Boundary between Stable and Unstable Regimes of Accretion. Ordered and Chaotic Unstable Regimes

A. A. Blinova⋆, M. M. Romanova†, R.V.E. Lovelace‡
Dept. of Astronomy, Cornell University, Ithaca, NY 14853

9 January 2015

ABSTRACT

We search for the boundary between stable and Rayleigh-Taylor unstable regimes of accretion to magnetized stars in a new set of high grid resolution simulations. We found that the boundary between stable and unstable regimes is mainly determined by the ratio of the corotation radius $r_{\text{cor}}$ (where the Keplerian angular velocity in the disc matches the angular velocity of the star) to the magnetospheric radius $r_{\text{m}}$ (where the magnetic stress in the magnetosphere matches the matter stress in the disc). Instability is stronger when $r_{\text{cor}}$ is larger with respect to $r_{\text{m}}$, that is, when the gravitational force is larger than the centrifugal force at the inner disc. In the cases of a small tilt of the magnetosphere, $\Theta = 5^\circ$, and a small $\alpha$-parameter of viscosity, $\alpha = 0.02$, the boundary is located at $r_{\text{cor}} \approx 1.4r_{\text{m}}$. Instability becomes stronger at higher values of viscosity, and occurs at lower values of $r_{\text{cor}}/r_{\text{m}}$. At higher values of $\Theta$, the variability associated with instability decreases.

Simulations show two types of unstable accretion: chaotic and ordered. In the chaotic regime, several tongues form randomly and the light-curve has a few peaks per dynamical time-scale. In the ordered unstable regime, one or two tongues form and rotate with the frequency of the inner disc. The ordered unstable regime is found in the cases of relatively small magnetospheres, $r_{\text{m}} \lesssim 4.2R_\star$, and when $r_{\text{cor}} \gtrsim 1.7r_{\text{m}}$.

Results of our simulations are applicable to accreting magnetized stars with relatively small magnetospheres: Classical T Tauri stars (CTTSs), some accreting magnetized white dwarfs and neutron stars. The variability associated with the unstable regimes may explain the quasi-periodic oscillations (QPOs) in different types of stars, such as accreting millisecond pulsars. In observations of young stars, this QPO frequency may be mistaken for the period of the star.

Key words: accretion, accretion discs; MHD; stars: neutron; stars: magnetic fields

1 INTRODUCTION

Magnetospheric accretion occurs in different types of stars, including Classical T Tauri stars (CTTSs) (e.g., Bouvier et al. 2007), magnetized cataclysmic variables (e.g., Warner et al. 1995, Hellier 2001), and accreting millisecond pulsars (e.g., van der Klis 2006). The dynamically-important magnetic field stops the accretion disc at some distance from the star, $r_{\text{m}}$ (called the magnetospheric radius), and subsequently, the magnetic field governs the flow of matter onto the star (e.g., Ghosh & Lamb 1978).

The disc-magnetosphere boundary is prone to the Kelvin-Helmholtz and magnetic Rayleigh-Taylor (RT) instabilities (e.g., Chandrasekhar 1961, Arons & Lea 1976). It was suggested in earlier studies that the instabilities at the disc-magnetosphere boundary may only lead to the mixing of plasma with the field in the external layers of the magnetosphere (Arons & Lea 1976). However, our global 3D MHD simulations show that the disc matter can deeply penetrate the magnetosphere in tongues which produce chaotic hot spots on the stellar surface and irregular light-curves. Numerical simulations show that an ordered magnetic field is not an obstacle for the magnetic Rayleigh-Taylor instability (e.g., Rastatter & Schindler 1999, Stone & Gardiner 2007a,b). These simulations and earlier simulations by Wang & Roberts (1984, 1985) show that the Rayleigh-Taylor instability leads to the formation of small-scale waves and filaments that merge to form much larger filaments, which then deeply penetrate the magnetically-dominated, low-density regions.

Global three-dimensional simulations of accretion onto a
star with a dipole magnetic field show that matter may accrete in either the stable or the unstable regime. In the stable regime, matter is lifted above the magnetosphere and accretes in two ordered funnel streams, forming two ordered hot spots on the surface of the star (Romanova et al. 2003; Kulkarni & Romanova 2005). In the unstable regime, matter penetrates through the magnetosphere in several equatorial “tongues” due to the magnetic Rayleigh-Taylor instability and forms several chaotic spots on the surface of the star (Kulkarni & Romanova 2006; Romanova et al. 2008; Bachetti et al. 2010; Kurosawa & Romanova 2013). The light-curves associated with the hot spots are expected to be periodic in the stable regime, and chaotic in the unstable regime, with a typical time-scale of a few peaks per rotational period of the inner disc (see also Kulkarni & Romanova 2009).

In our earlier simulations, we found that the unstable regime is associated with enhanced accretion rate, and searched for a boundary between stable and unstable regimes of accretion by varying the accretion rate in the disc. In these simulations, accretion rate was varied by the \( \alpha \) parameter of viscosity (Shakura & Sunyaev 1973) (with a few test simulations where it was varied by changing the density in the disc). We found that the boundary depends on accretion rate (e.g., Romanova et al. 2008; Kulkarni & Romanova 2008), with accretion through instabilities occurring at relatively high \( \alpha \) values, \( \alpha \gtrsim 0.04 \).

It is important to establish the boundary between stable and unstable regimes of accretion at different parameters of the star and the disc. In the present research, we reconsider this problem in a new set of simulations at a wider range of parameters, including different sizes of the magnetosphere and different rotation rates of the star. We noticed that accretion through instabilities can also occur at a low viscosity, and focused our attention on the cases with low viscosity values by using a small \( \alpha \) parameter \( \alpha = 0.02 \) in most simulation runs. We also noticed that at a higher grid resolution, used in the most recent simulation runs, instability occurs more readily compared with the coarser grid. It is therefore important to reconsider the problem at a higher grid resolution and lower viscosity. We also re-investigate the dependence of instability on \( \alpha \) viscosity.

Another important phenomenon found in the new simulations is the frequent presence of what we call the \textit{ordered unstable regime}, where the unstable tongues merge to form one or two “ordered” tongues that rotate with the frequency of the inner disc. This regime of accretion has been observed in earlier simulations (performed at a lower grid resolution), but only in the cases of very small magnetospheres and high \( \alpha \) parameters (Romanova & Kulkarni 2009). In the current simulations (at a higher grid resolution), we observed that this phenomenon is more common and is present at a wider range of parameter values, including larger-sized magnetospheres and at lower viscosities in the disc. The regime of ordered unstable accretion is important because it may explain different quasi-periodic oscillations (hereafter, QPOs) observed in accreting magnetized stars, such as the QPOs in accreting millisecond pulsars (see review by van der Klis 2006) or the phenomenon of temporary “drifting” periods in CTTSs (Ricininski et al. 2008; Siwak et al. 2011). On the other hand, many CTTSs show irregular photometric variability (e.g., Alencar et al. 2010; Stauffer et al. 2014), which may be connected with the \textit{chaotic unstable regime} of accretion, where several tongues form randomly per rotational period of the inner disc. These tongues are transient and irregular compared with the instabilities of the \textit{ordered unstable regime}.

In this paper, we show the results of new simulations of unstable accretion performed at a higher grid resolution and a wider range of parameters, find new boundaries between stable and unstable regimes of accretion and investigate the ordered and chaotic unstable regimes.

In Sec. 2 we describe the numerical model used in our simulations. In Sec. 3 we search for the boundary between stable and unstable regimes of accretion. In Sec. 4 we describe the ordered and chaotic regimes of unstable accretion. In Sec. 5 we discuss the dependence of instability trends on different parameters. We discuss the applications of results to different types of stars in Sec. 6. We conclude in Sec. 7.

### 2 NUMERICAL MODEL

We perform global 3D MHD simulations of disc accretion onto a rotating magnetized star. The model we use is the same
Table 1. Sample (representative) models for different values of parameters \( \mu, r_{\text{cor}}, \alpha, \text{and } \Theta \). The period of the star \( P_* \) and the periods of instabilities \( P_{\text{inst}} \) were obtained in the Fourier spectral analysis.

| Model          | \( \mu \) | \( r_{\text{cor}} \) | \( \alpha \) | \( \Theta \) | \( P_* \) | \( P_{\text{inst}} \) | Comments                              |
|----------------|----------|---------------------|----------|----------|--------|----------------|---------------------------------------|
| \( \mu = 0.3; \Theta = 0.02 \) | 0.3      | 1.8                 | 0.02     | 5        | 2.4    | 1.5            | intermediate (unstable chaotic/stable) |
| \( \mu = 0.3; \Theta = 0.02 \) | 0.3      | 2.5                 | 0.02     | 5        | 3.9    | 1.9            | unstable ordered                      |
| \( \mu = 0.3; \Theta = 0.02 \) | 0.3      | 5                   | 0.02     | 10       | 11.2   | 2.2            | unstable ordered/chaotic              |
| \( \mu = 0.5; \Theta = 0.02 \) | 0.5      | 1.5                 | 0.02     | 5        | 1.8    | 0.9            | unstable chaotic                     |
| \( \mu = 0.5; \Theta = 0.02 \) | 0.5      | 2                   | 0.02     | 5        | 2.8    | 2.4            | unstable chaotic                     |
| \( \mu = 0.5; \Theta = 0.02 \) | 0.5      | 3                   | 0.02     | 5        | 5.2    | 2.3, 2.7       | unstable ordered/chaotic              |
| \( \mu = 0.5; \Theta = 0.02 \) | 0.5      | 5                   | 0.02     | 5        | 11.2   | 3              | unstable ordered                     |
| \( \mu = 0.5; \Theta = 0.02 \) | 0.5      | 3                   | 0.1      | 5        | 5.2    | 1.6            | unstable chaotic                     |
| \( \mu = 0.5; \Theta = 0.02 \) | 0.5      | 5                   | 0.1      | 5        | 11.2   | 2.6            | unstable ordered                     |
| \( \mu = 2; \Theta = 0.02 \)   | 1        | 1.8                 | 0.02     | 5        | 2.4    | 0.85           | intermediate (unstable chaotic/stable) |
| \( \mu = 2; \Theta = 0.02 \)   | 1        | 2.5                 | 0.02     | 5        | 3.9    | 3              | unstable chaotic                     |
| \( \mu = 2; \Theta = 0.02 \)   | 1        | 3                   | 0.02     | 5        | 5.2    | 1.9, 3.8       | unstable ordered                     |
| \( \mu = 2; \Theta = 0.02 \)   | 1        | 1.5                 | 0.1      | 5        | 1.8    | 0.9            | unstable chaotic                     |
| \( \mu = 2; \Theta = 0.02 \)   | 1        | 3                   | 0.1      | 5        | 5.2    | 2.7            | unstable chaotic                     |
| \( \mu = 2; \Theta = 0.02 \)   | 2        | 2                   | 0.02     | 5        | 5.2    | 2.8            | stable                               |
| \( \mu = 2; \Theta = 0.02 \)   | 2        | 3                   | 0.02     | 5        | 5.2    | 2.8            | unstable chaotic                     |

Figure 2. Example of two simulation runs with same parameters (\( \mu = 2, \Theta = 5^\circ, \alpha = 0.02 \) at time \( t = 20 \)) but different corotation radii (models \( \mu = 2\Theta/50 = 0.02 \) and \( \mu = 2\Theta/50 = 0.02 \)). Left panels: xy-slice (top) and xz-slice (bottom) show the density distribution in case of \( r_{\text{cor}} = 2 \), where accretion is stable. Right panels: accretion becomes unstable when \( r_{\text{cor}} = 3 \). Red lines show sample magnetic field lines. Positions of the magnetospheric \( r_m \) and corotation radii are shown with the dashed lines.

Figure 3. Radial distribution of different parameters in the disc and terms of the Spruit criterion in the vicinity of \( r_m \) at time \( t = 6 \) (in the beginning of the unstable regime) for models shown in Fig. 2, which differ only in values of corotation radius: \( r_{\text{cor}} = 2 \) and 3. Top panels show that accretion is stable for the smaller corotation radius \( r_{\text{cor}} = 2 \), where \( \gamma_{\text{inst}} < \gamma_{\text{spr}} \) (left panel), and unstable for \( r_{\text{cor}} = 3 \), where \( \gamma_{\text{inst}} > \gamma_{\text{spr}} \) (right panel). Other panels show distribution of effective gravity \( g_{\text{eff}} \), angular velocity \( \Omega \), surface density \( \Sigma \) and magnetic field \( B_z \).

as in our earlier 3D MHD simulations of stable and unstable regimes (e.g., Romanova et al. 2008, Kulkarni & Romanova 2008), which has been described in our earlier papers (e.g., Koldoba et al. 2002, Romanova et al. 2003). Hence, we will only describe it briefly here.

Initial conditions: A star has a dipole magnetic moment \( \mu \), the axis of which makes an angle \( \Theta \) with the star’s rotational axis \( \Omega \). The rotational axes of the star and the accretion disc are aligned. A star is surrounded by an accretion disc and a corona. The disc is cold and dense, while the corona is hot and rarefied, and at the reference point (the inner edge of the disc in the disc plane) the temperature and density are \( T_* = 100T_d \) and \( \rho_* = 0.01\rho_d \), where subscripts ‘d’ and ‘c’ denote the disc and the corona. Initially, the disc and corona are in
rotational hydrodynamic equilibrium (see e.g., Romanov et al. 2002 for details). The disc is relatively thin, with the half-thickness to radius ratio $h/r \approx 0.1$.

**Boundary conditions.** At both the inner and outer boundaries, most of the variables $A_j$ are taken to have free boundary conditions, $\partial A_j/\partial r = 0$. The free boundary conditions on the hydrodynamic variables at the stellar surface mean that accreting gas can cross the surface of the star without creating a disturbance in the flow. These conditions neglect the complex physics of interaction between the accreting gas and the star. The magnetic field is frozen onto the surface of the star. That is, the normal component of the field, $B_n$, is fixed. The other components of the magnetic field vary. At the outer boundary, matter is not permitted to flow into the region. The simulation region is usually large enough for the disc to have enough mass to sustain accretion flow for the duration of the simulation run.

**MHD equations and numerical method.** To model accretion, the full set of 3D MHD equations is solved numerically using a Godunov-type numerical scheme, written in a “cubed-sphere” coordinate system which rotates with the star (Koldoba et al. 2002). The numerical approach is similar to that described in Powell (1999), where the seven wave Roe-type approximate Riemann solver is used. The energy equation is written in the form of entropy balance, and the equation of state is that of an ideal gas. Compared with Koldoba et al. (2002), viscosity terms are incorporated into the equations. Viscosity is modelled using the $\alpha$-model (Shakura & Sunyaev 1973), and is incorporated only into the disc, so that it controls the accretion rate through the disc. In contrast with our earlier studies, we use a small $\alpha$-parameter $\alpha = 0.02$ in most of the simulation runs, and use a larger value $\alpha = 0.1$ in the test runs.

A “cubed sphere” grid is used in the simulations. The grid consists of $N_r$ concentric spheres, where each sphere represents an inflated cube. Each of the six sides of the inflated cube has an $N \times N$ curvilinear grids which represent a projection of the Cartesian grid onto the sphere. The entire grid consists of $6 \times N_r \times N^2$ cells. The typical grid used in our simulations has $N_r = 140$ cells in the radial direction, and $N^2 = 61^2$ angular cells in each block. This grid is twice as fine as the $N^2 = 31^2$ grid used in our earlier studies of instabilities (e.g., Kulkarni & Romanova 2008).

**Dimensionalization.** Equations are written using dimensionless variables. The dimensionless value of every physical quantity $A_j$ is defined as $A_j = A_j/A_{j0}$, where $A_{j0}$ is the reference value for $A_j$. Appendix A shows how the reference values are determined, and lists the reference values for three classes of stars: classical T Tauri stars, white dwarfs and neutron stars. Subsequently, we drop the tildes above the dimensionless variables and show dimensionless values everywhere unless otherwise specified.

## 3 Boundary between Stable and Unstable Regimes of Accretion

One of the main goals of present research was to find a set of parameters which would determine the boundary between stable and unstable regimes of accretion.

According to theoretical studies (e.g., Chandrasekhar 1961), the occurrence of instability requires the presence of gravitational acceleration. In the case of a rotating disc, the inner disc can be RT-unstable if the effective gravity $g_{\text{eff}} = g + g_c$ is negative, where $g = -GM_*/r^2$ and $g_c = v^2_\text{circ}/r$ are gravitational and centrifugal acceleration, respectively.

In an ideal Keplerian disc, the gravitational and centrifugal forces balance each other and $g_{\text{eff}} = 0$. However, in the case of accretion to a magnetized star, the inner disc interacts with the magnetic field of the star and slows down due to the loss of angular momentum to the magnetic field lines, if the star rotates more slowly than the inner disc. Alternatively, the inner disc may rotate with super-Keplerian velocity if the star rotates more rapidly than the inner disc. In the first case, the effective gravity is negative and the unstable regime becomes probable. In other words, the condition of $g_{\text{eff}} < 0$ can be translated to the magnetospheric (truncation) radius of the disc, $r_m$, being smaller than the corotation radius $r_{\text{cor}}$, where by definition the Keplerian velocity of the disc matches the angular velocity of the star, $r_{\text{cor}} = [GM_*(P_*/2\pi)^2]^{1/3}$, where $P_*$ is the period of the star. It is now evident that the boundary between stable and unstable regimes of accretion may strongly depend on the relationship between $r_{\text{cor}}$ and $r_m$.

In a realistic situation, the interaction between the magnetosphere and the disc is more complex, and some factors act to oppose instability, such as the shear of angular velocity in the disc (e.g., Spruit et al. 1995). Nevertheless, we suggest that the main factor may be the sign and value of $g_{\text{eff}}$ and therefore the relationship between $r_{\text{cor}}$ and $r_m$, and we base our search for a new boundary on this assumption.

### 3.1 Set of performed simulations

To find the boundary between stable and unstable regimes of accretion, we performed a series of simulation runs, varying two different parameters. We kept the same initial con-
Boundary between Stable and Unstable

Figure 5. Left panel: 3D view of matter flow in a case where chaotic accretion in multiple tongues dominates, model $\mu_1c2.5\Theta=0.02$, at time $t=19$. One of the density levels is shown in color, selected magnetic field lines are shown in red. Middle panel: Same but in the face-on projection. Right panel: An equatorial slice of density distribution is shown in color.

Figure 6. An example of accretion in the chaotic unstable regime in the model $\mu_1c2.5\Theta=0.02$ (times $t=19−21$), where multiple tongues form. Top panels show consecutive 3D views of matter flow, where the color background represents one of the density levels and the lines are sample magnetic field lines. The axes show the directions for the angular momentum and magnetic momentum of the star. Two middle panels show consecutive $xy$ and $xz$ slices. The color background shows density distribution and the lines show where the kinetic plasma parameter $\beta_1 = 1$. Bottom panels show the light-curve from rotating spots calculated at an inclination angle $i = 45^\circ$ (left), Fourier transform (middle) and wavelet transform (right) obtained from analysis of the light-curve.

ditions for the density and pressure distribution in the disc and corona, but varied the magnetic moment $\mu$ of the star and the corotation radius $r_{\text{cor}}$ (which determines the period of the star: $P_*=2\pi r_{\text{cor}}^3/\sqrt{GM_\star}$). We used a small parameter of viscosity, $\alpha = 0.02$, in the main simulation runs to be sure that viscosity is not an essential factor in generating instability.

We ran two main sets of simulations for two different misalignment angles of the dipole moment: one at a relatively small angle, $\Theta = 5^\circ$, where instability is expected to be stronger, and the other at a larger angle, $\Theta = 20^\circ$, where instability is expected to be weaker and funnel stream accretion becomes more favorable (e.g., Kulkarni & Romanova 2008).

For each angle, we performed a series of simulations for stars with different magnetic moments $\mu$ in the interval from $\mu = 0.1$ (small-sized magnetospheres) to $\mu = 3$ (large-sized magnetospheres). We should note that in our dimensionless values, parameter $\mu$ determines the dimensionless size of the magnetosphere. We also varied the corotation radius in the range of $r_{\text{cor}} = 1.2−5$ (in dimensionless units), corresponding to $r_{\text{cor}} = (3.4−14.3)R_\star$ (in our dimensionless units, $R_\star = 0.35$, see Appendix A). Table 1 shows sample models used for finding the boundary. Many of these models are chosen for the table because they were used for detailed analysis. The total number of runs used for finding the boundary is much larger. The names of the models in the Table incorporate the parameters used in those models.

To check the main hypothesis (dependence of the bound-
ary on $r_{\text{cor}}$ and $r_{m}$, we calculated $r_{m}$ in each simulation run (see below) and marked whether accretion was stable or unstable at a given value of $r_{\text{cor}}$ (see Fig. 1). To find the truncation (magnetospheric) radius $r_{m}$ of the disc, we calculated the kinetic plasma parameter $\beta = (\rho v^{2} + p)/(B^{2}/8\pi) = 1$, using the values from simulations. Here, velocity $v$ is taken in a coordinate system rotating with the magnetosphere. This condition provides the radius where the inner disc density drops sharply and the magnetic pressure becomes dominant. The $\beta = 1$ line is not a smooth line: it reflects the inner parts of the disc as well as the matter-dominated unstable tongues. We determined $r_{m}$ using only the inner disc radius and ignoring the tongues.

The accretion rate and $r_{m}$ generally vary in time. In some cases, accretion is either stable or unstable throughout the entire simulation run, while in other cases it is only marginally stable, and can transition from stable to unstable and back to stable again. In each case, we measured $r_{m}$ from the simulations (using the position of $\beta = 1$ line) for a few moments in time, and marked whether the case is stable or unstable. All of these points were used for finding the boundary between stable and unstable regimes of accretion.

Fig. 1 (left panel) shows a diagram of several boundaries in the $r_{m} - r_{\text{cor}}$ parameter space obtained from multiple simulation runs for a small misalignment angle of the dipole, $\Theta = 5^\circ$. We can draw an approximate boundary between stable (red squares) and unstable (blue triangles and green x's) regimes of accretion. The plot shows that accretion is stable above this line, and is unstable otherwise. In this figure, the values of $r_{m}$ and $r_{\text{cor}}$ are given in radii of the star for user convenience. The best-fit boundary corresponds to the relationship $r_{\text{cor}}/R_{s} = 1.4(r_{m}/R_{s})^{0.9}$ (thin dashed line). However, for convenience, we suggest using $r_{\text{cor}} = 1.4r_{m}$ as the main boundary line (bold line in left panel of Fig. 1), which closely approximates the empirical boundary line. Moreover, both lines are likely to be within the error bars of our measurements, which are mainly connected with some uncertainty in our method of measuring $r_{m}$. As expected, instability occurs more easily when $r_{\text{cor}}$ is larger and $r_{m}$ is smaller, that is, when the ratio $r_{\text{cor}}/r_{m}$ is larger.

At a sufficiently large ratio of $r_{\text{cor}}/r_{m}$, another transition occurs and matter starts accreting in the ordered unstable regime. The boundary between chaotic and ordered unstable regimes approximately corresponds to the condition of $r_{\text{cor}} \approx 1.7r_{m}$, if the magnetosphere is not very large, $r_{m} \lesssim 4.2R_{s}$. At larger magnetospheres, $4.2R_{s} \lesssim r_{m} \lesssim 7R_{s}$, accretion has a tendency to be chaotic (see left panel of Fig. 1).

We should note that at sufficiently large magnetospheres, $r_{m} \gtrsim 10R_{s}$, chaotic instabilities only develop in the external regions of the magnetosphere, thus providing a mixing of plasma with the magnetic field of the external magnetosphere (as suggested by Arons & Lea 1976). In sample simulations with large-sized magnetospheres, $r_{m} \gtrsim 10R_{s}$, accretion through two funnel streams was observed. The corotation radius was larger than the magnetospheric radius, so the conditions for instability were favorable. In spite of this, the unstable tongues were present only at the disc-magnetosphere boundary and no unstable tongues reached the stellar surface (Romanova et al. 2014).

We should also note that when $r_{m} > r_{\text{cor}}$, the magnetosphere rotates more rapidly than the inner disc, and the propeller mechanism is expected. We observed from the simulations that when $r_{m} \approx r_{\text{cor}}$, the magnetosphere pushes the disc outward through the propeller mechanism. The parameter values under which the propeller mechanism is expected to dominate should be studied separately. In the present study, we draw the $r_{m} = r_{\text{cor}}$ line as a reminder that the propeller regime is expected near or above this line.

For the larger misalignment angle, $\Theta = 20^\circ$, instability also occurs, but at slightly larger values, $r_{\text{cor}} \gtrsim 1.5r_{m}$ (bold line in the right panel of Fig. 1). This is due to the fact that at larger misalignment angles the magnetic poles are closer to the accretion disc plane, which makes funnel stream formation more favorable than the RT-unstable tongues. The dash-dot-dot line shows the boundary corresponding to the case of $\theta = 5^\circ$. One can see that the boundaries differ, but not by much.

---

1. In all other plots, we use the units from our code, with $R_{s} = 0.35$ in dimensionless units.
2. The inner edge often has an elliptical or irregular shape, disturbed by the chaotic and ordered tongues, and therefore there is some uncertainty in choosing the value of $r_{m}$. 

---

**Figure 7.** Consecutive xy-slices and frequency analyses similar to those in Fig. 6 but for the case of larger viscosity, $\alpha = 0.1$ (model $\mu 1\leq 3\delta 0.1$, times $t = 16 - 18$).
3.2 Analysis of instability using theoretical stability criterion

In the simple case of a high-density fluid supported against gravity by a low-density fluid with a homogeneous magnetic field parallel to a plane boundary between them, the development of the Rayleigh-Taylor instability in the direction perpendicular to the field is not affected by the field — all perturbation modes are unstable (Chandrasekhar 1961). This suggests that for a star with a dipole field, azimuthal perturbations at the inner disc boundary should always be unstable. However, there are a few factors which act to suppress instability.

An important factor is the effect of the radial shear of angular velocity, $2 \left( \frac{d \Omega}{dr} \right)^2$, which can suppress instability by smearing out the perturbations. Spruit et al. (1995) have performed a general analysis of disc stability in the thin disc approximation, taking the velocity shear into account (see also earlier work by Kaisig et al. 1992). The disc has a surface density $\Sigma$ and is threaded by a magnetic field with the vertical component $B_z$. Their analytical criterion for the development of instability is:

$$\gamma^2 dB_z \equiv \left( -g_{eff} \right) \left| \frac{d}{dr} \ln \frac{\Sigma}{B_z} \right| > 2 \left( \frac{d \Omega}{dr} \right)^2 \equiv \gamma_0^2. \quad (1)$$

One can see that the sign and value of the effective gravitational acceleration $g_{eff}$ are important in this criterion. Namely, the disc-magnetosphere boundary is unstable if $g_{eff}$ is negative, that is, when the effective acceleration is directed towards the star. The term $\left| \frac{d}{dr} \ln \frac{\Sigma}{B_z} \right|$ characterizes the level of compression of the surface density and magnetic field in the disc. Instability occurs if the product of $(-g_{eff})$ and $\left| \frac{d}{dr} \ln \frac{\Sigma}{B_z} \right|$ is large enough to overcome the stabilizing effect of the velocity shear, $\gamma_0^2$. Our simulations show that this criterion, developed for a disc, also works well for the disc-magnetosphere boundary (e.g., Kulkarni & Romanova 2008).

To demonstrate the role of effective gravity $g_{eff}$, we compared two cases with the same magnetic moment, $\mu = 2$, but two different corotation radii, $r_{cor} = 2$ and $r_{cor} = 3$, which
correspond to two different stellar periods $P_*$ and different centrifugal accelerations $g_c$ in the formula $g_{cff} = g + g_c$. We found that for the same moment in time ($t = 11.5$) the case with the larger stellar period ($r_{cor} = 3$) is unstable, while the other case is stable (see Fig. 2). This is also consistent with our prediction that the ratio $r_{cor}/r_m$ is an important factor in determining the mode of accretion. This is just one example of two simulations at the same parameter values (model $\mu 0.3e2.5\Theta 5a0.02$, times $t = 30–32$).

We calculated the different terms of the Spruit criterion in the vicinity of $r_m$ (where the unstable perturbations can occur) for the stable and unstable cases shown in Fig. 2. To calculate these terms, we took the azimuthally-averaged values of relevant variables and plotted them as a function of radius in Fig. 3. Top panels show that in the stable regime (left panel, $r_{cor} = 2$), $\gamma_{\Omega} > -\gamma_B$; at the magnetospheric radius (vertical dashed line), while in the unstable regime (right panel, $r_{cor} = 3$), $\gamma_B > \gamma_{\Omega}$. In both cases, results of the simulations are consistent with the analytical prediction (Eq. 1). Middle panels show that in the unstable regime the effective gravity term $g_{eff}$ is about $0.15$, while in the stable regime it is slightly positive, $g_{eff} \approx 0.06$. The distribution of $\Omega$ is steeper in the stable regime, which yields a larger shear term $\gamma_{\Omega}$ in the top left panel of Fig. 3. In both cases, surface density $\Sigma$ decreases towards the star, but does so more rapidly in the unstable case (see bottom panels of Fig. 3). The magnetic field $B_z$ decreases with radius somewhat more rapidly in the unstable case than in the stable case. As a result, $\ln \frac{\Sigma}{\Sigma}$ increases faster with radius in the unstable case, and the compression factor $\left| \frac{d}{dr} \ln \frac{\Sigma}{\Sigma} \right|$ is somewhat larger. Therefore, the term characterizing the instability, $\gamma_B$, is larger in the unstable regime due to both a larger effective gravity term $-g_{eff}$ and a larger compression factor $\frac{d}{dr} \ln \frac{\Sigma}{\Sigma}$. The majority of our cases for both misalignment angles are consistent with the Spruit criterion.

We noticed that a coarse grid resolution is another factor that tends to suppress instability. Fig. 4 shows an example of two simulations at the same parameter values (model $\mu 0.5e1.5\Theta 5a0.02$ which is close to the border between stable and unstable regimes) but different angular grid resolutions. We observed that the case is stable at a coarser grid (left panels) and unstable at a finer grid (right panels). This model, taken at a coarser grid resolution, should become unstable at larger corotation radii. The mechanism of formation of the instabilities should be similar in the cases of finer and coarser grids. However, we observed that at a coarser grid the unstable regime occurs at a larger ratio $r_{cor}/r_m$. We suggest that at a coarser grid resolution numerical viscosity opposes the formation of unstable tongues.

4 TWO TYPES OF UNSTABLE REGIME: CHAOTIC AND ORDERED

Here, we describe the two types of unstable regime in greater detail, as observed in the simulations: (1) the chaotic unstable regime, where matter accretes in several chaotic, transient tongues, and (2) the ordered unstable regime, where one or two tongues form and rotate persistently with the frequency of the inner disc. Below, we discuss these two types of unstable regime for cases of a small tilt of the dipole, $\Theta = 5^\circ$.

4.1 Chaotic unstable regime

To demonstrate the chaotic unstable regime, we use the model $\mu 1e2.5\Theta 5a0.02$, where the magnetosphere is relatively large ($\mu = 1$) and the corotation radius is $r_{cor} = 2.5$, which corresponds to the period $P_* \approx 3.9$ in dimensionless units. We observed that matter accretes in several tongues, which form frequently and rapidly disappear. Fig. 5 (left two panels) shows a typical picture of matter flow through the chaotic unstable tongues, which are tall and narrow, and which penetrate the field lines by pushing them aside (right panel).

Fig. 5 shows the temporal evolution of tongues in several consecutive slices of density distribution, separated by the time interval of $\Delta t = 0.5$. Top two rows of panels show that the tongues are constantly modified, breaking or coalescing on a timescale of $\Delta t \approx 0.5 – 1$. The $xz-$ and 3D-panels

\footnote{Note that in our units $\Delta t = 1$ corresponds to the period of Keplerian rotation at $r = 1$. However, the inner disc usually rotates with sub-Keplerian velocity, because the disc interacts with the slowly rotating magnetosphere}
show accretion through chaotic instabilities, with the occasional formation of funnel streams.

The matter from unstable tongues accelerates due to gravity, falls onto the star and releases kinetic energy at the surface of the star, forming hot spots. We use a simple model for the radiation of the spots, suggesting that all kinetic energy of the falling matter is converted into radiation, which is distributed isotropically (see details in Romanova et al. 2004). We calculate the light-curve as seen by a remote observer located at an angle of $i = 45^\circ$ with respect to the rotational axis of the star (see bottom left panel of Fig. 6). The light-curve shows irregular variations on a time-scale $\approx 1$ times shorter than the period of the star, $P_\star = 3.9$. The bottom middle and right panels show the wavelet and the Fourier spectra of this light-curve. The wavelet shows different angular frequencies. One of them corresponds to the frequency of the star, $f_s = 0.26$ (corresponding to the period of $P_\star = 3.9$). Other frequencies are higher, with $f_{\text{inst}} \approx 0.3, 0.5$. The Fourier spectrum shows that there are several peaks in the interval of $P_{\text{inst}} = 2 - 3.2$, which correspond to the wavelet frequencies. These periods also correspond to the time-scale of variability observed in the light-curve. We suggest that these periods reflect the frequency of formation of the strongest tongues that reach the surface of the star.

Both the wavelet and the Fourier analysis show the presence of the stellar period, which can be associated with the fact that the entire set of unstable temporary spots is oriented around the magnetic pole. The magnetic pole is slightly tilted about the rotational axis (at $\Theta = 5^\circ$ in this case) and the rotation of the star modulates the light from the set of chaotic spots. This is a probable reason for why the stellar period is also observed. Another possible reason is that some matter accretes to the star in funnel streams above the magnetosphere and forms hot spots which tend to rotate with the star. In the present study, we chose an inclination angle of $i = 45^\circ$ for the analysis of instabilities. If this angle were smaller, then the stellar period would have had a smaller amplitude.

In another example of the chaotic unstable regime, we increase the $\alpha$-parameter of viscosity in the disc from $\alpha = 0.02$ to $\alpha = 0.1$, while keeping the other parameter values the same as in the first example, with the exception of a slight increase in the corotation radius from $r_{\text{cor}} = 2.5$ to $r_{\text{cor}} = 3$ (corresponding to the stellar period of $P_\star = 5.2$). The model is $\mu_1c3\Theta5\alpha0.1$. In this case, similar chaotic unstable accretion was observed (see Fig. 7 top panels). The light-curve from the hot spots shows similar but stronger irregular variability, with different time intervals between the peaks, including short time-scale intervals, with $\Delta t < 1$. The Fourier spectrum shows several peaks, with the strongest signal corresponding to $P_{\text{inst}} \approx 2.4$. The corresponding frequency $f_{\text{inst}} \approx 0.42$ is similar to the frequencies of enhanced amplitude in the wavelet. The wavelet spectrum also shows higher frequencies which probably correspond to the lower-period flares observed in the light-curve. The higher-frequency vertical “flares” in the wavelet probably correspond to the moments when the strongest tongues deposit matter onto the star and briefly rotate with the local Keplerian velocity. The period of the star is observed during times $t = 30 - 50$. Overall, instability is stronger in this case than in the case with lower viscosity, $\mu_1c2.5\Theta5\alpha0.02$. Analysis in Sec. 5.2.2 shows that at a larger viscosity, the higher radial velocity and compression of the magnetosphere by the inner disc matter are important factors that lead to a regime with stronger instabilities.

### 4.2 Ordered unstable regime: one- and two-tongue accretion

When the ratio $r_{\text{cor}}/r_m$ becomes sufficiently large, unstable accretion becomes ordered: multiple tongues merge, forming one or two ordered tongues. In some cases, two tongues carry a comparable amount of matter flux, while in other cases one of the two tongues may carry more mass. In all cases, the bases of the tongues rotate with the frequency of the inner disc.

Such a regular rotation of two tongues has been observed in earlier studies (Romanova & Kulkarni 2009), where simul-
lations were performed at the higher viscosity, \( \alpha = 0.1 \), and twice as coarse a grid. Only small-sized magnetospheres were considered. In this work, we can see that such ordered unstable accretion can occur at a wider range of parameter values, including larger-sized magnetospheres and smaller \( \alpha \)'s.

Fig. 8 shows a snapshot of unstable accretion in two tongues, model \( \mu 0.5c3\Theta 5c0.02 \). One can see that two ordered matter-dominated tongues push the magnetic field lines apart and penetrate into the deep layers of the magnetosphere, where they transition into regular funnel streams, but very close to the star. The top two panels of Fig. 9 show that the unstable tongues persist for a significant period of time, longer than in the chaotic regime of accretion. The figure also shows that initially there was only one tongue, and the second one formed later. One and two-tongue accretion modes are often observed in the same simulation run. Slices in the \( xz \)-projection show that only low-density matter accretes to the star through funnel streams.

The tongues deposit matter onto the stellar surface and form one or two hot spots that rotate more rapidly than the star. The light-curve shows regular peaks that reflect the frequency of rotation of these spots (not the frequency of stellar rotation). The Fourier analysis shows a large peak associated with the rotating spots, with a period of \( P_{\text{inst}} \approx 2 - 3 \). The period of the star, \( P_s = 5.2 \), is also visible, but with a smaller amplitude. The wavelet analysis shows frequencies in the range of \( f_{\text{inst}} \approx 0.3 - 0.6 \), which correspond to the peaks observed in the Fourier spectrum.

We show another example of the ordered unstable regime using the model \( \mu 0.3c2.5\Theta 5c0.02 \), with a somewhat smaller magnetosphere (\( \mu = 0.3 \)) and a slightly smaller period of rotation, \( P_s = 3.9 \). In this case, we observed persistent two-tongue accretion, though one of the tongues usually carried more matter than the other (see Fig. 10 top panels). The light-curve from the spots shows the frequent oscillations associated with the rotation of ordered spots along the surface of the star. Fourier analysis shows the period \( P_{\text{inst}} \approx 2 \) associated with these spots, which is lower than the period of the star. The corresponding frequency \( f_{\text{inst}} \approx 0.5 \) appears in the wavelet as the lowest frequency associated with unstable accretion. This frequency also corresponds to the frequency of the inner disc, which determines the rotation rate of the tongues. There are also strong, high-frequency “flares” in the wavelet associated with the propagation of the strongest tongues, which rotate with an increasingly higher angular velocity as they approach the star (to conserve angular momentum). These tongues form spots that rotate rapidly during a brief period of time, then slow down due to the slowly-rotating magnetosphere, and the rotation of the spots transmits through a wide range of high frequencies, forming nearly vertical features, or “flares”, in the wavelet.

In both models, the main period observed in the light-curve is associated with the rotation of the ordered unstable tongues and approximately corresponds to the period of inner disc rotation. The ordered unstable regime may be an important mechanism in the formation of quasi-periodic oscillations during episodes of enhanced accretion, when the magnetosphere is compressed and the magnetospheric radius is smaller than the corotation radius.

4.3 The boundary between chaotic and ordered unstable regimes

To find the boundary between the chaotic and ordered unstable regimes, we marked the cases in which one or two unstable tongues carry more mass than the other tongues and labelled these models as green 's in Fig. 1. The approximate position of this boundary corresponds to the line \( r_{\text{cor}} \approx 1.7r_m \) in the cases of relatively small magnetospheres, \( r_m \lesssim 4.2R_\star \). Ordered unstable accretion in one or two tongues dominates below this line. In the cases of larger-sized magnetospheres (\( r_m \gtrsim 4.2R_\star \)), only the chaotic unstable regime has been observed.

Simulations show that near the \( r_{\text{cor}} \approx 1.7r_m \) line accretion is mainly chaotic, although at some point in a simulation run one or two tongues may start to carry more matter than the other tongues. Accretion may also alternate between ordered and chaotic within the same simulation run. At larger values of \( r_{\text{cor}}/r_m \), accretion becomes systematically more ordered, and the frequency associated with the ordered tongues becomes stronger. This means that at larger values of the \( r_{\text{cor}}/r_m \) ratio the quality factor \( Q = f/\Delta f \) of the quasi-periodic oscillations associated with the ordered unstable tongues is higher. Such an increase in the quality factor is expected during accretion outbursts, when the disc moves inward towards the star.

The diagram in Fig. 11 summarizes the boundaries between the stable and unstable regimes, and between the chaotic and ordered unstable regimes.

5 INSTABILITY AT DIFFERENT PARAMETERS

In this section we show sample simulations and analyses of cases in which a wider range of parameters was varied, including higher tilts of the dipole \( \Theta \), different corotation radii \( r_{\text{cor}} \), and a higher \( \alpha \)-parameter of viscosity. We also compare relativistic and non-relativistic cases.

5.1 Instability in cases of higher \( \Theta \)

We performed test simulation runs at different tilts of the dipole: \( \Theta = 10^\circ, 15^\circ, 20^\circ, 30^\circ, 40^\circ \) and \( 60^\circ \). We focused on the ordered unstable regime and therefore used relatively small magnetospheres, \( \mu = 0.3 - 0.5 \) and large values of \( r_{\text{cor}} \). One of the questions was whether this type of unstable accretion will take place in the cases of larger tilts of the dipole.

First, we performed simulation runs for relatively small tilts, \( \Theta = 10^\circ, 15^\circ, 20^\circ, 30^\circ \). Fig. 12 shows \( xy \)-slices and frequency analysis of the hot spots for all of these cases, where the top to bottom panels correspond to the smallest to largest angles \( \Theta \), respectively.

The \( xy \)-slices of density distribution show ordered unstable accretion in one or two tongues in the cases of the two smallest tilts, \( \Theta = 10^\circ \) and \( 15^\circ \) (top two rows of panels of \( xy \)-slices). However, at larger misalignment angles, \( \Theta = 20^\circ \) and \( 30^\circ \), accretion becomes somewhat more chaotic (bottom two rows of panels of \( xy \)-slices).

The light-curve for the \( \Theta = 10^\circ \) case (second row from the top in Fig. 12) shows that the main source of variability is the rotation of unstable ordered spots. However, modulation by stellar rotation is also observed. The Fourier spectrum
Figure 12. Top two panels: consecutive xy-slices (times $t = 37 - 39$) and analysis of variability for a larger misalignment angle, $\Theta = 10^\circ$, model $\mu_0.35\Theta10\alpha0.02$. 3d and 4th panels: same, but for $\Theta = 15^\circ$, model $\mu_0.55\Theta15\alpha0.02$ (times $t = 86.5 - 88.5$). 5th and 6th panels: same, but for $\Theta = 20^\circ$, model $\mu_0.55\Theta20\alpha0.02$ (times $t = 30.75 - 32.75$). Bottom two panels: same, but for $\Theta = 30^\circ$, model $\mu_0.35\Theta30\alpha0.02$ (times $t = 32 - 34$).
shows a peak with a period of $P_{\text{inst}} \approx 2$, associated with the instability, and a peak associated with the period of the star, $P_{\ast} \approx 11.2$. The wavelet spectrum mainly shows the frequency $f_{\text{inst}} \approx 0.5$, associated with the instability. The light-curve was calculated for an observer’s inclination angle of $i = 45^\circ$, when the modulation by stellar rotation is expected to be the strongest.

The light-curve for the $\Theta = 15^\circ$ case (4th row from the top) shows the short time-scale variability associated with unstable accretion. However, the amplitude of variability associated with the rotation of the star is larger. The Fourier spectrum shows the period of instability, $P_{\text{inst}} \approx 3$, and the period of the star. The wavelet shows a strong signal for both the instability and the star.

In the cases of $\Theta = 20^\circ$ and $30^\circ$ the amplitude of the oscillations associated with instability is even smaller, and the light-curve is mainly determined by the rotation of the star, while the instabilities provide a high-frequency modulation of the main light-curve. The presence of instability is weak in the Fourier spectrum, but clearly visible in the wavelet spectrum.

In the cases of even larger tilts of the dipole, $\Theta = 40^\circ$ and $60^\circ$, the light-curve is still modulated by the unstable component of accretion, but the amplitude of the modulation is smaller than in the cases of $\Theta = 20^\circ$ and $30^\circ$.

5.2 Analysis of instability for different $r_{\text{cor}}$ and $\alpha$

In this section, we are interested in understanding the dependence of accretion mode on the corotation radius $r_{\text{cor}}$ and the $\alpha-$parameter of viscosity. Here, we use a different approach to investigate the boundary between stable and unstable regimes of accretion. We take one of the models at the boundary between stable and unstable regimes and recalculate it at different parameter values of $r_{\text{cor}}$ and $\alpha$. To understand the physics of transition between the stable and unstable regimes, we calculate for each model the different terms of the Spruit criterion (Eq. 1), with values taken at the disc-magnetosphere boundary, as described in Sec. 3.2.

5.2.1 Dependence of instability on $r_{\text{cor}}$

To understand the physics of transition between the stable and unstable regimes and its dependence on the $r_{\text{cor}}$ parameter, we take the model $\mu_1 c_1.5 \Theta \delta = 0.02$ and recalculate it at different corotation radii $r_{\text{cor}}$ while keeping all the other parameter values fixed ($\mu = 1$, $\Theta = 5^\circ$, $\alpha = 0.02$). We then calculate the main terms from Eq. 1, $\gamma_1^2$, and $\gamma_0^2$, which determine whether the model is stable or unstable (according to the analytical criterion and confirmed by our simulations). The term $\gamma_1^2$ represents the multiplication of two factors: the compression factor, $\frac{1}{\pi^2} \ln \frac{\Sigma}{\pi^2}$, and the effective gravity, $g_{\text{eff}}$.

To understand the physics of instability, we calculate these terms separately.

Fig. 13 (left panel) shows that effective gravity $g_{\text{eff}}$ decreases with $r_{\text{cor}}$ and then levels off at some negative value. It levels off because at large corotation radii the angular velocity is small and the centrifugal acceleration term in $g_{\text{eff}}$ becomes negligibly small compared with the gravitational acceleration. One can also see that the compression term $\frac{1}{\pi^2} \ln \frac{\Sigma}{\pi^2}$ is large, but does not vary systematically, and therefore the variation of the instability term $\gamma_1^2$ is mainly determined by $g_{\text{eff}}$. We conclude that the transition from the stable to the unstable regime occurs due to the variation in effective gravity. The transition occurs at $r_{\text{cor}} \approx 1.6$, when $\gamma_1^2 = \gamma_0^2$.

We should note that the point of transition, $r_{\text{cor}} \approx 1.6$, is close to the value of $r_{\text{cor}} \approx 1.5$, where $g_{\text{eff}}$ changes the sign from positive to negative. This means that accretion tends to be unstable whenever the effective gravity is negative at the disc-magnetosphere boundary, and the stabilizing shearing factor, $\gamma_0^2$, is not important in this set of simulations.

5.2.2 Dependence of instability on $\alpha$

In another set of experiments, we take the same model, $\mu_1 c_1.5 \Theta \delta = 0.02$, which is at the boundary between stable and unstable regimes, and vary parameter $\alpha$ in the range of $\alpha = 0.02 - 0.1$. We observed that this initially stable case becomes increasingly more unstable with increasing $\alpha$. We are interested in knowing why this transition happens and why the instability becomes stronger at higher values of $\alpha$.

Fig. 13 (right panel) shows results of our simulations. One can see that at larger $\alpha$, the regime switches from stable to unstable. The transition (where $\gamma_1^2 = \gamma_0^2$) occurs at $\alpha \approx 0.037$. At larger values of $\alpha$, the disc-magnetosphere boundary becomes more unstable due to both the decreasing effective gravity $g_{\text{eff}}$ and the increasing compression factor $\frac{1}{\pi^2} \ln \frac{\Sigma}{\pi^2}$. The shear $\gamma_0^2$ in the disc is relatively small and varies slowly.

Figure 13. Left Panel: Dependence of different terms from Spruit’s criterion on $\alpha-$ parameter of viscosity. Simulations were done for $\mu = 1$, $r_{\text{cor}} = 1.5$ and different $\alpha-$parameters of viscosity. $g_{\text{eff}}$ is multiplied by 10 for clarity. Right Panel: Same but for dependence on corotation radius $r_{\text{cor}}$. Simulations were performed for stars with $\mu = 1$, $\alpha = 0.02$ and different $r_{\text{cor}}$. 
compared with the $\gamma_{BS}^2$ term, thus having a smaller influence on the transition.

It is interesting to note that the compression term $\frac{d}{dr} \ln \frac{\Sigma}{\Sigma_0}$ increases with $\alpha$. This is probably due to the fact that the radial velocity of matter in the disc increases with $\alpha$: $v_r = \alpha c_s^2 / v_K$, where $c_s$ and $v_K$ are the local sound speed and Keplerian velocity, respectively. This means that higher gradients of surface density $\Sigma$ and magnetic field $B_s$ are expected, and the term $\Sigma / B_s$ varies more rapidly with radius in the cases of higher $\alpha$. On the other hand, the overall accretion rate increases with $\alpha$: $\dot{M} \sim v_r \sim \alpha$, which leads to a decrease in the magnetospheric radius: $r_m \sim 1 / \alpha^{2/7}$, and $g_{eff}$ becomes more negative. The transition between the stable and unstable regimes (where $\gamma_{BS}^2 = \gamma_{D}^2$) occurs at $\alpha \approx 0.037$, which is near the point $\alpha = 0.028$, where the effective gravity $g_{eff}$ changes the sign. Here, as in the case of varying $r_{cor}$, effective gravity plays an important role in the transition between stable and unstable regimes.

This set of simulations is similar to the simulations performed in our earlier studies, where we investigated the transition between stable and unstable regimes by changing parameter $\alpha$ to vary the accretion rate $\dot{M}$, which we took to be the main factor in determining the mode of accretion (e.g., Kulkarni & Romanova 2008; Romanova et al. 2008). In this paper, our approaches were aimed at a deeper understanding of the physics of transition between the stable and unstable regimes, and its dependence on different parameters.

In realistic discs, the angular momentum transport is probably provided by the magnetic turbulence, supported by the magneto-rotational instability (Balbus & Hawley 1991). The effective $\alpha$-parameter is determined by the ratio of magnetic stress to matter pressure. The $\alpha$-parameter may be relatively high in the inner parts of the disc, where the magnetic field of the turbulent cells is amplified by the rapid Keplerian rotation of inner disc matter (e.g., Hawley 2000 Armitage 2002). In addition, part of the stellar magnetic flux penetrates into the disc and increases the magnetic stress in the inner disc, so that the $\alpha$-parameter can be as high as $\alpha = 0.3 - 1$ (Romanova et al. 2011, 2012, Liu et al. 2014). This is why we also performed test simulation runs at a relatively high parameter of viscosity, $\alpha = 0.1$. These simulations have shown that instability is stronger when $\alpha = 0.1$ than in the cases of $\alpha = 0.02$, and that the boundary between stable and unstable regimes is expected to be higher, closer to the $r_{cor} / r_m \approx 1$ line, though additional research is required to find this line.

5.3 Comparison between relativistic and non-relativistic cases

The results of current simulations are relevant for non-relativistic stars. However, in case of neutron stars, we should take into account the relativistic gravitational potential, which provides a deeper gravitational well and may enhance instability, in particular when the unstable tongues move closer to the star. A relativistic potential has been used in most of our prior simulations (Kulkarni & Romanova 2008, 2009; Romanova et al., 2008, Bachetti et al. 2010).

In this paper, we consider non-relativistic stars to be sure that the instability mechanism is present even in the cases of shallower, non-relativistic potential wells. However, in application to neutron stars it is important to take into account the relativistic effects and to compare the differences between instabilities in non-relativistic and relativistic cases. It would be too time-consuming to recalculate all the cases for relativistic stars. Instead, we chose one representative model, $\mu_1c^{3950a0.02}$, and recalculated it taking into account the relativistic corrections. To model the relativistic effects we chose the Paczyński-Wiita (PW) pseudo-relativistic potential (Paczyński & Wiita 1980), $\Phi(r) = GM_s / (r - r_g)$, where $r_g = 2GM / c^2$ is the gravitational (Schwarzschild) radius.

For a typical neutron star with mass $M_s = 1.4M_\odot$ and radius $R_\star = 10$ km, the ratio $r_g / R_\star = 0.145$.

We found the results to be similar: instability of comparable strength has developed, with two ordered tongues forming and rotating with the frequency of the inner disc. Fig. 14 shows $xy$-slices of these two models for the same moment in time. The matter flux onto the star (bottom panel) is about 20% higher in the relativistic case. However, this does not affect the instability pattern qualitatively. We should note that the PW potential gives a steeper gravitational well than the fully relativistic approach, and therefore the relativistic effects will be even weaker in a more realistic relativistic case.

We conclude that the main effect of enhanced gravity is enhanced accretion rate to the star, $\dot{M}$. However, the difference in accretion rate between the non-relativistic and the PW pseudo-relativistic cases is $\sim 20\%$, which leads to a factor of only $1.2^{1/7} \approx 1.03$ difference in the magnetospheric radius $r_m$. This difference is small, which is why relativistic effects do not enhance instability significantly.

6 APPLICATION TO DIFFERENT MAGNETIZED STARS

The results of our simulations can be applied to different types of magnetized stars.
6.1 Application to CTTSs

Classical T Tauri stars are strongly magnetized (e.g., Johns-Krull 2007), so that the inner disc is disrupted by the magnetic field of the star at a distance of a few stellar radii, and the observational properties are determined by magnetospheric accretion (Bouvier et al. 2007). These stars show variability on different time-scales (Herbst et al. 1994).

A recent analysis of the light-curves from young stars in NCG 2264, obtained with the CoRoT satellite, shows that many stars have irregular variability (Alencar et al. 2010). Analysis of the variability shows that one of the important time-scales observed in CTTSs corresponds to that expected in the unstable regime of accretion, that is, a few peaks per rotational period of the inner disc (Stauffer et al. 2014; Cody et al. 2014). Similar variability is expected in the spectral lines originating in unstable tongues (Kurosawa & Romanova 2013).

Another important question is whether the period of rotation observed in CTTSs is the stellar period or the period associated with the unstable regime of accretion. In the case of the ordered unstable regime, one or two ordered tongues rotate with the frequency of the inner disc. For example, detailed observations of the photometric variability in TW Hya, performed by the MOST satellite, show that the star does not display a definite period, but instead a quasi-period that varies in time (Rucinski et al. 2008; Siwak et al. 2011). It is possible that TW Hya is accreting in the ordered unstable regime, and this quasi-period may correspond to the rotation of unstable tongues and hot spots along the surface of the star. The frequency of rotation varies when the inner disc changes its position, and therefore this quasi-period will vary with time. CTTSs are often observed during brief intervals of time, which only cover a few periods of stellar rotation, so that there may not be sufficient data to extract the period of the star, and it is easy to mistake the real period of the star for the period of the unstable tongues, which rotate more rapidly and may have a larger amplitude in the Fourier and wavelet spectra. Longer observations are required to separate the real period of the star from the quasi-period associated with unstable accretion.

The ordered unstable regime may also be important during periods of accretion outbursts, observed in EXor and FU Ori-type stars (e.g., Audard et al. 2014). In these stars, the accretion rate is greatly enhanced, the magnetosphere is compressed, and the magnetospheric radius may be much smaller than the corotation radius. Therefore, conditions become favorable for ordered unstable accretion in one or two ordered tongues. Recently, a very short period of ∼ 1 day was discovered in the protostar V1647 Ori during its two recent multiple-year accretion outbursts (Hamaguchi et al. 2012). Via time-series analysis, Hamaguchi et al. (2012) established that the puzzling short-term X-ray variability of V1647 Ori was due to rotational modulation. The ∼ 1 day X-ray period corresponds to rotation at near-breakup speed. Hamaguchi et al. (2012) demonstrated that a model consisting of two hot spots of high plasma density (> 10^10 cm^-3), located at or near the stellar surface, reproduced the X-ray rotational modulation signature well. We suggest that it is improbable that a star rotates so rapidly. Instead, the rapidly rotating regions of enhanced plasma found in the empirical model of Hamaguchi et al. (2012) may be connected with the ordered unstable tongues in the ordered unstable regime of accretion, which is highly probable during accretion outbursts.

6.2 Application to accreting white dwarfs

A few types of accreting white dwarfs have a strong magnetic field (e.g., Warner et al. 1995; Hellier 2001). In most of Intermediate Polars (hereafter IPs), the magnetosphere disrupts the disc at r_m ≳ 10R_⋆, which is larger than the radii considered in this paper. Simulations of accretion to large magnetospheres (Romanova et al. 2014) show that matter of the inner disc only partially penetrates the magnetosphere, and most of the matter flows onto the star in two funnel streams (above the magnetosphere), forming two ordered hot spots on the surface of the star. In these stars the light-curves reflect the period of stellar rotation and are expected to be ordered. This may explain why IPs have ordered light-curves.

There is another class of accreting white dwarfs - Dwarf Novae (DNs), which do not show a period, but instead only quasi-periodic oscillations are observed during accretion outbursts (Warner et al. 2004). The highest frequency oscillations (called Dwarf Novae Oscillations, or DNOs) have typical time-scales of tens of seconds. The nature of these oscillations has not yet been understood. It is possible that DNs are weakly magnetized white dwarfs in which the magnetosphere is strongly compressed during accretion outbursts, with r_m >> r_⋆. This condition is favorable for ordered unstable accretion, where one or two ordered tongues are the cause of quasi-periodic variability: variation in accretion rate leads to the variation in the inner disc position and frequency of the tongues, which may explain DNOs. Compared with CTTSs, the period of oscillations is very small, and many oscillations are observed during a single set of observations. This increases the quality factor of the oscillations (Q = f/Δf). At the same time, the large number of oscillations per observational set increases the probability of observing the period of the star, even if its amplitude is much smaller than the amplitude of DNOs. However, contrary to expectations, none of these DNs show the period of the star. We speculate that the period of the star may be found in the future, more precise observations of DNs.

6.3 Application to accreting neutron stars

In Low-mass X-ray Binaries (LMXBs), which are a sub-class of accreting neutron stars, quasi-periodic variability is observed during accretion outbursts (e.g., van der Klis 2006; Patruno & Watts 2012). The stellar period is observed in some LMXBs, and is associated with either magnetospheric accretion onto the star (in accreting millisecond pulsars, or AMPs) or with thermonuclear bursts. In LMXBs the conditions are favorable for unstable accretion, because the size of the magnetosphere is expected to be small during accretion outbursts (less than a few stellar radii), and the magnetospheric radius r_m is expected to be smaller than the corotation radius r_m for many instances.

We suggest that AMPs may switch between different regimes of accretion with time. When r_m ≈ r_m, stable accretion in two funnel streams dominates. When r_m decreases down to r_m/1.4, the chaotic unstable regime dominates. In this case, the whole set of chaotic spots rotates around the
magnetic pole with the frequency of the inner disc, and this may provide the QPO frequency. When the inner disc moves closer to the star and $r_{m}$ decreases further, one or two ordered tongues start dominating, and the quality factor $Q$ increases. When $r_{m}$ decreases down to $r_{cor}/1.7$, one or two tongues strongly dominate and the quality factor increases further due to the long-lived nature of the tongues in the ordered unstable regime. In summary, the frequency of the QPOs and the quality factor $Q$ increase when $r_{m}$ decreases. Observations show that the frequency of QPOs usually correlates with the frequency of the X-ray flux, which is, in most cases, proportional to the accretion rate (e.g., Papitto et al. 2007), and the quality factor increases with QPO frequency (e.g., van der Klis 2006).

In many LMXBs, no stellar period is detected, but instead only QPOs are observed. We speculate that the mechanism of ordered unstable accretion may be responsible for the formation of QPOs in these stars. However, in order for the stellar period to be “hidden”, the misalignment angle of the dipole should be very small, so that the QPO signal associated with unstable accretion is much stronger than the signal associated with the rotation of the star (see Lamb et al. 2009; Bachetti et al. 2010).

Usually, a pair of QPOs is observed in LMXBs (e.g., van der Klis 2006). Our simulations have shown the formation of one QPO associated with the rotation of unstable tongues. Detailed analyses of QPOs in models with very small tilts of the dipole (Θ = 2°) show that one of the QPO frequencies is associated with the unstable tongues, while the other one is associated with the funnel streams that rotate faster or slower than the star (Bachetti et al. 2010). These two frequencies were found in the spot-omega (phase) diagrams, which is another type of frequency analysis (e.g., Kulkarni & Romanova 2008). A more detailed analysis is required for finding the second QPO frequency in our current models.

We note that the formation of ordered unstable tongues may be connected with the formation of waves in the inner disc. In particular, the presence of one or two ordered unstable tongues may be connected with one- or two-armed density waves in the inner disc. Global simulations of interactions of the tilted dipole with the disc show the formation of density and bending waves in the inner disc, with two different frequencies that vary in time (Romanova et al. 2013).

In future research, we plan to investigate the correspondence between different waves in the inner disc and frequencies of rotating spots associated with the unstable tongues.

7 CONCLUSIONS

A new set of simulations has been performed with the goal of investigating the boundary between stable and unstable regimes of accretion and the properties of unstable accretion. Simulations were performed at twice as high a grid resolution as in our earlier studies (Kulkarni & Romanova 2008, 2009; Romanova et al. 2008). A low viscosity parameter in the disc ($\alpha = 0.02$) was used in most of the simulation runs. The main results of the new investigations are the following:

1. We found that the boundary between stable and unstable regimes strongly depends on the relationship between the corotation and magnetospheric radii. For a small misalignment angle of the dipole, $\Theta = 5^\circ$, and low viscosity in the disc, $\alpha = 0.02$, accretion is unstable if $r_{cor} \gtrsim 1.4r_{m}$, and is stable otherwise.

2. Below the $r_{cor} \approx 1.4r_{m}$ line, accretion proceeds in the chaotic unstable regime through several unstable tongues. The light-curve from the hot spots looks chaotic, and the spectral analysis shows several frequencies associated with the chaotic tongues. One of the frequencies often corresponds to the frequency of the inner disc, because the set of short-lived tongues rotates with the angular frequency of the inner disc. The frequency of the star is usually present in the frequency spectra if the simulation runs are sufficiently long.

3. If $r_{cor} \gtrsim 1.7r_{m}$, then matter accretes in the ordered unstable regime, where one or two ordered tongues form and rotate with the frequency of the inner disc. The tongues form ordered spots on the stellar surface that survive for longer periods of time than the chaotic spots. The light-curves show regular oscillations associated with the rotation of the spots. Both the Fourier and the wavelet spectra show high-amplitude frequency connected with the rotation of the spots. This frequency corresponds to the frequency of the inner disc. The ordered unstable regime has only been observed in the cases of relatively small magnetospheres, $r_{m} \lesssim 4.2R_{\star}$.

4. At a higher viscosity in the disc, $\alpha = 0.1$, chaotic instability becomes more irregular, and variability on different time-scales is observed in the light-curve. The frequency associated with the inner disc rotation is seen in both the Fourier and the wavelet spectra, while the frequency of the star has a much lower amplitude.

5. Analysis of the causes of instability in the borderline cases between stable and unstable regimes shows that:

(a) Increasing the corotation radius $r_{cor}$ (while fixing all other parameters) leads to the sign change of effective gravity $g_{eff}$ from positive to negative, causing the transition from the stable to the unstable regime of accretion. We believe the sign change of $g_{eff}$ to be the main factor in this transition.

(b) Increasing the viscosity parameter $\alpha$ also leads to the sign change of $g_{eff}$ from positive to negative values and to the transition from the stable to the unstable regime. In addition, at higher values of $\alpha$ the inner disc and the magnetosphere are more compressed, and a higher compression factor $\left| \frac{d}{dr} \ln \frac{\Sigma}{\Sigma_{c}} \right|$ leads to stronger instability.

6. For a larger misalignment angle, $\Theta = 20^\circ$, the boundary between stable and unstable regimes of accretion is slightly lower, at $r_{cor} = 1.5r_{m}$, compared with $\Theta = 5^\circ$. Stable accretion through regular funnel streams becomes slightly more favorable due to the larger tilt of the dipole.

7. Investigation of the ordered unstable regime at different $\Theta$’s shows that the high-frequency oscillations associated with the ordered unstable tongues determine the light-curve at $\Theta = 5 - 10^\circ$. However, at larger $\Theta$’s ($\Theta = 15 - 30^\circ$), the period of the star begins to have a higher amplitude than the instabilities, while the unstable tongues provide high-frequency modulation of the main light-curve (see Fig. 12). Signs of instability are seen in the wavelet spectra at different $\Theta$’s, including very large values, $\Theta = 40^\circ$ and $60^\circ$. The amplitude of the oscillations decreases with $\Theta$ because a strongly tilted dipole breaks the unstable tongues.

8. A comparison of relativistic and non-relativistic cases shows that instability is somewhat stronger in the relativistic cases. However, the relativistic potential is not the main factor in determining the mode of accretion.
The simulations were performed in dimensionless variables and should not be generalized to stars with large magnetospheres. Separate studies show that in the cases of large magnetospheres, instabilities are only present in the external parts of the magnetosphere, while matter accretes to the star in two ordered funnel streams (Romanova et al. 2014).

10. The results can be applied to different types of accreting magnetized stars. The main findings are the following:

(a) Accretion in ordered unstable tongues can be important during accretion outbursts, when the inner disc moves inward, compressing the magnetosphere, and the ratio \( r_{\text{cot}}/r_m \) is large. The frequency of the rotating tongues corresponds to the frequency of the inner disc and may be seen as a QPO feature in the frequency spectra. The QPO frequency increases when the inner disc moves inward. The quality factor also increases during the inward motion of the disc.

(b) When a star is observed for a few periods of stellar rotation, as in CTTs, the frequency of the oscillations associated with the ordered unstable tongues may be mistaken for the frequency of the star.

(c) A star may alternate between stable and unstable regimes of accretion, thus showing an intermittency in its pulsations.

In future research, accretion through instabilities should be investigated in more complex, turbulent discs.

Acknowledgments

Authors thank Alexander Koldoba for an earlier-developed ‘cubed sphere’ code. Resources supporting this work were provided by the NASA High-End Computing (HEC) Program through the NASA Advanced Supercomputing (NAS) Division at Ames Research Center and the NASA Center for Computational Sciences (NCCS) at Goddard Space Flight Center. The research was supported by NASA grants NNX11AF33G, NNX12AI85G and NSF grant AST-1211318.

APPENDIX A: REFERENCE VALUES

The simulations were performed in dimensionless variables \( \tilde{A} \). To obtain the physical dimensional values \( A \), the dimensionless values \( \tilde{A} \) should be multiplied by the corresponding reference values \( A_0 \) as \( A = \tilde{A}A_0 \). These reference values include: mass \( M_0 = M_\star \), where \( M_\star \) is the mass of the star; distance \( R_0 = R_\star/0.35 \), where \( R_\star \) is the radius of the star \(^4\); and velocity \( v_0 = (GM/R_0)^{1/2} \). The time scale is the period of rotation at \( r = R_0 \): \( P_0 = 2\pi R_0/v_0 \). Reference angular velocity is \( \omega_0 = v_0/R_0 \), reference frequency is \( \nu_0 = \omega_0/2\pi \). Let \( \mu_0 = B_0 R_0^3 \) be the magnetic moment of the star, where \( B_0 \) is the magnetic field at the magnetic equator. We determined the reference magnetic field \( B_0 \) and magnetic moment \( \mu_0 = B_0 R_0^3 \) such that \( \mu_0 = \mu_0/\tilde{\mu} \), where \( \tilde{\mu} \) is the dimensionless magnetic moment of the star, which is used as a parameter in the code to vary the size of the magnetosphere. The reference field is \( B_0 = \mu_0/(R_0^3 \tilde{\mu}) \) the reference density, which is \( \rho_0 = B_0^2/v_0^2 \) and the reference mass accretion rate is \( \dot{M}_0 = \rho_0 \nu_0 R_0^2 = \mu_0^2/(\tilde{\mu}^2 R_0^2 v_0) \). Reference surface density \( \sigma = \rho_0 R_0 \). The main reference values are given in Tab. A1 for \( \tilde{\mu} = 0.5 \).

REFERENCES

Alencar, S. H. P. et al., 2010, A&A, 519, A88
Altamirano D., Casella P., Patruno A., Wijnands R., van der Klis M., 2008, ApJ, 674, L45
Armitage, P. 2002, ApJ, 330, 895
Arons, J. & Lea, S.M. 1976, ApJ, 207, 914
Audard, M., Abraham, P., Dunham, M. M., Green, J.D., Grosso, N. et al. 2014, Accepted for publication as a review chapter in Protostars and Planets VI, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning
Bachetti, M., Romanova, M. M., Kulkarni, A., Burderi, L., di Salvo, T. 2010, MNras, 403, 1193
Balbus, S.A. & Hawley , J. F. 2000, ApJ, 528, 462
Barnes, J.E., Bate, M.R., Boden, D.M., 1998, MNRAS, 297, 125
Bowers, J., Lai, D., 2003, ApJ, 588, 1016
Bouvier J., 2000, ApJ, 534, 945
Bouvier J., Alencar S. H. P., Harries T. J., Johns-Krull C. M., Romanova M. M., Protostars and Planets V, Eds. Reipurth B., Jewitt D., Keil K. (University of Arizona Press, Tucson, 2007) 479
Chandrasekhar, S., 1961, Hydrodynamic and Hydromagnetic Stability. Clarendon, Oxford, p. 466
Cody A. M., Stauffer, J., Baglin, A., Micela, G., Rebull, L. M., Flaccomio, E., Morales-Caldern, M. et al., 2014, AJ, 147, 47 pp
Donati J.-F., Jardine, M. M., Gregory, S. G., et al., 2007, MNRAS, 380, 1297
Donati J.-F., Jardine, M. M., Gregory, S. G., et al., 2007, MNRAS, 380, 1297
Ghosh, P., Lamb, F. K., 1978, ApJ, 223, L83
Hamaguchi, K., Grosso, N., Kastner, J. H., Weintraub, D. A., Richmond, M. et al. 2012, ApJ, 754, 9pp
Hawley J. F. 2000, ApJ, 528, 462
Hellier, C. 2001, Cataclysmic variable stars. (Springer, Berlin 2001)
Hellier, C. 2014, Physics at the Magnetospheric Boundary, Geneva, Switzerland, Edited by E. Bozzo, P. Kretschmar; M. Audard; M. Falanga; C. Ferrigno; EPJ Web of Conferences, Volume 64, id.07001
Herbst, W., Herbst, D. K., Grossman, E. J., Weinstein, D. 1994, AJ, 108, 1906
Johns-Krull C. M., 2007, ApJ, 664, 975

\[
\begin{array}{ccc}
M (M_\odot) & 0.8 & 0.5 \\
R_\star & 2R_\odot & 5000 \text{ km} \\
B_0 (\text{G}) & 10^3 & 10^6 \\
R_0 (\text{cm}) & 4 \times 10^{11} & 1.4 \times 10^9 \\
v_0 (\text{cm s}^{-1}) & 1.6 \times 10^7 & 3 \times 10^8 \\
\rho_0 (\text{g cm}^{-3}) & 2.8 \times 10^{-11} & 7 \times 10^{-8} \\
\sigma_0 (\text{g cm}^{-2}) & 11.0 & 112.5 \\
\nu_0 & 0.25 \text{ day}^{-1} & 3.2 \times 10^{-2} \text{ Hz} \\
P_0 & 1.5 \text{ days} & 29 \text{ s} \\
\mu_0 (\text{Gcm}^3) & 2.7 \times 10^{36} & 1.2 \times 10^{32} \\
M_0 (M_\odot \text{yr}^{-1}) & 2.8 \times 10^{-7} & 1.9 \times 10^{-7} \\
\end{array}
\]

Table A1. Sample values of physical parameters for different types of stars. See Sec. [A] for a detailed description.

\(^4\) This value for the scale has been taken in the past models (e.g., Koldoba et al. 2002; Romanova et al. 2002), and now we keep it for consistency with earlier work. 

\(^5\) Subsequent reference values depend on \( \tilde{\mu} \).
Kaisig, M., Tajima, T., & Lovelace, R. V. E. 1992, ApJ., 386, 83
Koldoba, A. V., Romanova, M. M., Ustyugova, G. V., Lovelace, R. V. E. 2002, ApJ, 576, L53
Kulkarni, A. K., & Romanova, M. M., 2005, ApJ, 633, 349
Kulkarni, A., & Romanova, M.M. 2008, ApJ, 386, 673
Kulkarni, A., & Romanova, M.M. 2009, ApJ, 398, 1105
Kurosawa R. and Romanova M. M., MNRAS 2013, 431, 2673
Lamb, F. K., Boutloukos, S., Van Wassenhove, S., Chamberlain, R. T., Lo, K. H., Miller, M. C. 2009, MNRAS, 705, L36
Lii, P.S., Romanova, M.M., Ustyugova, G.V., Koldoba, A.V., Lovelace, R.V.E. 2014, MNRAS, 441, 86
Paczyński B. & Wiita P. J. 1980, A&A, 88, 23
Papitto, A., di Salvo, T., Burderi, L., Menna, M. T., Lavagetto, G., Riggio, A. 2007, MNRAS, 375, 971
Review to appear in “Timing neutron stars: pulsations, oscillations and explosions”, T. Belloni, M. Mendez, C.M. Zhang Eds., ASSL, Springer; arxiv1206.2727
Powell, K.G., Roe, P.L., Linde, T.J., Gombosi, T.I., & De Zeeuw, D.L. 1999, J. Comp. Phys., 154, 284
Rastatter, L. & Schindler, K. 1999, ApJ, 524, 361
Romanova, M.M., & Blinova, A.A. 2014, in prep.
Romanova, M.M. & Kulkarni, A.K. 2009, MNRAS, 398, 701
Romanova, M.M., Kulkarni, A.K., Lovelace, R.V.E. 2008, ApJ Letters, 273, L171
Romanova, M. M., Ustyugova, G. V., Koldoba, A. V., Lovelace, R.V.E., 2002, ApJ, 578, 420
Romanova, M. M., Ustyugova, G. V., Koldoba, A. V., Wick, J. V., Lovelace, R. V. E., 2003, ApJ, 595, 1009
Romanova, M. M., Ustyugova, G. V., Koldoba, A. V., Lovelace, R. V. E., 2004, ApJ, 610, 920
Romanova, M.M., Ustyugova, G.V., Koldoba, A.V., Lovelace, R.V.E., 2011, MNRAS, 416, 416
Romanova, M.M., Ustyugova, G.V., Koldoba, A.V., Lovelace, R.V.E., 2012, MNRAS, 421, 63
Romanova, M.M., Ustyugova, G.V., Koldoba, A.V., Lovelace, R.V.E., 2013, MNRAS, 430, 699
Romanova, M.M., Lovelave, R.V.E., Bachetti, M., Blinova, A.A., Koldoba, A.V., et al. 2014, Physics at the Magnetospheric Boundary, Geneva, Switzerland, Edited by E. Bozzo; P. Kretschmar; M. Audoard; M. Falanga; C. Perrigno; EPJ Web of Conferences, Volume 64
Rucinski S. M. et al., 2008, MNRAS, 391, 1913
Shakura, N.I., & Sunyaev, R.A. 1973, A&A, 24, 337
Stauffer, J., Cody, A.M., Baglin, A., Alencar, S., Rebull, L., Hillenbrand, L.A. et al. 2014, AJ, 147, 34pp
Siwak, M., Rucinski, S. M., Matthews, J. M., Pojmanski, G., Kuschnig, R. et al. 2011, MNRAS, 410, 2725
Spruit H. C., Stehle R., Papaloizou J. C. B., MNRAS 1995, 275, 1223
Stauffer J. et al., 2014, AJ, 147, 34
Stone J. M., Gardiner T. A., 2007a, Phys. Fluids, 19, 4104
Stone J. M., Gardiner T. A., 2007b, ApJ, 671, 1726
van der Klis M., Compact Stellar X-Ray Sources, Eds. Lewin W. H. G. and van der Klis M. (Cambridge Univ. Press, Cambridge, 2006) 39
Wang, Y.-M. & Robertson, J.A. 1984, A&A, 139, 93
Wang, Y.-M. & Robertson, J.A. 1985, ApJ, 299, 85
Warner B., 1995, Cataclysmic variable stars, (CUP, Cambridge 1995)
Warner B., PASP 2004, 116, 115