Lift of dilogarithm to partition identities

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Abstract

For the whole set of dilogarithm identities found recently using the thermodynamic Bethe-Ansatz for the $ADET$ series of purely elastic scattering theories we give partition identities which involve characters of those conformal field theories which correspond to the UV-limits of the scattering theories. These partition identities in turn allow to derive the dilogarithm identities using modular invariance and a saddle point approximation. We conjecture on possible generalizations of this correspondance, namely, a lift from dilogarithm to partition identities.
1 Introduction

Recently there has been quite some interest in the dilogarithm function, which satisfies a huge set of not obviously related identities. In Nahm et al (1992) a method of Richmond and Szekeres (1981) to derive dilogarithm identities from partition identities was resuscitated. Furthermore, it was shown that the partitions in question can be interpreted as characters of 2d conformal field theory (CFT) and that the dilogarithm identities follow from equating the asymptotic growth of the sum side of the partition identity to the known growth of a character of a CFT given by the central charge.

The example considered in Nahm et al (1992) is the set of all $(2, \text{odd})$ Virasoro minimal models, which corresponds to the tadpole series $T_n$ (sometimes called $A_{2n}^{(2)}$) of thermodynamic Bethe-Ansatz (TBA) equations and dilogarithm identities in Klassen, Melzer (1990). Their $A_n$ series in turn is related to the $\mathbb{Z}_n$ parafermion conformal field theories and the corresponding partition identities for the parafermion characters can be found in Lepowsky, Primc (1985) (cf. the appendix for these two series). The cases missing in the $ADET$ classification of Klassen, Melzer (1990) of purely elastic scattering theories related to perturbations of CFT are the exceptional ones $E_6$, $E_7$ and $E_8$, which correspond to the $(6,7)$, $(4,5)$ and $(3,4)$ Virasoro minimal models, and the $D_n$ series, which is believed to correspond to certain $U(1)/\mathbb{Z}_2$ orbifold models. However, none of the partition identities in the mathematical literature known to us (cf. Andrews (1976), (1986), Bressoud (1980)) do the job, namely, equating the asymptotic growth of the partitions (which if possible can be interpreted as characters of a CFT with appropriate central charge) yields the dilogarithm identities of Klassen, Melzer (1990).

One aim of this article is to close this gap: we will write down in chapter 2 the partition identities which give, using the method of Richmond, Szekeres (1981), the $D_n^{(1)}$, $E_6$, $E_7$ and $E_8$ dilogarithm identities of Klassen, Melzer (1990). Indeed, as expected, the partitions can be interpreted as characters of certain conformal field theories.

More than this result, we would like to stress the way it was achieved: Starting from a dilogarithm identity with corresponding TBA equations, a sum side of a partition identity was conjectured. Using computer algebra the other side was identified with characters of a conformal field theory. In chapter 4 we present a conjecture on how this program might generalize.
In chapter 3, we briefly comment on the invariance of the structure of the partition identities under the tensor product of CFTs.

2 The new identities

Let $C_n^{-1}$ be the inverse of the Cartan matrix $C_n$ of finite $D_n$ ($n \geq 3$)

$$C_n^{-1} = \begin{pmatrix}
1 & \ldots & 1 & 1/2 & 1/2 \\
1 & 2 & \ldots & 2 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & \ldots & n-2 & (n-2)/2 \\
1/2 & 1 & \ldots & (n-2)/2 & n/4 \\
1/2 & 1 & \ldots & (n-2)/2 & n/4
\end{pmatrix}.$$

Then the following identity seems to hold

$$q^{-1/24} \sum_{m_1,\ldots,m_n \geq 0} \frac{mC_n^{-1}m^t e^{\pi iz(m_n-m_{n-1})}}{(q)_{m_1} \cdots (q)_{m_n}} = \frac{1}{\eta(\tau)} (\Theta_{0,n} + \Theta_{n,n})(z, \tau, 0), \quad (2.1)$$

where $(q)_n = (1-q) \ldots (1-q^n)$ and the Jacobi theta functions are given by

$$\Theta_{l,n}(z, \tau, u) = e^{-2\pi i ku} \sum_{m \in \mathbb{Z}+l/2n} q^{nm^2} e^{-2\pi iznm}.$$

A product representation of the partitions above can be found easily from the theta function using the Jacobi triple product identity

$$\prod_{n=1}^{\infty} (1-q^{2n}) (1 + q^{2n} q^{2n-1}; q^{1/2}) \prod_{m=1}^{\infty} \left[ (1-q^{2mn}) (1 + q^{(2m-1)n+l} e^{-2\pi iz}) (1 + q^{(2m-1)n-l} e^{2\pi iz}) \right].$$

We should remark that if $n$ is an integer multiple of 4, the right hand side of (2.1) can also be written as $\Theta_{0,n/4}(z, \tau)$. 

2
Using the method of Richmond, Szekers (1981) and modular invariance (cf. Nahm et al. (1992)) one rederives from (2.1) at \( z = 0 \) the following dilogarithm identity given by Klassen, Melzer (1990)

\[
\frac{1}{L(1)} \left[ 2L \left( \frac{1}{n-1} \right) + \sum_{a=2}^{n-1} L \left( \frac{1}{a^2} \right) \right] = 1 = c_{U(1)}.
\]

(2.2)

As expected, both, the dilogarithm identity (2.2) as well as (2.1) show the connection to the \( U(1) \) CFT with central charge \( c = 1 \).

The corresponding partition identities for the exceptional cases are given by replacing the inverse of the Cartan matrix of \( D_n \) on the left hand side of (2.1) by the one of \( E_6, E_7, \) and \( E_8 \), setting \( z = 0 \) and replacing the right hand side by \( \chi_{1,1} + \chi_{1,5} + 2\chi_{1,3}, \chi_{1,1} + \chi_{1,3} \) and \( \chi_{1,1} \), where \( \chi_{r,s} \) are the Virasoro characters (cf. Rocha-Caridi (1984), Goddard et al. (1986)) of the \( (p, p') = (6, 7), (4, 5) \) and \( (3, 4) \) unitary minimal models with central charge \( 6/7, 7/10 \) and \( 1/2 \) respectively

\[
\chi_{r,s}^{(p,p')}(q) = \frac{1}{\eta(q)} \left( \Theta_{rp-sp',pp'} - \Theta_{rp+sp',pp'} \right)(0, \tau, 0).
\]

For the corresponding dilogarithm identities we refer once again to the paper of Klassen, Melzer (1990).

3 Tensor products of CFT

It is remarkable that the structure of the identities of type (2.1) is invariant under taking the product of two such identities: One obtains an identity which involves the inverse Cartan matrices \( C_1^{-1}, C_2^{-1} \) of the single theories in the combination \( C_1^{-1} \oplus C_2^{-1} = (C_1 \oplus C_2)^{-1} \) on the left hand side whereas on the right hand side we find characters of the tensor product of the two CFTs (up to field identification).

This compatibility of the structure of the identities with the tensor product of CFTs gives us additional evidence for the conjecture in the following chapter.
4 Lift of dilogarithm identities

Up to now we did not discuss another essential structural element connecting the partition and dilogarithm identities: the Bethe-Ansatz equations, which appear in the method of Richmond, Szekeres (1981) as saddle point conditions and which have to be solved to obtain the arguments of the dilogarithm functions. Also, these were conjectured in Nahm et al (1992) to be closely related to the fusion algebras of the corresponding CFTs.

In the cases considered in chapter 2 and the appendix, they take the form

\[ f_i = \prod_j (1 - f_j)^{B_{ij}}, \]  

(4.1)

where \( B \) is two times the inverse of the Cartan matrix of any of the \( X_l \) diagrams with \( X \in \{A, D, E, T\} \) and \( l \) the rank. This can also be written in the suggestive form

\[ B = C(A_1) \otimes C^{-1}(X_l), \]

where \( C(A_1) = 2 \) is the Cartan matrix of \( A_1 = SU(2) \).

The suggestion made is that as a generalization one should replace the Cartan matrix of \( A_1 \) by the Cartan matrix of any Lie algebra. Indeed, as far as the Bethe-Ansatz equations and corresponding dilogarithm identities are concerned, this generalization at least for \( X = A \) and \( A_1 \) replaced by simply-laced algebras does exist (Kirillov (1987), Kuniba, Nakanishi (1992), Nahm et al (1992) and references therein). We conjecture:

Given any matrix \( B = C(Y) \otimes C^{-1}(X) \), where \( C(X) \), \( C(Y) \) are the Cartan matrices corresponding to the Dynkin diagram of \( X, Y \in \{A, D, E, T\} \), the dilogarithm identities corresponding to the Bethe-Ansatz equations (4.1) lift to partition identities of the form

\[ q^{-c_{eff}/24} \sum_{m_1, \ldots, m_n \geq 0} q^{mBm^t/2+b_m^t} \frac{1}{(q)_{m_1} \cdots (q)_{m_n}} = \chi_{b}^{(X,Y)}(q), \]

where \( b \) is a certain vector and \( \chi_{b}^{(X,Y)}(q) \) a linear combination with integer coefficients of the characters of the CFT of effective central charge \( c_{eff} = \frac{1}{L(1)} \sum_i L(f_i) \), where the \( f_i \) are the real solutions of (4.1).

A few comments are in order: In the case \( X_l = A_l \) we would expect (cf. Kuniba, Nakanishi (1992)) that the CFT mentioned in the conjecture is the parafermionic theory corresponding to the \( Y_l \) WZW theory.
Concerning an extension of the conjecture to non-simply-laced Lie algebras, we would expect that also the denominators of the terms in the sum on the left hand side have to be adjusted.

5 Conclusion and Outlook

We believe that the mathematical structure encoded in the partition identities conjectured above is the one given by Lepowsky and Wilson (1981, 1984, 1985). For further possible applications in mathematics physics we refer to the paper of Nahm et al (1992).

We hope that understanding better the conjecture will lead to a better understanding of the general dilogarithm identities of Kuniba, Nakanishi (1992) and to the creation of many new ones.

It is obvious that a first physical interpretation is bound to be connected with perturbed CFT – this being one recent occurrence of dilogarithm identities. It seems that the structure of CFT knows about its integrable perturbations in the sense that it allows its characters to be written in a nice simple form which suggests connections to integrable perturbations (compare the examples $E_8$ and $A_1$ with Zamolodchikov (1989)). However, the exact structural correspondance is unclear to us. We refer to recent work of Klassen, Melzer (1992), Kedem, McCoy (1992), Zamolodchikov (1991), Ravanini, Tateo, Valleriani (1992) and references therein.

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After this work was completed, it came to my attention that a preprint (Kedem, Klassen, McCoy, Melzer (1992)) has recently appeared in which similar results were obtained.
Appendix

The case $T_n (= A_{2n}/\mathbb{Z}_2)$ ($n \geq 1$):

The partition identities are special cases of the Andrews-Gordon identities, which in turn are certain generalizations of the well known Rogers-Ramanujan identities. The one interesting for our application (cf. Andrews (1976), Bressoud (1980)) is

$$q^{-c_{\text{eff}}/24} \sum_{m_1, \ldots, m_n \geq 0} \frac{q^{mC_n^{-1}m^t}}{(q)_{m_1} \cdots (q)_{m_n}} = \chi_{1,n}^{(2,2n+3)}(q), \quad (A.1)$$

where $C_n^{-1}$ is the inverse of the 'Cartan' matrix ($2-$ incidence matrix) of the tadpole graph $T_n$ and $c_{\text{eff}} = 2n/(2n+3)$ is the effective central charge of the $(2, 2n+3)$ Virasoro minimal model. A generalization of the above identity for all the characters of this CFT exists. It involves in particular a linear correction to the exponent of the numerator on the left hand side.

The corresponding dilogarithm identity is given by

$$\frac{1}{L(1)} \frac{1}{n+1} \sum_{j=2}^{n+1} L \left( \frac{\sin^2\left(\frac{\pi}{2n+3}\right)}{\sin^2\left(\frac{\pi j}{2n+3}\right)} \right) = c_{\text{eff}}^n. \quad (A.2)$$

The arguments of the dilogarithm can be identified with certain quantum dimensions of the CFT.

The case $A_n$ ($n \geq 1$):

Here, the partition identities can be found in Lepowsky, Primc (1985)

$$q^{-c_n/24} \sum_{m_1, \ldots, m_n \geq 0} \frac{q^{mC_n^{-1}m^t}}{(q)_{m_1} \cdots (q)_{m_n}} = (q)_{\infty} \sum_{m=0}^{n-1} c_m^0(\tau), \quad (A.3)$$

where $C_n^{-1}$ is the inverse of the Cartan matrix of $A_n$ and $c_m^l$ are the so-called string functions at level $n+1$ (Kac, Peterson (1984))

$$c_m^l(\tau) = \eta(\tau)^{-3} \sum_{(x,y) \in \mathbb{R}^2} \text{sign}(x) q^{(k+2)x^2-ky^2}, \quad (A.4)$$

for $|x| < |y| \leq |x|$ or $(1/2-x, 1/2+y) \in (l+1)/2(k+2), m/2k + \mathbb{Z}^2$.
(and \( c_m^l(\tau) = 0 \) if \( l \neq m \mod 2 \)) which are the characters of the \( \mathbb{Z}_{n+1} \) parafermionic theory with central charge \( c_n = \frac{2n}{n+3} \). A similar representation exists for all characters of the parafermionic theory.

The corresponding dilogarithm identity is given by

\[
\frac{1}{L(1)} \sum_{j=2}^{n+1} L \left( \frac{\sin^2\left( \frac{\pi}{n+2} \right)}{\sin^2\left( \frac{\pi j}{n+2} \right)} \right) = c_n. \tag{A.5}
\]

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