Global Classical Solutions to the Mixed Initial-boundary Value Problem for a Class of Quasilinear Hyperbolic Systems of Balance Laws

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Abstract
It is proven that the mixed initial-boundary value problem for a class of quasilinear hyperbolic systems of balance laws with general nonlinear boundary conditions in the half space $\{(t,x) | t \geq 0, x \geq 0\}$ admits a unique global $C^1$ solution $u = u(t,x)$ with small $C^1$ norm, provided that each characteristic with positive velocity is weakly linearly degenerate. This result is also applied to the flow equations of a model class of fluids with viscosity induced by fading memory.

MSC: 35L45; 35L50; 35Q72.

Keywords: Mixed initial-boundary value problem; Global classical solution; Quasilinear hyperbolic; systems of balance laws; Weakly linearly degenerate characteristics

Introduction and Main Result
Consider the following quasilinear hyperbolic system of balance laws in one space dimension:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + Lu = 0$$  \hspace{1cm} (1.1)

where $L > 0$ is a constant; $u = (u^1, \ldots, u^n)^T$ is the unknown vector function of $(t, x)$, $f(u)$ is a given $C^3$ vector function of $u$.

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given $u$ on the domain under consideration, the Jacobian $A(u) = \nabla f(u)$ has $n$ real distinct eigenvalues.

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)$$  \hspace{1cm} (1.2)

Let $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ ($i = 1, \ldots, n$) (1.3) then we have

$$\det | l_i(u) | \neq 0 \quad \text{(Equivalently, } \det | l_i(u) | \neq 0 \text{).}$$  \hspace{1cm} (1.4)

Without loss of generality, we may assume that on the domain under consideration

$$l_i(u) r_j(u) = \delta_{ij} (i, j = 1, \ldots, n)$$  \hspace{1cm} (1.5)

And

$$r_i^T(u) r_j(u) = 1 \quad (i = 1, \ldots, n)$$  \hspace{1cm} (1.6)

Where $\delta_{ij}$ stands for the Kronecker's symbol.

Clearly, all $\lambda_i(u), l_i(u)$ and $r_i(u)$ ($i = 1, \ldots, n$) have the same regularity as $A(u)$, i.e., $C^2$ regularity.

We assume that on the domain under consideration, each characteristic with positive velocity is weakly linearly degenerate and the eigenvalues of $A(u) = \nabla f(u)$ satisfy the non-characteristic condition.

$$\lambda r(u) < 0 < \lambda s(u)$$  \hspace{1cm} (1.9)

$r = 1, \ldots, m; s = m+1, \ldots, n$)

We are concerned with the existence and uniqueness of global $C^1$ solutions to the mixed initial-boundary value problem for system (1.1) in the half space

$$D = \{(t,x) | t \geq 0, x \geq 0\}$$  \hspace{1cm} (1.11)

with the initial condition:

$$t = 0 : u = \varphi(x)(x \geq 0)$$  \hspace{1cm} (1.12)

and the nonlinear boundary condition:

$$x = 0 : v_i = G_i(\alpha(t), v^1, \ldots, v^m) + h_i(t), s = m+1, \ldots, n$$ (1.13)

Where

$$v_i = l_i(u) u = (i = 1, \ldots, n)$$  \hspace{1cm} (1.14)

And

$$\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$$

Here, $\varphi = (\varphi_1, \ldots, \varphi_r)^T$, $\alpha_i G_i$, and $h_i(s = m+1, \ldots, n)$ are all $C^1$ functions with respect to their arguments, which satisfy the conditions of $C^1$ compatibility at the point $(0; 0)$. Also, we assume that there exists a constant $\mu > 0$ such that

$$\theta \pm \max \sup_{x \geq 0} \{1+x\} |l_i(\varphi)(x)\varphi(x)| + |\varphi'(x)| \sup_{t \geq 0} (1+t) |h_i(t)| + \sup_{t \geq 0} |\alpha(t)| + |\alpha'(t)| + |h'(t)| < +\infty$$  \hspace{1cm} (1.15)

in which

$$h(t) = (h_{m+1}(t), \ldots, h_n(t))$$

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Without loss of generality, we assume that 
\[ G_i(\alpha(t),...,0) = 0 (s = m+1,...,n) \]  
(1.16)

For the special case where (1.1) is a quasilinear hyperbolic system of conservation laws, i.e., L = 0, such kinds of problems have been extensively studied (for instance, [1-8] and the references therein). In particular, Li and Wang proved the existence and uniqueness of global \( C^1 \) solutions to the mixed initial boundary value problem for first order quasilinear hyperbolic systems with general nonlinear boundary conditions in the half space \( \{(t,x) | t \geq 0, x \geq 0 \} \). On the other hand, for quasilinear hyperbolic systems of balance laws, many results on the existence of global solutions have also been obtained by Liu, et al., (for instance, see [8-14] and the references therein), and some methods have been established. So the following question arises naturally: when can we obtain the existence and uniqueness of semi-global \( C^1 \) solutions for quasilinear hyperbolic systems of balance laws?

It is well known that for first-order quasilinear hyperbolic systems of balance laws, generically speaking, the classical solution exists only locally in time and the singularity will appear in a finite time even if the data are sufficiently smooth and small [15-20]. However, in some cases global existence in time of classical solutions can be obtained. In this paper, we will generalize the results in [21] to a nonhomogeneous quasilinear hyperbolic system, the analysis relies on a careful study of the interaction of the nonhomogeneous term. Our main results can be stated as follows:

**Theorem 1.1.** Suppose that the non-characteristic condition (1.10) holds and system (1.1) is strictly hyperbolic. Suppose furthermore that for \( j = m + 1,...,n \); each \( j \)-characteristic field with positive velocity is weakly linearly degenerate. Suppose finally that \( \varphi, \alpha, G, h (s = m+1,...,n) \) are all \( C^1 \) functions with respect to their arguments, satisfying (1.15)-(1.16) and the conditions of \( C^1 \) compatibility at the point \( (0;0) \). Then there exists a sufficiently small \( \theta_0 > 0 \) such that for any given \( 0 \in [0,\theta_0] \), the mixed initial-boundary value problem (1.1) and (1.12)-(1.13) admits a unique global \( C^1 \) solution \( u = u(t; x) \) in the half space \( \{(t,x) | t \geq 0, x \geq 0 \} \).

The rest of this paper is organized as follows. In Section 2, we give the main tools of the proof that is several formulas on the decomposition of waves for system (1.1), which will play an important role in our discussion.

**Lemma 2.1.**
\[
\frac{d(e^{iw_j})}{dt} = \sum_{j,k=1}^{n} \delta_{jk} \gamma_{jk}(u) w_i w_k + \sum_{j,k=1}^{n} e^{iw_j} \gamma_{jk}(u) v_j v_k (i = 1,...,n) \]  
(2.6)

Where
\[
\gamma_{jk}(u) = (\lambda_i(u) - \lambda_j(u)) r_j^T (u) v_j (u) - \nabla \lambda_i(u) r_j(u) \delta_{ik} \]  
(2.7)

Hence, we have
\[
\gamma_{ik}(u) = 0 \forall j \neq i, i = 1,...,n \]  
(2.9)

Moreover, in the normalized coordinates, \( (\bar{u}(e_c), e_c) = 0, \forall | u | \) small, \( \forall i, j \).

while, when the \( i \)-characteristic \( \lambda_i(u) \) is weakly linearly degenerate, in the normalized coordinates, \( \gamma_{ik}(u,e_c) = 0, \forall | u | \) small, \( \forall i \).

**Lemma 2.2.**
\[
\frac{d(e^{iv_j})}{dt} = \sum_{j,k=1}^{n} \delta_{jk} \beta_{jk}(u) v_i v_k + \sum_{j,k=1}^{n} e^{iv_j} \beta_{jk}(u) v_j v_k (i = 1,...,n) \]  
(2.13)

Where
\[
\beta_{jk}(u) = (\lambda_i(u) - \lambda_j(u)) r_j^T (u) v_i (u) \]  
(2.14)

Thus, we have
\[
\beta_{ik}(u) = 0, \forall i, j(i, j = 1,...,n) \]  
(2.16)

Moreover, by (2.1), in the normalized coordinates we have

\[ w_i = li(u)(u) \]  
(2.17)
is given by (2.15) and we have on the existence domain of \( \frac{d}{dt} \tilde{\partial}_t (t, y) \) gives
\[
\frac{d}{dt} \tilde{\partial}_t (t, y) = \nabla \lambda (u(t, \tilde{x}(t, y)) \overline{\tilde{\partial}_t (t, y)} + \tilde{\partial}_t (t, y) \frac{\partial}{\partial y} 
\]
Thus, from (2.4), (2.7) and (2.22), we immediately get (2.19)-(2.22). The proof of Lemma 2.4 is finished.
Similarly, noting (2.4), by (2.13) and (2.31), we have
\[
\frac{d}{dt} \tilde{\partial}_t (t, y) = \tilde{\partial}_t (u(t, \tilde{x}(t, y)) \overline{\tilde{\partial}_t (t, y)} + \tilde{\partial}_t (t, y) \frac{\partial}{\partial y} 
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\]
Thus, from (2.4), (2.7) and (2.22), we immediately get (2.19)-(2.22). The proof of Lemma 2.4 is finished.
Proof of Theorem 1.1

By the existence and uniqueness of a local $C^1$ solution for quasilinear hyperbolic systems [22], there exists $T_0 > 0$ such that the mixed initial-boundary value problem (1.1) and (1.12)-(1.13) admits a unique $C^1$ solution $u = u(t, x)$ on the domain $D(T_0) = \{(t, x) | 0 \leq t \leq T, x(0) \in \Omega\}$. Thus, in order to prove Theorem 1.1 it suffices to establish a uniform a priori estimate for the $C^1$ norm of $u$ and $u_x$ on any given domain of existence.

Noting (1.2) and (1.10), we have

$$\lambda_i(0) < \cdots < \lambda_n(0) < \lambda_{m+1}(0) < \cdots < \lambda_n(0)$$

Thus, there exist sufficiently small positive constants $\delta$ and $\delta_0$ such that

$$\lambda_{i+1}(u) - \lambda_i(u) \geq 4 \delta_0 \quad \forall \ u \in \Omega, \ |v| \leq \delta(i-1, \ldots, n-1)$$

$$\lambda_i(u) - \lambda_i(u) \leq \frac{\delta}{2} \quad \forall \ u \in \Omega, \ |v| \leq \delta(i-1, \ldots, n)$$

And

$$|\lambda_i(u)| \geq \delta_0 \quad (i = 1, \ldots, n)$$

For the time being it is supposed that on the domain of existence of the $C^1$ solution $u = u(t, x)$ to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have

$$|u(t, x)| \leq \delta$$

At the end of the proof of Lemma 3.3, we will explain that this hypothesis is reasonable. Thus, in order to prove Theorem 1.1, we only need to establish a uniform a priori estimate for the piecewise $C^0$ norm of $v$ and $w$ defined by (1.14) and (2.1) on the domain of existence of the $C^1$ solution $u = u(t, x)$.

For any fixed $T > 0$, let

$$D^T = \{(t, x) | 0 \leq t \leq T, x \geq \lambda_{i+1}(0) + \delta_0 t\}$$

$$D^T = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq \lambda_m(0) + \delta_0 t\}$$

$$D^T = \{(t, x) | 0 \leq t \leq T, (\lambda_{i+1}(0) - \delta_0) t \leq x \leq (\lambda_i(0) + \delta_0) t\}$$

and for $i = m + 1, \ldots, n$, let

$$D^T = \{(t, x) | 0 \leq t \leq T, -[\delta_0 + n(\lambda_i(0) - \lambda_{m+1}(0))] t \leq x \leq -[\lambda_i(0) + (n+1) \delta_0] t\}$$

where $n > 0$ is suitably small (Figure 1).

Noting that $n > 0$ is small, by (3.3), it is easy to see that

$$D^T \cap D^T = \emptyset, \ \forall \ i \neq j$$

And

$$\bigcup_{i=m+1}^n D^T \subseteq D^T$$

By the definitions of $D^T$ and $D^T$, it is easy to get the following lemma. Lemma 3.1. For each $i = m + 1, \ldots, n$, on the domain $D^T / D^T$, we have

$$c t \leq x - \lambda_i(0) t \leq C t, \ c x \leq x - \lambda_i(0) t \leq C x$$

where $c$ and $C$ are positive constants independent of $T$.

Let

$$V(D^T) = \max_{i=m+1}^n \| (1 + t)^{\mu_i} u(t, x) \|_{L^2(D^T)}$$

$$W(D^T) = \max_{i=m+1}^n \| (1 + t)^{\mu_i} w(t, x) \|_{L^2(D^T)}$$

$$V(D^T) = \max_{i=m+1}^n \| (1 + t)^{\mu_i} u(t, x) \|_{L^2(D^T)}$$

$$W(D^T) = \max_{i=m+1}^n \| (1 + t)^{\mu_i} w(t, x) \|_{L^2(D^T)}$$

$$V_e(T) = \max \{ \max_{i=m+1}^n (1 + T)^{\mu_i} V_i(t, x) \}$$

$$\max \{ \max_{i=m+1}^n (1 + T)^{\mu_i} u(t, x) \}$$

$$W_e(T) = \max \{ \max_{i=m+1}^n (1 + T)^{\mu_i} W_i(t, x) \}$$

$$\max \{ \max_{i=m+1}^n (1 + T)^{\mu_i} u(t, x) \}$$

$$U_e(T) = \max \{ \max_{i=m+1}^n (1 + T)^{\mu_i} u(t, x) \}$$

$$\max \{ \max_{i=m+1}^n (1 + T)^{\mu_i} u(t, x) \}$$

$$\hat{V}(T) = \max \{ \max_{i=m+1}^n \int_{c_i}^{C_i} v_i(t, x) dt \}$$

Figure 1: Where $n > 0$ is suitably small.
with the straight line \( L_0(t) \). Then there exists a sufficiently small \( \bar{t} \leq \bar{t} \) such that for any fixed \( \bar{t} \leq \bar{t} \) and \( \bar{t} \leq \bar{t} \), the t-section of the whole characteristic through the point \( (0, y) \) is included in \( D \). Clearly, \( \bar{t} \) is equivalent to \( \bar{t} \). In the present situation, similar to the corresponding result in [24,30-33], we have

**Lemma 3.2.** Suppose that in a neighborhood of \( u_0 \): \( A(u) \in C^2 \) and \( (1.12) \) holds. Suppose furthermore that \( \varphi(x) \) satisfies (1.15). Then there exists a sufficiently small \( \bar{t} \) such that for any fixed \( \bar{t} \leq \bar{t} \) on any given existence domain \( (t, x) \in D \) of the C' solution \( u = u(t, x) \) to \( (1.1) \) and \( (1.12)-(1.13) \), we have the following uniform a priori estimates:

\[
V(D) \leq k_\bar{t}, \quad \bar{t} \geq k_\bar{t}
\]

where here and henceforth, \( k_{i_1, \ldots, i_n} \) are positive constants independent of \( \bar{t} \) and \( T \).

**Proof.** We first estimate \( V(D) \)

(i) For \( i = 1, \ldots, m \), let \( \xi = \varphi(y) \) be the \( i \)-th characteristic passing through any fixed point \( (t, x) \in D \) and intersecting the \( x \)-axis at a point \( (0, y) \). Noting (3.6), by (3.3)-(3.4), it is easy to see that the whole characteristic \( \xi = \varphi(y) \) is included in \( D \).

Noting (3.6), by (3.4) we have

\[
y(0) \leq \varphi(y) \leq \varphi(y) \quad \forall s \in [0, t]
\]

Let \( s \leq t \leq t_{\bar{t}} \) be the \( i \)-th characteristic passing through any fixed point \( (0, y) \) with the straight line \( x = y + \varphi(y) / 2 \) passing through the point \( (0, y) \). Clearly,}

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\[ W(D^+) \leq k_1 \theta \] (3.47)

\[ V(D^+) \leq k_1 \theta \] (3.48)

\[ U_n^{(T)}(T) \leq k_1 \theta \] (3.49)

\[ W_n^{(T)}(T), V_n^{(T)}(T) \leq k_1 \theta \] (3.50)

\[ W(T), W(T) \leq k_1 \theta \] (3.51)

\[ U_n^{(T)}(T), V_n^{(T)}(T) \leq k_1 \theta \] (3.52)

And

\[ W<T(T) \leq k_1 \theta \] (3.53)

Proof. We first estimate \( W(D^+) \)

For \( j = 1, \ldots, m \), passing through any fixed point \((t, x) \in D^+\), we draw the \( j \)-th characteristic \( C_j : \tilde{z} = \tilde{z}_j(s, t, x) \) which must intersect the boundary \( x = (\lambda_\theta_1(0) + \delta_\theta_1) t \) of \( D^+ \) at a point \((t_0, y)\).

Proposition 3.1. On this \( j \)-th characteristic \( C_j : \tilde{z} = \tilde{z}_j(s, t, x) \) it follows that

\[ t \geq t_0 \geq -\frac{\lambda_\theta_1(0) - \delta_\theta_1}{\lambda_\theta_1(0) + \delta_\theta_1} t \] (3.54)

Proof. Noting (3.4), it is easy to see that

\[ x \geq \frac{\lambda_\theta_1(0) - \delta_\theta_1}{\lambda_\theta_1(0) + \delta_\theta_1} t \] (3.55)

On the other hand, from (3.8), we have

\[ x \geq 0 \] (3.56)

Since

\[ y = (\lambda_\theta_1(0) + \delta_\theta_1) t \] (3.57)

we conclude from (3.55)-(3.57) that

\[ t_0 \geq -\frac{\delta_\theta_1}{\lambda_\theta_1(0) - \delta_\theta_1} t \] (3.58)

Noting the fact that \( t \geq t_0 \), we immediately get (3.54).

By integrating (2.6) along \( \xi = \tilde{z}_j(s, t, x) \) and noting (2.9) and (2.11), we have

\[ w(t, x) = e^{-L^{\omega,t} w(t_0, y)} \]

\[ + \int_0^t e^{-L^{\omega,t}} \left( \sum_{i=1}^m \lambda_i \sum_{j=1}^N \lambda_j + \sum_{i=1}^m \sum_{j=1}^N \lambda_j \sum_{i=1}^m \sum_{j=1}^N \lambda_j \right) \gamma_{ij}(u) w_i(s, \xi_j(s, t, x)) ds \]

\[ + \int_0^t e^{-L^{\omega,t}} \left( \sum_{i=1}^m \sum_{j=1}^N \sum_{i=1}^m \sum_{j=1}^N \lambda_j \sum_{i=1}^m \sum_{j=1}^N \lambda_j \right) \tilde{\gamma}_{ij}(u) w_i(s, \xi_j(s, t, x)) ds \] (3.59)

\[ + \int_0^t e^{-L^{\omega,t}} \left( \sum_{i=1}^m \sum_{j=1}^N \sum_{i=1}^m \sum_{j=1}^N \lambda_j \sum_{i=1}^m \sum_{j=1}^N \lambda_j \right) \gamma_{ij}(u) w_i(s, \xi_j(s, t, x)) ds \]

By using Lemma 3.2 and noting (3.54) and (3.57), it is easy to see that

\[ |w(t_0, y)| \leq k_1 \theta (1 + y)^{-(1 + \rho)} \leq C_1(1 + t)^{-(1 + \rho)} \leq C_1(1 + t)^{-(1 + \rho)} \] (3.60)

By Hadamard's formula, we have

\[ \tilde{\gamma}_{ij}(u) - \tilde{\gamma}_{ij}(u_0, e) = \int_0^t \sum_{i=1}^m \sum_{j=1}^N \sum_{i=1}^m \sum_{j=1}^N \sum_{i=1}^m \sum_{j=1}^N \lambda_j \sum_{i=1}^m \sum_{j=1}^N \lambda_j \gamma_{ij}(u) \] (3.61)

Thus, noting the fact that \( L^0 \geq 0 \), and using (3.13) and (3.54), we obtain from (3.59) that

\[ (1 + t)^{1+\rho} w(t, x) \leq C_1(1 + t)^{1+\rho} \] (3.62)

Similar to Lemma 3.2 in [21], differentiating the nonlinear boundary condition (1.13) with respect to \( t \), we get

\[ x = 0 : \frac{\partial v}{\partial t} = \sum_{i=1}^m \frac{\partial G}{\partial a_i} (\alpha(t), v_1, \ldots, v_m) \frac{\partial v_i}{\partial t} \] (3.63)

By (1.1), (1.3) and (2.4), it is easy to see that

\[ \frac{\partial v_i}{\partial t} = -\lambda_i(u) w_i + \sum_{i=1}^m \lambda_i(u) w_i - Lu^2 \] (3.64)

Where

\[ a_i(u) = -\lambda_i(u) \frac{\partial v_i}{\partial t} \] (3.65)

Therefore it follows from (3.63)-(3.65) that

\[ x = 0 : (I_{n-m} + B_1(u)) \begin{bmatrix} w_{m+1} \\ \vdots \\ w_m \end{bmatrix} = B_2 \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} - B_3 \] (3.66)

where \( B_1(u) \) is a matrix whose elements are all C1 functions of \( u \), which satisfy

\[ \text{In}-m \cdot B_1(u) \text{ is invertible}; \text{ for sufficiently small } |u| \]

\[ B_2 \text{ is an } (n - m) \times (n - m) \text{ matrix independent of } w(t(i = 1, \ldots, n)) \] (3.67)

\[ \tilde{B}_2 = \sum_{i=1}^m \lambda_i(u) \] (3.68)

in which \( F \) is continuous functions of \( t \) and \( u \).

Thus, noting (3.6), for \( \tilde{\partial} > 0 \) small enough, by (3.66)-(3.68) we easily get

\[ \begin{bmatrix} w_{m+1} \\ \vdots \\ w_m \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \] (3.69)
\( x = 0 : w_s = \sum_{j=1}^{n} \int_{t_j} w_j(t, u) \alpha_j(t) + \sum_{j=1}^{n} \int_{t_j} w_j(t, u) \alpha_j'(t) \) 
\[ (3.69) \]

Where \( f_j, \alpha_j, T_{t_j} \) and \( \bf{T}_{t_j} \) are continuous functions of \( t \) and \( u \).

For \( j = m+1; \ldots; n \), passing through any fixed point \((t, x) \in \mathbb{D}^T \)
we draw the \( j \)th characteristic \( c_j : \xi = \xi_j(s; t, x) \) which must intersect the \( t \)-axis at a point \((t_0, 0) \). Then, we have

Proposition 3.2. On this \( j \)th characteristic \( c_j : \xi = \xi_j(s; t, x) \) it follows that
\[ \lambda_j(0) - \lambda_{m+1}(0) + \frac{\delta_0}{2} t \geq t_0 \]  
\[ (3.70) \]

Proof. Noting (3.4), it is easy to see that
\[ x = (\lambda_j(0) - \frac{\delta_0}{2}) t \geq (\lambda_j(0) - \frac{\delta_0}{2}) t_0 \]  
\[ (3.71) \]

On the other hand, by (3.8), we have
\[ x \leq (\lambda_{m+1}(0) - \delta_0) t \]  
\[ (3.72) \]

Therefore, it follows from (3.71)-(3.72) that
\[ t_0 \geq \frac{\lambda_j(0) - \lambda_{m+1}(0) + \frac{\delta_0}{2}}{\frac{\delta_0}{2}} t \]  
\[ (3.73) \]

Noting the fact that \( t \geq t_0 \), we immediately get (3.70).

By integrating (2.6) along \( c_j : \xi = \xi_j(s; t, x) \) we have
\[ w_j(t, x) = e^{-\lambda_j(0) t} w_j(t_0, 0) + \int_{t_0}^{t} e^{-\lambda_j(0) s} \sum_{k=1}^{n} \int_{t_k}^{s} \int_{u(0)}^{u(s)} w_j(u, t_k) u(u) w_k(u) \, du \, w_k(u) \, ds \]  
\[ + \sum_{k=1}^{n} \int_{t_k}^{t} \int_{u(0)}^{u(s)} \alpha_j(u) w_j(u, t_k) \, du \, w_j(u, t_k) \, ds \]  
\[ (3.74) \]

By (3.69), we have
\[ w_j(t_0, 0) = \sum_{k=1}^{n} \int_{t_k}^{t} \int_{u(0)}^{u(s)} w_j(u, t_k) u(u) w_k(u) \, du \, w_k(u) \, ds \]  
\[ + \sum_{k=1}^{n} \int_{t_k}^{t} \int_{u(0)}^{u(s)} \alpha_j(u) w_j(u, t_k) \, du \, w_j(u, t_k) \, ds \]  
\[ (3.75) \]

By employing the same arguments as in (i), we can obtain
\[ (1 + t)^{\gamma_j} | w_j(t, x) | \leq C_1 (1 + (W(D)^2) + W(D)^2 + V(D)^2 + U(D)^2 + U(D)^2 + V(D)^2 + V(D)^2 + U(D)^2 + U(D)^2 + V(D)^2) \]  
\[ \geq (W(T)^2 + W(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2) \]  
\[ (3.76) \]

Thus, noting (1.15), (3.6) and (3.70), it follows from (3.75) and (3.76) that
\[ (1 + t)^{\gamma_j} | w_j(t, x) | \leq C_1 (1 + t)^{\gamma_j} | w_j(t, x) | \leq C_1 (1 + t)^{\gamma_j} | w_j(t, x) | \]  
\[ \leq C_1 (1 + (W(D)^2) + W(D)^2 + V(D)^2 + U(D)^2 + W(D)^2 + V(D)^2 + U(D)^2 + V(D)^2 + U(D)^2 + V(D)^2) \]  
\[ + W(T)^2 + W(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2) \]  
\[ + W(T)^2 + W(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2) \]  
\[ (3.77) \]

Hence, noting the fact that \( L(0) \), we obtain from (3.74) that
\[ (1 + t)^{\gamma_j} | w_j(t, x) | \leq C_2 (1 + W(D)^2 + W(D)^2 + V(D)^2 + U(D)^2) \]  
\[ + W(T)^2 + W(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2) \]  
\[ (3.78) \]

Combining (3.62) with (3.78), we get
\[ W(D)^2 \leq C_3 (1 + W(D)^2 + W(D)^2 + V(D)^2 + U(D)^2) \]  
\[ + W(T)^2 + W(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2 + V(T)^2 + U(T)^2 + U(T)^2 + V(T)^2) \]  
\[ (3.79) \]

We next estimate \( \tilde{W}(T) \)

Let \( \tilde{C}_j : x = x_j(t) \) be any given \( j \)th characteristic in \( \mathbb{D}^T (j \neq i, i = m+1, \ldots, n) \).

By Hadamard's formula and (2.11), we have
\[ f(t, u(t, 0)) h(t) = \sum_{i=1}^{n} \sum_{k=1}^{m} f_i (t, \alpha_i(t)) h(t) \]  
\[ (3.80) \]

which gives a one-to-one correspondence \( t = \theta(y) \) between the segment \( \tilde{O}(\tilde{D}_j) \) and \( \tilde{C}_j (t_j \leq t \leq t_j) \).

Thus, the integral on \( \tilde{C}_j \) with respect to \( t \) can be reduced to the integral with respect to \( y \).

Differentiating (3.80) with respect to \( t \) gives
\[ dt = \frac{1}{\lambda_j (t, x_j(t), y) - \lambda_j (t, x_j(t), y)} \frac{\partial x_j(t, y)}{\partial y} \, dy \]  
\[ (3.81) \]

in which \( t = \theta(y) \). Then, noting (3.3) and (3.6), it is easy to see that in order to estimate
\[ \int_{t_j}^{t} | w(t, x_j(t), y) | \, dt = \int_{t_j}^{t} | w(t, x_j(t), y) | \, dt \]
\[ = \int_{t_j}^{t} | w(t, x_j(t), y) | \, dt \]
\[ (3.82) \]

it suffices to estimate
\[ \int_{0}^{\tilde{t}_j} | q(t, x_j(t), y) | \, dy \text{ and } \int_{0}^{\tilde{t}_j} | q(t, x_j(t), y) | \, dy \]  
\[ (3.83) \]

We now estimate
\[ \int_{t_j}^{t} | q(t, x_j(t), y) | \, dy \text{ and } \int_{t_j}^{t} | q(t, x_j(t), y) | \, dy \]  
\[ (3.84) \]

By integrating (2.28) along \( \xi = \xi(s, y) \) and noting (2.30) and the fact that \( \tilde{y}(y/\lambda_j(0) + \delta_0) \), \( \lambda_j(y) \) we obtain
\[ q(t, x_j(t), y) |_{\gamma_j} = e^{-\lambda_j(y) t} \int_{\gamma_j}^{\gamma_j} w(t, x_j(t), y) |_{\gamma_j} \frac{\lambda_j(y/\lambda_j(0) + \delta_0) \lambda_j(y) + \delta_0}{\lambda_j(y) + \delta_0} \]  
\[ + \sum_{i=1}^{n} \sum_{k=1}^{m} f_i(t, \alpha_i(t)) h(t) \]  
\[ (3.85) \]

By Hadamard's formula and (2.11), we have

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Therefore we obtain which must intersect for fixing the idea we may assume that Clearly, we have and noting (2.9) and (2.11), which intersects one of the boundaries of point . We next estimate .

Similarly, we have

Thus, we obtain

We next estimate .

For , passing through any given point , we draw the th characteristic which intersects one of the boundary . Clearly, we have . Therefore we obtain

where and are shown in Figure 2.

Similar to (3.90), it follows from (3.91) that

We next estimate for \( r = 1; \ldots; m \), passing through any fixed point , we draw the th characteristic which must intersect the boundary at a point . Then, we have

Proposition 3.3. On this th characteristic it follows that

Proof. By (3.4), it is easy to see that

On the other hand, from (3.9), we have

Since

we conclude from (3.94)-(3.96) that

we immediately get (3.93).

By integrating (2.6) along and noting (2.9) and (2.11), we have

By Hadamard's formula, we have

Thus, noting (3.93) and the fact that , we obtain from (3.98) that by the definition of for fixing the idea we may assume that
which implies \( i < n \). Let \( \xi = \xi(s;t,x) \) be the \( i \)th characteristic passing through \((t,x)\), which intersects the boundary \( x = (\lambda_n(0) + \delta_i)t \) of \( \Omega^+ \) at a point \((0; y)\) (Figure 3).

Recalling (3.4), it is easy to see that

\[
x - \lambda_n(0)t > \left( \begin{array}{c} \xi \end{array} + n(\lambda_n(0) - \lambda_n(0)) \right) y \leq \frac{\delta_i}{2} y
\]

(3.102)

Thus, using (3.13), (3.108) and (3.113), and noting the fact that \( L > 0 \), it follows from (3.106) that

\[
\| w_t(x,t) \| \leq C_{\alpha} \theta(1 + t)^{\alpha - \alpha} \leq C_{\alpha} \theta(1 + t)^{\alpha - \alpha}
\]

(3.107)

By Hadamard’s formula, we have

\[
\hat{s}_\alpha(u) - \hat{s}_\alpha(u,c) = \int_0^1 \sum_{\tau = \alpha} \gamma_{\tau}(u,c) \hat{s}_\alpha(u,c) \phi(T) dT
\]

(3.108)

Thus, recalling (3.13) and (3.105), and noting the fact that \( L > 0 \), it follows from (3.106) that

\[
0 \leq |x - \lambda(0)t| \leq \left( \begin{array}{c} \xi \end{array} + n(\lambda(0) - \lambda_n(0)) \right) y \leq \frac{\delta_i}{2} y
\]

(3.109)

Next, we assume that

\[
x - \lambda(0)t < -\left( \begin{array}{c} \xi \end{array} + n(\lambda(0) - \lambda_n(0)) \right) y
\]

(3.110)

which implies \( i > m + 1 \). Let \( \xi = \xi(s; t, x) \) \((t, x)\), which intersects the boundary \( x = (\lambda_{m+1}(0) - \delta_i)t \) of \( \Omega^+ \) at a point \((0; y)\).

Recalling (3.4), it is easy to see that

\[
x - \lambda_n(0) - \delta_i t \geq y - (\lambda_n(0) - \delta_i) y
\]

(3.111)

Since

\[
y = (\lambda_n(0) - \delta_i) y
\]

(3.112)

noting (3.110) and the fact that \( t \geq t_0 \) it follows from (3.111) that

\[
t \geq t_0
\]

(3.113)

By integrating (2.6) along \( \xi = \xi(s; t, x) \) and noting (2.9) and (2.11), we have

\[
w_t(x, t) = e^{-\xi(t)} w(t, y)
\]

(3.114)

Combining (3.101) and (3.109), (3.116), we obtain

\[
W^+(T) \leq C_{\alpha} (1 + W(D^+)) + W^+(T) \hat{U}_t(T) + W^+(T) \hat{Y}_t(T) + W^+(T) \hat{U}_t(T) + W^+(T) \hat{Y}_t(T) + U_t(T) V_c(T) + U_t(T) V_c(T)
\]

(3.117)

We next estimate \( V(D^+) \) \((t) \) for \( j = 1; \ldots; m \), for any fixed point \((t, x) \in D^+ \) similar to (3.59), by integrating (2.13) along \( \xi = \xi(s; t, x) \) and noting (2.17)–(2.18), we have

\[
v_j(x, t) = e^{\xi(t)} v_j(t, x)
\]

(3.118)

By using Lemma 3.2 and noting (3.54) and (3.57), it is easy to see that
By Hadamard's formula, we have
\[ \beta_s(u) = \beta_s(u;d) - \beta_s(u_0; e) = \int_D \frac{\partial u}{\partial n}(x) \, d\mu(x) \]
and
\[ \beta_s(u) = \beta_s(u; d) - \beta_s(u_0; e) = \int_D \frac{\partial u}{\partial n}(x) \, d\mu(x) \]

Thus, noting the fact that \( L > 0 \), and using (3.13) and (3.54), we obtain from (3.118) that
\[ (1 + \lambda_0 y) + \frac{\partial u}{\partial y}(x) \leq C_0 \left[ \theta + (V(D') y)^2 + W(D') y \right] + \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) \right] \right] \right] \right] \]

(3.122)

For \( j = m + 1 \), for any fixed point \((t, x) \in D^T\) similar to (3.74), we have
\[ v_j(t, x) = e^{-L(t-x)} v_j(t_0, x) + \int_0^t e^{-L(t-u)} \sum_{k=1}^n [\beta_{sk}(u)] v_k(u, x) \, dT \]

(3.123)

Noting (1.16), by (1.13), it is easy to see that
\[ v_j(t_0, x) = \sum_{r=1}^n g_r(t_0) v_r(t_0, x) + h_j(t_0) \]

(3.124)

Where
\[ g_r(t_0) = \int_0^t \frac{\partial G_t}{\partial n}(a(t_0), x, t_0, 0, \ldots, t_0, 0, 0) dT \]

(3.125)

By employing the same arguments as in (1), we can obtain
\[ (1 + \lambda_0 y) + \frac{\partial u}{\partial y}(x) \leq C_0 \left[ \theta + (V(D') y)^2 + W(D') y \right] + \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) \right] \right] \right] \right] \]

(3.126)

Thus, noting (1.15) and (3.6) and (3.70), it follows from (3.124)-(3.126) that
\[ (1 + y) + \frac{\partial u}{\partial y}(x) \leq C_0 \left[ \theta + (V(D') y)^2 + W(D') y \right] + \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) \right] \right] \right] \right] \]

(3.127)

Hence, noting the fact that \( L > 0 \), we obtain from (3.123) that
\[ (1 + y) + \frac{\partial u}{\partial y}(x) \leq C_0 \left[ \theta + (V(D') y)^2 + W(D') y \right] + \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) \right] \right] \right] \right] \]

(3.128)

Combining (3.122) and (3.128), we get
\[ V_j(t) \leq C_0 \left[ \theta + (V(D') y)^2 + W(D') y \right] + \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) + \left[ \frac{\partial u}{\partial y}(x) \right] \right] \right] \right] \]

(3.129)

We next estimate \( V_j(T) \) and \( V(T) \).

For \( i = m + 1 \), for any given \( j \) characteristic \( C_j \) in \( D^T \) (\( j = i \)) as in the proof of (3.90), in order to estimate \( V_j(T) \) it suffices to estimate
\[ V_j(t, x) \leq k \theta (1 + y)^{d \alpha(t)} \leq C_0 \theta (1 + y)^{d \alpha(t)} \]
Then, using Hadamard’s formula, we have

\[ F_q(u) = F_{q_0}(u) - F_{q_0}(u_0, \xi), \]

for any given point \((t, x) \in D^T\) noting (2.19) and (2.20), similar to (3.98), we have}

\[ u(t, x) = e^{\lambda_0 t + \lambda_1 x} u(t_0, y) + \int_0^t \int_{\Omega(t, y)} e^{\lambda_0(t-s) + \lambda_1 x-s} \rho_{q_0}(u_0, u_1, u_2, \ldots, u_n)(x, y, ds) dy, \]

By using Lemma 3.2 and noting (3.93) and (3.96), it is easy to see that

\[ |V(t_0, y)| \leq C \theta(t_0 + t_*), \]

Thus, we obtain

\[ U_{\infty}(t) \leq C_0(\theta + V(t_0) U(t_0) + U_{\infty}(t) W_{\infty}(t) + W_{\infty}(t) U(t_0)), \]

\[ U_{\infty}(t) \leq C_0(\theta + V(t_0) U(t_0) + U_{\infty}(t) W_{\infty}(t) + W_{\infty}(t) U(t_0) + U_{\infty}(t) W(t)), \]

By using Lemma 3.2, similar to (3.88), it follows from (3.150) that

\[ \int_0^t \int_{\Omega(t, y)} e^{\lambda_0(t-s) + \lambda_1 x-s} \rho_{q_0}(u_0, u_1, u_2, \ldots, u_n)(x, y, ds) dy, \]

Hence, noting (3.6), (3.11), (3.13) and the fact that \(L > 0\) and \(\frac{\partial L(s, y)}{\partial y} > 0\) we obtain from (3.147) and (3.149) that

\[ z(t, x, y) \leq C_0(\theta + V(t_0) U(t_0) + U_{\infty}(t) W(t) + W_{\infty}(t) U(t)), \]

Similarly, we have

\[ U_{\infty}(t) \leq C_0(\theta + V(t_0) U(t_0) + U_{\infty}(t) W(t) + W_{\infty}(t) U(t_0)), \]

We next estimate \(U_{\infty}(T)\).

For \(i = m+1, \ldots, n\), for any fixed point \((t, x) \in D^T\) noting (2.19) and (2.20), similar to (3.98), we have

\[ u(t, x) = e^{\lambda_0 t + \lambda_1 x} u(t_0, y) + \int_0^t \int_{\Omega(t, y)} e^{\lambda_0(t-s) + \lambda_1 x-s} \rho_{q_0}(u_0, u_1, u_2, \ldots, u_n)(x, y, ds) dy, \]

By using Lemma 3.2 and noting (3.93) and (3.96), it is easy to see that

\[ |u(t_0, y)| \leq C_0 \theta, \]

Thus, we obtain

\[ U_{\infty}(t) \leq C_0(\theta + V(t_0) U(t_0) + U_{\infty}(t) W(t) + W_{\infty}(t) U(t)), \]

By using Lemma 3.2 and noting (3.93) and (3.96), it is easy to see that

\[ |u(t_0, y)| \leq C_0 \theta, \]

Thus, noting (3.93) and the fact that \(L > 0\), we obtain from (3.150) that

\[ z(t, x, y) \leq C_0(\theta + V(t_0) U(t_0) + U_{\infty}(t) W(t) + W_{\infty}(t) U(t)), \]

Similarly, we have

\[ U_{\infty}(t) \leq C_0(\theta + V(t_0) U(t_0) + U_{\infty}(t) W(t) + W_{\infty}(t) U(t)), \]

We next estimate \(U_{\infty}(T)\).
By integrating (2.13) along this characteristic and noting (2.16)-(2.18), we have
\[ \frac{\partial}{\partial t} v_i(x) + \sum_{j=1}^n A_{ij}(x) \frac{\partial}{\partial x_j} v_i(x) = 0, \quad v_i(x) \big|_{t=0} = v_{i0}(x). \]

Moreover, for \( \epsilon > 0 \) suitably small, we have
\[ \frac{\partial}{\partial t} v_i(x) + \sum_{j=1}^n A_{ij}(x) \frac{\partial}{\partial x_j} v_i(x) = 0, \quad v_i(x) \big|_{t=0} = v_{i0}(x). \]

Thus, by using Lemma 3.2 again, we have
\[ \frac{\partial}{\partial t} v_i(x) + \sum_{j=1}^n A_{ij}(x) \frac{\partial}{\partial x_j} v_i(x) = 0, \quad v_i(x) \big|_{t=0} = v_{i0}(x). \]

Noting Lemma 3.1 and Lemma 3.2, and using Hadamard’s formula, it follows from (3.160) that
\[ |v_i(x)| \leq C_{\epsilon} \theta + V(D^+), \quad |v_i(x)| \leq C_{\epsilon} \theta + V(D^+). \]

On the other hand, for \( m = 1, \ldots, n \), any fixed point \( (t, x) \notin D(T) = \{(t, x) | 0 \leq t \leq T, x \geq 0 \} \) but \( (t, x) \notin D^+, \) \( |v_i(x)| \) can be controlled by \( C_{\epsilon} \theta \) or \( V(D^+) \). Moreover, for \( i = 1, \ldots, m \), any fixed point \( (t, x) \notin D(T) = \{(t, x) | 0 \leq t \leq T, x \geq 0 \} \) \( |v_i(x)| \) can be controlled by \( C_{\epsilon} \theta \) or \( V(D^+) \) as well. Thus, by using Lemma 3.2 again, we have
\[ W_{\epsilon}^*(0), V_{\epsilon}^*(0) \leq C_{\epsilon} \theta \]
and
\[ T = 0: W(D^+), V(D^+) \leq C_{\epsilon} \theta. \]

Thus, by continuity there exist positive constants \( k_2; k_3; k_4; k_5; k_6; k_7; k_8 \) and \( k_0 \) independent of \( \mu \), such that (3.47)-(3.53) hold at least for \( 0 \leq T < 70 \) where \( T_0 \) is a small positive number. Hence, in order to prove (3.47)-(3.53) it suffices to show that we can choose \( k_2; k_3; k_4; k_5; k_6; k_7; k_8 \) in such a way that for any fixed \( T_0 < T \leq 70 \)
\[ W(D^+), V(D^+) \leq 2k_2 \theta \]
we have
\[ W(D^+) \leq k_2 \theta, \]
\[ V(D^+) \leq k_2 \theta, \]
\[ W_{\epsilon}^*(T_0), V_{\epsilon}^*(T_0) \leq k_2 \theta, \]
\[ W_{\epsilon}^*(T_0), V_{\epsilon}^*(T_0) \leq k_2 \theta, \]
\[ W_{\epsilon}^*(T_0), V_{\epsilon}^*(T_0) \leq k_2 \theta, \]
\[ W_{\epsilon}^*(T_0), V_{\epsilon}^*(T_0) \leq k_2 \theta, \]
\[ W_{\epsilon}^*(T_0) \leq k_2 \theta. \]
\[ V_\alpha (T_\theta) \leq 2 C_\alpha (1 + k_\alpha + k_\theta) \theta \] (3.195)

\[ W_\alpha (T_\theta) \leq 2 C_\alpha (1 + k_\alpha + k_\theta) \theta \] (3.196)

Hence, if \( k_32 > 2C_37; \ k_2 > 2C_13(1 + k_3); \ k_4 > 2C_57(1 + k_3); \ k_5 > 2 \max(C_29(1 + k_2); C_46(1 + k_3); k_6 \geq 2\max(C_1R(1 + k_2); C_19(1 + k_2); C_40(1 + k_3); C_41(1 + k_3); C_50(1 + k_3); C_51(1 + k_3); k_7 > 2C_59(1 + k_3 + k_5)) \) and \( k_8 > 2C_61(1 + k_2 + k_5) \), then we get (3.177)-(3.183). This proves (3.47)-(3.53).

Finally, we observe that when \( \mu > 0 \) is suitably small, by (3.52) we have

\[ U_\alpha(T) \leq k_\alpha, \theta \leq k_\alpha, \theta \leq \frac{\delta}{2} \] (3.197)

This implies the validity of hypothesis (3.6). The proof of Lemma 3.3 is finished.

Proof of Theorem 1.1. It suffices to prove Theorem 1.1 in the normalized coordinates. Under the assumptions of Theorem 1.1, by (3.52) and (3.53), we know that there is a sufficiently small \( \theta_1 > 0 \) such that for any fixed \( \theta \in (0, \theta_1) \) on any given domain of existence \( D(T) = \{(t, x) | 0 \leq t \leq T, x \geq 0\} \) of the CI solution \( u = u(t, x) \) to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have the following uniform priori estimate for the CI norm of the solution:

\[ \| u(t, \cdot) \| \phi^\theta \leq \| u_0(t, \cdot) \| \phi^\theta + \| u_1(t, \cdot) \| \phi^\theta \leq k_0 \theta \] (3.198)

Thus we immediately get the conclusion of Theorem 1.1. The proof of Theorem 1.1 is finished.

**Application**

Consider the following mixed initial-boundary value problem for the system of the flow equations of a model class of \( \alpha \)-fluids with viscosity induced by fading memory (cf. [7]):

\[
\begin{align*}
&\frac{\partial w}{\partial t} - \alpha w + w = 0 \\
&\frac{\partial v}{\partial t} - (\sigma(w)) x + v = 0
\end{align*}
\] (4.1)

with the initial condition

\[ t = 0 : w = w_0(x), v = v_0(x)(x \geq 0) \] (4.2)

and the boundary condition

\[ x = 0 : v = h(t)(t \geq 0) \] (4.3)

Here, \( w \) is the displacement gradient and \( v \) the velocity of a model class of fluids, the stress-strain function \( \sigma(w) \) is a suitably smooth function of \( w \) such that

\[ \sigma'(0) > 0 \] (4.4)

\( \sigma_0 \) is a constant, \((W_\alpha(x), V_\alpha(x)) \in C^1\) and we assume that there exists a constant \( \mu > 0 \) such that

\[ \sup_{x \in \mathbb{R}} (l + x)^{\alpha}(l + x)^{\alpha}(w_0(x)) + (v_0(x)) + |v_0(x)| + |v_0(x)| + |w_0(x)| + |v_0(x)|) < +\infty \] (4.5)

In addition, we assume that \( h(t) \in C^0 \)

\[ \sup_{t \in \mathbb{R}} (l + t)^{\alpha}(|h(t)| + |h(t)|) < +\infty \] (4.6)

Moreover, the conditions of CI compatibility are supposed to be satisfied at the point \( (0, 0) \).

Let

\[ u = \begin{pmatrix} w \\ v \end{pmatrix} \] (4.7)

By (4.4), it is easy to see that in a neighborhood of \( u_0 = \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \) system (4.1) is strictly hyperbolic and has the following two distinct real eigenvalues:

\[ \lambda_1(u) = -\sqrt{\sigma'(w)} < 0 < \lambda_2(u) = \sqrt{\sigma'(w)} \] (4.8)

The corresponding right eigenvectors are

\[ r_1(u) / (\sqrt{\sigma'(w)})^T, r_2(u) / (1 - \sqrt{\sigma'(w)})^T \] (4.9)

It is easy to see that in a neighborhood of \( u_0 = \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \) all characteristics are linearly degenerate, then weakly linearly degenerate, provided that

\[ \sigma''(w) \equiv 0, \forall |w| \quad \text{small} \] (4.10)

The corresponding left eigenvectors can be taken as

\[ l_1(u) = \sqrt{\sigma'(w)}(1), l_2(u) = (-\sqrt{\sigma'(w)}, 1) \] (4.11)

Let

\[ v = l_i(u) \] (4.12)

Then, the boundary condition (4.3) can be rewritten as

\[ x = 0: v_1 + v_2 = 2h(t) \triangleq H(t) \] (4.13)

By Theorem 1.1 we get

Theorem 5.1. Suppose that (4.4) and (4.10) hold. Suppose furthermore that \( v(t); v_0(x) \) are all CI functions with respect to their arguments, for which there is a constant \( \mu > 0 \) such that

\[ \theta \triangleq \max_{x \in \mathbb{R}} \sup_{t \geq 0} (l + x)^{\alpha}(l + x)^{\alpha}(w_0(x)) + |v_0(x)| + |w_0(x)| + |v_0(x)|) < +\infty \] (4.14)

Suppose finally that \( h(t) \in C^0 \) satisfies (4.14) and that the conditions of CI compatibility are satisfied at the point \( (0, 0) \). Then there is a sufficiently small \( \theta_2 > 0 \) such that for any given \( \theta \in [0, \theta_2] \) the mixed initial-boundary value problem (4.1)-(4.3) admits a unique global CI solution \( u = u(t, x) \) in the half space \( \{(t, x) | t \geq 0, x \geq 0\} \).

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