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Performing the $\sigma$-integration as above it remains to bound
\[
\frac{1}{(\tau + \xi^2)^{2a}} \left( \int \frac{\langle \eta \rangle \, d\eta}{(\tau + \xi^2 + \eta^2 - 2\xi \eta)^{2b}} \right)^{\frac{1}{2}} \leq c \left( \int \frac{d\eta}{(\tau + \xi^2 + 2\eta^2 - 2\xi \eta)^{2b}} \right)^{\frac{1}{2}} \leq c
\]
using $\langle \tau + \xi^2 \rangle^{2a} \geq c(\eta)^{2a} \geq c(\eta)^{\frac{1}{2}}$.

Next we consider the case $s \geq \frac{3}{2}$ in the region $E_1$. First of all, consider the subregion $|\xi| \geq \frac{3}{2} |\eta|$. In this case we get the following bound for (6) performing the $\sigma$-integration:
\[
c \left( \int \int \frac{\langle \eta \rangle \, d\sigma \, d\eta}{\langle \eta \rangle^{2s} \langle \sigma + \nu \eta \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \right)^{\frac{1}{2}} \leq c \left( \int \frac{\langle \eta \rangle \, d\eta}{(\tau + \xi^2 + \nu \eta \eta + \eta^2 - 2\xi \eta)^{2b}} \right)^{\frac{1}{2}} \leq c \left( \int \frac{d\eta}{(\tau + \xi^2 + 2\eta^2 - 2\xi \eta)^{2b}} \right)^{\frac{1}{2}} \leq c.
\]

In the subregion $|\xi| \leq \frac{3}{2} |\eta|$ we get the following bound for (6) performing the $\sigma$-integration:
\[
c \left( \int \int \frac{\langle \eta \rangle \, d\sigma \, d\eta}{\langle \eta \rangle^{2s} \langle \sigma + \nu \eta \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \right)^{\frac{1}{2}} \leq c \left( \int \frac{\langle \eta \rangle \, d\eta}{(\tau + \xi^2 + \eta^2 - 2\xi \eta)^{2b}} \right)^{\frac{1}{2}}.
\]

Substituting $\tau + \xi^2 + \nu \eta \eta + \eta^2 - 2\xi \eta = \tau + \xi^2 + 2\eta^2 - 2\xi \eta =: \eta'$ we have $|d\eta'| = |4\eta - 2\xi| = 2(|2\eta| - |\xi|) \geq 2(|2\eta| - \frac{3}{2}|\eta|) = |\eta| \sim \langle \eta \rangle$, and we get the bound
\[
c \left( \int \langle \eta \rangle^{-2b} d\eta' \right)^{\frac{1}{2}} < \infty.
\]

Next we have to prove (7). In the region $C$ it suffices to show
\[
\left\| \langle \eta \rangle^{\frac{1}{2s}} \left( \int \frac{\langle \xi \rangle^{2s} \chi_c \, d\tau \, d\xi}{(\tau + \xi^2)^2 |\eta| \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \right)^{\frac{1}{2}} \right\|_{L^{\infty} L^\infty} < \infty. \tag{9}
\]

The integration with respect to $\tau$ gives by [11], Lemma 2.5 (2.11) and Hölder:
\[
\int \frac{\langle \xi \rangle^{2s} \chi_c \, d\tau \, d\xi}{(\tau + \xi^2)^2 |\eta| \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \leq c \int \frac{\langle \xi \rangle^{2s} \chi_c \, d\xi}{\langle \sigma - \eta^2 + 2\xi \eta \rangle^{2a}} \]
\[
\leq c \left( \int_{|\xi| \leq \frac{3}{4} |\eta|} \langle \xi \rangle^{2s + \frac{2p}{p - 1}} d\xi \right)^{\frac{1}{q}} \left( \int \frac{d\xi}{\langle \sigma - \eta^2 + 2\xi \eta \rangle^{2a|q|}} \right)^{\frac{1}{q}}.
\]
Choosing $\frac{1}{q} = 2|a|$, $\frac{1}{p} = 1 + 2|a|$ and substituting $\eta' = \sigma - \eta^2 + 2\xi \eta$ we get the bound
\[
c \langle \eta \rangle^{\frac{2s + 2|a| - \frac{1}{q}}{2a|q|}} \left( \int \frac{d\eta'}{(\eta')^{1 + |a|}} \right)^{\frac{1}{q}} \leq c \langle \eta \rangle^{\frac{2s + 1}{p}} \frac{1}{p} \leq c \langle \eta \rangle^{2s + 4|a|} = c \langle \eta \rangle^{2s + 4|a|}.
\]
Thus we get the following bound for (9):

$$\langle \eta \rangle \frac{1}{2} - 2s + \frac{1}{2} - 2|a| + = \langle \eta \rangle \frac{1}{2} - 2|a| + \leq c$$

because $|a| > \frac{1}{2 - s}$.

Next we prove (7) in the region $E_2$. Using the estimate $\langle \xi \rangle \leq \langle \eta \rangle \langle \xi - \eta \rangle$ and performing the $\tau$-integration it is sufficient to get a bound on $E_2$ for

$$\langle \eta \rangle \frac{1}{2} ^\frac{1}{2} \left( \int \frac{d\xi}{\langle \sigma - \eta ^2 + 2\eta \xi \rangle^2|a|} \right) \frac{1}{2} \leq c \langle \sigma + \nu \eta |\eta| \rangle^{b} \left( \int \frac{d\eta}{\langle \sigma - \eta + 2\eta \xi \rangle^2|a|} \frac{1}{\langle \eta \rangle^{\frac{1}{2}} - |a|} \right) \frac{1}{2} \leq c \langle \sigma + \nu \eta |\eta| \rangle^{b} \leq c.$$

It remains to prove (8) in the region $\overline{R}_3$. Using the estimate $\langle \eta - \zeta \rangle \leq \langle \zeta \rangle \langle \eta \rangle$ and performing the $\sigma$-integration it is enough to give the following bound in $\overline{R}_3$:

$$\frac{1}{\langle \rho - \zeta ^2 \rangle^b} \left( \int \frac{\langle \eta \rangle d\eta}{\langle \rho - \zeta ^2 + \nu \eta |\eta| - \eta ^2 + 2\zeta \eta ^2|a|} \right) \frac{1}{2} \leq c \left( \int \frac{d\eta}{\langle \eta \rangle^{4b-1}} \right) ^\frac{1}{2} \leq c.$$

The case $\nu = -1$ can be treated in the same way by replacing $\eta < 0$ by $\eta > 0$ in the regions $B$ and $C$ and $\eta > 0$ by $\eta < 0$ in $D$ and $E$.

Moreover the following estimates for the nonlinearities are true (cf. [1], Cor. 3.3 and Lemma 3.4).

**Proposition 1.2**

1. For arbitrary $\nu$ and $s \geq 0$, $b > \frac{1}{2}$:

$$\|(u^2)_x\|_{X^{s, b}} \leq c u_{X^{s, b}}^2.$$

2. For $|\nu| \neq 1$ and $s \geq 0$, $b > \frac{1}{2}$:

$$\|uv\|_{X^{s, -\frac{b}{2}}} \leq c \|u\|_{X^{s, b}} \|v\|_{X^{s, -\frac{b}{2}}}.$$

These propositions imply by standard arguments the following local existence result.

**Theorem 1.1** Let $s > 0$ in the case $|\nu| = 1$, and $s \geq 0$ in the case $|\nu| \neq 1$. For any $(u_0, v_0) \in H^s(R) \times H^{s-\frac{1}{2}}(R)$ there exists $b > \frac{1}{2}$ and $\delta = \delta((\|u_0\|_{H^s}, \|v_0\|_{H^{s-\frac{1}{2}}}) > 0$ such that the Cauchy problem ([4], [2], [3]) has a unique solution $(u, v) \in X^{s, b}_\delta \times Y^{s-\frac{1}{2}, b}_\delta$ and $(u, v) \in C^0([0, \delta], H^s \times H^{s-\frac{1}{2}})$.
Applying the operator $I$ to the system (11), (12), we get the problem

$$
\begin{align*}
    i\partial_t u + I\partial_x^2 u &= \alpha I(uv) \quad (10) \\
    I\partial_t v + \nu I(\partial_x|\partial_x u) &= \beta I\partial_x(|u|^2) \quad (11) \\
    Iu(0) = Iu_0, \quad Iv(0) = Iv_0. \quad (12)
\end{align*}
$$

For this system the following modified local existence result holds:

**Proposition 1.3** Assume $1 \geq s > 0$, if $|\nu| = 1$, and $s \geq 0$ otherwise. For any $(u_0, v_0) \in H^s \times H^{\frac{5}{2} - s}$ there exists $\delta \leq 1$ and $\delta \sim (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^{\frac{5}{2}}})^{-\frac{s}{2}}$, if $|\nu| = 1$, and $\delta \sim (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^{\frac{5}{2}}})^{-4-}$, if $|\nu| \neq 1$, such that the system (10), (11), (12) has a unique local solution in the time interval $[0, \delta]$ with the property

$$
\|Iu\|_{X_{\delta}^{1, b}} + \|Iv\|_{Y_{\delta}^{4, b}} \leq \tilde{c}(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^{\frac{5}{2}}}),
$$

where $b = \frac{1}{2} +$.

**Proof:** We construct a fixed point of the mapping $S = (\tilde{S}_0, \tilde{S}_1)$ induced by the integral equations belonging to the system (10), (11), (12):

$$
\begin{align*}
    \tilde{S}_0(Iu, Iv)(t) &:= e^{it\partial_x^2}Iu_0 - i \int_0^t e^{i(t-s)\partial_x^2} \alpha I(u(s)v(s)) \, ds \\
    \tilde{S}_1(Iu, Iv)(t) &:= e^{-\nu t\partial_x|\partial_x|}Iv_0 + \int_0^t e^{-\nu(t-s)\partial_x|\partial_x|} \beta I(|u(s)|^2) \, ds.
\end{align*}
$$

The estimates for the nonlinearities in (11) and (12) carry over to corresponding estimates including the $I$-operators by the interpolation lemma of [7], namely

$$
\begin{align*}
    \|I(|u|^2)_x\|_{Y_{\delta}^{4, 0}} &\leq c\|Iu\|_{X_{\delta}^{1, b}}^2 \\
    \|I(uv)\|_{X_{\delta}^{4, -|a|}} &\leq c\|Iu\|_{X_{\delta}^{1, b}}\|Iv\|_{Y_{\delta}^{4, b}}
\end{align*}
$$

(with $|a| = \frac{1}{2} +$ if $|\nu| = 1$, and $|a| = \frac{1}{3}$ otherwise). This implies

$$
\begin{align*}
    \|\tilde{S}_0(Iu, Iv)\|_{X_{\delta}^{1, b}} &\leq c\|Iu_0\|_{H^1} + c|\alpha|\|Iu\|_{X_{\delta}^{1, b}}\|Iv\|_{Y_{\delta}^{4, \delta^{-|a|}-}} \\
    \|\tilde{S}_1(Iu, Iv)\|_{Y_{\delta}^{4, b}} &\leq c\|Iv_0\|_{H^{\frac{5}{2}}} + c|\beta|\|Iu\|_{X_{\delta}^{1, b}}^2\delta^{-\frac{1}{2}-|a|}.
\end{align*}
$$

This gives the desired bounds, provided

$$
c\delta^{-\frac{1}{2}-|a|}(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^{\frac{5}{2}}}) < 1.
$$

## 2 Conservation laws

Our system has the following conserved quantities:

$$
\begin{align*}
    M &:= \|u\| \\
    L(u, v) &:= -\frac{\alpha}{2\beta} \|v\|^2 - 3 \int u \bar{u} \, dx \\
    E(u, v) &:= \|u_x\|^2 - \frac{\alpha \nu}{2\beta} \|D_{\delta}^\frac{1}{2} v\|^2 + \alpha \int v|u|^2 \, dx.
\end{align*}
$$

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From now on, we assume $\nu > 0$ and $\frac{\alpha}{\beta} < 0$. Then $L$ and $E$ are controlled by $\|u\|_{H^1}$ and $\|v\|_{H^{\frac{1}{2}}}$, and vice versa, as one concludes as follows:

$$|L(u, v)| \leq c\|v\|^2 + M\|u_x\|$$  \(16\)

and

$$\|v\|^2 \leq c(|L| + M\|u_x\|).$$  \(17\)

Concerning $E$ we have by Gagliardo - Nirenberg

$$\int |v u^2| \, dx \leq \|v\|\|u\|^{\frac{2}{3}}\|u_x\|^{\frac{1}{3}} \leq c(|L|\|u\|^{\frac{2}{3}}\|u_x\|^{\frac{1}{3}} + M^2\|u_x\|)$$

and thus

$$\|u_x\|^2 + \left(\frac{\alpha \nu}{\beta}\right)\|D_x^\frac{1}{2}v\|^2 \leq |E| + c(|L|\|u\|^{\frac{2}{3}} + M^4) + \epsilon \|u_x\|^2,$$

consequently

$$\|u_x\|^2 + \|D_x^\frac{1}{2}v\|^2 \leq c(|E| + |L|\|u\|^{\frac{2}{3}} + M^4).$$  \(18\)

Similarly

$$|E| \leq c(\|u_x\|^2 + \|D_x^\frac{1}{2}v\|^2 + |L|\|u\|^{\frac{2}{3}} + M^4).$$  \(19\)

From (16) and (19) we get

$$|E| \leq c(\|u_x\|^2 + \|D_x^\frac{1}{2}v\|^2 + \|v\|^{\frac{2}{3}} + M^2\|u_x\|^{\frac{1}{3}} + M^4)$$

and thus

$$\|u_x\|^2 + \|D_x^\frac{1}{2}v\|^2 \leq c(|E| + |L|\|u\|^{\frac{1}{2}} + M^4).$$  \(20\)

From (17) and (18) we have

$$\|v\|^2 \leq c(|L| + M(|E|^{\frac{1}{2}} + |L|^{\frac{1}{2}} + M^2)) \leq c(|L| + M|E|^{\frac{1}{2}} + M^3 + 1).$$  \(21\)

Finally, from (18) and (21) we arrive at

$$\|u\|^2_{H^1} + \|v\|^2_{H^{\frac{1}{2}}} \leq c(|E| + |L|^{\frac{1}{2}} + M^4 + 1).$$  \(22\)

These estimates imply a-priori-bounds for the $H^1$-norm of $u$ and the $H^{\frac{1}{2}}$-norm of $v$ for data with finite energy $E$, finite $L$ and finite $\|u_0\|$. This is the case for $H^1$-data $u_0$ and $H^{\frac{1}{2}}$-data $v_0$. Thus our local existence result implies

**Theorem 2.1**  *For data $(u_0, v_0) \in H^1 \times H^{\frac{1}{2}}$ and $\nu > 0$, $\frac{\alpha}{\beta} < 0$ there exists $b > \frac{1}{2}$ such that \(17, 19, 16\) has a unique global solution $(u, v) \in X^{1, b} \times Y^{\frac{1}{2}, b}$ with $(u, v) \in C^0(\mathbb{R}^+, H^1 \times H^{\frac{1}{2}})$.*

In order to get a corresponding result for less regular data we consider the modified functionals $E(Iu, Iv)$ and $L(Iu, Iv)$.
Using the modified system (10), (11) an elementary calculation shows

\[
\frac{d}{dt} E(Iu, Iv) = \alpha v(I(|u|^2) - |Iu|^2, D_x Iv_x) + \alpha \beta \langle |u|^2 \rangle_x - (|Iu|^2)_x, |Iu|^2) \\
- 2\alpha^2 \Re \langle IvIu, I(vu) - IvIu \rangle - 2\alpha \beta \Re \langle Iu_x, I(vu)_x - (IvIu)_x \rangle \\
=: \sum_{j=1}^4 I_j
\]

and

\[
\frac{d}{dt} L(Iu, Iv) = -\alpha \langle (Iu, (|u|^2) - (|Iu|^2)_x) \rangle + 2\Re \langle I(vu) - IvIu, Iv \rangle .
\]

### 3 Almost conservation

**Proposition 3.1** If \((u, v)\) is a solution of (1), (2), (3) in \([0, \delta]\) in the sense of Proposition 3.3 then the following estimate holds for \(N \geq 1\) and \(s \geq \frac{1}{4}\) 

\[
|E(Iu(\delta), Iv(\delta)) - E(Iu(0), Iv(0))| + |L(Iu(\delta), Iv(\delta)) - L(Iu(0), Iv(0))| \\
\leq cN^{-1} \left( \|Iu\|^2_{X^s_b} + \|Iv\|^2_{X^s_b} + \|Iu\|^4_{X^s_b} + \|Iv\|^2_{X^s_b} \right).
\]

**Proof:** Integrating (23) over \(t \in [0, \delta]\) we have to estimate the various terms on the right hand side. We assume w.l.o.g. the Fourier transforms of all the functions to be nonnegative. We drop \(\delta\) from the notation \(X^s_{b,\mu}\) and \(Y^s_{b,\mu}\).

**Estimate of \(I_1\):** We have to show

\[
\int_0^\delta \int_s \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1) m(\xi_2)}{m(\xi_1) m(\xi_2)} \right| \|\xi_1 + \xi_2\| \|\widehat{u_1}(\xi_1, t)\widehat{u_2}(\xi_2, t)\|_{Y^s_{b,\mu}} \|\xi_3\| \|\widehat{v}(\xi_3, t)\|_{Y^s_{b,\mu}} d\xi dt \\
\leq cN^{-1} \|u_1\|_{X^s_b} \|u_2\|_{X^s_b} \|v\|_{Y^s_b}.
\]

Here and in the sequel * denotes integration over the set \(\sum \xi_i = 0\). We may assume \(|\xi_1| \geq N\) or \(|\xi_2| \geq N\), because otherwise the multiplier term vanishes, and also the two largest frequencies are equivalent.

**Case 1:** \(|\xi_1| \ll |\xi_2| \sim |\xi_3|\), \(|\xi_1| \leq N\), \(|\xi_2| \geq N\).

Using the mean value theorem the multiplier term is estimated by

\[
\left| \frac{m(\xi_1 + \xi_2) - m(\xi_1) m(\xi_2)}{m(\xi_2)} \right| \leq c \left| \frac{(\nabla m)(\xi_2) \xi_1}{m(\xi_2)} \right| \leq c \frac{|\xi_1|}{|\xi_2|} \leq c \frac{|\xi_1|}{N}.
\]

Thus by use of (10) the integral is bounded by

\[
\frac{c}{N} \int_0^\delta \int_s \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1) m(\xi_2)}{m(\xi_2)} \right| \|\xi_1 + \xi_2\|^2 \|\widehat{u_1}(\xi_1, t)\widehat{u_2}(\xi_2, t)\|_{X^s_b} \|\xi_3\|^2 \|\widehat{v}(\xi_3, t)\|_{Y^s_b} d\xi dt \\
\leq \frac{c}{N} \left( D_{x_1}^\frac{1}{2} \|D_x u_1 D_x \widehat{u_2}\|_{L_x^2} \|D_x^\frac{1}{2} v\|_{L_x^{2}} \right) \\
\leq \frac{c}{N} \|u_1\|_{X^s_b} \|u_2\|_{X^s_b} \|v\|_{Y^s_b}.
\]
Case 2: $|\xi_1| < |\xi_2| \sim |\xi_3|$, $|\xi_1|, |\xi_2| \geq N$.
The multiplier is bounded by $\frac{c}{m(\xi_1)} \leq c\frac{|\xi_1|}{N}$. Thus we can conclude as in Case 1.

Case 3: $|\xi_1| \sim |\xi_2| \geq cN$, $|\xi_1 + \xi_2| \leq 2N$ ($\implies |\xi_3| \leq c|\xi_1|, |\xi_2|$).
The multiplier is bounded by $\frac{c}{m(\xi_1)m(\xi_2)} \leq c\frac{|\xi_1||\xi_2|}{N^2}$. Thus we get the following bound for the integral using $\mathbf{H}$:

$$
\frac{c}{N^2} \int_0^\delta \int_s^\delta N|\xi_1|\nabla((\xi_1, t))|\xi_2||\nabla(\xi_2, t)||\xi_1 + \xi_2|^\frac{1}{2}|\nabla(\xi_3, t)| d\xi dt
\leq \frac{c}{N^2} |\nabla(\xi_1, t)\nabla(\xi_2, t)\nabla(\xi_3, t)| L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} d\xi dt
\leq \frac{c}{N} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} d\xi dt
\leq \frac{c}{N} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} d\xi dt
$$

Case 4: $|\xi_1| \sim |\xi_2| \geq cN$, $|\xi_1 + \xi_2| \geq 2N$.
The multiplier is bounded by

$$
\frac{m(\xi_1 + \xi_2)}{m(\xi_1)m(\xi_2)} + 1 \leq c\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}N^{1-s}}{N^{1-s}|\xi_1 + \xi_2|^{1-s}} = c\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}}{N^{1-s}|\xi_1 + \xi_2|^{1-s}}
$$

which gives the integral bound

$$
\frac{c}{N^{1-s}} \int_0^\delta \int_s^\delta |\xi_1 + \xi_2|^s|\xi_1|^{1-s}\nabla(\xi_1, t)|\xi_2|^{1-s}\nabla(\xi_2, t)|\xi_3|\nabla(\xi_3, t) d\xi dt
\leq \frac{c}{N^{1-s}} \int_0^\delta \int_s^\delta |\xi_1|\nabla(\xi_1, t)|\xi_2|\nabla(\xi_2, t)|\xi_3|\nabla(\xi_3, t) d\xi dt
\leq \frac{c}{N} \nabla(\xi_1, t)\nabla(\xi_2, t)\nabla(\xi_3, t) d\xi dt
\leq \frac{c}{N} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} d\xi dt
$$

Estimate of $f_4$: It is sufficient to show

$$
\int_0^\delta \int_s^\delta \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| |\xi_1 + \xi_2|\nabla(\xi_1, t)\nabla(\xi_2, t)\nabla(\xi_3, t) d\xi dt
\leq cN^{-1}||u_2||_{X^{1, \frac{1}{2}, x}}||u_3||_{X^{1, \frac{1}{2}, x}}||v||_{Y^{1, \frac{1}{2}, x}}
$$

Case 1: $|\xi_1| < |\xi_2| \sim |\xi_3|$, $|\xi_1|, |\xi_2| \geq N$ ($\implies |\xi_1 + \xi_2| \sim |\xi_3|$).

If $|\xi_1| \leq N$, the multiplier is bounded by

$$
\left| \frac{m(\xi_1 + \xi_2) - m(\xi_2)}{m(\xi_2)} \right| \leq c\left| \frac{\nabla m(\xi_2)\xi_1}{m(\xi_2)} \right| \leq c\frac{|\xi_1|}{|\xi_2|} \leq c\frac{|\xi_1|}{N}
$$

and, if $|\xi_1| \geq N$, we have the bound $\frac{c}{m(\xi_1)} \leq c\frac{|\xi_1|}{N}$, so that the integral is bounded by

$$
\frac{c}{N} \int_0^\delta \int_s^\delta |\xi_1||\xi_1 + \xi_2|\nabla(\xi_1, t)|\xi_2|\nabla(\xi_2, t)\nabla(\xi_3, t) d\xi dt
\leq \frac{c}{N} \int_0^\delta \int_s^\delta |\xi_1|^{1/2}|\xi_2 + \xi_3|^{1/2}\nabla(\xi_1, t)|\xi_2|\nabla(\xi_2, t)\nabla(\xi_3, t) d\xi dt
\leq \frac{c}{N} \nabla(\xi_1, t)\nabla(\xi_2, t)\nabla(\xi_3, t) d\xi dt
\leq \frac{c}{N} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} L^2_{|\xi_1 + \xi_2|^\frac{1}{2}} d\xi dt
$$
Similarly as in Case 1 the multiplier is bounded by $c\frac{|\xi|}{N}$ and the integral by

$$
\frac{c}{N} \int_{\delta}^{1} \int_{*}^{*} |\xi + \xi_{3}| |\xi_{3}| |\xi_{3}| |\xi_{3}| d\xi dt \\
\leq \frac{c}{N} \int_{0}^{\delta} \int_{0}^{\delta} |\xi + \xi_{3}| |\xi_{3}| |\xi_{3}| |\xi_{3}| d\xi dt ,
$$

the same bound as in Case 1, using $|\xi + \xi_{3}| \leq c|\xi| = c|\xi|^{\frac{1}{2}}|\xi + \xi_{3}|^{\frac{1}{2}}$ .

**Case 3:** $|\xi_{1}| \sim |\xi_{2}| \geq N$, $|\xi_{1} + \xi_{2}| \leq 2N$.

The multiplier bound $\frac{c}{m(\xi_{1})m(\xi_{2})} \leq c\frac{|\xi|}{N}$ implies the integral bound

$$
cN^{-2} \int_{\delta}^{1} \int_{*}^{*} |\xi_{1}| |\xi_{2}| |\xi_{3} + \xi_{4}| |\xi_{3}| |\xi_{4}| d\xi dt \\
\leq cN^{-2} \int_{0}^{\delta} \int_{0}^{\delta} |\xi_{1}|^{\frac{1}{2}} |\xi_{2} + \xi_{3}|^{\frac{1}{2}} |\xi_{2}|^{\frac{1}{2}} |\xi_{4}| |\xi_{3}| |\xi_{3}| |\xi_{4}| d\xi dt \\
\leq cN^{-1} \left\| D_{x}^{\frac{1}{2}} v \right\|_{L_{x}^{2}} \left\| D_{x}^{\frac{1}{2}} (D_{x} u_{2} D_{x} u_{2}) \right\|_{L_{x}^{2}} \\
\leq cN^{-1} \left\| v \right\|_{Y_{1}^{\frac{1}{2}}} \left\| u_{2} \right\|_{X_{1}^{\frac{1}{2}}} \left\| u_{3} \right\|_{X_{1}^{\frac{1}{2}}} .
$$

**Case 4:** $|\xi_{1}| \sim |\xi_{2}| \geq N$, $|\xi_{1} + \xi_{2}| \geq 2N$.

The multiplier is bounded by

$$
\frac{m(\xi_{1} + \xi_{2})}{m(\xi_{1})m(\xi_{2})} + 1 \leq c \frac{|\xi_{1}|^{1-s} |\xi_{2}|^{1-s} N^{1-s}}{N^{1-s} N^{1-s}} = c \frac{|\xi_{1}|^{1-s} |\xi_{2}|^{1-s}}{N^{1-s} |\xi_{1} + \xi_{2}|^{1-s}}
$$

and the integral by

$$
\frac{c}{N} \int_{\delta}^{1} \int_{*}^{*} |\xi_{1} + \xi_{2}| |\xi_{1}|^{1-s} |\xi_{1}|^{1-s} \tilde{v}(\xi_{1}, t) |\xi_{2}|^{1-s} |\xi_{3}| |\xi_{3}| d\xi dt \\
\leq \frac{c}{N} \int_{0}^{\delta} \int_{0}^{\delta} |\xi_{1}|^{\frac{1}{2}} |\xi_{2} + \xi_{3}|^{\frac{1}{2}} |\xi_{2}|^{\frac{1}{2}} |\xi_{3}| |\xi_{3}| |\xi_{3}| d\xi dt \\
\leq cN^{-1} \left\| D_{x}^{\frac{1}{2}} v \right\|_{L_{x}^{2}} \left\| D_{x}^{\frac{1}{2}} (D_{x} u_{2} D_{x} u_{2}) \right\|_{L_{x}^{2}} \\
\leq cN^{-1} \left\| v \right\|_{Y_{1}^{\frac{1}{2}}} \left\| u_{2} \right\|_{X_{1}^{\frac{1}{2}}} \left\| u_{3} \right\|_{X_{1}^{\frac{1}{2}}} 
$$

using $|\xi_{1} + \xi_{2}| |\xi_{1}|^{1-s} \leq c|\xi_{1}| = c|\xi_{1}|^{\frac{1}{2}}|\xi + \xi_{3}|^{\frac{1}{2}}$ and $|\xi_{2}|^{1-s} \leq |\xi_{2}| N^{1-s}$.

**Estimate of I2:** We have to show

$$
\int_{0}^{\delta} \int_{*}^{*} \frac{m(\xi_{1} + \xi_{2}) - m(\xi_{1})m(\xi_{2})}{m(\xi_{1})m(\xi_{2})} |\xi_{1} + \xi_{2}| \tilde{u}_{1}(\xi_{1}, t) \tilde{u}_{2}(\xi_{2}, t) \tilde{u}_{3}(\xi_{3}, t) \tilde{u}_{4}(\xi_{4}, t) d\xi dt \\
\leq cN^{-1} \prod_{i=1}^{4} \left\| u_{i} \right\|_{X_{1}^{\frac{1}{2}}} .
$$

The multiplier is bounded by $c|\xi_{1}| |\xi_{2}| N^{-2}$, if $|\xi_{1}|, |\xi_{2}| \geq N$, and the integral by

$$
cN^{-2} \int_{0}^{\delta} \int_{*}^{*} |\xi_{1}| \tilde{u}_{1}(\xi_{1}, t) |\xi_{2}| \tilde{u}_{2}(\xi_{2}, t) |\xi_{3}| \tilde{u}_{3}(\xi_{3}, t) \tilde{u}_{4}(\xi_{4}, t) + \tilde{u}_{3}(\xi_{3}, t) |\xi_{4}| \tilde{u}_{4}(\xi_{4}, t) d\xi dt
$$
using \(|\xi_1 + \xi_2| = |\xi_3 + \xi_4| \leq |\xi_3| + |\xi_4|\). Strichartz’ estimate gives the bound
\[
c N^{-2} \left\| D_x u_1 \right\|_{L^4_{xt}} \left\| D_x u_2 \right\|_{L^4_{xt}} \left( \left\| D_x u_3 \right\|_{L^4_{xt}} \left\| u_4 \right\|_{L^4_{xt}} + \left\| u_3 \right\|_{L^4_{xt}} \left\| D_x u_4 \right\|_{L^4_{xt}} \right)
\leq c N^{-2} \prod_{i=1}^4 \left\| u_i \right\|_{X^1, \frac{4}{4} +}.
\]

If however \(|\xi_1| \geq N\), \(|\xi_2| \leq N\), the multiplier bound \(c|\xi_1|N^{-1}\) similarly gives the bound \(c N^{-1} \prod_i^4 \left\| u_i \right\|_{X^1, \frac{4}{4} +}\).

**Estimate of \(I_3\):** It suffices to show
\[
\int_0^\delta \int_s m(\xi_1 + \xi_2) - m(\xi_1) m(\xi_2) \frac{\nu_1(\xi_1, t) \nu_2(\xi_2, t) \nu_3(\xi_3, t) \nu_4(\xi_4, t)}{m(\xi_1) m(\xi_2)} d\xi dt
\leq c N^{-1} \left\| v_1 \right\|_{Y^1, \frac{4}{4} +} \left\| u_2 \right\|_{X^1, \frac{4}{4} +} \left\| v_3 \right\|_{Y^1, \frac{4}{4} +} \left\| u_4 \right\|_{X^1, \frac{4}{4} +}.
\]

**Case 1:** \(|\xi_1| \geq N\), \(|\xi_2| \geq N\).

The multiplier bound
\[
c \left( \frac{|\xi_1|}{N} \right)^\frac{\alpha}{4} \left( \frac{|\xi_2|}{N} \right)^\frac{\alpha}{4} = c \frac{|\xi_1|^\frac{\alpha}{4} |\xi_2|^{\frac{\alpha}{4}} |\xi_3| |\xi_4|^{\frac{\alpha}{4}}}{N^\frac{\alpha}{4} N^\frac{\alpha}{4}}
\]
allows to estimate the integral by
\[
c N^{-\frac{\alpha}{4}} \int_0^\delta \int_s |\xi_1|^{\frac{\alpha}{4}} \nu_1(\xi_1, t) \langle \xi_2 \rangle^{\frac{\alpha}{4}} \nu_2(\xi_2, t) \langle \xi_3 \rangle^{\frac{\alpha}{4}} \nu_3(\xi_3, t) \langle \xi_4 \rangle^{\frac{\alpha}{4}} \nu_4(\xi_4, t) d\xi dt.
\]

Using Hölder’s inequality with exponent 4 and Strichartz’ estimate easily gives the desired bound.

**Case 2:** \(|\xi_1| \geq N\), \(|\xi_2| \leq N\) (or similarly \(|\xi_1| \leq N\), \(|\xi_2| \geq N\)).

The multiplier bound \(c|\xi_1|N^{-1} \leq c|\xi_1|^\frac{\alpha}{4} |\xi_2| + |\xi_3| |\xi_4|^\frac{\alpha}{4} N^{-1}\) allows to estimate the integral by
\[
c N^{-1} \int_0^\delta \int_s |\xi_1|^{\frac{\alpha}{4}} \nu_1(\xi_1, t) \langle \xi_2 \rangle^{\frac{\alpha}{4}} \nu_2(\xi_2, t) \langle \xi_3 \rangle^{\frac{\alpha}{4}} \nu_3(\xi_3, t) \langle \xi_4 \rangle^{\frac{\alpha}{4}} \nu_4(\xi_4, t) d\xi dt.
\]

Similarly as in Case 1 this gives the desired estimate.

Concerning the estimate for \(L\) we remark that the first term on the right hand side of \(21\) can be handled like \(I_1\) and the second term like \(I_4\) (with one derivative less). This completes the proof.

### 4 Global existence

One easily checks
\[
\|I_N u\|_{H^1} \leq c N^{1-s} \|u\|_{H^s}
\]
and also for \(0 < s \leq \frac{1}{2}\):
\[
\|I_N u\|_{L^2} \leq c N^{\frac{1}{2} - s} \|u\|_{H^s - \frac{1}{2}}.
\]

Trivially one has
\[
\|I_N u\|_{L^2} \leq c \|u\|_{L^2}.
\]
and also
\[ \|I_Nu\|_{L^4} \leq c\|u\|_{L^4} \]
by Mikhlin’s multiplier theorem. This implies immediately for \( 1 > s \geq \frac{1}{2} \):
\[
|E(I_Nu, I_Nv)| \leq c(\|I_Nu\|_{Iv}^2 + \|D_x^{\frac{1}{2}}I_Nv\|_{L^4}^2 + \|I_Nv\|_{L^2}2\|I_Nu\|_{H^s}^2)
\]
\[
\leq c\left[N^{2(1-s)}(\|u\|_{H^s}^2 + \|v\|_{H^{s-\frac{1}{2}}}^2) + \|v\|_{L^2}2\|u\|_{L^4}^2\right]
\]
and
\[
|L(I_Nu, I_Nv)| \leq c(\|I_Nv\|_{Iv}^2 + \|I_Nu\|\|I_Nu_{x}\|) \\
\leq c(\|v\|_{Iv}^2 + \|u\|N^{1-s}\|u\|_{H^s}).
\]
Similarly, for \( 0 < s \leq \frac{1}{2} \) we get
\[
|E(I_Nu, I_Nv)| \leq c\left[N^{2(1-s)}(\|u\|_{H^s}^2 + \|v\|_{H^{s-\frac{1}{2}}}^2) + N^{\frac{1}{2}-s}\|v\|_{H^{s-\frac{1}{2}}}2\|u\|_{L^4}^2\right]
\]
and
\[
|L(I_Nu, I_Nv)| \leq c(N^{2(\frac{1}{2}-s)}\|v\|_{H^{s-\frac{1}{2}}}^2 + \|u\|N^{1-s}\|u\|_{H^s}).
\]
We note that our system has a scaling invariance, i.e. if \((u, v)\) is a solution, then also
\[
u^{(\lambda)}(x, t) := \lambda^{-\frac{3}{2}}u\left(x, \frac{t}{\lambda^2}\right), \quad v^{(\lambda)}(x, t) := \lambda^{-2}v\left(x, \frac{t}{\lambda^2}\right)
\]
for \( \lambda > 0 \), as one easily checks. Then
\[
\|u_0^{(\lambda)}\|_{H^s} = \lambda^{-\frac{3}{2}}\|u_0\|_{H^s} = c\lambda^{-(s+1)}\|u_0\|_{H^s}
\]
and
\[
\|v_0^{(\lambda)}\|_{H^{s-\frac{1}{2}}} = \lambda^{-2}\|v_0\|_{H^{s-\frac{1}{2}}} = c\lambda^{-(s+1)}\|v_0\|_{H^{s-\frac{1}{2}}}
\]
(for \( s \geq \frac{1}{2} \))
as well as
\[
\|u_0^{(\lambda)}\|_{L^4} = c\lambda^{-\frac{3}{2}}\|u_0\|_{L^4}, \quad \|v_0^{(\lambda)}\|_{L^2} = c\lambda^{-\frac{3}{2}}\|v_0\|_{L^2}, \quad \|u_0^{(\lambda)}\|_{L^2} = c\lambda^{-1}\|u_0\|_{L^2}.
\]
We also need

**Lemma 4.1** For \( s \leq \frac{1}{2} \) and \( \lambda \geq 1 \) the following estimate holds:
\[
\|v_0^{(\lambda)}\|_{H^{s-\frac{1}{2}}} \leq c\lambda^{-(s+1)}\|v_0\|_{H^{s-\frac{1}{2}}}.
\]

**Proof:**
\[
\|v_0^{(\lambda)}\|_{H^{s-\frac{1}{2}}} = \lambda^{-2}\|v_0\|_{H^{s-\frac{1}{2}}} = \lambda^{-2}\|\xi^{s-\frac{1}{2}}\hat{v}_0(\xi)\|_{L^2}
\]
\[
= \lambda^{-1}\|\xi^{s-\frac{1}{2}}\hat{v}_0(\lambda\xi)\|_{L^2} = \lambda^{-\frac{3}{2}}\left(\int \left|\left(\frac{\xi}{\lambda}\right)^{s-\frac{1}{2}}\hat{v}_0(\eta)\right|^2 d\eta\right)^{\frac{1}{2}}
\]
\[
\leq c\lambda^{-\frac{3}{2}}\left[\left(\int_{|\eta| \leq 1} |\hat{v}_0(\eta)|^2 d\eta\right)^{\frac{1}{2}} + \left(\int_{|\eta| \geq 1} \left|\xi^{s-\frac{1}{2}}\hat{v}_0(\eta)\right|^2 d\eta\right)^{\frac{1}{2}}\right]
\]
\[
\leq c\lambda^{-\frac{3}{2}}(1 + \lambda^{-(s-\frac{1}{2})})\|v_0\|_{H^{s-\frac{1}{2}}} \leq c\lambda^{-(s+1)}\|v_0\|_{H^{s-\frac{1}{2}}}.
\]
This implies the following bounds for the modified functionals $E$ and $L$ for $\lambda \geq 1$:

a) In the case $1 \geq s \geq \frac{1}{2}$ we get

$$|E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \leq c \left( N^{2(1-s)} (\|u_0^{(\lambda)}\|_{H^s}^2 + \|v_0^{(\lambda)}\|_{H^{s-\frac{1}{2}}}^2) + \|v_0^{(\lambda)}\|_{L^2}^2 \right)$$

$$\leq c \left( N^{2(1-s)} \lambda^{-2(s+1)} (\|u_0\|_{H^s} + \|v_0\|_{H^{s-\frac{1}{2}}})^2 + \lambda^{-4} \|u_0\|_{L^4}^2 \right)$$

Thus

$$|E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \leq c_0^2 N^{2(1-s)} \lambda^{-2(s+1)} (1 + \|u_0\|_{H^s} + \|v_0\|_{H^{s-\frac{1}{2}}})^4.$$
because \( \| I_N u_0^{(\lambda)} \|_{L^2} \leq \| u_0^{(\lambda)} \|_{L^2} = \lambda^{-1} \| u_0 \|_{L^2} \leq \| u_0 \|_{L^2} \) for \( \lambda \geq 1 \). Thus the local existence theorem Proposition 2.3 gives a solution on a time interval of length \( \delta = \delta(\| u_0 \|_{L^2}) \) and
\[
\| I_N u^{(\lambda)} \|_{X^1_{\delta, b}} + \| I_N v^{(\lambda)} \|_{Y^1_{\delta, b}} \leq \tilde{c} \delta (2 + \| u_0 \|_{L^2})^{\frac{1}{2}}.
\] (26)

Thus Proposition 3.1 shows
\[
| E(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta)) | + | L(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta)) |
\leq C N^{-1} + | E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)}) | + | L(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)}) |
\]
where \( C = C(\| u_0 \|_{L^2}) \). We choose \( N \) large enough, so that
\[
| E(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta)) | + | L(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta)) | < 1
\]
and such that we can reapply the local existence theorem with time intervals of equal length (remark that \( \| u_0 \|_{L^2} \) is conserved) \( N^{1-s} \) times before the size of \( | E(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta)) | + | L(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta)) | \) reaches 1. During the whole iteration process the bounds for the iterated solutions on the right hand side of (25) and (26) can be chosen uniformly. Now, given any finite time \( T \) we are able to get a solution in this way on \([0, T]\), provided \( N^{1-s} \lambda^{-2} = T \), taking the scaling into account. Using the definition of \( \lambda \) above, this means that \( N^{1-s} \lambda^{-2} \frac{2(1-s)}{1+s} = T \). This can be fulfilled for a sufficiently large \( N \), provided \( 1 > \frac{2(1-s)}{1+s} \iff 1 + s > 2(1 - \frac{1}{s}) \iff s > \frac{1}{3} \).

Thus we have proven the following global existence result:

**Theorem 4.1** For \( 1 > s > \frac{1}{3} \) and \( (u_0, v_0) \in H^s \times H^{s-\frac{1}{2}} \) there exists \( b > \frac{1}{2} \) such that the Cauchy problem (1), (2), (3) has a unique global solution \( (u, v) \in X^{s,b} \times Y^{s-\frac{1}{2}, b} \) with \( (u, v) \in C^0_{\text{loc}}(\mathbb{R}^+, H^s \times H^{s-\frac{1}{2}}) \).

It is also possible to show that this global solution grows at most polynomially in \( t \). The procedure above namely shows
\[
| E(I_N u^{(\lambda)}(N^{1-s}, \delta), I_N v^{(\lambda)}(N^{1-s}, \delta)) | + | E(I_N u^{(\lambda)}(N^{1-s}, \delta), I_N v^{(\lambda)}(N^{1-s}, \delta)) | \leq 1.
\]
This implies by (22):
\[
\| I_N u^{(\lambda)}(N^{1-s}, \delta) \|_{H^1} + \| I_N v^{(\lambda)}(N^{1-s}, \delta) \|_{H^{1}^{1/2}} \leq c(1 + \| I_N u^{(\lambda)}(N^{1-s}, \delta) \|_{L^2}^4)
\]
\[
\leq c(1 + \| u^{(\lambda)}(N^{1-s}) \|_{L^2}^4) \leq c(1 + \| u_0^{(\lambda)} \|_{L^2}^4)
\]
\[
\leq c(1 + \lambda^{-4} \| u_0 \|_{L^2}^4) \leq c(1 + \| u_0 \|_{L^2}^4)
\]
for \( \lambda \geq 1 \). Thus we get
\[
\| u^{(\lambda)}(N^{1-s}) \|_{H^s} + \| v^{(\lambda)}(N^{1-s}) \|_{H^{s-\frac{1}{2}}} \leq c.
\]
But now
\[
\| u^{(\lambda)}(N^{1-s}) \|_{H^s} = \lambda^{-\frac{3}{4}} \| u(\frac{x}{\lambda}, T) \|_{H^s} \geq c \lambda^{-(1+s)} \| u(T) \|_{H^s}
\]
and similarly for $s \geq \frac{1}{2}$:

$$\|v^{(\lambda)}(N^{1-\delta})\|_{H^{s+\frac{1}{2}}} = \lambda^{-2}\|v^{(\lambda)}(\frac{x}{\lambda}, T)\|_{H^{s+\frac{1}{2}}} \geq c\lambda^{-(1+s)}\|v(T)\|_{H^{s+\frac{1}{2}}},$$

whereas for $s < \frac{1}{2}$:

$$\|v^{(\lambda)}(N^{1-\delta})\|_{H^{s+\frac{1}{2}}} \geq c\lambda^{-\frac{3}{2}}\|v(T)\|_{H^{s+\frac{1}{2}}}.$$

Because $\lambda \sim N^{\frac{1-s}{1+s}}$ and $T \sim N^{1-\lambda^{-2}} \sim N^{1-N^{-\frac{2(1-s)}{1+s}}} = N^{\frac{3s-1}{s+1}}$, thus $N \sim T^{\frac{1+s}{3s-1}}$, we get $\lambda \sim T^{\frac{1+s}{3s-1}}$, so that we have proven

**Theorem 4.2** The global solution of Theorem 4.1 fulfills for $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq c(1 + t^{\frac{(s+1)(1-s)}{3s-1}}) \quad \text{for} \quad 1 > s > \frac{1}{3}$$

and

$$\|v(t)\|_{H^{s+\frac{1}{2}}} \leq c(1 + t^{\frac{(s+1)(1-s)}{3s-1}}) \quad \text{for} \quad 1 > s \geq \frac{1}{2},$$

$$\|v(t)\|_{H^{s+\frac{1}{2}}} \leq c(1 + t^{\frac{3(1-s)}{3s-1}}) \quad \text{for} \quad \frac{1}{2} > s > \frac{1}{3}.$$
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