NO HYPERBOLIC SETS IN $J_\infty$ FOR INFINITELY RENORMALIZABLE QUADRATIC POLYNOMIALS

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Abstract. Let $f$ be an infinitely renormalizable quadratic polynomial and $J_\infty$ be the intersection of forward orbits of "small" Julia sets of its simple renormalizations. We prove that $J_\infty$ contains no hyperbolic sets.

1. Introduction

Let $f$ be a rational function of degree at least 2 considered as a dynamical systems $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere $\hat{\mathbb{C}}$. An $f$-invariant compact set $X \subset \hat{\mathbb{C}}$ is said to be hyperbolic if $f : X \to X$ is uniformly expanding, i.e., for some $C > 0$ and $\lambda > 1$, $|D(f^m)(x)| \geq C\lambda^m$ for all $x \in X$ and all $m \geq 0$ (here $D$ stands for the spherical derivative and $f^m$ is $m$-iterate of $f$). In particular, any repelling periodic orbit of $f$ is a hyperbolic set. The closure of all repelling periodic orbits of $f$ is the Julia set $J(f)$ of $f$. Hyperbolic sets of $f$ are contained in $J(f)$. Apart of repelling periodic orbits, $f$ admits plenty of infinite (Cantor) hyperbolic sets [30]. Attracting periodic orbits (if any) along with their basins are contained in the complement $\hat{\mathbb{C}} \setminus J(f)$ (which is called the Fatou set of $f$). See e.g. [2] for an introduction to complex dynamics and [31] for a recent survey.

If $J(f)$ is a hyperbolic set by itself, i.e., $f : J(f) \to J(f)$ is uniformly expanding, then $f$ is called a hyperbolic rational map. Equivalently, all critical points of $f$ are in basins of attracting cycles. Hyperbolic rational maps are analogous to Axiom A diffeomorphisms and their dynamics has been intensively studied and very well understood. The famous 'Density of Hyperbolicity Conjecture (DHC)' in holomorphic dynamics - sometimes also called the Fatou conjecture - asserts that any rational map (polynomial) can be approximated by hyperbolic rational maps (polynomials) of the same degree.

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In what follows \( f \) (unless mentioned explicitly) is a quadratic polynomial
\[ f_c(z) = z^2 + c. \] The DHC (as well as a more general MLC: Mandelbrot set Locally Connected) is widely open for the quadratic family \( f_c \), too (DHC for \( f_c \) as strongly believed accumulates in itself the essence of the general DHC). After a breakthrough work of Yoccoz [10] on the MLC, the only obstacle for proving DHC for quadratic polynomials are so-called infinitely renormalizable ones, see [28].

Somewhat informally, a quadratic polynomial \( f_c \) with connected Julia set is called renormalizable if, for some topological disks \( U, V \) around the critical point \( 0 \) of \( f_c \) and for some \( p \geq 2 \) (called period of the renormalization), the restriction \( f_c^p : U \to V \) is conjugate to another quadratic polynomial \( f_c' \) with connected Julia set (see [6] for exact definitions and the theory of polynomial-like mappings). The map \( F := f_c^p : U \to V \) is then a renormalization of \( f_c \) and the set \( K(F) = \{ z \in U : F^n(z) \in U \text{ for all } n \geq 1 \} \) is a "small" (filled in) Julia set of \( f_c \). If \( f_c \) is renormalizable by itself, then \( f_c \) is called twice renormalizable, etc. If \( f_c \) admits infinitely many renormalizations, it is called infinitely renormalizable. Recall that the renormalization \( F \) is simple if any two sets \( f^i(K(f)), f^j(K(F)) \), \( 0 \leq i < j \leq p - 1 \), are either disjoint or intersect each other at a unique point which does not separate either of them.

To state our main result - which is Theorems 1.1 - let \( f(z) = z^2 + c \) be infinitely renormalizable. Let \( 1 = p_0 < p_1 < ... < p_n < ... \) be the sequence of consecutive periods of simple renormalizations of \( f \) and \( J_n \) denotes the "small" Julia set of the \( n \)-renormalization (where \( J_0 = J(f) \)). Then \( p_{n+1}/p_n \) is an integer, \( f^{p_n}(J_n) = J_n \), for any \( n \), and \( \{J_n\}_{n=1}^{\infty} \) is a strictly decreasing sequence of continua without interior, all containing \( 0 \). Let
\[ J_{\infty} = \cap_{n \geq 0} \cup_{j=0}^{p_n-1} f^j(J_n) \]
be the intersection of orbits of the "small" Julia sets. \( J_{\infty} \) is a compact \( f \)-invariant set which contains the omega-limit set \( \omega(0) \) of \( 0 \). Each component of \( J_{\infty} \) is wandering, in particular, \( J_{\infty} \) contains no periodic orbits of \( f \). Note that a hyperbolic set in \( J_{\infty} \) (if existed) could not be repelling, that is any forward orbit of a point sufficiently close to this set must be in the set itself, since otherwise shadowing periodic orbits must be in \( J_{\infty} \).

It is shown in [23] that the low Lyapunov exponent of the critical value \( c \in J(f_c) \) is always non-negative. In the considered case, \( c \in J_{\infty} \). We prove:
Theorem 1.1. \( J_\infty \) contains no hyperbolic sets.

Combined with the Fatou-Mane theorem \[25\] Theorem 1.1 immediately implies

Corollary 1.1. \( \omega(x) \cap \omega(0) \neq \emptyset \), for the omega-limit set \( \omega(x) \) of every \( x \in J_\infty \).

The conclusion of Theorem 1.1 would obviously hold provided

\[
(1.1) \quad J_\infty \text{ is totally disconnected.}
\]

(1.1) is true indeed for many classes of maps (including real ones) where it follows from ‘complex bounds’ \[33\] (meaning roughly that the sequence of renormalizations is compact) \[17, 9, 24, 13, 14, 15\]. See also \[11, 12\]. However, (1.1) breaks down in general: see \[26, 32\] for the existence of such maps and \[18, 19, 20\] (see also \[5\]) for explicit combinatorial conditions on \( f_c \) for (1.1) to fail. Yoccoz \[35\] posed a problem to find a necessary and sufficient condition on the combinatorics of \( f_c \) for (1.1) to hold. At present, the gap between known sufficient and necessary conditions is still very big.

Another well-known open problem is to give necessary and sufficient conditions so that the Julia set \( J(f) \) is locally-connected. For example, if (1.1) does not hold then \( J(f) \) is not locally-connected. Theorem 1.1 implies

Theorem 1.2. Let \( f(z) = z^2 + c \) and \( f \) has no irrational indifferent periodic orbits. Then \( J(f) \) is locally-connected at every point of any hyperbolic set \( X \) of \( f \). In particular, there are at least one and at most finitely many external rays landing at each \( x \in X \).

Remark 1.2. The case that \( f \) does have an irrational cycle seems to be open and requires a separate consideration, see [4] though. Note also that Theorem 1.2 removes the only restriction in Proposition 2.11 of [11] for degree 2 polynomials without irrational cycles.

Theorem 1.2 has been known for the following quadratic maps \( f \). If \( f \) has an attracting cycle, then \( f \) is hyperbolic and the whole \( J(f) \) is locally-connected. The same conclusion holds if \( f \) has a parabolic cycle \[6\]. The first part of Yoccoz’s result (see e.g., \[10\]) says that \( J(f) \) is locally-connected if \( f \) has no indifferent irrational cycles and at most finitely many times renormalizable. This allows us to reduce the proof of Theorem 1.2 to the case of \( f \) as in Theorem 1.1 hence, by the latter, to the case when \( X \) is disjoint from \( J_\infty \) in which case it is well-known that Yoccoz puzzle pieces shrink to each
point of $X$ [26], [21]. This shows that $J(f)$ is locally connected at points of $X$. The last claim follows then from [16], see also [31] and [21].

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## 2. Preliminaries

Here we collect, for further references, necessary notations and general facts which are either well-known [28], [27] or follow readily from the known ones. Let $f(z) = z^2 + c$ be infinitely renormalizable. We keep the notations of the Introduction.

(A). Let $G$ be the Green function of the basin of infinity $A(\infty) = \{z|f^n(z) \to \infty, n \to \infty\}$ of $f$ with the standard normalization at infinity $G(z) = \ln|z| + O(1/|z|)$. The external ray $R_t$ of argument $t \in S^1 = \mathbb{R}/\mathbb{Z}$ is a gradient line to the level sets of $G$ that has the (asymptotic) argument $t$ at $\infty$. $G(z)$ is called the (Green) level of $z \in A(\infty)$ and the unique $t$ such that $z \in R_t$ is called the (external) argument (or angle) of $z$. A point $z \in J(f)$ is accessible if there is an external ray $R_t$ which lands at (i.e., converges to) $z$. Then $t$ is called an (external) argument (angle) of $z$.

Let $\sigma : S^1 \to S^1$ be the doubling map $\sigma(t) = 2t (mod 1)$. Then $f(R_t) = R_{\sigma(t)}$.

(B). Given a small Julia set $J_n$ containing 0, sets $f^j(J_n)$ ($0 \leq j < p_n$) are called small Julia sets of level $n$. Each $f^j(J_n)$ contains $p_{n+1}/p_n \geq 2$ small Julia sets $f^{j+kp_n}(J_{n+1})$, $0 \leq k < p_{n+1}/p_n$, of level $n+1$. We have $J_n = -J_n$. Since all renormalizations are simple, for $j \neq 0$, the symmetric companion $-f^j(J_n)$ of $f^j(J_n)$ can intersect the orbit $\text{orb}(J_n) = \cup_{j=0}^{p_n-1} f^j(J_n)$ of $J_n$ only at a single point which is preperiodic. On the other hand, since only finitely many external rays converge to each periodic point of $f$, the set $J_\infty$ contains no periodic points. In particular, each component $K$ of $J_\infty$ is wandering, i.e., $f^i(K) \cap f^j(K) = \emptyset$ for all $0 \leq i < j < \infty$. All this implies that $\{x, -x\} \subset J_\infty$ if and only if $x \in K_0 := \cap_{n=1}^{\infty} J_n$.

Given $x \in J_\infty$, for every $n$, let $j_n(x)$ be the unique $j \in \{0, 1, \ldots, p_n-1\}$ such that $x \in f^{j_n(x)}(J_n)$. Let $J_{x,n} = f^{j_n(x)}(J_n)$ be a small Julia set of level $n$ containing $x$ and $K_x = \cap_{n=0}^{\infty} J_{x,n}$, a component of $J_\infty$ containing $x$.

In particular, $K_0 = \cap_{n=0}^{\infty} J_n$ is the component of $J_\infty$ containing 0 and $K_c = \cap_{n=1}^{\infty} f(J_n)$, the component containing $c$. 
Moreover, for any \( n \in \mathbb{R} \), the rays \( s_n \) to note that \( S \) has the smallest length among all these arcs.

\[
\text{component of } f_{\sigma_n} \text{ has the smallest length among all these arcs.}
\]

(C). Given \( n \geq 0 \), the map \( f^n : f(J_n) \to f(J_n) \) has two fixed points: the separating fixed point \( \alpha_n \) (that is, \( f(J_n) \setminus \{\alpha_n\} \) has at least two components) and the non-separating \( \beta_n \) (so that \( f(J_n) \setminus \{\beta_n\} \) has a single component).

For every \( n > 0 \), there are two rays \( R_{t_n} \) and \( R_{\tilde{t}_n} \) (\( 0 < t_n < \tilde{t}_n < 1 \)) to the non-separating fixed point \( \beta_n \) of \( f^n \) such that the component \( \Omega_n \) of \( C \setminus (R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n) \) which does not contain \( 0 \) has two characteristic properties:

(i) \( \Omega_n \) contains \( c \) and contains no the forward orbit of \( \beta_n \),

(ii) for every \( 1 \leq j \leq p_n \), consider arguments (angles) of the external rays which land at \( f^{i-1}(\beta_n) \). The angles split \( S^1 \) into finitely many arcs. Then the arc

\[
S_{n,1} = [t_n, \tilde{t}_n] = \{t : R_t \subset \Omega_n\}
\]

has the smallest length among all these arcs.

Denote

\[
t'_n = t_n + \frac{\tilde{t}_n - t_n}{2p_n}, \quad \tilde{t}'_n = \tilde{t}_n - \frac{\tilde{t}_n - t_n}{2p_n}.
\]

The rays \( R_{t'_n}, R_{\tilde{t}'_n} \) land at a common point \( \beta'_n \in f^{-p_n}(\beta_n) \cap \Omega_n \). Introduce an (unbounded) domain \( U_n \) with the boundary to be two curves \( R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n \) and \( R_{t'_n} \cup R_{\tilde{t}'_n} \cup \beta'_n \). In other words, \( U_n \) is a component of \( f^{-p_n}(\Omega_n) \) which is contained in \( \Omega_n \). Then \( c \in U_n \) and \( f^n : U_n \to \Omega_n \) is a two-to-one branched covering so that

\[
(2.1) \quad f(J_n) = \{z | f^{kp_n}(z) \in U_n, G(f^{kp_n}(z)) \leq 10, k = 0, 1, \ldots\}.
\]

Moreover, for any \( n \), the closure of \( U_{n+1} \) is contained in \( U_n \). We denote

\[
s_{n,1} = [t_n, t'_n] \cup [\tilde{t}'_n, \tilde{t}_n].
\]

Then \( s_{n,1} \subset S_{n,1} \) and

\[
\sigma^n : s_{n,1} \to S_{n,1}
\]

so that \( \sigma^n \) is a homeomorphism of each component of \( s_{n,1} \) onto \( s_{n,1} \). End points \( t_n, \tilde{t}_n \) of \( S_{n,1} \) are fixed points of \( \sigma^n \). It’s important to note that \( S_{n+1,1} \subset S_{n,1} \), \( s_{n,1} \subset S_{n+1,1} \) for all \( n \) and the length \((\tilde{t}_n - t_n)/2p_n\) of each of the two components of \( s_{n,1} \) tends to zero as \( n \to \infty \) (while the length \(|t_n - \tilde{t}_n|\) of \( s_{n,1} \) can stay away from zero).

From now on, given a compact set \( Y \subset J(f) \) denote by \( \tilde{Y} \) the set of arguments of the external rays which have their limit sets in \( Y \).
For each $k \geq 0$ the boundary of the set $\{ z : f^{k\beta}(z) \in U_n \}$ consists of rays and (pre)periodic points. It follows that if a ray has at least one limit point in $f(J_n)$ then all its limit points are in $f(J_n)$, and (2.1) implies that

$$f(J_n) = \{ t | \sigma^m(t) \in s_{n,1}, j = 0, \ldots \}.$$  

So $\tilde{f}(J_n)$ is a Cantor set, in particular, closed. Let us show that

$$K_c = \cap_{n=1}^\infty s_{n,1}.$$  

Indeed, $t \in K_c$ implies $t \in \tilde{f}(J_n) \subset s_{n,1}$, for each $n$. Vice versa, let $t \in \cap_{n=1}^\infty s_{n,1}$. It is enough to show that $t \in \tilde{f}(J_n)$ for each $n$ (which would indeed imply that the limit set of the ray $R_t$ belongs to $f(J_n)$ for all $n$, i.e., $t \in K_c$). Fix $n$ and find a sequence $t_m \in \partial s_{n,1}$, such that $t_m \to t$ as $m \to \infty$. On the other hand, $\partial s_{n,1} \subset \tilde{f}(J_m) \subset \tilde{f}(J_n)$ because $f(J_m) \subset f(J_n)$ for $m > n$. As $\tilde{f}(J_n)$ is closed, $t \in \tilde{f}(J_n)$. This proves (2.3).

It implies that $K_c$ is either a single-point set or a two-point set. In particular, $K_c$ contains at most two different accessible points. As $f : K_0 \setminus \{0\} \to K_c \setminus \{c\}$ is two-to-one, $\tilde{K}_0 = \sigma^{-1}(K_c)$ consists of either 2 or 4 points.

Let us give, for completeness of the picture, a similar description of $\tilde{K}$ for each component $K$ of $J_\infty$ (Lemma 2.1 see below). It will be needed for the proof of Lemma 4.2 part (ii). Note, however, that part (ii) of Lemma 4.2 is not used in the proof of the main result.

Let

$$s_{n,j} = \sigma^{j-1}(s_{n,1}) = [t_{n,j}, t'_{n,j}] \cup [\tilde{t}'_{n,j}, \tilde{t}_{n,j}], 1 \leq j \leq p_n,$$

where $t_{n,j} = \sigma^{j-1}(t_n)$, $\tilde{t}_{n,j} = \sigma^{j-1}(\tilde{t}_n)$, $t'_{n,j} = \sigma^{j-1}(t'_n)$, and $\tilde{t}'_{n,j} = \sigma^{j-1}(\tilde{t}'_n)$. Then

$$t'_{n,j} - t_{n,j} = \tilde{t}_{n,j} - \tilde{t}'_{n,j} = \frac{\tilde{t}_n - t_n}{2p_{n-j+1}} < \tilde{t}_n - t_n < 1/2.$$  

So $\sigma^{j-1} : s_{n,1} \to s_{n,j}$ is a homeomorphism and $s_{n,j}$ has two components ("windows") $[t_{n,j}, t'_{n,j}]$ and $[\tilde{t}'_{n,j}, \tilde{t}_{n,j}]$ of equal length. However, $\sigma^{j}(S_{n,1})$ can cover the whole circle $S^1$ for some $j < p_n$. For this reason, an analogue of (2.1) breaks down if $j$ is big. For $j = 1, 2, \ldots, p_n$, let $U_{n,j} = f^{j-1}(U_n)$ and $\beta_{n,j} = f^{j-1}(\beta_n)$. The domain $U_{n,j}$ is bounded by two rays $R_{t_{n,j}} \cup R_{\tilde{t}_{n,j}}$ converging to $\beta_{n,j}$ and completed by $\beta_{n,j}$ along with two rays $R_{t'_{n,j}} \cup R_{\tilde{t}'_{n,j}}$ completed by their common
limit point \( f^{j-1}(\beta'_n) \). Let \( U_{n,j}^1 \) be a component of \( f^{-(p_n-j+1)}(U_n) \) which is contained in \( U_{n,j} \). Then
\[
(2.5) \quad f^{p_n} : U_{n,j}^1 \to U_{n,j}
\]
is a two-to-one branched covering, and
\[
(2.6) \quad f^{j-1}(J_n) = \{ z \mid f^{k_p_n}(z) \in \overline{U}_{n,j}^1, G(f^{k_p_n}(z)) \leq 10, k = 0, 1, \ldots \}.
\]
Note that this is consistent with (2.1) for \( j = 1 \). Similar to \( j = 1 \), if a ray has at least one limit point in \( f^{j-1}(J_n) \) then all its limit points are there.

Let \( s_{n,j}^1 \) be the set of arguments of rays entering \( \overline{U}_{n,j}^1 \). Then \( s_{n,j}^1 \subset s_{n,j} \) consists of two pairs of components (1-’windows’) where each pair is adjacent to two end points of one of the ’windows’ of \( s_{n,j} \). Moreover, \( \sigma^{p_n} : s_{n,j}^1 \to s_{n,j} \) is a two-to-one covering which maps each of four 1-’windows’ of \( s_{n,j}^1 \) homeomorphically onto one of the two ’windows’ of \( s_{n,j} \). Correspondingly, for each \( j = 1, 2, \ldots, p_n \),
\[
(2.7) \quad \tilde{f}^j(J_n) = \{ t \mid \sigma^{k_p_n}(t) \in s_{n,j}^1, k = 0, 1, \ldots \}.
\]
As (2.6) is consistent with (2.1) for \( j = 1 \), (2.7) is consistency with (2.2) for \( j = 1 \).

Given \( n, j \), denote by \( \Delta_{n,j} \) the length of each ’window’ of \( s_{n,j} \) and by \( \Delta_{n,j}^1 \) the length of each ’window’ of \( s_{n,j}^1 \). By (2.4), \( \Delta_{n,j} = \frac{L_n-j_n}{2^{p_n-j_n}} \) and \( \Delta_{n,j}^1 = \frac{\Delta_{n,j}}{2^{p_n}} \). So \( \Delta_{n,j}^1 < 2^{-p_n} \to 0 \) uniformly in \( j \) as \( n \to \infty \).

Let \( K \) be a component of \( J_\infty \):
\[
K = \cap_{n=1}^\infty f^{j_n}(J_n),
\]
where \( 1 \leq j_n \leq p_n \) and \( f^{j_n+1}(J_{n+1}) \subset f^{j_n}(J_n) \).

**Lemma 2.1.**
\[
(2.8) \quad \tilde{K} = \cap_{n=1}^\infty s_{n,j_n}^1
\]
where \( \{ s_{n,j_n}^1 \}_{n=1}^\infty \) is a decreasing sequence of compacts each consisting of four 1-’windows’ with equal lengths tending to zero as \( n \to \infty \). In particular, \( \tilde{K} \) consists of at most 4 points. There is an alternative (1)-(2):

1. either \( p_n - j_n \to \infty \) as \( n \to \infty \) so that the length of each ’window’ of \( s_{n,j_n} \) tends to zero, moreover,
\[
(2.9) \quad \tilde{K} = \cap_{n=1}^\infty s_{n,j_n}
\]
so that \( \# \tilde{K} \in \{ 1, 2 \} \) in this case.
2. there is \( N \geq 0 \) such that \( p_n - j_n = N \) for all \( n \), so that \( f^N(K) = K_0 \).
Proof. (2.8) is very similar to the proof of (2.3) and is left to the reader. As for the alternative (1)-(2), assume that there is $N \geq 0$ such that, for an infinite subsequence $(n') \subset \mathbb{N}$, $p_{n'} - j_{n'} = N$. Then $f^N(K) = \bigcap_{n'} f^{n'}(J_{n'}) = \bigcap_n f^n(J_n) = K_0$, hence, $p_n - j_n = N$ for all $n$. This explains why either $p_n - j_n \to \infty$ or $p_n - j_n = N$ for some $N \geq 0$ and all $n$. Consider the case $p_n - j_n \to \infty$. Then by (2.4) the length $\Delta_{n,j_n}$ of each 'window' of $s_{n,j_n}$ tends to zero uniformly in $j_n$. Repeating again the proof of (2.3) we get that $\cap_{n=1}^{\infty} s_{n,j_n} = \cap_{n=1}^{\infty} s'_{n,j_n}$. This settles the alternative. \qed

(D1). Consider in more detail the case when $K_0 = \{\tau_1, \tau_2\}$, $\tau_1 \neq \tau_2$. Let $S_c$ be the shortest arc in $S^1$ with the end points $\tau_1$, $\tau_2$. It follows from (C):

1. $\sigma^k(\tau_i) \notin S_c$, for $i = 1, 2$ and all $k > 0$,
2. for each $k > 0$, the length of arcs with the end points $\sigma^k(\tau_1)$, $\sigma^k(\tau_2)$ is bigger than or equal to the length of $S_c$,
3. (unlinking) for each positive $j \neq k$, one of the two arcs $S^1 \setminus \{\sigma^k(\tau_1), \sigma^k(\tau_2)\}$ contains both points $\sigma^j(\tau_1), \sigma^j(\tau_2)$. Furthermore, as $K_0 = \sigma^{-1}(\{\tau_1, \tau_2\})$, the set $K_0$ splits the unit circle $S^1$ into 4 arcs in such a way that, for each $j \geq 0$, both points $\sigma^j(\tau_1), \sigma^j(\tau_2)$ have to lie in one and only one of these arcs. In particular, $\sigma^j(\tau_1), \sigma^j(\tau_2)$ lie in one and only one of semicircles $S^1 \setminus \sigma^{-1}(\tau_i)$, for each $i = 1, 2$.

Remark 2.2. Let us put the 'unlinking' property in the context of Thurston's 'laminations' [34]. Consider a topological model of $J(f)$ by shrinking all components of $J_\infty$ as well as all their preimages to points. More formally, let’s build a lamination (a special equivalent relation on $S^1$) as follows [34], [16]: as all periodic points of $f$ are repelling, each (pre)periodic point is a landing point of at least one and at most finitely many rays. Let us identify arguments of such rays for each such point and take a closure of this partial relation to the whole $S^1$. The resulting relation is invariant under the map $\sigma : S^1 \to S^1$. For visualization, this relation is usually extended to the closed unit disk by taking the closed convex hull in the Euclidean plane of each equivalence class. Then obviously different convex hulls (classes) are disjoint. For example, $\sigma^{-1}(\{\tau_1, \tau_2\})$ is a single class, and $\sigma^i(\tau_1), \sigma^i(\tau_2)$ is the other one, for each $j \geq 0$, so their convex hulls are disjoint. The property $\# K \leq 2$ (which is proved in Lemma 2.1) for any component $K$ of $J_\infty$ other than preimages of $K_0$ is, in fact, a very particular case of the fundamental 'no wandering triangle' property of unicritical laminations [34], [21].
(D2). Given $\nu \in [0,1)$ there exists a unique minimal rotation set $\Lambda_\nu \subset S^1$ of $\sigma : S^1 \to S^1$ with the rotation number $\nu$ [2]. Recall that a closed subset $\Lambda$ of $S^1$ is a rotation set of $\sigma$ with the rotation number $\nu$ if $\sigma(\Lambda) \subset \Lambda$ and $\sigma : \Lambda \to \Lambda$ extends to a map of $S^1$ which lifts to an orientation preserving non-decreasing continuous map $F : \mathbb{R} \to \mathbb{R}$ with $F - \text{Id}$ to be 1-periodic and the fractional part of the rotation number of $F$ to be equal to $\nu$. Then [2] $\nu$ is irrational if and only if $\Lambda_\nu$ is infinite, in this case there is a unique closed semi-circle containing $\Lambda_\nu$ so that the end points of this semi-circle belong to $\Lambda_\nu$. Finally, any closed $\sigma$-invariant set $\Lambda \subset S^1$ which is contained in a semi-circle is a rotation set of $\sigma$.

3. Accessibility

We define a telescope following essentially [29]. Given $x \in J(f)$, $r > 0$, $\delta > 0$, $k \in \mathbb{N}$ and $\kappa \in (0, 1)$, an $(r, \kappa, \delta, k)$-telescope at $x \in J$ is collections of times $0 = n_0 < n_1 < \ldots < n_k = n$ and disks $B_l = B(f^{ni}(x), r), l = 0, 1, \ldots, k$ such that, for every $l > 0$: (i) $l/n_l > \kappa$, (ii) there is a univalent branch $g_{n_l} : B(f^{ni}(x), 2r) \to C$ of $f^{-ni}$ so that $g_{n_l}(f^{ni}(x)) = x$ and, for $l = 1, \ldots, k$, $d(f^{ni-1} \circ g_{n_l}(B_l), \partial B_{l-1}) > \delta$ (clearly, here $f^{ni-1} \circ g_{n_l}$ is a branch of $f^{-ni-1}$ that maps $f^{ni}(x)$ to $f^{ni-1}(x)$). The trace of the telescope is a collection of sets $B_{l,0} = g_{n_l}(B_l), l = 0, 1, \ldots, k$. We have: $B_{k,0} \subset B_{k-1,0} \subset \ldots \subset B_{1,0} \subset B_{0,0} = B_0 = B(x, r)$. By the first point of intersection of a ray $R_t$, or an arc of $R_t$, with a set $E$ we mean a point of $R_t \cap E$ with the minimal level (if it exists).

**Theorem 3.1.** [29] Given $r > 0$, $\kappa \in (0,1)$, $\delta > 0$ and $C > 0$ there exist $M > 0$, $\bar{l}, k \in \mathbb{N}$ and $K > 1$ such that for every $(r, \kappa, \delta, k)$-telescope the following hold. Let $k > \bar{k}$. Let $u_0 = u$ be any point at the boundary of $B_k$ such that $G(u) \geq C$. Then there are indexes $1 \leq l_1 < l_2 < \ldots < l_j = k$ such that $l_1 < \bar{l}, l_{i+1} < Kl_i, i = 1, \ldots, j-1$ as follows. Let $u_k = g_{n_k}(u) \in \partial B_{k,0}$ and let $\gamma_k$ be an infinite arc of an external ray through $u_k$ between the pint $u_k$ and $\infty$. Let $u_{k,k} = u_k$ and, for $l = 1, \ldots, k-1$, let $u_{k,l}$ be the first point of intersection of $\gamma_k$ with $\partial B_{l,0}$. Then, for $i = 1, \ldots, j$,

$$G(u_{k,l,i}) > M2^{-n_i}.$$

Applying Theorem 3.1 as in [29] (where the existence of a ray to $x$ is proved assuming $(r, \kappa, \delta, \infty)$-telescope to $x$) we obtain the following statement. See also [1] for a direct proof of part (1).

**Corollary 3.1.** Let $X$ be a hyperbolic set for $f$. (1) To every point $x \in X$ one can assign a non-empty set $A_x \subset S^1$ such that for every
t \in A_x\) the external ray \(R_t\) lands at \(x\). (2) The set \(A = \{(x,t) : x \in X, t \in A_x\}\) is closed in \(\mathbb{C} \times S^1\). (3) Moreover, for each \(\mu > 0\) there is \(C(\mu) > 0\) such that for all \(x \in X\) and all \(t \in A_x\), the first intersection of \(R_t\) with \(\partial B(x, \mu)\) has the level at least \(C(\mu)\).

Proof. As \(f : X \to X\) is expanding, there exist \(\lambda > 1, \rho > 0\) and \(j_0\) such that, for every \(x \in X\) and every \(k > 0\), there exists a (univalent) branch \(g_{k,x} : B(f^{kj_0}(x), \rho) \to \mathbb{C}\) of \(f^{-kj_0}\) with \(g_{k,x}(f^{kj_0}(x)) = x\) and \(|g_{k,x}'(y)/g_{k,x}'(z)| < 2\) for \(y, z \in B(f^{kj_0}(x), \rho)\). Moreover, \(|g_{k,x}'(z)| < \lambda^{-k}\) for all \(k > 0\) and \(x \in X\). Therefore, there are \(r > 0\), \(\delta > 0\) and \(\kappa = 1/j_0\) such that for any \(k > 1\), every point \(x \in X\) admits an \((r, \kappa, \delta, k)\)-telescope with \(n_k = kj_0\). In fact, \(n_i = i j_0\) for \(i = 0, 1, \ldots, k\). Let \(B_{k,0}(x) \subset B_{k-1,0}(x) \subset \cdots \subset B_{1,0}(x) \subset B_{0,0}(x)\) be the corresponding trace. Define \(C_0 = \inf_{y \in J(f)} \max_{z \in B(y, r)} G(z)\). It is easy to see that \(C_0 > 0\). For each \(x \in X\) we choose a point \(u(x) \in \partial B(x, r)\) such that \(G(u(x)) \geq C_0\). By Theorem 3.1 there are \(l\) and \(\tilde{k}\) such that for each \(k > \tilde{k}\) and each \(x \in X\) the following hold. There are \(1 \leq l_{i,k}(x) < l_{i+1,k}(x) < \cdots < l_{j_k^x,k}(x) = k\) such that \(l_{i,k}(x) < \tilde{l}, l_{i+1,k}(x) < Kl_{i,k}(x), i = 1, \ldots, j_k^x - 1\). Let \(\gamma_k(x)\) be an arc of an external ray between the point \(u_k(x) = g_{k,x}(u(f^{kj_0}(x)))\) and \(\infty\). Let \(u_{k,l}(x)\) be the first intersection of \(\gamma_k(x)\) with \(\partial B_{l,0}(x)\). Then, for \(i = 1, \ldots, j_k^x - 1\),

\[G(u_{k,l,i,k}(x)) \geq M 2^{-l_{i,k}(x)} j_0.\]

For all \(i = 1, \ldots, j^x_k - 1\),

\[i \leq l_{i,k}(x) < K^i \tilde{l}.\]

Denote by \(t_k(x)\) the argument of an external ray that contains the arc \(\gamma_k(x)\). It is enough to prove in the situation above

**Lemma 3.2.** (i) If \((x_m)_{m > 0} \subset X, x_m \to y\) and \(t_{k_m}(x_m) \to \tau\) for some \(k_m \to \infty\), then \(R_\tau\) lands at \(y\). (ii) Moreover, for each \(\mu > 0\) there is \(C(\mu) > 0\) such that for all pairs \((y, \tau)\) like this the first intersection of \(R_\tau\) with \(\partial B(y, \mu)\) has the level at least \(C(\mu)\).

Indeed, assuming this lemma, we can define \(A_x\) as the set of all angles \(t\) so that there exist \(x_m \in X\) and \(k_m \to \infty\) with \(x_m \to x\) and \(t_{k_m}(x_m) \to t\). The set \(A_x\) is not empty because one can take \(x_m = x\) for all \(m\) and \(k_m \to \infty\) such that \(\{t_{k_m}(x_m)\}\) converges. By Lemma 3.2(i), the ray \(R_t\) indeed lands at the point \(x\). It is then an elementary exercise to check that the set \(A\) is closed. Claim (ii) implies obviously the part (3).
So, let’s prove Lemma 3.2. Let \((y, \tau)\) be as in the lemma. Pick any \(\mu \in (0, r)\). Fix \(l_0\) so that \(\lambda^{-l_0}r < \mu/2\) and let

\[
C(\mu) = \frac{M}{2f_{j_0}K^{l_0}}.
\]

There is \(m_0\) such that for all \(m > m_0\) and all \(l > l_0\), \(B_{l_0, 0}(x_m) \subset B(y, \mu)\). Then, by (3.1)-(3.2), for every \(m > m_0\),

\[
G(u_{km, l_0, km}(x_m)) > C(\mu).
\]

Hence, for every \(m > m_0\) the first intersection of \(\gamma_{km}(x_m)\) with \(B(y, \mu)\) has level at least \(C(\mu)\). This implies that, given \(C \in (0, C(\mu))\), for any \(m > m_0\), an arc of the ray \(R_{t_m}(x_m)\) between the levels \(C\) and \(C(\mu)\) does not exit \(B(y, \mu)\). As this sequence of arcs tend uniformly, as \(m \to \infty\), to an arc of \(R_\tau\) between the levels \(C\) and \(C(\mu)\), this latter arc is contained in \(B(y, \mu)\). As \(C > 0\) can be chosen arbitrary small, the ray \(R_\tau\) must land at \(y\) and its first intersection with \(B(y, \mu)\) has the level at least \(C(\mu)\). Lemma 3.2 is proved.

4. A COMBINATORIAL PROPERTY

Let \(f\) be an infinitely-renormalizable quadratic polynomial. First, we prove the following combinatorial fact (a maximality property) about \(f\).

In the course of the further proofs the following well-known easy statement about expanding maps is used (for a more general theorem about expansive maps, see e.g. [8]):

**Proposition 4.1.** Let \(h\) be a homeomorphism of a compact metric space \((S, d)\) onto itself which is expanding in the following sense: there are \(\delta > 0\) and \(\lambda > 1\) such that \(d(h(x), h(x')) \geq \lambda d(x, x')\) whenever \(d(x, x') < \delta\). Then \(S\) is a finite set.

Let \(\omega(t)\) be the omega-limit set of \(t \in S^1\) under \(\sigma : t \mapsto 2t(mod1)\) and \(\omega(E) = \bigcup_{t \in E} \omega(t)\).

**Lemma 4.2.** (i) \(\sigma^{-1}(\tilde{K}_c) \subset \omega(\tilde{K}_c)\)

(ii) \(J_\infty = \omega(\tilde{K}_c)\).

**Remark 4.3.** Only part (i) is used in the proof of Theorem 1.1. Part (ii) seems to be of an independent interest and is included for completeness.

**Proof of (i).** (a) Consider \(\sigma : J_\infty \to J_\infty\). Each \(t \in J_\infty\) belongs to \(\tilde{K}\), for one and only one component \(K\) of \(J_\infty\). By (B), one and only
one component $K^{-1}$ of $f^{-1}(K)$ is a component of $J_\infty$, moreover, $f : K^{-1} \to K$ is a homeomorphism if $K \neq K_c$ and $f : K_0 \setminus \{0\} \to K_c \setminus \{c\}$ is two-to-one. Hence, $\sigma^{-1}(t) \cap \tilde{J}_\infty$ is a single point for $t \notin \tilde{K}_c$ and two different points for $t \in \tilde{K}_c$. On the other hand, for each $t \in S^1$, $\sigma$ maps $\omega(t)$ onto itself. Since $\sigma : \tilde{J}_\infty \to \tilde{J}_\infty$ has no periodic points, by Proposition \[1\] for each $t \in \tilde{J}_\infty$, the expanding map $\sigma : \omega(t) \to \omega(t)$ is not injective. Therefore, $\omega(t) \cap \tilde{K}_c \neq \emptyset$ for all $t \in \tilde{J}_\infty$.

(b) If $\tilde{K}_c$ consists of a single angle $\tau_c$, then (a) implies that there is a single point $t \in \omega(\tau_c)$ such that $\sigma^{-1}(t) \subset \omega(\tau_c)$ and, moreover, $t = \tau_c$. Therefore, $\sigma^{-1}(\tau_c) \subset \omega(\tau_c)$ and we are done in this case.

(c) It remains to deal with a two-point set $\tilde{K}_c = \{\tau_1, \tau_2\}$. Let us assume the contrary, i.e., $\sigma^{-1}(\tau_1) \cup \sigma^{-1}(\tau_2)$ is not a subset of $\omega(\tilde{K}_c) = \omega(\tau_1) \cup \omega(\tau_2)$. Hence, by (a) and by the assumption, either $\sigma^{-1}(\tau_1)$ or $\sigma^{-1}(\tau_2)$ is a subset of each $\omega(\tau_i)$, $i = 1, 2$. Let, say, $\sigma^{-1}(\tau_1) \subset \omega(\tau_1) \cap \omega(\tau_2)$. By the assumption, there is $\tau \in \sigma^{-1}(\tau_2)$ such that $\tau \notin \omega(\{\tau_1, \tau_2\})$. Let $L$ be a (open) semi-circles $S^1 \setminus \sigma^{-1}(\tau_1)$ that contains $\tau$. We claim that it is enough to show that for each $p_n$ and some $j_n > 0$ the arc $L$ contains $\sigma^{j_n p_n - 1}(\tau_1)$. Indeed, assume this is the case. Then, by (D1)3$_\infty$, Sect. \[2\] $L$ must contain one of the arcs $S^1 \setminus \{\sigma^{j_n p_n - 1}(\tau_1), \sigma^{j_n p_n - 1}(\tau_2)\}$ and, by (D1)2$_\infty$, the lengths of all such arcs are uniformly away from 0. Hence, there is a subsequence $n_i \to \infty$ such that the sequences $\sigma^{j_n p_n - 1}(\tau_1)$ and $\sigma^{j_n p_n - 1}(\tau_2)$ converge to points $a_1$ and $a_2$ respectively, where $a_1 \neq a_2$ and $a_1, a_2 \in \mathcal{L}$. On the other hand, $a_1, a_2 \in \tilde{K}_0$. But, from the assumption, $\tilde{K}_0 \cap L = \{\tau\}$. Therefore $\{a_1, a_2\} \subset \{\tau, \sigma^{-1}(\tau_1)\}$. Since $a_1, a_2 \in \omega(\{\tau_1, \tau_2\})$ while $\tau \notin \omega(\{\tau_1, \tau_2\})$, we conclude that $\{a_1, a_2\} = \sigma^{-1}(\tau_1)$. But then $\sigma^{j_n p_n}(\tau_1)$ and $\sigma^{j_n p_n}(\tau_2)$ converge to the same point $\tau_1$ which is a contradiction with (D1)2$_\infty$.

(d) To show that, for each $n$, the arc $L$ contains a point $\sigma^{j_n p_n - 1}(\tau_1)$, for some $j_n > 0$, let us assume the contrary. So, we fix $n > 0$ and assume that $\Lambda := \{\sigma^{j_n p_n - 1}(\tau_1) : j > 0\} \subset L' := S^1 \setminus L$. The idea is to show that the set $\Lambda$ corresponds to a rotation set $\Lambda_0$ of $\sigma : S^1 \to S^1$ and use a certain structure of later sets \[2\], see (D2) of Sect. \[2\] to arrive at a contradiction.

Following the notations of (C), let $\Omega^0_n = f^{-1}(\Omega_n)$ and $U^0_n = f^{-1}(U_n)$. To connect it to other notations introduced in (C), $\Omega^0_n = U^0_n = f^{p_n - 1}(U_n)$ and $U^0_n = U^0_n = U^0_n p_n$. Then $J_n = \{z | f^{j_n p_n}(z) \in U^0_n \cap G(f^{j_n p_n}(z)) \leq 10, j = 0, 1, \cdots\}$. The domain $\Omega^0_n$ is bounded by two bi-infinite curves $\Gamma_n, \Gamma'_n$ (components of $f^{-1}(\partial \Omega_n)$) and two angular (closed) "arcs at infinity" which are components $S_{n, 0}, S'_{n, 0}$
of $\sigma^{-1}(S_{n,1})$ where $S_{n,1} = [t_n, \bar{t}_n], 0 < t_n < \bar{t}_n < 1$. In fact, $S_{n,0}, S'_{n,0}$ are two components of $s_{n,pn} = \sigma^{pn-1}(s_{n,1})$. Arguments of rays entering $\overline{T_{\alpha}}_n$ form the set $s_{n,pn}^1$ consisting of four 1-"widows" such that $\sigma^{pn}$ is a homeomorphism of each of them onto either $S_{n,0}$ or $S'_{n,0}$. By (2.7), $J_n = f^{pn}(J_n) = \{t | \sigma^{kp_{n}}(t) \in s_{n,pn}^1, k = 0, 1, \ldots\}$.

Let us specify $\Gamma_n$ to be such component of $f^{-1}(\partial\Omega_n)$ that contains $f^{pn-1}(\beta_n)$ (in other words, $f^{pn}(\Gamma_n) = \Gamma_n$), and $S_{n,0}$ to be the first "arc" as one goes from $\Gamma_n$ to $\Gamma_n$ inside of $\Omega_n$ in the counterclockwise direction along the "circle at infinity". In other words, $S_{n,0}$ denotes the component of $\sigma^{-1}(S_{n,1})$ that contains $\sigma^{pn-1}(t_n)$ in its boundary. Now, let $\epsilon(t) = 0$ if $t \in S_{n,0}$ and $\epsilon(t) = 1$ if $t \in S'_{n,0}$. To every $t \in J_n$ we assign a point $\theta(t) \in S^1$ as follows:

$$\theta(t) = \sum_{j=0}^{\infty} \frac{\epsilon(\sigma^{jn}(t))}{2^{j+1}}.$$ 

Then $\overline{\theta(J_n)} = S^1$, $\theta \circ \sigma^{pn} = \sigma \circ \theta$, and $\theta : J_n \to S^1$ extends to a continuous monotone degree one map $S^1 \to S^1$, see [7, 27, 22 Theorem 3]. Moreover, $\theta$ is injective on a subset $T = \{t \in \overline{J_n} | \sigma^{pn}(t) \notin \partial(S_{n,0} \cup S'_{n,0}), k \geq 0\}$. Note that $\overline{J}_\infty \subset T$ where $\overline{J}_\infty$ is closed. Besides, $K_0 = \sigma^{-1}(\{\tau_1, \tau_2\})$ and $\Lambda = \{\sigma^{jn-1}(\tau_1) : j > 0\}$ are subsets of $\overline{J}_\infty \subset T$. If $\theta_0 := \theta(\tau_1/2)$ then $\theta_1 := \theta(\tau_1/2 + 1/2) = \theta_0 + 1/2$. Therefore, from the assumption, the set $\Lambda_0 := \theta(\Lambda)$ is a subset of a semi-circle $C_{\theta_0}$ with end points $\theta_0$ and $\theta_1$. As $\sigma^{pn}(\Lambda) \subset \Lambda$, then $\sigma(\Lambda_0) \subset \Lambda_0$. It follows [2] (see (D2) of Sect. 2 in the present paper), that the set $\overline{\Lambda_0}$ is a rotation set of $\sigma : S^1 \to S^1$. Let $E \subset \overline{\Lambda_0}$ be a closed subset such that $\sigma : E \to E$ is minimal. As $\overline{\Lambda} \subset J_n \subset T$ where $\overline{J}_\infty$ contains no periodic points of $\sigma^{pn}$, $E$ contains no periodic points of $\sigma$. Hence, $E$ is infinite. By [2], see (D2), for every closed infinite minimal rotation set of $\sigma$ there is a unique closed semi-circle containing it and in this case the end points of the semi-circle belong to the set. Thus $\theta_0, \theta_0+1/2 \in E \subset \overline{\Lambda_0} \subset C_{\theta_0}$. Coming back to the f-plane, we find two sequences $j_i, j'_i \to \infty$ so that $\sigma^{j_ipn-1}(\tau_1) \to \tau_1/2$ and $\sigma^{j'_ipn-1}(\tau_1) \to \tau_1/2 + 1/2$. But then $\sigma^{j_ipn}(\tau_1)$ and $\sigma^{j'_ipn}(\tau_1)$ both tend to $\tau_1$ from different sides, in a contradiction with (D1)1$_\infty$. This completes the proof of part (i) of the statement.

Let us prove part (ii). Obviously, $\omega(K_c) \subset \overline{J}_\infty$. To show the opposite, given $t \in \overline{J}_\infty$ let $K$ be a unique component of $\overline{J}_\infty$ such that $t \in K$. Let $K = \cap_{n\geq 1} J_n^{pn}(J_n)$. By Lemma 2.1 there is an
alternative: either (1) \( p_n - j_n \to \infty \) as \( n \to \infty \) and \( \tilde{K} = \cap_{n=1}^{\infty} s_{n,j_n} \), or (2) there is \( N \geq 0 \) such that \( f^N(K) = K_0 \), i.e., \( p_n - j_n = N \) for all \( n \). In case (2), \( f^N : K \to K_0 \) is a homeomorphism, which implies that \( \sigma^N : \tilde{K} \to \tilde{K}_0 \) is also a homeomorphism. Hence, if \( s \in \tilde{K}_0 \) is a point of \( \omega(\tilde{K}_c) \), then \( \sigma^N|_{\tilde{K}_0}^{-1}(s) \) is also a point of \( \omega(\tilde{K}_c) \). On the other hand, by part (i), \( \tilde{K}_0 = \sigma^{-1}(\tilde{K}_c) \subset \omega(\tilde{K}_c) \). Therefore, \( t \in \omega(\tilde{K}_c) \) in case (2). In case (1), \( \{t\} = \cap_{n \geq 1} s^{(t)}_{n,j_n} \) where \( s^{(t)}_{n,j_n} \) is one of the two 'windows' of \( s_{n,j_n} \), and \( \sigma^{j_n} : s^{(t)}_{n,1} \to s^{(t)}_{n,j_n} \) is a homeomorphism where of course \( s^{(t)}_{n,1} \) is one of the two 'windows' of \( s_{n,1} \). Hence, it's enough to show that each 'window' of \( s_{n,1} \) contains points of the orbit of \( \tilde{K}_c \). As \( \tilde{K}_0 = \cap_n s_{n,1} \), in the case \( \# \tilde{K}_c = 2 \), for all \( n \) big enough, each 'window' of \( s_{n,1} \) contains one and only one of the two points of \( \tilde{K}_c \), and we are done in this case. In the remaining case, \( \tilde{K}_c \) is a single point which lies in one of the two 'widows' of \( s_{n,1} \), for each \( n \). But, as \( \sigma^{p_n} \) maps each 'window' of \( s_{n,1} \) onto \( s_{n,1} \subset s_{n,1} \) homeomorphically it is obvious that each 'window' of \( s_{n,1} \) contains infinitely many points of the \( \sigma^{p_n} \)-orbit of any \( t_0 \in f(J_n) \) provided no point of this orbit hits \( \partial S_{n,1} \). In particular, this applies to points of \( \tilde{K} \). This completes the proof. \( \square \)

**Remark 4.4.** In case (1) we proved more: if \( t \in \tilde{J}_\infty \) is such that \( \sigma^N(t) \notin \tilde{K}_0 \) for all \( N \geq 0 \), then \( t \in \omega(\tau) \) for each \( \tau \in \tilde{K}_c \).

**5. Proof of Theorem 1.1**

1. Assume the contrary and let \( X \subset \tilde{J}_\infty \) be a compact \( f \)-invariant hyperbolic set. In particular, Corollary 3.1 applies.

2. Replacing \( X \) by its subset if necessary we can assume that \( f : X \to X \) is a minimal map.

3. \( 0 \notin X \), hence, \( c \notin X \), too.

4. As \( \tilde{J}_\infty \) contains no cycles, \( X \) is an infinite set. If we assume that \( f : X \to X \) is one-to-one, then \( f : X \to X \) is an expanding homeomorphism of a compact set, therefore, \( X \) is finite, a contradiction with Proposition 4.1.

5. Thus \( f : X \to X \) is not one-to-one. Then, by (B), \( X_c := X \cap K_c \neq \emptyset \). On the other hand, by step 3, \( c \notin X_c \).

6. By (C), \( \tilde{K}_c \) consists of either one or two arguments. As any point of \( X \) is accessible, \( X_c \) consists of either one or two different points. Let \( x_1 \in X_c \) and \( x_2 \in J(f) \) be any other point. Given \( r > 0 \) and \( n \), let \( W_n(x_1,r) \) be a component of \( B(x_1,r) \cap \Omega_n \) (see
(C), Sect. 2 where \( \Omega_n \) is defined) that contains the point \( x_1 \). We use repeatedly the following

**Claim 1.** Given \( \hat{r} > 0 \) and \( \hat{C} > 0 \), there is \( \hat{n} \in \mathbb{N} \) as follows. For \( k = 1, 2 \), suppose that, for some angles \( \hat{t}_k \), the ray \( R_{t_k} \) lands at \( x_k \) and let \( z_k \) be the first intersection of \( R_{t_k} \) with \( \partial B(x_k, \hat{r}/2) \). Assume:

(a) \( G(z_k) > \hat{C} \) for \( k = 1, 2 \), (b) \( x_2 \in \Omega_n \) and \( |x_1 - x_2| < \hat{r}/3 \), (c) one of the following holds: (i) \( t_n - \hat{t}_n \to 0 \) as \( n \to \infty \), or (ii) \( \hat{t}_1, \hat{t}_2 \) belong to a single component of \( s_{n,1} = [t_n, \hat{t}_n] \cup [\hat{t}_n, \hat{t}_n] \). Then \( x_2 \in W_n(x_1, \hat{r}) \) for each \( n > \hat{n} \).

Indeed, the length of each component of \( s_{n,1} \) goes to zero as \( n \to \infty \). Hence, as \( \hat{r} \) and \( \hat{C} \) are constants and \( n \) is big enough, condition (c) implies that a curve which consists of an arc of \( R_{\hat{t}} \), from \( x_1 \) to \( z_1 \), then the shortest arc of the equipotential containing \( z_1 \) from \( z_1 \) to the first intersection \( u_2 \) with \( R_{\hat{t}} \) and then back along \( R_{\hat{t}} \) from \( u_2 \) to \( x_2 \) belongs to \( \Omega_\hat{n} \) and \( B(x_1, \hat{r}) \) simultaneously.

7. Fix \( r > 0 \) small enough. Let \( a \in X_c \) and \( a_{-1} \in X \) be such that \( f(a_{-1}) = a \). As \( a_{-1} \in K_0 \cap X \), there is its uniquely defined backward orbit \( \{a_{-m}\}_{m=1}^\infty \subset X \), \( f(a_{-m}) = a_{-m} \), \( m \geq 1 \). Let \( a' \) be a limit point of the sequences \( a_{-p_n} \), i.e., \( a' = \lim_{n \to \infty} a_{-p_n} \). As \( a_{-p_n} \in f(J_n) \), \( a' \) belongs to \( K_c \) and \( X \) at the same time, that is, \( a' \in X_c \).

**Claim 2.** For all \( i \) large enough, \( a_{-p_{n_i}} \in W_n(a', r) \).

Indeed, by Corollary 3.1 there is a subsequence \( (n'_i) \) of \((n_i)\) and a converging sequence \( t_i \in A_{a_{-p_{n'_i}}} \) such that \( t := \lim_{i \to \infty} t_i \) and \( t \in A_{a'} \). We have: \( t_i \in s_{n'_i} \) for all \( i \). Then Claim 1 of Step 6 applies for each \( i \) big enough, with \( \hat{r} = r \), \( \hat{C} = C(r/2) \), \( x_1 = a' \), \( x_2 = a_{-p_{n'_i}} \), \( \hat{t}_1 = t \), \( \hat{t}_2 = t_i \) and \( z_1, z_2 \) defined by this data as in Claim 1. Indeed, (a) holds by Corollary 3.1(3) and (b) is obvious (note that \( a_{-p_n} \in f(J_n) \setminus \{\beta_n\} \subset \Omega_n \)). Moreover, if (c) breaks down, since \( t_i \to t \), then \( t_i \) and \( t \) must lie at the same component of \( s_{n'_i} \).

By the conclusion of Claim 1, \( a_{-p_{n'_i}} \in W_n(a', r) \) for each \( i \) big enough. Finally, as \( A_{a'} \) consists of at most four points (therefore, the sequence \((a_{-p_{n_i}})\) has at most four limit points), Claim 2 follows.

8. Consider the case \( X_c = \{a\} \). Let \( f^{-1}(a) = \{a_{-1}, a^*_{-1}\} \). By steps 2, 3 and 5, \( a_{-1} \neq a^*_{-1} \) and \( a_{-1}, a^*_{-1} \in X \). As \( X_c = \{a\} \), there is a subsequence \((n_i)\) such that backward images \( a_{-p_{n_i}}, a^*_{-p_{n_i}} \) of \( a_{-1}, a^*_{-1} \) respectively tend to the same point \( a \). By Claim 2 Step 7, for each \( i \) large, \( a_{-p_{n_i}}, a^*_{-p_{n_i}} \in W_n(a, r) \). Therefore, the following two sets (which are preimages of \( W_n(a, r) \) by \( f^{p_{n_i}} \)): \( V_{n_i} := \)
$g_{p_n, a_{-}(p_n)}(W_n(a, r))$ and $V_n^* := g_{p_n, a_{-}(p_n)}(W_n(a, r))$, are disjoint with their closures (because preimages of $B(a, r)$ along points of $X$ shrink exponentially) and both are contained in $W_n(a, r)$. Fix such $n = n_i$. Denote for brevity $J_{c,n} = f(J_n)$. Then we get that, for every $j > 0$, $2^j$ preimages of $a \in J_{c,n}$ by the map $f^{2^j} : J_{c,n} \to J_{c,n}$ are contained in the (disjoint) closures of $V_n$ and $V_n^*$. As the set of all those preimages are dense in $J_{c,n}$, we get a contradiction with the fact that $J_{c,n}$ is a continuum.

9. Consider the remaining case $X_c = \{a, b\}$, $a \neq b$. As $\#\tilde{K}_c \leq 2$, each point $a$ and $b$ is accessible by a single ray $R_{t(a)}$ and $R_{t(b)}$ respectively. Hence, any point $u$ from the grand orbits of $a$ and $b$ is a landing point of precisely one ray $R_{t_u}$. Let us prove that $f^{-1}(X_c) \subset X$. Let $f(w) = x \in X_c$. By Lemma 4.2(i), one can find $y \in X_c$ and $m_i \to \infty$ such that $\sigma^{m_i}(t_y) \to t_w$ and $f^{m_i}(y)$ tends to some point $\tilde{w} \in X$. By Corollary 3.1, $t_w \in A_{\tilde{w}}$. But $\sigma(t_w) = t_x$, hence, $f(\tilde{w}) = x$ and $A_{\tilde{w}} = \{t_w\}$. Thus $w = \tilde{w} \in X$.

10. We have just proved that $\{a_{-1}, a_{-1}^*\} = f^{-1}(a) \subset X$ and $\{b_{-1}, b_{-1}^*\} = f^{-1}(b) \subset X$. Now, we repeat the consideration as in Step 8. The sequences $a_{-(p_n)}$, $a_{-(p_n)}^*$, $b_{-(p_n)}$, $b_{-(p_n)}^*$ must have all their limit points in $X_c$. As $r > 0$ is small enough, $\overline{B(a, r)} \cap \overline{B(b, r)} = \emptyset$. By Claim 2 of Step 7, for each $n$ large, all 4 points $a_{-(p_n)}$, $a_{-(p_n)}^*$, $b_{-(p_n)}$, $b_{-(p_n)}^*$ are in $W_n(a, r) \cup W_n(b, r)$. Fixing $n$ large, for each disk $B(x, r)$, $x \in \{a, b\}$, there are two univalent branches of $f^{-p_n}$ which are defined in $B(x, r)$ such that they map $W_n(x, r)$ inside $W_n(a, r) \cup W_n(b, r)$. Hence, for every $j > 0$, $4^j$ preimages of $X_c \in J_{c,n}$ by the map $f^{2^j p_n} : J_{c,n} \to J_{c,n}$ are contained in the (disjoint) closures of $W_n(a, r)$ and $W_n(b, r)$. As the set of all those preimages are dense in $J_{c,n}$, we again get a contradiction with the fact that $J_{c,n}$ is a continuum.

Remark 5.1. The combinatorial property for quadratic polynomials of Lemma 4.2 implies that if $X \subset J_\infty$ is a minimal hyperbolic set then $f^{-1}(X) \cap J_\infty = X$ provided $f$ is quadratic, and this leads to a contradiction. Therefore, a small modification of the proof gives us the following claim for infinitely-renormalizable unicritical polynomial $f(z) = z^d + c$ with any $d \geq 2$: $J_\infty$ contains no hyperbolic sets $X$ such that $f : X \to X$ is minimal and $f^{-1}(X) \cap J_\infty = X$.

6. Final remarks

A hyperbolic set of a rational map always carries an invariant measure with a positive Lyapunov exponent. Conjecturally, $J_\infty$ as
in Theorem 1.1 never carries such a measure. We cannot prove this conjecture in the full generality so far, but we can easily prove at least that \( F := f|J_\infty \) is not "chaotic". Namely,

1. Every \( F \)-invariant probability measure \( \mu \) has zero entropy, \( h_\mu(F) = 0 \).

2. Topological entropy of \( F \) is zero, \( h_{\text{top}}(F) = 0 \).

Proving it we can assume \( \mu \) is ergodic due to Ergodic Decomposition Theorem, see e.g. \cite[Theorem 2.8.11]{30}. Start by observing that, for every \( n \) and \( 0 \leq j < p_n \), \( \mu(f^j(J_n)) = 1/p_n \), hence, \( \mu \) has no atoms and \( \mu(K) = 0 \) for every component \( K \) of \( J_\infty \). Therefore, if \( J'_\infty = J_\infty \setminus \bigcup_{n \in \mathbb{Z}} f^n(K_0) \) where \( K_0 \) is the component of \( J_\infty \) containing \( 0 \) (see (B) of Sect. 2), then \( F : J'_\infty \to J'_\infty \) is an automorphism. Suppose to the contrary that \( h_\mu(F) > 0 \). Then 1. follows from Rokhlin entropy formula, \cite[Theorem 2.9.7]{30}, saying that \( h_\mu(F) = \int \log \text{Jac}_\mu(F) d\mu \). Here \( \text{Jac}_\mu \) is Jacobian with respect to \( \mu \), equal to 1 \( \mu \)-a.e., since \( \mu \) must be supported on \( J'_\infty \subset J_\infty \) where \( F \) is an automorphism. A condition to be verified to apply Rokhlin formula is the existence of a one-sided countable generator of bounded entropy, proved to exist by Mañé, see e.g. \cite[Lemma 11.3.2]{30} and inclusion \cite[(11.4.8)]{30}, due to positive Lyapunov exponent \( \chi_\mu(F) := \int \log |F'| d\mu \geq \frac{1}{2} h_\mu(F) > 0 \) (Ruelle’s inequality). Thus \( h_\mu(F) > 0 \) has led to a contradiction.

2. follows from 1. by variational principle \( h_{\text{top}}(F) = \sup_\mu h_\mu(F) \).

Compare item 4 in Section 5. Here finiteness of \( X \) is replaced by zero entropy.

The same proof with obvious modifications holds for \( f(z) = z^d + c, \ d \geq 2 \).

References

[1] Benini, A., Lyubich, M.: Repelling periodic points and landing rays for post-singularity bounded exponential maps. Ann. Inst. Fourier (Grenoble) 64 (2014), 4, 1493-1520.

[2] Bullett, S., Sentenac, P.: Ordered orbits of the shift, square roots, and the devil’s staircase. Math. Proc. Cambridge Philos. Soc. 115 (1994), 3, 451-481.

[3] Carleson, L., Gamelin, T.: Complex Dynamics. Springer-Verlag, 1993

[4] Cheraghi, D.: Typical orbits of quadratic polynomials with a neutral fixed point: Brjuno type. Comm. Math. Phys. 322 (2013), 3, 999-1035.

[5] Cheraghi, D., Shishikura, M.: Satellite renormalization of quadratic polynomials. https://arxiv.org/abs/1509.07843

[6] Douady, A., Hubbard, J.H.: On the dynamics of polynomial-like mappings. Ann. Sci. Ec. Norm. Super. (4) 18 (1985), no. 2, 287-343.
[7] Douady, A.: Algorithms for computing angles in the Mandelbrot set. In: "Chaotic Dynamics and Fractals", ed. Barnsley Demko, Acad. Press (1986), pp. 155-168.
[8] W. H. Gottschalk and G. A. Hedlund: Topological Dynamics, Amer. Math. Soc. Colloquium Publications, vol. 36, Providence, 1955.
[9] Graczyk J, Świątek G.: Polynomial-like property for real polynomials. Topol. Proc. 21 (1996), 33-122.
[10] Hubbard, J.H.: Local connectivity of Julia sets and bifurcation coci: three theorems of J.-C. Yoccoz. In: "Topological Methods in Modern Mathematics." Houston, TX: Publish or Perish, 1993.
[11] Kozlovski, O., Shen, W., Van Strien, S.: Rigidity for real polynomials. Ann. Math. 165 (2007), no. 3, 749-841.
[12] Kozlovski, O., Shen, W., Van Strien, S.: Density of hyperbolicity in dimension one. Ann. Math. 166 (2007), no. 1, 145-182.
[13] Kahn, J.: A priori bounds for some infinitely renormalizable quadratics: I. Bounded primitive combinatorics. https://arxiv.org/abs/math/0609045
[14] Kahn, J., Lyubich, M.: A priori bounds for some infinitely renormalizable quadratics. II. Decorations. Ann. Sci. Ec. Norm. Super. (4) 41 (2008), no. 1, 57-84.
[15] Kahn, J., Lyubich, M.: A priori bounds for some infinitely renormalizable quadratics. III. Molecules. Complex dynamics, 229-254, A K Peters, Wellesley, MA, 2009.
[16] Kiwi, J.: Rational laminations of complex polynomials. Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), 111-154, Contemp. Math., 269, Amer. Math. Soc., Providence, RI, 2001.
[17] Levin, G., Van Strien, S.: Local connectivity of the Julia set of real polynomials. Ann. Math.147, 471-541 (1998)
[18] Levin, G.: Multipliers of periodic orbits of quadratic polynomials and the parameter plane. Israel Journal of Mathematics 170 (2009), 285-315.
[19] Levin, G.: Rigidity and non-local connectivity of Julia sets of some quadratic polynomials. Comm. Math. Phys. 304 (2011), 2, 295-328.
[20] Levin, G.: Addendum to: Rigidity and non-local connectivity of Julia sets of some quadratic polynomials. Comm. Math. Phys. 325 (2014), 3, 1171-1178.
[21] Levin, G.: On backward stability of holomorphic dynamical systems. Fund. Math. 158 (1998), no. 2, 97-107.
[22] Levin, G.: Rays to renormalizations. Bull. Polish Acad. Sci. Math., 68 (2020), no. 2.
[23] Levin, G., Przytycki, F., Shen, W.: The Lyapunov exponent of holomorphic maps. Invent. Math. 205 (2016), 363-382.
[24] Lyubich M, Yampolsky M.: Dynamics of quadratic polynomials: complex bounds for real maps. Ann. Inst. Fourier (Grenoble) 47 (1997), 1219-1255.
[25] Mane, M.: On a theorem of Fatou. Boletim da Sociedade Brasileira de Matemática, 24 (1993), 1–11.
[26] Milnor, J.: Local connectivity of Julia sets: expository lectures. The Mandelbrot set, theme and variations, 67-116, London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, Cambridge, 2000.
[27] Milnor, J.: *Periodic orbits, external rays and the Mandelbrot set: an expository account.* "Geometrie Complex et Systemes Dynamiques". Colloque en l'honneur d'Adrien Douady (Orsay, 1995). Asterisque 261, 277-331 (2000).
[28] McMullen, C: *Complex Dynamics and Renormalization.* Annals of Mathematical Studies, 135. Princeton Univ. Press, Princeton, New Jersey, 1994.
[29] Przytycki, F.: *Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps.* Fund. Math. 144 (1994), no.3, 259-278
[30] Przytycki, F., Urbański, M.: *Conformal Fractals: Ergodic Theory Methods.* Cambridge, 2010
[31] Rees, M.: *One hundred years of complex dynamics.* Proc. R. Soc. A 472 (2016)
[32] Sorensen, D.E.K.: *Infinitely renormalizable quadratic polynomials with non-locally connected Julia set.* J. Geom. Anal. 10 (2000), 169-208.
[33] Sullivan, D.: *Bounds, quadratic differentials and renormalization conjectures.* In: Mathematics into the Twenty-First Century, AMS Centennial Publications 2, Providence, RI: Amer. Math. Soc., 1991
[34] Thurston, W.: *On the combinatorics of iterated rational maps.* Princeton, NJ: Princeton University and IAS, 1985.
[35] Yoccoz, J.-Ch.: *Recent developments in dynamics.* Fields Medallists’ Lecture, Proc. Int. Congr. Math., 1994.

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