Modeling Crowd Dynamics through Hyperbolic – Elliptic Equations

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Abstract. Inspired by the works of Hughes [17, 18], we formalize and prove the well posedness of a hyperbolic–elliptic system whose solutions describe the dynamics of a moving crowd. The resulting model is here shown to be well posed and the time of evacuation from a bounded environment is proved to be finite. This model also provides a microscopic description of the individuals’ behaviors.

2010 Mathematics Subject Classification. Primary 35M11; Secondary 35L65, 35J60.

Keywords. Crowd dynamics; hyperbolic–elliptic systems.

1. Introduction

We consider the problem of describing how pedestrians exit an environment. From a macroscopic point of view, we identify the crowd through the pedestrians’ density, say \( \rho = \rho(t,x) \), and assume that the crowd behavior is well described by the continuity equation

\[
\partial_t \rho + \nabla \cdot (\rho \, V(x,\rho)) = 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega,
\]

where \( \Omega \subset \mathbb{R}^2 \) is the environment available to pedestrians, \( V = V(x,\rho) \in \mathbb{R}^2 \) is the velocity of the individual at \( x \), given the presence of the density \( \rho \). Several choices for the velocity function are available in the literature, see for instance [5, 6, 8, 9, 11, 17, 18, 19, 25] for velocities depending nonlocally on the density, and [20, Section 4.1] for velocities depending locally on the density. Here, we posit the following (local with respect to the density) assumption:

\[
V(x,\rho) = v(\rho) \, w(x),
\]

where \( v = v(\rho) \) is a smooth non-increasing scalar function, motivated by the common attitude of moving faster when the density is lower. A key role is played by \( w = w(x) \): this vector identifies the route followed by the individual at \( x \). It is reasonable to assume that the individual at \( x \) follows the shortest path from \( x \) towards the nearest exit. This naturally suggests to choose \( w \) parallel to \( \nabla \varphi \), the potential \( \varphi \) being the solution to the eikonal equation on \( \Omega \). Extending the results in [2, 12, 13] obtained in the 1-dimensional space to the 2-dimensional space, we consider the following elliptic regularization of the eikonal equation:

\[
\|\nabla \varphi\|^2 - \delta \Delta \varphi = 1, \quad x \in \Omega,
\]

where \( \delta \) is a fixed strictly positive parameter. Clearly, the resulting vector field \( \nabla \varphi \) depends only on \( \Omega \), namely only on the geometry of the environment available to the pedestrians, i.e., on the positions of the exits, on the possible presence of obstacles, and so on. We assume that the boundary \( \partial \Omega \) is partitioned in walls, say \( \Gamma_w \), exits, say \( \Gamma_e \), and corners, say \( \Gamma_c \): namely \( \partial \Omega = \Gamma_w \cup \Gamma_e \cup \Gamma_c \), the set \( \Gamma_e \), \( \Gamma_w \), \( \Gamma_c \) being two by two disjoint. \( \Gamma_c \) is a discrete subset of \( \partial \Omega \). Also \( \Gamma_e \) and \( \Gamma_w \) are subsets of \( \partial \Omega \) and they are open in the topology they inherit from \( \partial \Omega \). It is then natural to choose \( \varphi \) as solution to the elliptic equation

\[
\begin{cases}
\|\nabla \varphi\|^2 - \delta \Delta \varphi = 1 & x \in \Omega \\
\nabla \varphi(\xi) \cdot \nu(\xi) = 0 & \xi \in \Gamma_w \\
\varphi(\xi) = 0 & \xi \in \Gamma_c,
\end{cases}
\]

\( \nu(\xi) \) being the outward unit normal to \( \partial \Omega \) at \( \xi \). To select the direction \( w(x) \) followed by the pedestrian at \( x \) we set

\[
w = \mathcal{N}(-\nabla \varphi),
\]

the map \( \mathcal{N} \) being a regularized normalization, that is

\[
\mathcal{N}(x) = \frac{x}{\sqrt{\theta^2 + \|x\|^2}},
\]
for a fixed strictly positive parameter $\vartheta$. Finally, the evolution of the crowd density $\rho$ is then found solving the following scalar conservation law:

\[
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho v(\rho) w(x)) &= 0 & (t, x) &\in \mathbb{R}^+ \times \Omega \\
\rho(0, x) &= \rho_0(x) & x &\in \Omega \\
\rho(t, \xi) &= 0 & (t, \xi) &\in \mathbb{R}^+ \times \partial \Omega,
\end{aligned}
\]

where $\rho_0$ is the initial crowd distribution. In other words, for a given domain $\Omega$, from (1.1) we obtain the vector field $\nabla v$, that is used in (1.2) to define $w$ and then from (1.4) we obtain how the pedestrians’ density $\rho$ evolves in time starting from the initial density $\rho_0$.

Remark that the boundary condition $\rho(t, \xi) = 0$ has to be understood in the sense of conservation laws, see [4, 10] and Definition 2.3 below. Indeed, the choice in (1.4) allows a positive outflow from $\Omega$ through $\Gamma_e$ thanks to the definition of $w$, as proved in (E.2) of Proposition 2.2.

We prove below that the model consisting of (1.1)–(1.2)–(1.4) is well posed, i.e., it admits a unique solution which is a continuous function of the initial data. Moreover, we also ensure that the evacuation time is finite.

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The next two sections are devoted to the detailed formulation of the problem, to the statement of the well posedness result and of further qualitative properties of the model (1.1)–(1.2)–(1.4). All technical details are gathered in Section 4.

2. Well Posedness

Throughout, we denote $\mathbb{R}^+ = [0, \infty[$. For $x \in \mathbb{R}^2$ and $r > 0$, $B(x, r)$ stands for the open disk centered at $x$ with radius $r$. For any measurable subset $S$ of $\mathbb{R}^2$, we denote by $|S|$ its 2-dimensional Lebesgue measure. Recall that two (non-empty) subsets $A_1, A_2$ of $\mathbb{R}^2$ are separate whenever $\overline{A_1} \cap A_2 = \emptyset = A_1 \cap \overline{A_2}$.

A key role is played by the geometry of the domain $\Omega$. Here we collect the conditions necessary in the sequel, see Figure 1.

\begin{enumerate}
\item[(\Omega.1)] $\Omega \subset \mathbb{R}^2$ is non-empty, open, bounded and connected.
\item[(\Omega.2)] The boundary $\partial \Omega$ admits the disjoint decomposition $\partial \Omega = \Gamma_w \cup \Gamma_e \cup \Gamma_c$, where $\Gamma_w$ and $\Gamma_e$ are separate and are finite union of open 1-dimensional manifolds of class $C^3, \gamma$, for a given $\gamma \in [0, 1[$; $\Gamma_e$ is non-empty; $\Gamma_c$ is a discrete finite set and $\Gamma_w \cap \Gamma_e \subseteq \Gamma_c \subseteq \Gamma_w$.
\item[(\Omega.3)] For any $x \in \Gamma_e$, there exists an $\varepsilon > 0$ such that the intersection $B(x, \varepsilon) \cap \Omega$ is exactly a quadrant of the disk $B(x, \varepsilon)$.
\end{enumerate}

![Figure 1. Two examples of sets $\Omega$ with the notation used in (\Omega.2) and in (\Omega.3).](image)

The requirement (\Omega.1) is clear. In (\Omega.2) the term open has to be understood with respect to the topology inherited by $\partial \Omega$. Again concerning (\Omega.2), introduce the connected components of $\Gamma_w, \Gamma_e$ and $\Gamma_c$, i.e.,

\[
\Gamma_w = \bigcup_{i=1}^{n_w} \Gamma_w^i, \quad \Gamma_e = \bigcup_{i=1}^{n_e} \Gamma_e^i, \quad \text{and} \quad \Gamma_c = \bigcup_{i=1}^{n_c} \{J_i\}.
\]
Each of the $\Gamma^e_i$ is an exit, while the $J_i$ are points where the regularity of $\partial \Omega$ is allowed to be lower. Condition (\text{E.2}) implies that each $\Gamma^e_i$ and each $\Gamma^e_c$ is a $C^{3,\gamma}$ manifold. Since $\Gamma^e \subseteq \Gamma^w$, along the boundary $\partial \Omega$, between two different exits there is always a wall or, in other words, there can not be two exits separated only by a corner point. Condition (\text{E.3}) also implies that $n_{c} \geq 1$, so that there is at least one exit. Moreover, apart from the trivial case where $\partial \Omega = \Gamma^e$, the set $\Gamma^e$ may not be empty. Note also that any corner point $J_i$ in $\Gamma^e$ is either a doorjamb, if $J_i \in \Gamma^e_c$, or a wall corner, if $J_i \in (\Gamma^e \setminus \Gamma^e_c)$. Condition (\text{E.3}) says that the angles between each door and the walls are right and convex, and additionally that these contain straight segments. This is a technical assumption, related to the subtle mixed boundary conditions: Dirichlet and Neumann conditions meet at the doorjamb points. Condition (\text{E.3}) ensures the regularity of solutions in a neighborhood of these points, a property that might not hold for general angles.

Throughout, by solution to (1.1) we mean generalized solution in the sense of the following definition (see [13] Chapters 8 and 13).

\textbf{Definition 2.1} Let $\Omega$ satisfy (\text{E.1}) A function $\varphi \in H^1(\Omega; \mathbb{R})$ is a generalized solution to (1.1) if $\text{tr}_e \varphi = 0$ and
\[\delta \int_\Omega \nabla \varphi(x) \cdot \nabla \eta(x) \, dx + \int_\Omega \left( \| \nabla \varphi(x) \|^2 - 1 \right) \eta(x) \, dx = 0\]
for any $\eta \in H^1(\Omega; \mathbb{R})$ such that $\text{tr}_e \eta = 0$.

Above, $\text{tr}_e \eta$ denotes the trace of $\eta$ on $\Gamma^e$. We refer to [11] Chapter 5.5 for the definition and properties of the trace operator.

Note that no generalized solution to (1.1) can vanish a.e. on $\Omega$. The next proposition provides the basic existence result for the solutions to (1.1), together with some qualitative properties.

\textbf{Proposition 2.2 (Elliptic Problem)} Let $\Omega$ satisfy (\text{E.1}) (\text{E.2}) (\text{E.3}) Fix $\delta > 0$. Then, problem (1.1) admits a unique generalized solution $\varphi \in C^3(\overline{\Omega}; \mathbb{R})$ with the properties:

(\text{E.1}) For a.e. $x \in \Omega$, $\nabla \varphi(x) \neq 0$.

(\text{E.2}) For all $\xi \in \Gamma^e$, $-\nabla \varphi(\xi) \cdot \nu(\xi) > 0$.

(\text{E.3}) $\frac{|\Omega|}{\delta} \exp \left( - \frac{\max_{\partial \Omega} \varphi}{\delta} \right) \leq - \int_{\Gamma^e} \nabla \varphi(\xi) \cdot \nu(\xi) \, d\xi \leq \frac{|\Omega|}{\delta} \exp \left( \frac{\max_{\partial \Omega} \varphi}{\delta} \right)$.

The proof of the above proposition is postponed to Section 4. Here, we note that properties (\text{E.1}) (\text{E.2}) and (\text{E.3}) have clear consequences on the properties of the solutions to the full system (1.1)-(1.2)-(1.4).

Indeed, setting $w$ as in (1.2), property (\text{E.1}) implies that $w$ vanishes only on a set of measure 0; (\text{E.2}) ensures that $w$ is non zero and points outwards along exits; (\text{E.3}) can be used to provide bounds on the evacuation time.

In the hyperbolic problem (1.4), we use the following assumptions, which are standard in the framework of conservation laws:

(\text{C.1}) $v \in C^2([0, R_{\text{max}}]; [0, V_{\text{max}}])$ is weakly decreasing, $v(0) = V_{\text{max}}$ and $v(R_{\text{max}}) = 0$.

(\text{C.2}) $\rho_0 \in (BV \cap L^\infty)(\Omega; [0, R_{\text{max}}])$.

Above, $R_{\text{max}}$, respectively $V_{\text{max}}$, is the maximal density, respectively speed, possibly reached by the pedestrians.

We recall also the definition of entropy solution to (1.4), which originates in [27], see also [4] p. 1028. Here, we refer to [10] Definition 2.1.

\textbf{Definition 2.3} Let the conditions (\text{E.1}) (\text{E.2}) (\text{E.3}) (\text{C.1}) and (\text{C.2}) hold. Let $w \in C^2(\overline{\Omega}; \mathbb{R}(0,1))$. A function $\rho \in L^\infty(\Omega; \mathbb{R})$ is an entropy solution to the initial – boundary value problem (1.4) if for any test function $\zeta \in C^2([-\infty, T] \times \mathbb{R}^2; \mathbb{R}^+)$ and for any $k \in [0, R_{\text{max}}]$
\[
\int_0^T \int_{\Omega} \left\{ |\rho(t,x) - k| \partial_t \zeta(t,x) + \text{sign}(\rho(t,x) - k) \left( \rho(t,x) v(\rho(t,x)) - k v(k) \right) w(x) \cdot \nabla \zeta(t,x) \right\} \, dx \, dt \]
\[+ \int_{\Omega} |\rho_0(x) - k| \zeta(0,x) \, dx + \int_0^T \int_{\partial \Omega} \left( \text{tr}_{\partial \Omega} \rho(t,\xi) v(\text{tr}_{\partial \Omega} \rho(t,\xi)) - k v(k) \right) w(\xi) \cdot \nu(\xi) \zeta(t,\xi) \, d\xi \, dt \geq 0.
\]
As above, \( \text{tr} \partial u \) stands for the operator trace at \( \partial \Omega \) applied to the BV function \( u \), see for instance [14 § 5.5] or [10] Appendix]. Note that if the solution has bounded total variation in time, it has a trace at \( t = 0^+ \).

**Proposition 2.4 (Hyperbolic Problem)** Let the conditions \((\Omega.1)\) \((\Omega.2)\) and \((C.1)\) hold. Let \( w \in C^2(\Omega; \mathbb{R}(0,1)) \). Then, problem \((1.4)\) generates the map

\[
S : \mathbb{R}^+ \times (L^1 \cap \text{BV})(\Omega; [0, R_{\text{max}}]) \rightarrow (L^1 \cap \text{BV})(\Omega; [0, R_{\text{max}}])
\]

with the following properties:

**(H.1)** \( S \) is a semigroup.

**(H.2)** \( S \) is Lipschitz continuous with respect to the \( L^1 \)-norm, more precisely for any \( s, t \in [0, T] \)

\[
\| S_t \rho_o - S_s \rho_o \|_{L^\infty(\Omega; \mathbb{R})} \leq \left[ \sup_{\tau \in [s,t]} \text{TV}(S_\tau \rho_o) \right] |t - s|.
\]

**(H.3)** For any \( t \in [0, T] \)

\[
\| S_t \rho_o \|_{L^\infty(\Omega; \mathbb{R})} \leq \| \rho_o \|_{L^\infty(\Omega; \mathbb{R})} \exp(C_1 t), \quad \text{TV}(S_t \rho_o) \leq C_2 (1 + t + \text{TV}(\rho_o)) \exp(C_2 t),
\]

where the constants \( C_1, C_2 \) depend only on \( R_{\text{max}} \), \( \| v' \|_{W^{2,\infty}([0, R_{\text{max}}]; \mathbb{R})} \) and \( \| w \|_{W^{2,\infty}(\Omega; \mathbb{R}^2)} \).

**(H.4)** For any \( \rho_o \in (L^1 \cap \text{BV})(\Omega; [0, R_{\text{max}}]) \), the orbit \( t \mapsto S_t \rho_o \) is the unique solution to \((1.4)\) in the sense of Definition 2.3.

The proof of the above proposition is deferred to Section 4 where it is shown that the above statements follow from [10] Theorem 2.7.

We now give the definition of solution to \((1.1)-(1.2)-(1.4)\).

**Definition 2.5** Let the assumptions \((\Omega.1)-(\Omega.3)-(C.1)\) and \((C.2)\) hold. The pair of functions \((\varphi, \rho) \in H^1(\Omega; \mathbb{R}) \times (L^\infty \cap \text{BV})([0, T] \times \Omega; [0, R_{\text{max}}]) \) solves the problem \((1.1)-(1.2)-(1.4)\) if \( \varphi \) is a generalized solution to \((1.1)\) in the sense of Definition 2.1 and \( \rho \) is an entropy solution to \((1.4)\) in the sense of Definition 2.3 with \( w \) given by \((1.2)\).

The next theorem ensures the well posedness of the elliptic–hyperbolic model \((1.1)-(1.2)-(1.4)\).

**Theorem 2.6 (Mixed Problem)** Let the conditions \((\Omega.1)-(\Omega.3)-(\Omega.4)-(C.1)\) and \((C.2)\) hold. For any \( \delta, \vartheta > 0 \), the elliptic–hyperbolic problem \((1.1)-(1.2)-(1.4)\) generates a map

\[
M : \mathbb{R}^+ \times (L^1 \cap \text{BV})(\Omega; [0, R_{\text{max}}]) \rightarrow (L^1 \cap \text{BV})(\Omega; [0, R_{\text{max}}])
\]

with the following properties:

**(M.1)** \( M \) is a semigroup.

**(M.2)** \( M \) is Lipschitz continuous with respect to the \( L^1 \)-norm, more precisely for any \( s, t \in [0, T] \)

\[
\| M_t \rho_o - M_s \rho_o \|_{L^\infty(\Omega; \mathbb{R})} \leq \left[ \sup_{\tau \in [s,t]} \text{TV}(M_\tau \rho_o) \right] |t - s|.
\]

**(M.3)** For any \( t \in [0, T] \) we have that \((\varphi, \rho) = M_t \rho_o \) satisfies

\[
\| \rho \|_{L^\infty(\Omega; \mathbb{R})} \leq \| \rho_o \|_{L^\infty(\Omega; \mathbb{R})} \exp(C_1 t), \quad \text{TV}(\rho) \leq C_2 (1 + t + \text{TV}(\rho_o)) \exp(C_2 t),
\]

where \( C_1 \) is a positive constant depending on \( \| q \|_{W^1,\infty([0, R_{\text{max}}]; \mathbb{R})} \) and \( \| w \|_{W^2,\infty(\Omega; \mathbb{R}^2)} \), while the constant \( C_2 \) depends on \( \| q \|_{W^{2,\infty}(\Omega; \mathbb{R}^2)} \) and \( \| w \|_{W^2,\infty(\Omega; \mathbb{R}^2)} \), where as usual we set \( q(\rho) = \rho v(\rho) \).

**(M.4)** For all \( \rho_o \in (L^1 \cap \text{BV})(\Omega; [0, R_{\text{max}}]) \), the orbit \( t \mapsto M_t \rho_o \) is the unique solution to \((1.1)-(1.2)-(1.4)\) in the sense of Definition 2.5.

The above result is a direct consequence of Proposition 2.2 and Proposition 2.4.
3. Qualitative Properties

Here, we aim at further qualitative properties of the solutions to (1.1)–(1.2)–(1.4) that have a relevant meaning in the present setting.

Introduce for \( \hat{x} \in \Omega \) the path \( \hat{p}_x \) followed by those pedestrians that are at \( \hat{x} \) at time \( t = 0 \), i.e., the map \( \hat{p}_x \) is defined for \( t \geq 0 \) as the solution to the Cauchy problem

\[
\begin{cases}
\dot{x} = w(x) \\
 x(0) = \hat{x},
\end{cases}
\]

where \( w = \mathcal{N}(-\nabla \varphi) \).

Above, \( \mathcal{N} \) is defined in (1.3) and \( \varphi \) is the solution to (1.1).

Proposition 3.1 (Pedestrians’ Trajectories) Let \( \Omega \) satisfy (\( \Omega.1 \)), (\( \Omega.2 \)), (\( \Omega.3 \)) and call \( \varphi \) the solution to (1.1) provided by Proposition 2.2. Then:

(Q.1) For any \( \hat{x} \in \Omega \), there exists a unique globally defined path \( \hat{p}_x : I_{\hat{x}} \rightarrow \mathbb{R}^2 \) solving (3.1), \( I_{\hat{x}} \) being a suitable non trivial real interval.

(Q.2) Any two paths either coincide or do not intersect, in the sense that for any \( \hat{x}, \hat{y} \in \Omega \)

\[
\hat{p}_x(I_{\hat{x}}) \cap \hat{p}_y(I_{\hat{y}}) \neq \emptyset \implies \begin{cases} 
\text{either} & \hat{x} \in \hat{p}_y(I_{\hat{y}}) \text{ and } \hat{p}_x(I_{\hat{x}}) \subseteq \hat{p}_y(I_{\hat{y}}) \\
\text{or} & \hat{y} \in \hat{p}_x(I_{\hat{x}}) \text{ and } \hat{p}_y(I_{\hat{y}}) \subseteq \hat{p}_x(I_{\hat{x}}).
\end{cases}
\]

(Q.3) There exist a subset \( \hat{\Omega} \subset \Omega \) with \( |\hat{\Omega}| = 0 \) and a map \( T : \Omega \setminus \hat{\Omega} \rightarrow \mathbb{R}^+ \) such that \( I_{\hat{x}} = [0, T_{\hat{x}}] \) and \( \hat{p}_x(T_{\hat{x}}) \in \Gamma_e \) for all \( x \in \Omega \setminus \hat{\Omega} \).

The proof is deferred to Section 4. In other words, \( T_{\hat{x}} \) is the time that the pedestrian leaving from point \( \hat{x} \) needs to reach the exit. Property (Q.3) ensures that this time is finite for a.e. initial position \( \hat{x} \). Figure 2 shows that the set \( \hat{\Omega} \) in Proposition 3.1 is necessarily non-empty. In the room above, due to the presence of the two exits \( \Gamma_e \), the vector field \( w \) vanishes along the dotted segment \( \hat{\Omega} \).

4. Technical Details

We choose the following notation to denote a vector orthogonal to a given vector in \( \mathbb{R}^2 \):

if \( v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \), then \( v^\perp = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} \).

We frequently use the boundedness and Lipschitz continuity of the map \( \mathcal{N} \) as defined in (1.3), namely

\[
\begin{align*}
\|\mathcal{N}(x)\| &\leq 1 \quad \text{for all } x \in \mathbb{R}^2, \\
\|\mathcal{N}(x_1) - \mathcal{N}(x_2)\| &\leq \vartheta^{-1} \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in \mathbb{R}^2.
\end{align*}
\]

The Hopf-Cole transformation (see e.g. [14, Chapter 4.4.1])

\[
u = e^{-\varphi/\delta}
\]

transforms generalized solutions to (1.1) into generalized solutions to the linear problem

\[
\begin{align*}
\begin{cases}
 u = \delta^2 \Delta u & x \in \Omega \\
 \nabla u(\xi) \cdot \nu(\xi) = 0 & \xi \in \Gamma_w \\
u(\xi) = 1 & \xi \in \Gamma_e,
\end{cases}
\end{align*}
\]

whose precise definition (see e.g. [15, Chapter 8]) is here below.
Definition 4.1 A function \( u \in H^1(\Omega; \mathbb{R}) \) is a generalized solution to (4.3) on \( \Omega \) if \( \text{tr}_{\Gamma_c} u \equiv 1 \) and
\[
\delta^2 \int_{\Omega} \nabla u(x) \cdot \nabla \eta(x) \, dx + \int_{\Omega} u(x) \eta(x) \, dx = 0 \tag{4.4}
\]
for any \( \eta \in H^1(\Omega; \mathbb{R}) \) such that \( \text{tr}_{\Gamma_c} \eta \equiv 0 \).

The next Lemma collects various information on (4.3).

Lemma 4.2 Fix a positive \( \delta \) and let \( \Omega \) satisfy (\( \Omega.1 \)) and (\( \Omega.2 \)). Then,

1. Problem (4.3) admits a unique generalized solution \( u \in (H^1 \cap C^\infty)(\Omega; \mathbb{R}) \) in the sense of Definition 4.1. Moreover, \( u \in C^3(\Omega \setminus \Gamma_c; \mathbb{R}) \).

2. There exists a positive \( \varpi \) dependent only on \( \Omega \) such that \( u(x) \in ]\varpi, 1[ \) for all \( x \in \Omega \), so that \( u(x) \in ]\varpi, 1[ \) also for all \( x \in \overline{\Omega} \).

3. The solution \( u \) to (4.3) satisfies \( \nabla u(\xi) \cdot \nu(\xi) > 0 \) for all \( \xi \in \Gamma_c \).

4. The set \( \{ x \in \Omega : \nabla u(x) = 0 \} \) of critical points of \( u \) has measure 0.

If in addition \( \Omega \) satisfies (\( \Omega.3 \)) then:

5. \( u \in C^3(\overline{\Omega}; \mathbb{R}) \).

6. If \( \bar{x} \) is a critical point of \( u \), then the Hessian matrix \( D^2u(\bar{x}) \) has at least one positive eigenvalue.

Proof. Consider the different items above separately.

* (u.1) we use Lax–Milgram Lemma, see [14] Section 6.2.1. Introduce the Hilbert space \( H = \{ \eta \in H^1(\Omega; \mathbb{R}) : \text{tr}_{\Gamma_c} \eta = 0 \text{ a.e. on } \Gamma_c \} \) endowed with the usual scalar product and the coercive bilinear form
\[
a(u, \eta) = \delta^2 \int_{\Omega} \nabla u(x) \cdot \nabla \eta(x) \, dx + \int_{\Omega} u(x) \eta(x) \, dx.
\]
Note that \( H \) is a closed subspace of \( H^1(\Omega; \mathbb{R}) \) by the Trace Theorem [14] Chapter 5.5, Theorem 1. Indeed, if \( u^k \) is a sequence in \( H \) converging to \( u \) in \( H^1(\Omega; \mathbb{R}) \), then
\[
\|u\|_{L^2(\Gamma_c; \mathbb{R})} = \|u^k - u\|_{L^2(\Gamma_c; \mathbb{R})} \leq C \|u^k - u\|_{H^1(\Omega; \mathbb{R})} \to 0,
\]
for a constant \( C \) depending only on \( \Omega \), so that \( u \in H \). A function \( u \in H^1(\Omega; \mathbb{R}) \) is a generalized solution to (4.3) if and only if \( v = u - 1 \in H \) and \( a(v, \eta) = -\int_{\Omega} \eta(x) \, dx \) for all \( \eta \in H \). The map \( \eta \mapsto \int_{\Omega} \eta(x) \, dx \) is a linear functional over \( H \). By Lax–Milgram Lemma, we infer the existence and uniqueness of a generalized solution \( u \) to (4.3) such that \( u \in H \subset H^1(\Omega; \mathbb{R}) \). Moreover, \( u \in C^\infty(\Omega; \mathbb{R}) \) by Theorem 3 in Chapter 6.3 and Theorem 6 in Section 5.6.3. By (\( \Omega.1 \)) and (\( \Omega.2 \)) the results in [14] Theorem 9.3 ensure that \( u \in C^3(\overline{\Omega \setminus \Gamma_c}; \mathbb{R}) \).

* (u.2) note that, due to the boundary conditions along \( \Gamma_c \) and \( \Gamma_w \), no \( H^1 \) solution to (4.3) can be constant. The function \( \eta = (u - 1)^+ \), where \( (v)^+ = \max(v, 0) \), is in \( H^1(\Omega; \mathbb{R}) \) and inserting it in (4.4) we get
\[
\delta^2 \int_{\Omega} \|\nabla (u - 1)^+\|^2 + \int_{\Omega} \|u - 1\|^2 + \int_{\Omega} (u - 1)^+ = 0.
\]
This leads to \( (u - 1)^+ \equiv 0 \) a.e. in \( \Omega \), and, by the continuity of \( u \) on \( \overline{\Omega} \), \( u(x) \leq 1 \) for all \( x \in \overline{\Omega} \). The map \( u \) satisfies (4.3) in the strong sense everywhere in \( \Omega \). Hence, by the maximum principle [23] Chapter 2, Theorem 6 \( u(x) < 1 \) for all \( x \in \Omega \).

We show now that \( u > 0 \). As \( u \) is continuous in \( \overline{\Omega} \), it attains its minimum. Assume, by contradiction, that \( \min_{\partial\Omega} u = -m \) for some \( m \geq 0 \). Then, by applying the maximum principle to \( -u \), we know that there exists \( \xi \in \partial\Omega \) such that \( u(\xi) = -m \). We apply now Hopf’s Lemma, more precisely its extension from [21] to domains satisfying the cone condition (instead of the ball condition as in the original work by Hopf, see e.g., [23] Theorem 8 in Chapter 2), which implies that the normal derivative of \( u \) at \( \xi \) is positive, contradicting (4.3).

* (u.3) is an immediate consequence of (u.2) due to the boundary conditions in (4.3).

* (u.4) denote by \( D^2u \) the Hessian matrix of \( u \) and note that
\[
\{ x \in \Omega : \nabla u(x) = 0 \} = \{ x \in \Omega : \nabla u(x) = 0 \text{ and } \det D^2u(x) = 0 \} \cup \{ x \in \Omega : \nabla u(x) = 0 \text{ and } \det D^2u(x) \neq 0 \}.
\]
The former set has 2-dimensional measure zero by Sard Theorem applied to $\nabla u$. The latter set consists of isolated points all belonging to the compact set $\overline{\Omega}$, hence it is finite. Therefore, $\{x \in \Omega: \nabla u(x) = 0\} = 0$.

We verify that $u$ is $C^2$ at the points in $\Gamma_\circ$ under condition $(\Omega.3)$. To this aim, we adapt the arguments in [25, Theorem 3.1], there applied to Poisson equation.

Fix $x_o \in \Gamma_\circ \cap \Gamma_\iota$, i.e., $x_o$ is a doormb. Let $\varepsilon$ be as in $(\Omega.3)$, call $\ell = \varepsilon/2$ and choose $x_1 \in \Gamma_\circ \cap B(x_o, \ell)$ with $x_1 \neq x_o$. Let $v$ be a unit vector such that $v \cdot (x_1 - x_o) = 0$ and pointing outward $\Omega$ at $x_1$. Define $x_2 = x_1 - \ell v$ and $x_3 = x_o - \ell v$. Call $R$ the open rectangle with vertexes $x_o, x_1, x_2, x_3$, denote by $x_i x_j$ the open segment

$$x_i x_j = \{x \in \mathbb{R}^2: x = (1 - \vartheta) x_i + \vartheta x_j, \quad \vartheta \in [0, 1]\}$$

and by $\mathcal{S}$ the symmetry about the straight line including $x_o x_3$ and $R' = \mathcal{S}(R)$. Define the rectangle $R = R \cup x_o x_3 \cup R'$ and consider the problem

$$\begin{align*}
-\delta^2 \Delta w(x) + w(x) &= 0 & x \in R \\
w(\xi) &= 1 & \xi \in x_o x_1 \cup \mathcal{S}(x_o x_1) \\
(w(\xi) &= u(\xi) & \xi \in x_1 x_2 \cup x_2 x_3 \\
(w(\xi) &= w(\mathcal{S}(\xi)) & \xi \in \mathcal{S}(x_1 x_2 \cup x_2 x_3).
\end{align*}$$

Note that the boundary condition is of class $C^\infty$ by the regularity of $u$ proved above. Lax–Milgram Lemma ensures that the function $w$ exists, is unique and is in $C^\infty(R; \mathbb{R})$. By construction, $w$ is symmetric with respect to the straight line $x_o + R \nu$, in the sense that

$$w(x) = w(\mathcal{S}(x)) \quad \text{for all} \quad x \in R.$$ 

This in turn implies that

$$\nabla w(\xi) \cdot \nu(\xi) = 0 \quad \text{for all} \quad x \in x_o x_3.$$ 

Due to the $C^\infty$ regularity of the boundary of $R$ at $x_o$, $w$ is of class $C^\infty$ in a neighborhood of $x_o$. By uniqueness, $w = u$ on $\overline{R}$. Hence, $u$ is of class $C^\infty$ also in a neighborhood of $x_o$ restricted to $\Omega$.

If $x_o \in (\Gamma_\circ \setminus \overline{\Gamma_\iota})$, to prove the regularity of $u$ at $x_o$ we proceed as above, simply replacing the Dirichlet condition on $x_o x_1$ by a homogeneous Neumann one, applying again Lax-Milgram Lemma and concluding by symmetry and uniqueness.

The characteristic equation $\det(D^2 u(\bar{x}) - \lambda I) = 0$ in the case of a 2-dimensional problem is a quadratic equation with real solutions $\lambda_1(\bar{x}), \lambda_2(\bar{x})$ satisfying

$$\lambda_1(\bar{x}) \lambda_2(\bar{x}) = \det D^2 u(\bar{x}) = \lambda_1(\bar{x}) + \lambda_2(\bar{x}) = \Delta u(\bar{x}).$$

Note that by the $C^2$ regularity of $u$ proved at $\Omega.1$ the equation $u = \delta^2 \Delta u$ is satisfied in whole $\overline{\Omega}$. By $\Omega.2$

$$\lambda_1(\bar{x}) + \lambda_2(\bar{x}) = \delta^2 w(\bar{x}) > 0,$$ so that at least one of the eigenvalues has to be (strictly) positive.

Proof of Proposition 2.2. By $(4.2)$ and straightforward computations it is clear that $(4.1)$ has a solution if and only if $(4.3)$ has a solution which is positive a.e. in $\Omega$. Point $(\Omega.1)$ in Lemma 4.2 ensures the existence and uniqueness of a solution to $(4.3)$. Moreover, by $(\Omega.2)$ in Lemma 4.2 this solution is strictly positive a.e. in $\Omega$. This allows to define $\varphi = -\delta \ln u$. The remaining regularity statements and $(\Omega.1)$ follow again from Lemma 4.2 by $(4.2)$. So as to obtain $(\Omega.2)$ note first that $-\nabla \varphi \cdot \nu = \frac{\partial}{\partial} \nabla u \cdot \nu > 0$ everywhere on $\Gamma_\circ$ by $(4.2)$ and $(\Omega.3)$ in Lemma 4.2. Then, integrate $(4.3)$ on $\Omega$, use Green Theorem and again Lemma 4.2 to obtain $(\Omega.3)$.

Proof of Proposition 2.4. The present proof follows from [10, Theorem 2.7]. Indeed, referring to the notation therein, we define $q(\rho) = \rho \psi(\rho)$ and verify the necessary assumptions.

$(\Omega_3, \gamma)$ $\Omega$ is a bounded open subset of $\mathbb{R}^2$ with piecewise $C^3 \cap$ boundary $\partial \Omega$ by $(\Omega.1)$ and $(\Omega.2)$.

$(F)$ This condition is immediate since in the present case we have $F \equiv 0$.

$(f)$ In our case $f(t, x, \rho) = \rho \psi(\rho) w(x)$. By $(C.1)$ and the assumption that $w$ is in $(C^2 \cap W^{1, \infty}(\mathbb{R}; B(0, 1)))$, we have that $f$ is of class $C^2$ and moreover

$$\partial_\rho f(t, x, \rho) = q'(\rho) w(x), \quad \partial^2_{\rho \rho} f(t, x, \rho) = q''(\rho) w(x), \quad \partial_\rho \nabla \cdot f(t, x, \rho) = q'(\rho) \nabla \cdot w(x)$$

are all functions of class $L^\infty$ on $\mathbb{R}^+ \times \Omega \times [0, R_{\max}]$. 

This condition follows from (C.2) because in the present case $\rho_0 \equiv 0$.

We then obtain
\[
\|S_t \rho_0\|_{L^\infty(\Omega; \mathbb{R})} \leq \left( \|\rho_0\|_{L^\infty(\Omega; \mathbb{R})} + c_2 t \right) \exp(c_1 t)
\]
by [10] Formula (2.5)]
\[
\text{TV}(S_t \rho_0) \leq (A_1 + A_2 t + A_3 \text{ TV}(ho_0)) \exp(A_4 t)
\]
by [10] Formula (6.44)]
\[
\text{where, with reference to [10] Formula (5.1) and [10] § 6], the constants } c_1, c_2, A_1, \ldots, A_4 \text{ are estimated as follows:
\[
c_1 = 1 + \|q\|_{L^\infty([0, R_{\max}]; \mathbb{R})} \|\nabla \cdot w\|_{L^\infty(\Omega; \mathbb{R})} \leq 1 + \|q\|_{W^{1, \infty}([0, R_{\max}]; \mathbb{R})} \|w\|_{W^{1, \infty}(\Omega; \mathbb{R})},
\]
c_2 = 0,
\[
A_1 = O(1) \|Df\|_{L^\infty(\Omega \times [0, R_{\max}]; \mathbb{R}^n \times \mathbb{R}^{n \times (1+n)})} \leq O(1) \|q\|_{W^{1, \infty}([0, R_{\max}]; \mathbb{R})} \|w\|_{W^{1, \infty}(\Omega; \mathbb{R})},
\]
A_2 = O(1) \|Df\|_{W^{1, \infty}((\Omega \times [0, R_{\max}]; \mathbb{R}^n \times \mathbb{R}^{n \times (1+n)})} \leq O(1) \|q\|_{W^{2, \infty}((0, R_{\max}); \mathbb{R})} \|w\|_{W^{2, \infty}(\Omega; \mathbb{R})},
\]
A_3 = O(1) + \|q\|_{L^\infty([0, R_{\max}]; \mathbb{R})} \|w\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq O(1) + \|q\|_{W^{1, \infty}(0, R_{\max}; \mathbb{R})} \|w\|_{L^\infty(\Omega; \mathbb{R}^n)},
\]
A_4 = O(1) \left[ 1 + \|Df\|_{W^{1, \infty}((\Omega \times [0, R_{\max}]; \mathbb{R}^n \times \mathbb{R}^{n \times (1+n)})} \right] \leq O(1) \left[ 1 + \|q\|_{W^{2, \infty}((0, R_{\max}); \mathbb{R})} \|w\|_{W^{2, \infty}(\Omega; \mathbb{R})} \right]
\]
and the above norms of $q$ are bounded by (C.1) and by the adopted assumption on $w$. \hfill \Box

For technical reasons, below we fix an arbitrary open subset $\Omega'$ of $\mathbb{R}^2$ containing $\Omega$ and extend the unique generalized solution $\varphi \in C^3(\Omega'; \mathbb{R})$ of (1.1) given in Proposition 2.2 introducing a map $\tilde{\varphi} \in C^3_1(\mathbb{R}^2; \mathbb{R})$ such that $\tilde{\varphi} \equiv \varphi$ in $\Omega$ and $\tilde{\varphi} \equiv 0$ in $\mathbb{R}^2 \setminus \Omega'$. This is possible thanks to the regularity of $\varphi$ and to the following result.

Lemma 4.3 ([15] Lemma 6.37) Let $\Omega$ satisfy (Ω.1) (Ω.2), (Ω.3). For any open subset $\Omega'$ of $\mathbb{R}^2$ such that $\overline{\Omega'} \subset \Omega$, there exists a constant $C$ such that for any $f \in C^3(\Omega'; \mathbb{R})$, there exists a map $\tilde{f} \in C^3(\mathbb{R}^2; \mathbb{R})$ with
\[
\tilde{f}(x) = \begin{cases} 
    f(x) & \text{for all } x \in \Omega, \\
    0 & \text{for all } x \in \mathbb{R}^2 \setminus \Omega'
\end{cases}
\]
and
\[
\|\tilde{f}\|_{C^3(\mathbb{R}^2; \mathbb{R})} \leq C \|f\|_{C^3(\Omega'; \mathbb{R})}.
\]

Proof of Proposition 3.1 First, apply Lemma 4.3 and extend $\varphi$ to a $\tilde{\varphi} \in C^3(\mathbb{R}^2; \mathbb{R})$.

Define $\tilde{\varphi}(x) = \tilde{N}(\tilde{\varphi}(-x))$. By (1.1), Lemma 4.3 and Proposition 2.2 $\tilde{w} \in C^{0,1}(\mathbb{R}^2; \mathbb{R}^2)$. Hence, for any fixed $\hat{x} \in \mathbb{R}^2$, the Cauchy problem
\[
\dot{x} = \tilde{w}(x), \quad x(0) = \hat{x}
\]/admits a unique solution $\tilde{p}_\hat{x} : \mathbb{R} \to \mathbb{R}^2$. Define
\[
T_{\hat{x}} = \sup \left\{ t \in \mathbb{R}_+^\ast : \tilde{p}_\hat{x}(0, t) \subset \Omega \right\}
\]
and $p_{\hat{x}}(t) = \tilde{p}_\hat{x}(t)$ for $t \in [0, T_{\hat{x}}]$.

By construction, the map $p_{\hat{x}}$ solves (3.1). By the standard theory of ordinary differential equations, (Q.1) and (Q.2) are proved.

We consider now (Q.3). Note that (4.5) is dissipative in $\Omega$, in the sense that $\varphi$ is a (strict) Lyapunov function for (4.5) in $\Omega$, i.e., $\varphi$ decreases along the path $t \to p_{\hat{x}}(t)$ as long as $p_{\hat{x}}(t) \in \Omega$. In fact, as long as $p_{\hat{x}}(t) \in \Omega$
\[
\frac{d}{dt} \varphi(p_{\hat{x}}(t)) = \frac{d}{dt} \varphi(p_{\hat{x}}(t)) = - \left( \partial^2 + \|\nabla \varphi(p_{\hat{x}}(t))\|^2 \right) \frac{1}{2} \left( \nabla \varphi(p_{\hat{x}}(t)) \right)^2,
\]
which is strictly negative whenever $\hat{x}$ is not a critical point. By La Salle Principle [16] Theorem 9.22, see also Lemma 9.21 and Theorem 14.17, as $t$ goes to infinity, every bounded path $p_{\hat{x}}$ that remains in $\Omega$ is attracted towards the set of equilibria, i.e., of critical points of (4.5). More precisely, setting
\[
\omega(\hat{x}) = \left\{ x \in \mathbb{R}^2 : \text{ there exists } (t_n)_{n \in \mathbb{N}} \text{ such that } \lim_{n \to \infty} t_n = \infty \text{ and } \lim_{n \to \infty} p_{\hat{x}}(t_n) = x \right\}, \quad \mathcal{E}_D = \left\{ x \in D : \nabla \varphi(x) = 0 \right\}
\]
we proved that if $x \in \omega(\hat{x}) \cap \Omega$ for a $\hat{x} \in \Omega$, then $\nabla \varphi(x) = 0$.

Note that for any $\hat{x} \in \Omega$, the path $\tilde{p}_\hat{x}$ exiting $\hat{x}$ does not intersect $\Gamma_w$. Indeed, by the boundary condition imposed along $\Gamma_w$ in (1.1)
\[
\Gamma_w = \left\{ x \in \Gamma_w : \nabla \varphi(x) = 0 \right\} \cup \left\{ x \in \Gamma_w : \nabla \varphi(x) \neq 0 \text{ and } \nabla \varphi(x) \cdot \nu(x) = 0 \right\}.
\]
The former set above is clearly invariant, both positively and negatively, with respect to (4.5), hence it can not be reached by a path \( t \to p_2(t) \) starting in \( \Omega \). The latter consists of trajectories solving (4.5) that are entirely contained in \( \Gamma_w \), since \( \omega \) is parallel to \( \Gamma_w \). As a consequence, for any \( \hat{x} \in \Omega \), either the path \( t \to p_2(t) \) crosses \( \Gamma_e \), or it stays in \( \Omega \) and approaches a point in the set \( \mathcal{E}_\Theta \), namely \( \omega(\hat{x}) \subseteq \mathcal{E}_\Theta \).

It remains to determine the behaviour of the system near the critical points in \( \mathcal{E}_\Theta \). We proceed by linearisation around \( \bar{x} \), with \( \nabla \varphi(\bar{x}) = 0 \). Denote by \( A(\bar{x}) \) the first order total derivative of \( \nabla(-\varphi) \) computed at \( \bar{x} \in \mathcal{E}_\Theta \). By direct computations,

\[
A(\bar{x}) = D\nabla(-\nabla \varphi(\bar{x})) = -\partial^{-1}D^2\varphi(\bar{x}),
\]

thanks to \( \nabla \varphi(\bar{x}) = 0 \). Recall the map \( u \) given by (4.2). Due to (4.3) and (4.6) we have

\[
A(\bar{x}) = \frac{1}{\partial u(\bar{x})} \frac{\delta}{\partial u(\bar{x})} D^2 u(\bar{x}),
\]

proving that \( A(\bar{x}) \) is symmetric and diagonalizable. By (u.6) in Lemma 4.2, \( A(\bar{u}) \) has at least one strictly positive eigenvalue, say \( \lambda_2 > 0 \). Consider now two cases, depending on the value attained by the other eigenvalue \( \lambda_1 \):

\* \( \lambda_1 \neq 0 \): Then, by Hartman-Grobman Theorem, see e.g. [10] Theorem 9.35], depending on the sign of \( \lambda_1 \), \( \bar{x} \) is either a source or a saddle. In both cases, it is an isolated point of \( \mathcal{E}_\Theta \), so that \( \bar{x} \in \omega(\bar{x}) \) implies \( \{ \bar{x} \} = \omega(\bar{x}) \), by the connectedness of \( \omega(\bar{x}) \). This is possible only if \( \lambda_1 < 0 \), i.e., \( \bar{x} \) is a saddle, and \( \bar{x} \) belongs to the stable manifold consisting of two trajectories entering \( \bar{x} \), which is a set of measure zero.

\* \( \lambda_1 = 0 \): Then, \( \bar{x} \) is not necessarily an isolated point of \( \mathcal{E}_\Theta \). We use here the result of Palmer [22] about the local central manifold, which is an invariant 1-dimensional set containing all possible critical points in a neighborhood of \( \bar{x} \). This result can be seen as a generalization of the Hartman-Grobman Theorem, and gives the instability of the central manifold, see also [3] § 4, [7] § 9.2-9.3, [10] Theorem 10.14.

Let \( B \) be the change of coordinates matrix such that \( BA(\bar{x})B^{-1} \) is diagonal, with \( A(\bar{x}) \) given in (4.6). By means of the linear change of variables \( y(t) = B (\bar{x}t - \bar{x}) \), the differential equation in (4.5) can be written as

\[
\begin{align*}
\dot{y}_1 &= f_1(y_1, y_2), \\
\dot{y}_2 &= \lambda_2 y_2 + f_2(y_1, y_2),
\end{align*}
\]

where \( f \in \mathcal{C}^2(\mathbb{R}^2; \mathbb{R}^2) \) is bounded, see Lemma 4.3 and satisfies \( f(0) = 0 \). The dependence of \( B, f \) and \( \lambda_2 \) upon \( \bar{x} \) is here neglected. We obtain from [22] that there exist a Lipschitz continuous function \( h \) and a homeomorphism \( H : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2 \), such that the graph of \( h \) is the local central manifold and the map \( z(t) = H (t, y(t)) \), with \( H(t, 0) = 0 \), solves

\[
\begin{align*}
\dot{z}_1 &= f_1(z_1, h(t, z_1)), \\
\dot{z}_2 &= \lambda_2 z_2,
\end{align*}
\]

provided \( y \) solves (4.7). As a matter of fact, \( h \) can be proved to be also \( \mathcal{C}^2 \), see [3] Proposition 4.1 or [10] Theorem 10.14.

Then, by continuity of \( H \), there exists \( r_0 > 0 \) such that if \( \| y(t) \| < r_0 \), then \( |z_2(t)| = |H_2 (t, y(t))| < |z_2(0)| \). Solving the second equation in (4.8), we obtain that for \( y(0) \) such that \( z_2(0) = H_2 (0, \bar{y}(0)) \neq 0 \), there exists \( t_* > 0 \) such that \( \| y(t) \| > r_0 \) for all \( t > t_* \). Going back to the original \( x \)-variable, for any neighborhood \( \mathcal{O} \) of \( \bar{x} \) with \( \mathcal{O} \subseteq \mathbb{R}^2 \), introduce \( W = \{ x \in \mathcal{O}; H_2 (0, B(x - \bar{x})) = 0 \} \). We have obtained that if \( \bar{x} \in \mathcal{O} \setminus W \), then \( p_2(t) \) is outside \( \mathcal{O} \) for all \( t > t_* \). Thus, \( \bar{x} \) can be attractive only for the points lying on \( W \), which is clearly a 1-dimensional manifold and has 2-dimensional Lebesgue measure equal to 0. Moreover, \( W \) as a whole is repulsive.

Therefore, \( \omega(\bar{x}) \cap W \) is non-empty only if the path passing through \( \bar{x} \) lies inside \( W \). Therefore, the 1-dimensional Lebesgue measure of \( \omega(\bar{x}) \cap W \) is 0.

Finally, for almost all \( \bar{x} \), the path \( p_2(\mathbb{R}^+) \) given by (4.5) is not attracted by \( \mathcal{E}_\Theta \), hence it has to reach the exit \( \Gamma_e \), i.e., there exists a positive finite time \( T_{\bar{x}} \) such that \( p_2(T_{\bar{x}}) \in \Gamma_e \).

Acknowledgment: The authors were supported by the INDAM-GNAMPA project *Leggi di conservazione nella modellizzazione di dinamiche di aggregazione*. The last author was partially supported by ICM, UW.

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