Robust Control of a Multi-Degree-of-Freedom Electromechanical Plant with Adaptive Disturbance Compensation

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Abstract. This paper considers the problem of control of a multi-degree-of-freedom (multi-DOF) nonlinear electromechanical plant with elastic properties and an external unknown disturbance. The unknown disturbance is assumed to be deterministic and is represented as an output of a linear autonomous model with unknown constant parameters. To solve the problem of adaptive disturbance compensation, an adaptive observer of an unknown deterministic disturbance is applied. A nonlinear robust control of a multi-DOF elastic electromechanical plant, synthesized on the basis of the integrator backstepping method and combined with adaptive robust disturbance compensation, is also being developed. The simulation of the designed nonlinear control of a multi-DOF plant with adaptive robust compensation of disturbances is performed using the MATLAB / Simulink software.

Introduction
The paper considers the design of a nonlinear robust control system for a multi-DOF nonlinear electromechanical plant with elastic properties, effective within the conditions of immeasurable external disturbances. The increase in adaptive systems accuracy within the conditions of immeasurable external disturbances corresponds with the existing need for higher accuracy of technical plant control, therefore, the development of such systems is a topical research area. Since the late twentieth century, the methods for coarsening the adaptive and nonlinear control algorithms that provide exponential system stability in the absence of external disturbances and preserve the boundedness of all signals of their effects have been studied extensively. Among them are algorithms with the so-called σ-modification, namely, with quasi-negative feedback, with a dead zone and with switching [1–3], as well as projection algorithms [4, 5]. At the same time, coarsening algorithms are not meant for purposefully increasing the accuracy of adaptive systems, and in recent decades, methods for the external disturbance observer synthesis have been developed. Such observers act as a basis for solving the problems of adaptive disturbance compensation [2]. A similar approach is the internal model method that is used very widely. According to this method, if one assumes that the disturbances are deterministic, an external disturbance generator can be designed. Such a generator can act as a disturbance observer built into the control system that provides the solution for its compensation [2, 6]. However, compensation of disturbances with the help of observers is feasible with known parameters of the disturbance model and the plant, which contradicts the very formulation of the adaptive control problem. In the formulation of such problems, the assumption of parametric uncertainty of both the disturbance generators and the plant themselves is more natural. The research of the possibilities for extending the internal model methods to the classes of uncertain generators of
disturbances and plants are attributed to [6, 7]. This paper discusses the application of the methods described in [6, 7, 8] for designing adaptive observers of disturbances in terms of nonlinear multi-DOF elastic electromechanical plant control. Moreover, the method is used to design a nonlinear robust control system for the dynamics of a model of a multi-DOF electromechanical plant with elastic properties and unknown disturbance, represented by a five-stage structure adaptive integrator backstepping [1, 5, 8–10].

1. External Determined Disturbance Observer

The differential equations describing the motion of a multi-DOF electromechanical plant with elastic properties are the following:

\[
\begin{align*}
M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q) &= K_c (\theta - q) + d(t); \\
J\ddot{\theta} + K_c (\theta - q) &= I; \\
LI + RI + Kq &= u,
\end{align*}
\]  

where \( q \in \mathbb{R}^n \) – vector of joints rotation angles; \( \theta \in \mathbb{R}^n \) – vector of drive parts rotation angles; \( M(q) \in \mathbb{R}^{n\times n} \) – functional matrix of inertia, non-singular, symmetric and positive definite for all \( q \); \( C(q,\dot{q}) \in \mathbb{R}^{n\times n} \) – functional matrix of Coriolis and centrifugal forces; \( D(q) \in \mathbb{R}^n \) – vector function of gravitational forces; \( K_c \in \mathbb{R}^{n\times n}; J \in \mathbb{R}^{n\times n} \) – diagonal numerical matrices determined by the elastic coefficients of transmissions and the moments of drive parts inertia; \( d(t) \in \mathbb{R}^n \) – vector of unknown bounded external disturbances, \( \|d\| \leq \text{const} \); \( ||\| \) – the Euclidean norm; \( u_u \in \mathbb{R}^n, I_u \in \mathbb{R}^n \) – vectors of voltages and currents of the armature windings; \( L_a \in \mathbb{R}^{n\times n}, R_a \in \mathbb{R}^{n\times n}, M_m \in \mathbb{R}^{n\times n}, k_e \in \mathbb{R}^{n\times n} \) – diagonal matrices of inductances, active resistances of armature circuits and constant coefficients determined by the design data of electrical machines; \( n \) – the number of DOFs; \( I = k_M I_a; \quad L = L_a k_M^{-1}; \quad R = R_a k_M^{-1}; \quad K = k_e; \quad u = u_a – \text{synthesized control action.} \)

It is assumed that the plant is completely controllable and observable, and the components of the vectors \( q, \dot{q}, \theta, \dot{\theta}, I \) are measurable. Let \( x_1 = q, x_2 = \dot{q}, x_3 = \theta, x_4 = \dot{\theta}, x_5 = I \).

The system of equations of the plant (1) could be reduced to a cascade form as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2; \\
\dot{x}_2 &= M^{-1} [-C x_2 - D + K_c (x_3 - x_1) + d(t)]; \\
\dot{x}_3 &= x_4; \\
\dot{x}_4 &= J^{-1} (x_5 + K_c x_1 - K_c x_3); \\
\dot{x}_5 &= L^{-1} (u - R x_5 - K x_2),
\end{align*}
\]  

where \( M = M(x_1), \quad C = C(x_1, x_2), \quad D = D(x_1). \)

Let the deterministic model of the unknown disturbance \( d \) be considered as a vector whose components are bounded periodic functions of the form:

\[
d_i(t) = \sum_{j=1}^{k} b_{i,j} \sin(\omega_{i,j} t + \phi_{i,j}),
\]

With each of them being a sum \( k \) of harmonics with unknown frequencies \( \omega_{i,j} \), amplitudes \( b_{i,j} \) and initial phases \( \phi_{i,j} \). Hereinafter, the symbols \( j, i \) refer to the harmonics’ and DOFs’ numbers –
\( j = 1, k \) and \( i = 1, n \), respectively. According to [6,7], let us assume that the disturbance \( d \) is representable as the output of a linear finite-dimensional generator:

\[
\dot{x} = W \chi; \quad d = V \chi,
\]

where \( \chi \in \mathbb{R}^n \) – state space vector of the generator; \( W \in \mathbb{R}^{m \times m} \), \( V \in \mathbb{R}^{n \times m} \) – matrices of unknown coefficients, and the matrix \( W \) has all eigenvalues on the imaginary axis; the pair \((W, V)\) is completely observable. According to [6, 7], vector of the unknown disturbance \( d \) could be represented in the synthesis of the desired observer of an unknown disturbance as an output of the following parameterized model:

\[
\dot{\zeta} = G\zeta + Nd; \quad d = \Theta^T \zeta,
\]

where \( G \in \mathbb{R}^{m \times m} \) – arbitrary Hurwitz matrix; \( \Theta \in \mathbb{R}^{m \times n} \) – unknown matrix of constant coefficients, state vector \( \zeta \in \mathbb{R}^m \) related to the state vector \( \chi \in \mathbb{R}^n \) model (3) by the similarity ratio \( \zeta = H \chi \), \( H \in \mathbb{R}^{m \times m} \) – non-degenerate matrix that is the only solution to the matrix equation:

\[
HW - GH = NV,
\]

\( G, N \) – a completely controllable pair, \( \Theta = (VH^{-1})^T \).

A real non-linear observer for the inaccessible regressor \( \zeta \) is synthesized taking into account the following equations of a nonlinear multi-DOF elastic electromechanical plant (2) [6, 7, 10]:

\[
\dot{\zeta} = \eta + H M x_2,
\]

\[
\dot{\eta} = G \eta + G H M x_2 - H M x_2 + H [C x_2 + D - K_c (x_3 - x_1)],
\]

where \( \hat{\zeta} \) – vector \( \zeta \) estimate; \( \eta \) – auxiliary dynamic filter state vector (6).

Unknown disturbance vector \( \hat{d} \) could be represented as [6, 7]:

\[
\hat{d} = \Theta^T \zeta + e_d.
\]

Consequently, let us we introduce the vector of the regressor estimation error \( \hat{\zeta} \) (5):

\[
\hat{\zeta} = \zeta - \hat{\zeta}.
\]

Substituting \( \zeta = \hat{\zeta} + \hat{\zeta} \) in (4) and considering (7), we obtain \( e_d = \Theta^T \hat{\zeta} \). Differentiating (8) and considering (2), (4), (5) and (6), we obtain:

\[
\dot{\hat{\zeta}} = G \chi + Nd - \hat{\zeta} - N M x_2 - N M x_2 = G [\zeta - \eta - N M x_2] = G (\zeta - \hat{\zeta}) = G \hat{\zeta},
\]

whence, since the matrix \( G \) is Hurwitz, convergence follows \( \hat{\zeta} \rightarrow 0 \) while \( t \rightarrow \infty \).

Let us note that, within the framework of the considered problem, the matrices \( W \) and \( V \) of the generator (3) are unknown, and therefore the matrix \( \Theta \) is also unknown. Thus, the uncertainty of the external disturbance \( d \) is reduced to the uncertainty of the constant matrix of parameters \( \Theta \) of the parameterized disturbance model (4). The linear regression model of the form (4) with an unknown matrix of constant coefficients \( \Theta \) is widespread in problems of adaptive identification and adaptive control, therefore, to compensate for the disturbance represented in the form (4), the well-known methods of adaptive control could be applied [1].

Let us apply the integrator backstepping method to design an adaptive robust control system for a nonlinear elastic multi-DOF electromechanical plant with disturbance compensation [1, 7, 8, 10].
2. Adaptive Robust Control of a Multi-DOF Nonlinear Electromechanical Plant with Elastic Properties

Let us introduce the so-called virtual controls used in the iterative (step-by-step) synthesis of the integrator backstepping method \( \alpha_i \in R^n \), \( i = 1, 4 \). In addition, in the synthesis of the integrator backstepping method, the proposal expressed and substantiated in [11, 12] will be applied. Instead of the “pure” (physically unrealizable) derivatives of virtual controls appearing at each synthesis step \( \dot{\alpha}_i \), \( i = 2, 3, 4 \), starting from the second, let us use the “real” derivatives \( \dot{\alpha}_i^c \) (filtered analogues). This would make it possible to significantly reduce the volume of cumbersome calculations for replacing the “pure” derivatives with their analytical expressions designed according to the rules for complex function derivative calculations [1, 8, 9]. It was proposed in [11, 12] to implement such filtering of “pure” derivatives by a second-order filter having the form:

\[
\begin{align*}
\ddot{e}_1 &= e_2, \\
\ddot{e}_2 &= -2vbe_2 + b^2 (\alpha_i - e_i),
\end{align*}
\]

in which (with zero initial conditions \( e_i(0) = \alpha_i(0) \) and \( e_2(0) = 0 \)) will be \( e_i = \alpha_i^c \), \( e_2 = \dot{\alpha}_i^c \), where \( b \in R^+ \) – eigenfrequency of the filter, \( v \in R^+ \) – damping factor \( i = 2, 3, 4 \). Let us also introduce the notation for filtering errors \( c_i = \alpha_i^c - \alpha_i \).

**Step 1.** Let us introduce the new variables:

\[
z_1 = x_1 - x_d, \quad z_2 = x_2 - \alpha_i,
\]

where \( x_d \) – desired signal \( x_i \). Differentiating \( z_1 \), we obtain:

\[
\dot{z}_1 = \dot{x}_1 - \dot{x}_d = z_2 + \alpha_1 - \dot{x}_d.
\]

Let us consider a Lyapunov function of the form \( V_1 = 0.5z_1^Tz_1 \). Calculating its total derivative by the virtue of the equation in new variables (12), we obtain:

\[
\dot{V}_1 = -z_1^T \alpha_1 + z_1^T (z_2 + \alpha_1 - \dot{x}_d).
\]

Let us choose the virtual control \( \alpha_1 \) as:

\[
\alpha_1 = -k_1 z_1 + \dot{x}_d,
\]

where \( k_1 \in R^{n,n} \) is a positive definite symmetric matrix, we obtain:

\[
\dot{V}_1 = -z_1^T k_1 z_1 + z_1^T z_2,
\]

and the equation of the closed loop system in new variables at the first step will be the following:

\[
\dot{z}_1 = -k_1 z_1 + z_2.
\]

**Step 2.** Introducing a new variable:

\[
z_3 = x_3 - \alpha_i^c.
\]

Differentiating \( z_2 \) by (11) and taking into account (2), (7), (16) and the fact that \( \varepsilon_d = \Theta^T \xi \), we obtain:

\[
\dot{z}_2 = \dot{x}_2 - \dot{x}_1 = M^{-1} \left[ K_c (z_3 + \alpha_i^c - x_i) - M \dot{\alpha}_1 - Cx_2 - D + \Theta^T \dot{\xi} + \Theta^T \xi \right],
\]

where \( \dot{\alpha}_1 = -k_1 (x_2 - \dot{x}_d) + \dot{z}_d \). Let us consider a Lyapunov function of the form:

\[
V_2 = V_1 + 0.5z_2^T M z_2.
\]

Calculating the total derivative of the function \( \dot{V}_2 \) by virtue of equation (17) and taking into account (14), we obtain:

\[
\dot{V}_2 = -z_1^T k_1 z_1 + z_1^T z_2 + z_2^T \left[ K_c (z_3 + \alpha_i^c - x_i) - M \dot{\alpha}_1^c - C\dot{\alpha}_1 - D + \Theta^T \dot{\xi} + \Theta^T \xi \right] + 0.5z_2^T (M - 2C) z_2.
\]
It is known that $\hat{M} - 2C$ is a skew-symmetric matrix, therefore, $0.5z_2^T(\hat{M} - 2C)z_2 = 0$ for any vector $z_2$. Then:

$$V_2 = -z_1^T k_1 z_1 + z_1^T z_2 + z_2^T K_c (z_3 + \alpha_2^T + x_1) - M \alpha_1^T - C \alpha_1^T - D + \Theta^T \dot{\zeta} + \Theta^T \zeta.$$

Let us choose the virtual control law $\alpha_2$ as:

$$\alpha_2 = K_c^{-1} (-z_1 - k_2^* z_2 + M \alpha_1^T + C \alpha_1^T + D - \hat{\Theta}^T \dot{\zeta} + K_c x_1),$$  \hspace{1cm} (18)

where $k_2^* \in R^{n \times n}$ -- positive definite symmetric matrix, matrix $\hat{\Theta} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n]$ with columns $\hat{\theta}_i$ is the estimate of the unknown matrix $\Theta$. Consider the following adaptive robust algorithms with the so-called parametric projection for adjusting the columns of the evaluation matrix $\hat{\Theta}$ [4, 5]:

$$\hat{\theta}_i = \Gamma_i \text{Proj}(-z_{2,i} \dot{\zeta}, \hat{\theta}_i), i = \overline{1,n},$$  \hspace{1cm} (19)

where $\hat{\theta}_i, i = \overline{1,n} - i$-th columns of evaluation matrix $\hat{\Theta}$; $\Gamma_i = \Gamma_i^T \in R^{n \times m}$ -- symmetric positive definite matrices, $z_{2,i}$ -- $i$-th element of vector $z_2, i = \overline{1,n}$, $\text{Proj}(-z_{2,i} \dot{\zeta}, \hat{\theta}_i)$ -- projection operators for $i$-th columns $\hat{\theta}_i$ of matrix $\hat{\Theta}$, which are set as follows [4, 5]. Let us introduce the notations -- vectors $w = -z_{2,i} \dot{\zeta}; \theta = \theta_i$ for simplicity of further notation and let the vector of coefficients $\theta$ belong to the compact convex set $\Omega := \{\theta: \|\theta\| \leq \theta_0\}$, where $\theta_0$ -- known positive constant, $\hat{\theta} = \theta - \dot{\theta} - \hat{\theta}$ estimation error. Then the projection operator (19) is defined as follows:

$$\text{Proj}(w, \hat{\theta}) = \begin{cases} w, & \text{if} \quad p(\hat{\theta}) < 0; \\ w - \frac{p(\hat{\theta})\nabla p(\hat{\theta})\nabla p(\hat{\theta})^T}{\nabla p(\hat{\theta})^T}, & \text{if} \quad p(\hat{\theta}) \geq 0 \land \nabla p(\hat{\theta})^T w > 0; \\ w - \frac{\theta_0^2}{\varepsilon^2 + 2\varepsilon \theta_0}, & \text{if} \quad p(\hat{\theta}) \geq 0 \land \nabla p(\hat{\theta})^T w \leq 0, \end{cases}$$

where $\varepsilon$ -- arbitrary positive constant and $\nabla p(\hat{\theta})$ -- gradient of function $p$ by $\theta$. Let us also use the following property of the projection operator [5]:

$$-\hat{\theta}^T (\text{Proj}(w, \hat{\theta})) \leq -\hat{\theta}^T w.$$  \hspace{1cm} (20)

Taking into account (18), we write down the derivative $\dot{V}_{2a}$ as:

$$\dot{V}_2 = -z_1^T k_1 z_1 - z_1^T \dot{z}_2 + z_2^T K_c z_3 + z_2^T K_c \alpha_2 + z_2^T \Theta^T \dot{\zeta} + z_2^T \Theta^T \zeta.$$

Let us choose a positive definite Lyapunov function in the form:

$$V_{2a} = V_2 + 0.5 \sum_{i=1}^{n} \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i + \zeta^T \hat{Q} \zeta,$$

where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i, i = \overline{1,n}$, and matrix $\hat{Q} = \hat{Q}^T > 0$ is a solution to the matrix equation:

$$G^T \hat{Q} + \hat{Q} G = -\Theta \Theta^T.$$  \hspace{1cm} (23)

Calculating total derivative $\dot{V}_{2a}(t)$ and considering (19)-(21), we obtain:
\[ \dot{V}_{2a} = \dot{V}_2 + \sum_{i=1}^{n} \theta_i^T \Gamma_i^{-1} \dot{\theta}_i + \dot{\zeta}^T Q \dot{\zeta} + \dot{\zeta}^T \dot{Q} \dot{\zeta} \leq -z_1^T k_1 z_1 - z_2^T k_2 z_2 + z_2^T K_c z_3 + z_2^T K_c \alpha_2 + z_2^T \Theta \dot{\zeta} + \dot{\zeta}^T Q \dot{\zeta} + \dot{\zeta}^T \dot{Q} \dot{\zeta}. \]  

Using the Young's inequality [13], we obtain:

\[ z_2^T \Theta \dot{\zeta} \leq 0.25 z_2^T z_2 + \zeta^T (\Theta \Theta^T ) \zeta. \]  

From (9) and (23) we shall obtain:

\[ \zeta^T Q \dot{\zeta} + \dot{\zeta}^T Q \dot{\zeta} = \zeta^T G^T Q \zeta + \dot{\zeta}^T Q G \zeta = -\zeta^T (\Theta \Theta^T ) \zeta. \]  

Substituting (25) and (26) into (24), we obtain:

\[ \dot{V}_{2a} \leq -z_1^T k_1 z_1 - z_2^T k_2 z_2 + 0.25 z_2^T z_2 + z_2^T K_c z_3 + z_2^T K_c \alpha_2 + \zeta^T (\Theta \Theta^T ) \zeta - \zeta^T (\Theta \Theta^T ) \zeta = -z_1^T k_1 z_1 - z_2^T \left( k_2^* - 0.25 I_{n \times n} \right) z_2 + z_2^T K_c z_3 + z_2^T K_c \alpha_2. \]  

For generating \( \alpha_2^c \) and \( \alpha_2^c \), we pass \( \alpha_2 \) through a second-order filter of the form (10). Considering that \( x_3 = z_3 + \alpha_2^c \) and \( \tilde{\alpha}_2 = \alpha_2^c - \alpha_2 \) and considering (17), the equation of the closed loop system in new variables at Step 2, may be written as:

\[ \dot{z}_2 = M^{-1} \left[ -z_1 - k_2^* z_2 + K_c z_3 + K_c \tilde{\alpha}_2 + C \tilde{z}_2 + \tilde{\Theta}^T \dot{\zeta} + \Theta \dot{\zeta} \right]. \]  

**Step 3.** Introducing a new variable:

\[ z_4 = x_4 - \alpha_3^c. \]  

Differentiating \( z_3 \) by (16) and substituting the derivative \( \dot{x}_3 \) from the equation (2) and considering (29) we obtain:

\[ \dot{z}_3 = \dot{x}_3 - \dot{\alpha}_3^c = z_4 + \alpha_3^c - \dot{\alpha}_3^c. \]  

Let us consider a Lyapunov function of the form \( V_3 = V_{2a} + 0.5 z_3^T z_3 \). Calculating its total derivative by the virtue of equation (30) and taking into account (27), we obtain:

\[ \dot{V}_3 = \dot{V}_{2a} + z_3^T \dot{z}_3 \leq -z_1^T k_1 z_1 - z_2^T k_2 z_2 + z_2^T K_c z_3 + z_2^T K_c \tilde{\alpha}_2 + z_3^T (z_4 + \alpha_3^c - \dot{\alpha}_3^c), \]  

where \( k_2 = k_2^* - 0.25 I_{n \times n} \). Let us choose a virtual control law \( \alpha_3 \) as:

\[ \alpha_3 = -k_3 z_3 - K_c z_2 + \tilde{\alpha}_2^c, \]  

where \( k_3 \in R^{m \times n} \) – positive definite symmetric matrix. For generating \( \alpha_3^c \) and \( \alpha_3^c \), we pass \( \alpha_3 \) through a second-order filter of the form (10). Taking into account that \( \tilde{\alpha}_3 = \alpha_3^c - \alpha_3 \), let us rewrite the derivative \( \dot{V}_3 \) (31) and closed loop system equations (30) in new variables at Step 3 in the form:

\[ \dot{V}_3 \leq -\sum_{i=1}^{3} z_i^T k_i z_i + z_3^T z_4 + z_2^T K_c \tilde{\alpha}_2 + z_3^T \tilde{\alpha}_3; \]  

\[ \dot{z}_3 = z_4 - k_3 z_3 - K_c z_2 + \tilde{\alpha}_3. \]  

**Step 4.** Introducing a new variable:

\[ z_5 = x_5 - \alpha_4^c. \]  

Differentiating \( z_4 \) by (29) and substituting the derivative \( \dot{x}_4 \) from the equation (2) and considering (35) we obtain:
\[ \dot{z}_4 = \dot{x}_4 - \dot{\alpha}_3^c = J^{-1}(x_5 + K_c x_1 - K_c x_3) - \dot{\alpha}_3^c = J^{-1}(z_5 + \alpha_4^c + K_c x_1 - K_c x_3) - \dot{\alpha}_3^c. \]  

(36)

Let us consider a Lyapunov function of the form \( V_4 = V_3 + 0.5z_4^T Jz_4 \). Calculating its total derivative by the virtue of equation (36) and taking into account (33), we obtain:

\[ \dot{V}_4 = \dot{V}_3 + z_4^T J z_4 \leq -3 \sum_{i=1}^{3} z_i^T k_i z_i + z_3^T z_3 + z_3^T K_c \tilde{\alpha}_2 + z_3^T t_3^\alpha + z_4^T t_4^\alpha (z_5 + \alpha_4 + \tilde{\alpha}_4 - K_c (x_3 - x_1) - J \tilde{\alpha}_3^c). \]

Let us choose a virtual control law \( \alpha_4 \) as:

\[ \alpha_4 = -k_c z_4 - z_3 + K_c (x_3 - x_1) + J \dot{\alpha}_3^c. \]  

(37)

For generation \( \alpha_4^c \) and \( \dot{\alpha}_4^c \) we put \( \dot{\alpha}_4 \) through a second-order filter (10). Taking into account that \( \tilde{\alpha}_4 = \alpha_4^c - \alpha_4 \), let us rewrite the derivative \( \dot{V}_4 \) and closed loop system equations (36) in new variables at Step 4 in the form:

\[ \dot{V}_4 \leq -3 \sum_{i=1}^{4} z_i^T k_i z_i + z_4^T z_5 + z_5^T K_c \tilde{\alpha}_2 + z_3^T \tilde{\alpha}_3 + z_4^T \tilde{\alpha}_4; \]  

(38)

\[ \dot{z}_4 = J^{-1}(z_5 - k_c z_4 - z_3 + \tilde{\alpha}_4). \]  

(39)

**Step 5.** Differentiating \( z_5 \) by (35) and substituting the derivative \( \dot{x}_5 \) from the equation (2), we obtain:

\[ \dot{z}_5 = \dot{x}_5 - \dot{\alpha}_3^c = L^{-1}(u - Rx_5 - Kx_2 - L \dot{\alpha}_3^c). \]  

(40)

Let us consider a Lyapunov function of the form \( V_5 = V_4 + 0.5z_5^T L z_5 \). Calculating its total derivative by the virtue of equation (40) and taking into account (38), we obtain:

\[ \dot{V}_5 = \dot{V}_4 + z_5^T L z_4 \leq -4 \sum_{i=1}^{4} z_i^T k_i z_i + z_5^T z_5 + z_5^T K_c \tilde{\alpha}_2 + z_3^T \tilde{\alpha}_3 + z_4^T \tilde{\alpha}_4 + z_5^T (u - Rx_5 - Kx_2 - L \dot{\alpha}_3^c). \]

Let us choose the final control law \( u \) as:

\[ u = -k_c z_5 - z_4 + Rx_5 + Kx_2 + L \dot{\alpha}_3^c. \]  

(41)

Then:

\[ \dot{V}_5 \leq -4 \sum_{i=1}^{5} z_i^T k_i z_i + z_5^T K_c \tilde{\alpha}_2 + z_3^T \tilde{\alpha}_3 + z_4^T \tilde{\alpha}_4. \]  

(42)

For any \( \mu \in R^+ \) there is a fairly large \( b \) in (10), so that \( \tilde{\alpha}_i \leq \mu \) [12, 13]. Therefore, the inequality (42) could be rewritten as:

\[ \dot{V}_5 \leq -4 \sum_{i=1}^{5} z_i^T k_i z_i + K_c |\mu| z_5 \leq \|z_5\| + \mu \|z_4\|. \]  

(43)

Using the Jung's inequality [13], we obtain:

\[ \dot{V}_5 \leq -k_{c1} \|z_2\|^2 - k_{c2} \|z_2\|^2 / 2 - k_{c3} \|z_3\|^2 / 2 - k_{c4} \|z_4\|^2 / 2 - k_{c5} \|z_5\|^2 + \]

\[ + (k_{00})^2 / (2k_{c2}) + (\mu)^2 / (2k_{c3}) + (\mu)^2 / (2k_{c4}), \]

where \( k_{0} := \lambda_{\max}(K_c) ; k_{ci} := \lambda_{\min}(k_i), i = 1,5. \)

With \( k_c := \min \{2k_{c1}, k_{c2}, k_{c3}, k_{c4}, 2k_{c5}\} \), we obtain:

\[ \dot{V}_5 \leq -k_c \|z_2\|^2 / 2 + (k_0^2 + 2\mu^2) / (2k_c), z = col(z_i), i = 1,5. \]  

(45)

Considering (41), let us finally write down (40) as:
\[ \dot{z}_5 = \dot{x}_5 - \dot{a}_5^c = -L^{-1}k_5z_5 - L^{-1}z_4. \]  
\[ (46) \]

It could be shown using standard arguments that by choosing the parameter \( b \), characterizing the passband of the second order filter (10), admissible (limited) negative feedbacks of the matrices \( k_i \) and limited input program controls \( x_{id} \), all signals of the synthesized system will be bounded and its state variables, according to the well-known La Salle invariance principle [1], converge to the maximum invariant set determined by the zeros of the derivative \( V_\varepsilon \), which is equivalent to
\[ \lim_{x \to \infty} z_i(t) = 0, i = 1,5. \]  
In this case, the rate of the process convergence will be higher than \( \exp(-0.5k_\varepsilon t) \), where \( t \) – time, \( k_\varepsilon \in R_+ \), and increasing \( k_\varepsilon \) convergence, the speed increases.

Summarizing the results of the synthesis, we indicate the following main results:

a) step-by-step closed-loop equations in \( z_i, i = 1,5 \) in newcoordinates of the form (15), (28), (34), (39) and (46):
\[ \begin{aligned} 
\dot{z}_1 &= -k_1z_1 + z_2; \quad \dot{z}_2 = M^{-1} \left[ -z_1 - k_2^*z_2 + K_\varepsilon z_3 + K_\varepsilon \alpha_2 - Cz_2 + \Theta^T \ddot{z}^c + \Theta^T \ddot{c} \right]; \\
\dot{z}_3 &= z_4 - k_3z_3 - K_\varepsilon z_2 + \alpha_3; \quad \dot{z}_4 = J^{-1} (z_5 - k_4z_4 - z_3 + \alpha_4); \quad \dot{z}_5 = -L^{-1}k_5z_5 - L^{-1}z_4.
\end{aligned} \]
b) step-by-step virtual and final control laws of the form (18), (23), (33), (36) and (39):
\[ \begin{aligned} 
\alpha_1 &= -k_1z_1 + \dot{x}_d; \quad \alpha_1^c = \alpha_1; \quad \dot{\alpha}_1^c = -k_1(\dot{x}_1 - \dot{x}_d) + \ddot{x}_d; \\
\alpha_2 &= K_\varepsilon^{-1} (z_1 - k_2^*z_2 + M \alpha_1^c + Ca_1^c + D - \Theta^T \ddot{z}^c + K_\varepsilon x_1); \quad \ddot{\alpha}_2^c = -2v_\varepsilon \alpha_2^c + b^2 (\alpha_2 - \alpha_2^c); \\
\alpha_3 &= -k_3z_3 - K_\varepsilon z_2 + \alpha_3^c; \quad \ddot{\alpha}_3^c = -2v_\varepsilon \alpha_3^c + b^2 (\alpha_3 - \alpha_3^c); \\
\alpha_4 &= -k_4z_4 - z_3 + K_\varepsilon (x_3 - x_1) + J \dot{\alpha}_3^c; \quad \ddot{\alpha}_4^c = -2v_\varepsilon \alpha_4^c + b^2 (\alpha_4 - \alpha_4^c); \\
u &= -k_5z_5 - z_4 + Rx_5 + Ks_2 + L \dot{\alpha}_4^c.
\end{aligned} \]
c) adaptive robust adjustment algorithm of the form (19):
\[ \hat{\theta}_i = \Gamma_i \text{Proj} (-z_{2,i}^\varepsilon \hat{c}, \hat{\theta}_i), i = 1,n; \quad \hat{\Theta} = [\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_n] \]
d) non-linear disturbance observer (5), (6):
\[ \{ \hat{z}^c = \eta + H\dot{x}_2; \quad \dot{\eta} = G\eta + GH\dot{x}_2 - H\dot{\dot{x}}_2 + H [C\dot{x}_2 + D - K_\varepsilon (x_3 - x_1)] \].

3. The computer study of the resulting adaptive robust control system for a multi-DOF elastic electromechanical plant with disturbance compensation as exemplified by a four-DOF manipulation arm.

The simulation was carried out with the following numerical parameters of the manipulator joints (see Fig. 1): \( m_1=20 \) kg; \( m_2=15 \) kg; \( m_3=12 \) kg; \( m_4=8 \) kg; \( l_1=0.4 \) m; \( l_2=1.5 \) m; \( l_3=1.2 \) m; \( l_4=0.6 \) m; \( r_1=0.2 \) m; \( r_2=0.7 \) m; \( r_3=0.6 \) m; \( r_4=0.3 \) m; \( g=9.8 \) m/s\(^2\); \( L_{ai}=0.001 \) H; \( R_{ai}=1 \) Ohm; \( K_{li}=1 \) N.m/A; \( J=0.02 \) kg.m\(^2\); \( K_{ci}=10000 \) N.m/rad; \( k_{ei}=10 \) V.rad/s; \( i=1,4 \).

For an adaptive robust control system built on the basis of the integrator backstepping method with algorithms with parametric projection, we take:
\[ k_1 = k_1^T \in R^{4 \times 4}, \quad k_2^* = k_2^T \in R^{4 \times 4}, \quad k_3 = k_3^T \in R^{4 \times 4}, \quad k_4 = k_4^T \in R^{4 \times 4}, \quad k_5 = k_5^T \in R^{4 \times 4}, \quad \Gamma_i = \Gamma_i^T \in R^{16 \times 16} (i = 1,4), \quad \hat{\Theta} = [\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4] \in R^{16 \times 4}, \]
\[ \hat{\theta}_i = [\hat{\theta}_{1,i}, \hat{\theta}_{2,i}, ..., \hat{\theta}_{12,i}]^T \in R^{16} (i = 1,4) \quad \text{i-th column}; \]

Fig.1. Four-DOF manipulation arm model
The subsystem parameters for adaptive robust disturbance compensation with parametric projection: \( \Gamma_1 = 0.18I_{6 \times 16}; \Gamma_2 = 0.28I_{6 \times 16}; \Gamma_3 = 0.33I_{6 \times 16}; \Gamma_4 = 0.25I_{6 \times 16} \). The projection operator parameters for \( i = 1, 4 \): 
\[ \epsilon_{p_1} = \epsilon_{p_1} = \epsilon_{p_3} = \epsilon_{p_4} = 0.1; \theta_{01} = \theta_{02} = \theta_{03} = \theta_{04} = 30. \]

Filter parameters (10): 
\[ v = 1; \quad b = 0.05; \quad k_1 = diag(4, 4, 4, 4), \quad k_2 = diag(25, 25, 25, 25), \quad k_3 = diag(55, 55, 55, 55); \quad k_4 = diag(35, 35, 35, 35); \quad k_5 = diag(45, 45, 45, 45). \]

The equations for the linearly parameterized model of external disturbance:
\[
\begin{align*}
T_d(t) &= [d_1 \quad d_2 \quad d_3 \quad d_4]^T, \\
&= 4 + 4\sin(14\pi t) + 4\sin(15\pi t); \\
&= 8 + 8\sin(16\pi t) + 8\sin(18\pi t - \pi / 2); \\
T_d &= 4 + 4\sin(16\pi t) + 4\sin(15\pi t); \\
&= 6 + 6\sin(17\pi t - \pi / 2) + 8\sin(18\pi t).
\end{align*}
\]

Let us consider some of the results of computer studies shown in Fig. 2, 3.

Fig. 2 shows the transient joint rotation processes as a reaction to the step input action with the values for the elastic coefficients of transmissions (assumed to be identical for each joint) being in the range \( K_c = 2000; 500; 100 \) (Nm/rad). In Fig. 2, the solid lines refer to the step program input actions, which are identical for all joints, from the first to the fourth link, from top to bottom. The reactions of joints to step actions are shown as follows: the dashed lines (when \( K_c = 2000 \)), the dash-dotted lines (when \( K_c = 500 \)) and the dotted lines (when \( K_c = 100 \)). As could be seen in Fig. 2, the convergence time of the processes into a 5% tube is the following for all the joints: at \( K_c = 2000 \leq 0.09 \) s; at \( K_c = 500 \leq 0.24 \) s; at \( K_c = 100 \leq 0.42 \) s i.e. with a 20x change in the elastic joint parameter, with the \( \sqrt{20} \)-x change in the natural frequency of elastic deformations, a nonlinear robust control system effectively suppresses elastic vibrations, preventing their excitation.

Fig. 3 shows the processes that illustrate the effectiveness of adaptive robust compensation of an unknown disturbance based on the asymptotic estimate of the disturbance generated by a nonlinear observer (5), (6) of the unknown regressor of the linearly parameterized disturbance model (4).

In Fig. 3, the solid lines refer to real disturbances designed according to the models (47) and applied to all joints (from top to bottom), starting from the first one, and the dashed lines show the transient responses of asymptotic estimates of the real disturbance. As could be seen from Fig. 3, the convergence time of the adaptive robust estimation of the disturbance into a 5% tube is no more than 0.1 s for all joints with simultaneous disturbance compensation.
Conclusions
1. A new nonlinear robust control system for a multi-DOF nonlinear electromechanical plant has been designed and synthesized taking into account the elastic properties of joints and/or transmissions by the integrator backstepping method and filtering the “pure” (physically unrealizable) derivatives of virtual controls, which made it possible to exclude cumbersome analytical derivative calculations.

2. Five-step nonlinear control synthesis is combined with the design of a subsystem for adaptive robust compensation of unknown disturbances by the V.O. Nikiforov method [6, 7] of synthesizing a parameterized observer model of external deterministic disturbance. The final robust adjustment function of the unknown matrix parameters for the parameterized model is implemented using parametric adaptation algorithms.

3. The use of an iterative procedure for designing a nonlinear control system for a multi-DOF elastic electromechanical plant, combined with adaptive robust disturbance compensation, provides the possibility of decomposing the combined system into independent simplified nonlinear control subsystems of an elastic mechanical or rigid electromechanical/mechanical plant (neglecting electromagnetic dynamics), and ultimately to use just the adaptive robust disturbance compensation function.

4. A computer simulation program for the resulting combined system and simplified subsystems for controlling the dynamics of a four-DOF typical manipulation arm for industrial usage was developed and registered.

References
[1] Fradkov A.L., Miroshnik I.V., Nikiforov V.O. Nonlinear and adaptive control of complex systems. Dordrecht: Kluwer Academic Publisher, 1999, 510 p.
[2] Ioannou P.A., Kokotovic, P.V. Instability analysis and improvement of robustness of adaptive control // Automatica. 1984. Vol. 20. № 5. p. 583-594.
[3] His L., Costa R.R. Bursting phenomena in continuous-time adaptive systems with a \( \sigma \)-modification //IEEE Trans.on Autom. Control. 1987. Vol. 32. № 1. p. 84-86.
[4] Z. Cai, M.S. de Queiroz, D.M. Dawson. A sufficiently smooth projection operator// IEEE. Transactions on Automatic Control. Vol. 51, № 1, January, 2006.
[5] Ikhouane F.I., Krstic V. Adaptive backstepping which parameter projection: robustness and performance// Automatica. 1998. Vol. 34. № 4. p. 429-435.
[6] Nikiforov V.O. Observers of external deterministic disturbances. I. Objects with known parameters// Avtomatika I telemekhanika. 2004. № 10. pp. 13-24.
[7] Nikiforov V.O. Observers of external deterministic disturbances. II. Objects with known parameters // Avtomatika i telemekhanika. 2004, № 11, pp.40–48.
[8] Krstic, M., Kanellakopoulos, I., Kokotovic, P.V. Nonlinear and Adaptive Control Design. //N.Y.: John Wiley and Sons. NewYork. 1995. 576 p.
[9] Krstic, M., Kanellakopoulos, I., Kokotovic, P.V. Adaptive Nonlinear Control Without Over parametrization/Systems and Control Letters. 1992. Vol.19. P.177-185.
[10] Le Hong Quang, V.V. Putov, V.N. Sheludko. Control of Multi Degree-of-freedom Mechanical Plant with Adaptive Disturbance Compensation // Proceedings of 2020 23rd International Conference on Soft Computing and Measurements, SCM 2020 9198778, c. 242-245.
[11] Farrell, J.A., Polycarpou, M., Sharma, M., Dong, W. Command filtered backstepping. IEEE Trans. Autom. Control 54(6). 2009. p.1391–1395.
[12] W. Dong, J. A. Farrell, M. M. Polycarpou, V. Djapic, and M. Sharma. “Command filtered adaptive backstepping” IEEE Trans. Control Syst. Technol. May 2012. vol. 20. no. 3. p. 566–580.
[13] Elmer Tolsted. An elementary derivation of the Cauchy, Holder, and Minkowski inequalities from Young's inequality // Mathematics Magazine, Vol. 37, No.1, 1964, pp. 2-12.