A New Probability Distribution for Modeling Failure and Service Times: Properties, Copulas and Various Estimation Methods

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Abstract In this paper, a new generalization of the Pareto type II model is introduced and studied. The new density can be “right skewed” with heavy tail shape and its corresponding failure rate can be “J-shape”, “decreasing” and “upside down (or increasing-constant-decreasing).” The new model may be used as an “under-dispersed” and “over-dispersed” model. Bayesian and non-Bayesian estimation methods are considered. We assessed the performance of all methods via simulation study. Bayesian and non-Bayesian estimation methods are compared in modeling real data via two applications. In modeling real data, the maximum likelihood method is the best estimation method. So, we used it in comparing competitive models. Before using the the maximum likelihood method, we performed simulation experiments to assess the finite sample behavior of it using the biases and mean squared errors.

Keywords Pareto type II Distribution, Estimation, Simulations, Copulas, Renyi Copula, Farlie Gumbel Morgenstern Family, Ordinary Least Square, Bayesian Estimation.

AMS 2010 subject classifications 62E10, 62N01, 62N02

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1. Introduction

The Pareto type II (also called as Lomax) distribution is a heavy-tail probability distribution used in business, actuarial science, biological sciences, engineering, economics, income and wealth inequality, queueing theory, size of cities, and Internet traffic modeling (see Lomax [41]). It has been applied to model data obtained from income and wealth (Harris [34] and Atkinson and Harrison [10]), firm size (Corbellini et al. [31]) and reliability and life testing (Hassan and Al-Ghamdi [35]). The Pareto type II model is known as a special model form of Pearson type VI distribution and has also considered as a mixture of exponential (Exp) and gamma (Gam) distributions. The Pareto type II model belongs to the family of “decreasing” hazard rate function (HRF) and considered as a limiting model of residual lifetimes at great age (Balkema and de Hann [19] and Chahkandi and Ganjali [21]). The Pareto type II distribution has been suggested as heavy tailed alternative to the Exp, Weibull (W) and Gam distributions (Bryson [20]). For details about relation between the Pareto type II model and the Burr family and Compound Gamma (CGam) model see Tadikamalla [62] and Durbey [24].

The main aim of this work is to provide a flexible extension of the Pareto type II distribution using the odd Burr-G (OB-G) family defined by Alizadeh et al. [6]. Bayesian and Non-Bayesian estimation methods such as usual as the maximum likelihood estimation (MLE), ordinary least square estimation (OLSE), weighted least square estimation

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(WLSE) and the Kolmogorov estimation (KE) are considered. The Bayesian estimation method under the squared error loss function (SELF). The new model proved its ability in modeling

- The monotonically increasing hazard rate data.
- The bimodal left skewed real data sets.
- The bimodal right skewed real data sets.

A random variable (rv) $Z$ has the Pareto type II (P) distribution with two parameters $\alpha$ and $\beta$ if it has cumulative distribution function (CDF) (for $Z > 0$) given by

$$G_{\alpha, \beta}(z) = 1 - \left(1 + \frac{z}{\beta}\right)^{-\alpha}, \quad (1)$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. Then the corresponding probability density function (PDF) of (1) is

$$g_{\alpha, \beta}(z) = \frac{\alpha \beta}{\beta \left(1 + \frac{z}{\beta}\right)^{\alpha+1}}. \quad (2)$$

Due to Alizadeh et al. [6], the CDF of the OB-G family is given by

$$F_{v, \theta, \xi}(z) = 1 - G_{\xi}(\frac{z}{\theta})^{v\theta}, \quad (3)$$

where $G_{\xi}(z) = 1 - G_{\xi}(z)$. The PDF corresponding to (3) is given by

$$f_{v, \theta, \xi}(z) = \frac{v\theta g_{\xi}(z) G_{\xi}(z)^{v-1} G_{\xi}(z)^{v\theta-1}}{G_{\xi}(z)^{v} + G_{\xi}(z)^{v}}. \quad (4)$$

For $\theta = 1$, the OB-G family reduces to the Odd G (O-G) family (see Gleaton and Lynch [42]). For $v = 1$, the OB-G family reduces to the proportional reversed hazard rate (PRHR) family. The odd Burr Pareto type II (OBP) survival function (SF) is given by

$$S_{\xi}(z) \mid (\xi=v, \theta, \alpha, \beta) = \frac{\nabla_{\beta}(z)^{-\alpha v\theta}}{(1 - \nabla_{\beta}(z)^{-\alpha})^v + \nabla_{\beta}(z)^{-\alpha v})^\theta, \quad (5)$$

where $\nabla_{\beta}(z) = 1 + \frac{z}{\beta}$ and $S_{\xi}(z) = 1 - F_{\xi}(z) \mid (\xi=v, \theta, \alpha, \beta)$. For $\theta = 1$, the OBP reduces to the odd Pareto type II. For $v = 1$, the OBP reduces to the proportional reversed hazard rate Pareto type II. The PDF corresponding to (5) is given by

$$f_{\xi}(z) = \frac{v\theta \alpha}{\beta} \frac{\nabla_{\beta}(z)^{-\alpha(v\theta+1)} \left[1 - \nabla_{\beta}(z)^{-\alpha}\right]^{v-1}}{\left[1 - \nabla_{\beta}(z)^{-\alpha}\right]^v + \nabla_{\beta}(z)^{-\alpha v})^{1+\theta}. \quad (6)$$

The HRF for the new model can be derived from $f_{\xi}(z) / S_{\xi}(z)$. Many useful Pareto type II extensions can be found in Tahir et al. [64] (Weibull Pareto type II distribution), Cordeiro et al. [23] (the one parameter Pareto type II system of densities), Altun et al. [12] (Odd log-logistic Pareto type II), Altun et al. [12] (Zografos-Balakrishnan Pareto type II distribution), Elbiely and Yousof [33] (Weibull generalized Pareto type II, Rayleigh generalized Pareto type II and Exponential generalized Pareto type II distributions), Yousof et al. [66] (Topp Leone Poisson Pareto type II distribution), Goual and Yousof [43] (Pareto type II inverse Rayleigh), Gad et al. [27] (Burr type XII Pareto type II, Pareto type II Burr type XII and Pareto type II Pareto type II distributions), Yadav et al. [65] (Topp Leone Pareto type II distribution) and Ibrahim and Yousof [36] (Poisson Burr X generalized
Figure 1. PDF and HRF plots for some selected parameters value.

Pareto type II and Poisson Rayleigh generalized Pareto type II distributions). To illustrate the flexibility of the new PDF and its corresponding HRF we present Figure 1. Figure 1(a) gives some PDF plots for some selected parameters value. Figure 1(b) gives some HRF plots for some selected parameters value. Based on Figure 1(a) the OBP density can be right skewed with heavy tail shape. Based on Figure 1(b) the OBP HRF can be “J-shape” \((v = 100, \theta = 1, \alpha = 5, \beta = 0.55)\), “decreasing” \((v = 0.75, \theta = 7.5, \alpha = 0.1, \beta = 0.1)\) and “upside down (or increasing-constant-decreasing)” \((v = 1.5 = 1.5, \alpha = \beta = 1)\).

2. Extensions via copula

In this section, we derive some new bivariate type OBP (BOBP) models using Farlie Gumbel Morgenstern (FGM) copula (see Kotz and Johnson [50], Kotz and Johnson [51], Gumbel [44] and Gumbel [45]), modified FGM copula (see Rodriguez-Lallena and Ubeda-Flores [60]) and Clayton copula and Renyi entropy copula (Pougaza and Djafari [59]). The multivariate OBP (MOBP) type is also presented. However, future works may be allocated to study these new models. First, we consider the joint CDF of the FGM family, where \(H_{\rho}(\tau, \varsigma) = \tau \varsigma (1 + \rho \tau \varsigma)\) |\(x = 1 - \tau, \varsigma = 1 - \varsigma\) with the marginal functions \(\tau = F_{\xi_1}(z_1), \varsigma = F_{\xi_2}(z_2), \rho \in (-1, 1)\) is a dependence parameter and for every \(\tau, \varsigma \in (0, 1)\), \(H(\tau, 0) = H(0, \varsigma) = 0\) which is “grounded minimum” and \(H(\tau, 1) = \tau\) and \(H(1, \varsigma) = \varsigma\) which is ”grounded maximum”. \(H(\tau_1, \varsigma_1) + H(\tau_2, \varsigma_2) - H(\tau_1, \varsigma_2) - H(\tau_2, \varsigma_1) \geq 0\). For more details see Elgohari and Yousof [25], Elgohari and Yousof [26], Al-babtain et al. [3], Ali et al. [4] and Ali et al. [5].

2.1. BOBP type via FGM copula

A copula is continuous in \(\tau\) and \(\varsigma\); actually, it satisfies the stronger Lipschitz condition, where

\[
|H(\tau_2, \varsigma_2) - H(\tau_1, \varsigma_1)| \leq |\tau_2 - \tau_1| + |\varsigma_2 - \varsigma_1|.
\]

For \(0 \leq \tau_1 \leq \tau_2 \leq 1\) and \(0 \leq \varsigma_1 \leq \varsigma_2 \leq 1\), we have

\[
Pr(\tau_1 \leq \tau \leq \tau_2, \varsigma_1 \leq \varsigma \leq \varsigma_2) = H(\tau_1, \varsigma_1) + H(\tau_2, \varsigma_2) - H(\tau_1, \varsigma_2) - H(\tau_2, \varsigma_1) \geq 0.
\]
Then, setting $\bar{\tau} = 1 - F_{\xi_1}(z_1)|_{\tau=(1-\tau)\in(0,1)}$ and $\bar{\varsigma} = 1 - F_{\xi_2}(z_2)|_{\varsigma=(1-\varsigma)\in(0,1)}$, we can easily obtain the joint CDF of the FGM family. The joint PDF can then be derived from $c_{\rho}(\tau, \varsigma) = 1 + \rho \tau \varsigma |_{\tau = 1 - 2\tau}$ and $\varsigma = 1 - 2\varsigma$ or from $c(z_1, z_2) = H(F_{\xi_1}(z_1), F_{\xi_2}(z_2)) f_{\xi_1}(z_1) f_{\xi_2}(z_2)$.

### 2.2. BOBP type via modified FGM copula

The modified FGM copula is defined as

$$H_{\rho}(\tau, \varsigma) = \tau \varsigma [1 + \rho \beta(\tau) \Phi(\varsigma)] |_{\rho \in (-\infty, 1)}.$$  

Let $\tau \beta(\tau)$ and $\Phi(\varsigma)$ be two functional forms satisfy all the conditions stated earlier where $\beta(\tau) = \varsigma \Phi(\varsigma)$ and $\Phi(\varsigma) |_{\rho \in (-\infty, 1)} = \varsigma \Phi(\varsigma)$. Then, the corresponding BOBP-FGM (Type-I) can be derived from $H_{\rho}(\tau, \varsigma) = \tau \varsigma + \rho \tau \varsigma \Phi(\varsigma) |_{\rho \in (-\infty, 1)}$.

### Type-II

Let $\beta(\tau)$ and $\Phi(\varsigma)$ be two functional forms satisfy all the conditions stated earlier where $\beta(\tau) = \varsigma \Phi(\varsigma)$ and $\Phi(\varsigma) |_{\rho \in (-\infty, 1)} = \varsigma \Phi(\varsigma)$. Then, the corresponding BOBP-FGM (Type-II) can be derived from $H_{\rho}(\tau, \varsigma) = \tau \varsigma + \rho \tau \varsigma \Phi(\varsigma) |_{\rho \in (-\infty, 1)}$.

### Type-III

Let $\beta(\tau) = \tau \log (1 + \tau)$ and $\Phi(\varsigma) = \varsigma \log (1 + \varsigma)$ in this case, one can also derive a closed form expression for the associated CDF of the BOBP-FGM (Type-III) from $H_{\rho}(\tau, \varsigma) = \tau \varsigma + \rho \tau \varsigma \Phi(\varsigma) |_{\rho \in (-\infty, 1)}$.

### Type-IV

The CDF of the BOBP-FGM (Type-IV) model can be derived from $H(\tau, \varsigma) = \tau F_{\xi_1}(\varsigma) + \varsigma F_{\xi_2}(\tau) - F_{\xi_1}(\tau) F_{\xi_2}(\varsigma)$ where $F_{\xi_1}(\tau)$ and $F_{\xi_2}(\varsigma)$ can be easily derived from (5).

### 2.3. BOBP type via Ali–Mikhail–Haq copula

Under the stronger Lipschitz condition, the joint CDF of the archimedean Ali–Mikhail–Haq copula can expressed as

$$H(s_1, s_2) = \frac{s_1 s_2}{1 - \rho s_1 s_2 |_{\rho \in (-\infty, 1)}},$$

the corresponding joint PDF of the archimedean Ali–Mikhail–Haq copula can expressed as

$$h(s_1, s_2) = \frac{1 - \rho + 2 \rho \sqrt{s_1 s_2}}{[1 - s_1 s_2]^2 |_{\rho \in (-\infty, 1)}},$$

sitting $s_1 = 1 - F_{\xi_1}(z_1)$ and $s_2 = 1 - F_{\xi_2}(z_2)$ we can derive the joint CDF and the joint PDF of the BOBP type via Ali–Mikhail–Haq copula.

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and

Due to Alizadeh et al. (2017a), the PDF in (6) can be expressed as

3.2. Useful representations

Equation (7) is used for simulating the new model.

The asymptotics of CDF, PDF and HRF can be obtained for the new model. The asymptotics of the CDF, PDF and HRF can be described by

Asymptotic analysis is used for describing the limiting behavior of some functions. Asymptotics derivations for the CDF, PDF and HRF can be obtained for the new model. The asymptotics of the CDF, PDF and HRF can be obtained for the new model. The asymptotics of the CDF, PDF and HRF can be obtained for the new model.

3. Mathematical properties

3.1. Asymptotics and quantile function

In mathematical analysis, the asymptotic analysis is used for describing the limiting behavior of some functions.

Asymptotics derivations for the CDF, PDF and HRF can be obtained for the new model. The asymptotics of the CDF, PDF and HRF can be obtained for the new model.

For simulation of this new model, we obtain the quantile function (QF) of \( Z \), by inverting (5), say \( Z_u = F^{-1}(u) \), as

\[
Z_u = \beta \left\{ \left[ \left( 1 - \frac{1 - u^\frac{1}{\alpha}}{u} \right)^\frac{1}{\alpha} + \left( 1 - u \right)^\frac{1}{\beta} \right]^{-\frac{1}{\alpha}} - 1 \right\}.
\]

Equation (7) is used for simulating the new model.

3.2. Useful representations

Due to Alizadeh et al. (2017a), the PDF in (6) can be expressed as

\[
f(Z) = \sum_{s=0}^{\infty} \eta_s g_{s*, \alpha, \beta}(Z) |_{s*=1+s},
\]

where

\[
\eta_s = \frac{v^\theta}{s^\theta} \sum_{j_1, j_2=0}^{\infty} \sum_{j_3=s}^{\infty} (-1)^{j_2+k+s} \binom{- (1 + \theta)}{j_1} \binom{- v (1 + j_1) + 1}{j_2} \binom{v (1 + j_1) + j_2 + 1}{j_3} \binom{j_3}{s},
\]

\[\text{Stat., Optim. Inf. Comput. Vol. 9, September 2021}\]
and $g_{s^*, \alpha, \beta} (z)$ is the PDF of the F model with power parameter $s^*$. By integrating Equation (8), the CDF of $Z$ becomes

$$F(z) = \sum_{s=0}^{\infty} \eta_s G_{s^*, \alpha, \beta} (z),$$

(9)

where $G_{s^*, \alpha, \beta} (z)$ is the CDF of the Pareto type II distribution with power parameter $s^*$.

### 3.3. Moments and incomplete moments

The $r^{th}$ ordinary moment of $Z$ is given by

$$\mu_r = E(Z^r) = \int_{-\infty}^{\infty} z^r f(z) \, dz,$$

then we obtain

$$\mu_r = \sum_{s=0}^{\infty} \sum_{\kappa=0}^{r} \eta_s s^* \beta^r (-1)^\kappa \left( \frac{r}{\kappa} \right) B \left( s^*, 1 + \frac{1}{\alpha} (\kappa - r) \right) \big|_{(\kappa > r)},$$

(10)

where $B(w_1, w_2) = \int_{0}^{1} m^{w_1-1} (1 - m)^{w_2-1} \, dm$. Setting $r = 1, 2, 3$ and 4 in (10), we have

$$E(Z) = \sum_{s=0}^{\infty} \sum_{\kappa=0}^{1} \eta_s s^* \beta^1 (-1)^\kappa \left( \frac{1}{\kappa} \right) B \left( s^*, 1 + \frac{1}{\alpha} (\kappa - 1) + 1 \right) \big|_{(\kappa > 1)},$$

$$E(Z^2) = \sum_{s=0}^{\infty} \sum_{\kappa=0}^{2} \eta_s s^* \beta^2 (-1)^\kappa \left( \frac{2}{\kappa} \right) B \left( s^*, 1 + \frac{1}{\alpha} (\kappa - 2) + 1 \right) \big|_{(\kappa > 2)},$$

$$E(Z^3) = \sum_{s=0}^{\infty} \sum_{\kappa=0}^{3} \eta_s s^* \beta^3 (-1)^\kappa \left( \frac{3}{\kappa} \right) B \left( s^*, 1 + \frac{1}{\alpha} (\kappa - 3) + 1 \right) \big|_{(\kappa > 3)},$$

and

$$E(Z^4) = \sum_{s=0}^{\infty} \sum_{\kappa=0}^{3} \eta_s s^* \beta^4 (-1)^\kappa \left( \frac{4}{\kappa} \right) B \left( s^*, 1 + \frac{1}{\alpha} (\kappa - 4) + 1 \right) \big|_{(\kappa > 4)},$$

where $E(Z) = \mu_1$ is the mean of $Z$. The $r^{th}$ incomplete moment, say $I_r (t)$, of $Z$ can be expressed, from (9), as

$$I_{r, Z} (t) = \int_{-\infty}^{t} z^r f(z) \, dz = \sum_{s=0}^{\infty} \eta_s \int_{-\infty}^{t} z^r g_{s^*, \alpha, \beta} (z) \, dz$$

then

$$I_{r, Z} (t) = \sum_{s=0}^{\infty} \sum_{\kappa=0}^{r} \eta_s s^* \beta^r (-1)^\kappa \left( \frac{r}{\kappa} \right) B_\kappa \left( s^*, 1 + \frac{1}{\alpha} (\kappa - r) \right) \big|_{(\kappa > r)},$$

where $B_\kappa (w_1, w_2) = \int_{0}^{y} m^{w_1-1} (1 - m)^{w_2-1} \, dm$. The first incomplete moment given by (11) with $r = 1$ as

$$I_{1, Z} (t) = \sum_{s=0}^{\infty} \sum_{\kappa=0}^{1} \eta_s s^* \beta^1 (-1)^\kappa \left( \frac{1}{\kappa} \right) B_\kappa \left( s^*, 1 + \frac{1}{\alpha} (\kappa - 1) + 1 \right) \big|_{(\kappa > 1)},$$

The index of dispersion $I_{\text{Dis}}$ or the variance to mean ratio can derived as $I_{\text{Dis}} (z) = \mu_2 / \mu_1$. It is a measure used to quantify whether a set of observed occurrences are clustered or dispersed compared to a standard statistical model.

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3.4. Numerical and graphical analysis

By analyzing the mean \( \mathbb{E}(Z) \), variance \( \text{Var}(Z) \), skewness \( \text{Skew}(Z) \), kurtosis \( \text{Kur}(Z) \) and \( \text{IxDis}(Z) \) numerical in Table 1 we noted that, the skewness \( \text{Skew}(Z) \) of the OBP distribution can range in the interval \((-69216.87, 98983.16)\). The spread for the kurtosis \( \text{Kur}(Z) \) of the OBP model is ranging from 74.17124 to \( \infty \). The IxDis \( \text{IxDis}(Z) \) for the OBP model can be in \((0, 1)\) and also > 1 so it may be used as an “under-dispersed” and “over-dispersed” model. Figure 2 gives some three dimensional skewness plots for parameter \( \alpha \). The flexibility of the skewness of the new model are proved in Figures 2 and 3 using some some three dimensional plots. The flexibility of the kurtosis of the new model are proved in Figures 4 and 5 using some some three dimensional plots. Figure 3 gives some three dimensional skewness plots for parameter \( \theta \). Figure 4 gives some three dimensional kurtosis plots for parameter \( \alpha \). Finally, Figure 5 gives three dimensional kurtosis plots for parameter \( \theta \).

Table 1: \( \mathbb{E}(Z), \text{Var}(Z), \text{Skew}(Z) \) and \( \text{Kur}(Z) \) of the OBP model.

| \( v \) | \( \theta \) | \( \alpha \) | \( \beta \) | \( \mathbb{E}(Z) \) | \( \text{Var}(Z) \) | \( \text{Skew}(Z) \) | \( \text{Kur}(Z) \) | \( \text{IxDis}(Z) \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.5 | 10 | 1.5 | 0.1071429 | 0.01324568 | 2.4825320 | 13.63030 | 0.1236264 |
| 5 | 1.0003165 | 0.0005641241 | 0.4022094 | 4.077114 | 0.0056234 |
| 10 | 0.1033713 | 0.0001518643 | 0.0196449 | 3.829237 | 0.0014691 |
| 50 | 0.1066996 | 6.535887e-06 | -69216.87* | 8673587 | 6.126 \times 10^{-5} |
| 100 | 7.9602 \times 10^{-05} | 9.18712 \times 10^{-07} | 120.4033 | 14497.96 | 0.1154139 |
| 200 | 1.1781 \times 10^{-11} | 1.35973 \times 10^{-12} | 98983.16* | \infty * | 0.1154218 |
| 5 | 0.4836 | 0.01456 | 2.478782 | 16.67835 | 0.168561 |
| 10 | 0.4836 | 0.01456 | 2.478782 | 16.67835 | 0.168561 |
| 50 | 0.3618845 | 0.00549617 | -0.3233682 | 3.074851 | 0.019949 |
| 100 | 0.3201573 | 0.00438074 | -0.3796808 | 3.025701 | 0.013683 |
| 200 | 0.2828904 | 0.00348778 | -0.3712737 | 3.008953 | 0.012329 |
| 2.5 | 2.5 | 10 | 2.5 | 1.076590 | 0.22120710 | 0.729365 | 4.209933 | 0.2054701 |
| 50 | 0.101967 | 0.00179817 | 0.587341 | 3.757355 | 0.0176349 |
| 100 | 0.050832 | 0.00044447 | 0.579787 | 4.74781 | 0.0087438 |
| 200 | 0.025378 | 0.00011049 | 3.266505 | -14.47678 | 0.0043536 |
| 500 | 0.010142 | 1.7618 \times 10^{-5} | 10.98053 | -74.17124* | 0.0017371 |
| 1.5 | 3.5 | 1.5 | 0.119954 | 0.006565 | 1.215589 | 5.318137 | 0.054729 |
| 20 | 0.479814 | 0.105038 | 1.215591 | 5.318139 | 0.218914 |
| 50 | 1.199535 | 0.656487 | 1.215591 | 5.318139 | 0.547285 |
| 200 | 4.798140 | 10.50380 | 1.215591 | 5.318139 | 2.189139 |
| 500 | 11.99535 | 65.64873 | 1.215591 | 5.318139 | 5.472848 |
| 1000 | 23.99070 | 262.5949 | 1.215591 | 5.318139 | 10.94570 |

3.5. Some generating functions (GF)

The moment generating function (MGF) can be derived using (8) as

\[
M_Z(t) = \sum_{s=0}^{\infty} \eta_s \mathbb{M}_s(\eta, \beta),
\]
where $M_{\star}(t; \alpha, \beta)$ is the MGF of the ExpP model, then

$$M_{\star}(t) = \sum_{s=0}^{\infty} \sum_{r=0}^{\kappa} \sum_{\eta=0}^{r} \frac{t^r}{r!} \eta s^{\star} \beta^r (-1)^{\kappa} \left( \frac{\kappa}{\kappa} \right) B \left( s^{\star}, 1 + \frac{1}{\alpha} (\kappa - r) \right) |_{(\alpha > r)}.$$
The first $r$ derivatives of $M_Z(t)$, with respect to $t$ at $t = 0$, yield the first $r$ moments about the origin, i.e.,
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Figure 6. Three dimensional kurtosis plots for parameter $\theta$.

Figure 7. Three dimensional kurtosis plots for parameter $v$.

$$\mu_{r,z} = \mathbb{E}(z^r) = \frac{d^r}{dt^r} M_z(t) \mid (t=0 \text{ and } r=1,2,3,...),$$

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The generating function GF (CGF) is the logarithm of the MGF. Thus, $r^{th}$ cumulant, say $\kappa_r$, can be obtained from

$$
\kappa_{r|Z} = \frac{d^r}{dt^r} \log \left[ \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\kappa=0}^{\infty} \frac{t^r}{r!} \eta_s \beta^r (-1)^s \left( \frac{r}{\kappa} \right) B_s \left( s^*, 1 + \frac{1}{\alpha} (\kappa - r) \right) \right] \bigg|_{t=0, \text{and } r=1,2,3,...}.
$$

The 1st cumulant is the mean ($\kappa_{1|Z} = \mu_1$), the 2nd cumulant is the variance, and the 3rd cumulant is the same as the 3rd central moment $\kappa_{3|Z} = \mu_3$. But 4th and higher-order cumulants are not equal to central moments, that being said $\kappa_{4|Z} = \mu_4$, $\kappa_{2,2|Z} = \mu_{2,2} - \mu_2^2 = \mu_{2,2}$ and $\kappa_{3,2} = \mu_{3,2} - 3\mu_2 \mu_1 + 2\mu_3 = \mu_{3,2}$. In some cases theoretical treatments of problems in terms of cumulants are simpler than those using moments. In particular, when two or more RVs are statistically independent, the $r^{th}$ order cumulant of their sum is equal to the sum of their $r^{th}$ order cumulants. Moreover, the cumulants can be also obtained from $\kappa_{r|Z} |_{r \geq 1} = \mu_{r|Z} - \sum_{m=0}^{r-1} (r-1-m) \mu_{r-m,2} \kappa_{m,2}$.

### 3.6. Reversed residual life function

The $n^{th}$ moment of the reversed residual life, say $V_{n,Z}(t) = E \left[ (t - z)^n \right]_{t > 0 \text{ and } n = 1,2,...}$. Then, $V_{n,Z}(t)$ can written as

$$
V_{n,Z}(t) = \frac{1}{F_{v,\theta,\alpha}(t)} \int_0^t (t - z)^n dF_{v,\theta,\alpha}(z).
$$

Then, the $n^{th}$ moment of the reversed residual life of $Z$ becomes

$$
V_{n,Z}(t) = \frac{1}{F_{v,\theta,\alpha}(t)} \sum_{\kappa=0}^{\infty} \sum_{s=0}^{\infty} \sum_{h=0}^{\infty} \eta_s \beta^r (-1)^s \left( \frac{r}{\kappa} \right) B_s \left( s^*, 1 + \frac{1}{\alpha} (\kappa - r) \right) \bigg|_{\alpha > r},
$$

where

$$
\eta_s = \eta_0 \sum_{h=0}^{n} (-1)^h \binom{n}{h} t^{-h}.
$$

### 4. Estimation

In this section, non-Bayesian and Bayesian estimation methods are considered. In first subsection, we will consider four non-Bayesian estimation methods such as the MLE, OLSE, WLSE and the KE methods. In the second subsection, the Bayesian estimation method under the squared error loss function (SELF).

#### 4.1. Non-Bayesian estimation methods

##### 4.1.1. The MLE

Let $z_1, z_2, \ldots, z_n$ be a random sample from size $n$ from the OBP distribution with parameters $v, \theta, \alpha$ and $\beta$. Let $\xi^\top$ be the $4 \times 1$ parameter vector. For determining the MLE of $\xi$, we have the log-likelihood function

$$
\ell(\xi) = (v-1) \sum_{i=1}^{n} \log \left[ 1 - \nabla_\theta (z_i)^{-\alpha} \right] + n \log \left( v \theta \alpha \beta \right) - (\alpha v \theta + 1) \sum_{i=1}^{n} \nabla_\beta (z_i)
$$

$$
- (1 + \theta) \sum_{i=1}^{n} \log \left\{ (1 - \nabla_\beta (z_i)^{-\alpha}) v + \nabla_\beta (z_i)^{-\alpha} \right\}.
$$

The components of the score vector, $U \left( \xi \right) = \frac{\partial \ell(\xi)}{\partial \xi} = \left( \frac{\partial \ell(\xi)}{\partial v}, \frac{\partial \ell(\xi)}{\partial \theta}, \frac{\partial \ell(\xi)}{\partial \alpha}, \frac{\partial \ell(\xi)}{\partial \beta} \right)^\top$, are available if needed. Setting $U \left( v \right) = U \left( \theta \right) = U \left( \alpha \right) = U \left( \beta \right) = 0$ and solving them simultaneously yields the MLE $\hat{\xi} = (\hat{v}, \hat{\theta}, \hat{\alpha}, \hat{\beta})^\top$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell$. For interval estimation of the parameters, we obtain the $4 \times 4$ observed information matrix $J(\xi) = \left\{ \partial^2 \ell(\xi) / \partial m \partial w \right\}$ symbol \((m, w = v, \theta, \alpha, \beta)\).
4.1.2. OLS Let $F_{\xi}(z_{i,n})$ denotes the CDF of OBP model and let $z_1 < z_2 < \cdots < z_n$ be the $n$ ordered RS. The OLSEs are obtained upon minimizing

$$\text{OLSE}(\xi) = \sum_{i=1}^{n} \left[ F_{\xi}(z_{i,n}) - \tau_{(i,n)} \right]^2,$$

then, we have

$$\text{OLSE}(\xi) = \sum_{i=1}^{n} \left( 1 - \frac{\nabla_\beta (z_{i,n})^{-\alpha v \theta}}{\left\{ 1 - \nabla_\beta (z_{i,n})^{-\alpha} + \nabla_\beta (z_{i,n})^{-\alpha v} \right\}^{\theta} - \tau_{(i,n)} \right)^2,$$

where $\tau_{(i,n)} = \frac{i}{n+1}$. The LSEs are obtained via solving the following non-linear equations

$$0 = \sum_{i=1}^{n} \left( 1 - \frac{\nabla_\beta (z_{i,n})^{-\alpha v \theta}}{\left\{ 1 - \nabla_\beta (z_{i,n})^{-\alpha} + \nabla_\beta (z_{i,n})^{-\alpha v} \right\}^{\theta} - \tau_{(i,n)} \right) \varsigma_\theta(z_{i,n}, \xi),$$

$$0 = \sum_{i=1}^{n} \left( 1 - \frac{\nabla_\beta (z_{i,n})^{-\alpha v \theta}}{\left\{ 1 - \nabla_\beta (z_{i,n})^{-\alpha} + \nabla_\beta (z_{i,n})^{-\alpha v} \right\}^{\theta} - \tau_{(i,n)} \right) \varsigma_\lambda(z_{i,n}, \xi),$$

$$0 = \sum_{i=1}^{n} \left( 1 - \frac{\nabla_\beta (z_{i,n})^{-\alpha v \theta}}{\left\{ 1 - \nabla_\beta (z_{i,n})^{-\alpha} + \nabla_\beta (z_{i,n})^{-\alpha v} \right\}^{\theta} - \tau_{(i,n)} \right) \varsigma_\alpha(z_{i,n}, \xi),$$

$$0 = \sum_{i=1}^{n} \left( 1 - \frac{\nabla_\beta (z_{i,n})^{-\alpha v \theta}}{\left\{ 1 - \nabla_\beta (z_{i,n})^{-\alpha} + \nabla_\beta (z_{i,n})^{-\alpha v} \right\}^{\theta} - \tau_{(i,n)} \right) \varsigma_\beta(z_{i,n}, \xi),$$

where $\varsigma_\theta(z_{i,n}, \xi), \varsigma_\lambda(z_{i,n}, \xi), \varsigma_\alpha(z_{i,n}, \xi)$ and $\varsigma_\beta(z_{i,n}, \xi)$ defined above.

4.1.3. WLSE The WLSE are obtained by minimizing the function $\text{WLSE}(\xi)$ WRT $v, \theta, \alpha$ and $\beta$

$$\text{WLSE}(\xi) = \sum_{i=1}^{n} \omega_{(i,n)} \left[ F_{\xi}(z_{i,n}) - \tau_{(i,n)} \right]^2,$$

where $\omega_{(i,n)} = \left[ (1+n)(2+n) \right] / [i(1+n-i)]$. The WLSEs are obtained by solving

$$0 = \sum_{i=1}^{n} \omega_{(i,n)} \left( 1 - \frac{\nabla_\beta (z_{i,n})^{-\alpha v \theta}}{\left\{ 1 - \nabla_\beta (z_{i,n})^{-\alpha} + \nabla_\beta (z_{i,n})^{-\alpha v} \right\}^{\theta} - \tau_{(i,n)} \right) \varsigma_\theta(z_{i,n}, \xi),$$

$$0 = \sum_{i=1}^{n} \omega_{(i,n)} \left( 1 - \frac{\nabla_\beta (z_{i,n})^{-\alpha v \theta}}{\left\{ 1 - \nabla_\beta (z_{i,n})^{-\alpha} + \nabla_\beta (z_{i,n})^{-\alpha v} \right\}^{\theta} - \tau_{(i,n)} \right) \varsigma_\lambda(z_{i,n}, \xi),$$
0 = \sum_{i=1}^{n} \omega_{(i,n)} \left( 1 - \frac{\nabla_{\beta} \left( z_{[i:n]} \right)^{-\alpha v \theta}}{\left\{ 1 - \nabla_{\beta} \left( z_{[i:n]} \right)^{-\alpha} \right\}^v + \nabla_{\beta} \left( z_{[i:n]} \right)^{-\alpha v}} \right)^{\tau_{(i,n)}} \varsigma_{\alpha} \left( z_{[i:n]}, \xi \right),

and

0 = \sum_{i=1}^{n} \omega_{(i,n)} \left( 1 - \frac{\nabla_{\beta} \left( z_{[i:n]} \right)^{-\alpha v \theta}}{\left\{ 1 - \nabla_{\beta} \left( z_{[i:n]} \right)^{-\alpha} \right\}^v + \nabla_{\beta} \left( z_{[i:n]} \right)^{-\alpha v}} \right)^{\tau_{(i,n)}} \varsigma_{\beta} \left( z_{[i:n]}, \xi \right)

where \varsigma_{\alpha} \left( z_{[i:n]}, \xi \right), \varsigma_{\theta} \left( z_{[i:n]}, \xi \right), \varsigma_{v} \left( z_{[i:n]}, \xi \right) and \varsigma_{\beta} \left( z_{[i:n]}, \xi \right) defined above.

4.1.4. KE method The Kolmogorov estimates (KEs) \( \tilde{v}, \tilde{\theta}, \tilde{\alpha} \) and \( \tilde{\beta} \) of \( v, \theta, \alpha \) and \( \beta \) are obtained by minimizing the function

\[
KE = KE \left( v, \theta, \alpha, \beta \right) = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F_{\xi} \left( z_{[i:n]} \right) , F_{\xi} \left( z_{[i:n]} \right) - \frac{i - 1}{n} \right\}
\]

4.2. Bayesian estimation

Assume the gamma priors of the parameters \( v, \theta, \alpha \) and \( \beta \) of the following forms

\[
\pi_{1; \left( \zeta_1, \xi_1 \right)} \left( v \right) \sim \text{Gamma}(\zeta_1, \xi_1), \pi_{2; \left( \zeta_2, \xi_2 \right)} \left( \theta \right) \sim \text{Gamma}(\zeta_2, \xi_2)
\]

\[
\pi_{3; \left( \zeta_3, \xi_3 \right)} \left( \alpha \right) \sim \text{Gamma}(\zeta_3, \xi_3) and \pi_{4; \left( \zeta_4, \xi_4 \right)} \left( \beta \right) \sim \text{Uniform}(\zeta_4, \xi_4),
\]

Assume that the parameters are independently distributed. The joint prior distribution can be written as

\[
\pi \left( v, \theta, \alpha, \beta \right) = \frac{\xi_1 \xi_2 \xi_3 \xi_4}{\Gamma(\zeta_1) \Gamma(\zeta_2) \Gamma(\zeta_3) \Gamma(\zeta_4)} v^{\zeta_1 - 1} \theta^{\zeta_2 - 1} \alpha^{\zeta_3 - 1} \beta^{\zeta_4 - 1} \times \exp \left[ - \left( v \xi_1 + \theta \xi_2 + \alpha \xi_3 + \beta \xi_4 \right) \right]
\]

The posterior distribution \( \pi \left( v, \theta, \alpha, \beta | Z \right) \) of the parameters is defined as

\[
\pi \left( v, \theta, \alpha, \beta | Z \right) \propto \text{likelihood}(\xi | Z) \times \pi_{(\zeta, \xi)} \left( v, \theta, \alpha, \beta \right).
\]

Under squared error loss function, the Bayesian estimators of \( v, \theta, \alpha \) and \( \beta \) are the means of their marginal posteriors and defined as

\[
\tilde{v}_{\text{Bayesian}} = \int_{v, \theta, \alpha, \beta} v \pi \left( v, \theta, \alpha, \beta | Z \right) dvd\alpha d\theta dv, \tilde{\theta}_{\text{Bayesian}} = \int_{\theta, \alpha, v, \beta} \theta \pi \left( v, \theta, \alpha, \beta | Z \right) dv d\alpha d\theta
\]

\[
\tilde{\alpha}_{\text{Bayesian}} = \int_{\alpha, v, \theta, \beta} \alpha \pi \left( v, \theta, \alpha, \beta | Z \right) d\theta dv d\alpha and \tilde{\beta}_{\text{Bayesian}} = \int_{\beta, v, \theta, \alpha} \beta \pi \left( v, \theta, \alpha, \beta | Z \right) dv d\alpha dv d\beta.
\]

respectively. It is not possible to obtain the Bayesian estimates through the above formulae. So, the numerical approximation are needed. We propose the use of MCMC techniques namely Gibbs sampler and Metropolis Hastings (M-H) algorithm. Since the conditional posteriors of the parameters \( v, \theta, \alpha \) and \( \beta \) cannot be obtained in any standard forms, therefore, using a hybrid MCMC for drawing samples from the joint posterior of the parameters is suggested. the full conditional posteriors of \( v, \theta, \alpha \) and \( \beta \) can be easily derived. The simulation algorithm is given by

1. Provide the initial values, say \( v, \theta, \alpha \) and \( \beta \) then at \( i^{\text{th}} \) stage,
2. Using M-H algorithm, we generate \( v_{(i)} \sim \pi_{1} \left( v_{(i)} | \theta_{(i)}, \alpha_{(i)}, \beta_{(i)} ; Z \right) ;
3. Using M-H algorithm, we generate \( \theta_{(i)} \sim \pi_{2} \left( \theta_{(i)} | v_{(i)}, \alpha_{(i)}, \beta_{(i)}, Z \right) ;

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4. Using M-H algorithm, we generate $\alpha \sim \pi_2 \left(\alpha | v(i), \theta(i), \beta(i), Z\right)$;
5. Using M-H algorithm, we generate $\beta \sim \pi_3 \left(\beta | v(i), \theta(i), \alpha(i), Z\right)$;
6. Repeat steps 2–5, $M = 100000$ times to get the samples of size $M$ from the corresponding posteriors of interest. Obtain the Bayesian estimates of $v, \theta, \alpha$ and $\beta$ using the following formulae

$$
\hat{v}_{(\text{Bayesian})} = \frac{1}{M - M_0} \sum_{h=M_0+1}^{M} v[h], \quad \hat{\theta}_{(\text{Bayesian})} = \frac{1}{M - M_0} \sum_{h=M_0+1}^{M} \theta[h],
$$

$$
\hat{\alpha}_{(\text{Bayesian})} = \frac{1}{M - M_0} \sum_{h=M_0+1}^{M} \alpha[h] \quad \text{and} \quad \hat{\beta}_{(\text{Bayesian})} = \frac{1}{M - M_0} \sum_{h=M_0+1}^{M} \beta[h],
$$

respectively, where $M_0 (\approx 50000)$ is the burn-in period of the generated MCMC.

5. Simulation studies for comparing estimation methods

A numerical simulation is performed to compare the classical estimation methods. The simulation study is based on $N=1000$ generated data sets from the OBP version where $n = 50, 100, 150$ and $300$ and

|   | $\alpha$ | $\beta$ | $\theta$ | $v$ |
|---|---------|---------|---------|----|
| I | 0.6     | 0.5     | 0.8     | 2.0 |
| II| 1.2     | 0.9     | 0.4     | 0.9 |
| III| 1.5    | 1.5     | 1.5     | 1.5 |

The estimates are compared in terms of their
1-Bias $\text{BIAS}(\xi)$;
2-Root mean-standard error $\text{RMSE}(\xi)$;
3-the mean of the absolute difference between the theoretical and the estimates “$D$-abs” and
4-the maximum absolute difference between the true parameters and estimates “$D$-max”. Where

$$
\text{BIAS}(v) = \frac{1}{B} \sum_{i=1}^{B} (\bar{v}_i - v), \quad \text{BIAS}(\theta) = \frac{1}{B} \sum_{i=1}^{B} (\bar{\theta}_i - \theta),
$$

$$
\text{BIAS}(\alpha) = \frac{1}{B} \sum_{i=1}^{B} (\bar{\alpha}_i - \alpha) \quad \text{and} \quad \text{BIAS}(\beta) = \frac{1}{B} \sum_{i=1}^{B} (\bar{\beta}_i - \beta)
$$

$$
\text{RMSE}(v) = \sqrt{\frac{1}{B} \sum_{i=1}^{B} (\bar{v}_i - v)^2}, \quad \text{RMSE}(\theta) = \sqrt{\frac{1}{B} \sum_{i=1}^{B} (\bar{\theta}_i - \theta)^2},
$$

$$
\text{RMSE}(\alpha) = \sqrt{\frac{1}{B} \sum_{i=1}^{B} (\bar{\alpha}_i - \alpha)^2} \quad \text{and} \quad \text{RMSE}(\beta) = \sqrt{\frac{1}{B} \sum_{i=1}^{B} (\bar{\beta}_i - \beta)^2}
$$

$$
D_{-\text{abs}} = \frac{1}{nB} \sum_{i=1}^{B} \sum_{j=1}^{n} |F_{\bar{v},\bar{\theta},\bar{\alpha},\bar{\beta}}(w_{ij}) - F_{\bar{v},\bar{\theta},\bar{\alpha},\bar{\beta}}(t_{ij})|
$$

and

$$
D_{-\text{max}} = \frac{1}{B} \sum_{i=1}^{B} \max_{j} |F_{\bar{v},\bar{\theta},\bar{\alpha},\bar{\beta}}(w_{ij}) - F_{\bar{v},\bar{\theta},\bar{\alpha},\bar{\beta}}(t_{ij})|.
$$
From Tables 2, 3 and 4 we note that:
1-The BIAS($\xi$) tend to zero when $n$ increases which means that all estimators are non-biased.
2-The RMSE($\xi$) tend to zero when $n$ increases which means incidence of consistency property.

Table 2: Simulation results for blend I.

| n   | MLE  | OLS  | WLS  | KE   | Bayes |
|-----|------|------|------|------|-------|
| BIAS | 0.00080 | 0.01097 | 0.00975 | 0.02018 | 0.05360 |
| RMSE | 0.05360 | 0.05360 | 0.05360 | 0.05360 | 0.05360 |
| D  | 0.05360 | 0.05360 | 0.05360 | 0.05360 | 0.05360 |

Table 3: Simulation results for blend II.

| n   | MLE  | OLS  | WLS  | KE   | Bayes |
|-----|------|------|------|------|-------|
| BIAS | 0.00080 | 0.01097 | 0.00975 | 0.02018 | 0.05360 |
| RMSE | 0.05360 | 0.05360 | 0.05360 | 0.05360 | 0.05360 |
| D  | 0.05360 | 0.05360 | 0.05360 | 0.05360 | 0.05360 |

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Table 4: Simulation results for blend III.

| n  | MLE | OLS | WLS | KE | Bayes |
|----|-----|-----|-----|----|------|
| 50 | 0.00998 | 0.00977 | 0.00878 | 0.05577 | 0.29424 |
| 100 | 0.00977 | 0.00939 | 0.00983 | 0.02399 | 0.5525 |
| 150 | 0.00983 | 0.01089 | 0.01303 | 0.05571 | 0.02399 |
| 300 | 0.00939 | 0.00983 | 0.00983 | 0.02399 | 0.02506 |

Table 5: The values of estimators $A^\ast$ and $W^\ast$ under failure times.

| Method | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\beta}$ | $A^\ast$ | $W^\ast$ |
|--------|----------------|----------------|--------------|---------|---------|
| MLE    | 2.23386        | 32.84160       | 17.54380     | 262.208 | 0.56749 |
| OLS    | 3.30660        | 2.4718         | 0.94249      | 2.45219 | 1.24500 |
| WLS    | 3.77635        | 1.92859        | 0.90976      | 2.63173 | 0.82385 |
| KE     | 2.65987        | 1.56671        | 1.28125      | 3.97197 | 1.08694 |
| Bayes  | 2.03701        | 43.35538       | 5.03022      | 97.70641 | 0.63239 |

6. Applications for comparing classical methods

6.1. Comparing classical methods under failure times

The first real data set (data set I) represents the data on failure times of 84 aircraft windshield given in Murthy et al. [58]. The data are: 0.0400, 1.866, 2.3850, 3.443, 0.3010, 1.876, 2.4810, 3.467, 0.309, 1.8990, 2.610, 3.4780, 0.557, 1.9110, 2.625, 3.5780, 0.943, 1.9120, 2.632, 3.5950, 1.0700, 1.914, 2.6460, 3.699, 1.1240, 1.981, 2.661, 3.7790, 1.248, 2.0100, 2.688, 3.9240, 1.2810, 2.038, 2.820, 3.4035, 1.281, 2.0850, 2.890, 4.121, 1.3030, 2.089, 2.902, 4.167, 1.4320, 2.097, 2.934, 4.2400, 1.480, 2.135, 2.962, 4.2550, 1.505, 2.154, 2.9640, 4.278, 1.506, 2.190, 3.000, 4.3050, 1.568, 2.190, 4.376, 1.615, 2.2230, 3.114, 4.449, 1.6190, 2.224, 3.1170, 4.485, 1.652, 2.2290, 3.166, 4.570, 1.652, 2.2340, 3.344, 4.602, 1.7570, 2.324, 3.3760, 4.663. We consider the Cramér-Von Mises ($W^\ast$) and the Anderson-Darling ($A^\ast$) statistics. Figure 6 gives probability-probability (P-P) plots for comparing all methods under the failure times data set. From Table 5, the MLE method is the best method with $W^\ast=0.05840$ and $A^\ast=0.56749$ then Bayesian method with $W^\ast=0.06309$ and $A^\ast=0.63239$.

Table 5: The values of estimators $A^\ast$ and $W^\ast$ under failure times.

| Method | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\beta}$ | $A^\ast$ | $W^\ast$ |
|--------|----------------|----------------|--------------|---------|---------|
| MLE    | 2.23386        | 32.84160       | 17.54380     | 262.208 | 0.56749 |
| OLS    | 3.30660        | 2.4718         | 0.94249      | 2.45219 | 1.24500 |
| WLS    | 3.77635        | 1.92859        | 0.90976      | 2.63173 | 0.82385 |
| KE     | 2.65987        | 1.56671        | 1.28125      | 3.97197 | 1.08694 |
| Bayes  | 2.03701        | 43.35538       | 5.03022      | 97.70641 | 0.63239 |

6.2. Comparing classical methods under service times

The second real data set (data set II) represents the data on service times of 63 aircraft windshield given in Murthy et al. [58]. The data are: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.7190, 2.717, 0.2800, 1.794, 2.819, 0.3130, 1.915, 2.820, 0.389, 1.9200, 2.878, 0.487, 1.9630, 2.950, 0.622, 1.978, 3.0030, 0.9000, 2.053, 3.1020, 0.952, 2.065, 3.3040, 0.9960, 2.117, 3.483, 1.0030, 2.137, 3.500, 1.0100, 2.141, 3.6220, 1.085, 2.163,
Table 6: The values of estimators $A^*$ and $W^*$ under service times data.

| Method | $\widehat{\nu}$ | $\widehat{\theta}$ | $\widehat{\alpha}$ | $\widehat{\beta}$ | $A^*$ | $W^*$ |
|--------|------------------|---------------------|---------------------|-------------------|-------|-------|
| MLE    | 1.57877          | 17.0739             | 77.31319            | 1150.47           | 0.57283 | 0.09443 |
| OLS    | 1.176663         | 5.15567             | 1.59013             | 9.75342           | 0.71474 | 0.11726 |
| WLS    | 1.89100          | 5.97335             | 0.92384             | 5.14160           | 0.78311 | 0.12841 |
| ME     | 2.16031          | 1.38096             | 1.07536             | 2.52004           | 1.45874 | 0.23917 |
| Bayes  | 1.33126          | 304.370             | 0.93765             | 155.711           | 0.84289 | 0.13915 |

7. Graphical assessment for the MLEs

Graphically and using the biases and mean squared errors (MSEs), we can perform the simulation experiments to assess the finite sample behavior of the MLEs. The assessment was based on $n=1000$ replication for all $n\in\{n=50,100,...,500\}$. The following algorithm is considered:

1. Generate $n=1000$ samples of size $n\in\{n=50,100,...,500\}$ from the OBP distribution using (7);
2. Compute the MLEs for the $n=1000$ samples, say $\left[\hat{v}_n, \hat{\theta}_n, \hat{\alpha}_n, \hat{\beta}_n\right]|_{(n=1,2,...,1000)}$;
3. Compute the SEs of the MLEs for the 1000 samples, say $\left[S_{\hat{v}_n}, S_{\hat{\theta}_n}, S_{\hat{\alpha}_n}, S_{\hat{\beta}_n}\right]|_{(n=1,2,...,1000)}$.

The standard errors (SEs) were computed by inverting the observed information matrix.

4. Compute the biases and mean squared errors given for $\xi = v, \theta, \alpha, \beta$. We repeated these steps for $n\in\{n=50,100,...,500\}$ with $v = \theta = \alpha = \beta = 1$, so computing biases ($\text{Bias}_s(n)$), mean squared errors (MSEs) ($\text{MSE}_s(n)$) for $\xi = v, \theta, \alpha, \beta$ and $n\in\{n=50,100,...,500\}$ where

$$\text{Bias}_s(n)|_{(\xi=v,\theta,\alpha,\beta)} = \frac{1}{1000} \sum_{n=1}^{1000} \left(\hat{\xi}_n - \xi\right),$$

and

$$\text{MSE}_s(n)|_{(\xi=v,\theta,\alpha,\beta)} = \frac{1}{1000} \sum_{n=1}^{1000} \left(\hat{\xi}_n - \xi\right)^2.$$

Figures 8, 9, 10 and 11 gives the biases (left panels) and MSEs (right panels) for the parameters $v, \theta, \alpha$ and $\beta$ respectively.

The left panels from show how the four biases vary with respect to $n$. The right panels show how the four MSEs vary with respect to $n$. The broken line in red in Figure 8 corresponds to the biases being 0. From Figures 8, 9, 10 and 11 (left panels), the biases for each parameter are generally negative and tends to zero as $n \to \infty$. From Figures 8, 9, 10 and 11 (right panels), the MSEs for each parameter decrease to zero as $n \to \infty$. 
Figure 8. P-P plots for comparing classical methods under failure times.
Figure 9. P-P plots for comparing classical methods under service times data

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Figure 10. Biases and mean squared errors for the parameter $v$.

Figure 11. Biases and mean squared errors for the parameter $\theta$. 
Figure 12. Biases and mean squared errors for the parameter $\alpha$.

Figure 13. Biases and mean squared errors for the parameter $\beta$. 
8. Applications

In this section, we provide two real life applications to two real data sets to illustrate the importance and flexibility of the OBP model. We compare the fit of the OBP with some well-known competitive models (see Table 7)

| N. | Model                      | Abbreviation | Author                          |
|----|----------------------------|--------------|--------------------------------|
| 1  | Pareto type II             | P            | Lomax [41]                      |
| 2  | Exponentiated P            | ExpP         | Gupta et al. [30]               |
| 3  | Kumaraswamy P              | KumP         | Lemonte and Cordeiro [40]       |
| 4  | Macdonald P                | McP          | Lemonte and Cordeiro [40]       |
| 5  | Beta P                     | BP           | Lemonte and Cordeiro [40]       |
| 6  | Gamma P                    | GamP         | Cordeiro et al. [23]            |
| 7  | Transmuted Topp-Leone P    | TTLP         | Yousof et al. [68]              |
| 8  | Reduced TTLP               | RTTLP        | Yousof et al. [68]              |
| 9  | Odd log-logistic P         | OLLP         | Altun et al. [12]               |
| 10 | Reduced OLLP               | ROLLP        | Altun et al. [12]               |
| 11 | Reduced Burr-Hatke P       | RBHP         | Yousof et al. [73]              |
| 12 | Reduced OBP                | ROBP         | New                             |
| 13 | Proportional reversed hazard rate P | PRHRP   | New                             |
| 14 | Special generalized mixture P | SGMP      | Chesneaux and Yousof [22]      |

For checking the normality, the Quantile-Quantile (Q-Q) plot is sketched. For exploring the HRF for real data, the total time test (TTT) plot is provided. For exploring the initial shape of real data nonparametrically, kernel density estimation (KDE) is provided. Figures 12 and 13 give the box plot, normal Q-Q plot, TTT plot and nonparametric KDE for data set I and data set II respectively. Based on Figures 12(a) and 13(a), we note that no extreme values were found in the two real life data sets. Based on Figures 12(b) and 13(b), we note that the normality is nearly exists. Based on Figures 12(c) and 13(c), we note that the HRF is “increasing” for the two real life data sets. Figures 12(d) and 13(d) show nonparametric KDE for exploring the data.

We estimate the unknown parameters of each model by maximum likelihood using “L-BFGS-B” method and the goodness-of-fit statistics Akaike information criterion (AIC), Bayesian IC (BIC), Consistent AIC (CAIC), Hannan-Quinn IC (HQIC), Anderson-Darling (A*) and Cramér-von Mises (W*) are used to compare the five models. In general, the smaller the values of these statistics, the better the fit to the data. The required computations are obtained by using the “maxLik” and “goftest” sub-routines in R-software. For failure times data: the analysis results of are listed in Tables 8 and 9. Table 8 gives the MLEs and standard errors (SEs) for failure times data. Table 9 gives the − \ell and goodness-of-fits statistics for failure times data. For service times data: the analysis results of are listed in Tables 10 and 11. Table 10 gives the MLEs and SEs for service times data. Table 11 give the − \ell and goodness-of-fits statistics for the service times data. Based on Tables 9 and 11, we note that the OBP model gives the lowest values for the AIC, CAIC, BIC, HQIC, A* and W* among all fitted models. Hence, it could be chosen as the best model under these criteria.

Figure 14 gives the estimated PDF (EPDF) plot, estimated CDF (ECDF) plot, probability-probability (P-P) plot and estimated HRF (EHRF) plot for data set I. Figure 15 gives the EPDF plot, ECDF plot, P-P plot and EHRF plot for data set II. From Figures 14 and 15, it is noted that the OBP provides a very adequate fits to the empirical functions.
Figure 14. Box plot, normal Q-Q plot, TTT plot and nonparametric KDE for data set I.
Figure 15. Box plot, normal Q-Q plot, TTT plot and nonparametric KDE for data set II.
| Model            | Estimates       |
|------------------|-----------------|
| **OBP** ($v, \theta, \alpha, \beta$) | 2.27526, 57.35316, 6.576097, 119.4169 |
| McP ($v, \theta, c, \alpha, \beta$) | 2.1875, 119.1751, 12.4171, 19.9243, 75.6606 |
| TTLP ($v, \theta, \alpha, \beta$) | $-0.80750$, 2.47663, (15608.2), (38628.3) |
| KumP ($v, \theta, \alpha, \beta$) | 2.6150, 100.2756, 12.4171, 19.9243, 75.6606 |
| BP ($v, \theta, \alpha, \beta$) | 3.60360, 33.63870, 4.830700, 118.8374 |
| PRHRP ($\theta, \alpha, \beta$) | $3.73 \times 10^6$, $4.707 \times 10^{-1}$, $4.49 \times 10^6$ |
| RTTLPP ($\theta, \alpha, \beta$) | $-0.84732$, 5.52057, 1.15678 |
| SGMP ($v, \alpha, \beta$) | $-1.04 \times 10^{-1}$, $9.83 \times 10^6$, $1.18 \times 10^7$ |
| ROBP ($v, \alpha, \beta$) | $3.54793$, 30.63743, 0.24295 |
| OLLP ($v, \alpha, \beta$) | $2.32636$, (7.17 $\times 10^5$), (2.34 $\times 10^6$) |
| GamP ($v, \alpha, \beta$) | $3.58760$, 52001.49, 37029.66 |
| ExpP ($v, \alpha, \beta$) | $3.62610$, 20074.51, 26257.68 |
| ROLLP ($v, \alpha$) | $3.890564$, 0.57316 |
| RBHP ($\alpha, \beta$) | 10801754, 51367189 |
| P ($\alpha, \beta$) | 51425.35, 131789.8 |

Table 8: MLEs and SEs for failure times data.
Table 9: \(-\hat{\ell}\) and goodness-of-fits statistics for failure times data.

| Model  | \(-\hat{\ell}\) | AIC     | CAIC    | BIC     | HQIC    | \(A^*\) | \(W^*\) |
|--------|-----------------|---------|---------|---------|---------|---------|---------|
| OBP    | -129.5287       | 267.057 | 267.562 | 276.779 | 270.964 | 0.5675  | 0.0584  |
| McP    | -129.8023       | 269.6045| 270.3640| 281.8178| 274.5170| 0.6645  | 0.0658  |
| KumP   | -132.4048       | 272.8096| 273.3096| 282.5802| 276.7396| 0.6645  | 0.0658  |
| ROBP   | -134.3584       | 274.7169| 275.0169| 282.0093| 277.6848| 0.9439  | 0.1004  |
| OLLP   | -134.4235       | 274.8470| 275.1470| 282.1394| 277.7785| 0.9407  | 0.1009  |
| TTLP   | -135.5700       | 279.1400| 279.6464| 288.8633| 283.0487| 1.1257  | 0.1270  |
| GamP   | -138.4042       | 282.8083| 283.1046| 290.1363| 285.7559| 0.6645  | 0.0658  |
| BP     | -138.7177       | 285.4354| 285.9354| 295.2060| 289.3654| 1.4084  | 0.1680  |
| ExpP   | -141.3997       | 288.7994| 289.0957| 296.1273| 291.7469| 1.7435  | 0.2194  |
| ROLLP  | -142.8452       | 291.7400| 292.0400| 299.9900| 294.6199| 1.9566  | 0.2554  |
| SGMP   | -143.0874       | 292.1747| 292.4747| 300.4247| 295.1062| 1.3467  | 0.1578  |
| RTTLP  | -153.9809       | 313.9618| 314.2618| 321.2542| 316.8933| 3.7527  | 0.5592  |
| PRHRP  | -162.8770       | 331.7540| 332.0540| 339.0464| 334.6855| 1.3672  | 0.1609  |
| P      | -164.9884       | 333.9767| 334.1230| 338.8620| 335.9417| 1.3672  | 0.1609  |
| RBHP   | -168.6040       | 341.2081| 341.3562| 346.0697| 343.1624| 1.6711  | 0.2069  |

Table 10: MLEs and SEs for service times data.

| Model   | Estimates |
|---------|-----------|
| OBP\((v, \theta, \alpha, \beta)\) | 1.62783 (0.16217) | 318.092 (76.8271) | 2.21029 (4.5955) | 177.4062 (522.549) |
| KumP\((v, \theta, \alpha, \beta)\) | 1.6691 (0.2570) | 60.5673 (86.0131) | 2.56490 (4.7589) | 65.06400 (177.592) |
| BP\((v, \theta, \alpha, \beta)\) | 1.9218 (0.3184) | 31.2594 (316.841) | 4.9684 (50.528) | 169.5719 (339.207) |
| TTLP\((v, \theta, \alpha, \beta)\) | -0.6070 (0.21371) | 1.785780 (0.41522) | 2123.391 (163.915) | 4822.789 (200.009) |
| PRHRP\((\theta, \alpha, \beta)\) | 1.59 \times 10^6 (2.01 \times 10^5) | 3.93 \times 10^{-1} (0.0004 \times 10^{-1}) | 1.30 \times 10^6 (0.95 \times 10^6) |
| RTTLP\((v, \theta, \alpha)\) | -0.67145 (0.18746) | 2.74496 (0.6696) | 1.01238 (0.11405) |
| SGMP\((v, \alpha, \beta)\) | -1.04 \times 10^{-1} (4.1 \times 10^{-10}) | 6.45 \times 10^6 (3.21 \times 10^6) | 6.33 \times 10^6 (3.8573) |
| ROBP\((v, \theta, \alpha)\) | 2.35836 (0.2413) | 23.13999 (41.18194) | 0.20245 (0.1325) |
| OLLP\((v, \alpha, \beta)\) | 1.66419 (1.79 \times 10^{-1}) | 6.340 \times 10^5 (1.68 \times 10^4) | 2.01 \times 10^6 (7.22 \times 10^6) |
| GamP\((v, \alpha, \beta)\) | 1.9073 (0.3213) | 35842.433 (6945.074) | 39197.57 (151.653) |
| ExpP\((v, \alpha, \beta)\) | 1.9145 (0.3482) | 22971.154 (3209.533) | 32881.99 (162.230) |
| ROLLP\((v, \alpha)\) | 2.37233 (0.26825) | 0.69109 (0.04488) |
| RBHP\((\alpha, \beta)\) | 14055522 (422.005) | 53203423 (28.52323) |
| P\((\alpha, \beta)\) | 99269.78 (11863.5) | 207019.37 (301.2366) |

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Figure 16. EPDF, ECDF, P-P plot and EHRF for data set I.
9. Conclusions

A new lifetime model called the odd Burr Pareto type II (OBP) model is introduced and studied. The major motivation for the practicality of the OBP model is based on the wider importance of the standard Pareto type II model. The OBP density can be “right skewed” with heavy tail shape. The OBP failure rate can be “J-shape”, “decreasing” and “upside down (or increasing-constant-decreasing)”. The new OBP density can be expressed as a mixture of the exponentiated Pareto type II model. The skewness of the OBP distribution can range in the interval (−69216.87, 98983.16). The spread for the kurtosis of the OBP model is ranging from 74.17124 to ∞.

The index of dispersion for the OBP model can be in (0, 1) and also > 1 so it may be used as an “under-dispersed” and “over-dispersed” model. Bayesian and non-Bayesian estimation methods are considered. We assessed the performance of all methods via simulation study. Bayesian and many non-Bayesian estimation methods (such as the maximum likelihood estimation method, ordinary least square estimation method, weighted least square estimation method and the Kolmogorov estimation method) are compared in modeling real data via two applications. In modeling real data, the maximum likelihood method is the best estimation method. So, we used it in comparing competitive models. Before using the the maximum likelihood method, we performed simulation experiments to assess the finite sample behavior of it using the biases and mean squared errors. The OBP model could be chosen as the best model among Pareto type II, exponentiated Pareto type II, Kumaraswamy Pareto type II, Macdonald Pareto type II, beta Pareto type II, odd log-logistic Pareto type II, reduced odd log-logistic Pareto type II, reduced Burr-Hatke Pareto type II, reduced OBP and special generalized mixture Pareto type II distribution in modeling the “failure times” and the “service times” data sets.

As a future potential work we may consider the modified Nikulin-Rao-Robson goodness-of-fit test for distributional validation as presented by Abouelmagd et al. [1], Abouelmagd et al. [2], Ibrahim et al. [39], Goual et al. [28] and Goual et al. [29]. We may consider the modified Bagdonavičius-Nikulin Goodness-of-fit test for censored validation as presented by Mansour et al. ([53], [54], [56], [57], [52] and [55]), Yousof et al. [67] and Salah et al. [61]. Finally, we may characterize the OBP model using some characterizations theorems such as conditional expectation, truncated moment, hazard rate function, Mills ratio, certain functions of the random variable, first order statistic and conditional expectation of the record values as presented by Hamedani et al. ([46],[47] and [48]), Khalil et al. [49], Altun et al. ([13], [14], [15] and [17]), Alizadeh et al. ([7], [8] and [9]), Yousof et al. ([69], [70], [71], [72]), Altun et al. ([11] and [16]) and Ibrahim et al. [38].
Figure 17. EPDF, ECDF, P-P plot and EHRF for data set II.

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