A WEAK LAW OF LARGE NUMBERS FOR A LIMIT ORDER BOOK MODEL WITH FULLY STATE DEPENDENT ORDER DYNAMICS

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Abstract. This paper studies a one-sided limit order book (LOB) model, in which the order dynamics depend on both, the current best bid price and the current volume density function. For the joint dynamics of the best bid price and the standing buy volume density we derive a weak law of large numbers, which states that the LOB model converges to a continuous-time limit when the size of an individual order as well as the tick size tend to zero and the order arrival rate tends to infinity. In the scaling limit the standing buy volume density follows a non-linear PDE coupled with a non-linear ODE that describes the best bid price.

1. Introduction

While limit order books have extensively been discussed in the economic and econometric literature for some years (cf. for example [2, 6, 9, 15]), they have only recently gained increased attention by researchers in mathematical finance. One research objective is to specify a realistic discrete dynamics of a LOB which can be approximated by an analytically tractable continuous time model. This is achieved by introducing scaling parameters and passing to the high frequency limit, when the number of submitted orders gets large while the individual order size and the tick size tend to zero. Depending on the scaling assumptions the high frequency limit will either deterministic as in a law of large numbers or of (jump) diffusion type as in a functional central limit theorem.

Deterministic high frequency limits for LOB models were derived by [11] and [8]. In [11] a weak law of large numbers is established for a limit order book model with Markovian dynamics depending on prices only. In [8] the authors study a limit order book model, similar to ours but without any feedback effect, and derive a deterministic ODE limit using weak convergence in the space of positive measures on a compact interval. A diffusion limit for order book dynamics can be found in [4], where the top of the book is analyzed. The result was later generalized in [5]. In [13] a high frequency limit for a one-sided limit order book model is derived under the assumption that on average investors place their limit orders above the current best ask price. The opposite case when orders are placed in the spread with higher probability is analyzed in [12], where the authors use a coupling between a simple one-sided limit order book model and a branching random walk to characterize the diffusion limit, cf. also [16].

In this paper we adapt the model from [11] but allow for much more general order dynamics. There is considerable empirical evidence (see, e.g. [2, 3, 10] and references therein) that the state of the book, especially order imbalance at the top of the book, has a considerable impact on order dynamics. Our approach allows us to model fully state dependent order flow dynamics: the type of order (market order, limit order placement, cancellation), its size, and the price level at which the order is submitted can all depend on the current state of the limit order book, i.e. on prices as well as on the standing

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volumes. As a result, unlike in [11], the price process cannot be analyzed separately. Instead, we have to establish joint convergence of prices and volumes. The resulting scaling limit for a fully state dependent Markovian order book dynamics is the main result of this paper.

Our main theorem states that when the number of submitted orders goes to infinity over a fixed time horizon, while the proportion of active orders, the tick size, and the individual order size tend to zero, the dynamics of the price and the volume density function converge to the unique solution of a non-linear coupled ODE/PDE system. To show our main result we first construct a deterministic discrete non-linear approximation \( \tilde{S}^{(n)} \) to the random discrete order book dynamics \( S^{(n)} \). This is done using a weak law of large numbers for triangular martingale difference arrays as in [11], even though our method of approximation is different and more elegant, which allows us to handle this more general setting. In the next step we then construct an iteration towards the deterministic approximation for fixed \( n \), denoted \( \tilde{S}^{(n),m} \), and we prove that it approximates \( \tilde{S}^{(n)} \) almost uniformly. Afterwards it is shown that each iteration step in the prelimit converges as \( n \) goes to infinity to a continuous model \( \hat{S}^m \) solving a certain differential equation. Indeed, these models can be seen to be a fixed point iteration generated by a contraction mapping. The fixed point then gives a solution to our limiting coupled ODE/PDE system.

For the ease of notation we have chosen to model only the buy side of the order book together with the bid price in this paper. However, if one defines the sell side and ask price in an analogous way, the result can easily be extended to a two-sided order book with order dynamics depending on the whole limit order book, i.e. on bid and ask prices as well as the order volumes of both sides of the book. Especially, making the distribution of order types dependent on the bid-ask spread will ensure that the bid and ask price do not cross, cf. also [11]. Moreover, we assume that order arrival times are deterministic. However, one can easily generalize our main result allowing for randomly spaced arrival times by making use of the time change theorem as has been done in [11, 1].

The remainder of this paper is organized as follows: In Section 2 we define the dynamics of a sequence of discrete limit order book models, state our assumptions and the main result. We also give an example which satisfies all our assumptions. Section 3 is devoted to the analysis of the limiting coupled PDE/ODE system, while Section 4 contains the convergence proof of the discrete order book models to the high frequency limit.

2. Setup and main result

In this section we define for every \( n \in \mathbb{N} \) a model for the dynamics of a one-sided limit order book with tick size \( \Delta x^{(n)} \). Later we consider the scaling limit of these models when the tick size and the impact of a single order tend to zero, while the number of order placements and order cancelations over a given time horizon \([0, T]\) tends to infinity. Our modeling framework closely follows [11] but we allow for a much more general dependence of order arrivals on the state of the book. Throughout, all random variables are defined on a common complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

2.1. The model. The dynamics of the order book in the \( n \)-th model is described by a càdlàg stochastic process \( S^{(n)} = (S^{(n)}(t))_{0 \leq t \leq T} \) taking values in the Hilbert space

\[
E := \mathbb{R} \times L^2(\mathbb{R}), \quad \|\alpha\|_E := |\alpha_1| + \|\alpha_2\|_{L^2}.
\]
The state of the book changes due to arriving market and limit orders. In the \( n \)-th model there are \([T/\Delta t^{(n)}]\) such events taking place at times
\[
    t^{(n)}_k := k\Delta t^{(n)}, \quad k = 1, \ldots, \left\lfloor \frac{T}{\Delta t^{(n)}} \right\rfloor,
\]
where \( \Delta t^{(n)} \) denotes a scaling parameter converging to zero as \( n \to \infty \) and \( t^{(n)}_0 = 0 \).

The state of the book after \( k \) events is denoted \( s^{(n)}_k \). We put
\[
    s^{(n)}(t) := s^{(n)}_k := \left( B^{(n)}_k, v^{(n)}_k \right) \quad \text{for} \quad t \in \left[ t^{(n)}_k, t^{(n)}_{k+1} \right),
\]
where \( B^{(n)}_k \) and \( v^{(n)}_k \) denote the best bid price and the buy side volume density function relative to the best bid price (relative volume density function), respectively. To be precise, defining \( x^{(n)}_j := j\Delta x^{(n)} \) for \( j \in \mathbb{Z} \), \( n \in \mathbb{N} \),
\[
    \int_{x^{(n)}_{j-1}}^{x^{(n)}_j} v^{(n)}_k(x)dx
\]
represents the liquidity available for buying at a price which is \( j \in \mathbb{N}_0 \) ticks below the best bid price at time \( t^{(n)}_k \). In order to model placements of limit orders into the spread, the function \( v^{(n)}_k \), \( k \in \mathbb{N} \), will be defined on the whole real line. We refer to the buy volumes standing at positive distance from the best bid price as the shadow book. The idea of the shadow book is taken from [11]. The shadow book follows the same dynamics as the volumes of the visible book. The shadow book becomes part of the visible book through price changes. The working of the shadow book and its interaction with the visible book will be further explained below. For a detailed discussion we refer to [11].

At time \( t = 0 \) the state of the limit order book is deterministic for all \( n \in \mathbb{N} \) and denoted by
\[
    s^{(n)}_0 = \left( B^{(n)}_0, v^{(n)}_0 \right) \in \mathbb{R} \times L^2(\mathbb{R}).
\]
To state the convergence condition on the sequence of initial states we introduce for each \( n \in \mathbb{N} \) the translation operators \( T^+_n \) and \( T^-_n \), which act on functions \( f : \mathbb{R} \to \mathbb{R} \) in the following way:
\[
    T^+_n(f)(\cdot) := f(\cdot + \Delta x^{(n)}), \quad T^-_n(f)(\cdot) := f(\cdot - \Delta x^{(n)}).
\]
Furthermore, let us fix some constant \( M > 0 \) throughout.

**Assumption 2.1.** The initial volume function \( v^{(n)}_0 \) is a non-negative step-function on the grid \( \{x^{(n)}_j, j \in \mathbb{Z}\} \), which is uniformly bounded by \( M \) and has compact support in \([-M, M]\) for all \( n \in \mathbb{N} \). Moreover, there exists a non-negative continuously differentiable function \( v_0 \in L^2 \) with bounded derivative such that
\[
    \left\| v^{(n)}_0 - v_0 \right\|_{L^2} = o(1) \quad \text{and} \quad \left\| \left( T^+_n - I \right)(v^{(n)}_0) \right\|_{L^2} = O(\Delta x^{(n)}).
\]
Also there exists \( B_0 \in \mathbb{R}_+ \) such that \( B^{(n)}_0 \to B_0 \). We denote \( s_0 := (B_0, v_0) \in E \).

In our model there are three events that change the state of our limit order book. The (buy side) limit order book changes if:

- **(A):** a market sell order arrives. In this case the best bid price decreases by one tick and the relative volume density function shifts one tick to the right.
- **(B):** a buy limit order is placed in the spread one tick above the current best bid price. In this case the best bid price increases by one tick and the relative volume density function shifts one tick to the left.
• (C): a buy limit order placement of size \( \Delta \omega_k^{(n)} \) at price level \( P_k^{(n)} \) occurs. If \( \omega_k^{(n)} < 0 \), this corresponds to a cancelation of volume.

Here \( \Delta v^{(n)} \) is a scaling parameter that determines the size of an individual placement/cancelation. We refer to market orders and limit buy order placements in the spread (Types A, B) as active orders. They lead to price changes. Cancelations and limit order placements (Type C) do not lead to price changes. They are referred to as passive orders. The assumption that market orders match precisely against the volume standing at the top of the book and hence shift prices by exactly one tick is made for convenience. Our framework is flexible enough to allow for larger market orders. A market order that does not lead to a price change is equivalent to a cancelation of standing volume.

Event types are determined by a field of random variables \( (\phi_k^{(n)})_{k,n \in \mathbb{N}} \) taking values in the set \( \{A, B, C\} \). The size and the price level at which an order placement resp. cancelation takes place are determined by a field of random variables \( (\omega_k^{(n)}, \rho_k^{(n)})_{k,n \in \mathbb{N}_0} \) according to the following assumption.

**Assumption 2.2.** There exists a field of random variables \( (\pi_k^{(n)})_{k,n \in \mathbb{N}_0} \) taking values in the compact interval \([-M, M]\) almost surely and

\[
\rho_k^{(n)} := B_k^{(n)} + j\Delta x^{(n)} \quad \text{for} \quad \pi_k^{(n)} \in \left[ x_j^{(n)}, x_{j+1}^{(n)} \right].
\]

Furthermore, there exists a field of random variables \( (\omega_k^{(n)})_{k,n \in \mathbb{N}_0} \) such that \( \omega_k^{(n)} \in [-M, M] \) for all \( k, n \in \mathbb{N}_0 \).

The random variables \( \pi_k^{(n)}, k,n \in \mathbb{N}_0 \), determine the placement/cancelation price levels relative to the best bid price. If \( \rho_k^{(n)} = B_k^{(n)} \), then the placement/cancelation takes place at the best bid price; if \( \rho_k^{(n)} < B_k^{(n)} \), then it takes place deeper in the book; else it takes place in the shadow book. The shadow book interacts with the visible book through price changes which shift the relative volume density functions \( v^{(n)} \). For instance, if the \( k \)-th event is a limit order placement one tick above the best bid price into the shadow book, i.e.

\[
\phi_k^{(n)} = C, \quad \rho_k^{(n)} = B_k^{(n)} + \Delta x^{(n)} \quad \text{and} \quad \omega_k^{(n)} > 0,
\]

then this order becomes part of the visible book as soon as the best bid increases to \( B_k^{(n)} + \Delta x^{(n)} \), i.e. as soon as a limit order placement in the spread occurs.

Note that in general \( \omega_k^{(n)} \) is (even conditionally) dependent on \( \pi_k^{(n)} \), if one wants to avoid negatives volumes due to cancelations. The main contribution of this paper is that the conditional distribution of the random variables \( (\phi_k^{(n)}, \omega_k^{(n)}, \pi_k^{(n)})_{k,n \in \mathbb{N}_0} \) may depend on both, prices and volumes. This extends \( \Pi \) where prices are independent of volumes as well as \( \Pi \) where only the distributions of price increments depend on volumes.

To formulate the next assumption we need to introduce the space \( E' := \{ s = (B,v) \in E : v \in C^1 \} \).

**Assumption 2.3.**

(1) There are two Lipschitz continuous functions \( p^A, p^B : E \to [0,1] \) with Lipschitz constant \( L \) and a scaling parameter \( \Delta p^{(n)} \) such that for all \( n \in \mathbb{N}_0 \) and \( k \leq \lfloor T/\Delta t^{(n)} \rfloor \),

\[
\mathbb{P} \left( \phi_k^{(n)} = I \ \big| \ S_j^{(n)}, \ j \leq k \right) = \Delta p^{(n)} \mathbb{P} \left[ S_k^{(n)} \right] \quad \text{a.s.} \quad \text{for} \ I = A, B.
\]
We may interpret dynamics and standing volumes. The following example illustrates how our modeling framework allows for a dependence on the price depending on the choice of $h$ for Lipschitz continuous functions $f^{(n)} : E \rightarrow L^2$, $n \in \mathbb{N}_0$, with common Lipschitz constant $L > 0$ such that for all $k \leq \lfloor T/\Delta t^{(n)} \rfloor$,

$$f^{(n)} \left[ S_k^{(n)} \right](\cdot) = \frac{1}{\Delta t^{(n)}} \mathbb{E} \left( \omega_k^{(n)} \sum_{j \in \mathbb{Z}} 1_{\left\{ \tau_k^{(n)} \in \left[ x_j+\frac{1}{2}, x_{j+1} \right) \right\}}(\cdot) \mathbb{1}_{C} \left( \phi_k^{(n)} \right) \right) \left. S_j^{(n)} \right\vert j \leq k \big) \text{ a.s.}$$

and

$$\sup_{s \in E} \left\| \left( T_s^{(n)} - I \right) \left( f^{(n)}[s] \right) \right\|_{L^2} = O \left( \Delta x^{(n)} \right) \text{, } \sup_{s \in E} \left\| f^{(n)}[s](\cdot) \right\|_{\infty} \leq M.$$

Moreover, there exists a function $f : E \rightarrow L^2$ such that

$$\sup_{s \in E} \left\| f^{(n)}[s] - f[s] \right\|_{L^2} = o(1),$$

where $f[s](\cdot) : \mathbb{R} \rightarrow [-M, M]$ is continuously differentiable in $x$ for all $s \in E'$ with derivate being uniformly bounded in absolute value by $M$.

The following example illustrates how our modeling framework allows for a dependence on the price dynamics and standing volumes.

**Example 2.4.** Given any $h \in L^2$ define

$$H_k^{(n)} := \int_{\mathbb{R}^+} v_k^{(n)}(x)h(x)dx.$$

We may interpret $H_k^{(n)}$ as an indicator for the volume standing at the top of or deeper into the book, depending on the choice of $h$. Set

$$p^I \left[ S_k^{(n)} \right] := g^I \left( B_k^{(n)} , H_k^{(n)} \right), \quad I = A, B,$$

for Lipschitz continuous functions $g^I : \mathbb{R}^2 \rightarrow [0, 1]$. Then by the Cauchy-Schwarz inequality there exists $L < \infty$ such that

$$p^I \left[ S_k^{(n)} \right] - p^I \left[ S_k^{(n)} \right] \leq L \left\| S_k^{(n)} - S_k^{(n)} \right\|_{E},$$

Let us further assume that for all $n \in \mathbb{N}_0$, $k \leq \lfloor T/\Delta t^{(n)} \rfloor$, $I = A, B$,

$$\mathbb{P} \left( \phi_k^{(n)} = I \left\vert S_j^{(n)} \right. \right) = \Delta p^{(n)} p^I \left[ S_j^{(n)} \right] \text{ a.s.}$$

and that there exists a function $m : \mathbb{R}^2 \rightarrow L^2$ such that for all $k, n$ as above

$$\mathbb{P} \left( \pi_k^{(n)} \in dx \left\vert S_j^{(n)} \right. \right) = m \left[ B_k^{(n)} , H_k^{(n)} \right](x)dx,$$

where for each $(b, h) \in \mathbb{R}^2$ the function $m[b, h](\cdot)$ is uniformly bounded with bounded support in $[-M, M]$ and the mapping $(b, h, x) \mapsto m[b, h](x)$ is continuously differentiable with bounded derivatives. Moreover, suppose that there is a field $\left( \tilde{\omega}_k^{(n)} \right)$ of i.i.d. random variables with bounded density which has compact support in $[-M, M]$ and set

$$\omega_k^{(n)} = \tilde{\omega}_k^{(n)} \land \left( -v^{(n)} \left( \pi_k^{(n)} \right) + \varepsilon \right) \text{ for some } \varepsilon > 0.$$

If the random variables $\phi_k^{(n)}, \pi_k^{(n)}$, and $\tilde{\omega}_k^{(n)}$ are conditionally on $\{ S_j^{(n)} , j \leq k \}$ independent and if $\tilde{\omega}_k^{(n)}$ is also independent of $\{ S_j^{(n)} , j \leq k \}$, then Assumption $\mathbb{A}$ is satisfied.
2.2. Main result. We are now ready to define the full dynamics of the order book. For notational convenience we define for $I \in \{A, B, C\}$ and $k, n \in \mathbb{N}$, the event indicator variable

$$\mathbb{1}_{k}^{(n),I} := \mathbb{1}_{I}\left(\phi_{k}^{(n)}\right)$$

and introduce the short-hand notation $(I = A, B)$:

$$p^{(n),I}[j] := \Delta p^{(n),I}[j], \quad p^{(n),B-A} := p^{(n),B} - p^{(n),A}, \quad p^{B-A} := p^{B} - p^{A}, \quad \mathbb{1}_{k}^{(n),B-A} := \mathbb{1}_{k}^{(n),B} - \mathbb{1}_{k}^{(n),A}.$$ 

Definition 2.5. For each $n \in \mathbb{N}$ the dynamics of the state process $S^{(n)} = (B^{(n)}, v^{(n)})$ is given by $S^{(n)}_{0} := s^{(n)}_{0}$ and for $k = 1, \ldots, \left\lfloor \frac{T}{\Delta t^{(n)}} \right\rfloor$,

$$B_{k}^{(n)} = B_{k-1}^{(n)} + \Delta x^{(n)} I_{k-1}^{(n),B-A}$$

$$v_{k}^{(n)} = v_{k-1}^{(n)} + \left(T_{-}^{(n)} - I\right)\left(v_{k-1}^{(n)}\right) I_{k-1}^{(n),A} + \left(T_{+}^{(n)} - I\right)\left(v_{k-1}^{(n)}\right) I_{k-1}^{(n),B} + \Delta v^{(n)} M^{(n)}$$

where

$$M^{(n)}(\cdot) := \mathbb{1}_{k}^{(n),C} \frac{\Delta x^{(n)}_{k}}{\Delta t^{(n)}} \sum_{j \in \mathbb{Z}} \mathbb{1}_{\{x^{(n)}_{j} \in [x^{(n)}_{j-1}, x^{(n)}_{j})\}}(\cdot).$$

To derive a law of large numbers we need to make the right assumptions on the scaling parameters. Our choice of scaling introduces two time scales, a fast one for limit order arrivals and cancelations and a comparably slow one for market order arrivals and limit order placements in the spread.

Assumption 2.6. There exist constants $c_{0}, c_{1}, c_{2} > 0$ and $\beta \in (0, 1)$ such that

$$\lim_{n \to \infty} \frac{\Delta x^{(n)} \Delta p^{(n)}}{\Delta t^{(n)}} = c_{0}, \quad \lim_{n \to \infty} \frac{\Delta v^{(n)}}{\Delta t^{(n)}} = c_{1}, \quad \lim_{n \to \infty} \frac{\Delta x^{(n)}}{(\Delta t^{(n)})^{\beta}} = c_{2}.$$

W.l.o.g. we assume that $c_{0} = c_{1} = c_{2} = 1$ in the following.

Remark 2.7. While it is very natural to assume that $\Delta v^{(n)} \sim \Delta t^{(n)}$ for $n \to \infty$ in order to keep the total volume of orders in the limit order book of constant size, the assumption $\Delta x^{(n)} \Delta p^{(n)} \sim \Delta t^{(n)}$ is not so standard. However, note that this constitutes the critical (and interesting) case. Indeed, as can be easily seen from the proof of our main theorem, assuming that $\Delta x^{(n)} \Delta p^{(n)} = o(\Delta t^{(n)})$ would lead to a constant price in the high frequency limit. Such a result can be found in [8].

The following weak law of large numbers is the main result of this paper. It states that the state process converges in probability to a deterministic limit that can be described as the solution of a system of non-linear differential equations subject to an initial boundary condition.

Theorem 2.8. Under Assumptions 2.1 - 2.3 and 2.6 there exists a deterministic process $S : [0, T] \to E$ such that for all $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\left(\sup_{0 \leq t \leq T} \left\|S^{(n)}(t) - S(t)\right\|_{E} > \varepsilon\right) = 0.$$

The function $S = (B, v)$ is the unique classical solution to the following coupled ODE/PDE initial boundary value problem:

$$S(0) = s_{0},$$

$$dB(t) = p^{B-A}[S(t)]dt, \quad t \in [0, T],$$

$$v_{t}(t, x) = p^{B-A}[S(t)]v_{x}(t, x) + f[S(t)](x), \quad (t, x) \in [0, T] \times \mathbb{R}.$$
assumptions of Theorem 2.8 there exists a constant

\[ a \] solution via a fixed point iteration on a suitable Banach spa
t to prove existence. Under the

In this section we prove existence and uniqueness of a soluti
on to (2). First, we explicitly construct

is independent of any index involved.

Throughout the paper we will denote by

\[ C \]

Note that by definition and Assumptions 2.1, 2.2, and 2.3,

\[ \tilde{G} \]

Then, choosing \( \tilde{M} \) large enough, the ODE/PDE system (2) takes the special form

\[
S(0) = s_0, \\
dB(t) = p^{B-A}[B(t), H(t)]dt, \quad t \in [0, T], \\
v_t(t, x) = p^{B-A}[B(t), H(t)]v_x(t, x) + f_1[B(t), H(t)](x) - f_2[B(t), H(t)]v(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}.
\]

Throughout the paper we will denote by \( C > 0 \) a generic constant that may vary from line to line and
is independent of any index involved.

3. The limit model

In this section we prove existence and uniqueness of a solution to (2). First, we explicitly construct
a solution via a fixed point iteration on a suitable Banach space to prove existence. Under the
assumptions of Theorem 2.8 there exists a constant \( K < \infty \) such that

\[
\| E \|_E \leq B_0(n) + T \cdot \frac{\Delta p^{(n)}}{\Delta t^{(n)}} + \| v_0^{(n)} \|_{L^2} + T \cdot \frac{\Delta v^{(n)}}{\Delta t^{(n)}} \cdot \sup_{s \in E} \| f^{(n)}[s] \|_{L^2} \leq K
\]

for all \( k, n \in \mathbb{N} \). We thus choose as our Banach space the space \( \tilde{E} \) of functions \( g : [0, T] \rightarrow E \) which satisfy

\[
\sup_{t \in [0, T]} \| g(t) \|_E \leq K
\]

equipped with the norm \( \sup_{t \in [0, T]} \| g(t) \|_E \). Uniqueness will be shown using a standard Gronwall argument.

3.1. Fixed point iteration in the scaling limit. To construct a solution to (2) we perform a fixed point iteration for the function \( F : \tilde{E} \rightarrow \tilde{E} \) defined through \( F : g \mapsto G \), where \( G = (G^1, G^2) \) and

\[ G^1 : [0, T] \rightarrow \mathbb{R} \text{ and } G^2 : [0, T] \rightarrow L^2 \] are given by

\[
G^1(t) = B_0 + \int_0^t p^{B-A}[g(s)] ds, \\
G^2(t, x) = v_0 \left( x + \int_0^t p^{B-A}[g(s)] ds \right) + \int_0^t f[g(s)] \left( x + \int_s^t p^{B-A}[g(u)] du \right) ds.
\]

Note that by definition and Assumptions 2.1, 2.2, and 2.3 \( G^2(t, \cdot) \) has support in the compact interval
\( [-M - T, M + T] \) for all \( t \in [0, T] \).

We define \( \tilde{E}' = \{ g \in \tilde{E} \mid g : [0, T] \rightarrow \tilde{E}' \} \).
Lemma 3.1. For fixed \( g \in \tilde{E}' \) the function \( G = (G^1, G^2) \) satisfies
\[
\begin{align*}
G(0) & = s_0, \\
\partial_t G^1(t) & = p^{B-A}[g(t)] dt, \\
\partial_t G^2(t, x) & = p^{B-A}[g(t)] G^2(t, x) + f[g(t)](x) \quad \forall (t, x) \in [0, T] \times \mathbb{R}.
\end{align*}
\]
Moreover, in this case \( G \in \tilde{E}' \) and there exist two constants \( \bar{J}, \underline{L} < \infty \), which do not depend on \( g \), such that
\[
|G^2(t, x)| \leq \bar{J}, \quad |G^2(t, x)| \leq \underline{L} \quad \forall (t, x) \in [0, T] \times \mathbb{R}.
\]

Proof. It follows from the general theory of first-order PDEs that \( G \) solves the claimed PDE. Moreover, by Assumptions \( 2.1 \) and \( 2.3 \)
\[
\begin{align*}
|G^2_0(t, x)| & \leq \sup_{x \in \mathbb{R}} |v_0'(x)| + T \sup_{x \in \mathbb{R}} |(f[s])'(x)| =: \bar{J} < \infty, \\
|G^2_t(t, x)| & \leq |G^2_0(t, x)| + |f[g(t)](x)| \leq \bar{J} + M =: \underline{L} < \infty
\end{align*}
\]
as well as
\[
\sup_{t \in [0, T]} \|G(t)\|_E \leq B_0 + T + \|v_0\|_{L^2} + T \sup_{s \in E} \|f[s]\|_{L^2} \leq \underline{L}.
\]
\( \square \)

If we find a fixed point of \( F \) which lies in \( \tilde{E}' \), then Lemma 3.1 tells us that it must indeed solve \( (2) \). To do this, we will show that the function \( F \) is Lipschitz continuous to conclude with Banach’s fixed point theorem. In the following we write \( F_{B,t}(g) := G^1(t) \) and \( F_{v,t}(g) := G^2(t) \) for \( G \) defined as above. Then:
\[
|F_{B,t}(g) - F_{B,t}(\tilde{g})| \leq \int_0^t \|p^{B-A}[g(s)] - p^{B-A}[\tilde{g}(s)]\| ds \leq 2L \int_0^t \|g(s) - \tilde{g}(s)\|_E ds.
\]
Moreover,
\[
\|F_{v,t}(g) - F_{v,t}(\tilde{g})\|_{L^2} \leq \left\| v_0 \left( \cdot + \int_0^t p^{B-A}[g(s)] ds \right) - v_0 \left( \cdot + \int_0^t p^{B-A}[\tilde{g}(s)] ds \right) \right\|_{L^2} + \int_0^t \left| f[g(s)] \left( \cdot + \int_s^t p^{B-A}[g(u)] du \right) - f[\tilde{g}(s)] \left( \cdot + \int_s^t p^{B-A}[\tilde{g}(u)] du \right) \right|_{L^2} ds.
\]
Since \( v_0' \) is uniformly bounded with bounded support the mean value theorem along with Assumption \( 2.3 \) yields,
\[
\left\| v_0 \left( \cdot + \int_0^t p^{B-A}[g(s)] ds \right) - v_0 \left( \cdot + \int_0^t p^{B-A}[\tilde{g}(s)] ds \right) \right\|_{L^2} \leq C \int_0^t \|g(s) - \tilde{g}(s)\|_E ds.
\]
Similarly, as \( |(f[s])'(\cdot)| \) is uniformly bounded and has bounded support in \([-M, M]\) for all \( s \in E' \) and as \( f \) is Lipschitz continuous,
\[
\left\| f[g(s)] \left( \cdot + \int_s^t p^{B-A}[g(u)] du \right) - f[\tilde{g}(s)] \left( \cdot + \int_s^t p^{B-A}[\tilde{g}(u)] du \right) \right\|_{L^2} \leq C \int_s^t \left| p^{B-A}[g(u)] - p^{B-A}[\tilde{g}(u)] \right| du + \|f[g(s)] - f[\tilde{g}(s)]\|_{L^2} \leq 2CL \int_s^t \|g(u) - \tilde{g}(u)\|_E du + L \|g(s) - \tilde{g}(s)\|_E.
\]
Hence, there exists some $\hat{K} > 0$ such that for all $t \in [0, T]$,  
\[
\|F_t(g) - F_t(\tilde{g})\|_E \leq \hat{K} \int_0^t \|g(s) - \tilde{g}(s)\|_E \, ds.
\]

Now, the space $E$ is also a Banach space with respect to the equivalent weighted norm  
\[
\|g\|_\ast := \sup_{0 \leq t \leq T} e^{-\alpha t} \|g(t)\|_E
\]
for any $\alpha > 0$. Choosing $\alpha := 2\hat{K}$ we get  
\[
\|F_t(g) - F_t(\tilde{g})\|_E \leq \hat{K} \int_0^t e^{2\hat{K}s} \|g - \tilde{g}\|_\ast \, ds \leq \frac{1}{2}e^{2\hat{K}t} \|g - \tilde{g}\|_\ast
\]
and  
\[
\|F(g) - F(\tilde{g})\|_\ast \leq \frac{1}{2} \|g - \tilde{g}\|_\ast.
\]

Therefore, by Banach’s fixed point theorem there exists a unique fixed point $\hat{S}$ of $F$. As noted above $\hat{S}$ solves (2). Moreover, the sequence of continuous time models defined via $\hat{S}^0 \equiv s_0$ and $\hat{S}^{m+1} := F\left(\hat{S}^m\right)$, $m \in \mathbb{N}_0$, converges to $\hat{S}$. We have the following result.

**Theorem 3.2.** The fixed point $\hat{S}$ solves the ODE/PDE system (2) and  
\[
\lim_{m \to \infty} \sup_{t \in [0, T]} \left\| \hat{S}^m(t) - \hat{S}(t) \right\|_E = 0.
\]

The following lemma shows that $\hat{S}^m = \left(\hat{B}^m, \hat{v}^m\right)$ is Lipschitz continuous with respect to time.

**Lemma 3.3.** There exists a constant $\hat{L} > 0$ such that for all $m \in \mathbb{N}$ and all $s, t \in [0, T]$,  
\[
\left\| \hat{S}^m(t) - \hat{S}^m(s) \right\|_E \leq \hat{L}|t - s|.
\]

**Proof.** W.l.o.g. $s < t$. By the mean value theorem there exists some $u \in (s, t)$ and for every $x \in \mathbb{R}$ a point $u_x \in (s, t)$ such that  
\[
\left| \hat{B}^m(t) - \hat{B}^m(s) \right| = \left| \left(\hat{B}^m\right)'(u) \right| (t - s) = \| p^{B-A} \left[ \hat{S}^{m-1}(u) \right] \| (t - s) \leq t - s,
\]
\[
\left| \hat{v}^m(t, x) - \hat{v}^m(s, x) \right| = \left| \hat{v}_t^m(u_x, x) \right| (t - s) \leq \hat{T}(t - s),
\]

where the last inequality follows from Lemma 3.1. Since the function $\hat{v}^m(t, \cdot)$ and its partial derivatives have compact support in $[-M - T, M + T]$,  
\[
\| \hat{v}^m(t) - \hat{v}^m(s) \|_{L^2} = \left\| \mathbb{1}_{[-M - T, M + T]}(\cdot) \left| \hat{v}_t^{m+1}(u_x, x) \right| (t - s) \right\|_{L^2} \leq \sqrt{2(M + T)} \cdot \hat{T}(t - s).
\]
Setting $\hat{L} := 1 + \hat{T}\sqrt{2(M + T)}$ it follows that  
\[
\left\| \hat{S}^m(t) - \hat{S}^m(s) \right\|_E \leq \hat{L}|t - s| \quad \forall m \in \mathbb{N}.
\]

\[\square\]
3.2. Uniqueness. In order to show uniqueness of the solution to (2), we assume to the contrary that there exists another solution $S$ and define the shifted volume density processes

$$\tilde{\nu}(t, x) := \hat{v} \left( t, x - \int_0^t p^{B-A} \left[ \hat{S}(s) \right] ds \right), \quad \tilde{\psi}(t, x) := v \left( t, x - \int_0^t p^{B-A} \left[ S(s) \right] ds \right).$$

Then

$$\tilde{\nu}(t, x) = \hat{v}_t \left( t, x - \int_0^t p^{B-A} \left[ \hat{S}(s) \right] ds \right) - p^{B-A} \left[ \hat{S}(t) \right] \cdot \hat{v}_x \left( t, x - \int_0^t p^{B-A} \left[ \hat{S}(s) \right] ds \right)$$

and similarly for $\tilde{\psi}$. Integrating with respect to $t$ and using that $\tilde{\nu}(0) = \hat{v}(0) = v(0) = \tilde{\psi}(0)$ we obtain from Assumption 2.3 and the mean value theorem,

$$\|\tilde{\nu}(t) - \tilde{\psi}(t)\|_{L^2} \leq \int_0^t \left\| \left[ f \left[ \hat{S}(s) \right] \left( - \int_0^s p^{B-A} \left[ \hat{S}(u) \right] du \right) - f \left[ S(s) \right] \left( - \int_0^s p^{B-A} \left[ S(u) \right] du \right) \right] \right\|_{L^2} ds$$

Moreover, as $|\hat{v}_x(t, \cdot)|$ is uniformly bounded by $J$ with bounded support according to Lemma 3.1, again by the mean value theorem

$$\|\hat{v}(t, \cdot) - v(t, \cdot)\|_{L^2} = \|\hat{v} \left( t, \cdot - \int_0^t p^{B-A} \left[ S(s) \right] ds \right) - v \left( t, \cdot - \int_0^t p^{B-A} \left[ S(s) \right] ds \right) \|_{L^2}$$

Furthermore,

$$\hat{B}(t) - B(t) \leq \int_0^t \left\| p^{B-A} \left[ \hat{S}(s) \right] - p^{B-A} \left[ S(s) \right] \right\| ds \leq 2L \int_0^t \|\hat{S}(s) - S(s)\|_E ds.$$

Therefore,

$$\|\hat{S}(t) - S(t)\|_E \leq C \int_0^t \|\hat{S}(s) - S(s)\|_E ds$$

and the continuous version of Gronwall’s lemma, cf. Lemma [A.3] implies that $\hat{S} \equiv S$.

4. Convergence of the discrete order book models

The goal of this section is to prove Theorem 2.3. Note that as opposed to [11] we cannot treat the price process independently of the volume densities because the conditional event probabilities and order placements resp. cancelations do depend on both, prices and volumes.

In the following we set for all $n \in \mathbb{N}$ and $k \in \mathbb{R}$,

$$\left( T^{(n)}_+ \right)^k (f)(\cdot) := f \left( \cdot + k\Delta x^{(n)} \right).$$
Especially, this means that \( (T_+^{(n)})^{-1} \equiv T_-^{(n)} \). The following important Lemma deals with multiple applications of the translation operator. It will be used repeatedly in what follows.

**Lemma 4.1.** Let \( f \in L^2 \). Then for all \( n \in \mathbb{N} \) and \( k \in \mathbb{R} \):

a) the translation operator is isometric:

\[
\left\| \left( T_+^{(n)} \right)^k (f) \right\|_{L^2} = \| f \|_{L^2},
\]

b) If \( f \) satisfies \( f(x) = f \left(l \Delta x^{(n)}\right) \) for all \( x \in [l \Delta x^{(n)}, (l + 1) \Delta x^{(n)}] \) and all \( l \in \mathbb{N} \), then for all \( k \in \mathbb{R} \),

\[
\left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} \leq (|k| + 1) \left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2}.
\]

**Proof.** Part a) is trivial. To prove b) we first consider the case \( k \in \mathbb{N} \) and claim that in this case even

\[
\left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} \leq k \left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2}.
\]

Obviously, this is true for \( k = 1 \). Assuming that it is true for \( k - 1 \) we get

\[
\left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} \leq \left\| T_+^{(n)} \circ \left( T_+^{(n)} \right)^{k-1} - I \right\|_{L^2} + \left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2}
\]

\[
= \left\| \left( T_+^{(n)} \right)^{k-1} - I \right\|_{L^2} + \left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} \leq k \left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2}
\]

and hence the inequality follows by induction for all \( k \in \mathbb{N} \). Next for \( k \in (0, 1) \) either

\[
\left( T_+^{(n)} \right)^k - I \right) (f) = \left( T_+^{(n)} \right)^k (f) \quad \text{or} \quad \left( T_+^{(n)} \right)^k - I \right) (f) = 0, \quad x \in \mathbb{R}_+.
\]

Therefore, in this case

\[
\left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} \leq \left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2}.
\]

Now take any \( k \in \mathbb{R}_+ \). Then,

\[
\left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} \leq \left\| \left( T_+^{(n)} \right)^{\lfloor k \rfloor} - I \right\|_{L^2} + \left\| \left( T_+^{(n)} \right)^{k - \lfloor k \rfloor} - I \right\|_{L^2}
\]

\[
\leq \lfloor k \rfloor \left\| T_+^{(n)} - I \right\|_{L^2} + \left\| \left( T_+^{(n)} \right)^{k - \lfloor k \rfloor} - I \right\|_{L^2}
\]

\[
\leq (k + 1) \left\| T_+^{(n)} - I \right\|_{L^2}.
\]

Finally, the general case follows from the isometry property of the translation operator:

\[
\left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} = \left\| T_+^{(n)} \right\|_{L^2} - \left\| \left( T_+^{(n)} \right)^k - I \right\|_{L^2} = \left\| I - \left( T_+^{(n)} \right)^{-k} \right\|_{L^2}.
\]
4.1. A deterministic approximation of the discrete model for fixed \( n \in \mathbb{N} \). Recall that by definition
\[
v_k^{(n)} = \left( T_+^{(n)} \right) \sum_{j=0}^{k-1} 1_j^{(n),B-A} \left( v_0^{(n)} \right) + \Delta v^{(n)} \sum_{j=0}^{k-1} \left( T_+^{(n)} \right) \sum_{i=j}^{k-1} 1_i^{(n),B-A} \left( M_i^{(n)} \right).
\]

For each \( n \in \mathbb{N} \) we are now going to define two approximations to the discrete model dynamics \( S^{(n)} \), a deterministic non-linear approximation \( \tilde{S}^{(n)} \) in which the event indicator variables are replaced by their averages conditioned on the state of the approximating sequence and a random approximation \( S^{(n)} \) in which the event indicator variables are replaced by their averages conditioned on the random state of the original state sequence.

More precisely, we define for each \( n \in \mathbb{N} \) the process \( \tilde{S}^{(n)} \) through
\[
\tilde{B}_k^{(n)} := B_0^{(n)} + \Delta x^{(n)} \sum_{j=0}^{k-1} p^{(n),B-A} \left[ \tilde{S}_j^{(n)} \right]
\]
and
\[
\tilde{v}_k^{(n)} := \left( T_+^{(n)} \right) \sum_{j=0}^{k-1} 1_j^{(n),B-A} \left[ \tilde{S}_j^{(n)} \right] \left( v_0^{(n)} \right) + \Delta v^{(n)} \sum_{j=0}^{k-1} \left( T_+^{(n)} \right) \sum_{i=j}^{k-1} 1_i^{(n),B-A} \left[ \tilde{S}_i^{(n)} \right] \left( f^{(n)} \left[ \tilde{S}_j^{(n)} \right] \right)
\]
for \( k = 0, 1, \ldots, \left\lfloor \frac{T}{\Delta t^{(n)}} \right\rfloor \).

Moreover, we define a second process \( S^{(n)} \) through
\[
B_k^{(n)} := B_0^{(n)} + \Delta x^{(n)} \sum_{j=0}^{k-1} p^{(n),B-A} \left[ S_j^{(n)} \right]
\]
and
\[
v_k^{(n)} := \left( T_+^{(n)} \right) \sum_{j=0}^{k-1} 1_j^{(n),B-A} \left[ S_j^{(n)} \right] \left( v_0^{(n)} \right) + \Delta v^{(n)} \sum_{j=0}^{k-1} \left( T_+^{(n)} \right) \sum_{i=j}^{k-1} 1_i^{(n),B-A} \left[ S_i^{(n)} \right] \left( f^{(n)} \left[ S_j^{(n)} \right] \right)
\]
for \( k = 0, 1, \ldots, \left\lfloor \frac{T}{\Delta t^{(n)}} \right\rfloor \).

In a first step we are now going to show that the sequence \( S^{(n)} \) approximates \( S^{(n)} \). The proof uses a weak law of large numbers for triangular martingale difference arrays, which can be found in the appendix. Subsequently we show the desired convergence of \( \tilde{S}^{(n)} \) to the discrete model dynamics \( S^{(n)} \).

In what follows \( C > 0 \) denotes a generic constant that may vary from line to line and is independent of any index \( h, i, j, k, l, n \).

**Theorem 4.2.** Under the assumptions of Theorem 2.8, for all \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| S^{(n)}(t) - S^{(n)}(t) \right\|_E > \varepsilon \right) = 0.
\]

**Proof.** For the bid price component we have
\[
\left| B_k^{(n)} - \tilde{B}_k^{(n)} \right| = \Delta x^{(n)} \sum_{j=0}^{k-1} 1_j^{(n),B-A} - 1_j^{(n),A} - p^{(n),B} \left[ S_j^{(n)} \right] + p^{(n),A} \left[ S_j^{(n)} \right] .
\]
By definition the random variables
\[ Y_j^{(n)} := \Delta_x^{(n)} \left( l_j^{(n),B} - l_j^{(n),A} - p^{(n),B} \left[ S_j^{(n)} \right] + p^{(n),A} \left[ S_j^{(n)} \right] \right), \quad j \leq \left\lfloor \frac{T}{\Delta t^{(n)}} \right\rfloor, \quad n \in \mathbb{N}, \]
form a triangular martingale difference array. If we can show that there exists \( \alpha > \frac{1}{2} \) such that
\[
\sup_{j \leq T/\Delta t^{(n)}} \left( \frac{\mathbb{E} \left| Y_j^{(n)} \right|^2}{\left( \Delta t^{(n)} \right)^{2\alpha}} \right) < \infty,
\]
then Theorem A.1 will imply that
\[
\sup_{k \leq T/\Delta t^{(n)}} \left| \sum_{j=0}^{k-1} Y_j^{(n)} \right| = o(1) \quad \text{in probability.}
\]
Indeed, this follows immediately from Assumption 2.6 with \( \alpha := \frac{1 + \beta}{2} \), because
\[
\mathbb{E} \left| Y_j^{(n)} \right|^2 \leq 4 \left( \Delta x^{(n)} \right)^2 \mathbb{E} \left( p^{(n),A} \left[ S_j^{(n)} \right] + p^{(n),B} \left[ S_j^{(n)} \right] \right) \leq 4 \left( \Delta x^{(n)} \right)^2 \Delta p^{(n)} \leq 4C \left( \Delta t^{(n)} \right)^{1+\beta}.
\]
Next we consider the volume component:
\[
\left\| v_k^{(n)} - v_k^{(n)} \right\|_{L^2} \leq \left\| \left( T^{(n)} \right)^{\sum_{j=0}^{k-1} l_j^{(n),B}} - \left( T^{(n)} \right)^{\sum_{j=0}^{k-1} l_j^{(n),B-A}} \right\|_{L^2} + \Delta v^{(n)} \sum_{j=0}^{k-1} \left\| \left( T^{(n)} \right)^{\sum_{i=j}^{k-1} l_i^{(n),B-A}} - \left( T^{(n)} \right)^{\sum_{i=j}^{k-1} l_i^{(n),B-A}} \right\|_{L^2}.
\]
Let us first deal with the second term. Due to the norm invariance of the translation operator this term equals
\[
\left\| \Delta v^{(n)} \sum_{j=0}^{k-1} \left( T^{(n)} \right)^{\sum_{i=j}^{k-1} l_i^{(n),B-A}} \left( M_j^{(n)} - f^{(n)} \left[ S_j^{(n)} \right] \right) \right\|_{L^2}.
\]
The variables
\[ X_j^{(n)} := \Delta v^{(n)} \left( T^{(n)} \right)^{\sum_{i=0}^{j-1} l_i^{(n),B-A}} \left( M_j^{(n)} - f^{(n)} \left[ S_j^{(n)} \right] \right) \]
form a triangular martingale difference array. As \( M_j^{(n)} \) is bounded by \( M/\Delta x^{(n)} \) according to Assumption 2.2
\[
\mathbb{E} \left\| X_j^{(n)} \right\|_{L^2}^2 \leq 4 \left( \Delta v^{(n)} \right)^2 \mathbb{E} \left\| T^{(n)} \right\|_{L^2}^2 \left\| \sum_{i=0}^{j-1} l_i^{(n),B} \left( M_j^{(n)} \right) \right\|_{L^1}^2 = 4 \left( \Delta v^{(n)} \right)^2 \mathbb{E} \left\| M_j^{(n)} \right\|_{L^1}^2 \leq 4M^2 \left( \Delta v^{(n)} \right)^2/\Delta x^{(n)}.
\]
Therefore, for \( \tilde{\alpha} := \frac{2+\beta}{2} > \frac{1}{2} \) we have
\[
\sup_{j \leq T/\Delta t^{(n)}} \left( \frac{\mathbb{E} \left| X_j^{(n)} \right|^2}{\left( \Delta t^{(n)} \right)^{2\tilde{\alpha}}} \right) < \infty.
\]
and hence by Theorem A.1,

\[ \sup_{k \leq \left\lceil \frac{x}{\Delta t(n)} \right\rceil} \left\| \sum_{j=0}^{k-1} X_j^{(n)} \right\|_{L^2} = o(1) \quad \text{in probability.} \]

Regarding the third term note that by Lemma 4.1, Assumption 2.3, and Assumption 2.6, we have

\[ \Delta v^{(n)}(k-1) \sum_{j=0}^{k-1} \left\| \left( T_{j+1}^{(n)} \right) \sum_{i=j}^{k-1} \frac{1}{S_i^{(n)}} - \left( T_{j+1}^{(n)} \right) \sum_{i=j}^{k-1} \frac{1}{p_i^{(n),B-A}[S_i^{(n)}]} \right\|_{L^2} \]

\[ = \Delta v^{(n)}(k-1) \sum_{j=0}^{k-1} \left\| \left( T_{j+1}^{(n)} \right) \sum_{i=j}^{k-1} \frac{1}{S_i^{(n)}} - \left( T_{j+1}^{(n)} \right) \sum_{i=j}^{k-1} \frac{1}{p_i^{(n),B-A}[S_i^{(n)}]} - I \left( f^{(n)} \left[ S_j^{(n)} \right] \right) \right\|_{L^2} \]

\[ \leq \Delta v^{(n)}(k-1) \sum_{j=0}^{k-1} \left( \left\| \sum_{i=j}^{k-1} \frac{1}{S_i^{(n)}} - \frac{1}{p_i^{(n),B-A}[S_i^{(n)}]} \right\|_{L^2} + 1 \right) \left\| \left( T_{j+1}^{(n)} - I \right) \left( f^{(n)} \left[ S_j^{(n)} \right] \right) \right\|_{L^2} \]

\[ \leq C \sup_{j \leq k-1} \left( \left\| \sum_{i=j}^{k-1} \frac{1}{S_i^{(n)}} - \frac{1}{p_i^{(n),B-A}[S_i^{(n)}]} \right\|_{L^2} + 1 \right) \Delta x^{(n)}(k-1). \]

Thus, we may conclude as above for the price component that the term converges to zero in probability uniformly in \( k \leq T/\Delta t^{(n)} \). The convergence of the first term in the above decomposition follows analogously.

\[ \square \]

**Theorem 4.3.** Under the assumptions of Theorem 2.8 for all \( \varepsilon > 0 \),

\[ \lim_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| S^{(n)}(t) - \bar{S}^{(n)}(t) \right\|_E > \varepsilon \right) = 0. \]

**Proof.** We have

\[ \left\| S^{(n)} \left( t_k \right) - \bar{S}^{(n)} \left( t_k \right) \right\|_E \leq \left\| S^{(n)} \left( t_k \right) - \bar{S}^{(n)} \left( t_k \right) \right\|_E + \left\| \bar{S}^{(n)} \left( t_k \right) - \bar{S}^{(n)} \left( t_k \right) \right\|_E. \]

According to Theorem 4.2, the first term converges uniformly to zero in probability. In the following we use the Lipschitz continuity of the \( p' \)'s and \( f' \)'s formulated in Assumption 2.3 to derive an appropriate upper bound for the second term, which allows us to apply the discrete Gronwall lemma in order to prove the assertion. We start again with the bid price component:

\[ \left\| \bar{B}_k^{(n)} - \bar{B}_k^{(n)} \right\| = \Delta x^{(n)} \left\| k-1 \sum_{j=0}^{k-1} \frac{1}{p_j^{(n),B-A}[S_j^{(n)}]} - \frac{1}{p_j^{(n),B-A}[S_j^{(n)}]} \right\|_E \leq \Delta x^{(n)} \Delta p^{(n)}(k-1) \sum_{j=0}^{k-1} \left\| S_j^{(n)} - \bar{S}_j^{(n)} \right\|_E. \]

Moreover, using Lemma 4.1 and Assumption 2.3,

\[ \Delta v^{(n)}(k-1) \sum_{j=0}^{k-1} \left\| \left( T_{j+1}^{(n)} \right) \sum_{i=j}^{k-1} \frac{1}{S_i^{(n)}} - \left( T_{j+1}^{(n)} \right) \sum_{i=j}^{k-1} \frac{1}{p_i^{(n),B-A}[S_i^{(n)}]} \right\|_{L^2} \leq \Delta v^{(n)} L \sum_{j=0}^{k-1} \left\| \bar{S}_j^{(n)} - S_j^{(n)} \right\|_E. \]
Similarly to the proof of Theorem 4.2 using Lemma 4.1 and Assumption 2.3 we can derive the inequality
\[
\left\| \left( T^{(n)}_+ \right)^{k-1} p^{(n), B-A} \left[ S^{(n)}_i \right] - \left( T^{(n)}_+ \right)^{k-1} p^{(n), B-A} \left[ g^{(n)}_i \right] \right\|_{L^2} \leq C \left( \sum_{i=j}^{k-1} p^{(n), B-A} \left[ S^{(n)}_i \right] - p^{(n), B-A} \left[ S^{(n)}_i \right] \right) \Delta x^{(n)}
\]
\[
\leq C \Delta x^{(n)} \Delta p^{(n)} \sum_{i=0}^{k-1} \left\| S^{(n)}_i - S^{(n)}_i \right\|_E + C \Delta x^{(n)}.
\]
Relying on Assumption 2.1 instead of Assumption 2.2 we may replace \( f^{(n)} \left[ S^{(n)}_j \right] \) by \( v^{(n)}_0 \) in the above computations to get a similar estimate for the initial volume term.

Finally, putting everything together we have by Assumption 2.6
\[
\left\| \bar{v}_k^{(n)} - \bar{v}_k^{(n)} \right\|_{L^2} \leq \left\| \left( T^{(n)}_+ \right)^{k-1} p^{(n), B-A} \left[ S^{(n)}_j \right] - \left( T^{(n)}_+ \right)^{k-1} p^{(n), B-A} \left[ g^{(n)}_j \right] \right\|_{L^2} \leq C \Delta x^{(n)} \Delta p^{(n)} \sum_{i=0}^{k-1} \left\| S^{(n)}_i - S^{(n)}_i \right\|_E + C \Delta x^{(n)}.
\]

Therefore, for some sequence \( (a_n) \) converging to zero in probability and for all \( k \leq \frac{T}{\Delta t^{(n)}} \) we get the following uniform estimate by means of the discrete Gronwall Lemma A.2
\[
\left\| \bar{S}^{(n)}_k - \bar{S}^{(n)}_k \right\|_E \leq a_n + C \Delta t^{(n)} \sum_{l=0}^{k-1} \left\| \bar{S}^{(n)}_l - S_l^{(n)} \right\|_E \leq a_n + C \Delta x^{(n)} \sum_{l=0}^{k-1} \left\| \bar{S}^{(n)}_l - S_l^{(n)} \right\|_E \leq a_n + C \Delta x^{(n)} \sum_{l=0}^{k-1} \left\| \bar{S}^{(n)}_l - S_l^{(n)} \right\|_E \leq a_n.
\]

4.2. Almost uniform iteration to the discrete approximation. In this section we approximate for each \( n \in \mathbb{N} \) the model \( \bar{S}^{(n)} \) iteratively by an \( \tilde{E} \)-valued sequence \( \left( \tilde{S}^{(n),m}_l \right)_m \) of limit order book models. To this end, we define for each \( n \in \mathbb{N} \) a function \( F^{(n)} : \tilde{E} \to \tilde{E} \) via
\[
F^{(n)} : g \mapsto G^{(n)} = \left( G^{(n)}_{B,v}, G^{(n)}_{v} \right) \quad \text{with} \quad G^{(n)}(t) := G^{(n)}_j, \quad \text{if} \ t \in \left[ f^{(n)}_{i,j}, f^{(n)}_{j+1} \right),
\]
where \( C^{(n)}_0 := s^{(n)}_0 \) and for \( k \in \mathbb{N} \),
\[
C^{(n)}_{B,k} := G^{(n)}_{B,k-1} + \Delta x^{(n)} p^{(n), B-A} \left[ g^{(n)}_{k-1} \right]
\]
as well as
\[ G_{v,k}^{(n)} := \left( T_+^{(n)} \right)^{\sum_{j=0}^{k-1} p^{(n)} \cdot B - A} \left[ g_j^{(n)} \right] \left( v_0^{(n)} \right) + \Delta v^{(n)} \sum_{j=0}^{k-1} \left( T_+^{(n)} \right)^{\sum_{j=0}^{k-1} p^{(n)} \cdot B - A} \left[ g_j^{(n)} \right] \left( f^{(n)} \left[ g_j^{(n)} \right] \right) \]
with
\[ g_j^{(n)} := g \left( t_j^{(n)} \right). \]
We will write \( F_k^{(n)}(g) := G_k^{(n)} \) for \( G_k^{(n)} \) defined as above in what follows.

Using analogous arguments as in the proof of Theorem 4.3 one can find a constant \( K > 0 \) such that
\[ \left\| F_{k+1}^{(n)}(g) - F_{k+1}^{(n)}(\tilde{g}) \right\|_E \leq K \Delta v^{(n)} \sum_{j=0}^{k} \left\| g_j - \tilde{g}_j \right\|_E + K \Delta x^{(n)} \quad \text{for all } k \leq \frac{T}{\Delta t^{(n)}}, \ n \in \mathbb{N}. \]

Now as in Section 4.1 we define a weighted norm on \( \tilde{E} \) via
\[ \left\| g \right\|_{**} := \sup_{0 \leq t \leq T} e^{-3Kt} \left\| g(t) \right\|_E, \]
which allows us to get an estimate for the weighted norm with Lipschitz constant less than 1 up to an error of order \( \Delta x^{(n)} \). As for all \( n \in \mathbb{N}, \)
\[ \left\| F_{k+1}^{(n)}(g) - F_{k+1}^{(n)}(\tilde{g}) \right\|_E \leq K \Delta v^{(n)} \sum_{j=0}^{k} e^{3Kt^{(n)}_j} \left\| g - \tilde{g} \right\|_{**} + K \Delta x^{(n)} \]
\[ = K \Delta v^{(n)} \cdot \frac{e^{3K(k+1)\Delta t^{(n)}} - 1}{e^{3K\Delta t^{(n)}} - 1} \left\| g - \tilde{g} \right\|_{**} + K \Delta x^{(n)}, \]
there exists by Assumption 2.6 an \( N_0 \in \mathbb{N} \) such that for all \( n \geq N_0, \)
\[ e^{-3Kt^{(n)}_{k+1}} \left\| F_{k+1}^{(n)}(g) - F_{k+1}^{(n)}(\tilde{g}) \right\|_E \leq K \Delta v^{(n)} \frac{e^{3K\Delta t^{(n)}} - 1}{e^{3K\Delta t^{(n)}} - 1} \left\| g - \tilde{g} \right\|_{**} + K \Delta x^{(n)} \leq \frac{1}{2} \left\| g - \tilde{g} \right\|_{**} + K \Delta x^{(n)} \]
and therefore indeed
\[ \left\| F^{(n)}(g) - F^{(n)}(\tilde{g}) \right\|_{**} \leq \frac{1}{2} \left\| g - \tilde{g} \right\|_{**} + K \Delta x^{(n)} \quad \forall \ n \geq N_0. \]

W.l.o.g. we take \( N_0 = 1 \) in the following. For each \( n, m \in \mathbb{N}_0 \) we define a new discrete time model \( \tilde{S}^{(n),m} \) via \( \tilde{S}^{(n),0}(t) \equiv s_0^{(n)} \) as well as
\[ \tilde{S}^{(n),m+1} := F^{(n)} \left( \tilde{S}^{(n),m} \right). \]

**Theorem 4.4.**
\[ \lim_{n \to \infty} \sup_{0 \leq t \leq T} \left\| \tilde{S}^{(n),m}(t) - \tilde{S}^{(n)}(t) \right\|_E = 0. \]

**Proof.** First, note that by definition \( \tilde{S}^{(n)} \) is a fixed point of \( F^{(n)} \), i.e.
\[ F^{(n),m} \left( \tilde{S}^{(n)} \right) := \underbrace{F^{(n)} \circ \cdots \circ F^{(n)}}_{m \text{ times}} \left( \tilde{S}^{(n)} \right) = \tilde{S}^{(n)} \quad \forall \ m \in \mathbb{N}. \]
Hence, making use of the above computations we deduce that for every \( m, n \in \mathbb{N} \),
\[
\left\| \tilde{S}^{(n),m} - \tilde{S}^{(n)} \right\| \ast = \left\| F^{(n),m} \left( \tilde{S}^{(n),0} \right) - F^{(n),m} \left( \tilde{S}^{(n)} \right) \right\| \ast \\
\leq \left( \frac{1}{2} \right)^m \left\| \tilde{S}^{(n),0} - \tilde{S}^{(n)} \right\| \ast + \sum_{k=0}^{m-1} \left( \frac{1}{2} \right)^k K \Delta x^{(n)} \\
\leq \left( \frac{1}{2} \right)^m \sup_{0 \leq t \leq T} \left\| s_0^{(n)} - \tilde{S}^{(n)}(t) \right\| _E + 2K \Delta x^{(n)} \\
\leq \left( \frac{1}{2} \right)^{m-1} K + 2K \Delta x^{(n)}.
\]
Now the result follows from the equivalence of the weighted norm and the norm \( \sup_{0 \leq t \leq T} \left\| \cdot \right\| _E \). \hfill \Box

Especially, Theorem \ref{thm:4.4} implies that
\[
\lim_{m \to \infty} \lim_{n \to \infty} \sup_{0 \leq t \leq T} \left\| \tilde{S}^{(n),m}(t) - \tilde{S}^{(n)}(t) \right\| _E = \lim_{n \to \infty} \lim_{m \to \infty} \sup_{0 \leq t \leq T} \left\| \tilde{S}^{(n),m}(t) - \tilde{S}^{(n)}(t) \right\| _E = 0.
\]

4.3. Convergence of the discrete iteration to the continuous iteration. The goal of this section is to prove the following result.

**Theorem 4.5.** For all \( m \in \mathbb{N} \),
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left\| \tilde{S}^{(n),m}(t) - \hat{S}^m(t) \right\| _E = 0.
\]

**Proof.** To prove Theorem \ref{thm:4.5} we proceed by induction. Obviously, for \( m = 0 \) the claim holds by Assumption \ref{ass:2.1}. Now assume that the claim holds for \( m \) and consider the \((m+1)\)-th iteration: First, we show the convergence of the bid price process. Writing the integral as a limit of Riemann sums we have by Assumptions \ref{ass:2.1} and \ref{ass:2.6}
\[
\hat{B}^{m+1}(t) = \lim_{n \to \infty} B_0^{(n)}(t) + \lim_{n \to \infty} \sum_{j=0}^{\lfloor t/\Delta t^{(n)} \rfloor} \Delta x^{(n)} \Delta p^{(n)} p^{B-A} \left[ \hat{S}^m \left( t_j^{(n)} \right) \right].
\]
As \( p^{B-A} \) and \( \hat{S}^m \) are both Lipschitz continuous by Assumption \ref{ass:2.3} and Lemma \ref{lem:3.3} this convergence is uniform in \( t \in [0, T] \). Moreover by Assumption \ref{ass:2.3}
\[
\sup_{0 \leq t \leq T} \left| \sum_{j=0}^{\lfloor t/\Delta t^{(n)} \rfloor} \Delta x^{(n)} \Delta p^{(n)} \left( p^{B-A} \left[ \hat{S}^m \left( t_j^{(n)} \right) \right] - p^{B-A} \left[ \tilde{S}^{(n),m} \right] \right) \right| \\
\leq T \cdot \Delta x^{(n)} \Delta p^{(n)} \frac{\sup_{j \leq \lfloor t/\Delta t^{(n)} \rfloor} \left| p^{B-A} \left[ \hat{S}^m \left( t_j^{(n)} \right) \right] - p^{B-A} \left[ \tilde{S}^{(n),m} \right] \right|}{\Delta t^{(n)}} \\
\leq 2LT \cdot \Delta x^{(n)} \Delta p^{(n)} \frac{\sup_{0 \leq t \leq T} \left\| \hat{S}^m(t) - \tilde{S}^{(n),m}(t) \right\| _E}{\Delta t^{(n)}},
\]
which converges towards zero as \( n \to \infty \) by Assumption \ref{ass:2.6} and the induction hypothesis.

We now show the convergence of the buy side volume density function step by step. W.l.o.g. we only prove the convergence of the order placement / cancelation term. The convergence of the term involving the initial volume density function follows by analogous arguments. First note that a pointwise
Riemann sum approximation gives
\[
\int_0^t f \left[ \hat{S}^m(s) \right] \left( x + \int_s^t p^{B-A} \left[ \hat{S}^m(u) \right] du \right) ds
= \lim_{\Delta t(n) \to 0} \Delta t(n) \sum_{j=0}^{t/\Delta t(n)} \int_{t_j}^{t_{j+1}} f \left[ \hat{S}^m \left( t_j^{(n)} \right) \right] \left( x + \int_{t_j}^{t_{j+1}} p^{B-A} \left[ \hat{S}^m(u) \right] du \right).
\]

To show that the convergence also holds in \( L^2 \) observe that for all \( \mathbf{t} \leq \mathbf{t} \in [0,T] \) with \( |\mathbf{t} - \mathbf{t}| \leq \Delta t(n) \), making use of Assumption 2.3, Lemma 3.3 and the mean value theorem,
\[
\left\| f \left[ \hat{S}^m \left( \mathbf{t} \right) \right] \left( x + \int \mathbf{t} p^{B-A} \left[ \hat{S}^m(u) \right] du \right) - f \left[ \hat{S}^m \left( \mathbf{t} \right) \right] \left( x + \int \mathbf{t} p^{B-A} \left[ \hat{S}^m(u) \right] du \right) \right\|_{L^2} \\
\leq \left\| f \left[ \hat{S}^m \left( \mathbf{t} \right) \right] - f \left[ \hat{S}^m \left( \mathbf{t} \right) \right] \right\|_{L^2} + C \int \mathbf{t} p^{B-A} \left[ \hat{S}^m(u) \right] du \leq L \mathbf{t} + C |\mathbf{t} - \mathbf{t}| \leq C \Delta t(n).
\]

Therefore, the above convergence does indeed hold in \( L^2 \), uniformly in \( t \in [0,T] \).

Second, by similar arguments
\[
\Delta t(n) \sum_{j=0}^{t/\Delta t(n)} \left\| \left( T_{+}^{n} \right) \sum_{i=j}^{t/\Delta t(n)} p^{(n),B-A} \left[ \hat{S}^{(n),m}_i \right] \left( f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] - f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] \right) \right\|_{L^2} \\
\leq \sup_{j \leq t/\Delta t(n)} \left\| f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] \right\|_{L^2} \Delta t^{n} \Delta \mathbf{p}^{(n)} \sum_{i=j}^{t/\Delta t(n)} p^{(n),B-A} \left[ \hat{S}^{(n),m}_i \right] - \int \mathbf{t} p^{B-A} \left[ \hat{S}^m(u) \right] du \right\|_{L^2} \\
\leq C \Delta t(n) \Delta \mathbf{p}^{(n)} \sum_{i=j}^{t/\Delta t(n)} p^{B-A} \left[ \hat{S}^{(n),m}_i \right] - \int \mathbf{t} p^{B-A} \left[ \hat{S}^m(u) \right] du \right\|_{L^2}
\]
and as for the price component this term converges to zero uniformly in \( t \in [0,T] \).

Third, note that by Lemma 1.1 and Assumption 2.3 we have
\[
\Delta t(n) \sum_{j=0}^{t/\Delta t(n)} \left\| \left( T_{+}^{n} \right) \sum_{i=j}^{t/\Delta t(n)} p^{(n),B-A} \left[ \hat{S}^{(n),m}_i \right] \left( f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] - f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] \right) \right\|_{L^2} \\
\leq T \sup_{j \leq t/\Delta t(n)} \left\| f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] - f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] \right\|_{L^2} + \left\| f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] - f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] \right\|_{L^2} \\
\leq o(1) + L \sup_{j \leq t/\Delta t(n)} \left\| \hat{S}^{m} \left( t_j^{(n)} \right) - S^{m} \left( t_j^{(n)} \right) \right\|_{E^1},
\]
which converges to zero by the induction hypothesis.

Therefore, we proved that uniformly in \( t \in [0,T] \) the term
\[
\Delta t(n) \sum_{j=0}^{t/\Delta t(n)} \left( T_{+}^{n} \right) \sum_{i=j}^{t/\Delta t(n)} p^{(n),B-A} \left[ \hat{S}^{(n),m}_i \right] \left( f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] - f \left[ \hat{S}^{m} \left( t_j^{(n)} \right) \right] \right)
\]
converges in \( L^2 \) towards
\[
\int_0^t f \left[ \hat{S}^m(s) \right] \left( x + \int_s^t p^{B-A} \left[ \hat{S}^m(u) \right] du \right) ds.
\]
\[ \square \]
4.4. **Proof of Theorem 2.8.** Finally, let us put the partial convergence results proven in the previous subsections together to prove the convergence of the discrete limit order book models $S^{(n)}$ to $\hat{S}$. For this fix $\varepsilon > 0$. Then there exists by Theorem 3.2 an $M_1 = M_1(\varepsilon)$ such that for all $m \geq M_1$,

$$\left\| \hat{S}^m - \hat{S} \right\|_E < \frac{\varepsilon}{6}.$$ 

Also, by Theorem 4.4 there exist $M_2 = M_2(\varepsilon)$ and $N_1 = N_1(\varepsilon)$ such that for all $m \geq M_2$ and $n \geq N_1$, 

$$\left\| \hat{S}^{(n)} - \hat{S}^{(n),m} \right\|_E < \frac{\varepsilon}{6}.$$ 

We set $M_0 = M_0(\varepsilon) := M_1(\varepsilon) \lor M_2(\varepsilon)$. Then for all $n \geq N_1$, 

$$\left\| \hat{S}^{M_0} - \hat{S} \right\|_E + \left\| \hat{S}^{(n)} - \hat{S}^{(n),M_0} \right\|_E < \frac{\varepsilon}{3}.$$ 

Furthermore, Theorem 4.5 yields the existence of an $N_2 = N_2(M_0, \varepsilon) = N_2(M_0, \varepsilon) = N_2(\varepsilon)$ such that for all $n \geq N_2$, 

$$\left\| \hat{S}^{(n),M_0} - \hat{S}^{M_0} \right\|_E < \frac{\varepsilon}{6}.$$ 

Hence, for all $n \geq N_0 = N_0(\varepsilon) := N_1(\varepsilon) \lor N_2(\varepsilon)$, 

$$\left\| \hat{S}^{(n)} - \hat{S} \right\|_E \leq \left\| \hat{S}^{(n)} - \hat{S}^{(n),M_0} \right\|_E + \left\| \hat{S}^{(n),M_0} - \hat{S}^{M_0} \right\|_E + \left\| \hat{S}^{M_0} - \hat{S} \right\|_E < \frac{\varepsilon}{2}.$$ 

Finally, by Theorem 4.3 there exists for every $\delta \in (0,1)$ an $N_3 = N_3(\varepsilon, \delta)$ such that for all $n \geq N_3$, 

$$\mathbb{P}\left( \left\| \hat{S}^{(n)} - \hat{S} \right\|_E \leq \frac{\varepsilon}{2} \right) = 1 - \delta.$$ 

Setting $N = N(\varepsilon, \delta) := N_0(\varepsilon) \lor N_3(\varepsilon, \delta)$ we conclude that for all $n \geq N$, 

$$\mathbb{P}\left( \left\| \hat{S}^{(n)} - \hat{S} \right\|_E \leq \frac{\varepsilon}{2} \right) > 1 - \delta.$$ 

As $\varepsilon, \delta > 0$ were arbitrary, this completes the convergence proof.

\[\square\]

**Appendix A. Appendix**

A.1. **A weak law of large numbers for triangular martingale difference arrays.** The following weak law of large numbers for triangular martingale difference arrays relies on moment estimates from [14] and is shown in [11].

**Theorem A.1.** Let $(y^n_k, k = 1, \ldots, n; n \in \mathbb{N})$ be a triangular martingale difference array taking values in a real separable Hilbert space such that 

$$\sup_{m \leq n} \left( \sup_{n \in \mathbb{N}} (n^{2\alpha} \mathbb{E} |y^n_k|^2) \right) < \infty$$

for some $\alpha > \frac{1}{2}$. Then for all $\varepsilon > 0$, 

$$\lim_{n \to \infty} \mathbb{P}\left( \sup_{m \leq n} \left\| \sum_{k=1}^{m} y^n_k \right\| > \varepsilon \right) = 0.$$
A.2. Gronwall lemmas. For reference we cite the discrete and continuous version of Gronwall’s lemma which can for example be found in [7]:

**Lemma A.2.** Let $(y_m)_{m \geq 0}$, $(f_m)_{m \geq 0}$, and $(g_m)_{m \geq 0}$ be nonnegative sequences. If

$$y_m \leq f_m + \sum_{k=0}^{m-1} g_k y_k \quad \forall m,$$

then

$$y_m \leq f_m + \sum_{k=0}^{m-1} f_k g_k e^{\sum_{j=k+1}^{m-1} g_j}.$$

**Lemma A.3.** Let $\alpha, \beta, u$ be real-valued functions defined on some interval $I = [a, b]$, $a < b$, $a, b, \in \mathbb{R}$. Assume that $\beta$ and $u$ are continuous and that the negative part of $\alpha$ is integrable on every compact subinterval of $I$. If $\beta$ is non-negative and if $u$ satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds \quad \forall t \in I,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp \left( \int_s^t \beta(r)dr \right)ds, \quad t \in I.$$

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