TIME-DOMAIN ANALYSIS OF FORWARD OBSTACLE SCATTERING FOR ELASTIC WAVE

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ABSTRACT. This paper concerns a time-domain scattering problem of elastic plane wave by a rigid obstacle, which is immersed in an open space filled with homogeneous and isotropic elastic medium in two dimensions. A new compressed coordinate transformation is developed to reduce the scattering problem into an initial boundary value problem in a bounded domain over a finite time interval. The well-posedness is established for the reduced problem. This paper adopts Galerkin method to prove the uniqueness results and employs energy method to derive stability results for the scattering problem. Furthermore, we achieve a priori estimate with explicit time dependence.

1. Introduction. Elastic obstacle scattering problem, which concerns the scattering of an incident elastic wave by impenetrable obstacles, is an important research topic in scattering theory. It has continuously attracted much attention due to the significant applications in seismology and geophysics [17]. The elastic obstacle scattering problem can be divided into frequency domain problem with time-harmonic wave field and time-domain problem with time-dependent non-harmonic wave field.

For the frequency domain problem, it has been extensively investigated and lots of mathematical and numerical results are available. For example, in [2, 18, 19], the authors use transparent boundary condition (TBC) and Fredholm alternatives to prove the existence and uniqueness of solutions for penetrable and impenetrable obstacles. We refer to [5, 10] for perfectly matched layer (PML) method and [11] for boundary integral equation method to solve the forward scattering problem.

In many practical application, such as simulating more general and non-linear materials [6, 21], frequency domain data are usually difficult to measure, but the broadband signals in time-domain are easy to capture. Moreover, the time-domain data have more information than frequency domain data at discrete frequencies. Based on these factors, the time-domain scattering problems have received considerable attention in recent years. Compared with the time-harmonic scattering problems, the time-domain scattering problems are more difficult due to time dependence, so the research results are fewer. In the scattering problems of time-domain acoustic, elastic and electromagnetic waves, including various structures, the results...
are given in \cite{1, 3, 4, 7, 8, 9, 13, 14, 15, 16, 20, 22} for the forward scattering problem. In \cite{3, 13, 14, 15, 16}, the main idea is that by using TBC the model problem is reduced into an initial boundary value problem in a bounded domain, and it is transformed into \(s\)-domain by Laplace transform. Then the well-posedness of the reduced problem is considered in the \(s\)-domain, and uniqueness and existence results are derived in the time-domain through inverse Laplace transform. Finally, energy method is used to obtain the stability estimate for the solution in the time-domain.

In \cite{1, 20}, by using the fact that the scattered wave has a finite speed of propagation, a compressed coordinate transformation is proposed to reduce the problem into an initial boundary value problem in a bounded domain over a finite time interval. Based on the Galerkin method and energy estimate, the well-posedness and a priori estimate is obtained.

In this work, we consider the scattering of a time-domain elastic plane wave by a rigid obstacle which is embedded in an open space filled with homogeneous and isotropic elastic medium in two dimensions. Our aim is to prove the well-posedness and to obtain a priori estimates for the solution in the time-domain. A new compressed coordinate transformation with high order smoothness, which is performed on the Navier equation, is constructed to overcome the difficulty caused by a variable coefficient in the reduced equation. In addition, we adopt Galerkin method with modified subspace and energy method to establish the well-posedness analysis.

The paper is organized as follows. In Section 2, the mathematical model is introduced and a new compressed coordinate transformation is developed to reformulate the scattering problem into an initial boundary value problem in a bounded domain over a finite time interval. Section 3 devotes to establish the well-posedness for the reduced problem in the time-domain. In Section 4, a priori estimate is obtained with explicit dependence on the time.

2. Problem formulation. In this section, we introduce the mathematical model for the elastic problem and show some function spaces. A compressed coordinate transformation is presented to reduce the scattering problem into an initial boundary value problem.

2.1. Mathematical model. In the time-domain, consider the scattering of an elastic wave by a two-dimensional rigid obstacle, which is described by a bounded domain \(D \subset \mathbb{R}^2\) with Lipschitz continuous boundary \(\partial D\). Suppose that the infinite exterior domain \(\mathbb{R}^2 \setminus \overline{D}\) is filled with homogeneous and isotropic elastic medium with a unit mass density. It is known that the scattered wave has a finite speed of propagation in the time-domain, for any given time \(T > 0\), pick a sufficiently large \(R > 0\) such that the scattered wave still haven’t reach the surface \(\partial B_R = \{x \in \mathbb{R}^2 : |x| = R\}\) at time \(T\).

Let the obstacle be illuminated by a plane wave \(u^{\text{inc}}\), which satisfies the two-dimensional time-domain Navier equation:

\[
\partial_t^2 u^{\text{inc}} - \mu \Delta u^{\text{inc}} - (\lambda + \mu) \nabla \cdot u^{\text{inc}} = 0, \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \quad t > 0,
\]

where \(\lambda\) and \(\mu\) are the Lamé constants satisfying \(\mu > 0\) and \(\lambda + \mu > 0\). The incident wave can be either a compressional plane wave:

\[
u^{\text{inc}} = df (x \cdot d + c_1 t)
\]
or a shear plane wave:

\[ u^{\text{inc}} = d_{\perp} f (x \cdot d + c_2 t), \]

where \( d = (\cos \phi, \sin \phi)^T, \ d_{\perp} = (-\sin \phi, \cos \phi)^T, \ \phi \in [0, 2\pi) \) is the incident angle, and \( c_1 = (\lambda + 2\mu)/2, \ c_2 = \mu^{1/2} \) is the wave speed. We assume that at time zero the supports of \( u^{\text{inc}}(\cdot, 0) \) and \( \partial_t u^{\text{inc}}(\cdot, 0) \) do not intersect \( D \).

The total field \( u \) also satisfies the time-domain Navier equation:

\[ \partial_t^2 u - \mu \Delta u - (\lambda + \mu) \nabla \nabla \cdot u = 0, \quad \text{in} \ \mathbb{R}^2 \setminus \overline{D}, \ t > 0. \]

The obstacle is assumed to be elastically rigid, it holds that

\[ u = 0, \quad \text{on} \ \partial D. \]

Since the scattered wave has a finite speed of propagation, for any given time \( T > 0 \), we can pick a sufficiently large \( R > 0 \) such that

\[ u^s = 0, \quad \text{on} \ \partial B_R \times (0, T]. \]

That means

\[ u = u^{\text{inc}}, \quad \text{on} \ \partial B_R \times (0, T]. \]

In addition, \( u \) satisfies the initial condition

\[
\begin{cases}
\{ u|_{t=0} = u^{\text{inc}}|_{t=0} = u_0 \quad \text{in} \mathbb{R}^2 \setminus \overline{D}, \\
\partial_t u|_{t=0} = \partial_t u^{\text{inc}}|_{t=0} = v_0 \quad \text{in} \mathbb{R}^2 \setminus \overline{D}.
\end{cases}
\]

Then we have

\[
\begin{cases}
\partial_t^2 u - \mu \Delta u - (\lambda + \mu) \nabla \nabla \cdot u = 0 \quad \text{in} \ B_R \setminus \overline{D} \times (0, T], \\
u = u^{\text{inc}} \quad \text{on} \ \partial B_R \times (0, T], \\
u = 0 \quad \text{on} \ \partial D \times (0, T], \\
u|_{t=0} = u_0, \ \partial_t u|_{t=0} = v_0 \quad \text{in} \ B_R \setminus \overline{D}.
\end{cases}
\]

2.2. Function space. In order to describe the problem more precisely, we introduce some function spaces. Let \( L^2(\Omega)^2 = L^2(\Omega) \times L^2(\Omega) \) be the product space of \( L^2(\Omega) \) equipped with inner product and norm

\[ (u, v) = \int_{\Omega} u \cdot v \, dx, \quad \|u\|_{L^2(\Omega)^2} = (u, u)^{1/2}. \]

Let \( H^k(\Omega)^2 \) be the standard Sobolev space of square integrable functions with the order of derivatives up to \( k \)

\[ H^k(\Omega)^2 = \{D^\alpha u \in L^2(\Omega)^2 : \forall |\alpha| \leq k \}, \]

and the norm is given by

\[ \|u\|^2_{H^k(\Omega)^2} = \sum_{j=1}^2 \|u_j\|^2_{H^k(\Omega)} = \sum_{j=1}^2 \sum_{|\alpha| = 0}^k \|D^\alpha u_j(x)\|^2_{L^2(\Omega)}, \]

where \( u = (u_1, u_2)^T \). The 2-norm of the gradient tensor is defined by

\[ \|\nabla u\|^2_{L^2(\Omega)^2} := \sum_{j=1}^2 \|\nabla u_j\|^2_{L^2(\Omega)^2}. \]
2.3. Compressed coordinate transformation. In this section, denote by \( B_a = \{ x \in \mathbb{R}^2 : |x| < a \} \) with the boundary \( \partial B_a = \{ x \in \mathbb{R}^2 : |x| = a \} \), such that \( \mathcal{D} \subset B_a \), and assume that \( a \ll R \). Let \( b \) be an appropriate constant satisfying \( a < b \ll R \). We propose a new compressed coordinate transformation with high order smoothness which compresses the annulus \( \{ x \in \mathbb{R}^2 : a < |x| < R \} \) into a much smaller annulus \( \{ x \in \mathbb{R}^2 : a < |x| < b \} \) by mapping \( \partial B_R \) into \( \partial B_b \) while keeping \( \partial B_a \) unchanged, the elastic scattering problem (1) is reduced into an initial boundary value problem in a much smaller domain \( B_b \) over a finite time interval.

Consider the change of variables

\[
\rho = \zeta(r) = \begin{cases} r & r \in [0, a), \\ \eta(r) & r \in [a, b], \end{cases}
\]

where

\[
\eta(r) = Ar^4 + Br^3 + Cr^2 + Dr + E,
\]

\[
A = \frac{R - b}{(b - a)^4}, \quad B = -4Aa, \quad C = 6Aa^2, \quad D = 1 - 4Aa^3, \quad E = Aa^4.
\]

A simple calculation yields

\[
\eta(a) = a, \quad \eta(b) = R, \quad \eta'(a) = 1, \quad \eta''(a) = 12A(r-a)^2 \geq 0,
\]

which imply \( \zeta \in C^1[a, b] \) is positive and monotonically increasing, i.e., \( \zeta > 0 \) and \( \zeta' > 0 \). The transformation compresses the annulus \( B_R \setminus \mathcal{D} \) into the annulus \( B_b \setminus \mathcal{D} \) and keeps the ball \( B_a \) unchanged. Define \( \Omega = B_b \setminus \mathcal{D} \) and its boundary \( \partial \Omega = \partial D \cup \partial B_b \).

Let \( \mathbf{v} \) be the transformed total field of \( \mathbf{u} \) under the change of variables, then \( \mathbf{v} \) satisfies

\[
\begin{cases}
\beta \partial_r^2 \mathbf{v} - \mu \nabla \cdot (\nabla \mathbf{v} M) - (\lambda + \mu) K \nabla (\beta^{-1} \nabla \cdot (K \mathbf{v})) = 0 & \text{in } \Omega \times (0, T], \\
\mathbf{v} = \mathbf{v}^{\text{inc}} & \text{on } \partial B_b \times (0, T], \\
\mathbf{v} = 0 & \text{on } \partial D \times (0, T], \\
\mathbf{v}(t=0) = \mathbf{v}^{\text{inc}}(\cdot, 0), & \partial_t \mathbf{v}(t=0) = \partial_t \mathbf{v}^{\text{inc}}(\cdot, 0) & \text{in } \Omega,
\end{cases}
\]

where the operational rules of \( \nabla \) and \( \nabla \cdot \) are

\[
\nabla \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \nabla \cdot \nabla \mathbf{v} = \begin{bmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 \end{bmatrix}, \quad \nabla \cdot \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} \partial_{x_1} N_{11} + \partial_{x_2} N_{12} \\ \partial_{x_1} N_{21} + \partial_{x_2} N_{22} \end{bmatrix},
\]

and the symbols in the equation are

\[
\beta = \frac{\zeta \zeta'}{r}, \quad Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad M = Q \begin{bmatrix} \zeta \zeta' & 0 \\ 0 & \zeta \zeta' \end{bmatrix} Q^T, \quad K = Q \begin{bmatrix} \zeta & 0 \\ 0 & \zeta' \end{bmatrix} Q^T,
\]

where \((r, \theta)\) is the polar coordinates. Due to the high order smoothness of \( \zeta \), it is easy to verify that \( \beta \) is a continuous monotonous positive function and \( \beta' \) is a continuous bounded positive function. \( M \) and \( K \) are symmetric matrices and \( M \) is a positive definite matrix. The detailed process of transformation is given in the appendix.

For the given \( \mathbf{v}^{\text{inc}} \), there exists a smooth function \( \mathbf{U}_0 \) which has a compact support contained in \( \Omega \times (0, T] \) and satisfies the boundary conditions:

\[
\mathbf{U}_0|_{\partial B_b} = \mathbf{v}^{\text{inc}}|_{\partial B_b}, \quad \mathbf{U}_0|_{\partial D} = 0, \quad t \in (0, T].
\]
Let \( u = v - \tilde{U}_0 \), we may equivalently consider the following initial boundary value problem
\[
\begin{aligned}
\beta \partial_t^2 u - \mu \nabla \cdot (\nabla uM) - (\lambda + \mu) K \nabla \left( \beta^{-1} \nabla \cdot (Ku) \right) &= f & \text{in } \Omega \times (0, T], \\
u &= 0 & \text{on } \partial B_b \cup \partial D \times (0, T], \\
u|_{t=0} = g, & \partial_t u|_{t=0} = h & \text{in } \Omega,
\end{aligned}
\]
where \( f \in L^2(0, T; L^2(\Omega)^2), g \in H^1_b(\Omega)^2 := \{ u \in H^1(\Omega)^2 : u = 0 \text{ on } \partial B_b \cup \partial D \}, \ h \in L^2(\Omega) \).

3. Well-posedness. In this section, we give the well-posedness for the reduced initial boundary value problem (2) in the time-domain.

3.1. Uniqueness and existence. Taking the inner products in (2) with the test function \( v \in H^1_0(\Omega)^2 \), we arrive at the variational problem: to find \( u \in H^1_0(\Omega)^2 \) for all \( t > 0 \) such that
\[
A[u, v; t] = (f, v), \quad \forall \ v \in H^1_0(\Omega)^2,
\]
where
\[
A[u, v; t] = (\beta \partial_t^2 u, v) - \mu (\nabla \cdot (\nabla uM), v) - (\lambda + \mu) (K \nabla \left( \beta^{-1} \nabla \cdot (Ku) \right), v).
\]
Using the integration by parts and initial and boundary value conditions, we have
\[
A[u, v; t] = (\beta \partial_t^2 u, v) + \mu \int_{\partial B_b} (\nabla uM) \cdot v ds + \mu \int_{\partial D} (\nabla u) \cdot v ds \\
- \mu \int_{\partial B_b} (\beta^{-1} \nabla \cdot (Ku) \nabla \cdot (Kv)) ds + (\lambda + \mu) (\beta^{-1} \nabla \cdot (Ku), \nabla \cdot (Kv))
\]
\[
= (\beta \partial_t^2 u, v) + \mu (\nabla uM, \nabla v) + (\lambda + \mu) (\beta^{-1} \nabla \cdot (Ku), \nabla \cdot (Kv))
\]
\[
= (\beta \partial_t^2 u, v) + \mu (M^{1/2} \nabla u_1, M^{1/2} \nabla v_1) \\
+ \mu (M^{1/2} \nabla u_2, M^{1/2} \nabla v_2) + (\lambda + \mu) (\beta^{-1/2} \nabla \cdot (Ku), \beta^{-1/2} \nabla \cdot (Kv)),
\]
where \( u = (u_1, u_2)^T, v = (v_1, v_2)^T \) and \( A : B = \text{tr}(AB^T) \) is the Frobenius inner product of square matrices \( A \) and \( B \). Write
\[
a[u, v; t] = \mu (M^{1/2} \nabla u_1, M^{1/2} \nabla v_1) + \mu (M^{1/2} \nabla u_2, M^{1/2} \nabla v_2) \\
+ (\lambda + \mu) (\beta^{-1/2} \nabla \cdot (Ku), \beta^{-1/2} \nabla \cdot (Kv)).
\]
Assume that \( u \) is a smooth solution of (2) and define the associated mapping \( u : [0, T] \rightarrow H^1_0(\Omega)^2 \) by
\[
[u(t)](x) := u(x, t), \quad x \in \Omega, \ t \in [0, T].
\]
Analogously, we define the mapping \( u' : [0, T] \rightarrow L^2(\Omega)^2 \) and \( u'' : [0, T] \rightarrow H^{-1}(\Omega)^2 \) by
\[
[u'(t)](x) := \partial_t u(x, t), \quad [u''(t)](x) := \partial_t^2 u(x, t).
\]
In addition, we introduce the function \( f : [0, T] \rightarrow L^2(\Omega)^2 \) by
\[
[f(t)](x) := f(x, t), \quad x \in \Omega, \ t \in [0, T].
\]
We seek a weak solution $u(t)$ satisfying $u''(t) \in H^{-1}(\Omega)^2$ for a.e. $t \in [0, T]$. Hence the inner product $\langle \cdot, \cdot \rangle$ can also be interpreted as the pairing $\langle \cdot, \cdot \rangle$ which is defined between the dual spaces of $H^{-1}$ and $H^1$.

**Definition 3.1.** We say that the function $u \in L^2(0, T; H^1_0(\Omega)^2)$ with $u' \in L^2(0, T; L^2(\Omega)^2)$ and $u'' \in L^2(0, T; H^{-1}(\Omega)^2)$ is a weak solution of the initial boundary value problem (2) if it satisfies

1. $(\beta u''(t), v) + a[u, v; t] = (f(t), v)$, $\forall v \in H^1_0(\Omega)^2$, a.e. $t \in [0, T]$.
2. $u(0) = g, \quad u'(0) = h$.

We adopt the Galerkin method to construct the weak solution of the initial boundary value problem (2) by solving a finite dimensional approximation. We refer to [12] for the method to construct the weak solutions of the general second order parabolic and hyperbolic equations. The method begins with selecting smooth function $w_k, k \in \mathbb{N}$ by requiring that the smooth function $\{w_k\}_{k=1}^{\infty}$ is the standard orthogonal basis of $L^2(\Omega)^2$ and $\{w_k\}_{k=1}^{\infty}$ is also the orthogonal basis of $H^1_0(\Omega)^2$. In order to overcome the influence of $\beta$, we establish $u_m$ under the basis of $\{1/2w_k\}_{m}^{\infty}$ for positive integers $m$. That means, we find the solution $u_m$ in form of

$$u_m(t) := \sum_{k=1}^{m} u^k_m(t) \beta^{-1/2}w_k$$

satisfying the equation

$$(\beta u_m''(t), \beta^{-1/2}w_k) + a[u_m, \beta^{-1/2}w_k; t] = (f(t), \beta^{-1/2}w_k)$$

for $k = 1, \ldots, m$, $t \in [0, T]$, and the coefficient $u^k_m(t)$ should satisfy the initial conditions

$$u^k_m(0) = (g, \beta^{1/2}w_k), \quad \frac{du^k_m}{dt}(0) = (h, \beta^{1/2}w_k),$$

**Theorem 3.2.** For each $m \in \mathbb{N}$, there exists a unique solution $u_m$ in form of (3) satisfying (4)-(5).

**Proof.** Since $\{w_k\}_{k=1}^{\infty}$ is the standard orthogonal basis of $L^2(\Omega)^2$, from (3) we have

$$\langle \beta u''_m(t), \beta^{-1/2}w_k \rangle = \left( \beta \sum_{j=1}^{m} \frac{d^2u^j_m(t)}{dt^2} \beta^{-1/2}w_j, \beta^{-1/2}w_k \right) = \sum_{j=1}^{m} \frac{d^2u^j_m(t)}{dt^2} \int_{\Omega} w_j \cdot w_k dx = \frac{d^2u^j_m(t)}{dt^2},$$

\[ \text{for } k = 1, \ldots, m, \quad t \in [0, T], \quad \text{and the coefficient } u^k_m(t) \text{ should satisfy the initial conditions} \]

\[ u^k_m(0) = (g, \beta^{1/2}w_k), \quad \frac{du^k_m}{dt}(0) = (h, \beta^{1/2}w_k), \]
Lemma 3.3. It holds the estimate and

$$a[u_m, \beta^{-1/2} w_k; t]$$

$$= \mu \int_{\Omega} (\nabla u_m(t)M) : \left( \nabla \left( \beta^{-1/2} w_k \right) \right) dx$$

$$+ (\lambda + \mu) \int_{\Omega} \beta^{-1} \nabla \cdot (K u_m(t)) \nabla \cdot (K \beta^{-1/2} w_k) dx$$

$$= \mu \int_{\Omega} \left( \nabla \left( \sum_{j=1}^{m} u_m^j(t) \beta^{-1/2} w_j \right) \right) : \left( \nabla \left( \beta^{-1/2} w_k \right) \right) dx$$

$$+ (\lambda + \mu) \int_{\Omega} \beta^{-1} \nabla \cdot (K \beta^{-1/2} w_j) \nabla \cdot (K \beta^{-1/2} w_k) dx$$

$$= \sum_{j=1}^{m} u_m^j(t) \mu \int_{\Omega} (\nabla \beta^{-1/2} w_j M) : (\nabla (\beta^{-1/2} w_k)) dx$$

$$+ \sum_{j=1}^{m} u_m^j(t) (\lambda + \mu) \int_{\Omega} \beta^{-1} \nabla \cdot (K \beta^{-1/2} w_j) \nabla \cdot (K \beta^{-1/2} w_k) dx$$

$$= \sum_{j=1}^{m} u_m^j(t) a[\beta^{-1/2} w_j, \beta^{-1/2} w_k; t]$$

$$= \sum_{j=1}^{m} b_k^j u_m^j(t),$$

where $$b_k^j = a[\beta^{-1/2} w_j, \beta^{-1/2} w_k; t]$$, $$j, k = 1, \ldots, m$$. Define $$B = [b_k^j]_{m \times m}$$. Let

$$f^k(t) = (f^1(t), f^2(t) \cdots f_m(t))^T.$$ Define $$F(t) = (f^1(t), f^2(t) \cdots f_m(t))^T$$. Substituting (6)-(8) into (4), we obtain a linear system of second order equations

$$\frac{d^2 U_m(t)}{dt^2} + B U_m(t) = F,$$

subject to the initial conditions (5), where $$U_m = (u^1_m, \ldots, u^m_m)^T$$, it follows from the standard theory of the ordinary differential equations that there exists a unique $$C^2$$ function $$U_m(t)$$ consisting of $$u_m(t)$$ which satisfies (4)-(5) for $$t \in [0, T]$$.

The following lemma has been given in [18].

**Lemma 3.3.** It holds the estimate

$$\|w\|_{H^1_0(\Omega)}^2 \leq C \|\nabla w\|_{L^2(\Omega)}^2, \quad \forall w \in H^1_0(\Omega),$$

where $$C$$ is a positive constant.

**Theorem 3.4.** There exists a positive constant $$C$$ such that

$$\max_{t \in [0, T]} \left( \|u_m(t)\|_{H^1_0(\Omega)}^2 + \|u'_m(t)\|_{L^2(\Omega)}^2 \right) + \|\beta u''_m\|_{L^2(\Omega)}^2 \\
\leq C \left( \|f\|_{L^2(0, T; L^2(\Omega)^d)}^2 + \|g\|_{H^1(\Omega)^d}^2 + \|h\|_{L^2(\Omega)^d}^2 \right), \quad m = 1, 2, \ldots,$$
Proof. For each \( m \in \mathbb{N} \), \( \mathbf{u}_m(t) \in H_0^1(\Omega)^2 \), where \( \mathbf{u}_m(t) \in L^2(\Omega)^2 \), we have
\[
(\beta \mathbf{u}_m'(t), \mathbf{u}_m'(t)) + a[\mathbf{u}_m, \mathbf{u}_m'; t] = (f(t), \mathbf{u}_m'(t)), \quad \text{a.e. } t \in [0, T].
\] (9)
We observe that
\[
(\beta \mathbf{u}_m'(t), \mathbf{u}_m'(t)) = \frac{1}{2} \frac{d}{dt} \left( \frac{1}{2} \left\| \sqrt{\beta} \mathbf{u}_m'(t) \right\|_{L^2(\Omega)^2}^2 \right)
\] (10)
and
\[
a[\mathbf{u}_m, \mathbf{u}_m'; t] = \mu(M^{1/2} \nabla \mathbf{u}_{m,1}(t), M^{1/2} \nabla \mathbf{u}_{m,1}'(t)) + \mu(M^{1/2} \nabla \mathbf{u}_{m,2}(t), M^{1/2} \nabla \mathbf{u}_{m,2}'(t))
\]
\[
+ (\lambda + \mu) (\beta^{-1/2} \nabla \cdot (K \mathbf{u}_m(t)), \beta^{-1/2} \nabla \cdot (K \mathbf{u}_m'(t)))
\]
\[
= \frac{1}{2} \frac{d}{dt} \left( \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_{m,1}(t) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_{m,2}(t) \right\|_{L^2(\Omega)^2}^2
\]
\[
+ \left\| \sqrt{\lambda + \mu \beta^{-1/2} \nabla \cdot (K \mathbf{u}_m(t))} \right\|_{L^2(\Omega)}^2 \right).
\] (11)
where \( \mathbf{u}_m(t) = (\mathbf{u}_{m,1}(t), \mathbf{u}_{m,2}(t))^T \).

The matrix \( M \) is symmetric positive definite, \( \beta \) is a positive bounded function. There exists constants \( C_j, j = 1, \ldots, 4 \), depending only on \( \Omega, T \) and the coefficients of the initial boundary value problem (2) such that
\[
C_1 \left\| \nabla \mathbf{u}_{m,1}(t) \right\|_{L^2(\Omega)^2} \leq \left\| \mu M^{1/2} \nabla \mathbf{u}_{m,1}(t) \right\|_{L^2(\Omega)^2} \leq C_2 \left\| \nabla \mathbf{u}_{m,1}(t) \right\|_{L^2(\Omega)^2},
\] (12)
\[
C_3 \left\| \nabla \mathbf{u}_{m,2}(t) \right\|_{L^2(\Omega)^2} \leq \left\| \mu M^{1/2} \nabla \mathbf{u}_{m,2}(t) \right\|_{L^2(\Omega)^2} \leq C_4 \left\| \nabla \mathbf{u}_{m,2}(t) \right\|_{L^2(\Omega)^2} \cdot
\] (13)
For the norm of \( \left\| \sqrt{\lambda + \mu} \beta^{-1/2} \nabla \cdot (K \mathbf{u}_m(t)) \right\|_{L^2(\Omega)}^2 \), since \( K \) is symmetric, we derive that
\[
\nabla \cdot (K \mathbf{u}_m(t)) = (\nabla \cdot K) \cdot \mathbf{u}_m(t) + K : (\nabla \mathbf{u}_m(t)),
\]
where \( A : B = \text{tr}(AB^T) \) is the Frobenius inner product of square matrices \( A \) and \( B \). By calculating straightforwardly, we get \( \nabla \cdot K = 0 \). From the symmetric positive definiteness of \( K^2 \), there exists positive constants \( C_5 \) and \( C_6 \) such that
\[
C_5 \left\| \nabla \mathbf{u}_m(t) \right\|_{L^2(\Omega)^2} \leq \left\| \sqrt{\lambda + \mu} \beta^{-1/2} \nabla \cdot (K \mathbf{u}_m(t)) \right\|_{L^2(\Omega)}^2 \leq C_6 \left\| \nabla \mathbf{u}_m(t) \right\|_{L^2(\Omega)^2} \cdot
\] (14)
Combining (9)-(11), we obtain
\[
\alpha(t) = 2 \left( f(t), \mathbf{u}_m'(t) \right) \leq \left\| f(t) \right\|_{L^2(\Omega)^2}^2 + \left\| \mathbf{u}_m'(t) \right\|_{L^2(\Omega)^2}^2
\]
\[
\leq \delta(t) + \frac{1}{\beta_{\text{min}}} \alpha(t),
\]
where \( \delta(t) = \left\| f(t) \right\|_{L^2(\Omega)^2}^2 \) and
\[
\alpha(t) = \left\| \sqrt{\beta} \mathbf{u}_m'(t) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_{m,1}(t) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_{m,2}(t) \right\|_{L^2(\Omega)^2}^2
\]
\[
+ \left\| \sqrt{\lambda + \mu \beta^{-1/2} \nabla \cdot (K \mathbf{u}_m(t))} \right\|_{L^2(\Omega)}^2 \right).
\]
Based on the proof of Gronwall inequality, it can be derived that
\[
\alpha(t) \leq e^{\frac{4}{\beta_{\text{min}}}} \left( \alpha(0) + \int_0^t \delta(s) ds \right) \leq e^{\frac{4}{\beta_{\text{min}}}} \left( \alpha(0) + \left\| f \right\|_{L^2(0,T;L^2(\Omega)^2)}^2 \right), \quad t \in [0, T].
\]
Combining (12)-(14), we obtain

\[ 
\alpha(0) = \left\| \sqrt{\beta} u'_m(0) \right\|^2_{L^2(\Omega)^2} + \left\| \sqrt{\mu} M^{1/2} \nabla u_{m,1}(0) \right\|^2_{L^2(\Omega)^2} \\
+ \left\| \sqrt{\mu} M^{1/2} \nabla u_{m,2}(0) \right\|^2_{L^2(\Omega)^2} + \left\| \sqrt{\lambda + \mu} \beta^{-1} \nabla \cdot (K u_m(0)) \right\|^2_{L^2(\Omega)}
\]

\[ \leq 2 \max \{ \beta_{\max}, C_2, C_4, C_6 \} \left( \| h \|^2_{L^2(\Omega)^2} + \| g \|^2_{H^1(\Omega)^2} \right). \]

Thus

\[ \left( \| u'_m(t) \|^2_{L^2(\Omega)^2} + \| \nabla u_m(t) \|^2_{L^2(\Omega)^2} \right) \]

\[ \leq C_7 \alpha(t) \leq C_7 e^{\int_{\Omega} \alpha(0) + \| f \|^2_{L^2(0,T;L^2(\Omega)^2)}} \\
\leq C_8 \left( \| f \|^2_{L^2(0,T;L^2(\Omega)^2)} + \| h \|^2_{L^2(\Omega)^2} + \| g \|^2_{H^1(\Omega)^2} \right), \]

where

\[ C_7 = \min \{ \beta_{\min}, C_1, C_3, C_5 \}^{-1}, \]

\[ C_8 = C_7 e^{\int_{\Omega} \min \{ 1, 2 \max \{ \beta_{\max}, C_2, C_4, C_6 \} \}}. \]

By using Lemma 3.3, there exists a positive constant \( C_9 \) such that for any \( t \in [0, T] \) we have

\[ \| u'_m(t) \|^2_{L^2(\Omega)^2} + \| u_m(t) \|^2_{H^1_0(\Omega)^2} \leq C_9 \left( \| f \|^2_{L^2(0,T;L^2(\Omega)^2)} + \| h \|^2_{L^2(\Omega)^2} + \| g \|^2_{H^1(\Omega)^2} \right), \]

that means

\[ \max_{t \in [0,T]} \left\{ \| u'_m(t) \|^2_{L^2(\Omega)^2} + \| u_m(t) \|^2_{H^1_0(\Omega)^2} \right\} \]

\[ \leq C_9 \left( \| f \|^2_{L^2(0,T;L^2(\Omega)^2)} + \| h \|^2_{L^2(\Omega)^2} + \| g \|^2_{H^1(\Omega)^2} \right). \]

For any \( v \in H^1_0(\Omega)^2, \| v \|_{H^1(\Omega)^2} \leq 1 \), let \( v = v_1 + v_2 \), where

\[ v_1 = \sum_{k=1}^{m} \beta^{-1/2} v^k w_k, \quad v_2 = \sum_{k=m+1}^{\infty} \beta^{-1/2} v^k w_k. \]

It follows from the form of \( u_m(t) \) that

\[ \langle \beta u'_m(t), v \rangle = \langle \beta u'_m(t), v_1 \rangle + \langle \beta u'_m(t), v_2 \rangle \]

\[ = \left\langle \sum_{k=1}^{m} \frac{d^2 u^k_m(t)}{dt^2} w_k, \sum_{j=1}^{m} v^j \right\rangle + \left\langle \sum_{k=1}^{m} \frac{d^2 u^k_m(t)}{dt^2} w_k, \sum_{j=m+1}^{\infty} v^j \right\rangle \]

\[ = \left\langle \sum_{k=1}^{m} \frac{d^2 u^k_m(t)}{dt^2} w_k, \sum_{j=1}^{m} v^j \right\rangle = \langle \beta u'_m(t), v_1 \rangle. \]

Since \( \langle \beta^{1/2} v_1, \beta^{1/2} v_2 \rangle = 0 \), we have

\[ \| v_1 \|^2_{H^1(\Omega)^2} \leq C_{10} \| \beta^{1/2} v_1 \|^2_{H^1(\Omega)^2} \leq C_{10} \| \beta^{1/2} v \|^2_{H^1(\Omega)^2} \leq C_{11}. \]
It follows from the energy estimate in Theorem 3.4 that

\[ |\langle \beta u''_m(t), v \rangle| = |\langle \beta u''_m(t), v_1 \rangle| \leq |\langle f(t), v_1 \rangle| + |a[u_m, v_1; t]| \]

\[ \leq \|f(t)\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} + C_{12} \|u_m(t)\|_{H^1_0(\Omega)} \|v_1\|_{H^1(\Omega)}^2 \]

\[ \leq C_{13} \left( \|f(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{H^1_0(\Omega)} \right), \]

where \( C_{13} = C_{11} \max \{1, C_{12}\} \). Note that

\[ \|\beta u''_m(t)\|_{H^{-1}(\Omega)^2} = \sup_{\|v\|_{H^1(\Omega)^2} = 1} |\langle \beta u''_m(t), v \rangle|. \]

By using (15) and (16), we have

\[ \int_0^T \|\beta u''_m\|_{H^{-1}(\Omega)^2}^2 \, dt \leq C_{14} \left( \|f\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \|g\|_{H^1(\Omega)^2}^2 + \|h\|_{L^2(\Omega)^2}^2 \right). \] (17)

Combining (15) and (17), we get

\[ \max_{t \in [0,T]} \left( \|u_m(t)\|_{H^1_0(\Omega)}^2 + \|u'_m(t)\|_{L^2(\Omega)}^2 \right) + \|\beta u''_m\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 \]

\[ \leq C \left( \|f\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \|g\|_{H^1(\Omega)^2}^2 + \|h\|_{L^2(\Omega)^2}^2 \right), \quad m = 1, 2, \ldots, \]

the proof is completed. \(\square\)

**Theorem 3.5.** There exists a weak solution of the initial boundary value problem (2).

**Proof.** It follows from the energy estimate in Theorem 3.4 that

\[ \{u_m\}_{m=1}^\infty \text{ is bounded in } L^2(0,T;H^1_0(\Omega)^2), \]

\[ \{u'_m\}_{m=1}^\infty \text{ is bounded in } L^2(0,T;L^2(\Omega)^2), \]

\[ \{\beta u''_m\}_{m=1}^\infty \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)^2). \]

Therefore, there exists a subsequence still denoted as \( \{u_m\}_{m=1}^\infty \) and \( u \in L^2(0,T;H^1_0(\Omega)^2) \) with \( u' \in L^2(0,T;L^2(\Omega)^2) \) and \( \beta u'' \in L^2(0,T;H^{-1}(\Omega)^2) \) such that

\[ \begin{aligned}
  u_m &\rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1_0(\Omega)^2), \\
u'_m &\rightharpoonup u' \quad \text{weakly in } L^2(0,T;L^2(\Omega)^2), \\
\beta u''_m &\rightharpoonup \beta u'' \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)^2).
\end{aligned} \] (18)

Next we fix the integer \( n \) and choose functions \( v \in C^1([0,T];H^1_0(\Omega)^2) \) of the form

\[ v(t) = \sum_{k=1}^n v^k(t) \beta^{-1/2} w_k, \] (19)

where \( v_k, k = 1, \ldots, n \) are smooth functions. Letting \( m \geq n \), from (4) we have that

\[ \int_0^T \left( \langle \beta u''_m(t), v(t) \rangle + a[u_m, v; t] \right) dt = \int_0^T (f(t), v(t)) dt. \] (20)

Taking the limits \( m \to \infty \) in (20) and using (18) yield

\[ \int_0^T \left( \langle \beta u''(t), v(t) \rangle + a[u, v; t] \right) dt = \int_0^T (f(t), v(t)) dt, \] (21)
Proof. \( \text{It suffices to prove that} \) \\
\( \langle \beta u''(t), \tilde{v} \rangle + a[u, v; t] = (f(t), \tilde{v}) \) \\
and \\
\( u \in C([0, T]; H^1_0(\Omega)^2), \quad u' \in C([0, T]; L^2(\Omega)^2). \) \\
In the following, we verify that \\
\( u|_{t=0} = g, \quad u'|_{t=0} = h. \) \hspace{1cm} (22) \\
Choose any function \( v \in C^2([0, T]; H^1_0(\Omega)^2) \) with \( v(T) = v'(T) = 0. \) Using the integration by parts twice with respect to \( t \) in (21) yields \\
\[ \int_0^T (\beta v''(t), u(t)) + a[u, v; t]) dt = \int_0^T (f(t), v(t)) dt - (\beta u(0), v'(0)) \] \\
\[ + (\beta u'(0), v(0)). \] \hspace{1cm} (23) \\
Similarly, we have from (20) that \\
\[ \int_0^T (\beta v''(t), u_m(t)) + a[u_m, v; t]) dt = \int_0^T (f(t), v(t)) dt - (\beta u_m(0), v'(0)) \] \\
\[ + (\beta u_m'(0), v(0)). \] \hspace{1cm} (24) \\
Taking the limits \( m \to \infty \) in (24), using (5) and (18) yield \\
\[ \int_0^T (\beta v''(t), u(t)) + a[u, v; t]) dt = \int_0^T (f(t), v(t)) dt - (\beta g, v'(0)) + (\beta h, v(0)). \] \hspace{1cm} (25) \\
Comparing (23) and (25), we conclude (22) since \( v(0) \) and \( v'(0) \) are arbitrary. Hence \( u \) is a weak solution of the initial boundary value problem (2). \hspace{1cm} \( \square \) \\
**Theorem 3.6.** \hspace{1cm} The initial boundary value problem (2) has a unique weak solution. \\
**Proof.** It suffices to prove that \( u = 0 \) if \( f = g = h = 0. \) Fix \( 0 \leq t \leq T \) and let \\
\[ E(t) = \left\| \beta \partial_t u(t) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla u_1(t) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla u_2(t) \right\|_{L^2(\Omega)^2}^2 \] \\
\[ + \left\| (\lambda + \mu) \beta^{-2} \nabla \cdot (K u(t)) \right\|_{L^2(\Omega)}^2 \] \\
where \( u = (u_1, u_2)^T. \) Then for each \( t \in [0, T] \), using the integration by parts, we have \\
\[ E(t) - E(0) \] \\
\[ = \int_0^t E'(\tau) d\tau \] \\
\[ = 2 \int_0^t \int_{\Omega} \beta \partial_\tau^2 u \cdot \partial_\tau u dxd\tau + 2 \int_0^t \int_{\Omega} \mu (\nabla u M) \cdot (\nabla \partial_\tau u) dxd\tau \] \\
\[ + 2 \int_0^t \int_{\Omega} (\lambda + \mu) \beta^{-1} \nabla \cdot (K u) \nabla \cdot (K \partial_\tau u) dxd\tau \] \\
\[ = 2 \int_0^t \int_{\Omega} \left[ \beta \partial_\tau^2 u - \mu \nabla \cdot (\nabla u M) - (\lambda + \mu) K \nabla (\beta^{-1} \nabla \cdot (K u)) \right] \cdot \partial_\tau u dxd\tau \] \\
\[ = 2 \int_0^t \int_{\Omega} f \cdot \partial_\tau u dxd\tau. \]
If \( f = g = h = 0 \), we obtain that
\[
E(t) = E(0) = \left\| \sqrt{\beta} h \right\|_{L^2(\Omega)}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla g_1 \right\|_{L^2(\Omega)}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla g_2 \right\|_{L^2(\Omega)}^2 + \left\| \sqrt{\mu} (\lambda + \mu) \beta^{-1} \nabla \cdot (K \mathbf{u}) \right\|_{L^2(\Omega)}^2 = 0,
\]
which implies that
\[
\partial_t \mathbf{u} = \nabla \mathbf{u} = 0.
\]

Thus we have \( \mathbf{u} = 0 \) if \( f = g = h = 0 \), which completes the proof. \( \square \)

### 3.2. Stability

In this section, we introduce the stability estimate for the unique weak solution of the reduced initial boundary value problem (2).

**Theorem 3.7.** Let \( \mathbf{u} \) be the unique weak solution of the initial boundary value problem (2). Given \( f \in L^1(0, T; L^2(\Omega)^2) \), \( g \in H_0^1(\Omega)^2 \), \( h \in L^2(\Omega)^2 \), there exists a positive constant \( C \) such that
\[
\max_{t \in [0, T]} \left\{ \left\| \partial_t \mathbf{u} (\cdot, t) \right\|_{L^2(\Omega)^2}^2 + \left\| \nabla \mathbf{u} (\cdot, t) \right\|_{L^2(\Omega)^2}^2 \right\} \\
\leq \left( \left\| f \right\|_{L^1(0, T; L^2(\Omega)^2)}^2 + \left\| g \right\|_{H_0^1(\Omega)^2}^2 + \left\| h \right\|_{L^2(\Omega)^2}^2 \right).
\]

**Proof.** It follows from the discussion in previous section that the initial boundary value problem (2) has a unique weak solution \( \mathbf{u} \) satisfying
\[
\mathbf{u} \in L^2(0, T; H_0^1(\Omega)^2) \cap H^1(0, T; L^2(\Omega)^2).
\]

For any \( t \in [0, T] \), consider the energy function
\[
E(t) = \left\| \sqrt{\beta} \partial_t \mathbf{u} (\cdot, t) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_1 (\cdot, t) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_2 (\cdot, y) \right\|_{L^2(\Omega)^2}^2 + \left\| \sqrt{\mu} (\lambda + \mu) \beta^{-1} \nabla \cdot (K \mathbf{u} (\cdot, t)) \right\|_{L^2(\Omega)}^2
\]

According to Theorem 3.4, there exists \( C_j, j = 1, \ldots, 6 \) such that
\[
C_1 \left\| \nabla \mathbf{u}_1 (\cdot, t) \right\|_{L^2(\Omega)^2}^2 \leq \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_1 (\cdot, t) \right\|_{L^2(\Omega)^2}^2 \leq C_4 \left\| \nabla \mathbf{u}_1 (\cdot, t) \right\|_{L^2(\Omega)^2}^2,
\]
\[
C_2 \left\| \nabla \mathbf{u}_2 (\cdot, t) \right\|_{L^2(\Omega)^2}^2 \leq \left\| \sqrt{\mu} M^{1/2} \nabla \mathbf{u}_2 (\cdot, t) \right\|_{L^2(\Omega)^2}^2 \leq C_5 \left\| \nabla \mathbf{u}_2 (\cdot, t) \right\|_{L^2(\Omega)^2}^2,
\]
\[
C_3 \left\| \nabla \mathbf{u} (\cdot, t) \right\|_{L^2(\Omega)^2}^2 \leq \left\| \sqrt{(\lambda + \mu) \beta^{-1}} \nabla \cdot (K \mathbf{u} (\cdot, t)) \right\|_{L^2(\Omega)}^2 \leq C_6 \left\| \nabla \mathbf{u} (\cdot, t) \right\|_{L^2(\Omega)^2}^2.
\]
It follows from Theorem 3.6 and Young’s inequality that
\[
\min\{\beta_{\text{min}}, C_1, C_2, C_3\} \left( \|\partial_t u(\cdot, t)\|^2_{L^2(\Omega)^2} + \|\nabla u(\cdot, t)\|^2_{L^2(\Omega)^2 \times 2} \right) \\
\left\{ \begin{array}{l}
\leq E(t) = \int_0^t E'(\tau) d\tau + E(0) = 2 \int_0^t \int_\Omega f \cdot \partial_t u dx d\tau + E(0) \\
\leq 2 \max_{t \in [0, T]} \|\partial_t u(\cdot, t)\|_{L^2(\Omega)^2} \|\partial_t u(\cdot, t)\|_{L^2(\Omega)^2} d\tau + E(0)
\end{array} \right.
\]
\[
\leq 2 \max_{t \in [0, T]} \{ \|\partial_t u(\cdot, t)\|_{L^2(\Omega)^2} \}
\frac{1}{\epsilon} \left( \|f\|^2_{L^1(0, T; L^2(\Omega)^2)} + \|g\|^2_{L^2(\Omega)^2} + 2 \|\nabla g\|^2_{L^2(\Omega)^2 \times 2} \right) \\
+ 2 C_7 \left( \|h\|^2_{L^2(\Omega)^2} + \|\nabla g\|^2_{L^2(\Omega)^2} \right),
\]
where \(C_7 = 2 \max\{\beta_{\text{max}}, C_4, C_5, C_6\}\). Letting \(C_8 = \min\{\beta_{\text{min}}, C_1, C_2, C_3\}\) and taking \(\epsilon > 0\) to be sufficiently small such that \(C_8 - 2\epsilon > 0\), for example, \(\epsilon = \frac{C_8}{2}\). Hence, we have
\[
\max_{t \in [0, T]} \left\{ \|\partial_t u(\cdot, t)\|^2_{L^2(\Omega)^2} + \|\nabla u(\cdot, t)\|^2_{L^2(\Omega)^2 \times 2} \right\}
\leq C_9 \left( \|f\|^2_{L^1(0, T; L^2(\Omega)^2)} + \|h\|^2_{L^2(\Omega)^2} + \|g\|^2_{H^1(\Omega)^2} \right),
\]
where \(C_9 = \max\left\{ \frac{16}{C_8}, \frac{2 C_7}{C_8} \right\}\), which completes the proof. \(\square\)

4. A priori estimates. In this section, we derive a priori stability estimates for the elastic wave \(u\) with a minimum regularity requirement for the data and the explicit dependence on the time.

The variational problem of (2) is to find \(u(\cdot, t) \in H^1_0(\Omega)^2\) for \(t \in [0, T]\) such that
\[
\begin{aligned}
\int_\Omega \beta \partial_t^2 u(\cdot, t) \cdot wdx + \mu \int_\Omega (\nabla u(\cdot, t) M) : (\nabla w) dx \\
+ (\lambda + \mu) \int_\Omega \beta^{-1} \nabla \cdot (K u(\cdot, t)) \nabla \cdot (K w) dx = \int_\Omega f(\cdot, t) \cdot wdx
\end{aligned}
\tag{26}
\]
for any \(w \in H^1_0(\Omega)^2\).

Theorem 4.1. Let \(u\) be the unique weak solution of the initial boundary value problem (2). Given \(f \in L^1(0, T; L^2(\Omega)^2), g \in L^2(\Omega)^2, h \in L^2(\Omega)^2\), there exists positive constants \(C_1, C_2\) such that
\[
\|u\|^2_{L^\infty(0, T; L^2(\Omega)^2)} \leq K_1 \left( T^2 \|f\|^2_{L^1(0, T; L^2(\Omega)^2)} + \|g\|^2_{L^2(\Omega)^2} + T^2 \|h\|^2_{L^2(\Omega)^2} \right),
\]
and
\[
\|u\|^2_{L^2(0, T; L^2(\Omega)^2)} \leq K_2 \left( T^3 \|f\|^2_{L^1(0, T; L^2(\Omega)^2)} + T \|g\|^2_{L^2(\Omega)^2} + T^3 \|h\|^2_{L^2(\Omega)^2} \right).
\]

Proof. Let \(0 < s < T\) and define an auxiliary function
\[
\Phi(x, t) = \int_t^s u(x, \tau) d\tau, \quad x \in \Omega, \quad 0 \leq t \leq s.
\]
Note that
\[
\Phi(x, s) = 0, \quad \partial_t \Phi(x, t) = -u(x, t). \tag{27}
\]
For any $\Psi \in L^2([0, T]; L^2(\Omega))^2$, using the integration by parts and (27), we have

$$
\int_0^s \Psi(x, t) \cdot \Phi(x, t) \, dt = \int_0^s \left( \Psi(x, t) \cdot \int_t^s u(x, \tau) \, d\tau \right) \, dt
$$

$$
= \int_0^s \left[ \left( \int_0^t \Psi(x, \tau) \, d\tau \right) \cdot \left( \int_t^s u(x, \tau) \, d\tau \right) \right] \, dt
$$

$$
= \int_0^s \left( \int_0^t \Psi(x, \tau) \, d\tau \cdot u(x, t) \right) \, dt. \quad (28)
$$

Taking the test function $w = \Phi$ in (26) and integrating from 0 to $s$ with respect to $t$ yields

$$
\int_0^s \left( \int_\Omega \beta \partial_t^2 u \cdot \Phi \, dx \right) \, dt + \int_0^s \left( \int_\Omega \mu (\nabla u M) : (\nabla \Phi) \, dx \right) \, dt
$$

$$
+ \int_0^s \left( \int_\Omega (\lambda + \mu) \beta^{-1} \nabla \cdot (K u) \nabla \cdot (K \Phi) \, dx \right) \, dt \quad (29)
$$

$$
= \int_0^s \left( \int_\Omega f \cdot \Phi \, dx \right) \, dt.
$$

It follows from (27) that

$$
\int_0^s \left( \int_\Omega \beta \partial_t^2 u \cdot \Phi \, dx \right) \, dt
$$

$$
= \int_\Omega \beta \left[ \partial_t u \cdot \Phi |_{t=0} - \int_0^t \partial_t u \cdot \Phi \, dt \right] \, dx
$$

$$
= - \int_\Omega \beta h \cdot \Phi(x, 0) \, dx + \int_\Omega \frac{1}{2} \int_0^t \partial_t |u|^2 \, dx \, dt
$$

$$
= - \int_\Omega \frac{1}{2} \beta h \cdot \Phi(x, 0) \, dx + \int_\Omega \frac{1}{2} \left[ |u(x, s)|^2 - |u(x, 0)|^2 \right] \, dx
$$

$$
= \frac{1}{2} \left\| \sqrt{\beta} u(x, s) \right\|^2_{L^2(\Omega)^2} - \frac{1}{2} \left\| \sqrt{\beta} g \right\|^2_{L^2(\Omega)^2} - \int_\Omega \beta h \cdot \Phi(x, 0) \, dx, \quad (30)
$$

and

$$
\int_0^s \left( \int_\Omega \mu (\nabla u M) : (\nabla \Phi) \, dx \right) \, dt
$$

$$
= \frac{1}{2} \int_\Omega \mu (\nabla \Phi(x, 0) M) : (\nabla \Phi(x, 0)) \, dx \quad (31)
$$

$$
= \frac{1}{2} \left\| \sqrt{\mu} M^{1/2} \nabla \Phi_1(x, 0) \right\|^2_{L^2(\Omega)^2} + \frac{1}{2} \left\| \sqrt{\mu} M^{1/2} \nabla \Phi_2(x, 0) \right\|^2_{L^2(\Omega)^2},
$$

where $\Phi = (\Phi_1, \Phi_2)$. Also

$$
\int_0^s \left( \int_\Omega (\lambda + \mu) \beta^{-1} \nabla \cdot (K u) \nabla \cdot (K \Phi) \, dx \right) \, dt
$$

$$
= \frac{1}{2} \int_\Omega (\lambda + \mu) \beta^{-1} \nabla \cdot (K \Phi(x, 0)) \nabla \cdot (K \Phi(x, 0)) \, dx \quad (32)
$$

$$
= \frac{1}{2} \left\| \sqrt{(\lambda + \mu) \beta^{-1}} \nabla \cdot (K \Phi(x, 0)) \right\|^2_{L^2(\Omega)}. \]
It follows from the Cauchy-Schwarz inequality that
\[
\int \beta h \cdot \Phi(x,0) dx = \int \beta h \cdot \left( \int_0^s u(x,t) dt \right) dx \\
= \int_0^s \int \beta h \cdot u(x,t) dx dt \\
\leq \beta_{\text{max}} \| h \|_{L^2(\Omega)} \int_0^s \| u(\cdot,t) \|_{L^2(\Omega)}^2 dt.
\] (33)

For \(0 \leq t \leq s \leq T\), using (28) and Cauchy-Schwarz inequality, we have
\[
\int_0^s \left( \int_\Omega f \cdot \Phi dx \right) dt = \int \left( \int_0^s \left( \int_0^t f(\tau) d\tau \right) \cdot u(x,t) dt \right) dx \\
\leq \int_0^s \int_0^s \| f(\cdot,\tau) \|_{L^2(\Omega)^2} \| u(\cdot,t) \|_{L^2(\Omega)^2} dt dt \\
\leq \int_0^s \| f(\cdot,t) \|_{L^2(\Omega)^2} dt \int_0^s \| u(\cdot,t) \|_{L^2(\Omega)^2} dt.
\] (34)

Combining (29)-(34), we get
\[
\frac{1}{2} \| \beta u(x,s) \|^2_{L^2(\Omega)^2} + \frac{1}{2} \| \sqrt{\mu} \mathcal{M}^{1/2} \nabla \Phi_1(x,0) \|^2_{L^2(\Omega)^2} \\
+ \frac{1}{2} \| \sqrt{\mu} \mathcal{M}^{1/2} \nabla \Phi_2(x,0) \|^2_{L^2(\Omega)^2} + \frac{1}{2} \| \sqrt{\lambda + \mu} \beta^{-1} \nabla \cdot (K \Phi(x,0)) \|^2_{L^2(\Omega)} \\
\leq \frac{1}{2} \| \sqrt{\beta} g \|^2_{L^2(\Omega)^2} + \left( \int_0^s \| f(\cdot,t) \|_{L^2(\Omega)^2} dt + \beta_{\text{max}} \| h \|_{L^2(\Omega)^2} \right) \int_0^s \| u(\cdot,t) \|_{L^2(\Omega)^2} dt.
\]

Let \(C_1 = \frac{1}{2} \beta_{\text{min}}, C_2 = \frac{1}{2} \beta_{\text{max}}\), we have
\[
\| u(x,s) \|^2_{L^2(\Omega)^2} \\
\leq C_3 \left( \int_0^s \| f(\cdot,t) \|_{L^2(\Omega)^2} dt + \| h \|_{L^2(\Omega)^2} \right) \int_0^s \| u(\cdot,t) \|_{L^2(\Omega)^2} dt + C_3 \| g \|^2_{L^2(\Omega)^2},
\] (35)

where \(C_3 = \frac{\max\{1,2C_2\}}{C_1}\).

Taking the \(L^\infty\)-norm with respect to \(s\) on both sides of (35) and using Young’s inequality yields
\[
\| u \|^2_{L^\infty(0,T;L^2(\Omega)^2)} \leq C_3 T \left( \| f \|_{L^1(0,T;L^2(\Omega)^2)} + \| h \|_{L^2(\Omega)^2} \right) \| u \|_{L^\infty(0,T;L^2(\Omega)^2)} \\
+ C_3 \| g \|^2_{L^2(\Omega)^2} \\
\leq \frac{1}{\epsilon} C_3^2 T^2 \left( \| f \|_{L^1(0,T;L^2(\Omega)^2)} + \| h \|_{L^2(\Omega)^2} \right)^2 \\
+ \epsilon \| u \|^2_{L^\infty(0,T;L^2(\Omega)^2)} + C_3 \| g \|^2_{L^2(\Omega)^2}.
\]

Letting \(\epsilon\) to be a small enough positive constants, for example, \(\epsilon = \frac{1}{2}\), hence ,we have
\[
\| u \|^2_{L^\infty(0,T;L^2(\Omega)^2)} \leq 4C_3^2 T^2 \left( \| f \|_{L^1(0,T;L^2(\Omega)^2)} + \| h \|_{L^2(\Omega)^2} \right)^2 + 2C_3 \| g \|^2_{L^2(\Omega)^2} \\
\leq 8C_3^2 T^2 \left( \| f \|^2_{L^1(0,T;L^2(\Omega)^2)} + \| h \|^2_{L^2(\Omega)^2} \right) + 2C_3 \| g \|^2_{L^2(\Omega)^2} \\
\leq C_4 \left( T^2 \| f \|^2_{L^1(0,T;L^2(\Omega)^2)} + \| g \|^2_{L^2(\Omega)^2} + T^2 \| h \|^2_{L^2(\Omega)^2} \right),
\]
where \( C_1 = \max \{ 8C_3^2, 2C_3 \} \).

Integrating (35) with respect to \( s \) from 0 to \( T \) and using the Cauchy-Schwarz inequality and the Young’s inequality, we get

\[
\| u \|_{L^2(0,T; L^2_2(\Omega)^2)}^2 \\
\leq C_3 \left( \int_0^T \left( \| f \|_{L^1_1(0,T; L^2_2(\Omega)^2)} + \| h \|_{L^2_2(\Omega)^2} \right)^2 \, ds \right)^{\frac{1}{2}} \left( \int_0^T \left( \int_0^s \| u(x,t) \|_{L^2_2(\Omega)^2} \, dt \right)^2 \, ds \right)^{\frac{1}{2}} \\
+ C_3 T \| g \|_{L^2_2(\Omega)^2}^2
\]

\[
\leq C_3 T^{1/2} \left( \| f \|_{L^1_1(0,T; L^2_2(\Omega)^2)} + \| h \|_{L^2_2(\Omega)^2} \right) \left( \frac{1}{2} T^2 \int_0^T \| u(x,t) \|_{L^2_2(\Omega)^2}^2 \, dt \right)^{\frac{1}{2}} \\
+ C_3 T \| g \|_{L^2_2(\Omega)^2}^2
\]

\[
\leq \frac{1}{2} C_3^2 T^3 \left( \| f \|_{L^1_1(0,T; L^2_2(\Omega)^2)} + \| h \|_{L^2_2(\Omega)^2} \right)^2 + \epsilon \| u \|_{L^2_2(0,T; L^2_2(\Omega)^2)}^2 + C_3 T \| g \|_{L^2_2(\Omega)^2}^2.
\]

Similarly, taking \( \epsilon = \frac{1}{2} \), we obtain that

\[
\| u \|_{L^2_2(0,T; L^2_2(\Omega)^2)}^2 \\
\leq 2C_3^2 T^3 \left( \| f \|_{L^1_1(0,T; L^2_2(\Omega)^2)} + \| h \|_{L^2_2(\Omega)^2} \right)^2 + 2C_3 T \| g \|_{L^2_2(\Omega)^2}^2
\]

\[
\leq 4C_3^2 T^3 \left( \| f \|_{L^1_1(0,T; L^2_2(\Omega)^2)}^2 + \| h \|_{L^2_2(\Omega)^2}^2 \right) + 2C_3 T \| g \|_{L^2_2(\Omega)^2}^2
\]

\[
\leq C_3 \left( T^3 \| f \|_{L^1_1(0,T; L^2_2(\Omega)^2)}^2 + T \| g \|_{L^2_2(\Omega)^2}^2 + T^3 \| h \|_{L^2_2(\Omega)^2}^2 \right),
\]

where \( C_3 = \max \{ 4C_3^2, 2C_3 \} \), which completes the proof. \( \square \)

5. **Conclusion.** In this paper, we have studied the time-domain elastic wave scattering problem. We propose a new compressed coordinate transformation with high order smoothness to reduce the problem into an initial boundary value problem in a bounded domain over a finite time interval. We establish the well-posedness for the solution of the reduced problem by Galerkin method with modified subspace and energy estimates. A priori estimate with explicit time dependence is also obtained. Since the computational region of the equivalent reduced problem is small, the method of compressed coordinate transformation is suitable for numerical calculation.

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**Appendix A. Transformation process.** Transform the problem in Cartesian coordinates into polar coordinates: \( x = (x_1, x_2)^T, x_1 = \rho \cos \theta, x_2 = \rho \sin \theta \), the local orthonormal basis \( e_\rho, e_\theta \) in polar coordinates is

\[
e_\rho = (\cos \theta, \sin \theta)^T, \quad e_\theta = (-\sin \theta, \cos \theta)^T.
\]

Denote by \( \nabla_\rho \) and \( \nabla_\rho \) the divergence operator and the gradient operator in the old coordinates, respectively. We consider the Navier equation:

\[
\partial^2_t u - \mu \Delta_\rho u - (\lambda + \mu) \nabla_\rho \nabla_\rho \cdot u = 0,
\]

where \( \Delta_\rho \) is the Laplace operator.

Consider the change of variables \( \rho = \zeta(r) \), denote by \( \nabla_{r^*} \) and \( \nabla_r \) the divergence operator and the gradient operator in the new coordinates, respectively.
Lemma A.1. Let $u$ be a differentiable vector function, $v(r, \theta, t) = u(\rho, \theta, t) |_{\rho = \zeta(r)}$, then

$$
\nabla_{\rho} u(\rho, \theta, t) |_{\rho = \zeta(r)} = (\nabla_r, v) Q \begin{bmatrix} \frac{1}{\zeta} & 0 \\ 0 & \frac{\zeta'}{r} \end{bmatrix} Q^T,
$$

where $Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

Proof. Let $u = (u_1, u_2)^T$, then

$$
(\nabla_{\rho} u)^T = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial \rho} \cos \theta - \frac{1}{\rho} \frac{\partial u_1}{\partial \theta} \sin \theta & \frac{\partial u_2}{\partial \rho} \cos \theta - \frac{1}{\rho} \frac{\partial u_2}{\partial \theta} \sin \theta \\ \frac{\partial u_1}{\partial \rho} \sin \theta + \frac{1}{\rho} \frac{\partial u_1}{\partial \theta} \cos \theta & \frac{\partial u_2}{\partial \rho} \sin \theta + \frac{1}{\rho} \frac{\partial u_2}{\partial \theta} \cos \theta \end{bmatrix}
$$

$$
= Q \begin{bmatrix} \frac{1}{\zeta} & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \begin{bmatrix} \partial_r v_1 & \partial_r v_2 \\ \frac{1}{r} \partial_\theta v_1 & \frac{1}{r} \partial_\theta v_2 \end{bmatrix}
$$

$$
= Q \begin{bmatrix} \frac{1}{\zeta} & 0 \\ 0 & \frac{1}{r} \end{bmatrix} Q^T (\nabla_r, v)^T
$$

which completes the proof. \(\square\)

Lemma A.2. Let $M$ be a differentiable matrix function, $M(\rho, \theta, t) |_{\rho = \zeta(r)} = N(r, \theta, t)$, then

$$
\nabla_{\rho} \cdot M |_{\rho = \zeta(r)} = \frac{1}{\beta} \nabla_r \cdot \left( NQ \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{\zeta'}{r} \end{bmatrix} Q^T \right)
$$

where $\beta = \frac{\zeta''}{r}$.

Proof. Since

$$
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad \nabla \cdot M = \begin{bmatrix} \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} \\ \frac{\partial M_{21}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} \end{bmatrix}
$$

and

$$
\begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} = M^e_1 e_\rho + M^e_\theta e_\theta = N^e_1 e_r + N^e_\theta e_\theta,
$$

$$
\begin{bmatrix} M_{21} \\ M_{22} \end{bmatrix} = M^e_2 e_\rho + M^e_\theta e_\theta = N^e_2 e_r + N^e_\theta e_\theta.
$$

Then

$$
\nabla \cdot M = \begin{bmatrix} \frac{\partial}{\partial x_1} \left( M^e_1 \cos \theta - M^e_2 \sin \theta \right) + \frac{\partial}{\partial x_2} \left( M^e_1 \sin \theta + M^e_2 \cos \theta \right) \\ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho M^e_1 \cos \theta - \rho M^e_2 \sin \theta \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left( \rho M^e_1 \sin \theta + \rho M^e_2 \cos \theta \right) \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{1}{\zeta} \frac{\partial_r}{\partial r} \left( \frac{\zeta'}{r} N^e_1 \right) + \frac{1}{r} \partial_\theta \left( \zeta' N^e_2 \right) \\ \frac{1}{\zeta} \frac{\partial_r}{\partial r} \left( \frac{\zeta'}{r} N^e_1 \right) + \frac{1}{r} \partial_\theta \left( \zeta' N^e_2 \right) \end{bmatrix}
$$

$$
= \frac{r}{\zeta \zeta'} \nabla \cdot \tilde{N}
$$

where

$$
\tilde{N} = \begin{bmatrix} \frac{\zeta'}{r} N^e_1 \cos \theta - \zeta' N^e_2 \sin \theta \\ \frac{\zeta'}{r} N^e_1 \sin \theta + \zeta' N^e_2 \cos \theta \end{bmatrix}
$$

and then

$$
\tilde{N}^T = Q \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{\zeta'}{r} \end{bmatrix} Q^T \begin{bmatrix} N^e_1 & N^e_2 \\ N^e_2 & N^e_2 \end{bmatrix} = Q \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{\zeta'}{r} \end{bmatrix} Q^T N^T
$$
Thus, we have

\[ \nabla_\rho \cdot M|_{\rho = \zeta(r)} = \frac{1}{\beta} \nabla_r \cdot \left( NQ \begin{bmatrix} \xi & 0 \\ 0 & \xi' \end{bmatrix} Q^\top \right) \]

\[ \square \]

**Lemma A.3.** Let \( u \) be a differential vector function, \( v(r,\theta,t) = u(\rho,\theta,t)|_{\rho = \zeta(r)} \), then

\[ \Delta_\rho u|_{\rho = \zeta(r)} = \frac{1}{\beta(r)} \nabla_r \cdot (\nabla_r vM(r,\theta)) \]

where \( M(r,\theta) = Q \begin{bmatrix} \frac{\xi}{\sqrt{\xi'}} & 0 \\ 0 & \frac{\xi'}{\xi} \end{bmatrix} Q^\top \).

**Proof.** Using the last two lemmas

\[ \Delta_\rho u|_{\rho = \zeta(r)} = \nabla_r \cdot (\nabla_\rho u)|_{\rho = \zeta(r)} = \frac{1}{\beta} \nabla_r \cdot \left( \nabla_r vQ \begin{bmatrix} \frac{\xi}{\sqrt{\xi'}} & 0 \\ 0 & \frac{\xi'}{\xi} \end{bmatrix} Q^\top \right) \]

\[ = \frac{1}{\beta} \nabla_r \cdot \left( \nabla_r vQ \begin{bmatrix} \frac{\xi}{\sqrt{\xi'}} & 0 \\ 0 & \frac{\xi'}{\xi} \end{bmatrix} Q^\top \right) = \frac{1}{\beta} \nabla_r \cdot (\nabla_r vM) \]

which completes the proof. \[ \square \]

The following two lemmas are given in [20].

**Lemma A.4.** Let \( u \) be a differential vector function, \( v(r,\theta,t) = u(\rho,\theta,t)|_{\rho = \zeta(r)} \), then

\[ \nabla_\rho \cdot u|_{\rho = \zeta(r)} = \frac{1}{\beta} \nabla_r \cdot (Kv) \]

where \( K = Q \begin{bmatrix} \frac{\xi}{\sqrt{\xi'}} & 0 \\ 0 & \frac{\xi'}{\xi} \end{bmatrix} Q^\top \).

**Lemma A.5.** Let \( u \) be a differential scalar function, \( v(r,\theta,t) = u(\rho,\theta,t)|_{\rho = \zeta(r)} \), then

\[ \nabla_\rho u(\rho,\theta,t)|_{\rho = \zeta(r)} = A\nabla_r v, \]

where \( A = Q \begin{bmatrix} \frac{1}{\sqrt{\xi'}} & 0 \\ 0 & \frac{1}{\xi} \end{bmatrix} Q^\top \).

**Lemma A.6.** Let \( u \) be a differential vector function, \( v(r,\theta,t) = u(\rho,\theta,t)|_{\rho = \zeta(r)} \), then

\[ \nabla_\rho \nabla_\rho \cdot u|_{\rho = \zeta(r)} = A\nabla_r \left( \frac{1}{\beta} \nabla_r \cdot (Kv) \right). \]

**Lemma A.7.** Let \( u \) be a differential vector function and it satisfies

\[ \partial_t^2 u - \mu \Delta_\rho u - (\lambda + \mu) \nabla_\rho \nabla_\rho \cdot u = 0. \]

Let \( v(r,\theta,t) = u(\rho,\theta,t)|_{\rho = \zeta(r)} \), then

\[ \beta \partial_t^2 v - \mu \nabla_r \cdot (\nabla_r vM) - (\lambda + \mu) K \nabla_r \left( \frac{1}{\beta} \nabla_r \cdot (Kv) \right) = 0 \]

**Proof.** Combining Lemma A.3 and Lemma A.6, we have

\[ \partial_t^2 v - \frac{1}{\beta} \mu \nabla_r \cdot (\nabla_r vM) - (\lambda + \mu) A \nabla_r \left( \frac{1}{\beta} \nabla_r \cdot (Kv) \right) = 0. \]
Multiplying $\beta$ on both sides of the equation,

$$
\beta \partial^2 \! t \mathbf{v} - \mu \nabla_r \cdot (\nabla_r \mathbf{v} M) - (\lambda + \mu) \beta A \nabla_r \left( \frac{1}{\beta} \nabla_r \cdot (K \mathbf{v}) \right) = 0.
$$

A simple calculation yields

$$
\beta A = \frac{\zeta' r}{r} Q \left[ \begin{array}{cc}
\frac{1}{r} & 0 \\
0 & \zeta
\end{array} \right] Q^\top = Q \left[ \begin{array}{cc}
\frac{\zeta' r}{r} & 0 \\
0 & \zeta
\end{array} \right] Q^\top = K.
$$

which completes the proof. \qed

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