Error estimation of a discontinuous Galerkin method for time fractional subdiffusion problems with nonsmooth data

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Abstract
This paper is devoted to the numerical analysis of a piecewise constant discontinuous Galerkin method for time fractional subdiffusion problems. The regularity of weak solution is firstly established by using variational approach and Mittag-Leffler function. Then several optimal error estimates are derived with low regularity data. Finally, numerical experiments are conducted to verify the theoretical results.

Keywords Time fractional subdiffusion (primary) · Weak solution · Low regularity · Discontinuous Galerkin method · Optimal error estimate · Laplace transform

Mathematics Subject Classification 65M12 (primary) · 65M60 · 35S11

1 Introduction
This paper considers the lowest-order discontinuous Galerkin (DG) method for the time fractional subdiffusion equation:

\[
\begin{cases}
    u' - D_{0+}^{1-\alpha} \Delta u = f & \text{in } \Omega_T := \Omega \times (0, T), \\
    u = 0 & \text{on } \partial \Omega \times (0, T), \\
    u(\cdot, 0) = u_0 & \text{in } \Omega,
\end{cases}
\] (1.1)

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where \( \alpha \in (0, 1) \), \( T > 0 \) denotes the final time, \( \Omega \subset \mathbb{R}^d \) \((d = 1, 2, 3)\) is a convex polyhedral domain, \( f \) and \( u_0 \) are given data, and \( D_0^{1, -\alpha} \) is a left-sided Riemann–Liouville fractional differential operator (cf. Section 2).

Let us briefly review the efforts devoted to numerical treatments of problem (1.1). By now there are mainly two types of methods, according to how the time fractional derivative is approximated. The first type of schemes are based on finite difference formula, including L-type schemes [6, 12, 15, 46] and the Grünwald–Letnikov (GL) scheme [28, 44, 45]. The L1 method has the accuracy \( O(\tau^{1+\alpha}) \) for \( C^2 \) solutions. Utilizing the superconvergence property at some particular points of the GL formula, Gao et al. [7] constructed some finite difference schemes that achieve the accuracy \( O(\tau^2) \) for \( C^3 \) solutions. We also mention that for \( u_0 \in \mathring{H}^2(\Omega) \) and smooth \( f \), quadrature rules with exponential rate \( O(e^{-c\sqrt{N}}) \) can be found in [26, 27], where \( c > 0 \) is some constant and \( N \) stands for the number of quadrature points.

The second type of schemes adopt the time-stepping DG method [2, 36] and are combined with graded temporal grids to conquer the singularity. In [24], McLean et al. has applied the piecewise constant DG method to problem (1.1) and proved the error bound \( O(\tau + |\ln \tau| h^2) \) under the \( L^\infty(0, T; L^2(\Omega)) \)-norm, with initial data \( u_0 \in \mathring{H}^2(\Omega) \) and the following regularity assumption:

\[
\begin{align*}
\|u(t)\|_{\mathring{H}^2(\Omega)} + t \|u'(t)\|_{\mathring{H}^2(\Omega)} &\leq M \\
\tau^{2-\alpha} \|u'(t)\|_{\mathring{H}^2(\Omega)} + t^{3-\alpha} \|u''(t)\|_{\mathring{H}^2(\Omega)} &\leq M t^{\sigma-1} \quad 0 < t \leq T,
\end{align*}
\]  

(1.2)

where \( \sigma \) and \( M \) are two positive constants. Besides, we list more works using the piecewise linear DG method: Mustapha et al. [31] proved that the temporal convergence order under the \( L^\infty(0, T; L^2(\Omega)) \)-norm is \( O(\tau^{1+\alpha}) \); later in [33], the authors derived the improved bound \( O(\tau^{\min\{1.5+\alpha, 2\}}) \); in addition, they [32] proved the rate \( O(|\ln \tau| \tau^{1+2\alpha}) \), which yields superconvergence for \( \alpha \in (1/2, 1) \). We mention that the analyses in [32, 33] require stronger growth assumptions than (1.2). Mustapha [29] also proposed an \( hp \)-version DG method for problem (1.1) and established the suboptimal order \( O(\tau^{\max\{k, 2\} + (1-\alpha)/2}) \), where \( k \geq 1 \) is the polynomial degree.

It is worth to noticing an alternative form of (1.1):

\[
\begin{align*}
D_0^\alpha (u - u_0) - \Delta u &= \tilde{f} \quad \text{in } \Omega_T, \\
\quad u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\quad u(\cdot, 0) &= u_0 \quad \text{in } \Omega,
\end{align*}
\]  

(1.3)

where \( \tilde{f} = I_0^{1-\alpha} f \) with \( I_0^{1-\alpha} \) being a left-sided Riemann–Liouville fractional integral operator (cf. Section 2). For \( f = 0 \), both (1.1) and (1.3) share the same solution that can be represented by the Mittag-Leffler function. For solution regularity and numerical analysis of problem (1.3), especially for nonsmooth data, we refer the readers to [5, 14, 16–18, 20, 30, 37, 38, 41–43]

In a series of works [23, 26, 27], by using the Laplace transform, McLean et al. considered the regularity of problem (1.1) and proved some growth estimates. To our best knowledge, no work has been proposed for investigating the weak solution to
problem (1.1) by variational approach. In this paper, for the case $u_0 = 0$, $f \neq 0$, we introduce a weak solution to problem (1.1) via variational formulation, and prove that if $f \in L^2(0, T; \dot{H}^{-\beta}(\Omega))$ with $0 \leq \beta \leq 1$, then
\[
\|u\|_{0, H^1(0,T;\dot{H}^{-\beta}(\Omega))} + \|u\|_{0, H^{1-\alpha}(0,T;\dot{H}^{2-\beta}(\Omega))} \leq C_{\alpha} \|f\|_{L^2(0,T;\dot{H}^{-\beta}(\Omega))}.
\]
For nonhomogeneous case: $f = 0$, $u_0 \neq 0$, the weak solution is introduced and analyzed by Mittag-Leffler function. Here we note that, instead of proving the growth estimate like (1.2), we show what kind of vector-valued Sobolev space the weak solution belongs to.

As mentioned before, the error analyses of most existing numerical methods require either smooth property or growth estimate of the true solution. It will be more challenging to establish error estimates with given low regularity data. Let us summarize few work that aims to fill in this gap. For a temporal semi-discretization with $f = 0$, McLean and Mustapha [25] derived that
\[
\|(u - u_\tau)(t_j)\|_{L^2(\Omega)} \lesssim t_j^{-1} \tau \|u_0\|_{L^2(\Omega)},
\]
by using the Laplace transform. For a spatial semi-discretization, by energy argument, Karaa et al. [9] proved that, for $0 < t \leq T$,
\[
\|(u - u_h)(t)\|_{L^2(\Omega)} \lesssim h^{2-\alpha(2-\delta)/2} \left(\|u_0\|_{\dot{H}^\delta(\Omega)} + \sum_{i=0}^2 \int_0^T t^i \|f^{(i)}(t)\|_{\dot{H}^\delta(\Omega)} \, dt\right),
\]
where $0 \leq \delta \leq 2$.

In this work, we shall derive several optimal error estimates for a piecewise constant DG method with nonsmooth data:

- if $f = 0$ and $u_0 \in L^2(\Omega)$, then
  \[
  \|u - U\|_{L^2(\Omega_T)} \lesssim (\tau^{1/2} + h) \|u_0\|_{L^2(\Omega)}; \tag{1.4}
  \]
- if $u_0 = 0$ and $f \in L^2(\Omega_T)$, then
  \[
  (\tau^{1/2} + h)\|u - U\|_{\dot{H}^{1-\beta}(0,T;\dot{H}^{-\beta}(\Omega))} + \|u - U\|_{L^2(\Omega_T)} \lesssim (\tau + h^2) \|f\|_{L^2(\Omega_T)}; \tag{1.5}
  \]
Note that (1.5) is optimal with respect to the solution regularity (cf. Remark 3) but (1.4) is optimal only for temporal discretization. Moreover, for the case $u_0 = 0$ with uniform temporal grid, by means of Laplace transform, we prove the following quasi-optimal results:

- if $f \in L^2(\Omega_T)$, then
  \[
  \|u - U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \ln \tau \left(\tau^{1/2} + \epsilon_h h^{\min[2,1/\alpha]}\right) \|f\|_{L^2(\Omega_T)},
  \]
  where $\epsilon_h = 1$ if $\alpha \neq 1/2$ and $\epsilon_h = \sqrt{\ln h}$ if $\alpha = 1/2$.
• if \( f \in H^{1/2}(0, T; L^2(\Omega)) \), then
\[
\| u - U \|_{L^\infty(0,T;L^2(\Omega))} \lesssim |\ln \tau| \left( |\ln \tau| \tau + h^2 \right) \| f \|_{H^{1/2}(0,T;L^2(\Omega))}.
\]

The rest of this paper is organized as follows. Section 2 introduces some Sobolev spaces and fractional calculus operators. Section 3 defines the weak solution to problem (1.1) and investigates its regularity. Then Section 4 presents the DG discretization and lists the main results. Sections 5 and 6 establish detailed proofs of the error estimates. Finally, Section 7 conducts several numerical experiments to verify the theoretical results.

2 Notations

Firstly, let us make some conventions. For a Lebesgue measurable subset \( \omega \) of \( \mathbb{R}^l \) \((l = 1, 2, 3)\), we use \( H^\gamma(\omega) \) \((\gamma \in \mathbb{R})\) and \( H^\gamma_0(\omega) \) \((\gamma > 0)\) to denote two standard Sobolev spaces [39]. For a Lebesgue measurable subset \( \mathcal{O} \) of \( \mathbb{R}^l \) \((l = 1, 2, 3, 4)\), the symbol \( \langle p, q \rangle_\mathcal{O} \) means \( \int_\mathcal{O} pq \) for \( pq \in L^1(\mathcal{O}) \). If \( X \) is a Banach space, then \( X^* \) denotes its dual space and \( \langle \cdot, \cdot \rangle_X \) is the duality pairing between \( X^* \) and \( X \). For \( 0 < \theta < 1 \) and two Banach spaces \( X \) and \( Y \), \( [X, Y]_{\theta, 2} \) stands for the interpolation space constructed via the well-known \( K \)-method [39], with the norm
\[
\| v \|_{[X, Y]_{\theta, 2}} := \left( \int_0^\infty \left( t^{-\theta} K(t, v) \right)^2 \frac{dt}{t} \right)^{1/2}, \quad \forall v \in [X, Y]_{\theta, 2},
\]
where the functional \( K : (0, \infty) \times (X + Y) \to \mathbb{R} \) is defined as follows
\[
K(t, v) := \inf_{v = v_0 + v_1} \{ \| v_0 \|_X + t \| v_1 \|_Y \}.
\]
Moreover, if the symbol \( C \) has subscript(s), then it means a positive constant that depends only on its subscript(s), and its value may differ at each of its occurrence(s).

Secondly, we introduce some spaces constructed by the eigenvectors of \(-\Delta\). It is standard that [4, Theorem 1, §6.5.1] there exists an orthonormal basis \( \{ \phi_n : n \in \mathbb{N} \} \) of \( L^2(\Omega) \) such that \( \phi_n \in H^1_0(\Omega) \cap H^2(\Omega) \) and \(-\Delta \phi_n = \lambda_n \phi_n\), where \( \{ \lambda_n : n \in \mathbb{N} \} \) is a non-decreasing sequence and \( \lambda_n \to \infty \) as \( n \to \infty \). For any \( \gamma \in \mathbb{R} \), define
\[
\hat{H}^\gamma(\Omega) := \left\{ \sum_{n=0}^\infty c_n \phi_n : \sum_{n=0}^\infty c_n^2 \lambda_n^\gamma < \infty \right\},
\]
and equip this space with the inner product
\[
\left( \sum_{n=0}^\infty c_n \phi_n, \sum_{n=0}^\infty d_n \phi_n \right)_{\hat{H}^\gamma(\Omega)} := \sum_{n=0}^\infty \lambda_n^\gamma c_n d_n.
\]
for all $\sum_{n=0}^{\infty} c_n \phi_n, \sum_{n=0}^{\infty} d_n \phi_n \in \dot{H}^\gamma(\Omega)$, and we use $\|\cdot\|_{\dot{H}^\gamma(\Omega)}$ to denote the norm induced by this inner product. It is evident that $\dot{H}^\gamma(\Omega)$ is a Hilbert space with an orthonormal basis $\{\lambda_{n, \gamma/2} \phi_n : n \in \mathbb{N}\}$. In addition, $\dot{H}^{-\gamma}(\Omega)$ is the dual space of $\dot{H}^\gamma(\Omega)$ in the sense that

$$\left\langle \sum_{n=0}^{\infty} c_n \phi_n, \sum_{n=0}^{\infty} d_n \phi_n \right\rangle_{\dot{H}^\gamma(\Omega)} := \sum_{n=0}^{\infty} c_n d_n$$

for all $\sum_{n=0}^{\infty} c_n \phi_n \in \dot{H}^{-\gamma}(\Omega)$ and $\sum_{n=0}^{\infty} d_n \phi_n \in \dot{H}^\gamma(\Omega)$.

Thirdly, we introduce some interpolation spaces and vector-valued spaces. In what follows, assume $-\infty < a < b < \infty$. Following [22], for any $m \in \mathbb{N}$, define

$$0^{H^m}(a, b) := \{v \in H^m(a, b) : v^{(k)}(b) = 0, \ 0 \leq k < m, \ k \in \mathbb{N}\},$$

$$0^H(a, b) := \{v \in H^m(a, b) : v^{(k)}(a) = 0, \ 0 \leq k < m, \ k \in \mathbb{N}\},$$

where $v^{(k)}$ is the $k$-th weak derivative of $v$, and endow those two spaces with the following norms

$$\|v\|_{0^{H^m}(a, b)} := \|v^{(m)}\|_{L^2(a, b)} \quad \forall \ v \in 0^{H^m}(a, b),$$

$$\|v\|_{0^H(a, b)} := \|v^{(m)}\|_{L^2(a, b)} \quad \forall \ v \in 0^H(a, b).$$

Then for $\gamma > 0$, we define two interpolation spaces

$$0^{H^\gamma}(a, b) := [L^2(a, b), 0^{H^m}(a, b)]_{\theta, 2},$$

$$0^H(a, b) := [L^2(a, b), 0^H(a, b)]_{\theta, 2},$$

with corresponding interpolation norms defined by (2.1), where $0 < \theta < 1$ and $m \in \mathbb{N}$ such that $\gamma = m\theta$. By [21, Chapter 1] and [22], if $0 < \gamma < 1/2$, then $0^{H^\gamma}(a, b), 0^{H^\gamma}(a, b)$ and $H^\gamma(a, b)$ are equivalent and they share the same alternative norm

$$|v|_{H^\gamma(a, b)} := \left(\int_{\mathbb{R}} |\xi|^{2\gamma} |\mathcal{F}(v)(\xi)|^2 \ d\xi\right)^{1/2}, \quad \forall \ v \in H^\gamma(a, b), \quad (2.2)$$

where $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the Fourier transform and $Ev$ means extending $v$ to $\mathbb{R}\setminus(a, b)$ by zero; if $1/2 < \gamma < 1$, then

$$0^{H^\gamma}(a, b) = \{v \in H^\gamma(a, b) : v^{(m)}(a) = 0\},$$

$$0^H(a, b) = \{v \in H^\gamma(a, b) : v^{(m)}(b) = 0\};$$

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if \( \gamma = m + s \) with \( m \in \mathbb{N} \) and \( s \in (0, 1) \setminus \{1/2\} \), then
\[
0^H \gamma (a, b) = \left\{ v \in 0^H m (a, b) : v^{(m)} \in 0^H \gamma - m (a, b) \right\},
\]
\[
0^H \gamma (a, b) = \left\{ v \in 0^H m (a, b) : v^{(m)} \in 0^H \gamma - m (a, b) \right\};
\]
if \( \gamma = m + 1/2 \) with \( m \in \mathbb{N} \), then
\[
0^H \gamma (a, b) = \{ v \in 0^H m (a, b) : v^{(m)} \in H^{1/2} (a, b), (t - a)^{-1/2} v^{(m)} \in L^2 (a, b) \},
\]
\[
0^H \gamma (a, b) = \{ v \in 0^H m (a, b) : v^{(m)} \in H^{1/2} (a, b), (b - a)^{-1/2} v^{(m)} \in L^2 (a, b) \}.
\]
Moreover, by [21, Remark 11.5], the interpolation norms \( \|v\|_{0, H^{1/2} (a, b)} \) and \( \|v\|_{0, H^{1/2} (a, b)} \) are also equivalent to \( \|v\|_{H^{1/2} (a, b)} + \|(t - a)^{-1/2} v\|_{L^2 (a, b)} \) and \( \|v\|_{H^{1/2} (a, b)} + \|(b - t)^{-1/2} v\|_{L^2 (a, b)} \), respectively. Now let \( X \) be a separable Hilbert space with an inner product \((\cdot, \cdot)_X\) and an orthonormal basis \( \{e_i : i \in \mathbb{N}\} \).

For any \( \gamma \in \mathbb{R} \), define the vector-valued space
\[
H^\gamma (a, b; X) := \left\{ v \in L^2 (a, b; X) : \sum_{i=0}^{\infty} \| (v, e_i)_X \|_{H^\gamma (a, b)}^2 < \infty \right\},
\]
and endow this space with the norm
\[
\|v\|_{H^\gamma (a, b; X)} := \left( \sum_{i=0}^{\infty} \| (v, e_i)_X \|_{H^\gamma (a, b)}^2 \right)^{1/2}, \quad \forall v \in H^\gamma (a, b; X).
\]
For \( \gamma > 0 \), the two spaces \( 0^H \gamma (a, b; X) \) and \( 0^H \gamma (a, b; X) \) can be defined analogously.

Fourthly, we introduce the Riemann–Liouville fractional calculus operators. For \( \gamma > 0 \), define the fractional order integrals
\[
\left( I^\gamma_{a+} v \right) (t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t - s)^{\gamma - 1} v(s) \, ds, \quad t \in (a, b),
\]
\[
\left( I^\gamma_{b-} v \right) (t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s - t)^{\gamma - 1} v(s) \, ds, \quad t \in (a, b),
\]
for all \( v \in L^1 (a, b; X) \), where \( \Gamma(\cdot) \) denotes the well-known Gamma function. For \( j - 1 \leq \gamma < j \) with \( j \in \mathbb{N}_+ \), define the corresponding fractional derivatives
\[
D^\gamma_{a+} := D^j I^j_{a+} \gamma, \quad D^\gamma_{b-} := (-1)^j D^j I^j_{b-} \gamma,
\]
where \( D \) is the first-order differential operator in the distribution sense. Essentials of the fractional calculus are listed in Section 7.
3 Weak solution

3.1 The case \( u_0 = 0 \)

Define

\[
\mathcal{X} := 0^{H^{\alpha/2}}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
\]
\[
\mathcal{Y} := 0^{H^{1-\alpha/2}}(0, T; L^2(\Omega)) \cap 0^{H^{1-\alpha}}(0, T; H^1(\Omega)),
\]

and endow them with the following two norms

\[
\|\cdot\|_{\mathcal{X}} := \left(\|\cdot\|_{0^{H^{\alpha/2}}(0,T;L^2(\Omega))}^2 + \|\cdot\|_{L^2(0,T;H^1(\Omega))}^2\right)^{1/2},
\]
\[
\|\cdot\|_{\mathcal{Y}} := \left(\|\cdot\|_{0^{H^{1-\alpha/2}}(0,T;L^2(\Omega))}^2 + \|\cdot\|_{0^{H^{1-\alpha}}(0,T;H^1(\Omega))}^2\right)^{1/2}.
\]

Assuming that \( f \in \mathcal{Y}^* \), we call \( u \in \mathcal{X} \) a weak solution to problem (1.1) if

\[
\langle D^{\alpha}_{0+} u, v \rangle_{0^{H^{\alpha/2}}(0,T;L^2(\Omega))} + \langle \nabla u, \nabla v \rangle_{\Omega_T} = \langle f, I^{1-\alpha}_{T-} v \rangle_{\mathcal{Y}}, \quad \forall v \in \mathcal{X}. \tag{3.1}
\]

Since \( 0^{H^{\alpha/2}}(0, T; L^2(\Omega)) = H^{\alpha/2}(0, T; L^2(\Omega)) \) in the sense of equivalent norms and applying Lemma 14 implies that

\[
\| I^{1-\alpha}_{T-} v \|_{\mathcal{Y}} \leq C_{\alpha} \| v \|_{\mathcal{X}} \quad \text{for all} \ v \in \mathcal{X}, \tag{3.2}
\]

we readily conclude that the above weak solution is well-defined, according to Lemma 11 and the well-known Lax–Milgram theorem.

**Theorem 1** If \( f \in \mathcal{Y}^* \), then problem (1.1) admits a unique weak solution \( u \in \mathcal{X} \) satisfying \( \| u \|_{\mathcal{X}} \leq C_{\alpha} \| f \|_{\mathcal{Y}^*}. \)

**Remark 1** Our weak formulation (3.1) is motivated from that proposed in [14, 19] for (1.3). It looks quite different from the standard one for classical heat equation (cf. [36, Eq.(2.8)] or [21, Chapter 3]). A more natural formulation shall be stated as follows: for given \( f \in \mathcal{Y}^* \), find \( u \in \mathcal{X} \) such that

\[
- \langle u, w \rangle_{\mathcal{X}} + \left( \nabla u, D^{1-\alpha}_{T-} \nabla w \right)_{\Omega_T} = \langle f, w \rangle_{\mathcal{Y}}, \quad \forall w \in \mathcal{Y}. \tag{3.3}
\]

We claim that (3.3) boils down to our nonstandard one (3.1). Rigorous explanations are provided as follows.

First of all, the operator \( I^{1-\alpha}_{T-} : \mathcal{X} \rightarrow \mathcal{Y} \) is one-to-one, which means \( \mathcal{Y} = I^{1-\alpha}_{T-} \mathcal{X} \).

Indeed, from (3.2) follows the injection, and by [22, Lemmas 3.5 and 3.6], \( w = I^{1-\alpha}_{T-} D^{1-\alpha}_{T-} w \) with \( D^{1-\alpha}_{T-} w \in \mathcal{X} \), for all \( w \in \mathcal{Y} \).
We then prove that
\[
\left\langle D_{0+}^\alpha v, D_{T-}^{1-\alpha} w \right\rangle_{0 H^{\alpha/2}(0,T;L^2(\Omega))} = -\left\langle w', v \right\rangle_{0 H^{\alpha/2}(0,T;L^2(\Omega))},
\]
for all \( v \in \mathcal{X} \) and \( w \in \mathcal{Y} \). It is sufficient to consider the scalar case
\[
\left\langle D_{0+}^{\alpha/2} v, D_{T-}^{1-\alpha/2} w \right\rangle_{(0,T)} = -\left\langle w', v \right\rangle_{0 H^{\alpha/2}(0,T)}, \tag{3.4}
\]
for all \( v \in 0 H^{\alpha/2}(0,T) \) and \( w \in 0 H^{1-\alpha/2}(0,T) \). Define a linear functional \( s_w : 0 H^{\alpha/2}(0,T) \to \mathbb{R} \) by that
\[
s_w(v) := \left\langle D_{0+}^{\alpha/2} v, D_{T-}^{1-\alpha/2} w \right\rangle_{(0,T)}, \quad \forall v \in 0 H^{\alpha/2}(0,T),
\]
where \( w \in 0 H^{1-\alpha/2}(0,T) \) is fixed. For any \( v \in C_0^\infty(0,T) \), using integration by part, we find \( s_w(v) = \left\langle v', w \right\rangle_{(0,T)} \) and thus \( s_w = -w' \). Since \( C_0^\infty(0,T) \) is dense in \( 0 H^{\alpha/2}(0,T) \), (3.4) follows immediately.

Hence, by choosing “indirect” test function \( w = I_{T-}^{1-\alpha} v \) with \( v \in \mathcal{X} \), (3.3) agrees with (3.1). For investigating the weak solution, it is more convenient for us to treat the latter. This coincides with the same spirit as the weak form proposed for the fractional wave equation in [22]. However, for numerical discretization, the computational cost of (3.1) is larger than that of (3.3), since the former has one more fractional integral operator than the latter.

Then let us analyze the regularity of the weak solution to problem (1.1). We first consider the following problem: seek \( y \in 0 H^{\alpha/2}(0,T) \) such that
\[
\left\langle D_{0+}^\alpha y, z \right\rangle_{0 H^{\alpha/2}(0,T)} + \lambda \left\langle y, z \right\rangle_{(0,T)} = \left\langle g, I_{T-}^{1-\alpha} z \right\rangle_{0 H^{1-\alpha/2}(0,T)} \tag{3.5}
\]
for all \( z \in 0 H^{\alpha/2}(0,T) \), where \( g \in (0 H^{1-\alpha/2}(0,T))^* \) and \( \lambda > 0 \) is a constant. Again, by Lemmas 11 and 14 and the Lax–Milgram theorem, we conclude that problem (3.5) admits a unique solution \( y \in 0 H^{\alpha/2}(0,T) \) and there holds the estimate
\[
\|y\|_{0 H^{\alpha/2}(0,T)} + \lambda^{1/2} \|y\|_{L^2(0,T)} \leq C_\alpha \|g\|_{(0 H^{1-\alpha/2}(0,T))^*}.
\]

**Lemma 1** If \( g \in L^2(0,T) \), then the solution \( y \) to problem (3.5) satisfies \( y' + \lambda D_{0+}^{1-\alpha} y = g \) and
\[
\|y\|_{0 H^1(0,T)} + \lambda \|y\|_{0 H^{1-\alpha}(0,T)} \leq C_\alpha \|g\|_{L^2(0,T)}. \tag{3.6}
\]
In addition, if \( 1/2 \leq \alpha < 1 \), then for all \( 0 < \epsilon \leq 2 \),
\[ \lambda^{1/(2\alpha)-\sigma \epsilon/2} \| y \|_{C[0,T]} \leq \frac{C_{\alpha,T}}{\epsilon^{\sigma/2}} \| g \|_{L^2(0,T)}, \]  

(3.7)

where \( \sigma = 0 \) if \( 1/2 < \alpha < 1 \) and \( \sigma = 1 \) if \( \alpha = 1/2 \).

**Proof** Observe that the equality
\[ y = I_{0+}^{\alpha} (I_{0+}^{1-\alpha} g - \lambda y) = I_{0+} g - \lambda I_{0+}^{\alpha} y \]  

(3.8)

is contained in the proofs of [14, Lemmas 3.1 and 3.2], and using (3.8) recursively yields the relation
\[ y = (-\lambda)^n I_{0+}^{\alpha^n} y + \sum_{i=0}^{n-1} (-\lambda)^i I_{0+}^{1+i\alpha} g, \quad \forall n \in \mathbb{N}. \]

Then by Lemma 14, \( y \in \mathcal{O} H^1(0, T) \), and it follows from (3.8) that
\[ y' + \lambda D_{0+}^{1-\alpha} y = g. \]  

(3.9)

Next, let us prove (3.6). Multiplying both sides of (3.9) by \( y' \) and integrating over \((0, T)\) gives
\[ \| y' \|_{L^2(0,T)}^2 + \lambda \langle D_{0+}^{1-\alpha} y, y' \rangle_{(0,T)} = \langle g, y' \rangle_{(0,T)}, \]

so that, by Lemmas 11 and 14, the Cauchy–Schwartz inequality and Young’s inequality with \( \epsilon \), we obtain the estimate
\[ \| y' \|_{L^2(0,T)}^2 + \lambda \| y \|_{H^{1-\sigma/2}(0,T)}^2 \leq C_{\alpha} \| g \|_{L^2(0,T)}^2. \]  

(3.10)

Additionally, since \( y \in \mathcal{O} H^1(0, T) \), a straightforward computing gives
\[ y = I_{0+}^{1-\alpha} D_{0+}^{1-\alpha} y, \]  

(3.11)

which together with (3.9) and Lemma 14 implies
\[ \lambda \| y \|_{H^{1-\sigma}(0,T)} \leq C_{\alpha} \left( \| y' \|_{L^2(0,T)} + \| g \|_{L^2(0,T)} \right). \]  

(3.12)

Therefore, combining (3.10) and (3.12) proves (3.6).

It remains to prove (3.7). Thanks to (3.6) and Lemma 15, for \( 1/2 < \alpha < 1 \), we have
\[ \lambda^{1/(2\alpha)} \| y \|_{C[0,T]} \leq C_{\alpha,T} \| y \|_{H^1(0,T)}^{(\alpha-1/2)/\alpha} \left( \lambda \| y \|_{H^{1-\sigma}(0,T)} \right)^{1/(2\alpha)} \leq C_{\alpha,T} \| g \|_{L^2(0,T)}. \]
As for $\alpha = 1/2$, applying Lemma 15 once again yields that

$$\lambda^{1-\epsilon/2} \|y\|_{C[0,T]} \leq \frac{C_{\alpha,T}}{\sqrt{\epsilon}} \left( \lambda \|y\|_{0H^{1/2}(0,T)} \right)^{1-\epsilon/2} \|y\|_{0H^1(0,T)}^{\epsilon/2} \leq \frac{C_{\alpha,T}}{\sqrt{\epsilon}} \|g\|_{L^2(0,T)}$$

for all $0 < \epsilon \leq 2$. This completes the proof of this lemma.

Lemma 2 If $g \in 0H^\gamma(0,T)$ with $0 < \gamma \leq 1/2$, then the solution $y$ to problem (3.5) satisfies that

$$\|y\|_{0H^1(0,T)} + \lambda \|y\|_{0H^{1-\alpha}(0,T)} \leq C_{\alpha,\gamma} \|g\|_{0H^\gamma(0,T)}. \tag{3.13}$$

Proof Let us focus on the auxiliary problem: seek $w \in 0H^{\alpha/2}(0,T)$ such that

$$\langle D^\alpha_{0+} w, z \rangle_{0H^{\alpha/2}(0,T)} + \lambda \langle w, z \rangle_{(0,T)} = \langle f, I_1 - \alpha T - z \rangle_{(0,T)} \tag{3.14}$$

for all $z \in 0H^{\alpha/2}(0,T)$. By Lemmas 1 and 13, $w \in 0H^1(0,T)$ exists uniquely and satisfies

$$w' + \lambda D^{1-\alpha}_{0+} w = D^\gamma_{0+} g. \tag{3.15}$$

In addition, by (3.6) and Lemma 13, it holds that

$$\|w\|_{0H^1(0,T)} + \lambda \|w\|_{0H^{1-\alpha}(0,T)} \leq C_{\alpha} \|D^\gamma_{0+} g\|_{L^2(0,T)} \leq C_{\alpha,\gamma} \|g\|_{0H^\gamma(0,T)}. \tag{3.16}$$

Set $y := I^\gamma_{0+} w$, then using Lemma 14 gives (3.13). It remains to prove that $y = I^\gamma_{0+} w$ is the solution to problem (3.5). To this end, applying $I^\gamma_{0+}$ to both sides of (3.15), we obtain

$$(I^\gamma_{0+} w)' + \lambda D^{1-\alpha}_{0+} I^\gamma_{0+} w = g.$$  

This proves the desired result and completes the proof.

If $f \in L^2(0,T; \dot{H}^{-1}(\Omega))$, then similar to [22, Lemma 4.4] and [14, Theorem 3.1], we can prove that the weak solution $u$ to problem (1.1) is

$$u(t) = \sum_{n=0}^{\infty} y_n(t) \phi_n, \quad 0 < t \leq T,$$

where $y_n \in 0H^{\alpha/2}(0,T)$ satisfies that

$$\langle D^\alpha_{0+} y_n, z \rangle_{0H^{\alpha/2}(0,T)} + \lambda_n \langle y_n, z \rangle_{(0,T)} = \langle f, \phi_n \rangle_{\dot{H}^1(\Omega)}, \quad I^{1-\alpha}_{0+} z \rangle_{(0,T)},$$

for all $z \in 0H^{\alpha/2}(0,T)$. Therefore, the desired regularity result follows directly from Lemmas 1 and 2.
Theorem 2 Assume that \( f \in 0H^\gamma(0, T; \dot{H}^{-\beta}(\Omega)) \) with \( 0 \leq \gamma \leq 1/2 \) and \( 0 \leq \beta \leq 1 \). Then the weak solution \( u \) to problem (1.1) satisfies \( u' - D_{0+}^{1-\alpha} \Delta u = f \) in \( L^2(0, T; \dot{H}^{-\beta}(\Omega)) \) and
\[
\| u \|_{0H^{1+\gamma}(0, T; \dot{H}^{-\beta}(\Omega))} + \| u \|_{0H^{1+\gamma-\alpha}(0, T; H^{2-\beta}(\Omega))} \leq C_{\alpha, \gamma} \| f \|_{0H^\gamma(0, T; \dot{H}^{-\beta}(\Omega))} .
\]
In addition, if \( \gamma = 0 \) and \( 1/2 \leq \alpha < 1 \), then for all \( 0 < \epsilon \leq 2 \),
\[
\| y \|_{C([0, T]; 0H^{1/\alpha - \sigma - \epsilon - \beta}(\Omega))} \leq \frac{C_{\alpha, T}}{\epsilon^\alpha/2} \| f \|_{L^2(0, T; \dot{H}^{-\beta}(\Omega))} ,
\]
where \( \sigma = 0 \) if \( 1/2 < \alpha < 1 \) and \( \sigma = 1 \) if \( \alpha = 1/2 \).

For the dual problem of (1.1), we have the following theorem.

Theorem 3 Assume that \( q \in L^2(0, T; \dot{H}^{-\beta}(\Omega)) \) with \( 0 \leq \beta \leq 1 \). Then there exists a unique
\[
w \in \mathcal{G} := 0H^1(0, T; \dot{H}^{-\beta}(\Omega)) \cap 0H^{1-\alpha}(0, T; H^{2-\beta}(\Omega))
\]
such that \(-w' - D_{1-}^{1-\alpha} \Delta w = q \) and
\[
\| w \|_{0H^1(0, T; \dot{H}^{-\beta}(\Omega))} + \| w \|_{0H^{1-\alpha}(0, T; H^{2-\beta}(\Omega))} \leq C_{\alpha} \| q \|_{L^2(0, T; \dot{H}^{-\beta}(\Omega))} .
\]

3.2 The case \( f = 0 \)

For \( a, b > 0 \), recall the definition of the Mittag-Leffler function
\[
E_{a, b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + b)}, \quad z \in \mathbb{C}.
\]
Given \( \lambda, t > 0 \) and \( \gamma \in \mathbb{R}_+ \setminus \mathbb{N} \), we have the following facts (cf. [8]):
\[
|E_{a, b}(-t)| \leq \frac{C_{a, b}}{1 + t}, \quad (3.16)
\]
\[
D_{0+}^\gamma E_{a, 1}(-\lambda t^a) = t^{-\gamma} E_{a, 1-\gamma}(-\lambda t^a), \quad (3.17)
\]
\[
\frac{d}{dt} E_{a, 1}(-\lambda t^a) = -\lambda t^{a-1} E_{a, a}(-\lambda t^a). \quad (3.18)
\]
For any \( \lambda > 0 \) and \( y_0 \in \mathbb{R} \), by (3.17) and (3.18), it is easy to see that
\[
y(t) = y_0 E_{a, 1}(-\lambda t^a), \quad 0 \leq t \leq T,
\]
solves the equation
\[
y' + \lambda D_{0+}^{1-\alpha} y = 0, \quad 0 < t \leq T.
\]
with initial condition $y(0) = y_0$. Therefore, for $f = 0$ and $u_0 \in \dot{H}^{-2}(\Omega)$, it is natural to define a weak solution of problem (1.1) by (see [34]):

$$u(t) := \sum_{n=0}^{\infty} E_{\alpha, 1}(-\lambda_n t^\alpha) \langle u_0, \phi_n \rangle \dot{H}^2(\Omega) \phi_n, \quad 0 \leq t \leq T. \quad (3.19)$$

It follows from (3.16) that $u \in C([0, T]; \dot{H}^{-2}(\Omega))$ is well defined. In addition, we have $u(0) = u_0$ and

$$\|u\|_{C([0, T]; \dot{H}^{-2}(\Omega))} \leq C_\alpha \|u_0\|_{\dot{H}^{-2}(\Omega)}.$$

Since $u_0 \in \dot{H}^{-2}(\Omega)$, (3.19) shall be understood as the “very weak solution” by using the transposition method [21]; see also [22, Section 4.3]. In the following, we only consider the case $u_0 \in L^2(\Omega)$ and establish the weak formulation (3.20) and regularity estimate (3.21). Particularly, the formulation (3.20) tells us in what sense (3.19) is a weak solution to the original subdiffusion model (1.1).

**Theorem 4** If $u_0 \in L^2(\Omega)$, then the weak solution defined by (3.19) satisfies

$$\langle u', v \rangle_{H^{1-\alpha}/2(0, T; \dot{H}^1(\Omega))} + \left\langle D_{0+}^{1-\alpha} \nabla u, \nabla v \right\rangle_{H^{1-\alpha}/2(0, T; L^2(\Omega))} = 0, \quad (3.20)$$

for all $v \in H^{1-\alpha}/2(0, T; \dot{H}^1(\Omega))$, and we have the estimate

$$\|u'\|_{H^{1-\alpha}/2(0, T; \dot{H}^1(\Omega))} + \|u\|_{C([0, T]; L^2(\Omega))} + \|u\|_{H^{1-\alpha}/2(0, T; \dot{H}^1(\Omega))} + \epsilon_{\alpha, \gamma} \|u\|_{L^2(0, T; H^\gamma(\Omega))} \leq C_{\alpha, T} \|u_0\|_{L^2(\Omega)}, \quad \text{(3.21)}$$

where $\epsilon_{\alpha, \gamma} := \sqrt{2 - \gamma + \sqrt{2\alpha - 1}}$ with $\gamma = \min\{2, 1/\alpha\}$ if $\alpha \neq 1/2$ and $1 \leq \gamma < 2$ if $\alpha = 1/2$.

**Proof** By (3.17) and (3.18), a routine computation yields

$$t^{1-\alpha} \left\| D_{0+}^{1-\alpha} u(t) \right\|_{\dot{H}^1(\Omega)}^2 + t^{1-\alpha} \left\| u'(t) \right\|_{\dot{H}^1(\Omega)}^2 \leq C_\alpha \sum_{n=0}^{\infty} \frac{\lambda_n t^\alpha - 1}{(1 + \lambda_n t^\alpha)^2} \langle u_0, \phi_n \rangle^2_{\Omega},$$

and

$$\left\| D_{0+}^{1-\alpha/2} u(t) \right\|_{\dot{H}^1(\Omega)}^2 + \left\| I_{0+}^{1-\alpha/2} u'(t) \right\|_{\dot{H}^1(\Omega)}^2 \leq C_\alpha \sum_{n=0}^{\infty} \frac{\lambda_n t^\alpha - 1}{(1 + \lambda_n t^\alpha)^2} \langle u_0, \phi_n \rangle^2_{\Omega}.$$
we have that
\[
\|u\|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}^2 + \int_0^T t^{1-\alpha} \left\| D_{0+}^{1-\alpha} \nabla u(t) \right\|_{L^2(\Omega)}^2 \, dt \\
+ \int_0^T t^{1-\alpha} \left\| u'(t) \right\|_{\dot{H}^{-1}(\Omega)}^2 \, dt \leq C_\alpha \| u_0 \|_{L^2(\Omega)}^2.
\]
Since using [39, Lemma 16.3] implies
\[
\int_0^T t^{\alpha-1} \| v(t) \|_{\dot{H}^1(\Omega)}^2 \, dt \leq C_{\alpha,T} \| v \|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}^2,
\]
for all \( v \in H^{(1-\alpha)/2}(0, T; \dot{H}^1(\Omega)) \), it follows
\[
\| u' \|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} + \left\| D_{0+}^{1-\alpha} \nabla u \right\|_{H^{(1-\alpha)/2}(0,T;L^2(\Omega))} \leq C_{\alpha,T} \| u_0 \|_{L^2(\Omega)},
\]
and a direct calculation gives (3.20). Meanwhile, a similar manipulation accomplishes the estimate (3.21) and finishes the proof of this theorem.

4 Main results

Let us introduce the piecewise constant DG method proposed in [24]. Given \( J \in \mathbb{N}_{>0} \), let \( 0 = t_0 < t_1 < \cdots < t_J = T \) be a partition of \([0, T]\) with \( \tau := \max_{1 \leq j \leq J} (t_J - t_{j-1}) \), and set \( I_j := (t_{j-1}, t_j) \) for \( 1 \leq j \leq J \). Let \( K_h \) be a conventional conforming and quasi-uniform triangulation of \( \Omega \) consisting of \( d \)-simplexes, and we use \( h \) to denote the maximum diameter of the elements in \( K_h \). Define
\[
S_h := \left\{ v_h \in H^1_0(\Omega) : v_h|_K \in P_1(K), \quad \forall K \in K_h \right\},
\]
\[
\mathcal{X}_{\tau,h} := \left\{ V \in L^2(0, T; S_h) : V|_{I_j} \in P_0(I_j; S_h), \quad \forall 1 \leq j \leq J \right\},
\]
where \( P_1(K) \) is the set of all linear polynomials defined on \( K \), and \( P_0(I_j; S_h) \) is the set of all \( S_h \)-valued constant functions on \( I_j \). For each \( V \in \mathcal{X}_{\tau,h} \), we will use the following notations:
\[
V_j^+ := \lim_{t \to t_j^+} V(t) \quad \text{for } 0 \leq j < J, \quad \text{and } V_J^+ := 0;
\]
\[
V_j^- := \lim_{t \to t_j^-} V(t) \quad \text{for } 1 \leq j \leq J, \quad \text{and } V_0^- := 0;
\]
\[
\| V_j \| := V_j^+ - V_j^- \quad \text{for } 0 \leq j \leq J.
\]
Assuming that \( u_0 \in S_h^* \) and \( f \in \mathcal{X}_{\tau,h}^* \), the piecewise constant DG method defines a numerical solution \( U \in \mathcal{X}_{\tau,h} \) to problem (1.1) by that
\[
\mathcal{A}(U, V) = \langle f, V \rangle_{\mathcal{X}_{\tau,h}} + \left\{ u_0, V_0^+ \right\}_{S_h}, \quad \forall V \in \mathcal{X}_{\tau,h}, \quad (4.1)
\]
where
\[ A(W, V) := \sum_{j=0}^{J-1} \left( \|W_j\|_{L^2(\Omega)} + \|D_{0+}^{1-\alpha} \nabla W, \nabla V\right)_{\Omega_T} \]

for all \( W, V \in \mathcal{X}_{\tau,h} \). It is not hard to conclude from [40, Theorem 12.1] and Lemma 11 that if \( V \in \mathcal{X}_{\tau,h} \), then we have
\[ A(V, V \chi(a, b)) \geq \frac{1}{2} \left( \|V_j\|^2_{L^2(\Omega)} + \|V_0\|^2_{L^2(\Omega)} \right) + \sin \frac{\alpha \pi}{2} |V|_{H^{(1-\alpha)/2}(\Omega)}^2, \]

for all \( 1 \leq j \leq J \). Above and in what follows, \( \chi(a, b) \) denotes the indicator function of the interval \((a, b)\).

For convenience, in what follows we assume that \( u \) is the weak solution to problem (1.1) and \( U \) is its numerical approximation defined by (4.1). The notation \( a \lesssim b \) means that there exists a generic positive constant \( C \), independent of \( h, \tau \) and \( u \), such that \( a \leq Cb \). Moreover, \( a \sim b \) means \( a \lesssim b \lesssim a \).

The well-posedness of the solution \( U \) to (4.1) is firstly established in the following theorem.

**Theorem 5** If \( u_0 \in L^2(\Omega) \) and \( f \in (H^{(1-\alpha)/2}(0, T; H^1(\Omega)))^* \), then problem (4.1) admits a unique solution \( U \) such that
\[ \|U\|_{L^\infty(0,T;L^2(\Omega))} + |U|_{H^{(1-\alpha)/2}(0,T;H^1(\Omega))} \lesssim \|u_0\|_{L^2(\Omega)} + \|f\|_{(H^{(1-\alpha)/2}(0,T;H^1(\Omega)))^*}. \]

**Remark 2** We also refer the reader to [24, Theorem 1] for another stability estimate, which is derived in the case that \( u_0 \in L^2(\Omega) \) and \( f \in L^1(0,T;L^2(\Omega)) \).

Below, let us present our main error estimates.

**Theorem 6** If \( u_0 = 0 \) and \( f \in L^2(\Omega_T) \), then
\[ \|u - U\|_{L^2(\Omega_T)} \lesssim (h^2 + \tau) \|f\|_{L^2(\Omega_T)}, \]
\[ |u - U|_{H^{(1-\alpha)/2}(0,T;H^1(\Omega))} \lesssim (h + \tau^{1/2}) \|f\|_{L^2(\Omega_T)}. \]

**Theorem 7** Assume that \( f = 0 \). If \( u_0 \in L^2(\Omega) \), then
\[ \|u - U\|_{L^2(\Omega_T)} \lesssim (h + \tau^{1/2}) \|u_0\|_{L^2(\Omega)}. \]

**Remark 3** In view of Lemma 12, Theorems 2 and 4, we conclude that all the convergence rates in Theorem 6 are optimal with respect to the solution regularity while Theorem 7 only gives optimal estimate in temporal discretization. Indeed, ignoring the logarithm factor, the spatial accuracy in (4.6) should be \( \min\{2, 1/\alpha\} \); see numerical results in Table 6.
Remark 4 Although the half order \( O(\tau^{1/2}) \) in (4.6) is optimal with respect to the solution regularity (cf. Theorem 4) and this has been verified by the numerical test with uniform temporal grid (see Table 5), it is still possible to recover the first order accuracy \( O(\tau) \) by using graded temporal meshes; see [17, Theorem 3.1] for rigorous proof of the L1 scheme with smoother data \( u_0 \in \dot{H}^\sigma(\Omega), \ 0 < \sigma \leq 2 \). In the last experiment of Section 7, we investigate the performance of our DG scheme under nonuniform grids and obtain the rate \( O(\tau) \) for \( u_0 \in L^2(\Omega) \) with suitable graded meshes (cf. Table 7).

Moreover, if the temporal grid is equi-distributed, then quasi-optimal (including logarithm factors) error bounds under the \( L^\infty(0,T;L^2(\Omega)) \)-norm are derived.

Theorem 8 Assume \( u_0 = 0 \) and the temporal grid is uniform. If \( f \in L^2(\Omega_T) \), then

\[
\|u - U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim |\ln \tau| \left( |\ln \tau| + \epsilon_h h \min\{2, 1/\alpha\} \right) \|f\|_{L^2(\Omega_T)},
\]

(4.7)

where \( \epsilon_h = 1 \) if \( \alpha \neq 1/2 \) and \( \epsilon_h = \sqrt{|\ln h|} \) if \( \alpha = 1/2 \). Moreover, if \( f \in 0H^{1/2}(0,T;L^2(\Omega)) \), then

\[
\|u - U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim |\ln \tau| \left( |\ln \tau| \tau + h^2 \right) \|f\|_{0H^{1/2}(0,T;L^2(\Omega))}.
\]

(4.8)

5 Proofs of Theorems 5–7

5.1 Preliminaries

Given a Banach space \( X \), we introduce two interpolation operators as follows [40, Chapter 12]: given \( v \in C((0,T];X) \) and \( w \in C([0,T];X) \), define \( P_\tau v \) and \( Q_\tau w \) respectively by that

\[
(P_\tau v) |_{I_j} = v(t_j), \quad \forall \ 1 \leq j \leq J,
\]

\[
(Q_\tau w) |_{I_j} = w(t_{j-1}), \quad \forall \ 1 \leq j \leq J.
\]

Let \( P_h : L^2(\Omega) \rightarrow S_h \) be an \( L^2(\Omega) \)-orthogonal projection operator and \( R_h : \dot{H}^1(\Omega) \rightarrow S_h \) be the Ritz projection operator. Then by the theory of interpolation spaces [39] and the standard approximation estimates, we readily obtain that if \( v \in \dot{H}^r(\Omega) \) with \( 1 \leq r \leq 2 \), then

\[
\|(I - R_h)v\|_{L^2(\Omega)} + h \|(I - R_h)v\|_{\dot{H}^1(\Omega)} \lesssim h^r \|v\|_{\dot{H}^r(\Omega)},
\]

\[
\|(I - P_h)v\|_{L^2(\Omega)} + h \|(I - P_h)v\|_{\dot{H}^1(\Omega)} \lesssim h^r \|v\|_{\dot{H}^r(\Omega)}.
\]

Because the above two estimates are well known, we will use them implicitly for clarity.

Below, let us establish some nonstandard error estimates of \( P_\tau \) and \( Q_\tau \).
Lemma 3 If $0 \leq \beta < 1/2 < \gamma \leq 1$ and $v \in H^\gamma(0, T)$, then

$$
\|(I - P_\tau)v\|_{H^\beta(0, T)} + \|(I - Q_\tau)v\|_{H^\beta(0, T)} \leq C_{\beta, \gamma, T, \tau} \gamma - \beta \|v\|_{H^\gamma(0, T)}.
$$

(5.1)

Proof We first consider the estimate for $P_\tau$ and set $g := v - P_\tau v$. In view of the proof of [13, Lemma 4.3], we have

$$
\|g\|_{H^\beta(0, T)}^2 \leq C_{\beta, T} (I_1 + I_2),
$$

(5.2)

where

$$
\begin{align*}
I_1 & := \sum_{j=1}^J \int_{I_j} \int_{I_j} \frac{|v(s) - v(t)|^2}{|s - t|^{1+2\beta}} \, ds \, dt, \\
I_2 & := \sum_{j=1}^J \int_{I_j} g^2(t) \left((t_j - t)^{-2\beta} + (t - t_{j-1})^{-2\beta}\right) \, dt.
\end{align*}
$$

If $\gamma = 1$, then $v \in H^1(0, T)$ and

$$
I_1 = \sum_{j=1}^J \int_{I_j} \int_{I_j} \frac{|v(s) - v(t)|^2}{|s - t|^{1+2\beta}} \, ds \, dt = \sum_{j=1}^J \int_{I_j} \int_{I_j} \frac{|f'_s v'(r)dr|^2}{|s - t|^{1+2\beta}} \, ds \, dr
$$

$$
\leq \sum_{j=1}^J \left( \int_{I_j} |v'(r)|^2 \, dr \right) \left( \int_{I_j} \int_{I_j} |s - t|^{-2\beta} \, ds \, dt \right)
$$

$$
\leq C_{\beta} \sum_{j=1}^J \tau_j^{2-2\beta} \int_{I_j} |v'(r)|^2 \, dr \leq C_{\beta} \tau^{2-2\beta} \|v\|_{H^1(0, T)}^2.
$$

In addition, the term $I_2$ can be estimated similarly

$$
I_2 = \sum_{j=1}^J \int_{I_j} g^2(t) \left((t_j - t)^{-2\beta} + (t - t_{j-1})^{-2\beta}\right) \, dt
$$

$$
= \sum_{j=1}^J \int_{I_j} \left| \int_{t_j}^{t} v'(r) \, dr \right|^2 \left((t_j - t)^{-2\beta} + (t - t_{j-1})^{-2\beta}\right) \, dt
$$

$$
\leq C_{\beta} \sum_{j=1}^J \tau_j^{2-2\beta} \int_{I_j} |v'(r)|^2 \, dr \leq C_{\beta} \tau^{2-2\beta} \|v\|_{H^1(0, T)}^2.
$$

Plugging the above two estimates into (5.2) gives

$$
\|(I - P_\tau)v\|_{H^\beta(0, T)} \leq C_{\beta, T} \tau^{1-\beta} \|v\|_{H^1(0, T)}.
$$
Next, we consider $1/2 < \gamma < 1$. Observing [39, Lemma 36.1], we find that

$$\|I - P_{\mathcal{T}}v\|_{H^1(0, T)} \leq C_{\gamma, T} \tau^{2(\gamma-\beta)} \|v\|_{H^\gamma(0, T)}^2.$$  \hspace{1cm} (5.3)

We aim to establish

$$\|I - P_{\mathcal{T}}v\|_{L^\infty(0, T)} \leq C_{\gamma, T} \tau^{2(\gamma-\beta)} \|v\|_{H^\gamma(0, T)}^2.$$  \hspace{1cm} (5.4)

Since $0 \leq \beta < 1/2 < \gamma < 1$, by [11, Theorem 5.20], we have

$$\int_0^1 |w(t) - w(1)|^2 \left( t^{2\beta} + (1 - t)^{-2\beta} \right) dt \leq \int_0^1 |w(t) - w(1)|^2 \left( t^{2\gamma} + (1 - t)^{-2\gamma} \right) dt \leq C_{\gamma} \int_0^1 \int_0^1 \frac{|w(s) - w(t)|^2}{|s - t|^{1+2\gamma}} ds dt,$$

for all $w \in H^\gamma(0, 1)$. Therefore, a standard scaling argument implies

$$\|I - P_{\mathcal{T}}v\|_{L^\infty(0, T)} \leq C_{\gamma} \sum_{j=1}^J \tau_j^{2(\gamma-\beta)} \int_{I_j} \int_{I_j} \frac{|v(s) - v(t)|^2}{|s - t|^{1+2\gamma}} ds dt \leq C_{\gamma, T} \tau^{2(\gamma-\beta)} \|v\|_{H^\gamma(0, T)}^2.$$

This gives (5.3) and thus proves

$$\|I - P_{\mathcal{T}}v\|_{H^\beta(0, T)} \leq C_{\beta, \gamma, T} \tau^{\gamma-\beta} \|v\|_{H^1(0, T)},$$

where $1/2 < \gamma < 1$.

As the proof of the estimate for $Q_{\mathcal{T}}$ is similar, we omit it here and conclude the proof of this lemma. \hfill \square

Lemma 4 If $0 \leq \gamma < 1/2$ and $v \in \mathcal{H}^{1+\gamma}(0, T)$, then

$$\|I - P_{\mathcal{T}}v\|_{L^\infty(0, T)} \leq C_{\gamma} \tau^{\gamma+1/2} \|v\|_{\mathcal{H}^{1+\gamma}(0, T)},$$

where the implicit constant $C_{\gamma}$ is uniformly bounded whenever $\gamma \to 1/2$ or $\gamma \to 0$.

Proof The case $\gamma = 0$ is standard. Below we consider $0 < \gamma < 1/2$ and let $1 \leq j \leq J$ be arbitrary. Observe that

$$\|I - P_{\mathcal{T}}v\|_{L^\infty(I_j)}^2 \leq \tau_j \int_{I_j} |v'(t)|^2 dt \leq \tau_j^{1+2\gamma} \int_{I_j} (t - t_{j-1})^{-2\gamma} |v'(t)|^2 dt.$$
We extend \( v' \) to \( \mathbb{R} \setminus (t_{j-1}, T) \) by zero and denote it by \( E v' \). According to [13, Lemma 4.3] and [39, Lemma 16.3], it holds that
\[
\int_{I_j} (t - t_{j-1})^{-2\gamma} |v'(t)|^2 \, dt \\
\leq 2\gamma \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|(E v')(s) - (E v')(t)|^2}{|s - t|^{1+2\gamma}} \, ds \, dt \leq C |v'|_{H^\gamma(t_{j-1}, T)}^2,
\]
where \( C > 0 \) is independent of \( \gamma, v, j \) and \( T \). According to (2.2), it is not hard to find \( |v'|_{H^\gamma(t_{j-1}, T)} \leq |v'|_{H^\gamma(0, T)} \). Moreover, using Lemmas 11 and 13 gives
\[
|v'|_{H^\gamma(t_{j-1}, T)}^2 \leq \sec^2 \gamma \pi \| D_{0+}^\gamma v', D_T^\gamma v' \|_{L^2(0, T)}^2 \leq \frac{C_\gamma}{(1 - 2\gamma)^2} \| v \|_{H^{1+\gamma}(0, T)}^2,
\]
where the constant \( C_\gamma \) is uniformly bounded whenever \( \gamma \to 1/2 \) or \( \gamma \to 0 \). This establishes (5.4) and completes the proof.

**Lemma 5** If \( v \in L^2(0, T; \dot{H}^1(\Omega)) \) and \( v' \in L^2(0, T; \dot{H}^{-1}(\Omega)) \), then
\[
\langle v', V \rangle_{L^2(0, t_j; \dot{H}^1(\Omega))} = \left( v(t_j), V_j \right)_\Omega - \sum_{i=0}^{j-1} \left( [V_i], (Q \tau P_h v)_i^+ \right)_\Omega,
\]
\[
\langle v', V \rangle_{L^2(0, t_j; \dot{H}^1(\Omega))} = \sum_{i=0}^{j-1} \left( [P_\tau P_h v], V_i^+ \right)_\Omega - \langle v(0), V_0^+ \rangle_\Omega,
\]
for all \( V \in X_{\tau, h} \) and \( 1 \leq j \leq J \).

**Proof** By [4, Theorem 3 in §5.9.2], we have \( v \in C([0, T]; L^2(\Omega)) \) and the integration by parts formula holds
\[
\langle v', V \rangle_{L^2(t_{j-1}, t_j; \dot{H}^1(\Omega))} = \left( v(t_j), V_j \right)_\Omega - \left( v(t_{j-1}), V_{j-1}^+ \right)_\Omega.
\]
In view of the definitions of \( P_h, P_\tau \) and \( Q_\tau \), it is not hard to establish the desired results.

**Lemma 6** If \( u_0 = 0 \) and \( f \in L^2(0, T; \dot{H}^{-1}(\Omega)) \), then
\[
\| U - P_\tau P_h u \|_{L^\infty(0, T; L^2(\Omega))} + |U - P_\tau P_h u|_{H^{1-\alpha/2}(0, T; \dot{H}^1(\Omega))} \\
\leq C_\alpha \| (I - P_\tau P_h u) \|_{H^{1-\alpha/2}(0, T; \dot{H}^1(\Omega))}.
\]

**Proof** Set \( \theta = U - P_\tau P_h u. \) By Theorem 2, we have
\[
\langle u', V \rangle_{L^2(0, T; \dot{H}^1(\Omega))} + \left( D_{0+}^{1-\alpha} \nabla u, \nabla V \right)_{\Omega_T} = \langle f, V \rangle_{L^2(0, T; \dot{H}^1(\Omega))},
\]
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for all $V \in \mathcal{X}_{\tau,h}$. We use Lemma 5 to rewrite the first term and obtain

$$
\sum_{i=0}^{j-1} \langle (P_{\tau} P_h u)_i, \theta_i^+ \rangle \Omega + \left\langle D_{0+}^{1-a} \nabla u, \nabla V \right\rangle_{\Omega T} = \langle f, V \rangle_{L^2(0,T;\dot{H}^1(\Omega))},
$$

which gives the identity

$$
A(P_{\tau} P_h u, V) = \langle f, V \rangle_{L^2(0,T;\dot{H}^1(\Omega))} + \left\langle D_{0+}^{1-a} \nabla (P_{\tau} P_h u - u), \nabla V \right\rangle_{\Omega T}.
$$

This together with (4.1) yields the error equation

$$
A(\theta, V) = \left\langle D_{0+}^{1-a} \nabla (u - P_{\tau} P_h u), \nabla V \right\rangle_{\Omega T},
$$

for all $V \in \mathcal{X}_{\tau,h}$. Letting $V = \theta \chi_{(0,t_j)}$ and using (4.2) and Lemma 11, we get the estimate

$$
\| \theta_j \|^2_{L^2(\Omega)} + \| \theta_0^+ \|^2_{L^2(\Omega)} + \| \theta \|_{H^{(1-a)/2}(0,t_j;\dot{H}^1(\Omega))}^2 \\
\leq C_\alpha \| u - P_{\tau} P_h u \|_{H^{(1-a)/2}(0,t_j;\dot{H}^1(\Omega))} + \| \theta \|_{H^{(1-a)/2}(0,t_j;\dot{H}^1(\Omega))}.
$$

and using Young's inequality with $\epsilon$ proves (5.5).

\[\Box\]

### 5.2 Proof of Theorem 5

Let $1 \leq j \leq J$. Inserting $V = U \chi_{(0,t_j)}$ into (4.1) and applying (4.2) yield that

$$
\frac{1}{2} \left( \| U_j^- \|^2_{L^2(\Omega)} + \| U_0^+ \|^2_{L^2(\Omega)} \right) + \sin(\pi \alpha/2) \| U \|_{H^{(1-a)/2}(0,t_j;\dot{H}^1(\Omega))}^2 \\
\leq \langle f, U \chi_{(0,t_j)} \rangle_{H^{(1-a)/2}(0,T;\dot{H}^1(\Omega))} + \| U_0^+ \|^2_{L^2(\Omega)}.
$$

which further implies

$$
\| U_j^- \|^2_{L^2(\Omega)} + \sin(\pi \alpha/2) \| U \|_{H^{(1-a)/2}(0,t_j;\dot{H}^1(\Omega))}^2 \\
\leq \| U_0 \|^2_{L^2(\Omega)} + 2 \| f \|_{H^{(1-a)/2}(0,T;\dot{H}^1(\Omega))}^2 \| U \chi_{(0,t_j)} \|_{H^{(1-a)/2}(0,T;\dot{H}^1(\Omega))}.
$$

Noticing (2.2) we have

$$
\| \chi_{(0,t_j)} \|_{H^{(1-a)/2}(0,T;\dot{H}^1(\Omega))} \leq C_{\alpha,T} \| U \chi_{(0,t_j)} \|_{H^{(1-a)/2}(0,T;\dot{H}^1(\Omega))} \\
= C_{\alpha,T} \| U \|_{H^{(1-a)/2}(0,t_j;\dot{H}^1(\Omega))}.
$$
and invoking Young’s inequality with $\epsilon$, we obtain
\[
\left\| U^{-1}_j \right\|_{L^2(\Omega)}^2 + |U|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}^2 \leq C_\alpha T \left( \|u_0\|_{L^2(\Omega)}^2 + \|f\|_{(H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}^2 \right).
\]
Consequently, it follows that
\[
\|U\|_{L^\infty(0,T;L^2(\Omega))} + |U|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \lesssim \|u_0\|_{L^2(\Omega)} + \|f\|_{(H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}^*.
\]
This proves (4.3) and thus completes the proof. 

\section*{5.3 Proof of Theorem 6}

By Theorem 2, Lemmas 3, 4 and 12, a routine calculation gives the following estimates
\[
\|(I-P_h)u\|_{L^\infty(0,T;L^2(\Omega))} + |(I-P_h)u|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \lesssim h \|f\|_{L^2(\Omega_T)},
\]
\[
\|(I-P_T)u\|_{L^\infty(0,T;L^2(\Omega))} + |(I-P_T)P_hu|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \lesssim \tau^{1/2} \|f\|_{L^2(\Omega_T)}.
\]
Therefore, applying Lemma 6 yields that
\[
\|u - U\|_{L^\infty(0,T;L^2(\Omega))} + |u - U|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \lesssim \|u - P_T P_h u\|_{L^\infty(0,T;L^2(\Omega))} + \|P_T P_h u\|_{L^\infty(0,T;L^2(\Omega))} + |u - P_T P_h u|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} + |U - P_T P_h u|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \lesssim (h + \tau^{1/2}) \|f\|_{L^2(\Omega_T)}.
\]
This establishes (4.5).

Next, let us prove (4.4) by duality argument. By Theorem 3, there exists a unique $z \in G$ such that
\[
-\langle z', v \rangle_{L^2(0,T;\dot{H}^1(\Omega))} + \left\langle D_{T-}^{1-\alpha} \nabla z, \nabla v \right\rangle_{\Omega_T} = \langle u - U, v \rangle_{\Omega_T}
\]
for all $v \in L^2(0,T;\dot{H}^1(\Omega))$. Letting $v = u - U$ gives
\[
\|u - U\|_{L^2(\Omega_T)}^2 = -\langle z', u - U \rangle_{L^2(0,T;\dot{H}^1(\Omega))} + \left\langle D_{T-}^{1-\alpha} \nabla z, \nabla (u - U) \right\rangle_{\Omega_T} = \langle u', z \rangle_{L^2(0,T;\dot{H}^1(\Omega))} + \left\langle z', U \right\rangle_{L^2(0,T;\dot{H}^1(\Omega))} + \left\langle D_{0+}^{1-\alpha} \nabla (u - U), \nabla z \right\rangle_{\Omega_T},
\]
by integration by parts and Lemma 11. Moreover, setting \(Z = Q_{\tau} P_h z\) and combining (4.1), (5.6), and Lemma 5 yield that
\[
\langle u', Z \rangle_{L^2(0,T;\hat{H}^1(\Omega))} + \left\{ D_{0^+}^{1-\alpha} \nabla (u - U), \nabla Z \right\}_{\Omega_T}
= \sum_{j=0}^{J-1} \left\{ \| U_j \|, Z_j^+ \right\}_\Omega = - \langle z', U \rangle_{\Omega_T}.
\]
Consequently, we obtain
\[
\| u - U \|_{L^2(\Omega_T)}^2
= \langle u', z - Z \rangle_{L^2(0,T;\hat{H}^1(\Omega))} + \left\{ D_{0^+}^{1-\alpha} \nabla (u - U), \nabla (z - Z) \right\}_{\Omega_T}
\leq \| u' \|_{L^2(\Omega_T)} \| z - Z \|_{L^2(\Omega_T)}
+ |u - U|_{H^{(1-\alpha)/2}(0,T;\hat{H}^1(\Omega))} |z - Z|_{H^{(1-\alpha)/2}(0,T;\hat{H}^1(\Omega))},
\]
by the Cauchy–Schwartz inequality and Lemma 11. In view of Theorem 3, Lemmas 3 and 12, a direct computation implies
\[
\| z - Z \|_{L^2(\Omega_T)} \lesssim \left( h^2 + \tau \right) \| u - U \|_{L^2(\Omega_T)},
\]
\[
|z - Z|_{H^{(1-\alpha)/2}(0,T;\hat{H}^1(\Omega))} \lesssim \left( h + \tau^{1/2} \right) \| u - U \|_{L^2(\Omega_T)},
\]
and by (4.5) and Theorem 2, it follows that
\[
\| u - U \|_{L^2(\Omega_T)} \lesssim \left( h^2 + \tau \right) \| f \|_{L^2(\Omega_T)} + (h + \tau^{1/2})(h + \tau^{1/2}) \| f \|_{L^2(\Omega_T)}
\lesssim \left( h^2 + \tau \right) \| f \|_{L^2(\Omega_T)}.
\]
This proves (4.4) and concludes the proof of Theorem 6.

5.4 Proof of Theorem 7

According to Theorems 4 and 5 we have the stability estimate
\[
\| U \|_{L^\infty(0,T;L^2(\Omega))} + \| u' \|_{H^{(1-\alpha)/2}(0,T;\hat{H}^1(\Omega))}
+ \| u - U \|_{H^{(1-\alpha)/2}(0,T;\hat{H}^1(\Omega))} \lesssim \| u_0 \|_{L^2(\Omega)}.
\]
Recalling the proof of (4.4), we claim that there exists a unique
\[
w \in 0H^1(0,T;L^2(\Omega)) \cap 0H^{1-\alpha}(0,T;\hat{H}^2(\Omega))
\]
satisfying
\[ |w - W|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \lesssim (h + \tau^{1/2}) \|u - U\|_{L^2(\Omega_T)} \]
and
\[ \|u - U\|_{L^2(\Omega_T)}^2 \leq \|u'\|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}^* \|w - W\|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} + |u - U|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}^* |w - W|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))}. \]

where \( W = Q_t P_t w \). Consequently, we obtain (4.6) and complete the proof of Theorem 7.

\( \square \)

6 Proof of Theorem 8

In this section, based on some estimates established in [25], we aim to prove Theorem 8 under further assumption that temporal grid is uniform, i.e., \( \tau_j = \tau = T/J \) for all \( 1 \leq j \leq J \).

6.1 Auxiliary results

Assuming that \( y_0 \in \mathbb{R} \) and \( \lambda > 0 \) is a constant, we set \( Y_0 = y_0 \) and define a sequence \( \{Y_k\}_{k=1}^{\infty} \) as follows
\[
\mu \left( \sum_{j=1}^{k} (b_{k-j+2} - 2b_{k-j+1} + b_{k-j}) Y_j + b_1 Y_{k+1} \right) + Y_{k+1} - Y_k = 0, \tag{6.1}
\]
where \( \mu := \lambda \tau^\alpha \) and \( b_j := j^\alpha / \Gamma(1+\alpha) \) for all \( j \in \mathbb{N} \).

**Lemma 7** The sequence \( \{Y_k\}_{k=1}^{\infty} \) defined by (6.1) satisfies that
\[
|Y_k| + k |Y_{k+1} - Y_k| \leq C_{\alpha} |y_0|, \quad \forall k \geq 1, \tag{6.2}
\]
with some positive constant \( C_{\alpha} > 0 \).

To prove the above lemma, we shall introduce an auxiliary function [25]
\[
\psi(z) := \frac{(e^z - 1)}{2\pi i} \int_{-\infty}^{0+} \frac{w^{-1-\alpha}}{e^w - 1} \, dw, \quad z \in \mathbb{C} \setminus (-\infty, 0],
\]
where \( i \) denotes the imaginary unit and \( \int_{-\infty}^{0+} \) means the integration through the Hankel contour, i.e., a smooth and non-self-intersecting path enclosing the negative real axis and orienting counterclockwise, \( 0 \) and \( z + 2k\pi i, \, k \in \mathbb{Z}, \) lie on the different sides of this path.
By [25, Lemma 1], we have the expression
\[
\psi(z) = \frac{1}{2\pi i} \int_{-\infty}^{0+} \frac{w^{-\alpha}}{1 - e^{w-z}} \cdot e^w - 1 \, dw,
\]
and \(1 + \nu \psi(z) \neq 0\) for all \(\nu > 0\) and \(z \in \mathbb{C} \setminus (-\infty, 0]\). Since the integrand behaves like \(O(w^{-\alpha})\) as \(w \to 0\) and \(O(w^{-\alpha-1})\) as \(w \to -\infty\), we can shrink the Hankel contour on to \(\{s e^{\pi i} : 0 < s < \infty\}\) and \(\{s e^{-\pi i} : 0 < s < \infty\}\). This allows us to define
\[
\psi(s e^{\pi i}) := \lim_{z \to -s, \ Im z > 0} \psi(z), \quad \psi(s e^{-\pi i}) := \lim_{z \to -s, \ Im z < 0} \psi(z),
\]
where \(s e^{\pi i}\) and \(s e^{-\pi i}\) are identified as two different numbers.

**Proof of Lemma 7** We first prove \(|Y_k| \leq |y_0|\). Note that
\[
Y_{k+1} = \frac{Y_k}{1 + \mu b_1} - \frac{\mu}{1 + \mu b_1} \sum_{j=1}^{k} (b_{k-j+2} - 2b_{k-j+1} + b_{k-j}) Y_j.
\]  
(6.3)

Observing the fact
\[
0 < (j + 1)^\alpha - j^\alpha < j^\alpha - (j - 1)^\alpha \leq 1, \quad j > 0,
\]
we have the estimate
\[
\sum_{j=1}^{k} \left| b_{k-j+2} - 2b_{k-j+1} + b_{k-j} \right| = \sum_{j=1}^{k} \left( b_{k-j+1} - b_{k-j+2} + b_{k-j+1} - b_{k-j} \right)
\]
\[
= b_1 + b_k - b_{k+1} \leq b_1,
\]
for all \(k \in \mathbb{N}\). Since \(Y_0 = y_0\), by (6.3) and the above estimate, a standard deduction gives the stability result \(|Y_k| \leq |y_0|\).

Then let us establish (6.2). By [25, Theorem 3] we have the relation
\[
Y_{k+1} - Y_k = \frac{Y_0}{2\pi i} \int_{-\infty}^{0+} \frac{e^{kz}}{1 + \mu \psi(z)} \, dz,
\]
and using the same technique as that used to derive [25, (37)] yields
\[
Y_{k+1} - Y_k = \frac{Y_0}{2\pi i} \int_0^{\infty} e^{-ks} \left( \frac{1}{1 + \mu \psi(s e^{-\pi i})} - \frac{1}{1 + \mu \psi(s e^{\pi i})} \right) \, ds
\]
\[
= \frac{Y_0 \sin \alpha \pi}{\pi} \int_0^{\infty} \frac{\mu e^{-ks} s^{-\alpha} e^{-s} - 1}{\left| 1 + \mu \psi(s e^{\pi i}) \right|^2} \, ds.
\]
By [25, Lemma 9], there exists a positive constant $C_\alpha > 0$ such that

$$\frac{|1 + \mu \psi (s e^{+\pi i})|^2}{1 + \mu^2 s^{-2\alpha}} \geq C_\alpha, \quad 0 < s < \infty,$$

from which we obtain

$$|Y_{k+1} - Y_k| \leq |y_0| \frac{\sin \alpha \pi}{\pi C_\alpha} \int_0^\infty \frac{\mu e^{-ks}}{s^\alpha + \mu^2 s^{-\alpha}} \cdot \frac{1 - e^{-s}}{s} \, ds.$$

(6.4)

Observing the inequality $(1 - e^{-s})/s \leq 1$ for all $0 < s < \infty$, we estimate the integral as follows

$$\int_0^\infty \frac{\mu e^{-ks}}{s^\alpha + \mu^2 s^{-\alpha}} \cdot \frac{1 - e^{-s}}{s} \, ds \leq \int_0^\infty \frac{\mu e^{-ks}}{2\mu} \, ds = \frac{1}{2k}.$$

Putting this back to (6.4) proves (6.2) and thus completes the proof of this lemma. □

Define the discrete Laplacian operator $-\Delta_h : S_h \to S_h$ as follows

$$\langle -\Delta_h w_h, v_h \rangle_\Omega = \langle \nabla w_h, \nabla v_h \rangle_\Omega, \quad \forall v_h \in S_h,$$

for all $w_h \in S_h$. It is well-known that $-\Delta_h$ admits an orthonormal basis $\{\phi^n_h : 1 \leq n \leq |S_h|\}$ such that $-\Delta_h \phi^n_h = \lambda^n_h \phi^n_h$, where $|S_h| = \dim S_h$ and $\{\lambda^n_h : 1 \leq n \leq |S_h|\}$ is a non-decreasing positive sequence. Furthermore, we introduce the average interpolation operator $\Pi_\tau : L^1(0, T) \to \mathcal{X}_\tau$ by that

$$(\Pi_\tau v)|_{I_j} = \frac{1}{\tau_j} \int_{I_j} v(t) \, dt, \quad \forall 1 \leq j \leq J,$$

where $\mathcal{X}_\tau$ is given by

$$\mathcal{X}_\tau := \left\{ v_\tau \in L^2(0, T) : v_\tau|_{I_j} \in P_0(I_j), \quad \forall 1 \leq j \leq J \right\}.$$

For the operator $\Pi_\tau$, we have the commutativity

$$\int_0^T w(t)(\Pi_\tau v)(t) \, dt = \int_0^T v(t)(\Pi_\tau w)(t) \, dt,$$

(6.5)

for all $w, v \in L^1(0, T)$, and the following estimate is standard [1]: if $0 \leq \beta < 1/2$ and $\beta \leq \gamma < 3/2$, then

$$\| (I - \Pi_\tau) v \|_{H^\beta(0, T)} \lesssim \tau^{\gamma - \beta} \| v \|_{H^\gamma(0, T)}, \quad \forall v \in H^\gamma(0, T).$$

(6.6)

Given any $w_h \in S_h$, define $W \in \mathcal{X}_{\tau, h}$ such that

$$\mathcal{A}(W, V) = \langle w_h, V^+ \rangle_\Omega, \quad \forall V \in \mathcal{X}_{\tau, h}.$$
The well-posedness of the above problem follows directly from Theorem 5, and thanks to Lemma 7, we have a stability result which is crucial to our error analysis.

**Lemma 8** For any \( w_h \in S_h \), the unique solution \( W \in \mathcal{X}_{\tau,h} \) to (6.7) satisfies that

\[
\| W \|_{L^\infty(0,T;L^2(\Omega))} + |W|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \leq C_\alpha \| w_h \|_{L^2(\Omega)} ,
\]

\[
\sum_{j=1}^{J-1} \left\| W_j \right\|_{L^2(\Omega)} + \left\| \Pi_\tau D_{0+}^{1-\alpha} \Delta_h W \right\|_{L^1(0,T;L^2(\Omega))} \leq C_\alpha (1+\ln J) \| w_h \|_{L^2(\Omega)} .
\]

**Proof** Let us first prove (6.8). By (4.2), for any \( 1 \leq j \leq J \), inserting \( V = W \chi(I_{0j}) \) into (6.7) implies

\[
\left\| W_j \right\|_{L^2(\Omega)} + |W|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \leq C_\alpha \| w_h \|_{L^2(\Omega)} , \quad 1 \leq j \leq J.
\]

Therefore, (6.8) is obtained directly from the above estimate.

Then let us prove (6.9). It is not hard to find that the solution to (6.7) has explicit expression

\[ W = \sum_{n=1}^{\left| S_h \right|} w_n \phi_n. \]

Here \( w_n \in \mathcal{X}_\tau \) and \( w_n|_{I_j} = Y^n_j \) for all \( 1 \leq j \leq J \), where \( \{Y^n_j\}_{j=1} \) satisfies (6.1) with \( Y^n_0 = \langle w_h, \phi_n \rangle_\Omega \) and \( \mu = \lambda_h^n \tau^\alpha \). By Lemma 7, we have

\[
\left| Y^n_j - Y^n_{j-1} \right| \leq \frac{C_\alpha}{j} |Y^n_0| .
\]

Hence, it follows that

\[
\left\| W_j \right\|_{L^2(\Omega)} \leq \frac{C_\alpha}{j} \| w_h \|_{L^2(\Omega)} , \quad 1 \leq j \leq J,
\]

which yields the estimate

\[
\sum_{j=1}^{J-1} \left\| W_j \right\|_{L^2(\Omega)} \leq C_\alpha \sum_{j=1}^{J-1} \| w_h \|_{L^2(\Omega)} \leq C_\alpha (1+\ln J) \| w_h \|_{L^2(\Omega)} .
\]
for all $2 \leq j \leq J$. Hence, by (6.8) and (6.10), we obtain that

$$
\left\| \Pi_\tau D_0^{1-\alpha} \Delta_h W \right\|_{L^1(0,T;L^2(\Omega))} = \left\| W_0^+ - w_h \right\|_{L^2(\Omega)} + \sum_{j=1}^{J-1} \left\| W_j \right\|_{L^2(\Omega)} 
\leq C\alpha \left( 1 + \ln J \right) \| w_h \|_{L^2(\Omega)},
$$

which together with (6.10) establishes (6.9) and concludes the proof of this lemma. □

Symmetrically, we have the following lemma. As the proof is similar, we omit it here.

**Lemma 9** Given any $w_h \in S_h$, if $W \in \mathcal{X}_{\tau,h}$ satisfies that

$$
\mathcal{A}(V, W) = \langle w_h, V_j^- \rangle_{\Omega}, \quad \forall V \in \mathcal{X}_{\tau,h},
$$

then the following estimates hold:

$$
\| W \|_{L^\infty(0,T;L^2(\Omega))} + |W|_{H^{(1-\alpha)/2}(0,T;H^1(\Omega))} \leq C\alpha \| w_h \|_{L^2(\Omega)},
$$

$$
\sum_{j=1}^{J-1} \left\| W_j \right\|_{L^2(\Omega)} + \left\| \Pi_\tau D_1^{1-\alpha} \Delta_h W \right\|_{L^1(0,T;L^2(\Omega))} \leq C\alpha \left( 1 + \ln J \right) \| w_h \|_{L^2(\Omega)}.
$$

### 6.2 Main proof

We now arrive at a position for proving Theorem 8.

We first prove (4.7). To do this, we shall establish the estimate

$$
\| \theta \|_{L^\infty(0,T;L^2(\Omega))} \lesssim \left\| (I - \Pi_\tau)u \right\|_{H^{(1-\alpha)/2}(0,T;H^1(\Omega))} + \left( 1 + |\ln \tau| \right) \| R_h u - P_\tau P_h u \|_{L^\infty(0,T;L^2(\Omega))},
$$

(6.11)

where $\theta := U - P_\tau P_h u$. Let $W \in \mathcal{X}_{\tau,h}$ be defined as follows

$$
\mathcal{A}(V, W) = \langle \theta_j^-, V_j^- \rangle_{\Omega}, \quad \forall V \in \mathcal{X}_{\tau,h}.
$$

Plugging $V = \theta$ into the above equation and observing the error equation (5.7), we obtain the identity

$$
\| \theta_j^- \|_{L^2(\Omega)}^2 = \mathcal{A}(\theta, W) = \left\langle D_0^{1-\alpha} \nabla (u - P_\tau P_h u), \nabla W \right\rangle_{\Omega_T}
$$

$$
= \left\langle D_0^{1-\alpha} \nabla (R_h u - P_\tau P_h u), \nabla W \right\rangle_{\Omega_T}
$$

$$
= \left\langle P_\tau P_h u - R_h u, D_1^{1-\alpha} \Delta_h W \right\rangle_{\Omega_T}.
$$
Thanks to the commutativity (6.5), we have

\[ \| \theta_J^- \|^2_{L^2(\Omega)} = \left( P_T P_h u - R_h u, (I - \Pi \tau) D_T^{1-\alpha} \Delta_h W \right)_{\Omega_T} + \left( P_T P_h u - R_h u, \Pi \tau D_T^{1-\alpha} \Delta_h W \right)_{\Omega_T} = - \left( R_h u, (I - \Pi \tau) D_T^{1-\alpha} \Delta_h W \right)_{\Omega_T} + \left( P_T P_h u - R_h u, \Pi \tau D_T^{1-\alpha} \Delta_h W \right)_{\Omega_T} = \left( (I - \Pi \tau) R_h u, D_T^{1-\alpha} (-\Delta_h W) \right)_{\Omega_T} + \left( P_T P_h u - R_h u, \Pi \tau D_T^{1-\alpha} \Delta_h W \right)_{\Omega_T}. \]

Applying Lemmas 9 and 11 yields

\[
\left( P_T P_h u - R_h u, \Pi \tau D_T^{1-\alpha} \Delta_h W \right)_{\Omega_T} \lesssim \left\| \Pi \tau D_T^{1-\alpha} \Delta_h W \right\|_{L^1(0,T; L^2(\Omega))} \| R_h u - P_T P_h u \|_{L^\infty(0,T; L^2(\Omega))} \lesssim (1 + |\ln \tau|) \| \theta_J^- \|_{L^2(\Omega)} \| R_h u - P_T P_h u \|_{L^\infty(0,T; L^2(\Omega))}
\]

and

\[
\left( (I - \Pi \tau) R_h u, D_T^{1-\alpha} (-\Delta_h W) \right)_{\Omega_T} = \left( (I - \Pi \tau) \nabla R_h u, D_T^{1-\alpha} \nabla W \right)_{\Omega_T} \leq |W|_{H^{(1-\alpha)/2}(0,T; H^1(\Omega))} \left| (I - \Pi \tau) R_h u \right|_{H^{(1-\alpha)/2}(0,T; H^1(\Omega))} \lesssim \| \theta_J^- \|_{L^2(\Omega)} \left\| (I - \Pi \tau) u \right\|_{H^{(1-\alpha)/2}(0,T; H^1(\Omega))}. \]

Combining the above two estimates gives

\[
\| \theta_J^- \|_{L^2(\Omega)} \lesssim \left\| (I - \Pi \tau) u \right\|_{H^{(1-\alpha)/2}(0,T; H^1(\Omega))} + (1 + |\ln \tau|) \| R_h u - P_T P_h u \|_{L^\infty(0,T; L^2(\Omega))},
\]

and similarly one can prove this for \( \theta_J^- \) with \( 1 \leq j < J \). Therefore, the estimate (6.11) follows immediately.

We then estimate the right hand side terms in (6.11). By (6.6), Theorem 2, Lemmas 4 and 12, we have that

\[
\| (I - P_T) u \|_{L^\infty(0,T; L^2(\Omega))} + \| (I - \Pi \tau) u \|_{H^{1-\alpha/2}(0,T; H^1(\Omega))} \lesssim \tau^{1/2} \| f \|_{L^2(\Omega_T)}, \quad (6.12)
\]

and applying Theorem 2 and Lemmas 12 again implies

\[
\| (I - R_h) u \|_{L^\infty(0,T; L^2(\Omega))} + \| (I - P_h) u \|_{L^\infty(0,T; L^2(\Omega))} \lesssim \left\{ \begin{array}{ll}
\min^\left[2,1/\alpha\right] \| f \|_{L^2(\Omega_T)} & \text{if } \alpha \neq 1/2,
\frac{1}{\sqrt{\epsilon}} h^{2-\alpha} \| f \|_{L^2(\Omega_T)} & \text{if } \alpha = 1/2.
\end{array} \right. \quad (6.13)
\]
where $0 < \epsilon \leq 1$. For $\alpha = 1/2$, we choose $\epsilon = 1/(1 + |\ln h|)$ to obtain
\[
\|(I - R_h)u\|_{L^\infty(0,T;L^2(\Omega))} \leq (1 + \sqrt{|\ln h|}) h^2 \|f\|_{L^2(\Omega_T)}.
\] (6.14)

Plugging (6.12)–(6.14) into (6.11) and using the following estimate
\[
\|R_hu - P_T P_hu\|_{L^\infty(0,T;L^2(\Omega))} \leq \|(I - R_h)u\|_{L^\infty(0,T;L^2(\Omega))} + \|(I - P_h)u\|_{L^\infty(0,T;L^2(\Omega))}
\]
we find that
\[
\|\theta\|_{L^\infty(0,T;L^2(\Omega))} \leq (1 + |\ln \tau|) \left( \tau^{1/2} + \epsilon_h \ln h \right) \|f\|_{L^2(\Omega_T)},
\] (6.15)
where $\epsilon_h = 1$ if $\alpha \neq 1/2$ and $\epsilon_h = 1 + \sqrt{|\ln h|}$ if $\alpha = 1/2$. Consequently, (4.7) follows from (6.12)–(6.15) and the estimate
\[
\|u - U\|_{L^\infty(0,T;L^2(\Omega))} \leq \|\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|u - P_T P_hu\|_{L^\infty(0,T;L^2(\Omega))}
\]
In view of (6.6) and Lemma 4, we can establish the estimate (4.8) similarly. This ends the proof of Theorem 8. □

7 Numerical experiments

In this section, we present several numerical experiments to verify the theoretical results with $T = 1$ and $\Omega = (0, 1)$. We will use uniform grids both in time and space and introduce the following notations:
\[
E_1 := \|\hat{u} - U\|_{L^2(\Omega_T)},
\]
\[
E_2 := \|\hat{u} - U\|_{L^\infty(0,T;L^2(\Omega))},
\]
\[
E_3 := \sqrt{\left\|\left(D_0^{1-\alpha}(\nabla \hat{u} - \nabla U), \nabla \hat{u} - \nabla U\right)_{\Omega_T}\right.}
\]
where the reference solution $\hat{u}$ is the numerical solution with respect to $h = 2^{-10}$ and $\tau = 2^{-15}$. Note that, by Lemma 11,
\[
E_3 \sim \left\|D_0^{(1-\alpha)/2}(\nabla \hat{u} - \nabla U)\right\|_{L^2(\Omega_T)} \sim \|\hat{u} - U\|_{H^{1/2}(0,T;H^1(\Omega))}.
\]
Table 1  Spatial errors of Experiment 1 with $\tau = 2^{-15}$

| $h$   | $E_1$ | Order | $E_2$ | Order | $E_3$ | Order |
|-------|-------|-------|-------|-------|-------|-------|
| $\alpha = 0.8$ | $2^{-2}$ | 2.00e-02 | –     | 3.67e-02 | –     | 2.87e-01 | –     |
|       | $2^{-3}$ | 5.53e-03 | 1.85  | 1.48e-02 | 1.31  | 1.59e-01 | 0.85  |
|       | $2^{-4}$ | 1.50e-03 | 1.88  | 5.95e-03 | 1.31  | 8.68e-02 | 0.87  |
|       | $2^{-5}$ | 4.04e-04 | 1.90  | 2.42e-03 | 1.30  | 4.67e-02 | 0.89  |
| $\alpha = 0.2$ | $2^{-4}$ | 1.13e-03 | –     | 1.45e-03 | –     | 6.05e-02 | –     |
|       | $2^{-5}$ | 3.02e-04 | 1.90  | 3.88e-04 | 1.91  | 3.21e-02 | 0.91  |
|       | $2^{-6}$ | 7.99e-05 | 1.92  | 1.02e-04 | 1.92  | 1.69e-02 | 0.93  |
|       | $2^{-7}$ | 2.08e-05 | 1.94  | 2.67e-05 | 1.94  | 8.81e-03 | 0.94  |

With uniform temporal grids, the DG scheme (4.1) results in a block triangular Toeplitz-like with tri-diagonal block system, and we can adopt the fast direct method proposed in [10] to solve it efficiently with quasi-optimal complexity $O((\tau h)^{-1} \ln \tau^2)$. Moreover, $E_3$ can be computed via fast Fourier transform.

Experiment 1. Consider

$$u_0(x) := 0, \quad x \in \Omega,$$
$$f(x, t) := x^{-0.49}t^{-0.49}, \quad (x, t) \in \Omega_T.$$

To test the accuracy of spatial discretization, we fix temporal step size $\tau = 2^{-15}$. Since $f \in L^2(\Omega_T)$, according to Theorems 6 and 8, we have $E_1 = O(h^2)$, $E_2 = O(h^{\min\{2, 1/\alpha\}})$ and $E_3 = O(h)$. These coincide with the numerical results in Table 1.

Next, we consider temporal errors and choose $h = 2^{-10}$. In view of Table 2, we find that $E_1 = O(\tau)$, $E_2 = O(\tau^{1/2})$ and $E_3 = O(\tau^{1/2})$. Evidently, they match well the estimates given by Theorems 6 and 8.
Table 3 Spatial errors of Experiment 2 with $\tau = 2^{−15}$

| $h$   | $\alpha = 0.9$ | Order | $\alpha = 0.5$ | Order | $\alpha = 0.3$ | Order |
|-------|----------------|-------|----------------|-------|----------------|-------|
| $2^{−4}$ | 7.10e-04       | –     | 5.81e-04       | –     | 5.18e-04       | –     |
| $2^{−5}$ | 1.90e-04       | 1.91  | 1.55e-04       | 1.90  | 1.39e-04       | 1.90  |
| $2^{−6}$ | 5.01e-05       | 1.92  | 4.11e-05       | 1.92  | 3.66e-05       | 1.92  |
| $2^{−7}$ | 1.30e-05       | 1.94  | 1.07e-05       | 1.94  | 9.55e-06       | 1.94  |

Table 4 Temporal errors of Experiment 2 with $h = 2^{−10}$

| $\tau$ | $\alpha = 0.7$ | Order | $\alpha = 0.4$ | Order | $\alpha = 0.1$ | Order |
|--------|----------------|-------|----------------|-------|----------------|-------|
| $2^{−8}$ | 3.18e-04       | –     | 2.02e-04       | –     | 2.14e-04       | –     |
| $2^{−9}$ | 1.60e-04       | 1.00  | 1.00e-04       | 1.01  | 1.04e-04       | 1.04  |
| $2^{−10}$ | 7.95e-05      | 1.01  | 4.97e-05       | 1.01  | 5.03e-05       | 1.04  |
| $2^{−11}$ | 3.92e-05      | 1.02  | 2.46e-05       | 1.02  | 2.43e-05       | 1.05  |

Table 5 Temporal errors of Experiment 3 with $h = 2^{−10}$

| $\tau$ | $\alpha = 0.9$ | Order | $\alpha = 0.6$ | Order | $\alpha = 0.3$ | Order |
|--------|----------------|-------|----------------|-------|----------------|-------|
| $2^{−7}$ | 2.90e-02       | –     | 2.18e-02       | –     | 1.09e-02       | –     |
| $2^{−8}$ | 2.00e-02       | 0.54  | 1.46e-02       | 0.58  | 8.07e-03       | 0.44  |
| $2^{−9}$ | 1.37e-02       | 0.54  | 9.77e-03       | 0.58  | 5.82e-03       | 0.47  |
| $2^{−10}$ | 9.36e-03      | 0.55  | 6.53e-03       | 0.58  | 4.07e-03       | 0.52  |

**Experiment 2.** Consider

\[
\begin{align*}
  u_0(x) & := 0, & x & \in \Omega, \\
  f(x, t) & := x^{−0.49} t^{0.01}, & (x, t) & \in \Omega_T.
\end{align*}
\]

It is clear that $f \in \dot{H}^{1/2}(0, T; L^2(\Omega))$. In Tables 3 and 4, we observe the optimal convergence order $\mathcal{E}_2 = O(\tau + h^2)$, which agrees with Theorem 8.

**Experiment 3.** In third test, let us verify Theorem 7 and take

\[
\begin{align*}
  u_0(x) & := x^{−0.49}, & x & \in \Omega, \\
  f(x, t) & := 0, & (x, t) & \in \Omega_T.
\end{align*}
\]

The convergence rate $\mathcal{E}_1 = O(\tau^{1/2})$ in Table 5 coincides with Theorem 7. However, as we mentioned in Remark 3, Theorem 7 only gives suboptimal spatial rate $\mathcal{E}_1 = O(h)$. 

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### Table 6  Spatial errors of Experiment 3 with $\tau = 2^{-15}$

| $h$   | $\alpha = 0.8$ $\mathcal{E}_1$ Order | $\alpha = 0.5$ $\mathcal{E}_1$ Order | $\alpha = 0.2$ $\mathcal{E}_1$ Order |
|-------|-----------------|-----------------|-----------------|
| $2^{-2}$ | 3.37e-02 – | 1.54e-02 – | 1.10e-02 – |
| $2^{-3}$ | 1.36e-02 1.31 | 4.49e-03 1.78 | 3.03e-03 1.86 |
| $2^{-4}$ | 5.31e-03 1.36 | 1.27e-03 1.82 | 8.20e-04 1.89 |
| $2^{-5}$ | 1.90e-03 1.48 | 3.48e-04 1.86 | 2.19e-04 1.90 |

### Table 7  Temporal accuracy of Case 1 in Experiment 4

| $\alpha = 0.3$ | $\sigma$ | $J$ | $\mathcal{E}_2$ Order | $\alpha = 0.9$ | $\sigma$ | $J$ | $\mathcal{E}_2$ Order |
|---------------|--------|----|-----------------|---------------|--------|----|-----------------|
| $2$ | 2$^5$ | 9.91e-01 – | 1.5 | 2$^5$ | 1.17e-00 – | 2$^6$ | 8.59e-01 0.21 | 2$^6$ | 9.80e-01 0.26 |
| | 2$^7$ | 7.26e-01 0.24 | 2$^7$ | 7.77e-01 0.33 | | 2$^8$ | 5.98e-01 0.28 | 2$^8$ | 5.68e-01 0.45 |
| | 2$^5$ | 1.09e-00 – | 2.5 | 2$^5$ | 5.17e-01 – | | 2$^6$ | 8.73e-01 0.32 | 2$^6$ | 3.09e-01 0.74 |
| | 2$^7$ | 6.42e-01 0.44 | 2$^7$ | 1.67e-01 0.89 | | 2$^8$ | 4.16e-01 0.63 | 2$^8$ | 8.62e-02 0.95 |
| | 2$^5$ | 3.81e-01 – | 4 | 2$^5$ | 2.38e-01 – | | 2$^6$ | 2.03e-01 0.91 | 2$^6$ | 1.25e-01 0.93 |
| | 2$^7$ | 1.02e-01 0.99 | 2$^7$ | 6.24e-02 1.00 | | 2$^8$ | 4.98e-02 1.03 | 2$^8$ | 3.09e-02 1.00 |

### Table 8  Temporal accuracy of Case 2 in Experiment 4

| $\sigma$ | $J$ | $\alpha = 0.2$ $\mathcal{E}_2$ Order | $\alpha = 0.4$ $\mathcal{E}_2$ Order | $\alpha = 0.8$ $\mathcal{E}_2$ Order |
|----------|----|-----------------|-----------------|-----------------|
| 1.5 | 2$^5$ | 3.94e-02 – | 7.99e-02 – | 1.75e-01 – |
| | 2$^6$ | 2.67e-02 0.56 | 5.81e-02 0.46 | 1.16e-01 0.60 |
| | 2$^7$ | 1.78e-02 0.58 | 4.02e-02 0.53 | 7.33e-02 0.66 |
| | 2$^8$ | 1.16e-02 0.62 | 2.64e-02 0.61 | 4.49e-02 0.71 |
| 2.5 | 2$^5$ | 1.16e-02 – | 2.56e-02 – | 4.90e-02 – |
| | 2$^6$ | 5.93e-03 0.97 | 1.31e-02 0.97 | 2.46e-02 0.99 |
| | 2$^7$ | 2.94e-03 1.01 | 6.62e-03 0.98 | 1.22e-02 1.01 |
| | 2$^8$ | 1.46e-03 1.01 | 3.26e-03 1.02 | 6.01e-03 1.02 |
| $\alpha$ | $J$   | $\sigma = 1.5$ | $\sigma = 2.5$ | $\sigma = 5$ | $\sigma = 10$ |
|---------|-------|----------------|----------------|-------------|--------------|
|         |       | $\mathcal{E}_2$ | $\mathcal{E}_2$ | $\mathcal{E}_2$ | $\mathcal{E}_2$ |
|         |       | Order           | Order           | Order       | Order        |
| 0.1     | $2^8$ | 2.44e-02        | 3.03e-02        | 4.06e-02    | 5.45e-02     |
|         | $2^9$ | 1.80e-02, 0.43  | 2.24e-02, 0.43  | 3.01e-02, 0.43 | 4.04e-02, 0.43 |
|         | $2^{10}$ | 1.32e-02, 0.45 | 1.64e-02, 0.45 | 2.21e-02, 0.45 | 2.96e-02, 0.45 |
|         | $2^{11}$ | 9.56e-03, 0.47 | 1.19e-02, 0.47 | 1.60e-02, 0.47 | 2.14e-02, 0.47 |
| 0.2     | $2^9$ | 2.58e-02, 0.46  | 3.08e-02, 0.46  | 3.92e-02, 0.37 | 4.98e-02, 0.37 |
|         | $2^{10}$ | 1.99e-02, 0.37 | 2.39e-02, 0.37 | 3.04e-02, 0.37 | 3.87e-02, 0.36 |
|         | $2^{11}$ | 1.51e-02, 0.39 | 1.82e-02, 0.39 | 2.32e-02, 0.39 | 2.96e-02, 0.39 |
|         | $2^{12}$ | 1.10e-02, 0.46 | 1.33e-02, 0.45 | 1.70e-02, 0.45 | 2.18e-02, 0.44 |
The optimal order of spatial discretization should be $E_1 = O(h^{\min(2,1/\alpha)})$, which can be observed from Table 6.

**Experiment 4.** Although the rate $O(\tau^{1/2})$ established in Theorem 6 is optimal with respect to the Sobolev regularity, it can be further improved via graded grids, provided that the solution possesses some growth estimates like (1.2).

To this end, let us investigate the performance of the DG scheme (4.1) under graded temporal grid $t_j = (j/J)^\sigma$, $j = 0, 1, \ldots, J$, with $\sigma > 1$. For simplicity we pay attention to the quantity $E_2$, which corresponds to the $L^\infty(0,T;L^2(\Omega))$-norm, and consider three cases:

- **Case 1:** $u_0(x) = x^{-0.49}$, $f(x,t) = 0$;
- **Case 2:** $u_0(x) = 0$, $f(x,t) = x^{-0.49} t^{-0.49}$;
- **Case 3:** $u_0(x) = 0$, $f(x,t) = x^{-0.49} |1 - 2t|^{-0.49}$.

Note that for all cases we have $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega_T)$. According to [23], one can obtain growth estimates for the first two cases and the first order accuracy $E_2 = O(\tau)$ is maintained with suitable parameter $\sigma > 1$; see Tables 7 and 8.

However, for the last case, it seems hard (or even impossible) to obtain growth estimate of the solution, and the accuracy $E_2 = O(\tau^{1/2})$ can not be improved; see Table 9.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Appendix A. Some properties of fractional calculus operators**

**Lemma 10** ([35]) If $0 < \alpha, \beta < \infty$, then

$$I_0^\beta I_0^\alpha v = I_0^{\beta+\alpha} v, \quad I_1^- I_1^- v = I_1^{\beta+\alpha} v,$$

for all $v \in L^1(0,1)$, and if $0 < \alpha < \beta < \infty$, then

$$D_0^\beta I_0^\alpha v = D_0^{\beta-\alpha} v, \quad D_1^- I_1^- v = D_1^{\beta-\alpha} v,$$

for all $v \in L^1(0,1)$. Moreover, for all $v, w \in L^2(0,1)$,

$$\left\langle I_0^\beta v, w \right\rangle_{(0,1)} = \left\langle v, I_1^\beta w \right\rangle_{(0,1)}.$$
Lemma 11 ([3]) If $0 < \gamma < 1/2$ and $v, w \in H^\gamma(0, T)$, then

\[
\begin{align*}
[D^\gamma_{0+} v, D^\gamma_{T-} w]_{0,T} &= \cos \gamma \pi |v|^2_{H^\gamma(0,T)}, \\
[D^\gamma_{0+} v, D^\gamma_{T-} w]_{0,T} &= \left( D^\gamma_{0+} w, D^\gamma_{T-} v \right)_{H^\gamma(0,T)} = \left( D^\gamma_{T-} w, D^\gamma_{0+} v \right)_{H^\gamma(0,T)}, \\
\cos \gamma \pi \|1^\gamma_{0+} v\|^2_{L_2^2(0,T)} &\leq \left(1^\gamma_{0+} v, 1^\gamma_{T-} v\right)_{(0,T)} \leq \sec \gamma \pi \|1^\gamma_{0+} v\|^2_{L_2^2(0,T)}, \\
\cos \gamma \pi \|D^\gamma_{0+} v\|^2_{L_2^2(0,T)} &\leq \left(D^\gamma_{0+} v, D^\gamma_{T-} v\right)_{(0,T)} \leq \sec \gamma \pi \|D^\gamma_{0+} v\|^2_{L_2^2(0,T)}.
\end{align*}
\]

Lemma 12 ([22]) If $v \in 0 H^\beta(0, 1; \dot{H}^r(\Omega)) \cap 0 H^\gamma(0, 1; \dot{H}^s(\Omega))$ with $\gamma, \beta \geq 0$ and $s, r \in \mathbb{R}$, then for all $0 < \theta < 1$,

\[
\|v\|_{0 H^{\beta+(1-\theta)\gamma}(0,1;\dot{H}^{\theta r+(1-\theta)\gamma}(\Omega))} \leq C_{\beta, \gamma, \theta} \left( \|v\|_{0 H^{\beta}(0,1;\dot{H}^r(\Omega))} + \|v\|_{0 H^{\gamma}(0,1;\dot{H}^s(\Omega))} \right).
\]

Similarly, if $v \in 0 H^\beta(0, 1; \dot{H}^r(\Omega)) \cap 0 H^\gamma(0, 1; \dot{H}^s(\Omega))$ with $\gamma, \beta \geq 0$ and $s, r \in \mathbb{R}$, then for all $0 < \theta < 1$,

\[
\|v\|_{0 H^{\beta+(1-\theta)\gamma}(0,1;\dot{H}^{\theta r+(1-\theta)\gamma}(\Omega))} \leq C_{\beta, \gamma, \theta} \left( \|v\|_{0 H^\beta(0,1;\dot{H}^r(\Omega))} + \|v\|_{0 H^\gamma(0,1;\dot{H}^s(\Omega))} \right).
\]

Lemma 13 ([22]) If $\beta \geq \gamma > 0$, then

\[
\begin{align*}
\|D^\gamma_{T-} v\|_{0 H^{\beta-\gamma}(0,T)} &\leq C_1 \|v\|_{0 H^\beta(0,T)} \forall v \in 0 H^\beta(0,T), \\
\|D^\gamma_{0+} v\|_{0 H^{\beta-\gamma}(0,T)} &\leq C_2 \|v\|_{0 H^\beta(0,T)} \forall v \in 0 H^\beta(0,T),
\end{align*}
\]

where $C_1$ and $C_2$ depend only on $\gamma$ and $\beta$.

Lemma 14 ([22]) If $\beta, \gamma \geq 0$, then

\[
\begin{align*}
C_1 \|v\|_{0 H^\beta(0,T)} &\leq \|1^\gamma_{T-} v\|_{0 H^{\beta+\gamma}(0,T)} \leq C_2 \|v\|_{0 H^\beta(0,T)} \forall v \in 0 H^\beta(0,T), \\
C_3 \|v\|_{0 H^\beta(0,T)} &\leq \|1^\gamma_{0+} v\|_{0 H^{\beta+\gamma}(0,T)} \leq C_4 \|v\|_{0 H^\beta(0,T)} \forall v \in 0 H^\beta(0,T).
\end{align*}
\]

where $C_1$, $C_2$, $C_3$ and $C_4$ depend only on $\gamma$ and $\beta$.

Lemma 15 ([22]) If $0 < \gamma < 1/2$, then for all $v \in 0 H^1(0, 1)$,

\[
\|v\|_{C[0,1]} \leq C_\gamma \|v\|_{0 H^1(0,1)}^{1/2 - \gamma} \|v\|_{0 H^\gamma(0,1)}^{1/2 - 2\gamma}.
\]

Moreover, if $v \in 0 H^\gamma(0, 1)$ with $1/2 < \gamma \leq 1$, then for all $0 < \epsilon \leq 1$,

\[
\|v\|_{C[0,1]} \leq \frac{C_\gamma}{\sqrt{\epsilon}} \|v\|_{0 H^{1/2}(0,1)}^{1-\epsilon} \|v\|_{0 H^\gamma(0,1)}^\epsilon.
\]
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