Abstract

Given a spectrally negative Lévy process $X$ drifting to infinity, (inspired on the early ideas of Shiryaev (2002)) we are interested in finding a stopping time that minimises the $L^p$ distance ($p > 1$) with $g$, the last time $X$ is negative. The solution is substantially more difficult compared to the case $p = 1$, for which it was shown in Baurdoux and Pedraza (2020b) that it is optimal to stop as soon as $X$ exceeds a constant barrier. In the case of $p > 1$ treated here, we prove that solving this optimal prediction problem is equivalent to solving an optimal stopping problem in terms of a two-dimensional strong Markov process that incorporates the length of the current excursion away from 0. We show that an optimal stopping time is now given by the first time that $X$ exceeds a non-increasing and non-negative curve depending on the length of the current excursion away from 0. We further characterise the optimal boundary and the value function as the unique solution of a non-linear system of integral equations within a subclass of functions. As examples, the case of a Brownian motion with drift and a Brownian motion with drift perturbed by a Poisson process with exponential jumps are considered.

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1 Introduction

In recent years last passage times have received considerable attention in the literature. For instance, in risk theory, the capital of an insurance company over time is studied. In the classical risk theory this is modelled by the Cramér–Lundberg process, defined as a compound Poisson process with drift. In more recent literature, this process has been replaced by a more general spectrally negative Lévy process. A key quantity of interest is the moment of ruin, which is classically defined as the first passage time below zero. Consider instead the situation where after the moment of ruin the company may have funds to endure a negative capital for some time. In that case, the last passage time below zero becomes an important quantity to be studied. In this framework, in Chiu and Yin (2005) the Laplace transform of the last passage time is derived.

Secondly, Paroissin and Rabehasaina (2013) consider spectrally positive Lévy processes as a degradation model. In a traditional setting, the failure time of a device is the first time the model hits a certain critical level $b$. However, another approach has been considered in the literature. For example, in Barker and Newby (2009) they considered the failure time as the last time the process is below $b$. After the last passage time, the process can never go back to this level meaning that the device is “beyond repair”.

Thirdly, Egami and Kevkhisvili (2020) studied the last passage time of a general time-homogeneous transient diffusion with applications to credit risk management. They proposed the leverage process (the ratio of a company asset process over its debt) as a geometric Brownian motion over a process that grows at a risk-free rate. It is shown there that the last passage time of the leverage ratio is equivalent to a last passage
time of a Brownian motion with drift. In this setting, the last passage represents the situation where the company cannot recover to normal business conditions after this time has occurred.

An important feature of last passage times is that they are random times that are not stopping times. In the recent literature, the problem of finding a stopping time that approximates last passage times has been solved. There are for example various papers in which the approximation is in $L_1$ sense. To mention a few: du Toit et al. (2008) predicted the last zero of a Brownian motion with drift in a finite horizon setting; du Toit and Peskir (2008) predicted the time of the ultimate maximum at time $t = 1$ for a Brownian motion with drift is attained; Shiryaev (2009) focused on the last time of the attainment of the ultimate maximum of a Brownian motion and proceeded to show that it is equivalent to predicting the last zero of the process in this setting; Glover et al. (2013) predicted the time in which a transient diffusion attains its ultimate minimum; Glover and Hulley (2014) predicted the last passage time of a level $z > 0$ for an arbitrary nonnegative time-homogeneous transient diffusion; Baurdoux and van Schaik (2014) predicted the time at which a Lévy process attains its ultimate supremum and Baurdoux et al. (2016) predicted when a positive self-similar Markov process attain its path-wise global supremum or infimum before hitting zero for the first time and Baurdoux and Pedraza (2020b) predicted the last zero of a spectrally negative Lévy process.

Note that in Shiryaev (2002), the author states some general optimal prediction problems that are natural for the “technical analysis” of the financial data. In particular, among other problems, it is proposed to predict the time in which a process attains its maximum (over a finite interval) in an $L_p$ sense. However, no solution to the problem is provided. Moreover, to the best of our knowledge optimal prediction problems for last passage times have been only solved in an $L_1$ sense. Hence, inspired by this, we consider the problem of predicting the last zero of a spectrally negative Lévy process (drifting to infinity) in an $L^p$ sense, i.e. we are interested in solving

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}(|\tau - g|^p),$$

where $g = \sup\{t \geq 0 : X_t \leq 0\}$ is the last time a spectrally negative Lévy process drifting to infinity is below the level zero and $p > 1$. The case when $p = 1$ was solved in Baurdoux and Pedraza (2020b) for the spectrally negative case. An optimal stopping time in this case is the first time the process crosses above a fixed level $a^* \geq 0$ which is characterised in terms of the distribution function of the infimum of the process. The case $p > 1$ is substantially more complex, as an optimal stopping time now depends on the length of the current excursion above the level zero given by $U_t = t - \sup\{0 \leq s \leq t : X_s \leq 0\}$. The process $(U, X)$ is a Markov process taking values in $E = [(0, \infty) \times (0, \infty)] \cup \{0\} \times (-\infty, 0)]$.

We show that an optimal stopping time (when $p > 1$) is given by $\tau_D = \inf\{t > 0 : (U_t, X_t) \in D\} = \inf\{t \geq 0 : X_t \geq b(U_t)\}$, where $b$ is a non-negative, non-increasing and continuous curve. That is, is not optimal to stop when $(U, X)$ is in the (continuation) set $C := E \setminus D$ whilst we should stop as soon as the process enters the (stopping) set $D$ (see Figure 1). In other words, given the strong dependence of $U$ on $X$, the latter has the following interpretation in terms of the sample paths of $X$: It is optimal to stop when $X$ is sufficiently large or has stayed for a sufficiently large amount of time above zero, whereas we will never stop when $X$ is in the negative half-line (see Figure 1).
Figure 1: Stopping and continuation set in the \((U, X)\) plane

In the figure below we include a plot of a sample path of \(X_t\) and \(b(U_t)\), where we calculated numerically the function \(b\) for the Brownian motion with drift case (see Section 6.1 and Figure 3).

Figure 2: Black line: \(t \mapsto X_t\); Blue line: \(t \mapsto b(U_t)\).

The paper is organised as follows. Section 2 gives a short overview of main results and notation on the fluctuation theory of spectrally negative Lévy processes. In Section 3 we formulate the optimal prediction problem and we show that it is equivalent to an optimal stopping problem which solution is described in Theorem 3.3. Since the proof of Theorem 3.3 is rather long, we dedicate Section 4 for that purpose. In particular, we show that an optimal stopping time is given by the first time \(X\) exceeds a boundary \(b\) which depends on the length of the current excursion above zero. We derive various properties of \(b\). For example,
in Lemma 4.15 we show that $b$ is continuous and in Lemma 4.17 we show that smooth fit holds at the boundary. The main result, Theorem 4.18, provides a characterisation of $b$ and the value function of the optimal stopping problem. In Section 6 we provide two numerical examples: Firstly, when $X$ is a Brownian motion with drift, and secondly when $X$ is a Brownian motion perturbed by a compound Poisson process with exponential jumps. Finally, some of the more technical proofs are deferred to the Appendix A.

2 Preliminaries

A Lévy process $X = \{X_t, t \geq 0\}$ is an almost surely càdlàg process that has independent and stationary increments such that $\mathbb{P}(X_0 = 0) = 1$. We take it to be defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by $X$ which is naturally enlarged (see Definition 1.3.38 of Bichteler (2002)) From the stationary and independent increments property the law of $\mathbb{F}$ increments such that $P$.

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We state now some properties and facts about Lévy processes. The reader can refer, for example, to Bertoin (1998), Sato (1999) and Kyprianou (2014) for more details. Every Lévy process $X$ is also a strong Markov $\mathbb{F}$-adapted process. For all $x \in \mathbb{R}$, denote $\mathbb{P}_x$ as the law of $X$ when started at the point $x \in \mathbb{R}$, that is, $\mathbb{E}_x(\cdot) = \mathbb{E}(\cdot | X_0 = x)$. Due to the spatial homogeneity of Lévy processes, the law of $X$ under $\mathbb{P}_x$ is the same as that of $X + x$ under $\mathbb{P}$.

The process $X$ is a spectrally negative Lévy process if it has no negative jumps ($\Pi(0_+) = 0$) implies that $\Phi$ is the usual inverse with $\Phi(0) = 0$. The process $X$ drifts to infinity, i.e., $\lim_{t \to \infty} X_t = \infty$ when $\psi'(0+) < 0$, $X$ drifts to minus infinity and the condition $\psi'(0+) = 0$ implies that $X$ oscillates, that is, $\limsup_{t \to \infty} X_t = -\liminf_{t \to \infty} X_t = \infty$. We denote by $\Phi$ the right-inverse of $\psi$, i.e.

$$\Phi(q) = \sup\{\beta \geq 0 : \psi(\beta) = q\}, \quad q \geq 0.$$ 

In the particular case that $X$ drifts to infinity, we have that $\psi'(0+) > 0$ which implies that $\psi$ is strictly increasing and then $\Phi$ is the usual inverse with $\Phi(0) = 0$. The process $X$ has paths of finite variation if and only if $\sigma = 0$ and $\int_{(-\infty,0]} |x| \Pi(dx) < \infty$, otherwise $X$ has paths of infinite variation. Denote by $\tau^+_a$ the first passage time above the level $a > 0$,

$$\tau^+_a = \inf\{t > 0 : X_t > a\}.$$ 

The Laplace transform of $\tau^+_a$ is given by

$$\mathbb{E}(e^{-a \tau^+_a} \mathbb{1}_{\{\tau^+_a < \infty\}}) = e^{-\Phi(q)a}, \quad a > 0.$$ (1)
An important family of functions for spectrally negative Lévy processes consists of the scale functions, usually denoted by $W^{(q)}$ and $Z^{(q)}$. For all $q \geq 0$, the scale function $W^{(q)} : \mathbb{R} \to \mathbb{R}_+$ is such that $W^{(q)}(x) = 0$ for all $x < 0$ and it is characterised on the interval $[0, \infty)$ as the strictly and continuous function with Laplace transform given by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \, dx = \frac{1}{\psi(\beta) - q}, \quad \text{for } \beta > \Phi(q).$$

The function $Z^{(q)}$ is defined for all $q \geq 0$ by

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) \, dy, \quad \text{for } x \in \mathbb{R}.$$

For the case $q = 0$ we simply denote $W = W^{(0)}$. When $X$ has paths of infinite variation, $W^{(q)}$ is continuous on $\mathbb{R}$ and $W^{(q)}(0) = 0$ for all $q \geq 0$, otherwise $W^{(q)}(0) = 1/d$ for all $q \geq 0$, where

$$d := -\mu - \int_{(-1,0)} x \Pi(dx).$$

Note that since processes with monotone paths are excluded from the definition of spectrally negative processes we necessarily have that $d > 0$ when $X$ is of bounded variation.

For all $q \geq 0$, $W^{(q)}$ has left and right derivatives. Moreover, when $X$ is of infinite variation we have that $W^{(q)} \in C^1((0, \infty))$ with derivative at zero given by $W^{(q)}'(0) = 2/\sigma^2$. When $X$ is of finite variation $W^{(q)} \in C^1((0, \infty))$ when $W$ has no atoms. Henceforth, we will assume that when $X$ is of finite variation the Lévy measure $\Pi$ has no atoms. Furthermore, for each $x \geq 0$ and $q \geq 0$, $W^{(q)}$ has the following representation

$$W^{(q)}(x) = \sum_{k=0}^{\infty} q^k W^{*^{(k+1)}}(x),$$

where $W^{*^{(k+1)}}$ is the $(k + 1)$-th convolution of $W$ with itself. Various fluctuation identities for spectrally negative Lévy processes have been found in terms of the scale functions. Here we list some that will be useful in later sections. Denote by $\tau^-_x$ the first passage time below the level $x \leq 0$, i.e.,

$$\tau^-_x = \inf\{t > 0 : X_t < x\}.$$ 

Then for any $q \geq 0$ and $x \leq a$ we have

$$\mathbb{E}_x\left(e^{-q \tau^+_x} \mathbb{1}_{\{\tau^-_x > \tau^+_x\}}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}. \quad (3)$$

For any $x \in \mathbb{R}$ and $q \geq 0,$

$$\mathbb{E}_x(e^{-q \tau^-_0} \mathbb{1}_{\{\tau^-_0 < \infty\}}) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \quad (4)$$

where we understand $q/\Phi(q)$ in the limiting sense when $q = 0$. Since $X$ has only negative jumps we have that only creeps upwards, that is,

$$\mathbb{P}(X_{\tau^+_x} = x, \tau^+_x < \infty) = 1 \quad \text{for } x > 0.$$ 

Moreover, $X$ creeps downwards if and only $\sigma > 0$ with probability given by

$$\mathbb{P}_x(X_{\tau^-_0} = 0, \tau^-_0 < \infty) = \frac{\sigma^2}{2} \left(W'(x) - \Phi(0)W(x)\right) \quad \text{for } x > 0.$$ 

for any $x > 0.$
Denote by $\sigma_{x} = \inf_{0 \leq s \leq t} X_s$ and $\overline{X}_{t} = \sup_{0 \leq s \leq t} X_s$ the running infimum and running maximum of the process $X$ up to time $t > 0$, respectively. For $q \geq 0$, let $e_q$ be an exponential random variable with mean $1/q$ independent of $X$, where we understand that $e_q = \infty$ almost surely when $q = 0$. Then $\overline{X}_{e_q}$ is exponentially distributed with parameter $\Phi(q)$ and the Laplace transform of $\overline{X}_{e_q}$ is given by

$$\mathbb{E}(e^{\beta \overline{X}_{e_q}}) = \frac{q - \Phi(q) - \beta}{\Phi(q) - q - \psi(\beta)}, \quad \beta \geq 0.$$ \hspace{2cm} (7)

Denote by $\sigma_x^-$ the first time the process $X$ is below or equal to the level $x$, i.e.

$$\sigma_x^- = \inf\{t > 0 : X_t \leq x\}.$$

It is easy to show that the mapping $x \mapsto \sigma_x^-$ is non-increasing, right-continuous with left limits. The left limit is given by $\lim_{h \downarrow 0} \sigma_x^{-h} = \tau_x^-$ for all $x \in \mathbb{R}$. Moreover, since

$$\mathbb{E}(e^{-q \sigma_x^- 1_{\{\sigma_x^- < \infty\}}}) = \mathbb{P}(e_q > \sigma_x^-) = \mathbb{P}(\overline{X}_{e_q} \leq -x)$$

for all $x \leq 0$ and the fact that the random variable $\overline{X}_{e_q}$ is continuous on $(-\infty, 0)$, we have that, for any $x > 0$, the stopping times $\sigma_x^-$ and $\tau_x^-$ have the same distribution. When $X$ is of infinite variation, $X$ enters instantly to the set $(\infty, 0)$ whilst in the finite variation case there is a positive time before the process enters it. That implies that in the infinite variation case $\tau_0^+ = \sigma_0^- = 0$ a.s. whereas in the finite variation case, $\sigma_0^- = 0$ and $\tau_0^+ > 0$.

Let $q > 0$ and $a \in \mathbb{R}$. The $q$-potential measure of $X$ killed on exiting $[0, a]$, \begin{equation*} \int_0^\infty e^{-qt} \mathbb{P}(X_t \in dy, t < \tau_a^+ \wedge \tau_0^-)dt \end{equation*} is absolutely continuous with respect to Lebesgue measure and it has a density given by

$$\frac{W(q)(x)W(q)(a-y)}{W(q)(a)} - W(q)(x-y), \quad x, y \in [0, a].$$ \hspace{2cm} (8)

Similarly, the $q$-potential measure of $X$ killed on exiting $(-\infty, a]$ and the $q$-potential measure of $X$ are absolutely continuous with respect to Lebesgue measure with a density given by

$$e^{-\Phi(q)(a-x)}W(q)(a-y) - W(q)(x-y), \quad x, y \leq a,$$ \hspace{2cm} (9)

and

$$\Phi'(q)e^{-\Phi(q)(y-x)}W(q)(x-y), \quad x, y \in \mathbb{R},$$ \hspace{2cm} (10)

respectively. In the case when $X$ drifts to infinity these expression are also valid for $q = 0$.

For any $t \geq 0$ and $x \in \mathbb{R}$, we denote by $g_t^{(x)}$ the last time that the process is below $x$ before time $t$, i.e.,

$$g_t^{(x)} = \sup\{0 \leq s \leq t : X_s \leq x\},$$ \hspace{2cm} (11)

with the convention $\sup \emptyset = 0$. We simply denote $g_t := g_t^{(0)}$ for all $t \geq 0$. Note that when $\mathbb{P}(X_t \geq 0) = \rho$ for some $\rho \in (0, 1)$, then $g_t/t$ follows the generalised arcsine law with parameter $\rho$, see Theorem 13 in Bertoin (1998). The last hitting time of zero is of key importance in the study of Azéma’s martingale (see Azéma and Yor (1989)). We also define, for each $t \geq 0$, $U_t^{(x)}$ as the time spent by $X$ above the level $x$ before time $t$ since the last visit to the interval $(-\infty, x]$, i.e.,

$$U_t^{(x)} := t - g_t^{(x)} \quad t \geq 0.$$
It turns out that for our optimal prediction problem
\[
\inf_{\tau \in \mathcal{T}} \mathbb{E}(|\tau - g|^p)
\]
the process \( U_t = U_t^{(0)} \) plays a vital role. It can be readily seen that for all \( x \in \mathbb{R} \), the process \( \{U_t^{(x)}, t \geq 0\} \) is not a Markov process. We now list a number of results from Baurdoux and Pedraza (2020a) concerning \( U \). The strong Markov property holds for the two dimensional process \( \{(U_t, X_t), t \geq 0\} \) with respect to the filtration \( \{\mathcal{F}_t, t \geq 0\} \) and state space given by
\[
E = \{(u, x) : u > 0 \text{ and } x > 0\} \cup \{(u, x) : u = 0 \text{ and } x \leq 0\}.
\]
Then there exists a family of probability measures \( \{\mathbb{P}_{u,x}, (u, x) \in E\} \) such that for any \( A \in \mathcal{B}(E) \), Borel set of \( E \), we have that \( \mathbb{P}_{u,x}(E_{t+s}, X_{t+s}) = \mathbb{P}_{u,x}(E_t, X_t) \) for all \( (u, x) \in E \). For each \( (u, x) \in E \), \( \mathbb{P}_{u,x} \) can be written in terms of \( \mathbb{P}_x \) via
\[
\mathbb{E}_{u,x}(h(U_s, X_s)) := \mathbb{E}_x(h(u + s, X_s) \mathbb{I}_{\{\sigma > s\}}) + \mathbb{E}_x(h(U_s, X_s) \mathbb{I}_{\{\sigma \leq s\}}),
\]
for any positive measurable function \( h \). Let \( F \) a \( C^{1,2}(E) \) real-valued function. In addition, in the case that \( \sigma > 0 \), assume that \( \lim_{u \downarrow 0} F(u, h) = F(0, 0) \) for all \( u > 0 \). Then we have the following version of Itô formula
\[
F(U_t, X_t) = F(U_0, X_0) + \int_0^t \frac{\partial}{\partial u} F(U_s, X_s) \mathbb{I}_{\{\tau > 0\}} ds + \int_0^t \frac{\partial}{\partial x} F(U_s, X_s) dX_s + \frac{1}{2} \sigma^2 \int_0^t \frac{\partial^2}{\partial x^2} F(U_s, X_s) ds
\]
\[
+ \int_{[0,t]} \int_{(-\infty,0)} \left( F(U_s, X_s^- + y) - F(U_s, X_s^-) - y \frac{\partial}{\partial x} F(U_s, X_s^-) \right) N(ds \times dy)
\]
Moreover, if \( f \) is a \( C^{1,2}(E) \) bounded function, the infinitesimal generator \( A_{U,X} \) of the process \( (U, X) \) is given by
\[
A_{U,X}(f)(u, x) = \frac{\partial}{\partial u} f(u, x) \mathbb{I}_{\{x > 0\}} - \mu \frac{\partial}{\partial x} f(u, x) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} f(u, x)
\]
\[
+ \int_{(-\infty,0)} \left( f(u, x + y) - f(u, x) - y \frac{\partial}{\partial x} f(u, x) \right) \mathbb{I}_{\{y > 0\}} \Pi(dy)
\]
\[
+ \int_{(-\infty,0)} \left( f(0, x + y) - f(0, x) - y \frac{\partial}{\partial x} f(0, x) \right) \mathbb{I}_{\{y \leq 0\}} \Pi(dy)
\]
\[
+ \int_{(-\infty,0)} \left( f(0, x + y) - f(u, x) - y \frac{\partial}{\partial x} f(u, x) \right) \mathbb{I}_{\{0 < x < y < 0\}} \Pi(dy)
\]
\[
= \frac{\partial}{\partial u} \tilde{f}(u, x) - \mu \frac{\partial}{\partial x} \tilde{f}(u, x) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} \tilde{f}(u, x)
\]
\[
+ \int_{(-\infty,0)} \left( \tilde{f}(u, x + y) - \tilde{f}(u, x) - y \frac{\partial}{\partial x} \tilde{f}(u, x) \right) \Pi(dy).
\]
where \( \tilde{f} \) is a function that extends \( f \) to the set \( \mathbb{R}_+ \times \mathbb{R} \) given by
\[
\tilde{f}(u, x) = \begin{cases} 
  f(u, x) & u > 0 \text{ and } x > 0, \\
  f(0, x) & u \geq 0 \text{ and } x \leq 0, \\
  f(0, 0) & u = 0 \text{ and } x > 0.
\end{cases}
\]
\( u \mapsto C(u, x) \) is a monotone function for all \( x \in \mathbb{R}, \) \( |K(u, s)| \leq C(u, x) \) and \( \mathbb{E}_{u,x} \left( \int_0^\infty e^{-q\tau} C(U_r, X_r + y) dr \right) < \infty \) for all \( (u, x) \in E \) and \( y \in \mathbb{R}. \) Then we have that
\[
\mathbb{E} \left( \int_0^\infty e^{-q\tau} K(U_r, X_r) dr \right) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_x \left( \int_{\tau_0^- < \infty} e^{-q\tau_0^-} K^-(X_{\tau_0^-} - \varepsilon) \right) + K^+(0, \varepsilon)}{\psi'(\Phi(q))W^{(q)}(\varepsilon)},
\]
where \( K^+ \) and \( K^- \) are given by
\[
K^+(u, x) = \mathbb{E}_x \left( \int_0^{\tau_0^-} e^{-q\tau} K(u + r, X_r) dr \right) \quad \text{and} \quad K^-(x) = \mathbb{E}_x \left( \int_{\tau_0^-}^{\tau_0^+} e^{-q\tau} K(0, X_r) dr \right),
\]
for all \( (u, x) \in E. \) As a direct application of the aforementioned formula we can calculate a density the \( q \)-potential measure of \( (U, X). \) For any \( v, y > 0 \) we have that
\[
\int_0^\infty e^{-q\tau} \mathbb{P}(U_r \in dv, X_r \in dy) dr = \Phi'(q) \frac{y}{v} e^{-qv\tau} \mathbb{P}(X_0 \in dy) dv.
\]
We conclude this section by collecting some additional results about the last passage time
\[
g = g_\infty = \sup\{t \geq 0 : X_t \leq 0\}.
\]
The Laplace transform of \( g \) was found in Chiu and Yin (2005) as
\[
\mathbb{E}_x(e^{-qg}) = e^{\Phi(q)\psi'(0^+)} + \psi'(0^+)W(x) - W^{(q)}(x), \quad q \geq 0.
\]
The distribution function of \( g \) under \( \mathbb{P}_x \) is found by observing that
\[
\mathbb{P}_x(g \leq \gamma) = \mathbb{P}_x(X_{u+\gamma} > 0 \text{ for all } u \in (0, \infty)) \quad \text{if } \gamma \geq 0
\]
\[
= \mathbb{E}_x(\mathbb{P}_x(X_{u+\gamma} > 0 \text{ for all } u \in (0, \infty)|\mathcal{F}_\gamma))
\]
\[
= \mathbb{E}_x(\mathbb{P}_x(\tau_0^- = \infty))
\]
\[
= \mathbb{E}_x(\psi'(0^+)W(X_\gamma)),
\]
where we used the tower property of conditional expectation in the second equality and the Markov property of Lévy processes in the third. Note that the law of \( g \) under \( \mathbb{P}_x \) may have an atom at zero given by
\[
\mathbb{P}_x(g = 0) = \mathbb{P}_x(\tau_0^- = \infty) = \psi'(0^+)W(x).
\]
For our optimal prediction problem we require the \( p \)-th moment of \( g \) to be finite. The following result is from Doney and Maller (2004) (see Theorem 1, Theorem 4, Theorem 5 and Remark (ii)).

**Lemma 2.1.** Let \( X \) be a spectrally negative Lévy process drifting to infinity. Then, for a fixed \( p > 0, \) the following are equivalent:

1. \( \mathbb{E}_x(g^p) < \infty \) for some (hence every) \( x \leq 0; \)
2. \( \int_{(-\infty, -1]} |x|^{1+p}\Pi(dx) < \infty; \)
3. \( \mathbb{E}(\langle X^p \rangle_{\tau_0^-}) < \infty; \)
4. \( \mathbb{E}_x((\tau_0^+)^{1+p+1}) < \infty \) for some (hence every) \( x \leq 0; \)
5. \( \mathbb{E}_x((\tau_0^-)^{1+p}(\tau_0^- < \infty)) < \infty \) for some (hence every) \( x \geq 0. \)

The next lemma states that when \( \tau_0^+ \) has finite \( p \)-th moment under \( \mathbb{P}_x, \) then the function \( \mathbb{E}_x((\tau_0^+)^p) \) has a polynomial bound in \( x. \) It will be of use later to deduce a lower bound for our optimal prediction problem and its proof can be found in the Appendix A.
Lemma 2.2. Let $p > 0$ and suppose $E_x((\tau_0^n)^{p+1}) < \infty$ for some $x \leq 0$. Then, for each $0 \leq r \leq p$, there exist non-negative constants $A_r$ and $C_r$ such that

$$E_x((\tau_0^n)^r) \leq A_r + C_r|x|^r \quad \text{and} \quad E_x(g^r) \leq 2^r[E(g^r) + A_r] + 2^rC_r|x|^r, \quad x \leq 0.$$  

Here $\lfloor p \rfloor$ denotes the integer part of $p$.

The following lemma shows some properties of the function $x \mapsto E_x(g^p)$. The proof is included in the Appendix A.

Lemma 2.3. Let $p > 0$ and assume that $\int_{(-\infty, -1]} |x|^{p+1} \Pi(dx) < \infty$. Then $x \mapsto E_x(g^p)$ is a non-increasing, non-negative and continuous function. Moreover,

$$\lim_{x \to -\infty} E_x(g^p) = \infty \quad \text{and} \quad \lim_{x \to \infty} E_x(g^p) = 0.$$  

We conclude this section with a technical result extracted from Baurdoux and van Schaik (2014) (see Lemma 5) that will be useful later.

Lemma 2.4. Let $X$ be any Lévy process drifting to $-\infty$. Denote $T_+(0) = \inf\{t \geq 0 : X_t \geq 0\}$ Consider, for $a > 0$ and $b < 0$, the optimal stopping problem

$$P(x) = \inf_{\tau \in T} E_x[a\tau + 1_{\{\tau \geq T_+(0)\}}b] \quad \text{for } x \in \mathbb{R}.$$  

Then there is an $x_0 \in (-\infty, 0)$ so that $P(x) = 0$ for all $x \leq x_0$.

3 Optimal prediction problem

Denote by $V_*$ the value of the optimal prediction problem, i.e.

$$V_* = \inf_{\tau \in T} E(|\tau - g|^p), \quad (21)$$

where $T$ is the set of all stopping times with respect to $\mathcal{F}$, $p > 1$ and $g$ is the last zero of $X$ given in (18). Since $g$ is only $\mathcal{F}$ measurable standard techniques of optimal stopping times are not directly applicable. However, there is an equivalence between the optimal prediction problem (21) and an optimal stopping problem. The next lemma, inspired in the work of Urusov (2005), states such equivalence.

Lemma 3.1. Let $p > 1$ and let $X$ be a spectrally negative Lévy process drifting to infinity such that $\int_{(-\infty, -1]} |x|^{p+1} \Pi(dx) < \infty$. Consider the optimal stopping problem

$$V = \inf_{\tau \in T} E\left(\int_0^\tau G(s - g, X_s)ds\right), \quad (22)$$

where the function $G$ is given by

$$G(u, x) = u^{p-1}\psi'(0+)W(x) - E_x(g^{p-1})$$

for $u \geq 0$ and $x \in \mathbb{R}$. Then we have that $V_* = pV + E(g^p)$ and a stopping time minimises (21) if and only if it minimises (22).

Proof. Let $\tau \in T$. Then the following equality holds

$$|\tau - g|^p = \int_0^\tau g(s - g)ds + g^p, \quad (23)$$
where the function $g$ is defined by

$$g(x) = p \left[ \frac{(-x)^p}{x} I_{\{x<0\}} + x^{p-1} I_{\{x \geq 0\}} \right].$$

Taking expectations in equation (23) and then using Fubini’s theorem and the tower property for conditional expectation we obtain

$$\mathbb{E}(|\tau - g|^p) = \int_0^\infty \mathbb{E} \left( g(s - g) I_{\{s \leq \tau\}} \right) ds + \mathbb{E}(g^p)$$

$$= \int_0^\infty \mathbb{E} \left[ I_{\{s \leq \tau\}} \mathbb{E}(g(s - g) | \mathcal{F}_s) \right] ds + \mathbb{E}(g^p)$$

$$= \mathbb{E} \left( \int_0^\tau \mathbb{E}(g(s - g) | \mathcal{F}_s) ds \right) + \mathbb{E}(g^p).$$

To evaluate the conditional expectation inside the last integral, note that for all $t \geq 0$ we can write the the time $g$ as

$$g = g_t \lor \sup \{ s \in (t, \infty) : X_s \leq 0, \}$$

recalling that $g_t = g_t^{(0)}$ defined in (11). Hence, using the Markov property for Lévy processes and the fact that $g_s$ is $\mathcal{F}_s$ measurable we have that

$$\mathbb{E}(g(s - g) | \mathcal{F}_s) = \mathbb{E}(g(s - g_t \lor \sup \{ r \in (s, \infty) : X_r \leq 0, \}) | \mathcal{F}_s)$$

$$= g(s - g_t) \mathbb{P}(X_r > 0 \text{ for all } r \in (s, \infty)) | \mathcal{F}_s$$

$$+ \mathbb{E}(g(s - \sup \{ r \in (s, \infty) : X_r \leq 0, \}) I_{\{X_r \leq 0 \text{ for some } r \in (s, \infty)\}} | \mathcal{F}_s)$$

$$= g(s - g_t) \mathbb{P}(X_r > 0) + \mathbb{E}_X \left( (g - g_t) I_{\{X_r \leq 0, \}} \right)$$

$$= p(s - g_t)^p - 1 \psi'(0+) W(X_s) - p \mathbb{E}_X ((g - g_t)^p - 1).$$

Then we have that

$$\mathbb{E}(|\tau - g|^p) = p \mathbb{E} \left( \int_0^\tau G(s - g_t, X_s) ds \right) + \mathbb{E}(g^p).$$

\[\square\]

**Remark 3.2.** A close inspection of the proof of Lemma 3.1 tells us that the function $g$ corresponds to the right derivative of the function $f(x) = |x|^p$. Therefore, using similar arguments we can actually extend the result to any convex function $d : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$. That is, under the assumption that $\mathbb{E}(d(0, g)) < \infty$, the optimal prediction problem

$$V_d = \inf_{\tau \in \mathcal{T}} \mathbb{E}(d(\tau, g))$$

is equivalent to the optimal stopping problem

$$\inf_{\tau \in \mathcal{T}} \mathbb{E} \left[ \int_0^\tau G_d(g_t, s, X_s) ds \right],$$

where $G_d(\gamma, t, x) = \varrho_d(s, \gamma) \psi'(0+) W(x) + \mathbb{E}_X (\varrho_d(s, g + s) I_{\{g > 0\}})$ and $\varrho_d$ is the right derivative with respect the first argument of $d$.

The next theorem states the solution of the optimal prediction problem. Note that its proof is rather lengthy so the next section is entirely dedicated on that purpose.
Theorem 3.3. Let $p > 1$ and let $X$ be a spectrally negative Lévy process drifting to infinity such that $\Pi$ has no atoms and that $\int_{(0, \infty)} |x|^{p+1} \Pi(dx) < \infty$. Then there exists a non-decreasing and continuous function $b : (0, \infty) \mapsto [0, \infty)$ such that $b(u) \geq h(u) := \inf\{x \in \mathbb{R} : G(u, x) \geq 0\}$ for all $u \geq 0$, $\lim_{u \to 0} b(u) = \infty$, $\lim_{u \to \infty} b(u) = 0$ and the infimum in (22) (and hence in (21)) is attained by

$$\tau_D = \inf\{t > 0 : X_t \geq b(U_t)\}.$$  \hspace{1cm} (24)

Moreover, the function $b$ is uniquely characterised as in Theorem 4.18.

4 Solution to the optimal stopping problem

Throughout this section we are going to assume that $p > 1$ and that $X$ is a spectrally negative Lévy process drifting to infinity such that $\Pi$ has no atoms and $\int_{(-\infty, -1]} |x|^{p+1} \Pi(dx) < \infty$. In order to solve the optimal stopping problem (22) using the general theory of optimal stopping (see e.g. Peskir and Shiryaev (2006)) we have to extend it to an optimal stopping problem driven by a strong Markov process. For every $(u, x) \in E$, we define the optimal stopping problem

$$V(u, x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{u,x} \left[ \int_0^\tau G(U_s, X_s) \, ds \right],$$  \hspace{1cm} (25)

where the function $G$ is given by $G(u, x) = u^{p-1}\psi'(0+)W(x) - \mathbb{E}_x(g^{p-1})$ for any $u \geq 0$ and $x \in \mathbb{R}$. Therefore we have that $V_\ast = pV(0, 0) + \mathbb{E}(g^p)$. Note that using the definition of $\mathbb{E}_{u,x}$ we have that (25) takes the form

$$V(u, x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left( \int_0^\tau \left\{ G(u + s, X_s)I_{\{\xi_1 > s\}} + G(U_s, X_s)I_{\{\xi_2 \leq s\}} \right\} \, ds \right).$$  \hspace{1cm} (26)

As a consequence of Lemma 2.3 we have the following behaviour of the function $G$. For all $x \in \mathbb{R}$, the function $u \mapsto G(u, x)$ is non-decreasing. In particular when $x < 0$, $u \mapsto G(u, x) = -\mathbb{E}(g^{p-1})$ is a strictly negative constant. For fixed $u \geq 0$, $x \mapsto G(u, x)$ is a non-decreasing right-continuous function which is continuous everywhere apart from possibly at $x = 0$ (since $W$ is discontinuous at zero when $X$ is of finite variation) such that for all $u \geq 0$,

$$\lim_{x \to -\infty} G(u, x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} G(u, x) = u^{p-1} \geq 0.$$

Moreover, we have that $\lim_{u \to \infty} G(u, x) = \infty$ and $G(0, x) = -\mathbb{E}(g^{p-1}) < 0$ for all $x \geq 0$. Recall that for any $u \geq 0$,

$$h(u) = \inf\{x \in \mathbb{R} : G(u, x) \geq 0\}. \hspace{1cm} (27)$$

From the description of $G$ above we have that $h$ is a non-negative and non-decreasing function such that $h(u) < \infty$ for all $u \in (0, \infty)$, $h(0) = \infty$ and $\lim_{u \to \infty} h(u) = 0$. Moreover, since $W$ is strictly increasing on $(0, \infty)$, the function

$$T(x) := \frac{\mathbb{E}_x(g^{p-1})}{\psi'(0+)W(x)}$$

is continuous and strictly decreasing on $[0, \infty)$. Then there exists an inverse function $T^{-1}$ which is continuous and strictly decreasing on $(0, u_h^*)$ with

$$u_h^* := \frac{\mathbb{E}(g^{p-1})}{\psi'(0+)W(0)}, \hspace{1cm} (28)$$

where we understand $1/0 = \infty$ when $X$ is of infinite variation. Hence we can write

$$h(u) = \begin{cases} T^{-1}(u^{p-1}) & u < (u_h^*)^{\frac{1}{p-1}} \\ 0 & u \geq (u_h^*)^{\frac{1}{p-1}} \end{cases}.$$
Therefore, since \( T^{-1}(u^*_t) = 0 \), we conclude that \( h \) is a continuous function on \([0, \infty)\). From the definition of \( h \) we clearly have that \( G(u, x) \geq 0 \) if and only if \( x \geq h(u) \).

The facts above give us some intuition about the optimal stopping rule for the optimal stopping problem (25). Since we are dealing with a minimisation problem, before stopping we want the process \((U, X)\) to be in the set in which \( G \) is negative as much as possible. Then the fact that \( G(U_t, X_t) \) is strictly negative when \( X_t < h(U_t) \) suggests that it is never optimal to stop on this region. When \( X_t > h(U_t) \) we have that \( G(U_t, X_t) \geq 0 \) but with strictly positive probability \((U, X)\) can enter the set in which \( G \) is negative. Moreover, \( t \mapsto U_t \) is strictly increasing when \( X \) is in the positive half line so that \( t \mapsto h(U_t) \) gets closer to zero when the current excursion away from \((-\infty, 0] \) sufficiently large, then \( G(U_t, X_t) \geq 0 \) even when \( X_t \) is relatively close to zero. That suggests that it is optimal to stop when the current excursion away from \((-\infty, 0] \) is large or \( X \) takes a sufficiently large values. Then we infer the existence of a non-negative curve \( b \geq h \) such that it is optimal to stop when \( X \) crosses above \( b(U_t) \). We will formally show in the next Lemmas the existence of such boundary.

Note that if there exists a stopping time \( \tau \) for which the expectation of the right hand side of (25) is minus infinity then \( V \) would also be minus infinity. The next Lemma provides the finiteness of a lower bound of \( V \) that will ensure that \( V \) only takes finite values, its proof is included in the Appendix.

**Lemma 4.1.** We have that

\[
0 \leq \mathbb{E}_x \left( \int_0^\infty \mathbb{E}_{X_t}(g^{p-1}) ds \right) < \infty \quad \text{for all } x \in \mathbb{R}.
\]

We now prove the finiteness of the function \( V \).

**Lemma 4.2.** For every \((u, x) \in E \) we have that \( V(u, x) \in (-\infty, 0] \). In particular \( V(u, x) < 0 \) for \((u, x) \in B := \{ (u, x) \in E : x < h(u) \} \), where \( h \) is defined in (27).

**Proof.** By taking the stopping time \( \tau = 0 \) we deduce that for all \((u, x) \in E \), \( V(u, x) \leq 0 \). In order to check that \( V(u, x) \to -\infty \) we use that \( G(u, x) \geq -\mathbb{E}_x(g^{p-1}) \) to get

\[
V(u, x) = \inf_{\tau \in T} \mathbb{E}_{u, x} \left[ \int_0^\tau G(U_s, X_s) ds \right] \geq -\sup_{\tau \in T} \mathbb{E}_x \left[ \int_0^\tau \mathbb{E}_{X_s}(g^{p-1}) ds \right],
\]

for all \((u, x) \in E \). Hence by Lemma 4.1 we have that

\[
V(u, x) \geq -\mathbb{E}_x \left[ \int_0^\infty \mathbb{E}_{X_t}(g^{p-1}) ds \right] > -\infty, \quad (u, x) \in E.
\]

Using standard arguments we can prove that \( V(u, x) < 0 \) when \((u, x) \in B \).

**Remark 4.3.** Note that we have that \( h(0) = \infty \) which implies that \((0, 0) \in B \) and then, from the Lemma above, \( V(0, 0) < 0 \). Moreover, from Lemma 3.1 we have that \( pV(0, 0) + \mathbb{E}(g^{p-1}) = V_x \geq 0 \) which implies that

\[
-\frac{\mathbb{E}(g^{p-1})}{p} \leq V(0, 0) < 0.
\]

Now we prove some basic properties of \( V \).

**Lemma 4.4.** We have the following monotonicity property of \( V \). For all \((u, x), (v, y) \in E \) such that \( u \leq v \) and \( x \leq y \) we have that \( V(u, x) \leq V(v, y) \).

**Proof.** From equation (26) we have that

\[
V(u, x) = \inf_{\tau \in T} \mathbb{E}_x \left( \int_0^\tau \left\{ G(u + s, X_s)^{1_{\{\sigma^0 \geq s\}}} + G(U_s, X_s)^{1_{\{\sigma^0 \leq s\}}} \right\} ds \right)
\]

\[
= \inf_{\tau \in T} \mathbb{E}_x \left( \int_0^\tau \left\{ G(u + s, X_s + x)^{1_{\{\sigma^0 - \sigma \geq s\}}} + G(U_s, X_s + x)^{1_{\{\sigma^0 \leq s\}}} \right\} ds \right),
\]
where $\sigma_x^+ = \inf\{t \geq 0 : X_t \leq -x\}$ and $U_x^{(-x)} = s - \sup\{t \geq 0 : X_t \leq -x\}$. Recall that for all $s \geq 0$, $x \mapsto U_x^{(-x)}$ and $x \mapsto \sigma_x^-$ are non-decreasing and that the function $G$ is non-decreasing in each argument. Define the function

$$G^*(u, x) := G(u + s, X_s + x[\sigma_x^- > s]) + G(U_s^{(-x)}, X_s + x[I_{\{\sigma_x^- \leq s\}}].$$

We show by cases that the function $G^*$ is non-decreasing in each argument. Take $x \leq y$ and $0 \leq u \leq v$. First we suppose that $\omega \in \{\sigma_x^- > s\} \subset \{\sigma_y^- > s\}$. Since $G$ is non-decreasing in each argument we then have

$$G^*(u, x)(\omega) = G(u + s, X_s(\omega) + x) \leq G(v + s, X_s(\omega) + y) = G^*(v, y)(\omega).$$

Similarly, if $\omega \in \{\sigma_x^- \leq s\} \cap \{\sigma_y^- > s\}$ we have that

$$G^*(u, x)(\omega) = G(U_s^{(-x)}(\omega), X_s(\omega) + x) \leq G(U_s^{(-y)}(\omega), X_s(\omega) + y) = G^*(v, y)(\omega).$$

Lastly, take $\omega \in \{\sigma_x^- \leq s\} \cap \{\sigma_y^- \leq s\}$. Then using the fact that $U_s^{(-x)} = s - g_x^{(-x)} \leq s \leq v + s$ and the monotonicity of $G$ we get

$$G^*(u, x)(\omega) = G(U_s^{(-x)}(\omega), X_s(\omega) + x) \leq G(v + s, X_s(\omega) + y) = G^*(v, y)(\omega).$$

All this together implies that the function $G^*(u, x)$ is non-decreasing in each argument for all $u \geq 0$ and $x \in \mathbb{R}$, in particular for all $(u, x) \in E$ and hence the claim on $V$ holds.

In the next Lemma we give an expression for $V(0, x)$ when $x < 0$ in terms of $V(0, 0)$ and we use it to give a lower bound for $V$.

**Lemma 4.5.** For any $x \leq 0$ we have that

$$V(0, x) = \mathbb{E}_x \left( \int_0^{t_0^+} G(0, X_s)ds \right) + V(0, 0) = -\int_0^{-x} \int_{(0, \infty)} \mathbb{E}_{u - x}(g^{p-1})W(du)dz + V(0, 0). \quad (30)$$

Moreover, for all $(u, x) \in E$ we have that there exist non-negative constants $A_{p-1}'$ and $C_{p-1}'$ such that

$$V(u, x) \geq -A_{p-1}' - C_{p-1}'|x|^p + V(0, 0). \quad (31)$$

**Proof.** Let $x < 0$, using the Markov property and a dynamic programming argument we can write for all $x < 0$,

$$V(0, x) = \inf_{\tau \in \mathcal{F}_x} \mathbb{E}_x \left( \int_0^{\tau \wedge t_0^+} G(0, X_s)ds + \mathbb{I}_{\{\tau_0^+ < \tau\}} \int_{\tau_0^+}^{\tau} G(U_s, X_s)ds \right)$$

$$= \inf_{\tau \in \mathcal{F}_x} \mathbb{E}_x \left( \int_0^{\tau \wedge t_0^+} G(0, X_s)ds + \mathbb{I}_{\{\tau_0^+ < \tau\}} V(0, 0) \right)$$

$$= \mathbb{E}_x \left( \int_0^{t_0^+} G(0, X_s)ds \right) + V(0, 0),$$

where the last equality follows since $G(0, x) \leq 0$ for all $x \leq 0$ and $V(0, 0) \leq 0$ and hence the infimum is attained for any $\tau \geq t_0^+$. Using the fact that $G(0, x) = -\mathbb{E}_x(g^{p-1})$ for all $x < 0$ and Fubini’s theorem we get that

$$V(0, x) = -\mathbb{E}_x \left( \int_0^{t_0^+} \mathbb{E}_x(g^{p-1})ds \right) + V(0, 0)$$

$$= -\int_{(-\infty, 0)} \mathbb{E}_x(g^{p-1}) \int_0^{\infty} \mathbb{P}_\omega(X_s \in dz, s < t_0^+))ds + V(0, 0).$$
Using the 0-potential measure of $X$ killed on exiting the interval $(-\infty,0]$ (see equation (9)) and Fubini’s theorem we obtain that

$$
V(0,x) = -\int_0^\infty \mathbb{E}_{-z}(g^{p-1}|W(z) - W(x + z)|)dz + V(0,0)
$$

$$
= -\int_0^\infty \mathbb{E}_{-z}(g^{p-1}W(du))dz + V(0,0)
$$

$$
= -\int_0^{\infty} W(du)\int_{u-x}^{u} \mathbb{E}_{-z}(g^{p-1})dz + V(0,0)
$$

$$
= -\int_0^{-x} \int_{(0,\infty)} \mathbb{E}_{-u-z}(g^{p-1})W(du)dz + V(0,0).
$$

From equation (30) and the fact that $\chi_{\{x > 0\}}$ is non-increasing and bounded from above by a polynomial (see Lemmas 2.2 and 2.3) we have the inequalities for $x < 0$,

$$
V(0,x) \geq \int_{[0,\infty]} \mathbb{E}_{x-u}(g^{p-1})W(du) + V(0,0)
$$

$$
\geq \frac{1}{\psi'(0+)}2^{p-1}[\mathbb{E}(g^{p-1}) + A_{p-1}x] + \frac{1}{\psi'(0+)}2^{p-1}C_{p-1}x\mathbb{E}(|x + \chi_\infty |^{p-1}) + V(0,0)
$$

$$
\geq \frac{1}{\psi'(0+)}2^{p-1}[\mathbb{E}(g^{p-1}) + A_{p-1} + 2^{p-1}C_{p-1}\mathbb{E}(\chi_\infty^{p-1})]x \geq \frac{1}{\psi'(0+)}2^{p-1}C_{p-1}x|^{p+1} + V(0,0).
$$

Hence (31) follows for $x < 0$. The general statement holds since $V$ is non-decreasing in each argument.

Define the set $D := \{(u,x) \in E : V(u,x) = 0\}$. From Lemma 4.2 we know that $V(u,x) < 0$ for all $(u,x) \in E$ such that $x < h(u)$. Hence if $(u,x) \in D$ we have that $x \geq h(u) \geq 0$. We then define the function $b : (0,\infty) \mapsto \mathbb{R}$ by

$$
b(u) = \inf\{x > 0 : V(u,x) = 0\},
$$

where $\inf \emptyset = \infty$ and $\inf(0,\infty) = 0$. Then it directly follows that $b(u) \geq h(u) \geq 0$ for all $u > 0$. Moreover, since $h(0) = \infty$ we have that $\lim_{u \downarrow 0} b(u) = \infty$. Furthermore, since $V$ is monotone in each argument we deduce that $u \mapsto b(u)$ is non-increasing and $V(u,x) = 0$ for all $x > b(u)$. We then have the following Lemma.

**Lemma 4.6.** The function $b : \mathbb{R}_+ \mapsto \mathbb{R}$ is non-increasing with $0 \leq h(u) \leq b(u)$. We have that $\lim_{u \downarrow 0} b(u) = \infty$ and $b(u) \downarrow \infty$ for all $u > 0$.

**Proof.** We show that for each $u > 0$, $b(u) < \infty$. Fix $u > 0$ and take $x > y > 0$. By a dynamic programming argument we obtain that

$$
V(u,x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{u,x} \left( \int_0^{\tau \wedge \sigma_y^\tau} G(u + s, X_s)ds + \mathbb{I}_{\{\sigma_y^\tau < \tau\}} V(U_{\sigma_y^\tau}, X_{\sigma_y^\tau}^-) \right)
$$

$$
\geq \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left( \int_0^{\tau \wedge \sigma_y^\tau} G(u + s, X_s)ds + \mathbb{I}_{\{\sigma_y^\tau < \tau\}} V(0,0) + \mathbb{I}_{\{\sigma_y^\tau < \tau, X_{\sigma_y^\tau} < 0\}} V(0, X_{\sigma_y^\tau}^-) \right),
$$

where the inequality follows since $V$ is non-positive and non decreasing. By the compensation formula for Poisson random measures we have that for any stopping time $\tau$ (we assume without loss of generality that $\tau < \infty$ a.s.),

$$
\mathbb{E}_x \left( \mathbb{I}_{\{\sigma_y^\tau < \tau, X_{\sigma_y^\tau} < 0\}} V(0, X_{\sigma_y^\tau}^-) \right) = \mathbb{E}_x \left( \int_0^{\tau \wedge \sigma_y^\tau} \int_{(-\infty,0)} V(0, X_{s+} + z)\mathbb{I}_{\{s < \tau \wedge \sigma_y^\tau\}} N(ds,dz) \right)
$$

$$
= \mathbb{E}_x \left( \int_0^{\tau \wedge \sigma_y^\tau} \int_{(-\infty,0)} V(0, X_{s+} + z)\mathbb{I}_{\{X_{s+} + z < 0\}} N(dz)ds \right),
$$
Hence, from the equation above, since $G$ and $V$ are non-decreasing in each argument, $V \leq 0$ and $X_s \geq y$ for all $s \geq \sigma_y^-$ we have that
\[
V(u, x) \geq \inf_{\tau \in T} \mathbb{E}_x \left( (\tau \wedge \sigma_y^-) \left[ G(u, y) + \int_{(-\infty, -y]} V(0, z)dz \right] + \mathbb{1}_{\{\sigma_z^- < \tau\}} V(0, 0) \right).
\]
Note that from equation (31) and Lemma 2.1 the integral with respect to $\Pi(dz)$ above is finite so we can take $y$ sufficiently large such that $a = G(u, y) + \int_{(-\infty, -y]} V(0, z)\Pi(dz) \geq 0$. Then from Lemma 2.4 we have that (since $V(0, 0) \leq 0$ and $-X$ drifts to $-\infty$) that there exists a value $x_0(u) < 0$ such that the right hand side of the equation above vanishes for all $y - x \leq x_0(u)$. Hence, we have that $V(u, x) = 0$ for all $x \geq y - x_0(u)$ and then $b(u) < \infty$.

Let $(u, x) \in E$. We define, under the measure $\mathbb{P}_{u,x}$, the stopping times
\[
\tau_D = \inf\{t \geq 0 : (U_t, X_t) \in D\} = \inf\{t \geq 0 : X_t \geq b(U_t)\},
\]
\[
\tau_{b,y} = \inf\{t > 0 : X_t + y \geq b(U_t - y)\}, \quad y > 0 \text{ and } y \in \mathbb{R},
\]
(32) and for any $x \in \mathbb{R}$, under the measure $\mathbb{P}_x$, the stopping time
\[
\tau_{b,y} = \inf\{t > 0 : X_t + y \geq b(U_t - y)\}, \quad y \in \mathbb{R}.
\]
(33)
Note that for any $y \in \mathbb{R}$ and $v > 0$, the stopping time $\tau_{b,y}$ does not depend on the process $U$ and hence for any measurable function $f$, we have that $\mathbb{E}_{u,x}(f(\tau_{b,y})) = \mathbb{E}_x(f(\tau_{b,y}))$. Hence, for any $(u, x) \in E$ and any measurable function $f$ it can be seen that
\[
\mathbb{E}_{u,x}(f(\tau_D)) = \mathbb{E}_x(f(\tau_{0,0}^{u,0}\{\tau_{0,0} \leq \sigma_0\})) + \mathbb{E}_x(f(\tau_{0,0}^{u,0}\{\tau_{0,0} > \sigma_0\})).
\]
Now we introduce a technical lemma that ensures that the stopping time $\tau_D$ has moments of order $p$. The proof can be found in the Appendix A.

**Lemma 4.7.** For all $(u, x) \in E$ we have that
\[
\mathbb{E}_{u,x}((\tau_D)^p) < \infty.
\]

Now we are ready to show (using the general theory of optimal stopping) that $\tau_D$ is an optimal stopping time for (25) in terms of the set $D$.

**Lemma 4.8.** An optimal stopping time for (25) is given by $\tau_D$, the first entrance of $(U, X)$ to the closed set $D$, i.e.
\[
\tau_D = \inf\{t \geq 0 : (U_t, X_t) \in D\}.
\]
Then the function $V$ takes the form
\[
V(u, x) = \mathbb{E}_{u,x} \left( \int_0^{\tau_D} G(U_s, X_s)ds \right), \quad (u, x) \in E.
\]

**Proof.** Note that it follows from Lemma 4.7 that $\mathbb{P}_{u,x}(\tau_D < \infty) = 1$ for all $(u, x) \in E$. Then using a dynamic programming argument we deduce that
\[
V(u, x) = \inf_{\tau \in T} \mathbb{E}_{u,x} \left( \int_0^{\tau \wedge \tau_D} G(U_s, X_s)ds + \mathbb{1}_{\{\tau_D < \tau\}} V(U_{\tau_D}, X_{\tau_D}) \right)
\]
\[
= \inf_{\tau \in T} \mathbb{E}_{u,x} \left( \int_0^{\tau \wedge \tau_D} G(U_s, X_s)ds \right),
\]
where in the last equality we used that $V(u, x) = 0$ on $D$. 


Since \( W(x) \leq 1/\psi'(0+) \) for all \( x \in \mathbb{R} \) we have that \( |G(u, x)| \leq u^{p-1} + \mathbb{E}_x(g^{p-1}) \). Then for any \((u, x) \in E\) we deduce that

\[
\mathbb{E}_{u,x} \left[ \sup_{t \geq 0} \int_0^{t \land \tau_D} G(U_s, X_s) \, ds \right] \leq \mathbb{E}_{u,x} \left[ \int_0^{\tau_D} [(U_s)^{p-1} + \mathbb{E}_x(g^{p-1})] \, ds \right]
\]

\[
\leq \mathbb{E}_{u,x} \left( \int_0^{\tau_D} (u + s)^{p-1} \, ds \right) + \mathbb{E}_{u,x} \left( \int_0^{\tau_D} \mathbb{E}_x(g^{p-1}) \, ds \right)
\]

\[
\leq 2^{p-1}[u^{p-1} + \frac{1}{p} \mathbb{E}_{u,x}[(\tau_D)^p]] + \mathbb{E}_{x} \left( \int_0^{\tau_D} \mathbb{E}_x(g^{p-1}) \, ds \right) < \infty,
\]

where in the second inequality we used that \( U_s \leq u + s \) for all \( s \geq 0 \) under the measure \( \mathbb{P}_{u,x} \) and on the last equality follows from Lemmas 4.1 and 4.7.

Note that from equation (30) we have that \( \tau \) is optimal we can then give a representation of \( \tau^*_s \) in terms of the original measure \( \mathbb{P} \) and the stopping times \( \tau^*_u^x \) and \( \tau^*_0^x \) defined in (32) and (33), respectively. For any \((u, x) \in E\) we can write

\[
V(u, x) = \mathbb{E}_{u,x} \left( \int_0^{\tau_D} G(U_s, X_s) \, ds \right)
\]

\[
= \mathbb{E} \left( \int_0^{\tau^*_u^x} G(u + s, X_s + x) \, ds + \mathbb{I}_{\{\tau^*_u^x \leq \tau^*_0^x\}} \int_{\tau^*_u^x}^{\tau^*_0^x} G(U_s(-x), X_s + x) \, ds \right)
\]

\[
= \mathbb{E} \left( \int_0^{\tau^*_u^x} G(u + s, X_s + x) \, ds + \mathbb{I}_{\{\tau^*_u^x \leq \tau^*_0^x\}} V(0, X_{\tau^*_0^x} - x) \right). \tag{34}
\]

Note that in the last equality we do not longer have explicitly the process \( \{U_t(-x), t \geq 0\} \). This alternative representation of \( V \) in terms of the original measure \( \mathbb{P} \) will be useful to prove further properties of \( b \) and \( V \). The next lemma describes the limit behaviour of the function \( b \).

**Lemma 4.9.** We have that

\[
\lim_{u \to \infty} b(u) = 0.
\]

**Proof.** Note that, since \( b \) is non-increasing and it is bounded from below by \( \lim_{u \to \infty} h(u) = 0 \), the limit \( b^* := \lim_{u \to \infty} b(u) \) exists and \( b^* \geq 0 \). We prove by contradiction that \( b^* = 0 \). Suppose \( b^* > 0 \) and define the stopping time

\[
\sigma_\ast = \inf \{ t \geq 0 : X_t \notin (0, b^*) \}.
\]

Take \( u > 0 \) and \( x \in (0, b^*) \). From the fact that \( b(u) \geq b^* > 0 \) we have that \( \sigma_\ast \leq \tau_D \land \sigma_0^- \) under \( \mathbb{P}_{u,x} \). Then
we have that
\[
V(u, x) = \mathbb{E}_{u,x} \left( \int_0^{\tau_D} G(U_s, X_s) \, ds \right)
\]
\[
= \mathbb{E}_x \left( \int_0^{\tau} G(u + s, X_s) \, ds \right) + \mathbb{E}_{u,x} \left( V(U_{\sigma_s}, X_{\sigma_s}) \right)
\]
\[
= \mathbb{E}_x \left( \int_0^{\tau} G(u + s, X_s) \, ds \right) + \mathbb{E}_x \left( V(u + \sigma_s, X_{\sigma_s})1_{\{X_{\sigma_s} > 0\}} \right) + \mathbb{E}_x \left( V(0, X_{\sigma_s})1_{\{X_{\sigma_s} \leq 0\}} \right),
\] (35)

where in the last equality we used the Markov property of the two dimensional process \(\{(U_t, X_t), t \geq 0\}\). For a fixed \(x \in \mathbb{R}\), the function \(u \mapsto V(u, x)\) is non-decreasing and bounded from above by zero, thus we have that \(\lim_{u \to \infty} V(u, x)\) exists and \(-\infty < \lim_{u \to \infty} V(u, x) \leq 0\) for all \(x \in \mathbb{R}\). By the dominated convergence theorem we also conclude that \(-\infty < \lim_{u \to \infty} \mathbb{E}_x \left( V(u + \sigma_s, X_{\sigma_s})1_{\{X_{\sigma_s} > 0\}} \right) \leq 0\). Moreover, using the general version of Fatou’s lemma and the fact that \(\lim_{u \to \infty} G(u, x) = \infty\) we have that
\[
\lim_{u \to \infty} \mathbb{E}_x \left( \int_0^{\tau} G(u + s, X_s) \, ds \right) = \infty.
\]

Therefore, taking \(u \to \infty\) in (35) we get that
\[
\lim_{u \to \infty} V(u, x) = \infty.
\]

Which yields the desired contradiction. Therefore we conclude that \(b^* = 0\).

In the following, we proceed to analyse continuity properties of \(b\) and \(V\). Note that, by using standard arguments (from the fact that \(D\) is a closed set) we can show that the function \(b\) is right continuous. It turns out that \(b\) is continuous, the proof of this fact makes use of a variational inequality and will be proved later.

Now we are ready to show the continuity of the value function \(V\). The proof is rather long and technical so is included in the Appendix A.

**Lemma 4.10.** The function \(V\) is continuous on \(E\). Moreover, in the case that \(X\) is of infinite variation we have that
\[
\lim_{h \downarrow 0} V(u, h) = V(0, 0)
\]
for all \(u > 0\).

We know from the fact that \(D\) is a closed set that \(b\) is a right-continuous function. In order to show left continuity we make use of a variational inequality that is satisfied by the value function \(V\). The oncoming paragraphs will be dedicated on introducing that.

It is well known that for every optimal stopping problem there is a free boundary problem which is stated in terms of the infinitesimal generator (see e.g. Peskir and Shiryaev (2006), Chapter III). In this particular case, provided that the value function is smooth enough, we have that \(V\) solves the Dirichlet/Poisson problem. That is,
\[
\mathcal{A}_{U,X}(V) = \frac{\partial}{\partial u} V + \mathcal{A}_X(V) = -G \quad \text{in } E \setminus D,
\]
where \(\mathcal{A}_{U,X}\) corresponds to the infinitesimal generator of the process \((U, X)\) given in (14) and \(\mathcal{A}_X\) is the infinitesimal generator of \(X\) given by
\[
\mathcal{A}_X(V) = -\mu \frac{\partial}{\partial x} V(u, x) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} F(u, x) + \int_{(-\infty,0)} \left( \hat{V}(u, x + y) - \hat{V}(u, x) - y1_{\{y > 1\}} \frac{\partial}{\partial x} \hat{V}(u, x) \right) \Pi(dy),
\]
whilst $\tilde{V}$ is the extension of $V$ to the set $\mathbb{R}_+ \times \mathbb{R}$ given by

$$\tilde{V}(u, x) = \begin{cases} V(u, x) & u > 0 \text{ and } x > 0, \\ V(0, x) & u \geq 0 \text{ and } x \leq 0, \\ V(0, 0) & u = 0 \text{ and } x > 0. \end{cases}$$ (36)

However, in our setting turns out to be challenging to show that $V$ is a $C^{1,2}$ function. Lamberton and Mikou (2008) showed that we can state an analogous (in)equality in the sense of distributions. In particular, since $V$ is continuous on $E$ we have that $\tilde{V}$ is a locally integrable function in $\mathbb{R}_+ \times \mathbb{R}$ (note that $\tilde{V}$ may be discontinuous at points of the form $(u, 0)$ for $u > 0$ when $X$ is of finite variation) so we can define $\tilde{V}$ as a distribution in any open set $O \subset \mathbb{R}_+ \times \mathbb{R}$ via

$$\langle \tilde{V}, \varphi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \tilde{V}(u, x) \varphi(u, x) du dx$$

for any test function $\varphi$ with compact support in $O$. Then the derivatives of the distribution $\tilde{V}$ are defined as

$$\langle \partial_j^{i+j} \tilde{V}, \varphi \rangle = (-1)^{i+j} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \tilde{V}(u, x) \partial_j^{i+j} \varphi(u, x) du dx.$$

Moreover, provided that the function $(u, x) \mapsto \int_{(-\infty, -1)} \tilde{V}(u, x+y)\Pi(dy)$ is locally integrable in $\mathbb{R}_+ \times \mathbb{R}$, the operator $B_X$, defined for any test function $\varphi$, with compact support in $O$, by

$$\langle B_X(\tilde{V}), \varphi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \tilde{V}(u, x) B_X^\ast(\varphi)(u, x) du dx dy,$$

defines a distribution on $O$, where

$$B_X^\ast(\varphi)(u, x) = \int_{(-\infty, 0)} [\varphi(u, x-y) - \varphi(u, x) + y \partial_x \varphi(u, x)] \Pi(y \geq 1) dy.$$

We have the following Lemma that ensures that the integrability conditions for $\tilde{V}$ are satisfied so then $B_X(\tilde{V})$ is indeed a distribution.

**Lemma 4.11.** The function

$$(u, x) \mapsto \int_{(-\infty, -1)} \tilde{V}(u, x+y)\Pi(dy)$$

is locally integrable in $\mathbb{R}_+ \times \mathbb{R}$.

**Proof.** First note that from equation (31) we have that for any $x \leq 0$,

$$\int_{(-\infty, -1)} V(0, x+y)\Pi(dy) \geq -A_{p-1}' \Pi(-\infty, -1] - C_{p-1}' \int_{(-\infty, -1]} |x+y|^p \Pi(dy)$$

$$> -\infty,$$

where we used the fact that $\Pi(-\infty, -1] < \infty$ and Lemma 2.1. Moreover, since $V$ is non-decreasing in each argument we have that for any $u > 0$ and $x > 0$ that

$$\int_{(-\infty, -1)} \tilde{V}(u, x+y)\Pi(dy) \geq \int_{(-\infty, -1)} V(0, 0)\Pi(dy) > -\infty.$$

Hence we conclude that $\int_{(-\infty, -1)} \tilde{V}(u, x+y)\Pi(dy) > -\infty$ for any $(u, x) \in \mathbb{R}_+ \times \mathbb{R}$. Since $V$ is continuous on $E$ and the definition of $\tilde{V}$ we have that the mapping $(u, x) \mapsto \int_{(-\infty, -1)} \tilde{V}(u, x+y)\Pi(dy)$ is locally integrable. 

\[ \square \]
Hence, we can define the operator $\mathcal{A}_X$ in the sense of distributions by

$$\mathcal{A}_X(\tilde{V}) = -\mu \frac{\partial}{\partial x} \tilde{V} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} \tilde{V} + B_X(\tilde{V}).$$

The next lemma is an extension of Proposition 2.5 in Lamberton and Mikou (2008).

**Lemma 4.12.** The distribution $\frac{\partial}{\partial u} \tilde{V} + \mathcal{A}_X(\tilde{V}) + G$ is a non-negative distribution on $(0, \infty) \times (0, \infty)$. Moreover, we have $\frac{\partial}{\partial u} \tilde{V} + \mathcal{A}_X(\tilde{V}) + G = 0$ on the set $C^+ := \{(u, x) \in (0, \infty) \times (0, \infty) : 0 < x < b(u)\}$ and $\mathcal{A}_X(V(0, \cdot)) + G(0, \cdot) = 0$ on $(-\infty, 0)$ in the sense of distributions.

**Proof.** The proof relies on the fact that under the measure $\mathbb{P}_{t,u,x}$

$$V(U_{t\wedge \sigma^0}, X_{t\wedge \sigma^0}) = V(u + t, X_t|\{t < \sigma^0\}) + V(0, X_{\sigma^0})|\{t \geq \sigma^0\} = \tilde{V}(u + t \wedge \sigma^0, X_{t\wedge \sigma^0}),$$

for any $u, x > 0$ and $t \geq 0$ and that for any $x < 0$,

$$V(U_{t\wedge \sigma^0}^+, X_{t\wedge \sigma^0}^+) = V(0, X_{t\wedge \sigma^0}^+).$$

The main details are analogous to Proposition 2.5 in Lamberton and Mikou (2008) so they are omitted.

**Remark 4.13.**

i) In Lamberton and Mikou (2008) the definition of the infinitesimal generator in the sense of distributions assumes that the value function is a bounded Borel measurable function. In our setting such condition can be relaxed by the fact that $\Lambda$ is continuous on the interior of $D$.

ii) We note that similar as in (14) the infinitesimal generator of $(U, X)$ can be defined as $\mathcal{A}_{U, X}(V) := \frac{\partial}{\partial u} \tilde{V} + \mathcal{A}_X(\tilde{V})$ in the sense of distributions, where $\mathcal{A}_X$ corresponds to the infinitesimal generator of $X$ (seen as a distribution).

Let $\text{int}(D)$ be the interior of the set $D$. For $(u, x) \in \text{int}(D)$ we define the function

$$\Lambda(u, x) := \int_{(-\infty, 0]} V(u, x + y) \Pi(dy) + G(u, x), \quad x > b(u).$$

The next lemma states some basic properties of the function $\Lambda$.

**Lemma 4.14.** The function $\Lambda$ is a non-decreasing (in each argument) function such that $0 < \Lambda(u, x) < \infty$ for all $(u, x) \in \text{int}(D)$. Moreover, is strictly increasing in each argument and continuous in the interior of the set $D$. Furthermore, $\Lambda = \frac{\partial}{\partial u} \tilde{V} + \mathcal{A}_X(\tilde{V}) + G$ on $\text{int}(D)$ in the sense of distributions.

**Proof.** It follows from Lemma 4.11 and the fact that $V$ vanishes in $D$ that $|\Lambda(u, x)| < \infty$ for all $(u, x) \in \text{int}(D)$. The fact that $\Lambda$ is continuous on the interior of $D$ follows from the continuity of $V$ and $G$, the dominated convergence theorem. Moreover, $\Lambda$ is strictly increasing in each argument on $\text{int}(D)$ since $V$ is non-decreasing in each argument and $G$ is strictly increasing in each argument on $D$. It can be seen from Lemma 4.12 that $\frac{\partial}{\partial u} \tilde{V} + \mathcal{A}_X(\tilde{V}) + G = \Lambda$ on in the interior of $D$ and that $\Lambda(u, x) > 0$ for all $u > 0$ and $x > b(u)$.

It turns out that the function $b$ is continuous, the proof is analogous to the one presented in Lamberton and Mikou (2008) (see Theorem 4.2) in the American option setting so is omitted.

**Lemma 4.15.** The function $b$ is continuous.

From Lemma 4.9 we know that $b$ converges to zero when $u$ tends to infinity. Moreover, from the discussion about $h$ after equation (27) we know that in case that $X$ is of finite variation, there exists a value $u_h^* < \infty$ for which $h(u) = 0$ for all $u \geq u_h^*$. That suggests a similar behaviour for $b$, the next lemma addresses this conjecture.

**Lemma 4.16.** Define $u_b = \inf\{u > 0 : b(u) = 0\}$. If $X$ is of infinite variation or finite variation and infinite activity we have that $u_b = \infty$. Otherwise $u_b = u^*$, where $u^*$ is the unique solution to

$$G(u, 0) + \int_{(-\infty, 0]} V(0, y) \Pi(dy) = 0. \quad (37)$$
Proof. From the fact that \( h(u) > 0 \) for all \( u > 0 \) when \( X \) is of infinite variation and inequality \( b(u) \geq h(u) \) we have that assertion is true for this case. Suppose that \( X \) has finite variation with infinite activity, that is, \( \Pi(-\infty, 0) = \infty \), and assume that \( u_b < \infty \). Then since \( b \) is non-increasing we have that \( b(u) = 0 \) for all \( u > u_b \) and then \( V(u, x) = 0 \) for all \( x > 0 \) and \( u > u_b \). From Lemma 4.14 we have that

\[
G(u, x) + \int_{(-\infty, -x)} V(0, x + y)\Pi(dy) \geq 0
\]

for all \( x > 0 \) and for all \( u > u_b \). Taking \( x \downarrow 0 \) in the equation above and using the expression for \( V(0, z) \) for \( z < 0 \) given in (30) we have that for any \( u > u_b \),

\[
0 \leq G(u, 0) - \lim_{x \downarrow 0} \int_{(-\infty, 0)} \int_{0}^{\infty} \mathbb{E}_{u \rightarrow x}(g^{u-1})W(du)dz\Pi(dy) + \lim_{x \downarrow 0} V(0, 0)\Pi(-\infty, -x)
\]

which is a contradiction and then \( u_b = \infty \). Now assume that \( X \) has finite variation with \( \Pi(-\infty, 0) < \infty \). Assume that \( b(u^*) > 0 \), then \( V(u^*, x) < 0 \) for \( x \in (0, b(u^*)) \). Moreover, since \( V \leq 0 \) and using the compensation formula for Poisson random measures we have that for all \( u > 0 \) and \( x < b(u) \),

\[
\mathbb{E}_{u, x}(V(0, X_{\tau_0})\mathbb{I}_{\{\tau_0 < \tau_D\}}) = \mathbb{E}_{u, x} \left( \int_{[0, \infty)} \int_{(-\infty, 0)} V(0, X_{s+} + y)\mathbb{I}_{\{X_{s+} > 0\}}\mathbb{I}_{\{X_{s+} + y < 0\}}\mathbb{I}_{\{X_r \leq b(U_r) \text{ for all } r < s\}}N(ds, dy) \right)
\]

Then from the Markov property we have that for all \( x < b(u^*) \)

\[
V(u^*, x) = \mathbb{E}_{u^*, x} \left( \int_{0}^{\tau_D \land \tau_0} G(u^* + s, X_s)ds \right) + \mathbb{E}_{u^*, x}(V(0, X_{\tau_0})\mathbb{I}_{\{\tau_0 < \tau_D\}})
\]

\[
= \mathbb{E}_{u^*, x} \left( \int_{0}^{\tau_D \land \tau_0} \left[ G(u^* + s, X_s) + \int_{(-\infty, 0)} V(0, X_s + y)\mathbb{I}_{\{X_s + y < 0\}}\Pi(dy) \right] ds \right) > 0,
\]

where the strict inequality follows from the fact that \( X \) is of finite variation and then \( \tau_D \land \tau_0 > 0 \), the definition of \( u^* \) and the fact that \( G \) and \( V \) are non-decreasing in each argument. Then we are contradicting the fact that \( V(u^*, x) < 0 \) and we conclude that \( b(u^*) = 0 \) and \( u_b \leq u^* \). Moreover, from Lemma 4.14 we know that for all \( u > u_b \)

\[
G(u, x) + \int_{(-\infty, -x)} V(0, x + y)\Pi(dy) \geq 0 \text{ for all } x > 0.
\]

Taking \( x \downarrow 0 \) we get that for all \( u \geq u_b \), \( G(u, 0) + \int_{(-\infty, 0)} V(0, y)\Pi(dy) \geq 0 \). The latter implies that \( u^* \leq u_b \) (since \( u \mapsto G(u, 0) \) is strictly increasing). Therefore we conclude that \( u^* = u_b \) and the proof is complete. \( \square \)

As we mentioned before it is challenging to prove the existence of the derivatives of \( V \). However, it is possible to show that the derivatives of \( V \) at the boundary exist and are equal to zero. Recall from Lemma 4.16 that when \( X \) is of infinite variation or finite variation with infinite activity we have that \( b(u) > 0 \) for all \( u > 0 \). In the case that \( X \) is of finite variation we have that \( b(u) > 0 \) only if \( u < u_b \) where \( u_b \) is the solution to (37). In such cases we can guarantee that the derivatives of \( V \) exist at the boundary and are equal to zero, which is proven in the following Theorem. Since the proof is rather long and technical, it can be found in the Appendix.
**Lemma 4.17.** Suppose that $u > 0$ is such that $b(u) > 0$. Then the first partial derivatives of $V(u, x)$ exist at the point $x = b(u)$ and

$$
\frac{\partial}{\partial x} V(u, b(u)) = 0 \quad \text{and} \quad \frac{\partial}{\partial u} V(u, b(u)) = 0.
$$

Recall from equation (30) that when $x < 0$,

$$
V(0, x) = -\int_{0}^{x} \int_{(0, \infty)} \mathbb{E}_{u-z}(g^{p-1})W(du)dz + V(0, 0).
$$

Note that the first term on the right-hand side of the equation above does not depend on the boundary $b$. Then, for $x < 0$, the value function $V(0, x)$ is characterised by the value $V(0, 0)$. Moreover, from Lemma 4.16 we know that when $X$ is of finite variation with $\Pi(-\infty, 0) < \infty$, the value $u_b$ is the unique solution to

$$
G(u, 0) - \int_{(-\infty, 0)} \int_{0}^{y} \mathbb{E}_{u-z}(g^{p-1})W(du)dz\Pi(dy) + V(0, 0)\Pi(-\infty, 0) = 0,
$$

otherwise, $u_b = \infty$. Then if $X$ is of finite variation with finite activity, $u_b$ is also characterised by the value $V(0, 0)$, where we know from Remark 4.3 that

$$
-\frac{\mathbb{E}(g^{p-1})}{p} \leq V(0, 0) < 0.
$$

The next theorem gives a characterisation of the value function $V$ on the set $(0, \infty) \times (0, \infty)$, the boundary $b$ and the values $V(0, 0)$ and $u_b$ as unique solutions of a system of non-linear integral equations within a class of functions. The method of proof is deeply inspired on the ideas of du Toit et al. (2008). However, the presence of jumps adds an important level of difficulty. In particular, when $\Pi \neq 0$, the inequality

$$
G(u, x) + \int_{(-\infty, 0)} \tilde{V}(u, x + y)\Pi(dy) > 0
$$

for all $(u, x) \in D$ is a necessary condition for the process \{$V(U_t, X_t) + \int_{0}^{t} G(U_s, X_s)ds, t \geq 0$\} to be a submartingale.

**Theorem 4.18.** Let $p > 1$ and $X$ be a spectrally negative Lévy process drifting to infinity such that its Lévy measure $\Pi$ has no atoms and $\int_{(-\infty, -1]} |x|^{p+1}\Pi(dx) < \infty$. For all $u > 0$ and $x > 0$, the function $V$ can be written as

$$
V(u, x) = V(0, 0)\frac{\sigma^2}{2}W'(x) - \mathbb{E}_x \left( \int_{0}^{\tau_0} \int_{(-\infty, 0)} V(u + s, X_s + y)I_{\{0 < X_s + y < b(u + s)\}}\Pi(dy)I_{\{X_s > b(u + s)\}}ds \right)
$$

$$
+ \mathbb{E}_x \left( \int_{0}^{\tau_0} G(u + s, X_s) + \int_{(-\infty, 0)} V(0, X_s + y)I_{\{X_s + y < 0\}}\Pi(dy)I_{\{X_s < b(u + s)\}}ds \right), \quad (38)
$$

the value $V(0, 0)$ satisfies

$$
V(0, 0) = -\frac{1}{\psi'(0+)} \int_{0}^{\infty} \mathbb{E}_{z}(g^{p-1})(1 - \psi'(0+)W(z)dz + \frac{1}{\psi'(0+)} \mathbb{E} \left( \int_{0}^{\infty} G(s, X_s)\frac{X_s}{s}I_{\{0 < X_s < b(s)\}}ds \right)
$$

$$
- \frac{1}{\psi'(0+)} \mathbb{E} \left( \int_{0}^{\infty} \int_{(-\infty, 0)} V(s, X_s + y)I_{\{0 < X_s + y < b(s)\}}\Pi(dy)\frac{X_s}{s}I_{\{X_s > b(s)\}}ds \right)
$$

$$
- \frac{1}{\psi'(0+)} \mathbb{E} \left( \int_{0}^{\infty} \int_{(-\infty, 0)} V(0, X_s + y)I_{\{X_s + y \leq 0\}}\Pi(dy)\frac{X_s}{s}I_{\{X_s > b(s)\}}ds \right), \quad (39)
$$

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whilst the curve \( b \) satisfies the equation

\[ 0 = V(0, 0) \frac{\sigma^2}{2} W'(b(u)) - \mathbb{E}_b(u) \left( \int_0^{b(u)} \int_{(-\infty,0)} V(u+s, X_s + y) I_{1(0<X_s+y<b(u+s))} \Pi(dy) \Pi(X_s>b(u+s)) \, ds \right) \]

\[ + \mathbb{E}_b(u) \left( \int_0^{b(u)} \left[ G(u+s, X_s) + \int_{(-\infty,0)} V(0, X_s + y) I_{\{X_s+y\leq 0\}} \Pi(dy) \right] I_{\{X_s<b(u+s)\}} \, ds \right) \]

(40)

for all \( u < u_b \), where for \( x \leq 0 \), the function \( V(0, x) \) depends on \( V(0, 0) \) via (30). For \( u \geq u_b \) we have \( b(u) = 0 \), where \( u_b = \infty \) in the case \( X \) is of infinite variation or finite variation with \( \Pi(-\infty,0) = \infty \). Otherwise, \( u_b \) is the unique solution to

\[ G(u, 0) - \int_{(-\infty,0)} \int_0^y \mathbb{E}_{-u-z}(g^{p-1}) W(du) dz \Pi(dy) + V(0, 0) \Pi(-\infty, 0) = 0. \]

(41)

Moreover, in the case that there is a Brownian motion component (i.e. \( \sigma > 0 \)) we have that (39) is equivalent to

\[ \frac{\partial}{\partial x} V_+(0, 0) = \frac{\partial}{\partial x} V_-(0, 0), \]

(42)

where \( \frac{\partial}{\partial x} V_+(0, 0) \) and \( \frac{\partial}{\partial x} V_-(0, 0) \) are the right and left derivatives of \( x \mapsto V(u, x) \) and \( x \mapsto V(0, x) \) at zero, respectively and \( \frac{\partial}{\partial x} V_+(0, 0) = \lim_{u \downarrow 0} \frac{\partial}{\partial x} V_+(u, 0) \).

Furthermore, the quadruplet \((V, b, V(0, 0), u_b)\) is uniquely characterised by the equations above, where \( V \) is considered in the class of non-positive continuous functions such that

\[ \int_{(-x-b(u), -x)} V(u, b(u) + x + y) \Pi(dy) + \int_{(-\infty, -x-b(u))} V(0, b(u) + x + y) \Pi(dy) + G(u, x + b(u)) \geq 0 \]

(43)

for all \( u < u_b \) and \( x > 0 \) and \( b \) is considered in the class of non-increasing functions with \( b \geq h \) whereas \(-\frac{1}{p} \mathbb{E}(q^p) \leq V(0, 0) < 0 \).

Since the proof of Theorem 4.18 is rather long we break it in a series of Lemmas. Next section is entirely dedicated to that purpose.

## 5 Proof of Theorem 4.18

First, we show that the relevant quantities are integrable. The Lemma is long so is included in the Appendix A.

**Lemma 5.1.** We have that for all \((u, x) \in E\),

\[ \mathbb{E}_{u,x} \left( \int_0^\infty |G(U_s, X_s)| I_{\{X_s < b(U_s)\}} \, ds \right) < \infty, \]

(44)

\[ \mathbb{E}_{u,x} \left( \int_0^\infty \int_{(-\infty,0)} V(U_s, X_s + y) I_{\{X_s+y\leq 0\}} \Pi(dy) I_{\{X_s>b(U_s)\}} \, ds \right) > -\infty. \]

(45)

Moreover, we have that

\[ \lim_{u,x \to \infty} \mathbb{E}_{u,x} \left( \int_0^\infty G(U_s, X_s) I_{\{X_s<b(U_s)\}} \, ds \right) = 0, \]

(46)

\[ \lim_{u,x \to \infty} \mathbb{E}_{u,x} \left( \int_0^\infty \int_{(-\infty,0)} V(U_s, X_s + y) I_{\{X_s+y\leq 0\}} \Pi(dy) I_{\{X_s>b(U_s)\}} \, ds \right) = 0. \]

(47)

Next, we show that \( V \) satisfies the alternative representation mentioned in the infinite variation case.
Lemma 5.2. Suppose that \( X \) is of infinite variation. Then we have that \( V \) and \( b \) satisfy equations (38) and (40).

Proof. Recall that \( V \) is continuous on \( E \) and, in this case, (see Lemma 4.10) we have that for any \( u > 0 \), \( \lim_{t\to0} V(u,x) = V(0,0) \) implying that \( \tilde{V} \) is continuous on \( \mathbb{R}_+ \times \mathbb{R} \). We follow an analogous argument as Lamberton and Mikou (2013) (see Theorem 3.2). Let \( \rho \) be a positive \( C^\infty \) function with support in \([0,1] \times [0,1] \) and \( \int_0^\infty \int_0^\infty \rho(v,y) dv dy = 1 \). For \( n \geq 1 \), define \( \rho_n(v,y) = n^2 \rho(nv, ny) \), then \( \rho_n \) is \( C^\infty \) and has compact support in \([0,1/n] \times [0,1/n] \). The function defined by \( \tilde{V}_n(u,x) := (\tilde{V} \ast \rho_n)(u,x) = \int_0^\infty \int_0^\infty \tilde{V}(u-v, x-y) \rho_n(v,y) dv dy \) is of infinite variation (see Sato (1999), Theorem 27.4). Note that, since \( \rho_n \) is of infinite variation (see Sato (1999), Theorem 27.4) we can show that for all \((u,x)\) \( V \) is continuous on \([0,1] \times [0,1] \) and \( \rho_n \) is finite for all \( n \) sufficiently large. Hence,

\[
\frac{\partial}{\partial u} \tilde{V}_n(u,x) + A_X(\tilde{V}_n)(u,x) = -(G * \rho_n)(u,x),
\]

where \( A_X \) is the infinitesimal generator of the process \( X \). On the other hand, note that if \((u,x)\) \( \in D \) we have that \( V(u,x) = 0 \) and hence \( \tilde{V}_n(u,x) = 0 \) for \( n \) sufficiently large. Hence,

\[
\frac{\partial}{\partial u} \tilde{V}_n(u,x) + A_X(\tilde{V}_n)(u,x) = \int_{(-\infty,0)} \tilde{V}_n(u,x+y) \Pi(dy).
\]

Therefore, by the dominated convergence theorem we have that,

\[
\lim_{n \to \infty} \left[ \frac{\partial}{\partial u} \tilde{V}_n(u,x) + A_X(\tilde{V}_n)(u,x) \right] = \int_{(-\infty,0)} \tilde{V}(u,x+y) \Pi(dy).
\]

for any \((u,x)\) \( \in D \).

Next, let \( u > 0 \) and \( x > 0 \) fixed and take \( n > 0 \) and \( k > 0 \) such that \( u > 1/n > 0 \) and \( x > k > 1/n > 0 \). We apply Itô formula to \( \tilde{V}_n(u + t \wedge \tau^-_k, X_{t \wedge \tau^-_k} - x) \) to get

\[
\tilde{V}_n(u + t \wedge \tau^-_k, X_{t \wedge \tau^-_k} - x) = \tilde{V}_n(u,x) + M_t + \int_0^{t \wedge \tau^-_k} \left[ \frac{\partial}{\partial u} \tilde{V}_n(u+s, X_s - x) + A_X(\tilde{V}_n)(u+s, X_s + x) \right] ds,
\]

where \( \{M_t, t \geq 0\} \) is a zero mean martingale. Taking expectations we get that

\[
E_x \left( \tilde{V}_n(u + t \wedge \tau^-_k, X_{t \wedge \tau^-_k}) \right) = \tilde{V}_n(u,x) + E_x \left( \int_0^{t \wedge \tau^-_k} \left[ \frac{\partial}{\partial u} \tilde{V}_n(u+s, X_s) + A_X(\tilde{V}_n)(u+s, X_s) \right] ds \right),
\]

\[
= \tilde{V}_n(u,x) - E_x \left( \int_0^{t \wedge \tau^-_k} (G * \rho_n)(u+s, X_s) I_{\{X_s < b(u+s)\}} ds \right) + E_x \left( \int_0^{t \wedge \tau^-_k} \frac{\partial}{\partial u} \tilde{V}_n(u+s, X_s) + A_X(\tilde{V}_n)(u+s, X_s) I_{\{X_s > b(u+s)\}} ds \right),
\]

where we used the fact that \( b \) is finite for all \( u > 0 \) and that \( E_x(X_s = b(u+s)) = 0 \) for all \( s > 0 \) and \( x \in \mathbb{R} \) when \( X \) is of infinite variation (see Sato (1999), Theorem 27.4). Note that, since \( X_t \geq X_\infty \) for all \( t > 0 \) and \( V \) is non-decreasing in each argument, we have that

\[
0 \geq E_x \left( \tilde{V}_n(u + t \wedge \tau^-_k, X_{t \wedge \tau^-_k}) \right) \geq -A_{p-1}' - C_{p-1}' E_x((-X_\infty)^p) + V(0,0) > -\infty,
\]

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where the second inequality follows from equation (31) and the last quantity is finite by Lemma 2.1. Therefore by the dominated convergence theorem we have that letting $n, t \to \infty$ and $k \downarrow 0$,

$$
\mathbb{E}_x \left( \tilde{V}(u + \tau_0^-, X_{\tau_0^-}) \right) = V(u, x) - \mathbb{E}_x \left( \int_0^{\tau_0^-} G(u + s, X_s) \mathbb{I}_{\{X_s < b(u+s)\}} ds \right) \\
+ \mathbb{E}_x \left( \int_0^{\tau_0^-} \int_{(-\infty,0)} \tilde{V}(u + s, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > b(u+s)\}} ds \right)
$$

(49)

for all $u > 0$ and $x > 0$. Note that, since $\lim_{u \to \infty} b(u) = 0$, we have that $\lim_{x \to \infty} \tilde{V}(u, x) = V(u, x) = 0$. Hence, since $\tilde{V}(u, x) = V(0, x)$ for any $u \geq 0$ and $x \leq 0$ and $X$ drifts to infinity we get that

$$
\mathbb{E}_x \left( \tilde{V}(u + \tau_0^-, X_{\tau_0^-}) \right) = \mathbb{E}_x \left( V(0, X_{\tau_0^-}) \mathbb{I}_{\{\tau_0^- < \infty\}} \right)
$$

$$
= V(0, 0) \mathbb{P}_x(X_{\tau_0^-} = 0, \tau_0^- < \infty) + \mathbb{E}_x \left( V(0, X_{\tau_0^-}) \mathbb{I}_{\{X_{\tau_0^-} < 0\}} \right) \\
= V(0, 0) \frac{\sigma^2}{2} W'(x) + \mathbb{E}_x \left( \int_0^{\tau_0^-} V(0, X_{s^-} + y) \mathbb{I}_{\{X_{s^-} + y < 0\}} \mathcal{N}(ds, dy) \right) \\
= V(0, 0) \frac{\sigma^2}{2} W'(x) + \mathbb{E}_x \left( \int_0^{\tau_0^-} V(0, X_s + y) \mathbb{I}_{\{X_s + y < 0\}} ds \mathbb{I}(dy) \right),
$$

where in the second last equality we used the probability of creeping given in (6) (note that $\Phi(0) = 0$ since $X$ drifts to infinity) and in the last the compensation formula for Poisson random measures. Then from above and equation (49) we see that

$$
V(u, x) = \mathbb{E}_x \left( \tilde{V}(u + \tau_0^-, X_{\tau_0^-}) \right) + \mathbb{E}_x \left( \int_0^{\tau_0^-} G(u + s, X_s) \mathbb{I}_{\{X_s < b(u+s)\}} ds \right) \\
- \mathbb{E}_x \left( \int_0^{\tau_0^-} \int_{(-\infty,0)} \tilde{V}(u + x, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > b(u+s)\}} ds \right)
$$

$$
= V(0, 0) \frac{\sigma^2}{2} W'(x) - \mathbb{E}_x \left( \int_0^{\tau_0^-} \int_{(-\infty,0)} V(u + s, X_s + y) \mathbb{I}_{\{0 < X_s + y < b(u+s)\}} \Pi(dy) \mathbb{I}_{\{X_s > b(u+s)\}} ds \right) \\
+ \mathbb{E}_x \left( \int_0^{\tau_0^-} \left[ G(u + s, X_s) + \int_{(-\infty,0)} V(0, X_s + y) \mathbb{I}_{\{X_s + y < 0\}} \Pi(dy) \right] \mathbb{I}_{\{X_s < b(u+s)\}} ds \right),
$$

where in the last equality we used that $V(u + s, X_s + y) = 0$ when $X_s + y \geq b(u+s)$. Moreover, we have that (40) follows directly from the equation above since $V(u, b(u)) = 0$ for all $u > 0$. \hfill \square

We define an auxiliary function. For all $(u, x) \in E$, we define

$$
R(u, x) = \mathbb{E}_{u,x} \left( \int_0^\infty G(U_s, X_s) \mathbb{I}_{\{X_s < b(U_s)\}} ds \right) \\
- \mathbb{E}_{u,x} \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > b(U_s)\}} ds \right).
$$

Note from Lemma 5.1 that $R$ is well defined and

$$
\lim_{u, x \to \infty} R(u, x) = 0.
$$

The next Lemma shows that $R$ coincides with $V$. 

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Lemma 5.3. For any \((u, x) \in E\) we have that

\[
V(u, x) = \mathbb{E}_{u, x} \left( \int_0^\infty G(U_s, X_s) dI_{\{X_s < b(U_s)\}} ds \right) - \mathbb{E}_{u, x} \left( \int_0^\infty \int_{(-\infty, 0)} \tilde{V}(U_s, X_s + y) \Pi(dy) dI_{\{X_s > b(U_s)\}} ds \right).
\] (50)

Proof. First, we assume that \(X\) is of infinite variation. Let \((u, x) \in E\), from the Markov property applied to the stopping time \(\tau_0^+\), the fact that \(b\) is non-negative and equation (30) we get that for all \(x < 0\),

\[
R(0, x) = \mathbb{E}_x \left( \int_0^{\tau_0^+} G(0, X_s) ds \right) + R(0, 0) = V(0, x) + R(0, 0) - V(0, 0).
\]

Similarly, using the Markov property at time \(\tau_0^-\) we get that for any \(u > 0\) and \(x > 0\) that

\[
R(u, x) = \mathbb{E}_x \left( \int_0^{\tau_0^-} G(0, X_s) ds \right) + \mathbb{E}_x \left( \int_0^{\tau_0^-} G(u + s, X_s) dI_{\{X_s < b(u + s)\}} ds \right) - \mathbb{E}_x \left( \int_0^{\tau_0^-} \int_{(-\infty, 0)} \tilde{V}(u + s, X_s + y) \Pi(dy) dI_{\{X_s > b(u + s)\}} ds \right)
\]

\[
= V(u, x) + \mathbb{E}_x [\{R(0, 0) - V(0, 0)\} \mathbb{P}_{x}(\tau_0^- < \infty)]
\]

where the second equality follows from equation (49) and the last from the expression for \(R(0, x)\) deduced above. Then applying the strong Markov property at time \(\tau_D\), the fact that for any \(s < \tau_D\) we have that \(X_s < b(U_s)\) and the equation above we get that for \(u > 0\) and \(x < b(u)\),

\[
R(u, x) = \mathbb{E}_{u, x} \left( \int_0^{\tau_D} G(U_s, X_s) ds \right) + \mathbb{E}_{u, x} (R(U_{\tau_D}, X_{\tau_D}))
\]

\[
= \mathbb{E}_x \left( \int_0^{\tau_0^-} G(u + s, X_s) dI_{\{X_s < b(u + s)\}} ds \right) - \mathbb{E}_x \left( \int_0^{\tau_0^-} \int_{(-\infty, 0)} \tilde{V}(u + s, X_s + y) \Pi(dy) dI_{\{X_s > b(u + s)\}} ds \right)
\]

\[
= \mathbb{E}_x \left( \int_0^{\tau_0^-} G(u + s, X_s) dI_{\{X_s < b(u + s)\}} ds \right) + \mathbb{E}_x [\{R(0, U_{\tau_D}, X_{\tau_D}) - V(0, 0)\} \mathbb{P}_{x}(\tau_0^- < \infty)]
\]

where in the first equality we used that \(\tau_D\) is optimal for \(V\) and in the last we used that \(V\) vanishes on \(D\) and that \(X\) creeps upwards. Taking \(u = 0\) and \(x = 0\) we conclude that

\[
0 = \mathbb{E}_x \left( \int_0^{\tau_0^-} G(u, X_s) dI_{\{X_s < b(u)\}} ds \right) - \mathbb{E}_x \left( \int_0^{\tau_0^-} \int_{(-\infty, 0)} \tilde{V}(u, X_s + y) \Pi(dy) dI_{\{X_s > b(u)\}} ds \right)
\]

\[
= \mathbb{E}_x \left( \int_0^{\tau_0^-} G(u, X_s) dI_{\{X_s < b(u)\}} ds \right) + \mathbb{E}_x [\{R(0, 0) - V(0, 0)\} \mathbb{P}_{x}(\tau_0^- = \infty)].
\]

Since \(b(u) > 0\) for all \(u > 0\) and \(\mathbb{P}_x(\tau_0^- = \infty) > 0\) for all \(x > 0\), the equation above implies that \(R(0, 0) = V(0, 0)\) and then \(V(u, x) = R(u, x)\) in the infinite variation case. For the finite variation case consider the sequence of stopping times,

\[
\tau_b^{(1)} = \inf \{ t \geq 0 : X_t \geq b(U_t) \},
\]

and for \(k = 1, 2, \ldots\)

\[
\sigma_b^{(k)} = \inf \{ t \geq \tau_b^{(k)} : X_t < b(U_t) \},
\]

\[
\tau_b^{(k+1)} = \inf \{ t \geq \sigma_b^{(k)} : X_t \geq b(U_t) \}.
\]

Since \(X\) is of finite variation we have that \(\tau_b^{(k)} < \sigma_b^{(k)} < \tau_b^{(k+1)}\) for all \(k \geq 1\). Let \(u > 0\) and \(x \geq b(u)\), by the
Markov property applied to time $\tau_b^{(2)}$ we get that

$$R(u, x) = -\mathbb{E}_{u, x} \left( \int_0^{\tau_b^{(1)}} \int_{(-\infty, 0)} \tilde{V}(U_s, X_s + y) \Pi(dy) ds \right)$$

$$+ \mathbb{E}_{u, x} \left( \mathbb{I}_{\{\tau_b^{(1)} < \infty\}} \int_{\tau_b^{(1)}}^{\tau_b^{(2)}} G(U_s, X_s) ds \right) + \mathbb{E}_{u, x} (R(U_{\tau_b^{(2)}}, X_{\tau_b^{(2)}}) \mathbb{I}_{\{\tau_b^{(2)} < \infty\}})$$

$$= -\mathbb{E}_{u, x} \left( \int_0^{\tau_b^{(1)}} \int_{(-\infty, 0)} \tilde{V}(U_s, X_s + y) \Pi(dy) ds \right) + \mathbb{E}_{u, x} \left( \mathbb{I}_{\{\tau_b^{(1)} < \infty\}} V(U_{\tau_b^{(1)}}, X_{\tau_b^{(1)}}) \right)$$

$$+ \mathbb{E}_{u, x} (R(U_{\tau_b^{(2)}}, X_{\tau_b^{(2)}}) \mathbb{I}_{\{\tau_b^{(2)} < \infty\}})$$

where in the second inequality we used the Markov property at time $\tau_b^{(1)}$, the definition of $V$ in terms of the stopping time $\tau_D$ and in the last equality we used the compensation formula for Poisson random measures. Using an induction argument we can verify that for all $x \geq b(u)$ and $n \geq 1$,

$$R(u, x) = \mathbb{E}_{u, x} (R(U_{\tau_b^{(n)}}, X_{\tau_b^{(n)}}) \mathbb{I}_{\{\tau_b^{(n)} < \infty\}}).$$

It can be shown that for any $(u, x) \in E$, $\lim_{n \to \infty} \tau_b^{(n)} = \infty$ a.s. Hence, by the dominated convergence theorem, the fact that $\lim_{n \to \infty} R(u, x) = 0$ (see (46) and (47)), that $\lim_{t \to \infty} U_t = t - g_t \geq \lim_{t \to \infty} t - g = \infty$ and that $X$ drifts to infinity we get that

$$R(u, x) = \lim_{n \to \infty} \mathbb{E}_{u, x} (R(U_{\tau_b^{(n)}}, X_{\tau_b^{(n)}}) \mathbb{I}_{\{\tau_b^{(n)} < \infty\}}) = 0$$

for all $u > 0$ and $x \geq b(u)$. Now take $x < b(u)$, applying the strong Markov property and using that $\tau_b^{(1)}$ is optimal for $V$ we get that

$$R(u, x) = \mathbb{E}_{u, x} \left( \int_0^{\tau_b^{(1)}} G(U_s, X_s) ds \right) + \mathbb{E}_{u, x} (R(U_{\tau_b^{(1)}}, X_{\tau_b^{(1)}})) = V(u, x).$$

Hence, we conclude that for all $(u, x) \in E$,

$$V(u, x) = R(u, x).$$

Now we are ready to show that in either case regarding the variation of $X$, the equations stated in Theorem 4.18 hold.

**Lemma 5.4.** The quadruplet $(V, b, V(0, 0), u_b)$ satisfy equations (38)-(41) and equation (43).

**Proof.** We know from Lemma 5.2 that equations (38) and (40) hold in the infinite variation case. Then suppose that $X$ is of finite variation. The strong Markov property applied at time $\tau_b^{(1)}$ in (50) imply that (49) also holds in the finite variation case. Then proceeding as in Lemma 5.2 (see argument below equation (49)) we see that equations (38) and (40) also hold in the finite variation case. Moreover, the assertions about $u_b$ and equation (41) follow from Lemma 4.16, the lower bound for $V(0, 0)$ follows from Remark 4.3 and (43) holds due to Lemma 4.14.

We now proceed to show that (39) is satisfied for $V(0, 0)$. Taking $u = x = 0$ in (50) and using Fubini’s
where for all $\delta > 0$,

$$V(0, 0) = \mathbb{E} \left( \int_0^\infty G(U_s, X_s)I_{\{X_s < b(U_s)\}}ds \right) - \mathbb{E} \left( \int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s + y)\Pi(dy)I_{\{X_s > b(U_s)\}}ds \right)$$

$$= \mathbb{E} \left( \int_0^\infty G(0, X_s)I_{\{X_s \leq 0\}}ds \right) + \mathbb{E} \left( \int_0^\infty G(U_s, X_s)I_{\{0 < X_s < b(U_s)\}}ds \right) - \mathbb{E} \left( \int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s + y)\Pi(dy)I_{\{X_s > b(U_s)\}}ds \right)$$

$$= \int_{(-\infty,0)} G(0, z) \int_0^\infty \mathbb{P}(X_s \in dz)ds + \int_{(0,\infty)} G(u, z) \int_0^\infty \mathbb{P}(U_s \in du, X_s \in dz)ds - \int_{(0,\infty)} \int_{(b(u),\infty)} \int_{(-\infty,0)} \widetilde{V}(u, z + y)\Pi(dy) \int_0^\infty \mathbb{P}(U_s \in du, X_s \in dz)ds,$$

where in the first equality we used the fact that $b$ is non-negative and that $U_s = 0$ if and only if $X_s \leq 0$. From the fact that $G(0, z) = -E_x(g^{p-1})$ for any $z < 0$ and the formulas for the 0-potential density of $X$ and $(U, X)$ (see equations (10) and (17)), respectively, we obtain that

$$V(0, 0) = -\frac{1}{\psi'(0+) \int_0^\infty \mathbb{E}_x(g^{p-1})[1 - \psi'(0+)W(z)]dz} + \frac{1}{\psi'(0+) \int_{(0,b(s))} \mathbb{E}_s \mathbb{P}(X_s \in dz)ds} - \frac{1}{\psi'(0+) \int_{(b(s),\infty)} \int_{(-\infty,0)} \mathbb{V}(s, z + y)\Pi(dy) \mathbb{P}(X_s \in dz)ds}$$

$$= -\frac{1}{\psi'(0+) \int_0^\infty \mathbb{E}_x(g^{p-1})[1 - \psi'(0+)W(z)]dz} + \frac{1}{\psi'(0+) \int_{(0,b(s))} \mathbb{E}_s \mathbb{P}(X_s \in dz)ds} - \frac{1}{\psi'(0+) \int_{(-\infty,0)} \mathbb{V}(s, z + y)\Pi(dy) \mathbb{P}(X_s \in dz)ds}.$$

Then equation (39) holds by recalling that $\mathbb{V}(u, x) = V(u, x)$ when $u > 0$ and $x > 0$ and $\mathbb{V}(u, x) = V(0, x)$ when $x \leq 0$ for any $u \geq 0$.

We finish the first part of the proof by showing that the derivative of $V$ at $(0, 0)$ exists when there is a Brownian motion component.

**Lemma 5.5.** The function $V$ satisfies equation (42) when $\sigma > 0$.

**Proof.** From equation (50) and the dominated convergence theorem we obtain that

$$V(0, 0) = \mathbb{E} \left( \int_0^\infty G(U_s, X_s)I_{\{X_s < b(U_s)\}}ds \right) - \mathbb{E} \left( \int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s + y)\Pi(dy)I_{\{X_s > b(U_s)\}}ds \right)$$

$$= \lim_{\delta \downarrow 0} \left\{ \mathbb{E} \left( \int_0^\infty K_1(U_s + \delta, X_s)ds \right) - \mathbb{E} \left( \int_0^\infty K_2(U_s + \delta, X_s)ds \right) \right\},$$

where $K_1(u, x) := G(u, x)I_{\{x < b(u)\}}$ and $K_2(u, x) := \int_{(-\infty,0)} \widetilde{V}(u, x + y)\Pi(dy)I_{\{x > b(u)\}}$ for all $(u, x) \in E$. Note since $b$ is non-increasing we have that $u \mapsto K_2(u, x)$ is non-decreasing for all $x \in \mathbb{R}$ and $\delta > 0$ and $|K_1(u + \delta, x)| \leq (u + \delta)^{p-1}I_{\{u < b(\delta)\}} + \mathbb{E}_x(g^{p-1})$. Hence, using formula (16) applied to the functions $K_1$ and $K_2$ (with $u = x = 0$) (note from equations (44) and (45) that the integrability conditions are satisfied) and above we get that

$$V(0, 0) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left( \frac{\mathbb{I}_{\{\tau^+_0 < \infty\}}K^-(\delta, X_{\tau^-_0} - \varepsilon) + K^+(\delta, \varepsilon)}{\psi'(0+)W(\varepsilon)} \right),$$

where for all $\delta > 0$ and $x \leq 0$,

$$K^-(\delta, x) = \mathbb{E}_x \left( \int_0^{\tau^+_0} [K_1(\delta, X_r) - K_2(\delta, X_r)]dr \right)$$

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and for all $\delta, x > 0$,

$$K^{+}(\delta, x) = \mathbb{E}_{x} \left( \int_{0}^{\tau_{0}^{-}} [K_{1}(\delta + s, X_{r}) - K_{2}(\delta + s, X_{r})]dr \right).$$

Using the fact that $b$ is non-negative and $W(x) = 0$ for all $x < 0$ (and then $G(\delta, x) = G(0, x)$ for all $x < 0$) we have that for all $x < 0$,

$$K^{-}(\delta, x) = \mathbb{E}_{x} \left( \int_{0}^{\tau_{0}^{-}} G(\delta, X_{s})ds \right) = V(0, x) - V(0, 0),$$

where the last equality follows from the expression of $V$ in terms of the stopping time $\tau_{D}$. Moreover for all $\delta > 0$ and $x > 0$ we have that from equation (49) that

$$K^{+}(\delta, x) = V(\delta, x) - \mathbb{E}_{x}(V(0, X_{\tau_{0}^{-}})I_{\{\tau_{0}^{-} < \infty\}}).$$

Hence, substituting the expressions for $K^{+}$ and $K^{-}$, rearranging the terms and by dominated convergence theorem we have that

$$V(0, 0) = \lim_{\delta \downarrow 0} \lim_{x \downarrow 0} \frac{\mathbb{E}(V(0, X_{\tau_{0}^{-}})I_{\{\tau_{0}^{-} < \infty\}}) - \mathbb{E}(V(0, X_{\tau_{0}^{-}} + \epsilon)I_{\{\tau_{0}^{-} < \infty\}}) + V(\delta, x) - V(0, 0)\mathbb{P}_{x}(\tau_{0}^{-} < \infty)}{\psi'(0+)W(\epsilon)}$$

$$= \frac{\sigma^{2}}{2\psi'(0+)} \left[ -\frac{\partial}{\partial x}V_{-}(0, 0) + \frac{\partial}{\partial x}V_{+}(0, 0) \right] + V(0, 0),$$

where in the last equality we used that $\mathbb{P}_{x}(\tau_{0}^{-} < \infty) = 1 - \psi'(0+)W(\epsilon)$ (see equation (4)) and the fact that $W'(0) = 2/\sigma^{2}$. Therefore we conclude that (42) holds and the proof is now complete. □

Now we show the uniqueness claim. Suppose that there exist continuous functions $H$ and $c$ on $E$ and $\mathbb{R}_{+}$, respectively, and real numbers $H_{0} < 0$ and $u_{H} > 0$ such that the conclusions of the theorem hold. Specifically, suppose that $H$ is a non-positive continuous real valued function on $E$, $c$ is a continuous real valued function on $(0, \infty)$ such that $c \geq h \geq 0$ and $H_{0} \in (-\frac{1}{2}\mathbb{E}(g^{p}), 0)$ such that equations (38)-(40) hold. That is, we assume that $H$, $H_{0}$ and $c$ are solutions to the equations

$$H(u, x) = H_{0} \frac{\sigma^{2}}{2} W'(x) - \mathbb{E}_{x} \left( \int_{0}^{\tau_{0}^{-}} \int_{(-\infty, 0)} H(u + s, X_{s} + y)I_{\{0 < X_{s} + y < b(u+s)\}}(dy)I_{\{X_{s} > c(u+s)\}}ds \right)$$

$$+ \mathbb{E}_{x} \left( \int_{0}^{\tau_{0}^{-}} \left[ G(u + s, X_{s}) + \int_{(-\infty, 0)} H(0, X_{s} + y)I_{\{X_{s} + y < c\}}(dy)I_{\{X_{s} < c(u+s)\}}ds \right] I_{\{X_{s} < c(u+s)\}}ds \right), \quad (51)$$

for $u > 0$ and $x > 0$,

$$H_{0} = -\frac{1}{\psi'(0+)} \int_{0}^{\infty} \mathbb{E}_{-z} (g^{p-1})[1 - \psi'(0+)W(z)]|dz + \frac{1}{\psi'(0+)} \int_{0}^{\infty} \mathbb{E} \left( \int_{0}^{\infty} H(s, X_{s} + y)I_{\{0 < X_{s} + y < c\}}(dy) \frac{X_{s}}{s} I_{\{X_{s} > c\}}ds \right), \quad (52)$$

and

$$0 = H_{0} \frac{\sigma^{2}}{2} W'(c(u)) - \mathbb{E}_{c(u)} \left( \int_{0}^{\tau_{0}^{-}} \int_{(-\infty, 0)} H(u + s, X_{s} + y)I_{\{0 < X_{s} + y < c(u+s)\}}(dy)I_{\{X_{s} > c(u+s)\}}ds \right)$$

$$+ \mathbb{E}_{c(u)} \left( \int_{0}^{\tau_{0}^{-}} \left[ G(u + s, X_{s}) + \int_{(-\infty, 0)} H(0, X_{s} + y)I_{\{X_{s} + y < c\}}(dy)I_{\{X_{s} < c(u+s)\}}ds \right] I_{\{X_{s} < c(u+s)\}}ds \right), \quad (53)$$

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for $u < u_H$, where for any $x \leq 0$,
\[
H(0, x) = -\int_0^x \int_{[0, \infty]} E_{-u-z}(g^{p-1})W(du)dz + H_0. \tag{54}
\]

The value $u_H$ is such that $u_H = \infty$ when $X$ is of infinite variation or $X$ is of finite variation with infinite activity. Otherwise, let $u_H$ be the solution of
\[
G(u, 0) - \int_{(-\infty, 0)} \int_0^y \int_{[0, \infty]} E_{-u-z}(g^{p-1})W(du)dz\Pi(dy) + H_0\Pi(-\infty, 0) = 0. \tag{55}
\]

Moreover, assume that $c(u) > 0$ for all $u < u_H$ and $c(u) = 0$ for all $u \geq u_H$ and that
\[
\int_{(-\infty, -x)} \bar{H}(u, x + c(u) + y)\Pi(dy) + G(u, c(u) + x) \geq 0 \tag{56}
\]
for all $u < u_H$ and $x > 0$, where $\bar{H}$ is the extension of $H$ to the set $\mathbb{R}^+ \times \mathbb{R}$ as in (15). That is,
\[
\bar{H}(u, x) = \begin{cases}
H(u, x) & u > 0 \text{ and } x > 0, \\
H(0, x) & u \geq 0 \text{ and } x \leq 0, \\
H(0, 0) & u = 0 \text{ and } x > 0.
\end{cases} \tag{57}
\]

Note that using the same arguments as the ones used in Lemma 5.2 (see argument below equation (49)) that (51) and (53) are equivalent to
\[
H(u, x) = \mathbb{E}_x(H(0, X_{r_0}^{-})\mathbb{I}_{\{\tau_0^- < \infty\}}) + \mathbb{E}_x\left(\int_0^{\tau_0^-} G(u + s, X_s)\mathbb{I}_{\{X_s < c(u+s)\}}ds\right)
- \mathbb{E}_x\left(\int_0^{\tau_0^-} \int_{(-\infty, 0)} \bar{H}(u + s, X_s + y)\mathbb{I}_{\{X_s+y < c(u+s)\}}\Pi(dy)\mathbb{I}_{\{X_s > c(u+s)\}}ds\right) \tag{58}
\]
for all $(u, x) \in E$ and
\[
\mathbb{E}_{c(u)}(H(0, X_{r_0}^{-})\mathbb{I}_{\{\tau_0^- < \infty\}}) + \mathbb{E}_{c(u)}\left(\int_0^{\tau_0^-} G(u + s, X_s)\mathbb{I}_{\{X_s < c(u+s)\}}ds\right)
= \mathbb{E}_{c(u)}\left(\int_0^{\tau_0^-} \int_{(-\infty, 0)} \bar{H}(u + s, X_s + y)\mathbb{I}_{\{X_s+y < c(u+s)\}}\Pi(dy)\mathbb{I}_{\{X_s > c(u+s)\}}ds\right) \tag{59}
\]
for any $u < u_H$. Following a similar proof than du Toit and Peskir (2008) we are going to show that $c = b$ which implies that $H = V$, $H_0 = V(0, 0)$ and $u_H = u_0$.

First, we show that $H$ has an alternative representation.

**Lemma 5.6.** For all $(u, x) \in E$ we have that
\[
H(u, x) = \mathbb{E}_{u,x}\left(\int_0^\infty G(U_s, X_s)\mathbb{I}_{\{X_s < c(U_s)\}}ds\right)
- \mathbb{E}_{u,x}\left(\int_0^\infty \tilde{H}(U_s, X_s + y)\mathbb{I}_{\{X_s+y < c(U_s)\}}\Pi(dy)\mathbb{I}_{\{X_s > c(U_s)\}}ds\right). \tag{60}
\]

Moreover, the same conclusion holds if, in the case that $\sigma > 0$, instead of (52) we assume that
\[
\frac{\partial}{\partial x}H_+(u, 0, 0) = \frac{\partial}{\partial x}H_-(0, 0, 0), \tag{61}
\]
where $\frac{\partial}{\partial x}H_+(u, 0)$ and $\frac{\partial}{\partial x}H_-(0, 0)$ are the right and left derivatives of $x \mapsto H(u, x)$ and $x \mapsto H(0, x)$ at zero, respectively and $\frac{\partial}{\partial x}H_+(0, 0) = \lim_{u \downarrow 0} \frac{\partial}{\partial x}H_+(u, 0)$. 

Proof. Define for all \((u, x) \in E\) the function

\[
K(u, x) = E_{u,x} \left( \int_0^\infty G(U_s, X_s)I_{\{X_s < c(u_s)\}} ds \right)
- E_{u,x} \left( \int_0^\infty \int_{(-\infty, 0]} H(U_s, X_s + y)I_{\{X_s + y < c(u_s)\}} \Pi(dy)I_{\{X_s > c(u_s)\}} ds \right).
\]

In a analogous way as in Lemma 5.4, from (10) and (17), we have that for any spectrally negative Lévy process \(X\),

\[
K(0, 0) = E \left( \int_0^\infty G(U_s, X_s)I_{\{X_s < c(u_s)\}} ds \right)
- E_{u,x} \left( \int_0^\infty \int_{(-\infty, 0]} H(U_s, X_s + y)I_{\{X_s + y < c(u_s)\}} \Pi(dy)I_{\{X_s > c(u_s)\}} ds \right)
= \frac{1}{\psi'(0+)} \int_0^\infty E_{-z}(g_{\psi^{-1}}(1 - \psi'(0+)W(z))]dz + \frac{1}{\psi'(0+)} E \left( \int_0^\infty G(s, X_s)\frac{X_s}{s}I_{\{0 < X_s < c(s)\}} ds \right)
- \frac{1}{\psi'(0+)} E \left( \int_0^\infty \int_{(-\infty, 0]} H(s, X_s + y)I_{\{X_s + y < c(s)\}} \Pi(dy)\frac{X_s}{s}I_{\{X_s > c(s)\}} ds \right)
= H_0
= H(0, 0).
\]

Moreover, for \(u = 0\) and \(x < 0\) we have that by the Markov property, the fact that \(X\) creeps upwards, \(c\) is a nonnegative curve and the definition of \(H(0, x)\), for \(x < 0\) (see (54)) that

\[
K(0, x) = E_x \left( \int_{0}^{\tau^-_0} G(U_s, X_s) ds \right) + K(0, 0) = H(0, x).
\]

Then, taking \(u > 0\) and \(x > 0\), by the strong Markov property at time \(\tau^-_0\) and equation (58),

\[
K(u, x) = E_x(K(0, X_{\tau^-_0})I_{\{\tau^-_0 < c\}}) + E_x \left( \int_{0}^{\tau^-_0} G(u + s, X_s)I_{\{X_s < c(u+s)\}} ds \right)
- E_x \left( \int_{0}^{\tau^-_0} \int_{(-\infty, 0]} H(u + s, X_s + y)I_{\{X_s + y < c(u+s)\}} \Pi(dy)I_{\{X_s > c(u+s)\}} ds \right)
= H(u, x).
\]

If in the case that \(\sigma > 0\) we assume that \(H\) and \(c\) satisfy equations (54), (58), (59) and (61). From formula (16) (in a similar way as in the proof of Lemma 5.4) we obtain that

\[
K(0, 0) = \frac{\sigma^2}{2\psi'(0+)} \left[ -\frac{\partial}{\partial x} H_- (0, 0) + \frac{\partial}{\partial x} H_+ (0, 0) \right] + H(0, 0)
= H(0, 0).
\]

The rest of the proof remains unchanged. \(\square\)

Define the set \(D_c = \{(u, x) \in E : x \geq c(u)\}\). We show in the following lemma that \(H\) vanishes in \(D_c\) so that \(D_c\) corresponds to the “stopping set” of \(H\).

**Lemma 5.7.** We have that \(H(u, x) = 0\) for all \((u, t) \in D_c\).
Proof. Note that from equations (58) and (59) we know that $H(u, c(u)) = 0$ for all $u \in (0, u_H)$. Let $(u, x) \in D_c$ such that $x > c(u)$ and define $\sigma_c$ as the first time that $(U, X)$ exits $D_c$, that is,

$$\sigma_c = \inf\{s \geq 0 : X_s < c(U_s)\}.$$ 

From the fact that $X_r \geq c(U_r)$ for all $r < \sigma_c$ we have that from the Markov property and representation (60) of $H$,

$$H(u, x) = \mathbb{E}_{u,x}(H(U_{\sigma_c}, X_{\sigma_c})\mathbb{I}_{\{\sigma_c < \infty\}}) + \mathbb{E}_{u,x} \left( \int_{0}^{\sigma_c} G(U_s, X_s)\mathbb{I}_{\{X_s < c(U_s)\}}ds \right)$$

$$- \mathbb{E}_{u,x} \left( \int_{0}^{\sigma_c} \int_{(-\infty,0)} \tilde{H}(U_s, X_s + y)\mathbb{I}_{\{X_s + y < c(U_s)\}}\Pi(dy)\mathbb{I}_{\{X_s < c(U_s)\}}ds \right)$$

$$= \mathbb{E}_{u,x}(H(U_{\sigma_c}, X_{\sigma_c})\mathbb{I}_{\{\sigma_c < \infty\}}\mathbb{I}_{\{X_{\sigma_c} < c(U_{\sigma_c})\}})$$

$$- \mathbb{E}_{u,x} \left( \int_{0}^{\sigma_c} \int_{(-\infty,0)} \tilde{H}(U_s, X_s + y)\mathbb{I}_{\{X_s + y < c(U_s)\}}\Pi(dy)ds \right).$$

where the last equality follows from the fact that $\mathbb{P}_x(X_{\sigma_c} = c(u + \sigma_c)) > 0$ only when $\sigma > 0$ and then $U(u, c(u)) = 0$ for all $u > 0$ (since $u_H = \infty$ in this case). Then, since $H \leq 0$, applying the compensation formula for Poisson random measures and the fact that $\sigma_c \leq \tau_0^-$ (since $c(u) \geq 0$ for all $u > 0$) we get

$$\mathbb{E}_{u,x}(H(U_{\sigma_c}, X_{\sigma_c})\mathbb{I}_{\{\sigma_c < \infty\}}\mathbb{I}_{\{X_{\sigma_c} < c(U_{\sigma_c})\}})$$

$$= \mathbb{E}_x \left( \int_{0}^{\infty} \int_{(-\infty,0)} \mathbb{I}_{\{X_s \geq c(u + s)\text{ for all } r < s\}}\mathbb{I}_{\{X_s + y < c(u + s)\}}\tilde{H}(u + s, X_s + y)N(ds, dy) \right)$$

$$= \mathbb{E}_x \left( \int_{0}^{\infty} \int_{(-\infty,0)} \mathbb{I}_{\{X_s \geq c(u + s)\text{ for all } r < s\}}\mathbb{I}_{\{X_s + y < c(u + s)\}}\tilde{H}(u + s, X_s + y)\Pi(dy)ds \right)$$

$$= \mathbb{E}_{u,x} \left( \int_{0}^{\sigma_c} \int_{(-\infty,0)} \tilde{H}(U_s, X_s + y)\mathbb{I}_{\{X_s < c(U_s)\}}\Pi(dy)ds \right).$$

Hence we have that $H(u, x) = 0$ for all $(u, x) \in D_c$ as claimed.

The following Lemma states that $H$ dominates the value function $V$. That suggest that $H$ is the largest function with $H \leq 0$ that makes the process $\{H(U_s, X_s) + \int_0^s G(U_s, X_s)ds, t \geq 0\}$ a $\mathbb{P}_{u,x}$-submartingale. The latter assertion will be shown indirectly on the upcoming lemmas.

**Lemma 5.8.** We have that $H(u, x) \geq V(u, x)$ for all $(u, x) \in E$.

*Proof.* If $(u, x) \in D_c$ we have the inequality

$$H(u, x) = 0 \geq V(u, x).$$

Now we show that the inequality also holds in $E \setminus D_c$. Consider the stopping time

$$\tau_c = \inf\{s \geq 0 : X_s \geq c(U_s)\}.$$

Then using the Markov property and equation (60) we get that for all $(u, x) \in E$ with $x < c(u),$

$$H(u, x) = \mathbb{E}_{u,x}(H(U_{\tau_c}, X_{\tau_c})) + \mathbb{E}_{u,x} \left( \int_{0}^{\tau_c} G(U_s, X_s)\mathbb{I}_{\{X_s < c(U_s)\}}ds \right)$$

$$- \mathbb{E}_{u,x} \left( \int_{0}^{\tau_c} \int_{(-\infty,0)} \tilde{H}(U_s, X_s + y)\mathbb{I}_{\{X_s + y < c(U_s)\}}\Pi(dy)\mathbb{I}_{\{X_s < c(U_s)\}}ds \right)$$

$$= \mathbb{E}_{u,x}(H(U_{\tau_c}, c(U_{\tau_c}))) + \mathbb{E}_{u,x} \left( \int_{0}^{\tau_c} G(U_s, X_s)ds \right),$$
where in the second equality we used the fact \( X \) creeps upwards and \( \tau_c < \infty \). Note that since \( X_t > 0 \) if and only if \( U_t > 0 \) for all \( t > 0 \) and that \( c(u) > 0 \) for all \( u \) sufficiently small we have that \( c(U_{\tau_c}) > 0 \) and hence \( H(U_{\tau_c}, c(U_{\tau_c})) = 0 \). Therefore

\[
H(u, x) = \mathbb{E}_{u,x} \left( \int_0^{\tau_c} G(U_s, X_s)ds \right) \geq V(u, x),
\]

where the inequality follows from the definition of \( V \) as per (25).

It turns out that the fact that \( H \) dominates \( V \) implies that \( b \) dominates the curve \( c \). This fact is shown in the following Lemma.

**Lemma 5.9.** We have that \( b(u) \geq c(u) \) for all \( u > 0 \).

**Proof.** Note that in the case that \( X \) is of finite variation with \( \Pi(\infty, 0) < \infty \) we have that \( c(u) = 0 \leq b(u) \) for all \( u > u_H \). We proceed by contradiction. Suppose that there exists \( u_0 > 0 \) such that \( b(u_0) < c(u_0) \). Then in the case that \( X \) is of finite variation with \( \Pi(\infty, 0) < \infty \), it holds that \( u_0 < u_H \). Take \( x > c(u_0) \) and consider the stopping time

\[
\sigma_b = \inf\{s > 0 : X_s < b(U_s)\}.
\]

Then from the Markov property and the representation of \( H \) given in (60) we have that

\[
H(u_0, x) = \mathbb{E}_{u_0,x} H(U_{\sigma_b^-}, X_{\sigma_b^-}) + \mathbb{E}_{u_0,x} \left( \int_0^{\sigma_b^-} G(U_s, X_s)\mathbb{I}_{[X_s < c(U_s^+)]} ds \right)
\]

\[
- \mathbb{E}_{u_0,x} \left( \int_0^{\sigma_b^-} \int_{(-\infty,0)} \bar{H}(U_s, X_s + y)\Pi(dy)\mathbb{I}_{[X_s + y < c(U_s^+)]} ds \right).
\]

Moreover, since \( V(u, x) = 0 \) for \( (u, x) \in D \) and \( 0 \geq H \geq V \) we have that

\[
\mathbb{E}_{u_0,x} H(U_{\sigma_b^-}, X_{\sigma_b^-}) = \mathbb{E}_{u_0,x} H(U_{\sigma_b^-}, X_{\sigma_b^-})\mathbb{I}_{[X_{\sigma_b^-} < b(U_{\sigma_b^-})]} \]

\[
= \mathbb{E}_{u_0,x} \left( \int_0^{\sigma_b^-} \int_{(-\infty,0)} \bar{H}(U_s, X_s + y)\mathbb{I}_{[X_s + y < b(U_s^+)]} \Pi(dy) ds \right)
\]

\[
= \mathbb{E}_{u_0,x} \left( \int_0^{\sigma_b^-} \int_{(-\infty,0)} \bar{H}(U_s, X_s + y)\Pi(dy) ds \right),
\]

where the second equality follows from the compensation formula for Poisson random measures. Hence, combining the two equations above and from the fact that \( x > c(u_0) \) and then \( H(u_0, x) = 0 \) we get

\[
0 = \mathbb{E}_{u_0,x} \left( \int_0^{\sigma_b^-} \left[ G(U_s, X_s) + \int_{(-\infty,0)} \bar{H}(U_s, X_s + y)\Pi(dy) \right] \mathbb{I}_{[X_s < c(U_s^+)]} ds \right).
\]

Due to the continuity of \( b \) and \( c \) we have that there exists a value \( u_1 \) sufficiently small such that \( c(v) > b(v) \) for all \( v \in [u_1, u_1] \). Thus, from Lemma 4.14, the fact that \( u \rightarrow G(u, x) \) is strictly increasing when \( x > 0 \) and the inequality \( U \geq V \) (see Lemma 5.8) we have that for all \( u > 0 \) and \( x > b(u) \),

\[
G(u, x) + \int_{(-\infty,0)} \bar{H}(u, x + y)\Pi(dy) \geq G(u, x) + \int_{(-\infty,0)} \bar{V}(u, x + y)\Pi(dy) > 0,
\]

where the strict inequality follows from Lemma 4.14. Note that taking \( x \) sufficiently large we have that, under the measure \( \mathbb{P}_{u_0,x} \), \( X \) spends a positive amount of time between the curves \( b(u) \) and \( c(u) \) for \( u \in [u_0, u_1] \) with positive probability. Thus, since \( \sigma_c < \tau_0 \) the above facts imply that
Lemma 5.10. We have that then \( c(u) = b(u) \) for all \( u > 0 \) and \( V(u, x) = H(u, x) \) for all \( (u, x) \in E \).

Proof. Suppose that there exists \( u > 0 \) such that \( c(u) < b(u) \) and take \( x \in (c(u), b(u)) \). Then we have by the Markov property and representation (60) that

\[
H(u, x) = \mathbb{E}_{u, x} \left( H(U_{\tau_D}, X_{\tau_D}) \right) + \mathbb{E}_{u, x} \left( \int_{0}^{\tau_D} G(U_s, X_s) \mathbb{I}_{\{X_s < c(U_s)\}} \, ds \right) - \mathbb{E}_{u, x} \left( \int_{0}^{\tau_D} \int_{(-\infty, 0)} \tilde{H}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > c(U_s)\}} \, ds \right),
\]

where \( \tau_D = \inf\{t > 0 : X_t \geq b(U_t)\} \). On the other hand, we have that

\[
V(u, x) = \mathbb{E}_{u, x} \left( \int_{0}^{\tau_D} G(U_s, X_s) \, ds \right).
\]

Hence, since \( X_{\tau_D} = b(U_{\tau_D}) \geq c(U_{\tau_D}) \) and Lemma 5.7 we have that \( H(U_{\tau_D}, X_{\tau_D}) = 0 \). Moreover, using the inequality \( H \geq V \) (see Lemma 5.8) we obtain that

\[
0 \geq \mathbb{E}_{u, x} \left( \int_{0}^{\tau_D} \left[ G(U_s, X_s) + \int_{(-\infty, 0)} \tilde{H}(U_s, X_s + y) \Pi(dy) \right] \mathbb{I}_{\{X_s > c(U_s)\}} \, ds \right) > 0,
\]

where the strict inequality follows by the inequality (56) and the continuity of \( b \) and \( c \). This contradiction allows us to conclude that \( c(u) = b(u) \) for all \( u > 0 \) and \( H(u, x) = V(u, x) \) for all \( (u, x) \in E \). \( \square \)

Remark 5.11. A close inspection of the proof tells us that the assumption \( H \leq 0 \) can be dropped when \( \Pi \equiv 0 \).

6 Examples

6.1 Brownian Motion with drift example

Suppose that \( X_t \) is given by

\[
X_t = \mu t + \sigma B_t,
\]

where \( \mu > 0, \sigma > 0 \) and \( B = \{B_t, t \geq 0\} \) is a standard Brownian motion. Here, we consider the case \( p = 2 \). Then

\[
G(u, x) = u \psi'(0+) W(x) - \mathbb{E}_x(g).
\]

It is well known that for \( \beta \geq 0 \) and \( q \geq 0, \)

\[
\psi(\beta) = \frac{\sigma^2}{2} \beta^2 + \mu \beta \quad \text{and} \quad \Phi(q) = \frac{1}{\sigma^2} \left[ \sqrt{\mu^2 + 2\sigma^2 q} - \mu \right].
\]
Thus, $\psi'(0+) = \mu$, $\Phi'(0) = \frac{1}{\mu}$, $\Phi''(0) = -\frac{e^x}{\mu^3}$ and $\Phi'''(0) = 3\sigma^4/\mu^5$. The scale function is (see e.g. Kuznetsov et al. (2013), on p. 102) given by

\[ W(x) = \frac{1}{\mu} \left( 1 - \exp(-2\mu x/\sigma^2) \right), \quad x \geq 0. \]

An easy calculation shows that $W^{*}(2)$ is given by

\[ W^{*}(2)(x) = \frac{1}{\mu^2} x \left[ 1 + \exp(-2\mu x/\sigma^2) \right] - \frac{\sigma^2}{\mu^3} \frac{1}{\mu} \left( 1 - \exp(-2\mu x/\sigma^2) \right), \quad x \geq 0. \]

For all $x \in \mathbb{R}$, the value $E_x(g)$ can be calculated from (19) via differentiation to have

\[ E_x(g) = -\psi'(0+)\Phi''(0+) + x\Phi'(0) + \psi'(0+)W^{*}(2)(x) \]

\[ = \begin{cases} \frac{\sigma^2}{\mu^2} - \frac{x}{\mu^2}, & x < 0, \\ \frac{\sigma^2}{\mu^2} \exp(-2\mu x/\sigma^2) + \frac{z}{\mu} \exp(-2\mu x/\sigma^2), & x \geq 0. \end{cases} \]

and $E(g^2) = \Phi'''(0)\psi'(0+) = 3(\sigma/\mu)^4$. Moreover, it is well known that $X_r \sim N(\mu r, \sigma^2 r)$ for any $r \geq 0$ and

\[ P_x(X_r \in dz, X_r \geq 0) = e^{-\frac{2ux}{\sigma^2 r}} \frac{1}{\sqrt{2\pi\sigma^2 r}} \phi \left( \frac{z - x - \mu r}{\sqrt{\sigma^2 r}} \right) dz, \]

for any $x \geq 0$ and $z \geq 0$, where $\phi$ is the density of a standard normal distribution. Hence we have that for any $x \geq 0$ and $z \geq 0$,

\[ P_x(X_r \in dz, X_r \geq 0) = \frac{1}{\sqrt{2\pi\sigma^2 r}} \left[ \phi \left( \frac{z - x - \mu r}{\sqrt{\sigma^2 r}} \right) - e^{-\frac{2ux}{\sigma^2 r}} \phi \left( \frac{z - x - \mu r}{\sqrt{\sigma^2 r}} \right) \right] dz. \]

Then we calculate for any $u > 0$

\[ E_x \left( \int_0^{\tau_0} \left[ (r + u)\psi'(0+) W(X_r) - E_x(g) \right] I_{X_r < b(r + u)} \right) \]

\[ = \int_0^\infty \int_0^{b(r + u)} \left[ (r + u)\psi'(0+) W(z) - E_x(g) \right] P_x(X_r \in dz, X_r \geq 0) dr \]

\[ = \int_0^\infty \left( H(r, x, b(r + u)) - e^{-2u/\sigma^2 x} H(r, u, -x, b(r + u)) \right) dr, \]

where a lengthy but straightforward calculation gives

\[ H(r, t, x, b) = \int_0^b \left[ (r + t)\psi'(0+) W(z) - E_x(g) \right] \frac{1}{\sqrt{2\pi\sigma^2 r}} \phi \left( \frac{z - x - \mu r}{\sqrt{\sigma^2 r}} \right) dz \]

\[ = (r + t) \left[ \Psi \left( \frac{b - x - \mu r}{\sigma\sqrt{r}} \right) - \Psi \left( \frac{-x - \mu r}{\sigma\sqrt{r}} \right) \right] \]

\[ - \frac{\sigma^2}{\mu^2} e^{-2\mu/\sigma^2 x} \left[ \Psi \left( \frac{b - x + \mu r}{\sigma\sqrt{r}} \right) - \Psi \left( \frac{-x + \mu r}{\sigma\sqrt{r}} \right) \right] \]

\[ + \frac{\sigma\sqrt{r}}{\mu} e^{-2\mu/\sigma^2 x} \left[ \phi \left( \frac{b - x + \mu r}{\sigma\sqrt{r}} \right) - \phi \left( \frac{-x + \mu r}{\sigma\sqrt{r}} \right) \right]. \]

From formula (30) we know that

\[ V(0, x) = -\int_0^{-x} \int_{(0, \infty)} E_{u - z}(g) W(du)dz + V(0, 0) \]

\[ = 3\frac{\sigma^2}{2\mu^3} x - \frac{1}{2\mu^2} x^2 + V(0, 0). \]
Then,
\[
\frac{\partial}{\partial x} V_-(0,0) = \frac{3\sigma^2}{2\mu^3}.
\]

From Theorem 4.18 we have that for \( u > 0 \) and \( x > 0 \),
\[
V(u, x) = V(0, 0)[1 - \psi'(0+)W(x)] + \int_0^\infty \left\{ H(r, u, x, b(r + u)) - e^{-2\mu/\sigma^2 x} H(r, u, -x, b(r + u)) \right\} \, dr.
\]

Therefore the curve \( b(u) \) and \( V(0,0) \) satisfy the equations
\[
0 = \int_0^\infty \left\{ H(r, u, x, b(r + u)) - e^{-2\mu/\sigma^2 x} H(r, u, -x, b(r + u)) \right\} \, dr + V(0,0)[1 - \psi'(0+)W(b(u))],
\]
\[
0 = \frac{3\sigma^2}{2\mu^3} - \frac{\partial}{\partial x} V_+(0,0),
\]
for all \( u > 0 \), where
\[
\frac{3\sigma^4}{2\mu^4} \leq V(0,0) < 0.
\]

Note that \( \frac{\partial}{\partial x} V_+(0,0) \) can be estimated via \( [V(h_0,h_0) - V(0,0)]/h_0 \) for \( h_0 \) sufficiently small.

We can approximate the integrals above by Riemann sums so a numerical approximation can be implement. Indeed, take \( n \in \mathbb{Z}_+ \) and \( T > 0 \) sufficiently large such that \( h = T/n \) is small. For each \( k \in \{0,1,2,\ldots,n\} \), we define \( u_k = kh \). Then the sequence of times \( \{u_k, k = 0,1,\ldots,n\} \) is a partition of the interval \([0,T]\). For any \( x \in \mathbb{R} \) and \( u \in [u_k,u_{k+1}) \), we approximate \( V(u,x) \) by
\[
V_h(u_k,x) = V_0[1 - \psi'(0+)W(x)] + \sum_{i=k}^{n-1} \left[ H(u_{i-k+1},u_k,x,b_i) - e^{-2\mu/\sigma^2 x} H(u_{i-k+1},u_k,-x,b_i) \right] h,
\]
where the sequence \( \{b_k, k = 0,1,\ldots,n-1\} \) and \( V_0 \) are solutions to
\[
0 = V_0[1 - \psi'(0+)W(b_k)] + \sum_{i=k}^{n-1} \left[ H(u_{i-k+1},u_k,b_i,b_i) - e^{-2\mu/\sigma^2 x} H(u_{i-k+1},u_k,-b_i,b_i) \right] h,
\]
\[
0 = \frac{3\sigma^2}{2\mu^3} - \frac{V_h(h_0,h_0) - V_0}{h_0},
\]
for each \( k \in \{0,1,\ldots,n-1\} \). Note that, for \( T \) and \( n \) sufficiently large such that \( h \) is sufficiently small, the sequence \( \{b_k, k = 0,1,\ldots,n\} \) is a numerical approximation to the sequence \( \{b(t_k), k = 0,1,\ldots,n\} \) and can be calculated by using backwards for a fixed value \( V_0 \). Indeed, a method for solving the system is: fix \( V_0 \) and calculate the sequence \( \{b_k^{V_0}, k = 0,1,\ldots,n\} \) by using the first equation above. If the curve obtained and the value \( V_0 \) satisfy the second equation above then we have that \( V_0 = V(0,0) \) and \( \{b_k^{V_0}, k = 0,1,\ldots,n\} = \{b_k, k = 0,1,\ldots,n\} \). Otherwise, vary the quantity \( V_0 \) and recalculate until both equations are satisfied. We show in Figure 3 a numerical calculation of the optimal boundary and the value function using the equations above. The case considered is when \( \mu = 1/2 \) and \( \sigma = 1. \)
6.2 Brownian motion with exponential jumps example

Consider the case in which \( p = 2 \) and \( X \) a Brownian motion with drift and exponential jumps and the case, this is, \( X = \{X_t, t \geq 0\} \) with

\[
X_t = \mu t + \sigma B_t - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,
\]

where \( \sigma > 0, \mu > 0, B = \{B_t, t \geq 0\} \) is a standard Brownian motion, \( N = \{N_t, t \geq 0\} \) is an independent Poisson process with rate \( \lambda > 0 \) and \( \{Y_i, i \geq 1\} \) is a sequence of independent exponential distributed random variables with parameter \( \rho > 0 \) independent of \( B \) and \( N \). We further assume that \( \mu \rho > \lambda \) so \( X \) drifts to infinity. The Laplace exponent is given by for \( \beta \geq 0 \) by

\[
\psi(\beta) = \mu \beta + \frac{\sigma^2}{2} \beta^2 - \frac{\lambda \beta}{\rho + \beta},
\]

where \( \mu \) is a positive constant. In this case the Lévy measure is given by \( \Pi(dx) = \lambda \rho e^{\rho x} dx \), for all \( x < 0 \). An easy calculation leads to \( \psi'(0+) = \mu - \lambda / \rho \),

\[
\Phi'(0+) = \frac{\rho}{\mu \rho - \lambda} \quad \text{and} \quad \Phi''(0+) = -\frac{\sigma^2 \rho^3 + 2 \lambda \rho}{[\mu \rho - \lambda]^3}.
\]

It is known that (see e.g. Kuznetsov et al. (2013), on p. 101) the scale function \( W \) is given by

\[
W(x) = \frac{1}{\psi'(0+)} + \frac{e^{\zeta_1 x}}{\psi'(\zeta_1)} + \frac{e^{\zeta_2 x}}{\psi'(\zeta_2)}
\]

for \( x \geq 0 \), where

\[
\zeta_1 = \frac{-\left(\frac{\sigma^2}{2} \rho + \mu\right) + \sqrt{(\frac{\sigma^2}{2} \rho - \mu)^2 + 2 \sigma^2 \lambda}}{\sigma^2}
\]
Similarly, we have that for all 

\[ \zeta_2 = -\left(\frac{\sigma^2}{2}\rho + \mu\right) - \sqrt{\left(\frac{\sigma^2}{2}\rho - \mu\right)^2 + 2\sigma^2\lambda} \, . \]

Then differentiating (19) we have that

\[
\mathbb{E}_x(g) = -\psi'(0+)\Phi''(0) + x\Phi'(0)^2 + \psi'(0+)W^*(x)
\]

\[
= \begin{cases} \frac{e^{\frac{\sigma^2}{2}\lambda}}{\mu - \lambda} - \frac{\mu - \lambda}{\mu + \lambda}x, & x < 0, \\ \frac{e^{\frac{\sigma^2}{2}\lambda}}{\mu - \lambda} - \frac{\mu - \lambda}{\mu + \lambda}x + (\mu - \lambda)W^*(x), & x \geq 0. \end{cases}
\]

For \( x < 0 \), the value function is then given by

\[
V(0, x) = -\int_0^-x \int_{[0, \infty)} \mathbb{E}_{x-z}(g)W(du)dz + V(0, 0)
\]

\[
= \int_0^-x \int_{[0, \infty)} [\Phi''(0) + \Phi'(0)^2(-u - z)] \psi'(0+)W(du)dz + V(0, 0)
\]

\[
= \Phi''(0)\psi'(0+) + \Phi'(0)^2E(X_\infty)(-x) - \Phi'(0)^2x^2/2 + V(0, 0),
\]

where in the last equality we used that \( \psi'(0+)W(x) = \mathbb{P}(\tau^- = \infty) = \mathbb{P}(X_\infty \leq x) \) and hence \( \psi'(0+)W(du) \) is the density of the random variable \( X_\infty \). From (7) we know that for any \( \beta \geq 0 \),

\[
\mathbb{E}(e^{\beta X_\infty}) = \psi'(0+) \frac{\beta}{\psi(\beta)}.
\]

Hence, by differentiating and using the fact that \( \Phi'(q) = 1/\psi'(\Phi(q)) \) we can see that

\[
\mathbb{E}(X_\infty) = \frac{\Phi'(0)}{2\Phi'(0)^2}.
\]

Hence,

\[
V(0, x) = -\left[\frac{3}{2}\Phi''(0)x + \Phi'(0)^2x^2/2\right] + V(0, 0)
\]

for any \( x < 0 \). Next, we calculate for any \( x > 0 \),

\[
\int_{(-\infty, 0)} V(0, x + y)1_{y < 0} \Pi(dy) = \int_{-\infty}^x \left[ \int_0 y \Phi''(0)(x + y) - \Phi'(0)^2(x + y)^2/2 + V(0, 0) \right] \lambda e^{-\rho y} dy
\]

\[
= \lambda e^{-\rho x} \int_{-\infty}^0 \left[ -\frac{3}{2}\Phi''(0)y - \Phi'(0)^2y^2/2 + V(0, 0) \right] e^{-\rho y} dy
\]

\[
= \lambda e^{-\rho x} \left[ \frac{3\Phi'(0)}{2\rho} - \frac{\Phi'(0)^2}{\rho^2} + V(0, 0) \right].
\]

Similarly, we have that for all \( u > 0 \) and \( x > b(u) \),

\[
\int_{(-\infty, 0)} V(u, x + y)1_{y < 0} \Pi(dy) = e^{-\rho x} \int_0^{b(u)} V(u, y) \lambda e^{-\rho y} dy
\]

\[
= e^{-\rho(x - b(u))} \int_{(-b(u), 0)} V(u, y + b(u)) \Pi(dy).
\]
Therefore, for any \( u, x > 0 \), equation (38) reads as

\[
V(u, x) = V(0, 0)\frac{\sigma^2}{2}W'(x) - \mathbb{E}_x\left(\int_0^{T_0} e^{-\rho(X_s-b(u+s))}\mathbb{I}_{\{X_s\geq b(u+s)\}}(u+s, y + b(u + s))\Pi(dy)ds\right)
\]

\[
+ \mathbb{E}_x\left(\int_0^{T_0} \left[ G(u + s, X_s) + \int_{(-\infty,0)} V(0, X_s + y)\mathbb{I}_{\{X_s+y<0\}}\Pi(dy) \right] \mathbb{I}_{\{X_s<b(u+s)\}}ds\right)
\]

\[
= V(0, 0)\frac{\sigma^2}{2}W'(x) - \int_0^{\infty} \mathbb{E}_x\left(\int_{(-b(u+s),0)} V(0, X_s + y)\mathbb{I}_{\{X_s+y<0\}}\Pi(dy) \right] \mathbb{I}_{\{X_s<b(u+s)\}}ds \right) \right) ds
\]

\[
= V(0, 0)\frac{\sigma^2}{2}W'(x) - \int_0^{\infty} \mathbb{E}_x\left(\int_{(-b(u+s),0)} V(0, X_s + y)\mathbb{I}_{\{X_s+y<0\}}\Pi(dy) \right] \mathbb{I}_{\{X_s<b(u+s)\}}ds \right) \right) ds
\]

where for any \( s, u, x, b > 0 \)

\[
F_1(s, u, x, b) = \mathbb{E}\left( G(u + s, X_s + x)\mathbb{I}_{\{X_s+x<b, X_s+x\geq 0\}} \right)
\]

\[
+ \mathbb{E}\left( \lambda e^{-\rho(X_s+x)} \left[ \frac{3\Phi''(0)}{2\rho} - \frac{\Phi'(0)^2}{\rho^2} + V(0, 0) \right] \mathbb{I}_{\{X_s+x<b, X_s+x\geq 0\}} \right),
\]

\[
F_2(s, u, x, b) = \mathbb{E}\left( e^{-\rho(X_s+x-b)}\mathbb{I}_{\{X_s+x>b, X_s+x\geq 0\}} \right),
\]

\[
V(u, b) = \int_{(-b,0)} (u + y + b)\Pi(dy).
\]

In summary, we have that \( V, b \) and \( V(0, 0) \) satisfy the equations

\[
V(u, x) = V(0, 0)\frac{\sigma^2}{2}W'(x) + \int_0^{\infty} F_1(s, u, x, b(u+s))ds - \int_0^{\infty} V(u + s, b(u + s))F_2(s, x, b(u + s))ds,
\]

\[
0 = V(0, 0)\frac{\sigma^2}{2}W'(b(u)) + \int_0^{\infty} F_1(s, u, b(u), b(u+s))ds
\]

\[
- \int_0^{\infty} V(u + s, b(u + s))F_2(s, b(u), b(u + s))ds,
\]

\[
0 = \frac{3}{2}\Phi''(0) + \frac{\partial}{\partial x} V_+(0, 0),
\]

for all \( u, x > 0 \).

We can approximate the integrals above by Riemann sums so a numerical approximation can be implemented. Indeed, take \( n \in \mathbb{Z}_+ \) and \( T > 0 \) sufficiently large such that \( h = T/n \) is small. For each \( k \in \{0, 1, 2, \ldots, n\} \), we define \( u_k = kh \). Then the sequence of times \( \{u_k, k = 0, 1, \ldots, n\} \) is a partition of the interval \([0, T]\). For any \( x \in \mathbb{R} \) and \( u \in [u_k, u_{k+1}] \), we approximate \( V(u, x) \) by

\[
V_h(u_k, x) = V_0\frac{\sigma^2}{2}W'(x) + \sum_{i=k}^{n-1} [F_1(u_{i-k+1}, u_k, x, b_i) - V_h(u_{i+1}, b_{i+1})F_2(u_{i-k+1}, x, b_i)]h,
\]

where \( V(u_n, b_n) = 0 \) and

\[
V_h(u_i, b_i) = \sum_{j=1}^{[b_i/h]} V_h(u_i, jh)\lambda e^{\rho hj}h.
\]

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for any \( i \in \{1, 2, \ldots, n - 1\} \). The sequence \( \{b_k, k = 1, \ldots, n - 1\} \) and \( V_0 \) are solutions to

\[
V_0 \frac{\sigma^2}{2} W''(b_k) + \sum_{i=k}^{n-1} [F_1(u_{i-k+1}, u_k, b_k, b_i) - V_h(u_{i+1}, b_{i+1})F_2(u_{i-k+1}, b_k, b_i)]h = 0, \tag{63}
\]

for each \( k \in \{0, 1, \ldots, n - 1\} \). The functions \( F_1 \) and \( F_2 \) can be estimated by simulating the process \( \{(X_t, X_{\tau_t}), t \geq 0\} \) (see e.g. Kuznetsov et al. (2011), Theorem 4 and Remark 3). Note that, for \( n \) and \( T \) sufficiently large, the sequence \( \{b_k, k = 1, \ldots, n\} \) is a numerical approximation to the sequence \( \{b(t_k), k = 1, \ldots, n\} \) and can be calculated by using backwards induction. Indeed, with a fixed value \( V_0 \) and the condition \( V(u_n, b_n) = 0 \), we can first obtain \( b_{n-1} \) using equation (63). This allows us to compute \( V_h(u_{n-1}, x) \) which in turn gives us \( V_h(u_{n-1}, b_{n-1}) \). We can then finally obtain \( b_{n-2}, V_h(u_{n-2}, b_{n-2}), b_{n-3}, V_h(u_{n-3}, b_{n-3}), \ldots, b_1 \) by repeating the aforementioned steps. With these values, we can calculate \( V_h(h_0, h_0) \) and repeat the procedure for different values of \( V_0 \) until (64) is satisfied.

We show in Figure 4 a numerical calculation of the optimal boundary and the value function using the parametrisation \( \mu = 3, \sigma = 1, \lambda = 1 \) and \( \rho = 1 \). The functions \( F_1 \) and \( F_2 \) above were estimated using Monte Carlo simulations accordingly to the algorithm given in Kuznetsov et al. (2011).

![Figure 4: Numeric calculation of the optimal boundary and value function V for the Brownian motion with exponential jumps case.](image-url)
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A Appendix

Proof of Lemma 2.2. From equation (1) we know that

\[ F(\theta, x) := \mathbb{E}_x(e^{-\theta \tau^+}) = e^{\Phi(\theta)}x, \quad x \leq 0. \]

Then using the formula of Faà di Bruno (see for example Spindler (2005)) we have that for any \( n \geq 1, \)

\[ \frac{\partial^n}{\partial \theta^n} F(\theta, x) = \sum_{k=1}^{n} \frac{n!}{k_1!k_2!...k_n!} \left( \frac{\Phi'\left(\theta\right)}{1!} \right)^{k_1} \left( \frac{\Phi''\left(\theta\right)}{2!} \right)^{k_2} ... \left( \frac{\Phi^{(n)}\left(\theta\right)}{n!} \right)^{k_n}. \]

Then evaluating at zero the above equation, using \( \Phi(0) = 0 \) and the fact that \( \Phi^{(i)}(0) < \infty \) for \( i = 1, \ldots, [p] + 1, \) we can find constants \( A_r, C_r \geq 0 \) such that \( \mathbb{E}_x((\tau^+_0)^r) \leq A_r + C_r|x|^r \) for any \( r \in \{1, \ldots, [p] + 1\}. \) For any non integer \( r < [p] + 1 \) we can use Hölder’s inequality to obtain

\[ \mathbb{E}_x((\tau^+_0)^r) \leq \mathbb{E}_x((\tau^+_0)^{[r]+1}) \frac{1}{r+1} \leq (A_{[r]+1} + C_{[r]+1}|x|^{[r]+1}) \frac{1}{r+1}. \]

The result follows from the inequality \((a+b)^q \leq 2^q(a^q+b^q) \) which is true for any \( q > 0 \) and \( a, b > 0. \) Now we show that the second inequality holds. From the strong Markov property we get that for any \( x < 0, \)

\[ \mathbb{E}_x(g^r) \leq 2^r \mathbb{E}_x(g^r) + 2^r \mathbb{E}_x((\tau^+_0)^r) \leq 2^r[\mathbb{E}(g^r) + A_r] + 2^rC_r|x|^r. \]

\[ \square \]

Proof of Lemma 2.3. It follows from the definition of \( g \) that \( x \mapsto \mathbb{E}_x(g^p) = \mathbb{E}(g^{(-x)}) \) is non-negative and non-increasing. In order to check continuity notice that by integration by parts we get

\[ \mathbb{E}_x(g^p) = p \int_0^\infty s^{p-1} \mathbb{P}_x(g > s)ds = p \int_0^\infty s^{p-1} \mathbb{E}_x(1 - \psi'(0+)W(X_s))ds, \]

where the last equality follows from (20). Take \( x \in \mathbb{R} \) and \( \delta \in \mathbb{R}. \) Then using the equation above we have that

\[ |\mathbb{E}_x(g^p) - \mathbb{E}_{x+\delta}(g^p)| \leq p\psi'(0+)\mathbb{E} \left( \int_0^\infty s^{p-1}|W(X_s + x + \delta) - W(X_s + x)|ds \right). \] (65)

First, suppose that \( X \) is of infinite variation and thus \( W \) is continuous on \( \mathbb{R}. \) From the fact that \( X \) drifts to \( \infty \) we know that \( W(\infty) = 1/\psi'(0+) \) and therefore it follows that \( s^{p-1}(1 - \psi'(0+)W(X_s)) \) is integrable with respect to the product measure \( \mathbb{P}_x \times \lambda([0, \infty)), \) where \( \lambda \) denotes Lebesgue measure. We can now invoke the dominated convergence theorem to deduce that \( x \mapsto \mathbb{E}_x(g^p) \) is continuous.

Next, in the case that \( X \) is of finite variation we have that \( W \) has a discontinuity at zero. However, the set \{ \( s \geq 0 : X_s = x \) \} is almost surely countable and thus has Lebesgue measure zero. We can again use the dominated convergence theorem to conclude that continuity also holds in this case.

We prove now the asymptotic behaviour of \( \mathbb{E}_x(g^p). \) Note that when \( x \) tends to \( -\infty \) the random variable \( g^{(-x)} \to \infty. \) Then using Fatou’s lemma

\[ \liminf_{x \to -\infty} \mathbb{E}(g^p) \geq \mathbb{E}((\liminf_{x \to -\infty} g^{(-x)})^p) = \infty. \]
Therefore, \( \lim_{x \to -\infty} \mathbb{E}_x(g^p) = \infty \). In the other hand, note that for \( x > 0 \),

\[
\mathbb{P}_x(g^p = 0) = \mathbb{P}_x(g = 0) = \mathbb{P}_x(\tau_0^- = \infty) = \psi'(0+)W(x) \xrightarrow{x \to \infty} 1.
\]

Hence we deduce that the sequence \( \{(g^{(-n)})^p\}_{n \geq 1} \) converges in probability to 0 (under the measure \( \mathbb{P} \)) when \( n \) tends to infinity. Moreover, since the sequence \( \{\mathbb{E}((g^{-n})^p)\}_{n \geq 1} \) is a non-increasing sequence of positive numbers we get that

\[
\sup_{n \geq 1} \mathbb{E}((g^{-n})^p) \leq \mathbb{E}(g^p) < \infty,
\]

where the last inequality holds due to Lemma 2.1 and by assumption. Then \( \{(g^{(-n)})^p\}_{n \geq 1} \) is an uniformly integrable family of random variables. Then, together with the convergence in probability, allows us to conclude that \( \mathbb{E}_x(g^p) \to 0 \) when \( x \to \infty \) as claimed.

**Proof of Lemma 4.1.** First, notice that due to the spatial homogeneity of Lévy processes and that \( x \mapsto \mathbb{E}_x(g^{p-1}) \) is non-increasing it suffices to prove the assertion for \( x \leq 0 \). Using Fubini’s theorem we have that for all \( x \leq 0 \),

\[
\mathbb{E}_x \left( \int_0^\infty \mathbb{E}_{X_s}(g^{p-1})ds \right) = \int_{(-\infty,\infty)} \mathbb{E}_x(g^{p-1}) \int_0^\infty \mathbb{P}_x(X_s \in dz).
\]

Since \( X \) drifts to infinity we can use the density for the 0-potential measure of \( X \) without killing (see equation (10)) to obtain

\[
\mathbb{E}_x \left( \int_0^\infty \mathbb{E}_{X_s}(g^{p-1})ds \right) = \int_{-\infty}^\infty \mathbb{E}_x(g^{p-1}) \left[ \frac{1}{\psi'(0+)} - W(x-z) \right]dz
\]

\[
= \frac{1}{\psi'(0+)} \int_{-\infty}^x \mathbb{E}_x(g^{p-1}) \left[ 1 - \psi'(0+)W(x-z) \right]dz + \frac{1}{\psi'(0+)} \int_x^\infty \mathbb{E}_x(g^{p-1})dz. \tag{67}
\]

Now we prove that the above two integrals are finite for all \( x \leq 0 \). From the fact that \( z \mapsto \mathbb{E}_z(g^{p-1}) \) is continuous on \( \mathbb{R} \) and \( W \) is continuous on \((0,\infty)\) we can assume without loss of generality that \( x = 0 \).

First, we show that the first integral on the right hand side of (67) is finite. From Lemma 2.2 we have that

\[
\int_0^\infty \mathbb{E}_{-z}(g^{p-1}) \left[ 1 - \psi'(0+)W(z) \right]dz \leq 2^{p-1} \mathbb{E}(-X_{\infty})[\mathbb{E}(g^{p-1}) + A_{p-1} + \frac{2^{p-1}}{p} C_{p-1} \mathbb{E}((-X_{\infty})^p),
\]

where \( A_{p-1} \) and \( C_{p-1} \) are non-negative constants. In the equality above we relied on the fact that \( z \mapsto \psi(0+)W(z) \) corresponds to the distribution function of the random variable \(-X_{\infty}\). We conclude from Lemma 2.1 that

\[
\int_0^\infty \mathbb{E}_{-z}(g^{p-1}) \left[ 1 - \psi'(0+)W(z) \right]dz < \infty.
\]

Now we proceed to check the finiteness of the second integral in (67) when \( x = 0 \). Using the strong Markov property we have that

\[
\int_0^\infty \mathbb{E}_x(g^{p-1})dz = \int_0^\infty \mathbb{E}_x(g^{p-1} 1_{\{\tau_0^- < \infty\}})dz
\]

\[
\leq 2^{p-1} \int_0^\infty \mathbb{E}_x((\tau_0^-)^{p-1} 1_{\{\tau_0^- < \infty\}})dz + 2^{p-1} \int_0^\infty \mathbb{E}_x(\mathbb{E}_{X_{\tau_0^-}}(g^{p-1}) 1_{\{\tau_0^- < \infty\}})dz
\]

\[
\leq 2^{p-1} \int_0^\infty \mathbb{E}_x((\tau_0^-)^{p-1} 1_{\{\tau_0^- < \infty\}})dz + 2^{p-1} \int_0^\infty \mathbb{E}_x(\mathbb{E}_{X_{\infty}}(g^{p-1}) 1_{\{X_{\infty} < 0\}})dz,
\]

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Proof of Lemma 4.7. Let $x \leq 0$ and take $\delta > 0$. Then

$$E_{0, x}((\tau_D)^p) = E_x((\tau_b^{0, 0})^p) \leq E_x((\tau_b^{0, 0})^p \mathbb{I}_{(g + \delta < \tau_b^{0, 0})}) + E_x((g + \delta)^p \mathbb{I}_{(g + \delta > \tau_b^{0, 0})}).$$

Note that on the event \( \{g + \delta < \tau_b^{0, 0}\} \) we have that

$$\tau_b^{0, 0} = \inf\{t > g + \delta : X_t \geq b(U_t)\} = \inf\{t > 0 : X_{t+g+\delta} \geq b(t+\delta)\} + g + \delta \leq \inf\{t > 0 : X_{t+g+\delta} \geq b(\delta)\} + g + \delta.$$
where the second equality follows from the fact that after $g$, the process $X$ never goes back below zero and the last inequality holds since $b$ is non-increasing. We have that the law of the process $\{X_{t+g}, t \geq 0\}$ is the same as that of $\mathbb{P}^\dagger$ where $\mathbb{P}^\dagger = \mathbb{P}_0^\dagger$ and $\mathbb{P}^\dagger_\mathbb{U}$ corresponds to the law of $X$ starting at $x$ conditioned to stay positive (see Bertoin (1998) Corollary VII.4.19). Using the Markov property and equation VII.3.(6) in Bertoin (1998) we get

$$
E_x((\tau_b^\dagger)^p) \leq 2^p E^\dagger E_X^\dagger ((\tau_b^\dagger)^p) + (2^p + 1) E_x((g + \delta)^p)
$$

$$
= 2^p E^\dagger \left( \frac{W(b(\delta))}{W(x)} E_{X_\delta}((\tau_b^\dagger)^p \mathbb{1}_{\{\tau_0^\dagger > \tau_b^\dagger\}}) \right) + (2^p + 1) E_x((g + \delta)^p)
$$

$$
\leq 2^p E((\tau_b^\dagger)^p) E^\dagger \left( \frac{W(b(\delta))}{W(X_\delta)} \right) + (2^p + 1) E_x((g + \delta)^p)
$$

$$
= 2^p E((\tau_b^\dagger)^p) \int_{(0, \infty)} \frac{W(b(\delta))}{W(z)} \mathbb{P}^\dagger(X_\delta \in dz) + (2^p + 1) E_x((g + \delta)^p),
$$

where the second inequality follows from the fact that $E_x((\tau_b^\dagger)^p) \leq E((\tau_b^\dagger)^p)$ for all $0 \leq x \leq a$ and $X_\delta > 0$ under $\mathbb{P}^\dagger$. Thus, using that $\mathbb{P}^\dagger(X_\delta \in dz) = [z W(z)/\delta] \mathbb{P}(X_\delta \in dz)$ (see e.g. Corollary VII.3.16 in Bertoin (1998)) we have that

$$
E_x((\tau_b^\dagger)^p) \leq 2^p E((\tau_b^\dagger)^p) \int_{(0, \infty)} \frac{W(b(z))}{W(z)} \mathbb{P}^\dagger(X_\delta \in dz) + (2^p + 1) E_x((g + \delta)^p)
$$

$$
= 2^p E((\tau_b^\dagger)^p) \frac{W(b(z))}{\delta} \mathbb{E}(X_\delta^\dagger) + 2^p (2^p + 1) \delta^p + 2^p (2^p + 1) E_x((g)^p),
$$

(68)

where $X_\delta^\dagger$ is the positive part of $X_\delta$. Thus from Lemma 2.1 we have that $E_x((\tau_b^\dagger)^p)$ is finite.

Next, we show that $E_{u,x}((\tau_D)^p) < \infty$ when $u, x > 0$. From the Markov property of Lévy processes we have that

$$
E_{u,x}((\tau_D)^p) = E_x((\tau_u^u,0)^p \mathbb{1}_{\{\tau_u^u,0 < \sigma_0^+\}}) + E_x((\tau_b^\dagger,0)^p \mathbb{1}_{\{\tau_b^\dagger,0 > \sigma_0^+\}})
$$

$$
\leq E_x((\tau_u^u,0)^p) + 2^p E_x((\sigma_0^+)^p \mathbb{1}_{\{\sigma_0^+ < \infty\}}) + E_x(\mathbb{1}_{\{\sigma_0^+ < \infty\}} E_{X_{\sigma_0^+}}((\tau_b^\dagger,0)^p))
$$

Using (68), the inequality $|X_{\sigma_0^+} - \sigma_0^+| \leq |X_\infty|$ under the event $\{\sigma_0^+ < \infty\}$ and Lemmas 2.1 and 2.2 we deduce that $E_{u,x}((\tau_D)^p) < \infty$ and the proof is complete.

Using that $b$ is a right-continuous and a non-decreasing function and that $X$ creeps upwards, it can be shown that for any $u \geq 0$ and $x \in \mathbb{R}$,

$$
\lim_{h \to 0} \tau_b^{u,x+h} = \tau_b^{u,x} \quad \text{a.s. and} \quad \lim_{(h_1,h_2) \to (0,0)} \tau_b^{u+h_1,x+h_2} = \tau_b^{u,x} \quad \text{a.s.}
$$

These facts will be useful in the proof of the continuity of the function $V$.

**Proof of Lemma 4.10.** First, we show that the function $u \mapsto V(u, x)$ is continuous for all $x > 0$ fixed. Take $u_1, u_2 > 0$ and $x > 0$, then since the stopping time $\tau_{(u_1,x)} := \tau_b^{u_1,x} \mathbb{1}_{\{\tau_b^{u_1,x} < \sigma_0^-\}} + \tau_b^{u_2,x} \mathbb{1}_{\{\tau_b^{u_2,x} \geq \sigma_0^-\}}$ is optimal for $V(u_1, x)$ (under $\mathbb{P}$) we have that

$$
V(u_1, x) = \mathbb{E} \left( \int_0^{\tau_b^{u_1,x} \wedge \tau_b^{u_2,x}} G(u_1 + s, X_s + x)ds + \mathbb{1}_{\{\tau_b^{u_1,x} \geq \sigma_0^-\}} V(0, X_{\sigma_0^-} + x) \right)
$$

$$
= \mathbb{E}_x \left( \int_0^{\tau_b^{u_1,x} \wedge \tau_b^{u_2,x}} G(u_1 + s, X_s)ds + \mathbb{1}_{\{\tau_b^{u_1,x} \geq \sigma_0^-\}} V(0, X_{\sigma_0^-}) \right).
$$

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On the other hand, from (26) we get
\[ V(u_2, x) \leq \mathbb{E} \left( \int_0^{\tau_{(u_1,x)}^+} \left\{ G(u_2 + s, X_s + x) I_{\{\sigma_{u_2} > s\}} + G(U_s^{(x)}, X_s + x) I_{\{\sigma_{u_2} > s\}} \right\} ds \right) \]
\[ = \mathbb{E}_x \left( \int_0^{\tau_{(u_1,x)}^+} \left\{ G(u_2 + s, X_s) I_{\{\sigma_{u_2} > s\}} + G(U_s, X_s) I_{\{\sigma_{u_2} > s\}} \right\} ds \right) \]
\[ + \mathbb{E}_x \left( \int_0^{\tau_{(u_1,x)}^+} \left\{ G(u_2 + s, X_s) I_{\{\sigma_{u_2} > s\}} + G(U_s, X_s) I_{\{\sigma_{u_2} < s\}} \right\} ds \right) \]
\[ = \mathbb{E}_x \left( \int_0^{\tau_{(u_1,x)}^+} \left\{ G(u_2 + s, X_s) ds \right\} + \mathbb{E}_x \left( \int_0^{\tau_{(u_1,x)}^+} G(U_s, X_s)ds \right) \right) \]
\[ = \mathbb{E}_x \left( \int_0^{\tau_{(u_1,x)}^+} G(u_2 + s, X_s) ds \right) + \mathbb{E}_x \left( \int_0^{\tau_{(u_1,x)}^+} G(U_s, X_s)ds \right), \]
where in the first equality we used the definition of \( \tau_{(u_1,x)}^+ \) given above, in the second equality that \( \tau_{b_0}^{u_1,0} \leq \tau_{b_0}^{u_0} \) and the last equality follows from the strong Markov property applied at time \( \sigma_{u_2}^0 \). Hence, we have that for any \( x > 0 \) fixed and \( u_1, u_2 > 0 \),
\[ |V(u_2, x) - V(u_1, x)| \leq \mathbb{E}_x \left( \int_0^{\tau_{(u_1,x)}^+ \wedge \tau_{b_0}^{u_1,0}} |G(u_2 + s, X_s) - G(u_1 + s, X_s)|ds \right) \]
\[ \leq \mathbb{E}_x \left( \int_0^{\tau_{b_0}^{u_1} \wedge \tau_{u_1}^+} |G(u_2 + s, X_s) - G(u_1 + s, X_s)|ds \right) \]
\[ \leq \mathbb{E} \left( \int_0^{\tau_{b_0}^{u_1} + \tau_{b_0}^{u_2}} |(u_2 + s)^{p-1} - (u_1 + s)^{p-1}|ds \right) \]
\[ = \frac{1}{p} \mathbb{E}((\tau_{b_0}^{u_1} + u_2)^p) - \mathbb{E}((\tau_{b_0}^{u_1} + u_1)^p) - |u_2 - u_1|^p], \]
where \( \tau_{b_0}^{u_1} = \inf\{ t \geq 0 : X_t > b(u_1) \} \). Thus tending \( u_2 \to u_1 \), with the dominated convergence theorem and the fact that \( \mathbb{E}((\tau_{a_0}^u + u)^p) < \infty \) for all \( u, a \geq 0 \) we get that \( u \mapsto V(u, x) \) is continuous uniformly over all \( x > 0 \).

Now we show that \( x \mapsto V(u, x) \) is continuous. From equation (30) we easily deduce that \( x \mapsto V(0, x) \) is a continuous function on \((-\infty, 0]\). Next, suppose that \( u > 0 \) and \( x > 0 \). Recall from equation (34) that we can write
\[ V(u, x) = \mathbb{E} \left( \int_0^{\sigma_{u_2}^+ \wedge \tau_{b_0}^{u_1}} G(u + s, X_s + x) ds \right) + \mathbb{E}(V(0, X_{\sigma_{u_2}^+} + x) I_{\{\sigma_{u_2}^+ \leq \tau_{b_0}^{u_1} \}}). \]

Note that for all \( s \leq \tau_{b_0}^{u,0} \wedge \sigma_{u_2}, \) it holds that \( 0 < X_s + x \leq b(u + s) \leq b(u) \) and for all \( x \in \mathbb{R} \) (see equation (31)), \( V(0, X_{\sigma_{u_2}^+} + x) I_{\{\sigma_{u_2}^+ \leq \tau_{b_0}^{u,0} \}} \geq V(0, X_{\sigma_{u_2}^+} + x) \geq -A_{p-1} - C_{p-1} |X_{\sigma_{u_2}^+} + x|^p + V(0, 0) \), where the latter expression is integrable from Lemma 2.1. Moreover, it can be shown that \( \lim_{h \to 0} \sigma_{u_2}^+ = \sigma_{u_2}^+ \) a.s. and that \( \lim_{h \to 0} \tau_{b_0}^{u,x+h} = \tau_{b_0}^{u,x} \) a.s. for any \( x \in \mathbb{R} \). Then by the dominated convergence theorem and the fact that \( V \) is continuous on \((-\infty, 0]\) and \( \mapsto G(u, x) \) is continuous on \((0, \infty)\) we conclude that, for each \( u > 0 \), the mapping \( x \mapsto V(u, x) \) is continuous on \((0, \infty)\). Note that when \( X \) is of infinite variation, \( \lim_{h \to 0} \sigma_{u_2}^+ = \tau_{0}^- = 0 \) a.s. and the latter argument also tells us that for all \( u > 0 \),
\[ \lim_{h \to 0} V(u, h) = V(0, 0). \]
Note that the limit above implies that \( \lim_{(u,x) \to (0,0)^+} V(u, x) = V(0, 0) \) in the infinite variation case. Then we proceed to prove that this also holds when \( X \) is of finite variation. In this case we know that \( \sigma_0^- = 0 \) and \( \tau_0^- > 0 \). Due to the strong Markov property,

\[
V(0, 0) = \mathbb{E} \left( \int_0^{\tau_0^0 \wedge \tau_0^-} G(s, X_s) ds \right) + \mathbb{E}(\mathbb{1}_{\{\tau_0^- < \tau_0^0\}} V(0, X_{\tau_0^-})),
\]

where \( \tau_0^0 = \inf\{t > 0 : X_t \geq b(s)\} \). Taking limits in (34) we have from the dominated convergence theorem,

\[
\begin{align*}
\lim_{(u,x) \to (0,0)^+} V(u, x) &= \lim_{(u,x) \to (0,0)^+} \mathbb{E} \left( \int_0^{\tau_u^0 \wedge \tau_b^-} G(u + s, X_s + x) ds \right) \\
&\quad + \lim_{(u,x) \to (0,0)^+} \mathbb{E}(V(0, X_{\tau_u^-} + x) \mathbb{1}_{(\tau_u^- \leq \tau_b^-)}) \\
&= \mathbb{E} \left( \int_0^{\tau_0^0 \wedge \tau_0^-} G(s, X_s) ds \right) + \mathbb{E}(\mathbb{1}_{\{\tau_0^- < \tau_0^0\}} V(0, X_{\tau_0^-})) \\
&= V(0, 0),
\end{align*}
\]

where we used that \( \lim_{x \to 0} \sigma_u^- = \tau_u^- \) and \( \lim_{(u,x) \to (0,0)^+} \tau_b^- = \tau_0^0 \) a.s. Therefore \( V \) is continuous on the set \( E \).

Before proving Lemma 4.17 we first consider a technical lemma involving the derivative of the potential measure. More specifically, for fixed \( a > 0 \), \( x \in (0, a) \) and \( r \in \mathbb{N} \cup \{0\} \) denote by \( U_r(a, x, dy) \) as the measure

\[
U_r(a, x, dy) = \int_0^\infty t^r \mathbb{P}_x(X_t \in dy, t < \sigma_0^- \wedge \tau_a^+) dt.
\]

**Lemma A.1.** Let \( q \in \mathbb{N} \cup \{0\} \) such that \( \int_{(-\infty, -1]} |x| q(dx) < \infty \). Fix \( a > 0 \) and \( 0 \leq x \leq a \). We have that for all \( r \in \{0, 1, \ldots, q\} \) the measure \( U_r(a, x, dy) \) is absolutely continuous with respect to the Lebesgue measure. It has a density \( u_r(a, x, y) \) given by

\[
u_r(a, x, y) = \lim_{q \to 0} (-1)^r \frac{\partial^r}{\partial q^r} \left[ \frac{W(q)(x)W(q)(a-y)}{W(q)(a)} - W(q)(x-y) \right],
\]

for \( y \in (0, a] \). Moreover, for a fixed \( a > 0 \) the functions \( x \mapsto \mathbb{E}_x((\tau_a^+)^r \mathbb{1}_{(\sigma_0^- < \tau_a^+)}) \) and \( x \mapsto u_r(a, x, y) \) are differentiable on \((0, a)\) and have finite left derivative at \( x = a \) for all \( y \in (0, a) \) and \( r \in \{0, 1, \ldots, q\} \).

**Proof.** Let \( a > 0 \) and \( x \in (0, a) \). First we show that for all \( r \in \{0, 1, \ldots, q\} \) the measure \( U_r(a, x, dy) \) is absolutely continuous with respect to the Lebesgue measure. Take any measurable set \( A \subset (0, a) \), thus by Fubini’s theorem

\[
\int_A U_r(a, x, dy) = \int_0^\infty t^r \mathbb{P}_x(X_t \in A, t < \sigma_0^- \wedge \tau_a^+) dt \\
= \mathbb{E}_x \left( \int_0^{\tau_a^+ \wedge \sigma_0^-} t^r \mathbb{1}_{(X_t \in A)} dt \right).
\]

From Lemma 2.1 we know that \( \mathbb{E}_x((\tau_a^+)^r) < \infty \) for all \( r \in \{0, 1, \ldots, q\} \). Then by dominated convergence theorem we have that

\[
\int_A U_r(a, x, dy) = \lim_{q \to 0} \mathbb{E}_x \left( \int_0^{\tau_a^+ \wedge \sigma_0^-} t^r e^{-qt} \mathbb{1}_{(X_t \in A)} dt \right) \\
= \int \lim_{q \to 0} (-1)^r \frac{\partial^r}{\partial q^r} \left[ \frac{W(q)(x)W(q)(a-y)}{W(q)(a)} - W(q)(x-y) \right] dy,
\]
where the last equality follows from (8). From the convolution representation of $W^{(q)}$ (see equation (2)) the derivatives in the last equation above exist and indeed $u_\ast(a, x, y)$ is a density of $U_\ast(a, x, dy)$ for all $y \in (0, a)$. Now we proceed to show the differentiation statements. Note that from equations (1) and (3) we have that

\[ f_x(q) := \mathbb{E}_x(e^{-q \tau_+} \mathbb{1}_{(\sigma_0 < \tau_+)} = e^{\Phi(q)(x-a)} - \frac{W^{(q)}(x)}{W^{(q)}(a)}, \]

for any $x \in (0, a)$. Since $W$ is differentiable, the proof follows by induction and implicit differentiation. A similar argument works for the function $x \mapsto u_\ast(a, x, y)$. \qed

We are ready to proof that the partial derivatives of $V$ at $(u, b(u))$ exist and are equal to zero.

**Proof of Lemma 4.17.** We first show that for all $u > 0$ such that $b(u) > 0$,

\[ \frac{\partial}{\partial u} V(u, b(u)) = 0. \]

From the proof of Lemma 4.10 we know that for any $h > 0$

\[ 0 \leq \frac{V(u, b(u)) - V(u - h, b(u))}{h} \leq \mathbb{E}_{b(u)} \left( \int_0^{\tau_{h(u)}} \left[ (u + s)^{p-1} - (u - h + s)^{p-1} \right] ds \right). \]

The result then follows taking $h \downarrow 0$ and from the fact that the function $u \mapsto u^p$ is differentiable on $[0, \infty)$ for all $p > 1$, the dominated convergence theorem and the fact that $b$ is continuous.

Next we proceed to show the smooth fit condition on the spatial argument, that is,

\[ \frac{\partial}{\partial x} V(u, b(u)) = 0. \]

Let $x > 0$, $u > 0$ and $0 < \varepsilon < 1$ such that $x - \varepsilon > 0$ and $b(u) > 0$. From equation (34) we know that

\[ V(u, x - \varepsilon) = \mathbb{E}_x \left( \int_0^{\tau_{h}^+ \wedge \sigma_0^-} G(u + s, X_s + x - \varepsilon) ds \right) \]

\[ + \mathbb{E}_x \left( \mathbb{1}_{(\sigma_0^- < \tau_{h}^+ \wedge \sigma_0^-)} \int_{\sigma_0^-}^{\tau_{h}^+} G(U_s(\varepsilon), X_s + x - \varepsilon) ds \right) \]

\[ = \mathbb{E}_x \left( \int_0^{\tau_{h}^+ \wedge \sigma_0^-} G(u + s, X_s) ds \right) + \mathbb{E}_x \left( \mathbb{1}_{(\sigma_0^- < \tau_{h}^+ \wedge \sigma_0^-)} \int_{\sigma_0^-}^{\tau_{h}^+} G(U_s(\varepsilon), X_s - \varepsilon) ds \right) \]

\[ + \mathbb{E}_x \left( \mathbb{1}_{(\sigma_0^- < \tau_{h}^- \wedge \sigma_0^-)} \int_{\sigma_0^-}^{\tau_{h}^-} G(U_s(\varepsilon), X_s - \varepsilon) ds \right), \]

where in the last inequality we used that $\sigma_0^- < \sigma_0$ under the measure $\mathbb{P}_x$. On the other hand, define the stopping time $\tau_* := \tau_{h}^+ \wedge \tau_{h}^- \mathbb{1}_{(\sigma_0 > \tau_{h}^- \wedge \sigma_0^-)} + \tau_{h}^- \mathbb{1}_{(\sigma_0^- < \tau_{h}^- \wedge \sigma_0^-)}$. From equation (26) we have that

\[ V(u, x) \leq \mathbb{E}_x \left( \int_0^{\tau_* \wedge \sigma_0^-} G(u + s, X_s) ds \right) + \mathbb{E}_x \left( \mathbb{1}_{(\sigma_0^- < \tau_*)} \int_{\sigma_0^-}^{\tau_*} G(U_s, X_s) ds \right) \]

\[ = \mathbb{E}_x \left( \int_0^{\tau_{h}^- \wedge \sigma_0^-} G(u + s, X_s) ds \right) + \mathbb{E}_x \left( \mathbb{1}_{(\sigma_0^- < \tau_{h}^- \wedge \sigma_0^-)} \int_{\sigma_0^-}^{\tau_{h}^-} G(u + s, X_s) ds \right) \]

\[ + \mathbb{E}_x \left( \mathbb{1}_{(\sigma_0^- < \tau_{h}^- \wedge \sigma_0^-)} \int_{\sigma_0^-}^{\tau_{h}^-} G(U_s, X_s) ds \right), \]

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where we again used that $\sigma_{\varepsilon}^- \leq \sigma_0^-$. Hence for any $u > 0$, $0 < x \leq b(u)$ and $0 < \varepsilon < 1$ such that $x - \varepsilon > 0$ and $b(u) > 0$, 

$$0 \leq \frac{V(u, x) - V(u, x - \varepsilon)}{\varepsilon} \leq R_1^{(c)}(u, x) + R_2^{(c)}(u, x) + R_3^{(c)}(u, x),$$

where

$$R_1^{(c)}(u, x) := \frac{1}{\varepsilon} E_x \left( \int_0^{\tau_{b, +} \wedge \sigma_{\varepsilon}^-} [G(u + s, X_s) - G(u + s, X_s - \varepsilon)] ds \right) \geq 0,$$

$$R_2^{(c)}(u, x) := \frac{1}{\varepsilon} E_x \left( \int_{\sigma_0^-}^{\tau_{b, +} \wedge \sigma_{\varepsilon}^-} [G(u + s, X_s) - G(U_s^{(c)}, X_s - \varepsilon)] ds \right) \geq 0,$$

$$R_3^{(c)}(u, x) := \frac{1}{\varepsilon} E_x \left( \int_{\sigma_0^-}^{\tau_{b, +} \wedge \sigma_{\varepsilon}^-} [G(U_s, X_s) - G(U_s^{(c)}, X_s - \varepsilon)] ds \right) \geq 0.$$

We will show that $\lim_{\varepsilon \downarrow 0} R_i^{(c)}(u, x) = 0$ for $i = 1, 2, 3$. From the fact that $b$ is non-increasing we have that $\tau_{b, +}^{u, -} \leq \tau_{b(u) + \varepsilon}$ and then for all $u > 0$ and $x = b(u)$ we have that

$$R_1^{(c)}(u, b(u)) \leq \frac{1}{\varepsilon} E_{b(u)} \left( \int_0^{\tau_{b(u) + \varepsilon} \wedge \sigma_{\varepsilon}^-} (u + s)^{p-1}\psi'(0+) [W(X_s) - W(X_s - \varepsilon)] ds \right)$$

$$- \frac{1}{\varepsilon} E_{b(u) - \varepsilon} \left( \int_0^{\tau_{b(u) + \varepsilon} \wedge \sigma_{\varepsilon}^-} [E_{X_s + \varepsilon}(g^{p-1}) - E_{X_s}(g^{p-1})] ds \right)$$

$$= \frac{1}{\varepsilon} E_{b(u)} \left( \int_0^{\tau_{b(u) + \varepsilon} \wedge \sigma_{\varepsilon}^-} (u + s)^{p-1}\psi'(0+) [W(X_s) - W(X_s - \varepsilon)] ds \right)$$

$$- \frac{1}{\varepsilon} \int_{(0,b(u))} [E_{z + \varepsilon}(g^{p-1}) - E_z(g^{p-1})] \int_0^\infty P_{b(u) - \varepsilon}(X_s \in dz, t < \tau_{b(u) + \varepsilon}^-) ds.$$

Using the density of the 0-potential measure of $X$ exiting the interval $[0, b(u)]$ given in equation (8) we obtain that

$$R_1^{(c)}(u, b(u)) \leq E_{b(u)} \left( \int_0^{\tau_{b(u) + \varepsilon} \wedge \sigma_{\varepsilon}^-} (u + s)^{p-1}\psi'(0+) \frac{W(X_s) - W(X_s - \varepsilon)}{\varepsilon} ds \right)$$

$$- \int_0^{\tau_{b(u) + \varepsilon}^-} \frac{[W(b(u) - \varepsilon)W(b(u) - z)]}{W(b(u))} \int_0^z [W(b(u) - \varepsilon)W(b(u) - z)] dz.$$

Note that for all $s < \tau_{b(u) + \varepsilon}^- \wedge \sigma_{\varepsilon}^-$, we have $X_s \in (\varepsilon, b(u) + \varepsilon)$. Then using the fact that $W \in C^1((0, \infty))$, the function $z \mapsto E_z(g^{p-1})$ is continuous, $\lim_{\varepsilon \downarrow 0} \tau_{b(u) + \varepsilon}^- \wedge \sigma_{\varepsilon}^- = \tau_{b(u)}^- \wedge \sigma_0^- = 0$ a.s. under $P_{b(u)}$ and the dominated convergence theorem we conclude that

$$\lim_{\varepsilon \downarrow 0} R_1^{(c)}(u, b(u)) = 0.$$

Now we show that $\lim_{\varepsilon \downarrow 0} R_2^{(c)}(u, b(u)) = 0$. Take $0 < x \leq b(u)$. Then using the inequality $G(u, x) \leq u^{p-1}$, the fact that for $s < \sigma_0^-, X_s > 0$ (then $-E_-(g^{p-1}) = G(0, -1) \leq G(U_s^{(c)}, X_s - \varepsilon)$) and the strong Markov property at time $\sigma_{\varepsilon}^-$ we get that

$$\lim_{\varepsilon \downarrow 0} R_2^{(c)}(u, b(u)) = 0.$$
\[ R_2^c(u, x) \leq \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} [\tau_b^{g,-\varepsilon} \wedge \sigma_0^- - \sigma_0^-] (u + \tau_b^{g,-\varepsilon} \wedge \sigma_0^-)^{p-1} + \mathbb{E}_1 (g^{p-1}) \right) \]
\[ \leq \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} f(\sigma^0_-, X_{\sigma^0_-}) \right), \]

where \( f \) is given for all \( t \geq 0 \) and \( x \in \mathbb{R} \) by
\[ f(t, x) := [2^{p-1}(u + t)^{p-1} + \mathbb{E}_1 (g^{p-1})] \mathbb{E}_x (\tau_b^{g,-\varepsilon} \wedge \sigma_0^-) + 2^{p-1} \mathbb{E}_x ((\tau_b^{g,-\varepsilon} \wedge \sigma_0^-)^p) < \infty, \]
due to Lemma 4.7. Note that \( \mathbb{E}_x (\tau_b^{g,-\varepsilon} \wedge \sigma_0^-) = \mathbb{E}_x ((\tau_b^{g,-\varepsilon} \wedge \sigma_0^-)^p) = 0 \) for all \( x \leq 0 \). Thus, from (68) there exists \( M > 0 \) such that
\[ \max(\mathbb{E}_x (\tau_b^{g,-\varepsilon} \wedge \sigma_0^-), \mathbb{E}_x ((\tau_b^{g,-\varepsilon} \wedge \sigma_0^-)^p)) \leq M \]
for all \( x \leq \varepsilon \). Hence from the compensation formula for Poisson random measures we get that
\[ R_2^c(u, x) \]
\[ \leq \max(\mathbb{E}_x (\tau_b^{g,-\varepsilon} \wedge \sigma_0^-), \mathbb{E}_x ((\tau_b^{g,-\varepsilon} \wedge \sigma_0^-)^p)) \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} [2^{p-1}(u + \tau_b^{g,-\varepsilon})^{p-1} + \mathbb{E}_1 (g^{p-1}) + 2^{p-1}] \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \right) \]
\[ + M \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} [2^{p-1}(u + \tau_b^{g,-\varepsilon})^{p-1} + \mathbb{E}_1 (g^{p-1}) + 2^{p-1}] \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \right) \]
\[ = \max(\mathbb{E}_x (\tau_b^{g,-\varepsilon} \wedge \sigma_0^-), \mathbb{E}_x ((\tau_b^{g,-\varepsilon} \wedge \sigma_0^-)^p)) \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} [2^{p-1}(u + \tau_b^{g,-\varepsilon})^{p-1} + \mathbb{E}_1 (g^{p-1}) + 2^{p-1}] \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \right) \]
\[ + M \frac{1}{c} \mathbb{E}_x \left( \int_{-\infty, 0} [2^{p-1}(u + t)^{p-1} - G(0, -1) + 2^{p-1}] \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \Pi(dy)dt \right) \]
\[ = \max(\mathbb{E}_x (\tau_b^{g,-\varepsilon} \wedge \sigma_0^-), \mathbb{E}_x ((\tau_b^{g,-\varepsilon} \wedge \sigma_0^-)^p)) \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} [2^{p-1}(u + \tau_b^{g,-\varepsilon})^{p-1} + \mathbb{E}_1 (g^{p-1}) + 2^{p-1}] \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \right) \]
\[ + \int_0^{b(u)} \int_{(-\varepsilon, -\varepsilon)}^{\infty} [2^{p-1}(u + t)^{p-1} + \mathbb{E}_1 (g^{p-1}) + 2^{p-1}] \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \Pi(dy)dt \]

Letting \( x = b(u) \) and tending \( \varepsilon \downarrow 0 \) we get from Lemma A.1 that
\[ \lim_{\varepsilon \downarrow 0} R_2^c(u, b(u)) = 0. \]

Lastly, using the Markov property at time \( \sigma_0^- \) and the fact that \( \tau_b^{g,0} \leq \tau_b^{g,-\varepsilon} \) we get that
\[ R_3^c(u, x) = \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \mathbb{E}_{X_{\sigma^0_-}} \left[ \int_{0}^{\tau_b^{g,-\varepsilon}} [G(U_s, X_s) - G(U_s^{(\varepsilon)}(\varepsilon), X_s - \varepsilon)] ds \right] \right) \]
\[ \leq \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \mathbb{E}_{X_{\sigma^0_-}} \left[ \int_{0}^{\tau_b^{g,-\varepsilon}} [G(U_s, X_s) + \mathbb{E}_0 (g^{p-1})] ds \right] \right) \]
\[ \leq \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \mathbb{E}_{X_{\sigma^0_-}} \left[ \int_{0}^{\tau_b^{g,-\varepsilon}} [G(U_s, X_s) + \mathbb{E}_0 (g^{p-1})] ds \right] \right) \]
\[ \leq \frac{1}{c} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma^0_- < \tau_b^{g,-\varepsilon}\}} \mathbb{E}_{X_{\sigma^0_-}} \left[ \int_{0}^{\tau_b^{g,-\varepsilon}} [G(U_s, X_s) + \mathbb{E}_0 (g^{p-1})] ds \right] \right), \]

where we used the fact that \( G(U_s, X_s) \leq s^{p-1} \leq (\tau_b^{g,-\varepsilon})^{p-1} \) for all \( s \in [\tau_b^{g,0}, \tau_b^{g,-\varepsilon}] \). We can easily deduce from (30) that for any \( x < 0 \),
\[
0 \leq \frac{\partial}{\partial x} V(0, x) = \int_{[0,\infty)} \mathbb{E}_{x-u}(g^{p-1})W(du).
\]

Then for all \( x < 0, x \mapsto V(0, x) \) is differentiable and has left derivative at zero. Using Lemma 2.2 and the fact that \( \mathbb{P}(-\infty \in du) = \psi'(0+)W(du) \) we get that for all \( x < 0, \)
\[
\frac{\partial}{\partial x} V(0, x) \leq \frac{2p-1[E(g^{p-1}) + A_{p-1}] + 4p-1C_{p-1}E((-X_{\infty})^{p-1})}{\psi'(0+)} + \frac{4p-1C_{p-1}|x|^{p-1}}{\psi'(0+)}. \]

Thus since \( |X_{\sigma_{0}}| \leq |X_{\infty}| \) and \( \mathbb{E}_{x}((-X_{\infty})^{p-1}) < \infty \) for all \( x \in \mathbb{R} \) we have that \( \mathbb{E}_{x} \left( \frac{\partial}{\partial x} V(0, X_{\sigma_{0}}) \right) \) is locally bounded. Moreover, by the dominated convergence theorem we can also conclude that for each \( x < 0, \)
\[
\frac{\partial}{\partial x} V(0, x)
\]
is continuous. Hence, by the dominated convergence theorem and the right continuity of \( b \) we have that
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}_{x} \left( \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{-}\}} |V(0, X_{\sigma_{0}}) - V(0, X_{\sigma_{0}} - \varepsilon)| \right) = \mathbb{E}_{x} \left( \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{-}\}} \frac{\partial}{\partial x} V(0, X_{\sigma_{0}}) \right). \]

In particular taking \( x = b(u) \) we have that equation above is equal to zero. On the other hand, conditioning on \( \sigma_{-} \) we have that
\[
\frac{1}{\varepsilon} \mathbb{E}_{x} \left( \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{-}\}} \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{-}\}} \mathbb{E}_{X_{\sigma_{0}}} \left( (\tau_{b_{y}^{\sigma_{-}}} - \tau_{b}^{0}) (\tau_{b_{y}^{\sigma_{-}}} - \varepsilon)^{p-1} \right) \right)
\]
\[
= \frac{1}{\varepsilon} \mathbb{E}_{x} \left( \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{-}\}} f_{2}(\varepsilon, X_{\sigma_{-}}) \right),
\]
where
\[
0 \leq f_{2}(\varepsilon, x) = \mathbb{E}_{x} \left( \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{-}\}} \mathbb{E}_{X_{\sigma_{0}}} \left( (\tau_{b_{y}^{\sigma_{-}}} - \tau_{b}^{0}) (\tau_{b_{y}^{\sigma_{-}}} - \varepsilon)^{p-1} \right) \right).
\]

We show that \( f_{2} \) is finite function. For all \( y \leq 0 \) we have that conditioning with respect to \( \tau_{y}^{+} \) and the strong Markov property of Lévy processes
\[
\mathbb{E}_{y} \left( (\tau_{b_{y}^{\sigma_{-}}} - \tau_{b}^{0}) (\tau_{b_{y}^{\sigma_{-}}} - \varepsilon)^{p-1} \right) \leq 2^{p} \mathbb{E}_{y}((\tau_{b_{y}^{\sigma_{-}}} - \varepsilon)^{p}) + 2^{p} \mathbb{E}_{y}((\tau_{y}^{+})^{p}) \leq 2^{p} \mathbb{E}((\tau_{b_{y}^{\sigma_{-}}} - \varepsilon)^{p}) + 2^{p} A_{p} + 2^{p} C_{p} |y|^{p},
\]
where the last inequality follows from Lemma 2.2. Hence, since \( |X_{\sigma_{0}}| \leq |X_{\infty}| \) under the event \( \{\sigma_{0} < \infty\} \) we have that
\[
f_{2}(\varepsilon, x) \leq \left\{ \begin{array}{ll}
2^{p} \mathbb{E}((\tau_{b_{y}^{\sigma_{-}}} - \varepsilon)^{p}) + 2^{p} A_{p} + 2^{p} C_{p} |x|^{p} & x > 0,
2^{p} \mathbb{E}((\tau_{b_{y}^{\sigma_{-}}} - \varepsilon)^{p}) + 2^{p} A_{p} + 2^{p} C_{p} |x|^{p} & x \leq 0.
\end{array} \right.
\]

(69)

From Lemmas 2.1 and 4.7 we conclude that \( f_{2}(\varepsilon, x) \) is a finite function. Moreover from the fact that \( b \) is continuous and \( x \mapsto U_{t}^{(x)} \) is right continuous we can show that \( \lim_{\varepsilon \downarrow 0} \tau_{b}^{\sigma_{-}} = \tau_{b}^{0} \) a.s. and then by the dominated convergence theorem, \( \lim_{\varepsilon \downarrow 0} f_{2}(\varepsilon, x) = 0 \) for all \( x \in \mathbb{R} \). Moreover, using the compensation formula for Poisson random measures we get that
\[
\frac{1}{\varepsilon} \mathbb{E}_{x} \left( \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{-}\}} f_{2}(\varepsilon, X_{\sigma_{-}}) \right)
\]
\[
= \frac{1}{\varepsilon} \mathbb{E}_{x} \left( \mathbb{1}_{\{\sigma_{-} < \tau_{y}^{+}\}} f_{2}(\varepsilon, X_{\sigma_{-}}) \right)
\]
\[
= f_{2}(\varepsilon, x) \frac{\mathbb{P}_{x}(\sigma_{-} < \tau_{0(u)}^{+}, X_{\sigma_{-}} = x)}{\varepsilon}
\]
\[
+ \frac{1}{\varepsilon} \mathbb{E}_{x} \left( \int_{[0,\infty)} \int_{(-\infty,0)} f_{2}(\varepsilon, X_{t-} + y) \mathbb{1}_{\{X_{t-} < b(u)\varepsilon\}} \mathbb{1}_{\{X_{t-} > \varepsilon\}} \mathbb{1}_{\{X_{t-} + y \leq \varepsilon\}} N(dt, dy) \right)
\]
\[
\leq f_{2}(\varepsilon, x) \frac{\mathbb{P}_{x}(\sigma_{-} < \tau_{0(u)}^{+})}{\varepsilon}
\]
\[
+ \frac{1}{\varepsilon} \mathbb{E}_{x} \left( \int_{\infty} \int_{(-\infty,0)} f_{2}(\varepsilon, X_{t-} + \varepsilon + y) \mathbb{1}_{\{t < \tau_{0(u)}^{+} \wedge \sigma_{0}^{-}\}} \mathbb{1}_{\{X_{t-} + y \leq \varepsilon\}} \Pi(dy)dt \right). \]
From the 0-potential density of the process killed on exiting $[0, b(u)]$ (see equation (8)) and from equation (3) we obtain
\[
\frac{1}{\varepsilon} \mathbb{E}_x \left( \mathbb{I}_{\left( \tau_0^- < \tau_0^- \right)} f_2(\varepsilon, X_{\tau_0^-}) \right) \\
\leq f_2(\varepsilon, x) \frac{W(b(u)) - W(x - \varepsilon)}{\varepsilon W(b(u))} \\
+ \frac{1}{\varepsilon} \int_{(0,b(u))} \int_{(-\infty,0)} f_2(\varepsilon, z + \varepsilon + y) \mathbb{I}_{\left( z + y \leq 0 \right)} \Pi(dy) \int_0^\infty \mathbb{P}_{x-\varepsilon}(X_t \in dz, t < \tau^+_{b(u)} \wedge \sigma^0) dt \\
= f_2(\varepsilon, x) \frac{W(b(u)) - W(x - \varepsilon)}{\varepsilon W(b(u))} \\
+ \frac{1}{\varepsilon} \int_{(x-\varepsilon)^+ \setminus (x-\varepsilon)^0} \left[ \frac{W(x - \varepsilon) W(b(u) - z)}{W(b(u))} - W(x - \varepsilon - z) \right] \int_{(-\infty, -z)} f_2(\varepsilon, z + \varepsilon + y) \Pi(dy) dz \\
+ \frac{1}{\varepsilon} \int_{(x-\varepsilon)^0} \left[ \frac{W(x - \varepsilon) W(b(u) - z)}{W(b(u))} - W(x - \varepsilon - z) \right] \int_{(-\infty, -z)} f_2(\varepsilon, z + \varepsilon + y) \Pi(dy) dz.
\]

Note that since $\Pi$ is finite on sets of the form $(-\infty, -\delta)$ for all $\delta > 0$, Lemma 2.1 and equation (69) we have that the integrals above with respect to $\Pi$ are finite and bounded. Hence, taking $x = b(u)$ and from the dominated convergence theorem we conclude that
\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_{b(u)} \left( \mathbb{I}_{\left( \tau_0^- < \tau_0^- \right)} g(\varepsilon, X_{\tau_0^-}) \right) \leq 0.
\]

Hence, we also have that
\[
\lim_{\varepsilon \downarrow 0} R_3(\varepsilon)(u, b(u)) = 0
\]
and the conclusion of the Lemma holds.

\textit{Proof of Lemma 5.1.} Let $(u, x) \in E$, we first show that (44) is satisfied. Indeed, using that $|G(u, x)| < u^{p-1} - \mathbb{E}_x(g^{p-1})$ and that $U_s \leq s$ we have that
\[
\mathbb{E}_{u,x} \left( \int_0^\infty |G(U_s, X_s)| \mathbb{I}_{\left( X_s < b(U_s) \right)} ds \right) \leq \mathbb{E}_{u,x} \left( \int_0^\infty s^{p-1} \mathbb{I}_{\left( X_s < b(U_s) \right)} ds \right) + \mathbb{E}_x \left( \int_0^\infty \mathbb{E}_{X_s} \left( g^{p-1} \right) ds \right).
\]

From Lemma 4.1 we know that the second integral above is finite. Now we check that the first integral above is also finite. Consider $\delta > 0$ and consider $g^{(b(\delta))}$, the last time $X$ is below the level $b(\delta)$, then $g^{(b(\delta))} \geq g$ and $X_{s+g^{(b(\delta))}+\delta} \geq b(\delta) \geq b(U_s)$ for all $s \geq 0$. Hence, since $b$ is non-increasing we get
\[
\mathbb{E}_{u,x} \left( \int_0^\infty s^{p-1} \mathbb{I}_{\left( X_s < b(U_s) \right)} ds \right) = \mathbb{E}_x \left( \int_0^{(g^{(b(\delta))} + \delta)} s^{p-1} \mathbb{I}_{\left( X_s < b(U_s) \right)} ds \right) \leq \mathbb{E}_x \left( ((g^{(b(\delta))} + \delta) \right) < \infty,
\]
where the last expectation is finite by Lemma 2.1. Therefore we conclude that (44) holds. Moreover, from the fact that $x \mapsto \mathbb{E}_x(g^p)$ is non increasing, that $\lim_{x \to \infty} \mathbb{E}_x(g^p) = 0$ (see Lemma 2.3) and the dominated convergence theorem that
\[
\lim_{u, x \to \infty} \left| \mathbb{E}_{u,x} \left( \int_0^\infty G(U_s, X_s) \mathbb{I}_{\left( X_s < b(U_s) \right)} ds \right) \right| \leq \lim_{u, x \to \infty} \mathbb{E}_{u,x} \left( \int_0^\infty |G(U_s, X_s)| \mathbb{I}_{\left( X_s < b(U_s) \right)} ds \right) \\
\leq \lim_{x \to \infty} \mathbb{E}_x \left( ((g^{(b(\delta))} + \delta) \right) + \lim_{x \to \infty} \mathbb{E}_x \left( \int_0^\infty \mathbb{E}_{X_s} \left( g^{p-1} \right) ds \right) \\
= \delta^p
\]
for any $\delta > 0$. Hence we conclude that
\[
\lim_{u, x \to \infty} \mathbb{E}_{u,x} \left( \int_0^\infty G(U_s, X_s) \mathbb{I}_{\left( X_s < b(U_s) \right)} ds \right) = 0.
\]
Next we prove that (45) also holds. Since $V$ is non-decreasing in each argument we have that is enough to show that (45) holds for $u = 0$ and $x \leq 0$. Let $N > 0$ any positive number, then we have that

$$\mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > b(U_s)\}} ds \right)$$

$$= \mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{U_s \leq N\}} \mathbb{I}_{\{X_s > b(U_s)\}} ds \right)$$

$$+ \mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{U_s > N\}} \mathbb{I}_{\{X_s > b(N)\}} ds \right)$$

$$+ \mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{U_s > N\}} \mathbb{I}_{\{b(N) \geq X_s > b(U_s)\}} ds \right).$$

Hence, we next show that the three expectations above are finite. Using the fact that $\int_{(-\infty,0)} V(u, x + y) \Pi(dy) + G(u, x) \geq 0$ for all $u > 0$ and $x > b(u)$ (see Lemma 4.14), that $G(u, x) \leq w^{p-1}$ for all $(u, x) \in E$ and that $b$ is non increasing we get that

$$\mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{U_s \leq N\}} \mathbb{I}_{\{X_s > b(U_s)\}} ds \right) \geq -\mathbb{E}_x \left( \int_0^\infty \tilde{G}(U_s, X_s) \mathbb{I}_{\{U_s \leq N, X_s > b(U_s)\}} ds \right)$$

$$\geq -N^{p-1} \mathbb{E}_x \left( \int_0^\infty \mathbb{I}_{\{U_s \leq N\}} ds \right)$$

$$\geq -N^{p-1} \mathbb{E}_x (g(N) + N)$$

$$> -\infty,$$

where in the second last inequality we used the fact that $U_s > N$ for all $s \geq g + N$. In a similar way we have that

$$\mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{U_s > N\}} \mathbb{I}_{\{b(N) \geq X_s > b(U_s)\}} ds \right)$$

$$\geq -\mathbb{E} \left( \int_0^\infty s^{p-1} \mathbb{I}_{\{U_s > N\}} \mathbb{I}_{\{b(N) \geq X_s > b(U_s)\}} ds \right)$$

$$= -\mathbb{E} \left( \int_0^\infty s^{p-1} \mathbb{I}_{\{U_s > N\}} \mathbb{I}_{\{b(N) \geq X_s > b(U_s)\}} ds \right)$$

$$\geq -\frac{1}{p} \mathbb{E}((g^{(b(N)))^p})$$

$$> -\infty,$$

where we used that $U_s \leq s$ and that $g^{(b(N))} = \sup\{t \geq 0 : X_t \leq b(N)\}$ has moments of order $p$ (see Lemma 2.1). Lastly, since $V$ is non increasing in each argument and $b$ is non decreasing we have that by Fubini’s theorem that

$$\mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{U_s > N\}} \mathbb{I}_{\{X_s > b(N)\}} ds \right)$$

$$\geq \mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(N, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > b(N)\}} ds \right)$$

$$= \int_{(b(N), \infty)} \int_{(-\infty,0)} \tilde{V}(N, z + y) \Pi(dy) \int_0^\infty \mathbb{P}_x(X_s \in dz) ds$$

$$= \Phi'(0) \int_{b(N)}^\infty \int_{(-\infty,0)} \tilde{V}(N, z + y) \Pi(dy) dz,$$
where in the last equality we used a density of the 0-potential measure of $X$ without killing (see (10)) and the fact that $W$ vanishes on $(-\infty,0)$. From Fubini’s theorem and since $V$ is non-decreasing function in each argument that vanishes on $D$ we obtain that

\[
\begin{align*}
\mathbb{E}_x \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{U_s > b(0)\}} \mathbb{I}_{\{X_s > b(N)\}} \right)
\geq \Phi'(0) \int_{b(N)}^{b(N)+1} \int_{(-\infty,0)} \tilde{V}(N, z + y) \Pi(dy) dz + \Phi'(0) \int_0^\infty \int_{b(N)+1}^{\infty} \tilde{V}(N, z + y) \Pi(dy) dz
\geq \Phi'(0) \int_{(-\infty,0)} \tilde{V}(N, b(N) + y) \Pi(dy) + \Phi'(0) \int_{(-\infty,-1)} b(N) \tilde{V}(N, z + y) dz \Pi(dy)
\geq \Phi'(0) \int_{(-\infty,0)} \tilde{V}(N, b(N) + y) \Pi(dy) - \Phi'(0) \int_{(-\infty,-1)} (y + 1) \tilde{V}(0, y) \Pi(dy)
> -\infty,
\end{align*}
\]

where the finiteness of the last integrals follow from Lemmas 2.1 and 4.14 and equation (31). Moreover, from the dominated convergence theorem we have that

\[
\lim_{u,x \to \infty} \mathbb{E}_{u,x} \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > b(U_s)\}} \right)
= \lim_{u,x \to \infty} \mathbb{E} \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(u + s, X_s + x + y) \Pi(dy) \mathbb{I}_{\{X_s + x > b(u+s)\}} \mathbb{I}_{\{s < \sigma^-_{x}\}} \right)
+ \lim_{x \to \infty} \mathbb{E} \left( \int_0^\infty \int_{ (-\infty,0)} \tilde{V}(U_s^{-x}, X_s + x + y) \Pi(dy) I_{\{X_s + x > b(U_s^{-x})\}} \mathbb{I}_{\{s \geq \sigma^-_{x}\}} \right)
\geq \mathbb{E} \left( \int_0^\infty \int_{(-\infty,0)} \lim_{u,x \to \infty} \tilde{V}(u + s, X_s + x + y) \Pi(dy) \mathbb{I}_{\{X_s + x > b(u+s)\}} \right)
+ \mathbb{E} \left( \int_0^\infty \int_{(-\infty,0)} \lim_{x \to \infty} \tilde{V}(U_s^{-x}, X_s + x + y) \Pi(dy) I_{\{X_s + x > b(U_s^{-x})\}} \mathbb{I}_{\{s \geq \sigma^-_{x}\}} \right).
\]

Note that $b$ is a decreasing function and then $\lim_{u,x \to \infty} V(u,x) = 0$ and $\lim_{x \to \infty} V(u,x) = 0$ for any $u > 0$. Moreover, for any $s \geq 0$, $x \mapsto U_s^{-x}$ is increasing and bounded so then $\lim_{x \to \infty} U_s^{-x}$ exists. Then we have that

\[
\lim_{u,x \to \infty} \mathbb{E}_{u,x} \left( \int_0^\infty \int_{(-\infty,0)} \tilde{V}(U_s, X_s + y) \Pi(dy) \mathbb{I}_{\{X_s > b(U_s)\}} \right) = 0
\]

as claimed.

\[\square\]

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