Pairs of MOLS of order ten satisfying non-trivial relations

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Abstract

A relation on a $k$-net$(n)$ (or, equivalently, a set of $k - 2$ mutually orthogonal Latin squares of order $n$) is an $\mathbb{F}_2$ linear dependence within the incidence matrix of the net. Dukes and Howard (2014) showed that any 6-net(10) satisfies at least two non-trivial relations, and classified the relations that could appear in such a net. We find that, up to equivalence, there are 18 526 320 pairs of MOLS satisfying at least one non-trivial relation. None of these pairs extend to a triple. We also rule out one other relation on a set of 3-MOLS from Dukes and Howard’s classification.

1 Introduction

A partial linear space $(P, L)$ consists of a set $P$ of points and a set $L$ of subsets of $P$ called lines, such that distinct lines share at most one point. A line $\ell \in L$ and a point $p \in P$ are incident if $p \in \ell$. Two lines $\ell_1$ and $\ell_2$ are parallel if $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$. Similarly, two lines $\ell_1$ and $\ell_2$ are orthogonal if $|\ell_1 \cap \ell_2| = 1$. A $k$-net$(n)$ is a partial linear space $N = (P, L)$ consisting of a set $P$ of $n^2$ points and a set $L$ of $kn$ lines, such that each line is incident with $n$ points and each point is incident with $k$ lines. The lines of $N$ should partition into $k$ parallel classes with each parallel class containing $n$ pairwise parallel lines. Every pair of lines that are not in the same parallel class must be orthogonal. Two $k$-nets, $N_1 = (P_1, L_1)$ and $N_2 = (P_2, L_2)$, of order $n$ are isomorphic if there exists a bijection $\phi : P_1 \to P_2$ such that $\ell \in L_1$ if and only if $\phi(\ell) \in L_2$. A $k'$-net$(n)$ $N_1 = (P, L_1)$ is a subnet of a $k$-net$(n)$ $N_2 = (P, L_2)$ if $L_1 \subseteq L_2$.

Let $N = (P, L)$ be a $k$-net$(n)$. The incidence matrix $M(N)$ is an $n^2 \times nk$ array over the finite field $\mathbb{F}_2$ with rows and columns indexed by the points and lines of $N$, respectively. The entry $M_{i,j} = 1$ if the point $i$ is contained in line $j$ and $M_{i,j} = 0$ otherwise. For a line $\ell$ in the net, we define $\alpha(\ell)$ to be the column vector in $M(N)$ corresponding to the line $\ell$. A relation $R \subseteq L$ is a set of lines such that $\sum_{\ell \in R} \alpha(\ell) = 0$, the all zero $\mathbb{F}_2$-vector. In other words, a relation is an $\mathbb{F}_2$-linear dependence among the columns of $M(N)$. It is immediate from this definition that the symmetric difference of two relations is a relation. Every parallel class $\Pi$ has $\sum_{\ell \in \Pi} \alpha(\ell) = 1$, the all one $\mathbb{F}_2$-vector, so the union of any pair of distinct parallel classes is a relation. Any relation that is the union of any number of parallel classes is considered trivial.

Each relation on $N$ corresponds to a null vector of $M(N)$. We say that a set of relations is linearly independent if the corresponding set of null vectors is linearly independent. We define the dimension of $N$, written $\dim(N)$, to be the rank of $M(N)$. Throughout, we will use $\Pi_0, \Pi_1, \ldots, \Pi_{k-1}$ to denote the parallel classes of a $k$-net $N$. The set $\{\Pi_0 \cup \Pi_1, \Pi_0 \cup \Pi_2, \ldots, \Pi_0 \cup \Pi_{k-1}\}$ is a set of $k - 1$ linearly independent relations on $N$ which span the trivial relations. It follows that

$$\dim(N) \leq nk - k + 1,$$

with equality if and only if there are no non-trivial relations on $N$.

Let $N = (P, L)$ be a $k$-net$(n)$ and let $\mathcal{R}$ be a relation on $N$. We say that a line $\ell \in L$ is relational if $\ell \in \mathcal{R}$. The weight $w(p)$ of a point $p$ is the number of relational lines incident to $p$ in $N$. The fact that $\mathcal{R}$ is a relation is equivalent to the statement that $w(p) \equiv 0 \pmod{2}$ for all points $p \in P$. The number of relational
lines in a parallel class II is the \textit{weight} of II. The \textit{type} of a relation indicates the weight of each parallel class in \( N \). For simplicity, we will often use power notation, with \( a^b \) indicating that \( b \) parallel classes have weight \( a \). For example, a relation on a 6-net(\( n \)) with 3 parallel classes containing 2 relational lines, 2 parallel classes containing 4 relational lines and 1 parallel class containing 6 relational lines has type \((2, 2, 4, 4, 4, 6)\) or \(2^44^26\).

Dukes and Howard \[8\], building on earlier work by Stinson \[13\] and Dougherty \[7\], proved the following necessary condition on the set of relations on a 6-net(10), if such a net exists.

\textbf{Theorem 1.1.} \textit{Any} 6-net(10) \textit{satisfies at least two linearly independent non-trivial relations.}

Additionally, Dukes and Howard \[8\] showed that any non-trivial relation on a 6-net(10) has one the following types (up to taking disjoint union with a trivial relation): \(4^4, 2^34^2, 2^43^2, 4^26, 2^42^24^2, 2^34^26, 2^24^4, 24^6, 4^6\). In this paper we report on two computations on nets of order 10. The first found all 4-nets containing \( 4 \) relational lines and \( 1 \) parallel class containing \( 6 \) relational lines has type \((2, 2, 4, 4, 4, 6)\) or \(2^44^26\).

The structure of this paper is as follows. In Section \[2\] we give the necessary background and definitions, including the interpretation of our problem in terms of mutually orthogonal Latin squares. In Section \[3\] we prove a general result on the structure of relations in a 6-net(\( n \)) for even \( n \). We also give some structural lemmas for nets with relations of type \(4^4\) and \(2^34^2\). In Section \[4\] we describe algorithms that generate a catalogue of nets that satisfy a given relation. Sections \[5\] and \[6\] report on the computational results we obtained using those algorithms for relations of type \(4^4\) and \(2^34^2\), respectively. For the purposes of crosschecking, all computational results reported in this paper were obtained independently by the two authors.

\section{Background and Definitions}

A \textit{frequency square} of order \( n \), with \( s \) symbols, and frequencies \((\lambda_0, \ldots, \lambda_{s-1})\) is an \( n \times n \) array with a symbol multiset \( S \) containing \( \lambda_i \) copies of symbol \( i \), such that each row and each column is a permutation of \( S \). A frequency square is \textit{balanced} if each symbol has the same multiplicity in \( S \). A balanced frequency square for which each symbol occurs once in \( S \) is a \textit{Latin square}.

We index the rows and columns of \( n \times n \) matrices using \( \{0, 1, \ldots, n - 1\} \). Two frequency squares \( F_1 \) and \( F_2 \) with frequencies \((\lambda_0, \ldots, \lambda_{s-1})\) and \((\mu_0, \ldots, \mu_{s-1})\) are \textit{orthogonal} if for each \( \ell \in \{0, \ldots, s - 1\} \) and \( m \in \{0, \ldots, t - 1\} \) there are \( \lambda_{m\ell} \) ordered pairs \((F_1[i, j], F_2[i, j]) = (\ell, m)\) with \( i, j \in \{0, \ldots, n - 1\} \). A set of \( k \) frequency squares of order \( n \) is \textit{mutually orthogonal} if every pair of squares in the set is orthogonal. In this case we write that the frequency squares form a set of \( k \)-MOFS(\( n \)). If every square in the set of \( k \)-MOFS(\( n \)) is Latin, then the set is a set of \( k \) \textit{mutually orthogonal Latin squares}, abbreviated \( k \)-MOLS(\( n \)). Although it is traditional to talk about \textit{sets} of MOFS or MOLS, in this paper we will often care about the order of the elements within that set. Hence we will often think of \textit{lists} of MOFS or MOLS.

It is well known that a \( k \)-net(\( n \)) is equivalent to a set of \((k - 2)\)-MOLS(\( n \)). Under this correspondence, the lines of two arbitrary parallel classes of the net correspond to the rows and the columns, while each of the remaining \( k - 2 \) parallel classes correspond to the symbols in a different square. A \textit{transversal} \( t \) in a \( k \)-net(\( n \)) \( N \) is a set of \( n \) points in \( N \) such that \( t \) is orthogonal to every line in \( N \). In a related notion, a \textit{transversal} in a Latin square is defined to be a set of \( n \) cells that have no rows nor columns in common, such that every symbol occurs exactly once within the cells. For a set \( L \) of \((k - 2)\)-MOLS(\( n \)), a \textit{common transversal} is a set \( t' \) of \( n \) cells such that \( t' \) is a transversal in each of the \( k - 2 \) Latin squares of \( L \). A common transversal in a set of MOLS is equivalent to a transversal in the corresponding net. A \( k \)-net(\( n \)) \( N \) has \( n \) disjoint transversals if and only if \( N \) is a subnet of some \((k + 1)\)-net(\( n \)).

Let \((s_0, \ldots, s_{t-1})\) be a list of positive integers, \((S_0, \ldots, S_{t-1})\) be a list of multisets such that, for \( 0 \leq i < t \), the support of \( S_i \) is \( \{0, \ldots, s_i - 1\} \) and let \( \lambda_i^j \) be the multiplicity of the symbol \( k \) in the multiset \( S_i \). A \textit{mixed type orthogonal array} (MOA) with multisets \((S_0, \ldots, S_{t-1})\) is an \( s \times t \) array in which the \( i \)-th column is a permutation of the multiset \( S_i \), for \( 0 \leq i < t \). Also, for every pair of disjoint columns \( i \) and \( j \) and symbols \( k \in S_i \) and \( \ell \in S_j \) there should be exactly \( \lambda_i^j \lambda_i^\ell / s \) rows where \( k \) occurs in column \( i \) and \( \ell \) occurs in column \( j \). Mixed type orthogonal arrays generalise MOFS. From any \((t - 2)\)-MOFS(\( n \)) \( F_0, \ldots, F_{t-3} \) we can build an \( n^2 \times t \) MOA in which each row records the index \( i \) of a row, the index \( j \) of a column and then
the symbols \( F_0[i,j], F_1[i,j], \ldots, F_{i-3}[i,j] \). Moreover, if a MOA has \( s_0 = s_1 = n \) and \( \lambda^0_k = \lambda^1_k = n \) for each symbol \( k \in \{0, \ldots, n - 1\} \) then the MOA can be formed from a set of MOFS in this way. If \( \lambda^0_k = n \) for all \( i \) and \( k \) then the MOA corresponds to a set of MOLS.

We do not attach any significance to the order of the rows in a MOA, so we consider two MOAs to be the same if one can be obtained from the other by any permutation of the rows. A MOA \( O_1 \) is a conjugate of another MOA \( O_2 \) if \( O_2 \) can be obtained by permuting the columns of \( O_1 \). Two MOAs \( O_1 \) and \( O_2 \) are isotopic if \( O_1 \) can be obtained from \( O_2 \) by, for each \( i \), applying some permutation of \( \{0, \ldots, s_i - 1\} \) to the symbols within column \( i \). Two MOAs \( O_1 \) and \( O_2 \) are paratopic, or belong to the same species, if \( O_2 \) is isotopic to a conjugate of \( O_1 \). Any \( s \times t \) MOA has \( t! \) conjugates. In this paper we are only interested in MOAs that represent MOFS. For that reason we will only consider \( n^2 \times t \) MOAs with \( t \geq 3 \) and multisets \( S_0 \) and \( S_1 \) that have support \( \{0, \ldots, n - 1\} \), with multiplicity \( n \) for each symbol in \( S_0 \) and \( S_1 \). The multisets \( S_0 \) and \( S_1 \) will represent the rows and columns of the MOFS, respectively. Furthermore, for any MOA \( O \), with multisets \( (S_0, \ldots, S_{t-1}) \), we only consider conjugates of \( O \) that respect the symbol sets in the sense that the multisets for the conjugate also equal \((S_0, \ldots, S_{t-1})\).

Let \( F \) be a frequency square with symbol set of size \( s \) and let \( t \leq s \). A symbol homotopy \( \theta : \{0, \ldots, s-1\} \to \{0, \ldots, t-1\} \) is a mapping from the symbol set of \( F \) to a symbol set of size \( t \) resulting in a frequency square obtained by identifying symbols in \( F \). Let \( F' \) be a frequency square with symbol set of size \( t \). The frequency square \( F \) is a refinement of \( F' \) if a symbol homotopy \( \theta \) exists such that \( F' = \theta(F) \). A symbol homotopy of a list of frequency squares \((F_0, \ldots, F_{k-1})\) is a list of functions \((\theta_0, \ldots, \theta_{k-1})\) such that \( \theta_i \) is a symbol homotopy of \( F_i \), for \( 0 \leq i < k \). A list of frequency squares \((G_0, \ldots, G_{k-1})\) is a refinement of \((F_0, \ldots, F_{k-1})\) if \((F_0, \ldots, F_{k-1}) = (\theta_0(G_0), \ldots, \theta_{k-1}(G_{k-1}))\).

Let \( R_1 \) and \( R_2 \) be two relations on a \( k \)-net \((n)\), and suppose that \( R_2 \) is a trivial relation. Then the symmetric difference of \( R_1 \) and \( R_2 \) is a relation obtained by complementing the relational lines of \( R_1 \) relative to each of the parallel classes that are included in \( R_2 \). For example, by complementing two parallel classes of weight \( 2 \) in a relation of type \( 2^4 \times 2^6 \) we obtain a relation of type \( 2^4 \times 68^2 \). An odd relation can be obtained from a relation by complementing a set of relational lines contained in a single parallel class. Equivalently, an odd relation is any set of lines \( R_0 \subseteq L \) satisfying the equation \( \sum_{x \in R_0} \alpha(x) = 1 \). Each point \( p \in P \) in an odd relation has weight \( w(p) \equiv 1 \pmod{2} \).

The cardinality \( N(n) \) of a largest set of MOLS \((n)\) has been of interest to the mathematical community for centuries. Euler, while looking for new ways to construct magic squares, famously conjectured that \( N(4t - 2) = 1 \) for all \( t \geq 1 \). Tarry \[16\] used exhaustive case analysis to prove \( N(6) = 1 \), confirming Euler’s conjecture in that case. However, Bose and Shrikhande \[2\] discovered a counterexample to the conjecture of order 22. Later the same year Parker \[14\] discovered a counterexample of order 10. Bose, Shrikhande, and Parker \[3\] provided a construction that shows that \( N(n) \geq 2 \) for all \( n \geq 7 \). Egan and Wanless \[6\] counted and classified all sets of MOLS \((n)\) for \( n \leq 9 \). However, even the value of \( N(10) \) has proved elusive despite considerable effort (see McKay, Meynert and Myrvold \[12\]). This context provides the motivation for the present work.

It is easily seen that \( N(n) \leq n - 1 \). A set of \((n-1)\)-MOLS \((n)\) is said to be complete. A complete set of MOLS \((n)\) exists if and only if there is a projective plane of order \( n \). As there is a Desarguesian plane for all prime power orders, a complete set of MOLS exists for each prime power order. Bruck \[3\] used geometric arguments to bound the size of the largest set of MOLS when no complete set exists, showing that if \( N(n) < n - 1 \) then \( N(n) < n - 1 - (2n)^{1/4} \). This result was later improved by Metsch \[13\].

Stinson \[13\] used the theory of transversal designs to give a short proof of the non-existence of a pair of MOLS of order 6. This was a significant step given the exhaustive cases analysis required by Tarry \[16\]. Later, Dougherty \[7\] combined net and coding theoretic techniques to give another short proof that \( N(6) = 1 \). Lam, Thiel, and Swiercz \[10\] proved the non-existence of a projective plane of order 10 using an extensive computational search. Thus there does not exist a complete set of 9-MOLS(10) and it follows from the theorems of Bruck \[3\] and Metsch \[13\] that \( N(10) \leq 6 \). If all of the types of relations in Dukes and Howard’s catalogue could be ruled out, then Theorem \[1.1\] would show that \( N(10) \leq 3 \). This paper takes a first step in that direction, ruling out two of the types.
3 Templates

In this section we introduce a structure called a template, which will assist in our study of relations. Throughout this section $n$ will be an even positive integer.

A template $T$ of order $n$ and type $(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$, with $\lambda_i$ even, is a list $(F_0, F_1, \ldots, F_{k-3})$ of $(k-2)$-MOFS$(n)$ with the following properties.

1. $T$ has rows $(0,1,\ldots,n-1)$ and columns $(0,1,\ldots,n-1)$. The rows $(0,1,\ldots,\lambda_0-1)$ and the columns $(0,1,\ldots,\lambda_1-1)$ are deemed to be relational.

2. For each $t \in \{2,\ldots,k-1\}$ the binary frequency square $F_{t-2}$ has frequencies $(n-\lambda_t, \lambda_t)$.

3. The type of a point $(i,j)$ is the $k$-tuple $(x, y, F_0[i,j], F_1[i,j], \ldots, F_{k-3}[i,j])$ where $x = 1$ if row $i$ is relational and 0 otherwise, and $y = 1$ if column $j$ is relational and 0 otherwise. A point $(i,j)$ is relational relative to a frequency square $F_\ell$ if $F_\ell[i,j] = 1$.

4. The weight of a point is the sum (in $\mathbb{Z}$) of the elements of its type $k$-tuple. All points $p$ in $T$ have weight $w(p) \equiv \chi \pmod{2}$, where $\chi = (\sum_i \lambda_i)/2$.

For simplicity we represent the type of a point in a template as a binary string (i.e. we omit the commas and parentheses). Two templates are of equivalent type if there is a permutation of one type that is equal to the other. Two templates of equivalent type are isomorphic if one template can be obtained from the other by applying any sequence of the following operations:

1. Uniformly permuting the rows of the frequency squares in the template. Such a permutation must map relational rows to relational rows.

2. Uniformly permuting the columns of the frequency squares in the template. Such a permutation must map relational columns to relational columns.

3. Transposing all of the frequency squares in the template.

4. Reordering the frequency squares in the template.

A $k$-net$(n)$ $N$ is said to be a refinement of a template $T$ if the list of $(k-2)$-MOLS$(n)$ that is equivalent to $N$ is a refinement of $T$. Implicit in this statement is the need for us to have fixed the ordering of the parallel classes of $N$, say by indexing them as $\Pi_0, \ldots, \Pi_{k-1}$. Under this ordering, the parallel class $\Pi_0$ corresponds to the rows in $T$, the parallel class $\Pi_1$ corresponds to the columns of $T$ and each remaining parallel class $\Pi_\ell$ corresponds to the frequency square $F_{\ell-2}$. The properties imposed on the frequency squares of the template guarantee that if a net is a refinement of the template then the net satisfies a relation (or odd relation) $\mathcal{R}$ of type $(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$. Whether $\mathcal{R}$ is an odd relation depends on the parity of $\chi$. Each relational line in $\mathcal{R}$ corresponds to a relational row, relational column, or a set of $n$ relational points relative to a frequency square in the template. For every net $N$ with a non-trivial relation, there exists a template such that $N$ is a refinement of the template. This can be seen by considering the list of MOLS equivalent to $N$ and applying a symbol homotopy that maps each symbol in each square to 0 or 1 according to whether the line corresponding to that symbol is non-relational or relational in $N$, respectively. However, starting with a template, it may or may not be possible to find a net that refines it. In §3.2 we will give a template that is not refined by any net. However, we are only really interested in templates that have a refinement to a net. So in the following sections we will often hypothesise the existence of such a refinement when studying the structure of templates.

3.1 Template Structures

Let $B^k$ be the set of all length $k$ binary strings containing an even number of ones. Let $B^k_{i,j}$ be the subset of $B^k$ such that $b \in B^k_{i,j}$ if and only if both the $i$th and $j$th bits of $b$ are 1. Similarly, let $\bar{B}^k_{i,j}$ be the subset of $B^k$ such that $b \in \bar{B}^k_{i,j}$ if and only if the $i$th bit of $b$ is 1 and the $j$th bit of $b$ is 0. For a particular template $T$, let $t_b$ be the number of points in $T$ of type $b$. The bitwise complement of a type $b$ is denoted $\bar{b}$. 
Let $N$ be a $k$-net that admits a relation $R$ of type $(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$. By considering the orthogonality of lines in $N$ we can derive a set of equations that the points of a template must satisfy if the template is to be refined by $N$. For later simplicity, let $g_i = \frac{n}{2} - \lambda_i$, for $0 \leq i < k$. We have

\[ \sum_{b \in B_c} t_b = n^2, \quad (2) \]
\[ \sum_{b \in B^c_{i,j}} t_b = \left( \frac{n}{2} - g_i \right) \left( \frac{n}{2} - g_j \right) \quad \text{for all } i < j, \quad (3) \]
\[ \sum_{b \in B^c_{i,j}} t_b = \left( \frac{n}{2} + g_i \right) \left( \frac{n}{2} + g_j \right) \quad \text{for all } i < j, \quad (4) \]
\[ \sum_{b \in B^c_{i,j}} t_b = \left( \frac{n}{2} - g_i \right) \left( \frac{n}{2} + g_j \right) \quad \text{for all } i \neq j. \quad (5) \]

Equation (2) simply states that a net of order $n$ contains $n^2$ points. The set of equations (3) are derived from the fact that for every pair of parallel classes $\Pi_i$ and $\Pi_j$ in $N$, each relational line from $\Pi_i$ shares exactly one point with each relational line in $\Pi_j$. Similarly, the equations (4) come from the fact that every non-relational line in $\Pi_i$ shares exactly one point with each non-relational line in $\Pi_j$. Finally the equations (5) are a consequence of every relational line in $\Pi_i$ sharing exactly one point with each non-relational line in $\Pi_j$.

The equations (2) – (5) hold for all 6-nets of even order. Let $(\lambda_0, \lambda_1, \ldots, \lambda_5)$ be the type of a relation. The linear system consists of 61 equations with a variable for each of the 32 point types. Unfortunately, the resulting system is underdetermined, with 10 degrees of freedom. Table 1 gives the number of nonnegative integer solutions to the equations (2) – (5) for certain relations on a 6-net(10). Dukes and Howard [8] showed that every relation on a 6-net(10) that has $0 < \lambda_i < 10$ for $0 \leq i \leq 5$ is equivalent to one of those listed in the table. The same paper demonstrates a restriction which is not included in the system of equations (2) – (5). In particular, they show that, for $i \in \{0, \ldots, 5\},$

\[ t_{111111} \leq \lambda_i \left\lfloor \sum_{j \neq i} \lambda_j - \frac{10}{4} \right\rfloor. \quad (6) \]

An easy way to deduce (6) is to consider a relational line $\ell$ in the $i$-th parallel class. All ten points in $\ell$ must meet an odd number of the relational lines that are orthogonal to $\ell$. Allocating one such line to each point, we then have $\sum_{j \neq i} \lambda_j - 10$ lines as yet unallocated, and each weight 6 point on $\ell$ uses four of them. It follows that (6) holds, since we have $\lambda_i$ choices for $\ell$ and every weight 6 point must lie on one of them. In practice (6) only removes 16 solutions for relations of type $2^41^2$ and 16 solutions for relations of type $4^6$. The only solution for relations of type $2^56$, marked with a †, satisfies (6) but can be eliminated by [8, Prop. 2.9].

| Relation type | Solutions to (2) | Solutions to (2) – (6) |
|---------------|-----------------|------------------------|
| $2^6$         | 1               | 1                      |
| $2^56^1$      | $1^\dagger$     | $1^\dagger$            |
| $2^42^1$      | 146             | 130                    |
| $2^34^26^1$   | 1302            | 1302                   |
| $2^24^4$      | 5286            | 5286                   |
| $2^14^46^1$   | 61 113          | 61 113                 |
| $4^6$         | 1 832 069       | 1 832 053              |

Table 1: The number of solutions to the constraints on the $t_b$.

Although our system of equations usually has many solutions, there is a relationship between the counts of complementary point types:
Theorem 3.1. Let \( n \) be even and let \( N = (P, L) \) be a 6-net\((n)\) with relation \( R \) of type \((\lambda_0, \lambda_1, \ldots, \lambda_5)\). Then,

\[
t_b + t_b = \frac{1}{16} n^2 + \frac{1}{4} \sum_{i<j} (-1)^{b_i+b_j} g_i g_j, \quad \text{for all } b \in B^6.
\] (7)

Proof. Let \( z \) be the point type of weight 0 and \( \bar{z} \) be its complement. Equation (2) contains every point type exactly once and every point type \( b \in B^6 \setminus \{z, \bar{z}\} \) is in exactly 8 equations from (5). It follows that the equation obtained from (2) by subtracting \( 1/8 \)th of each equation in (5) gives,

\[
t_z + t_{\bar{z}} = \frac{1}{16} n^2 + \frac{1}{4} \sum_{i<j} g_i g_j.
\] (8)

Now consider a point type \( z' \) of weight 2 in \( R \) with \( z'_a = z'_b = 1 \), where \( a \neq b \). Complementing the relational lines in \( \Pi_a \) and \( \Pi_b \) gives a new relation \( R' \) for which \( g' a = -g a \) and \( g' b = -g b \) and \( g' k = g_k \) for all \( k \in \{0, \ldots, 5\} \setminus \{a, b\} \). Moreover, points of type \( z' \) in \( R \) have weight 0 in \( R' \) and vice versa. Thus, there are \( t_{z'} \) points of weight 0 and \( t_{\bar{z}} \) points of weight 6 in \( R' \) (where each \( t_b \) refers to counts in \( R \), not in \( R' \)). So from (8), we have

\[
t_{z'} + t_{\bar{z}'} = \frac{1}{16} n^2 + \frac{1}{4} \sum_{i<j} g'_i g'_j = \frac{1}{16} n^2 + \frac{1}{4} \sum_{i<j} (-1)^{z_i+z_j} g_i g_j.
\]

We have thus shown that (7) holds when \( b \) has weight 0 or 2 in \( R \). Moreover, the truth of (7) is clearly preserved if we replace \( b \) by its complement. Since every point type in \( B^6 \) has weight 0, 2, 4, or 6, the result follows.

A relation \( R \) in a 6-net\((n)\) that ignores a parallel class, e.g. of type \((\lambda_0, \lambda_1, \ldots, \lambda_4, 0)\), is a relation on a 5-net\((n)\) that is a subnet of the 6-net\((n)\). In some ways, every relation on a 5-net\((n)\) can be thought of in this way, regardless of whether the 5-net\((n)\) embeds in a 6-net\((n)\). The reason is that the equations (2) – (5) still hold for any 5-net\((n)\) if we invent a sixth parallel class that has zero relational lines (it does not matter whether such a parallel class is actually achievable). By taking \( \lambda_5 = 0 \) we ensure that there can be no points incident with a relational line in the sixth parallel class. So for any \( b \in B^6 \) one of \( b \) or \( \bar{b} \) will necessarily be zero, and the other can be determined from Theorem 3.1. In other words, the number of points of each type is completely determined. This gives the following equations for a 5-net\((n)\):

\[
t_b = \frac{1}{16} n^2 + \frac{1}{8} n \sum_i (-1)^{b_i} g_i + \frac{1}{4} \sum_{i<j} (-1)^{b_i+b_j} g_i g_j \quad \text{for all } b \in B^5.
\] (9)

The same approach can be used when considering a 4-net\((n)\). In this case we can take \( \lambda_4 = \lambda_5 = 0 \), implying that there are no points incident with a relational line in either \( \Pi_4 \) or \( \Pi_5 \). This gives the following solution for the counts, by type, of all points in a 4-net\((n)\):

\[
t_b = \frac{1}{8} n^2 + \frac{1}{4} n \sum_i (-1)^{b_i} g_i + \frac{1}{4} \sum_{i<j} (-1)^{b_i+b_j} g_i g_j \quad \text{for all } b \in B^4.
\] (10)

Dukes and Howard [8] eliminated the following types of relations on 4 parallel classes: \( 2^4 \), \( 2^36 \), \( 2^24^2 \) and \( 24^26 \). In the first three of these cases, (10) gives \( t_{1111} < 0 \), providing an alternative way to see that such relations are impossible. Dukes and Howard also eliminated a relation on 5 parallel classes of type \( 2^44 \). In this case with \( b \in \{11101, 11011, 10111, 01111\} \) gives \( t_b < 0 \).

3.2 Templates for Relations of Type \( 4^4 \)

This subsection describes the structure of a template of type \( 4^4 \) that has a refinement to a 4-net\((10)\). The relational rows and relational columns of a template impose a quadrant structure on the frequency squares in the template. The 4 subarrays are defined as follows:

- \( Q_1 \) contains all points that are in relational rows and also in relational columns.
• $Q_2$ contains all points that are in relational rows and non-relational columns.
• $Q_3$ contains points that are in relational columns and non-relational rows.
• $Q_4$ contains points that are in non-relational rows and also in non-relational columns.

From (10) we know that $t_{0000} = 24$, $t_{1010} = t_{0110} = t_{1001} = t_{0101} = t_{0011} = 12$, and $t_{1111} = 4$. As every point in $Q_1$ is contained in a relational row and a relational column, $Q_1$ contains points of type 1100 and 1111 representing points of weight 2 and 4. Similarly, $Q_2$ contains points of type 1001 and 1010 and $Q_3$ contains points of type 0101 and 0110 representing the points of weight 2 contained in either a relational row or a relational column but not both. Lastly, the points in $Q_4$ are in non-relational rows and columns and thus have weight 0 or 2 and type 0000 or 0011. In the following lemmas, we will use the symbol $\ast$ as a wildcard which could represent either a 1 or 0 as required.

**Lemma 3.2.** Let $N$ be a 4-net(10) that is a refinement of a template $T$ of type $4^4$.

(i) The four points of weight 4 contained in template $T$ do not share any row or column.

(ii) Each row in $Q_2$ and column in $Q_3$ contains three points of type $\ast\ast01$ and three points of type $\ast\ast10$.

(iii) Each relational line in $\Pi_2 \cup \Pi_3$ is incident to exactly one point in $Q_1$ and three points in each of $Q_2$, $Q_3$, and $Q_4$.

(iv) Each column in $Q_2$ and row in $Q_3$ contains two points of type $\ast\ast01$ and two points of type $\ast\ast10$. Moreover, each row and column in $Q_4$ contains two points of type 0011 and four points of type 0000.

**Proof.** Consider a relational line $\ell$ with respect to a relation of type $4^4$ in a 4-net(10). The ten points on $\ell$ each have weight 2 or 4, and the total of their weights is $10 + 3 \times 4 = 22$. Hence there must be one weight 4 point and nine weight 2 points on $\ell$. Parts (i), (ii) and (iii) of the Lemma now follow easily.

For part (iv) we consider instead a non-relational line $\ell'$. Its points all have weight 0 or 2, with a total weight of $3 \times 4 = 12$. Hence it has six points of weight 2 and four points of weight 0. Each parallel class orthogonal to $\ell'$ contains six non-relational lines. Four of these six lines hit points of weight zero in $\ell'$ and the other two must hit points of weight 2. Part (iv) follows.

![Figure 1: A template of order 10, type $4^4$.](image)

Exhaustive computations found that there are 6,965 isomorphism classes of templates of type $4^4$ satisfying conditions (i), (ii) and (iv) of Lemma 3.2. See Figure 1 for one example. We stress that to generate this
catalogue we did not impose an assumption that templates must have a refinement to a net. Indeed, our next result will show that some of the 6,965 templates cannot be refined to a net.

**Lemma 3.3.** Let $T$ be the template given in Figure 1. There does not exist a net $N$ that is a refinement of $T$.

**Proof.** Suppose for the sake of contradiction that there exists a 4-net(10) $N$ that is a refinement of $T$. Let $\ell \in \Pi_2$ be a relational line in $N$ incident to the point $(0,0)$. By Lemma 3.2(iii), $\ell$ is incident to no other point in $Q_1$. It follows that $\ell$ is incident with one point in rows 4–6 of column 1. This implies that $\ell$ is incident with two points in rows 4–6 in $Q_4$. Similarly, $\ell$ is incident with two points in columns 4–6 in $Q_2$ and thus columns 4–6 contain one point incident to $\ell$ in $Q_4$. Thus point $(6,7)$ is incident to $\ell$.

Now let $\ell' \in \Pi_2$ be a relational line such that $\ell$ is incident to the point $(1,1)$. By the same logic used above, $\ell'$ is incident to point $(6,7)$. This is a contradiction as both $\ell$ and $\ell'$ are in the same parallel class and $\ell \neq \ell'$. It follows that there does not exist a net $N$ that is a refinement of $T$. □

The key point, guaranteed by Lemma 3.2, is that any 4-net(10) that satisfies a relation of type $4^2$ is isomorphic to a net that is a refinement of one of our 6,965 templates. The computations that we report in §5 will rely on this fact.

### 3.3 Templates for Odd Relations of Type $4^22^3$

In this subsection we consider templates for odd relations of type $4^22^3$. A significant difference between our task here and the one in the previous subsection is that the parallel classes now do not all have the same weight. Therefore, the ordering of the parallel classes makes a material difference. In a template the rows and columns play a different role to the other parallel classes as represented by the frequency squares. Hence, we need to consider three equivalent types, namely $2^34^2$, $242^4$, and $4^22^3$. A simple recursive backtracking algorithm, with isomorphism rejection, can be used to find the set of templates of each type. The sets of templates generated for $2^34^2$ and $242^4$ were prohibitively large, with the enumeration of templates of type $2^34^2$ not completing after producing billions of templates. However, we found that there are only 30 isomorphism classes of odd templates of type $4^22^3$. See Figure 2 for one example.

![Figure 2: A template of order 10, type $4^22^3$ corresponding to † in Table 2](image)

The following lemma describes the structure of a template $T$ of type $4^22^3$, and assisted in the generation of the 30 templates. The equation (9) for the number of points with a given type in a relation can be converted into an equation for the number of points of a given type in an odd relation by flipping the rightmost bit.
in the point types and exchanging all instances of \( g_5 \) with \(-g_5\). Doing so, we find that \( t_{10000} = t_{01000} = 24, t_{00100} = t_{00010} = t_{00001} = 12, \) and \( t_{11100} = t_{11010} = t_{11001} = t_{11011} = 4 \) for an odd relation of type \( 4^22^3 \). Using similar notation as in the previous subsection we then have:

**Lemma 3.4.** Consider a template of order 10, type \( 4^22^3 \).

(i) The points in quadrants \( Q_2 \) and \( Q_3 \) are of type \(*\star000\).

(ii) Every point in \( Q_1 \) is of type \( 11001, 11010, 11100, \) or \( 11111 \) with each of these types contained in each row and column once.

(iii) Every point in \( Q_4 \) is of type \( 00001, 00100, \) or \( 00100 \) with each of these types appearing twice in each row and column.

**Proof.** Part (i) follows immediately from the counts of each point type.

Next, consider the four points within a row or column of \( Q_1 \). These four points have total weight \( 4 + 4 + 3 \times 2 = 14 \). Hence they must consist of one point of weight 5 and three points of weight 3. Also, the four points must contain two which are relational for \( \Pi_i \), for \( 2 \leq i \leq 4 \). Part (ii) follows.

Part (iii) follows from the requirement for each non-relational row or column to meet each relational line in \( \Pi_2, \Pi_3 \) and \( \Pi_4 \).

Table 2 lists all 30 templates of type \( 4^22^3 \). Each template can be obtained by replacing quadrant \( Q_4 \) of Figure 2 with the \( 6 \times 6 \) subarray encoded in the appropriate column of Table 2. To decode, interpret each six digit number in the table as one row of the subarray and replace 1 with 100, 2 with 010 and 3 with 001. Other features of Table 2 will be explained when we consider it further, in Section 6.

4 Refinements of Templates

In this section we will discuss the computational techniques used to find all possible refinements of a template \( T \) to a net \( N \). We will first formalise the notion of a partial \( k \)-net \((n)\), which allows for a stepwise refinement by adding one line at a time.

A partial \( k \)-net \( \mathcal{P} = (P, L) \) of order \( n \) is an incidence structure consisting of a set \( P \) of \( n^2 \) points and a set \( L \) of at most \( kn \) lines satisfying the following conditions.

1. Every line is incident to \( n \) points.
2. Every point is incident to at most \( k \) lines.
3. Lines are either parallel or orthogonal.
4. The set of lines partition into \( k \) (possibly empty) sets of parallel lines.

It is easy to see that partial nets generalise nets as each set of parallel lines contains at most \( n \) lines. For a given partial net \( \mathcal{P} = (P, L) \), a set of parallel lines \( \Pi \subseteq L \) is a partial parallel class if \(|\Pi| < n\), otherwise the set is a parallel class. Let \( \mathcal{L}^\Pi \) denote the set of lines in \( \binom{n}{k} \) not contained in \( \Pi \) that are both orthogonal to all lines in \( L \setminus \Pi \) and parallel to every line in \( \Pi \). For any line \( \ell \in \mathcal{L}^\Pi \) we can construct a new partial net \( \mathcal{P}' = (P, L \cup \{\ell\}) \). Any partial net \( \mathcal{P}' = (P, L') \) with \( L \subseteq L' \) is an extension of \( \mathcal{P} \).

As a partial net can have empty partial parallel classes, a partial \( k \)-net is also a partial \((k + 1)\)-net, where the last partial parallel class is empty. A partial net is trivial if the partial net has exactly two parallel classes and every partial parallel class is empty. Up to isomorphism, the \( n \times n \) grid is the only trivial partial net of order \( n \). For \( k \geq 3 \), every \( k \)-net \((n)\) is an extension of the trivial partial \( k \)-net of order \( n \).

We now describe a simple exhaustive backtracking algorithm for generating all possible \( k \)-nets of order \( n \) using lines from within some predetermined set of lines. Starting with the trivial partial \( k \)-net \((n) \) \( N \), for each partial parallel class \( \Pi_i \) we first determine a set \( \mathcal{L}_{0,i} \subseteq \mathcal{L}^\Pi_i \). Elements of \( \bigcup_{i} \mathcal{L}_{0,i} \) will be referred to as candidate lines. We will say more about the choice of candidate lines below. Let \( \mathcal{L}_0 = (\mathcal{L}_{0,2}, \ldots, \mathcal{L}_{0,k-1}) \). Let \( S \) contain a list of indices of the partial parallel classes. Each index should be repeated in \( S \) as many times as the number of lines we wish to add to the partial parallel class of that index. The list will be used
The ordering of the elements within the search strategy has algorithmic implications that will be discussed below.

Algorithm 1 proceeds via recursion. At each stage $i$, we have a partial net $\mathcal{P}_i = (P, L_i)$ and a tuple $\mathcal{L}_i = (\mathcal{L}_{i,2}, \ldots, \mathcal{L}_{i,k-1})$ of sets of lines available for forming extensions of $\mathcal{P}_i$. In stage $i$ we select a new line from $\mathcal{L}_{i,S[i]}$ and add it to the partial parallel class $\Pi_{S[i]}$. For example, if $S = [2, 3, 2, 3, 2, 3, \ldots, 2, 3]$ then we would seek a 4-net$(n)$ by alternately extending the two partial parallel classes $\Pi_2$ and $\Pi_3$. We are only interested in partial nets that might extend to a $k$-net$(n)$, and in such a net every point must be incident with a line in $\Pi_{S[i]}$. So, our recursive step need only consider the set of lines that are incident to a single point $p$ that is not yet contained in a line of the partial parallel class $\Pi_{S[i]}$. To limit the branching factor of our recursive step we choose $p$ such that the number of lines incident to $p$ in $\mathcal{L}_{i,S[i]}$ is a minimum. The same heuristic was used by Mathon [1] and Best [1], and easily justifies the time required to identify $p$ in terms of the speed-up it achieves.

Let $j = S[i]$. If $p$ is not contained in any line of $\mathcal{L}_{i,j}$, then the point $p$ cannot be covered by any line and thus the partial parallel class $\Pi_i$ cannot be completed. Otherwise, for each line $\ell \in \mathcal{L}_{i,j}$ that is incident to $p$, we obtain a partial net $\mathcal{P}_{i+1} = (P, L_i \cup \{\ell\})$. Then, we obtain the set $\mathcal{L}_{i+1,j} \subseteq \mathcal{L}_{i,j}$ by taking every line in $\mathcal{L}_{i,j}$ that is disjoint from $\ell$. Also, for every partial parallel class $\Pi_m$ with $m \neq j$, we obtain the set $\mathcal{L}_{i+1,m} \subseteq \mathcal{L}_{i,m}$ by taking every line in $\mathcal{L}_{i,m}$ that is orthogonal to $\ell$. This gives the tuple $\mathcal{L}_{i+1} = (\mathcal{L}_{i+1,2}, \ldots, \mathcal{L}_{i+1,k-1})$. A recursive call on $\mathcal{P}_{i+1}$ and $\mathcal{L}_{i+1}$ is then used to determine if further extensions of the partial net $\mathcal{P}_{i+1}$ exist.

Table 2: The 30 odd templates of type $4^22^3$.

| Symmetry type | $|\text{Aut}(T)|$ | Encoding of $Q_4$ | $|\Omega'(T)|$ |
|--------------|-----------------|-------------------|---------------|
| $S_3$        | 24              | 112233 121323 223131 233112 312321 331212 1392 |               |
| $S_3$        | 48              | 112233 112323 231132 231312 321321 323211 5967 |               |
| $S_3$        | 96              | 112233 121323 231132 233112 312321 332112 361  |               |
| $S_3$        | 96              | 112233 121323 233112 231312 321321 331122 2932 |               |
| $S_3$        | 144             | 112233 121323 233112 232131 313212 331122 262  |               |
| $S_3$        | 192             | 112233 112323 231132 233112 312321 332112 1513 |               |
| $S_3$        | 288             | 112233 122313 233112 231122 331122 143 |               |
| $S_3$        | 1152            | 112233 112323 231132 231312 321321 323211 1887 |               |
| $S_3$        | 9216            | 112233 112323 231132 233112 321321 331122 342  |               |
| $C_2$        | 8               | 112233 232113 311232 321321 323121 123312 2943 |               |
| $C_2$        | 8               | 112233 112323 231132 233211 312321 331212 5924 |               |
| $C_2$        | 8               | 112233 112323 233112 231312 313212 331212 6027 |               |
| $C_2$        | 8               | 112233 132213 213213 231312 313212 11654 |               |
| $C_2$        | 16              | 112233 232113 311232 321321 323121 123321 1460 |               |
| $C_2$        | 16              | 112233 113322 231123 232311 312321 331212 2863 |               |
| $C_2$        | 16              | 112233 112323 231132 233211 312321 331212 2971 |               |
| $C_2$        | 16              | 112233 112323 231132 233211 312321 331212 3037 |               |
| $C_2$        | 32              | 112233 112323 231321 233112 321321 331212 2842 |               |
| $C_2$        | 32              | 112233 121323 212331 233211 312312 331122 2986 |               |
| $C_2$        | 48              | 112233 112323 231312 233121 321321 331212 1961 |               |
| $C_2$        | 64              | 112233 113322 232131 232131 312321 331212 2821 |               |
| $C_2$        | 64              | 112233 112323 231312 231321 323121 323121 6018 |               |
| $C_2$        | 96              | 112233 232113 312312 321321 331212 123321 926  |               |
| $C_2$        | 128             | 112233 112323 213213 231312 313212 331212 3048 |               |
| $C_2$        | 192             | 112233 113322 231213 233121 321321 332131 507  |               |
| $C_2$        | 192             | 112233 113322 232131 232131 312321 331212 1898 |               |
| –             | 8               | 112233 121323 213132 231321 323211 332112 2803 |               |
| –             | 8               | 112233 121323 231231 233112 323211 332112 5649 |               |
| –             | 8               | 112233 121323 231321 233112 323211 332112 11822 |              |
| –             | 16              | 112233 113322 231123 231231 322311 323112 5867 |               |
If every element of $S$ has been exhausted then we have reached the desired partial net and the result is recorded.

**Algorithm 1** Exhaustively search for partial nets within a given family of lines.

```plaintext
function Add_line($i, P_i = (P_i, L_i), L_i, S$)
  if $i \geq |S|$ then
    Record $P$
  else
    Let $\Pi$ be the partial parallel class indexed by $S[i]$
    Choose $p \in P$ not incident to any line in $\Pi$ and incident to the fewest lines in $L_i, S[i]$
    for $\ell \in L_i, S[i]$ incident to $p$ do
      Let $L_{i+1, S[i]}$ be the set of lines in $L_{i, S[i]}$ that are disjoint from $\ell$
      for all $j \neq S[i]$ do
        Let $L_{i+1, j}$ be the set of lines in $L_{i, j}$ that are orthogonal to $\ell$
      end for
      Let $L_{i+1} = (L_{i+1, 2}, \ldots, L_{i+1, k-1})$
      Add_line($i + 1, (P, L_i \cup \{\ell\}), L_{i+1}, S$)
    end for
  end if
end function
```

The algorithm as described thus far is quite general. For our specific purposes we will want to restrict the class of partial nets it generates by choosing candidate lines that respect a given template $T$ in a sense we now describe. If a $k$-net in $N$ refines $T$, then for each frequency square $F$ in $T$ the corresponding lines in $N$ hit points that have a constant value in $F$. Thus, for each given parallel class $\Pi_i$ and its assigned frequency square $F_i$, we restrict $L_{0,i}$ to be the set of lines in $L_i^{II}$ that contain only relational points or contain only non-relational points, relative to $F_i$. It follows that any net resulting from running Algorithm 1 with these candidate lines must satisfy the relation indicated by the template $T$. Similarly, if no net is returned from Algorithm 1 then there does not exist any net $N$ that refines $T$.

### 4.1 Symmetry Breaking

Algorithm 1 is sufficient for enumerating all 2-MOLS(10) that satisfy a relation, but it is too slow for dealing with symmetries of a template $T$ to reduce the computation time. Let $\text{Aut}(T)$ be the group of all automorphisms of $T$ induced by the template isomorphism relation. Suppose $L_0 = (L_{0,2}, \ldots, L_{0,k-1})$ and the candidate lines in $L = L_i \cup L_{0,i}$ have been selected with respect to $T$. Consider the induced action of $\text{Aut}(T)$ on $L_i$. For a line $\ell \in L_{0,2}$ we let $O_\ell = \{ \ell' \in L : \phi(\ell) = \ell' \text{ for some } \phi \in \text{Aut}(T) \}$ denote the orbit of $\ell$ under this action. As template isomorphism allows for the reordering of frequency squares, the orbit $O_\ell$ may or may not contain lines from $L_{0,j}$ where $j > 2$. Let $\ell^*$ denote the lexicographically least element of $L_{0,2} \cap O_\ell$, which we call the *orbit leader*. Suppose that a net $N$ is a refinement of $T$. By definition, if $N$ includes $\ell$ then $N$ is isomorphic to a refinement of $T$ that includes $\ell^*$. We therefore lose no generality in assuming that $N$ includes $\ell^*$. Moreover, we can impose an arbitrary ordering on $O = \{ O_\ell : \text{non-relational } \ell \in L_{0,2} \}$ and then assume that $\ell$ was chosen from the first orbit in $O$ that contains any non-relational line in $N$. Algorithm 2 enumerates partial nets that might extend to a net $N$ satisfying these additional assumptions. In particular, for each orbit in $O_\ell \in O$ we do one computation using Algorithm 1 where we first discard all orbits that are earlier in the ordering of $O$ and also assume that our partial net includes the orbit leader $\ell^*$ from $O_\ell$. This produces a dramatic speed-up compared to using Algorithm 1 on its own, especially for orbits which occur late in the ordering of $O$. The search strategy as specified by the list $S$, is chosen from a set of search strategies $S$ before invoking Algorithm 1. This choice, which affects efficiency but not correctness, will be described in more detail in Section 6. For the time being, we merely remark that there are reasons to choose different strategies at different stages of the computation.
Algorithm 2 Symmetry breaking

function Sym\_Break(\(P = (P, L), \mathcal{O}, \mathcal{L}, \mathcal{S}\))

for all \(O_i \in \mathcal{O}\) do

Let \(\ell^*\) be the orbit leader of \(O_i\)

Let \(P_0 = (P, L \cup \{\ell^*\})\)

for \(k \geq 2\) do

\(\mathcal{L}_{0,k}' = \mathcal{L}_{0,k} \setminus \bigcup_{j<i} O_j\)

end for

Let \(\mathcal{L}_{0,2}'\) be the set of lines in \(\mathcal{L}_{0,2}\) that are disjoint from \(\ell^*\)

for all \(k > 2\) do

Let \(\mathcal{L}_{0,k}'\) be the set of lines in \(\mathcal{L}_{0,k}'\) that are orthogonal to \(\ell^*\)

end for

\(\mathcal{L}^*_0 = (\mathcal{L}^*_{0,2}, \ldots, \mathcal{L}^*_{0,k-1})\)

Select appropriate \(S \in \mathcal{S}\)

AddLine(0, \(P_0, \mathcal{L}^*_0, S\))

end for

end function

5 Nets with Relations of Type 4^4

By construction, any 4-net(10) that satisfies a relation of type 4^4 must be isomorphic to one that is a refinement of one of the 6965 templates constructed in §3.2.

For each template \(T\), the set of candidate lines was computed using a simple recursive backtracking algorithm. This computation produced approximately 14,000 lines for each of the two frequency squares in \(T\). Algorithm \(\text{H}\) was then used to search for extensions of the trivial partial 4-net(10). Finally, the output of Algorithm \(\text{H}\) was screened for isomorphs. In this way we obtained a set \(\Omega\) of species representatives of all 2-MOLS(10) satisfying a relation of type 4^4. The search strategy \(S\) was chosen such that Algorithm \(\text{H}\) alternated between the partial parallel classes in the partial net. This choice of \(S\) proved much faster than attempting to complete the partial parallel classes one at a time. All templates that had a refinement into some net had refinements to at least 31 non-isomorphic nets and at most 46,161 non-isomorphic nets. Of the 6965 templates, only ten had no refinement to a net, including the template discussed in Lemma 3.3. By enumerating \(\Omega\), we found:

Lemma 5.1. There are 18,526,320 isomorphism classes of 4-nets of order 10 satisfying at least one non-trivial relation.

The dimension of each pair of MOLS in \(\Omega\) was computed. These results confirmed the results of Delisle [6], who, under the supervision of Wendy Myrvold, computed the set of all 2-MOLS(10) satisfying at least two linearly independent non-trivial relations (dimension at most 35). The number of 2-MOLS(10) with dimension at most 36 is listed in Table 3.

| Dimension | Species |
|-----------|---------|
| 34        | 6       |
| 35        | 85      |
| 36        | 18,526,229 |

Table 3: The pairs of MOLS in \(\Omega\) classified by dimension.

For each pair of MOLS in \(\Omega\) we computed the set of common transversals. No pair contained more than 5 common disjoint transversals. Table 3 shows the number of pairs of MOLS generated, partitioned according to the number of disjoint common transversals.

As no pair of MOLS in \(\Omega\) was extendible to a triple, we have the following theorem.
Disjoint Transversals | Transversals | Species |
--- | --- | --- |
0 | 0 | 14389542 |
1 | 1 | 3634655 |
1 | 2 | 274766 |
1 | 3 | 8601 |
1 | 4 | 164 |
1 | 5 | 3 |
1 | 6 | 6 |
1 | 7 | 8 |
1 | 8 | 12 |
1 | 9 | 1 |
1 | 10 | 8 |
1 | 11 | 1 |
1 | 12 | 5 |
1 | 13 | 1 |
1 | 16 | 1 |
2 | 2 | 185303 |
2 | 3 | 28395 |
2 | 4 | 1845 |
2 | 5 | 73 |
2 | 6 | 11 |
2 | 7 | 5 |
2 | 8 | 6 |
2 | 9 | 2 |
3 | 3 | 2339 |
3 | 4 | 466 |
3 | 5 | 45 |
3 | 6 | 15 |
3 | 7 | 6 |
3 | 8 | 6 |
3 | 10 | 1 |
3 | 12 | 1 |
3 | 17 | 1 |
3 | 18 | 1 |
3 | 21 | 4 |
4 | 4 | 11 |
4 | 5 | 3 |
4 | 7 | 1 |
4 | 8 | 2 |
4 | 10 | 2 |
5 | 7 | 1 |
5 | 19 | 1 |

Table 4: The MOLS in $\Omega$ classified by number of transversals and disjoint transversals.

**Theorem 5.2.** Let $N$ be a 4-net(10) that is a subnet of a 5-net(10). Then $N$ does not satisfy a non-trivial relation.

Inferring that we achieve equality in (1) in this situation, we have:

**Corollary 5.3.** Let $N$ be a 4-net(10) that is a subnet of a 5-net(10). Then $\dim(N) = 37$.

Two species in $\Omega$ have 5 disjoint common transversals, see Figures 3 and 4. Delisle [6] found 4 species of 2-MOLS(10) with 3 disjoint common transversals. Brown, Hedayat, and Parker [4] found a species of 2-MOLS(10) with 4 disjoint common transversals. Only one species of 2-MOLS(10) with more than 4 disjoint common transversals was previously known, namely the pair with 7 common disjoint transversals found by Egan and Wanless [9].
Figure 3: A 4-net(10) satisfying a relation of type $4^4$ with 7 common transversals and a set of 5 disjoint common transversals (coloured).

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |   |
| 1 | 2 | 5 | 8 | 7 | 9 | 4 | 3 | 6 | 0 |   |
| 2 | 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 |   |
| 3 | 4 | 7 | 1 | 5 | 8 | 3 | 9 | 0 | 2 |   |
| 4 | 5 | 1 | 2 | 6 | 0 | 9 | 8 | 7 | 3 |   |
| 5 | 6 | 9 | 8 | 2 | 0 | 4 | 1 | 7 |   |   |
| 6 | 7 | 4 | 0 | 9 | 1 | 8 | 6 | 3 | 5 |   |

Figure 4: A 4-net(10) satisfying a relation of type $4^4$ with 19 common transversals and a set of 5 disjoint common transversals (coloured).

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |   |
| 1 | 2 | 5 | 8 | 7 | 9 | 4 | 3 | 6 | 0 |   |
| 2 | 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 |   |
| 3 | 4 | 7 | 1 | 5 | 8 | 3 | 9 | 0 | 2 |   |
| 4 | 5 | 1 | 2 | 6 | 0 | 9 | 8 | 7 | 3 |   |
| 5 | 6 | 9 | 8 | 2 | 0 | 4 | 1 | 7 |   |   |
| 6 | 7 | 4 | 0 | 9 | 1 | 8 | 6 | 3 | 5 |   |

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Figure 5: A 4-net(10) contained in $\Omega'$ with 4 common transversals that are all disjoint (coloured) obtained from Figure 2. The 0 and 1 symbol lines are relational in the first square and the 0 and 2 symbol lines are relational in the second.

6 Nets with Odd Relations of Type $4^22^3$

While the set of templates of type $4^22^3$ is small, each template contains three frequency squares with approximately 74,000 candidate lines per frequency square. As the complexity of Algorithm 1 is dependent on the number of candidate lines, the time required to find all nets that are extensions of the trivial partial net using the candidate lines is prohibitively large. Since it is conjectured that no 5-net(10) exists, the computation of nets containing an odd relation of type $4^22^3$ is likely to return no output. For these two reasons we instead enumerated a certain set $\Omega'$ of 2-MOLS(10). This set is such that if there is a 5-net(10) satisfying an odd relation of type $4^22^3$ then it must contain a subnet isomorphic to a pair of MOLS in $\Omega'$. Given such a set, if no pair of MOLS in $\Omega'$ is extendible to a triple, then no 5-net(10) satisfies an odd relation of type $4^22^3$.

As $\Omega'$ will contain pairs of MOLS, we need to select 2 frequency squares and search for all possible refinements of these frequency squares. The choice of the frequency squares affects not just the efficiency of computing $\Omega'$, it affects the content of the set itself. For example, consider Figures 5 and 6 both of which show pairs of MOLS that are refinements of the frequency squares from the template in Figure 2. Figure 5 is a refinement of the first and second frequency squares, whereas Figure 6 is a refinement of the second and third frequency squares. As a consequence, it will turn out that the MOLS in Figure 5 were in our set $\Omega'$, but the MOLS in Figure 6 were not paratopic to any pair of MOLS in $\Omega'$.

To construct the set $\Omega'$ we first selected an ordering of the three frequency squares in each template. Then we determined the set of candidate lines with respect to the template. Next, we used Algorithm 2 to extend the trivial partial net to obtain pairs of MOLS that refine the first and second frequency squares. The construction of the orbits for the candidate lines, as described in Section 4.1, is heavily dependent on the parallel class $\Pi_2$. This is one reason why the ordering of the squares in the template affects the efficiency of the computation.

All search strategies supplied to Algorithm 2 first extended the trivial partial net by selecting the two relational candidate lines in $\Pi_4$; this is fast and greatly restricts the lines that are available for $\Pi_2$ and $\Pi_3$. The remaining details of the search strategy depended on the automorphism group of the template concerned, and will be discussed in detail below. The result of this computation is a set of partial 5-nets of order 10. The partial parallel class $\Pi_5$ was then discarded to give a set of pairs of MOLS. This set was screened to obtain a set $\Omega'(T)$ of species representatives of the MOLS that we obtained from the template $T$. Finally, the set $\Omega'$ was obtained by taking the union over all sets $\Omega'(T)$ for the 30 choices of template $T$ given in Table 2. It is clear from this process that if there exists a 5-net(10) $N$ satisfying an odd relation of type $4^22^3$.
Figure 6: A 4-net(10) that is a refinement of the second and third squares of the template in Figure 2. The pair is not contained in $\Omega'$ and contains 13 common transversals and a set of 3 disjoint common transversals (coloured). The 0 and 2 symbol lines are relational in the first square and the 0 and 3 symbol lines are relational in the second.

then there is a 4-net(10) in $\Omega'$ that is isomorphic to a subnet of $N$.

### 6.1 Symmetries in Templates

We say that a template of type $4^22^3$ has $S_3$ symmetry if the automorphism group of the template can induce any reordering of the three frequency squares in the template. A template admits a $C_2$ symmetry if there is an automorphism that exchanges two of the frequency squares but the template does not have $S_3$ symmetry. Of the 30 templates of type $4^22^3$, there are 9 possessing $S_3$ symmetry, 17 with a $C_2$ symmetry and the remaining 4 templates admit no automorphism that reorders the frequency squares.

Table 2 lists the templates that we used to compute $\Omega'$. For templates other than those with $C_2$ symmetry we had no reason to choose any ordering of the frequency squares over any other ordering, so we simply used them in the order that we had generated them. The situation for $C_2$ symmetry is more interesting. After some experimenting we discovered that it is best to use an ordering for which the $C_2$ symmetry exchanges the second and third frequency squares. However, we did not realise this until after we had computed all nets (up to isomorphism) that are refinements of the template marked with a * in Table 2 so for that template only we had the $C_2$ symmetry exchanging the first two frequency squares.

The reason it is better to have the $C_2$ exchanging the last two squares is related to the size and position of the orbits of candidate lines. Each of these orbits typically includes twice as many lines in the square that is fixed by the $C_2$ compared to each of the other two squares (individually). We found it was quicker to have larger orbits in the first square compared to orbits of the same size split between the first two squares. The worst option of all is to position the $C_2$ so that it exchanges the first and third square, since then half of each orbit is wasted because of the fact that we only ever add two lines to $\Pi'$. We used two different search strategies. Both found extensions of the trivial partial net by selecting two relational candidate lines in $\Pi'$ first, but after that they diverged. One strategy proceeded by alternating between the first two frequency squares. The other found a refinement of the first square to a parallel class before turning to the second frequency square. Our limited experimentation suggests that the alternating strategy is faster whenever the two squares have similar numbers of candidate lines remaining. This is always true at the start of the search, and it remains true throughout the search when the template has $S_3$ symmetry. For templates without $S_3$ symmetry (assuming we implement our preference above in the case of $C_2$ symmetry), the number of candidate lines becomes unbalanced as orbits are discarded from the first
At some point it becomes faster to switch to the non-alternating search strategy. The exact point at which this happens is unclear and was only roughly approximated. Our two implementations chose to switch strategies at different points.

6.2 Computational Results

Our computation of $\Omega'$ found $100,826$ species of 2-MOLS(10). For each pair of MOLS the common transversals were computed. No pair of MOLS had more than four disjoint common transversals. As the computation of $\Omega'$ began by adding the two relational lines to $\Pi_4$, each pair of MOLS has at least two disjoint common transversals. Table 5 lists the number of pairs of MOLS classified by number of common transversals and disjoint common transversals. Every pair of MOLS in $\Omega'$ has dimension 37, satisfying no non-trivial relations. For each pair of MOLS in $\Omega'$ we attempted to extend the pair to a triple in all possible ways (not just those consistent with the third frequency square). No extension was found, giving our second main result:

**Theorem 6.1.** Let $N$ be a 5-net(10). Then $N$ does not satisfy an odd relation of type $2^3 4^2$.

| Disjoint Transversals | Transversals | Species |
|-----------------------|--------------|---------|
| 2                     | 2            | 88,611  |
| 2                     | 3            | 8,317   |
| 2                     | 4            | 401     |
| 2                     | 5            | 11      |
| 3                     | 3            | 3,062   |
| 3                     | 4            | 387     |
| 3                     | 5            | 11      |
| 3                     | 6            | 1       |
| 3                     | 8            | 1       |
| 4                     | 4            | 22      |
| 4                     | 5            | 2       |

Table 5: The MOLS in $\Omega'$ classified by the number of transversals and disjoint transversals.

7 Concluding remarks

We have enumerated all 2-MOLS(10) that satisfy a non-trivial relation and found that none of them extend to 3-MOLS(10). In Theorem 6.1 we have also eliminated the possibility of 3-MOLS(10) satisfying a relation of type $22246$. There are nine other plausible types of relations on 4-MOLS(10) listed by Dukes and Howard [8]. If all nine could be eliminated then, by Theorem 1.1 it would follow that $N(10) \leq 3$. Unfortunately, that is not possible with our method. Based on evidence including Table 4 it seems a general rule that relations have greater flexibility (and hence require more computation to eliminate) if they have more of their $\lambda_i$ closer to $n/2$. In other words, we have eliminated only the easiest of the plausible relations on 3-MOLS(10), and we believe type $4^5$ to be very much harder to eliminate than type $22246$. Furthermore, we expect that eliminating relations on 4-MOLS(10) will be significantly harder again.

Nevertheless, we have made a useful start. As a glimpse of the possibilities our work might open up, consider the following situation. Suppose that $N_8$ is an 8-net(10) and that $N_6$ is the sub 6-net of largest dimension. Let $N_7$ be either of the sub 7-nets of $N_8$ that contain $N_6$. By [8, Prop. 2.5] we know that $\dim(N_7) \leq 53$. However, $N_7$ is not maximal, so by [7, Cor. 2.2], we have $\dim(N_6) < \dim(N_7)$. This means that $N_6$ has at least $t$ independent non-trivial relations on it, for some $t \geq 3$. By choice of $N_6$, every sub 6-net of $N_8$ has the same property.

By Theorem 5.2 we know that no sub 4-net of $N_8$ has a non-trivial relation. Each of the $\binom{8}{6} = 28$ sub 6-nets of $N_8$ has $t$ independent non-trivial relations on it, and each such relation appears on at most three different sub 6-nets. Hence, we have identified at least $28t/3$ different non-trivial relations on $N_8$. We can
see no reason why these relations should be independent. However, if they were, it would show that
\[55 - t = \dim(N_6) \leq \dim(N_8) \leq 73 - 28t/3.\]
This would contradict \( t \geq 3 \), showing that \( N_8 \) could not exist and demonstrating that \( N(10) \leq 5 \).

Finally, we note that the MOLS in \( \Omega \) and \( \Omega' \) can be downloaded from [17], as can the templates used to generate them.

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