NEW INEQUALITIES OF THE KANTOROVICH TYPE WITH TWO NEGATIVE PARAMETERS

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ABSTRACT. We show the following result: Let $A, B \in \mathcal{B}(\mathcal{H})$ be two strictly positive operators such that $A \leq B$ and $m\mathbb{1}_\mathcal{H} \leq B \leq M\mathbb{1}_\mathcal{H}$ for some scalars $0 < m < M$. Then

$$B^p \leq \exp\left(\frac{M\mathbb{1}_\mathcal{H} - B}{M - m} \ln m^p + \frac{B - m\mathbb{1}_\mathcal{H}}{M - m} \ln M^p\right) \leq K(m, M, p, q) A^q \quad \text{for } p \leq 0, q \in \mathbb{R} \setminus (0, 1)$$

where $K(m, M, p, q)$ is the generalized Kantorovich constant with two parameters. In addition, we obtain Kantorovich type inequalities for the chaotic order.

1. Introduction and Preliminaries

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator $A$ is said to be positive (denoted by $A \geq 0$ ) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and also an operator $A$ is said to be strictly positive (denoted by $A > 0$) if $A$ is positive and invertible. Here $\mathbb{1}_\mathcal{H}$ stands for the identity operator on $\mathcal{H}$. $Sp(A)$ denotes the usual spectrum of $A$. If a positive function $f : I \subseteq \mathbb{R} \rightarrow (0, \infty)$ satisfies

$$f((1-v)x+vy) \leq [f(x)]^{1-v}[f(y)]^v,$$

for all $x, y \in I$ and $v \in [0,1]$, then we say that $f$ is a logarithmically convex (or simply, log-convex) function on $I$. The weighted arithmetic-geometric mean inequality readily yields that every log-convex function is also convex. It is worth emphasizing that the function $f(t) = t^p$ is log-convex for $p \leq 0$ on $(0, \infty)$.

The “Löwner-Heinz inequality” asserts that $0 \leq A \leq B$ ensures $A^p \leq B^p$ for any $p \in [0, 1]$. As is well-known, the Löwner-Heinz inequality does not always hold for $p > 1$. The following theorem due to Furuta [8, Theorem 2.1] (see also [9, Theorem 4.1]) is the starting point for our discussion.

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Theorem 1.1. Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be two strictly positive operators such that \( A \leq B \) and \( m \mathbf{1}_\mathcal{H} \leq A \leq M \mathbf{1}_\mathcal{H} \) for some scalars \( 0 < m < M \). Then
\[
A^p \leq K(m, M, p) B^p \leq \left( \frac{M}{m} \right)^{p-1} B^p \quad \text{for } p \geq 1,
\]
where \( K(m, M, p) \) is a generalized Kantorovich constant in the sense of Furuta [7]:
\[
(1.2) \quad K(m, M, p) = \left( \frac{mM^p - Mm^p}{(p-1)(M - m)} \right)^{p} \left( \frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^{p} \quad \text{for } p \in \mathbb{R}.
\]

In [14, Theorem 2.1], Mićić, Pečarić and Seo proved some fascinating results about the function preserving the operator order, under a general setting:

Theorem 1.2. Let \( A \) and \( B \) be two strictly positive operators on a Hilbert space \( \mathcal{H} \) satisfying \( m \mathbf{1}_\mathcal{H} \leq A \leq M \mathbf{1}_\mathcal{H} \) for some scalars \( 0 < m < M \). Let \( f : [m, M] \to \mathbb{R} \) be a convex function and \( g : I \to \mathbb{R} \), where \( I \) be any interval containing \( \text{Sp}(B) \cup [m, M] \). Suppose that either of the following conditions holds: (i) \( g \) is increasing convex on \( I \), or (ii) \( g \) is decreasing concave on \( I \). If \( A \leq B \), then for a given \( \alpha > 0 \) in the case (i) or \( \alpha < 0 \) in the case (ii)
\[
f(A) \leq \alpha g(B) + \beta \mathbf{1}_\mathcal{H},
\]
holds for
\[
(1.3) \quad \beta = \max_{m \leq t \leq M} \left\{ a_f t + b_f - \alpha g(t) \right\},
\]
where
\[
a_f \equiv \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad b_f \equiv \frac{Mf(m) - mf(M)}{M - m}.
\]

The following converse of Theorem 1.2 was proven in [16, Theorem 2.1]:

Theorem 1.3. Let \( A \) and \( B \) be two strictly positive operators on a Hilbert space \( \mathcal{H} \) satisfying \( m \mathbf{1}_\mathcal{H} \leq B \leq M \mathbf{1}_\mathcal{H} \) for some scalars \( 0 < m < M \). Let \( f : [m, M] \to \mathbb{R} \) be a convex function and \( g : I \to \mathbb{R} \), where \( I \) be any interval containing \( \text{Sp}(A) \cup [m, M] \). Suppose that either of the following conditions holds: (i) \( g \) is decreasing convex on \( I \), or (ii) \( g \) is increasing concave on \( I \). If \( A \leq B \), then for a given \( \alpha > 0 \) in the case (i) or \( \alpha < 0 \) in the case (ii)
\[
f(B) \leq \alpha g(A) + \beta \mathbf{1}_\mathcal{H},
\]
holds with \( \beta \) as (1.3).

This paper has been divided into four sections. The proof of our main result, Theorem 2.1, is given in Section 2. The essential idea is to consider the log-convex function instead of the convex function in Theorem 1.3. As applications, in Section 3, we show some characterizations of the chaotic order. Further results based on the Mond-Pečarić method are given in Section 4.
 FUNCTIONS REVERSING THE OPERATOR ORDER

In the sequel, \( a_f \) and \( b_f \) will refer to those of Theorem 1.2.

Our principal result is the following theorem. The role of (1.1) is clearly brought out in our proof.

**Theorem 2.1.** Let \( A, B \in \mathfrak{B}(\mathcal{H}) \) be two self-adjoint operators such that \( m \mathbf{1}_{\mathcal{H}} \leq B \leq M \mathbf{1}_{\mathcal{H}} \) for some scalars \( m < M \). Let \( f : [m, M] \to (0, \infty) \) be a log-convex function and \( g : I \to \mathbb{R} \), where \( I \) be any interval containing \( \text{Sp}(A) \cup [m, M] \). Suppose that either of the following conditions holds: (i) \( g \) is decreasing convex on \( I \), or (ii) \( g \) is increasing concave on \( I \). If \( A \leq B \), then for a given \( \alpha > 0 \) in the case (i) or \( \alpha < 0 \) in the case (ii)

\[
\tag{2.1}
 f(B) \leq \exp \left( \frac{M \mathbf{1}_{\mathcal{H}} - B}{M - m} \ln f(m) + \frac{B - m \mathbf{1}_{\mathcal{H}}}{M - m} \ln f(M) \right) \leq \alpha g(A) + \beta \mathbf{1}_{\mathcal{H}},
\]

holds with \( \beta \) as (1.3).

**Proof.** We prove the inequalities (2.1) under the assumption (i). It is immediate to see that

\[
\tag{2.2}
 f(t) \leq [f(m)]^{\frac{M - t}{M - m}} [f(M)]^{\frac{t - m}{M - m}} \leq L(t) \quad \text{for } m \leq t \leq M,
\]

where

\[
L(t) = \frac{M - t}{M - m} f(m) + \frac{t - m}{M - m} f(M) = a_f t + b_f.
\]

By applying the standard operational calculus of self-adjoint operator \( B \) to (2.2), we obtain for every unit vector \( x \in \mathcal{H} \),

\[
\langle f(B)x, x \rangle \leq \exp \left( \frac{M \mathbf{1}_{\mathcal{H}} - B}{M - m} \ln f(m) + \frac{B - m \mathbf{1}_{\mathcal{H}}}{M - m} \ln f(M) \right) x, x \rangle \leq a_f \langle Bx, x \rangle + b_f,
\]

and from this it follows that

\[
\langle f(B)x, x \rangle - \alpha g(\langle Bx, x \rangle)
\]

\[
\leq \exp \left( \frac{M \mathbf{1}_{\mathcal{H}} - B}{M - m} \ln f(m) + \frac{B - m \mathbf{1}_{\mathcal{H}}}{M - m} \ln f(M) \right) x, x \rangle - \alpha g(\langle Bx, x \rangle)
\]

\[
\leq a_f \langle Bx, x \rangle + b_f - \alpha g(\langle Bx, x \rangle)
\]

\[
\leq \max_{m \leq t \leq M} \{a_f t + b_f - \alpha g(t)\}.
\]

Here we put \( t = \langle Bx, x \rangle \), then \( m \leq t \leq M \). Whence

\[
\langle f(B)x, x \rangle \leq \exp \left( \frac{M \mathbf{1}_{\mathcal{H}} - B}{M - m} \ln f(m) + \frac{B - m \mathbf{1}_{\mathcal{H}}}{M - m} \ln f(M) \right) x, x \rangle
\]

\[
\leq \alpha g(\langle Bx, x \rangle) + \beta
\]

\[
\leq \alpha g(\langle Ax, x \rangle) + \beta \quad \text{(since } A \leq B \text{ and } g \text{ is decreasing)}
\]

\[
\leq \alpha \langle g(A) x, x \rangle + \beta \quad \text{(since } g \text{ is convex)}
\]
and the assertion follows. \[\Box\]

The following corollary improves the result in [16, Corollary 2.5]. In fact, if we put \( f(t) = t^p \) and \( g(t) = t^q \) with \( p \leq 0 \) and \( q \in \mathbb{R} \setminus (0, 1) \), we get:

**Corollary 2.1.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be two strictly positive operators such that \( A \leq B \) and \( m1_H \leq B \leq M1_H \) for some scalars \( 0 < m < M \). Then for a given \( \alpha > 0 \),

\[
(2.3) \quad B^p \leq \exp \left( \frac{M1_H - B}{M - m} \ln m^p + \frac{B - m1_H}{M - m} \ln M^p \right) \leq \alpha A^q + \beta 1_H,
\]

holds, where \( \beta \) is defined as

\[
(2.4) \quad \beta = \begin{cases} 
\alpha (q - 1) \left( \frac{M^p - m^p}{m(q(M - m))} \right)^\frac{1}{q-1} + \frac{Mm^p - mM^p}{m-m} & \text{if } m \leq \left( \frac{M^p - m^p}{m(q(M - m))} \right)^\frac{1}{q-1} \leq M \\
\max \{ m^p - \alpha m^q, M^p - \alpha M^q \} & \text{otherwise}
\end{cases}
\]

Especially, by setting \( p = q \) in (2.3), we reach

\[
B^p \leq \exp \left( \frac{M1_H - B}{M - m} \ln m^p + \frac{B - m1_H}{M - m} \ln M^p \right) \leq \alpha A^p + \beta 1_H \quad \text{for } p \leq 0,
\]

where

\[
(2.5) \quad \beta = \begin{cases} 
\alpha (p - 1) \left( \frac{M^p - m^p}{m(q(M - m))} \right)^\frac{1}{p-1} + \frac{Mm^p - mM^p}{m-m} & \text{if } m \leq \left( \frac{M^p - m^p}{m(q(M - m))} \right)^\frac{1}{p-1} \leq M \\
\max \{ m^p - \alpha m^q, M^p - \alpha M^q \} & \text{otherwise}
\end{cases}
\]

If we choose \( \alpha \) such that \( \beta = 0 \) in Theorem 2.1, then we obtain the following corollary. For completeness, we sketch the proof.

**Corollary 2.2.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be two strictly positive operators such that \( A \leq B \) and \( m1_H \leq B \leq M1_H \) for some scalars \( 0 < m < M \). Let \( f : [m, M] \to (0, \infty) \) be a log-convex function and \( g : I \to \mathbb{R} \) be a continuous function, where \( I \) is an interval containing \( Sp(A) \cup [m, M] \). If \( g \) is a non-negative decreasing convex on \([m, M]\), then

\[
(2.6) \quad f(B) \leq \exp \left( \frac{M1_H - B}{M - m} \ln f(m) + \frac{B - m1_H}{M - m} \ln f(M) \right) \leq \max_{m \leq t \leq M} \left\{ \frac{a_f t + b_f}{g(t)} \right\} g(A).
\]

Moreover

\[
(2.7) \quad B^p \leq \exp \left( \frac{M1_H - B}{M - m} \ln m^p + \frac{B - m1_H}{M - m} \ln M^p \right) \leq K (m, M, p, q) A^q,
\]

where \( K (m, M, p, q) \) is defined as

\[
(2.8) \quad K (m, M, p, q) = \begin{cases} 
\left( \frac{m(M^p - m^p)}{(q-1)(M-m)} \right) \left( \frac{m^p - q}{q - m} \right)^{\frac{q-1}{M^p - m^p}} & \text{if } m \leq \frac{q(m^p - q)}{(q-1)(M^p - m^p)} \leq M \\
\max \{ m^{p-q}, M^{p-q} \} & \text{otherwise}
\end{cases}
\]

(We emphasize that \( K(m, M, p, q) \) was given in [14, Theorem 3.1].)
In particular, if \( p = q \) in (2.7), we get

\[
B^p \leq \exp \left( \frac{M \mathbf{1}_H - B}{M - m} \ln m^p + \frac{B - m \mathbf{1}_H}{M - m} \ln M^p \right) \leq K (m, M, p) A^p \quad \text{for } p \leq 0,
\]

where \( K (m, M, p) \) is defined as (1.2).

\textbf{Proof.} From the condition on the function \( g \), we have \( \beta = \max_{m \leq t \leq M} \{ a_t + b_t \} - \alpha \min_{m \leq t \leq M} \{ g(t) \} \).

When \( \beta = 0 \), we have \( \alpha = \max_{m \leq t \leq M} \left\{ \frac{a_t + b_t}{g(t)} \right\} = \max_{m \leq t \leq M} \left\{ \frac{a_t + b_t}{g(t)} \right\} \). Thus we have the inequalities (2.6).

If we take \( f(t) = t^p \) and \( g(t) = t^q \) with \( p \leq 0 \) and \( q \in \mathbb{R} \setminus (0, 1) \) in (2.6), we have \( \alpha = \max_{m \leq t \leq M} \{ a_t t^{1-q} + b_t t^{-q} \} \). We find \( \alpha = \left( \frac{b_p}{1-q} \right) \left( \frac{1-q}{q \phi_p} \right)^q \) when \( t_0 = \frac{q b_p}{1-q} \alpha \) satisfies \( m \leq t_0 \leq M \). Thus we have \( \alpha = K (m, M, p, q) \) by simple calculations with \( a_t = \frac{M^p - m^p}{M-m} \), \( b_t = \frac{M m^p - M^p}{M-m} \) and the other cases are trivial. Thus we have the inequalities (2.7) and (2.9). \( \square \)

Observe that Corollary 2.2 gives a refinement of [16, Corollary 2.6].

The last result in this section, which is a refinement of [10, Corollary 2.2] (see also [12, Corollary 1]) can be stated as follows.

\textbf{Corollary 2.3.} Let \( A, B \in \mathfrak{B} (\mathcal{H}) \) be two strictly positive operators such that \( A \leq B \) and \( m \mathbf{1}_H \leq B \leq M \mathbf{1}_H \) for some scalars \( 0 < m < M \). Then

\[
B^p \leq \exp \left( \frac{M \mathbf{1}_H - B}{M - m} \ln m^p + \frac{B - m \mathbf{1}_H}{M - m} \ln M^p \right) \leq C (m, M, p, q) \mathbf{1}_H + A^q \quad \text{for } p, q \leq 0,
\]

where \( C (m, M, p, q) \) is the Kantorovich constant for the difference with two parameters and defined by

\[
C (m, M, p, q) = \begin{cases} 
\frac{M m^p - m M^p}{M-m} + (q-1) \left( \frac{M^p - m^p}{q(M-m)} \right)^{\frac{1}{q-1}} & \text{if } m \leq \left( \frac{M^p - m^p}{q(M-m)} \right)^{\frac{1}{q-1}} \leq M, \\
\max \{ M^p - M^q, m^p - m^q \} & \text{otherwise} 
\end{cases}
\]

\textbf{Proof.} If we put \( \alpha = 1 \), \( f(t) = t^p \) for \( p \leq 0 \) and \( g(t) = t^q \) for \( q \leq 0 \) in Theorem 2.1, then we have \( \beta = \max_{m \leq t \leq M} \{ a_t t + b_t t^{-q} \} \). By simple calculations, we have \( \beta = (q-1) \left( \frac{a_p}{q} \right)^{\frac{1}{q-1}} + b_p \)

when \( t_0 = \left( \frac{a_p}{q} \right)^{\frac{1}{q-1}} \) satisfies \( m \leq t_0 \leq M \). The other cases are trivial. Thus we have the desired conclusion, since \( a_t = \frac{M^p - m^p}{M-m} \) and \( b_t = \frac{M m^p - M^p}{M-m} \). \( \square \)

3. Application to the Chaotic Order

In this section, we show some inequalities on chaotic order (i.e., \( \log A \leq \log B \) for \( A, B > 0 \)). To achieve our next results, we need the following lemma. Its proof is standard but we provide a proof for the sake of completeness.
Lemma 3.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be two strictly positive operators. Then the following statements are equivalent:

(i) $\log A \leq \log B$.

(ii) $B^r \leq \left( B^{\frac{p}{p+r}} A^p B^{\frac{r}{p+r}} \right)^{\frac{p}{p+r}}$ for $p \leq 0$ and $r \leq 0$.

Proof. From the well-known “chaotic Furuta inequality” (see, e.g., [2, 3, 6]) the order $\log A \geq \log B$ is equivalent to the inequality $(B^r A^p B^r)^{\frac{2}{p+r}} \geq B^r$ for $p, r \geq 0$ and $A, B > 0$. The assertion (i) is equivalent to the order $\log B^{-1} \leq \log A^{-1}$. By the use of chaotic Furuta inequality, the order $\log B^{-1} \leq \log A^{-1}$ is equivalent to the inequality

$$B^{-r} \leq \left( B^{\frac{-r}{p}} A^{-p} B^{\frac{-r}{p}} \right)^{\frac{p}{p+r}}$$

for $p, r \geq 0$.

This is equivalent to the inequality

$$B^{r'} \leq \left( B^{\frac{r'}{p}} A^{p'} B^{\frac{r'}{p}} \right)^{\frac{p'}{p+r}}$$

for $p', r' \leq 0$, by substituting $p' = -p$ and $r' = -r$ in (3.1). We thus obtain the desired conclusion. □

As an application of Corollary 2.2, we have the following result:

Corollary 3.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be two strictly positive operators such that $m \mathbf{1}_\mathcal{H} \leq B \leq M \mathbf{1}_\mathcal{H}$ for some scalars $0 < m < M$ and $\log A \leq \log B$. Then for $p \leq 0$ and $-1 \leq r \leq 0$,

$$B^p \leq B^{-r} \exp \left( \frac{M \mathbf{1}_\mathcal{H} - B}{M - m} \ln m^{p+r} + \frac{B - m \mathbf{1}_\mathcal{H}}{M - m} \ln M^{p+r} \right) \leq K(m, M, p + r) A^p.$$

Proof. The idea of proof is similar to the one in [17, Theorem 1]. Thanks to Lemma 3.1, the chaotic order $\log A \leq \log B$ is equivalent to $B^r \leq \left( B^{\frac{p}{p+r}} A^p B^{\frac{r}{p+r}} \right)^{\frac{p}{p+r}}$ for $p, r \leq 0$. Putting $B_1 = B$ and $A_1 = \left( B^{\frac{p}{p+r}} A^p B^{\frac{r}{p+r}} \right)^{\frac{p}{p+r}}$ in the above, then $0 < A_1 \leq B_1$ and $m \mathbf{1}_\mathcal{H} \leq B_1 \leq M \mathbf{1}_\mathcal{H}$. Thus we have for $p_1 \leq 0$

$$B^{p_1} = B_1^{p_1} \leq \exp \left( \frac{M \mathbf{1}_\mathcal{H} - B}{M - m} \ln m^{p_1} + \frac{B - m \mathbf{1}_\mathcal{H}}{M - m} \ln M^{p_1} \right) \leq K(m, M, p_1) A_1^{p_1} = K(m, M, p_1) \left( B^{\frac{p}{p+r}} A^p B^{\frac{r}{p+r}} \right)^{\frac{p_1}{p+r}},$$

by (2.9). By setting $p_1 = p + r \leq 0$ and multiplying $B^{-r}$ to both sides, we obtain the desired conclusion. □

In a similar fashion, one can prove the following result:
Corollary 3.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be two strictly positive operators such that $m1_\mathcal{H} \leq B \leq M1_\mathcal{H}$ for some scalars $0 < m < M$ and $\log A \leq \log B$. Then for $p \leq 0$ and $-1 \leq r \leq 0$,

$$B^p \leq B^{-r} \exp \left( \frac{M1_\mathcal{H} - B}{M-m} \ln m^{p+r} + \frac{B - m1_\mathcal{H}}{M-m} \ln M^{p+r} \right) \leq C (m, M, p + r) 1_\mathcal{H} + A^p.$$  

Proof. If we set $p = q$ in Corollary 2.3, we have the following inequalities for $p \leq 0$

$$B^p \leq \exp \left( \frac{M1_\mathcal{H} - B}{M-m} \ln m^p + \frac{B - m1_\mathcal{H}}{M-m} \ln M^p \right) \leq C (m, M, p) 1_\mathcal{H} + A^p,$$

where

$$C (m, M, p) = \begin{cases} \frac{Mm^p - mM^p}{M-m} + (p-1) \left( \frac{M^p - m^p}{p(M-m)} \right) \frac{1}{m^r} & \text{if } m \leq \left( \frac{M^p - m^p}{p(M-m)} \right)^{\frac{1}{1-r}} \leq M, \\ 0 & \text{otherwise} \end{cases}$$

Thanks to Lemma 3.1, the chaotic order $\log A \leq \log B$ is equivalent to $B^r \leq (B^\frac{r}{2} A^p B^{\frac{r}{2}})^{\frac{1}{r}}$ for $p, r \leq 0$. Putting $B_1 = B$ and $A_1 = (B^\frac{r}{2} A^p B^{\frac{r}{2}})^{\frac{1}{r}}$ in the above, then $0 < A_1 \leq B_1$ and $m1_\mathcal{H} \leq B_1 \leq M1_\mathcal{H}$. Thus we have for $p_1 \leq 0$

$$B_1^{p_1} \leq \exp \left( \frac{M1_\mathcal{H} - B_1}{M-m} \ln m^{p_1} + \frac{B_1 - m1_\mathcal{H}}{M-m} \ln M^{p_1} \right) \leq C (m, M, p_1) 1_\mathcal{H} + A_1^{p_1},$$

by (3.2). Putting $p_1 = p + r \leq 0$ and multiplying $B^{-\frac{r}{2}}$ to both sides, we obtain the desired conclusion.  

$\square$

4. Miscellanea

By the similar way presented in this article, it is also possible to improve the results which previously obtained by employing the Mond-Pečarić method.

As a multiple operator version of the celebrated “Davis-Choi-Jensen inequality” [1], Mond and Pečarić in [15, Theorem 1] proved the inequality

$$f \left( \sum_{i=1}^n w_i \Phi_i (A_i) \right) \leq \sum_{i=1}^n w_i \Phi_i (f (A_i)),$$

for operator convex function $f$ defined on an interval $I$, where $\Phi_i$ ($i = 1, \ldots, n$) are normalized positive linear mappings from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$, $A_1, \ldots, A_n$ are self-adjoint operators with spectra in $I$ and $w_1, \ldots, w_n$ are non-negative real numbers with $\sum_{i=1}^n w_i = 1$.

In a reverse direction to that of inequality (4.1) we have the following:

Theorem 4.1. Let $\Phi_i$ be normalized positive linear maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$, $A_i \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators with $m1_\mathcal{H} \leq A_i \leq M1_\mathcal{H}$ for some scalars $m < M$ and $w_i$ be positive
numbers such that \( \sum_{i=1}^{n} w_i = 1 \). If \( f \) is a log-convex function and \( g \) is a continuous function on \([m, M]\), then for a given \( \alpha \in \mathbb{R} \)

\[
\sum_{i=1}^{n} w_i \Phi_i (f(A_i)) \leq \sum_{i=1}^{n} w_i \Phi_i \left( \exp \left( \frac{M1_H - A_i}{M - m} \ln f (m) + \frac{A_i - m1_H}{M - m} \ln f (M) \right) \right) \\
\leq \alpha g \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right) + \beta 1_K,
\]

holds with \( \beta \) as (1.3).

Proof. Thanks to (2.2), we get

\[
f(A_i) \leq \exp \left( \frac{M1_H - A_i}{M - m} \ln f (m) + \frac{A_i - m1_H}{M - m} \ln f (M) \right) \leq a_f A_i + b_f 1_K.
\]

The hypotheses on \( \Phi_i \) and \( w_i \) ensure the following:

\[
\sum_{i=1}^{n} w_i \Phi_i (f(A_i)) \leq \sum_{i=1}^{n} w_i \Phi_i \left( \exp \left( \frac{M1_H - A_i}{M - m} \ln f (m) + \frac{A_i - m1_H}{M - m} \ln f (M) \right) \right) \\
\leq a_f \sum_{i=1}^{n} w_i \Phi_i (A_i) + b_f 1_K.
\]

Using the fact that \( m1_K \leq \sum_{i=1}^{n} w_i \Phi_i (A_i) \leq M1_K \), we can write

\[
\sum_{i=1}^{n} w_i \Phi_i (f(A_i)) - \alpha g \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right) \\
\leq \sum_{i=1}^{n} w_i \Phi_i \left( \exp \left( \frac{M1_H - A_i}{M - m} \ln f (m) + \frac{A_i - m1_H}{M - m} \ln f (M) \right) \right) - \alpha g \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right) \\
\leq a_f \sum_{i=1}^{n} w_i \Phi_i (A_i) + b_f 1_K - \alpha g \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right) \\
\leq \max_{m \leq t \leq M} \{ a_f t + b_f - \alpha g (t) \} 1_K,
\]

which is, after rearrangement, equivalent to (4.2). So the proof is complete. \( \square \)

It is worth mentioning that, Theorem 4.1 is stronger than what appears in [13, Theorem 2.2].

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function and \( A, B \in \mathbb{B} (\mathcal{H}) \) be two strictly positive operators such that \( Sp \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) \subseteq I \). Then the operator \( \sigma_f \) given by

\[
A\sigma_f B = A^{\frac{3}{2}} f \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) A^{\frac{1}{2}},
\]

is called \( f \)-connection (cf. [11]). We shall show the following result involving \( f \)-connection of strictly positive operators.
Theorem 4.2. Let $\Phi$ be a normalized positive linear map from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{K})$ and $A, B \in \mathbb{B}(\mathcal{H})$ be two strictly positive operators such that $mA \leq B \leq MA$ for some scalars $0 < m < M$. If $f$ is a log-convex function on $[m, M]$, then for a given $\alpha \in \mathbb{R}$

\begin{equation}
\Phi (A \sigma_f B) \leq \Phi \left( A^{1/2} \exp \left( \frac{M1_\mathcal{H} - A^{-1/2}BA^{-1/2}}{M-m} \ln f(m) + \frac{A^{-1/2}BA^{-1/2} - m1_\mathcal{H}}{M-m} \ln f(M) \right) A^{1/2} \right)
\end{equation}

\begin{equation}
\leq \beta \Phi (A) + \alpha \left( \Phi (A) \sigma_f \Phi (B) \right),
\end{equation}

holds with $\beta$ as (1.3).

Proof. We give a sketch of long but routine calculations. It follows from Theorem 4.1 that

\begin{equation}
\Psi \left( f \left( A^{-1/2}BA^{-1/2} \right) \right) \leq \Psi \left( \exp \left( \frac{M1_\mathcal{H} - A^{-1/2}BA^{-1/2}}{M-m} \ln f(m) + \frac{A^{-1/2}BA^{-1/2} - m1_\mathcal{H}}{M-m} \ln f(M) \right) \right)
\end{equation}

\begin{equation}
\leq \beta 1_\mathcal{K} + \alpha f \left( \Psi \left( A^{-1/2}BA^{-1/2} \right) \right),
\end{equation}

where $\Psi$ is a normalized positive linear map from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{K})$.

By taking $\Psi (X) := \Phi( A^{-1/2}X A^{1/2} ) \Phi(A)^{-1/2}$, where $\Phi$ is an arbitrary normalized positive linear map in (4.4), we obtain the desired result (4.3).

In the sequel, we use the notation $A^{\#} v = A^{1/2} \left( A^{-1/2}BA^{-1/2} \right)^v A^{1/2} \ (v \in \mathbb{R})$. The following corollary follows by setting $f(t) = t^p \ (p \leq 0)$ in the previous theorem.

Corollary 4.1. Let $\Phi$ be a normalized positive linear map from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{K})$ and $A, B \in \mathbb{B}(\mathcal{H})$ be two strictly positive operators such that $mA \leq B \leq MA$ for some scalars $0 < m < M$. Then for a given $\alpha \in \mathbb{R}$,

\begin{equation}
\Phi \left( A^{\#} p B \right) \leq \Phi \left( A^{1/2} \exp \left( \frac{M1_\mathcal{H} - A^{-1/2}BA^{-1/2}}{M-m} \ln m^p + \frac{A^{-1/2}BA^{-1/2} - m1_\mathcal{H}}{M-m} \ln M^p \right) A^{1/2} \right)
\end{equation}

\begin{equation}
\leq \beta \Phi (A) + \alpha \left( \Phi (A)^{\#} \Phi (B) \right),
\end{equation}

holds for $p \leq 0$, where $\beta$ is defined as (2.5).

Fujii and Seo [4, Theorem 2.2] showed the following operator inequality: Let $A, B \in \mathbb{B}(\mathcal{H})$ be two positive operators and $\Phi$ be a normalized positive linear map, then

\begin{equation}
\Phi \left( A^{\#} p \Phi (B) \right) \leq \Phi \left( A^{\#} p B \right) \quad \text{for } p \in [-1, 0).
\end{equation}

The following corollary is a complementary result for (4.5). The proof is immediate by using Corollary 4.1.
Corollary 4.2. Let $\Phi$ be a normalized positive linear map from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{K})$ and $A, B \in \mathbb{B}(\mathcal{H})$ be two strictly positive operators such that $mA \leq B \leq MA$ for some scalars $0 < m < M$ and $p \leq 0$.

(i) As a ratio type reverse of inequality (4.5) we have:

$$\Phi (A^p B) \leq \Phi \left( A^{\frac{1}{p}} \exp \left( \frac{M1_H - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M - m} \ln m^p + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - m1_H}{M - m} \ln M^p \right) A^{\frac{1}{2}} \right)$$

$$\leq K(m, M, p) (\Phi (A)^p p \Phi (B)),$$

where $K(m, M, p)$ is defined as (1.2).

(ii) As a difference type reverse of inequality (4.5) we have:

$$\Phi (A^p B) \leq \Phi \left( A^{\frac{1}{p}} \exp \left( \frac{M1_H - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M - m} \ln m^p + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - m1_H}{M - m} \ln M^p \right) A^{\frac{1}{2}} \right)$$

$$\leq C(m, M, p) \Phi (A) + \Phi (A)^p p \Phi (B),$$

where $C(m, M, p)$ is defined as (3.3).

We close this paper by presenting a result on the inequalities for the Tsallis relative operator entropy. The Tsallis relative operator entropy with negative parameter introduced in [5] as

$$T_p (A|B) = \frac{A^p B - A}{p} \text{ for } p < 0.$$  

Research in this field includes obtaining new inequalities and refining existing ones. For example, in [4, Theorem 3.1 (2')], the following inequality has been already shown:

$$\Phi (T_p (A|B)) \leq T_p (\Phi (A) | \Phi (B)) \text{ for } p \in [-1, 0).$$  

We shall give complementary inequalities to the inequality (4.7), thanks to Corollary 4.2.

Theorem 4.3. Let $\Phi$ be normalized positive linear map from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{K})$ and $A, B \in \mathbb{B}(\mathcal{H})$ be two strictly positive operators such that $mA \leq B \leq MA$ for some scalars $0 < m < M$. Then for $p \in [-1, 0)$,

$$\Phi (T_p (A|B))$$

$$\geq \frac{1}{p} \Phi \left( A^{\frac{1}{2}} \exp \left( \frac{M1_H - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M - m} \ln m^p + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - m1_H}{M - m} \ln M^p \right) A^{\frac{1}{2}} - A \right)$$

$$\geq T_p (\Phi (A) | \Phi (B)) - \left( \frac{1 - K(m, M, p)}{p} \right) (\Phi (A)^p p \Phi (B)),$$
and
\[
\Phi(T_p (A|B)) \geq \frac{1}{p} \Phi \left( A^{\frac{1}{2}} \exp \left( \frac{M1_H - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M-m} \ln m^p + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - m1_H \ln M^p}{M-m} \right) A^{\frac{1}{2}} - A \right) \\
\geq T_p (\Phi (A) | \Phi (B)) + \frac{C (m, M, p) \Phi (A)}{p}.
\]

**Proof.** It follows from Corollary 4.2 (i) that
\[
\Phi \left( \frac{A^{\frac{1}{2}} p B - A}{p} \right) \\
\geq \frac{1}{p} \Phi \left( A^{\frac{1}{2}} \exp \left( \frac{M1_H - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M-m} \ln m^p + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - m1_H \ln M^p}{M-m} \right) A^{\frac{1}{2}} - A \right) \\
\geq \frac{K (m, M, p) (\Phi (A) ^{\frac{1}{2}} p \Phi (B)) - \Phi (A)}{p}.
\]

On account of (4.6) the inequalities in (4.10) are equivalent to (4.8). From Corollary 4.2 (ii) we have
\[
\Phi \left( \frac{A^{\frac{1}{2}} p B - A}{p} \right) \\
\geq \frac{1}{p} \Phi \left( A^{\frac{1}{2}} \exp \left( \frac{M1_H - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M-m} \ln m^p + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - m1_H \ln M^p}{M-m} \right) A^{\frac{1}{2}} - A \right) \\
\geq \frac{C (m, M, p) \Phi (A) + \Phi (A) ^{\frac{1}{2}} p \Phi (B) - \Phi (A)}{p},
\]
which is equivalent to (4.9). \qed

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