The Schwinger Model on the Torus

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Abstract

The classical and quantum aspects of the Schwinger model on the torus are considered. First we find explicitly all zero modes of the Dirac operator in the topological sectors with nontrivial Chern index and its spectrum. In the second part we determine the regularized effective action and discuss the propagators related to it.

Finally we calculate the gauge invariant averages of the fermion bilinears and correlation functions of currents and densities. We show that in the infinite volume limit the well-known result for the chiral condensate can be obtained and the clustering property can be established.

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Introduction

The Schwinger Model (SM) or quantum electrodynamics with massless fermions in two-dimensional space-time is one of the exactly solvable models of quantum field theory (QFT). This was shown some time ago by using operator methods and the path integral approach. The SM usually serves for the illustration of such phenomena in particle physics as: spontaneous breakdown of local gauge symmetry through axial anomaly, mass generation, charge screening and quark confinement, vacuum structure and the realization of gauge transformations.

Most of the mathematical problems of QFT, related to these phenomena can be treated more exactly and rigorously in a compactified version of Euclidean space-time, where the spectrum of the Dirac operator becomes exact, and a precise definition of topological sectors together with corresponding zero modes can be given. Such compactification in the SM on the manifold without boundary was considered for the first time by C. Jayewardena, who used two-dimensional sphere as a compact Euclidean space-time.

In this work we present the detailed calculations concerning classical and quantum aspects of the SM on the torus.

Compactification on the torus ($T$) is much better than the compactification on the sphere because it allows to find finite temperature (size) effects and is appropriate to the systematic analysis of the lattice approximation on which the numerical calculations are performed (on the lattice, usually periodic (torus) boundary conditions are considered). Moreover, from the results obtained for this model one can extract the information concerning the SM on a cylinder, which also has attracted attention recently.

We show that the torus compactification on the SM, as in the case of sphere compactification, does not destroy the solvability of the model.

The SM on the symmetric torus (where the lengths of both circumferences are equal) has been considered in and the SM with Dirac-Kähler fermions (the geometric SM, which is equivalent to the SM with two flavors) on the torus was investigated in and [11]. I.Sachs and A.Wipf discussed the role of the zero modes in the SM on the torus, derived some relevant Green’s functions and found the analytic form of the chiral condensate.

Finite temperature SM has been considered in [13].

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1 The discussion of the SM are so numerous that it is not possible to refer to all of them.
2 Notice that technically compactification allows to avoid the infrared divergences, which sometimes plague the analysis of two-dimensional gauge theories.
The paper is organized as follows.

The first part contains the definition of the model and the discussion of some peculiarities due to the fact that it is defined on the compact space-time. We find the general expression for the normalized zero modes of the Dirac operator in the sector with any topological charge. Then we calculate the spectrum of this operator and the corresponding eigenfunctions.

In the second part we discuss the general path integral formula which can be used for the calculation of the quantum mechanical vacuum expectation values of observables in the case where zero modes are present. The rest of this part is devoted to the calculation of the regularized effective action and the propagators related to it.

In the third part we consider the objects of physical significance, i.e. the expectation values of various operators. We calculate separately the contributions to them from different topological sectors. Taking the infinite volume limit we obtain the well-known result for the chiral condensate and establish the clustering property.

In the appendices the details of the derivations of some important formulas used in the main text are presented.

1 Classical theory

1.1 The action

\[ S = \int_{T=S_1 \times S_1'} d^2x \left\{ \frac{1}{2} F_{12}^2(x) + \bar{\psi}(x) \gamma_\mu (\partial_\mu - ieA_\mu) \psi(x) \right\}, \quad (1.1) \]

where \( 0 \leq x_\mu \leq L_\mu, \mu = 1, 2, L_1 \) and \( L_2 \) are lengths of the large and small circumferences of the torus, respectively, and the field strength \( F_{12}(x) = \partial_1 A_2(x) - \partial_2 A_1(x) \).

Our \( \gamma \)-matrices satisfy:

\[ \{ \gamma_\mu, \gamma_\nu \} = 2 \delta_\mu_\nu, \gamma_1 \gamma_2 = i \gamma_5, \gamma_\mu^\dagger = \gamma_\mu, \]

which implies in two dimensions \( \gamma_\mu \gamma_5 = -i \epsilon_{\mu \nu} \gamma_\nu \), where \( \epsilon_{12} = -\epsilon_{21} = 1 \).

The geometry of fields on the torus requires certain periodicity conditions:

\[ \psi(x + L_\nu \hat{\nu}) = \Lambda_\nu(x) \psi(x), \quad (1.2a) \]

\[ \bar{\psi}(x + L_\nu \hat{\nu}) = \bar{\psi}(x) \Lambda^{-1}_\nu(x), \quad (1.2b) \]

\[ A_\mu(x + L_\nu \hat{\nu}) = A_\mu(x) - \frac{i}{e} \Lambda^{-1}_\nu(x) \partial_\mu \Lambda_\nu(x), \quad (1.3) \]

where \( \hat{\nu} \) is a unit vector in \( \nu \)th direction. The transition functions \( \Lambda_\nu(x) \) satisfy the cocycle condition

\[ \Lambda_\mu(x) \Lambda_\nu(x + L_\mu \hat{\mu}) = \Lambda_\nu(x) \Lambda_\mu(x + L_\nu \hat{\nu}). \quad (1.4) \]

It is well-known [14] that under these requirements the gauge field configurations fall into classes \( \mathcal{CH}^{(k)} \) (topological sectors, Chern classes) characterized by the Pontriyagin (Chern) index (topological charge, topological quantum number, winding number)

\[ k = \frac{e}{2\pi} \int_{T} F_{12} d^2x, \quad k = 0, \pm 1, \pm 2, \cdots \quad (1.5) \]
As a special representative we choose a field with a constant field strength:

\[ F_{\mu\nu} = B\epsilon_{\mu\nu} = \frac{2\pi k}{eL_1L_2}\epsilon_{\mu\nu}. \]  (1.6)

In the Lorentz gauge a corresponding potential is

\[ C^{(k)}_{\mu}(x) = -\frac{B}{2}\epsilon_{\mu\nu}x_\nu = -\frac{\pi k}{eL_1L_2}\epsilon_{\mu\nu}x_\nu. \]  (1.7)

In the axial gauge \( \partial_1 A_\mu(x) = 0 \) we may choose the representative

\[ \tilde{C}^{(k)}_{\mu}(x) = \begin{cases} 
-\frac{2\pi k}{eL_1L_2}x_2, & \text{for } \mu = 1 \\
0, & \text{for } \mu = 2.
\end{cases} \]

In this case

\[ \tilde{F}^{(k)}_{12} = \partial_1\tilde{C}^{(k)}_2(x) - \partial_2\tilde{C}^{(k)}_1(x) = \frac{2\pi}{eL_1L_2}k, \]

and we see that again

\[ \frac{e}{2\pi} \int_{\mathcal{T}} \tilde{F}^{(k)}_{12} d^2x = k. \]

The periodicity condition is

\[ \tilde{C}^{(k)}_{\mu}(x + L_\nu\tilde{\nu}) = \tilde{C}^{(k)}_{\mu}(x) - \frac{i}{e}\Lambda^{-1}\nu(x)\partial_\mu\Lambda_\nu(x), \]

where

\[ \Lambda_\nu(x) = \begin{cases} 
1, & \text{for } \nu = 1 \\
e^{-2\pi ik\frac{x_2}{L_2}}, & \text{for } \nu = 2.
\end{cases} \]

The relation to the Lorentz gauge is

\[ C^{(k)}_{\mu}(x) = \tilde{C}^{(k)}_{\mu}(x) - \frac{i}{e}E^{-1}(x)\partial_\mu E(x), \]

where

\[ E(x) = e^{i\pi k\frac{x_1}{L_1}\frac{x_2}{L_2}}. \]

In what follows we will work in the Lorentz gauge.

In the topological sector \( A^{(k)} \) a general gauge potential has the form

\[ A^{(k)}_{\mu}(x) = A^{(0)}_{\mu} + C^{(k)}_{\mu}(x), \]  (1.8)

\( A^{(0)}_{\mu}(x) \) is a single valued ‘continuous’ function on \( \mathcal{T} \). Thus we may apply the Hodge decomposition theorem [14]

\[ A^{(0)}_{\mu}(x) = \partial_\mu a(x) + t_\mu + \epsilon_{\mu\nu}\partial_\nu b(x), \]  (1.9)

where \( \partial_\mu a(x) \) is a ‘pure gauge’, \( t_\mu \) is a (constant) toron field restricted to \( 0 \leq t_\mu < 2\pi/eL_\mu \), \( \epsilon_{\mu\nu}\partial_\nu b(x) \) is a curl and \( a(x) \) and \( b(x) \) are continuous on \( \mathcal{T} \) and orthogonal to the constant, \( \int_{\mathcal{T}} a(x)d^2x = \int_{\mathcal{T}} b(x)d^2x = 0 \).

The toron field is a gauge invariant up to large gauge transformations, when \( \Lambda(x) = \exp \left\{ 2\pi i \left( m_1\frac{x_1}{L_1} + m_2\frac{x_2}{L_2} \right) \right\} \), and \( m_\mu \) is integer: \( t_\mu \Lambda^{-1}t_\mu + \frac{2\pi}{eL_\mu}m_\mu \), in this case \( a(x) \Lambda a(x) \) and \( b(x) \Lambda b(x) \). Under small gauge transformations \( \Lambda(x) = e^{i\lambda(x)}, a(x) \Lambda a(x) + \frac{1}{e}\lambda(x), b(x) \Lambda b(x), t_\mu \Lambda t_\mu \).
1.2 Dirac equation (DE)

\[ D\chi(x) \equiv \gamma_\mu (\partial_\mu - ieA_\mu)\chi(x) = 0, \quad (1.10) \]

where

\[ \chi(x) = \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix} \]

is a two-component complex spinor.

Formally the solution of DE (1.10) with the potential (1.8), (1.9) is

\[ \chi(x) = e^{ie[a(x)+t_\mu x_\mu-i\gamma_5(b(x)-\frac{x^k}{2x_1x_2}l_2)]} \chi_0(x) \quad (1.11) \]

and \( \chi_0(x) \) satisfies the free DE

\[ \gamma_\mu \partial_\mu \chi_0(x) = 0. \quad (1.12) \]

The main problem is to find such \( \chi_0(x) \)'s, which satisfy the periodicity condition (1.2) for \( \chi(x) \).

In the future we will consider also the operator

\[ D_0 = D|_{a=b=0} = \gamma_\mu (\partial_\mu - ie(t_\mu + C^{(k)}(x))). \quad (1.13) \]

Then

\[ \hat{\chi}_i(x) = e^{ie\gamma_5b(x)+iea(x)}\chi_i(x), \]

where \( \hat{\chi}_i(\chi_i) \) is a zero mode of the \( D(D_0) \) operator.

1.3 General remarks

The operator \( iD \) is the Hermitian operator in the space of two-component spinors, with a scalar product

\[ (\psi, \chi) = \int_T d^2x \bar{\psi}(x)\chi(x). \]

It is an elliptic operator on the compact manifold \( T \) and its spectrum is discrete. \( \{\psi_\nu\}(\nu = 1, 2, \cdots) \) is a set of independent eigenfunctions of the \( iD \) with positive eigenvalues \( E_\nu \). Since \( iD \) anticommutes with \( \gamma_5 \), \( \{\gamma_5\psi_\nu\}(\nu = 1, 2, \cdots) \) is a set of independent eigenfunctions of the \( iD \) with eigenvalues \( -E_\nu \). Denoting \( \psi_+ = \gamma_5\psi_\nu \) and \( \psi_- = -\psi_\nu \) we have, that \( \{\psi_\nu\}, \nu = \pm 1, \pm 2, \cdots \) is a complete set of the eigenfunctions of \( iD \). Again, since \( \{D, \gamma_5\} = 0 \), we can choose zero modes to have a definite chirality.

**The Index theorem.** Each zero mode \( \chi_i(x) \) has a definite chirality \( \gamma_5\chi_i = \pm \chi_i \) and a number of the zero modes \( n = n_++n_- \) \( n_+ \) \( n_- \) is a number of the zero modes with a positive (negative) chirality] satisfies the following rule

\[ n_+ = k, \quad n_- = 0, \quad \text{if } k \geq 0, \]

\[ n_+ = 0, \quad n_- = |k|, \quad \text{if } k \leq 0. \]

In the trivial sector \( (k = 0) \) there are no zero modes \[15\].
1.4 The zero modes of the $D_0$ operator in the nontrivial sector

Let us introduce the complex (dimensionless) variables

$$z = \frac{x_1 + ix_2}{L_1}, \quad \bar{z} = \frac{x_1 - ix_2}{L_1}. \tag{1.14}$$

Then

$$\partial_1 = \frac{1}{L_1}(\partial_z + \partial_{\bar{z}}), \quad \partial_2 = \frac{i}{L_1}(\partial_z - \partial_{\bar{z}}), \tag{1.15}$$

and the free Dirac equation (1.12) has the following form

$$2 \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \chi_1(z, \bar{z}) \\ \chi_2(z, \bar{z}) \end{pmatrix} = 0. \tag{1.16}$$

From this equation we see that

$$\chi_1(z, \bar{z}) = \chi_1(z), \tag{1.17a}$$

$$\chi_2(z, \bar{z}) = \chi_2(\bar{z}). \tag{1.17b}$$

The vector-potential is chosen as follows

$$A_\mu(x) = t_\mu + C^{(k)}_\mu(x) = t_\mu - \frac{\pi k}{eL_1L_2} \varepsilon_{\mu\nu}x_\nu, \tag{1.18}$$

and in accordance with the general solution (1.11) we get

$$\chi_1(z, \bar{z}) = e^{i\frac{\pi}{2}L_1(t_1 + t_2) - \frac{\pi}{2}k |z|^2} \chi_1(z), \tag{1.19a}$$

$$\chi_2(z, \bar{z}) = e^{i\frac{\pi}{2}L_1(t_1 + t_2) + \frac{\pi}{2}k |\bar{z}|^2} \chi_2(\bar{z}), \tag{1.19b}$$

where $\chi_1(z, \bar{z})$ and $\chi_2(z, \bar{z})$ are the components of the spinor

$$\chi(z, \bar{z}) = \begin{pmatrix} \chi_1(z, \bar{z}) \\ \chi_2(z, \bar{z}) \end{pmatrix}, \tag{1.20}$$

which is the solution of the equation

$$D_0 \chi(z, \bar{z}) = 0, \tag{1.21}$$

$t_\pm = t_1 \pm it_2$.

Using the expression (1.18) and taking into account (1.2) we get the periodicity conditions for the function $\chi(z, \bar{z}), \tau = i\frac{\pi}{L_1}$

$$\chi(z + 1, \bar{z} + 1) = e^{\frac{\pi k}{2|\tau|}(z - \bar{z})} \chi(z, \bar{z}), \tag{1.22a}$$

$$\chi(z + \tau, \bar{z} + \bar{\tau}) = e^{-i\frac{\pi k}{2}(z + \bar{z})} \chi(z, \bar{z}). \tag{1.22b}$$
Then from (1.19) and (1.22) it follows that $^{0}\chi_{1}(z)$ and $^{0}\chi_{2}(\bar{z})$ have the following periodicity properties (for simplicity we put $L_{1} = 1$):

$$^{0}\chi_{1}(z + 1) = e^{\frac{\pi i k z}{2|\tau|} + \frac{\pi i}{2}(t_{+} + t_{-})} ^{0}\chi_{1}(z),$$  
(1.23a)

$$^{0}\chi_{1}(z + \tau) = e^{-i\pi k z + \frac{\pi i}{2}|\tau| - \frac{e|\tau|}{2}(t_{+} + t_{-})} ^{0}\chi_{1}(z),$$  
(1.23b)

$$^{0}\chi_{2}(\bar{z} + 1) = e^{-\frac{\pi i k}{2} - \frac{\pi i}{2}(t_{+} + t_{-})} ^{0}\chi_{2}(\bar{z}),$$  
(1.24a)

$$^{0}\chi_{2}(\bar{z} + \tau) = e^{-i\pi k \bar{z} - \frac{\pi i}{2}|\tau| - \frac{e|\tau|}{2}(t_{+} + t_{-})} ^{0}\chi_{2}(\bar{z}).$$  
(1.24b)

For $k \geq 1$ we shall use the following ansatz

$$^{0}\chi_{1}(z) \sim e^{\alpha z + \beta z} \vartheta_{3}(kz + \gamma | k\tau),$$  
(1.25)

where $\vartheta_{3}(z | \tau)$ is the Jacobi’s theta function [16] and $\alpha, \beta$ and $\gamma$ are constants to be found from (1.23) and (1.24). From (1.25)

$$^{0}\chi_{1}(z + 1) \sim e^{\beta + 2\alpha z + \alpha} ^{0}\chi_{1}(z)$$  
(1.26a)

and

$$^{0}\chi_{1}(z + \tau) \sim e^{2z\alpha + \alpha \tau^{2} + \beta \tau - i\pi(2\gamma + k\tau)} ^{0}\chi_{1}(z).$$  
(1.26b)

Comparing (1.26) with (1.23), we have

$$\alpha + \beta = \frac{\pi k}{2|\tau|} - \frac{ie}{2}(t_{+} + t_{-}) + 2\pi in,$$  
(1.27)

$$2\alpha = \pi k |\tau|,$$  
(1.28)

$$\alpha \tau^{2} + \beta \tau - i\pi(2\gamma + k\tau) = \frac{\pi}{2} k|\tau| - \frac{e|\tau|}{2}(t_{+} - t_{-}) + 2\pi im,$$  
(1.29)

where $n$ and $m$ are arbitrary integers.

Then

$$\alpha = \frac{\pi k}{2|\tau|},$$  
(1.30)

$$\beta = -\frac{ie}{2}(t_{+} + t_{-}) + 2\pi in$$  
(1.31)

and

$$\gamma = n\tau - \frac{ie t_{+}}{2\pi} |\tau| - m.$$  
(1.32)

Since $\vartheta_{3}(z + 1 | \tau) = \vartheta_{3}(z | \tau)$, we get the following expression for $k$ zero modes of the free Dirac equation in the sector with the topological charge $k \geq 1$ (in accordance with the Index theorem) up to a normalization factor

$$^{0} \chi_{1}^{(n)}(z) = e^{\frac{\pi i k z^{2}}{2|\tau|} - \frac{e|\tau|}{2}(t_{+} + t_{-})z + 2\pi inz} \vartheta_{3}\left(kz - \frac{ie t_{+}}{2\pi} |\tau| + n\tau |k\tau| \right),$$  
(1.33)

$n = 0, 1, \cdots, k - 1$. 

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Reconstructing $L_1$ and introducing the normalization factor $1/N(n)$ we obtain from (1.19a) and (1.20)

$$\chi^{(n)}(z, \bar{z}) = \frac{1}{N(n)} e^{i\frac{2\pi}{L_2} t_1 + \frac{2\pi}{L_2} k (z^2 - z'^2) + 2\pi i n z'} \vartheta_3(k z' + n \tau \mid k \tau) \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \quad (1.34)$$

where

$$z' = \frac{1}{L_1} (x_1 + i x_2) - \frac{i e L_1 |\tau|}{2\pi k} t_+. \quad (1.35)$$

From the definition of the $\vartheta_3$ function it follows that $\chi^{(n)}(z, \bar{z})$'s for different values of $n$ are orthogonal

$$\int_0^{L_1} \int_0^{L_2} \bar{\chi}^{(n_1)}(z, \bar{z}) \chi^{(n_2)}(z, \bar{z}) dx_1 dx_2 = 0, \quad (1.36)$$

if $n_1 \neq n_2$, and in order to get the normalized zero modes one should find the normalization factor $1/N(n)$.

For the product $\bar{\chi}^{(n)}(z, \bar{z}) \chi^{(n)}(z, \bar{z})$ we have from (1.34)

$$\bar{\chi}^{(n)}(z, \bar{z}) \chi^{(n)}(z, \bar{z}) = \frac{1}{N^2(n)} e^{\frac{2\pi}{L_2} t_1} \vartheta_3(X - i Y \mid \tau') \vartheta_3(X + i Y \mid \tau'), \quad (1.37)$$

where

$$\tau' \equiv k \tau, \quad X \equiv \frac{k x_1}{L_1} + \frac{e L_1 |\tau|}{2\pi} t_2, \quad Y \equiv \frac{k x_2}{L_1} - \frac{e L_1 |\tau|}{2\pi} t_1 + n |\tau|. \quad (1.38)$$

Using relations among the $\vartheta$- functions (see [18])

$$\vartheta_3(X - i Y \mid \tau') \vartheta_3(X + i Y \mid \tau') = \frac{1}{\vartheta_4^2(0 \mid \tau')} \left\{ \vartheta_3^2(X \mid \tau') \vartheta_4^2(i Y \mid \tau') - \vartheta_2^2(X \mid \tau') \vartheta_4^2(i Y \mid \tau') \right\}, \quad (1.39)$$

and

$$\vartheta_3(i Y \mid \tau') = \frac{1}{\sqrt{|\tau'|}} e^{\frac{2\pi}{|\tau'|} y^2} \vartheta_1 \left(\frac{Y}{|\tau'|} \mid -\frac{1}{\tau'}\right),$$

$$\vartheta_3(i Y \mid \tau') = \frac{1}{\sqrt{|\tau'|}} e^{\frac{2\pi}{|\tau'|} y^2} \vartheta_1 \left(\frac{Y}{|\tau'|} \mid -\frac{1}{\tau'}\right).$$

from [19] (13.22 (8)) we get

$$\bar{\chi}^{(n)}(z, \bar{z}) \chi^{(n)}(z, \bar{z}) = \frac{1}{N^2(n)} e^{\frac{2\pi}{L_2} t_1} \vartheta_3^2(X \mid \tau') \vartheta_4^2 \left(\frac{Y}{|\tau'|} \mid -\frac{1}{\tau'}\right)$$

$$+ \vartheta_2^2(X \mid \tau') \vartheta_4^2 \left(\frac{Y}{|\tau'|} \mid -\frac{1}{\tau'}\right) \quad (1.38)$$

With the help of Landen’s transformation [19] (13.23(16)) we can rewrite it in the following form.
\[ \chi^{(n)}(z, \bar{z}) = \frac{e^{\frac{2\pi n^2|z'|}{k}}}{|\tau'|^{1/2}} \left\{ [\vartheta_3(2X \mid 2\tau') | 2\tau'] (A_2 - A_3)B_3 + [\vartheta_2(2X \mid 2\tau') | 2\tau'] (A_2 + A_3)B_2 \right\}, \] (1.39)

where \( A_i = \vartheta_i(0 \mid 2\tau'), B_i = \vartheta_i(0 \mid -\frac{2}{\tau'}), i = 2, 3. \) From the normalization condition

\[ \int_{L_1}^{L_2} \int_{L_1}^{L_2} \chi^{(n)}(z, \bar{z}) \chi^{(n)}(z, \bar{z}) dx_1 dx_2 = 1 \] (1.40)

we get

\[ \mathcal{N}^2_{(n)} = \frac{L_2^2 e^{\frac{2\pi n^2|z'|}{k}}}{k|\tau|^{1/2}} (A_2 + A_3) \]

\[ = \frac{L_2^2 e^{\frac{2\pi n^2|z'|}{k}}}{\sqrt{2k}} \sqrt{|\tau|} = \frac{L_1 L_2 e^{\frac{2\pi n^2|z'|}{k}}}{\sqrt{2k|\tau|}}. \] (1.41)

Thus the normalized zero modes in the sector with \(|k| \geq 1\) are:

for \( k \geq 1 \)

\[ \chi^{(n)}(z, \bar{z}) = \left( \frac{2k}{|\tau|} \right)^{1/4} \frac{1}{L_1} e^{i\frac{2\pi x_i t_\mu + \frac{x_i^2}{2|\tau|}(z'^2 - z'^2)}{2\pi n^2|z'| + 2\pi i n z'}} \]

\[ \times \vartheta_3(kz' + n\tau \mid k\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] (1.42)

\[ n = 0, 1, \cdots, k - 1, \quad z' = \frac{1}{L_1} (x_1 + i x_2) - \frac{i e L_1 |\tau|}{2\pi k} t_+, \]

for \( k \leq -1 \)

\[ \phi^{(n)}(z, \bar{z}) = \left( \frac{2|k|}{|\tau|} \right)^{1/4} \frac{1}{L_1} e^{i\frac{2\pi x_i t_\mu + \frac{x_i^2}{2|\tau|}(z''^2 - z''^2)}{2\pi n^2|k| + 2\pi i n z''}} \]

\[ \times \vartheta_3(|k|z'' + n\tau \mid |k|\tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] (1.43)

\[ n = 0, 1, \cdots, |k| - 1; \quad z'' = \frac{1}{L_1} (x_1 - i x_2) - \frac{i e L_1 |\tau|}{2\pi |k|} t_- . \]
1.5 Spectrum of the $D_0$ operator

1.5.1 Trivial sector $k = 0$.

We have the following eigenvalue and eigenfunction equation for the $\tilde{D}_0 \equiv L_1 D_0$ operator

$$\tilde{D}_0^{(k=0)} \psi(x) = L_1 \begin{pmatrix} 0 & \partial_1 - i \partial_2 - i e(t_1 - it_2) \\ \partial_1 + i \partial_2 - i e(t_1 + it_2) & 0 \end{pmatrix} \psi(x) = \varepsilon \psi(x).$$

(1.44)

In our case, when $A_\mu = t_\mu$ and $t_\mu$ is a constant, the functions $\Lambda_\nu(x)$ from (1.3) are constants. So the periodicity conditions for the normalized solutions $s$ are as follows

$$\psi(x + L_\nu \hat{\nu}) = \psi(x).$$

(1.45)

Then one can look for the solution of (1.44) using the ansatz

$$\psi(x) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{2\pi i (\frac{n_1}{L_1} x_1 + \frac{n_2}{L_2} x_2)},$$

(1.46)

where $n_\nu = 0, \pm 1, \pm 2, \ldots$, and $a_1$ and $a_2$ are some complex numbers which should be found from normalization condition.

We get for the eigenvalues

$$\varepsilon = \pm 2\pi \sqrt{(n_1 - \tilde{t}_1)^2 + \frac{1}{|\tau|^2} (n_2 - \tilde{t}_2)^2} = \pm i \sqrt{n_+ n_-},$$

(1.47)

where $\tilde{t}_\nu = \frac{eL_\nu}{2\pi} t_\nu$, $\bar{n}_\pm = 2\pi [(n_1 \pm \frac{i}{|\tau|} n_2) - (\tilde{t}_1 \pm \frac{i}{|\tau|} \tilde{t}_2)]$.

Using the normalization condition

$$\int_0^{L_1} dx_1 \int_0^{L_2} dx_2 \bar{\psi}(x)\psi(x) = 1,$$

(1.48)

we get, that $a_1$ and $a_2$ can be any complex numbers subject the condition: $|a_1|^2 + |a_2|^2 = \frac{1}{L_1 L_2}$.

1.5.2 Nontrivial sector $k \neq 0$.

Using the form (1.18) for the vector potential in the case $k \neq 0$ and complex variables, we get (for dimensionless operator)

$$\tilde{D}_0^{(k \neq 0)} \equiv L_1 D_0^{(k \neq 0)} = 2 \begin{pmatrix} 0 & \partial_1 - i \partial_2 - i eL_1 \frac{z}{4} \\ \partial_1 + i \partial_2 - i eL_1 \frac{z}{4} + \frac{i eL_1}{2} t_- \end{pmatrix}$$

$$= \sqrt{2e|B|L_1^2} \begin{pmatrix} 0 & d_+ \\ d_- & 0 \end{pmatrix},$$

(1.49)

where $B \equiv \frac{2\pi k}{eL_1 L_2}$, $d_+ \equiv \frac{1}{\sqrt{2e|B|L_1}} (\partial_1 - \frac{eB L_1^2}{4} z - \frac{i e L_1}{2} t_-)$ and $d_- \equiv \frac{1}{\sqrt{2e|B|L_1}} (\partial_1 + \frac{eB L_1^2}{4} z - \frac{i e L_1}{2} t_+)$. Note that $(d_+)^\dagger = -d_-$ and

$$[d_+, d_-] = \frac{k}{|k|}.$$

(1.50)
Then
\[ \tilde{D}_0^\dagger = -\sqrt{2e|B|L_1^2} \begin{pmatrix} 0 & d_+ \\ d_- & 0 \end{pmatrix} \] (1.51)
and
\[ \tilde{D}_0^\dagger \tilde{D}_0 = -2e|B|L_1^2 \begin{pmatrix} d_+ d_- & 0 \\ 0 & d_- d_+ \end{pmatrix} \] (1.52)

Let us consider the cases of positive and negative \( k \) separately.

**A. \( k > 0 \).**

First let us define the vacuum states in the Fock space with creation and annihilation operators \( d_+ \) and \( d_- \) functions, which are solutions of the equation
\[ d_- \chi_1(z, \bar{z}) = 0 \] (1.53)
We already know that there should be exactly \( k \) such functions. Equation (1.53) is the equation for the first component of the spinor \( \chi(z, \bar{z}) \) in (1.20) with the normalized solutions given in (1.42).

Then the eigenfunctions of the \( \tilde{D}_0^\dagger \tilde{D}_0 \) operator are the functions
\[ \begin{pmatrix} (d_+)^m \chi_1^{(n)}(z, \bar{z}) \\ 0 \end{pmatrix} \] and \[ \begin{pmatrix} 0 \\ (d_+)^{m-1} \chi_1^{(n)}(z, \bar{z}) \end{pmatrix} \] (1.54)
which correspond to eigenvalue \( \varepsilon_m^2 = \frac{4\pi k}{|\tau|} m \), where \( m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots, k - 1 \).

Let us check this for \( m > 0 \)
\[ \tilde{D}_0^\dagger \tilde{D}_0 \begin{pmatrix} (d_+)^m \chi_1^{(n)}(z, \bar{z}) \\ 0 \end{pmatrix} = -\frac{4\pi k}{|\tau|} \begin{pmatrix} d_+ d_- & 0 \\ 0 & d_- d_+ \end{pmatrix} \begin{pmatrix} (d_+)^m \chi_1^{(n)}(z, \bar{z}) \\ 0 \end{pmatrix} \] (1.55)
\[ = -\frac{4\pi k}{|\tau|} \begin{pmatrix} d_+ d_- (d_+)^m \chi_1^{(n)}(z, \bar{z}) \\ 0 \end{pmatrix} \]
From (1.50) we have \( d_- (d_+)^m = (d_+)^m d_- - m(d_+)^{m-1} \). Then
\[ \tilde{D}_0^\dagger \tilde{D}_0 \begin{pmatrix} (d_+)^m \chi_1^{(n)}(z, \bar{z}) \\ 0 \end{pmatrix} = \frac{4\pi km}{|\tau|} \begin{pmatrix} (d_+)^m \chi_1^{(n)}(z, \bar{z}) \\ 0 \end{pmatrix} \] (1.56a)

Similarly
\[ \tilde{D}_0^\dagger \tilde{D}_0 \begin{pmatrix} 0 \\ (d_+)^{m-1} \chi_1^{(n)}(z, \bar{z}) \end{pmatrix} = -\frac{4\pi k}{|\tau|} \begin{pmatrix} 0 \\ (d_- d_+)(d_+)^{m-1} \chi_1^{(n)}(z, \bar{z}) \end{pmatrix} \] (1.56b)
\[ = \frac{4\pi km}{|\tau|} \begin{pmatrix} 0 \\ (d_+)^{m-1} \chi_1^{(n)}(z, \bar{z}) \end{pmatrix} \]

Since the index \( i \) takes \( k \) values, the spectrum of the \( \tilde{D}_0^\dagger \tilde{D}_0 \) operator has a \( 2k \) fold degeneracy for \( m > 0 \), and a \( k \) fold degeneracy for \( m = 0 \).
One should remember that $\tilde{D}_0^\dagger = -\tilde{D}_0$, so for the eigenvalues of the $\tilde{D}_0$ operator we have: $\varepsilon = 0, \pm i \sqrt{\frac{4\pi km}{|m|}}$ and each value has a $k$-fold degeneracy.

The eigenfunctions of this operator have the following form ($k > 0$)

$$\psi_{+m}^{(n)}(x) = \frac{1}{N_{m,+}^{(n)}} \left( \frac{(d_+)^m\chi_1^{(n)}(z, \bar{z})}{i\sqrt{m}(d_+)^{m-1}\chi_1^{(n)}(z, \bar{z})} \right)$$

(1.57a)

for $\varepsilon = +i \sqrt{\frac{4\pi km}{|m|}}$, and

$$\psi_{-m}^{(n)}(x) = \frac{1}{N_{m,-}^{(n)}} \left( \begin{array}{c} (d_+)^m\phi_2^{(n)}(z, \bar{z}) \\ -i\sqrt{m}(d_+)^{m-1}\phi_2^{(n)}(z, \bar{z}) \end{array} \right)$$

(1.57b)

for $\varepsilon = -i \sqrt{\frac{4\pi km}{|m|}}$, $1/N_{m, \pm}^{(n)}$ is a normalization factor.

**B. $k < 0$.**

Now the vacuum state is defined as follows

$$d_+\phi_2(z, \bar{z}) = 0$$

(1.58)

There are $|k|$ vacuum states in accordance with the index theorem. They are given in (1.43).

The eigenvalues of the $\tilde{D}_0^\dagger \tilde{D}_0$ operator $\varepsilon_m^2 = -\frac{4\pi km}{|m|}$ are $2|k|$ fold degenerate for $m > 0$, and $|k|$ fold degenerate for $n = 0$, and its eigenfunctions are as follows

$$\begin{pmatrix} 0 \\ (d_-)^m\phi_2^{(n)} \end{pmatrix}$$

for $m > 0$.

(1.59)

In this case, for the eigenvalues of the $\tilde{D}_0$ operator we have: $\varepsilon = 0, \pm i \sqrt{\frac{4\pi km}{|m|}}$, $m = 1, 2, \cdots$, with a $|k|$ fold degeneracy and corresponding eigenfunctions ($n = 0, \cdots, |k| - 1$)

$$\tilde{\psi}_{+m}^{(n)}(x) = \frac{1}{N_{m,+}^{(n)}} \left( \begin{array}{c} i\sqrt{m}(d_-)^{m-1}\phi_2^{(n)}(z, \bar{z}) \\ (d_-)^m\phi_2^{(n)}(z, \bar{z}) \end{array} \right)$$

(1.60a)

for $\varepsilon = i \sqrt{\frac{4\pi |k|}{|m|}}$, and

$$\tilde{\psi}_{-m}^{(n)}(x) = \frac{1}{N_{m,-}^{(n)}} \left( \begin{array}{c} i\sqrt{m}(d_-)^{m-1}\phi_2^{(n)}(z, \bar{z}) \\ -(d_-)^m\phi_2^{(n)}(z, \bar{z}) \end{array} \right)$$

(1.60b)

for $\varepsilon = -i \sqrt{\frac{4\pi |k|}{|m|}}$.

### 2 Quantum theory

The calculation of the quantum mechanical vacuum expectation values (VEV) of observables $\Omega(A, \bar{\psi}, \bar{\psi})$ is performed with the help of the path integral formula

$$\langle \Omega(A, \bar{\psi}, \bar{\psi}) \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S[A, \bar{\psi}, \bar{\psi}]} \Omega(A, \bar{\psi}, \bar{\psi}).$$

(2.1)
Since the action (1.1) is quadratic in fermion fields the fermionic integration can easily be done. Recollecting that there are different topological sectors of gauge field configurations we have

\[ \langle \Omega(A, \bar{\psi}, \psi) \rangle = \frac{1}{Z} \sum_{k = -\infty}^{\infty} \int_{A_k} \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S[A] - \int_{\mathcal{T}} d^2\bar{\psi}D\psi} \times \Omega(A, \bar{\psi}, \psi), \]  

(2.2)

where the partition function

\[ Z = \sum_{k = -\infty}^{\infty} \int_{A_k} \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S[A] - \int_{\mathcal{T}} d^2\bar{\psi}D\psi}. \]  

(2.3)

The result of the fermionic integration depends crucially on the number \(|k|\) of zero modes of the operator \(\tilde{D}[A]\) dependent of the gauge field \(A_\mu\) from the topological sector \(A_k\) and we obtain

\[ \langle \Omega(A) \rangle = Z^{-1} \int_{A_0} \mathcal{D}A \det \tilde{D} e^{-S[A]} \Omega(A) \]  

(2.4)

\[ \langle \psi_{\alpha_1}(x_1) \bar{\psi}_{\beta_1}(y_1) \cdots \psi_{\alpha_N}(x_N) \bar{\psi}_{\beta_N}(y_N) \rangle = Z^{-1} \sum_{k = 0, \pm 1, \ldots, \pm N} L_{\alpha_1}^{[k]} \int_{A_k} \mathcal{D}A e^{-S[A]} \det'[\tilde{D}] \times \sum_P (-1)^p \chi_{\alpha_1}^{(1)}(x_{i_1}) \cdots \chi_{\alpha_k}^{(k)}(x_{i_k}) \bar{\chi}_{\beta_1}^{(1)}(y_{j_1}) \cdots \bar{\chi}_{\beta_k}^{(k)}(y_{j_k}) \times S_{\alpha_1 \beta_1, \ldots, \alpha_k \beta_k}^{(k)}(x_{i_1}, y_{j_1}; A) \cdots S_{\alpha_N \beta_N}^{(k)}(x_{i_N}, y_{j_N}; A), \]  

(2.5)

\[ Z = \int_{A_0} \mathcal{D}A e^{-S[A]} \det[\tilde{D}], \]  

(2.6)

\((\chi^{(n)}(x), \bar{\chi}^{(n)}(y))\ n = 1, \ldots, |k|\) is an orthonormal set of the zero mode wave functions and \(S^{(k)}(x, y; A)\) is a Green’s function of the \(\tilde{D}[A]\) operator with a gauge field \(A_\mu\) from the topological sector \(A_k\), \(\det'[\tilde{D}]\) is a product of nonzero eigenvalues of the dimensionless \(\tilde{D} = L_1D\) operator. The sum is taken over all possible permutations \(P = (i_1, \ldots, i_N)\) of \((1, \ldots, N), (-1)^p\) is a parity of the permutation \(P\)\(^3\).

All these expressions are formal and need regularization. We will use the Pauli-Villars regularization. It means that we will add to the action a sum

\[ \sum_{i = 1}^{r} \int d^2x \bar{\psi}_i(D - M_i) \psi_i, \]  

(2.7)

where the regulator masses \(M_i > 0, i = 1, \ldots, r\) satisfy the regulator conditions \(\sum_{i = 1}^{r} e_i(M_iL_1)^{2p} = 0, p = 1, \ldots, r - 1, \sum_{i = 1}^{r} e_i = -1\) and \(e_i = +1(-1)\), if \(\psi_i\) is a fermion (boson) field, \(r\) is a number of regular fields, sufficient to regularize all singularities which appeared in the theory\(^4\).

In what follows we will use only dimensionless operators \(\tilde{D}, \tilde{D} \bar{D}^\dagger\) etc. and will not write tilde. Then

\[ \det' D \rightarrow \det' \tilde{D} \prod_{i = 1}^{r} \det(D - M_iL_1)^{e_i} \equiv \exp \frac{1}{2} \Gamma_{reg}^{(k)}[A]. \]  

(2.8)

\(^3\)Note that \([\chi] = [\bar{\chi}] = [S^{(k)}] = [l]^{-1}.\)
It is more convenient to work with the operator 
\(DD^\dagger\) which has a nonnegative spectrum. Then (see Appendix A)

\[
\Gamma^{(k)}_{\text{reg}}[A] = 2\pi i |k| + \text{Tr} \ln DD^\dagger + \text{Tr} \sum_{i=1}^{r} e_i \ln (DD^\dagger + M_i^2 L_1^2) \tag{2.9}
\]

The prime on the trace means that the zero modes are excluded. The regularization allows the following mathematical operations leading to an evaluation of the effective action \(\Gamma^{(k)}_{\text{reg}}[A]\).

a) First, we will find the variation of \(\Gamma^{(k)}_{\text{reg}}[A]\) under a variation of \(b(x)\) in the expression of the potential \(A_\mu\) (see (1.9)), then by solving the variational equation we will obtain \(\Gamma^{(k)}_{\text{reg}}[A]\) in the following form

\[
\Gamma^{(k)}_{\text{reg}}[A] = F[b] + \alpha(k, M_i L_1), \tag{2.10}
\]

where \(\alpha(k, M_i L_1)\) does not depend on \(b(x)\).

b) Second, we will calculate \(\alpha(k, M_i L_1)\).

The regularized action (2.9) may be rewritten as follows

\[
\Gamma^{(k)}_{\text{reg}}[A] = 2\pi i |k| + |k| \sum_{i=1}^{r} e_i \ln (M_i^2 L_1^2) - \int_{0}^{\infty} \frac{dt}{t} (\text{Tr} e^{-tDD^\dagger} - |k|) \left( 1 + \sum_{i=1}^{r} e_i e^{-tM_i^2 L_1^2} \right). \tag{2.11}
\]

Therefore under the variation of \(b(x)\) we get

\[
\delta \Gamma^{(k)}_{\text{reg}}[A] = - \int_{0}^{\infty} \delta(\text{Tr} e^{-tDD^\dagger}) \left( 1 + \sum_{i=1}^{r} e_i e^{-tM_i^2 L_1^2} \right) \frac{dt}{t}. \tag{2.12}
\]

In Appendix B it is shown that

\[
\delta(\text{Tr} e^{-tDD^\dagger}) = 4e \frac{d}{dt} \text{Tr}(\gamma_5 \delta be^{-tDD^\dagger}). \tag{2.13}
\]

Substituting this formula into (2.12), we have

\[
\delta \Gamma^{(k)}_{\text{reg}}[A] = -4e \text{Tr}(\gamma_5 \delta be^{-tDD^\dagger}) \left( 1 + \sum_{i=1}^{r} e_i e^{-tM_i^2 L_1^2} \right) \bigg|_{0}^{\infty} - 4e \sum_{i=1}^{r} e_i M_i^2 L_1^2 \int_{0}^{\infty} dt \text{Tr}(\gamma_5 \delta be^{-tDD^\dagger}) e^{-tM_i^2 L_1^2}. \tag{2.14}
\]

The first term in (2.14) contributes only on the upper limit, if the \(DD^\dagger\) operator has zero modes. So when \(t \to \infty\)

\[
\text{Tr}(\gamma_5 \delta be^{-tDD^\dagger}) \to \text{Tr}(\gamma_5 \delta b P_0), \tag{2.15}
\]

where \(P_0\) is the projection operator on the subspace spanned by the zero modes. To calculate the second term we will use the heat kernel expansion of \(e^{-tDD^\dagger}\) for small \(t\)

\[
\langle x \mid e^{-tDD^\dagger} \mid x \rangle = \frac{1}{4\pi t} [1 + e\gamma_5 F_{12}(x) t + \cdots] \tag{2.16}
\]
and for $M_i \to \infty$
\[
\int_0^\infty dt \text{Tr}(\gamma_5 \delta b e^{-tDD_1^\dagger}) e^{-tM_i^2L_i^2} = \frac{1}{2\pi M_i^2L_i^2} \int_T d^2x \delta b e F_{12}(x). \tag{2.17}
\]

Finally we obtain
\[
\delta \Gamma_{\text{reg}}^{(k)}[A] = -4e \text{Tr}(\gamma_5 \delta b \mathcal{P}_0) - \frac{2e^2}{\pi} \int_T d^2x \delta b(x) F_{12}(x) = \delta 2(\ln \det \mathcal{N}_A^{(k)}) + \delta \left( \frac{e^2}{\pi} \int_T d^2xb(x) \Box b(x) \right), \tag{2.18}
\]

where $\Box = \partial_1^2 + \partial_2^2$ and $\mathcal{N}_A^{(k)}$ is a $k \times k$ matrix of the scalar products of the zero modes
\[
\mathcal{N}_A^{(k)} = ||(\chi^{(n)}, \chi^{(n')})||, \quad n, n' = 0, \ldots, k - 1.
\]

The variational equation (2.18) has the solution
\[
\Gamma_{\text{reg}}^{(k)}[A] = 2 \ln \det \mathcal{N}_A^{(k)} + \frac{e^2}{\pi} \int_T d^2x b(x) \Box b(x) + \alpha(k, M_iL_1), \tag{2.19}
\]

where $\alpha(k, M_iL_1)$ is the integration constant.

Now let us find $\alpha(k, M_iL_1)$. From (2.9) and (2.10) it follows that
\[
\alpha(k, M_iL_1) = \Gamma_{\text{reg}}^{(k)}[A] |_{b=0} = 2\pi i |k| + |k| \sum_{i=1}^r e_i \ln M_i^2L_i^2 + \text{Tr}' \left\{ \sum_{i=1}^r e_i \ln (D_0D_0^\dagger + M_i^2L_i^2) + \ln(D_0D_0^\dagger) \right\}. \tag{2.20}
\]

To calculate the trace in the second line of (2.20) we use the Lüscher’s formula (see Appendix C) and
\[
\text{Tr}' \left\{ \sum_{i=1}^r e_i \ln (D_0D_0^\dagger + M_i^2L_i^2) + \ln(D_0D_0^\dagger) \right\} \tag{2.21}
\]
\[
= \sum_{i=1}^r e_i \left( -r_1(D_0D_0^\dagger) M_i^2L_i^2 \ln M_i^2L_i^2 + \zeta(0 \mid D_0D_0^\dagger) \ln M_i^2L_i^2 \right) - \zeta'(0 \mid D_0D_0^\dagger),
\]

where terms vanishing for $M_i \to \infty$ have been neglected, $\zeta(s) \equiv \zeta(s \mid D_0D_0^\dagger)$ is a $\zeta$-function of the $D_0D_0^\dagger$ operator
\[
\zeta(s \mid D_0D_0^\dagger) = \sum_m (\lambda_m)^{-s} d_m \tag{2.22}
\]
is defined for sufficiently large real part Res and a sum is taken over all nonzero eigenvalues $\lambda_m$ of $D_0D_0^\dagger$ operator with multiplicity $d_m$. This function has a pole at $s = 1$ with the residue $r_1(D_0D_0^\dagger)$
\[
\zeta(s \mid D_0D_0^\dagger) = \frac{r_1(D_0D_0^\dagger)}{s - 1} + \cdots. \tag{2.23}
\]

Thus we see, that in order to find $\Gamma_{\text{reg}}^{(k)}[A]$, we should know $r_1(D_0D_0^\dagger)$, $\zeta(0 \mid D_0D_0^\dagger)$ and $\zeta'(0 \mid D_0D_0^\dagger)$. But the spectrum of the $D_0D_0^\dagger$ operator or the eigenvalues $\lambda_m$ and their
multiplicity is known and after some calculations connected with summation of infinite series (see Appendix D) we get the final result

\[
\Gamma^{(k)}_{\text{reg}}[A] = \frac{e^2}{\pi} \int_{\mathcal{T}} d^2 x b(x) \Box b(x) + 2 \ln \text{det} \mathcal{N}^{(k)}_A - |k| \ln \frac{2|k|}{|\tau|} + \delta_{0k} \left\{ - \frac{2 \ln \theta_1(t | \tau) \eta^{-1}(\tau)^2 - \frac{4\pi}{|\tau| t_1^2}}{2\pi} \right\} + \sum_{j=1}^{r} e_j \left( - \frac{|M_j^2 L_1^2|}{2\pi} \ln(M_j^2 L_1^2) + 2\pi i |k| \right).
\]

We want to discuss shortly the implications of this result for the path integral formula after fermion integration, Eq. (2.5). The normalization factor \(Z\) follows from the condition \(\langle 1 | 1 \rangle = 1\) and therefore is determined by the integration over the trivial sector \(k = 0\) only. The \(k\) independent parts of \(\Gamma^{(k)}_{\text{reg}}[A]\) can be factorized out of the sum over the gauge sectors. These are the regulator mass dependent term and the term with \(b(x)\). Therefore the regulator mass term in \(\Gamma^{(k)}_{\text{reg}}[A]\) cancels against that of \(Z\) and the normalized result is independent of these masses as required for a consistent renormalization scheme. Also the integration over the pure gauge component \(Da\) factorizes from all other integrations. Thus we let this contribution cancel with the corresponding of \(Z\) without going into further details of gauge fixing.

After fermion integration the gauge field dependent part of the action in the trivial sector is

\[
S'[b(x)] = \frac{1}{2} \int_{\mathcal{T}} d^2 x \left( F_{12}(x) J_{12}(x) - \frac{e^2}{\pi} b(x) \Box b(x) \right)
= \frac{1}{2} \int_{\mathcal{T}} d^2 x b(x) \Box (\Box - m^2) b(x),
\]

where \(m^2 = \frac{e^2}{\pi}\). As we see this action is bilinear in \(b(x)\) and hence describes free particles. The generating functional of its correlation functions can be calculated by Gaussian integration:

\[
\frac{1}{Z_0} \int \mathcal{D}b e^{-S'[b]} e^\int_{\mathcal{T}} d^2 x J(x) b(x) = e^{\frac{1}{2} \int_{\mathcal{T}} d^2 x \int_{\mathcal{T}} d^2 x' J(x) G(x-x') J(x')}
\]

and

\[
Z_0 = \int \mathcal{D}b e^{-S'[b]}.
\]

The propagator \(G(x - x')\) related to \(S'[b]\) obeys the following equation

\[
\Box (\Box - m^2) G(x - x') = \delta^{(4)}(x - x') \equiv \delta(x - x') - \frac{1}{L_1 L_2}
\]

and it plays an important role in future calculations. We may express \(G(x)\) by the eigenfunctions and eigenvalues of the Laplacian \(\Box\) on \(\mathcal{T}\):

\[
G(x) = \frac{1}{m^2 L_1 L_2} \sum_n \left\{ m^{2n} e^{\frac{2\pi}{L_2} (n_1 x_1 + |\tau|^{-1} n_2 x_2)} \left( \frac{2\pi}{L_1} \right)^2 (n_1^2 + |\tau|^{-2} n_2^2) - m^{2n} e^{\frac{2\pi}{L_1} (n_1 x_1 + |\tau|^{-1} n_2 x_2)} \left( m^2 + \frac{2\pi}{L_1} \right)^2 (n_1^2 + |\tau|^{-2} n_2^2) \right\} \equiv \frac{1}{m^2} \{ G_0(x) - G_m(x) \}.
\]
The summation $\sum'_{n_1}$ excludes $n_1 = n_2 = 0$. Therefore $G_0(x)$ is the Green’s function of the Laplacian on $\mathcal{T}$, which as an integral operator transforms a constant function into zero $G_0 \ast (\text{const}) = 0$. It has a representation (see Appendix E)

$$G_0(x) = \frac{1}{L_1 L_2} \sum'_{n_1} \frac{e^{n_1 L_1}}{(\frac{2\pi n_1}{L_1})^2 (\frac{2\pi n_1}{L_1})^2}$$

$$= -\frac{1}{2\pi} \ln \left| \theta_1(z | \tau) \right| + \frac{x^2}{2L_1 L_2} + \frac{\tau}{12} - \frac{1}{2\pi} \ln 2\pi \eta_0^2(\tau)$$

$$= -\frac{1}{2\pi} \ln \left( 2\pi \eta^2(\tau) e^{-\frac{\pi^2 x^2}{\tau^2}} \right) \left| \theta_1(z | \tau) \right| \theta_1'(0 | \tau) \right).$$

(2.30)

Note that $\theta_1'(0 | \tau) = 2\pi \eta^3(\tau)$. With the help of the formula

$$\sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n x}}{a^2 + n^2} = \frac{\pi \cosh(\pi a[1 - 2|x|])}{a \sinh(\pi a)}$$

(2.31)

we can do one summation in $G_m(x)$

$$G_m(x) + \frac{1}{L_1 L_2 m^2} = G_m(x) = \sum_n \frac{e^{2\pi i n x_1}}{2\xi(n) \sinh(\pi \xi(n)/2)} \cosh[\tau \xi(n)(1/2 - x_2/L_2)]$$

$$= \sum_n \frac{e^{2\pi i n x_1 - \xi(n) \frac{x_1}{L_1}}}{2\xi(n) \sinh(\pi \xi(n)/2)}$$

(2.32)

$$= \frac{1}{2\pi} K_0(m|x|) = \frac{1}{4\pi^2} \int \int \frac{d^2pe^{ipx}}{p^2 + m^2},$$

where $\xi(n) = \sqrt{4\pi^2 n^2 + \frac{L_1^2 m^2}{L_2^2 m^2}}$ and $\frac{1}{2\pi} K_0(m|x|)$ is the Green’s function of the free particle with a mass $m$ in the infinite two dimensional Euclidean space-time (to find the limit $L_1 \to \infty$ we used Euler’s summation formula) which has the following asymptotics [24]

$$K_0(m|x|) \to -\left[ \ln \frac{m|x|}{2} + \gamma \right], \quad |x| \to 0$$

(2.33)

$$K_0(m|x|) \to \frac{\sqrt{\pi}}{2m|x|} e^{-m|x|} \left[ 1 + O \left( \frac{1}{m|x|} \right) \right], \quad |x| \to \infty$$

(2.34)

3 Applications

3.1 Average of fermion bilinears

Now we calculate the vacuum expectation value of the gauge invariant fermion bilinear, using the general formula [25] (we do gauge invariant point-splitting)

$$\langle M(x) \rangle = \langle \bar{\psi}(x + \zeta) \Gamma e^{ix \zeta} \psi(x - \zeta) \rangle$$

$$= \frac{1}{Z} \sum_k \int_{A_k} D\bar{\psi} D\psi D\mathcal{A} e^{-\mathcal{S}[\mathcal{A}] - \mathcal{S}_F[\bar{\psi}, \psi, \mathcal{A}]} \bar{\psi}(x + \zeta) \Gamma \psi(x - \zeta)$$

$$= \frac{1}{Z} \sum_k \int_{A_k} D\mathcal{A} e^{-\mathcal{S}[\mathcal{A}]} I_k(x, \zeta; \Gamma),$$

(3.1)
there is the contribution only from the topological sectors $\Gamma = \Gamma_{\text{trivial}}$, and for scalar $\Gamma = \Gamma_{\text{vector}}$ configurations

For the sector $\Gamma = \Gamma_{\text{vector}}$ or pseudoscalar $\Gamma = \Gamma_{\text{pseudoscalar}}$ currents there is the contribution only from the trivial sector, and for scalar $\Gamma = \Gamma_{\text{scalar}}$ or pseudovector $\Gamma = \Gamma_{\text{pseudovector}}$ currents there is the contribution only from the topological sectors $|k| = 1$.

We see that for the vector $\Gamma = \Gamma_{\text{vector}}$ or pseudovector $\Gamma = \Gamma_{\text{pseudovector}}$ currents there is the contribution only from the trivial sector, and for scalar $\Gamma = \Gamma_{\text{scalar}}$ or pseudoscalar $\Gamma = \Gamma_{\text{pseudoscalar}}$ there is the contribution only from the topological sectors $|k| = 1$.

We get, using (3.1) and (3.10)

\[
\langle \tilde{\psi}(x + \zeta) \gamma_{\alpha} e^{ie \int_{x-\zeta}^{x+\zeta} A_{\beta}(y) dy} \psi(x - \zeta) \rangle \approx \frac{2}{\pi e^{I_{2}(\zeta)}} \left[ \delta_{\mu\alpha} \cos I_{1}(\zeta) + \epsilon_{\mu\alpha} \sin I_{1}(\zeta) \right] \times \frac{\zeta_{\mu}}{2|\zeta|^{2}} - \langle K_{\mu}(t) \rangle^{(0)}_{t},
\]

where

\[
I_{1}(\zeta) \equiv e^{2} \epsilon_{\alpha\beta} \int_{-\zeta}^{\zeta} [\partial_{\beta} G(y + \zeta) - \partial_{\beta} G(y - \zeta)] dy_{\alpha},
\]

\[
I_{2}(\zeta) \equiv \frac{e^{2}}{2} \epsilon_{\alpha\beta} \epsilon_{\gamma\nu} \int_{-\zeta}^{\zeta} \partial_{\beta} \partial_{\nu} G(x - y) dx_{\alpha} dy_{\gamma},
\]

$G(x)$ is the propagator of the $b$-field (2.29) and $\langle \ldots \rangle^{(0)}$ is the averaging over the toron configurations

\[
\langle \ldots \rangle^{(0)}_{t} = \frac{1}{Z_{\text{toron}}} \int_{0}^{T_{1}} dt_{1} \int_{0}^{T_{2}} dt_{2} e^{\frac{1}{2} \Gamma^{(0)}(t)} \ldots,
\]
and

\[
Z_{toron} = \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 e^{\frac{\pi}{2} \Gamma(0)(t)} \text{nonumber} \tag{3.12}
\]

\[
= \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 |\vartheta_1(i\tilde{t}|\tau)|^2 e^{-\frac{\pi t^2}{e^2 \sqrt{2|\tau|L_1L_2}}}.
\tag{3.13}
\]

Using the relation \(K_\mu(t) = \frac{i\pi}{2eL_1L_2} \frac{\partial}{\partial t} \Gamma(0)(t)\) (see (F.11)-(F.12)), one can easily prove that \(\langle K_\mu(t) \rangle |^{(0)}_t = 0\).

We have the singular behavior for the current

\[
\langle \bar{\psi}(x + \zeta) \gamma_{\alpha} e^{\frac{i e}{\sqrt{\pi}} \int_{x-\zeta}^{x+\zeta} A_\sigma(y) dy} \psi(x - \zeta) \rangle \sim \frac{\zeta_{\alpha}}{\zeta \to 0 |\zeta|^2} + o(|\zeta|). \tag{3.14}
\]

For the scalar \((\Gamma = 1)\) in (3.1) (in the limit \(\zeta \to 0\)) , using (3.2),(2.24) and (3.11) we get

\[
\langle \bar{\psi}(x) \psi(x) \rangle = \frac{2\eta^2(\tau)}{L_1} \frac{e^{2G(0)-2\pi^2/e^2L_1L_2}}{L_1} = \langle \bar{\psi}(x) \gamma_5 \psi(x) \rangle \tag{3.15}
\]

(see [9] and [12]).

In the limit \(L_1 = L_2 = L \to \infty\) from (2.29)-(2.32) it follows that

\[
2e^2G(0) = \ln \eta^2(\tau) + \ln \frac{m}{4\pi} + \gamma + \ln L_1, \tag{3.16}
\]

where \(m = \frac{e}{\sqrt{\pi}}\) and \(\gamma\) is the Euler constant, and

\[
\langle \bar{\psi}(x) \psi(x) \rangle = \frac{e^{\gamma}}{\sqrt{\pi \cdot 2\pi}} \tag{3.17}
\]

3.2 Correlation functions for fermion bilinears

Using our definition of \(M(x)\) we will consider the correlation functions for gauge invariant (regularized) bilinears

\[
\langle M(x)M'(y) \rangle = \lim_{\zeta \to 0} \frac{|\tilde{\psi}(x + \zeta) \bar{\Gamma} \psi(x - \zeta) \bar{\psi}(y - \zeta') \Gamma' \psi(y') \rangle}{\langle \bar{\psi}(x + \zeta) \bar{\psi}(y + \zeta') \rangle} \tag{3.18}
\]

\[
- \langle \bar{\psi}(x + \zeta) \bar{\psi}(y + \zeta') \bar{\psi}(y - \zeta') \rangle \langle \bar{\psi}(y + \zeta') \Gamma' \psi(y - \zeta') \rangle\}
\]

where \(\Gamma = \Gamma \exp \left\{ \frac{ie}{x-\zeta} A_\alpha(x') dx' \right\}, \bar{\Gamma'} = \bar{\Gamma'} \exp \left\{ \frac{ie}{y-\zeta'} A_\delta(y') dy' \right\}\),

\[
\Gamma, \bar{\Gamma'} = \gamma_\mu, \gamma_5, 1.
\]
From the general formula (2.5) we have

\[
\langle \bar{\psi}(x + \zeta) \bar{\Gamma} \psi(x - \zeta) \bar{\psi}(y + \zeta') \Gamma' \psi(y - \zeta') \rangle = \frac{1}{Z} \sum_{k} \int_{A_k} \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A e^{-S[A] - S_{\psi,\bar{\psi},A}} \times \bar{\psi}(x + \zeta) \bar{\Gamma} \psi(x - \zeta) \bar{\psi}(y + \zeta') \Gamma' \psi(y - \zeta')
\]

\[
= \frac{1}{Z} \sum_{k} \int_{A_k} \mathcal{D}A e^{-S[A]} I_{(k)}[x, \zeta; y, \zeta'; \Gamma, \Gamma']
\]

and only gauge sectors \( C^\mathcal{H}^{(k)} \), with \( k = 0, \pm 1, \pm 2 \) contribute.

The trivial sector \( k = 0 \). For this sector

\[
I_{(0)}[x, \zeta; y, \zeta'; \Gamma, \Gamma'] = \left\{ -T_{\mu\nu}[a, b, x, \zeta, y, \zeta'; \Gamma, \Gamma'] S_t^{(\mu)}(x - \zeta - y - \zeta') S_t^{(\nu)}(y - \zeta' - x - \zeta) + T_{\mu}[a, b, x, \zeta; \Gamma] T_{\nu}[a, b, y, \zeta'; \Gamma'] S_t^{(\mu)}(-2\zeta) S_t^{(\nu)}(-2\zeta') \right\} e^{\frac{i}{2} \Gamma^{(0)}[A]},
\]

where

\[
T_{\mu\nu}[a, b, x, \zeta, y, \zeta'; \Gamma, \Gamma'] \\
\equiv e^{-i \Delta a(x) - i \Delta a(y)} \left\{ \text{Tr} (\gamma_\mu \Gamma' \gamma_\nu \Gamma) \cosh b_1 \cosh b_2 + \text{Tr} (\gamma_\mu \Gamma' \gamma_5 \Gamma) \sinh b_1 \cos b_2 + \text{Tr} (\gamma_\mu \Gamma' \gamma_5 \gamma_\nu \Gamma) \sinh b_1 \sinh b_2 \right\}
\]

contributes to the connected part of the corresponding diagram and \( T_{\mu}[a, b, x, \zeta; \Gamma] \), given in (3.4), contributes to the disconnected part of this diagram.

\[
\Delta a(x) \equiv a(x + \zeta) - a(x - \zeta),
\]

\[
\Delta a(y) \equiv a(y + \zeta') - a(y - \zeta'),
\]

\[
b_1 \equiv eb(x - \zeta) - eb(y + \zeta'),
\]

\[
b_2 \equiv eb(y - \zeta') - eb(x + \zeta).
\]

Sectors \( k = \pm 1 \). For the sector \( k = 1 \)

\[
I_{(1)}[x, \zeta; y, \zeta'; \Gamma, \Gamma'] = L_1 e^{\frac{1}{2} \Gamma^{(1)}[A]} (\det \mathcal{N}^{(1)}_A)^{-1} \times \left\{ \hat{\chi}(x + \zeta) \hat{\Gamma} \hat{\chi}(x - \zeta) T_{\mu}[a, b; y, \zeta', \Gamma'] S_t^{(1)(\mu)}(y - \zeta', y + \zeta') + T_{\mu}[a; b, x, \zeta; \Gamma] S_t^{(1)(\mu)}(x - \zeta, x + \zeta) \hat{\chi}(y + \zeta') \Gamma' \hat{\chi}(y - \zeta') - \hat{\chi}(y + \zeta') \Gamma e^{ia(x - \zeta)} \gamma_\mu e^{-ia(x + \zeta)} \Gamma' \hat{\chi}(x - \zeta) S_t^{(1)(\mu)}(y - \zeta', x + \zeta) - \hat{\chi}(x + \zeta) \Gamma e^{ia(x - \zeta)} \gamma_\mu e^{-ia(x + \zeta')} \Gamma' \hat{\chi}(y - \zeta') S_t^{(1)(\mu)}(x - \zeta, y + \zeta') \right\},
\]

\[
\hat{\chi}(x) = e^{ie[a(x) - ib(x)]} \chi(x) \quad \text{and} \quad \chi(x) \text{ is a zero mode in the sector } k = 1 \text{ given in Eq. (1.34)} \]

and a similar expression for \( I_{(-1)}[x, \zeta; y, \zeta'; \Gamma, \Gamma'] \) for the sector \( k = -1 \).

Sectors \( k = \pm 2 \). For \( k = 2 \) we have
\[ I_{(2)}[x, \zeta; \zeta', \Gamma, \Gamma'] = L_2^2 e^{\frac{i\Gamma z}{2}[A]}(\det N_A^{(2)})^{-1} \times \left\{ \hat{\chi}^{(0)}(x+\zeta) \Gamma \hat{\chi}^{(0)}(x-\zeta) \hat{\chi}^{(1)}(y+\zeta') \Gamma' \hat{\chi}^{(1)}(y-\zeta') \right. \\
\left. + \hat{\chi}^{(1)}(x+\zeta) \Gamma \hat{\chi}^{(1)}(x-\zeta) \hat{\chi}^{(0)}(y+\zeta') \Gamma' \hat{\chi}^{(0)}(y-\zeta') \right. \\
\left. - \hat{\chi}^{(0)}(x+\zeta) \Gamma \hat{\chi}^{(1)}(x-\zeta) \hat{\chi}^{(0)}(y+\zeta') \Gamma' \hat{\chi}^{(0)}(y-\zeta') \right. \\
\left. - \hat{\chi}^{(1)}(x+\zeta) \Gamma \hat{\chi}^{(0)}(x-\zeta) \hat{\chi}^{(0)}(y+\zeta') \Gamma' \hat{\chi}^{(0)}(y-\zeta') \right\}, \tag{3.27} \]

where \( \hat{\chi}^{(0)} \) and \( \hat{\chi}^{(1)} \) are two independent zero modes of \( D_A \) operator for \( k = 2 \) and a similar expression for \( I_{(-2)}[x, \zeta; \zeta', \Gamma, \Gamma'] \) for the sector \( k = -2 \).

In these formulas the vector-potential of the electromagnetic field from the topological sector \( A_k \), is given in \((1.8)\),

\[ S^{(k)}_t(x,y) = \gamma_\mu S^{(k)(\mu)}_t(x,y) \tag{3.28} \]

is the Green’s function of the \( D_0 \) operator given in \((1.13)\) \( (S^{(0)(\mu)}_t(x,y) \equiv S^{(\mu)}_t(x-y)) \). In general, the propagator \( S^{(k)}(x,y;A) \) of the fermions in the background field \( A_\mu(x) \) from \((1.8)\) can be expressed as follows

\[ S^{(k)}(x,y;A) = e^{ie\alpha(x)}S^{(k)}_t(x,y)e^{-ie\alpha'(y)}, \tag{3.29} \]

where \( \alpha(x) = a(x) - i\gamma_5 b(x) \), \( \alpha'(y) = a(y) + i\gamma_5 b(y) \), (see Appendix G)

### 3.2.1 Currents \( \Gamma = i\gamma_\alpha, \Gamma' = i\gamma_\beta \). \( (M(x) \equiv j_\alpha(x), M(y) \equiv j_\beta(y)) \)

In this case we have contribution only from the trivial sector \( k = 0 \). The evaluation of the connected and disconnected parts is straightforward (see Appendix F):

\[ \langle j_\alpha(x)j_\beta(y) \rangle_c = 2(S^{(\beta)}_t(x-y)S^{(\alpha)}_t(y-x) + S^{(\alpha)}_t(x-y)S^{(\beta)}_t(y-x)) - \delta^{\alpha\beta}S^{(\mu)}_t(x-y)S^{(\mu)}_t(y-x), \tag{3.30} \]

\[ \langle j_\alpha(x)j_\beta(y) \rangle_d = \left\{ \frac{1}{\pi^2} \frac{\zeta_\alpha \zeta_\beta}{|\zeta|^2} \right\} + \frac{e^2}{\pi^2} \epsilon_{\alpha\rho} \epsilon_{\beta\sigma} \partial_\rho \partial_\sigma G(x-y) + R_{\alpha\beta} \right\}, \tag{3.31} \]

\[ R_{11} = \frac{1}{2\pi^2} [\langle K_1(t)K_1(t) \rangle_t^{(0)} - \langle K_2(t)K_2(t) \rangle_t^{(0)}], \tag{3.32} \]

\[ R_{12} = \frac{1}{\pi^2} \langle K_1(t)K_2(t) \rangle_t^{(0)}. \tag{3.33} \]

Applying an ‘addition theorem’ of the \( \partial \)-functions

\[ \partial_1(z+w)\partial_1(z-w) = \frac{1}{\partial_1^2(0)}(\partial_1^2(z)\partial_1^2(w) - \partial_1^2(z)\partial_1^2(w)) \tag{3.34} \]

and using the ‘light cone components’ \( j_\pm(x) = j_1(x) \mp ij_2(x) \), we get for the ‘connected’ part a simple result in terms of the Weierstrass function \( \wp(z) \) \[ [18] \ [19] \]
\begin{align*}
\langle j_+(x)j_+(0) \rangle_c &= \langle j_-(x)j_-(0) \rangle_c^* = -\frac{1}{\pi^2 L_1^2} \{ \varphi(z) - \langle \varphi(\bar{t}) \rangle_t^{(0)} \}, \\
\langle j_+(x)j_-(0) \rangle_c &= 0, \\
\langle j_+(x)j_-(0) \rangle_c &= -\frac{1}{\pi^2 L_1^2} \frac{d^2}{dz^2} \ln \delta_1(z | \tau) + \frac{1}{3} \frac{\varphi''''(0 | \tau)}{\varphi'(0 | \tau)},
\end{align*}

or returning to Cartesian components

\begin{align*}
\langle j_1(x)j_1(0) \rangle_c &= -\frac{1}{4\pi^2 L_1^2} \{ \varphi(z) + \varphi(\bar{z}) - \langle \varphi(\bar{t}) \rangle_t^{(0)} - \langle \varphi(t) \rangle_t^{(0)} \} \\
&= -\langle j_2(x)j_2(0) \rangle_c, \\
\langle j_1(x)j_2(0) \rangle_c &= \frac{i}{4\pi^2 L_1^2} \{ \varphi(z) - \langle \varphi(t) \rangle_t^{(0)} - \varphi(\bar{z}) + \langle \varphi(\bar{t}) \rangle_t^{(0)} \} \\
&= \langle j_2(x)j_1(0) \rangle_c.
\end{align*}

The averaging over the toron configuration results in

\begin{align*}
\langle \varphi(t) \rangle_t^{(0)} &= L_1^2 \{ [K_1(t) + iK_2(t)] \}^{(0)} + \frac{\pi}{|\tau|} + \frac{1}{3} \frac{\varphi''''(0 | \tau)}{\varphi'(0 | \tau)} \\
&= \langle \varphi(\bar{t}) \rangle_t^{(0)}
\end{align*}

and

\begin{align*}
\langle j_1(x)j_1(0) \rangle_c &= \frac{1}{\pi} \partial_2^2 G_0(x) + \frac{1}{2\pi^2} \{ [K_1(t)K_1(t)] \}^{(0)} - \langle K_2(t)K_2(t) \rangle_t^{(0)}], \\
\langle j_1(x)j_2(0) \rangle_c &= \frac{1}{\pi} \partial_1 \partial_2 G_0(x) + \frac{1}{\pi^2} \langle K_1(t)K_2(t) \rangle_t^{(0)}.
\end{align*}

Adding the connected and disconnected parts we get for the correlation function of the currents

\begin{align*}
\langle j_\alpha(x)j_\beta(y) \rangle = \frac{1}{\pi} \epsilon_{\alpha\rho} \epsilon_{\beta\sigma} \partial_\rho \partial_\sigma G_m(x - y).
\end{align*}

### 3.2.2 Clustering

The clustering property means that the vacuum matrix element of a product of local operators factorizes when their space-like separation becomes large. Usually this property is well established in the theories, where we have massive particles from the beginning. We will prove that in the Schwinger model (where massless fermions interact with a gauge field) this property also holds.

In order to do that we should calculate the four point function \( \langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0) \rangle \), and as we have demonstrated in the previous section for this aim one should consider all topological sectors with \(|k| \leq 2\). In fact, in this case only the sectors \( k = 0 \) and \(|k| = 2\) contribute (see Eq.(3.18)). Then the \( L_1 = L_2 = L \to \infty \) limit
should be taken, and we will prove that as in the case of the Schwinger model of the sphere [22] the trivial sector gives only a half of the value expected, the other half comes from the sector $|k| = 2$.

From the general expression (3.19) by calculating traces we obtain:

$$
\langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0) \rangle = -2R(x, 0)\langle S_t^{(\mu)}(x)S_t^{(\mu)}(-x) \rangle_{t}^{(0)}
$$

$$
+ V(x, 0) \left\{ (\bar{x}^{(0)}(x)\chi^{(0)}(x)\bar{\chi}^{(1)}(0)\chi^{(1)}(0) + \bar{\chi}^{(1)}(x)\chi^{(1)}(x)\bar{\chi}^{(0)}(0)\chi^{(0)}(0)

- \bar{x}^{(0)}(x)\chi^{(1)}(x)\bar{\chi}^{(1)}(0)\chi^{(0)}(0) - \bar{\chi}^{(1)}(x)\chi^{(0)}(x)\bar{\chi}^{(0)}(0)\chi^{(1)}(0))_{t}^{(2)}

+ \langle \bar{\phi}(0)(x)\phi^{(0)}(x)\bar{\phi}^{(1)}(0)(x)\phi^{(1)}(0) + \bar{\phi}^{(1)}(x)\phi^{(1)}(x)\bar{\phi}^{(0)}(0)(x)\phi^{(0)}(0)

- \bar{\phi}^{(0)}(x)\phi^{(1)}(x)\bar{\phi}^{(1)}(0)(x)\phi^{(0)}(0) - \bar{\phi}^{(1)}(x)\phi^{(0)}(x)\bar{\phi}^{(0)}(0)(x)\phi^{(1)}(0))_{t}^{(2)} \right\}
$$

where $R(x, 0) = e^{4\varepsilon [G(0) - G(x)]}$ and $V(x, 0) = \frac{\eta^2(\tau)\tau|L_2^2|}{4}e^{4\pi^2[G(0) + G(x)]}e^{-\frac{8\pi z}{\varepsilon_1\varepsilon_2\varepsilon_2}}$

One can easily show that the contributions from the sectors with $k = +2$ and $k = -2$ are equal.

So

$$
\langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0) \rangle = -2R(x, 0)\langle S_t^{(\mu)}(x)S_t^{(\mu)}(-x) \rangle_{t}^{(0)}
$$

$$
+ V(x, 0)\langle |\chi_1^{(0)}(x)\chi_1^{(1)}(0) - \chi_1^{(1)}(x)\chi_1^{(0)}(0)\rangle_{t}^{(2)},
$$

where $\chi_1^{(n)}(x)$ is the first component of the zero mode $\chi^{(n)}(x)$ with the positive chirality.

For the $t$-averaging in the trivial sector $(k = 0)$ we get

$$
\langle S_t^{(\mu)}(x)S_t^{(\mu)}(-x) \rangle_{t}^{(0)} = -\frac{\eta^6(\tau)}{|L_1\vartheta_1(z|\tau)|^2}e^{\frac{2\pi}{\varepsilon_1\varepsilon_2\varepsilon_2}}
$$

and

$$
R(x, 0) = e^{4\varepsilon^2 G(0)}e^{-4\pi G_0(x)}e^{4\pi G_m(x)}.
$$

From the general expression (3.42) for the zero modes with the positive chirality in the sector $k = 2$ we have two of them

$$
\chi_1^{(0)}(x) = \left(\frac{4}{|\tau|}\right)^{1/4}\frac{1}{L_1}e^{i\frac{x}{2}\mu t_\mu + 2\pi i z^2 - z^2} \vartheta_3(2z'|2\tau).
$$

and

$$
\chi_1^{(1)}(x) = \left(\frac{4}{|\tau|}\right)^{1/4}\frac{1}{L_1}e^{i\frac{x}{2}\mu t_\mu + 2\pi i z^2 - z^2 - \frac{|z|}{2} + 2\pi i z'} \vartheta_3(2z' + \tau|2\tau)
$$

Using the relation between $\vartheta$-functions [13]

$$
\vartheta_3(z + \frac{\tau}{2}|\tau) = e^{-i\pi(z + \frac{1}{2})}\vartheta_2(z|\tau),
$$

$\chi_1^{(1)}(x)$ can be rewritten as follows

$$
\chi_1^{(1)}(x) = \left(\frac{4}{|\tau|}\right)^{1/4}\frac{1}{L_1}e^{i\frac{x}{2}\mu t_\mu + 2\pi i z^2 - z^2} \vartheta_2(2z'|2\tau).
$$
With the help of the formula
\[ \vartheta_3(2v|2\tau)\vartheta_2(2u|2\tau) - \vartheta_2(2v|2\tau)\vartheta_3(2u|2\tau) = \vartheta_1(v + u|\tau)\vartheta_1(v - u|\tau), \] (3.52)
which follows from formulas (118) and (128) in [22], we obtain

\[ \chi_1^{(0)}(x)\chi_1^{(1)}(0) - \chi_1^{(1)}(x)\chi_1^{(0)}(0) = \frac{2}{|\tau|^2 L_1^2} e^{\frac{i\pi x_1^2 + \frac{\pi}{2}\tau(z^2 - \bar{z}^2)}{2|\tau|}} \times \vartheta_1(z|\tau)\vartheta_1(z + \bar{t}|\tau)e^{\frac{2\pi x_1^2}{L_1}} \] (3.53)

and

\[ \langle |\chi_1^{(0)}(x)\chi_1^{(1)}(0) - \chi_1^{(1)}(x)\chi_1^{(0)}(0)|^2 \rangle_\tau = \frac{4}{|\tau|L_1^2} e^{\frac{4\pi x_1^2}{L_1}} \times \vartheta_1(z|\tau)^2 e^{-2\pi|\tau|^2 + \frac{4\pi x_1^2}{L_1}} \] (3.54)

One can easily check that in the last formula
\[ \langle...\rangle_\tau = e^{\frac{2\pi x_1^2}{L_1}}. \] (3.55)

So
\[ \langle |\chi_1^{(0)}(x)\chi_1^{(1)}(0) - \chi_1^{(1)}(x)\chi_1^{(0)}(0)|^2 \rangle_\tau = \frac{4}{|\tau|L_1^2} e^{\frac{2\pi x_1^2}{L_1}} \vartheta_1(z|\tau)^2. \] (3.56)

From (3.53)-(3.57) and (3.56) we get
\[ \langle \tilde{\psi}(x)\psi(x)\tilde{\psi}(0)\psi(0) \rangle = \frac{2\eta^4(\tau)}{L_1^2} e^{4\pi G(0)} [e^{4\pi G_m(x)} + e^{-4\pi G_m(x)} - \frac{8\pi}{L_1^2}] \] (3.57)
and from (2.32) and (3.13)
\[ \langle \tilde{\psi}(x)\psi(x)\tilde{\psi}(0)\psi(0) \rangle = \langle \tilde{\psi}(x)\psi(x) \rangle^2 \cosh 4\pi \tilde{G}_m(x). \] (3.58)

So in the flat space limit we have, using (3.17)
\[ \langle \tilde{\psi}(x)\psi(x)\tilde{\psi}(0)\psi(0) \rangle_{L_1=L_2=\infty} \to \frac{e^{2\gamma}}{\pi} \frac{e^2}{8\pi^2} \left\{ e^{2K_0(m|x|)} + e^{-2K_0(m|x|)} \right\} \] (3.59)
and when \(|x| \to \infty\) with the help of (2.33)
\[ \langle \tilde{\psi}(x)\psi(x)\tilde{\psi}(0)\psi(0) \rangle \to \langle \tilde{\psi}(x)\psi(x) \rangle^2 \to \infty \left( \frac{e^{e^\gamma}}{\sqrt{\pi} 2\pi} \right)^2 = \langle \tilde{\psi}\psi \rangle^2. \] (3.60)

Thus the clustering property is indeed satisfied.
Conclusions

Using the path integral approach we have performed a complete analysis of the Schwinger model on the torus. This model is not just a generalization to the ordinary SM in the infinite space time. The compactification enables to investigate the model in a mathematically more satisfactory way, keeping all singularities under control. The relevant differential operators have discrete spectra and the path integrals in the quantum theory can be properly defined if one uses their eigenfunctions.

All configurations of the electromagnetic potential (abelian gauge field) defined on the torus can be classified according to their topological charge. The nontrivial gauge field topology implies the occurrence of fermionic zero modes, which need a special treatment in the quantum theory and contribute to correlation functions of fermionic fields. For gauge fields of the topological charge $k$, the massless Dirac operator possesses exactly $|k|$ zero modes. We have obtained explicit expressions for them and calculated the spectrum of the Dirac operator in any topological sector.

In quantum theory using the Pauli-Villars regularization to remove the ultraviolet divergences occurring in the fermionic path integral, we have calculated the part of the gauge field effective action, which appears due to the fermion integration (Eq.\(\text{(2.24)}\)). Effect of the torons (zero modes of the gauge potential) on the fermion fields reveals itself in the toron effective action $\Gamma^{(0)}(t)$ (Eq. \(\text{(3.13)}\)), which rules their dynamics and controls infrared singularities.

We have found the propagator of the gauge field on the torus, which is expressed in terms of massless and massive Green's functions of the Laplacian on the torus, and calculated the fermionic propagators in the background toron field and in topologically nontrivial gauge field (Appendices F and G).

Finally, we have explicitly calculated several expectation values of physical interest. Toron averaging (Eq.\(\text{(3.11)}\)) assures a translational invariant distribution of the symmetry breaking zero modes in the topologically non-trivial sectors. For the two- and four-point functions of fermion fields sectors with $|k| \leq 1$ and $|k| \leq 2$, respectively, contribute. For two point function the correct result (found before by operator methods) is obtained only, if the presence of the zero modes are properly accounted for and their role in the chiral symmetry breaking by an anomaly becomes particularly transparent. We have calculated the four-point function $\langle \bar{\psi}(x)\psi(0)\rangle$ and proved the clustering property.

Comparing the torus compactification of the geometric \([11]\) with the ordinary SM, which is considered in the present work, one can find many similarities. But there are also some differences. The effective action in the geometric SM has a factor 2, which implies a factor $\sqrt{2}$ in the mass of the isoscalar particle (in the geometric SM we have additional internal symmetry and isospin multiplet of massless particles). The factor 2 in the toron action changes the character of the integration over the torons considerably, and hence the dynamical role of the torons in these cases.

Nowadays there are very intensive discussions in the literature on finite temperature (size) effects in quantum field-theoretical models. To study them in the present context one should let the spacial (‘temporal’) extension of $L_2(L_1)$ tend to infinity.

It would be interesting to extend this investigation to the cases of the chiral Schwinger, massive Schwinger and Thirring models.

These problems remain to be considered in the future.
Note added After finalizing this work I became aware of a recent paper by Fayyazud-din et al \[23\], where the same result for the four point fermionic function (Eq. (3.58)) was obtained. I thank A.Wipf for bringing this reference to my attention.

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Appendix A. Derivation of the Eq.(2.9).

As it has been mentioned the nonzero eigenvalues of the $D$ operator appear in pairs: $iE_{\nu}$ and $-iE_{\nu}$($E_{\nu}$ is real). Therefore

$$\det' D = \prod_{\nu}(iE_{\nu})(-iE_{\nu}) = \prod_{\nu} E_{\nu}^2$$

and

$$\det' D^\dagger = \prod_{\nu}(-iE_{\nu})(iE_{\nu}) = \prod_{\nu} E_{\nu}^2.$$ (A.2)

Thus

$$\det' D = \sqrt{\det' DD^\dagger}. $$ (A.3)

Furthermore

$$\det(D - M_iL_1) = \prod_{\nu}(-iE_{\nu} - M_iL_1)(iE_{\nu} - M_iL_1)(-M_iL_1)^{|k|}$$

$$= \prod_{\nu}(E_{\nu}^2 + M_i^2 L_1^2)(-M_iL_1)^{|k|},$$ (A.4)

since in the sector with Pontriyagin index $k$ the $D$ operator has $|k|$ zero modes.

On the other hand

$$\det(DD^\dagger + M_i^2 L_1^2) = \det[(D - M_iL_1)(-D - M_iL_1)]$$

$$= \det(D - M_iL_1) \det(-D - M_iL_1)$$

$$= \prod_{\nu}(E_{\nu}^2 + M_i^2 L_1^2)^2(-M_iL_1)^{2|k|}. $$ (A.5)
Second, let us find the variation \( \delta D \) any function \( f \) where we have not written the terms which do not vary under the variation of \( f \). We have
\[
\delta(D - M_i L_1) = (-1)^k \det(DD^\dagger + M_i^2 L_1^2)^{1/2}
\] (A.6)
and from (2.8)
\[
\exp \frac{1}{2} \Gamma_{reg}^{(k)}[A] = (-1)^{|k|} (\det DD^\dagger)^{1/2} \prod_{i=1}^r \det(DD^\dagger + M_i^2 L_1^2)^{\frac{1}{2r}}.
\] (A.7)

**Appendix B. Derivation of the Eq.(2.13).**

First, let us prove that for any operator \( \hat{A} \)
\[
\delta \text{Tr}(e^{-t\hat{A}}) = -t \text{Tr}(\delta A e^{-t \hat{A}}).
\] (B.1)
We have
\[
\delta e^{-t\hat{A}} = e^{-t(\hat{A} + \delta \hat{A})} - e^{-t\hat{A}} = e^{-t\hat{A}} e^{t\delta \hat{A}} e^{t(\delta \hat{A})} - e^{-t\hat{A}}
\]
\[
+ O((\delta A)^2) = e^{-t\hat{A}} (-t\delta \hat{A} + t[\hat{A}, \delta \hat{A}] + O((\delta A)^2))
\] (B.2)
Since \( \text{Tr} \) is a linear operation
\[
\delta \text{Tr}(e^{-t\hat{A}}) = \text{Tr}(\delta e^{-t\hat{A}}) = -t \text{Tr}(\delta A e^{-t \hat{A}}).
\] (B.3)
Second, let us find the variation \( \delta D \) under the variation of \( b(x) \)
\[
D = L_1 \gamma_\mu (\partial_\mu - ie A_\mu) = L_1 \gamma_\mu \partial_\mu + e L_1 \gamma_\mu \gamma_5 \partial_\mu b(x) + \cdots,
\]
where we have not written the terms which do not vary under the variation of \( b(x) \). For any function \( f(x) \) we have
\[
(\delta D f)(x) = (D(b + \delta b)f)(x) - (Df)(x) = e L_1 \gamma_\mu \gamma_5 (\partial_\mu b(x) + \partial_\mu \delta b(x)) f(x) - e L_1 \gamma_\mu \gamma_5 \partial_\mu b(x) f(x) = e L_1 \gamma_\mu \gamma_5 (\partial_\mu \delta b(x)) f(x),
\] (B.4)
since
\[
\gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu.
\]
Thus
\[
\delta D = -e \gamma_5 [D, \delta b(x)].
\] (B.5)
Now let us find the variation \( \delta (DD^\dagger) \):
\[
\delta(DD^\dagger) = -(\delta D^2) = -D\delta D - (\delta D)D = e[D \gamma_5 D \delta b - D \gamma_5 \delta b D + \gamma_5 D \delta b D - \gamma_5 \delta b D D] = e \gamma_5 [DD^\dagger \delta b + \delta b DD^\dagger + 2D \delta b D],
\] (B.6)
since
\[
D \gamma_5 = -\gamma_5 D = \gamma_5 D^\dagger.
\] (B.7)
Finally from (B.3) and (B.7) we get
\[
\delta(\text{Tr} e^{-tDD^\dagger}) = -t \text{Tr}(\delta (DD^\dagger) e^{-tDD^\dagger}) = -4et Tr(\gamma_5 \delta b D D^\dagger e^{-tDD^\dagger}) = 4et \frac{d}{dt} (\gamma_5 \delta b e^{-tDD^\dagger}).
\] (B.8)

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The second term disappears in the limit $M_i \to \infty$ and only the first term gives ultraviolet divergencies when $M_i \to \infty$. So we should calculate the integral in (C.7) only for $0 < t < 1$. In this case we can close the contour in (C.3) on the left, take for the integrand its expression (C.5) and use the residuum formula.

Then

$$\text{Tr}'(e^{-tD_0D_0^\dagger}) = \frac{r_1(D_0D_0^\dagger)}{t} - r_1(D_0D_0^\dagger) + A + \cdots$$  \hspace{1cm} (C.8)
where \( \cdots \) is a sum of the positive powers of \( t \) which also do not contribute to the integral in the limit \( M_i \to \infty \). So for the first sum in the second line of (2.20) we have

\[
\sum_{i=1}^{r} e_i \int_0^1 \frac{dt}{t} \left( \frac{r_1(D_0 D_0^\dagger)}{t} + \zeta(0 \mid D_0 D_0^\dagger) \right) e^{-t M_i^2 L_i^2} \tag{C.9}
\]

up to terms, which disappear in the limit \( M_i \to \infty \). The expression in (C.9) is equal to

\[
\sum_{i=1}^{r} e_i (r_1(D_0 D_0^\dagger) M_i^2 L_i^2 \ln M_i^2 L_i^2 + \zeta(0 \mid D_0 D_0^\dagger) \ln M_i^2 L_i^2). \tag{C.10}
\]

So from (C.1) and (C.7)–(C.10), we obtain (2.21).

**Appendix D. Calculation of** \( r_1(D_0 D_0^\dagger) \), \( \zeta(0 \mid D_0 D_0^\dagger) \) **and** \( \zeta'(0 \mid D_0 D_0^\dagger) \).

1. **Trivial sector** \( (k = 0) \). In this case the background electromagnetic field is represented only by the toron field. The spectrum \( \lambda_{n_1,n_2} \) of the \( D_0 D_0^\dagger \) operator with 2-fold degeneracy is known (see (1.47)) and

\[
\zeta(s \mid D_0 D_0^\dagger) = 2(4\pi^2)^{-s} \sum_{n_1,n_2} \left[ (n_1 - \tilde{t}_1)^2 + \frac{1}{|\tau|^2} (n_2 - \tilde{t}_2)^2 \right]^{-s} = 2(4\pi^2)^{-s} Z \begin{vmatrix} -\tilde{t}_1 & -\tilde{t}_2 \\ 0 & 0 \end{vmatrix}^s, \tag{D.1}
\]

where \( Z \mid \varphi^{(s)} \) is Epstein \( \zeta \)-function (see [19] (17.93), in our case \( p = 2 \), \( \varphi(x) = x_1^2 + \frac{1}{|\tau|^2} x_2^2 \), \( g_1 = -\tilde{t}_1 \), \( g_2 = -\tilde{t}_2 \), \( n = 0 \)).

This function has a pole at \( s = 1 \) with the residue \( \tilde{r}_1 = \pi \Delta^{-1/2} \Gamma(2) \), where \( \Delta = \det A \) and \( A \) is a matrix of the quadratic form \( \varphi(x) = \sum_{i,j} a_{ij} x_i x_j \), \( A = \parallel a_{ij} \parallel \). In our case \( \Delta = \frac{1}{|\tau|^2} \), so \( \tilde{r}_1 = \pi |\tau| \). Thus

\[
r(D_0 D_0^\dagger) = \frac{2\pi |\tau|}{4\pi^2} = \frac{|\tau|}{2\pi}. \tag{D.2}
\]

In order to find \( \zeta(0 \mid D_0 D_0^\dagger) \) and \( \zeta'(0 \mid D_0 D_0^\dagger) \), i.e. \( Z(0) \) and \( Z'(0) \), we will use the following method[2]

\[
Z(s) = \sum_{n_1,n_2} \left( \tilde{n}_1^2 + \frac{1}{|\tau|^2} \tilde{n}_2^2 \right)^{-s} = \frac{1}{\Gamma(s)} \sum_{n_1,n_2} \int_0^\infty \frac{dt}{t^s} e^{-t \tilde{n}_1^2 - t \frac{1}{|\tau|^2} \tilde{n}_2^2 t} = \frac{1}{\Gamma(s)} J(s), \tag{D.3}
\]

where \( \tilde{n}_i \equiv n_i - \tilde{t}_i \), \( i = 1, 2 \).

Using the Poisson summation formula

\[
\sum_n f(n) = \sum_p \int_{-\infty}^\infty dx f(x)e^{-2\pi ipx}, \tag{D.4}
\]

[1] We should remind the reader that now we are considering dimensionless operators \( D_0 \) and \( D_0^\dagger \) obtained by the multiplication of \( L_1 \).

[2] This method was suggested by A.Coste.
we obtain for the special case
\[
J(s) = \sum_p \sum_{n_2} \int_0^\infty \frac{dt}{t} t^s \int_{-\infty}^{\infty} dx e^{-2\pi i px - \left[(x-i\tilde{t})^2 + \frac{1}{|\tau|^2} \tilde{t}^2\right]t} \tag{D.5}
\]
Doing the Gaussian integration over \(x\) we get
\[
J(s) = \sqrt{\pi} \sum_p \sum_{n_2} \int_0^\infty \frac{dt}{t} t^s \frac{t^{\tilde{t}^2}}{\sqrt{t}} e^{-t\left(\tilde{t}^2 + \frac{1}{|\tau|^2} \tilde{t}^2\right)} e^{-t\left(\tilde{t}^2 + \frac{1}{|\tau|^2} \tilde{t}^2\right)} = \sum_p J_p(s). \tag{D.6}
\]
For the calculation of the value at \(s = 0\) we will consider two cases separately
\(\text{a)} p = 0\)
\[
J_0(s) = \sqrt{\pi} \sum_{n_2} \int_0^\infty \frac{dt}{t} t^s e^{-\tilde{t}^2} = \sqrt{\pi} \sum_{n_2} \int_0^\infty dt t^{s-\frac{3}{2}} e^{-t(\tilde{t})^2} \tag{D.7}
\]
\[
J_0(s) = \sqrt{\pi} \Gamma(s - \frac{1}{2}) |\tau|^{2s-1} \sum_{n_2} |n_2 - \tilde{t}|^{-2s+1} \tag{D.7}
\]
\[
J_0(s) = \sqrt{\pi} \Gamma(s - \frac{1}{2}) |\tau|^{2s-1} \left\{ \zeta_R(2s - 1, \tilde{t}_2) + \zeta_R(2s - 1, 1 - \tilde{t}_2) \right\},
\]
since \(0 < \tilde{t}_2 < 1\). \(\zeta_R(s \mid q)\) is a generalized Riemann’s \(\zeta\)-function
\[
\zeta_R(s \mid q) = \sum_{n=0}^\infty \frac{1}{(q + n)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-qt}}{1 - e^{-t}} dt, \quad \text{Res} > 0. \tag{D.8}
\]
If \(n\) is nonnegative integer,
\[
\zeta_R(-n \mid q) = -\frac{B_{n+1}(q)}{n+1}, \tag{D.9}
\]
therefore
\[
J_0(0) = -\frac{2\pi}{|\tau|} \left\{ -\frac{B_2 (1 - \tilde{t}_2)}{2} - \frac{B_2 (1 - \tilde{t}_2)}{2} \right\} = \frac{2\pi}{|\tau|} B_2 (1 - \tilde{t}_2) = \frac{2\pi}{|\tau|} \left( \tilde{t}_2^2 - \tilde{t}_2 + \frac{1}{6} \right). \tag{D.10}
\]
\(b) p \neq 0\)
\[
J_p(s) = \sqrt{\pi} e^{-2\pi i \tilde{t}_1} \sum_{n_2} \int_0^\infty \frac{dt}{t} t^s \frac{t^{\tilde{t}^2}}{\sqrt{t}} e^{-t\left(\tilde{t}^2 + \frac{1}{|\tau|^2} \tilde{t}^2\right)} \tag{D.11}
\]
Using the new variables \(u = \frac{1}{|\tau||p|} t\), \(k = \frac{|n_2|}{|\tau|}|p|\) and taking into account that
\[
\int_0^\infty du u^{-3/2} e^{-\pi k (u + \frac{1}{2})} = \frac{1}{\sqrt{k}} e^{-\frac{2\pi k}{3}}, \tag{D.12}
\]
we obtain for the special case \(s = 0\)
\[
J_p(0) = e^{-2\pi i \tilde{t}_1} \frac{1}{|p|} e^{-2\pi \frac{|n_2|}{|\tau||p|}}. \tag{D.13}
\]
Thus

\[ J(0) = J_0(0) + \sum_{\mu \neq 0} J_\mu(0) = J_0(0) \]

\[- \sum_{n_2} \ln \left\{ (1 - e^{-2\pi|\mu|t_2 + i\vec{t}_1})(1 - e^{-2\pi|\mu|t_2 - i\vec{t}_1}) \right\} \]

\[ = J_0(0) - \sum_{n=1}^{\infty} \ln \left\{ (1 - e^{-\frac{2\pi}{\tau} (n - \bar{t}_2 + i|\tau|\vec{t}_1)})(1 - e^{-\frac{2\pi}{\tau} (n + \bar{t}_2 - i|\tau|\vec{t}_1)}) \right\} \]

\[- \ln(1 - e^{-\frac{2\pi}{\tau} (\bar{t}_2 + i|\tau|\vec{t}_1)}) - \sum_{n=1}^{\infty} \ln \left\{ (1 - e^{-\frac{2\pi}{\tau} (n - \bar{t}_2 + i|\tau|\vec{t}_1)})(1 - e^{-\frac{2\pi}{\tau} (n + \bar{t}_2 - i|\tau|\vec{t}_1)}) \right\} \]

\[- \ln(1 - e^{-\frac{2\pi}{\tau} (\bar{t}_2 - i|\tau|\vec{t}_1)}) = J_0(0) - \ln \left\{ \prod_{n=1}^{\infty} (1 - e^{-\frac{2\pi}{\tau} (n - \bar{t})})(1 - e^{-\frac{2\pi}{\tau} (n + \bar{t})}) \right\} \]

\[- \ln \left\{ \prod_{n=1}^{\infty} (1 - e^{-\frac{2\pi}{\tau} (n - \bar{t})})(1 - e^{-\frac{2\pi}{\tau} (n + \bar{t})}) \right\} \]

\[- \ln \left\{ \prod_{n=1}^{\infty} \left( 1 - 2e^{-\frac{2\pi}{\tau} n} \cosh \frac{2\pi}{|\tau|} \bar{t} + e^{-\frac{4\pi}{|\tau|} n} \right) \right\} \]

\[- \ln \left\{ \prod_{n=1}^{\infty} \left( 1 - 2e^{-\frac{2\pi}{\tau} n} \cosh \frac{2\pi}{|\tau|} \bar{t} + e^{-\frac{4\pi}{|\tau|} n} \right) \right\} . \]

Now let us recollect the definition of the \( \vartheta \) Jacobi's \( \vartheta \) function \([13]\)

\[ \vartheta_1(t \mid q) = 2q_0q^{1/4} \sin \pi t \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi t + q^{4n}) , \]

where \( q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}) \).

In our case we will take \( q = e^{-\frac{\pi}{|\tau|}} = e^{-i\frac{\pi}{\tau}} \). Then

\[ J(0) = J_0(0) - \ln \left| \vartheta_1 \left( \frac{t}{|\tau|} \left| - \frac{1}{\tau} \right) \right| ^2 + 2 \ln(2q_0 e^{-\frac{\pi}{|\tau|}}) \]

\[ + \ln \left( \sinh \frac{\pi}{|\tau|} t \sinh \frac{\pi}{|\tau|} \bar{t} \right) - \ln \left\{ (1 - e^{-\frac{2\pi}{|\tau|} t})(1 - e^{-\frac{2\pi}{|\tau|} \bar{t}}) \right\} , \quad (D.14) \]

where \( t = \bar{t}_2 + i|\tau|\vec{t}_1 \). For the last term we have

\[ \ln \left\{ (1 - e^{-\frac{2\pi}{|\tau|} t})(1 - e^{-\frac{2\pi}{|\tau|} \bar{t}}) \right\} = \ln \left\{ e^{-\frac{\pi}{|\tau|} t}(e^{\frac{\pi}{|\tau|} t} - e^{-\frac{\pi}{|\tau|} \bar{t}})e^{-\frac{\pi}{|\tau|} \bar{t}} \right\} \]

\[ \times (e^{\frac{\pi}{|\tau|} \bar{t}} - e^{-\frac{\pi}{|\tau|} \bar{t}}) \right\} = \ln \left\{ 4e^{-\frac{\pi}{|\tau|} (t + \bar{t})} \right\} + \ln \left\{ \sinh \frac{\pi}{|\tau|} t \sinh \frac{\pi}{|\tau|} \bar{t} \right\} . \]

So

\[ J(0) = \frac{2\pi}{|\tau|} \left( \bar{t}_2 - \bar{t}_2 + \frac{1}{6} \right) - \ln \left| \vartheta_1 \left( \frac{t}{|\tau|} \left| - \frac{1}{\tau} \right) \right| ^2 + \ln q_0^2 - \frac{\pi}{2|\tau|} - \ln e^{-\frac{2\pi}{|\tau|} \bar{t}_2} \quad (D.15) \]
and using the relation
\[ \vartheta_1 \left( \frac{z}{\tau} \right) = i \sqrt{|\tau|} e^{-\frac{\pi^2}{|\tau|}} \vartheta_1(iz \mid \tau), \]
we finally get
\[ J(0) = \frac{2\pi}{|\tau|} \left( \bar{t}_1^2 - \frac{1}{12} \right) - \ln |\tau| - \frac{\pi}{|\tau|} (t^2 + \bar{t}^2) \]
\[ - \ln |\vartheta_1(t \mid \tau)|^2 + \ln \vartheta_0^2 \left( \frac{1}{\tau} \right) \]
\[ = \frac{2\pi}{|\tau|} \left( \bar{t}_1^2 - \frac{1}{12} \right) - \ln |\vartheta_1(t \mid \tau)|^2 + \ln \vartheta_0^2 \left( \frac{1}{\tau} \right) - \ln |\tau|, \] (D.16)
where
\[ \bar{t}_1 = |\tau| \tilde{t}_1 = \frac{eL_2t_1}{2\pi}. \]

From (D.1) and (D.3) we have
\[ \zeta(0 \mid D_0D_0^\dagger) = 2 \frac{1}{\Gamma(0)} J(0) = 0, \text{ since } \frac{1}{\Gamma(0)} = 0 \] (D.17)
and
\[ \zeta'(0 \mid D_0D_0^\dagger) = 2 J(0) \]
\[ = \frac{4\pi}{|\tau|} \left( \bar{t}_1^2 - \frac{1}{12} \right) - 2 \ln |\vartheta_1(t \mid \tau)|^2 + 2 \ln \vartheta_0^2 \left( \frac{1}{\tau} \right) - 2 \ln |\tau| \] (D.18)
\[ = \frac{4\pi}{|\tau|} \bar{t}_1^2 - 2 \ln |\vartheta_1(t \mid \tau)\eta^{-1}(\tau)|^2, \] (D.19)
where \( \eta(\tau) \) is Dedekind’s function.

2. Nontrivial sector \((k \neq 0)\)

In this case the spectrum of the \( D_0D_0^\dagger \) operator: \( \lambda_m = \frac{4\pi|k|}{|\tau|} n \) with \( 2|k| \) fold degeneracy and
\[ \zeta(s \mid D_0D_0^\dagger) = \frac{2|k||\tau|^s}{(4\pi|k|)^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{2|k||\tau|^s}{(4\pi|k|)^s} \zeta_R(s), \] (D.20)
where \( \zeta_R(s) \) is Riemann’s \( \zeta \) function which has a pole at \( s = 1 \) with the residue \( 1 \) and
\[ \zeta_R(0) = -\frac{1}{2}, \quad \zeta_R'(0) = -\frac{1}{2} \ln 2\pi. \]
Thus
\[ \zeta(0 \mid D_0D_0^\dagger) = -|k|, \quad r(D_0D_0^\dagger) = \frac{|\tau|}{2\pi}, \] (D.21)
\[ \zeta'(s \mid D_0D_0^\dagger) = 2|k| \left\{ - \left( \ln \frac{4\pi|k|}{|\tau|} \right) \left( \frac{|\tau|}{4\pi|k|} \right)^s \zeta_R(s) + \frac{|\tau|^s}{(4\pi|k|)^s} \zeta_R'(s) \right\}, \] (D.22)
\[ \zeta'(0 \mid D_0D_0^\dagger) = -|k| \ln \frac{4\pi|k|}{|\tau|} + |k| \ln 2\pi = -|k| \ln \frac{2|k|}{|\tau|}. \] (D.23)
Appendix E. The Green’s function of the Laplacian on the Torus.

For the function $G_0(x)$ we have the following equation

$$-\square G_0(x) = \delta(x) - \frac{1}{L_1 L_2},$$

(E.1)

where $\delta(x)$ is a $\delta$-function on the torus

$$\delta(x) = \frac{1}{L_1 L_2} \sum_{n_1, n_2} e^{\frac{2\pi i}{L_1} (n_1 x_1 + |\tau|^{-1} n_2 x_2)}.$$  

(E.2)

Let us introduce the function

$$\tilde{G}(x) = G_0(x) - x^2 / 4 L_1 L_2.$$  

(E.3)

Then from (E.1) it follows that

$$-\square \tilde{G}(x) = \delta(x)$$  

(E.4)

Since the $G_0(x)$ function obeys the periodicity condition

$$G_0(x + \hat{\nu} L_\nu) = G_0(x),$$

(E.5)

for the $\tilde{G}(x)$ we have

$$\tilde{G}(x + \hat{\nu} L_\nu) = G_0(x) - \frac{(x + \hat{\nu} L_\nu)^2}{4 L_1 L_2} = \tilde{G}(x) - \frac{x_\nu L_\nu}{2 L_1 L_2} - \frac{L_\nu^2}{4 L_1 L_2}.$$  

(E.6)

For the $\tilde{G}(x)$ function we will choose the following ansatz

$$\tilde{G}(x) = -\frac{1}{2\pi} \text{Re} \ln \sigma(z),$$

(E.7)

where $z$ is defined in (1.14), and $\sigma(z)$ is analytic in the box $(1, \tau)$ and has a single zero at $z = 0$

$$\sigma(z) = z + \cdots.$$  

(E.8)

So

$$\tilde{G}(x) = -\frac{1}{2\pi} \ln |z| + \cdots, \text{ when } |z| \to 0.$$  

(E.9)

Taking into account the behaviour of the $\sigma(z)$ function for small $|z|$ and the periodicity condition (E.5), we choose it in the following form

$$\sigma(z) = ce^{az^2 + bx} \vartheta_1(z \mid \tau).$$  

(E.10)

Since the $\vartheta_1(z \mid \tau)$ function obeys the following periodicity conditions

$$\vartheta_1(z + 1 \mid \tau) = -\vartheta_1(z \mid \tau),$$

$$\vartheta_1(z + 1 \mid \tau) = -e^{-i\pi(2z + \tau)} \vartheta_1(z \mid \tau),$$

we get the following equations for the constants $a$ and $b$
\[ \ln |\sigma(z + 1)| = \text{Re}(2az + a) + \text{Re} b + \ln |\sigma \left(\frac{z}{L_1}\right)| = \frac{\pi x_1}{L_2} + \frac{\pi}{\tau} + \ln |\sigma(z)|, \]
\[ \ln |\sigma(z + \tau)| = \text{Re}(2ai\tau|z| - a|\tau|^2) - \text{Im}b + 2\pi \frac{x_2}{L_1} + \pi|\tau| + \ln |\sigma(z)| \]
\[ = \frac{\pi x_2}{L_1} + \frac{\pi}{2}|\tau| + \ln |\sigma(z)|. \]

From these equations it follows that
\[ a = \frac{\pi}{2|\tau|}, \quad \text{and} \quad b = 0. \quad (E.11) \]

The constant \( c \) is determined from condition (E.8). Thus
\[ \sigma(z) = e^{\frac{\pi x_1^2}{4|\tau|}} \frac{\vartheta_1(z | \tau)}{\vartheta'_1(0 | \tau)} \quad (E.12) \]
and
\[ \tilde{G}(x) = -\frac{1}{2\pi} \ln \left| \frac{e^{\frac{\pi x_1^2}{4|\tau|}} \vartheta_1(z | \tau)}{\vartheta'_1(0 | \tau)} \right| + \tilde{C}, \quad (E.13) \]
where the constant \( \tilde{C} \) can be found from the condition
\[ \int_0^{L_1} \int_0^{L_2} G_0(x)dx_1dx_2 = 0, \]
or we can choose the other possibility
\[ \int_0^{L_1} G_0(x_1, 0)dx_1 = \frac{L_1}{4\pi^2|\tau|} \sum_{n_1, n_2} \frac{e^{2\pi i x_1}}{n_1^2 + |\tau|^{-2}n_2^2} = \frac{L_1|\tau|}{4\pi^2} \left( \sum_{n_2 = -\infty}^{\infty} \frac{1}{n_2^2} + \sum_{n_2 = 1}^{\infty} \frac{1}{n_2^2} \right) = \frac{L_1|\tau|}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{L_1|\tau|}{12}. \quad (E.14) \]

From (E.3) and (E.13) we have
\[ G_0(x) = -\frac{1}{2\pi} \ln \left| \frac{\vartheta_1(z | \tau)}{\vartheta'_1(0 | \tau)} \right| + \frac{x_2^2}{2L_1L_2} + \tilde{C}. \quad (E.15) \]

Thus from (E.14) and (E.13) we will obtain
\[ \tilde{C} = \frac{1}{2\pi} \int_0^1 dx \ln \left| \frac{\vartheta_1(x | \tau)}{\vartheta'_1(0 | \tau)} \right| + \frac{|\tau|}{12}. \quad (E.16) \]

From [19] (13.19(17)) we have
\[ \ln \left| \frac{\vartheta_1(x | \tau)}{\vartheta'_1(0 | \tau)} \right| = -\ln \pi + \ln \sin \pi x + 4 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \frac{1 - \cos 2\pi mx}{2m}, \]
where
where \( q = e^{i\pi \tau} \).

Then
\[
\int_0^1 dx \ln \left| \frac{\varphi_1(x \mid \tau)}{\varphi_1(0 \mid \tau)} \right| = -\ln \pi - \ln 2 - \ln q_0^2(\tau), \tag{E.17}
\]
where \( q_0^2(\tau) = \prod_{n=1}^\infty (1 - q^{2n}) \).

So
\[
\tilde{C} = \frac{|\tau|}{12} - \frac{1}{2\pi} \ln (2\pi q_0^2(\tau)). \tag{E.18}
\]

Finally we get
\[
G_0(x) = -\frac{1}{2\pi} \ln \left| \frac{\varphi_1(z \mid \tau)}{\varphi_1(0 \mid \tau)} \right| + \frac{x_2^2}{2L_1L_2} + \frac{|\tau|}{12} - \frac{1}{2\pi} \ln (2\pi q_0^2(\tau)). \tag{E.19}
\]

**Appendix F. The propagator of fermions in the background toron field.**

The fermion propagator \( S_t(x) \) in the background toron field is defined as a solution of the equation
\[
D_t S_t(x) \equiv \gamma_\mu (\partial_\mu - i e t_\mu) S_t(x) = -\delta(x) \tag{F.1}
\]
and is periodic in the Euclidean space-time
\[
S_t(x_1 + k_1L_1, x_2 + k_2L_2) = S_t(x), \tag{F.2}
\]
where \( k_\mu (\mu = 1, 2) \) are some integers, and is quasiperiodic in \( t \)-space
\[
S_{t_{\mu_1} + \frac{2\pi}{e t_\mu} m_\mu}(x) = e^{\frac{2\pi i}{L_1} m_1 x_1 + \frac{2\pi i}{L_2} m_2 x_2} S_t(x), \tag{F.3}
\]
where \( m_\mu (\mu = 1, 2) \) are some integers, too. Note that \( [S_t(x)] = [t]^{-1} \). One can easily find that the solution of (F.1), which obeys the conditions (F.2) and (F.3) has the following form
\[
S_t(x) = -D_t G_t(x) = \gamma_\mu S_t^{(\mu)}(x), \tag{F.4}
\]
where
\[
G_t(x) = \frac{1}{L_1L_2} \sum_\mu \frac{e^{\frac{2\pi i}{L_1} (n_1 x_1 + |\tau|^{-1} n_2 x_2)}}{(\frac{2\pi}{L_1})^2 [(n_1 - \tilde{t}_1)^2 + |\tau|^{-2} (n_2 - \tilde{t}_2)]}. \tag{F.5}
\]

It can be shown (see [21]) that this function has a following representation
\[
G_t(x) = \frac{1}{4\pi} \sum_{\mu \nu} e^{-2\pi i (-\frac{x_2}{L_2} + \nu)(\mu - \tilde{t}_1)||\tau|} e^{2\pi i (\mu L_1 + \nu L_2)} e^{2\pi i \tilde{t}_1 \tilde{t}_2}, \quad \text{if } \tilde{t}_1 \in Z
\]
\[
+ \frac{|\tau|}{4\pi^2} S_0 (-\tilde{t}_2, -\frac{x_2}{L_2}, 1) e^{2\pi i x_2 \tilde{t}_1}, \quad \text{if } \tilde{t}_1 \in Z \tag{F.6}
\]
\[
+ \frac{1}{4\pi^2} S_0 (-\tilde{t}_1, -\frac{x_1}{L_1}, \frac{x_2}{L_2}), \quad \text{if } \frac{x_2}{L_2} \in Z,
\]
where $\tilde{t}_i = \frac{\nu L_i}{2\pi} (i = 1, 2)$, prime in the sum means that the terms with $\mu = \tilde{t}_1$, if $\tilde{t}_1 \in \mathbb{Z}$ and $\nu = \frac{\nu_2}{L_2}$, if $\frac{\nu_2}{L_2} \in \mathbb{Z}$ should be omitted.

\[
S_0 \left(-\tilde{t}_2, -\frac{x_2}{L_2}, 1\right) = \sum_{\mu} e^{2\pi i \frac{x_2}{L_2} \mu} (\tilde{t}_2 - \mu)^2
\]

\[
S_0 \left(-\tilde{t}_1, -\frac{x_1}{L_1}, \frac{1}{2}\right) = \sum_{\mu} | -\tilde{t}_1 + \mu|^{-1} e^{2\pi i \frac{x_1}{L_1} \mu}
\]

for $0 \leq \tilde{t}_1 < 1$, and

\[
S_0 \left(-\tilde{t}_1 - \frac{x_1}{L_1}, \frac{1}{2}\right) = \ln \left| 2 \sin \frac{\pi x_1}{L_1} \right|^2 + \frac{1}{\tilde{t}_1} + \tilde{t}_1 \sum_{\mu=1}^{\infty} \left( \frac{e^{2\pi i \frac{x_1}{L_1} \mu}}{\mu(\mu - \tilde{t}_1)} + \frac{e^{-2\pi i \frac{x_1}{L_1} \mu}}{\mu(\mu + \tilde{t}_1)} \right)
\]

for $0 < \tilde{t}_1 < 1$. Star in the sum means that if $\tilde{t}_2(\tilde{t}_1)$ is an integer the term with $\mu = \tilde{t}_2(\mu = \tilde{t}_1)$ should be omitted.

One can prove that the $S_{\tilde{t}}(\mu)(x)$ functions defined in (F.4) have the following short distance behaviour

\[
S_{\tilde{t}}(\mu)(x) = \frac{e^{iet_\mu x_\mu}}{2\pi} \left( \frac{x_\mu}{|x|^2} + K_\mu(t) \right) + O(|x|),
\]

where

\[
K_1 = -iet_1 + \frac{1}{2L_1} \left( \frac{\vartheta_1'(\tilde{t})}{\vartheta_1(\tilde{t})} - \frac{\vartheta_1'(t)}{\vartheta_1(t)} \right),
\]

\[
K_2 = \frac{i}{2L_1} \left( \frac{\vartheta_1'(\tilde{t})}{\vartheta_1(\tilde{t})} + \frac{\vartheta_1'(t)}{\vartheta_1(t)} \right)
\]

and $t = \tilde{t}_2 + i|\tau|\tilde{t}_1$, $\tilde{t} = \tilde{t}_2 - i|\tau|\tilde{t}_1$.

It can easily be checked that the functions which obey the periodicity conditions (F.2), (F.3) and short distance behaviour (F.10) have the following form

\[
S_{\tilde{t}}^{(1)}(x) = \frac{1}{4\pi L_1} \left\{ e^{ext_+} \frac{\vartheta_1'(0)}{\vartheta_1(z)} \vartheta_1(z + \tilde{t}) + e^{-ext_-} \frac{\vartheta_1'(0)}{\vartheta_1(t - \tilde{z})} \vartheta_1(t) \right\}
\]

\[
S_{\tilde{t}}^{(2)}(x) = \frac{1}{4\pi i L_1} \left\{ e^{-ext_-} \frac{\vartheta_1'(0)}{\vartheta_1(t)} \vartheta_1(t - \tilde{z}) - e^{ext_+} \frac{\vartheta_1'(0)}{\vartheta_1(z)} \vartheta_1(z + \tilde{t}) \right\}
\]

where $t_{\pm} = t_1 \pm it_2$, or in the matrix form

\[
S_{\tilde{t}}(x) = \frac{e^{iet_\mu x_\mu}}{2\pi L_1} \begin{pmatrix}
0 & \frac{\vartheta_1'(0)\vartheta_1(z + \tilde{t})}{\vartheta_1(t)\vartheta_1(z)} e^{-ieL_1t_1\tilde{t}} \\
-\frac{\vartheta_1'(0)\vartheta_1(z - \tilde{t})}{\vartheta_1(t)\vartheta_1(z)} e^{ieL_1t_1\tilde{t}} & 0
\end{pmatrix}
\]

Note that $S_{\tilde{t}}(x)$ becomes singular for $t = 0$. This singularity is caused by the constant solution of the Dirac equation with $t = 0$, which represents zero modes in the trivial sector. In the path integral it is compensated by the zero of the Boltzmann factor of the induced toron action $Z_{tor}(t)$. 

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Appendix G. The propagator of fermions in the external field from the topologically nontrivial class ($k \neq 0$).

Fermion propagator in the background field $A^{(k)}_\mu$

$$S^{(k)}_{\alpha\beta}(x, y; A) = \langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle^{(k)}$$

is a solution of the equation

$$DS^{(k)}(x, y; A) = -\delta^{(2)}(x - y) + \mathcal{P}_0(x, y),$$

where $\mathcal{P}_0(x, y)$ is the projector operator onto the space of the zero modes.

Using (1.7)-(1.9) the Dirac operator $D$ can be written as follows

$$D = e^{i\beta_+(x)} \gamma_\mu \partial_\mu e^{-i\beta_-(x)}$$

with

$$\beta_\pm(x) = e\{a(x) + t_\mu x_\mu \pm i\gamma_5[b(x) - \frac{\pi k x^2}{2eL_1L_2}]\}. \hspace{1cm} (G.4)$$

Then the propagator $S^{(k)}(x, y)$ can be expressed in the following form

$$S^{(k)}(x, y; A) = e^{i\beta_-(y)} Q^{(k)}(x, y) e^{-i\beta_+(x)}$$

and $Q^{(k)}$ obeys the equation

$$\gamma_\mu \partial_\mu Q^{(k)}(x, y) = -\delta^{(2)}(x - y) + \tilde{\mathcal{P}}_0(x, y),$$

where

$$\tilde{\mathcal{P}}_0(x, y) = e^{-i\beta_+(x)} \mathcal{P}_0(x, y) e^{i\beta_+(y)}. \hspace{1cm} (G.7)$$

Further, we consider in detail only the case $k > 0$, because for $k < 0$ everything is similar. For the trasformed projection operator we have

$$\tilde{\mathcal{P}}_0(x, y) = \sum_{n=0}^{k-1} \tilde{\chi}^{(n)}(x) \bar{\tilde{\chi}}^{(n)}(y), \hspace{1cm} (G.8)$$

where $\tilde{\chi}^{(n)}$ are transformed zero modes with positive chirality.

One can introduce a new spinor $\eta^{(n)}(x)$ such that

$$\gamma_\mu \partial_\mu \eta^{(n)}(x) = \tilde{\chi}^{(n)}(x).$$

Then we define a new function

$$\tilde{Q}^{(k)}(x, y) = Q^{(k)}(x, y) + \sum_{n=0}^{k-1} \eta^{(n)}(x) \bar{\tilde{\chi}}^{(n)}(y)$$

(G.10)
and from \((G.4),(5.8)\) and \((3.9)\) obtain for it

\[
\gamma_\mu \partial_\mu \tilde{Q}^{(k)}(x,y) = -\delta^{(2)}(x-y).
\]  \hspace{1cm} (G.11)

Since \(\{D,\gamma_5\} = 0, \gamma_5 S^{(k)} \gamma_5 = -S^{(k)}\) and \(\gamma_5 \tilde{Q}^{(k)} \gamma_5 = -\tilde{Q}^{(k)}\) the matrix \(\tilde{Q}^{(k)}\) has only nonzero off-diagonal elements

\[
\tilde{Q}^{(k)} = \begin{pmatrix}
0 & \tilde{Q}^{(k)}_{12} \\
\tilde{Q}^{(k)}_{21} & 0
\end{pmatrix}.
\]  \hspace{1cm} (G.12)

In terms of complex variables \(z = \frac{\eta + i\pi}{L_1}\) and \(w = \frac{\mu + i\pi}{L_1}\) this equation can be rewritten as follows

\[
2L_1 \partial_z \tilde{Q}^{(k)}_{21}(z,w) = -\delta^{(2)}(z-w),
\]  \hspace{1cm} (G.13)

\[
2L_1 \partial_z \tilde{Q}^{(k)}_{12}(z,w) = -\delta^{(2)}(z-w).
\]  \hspace{1cm} (G.14)

Let us consider the function \(\tilde{Q}^{(k)}_{12}(z,w)\) (for \(\tilde{Q}^{(k)}_{21}(z,w)\) everything can be done in the same way). From the definition of the fermion propagator \((G.1)\), from relations \((G.5)\), \((G.10)\) and from the periodicity conditions \((1.2)\) for Fermi fields, which in this particular case have the form given in \((1.22)\), the periodicity conditions on \(\tilde{Q}^{(k)}_{12}(z,w)\) can be found to be

\[
\tilde{Q}^{(k)}_{12}(z+1,\bar{z}+1;w,\bar{w}) = e^{-iet_1 L_1 - \pi k z + \frac{\pi k}{2} |\tau|} \tilde{Q}^{(k)}_{12}(z,\bar{z};w,\bar{w}),
\]  \hspace{1cm} (G.15a)

\[
\tilde{Q}^{(k)}_{12}(z+\tau,\bar{z}+\bar{\tau};w,\bar{w}) = e^{-iet_2 L_2 - \pi k z + \frac{\pi k}{2} |\tau|} \tilde{Q}^{(k)}_{12}(z,\bar{z};w,\bar{w}),
\]  \hspace{1cm} (G.15b)

\[
\tilde{Q}^{(k)}_{12}(z,\bar{z};w+1,\bar{w}+1) = e^{iet_1 L_1 - \pi k z - \frac{\pi k}{2} |\tau|} \tilde{Q}^{(k)}_{12}(z,\bar{z};w,\bar{w}),
\]  \hspace{1cm} (G.15c)

\[
\tilde{Q}^{(k)}_{12}(z,\bar{z};w+\tau,\bar{w}+\bar{\tau}) = e^{-iet_2 L_2 + \pi k w - \frac{\pi k}{2} |\tau|} \tilde{Q}^{(k)}_{12}(z,\bar{z};w,\bar{w}).
\]  \hspace{1cm} (G.15d)

The general form of \(\tilde{Q}^{(k)}_{12}(z,w)\) that satisfies \((G.14)\) is

\[
\tilde{Q}^{(k)}_{12}(z,w) = \frac{1}{2\pi L_1} \frac{\vartheta_1'(0)}{\vartheta_1(z-w)} q(z,w),
\]  \hspace{1cm} (G.16)

where \(q(z,w)\) should be chosen to obey the periodicity conditions in \((G.15)\), to have no poles for \(z-w\) and to be 1 at \(z = w\). A solution that satisfies these conditions is

\[
\tilde{Q}^{(k)}_{12}(z,w) = \frac{1}{2\pi L_1} \frac{\vartheta_1'(0)}{\vartheta_1(z-w)} \vartheta_1(z-w+t) \vartheta_3(kz|k\tau) \\
\times e^{-iet_1 t_1(z-w)+\frac{\pi k}{2}|\tau|z^2-w^2}. \hspace{1cm} (G.17)
\]

For \(\tilde{Q}^{(k)}_{21}(z,w)\) one obtains the similar expression

\[
\tilde{Q}^{(k)}_{21}(z,w) = - \frac{1}{2\pi L_1} \frac{\vartheta_1'(0)}{\vartheta_1(z-w)} \vartheta_1(z-w+t) \vartheta_3(k\bar{z}|k\tau) \\
\times e^{-iet_1 t_1(z-w)+\frac{\pi k}{2}|\tau|z^2-w^2}. \hspace{1cm} (G.18)
\]
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