On the Deformation of $\Lambda$-Symmetry in B-field Background

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Abstract

In this note we will show that the $\Lambda$ symmetry, namely the $U(1)$ symmetry of the open string sigma model which relates the B-field and the $U(1)$ gauge field of a brane to each other, is deformed to a noncommutative version in a constant B-field background.

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Introduction

Recently open strings in a constant B-field background have been studied extensively \cite{1,2,3,4,5,6,7,8,9,10,11}. The main result obtained in these papers is that the world volume of the branes in a B-field background is a noncommutative space in terms of Connes noncommutative geometry.

The classical $\sigma$-model action in the B-field background,

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma [g_{ij}\partial_a X^i \partial^a X^j + \epsilon^{ab} B_{ij} \partial_a X^i \partial_b X^j + \frac{1}{2\pi\alpha'} \oint_{\partial \Sigma} d\tau A_i \partial_{\tau} X^i],$$

\label{1}

where $A_\mu$, is the $U(1)$ gauge field living on the D-brane, enjoys two $U(1)$ symmetries, \lambda-symmetry:

$$A \rightarrow A + d\lambda,$$

\label{2}

and $\Lambda$-symmetry:

$$\begin{cases} B \rightarrow B + d\Lambda \\ A \rightarrow A + \Lambda \end{cases}$$

\label{3}

It has been argued that $\lambda$-symmetry in the case of $N$ coincident D-branes is enhanced to the $U(N)$ symmetry, while the $\Lambda$-symmetry is believed to remain a $U(1)$ which acts on the $U(1)$ part of the $U(N)$ symmetry mentioned earlier \cite{12}. In the case of the non-zero B-field background it has been argued that at the quantum level the $\lambda$-symmetry will be deformed to a Moyal gauge symmetry \cite{1,3,4}, however the $\Lambda$-symmetry, has not been well studied in these backgrounds. In all of the works mentioned above the noncommutativity was obtained under the assumption of vanishing field strength for the $A$ field at infinity. However, in \cite{13}, where a non-zero constant field strength for the $A$ field was considered, it was shown that the deformation parameter, $\theta$, is not invariant under this classical $\Lambda$-symmetry, in other words, one should revise this symmetry at quantum level.

In this note, we would like to elaborate more on this symmetry. We show that this symmetry just like the $\lambda$-symmetry, will be deformed to a Moyal symmetry at quantum level. It is worth noting that similar to the arguments of \cite{4}, this is the result of the point splitting regularization we use.

Calculations

To study the $\Lambda$-symmetry at quantum level we use the Path integral formulation and first we build the necessary tools for this calculation.
It was shown in [14] that the two point function of open strings, \( X^i(z) \) showing a point in the upper half plane, constrained to the mixed boundary conditions,
\[
g_{ij}(\partial - \bar{\partial})X^j + B_{ij}(\partial + \bar{\partial})X^j|_{z=\bar{z}} = 0, \tag{4}
\]
in the decoupling limit, i.e. \( \alpha' \to 0 \) and other parameter properly scaled [9], is
\[
\langle X^i(z)X^j(z') \rangle = -\frac{1}{2\pi} \theta^{ij} \log \frac{z - z'}{\bar{z} - z'} + D^{ij}, \tag{5}
\]
where
\[
\theta^{ij} = (\frac{1}{B})^{ij}, \tag{6}
\]
and \( D^{ij} \) is a constant which does not depend on \( z, z' \) and plays no essential role so we will set it to zero. For the cases in which one of the operators is on the boundary, i.e. \( z' = \bar{z}' = \tau' \), and the other at \( z = \tau + i\sigma, \ \sigma \geq 0 \), the above propagator reads as:
\[
\langle X^i(z)X^j(z') \rangle = -\frac{i}{\pi} \theta^{ij} \tan^{-1} \frac{\sigma}{\tau - \tau'} \equiv \frac{i}{\pi} \theta^{ij} K(\sigma; \tau; \tau'). \tag{7}
\]
The equation (7) for the values of \( |\tau - \tau'| < \delta \) in the \( \delta \to 0 \) limit, does not dependent on \( \sigma \) and
\[
\langle X^i(z)X^j(z') \rangle = -i\theta^{ij} \epsilon(\tau - \tau'), \tag{8}
\]
where \( \epsilon(x) \) is 1 or -1 for positive or negative \( x \), respectively.

Let us consider the action (1),
\[
S(B, A) = S_0 + \int_{\partial \Sigma} d\tau A_i \partial_\tau X^i, \tag{9}
\]
where \( S_0 \) is the part of the action containing the string kinetic energy and the B-field. The propagator (7) has been calculated for this part. Under the \( \Lambda \)-symmetry, \( S(B, A) \) is transformed as
\[
\delta S = S(B + d\Lambda, A + \Lambda) - S(B, A) = \int_{\Sigma} d\tau d\sigma \partial_\tau \left( \Lambda_i \partial_\sigma X^i \right). \tag{9}
\]
As we expected it is a total time derivative, this is a classical symmetry.

In order to explore the \( \Lambda \)-symmetry at quantum level, we study the partition function of the theory under the \( \Lambda \)-transformations:
\[
Z(B, A) = \int DX e^{S_0 + \int_{\partial \Sigma} d\tau A_i \partial_\tau X^i}, \tag{10}
\]
\( \dagger \)Going to the decoupling limit makes the calculations and the results more clean and clear.
and hence
\[ Z + \delta Z = \int DX e^{S_0 + \int d\Omega^2 \frac{d\tau}{4} A_i \partial_{\tau} X^i + \delta S}, \]  
(11)

Expanding (11) up to the first order in \( A \) and \( \Lambda \), we obtain
\[
\delta Z = \int DX e^{S_0} \left( \int d\tau A_i \partial_{\tau} X^i \cdot \int d\tau' d\sigma \partial_{\tau'} (\Lambda_i \partial_{\sigma} X^i) \right) 
=: \int d\tau A_i \partial_{\tau} X^i : : \int d\tau' d\sigma \partial_{\tau'} (\Lambda_i \partial_{\sigma} X^i) : 

= \int d\tau d\sigma d\tau' \partial_{\tau'} \left( : A_i(x(\tau)) \Lambda_j(x(\sigma, \tau')) :: \partial_{\tau} X^i(\tau) \partial_{\sigma} X^j(\sigma, \tau') : + 
A_i(x(\tau)) \partial_{\sigma} X^i(\sigma, \tau') :: \partial_{\tau} X^j(\tau) \Lambda_j(\sigma, \tau') : \right).
\]
(12)

The above OPE’s can be evaluated by means of (7), however performing the integration in \( \tau' \), because of (8), one should note the time ordering of the operators. In the field theory language this corresponds to regularizing the field products by the point splitting method [9]. Since we are only interested in the constant \( B \) and \( dA \) backgrounds, for simplicity we consider the \( \Lambda \) transformations which are linear in \( X^i \),
\[ \Lambda_i = \frac{1}{2} f_{ij} X_j, \]
(13)
where \( f \) is an arbitrary constant anti-symmetric two form. Inserting (13) into (12) we find
\[
\delta Z = \frac{1}{2} \theta^{ij} \theta^{kl} f_{jl} \int d\tau d\sigma d\tau' \partial_{\tau'} \left( \partial_i A_k K \partial_{\sigma} \partial_{\tau} K + \partial_k A_i \partial_{\sigma} K \partial_{\tau} K \right).
\]
(14)

Integrating over sigma by parts, we obtain
\[
\delta Z = \frac{1}{2} \theta^{ij} \theta^{kl} f_{jl} \int d\tau d\sigma d\tau' \partial_{\tau'} \left( F_{ik} K \partial_{\sigma} \partial_{\tau} K \right),
\]
(15)
with
\[ F_{ik} = \partial_k A_i - \partial_i A_k. \]

The equation (15) can be regularized by the point splitting and then we can perform the integration in \( \tau' \). The answer expressed in terms of the Moyal bracket is
\[
\delta Z = i : \int d\tau d\sigma \{ A_i, \Lambda_j \}_{M.B.} \partial_{\tau} X^i(\tau) \partial_{\sigma} X^j(\sigma, \tau) : ,
\]
(16)
where \( A \) is boundary valued.

To cancel this extra term we should deform the \( \Lambda \)-symmetry as the following:
\[
\begin{align*}
B & \to B + d\Lambda + i \{ \Lambda_i, A_j \}_{M.B.} \\
A & \to A + \Lambda
\end{align*}
\]
(17)
Although we have presented the calculations only up to the first order in the $A$ field, along the lines of [4] it can be checked that the variations in any arbitrary power of $A$ will cancel out in lights of (17).

Discussion

Here we have shown that the $\Lambda$-symmetry should be modified in the presence of the $B$-field, similarly to the $\lambda$-symmetry. The interesting point is that the $\theta$ parameter is a function of the background $B$-field, and therefore is changed under a $\Lambda$-transformation. Hence the parameter $\theta$ is not invariant under this $\Lambda$-symmetry.

Another point we should note is that in order to use the noncommutative description the background $A$ field should be set to zero at infinity, by a proper $\Lambda$-transformation, i.e. $\Lambda = -A$. Under this $\Lambda$-transformation the $B$-field will transform as:

$$B \rightarrow \hat{B} = B - dA - i\{A,A\};$$
(18)

recalling that $dA = F$, we find

$$\hat{B} = B - (F - \frac{1}{4} F\theta F).$$
(19)

In the above relation the space-time indices have been summed over like matrix products. The (19) is the same as the Seiberg-Witten map which relates the commutative and non-commutative descriptions. In other words the results of different regularization methods are related by this modified $\Lambda$-symmetry. So, there is some hope that the deformed $\Lambda$-symmetry sheds light on the Seiberg-Witten map. Studying this point and other physical consequences of this symmetry is postponed to future works [15].

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