Information causality and non-locality swapping are equivalent from emergence of quantum correlations

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Abstract

Is information causality a new physical principle? To answer this question, we first analytically derive the criteria of emergence of quantum correlations from information causality. Then it is shown that, as emergence criteria of quantum correlations, information causality and uselessness of coupler-based non-locality swapping can be regarded equivalent. Therefore, incapability of non-locality swapping using a coupler is as powerful as information causality in the single-out of quantum physics from generalized non-signalling models.
Introduction. Quantum theory and relativity are the two foundations of modern physics. Quantum theory is non-local, and relativity does not allow for superluminal signalling. From the perspective of gravity, proving the coexistence of these two theories is a great challenge for theoretical physicists. In the non-relativistic case, quantum theory must be a non-signalling theory, since one can access no information of any distant party only by local operations. To test the non-local correlations, a lot of Bell-type inequalities are proposed [1]. In general, except stabilizer-type (or GHZ-Mermin-type) inequalities [2], a Bell-type inequality may be violated by an entangled state, but never be violated maximally by any quantum state. For example, quantum correlations can violate the celebrated Clauser-Horn-Shimony-Holt (CHSH) inequality [3] up to $2\sqrt{2}$ rather than 4. As proved by Tsirelson, the amount of non-locality allowed by quantum mechanics is limited [4]. Furthermore, Popescu and Rohrlich (PR) pointed out that such a limitation does not result from the request of relativity [5]. In other words, the CHSH inequality can be maximally violated even without superluminal signalling, and as a result of this, an interesting question arises, which is: Is there a more fundamental principle bounding the non-locality in quantum mechanics?

On the other hand, why not can quantum theory be more non-local? As a significant property of quantum mechanics, non-locality plays an essential role in quantum information science. Being regarded as physical resource of information processing, non-local correlations embedded in quantum entanglement can be exploited for efficient computation, secure communication, and some other tasks, such as teleportation [6] and dense coding [7], which cannot be physically realized without quantum systems. If quantum correlations were more non-local, quantum information processing would be more efficient. For example, if the CHSH inequality were maximally violated, which can be achieved by the PR box (see later), some unconditional secure computations, such as one-out-of-two oblivious transfer, can be performed perfectly [8]. In addition, post-quantum correlations take more advantage in non-local computation than quantum ones [9]. Even in the noisy PR box, the non-local correlations slightly stronger than quantum ones would collapse communication complexity, which are strongly believed to be non-trivial [10]. Therefore, from the perspective of information science, understanding what bounds the non-local correlations in the quantum mechanics is also important and interesting.

As previously mentioned, the non-signalling principle cannot be the answer. Several works focused on the generalized non-signalling models. Wherein, their probabilistic predic-
tions allow for many common features with quantum mechanics, such as no-cloning [11, 12], no-broadcasting [12, 13], and secret correlations [11, 14]. In addition, like quantum mechanics, the non-signalling principle can be exploited for secure key distribution [15, 16]. However, these features do not distinguish quantum correlations from other non-local correlations [17]. Recently, it was shown that part of the boundary between quantum and post-quantum correlations partially emerge from coupler-based non-locality swapping, the analogue of quantum entanglement swapping [18]. Therein, an imaginary device, called a coupler, is exploited for joint measurements.

Another potential principle is causality. Initially, the relation between relativistic causality and Tsirelson bounds was investigated [19, 20]. Therein, it is shown that some joint reversible unitary quantum evolutions can be exploited for entanglement generation and even for signalling.

In this Letter, we focus on another kind of causality: information causality, which is proposed as a new physical principle by Pawlowski et al. [21]. Briefly, information causality states that the information gain cannot exceed the amount of the classical communication, even if non-local correlations achievable in quantum mechanics can be used as physical resource. On the other hand, any non-local correlations stronger than quantum ones must violate information causality. Information causality can therefore be used to specify quantum theory from other unphysical non-signalling theories.

Is information causality at the very root of quantum theory? The answer is partially answered by Allock et al. [17]. Based on the numerical simulation, it is shown that part of the boundary of quantum correlations actually emerges from information causality. In this Letter, we study the relation between information causality and quantum correlations. We analytically derive the criteria of emergence of quantum correlations from information causality. Interestingly, information causality, as well as incapability of non-locality swapping using a coupler [18], leads to a quadratic Bell-type inequality. We will show the equivalence between these two emergence criteria of quantum correlations.

**The bipartite correlation boxes.** Before proceeding further, we briefly review the 8-dimensional convex no-signalling polytope, which comprises 24 vertices [11, 22]. Each vertex corresponds to a bipartite correlation box (hereafter, just “box”). Here a box is defined by a set of two possible inputs for each of spatially separate Alice and Bob, and a set of two possible outputs for each. Alice’s and Bob’s inputs are denoted by $x$ and $y$ respectively, and
their outputs by \(a\) and \(b\). A joint probability of getting a pair of outputs \(a\) and \(b\) given a pair of inputs \(x\) and \(y\) is \(P(a, b|x, y)\), which is definitely positive. In the following, \(a\), \(b\), \(x\), and \(y\) \(\in\) \{0, 1\}. For a non-signalling box,

\[
\sum_b P(a, b|x, y = 0) = \sum_b P(a, b|x, y = 1) = P(a|x),
\]

and

\[
\sum_a P(a, b|x = 0, y) = \sum_a P(a, b|x = 1, y) = P(b|y).
\]

The probability distribution is unbiased if the marginal probabilities \(P(a|x) = P(b|y) = \frac{1}{2}\), \(\forall a, b, x, y\). There are eight extreme non-local boxes, which have the form:

\[
P_{NL}^{\mu\nu\sigma}(a, b|x, y) = \begin{cases} 
\frac{1}{2} & \text{if } a \oplus b = xy \oplus \mu x \oplus \nu y \oplus \sigma \\
0 & \text{otherwise},
\end{cases} \quad (1)
\]

where \(\mu, \nu, \sigma \in \{0, 1\}\) (PR box is the extreme non-local box with \(\mu = \nu = \sigma = 0\)). The sixteen local deterministic boxes are denoted by \(P_L^{\mu\nu\sigma\tau}\), which have the form

\[
P_L^{\mu\nu\sigma\tau}(a, b|x, y) = \begin{cases} 
1 & \text{if } a = \mu x \oplus \nu b = \sigma y \oplus \tau \\
0 & \text{otherwise},
\end{cases} \quad (2)
\]

where \(\mu, \nu, \sigma\) and \(\tau \in \{0, 1\}\). Given a pair of inputs \(x\) and \(y\), the corresponding correlator is denoted by

\[
C_{xy} = \sum_{a^\prime, b^\prime} P(a^\prime, b^\prime|x, y) - \sum_{a^\prime \neq b^\prime} P(a^\prime, b^\prime|x, y).
\]

In addition, define \(B_{xy} = \sum_{x^\prime, y^\prime} C_{x'y'} - 2C_{xy}\) and \(B = \max\{B_{00}, B_{01}, B_{10}, B_{11}\}\). According to the CHSH scenario, a mixture of local deterministic boxes must satisfy

\[
B_{xy} \leq 2, \quad \forall x, y.
\]

As for an extreme non-local box, we have

\[
B = 4. \quad (4)
\]

In other words, the CHSH inequalities can be maximally violated by these non-local boxes.

As for the correlations which can be obtained by performing local measurements on a bipartite quantum system, we have
In between the non-signalling polytope, the quantum correlations form a body with a smooth convex curve as its boundary. Given a two-input-two-output probabilistic distribution, can it be physically realized by quantum systems? The interesting question was answered independently by Tsirelson, Landau and Masanes (TLM) \[23–25\]. These proposed criteria on quantumness are equivalent \[26, 27\]. Here we exploit Landau’s criterion, which can be stated as follows. If a set of correlators $C_{xy}$ is admitted by a quantum description with unbiased marginals, we have

\[
A = |C_{00}C_{10} - C_{01}C_{11}| \leq \sum_{j=0,1} \sqrt{(1 - C^2_{0j})(1 - C^2_{1j})}. \tag{6}
\]

It is worthy noting that Navascues et al. recently proposed the criteria on quantum-obtained correlators $C_{xy}$ with biased marginals \[28, 29\].

**Information causality.** Information causality considers the following communication scenario. Spatially separate Alice and Bob share the non-local, no-signalling and accessible resources. In addition, Alice is given $N$ random bits $\vec{a} = (a_1, a_2, \ldots, a_N)$ and Bob is given a random variable $b \in \{1, 2, \ldots, N\}$. The task for Bob is to guess the bit $a_b$. Wherein, Alice is allowed to perform any local operation on the resource at hand. Then she sends $m$ classical bits to Bob via one-way classical communication. As for Bob, he can also perform any local operation on the accessible resource in his information processing. Finally, Bob outputs $b$ as his answer. To optimize the probability of successful guessing, the proposed protocol in Ref. \[21\] can be regarded as the extension of the van Dam’s protocol \[10\], which is originally proposed for 1-out-of-2 oblivious transfer \[8\].

Information causality is fulfilled if

\[
\sum_{K=1}^{N} I(a_K : b | b = K) \leq m, \tag{7}
\]

where $I(a_K : b | b = K)$ is Shannon mutual information between $a_K$ and $\beta$, given $b$ equal to $K$. Equivalently, Eq. (7) can be restated in terms of correlators. That is, information causality is fulfilled if

\[
S = (C_{00} + C_{10})^2 + (C_{01} - C_{11})^2 \leq 4. \tag{8}
\]
Interestingly, Eq. (8) is equivalent to the Uffink’s inequality, which is a bipartite quadratic Bell-type inequality [30]. Notably, Eq. (8) cannot be exploited for entanglement testing. It is easy to verify that $S$ can be maximally violated by PR box. Later it will be shown that Eq. (8) is a weaker form of TLM criteria [27].

Emergence of quantum correlations from information causality. In the following, without loss of generality, we assume $C_{00}, C_{10}, C_{01} \geq 0$ and $C_{11} \leq 0$. As a result, $S$ can be optimized in Eq. (8) given a set of the absolute values of correlators $|C_{xy}|$. As shown in Fig. 1, in a two-dimensional Cartesian plane, we define $\vec{r}_1 = (C_{00}, C_{01}), \vec{r}_2 = (-C_{10}, C_{11})$, and $\vec{r}_3 = (C_{11}, C_{10})$, which respectively belong to the first, third and second quadrants. Note that $\vec{r}_2 \perp \vec{r}_3$ and $|\vec{r}_2| = |\vec{r}_3|$. The angles between vector pairs $(\vec{r}_1, \vec{r}_2)$ and hence $(\vec{r}_1, \vec{r}_3)$ are $\frac{\pi}{2} + \phi$ and $\phi$, respectively. As a result, $S$ can be regarded as the area of the square with the side length equal to $|\vec{r}_1 - \vec{r}_2|$. On the other hand, $A$ in Eq. (6) is equal to the area of a parallelogram spanned by $\vec{r}_1$ and $\vec{r}_3$. The relation between $S$ and $A$ can be revealed by the law of cosine. Namely,

$$S = |\vec{r}_1 - \vec{r}_2|^2$$

$$= r_1^2 + r_2^2 - 2r_1r_2 \cos(\frac{\pi}{2} + \phi)$$

$$= r_1^2 + r_2^2 + 2r_1r_3 \sin \phi$$

$$= r_1^2 + r_2^2 + 2A$$

$$= \sum_{xy} C_{xy}^2 + 2A$$

(9)

where $r_1^2 + r_2^2 = \sum_{xy} C_{xy}^2$. From Ineq. (8), we have

$$S \leq \sum_{i,j=0,1} C_{ij}^2 + 2 \sum_{j=0,1} \sqrt{(1 - C_{0j}^2)(1 - C_{1j}^2)}$$

(10)

Now we compare RHS of Ineqs. (8) and (10). Set $\omega_{00} = 1 - C_{00}^2, \omega_{10} = 1 - C_{10}^2, \omega_{01} = 1 - C_{01}^2, \omega_{11} = 1 - C_{11}^2$, where $\omega_{ij} \geq 0 \forall i, j \in \{0, 1\}$. We have

$$4 - (\sum_{i,j=0,1} C_{ij}^2 + 2 \sum_{j=0,1} \sqrt{(1 - C_{0j}^2)(1 - C_{1j}^2)})$$

$$= \omega_{00} + \omega_{10} + \omega_{01} + \omega_{11} - 2\sqrt{\omega_{00}\omega_{10}} - 2\sqrt{\omega_{01}\omega_{11}} \geq 0.$$

(11)
The inequality in (11) holds since the arithmetic average is not smaller than the geometric average. According to Ineq. (11), Ineq. (10) is stronger than Ineq. (8) and hence Uffink’s quadratic inequality is weaker than TLM criteria [27].

Furthermore, Ineqs. (8) and (11) are equivalent when the equality of the inequality in (11) holds. In other words, if

\[ \omega_{00} = \omega_{10} \quad \text{and} \quad \omega_{01} = \omega_{11}, \]

or equivalently,

\[ \vec{r}_1 + \vec{r}_2 = 0 \iff C_{00} = C_{10} \quad \text{and} \quad C_{01} = -C_{11}. \]  

the criteria of the quantumness and information causality in Ineqs. (8) and (10) coincide. As a result, these two boundaries merge.

Now we consider a mixture of one non-local box and a box \( B \) with completely depolarizing noise as follows [17]

\[ P_{\lambda, \eta} = \lambda P_{NL}^{\mu\nu\sigma} + \eta B + (1 - \lambda - \eta)I, \]  

(13)

where \( \lambda, \eta \geq 0, \quad \text{and} \quad 0 \leq 1 - \lambda - \eta \leq 1. \)

Case (a) : \( B \) is a non-local box. That is, \( B = P_{NL}^{\mu'\nu'\sigma'}. \) It is easy to verify that \( C_{00} = (-1)^{\sigma} \lambda + (-1)^{\nu'} \eta, \quad C_{10} = (-1)^{\mu' + \sigma} \lambda + (-1)^{\nu' + \sigma'} \eta, \quad C_{01} = (-1)^{\nu + \sigma} \lambda + (-1)^{\nu' + \sigma'} \eta, \quad \text{and} \quad C_{11} = (-1)^{\nu + \mu + \sigma + 1} \lambda + (-1)^{\nu' + \mu + \sigma' + 1} \eta. \) As a result, Eq. (12) is satisfied if \( \mu = \mu' = 0. \) That is, if the non-local correlations of \( P_{\lambda, \eta} \) can be obtained by performing local measurements on a quantum state, the information causality is automatically fulfilled. Therefore, the boundaries of quantumness and information causality merge. Obviously, a mixed box \( PR = \sum_{\nu, \sigma} \lambda_{\nu\sigma} P_{NL}^{\nu\sigma} + (1 - \sum_{\nu, \sigma} \lambda_{\nu\sigma})I, \) where \( 0 \leq \lambda_{\nu\sigma} \quad \forall \nu, \sigma \) and \( 0 \leq \sum_{\nu, \sigma} \lambda_{\nu\sigma} \leq 1, \) can also lead to the boundary-merging phenomenon.

Case (b) : \( B \) is a local box. That is, \( B = P_{L}^{\mu'\nu'\sigma'\sigma'}. \) These four correlators are \( C_{00} = (-1)^{\sigma} \lambda + (-1)^{\nu'} \sigma' \eta, \quad C_{10} = (-1)^{\mu' + \sigma} \lambda + (-1)^{\nu' + \sigma' + \mu'} \eta, \quad C_{01} = (-1)^{\nu + \sigma} \lambda + (-1)^{\nu' + \sigma' + \mu'} \eta, \quad \text{and} \quad C_{11} = (-1)^{\nu + \mu + \sigma + 1} \lambda + (-1)^{\nu' + \sigma' + \mu' + \sigma'} \eta. \) The conditions \( C_{00} = C_{10} \quad \text{and} \quad C_{01} = -C_{11} \) lead to \( \mu = \mu' = 0 \) and \( \mu = \mu' = 1, \) respectively, which contradict each other. Therefore, two boundaries never merge in this case.

The reader can refer to Ref. [17] for sketching the slices of the non-signalling polytope. Wherein, the boundary-merge is demonstrated.

**Discussions.** In Ref [18], Skrzypczyk et al. showed the uselessness of quantum correlations for non-locality swapping. Wherein, the coupler of two non-local boxes \( PR_{\lambda, \eta} = \)
\(\lambda P_{NL}^{00} + \eta P_{NL}^{10} + (1 - \lambda - \eta)1\) is studied. The emergence of quantum correlations from \(PR_{\lambda,\eta}\) uselessness of non-locality leads to a quadratic Bell-type inequality,

\[(C_{00} + C_{01})^2 + (C_{10} - C_{11})^2 \leq 4.\]  \(14\)

Obviously, Ineq. \(8\) is equal to Ineq. \(14\) under the permutation of two subscripts. In other words, once Alice and Bob exchange the roles in the protocol, information causality can therefore be fulfilled by Ineq. \(14\). On the other hand, as for coupler-based non-locality swapping in Ref [18], if the two particles on which the joint measurement is performed are interchanged with the other two particles, inability of non-locality swapping is fulfilled by Ineq. \(8\). Information causality and uselessness of non-locality swapping are therefore equivalent from the perspective of emergence of quantum correlations. However, the connection between information causality and non-locality swapping is unclear.

In conclusion, the merging criterion of information causality and quantum mechanics is analytically derived. These boundaries can be merged with the unbiased marginals when the quantum correlations can be described by the depolarized mixture of the \(\mu = 0\) non-local boxes. In addition, information causality is equivalent to non-locality swapping from emergence of quantum correlations.

Notably, the proposed protocol in information causality and the scheme for coupler-based non-locality swapping are asymmetrical. These criteria each cannot lead to the third quadratic Bell-type inequality, which reads

\[(C_{10} + C_{01})^2 + (C_{00} - C_{11})^2 \leq 4.\]  \(15\)

Can this inequality be derived from some other principle? It is an open question. If such principle exists, the employed protocol or scheme should be symmetrical.

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I. FIGURE CAPTION:

Fig. (1) : The geometrical description of $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, $S$, and $A$. 