Probabilistic behavior of hash tables

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Abstract

We extend a result of Goldreich and Ron about estimating the collision probability of a hash function. Their estimate has a polynomial tail. We prove that when the load factor is greater than a certain constant, the estimator has a gaussian tail. As an application we find an estimate of an upper bound for the average search time in hashing with chaining, for a particular user (we allow the overall key distribution to be different from the key distribution of a particular user). The estimator has a gaussian tail.

1 Introduction

Hash tables have many applications in computer science [1], [3]. We especially mention data bases, where hash tables are used for storing values of an attribute; see chapter 12 of [4]. Following the notation of [1], a hash function is a function \( h : U \rightarrow T \), where both the domain \( U \) and the range \( T \) are finite. Traditionally, \( U \) is called the key space or the “universe”, and elements \( x \in U \) are called keys. The set \( T \) is called the table, and its elements are called the table slots. When \( h(x) = i \) we say that \( h \) hashes the key \( x \) into the slot \( i \). We shall denote by \( n \) the cardinality of \( T \) and we will simply assume that \( T = \{1, \ldots, n\} \). We assumed that \( U \) is (very much) larger than \( T \).

We assume that a probability measure \( q \) has been defined on \( U \). The probability of \( S (\subset U) \) is denoted by \( P(S) = \sum_{x \in S} q(x) \). We also put the product measure on \( U \times U \) and on \( U^m \) (for any positive integer \( m \)); using the product measure amounts to saying that in a sequence of \( m \) keys, all the keys are independent.

The probability on \( U \) induces a probability measure on \( T \): The probability that some key hashes to slot \( i \) (\( \in T \)) is \( p_i = \sum_{x \in h^{-1}(i)} q(x) = P(h^{-1}(i)) \).

If two keys \( x_1, x_2 \in U \) have the same hash value, these keys are said to collide. The collision probability of the hash function \( h \) is defined to be \( P\{(x_1, x_2) \in U \times U : h(x_1) = h(x_2)\} \) (in short-hand this is denoted by \( P(h(x_1) = h(x_2)) \)). Here we use the product measure (i.e., keys

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are “chosen independently”). A true collision corresponds to keys $x_1, x_2 \in U$ such that $x_1 \neq x_2$ and $h(x_1) = h(x_2)$.

Throughout this paper, $\|\cdot\|$ denotes euclidean norm. It is straightforward to prove the following.

**Proposition 1.1** The collision probability of $h$ is equal to $\sum_{i=1}^{n} p_i^2$ ($= \|p\|^2$).

Moreover, we always have $\sum_{i=1}^{n} p_i^2 \geq \frac{1}{n}$, and equality holds iff $p_i = \frac{1}{n}$ for all $i \in T$.

Similarly, the probability that two independently chosen keys are equal is $\sum_{x \in U} q(u)^2$. Hence, the probability of true collisions for $h$ is $\sum_{i=1}^{n} p_i^2 - \sum_{x \in U} q(u)^2$.

Note that $\sum_{x \in U} q(u)^2$ will usually be very small assuming that $U$ is very large (compared to $n$ and compared to the length $m$ of key sequences used), and assuming that the probability distribution $q$ on $U$ is not very concentrated. Therefore, the difference between the collision probability $\|p\|^2$ and the probability of true collisions is usually quite small.

In this paper we assume that collisions are resolved by some form of chaining; i.e., all the keys that are hashed into one slot are stored in that slot. For a hash table with chaining, we will simply assume that the search time (for both successful or unsuccessful search) in a slot $i$ is proportional to the number of keys stored in that slot; for simplicity, we simply identify search time in a slot and chain length in the slot.

**Notation “$k_i(x)$”:** Let $x = (x_1, \ldots, x_m)$ be a sequence of $m$ keys that are inserted into our hash table, and let $i$ be a slot ($i = 1, \ldots, n$). We let $k_i(x)$ denote the number of keys (counted with multiplicities) inserted into slot $i$. (“With multiplicities” means that if a key occurs several times in $x$ it is counted as many times as it occurs.)

Since in $k_i(x)$ we count keys with multiplicities, $k_i(x)$ is an upper bound on the number of different keys stored in slot $i$.

**Proposition 1.2** For a sequence of keys $x = (x_1, \ldots, x_m)$ that are inserted, the number of collisions between keys in $x$ is

$$\sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{2}.$$ 

The proof is straightforward. Recall that we count pairs of equal keys in the sequence $x$ as collisions. Since there are $\frac{m(m-1)}{2}$ unordered pairs of key insertions in $x$, we call

$$\sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)}$$ 

the empirical collision probability of $x$. This concept, and its relation with the collision probability $\|p\|^2$, were first studied by Goldreich and Ron [2].
In this paper we obtain two results, in the form of deviation bounds. (1) We give an estimation of the collision probability. (2) We give a deviation bound for an upper bound on the average search time.

In the second result we assume that the load factor is $> 9$ (see later for the exact assumptions). Applications in data bases often lead to hash tables with large load factor ([4], Chapter 12). We allow arbitrary key distributions.

**Estimation of the collision probability**

Our first result extends a result of Goldreich and Ron [2], namely that $\sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)}$ is a very good estimator for the collision probability $\|p\|^2$. How good the estimator is can be measured by the relative error $\left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} \cdot \frac{1}{\|p\|^2} - 1 \right|$. Their result, as well as ours, gives a deviation bound for this relative error. Goldreich and Ron [2] proved a polynomial deviation bound for the estimator $\sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)}$. Their goal was to find sublinear-time algorithms for testing expansion properties of bounded-degree graphs.

**Theorem 1.3** (Goldreich and Ron [2]). For all $\beta > 0$, $\lambda \geq 0$, if $m = n^{1/2+\beta+\lambda}$ then

$$
P \left\{ \left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} \cdot \frac{1}{\|p\|^2} - 1 \right| \leq \frac{3}{n^{3/2}} \right\} \geq 1 - \frac{4}{9n^\lambda}.
$$

We extend the theorem of Goldreich and Ron as follows:

**Theorem 1.4** For all $n > 24$, $\frac{1}{3} > \epsilon > 0$, $\delta > 0$, $s > 0$, if $m = \epsilon^{-2}n^{1+\delta}$ we have

$$
P \left\{ \left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} \cdot \frac{1}{\|p\|^2} - 1 \right| \leq \epsilon \left( 3 + \frac{6s}{n^{\delta/2}} + \frac{5s^2\epsilon}{n^\delta} \right) \right\} \geq 1 - \frac{10}{9} e^{-s^2/4}.
$$

By taking $s = 2 n^{\delta/2}$, the expression $3 + \frac{6s}{n^{\delta/2}} + \frac{5s^2\epsilon}{n^\delta}$ becomes $3 + 12 + 20 \epsilon (< 22)$; here we use $\epsilon < \frac{1}{3}$. Therefore,

**Corollary 1.5** For all $n > 24$, $\frac{1}{3} > \epsilon > 0$, $\delta > 0$, if $m = \epsilon^{-2}n^{1+\delta}$ we have

$$
P \left\{ \left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} \cdot \frac{1}{\|p\|^2} - 1 \right| \leq 22 \epsilon \right\} \geq 1 - \frac{10}{9} e^{-n^\delta}.
$$

Writing $\delta = \frac{\log C}{\log n}$, for $C > 1$, we obtain $n^\delta = C$, and $m = \epsilon^{-2}Cn$, i.e., the load factor is $L = C\epsilon^{-2}$. Therefore,

**Corollary 1.6** For all $n > 24$, $\frac{1}{3} > \epsilon > 0$, and all $m$ such that $L = \frac{m}{n} > \epsilon^{-2}$ ($> 9$) we have

$$
P \left\{ \left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} \cdot \frac{1}{\|p\|^2} - 1 \right| \leq 22 \epsilon \right\} \geq 1 - \frac{10}{9} e^{-L\epsilon^2}.
$$

3
Note that the assumptions of this Corollary impose the following relation between $L$ and $\epsilon$: $\frac{1}{3} > \epsilon > \frac{1}{\sqrt{L}}$; equivalently, $L = \frac{m}{n} > \epsilon^{-2}$ ($>9$).

To compare with the result of Goldreich and Ron, let us pick $\epsilon = n^{-\beta/2}$ in Corollary 1.5. Then $n^{1/2+\beta+\lambda} = m = \epsilon^{-2}n^{1+\delta}$ implies $\delta = \lambda - \frac{1}{2}$. Hence our Corollary becomes:

**Corollary 1.7** For all $n > 24$, $\beta > \frac{\log 3}{\log n}$, $\lambda > \frac{1}{2}$, if $m = n^{1/2+\beta+\lambda}$ we have

$$P \left\{ \left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m - 1)} \cdot \frac{1}{\|p\|^{2}} - 1 \right| \leq \frac{22}{5} n^{-\beta/2} \right\} \geq 1 - \frac{10}{9} e^{-n^{\lambda - \frac{1}{2}}}.$$

Comparing 1.7 with the theorem of Goldreich and Ron: Our theorem gives a much better deviation bound (it is exponential, as opposed to the polynomial bound of Goldreich and Ron); but it applies only when the load factor $L$ is $>9$ (whereas in the result of Goldreich and Ron, the load factor $L = n^{\beta+\lambda-1/2}$ can be arbitrarily small, depending on $n$).

**The average search time for a particular user**

In order to analyze the efficiency of a hash table one considers the overall usage statistics of the keys (over all users). By “user” we mean a person or a process. For every user we introduce a vector $v = (v_1, \ldots, v_n)$, where $v_i$ is the frequency of the user’s access (for search) to slot $i$. More precisely, $v_i$ is the number of searches at slot $i$, divided by the total number of searches in the table, for this user. Then $0 \leq v_i \leq 1$ and $\sum_{i=1}^{n} v_i = 1$. We shall call $v$ the user’s *access pattern*. Traditional analysis of the average search time assumes that the accesses pattern of a user is the same as the key distribution (see e.g., [1]).

We let $\text{AST}(v, x)$ denote the average search time for a user with access pattern $v$, under the condition that a sequence $x$ of $m$ independent keys was previously inserted into the hash table. Clearly, we have the following upper bound:

$$\text{AST}(v, x) \leq \sum_{i=1}^{n} v_i \cdot k_i(x).$$

The difference between $\text{AST}(v, x)$ and $\sum_{i=1}^{n} v_i \cdot k_i(x)$ is caused by the possibility of pseudo-collisions. Here we are only concerned with upper bounds on $\text{AST}(v, x)$, so we can use $\sum_{i=1}^{n} v_i \cdot k_i(x)$.

We write $m$ as $m = Ln$, where $L$ is called the *load factor*. We do not assume that $L$ is a constant. Applying Theorem 1.4 we show

**Corollary 1.8** For all $n > 24$, $s > 0$, $L > 9$, and $m = Ln$ we have

$$P \left\{ \text{AST}(v, x) \leq Ln\|v\|\|p\| \sqrt{1 + \frac{3 + 6s}{\sqrt{L}} + \frac{5s^2}{L} + 1} \right\} \geq 1 - \frac{10}{9} e^{-s^2/4}.$$

Noting that $\sqrt{1 + \frac{3 + 6s}{\sqrt{L}} + \frac{5s^2}{L}} < 1 + \frac{4s}{\sqrt{L}}$ and letting $\epsilon = \frac{s}{2\sqrt{L}}$ we obtain
Corollary 1.9  For all  $n > 24$,  $\epsilon > 0$, $L > 9$, and $m = Ln$ we have
\[
P \{ \text{AST}(v, x) \leq Ln \|v\| \|p\| (1 + 8\epsilon) + 1 \} \geq 1 - \frac{10}{9} e^{-L\epsilon^2}.
\]

One notices that the probability bound is only interesting when $L$ is significantly larger than $\epsilon^2$. Also, the error bound is interesting only when $\epsilon$ is less than 1/8; this means that the load factor has to be at least 100 for our results to be interesting. In that sense, the results are theoretical, and show just what type of behavior to expect, up to big-O.

In [1] (chapt. 12, exercise 12-3) the expected search time (for every user) was found to be $\Theta(L)$, under the assumption that both the key distribution and the distribution of user’s accesses are uniform. Our Corollary implies that if $\|p\|^2 = \Theta(\frac{1}{n})$ and $\|v\|^2 = \Theta(\frac{1}{n})$ (which is much more relaxed than the assumption of a uniform distribution), then with exponentially high probability, the average search time is $O(L)$ for a user with access pattern $v$.

Example 1

Suppose that a hash table, designed for a certain population of users, has collision probability $\|p\| \leq \frac{c}{\sqrt{n}}$ (for the overall population of users); $c$ is a positive constant. The keys in the hash table are independent random samples. Now consider an individual user who accesses a subset of cardinality $\alpha n$ (where $0 < \alpha \leq 1$) of the $n$ slots of the hash table, with uniform probability $\frac{1}{\alpha n}$, and who does not access the other $(1 - \alpha)n$ slots of the hash table at all (i.e., those slots have probability 0 for this user). Then the question is: What is the average search time for this user and this table, and what is the deviation bound?

Since the user accesses a fraction $\alpha$ of the slots uniformly, we have $\|v\| = \frac{1}{\sqrt{\alpha n}}$. By Corollary 1.9
\[
P \{ \text{AST}(v, x) \leq \frac{cL}{\sqrt{\alpha}} (1 + 8\epsilon) + 1 \} \geq 1 - \frac{10}{9} e^{-L\epsilon^2}.
\]

So, the average search time is at most $1 + \frac{cL}{\sqrt{\alpha}} (1 + 8\epsilon)$, with smaller error bound (namely $\frac{cL}{\sqrt{\alpha}} 8\epsilon$), and with probability close to 1 (namely $1 - \frac{10}{9} e^{-L\epsilon^2}$).

One observes that when the fraction $\alpha$ of the table used by the user becomes smaller, the upper bound on the average search time for this user increases, as does the error bound. This is not surprising; hashing works best when the keys are spread over the table as evenly as possible. Interestingly, our probability bound does not depend on $\alpha$.

Some possible numerical values: For $c = 5$,  $\alpha = 0.1$, $\epsilon = 0.05$, $L = 1000$, we get $\text{AST}(v, x) \leq 15811 \pm 6324$, with probability at least $1 - \frac{10}{9} e^{-L\epsilon^2} = 0.909$. For $c = 5$,  $\alpha = 0.1$, $\epsilon = 0.05$, $L = 10000$, we get $\text{AST}(v, x) \leq (1.58 \pm 0.64) \cdot 10^5$, with probability at least $1 - 1.54 \cdot 10^{-11}$.

Example 2

Let us consider the situation in which a query consists of two subqueries, $Q_1$ and $Q_2$. This happens very commonly (e.g., in a “three-tier architecture”); see [4]. The two subqueries can be viewed as two users with access patterns $v^{(1)}$ and $v^{(2)}$. Assume, for this example, that each
of \( Q_1 \) and \( Q_2 \) behaves like the user in Example 1 above. In particular, for \( Q_i \) \((i = 1, 2)\) we have 
\[
\|v^{(i)}\| = \frac{1}{\sqrt{\alpha_i n}},
\]
and
\[
P\{\text{AST}_i(v^{(i)}, x) \leq \frac{c_L}{\sqrt{\alpha_i}} (1 + 8\epsilon) + 1 \} \geq 1 - \frac{10}{9} e^{-L^2}.
\]
Hence, for the combined query the average search time is a weighted sum
\[
\text{AST} = w_1 \cdot \text{AST}_1 + w_2 \cdot \text{AST}_2, \quad \text{with} \ w_1 + w_2 = 1.
\]
Let 
\[
a_i = \frac{c_L}{\sqrt{\alpha_i}} (1 + 8\epsilon) + 1.
\]
Then
\[
P\{\text{AST} \leq w_1 a_1 + w_2 a_2 \} \geq P\{\text{AST}_1 \leq \max\{a_1, a_2\}, \ \text{AST}_2 \leq \max\{a_1, a_2\}\}
\geq 1 - 2 \frac{10}{9} e^{-L^2}.
\]
Therefore, the average search time \( \text{AST}(v^{(1)}, v^{(2)}, x) \) of the combined query satisfies
\[
P\{\text{AST}(v^{(1)}, v^{(2)}, x) \leq \frac{c_L}{\sqrt{\min\{\alpha_1, \alpha_2\}}} (1 + 8\epsilon) + 1 \} \geq 1 - \frac{20}{9} e^{-L^2}.
\]
Hence, when the load factor is large (compared to \( \epsilon^2 \)) we obtain a very reliable upper bound on the average search time for the combined query. The knowledge of this upper bound enables various processes (that wait for the completion of this query) to be scheduled in a predictable way.

The constants in our results are rather large. This is due to the generality of our results. In a precise practical situation, our results could be used for the format of the probabilistic behavior, with constants to be determined empirically.

The next section contains the proofs of our theorems.

2 Proofs

2.1 A deviation bound for the empirical collision probability: Proof of Theorem 1.4

Our main technique will be Talagrand’s isoperimetric theory, developed by Talagrand in the mid 1990s [6]. It has had a profound impact on the probabilistic theory of combinatorial optimization [5] (see Sections 6 - 13 of [6] and chapter 6 of [5]).

Let \((\Omega, \mu)\) be a probability space, and let \((\Omega^m, \mu^m)\) be the product space. For \(x \in \Omega^m\) and \(A \subset \Omega^m\), Talagrand’s convex distance \(d_T(x, A)\) is defined by
\[
d_T(x, A) = \sup_{\alpha} \left\{ z_{\alpha} \right\} = \inf_{y \in A} \left\{ \sum_{j=1}^{m} \alpha_i \cdot 1(x_j \neq y_j) : \alpha = (\alpha_1, \ldots, \alpha_m), \sum_{j=1}^{m} \alpha_j^2 \leq 1 \} ,
\]
where \(x = (x_1, \ldots, x_m), \ y = (y_1, \ldots, y_m)\). Here, \(1(x_i \neq y_i) = 1\) if \(x_i \neq y_i\), and it is 0 otherwise.
Theorem 2.1 (Talagrand 1995) For every \( A \subset \Omega^m \) with \( \mu^m(A) > 0 \), we have
\[
\int_{\Omega^m} \exp \left( \frac{1}{4} d_T(x, A)^2 \right) \, d\mu^m(x) \leq \frac{1}{\mu^m(A)},
\]
and consequently, we have for all \( s > 0 \),
\[
\Pr \{ d_T(x, A) \geq s \} \leq \frac{e^{-s^2/4}}{\mu^m(A)}.
\]

To apply Talagrand’s theorem to our situation we define a set \( A \subset U^m \) by
\[
A = \left\{ y \in U^m : \sum_{i=1}^{n} k_i(y)(k_i(y) - 1) \cdot \frac{1}{\|p\|^2} - 1 \leq 3\epsilon \right\}.
\]

Lemma 2.2 For all \( n > 24 \) we have \( \Pr(A) \geq \frac{9}{10} \).

Proof. Recall that \( m = \epsilon^{-2} n^{1+\delta} \) with \( \frac{1}{3} > \epsilon > 0 \), \( \delta > 0 \). Letting \( \beta = \frac{-2\log \epsilon}{\log n} \) and \( \lambda = 1/2 + \delta \), we rewrite \( m \) as \( n^{1/2+\beta+\lambda} \). Then the lemma follows from Theorem of Goldreich and Ron.

For every \( s > 0 \) we define a set \( C_s \subset U^m \) by
\[
C_s = \{ x \in U^m : d_T(x, A) < s \}.
\]

By Theorem 2.1 and Lemma 2.2 we have for all \( n > 24 \) and all \( s > 0 \)
\[
\Pr(C_s) \geq 1 - \frac{10}{9} e^{-s^2/4}. \tag{1}
\]

Lemma 2.3 For every \( x = (x_1, \ldots, x_m) \in C_s \) there is \( y = (y_1, \ldots, y_m) \in A \) such that
\[
\sum_{j=1}^{m} 1(x_j \neq y_j) \leq sm^{1/2}.
\]

Proof. Assume, by contradiction, that there is \( x \in C_s \) such that for all \( y \in A \),
\[
\sum_{j=1}^{m} 1(x_j \neq y_j) > sm^{1/2}.
\]
Now, if we take \( \alpha = (\alpha_1, \ldots, \alpha_m) = (m^{-1/2}, \ldots, m^{-1/2}) \) in the definition of the Talagrand distance \( d_T \), the inequality above implies \( d_T(x, A_1) \geq s \). But since \( x \in C_s \), we also have \( d_T(x, A_1) < s \), a contradiction.

Recall that for any \( x = (x_1, \ldots, x_m) \), \( y = (y_1, \ldots, y_m) \in U^m \), we defined \( k_i(x) \) (resp. \( k_i(y) \)) to be the number of the keys (with multiplicity) that are hashed into the slot \( i \) for input sequence \( x \), resp. \( y \). We define integers \( s_i \) (\( 1 \leq i \leq n \)) by
\[
k_i(x) = k_i(y) + s_i.
\]
Lemma 2.4  For all $x, y \in U^n$,
\[
\sum_{i=1}^{n} |s_i| \leq 2 \sum_{j=1}^{m} 1(x_j \neq y_j).
\]

Proof. We prove the lemma by induction on $\sum_{i=1}^{m} 1(x_i \neq y_i)$.

(0) $\sum_{j=1}^{m} 1(x_j \neq y_j) = 0$:
Then we have $x_j = y_j$ for all $j = 1, \ldots, m$, and hence, $k_i(x) = k_i(y)$ for all $i = 1, \ldots, n$. Thus, we have $\sum_{i=1}^{n} |s_i| = 0$, finishing the base case.

(Inductive step) Assume $\sum_{j=1}^{m} 1(x_j \neq y_j) > 0$:
Without loss of generality we assume that $x_m \neq y_m$. Now, consider $\bar{x} = (x_1, \ldots, x_{m-1}, y_m)$. We write $k_i(\bar{x}) = k_i(y) + s_i$ for $i = 1, \ldots, n$. By the induction hypothesis we have
\[
\sum_{i=1}^{n} |\bar{s}_i| \leq 2 \sum_{j=1}^{m} 1(\bar{x}_j \neq y_j). \tag{2}
\]
Since $x$ differs from $\bar{x}$ only in its last component, we either have $h(x_m) = h(y_m)$, in which case $\bar{s}_i = s_i$ for all $i = 1, \ldots, n$. Or we have $h(x_m) \neq h(y_m)$; let $i_1 = h(x_m)$ and $i_2 = h(y_m)$. Then $\bar{s}_{i_1} = s_{i_1} + 1$, $\bar{s}_{i_2} = s_{i_2} - 1$, and $\bar{s}_i = s_i$ for all $i \in \{1, \ldots, n\} \setminus \{i_1, i_2\}$. In both cases,
\[
\left| \sum_{i=1}^{n} |\bar{s}_i| - \sum_{i=1}^{n} |s_i| \right| \leq 2. \tag{3}
\]
On the other hand,
\[
\sum_{j=1}^{m} 1(x_j \neq y_j) = \sum_{j=1}^{m} 1(\bar{x}_j \neq y_j) + 1.
\]
Combining this, (2), and (3), completes the proof for the inductive step. \qed

Lemma 2.5  For every $x \in C_s$ there is $y \in A$ such that for all $n > 24$, $0 < \epsilon < 1/3$, $s > 0$, and $m = \epsilon^{-2} n^{1+\delta}$, we have
\[
\left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} - \sum_{i=1}^{n} \frac{k_i(y)(k_i(y) - 1)}{m(m-1)} \right| \leq \epsilon \|p\|^2 \left( \frac{6s}{n^{\delta/2}} + \frac{5s^2 \epsilon}{n^{3\delta}} \right).
\]

Proof. For any fixed $x \in C_s$ we take $y \in A$ according to Lemma 2.3. That is,
\[
\sum_{j=1}^{m} 1(x_j \neq y_j) \leq sm^{1/2}. \tag{4}
\]
As in the proof for Lemma 2.4 we use the notation $k_i(x)$, $k_i(y)$, and $s_i$ ($i = 1, \ldots, n$). We will leave the common denominator $m(m-1)$ out of the computations until the end:
\[ | \sum_{i=1}^{n} k_i(x)(k_i(x) - 1) - \sum_{i=1}^{n} k_i(y)(k_i(y) - 1)| \]
\[ = | \sum_{i=1}^{n} (k_i(y) + s_i)(k_i(y) + s_i - 1) - \sum_{i=1}^{n} k_i(y)(k_i(y) - 1)| \]
\[ = | \sum_{1 \leq i \leq n, k_i(y) \geq 1} [(k_i(y) + s_i)(k_i(y) + s_i - 1) - k_i(y)(k_i(y) - 1)] \]
\[ + \sum_{1 \leq i \leq n, k_i(y) = 0} s_i(s_i - 1)| \]
\[ \leq \sum_{1 \leq i \leq n, k_i(y) \geq 1} 2 |s_i| (k_i(y) - 1) + \sum_{1 \leq i \leq n, k_i(y) \geq 1} (s_i^2 + |s_i|) + | \sum_{1 \leq i \leq n, k_i(y) = 0} s_i(s_i - 1)| \]
\[ \leq \sum_{1 \leq i \leq n, k_i(y) \geq 1} 2 |s_i| (k_i(y) - 1) + \sum_{i=1}^{n} (s_i^2 + |s_i|) \]

By the Cauchy-Schwarz inequality, this is bounded by
\[ \leq 2 (\sum_{1 \leq i \leq n, k_i(y) \geq 1} s_i^2)^{1/2} (\sum_{1 \leq i \leq n, k_i(y) \geq 1} (k_i(y) - 1)^2)^{1/2} + \sum_{i=1}^{n} (s_i^2 + |s_i|) \]
\[ \leq 2 (\sum_{i=1}^{n} s_i^2)^{1/2} (\sum_{i=1}^{n} k_i(y)(k_i(y) - 1)^2)^{1/2} + \sum_{i=1}^{n} (s_i^2 + |s_i|) \]

By Lemma 2.4 and 1 we have
\[ \sum_{i=1}^{n} s_i^2 \leq \left( \sum_{i=1}^{n} |s_i| \right)^2 \leq \left( 2 \sum_{j=1}^{m} |x_j \neq y_j| \right)^2 \leq 4s^2m \]  \hspace{1cm} (5)

Since \( y \in A \) we have
\[ \sum_{i=1}^{n} k_i(y)(k_i(y) - 1) \leq \|p\|^2 (1 + 3\epsilon) \]

Hence, by all the above:
\[ \left| \sum_{i=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} - \sum_{i=1}^{n} \frac{k_i(y)(k_i(y) - 1)}{m(m-1)} \right| \]
\[ \leq \frac{4s}{(m-1)^{1/2}} \cdot \|p\| (1 + 3\epsilon)^{1/2} + \frac{4s^2}{m-1} + \frac{2s}{m^{1/2}(m-1)} \]

By calculating, and using the fact that \( \|p\|^2 \geq \frac{1}{n}, 0 < \epsilon < 1/3 \), and \( m = \epsilon^{-2} n^{1+\delta} \), we find the following upper bound for \( \left| \sum_{y=1}^{n} \frac{k_i(x)(k_i(x) - 1)}{m(m-1)} - \sum_{i=1}^{n} \frac{k_i(y)(k_i(y) - 1)}{m(m-1)} \right| : \)
\[ \frac{s\epsilon}{n^{\delta/2}} \|p\|^2 \frac{4(1 + 3\epsilon)^{1/2} n^{1/2}}{(n - \epsilon^2 n^{-\delta})^{1/2}} + \frac{s\epsilon}{n^{\delta/2}} \|p\|^2 \frac{2\epsilon^2 n}{n^{1/2}(n^{1+\delta} - \epsilon^2)} + \frac{s^2\epsilon^2}{n^{\delta/2}} \|p\|^2 \frac{4n}{n^{1+\delta} - \epsilon^2} \]

Combining this and using \( n > 24 \) we obtain the upper bound
\[ \epsilon \|p\|^2 \left( \frac{6s}{n^{\delta/2}} + \frac{5s^2\epsilon}{n^\delta} \right) \]

\[ \square \]

**Proof of Theorem 1.4** The theorem follows from the definition of \( A \), inequality (1), and Lemma 2.5. \( \square \)
2.2 Average search time for a particular user

Proof of Corollary 1.8. Recall that the average search time $\text{AST}(v, x)$ is bounded from above by $\sum_{i=1}^{n} v_i \cdot k_i(x)$. In Theorem 1.4 let us write $m = L_1 L_2 n$, and choose

$$\epsilon = \frac{1}{\sqrt{L_1}} \quad \text{and} \quad \delta = \frac{\log L_2}{\log n}.$$ 

Note that for all $i$,

$$k_i(x) - 1 \leq \sqrt{k_i(x)(k_i(x) - 1)}$$

since the left side is 0 when $k_i(x) = 0$ or 1. Therefore,

$$\text{AST}(x, v) \leq \sum_{i=1}^{n} v_i \cdot k_i(x) = \sum_{i=1}^{n} v_i(k_i(x) - 1) + 1 \leq \sqrt{\sum_{i=1}^{n} v_i^2} \sqrt{\sum_{i=1}^{n} (k_i(x) - 1)^2} + 1$$

$$\leq \sqrt{\sum_{i=1}^{n} v_i^2} \sqrt{\sum_{i=1}^{n} k_i(x)(k_i(x) - 1)} + 1 \leq \|v\| \|p\| m(m-1) \sqrt{\sum_{i=1}^{n} \frac{k_i(x)(k_i(x)-1)}{m(m-1)}} \frac{1}{\|p\|^2}.$$ 

The corollary follows from this and Theorem 1.3. 

Remark. Our proof method depends crucially on Talagrand’s theorem. Many readers, more familiar with techniques like the Chernoff bound, or more generally, the Hoeffding inequality for martingale differences (from which the Chernoff bound follows directly), may wonder whether these simpler techniques don’t work here. In order to apply Hoeffding’s inequality we could view $\sum_{i=1}^{n} v_i \cdot k_i(x)$ as a weighted sum of the random variables $k_i(x)$; to apply Hoeffding one needs to bound $|k_i(x)|$, but we don’t have good bounds a priori; finding good bounds on $|k_i(x)|$ seems harder and less promising than our method, based on Talagrand’s theorem. See, e.g., Michael Steele’s book [5], which discusses the advantages of applying Talagrand’s theorem at length.

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