WELL-POSEDNESS FOR THE 1D CUBIC NONLINEAR SCHRÖDINGER EQUATION IN $L^p, p > 2$

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ABSTRACT. In this paper, local well-posedness is shown for the one dimensional cubic nonlinear Schrödinger equation in $L^p$-spaces for $2 < p < 4$, which generalizes a classical result for $p = 2$ by Y. Tsutsumi and recent work for $1 < p < 2$ by Y. Zhou. As a consequence, a local theory of solutions is developed for a class of data which decay more slowly than square integrable functions. Regularity properties of the local solutions in the $L^p$-based Sobolev spaces and Stricharz spaces are also proved.

1. Introduction

We consider the Cauchy problem for the one dimensional cubic nonlinear Schrödinger equation:

\begin{equation}
(1.1) \quad iu_t + u_{xx} + |u|^2 u = 0, \quad u|_{t=0} = \phi.
\end{equation}

It is well known that (1.1) is locally well posed in $L^2(\mathbb{R})$. Here the “well-posedness” means the existence of a local solution $u : [0, T] \times \mathbb{R} \to \mathbb{C}$, $T \triangleq T(||\phi||_{L^2}) > 0$ for any $\phi \in L^2$, uniqueness (in a suitable solution space), continuous dependence on data, and the persistence property of the solution—i.e., $u \in C([0, T]; L^2)$. This is a classical result by Y. Tsutsumi [17]. Our interest in this paper is to extend this standard well-posedness in $L^2$ into $L^p, p \neq 2$ in a natural manner. As far as the author knows, there are not many studies on (1.1) in this direction. One reason for that is it is widely believed that (1.1) is not locally well posed in $L^p$ if $p \neq 2$. In fact, as early as the 1960s, it was already proved that the initial value problem for the corresponding linear equation

\begin{equation}
(1.2) \quad iu_t + u_{xx} = 0, \quad u|_{t=0} = \phi,
\end{equation}

is not well posed in $L^p(\mathbb{R})$ unless $p = 2$ (see [12]). In particular, if $\phi \in L^p, p \neq 2$ we cannot expect the solution $u(t)$ of (1.2) to belong to $L^p$. Recently, though, there are several works that treat (1.1) in data spaces whose norm are characterized by some kind of $p$-th integrability with $p \neq 2$. For example, Grünrock [10] proved that (1.1) is locally well posed in $\hat{L}^p$ for $1 < p < \infty$, where

\begin{equation}
\hat{L}^p \triangleq \{ \phi \in S'(\mathbb{R}) \mid \hat{\phi} \in L^p \}.
\end{equation}

Notice that by the Hausdorff–Young inequality

\[ L^p \subset \hat{L}^p, \quad \text{if } p \leq 2, \quad \hat{L}^p \subset L^p, \quad \text{if } p \geq 2. \]

Thus (1.1) can be well posed in spaces which are similar to the $L^p$-spaces even if $p \neq 2$. Moreover, one remarkable work was done by Zhou. In [18] he consider the corresponding integral equation

\begin{equation}
(1.3) \quad u(t) = U(t)\phi + i \int_0^t U(t-s)|u(s)|^2 u(s)ds,
\end{equation}

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where $U(t)$ denotes the free Schrödinger group, and introduced the twisted variable $v(t) = U(-t)u(t)$ to rewrite (1.3) as

$$
(1.4) \quad v(t) = \phi + i \int_0^t U(-s) \left[ |U(s)v(s)|^2(U(s)v(s)) \right] \, ds.
$$

Then he discovered that (1.3) is locally well posed in $L^p$, $1 < p < 2$ in the sense that a local solution $v \in C([0, T]; L^p)$ of (1.4) exists for any data $\phi \in L^p$ with uniqueness and continuous dependence on data. This result suggests that (1.1) can be locally well posed in $L^p$, $p \neq 2$ if one consider the “twisted” persistence property $U(-t)u(t) \in C([0, T]; L^p)$ of the solution in place of the usual persistency. Note that by the unitarity property $u(t) \in C(I; L^2)$ if and only if $U(-t)u(t) \in C(I; L^2)$ for any $I \subset R$. Thus this kind of well-posedness in $L^p$ can be regarded as a natural extension of the one in $L^2$ in the usual sense. Now one question arises: does the similar well-posedness hold true for $p > 2$? As far as the author knows there is no previous work that addresses this problem. But $L^p$-spaces for $p > 2$ are most typical spaces whose functions decay at $|x| \to \infty$ more slowly than square integrable functions, and we believe it is very interesting to establish a theory of solutions to (1.1) for such a class of data. As a study in a similar direction, we refer to the most recent work [5, 15, 16] where the authors consider Bessel potential spaces $H^{s,p}$ for $p > 2$ with $s > 0$. Here in this paper we consider mere $L^p$-spaces – without any smoothness or other additional assumptions. Throughout the paper we use the following notations. For $r \in [1, \infty]$, $r'$ denotes the conjugate of $r$: $1/r + 1/r' = 1$. $F$ and $F^{-1}$ denote the Fourier transform and the inverse Fourier transform respectively. We also use $\hat{\phi}$ to denote the Fourier transform of $\phi$. $c, C$ are positive constants that may vary line to line. In particular, we use $C_{A,B,\ldots}$ when we want to emphasize that the constant depends on the parameters $A, B, \ldots$. Let $I \subset R$ be an interval and let $X$ be a Banach space of complex valued functions on $R$. The Bessel potential space is denoted by $H^{s,p}$. We define the space $C_\Theta(I; X)$ by

$$
C_\Theta(I; X) \triangleq \{ u : I \times R \to C \mid U(-t)u(t) \in C(I; X) \}.
$$

Now we give the definition of local well-posedness introduced by Zhou[18].

**Definition 1.1.** Let $X$ be a Banach space of $C$-valued functions on $R$. We say that (1.1) is locally well posed in $X$ if, for any $\phi \in X$ there are $T > 0$ and a unique solution $u : [0, T] \times R \to C$ of (1.1) such that

$$
u \in C_\Theta([0, T]; X) \cap Z_T \triangleq S_T,
$$

where $Z_T$ is some auxiliary space. Moreover, the map $\phi \mapsto u$ is locally Lipschitz from $L^p$ to $S_T$.

The main result of the present paper is a local well-posedness in $L^p$ with $p > 2$ in the above sense.

**Theorem 1.2.** For $2 \leq p < 4$ Cauchy problem (1.1) is locally well posed in $L^p(R)$ in the sense of Definition 1.1.

**Remark 1.3.** It is obvious that if $X$ has the unitarity property i.e. $\|U(t)\phi\|_X = \|\phi\|_X \forall t \in I$, then $C_\Theta(I; X) = C(I; X)$. In such cases, local well-posedness in the sense of Definition 1.1 is equivalent to the usual one equipped with the persistence property of solution. In particular, the classical result [17] says (1.1) is locally well posed in $L^2$ in the sense of Definition 1.1.

We prove Theorem 1.2 by constructing a local solution in a space $Y^p_{q,\theta}(T)$ of functions defined on $[0, T] \times R$ such that

$$
Y^p_{q,\theta}(T) \hookrightarrow C_\Theta([0, T]; L^p).
$$
The definition of $Y^p_{q,\theta}$ is given at the end of this section. Below we state our local well-posedness result in a more precise manner. For $M > 0$ and $1 \leq p \leq \infty$ we set

\begin{equation}
\mathcal{B}^p_M \triangleq \{ \phi \in L^p(\mathbb{R}) \mid \| \phi \|_{L^p} \leq M \}.
\end{equation}

**Proposition 1.4.** Let $2 \leq p < 4$. For any $M > 0$ there is a $T_M > 0$ with $\lim_{M \to \infty} T_M = \infty$ such that: for any $\phi \in \mathcal{B}^p_M$ there is a unique solution $u \in Y^p_{q,\theta}(T_M)$ of (1.1). Moreover, the map $\phi \mapsto u$ is Lipschitz from $\mathcal{B}^p_M$ to $Y^p_{q,\theta}(T_M)$.

We also present miscellaneous results on the solution given by Proposition 1.4. It would be of interest to pursue the regularity of the solution at the $L^p$-level. Unfortunately, we cannot expect the solution to be in $L^p$ in general. This is not unexpected from the well known ill-posed result for the corresponding linear equation. For the regularity in the $L^p$-framework we have the following results: let $2 \leq p < 4$ and $M > 0$. For $\phi \in \mathcal{B}^p_M$ we let $\mathcal{S}_M \phi \in Y^p_{q,\theta}(T_M)$ be the (unique) solution to (1.1) given by Proposition 1.4.

**Corollary 1.5.** Let $2 \leq p < 4$ and $s > -(1 - 2/p)$. Then (1.1) is locally well posed from $L^p$ to $H^{s,p}$ in the following sense: for any $M > 0$ one has

1. $\mathcal{S}_M \phi \in C([0, T_M]; H^{s,p}(\mathbb{R}))$ for every $\phi \in \mathcal{B}^p_M$.
2. There exists $C_M > 0$ such that

$$\sup_{t \in [0, T_M]} \| \mathcal{S}_M \phi_1 - \mathcal{S}_M \phi_2 \|_{H^{s,p}(\mathbb{R})} \leq C_M \| \phi_1 - \phi_2 \|_{L^p}$$

for any $\phi_1, \phi_2 \in \mathcal{B}^p_M$.

On the other hand, if $s > -(1 - 2/p)$ the Cauchy problem (1.1) is not locally well posed from $L^p$ to $H^{s,p}$ in the above sense. So in general, from a standpoint of the well-posedness with the $L^p$-based regularity, there is a loss of smoothness of order $1 - 2/p$. More precisely, we have the following result:

**Corollary 1.6.** Let $2 < p \leq 3$ and $s > -(1 - 2/p)$. Then Cauchy problem (1.1) is not locally well posed from $L^p$ to $H^{s,p}$ in the sense of Corollary 1.5.

**Remark 1.7.** The case $3 < p < 4$ is excluded from the statement of Corollary 1.6. This is due to the availability of key Strichartz type estimates which lead to the ill-posedness results.

Our next interest is the regularity in the so-called Strichartz space. It is well known that the solution $u : I \times \mathbb{R} \to \mathbb{C}$ of (1.1) for $\phi \in L^2$ belongs to the space $L^\rho(I; L^r(\mathbb{R}))$ for $\rho, r \in [2, \infty]$ satisfying

$$\frac{2}{\rho} + \frac{1}{r} = \frac{1}{2}.$$

Recall that such a pair of exponents $(\rho, r)$ is called admissible. Our next regularity result shows that this kind of property for the $L^2$-solution can be extended to the $L^p$-setting for $p > 2$ in a very natural manner. For $1 \leq \rho, r < \infty$, $\alpha \in \mathbb{R}$ and $I \subset [0, \infty)$ we define the space of functions $L^\rho(I, t^\alpha dt; L^r(\mathbb{R}))$ by

$$L^\rho(I, t^\alpha dt; L^r(\mathbb{R})) \triangleq \{ u : I \times \mathbb{R} \to \mathbb{C} \mid \| u \|_{L^\rho(I, t^\alpha dt; L^r(\mathbb{R}))} < \infty \},$$

where

$$\| u \|_{L^\rho(I, t^\alpha dt; L^r(\mathbb{R}))} \triangleq \left( \int_I \left( \int_{\mathbb{R}} |u(t, x)|^r dx \right)^{\frac{\rho}{r}} t^\alpha dt \right)^{\frac{1}{\rho}}.$$

We have the following regularity property for the solution $u$ to (1.1) given by Theorem 1.2.
Corollary 1.8. Let $M > 0$ and let $2 \leq p < 4$ and $2 < \rho, r < \infty$. Assume that
\[
\frac{2}{\rho} + \frac{1}{r} + \frac{1}{p} = 1.
\]
and $q, r$ satisfy either of (i), (ii) below:

(i) $0 \leq \frac{1}{\rho} < \min \left( \frac{1}{4}, \frac{1}{2} - \frac{1}{r} \right)$,

(ii) $4 < r \leq \infty$ and $\rho = \frac{1}{4}$.

Then
\[
(1.6) \quad \mathcal{S}_M \phi \in L^p([0, T_M], t^{\frac{1}{p} - \frac{1}{2}} dt; L^r(\mathbb{R}))
\]
for any $\phi \in \mathcal{B}_M$.

Now we give the definition of the space $Y_{q, \theta}^p$ along with related spaces. These spaces were first introduced by Zhou in [18] to show the well-posedness of (1.1) in $L^p$, $1 < p < 2$.

Definition 1.9. Let $T > 0$ and let $1 \leq p, q < \infty$ and $\theta \in \mathbb{R}$.

(i) The space $\tilde{X}_{q, \theta}^p(T)$ is defined by
\[
\tilde{X}_{q, \theta}^p(T) \triangleq \{ v : [0, T] \times \mathbb{R} \rightarrow \mathbb{C} \mid \| v \|_{\tilde{X}_{q, \theta}^p(T)} < \infty \},
\]
where
\[
\| v \|_{\tilde{X}_{q, \theta}^p(T)} \triangleq \left( \int_0^T \left( s^\theta \| (\partial_s v)(s, \cdot) \|_{L^p} \right)^q ds \right)^{\frac{1}{q}},
\]
and $X_{q, \theta}^p(T)$ by
\[
X_{q, \theta}^p(T) \triangleq \{ v \in \tilde{X}_{q, \theta}^p(T) \mid v(0) \in L^p \}
\]
equipped with the norm
\[
\| v \|_{X_{q, \theta}^p(T)} \triangleq \| v(0) \|_{L^p} + \| v \|_{\tilde{X}_{q, \theta}^p(T)}.
\]

(ii) The space $\tilde{Y}_{q, \theta}^p(T)$ is defined by
\[
\tilde{Y}_{q, \theta}^p(T) \triangleq \{ u : [0, T] \times \mathbb{R} \rightarrow \mathbb{C} \mid U(-t)u(t) \in \tilde{X}_{q, \theta}^p(T) \}
\]
with
\[
\| u \|_{\tilde{Y}_{q, \theta}^p(T)} \triangleq \| U(-t)u(t) \|_{\tilde{X}_{q, \theta}^p(T)}.
\]
The space $Y_{q, \theta}^p(T)$ is defined by
\[
Y_{q, \theta}^p(T) \triangleq \{ u : [0, T] \times \mathbb{R} \rightarrow \mathbb{C} \mid U(-t)u(t) \in X_{q, \theta}^p(T) \},
\]
equipped with the norm
\[
\| u \|_{Y_{q, \theta}^p(T)} \triangleq \| U(-t)u(t) \|_{X_{q, \theta}^p(T)}.
\]

We easily get the following embedding result:

Lemma 1.10. (See e.g. [18, Lemma 2.1]) Let $T > 0$ and let $1 \leq p, q < \infty$ and $\theta \in \mathbb{R}$. Suppose that $-\infty < q' \theta < 1$. Then
\[
X_{q, \theta}^p(T) \hookrightarrow C([0, T]; L^p(\mathbb{R})).
\]
In particular, the embedding
\[
Y_{q, \theta}^p(T) \hookrightarrow C(\Theta([0, T]; L^p(\mathbb{R})))
\]
holds.
2. **Key Lemmata**

2.1. **Generalized Strichartz type inequality.** The key to our local well-posedness results is generalized Strichartz type estimates:

**Lemma 2.1.** Let $2 \leq p < 4$. Then the estimate

$$\|U(t)\phi\|_{L^3_t(L^p_x(R^2))} \leq C\|\hat{\phi}\|_{L^p(R)}$$

holds true.

In this paper we use estimate (2.1) in the following form:

**Corollary 2.2.** Let $2 \leq p < 4$. Then the estimate

$$\|t^{-\frac{3}{2p'}}U(1/4t)\phi\|_{L^{3p'}([0,T];L^{p'}(R))} \leq C\|\hat{\phi}\|_{L^p(R)}$$

holds true.

**Remark 2.3.** Here are some comments on the above estimates.

- The estimates can be regarded as a generalization of the well-known Strichartz estimate for $\phi \in L^2$ and can be traced back to [6]. See also [4] and introduction in [9]. The proof can be found in e.g. [14].
- It is well known that the local in time Strichartz estimate

$$\|U(t)\phi\|_{L^6([0,T];L^6(R))} \leq C\|\phi\|_{L^2(R)}$$

is exploited to prove the existence of an $L^2$-solution to (1.1) for a sufficiently small time $T > 0$. In our analysis we essentially use (2.1) of the following form to establish a local solution on $[0,T]$:

$$\|U(t)\phi\|_{L^{3p}_{x,t}([T^{-1},\infty) \times R)} \leq C\|\hat{\phi}\|_{L^p(R)}.$$\(2.3)\)

This implies that one needs a global in time version of Strichartz type estimates, even in proving the existence of a “local” solution for data $\phi \in L^p, p > 2$.

2.2. **Factorization of $U(t)$ and cubic nonlinearity.** In order to construct a solution $u$ of (1.1) with twisted persistence property $U(-t)u(t) \in L^p$ we rewrite the cubic nonlinearity in terms of $v(t) \triangleq U(-t)u(t)$. This is done via factorization of the free Schrödinger group $U(-t)$. This kind of expression is known and has been used in [18] and in much earlier studies. See e.g. [11]. Here we present the details for completeness and for convenience of the reader. To this end we introduce several operators. The phase modulation operator $M_t$ is defined by

$$M_t : w \mapsto e^{i\frac{x^2}{4t}}w.$$\[M_t]

The dilation operator $D_t$ is defined by

$$(D_tw)(x) \triangleq (4\pi it)^{-\frac{1}{2}}w\left(\frac{x}{4\pi it}\right).$$\[D_t]

$R$ is the reflection operator: $(Rw)(x) \triangleq w(-x)$. Using these operators we get a factorization of $U(t)$ and $U(-t)$ as follows (see [3, Chapter 4]):

$$U(t) = M_tD_tF, \quad U(-t) = M_t^{-1}F^{-1}D_t^{-1}M_t^{-1}.$$\[U_t]

This leads to our next key lemma:
Lemma 2.4. For $t \neq 0$ the following equality holds:

$$U(-t)[u_1(t)u_2(t)u_3(t)] = ct^{-1}M_t^{-1}(M_tU(-t)u_1(t)) * (RM_tU(-t)u_2(t)) * (M_tU(-t)u_3(t)),$$

where $*$ denotes the convolution with respect to the space variable and $c$ is an absolute constant.

Proof. By the factorization of $U(-t)$, we have

$$M_tU(-t)u_1(t)u_2(t)u_3(t) = \mathcal{F}^{-1}D_t^{-1}M_t^{-1}u_1(t)u_2(t)u_3(t),$$

where we have also used the following equalities

$$D_t^{-1}(fgh) = (4\pi it)^{-1}(D_t^{-1}f)(D_t^{-1}g)(D_t^{-1}h), \quad (\mathcal{F}^{-1}f)(x) = (R\mathcal{F}^{-1}f)(x).$$

3. Proof of the well-posedness results

3.1. Non-linear estimates. We first prove a key trilinear estimate in $X^p_{q, \theta}$-spaces from which we deduce the desired local well-posedness result via the fixed point theorem. We introduce the trilinear form $\mathcal{D}(v_1, v_2, v_3)$ by

$$\mathcal{D}(v_1, v_2, v_3) \triangleq \int_0^t s^{-1}M_s^{-1}[(M_s v_1(s)) * (RM_s v_2(s)) * (M_s v_3(s))]ds.$$

Proposition 3.1. Let $T > 0$. Assume that $2 \leq p < 4$. Then

$$\|\mathcal{D}(v_1, v_2, v_3)\|_{\dot{X}^p_{q, \theta}(T)} \leq C \prod_{j=1}^3 \|v_j\|_{X^p_{q, \theta}(T)}$$

with

$$q = \frac{p}{p - 1} (= p'), \quad \theta = -(1 - \frac{2}{p}).$$

Proof. By the Hausdorff–Young and Hölder inequalities, we have

$$t^\theta \|\partial_t \mathcal{D}(v_1, v_2, v_3)\|_{L^p} = t^{\theta - 1} \left\|\left(M_t v_1(t)\right) * \left(RM_t v_2(t)\right) * \left(M_t v_3(t)\right)\right\|_{L^p} \leq C \prod_{j=1}^3 \left(t^{-\frac{\theta - 1}{\theta}} \left\|\mathcal{F}^{-1}M_t v_j(t)\right\|_{L^{3p'}}\right),$$

where we have also used the identity $\mathcal{F}^{-1}RF_j = \mathcal{F}^{-1}f$. Taking $L^q([0, T])$-norm of both sides and using Hölder’s inequality in the time variable, and the fact that $U(-1/4t) = \mathcal{F}^{-1}M_t \mathcal{F}$, $t \neq 0$, we get

$$\|\mathcal{D}(v_1, v_2, v_3)\|_{\dot{X}^p_{q, \theta}(T)} \leq C \prod_{j=1}^3 \left\|t^{-\frac{\theta - 1}{\theta}} U(-1/4t)F^{-1}v_j(t)\right\|_{L^{3q}([0, T]; L^{3p'})}.$$

We estimate the right hand side. Observe that for each $j = 1, 2, 3$

$$\left\|t^{-\frac{\theta - 1}{\theta}} U(-1/4t)F^{-1}v_j(t)\right\|_{L^{3q}([0, T]; L^{3p'})} = \left\|t^{-\frac{\theta - 1}{\theta}} U(1/4t)F^{-1}v_j(t)\right\|_{L^{3q}([0, T]; L^{3p'})} = \left\|t^{-\frac{\theta - 1}{\theta}} U(-1/4t)F^{-1}v_j(t)\right\|_{L^{3q}([0, T]; L^{3p'})}.$$
Now we write
\[ \overline{v_j(t)} = \overline{v_j(0)} + \int_0^t (\partial_s \overline{v_j}) (s) ds. \]

Using the symbol \( W(t) \triangleq t^{-\frac{q}{2}} U(1/4t) \mathcal{F}^{-1} \) we have
\[
\| W(t) \overline{v_j(t)} \|_{L^{3q}([0,T] ; L^{3q'})} \leq \left\| W(t) \overline{v_j(0)} \right\|_{L^{3q}([0,T] ; L^{3q'})} + \left\| \int_0^t W(t) (\partial_s \overline{v_j}) (s) ds \right\|_{L^{3q}([0,T])}
\leq \left\| W(t) \overline{v_j(0)} \right\|_{L^{3q}([0,T] ; L^{3q'})} + \left\| \int_0^T W(t) (\partial_s \overline{v_j}) (s) ds \right\|_{L^{3q}([0,T])}
\leq \left\| W(t) \overline{v_j(0)} \right\|_{L^{3q}([0,T] ; L^{3q'})} + \int_0^T \| W(t) (\partial_s \overline{v_j}) (s) \|_{L^{3q}([0,T])} ds.
\]

Now we take \( q, \theta \) as in (3.2). Then by (2.2) the right hand side of the above inequalities is smaller than
\[
C \left( \| v_j(0) \|_{L^p} + \int_0^T \| \partial_s v_j(s) \|_{L^p} ds \right) = C \left( \| v_j(0) \|_{L^p} + \int_0^T \| \partial_s v_j(s) \|_{L^p} ds \right) = C \| v_j \|_{X^q_{1,0}(T)}.
\]
This proves the nonlinear estimate in question.

\( \square \)

3.2. Proof of Proposition 1.4. Now we prove the main local well-posedness result. Consider the integral equation
\[
u(t) = U(t) \phi + i \int_0^t U(t - s) |u(s)|^2 u(s) ds. \tag{3.3}\]

Let \( q, \theta \) be as in (3.2). Using the nonlinear estimate (3.1), we want to find a fixed point of the operator
\[
(\Phi u)(t) \triangleq U(t) \phi + i \int_0^t U(t - s) |u(s)|^2 u(s) ds
\]
in a closed subset of \( Y^p_{q, \theta}(T) \) for a suitable \( T > 0 \). Throughout the proof we use the convention that \( v(t) = U(-t) u(t), v_j(t) = U(-t) u_j(t) \). Using these notations and Lemma 2.4 we have
\[
(\Phi u)(t) = \phi + \mathcal{E}(v, v, v).
\]

For \( a > 0 \) we define \( \mathcal{V}(a) \) by
\[
\mathcal{V}(a) \triangleq \{ u \in Y^p_{q, \theta}(T) \mid u(0) = \phi, \| u \|_{Y^p_{q, \theta}(T)} \leq a \}
\]
equipped with the distance
\[
d(u_1, u_2) \triangleq \| u_1 - u_2 \|_{Y^p_{q, \theta}(T)}.\]

We first estimate \( \Phi u \) for \( u \in \mathcal{V}(a) \). By (3.4) and (3.1), we have
\[
\| \Phi u \|_{Y^p_{q, \theta}(T)} \leq \| U(-t) \Phi u \|_{Y^p_{q, \theta}(T)} \leq C \| v \|_{X^p_{1,0}(T)} \leq C \| u \|_{Y^p_{1,0}(T)}.
\]
By Hölder’s inequality we get
\[
\|u\|_{Y^0_T}^3 \leq \left( \|\phi\|_{L^p} + T^{1 - \frac{1}{p}} \|\phi\|_{\tilde{Y}^p_{q,0}(T)} \right)^3 
\leq 8\|\phi\|_{L^p}^3 + 8T^{3(1 - \frac{1}{p})}\|u\|_{\tilde{Y}^p_{q,0}(T)}^3.
\]
Therefore, \(\Phi : \mathcal{V}(a) \to \mathcal{V}(a)\) is well defined if we choose \(a, T\) so that
\[
(3.7) \quad 8C\|\phi\|_{L^p}^3 \leq \frac{a}{2}, \quad 8CT^{3(1 - \frac{1}{p})}a^3 \leq \frac{a}{2}.
\]
Similarly, for \(u_1, u_2 \in \mathcal{V}(a)\) we have
\[
\|\Phi u_1 - \Phi u_2\|_{\tilde{Y}^p_{q,0}(T)} = \|\mathcal{D}(v_1, v_1, v_1) - \mathcal{D}(v_2, v_2, v_2)\|_{\tilde{X}^p_{q,0}(T)} 
\leq \|\mathcal{D}(v_1 - v_2, v_1, v_1)\|_{\tilde{X}^p_{q,0}(T)} + \|\mathcal{D}(v_2, v_1 - v_2, v_1)\|_{\tilde{X}^p_{q,0}(T)} 
+ \|\mathcal{D}(v_2, v_2, v_1 - v_2) - \mathcal{D}(v_2, v_2, v_2)\|_{\tilde{X}^p_{q,0}(T)} 
\leq CT^{1 - \frac{1}{p}}\|v_1 - v_2\|_{\tilde{X}^p_{q,0}(T)} 
\times \sum_{1 \leq j, k \leq 2} (\|\phi\|_{L^p} + T^{1 - \frac{1}{p}}\|v_j\|_{\tilde{X}^p_{q,0}(T)}) (\|\phi\|_{L^p} + T^{1 - \frac{1}{p}}\|v_k\|_{\tilde{X}^p_{q,0}(T)}) 
\leq 8CT^{1 - \frac{1}{p}}(\|\phi\|_{L^p}^2 + T^{2(1 - \frac{1}{p})}a^2)\|u_1 - u_2\|_{\tilde{Y}^p_{q,0}(T)}.
\]
Thus \(\Phi\) is a contraction mapping if
\[
(3.6) \quad 8CT^{1 - \frac{1}{p}}(\|\phi\|_{L^p}^2 + T^{2(1 - \frac{1}{p})}a^2) < \frac{1}{2}.
\]

Now we prove the existence of a local solution by the fixed point argument. Let \(M > 0\). For any \(\phi \in \mathcal{B}_M\) we put
\[
(3.7) \quad a = 16CM^3, \quad T = \varepsilon M^{-\frac{2p}{p-1}} (\triangleq T_M),
\]
where \(\varepsilon > 0\) is a constant independent of \(a, M\). Then it is easy to see that \(a\) and \(T\) defined by
\[(3.7)\] satisfy \[(3.5)\] and \[(3.6)\] if \(\varepsilon\) is sufficiently small. We choose such an \(\varepsilon\). Then \(\Phi : \mathcal{V}(a) \to \mathcal{V}(a)\) is well defined and is a contraction mapping. By the fixed point theorem, there exists a solution \(u \in Y^p_{q,0}(T_M)\) of the integral equation \[(3.3)\]. Moreover, the uniqueness in \(Y^p_{q,0}(T_M)\) and continuous dependence on data follows from a similar difference estimate as above. Consequently, the desired local well-posedness result has been proved.

4. PROOF OF THE REGULARITY RESULTS

4.1. Proof of Corollary 1.5. We recall some classical results on the \(L^p\)-regularity for the solution to the linear Schrödinger equation. Denote \(B^s_{p,r}\) by the Besov space of order \(s\). For the definition of the Besov space see e.g. [1].

**Lemma 4.1.** [2, 6.1 Theorem 1] Let \(1 \leq p \leq \infty\) and \(T > 0\). The estimate
\[
(4.1) \quad \sup_{t \in [0, T]} \|U(t)\phi\|_{L^p(\mathbb{R})} \leq C_T \|\phi\|_{B^s_{p,1}(\mathbb{R})}, \quad \forall \phi \in B^s_{p,1}(\mathbb{R})
\]
holds true for some \(C_T > 0\) if and only if \(s \geq 2|1/p - 1/2|\).
In particular, by the inclusion relation
\[ H^{s,p} \subset B^{s,1}_p \subset H^{s,p} \]
for \( s < s' \), it follows that the estimate
\[ \sup_{t \in [0,T]} \| U(t)\phi \|_{H^{s,p}(\mathbb{R})} \leq C_T \| \phi \|_{L^p}, \quad \forall \phi \in L^p \]
holds true if \( s < -(1 - 2/p) \) and fails if \( s > -(1 - 2/p) \).

**Lemma 4.2.** Let \( T > 0 \). Let \( 2 < p < \infty \) and \( s < -(1 - 2/p) \). Then \( U(t)\phi \in C([0,T]; H^{s,p}(\mathbb{R})) \) for any \( \phi \in L^p(\mathbb{R}) \).

**Proof.** We prove the continuity. Take \( \varepsilon > 0 \) arbitrarily and fix it. For \( t, t' \in [0,T] \) we write
(4.2) \[ \| U(t)\phi - U(t')\phi \|_{H^{s,p}} \leq \| U(t)(\phi - \tilde{\phi}) \|_{H^{s,p}} + \| U(t')\tilde{\phi} \|_{H^{s,p}} + \| U(t'')(\phi - \tilde{\phi}) \|_{H^{s,p}} \]
for some \( \tilde{\phi} \in C_0^{\infty}(\mathbb{R}) \). By Lemma 4.1 and density, we may choose \( \tilde{\phi} \in C_0^{\infty}(\mathbb{R}) \) so that
\[ \max\left( \| U(t)(\phi - \tilde{\phi}) \|_{H^{s,p}}, \| U(t')(\phi - \tilde{\phi}) \|_{H^{s,p}} \right) \leq C_T \| \phi - \tilde{\phi} \|_{L^p} \leq \frac{\varepsilon}{3}. \]
For the second term in the right hand side of (4.2), we have
\[ \| U(t)\tilde{\phi} - U(t')\tilde{\phi} \|_{H^{s,p}} \leq \| U(t)\tilde{\phi} - U(t')\tilde{\phi} \|_{L^p} \leq C\| (e^{it|.|^2} - e^{it'|.|^2})F \tilde{\phi} \|_{L^{p'}} \leq C|t - t'| \| \cdot |^2 F \tilde{\phi} \|_{L^{p'}}. \]
Therefore, there is \( \delta = \delta(\varepsilon) > 0 \) such that
\[ \| U(t)\tilde{\phi} - U(t')\tilde{\phi} \|_{H^{s,p}} \leq \frac{\varepsilon}{3} \]
for any \( t, t' \in [0,T] \) with \( |t - t'| < \delta(\varepsilon) \). Now the desired continuity assertion follows from the elementary \( \varepsilon/3 \)-argument.

\[ \square \]

**Proof of Corollary 4.3** It is enough to show the embedding
(4.3) \[ Y^{p}_{q,\theta}(T) \hookrightarrow C([0,T]; H^{s,p}(\mathbb{R})). \]
To prove this we first check that
(4.4) \[ Y^{p}_{q,\theta}(T) \hookrightarrow L^{\infty}([0,T]; H^{s,p}(\mathbb{R})). \]
Let \( u \in Y^{p}_{q,\theta}(T) \) and write \( u(t) = U(t)v(t), v(t) = U(-t)u(t) \). We have
\[ U(t)v(t) = U(t)v(0) + \int_0^t U(t)(\partial_s v)(\tau)d\tau. \]
Taking \( H^{s,p} \)-norm of both sides and using Lemma 4.1 we have
\[ \| u(t) \|_{H^{s,p}} \leq \| U(t)v(0) \|_{H^{s,p}} + \int_0^t \| U(t)(\partial_s v)(\tau) \|_{H^{s,p}}d\tau \]
\[ \leq C_T \| v(0) \|_{L^p} + C_T \int_0^t \| (\partial_\tau v)(\tau) \|_{L^p}d\tau \]
\[ \leq C_T \| v \|_{X^{p}_{q,\theta}(T)} \leq C_T \| v \|_{X^{p}_{q,\theta}(T)} = \| u \|_{Y^{p}_{q,\theta}(T)} \]
for any \( t \in [0,T] \). This proves (4.4). Now it is enough to check the continuity of the map \( t \mapsto u(t) \) from \([0,T]\) to \( H^{s,p} \) to show (4.3).
For \( t, t' \in [0, T] \) we write
\[
\begin{align*}
\mathcal{H}(t) - \mathcal{H}(t') &= U(t)v(0) + \int_0^t U(t)(\partial_x v)(s)ds - U(t')v(0) - \int_0^{t'} U(t') (\partial_x v)(\tau)d\tau \\
&= \left[ U(t)v(0) - U(t')v(0) \right] + \left[ \int_0^t (U(t)(\partial_x v)(\tau) - U(t')(\partial_x v)(\tau))d\tau \right] + \int_t^{t'} U(t')(\partial_x v)(\tau)d\tau \\
&= I_1 + I_2 + I_3.
\end{align*}
\]
Now taking \( H^{s,p} \)-norm and letting \( t' \) tend to \( t \), we see that \( \|I_1\|_{H^{s,p}} \) converges to 0 by Lemma 4.2. Similarly, \( I_2 \) also tends to 0 in \( H^{s,p} \) since
\[
\|I_2\|_{H^{s,p}} \leq \int_0^T \|U(t)(\partial_x v)(\tau) - U(t')(\partial_x v)(\tau)\|_{H^{s,p}}d\tau
\]
and
\[
\|U(t)(\partial_x v)(\tau)\|_{L^q([0,t];H^{s,p})} \leq C_T \|u\|_{Y_{p,q}^{s,\theta}}, \quad \forall t \in [0,T]
\]
by Lemma 4.1. For \( I_3 \) we have
\[
\|I_3\|_{H^{s,p}} \leq \int_t^{t'} \|U(t')(\partial_x v)(\tau)\|_{H^{s,p}}d\tau
\]
\[
\leq C_T \int_t^{t'} \|\partial_x v(\tau)\|_{L^p}d\tau
\]
\[
\leq C_T |t - t'|^{q/2 + 1}/\tilde{X}_p^{s,\theta}(T),
\]
from which it follows that \( I_3 \) converges to 0 in \( H^{s,p} \) as \( t' \to t \). Consequently, we see that \( (t \mapsto u(t)) \in C([0,T];H^{s,p}) \).

\[\square\]

4.2. \textbf{Proof of Corollary 1.6}. We need an off-diagonal generalization of the generalized Strichartz estimate (2.1).

\textbf{Lemma 4.3. (14)} Let \( 2 \leq p < 4 \) and let \( q, r \) be such that
\[
\frac{2}{q} + \frac{1}{r} = \frac{1}{p'}.
\]
Moreover, assume either of (i),(ii) below:
\[
(i) \quad 0 \leq \frac{1}{q} < \min \left( \frac{1}{4}, \frac{1}{2} - \frac{1}{r} \right).
\]
\[
(ii) \quad 4 < r \leq \infty \quad \text{and} \quad q = \frac{1}{4}.
\]
Then the estimate
\[
\|U(t)\phi\|_{L^q(\mathbb{R};L^r(\mathbb{R}))} \leq C\|\hat{\phi}\|_{L^p}.
\]
holds true. In particular, the estimate
\[
\|t^{-\frac{1}{4}}U(1/4t)\phi\|_{L^q(\mathbb{R}^+;L^r(\mathbb{R}))} \leq C\|\hat{\phi}\|_{L^p}.
\]
holds true.

The key to the ill-posedness result is an "\( L^2 \)-smoothing" for the Duhamel contribution of the solution.
Lemma 4.4. Let $2 < p \leq 3$ and let $M > 0$. Then

$$S_M \phi - U(t) \phi \in C([0, T_M]; L^2(\mathbb{R}))$$

for any $\phi \in \mathscr{B}_M^p$. In particular,

$$\sup_{t \in [0, T_M]} \|S_M \phi - U(t) \phi\|_{L^2} \leq CM^{\frac{1}{p-1}}.$$ 

Proof. Let $q_0 \geq 1$. Observe first that

$$Y_{q_0,0}^2(T_M) \hookrightarrow C([0, T_M]; L^2(\mathbb{R}))$$

by Plancherel’s identity. So we estimate $Y_{q_0,0}^2$-norm of the Duhamel terms of the solution. Arguing as in the proof of Proposition 3.1 we have

$$\|S_M \phi - U(t) \phi\|_{Y_{q_0,0}^2(T_M)} = \|S_M \phi - U(t) \phi\|_{Y_{q_0,0}^2(T_M)} = \left\| \int_0^t U(t-s) |u(s)|^2 u(s) ds \right\|_{Y_{q_0,0}^2(T_M)} = CT_{\mathcal{H}}^2 \|v\|_{L^p}.$$ 

Now we put $q_0 = \frac{4p}{3p-6}$. Then $(q, r) = (3q_0, 6)$ satisfies either (i) or (ii) in the statement of Lemma 4.3 as long as $2 \leq p \leq 3$. Thus arguing as in the proof of Proposition 3.1 we have

$$\|t^{-\frac{1}{p}} U(-1/4t) \mathcal{F}^{-1} v(t)\|_{L^{3q_0}([0, T_M]; L^6)} \leq CM^{\frac{2-q_0}{r}} \|v\|_{L^p}.$$ 

Consequently, for $T_M$ and $a$ as in (3.7) we have

$$\|S_M \phi - U(t) \phi\|_{Y_{q_0,0}^2(T_M)} \leq CT_{\mathcal{H}}^{\frac{2-q_0}{r}} \|v\|_{L^p}^{3} \leq C T_{M}^{\frac{p-q_0}{r}} a = CM^{\frac{1}{p-1}}.$$ 

□

We present two proofs of Corollary 1.6

First proof of Corollary 1.6. Assume $s > -(2/p - 1)$. Let $M > 0$. We show that there is a sequence of data $(\phi_n)_{n \geq 1} \subset \mathscr{B}_M^p(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ such that $\lim_{n \to \infty} \sup_{t \in [0, T_M]} \|S_M \phi_n\|_{L^p} = \infty$. We may assume that $s$ is sufficiently close to $-(1 - 2/p)$, say $s < -(1/2 - 1/p)$, so that Sobolev’s embedding

$$L^2(\mathbb{R}) \hookrightarrow H^{s, p}(\mathbb{R})$$

holds. Let $M > 0$, then by the “only if” part of Lemma 4.1 we can take a sequence $(\phi_n)_{n \geq 1} \subset \mathscr{B}_M^p \cap \mathcal{S}(\mathbb{R})$ such that

$$\sup_{t \in [0, T_M]} \|U(t) \phi_n\|_{H^{s, p}} > n.$$
Otherwise one may establish estimate (4.1) by density and the Banach–Steinhaus theorem. Now we write \( U(t)\phi_n = S_M\phi_n + (U(t)\phi_n - S_M\phi_n) \) and apply Lemma 4.4 to obtain
\[
\sup_{t \in [0,T_M]} \|U(t)\phi_n\|_{H^{s,p}} \leq \sup_{t \in [0,T_M]} \|S_M\phi_n\|_{H^{s,p}} + \sup_{t \in [0,T_M]} \|S_M\phi_n - U(t)\phi_n\|_{H^{s,p}}
\]
\[
\leq \sup_{t \in [0,T_M]} \|S_M\phi_n\|_{H^{s,p}} + \sup_{t \in [0,T_M]} \|S_M\phi_n - U(t)\phi_n\|_{L^2}
\]
\[
\leq \sup_{t \in [0,T_M]} \|S_M\phi_n\|_{H^{s,p}} + CM^{1/2}.
\]

Letting \( n \to \infty \) we see that
\[
\lim_{n \to \infty} \sup_{t \in [0,T_M]} \|S_M\phi_n\|_{H^{s,p}} = \infty.
\]

This implies that property (ii) in Corollary 1.5 does not hold.

\[\square\]

**Second proof of Corollary 1.6** Let \( M > 0 \). In the second proof we show that assertion (i) in the statement of Corollary 1.5 fails by showing the existence of data \( \phi \in \mathcal{B}^p_M \) such that \( \langle S_M\phi(t_0) \rangle \notin H^{s,p} \) for some \( t_0 \in (0,T_M) \) if \( 2 < p \leq 4 \) and \( s > 2/p - 1 \). We first recall the following result on the \( L^p \)-regularity for the homogeneous data:

**Lemma 4.5.** \([3, Theorem 2.6.1]\) Let \( 0 < a < 1 \) and \( \psi_a(x) = |x|^{-a} \), \( x \in \mathbb{R} \). Then
\[
U(t)\psi_a \in L^p(\mathbb{R})
\]
for all \( t > 0 \) and for any \( p \) such that
\[
p > \max \left( \frac{1}{a}, \frac{1}{1-a} \right).
\]

Observe that when \( p > 2 \) we may choose \( a \) such that
\[
\frac{1}{p} < a < 1 - \frac{1}{p}.
\]

We take such an \( a \) and for \( t_0 \in (0,T_M) \) and we set
\[
\phi_a \triangleq cU(-t_0)\psi_a = \overline{cU(t_0)\psi_a},
\]
which belongs to \( L^p \) for any \( p \) satisfying (4.7). The constant \( c \) can be varied depending on the size of \( M \). Here we assume \( c = 1 \) for simplicity. Clearly, \( U(t_0)\phi = \psi_a = |x|^{-a} \). We show that there is an \( a \) such that \( \langle S_M\phi_a(t_0) \rangle \notin H^{s,p} \) if \( s > 2/p - 1 \) and \( 2 < p \leq 3 \). As in the first proof, we may assume that \( s < 1/p - 1/2 \). Then by the \( L^2 \)-smoothing, \( (S_M\phi_a)(t_0) - U(t_0)\phi_a \in L^2 \subset H^{s,p} \) and it is enough to check that \( U(t_0)\phi_a \notin H^{s,p} \) to conclude that \( \langle S_M\phi_a(t_0) \rangle \notin H^{s,p} \). We estimate
\[
\langle D \rangle^s U(t_0)\phi_a \triangleq \mathcal{F}^{-1}(\cdot)^s \hat{U}(t_0)\hat{\phi}_a, \text{ where } (\xi)^s \triangleq (1 + |\xi|^2)^{1/2}.
\]
It is known ([3, Proposition 1.2.5]) that \( \mathcal{F}^{-1}(\cdot)^s \) is strictly positive and satisfies
\[
|\mathcal{F}^{-1}(\cdot)^s(x)| \geq C|x|^{-s-1} + O(|x|^{-s+1})
\]
for \( |x| \leq 2 \). In particular, for a sufficiently small \( \delta > 0 \) one has
\[
|\mathcal{F}^{-1}(\cdot)^s(x)| \geq c|x|^{-s-1}
\]
Indeed, once (4.10) is verified, the desired embedding follows arguing as in the proof of (4.4).

\[(4.10) \quad \|x\|^{-s-a} \geq C \int_{y \in x+I_\delta} |x-y|^{-1-s}|y|^{-a}dy + C \int_{y \notin x+I_\delta} |F^{-1}(\cdot)^s|(x-y)|y|^{-a}dy \]

Now we can easily verify that the function in the right hand side is not in \(L^p([-\delta/4, \delta/4])\) as follows. We may write

\[C|x|^{-s-a} = (| \cdot |^{-1-s} * | \cdot |^{-a})(x) = \int_{y \in x+I_\delta} |x-y|^{-1-s}|y|^{-a}dy + \int_{y \notin x+I_\delta} |x-y|^{-1-s}|y|^{-a}dy \]

where we have used some basic facts (see e.g. [7]) on the convolution and the Fourier transform of the homogeneous functions. Clearly, \(| \cdot |^{-s-a} = H_1 + H_2 \notin L^p_{loc}(\mathbb{R})\) if \(a \geq 1/p - s\). In view of (4.8), such an \(a\) exists if \(1/p - s < 1 - 1/p\), which is equivalent to \(s > 2/p - 1\). On the other hand, we see that \(|x| < \delta/4\) and \(|x-y| > \delta\) implies \(|y| \geq (3/4)\delta\) and \((2/3)|y| \leq |x-y|\). Thus we have

\[H_2(x) \leq (2/3)^{-s} \int_{|y|>(3/4)\delta} |y|^{-1-s-a}dy = C_{\delta} < \infty \]

for any \(x \in [-\delta/4, \delta/4]\). Hence \(H_1 \notin L^p(\mathbb{R})\) and consequently, we see that \(U(t_0)\phi_a \notin H^{s,p}\). \(\Box\)

4.3. Proof of Corollary 1.8. Finally, we prove the result on the Strichartz regularity for the local solution. It suffices to prove that the embedding

\[(4.9) \quad Y_{q,0}^{p}(T_M) \hookrightarrow L^p([0, T_M], t^{\frac{s}{p} - \frac{s}{2}}dt; L^r(\mathbb{R}))\]

holds. To prove this inclusion relation it is enough to show the following Strichartz type estimate:

\[(4.10) \quad \|U(t)\phi\|_{L^q(\mathbb{R}, t^{1/p - 1/2}dt; L^r(\mathbb{R}))} \leq C\|\phi\|_{L^p}.\]

Indeed, once (4.10) is verified, the desired embedding follows arguing as in the proof of (4.4).

Essentially, the estimate is equivalent to (4.5). We prove (4.10) by showing this. Recall the factorization of \(U(t)\) in Section 2. For \(f \in S(\mathbb{R})\) we have

\[|U(t)f| = |D_t F M_t F^{-1}Ff| = |D_t F M_t F^{-1} \overline{Ff}| = |D_t F^{-1} \overline{M_t F F^{-1}f}| = |D_t U(1/4t)F^{-1}\overline{f}|.\]

Now we substitute \(\phi = F^{-1}\overline{f}\) (i.e. \(f = F^{-1}\phi\)) into the above equality and apply (4.5) to obtain

\[\|D_t U(1/4t)\phi\|_{L^q(\mathbb{R}, L^r(\mathbb{R}))} = \|U(t)F^{-1}\phi\|_{L^q(\mathbb{R}, L^r(\mathbb{R}))} \leq C\|\phi\|_{L^p} = C\|\phi\|_{L^p}.\]

Finally, a suitable change of the space and time variables in the left hand side of the above inequality yields (4.10). \(\Box\)
References

[1] J. Bergh and J. L"ofstr"om, *Interpolation Spaces: An Introduction*, Springer-Verlag, 1976.

[2] P. Brenner, V. Thom"ee and L.B. Wahlbin, *Besov Spaces and Applications to Difference Methods for Initial Value Problems*, Lecture Notes in Math. Springer 434.

[3] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lect. Notes Math. 10, New York Univ., Courant Inst. Math. Sci., New York, 2003.

[4] T. Cazenave, L. Vega, and M.C. Vilela, *A note on the nonlinear Schrödinger equation in weak $L^p$ spaces*, Communications in contemporary Mathematics, Vol. 3, No.1 (2001),153–162.

[5] B. Dodson, A. Soffer, and T. Spencer, *Global well-posedness for the cubic nonlinear Schrödinger equation with initial data lying in $L^p$-based Sobolev spaces*, J. Math. Phys. 62 (2021), 071507.

[6] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), 9-36.

[7] L. Grafakos, *Classical Fourier Analysis*, Third edition, Graduate Texts in Math. 249, Springer, New York, (2014).

[8] L. Grafakos, *Modern Fourier Analysis*, Third edition, Graduate Texts in Math. 250, Springer, New York, (2014).

[9] A. Gr"unrock, *An improved local well-posedness result for the modified KdV equation*, Int. Math. Res. Not., 41 (2004) 3287-3308.

[10] A. Gr"unrock, *Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS*, Int. Math. Res. Not., 41 (2005), 2525-2558.

[11] N. Hayashi and P. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. 120 (1998), 369-389.

[12] L. H"ormander, *Estimates for translation invariant operators in $L^p$ spaces*, Acta Math., 104 (1960), 141–164.

[13] R. Hyakuna, *Global solutions to the Hartree equation for large $L^p$-initial data*, Indiana Univ. Math. J. 68 (2019), 1149–1172.

[14] R. Hyakuna and M. Tsutsumi, *On existence of global solutions of Schrödinger equations with subcritical nonlinearity for $L^p$-data*, Proc. Am, Math. Soc. 140 (2012), 3905–3920.

[15] R. Schippa, *On smoothing estimates in modulation spaces and the nonlinear Schrödinger equation with slowly decaying initial data*, J. Funct. Anal. 282 (5) (2022), 109352.

[16] R. Schippa, *Infinite-energy solutions to energy-critical nonlinear Schrödinger equations in modulation spaces*, arXiv:2204.01001.

[17] Y. Tsutsumi, *$L^2$ -solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac., 30 (1987), 115-125.

[18] Y. Zhou, *Cauchy problem of nonlinear Schrödinger equation with initial data in Sobolev space $W^{s,p}$ for $p < 2$*, Trans. Amer. Math. Soc., 362 (2010), 4683-4694.

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