We compute the dynamical Green function and density-density correlation in the Calogero-Sutherland model for all integer values of the coupling constant. An interpretation of the intermediate states in terms of quasi-particles is found.
1. Introduction

The Calogero-Sutherland Hamiltonian was defined in [1] as a model of particles on a circle interacting with a long range potential. The positions of the particles are denoted by \( x_i, 1 \leq i \leq N, 0 \leq x_i \leq L \). The total momentum and the hamiltonian which give their dynamics are respectively given by:

\[
\hat{P} = \sum_{j=1}^{N} \frac{1}{i} \frac{d}{dx_i} \tag{1.1}
\]

\[
\hat{H} = -\sum_{j=1}^{N} \frac{1}{2} \frac{d^2}{dx_i^2} + \beta (\beta - 1) \frac{\pi^2}{L^2} \sum_{i<j} \frac{1}{\sin^2(\pi(x_i - x_j)/L)} \tag{1.2}
\]

\( \beta \) is a positive constant which we shall take to be an integer. The wave functions solutions of the equation \( \hat{H}\Psi = E\Psi \) have the following structure

\[
\Psi(x) = \Delta^\beta(x)\Phi(x) \tag{1.3}
\]

where

\[
\Delta(x) = \prod_{i<j} \sin \left( \frac{\pi(x_i - x_j)}{L} \right) \tag{1.4}
\]

and \( \Phi(x) \) denotes a polynomial in the variables \( z_j = e^{i2\pi x_j/L} \) and \( z_j^{-1} \) symmetric under the permutations of the indices. An eigenstate of \( \hat{H} \) is characterized by the highest weight monomial of \( \Psi(x) \),

\[
\exp \left( \frac{2\pi i}{L} \sum_{j=1}^{N} k_j x_j \right) \tag{1.5}
\]

where the \( k_j \)'s define an increasing sequence of (not necessarily positive) integers subject to the constraint

\[
k_{j+1} - k_j \geq \beta \tag{1.6}
\]

The eigenvalues of \( \hat{P} \) and \( \hat{H} \) on this eigenstate are given by:

\[
\hat{P}|k_1...k_N> = \sum_{j=1}^{N} \frac{2\pi k_j}{L} |k_1...k_N> \tag{1.7}
\]

\[
\hat{H}|k_1...k_N> = \sum_{j=1}^{N} \frac{1}{2} \left( \frac{2\pi k_j}{L} \right)^2 |k_1...k_N> \tag{1.7}
\]

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1. For simplicity we take \( \beta(N - 1) \) to be an even integer so that the ground state is non-degenerate.
The wave function for these states will be described in detail in the next section.

The results reported here are the expression of the retarded Green function and the density-density dynamical correlation function at arbitrary integer value of the coupling constant $\beta$. Here, we give the result in the thermodynamic limit, $N \to \infty, L \to \infty$, keeping the density $\rho = N/L$ fixed. The exact expressions for a finite number of particles are given in section 3 and 4.

The retarded Green function is defined by:

$$
\eta(x,t; x', t') = \langle 0 | \Psi(x,t) \Psi(x', t') | 0 \rangle
$$

$$
= N^{-1} \rho \left( \prod_{k=1}^{N} \int_{0}^{L} dy_k \right) \prod_{i} \delta^{(N-1)/2} \Delta^{\beta}(x', y_k) e^{-i(\tilde{H} - E_0)(t' - t)} \Delta^{\beta}(x, y_k) \prod_{l} z^{-\beta(N-1)/2}
$$

(1.8)

with

$$
\mathcal{N} = \int_{0}^{L} dx \prod_{k=1}^{N} dy_k \Delta^{2\beta}(x, y_k)
$$

(1.9)

and it is equal to:

$$
\eta(x, t, x', t') = \frac{\rho}{2} \left( \prod_{i=1}^{N} \frac{\Gamma(1 + 1/\beta)}{\Gamma(i/\beta)^2} \int_{-1}^{1} dv_i \right) f_{\beta}(v_i) e^{i(Q(x-x') - E(t-t'))}
$$

(1.10)

$$
f_{\beta}(v_i) = \prod_{i=1}^{N} (1 - v_i^2)^{-1+1/\beta} \times \prod_{i<j} |v_i - v_j|^{\frac{2}{\beta}}
$$

(1.11)

The energy and momentum are given by

$$
E = -\rho^2 \pi^2 \beta \sum_{i=1}^{N} v_i^2, \quad Q = \pi \rho \sum_{i=1}^{N} v_i
$$

(1.12)

The case of physical interest is for $\beta = 2$, $\rho = 1/2$, for which $\eta$ gives the spin-spin correlation function of the Haldane-Shastry chain. In this case, the spin operator creates 2 holes which propagate freely with velocities $v_i$.

The density-density correlation function is given by:

$$
g_{\beta}(x, t) = \langle 0 | \rho(x, t) \rho(0, 0) | 0 \rangle
$$

(1.13)
where \( \rho(x) = \sum_{i=1}^{N} \delta(x - x_i) \). We have obtained the following expression in the thermodynamic limit:

\[
A(\beta) \int_1^\infty dw \int_{-1}^1 dv_1 \cdots \int_{-1}^1 dv_\beta \frac{(w^2 - 1)^{\beta-1} \prod_{i=1}^\beta (1 - v_i^2)^{-1+1/\beta} \times}{\prod_{i<j} |v_i - v_j|^{2/\beta} \times \frac{(\sum_i v_i - \beta w)^2}{\prod_{i=1}^\beta (v_i - w)^2} e^{-\frac{1}{\beta} Et} \cos(Qx)}
\]

(1.14)

with the constant:

\[
A(\beta) = \frac{\rho^2}{2\beta} \prod_i \frac{\Gamma(1 + 1/\beta)}{\Gamma(i/\beta)^2}
\]

(1.15)

and the energy and momentum:

\[
E = \frac{\pi^2 \rho^2 \beta}{2} \left( \beta w^2 - \sum_{i=1}^{\beta} v_i^2 \right), \quad Q = \pi \rho \left( \sum_{i=1}^{\beta} v_i - \beta w \right).
\]

(1.16)

The physical interpretation of this result is that the density operator \( \rho(x) \) creates excitations composed of \( \beta \) holes with velocity \( v_i \) and a particle with velocity \( w \) and mass \( \beta \) upon acting on the vacuum.

The method followed here to compute a correlation function consists in inserting a complete set of eigenstates in between the two operators evaluated at time \( t \) and at time 0. Before going to the precise calculation, let us determine the states which propagate and give their interpretation as quasiparticles. In the absence of second quantization our argument is heuristic but in complete agreement with the exact results.

Let us describe the ground state as a filled Fermi sea occupied with particles of momenta \( k_i \) subject to the constraint (1.6). Then, in the ground state we have:

\[
k_i = \beta \left( i - \frac{N + 1}{2} \right), \quad 1 \leq i \leq N
\]

(1.17)

and the Fermi momentum is given by \( k_F = \beta \frac{N - 1}{2} \). This is described in figure 1 for \( \beta = 2, N = 5 \). The ones stand for the occupied momenta and the zeroes for the empty momenta. Let us now consider the case of the Green function. The action of the operator \( \Psi(x) \) is to remove a particle from the Fermi sea without changing the Fermi level. Therefore, the intermediate states which can propagate are given by all the possible arrangements of
Fig. 1: a) The ground state, b) an excitation created by $\Psi(x)$, c) an excitation created by $\rho(x)$.

$N - 1$ momenta $k_i$ inside the Fermi sea ($-k_F \leq k_i \leq k_F$) and subject to the constraint (1.6).

A typical configuration is shown in figure 1 b) for $\beta = 2$. It is completely characterized by the positions of the $\beta$ sequences of $\beta$ consecutive zeroes (denoted by crosses on the figure). We shall see that these excitations can be interpreted as $\beta$ holes whose dynamics is governed by a Calogero-Sutherland Hamiltonian at coupling $1/\beta$. In the case of the Haldane-Shastry chain ($\beta = 2$), this picture coincides precisely with the low excitations found for the XXX chain [3]. This is not surprising since the spectrum of the Haldane-Shastry chain can be obtained as a limit of the Bethe-Ansatz spectrum [4].

The density operator $\rho(x) = \Psi^\dagger(x)\Psi(x)$ can be handled similarly. $\Psi^\dagger\Psi(x)$ moves a particle out of the Fermi-sea. Thus, the intermediate states can be described with $\beta$ holes propagating inside the Fermi sea and 1 particle propagating outside (Fig. 1 c).

The result for the Green function at $\beta = 2$ is in accordance with the expression previously obtained by Haldane and Zirnbauer [5]. If $\beta > 2$, we get the expressions conjectured by Haldane for the Green function [5] and which have been derived at equal time by Forrester [7]. The density-density correlation was computed for the values $\beta = 1/2, 1, 2$ by Simons, Lee and Altshuler [8] and conjectured by Haldane for all rational values of $\beta$ [9]. Here we prove the conjecture for $\beta$ positive integer. Forrester has obtained the equal time limit of this result, his interpretation however is different; he computes $<0|\Psi^\dagger\Psi^\dagger\Psi\Psi|0>$ instead of $<0|\Psi^\dagger\Psi\Psi^\dagger\Psi|0>$. Thus, he obtains $2\beta$ holes propagating in the intermediate states where we obtain $\beta$ holes and a particle.

Although there are conjectured expressions for the density-density correlation for rational $\beta$ [5, 7], we did not manage to prove them using these techniques. Also a generalisation to the $Su(N)$ spin chains and Calogero-Sutherland models [10] could be done along the
same line, but then the properties of the Jack polynomials studied in [11], [12] must be
generalized to the wave functions of these systems.

2. Jack Symmetric Functions

2.1. Generalities.

Before introducing Jack symmetric functions we shall define a few quantities needed in
the following [13]. We denote a partition \( \{ \lambda \} \) of the integer \( |\lambda| \) by its parts \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \)
and such a partition can be encoded in a Young tableau (Fig. 2). We define a partial
ordering by \( \{ \lambda \} \geq \{ \mu \} \) if \( |\lambda| = |\mu| \) and \( \lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i \) for all \( i \geq 1 \).

![Young tableau](image)

**Fig. 2:** Young tableau

The conjugate of a partition is found by interchanging the rows and columns in the
corresponding Young tableau, it is denoted by primed letters (eg. \( \{ \lambda' \} \)). The number of
parts of a partition is called the length, \( l(\lambda) \), and is equal to \( \lambda'_1 \). The squares in the tableau
are given the coordinates \( (i,j), 1 \leq j \leq \lambda_i \). The coordinate \( j \) increases towards the right
starting from 1, the other coordinate, \( i \), increases going downwards. For each of these
squares, we define the arm-length, \( a(s) \), leg-length, \( l(s) \), arm-colength, \( a'(s) \), leg-colength,
\( l'(s) \), in the following way :

\[
\begin{align*}
    a(s) &= \lambda_i - j, & a'(s) &= j - 1 \\
    l(s) &= \lambda'_j - i, & l'(s) &= i - 1.
\end{align*}
\]

These are the number of squares right, down, left and up a square \( s \equiv (i,j) \) in a Young
tableau. The **hook-length** of a square is \( a(s) + l(s) + 1 \) and is the number of squares below
and to the right of it plus itself.
We denote by $\Lambda_n$ the ring of symmetric polynomials in $n$ indeterminates $z_1, \ldots, z_n$. There is a natural inclusion of $\Lambda_n$ in $\Lambda_{n+1}$ which consists in equating to zero the last variable in $\Lambda_{n+1}$. It is convenient to consider the formal limit $\Lambda$ of $\Lambda_n$ where we let $n$ go to infinity. Natural basis of $\Lambda$ are given by the elementary symmetric function in infinitely many variables. Let us here define the symmetric functions we use in the following.

The power-sums are defined as:

$$ p_i = \sum_k z_k^i $$

and we can use them to obtain a basis of $\Lambda$ formed by the power sum symmetric functions:

$$ p_{\{\lambda\}} = p_{\lambda_1} p_{\lambda_2} \cdots $$

(2.1)

To a partition $\{\lambda\}$ we can also associate the monomial symmetric function $m_{\{\lambda\}}$ in these indeterminates. It is the sum of all distinct permutations of the monomial $z_1^{\lambda_1} z_2^{\lambda_2} \cdots$. For example $m_{\{3,1\}} = \sum z_i^3 z_j$ for $i \neq j$. If the number of indeterminates is less than the length of the partition, the corresponding function $m_{\{\lambda\}}(z_i)$ is equal to zero. A symmetric function $f \in \Lambda$ can be expressed in terms of linear combinations of $m_{\{\lambda\}}$’s. Another basis of $\Lambda_n$ is given by the Schur functions, $s_{\{\lambda\}}$, homogeneous polynomials of degree $|\lambda|$. It is defined as:

$$ s_{\{\lambda\}} = \frac{\det(z_i^{\lambda_j+n-j})}{\det(z_i^{n-j})} $$

(2.2)

where the denominator is recognized to be the Vandermonde determinant. It can be seen that the definition of $s_{\{\lambda\}}$ is compatible with the inclusion of $\Lambda_n$ into $\Lambda_{n+1}$ and therefore the $s_{\{\lambda\}}$ can be defined as a basis of $\Lambda$.

2.2. Definition and properties of Jack Symmetric functions.

To a partition $\{\lambda\}$ having the part 1 with multiplicity $m_1$, the part 2 with $m_2$, etc... we associate the number: $z_{\{\lambda\}}$

$$ z_{\{\lambda\}} = 1^{m_1} m_1! 2^{m_2} m_2! \cdots $$

(2.3)

We define a scalar product on $\Lambda$ given by:

$$ < p_{\{\lambda\}}, p_{\{\mu\}} > = \delta_{\{\lambda\},\{\mu\}} z_{\{\lambda\}} \beta^{-l(\lambda)} $$

(2.4)
Then the Jack symmetric functions are symmetric functions in \( \Lambda \), uniquely defined by the following conditions:

\[
\begin{align*}
&i) \quad J_{\{\lambda\}}(z_i; \beta) = m_{\{\lambda\}} + \sum_{\{\mu\} \subset \{\lambda\}} v_{\{\mu\}, \{\lambda\}}(\beta) \ m_{\{\mu\}} \\
&ii) \quad < J_{\{\lambda\}}, J_{\{\mu\}} > = 0, \text{ if } \{\lambda\} \neq \{\mu\}
\end{align*}
\]

(2.5)

where the value of \( \beta \) is the same in the scalar product and in the Jack polynomial. For \( \beta = 1 \) the Jack symmetric functions coincide with the Schur functions. It follows from \( i) \) that \( J_{\{\lambda\}}(z_1, \ldots, z_n; \beta) = 0 \) if \( n < l(\lambda) \). Up to now there is no explicit expression for these functions. The first ones are equal to:

\[
\begin{align*}
\{0\} = 1 \\
\{1\} &= m_{\{1\}} \\
\{2\} &= m_{\{2\}} + \frac{2\beta}{\beta + 1} m_{\{1^2\}} \\
\{3\} &= m_{\{3\}} + \frac{3}{\beta + 2} m_{\{2,1\}} + \frac{6\beta}{(\beta + 1)(\beta + 2)} m_{\{1^3\}} \\
\{2,1\} &= m_{\{2,1\}} + \frac{6\beta}{2\beta + 1} m_{\{1^3\}}
\end{align*}
\]

When the number of indeterminates is equal to \( N \), they are the eigenfunctions of a differential operator, \( \tilde{H}(\beta) \), which is a gauge transformed version of the Calogero-Sutherland Hamiltonian:

\[
\tilde{H} = \sum_{i=1}^{N} (z_i \frac{\partial}{\partial z_i})^2 + \beta \sum_{i \neq j} \frac{z_i + z_j}{z_i - z_j} \frac{\partial}{\partial z_i}
\]

(2.6)

with the eigenvalue given by:

\[
e_{\{\lambda\}}(\beta) = \sum_{i} (\lambda_i + \beta(N - i))^2
\]

(2.7)

This follows from the fact that \( \tilde{H} \) is hermitian and triangular in the \( m_{\{\lambda\}} \) basis [12]. It is then not difficult to show that if one sets \( z_k = e^{i2\pi x_k/L} \), the eigenfunctions of \( \tilde{H} \), \( \tilde{P} \) defined in (1.1) and (1.2) are given by:

\[
\psi_{\{k\}}(x) = J_{\{\lambda\}}(z_i) \left( \prod_{i=1}^{N} z_i \right)^{q-(N-1)\beta/2} \times \prod_{i<j} (z_i - z_j)^{\beta}
\]

(2.8)
The momenta \( k_i \) defined in (1.7) are related to the partition \( \{ \lambda \} \) and to \( q \) (positive or negative integer) by the formula:

\[
k_i = \lambda_{N-i+1} + q + \beta \left( i - \frac{N+1}{2} \right) \tag{2.9}
\]

Using the explicit form of the Hamiltonian and the corresponding eigenvalue, it is easy to see that:

\[
\left( \prod_{i=1}^{N} z_i \right) J_{\{\lambda\}}(z_1...z_N; \beta) = J_{\{\lambda+1\}}(z_1...z_N; \beta) \tag{2.10}
\]

where by \( \{ \lambda + 1 \} \) we understand the partition in which we have added one column of \( N \) boxes. So, to avoid double counting of states in (2.8), \( \{ \lambda \} \) must be such that \( l(\lambda) < N \) (\( \lambda_N = 0 \)).

These functions have many interesting properties, one of them is the duality which relates polynomials with coupling \( \beta \) and \( 1/\beta \) [11] [14]:

\[
\prod_{i,j} (1 + z_i w_j) = \sum_{\lambda} J_{\{\lambda\}}(z_i, \beta) J_{\{\lambda'\}}(w_i, 1/\beta). \tag{2.11}
\]

To derive the results we rely on a few theorems which we now state. The first theorem is an extension of the constant term conjectures of Dyson and Macdonald. We define a second norm, \( (.,.) \) by:

\[
(X,Y)_{\beta} = \int \frac{d\theta_1}{2\pi} ... \int \frac{d\theta_N}{2\pi} (\Delta(z_i)\overline{\Delta(z_i)})^{\beta} \times X Y. \tag{2.12}
\]

where \( \Delta(z_i) = \prod_{i<j}(z_i - z_j) \) and \( z_k = e^{i\theta_k} \) is a variable taking value on the unit circle. We then have the following theorem.

**Theorem 1** (Macdonald [12])

\[
(J_{\{\lambda\}}, J_{\{\mu\}})_{\beta} = \delta_{\{\lambda\},\{\mu\}} c_N \prod_{s \in \lambda} \frac{a'(s) + \beta(N-l'(s))}{a'(s)+1 + \beta(N-l'(s)-1)} \times \frac{h^*_{\lambda}(s)}{h^*_\lambda(s)} \tag{2.13}
\]

where:

\[
c_N = \frac{(N\beta)!}{(\beta!)^N}
\]

and the quantities \( h^*_\lambda \) and \( h^*_{\lambda} \) are the upper hook-length and lower hook-length, defined as:

\[
h^*_\lambda(s) = \beta l(s) + a(s) + 1,
\]

\[
h^*_\lambda(s) = \beta(l(s)+1) + a(s). \tag{2.14}
\]
The second theorem gives the value of the Jack polynomial when $p$ variables are equal to 1 and all other vanish.

**Theorem 2** (Stanley [11], Macdonald [12])

$$J_{\{\lambda\}}(z_1 = \ldots = z_p = 1, z_{p+1} = \ldots = 0; \beta) = \prod_{s \in \lambda} \frac{a'(s) + \beta(p - l'(s))}{a(s) + \beta(l(s) + 1)}$$ (2.15)

We can also, in parallel to Schur functions, define the notion of *skew Jack polynomial*, $J_{\{\lambda/\mu\}}(z_i; \beta)$, through:

$$< J_{\{\lambda/\mu\}}, J_{\{\nu\}} > = < J_{\{\lambda\}}, J_{\{\mu\}}J_{\{\nu\}} > .$$ (2.16)

Using these functions, given 2 sets of variables $z_i$ and $w_i$, one can show that [11] [12] :

$$J_{\{\lambda\}}(z_i, w_i; \beta) = \sum_{\{\nu\}} J_{\{\nu\}}(z_i; \beta)J_{\{\lambda/\nu\}}(w_i; \beta)$$ (2.17)

In the specialized form where the $w_i$'s are represented by only one variable, $u$, we have:

$$J_{\{\lambda\}}(u, z_i; \beta) = \sum_{\{\nu\}} J_{\{\lambda/\nu\}}(u; \beta)J_{\{\nu\}}(z_i; \beta)$$ (2.18)

with the skew function different from zero only when $\{\lambda - \nu\}$ is a horizontal strip [11], i.e., when all $\lambda'_i - \nu'_i = 0, 1$. In other words, the diagram $\{\lambda\}$ contains that of $\{\nu\}$ ($\lambda_i \geq \nu_i$ for all $i$) and the difference denoted $\{\lambda - \nu\}$ has at most one box in each columns. If $\{\lambda - \nu\}$ is a horizontal strip, we define $C_{\{\lambda/\nu\}}$ to be the set of the columns for which $\lambda'_i - \nu'_i = 0$. An explicit expression for $J_{\{\lambda/\nu\}}(u; \beta)$ is given by:

**Theorem 3** (Stanley [11])

$$J_{\{\lambda/\nu\}}(u; \beta) = u^{\lambda - \nu} \prod_{s \in C_{\{\lambda/\nu\}}} \frac{h^*(s)}{h_*(s)} \lambda \left( \frac{h_*(s)}{h^*(s)} \right)_\nu$$ (2.19)

where the notation $(...)_\lambda$ means that we evaluate the upper and lower hook-length with respect to the partition $\{\lambda\}$. 

9
3. Retarded Green function.

The retarded single particle Green function, \(<0|\bar{\Psi}(x,t)\Psi(0,0)|0>\), can be simply computed at integer coupling. From now on we will use variables \(z_k = e^{2i\pi x_k/L}\). In these variables the ground state is given by:

\[
|0\rangle_{N+1} = \Delta_{N+1}^\beta (z_i) \prod_i z_i^{-\beta N/2}.
\] (3.1)

When acting upon it with the operator \(\Psi(0,0)\) we fix the variable \(z_{N+1}\) to be equal to 1 and obtain:

\[
\Psi(0,0)|0\rangle_{N+1} = \prod_{i=1}^{N} (z_i - 1)^\beta \prod_{i=1}^{N} z_i^{-\beta/2} |0\rangle_N .
\] (3.2)

Using:

\[
\bar{\Psi}(x,t) = e^{i(\hat{H}t - \hat{P}x)} \bar{\Psi}(0,0) e^{-i(\hat{H}t - \hat{P}x)}
\] (3.3)

the correlation we are seeking can be written as:

\[
<0|\bar{\Psi}(x,t)\Psi(0,0)|0> = c_{N+1}^\beta (\beta)^{-1} <0\prod_{i=1}^{N} z_i^{\beta/2} (1 - z_i)^\beta e^{-i((\hat{H} - E_0)t + (\hat{P} - P_0)x)} \prod_{i=1}^{N} z_i^{-\beta/2} (1 - z_i)^\beta |0\rangle_N .
\] (3.4)

The notation \(<0|...|0\rangle_N\) means that we integrate over the angles \(\theta_i = 2\pi x_i/L\) with \(i = 1,..N\), which amounts to compute the second norm defined in the last section (\(c_{N+1}\) is simply the normalization \(<0|0\rangle_{N+1}\)).

It can be done by expanding the product on the Jack symmetric functions using (2.11) where we set \(w_j = -1\) for \(1 \leq j \leq \beta\):

\[
\prod_{i=1}^{N} (1 - z_i)^\beta = \sum_{\{\lambda\} \leq \beta, l(\lambda) \leq N} J_{\{\lambda\}}(z_i; \beta) J_{\{\lambda\}'}(w_1 = ... = w_\beta = -1, 0; \frac{1}{\beta})
\]

\[
= \sum_{\{\lambda\} \leq \beta, l(\lambda) \leq N} (-1)^{|\lambda|} J_{\{\lambda\}}(z_i; \beta) J_{\{\lambda\}'}(w_1 = ... = w_\beta = 1; \frac{1}{\beta})
\] (3.5)

The length of \(\{\lambda\}\) and \(\{\lambda'\}\), \(l(\lambda)\) and \(l(\lambda')\) are respectively less or equal than \(N\) and \(\beta\) because the number of variables in \(J_{\{\lambda\}}\) and \(J_{\{\lambda\}'}\) is respectively equal to \(N\) and \(\beta\). As shown before, the wave functions defined by (3.5) are eigenstates of the hamiltonian and

\[\text{We put } N + 1 \text{ particles in the ground state to ease the forthcoming notation.}\]
the momentum operators. Let us denote by $E_{\lambda}$ the eigenvalue of $\hat{H} - E_0$ where $E_0$ is the ground state energy (with $N + 1$ particles) and by $Q_{\lambda}$ the eigenvalue of $\hat{P}$. Expanding both products in (3.4) in that way and using the fact that the Jack symmetric functions are orthogonal to each other (2.12), we obtain:

$$<0|\Psi^\dagger(x,t)\Psi(0,0)|0> = \sum_{i(\lambda') \leq \beta, i(\lambda) \leq N} e^{-iE_{\lambda}t + iQ_{\lambda}x} N_{\lambda} \left( J_{\lambda'}(w_1 = ... = w_\beta = 1, 0; \frac{1}{\beta}) \right)^2.$$

(3.6)

It remains to evaluate the norm, $N_{\lambda}$ and the Jack polynomial $J_{\lambda'}$ at $w_1 = w_2 = ... = w_\beta = 1$. These two quantities derive from the theorems 1 and 2. As an example we show on Fig.3 how to evaluate $J_{\lambda'}(1,1,0,0,...; 1/2)$ for $\beta = 2$, that is:

$$\prod_{s \in \lambda'} \frac{j - i/2 + 1/2}{\lambda'_j - j - i/2 + \lambda'_j/2 + 1/2}$$

(3.7)

where the partition $\{\lambda'\}$ has only two rows.

| $\lambda'_1$ | $\lambda'_2$ |
|-------------|-------------|
| $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\frac{3}{2}$ | $\frac{3}{2}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ |

**Fig. 3:** Evaluation of $J_{\lambda'}(1,1,0,...; 1/2)$

The result is:

$$J_{\lambda_1,\lambda_2}(1,1,0,...; 1/2) = \Gamma(1/2) \frac{\Gamma(\lambda'_1 - \lambda'_2 + 1)}{\Gamma(\lambda'_1 - \lambda'_2 + 1/2)}.$$

(3.8)

The generalisation of this expressions to an arbitrary value of $\beta$ and the norms are given in the appendix A. We give here the final result for the correlation function (3.6):

$$\eta(x,t,0,0) = \frac{\beta!}{N + 1} \sum_{0 \leq \lambda'_0 \leq \lambda'_1 \leq N} \prod_{i < j} \frac{\Gamma(\lambda'_i - \lambda'_j + \frac{j-i+1}{\beta}) \Gamma(\lambda'_i - \lambda'_j + 1 + \frac{j-i}{\beta})}{\Gamma(\lambda'_i - \lambda'_j + \frac{j-i}{\beta}) \Gamma(\lambda'_i - \lambda'_j + 1 + \frac{j-i-1}{\beta})} \times \left( \prod_{i=1}^{\beta} \frac{\Gamma(1 + \frac{1}{\beta}) \Gamma(\lambda'_i + 1 + \frac{i-1}{\beta}) \Gamma(N - \lambda'_i + i/\beta)}{\Gamma(i/\beta)^2 \Gamma(N - \lambda'_i + 1 + \frac{i-1}{\beta})} \right) e^{-iE_{\lambda}t + iQ_{\lambda}x}$$

(3.9)
The values of the energy and momentum are:

\[
E_{\{\lambda\}} = \sum_{i=1}^{N} \left( \lambda_i + \beta \left( \frac{N + 2}{2} - i \right) \right)^2 - \sum_{i=1}^{N+1} \beta^2 \left( \frac{N + 2}{2} - 1 \right)^2
= \frac{2\pi^2 \beta}{L^2} \left( \sum_{i=1}^{\beta} \lambda_i' \left( N + 1 - \lambda_i' + \frac{2i - 1}{\beta} \right) - \beta \frac{N^2}{4} \right)
\]  

(3.10)

and:

\[
Q_{\{\lambda\}} = \frac{2\pi^2}{L} \sum_{i} \left( \lambda_i' - \frac{N}{2} \right)
\]  

(3.11)

In the thermodynamic limit, \( N \to \infty \) , \( L \to \infty \), we define the variables:

\[
v_i = \frac{2}{N} \left( \lambda_i' - \frac{N}{2} \right)
\]  

(3.12)

and use the Stirling formula:

\[
\Gamma(N + 1)|_{N \to \infty} \simeq \sqrt{2\pi N^{N+1/2}} e^{-N}
\]  

(3.13)

We also write the sums as integrals and remove the order on partitions:

\[
\frac{2}{N} \sum_{0 \leq \lambda_i' \leq \lambda'_{i-1}} \to \int_{-1}^{v_{i-1}} dv_i
\]  

(3.14)

We finally obtain:

\[
<0|\Psi^\dagger(x,t)\Psi(0,0)|0> = \frac{\rho}{2} \left( \prod_{i=1}^{\beta} \int_{-1}^{1} dv_i \right) e^{i(Qx-Et)} \prod_{i=1}^{\beta} \frac{\Gamma(1 + \frac{1}{\beta})}{\Gamma(i/\beta)^2} (1 - v_i^2)^{-1+1/\beta} \times \prod_{i<j} \left| v_i - v_j \right|^{2/\beta}
\]  

(3.15)

In this limit the energy and momentum eigenvalues are given by:

\[
E = -\beta \rho^2 \pi^2 \sum_{i=1}^{\beta} v_i^2.
\]  

(3.16)

\[
Q = \rho \pi \sum_{i=1}^{\beta} v_i.
\]  

(3.17)
4. Density-Density Correlation

The density-density correlation function can be expressed as:

\[
< 0 | \rho(x,t) \rho(0,0) | 0 > = \sum_\alpha \frac{1}{N_{(0)}(\beta)N_\alpha(\beta)} < 0 | \rho(x,t) | \alpha > < \alpha | \rho(0,0) | 0 > = \sum_\alpha \frac{e^{iQ_\alpha x - E_\alpha t}}{N_{(0)}(\beta)N_\alpha(\beta)} < 0 | \rho(0,0) | \alpha >^2.
\]

(4.1)

where \( | \alpha > \) stands for a complete set of states characterized by their momenta \( k_i \) as explained in the introduction. Let us first consider the intermediate states which have all the momenta \( k_i \geq -k_F \) where \( k_F \) is the Fermi momentum. They can be labeled by the partition of the Jack polynomial, \( \{ \lambda \} \).

Consider the ground state with \( N+1 \) particles, then the density operator \( \rho(0,0) = \sum_i \delta(x_i) \) has a form factor equal to:

\[
< 0 | \rho(0,0) | \{ \lambda \} > = \int \frac{d\theta_1}{2\pi} ... \frac{d\theta_N}{2\pi} (\Delta(z_i)\overline{\Delta(z_i)})^\beta \times \prod_{i=1}^N (1 - z_i)^\beta (1 - \overline{z_i})^\beta J_{\{\lambda\}}(1, z_1, ..., z_N; \beta)
\]

(4.2)

Since we are working on the unit circle, \( z_i = 1/\overline{z_i} \) and we can rewrite:

\[
\prod_i (1 - z_i)^\beta (1 - \overline{z_i})^\beta = (-1)^{\beta N} \prod_i \frac{(1 - z_i)^{2\beta}}{z_i^\beta}
\]

(4.3)

Expanding the product on the numerator with (2.11) and using (2.18) to separate the first variable in the Jack polynomial we get the expression:

\[
< 0 | \rho(0,0) | \{ \lambda \} > = \int \frac{d\theta_1}{2\pi} ... \frac{d\theta_N}{2\pi} (\Delta(z_i)\overline{\Delta(z_i)})^\beta \sum_{\{\mu\},\{\nu\}} (-1)^{|\mu|+\beta N} J_{\{\mu\}}(z_i; \beta)J_{\{\nu\}}(z_i; \beta) \times J_{\{\mu'\}}(w_1 = ... w_2 = 1, 0; 1/\beta) J_{\{\lambda/\nu\}}(1; \beta) \prod_i z_i^\beta.
\]

(4.4)

The product over all variables \( z_i \) can be incorporated into one of the Jack polynomials using (2.10); after integration we obtain the final expression:

\[
< 0 | \rho(0,0) | \{ \lambda \} > = \sum_{\{\mu\}} N_{\{\mu\}}(\beta)(-1)^{|\mu|} J_{\{\mu+\beta\}}(w_1 = ... = w_{2\beta} = 1, 0; 1/\beta) J_{\{\lambda/\mu\}}(1; \beta)
\]

(4.5)
where the partitions \( \{\mu + 2\} \) and \( \{\lambda\} \) are illustrated in the case \( \beta = 2 \) in figure 4.

In (4.5) the evaluation of the norm, and that of the Jack polynomial at 1 can be done in exactly the same manner as precedently. We need to identify which partitions \( \{\mu\} \) contribute to the sum and for these partitions evaluate \( J_{\{\lambda/\mu\}}(1; \beta) \). It follows from theorem 3, that all partitions \( \{\mu\} \) are included in \( \{\lambda\} \) and differ from it by at most one box at the bottom of its columns, that is all possibilities \( \lambda'_i - \mu'_i = 0,1 \).

From the expansion of (4.3) it follows that only partitions \( \{\mu\} \) with \( \beta \) columns contribute to the sum. Then the non vanishing contributions from partitions \( \{\lambda\} \) are those with \( \beta \) legs and 1 arm of length \( \lambda_1 = p + \beta \).

For example, when \( \beta = 2 \) (Fig. 4) we have four possibilities, a) when both dark shaded boxes are present, b), c) when one of the dark shaded box is present, d) when no dark shaded boxes is present. Then theorem 3 states that the corresponding value of \( J_{\{\lambda/\mu\}}(1; \beta) \) is the product over squares of columns having shaded boxes of the following quantity:

\[
\left( \frac{h^*(s)}{h(s)} \right)^\lambda \left( \frac{h^*(s)}{h(s)} \right)^\mu
\]

(4.6)

where the upper and lower hook-length are taken with respect to the partition \( \{\lambda\} \) or \( \{\mu\} \) respectively.

Let us denote by \( I \) the ensemble of columns of \( \{\mu\} \) equal to the corresponding ones in \( \{\lambda\} \). Then the sum over partitions \( \{\mu\} \) can be replaced by a sum over the ensembles \( I \). Since the partition \( \{\mu\} \) must be included in the partition \( \{\lambda\} \) let us factor out of the
sum the terms for which all parts $\mu'_i = \lambda'_i - 1$; we obtain the following expression for the form factor:

$$< 0|\rho(0,0)|\{\lambda\} > = c_N(\beta) \left( \prod_{i=1}^{\beta} \frac{\Gamma(N+1+i/\beta)}{\Gamma(1+i/\beta)\Gamma(N+i/\beta)} \right) (-1)^{|\lambda| - \beta - p} \times \left( \prod_{i=1}^{\beta} \frac{\Gamma(N - \lambda'_i + 2 + i/\beta)\Gamma(\lambda'_i - i/\beta)}{\Gamma(N - \lambda'_i + 2 + i/\beta)\Gamma(\lambda'_i + 1 - i/\beta)} \right) \times \prod_{\lambda < j} \frac{\Gamma(\lambda'_i - \lambda'_j + 1 + \frac{i-1}{\beta})}{\Gamma(\lambda'_i - \lambda'_j + 1 + \frac{i-1}{\beta})} \times \sum_{\{\mu\}} Q(\{\mu\}).$$

(4.7)

with the sum over the partitions $\Sigma_{\{\mu\}} Q(\{\mu\})$ being equal to the following sum over the ensembles $I$:

$$\sum_{I} (-1)^{\text{card}(I)} \prod_{i \in I} \frac{p - i + 1 + \beta\lambda'_i}{p - i + \beta(\lambda'_i + 1)} \times \frac{N - \lambda'_i + 1 + \frac{i-1}{\beta}}{N - \lambda'_i + 1 + i/\beta} \times \prod_{j \neq I} \frac{\lambda'_i - \lambda'_j + \frac{i-1}{\beta}}{\lambda'_i - \lambda'_j + \frac{i-1}{\beta}}.$$  (4.8)

We evaluate this sum in appendix B, the result is:

$$\frac{\left(\sum_{i} \lambda'_i + p\right) \beta! \left(\frac{\beta N + p + 2\beta - 1}{\beta - 1}\right) (\beta - 1)!}{\beta^2 \prod_{i=1}^{\beta} (p - i + \beta(\lambda'_i + 1))(N - \lambda'_i + 1 + i/\beta)}$$

(4.9)

where in the general case $p$ is $\lambda_1 - \beta$ (see Fig. 4). The norm $N_{\{\lambda\}}$ needed to evaluate (4.1) is given in the appendix A. We also need to take into account the states for which the particle has a momentum less than $-k_F$, which amounts to replace $e^{iQ_{\{\lambda\}}x}$ by $2\cos(Q_{\{\lambda\}}x)$ in the final expression. The result is:

$$g_{N,\beta}(x, t) = \frac{2\beta!}{(N + 1)^2} \prod_{i=1}^{\beta} \frac{\Gamma(1 + 1/\beta)}{\Gamma^2(i/\beta)} \sum_{\text{o} \leq \lambda'_o \leq \cdots \leq \lambda'_N \leq \text{o} \leq \infty} \cos(Q_{\{\lambda\}}x)e^{-iE_{\{\lambda\}}t}$$

$$G_{\beta}\left(\frac{\Gamma(N - \lambda'_i + 1 + i/\beta)\Gamma(\lambda'_i + \frac{i-1}{\beta})}{\Gamma(N - \lambda'_i + 2 + \frac{i-1}{\beta})\Gamma(\lambda'_i + 1 - i/\beta)}\right) \prod_{\lambda < j} \frac{\Gamma(\lambda'_i - \lambda'_j + 1 + \frac{i-1}{\beta})\Gamma(\lambda'_i - \lambda'_j + \frac{i-1}{\beta})}{\Gamma(\lambda'_i - \lambda'_j + 1 + \frac{i-1}{\beta})\Gamma(\lambda'_i - \lambda'_j + \frac{i-1}{\beta})} \times \frac{\Gamma(p + \beta)\Gamma(\beta N + 2\beta + p)(p + \Sigma \lambda'_i)^2}{\Gamma(p + 1)\Gamma(\beta N + \beta + p + 1)\prod_{i}(p + \beta \lambda'_i + \beta - i)(p + \beta \lambda'_i - i + 1)}$$

(4.10)

with the explicit form of the energy and momentum given by:

$$E_{\{\lambda\}} = \frac{2\pi^2}{L^2} \left( \sum_{i=1}^{\beta} \lambda'_i \beta(N + 1 + \frac{2i - 1}{\beta} - \lambda'_i) \right) + p^2 + \beta(N + 2)p$$

(4.11)

$$Q_{\{\lambda\}} = \frac{2\pi}{L} |\lambda|.$$
In the thermodynamic limit though everything simplifies drastically. As in the last section, we use Stirling’s formula and the change of variables:

\[
\frac{p}{\beta N} = -\frac{w + 1}{2}, \quad \frac{\lambda_i}{N} = \frac{v_i + 1}{2}
\]  \(\text{(4.12)}\)

and the following limits for the energy and momentum:

\[
E = \frac{\beta \pi^2 \rho^2}{2} \left( -\sum_i v_i^2 + \beta w^2 \right)
\]  \(\text{(4.13)}\)

\[
Q = \pi \rho (\sum_i v_i - \beta w)
\]  \(\text{(4.14)}\)

We then find for the density-density correlation function:

\[
A(\beta) \int_{-\infty}^{-1} dw \int_{-1}^{1} dv_1 \ldots \int_{-1}^{1} dv_{\beta} \left( w^2 - 1 \right)^{\beta-1} \prod_{i=1}^{\beta} \left( 1 - v_i^2 \right)^{-1+1/\beta} \times \frac{\left( \sum_i v_i - \beta w \right)^2}{\prod_{i=1}^{\beta} (v_i - w)^2} e^{-iEt} \cos(Qx)
\]  \(\text{(4.15)}\)

with the constant:

\[
A(\beta) = \frac{\rho^2}{2\beta} \prod_{i=1}^{\beta} \frac{\Gamma(1+1/\beta)}{\Gamma(i/\beta)^2}.
\]  \(\text{(4.16)}\)

Note: We have just received a paper by Polychronakos and Minahan [15] who also conjecture this formula and also a paper by Z.N.C. Ha [16] who announces the same results.

5. Acknowledgements

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Appendix A. Norms and Evaluations of Jack polynomials at specific values.

We list in this appendix the results found for the norms and evaluation of the Jack polynomials at specialized values.

In section 3 we need to evaluate the Jack polynomial defined on a partition \( \{ \lambda \} = \{ \lambda_1, ..., \lambda_\beta \} \) (where we defined the partition by its conjugate) at \( z_1 = ... = z_\beta = 1 \). The result is:

\[
J_{\{ \lambda \}}(z_1 = ... = z_\beta = 1, 0, 0, ...; \frac{1}{\beta}) = \beta^{-1} \prod_{i=0}^{\beta-1} \frac{\Gamma(1/\beta)}{\Gamma(1-i/\beta)} \times \prod_{i<j} \frac{\Gamma(\lambda'_i - \lambda'_j + \frac{i-j+1}{\beta})}{\Gamma(\lambda'_i - \lambda'_j + \frac{i-j-1}{\beta})} \tag{A.1}
\]

We also need the norm of the same partition, \( N_{\{ \lambda \}}(\beta) \), it is given by:

\[
N_{\{ \lambda \}}(\beta) = c_N(\beta) \prod_{i=1}^\beta \frac{\Gamma(N + i - \frac{1}{\beta})}{\Gamma(N + i/\beta)} \times \frac{\Gamma(\lambda'_i + \frac{i}{\beta} + \frac{i-1}{\beta})}{\Gamma(\lambda'_i + 1 + \frac{i-1}{\beta})} \times \prod_{i<j} \frac{\Gamma(\lambda'_i - \lambda'_j + \frac{i-j-1}{\beta})}{\Gamma(\lambda'_i - \lambda'_j + 1 + \frac{i-j-1}{\beta})} \tag{A.2}
\]

In section 4 we need to evaluate the norm of two Jack polynomials on a partition of type \( \{ \lambda \} = \{ p; \lambda'_1, ..., \lambda'_\beta \} \) where the \( \lambda'_i \)'s denote the first \( \beta \) legs of the partition and \( p \) denote \( \lambda_1 - \beta \). The norm is found to be:

\[
c_{\lambda+1}(\beta) \prod_{i=1}^\beta \frac{\Gamma(N + 1 + \frac{i-1}{\beta})\Gamma(N - \lambda'_i + \frac{i}{\beta} + \frac{i+1}{\beta})}{\Gamma(N + \frac{i}{\beta} + 1)\Gamma(N - \lambda'_i + \frac{i}{\beta} + 1 + \frac{i}{\beta})\Gamma(\lambda'_i + \frac{i}{\beta} + 1)} \times \frac{\Gamma(p + 1)\Gamma(\beta(N + 1) + 1))\Gamma(p + \beta(N + 2))}{\Gamma(p + \beta(N + 2))\Gamma(p + \beta(N + 1) + 1)} \tag{A.3}
\]

Another quantity we used to calculate the density-density correlation function is the polynomial associated to the partition with \( \beta \) arms of length \( N \) and \( \beta \) arms of length \( \lambda'_1, ..., \lambda'_\beta \) evaluated at \( z_1 = ... = z_{2\beta} = 1 \). The result is:

\[
J_{\{ (\beta+\lambda') \}}(z_1 = ... = z_{2\beta} = 1, 0, ...; \frac{1}{\beta})
\]

\[
= \prod_{i=1}^\beta \beta \left( N - \lambda'_i + \frac{i}{\beta} \right) \times J_{\{ \lambda \}}(z_1 = ... = z_\beta = 1, 0, 0, ...; \frac{1}{\beta}) \tag{A.4}
\]
Appendix B. Summation of (1.8).

We wish to evaluate the following expression,

\[
\sum_{I} (-1)^{\text{card}(I)} \prod_{i \in I} \frac{p - i + 1 + \beta \lambda_i'}{p - i + \beta(\lambda_i' + 1)} \times \frac{N - \lambda'_i + 1 + \frac{i - 1}{\beta}}{N - \lambda'_i + 1 + i/\beta} \times \prod_{j \notin I} \lambda'_j - \lambda'_j + \frac{i - 1 + i}{\beta}. \tag{B.1}
\]

where \( I \subset \{1, \ldots, \beta\} \) and \( J \) is the complement of \( I \). Changing variables and writing \( u_i = \lambda_i' - i/\beta + p/\beta \) and \( y = N + p/\beta + 1 \) we obtain the simpler expression:

\[
\sum_{I} (-1)^{\text{card}(I)} \prod_{i \in I} \frac{u_i + 1/\beta}{u_i + 1} \times \frac{y - u_i - 1/\beta}{y - u_i} \times \prod_{j \notin I} \frac{u_i - u_j + 1/\beta}{u_i - u_j}. \tag{B.2}
\]

By considering the 2 terms with \( i \in I, j \in J \) and \( i \in J \) and \( j \in I \), all other indices being fixed, one sees that this expression has no pole at \( u_i = u_j \). It is symmetric under permutations of the indices. Thus, putting everything on the same denominator, the numerator is a symmetric polynomial of degree at most two in each variables, \( u_i \). By considering the two terms with \( i \in I \) and \( i \in J \) all other indices being kept fixed, one sees that the numerator is in fact of degree one in each variables \( u_i \). It is therefore determined by the \( \beta + 1 \) coefficients of the symmetric polynomials \( m_{\{1^k\}}, 0 \leq k \leq \beta \). If we specialize the variables to \( u_i = u - i/\beta \), these coefficients are determined by considering the polynomial in \( u \) at the numerator. In that case the only terms contributing to the sum are those for which the first \( p \) terms are in the ensemble \( I \) and the \( \beta - p \) last terms are in the complement \( J \). The sum is then written as:

\[
\sum_{p=0}^{\beta} (-1)^p \binom{\beta}{p} \prod_{j=1}^{p} \frac{u + \frac{1-i}{\beta}}{u + 1 - j/\beta} \times \frac{y - u + \frac{j-1}{\beta}}{y - u + j/\beta}. \tag{B.3}
\]

making use of the identity\(^3\) [17]:

\[
\sum_{p=0}^{\beta} (-\beta)_p (c)_p (a + b - c + \beta - 2)_p \frac{p! (a)_p (b)_p}{(a - c - 1)_\beta (b - c - 1)_\beta (a - c - 1) (b - c - 1)} = \frac{(a - c - 1)\beta (b - c - 1)\beta}{(a)_\beta (b)_\beta (a - c - 1) (b - c - 1)} [(a - c - 1)(b - c - 1) + \beta (a + b - c + \beta - 2)]
\]

(B.4) becomes:

\[
\frac{(\beta y + 1)\beta \Gamma(\beta y)}{\prod_{i=1}^{\beta} (\beta (y - u) + i)(\beta u + \beta - i)}. \tag{B.5}
\]

We finally find the sum (B.1) to be:

\[
\frac{\prod_{i=1}^{\beta} (\beta^N + p + 2\beta - 1)(\beta - 1)!}{\beta^p \prod_{i=1}^{\beta} (p - i + \beta(\lambda'_i + 1))(N - \lambda'_i + 1 + i/\beta). \tag{B.6}
\]

\(^3\) We denote \( (a)_\beta = \frac{\Gamma(a+\beta)}{\Gamma(a)} \)
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