MULTIPLIER BIALGEBRAS IN BRAIDED MONOIDAL CATEGORIES

GABRIELLA BÖHM AND STEPHEN LACK

ABSTRACT. Multiplier bimonoids (or bialgebras) in arbitrary braided monoidal categories are defined. They are shown to possess monoidal categories of comodules and modules. These facts are explained by the structures carried by their induced functors.

1. Introduction

A bimonoid (or bialgebra) $A$ in a braided monoidal category can equivalently be described without referring separately to the multiplication $m : A.A \to A$ and comultiplication $d : A \to A.A$ but only the unit, the counit, and the so-called fusion morphism $t_1 := (A.A \xrightarrow{d.1} A.A.A \xrightarrow{1.m} A.A)$. In terms of these data, the axioms are given by the fusion equation, expressed by the commutativity of

\[
\begin{array}{c}
A^3 \xrightarrow{1.t_1} A^3 \xrightarrow{h.1} A^3 \xrightarrow{1.t_1} A^3 \xrightarrow{b^{-1}.1} A^3 \\
A^3 \xrightarrow{1.t_1} A^3
\end{array}
\]

(where $b$ stands for the braiding) and some compatibilities of $t_1$ with the unit and the counit. This is analyzed in [6, Section 2] in terms of augmented lax tricocycloids: a morphism $t_1$ satisfies the fusion equation if and only if $t' = b^{-1}t_1$ satisfies the cocycle condition $(1.t')(t'.1)(1.t') = (t'.1)(1.b^{-1})(t'.1)$; an object $A$ equipped with a morphism $A.A \to A.A$ satisfying the cocycle condition is called a lax tricocycloid, or an augmented lax tricocycloid if it is also unital and counital. The existence of an antipode; that is, the Hopf condition on the bimonoid $A$, is equivalent to the invertibility of the fusion morphism $t_1$.

The notion of bimonoid is self-dual; that is, symmetric under reversing the arrows in the diagrams encoding the axioms. However, the above description in terms of the fusion morphism $t_1$ is not self-dual. Instead, there is an equivalent dual description in terms of the morphism

\[
t_2 := (A.A \xrightarrow{1.d} A.A.A \xrightarrow{m.1} A.A)
\]
which satisfies a modified fusion equation. In fact, using the braiding one can construct two further variants

\[
\begin{align*}
t_3 & := (A.A \xrightarrow{d.1} A.A.A \xrightarrow{1.b^{-1}} A.A \xrightarrow{1.m} A.A) \\
t_4 & := (A.A \xrightarrow{1.d} A.A.A \xrightarrow{b^{-1}.1} A.A.A \xrightarrow{m.1} A.A)
\end{align*}
\]

of the fusion morphism. Each of the morphisms \(t_1, t_2, t_3\) and \(t_4\) can be expressed in terms of each of the others with the help of the unit and the counit of \(A\).

Using the language of fusion morphisms, a (say, right) comodule \(V\) over the comonoid \(A\) can be described equivalently in terms of a morphism \(v : V.A \to V.A\). Coassociativity and counitality of the coaction \(V \to V.A\) translate to commutativity of the respective diagrams

\[
\begin{align*}
V.A.A & \xrightarrow{v.1} V.A.A & V.A.A & \xrightarrow{V.A.V} V.A.A \\
V.A.A & \xrightarrow{1.t_1} V.A.A & V.A.A & \xrightarrow{V.A.A} V.A.A
\end{align*}
\]

(where \(e\) denotes the counit of \(A\)). By duality, there is a symmetric description, in terms of \(t_1\), of left modules over the monoid \(A\).

A key example of a bialgebra (in the symmetric monoidal category \(\text{vec}\) of vector spaces over a field \(k\)) is given by the algebra \(k^G\) of all \(k\)-valued functions on a finite monoid \(G\). In the case of functions of finite support on an infinite monoid \(G\), we still have the multiplication and the counit of \(k^G\), but not the unit or comultiplication; instead of a bialgebra, under a mild cancellation assumption we have a multiplier bialgebra. Motivated by this, we consider various weakenings of the notion of bimonoid in a braided monoidal category.

First we ask what happens to the above picture if we give up self-duality and consider a counital, but no longer unital fusion morphism \(t_1 : A.A \to A.A\)? By this we mean that \(t_1\) satisfies the fusion equation (1.1) and is equipped with a morphism (the counit) \(e\) from \(A\) to the monoidal unit; such that \((1.e)t_1 = 1.e\). We do not require the existence of a unit and drop all axioms in [6] that involve the unit. Then composing \(t_1\) with \(e.1\), we can still equip \(A\) with a (no longer unital but still associative) multiplication \(A.A \to A\). But in the absence of a unit, there is no longer a comultiplication \(A \to A.A\).

Our study of counital but not necessarily unital fusion morphisms is motivated by Van Daele’s approach to \((\text{regular})\) multiplier Hopf algebra [9] (over a field) — based on generalizations of the fusion morphisms \(t_1\) and \(t_2\) (together with \(t_3\) and \(t_4\) in the regular case) which are in turn counital but non-unital fusion morphisms (in the symmetric monoidal category of vector spaces). In the absence of a unit for the algebra \(A\), none of the morphisms \(t_1, t_2, t_3\) and \(t_4\) determines the others. All of them are needed to formulate the axioms. Although in the absence of a unit none of the fusion morphisms \(t_1, t_2, t_3\) and \(t_4\) determines a comultiplication \(A \to A.A\); thanks to their compatibility axioms any pair of them determines a generalized comultiplication taking values in the multiplier algebra [4] of \(A.A\).

The main aim of this paper is to extend the definition of \((\text{regular})\) multiplier bialgebra to any braided monoidal category. In formulating the definition, any reference to multipliers is completely avoided. If applying it to the symmetric monoidal category of vector spaces,
our definition covers the notion of multiplier bialgebra in [2, Theorem 2.11] but it is slightly more general.

We build up gradually to the full structure of regular multiplier bimonoid, first developing in Section 2.1 the theory of a single (counital) fusion morphism $t_1$; then considering in Section 3.1 multiplier bimonoids, involving a pair of these, such as $t_1$ and $t_2$ or $t_3$ and $t_4$; before finally considering in Section 3.2 regular multiplier bimonoids, which involve all four fusion morphisms. We observed above that, unlike the situation with bimonoids, the notion of multiplier bimonoid is not stable under passage from a braided monoidal category to its opposite. But there are other duality principles available in braided monoidal category: one can reverse the multiplication, one can replace the braiding by its inverse, or one can do both. The resulting four variants in some sense correspond to the four fusion morphisms in a regular multiplier bimonoid. Our proofs often rely on these duality principles.

At each stage we study the corresponding notions of module and comodule. The unit of the monoid $A$ does not occur in the diagrams in (1.2). Hence they can be used to define comodules also for only counital fusion morphisms. We study this situation in Section 2.2, where we show that such comodules constitute a monoidal category admitting a strict monoidal forgetful functor to the base category. Comodules over a bimonoid (i.e. over a unital and counital fusion morphism) constitute a monoidal category because they induce bicomonads (that is, monoidal comonads). This is no longer true for a counital but non-unital fusion morphism. We describe a generalization of the notion of monoidal comonad which can be used to explain the monoidal structure in this case.

It is somewhat more delicate what should be a module over a counital but non-unital fusion morphism. Replacing an associative (left) action, we require the existence of a morphism $q : A.Q \rightarrow A.Q$ satisfying an appropriate fusion type equation. But if $A$ has no unit, it is not immediate what should replace the unitality of an action. One possibility is to require that the morphism

$$A.Q \xrightarrow{q} A.Q \xrightarrow{e_1} Q$$

be an epimorphism. But in order to develop the theory, we need this epimorphism to be preserved by tensoring on either side as well as by the various fusion morphisms. So for simplicity we suppose that it is preserved by all functors; equivalently, that it is a split epimorphism. (When the fusion morphism is unital as well as counital, and so corresponds to a bimonoid, this is equivalent to the usual unitality condition for an action.) In Section 2.3 we investigate additional assumptions on a counital fusion morphism, weaker than unitality, under which such modules constitute a monoidal category admitting a strict monoidal forgetful functor to the base category. Again, modules over a bimonoid (i.e. over a unital and counital fusion morphism) constitute a monoidal category because they induce bimonads (that is, opmonoidal monads). This is no longer true for a counital but non-unital fusion morphism. We describe a generalization of the notion of opmonoidal monad which can be used to explain the monoidal structure in this case.

Assuming regularity of a multiplier bimonoid in a braided monoidal category, its comodules and modules are defined in the respective Sections 4 and 5 as objects carrying compatible (co)module structures over appropriate pairs of the fusion morphisms $t_1, t_2, t_3$ and $t_4$. They are shown to constitute monoidal categories. This is explained by the structure carried by their induced functors. The definition of (co)modules in terms of pairs of the fusion morphisms $t_1, t_2, t_3$ and $t_4$ goes back to [10]. This approach becomes particularly important in the generalization [1] to weak multiplier bialgebras [2], when both
(co)actions are needed to equip any (co)module with the structure of a bimodule over the so-called base algebra.

One could also define multiplier Hopf monoids in a braided monoidal category as multiplier bimonoids whose constituent fusion morphisms \( t_1 \) and \( t_2 \) are isomorphisms. In the symmetric monoidal category of vector spaces this is known to be equivalent to the existence of an antipode taking values in the multiplier algebra \([9]\). In our general setting, however, the notion of multipliers is not available. The abstract categorical treatment of multipliers requires the additional assumption that our braided monoidal category is in fact closed. We plan to expound this construction, and the resulting theory of multiplier Hopf monoids, in a subsequent paper \([3]\).

**Notation.** Throughout, \( \mathcal{C} \) is a braided monoidal category (unless otherwise stated). We denote the monoidal product by \( ; \); the monoidal unit by \( I \); and the braiding by \( b \). For the monoidal product of several copies of the same object also the power notation is used: \( A^2 \). Composition of morphisms is denoted by juxtaposition and the identity morphism (at any object) is denoted by \( 1 \). We do not assume that the monoidal structure is strict but — relying on coherence — usually we omit explicitly denoting the associativity and unit isomorphisms. For a braided monoidal category \( \mathcal{C} \), we denote by \( \overline{\mathcal{C}} \) the same monoidal category \( \mathcal{C} \) with the inverse braiding \((b^{-1})_{Y,X}:Y.X \to X.Y\). The reverse \( \mathcal{C}^{rev} \) of \( \mathcal{C} \) means the same category \( \mathcal{C} \) with the opposite monoidal product \((X,Y)\mapsto Y.X\) (thus the same monoidal unit \( I \)) and the braiding \( b_{Y,X}:Y.X \to X.Y \). The braided monoidal categories \( \overline{\mathcal{C}}^{rev} \) and \( \overline{\mathcal{C}}^{rev} \) clearly coincide.

2. Counital fusion morphisms

Motivated by our definition in Section 3 of multiplier bimonoid in a braided monoidal category, in this section we study counital fusion morphisms possibly without unit. To any such creature we associate a monoidal category of appropriately defined comodules. Under further assumptions — about certain morphisms being epimorphic — we associate to it a second monoidal category of suitable modules. The monoidality of these categories is explained by the structure of the functors induced by a counital fusion morphism.

2.1. Definition and properties. Lax tricocycloids — corresponding bijectively to fusion morphisms — were defined in \([6]\); where they were equipped both with a counit and a unit. We need the following weakening.

**Definition 2.1.** A counital fusion morphism in a braided monoidal category \( \mathcal{C} \) is given by a pair of morphisms \( t:A^2 \to A^2 \) (called a fusion morphism) and \( e:A \to I \) (called the counit) such that the following diagrams commute.

\[
\begin{array}{cccccc}
A^3 & \xrightarrow{1.t} & A^3 & \xrightarrow{b.1} & A^3 & \xrightarrow{1.t} & A^3 \\
\downarrow{t.1} & & & & & \downarrow{t.1} & & \downarrow{1.t} & & \downarrow{1.e} & & \downarrow{1.e} \\
A^3 & & & & & A^3 & & & & \to & A \\
\end{array}
\]

We refer to the first condition as the fusion equation and to the second one as the counitality condition.

A class of examples, albeit unital ones, comes from bimonoids in braided monoidal categories:
Example 2.2. Consider a bimonoid $A$ in a braided monoidal category $C$. Denote its monoid structure by $(m : A^2 \to A, u : I \to A)$ and denote the comonoid structure by $(d : A \to A^2, e : A \to I)$. Then

$$t := (A^2 \xrightarrow{d} A^3 \xrightarrow{1} A^2)$$

is a counital fusion morphism. This is easiest to see using string diagrams; when we draw

$$m = \quad d = \quad e = \quad b = \quad .$$

The fusion equation follows by the calculation

where in the first equality we used naturality of the braiding, associativity of the multiplication, and coassociativity of the comultiplication. In the second equality we used multiplicativity of the comultiplication. The counitality condition follows by the calculation

where the first equality uses multiplicativity of the counit and the second one counitality of the comultiplication.

Further examples, no longer unital, come from multiplier bialgebras over a field:

Example 2.3. By a multiplier bialgebra over a field $k$ we mean the structure in [2, Theorem 2.11]. Based on [1, Theorem 1.2] and [2, Proposition 2.6], it can be described as follows. A multiplier bialgebra is given by a vector space $A$ equipped with an associative but not necessarily unital multiplication $m : A^2 \to A$; which is required to be surjective and non-degenerate in the sense that both maps $A \to \text{End}(A)$, $a \mapsto m(a,-)$ and $a \mapsto m(-,a)$ are injective. Furthermore, the existence of linear maps $t_1, t_2 : A^2 \to A^2$ and $e : A \to k$ is required such that the following axioms hold (where $b$ denotes the symmetry in $\text{vec}$; of course its components satisfy $b^2 = 1$).

(a) $t_1(m.1) = (m.1)(b.1)(1.t_1)(b.1)(1.t_1)$; equivalently,

$$t_2(1.m) = (1.m)(1.b)(t_2.1)(1.b)(t_2.1).$$

(b) Both maps $(m.1)(b.1)(1.t_1)$ and $(1.m)(1.b)(t_2.1)$ are surjective.

(c) $em = e.e$.

(d) $(t_2.1)(1.t_1) = (1.t_1)(t_2.1)$.

(e) $(e.1)t_1 = m = (1.e)t_2$.

We claim that $t_1$ is then a counital fusion morphism in $\text{vec}$. Indeed, by (e) and (d),

$$(2.2) \quad (m.1)(1.t_1) = (1.m)(t_2.1).$$
Postcomposing (2.2) with \(1.e\), we obtain by (c) and (e)

\[
m[1.(1.e)t_1] = (1.e)t_2.e = m(1.1.e),
\]

from which we conclude by the non-degeneracy of \(m\) that the counitality condition \((1.e)t_1 = 1.e\) holds. Furthermore,

\[
t_1 t_1 t_1 (2.2) = t_1 t_1 t_2 (d) = t_1 t_1 t_2 (a) = t_1 t_2 (2.2) =
\]

from which we conclude by the non-degeneracy of \(m\) that the fusion equation \((t_1.1)(t_1.1)^{13}(1.t_1) = (1.t_1)(t_1.1)\) holds. Dually, \(t_2\) is a counital fusion morphism in \(\text{vec}^{\text{rev}}\).

**Proposition 2.4.** For a counital fusion morphism \((t : A^2 \to A^2, e : A \to I)\) in a braided monoidal category \(\mathcal{C}\), the following assertions hold.

1. There is an associative multiplication \(m := (A^2 \xrightarrow{t} A^2 \xrightarrow{e.1} A)\).
2. \((1.m)(t.1) = t(1.m)\).
3. \((m.1)(b^{-1}.1)(1.t)(b.1)(1.t) = t(m.1)\).
4. \(em = e.e\).

**Proof.** In the following diagram, the top region commutes by the fusion equation and the triangular region commutes by the counitality condition. The bottom left region commutes by functoriality of the monoidal product and coherence of the braiding.

![Diagram](image)

This proves part (2) and postcomposing both paths by \(e.1\) we obtain part (1) by functoriality of the monoidal product. In order to prove (3), postcompose both sides of the fusion equation by \(e.1.1\) and use functoriality of the monoidal product. For (4), postcompose both sides of the counitality condition by \(e\) and use functoriality of the monoidal product. \(\Box\)

### 2.2. Comodules and right multiplier bicomonads

Our approach to the study of comodules over a counital fusion morphism is based on the use of the following notion.

**Definition 2.5.** A right multiplier bicomonad on a monoidal category \(\mathcal{C}\) is a functor \(G : \mathcal{C} \to \mathcal{C}\) equipped with natural transformations \(G_2 : GXGY \to G(GX.Y)\) and \(\varepsilon : GX \to X\).
such that the diagrams
\[(2.4)\]
\[
\begin{array}{c}
G.X.GY.GZ \xrightarrow{\overrightarrow{G_2}} G.X.G(Y.Z) \xrightarrow{\overrightarrow{G_2}} G(G.X.GY.Z) \\
\downarrow \overrightarrow{G_2.1} \\
G(G(X.Y).GZ) \xrightarrow{\overrightarrow{G_2}} G(G(G(X.Y).Z))
\end{array}
\]
\[
\begin{array}{c}
GX.GY \xrightarrow{\overrightarrow{G_2}} G(GX.Y) \\
\downarrow 1.\epsilon \\
GX.Y
\end{array}
\]

commute for any objects \(X, Y, Z\).

A left multiplier bicomonad on \(\mathcal{C}\) is a right multiplier bicomonad on the reverse monoidal category \(\mathcal{C}^{rev}\).

To a right multiplier bicomonad \(G\), we can associate a natural transformation
\[
GX.GY \xrightarrow{\overrightarrow{G_2}} G(GX.Y) \xrightarrow{G(\epsilon.1)} G(X.Y).
\]

It satisfies the same associativity condition as the binary part of a monoidal functor \(G\) but it has no nullary part; thus it makes \(G\) what might be called a semimonoidal functor. A right multiplier bicomonad is indeed a generalization of monoidal comonad (also known as ‘bicomonad’) as the following example shows:

**Example 2.6.** Let \((G, \delta, \epsilon)\) be a monoidal comonad on a monoidal category \(\mathcal{C}\); that is, assume the existence of a monoidal structure \((G_2 : G(-).G(-) \to G(-.-), G_0 : I \to GI)\) on the functor \(G\), and the monoidality of the coassociative comultiplication \(\delta : G \to G^2\) and of the counit \(\epsilon : G \to 1\). Consider
\[
\overrightarrow{G_2} := (GX.GY \xrightarrow{\delta.1} G^2.X.GY \xrightarrow{G_2} G(GX.Y)) .
\]

Then the diagrams in \((2.4)\) commute by the coassociativity of \(\delta\), the associativity of \(G_2\), and by the monoidality of \(\delta\) on one hand; and by the monoidality of \(\epsilon\) and the counitality of \(\delta\) on the other hand.

Further examples are provided by the functors induced by counital fusion morphisms:

**Example 2.7.** Let \((t : A^2 \to A^2, e : A \to I)\) be a counital fusion morphism in a braided monoidal category \(\mathcal{C}\). Consider the functor \(G := (-).A : \mathcal{C} \to \mathcal{C}\) with the natural transformations \(\epsilon := 1.e\) and
\[
\overrightarrow{G_2} := (X.A.Y.A \xrightarrow{1.b.1} X.Y.A^2 \xrightarrow{1.1.t} X.Y.A^2 \xrightarrow{1.b^{-1}.1} X.A.Y.A) .
\]

Then the diagrams in \((2.4)\) commute by the fusion equation and by the counitality condition on \(t\), respectively.

**Definition 2.8.** Consider a right multiplier bicomonad \(G\) on a monoidal category \(\mathcal{C}\). A \(G\)-comodule is an object \(V\) together with a natural transformation \(\overrightarrow{\varphi} : V.G(-) \to G(V.-)\) rendering commutative, for any objects \(Y\) and \(Z\), the diagrams
\[(2.5)\]
\[
\begin{array}{c}
V.GY.GZ \xrightarrow{1.\overrightarrow{G_2}} V.G(GY.Z) \xrightarrow{\overrightarrow{\varphi.1}} G(V.GY.Z) \\
\downarrow \overrightarrow{G_2} \\
G(V(Y).GZ) \xrightarrow{\overrightarrow{G_2}} G(G(V(Y).Z))
\end{array}
\]
\[
\begin{array}{c}
V.GY \xrightarrow{\overrightarrow{\varphi}} G(V.Y) \\
\downarrow G(\overrightarrow{\varphi.1}) \\
\downarrow 1.\epsilon \\
V.Y
\end{array}
\]

A morphism of \(G\)-comodules is a morphism \(f : V \to W\) such that \(G(f.1)\overrightarrow{\varphi} = \overrightarrow{\varphi}(f.1)\).
Comodules over a left multiplier bicomonad are defined as comodules over the corresponding right multiplier bicomonad on \( C^{\text{rev}} \).

These comodules behave well with respect to the monoidal structure of the base category:

**Theorem 2.9.** Consider a right multiplier bicomonad \( G \) on a monoidal category \( C \). Then the \( G \)-comodules of Definition 2.8 and their morphisms constitute a monoidal category such that the evident forgetful functor to \( C \) is strict monoidal.

**Proof.** The monoidal unit \( I \) is a \( G \)-comodule via the composite

\[
I.G(-) \xrightarrow{\cong} G \xrightarrow{\cong} G(I.-)
\]

of the unit isomorphisms. For \( G \)-comodules \( V \xrightarrow{\vartheta} V.G(-) \rightarrow G(V.-) \) and \( W \xrightarrow{\vartheta'} : W.G(-) \rightarrow G(W.-) \), also \( V.W \) is a \( G \)-comodule via

\[
V.W.G(-) \xrightarrow{1.G_0} V.G(W.-) \xrightarrow{\vartheta} G(V.W.-).
\]

The monoidal product of \( G \)-comodule morphisms, as well as the unit and associativity isomorphisms, are evidently morphisms of \( G \)-comodules. \( \Box \)

**Example 2.10.** We claim that applying Definition 2.8 to the right multiplier bicomonad \( G \) in Example 2.6 we obtain a category of comodules which is equivalent (in fact, isomorphic) to the usual Eilenberg-Moore category of comodules (or coalgebras). Therefore Theorem 2.9 generalizes the well-known result [7] that the monoidal structure of the base category lifts to the Eilenberg-Moore category of a monoidal comonad.

Consider the category of \( G \)-comodules in Definition 2.8 and the forgetful functor to \( C \) sending \((V, \vartheta) \) to \( V \). We shall show that for a right multiplier bicomonad \( G \) as in Example 2.6, the forgetful functor \( U \) has a right adjoint \( F \) such that \( G = UF \) as comonads. Indeed, let \( F \) take an object \( X \) to the \( G \)-comodule \((GX, \vartheta) \). The counit of the adjunction is \( \varepsilon : UF = G \rightarrow C \) and the unit \( \eta \) evaluated at a \( G \)-comodule \((V, \vartheta) \) is

\[
V \xrightarrow{1.G_0} V.GI \xrightarrow{\vartheta} GV.
\]

This is a morphism of \( G \)-comodules, in the sense of Definition 2.8 by commutativity of the diagram

\[
\begin{array}{ccc}
V.GX & \xrightarrow{1.G_0} & V.GI.GX \\
\downarrow{1.G_0} & & \downarrow{1.G_0} \\
V.GI.GX & \xrightarrow{1.G^2} & V.G^2I.GX \\
\downarrow{1.G_2} & & \downarrow{1.G_2} \\
V.GX & \xrightarrow{1.G(G_0)} & V.G(GI.X) \\
\downarrow{\vartheta} & & \downarrow{\vartheta} \\
G(V.X) & \xrightarrow{G(G_0)} & G(V.GI.X)
\end{array}
\]

for any object \( X \) of \( C \). The region on the right commutes by the fusion equation on \( \vartheta \); the top-left square commutes by the monoidality of \( \delta \), and the regions below it commute by the naturality of \( G_2 \) and \( \vartheta \), respectively. The leftmost region commutes by the unitality
of the monoidal structure \((G_2, G_0)\). Evaluating \(\eta\) at a \(G\)-comodule of the form \(GX = (GX, \overrightarrow{G})\), we obtain \(G_2(\delta \cdot 1)(1, G_0) = \delta\) (where the equality follows by the functoriality of the monoidal product and the unitality of \((G_2, G_0)\)). Hence the first triangle condition on the adjunction \(U \dashv F\) follows by the counitality of \(\delta\). The other triangle condition holds by the counitality of \(\overrightarrow{\varepsilon}\) and the monoidality of \(\varepsilon\). In order to see that the evidently conservative left adjoint functor \(U\) is comonadic, we need to prove that it creates \(U\)-absolute equalizers. Suppose then that \(f\) and \(g\) are morphisms \((V, \overrightarrow{v}) \rightarrow (W, \overrightarrow{w})\) and that

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & V \\
& & \xrightarrow{f} \\
& & \xrightarrow{g} W
\end{array}
\]

is an absolute equalizer in \(\mathcal{C}\). Then in the solid part of the diagram

\[
\begin{array}{ccc}
Z.GX & \xrightarrow{h.1} & V.GX \\
\xrightarrow{\overrightarrow{v}} & & \xrightarrow{f.1} \\
G(Z.X) & \xrightarrow{G(h.1)} & G(V.X)
\end{array}
\]

the rows are equalizers, and the parallel pairs commute serially with the verticals, thus there is a unique induced morphism \(\overrightarrow{z}\) as in the dashed part of the diagram. The axioms for \((Z, \overrightarrow{z})\) to be a comodule follow easily from the corresponding axioms for \((V, \overrightarrow{v})\) and the fact that \(h\) is an absolute monomorphism. The morphism \(h\) preserves the comodule structure by construction, and the universal property of \((Z, \overrightarrow{z})\) follows from the universal property of \(Z\) along with the fact that \(G(h.1)\) is a monomorphism.

Explicitly, the inverse of the comparison functor – from the category of \(G\)-comodules in Definition \([2.8]\) to the category of Eilenberg-Moore \(G\)-comodules – sends an Eilenberg-Moore comodule \(v : V \rightarrow GV\) to the equalizer of the comodule morphisms \(\delta, Gv : (GV, \overrightarrow{G}) \rightarrow (G^2V, \overrightarrow{G})\); that is, to the comodule

\[
V. G(-) \xrightarrow{v.1} GV. G(-) \xrightarrow{G_2} G(V.-).
\]

Further examples of the situation in Definition \([2.8]\) are provided by the following.

**Definition 2.11.** Consider a counital fusion morphism \((t : A^2 \rightarrow A^2, e : A \rightarrow I)\) in a braided monoidal category \(\mathcal{C}\). A **comodule** over it is an object \(V\) together with a morphism \(v : V.A \rightarrow V.A\) in \(\mathcal{C}\) rendering commutative the following diagrams.

\[
\begin{array}{ccc}
V.A^2 & \xrightarrow{1.t} & V.A^2 \\
\xrightarrow{v.1} & & \xrightarrow{1.t} V.A^2
\end{array}
\]

\[
\begin{array}{ccc}
V.A^2 & \xrightarrow{t} & V.A^2 \\
\xrightarrow{v.1} & & \xrightarrow{v.1} V.A^2
\end{array}
\]

\[
\begin{array}{ccc}
V.A & \xrightarrow{v} & V.A \\
\xrightarrow{1.e} & & \xrightarrow{1.e} V.A
\end{array}
\]

A **morphism** of comodules is a morphism \(f : V \rightarrow W\) in \(\mathcal{C}\) such that \(w(1.f) = (1.f)v\).

**Remark 2.12.** A comodule \(v : V.A \rightarrow V.A\) as in Definition \([2.11]\) induces a comodule over the right multiplier bicomonad \((-).A\) in Example \([2.7]\) by putting

\[
\begin{array}{ccc}
V.X.A & \xrightarrow{1.v} & X.V.A \\
\xrightarrow{1.v} & & \xrightarrow{1.v} X.V.A
\end{array}
\]

This is in fact the object-part of a fully faithful functor from the category of comodules for \(A\) to the category of comodules for the induced comonad \(G\). It’s not hard to see...
that the functor is injective on objects; in many concrete cases, such as \( C = \text{vec} \), it is an isomorphism of categories.

Hence from Theorem 2.9 we have the following.

**Corollary 2.13.** Consider a counital fusion morphism \((t : A^2 \to A^2, e : A \to I)\) in a braided monoidal category \( C \). Its category of comodules, as in Definition 2.11, is monoidal in such a way that the evident forgetful functor to \( C \) is strict monoidal.

**Proof.** It suffices to observe that the full subcategory of \( G \)-comodules consisting of the comodules for \( A \) is closed under the monoidal structure. Explicitly, the monoidal unit \( I \) is a comodule via the identity morphism \( A \to A \) and the monoidal product of comodules \( v : V.A \to V.A \) and \( w : W.A \to W.A \) is a comodule via

\[
V.W.A \xrightarrow{v \cdot w} V.W.A \xrightarrow{b \cdot 1} W.V.A \xrightarrow{b^{-1} \cdot 1} V.W.A.
\]

\( \Box \)

### 2.3. Modules and left multiplier bimonads

In studying modules over a counital fusion morphism, we rely on the following notion.

**Definition 2.14.** A left multiplier bimonad on a monoidal category \( C \) is a functor \( T : C \to C \) equipped with a natural transformation \( \mu : T(X.T Y) \to T(X).T Y \) and a morphism \( T_0 : TI \to I \) such that the diagrams

\[
\begin{array}{ccc}
T(X.T(Y.TZ)) & \xrightarrow{T(1.T_2)} & T(X.TY.TZ) \\
\llap{T_2} & & \rlap{T_2} \\
T(X.T(Y.TZ)) & \xrightarrow{1.T_2} & T.X.TY.TZ & \rlap{T(1.T_0)} \\
\uparrow{1.T_2} & & \uparrow{T_2.1} & \uparrow{1.T_0} \\
T.X(T(Y.TZ)) & \xrightarrow{1.\overline{T_2}} & T.X.TY.TZ & TX & \rlap{T_2}
\end{array}
\]

commute for any objects \( X, Y, Z \).

A right multiplier bimonad on \( C \) is a left multiplier bimonad on the reverse monoidal category \( C^{\text{rev}} \).

To a left multiplier bimonad \( T \), we can associate a natural transformation

\[
\mu := \left( T^2 X \overset{T_2}{\longrightarrow} T.I.TX \overset{T_0.1}{\longrightarrow} TX \right).
\]

This yields an associative but non-unital multiplication. Left multiplier bimonads are indeed a generalization of opmonoidal monads (also known as ‘bimonads’) as the following example shows:

**Example 2.15.** Let \((T, \mu, \eta)\) be an opmonoidal monad on a monoidal category \( C \); that is, assume the existence of an opmonoidal structure \((T_2 : T(-,-) \to T(-), T_0 : TI \to I)\) and the opmonoidality of the associative multiplication \( \mu : T^2 \to T \) and of the unit \( \eta : 1 \to T \). Consider

\[
\overline{T_2} := \left( T(X.TY) \overset{T_2}{\longrightarrow} T.X.T^2Y \overset{1.\mu}{\longrightarrow} T.X.TY \right).
\]

Then the diagrams in \((2.7)\) commute by the associativity of \( \mu \), the coassociativity of \( T_2 \), and the opmonoidality of \( \mu \) on one hand, and by the opmonoidality of \( \mu \) and the counitality of \( T_2 \) on the other hand.

Further examples are provided by the functors induced by counital fusion morphisms:
Example 2.16. Let \((t : A^2 \to A^2, e : A \to I)\) be a counital fusion morphism in a braided monoidal category \(C\). Consider the functor \(T := A.(-) : C \to C\) with the morphism \(T_0 := e : A \to I\) and the natural transformation

\[
\begin{align*}
\overset{T_2}{\longrightarrow} & := (A.X.A.Y \xrightarrow{1.b.1} A.A.X.Y \xrightarrow{t.1.1} A.A.X.Y \xrightarrow{1.b.1} A.X.A.Y).
\end{align*}
\]

Then the diagrams in (2.7) commute by the fusion equation and by the counitality condition on \(t\), respectively. In this example, the multiplication (2.8) is induced by the multiplication in Proposition 2.4 (1).

Definition 2.17. Consider a left multiplier bimonad \(T\) on a monoidal category \(C\). A \(T\)-module is an object \(Q\) together with a natural transformation \(\overset{q}{\longrightarrow} : T(\cdot.Q) \to T(\cdot).Q\) such that the diagram

\[
\begin{array}{ccc}
T(X.T(Y.Q)) & \xrightarrow{T(1.q)} & T(X.TY).Q \\
\phantom{T(X.T(Y.Q))} & \overset{T_2}{\Downarrow} & \phantom{T(X.TY).Q} \\
TX.T(Y.Q) & \xrightarrow{1.q} & TX.TY.Q
\end{array}
\]

commutes for any objects \(X\) and \(Y\); and the morphism \(TQ \xrightarrow{q} TI.Q \xrightarrow{T_0.1} Q\) — which is in fact an associative action with respect to the multiplication (2.8) — is a split epimorphism. (Note that an associative action \(TQ \to Q\) by a unital monad \(T\) is a split epimorphism if and only if it is unital.)

A morphism of modules from \((Q, q)\) to \((R, r)\) is a morphism \(Q \to R\) in \(C\) satisfying the evident compatibility condition.

A module over a right multiplier bimonad on \(C\) is defined as a module over the corresponding left multiplier bimonad on \(C^{rev}\).

Theorem 2.18. Consider a left multiplier bimonad \(T\) on a monoidal category \(C\). Assume that both \(T_0\) and the morphisms

\[
\begin{align*}
T(T.X.TY) & \xrightarrow{T_2} T^2.X.TY \xrightarrow{T_2.1} TI.T.X.TY \xrightarrow{T_0.1.1} TX.TY,
\end{align*}
\]

for any objects \(X\) and \(Y\), are split epimorphisms. Then the \(T\)-modules of Definition 2.17 constitute a monoidal category such that the evident forgetful functor to \(C\) is strict monoidal.

Proof. The monoidal unit \(I\) is a \(T\)-module via the composite of the unit constraints

\[
\begin{align*}
T(-.I) & \xrightarrow{\cong} T \xrightarrow{\cong} T(-).I.
\end{align*}
\]

Indeed, the fusion equation holds by coherence (and naturality of \(\overset{T_2}{\longrightarrow}\) and of the unit isomorphisms) and \(T_0\) is a split epimorphism by assumption. For \(T\)-modules \(\overset{q}{\longrightarrow} : T(-.Q) \to T(-).Q\) and \(\overset{p}{\longrightarrow} : T(-.P) \to T(-).P\), we consider the candidate module structure

\[
\begin{align*}
T(-.P.Q) & \xrightarrow{q} T(-.P).Q \xrightarrow{p.1} T(-).P.Q.
\end{align*}
\]
It satisfies the fusion equation since both \( \tilde{\gamma} \) and \( \tilde{\rho} \) do: for any objects \( X \) and \( Y \) the diagram

\[
\begin{array}{c}
T(X.T(Y.P.Q)) \xrightarrow{T(\tilde{\gamma})} T(X.T(Y.P).Q) \xrightarrow{T(\tilde{\theta})} T(X.TY.P.Q) \\
\downarrow \tilde{\gamma} \downarrow \tilde{\theta} \\
T(X.T(Y.P)).Q \xrightarrow{T(\tilde{\theta}).1} T(X.TY).P.Q \\
\downarrow \tilde{\theta}.1 \\
TX.T(Y.P).Q \xrightarrow{1.\tilde{\gamma}} TX.T(Y.P).Q \xrightarrow{1.\tilde{\theta}.1} TX.TY.P.Q
\end{array}
\]

commutes. In the diagram

\[
\begin{array}{c}
T(TP.TQ) \xrightarrow{T(\tilde{\gamma}).1} T(TI.P.TQ) \xrightarrow{T(\tilde{\theta}).1} T(TQ) \xrightarrow{T(\tilde{\theta}).1} T(PQ) \\
\downarrow \tilde{\gamma} \downarrow \tilde{\theta} \downarrow \tilde{\theta}.1 \\
T^2TP.TQ \xrightarrow{T(\tilde{\gamma}).1} T(TI.P).TQ \xrightarrow{T(\tilde{\theta}).1} TTP.TQ \xrightarrow{1.\tilde{\gamma}} TP.TI.Q \xrightarrow{1.\tilde{\theta}.1} T.P.Q \\
\downarrow \tilde{\gamma}.1 \downarrow \tilde{\theta}.1 \downarrow \tilde{\theta}.1 \\
T.ITP.Q \xrightarrow{1.\tilde{\gamma}.1} TIT.I.P.TQ \xrightarrow{1.\tilde{\theta}.1} TITI.P.TQ \xrightarrow{T\tilde{\theta}.1} TITI.P.Q \\
\downarrow \tilde{\gamma}.1 \downarrow \tilde{\theta}.1 \downarrow \tilde{\theta}.1 \\
TP.TQ \xrightarrow{1.\tilde{\gamma}.1} T.P.TQ \xrightarrow{T\tilde{\theta}.1} P.TQ \xrightarrow{1.\tilde{\gamma}} P.TI.Q \xrightarrow{1.\tilde{\theta}.1} PQ
\end{array}
\]

the left column and the bottom row are split epimorphisms by assumption. Hence also the top-right path is a split epimorphism proving that so is the morphism in the right column. In this diagram the unlabelled regions commute by naturality of \( \tilde{\gamma} \) and \( \tilde{\theta} \) and functoriality of the monoidal product. The regions marked by \( (*) \) commute since for any \( T \)-module \( \tilde{\gamma} : T(-) \to T(-).Q \) and any object \( X \), the diagram

\[
\begin{array}{c}
T(X.TQ) \xrightarrow{T(\tilde{\gamma})} T(X.TI.Q) \xrightarrow{T(\tilde{\theta}).1} T(X.Q) \\
\downarrow \tilde{\gamma} \downarrow \tilde{\theta}.1 \\
T(X.TI).Q \xrightarrow{T(\tilde{\theta}).1} TX.Q \\
\downarrow \tilde{\theta}.1 \\
TX.TQ \xrightarrow{1.\tilde{\gamma}} TX.TI.Q \xrightarrow{1.\tilde{\theta}.1} TX.Q
\end{array}
\]

commutes by naturality of \( \tilde{\gamma} \), by the counitality condition in \( (2.7) \), and by the fusion equation on \( \tilde{\gamma} \). The monoidal product of \( T \)-module morphisms, as well as the unit and associativity isomorphisms are evidently morphisms of \( T \)-modules. \( \square \)

**Example 2.19.** We claim that applying Definition 2.17 to the functor \( T \) in Example 2.15 we obtain a category of modules which is equivalent (in fact, isomorphic) to the usual Eilenberg-Moore category of modules (or algebras). Therefore Theorem 2.18 generalizes
the well-known result [7] that the monoidal structure of the base category lifts to the Eilenberg-Moore category of an opmonoidal monad.

The reasoning is similar to Example 2.10. Consider the category of $T$-modules in Definition 2.17 and the forgetful functor $U$ from it to $\mathcal{C}$. For a left multiplier bimonad $T$ as in Example 2.15, $U$ possesses a left adjoint $F$ such that $UF = T$ as monads: $F$ takes an object $X$ to the $T$-module $(TX, T_2)$. (It obeys the fusion equation by assumption and (2.8) is an epimorphism split by $\eta$.) The unit of the adjunction is $\eta : \mathcal{C} \to T = UF$ and the counit, evaluated at a $T$-module $(Q, \theta_q)$, is $\eta := (T_0.1)^{\theta_q} : TQ \to Q$. (This is a morphism of $T$-modules, in the sense of Definition 2.17 by the fusion equation on $\theta_q$, the counitality of $T_2 = (1.\mu)T_2$, and the naturality of $\theta_q$.) Then the counit at an object $FX = (TX, \theta_{T_2})$ is equal to

$$
\left( T^2X \xrightarrow{T_2} TI.T^2X \xrightarrow{1.\mu} TI.TX \xrightarrow{T_0.1} TX \right) = \mu
$$

hence the first triangle condition follows by the unitality of $\mu$. Since for a $T$-module $(Q, \theta_q)$, $q$ is an associative action which is epi by assumption, the other triangle condition follows by the naturality of $\eta$, the associativity of $q$, and the unitality of $\mu$ again:

$$
q\eta q = q(Tq)\eta = q\mu\eta = q.
$$

In order to see that the obviously conservative right adjoint functor $U$ is monadic, we need to prove that it creates $U$-absolute coequalizers. Suppose then that $f$ and $g$ are morphisms $(Q, \theta_q) \to (R, \theta_r)$, and that

$$
Q \xrightarrow{f} R \xrightarrow{h} S
$$

is an absolute coequalizer in $\mathcal{C}$. Then in the solid part of the diagram

$$
\begin{array}{ccc}
T(X.Q) & \xrightarrow{T(1,f)} & T(X,R) \\
\xrightarrow{\theta_q} & \xrightarrow{\theta_r} & \xrightarrow{\theta_r} \\
TX.Q & \xrightarrow{1.f} & TX.R \\
\xrightarrow{1.g} & & \xrightarrow{1.h} \\
& & TX.S
\end{array}
$$

the rows are coequalizers, and the parallel pairs commute serially with the verticals, thus there is a unique induced morphism $\theta_S$ as in the dashed part of the diagram. The axioms for $(S, \theta_S)$ to be a module follow easily from the corresponding axioms for $(R, \theta_r)$ and the fact that $h$ is an absolute (thus split) epimorphism. The morphism $h$ preserves the module structure by construction, and the universal property of $(S, \theta_S)$ follows from the universal property of $S$ and the fact that $T(1.h)$ is an epimorphism. Explicitly, the inverse of the comparison functor – from the category of $T$-modules in Definition 2.17 to the category of Eilenberg-Moore $T$-modules – sends an Eilenberg-Moore module $q : TQ \to Q$ to the coequalizer of the $T$-module morphisms $\mu, Tq : (T^2Q, \theta_{T_2}) \to (TQ, \theta_{T_2})$; that is, to the $T$-module

$$
T(\cdot.Q) \xrightarrow{T_2} T(\cdot).TQ \xrightarrow{1.q} T(\cdot).Q.
$$

Further examples of the situation in Definition 2.17 are provided by the following.

**Definition 2.20.** Consider a counital fusion morphism $(t : A^2 \to A^2, e : A \to I)$ in a braided monoidal category $\mathcal{C}$. A module over it is an object $Q$ together with a morphism
$q : A . Q \to A . Q$ in $C$ such that the diagram

$$
\begin{array}{c}
A^2 . Q \xrightarrow{1,q} A^2 . Q \quad \xrightarrow{b_1} A^2 . Q \\
\downarrow{t.1} \quad \downarrow{1,q} \\
A^2 . Q \quad \to A^2 . Q
\end{array}
$$

commutes and $A . Q \xrightarrow{q} A . Q \xrightarrow{\varepsilon} Q$ is a split epimorphism. A morphism of modules is a morphism $f : Q \to R$ in $C$ such that $(1 . f) q = r (1 . f)$.

**Example 2.21.** Consider a multiplier bialgebra $A$ over a field. By Example 2.3 there is an isomorphism $\Phi$ from the category of modules in Definition 2.20 to the following category. The objects are vector spaces $Q$ equipped with an associative $A$-action $q : A . Q \to Q$ which is in addition a surjective map. The morphisms are the linear maps which commute with the actions.

Suppose then that $q_1 : A . Q \to A . Q$ is a module as in Definition 2.20. Composing the fusion equation (2.9) with $1 . e . t$, and writing $q$ for the surjection $(e . t) q_1$, we see that the diagram

$$
\begin{array}{c}
A^2 . Q \xrightarrow{t_1 . 1} A^2 . Q \\
\downarrow{1,q} \quad \downarrow{1,q} \\
A . Q \xrightarrow{q_1} A . Q
\end{array}
$$

commutes. Composing this with $e . t$, we see that $q$ is associative. This defines the value on objects of a functor $\Phi$, which acts as the identity on morphisms.

Since $1 . q$ is, like $q$, surjective, we can recover $q_1$ from $q$, and so $\Phi$ is injective on objects. Using surjectivity of $1 . q$ once again, one deduces that $\Phi$ is full.

Thus it remains only to show that $\Phi$ is surjective on objects. To do this, let $q : A . Q \to Q$ be a surjective associative action, and use the fact that in $\text{vec}$ every surjective map — so in particular the map $1 . q$ — is the cokernel of its kernel. So to deduce the existence of a map $q_1$ making the diagram in (2.10) commute, it will suffice to show that $(1 . q) (t_1 . 1)$ vanishes on the kernel of $1 . q$.

By (2.2) and by the associativity of $q$,

$$(1 . q) (m . 1 . 1) (t_1 . 1) = (1 . q) (1 . m . 1) (t_2 . 1 . 1) = (1 . q) (1 . 1 . q) (t_2 . 1 . 1) = (1 . q) (t_2 . 1) (1 . 1 . q)$$

so that $(m . 1) (1 . 1 . q) (t_1 . 1)$ vanishes on $\ker (1 . 1 . q) = A . \ker (1 . q)$. By the non-degeneracy of $m$ this proves that $(1 . q) (t_1 . 1)$ vanishes on $\ker (1 . q)$, so that $q_1$ is indeed well-defined. Using the fact that $q$ is surjective, the fusion equation (2.9) for $q_1$ follows by

$$(t_1 . 1) q_1^{13} (1 . q_1) (1 . 1 . q) = (t_1 . 1) q_1^{13} (1 . 1 . q) (t_1 . 1) = (t_1 . 1) (1 . 1 . q) (t_1^{13} . 1) (t_1 . 1)$$

$$= (1 . 1 . q) (t_1 . 1 . 1) t_1^{13} (t_1 . 1) = (1 . 1 . q) (t_1 . 1 . 1) (t_1 . 1 . 1)$$

$$= (1 . q_1) (1 . 1 . q) (t_1 . 1 . 1) = (1 . q_1) (t_1 . 1) (1 . 1 . q).$$

In the fourth equality we used the fusion equation on $t_1$. Again by the surjectivity of $q$,

$$(e . 1) q_1 (1 . q) = (e . 1) (1 . q) (t_1 . 1) = q (m . 1) = q (1 . q)$$
implies (e.1)q_{1} = q which is surjective (hence a split epimorphism) by assumption. In the penultimate equality we used the functoriality of the monoidal product and axiom (e) in Example 2.3 and in the last equality we used the associativity of q.

**Remark 2.22.** A module q : A.Q → A.Q as in Definition 2.20 induces a module over the left multiplier bimonad A.(−) in Example 2.16 by putting

\[
A.X.Q \xrightarrow{1.b^{-1}} A.Q.X \xrightarrow{q.1} A.Q.X \xrightarrow{1.b} A.X.Q.
\]

Once again, this defines the object-part of a fully faithful injective functor from the category of A-modules to the category of T-modules for the induced monad T; once again this will be an isomorphism in many concrete cases, such as C = vec.

Hence from Theorem 2.18 we have the following.

**Corollary 2.23.** Consider a counital fusion morphism \((t : A^2 \to A^2, e : A \to I)\) in a braided monoidal category C. Assume that e and

\[
A^3 \xrightarrow{1.t} A^3 \xrightarrow{b^{-1}.1} A^3 \xrightarrow{1.m} A^2
\]

are split epimorphisms. Its category of modules, as in Definition 2.20 is monoidal in such a way that the evident forgetful functor to C is strict monoidal.

The monoidal unit I is a module via the identity morphism A → A, and the monoidal product of modules q : A.Q → A.Q and p : A.P → A.P is a module via

\[
A.P.Q \xrightarrow{b.1} P.A.Q \xrightarrow{1.q} P.A.Q \xrightarrow{b^{-1}.1} A.P.Q \xrightarrow{p.1} A.P.Q.
\]

In particular, in view of Example 2.21 we conclude that for a multiplier bialgebra A over a field, the category of associative A-modules with a surjective action is monoidal via the tensor product of vector spaces.

### 3. Multiplier bimonoids in braided monoidal categories

In this section we define the central object studied in the paper — *multiplier bimonoids* in a braided monoidal category — using the theory of counital fusion morphisms developed in the previous section. We discuss their further properties like *regularity* and *non-degeneracy* of the multiplication (in Proposition 2.4 (1)). We show how the motivating examples — bimonoids in braided monoidal categories and multiplier bialgebras over a field — are covered by our definition.

#### 3.1. Multiplier bimonoid

A multiplier bimonoid is defined in terms of a compatible pair of fusion morphisms:

**Definition 3.1.** A *multiplier bimonoid* in a braided monoidal category C consists of a fusion morphism \(t_1 : A^2 \to A^2\) in C and a fusion morphism \(t_2 : A^2 \to A^2\) in \(C^{\text{rev}}\) possessing a common counit \(e : A \to I\) such that the following diagrams commute.

\[
\begin{align*}
A^3 & \xrightarrow{t_2.1} A^3 \\
& \downarrow 1.t_1 \\
A^3 & \xrightarrow{t_2.1} A^3
\end{align*}
\quad
\begin{align*}
A^2 & \xrightarrow{t_1} A^2 \\
& \downarrow t_2 \\
A^2 & \xrightarrow{1.e} A
\end{align*}
\]

\[
\begin{align*}
A^2 & \xrightarrow{t_1} A^2 \\
& \downarrow t_2 \\
A^2 & \xrightarrow{1.e} A
\end{align*}
\]
The second diagram in Definition 3.1 expresses the requirement that the multiplications, corresponding as in Proposition 2.4 (1) to the counital fusion morphisms \( t_1 \) and \( t_2 \), coincide. In the axioms in Definition 3.1 the roles of \( t_1 \) and \( t_2 \) are symmetric: \((t_1, t_2, e)\) is a multiplier bimonoid in \( C \) if and only if \((t_2, t_1, e)\) is a multiplier bimonoid in \( C^{rev} \).

As the name suggests, this is a common generalization of bimonoids in braided monoidal categories and of multiplier bialgebras over a field:

**Example 3.2.** A bimonoid \( A \) in a braided monoidal category \( C \) determines a multiplier bimonoid in \( C \) by

\[
\begin{align*}
t_1 &:= \begin{pmatrix} A^2 & d_1 & A^3 & l_m & A^2 \end{pmatrix} \\
t_2 &:= \begin{pmatrix} A^2 & l_d & A^3 & m_1 & A^2 \end{pmatrix}.
\end{align*}
\]

Indeed, \( t_1 \) is a counital fusion morphism in \( C \) by Example 2.2 and \( t_2 \) is a counital fusion morphism in \( C^{rev} \) by symmetry. The first diagram in Definition 3.1 commutes by the functoriality of the monoidal product and the coassociativity of the comultiplication. The second diagram in Definition 3.1 commutes by the functoriality of the monoidal product and the counitality of the comultiplication.

**Example 3.3.** For a multiplier bialgebra over a field, the maps \( t_1 \) and \( t_2 \) in Example 2.3 constitute a multiplier bimonoid in \( vec \); see axioms (d) and (e) in Example 2.3.

Also a certain converse holds:

**Proposition 3.4.** Consider a multiplier bimonoid \((t_1, t_2 : A^2 \to A^2, e : A \to I)\) in \( vec \) and denote \( m := (e.1)t_1 = (1.e)t_2 \) (it is an associative multiplication by Proposition 2.4 (1)). If

- \( m \), \((m.1)(b.1)(1.t_1)\) and \((1.m)(1.b)(t_2.1)\) are surjective and
- \( m \) is non-degenerate,

then \( A \) is a multiplier bialgebra (in the sense recalled in Example 2.3).

**Proof.** Axiom (c) in Example 2.3 holds by Proposition 2.4 (4). Axiom (a) in Example 2.3 follows by postcomposing the fusion equation on \( t_1 \) by \( e.1.1 \) (or postcomposing the fusion equation on \( t_2 \) by \( 1.1.e \)). \(\square\)

Examples of multiplier bimonoids in \( vec \) which are not multiplier bialgebras, however, can be obtained as linear spans of semigroups \( S \) (that is, non-unital monoids \( S \) in \( set \)). In this case the fusion maps are given by

\[
\begin{align*}
t_1 : a.b &\mapsto a.ab \\
t_2 : a.b &\mapsto ab.b
\end{align*}
\]

on the linear basis \( \{a.b \mid a, b \in S\} \) — where juxtaposition denotes the multiplication in \( S \) — and the counit is the linear map sending any element of \( S \) to the unit element of the base field. These need not be multiplier bialgebras because the multiplication need not be non-degenerate, and the surjectivity condition (b) of Example 2.3 need not hold.

Non-degeneracy of the multiplication can be formulated also in our context:

**Definition 3.5.** Consider a morphism \( m : A^2 \to A \) in a monoidal category. We say that \( m \) is non-degenerate if for any objects \( X \) and \( Y \), both maps

\[
\begin{align*}
\mathcal{C}(X, Y.A) &\to \mathcal{C}(X.A, Y.A), \\
\mathcal{C}(X, A.Y) &\to \mathcal{C}(A.X, A.Y),
\end{align*}
\]

\[
\begin{align*}
f &\mapsto (X.A \xrightarrow{f_1} Y.A^2 \xrightarrow{l_m} Y.A) \\
g &\mapsto (A.X \xrightarrow{l_g} A^2.Y \xrightarrow{m_1} A.Y)
\end{align*}
\]

are injective.
Clearly, if $m$ is a unital multiplication then it is non-degenerate. If the multiplication of a multiplier bimonoid is non-degenerate, then some of the axioms in Definition 3.1 become redundant.

**Remark 3.6.** Consider morphisms $t_1, t_2 : A^2 \to A^2$ and $e : A \to I$ in a braided monoidal category $C$ such that $(t_2.1)(1.t_1) = (1.t_1)(t_2.1)$ and the morphisms $(e.1)t_1$ and $(1.e)t_2$ are equal and non-degenerate. Observe that, by the argument given in (2.3), the fusion equation for $t_1$ follows from the “short” fusion equation on the left in (3.1)

$$t_1 \circ t_1 = t_2 \circ t_2 = t_2.$$

On the other hand the short fusion equation follows from the fusion equation by composing with counit on the first string, thus the two equations are equivalent. Dually, the fusion equation for $t_2$ is equivalent to its short version appearing on the right in (3.1).

**Proposition 3.7.** Consider morphisms $t_1, t_2 : A^2 \to A^2$ and $e : A \to I$ in a braided monoidal category $C$ such that $(e.1)t_1 = (1.e)t_2$ and $(t_2.1)(1.t_1) = (1.t_1)(t_2.1)$. Assume that $m := (e.1)t_1 = (1.e)t_2$ is non-degenerate.

1. The following assertions are equivalent to each other.
   - (i) $t_1$ is a fusion morphism in $C$.
   - (ii) $t_2$ is a fusion morphism in $C^{\text{rev}}$.

2. The following assertions are also equivalent to each other.
   - (i) $(1.e)t_1 = 1.e$.
   - (ii) $(e.1)t_2 = e.1$.
   - (iii) $em = e.e$.

The datum $(t_1, t_2, e)$ is a multiplier bimonoid in $C$, equivalently, $(t_2, t_1, e)$ is a multiplier bimonoid in $C^{\text{rev}}$, if and only if the assertions in parts (1) and (2) hold.

**Proof.** Let us again use the string notation

$$t_1 = \begin{array}{c}
\{1\}
\end{array}, \quad m = \begin{array}{c}
\cup
\end{array}, \quad e = \begin{array}{c}
\circ
\end{array}, \quad b = \begin{array}{c}
\times
\end{array}$$

for any $i = 1, 2, 3, 4$. We repeatedly use

$$m = \begin{array}{c}
\cup
\end{array}, \quad e = \begin{array}{c}
\circ
\end{array}, \quad b = \begin{array}{c}
\times
\end{array}$$

for any $i = 1, 2, 3, 4$. We repeatedly use

$$m = \begin{array}{c}
\cup
\end{array}, \quad e = \begin{array}{c}
\circ
\end{array}, \quad b = \begin{array}{c}
\times
\end{array}$$

which follows immediately from the axiom $(t_2.1)(1.t_1) = (1.t_1)(t_2.1)$ on composing with $1.e.1$.

1. By Remark 3.6 it will suffice to prove the equivalence of the short fusion equations. This follows via non-degeneracy from the following calculations, which use (3.2) and associativity.

$$t_1 = \begin{array}{c}
\{1\}
\end{array}, \quad m = \begin{array}{c}
\cup
\end{array}, \quad e = \begin{array}{c}
\circ
\end{array}, \quad b = \begin{array}{c}
\times
\end{array}$$

$$t_1 = \begin{array}{c}
\{1\}
\end{array}, \quad m = \begin{array}{c}
\cup
\end{array}, \quad e = \begin{array}{c}
\circ
\end{array}, \quad b = \begin{array}{c}
\times
\end{array}$$
Definition 3.8. A regular multiplier bimonoid in a braided monoidal category \( \mathcal{C} \) consists of a multiplier bimonoid \((t_1, t_2)\) in \( \mathcal{C} \) and a multiplier bimonoid \((t_3, t_4)\) in \( \overline{\mathcal{C}} \) with a common counit \( e : A \to I \) such that the following diagrams commute.

\[
\begin{array}{cccc}
A^3 & b_1 & A^3 & t_2 \cdot 1 \\
\downarrow 1 \cdot t_1 & (A) & \downarrow 1 \cdot m & \downarrow t_4 \cdot 1 \\
A^3 & b_1 & A^3 & 1 \cdot m \\
\downarrow t_2 \cdot 1 & (A^{\text{rev}}) & \downarrow m \cdot 1 & \downarrow (B^{\text{rev}}) \\
A^3 & b_1 & A^3 & m_1 \\
\downarrow 1 \cdot b & (B) & \downarrow 1 \cdot t_3 & \downarrow 1 \cdot t_3 \\
A^2 & t_3 \cdot 1 & A^2 & e \cdot 1 \\
\downarrow t_1 & (C) & \downarrow e \cdot 1 & \downarrow A \\
A^2 & t_1 & A^2 & A \\
\end{array}
\]

Proposition 3.9. Given a multiplier bimonoid \((t_1, t_2, e)\) in \( \mathcal{C} \) and a multiplier bimonoid \((t_3, t_4, e)\) in \( \overline{\mathcal{C}} \), the following conditions are equivalent:

1. \((t_1, t_2, t_3, t_4, e)\) is a regular multiplier bimonoid in \( \mathcal{C} \);
2. \((t_2, t_1, t_4, t_3, e)\) is a regular multiplier bimonoid in \( \mathcal{C}^{\text{rev}} \);
3. \((t_3, t_4, t_1, t_2, e)\) is a regular multiplier bimonoid in \( \overline{\mathcal{C}} \);
4. \((t_4, t_3, t_2, t_1, e)\) is a regular multiplier bimonoid in \( \mathcal{C}^{\text{rev}} \).

Proof. We have seen that \((t_1, t_2, e)\) is a multiplier bimonoid in \( \mathcal{C} \) just when \((t_2, t_1, e)\) is a multiplier bimonoid in \( \mathcal{C}^{\text{rev}} \); thus similarly \((t_3, t_4, e)\) is a multiplier bimonoid in \( \overline{\mathcal{C}} \) just when \((t_4, t_3, e)\) is a multiplier bimonoid in \( \mathcal{C}^{\text{rev}} \). Under this duality, the axioms (A) and (B) correspond, respectively, to the axioms \((A^{\text{rev}})\) and \((B^{\text{rev}})\). Condition (C) says that...
the multiplication \( m \) for the multiplier bimonoid \((t_1, t_2)\) is related to the multiplication \( \overline{m} \) for the multiplier bimonoid \((t_3, t_4)\) via the equation \( m = \overline{m}b \); this condition is self-dual. Thus (1) is equivalent to (2), and (3) is equivalent to (4). In applying the \( C \text{-} \overline{C} \) duality, one must also replace \( m \) by \( \overline{m} = m b^{-1} \). Once again, this duality interchanges (B) and (B'\text{rev}), and leaves (C) unchanged. Applied to (A) it gives an equivalent diagram (obtained by composing both sides with various braid isomorphisms); the case of (A'\text{rev}) is similar. □

Remark 3.10. The definition given above is in some sense a minimal one: we assume only those axioms which will be needed to prove our results about modules and comodules in the following sections. There are many further relationships between the \( t_i \) that follow from these in the non-degenerate case in which we are primarily interested, and it may well be that some of these are needed for the further development of the theory in the absence of non-degeneracy. In particular, one might consider commutativity of diagrams such as the following (or various dualizations).

We now describe further simplifications which are possible in the non-degenerate case.

Proposition 3.11. Let \((t_1, t_2, e)\) define a multiplier bimonoid in the braided monoidal category \( C \), and suppose that the corresponding multiplication \( m : A^2 \to A \) is non-degenerate. Then morphism \( t_3, t_4 : A^2 \to A^2 \) define a regular multiplier bimonoid \((t_1, t_2, t_3, t_4, e)\) if and only if the diagrams (A) and (A’\text{rev}) commute.

Proof. We need to prove the commutativity of (B), (B'\text{rev}), and (C), as well as the fact that \((t_3, t_4, e)\) defines a multiplier bimonoid in \( \overline{C} \); for the latter, we shall use Proposition 3.7.

Applying \( e.1 \) to either side of (A) and using non-degeneracy, we deduce (C). Similarly, applying \( 1.e \) to either side of (A'\text{rev}) and using non-degeneracy, we see that \((1.e)t_4.b = (1.e)t_2 = m\), and so that \((1.e)t_4 = \overline{m} = (e.1)t_3\). Since \( \overline{m} \) is equal to \( mb^{-1} \), it is non-degenerate; giving another one of the hypotheses of Proposition 3.7.

As for (B), we have

where the first and last equalities use (A'\text{rev}) and the middle one uses \((t_2.1)(1.t_1) = (1.t_1)(t_2.1)\); now (B) follows by non-degeneracy, and dually (B'\text{rev}) also holds.

A similar (dual) argument applied to (B) shows that \((t_4.1)(1.t_3) = (1.t_3)(t_4.1)\) holds; thus we are in a position to apply Proposition 3.7. Now \( e\overline{m} = emb^{-1} = (e.e)b^{-1} = e.e \), and so \( e \) is multiplicative with respect to the multiplication \( \overline{m} \).
It remains to check that \( t_3 \) is a fusion morphism in \( \mathcal{C} \); furthermore, by Remark 3.6 it suffices to check the short fusion equation. In the calculation
\[
\begin{align*}
\begin{array}{c}
\text{(first equality)} \\
\text{(second and third by (A))}
\end{array}
\end{align*}
\]
the first equality holds by naturality of the braiding, and the second and third by (A); while
\[
\begin{align*}
\begin{array}{c}
\text{(first equality)} \\
\text{(second by (A))}
\end{array}
\end{align*}
\]
holds by (A) once again. Since the left hand sides of the two displayed calculations agree by the short fusion equation for \( t_1 \), the right hand sides must also agree. By non-degeneracy we may cancel the right-most input strings, and finally composing with suitably chosen braid isomorphisms gives
\[
\begin{align*}
\begin{array}{c}
\text{(first equality)} \\
\text{(second by (A))}
\end{array}
\end{align*}
\]
which is the short fusion equation for \( t_3 \).

Just as for multiplier bialgebras over fields, for a regular multiplier bimonoid \((t_1, t_2, t_3, t_4)\), any one of the maps \( t_1, t_2, t_3, \) or \( t_4 \) determines each of the others whenever the multiplication is non-degenerate; cf. axioms (A) and (A\(^{rev}\)) in Definition 3.8.

From [1, Theorem 1.2] we immediately obtain the following.

**Example 3.12.** A multiplier bialgebra over a field is regular (in the sense of [1, Definition 1.1]) if and only if the corresponding multiplier bimonoid in \( \text{vec} \) in Example 3.3 extends to a regular multiplier bimonoid.

Another class of examples is provided by bimonoids in braided monoidal categories:

**Example 3.13.** The multiplier bimonoid induced by a bimonoid \( A \) in a braided monoidal category \( \mathcal{C} \) in Example 3.2 can be supplemented with the morphisms
\[
t_3 := (A^2 \xrightarrow{d_1} A^3 \xrightarrow{b^{-1}} A^3 \xrightarrow{1_m} A^2) \]
\[
t_4 := (A^2 \xrightarrow{1_d} A^3 \xrightarrow{b^{-1} \cdot 1} A^3 \xrightarrow{m \cdot 1} A^2).
\]

Very similar computations to those in Example 2.2 and Example 3.2 show that it yields a regular multiplier bimonoid.
4. Comodules and multiplier bicomonads

As we have seen in Section 2 in the case of counital fusion morphisms, the best way to investigate the behavior of modules and comodules is to study the induced functors. In this section, therefore, we generalize ‘bicomonads’ (that is, monoidal comonads) to multiplier bicomonads on arbitrary monoidal categories. We show that the monoidal structure of the base category lifts to a suitably defined category of comodules. Proving that any regular multiplier bimonoid induces a multiplier bicomonad, we conclude that their comodules (in the appropriate sense) constitute a monoidal category admitting a strict monoidal forgetful functor to the base category.

4.1. Multiplier bicomonad. Based on the considerations in Section 2.2, we start with the following.

Definition 4.1. A multiplier bicomonad on a monoidal category $\mathcal{C}$ is a functor $G : \mathcal{C} \to \mathcal{C}$ equipped with natural transformations $\bar{\eta}_2 : GX.GY \to G(X.Y)$ and $\varepsilon : GX \to X$ such that $(\bar{\eta}_2, \varepsilon)$ makes $G$ a right multiplier bicomonad on $\mathcal{C}$, $(\bar{\eta}_2, \varepsilon)$ makes $G$ a left multiplier bicomonad on $\mathcal{C}$, and the following diagrams, expressing their compatibility, commute for any objects $X, Y, Z$. First,

\[
\begin{array}{c}
GX.GY \\ \downarrow \bar{\eta}_2 \\
G(X.Y)
\end{array}
\xrightarrow{G(\varepsilon.1)}
\begin{array}{c}
G(\varepsilon.1) \\
G(X.Y)
\end{array}
\]

The common diagonal in this diagram satisfies the same associativity condition as the binary part of a monoidal functor (see Section 2.2). For this reason — although in general it does not admit for a nullary part — we use the notation $G_2 : GX.GY \to G(X.Y)$ for it. We also require this $G_2$ to satisfy the second compatibility condition

\[
\begin{array}{c}
GX.GY.GZ \\ \downarrow G_{2.1} \\
G(X.GY).GZ
\end{array}
\xrightarrow{G_2}
\begin{array}{c}
G_2 \\
G(X.GY).Z
\end{array}
\]

Example 4.2. Consider a monoidal comonad $(G, \delta, \varepsilon)$ on a monoidal category $\mathcal{C}$. We know from Example 2.6 that

\[
\bar{\eta}_2 := (GX.GY \xrightarrow{\delta} G^2X.GY \xrightarrow{G_2} G(X.Y))
\]

(together with $\varepsilon$) makes $G$ into a right multiplier bicomonad, and similarly

\[
\bar{\eta}_2 := (GX.GY \xrightarrow{1.\delta} G.X.G^2Y \xrightarrow{G_2} G(X.GY))
\]
makes $G$ into a left multiplier bicomonad. We claim that they constitute a multiplier bicomonad. Indeed, by the naturality of $G_2$ and by the counitality of $\delta$, also

$$
\begin{array}{ccc}
GX.GY & \xrightarrow{1.\delta} & GX.G^2Y \\
\downarrow & & \downarrow 1.Gz \\
G^2X.GY & \xrightarrow{Gz.1} & GX.GY \\
\downarrow G_2 & & \downarrow G_2 \\
G(GX.Y) & \xrightarrow{G(\varepsilon.1)} & G(X.Y)
\end{array}
$$

commutes and so does the second compatibility diagram in Definition 4.1 by the associativity of $G_2$.

**Example 4.3.** Consider a regular multiplier bimonoid $(t_1, t_2, t_3, t_4: A^2 \to A^2, e: A \to I)$ in a braided monoidal category $C$; and the induced functor $G = (-).A: C \to C$. We know from Example 2.7 that $\varepsilon := 1.e$ and

$$
\begin{array}{ccc}
X.A.Y.A & \xrightarrow{1.b.1} & X.Y.A^2 \\
\downarrow & & \downarrow 1.1.t_1 \\
X.Y.A^2 & \xrightarrow{1.b^{-1}.1} & X.A.Y.A
\end{array}
$$

make $G$ into a right multiplier bicomonad. We claim that together with

$$
\begin{array}{ccc}
X.A.Y.A & \xrightarrow{1.b.1} & X.Y.A^2 \\
\downarrow & & \downarrow 1.1.b \\
X.Y.A^2 & \xrightarrow{1.1.t_3} & X.Y.A^2
\end{array}
$$

they constitute a multiplier bicomonad. Certainly $\overleftarrow{G}_2$ provides a left multiplier bicomonad structure; the first compatibility condition in Definition 4.1 follows from axiom (C) in Definition 3.8 (together with the naturality and the coherence of the braiding) and the second one follows by axiom (A) in Definition 3.8.

4.2. **The category of comodules.** Based on the notion of comodule in Definition 2.8, we introduce the following.

**Definition 4.4.** By a *comodule* over a multiplier bicomonad $G$ on a monoidal category $C$ we mean the following. It is an object $V$ in $C$ equipped with the structure $\overrightarrow{V}: V.G(-) \to G(V.-)$ of a comodule over the right multiplier bicomonad $(G, \overrightarrow{G}_2, \varepsilon)$, and also with the structure $\overleftarrow{V}: G(-).V \to G(-.V)$ of a comodule over the left multiplier bicomonad $(G, \overleftarrow{G}_2, \varepsilon)$ such that the compatibility diagram

$$
\begin{array}{ccc}
GX.V.GY & \xrightarrow{1.\overrightarrow{V}} & GX.G(V.Y) \\
\downarrow \overleftarrow{V}.1 & & \downarrow G_2 \\
G(X.V).GY & \xrightarrow{G_2} & G(X.V.Y)
\end{array}
$$

commutes for any objects $X, Y$ (where $G_2$ is the natural transformation introduced in Definition 4.1). A *morphism* of comodules is a morphism $f: V \to W$ in $C$ such that both diagrams

$$
\begin{array}{ccc}
V.GX & \xrightarrow{\overrightarrow{V}} & G(V.X) \\
\downarrow f.1 & & \downarrow G(f.1) \\
W.GX & \xrightarrow{\overrightarrow{W}} & G(W.X)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
GX.V & \xrightarrow{\overleftarrow{V}} & G(X.V) \\
\downarrow 1.f & & \downarrow G(1.f) \\
GX.W & \xrightarrow{\overleftarrow{W}} & G(X.W)
\end{array}
$$
Theorem 4.5. Consider a multiplier bicomonad $G$ on a monoidal category $C$. The monoidal structure of $C$ lifts to the category of $G$-comodules in Definition 4.4.

Proof. By Theorem 2.9, the monoidal unit $I$ carries coassociative and counital comodule structures

$I.G(-) \xrightarrow{\cong} G \xrightarrow{\cong} G(I.-) \quad G(-).I \xrightarrow{\cong} G \xrightarrow{\cong} G(-.I),$

built up from the unit isomorphisms in $C$. The diagram in Definition 4.4 commutes for this $G$-comodule $I$ by naturality of $G_2$ and the coherence in $C$. By Theorem 2.9, for any two $G$-comodules $V$ and $W$, there are coassociative and counital comodule structures

$V.W.G(-) \xrightarrow{v.1} V.G(W.-) \quad G(-).V.W \xrightarrow{v.1} G(-.V).W \quad G(-.V.W),$

Since both $V$ and $W$ satisfy the compatibility condition in Definition 4.4, it follows by the functoriality of the monoidal product that also

$GX.V.W.GY \xrightarrow{1.1.v} GX.V.G(W.Y) \xrightarrow{1.v} GX.V(W.W.Y) \quad G(X.V).W.GY \xrightarrow{1.v} G(X.V).G(W.Y) \quad G(X.V.W).GY \xrightarrow{G_2} G(X.V.W.Y),$

commutes for any objects $X, Y$. In light of Theorem 2.9 this completes the proof. □

Example 4.6. Consider the multiplier bicomonad on a monoidal category $C$ induced by a bicomonad $(G, \delta, \varepsilon)$ as in Example 4.2. We claim that the category of comodules over it, in the sense of Definition 4.4, is isomorphic to the Eilenberg-Moore category of comodules over the comonad $G$; hence Theorem 4.5 extends the result about the lifting of the monoidal structure of $C$ to the Eilenberg-Moore category of $G$ (see e.g. [7]).

The stated isomorphism acts on the morphisms as the identity map and its object map is the following. Let us begin with an Eilenberg-Moore comodule $v : V \to GV$ and put

$\overrightarrow{v} := (V.GX \xrightarrow{v.1} GV.GX \xrightarrow{G_2} G(V.X)) \quad \overleftarrow{v} := (GX.V \xrightarrow{1.\overrightarrow{v}} GX.GV \xrightarrow{G_2} G(X.V)).$

They satisfy the conditions in (2.23) and their opposites, respectively, see Example 2.10 and they obey the compatibility condition in Definition 4.4 by the associativity of $G_2$. Conversely, let $(V, \overrightarrow{v}, \overleftarrow{v})$ be a $G$-comodule as in Definition 4.4. We know from Example 2.10 that the natural transformation $\overrightarrow{v}$ corresponds bijectively to the Eilenberg-Moore coaction

$V \xrightarrow{1.G_0} V.GI \xrightarrow{\overrightarrow{v}} GV,$

and the natural transformation $\overleftarrow{v}$ corresponds bijectively to the Eilenberg-Moore coaction

$V \xrightarrow{G_0.1} GI.V \xrightarrow{\overleftarrow{v}} GV.$
Precomposing by $V \xrightarrow{G_0.1.G_0} GI.V.GI$ both paths around the compatibility diagram in Definition 4.4 at $X = Y = I$, and using the unitality of the monoidal structure of $G$, we conclude that these Eilenberg-Moore coactions coincide.

Further examples of the situation in Definition 4.4 are provided by the following.

**Definition 4.7.** A *comodule* over a regular multiplier bimonoid $(t_1, t_2, t_3, t_4 : A^2 \to A^2, e : A \to I)$ in a braided monoidal category $C$ is an object $V$ of $C$ equipped with the structure $v_1 : V.A \to V.A$ of a comodule over the counital fusion morphism $t_1$ in $C$ and the structure $v_3 : V.A \to V.A$ of a comodule over the counital fusion morphism $t_3$ in $C$ such that the following diagram commutes.

$$
\begin{array}{ccc}
A.V.A & \xrightarrow{1.v_1} & A.V.A \\
\downarrow^{b.1} & & \downarrow^{1.m} \\
V.A^2 & \xrightarrow{v_3.1} & V.A \\
\end{array}
$$

where $m$ is the multiplication in Proposition 2.4 (1) (associated to $t_1$ or $t_2$, cf. the second diagram in Definition 3.1 differing by the braiding from the multiplication associated to $t_3$ or $t_4$; cf. axiom (C) in Definition 3.8). A *morphism* of comodules is a morphism $f : V \to W$ in $C$ such that the following diagrams commute.

$$
\begin{array}{ccc}
V.A & \xrightarrow{f.1} & V.A \\
\downarrow^{v_1} & & \downarrow^{v_3} \\
W.A & \xrightarrow{w_1} & W.A \\
\end{array}
$$

It follows immediately from the compatibility condition in Definition 4.7 that, in a comodule $(V, v_1, v_3)$ over a regular multiplier bimonoid with non-degenerate multiplication, $v_3$ is uniquely determined by $v_1$. Equivalently, $v_1$ is uniquely determined by $v_3$. Furthermore, in this case, both diagrams in Definition 4.7 defining morphisms of comodules become equivalent to each other: morphisms of comodules can be defined by either one of them.

For a regular multiplier bimonoid in $\text{vec}$, induced by a regular multiplier bialgebra over a field as in Example 3.12, we recover the notion of comodule in [10, Definition 2.7].

A comodule $(V, v_1, v_3)$ as in Definition 4.7 induces a comodule over the multiplier bi-comonad in Example 4.3 by putting

$$
\overrightarrow{v} := (V.X.A \xrightarrow{b.1} X.V.A \xrightarrow{1.v_1} X.V.A \xrightarrow{1.v_3} X.V.A)
$$

$$
\overleftarrow{v} := (X.A.V \xrightarrow{1.b} X.V.A \xrightarrow{1.v_3} X.V.A).
$$

Hence from Theorem 4.5 we obtain the following.

**Corollary 4.8.** For any regular multiplier bimonoid $(t_1, t_2, t_3, t_4 : A^2 \to A^2, e : A \to I)$ in a braided monoidal category $C$, the monoidal structure of $C$ lifts to the category of comodules in Definition 4.7.
For any comodules \((V, v_1, v_3)\) and \((W, w_1, w_3)\), \(V.W\) is again a comodule via

\[
V.W.A \xrightarrow{1_{W}} V.W.A \xrightarrow{b.1} W.V.A \xrightarrow{1_{V}} W.V.A \xrightarrow{b^{-1}.1} V.W.A \\
V.W.A \xrightarrow{1_{V}.b^{-1}} V.A.W \xrightarrow{v_3.1} V.A.W \xrightarrow{1_{V}.b} W.V.A \xrightarrow{1_{W}} V.W.A.
\]

5. Modules and multiplier bimads

This section is devoted to the study of the modules over a regular multiplier bimonoid in a braided monoidal category. As in the case of comodules in the previous section, this will be done by investigating the induced functors. To this end, we define \textit{multiplier bimonads} on arbitrary monoidal categories, which generalize ‘bimonads’; that is, opmonoidal monads. Under some further assumptions (of certain morphisms being split epimorphisms) we show that the monoidal structure of the base category lifts to the category of suitably defined modules. Showing that any regular multiplier bimonoid induces a multiplier bimonad, we draw conclusions about the categories of their modules.

5.1. Multiplier bimonad. Based on the considerations in Section 2.3, we introduce the following notion.

\textbf{Definition 5.1.} A \textit{multiplier bimonad} on a monoidal category \(C\) is a functor \(T : C \to C\) equipped with natural transformations

\[
\begin{align*}
\Rightarrow & : T(X.TY) \to TX.TY, \quad \Rightarrow : T(TX.Y) \to TX.TY \\
\Rightarrow_0 : TI \to I
\end{align*}
\]

and a morphism \(\Rightarrow_0 : T I \to I\) such that \((\Rightarrow_0, \Rightarrow)\) makes \(T\) a left multiplier bimonad on \(C\), \((\Rightarrow, \Rightarrow_0)\) makes \(T\) a right multiplier bimonad on \(C\), and these structures are compatible in the sense of the following commutative diagrams.

\[
\begin{align*}
T(TX.Y.TZ) & \xrightarrow{T \Rightarrow} TX.T(Y.TZ) & T^2X & \xrightarrow{T \Rightarrow} T.I.TX \\
\Rightarrow & \downarrow & \Rightarrow & \downarrow \\
T(TX.Y).TZ & \xrightarrow{1.T \Rightarrow} TX.TY.TZ & TX.TI & \xrightarrow{1.T_0} TX \\
\Rightarrow_0 & \downarrow & \Rightarrow_0 & \\
TX.T^2Y & \xrightarrow{1.T \Rightarrow} TX.TI.TY & T_0.1 & \\
\Rightarrow_0 & \downarrow & \\
TX.TY.TI & \xrightarrow{1.1.T_0} TX.TY & \\
\Rightarrow_0 & \\
\Rightarrow_0 &
\end{align*}
\]

The common diagonal in the second diagram in Definition 5.1 is an associative (though in general non-unital) multiplication that we denote by \(\mu : T^2 \to T\).

By the compatibility diagrams in Definition 5.1, the following diagram commutes for any multiplier bimonad \(T\) and any objects \(X, Y\).

\[
\begin{align*}
T(TX.TY) & \xrightarrow{T \Rightarrow} T^2X.TY & \xrightarrow{T \Rightarrow_0} T.I.TX.TY \\
\Rightarrow & \downarrow & \Rightarrow_0 & \downarrow \\
TX.T^2Y & \xrightarrow{1.T \Rightarrow} TX.TI.TY & T_0.1 & \\
\Rightarrow & \downarrow & \\
TX.TY.TI & \xrightarrow{1.1.T_0} TX.TY & \\
\Rightarrow & \\
\Rightarrow_0 &
\end{align*}
\]

\textbf{Example 5.2.} Consider an opmonoidal monad \((T, \mu, \eta)\) on a monoidal category \(C\). We know from Example 2.15 that

\[
\Rightarrow_0 := (T(X.TY) \xrightarrow{T} TX.T^2Y \xrightarrow{1.T} T.X.TY)
\]
(together with the nullary part $T_0 : TI \to I$ of the opmonoidal structure) makes $T$ into a left multiplier bimonad, and similarly
\[
\overrightarrow{T_2} := (T(TX.Y) \xrightarrow{T_2} T^2 X.TY \xrightarrow{\mu_1} TX.TY)
\]
and $T_0$ make $T$ into a right multiplier bimonad. They obey the compatibility conditions in Definition 5.1 by the coassociativity and the counitality of the opmonoidal structure $(T_2, T_0)$ and the functoriality of the monoidal product.

**Example 5.3.** Consider a regular multiplier bimonoid $(t_1, t_2, t_3, t_4 : A^2 \to A^2, e : A \to I)$ in a braided monoidal category $C$ and the induced functor $T = (\cdot) : C \to C$. We know from Example 2.16 that $T_0 = e$ and
\[
\overrightarrow{T_2} := (A.X.Y \xrightarrow{1.b^{-1}.1} A^2.X.Y \xrightarrow{t_1.1.1} A^2.X.Y \xrightarrow{1.b.1} A.X.A.Y)
\]
make $T$ into a left multiplier bimonad. We claim that together with
\[
\overleftarrow{T_2} := (A^2.X.Y \xrightarrow{b.1.1} A^2.X.Y \xrightarrow{t_4.1.1} A^2.X.Y \xrightarrow{1.b.1} A.X.A.Y)
\]
they constitute a multiplier bimonad. Certainly $\overrightarrow{T_2}$ provides a right multiplier bimonad structure; the compatibility conditions in Definition 5.1 hold by axiom (B), and the equivalent form $(1.e)t_4b = (e.1)t_1$ of axiom (C) in Definition 3.8 respectively.

5.2. **The category of modules.** Based on the notion of module in Definition 2.17, we introduce the following.

**Definition 5.4.** A *module* over a multiplier bimonad on a monoidal category $C$ is an object $Q$ in $C$ equipped with the structure $\overrightarrow{q} : T(-.Q) \to T(-)Q$ of a module over the left multiplier bimonad $(T, \overrightarrow{T_2}, T_0)$, and also with the structure $\overrightarrow{q} : T(Q,-) \to Q.T(-)$ of a module over the right multiplier bimonad $(T, \overleftarrow{T_2}, T_0)$ and these structures are compatible in the sense of the following commutative diagrams (for any objects $X, Y$).

\[
\begin{align*}
T(X.Y.Q) &\xrightarrow{\overrightarrow{q}} T(TX.Y).Q \\
\overrightarrow{T_2} &\downarrow \quad \overrightarrow{T_2} \downarrow \quad \overrightarrow{T_2} \\
TX.T(Y.Q) &\xRightarrow{1.\overrightarrow{q}} TX.TY.Q \\
1.\overrightarrow{T_2} &\downarrow \quad 1.\overrightarrow{T_2} &\downarrow \quad 1.\overrightarrow{T_2}
\end{align*}
\]
\[
\begin{align*}
TQ &\xrightarrow{\overrightarrow{q}} TI.Q \\
\overrightarrow{q} &\downarrow \quad T_0.1 \\
Q.TI &\xRightarrow{1.T_0} Q.
\end{align*}
\]

A *morphism* of modules is a morphism $f : Q \to P$ in $C$ rendering commutative the diagrams

\[
\begin{align*}
T(X.Q) &\xrightarrow{\overrightarrow{q}} TX.Q \\
T(1.f) &\downarrow \quad 1.f \\
T(X.P) &\xRightarrow{\overrightarrow{p}} TX.P
\end{align*}
\]
\[
\begin{align*}
T(Q.X) &\xrightarrow{\overrightarrow{q}} Q.TX \\
T(f.1) &\downarrow \quad f.1 \\
T(P.X) &\xRightarrow{\overrightarrow{p}} P.TX.
\end{align*}
\]

for any object $X$. 

The common diagonal in the third diagram in Definition 5.4 is an associative action (with respect to the associative multiplication $\mu : T^2 \to T$) that we denote by $q : TQ \to Q$. It is a split epimorphism by Definition 2.17.

**Theorem 5.5.** Consider a multiplier bimonad $T$ on a monoidal category $\mathcal{C}$. Assume that $T_0$ and, for any objects $X$ and $Y$, the equal morphisms (cf. (5.1))

$$(T(X.TY) \xrightarrow{\overline{T}_2} T^2X.TY \xrightarrow{\overline{T}_2,1} TI.X.TY \xrightarrow{T_0,1} TX.TY) =$$

$$(T(X.TY) \xrightarrow{\overline{T}_2} TX.T^2Y \xrightarrow{1,\overline{T}_2} T.X.TY.II \xrightarrow{1,1.T_0} TX.TY)$$

are split epimorphisms. Then the category of $\mathcal{T}$-modules of Definition 5.4 is monoidal in such a way that the forgetful functor to $\mathcal{C}$ is strict monoidal.

**Proof.** By Theorem 2.18 and by the coherence in $\mathcal{C}$, the monoidal unit $I$ of $\mathcal{C}$ carries a $T$-module structure (via the natural transformations built up from the unit isomorphisms). Also by Theorem 2.18, the monoidal product of any $\mathcal{T}$-modules $P$ and $Q$ is a $T$-module via the natural transformations

$$T(-.P.Q) \xrightarrow{\overline{q}} T(-.P).Q \xrightarrow{\overline{q}.1} TX.P.Q$$

$$T(P.Q.X) \xrightarrow{\overline{p}} P.T(Q.X) \xrightarrow{1,\overline{p}} P.Q.TX.$$ 

These natural transformations clearly obey the first two compatibility conditions in Definition 5.4 whenever $(\overline{q}, \overline{q})$ and $(\overline{p}, \overline{p})$ do. Since $q = (T_0,1)\overline{q}$ is a split epimorphism, the equal morphisms in the top row and in the left column of the diagram

$$T(P.TQ) \xrightarrow{\overline{T}_1} T(P.TQ) \xrightarrow{T(1,\overline{q})} T(P.Q)$$

are epimorphisms. The regions labelled by (a) and (b) commute by the naturality of $\overline{p}$ and of $\overline{q}$, respectively. Regions (c) commute by the counitality of $\overline{T}_2$. Regions (d) commute by the fusion equation on $\overline{q}$ and (e) commutes since the second compatibility condition in Definition 5.4 holds on $P$. The unlabelled regions commute by the functoriality of the monoidal product. This proves that the morphisms in the right column and the bottom
row are equal; that is, $(1_{\bar{q}} \bar{p}) = (\bar{p} \cdot 1_{\bar{q}})$. Using this identity and the assumption that $P$ and $Q$ obey the third compatibility condition in Definition 5.4, we conclude that also $P \cdot Q$ obeys the third compatibility condition in Definition 5.4:

$$
\begin{align*}
T(P \cdot Q) \xrightarrow{\bar{q}} T \cdot P \cdot Q \xrightarrow{\bar{p} \cdot 1} T \cdot I \cdot P \cdot Q \\
\downarrow \bar{p} \downarrow \quad \downarrow \bar{p} \downarrow \quad \downarrow 1 \cdot T_0 \downarrow \\
P \cdot T Q \xrightarrow{1_{\bar{q}}} P \cdot T I \cdot Q \\
\downarrow 1_{\bar{q}} \downarrow \\
P \cdot Q \cdot T I \xrightarrow{1 \cdot T_0} P \cdot Q.
\end{align*}
$$

In light of Theorem 2.18, this completes the proof. \qed

**Example 5.6.** Consider an opmonoidal monad $(T, \mu, \eta)$ on a monoidal category $C$ and the induced multiplier bimonad in Example 5.2. We claim that its category of modules in the sense of Definition 5.4 is isomorphic to the usual Eilenberg-Moore category of modules. Hence Theorem 5.5 generalizes the fact (see e.g. [7]) that the monoidal structure of the base category lifts to the Eilenberg-Moore category of a bimonad.

The stated isomorphism acts on the morphisms as the identity map. It takes an Eilenberg-Moore module $q : T \cdot Q \to Q$ to the module $T(-) \cdot Q$ in the sense of Definition 5.4. They obey the conditions in Definition 2.17 and their opposites, respectively, see Example 2.19. The compatibility conditions in Definition 5.4 hold by the coassociativity and counitality of the opmonoidal structure $(T_2, T_0)$ and the functoriality of the monoidal product. In the opposite direction, a module $(\bar{q} : T(-) \cdot Q \to T(-) \cdot Q, \bar{q} : T(Q(-)) \to Q(-))$ in the sense of Definition 5.4 is taken to the Eilenberg-Moore comodule $q := (T_0, 1) \bar{q} = (1, T_0) \bar{q}$. We know from Example 2.19 that this is an associative and unital action and these constructions yield mutually inverse bijections.

Further examples of the situation in Definition 5.4 are obtained from the following.

**Definition 5.7.** A module over a regular multiplier bimonoid $(t_1, t_2, t_3, t_4 : A^2 \to A^2, e : A \to I)$ in a braided monoidal category $C$ is an object $Q$ of $C$ equipped with morphisms $q_1 : A \cdot Q \to A \cdot Q$ and $q_4 : Q \cdot A \to Q \cdot A$ in $C$ such that $(Q, q_1)$ is a module (in the sense of Definition 2.20) over the comonital fusion morphism $t_1$ in $C$; $(Q, q_4)$ is a module over the counital fusion morphism $t_4$ in $C^{rev}$ and these structures are compatible in the sense of the following diagrams.

$$
\begin{align*}
A^2 \cdot Q \xrightarrow{1_{q_1}} A^2 \cdot Q \\
\downarrow t_4^{-1} \quad \downarrow t_4^{-1} \quad \downarrow 1 \cdot t_1 \quad \downarrow 1 \cdot t_1 \\
A^2 \cdot Q \xrightarrow{1 \cdot q_1} A^2 \cdot Q
\end{align*}
$$

(And that the common diagonal of the last diagram is a split epimorphism by Definition 2.20)
A morphism of modules is a morphism $f : Q \to P$ in $\mathcal{C}$ such that the following diagrams commute.

\[
\begin{array}{ccc}
A.Q & \xrightarrow{q_1} & A.Q \\
\downarrow 1.f & & \downarrow 1.f \\
A.P & \xrightarrow{p_1} & A.P
\end{array}
\quad
\begin{array}{ccc}
Q.A & \xrightarrow{q_4} & Q.A \\
\downarrow f.1 & & \downarrow f.1 \\
P.A & \xrightarrow{p_4} & P.A
\end{array}
\]

By the first and the last compatibility conditions on a module $(q_1, q_4)$ in Definition 5.7, the diagram

\[
\begin{array}{ccc}
A.Q & \xrightarrow{1.q_1} & A^2.Q \\
\downarrow t_1 & & \downarrow t_1 \\
A^2.Q & \xrightarrow{1.q_1} & A^2.Q
\end{array}
\quad
\begin{array}{ccc}
A^2.Q & \xrightarrow{1.q_1} & A^2.Q \\
\downarrow 1.q_4 b & & \downarrow 1.q_4 b \\
A.Q & \xrightarrow{1.1.e} & A.Q
\end{array}
\quad
\begin{array}{ccc}
A^2.Q & \xrightarrow{1.t_4} & Q.A^2 \\
\downarrow 1.q & & \downarrow 1.q \\
Q.A & \xrightarrow{q_4} & Q.A
\end{array}
\]

commutes. Hence if the multiplication $m$ is non-degenerate, then $q_1$ is uniquely determined by $q_4$. Equivalently, using the second and the last compatibility conditions in Definition 5.7, $q_4$ is uniquely determined by $q_1$. Furthermore, in this case, both diagrams in Definition 5.7 defining morphisms of modules become equivalent to each other: morphisms of modules can be defined by either one of them.

**Example 5.8.** Consider a regular multiplier bimonoid in $\text{vec}$ induced by a regular multiplier bialgebra over a field as in Example 3.12. We claim that its category of modules is isomorphic to the following category. The objects are vector spaces $Q$ equipped with an associative $A$-action $q : A.Q \to Q$ which is in addition a surjective map. The morphisms are the linear maps which commute with the actions.

The stated isomorphism acts on the morphisms as the identity map. It takes a module $(q_1, q_4)$ in Definition 5.7 to the associative and surjective action $(e.1)q_1 = (1.e)q_4 b$ (see Example 2.21), where $b$ stands for the symmetry in $\text{vec}$. In the opposite direction, we know from Example 2.21 that associative and surjective actions $q : A.Q \to Q$ are in a bijective correspondence with $t_1$-modules $q_1 : A.Q \to A.Q$, and also with $t_4$-modules $q_4 : Q.A \to Q.A$, rendering commutative the respective diagrams

\[
\begin{array}{ccc}
A^2.Q & \xrightarrow{t_1.1} & A^2.Q \\
\downarrow 1.q & & \downarrow 1.q \\
A.Q & \xrightarrow{q_1} & A.Q
\end{array}
\quad
\begin{array}{ccc}
Q.A^2 & \xrightarrow{1.t_4} & Q.A^2 \\
\downarrow q_4 b & & \downarrow q_4 b \\
Q.A & \xrightarrow{q_4} & Q.A
\end{array}
\]

We only need to show that $q_1$ and $q_4$ satisfy the compatibility conditions in Definition 5.7. The last one holds since the common diagonal in the last diagram is the associative and surjective action $q : A.Q \to Q$. Since $q$ is surjective, the first compatibility condition follows by

\[
(t_4.1)(1.q_1)(1.1.q) = (t_4.1)(1.1.q)(1.t_1.1) = (1.1.q)(t_4.1.1)(1.t_1.1) = (1.1.q)(1.t_1.1)(t_4.1.1) = (1.q_1)(1.1.q)(t_4.1.1) = (1.q_1)(t_4.1)(1.1.q).
\]

In the third equality we used axiom (B) in Definition 3.8. The second compatibility condition in Definition 5.7 follows symmetrically.
A module \((Q, q_1, q_4)\) in Definition 5.7 induces a module over the induced multiplier bimonad \(A.(-)\) in Example 5.3 by putting
\[
\Delta_q := (A.X.Q \xrightarrow{b^{-1}} A.Q.X \xrightarrow{q_1^{-1}} A.Q.X \xrightarrow{b} A.X.Q)
\]
\[
\nabla_q := (A.Q.X \xrightarrow{b.1} Q.A.X \xrightarrow{q_4^{-1}} Q.A.X).
\]
Hence from Theorem 5.5 we obtain the following.

**Corollary 5.9.** Consider a regular multiplier bimonoid \((t_1, t_2, t_3, t_4 : A^2 \to A^2, e : A \to I)\) in a braided monoidal category \(\mathcal{C}\). Assume that \(e\) and the equal morphisms (cf. the composite of axiom (B) in Definition 3.8 with \(1.e\).
\[
(A^3 \xrightarrow{1.t_4} A^3 \xrightarrow{b^{-1}.1} A^3 \xrightarrow{1.m} A^2) = (A^3 \xrightarrow{t_4.1} A^3 \xrightarrow{1.m} A^2)
\]
are split epimorphisms. Then the category of modules of Definition 5.7 is monoidal in such a way that the forgetful functor to \(\mathcal{C}\) is strict monoidal.

The monoidal product of modules \(P\) and \(Q\) is again a module via the morphisms
\[
A.P.Q \xrightarrow{b^{-1}} A.Q.P \xrightarrow{q_1^{-1}} A.Q.P \xrightarrow{b} A.P.Q
\]
\[
P.Q.A \xrightarrow{b^{-1}} P.A.Q \xrightarrow{q_4^{-1}} P.A.Q \xrightarrow{b} P.Q.A.
\]

**References**

[1] G. Böhm, *Comodules over weak multiplier bialgebras*, Int. J. Math. 25 (2014), 1450037.
[2] G. Böhm, J. Gómez-Torrecillas and E. López-Centella, *Weak multiplier bialgebras*, Trans. Amer. Math. Soc., in press. Preprint available at [http://arxiv.org/abs/1306.1466](http://arxiv.org/abs/1306.1466).
[3] G. Böhm and S. Lack, *Multiplier Hopf monoids in closed braided monoidal categories*, in progress.
[4] J. Dauns, *Multiplier rings and primitive ideals*, Trans. Amer. Math. Soc. 145 (1969), 125–158.
[5] K. Janssen and J. Vercruysse, *Multiplier Hopf and bi-algebras*, J. Algebra Appl. 9 (2010), no. 2, 275–303.
[6] S. Lack and R. Street, *Skew monoidales, skew warpings and quantum categories*, Theory Appl. Categ. 26 (2012), 385–402.
[7] P. McCrudden, *Opmonoidal monads*, Theory Appl. Categ. 10 (2002), no. 19 469–485.
[8] R. Street, *Fusion operators and cocycloids in monoidal categories*, Applied Categorical Structures 6(2) (Special Issue on Quantum Groups, Hopf Algebras and Category Theory, ed. A. Verschoren, 1998) 177–191.
[9] A. Van Daele, *Multiplier Hopf algebras*, Trans. Amer. Math. Soc. 342 (1994), no. 2, 917–932.
[10] A. Van Daele and Y. Zhang, *Corepresentation theory of multiplier Hopf algebras I.*, Int. J. Math. 10 (1999), 503–539.

WIGNER RESEARCH CENTRE FOR PHYSICS, H-1525 BUDAPEST 114, P.O.B. 49, HUNGARY

E-mail address: bohm.gabriella@wigner.mta.hu

DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY NSW 2109, AUSTRALIA

E-mail address: steve.lack@mq.edu.au