The role of complex structures in $w$-symmetry

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Abstract

In a symplectic framework, the infinitesimal action of symplectomorphisms together with suitable reparametrizations of the two dimensional complex base space generate some type of $W$-algebras. It turns out that complex structures parametrized by Beltrami differentials play an important role in this context. The construction parallels very closely two dimensional Lagrangian conformal models where Beltrami differentials are fundamental.

Key-Words: Symplectic geometry, Beltrami parametrisation of complex structures in 2-D, $W$-algebras, Lagrangian quantum field theory.

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1 Introduction

In the last decade, $W$-algebras have provided a unifying landscape for various topics like integrable systems, conformal field theory, as well as uniformization of 2-dimensional gravity. They were originally discovered as a natural extension of the Virasoro algebra by Zamolodchikov [1] and later implicitly by Drinfeld and Sokolov [2]. The latter obtained the classical $W$-algebras by equipping the coefficients of first order matrix differential operators with the second Gel’fand-Dickey Poisson bracket [3].

In the study of two dimensional conformal models in the so-called conformal gauge $W$-gravity can be defined as a generalization of the reparametrization invariance such that in the conformal gauge two copies of the corresponding $w$-algebra are obtained. Moreover, it has be found that the matrix differential operator has to be supplemented by another equation which is usually referred as the Beltrami equation [2].

Several attempts have been made in order to give a geometric picture of this very rich structure provided by $w$-algebras. Various aspects of this issue have been tackled independently by many authors and it is by now universally referred to as $w$-geometry [4, 5, 6]. This geometry is naturally related to $W$-gravity.

On the one hand, the geometric structure turns out to be related to the uniformization of Riemann surfaces; for instance in [7] a uniformization in higher dimensions was shown to be related to the Teichmüller spaces constructed by Hitchin [8]. However, the Beltrami differentials which naturally occur there are related to KdV flows but not to the complex structures underlying the $w$-geometry, in contrast to what it will be shown in the present paper.

On the other hand, Witten [9] and Hull [10] both pointed out that the use of Poisson bracket induces symplectic forms on certain manifolds and, in doing so, they proposed to study the role of symplectic diffeomorphisms in the construction of the $w$-algebras. Symplectic diffeomorphisms (or symplectomorphisms) form a class of diffeomorphisms on the cotangent bundle over the configuration space (phase space) which leave the canonical symplectic form invariant.

This type of invariance is very rich in the sense that it is the infinitesimal action of symplectomorphisms on a numerable set (may be even infinite) of very peculiar smooth changes of coordinates on the base Riemann surface which generates $w$-algebras as it has be shown quite recently [11]. Similarly to the fact that the moduli space of Riemann surfaces plays an important role in 2d-conformal field theory coupled to gravity, it is expected that the same ought to hold for the $w$-symmetry [14].
Our treatment gives in an explicit way the infinitesimal mappings, so we can automatically give a set of Beltrami parametrizations \[15\] of them which can represent this moduli space. These problems, mostly topological, have an important counterpart within Lagrangian Quantum Field Theory, involving locality: indeed the occurrence of a numerable set of \textit{"Beltrami equations"} imposes a non local dependence of some fields on other ones, spoiling the fundamental requirements of local Quantum Field Theory. We do not realize the link between our reparametrizations and the gauge transformations of the flat \(SL(n, \mathbb{C})\) vector bundles canonically associated to the generalized projective structures, as proposed by Zucchini \[16\], and the embedding transformations proposed in References \[17, 18, 19\], but we hope that the absolute general statement of our transformations could shed some new light on the geometrical nature of \(w\)-symmetry.

Other examples of \(w\) algebras, where the relevance of the complex structure occurs, were considered in many References \[20, 21, 22, 23, 24, 26, 27\]. The former can be reconstructed by a partial breaking of the reparametrization invariance, while a set of consistency conditions controls the spoiling of the breaking of the symmetry under reparametrizations \[11\].

Moreover our approach differs slightly from these ones since it is grounded on a set of complex structures parametrized by Beltrami differentials, whose expressions give a geometrical meaning to quantities introduced in these References.

We stress that our philosophy relies on a deep connection between reparametrization invariance and \(w\)-symmetry, so if we want to investigate the physical aspects of this problem, a correct geometrical formulation will prove to be of great help.

In this context we consider a Quantum Extension of the theory, within the Lagrangian framework. However, we shall see that the locality (in the fields) assumption of a common Lagrangian Theory cannot be fulfilled; anyhow the dismissal of this requirement will provide the mechanism for the compensation of the Quantum Anomalies and the improvement at the quantum level of physical quantities (Energy Momentum tensors) with definite holomorphic properties involving well defined complex structures.

Section 2 recalls some of the main results obtained in a previous work \[11\] with emphasis on the notation and the basic symplectic geometry underlying our problem.

In Section 3 we study the geometrical properties of the spaces, whose mappings toward a common background space define the transformations leading to \(w\) symmetry.

In Section 4 we find, within a B.R.S. framework, the \(w\) algebra transformations.
In Section 5 we build a very general Lagrangian Quantum Field theory, and using the deep connections between reparametrization invariance and $w$ symmetry we improve its B.R.S. Quantum extension. An Appendix is devoted to the cohomological problems and and Spectral Sequences calculations.

2 Geometrical Approach

The starting point of a symplectic structure is the definition on a manifold of the canonical 1-form on $T^*(z, y)$ $\theta$ defined in a local chart frame $\mathcal{U}_{(z,y)}$

$$\theta|_{\mathcal{U}_{(z,y)}} = [y_zdz + \overline{y}_zd\overline{z}]$$ (2.1)

The $(2,0)-(0,2)$ form $\Omega \equiv d\theta$

$$\Omega|_{\mathcal{U}_{(z,y)}} \equiv [dy_z \wedge dz + d\overline{y}_z \wedge d\overline{z}]$$ (2.2)

is closed and

$$\int_{\Sigma} \Omega = \int_{\partial\Sigma} \theta$$ (2.3)

Let us consider now a frame $\mathcal{U}_{(Z,Y)} : \theta$ will take the form on $T^*(Z,Y)$

$$\theta|_{\mathcal{U}_{(Z,Y)}} = [Y_ZdZ + \overline{Y}_Zd\overline{Z}]$$ (2.4)

$\Omega$ is globally defined and in $\mathcal{U}_{(Z,Y)}$ is defined as:

$$\Omega|_{\mathcal{U}_{(Z,Y)}} = d\theta = [dY_Z \wedge dZ + d\overline{Y}_Z \wedge d\overline{Z}]$$ (2.5)

If we require the invariance of the theory under diffeomorphisms, we have to impose that the local change of frame will generate a canonical transformation.
The change of charts will be canonical if in \( \mathcal{U}(z,y) \cap \mathcal{U}(Z,Y) \)

\[
\Omega|_{\mathcal{U}(z,y)} = \Omega|_{\mathcal{U}(Z,Y)}
\]  

(2.6)

this would imply:

\[
\theta|_{\mathcal{U}(z,y)} - \theta|_{\mathcal{U}(Z,Y)} = dF
\]

(2.7)

\( F \) is a function on \( \mathcal{U}(z,y) \cap \mathcal{U}(Z,Y) \). In the \((z,Y)\) plane we can define the function \( \Phi(z,Y) \) as:

\[
d\Phi(z,Y) \equiv d\left(F + (Y_Z Z + \overline{\partial}_Z Z)\right) = \left(y_z dz + \overline{\partial}_z Z\right) + \left(dY_Z Z + d\overline{\partial}_Z Z\right)
\]

(2.8)

The function \( \Phi(z,Y) \) is the generating function for the change of charts and it is defined (up to a total differential) on the space \( \mathcal{U}(z,y) \) and has non vanishing Hessian, \( ||\frac{\partial^2 \Phi}{\partial z \partial Y}|| \).

In the region \( \mathcal{U}(z,Y) \) the differential operator takes the form

\[
d = dz \partial_z + d\overline{\partial}_z + dY_Z \frac{\partial}{\partial Y_Z} + d\overline{\partial}_Z \frac{\partial}{\partial \overline{\partial}_Z} \equiv (dz + dY_Z)
\]

(2.9)

and \( d^2 = 0 \) will imply:

\[
\left[ \partial_z, \frac{\partial}{\partial Y_Z} \right] = 0 \quad \left[ \overline{\partial}_z, \frac{\partial}{\partial Y_Z} \right] = 0
\]

(2.10)

(and the c.c. commutators).

If we impose \( d^2 \Phi = 0 \) we get the important properties:

\[
\overline{\partial} y_z = \partial \overline{\partial} z \quad \frac{\partial}{\partial Y_Z} Z = \frac{\partial}{\partial Y_Z} \overline{\partial}_Z
\]

\[
\frac{\partial}{\partial Y_Z} y_z = \partial Z \quad \frac{\partial}{\partial \overline{\partial}_Z} y_z = \partial Z
\]

(2.11)

and their c.c., which yield that the mappings:

\[
y_z(z,Y) = \partial \Phi(z,Y), \quad Z(z,Y) = \frac{\partial}{\partial Y_Z} \Phi(z,Y)
\]

(2.12)
are canonical. In particular for an arbitrary surface \( Y_Z = \text{const} \) we can construct a local change of coordinates:

\[
(z, \varpi) \rightarrow (Z, \overline{Z}) \quad \text{for} \quad \Phi(z, Y)|_{Y_Z=\text{const}}
\]

(2.13)

In the following, for writing convenience, we shall choose this constant equal to zero. Going on we can verify that:

\[
\theta|_{U(z, \varpi)} = d_z \Phi(z, Y)
\]

(2.14)

So \( \Omega|_{U(z, \varpi)} \) takes the elementary form:

\[
\Omega|_{U(z, \varpi)} = d\theta|_{U(z, \varpi)} = dY_Z \wedge dz \frac{\partial}{\partial Y_Z} \frac{\partial}{\partial z} \Phi(z, Y) + dY_Z \wedge dz \frac{\partial}{\partial Y_Z} \frac{\partial}{\partial z} \Phi(z, Y)
\]

\[
+ dY_Z \wedge d\varpi \frac{\partial}{\partial Y_Z} \frac{\partial}{\partial \varpi} \Phi(z, Y) + dY_Z \wedge d\varpi \frac{\partial}{\partial Y_Z} \frac{\partial}{\partial \varpi} \Phi(z, Y)
\]

\[
= dY_Z \wedge d_z Z + dY_Z \wedge d_z \overline{Z} = dY_Z \wedge dz + d\overline{Y}\wedge d\varpi
\]

\[
= dY_Z \wedge d_z \Phi(z, Y)
\]

(2.15)

We shall now introduce some quantities which will be useful for our treatment. Let us call

\[
\lambda(z, Y) = \partial_z \frac{\partial}{\partial Y} \Phi(z, Y) \quad \mu(z, Y) = \partial_z \frac{\partial}{\partial \varpi} \Phi(z, Y)
\]

\[
\overline{\lambda}(z, Y) = \partial_{\overline{z}} \frac{\partial}{\partial Y} \Phi(z, Y) \quad \overline{\mu}(z, Y) = \partial_{\overline{z}} \frac{\partial}{\partial \varpi} \Phi(z, Y)
\]

(2.16)

the above expression will take the form:

\[
\Omega|_{U(z, \varpi)} = \left[ dz \wedge \left( \lambda dY_Z + \overline{\lambda}(z, Y)\overline{\mu}(z, Y)dY_Z \right) + d\varpi \wedge \left( \overline{\lambda}(z, Y)dY_Z + \lambda(z, Y)\mu(z, Y)dY_Z \right) \right]
\]

\[
= \left[ \lambda(z, Y) \left( dz + \mu(z, Y)d\varpi \right) \wedge dY_Z + \overline{\lambda}(z, Y) \left( d\varpi + \overline{\mu}(z, Y)d\overline{z} \right) \wedge d\overline{Y} \right] \quad (2.17)
\]
due to the global definition of $\Omega_{|U(z,Y)}$ we can derive:

$$d_z Z(z, Y) = \lambda(z, Y) \left( dz + \mu(z, Y) d\bar{z} \right)$$  

(2.18)

$$d_Y y_z(z, Y) = \lambda(z, Y) \left( dY_Z + \frac{\lambda(z, Y) \overline{\mu}(z, Y)}{\lambda(z, Y)} d\overline{Y}_Z \right)$$  

(2.19)

(and their c.c.) which reveal a complex structure parametrized by an ordinary Beltrami multiplier $\mu(z, Y)$ in the $(z, \bar{z})$ plane by $\frac{\lambda \overline{\mu}}{\lambda}$ in the $(Y_Z, \overline{Y}_Z)$ one.

So the $Z(z, Y)$ and $y(z, Y)$ coordinate systems can be defined in terms of a given $\mu(z, Y)$, by means of the Equations:

$$\left( \bar{D} - \mu(z, Y) \partial \right) Z(z, Y) = 0$$  

(2.20)

$$\left( \frac{\partial}{\partial Y_Z} - \frac{\lambda(z, Y) \overline{\mu}(z, Y)}{\lambda(z, Y)} \frac{\partial}{\partial \overline{Y}_Z} \right) y_z(z, Y) = 0$$  

(2.21)

From the previous equations the Liouville theorem will follow:

$$\det \left| \frac{\partial Z(z, Y)}{\partial z} \right| = \lambda(z, Y) \lambda(z, Y) (1 - \mu(z, Y) \overline{\mu}(z, Y)) = \det \left| \frac{\partial y_z(z, Y)}{\partial Y} \right|$$  

(2.22)

Using the previous parametrization, as is well known, we can write the derivative operators $\partial_Z, \frac{\partial}{\partial Y_Z}$ as:

$$\partial_Z = \frac{\partial_z - \mu(z, Y) \bar{D}}{\lambda(z, Y) (1 - \mu(z, Y) \overline{\mu}(z, Y))}$$  

(2.23)

$$\frac{\partial}{\partial y_z(z, Y)} = \frac{D_z - \mu(z, Y) D}{1 - \mu(z, Y) \overline{\mu}(z, Y)}$$  

(2.24)

(and their c.c) where we have introduced:

$$D_z(z, Y) = \frac{1}{\lambda(z, Y)} \frac{\partial}{\partial Y_Z}$$  

(2.25)

Finally, if we work in the $U(z, Y)$ space, taking $z$ and $Y_Z$ as passive coordinates, the condition $d\Omega_{|U(z,Y)} = 0$ will give:

$$d\Omega_{|U(z,Y)} = \partial \lambda(z, Y) d\bar{z} \wedge dz \wedge dY_Z + \partial (\lambda(z, Y) \mu(z, Y)) dz \wedge d\bar{z} \wedge dY_Z$$
\[ + \partial \lambda(z, Y) dz \land d\bar{\tau} \land dY_Z + \bar{\partial}(\lambda(z, Y)\overline{\mu}(z, Y)) d\bar{\tau} \land dz \land dY_Z \]
\[ + dY_Z \land \frac{\partial}{\partial Y_Z} \left( \lambda(z, Y) \left( dz + \mu(z, Y) d\bar{\tau} \right) \right) \land dY_Z \]
\[ + dY_Z \land \frac{\partial}{\partial Y_Z} \left( \overline{\lambda}(z, Y) \left( d\bar{\tau} + \overline{\mu}(z, Y) dz \right) \right) \land dY_Z = 0 \quad (2.26) \]

which gives rise to the Beltrami identities:

\[
\overline{\partial} \lambda(z, Y) = \partial(\mu(z, Y)\lambda(z, Y)), \quad \frac{\partial}{\partial Y_Z} \lambda(z, Y) = \frac{\partial}{\partial Y_Z} \left( \overline{\lambda}(z, Y)\overline{\mu}(z, Y) \right) \quad (2.27)
\]

and their c.c. It is important to remark that from Eq (2.27) one has,

\[
\overline{\partial} \lambda(z, Y) = \partial(\mu(z, Y)\lambda(z, Y)), \quad \frac{\partial}{\partial Y_Z} \lambda(z, Y) = \frac{\partial}{\partial Y_Z} \left( \overline{\lambda}(z, Y)\overline{\mu}(z, Y) \right)
\]

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\[
\overline{\partial} \lambda(z, Y) = \partial(\mu(z, Y)\lambda(z, Y)), \quad \frac{\partial}{\partial Y_Z} \lambda(z, Y) = \frac{\partial}{\partial Y_Z} \left( \overline{\lambda}(z, Y)\overline{\mu}(z, Y) \right)
\]

and its c.c. From Eq (2.10) we also get:

\[
\left[ \partial_z, D^z \right] = -\partial_z \log \lambda(z, Y) D^z
\]
\[
\left[ \bar{\partial}_{\overline{\tau}}, D^z \right] = -\overline{\partial}_{\overline{\tau}} \log \lambda(z, Y) D^z = -\left( \partial \mu(z, Y) D^z - \mu(z, Y) \left[ \partial_z, D^z \right] \right)
\]

(and their c.c.), and the commutator,

\[
\left[ D^z, \overline{D}^z \right] = (D^z \overline{\mu}(z, Y)) \frac{\partial}{\partial y_z(z, Y)} - (\overline{D}^z \mu(z, Y)) \frac{\partial}{\partial \overline{\tau}(z, Y)}
\]

from which it follows:

\[
\left[ \frac{\partial}{\partial y_z(z, Y)}, \frac{\partial}{\partial \overline{\tau}(z, Y)} \right] = 0
\]

At this stage, one remark is in order. In Eq (2.13) the terms \( d_y Z \land dY_Z + d_{Y_Z} Z \land dY_Z \) and \( d_z y_z \land dz + d_{\overline{\tau}} Y_Z \land d\overline{\tau} \) will identically vanish in \( \Omega \). Accordingly, we can state the important Theorem [11]:

**Theorem 2.1** On the smooth trivial bundle \( \Sigma \times \mathbb{R}^2 \), the vertical holomorphic change of local coordinates,

\[
Z((z, \bar{\tau}), Y_Z, Y_Z) \rightarrow Z((z, \bar{\tau}), \mathcal{F}(Y_Z), Y_Z),
\]
where $F$ is a holomorphic function in $Y_Z$, while the horizontal holomorphic change of local coordinates,

$$y_z(z, \overline{z}, (Y_Z, \overline{Y_Z})) \rightarrow y_z(f(z), \overline{z}, (Y_Z, \overline{Y_Z})), \quad (2.33)$$

where $f$ is a holomorphic function in $z$, are both canonical transformations.

In the first case an infinitesimal variation of $Z((z, Y))$ in $Y_Z$ does not modify, for fixed $(z, \overline{z})$ the $\Omega$ form.

So the diffeomorphisms $(z) \rightarrow Z((z, \overline{z}), Y_Z); z \rightarrow Z((z, \overline{z}), Y_Z + dY_Z)$, will be related to the same two form $\Omega$.

If we make the expansion around, say, $Y_Z = 0, \overline{Y_Z} = 0$ the generating function $\Phi$ will be written as the series:

$$\Phi(z, Y) = \sum_{n=1}^{\max} \frac{1}{n!} \left[ Y^n_Z \left( \frac{\partial}{\partial Y_Z} \Phi_1((z, \overline{z}), Y_Z) \right) \right]_{Y_Z=0, \overline{Y_Z}=0} + \sum_{n=1}^{\max} \frac{1}{n!} \left[ \overline{Y^n_Z} \left( \frac{\partial}{\partial \overline{Y_Z}} \Phi_1((z, \overline{z}), \overline{Y_Z}) \right) \right]_{Y_Z=0, \overline{Y_Z}=0} \equiv \sum_{n=1}^{\max} \left[ Y^n_Z Z^{(n)}(z, \overline{z}) \right] + \sum_{n} \left[ \overline{Y^n_Z} Z^{(n)}(z, \overline{z}) \right] \quad (2.34)$$

where:

$$Z^{(n)}(z, \overline{z}) \equiv \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_Z} \right)^n \Phi_1(z, Y) \right]_{Y_Z=0, \overline{Y_Z}=0} \quad (2.35)$$

And we can reconstruct $Z(z, Y)$ as:

$$Z(z, Y) = \sum_{n} nZ^{(n)}(z, \overline{z})Y_Z^{n-1} \quad (2.36)$$

We shall see in the following that the family of reparametrizations:

$$(z, \overline{z}) \rightarrow (Z^{(n)}, \overline{Z^{(n)}}) \quad (2.37)$$

will be the origin of the $w$ algebras symmetry transformations. Obviously the choice of the $(Y_Z, \overline{Y_Z})$ origin as starting point does not alter the treatment
The symplectic form will then be written:

\[ \Omega_{U(z,Y)} = d_Y d_z \Phi(z,Y) = \sum_n \left[ d_Y Y^*_n \wedge d_z Z^{(n)}(z,\overline{z}) + d_Y \overline{Y}^*_n \wedge d_z \overline{Z}^{(n)}(z,\overline{z}) \right] \] (2.38)

Note that the conjugate momenta to the complex coordinates \( Z^{(n)} \), \( \forall n \neq 1 \) are related to the \( n \)-th power of the conjugate momenta of the coordinate \( Z^{(1)}(z,\overline{z}) \).

3 The geometry of the \( Z^{(n)} \)-spaces

In the previous chapter we introduced the mappings:

\[ (z,\overline{z}) \rightarrow (Z^{(n)},\overline{Z}^{(n)}) \quad \forall n = 1 \cdots n \] (3.1)

foreseeing that they will be fundamental for our purposes.

Before investigating their role in the construction of \( w \)-algebras, we shall derive down below the most important properties of the \( (Z^{(n)},\overline{Z}^{(n)}) \)-spaces.

3.1 Generalities on \( (Z^{(n)},\overline{Z}^{(n)}) \)-spaces

The \( Z^{(n)}(z,\overline{z}) \) coordinates are defined as:

\[ Z^{(n)}(z,\overline{z}) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y^*_n} \right)^n (\Phi(z,Y)) \right]|_{Y^*_n,\overline{Y}^*_n=0} \]

\[ = \sum_{j=1,\ldots,n} j! \prod_{i=1,\ldots,m_j} \left[ \frac{(\partial^n Z^{(n)}(z,\overline{z}))^{a_i}}{a_i!} \right] \left\{ \sum_{i:a_i=j} \left( \sum_{p_1>p_2>\cdots>p_{m_j}} a_i^{p_1} \right) \right\} M^{(j)}(z,\overline{z}) \] (3.2)

where the functions \( M^{(j)}(z,\overline{z}) \) are given by:

\[ M^{(j)}(z,\overline{z}) = \frac{1}{j!} \left( \frac{D}{D^j} \Phi(z,Y) \right)|_{Y^*_n,\overline{Y}^*_n=0} \] (3.3)

Note that the \( Z^{(n)}(z,\overline{z}) \) coordinate is no more independent from \( Z^{(r)} \), for \( r = 1,\ldots,n-1 \); by the way it obeys differential consistency conditions induced by the \( M^{(j)}(z,\overline{z}) \) functions: we
shall sketch the above system for further convenience:

\[
Z(z, \bar{z}) = \partial Z \mathcal{M}^{(1)}(z, \bar{z})
\]

\[
Z^{(2)}(z, \bar{z}) = \partial Z^{(2)}(z, \bar{z}) \mathcal{M}^{(1)}(z, \bar{z}) + (\partial Z)^2 \mathcal{M}^{(2)}(z, \bar{z})
\]

\[
Z^{(3)}(z, \bar{z}) = \partial Z^{(3)}(z, \bar{z}) \mathcal{M}^{(1)}(z, \bar{z}) + 2\partial Z(z, \bar{z})\partial Z^{(2)}(z, \bar{z}) \mathcal{M}^{(2)}(z, \bar{z}) + \partial Z(z, \bar{z})^3 \mathcal{M}^{(3)}(z, \bar{z})
\]

\[\vdots\]

\[
Z^{(N)}(z, \bar{z}) = \partial Z^{(N)}(z, \bar{z}) \mathcal{M}^{(1)}(z, \bar{z}) + \cdots + \left(\partial Z(z, \bar{z})\right)^N \mathcal{M}^{(N)}(z, \bar{z})
\]  

(3.4)

where the first equation can be solved by

\[
\ln Z(z, \bar{z}) = \int_{\tilde{z}}^{z} \frac{1}{\mathcal{M}^{(1)}(z', \bar{z})} + \ln Z(\tilde{z}, \bar{z})
\]  

(3.5)

where \(\ln Z(\tilde{z}, \bar{z})\) takes into account the boundary conditions. Thus, one has

\[
Z(z, \bar{z}) = Z(\tilde{z}, \bar{z}) \exp \int_{\tilde{z}}^{z} \frac{1}{\mathcal{M}^{(1)}(z', \bar{z})}
\]  

(3.6)

which plugged into the second equation gives \(Z^{(2)}\) as an integral relation between \(Z, \mathcal{M}^{(1)}(z, \bar{z})\) and \(\mathcal{M}^{(2)}(z, \bar{z})\);

\[
Z^{(2)}(z, \bar{z}) = Z(z, \bar{z}) \left[\int_{\tilde{z}}^{z} dz'' \frac{\mathcal{M}^{(2)}(z'', \bar{z}) Z(z'', \bar{z})}{(\mathcal{M}^{(1)}(z'', \bar{z}))^3}\right]
\]  

(3.7)

In full generality, one gets,

\[
Z^{(N)}(z, \bar{z}) = Z(z, \bar{z}) \left[\int_{\tilde{z}}^{z} dz'' \frac{1}{Z(z'', \bar{z})} \mathcal{F}^N \left(Z^{(j)}(z'', \bar{z}) \mathcal{M}^{(k)}(z'', \bar{z})_{k \leq j}\right)\mathcal{M}^{(i)}(z'', \bar{z})\right]
\]  

(3.8)

where \(\mathcal{M}^{(i)}(z, \bar{z})\quad i = 1, 2, \cdots, n\) are fixed and \(Z^{(j)}(z'', \bar{z})\quad j < n\) have been calculated at the preceding orders. So we can state the following

**Theorem 3.1** The set of functions \(\mathcal{M}^{(i)}(z, \bar{z}), i = 1, \cdots, n\) completely identify the set of coordinates \(Z^{(j)}(z, \bar{z}), j = 1, \cdots, n\)

Now we want to solve another problem: given a local change of coordinates \((z, \bar{z}) \rightarrow (Z(z, \bar{z}), \overline{Z}(z, \bar{z}))\) is it possible to consider it as an element of arbitrary n-th order of a w hierarchy
\[(z, \overline{z}) \rightarrow (Z^{(n)}, \overline{Z}^{(n)})\], and to find a construction of the underlying \((Z^{(i)}, \overline{Z}^{(i)}) \ i = 1, \cdots, n-1\) in order to get a \(w\) description of this local change?

The answer is positive, since using a standard construction of the \((Z^{(i)}, \overline{Z}^{(i)})\), \(i = 1, \cdots, n-1\) spaces (which do not interfere with \((Z^{(n)}, \overline{Z}^{(n)})\)), we can use it in the last equation of (3.4) and get in an algebraic way the suitable solution \(M^N(z, \overline{z})\). So we can state the Theorem:

**Theorem 3.2** For an arbitrary local change: \((z, \overline{z}) \rightarrow (Z(z, \overline{z}), \overline{Z}(z, \overline{z}))\) it is possible to generate a space hierarchy \((z, \overline{z}) \rightarrow (Z^{(r)}, \overline{Z}^{(r)}), \ r = 1, \cdots, n-1\).

Finally we explore the \((Z^{(n)}, \overline{Z}^{(n)})\) spaces with respect to the \((z, \overline{z})\) background. As in [28], we shall introduce:

\[
d_z Z^{(n)}(z, \overline{z}) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_Z} \right)^n \partial \Phi(z, Y) dz + \frac{1}{n!} \left( \frac{\partial}{\partial Y_{\overline{z}}} \right)^n \overline{\partial} \Phi(z, Y) d\overline{z} \right] |_{Y_Z=0, Y_{\overline{z}}=0}
\]

\[
= \lambda^Z_z(n, (z, \overline{z})) d\overline{z} + \mu^Z_z(n, (z, \overline{z})) d\overline{z}
\] (3.9)

where:

\[
\lambda^Z_z(n, (z, \overline{z})) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_Z} \right)^n \partial \Phi(z, Y) \right] |_{Y_Z=0, Y_{\overline{z}}=0} \equiv \partial Z^{(n)}(z, \overline{z})
\]

\[
\lambda^Z_z(n, (z, \overline{z})) \mu^Z_z(n, (z, \overline{z})) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_Z} \right)^n \overline{\partial} \Phi(z, Y) \right] |_{Y_Z=0, Y_{\overline{z}}=0} \equiv \overline{\partial} Z^{(n)}(z, \overline{z}).
\] (3.10)

So we shall define, for all \(n\):

\[
\frac{\partial}{\partial Z^{(n)}} = \frac{\partial - \mu(n, (z, \overline{z})) \overline{\partial}}{(1 - \mu(n, (z, \overline{z})) \overline{\partial}(n, (z, \overline{z}))}
\]

(3.11)

In particular we get from Eq (3.2):

\[
\frac{\partial}{\partial Z^{(n)}} \left( \sum_{j=1}^n \frac{m_j}{a_1!} \prod_{i=1}^j \left[ \lambda^Z_z(n, (z, \overline{z})) \right]^{a_i} \right) \left\{ \sum_{a_1 + \cdots + a_{m_j} = n} \right\} \left\{ \sum_{p_1 > p_2 > \cdots > p_{m_j}} \right\} M^{(j)}(z, \overline{z}) = 0.
\]

(3.12)

This identity which now appears trivial, will acquire a particular meaning in the following.

It is so evident that the quantity \(\mu^Z_z(n, (z, \overline{z}))\) will label the complex structure of the space \(Z^{(n)}\) in the \((z, \overline{z})\) background and increasing the order \(n\) this complex structure will explore the complex structure of all the \((z, Y)\) space.
More precisely we get:

\[
\lambda(z, Y) \mu(z, Y) = \sum_n n \lambda_z^{(n)}(z, \bar{z}) \mu(n, (z, \bar{z})) Y Z^{n-1}
\]

\[
\lambda(z, Y) = \sum_n n \lambda_z^{(n)}(z, \bar{z}) Y Z^{n-1}
\]  

(3.13)

The symplectic form will reduce to:

\[
\Omega|_{U(z,Y)} = \sum_{n=1}^{\text{max}} \left[ \lambda_z^{(n)}(z, \bar{z}) dY Z^n + \lambda_z^{(n)}(z, \bar{z}) \mu(n, (z, \bar{z})) dY Z^n \right] \wedge dz
\]

\[
+ \sum_{n=1}^{\text{max}} \left[ \lambda_z^{(n)}(z, \bar{z}) dY Z^n + \lambda_z^{(n)}(z, \bar{z}) \mu(n, (z, \bar{z})) dY Z^n \right] \wedge d\bar{z}
\]  

(3.14)

From the very definition, the Beltrami parameter will take the general form:

\[
\mu_z^{(n)}(z, \bar{z}) \equiv \partial_z \Phi^{(n)}(z, Y) \left|_{Y Z=0} \right. \]

where we have introduced:

\[
\mu_z^{(n)}(z, \bar{z}) = \left[ \frac{1}{(n)!} \left( D^z \right)^n \Phi(z, Y) \right]_{Y Z=0}
\]  

(3.16)

We remark that, due to Eq. (2.19) the presence of \(\lambda\)'s in the \(Y Z\) derivative compromises the locality requirements but the parameters in Eq (3.16) introduce a suitable parametrization for a local Lagrangian Quantum Field Theory use. Furthermore these ones have to be considered as the least common factors for all the Beltrami factors of the spaces \(Z^{(r)}(z, \bar{z}) \quad r = 1, \cdots, n\)

Note that the Beltrami multiplier \(\mu_z^{(n)}(z, \bar{z})\) will depend on the \(\lambda\)'s parameters of the spaces parametrized by the \(Z^{(i)}\) coordinates with \(i \leq n\).

Under a change of background coordinates the Beltrami multiplier transforms as:

\[
\mu_z^{(n)}(n, (z, \bar{z})) = \mu_z^{(n)}(n, (z', \bar{z}))(\partial' z)(\partial' \bar{z})
\]  

(3.17)

so we can derive:

\[
\mu_z^{(n)}(z, \bar{z}) = \mu_z^{(n)}(z', \bar{z}')(\partial' z)\bar{z}'(\partial' \bar{z})
\]  

(3.18)
giving a well-defined geometrical status to $\mu_z^\sharp$ as a $(-n, 1)$-conformal field.

A Beltrami identity is immediately recovered for each (and any) $n$ as a consequence of $d_z^2 = 0$,

$$\left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_Z} \right)^n \partial \overline{\partial} \Phi (z, Y) \right]_{Y_Z = 0, \overline{Y}_Z = 0} = \overline{\partial} \lambda_{Z}^{Z(n)} (z, \overline{z}) = \partial (\lambda_{Z}^{Z(n)} (z, \overline{z}) \mu_{Z}^\sharp (n, (z, \overline{z}))) \quad (3.19)$$

It is not only obvious that $\lambda_{Z}^{Z(n)}$ is a non local object on $\mu_{Z}^\sharp (n, (z, \overline{z}))$, but, due to Eq. (3.15) the parameters $\lambda_{Z}^{Z(n)}$ is not local on the parameters $\mu_{Z}^{(i)} (z, \overline{z})$ with $i \leq n$. Furthermore from Eq (3.16) we can realize that the Beltrami multiplier $\mu_{Z}^\sharp (n, (z, \overline{z}))$ (see Eqs (3.19) (3.16)) is sensible to the complex structures of the inner subspaces; so we can state:

**Statement 3.1**

a) The $\lambda_{Z}^{Z(n)}$ (for fixed $n$) are non local functions of the parameters $\mu_{Z}^{(r)}$, and will contain all of them with order $r \leq n$.

b) The complex structure of the $(Z^{(n)}, \overline{Z}^{(n)})$ spaces can be described by parameters $\mu_{Z}^\sharp (n, (z, \overline{z}))$ which extend to these spaces the Beltrami multipliers. For a given $n$, $\mu_{Z}^\sharp (n, (z, \overline{z}))$ will depend, through $\lambda_{Z}^{Z(r)}$, $r < n$, in a non local way on the $\mu_{Z}^{(j)}$'s with $j \leq n$.

c) As a consequence of the previous statements and of Eq (3.9) the coordinate $Z^{(n)} (z, \overline{z})$ is a non local function of $\mu_{Z}^{(j)} (z, \overline{z})$ with $j \leq n$

These are important geometrical statements of the work and are the basis for the physical discussion of the problem. So the mappings

$$(z, \overline{z}) \rightarrow (Z^{(n)}, \overline{Z}^{(n)}) \quad \forall n \quad (3.20)$$

are non holomorphic and the non holomorphicity depends on $n$.

The only possibility to get local Beltrami multipliers $\mu_{Z}^\sharp (n, (z, \overline{z}))$ is to have $\mu_{Z}^{(r)} (z, \overline{z}) = 0 \quad \forall r > 1$. In this case the complex structure of all the $(Z^{(n)}, \overline{Z}^{(n)})$ space will coincide with the $(Z^{(1)}, \overline{Z}^{(1)})$ one. So necessarily $\forall r \geq 2$:

$$\left[ D_z^r \overline{\partial} \Phi (z, Y) \right]_{Y_Z = 0, \overline{Y}_Z = 0} = \left[ \frac{1}{\lambda(z, \overline{Y})} \frac{\partial}{\partial Y_Z} \right]^r \sum_n Y_Z^n \overline{\partial} Z^{(n)} (z, \overline{z}) \right]_{Y_Z = 0, \overline{Y}_Z = 0} = 0, \quad (3.21)$$
It is easy to see that the previous equation leads to:

\[
\frac{\partial Z^{(r)}}{\partial Z^{(r)}} = \frac{\partial Z(z, \bar{z})}{\partial Z(z, \bar{z})} \equiv \mu(z, \bar{z}), \quad \forall r \geq 2
\]

(3.22)

which, expressed in terms of the \((Z, \bar{Z})\) background, gives:

\[
(\bar{\partial} - \mu(z, \bar{z})\partial)Z^{(r)} = 0, \quad \forall r \geq 2,
\]

(3.23)

showing that \(Z^{(r)}\) is an holomorphic function of \(Z\).

This result is straightforwardly generalized. Imposing \(\forall r \geq l + 1:\)

\[
\left[\mathcal{D}^z\right]^r \Phi(z, Y)|_{Y_z, \bar{Y}_z=0} = \left[\frac{1}{\lambda(z, Y)} \frac{\partial}{\partial Y_z}\right]^r \sum_n \left[Y^n_Z \bar{\partial}Z^{(n)}(z, \bar{z})\right]|_{Y_z, \bar{Y}_z=0} = 0,
\]

(3.24)

one recovers

\[
\frac{\partial Z^{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} = \frac{\partial Z^{(l)}(z, \bar{z})}{\partial Z^{(l)}(z, \bar{z})} \equiv \mu(l, (z, \bar{z})), \quad \forall r \geq l + 1
\]

(3.25)

which leads to:

\[
\bar{\partial}_Z Z^{(r)} = 0, \quad \forall r \geq l + 1.
\]

(3.26)

Thus \(Z^{(r)}\) is holomorphic in \(Z^{(l)}\). Obviously the inverse is always true; so we can state:

**Theorem 3.3** The set of conditions: \(\mu^{(r)} \equiv 0, r = l + 1, \ldots, n\) will imply that \(Z^{(n)}\) is an holomorphic function of \(Z^{(l)}\) and vice-versa.

### 3.2 Complex structures in \((Z^{(r)}, \bar{Z}^{(r)})\) backgrounds

It is already interesting to analyze the complex structure of \((Z^{(n)}, \bar{Z}^{(n)})\)-space with respect to different backgrounds; indeed setting

\[
dZ^{(n)} = \Lambda^{Z^{(n)}}_{\bar{Z}^{(r)}}(Z^{(r)}, \bar{Z}^{(r)}) \left[dZ^{(r)} + \Xi^{Z^{(r)}}_{\bar{Z}^{(r)}}(n, (Z^{(r)}, \bar{Z}^{(r)}))d\bar{Z}^{(r)}\right]
\]

(3.27)

where:

\[
\Lambda^{Z^{(n)}}_{\bar{Z}^{(r)}}(Z^{(r)}, \bar{Z}^{(r)}) \equiv \frac{\partial Z^{(n)}(Z^{(r)}, \bar{Z}^{(r)})}{\partial Z^{(r)}}
\]

\[
\Lambda^{Z^{(n)}}_{\bar{Z}^{(r)}}(Z^{(r)}, \bar{Z}^{(r)}) \Xi^{Z^{(r)}}_{\bar{Z}^{(r)}}(n, (Z^{(r)}, \bar{Z}^{(r)})) \equiv \frac{\partial Z^{(n)}(Z^{(r)}, \bar{Z}^{(r)})}{\partial Z^{(r)}}
\]

(3.28)
so that the quantity \( \Xi^{(r)}(r, Z^{(r)}) \) is the Beltrami multiplier of the coordinates \( Z^{(n)} \) in the \( (Z^{(r)}, \overline{Z}^{(r)}) \) background \( \forall n \). So we can relate these quantities to the corresponding objects relatively to the \((z, \overline{z})\) background, \( \forall n \) and \( r \):

\[
\Lambda_{Z^{(n)}}^{(r)}(Z^{(r)}, \overline{Z}^{(r)})|_{(z, \overline{z})} = \frac{\lambda^{Z^{(r)}}(z, \overline{z})}{\lambda^{Z^{(n)}}(z, \overline{z})} \left( 1 - \mu(r, (z, \overline{z}))\overline{\mu}(n, (z, \overline{z})) \right)
\]

(3.29)

and:

\[
\Xi^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)}))|_{(z, \overline{z})} = \frac{\lambda^{Z^{(r)}}(z, \overline{z})}{\lambda^{Z^{(r)}}(z, \overline{z})} \left( 1 - \mu(r, (z, \overline{z}))\overline{\mu}(n, (z, \overline{z})) \right)
\]

(3.30)

also we derive:

\[
\left[ \Lambda_{Z^{(n)}}^{(r)}(Z^{(r)}, \overline{Z}^{(r)})\overline{\Lambda}_{Z^{(n)}}^{Z^{(r)}}(Z^{(r)}, \overline{Z}^{(r)}) (1 - \Xi^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)}))\Xi^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)}))) \right]|_{(z, \overline{z})} = \frac{\lambda^{Z^{(n)}}(z, \overline{z})}{\lambda^{Z^{(r)}}(z, \overline{z})} \left( 1 - \mu(r, (z, \overline{z}))\overline{\mu}(n, (z, \overline{z})) \right)
\]

(3.31)

4 \quad \text{B.R.S. algebras}

The previous construction introduces to a BRS derivation of \( w \)-symmetry as shown in [11]. Our aim is to construct a BRS differential which considers, in an infinitesimal approach all the mappings \((z, \overline{z}) \rightarrow (Z^{(n)}, \overline{Z}^{(n)})\), for all \( n \), on the same footing.

Consider first the infinitesimal variations \( \Lambda(z, Y) \) of the generating function \( \Phi(z, Y) \) under the diffeomorphism action of the cotangent bundle. Then by taking the expansion (2.34) one can proceed as follows [11]. Let \( S \) be the BRS diffeomorphism operator acting on the \((z, \overline{z})\) basis and defined for each \( n \) by

\[
SZ^{(n)}(z, \overline{z}) \equiv Y^{(n)}(z, \overline{z}) \equiv \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y^{(r)}} \right)^n \Lambda(z, Y) \right] |_{Y^{(r)} = 0} \quad (4.1)
\]

consequently the \( S \) nilpotency will give:
\[ S\mathcal{Y}^{(n)}(z, \bar{z}) = 0 \]  

we shall decompose:

\[ \mathcal{Y}^{(n)}(z, \bar{z}) \equiv \left[ \lambda_{z}^{Z^{(n)}}(z, \bar{z}) \kappa_{n}^{z}(z, \bar{z}) \right] \]  

and we get from the definition of \( \lambda_{z}^{Z^{(n)}}(z, \bar{z}) \):

\[ S\lambda_{z}^{Z^{(n)}}(z, \bar{z}) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y} \right)^{n} \partial \Lambda(z, Y) \right]|_{Y=0, \bar{Y}=0} \]

\[ \equiv \left[ \partial \left( \frac{1}{n!} \left( \frac{\partial}{\partial Y} \right)^{n-1} (\lambda(z, Y)\mathcal{C}(z, Y)) \right) \right]|_{Y=0, \bar{Y}=0} \]

\[ = \partial \left( \lambda_{z}^{Z^{(n)}}(z, \bar{z}) \kappa_{n}^{z}(z, \bar{z}) \right) \]  

and consequently:

\[ SK_{n}^{z}(z, \bar{z}) = \kappa_{n}^{z}(z, \bar{z}) \partial \kappa_{n}^{z}(z, \bar{z}) \]  

Expanding the calculations yields:

\[ K_{n}^{z}(z, \bar{z}) = \sum_{j=1}^{n} j! \prod_{i=1}^{m_{j}} \left[ \frac{\left( \lambda_{z}^{Z^{(p_{i})}}(z, \bar{z}) \right)^{a_{i}}}{a_{i}! \lambda_{z}^{Z^{(n)}}} \right] \left\{ \sum_{a_{i} = j} \sum_{p_{1} > p_{2} > \cdots > p_{m_{j}}} \mathcal{C}^{(j)}(z, \bar{z}) \right\} \]  

where we have introduced:

\[ \mathcal{C}^{(n)}(z, \bar{z}) = \left[ \frac{1}{n!} \left( D^{z} \right)^{n} \Lambda(z, Y) \right]|_{Y=0, \bar{Y}=0}, \quad n = 1, 2 \cdots \]  

which, for all \( n \), provide, in Eq (4.6), a geometric expansion with the same non local coefficients as in Eq (3.15).

It is quite easy, from the very definition, to derive that these ghosts transform as:

\[ SC^{(n)} = \sum_{r=1}^{n} \left( r \mathcal{C}^{(r)} \partial \mathcal{C}^{(n-r+1)} \right) \]  

revealing the holomorphic \( w \) character of these ghosts. We remark that the B.R.S. variations of \( \mathcal{C}^{(n)}(z, \bar{z}) \) depends on the fields \( \mathcal{C}^{(r)}(z, \bar{z}) \) with \( r \leq n \). The upper limit of the indices of these
ghosts coincides with the upper index of the expansion Eq (2.34), and will characterize this symmetry: we do not fix it, and our conclusions hold their validity for any value (finite or infinite) of this index.

The connection between Eqs (4.6) and (3.15) spreads new light on the connection between diffeomorphisms and $w$ algebras by putting into the game the complex structures.

Now the coefficients of these expansions are essentially geometrical factors.

On the other hand the quantities $\mu^\mu_\sigma(n, (z, \bar{z}))$ in Eq (3.16) will have the B.R.S variations:

$$S_{\mu^\mu_\sigma(n, (z, \bar{z}))} = \mathcal{O}_C(n, (z, \bar{z})) - \sum_{r=1}^{n} \left[ r \mu^\mu_\sigma (z, \bar{z}) \partial C(n-r+1)(z, \bar{z}) - r C^{(r)}(z, \bar{z}) \partial \mu^\mu_\sigma(n-r+1)(z, \bar{z}) \right]$$

(4.9)

and:

$$S\left[ \chi^{Z(n)}(z, \bar{z}) \mu^\mu_\sigma(n, (z, \bar{z})) \right] = \partial \Upsilon^{(p)}$$

(4.10)

so that:

$$S \mu^\mu_\sigma(n, (z, \bar{z})) = \mathcal{K}^{\mu_\sigma}(z, \bar{z}) \partial \mu^\mu_\sigma(n, (z, \bar{z})) - \mu^\mu_\sigma(n, (z, \bar{z})) \partial \mathcal{K}^{\mu_\sigma}(z, \bar{z}) + \bar{\partial} \mathcal{K}^{\mu_\sigma}(z, \bar{z})$$

(4.11)

So for each $n$ a diffeomorphic structure with a ghost $\mathcal{K}^{\mu_\sigma}(z, \bar{z})$ (non local in the complex structure parameter) can be put into evidence. From Eqs (3.13), it is easy to find the form of these variations in terms of $w$ components. We shall introduce:

$$\kappa^\mu_\sigma(z, \bar{z}) \equiv \frac{\mathcal{K}^{\mu_\sigma}(z, \bar{z}) - \mu(n, (z, \bar{z})) \mathcal{K}^{\mathcal{O}}(z, \bar{z})}{1 - \mu(n, (z, \bar{z})) \mu(n, (z, \bar{z}))}$$

(4.12)

for which:

$$\Upsilon^{(n)}(Z^{(n)}, \overline{Z}^{(n)}) \partial_{Z^{(n)}} + \overline{\Upsilon}^{(n)}(Z^{(n)}, \overline{Z}^{(n)}) \partial_{\overline{Z}^{(n)}} = \kappa^\mu_\sigma(z, \bar{z}) \partial + \kappa^\mu_\sigma(z, \bar{z}) \overline{\partial}$$

(4.13)

so:

$$\mathcal{K}^{\mu_\sigma}(z, \bar{z}) = \kappa^\mu_\sigma(z, \bar{z}) + \mu(n, (z, \bar{z})) \mathcal{K}^{\mu_\sigma}(z, \bar{z})$$

(4.14)
In this base we have:

\[ S \kappa_n^z(z, \bar{z}) = \kappa_n^z(z, \bar{z}) \partial \kappa_n^z(z, \bar{z}) + \overline{\kappa_n^z}(z, \bar{z}) \partial \kappa_n^z(z, \bar{z}) \] (4.15)

\[ S \mu(n, (z, \bar{z})) = (\kappa_n^z \partial + \overline{\kappa_n^z} \partial) \mu(n, (z, \bar{z})) - \mu(n, (z, \bar{z})) (\partial \kappa_n^z + \mu(n, (z, \bar{z})) \partial \overline{\kappa_n^z}) \] (4.16)

\[ S \lambda_z^{Z(n)}(z, \bar{z}) = (\kappa_n^z \partial + \overline{\kappa_n^z} \partial) \lambda_z^{Z(n)}(z, \bar{z}) + \lambda_z^{Z(n)}(z, \bar{z}) (\partial \kappa_n^z + \mu(n, (z, \bar{z})) \partial \overline{\kappa_n^z}) \] (4.17)

Note that the condition:

\[ S Y^{(n)}(z, \bar{z}) = 0 \] (4.18)

is verified only if the Beltrami condition Eq(3.19) holds.

In this approach we can also derive:

\[ C^{(j)}(z, \bar{z}) = \sum_{r, s = 0}^{j} \left[ r! (\mu_{\overline{\kappa_n^z}}^{(k)})_{(z, \bar{z})} \right]_{k_i} \left\{ \sum_{r + s = 1}^{r, s} (\sum_{l, k_i = j}) \right\} c^{(r,s)}(z, \bar{z}) \] (4.19)

where we have introduced, in the spirit of Eq (2.24) the new ghosts:

\[ c^{(p,q)}(z, \bar{z}) = \left[ \frac{1}{p! q!} \left( \frac{\partial}{\partial y_z(z, Y)} \right)^p \left( \frac{\partial}{\partial \overline{\kappa_n^z}(z, Y)} \right)^q \Lambda(z, Y) \right] \bigg|_{Y, \overline{Y} = 0} \] (4.20)

of conformal weight \((-p, -q)\) and which transform as:

\[ S c^{(p,q)}(z, \bar{z}) = \sum_{r, s = 0}^{r=p, s=q} \left( r c^{(r,s)}(z, \bar{z}) \partial_{\overline{\kappa_n^z}} c^{(p-r+1,q-s)}(z, \bar{z}) \right) \] (4.21)
We also remark that the variation of $c^{(p,q)}$ contains the ghost fields $c^{(r,s)}$ with lower degrees, $r \leq p; s \leq q$.

Combining together Eqs.(4.7) and (4.19) we shall write:

$$K^n_z(z,\overline{z}) = \sum_{r,s=0}^{n} \eta^z_{(r,s)}(n, (z,\overline{z})) c^{(r,s)}(z,\overline{z})$$  \hspace{1cm} (4.22)

where

$$\eta^z_{(r,s)}(n, (z,\overline{z})) = \sum_{j=\max(r,s)}^{n} j! \prod_{i=1}^{m_i} \left( \frac{\lambda^Z(p_i) (z,\overline{z})}{a_i! \lambda^Z(n)} \right)$$  \hspace{1cm} (4.23)

$$\times \left[ r! s! \left( \frac{\mu^Z(l_i)}{k_i!} \right) \right] \left\{ \begin{array}{l} \sum_{i,k_i=s}^{r} \sum_{i,k_i=j}^{s} \sum_{a_i=a_1=j}^{r}\sum_{a_i,p_i=n}^{s} \sum_{p_1 > p_2 > \cdots > p_m} \end{array} \right\}$$

in particular:

$$\eta^z_{(1,0)}(n, (z,\overline{z})) = 1$$

$$\eta^z_{(0,1)}(n, (z,\overline{z})) = \mu^Z(n, (z,\overline{z}))$$  \hspace{1cm} (4.24)

The same can be written in terms of the $\kappa_n(z,\overline{z})$ ghosts, getting:

$$\kappa^n_z(z,\overline{z}) = c^{(1,0)}(z,\overline{z})$$

$$+ \sum_{r,s=0}^{n} \frac{\eta^z_{(r,s)}(n, (z,\overline{z})) - \mu(n, (z,\overline{z})\eta^z_{(r,s)}(z,\overline{z}))}{1 - \mu(n, (z,\overline{z}))\mu(n, (z,\overline{z}))} c^{(r,s)}(z,\overline{z}).$$  \hspace{1cm} (4.25)

4.1 The introduction of matter field sectors in $w_n$-algebras

The introduction of matter in $w$ invariant models, and in particular $w$ gravity, have been treated in the literature in different scenarios according to the different attempts to reach $w$ algebras, e.g [10].
Our point of view, which relates in a geometrical fashion $w$ algebras to two dimensional conformal field theory, heavily supports the methods which introduce matter in conformal models. A proper $(\alpha, \beta)$-differential has to be invariant under holomorphic changes of charts, thus induces a local rescaling by the $\lambda Z(n)^z$'s,

\[
\phi_{(\alpha, \beta)}(Z^{(n)}, \overline{Z}^{(n)}) (dZ^{(n)})^\alpha (d\overline{Z}^{(n)})^\beta = (\lambda Z(n)^z(z, \overline{z}))^\alpha (\overline{\lambda Z(n)^z}(z, \overline{z}))^\beta \phi_{(\alpha, \beta)}(Z^{(n)}, \overline{Z}^{(n)})
\times (dz + \mu(n, (z, \overline{z}))d\overline{z})^\alpha (d\overline{z} + \overline{\mu}(n, (z, \overline{z}))dz)^\beta
\equiv \varphi_{(\alpha, \beta)}(z, \overline{z})(dz + \mu(n, (z, \overline{z}))d\overline{z})^\alpha (d\overline{z} + \overline{\mu}(n, (z, \overline{z}))dz)^\beta
\] (4.26)

The field will be said scalar if $(\alpha, \beta) = (0, 0)$, namely,

\[
\phi(Z^{(n)}, \overline{Z}^{(n)}) \equiv \varphi(z, \overline{z}), \quad \forall n
\] (4.27)

and will transform with only under point displacement

\[
S\phi(Z^{(n)}, \overline{Z}^{(n)}) = \left( \gamma^{(n)} \partial_{Z^{(n)}} + \overline{\gamma} \overline{\partial}_{\overline{Z}^{(n)}} \right) \phi(Z^{(n)}, \overline{Z}^{(n)})
\] (4.28)

Going now to background $(z, \overline{z})$ we have

\[
S\varphi(z, \overline{z}) = \left( \kappa(z, \overline{z}) \partial + \overline{\kappa} \overline{\partial} \right) \varphi(z, \overline{z})
\] (4.29)

Now each $(Z^{(n)}, \overline{Z}^{(n)})$ space has a different complex structure, so using the background representation each field living in this space carries into its transformation the imprinting of this space.

\[
S\phi_{(\alpha, \beta)}(Z^{(n)}, \overline{Z}^{(n)}) = \left( \gamma^{(n)} \partial_{Z^{(n)}} + \overline{\gamma} \overline{\partial}_{\overline{Z}^{(n)}} \right) \phi_{(\alpha, \beta)}(Z^{(n)}, \overline{Z}^{(n)})
= \left( \kappa^{\alpha}_\beta \partial + \overline{\kappa}^\beta_{\alpha} \overline{\partial} \right) \varphi_{(\alpha, \beta)}(z, \overline{z})
\] (4.30)

The same can be done with respect to the background system of coordinates

\[
S\varphi_{(\alpha, \beta)}(z, \overline{z}) = \left( \kappa^{\alpha}_\beta \partial + \overline{\kappa}^\beta_{\alpha} \overline{\partial} \right) \varphi_{(\alpha, \beta)}(z, \overline{z})
+ \alpha(\overline{\partial}\kappa^{z}_n + \mu(n, (z, \overline{z}))\overline{\partial}\kappa_{\overline{z}n}) \varphi_{(\alpha, \beta)}(z, \overline{z})
+ \beta(\overline{\partial}\kappa^{\overline{z}}_n + \overline{\mu}(n, (z, \overline{z}))\overline{\partial}\kappa_{z n}) \varphi_{(\alpha, \beta)}(z, \overline{z})
\] (4.31)
In conclusion the above ghost decompositions Eqs (4.6) (4.2 2) clarify our strategy towards a treatment of $w$ algebras in a Lagrangian Quantum Field Theory framework; since from the former it is quite straightforward to derive in combination with the canonical construction of the diffeomorphism B.R.S. operator, the one induced by the $w$ ordinary algebras. This will be very useful in the next Section.

The diffeomorphism variations of the "matter fields" $\phi(\alpha,\beta)(Z^{(n)}, \overline{Z}^{(n)})$, fix, from our point of view, their $w$ transformations, since it will be provided by the decomposition of ghosts $\kappa^z_n(z, \overline{z})$ in terms of the true $c^{(r,s)}(z, \overline{z})$ symplectomorphism ghost fields.

In particular, for the scalar field, the B.R.S. variation Eq (4.31) is rewritten as:

$$S\phi(z, \overline{z}) = \sum_{r, s = 0}^{n} c^{(r,s)}(z, \overline{z}) \left( \tau^z_{n,(r,s)} \partial + \tau^{\overline{z}}_{n,(r,s)} \overline{\partial} \right) \phi(z, \overline{z}) \quad (4.32)$$

We remark that this description is totally different from the various approach to $w$ matter found in the literature e.g. [10]. Moreover, according to this viewpoint, it gives completely trivial the problem.

## 5 Lagrangian Field Theory building

The approach we have given before to $w$ algebras, and in particular the relevance of complex structures in their construction, suggests to investigate the role played by these symmetries in a Lagrangian Field Theory.

In the previous Sections we have emphasized the linking points between $w$ algebras and two dimensional conformal transformations, so our discussion starts with an example of conformal invariant models.

As it is well known, two dimensional conformal symmetry means reparametrization invariance: in our $w$ scheme we have to improve the symmetry of a wide class of reparametrization mappings, so a lot of care is required in order to respect the Lagrangian Field Theory prescriptions.

We shall deal with a common conformal model built on a two dimensional space manifold $\mathcal{Z}(z, \overline{z}), \overline{\mathcal{Z}}(z, \overline{z})$. 
In order to construct a properly defined local Lagrangian theory we have to take care of:
1) a well definition of the Action integral,
2) the locality on the constituent fields,
3) the symmetry constraints.

We consider the scalar field case:

\[ \Gamma_{\text{scalar}} = \int d\Gamma \partial_{\Gamma} \phi(\Gamma, \overline{\Gamma}) \land d\overline{\Gamma} \partial_{\overline{\Gamma}} \phi(\Gamma, \overline{\Gamma}) \]

\[ \equiv \int d\Gamma^{(n)} \partial_{\Gamma^{(n)}} \phi(\Gamma^{(n)}, \overline{\Gamma}^{(n)}) \land d\overline{\Gamma}^{(n)} \partial_{\overline{\Gamma}^{(n)}} \phi(\Gamma^{(n)}, \overline{\Gamma}^{(n)}) \quad (5.1) \]

So we shall start from a model which is invariant under a reparametrization \((z, \overline{z}) \rightarrow (Z(z, \overline{z}), \overline{Z}(z, \overline{z}))\), which is well defined but has quantum anomalies.

Our strategy for the realization of a \(w\) symmetry in this model will be to consider the \(Z(z, \overline{z}), \overline{Z}(z, \overline{z})\) space as an \(n\)-th element of a \(w\) space hierarchy as in Eq(3.4).

A positive answer for our purposes comes from Theorem (3.2) but more care has to be exercised.

For this reason the model is to be “well defined” with respect all the possible backgrounds; indeed the Lagrangian in Eq (5.1) written in terms of the \((z, \overline{z})\) background takes the form:

\[ \Gamma_{\text{scalar}} \equiv \int dz \land d\overline{z} L_{z, \overline{z}}(z, \overline{z}) \]

\[ = \int dz \land d\overline{z} \left[ \partial - \overline{\mu}(n, (z, \overline{z})\overline{\partial}) \phi(z, \overline{z}) \left[ \overline{\partial} - \mu(n, (z, \overline{z})\partial) \right] \phi(z, \overline{z}) \right] \left( 1 - \mu(n, (z, \overline{z})\overline{\mu}(n, (z, \overline{z})) \right) \right) \quad (5.2) \]

Moreover the model is well defined in each \((Z^{(r)}, \overline{Z}^{(r)})\) frame (\(\forall r\)) since:

\[ \Gamma_{\text{scalar}} = \int d\Gamma^{(n)} \partial_{\Gamma^{(n)}} \phi(\Gamma^{(n)}, \overline{\Gamma}^{(n)}) \land d\overline{\Gamma}^{(n)} \partial_{\overline{\Gamma}^{(n)}} \phi(\Gamma^{(n)}, \overline{\Gamma}^{(n)}) \]

\[ = \int d\Gamma^{(r)} \land d\overline{\Gamma}^{(r)} \left( 1 - \Xi Z^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)})) \Xi_{Z}^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)})) \right)^{-1} \]

\[ \times \left[ \partial_{Z^{(r)}} - \Xi_{Z}^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)})) \Xi_{Z}^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)})) \partial_{Z^{(r)}} \right] \phi(z, \overline{z}) \left[ \overline{\partial}_{\overline{Z}^{(r)}} - \Xi_{\overline{Z}}^{(r)}(n, (Z^{(r)}, \overline{Z}^{(r)})) \partial_{\overline{Z}^{(r)}} \right] \phi(z, \overline{z}) \quad (5.3) \]

This means that in this framework we can assume as symmetry transformations the changes of charts:

\((z, \overline{z}) \rightarrow (Z^{(r)}(z, \overline{z}), \overline{Z}^{(r)}(z, \overline{z})) \quad r = 1 \cdots n \quad \forall n \quad (5.4)\)
just defined in Eq (3.4); and the dynamics of the particle, which is free and scalar in the space \((Z(n), Z^{(n)})\), if described by means of the background of the underlying complex spaces \((Z^{(r)}(z, \zbar), Z^{(r)}(z, \zbar))\), \(r = 1 \cdots n-1\) need the parametrization of the Beltrami multiplier \(\mu_z(n(z, \zbar))\) just found in the Eq (3.15).

So at the light of previous arguments and of Theorem (3.2)

Statement 5.1 A two dimensional conformal model admits, at the classical limit, a \(w\)-symmetry of arbitrary order.

Anyhow the quantum extension requires some care.

Indeed the \(\lambda\)’s, are non local functions of the \(\mu_z^{(r)}(z, \zbar)\), so in a local Lagrangian Quantum Field theory approach, they are not primitive, but, they are essential for the geometrical meaning of our \(w\) construction.

We have just seen in the preceding Lagrangian construction that, if we want to maintain the "well definition" of the Lagrangian with respect all the \((Z^{(r)}, Z^{(r)})\) frames, they are contained in the Beltrami \(\Xi^{(r)}(n, (Z^{(r)}, Z^{(r)}))\) due to Eqs (3.3), (3.15).

If we want to analyze how the underlying complex structure contributes to the dynamics the price to pay is to put into the game all the \(\lambda_z^{(r)}(z, \zbar)\) fields, \(r = 1 \cdots n\) induced by the decomposition of the \((Z^{(r)}, Z^{(r)})\)-spaces Eq.(3.2). These fields (even if local in the \((z, \zbar)\) background) are non local in the \(\mu_z^{(r)}(z, \zbar)\) \(r = 1 \cdots n\) fields due to the Beltrami equations (3.19) in each \((Z^{(r)}, Z^{(r)})\), \(r = 1 \cdots n\) sectors. So, according to this point of view, the model becomes intrinsically non local in the fields (unless \(n = 1\)).

We have now to choose the set of fields which exhausts the dynamical configuration space: so our coordinates will be the fields: \(\varphi, c^{(r,s)}, \mu_z^{(r)}(z, \zbar), \lambda_z^{(r)}, r = 1, \cdots, n\) and their derivatives. This means that all the Classical B.R.S. diffeomorphism transformations of these fields have to be written in terms of the \(c^{(p,q)}\) ghosts using the expansion of the various ghosts \(C^{(j)}, K_n^z, \kappa_n^z\) as written previously.

So we define as “naive” BRS functional operator the following \(\delta_c\):

\[
\delta_c = \sum_{n=1}^{\text{max}} \int dz \wedge d\zbar \left( \lambda_z^{(n)}(z, \zbar) (\kappa_n^z + \mu(n, (z, \zbar)) \kappa_n^z(z, \zbar)) \delta \delta Z^{(n)}(z, \zbar) \right) \right. \\
+ \sum_{r,s=0 \atop r+s \geq 1}^{n} \left( r c^{(r,s)}(z, \zbar) \partial_z c^{(p-r+1,q-s)}(z, \zbar) + s c^{(r,s)}(z, \zbar) \overline{\partial_z c^{(r,s,q-s+1)}(z, \zbar)} \right) \delta \delta c^{(p,q)}(z, \zbar)
\]
\( + \left( \partial C^{(n)}(z, \overline{z}) - \sum_{r=1}^{n} \left[ r \mu^{(r)}(z, \overline{z}) \partial C^{(n-r+1)}(z, \overline{z}) - r C^{(r)}(z, \overline{z}) \partial \mu^{(n-r+1)}(z, \overline{z}) \right] \right) \frac{\delta}{\delta \mu^{(n)}(z, \overline{z})} \\
+ \left( \kappa^n \partial + \kappa^{(n)} \partial \right) \lambda^{(n)}(z, \overline{z}) + \lambda^{(n)}(z, \overline{z}) (\partial \kappa^n + \mu(n, (z, \overline{z})) \partial \lambda^{(n)}) \right) \frac{\delta}{\delta \lambda^{(n)}(z, \overline{z})} \\
+ \left[ \sum_{r, s = 0}^{n} c^{(r,s)}(z, \overline{z}) \left( \tau_{n, (r,s)} \partial + \tau_{n, (r,s)} \overline{\partial} \right) \varphi(z, \overline{z}) \right] \frac{\delta}{\delta \varphi(z, \overline{z})} + c.c. \right] \\
(5.5)\)

where both the ghosts \( \kappa^n \), \( C^{(j)} \) have been expressed in terms of the \( c^{(p,q)} \) ghosts according to Eqs. (4.23) (4.19) respectively.

This is the ordinary diffeomorphism BRS operator, and its nilpotency is verified if the Beltrami conditions (3.19) hold for all the \( \lambda \)'s (28).

So the invariance of the Lagrangian \( \Gamma_{\text{Scalar}} \) in Eq (5.1) is written in a local form

\[ \delta_c \mathcal{L}_{\overline{z}}(z, \overline{z}) = \partial (\kappa^n(z, \overline{z}) \mathcal{L}_{\overline{z}}(z, \overline{z})) + \overline{\partial} (\kappa^n(z, \overline{z}) \mathcal{L}_{\overline{z}}(z, \overline{z})) \]  
(5.6)

We now define a set of local operators of zero F-P charge by:

\[ \delta_c = \int dz \wedge d\overline{z} \sum_{p, q = 0}^{n_{\max}} \left( c^{(p,q)}(z, \overline{z}) \mathcal{T}_{(p,q)}(z, \overline{z}) + \mathcal{S}^{c^{(p,q)}}(z, \overline{z}) \right) \delta \frac{\delta}{\delta c^{(p,q)}(z, \overline{z})} \]  
(5.7)

then thanks to both \( \{ \delta_c, \delta_c \} = 0 \), and Eq. (4.21), it turns out that the \( \mathcal{T}_{(p,q)}(z, \overline{z}) \)'s fulfill commutation rules of \( w \)-algebra type, see e.g. (23) and references therein :

\[ \mathcal{T}_{(p,q)}(z, \overline{z}), \mathcal{T}_{(r,s)}(z', \overline{z}') = \]  
\[ = p \partial_z \delta^{(2)}(z' - z) \mathcal{T}_{(p+r-1,q+s)}(z, \overline{z}) - r \partial_{\overline{z}} \delta^{(2)}(z - z') \mathcal{T}_{(p+r-1,q+s)}(z', \overline{z}) \]  
\[ + q \partial_{\overline{z}} \delta^{(2)}(z' - z) \mathcal{T}_{(p+r,q+s-1)}(z, \overline{z}) - s \partial_z \delta^{(2)}(z - z') \mathcal{T}_{(p+r,q+s-1)}(z', \overline{z}). \]  
(5.8)

Note however that the obtained \( w \)-algebra with respect to the \( c^{(p,q)} \) ghosts is not chiral, but in the vacuum sector where the one relative to the \( C^{(j)} \) ghosts is chiral.

**Statement 5.2** The ordinary diffeomorphism BRS transformations will induce, from Eqs (4.4) (4.23) \( w \)-algebra symmetry transformations. The BRS functional operator to be used for the
Field Theory quantum extension the diffeomorphism symmetry (in terms of the $\mathcal{K}_n, \kappa_n$ ghosts), will give, if written in terms of the $c^{(r,s)}$ ghosts, a BRS differential for $w$-algebras.

The ordinary procedure for Quantum extension suggests to introduce the anti-fields in the Lagrangian term:

$$
L_{antifields} = \int dz \wedge d\overline{z} \left( \sum_{r,s} (\xi_{(r+1,s+1)}(z,\overline{z}) Sc^{(r,s)}(z,\overline{z})) + \nu_{(s+1)}(z,\overline{z}) S\mu_{(s)}^{(s)}(z,\overline{z}) + \chi_{(s)}(z,\overline{z}) S\varphi(z,\overline{z}) + \sum_r (\rho_{(r)}(z,\overline{z}) S\lambda_{(r)}^{(r)}(z,\overline{z})) + \text{c.c.} \right) \quad (5.9)
$$

So the complete Classical Action becomes

$$
\Gamma^{\text{Classical}} = \Gamma^{\text{scalar}} + \Gamma^{\text{antifields}} \quad (5.10)
$$

and at the classical level we get:

$$
\delta \Gamma^{\text{(Classical)}} = \int dz \wedge d\overline{z} \left[ \frac{\delta \Gamma^{\text{(Classical)}}}{\delta \chi_{(s)}^{(s)}(z,\overline{z})} \frac{\delta \Gamma^{\text{(Classical)}}}{\delta \varphi(z,\overline{z})} + \frac{\delta \Gamma^{\text{(Classical)}}}{\delta \xi_{(r+1,s+1)}(z,\overline{z})} \frac{\delta \Gamma^{\text{(Classical)}}}{\delta c^{(r,s)}(z,\overline{z})} 
+ \frac{\delta \Gamma^{\text{(Classical)}}}{\delta \nu_{(s+1)}(z,\overline{z})} \frac{\delta \Gamma^{\text{(Classical)}}}{\delta \mu_{(s)}^{(s)}(z,\overline{z})} + \frac{\delta \Gamma^{\text{(Classical)}}}{\delta \rho_{(r)}(z,\overline{z})} \frac{\delta \Gamma^{\text{(Classical)}}}{\delta \lambda_{(r)}^{(r)}(z,\overline{z})} + \text{c.c.} \right] = 0 \quad (5.11)
$$

By the way if we want to reproduce here one of the outstanding feature of conformal models, that is the holomorphic properties of the object coupled in an invariant way to Beltrami fields (i.e. the Energy Momentum tensor) the task is not so simple.

This fact is, in this context, particularly fruitful: the presence of $n$ independent complex structures (and then $n$ independent Beltrami fields) means that we can derive at least $n$ energy-momentum tensors and their related holomorphic properties.

The problem is that the Beltrami multipliers are non local objects, so the Energy-Momentum tensor cannot be defined in terms of functional derivatives except for the case $n = 1$.

We can provide a solution by the following shortcut: introduce the following lower triangular $n_{\text{max}} \times n_{\text{max}}$ matrix $A$ with entries $(r-1,0)$-differentials valued bilocal kernels - but highly non
local in the $\mu^{(j)}$s,

$$
A(n, r; (z, \overline{z}), (z', \overline{z}'))(r-1) \equiv \frac{\delta \mu^{(r)}(n, (z, \overline{z}))}{\delta \mu^{(r)}(z', \overline{z}')}, \quad n = 1 \cdots n_{\text{max}}, \quad 0 \leq r \leq n,
$$

(5.12)

such that (compare with the expansion (3.15)),

$$
\mu^{(r)}(n, (z, \overline{z})) = \int dz' \wedge d\overline{z}' \sum_{r=1}^{n_{\text{max}}} A(n, r; (z, \overline{z}), (z', \overline{z}'))(r-1) \mu^{(r)}(z', \overline{z}')
$$

(5.13)

We shall suppose that $A$ has an inverse $B$ with entries $(2 - r, 1)$-differentials valued bilocal kernels, such that everywhere,

$$
\int dz'' \wedge d\overline{z}'' \sum_{r=1}^{n_{\text{max}}} B(n, r; (z, \overline{z}),(z'', \overline{z}''))(2-r,1)A(r, n'; (z'', \overline{z}''))(z', \overline{z}')(r-1,0) = \delta_{n,n'}\delta^{(2)}(z - z').
$$

(5.14)

If we define:

$$
P^{\overline{z}}_z(n, (z, \overline{z})) \equiv \int dz' \wedge d\overline{z}' \sum_{r} B(n, r; (z, \overline{z}),(z', \overline{z}'))(2-r,1)\frac{\delta}{\delta \mu^{(r)}(z', \overline{z}')}
$$

(5.15)

one thus has

$$
P^{\overline{z}}_z(n, (z, \overline{z}))\mu(n', (z', \overline{z}')) = \delta_{n,n'}\delta^{(2)}(z - z').
$$

(5.16)

The $P$'s play the role of the "functional derivative operators" with respect the Beltrami parameters; they will be (as well as these last) non local (in $\mu^{(r)}(z, \overline{z})$ ) functional operators and have a fundamental role in our context. If $\mu(n, (z, \overline{z}))$ is coupled at the tree approximation with a local field $\Theta^{(\text{Classical})}(n, (z, \overline{z}))$ in an invariant way, for each $n n = 1 \cdots n_{\text{max}}$ we have

$$
\Theta^{(\text{Classical})}(n, (z, \overline{z})) = P^{\overline{z}}_z(n, (z, \overline{z}))\Gamma^{(\text{Classical})}
$$

(5.17)

so that the latter will transform at the classical level as:

$$
S\Theta^{(\text{Classical})}(n, (z, \overline{z})) = \mathcal{K}^{n}(z, \overline{z})\partial \Theta^{(\text{Classical})}(z, \overline{z}) + 2\Theta^{(\text{Classical})}(z, \overline{z})\partial \mathcal{K}^{n}(z, \overline{z})
$$

(5.18)

By the anticommutator $\overline{\mathcal{S}} = \{S, \frac{\partial}{\partial \kappa_n}\}$ one gets :

$$
\left(\overline{\mathcal{S}} - \mu(n, (z, \overline{z}))\partial - 2\partial \mu(n, (z, \overline{z}))\right)\Theta^{(\text{Classical})}(n, (z, \overline{z})) = 0
$$

(5.19)
Defining the $(2,0)$-differential

\[ J_{Z(n)Z(n)}^{(\text{Classical})}(n, (Z^{(n)}, \overline{Z}^{(n)})) \equiv \left[ \frac{1}{\lambda_z^{Z(n)}Z} \Theta_{(zz)}^{(\text{Classical})}(n, (z, \overline{z})) \right] \tag{5.20} \]

the previous equation leads to:

\[ \frac{\partial}{\partial Z^{(n)}} J_{Z(n)Z(n)}^{(\text{Classical})}(n, (Z^{(n)}, \overline{Z}^{(n)})) = 0 \tag{5.21} \]

In particular we remark that in the $(z, \overline{z})$ background:

\[ dJ_{Z(n)Z(n)}^{(\text{Classical})}(Z^{(n)}(z, \overline{z})) = \left[ \frac{\partial J_{Z(n)Z(n)}^{(\text{Classical})}(Z^{(n)})}{\partial Z^{(n)}} \right] (z, \overline{z}) \lambda_z^{Z^{(n)}}(z, \overline{z}) \left[ dz + \mu(n, (z, \overline{z}))d\overline{z} \right] \tag{5.22} \]

This means that:

**Statement 5.3** *The conserved current \( J_{Z(n)Z(n)}^{(\text{Classical})} \) is a non local function of \( \mu(n, (z, \overline{z})) \). It will imply that this object is a nonlocal function of \( \mu^{(j)}_{z}(z, \overline{z}) \) for \( j \leq n \).*

Moreover we can rewrite the B.R.S transformation of the current as:

\[ \mathcal{S} J_{Z(n)Z(n)}^{(\text{Classical})}(Z^{(n)}) = \Upsilon^n \partial_{Z^{(n)}} J_{Z(n)Z(n)}^{(\text{Classical})}(Z^{(n)}) \tag{5.23} \]

so that we can define a set of invariant charges:

\[ Q_n^{(\text{Classical})} = \int J_{Z(n)Z(n)}^{(\text{Classical})}(Z^{(n)})dZ^{(n)} \tag{5.24} \]

which are functional depending on the local parameters \( \mu^{(j)}_{z}(z, \overline{z}) \) for \( j \leq n \),

\[ \mathcal{S} Q_n^{(\text{Classical})} = 0, \quad \forall n = 1 \cdots n_{\max} \tag{5.25} \]

and *a fortiori*:

\[ \mathcal{T}_{(p,q)} Q_n^{(\text{Classical})} = 0, \quad \forall p, q \leq n = 1 \cdots n_{\max}, \tag{5.26} \]

that is the charges \( Q_n \) are invariant under both diffeomorphism and \( w \) action.
Even if we have stressed the "non local" nature of our theory, we can ask whether some noteworthy property is hidden in the pure local sector of the model. 

The local counterpart of the Energy-momentum tensor (which is invariantly coupled to the Beltrami fields) are the quantities which are coupled in an invariant way to the $\mu^{(j)}$'s. 

We have already pointed out that in this context all the geometrical architecture of our building cannot be appreciated; anyhow a relic of $w$ algebras still appears: we now show, that their OPE's will generate a $w$ expansion, as it has already been shown in the Literature [4]. Indeed introducing, at the Classical level, the Ward identity for the appropriate partition function $Z^{(\text{Classical})}(\mu)$ of the vacuum sector is:

$$\frac{\partial}{\partial \mu^{(s)}(z, \bar{z})}\delta Z^{(\text{Classical})}(\mu) = 0$$

(5.27)

from which we derive by multiplying by $\pi \frac{1}{(z-z')}$ and integration,

$$\pi^2 \frac{\partial Z^{(\text{Classical})}(\mu)}{\partial \mu^{(s)}(z', \bar{z}')} + \sum_j^{n_{\text{max}}-s} \int d\bar{z} \frac{(s+j+1)\partial \mu^{(j+1)} + (j+1)\mu^{(j+1)} \partial}{\mu^{(j+s)}(z, \bar{z})} = 0$$

(5.28)

Setting all the $\mu^{(j)}$'s to zero and by quantum action principle one thus gets

$$\pi^2 \frac{\delta Z^{(\text{Classical})}(\mu)}{\delta \mu^{(s)}(z', \bar{z}')} \bigg|_{\mu=0} + \sum_j^{n_{\text{max}}-s} \delta^{(r+s-1)} \left( \frac{s+r}{(z-z')^2} + \frac{s}{(z-z')^2} \right) \frac{\delta Z^{(\text{Classical})}(\mu)}{\delta \mu^{(r+s-1)}(z, \bar{z})} \bigg|_{\mu=0} = 0$$

(5.29)

which gives the OPE for the tensors coupled to these objects.

This is valid only at the classical level: the local theory display at the quantum level anomalies, while the "non local" approach admits, as we shall see in the next Section, a rather painless cancellation mechanism.

28
5.1 Quantum extension and Anomalies

The difficulties avoided using a local \( w \) algebra using \( C^{(n)}(z, \overline{z}) \) or \( c^{(r,s)}(z, \overline{z}) \) ghosts will create other problems for the Quantum extension of the model.

Due to quantum perturbative corrections the Action functional may violate the Ward identities, and according to the usual Lagrangian framework, one introduces the corresponding linearized BRS operator,

\[
\delta \Gamma = \Delta
\]  

(5.30)

so only by counter-terms inclusion the symmetry will be restored at each order of the perturbative expansion. As is well-known, this requires a cohomological approach and if the cohomology is empty, the symmetry is restored at the quantum level.

This calculation is performed in the Appendix, where we show that the cohomology sector in the space of the local functions is isomorphic to the tensor product of the cohomologies of the ordinary disjoint smooth changes of coordinates \( (z, \overline{z}) \rightarrow (Z^{(r)}(z, \overline{z}), \overline{Z}^{(r)}(z, \overline{z})) \), \( \forall r = 1 \cdots n \).

This result shifts the problem to the quantum extension of a theory whose symmetry is provided by \( n \) disjoint ordinary changes of coordinates, for which many known results are at our disposal [28] [29]

In particular if we add to our field content all the \( \ln \lambda^{Z^{(r)}(z, \overline{z})} \) \( r = 1 \cdots n_{\text{max}} \) the cohomology becomes empty.

The implicit “non locality on the fields” of our model softens the possible disappointment coming from the introduction of logarithms.
In the usual Quantum Field Theory the local anomaly is a cocycle which has a coboundary term which is \( \log \lambda \) dependent and derives from the usual transgression formulas coming from the Gel’fand-Fuchs cocycle, which becomes coboundary if we put \( \ln \lambda \)'s into the game: in our case we get:

\[
\Delta^\natural(z, \overline{z}) = \sum_{r=1}^{n} c_r \mathcal{K}_r(z, \overline{z}) \partial \mathcal{K}_r(z, \overline{z}) \partial^2 \mathcal{K}_r(z, \overline{z})
\]

(modulo coboundary terms and total derivatives; the anomaly in the space of local functionals is recovered by using the techniques of Ref \[28, 30\].)

\[
\Delta(z, \overline{z}) = \sum_{r=1}^{n_{\text{max}}} c_r \mathcal{K}_r^\natural(z, \overline{z}) \partial^3 \mu_z^\natural(r, (z, \overline{z}))
\]

(5.33)

Anyhow the B.R.S diffeomorphism operator is deeply related to the one of \( w \) symmetry when we render explicit the \( \mathcal{K}_r(z, \overline{z}) \) ghosts in terms of \( c^{(r,s)}(z, \overline{z}) \) ones; this allows to calculate the \( w \) local anomalies:

\[
\mathcal{T}_{(r,s)}(z, \overline{z}) \Gamma = \sum_{n=\text{max}(r,s)}^{n_{\text{max}}} c_n \eta_{(r,s)}(n, (z, \overline{z})) \partial^3 \mu(n, (z, \overline{z})).
\]

(5.34)

From the usual construction \[28\]

\[
\mathcal{K}_r(z, \overline{z}) \partial \mathcal{K}_r(z, \overline{z}) \partial^2 \mathcal{K}_r(z, \overline{z}) = \delta(\mathcal{K}_r(z, \overline{z}) \partial \mathcal{K}_r(z, \overline{z}) \partial \ln \lambda_z^{Z(r)}(z, \overline{z}))
\]

(5.35)

so the non local Action compensating terms will be:

\[
\Gamma^{(\text{Polyakov})} = \sum c_r \int dz \wedge d\overline{z}(\mu_z^\natural(r, (z, \overline{z})) \partial^2 \ln \lambda_z^{Z(r)}(z, \overline{z}))
\]

(5.36)

and the corresponding e-m tensor

\[
\Theta_{(z z)}^{(\text{Polyakov})}(n, (z, \overline{z})) = P_z^\natural(n, (z, \overline{z})) \Gamma^{(\text{Polyakov})} = -2c_n S_{zz}(Z^{(n)}(z, \overline{z}))
\]

(5.37)
where, as usual:

\[
S_{zz}(Z^{(n)}(z, \bar{z})) \equiv \partial^2 \ln \lambda_{z}^{Z^{(n)}(z, \bar{z})} - \frac{1}{2} \left( \partial \ln \lambda_{z}^{Z^{(n)}(z, \bar{z})} \right)^2 \tag{5.38}
\]

So we can define:

\[
\Theta_{zz}(n, (z, \bar{z})) = P_{z}^{\bar{z}}(n, (z, \bar{z})) \left[ \Gamma - \Gamma^{(Polyakov)} \right] \tag{5.39}
\]

So the anomaly compensation mechanism allows to construct at the Quantum level an energy momentum tensor which verifies the same symmetry properties as the classical energy momentum tensor.

At this stage it is trivial to define a current

\[
J_{Z^{(n)}Z^{(n)}}(n, (Z^{(n)}, \bar{Z}^{(n)})) \equiv \frac{1}{\lambda_{z}^{Z^{(n)}2}} \left[ P_{z}^{\bar{z}}(n, (z, \bar{z})) \Gamma + 2 c_{n} S_{zz}(Z^{(n)}(z, \bar{z})) \right] \tag{5.40}
\]

which is the Quantum extension of the classical (2, 0)-covariant tensor \( J^{(Classical)}_{Z^{(n)}Z^{(n)}}(n, (Z^{(n)}, \bar{Z}^{(n)})) \).

Note that due to the presence of the Schwarzian derivative the former is no longer a tensor.

Moreover we can also define, for each \( n \leq n_{max} \):

\[
Q_{n} = \int J_{Z^{(n)}Z^{(n)}}(Z^{(n)})dZ^{(n)} \tag{5.41}
\]

which will be invariant even in the Quantum level.

\section{6 Conclusions}

The many aspects of two dimensional reparametrization invariance provide a further geometrical description of \( w \)-algebras. We have addressed the question of introducing local \((-n, 1)\)-conformal fields generalizing the usual Beltrami differential appearing in \( w \)-gravity. It was shown that the way out is based on the infinitesimal action of symplectomorphisms on coordinate transformations dictated by very special canonical transformations.

Also, it is both interesting and intriguing to note how intermingled the symplectic and conformal geometries are relevant for all the present treatment. The combination of Beltrami parametrization of complex structures, canonical transformations and symplectomorphisms yields to a BRS formulation of \( w \)-algebras.
However, although the locality requirements are fundamental for the physical contents within a Lagrangian field Theory, we have overcome them in order to take ever present the geometrical aspect of the problem. But we aim to treat the former in order to understand better the role of the Quantum local anomalies \([10, 23, 24, 26]\) in relation to the point of view expressed in the present paper.

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7 Appendix

The purpose of this Appendix is to show that the cohomology space of our BRS operator \(\delta\) in the space of local functions, is isomorphic to the one of the \(n\) independent reparametrizations \((z, \overline{z}) \rightarrow (Z^{(n)}, \overline{Z}^{(n)})\).

We have shown in [23] that the cohomology space in the functional of the BRS operator \(\delta\) will coincide with the local function cohomology of the nilpotent BRS operator \(\delta - c^{(1,0)} \partial - c^{(0,1)} \overline{\partial}\).

This cohomology space will be computed by using the spectral sequences method. Let us filter with:

\[
\nu = \sum_{p,q,m,n} \left( p + q \right) \partial^m \overline{\partial}^n c^{(p,q)}(z, \overline{z}) \frac{\partial}{\partial \partial^m \overline{\partial}^n c^{(p,q)}(z, \overline{z})}. \tag{7.1}
\]

At the zero eigenvalue the following operator is obtained,

\[
\delta_0 \equiv \int dz \wedge d\overline{z} \left[ \frac{\delta \Gamma^{(\text{Classical})}}{\delta \phi(z, \overline{z})} \frac{\delta}{\delta \chi(z, \overline{z})} + \sum_s \left( \frac{\delta \Gamma^{(\text{Classical})}}{\delta \mu_s(z, \overline{z})} \frac{\delta}{\delta \nu_s(z, \overline{z})} \right) \right]
+ \sum_{r,s} \left( \frac{\delta \Gamma^{(\text{Antifields})}}{\delta c^{(r,s)}(z, \overline{z})} \right)_{c=0} \left[ \frac{\delta}{\delta \zeta_{(r+1,s+1)}(z, \overline{z})} \right]
+ \sum_r \left( \frac{\delta \Gamma^{(\text{Classical})}}{\delta \lambda^{(r)}(z, \overline{z})} \frac{\delta}{\delta \rho_{r,z}^{(r)}(z, \overline{z})} \right) \tag{7.2}
\]

where \(\frac{\delta \Gamma^{(\text{Antifields})}}{\delta c^{(p,q)}(z, \overline{z})} \bigg|_{c=0}\) is the \(c\) independent part of the BRS variation of the anti-fields \(\zeta\) induced by the linearization of \(\delta\).

This operator is clearly nilpotent due to the \(\Phi\)-\(\Pi\) neutrality of the \(\Gamma^{\text{Classical}}\) terms. Its cohomology space can be calculated using again the spectral sequences method. Its adjoint can be defined upon using the Dixon procedure [31] and the Laplacian kernel is isomorphic to the
cohomology space. The upshot of this calculation does not modify the final result; anyhow for the sake of completeness we can calculate this space by first filtrating this operator with the field operator counter, and by calculating the kernel of the Laplacian. It is easy to convince one self that the cohomology space will be independent on the anti-fields $\rho_{z,\bar{z}}(r, (z, \bar{z})), \nu_{\alpha}(s+1)(z, \bar{z})$, $\chi_{z,\bar{z}}(z, \bar{z})$, $\zeta_{r+1,s+1}(z, \bar{z})$ and complicated combinations in the matter fields and $\lambda$’s and $\mu$’s.

The fundamental step takes place in the analysis of the action on $\delta$ of the filtering operator (7.1) at the eigenvalue equal to one. In this case we have to calculate the kernel of this operator (and its adjoint) on the space previously calculated with $\delta_0$. It is easy to derive that this operator is nothing else but the sum of the operators $\delta - c^{(1,0)}_0 \partial - c^{(0,1)}_0 \partial_s$ (where $\delta$ is the ordinary diffeomorphism operator of the $\Phi-\Pi$ neutral fields containing the ghosts $c^{(1,0)}$ and $c^{(0,1)}$) plus the total variation of $c^{(p,q)}$.

This operator is still nilpotent so we can filter it again. We shall choose as filtering operator the one which counts the $c^{(1,0)}$ and $c^{(0,1)}$ ghost fields, namely,

$$\nu' = \sum_{p,q,m,n} \partial^m \partial^n c^{(1,0)}(z, \bar{z}) \frac{\partial}{\partial \partial^n \partial^m c^{(1,0)}(z, \bar{z})} + \partial^m \partial^n c^{(0,1)}(z, \bar{z}) \frac{\partial}{\partial \partial^n \partial^m c^{(0,1)}(z, \bar{z})}$$

(7.3)

At zero eigenvalue we find:

$$\delta'_0 = \sum_{j, l, m, n, r, s, p, q} \frac{\partial^{n+l}}{n! l!} \left[ \partial^l \partial^j c^{(r,s)}(z, \bar{z}) \left( r \partial^{(n-l+1)} \partial^{m-j} c^{(p-r+1,q-s)}(z, \bar{z}) ight) + s \partial^{(n-l)} \partial^{m-j} c^{(p-r,q-s+1)}(z, \bar{z}) \right] \frac{\partial}{\partial \partial^n \partial^m c^{(p,q)}(z, \bar{z})}$$

(7.4)

After defining its adjoint according to the Dixon procedure it is easy to find that the cohomology does not depend on the ghost fields $c^{(p,q)}$ and their derivatives, with the condition $(p + q) > 1$.

At the end we are left with the BRS operator induced by the following transformation rules, for any $n$:

$$S\mu^{(n)}_x(z, \bar{z}) = c^{(1,0)}(z, \bar{z}) \partial \mu^{(n)}_x(z, \bar{z}) + \partial \left( \mu^{(n)}_x(z, \bar{z}) c^{(0,1)}(z, \bar{z}) + \partial c^{(1,0)}(z, \bar{z}) \partial_n \right)$$

$$- n \mu^{(n)}_x(z, \bar{z}) \partial c^{(1,0)}(z, \bar{z}) - \left( \sum_r r \mu^{(r)}_x(z, \bar{z}) \mu^{(n-r+1)}_x(z, \bar{z}) \right) \partial c^{(0,1)}(z, \bar{z})$$

(7.5)
$$S\lambda^Z_{(n)}(z,\bar{z}) = \left( c^{(1,0)}(z,\bar{z})\partial + c^{(0,1)}(z,\bar{z})\bar{\partial} \right) \lambda^Z_{(n)}(z,\bar{z})$$

$$+ \lambda^Z_{(n)}(z,\bar{z}) \left( \partial c^{(1,0)}(z,\bar{z}) + \mu(n,(z,\bar{z}))\partial c^{(0,1)}(z,\bar{z}) \right)$$

(7.6)

where $\mu(n,(z,\bar{z}))$ must be written according to the expansion Eq.(3.15), we thus get the transformations for the Beltrami differentials at any level $n$:

$$S\mu(n,(z,\bar{z})) = \left( c^{(1,0)}(z,\bar{z})\partial + c^{(0,1)}(z,\bar{z})\bar{\partial} \right) \mu(n,(z,\bar{z}))$$

$$+ \bar{\partial} c^{(1,0)}(z,\bar{z}) + \mu(n,(z,\bar{z}))\partial c^{(0,1)}(z,\bar{z})$$

$$- \mu(n,(z,\bar{z})) \left( \partial c^{(1,0)}(z,\bar{z}) + \mu(n,(z,\bar{z}))\partial c^{(0,1)}(z,\bar{z}) \right)$$

(7.7)

while for the scalar matter field,

$$S\varphi(z,\bar{z}) = \left( c^{(1,0)}(z,\bar{z})\partial + c^{(0,1)}(z,\bar{z})\bar{\partial} \right) \varphi(z,\bar{z})$$

(7.8)

and the ghost field $c^{(1,0)}$ (and the c.c. for $c^{(0,1)}$),

$$S c^{(1,0)}(z,\bar{z}) = \left( c^{(1,0)}(z,\bar{z})\partial_z + c^{(0,1)}(z,\bar{z})\bar{\partial}_\bar{z} \right) c^{(1,0)}(z,\bar{z}).$$

(7.9)

All of these are ordinary diffeomorphism transformations.

So the $\mathfrak{w}$ algebra reduces to a tensor product of $n$ independent diffeomorphisms of level equal to one $(z,\bar{z}) \longrightarrow (Z^{(n)},\bar{Z}^{(n)})$, $\forall n = 1, \cdots, n_{\text{max}}$.

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