SANDWICH CLASSIFICATION FOR $GL_n(R)$, $O_{2n}(R)$ AND $U_{2n}(R,\Lambda)$ REVISITED

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ABSTRACT. Let $n$ be a natural number greater or equal to 3, $R$ a commutative ring and $\sigma \in GL_n(R)$. We show that $t_{kl}(\sigma_{ij})$ (resp. $t_{kl}(\sigma_{ij} - \sigma_{jj})$) where $i \neq j$ and $k \neq l$ can be expressed as a product of 8 (resp. 24) matrices of the form $\epsilon \sigma^\pm 1$ where $\epsilon \in E_n(R)$. We prove similar results for the orthogonal groups $O_{2n}(R)$ and the hyperbolic unitary groups $U_{2n}(R,\Lambda)$ under the assumption that $R$ is commutative and $n \geq 3$. This yields new, very short proofs of the Sandwich Classification Theorems for the groups $GL_n(R)$, $O_{2n}(R)$ and $U_{2n}(R,\Lambda)$.

1. Introduction

Let $n$ be a natural number greater or equal to 3 and $R$ a commutative ring. Let $\sigma \in GL_n(R)$ and set $H := E_n(R)\sigma$, i.e. $H$ is the smallest subgroup of $GL_n(R)$ which contains $\sigma$ and is normalized by $E_n(R)$. Let $I$ be the ideal of $R$ defined by $I := \{x \in R \mid t_{12}(x) \in H\}$. Then clearly $E_n(R,I) \subseteq H$. By the Sandwich Classification Theorem (SCT) for $GL_n(R)$ one also has $H \subseteq C_n(R,I)$. It follows that $\sigma_{ij}, \sigma_{ii} - \sigma_{jj} \in I$ for any $i \neq j$, i.e. the matrices $t_{12}(\sigma_{ij})$ and $t_{12}(\sigma_{ii} - \sigma_{jj})$ can be expressed as products of matrices of the form $\epsilon \sigma^\pm 1$ where $\epsilon \in E_n(R)$. We show how one can use the theme of the paper [4] in order to find such expressions and give boundaries for the number of factors, see Theorem 12. This yields a new, very simple proof of the SCT for $GL_n(R)$.

Further we prove an orthogonal and a unitary version of Theorem 12 (cf. Theorem 27 and Theorem 49). The proof of the orthogonal version is very simple. The proof of the unitary version is a bit more complicated, but still it is much shorter than the proof of the SCT for the groups $U_{2n}(R,\Lambda)$ given in [3] (on the other hand, in [3] the ring $R$ is only assumed to be quasi-finite and hence the result is a bit more general). For the hyperbolic unitary groups $U_{2n}(R,\Lambda)$ this yields the first proof of the SCT which does not use localization.

This paper is organized as follows. In Section 2 we recall some standard notation which will be used throughout the paper. In Section 3 we state two lemmas which will be used in the proofs of the main theorems 12, 27 and 49. In Section 4 we recall the definitions of the general linear group $GL_n(R)$ and some important subgroups, in Section 5 we prove Theorem 12. In Section 6 we recall the definitions of the (even-dimensional) orthogonal group $O_{2n}(R)$ and some important subgroups, in Section 7 we prove Theorem 27. In Section 8 we recall the definitions of A. Bak’s hyperbolic unitary group $U_{2n}(R,\Lambda)$ and some important subgroups and in the last section we prove Theorem 49.

2. Notation

By a natural number we mean an element of the set $\mathbb{N} := \{1, 2, 3, \ldots\}$. If $G$ is a group and $g, h \in G$, we let $hg := hgh^{-1}$ and $[g, h] := ghg^{-1}h^{-1}$. By a ring we will always mean an associative ring with 1 such that $1 \neq 0$. Ideal will mean two-sided ideal. If $X$ is a subset of a ring $R$, then we denote by $I(X)$ the ideal of $R$ generated by $X$. If $X = \{x\}$, then we may write $I(x)$ instead of $I(X)$. The set of all invertible elements in a ring $R$ is denoted by $R^*$. If $m$ and $n$ are natural numbers and $R$ is a ring, then the set of all $m \times n$ matrices with entries in $R$ is denoted by $M_{m \times n}(R)$. If $a \in M_{m \times n}(R)$, we denote the transpose of $a$ by $a^t$ and the entry of $a$ at position $(i,j)$ by $a_{ij}$. We denote the $i$-th row of $a$ by $a_{i\ast}$ and its $j$-th
column by $a_{ij}$. We set $M_n(R) := M_{n \times n}(R)$. The identity matrix in $M_n(R)$ is denoted by $e$ or $e^{n \times n}$ and the matrix with a 1 at position $(i,j)$ and zeros elsewhere is denoted by $e_{ij}$. If $a \in M_n(R)$ is invertible, the entry of $a^{-1}$ at position $(i,j)$ is denoted by $a'_{ij}$, the $i$-th row of $a^{-1}$ by $a'_i$, and the $j$-th column of $a^{-1}$ by $a'_j$. Further we denote by $R$ the set of all rows $v = (v_1, \ldots, v_n)$ with entries in $R$ and by $R^n$ the set of all columns $u = (u_1, \ldots, u_n)$ with entries in $R$. We consider $R$ as left $R$-module and $R^n$ as right $R$-module.

3. Preliminaries

The following two lemmas are easy to check.

**Lemma 1.** Let $G$ be a group and $a, b, c \in G$. Then $b^{-1}[a, bc] = [b^{-1}, a][a, c]$.

**Lemma 2.** Let $G$ be a group, $E$ a subgroup and $a \in G$. Suppose that $b \in G$ is a product of $n$ elements of the form $e^{a^\pm 1}$ where $e \in E$. Then

(i) $e'b$ is a product of $n$ elements of the form $e^{a^\pm 1}$

(ii) $[e', b]$ is a product of $2n$ elements of the form $e^{a^\pm 1}$

for any $e' \in E$.

Lemma 2 will be used in the proofs of the main theorems without explicit reference.

4. The general linear group $GL_n(R)$

In this section $n$ denotes a natural number, $R$ a ring and $I$ an ideal of $R$. We shall recall the definitions of the general linear group $GL_n(R)$ and the following subgroups of $GL_n(R)$: the elementary subgroup $E_n(R)$, the preelementary subgroup $E_n(I)$ of level $I$, the elementary subgroup $E_n(R, I)$ of level $I$, the principal congruence subgroup $GL_n(R, I)$ of level $I$ and the full congruence subgroup $C_n(R, I)$ of level $I$.

4.1. The general linear group.

**Definition 3.** $GL_n(R) := (M_n(R))^*$ is called general linear group.

4.2. The elementary subgroup.

**Definition 4.** Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$ and $x \in R$. Then $t_{ij}(x) := e + xe^{ij}$ is called an elementary transvection. The subgroup of $GL_n(R)$ generated by all elementary transvections is called elementary subgroup and is denoted by $E_n(R)$. An elementary transvection $t_{ij}(x)$ is called $I$-elementary if $x \in I$. The subgroup of $GL_n(R)$ generated by all $I$-elementary transvections is called preelementary subgroup of level $I$ and is denoted by $E_n(I)$. Its normal closure in $E_n(R)$ is called elementary subgroup of level $I$ and is denoted by $E_n(R, I)$.

**Lemma 5.** The relations

$$t_{ij}(x)t_{ij}(y) = t_{ij}(x + y), \quad (R1)$$

$$[t_{ij}(x), t_{ik}(y)] = e \text{ and } (R2)$$

$$[t_{ij}(x), t_{jk}(y)] = t_{ik}(xy) \quad (R3)$$

hold where $i \neq k, j \neq h$ in (R2) and $i \neq k$ in (R3).

**Proof.** Straightforward computation. □

**Definition 6.** Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. Define $p_{ij} := e + e^{ij} - e^{ji} - e^{ii} - e^{jj} = t_{ij}(1)t_{ji}(-1)t_{ij}(1) \in E_n(R)$. It is easy show that $p_{ij}^{-1} = p_{ji}$.

**Lemma 7.** Let $x \in R$ and $i, j, k \in \{1, \ldots, n\}$ be pairwise distinct indices. Then
(i) $p_{ij}t_{ij}(x) = t_{kj}(x)$ and 
(ii) $p_{ij}t_{ij}(x) = t_{ik}(x)$.

**Proof.** Follows from the relations in Lemma 5.

### 4.3. Congruence subgroups.

**Definition 8.** The kernel of the group homomorphism $GL_n(R) \rightarrow GL_n(R/I)$ induced by the canonical map $R \rightarrow R/I$ is called the principal congruence subgroup of level $I$ and is denoted by $GL_n(R, I)$. Obviously $GL_n(R, I)$ is a normal subgroup of $GL_n(R)$.

**Definition 9.** The preimage of $Center(GL_n(R/I))$ under the group homomorphism $GL_n(R) \rightarrow GL_n(R/I)$ induced by the canonical map $R \rightarrow R/I$ is called the full congruence subgroup of level $I$ and is denoted by $C_n(R, I)$. Obviously $GL_n(R, I) \subseteq C_n(R, I)$ and $C_n(R, I)$ is a normal subgroup of $GL_n(R)$.

**Theorem 10.** If $n \geq 3$ and $R$ is almost commutative (i.e., module finite over its center), then the equalities 

\[ [C_n(R, I), E_n(R)] = [E_n(R, I), E_n(R)] = E_n(R, I) \]

hold.

**Proof.** See [5], Corollary 14.

### 5. Sandwich classification for $GL_n(R)$

In this section $n$ denotes a natural number greater or equal to 3 and $R$ a commutative ring.

**Definition 11.** Let $\sigma \in GL_n(R)$. Then a matrix of the form $^\epsilon \sigma^{\pm 1}$ where $\epsilon \in E_n(R)$ is called an elementary $\sigma$-conjugate.

**Theorem 12.** Let $\sigma \in GL_n(R)$, $i \neq j$ and $k \neq l$. Then

(i) $t_{kl}(\sigma_{ij})$ is a product of 8 elementary $\sigma$-conjugates and
(ii) $t_{kl}(\sigma_{ii} - \sigma_{jj})$ is a product of 24 elementary $\sigma$-conjugates.

**Proof.** (i) Set $\tau := t_{21}(-\sigma_{23})t_{31}(\sigma_{22})$. One checks easily that the second row of \( \sigma \tau^{-1} \) equals the second row of $\sigma$ and hence the second row of $\xi := \sigma \tau^{-1}$ is trivial. Set

\[ \zeta := \tau^{-1}[t_{32}(1), \tau, \sigma] = \tau^{-1}[t_{32}(1), \tau \xi] = \tau^{-1}[t_{32}(1), \xi]. \]

One checks easily that $[\tau^{-1}, t_{32}(1)] = t_{31}(-\sigma_{23})$ and $[t_{32}(1), \xi] = \prod_{i \neq 2} t_{22}(x_i)$ for some $x_1, x_3, x_4, \ldots, x_n \in R$.

Hence $\zeta = t_{31}(-\sigma_{23}) \prod_{i \neq 2} t_{22}(x_i)$. It follows that $[t_{12}(1), \zeta] = t_{32}(\sigma_{23})$. Hence we have shown

\[ [t_{12}(1), t_{21}(-\sigma_{23})t_{31}(-\sigma_{22})[t_{32}(1), [t_{21}(-\sigma_{23})t_{31}(\sigma_{22}), \sigma]]] = t_{32}(\sigma_{23}). \]

This implies that $t_{32}(\sigma_{23})$ is a product of 8 elementary $\sigma$-conjugates. It follows from Lemma 7 that $t_{kl}(\sigma_{23})$ is a product of 8 elementary $\sigma$-conjugates. Since one can bring $\sigma_{ij}$ to position $(2, 3)$ by conjugating monomial matrices in $E_n(R)$ (see Definition 6) to $\sigma$, the assertion of (i) follows.

(ii) Clearly the entry of $t_{ij}(1)\sigma$ at position $(j, i)$ equals $\sigma_{ii} - \sigma_{jj} + \sigma_{ji} - \sigma_{ij}$. Applying (i) to $t_{ij}(1)\sigma$ we get that $t_{kl}(\sigma_{ii} - \sigma_{jj} + \sigma_{ji} - \sigma_{ij})$ is a product of 8 elementary $\sigma$-conjugates (note that any elementary $t_{ij}(1)\sigma$-conjugate is also an elementary $\sigma$-conjugate). Applying (i) to $\sigma$ we get that $t_{kl}(\sigma_{ij} - \sigma_{ji}) = t_{kl}(\sigma_{ij})t_{kl}(-\sigma_{ji})$ is a product of 16 elementary $\sigma$-conjugates. It follows that $t_{kl}(\sigma_{ii} - \sigma_{jj}) = t_{kl}(\sigma_{ii} - \sigma_{jj} + \sigma_{ji} - \sigma_{ij})t_{kl}(\sigma_{ij} - \sigma_{ji})$ is a product of 24 elementary $\sigma$-conjugates.

As a corollary we get the Sandwich Classification Theorem for $GL_n(R)$. Note that if $\sigma \in GL_n(R)$ and $I$ is an ideal of $R$, then $\sigma \in C_n(R, I)$ if and only if $\sigma_{ij}, \sigma_{ii} - \sigma_{jj} \in I$ for any $i \neq j$. 

Corollary 13. Let $H$ be a subgroup of $GL_n(R)$. Then $H$ is normalized by $E_n(R)$ if and only if
\[ E_n(R, I) \subseteq H \subseteq C_n(R, I) \] for some ideal $I$ of $R$.

Proof. First suppose that $H$ is normalized by $E_n(R)$. Let $I$ be the ideal of $R$ defined by $I := \{ x \in R \mid t_{12}(x) \in I \}$. Then clearly $E_n(R, I) \subseteq H$. It remains to show that $H \subseteq C_n(R, I)$, i.e. that if $\sigma \in H$, then $\sigma_{ij} - \sigma_{ji} \in I$ for any $i \neq j$. But that follows from the previous theorem. Suppose now that (1) holds for some ideal $I$. Then it follows from the standard commutator formulas in Theorem 10 that $H$ is normalized by $E_n(R)$. □

6. The even-dimensional orthogonal group $O_{2n}(R)$

In this section $n$ denotes a natural number, $R$ a commutative ring and $I$ an ideal of $R$. We shall recall the definitions of the even-dimensional orthogonal group $O_{2n}(R)$ and the following subgroups of $O_{2n}(R)$: the elementary subgroup $EO_{2n}(R)$, the preelementary subgroup $EO_{2n}(I)$ of level $I$, the elementary subgroup $EO_{2n}(R, I)$ of level $I$, the principal congruence subgroup $O_{2n}(R, I)$ of level $I$, and the full congruence subgroup $CO_{2n}(R, I)$ of level $I$.

6.1. The even-dimensional orthogonal group.

Definition 14. Set $V := R^{2n}$. We use the following indexing for the elements of the standard basis of $V$: $(e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1})$. That means that $e_i$ is the column whose $i$-th coordinate is one and all the other coordinates are zero if $1 \leq i \leq n$ and the column whose $(2n + 1 + i)$-th coordinate is one and all the other coordinates are zero if $-n \leq i \leq -1$. Let $p \in M_n(R)$ be the matrix with ones on the skew diagonal and zeros elsewhere. We define the quadratic form
\[ q : V \to R \]
\[ v \mapsto v^t \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} v. \]

The subgroup $O_{2n}(R) := \{ \sigma \in GL_{2n}(R) \mid q(\sigma v) = q(v) \ \forall v \in V \}$ of $GL_{2n}(R)$ is called (even-dimensional) orthogonal group.

Remark 15. The even-dimensional orthogonal groups are special cases of the hyperbolic unitary groups, cf. Example 32.

Definition 16. We define $\Omega := \{1, \ldots, n, -n, \ldots, -1\}$.

Lemma 17. Let $\sigma \in GL_{2n}(R)$. Then $\sigma \in O_{2n}(R)$ if and only if
\begin{enumerate}
  \item $\sigma_{ij}' = \sigma_{-j, -i} \ \forall i, j \in \Omega$ and
  \item $q(\sigma_{ij}) = 0 \ \forall j \in \Omega$.
\end{enumerate}

Proof. See [2], p.167. □

Lemma 18. Let $\sigma \in O_{2n}(R)$, $x \in R^*$ and $k \in \Omega$. Then the statements below are true.
\begin{enumerate}
  \item If the $k$-th column of $\sigma$ equals $e_k x$ then the $(-k)$-th row of $\sigma$ equals $x^{-1} e_k'$.
  \item If the $k$-th row of $\sigma$ equals $x e_k'$ then the $(-k)$-th column of $\sigma$ equals $e_{-k} x^{-1}$.
\end{enumerate}

Proof. Follows from (i) in the previous lemma. □
6.2. The elementary subgroup.

**Definition 19.** If \( i, j \in \Omega \) such that \( i \neq \pm j \) and \( x \in R \), then the matrix 
\[
T_{ij}(x) := e + xe^{ij} - xe^{-j,-i} \in O_{2n}(R)
\]
is called an *elementary orthogonal transvection*. The subgroup of \( O_{2n}(R) \) generated by all elementary orthogonal transvections is called *elementary orthogonal group* and is denoted by \( EO_{2n}(R) \). An elementary orthogonal transvection \( T_{ij}(x) \) is called \( I \)-elementary if \( x \in I \). The subgroup of \( O_{2n}(R) \) generated by all \( I \)-elementary orthogonal transvections is called *preelementary subgroup of level \( I \)* and is denoted by \( EO_{2n}(R, I) \). Its normal closure in \( EO_{2n}(R) \) is called *elementary subgroup of level \( I \)* and is denoted by \( EO_{2n}(R, I) \).

**Lemma 20.** The relations 
\[
T_{ij}(x) = T_{-j,-i}(-x), \quad (R1)
\]
\[
T_{ij}(x)T_{ij}(y) = T_{ij}(x + y), \quad (R2)
\]
\[
[T_{ij}(x), T_{hk}(y)] = e, \quad (R3)
\]
\[
[T_{ij}(x), T_{jk}(y)] = T_{ik}(xy), \quad (R4)
\]
\[
[T_{ij}(x), T_{j,-i}(y)] = e \quad (R5)
\]
hold where \( h \neq j, -i \) and \( k \neq i, -j \) in (R3) and \( i \neq \pm k \) in (R4).

*Proof.* Straightforward calculation. \qed

**Definition 21.** Let \( i, j \in \Omega \) such that \( i \neq \pm j \). Define \( P_{ij} := e + e^{ij} - e^{ji} - e^{-i,-j} - e^{-j,i} - e^{ij} - e^{-i,-i} - e^{-j,-j} = T_{ij}(1)T_{ji}(-1)T_{ji}(1) \in EO_{2n}(R) \). It is easy to show that \((P_{ij})^{-1} = P_{ji} \).

**Lemma 22.** Let \( x \in R \) and \( i, j, k \in \Omega \) such that \( i \neq \pm j \) and \( k \neq \pm i, \pm j \). Then
\[
(i) \quad P_{ki}T_{ij}(x) = T_{kj}(x) \text{ and } \quad (ii) \quad P_{kj}T_{ij}(x) = T_{ik}(x).
\]

*Proof.* Follows from the relations in Lemma 20. \qed

6.3. Congruence subgroups.

**Definition 23.** The kernel of the group homomorphism \( O_{2n}(R) \rightarrow O_{2n}(R/I) \) induced by the canonical map \( R \rightarrow R/I \) is called *principal congruence subgroup of level \( I \)* and is denoted by \( O_{2n}(R, I) \). Obviously \( O_{2n}(R, I) \) is a normal subgroup of \( O_{2n}(R) \).

**Definition 24.** The preimage of \( Center(O_{2n}(R/I)) \) under the group homomorphism \( O_{2n}(R) \rightarrow O_{2n}(R/I) \) induced by the canonical map \( R \rightarrow R/I \) is called *full congruence subgroup of level \( I \)* and is denoted by \( CO_{2n}(R, I) \). Obviously \( O_{2n}(R, I) \subseteq CO_{2n}(R, I) \) and \( CO_{2n}(R, I) \) is a normal subgroup of \( O_{2n}(R) \).

**Theorem 25.** If \( n \geq 3 \), then the equalities
\[
[CO_{2n}(R, I), EO_{2n}(R)] = [EO_{2n}(R, I), EO_{2n}(R)] = EO_{2n}(R, I)
\]
hold.

*Proof.* See [2], Theorem 1.1 and Lemma 5.2. \qed

7. Sandwich classification for \( O_{2n}(R) \)

In this section \( n \) denotes a natural number greater or equal to 3 and \( R \) a commutative ring.

**Definition 26.** Let \( \sigma \in O_{2n}(R) \). Then a matrix of the form \( ^{c} \sigma \epsilon \) where \( \epsilon \in EO_{2n}(R) \) is called an *elementary (orthogonal) \( \sigma \)-conjugate*.

**Theorem 27.** Let \( \sigma \in O_{2n}(R), i \neq \pm j \) and \( k \neq \pm l \). Then
(i) $T_{kl}(s_{ij})$ is a product of 8 elementary orthogonal $\sigma$-conjugates,
(ii) $T_{kl}(s_{i,-i})$ is a product of 16 elementary orthogonal $\sigma$-conjugates,
(iii) $T_{kl}(s_{ii} - s_{jj})$ is a product of 24 elementary orthogonal $\sigma$-conjugates and
(iv) $T_{kl}(s_{ii} - s_{-i,-i})$ is a product of 48 elementary orthogonal $\sigma$-conjugates.

Proof. (i) Set $\tau := T_{21}(-\sigma_{23})T_{31}(-\sigma_{22})T_{2,-3}(\sigma_{2,-1})$. One checks easily that the second row of $\sigma\tau^{-1}$ equals the second row of $\sigma$ and hence the second row of $\xi := \sigma\tau^{-1}$ is trivial. By Lemma 18 the second last column of $\xi$ also is trivial. Set
$$\zeta := \tau^{-1}[T_{32}(1),[\tau,\sigma]] = \tau^{-1}[T_{32}(1),\tau\xi] = \tau^{-1}[T_{32}(1)]T_{32}(1), \zeta].$$
One checks easily that $[\tau^{-1},T_{32}(1)] = T_{31}(-\sigma_{23})$ and $T_{32}(1), \xi] = \prod_{i \neq \pm 2} T_{i2}(x_i)$ for some $x_i \in R$ ($i \neq \pm 2$).

Hence $\zeta = T_{31}(-\sigma_{23}) \prod_{i \neq \pm 2} T_{i2}(x_i)$. It follows that $[T_{12}(1), \zeta] = T_{32}(\sigma_{23})$. Hence we have shown
$$[T_{12}(1), T_{21}(\sigma_{23})T_{31}(\sigma_{22})T_{2,-3}(\sigma_{2,-1})]T_{32}(1), [T_{21}(-\sigma_{23})T_{31}(\sigma_{22})T_{2,-3}(\sigma_{2,-1}), \sigma]] = T_{32}(\sigma_{23}).$$
This implies that $T_{32}(\sigma_{23})$ is a product of 8 elementary $\sigma$-conjugates. It follows from Lemma 22 that $T_{kl}(\sigma_{23})$ is a product of 8 elementary $\sigma$-conjugates. Since one can bring $\sigma_{ij}$ to position $(2,3)$ by conjugating monomial matrices in $EO_{2n}(R)$ (see Definition 21) to $\sigma$, the assertion of (i) follows.

(ii) Clearly the entry of $T_{ji}(\sigma)$ at position $(j, i)$ equals $s_{i,-i} + s_{j,-i}$. Applying (i) to $T_{ji}(\sigma)$ we get that $T_{kl}(s_{i,-i} + s_{j,-i})$ is a product of 8 elementary $\sigma$-conjugates (note that any elementary $T_{ji}(\sigma)$-conjugate is also an elementary $\sigma$-conjugate). Applying (i) to $\sigma$ we get that $T_{kl}(s_{j,-i})$ is a product of 8 elementary $\sigma$-conjugates. It follows that $T_{kl}(s_{i,-i}) = T_{kl}(s_{i,-i} + s_{j,-i})T_{ji}(-s_{j,-i})$ is a product of 16 elementary $\sigma$-conjugates.

(iii) Clearly the entry of $T_{ji}(\sigma)$ at position $(j, i)$ equals $s_{ii} - s_{jj} + s_{ji} - s_{ij}$. Applying (i) to $T_{ji}(\sigma)$ we get that $T_{kl}(s_{ii} - s_{jj} + s_{ji} - s_{ij})$ is a product of 8 elementary $\sigma$-conjugates (note that any elementary $T_{ji}(\sigma)$-conjugate is also an elementary $\sigma$-conjugate). Applying (i) to $\sigma$ we get that $T_{kl}(s_{ij} - s_{ji}) = T_{kl}(s_{ij})T_{kl}(-s_{ji})$ is a product of 16 elementary $\sigma$-conjugates. It follows that $T_{kl}(s_{ii} - s_{jj}) = T_{kl}(s_{ii} - s_{jj} + s_{ji} - s_{ij})T_{kl}(s_{ij} - s_{ji})$ is a product of 24 elementary $\sigma$-conjugates.

(iv) Follows from (iii) since $T_{kl}(s_{ii} - s_{-i,-i}) = T_{kl}(s_{ii} - s_{jj})T_{kl}(s_{jj} - s_{-i,-i})$.

□

As a corollary we get the Sandwich Classification Theorem for $O_{2n}(R)$. Note that if $\sigma \in O_{2n}(R)$ and $I$ is an ideal of $R$, then $\sigma \in CO_{2n}(R, I)$ if and only if $s_{ij}, s_{ji} - s_{jj} \in I$ for any $i \neq j$.

Corollary 28. Let $H$ be a subgroup of $O_{2n}(R)$. Then $H$ is normalized by $EO_{2n}(R)$ if and only if
$$EO_{2n}(R, I) \subseteq H \subseteq CO_{2n}(R, I)$$
for some ideal $I$ of $R$.

Proof. First suppose that $H$ is normalized by $EO_{2n}(R)$. Let $I$ be the ideal of $R$ defined by $I := \{x \in R \mid T_{12}(x) \in H\}$. Then clearly $EO_{2n}(R, I) \subseteq H$. It remains to show that $H \subseteq CO_{2n}(R, I)$, i.e. that if $\sigma \in H$, then $s_{ij}, s_{ji} - s_{jj} \in I$ for any $i \neq j$. But that follows from the previous theorem. Suppose now that (2) holds for some ideal $I$. Then it follows from the standard commutator formulas in Theorem 25 that $H$ is normalized by $EO_{2n}(R)$.

□
8. Bak’s unitary group $U_{2n}(R,\Lambda)$

In order to classify the subgroups of a general linear group (resp. an even-dimensional orthogonal group) which are normalized by the elementary subgroup (resp. the elementary orthogonal group), the notion of an ideal of a ring is sufficient. Bak’s dissertation [1] showed that the notion of an ideal by itself was not sufficient to solve the analogous classification problem for unitary groups, but that a refinement of the notion of an ideal, called a form ideal, was necessary. This led naturally to a more general notion of unitary group, which was defined over a form ring instead of just a ring and generalized all previous concepts. We describe form rings $(R,\Lambda)$ and form ideals $(I,\Gamma)$ first, then the hyperbolic unitary group $U_{2n}(R,\Lambda)$ and its elementary subgroup $EU_{2n}(R,\Lambda)$ over a form ring $(R,\Lambda)$. For a form ideal $(I,\Gamma)$, we recall the definitions of the following subgroups of $U_{2n}(R,\Lambda)$; the preelementary subgroup $EU_{2n}(I,\Gamma)$ of level $(I,\Gamma)$, the elementary subgroup $EU_{2n}((R,\Lambda),(I,\Gamma))$ of level $(I,\Gamma)$, the principal congruence subgroup $U_{2n}((R,\Lambda),(I,\Gamma))$ of level $(I,\Gamma)$, and the full congruence subgroup $CU_{2n}((R,\Lambda),(I,\Gamma))$ of level $(I,\Gamma)$.

8.1. Form rings and form ideals.

**Definition 29.** Let $R$ be a ring and $\bar{\cdot}: R \to R$

\[ x \mapsto \bar{x} \]

an involution on $R$, i.e. $\bar{x+y} = \bar{x} + \bar{y}$, $\bar{xy} = \bar{y}\bar{x}$ and $\bar{x} = x$ for any $x, y \in R$. Let $\lambda \in center(R)$ such that $\lambda^2 = 1$ and set $\Lambda_{\min} := \{x - \lambda\bar{x} \mid x \in R\}$ and $\Lambda_{\max} := \{x \in R \mid x = \lambda\bar{x}\}$. An additive subgroup $\Lambda$ of $R$ such that

- $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$
- $x\Lambda \subseteq \Lambda \forall x \in R$

is called a form parameter for $R$. If $\Lambda$ is a form parameter for $R$, the pair $(R,\Lambda)$ is called a form ring.

**Definition 30.** Let $(R,\Lambda)$ be a form ring and $I$ an ideal such that $I = I$. Set $\Gamma_{\max} = I \cap \Lambda$ and $\Gamma_{\min} = \{x - \lambda\bar{x} \mid x \in I\} \cup \{(xy\bar{x} \mid x \in I, y \in \Lambda)\}$. An additive subgroup $\Gamma$ of $I$ such that

- $\Gamma_{\min} \subseteq \Gamma \subseteq \Gamma_{\max}$
- $x\Gamma \subseteq \Gamma \forall x \in R$

is called a relative form parameter of level $I$. If $\Gamma$ is a relative form parameter of level $I$, then $(I,\Gamma)$ is called a form ideal of $(R,\Lambda)$.

Until the end of section 8 let $n \in \mathbb{N}$, $(R,\Lambda)$ a form ring and $(I,\Gamma)$ a form ideal of $(R,\Lambda)$.

8.2. The hyperbolic unitary group.

**Definition 31.** Set $V := R^{2n}$. We use the following indexing for the elements of the standard basis of $V$:

$(e_1,\ldots, e_n, e_{-n},\ldots,e_{-1})$. That means that $e_i$ is the column whose $i$-th coordinate is one and all the other coordinates are zero if $1 \leq i \leq n$ and the column whose $(2n+1+i)$-th coordinate is one and all the other coordinates are zero if $-n \leq i \leq -1$. Let $p \in M_n(R)$ be the matrix with ones on the skew diagonal and zeros elsewhere. We define the maps

\[ f : V \times V \to R \]

\[ (v, w) \mapsto \bar{v}' \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} w, \]

\[ h : V \times V \to R \]

\[ (v, w) \mapsto \bar{v}' \begin{pmatrix} 0 & p \\ \lambda p & 0 \end{pmatrix} w, \]

\[ q : V \to R/\Lambda \]

\[ v \mapsto f(v, v) + \Lambda \]

where $\bar{v}$ is obtained from $v$ by applying $\bar{\cdot}$ to each entry of $v$. For any $v \in V$, $f(v, v)$ is called the value of $v$ and is denoted by $|v|$. The subgroup $U_{2n}(R,\Lambda) := \{\sigma \in GL_{2n}(R) \mid h(\sigma u, \sigma v) = h(u, v) \wedge q(\sigma v) = q(v) \ \forall u, v \in V\}$ of $GL_{2n}(R)$ is called hyperbolic unitary group.
Example 32. If $R$ is commutative, $- = id$, $\lambda = -1$ and $\Lambda = \Lambda_{\text{max}} = R$, then $U_{2n}(R, \Lambda)$ equals the symplectic group $Sp_{2n}(R)$. If $R$ is commutative, $- = id$, $\lambda = 1$ and $\Lambda = \Lambda_{\text{min}} = \{0\}$, then $U_{2n}(R, \Lambda)$ equals the orthogonal group $O_{2n}(R)$.

**Definition 33.** We define $\Omega_+ := \{1, \ldots, n\}$, $\Omega_- := \{-n, \ldots, -1\}$, $\Omega := \Omega_+ \cup \Omega_-$ and 
$$
\epsilon : \Omega \to \{-1, 1\} \quad i \mapsto \epsilon(i) := \begin{cases} 
1, & \text{if } i \in \Omega_+, \\
-1, & \text{if } i \in \Omega_-. 
\end{cases}
$$

Further if $i, j \in \Omega$, we write $i < j$ iff either $i, j \in \Omega_+ \land i < j$ or $i, j \in \Omega_- \land i < j$ or $i \in \Omega_+ \land j \in \Omega_-$. 

**Lemma 34.** Let $\sigma \in GL_{2n}(R)$. Then $\sigma \in U_{2n}(R, \Lambda)$ if and only if 

(i) $\sigma_{ij}' = \lambda^{(e(i)-e(j))}/2 \sigma_{-j,-i}$ $\forall i, j \in \Omega$ and 

(ii) $|\sigma_{ij}| \in \Lambda \forall j \in \Omega$.

**Proof.** See [2], p.167. \qed

**Lemma 35.** Let $\sigma \in U_{2n}(R, \Lambda)$, $x \in R^*$ and $k \in \Omega$. Then the statements below are true. 

(i) If the $k$-th column of $\sigma$ equals $e_k x$ then the $(-k)$-th row of $\sigma$ equals $x^{-1} e_{-k}'$. 

(ii) If the $k$-th row of $\sigma$ equals $xe_k'$ then the $(-k)$-th column of $\sigma$ equals $e_{-k} x^{-1}$. 

**Proof.** Follows from (i) in the previous lemma. \qed

8.3. Polarity map.

**Definition 36.** The map 
$$
\tilde{\:} : V \to 2^n R \\
v \mapsto (\lambda_1 v_1 \ldots \lambda_n v_n \ldots \lambda_{-n} v_{-1})
$$
is called polarity map. One checks easily that $h(u, v) = \tilde{u}v$ for any $u, v \in V$ and that $\tilde{\:}$ is involutory linear, i.e. $u + v = \tilde{u} + \tilde{v}$ and $\tilde{uv} = \tilde{v}\tilde{u}$ for any $u, v \in V$ and $x \in R$.

**Lemma 37.** If $\sigma \in U_{2n}(R, \Lambda)$ and $v \in V$, then $\tilde{\sigma}v = \tilde{v}\sigma^{-1}$. 

**Proof.** See [2, Lemma 2.5]. \qed

8.4. The elementary subgroup.

**Definition 38.** If $i, j \in \Omega$ such that $i \neq \pm j$ and $x \in R$, then the matrix 
$$
T_{ij}(x) := e + xe_i e_j = \lambda^{(e(i)-e(j))}/2 e_{-j,-i} \in U_{2n}(R, \Lambda)
$$
is called an elementary short root transvection. If $i \in \Omega$ and $y \in \lambda^{-(e(i)+1)/2} \Lambda$, then the matrix 
$$
T_{i,-i}(y) := e + ye_{-i} \in U_{2n}(R, \Lambda)
$$
is called an elementary long root transvection. If $\sigma \in U_{2n}(R, \Lambda)$ is an elementary short root transvection or an elementary long root transvection, it is called an elementary unitary transvection. The subgroup of $U_{2n}(R, \Lambda)$ generated by all elementary unitary transvections is called elementary unitary group and is denoted by $EU_{2n}(R, \Lambda)$. An elementary unitary transvection $T_{ij}(x)$ is called $(I, \Gamma)$-elementary if $i \neq -j \land x \in I$ or $i = -j \land x \in \lambda^{-(e(i)+1)/2} \Gamma$. The subgroup of $U_{2n}(R, \Lambda)$ generated by all $(I, \Gamma)$-elementary transvections is called preelementary subgroup of level $(I, \Gamma)$ and is denoted by $EU_{2n}(I, \Gamma)$. Its normal closure in $EU_{2n}(R, \Lambda)$ is called elementary subgroup of level $(I, \Gamma)$ and is denoted by $EU_{2n}(R, \Lambda)(I, \Gamma)$.
Lemma 39. The relations

\[ T_{ij}(x) = T_{-j,-i}(-\lambda^{(e(j)-e(i))/2}x), \]  
\[ T_{ij}(x)T_{ij}(y) = T_{ij}(x + y), \]  
\[ [T_{ij}(x), T_{hk}(y)] = e, \]  
\[ [T_{ij}(x), T_{jk}(y)] = T_{ik}(xy), \]  
\[ [T_{ij}(x), T_{-i,-j}(y)] = T_{ij}(xy)T_{-j,-i}(x) - \lambda^{-e(i)-e(j)/2}xy \]

hold where \( h \neq j, -i \) and \( k \neq i, -j \) in (R3), \( i, k \neq \pm j \) and \( i \neq \pm k \) in (R4) and \( i \neq \pm j \) in (R5) and (R6).

Proof. Straightforward calculation.

Definition 40. Let \( v \in V \) be isotropic (i.e. \( q(v) = 0 \)) such that \( v_{-1} = 0 \). Then we denote the matrix

\[
\begin{pmatrix}
1 & -v_2 & \cdots & -v_n \\
& & \ddots & \\
& & & 1 \\
-\bar{\lambda}v_n & \cdots & -\bar{\lambda}v_2 & v_1 - \bar{\lambda}v_1 \\
& & & \\
& & & v_n \\
& & & \\
& & & v_1 - \bar{\lambda}v_1
\end{pmatrix}
\]

\[
e + ve_{-1} - e_1 \bar{\lambda}v = T_{1,-1}(\bar{\lambda}|v| + v_1 - \bar{\lambda}v_1) \prod_{i=2}^{n} T_{i,-1}(v_i) \in EU_{2n}(R, \Lambda)
\]

by \( T_{s,-1}(v) \). Clearly \( T_{s,-1}(v)^{-1} = T_{s,-1}(-v) \) (note that \( \bar{v}v = 0 \) since \( v \) is isotropic) and

\[
\sigma T_{s,-1}(v) = e + \sigma v \sigma_{-1,s} - \sigma_{s1} \bar{\lambda}v \sigma^{-1} \equiv e + \sigma v \bar{\sigma}_s - \sigma_{s1} \bar{\lambda}\bar{\sigma}v
\]

(3)

for any \( \sigma \in U_{2n}(R, \Lambda) \).

Definition 41. Let \( i, j \in \Omega \) such that \( i \neq \pm j \). Define \( P_{ij} := e + e^{ij} - e^{ji} + \lambda^{(e(i)-e(j))/2}e^{-i,-j} - \lambda^{(e(j)-e(i))/2}e^{-j,-i} - e^{ij} - e^{ji} - e^{-i,-i} - e^{-j,-j} = T_{ij}(1)T_{ji}(-1)T_{ij}(1) \in EU_{2n}(R, \Lambda) \). It is easy show that \( (P_{ij})^{-1} = P_{ji} \).

Lemma 42. Let \( x \in R \) and \( i, j, k \in \Omega \) such that \( i \neq \pm j \) and \( k \neq \pm i, \pm j \). Further let \( y \in \lambda^{-(e(i)+1)/2} \Lambda \).

Then

(i) \( P_{s1}T_{ij}(x) = T_{kj}(x) \),
(ii) \( P_{s1}T_{ij}(x) = T_{ik}(x) \) and
(iii) \( P_{-i,-j}T_{i,-i}(y) = T_{k,-k}(\lambda^{(e(i)-e(k))/2}y) \).

Proof. Follows from the relations in Lemma 39.

Lemma 43. Let \( \sigma \in U_{2n}(R, \Lambda) \) and \( i, j \in \Omega \) such that \( i \neq \pm j \). Set \( \bar{\sigma} := P_{ij} \sigma \). Then

\[
|\bar{\sigma}_s| = \begin{cases}
|\sigma_{s_j}|, & \text{if } e(i) = e(j), \\
|\sigma_{s_j}| + \sigma_{j}s_{i-j} + \lambda \sigma_{i,j}s_{j}, & \text{if } e(i) = 1, e(j) = -1, \\
|\sigma_{s_j}| - \sigma_{i,j}s_{i-j} + \lambda \sigma_{i,j}s_{j}, & \text{if } e(i) = -1, e(j) = 1.
\end{cases}
\]

Proof. Straightforward computation.
8.5. Congruence subgroups.

**Definition 44.** The group consisting of all \( \sigma \in U_{2n}(R, \Lambda) \) such that \( \sigma \equiv e \mod I \) and \( f(\sigma v, \sigma v) \equiv f(v, v) \mod \Gamma \) \( \forall v \in V \) is called principal congruence subgroup of level \((I, \Gamma)\) and is denoted by \( U_{2n}((R, \Lambda), (I, \Gamma)) \).

By a theorem of Bak [1], 4.1.4, cf. [2], 4.4, it is a normal subgroup of \( U_{2n}(R, \Lambda) \).

**Lemma 45.** Let \( \sigma \in U_{2n}(R, \Lambda) \). Then \( \sigma \in U_{2n}((R, \Lambda), (I, \Gamma)) \) if and only if

(i) \( \sigma \equiv e \mod I \) and

(ii) \( |\sigma_{ij}| \in \Gamma \forall j \in \Omega \).

**Proof.** [2], p.174. \( \square \)

**Definition 46.** The subgroup

\[
\{ \sigma \in U_{2n}(R, \Lambda) \mid [\sigma, EU_{2n}(R, \Lambda)] \subseteq U_{2n}((R, \Lambda), (I, \Gamma)) \}
\]

of \( U_{2n}(R, \Lambda) \) is called full congruence subgroup of level \((I, \Gamma)\) and is denoted by \( CU_{2n}((R, \Lambda), (I, \Gamma)) \). Obviously \( U_{2n}((R, \Lambda), (I, \Gamma)) \subseteq CU_{2n}((R, \Lambda), (I, \Gamma)) \).

If \( EU_{2n}(R, \Lambda) \) is a normal subgroup of \( U_{2n}(R, \Lambda) \) (which for example is true if \( n \geq 3 \) and \( R \) is almost commutative, see [2, Theorem 1.1]), then \( CU_{2n}((R, \Lambda), (I, \Gamma)) \) is a normal subgroup of \( U_{2n}(R, \Lambda) \).

**Theorem 47.** If \( n \geq 3 \) and \( R \) is almost commutative (i.e. module finite over its center), then the equalities

\[
[CU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] = [EU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] = EU_{2n}((R, \Lambda), (I, \Gamma))
\]

hold.

**Proof.** By [2, Theorem 1.1]), \( EU_{2n}((R, \Lambda), (I, \Gamma)) \) is normal in \( U_{2n}(R, \Lambda) \) and

\[
[U_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] \subseteq EU_{2n}((R, \Lambda), (I, \Gamma))
\]

(note that in [2] the full congruence subgroup is defined a little differently). By [2, Lemma 5.2],

\[
[EU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] = EU_{2n}((R, \Lambda), (I, \Gamma)).
\]

Hence

\[
[CU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] =\]

\[
[EU_{2n}(R, \Lambda), CU_{2n}((R, \Lambda), (I, \Gamma))] =\]

\[
[[EU_{2n}(R, \Lambda), EU_{2n}(R, \Lambda)], CU_{2n}((R, \Lambda), (I, \Gamma))] \subseteq EU_{2n}((R, \Lambda), (I, \Gamma))
\]

by the definition of \( CU_{2n}((R, \Lambda), (I, \Gamma)) \), (4) and the three subgroups lemma. (5) and (6) imply the assertion of the theorem. \( \square \)

9. Sandwich classification for \( U_{2n}(R, \Lambda) \)

In this section \( n \) denotes a natural number greater or equal to 3 and \( (R, \Lambda) \) a form ring where \( R \) is commutative.

**Definition 48.** Let \( \sigma \in U_{2n}(R, \Lambda) \). Then a matrix of the form \( e^\sigma \) where \( e \in EU_{2n}(R, \Lambda) \) is called an elementary (unitary) \( \sigma \)-conjugate.

**Theorem 49.** Let \( \sigma \in U_{2n}(R, \Lambda) \), \( k \neq \pm l \) and \( i \neq \pm j \). Then

(i) \( T_{kl}(\sigma_{ij}) \) is a product of 160 elementary unitary \( \sigma \)-conjugates,

(ii) \( T_{kl}(\sigma_{i,-i}) \) is a product of 320 elementary unitary \( \sigma \)-conjugates,

(iii) \( T_{kl}(\sigma_{ii} - \sigma_{jj}) \) is a product of 480 elementary unitary \( \sigma \)-conjugates,

(iv) \( T_{kl}(\sigma_{ii} - \sigma_{-i,-i}) \) is a product of 960 elementary unitary \( \sigma \)-conjugates and

(v) \( T_{k,-k}(\lambda^{-(e(k)+1)/2} |\sigma_{ij}|) \) is a product of 1600n + 4004 elementary unitary \( \sigma \)-conjugates.
Proof. (i) In step 1 below we show that \( T_{kl}(x\sigma_23\sigma_{2,-1}) \) where \( x \in R \) is a product of 16 elementary \( \sigma \)-conjugates. In step 2 we show that \( T_{kl}(x\sigma_23\sigma_{21}) \) where \( x \in R \) is a product of 16 elementary \( \sigma \)-conjugates. In step 3 we show that \( T_{kl}(x\sigma_23\sigma_{22}) \) is a product of 32 elementary \( \sigma \)-conjugates. In step 4 we use steps 1-3 in order to prove (i).

**step 1** Set \( \tau := T_{21}(\overline{\sigma_23}\sigma_{23})T_{31}(-\overline{\sigma_23}\sigma_{21})T_{3,-2}(\overline{\sigma_23}\sigma_{2,-1})T_{3,-3}(-\overline{\sigma_22}\sigma_{2,-1} + \lambda\overline{\sigma_21}\sigma_{22}) \). One checks easily that the second row of \( \sigma \tau^{-1} \) equals the second row of \( \sigma \) and hence the second row of \( \xi := \sigma \tau^{-1} \) is trivial. By Lemma 35 the second last column of \( \xi \) also is trivial. Set

\[
\zeta := \tau^{-1}[T_{-1,2}(1), [\tau, \sigma]] = \tau^{-1}[T_{-1,2}(1), \tau\xi] \overset{L.1}{=} \tau^{-1}, T_{-1,2}(1)[T_{-1,2}(1), \xi].
\]

One checks easily that \( \tau^{-1}, T_{-1,2}(1) = T_{31}(\lambda\overline{\sigma_23}\sigma_{2,-1})T_{-1,1}(z) \) for some \( z \in \Lambda \) and \( [T_{-1,2}(1), \xi] = \prod_{i \neq 2} T_{i2}(x_i) \) for some \( x_i \in R (i \neq 2) \). Hence \( \zeta = T_{31}(\lambda\overline{\sigma_23}\sigma_{2,-1})T_{-1,1}(z) \prod_{i \neq 2} T_{i2}(x_i) \). It follows that \( [T_{-1,3}(-x\lambda), T_{12}(1), \xi] = T_{-1,2}(x\overline{\sigma_23}\sigma_{2,-1}) \) for any \( x \in R \). Hence we have shown

\[
[T_{-1,3}(-x\lambda), T_{12}(1), \tau^{-1}[T_{-1,2}(1), [\tau, \sigma]]] = T_{-1,2}(x\overline{\sigma_23}\sigma_{2,-1}).
\]

This implies that \( T_{-1,2}(x\overline{\sigma_23}\sigma_{2,-1}) \) is a product of 16 elementary \( \sigma \)-conjugates. It follows from Lemma 42 that \( T_{kl}(x\overline{\sigma_23}\sigma_{2,-1}) \) is a product of 16 elementary \( \sigma \)-conjugates.

**step 2** Set \( \tau := T_{1,-2}(\overline{\sigma_23}\sigma_{2})T_{3,-2}(-\overline{\sigma_23}\sigma_{21})T_{3,-1}(\overline{\sigma_22}\sigma_{22})T_{3,-3}(\overline{\sigma_22}\sigma_{21} - \lambda\overline{\sigma_21}\sigma_{22}) \). One checks easily that the second row of \( \sigma \tau^{-1} \) equals the second row of \( \sigma \) and hence the second row of \( \xi := \sigma \tau^{-1} \) is trivial. By Lemma 35 the second last column of \( \xi \) also is trivial. Set

\[
\zeta := \tau^{-1}[T_{-2,-1}(1), [\tau, \sigma]] = \tau^{-1}[T_{-2,-1}(1), \tau\xi] \overset{L.1}{=} \tau^{-1}, T_{-2,-1}(1)[T_{-2,-1}(1), \xi].
\]

One checks easily that \( \tau^{-1}, T_{-2,-1}(1) = T_{3,-1}(\overline{\sigma_23}\sigma_{21})T_{1,-1}(z) \) for some \( z \in \overline{\Lambda} \) and \( [T_{-2,-1}(1), \xi] = \prod_{i \neq 2} T_{i2}(x_i) \) for some \( x_i \in R (i \neq 2) \). Hence \( \zeta = T_{3,-1}(\overline{\sigma_23}\sigma_{21})T_{1,-1}(z) \prod_{i \neq 2} T_{i2}(x_i) \). It follows that \( [T_{-1,3}(-x), T_{-2,3}(1), \xi] = T_{-2,3}(x\overline{\sigma_23}\sigma_{21}) \) for any \( x \in R \). Hence we have shown

\[
[T_{-1,3}(-x), T_{-2,3}(1), \tau^{-1}[T_{-2,-1}(1), [\tau, \sigma]]] = T_{-2,3}(x\overline{\sigma_23}\sigma_{21}).
\]

This implies that \( T_{-2,3}(x\overline{\sigma_23}\sigma_{21}) \) is a product of 16 elementary \( \sigma \)-conjugates. It follows from Lemma 42 that \( T_{kl}(x\overline{\sigma_23}\sigma_{21}) \) is a product of 16 elementary \( \sigma \)-conjugates.

**step 3** Set \( \tau := T_{21}(\overline{\sigma_22}\sigma_{23})T_{31}(\overline{\sigma_22}\sigma_{22})T_{2,-3}(\overline{\sigma_22}\sigma_{2,-1})T_{2,-2}(-\overline{\sigma_22}\sigma_{2,-1} + \lambda\overline{\sigma_21}\sigma_{23}) \). One checks easily that the second row of \( \sigma \tau^{-1} \) equals the second row of \( \sigma \) and hence the second row of \( \xi := \sigma \tau^{-1} \) is trivial. By Lemma 35 the second last column of \( \xi \) also is trivial. Set

\[
\zeta := \tau^{-1}[T_{32}(1), [\tau, \sigma]] = \tau^{-1}[T_{32}(1), \tau\xi] \overset{L.1}{=} \tau^{-1}, T_{32}(1)[T_{32}(1), \xi].
\]

One checks easily that \( \tau^{-1}, T_{32}(1) = T_{32}(\overline{\sigma_22}\sigma_{23})T_{3,-3}(y)T_{3,-2}(z) \) for some \( y \in \overline{\Lambda} \) and \( z \in R \) and \( \theta := [T_{32}(1), \xi] = \prod_{i \neq 2} T_{i2}(x_i) \) for some \( x_i \in R (i \neq 2) \). Set

\[
\chi := \psi^{-1}[T_{12}(1), \xi] = \psi^{-1}[T_{12}(1), \psi\theta] \overset{L.1}{=} \psi^{-1}, T_{12}(1)[T_{12}(1), \theta].
\]

One checks easily that \( \psi^{-1}, T_{12}(1) = T_{32}(\overline{\sigma_22}\sigma_{23})T_{3,-3}(a)T_{3,-2}(b) \) for some \( a \in \overline{\Lambda} \) and \( b \in R \) and \( [T_{12}(1), \theta] = T_{-2,2}(d) \) for some \( d \in \Lambda \). Hence \( \chi = T_{32}(\overline{\sigma_22}\sigma_{23})T_{3,-3}(a)T_{3,-2}(b)T_{-2,2}(d) \). It follows that \( [T_{-2,3}(\overline{x}), T_{2,-1}(1), \xi] = T_{-2,1}(-\overline{x}\overline{\sigma_22}\sigma_{23}) \overset{(R1)}{=} T_{12}(x\overline{\sigma_23}\sigma_{22}) \) for any \( x \in R \). Hence we have shown

\[
[T_{-2,3}(\overline{x}), T_{2,-1}(1), \psi^{-1}[T_{12}(1), \tau^{-1}[T_{32}(1), [\tau, \sigma]]]] = T_{12}(x\overline{\sigma_23}\sigma_{22}).
\]

This implies that \( T_{12}(x\overline{\sigma_23}\sigma_{22}) \) is a product of 32 elementary \( \sigma \)-conjugates. It follows from Lemma 42 that \( T_{kl}(x\overline{\sigma_23}\sigma_{22}) \) is a product of 32 elementary \( \sigma \)-conjugates.
step 4 Set \( I := I(\{\bar{\sigma}_{23}\bar{\sigma}_{22}, -1, \sigma_{23}\sigma_{21}\}) \), \( J := I(\{\bar{\sigma}_{23}\bar{\sigma}_{22}, \sigma_{23}\sigma_{21}, \sigma_{23}\sigma_{22}\}) \) and

\[
\tau := [\bar{\sigma}^{-1}, T_{12}(\bar{\sigma}_{23})] = (e - \sigma_{31}^{3} \sigma_{23}\sigma_{21} + \sigma_{32}^{1} \sigma_{23}\sigma_{21}) T_{12}(\bar{\sigma}_{23}).
\]

One checks easily that \( \tau_{11} \equiv 1 \mod I \) and \( \tau_{12} \equiv \bar{\sigma}_{23} \mod J \). Set \( \zeta := P_{13}P_{21} \tau \). Then \( \zeta_{22} = \tau_{11} \) and \( \zeta_{23} = \tau_{12} \) and hence \( \zeta_{23}\zeta_{22} \equiv \sigma_{23} \mod I + J \). Applying step 3 above to \( \zeta \), we get that \( T_{kl}(\zeta_{23}\zeta_{22}) \) is a product of 32 elementary \( \zeta \)-conjugates. Since any elementary \( \zeta \)-conjugate is a product of 2 elementary \( \sigma \)-conjugates, it follows that \( T_{kl}(\zeta_{23}\zeta_{22}) \) is a product of 64 elementary \( \sigma \)-conjugates. Thus, by steps 1-3, \( T_{kl}(\sigma_{23}) \) is a product of \( 64 + 16 + 16 + 16 + 16 + 32 = 160 \) elementary \( \sigma \)-conjugates. Since one can bring \( \sigma_{ij} \) to position (3, 2) by conjugating monomial matrices in \( EU_{2n}(R, \Lambda) \), the assertion of (i) follows.

(ii)-(iv) See the proof of Theorem 27.

(v) Set \( m := 160 \). In step 1 we show that \( T_{k-l}(\lambda^{-2}x_{k+l+1}/2 \sigma_{11}|\sigma_{11}|x_{1}) \) where \( x \in R \) is a product of \( (2n + 17)m + 4 \) elementary \( \sigma \)-conjugates. In step 2 we use step 1 in order to prove (v).

step 1 Set \( v' := (0 \ldots 0 \sigma_{23} - \sigma_{23})^t = (0 \ldots 0 \bar{\sigma}_{11} \bar{\sigma}_{23})^t \in V \) and \( v := \sigma^{-1}v' \in V \). Then clearly \( v_{-1} = 0 \). Further \( q(v) = q(\sigma^{-1}v') = q(v') = 0 \) and hence \( v \) is isotropic. Set

\[
\xi := \sigma T_{\sigma^{-1}}(v) = e - \sigma v \bar{\sigma}_{31} + \sigma_{31} \bar{\lambda} \bar{\sigma} v = e - v' \bar{\sigma}_{31} + \sigma_{31} \bar{\lambda} v'.
\]

Then

\[
\xi = \begin{pmatrix}
* & \sigma_{11}\sigma_{11} & 1 \\
* & 1 + \sigma_{21}\sigma_{11} & \\
* & \sigma_{31}\sigma_{11} & 1 \\
* & \sigma_{41}\sigma_{11} & 1 \\
\vdots & \vdots & \ddots \\
* & \sigma_{n1}\sigma_{11} & 1 \\
-\sigma_{-3,1}\sigma_{21} & \sigma_{-3,1}\sigma_{21} & \\
* & \alpha & * & \ldots & * & \ldots & * & -\bar{\sigma}_{11}\bar{\sigma}_{31} & * & * \\
* & \beta & * & \ldots & * & \ldots & * & \bar{\sigma}_{21}\bar{\sigma}_{31} & * & * \\
\end{pmatrix}
\]

where \( \alpha = \sigma_{-2,1}\sigma_{11} - \lambda \bar{\sigma}_{11}\bar{\sigma}_{-2,1} \) and \( \beta = \sigma_{-1,1}\sigma_{11} + \lambda \bar{\sigma}_{21}\bar{\sigma}_{-1,1} \). Set

\[
\tau := T_{-3,1}(\sigma_{-3,1}\sigma_{21}) T_{-3,2}(\sigma_{-3,1}\sigma_{11}).
\]
It follows from (i) that $\tau$ is a product of $2m$ elementary $\sigma$-conjugates. Clearly

$$
\xi_{\tau} = \begin{pmatrix}
* & \sigma_{11}\sigma_{11} & 1 & 1 \\
* & 1 + \sigma_{21}\sigma_{11} & 1 & 0 \\
* & \sigma_{31}\sigma_{11} & 1 & 1 \\
* & \sigma_{41}\sigma_{11} & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots \\
* & \sigma_{n1}\sigma_{11} & 1 & 0 \\
* & \sigma_{-n,1}\sigma_{11} & 0 & 1 \\
\end{pmatrix}
$$

where $\gamma = \alpha + \sigma_{11}\sigma_{31}\sigma_{-3,1}\sigma_{11}$ and $\delta = \beta - \sigma_{21}\sigma_{31}\sigma_{-3,1}\sigma_{11}$. Let $x \in R$ and set

$$
\zeta := T_{x,-1}(-v)[T_{x,-1}(y), [T_{x,-1}(v), \sigma]\tau] = [T_{x,-1}(y), [T_{x,-1}(v), \xi_{\tau}]] = [T_{x,-1}(y), [T_{x,-1}(v), T_{x,-1}(\tau)]]
$$

Clearly $\zeta$ is a product of $4m + 4$ elementary $\sigma$-conjugates. One checks easily that

$$
[T_{x,-1}(y), T_{x,-3}(z)] = T_{1,-1}(\lambda)T_{1,-2}(\lambda T_{-2}(y))T_{1,-1}(b + c)T_{1,-3}(x\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21})
$$

for some $a, b \in I(\sigma_{21}), c \in I(\sigma_{23})$. Further

$$
[T_{x,-1}(y), T_{x,-3}(z)] = \left( \prod_{p=2}^{4} T_{p,-1}(x\sigma_{p1}\sigma_{11}) \right) T_{-2,-3}(y) T_{-1,-3}(x\delta) T_{3,-3}(y)
$$

where $y = \lambda(\sigma_{11}|\sigma_{11}|x + d - \lambda\alpha + e - \lambda\delta)$ for some $d \in I(\sigma_{31}), e \in I(\sigma_{-3,1})$. Hence

$$
\zeta = T_{1,-1}(\lambda)T_{1,-2}(b + c)T_{1,-3}(x\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21})
$$

$$
\cdot \left( \prod_{p=2}^{4} T_{p,-1}(x\sigma_{p1}\sigma_{11}) \right) T_{-2,-3}(y) T_{-1,-3}(x\delta) T_{3,-3}(y).
$$

It follows from (i), (ii) and (iii) and relation (R5) in Lemma 39 that $T_{x,-3}(y)$ is a product of $4m + 4 + 2m + 2m + (2n - 5)m + 3m + 3m = (2n + 13)m + 4$ elementary $\sigma$-conjugates. By (i) and relation (R5) in Lemma 39, $T_{x,-3}(\lambda(d - \lambda d))$ and $T_{x,-3}(\lambda(e - \lambda e))$ each are a product of $2m$ elementary $\sigma$-conjugates. Hence $T_{x,-3}(\lambda(\sigma_{11}|\sigma_{11}|x)) = T_{x,-3}(y)T_{x,-3}(\lambda(d - \lambda d))T_{x,-3}(\lambda(e - \lambda e))$ is a product of $(2n + 17)m + 4$ elementary $\sigma$-conjugates. It follows from Lemma 42 that $T_{x,-3}(\lambda^{-1}(k + 1)^{2}\sigma_{11}|\sigma_{11}|x)$ is a product of $(2n + 17)m + 4$ elementary $\sigma$-conjugates.
step 2 Clearly

\[ T_{k,-k}(λ^{-(e(k)+1)/2}|σ_{s1}) = T_{k,-k}(λ^{-(e(k)+1)/2} \sum_{q, r ∈ Ω} σ_1 σ_1 σ_1 σ_1 ) = T_{k,-k}(λ^{-(e(k)+1)/2} \sum_{q, r ∈ Ω} σ_1 σ_1 σ_1 σ_1 ) = T_{k,-k}(λ^{-(e(k)+1)/2} \sum_{q, r ∈ Ω} σ_1 σ_1 σ_1 σ_1 ) = T_{k,-k}(λ^{-(e(k)+1)/2} \sum_{q, r ∈ Ω} σ_1 σ_1 σ_1 σ_1 ) \]

since \( |σ_{s1}| \in Λ \subseteq Λ_{max} \). By step 1, \( T_{k,-k}(λ^{-(e(k)+1)/2}σ_{s1}|σ_{s1}|σ_{s1}|σ_{s1}|) \) is a product of \((2n + 17)m + 4\) elementary \( σ \)-conjugates. By (i), (ii) and relation (R6) in Lemma 39, \( T_{k,-k}(λ^{-(e(k)+1)/2}σ_{q}|σ_{q}|σ_{q}|σ_{q}|) \) is a product of \( 3m \) elementary \( σ \)-conjugates if \( q \neq ±1 \) resp. a product of \( 6m \) elementary \( σ \)-conjugates if \( q = 1 \). Hence \( A \) is a product of \((2n + 17)m + 4 + (2n - 2)·3m + 6m = (8n + 17)m + 4\) elementary \( σ \)-conjugates. On the other hand \( B = T_{k,-k}(λ^{-(e(k)+1)/2}(x - λx)) \) where \( x ∈ I(σ_{s1}) \). Since \( |σ_{s1}| = \sum_{i ∈ Ω} σ_{i1} σ_{i1} σ_{i1} σ_{i1} \), it follows from (i), (ii) and relation (R5) in Lemma 39 that \( B \) is a product of \((4m + (n - 1)·2m = (2n + 2)m\) elementary \( σ \)-conjugates. Hence \( T_{k,-k}(λ^{-(e(k)+1)/2}σ_{s1}) \) is a product of \((10m + 19)m + 4 = 1600n + 3044\) elementary \( σ \)-conjugates. The assertion of (v) follows now from Lemma 43.

As a corollary we get the Sandwich Classification Theorem for \( U_{2n}(R, Λ) \).

**Corollary 50.** Let \( H \) be a subgroup of \( U_{2n}(R, Λ) \). Then \( H \) is normalized by \( EU_{2n}(R, Λ) \) if and only if

\[ EU_{2n}((R, Λ), (I, Γ)) \subseteq H \subseteq CU_{2n}((R, Λ), (I, Γ)) \]

for some form ideal \((I, Γ)\) of \((R, Λ)\).

**Proof.** First suppose that \( H \) is is normalized by \( EU_{2n}(R, Λ) \). Let \((I, Γ)\) be the form ideal of \((R, Λ)\) defined by \( I := \{ x ∈ R | T_{2}(x) ∈ H \} \) and \( Γ := \{ y ∈ Λ | T_{-1}(y) ∈ H \} \). Then clearly \( EU_{2n}((R, Λ), (I, Γ)) \) \( H \). It remains to show that \( H \subseteq CU_{2n}((R, Λ), (I, Γ)) \), i.e. that \( σ \in H \) and \( ε \in EU_{2n}(R, Λ) \), then \( [σ, ε] ∈ U_{2n}((R, Λ), (I, Γ)) \). By Lemma 45 it suffices to show that if \( σ \in H \) and \( ε \in EU_{2n}(R, Λ) \), then \( [σ, ε] \equiv e \mod I \) and \([σ, ε]_{x} \) \( Γ \) for any \( j ∈ Ω \). But that follows from the previous theorem (applying the theorem to \( σ \)) we get that \( σ \equiv \operatorname{diag}(x, \ldots, x) \mod I \) for some \( x ∈ R \) and hence \([σ, ε] \equiv e \mod I \); applying it to \([σ, ε] \) we get that \([σ, ε]_{x} \) \( Γ \) for any \( j ∈ Ω \). Suppose now that (7) holds for some form ideal \((I, Γ)\). Then it follows from the standard commutator formulas in Theorem 47 that \( H \) is normalized by \( EU_{2n}(R, Λ) \).

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