NEW METRICS FOR RISK ANALYSIS

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ABSTRACT. This paper introduces a new framework for risk analysis for distributions of finite mean, building on metrics $\mu_s$ indexed by $s \in \mathbb{R}$. The neutral metric $\mu_0$ can be written as a simple linear combination of the mean and the cumulative entropy. The sequence $\{\mu_n, n \geq 1\}$ characterizes distributions up to translation. The order derived from these metrics respects the usual stochastic order. The range of the metric is described explicitly for positive random variables and in the case of finite variance, with a unique maximizer up to affine transformation. Along the way, we obtain a characterization of the logistic and the exponential distribution by the cumulative entropy. Our metrics are then embedded in a generic risk analysis framework that entails dual properties and provides for interval screening and operators variations. Contrary to the existing literature, this framework does not parametrize risk with a quantile. This instead integrates information along all possible risk levels and assigns weight to each of them, yielding an alternative approach to risk understanding.

1. INTRODUCTION

Real random variables are a natural representation for uncertainties that yield gains or losses. A specific objective of risk theory is to design a function $\mu$ that maps the law of such real random variable $X$ onto a real number $\mu(X)$ measuring the underlying risk, and which we may call a metric. A metric allows the comparison of any two random variables in terms of risks (and opportunities alike). This means both ranking them and figuring out “how far away” they are one another in terms of risk. By the same token, a metric gives rise to an order on random variables which is simply defined by $X \preceq_{\mu} Y$ for $\mu(X) \leq \mu(Y)$. There are two basic and well-known orders already worth considering for such purposes:

- The stochastic dominance order $\preceq_{\text{st}}$, that is $X \preceq_{\text{st}} Y$ if $\mathbb{P}[X > x] \leq \mathbb{P}[Y > x]$ for all $x \in \mathbb{R}$;
- The convex order $\preceq_{\text{cx}}$, that is $X \preceq_{\text{cx}} Y$ if $\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)]$ for all $\varphi$ convex.

Neither the stochastic dominance nor the convex order are really satisfying unfortunately. The stochastic dominance order is very natural but also quite stringent since it prevents cumulative functions from crossing each other. The convex order, which implies the equality of expectations, can be viewed as an ordering of the second kind since it gives an inequality on variances, but is also stringent since it prevents integrated cumulative functions and integrated tail distribution functions from crossing one another - see Chapter 3 in [10]. A less stringent order is the lexical order $\preceq_L$: the random variable of highest expectation is dominant; in case of equality, take that of smallest variance; if equality, take that of smallest centred moment of order 3, etc. However, the latter does not work well for at least three reasons:

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• it can select a variable over another with an arbitrarily smaller superior expectation and an arbitrarily higher variance, which is hardly a sound risk management approach;
• The integer moments fully characterize the law of a compact-valued random variable; thus the neutral lines of a given distribution for $\prec_L$ is reduced to that distribution only. With a trivial efficient frontier there is no possible trade-off between risks and opportunities, which strongly hints a methodological flaw;
• Practical implementation can be challenging: estimate of moments of order 3 and above, even with good computational power, is often poor and hence a risk of error.

This motivates the research of new metrics leading to orders that comply with three specifications:

• The induced order should be total, modulo non-trivial neutral lines (i.e. equivalent classes) giving rise to a possible “trade-off” between risks and opportunities;
• It should respect the stochastic dominance where it exists (monotonicity);
• It should leave some room $I$ for parametrization for practical applications: specific fields and contexts require the flexibility to design bespoke risk metrics.

We consider the above properties a must for sound risk management practice. Below are two other properties we deem desirable:

• $\mu = \{\mu_i, i \in I\}$ characterizes the law (thoroughness);
• For all $i \in I$, $\mu_i$ is well-defined on laws having a finite moment of fractional order $\alpha$ for some $\alpha > 1$, and not necessarily a variance.

When the fourth property does not hold true, this intuitively means that $\mu$ fails to capture part of the risk information of the underlying random variable, hence the lack of thoroughness. As for the fifth property, it is a practical matter: many laws naturally occurring and worth of consideration for risk analysis such as Pareto, Power law or Generalized Extreme Values are in $L_\alpha$ for some $\alpha \in (1, 2)$ only.

These properties do not recourse to any concept or exogenous variable relevant to a particular field (contrary to the Capital Asset Pricing Model e.g.) and hence the wide spectrum for applications. The only requirement is that some real random variables are acceptable surrogates for the outcomes of interests. Examples range from Net Present Value (NPV) in finance and economics to the concentration of a chemical agent in environmental and health sciences, or the lifespan of an equipment in engineering, etc. There is already a large literature on risk metrics, essentially attached to quantitative finance and actuarial science, where the Value at Risk (VaR) and the Tail Value at Risk (TVaR) are emblematic examples. This literature suggests specifications for risk metrics and efforts to characterize them. Some of them are compliant with the above stated conditions. This is the case of risk coherent measures, which even comply with further desirable properties in the finance industry. These also happen to be conveniently characterized as expectations over a particular probability measure – dual or robust representation [1].
All these metrics are parametrized by a given quantile of the underlying random variable, that is \( T = [0, 1] \). Instead of choosing a quantile, we use all of them, ranging across all levels of risks, and merge the result into one metric. The parametrization of this metric occurs through weighting functions instead and puts a differential emphasis on certain sections of the distribution support, and so yielding an alternative approach to risk understanding.

The paper is organized as follows: in Section 2 we introduce definitions and concepts, show their compliance with the above five design criteria in Section 3, then explore further properties of these metrics in Section 4. Section 5 discusses the paper findings through an extension of the proposed framework, some practical illustrations, an example and concluding remarks.

2. Main notations and definitions

Our framework will be the class, which we denote by \( \mathbb{D} \), of real absolutely continuous random variables with finite expectation and whose support is an interval, possibly unbounded. For \( X \in \mathbb{D} \) and \( x \in \mathbb{R} \), we call \( f_X \) its density function, \( F_X(x) = \mathbb{P}[X \leq x] \) its distribution function, \( \bar{F}(x) = 1 - F_X(x) = \mathbb{P}[X > x] \) its survival function and \( F_X^{-1}(x) = \inf\{y \in \mathbb{R}, F_X(y) \geq x\} \) its inverse distribution function. Since \( X \) has finite expectation, we can also define

\[
G_X(x) = \int_{-\infty}^{x} F_X(y) \, dy = \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(z) \, dz \, dy
\]

for all \( x \in \mathbb{R} \), with \( G_X'(x) = F_X(x) \) and \( G_X''(x) = f_X(x) \). When there is no ambiguity we will set \( f, F, \bar{F}, F^{-1} \) and \( G \) respectively for \( f_X, F_X, \bar{F}_X, F_X^{-1} \) and \( G_X \). Recall the definitions of common risk metrics of level \( \alpha \in (0, 1) \) for a random variable \( X \in \mathbb{D} \):

- **Value at Risk:**
  \[
  \text{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{y \in \mathbb{R}, F_X(y) \geq \alpha\}
  \]

- **Tail value at Risk:**
  \[
  TV\text{VaR}_\alpha(X) = \mathbb{E}[-X \mid X \leq F_X^{-1}(\alpha)] = -\frac{1}{\alpha} \int_{0}^{\alpha} F_X^{-1}(y) \, dy.
  \]

Notice that either one of the two families \( \{\text{VaR}_\alpha(X), \alpha \in (0, 1)\} \) and \( \{TV\text{VaR}_\alpha(X), \alpha \in (0, 1)\} \) determines the distribution of \( X \in \mathbb{D} \). Throughout, we will call repeatedly on the property of \( \mathbb{D} \) that \( F_X \) and \( F_X^{-1} \) are increasing bicontinuous bijections between \( [0, 1] \) and \( \text{Supp} \, X \), where \( \text{Supp} \, X \) denotes the closed support of \( X \in \mathbb{D} \). This is a consequence of the fact that \( \text{Supp} \, X \) is assumed to be an interval. In particular, one has \( F_X \circ F_X^{-1}(x) = x \) for all \( x \in (0, 1) \) and \( F_X^{-1} \circ F_X(x) = x \) for all \( x \in \text{Supp} \, X \). For any \( X \in \mathbb{D} \) and \( x \in \text{Supp} \, X \), we will set \( X_x \) for the random variable \( X \) truncated at \( x \), with density \( f_X(\cdot)/F_X(x) \) on \( (-\infty, x] \) and \( 0 \) on \( (x, \infty) \). It is clear by definition that \( X_x \) is integrable with \( \mathbb{E}[X_x] \leq x \) for all \( x \in \mathbb{R} \) and \( \mathbb{E}[X_x] \to \mathbb{E}[X] \) as \( x \to \infty \). Since \( x \mapsto \mathbb{E}[X_x] \) is continuous, this implies that it is also bounded from above on \( \mathbb{R} \). Let \( W \) be the set of measurable functions from \( [0, 1] \) to \( \mathbb{R}^+ \) whose integral equals 1. For any \( w \in W \), we consider the following
function from $\mathbb{D}$ to $\mathbb{R} \cup \{-\infty\}$:

$$
\mu_w(X) = \int_0^1 w(t) \mathbb{E}[X_{F^{-1}(t)}] \, dt,
$$

which we call the $\mu$-metric of $X$ with weight $w$. For any $s \geq 0$, the functions $p_s : t \mapsto (1 + s)t^s$ and $q_s : t \mapsto (1 + s)(1 - t)^s$ belong to $W$ and for the sake of clarity, we will set $\mu_s$ for $\mu_{p_s}$ and $\mu_{-s}$ for $\mu_{q_s}$ and call it the $\mu$-metric of order $s$, respectively $-s$. The full $\mu$-order $\prec_\mu$ is then defined as follows:

$$
X \prec_\mu Y \text{ if } \mu_s(X) \leq \mu_s(Y) \text{ for all } s \in \mathbb{R}.
$$

The following functional acting on $\mathbb{D}$ and indexed by $s \in \mathbb{R}$:

$$
\Delta_s(X) = \mathbb{E}[X] - \mu_s(X) \in \mathbb{R} \cup \{\infty\},
$$

will be of central use throughout the paper. We also define the dual metric

$$
\tilde{\mu}_s(X) = -\mu_s(-X) = \mathbb{E}[X] + \tilde{\Delta}_s(X) = \mathbb{R} \cup \{\infty\}
$$

acting on $\mathbb{D}$ for all $s \in \mathbb{R}$, with the notation $\tilde{\Delta}_s(X) = \Delta_s(-X)$.

3. General properties

3.1. Integral formulas. In this paragraph we obtain some closed integral formulas for the central quantities $\Delta_s$, which have a different shape according as $s > 0$, $s = 0$ or $s < 0$. Our first result establishes a direct connection between the functional $\Delta_0(X)$ and the so-called cumulative entropy introduced in [4].

**Theorem 1.** For any $X \in \mathbb{D}$ one has

$$
\Delta_0(X) = -\int_\mathbb{R} F(x) \log F(x) \, dx \in (0, \infty].
$$

**Proof.** Since $X$ has finite expectation, for all $x \geq 0$ one has

$$
xF(-x) \leq -\int_{-\infty}^0 uf(u) \, du \to 0 \quad \text{and} \quad x\tilde{F}(x) \leq \int_x^\infty uf(u) \, du \to 0 \quad (1)
$$

as $x \to \infty$. An integration by parts and the first limit in (1) gives

$$
G(x) = xF(x) - \int_{-\infty}^x uf(u) \, du \quad (2)
$$

for all $x \in \mathbb{R}$ and this implies

$$
\frac{f(x)}{F(x)} G(x) = f(x) \left( x - \int_{-\infty}^x \frac{uf(u)}{F(x)} \, du \right) = f(x) (x - \mathbb{E}[X_x]).
$$

Since $x - \mathbb{E}[X_x] \geq 0$ for all $x \in \mathbb{R}$, we can integrate on both sides and obtain

$$
\int_\mathbb{R} \frac{f(x)}{F(x)} G(x) \, dx = \mathbb{E}[X] - \int_\mathbb{R} f(x) \mathbb{E}[X_x] \, dx
$$

with both integrals well-defined. Suppose first that the cumulative entropy is finite, in other words

$$
-\int_\mathbb{R} F(x) \log F(x) \, dx < \infty.
$$
Then, another integration by parts implies
\[ \int_{\mathbb{R}} \frac{f(x)}{F(x)} G(x) \, dx = - \int_{\mathbb{R}} F(x) \log F(x) \, dx + [G(x) \log F(x)]_{\pm \infty} \]
where the bracket is zero because \( G(x) \log F(x) \sim -G(x) \bar{F}(x) \sim -x \bar{F}(x) \to 0 \) as \( x \to \infty \) by the second limit in \([\text{I}]\), and
\[ 0 \leq -G(x) \log F(x) \leq - \int_{-\infty}^{x} F(u) \log F(u) \, du \to 0 \]
as \( x \to -\infty \). Putting everything together, we have shown that
\[ - \int_{\mathbb{R}} F(x) \log F(x) \, dx = \mathbb{E}[X] - \int_{\mathbb{R}} \mathbb{E}[X_{x}] f(x) \, dx = \mathbb{E}[X] - \int_{0}^{1} \mathbb{E} [X_{F^{-1}(u)}] \, du = \Delta_0(X) \]
as required, where in the second equality we have made the change of variable \( x = F^{-1}(t) \). Finally, if the cumulative entropy is infinite, then the equality
\[ - \int_{x}^{\infty} F(u) \log F(u) \, du - G(x) \log F(x) = \int_{x}^{\infty} \frac{f(u)}{F(u)} G(u) \, dx \]
for every \( x \in \mathbb{R} \), which is obtained similarly as above, leads to
\[ \Delta_0(X) = \lim_{x \to -\infty} \int_{x}^{\infty} \frac{f(u)}{F(u)} G(u) \, dx \geq \lim_{x \to -\infty} - \int_{x}^{\infty} F(u) \log F(u) \, du = \infty. \]
\[ \square \]

**Remark 1.** In the next section, we will display several upper bounds on \( \Delta_0(X) \). The result also shows that for every \( X \in \mathbb{D} \), one has \( \Delta_0(X) < \infty \) if and only if the cumulative entropy of \( X \) is finite. In the following, we will set
\[ \mathbb{D}_e = \{X \in \mathbb{D} \mid \Delta_0(X) < \infty\}. \]
Clearly, a random variable \( X \in \mathbb{D} \) is in \( \mathbb{D}_e \) if and only if \( F_X \log F_X \) is integrable at \(-\infty\), and in particular, one has \( \mathbb{D} \cap \mathcal{L}_\alpha \subset \mathbb{D}_e \) for every \( \alpha > 1 \) where, here and throughout, \( \mathcal{L}_\alpha \) is the set of real random variable having finite \( \alpha \)-th moment. Observe also that all distributions in \( \mathbb{D} \) with bounded support from below are in \( \mathbb{D}_e \). In particular, all positive distributions in \( \mathbb{D} \) belong to \( \mathbb{D}_e \).

We next handle the positive case and obtain a simple formula which is here finite for all \( X \in \mathbb{D} \) and not only for \( X \in \mathbb{D}_e \).

**Theorem 2.** For every \( X \in \mathbb{D} \) and \( s > 0 \), one has
\[ \Delta_s(X) = \frac{1}{s} \int_{\mathbb{R}} F(x)(1 - F(x)^s) \, dx \in (0, \infty). \]

**Proof.** The argument is similar to that of Theorem [\(\text{I}\)]. Since \( F(x)(1 - F(x)^s) \sim s \bar{F}(x) \) as \( x \to \infty \) and \( F(x)(1 - F(x)^s) \sim F(x) \) as \( x \to -\infty \), the integral on the right-hand side is finite and an integration by parts implies on the one hand
\[ \frac{1}{s} \int_{\mathbb{R}} F(x)(1 - F(x)^s) \, dx = \int_{\mathbb{R}} G(x) F(x)^{s-1} f(x) \, dx + [G(x) F(x)^s]_{\pm \infty} \]
\[ = \int_{\mathbb{R}} G(x) F(x)^{s-1} f(x) \, dx \]
\[ = \int_{\mathbb{R}} x F^s(x) f(x) \, dx - \int_{\mathbb{R}} h(x) F^s(x) f(x) \, dx \]
with \( h(x) = \mathbb{E}[X_x] \), where in the third equality we have used \( \int \). On the other hand, the change of variable \( x = F^{-1}(t) \) and another integration by part yield

\[
\mu_s(X) = (s + 1) \int_{\mathbb{R}} h(x) F(x)^s f(x) \, dx
\]

\[
= [h(x)F(x)^{s+1}]_{s=\infty} - \int_{\mathbb{R}} h'(x)F^{s+1}(x) \, dx
\]

\[
= \mathbb{E}[X] + \int_{\mathbb{R}} h(x) F^s(x) f(x) \, dx - \int_{\mathbb{R}} x F^s(x) f(x) \, dx
\]

where in the third equality we have used the fact that \( h(x)F^{s+1}(x) \) tends to \( \mathbb{E}[X] \) as \( x \to \infty \) resp. to 0 as \( x \to -\infty \) since \( s > 0 \). Putting everything together, we obtain

\[
\Delta_s(X) = \mathbb{E}[X] - \mu_s(X) = \frac{1}{s} \int_{\mathbb{R}} F(x)(1 - F(x)^s) \, dx.
\]

\[\square\]

**Remark 2.** (a) The above proof shows the alternative expression

\[
\Delta_s(X) = \int_{\mathbb{R}} G(x) F(x)^{s-1} f(x) \, dx
\]

for every \( s > 0 \), which has a more complicated character since it also involves the density \( f \).

(b) For every \( X \in \mathbb{D} \), the mapping \( s \mapsto \Delta_s(X) \) is non-increasing on \((0, \infty)\) from \( \Delta_0(X) \) to 0. Indeed, for every \( t = F(x) \in (0,1) \), the derivative of \( s \mapsto s^{-1}(1 - t^s) \) is

\[
t^s(\log(t^{-s}) + 1 - t^{-s}) - \frac{t^s}{s^2} < 0,
\]

and the limits of \( \Delta_s(X) \) at 0 and \( \infty \) are obtained by monotone convergence.

We next prove the important property that the sequence \( \{\Delta_n(X), n \geq 1\} \) determines the law of \( X \in \mathbb{D} \), up to translation. This is a sharp difference with the characterization of laws through the families \( \text{VaR}_\alpha \) or \( \text{TVaR}_\alpha \), which requires the continuum \( \alpha \in (0,1) \). A consequence of the following theorem is also that for centered random variables \( X \in \mathbb{D} \), the family \( \{\mu_n(X), n \geq 1\} \) characterizes the law of \( X \).

**Theorem 3.** Let \( X, Y \in \mathbb{D} \) such that \( \Delta_n(X) = \Delta_n(Y) \) for all \( n \geq 1 \). Then, \( X \) and \( Y \) have the same law up to translation.

**Proof.** Since \( X \in \mathbb{D} \subset \mathcal{L}_1 \), the positive measure \( dF_X^{-1}(x) \) on \((0,1)\) is such that

\[
\int_0^1 x(1 - x) dF_X^{-1}(x) = \int_{\mathbb{R}} F(z) \bar{F}(z) \, dz < \infty
\]

and the same holds for \( dF_Y^{-1}(x) \). Set \( d\bar{F}_X^{-1}(x) = x(1 - x) dF_X^{-1}(x) \), a finite positive measure on \((0,1)\) which clearly determines \( dF_X^{-1} \) and hence the law of \( X \), up to translation. Similarly, we set \( d\bar{F}_Y^{-1}(x) = x(1 - x) dF_Y^{-1}(x) \). For every \( n \geq 1 \), the equality \( \Delta_n(X) = \Delta_n(Y) \) reads

\[
\int_0^1 x(1 - (1 - (1 - x))^n) dF_X^{-1}(x) = \int_0^1 x(1 - (1 - (1 - x))^n) dF_Y^{-1}(x)
\]
which, expanding the polynomial, implies
\[ \sum_{k=1}^{n} (-1)^k \binom{n}{k} \int_{0}^{1} (1 - x)^{k-1} d\tilde{F}_X^{-1}(x) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \int_{0}^{1} (1 - x)^{k-1} d\tilde{F}_Y^{-1}(x). \]
This being true for every \( n \geq 1 \), it is easy to deduce from the above triangular array that
\[ \int_{0}^{1} (1 - x)^{n} d\tilde{F}_X^{-1}(x) = \int_{0}^{1} (1 - x)^{n} d\tilde{F}_Y^{-1}(x) \]
also holds true for all \( n \geq 0 \). Recall now that every positive finite measure \( \mu \) on \((0, 1)\) is determined by its integer moments since by Fubini’s theorem
\[ \int_{0}^{1} e^{sx} d\mu(x) = \sum_{n \geq 0} \left( \int_{0}^{1} x^n d\mu(x) \right) \frac{s^n}{n!} \]
for all \( s \in \mathbb{R} \) and the Laplace transform on the left-hand side characterizes \( \mu \). Putting everything together, we have shown that \( d\tilde{F}_X^{-1}(1 - x) = d\tilde{F}_Y^{-1}(1 - x) \) or equivalently that
\[ dF_X^{-1}(x) = dF_Y^{-1}(x) \]
for all \( x \in (0, 1) \), which completes the argument. \( \square \)

Finally, we handle the negative case. Here the formula is given also in terms of \( f \) and \( G \). See however the formula (3) below for an expression depending solely on \( F \).

**Theorem 4.** For every \( X \in \mathbb{D} \) and \( s > 0 \), one has
\[ \Delta_s(X) = \int_{\mathbb{R}} \frac{(1 - \bar{F}(x)^{s+1})}{F^2(x)} G(x) f(x) dx \in (0, \infty]. \]
Moreover, one has
\[ \Delta_s(X) < \infty \iff X \in \mathbb{D}_e. \]

**Proof.** Similarly as above, we have
\[ \mu_s(X) = (s + 1) \int_{\mathbb{R}} h(x) \bar{F}(x)^s f(x) dx \]
\[ = \left[ h(x)(1 - \bar{F}(x)^{s+1}) \right] \pm \int_{\mathbb{R}} h'(x)(1 - \bar{F}(x)^{s+1}) dx \]
\[ = \mathbb{E}[X] - \int_{\mathbb{R}} \frac{(1 - \bar{F}(x)^{s+1})}{F^2(x)} G(x) f(x) dx \]
where in the last equality we have used (2) which implies
\[ h'(x) = \frac{G(x)f(x)}{F(x)}. \]
This implies
\[ \Delta_s(X) = \int_{\mathbb{R}} \frac{(1 - \bar{F}(x)^{s+1})}{F^2(x)} G(x) f(x) dx \in (0, \infty]. \]
Finally, the easily established fact that \( x \mapsto (1 - \bar{F}(x)^{s+1})/F(x) \) decreases from \( s + 1 \) to \( 1 \) implies that the right-hand side is finite if and only if
\[ \int_{\mathbb{R}} \frac{G(x)f(x)}{F(x)} dx = \Delta_0(X) < \infty. \]
\( \square \)
Remark 3. (a) Letting $s \to 0$, we obtain by dominated convergence
\[
\Delta_{-s}(X) \to \int_{\mathbb{R}} \frac{G(x)f(x)}{F(x)} \, dx = - \int_{\mathbb{R}} F(x) \log F(x) \, dx
\]
and retrieve the statement of Theorem 1.

(b) The above result shows that for every $X \in \mathbb{D}$, the mapping $s \mapsto \Delta_s(X)$ is non-increasing on $(-\infty, 0)$ since for every $t = \bar{F}(x) \in (0, 1)$, the mapping $s \mapsto (1 - t^{s+1})$ is clearly non-decreasing. Moreover, it can be checked that
\[
\Delta_{-s}(X) \to \mathbb{E}[X] - \min(X)
\]
as $s \to \infty$.

(c) If $X$ is a random variable in $\mathbb{D}$ resp. in $\mathbb{D}_e$ which is symmetric around some point $x$, an easy consequence of Theorems 1, 2 and 4 is that $\Delta_s(X) = \bar{\Delta}_s(X)$ for all $s \in \mathbb{R}$. In particular, one has
\[
\mu_s(X) = x - \Delta_s(X) \quad \text{and} \quad \bar{\mu}_s(X) = x + \Delta_s(X).
\]
Contrary to Theorem 2, the expression of $\Delta_{-s}(X)$ in Theorem 4 is not given in terms of $F$ only. It is actually possible to obtain such an expression as an infinite series, by the generalized binomial theorem expanding $1 - \bar{F}(x)^{s+1} = 1 - (1 - F(x))^{s+1}$. Skipping details, for all $X \in \mathbb{D}_e$ we obtain
\[
\Delta_{-s}(X) = (1 + s) \left( \Delta_0(X) - s \sum_{n \geq 0} \frac{(1 - s)_n}{(n + 1)(n + 2)!} \int_{\mathbb{R}} F(x)(1 - F^{n+1}(x)) \, dx \right)
\]
\[
= (1 + s) \left( \Delta_0(X) - s \sum_{n \geq 0} \frac{(1 - s)_n}{(n + 2)!} \Delta_{n+1}(X) \right)
\]
\[
= (1 + s) \sum_{n \geq 0} \frac{(-s)_n}{(n + 1)!} \Delta_n(X) \tag{3}
\]
with the standard Pochhammer notation $(t)_0 = 1$ and $(t)_n = t(t+1)\ldots(t+n-1)$, $n \geq 1$, for the ascending factorial. This formula can be written
\[
\Delta_{-s}(X) - (s + 1)\Delta_0(X) = (1 + s) \sum_{n \geq 1} \frac{(-s)_n}{(n + 1)!} \Delta_n(X).
\]
Notice that the series on the right-hand side remains well-defined for every $X \in \mathbb{D}$ by the binomial theorem, since $\Delta_n(X) \to 0$ as $n \to \infty$. This is in accordance with
\[
\Delta_{-s}(X) < \infty \iff \Delta_0(X) < \infty \iff X \in \mathbb{D}_e
\]
for every $X \in \mathbb{D}$ and $s > 0$. In the case when $s = k$ is an integer we have the finite formula
\[
\Delta_{-k}(X) = \sum_{n=0}^{k} \binom{k+1}{n+1} (-1)^n \Delta_n(X)
\]
which is self-invertible, that is
\[
\Delta_k(X) = \sum_{n=0}^{k} \binom{k+1}{n+1} (-1)^n \Delta_{-n}(X).
\]
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The latter can then be generalized to all $s > 0$ by
\[
\Delta_s(X) = (1 + s) \sum_{n=0}^{\infty} \frac{(-s)_n}{(n+1)!} \Delta_{-n}(X)
\]
with an argument analogous to (3). Putting together Theorem 3 and (4), we get the following

**Corollary 1.** For every $X \in \mathbb{D}$, the sequence $\{\Delta_{-n}(X), n \geq 0\}$ determines the law of $X$ up to translation.

### 3.2. Relations with the stochastic order.

In this paragraph we establish the property that the metrics $\mu_s$ preserve the usual stochastic order $\prec_{st}$. We also observe in Remark 4 (c) below that the converse is not true. See [10] for a classic account on stochastic orderings. We also observe that the dual metric $\bar{\mu}_s$ is for all $s \in \mathbb{R}$ a coherent risk measure.

**Proposition 1.** Let $X, Y \in \mathbb{D}$. If $X \prec_{st} Y$, then $\mu_s(X) \leq \mu_s(Y)$ for every $s \in \mathbb{R}$.

**Proof.** Suppose first $s > 0$. For all $X \in \mathbb{D}$, a combination of the formula
\[
\mathbb{E}[X] = \int_{0}^{\infty} (1 - F_X(x)) \, dx - \int_{-\infty}^{0} F_X(x) \, dx
\]
and Theorem 2 implies
\[
\mu_s(X) = \int_{0}^{\infty} (1 - h_s(F_X(x))) \, dx - \int_{-\infty}^{0} h_s(F_X(x)) \, dx
\]
with the notation
\[
h_s(t) = t + \frac{(1 - t^s)}{s}
\]
for all $t \in [0, 1]$ and $s > 0$, an increasing function in $t$ since $sh'_s(t) = (s + 1)(1 - t^s) > 0$ on $[0, 1)$. Hence, $X, Y \in \mathbb{D}$ are such that $X \prec_{st} Y$, then $F_X(x) \geq F_Y(x)$ for all $x \in \mathbb{R}$ and the above shows $\mu_s(X) \leq \mu_s(Y)$ for all $s > 0$.

The case with negative parameter goes along the same way but requires some more effort. For every $s > 0$, we first write
\[
\mu_{-s}(X) = \mathbb{E}[X] - \Delta_{-s}(X)
\]
\[
= (1 + s) \sum_{n=0}^{\infty} \frac{(-s)_n}{(n+1)!} (\mathbb{E}[X] - \Delta_n(X))
\]
\[
= (1 + s) \sum_{n=0}^{\infty} \frac{(-s)_n}{(n+1)!} \mu_n(X)
\]
where in the second equality we have used (3) and the binomial theorem. By Theorem 2 and Fubini’s theorem, this leads to
\[
\mu_{-s}(X) = \int_{0}^{\infty} (1 - h_{-s}(F_X(x))) \, dx - \int_{-\infty}^{0} h_{-s}(F_X(x)) \, dx
\]
with the notation
\[
h_{-s}(t) = t \left( 1 + (1 + s) \sum_{n=0}^{\infty} \frac{(-s)_n(1 - t^n)}{n(n+1)!} \right)
\]
\[
= t \left( 1 + (1 + s) (h_{-s}(1) - h_{-s}(t) - \log t) \right)
where we have used the further notation
\[ H_{-s}(t) = \sum_{n \geq 1} \frac{(-s)_n t^n}{n(n+1)!}. \]

We next compute
\[
h'_{-s}(1) = -(s + 1)H'_{-s}(1) - s = -(s + 1) \sum_{n \geq 1} \frac{(-s)_n}{(n + 1)!} - s
\]
\[= 1 - (s + 1) \sum_{n \geq 0} \frac{(-s)_n}{(n+1)!} = 0,
\]
and
\[
h''_{-s}(t) = -(s + 1) t^{-1} \sum_{n \geq 0} \frac{(-s)_n}{n!} t^n = -(s + 1) t^{-1} (1 - t)^s
\]
where the second equality follows from some easy simplifications left to the reader, and the third one from again the binomial theorem. This implies \(h''_{-s}(t) < 0\) and hence \(h'_{-s}(t) > 0\) for all \(t \in (0,1)\) so that \(h_{-s}\) increases on \([0,1]\) and we can conclude as in the case \(s > 0\).

**Corollary 2.** For every \(s \in \mathbb{R}\), the dual metric \(\bar{\mu}_s\) is a coherent risk measure.

**Proof.** It is clear from the integral formulas for \(\Delta_s(X)\) and the linearity of the expectation that
\[\bar{\mu}_s(aX + b) = a\bar{\mu}_s(X) + b\]
for all \(a > 0, b \in \mathbb{R}\) and \(X \in \mathbb{D}\) resp. in \(\mathbb{D}_e\). Moreover, we have \(\bar{\mu}_s(X) \leq \bar{\mu}_s(Y)\) for \(X \prec_{st} Y\) directly from Proposition \[4] Finally, using \(F_X(x) = \bar{F}_X(-x)\), the proof of Proposition \[1\] shows that
\[\bar{\mu}_s(X) = -\mu_s(-X) = \int_0^\infty h_s(F_X(x)) \ dx - \int_{-\infty}^0 (1 - h_s(F_X(x))) \ dx
\]
where \(h_s\) is increasing concave on \((0,1)\) for all \(s \in \mathbb{R}\). By Theorem 10 in [11], this implies that \(\bar{\mu}_s(X + Y) \leq \bar{\mu}_s(X) + \bar{\mu}_s(Y)\) regardless of the dependency relation between \(X\) and \(Y\). Putting all these properties shows that \(\bar{\mu}_s\) is a coherent risk measure. \(

**Remark 4.** (a) Setting \(g_s(t) = 1 - h_s(1 - t)\) and \(g_{-s}(t) = 1 - h_{-s}(1 - t)\) for \(s > 0\) and \(t \in [0,1]\), we also deduce from the above proof the representation
\[\mu_s(X) = \int_0^\infty g_s(F_X(x)) \ dx - \int_{-\infty}^0 (1 - g_s(F_X(x))) \ dx
\]
with \(g_s\) an increasing convex function on \((0,1)\) With the terminology of Section 3.2.6 in [3], the increasing character of \(g_s\) on \((0,1)\) implies that \(\mu_s(X)\) is a monetary risk measure.

(b) Consider the general metric
\[\mu_{a,b}(X) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}(1-t)^{b-1} \mathbb{E} \left[X_{F_X^{-1}(t)}\right] \ dt
\]
with $a, b > 0$ and $\Gamma$ the classical Gamma function. It can be shown that $\mu_{a,b}$ is well-defined on $\mathbb{D}_c \cap \mathcal{L}_c$ with $c = \max\{a^{-1}, b^{-1}\}$. Observe that with the above notation, one has $\mu_s = \mu_{1+s, 1}$ and $\mu_{-s} = \mu_{1,1+s}$. Further computations, which we shall not include here, show that

$$\mu_{a,b}(X) = \int_0^\infty g_{a,b}(\bar{F}_X(x)) \, dx - \int_0^0 (1-g_{a,b}(\bar{F}_X(x))) \, dx$$

with $g'_{a,b}(0) = 0$ and

$$g''_{a,b}(t) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (1-t)^{a-2} t^{b-1} > 0.$$ 

This proves that $\nu_{a,b}(X)$ is a monetary risk measure for all $a, b > 0$. Similarly as above, we can also show that the dual metric $\bar{\mu}_{a,b}$ is a coherent risk measure for all $a, b > 0$.

(c) The converse result to Proposition 3.3.1 is not true in general, that is one may have $\mu_s(X) \leq \mu_s(Y)$ for all $s \in \mathbb{R}$ but no stochastic dominance between $X$ and $Y$. For example, if $X_{a,L}$ is uniformly distributed on $(a, a + L)$, then some computations using Propositions 2 and 4 (see also Paragraph 3.3.1 below) imply

$$\mu_s(X_{a,L}) = a + \frac{L(1+s_+)}{2(2+|s|)}$$

for all $s \in \mathbb{R}$ with the notation $s_+ = \sup(0, s)$. Hence, if we choose $b > a$ and $M < L$ such that

$$\frac{L - M}{2} < b - a < L - M,$$

then we will have $\mu_s(X_{b,M}) > \mu_s(X_{a,L})$ for all $s \in \mathbb{R}$ but clearly there is no stochastic dominance between $X_{a,L}$ and $X_{b,M}$.

3.3. Examples. In this paragraph we display some random variables in $\mathbb{D}_c$ where the quantities $\Delta_s$ and $\mu_s$ can be computed in closed form, sometimes in terms of special functions. The list is not exhaustive, and for the sake of concision we will not give the full details behind the computations, which are available upon request. Some formulas will be used in the next section where the range of certain metrics will be investigated. We will mostly consider explicit transformations of $U$ the uniform random variable on $(0,1)$ and $L$ the standard exponential random variable with density $e^{-x}$ on $(0, \infty)$. Throughout, we will use the easily established fact that for all $X \in \mathbb{D}_c, a > 0$ and $b, s \in \mathbb{R}$ one has

$$\Delta_s(aX + b) = a\Delta_s(X) \quad \text{and} \quad \mu_s(aX + b) = a\mu_s(X) + b$$

On the other hand, there is no simple formula relating $\Delta_s(X)$ to $\bar{\Delta}_s(X) = \Delta_s(-X)$ in general. We begin with random variables with compact support.

3.3.1. $X = U^{1/\beta}, \beta > 0$. The density function is $f(x) = \beta x^{\beta-1} 1_{(0,1)}(x)$. For all $s > 0$, we have

- $\Delta_s = \frac{\beta}{(\beta + 1)(\beta(1+s) + 1)}$ and $\mu_s = \frac{\beta^2(1+s)}{(\beta + 1)(\beta(1+s) + 1)}$
- $\Delta_{-s} = \frac{\beta}{\beta + 1} \left(1 - \frac{\Gamma(1/\beta + 1)\Gamma(s + 2)}{\Gamma(1/\beta + s + 2)}\right)$ and $\mu_{-s} = \frac{\Gamma(1/\beta)\Gamma(s + 2)}{(\beta + 1)\Gamma(1/\beta + s + 2)}$. 
Letting $s \to 0$ from above or from below, we get
\[
\Delta_0 = \frac{\beta}{(\beta + 1)^2} \quad \text{and} \quad \mu_0 = \frac{\beta^2}{(\beta + 1)^2}.
\]
It is easy to check that $s \mapsto \mu_s$ is increasing on $\mathbb{R}$ from $0 = \min(X)$ to $\beta/(\beta + 1) = \mathbb{E}[X]$ and that for all $s \in \mathbb{R}$ the mapping $\beta \mapsto \mu_s(U^{1/\beta})$ also increases on $(0, \infty)$, in accordance with Proposition 1 and the fact that $\beta \mapsto U^{1/\beta}$ increases for the stochastic order.

3.3.2. $X = 1 - U^{1/\beta}$, $\beta > 0$. The density function is $f(x) = \beta(1 - x)^{\beta - 1}1_{(0,1)}(x)$. For all $s > 0$, we have
\[
\begin{align*}
\Delta_s &= \frac{\beta}{s(\beta + 1)} \left(1 - \frac{\Gamma(1/\beta + 2)\Gamma(s + 2)}{\Gamma(1/\beta + s + 2)}\right) \\
\mu_s &= \frac{1}{\beta + 1} + \frac{\beta}{s(\beta + 1)} \left(\frac{\Gamma(1/\beta + 2)\Gamma(s + 2)}{\Gamma(1/\beta + s + 2)} - 1\right).
\end{align*}
\]
For the negative parameters, we first have an integral formulation which, for every $s > 0$, reads
\[
\Delta_{-s} = \int_0^1 \frac{(1 - t^{s+1})(1 - t^{1/\beta})}{(1 - t)^2} dt - \frac{1}{\beta + 1} \int_0^1 \frac{(1 - t^{s+1})(1 - t^{1/\beta+1})}{(1 - t)^2} dt
\]
and leads after some remarkable simplifications to
\[
\begin{align*}
\Delta_{-s} &= \frac{\beta((s + 1)(\psi(1/\beta + 2 + s) - \psi(s + 1)) - 1)}{\beta + 1} \\
\mu_{-s} &= 1 + \frac{\beta(s + 1)(\psi(s + 1) - \psi(1/\beta + 2 + s))}{\beta + 1}
\end{align*}
\]
where
\[
\psi(z) = -\gamma + \sum_{n \geq 0} \left(\frac{1}{n + 1} - \frac{1}{n + z}\right) = -\gamma + \int_0^1 \frac{1 - t^{z-1}}{1 - t} dt
\]
is the standard Digamma function. Letting $s \to 0$ from above or from below, we get
\[
\Delta_0 = \frac{\beta(\psi(1/\beta + 2) - \psi(2))}{\beta + 1} \quad \text{and} \quad \mu_0 = 1 + \frac{\beta(\psi(1) - \psi(1/\beta + 2))}{\beta + 1}.
\]
When $\beta = 1/n$ the reciprocal of an integer, the concatenation formula for the Digamma function gives the simple expression
\[
\Delta_0 = \frac{1}{n + 1} \left(\frac{1}{2} + \cdots + \frac{1}{n + 1}\right)
\]
In particular, the case $\beta = 1$ gives $\Delta_0 = 1/4$, which was recently evaluated in Example 2 of [2] by other methods. Using convexity properties of the Gamma and Digamma function, one can check that $s \mapsto \mu_s$ increases from $0 = \min(X)$ to $1/(\beta + 1) = \mathbb{E}[X]$ and that $\beta \mapsto \mu_s(1 - U^{1/\beta})$ decreases for all $s \in \mathbb{R}$, in accordance with the stochastic order.

**Remark 5.** This example can be used to compute the metrics of the exponential distribution $\textbf{L}$ which is the limit in law of $\beta(1 - U^{1/\beta})$ as $\beta \to \infty$. Using [5], we obtain for every $s > 0$
\[
\Delta_s(\textbf{L}) = \frac{\psi(s + 2) - \psi(2)}{s} \quad \text{and} \quad \Delta_{-s}(\textbf{L}) = (s + 1)\psi'(s + 2).
\]
Observe that both quantities converge as $s \to 0$ to
\[
\Delta_0(\textbf{L}) = \psi'(2) = \frac{\pi^2}{6} - 1,
\]
which was recently evaluated in Example 3 of [2] by other methods.
We next consider some distributions with non-compact support.

3.3.3. $X = U^{-1/\beta} - 1$. The density function is $f(x) = \beta(1 + x)^{-\beta - 1}1_{(0,\infty)}(x)$. We recognize the Lomax distribution, or Pareto distribution of type II, which is the prototype of a power law distribution. We will consider only the case with finite expectation, that is $\beta > 1$. For all $s > 0$ one has the integral formula

$$\Delta_s = \frac{1}{\beta s} \int_0^1 x - x^{s+1} (1 - x)^{1/\beta + 1} dx = \frac{1}{\beta \sum_{n \geq 0} \frac{(1/\beta + 1)_n}{n!(n + s + 2)(n + 2)}}.$$

The series on the right-hand side can be written as an hypergeometric functions and a consequence of Thomae’s relationship for $3F_2(1)$ and Gauss’ formula for $2F_1(1)$ is

- $\Delta_s = \frac{\beta}{s(\beta - 1)} \left( \frac{\Gamma(2 - 1/\beta)\Gamma(s + 2)}{\Gamma(s + 2 - 1/\beta)} - 1 \right)$
- $\mu_s = \frac{1}{\beta - 1} + \frac{\beta}{s(\beta - 1)} \left( 1 - \frac{2 - 1/\beta)\Gamma(s + 2)}{\Gamma(s + 2 - 1/\beta)} \right)$,

which can also be obtained from the results in Paragraph 3.3.2 through an analytic continuation at $\beta = \infty$. For negative parameters, the starting point is the integral formula

$$\Delta_{-s} = \int_0^1 (1 - t^{s+1})/(t^{1/\beta + 1} - 1) \cdot \frac{(1 - t^{s+1})(t^{1/\beta + 1} - 1)}{(1 - t)^2} dt$$

for all $s > 0$, which leads similarly as in Paragraph 3.3.2 to

- $\Delta_{-s} = \frac{\beta(s + 1)(\psi(s + 1) + 1 - \psi(2 + s - 1/\beta))}{\beta - 1}$
- $\mu_{-s} = -1 + \frac{\beta(s + 1)(\psi(2 + s - 1/\beta) - \psi(s + 1))}{\beta - 1}$.

Letting $s \to 0$ from above or from below, we get

$$\Delta_0 = \frac{\beta(\psi(2) - \psi(2 - 1/\beta))}{\beta - 1} \quad \text{and} \quad \mu_0 = -1 + \frac{\beta(\psi(2 - 1/\beta) - \psi(1))}{\beta - 1}.$$

Observe from (5) that since $\beta(U^{-1/\beta} - 1)$ converges in law to $L$ as $\beta \to \infty$, we can also deduce the formulas of Remark 5 from the above computations on Lomax.

3.3.4. $X = 1 - U^{-1/\beta}$. The density function is $f(x) = \beta(1 - x)^{-\beta - 1}1_{(-\infty,0)}(x)$. This random variable can be viewed as a negative Lomax. Again, we consider only the case with finite expectation, that is $\beta > 1$. For every $s > 0$, computations analogous to Paragraphs 3.3.1 and 3.3.3 give

- $\Delta_s = \frac{\beta}{(\beta - 1)(\beta(1 + s) - 1)}$ and $\Delta_{-s} = \frac{\beta}{\beta - 1} \left( \frac{\Gamma(1 - 1/\beta)\Gamma(s + 2)}{\Gamma(s + 2 - 1/\beta)} - 1 \right)$

and the corresponding $\mu_s$ and $\mu_{-s}$. Letting $s \to 0$ from above or from below, we get

$$\Delta_0 = \frac{\beta}{(\beta - 1)^2}.$$

Remark 6. (a) Letting $\beta \to \infty$ gives from (5) gives the metrics for the negative exponential: for every $s > 0$, one finds

$$\Delta_s(-L) = \frac{1}{s + 1} \quad \text{and} \quad \Delta_{-s}(-L) = \psi(s + 2) + \gamma,$$
which both converge to 1 as \( s \to 0 \). See also Example 1 in [2] for another proof of \( \Delta_0(-L) = 1 \).

(b) The above computations display some cases of a random variable \( X \) whose metrics \( \mu_s(X) \) and \( \mu_s(-X) \) are explicit but very different, contrary to the affine relationship [5]. In the case \( s = 0 \), this amounts to computing both the cumulative entropy and the cumulative residual entropy of \( X \), which is

\[
- \int_{\mathbb{R}} \bar{F}_X(x) \log \bar{F}_X(x) \, dx.
\]

We refer to [2] for further joint computations on cumulative and cumulative residual entropies.

3.3.5. \( X = L^{-1/\beta} \). The density function is \( f(x) = \beta x^{-\beta-1} e^{-x^\beta} 1_{(0,\infty)}(x) \). This is a Fréchet distribution, or type II extreme value distribution, and another example of a power law distribution. We consider the case of finite expectation only, that is \( \beta > 1 \). For all \( s > 0 \) one has the Frullani-type integral

\[
\Delta_s = \frac{1}{\beta s} \int_0^\infty \frac{e^{-x} - e^{-(s+1)x}}{x^{1/\beta+1}} \, dx,
\]

which leads to

\[
\Delta_s = \Gamma(1 - 1/\beta) \left( \frac{(s+1)^{1/\beta} - 1}{s} \right) \quad \text{and} \quad \mu_s = \Gamma(1 - 1/\beta) \left( \frac{(s+1) - (s+1)^{1/\beta}}{s} \right).
\]

Letting \( s \to 0 \), we get \( \Delta_0 = -\Gamma(1 - 1/\beta) \) and \( \mu_0 = \Gamma(2 - 1/\beta) \). For the negative parameters, the formula (3) yields

\[
\begin{align*}
\bullet \, \Delta_{-s}(X) &= \frac{s + 1}{\beta} \Gamma(1 - 1/\beta) + (s + 1) \Gamma(1 - 1/\beta) \sum_{n \geq 1} (-s)_n \frac{(n + 1)^{1/\beta} - 1}{n} \\
\bullet \, \mu_{-s}(X) &= \left( 1 - \frac{s + 1}{\beta} \right) \Gamma(1 - 1/\beta) + (s + 1) \Gamma(1 - 1/\beta) \sum_{n \geq 1} (-s)_n \frac{1 - (n + 1)^{1/\beta}}{n} \end{align*}
\]

which neither seem to have a more explicit expression. Observe that considering \( \beta (L^{-1/\beta} - 1) \) with \( \beta \to \infty \) and using [5], we can compute the metrics associate with the Gumbel distribution \(-\log L\) with density \( e^{-(x+e^{-x})} \) on \( \mathbb{R} \):

\[
\Delta_s(-\log L) = \frac{\log(s + 1)}{s} \quad \text{and} \quad \Delta_{-s}(-\log L) = (s + 1) \left( 1 + \sum_{n \geq 1} (-s)_n \frac{\log(1 + n)}{(n + 1)! n} \right)
\]

for all \( s > 0 \), which both imply \( \Delta_0(-\log L) = 1 \).

3.3.6. \( X = \log(U^{-1} - 1) \). The density function is \( e^x / (1+e^x)^2 \) over \( \mathbb{R} \) and is known in the literature as the logistic distribution. The random variable \( X \) is symmetric and we hence have \( \mu_s(X) = -\Delta_s(X) \).

For every \( s > 0 \) we have

\[
\Delta_s = \frac{\psi(s + 1) + \gamma}{s} \quad \text{and} \quad \Delta_{-s} = \gamma + \psi(s + 1) + (s + 1) \psi'(s + 1),
\]

with both quantities converging as \( s \to 0 \) to

\[
\Delta_0 = \psi'(1) = \frac{\pi^2}{6}.
\]
It is interesting to compare this formulas with those for $L = -\log U$ obtained in Remark 5: for every $s \geq 0$, one has
\[ \Delta_s(X) = \frac{1}{s+1} + \Delta_s(L) \quad \text{and} \quad \Delta_{-s}(X) = 1 + \Delta_{-s}(L) + s\Delta_s(L). \]

4. Description of the range

In this section, we investigate some properties of the closed range of the mapping $X \mapsto \Delta_s(X)$ for $s \geq 0$, extending some results previously obtained in [2, 4]. We will deal with the positive case, the case with finite variance, and the symmetric case with finite variance. For every $s \geq 0$, we set $\Delta_e s(X) = \Delta s(X)/\mathbb{E}[X]$ for $X \in \mathbb{D}$ and $\Delta_\sigma s(X) = \Delta s(X)/\sigma_X$ for $X \in \mathbb{D} \cap L_2$, with the notation $\sigma_X = \sqrt{\text{Var}X}$. This positive linear renormalization is natural in view of the affine relationship (5).

We will consider the intervals
\[ R_{s,+} = \{ \Delta_e s(X), X \in \mathbb{D} \text{ and } X > 0 \}, \quad R_{s,2} = \{ \Delta_\sigma s(X), X \in \mathbb{D} \cap L_2 \} \]
and
\[ R_{s,\text{sym}} = \{ \Delta_\sigma s(X), X \in \mathbb{D} \cap L_2 \text{ and } X \overset{d}{=} -X \}. \]

We first handle the case of the cumulative entropy $s = 0$ and show the following, which can be viewed as a generalization of the inequality (21) in [4] in the positive case - see also the examples given at the end of Section 2 in [8]. In the $L_2$ case, our optimal bounds also improve on all the results of Section 3 in [2].

Proposition 2. One has
\[ R_{0,+} = R_{0,2} = [0, 1] \quad \text{and} \quad R_{0,\text{sym}} = [0, \pi/2\sqrt{3}]. \]

Proof. We start with the positive case. It is clear that
\[ 0 \leq \Delta_0(X) = \int_0^1 \mathbb{E} \left[ X - X_{F_X^{-1}(t)} \right] dt \leq \mathbb{E}[X]. \]
Moreover, if we consider $X = U^{1/\beta}$ for some $\beta > 0$, we have seen in 3.3.1 above that
\[ \Delta_0(X) = \frac{\beta}{(\beta + 1)^2} \quad \text{and} \quad \mathbb{E}[X] = \frac{\beta}{\beta + 1} \]
and this implies $\Delta_0(X)/\mathbb{E}[X] = 1/(\beta + 1)$, whose closed range is $[0, 1]$ as $\beta$ varies from 0 to $\infty$. We next consider the square integrable case. Similarly as in the proof of Theorem 1 we get
\[ \int_0^1 \mathbb{E} \left[ X_{F_X^{-1}(t)} \right] dt = \int_\mathbb{R} f(x) \mathbb{E}[X_x] \, dx \]
\[ = \int_\mathbb{R} f(x) \frac{F(x)}{F(x)} \left( \int_{-\infty}^x u f(u) \, du \right) \, dx = -\int_\mathbb{R} x f(x) \log F(x) \, dx \]
where the integration by parts in the third equality is justified by (2) and the fact that both $xF(x) \log F(x)$ and $G(x) \log F(x)$ tend to zero at both infinities since $X$ has finite variance. Hence,
we get the alternative representation

\[ \Delta_0(X) = \int_{\mathbb{R}} x f(x) (1 + \log F(x)) \, dx \quad (6) \]

which is interesting in its own right. Now, the Cauchy-Schwartz inequality implies

\[
\Delta_0(X)^2 \leq \left( \int_{\mathbb{R}} x^2 f(x) \, dx \right) \times \left( \int_{\mathbb{R}} (1 + \log F(x))^2 f(x) \, dx \right)
= \text{Var}X \times \left( \int_{\mathbb{R}} (1 + \log F(x))^2 f(x) \, dx \right) = \text{Var}X \times \left( \int_0^1 (1 + \log u)^2 \, du \right) = \text{Var}X
\]

where the computation on the third equality follows from the change of variable \( u = e^{-x} \). Considering again \( X = U^{1/\beta} \) for \( \beta > 0 \), we have

\[ \text{Var}X = \frac{\beta}{(\beta + 1)^2(\beta + 2)} \quad \text{and} \quad \frac{\Delta_0(X)}{\sigma_X} = \frac{\sqrt{\beta(\beta + 2)}}{\beta + 1} \]

whose closed range is again \([0, 1] \) as \( \beta \) varies from 0 to \( \infty \). We finally consider the symmetric case with finite variance, which is slightly more involved. We first start with yet another formulation of the cumulative entropy in the symmetric case, which reads

\[ \Delta_0(X) = \int_{0}^{\infty} x f(x) \log \left( \frac{F(x)}{\bar{F}(x)} \right) \, dx \quad (7) \]

and is a direct consequence of (6) and the fact that \( F(x) = \bar{F}(-x) \) for all \( x \geq 0 \). Applying now the Cauchy-Schwarz inequality to (7), we get

\[
\Delta_0(X)^2 \leq \left( \int_{0}^{\infty} x^2 f(x) \, dx \right) \times \left( \int_{0}^{\infty} \log^2 \left( \frac{F(x)}{\bar{F}(x)} \right) f(x) \, dx \right)
= \frac{\text{Var}X}{2} \times \left( \int_{1/2}^{1} \log^2 \left( x/(1 - x) \right) \, dx \right).
\]

Making the change of variable \( u = \log(x/(1 - x)) \), we next evaluate by Fubini’s theorem and the Basel problem

\[
\int_{1/2}^{1} \log^2 \left( \frac{x}{1 - x} \right) \, dx = \int_{0}^{\infty} \frac{u^2 e^{-u}}{(1 + e^{-u})^2} \, du
= \sum_{n \geq 1} (-1)^{n-1} \left( \int_{0}^{\infty} nu^2 e^{-nu} \, du \right) = 2 \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{6}.
\]

Putting everything together, we obtain the required upper bound

\[ \frac{\Delta_0(X)}{\sigma_X} \leq \frac{\pi}{2\sqrt{3}} = 0.9069... < 1. \]

To show the full range, we consider the symmetric powers of the logistic random variable \( X = \log(U^{-1} - 1) \). More precisely, for every \( \beta \in (0, 1] \) the function

\[ F_\beta(x) = \begin{cases} 
\frac{e^{x\beta}}{1 + e^{x\beta}} & \text{if } x \geq 0 \\
\frac{1}{1 + e^{x|\beta|}} & \text{if } x \leq 0
\end{cases} \]
is the distribution function of a symmetric random variable $X_\beta$ with finite variance, which is the logistic random variable for $\beta = 1$. On the one hand its variance is computed, by Fubini’s theorem, as

$$\text{Var}X_\beta = 4\int_0^\infty x \tilde{F}_\beta(x) \, dx = 4 \int_0^\infty \frac{x}{1 + e^{x\beta}} \, dx = \frac{4}{\beta} \int_0^\infty \frac{u^{2/\beta-1}e^{-u}}{1 + e^{-u}} \, du$$

$$= \frac{4}{\beta} \sum_{n \geq 1} (-1)^{n-1} \left( \int_0^\infty u^{2/\beta-1}e^{-nu} \, du \right)$$

$$= \frac{4\Gamma(2/\beta)}{\beta} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^{2/\beta}}$$

$$= 2 \Gamma(2/\beta + 1) \eta(2/\beta)$$

with the standard notation for Riemann’s eta function. On the other hand, its cumulative entropy reads

$$\Delta_0(X_\beta) = \int_0^\infty x^{\beta+1} f_\beta(x) \, dx = \beta \int_0^\infty \frac{x^{2\beta} e^{x\beta}}{(1 + e^{x\beta})^2} \, dx = \int_0^\infty \frac{u^{1/\beta+1}e^{-u}}{(1 + e^{-u})^2} \, du$$

$$= \sum_{n \geq 1} (-1)^{n-1} \left( \int_0^\infty nu^{1/\beta+1}e^{-nu} \, du \right)$$

$$= \Gamma(1/\beta + 2) \eta(1/\beta + 1),$$

and we get

$$\frac{\Delta(X_\beta)}{\sqrt{\text{Var}X_\beta}} = \frac{\Gamma(1/\beta + 2) \eta(1/\beta + 1)}{\sqrt{2\Gamma(2/\beta + 1) \eta(2/\beta)}} \sim \frac{\Gamma(1/\beta + 2)}{\sqrt{2\Gamma(2/\beta + 1)}} \sim \pi^{1/4} \beta^{-5/4} 2^{-(1/2+1/\beta)} \to 0$$

as $\beta \to 0$, by Stirling’s formula. Moreover, it is clear that the function

$$\beta \mapsto \frac{\Delta(X_\beta)}{\sqrt{\text{Var}X_\beta}}$$

is continuous on $(0,1]$ and attains the value $\sqrt{\eta(2)} = \pi/2\sqrt{3}$ at $\beta = 1$, which completes the proof by continuity. \hfill \square

**Remark 7.** (a) The formula (6) can be extended to any $X \in \mathcal{L}_2$ not necessarily absolutely continuous: the same integration by parts shows

$$\Delta_0(X) = \int_\mathbb{R} x (1 + \log F(x)) \, P_X(dx)$$

where $P_X$ stands for the law of any $X \in \mathcal{L}_2$. Applying the Cauchy-Schwartz inequality with respect to the finite measure $P_X$ implies

$$- \int_\mathbb{R} F_X(x) \log F_X(x) \, dx \leq \sqrt{\text{Var}X}$$

for any $X \in \mathcal{L}_2$, an optimal bound which improves on Theorem 1 in [2]. Similarly, an extension of (7) to the whole $\mathcal{L}_2$ yields the optimal bound

$$- \int_\mathbb{R} F_X(x) \log F_X(x) \, dx \leq \frac{\pi \sqrt{\text{Var}X}}{2\sqrt{3}}$$

for any $X \in \mathcal{L}_2$ symmetric, which improves on Theorems 4 and 5 in [2].
(b) It is noticeable that the maximum of $R_{0,\text{sym}}$ is attained by the logistic random variable. Observe also from the case $s = 0$ in Remark 6 that the maximum of $R_{0,2}$ is also attained by the negative exponential $-\mathbf{L}$, whose variance is 1. We will see soon afterwards that this maximum property characterizes both distributions, in other words that both maxima are uniquely attained. On the other hand, the maximum of $R_{0,+}$ is not attained because
\[
\Delta_0(X) \bigg/ \mathbb{E}[X] = 1 - \frac{\mu_0(X)}{\mathbb{E}[X]} < 1
\]
for all $X \in \mathbb{D}$ positive.

(c) As $\beta \to \infty$, one has
\[
\frac{\Delta(X_\beta)}{\sqrt{\text{Var} X_\beta}} \sim \frac{\eta(1/\beta + 1)}{\sqrt{2 \eta(2/\beta)}} = \frac{\eta(1/\beta + 1)}{\sqrt{2 (1 - 2^{-2/\beta}) \zeta(2/\beta)}} \to \log 2 = 0.6931... < \pi/2\sqrt{3}
\]
by standard properties of Riemann’s zeta function. Simulations show that the function
\[
\beta \mapsto \frac{\Delta(X_\beta)}{\sqrt{\text{Var} X_\beta}}
\]
is unimodal on $(0, \infty)$.

As a consequence of the proof of Proposition 2 we have the following interesting characterization of the logistic distribution in terms of the cumulative entropy.

**Corollary 3.** Up to linear transformation, the logistic random variable $\log(U^{-1} - 1)$ is the unique maximizer of $X \mapsto \Delta_0(X)/\sigma_X$ among symmetric distributions in $\mathbb{D} \cap \mathbb{L}_2$.

**Proof.** By the proof of Proposition 2 and the case of equality in the Cauchy Schwartz inequality, a symmetric random variable $X \in \mathbb{D} \cap \mathbb{L}_2$ reaches the maximum of $\mathcal{R}_{0,\text{sym}}$ if and only if there exists some constant $\lambda > 0$ such that
\[
\lambda x \sqrt{f_X(x)} = \log \left( \frac{F_X(x)}{F_X(-x)} \right) \sqrt{f_X(x)}
\]
for almost every $x \in \mathbb{R}^+$, which is by symmetry and continuity equivalent to
\[
\frac{F_X(x)}{1 - F_X(x)} = e^{\lambda x}
\]
for every $x \in \mathbb{R}$, that is $X \overset{d}{=} \frac{1}{X} \log(U^{-1} - 1)$ as required.

Similarly, we have the following characterization of the exponential distribution by the cumulative residual entropy, which seems unnoticed in the literature. We will use the notation
\[
\bar{\Delta}_0(X) = -\int_{\mathbb{R}} \bar{F}_X(x) \log \bar{F}_X(x) \, dx.
\]

**Corollary 4.** Up to affine transformation, the exponential random variable $\mathbf{L}$ is the unique maximizer of $X \mapsto \bar{\Delta}_0(X)/\sigma_X$ among distributions in $\mathbb{D} \cap \mathbb{L}_2$. 
Proof. By the same argument as above, up to translation a random variable $X \in \mathbb{D} \cap \mathcal{L}_2$ reaches the maximum of $\mathcal{R}_{0, \text{sym}}$ if and only if there exists some constant $\lambda > 0$ such that

$$\lambda x \sqrt{f_X(x)} = (1 + \log F_X(x)) \sqrt{f_X(x)}$$

for almost every $x \in \mathbb{R}$. This amounts to $F_X(x) = e^{\lambda x - 1} 1_{(-\infty, 1/\lambda]}(x)$ that is $X \overset{d}{=} \frac{1 - U}{\lambda}$. Since $\Delta_0(X) = \Delta_0(-X)$, this completes the proof.

\[\square\]

We next consider the case $s > 0$. In the symmetric case, the upper bound of the range shares some similarities with $\Delta_s(1 - U^s)$ as computed in Paragraph 3.3.2 and we will see during the proof that it is actually reached for $(1 - U)^s - U^s$, which is up to normalization a finite range approximation of the logistic distribution. Observe also the similarity between our optimal bound and the general term of the series appearing in Theorem 5 of [2].

**Proposition 3.** For every $s > 0$, one has $\mathcal{R}_{s,+} = [0, 1], \mathcal{R}_{s,2} = [0, 1/\sqrt{2s + 1}]$ and

$$\mathcal{R}_{s,\text{sym}} = \left[0, \frac{s + 1}{\sqrt{2s}} \times \sqrt{\frac{1}{2s + 1} - \frac{\Gamma^2(s + 1)}{\Gamma(2s + 2)}}\right].$$

**Proof.** We begin with the positive case. It is clear that $\mathcal{R}_{s,+} \subset \mathcal{R}_{0,+} \subset [0, 1]$ because $0 \leq \Delta_s(X) \leq \Delta_0(X)$ for all $s > 0$. Moreover, again for $X = U^{1/\beta}$, with $\beta > 0$ we have

$$\frac{\Delta_s(X)}{\mathbb{E}[X]} = \frac{1}{\beta(s + 1) + 1}$$

whose closed range is $[0, 1]$ as $\beta$ varies from 0 to $\infty$. For $\mathbb{D} \cap \mathcal{L}_2$, we start with the alternative representation

$$\Delta_s(X) = \frac{1}{s} \int_{\mathbb{R}} x f(x) \left((s + 1) F^s(x) - 1\right) dx \quad (8)$$

which follows similarly as in the case $s = 0$ from the integration by parts

$$\int_0^1 t^s \mathbb{E} \left[ X F^{-1}_X(t) \right] dt = \int_{\mathbb{R}} F^{s-1}(x) f(x) \left( \int_{-\infty}^x u f(u) du \right) dx = \frac{1}{s} \int_{\mathbb{R}} x f(x) (1 - F^s(x)) dx.$$

Applying the Cauchy-Schwartz inequality to (8), we obtain the required upper bound:

$$\Delta_s^2(X) \leq \frac{\text{Var} X}{s^2} \times \int_{\mathbb{R}} \left((s + 1) F^s(x) - 1\right)^2 f(x) dx = \frac{\text{Var} X}{s^2} \times \int_0^1 \left((s + 1) u^s - 1\right)^2 du = \frac{\text{Var} X}{2s + 1},$$

Finally, setting $X = U^{1/\beta}$, we obtain

$$\frac{\Delta_s(X)}{\sigma_X} = \frac{\sqrt{\beta(\beta + 2)}}{\beta(s + 1) + 1}$$

which is a unimodal function in $u \in (0, \infty)$ from 0 to $1/(s + 1)$ reaching its maximum $1/\sqrt{2s + 1}$ at $\beta = 1/s$, which shows that $\mathcal{R}_{s,2} = [0, 1/\sqrt{2s + 1}]$. To describe $\mathcal{R}_{2,\text{sym}}$, we use the formula

$$\Delta_s(X) = \frac{s + 1}{s} \int_0^\infty x f(x) \left(F^s(x) - F^s(0)\right) dx \quad (9)$$
for all \( X \in \mathbb{D} \cap \mathcal{L}_2 \) symmetric, which is a direct consequence of (9), and the Cauchy-Schwartz inequality, to obtain

\[
\frac{\Delta_s(X)^2}{\text{Var}X} \leq \frac{(s + 1)^2}{2s^2} \int_{1/2}^{1} (x^s - (1 - x)^s)^2 \, dx = \frac{(s + 1)^2}{4s^2} \int_{0}^{1} (x^s - (1 - x)^s)^2 \, dx
\]

\[
= \frac{(s + 1)^2}{2s^2} \left( \frac{1}{2s + 1} - \int_{0}^{1} x^s(1 - x)^s \, dx \right)
\]

\[
= \frac{(s + 1)^2}{2s^2} \left( \frac{1}{2s + 1} - \frac{\Gamma^2(s + 1)}{\Gamma(2s + 2)} \right)
\]

as required for the upper bound. To show the full range, we consider the function

\[
\phi_s(x) = x^s - (1 - x)^s
\]

which defines an increasing bijection from \([1/2, 1]\) onto \([0, 1]\), and for every \( \beta \in (0, 1] \) the symmetric random variable \( X_{s,\beta} \) on \([-1, 1]\) with distribution function

\[
F_{X_{s,\beta}}(x) = \phi_s^{-1}(x^\beta), \quad x \in [0, 1].
\]

From the easily established identity \( X_{s,\beta} \overset{d}{=} \varepsilon|X_{s,1}|^{1/\beta} \) with \( \varepsilon \) an independent random variable such that \( \mathbb{P}[\varepsilon = 1] = \mathbb{P}[\varepsilon = -1] = 1/2 \), we have

\[
\text{Var}X_{s,\beta} = \mathbb{E}[|X_{s,1}|^{2/\beta}] = 2 \int_{0}^{1} x^{2/\beta} f_s(x) \, dx = \beta \int_{0}^{\infty} e^{-u(1+\beta/2)} f_s(e^{-\beta u/2}) \, du
\]

where \( f_s \) stands for the density of \( X_{s,1} \). As \( \beta \to 0 \), this gives the asymptotics

\[
\text{Var}X_{s,\beta} \sim \beta f_s(1) = \frac{\beta}{s}
\]

where the equality is an easy consequence of (10). On the other hand, it follows from (9) and (10) that

\[
\Delta_s(X_{s,\beta}) = \frac{s + 1}{s} \int_{0}^{1} x^{\beta+1} f_{X_{s,\beta}}(x) \, dx = \frac{\beta(s + 1)}{s} \int_{0}^{1} x^{2\beta} f_s(x) \, dx \sim \frac{\beta(s + 1)}{2s}
\]

as \( \beta \to 0 \), which implies

\[
\Delta_s^\beta(X_{s,\beta}) \sim \frac{\sqrt{\beta}(s + 1)}{2\sqrt{s}} \to 0 \quad \text{as} \ \beta \to 0.
\]

For \( \beta = 1 \), the above computation also gives

\[
\frac{\Delta_s(X_{s,1})^2}{\text{Var}X_{s,1}} = \frac{(s + 1)^2 \text{Var}X_{s,1}}{4s^2} = \frac{(s + 1)^2}{2s^2} \int_{0}^{1} x^2 f_s(x) \, dx
\]

\[
= \frac{(s + 1)^2}{2s^2} \int_{1/2}^{1} (x^s - (1 - x)^s)^2 \, dx
\]

\[
= \frac{(s + 1)^2}{2s^2} \left( \frac{1}{2s + 1} - \frac{\Gamma^2(s + 1)}{\Gamma(2s + 2)} \right),
\]

showing that the upper bound is attained. We can now conclude as in Proposition 2 by the continuity of \( \beta \to \Delta_s^\beta(X_{s,\beta}) \) on \((0, 1]\).

\[\square\]

Remark 8. (a) As for the case \( s = 0 \), the upper bounds of \( \mathcal{R}_{s,2} \) and \( \mathcal{R}_{s,sym} \) are attained, whereas that of \( \mathcal{R}_{s,\ast} \) is not attained. The unique maximizer of \( \mathcal{R}_{s,2} \) is \( U^s \) up to affine transformation, and
the unique maximizer of $\mathcal{R}_{s,\text{sym}}$ is

$$X_{s,1} \overset{d}{=} (1 - U)^s - U^s$$

up to linear transformation. Observe that $s^{-1}X_{s,1}$ converges in law to the logistic distribution as $s \to 0$. The density of $X_{s,1}$ does not seem explicit in general, save for $s = 1, 2$ where $X_{1,1}$ and $X_{2,1}$ are uniform on $[-1, 1]$, and for $s = 1/2$ where $X_{1/2,1}$ has density

$$\frac{1 - x^2}{\sqrt{2 - x^2}} 1_{(-1,1)}(x).$$

(b) The non-increasing character of $s \mapsto \Delta s(X)$ implies that $\{\mathcal{R}_{s,2}, s \geq 0\}$ and $\{\mathcal{R}_{2,\text{sym}}, s \geq 0\}$ are non-increasing families of intervals, shrinking to $\{0\}$ as $s \to \infty$. Considering the upper bound of $\mathcal{R}_{2,\text{sym}}$ shows that the mapping

$$s \mapsto \frac{(s + 1)^2}{2s^2(2s + 1)} \left(1 - \frac{\Gamma^2(s + 1)}{\Gamma(2s + 1)}\right)$$

decreases on $(0, \infty)$ from $\pi^2/6$ to 0.

(c) By translation invariance, the above result implies the following deviation bound for every $X \in \mathcal{D} \cap L_2$ which is symmetric around its mean $\mathbb{E}[X]$. For every $s > 0$, one has

$$\mathbb{E}[X] - \mu_s(X) \in \left[0, \frac{s + 1}{\sqrt{2} s} \times \sqrt{\left(\frac{1}{2s + 1} - \frac{\Gamma^2(s + 1)}{\Gamma(2s + 2)}\right)} \times \text{Var}X\right].$$

We end this section with a new and non-trivial inequality for the Gamma function, which is in the case $s \in (-1/2, 1)$ a consequence of Proposition\footnote{Proposition 3.}

**Corollary 5.** For every $s \geq -3/2$, one has

$$\frac{\Gamma^2(s + 2)}{\Gamma(2s + 1)} \geq 1 + 2s - s^2 \quad (11)$$

Moreover, the inequality is strict except at $s = 0, 1$.

**Proof.** The equality is plain for $s = 0$ or $s = 1$, and the strict inequality is also straightforward for $s \in [-3/2, -1] \cup (1 + \sqrt{2}, \infty)$ since then the right-hand side is negative and the left-hand side is non-negative. The strict inequality for $s \in (-1, -1/2]$ is obtained directly from the equivalent formulation

$$\frac{\sqrt{\pi}(s + 1)\Gamma(s + 2)}{4s\Gamma(s + 1/2)} \geq 1 + 2s - s^2 \quad (12)$$

given by the Legendre duplication formula: by the change of variable $s = -1/2 - t$, (12) amounts indeed to

$$\frac{\sqrt{\pi}4t\Gamma(3/2 - t)}{\Gamma(1 - t)} \times t(1 - 2t) \leq 1/4 + 3t + t^2$$

for all $t \in [0, 1/2)$, whose left-hand side is bounded by $\pi t(1 - 2t) \leq 1/4 + 3t + t^2$ by log-convexity of the Gamma function and an elementary trinomial analysis.
We next consider the strict inequality for \( s \in (1, 1 + \sqrt{2}) \). Taking the logarithmic derivatives on both sides, we are reduced to show that

\[
\psi(s + 2) - \psi(2s + 1) > \frac{1 - s}{1 + 2s - s^2}
\]

for all \( s \in (1/(\sqrt{2} - 1)) \). This amounts to

\[
\frac{1}{1 + 2s - s^2} - \sum_{n \geq 1} \frac{1}{(n + 2s)(n + s + 1)} > 0,
\]

which holds true since the LHS equals \( 9/4 - \pi^2/6 > 0 \) at \( s = 1 \) and increases on \((1, 1 + \sqrt{2})\).

We finally show the strict inequality for \( s \in (-1/2, 0) \cup (0, 1) \), which cannot seem to be handled neither directly nor with classical monotonicity or convexity arguments. Instead, we consider the equivalent formulation

\[
\frac{s + 1}{\sqrt{2}s} \times \sqrt{\frac{1}{2s + 1} - \frac{\Gamma^2(s + 1)}{\Gamma(2s + 2)}} < \frac{1}{\sqrt{2s + 1}}
\]

for \( s \in (-1/2, 1) \). For \( s \in (0, 1) \), this is tantamount to \( \Delta_s^\sigma((1 - U)^s - U^s) < \Delta_s^\sigma(U^s) \) by the proof of Proposition \( \text{3} \) and it is clear that

\[
\Delta_s^\sigma((1 - U)^s - U^s) \leq \max\{\Delta_s^\sigma(X), X \in \mathcal{D} \cap \mathcal{L}_2\} = \Delta_s^\sigma(U^s)
\]

and that the inequality is strict by uniqueness, since any affine transformation of \( U^s \) is not symmetric for \( s \in (0, 1) \) and hence cannot be distributed as \((1 - U)^s - U^s\). The final case \( s \in (-1/2, 0) \) can be handled with the same argument. More precisely, the functional \( \text{9} \) can be extended to \( s \in (-1/2, 0) \) for all \( X \in \mathcal{D} \cap \mathcal{L}_2 \) symmetric since

\[
(s + 1) \int_0^\infty x f(x) \hat{F}^s(x) \, dx = \left[-x \hat{F}^{s+1}(x)\right]_0^\infty + \int_0^\infty \hat{F}^{s+1}(x) \, dx = \int_0^\infty \hat{F}^{s+1}(x) \, dx
\]

with \( \hat{F}^{s+1}(x) = o(x^{-2(s+1)}) \) as in \( \text{1} \). Similarly, the functional \( \text{8} \) can be extended to \( s \in (-1/2, 0) \) for all \( X \in \mathcal{D} \cap \mathcal{L}_2 \), with the same maximizer \((1 - U)^s - U^s \) among symmetric distributions resp. \( U^s \) among general distributions. The inequality \( \text{11} \) follows then exactly as for \( s \in (0, 1) \).

\[\square\]

**Remark 9.** (a) The inequality \( \text{12} \) can be rewritten as

\[
\frac{\Gamma(s + 1)}{\Gamma(s + 1/2)} \geq \frac{4^s(1 + 2s - s^2)}{\sqrt{\pi}(1 + s)^2}
\]

for all \( s > -1/2 \), whose right-hand side can be shown to be greater than \( \sqrt{s + 1/4} \) for all \( s \in [-1/4, 1] \) by an elementary monotonicity argument. In particular, \( \text{11} \) can be viewed as an improvement on Watson’s inequality \( \text{12} \). We refer to \( \text{3, 9} \) and the references therein for a collection of classical inequalities for the Gamma function, none of which seems to imply \( \text{11} \) directly. We stress that the inequality \( \text{11} \) is rather sharp on \((0, 1)\): simulations show that

\[
s \mapsto \frac{\Gamma^2(s + 2)}{\Gamma(2s + 1)} - 1 - 2s + s^2
\]

is unimodal on \((0, 1)\) from 0 to 0, with a small maximum value 0.0172962.. attained at \( s = 0.4671.. \).
(b) The inequality (11) remains true in the immediate left vicinity of $s = -3/2$ but it is clearly false as $s \to -2$ because the left-hand side tends to $-\infty$. Simulations show that (13) increases on $(-2, -3/2)$, which implies that (11) holds for all $s \geq s_*$ where $s_* = -1.6609\ldots$ stands for the unique root of (13) on $(-2, -3/2)$.

(c) For every $t \in (0, 1)$, the two functionals in (8) and (9) can be extended to $s \in (-t, 0)$ for all $X \in D \cap L^{1/(1-t)}$. 

5. Discussion

5.1. A possible generalization. The framework of this paper depends on a truncation on the right, considering cut-off random variables on $(-\infty, x]$. We can extend this by considering truncations on $[u, v]$ with $u, v \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. More precisely, consider $X_{p,q}$ indexed by $0 \leq p \leq q \leq 1$ with density $f_X/(F_X^{-1}(q) - F_X^{-1}(p))$ on $[F_X^{-1}(p), F_X^{-1}(q)]$. Setting $A = (0, 0)$ and $B = (0, 1)$ on the unit square, $\mu_s$ appears as a path integral along $[A, B]$ with speed $p_s$ or $q_s$, accordingly.

Travelling on $(BC)$ at speed $p_s$ or $q_s$ provides the dual metric $\bar{\mu}_s$ where the focus is on the right tail instead.

$$\bar{\mu}_s(X) = (s + 1) \int_0^1 t^s \mathbb{E} \left[ X_{F_X^{-1}(t)} \right] \, dt \quad \text{and} \quad \bar{\mu}_{-s}(X) = (s + 1) \int_0^1 (1 - t)^s \mathbb{E} \left[ X_{F_X^{-1}(t)} \right] \, dt$$

for all $s \geq 0$, where $\bar{X}_x$ has density $f_X/F_X$ on $[x, \infty)$ and zero otherwise. Further note that for any $x \in \text{Supp}X$ we have:

$$F(x) \mathbb{E}[X_x] + \bar{F}(x) \mathbb{E}[\bar{X}_x] = \mathbb{E}[X] \quad \text{and} \quad \bar{X}_x = (-X)_{-x}.$$

The construction of $\mu_w(X)$ involves the use of the mean operator ($\mathbb{E}$) twice: pointwise on $[A,B]$ as each point is an interval (operator 1), and another integration of these values along $[A,B]$ itself (operator 2 - and likewise on $[B,C]$ for $\bar{\mu}_w(X)$). We can entertain other operators than the mean to broaden the expressivity of the framework, such as variance, entropy or even any of the $\mu_s$. Note that the representation is not unique: when the path is trivial (say $A'$ as in VaR), it can be replaced by any path containing $A'$ using a Dirac on $\alpha$ as weight function and any path can be travelled backwards if either operator switches sign.

Many metrics can be represented in this framework.
• The integral along \([A', C']\) with \(A' = (\alpha, \alpha)\) and \(B' = (\beta, \beta)\) yields \(E[X_{\alpha,\beta}]\) and thus \(\text{VaR}_\alpha(X)\).

• The integral along \([D, D]\) with \(D = [0, \alpha]\) yields \(\text{TVaR}_\alpha(X)\).

• Any law invariant risk measure, by the general results of [7, 6] showing that they can be represented as a distorted version of either \(\text{VaR}\) or \(\text{TVaR}\).

The representation capacity of this framework eludes us however in full generality. In terms of practical applications, and restricting the focus on the path \((AB)\) and for a given \((s, X) \in \mathbb{R} \times \mathbb{D}_e\) it amounts to finding a function \(g\) and a weight \(w\) such that \(\mu_w(X) = \mu_s(g(X))\) where the computation of \(\mu_s(g(X))\) is possible as in Paragraph 3.3 - see the illustrations below.

5.2. Numerical illustration. In this section we will compare the mappings \(s \rightarrow \mu_s\) when \(s\) ranges in the centered interval \([-30, 30]\) for random variables of same support and expectation, respectively \(\mathbb{R}^-\) (negative Lomax and negative exponential), \(\mathbb{R}^+\) (Fréchet and Lomax) and \(\mathbb{R}\) (Normal, Logistic and Uniform). The point is also to allow the comparison with the cumulative distribution function.

5.2.1. Negative support. We first consider negative Lomax and negative exponential both of mean \(-1\), i.e. \(\beta = 2\) in 3.3.4 and Remark 6 (a) for the negative exponential. It is expected from the cumulative distributions that the \(\mu_s\) of the negative exponential are larger than those of the negative Lomax. Given that the cumulative curves intersect, it is a priori undecided whether \(\mu_s\) curves intersect for positive values of \(s\) - recall Remark 4(c): the \(\mu_s\)'s need not to cross as many times as the cumulative functions. And indeed it is clear from the closed formulae that they do not in this case. We know from Remark 7 (b) that the negative exponential maximizes \(R_{0.2}\) but \(\beta = 2\) is the switching value for which the variance is not defined for Lomax so it is somewhat expected that the negative exponential outperforms negative Lomax for any \(s\) as it entails less risks.

5.2.2. Positive support. We next consider Lomax and Fréchet, using the formulas in Paragraphs 3.3.3 and 3.3.5. Both have an expectation equal to 2, i.e. \(\beta = 3/2\) and \(\Gamma(1 - 1/\beta) = 2\) viz. \(\beta \sim 1.79\) respectively. Note that none of these have variance, making them impossible to compare using this common metric. The cumulative distributions also intersect, and similar to the above, this is not the case for the metric \(\mu\) - again the closed formulae are convenient to check that this is the case with arbitrary large values for \(s\).
5.2.3. Real support. We finally consider Normal, Logistics, and Uniform distributions. We take all three of them centered and with the standard deviation of $\pi/\sqrt{3}$. We compute the values for the uniform distribution using $\beta = 1$ in Paragraph 3.3.1 and (5) with support $[-\pi, \pi]$; for Logistic, we use 3.3.6; in the absence of closed formulas for the Normal distribution, $\mu_\alpha$ is directly estimated from its integral definition using the Simpson-Jones approximation. The equality of the two first moments make it challenging to rank these random variables in terms of risks and opportunities using the cumulative curves only.

It is intuitive that uniform performs better for $s < 0$ since it is bounded. It is also expected that Logistics lags behind at $s = 0$ given Corollary 3 - see also Remark 8 (a). For $s > 0$, the performance is very similar across distributions, though uniform seem to keep an edge over the two others. This is not quite the case, as shown in the plot below where the $\mu_\alpha$’s of the Normal and the Logistic random variables are shown relative to the $\mu_\alpha$’s of the Uniform distribution (for visualization purpose). We also plot the envelope of the $\mu_\alpha$’s (lower bound) as given in proposition 3 (dashed black line).

Close to 1, both Normal and Logistic outperform the uniform on a small interval - Normal on $[0.7, 2.2]$ and Logistic on $[0.2, 4.2]$. The positive difference with Uniform is maximized at $s = 1$, which is not surprising given the intermediary result of the proof of proposition 3: for a given standard deviation, the maximum for $\Delta_\alpha$ for $U^{1/\beta}$ is reached at $\beta = 1/s$, which is 1 for the uniform distribution. Then uniform dominates again, but from $s = 135$ onwards, Normal eventually takes over: the unbounded support on the right “pays off” and in this case there are as many crossing in the cumulative distributions and in the $\mu_\alpha$’s. This does not happen with the logistic distribution (this was checked up to $s = 10$ billions).
Logistic dominates Normal on $[-0.8, 29.4]$ i.e. two intersections hence one less than the cumulative distributions (in $-\pi, 0$ and $\pi$). Logistic and Normal are very similar in shape – in practical applications it is commonplace to replace the latter by the former because it has a closed formulae for the cumulative - and it is thus noteworthy that the behaviour of $\mu_s$ clearly differentiate these two random variables with respect to the Uniform, even if moments of order 1 and 2 are equal.

On balance, $\mu$ can provide a somewhat convenient and simplified assessment of the risk and performance profile of random variables, either because they do not have a variance or because $\mu$ allows for a comparison and order that cumulative functions do not. What is remarkable is that this form of simplification occurs without any loss of information since $\mu$ characterizes laws by Theorem 3. In the last illustration, this is not straightforward: for $s$ a negative integer, we have a strict ordering of the $\mu_s$ for Normal and Uniform, where the latter strictly dominates the former. Yet, these fully-ordered numbers fully encode the three crossing points occurring on $\mathbb{R}^+$ (at 0.7, 2.2 and 135 respectively) - see Corollary 1. Such representations also suggest to contemplate a possible distance between random variables of the form

$$d_\theta(X, Y) = \int_{\mathbb{R}} |\mu_s(X) - \mu_s(Y)|^\theta ds.$$ 

Given that some convergences are rather slow, closed formulas of the metric come in handy - see how Fréchet or negative exponential are slow to converge towards 0 and $-\infty$ respectively. For valuable analytical formulas are, a caveat is worth mentioning: for not so large values of $|s|$, computation of large Pochhammer series can be challenging. In such cases, direct estimates, as done with the Normal distribution above, seems the only alternative. The two approaches thus complement each other nicely.

5.3. **An example.** Let us finally outline an example to illustrate how the above findings can be practically applied with no or very limited recourse to computation. Say we want to assess the economic value of a large-scale investment, such as an infrastructure, by means of a Cost Benefit Analysis (CBA). We shall estimate the project Net Present Value (NPV), which consists of the sum of the discounted economic flows throughout the project time horizon, with the classical view that the project is worth funding if and only if its NPV is non-negative. We assume that the random variables associated with each year economic flow are independent and since infrastructure projects
are expected to last long, it is acceptable to model the NPV with a Normal law. Let us finally assume we have an estimate for both the mean and the standard deviation of the investment, either through Monte Carlo computations or using a deterministic model for the mean and an estimate of standard deviation derived from either of relevant hard data or subject matter expert elicitation.

Suppose that the mean is positive. If there were no uncertainty it would be sufficient to proceed with the investment. But a part of the distribution is negative, so it is unclear whether the risk is worth taking. As common practice, we want to avoid widely optimistic views and hence indulge a bias for conservativeness. In standard risk management, this does not really help to set a quantile and require that the value at this quantile should be non-negative: arguably it is less than one half, but that leaves too large of a choice left. Adjusting in terms of standard deviation is also natural, but we have no benchmark either about how many standard deviations the mean should be penalized by.

Building on the approach developed in this paper, we use $\mu$ because by truncating right tails (most favourable cases) it complies with a conservative perspective (as opposed to $\bar{\mu}$ in this case). In absence of further insights, the default natural choice is $\mu_0$. For a Normal law this leads to scoring the distribution at the 18th quantile approximately, thus providing some answer which the classic framework leaves open. In terms of efficient frontier, this means that the investment is exactly as attractive as one where the outcome would be the 18-th quantile without any uncertainty. Also note that we can also use a very simple yet accurate (slightly conservative) estimate using the upper bound for symmetric variables established in Section 4: $\mu_0(X) \approx E[X] - \pi \sigma_X / 2\sqrt{3}$. Put differently, the default conservative approach would be to penalize the mean by about 0.9 standard deviation a normal distribution. For other common symmetric laws (PERT and triangle symmetric, logistic, uniform), the quantile level under the same conditions would range between 16th and 25th (the spread is narrowing as $|s|$ increases in $\mu_s$, as expected). It is also possible to take conservative values of $\mu_s$ using the upper bound of Remark 8 (c) in this case for $s > 0$.

5.4. Conclusion. This paper connects to a two-pronged literature: on the one hand the study of cumulative entropy which links to reliability theory and survival analysis (i.e. when random variables represent time), and risk analysis as entertained in the finance industry and actuarial sciences (when random variables quantify assets) on the other hand.

We discuss relevant criteria to analyse risk (part 1) and introduce a new family of metrics featured by the emphasis put on specific parts of the distribution using weight functions throughout the support of the underlying random variable (part 2). This approach deviates from standard risk analysis where risk metrics are parametrized by quantile values in $[0,1]$ - the selection of which is not always straightforward in practice.

We study a dense subset of this family parametrized by a real number $\{\mu_s, s \in \mathbb{R}\}$ and show its compliance with properties we deem either necessary or desirable for sound risk analysis (part 3): (i) metrics are defined over a large set of real random variables containing $\mathcal{L}_\alpha$ for all $\alpha > 1$ and including most of those likely to appear in practice; (ii) the family is thorough i.e. the whole
range of risk metrics across the family fully characterizes the underlying distribution (distributions are even characterized for values of the metrics on either of $N$ or $-N$ only); (iii) all metrics induce a total order consistent with the stochastic order which yield non-trivial risk-neutral lines, thus reflecting trade-offs between risks and opportunities. These findings are established together with closed formulae of the metrics in great generality. Specific formulas are also provided for most common laws, sometimes in terms of special functions.

Using Wang’s distortion representation, we further established that the metrics are monetary but not coherent. We do not consider this penalizing so long as metrics are fit for their purpose. As it happens, $\mu_s$ typically comes in handy (and was originally designed) to entertain performance or opportunity while factoring in risks - which is conveniently interpreted $E[X]$ being “adjusted” of $\Delta_s(X)$. As discussed in the introduction, losses and gains are subjective and relative to a reference outside of the mathematical framework. It boils down to the risk practitioner to decide what metric is most convenient in context, which includes deciding whether to consider $X$ or $-X$, left or right tails, etc.

In Section 4, we then shed further light on renormalized version of $\Delta_s(X)$ by $\sigma_X$ and $E[X]$ respectively for restrictions of $\mu$’s domain of definition on random variables of interest - symmetric within $L_2$ and positive respectively. These ratios can be interpreted as relative “risk penalties” inflicted on the “performance” of an uncertain outcome and thus provide alternative insight on how a given (parametrized) distribution encodes risk as assessed through the $\mu_s$ lens. This yields alternative distribution characterization (as maximizer for a given metric level $s$) and the established bounds can be used to cap opportunity or risk where $\mu_s$ is not easily tractable (e.g. symmetric distributions).

In Paragraph 5.1 we finally represented the metrics as a specific baseline case of a more general framework where metric definition arises along three dimensions: (i) a path that encodes an interval screening strategy; (ii) a travel speed along this path - interpretation of the weight functions; (iii) two operators - one pointwise and one along the path. Altogether it yields an expressive framework for risk metric design and risk analysis, accounting for either of theoretical or practical considerations. Numerical illustrations and a notional practical example are provided in Paragraphs 5.2 and 5.3.

Some points remain open for further developments: compilation of the $\mu_s$ and $\Delta_s$ for further laws and key properties and metrics from within the risk framework generalization.

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