The interrelationship of integrable equations,
differential geometry and the geometry of their
associated surfaces

Paul Bracken
Department of Mathematics
University of Texas
Edinburg, TX 78540 USA *

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Abstract

A survey of some recent and important results which have to do with integrable
equations and their relationship with the theory of surfaces is given. Some new
results are also presented. The concept of the moving frame is examined, and it is
used in several subjects which are discussed. Structure equations are introduced in
terms of differential forms. Forms are shown to be very useful in relating geometry,
equations and surfaces, which appear in many sections. The topics of the chapters
are different and separate, but joined together by common themes and ideas. Sev-
eral subjects which are not easy to access are reviewed and elaborated upon. These
topics include Maurer-Cartan cocycles and recent results with regard to general-
izations of the Weierstrass-Enneper system for generating constant mean curvature
surfaces in three and higher dimensional Euclidean spaces.

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1. INTRODUCTION

The origins of soliton theory go back to the early part of the nineteenth century, in particular, to the observation of John Scott Russell in 1834 of a solitary bump-shaped wave moving along a canal near Edinburgh. However, it wasn’t until 1965 that this type of phenomena was rediscovered, in particular, by Kruskal and Zabusky [1,2] in the context of the Fermi-Pasta-Ulam [3] problem. It was they that coined the term soliton. In 1895, two Dutch mathematicians, Korteweg and de-Vries derived a nonlinear wave equation which now bears their name. It models long wave propagation in a rectangular channel and has a traveling wave solution which resembles the solitary wave observed by Russell. In fact, a pair of equations equivalent to the KdV equation appeared even earlier in a work by Boussinesq. It was not until the mid-twentieth century that the equation reappeared in work by such researchers as Kruskal and Zabusky and Gardener and Morikawa in 1960 in an analysis of the transmission of hydromagnetic waves. There continues to be ongoing interest in such nonlinear equations which arise in a diversity of systems such as the theory of solids, liquids and gases [4,5]. Self-localized nonlinear excitations are fundamental and inherent features of quasi-one-dimensional conducting polymers. In 1962, Perring and Skyrme solved the sine-Gordon equation numerically while using it in an elementary particle model. The results generated from this equation were found not to disperse and two solitary waves were seen to keep their original shapes and velocities despite collisions.

In the pioneering work of Kruskal and Zabusky, the KdV equation was obtained as a continuum limit of an anharmonic lattice model with cubic nonlinearity. The model displays the existence of solitary waves. These waves have the remarkable property that they preserve both their amplitude and speed upon interaction. These properties are the main reason for the use of the term soliton. A soliton or solitary wave can be regarded as a solution to any number of a variety of nonlinear partial differential equation. In a more physical language, solitons have the following striking properties. Energy is localized within a small region and an elastic scattering phenomenon exists in the interaction of two solitons. To put it another way, the shape and velocity of the wave are recovered after an interaction between such solutions. Solitons seem to behave as both particle and wave. Originally they arose in the area of fluid mechanics, and their study has extended into the areas of plasma physics, nonlinear optics and classical and quantum field theory. In large part, this is precisely due to the aforementioned properties. What is more, many branches of mathematics and physics provide important tools for the study of solitons. It will be seen here that the development of the study of solitons has resulted in reciprocal advances in many areas of mathematics as well.

Moreover, there is a deep connection between many of these equations, the theory of surfaces [6] and integrable systems [7,8]. It is the intention here to explore this interrelationship between the theory of these equations and the surfaces that can be determined.
by them. It will be seen that many important ideas from the area of differential geometry are applicable to the subjects studied here and give the subject a unified perspective.

A generic method for the description of soliton interaction begins with a transformation which was originally introduced by Bäcklund to generate pseudospherical surfaces. Later Bianchi showed that the Bäcklund transformation admits a commutativity property, a consequence of which is a nonlinear superposition principle which is referred to as the permutability theorem. As an example, both KdV and MKdV equations reside in hierarchies which admit auto-Bäcklund transformations, nonlinear superposition principles as well as multi-soliton solutions.

This article is a review of recent results by the author as well as by other researchers. Let us begin with a brief overview of the contents which follow. Although there are numerous ideas and themes which run throughout the article, each chapter is separate and can be read on its own independently of the others. First, a review of surface theory from the classical point of view will be presented [6,9]. The next section is a novel and active area of interest which should appeal to those with an interest in this area. The subject of the immersion of a two-dimensional surface into a three-dimensional Euclidean space, as well as the $n$-dimensional generalization, has been related to the problem of investigating surfaces in Lie groups and in Lie algebras as well [10,11]. This gives an interesting correspondence between the Lax pair of an integrable equation and their integrable surfaces. Using the formulation of the immersion of a two-dimensional surface into three-dimensional Euclidean space, it will be shown that a mapping from each symmetry of integrable equations to surfaces in $\mathbb{R}^3$ can be established.

Next a differential forms approach to surfaces will be presented [12-14]. The problem of identifying whether a given nonlinear partial differential equation admits a linear integrable system is studied here by means of this differential geometric formalism [15]. It is shown that the fundamental equations of surface theory can be used to reproduce the compatibility conditions obtained from a linear system in matrix form corresponding to a number of different Lie algebras. In fact, the system of equations which has been obtained from the linear matrix problem is derived from a system of differential forms and in combination with the first and second fundamental forms leads to a link with surface theory in differential geometry [16].

The subject of non-linear evolution equations and Maurer-Cartan cocycles on $\mathbb{R}^2$ is introduced next. Maurer-Cartan cocycles are defined and some general theoretical information about them is provided. By using Maurer-Cartan cocycles and Cartan prolongations for individual nonlinear equations, it is shown how Bäcklund transformations can be calculated for specific equations. As an example, the Bäcklund transformation for the sine-Gordon equation will be derived.

Finally, the subject of constant mean curvature surfaces has been of great interest recently. From what has been discussed already, the theory of surfaces has many applications in a great number of areas of physical science. The theory of constant mean
curvature surfaces has had a great impact on many problems which have physical applications. In particular, there are many applications to such areas as two-dimensional gravity, quantum field theory, statistical physics and fluid dynamics [17,18]. An application of recent interest is the propagation of a string through space-time, in which the particle describes a surface called its world sheet. Thus, the subject of generalized Weierstrass representations, in particular, the generalization of the Weierstrass-Enneper approach due to B. Konopelchenko [19] will be discussed in detail. There exists a correspondence between this representation and the two-dimensional nonlinear sigma model. Both of these systems have been shown to be integrable, and their symmetry groups have been calculated [20]. These symmetries have lead to the calculation in closed form of many explicit solutions of the system, and the determination of their soliton surfaces, as will be seen.

2. CLASSICAL THEORY OF SURFACES

It is perhaps useful at this point to introduce some classical results concerning surfaces, which arise out of classical differential geometry. This will give a basic review of surface theory and some preparation for what is to follow.

Let \( r = r(u, v) \) denote the position vector of a generic point \( P \) on a surface \( \Sigma \) in \( \mathbb{R}^3 \). Then the vectors \( r_u \) and \( r_v \) are tangential to \( \Sigma \) at \( P \) and at such points at which they are linearly independent, the quantity

\[
N = \frac{r_u \times r_v}{|r_u \times r_v|}
\]

(2.1)

determines the unit normal to \( \Sigma \). The first and second fundamental forms of \( \Sigma \) are defined by

\[
I = ds^2_I = dr \cdot dr = E \, du^2 + 2F \, dudv + G \, dv^2,
\]

\[
II = ds^2_{II} = -dr \cdot dN = e \, du^2 + 2f \, dudv + g \, dv^2.
\]

(2.2)

In (2.2), the coefficients are defined by

\[
E = r_u \cdot r_u, \quad F = r_u \cdot r_v, \quad G = r_v \cdot r_v,
\]

\[
e = -r_u \cdot N_u, \quad g = -r_v \cdot N_v, \quad f = -r_u \cdot N_v = -r_v \cdot N_u.
\]

(2.3)

There is a classical result of Bonnet which states that \( \{E, F, G, e, f, g\} \) determines the surface \( \Sigma \) up to its position in space. The Gauss equations associated with \( \Sigma \) are given as

\[
r_{uu} = \Gamma^1_{11} r_u + \Gamma^2_{11} r_v + eN, \quad r_{uv} = \Gamma^1_{12} r_u + \Gamma^2_{12} r_v + fN, \quad r_{vv} = \Gamma^1_{22} r_u + \Gamma^2_{22} r_v + gN,
\]

(2.4)
while the Weingarten equations are given by
\[ N_u = \frac{f F - e G}{H^2} r_u + \frac{e F - f E}{H^2} r_v, \quad N_v = \frac{g F - f G}{H^2} r_u + \frac{f F - g E}{H^2} r_v, \quad (2.5) \]
where \( H^2 = |r_u \times r_v|^2 = E G - F^2 \). The \( \Gamma^i_{jk} \) are the Christoffel symbols and since the derivatives of all the \( \{ E, F, G, e, f, g \} \) with respect to \( u \) and \( v \) can be calculated from (2.3) and (2.4), the derivatives of all the \( \Gamma^i_{jk} \) can be calculated as well.

Thus, using these derivatives, the compatibility conditions \( r_{uu} = (r_{uv})_v \) and \( (r_{uv})_v = (r_{vv})_u \) applied to the linear Gauss system (2.4) produces the nonlinear Mainardi-Codazzi system
\[ e_v - f_u = e \Gamma^1_{12} + f (\Gamma^2_{12} - \Gamma^1_{11}) - g \Gamma^2_{11}, \quad f_v - g_u = e \Gamma^1_{22} + f (\Gamma^2_{22} - \Gamma^1_{12}) - g \Gamma^2_{12}. \quad (2.6) \]

The Theorema egregium of Gauss provides an expression for the Gaussian or total curvature
\[ K = \frac{e g - f^2}{E G - F^2}, \quad (2.7) \]
or in terms of \( E, F, \) and \( G \) alone in Liouville’s representation
\[ K = \frac{1}{H} \left[ \left( \frac{H}{E} \Gamma^2_{11} \right)_v - \left( \frac{H}{E} \Gamma^2_{12} \right)_u \right]. \quad (2.8) \]
If the total curvature of \( \Sigma \) is negative, that is, if \( \Sigma \) is a hyperbolic surface, then the asymptotic lines on \( \Sigma \) may be taken as parametric curves. Then \( e = g = 0 \) and the Mainardi-Codazzi equations reduce to
\[ \left( \frac{f}{H} \right)_u + 2 \Gamma^2_{12} \frac{f}{H} = 0, \quad \left( \frac{f}{H} \right)_v + 2 \Gamma^1_{12} \frac{f}{H} = 0. \quad (2.9) \]
Moreover, we have
\[ \frac{f^2}{H^2} = -\frac{1}{\rho^2}, \quad \Gamma^1_{12} = \frac{G E_u - F G_u}{2 H^2}, \quad \Gamma^2_{12} = \frac{E G_u - F E_v}{2 H^2}. \quad (2.10) \]
The angle between the parametric lines is such that
\[ \cos \omega = \frac{F}{\sqrt{E G}}, \quad \sin \omega = \frac{H}{\sqrt{E G}}, \quad (2.11) \]
and since \( E, G > 0 \), we may take without loss of generality
\[ E = \rho^2 a^2, \quad G = \rho^2 a^2, \quad f = ab \rho \sin \omega. \quad (2.12) \]
Then the Christoffel symbols are given by

\[ \Gamma^1_{12} = \frac{\rho_v a + \rho a_v - \cos \omega (\rho_u b + \rho b_u)}{\rho a \sin^2 \omega}, \quad \Gamma^2_{12} = \frac{\rho_u b + \rho b_u - \cos \omega (\rho_v a + \rho a_v)}{\rho b \sin^2 \omega}. \] (2.13)

Substituting (2.13) into the pair (2.9), there results

\[ 2\rho_v a + 2\rho a - 2 \cos \omega (\rho_u b + \rho b_u) - \rho_v a \sin^2 \omega = 0, \quad 2\rho b_u + 2\rho_u b - 2 \cos \omega (\rho_v a + \rho a_v) - \rho_u b \sin^2 \omega = 0. \] (2.14)

Solving the linear system in (2.14) for \( a_v \) and \( b_u \), we obtain

\[ a_v + \frac{\rho}{2} \rho \frac{\rho_u}{\rho} \cos \omega = 0, \quad b_u + \frac{\rho}{2} \rho \frac{\rho_u}{\rho} \cos \omega = 0. \] (2.15)

The representation for the total curvature is

\[ \omega_{uv} + \frac{1}{2} \left( \frac{\rho_u}{\rho} \frac{\rho v}{\rho} \sin \omega \right)_u + \frac{1}{2} \left( \frac{\rho u}{\rho} \frac{\rho v}{\rho} \sin \omega \right)_v - ab \sin \omega = 0. \] (2.16)

For the particular case in which \( K = -1/\rho^2 < 0 \) is a constant, \( \Sigma \) is referred to as a pseudospherical surface. Then (2.15) implies that \( a = a(u) \), \( b = b(v) \), and if \( \Sigma \) is now parametrized by arc length along asymptotic lines, then

\[ ds^2_I = du^2 + 2 \cos \omega du dv + dv^2, \quad ds^2_{II} = \frac{2}{\rho} \sin \omega du dv. \] (2.17)

Equation (2.16) then reduces to the sine-Gordon equation

\[ \omega_{uv} = \frac{1}{\rho^2} \sin \omega. \] (2.18)

Thus, there is a clear indication of a relationship between surfaces and an integrable equation.

### 3. SURFACES ON LIE ALGEBRAS, LIE GROUPS AND INTEGRABILITY

There have been some interesting developments recently related to the problem of the immersion of a 2-dimensional surface into a 3-dimensional Euclidean space, as well as the \( n \)-dimensional generalization [21,22,23]. These will be reviewed here. This subject has been shown to be related to the problem of studying surfaces in Lie groups and Lie algebras [24]. It has been found useful for investigating integrable surfaces, or surfaces which are described by integrable equations. Starting from a suitable Lax pair, which implies a suitable integrable equation, it is possible to construct explicitly large classes of integrable surfaces.
Let \( F = (F_1, F_2, F_3) : \pi \to \mathbb{R}^3 \) be an immersion of a domain \( \pi \subset \mathbb{R}^2 \) into 3-dimensional Euclidean space. For \((u, v) \in \pi\), the Euclidean metric induces a metric with coefficients \( g_{ij}(u, v) \) on the surface. These functions and \( d_{ij}(u, v) \), which define the second fundamental form, satisfy a system of three nonlinear equations known as the Gauss-Codazzi equations, which are the compatibility condition of the Gauss-Weingarten system. There exist two geometrical characteristics on such a surface known as the Gauss curvature \( K \) and the mean curvature \( H \). Some results will be given in other sections which correspond to constant \( K \) and constant \( H \).

A surface will be called integrable if and only if its Gauss-Codazzi equations are integrable. Integrable equations also arise as the compatibility condition of a pair of linear equations, which is usually referred to as a Lax pair. Here we want to show this problem is closely related to the problem of studying surfaces in Lie groups and Lie algebras.

Let \( G \) be a group and \( \mathcal{G} \) the Lie algebra of \( G \). Assume there exists an invariant scalar product in \( \mathcal{G} \). The scalar product will not be degenerate so there exists an orthonormal basis \( \{ e_i \} \) in \( \mathcal{G} \) such that \( \langle e_i, e_j \rangle = \delta_{ij} \). To introduce a surface in \( \mathcal{G} \), let \( \Phi(u, v) \in \mathcal{G} \) for every \((u, v)\) in some neighborhood of \( \mathbb{R}^2 \). There exists a canonical map from the tangent space of \( \mathcal{G} \) to the Lie algebra \( \mathcal{G} \). If \( \Phi_u \) and \( \Phi_v \) are the tangent vectors of \( \Phi \) at the point \((u, v)\), this map is defined by

\[
\frac{\partial \Phi}{\partial u} \Phi^{-1} = U_je_j, \quad \frac{\partial \Phi}{\partial v} \Phi^{-1} = V_je_j, \quad (3.1)
\]

where \( U_j \) and \( V_j \) are some functions of \((u, v)\) and \( j = 1, \cdots, n \). Equations (3.1) define \( \Phi \) through its value in the Lie algebra. Suppose the structure constants of \( \mathcal{G} \) satisfy

\[
[e_k, e_m] = c_{kmj}e_j, \quad (3.2)
\]

with summation implied. Differentiating the first equation of (3.1) with respect to \( v \) and the second with respect to \( u \), then upon subtracting these we have

\[
\frac{\partial U_j}{\partial v}e_j \Phi + U_je_j \frac{\partial \Phi}{\partial v} - \frac{\partial V_j}{\partial u}e_j \Phi - V_je_j \frac{\partial \Phi}{\partial u} = (\frac{\partial U_j}{\partial v} - \frac{\partial V_j}{\partial u})e_j \Phi + (U_m\Phi V_s e_m e_s - V_s \Phi U_m e_s e_m) \Phi = 0.
\]

Expression (3.3) implies that

\[
(\frac{\partial U_j}{\partial v} - \frac{\partial V_j}{\partial u})e_j + U_m\Phi V_s e_m e_s = 0.
\]

which when written just in terms of \( U \) and \( V \), this is written

\[
\frac{\partial U}{\partial v} - \frac{\partial V}{\partial u} + [U, V] = 0. \quad (3.4)
\]
This result can be summarized next.

**Proposition 3.1.** Let \( \Phi(u, v) \in G \) be a differentiable function of \( u, v \) for every \( (u, v) \) in some neighborhood of \( \mathbb{R}^2 \). Then \( \Phi \) defined by (3.1) exists if and only if the functions \( U_j \) and \( V_j \) satisfy (3.4).

To introduce a surface in \( G \), let \( F(u, v) \in G \) for every \( (u, v) \) in a neighborhood of \( \mathbb{R}^2 \). The first fundamental form of \( F \) is defined by

\[
\left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \right\rangle du^2 + 2 \left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \right\rangle dudv + \left\langle \frac{\partial F}{\partial v}, \frac{\partial F}{\partial v} \right\rangle dv^2. \tag{3.5}
\]

Let \( N^{(s)} \in G, s = 1, \ldots, n - 2 \) be the elements of \( G \) defined by \( \left\langle N^{(s)}, N^{(t)} \right\rangle = 1, \left\langle F_u, N^{(t)} \right\rangle = \left\langle F_v, N^{(t)} \right\rangle = 0 \). Then the second fundamental forms of \( F \) are defined by

\[
\left\langle \frac{\partial^2 F}{\partial u^2}, N^{(s)} \right\rangle du^2 + 2 \left\langle \frac{\partial^2 F}{\partial u \partial v}, N^{(s)} \right\rangle dudv + \left\langle \frac{\partial^2 F}{\partial v^2}, N^{(s)} \right\rangle dv^2, \tag{3.6}
\]

for \( s = 1, \ldots, n - 2 \). Surfaces in \( G \) can be related to surfaces in \( \mathcal{G} \) by using the adjoint representation to write

\[
\frac{\partial F}{\partial u} = \Phi^{-1}a_je_j\Phi, \quad \frac{\partial F}{\partial v} = \Phi^{-1}b_je_j\Phi, \tag{3.7}
\]

where \( a_j \) and \( b_j \) are some functions of \( (u, v) \). Differentiating the first expression in (3.7) with respect to \( v \) and using the fact that \( (\Phi^{-1})_v = -\Phi^{-1}\Phi^{-1} \), then modulo (3.7), we obtain

\[
\frac{\partial^2 F}{\partial u \partial v} = -\Phi^{-1}V_s a_je_j\Phi + \Phi^{-1}b_je_j\Phi + \Phi^{-1}a_je_j V_s e_s \Phi = \Phi^{-1}(\frac{\partial a_j}{\partial v} e_j - V_s a_m c_{smj} e_j)\Phi. \tag{3.8}
\]

In a similar way, differentiating \( F_v \) with respect to \( u \), we have

\[
\frac{\partial^2 F}{\partial u \partial v} = \Phi^{-1}(\frac{\partial b_j}{\partial u} e_j - U_s b_m c_{smj} e_j)\Phi. \tag{3.9}
\]

Requiring that the derivatives in (3.8) and (3.9) match gives the following result.

**Proposition 3.2.** Let \( \Phi(u, v) \in G \) be a surface defined by (3.1). Let \( F(u, v) \in G \) be a differentiable function of \( u \) and \( v \) for every \( (u, v) \) in some neighborhood of \( \mathbb{R}^2 \). Then (3.7) defines a surface \( F(u, v) \in G \) if and only if \( a_j \) and \( b_j \) satisfy

\[
\frac{\partial a_j}{\partial v} + a_k V_m c_{kmj} = \frac{\partial b_j}{\partial u} + b_k V_m c_{kmj}, \quad k, m, j = 1, \ldots, n. \]

It is often possible to calculate \( a_j, b_j \) and \( F \) explicitly.

**Theorem 3.1.** Let \( U_j(u, v) \) and \( V_j(u, v), j = 1, \ldots, n \) be differentiable functions of \( u \) and \( v \) for every \( (u, v) \) in some neighborhood of \( \mathbb{R}^2 \). Let \( \{e_j\}_{j=1}^n \) be an orthonormal basis in the Lie algebra \( \mathcal{G} \) of the Lie group \( G \).
Suppose that \( U_j \) and \( V_j \) depend on a parameter \( \lambda \) and satisfy (3.3), where \( c_{kmj}, k, m, j = 1, \ldots, n \) are the structure constants associated with \( \mathcal{G} \), but \( \lambda \) does not appear explicitly in (3.4). Define \( U \) and \( V \) as follows
\[
U = U_j e_j, \quad V = V_j e_j. \tag{3.10}
\]

(i) If \( A \) and \( B \) are defined to be
\[
A = a_j e_j = \alpha_1 \frac{\partial U}{\partial u} + \alpha_2 \frac{\partial U}{\partial v} + \alpha_3 \frac{\partial U}{\partial \lambda} + \alpha_4 \frac{\partial}{\partial u}(uU) + \alpha_5 \frac{\partial U}{\partial v} + [U, M],
\]
\[
B = b_j e_j = \alpha_1 \frac{\partial V}{\partial u} + \alpha_2 \frac{\partial V}{\partial v} + \alpha_3 \frac{\partial V}{\partial \lambda} + \alpha_4 u \frac{\partial V}{\partial u} + \alpha_5 \frac{\partial}{\partial v}(vV) + [V, M], \tag{3.11}
\]
where \( M = m_j e_j \) and \( \alpha_1, \ldots, \alpha_5, m_1, \ldots, m_n \) are constant scalars, then the equations
\[
\frac{\partial F}{\partial u} = \Phi^{-1} A, \quad \frac{\partial F}{\partial v} = \Phi^{-1} B, \tag{3.12}
\]
are compatible, and can be used to define a surface \( F(u, v) \in \mathcal{G} \).

(ii) The solution of (3.12) where \( A \) and \( B \) are defined by (3.11) is, to within an additive constant, given by
\[
F = \alpha_1 \Phi^{-1} U \Phi + \alpha_2 \Phi^{-1} V \Phi + \alpha_3 \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} + \alpha_4 u \Phi^{-1} U \Phi + \alpha_5 v \Phi^{-1} V \Phi - \Phi^{-1} M \Phi. \tag{3.13}
\]

Proof: The equations (3.1) are compatible if and only if (3.4) holds. From the equations for \( F \), we determine that
\[
\frac{\partial^2 F}{\partial v \partial u} = \frac{\partial \Phi^{-1}}{\partial v} A \Phi + \Phi^{-1} \frac{\partial A}{\partial v} \Phi + \Phi^{-1} A \frac{\partial \Phi}{\partial v} = -\Phi^{-1} V m e_m A \Phi + \Phi^{-1} \frac{\partial A}{\partial v} \Phi + \Phi^{-1} A V m e_m \Phi
\]
\[
= \Phi^{-1} \left( \frac{\partial A}{\partial v} - [V, A] \right) \Phi. \tag{3.14}
\]
Similarly,
\[
\frac{\partial^2 F}{\partial u \partial v} = \Phi^{-1} \left( \frac{\partial B}{\partial u} + [B, U] \right) \Phi. \tag{3.15}
\]
Upon equating the derivatives in (3.14) and (3.15) and moving all terms to the same side, it follows that
\[
\frac{\partial A}{\partial v} - \frac{\partial B}{\partial u} + [A, V] + [U, B] = 0. \tag{3.16}
\]
Suppose \( A \) and \( B \) are defined by (3.11) and \( U \) and \( V \) satisfy (3.4), then by direct calculation, we have
\[
\frac{\partial A}{\partial v} - \frac{\partial B}{\partial u} = \alpha_1 \frac{\partial U}{\partial v} - \frac{\partial V}{\partial u} + \alpha_2 \frac{\partial U}{\partial v} - \frac{\partial V}{\partial u} + \alpha_3 \frac{\partial^2 U}{\partial v \partial \lambda} - \frac{\partial^2 V}{\partial u \partial \lambda} + \alpha_4 \left( \frac{\partial (uU)}{\partial u} - \frac{\partial V}{\partial u} \right).
\]
\[ +\alpha_5 \frac{\partial}{\partial v} \left( v \frac{\partial U}{\partial v} - \frac{\partial}{\partial u} (vV) \right) + \left[ \frac{\partial U}{\partial v}, M \right] - \left[ \frac{\partial V}{\partial u}, M \right] \]

\[ = \alpha_1 \frac{\partial}{\partial u} [U, V] - \alpha_2 \frac{\partial}{\partial v} [U, V] - \alpha_3 \frac{\partial}{\partial \lambda} [U, V] - \alpha_4 \frac{\partial}{\partial u} (u[M, V]) - \alpha_5 \frac{\partial}{\partial v} (v[U, V]) \quad (3.17) \]

Using \( A \) and \( B \) given in (3.11), let us work out the terms of \([A, V] + [U, B]\) according to each coefficient \( \alpha_j \) one at a time,

\[ \alpha_1 \frac{\partial U}{\partial u}[V] + \alpha_1 [U, \frac{\partial V}{\partial u}] = \alpha_1 \frac{\partial}{\partial u} [U, V], \]

\[ \alpha_2 \frac{\partial U}{\partial v}[V] + \alpha_2 [U, \frac{\partial V}{\partial v}] = \alpha_2 \frac{\partial}{\partial v} [U, V], \]

\[ \alpha_3 \frac{\partial U}{\partial \lambda}[V] + \alpha_3 [U, \frac{\partial V}{\partial \lambda}] = \alpha_3 \frac{\partial}{\partial \lambda} [U, V], \]

\[ \alpha_4 \left( \frac{\partial}{\partial u} (uU), V \right) + \alpha_4 [U, \frac{\partial V}{\partial u}] = \alpha_4 \frac{\partial}{\partial u} (u[U, V]), \]

\[ \alpha_5 [v \frac{\partial U}{\partial v}, V] + \alpha_5 [U, \frac{\partial V}{\partial v}] = \alpha_5 \frac{\partial}{\partial v} (v[U, V]). \]

Finally, using Jacobi’s identity, we can write

\[ [[U, M], V] + [U, [V, M]] = [V, [M, U]] + [U, [V, M]] = [[U, V], M] = -\left[ \frac{\partial U}{\partial v} - \frac{\partial V}{\partial u}, M \right]. \]

Substituting all of these results for the brackets (3.18) as well as for \( A_v - B_u \) from (3.17) into the left-hand side of (3.16), it can be seen that (3.16) is satisfied identically.

(ii) To prove that \( F \) given by (3.13) satisfies (3.12), differentiate \( F \) with respect to \( u \) to obtain

\[ \frac{\partial F}{\partial u} = -\alpha_1 \Phi^{-1} U U \Phi + \alpha_1 \Phi^{-1} \frac{\partial U}{\partial u} \Phi + \alpha_1 \Phi^{-1} U U \Phi - \alpha_2 \Phi^{-1} U V \Phi + \alpha_2 \Phi^{-1} \frac{\partial V}{\partial u} \Phi + \alpha_2 \Phi^{-1} V U \Phi \]

\[ -\alpha_3 \Phi^{-1} U \frac{\partial \Phi}{\partial \lambda} + \alpha_4 \Phi^{-1} \frac{\partial^2 \Phi}{\partial u \partial \lambda} + \alpha_4 \Phi^{-1} U \Phi - \alpha_4 u \Phi^{-1} U U \Phi + \alpha_4 u \Phi^{-1} \frac{\partial U}{\partial u} \Phi + \alpha_4 u \Phi^{-1} U U \Phi \]

\[ -\alpha_5 v \Phi^{-1} U V \Phi + \alpha_5 v \Phi^{-1} \frac{\partial V}{\partial u} \Phi + \alpha_5 v \Phi^{-1} V U \Phi + \Phi^{-1} U M \Phi - \Phi^{-1} M U \Phi \]

\[ = \Phi^{-1} \left\{ \alpha_1 \frac{\partial U}{\partial u} + \alpha_2 \left( \frac{\partial V}{\partial u} - [U, V] \right) - \alpha_3 \frac{\partial U}{\partial \lambda} + \alpha_4 \frac{\partial}{\partial u} (uU) + \alpha_5 (v \frac{\partial V}{\partial u} - v[U, V]) + [U, M] \right\} \Phi. \]
Using (3.4), this simplifies to the form
\[ \frac{\partial F}{\partial u} = \Phi^{-1} \{ \alpha_1 \frac{\partial U}{\partial u} + \alpha_2 \frac{\partial V}{\partial v} - \alpha_3 \frac{\partial U}{\partial \lambda} + \alpha_4 \frac{\partial}{\partial u}(uU) + \alpha_5 v \frac{\partial U}{\partial v} + [U, M] \} \Phi = \Phi^{-1} A \Phi, \]
as required. Similarly, the derivative of \( F \) with respect to \( v \) is calculated in the same way, and the second equation of (3.12) then results.

Using a variation of parameter, it follows that this \( F \) is unique to within a constant matrix. ♣

This is really a consequence of the fact that (3.16) is the variational equation of (3.4). In fact, if \( U \) and \( V \) are replaced by \( U + \epsilon A \) and \( V + \epsilon B \), then the \( O(\epsilon) \) term of (3.4) yields (3.16). This means that every symmetry of (3.4) implies a solution of (3.16).

It will be useful and instructive to write down the previous Theorem for the case in which the group \( G \) is \( SU(2) \). In this case, \( e_j = -i\sigma_j \), for \( j = 1, 2, 3 \) where \( \sigma_j \) are the Pauli matrices given by
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (3.19)
and the structure constants are given by \( c_{ijk} = 2\epsilon_{ijk} \), where \( \epsilon_{ijk}, i, j, k = 1, 2, 3 \) is the usual antisymmetric tensor.

To the vector \( F = (F_1, F_2, F_3)^T \in \mathbb{R}^3 \), we associate the matrix \( F = F \sigma_j \in su(2) \), which we write
\[ F = -iF_j e_j = \begin{pmatrix} -iF_3 & -F_2 + iF_1 \\ F_2 - iF_1 & iF_3 \end{pmatrix}. \] (3.20)
The problem of immersing the 2-dimensional surface \( x_j = F_j(u, v), j = 1, 2, 3 \) into 3-dimensional space becomes the problem of studying the relationship between the 3-dimensional sphere \( \Phi(u, v) \in SU(2) \) and the two-dimensional surface \( F(u, v) \in su(2) \). Thus taking \( e_j = -i\sigma_j \) and \( c_{ijk} = 2\epsilon_{ijk} \) in Theorem 3.1, this theorem can be restated for the case of \( SU(2) \).

**Theorem 3.2.** Let \( U(u, v) \) and \( V(u, v) \in su(2) \) be differentiable functions of \( u \) and \( v \) for every \( (u, v) \) in some neighborhood of \( \mathbb{R}^2 \). Assume that the functions \( U \) and \( V \) satisfy equation (3.4). Then the equations
\[ \frac{\partial \Phi}{\partial u} = U \Phi, \quad \frac{\partial \Phi}{\partial v} = V \Phi, \] (3.21)
define a 2-dimensional surface \( \Phi(u, v) \in SU(2) \).

Let \( A(u, v) \) and \( B(u, v) \in su(2) \) be real, differentiable functions of \( u \) and \( v \) for every \( (u, v) \) in some neighborhood of \( \mathbb{R}^2 \). In addition to this, assume that these functions satisfy (3.16). Then equations (3.12) together with \( F = -iF \sigma_j \) define a 2-dimensional
surface \( x_j = F_j(u, v) \in \mathbb{R}, \; j = 1, 2, 3 \) in a 3-dimensional Euclidean space. The first and second fundamental forms of this surface are

\[ \langle A, A \rangle \, du^2 + 2 \langle A, B \rangle \, dudv + \langle B, B \rangle \, dv^2, \quad (3.22) \]

and

\[ \langle \frac{\partial A}{\partial u} + [A, U], C \rangle \, du^2 + 2 \langle \frac{\partial A}{\partial v} + [A, V], C \rangle \, dudv + \langle \frac{\partial B}{\partial v} + [B, V], C \rangle \, dv^2, \quad (3.23) \]

respectively. In (3.22) and (3.23), we have

\[ \langle A, B \rangle = -\frac{1}{2} \text{tr} (A, B), \quad C = \frac{[A, B]}{|[A, B]|}, \quad |A| = \sqrt{\langle A, A \rangle}. \quad (3.24) \]

Let us consider the following example. In Theorem 3.2, let us put \( A = -ia\sigma_1, \quad B = -i(b_1\sigma_1 + b_2\sigma_2), \quad U = -\frac{i}{2}U_j\sigma_j \) and \( V = -\frac{i}{2}V_j\sigma_j \) into (3.16). After using the commutation relations, we obtain

\[ (-\frac{\partial a}{\partial v} + \frac{\partial b_1}{\partial u} + b_2U_3)\sigma_1 + (\frac{\partial b_2}{\partial u} + aV_3 - b_1U_3)\sigma_2 + (-aV_2 + b_1U_2 - b_2U_1)\sigma_3 = 0. \quad (3.25) \]

For this to hold, the coefficients of \( \sigma_j \) must vanish giving the system of equations

\[ \frac{\partial a}{\partial v} - \frac{\partial b_1}{\partial u} - b_2U_3 = 0, \quad \frac{\partial b_2}{\partial u} + aV_3 - b_1U_3 = 0, \quad aV_2 - b_1U_2 + b_2U_1 = 0. \quad (3.26) \]

The first and third equations of (3.26) imply that

\[ U_3 = \frac{1}{b_2}(\frac{\partial a}{\partial v} - \frac{\partial b_1}{\partial u}), \quad V_2 = \frac{1}{a}[b_1U_2 - b_2U_1]. \]

Putting these results in the second equation of (3.26) gives

\[ V_3 = \frac{1}{ab_2}[b_1(\frac{\partial a}{\partial v} - b_1\frac{\partial b_1}{\partial u} - b_2\frac{\partial b_2}{\partial u})]. \]

Let \( \Phi \in SU(2) \), then the equations for \( F \) are obtained by substituting \( A \) and \( B \). They take the form

\[ \frac{\partial F}{\partial u} = -\Phi^{-1}a\sigma_1\Phi, \quad \frac{\partial F}{\partial v} = -i\Phi^{-1}(b_1\sigma_1 + b_2\sigma_2)\Phi. \quad (3.27) \]

Then \( F \) defines a 2-dimensional surface in \( \mathbb{R}^3 \). Moreover,

\[ \langle A, A \rangle = a^2, \quad \langle A, B \rangle = ab_1, \quad \langle B, B \rangle = b_1^2 + b_2^2. \]
give the coefficients of the first fundamental form (3.22), which can be written
\[
I = a^2 \, du^2 + 2ab_1 \, dudv + (b_1^2 + b_2^2) \, dv^2. \tag{3.28}
\]
With \(A = -i\sigma_1\) and \(B = -i(b_1\sigma_1 + b_2\sigma_2)\), we calculate \([A, B] = -[a\sigma_1, b_1\sigma_1 + b_2\sigma_2] = -2ab_2\sigma_3\) and \(C = -\sigma_3\). Therefore, \([A, U] = -\frac{1}{2}aU_1[\sigma_1, \sigma_j] = -aU_2\sigma_3\) and
\[
\langle \frac{\partial A}{\partial u} + [A, U], C \rangle = \langle -i\frac{\partial a}{\partial u}\sigma_1 - aU_2\sigma_3, -\sigma_3 \rangle = aU_2.
\]
Similarly, it follows that \([A, V] = -a(\sigma_3 V_2 - \sigma_2 V_3)\), and we must have
\[
\langle \frac{\partial A}{\partial v} + [A, V], C \rangle = aV_2, \quad \langle \frac{\partial B}{\partial v} + [B, V], C \rangle = b_1V_2 - b_2V_1.
\]
Putting these together in (3.23), the second fundamental form is given by
\[
II = aU_2 \, du^2 + 2aV_2 \, dudv + (b_1V_2 - b_2V_1) \, dv^2. \tag{3.29}
\]
In terms of matrices, these fundamental forms are given by
\[
I = \begin{pmatrix} a^2 & ab_1 \\ ab_1 & b_1^2 + b_2^2 \end{pmatrix}, \quad II = \begin{pmatrix} aU_2 & aV_2 \\ aV_2 & b_1V_2 - b_2V_1 \end{pmatrix}. \tag{3.30}
\]
The Gauss and mean curvature are defined by
\[
K = \det(II \cdot I^{-1}) = -(\frac{U_1}{a})^2 + \frac{U_2}{a} (\frac{b_1U_1 - aV_1}{ab_2}), \quad H = -\text{tr}(II \cdot I^{-1}) = -\frac{U_2}{a} \frac{b_1U_1 - aV_1}{ab_2}. \tag{3.31}
\]
It can be shown the surface \(F\) is unique up to position in space. Given the fundamental forms (3.30), \(U_2, V_2\) and \(V_1\) can be solved for. Since these functions satisfy the Gauss-Codazzi equations (3.4), \(\Phi \in SU(2)\) can be defined by (3.21) to within three constants. Equations (3.12) imply \(F \in su(2)\) within three additional constants. These six arbitrary constants correspond to arbitrary motions of the surface in \(\mathbb{R}^3\). Indeed, the transformations \(\bar{F} = FF^{-1} + \bar{A}, \bar{N} = FNf^{-1}, \bar{\Phi} = \Phi f, f \in SU(2), \bar{A} \in su(2)\) leave (3.21) and the fundamental forms invariant. The constants of \(\bar{A}\) introduce a translation while the constants of \(f\) introduce a rotation. Therefore, six arbitrary constants appear.

Let us summarize this collection of results in Theorem 3.3.

**Theorem 3.3.** Let \(U_1, U_2, V_1, a, b_1\) and \(b_2\) such that \(a \neq 0, b_2 \neq 0\) be real differentiable functions of \(u\) and \(v\) for every \((u, v)\) in some neighborhood of \(\mathbb{R}^2\). Assume that these functions satisfy the Gauss-Codazzi equations (3.4), where \(U_3, V_2\) and \(V_3\) are defined by
\[
U_3 = \frac{1}{b_2} (\frac{\partial a}{\partial v} - \frac{\partial b_1}{\partial u}), \quad V_2 = \frac{1}{a} (b_1U_2 - b_2U_1), \quad V_3 = \frac{1}{ab_2} (b_1 \frac{\partial a}{\partial v} - b_1 \frac{\partial b_1}{\partial u} - b_2 \frac{\partial b_2}{\partial u}). \tag{3.32}
\]
Let \( \Phi \in SU(2) \) be defined by (3.21). Then the equations
\[
\frac{\partial F}{\partial u} = -i\Phi^{-1}a\sigma_1 \Phi, \quad \frac{\partial F}{\partial v} = -i\Phi^{-1}(b_1\sigma_1 + b_2\sigma_2) \Phi, \tag{3.33}
\]
where \( \sigma_j \) are the Pauli matrices (3.19), define a 2-dimensional surface
\[
x_j = F_j(u,v), \quad j = 1, 2, 2, \tag{3.34}
\]
in \( \mathbb{R}^3 \). Its first and second fundamental forms are given in (3.28) and (3.29). The Gauss and mean curvatures are given in (3.31). This surface is unique to within position in space.

To close this section, a final result and an application along these lines is presented below and gives an explicit construction of functions \( A \) and \( B \) as well as the immersion function \( F \) based on the symmetries of (3.4) and (3.1) [25].

**Theorem 3.4.** Suppose that \( U(u,v) \) and \( V(u,v) \) can be parametrized in terms of \( \lambda \) and a scalar function \( \theta(u,v) \) in such a way that (3.4) is equivalent to a single partial differential equation for \( \theta(u,v) \) independent of \( \Lambda \). This equation, which by definition is called an integrable partial differential equation, possesses the Lax pair defined by (3.21). Define the \( su(2) \) valued functions \( A(u,v,\lambda) \) and \( B(u,v,\lambda) \) by
\[
A = \alpha \frac{\partial U}{\partial \lambda} + \frac{\partial M}{\partial u} + [M, U] + U'\phi, \tag{3.35}
\]
\[
B = \alpha \frac{\partial V}{\partial \lambda} + \frac{\partial M}{\partial v} + [M, V] + V'\phi, \tag{3.36}
\]
where \( \alpha(\lambda) \) is an arbitrary scalar function of \( \lambda \). Also, \( M(u,v;\lambda) \) is an \( su(2) \) valued arbitrary function of \( u, v \) and \( \lambda \) and the scalar \( \phi \) is a symmetry of the partial differential equation satisfied by the function \( \theta(u,v) \). The prime denotes Fréchet differentiation. Then there exists a surface with immersion \( F(u,v;\lambda) \) defined in terms of \( A, B, \Phi \) by (3.35) and (3.36). Furthermore, \( F \) to within an additive constant, is given by
\[
F = \Phi^{-1}(\alpha \frac{\partial \Phi}{\partial \lambda} + M\Phi + \Phi'\phi). \tag{3.37}
\]

**Proof:** This is similar to Theorem 3.2, so we just verify (3.16),
\[
\frac{\partial A}{\partial v} = \alpha \frac{\partial}{\partial \lambda}(V_u - [U, V]) + \frac{\partial^2 M}{\partial v \partial u} + \frac{\partial M}{\partial v} - [M, U] + [M, V_u - [U, V]] + (U'\phi)_v,
\]
\[
\frac{\partial B}{\partial u} = \alpha \frac{\partial}{\partial \lambda}V_u + \frac{\partial^2 M}{\partial u \partial v} + [M_u, V] + [M, V_u] + (V'\phi)_u.
\]
Then substituting these into (3.16) and simplifying, we find that
\[
\frac{\partial A}{\partial v} - \frac{\partial B}{\partial u} + [A, V] + [U, B] = -[M, [U, V]] + [M, U], V] + [U, [M, V]]
\]
\[
+ (U' \phi)_v - (V' \phi)_u + [U' \phi, V] + [U, V' \phi].
\]
Using Jacobi’s identity, the first three terms combine to give zero, and the last terms are the Fréchet derivative of (3.4), so (3.16) holds.

Let us apply Theorem 3.4 to the case of the sine-Gordon equation which is given by
\[
\vartheta_{uv} = \sin \vartheta.
\]
(3.38)
In (3.38), \(\vartheta(u, v)\) is a real, scalar function and time is denoted by \(v\). Define \(U(u, v, \lambda)\) and \(V(u, v, \lambda)\) in terms of the Pauli matrices as
\[
U = \frac{i}{2}(-\vartheta_u \sigma_1 + \lambda \sigma_3), \quad V = \frac{i}{2\lambda}(\sin \vartheta \sigma_2 - \cos \vartheta \sigma_3).
\]
(3.39)
Let \(\varphi\) be a solution of the equation
\[
\varphi_{uv} = \varphi \cos \vartheta,
\]
(3.40)
so \(\varphi\) is considered to be a symmetry of (3.38), and solutions of (3.40) contain the geometrical and generalized symmetries of (3.38). For each \(\varphi\), Theorem 3.4, with \(\alpha = 0\), \(M = 0\) implies the surface constructed from
\[
A = \frac{i}{2} \frac{\partial \varphi}{\partial u} \sigma_1, \quad B = \frac{i}{2\lambda} \varphi (\cos \vartheta \sigma_2 + \sin \vartheta \sigma_3),
\]
(3.41)
has the immersion function given by \(F = \Phi^{-1} \Phi'(\varphi)\). Sine-Gordon equation (3.38) is an integrable equation and hence admits infinitely many symmetries, which are referred to as generalized symmetries.

Let \(S\) be the surface generated by \(U, V, A, B\) defined by (3.39)-(3.41). The first and second fundamental forms, Gaussian curvature and mean curvatures of this surface are given by
\[
I = \frac{1}{4}(\varphi_u^2 \, du^2 + \frac{1}{\lambda^2} \varphi_v^2 \, dv^2), \quad II = \frac{1}{2}(\lambda \varphi_u \sin \vartheta \, du^2 + \frac{1}{\lambda} \varphi_v \, dv^2),
\]
(3.42)
\[
K = \frac{4\lambda^2 \varphi_u \sin \vartheta}{\varphi \varphi_u}, \quad H = \frac{2\lambda(\varphi_u \varphi_v + \varphi \sin \vartheta)}{\varphi \varphi_u}.
\]
(3.43)

**Theorem 3.5.** Let \(S\) be a regular surface defined by (3.42) and (3.43) in terms of a generalized symmetry of sine-Gordon equation (3.38). If \(S\) is an oriented, compact and connected surface, then it is homeomorphic to a sphere.
Proof: All compact, connected surfaces with the same Euler-Poincaré characteristic are homeomorphic. For compact surfaces, the Euler-Poincaré characteristic $\chi$ is given by

$$\chi = \frac{1}{2\pi} \int \int_{\Omega} \sqrt{\det(g)} K \, dudv. \quad (3.44)$$

From (3.43), the integrand can be worked out to be

$$\sqrt{g}K = \lambda \vartheta_v \sin \vartheta = -\lambda (\cos \vartheta)_v. \quad (3.45)$$

Hence, $\chi$ is independent of the deformations $\varphi$, and putting (3.45) into (3.44), we obtain

$$\chi = -\frac{\lambda}{2\pi} \int \int_{\Omega} (\cos \vartheta)_v \, dudv. \quad (3.46)$$

This implies that $\chi$ has the same value for all generalized symmetries and hence for all sine-Gordon deformed surfaces. It suffices to take a simple case to calculate $\chi$. With $\varphi = \vartheta_v$, this is a sphere with $\chi = 2$. Hence, all deformed surfaces have the Euler-Poincaré characteristic $\chi = 2$.

4. DIFFERENTIAL FORMS, MOVING FRAMES AND SURFACES

4.1. Introduction to Moving Frames.

The use of moving frames and exterior differentiation together has become a powerful tool in differential geometry [16]. Suppose $f : M \rightarrow \mathbb{R}^N$ is an embedding of an $m$-dimensional oriented smooth submanifold in $\mathbb{R}^N$. The range of values for the indices is $1 \leq i, j, k, l \leq m$, $m + 1 \leq A, B, C, D \leq N$, $1 \leq \alpha, \beta, \gamma, \delta \leq N$. Attach an orthogonal frame $(p; e_1, \ldots, e_N)$ to every point in $M$ such that $e_i$ is a tangent vector of $M$ at $p$, $e_A$ is a normal vector of $M$ at $p$ and $(e_1, \ldots, e_m)$ and $(e_1, \ldots, e_N)$ have the same orientation as a fixed frame $(0, \delta_1, \ldots, \delta_N)$ in $\mathbb{R}^N$. Suppose there is a frame field on an open neighborhood $U$ of $M$, which depends continuously and smoothly on the local coordinates of $U$. Then we usually call such a local orthogonal frame a Darboux frame on the submanifold $M$. There always exists a Darboux frame in a sufficiently small neighborhood of every point in $M$, and the following transformations apply

$$e_i' = \sum_{j=1}^{m} a_{ij} e_j, \quad e_A' = \sum_{B=m+1}^{N} a_{AB} e_B, \quad (4.1)$$

where $a_{ij}$, $a_{AB}$ are smooth functions on $U$ such that $(a_{ij}) \in SO(m, \mathbb{R})$, $(a_{AB}) \in SO(N-m; \mathbb{R})$. 

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Denote by \( \omega_\alpha, \omega_\alpha\beta \) the differential 1-forms obtained by pulling the relative components of moving frames in \( \mathbb{R}^N \) back to \( U \) by \( f^* \). Obviously, these 1-forms on \( U \) still satisfy the structure equations

\[
d\omega_\alpha = \sum_{\gamma=1}^{N} \omega_\beta \wedge \omega_{\beta\alpha}, \quad d\omega_{\alpha\beta} = \sum_{\gamma=1}^{N} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \tag{4.2}
\]

Since the origin \( p \) of the Darboux frame is in \( M \), and \( e_i \) is a tangent vector of \( M \) at \( p \), we have

\[
dp = \sum_{i=1}^{m} \omega_i e_i, \quad \omega_A = 0, \tag{4.3}
\]

and the \( \omega_i, 1 \leq i \leq m \) are linearly independent everywhere. Suppose

\[
I = dp \cdot dp = \sum_{i=1}^{m} (\omega_i)^2, \quad dA = \omega_1 \wedge \cdots \wedge \omega_m. \tag{4.4}
\]

These quantities are independent of the transformation of Darboux frame, so they are defined on the whole manifold \( M \). They are referred to as the first fundamental form and the area element of \( M \). With \( I \) as the Riemannian metric, the manifold \( M \) becomes a Riemannian manifold, so \( M \) has a Riemannian metric induced from \( \mathbb{R}^N \). The equations of motion for a Darboux frame can be written

\[
de_i = \sum_{j=1}^{m} \omega_{ij} e_j + \sum_{A=m+1}^{N} \omega_{iA} e_A, \quad de_B = \sum_{j=1}^{m} \omega_{Bj} e_j + \sum_{A=m+1}^{N} \omega_{BA} e_A, \tag{4.5}
\]

where \( \omega_\alpha, \omega_\alpha\beta = -\omega_{\beta\alpha} \) are the relative components which satisfy the structure equations

\[
d\omega_i = \sum_{j=1}^{m} \omega_j \wedge \omega_{ji}, \quad 0 = \sum_{j=1}^{m} \omega_j \wedge \omega_{jA}, \tag{4.6}
\]

\[
d\omega_{ij} = \sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj} + \sum_{A=m+1}^{N} \omega_{iA} \wedge \omega_{Aj},
\]

\[
d\omega_{iB} = \sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kB} + \sum_{A=m+1}^{N} \omega_{iA} \wedge \omega_{AB}, \tag{4.7}
\]

\[
d\omega_{AB} = \sum_{k=1}^{m} \omega_{Ak} \wedge \omega_{kB} + \sum_{C=m+1}^{N} \omega_{AC} \wedge \omega_{CB}.
\]
By the Fundamental Theorem of Riemannian Geometry, the first formula of (4.6) and the skew-symmetry $\omega_{ij} + \omega_{ji} = 0$ together imply that $\omega_{ij}$ is the Levi-Civita connection on the Riemannian manifold $M$,

\[
De_j = \sum_{j=1}^{m} \omega_{ij} e_j. \tag{4.8}
\]

By the first formula in (4.5), we have that $De_i$ is the orthogonal projection of $de_i$ on a tangent plane of $M$.

By Cartan’s lemma [16], it follows from the second equation of (4.6) that

\[
\omega_{jA} = \sum_{i=1}^{m} h_{Aji} \omega_i, \quad h_{Aji} = h_{Aij}. \tag{4.9}
\]

Let us put

\[
II = \sum_{i,A} \omega_i \omega_i A e_A = \sum_{A=m+1}^{N} \left( \sum_{i,j=1}^{m} h_{Aij} \omega_i \omega_j \right) e_A. \tag{4.10}
\]

Then $II$ is independent of transformation of Darboux frame. It is a differential 2-form defined on the whole manifold $M$, and taking values on the space of normal vectors to $M$. It is called the second fundamental form of the submanifold $M$.

The curvature form of the Levi-Civita connection on $M$ is

\[
\Omega_{ij} = d\omega_{ij} - \sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj} = \frac{1}{2} \sum_{k,l=1}^{m} R_{ijkl} \omega_k \wedge \omega_l, \tag{4.11}
\]

where $R_{ijkl}$ is the curvature tensor. From the first formula in (4.7), we obtain

\[
R_{ijkl} = \sum_{A=m+1}^{N} \left( h_{Ail} h_{Ajk} - h_{Aik} h_{Ajl} \right). \tag{4.12}
\]

This is the Gauss equation for the submanifold $M$. The last two formulas in (4.7) are the Codazzi equations from the theory of surfaces. For hypersurfaces in a Euclidean space, the above formulas can be greatly simplified.

The reason for the preceding introduction is to address the following objectives. Let us show how the structure equations for surfaces in $\mathbb{R}^3$ can be used to generate integrable equations under a suitable choice of the differential forms. This can be used to make a connection between these equations and the theory of surfaces. This is on account of the fundamental theorem for hypersurfaces in $\mathbb{R}^{m+1}$. 

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Proposition 4.1. Suppose there exist two differential 2-forms
\[ I = \sum_{i=1}^{m} (\omega_i)^2, \quad II = \sum_{i,j=1}^{m} h_{ij} \omega_i \omega_j, \]
where the \( \omega_i \) (1 ≤ i ≤ m) are linearly independent differential 1-forms depending on m variables and \( h_{ij} = h_{ji} \) are functions of these m variables. Then a necessary and sufficient condition for a hypersurface to exist in \( \mathbb{R}^{m+1} \) with I and II as its first and second fundamental forms is: I and II satisfy the Gauss-Codazzi equations (4.12) and (4.7). Moreover, any two such hypersurfaces in \( \mathbb{R}^{m+1} \) are related by a rigid motion. ♣

Now let us show how the structure equations for surfaces in \( \mathbb{R}^3 \) can be used to generate integrable equations by choosing the differential forms appropriately. This will allow us to make a connection between integrable equations and the theory of surfaces by means of Proposition 4.1. Moreover, it will be shown that these same partial differential equations will result from the integrability condition of a particular linear system of equations [26]. Thus, the idea of a Lax pair has a geometrical connotation as well [27]. It will be seen that very many integrable equations which are of interest in theoretical physics can be generated in this way [28]. It has also been shown that a system of differential forms can reproduce the complete set of differential equations generated by an \( SO(m) \) matrix Lax pair [29].

Here it will be of interest to study how the fundamental equations of surface theory can be used to reproduce the compatibility conditions obtained from a linear system in matrix form. The approach will be from the geometrical point of view using the structure equations and particular choices for the differential forms which appear in them. Concurrently, the linear matrix problem will be worked out alongside so the equations obtained each way can be compared. The coefficient matrices for the linear systems of interest will be based on the Lie algebras \( so(3) \) and \( so(2,1) \), which are isomorphic to the Lie algebras \( su(2) \) and \( sl(2,\mathbb{R}) \). It will be seen that a nonlinear partial differential equation which admits an \( SO(3) \) or \( SO(2,1) \) Lax pair must be the Gauss equation of the unit sphere in Euclidean space \( \mathbb{R}^3 \) or Minkowski space.

4.2. \( SO(3) \) Lax Pair.

Let us consider the \( so(3) \) algebra first. The general form for a differential equation in terms of two independent variables and a single unknown function \( \varphi \) can be given in the form
\[ G(\varphi, \varphi_x, \varphi_t, \varphi_{xx}, \varphi_{xt}, \varphi_{tt}, \cdots) = 0. \quad (4.13) \]
In (4.13), \( \varphi = \varphi(x, t) \) and \( \varphi_\alpha, \varphi_{\alpha\beta}, \cdots \) with \( \alpha, \beta \in \{t, x\} \) are the partial derivatives of \( \varphi \) with respect to \( x \) and \( t \). For the case of an \( SO(3) \) Lax pair, it is required that there exist
two three by three antisymmetric matrices which can be expressed in the form

\[
U = \begin{pmatrix}
0 & u_{12} & u_{13} \\
-u_{12} & 0 & u_{23} \\
-u_{13} & -u_{23} & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & v_{12} & v_{13} \\
-v_{12} & 0 & v_{23} \\
-v_{13} & -v_{23} & 0
\end{pmatrix}.
\] (4.14)

such that the two linear systems

\[
\Phi_t = U \Phi, \quad \Phi_x = V \Phi,
\] (4.15)

are completely integrable when \( \varphi \) satisfies (4.13). It is said that (4.13) is a partial differential equation admitting an \( SO(3) \) Lax pair (4.14). The elements \( u_{ij} \) and \( v_{ij} \) which appear in (4.14) will depend on \( \varphi \) and its derivatives up to a certain order. The function \( \Phi \) which appears in (4.15) can be thought of as a function in \( \mathbb{R}^3 \) or \( SO(3) \). In fact, all possible partial differential equations of the form (4.13) which do admit such Lax pairs can be determined. The integrability condition for (4.15) in terms of \( U \) and \( V \) is given as

\[
U_x - V_t + [U, V] = 0.
\] (4.16)

Theorem 4.1. With respect to the components of the matrices \( U \) and \( V \) defined by the matrices in (4.14), the independent component equations of (4.16) take the form

\[
\begin{align*}
0_{12,x} - v_{12,t} + u_{23} v_{13} - u_{13} v_{23} &= 0, \\
u_{13,x} - v_{13,t} + u_{12} v_{23} - u_{23} v_{12} &= 0, \\
u_{23,x} - v_{23,t} + u_{13} v_{12} - u_{12} v_{13} &= 0.
\end{align*}
\] (4.17)

Equations (4.16) follow by using (4.14) in (4.16) and working out all the operations to obtain the independent components of the final matrix in (4.16). It may now be asked to what extent can the equations given in (4.17) be obtained from the structure equations given earlier which govern the Darboux frame for a manifold or immersed surface \( M \subset \mathbb{R}^3 \). The method can be specialized to the case of a surface in \( \mathbb{R}^3 \). Let \( \{p; e_1, e_2, e_3\} \) be a Darboux frame with origin \( p \) in \( M \). A set of differential one-forms must be written down which depend on the functions \( \{u_{ij}\} \) and \( \{v_{ij}\} \). First, the one-forms \( \omega_i \) are defined to be

\[
\omega_1 = u_{12} \ dt + v_{12} \ dx, \quad \omega_2 = u_{13} \ dt + v_{13} \ dx, \quad \omega_3 = 0.
\] (4.18)

The forms which specify the connection are written as

\[
\begin{align*}
\omega_{12} &= u_{23} \ dt + v_{23} \ dx, \\
\omega_{13} &= u_{12} \ dt + v_{12} \ dx, \\
\omega_{23} &= u_{13} \ dt + v_{13} \ dx.
\end{align*}
\] (4.19)

The forms given in (4.19) satisfy \( \omega_{ij} + \omega_{ji} = 0 \), hence the connection in (4.19) is Riemannian. Therefore, it follows that

\[
dp = \omega_1 e_1 + \omega_2 e_2 = (u_{12} \ dt + v_{12} \ dx)e_1 + (u_{13} \ dt + v_{13} \ dx)e_2.
\] (4.20)
The frame vectors \( \{e_i\} \) must satisfy the equations
\[
de_i = \sum_{j=1}^{3} \omega_{ij} e_j. \tag{4.21}
\]

**Theorem 4.2.** The structure equations
\[
d\omega_1 = \omega_2 \wedge \omega_{21}, \quad d\omega_2 = \omega_1 \wedge \omega_{12}, \quad d\omega_{12} = \omega_{12} \wedge \omega_{32}, \tag{4.22}
\]
and the differential forms given in (4.18) and (4.19) imply the system of equations (4.17).

**Proof:** From (4.18) and (4.19), it follows that \( d\omega_1 = (u_{12,x} - v_{12,t}) \, dx \wedge dt \) and \( \omega_2 \wedge \omega_{21} = (u_{13,v_{23}} - v_{13,u_{23}}) \, dx \wedge dt, \) \( d\omega_2 = (u_{13,x} - v_{13,t}) \, dx \wedge dt \) and \( \omega_1 \wedge \omega_{12} = (u_{23,v_{23}} - u_{12,v_{23}}) \, dx \wedge dt, \) and finally, \( d\omega_{12} = (u_{23,x} - v_{23,t}) \, dx \wedge dt, \omega_{13} \wedge \omega_{32} = (u_{12,v_{13}} - u_{13,v_{12}}) \, dx \wedge dt. \) Substituting these results into (4.22), it is found that system (4.17) results. In fact, it can be seen that the two remaining structure equations \( d\omega_{13} = \omega_{12} \wedge \omega_{23} \) and \( d\omega_{23} = \omega_{21} \wedge \omega_{13} \) simply reproduce two of the equations already given in (4.17). Since \( \omega_1 \wedge \omega_{13} = 0 \) and \( \omega_2 \wedge \omega_{23} = 0, \) automatically it follows that \( \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0. \)

Using these results for \( \omega_i \) and \( \omega_{ij} \), the fundamental forms can be written down in terms of the \( u_{ij} \) and \( v_{ij} \) as follows
\[
I = \omega_1^2 + \omega_2^2 = (u_{12}^2 + u_{13}^2) \, dt^2 + 2(u_{12} v_{12} + u_{13} v_{13}) \, dt \, dx + (v_{12}^2 + v_{13}^2) \, dx^2, \\
II = h_{11} \omega_1^2 + 2h_{12} \omega_1 \omega_2 + h_{22} \omega_2^2 = \omega_1 \omega_{13} + \omega_2 \omega_{23} = I, \\
III = \omega_1^2 + \omega_2^2 = I.
\tag{4.23}
\]

Now \( \omega_{13} = h_{11} \omega_1 + h_{12} \omega_2 \) and \( \omega_{23} = h_{21} \omega_1 + h_{22} \omega_2, \) and since \( \omega_{13} = \omega_1 \) and \( \omega_{23} = \omega_2, \) the components \( h_{ij} \) of \( II \) must be \( h_{11} = h_{22} = 1, \) \( h_{12} = h_{21} = 0. \) In this case, the two expressions for \( II \) in (4.23) exactly coincide. Using this information about \( h_{ij} \) and the definition of principle curvature, it follows that \( \kappa_1 = \kappa_2 = 1. \) Therefore, every point of an associated surface is an umbilical point of \( M. \) If \( M \) is a connected surface on which every point is an umbilical point, then \( M \) must be a sphere or a plane. It follows that the mean curvature and the Gaussian curvature have the values
\[
H = \frac{1}{2}(h_{11} + h_{22}) = \frac{1}{2}(\kappa_1 + \kappa_2) = 1, \quad K = h_{11}h_{22} - h_{12}^2 = \kappa_1 \kappa_2 = 1. \tag{4.24}
\]

The Gauss equation for the sphere can be obtained from (4.17). Solving the first two equations for \( u_{23} \) and \( v_{23} \) in (4.17), we obtain
\[
u_{23} = \frac{1}{u_{12}v_{13} - u_{13}v_{12}}[\{(v_{12,t} - u_{12,x})u_{12} + (v_{13,t} - u_{13,x})u_{13}\}]
\tag{4.25}
\]
\[
v_{23} = \frac{1}{u_{12}v_{13} - u_{13}v_{12}}\[\{(v_{13,t} - u_{13,x})v_{13} + (v_{12,t} - u_{12,x})v_{12}\}]
\]
Substituting $u_{23}$ and $v_{23}$ from (4.25) into the third equation of (4.17) gives the following second order partial differential equation

$$
\left( \frac{v_{12,t} - u_{12,x}}{u_{12}v_{13} - u_{13}v_{12}} + \frac{(v_{13,t} - u_{13,x})u_{12}}{u_{12}v_{13} - u_{13}v_{12}} \right)_x - \left( \frac{(v_{13,t} - u_{13,x})v_{13}}{u_{12}v_{13} - u_{13}v_{12}} + \frac{(v_{12,t} - u_{12,x})v_{12}}{u_{12}v_{13} - u_{13}v_{12}} \right)_t + u_{13}v_{12} - u_{12}v_{13} = 0. 
$$

(4.26)

This is an equation that is of the form (4.13). Equation (4.26) is the Gauss equation for the sphere $S^2$. Therefore, the nonlinear partial differential equation (4.26) admits an $SO(3)$ Lax pair corresponding to an equation of the type (4.13).

It is convenient to refer to a partial differential equation $Q$ as a subequation of another equation $G(\varphi, \varphi_t, \varphi_x, \cdots) = 0$ if every solution of $Q = 0$ also satisfies $G = 0$. Clearly, if $Q = 0$ admits a Lax pair, then $Q = 0$ must be a subequation of each equation of (4.16). Conversely, if for given $u_{13}, u_{12}, v_{12}, v_{13}$ with $u_{12}v_{13} - u_{13}v_{12} \neq 0$, $Q = 0$ is a subequation of (4.26), then $Q = 0$ admits a Lax Pair in which $u_{23}, v_{23}$ are defined by (4.25). In this sense, all possible equations admitting $SO(3)$ Lax pairs with $u_{12}v_{13} - u_{13}v_{12} \neq 0$ have been determined.

Defining the matrix

$$
M = \begin{pmatrix}
  u_{23} & u_{13} & u_{12} \\
  v_{23} & v_{13} & v_{12}
\end{pmatrix},
$$

(4.27)

then if $\text{rank}(M) = 2$, we can assume that $v_{13}u_{12} - u_{13}v_{12} \neq 0$. When $\text{rank}(M) = 1$, the second row of (4.27) must be a multiple of the first row. In this case, we have

$$
v_{23} = \sigma u_{23}, \quad v_{13} = \sigma u_{13}, \quad v_{12} = \sigma u_{12}.
$$

(4.28)

Substituting (4.28) into the compatibility conditions (4.17), the following conservation laws result

$$
u_{23,x} - (\sigma u_{23})_t = 0, \quad u_{13,x} - (\sigma u_{13})_t = 0, \quad u_{12,x} - (\sigma u_{12})_t = 0.
$$

(4.29)

Since the integrability condition (4.16) consists of only one equation, we suppose (4.13) is the first equation here, namely $u_{23,x} - (\sigma u_{23})_t = 0$. This is the integrability condition of the system

$$
\psi_t = u_{23}\psi, \quad \psi_x = \sigma u_{23}\psi.
$$

(4.30)

In (4.30), $\psi$ is a real function and (4.30) is a $U(1)$ Lax pair. These results can be summarized in the form of the following Theorem.

**Theorem 4.3.** All nonlinear partial differential equations admitting $SO(3)$ integrable systems can be obtained in the following ways:

(i) When $\text{rank}(M) = 2$, the nonlinear equation is the Gauss equation of $S^2 \subset \mathbb{R}^3$ or its subequation and $u_{12}, u_{13}, v_{12}, v_{13}$ in (4.27) are any given functions of $\varphi$ and derivatives of $\varphi$ up to a certain order.
(ii) When rank \((M) = 1\), the nonlinear equation can be chosen to be the equation of a conservation law \(M_t + N_x = 0\), where \(N \neq 0\).

If \(u_{12}, u_{13}, v_{12}\) and \(v_{13}\) are given functions of \(\varphi\) and derivatives of \(\varphi\) up to a certain order such that \(u_{13}v_{13} - u_{12}v_{12} \neq 0\), then Theorem 4.3 gives a straightforward way of building all nonlinear partial differential equations which admit \(SO(3)\) Lax pairs. Substituting this set of functions into (4.26), the corresponding nonlinear equation (4.13) is obtained. Some examples in which this is done will be presented now.

Example 1: Let \(u_{13} = v_{12} = 0, u_{12} = \cos(\varphi/2), v_{13} = \sin(\varphi/2)\). Putting these in (4.26) gives

\[
\varphi_{tt} - \varphi_{xx} = -\sin(\varphi).
\]

Example 2: Let \(u_{13} = v_{12} = 0, u_{12} = \cosh(\varphi/2), v_{13} = \sinh(\varphi/2)\) in (4.26) gives the equation

\[
\varphi_{tt} + \varphi_{xx} = -\sinh(\varphi).
\]

Example 3: Let \(u_{13} = v_{12} = 0, u_{12} = v_{13} = e^\varphi\), then the Liouville equation is obtained

\[
\varphi_{tt} + \varphi_{xx} = -e^\varphi.
\]

Example 4: Let \(u_{13} = v_{12} = 0, u_{12} = \varphi_t\) and \(u_{13} = \varphi^2\), then (4.26) gives

\[
(2 + \varphi^2) \varphi_t + \frac{\varphi_{xt}}{\varphi^2} - 2\frac{\varphi_x \varphi_{xt}}{\varphi^3} = 0.
\]

4.3. Nonlinear Partial Differential Equations Admitting \(SO(2, 1)\) Lax Pairs.

Consider nonlinear partial differential equations of the form (4.13) which now admit the \(SO(2, 1)\) Lax pair with structure identical to (4.15), but with matrices \(U\) and \(V\) taking values in the Lie algebra \(so(2, 1)\). The case in which the integrability condition for (4.15) is the Gauss equation for \(H \subset \mathbb{R}^{2,1}\) will be examined. The case of \(S^{1,1} \subset \mathbb{R}^{2,1}\) has been examined [27].

Let us consider the case in which the relevant matrices \(U\) and \(V\) are given by

\[
U = \begin{pmatrix}
0 & u_{12} & u_{13} \\
u_{12} & 0 & u_{23} \\
u_{13} & -u_{23} & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & v_{12} & v_{13} \\
v_{12} & 0 & v_{23} \\
v_{13} & -v_{23} & 0
\end{pmatrix}.
\] (4.31)

The compatibility condition (4.16) leads to the following Theorem.

Theorem 4.4. In terms of the components of the matrices \(U\) and \(V\) defined by (4.31), the independent components of (4.16) take the form

\[
\begin{align*}
u_{12,x} - v_{12,t} + u_{23}v_{13} - u_{13}v_{23} &= 0, & u_{13,x} - v_{13,t} + u_{12}v_{23} - u_{23}v_{12} &= 0, \\
u_{23,x} - v_{23,t} + u_{12}v_{13} - u_{13}v_{12} &= 0.
\end{align*}
\] (4.32)
These same equations can be obtained directly from the structure equations by specifying a set of differential forms. First, the one forms \( \omega_i \) are defined to be
\[
\omega_1 = u_{12} \, dt + v_{12} \, dx, \quad \omega_2 = u_{13} \, dt + v_{13} \, dx, \quad \omega_3 = 0. \tag{4.33}
\]
The forms which specify the connection are given by
\[
\omega_{12} = u_{23} \, dt + v_{23} \, dx, \quad \omega_{13} = u_{12} \, dt + v_{12} \, dx, \quad \omega_{23} = u_{13} \, dt + v_{13} \, dx, \tag{4.34}
\]
which satisfy \( \omega_{ij} = -\omega_{ji} \) and \( \omega_{23} = \omega_{32} \), so the connection is quasi-Riemannian.

**Theorem 4.5.** For the space \( H^2 \subset \mathbb{R}^{2,1} \), the structure equations (4.22) and the differential forms (4.33) and (4.34) imply the system of equations (4.32).

**Proof:** From (4.33) and (4.34), it follows that \( d\omega_1 = (u_{12,x} - v_{12,t}) \, dx \wedge dt \) and \( \omega_2 \wedge \omega_{21} = (u_{13}v_{23} - u_{23}v_{13}) \, dx \wedge dt \), moreover \( d\omega_2 = (u_{13,x} - v_{13,t}) \, dx \wedge dt \) and \( \omega_1 \wedge \omega_{12} = (u_{23}v_{12} - u_{12}v_{23}) \, dx \wedge dt \), and finally \( d\omega_{12} = (u_{23,x} - v_{23,t}) \, dx \wedge dt \), with \( \omega_{13} \wedge \omega_{32} = (u_{13}v_{12} - u_{12}v_{13}) \, dx \wedge dt \). Substituting these results into (4.22), the system of equations (4.32) results. The remaining two structure equations which go with (4.22) simply reproduce two of the equations present in (4.32). Since both \( \omega_1 \wedge \omega_{13} = 0 \) and \( \omega_2 \wedge \omega_{23} = 0 \), it follows that \( \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0. \)

The fundamental forms can be calculated according to (4.23), and we have
\[
I = \omega_1^2 + \omega_2^2 = (u_{12}^2 + u_{13}^2) \, dt^2 + 2(u_{12}v_{12} + u_{13}v_{13}) \, dx dt + (v_{12}^2 + v_{13}^2) \, dx^2. \tag{4.35}
\]
It is found that \( h_{11} = 1, h_{22} = 1, h_{12} = h_{21} = 0 \), hence \( H = 1 \) and \( K = 1 \). Solving the first two equations of (4.32) for \( u_{23} \) and \( v_{23} \), we obtain
\[
\begin{align*}
  u_{23} &= \frac{1}{u_{12}v_{13} - u_{13}v_{12}} \left[ u_{12}(v_{12,t} - u_{12,x}) + u_{13}(v_{13,t} - u_{13,x}) \right], \\
  v_{23} &= \frac{1}{u_{12}v_{13} - u_{13}v_{12}} \left[ v_{13}(v_{13,t} - u_{13,x}) + v_{12}(v_{12,t} - u_{12,x}) \right].
\end{align*} \tag{4.36}
\]
Using these results in the third equation of (4.32), the Gauss equation of \( H^2 \subset \mathbb{R}^{2,1} \) is obtained
\[
\left( \frac{v_{12,t} - u_{12,x}}{u_{12}v_{13} - u_{13}v_{12}} \right) u_{12} + \left( \frac{(v_{13,t} - u_{13,x})u_{13}}{u_{12}v_{13} - u_{13}v_{12}} \right) v_{12} + u_{12}v_{13} - u_{13}v_{12} = 0. \tag{4.37}
\]
Let us summarize these results in the last Theorem of this section.

**Theorem 4.6.** A nonlinear partial differential equation which admits an \( SO(2,1) \) Lax pair with \( u_{12}v_{13} - u_{13}v_{12} \neq 0 \) is equation (4.37) or a subequation. Equation (4.37) is the
Gauss equation for $H^2 \subset R^{2,1}$, and $u_{12}, u_{13}, v_{12}$ and $v_{13}$ are given functions of $\varphi$ and the partial derivatives of $\varphi$ up to a certain order.

Several examples of equations which are given by (4.37) after picking the $u_{ij}$ and $v_{ij}$ will be given to finish the Section.

Example 1: Let $u_{13} = v_{12} = 0, u_{12} = \cos(\varphi/2), v_{13} = \sin(\varphi/2)$, then (4.37) gives

$$\varphi_{tt} - \varphi_{xx} = \sin(\varphi).$$

Example 2: Taking $u_{13} = v_{12} = 0, u_{12} = \cosh(\varphi/2), v_{13} = \sinh(\varphi/2)$, then (4.37) gives

$$\varphi_{tt} + \varphi_{xx} = \sinh(\varphi).$$

Example 3: With $u_{13} = v_{12} = 0, u_{12} = v_{13} = e^{\varphi}$. we have

$$\varphi_{tt} + \varphi_{xx} = e^{2\varphi}.$$
If we take as a basis of $\Omega^1_{MC}(G)$ the 1-forms $\omega^1, \cdots, \omega^n$, then every form $\omega$ on $G$ is of the form

$$\omega = \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \cdots i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}, \quad (5.3)$$

where $\alpha_{i_1 \cdots i_p}$ are $C^\infty$-functions on $G$ and the $p$-form $\omega$ is left-invariant if and only if the functions $\alpha_{i_1 \cdots i_p}$ are constant. Thus, $\Omega_{MC}(G)$ is the exterior algebra over $\mathbb{R}$ generated by $\Omega^1_{MC}(G)$. By the Frobenius Theorem, $\Omega_{MC}(G)$ is closed in the following sense.

**Definition 5.2.** Let $\Gamma^1$ be a vector space of finite dimension $n$ on which an exterior differential operator $d : \Gamma^1 \to \Gamma^1 \wedge \Gamma^1$ is given. The exterior algebra $\Gamma$ generated by $\Gamma^1$ and $d$ will be called a Maurer-Cartan algebra. On $\Gamma$, we have that $dd\omega = 0$, $d(\omega + \eta) = d\omega + d\eta$, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{k} \omega \wedge d\eta$, where $\omega$ is a $k$-form.

Therefore, $\Omega_{MC}(G)$ is a Maurer-Cartan algebra and will be called the Maurer-Cartan algebra of the Lie group $G$. If $\omega^i$ is a basis in $\Gamma^i$, then we have

$$d\omega^i = c^i_{jk} \omega^j \wedge \omega^k, \quad c^i_{jk} = -c^i_{kj}. \quad (5.4)$$

The $c^i_{jk}$ are called structure constants of the Maurer-Cartan algebra $\Gamma$ with respect to this basis.

**Definition 5.3.** Let $\Gamma$ be a Maurer-Cartan algebra generated by a vector space $\Gamma^1$ of dimension $n$ and $M$ a connected manifold also of dimension $n$. Let $\Omega(M)$ be the set of exterior differential forms on $M$ and $T^*_m$ the cotangent space of $M$ at $m \in M$. Then $M$ is called a Maurer-Cartan space if there exists a morphism $\varphi : \Gamma \to \Omega(M)$, such that at every point $m \in M$, the mapping $\varphi|_{\Gamma^1} : \Gamma \to T^*_m$ is a bijection.

It follows that all co-tangent spaces of a Maurer-Cartan space are isomorphic, and hence every connected open sub-manifold of a Maurer-Cartan space is again a Maurer-Cartan space.

If we have a Maurer-Cartan basis $\{\omega^1, \cdots, \omega^n\}$ of $\Gamma$ with structure constants $c^i_{jk}$, then in the $\Gamma$-space $M$ we have, due to the morphism $\varphi$, $n$ 1-forms $\tau^1, \cdots, \tau^n$ on $M$ which form a basis of $T^*_m$ at every point $m$ of $M$ such that $d\tau^i = c^i_{jk} \tau^j \wedge \tau^k$. The 1-forms $\tau^i = \varphi(\omega^i)$, $i = 1, \cdots, n$ satisfy the same structural equations as those of the Maurer-Cartan algebra $\Gamma$. The idea of $\Gamma$-cocycles play an important role in the treatment of evolution equations.

**Definition 5.4.** Let $M$ be a manifold. A $\Gamma$-cocycle is a morphism $\chi : \Gamma \to \Omega(M)$.

This need not be an injection or surjection. A $\Gamma$ cocycle means only that we have on $M$ the set of 1-forms $\sigma^1, \cdots, \sigma^n$ such that $\sigma^i = \chi(\omega^i)$ which satisfy $d\sigma^i = c^i_{jk} \sigma^j \wedge \sigma^k$. These 1-forms need not necessarily be independent. For example, the trivial $\Gamma$-cocycle is given by $\sigma^1 = \cdots = \sigma^n = 0$, and every $\Gamma$ space $M$ has the injection $\varphi : \Gamma \to \Omega(M)$ as a $\Gamma$-cocycle. These cocycles can be referred to as representative cocycles on account of the following Theorem.
**Theorem 5.2.** Let $M$ be a $\Gamma$-space with representative cocycle $\tau^1, \cdots, \tau^n$ and let $N$ be a manifold with $\Gamma$-cocycle $\sigma^1, \cdots, \sigma^n$. Then on $M \times N$, the system \( \tilde{\mu}^1 = \tilde{\tau}^1 - \tilde{\sigma}^1 = 0, \cdots, \tilde{\mu}^n = \tilde{\tau}^n - \tilde{\sigma}^n = 0 \) is completely integrable.

**Proof:** This result is actually a straightforward application of the Frobenius Theorem and the fact that $d\tau^i = c_{jk}^i \tau^j \wedge \tau^k$ and $d\sigma^i = c_{jk}^i \sigma^j \wedge \sigma^k$ with $i = 1, \cdots, n$. Thus

\[
d\tilde{\mu}^i = (c_{jk}^i \tilde{\tau}^j - c_{kj}^i \tilde{\tau}^j) \wedge \tilde{\mu}^k \equiv 0, \quad \text{mod}(\tilde{\mu}^k).
\]

From the fact that the forms \( \{\tau^i\}_1^n \) are independent, it follows that the forms $\mu^i$ are independent and hence determine a foliation of codimension $n$.

At this point, the complete $\Gamma$-space $M$ will be a subgroup of the linear group $GL(n, \mathbb{R})$, $n \in \mathbb{N}$ with Lie algebra $gl(n, \mathbb{R})$, which is isomorphic to the space of all $n \times n$ matrices, which will be called $M(n, \mathbb{R})$.

To find the Maurer-Cartan algebra of $GL(n, \mathbb{R})$, we consider the left-invariant forms $\omega^i_j$ as the elements of a matrix, namely

\[
\omega = (\omega^i_j) = X^{-1} dX,
\]

where $X$ is the natural embedding of the group into $\mathbb{R}^{n^2}$, and $X^{-1} dX$ will be called the Maurer-Cartan form. If we define

\[
d\omega = (d\omega^i_j),
\]

and regard $\omega \wedge \omega$ as the matrix product with exterior multiplication

\[
(\omega \wedge \omega)^i_j = \omega^k_i \wedge \omega^j_k.
\]

Upon differentiation, it is easy to see that

\[
d\omega = d(X^{-1} dX) = -X^{-1} dXX^{-1} \wedge dX = -(X^{-1} dX) \wedge (X^{-1} dX) = -\omega \wedge \omega.
\]

The Maurer-Cartan equation or algebra can be specified as

\[
d\omega + \omega \wedge \omega = 0.
\]

For subgroups of $GL(n, \mathbb{R})$, not all $\omega^i_j$ will be independent, but with the natural embedding, equation (5.9) still applies. For example, using $SL(2, \mathbb{R}) = \{ X \in GL(2, \mathbb{R}) | \det(X) = 1 \}$, we can use the exponential mapping to obtain the Maurer-Cartan algebra. The exponential map is given by

\[
\exp : gl(2, \mathbb{R}) \rightarrow GL(2, \mathbb{R}), \quad A \rightarrow e^{tA}, \quad t \in \mathbb{R}.
\]
Since $e^{tA}$ should be an element of $SL(2, \mathbb{R})$, it will hold that $\det(e^{tA}) = 1$. Hence, using $\det(e^{tA}) = e^{tr(tA)}$, which holds for all $t$, $tr(A) = 0$. This implies that $\omega_1^1 + \omega_2^2 = 0$, so the matrix $\omega$ in (5.5) is traceless. The Maurer-Cartan equation then takes the form,

$$d \begin{pmatrix} \omega_1^1 & \omega_2^2 \\ \omega_2^3 & -\omega_1^1 \end{pmatrix} + \begin{pmatrix} \omega_1^1 & \omega_2^2 \\ \omega_2^3 & -\omega_1^1 \end{pmatrix} \wedge \begin{pmatrix} \omega_1^1 & \omega_2^2 \\ \omega_2^3 & -\omega_1^1 \end{pmatrix} = 0. \quad (5.11)$$

The Maurer-Cartan algebra of the group $SL(2, \mathbb{R})$ is then given by (5.11) as

$$d\omega_1^1 + \omega_1^2 \wedge \omega_2^2 = 0, \quad d\omega_2^2 + 2\omega_1^1 \wedge \omega_1^2 = 0, \quad d\omega_2^3 + 2\omega_1^1 \wedge \omega_1^3 = 0. \quad (5.12)$$

The Frobenius Theorem can be applied to the last pair in (5.12) to conclude that $\omega_1^2 = 0$ and $\omega_2^2 = 0$ are integrable. Hence, $\omega_1^1 = 0$ and $\omega_2^3 = 0$ determine two foliations of $SL(2, \mathbb{R})$. If we put $\omega_2^3 = 0$, then (5.12) becomes the Maurer-Cartan algebra of a subgroup of $GL(2, \mathbb{R})$,

$$d\omega_1^1 = 0, \quad d\omega_2^3 + 2\omega_1^1 \wedge \omega_1^3 = 0. \quad (5.13)$$

This subgroup will be called $BL(2, \mathbb{R})$ here, which, using the exponential mapping, consists of matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} \subset GL(2, \mathbb{R}), \quad \beta \in \mathbb{R}, \quad \alpha \in \mathbb{R}^+. \quad (5.14)$$

By taking $\omega_2^3 = 0$, the algebra reduces to $d\omega_1^1 = 0$. Let us now define new forms in order to put (5.12) in a more convenient form. Introduce the forms $\omega^1, \omega^2$ and $\omega^3$ such that

$$\omega = \begin{pmatrix} \omega_1^1 & \omega_2^2 \\ \omega_2^3 & -\omega_1^1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \omega_1^1 & \omega_3^3 - \omega_2^2 \\ -\omega_3^3 - \omega_2^2 & -\omega_1^1 \end{pmatrix}. \quad (5.15)$$

Therefore, the $\{\omega^i\}$ satisfy the Maurer-Cartan equations

$$d\omega^1 = \omega^3 \wedge \omega^2, \quad d\omega^2 = \omega^1 \wedge \omega^3, \quad d\omega^3 = \omega^1 \wedge \omega^2. \quad (5.16)$$

It is remarkable that in the case of linear prolongation structures for exterior differential systems $\{\alpha^i\} = 0$, the prolongation condition

$$d\eta + \frac{1}{2}[\eta, \eta] = 0, \quad (5.17)$$

determines Maurer-Cartan cocycles on $\mathbb{R}^2$. Along transversal integral manifolds which are solutions of $\{\alpha^i\} = 0$, we have from (5.17)

$$d\eta = d_M \eta^i \frac{\partial}{\partial y^i}, \quad \frac{1}{2}[\eta, \eta] = (\eta^j \wedge \frac{\partial \eta^i}{\partial y^j}) \frac{\partial}{\partial y^i}. \quad (5.18)$$

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The vector valued 1-form \( \eta \) has been expressed as
\[
\eta = \eta^i \frac{\partial}{\partial y^i}, \quad \eta^i = A^i \, dx + B^i \, dt.
\] (5.19)

Therefore, \( \eta \) takes the form
\[
\eta = A^i \frac{\partial}{\partial y^i} \, dx + B^i \frac{\partial}{\partial y^i} \, dt = (A^i \, dx + B^i \, dt) \frac{\partial}{\partial y^i}.
\] (5.20)

Now \( A \) and \( B \) in their turn can be written as combinations of vertical vector fields \( X_i(y) \) with \( i = 1, 2, 3 \) such that the coefficients depend on the variables in the base manifold \( M \), but not on the variables \( y = (y^1, \cdots, y^n) \) in the fibre.

These vector fields satisfy a complete Lie algebra structure and we write
\[
\eta = \tilde{\sigma}^i X_i, \quad \tilde{\sigma}^i = \tilde{\sigma}^i_1 \, dx + \tilde{\sigma}^i_2 \, dt.
\] (5.21)

with \( \tilde{\sigma}^i \) 1-forms \( \tilde{\sigma}^i_1 \, dx + \tilde{\sigma}^i_2 \, dt \). The vector fields \( X_i \) can be written as linear fields given by
\[
X_i = \alpha_{ij}^k \, y^j \frac{\partial}{\partial y^k}, \quad i = 1, 2, 3.
\] (5.22)

For this to match \( \eta = \eta^i \frac{\partial}{\partial y^i} = \tilde{\sigma}^i X_i \), we require that
\[
\eta^i = \tilde{\sigma}^i_1 \, y^j, \quad \frac{1}{2} [\eta, \eta] = (\eta^i \wedge \frac{\partial \eta^i}{\partial y^j}) \frac{\partial}{\partial y^j}.
\] (5.23)

Therefore, we have
\[
d\eta = d\tilde{\sigma}^i_1 \, y^j \frac{\partial}{\partial y^i}, \quad \frac{1}{2} [\eta, \eta] = (\tilde{\sigma}^i_1 \, y^j \wedge \tilde{\sigma}^i_2 \, y^j) \frac{\partial}{\partial y^i}.
\] (5.24)

Combining these, we obtain
\[
d\eta + \frac{1}{2} [\eta, \eta] = (d\tilde{\sigma}^i_1 \, y^j + \tilde{\sigma}^i_2 \, y^j \wedge \tilde{\sigma}^i_2 \, y^j) \frac{\partial}{\partial y^i} = 0.
\] (5.25)

It follows that
\[
d\tilde{\sigma}^i_1 \, y^j + \tilde{\sigma}^i_2 \, y^j \wedge \tilde{\sigma}^i_2 \, y^j = 0.
\] (5.26)

If we simply put \( \tilde{\sigma}^i_1 \alpha_{ij}^k = \sigma^k_j \), then equation (5.25) becomes
\[
d\sigma^k_j + \sigma^m_j \wedge \sigma^k_m = 0.
\] (5.27)

Now if we set \( \sigma^k_j = (\sigma^k_j) \), then (5.27) is the Maurer-Cartan algebra that has been discussed and must hold along the integral transversal manifolds \( \{ \alpha^i \}_i \). Some Maurer-Cartan cocycles can be derived now for some equations by using their prolongations.
First, consider the $sl(2, \mathbb{R})$ prolongation of the sine-Gordon equation. A different application of this formalism has been considered [30]. Without deriving the prolongation, we simply present here the required results,

$$ \eta = A \, dx + B \, dt, \quad A = X_1 + pX_2, \quad B = X_1 \cos u + X_3 \sin u, \quad (5.28) $$

with $p = u_x$, and where the $X_j$ satisfy the algebra

$$ [X_1, X_2] = X_3, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = X_1. \quad (5.29) $$

Let us take the following basis for the $sl(2, \mathbb{R})$ algebra in terms of matrices

$$ \tilde{X}_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tilde{X}_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tilde{X}_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (5.30) $$

which satisfy the brackets (5.29). In terms of this basis, the prolongation (5.28) can be written

$$ \tilde{A} = \frac{1}{2} \begin{pmatrix} 0 & -1 + u_x \\ -1 - u_x & 0 \end{pmatrix}, \quad \tilde{B} = \frac{1}{2} \begin{pmatrix} \sin u & \cos u \\ -\cos u & -\sin u \end{pmatrix}. \quad (5.31) $$

Therefore, the Maurer-Cartan structure is

$$ \sigma = \frac{1}{2} \begin{pmatrix} 0 & -1 + u_x \\ -1 - u_x & 0 \end{pmatrix} \, dx + \frac{1}{2} \begin{pmatrix} \sin u & \cos u \\ -\cos u & -\sin u \end{pmatrix} \, dt. \quad (5.32) $$

From this, the components of $\sigma$ are given as

$$ \sigma^1 = \sin u \, dt, \quad \sigma^2 = dx + \cos u \, dt, \quad \sigma^3 = u_x \, dx. \quad (5.33) $$

Differentiating these, we obtain

$$ d\sigma^1 = (\cos u) u_x \, dx \wedge dt, \quad d\sigma^2 = -\sin u \, u_x \, dx \wedge dt, \quad d\sigma^3 = u_{xt} \, dt \wedge dx. \quad (5.34) $$

Using these forms, it can be seen that the first two equations of the Maurer-Cartan algebra hold identically, and the third holds provided that $u$ satisfies the sine-Gordon equation

$$ u_{xt} = \sin(u). \quad (5.35) $$

There exists an $sl(2, \mathbb{R})$ prolongation of the KdV equation which is given by

$$ A = X_1 + uX_2, \quad B = 2uX_1 + (2u^2 - q)X_2 + pX_3, \quad p = u_x, \quad q = u_{xx}. \quad (5.36) $$

As a basis for the algebra $sl(2, \mathbb{R})$, the following matrices can be taken

$$ \tilde{X}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{X}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{X}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.37) $$
It is found from (5.37) that
\[
\sigma = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} dx + \begin{pmatrix} u_x & 2u \\ 2u^2 - u_{xx} & -u \end{pmatrix} dt. \tag{5.38}
\]
The components of \(\sigma\) are then given by
\[
\sigma^1 = 2u_x dt, \quad \sigma^2 = -(1 + u) dx - (2u + 2u^2 - u_{xx}) dt,
\]
\[
\sigma^3 = (1 - u) dx + (2u - 2u^2 + u_{xx}) dt. \tag{5.39}
\]
Again, the first Maurer-Cartan equation holds identically, and the last two hold provided that the function \(u\) satisfies the KdV equation
\[
u_t - 6uu_x + u_{xxx} = 0. \tag{5.40}
\]
There is a link between these cocycles and Bäcklund transformations. The following theorem implies that there exists a function \(f : \mathbb{R}^2 \to SL(2, \mathbb{R})\) with \(f^*(\omega^i) = \sigma^i\) for \(i = 1, 2, 3\) determined up to a left-multiplication by \(A \in SL(2, \mathbb{R})\).

**Theorem 5.3.** Let \(M\) be a complete \(\Gamma\)-space with representation cocycles \(\tau^1, \cdots, \tau^n\) and let \(N\) be a simply connected manifold with \(\Gamma\)-cocycle \(\sigma^1, \cdots, \sigma^n\). Then there exists a function \(f : N \to M\) such that \(f^*(\tau^i) = \sigma^i\) for \(i = 1, \cdots, n\) which is determined up to a left-factor \(A \in Aut_\Gamma(M)\).

Thus, with every solution, or surface, in \(\mathbb{R}^3\) of the original evolution equation, there corresponds a two-dimensional surface in the group \(SL(2, \mathbb{R})\) which may be parametrized with the coordinates \((x, t) \in \mathbb{R}^2\). This fact also leads to the idea of Bäcklund transformations. To develop this idea, begin with the fact that \(SL(2, \mathbb{R})\) can be written as a product \(SL(2, \mathbb{R}) = BL(2, \mathbb{R}) \cdot SO(2)\) so that, for all \(X \subset SL(2, \mathbb{R})\), we have
\[
X = A \cdot B, \quad A \in BL(2, \mathbb{R}), \quad B \in SO(2). \tag{5.41}
\]
In terms of components, this can be expressed as,
\[
X = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & 1/\alpha \end{pmatrix} \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} = \begin{pmatrix} \alpha \cos \gamma & \alpha \sin \gamma \\ \beta \cos \gamma - \frac{1}{\alpha} \sin \gamma & \beta \sin \gamma + \frac{1}{\alpha} \cos \gamma \end{pmatrix}. \tag{5.42}
\]
By identifying corresponding terms, (5.42) implies that
\[
\xi_1^1 = \alpha \cos \gamma, \quad \xi_1^2 = \alpha \sin \gamma, \quad \xi_2^1 = \beta \cos \gamma - \frac{1}{\alpha} \sin \gamma, \quad \xi_2^2 = \beta \sin \gamma + \frac{1}{\alpha} \cos \gamma. \tag{5.43}
\]
From the first two equations in (5.43), we obtain
\[ \alpha = \sqrt{(\xi_1^2 + (\xi_1^*)^2)}, \quad \beta \cos \gamma = \xi_2^1 + \frac{\xi_1^2}{(\xi_1^2 + (\xi_1^*)^2)}, \quad \beta \sin \gamma = \xi_2^2 - \frac{\xi_1^1}{(\xi_1^2 + (\xi_1^*)^2)}. \]

From the results in (5.44), we obtain \( \beta \) in the form
\[ \beta = \frac{\xi_2^1}{(\xi_1^*)^2} - \frac{\xi_1^1}{\xi_1^2} = \frac{\xi_1^3}{\xi_1^2} \alpha - \frac{\xi_1^1}{\xi_1^2} \alpha, \]  
\[ \text{(5.45)} \]
and also,
\[ \beta = \xi_1^2, \text{ if } \xi_1^2 = 0, \quad \beta = \xi_2^2, \text{ if } \xi_1^1 = 0. \]

Finally, \( \gamma \) is defined by the equation
\[ \cos \gamma = \frac{\xi_1^1}{\alpha}, \quad \sin \gamma = \frac{\xi_2^1}{\alpha}. \]  
\[ \text{(5.46)} \]

With this type of decomposition for \( SL(2, \mathbb{R}) \) as a product, the Maurer-Cartan form can be written as
\[ \omega = X^{-1}dX = B^{-1}(A^{-1}dA) \cdot B + B^{-1} \cdot dB. \]  
\[ \text{(5.47)} \]
An interesting identification can be made on the basis of (5.47). Here \( A^{-1}dA \) can be regarded as the Maurer-Cartan form of \( BL(2, \mathbb{R}) \) and \( B^{-1}dB \) that of \( SO(2) \). For the particular choice of \( A \) and \( B \) given in (5.42), these can be calculated exactly,
\[ A^{-1}dA = \begin{pmatrix} \frac{1}{\alpha} d\alpha & 0 \\ -\beta d\alpha + \alpha d\beta & -\frac{1}{\alpha} d\alpha \end{pmatrix}, \quad B^{-1}dB = \begin{pmatrix} 0 & d\gamma \\ -d\gamma & 0 \end{pmatrix}. \]  
\[ \text{(5.48)} \]
Substituting (5.48) into \( \omega \) in (5.47), we obtain
\[ \omega = \begin{pmatrix} \cos 2\gamma \cdot \frac{1}{\alpha} d\alpha + \frac{1}{2} \sin 2\gamma \cdot (\beta \alpha - d\beta) & \sin 2\gamma \cdot \frac{1}{\alpha} d\alpha + \sin^2 \gamma \cdot (\beta \alpha - d\beta) \\ \sin 2\gamma \cdot \frac{1}{\alpha} d\alpha - \cos 2\gamma (\beta \alpha - d\beta) & -\cos 2\gamma \cdot \frac{1}{\alpha} d\alpha - \frac{1}{2} \sin 2\gamma \cdot (\beta \alpha - d\beta) \end{pmatrix} + \begin{pmatrix} 0 & d\gamma \\ -d\gamma & 0 \end{pmatrix}. \]  
\[ \text{(5.49)} \]
Taking \( \omega \) to be of the form (5.15), the \( \omega^j \) can be solved for and must be given by
\[ \omega^1 = 2 \cos 2\gamma \cdot \frac{1}{\alpha} d\alpha + \sin 2\gamma \cdot (\beta \alpha - d\beta), \quad \omega^2 = -2 \sin 2\gamma \cdot \frac{1}{\alpha} d\alpha + \cos 2\gamma \cdot (\beta \alpha - d\beta), \]
\[ \omega^3 = \beta \, d\alpha - \alpha \, d\beta + d(2\gamma). \] (5.50)

This set of forms can be written in a much more compressed form if we introduce forms \( \tau^i \) defined as
\[ \tau^1 = \frac{2}{\alpha} \, d\alpha, \quad \tau^2 = \beta \, d\alpha - \alpha \, d\beta, \quad \psi = 2\gamma. \] (5.51)

Then the set (5.50) reduces to the form
\[ \left( \begin{array}{c} \omega^1 \\ \omega^2 \end{array} \right) = \left( \begin{array}{cc} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{array} \right) \left( \begin{array}{c} \tau^1 \\ \tau^2 \end{array} \right), \quad \omega^3 = \tau^2 + d\psi. \] (5.52)

Now \( \tau^1 \) and \( \tau^2 \) can be thought of as forms on \( BL(2, \mathbb{R}) \) satisfying the Maurer-Cartan algebra
\[ d\tau^1 = 0, \quad d\tau^2 = \tau^1 \wedge \tau^2. \] (5.53)

There exists a function \( f : \mathbb{R}^2 \to SL(2, \mathbb{R}) \) such that \( f^*(\omega^i) = \sigma^i, \; i = 1, 2, 3 \). Consequently, it follows from (5.53) that
\[ \left( \begin{array}{c} \sigma^1 \\ \sigma^2 \end{array} \right) = \left( \begin{array}{cc} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{array} \right) \left( \begin{array}{c} \tilde{\sigma}^1 \\ \tilde{\sigma}^2 \end{array} \right), \quad \sigma^3 = \tilde{\sigma}^2 + d\psi, \] (5.54)

where \( \psi = \psi(x, t) \) and \( \tilde{\sigma}^i = f^*(\tau^i), \; i = 1, 2 \). Of course, the relations \( f^*(d\tau) = df^*(\tau) \) and \( f^*(\tau^1) \wedge f^*(\tau^2) \) also hold. Using these, the Maurer-Cartan algebra is transformed into
\[ d\tilde{\sigma}^1 = 0, \quad d\tilde{\sigma}^2 = \tilde{\sigma}^1 \wedge \tilde{\sigma}^2. \] (5.55)

Moreover, it follows that (5.54) can be inverted to the form
\[ \left( \begin{array}{c} \tilde{\sigma}^1 \\ \tilde{\sigma}^2 \end{array} \right) = \left( \begin{array}{cc} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{array} \right) \left( \begin{array}{c} \sigma^1 \\ \sigma^2 \end{array} \right), \quad \tilde{\sigma}^2 = \sigma^3 - d\psi. \] (5.56)

Eliminating \( \tilde{\sigma}^2 \), an expression for \( d\psi \) results
\[ d\psi = \sigma^3 - \sin \psi \, \sigma^1 - \cos \psi \, \sigma^2. \] (5.57)

If we suppose that \( u \) satisfies the sine-Gordon equation, the cocycle of the sine-Gordon equation (5.33) can be substituted into (5.57) to give
\[ d\psi = u_x \, dx - \sin \psi \, sin \, u \, dt - \cos \psi \, dx - \cos \psi \, \cos \, u \, dt. \] (5.58)

Collecting coefficients of \( dx \) and \( dt \), this implies by using \( d\psi = \psi_x \, dx + \psi_t \, dt \) that the \( \psi \) derivatives are determined as
\[ \psi_x = u_x - \cos \psi, \quad \psi_t = -\sin \psi \, sin \, u - \cos \psi \, \cos \, u = -\cos(\psi - u). \] (5.59)
This work has led to a very important result. Equation (5.59) is an example of a Bäcklund transformation. This transforms solutions \( u \) of the sine-Gordon equation into the solutions of another equation. To write the other equation, \( u = u(x,t) \) is eliminated from (5.59). Differentiating the first equation in (5.59) with respect to time, we have

\[
\psi_{xt} = u_{xt} + \sin \psi \psi_t = \sin u + \sin \psi \psi_t,
\]

which implies that

\[
\sin u = \psi_{xt} - \sin \psi \psi_t. \tag{5.60}
\]

Substituting this into the second equation in (5.59), we can obtain \( \cos u \) as

\[
\cos u = -\tan \psi \psi_{xt} - \cos \psi \psi_t. \tag{5.61}
\]

Squaring (5.60) and (5.61) and then adding the results, all dependence on \( u \) goes and we are left with an equation for the function \( \psi \)

\[
\psi_{xt}^2 = \cos^2 \psi (1 - \psi_t^2). \tag{5.62}
\]

An auto-Bäcklund transformation can also be constructed, such that the transformed equation is again the sine-Gordon equation. To this end, we substitute \( \tilde{u} = u - 2\psi + \pi \) into (5.59) to obtain

\[
\left( \frac{\tilde{u} + u}{2} \right)_x = \sin \left( \frac{\tilde{u} - u}{2} \right), \quad \left( \frac{\tilde{u} - u}{2} \right)_t = \sin \left( \frac{\tilde{u} + u}{2} \right). \tag{5.63}
\]

If \( u \) is eliminated from (5.63), it is found that \( \tilde{u} \) again satisfies the sine-Gordon equation \( \tilde{u}_{xt} = \sin (\tilde{u}) \).

6. THE GENERALIZED WEIERSTRASS SYSTEM INDUCING SURFACES OF CONSTANT AND NONCONSTANT MEAN CURVATURE

6.1. Generalized Weierstrass Representations.

The theory of immersion and deformations of surfaces has been an important part of classical differential geometry, and many methods have been used to describe immersions and types of deformations as well. The generalized Weierstrass representation put forward first by Konopelchenko and Taimanov [31] is particularly useful in considering these particular kinds of problems, which will be of interest here.

Surfaces and their dynamics are very important ingredients in a great number of phenomena in physics and applied mathematics as mentioned in the Introduction [32]. They appear in the study of surface waves, shock waves, deformations of membranes, and many problems in hydrodynamics connected with the motion of boundaries between regions of differing densities and viscosities [33-36]. Of special interest is the
case of surfaces which have zero mean curvature and such surfaces are referred to as minimal surfaces. The most general method for constructing minimal surfaces in three-dimensional Euclidean space was introduced by Weierstrass, and we begin by reviewing this [37-38].

Let us take a pair of functions \((\psi_1, \psi_2)\) such that \(\psi_1\) is antiholomorphic and \(\psi_2\) is holomorphic. Let us suppose that these functions are defined in the same simply connected domain \(S\) in the complex plane [39]. We have the system of equations

\[
\partial \psi_1 = 0, \quad \bar{\partial} \psi_2 = 0. \quad (6.1)
\]

The bar denotes complex conjugation and the derivatives are abbreviated \(\partial = \partial / \partial z\) and \(\bar{\partial} = \partial / \partial \bar{z}\). In terms of these functions, let us define the mapping \(T\) by the following formulas

\[
T : S \rightarrow \mathbb{R}^3, \quad z \in S \rightarrow (X_1(z, \bar{z}), X_2(z, \bar{z}), X_3(z, \bar{z})) \in \mathbb{R}^3,
\]

where the \(X_j\) are determined by

\[
\begin{align*}
X_1 + iX_2 &= i \int_{\Gamma}(\bar{\psi}_2^2 dz' - \bar{\psi}_1^2 d\bar{z}'), \quad X_1 - iX_2 = i \int_{\Gamma}(\psi_2^2 dz' - \psi_1^2 d\bar{z}'), \\
X_3 &= -\int_{\Gamma}(\bar{\psi}_1 \psi_2 dz' + \psi_1 \bar{\psi}_2 d\bar{z}').
\end{align*}
\]

The integrals are taken over any path \(\Gamma\) which lies in \(S\) and connects the point \(z\) to some initial point \(z_0\). From (6.1), it follows that the integrands are closed forms and hence the values of the integrals do not depend on the choice of the path \(\Gamma\). Weierstrass showed that the surface \(T(S)\) is minimal in the sense that its mean curvature vanishes everywhere.

To begin to generalize this, suppose the functions \(\psi_1\) and \(\psi_2\) satisfy the more general system of equations

\[
\begin{align*}
\partial \psi_1 &= \frac{1}{2} p(z, \bar{z}) H \psi_2, \quad \bar{\partial} \psi_2 = -\frac{1}{2} p(z, \bar{z}) H \psi_1, \quad (6.3)
\end{align*}
\]

and their complex conjugates, with real potential \(p(z, \bar{z})\). The integrals (6.2) then define the coordinates of a surface in three-dimensional Euclidean space. This was first put forward by Konopelchenko and Taimanov [29]. The mean curvature function is \(H(z, \bar{z})\) in (6.3). It will be seen here how (6.3) can arise. The coordinates \((z, \bar{z})\) are conformal and in terms of these, the metric and Gaussian curvature are given by

\[
\begin{align*}
p(z, \bar{z})^2 dz d\bar{z}, \quad K &= -\frac{1}{p^2} \partial \bar{\partial} \log p. \quad (6.4)
\end{align*}
\]
We can now ask how wide is the class of surfaces represented by the Weierstrass formulas (6.3).

Let $F : \Sigma \to \mathbb{R}^3$ be a regular mapping of the domain $\Sigma$ of the complex plane with coordinates $(z, \bar{z})$ into three-dimensional Euclidean space, and metric tensor given by (6.4) [39]. In this case, the vector

$$G(z) = (\partial F_1, \partial F_2, \partial F_3),$$

(6.5)
satisfies the equation

$$(\partial F_1)^2 + (\partial F_2)^2 + (\partial F_3)^2 = 0.$$  

(6.6)

Therefore,

$$(F_x - iF_y, F_x - iF_y) = (F_x, F_x) - (F_y, F_y) = 0.$$

This immediately follows from the formula $G(z) = \partial F = \frac{1}{2}(F_x - iF_y)$ and the condition that the metric is conformally Euclidean $(F_x, F_x) = (F_y, F_y), (F_x, F_y) = 0$. The subvariety $Q \subset \mathbb{C}P^1$ is defined in terms of the homogeneous coordinates $(\phi_1, \phi_2, \phi_3)$ by

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0.$$

It is diffeomorphic to the Grassmann manifold $G_{3,2}$ formed by two-dimensional subspaces of $\mathbb{R}^3$. This diffeomorphism is given by the mapping $G_{3,2} \to Q$, which assigns the point $(a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) \in Q$ to the plane generated by the pair of unit vectors $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$. Thus, $G$ can be regarded as the Gauss map. The Gauss map defined in this way for the surface (6.2) takes the form

$$G(z) = \left(\frac{i}{2}(\bar{\psi}_1^2 + \psi_2^2), \frac{1}{2}(\bar{\psi}_1^2 - \psi_2^2), -\bar{\psi}_1 \psi_2\right).$$

(6.7)

Solving (6.6) and (6.7) for $\psi_1^2$ and $\psi_2^2$, we obtain

$$\psi_1^2 = \bar{\partial}F_2 + i\bar{\partial}F_1, \quad \psi_2^2 = -\bar{\partial}F_2 - i\bar{\partial}F_1.$$

These results give rise to the following Proposition.

**Proposition 6.1.** Every regular conformally Euclidean immersion of a surface into three-dimensional Euclidean space is locally defined by the generalized Weierstrass formulas (6.2)-(6.3).

**Proof:** Assume that $\partial F_3 \neq 0$, otherwise change coordinates in $\mathbb{R}^3$. Let us compare $G(z) = \frac{1}{2}(F_x - iF_y)$ with $G$ in the form of the Gauss map and define the functions

$$\varphi_1^2 = \bar{\partial}F_2 + i\bar{\partial}F_1, \quad \varphi_2^2 = -\bar{\partial}F_2 - i\bar{\partial}F_1,$$

(6.8)

and their conjugates. In fact, these imply that $-(\partial F_1)^2 - (\partial F_2)^2 = \varphi_1^2 \varphi_2^2$ and therefore,

$$(\partial F_1)^2 + (\partial F_2)^2 + (\partial F_3)^2 = 0.$$
Also (6.8) can be solved for \( \varphi_1 \) and \( \varphi_2 \) as square roots. Recall the definition of the second fundamental form \( h_{ij} \). Let the metric tensor on the surface \( F : \Sigma \to \mathbb{R}^3 \) be given by (6.4). Take an orthonormal basis in the tangent plane at the point \( z \),

\[
e_1 = \frac{1}{p} F_x, \quad e_2 = \frac{1}{p} F_y,
\]

and extend it to a basis in \( \mathbb{R}^3 \) by including a unit normal vector

\[
e_3 = e_1 \times e_2.
\]

Components of the curvature tensor are defined by the decomposition formulas

\[
F_{xx} = p_x e_1 - p_y e_2 + p^2 h_{11} e_3, \quad F_{xy} = p_y e_1 + p_x e_2 + p^2 h_{12} e_3, \quad F_{yy} = -p_x e_1 + p_y e_2 + p^2 h_{22} e_3.
\]

(6.9)

Given the system (6.9), we derive the associated system satisfied by \( (\varphi_1, \varphi_2) \). We make use of the identity

\[
\partial \bar{\varphi} = \frac{1}{4} (\partial_x^2 + \partial_y^2).
\]

Differentiating \( \varphi_1 = \bar{\partial} F_2 + i \bar{\partial} F_1 \) with respect to \( \partial \), we obtain

\[
2 \varphi_1 \partial \varphi_1 = \partial \bar{\partial} F_2 + i \partial \bar{\partial} F_1 = \frac{1}{4} (\partial_x^2 F_2 + \partial_y^2 F_2) + \frac{i}{4} (\partial_x^2 F_1 + \partial_y^2 F_1).
\]

(6.10)

An explicit formula for \( e_3 \) is required and can be obtained by starting with the representations

\[
e_1 = \frac{1}{p} F_x = \frac{1}{p} (F_{1,x}, F_{2,x}, F_{3,x}), \quad e_2 = \frac{1}{p} F_y = \frac{1}{p} (F_{1,y}, F_{2,y}, F_{3,y}).
\]

Taking the cross product,

\[
e_1 \times e_2 = \frac{1}{p^2} (F_{2,x} F_{3,y} - F_{3,x} F_{2,y}, -(F_{1,x} F_{3,y} - F_{1,y} F_{3,x}), F_{1,x} F_{2,y} - F_{1,y} F_{2,x}).
\]

Thus,

\[
8 \varphi_1 \partial \varphi_1 = \frac{p_x}{p} F_{2,x} - \frac{p_y}{p} F_{2,y} + h_{11} (-F_{1,x} F_{3,y} + F_{1,y} F_{3,x}) - \frac{p_x}{p} F_{2,x} + \frac{p_y}{p} F_{2,y} + h_{22} (-F_{1,x} F_{3,y} + F_{1,y} F_{3,x})
\]

\[
+ i \left( \frac{p_x}{p} F_{1,x} - \frac{p_y}{p} F_{1,y} + h_{11} (F_{2,x} F_{3,y} - F_{3,x} F_{2,y}) - \frac{p_x}{p} F_{1,x} + \frac{p_y}{p} F_{1,y} + h_{22} (F_{2,x} F_{3,y} - F_{3,x} F_{2,y}) \right)
\]

\[
= (h_{11} + h_{22}) (-F_{1,x} F_{3,y} + F_{1,y} F_{3,x}) + i (h_{11} + h_{22}) (F_{2,x} F_{3,y} - F_{3,x} F_{2,y})
\]

\[
= (h_{11} + h_{22}) [i F_{3,y} (i F_{1,x} + F_{2,x}) - F_{3,x} (i F_{2,y} - F_{1,y})].
\]

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Finally, it is required to replace the derivatives of $F_j$ in terms of the functions $\varphi_j$ and their complex conjugates. To do this, we write explicitly,

$$\varphi_1^2 = \frac{1}{2}(F_{2,x} + iF_{2,y} + iF_{1,x} - iF_{1,y}), \quad \varphi_2^2 = \frac{1}{2}(-F_{2,x} + iF_{2,y} - iF_{1,x} - F_{1,y}), \quad (6.12)$$

and their complex conjugate equations. From (6.12), it follows that

$$\varphi_1^2 - \varphi_2^2 = F_{2,x} + iF_{1,x}, \quad \varphi_1^2 + \varphi_2^2 = iF_{2,y} - F_{1,y}.$$  

Moreover,

$$\partial F_3 = \frac{1}{2}(\partial_x - i\partial_y)F_3 = -\bar{\varphi}_1 \varphi_2, \quad \bar{\partial} F_3 = \frac{1}{2}(\partial_x + i\partial_y)F_3 = -\varphi_1 \bar{\varphi}_2,$$

$$\partial_x F_3 = -(\bar{\varphi}_1 \varphi_2 + \varphi_1 \bar{\varphi}_2), \quad i\partial_y F_3 = \bar{\varphi}_1 \varphi_2 - \varphi_1 \bar{\varphi}_2.$$  

Substituting these results into the final expression produced in (6.11), we obtain

$$8\varphi_1 \partial \varphi_1 = (h_{11} + h_{22})[iF_{3,y}(\varphi_1^2 - \varphi_2^2) - F_{3,x}(\varphi_1^2 + \varphi_2^2)]$$

$$= (h_{11} + h_{22})[(\bar{\varphi}_1 \varphi_2 - \varphi_1 \bar{\varphi}_2)(\varphi_1^2 - \varphi_2^2) + (\bar{\varphi}_1 \varphi_2 + \varphi_1 \bar{\varphi}_2)(\varphi_1^2 + \varphi_2^2)]$$

$$= 2(h_{11} + h_{22})(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 \varphi_2.$$  

Solving for $\partial \varphi_1$ and using the definition of mean curvature in terms of $h_{ij}$, we have the first equation in (6.3). Similarly, we can work out

$$8\varphi_2 \bar{\partial} \varphi_2 = -(h_{11} + h_{22})(-F_{1,x}F_{3,y} + F_{1,y}F_{3,x} + iF_{2,x}F_{3,y} - iF_{3,x}F_{2,y}).$$

The right-hand side of this result is identical except for sign to what was worked out in the previous case, hence,

$$2\varphi_2 \bar{\partial} \varphi_2 = -2\varphi_1 \varphi_2(h_{11} + h_{22})p.$$  

This is the second equation in (6.3). This finishes the proof.

Thus, Konopelchenko [19, 31, 37] has established a connection between certain classes of constant mean curvature surfaces and the trajectories of an infinite-dimensional Hamiltonian system of the form (6.3). He considered the nonlinear Dirac-type system of equations in terms of two complex valued functions $\psi_1$ and $\psi_2$ which, after absorbing constants into the derivative variables, satisfy the set,

$$\partial \psi_1 = p\psi_2, \quad \bar{\partial} \psi_1 = p\bar{\psi}_2, \quad \bar{\partial} \psi_2 = -p\bar{\psi}_1, \quad \partial \psi_2 = -p\psi_1,$$

$$p = |\psi_1|^2 + |\psi_2|^2. \quad (6.13)$$
System (6.13) has been referred to as the generalized Weierstrass (GW) system in the literature recently [19]. Using (6.13), it can be verified that the following conservation laws hold

\[ \partial(\psi_1^2) + \bar{\partial}(\bar{\psi}_2^2) = 0, \quad \bar{\partial}(\bar{\psi}_1^2) + \partial(\psi_2^2) = 0, \quad \partial(\psi_1 \bar{\psi}_2) + \bar{\partial}(\bar{\psi}_1 \psi_2) = 0. \tag{6.14} \]

Making use of these conserved quantities, there exist three real-valued quantities \( X_i(z, \bar{z}) \) which are completely determined by the following path integrals

\[ X_1 + iX_2 = 2i \int_\Gamma (\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}'), \quad X_1 - iX_2 = 2i \int_\Gamma (\psi_1^2 dz' - \psi_2^2 d\bar{z}'), \]

\[ X_3 = -2 \int_\Gamma (\bar{\psi}_1 \psi_2 dz' + \psi_1 \bar{\psi}_2 d\bar{z}'). \tag{6.15} \]

On account of conservation laws (6.14), these integrals are found to be independent of the path \( \Gamma \) chosen. The functions \( X_i(z, \bar{z}) \) can be treated as the coordinates of a surface immersed in \( \mathbb{R}^3 \). The Gaussian curvature and the first fundamental form of the surface are given by

\[ K = -\frac{\partial \bar{\partial}(\log p)}{p^2}, \quad \Omega = 4p^2 dzd\bar{z}, \]

in isothermic coordinates. There is also a current which is conserved and given by

\[ J = \bar{\psi}_1 \partial \psi_2 - \psi_2 \bar{\partial} \bar{\psi}_1. \tag{6.16} \]

The current (6.16) satisfies \( \bar{\partial} J = 0 \) modulo (6.13). The integrability of system (6.13) has been examined extensively [40-43] by using Cartan’s theorem on systems in involution using a set of differential forms which are equivalent to system (6.13). A Bäcklund transformation has also been determined for GW system (6.13) [44].

At this point, a correspondence between system (6.13) and the two-dimensional non-linear sigma model can be made. Introduce the new variable \( \rho \) which is defined in terms of the \( \psi \) as

\[ \rho = \frac{\psi_1}{\psi_2}. \tag{6.17} \]

Using (6.13), it can be seen that

\[ \partial \rho = \frac{\partial \psi_1}{\psi_2} - \frac{\psi_1}{\psi_2} \bar{\partial} \bar{\psi}_2 = (1 + |\rho|^2) \psi_2^2. \tag{6.18} \]

Solving for \( \psi_2^2 \) in (6.18), using (6.17) to get \( \psi_1 \), the following transformation from \( \rho \) to the set of \( \psi \) is produced

\[ \psi_1 = \epsilon \rho \frac{(\bar{\rho})^{1/2}}{1 + |\rho|^2}, \quad \psi_2 = \epsilon \frac{(\partial \rho)^{1/2}}{1 + |\rho|^2}, \quad \epsilon = \pm 1. \tag{6.19} \]
Proposition 6.2. If $\psi_1$ and $\psi_2$ are solutions of GW system (6.13), then the function $\rho$ defined by (6.17) is a solution of the second order sigma model system

$$\partial \bar{\partial} \rho - \frac{2\rho}{1+|\rho|^2} \partial \rho \bar{\partial} \rho = 0, \quad \bar{\partial} \partial \rho - \frac{2\rho}{1+|\rho|^2} \bar{\partial} \rho \partial \rho = 0.$$  \hspace{1cm} (6.20)

Proposition 6.3. If $\rho$ is a solution to sigma model system (6.20), then the functions $\psi_1$ and $\psi_2$ defined in terms of $\rho$ by the expressions

$$\psi_1 = \epsilon \rho \frac{(\bar{\partial} \rho)^{1/2}}{1+|\rho|^2}, \quad \psi_2 = \epsilon \frac{(\partial \rho)^{1/2}}{1+|\rho|^2},$$  \hspace{1cm} (6.21)
satisfy GW system (6.13).

Equivalently, given a solution to sigma model (6.20), a surface can be obtained by calculating the $\psi_i$ by means of (6.21) in Proposition 6.3 and then substituting the $\psi_i$ into (6.15) to obtain the coordinates of the corresponding surface. This may seem involved, but the classical symmetry group, and integrability, of system (6.20) has been calculated explicitly [20,41]. The symmetry structure of (6.20) is complicated enough to be able to generate a great variety of solutions $\rho$, and by means of (6.19) to GW system (6.13) as well. Thus, the procedure produces useful solutions in this way which. Once the $\psi_i$ have been calculated from (6.19), the coordinates of a surface follow from (6.15). Another type of solution to the sigma model system has also been discussed [20,41], and will be given here as an example.

Proposition 6.4. Suppose that for each $i = 1, \cdots, N$ the complex valued functions $\rho_i$ satisfy sigma model system (6.20) as well as the conditions $|\rho_i|^2 = 1$. Then the product of the functions $\rho_i$

$$\rho = \prod_{i=1}^N \rho_i,$$  \hspace{1cm} (6.22)
is also a solution to system (6.20).

Let us now discuss the calculation of an algebraic multi-soliton solution of (6.13) and associated surface based on Proposition 6.4. First, we look for a particular class of rational solutions to (6.20) which admit simple poles at $\bar{z} = \bar{a}_j$ given by

$$\rho_j = \frac{z - a_j}{\bar{z} - \bar{a}_j}, \quad a_j \in \mathbb{C}, \quad j = 1, \cdots, N.$$  \hspace{1cm} (6.23)

A more general class of rational solution to (6.20) admitting simple poles by Proposition 6.4 is given by

$$\rho = \prod_{j=1}^N \frac{z - a_j}{\bar{z} - \bar{a}_j}.$$  \hspace{1cm} (6.24)

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This function satisfies $\partial \bar{\partial} \rho \neq 0$ and $|\rho|^2 = 1$ as well. The first derivatives of $\rho$ are given as

$$
\partial \rho = \sum_{j=1}^{N} \frac{\rho}{z - a_j} = F(z) \rho, \quad \bar{\partial} \rho = -\sum_{j=1}^{N} \frac{\rho}{\bar{z} - \bar{a}_j} = -\bar{F}(\bar{z}) \rho. \quad (6.25)
$$

Moreover, $p$ and current $J$ given by (6.15) are calculated to be

$$
p = \frac{1}{2} \left| \sum_{j=1}^{N} \frac{1}{z - a_j} \right|, \quad J = \frac{1}{4} \left( \sum_{j=1}^{N} \frac{1}{z - a_j} \right)^2. \quad (6.26)
$$

For the case $N = 1$, the functions $\psi_1$ and $\psi_2$ can be substituted into relations (6.15) which give the coordinates $X_i$ of a surface. The corresponding constant mean curvature surface is then given by the algebraic relation

$$
((X_1)^2 + (X_2)^2) - (2 + \frac{a^2}{4} e^{2X_3})(X_1^2 + X_2^2) + \frac{a^2}{2} e^{2X_3} X_2 + 1 - \frac{a^2}{4} e^{2X_3} = 0. \quad (6.27)
$$

This type of inducing can be extended to higher dimensional spaces, in particular, 4-dimensional Euclidean space and Minkowski spaces. This was first proposed by Konopelchenko and Landolfi [36]. They consider a first order nonlinear system of two-dimensional Dirac-type equations in terms of four complex valued functions $\psi_\alpha$ and $\varphi_\alpha$, with $\alpha = 1, 2$. This system can be written as follows

$$
\partial \psi_\alpha = p \varphi_\alpha, \quad \bar{\partial} \varphi_\alpha = -p \psi_\alpha, \quad \alpha = 1, 2, \quad (6.28)
$$

$$
p = \sqrt{u_1 u_2}, \quad u_\alpha = |\psi_\alpha|^2 + |\varphi_\alpha|^2,
$$

as well as the complex conjugate equations of (6.28). The system (6.28) possesses several conservation laws, such as

$$
\partial (\psi_\alpha \psi_\beta) + \bar{\partial} (\varphi_\alpha \varphi_\beta) = 0, \quad \partial (\psi_\alpha \bar{\varphi}_\beta) - \bar{\partial} (\varphi_\alpha \psi_\beta) = 0. \quad (6.29)
$$

As a consequence of these conserved quantities, there exist four real-valued functions $X_i(z, \bar{z})$, $i = 1, \cdots, 4$ which can be interpreted as the coordinates of a surface immersed in Euclidean 4-space. The coordinates of the position vector $X = (X_1, X_2, X_3, X_4)$ of a constant mean curvature surface in $\mathbb{R}^4$ are determined by the integrals

$$
X_1 = \frac{i}{2} \int_{\Gamma} \left[ (\bar{\psi}_1 \bar{\psi}_2 + \varphi_1 \varphi_2) \, dz' - (\psi_1 \psi_2 + \bar{\varphi}_1 \bar{\varphi}_2) \, d\bar{z}' \right],
$$

$$
X_2 = \frac{1}{2} \int_{\Gamma} \left[ (\bar{\psi}_1 \bar{\psi}_2 - \varphi_1 \varphi_2) \, dz' + (\psi_1 \psi_2 - \bar{\varphi}_1 \bar{\varphi}_2) \, d\bar{z}' \right],
$$

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\[ X_3 = -\frac{1}{2} \int_\Gamma [(\bar{\psi}_1 \varphi_2 + \psi_2 \varphi_1) \, dz' + (\psi_1 \bar{\varphi}_2 + \varphi_2 \bar{\varphi}_1) \, d\bar{z}'], \quad (6.30) \]
\[ X_4 = \frac{i}{2} \int_\Gamma [(\bar{\psi}_1 \varphi_2 - \psi_2 \varphi_1) \, dz' - (\psi_1 \bar{\varphi}_2 - \varphi_2 \bar{\varphi}_1) \, d\bar{z}']. \]

In (6.30), \( \Gamma \) is any contour in the complex plane. The integrals depend only on the endpoints of the contour on account of conservation laws (6.29). Many results for the 4-dimensional case have been found and given in [43].

### 6.2. A Physical Application Involving Nonlinear Sigma Models.

Here is a physical example which should give the previous considerations a physical perspective. Other interesting applications can be found in [45, 46]. Consider the classical spin vector \( \mathbf{S} = (S_1, S_2, S_3) \), where each \( S_j \) depends on the variable \( t = x_0 \) as well as two spatial degrees of freedom \( x_1 \) and \( x_2 \). The \( S_j \) are real functions which satisfy
\[ S_3^2 + \kappa^2 (S_1^2 + S_2^2) = 1, \quad (6.31) \]
and \( \kappa^2 = \pm 1 \) represents the curvature of spin phase space. It is associated with the sphere \( S^2 \) when \( \kappa^2 = 1 \), or the pseudosphere when \( \kappa^2 = -1 \).

The Landau-Lifshitz equation describes the time evolution of the spin vector and is given by
\[ \partial_t \mathbf{S} = \mathbf{S} \times \nabla^2 \mathbf{S}. \quad (6.32) \]

Let \( \mathbf{S} \) be a matrix defined by
\[ \mathbf{S} = \begin{pmatrix} S_3 & \kappa \bar{S}_+ \hfill \\ \kappa S_+ & -S_3 \hfill \end{pmatrix}, \quad (6.33) \]
where \( S_{\pm} = S_1 \pm iS_2 \) and the bar denotes complex conjugation. In terms of the matrix \( \mathbf{S} \), the Landau-Lifshitz equation can be written as
\[ \partial_t \mathbf{S} = \frac{1}{2\kappa} [\mathbf{S}, \nabla^2 \mathbf{S}]. \quad (6.34) \]

Introduce the variable \( r \) defined in terms of the \( S_j \) as follows
\[ r = \frac{S_1 + iS_2}{1 + S_3}. \quad (6.35) \]

The Cartesian components of the magnetization for \( \kappa^2 = 1 \) can be shown to be
\[ S_1 = \frac{r + \bar{r}}{1 + |r|^2}, \quad S_2 = \frac{r - \bar{r}}{i(1 + |r|^2)}, \quad S_3 = \frac{1 - |r|^2}{1 + |r|^2}. \quad (6.36) \]

It is clear that the components \( S_j \) in (6.36) satisfy the constraint (6.31) such that \( \Delta \mathbf{S} \) and \( \dot{\mathbf{S}} \) can be evaluated using (6.36). Substituting the derivatives into the second form of the
Landau-Lifshitz equation, two independent equations in terms of the derivatives $\dot{r}$ and $\bar{\dot{r}}$ are obtained. They are given explicitly by

$$i\dot{r} = -\Delta r + 2\bar{r} \frac{\nabla r}{1 + |r|^2}, \quad -i\bar{\dot{r}} = -\Delta \bar{r} + 2r \frac{\nabla \bar{r}}{1 + |r|^2}. \quad (6.37)$$

Transforming the derivatives into complex form, we obtain

$$\frac{i}{4} \dot{r} + \bar{\partial} \bar{r} = 2\bar{r} \frac{\partial r \partial \bar{r}}{1 + |r|^2}, \quad -\frac{i}{4} \bar{\dot{r}} + \partial \partial \bar{r} = 2r \frac{\partial \bar{r} \partial \bar{r}}{1 + |r|^2}. \quad (6.38)$$

The related case $\kappa^2 = -1$ can be analyzed in a similar way. The stereographic projection of the pseudosphere onto the $(S_1, S_2)$ plane is used

$$S_1 = \frac{\xi + \bar{\xi}}{1 - |\xi|^2}, \quad S_2 = \frac{\xi - \bar{\xi}}{i(1 - |\xi|^2)}, \quad S_3 = \frac{1 + |\xi|^2}{1 - |\xi|^2}. \quad (6.39)$$

The equations of motion that follow are given as

$$i\dot{\xi} = -\Delta \xi - 2\xi \frac{\nabla \xi}{1 - |\xi|^2}, \quad -i\dot{\bar{\xi}} = -\Delta \bar{\xi} - 2\bar{\xi} \frac{\nabla \bar{\xi}}{1 - |\xi|^2}. \quad (6.40)$$

Regarding $t$ as a time variable, then if $t$ is held fixed, or if we consider $r$ or $\xi$, depending on the case, to be independent of $t$, then systems (6.38) and (6.40) reduce to exactly the nonlinear sigma model equations of the form (6.20).

### 6.3. Non-Constant Mean Curvature Surfaces.

Consider next the case in which the mean curvature $H$ is not constant [47]. Up to this point, constant $H$ with factor $1/2$ has been absorbed into the spatial coordinates. Putting the numerical factor in the coordinates $(z, \bar{z})$, the system of equations satisfied by the $\psi_\alpha$ which determine a surface with mean curvature function $H$ become

$$\partial \psi_1 = p H \psi_2, \quad \bar{\partial} \bar{\psi}_2 = -p H \bar{\psi}_1,$$

$$\bar{\partial} \bar{\psi}_1 = p H \bar{\psi}_2, \quad \partial \bar{\psi}_2 = -p H \bar{\psi}_1. \quad (6.41)$$

The function $H(z, \bar{z})$ denotes the mean curvature the surface will have. Versions of Propositions 6.2-6.3 can be obtained starting with system (6.41).

**Proposition 6.5.** If $\psi_1$ and $\psi_2$ are solutions of the system (6.41) and $\rho$ is given by (6.17), then $\psi_1$ and $\psi_2$ are obtained from $\rho$ by means of the equations

$$\psi_1 = \epsilon \rho \frac{(\bar{\partial} \rho)^{1/2}}{H^{1/2}(1 + |\rho|^2)}, \quad \psi_2 = \epsilon \frac{(\partial \rho)^{1/2}}{H^{1/2}(1 + |\rho|^2)}, \quad \epsilon = \pm 1. \quad (6.42)$$
Moreover, $\rho$ is a solution to the following second order system,
\[
\partial \bar{\partial} \rho - \frac{2\bar{\rho}}{1 + |\rho|^2} \partial \rho \bar{\partial} \rho = \bar{\partial} (\ln H) \partial \rho, \quad \partial \bar{\partial} \bar{\rho} - \frac{2\rho}{1 + |\rho|^2} \bar{\partial} \rho \partial \bar{\rho} = \partial (\ln H) \bar{\partial} \bar{\rho}. \quad (6.43)
\]

The converse of Proposition 6.5 holds as well. Note the close similarity between (6.43) and (6.20). Moreover, (6.41) implies that the $\psi_\alpha$ satisfy the set of conservation laws (6.14). Surfaces can be induced by using solutions of (6.41) and then substituting into (6.15), or alternatively, given a solution of (6.43), it can be put in (6.42) to obtain the $\psi_\alpha$, which are then used in (6.15). Several results concerning this system and details concerning the proofs can be found in [47].

**Proposition 6.6.** (i) If $\psi_1$ and $\psi_2$ are solutions of (6.41) given in terms of $\rho$ in (6.17) by (6.42), then $J$ defined by (6.16) in terms of the function $\rho$ takes the form
\[
J(z, \bar{z}) = -\frac{\partial \rho \partial \bar{\rho}}{H(1 + |\rho|^2)^2}. \quad (6.44)
\]

(ii) Let $J$ be defined by (6.16), then the quantity $\mathcal{J}$ defined by
\[
\mathcal{J} = J + \int_{z_0}^{\bar{z}} p^2(z, \tau) \partial H(z, \tau) d\tau, \quad (6.45)
\]
is conserved under differentiation with respect to $\bar{z}$,
\[
\bar{\partial} \mathcal{J} = 0. \quad (6.45)
\]

**Proof:** (i) Substituting $\psi_\alpha$ from (6.42) into (6.16) and differentiating using the product rule, we obtain
\[
J = -\bar{\rho} \frac{\partial \rho}{2H^2(1 + |\rho|^2)^2} \partial H + \bar{\rho} \frac{(\partial \rho)^{1/2}}{H(1 + |\rho|^2)} \partial (\bar{\rho}(\partial \rho)^{1/2})
+ \frac{\partial \rho}{2H^2(1 + |\rho|^2)^2} \partial H - \frac{(\partial \rho)^{1/2}}{H(1 + |\rho|^2)} \partial (\bar{\rho}(\partial \rho)^{1/2})
- \frac{(\partial \rho)^{1/2}}{H(1 + |\rho|^2)^2} \partial \bar{\rho} \partial (\partial \rho)^{1/2}
- \frac{(\partial \rho)^{1/2}}{H(1 + |\rho|^2)^2} \partial \rho \partial (\partial \rho)^{1/2}.
\]

(ii) A proof can be found in [46].
Proposition 6.7. With $p$ defined in (6.41), and $J$ in (6.16), then $p$ satisfies a second order differential equation which involves $p$, $J$ and the mean curvature function $H$. The equation is given by

$$\partial \bar{\partial} \ln p = \frac{|J|}{p^2} - H^2 p^2.$$  \hfill (6.46)

It has been shown [39] that when $H$ is constant, there is a connection between the time-independent Landau-Lifshitz equation, which can be expressed as

$$[S, \partial \bar{\partial} S] = 0,$$  \hfill (6.47)

and the two-dimensional nonlinear sigma model. The matrix $S$ will be referred to as the spin matrix. In terms of the sigma model quantity $\rho$, the matrix $S$ is given by

$$S = \frac{1}{1 + |\rho|^2} \begin{pmatrix} 1 - |\rho|^2 & \frac{2\bar{\rho}}{2\rho} \\ \frac{2\rho}{2\rho} & -1 + |\rho|^2 \end{pmatrix}.$$  \hfill (6.48)

Define $f$ and $\bar{f}$ to be the $\rho$-dependent factors on the left-hand side of the sigma model equations given in (6.20), so in fact (6.20) can be written in the form $f = 0$, $\bar{f} = 0$. In terms of $f$ and $\bar{f}$, the matrix generated by (6.47) is of the form

$$[S, \partial \bar{\partial} S] = \frac{4}{(1 + |\rho|^2)^2} \begin{pmatrix} \bar{\rho} f - \rho \bar{f} & \rho^2 f - \bar{\rho} \bar{f} \\ \rho^2 f - \bar{\rho} \bar{f} & \rho \bar{\rho} f - \bar{\rho} \rho \bar{f} \end{pmatrix}.$$  \hfill (6.49)

These results can be summarized as follows.

Proposition 6.8. If $\rho$ is a solution of the nonlinear sigma model system (6.20), then the spin matrix $S$ defined by (6.48) is a solution of the Landau-Lifshitz equation (6.47).

Proposition 6.8 and equation (6.47) can be modified to include the case in which the mean curvature is not constant. Define the matrices $R$ and $H$ as follows,

$$R = \frac{4}{(1 + |\rho|^2)^2} \begin{pmatrix} -\bar{\rho} \partial \rho & \rho \bar{\rho} \partial \rho \\ -\rho^2 \partial \bar{\rho} & -\rho^2 \partial \bar{\rho} \end{pmatrix}, \quad H = \begin{pmatrix} \partial \ln(H) & \bar{\rho} \partial \ln(H) \\ \partial \ln(H) & \frac{1}{\rho} \partial \ln(H) \end{pmatrix}.$$  \hfill (6.50)

The matrix $R$ depends only on the variable $\rho$. The following generalization of Proposition 6.8 can be formulated.

Proposition 6.9. If $\rho$ is a solution of the sigma model equations (6.43) and the matrices $R$ and $H$ are defined in (6.50), then spin matrix $S$ given by (6.48) is a solution of the nonhomogeneous Landau-Lifshitz equation

$$[S, \partial \bar{\partial} S] + R H = 0,$$  \hfill (6.51)

modulo (6.43).
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