ASCENDING CHAINS OF FREE GROUPS IN 3-MANIFOLD GROUPS

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ABSTRACT. Takahasi and Higman independently proved that any ascending chain of subgroups of constant rank in a free group must stabilize. Kapovich and Myasnikov gave a proof of this fact in the language of graphs and Stallings folds. Using profinite techniques, Shusterman extended this constant-rank ascending chain condition to limit groups, which include closed surface groups. Motivated by Kapovich and Myasnikov’s proof we provide two new proofs of this ascending chain condition for closed surface groups, and establish the ascending chain condition for free subgroups of constant rank in a closed (or finite-volume hyperbolic) 3-manifold group. Hyperbolic geometry, geometrization, and graphs-of-groups decompositions all play a role in our proofs.

1. INTRODUCTION

When does a set of free subgroups of a group satisfy the ascending chain condition? If a group $G$ contains a free subgroup, then it contains a subgroup isomorphic to the free group $F(a, b)$. The subgroups $\langle a \rangle < \langle a, bab^{-1} \rangle < \langle a, bab^{-1}, b^2ab^{-2} \rangle < \cdots$ are a proper ascending chain of unbounded rank. Imposing a bound on the rank of each subgroup under consideration removes this obvious chain and it is a classical result due independently to Takahasi [27, Theorem 1] and Higman [10, Lemma] that any ascending chain of subgroups of constant rank in a fixed free group must stabilize.

Kapovich and Myasnikov [14, Theorem 14.1] give a proof of this ascending chain condition using finite graphs following Stallings. To sketch the argument, suppose $G$ is a fixed finite-rank free group and $H_1 \leq H_2 \leq H_3 \cdots \leq G$ is an ascending chain of subgroups of constant rank $r$. Fix a finite graph $\Gamma$ such that $G = \pi_1(\Gamma)$. Its universal cover $\tilde{\Gamma}$ is a tree. Each $H_i$ acts on $\tilde{\Gamma}$ and has a minimal non-empty invariant subtree $\tilde{\Gamma}_i$ with quotient a finite graph $\Gamma_i$. The nesting of subgroups induces graph maps $\Gamma_i \to \Gamma_{i+1}$. One may assume without loss of generality (Lemma 2.1) that for all $i$ the subgroup $H_i$ is not contained in a proper free factor of $H_{i+1}$. This implies that each map $\Gamma_i \to \Gamma_{i+1}$ is surjective. However, each $\Gamma_i$ must have at least $r$ edges. Therefore the inclusion $H_i \leq H_{i+1}$ can only be proper a finite number of times and the chain stabilizes.

Inspired by this argument we establish an ascending chain condition for free groups of constant rank in closed 3-manifold groups.

Theorem 1.1. Closed 3-manifold groups do not admit proper ascending chains of free groups of constant rank.

2020 Mathematics Subject Classification. 57M05, 20E07, 57K20, 57K32.
EB was supported by the Azrieli foundation.
NL was supported by the Israel Science Foundation (grant no. 1562/19).
In the setting of surface groups, this constant-rank chain condition was proved by Shusterman using the theory of limit groups and profinite completions \[24\, \text{Theorem 1.2}\]. For 3-manifolds Shusterman’s limit group techniques do not apply: the only 3-manifold groups that are limit groups are free products of \(\mathbb{Z}\) and \(\mathbb{Z}^3\). We first prove the following combination theorem using a generalization of the graphical argument to graphs of groups. Closed surface groups split as cyclic amalgams of free groups, so the constant-rank chain condition for closed surfaces follows from our combination theorem and the constant-rank chain condition in free groups; this argument is the first of two new proofs of the ascending chain condition for constant rank free subgroups of closed surface groups. The proof of our combination result should be compared to Shusterman \[24\, \text{Proposition 3.2}\].

**Theorem 1.2.** If a group \(G\) splits over abelian groups such that no vertex group admits a proper ascending chain of free groups of constant rank, then \(G\) does not admit a proper ascending chain of free groups of constant rank.

To prove Theorem 1.1 we appeal to the prime decomposition, JSJ decomposition, and geometrization theorems to break up a general closed 3-manifold along tori and spheres into Seifert fiber spaces and finite-volume hyperbolic 3-manifolds. This decomposition induces a graph of groups decomposition of the fundamental group with abelian edge groups. We prove Theorem 1.1 for Seifert fiber spaces in Corollary 5.2 and finite-volume hyperbolic 3-manifolds in Proposition 6.1, and the general conclusion follows from Theorem 1.2.

To motivate the proof of Proposition 6.1 we give a second new proof of this ascending chain condition in the closed surface setting. Our second surface proof uses only tools from surface theory: Mumford’s compactness criterion for the moduli space of hyperbolic metrics replaces edge-counting as the source of finiteness which forces any chain to stabilize, as detailed in Theorem 4.2.

In general, subgroups of \(\text{Isom}^+(\mathbb{H}^3)\) do not satisfy the ascending chain condition on bounded rank free groups: Calegari and Dunfield produced a non-discrete free group of rank 6 in \(\text{Isom}^+(\mathbb{H}^3)\) that is conjugate to a proper subgroup of itself \[4\]. The union of these nested conjugates has a proper ascending chain of free subgroups of rank 6.

There is an alternative definition that excludes the “obvious” ascending chains in a free group: Consider only ascending chains of finitely generated free groups in which each inclusion is not into a proper free factor. Higman proved a locally free group (a group where every finitely generated subgroup is free) is countably free if and only if every ascending chain of finitely generated free groups with no proper free factor inclusions stabilizes \[9, \text{Theorem 1}\]. Countable free groups are the only countably, countably free groups. Thus, Higman’s criterion provides a very strong constraint on groups where every factor-avoiding chain of finitely generated free subgroups stabilizes: every locally free subgroup must be free. This is true for surface groups, and one can verify the factor-avoiding ascending chain condition for surface groups using an argument similar to Higman’s.

However, 3-manifold groups can contain non-free locally free subgroups, even in geometric settings. Kurosh \[16\] proved that the group \(K = \langle a, b, t | tata^{-1} = [a, b] \rangle\) has a locally free subgroup which is not free. Thus, by Higman’s characterization, \(K\)

\[1\]Higman uses the shorthand \(n\)-subgroup for finitely generated subgroup in this paper \[9\, \text{p. 285}\].
must contain a proper ascending chain of finitely generated free groups where no inclusion is into a proper free factor. One can construct a space \( X \) with \( \pi_1(X) = K \) by gluing together an infinite string of one-holed tori according to the presentation relation, by thickening \( X \) one obtains a 3-manifold. Hyperbolic geometry cannot rule out such phenomena: Maskit [17, Chapter VIII.E.9] realized \( K \) as a Kleinian group. Building on Maskit, Anderson [2] used \( K \) to construct infinitely many non-commensurable finite-volume hyperbolic 3-manifolds with \( K \) as a subgroup of the fundamental group.

Acknowledgements. We are grateful to Michah Sageev, particularly in the early stages of this work, and Henry Wilton, both for helpful conversations.

2. Avoiding splitting factors

In each of our proofs the first essential reduction is from general chains to chains where \( H_i \) is not contained in a proper free factor of \( H_{i+1} \). This condition is used to ensure that a corresponding map of topological objects is essential. A version of lemma exists, in different language, as the first part of Kapovich and Myasnikov’s proof, but it is not stated explicitly [14, Proof of Theorem 14.1].

Lemma 2.1. Let \( G \) be a group. Suppose \( H_1 \leq H_2 \leq \ldots \leq G \) is an ascending chain of free subgroups of constant rank. Then, there exists \( K_1 \leq K_2 \leq \ldots \leq G \) an ascending chain of free subgroups of constant rank such that for all \( i \) the group \( K_i \) is not contained in a proper subgroup \( L \leq K_{i+1} \) where \( \text{rk}(L) < \text{rk}(K_i) \). In particular, \( K_i \) is not contained in a proper free factor of \( K_{i+1} \) for all \( i \). Moreover, if the chain \( H_i \) does not stabilize, then the chain \( K_i \) does not stabilizes.

Proof. We prove the lemma by induction on the constant rank \( r \) of \( H_i \). If \( r = 1 \), there is nothing to prove. Let \( I \subset \mathbb{N} \) be the subset of indices such that \( H_i \) is contained in a subgroup of \( H_{i+1} \) with rank less than \( r \). If \( I \) is finite, then the tail \( K_i = H_{N+i} \) where \( N = \max I \) has the required property. If \( I \) is infinite, then for each \( i \in I \) we get \( H_i \leq K_i \leq H_{i+1} \) where \( K_i \) is a proper subgroup of \( H_{i+1} \) and \( \text{rk}(K_i) < \text{rk}(H_{i+1}) = r \). Thus, we get an ascending chain \( \{K_i\}_{i \in I} \) of strictly smaller rank where each inclusion is proper. By the pigeonhole principle there is a subsequence \( \{K_j\}_{j \leq \lambda} \subseteq \{K_i\} \) with constant rank strictly smaller than \( r \). Applying the induction hypothesis to \( \{K_j\}_j \) completes the proof.

Lemma 2.2. Let \( G \) be a group. Suppose \( H_1 \leq H_2 \leq \ldots \leq G \) is an ascending chain of free subgroups of constant rank and for all \( i \) the group \( H_i \) is not contained in a proper free factor of \( H_{i+1} \). Then for all \( i \) the group \( H_1 \) is not contained in a proper free factor of \( H_i \).

Proof. We proceed by induction on \( i \). When \( i = 1, 2 \) the claim is true by hypothesis. So suppose \( H_1 \) is not contained in a proper free factor of \( H_{i-1} \). Suppose \( H_1 = K \ast L \) is a non-trivial free splitting of \( H_i \). By hypothesis \( H_{i-1} \) is not contained in \( K \) or \( L \), so every factor in the Kurosh decomposition of \( H_{i-1} = F \ast K_{i-1} \ast L_{i-1} \) is non-trivial; here where \( F \) is free, and \( K_{i-1}, L_{i-1} \) are free products of subgroups of \( K \) and \( L \) respectively. By the induction hypothesis \( H_1 \) is not contained in \( F \), \( K_{i-1} \), or \( L_{i-1} \), and therefore \( H_1 \) is not contained in any conjugate of \( K \) or \( L \).

Given a graph of groups \( \mathcal{G} = \langle \Gamma, \{G_v\}_{v \in V_T}, \{G_e\}_{e \in E_T} \rangle \) and a finitely generated subgroup \( H \leq \pi_1(\mathcal{G}) \), let \( \mathcal{G}^H = \langle \Gamma^H, \{H_v\}_{v \in V_T}^H, \{H_e\}_{e \in E_T}^H \rangle \) denote the graph of
groups corresponding to the minimal subtree for the action of \( H \) on the Bass-Serre tree of \( G \), and denote its underlying graph \( \Gamma^H \). For any two nested subgroups \( K \leq H \leq \pi_1(\mathcal{G}) \) the inclusion of minimal subtrees induces a map of underlying graphs \( \Gamma^K \to \Gamma^H \).

For chains inside of a fixed free group we can represent the chain as a chain of graph maps—the conclusion of Lemma 2.1 implies each graph map in a chain satisfying this conclusion is surjective. When working with a chain of free subgroups in a group with an abelian splitting an analogous surjectivity can be imposed on the induced maps of graphs underlying the splittings of the chain elements using a similar argument.

**Lemma 2.3.** Suppose \( \mathcal{G} = (\Gamma, \{G_e\}_{e \in \mathcal{V}_\Gamma}, \{G_e\}_{e \in \mathcal{E}_\Gamma}) \) is a finite graph of groups with abelian edge groups with fundamental group \( G = \pi_1(\mathcal{G}) \). Suppose \( H_1 \leq H_2 \leq \ldots \leq H \) is an ascending chain of free subgroups of constant rank. Then, there exists \( K_1 \leq K_2 \leq \ldots \leq G \) an ascending chain of free subgroups of constant rank such that the induced map \( \Gamma^{K_1} \to \Gamma^{K_{i+1}} \) is surjective for all \( i \). Moreover, if the chain \( H_i \) does not stabilize, then the chain \( K_i \) does not stabilize.

**Proof.** First observe that for every free subgroup \( H \leq G \) all edge groups of \( \mathcal{G}^H \) are abelian and hence trivial or infinite cyclic. For a finitely generated free subgroup \( H \leq G \) define its complexity to be \( c(H) = \beta_1(\Gamma^H) + \operatorname{rk}(H) \in \mathbb{N} \), where \( \beta_1 \) is the first betti number. Note that \( c(H) \leq 2\operatorname{rk}(H) \). Fix a finitely generated free subgroup \( H \leq G \). Let \( \Lambda \) be a connected component of \( \Gamma^H \) and \( \mathcal{L} \) be the restriction of \( \mathcal{G}^H \) to \( \Lambda \), and let \( L = \pi_1(\mathcal{L}) \).

**Claim 2.4.** \( c(L) < c(H) \).

**Proof of claim.** It is immediate that \( \beta_1(\Gamma^H) \leq \beta_1(L) \leq \beta_1(\Gamma^H) \).

If \( \epsilon \) is separating, let \( \Xi \) be the connected component other than \( \Lambda \). Since \( \mathcal{G}^H \) is minimal, the group \( X = \pi_1(\mathcal{G}^H\Xi) \) is non-trivial and properly contains the edge group \( H_e \). There are two cases coming from the possible edge groups of \( e \). If \( H_e \) is trivial, then \( H = \Lambda \times X \), hence \( \operatorname{rk}(L) < \operatorname{rk}(H) \). If \( H_e \) is cyclic, then it follows from Shenitzer’s theorem on cyclic splittings of free groups that \( \operatorname{rk}(H) = \operatorname{rk}(L) + \operatorname{rk}(X) - 1 \geq 2 \). Since \( X \) properly contains the cyclic edge group \( H_e \), \( \operatorname{rk}(X) \geq 2 \), thus \( \operatorname{rk}(L) < \operatorname{rk}(H) \).

If \( \epsilon \) is non-separating then \( \beta_1(L) < \beta_1(\Gamma^H) \). By collapsing \( \Lambda \) we obtain \( H \) as an HNN extension of \( L \). If the edge group \( H_e \) is trivial, \( \operatorname{rk}(L) < \operatorname{rk}(H) \), and if the edge group \( H_e \) is cyclic it follows from Swarup’s theorem on cyclic HNN extensions of free groups that \( \operatorname{rk}(L) = \operatorname{rk}(H) \).

Note that \( \mathcal{L} \) may not be minimal, but a minimal graph of groups can be obtained by restricting to a finite subgraph.

We now prove the lemma by induction on the maximum complexity \( c = \max c(H_i) \); this maximum exists since the rank of \( H_i \) is assumed to be constant. If \( c = 1 \) then there is nothing to prove. Let \( I \) be the subset of indices such that the induced map \( f_i : \Gamma^{H_i} \to \Gamma^{H_{i+1}} \) is not surjective. If \( I \) is finite, then the tail \( K_i = H_{N+i} \) where \( N = \max I \) has the required property. If \( I \) is infinite then for each \( i \in I \) we define a subgroup \( H_i < K_i < H_{i+1} \) with \( c(K_i) < c(H_i) \) as follows. Let \( L_i = f_i(\Gamma^{H_i}) \), \( \mathcal{L}_i = \mathcal{G}^{H_{i+1}}|_{\Lambda_i} \), and set \( K_i = \pi_1(\mathcal{L}_i) \). The group \( K_i \) can be obtained from \( \Gamma^{H_{i+1}} \) by

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2Stallings gives a unified account of the two splitting theorems used here [25].
deleting a finite list of edges $e_1, \ldots, e_k$ in sequence. Let $A^j_i$ be the connected component of the intermediate graph where $e_1, \ldots, e_j$ have been deleted that contains $A_i$, and set $L_j = \pi_1(A^j_i)$. Either $L_j = L_i+1$, or $A^j_i$ is minimal for $L_j$ and by Claim 2.4 $c(L_{j+1}) < c(L_j)$. Moreover by Claim 2.4 since $\Gamma^{H_{i+1}}$ is minimal $c(L_i) < c(H_{i+1})$. We conclude $c(K_i) < c(H_{i+1}) \leq c$. By the pigeonhole principle there is a subsequence $\{K_j\} \subseteq \{K_i\}$ of constant rank. Applying the induction hypothesis to $\{K_j\}$ completes the proof. □

Proof of Theorem 1.2. Let $\mathcal{G} = (\Gamma, \{G_v\}_{v \in \mathcal{V}}, \{G_e\}_{e \in \mathcal{E}})$ be a finite graph of groups with abelian edge groups. Let $G = \pi_1(\mathcal{G})$. Let $H_1 \leq H_2 \leq \ldots$ be an ascending chain of free subgroups of constant rank. Since each $H_i \leq G$ is a finitely generated free group it decomposes as well as a finite graph of groups $H_i = \mathcal{G}^H_i$ with defining graph $\Lambda_i = \Gamma^{H_i}$ and trivial or infinite cyclic edge groups. The inclusions give a chain of morphisms $H_i \to H_{i+1} \to \ldots \to G$.

The morphism $H_i \to H_{i+1}$ induces a morphism on the underlying graphs $\Lambda_i \to \Lambda_{i+1}$. By Lemma 2.3 we may assume each morphism is surjective and thus eventually the sequence of graphs the $\Lambda_i$ stabilizes to a graph $\Lambda$. Without loss of generality assume $\Lambda_i = \Lambda$ for all $i$ and that the morphism $H_i \to \mathcal{G}$ induces a fixed map $f: \Lambda \to \Gamma$ on the underlying graph. Let $v$ be some vertex of $\Lambda$. The vertex groups $(H_i)_v$ form an ascending chain of free groups in the vertex group $G_v$. By assumption they must stabilize. Since there are only finitely many $v \in \mathcal{V}$, all vertex groups appearing in $\mathcal{G}^{H_i}$ eventually stabilize. Similarly, all edge groups appearing in $\mathcal{G}^{H_i}$ must eventually stabilize. It follows that $H_i$ eventually stabilizes. □

3. Hyperbolic isometries and total displacement

In this section we record some notions from hyperbolic geometry, which are common to our surface and hyperbolic 3-manifold arguments.

Given a subset $X \subseteq \mathbb{H}^n \cup S_{\infty}^{n-1}$ its convex hull is denoted $\text{Hull}(X)$. For $g \in \text{Isom}(\mathbb{H}^n)$ we let $\tau(g)$ denote the minimum translation length; recall $\tau(g)$ is a conjugacy invariant. An isometry $g \in \text{Isom}(\mathbb{H}^n)$ is parabolic if $\tau(g) = 0$ but $g$ fixes no point of $\mathbb{H}^n$. A subgroup $\text{Isom}(\mathbb{H}^n)$ is parabolic if all of its elements are parabolic. When $G \leq \text{Isom}(\mathbb{H}^n)$ is discrete, torsion-free, and non-abelian we denote the limit set by $\partial G \subseteq S_{\infty}^{n-1}$ and the convex core by $\text{CC}(G) = \text{Hull}(\partial G)/G$. Suppose $A \subset \text{Isom}^+(\mathbb{H}^n)$ is a finite set of isometries. Define the minimum total displacement $\tau(A)$ to be

$$\tau(A) = \inf_{x \in \mathbb{H}^n} \sum_{a \in A} d(x, a.x).$$

Note that if $(A)$ is not parabolic then $\tau(A) > 0$. This agrees with the previous definition when $A$ is a singleton. For a finitely generated group of isometries we define the minimum total displacement of $G$ to be the infemum over that of finite generating sets

$$\tau(G) = \inf \{ \tau(A) \mid A \text{ is a finite generating set of } G \}$$

and as a short hand if $M$ is a hyperbolic $n$-manifold we write $\tau(M) = \tau(\pi_1(M))$.

Lemma 3.1. Suppose $M$ is a finite-volume hyperbolic $n$-manifold. Given $C \in \mathbb{R}$ there are finitely many conjugacy classes of subsets $A \subseteq \pi_1(M) < \text{Isom}^+(\mathbb{H}^n)$ such that $(A)$ is non-parabolic and $\tau(A) \leq C$. 

Proof. Choose a thick-thin decomposition of $M$ such that for the lift $\tilde{N}$ of the thick part $M_{\geq \tau}$ to $\mathbb{H}^n$ the distance between distinct horosphere boundary components of $\tilde{N}$ is at least $C$ and the interior of $M_{\geq \tau}$ is diffeomorphic to $M$. Fix a basepoint $* \in M_{\geq \tau}$ so that $\tilde{N}$ is the based lift of $M_{\geq \tau}$ to $\mathbb{H}^n$ at a chosen origin $o$. Let $A \subset \pi_1(M)$ be a set of elements such that $\tau(A) \leq C$ and $\langle A \rangle$ is non-parabolic. Let $x \in \mathbb{H}^n$ be the point realizing $\tau(A)$. Since $\tau(A) \leq C$ and $\langle A \rangle$ is non-parabolic, we must have $x \in \tilde{N}$; thus there is some element $h \in \pi_1(M)$ such that $d(t,x,o) \leq \text{diam}(M_{\geq \tau}) = D$. Conjugating by $h$, for all $a \in A^h$ we have $d(o,a,o) \leq 2D+C$. The action of $\pi_1(M)$ is properly discontinuous, so we conclude $A^h$ is contained in the finite set $\{g \in \pi_1(M) | d(o,go) \leq 2D+C\}$. 

In the sequel, we will apply Lemma 3.1 in combination with the following theorem of Ohshika and Potyagailo (in light of the tameness theorem 1.5 we remove the topologically tame hypothesis in our quotation).

**Theorem 3.2** ([20] Theorem 1.3(b)). Suppose $G \leq \text{Isom}^+ (\mathbb{H}^3)$ is a discrete, non-elementary, torsion-free finitely generated group. Then for all $\alpha \in \text{Isom}^+ (\mathbb{H}^3)$, the inclusion $\alpha G \alpha^{-1} \subset G$ implies $\alpha G \alpha^{-1} = G$.

Note that this implies the analogous statement for subgroups of $\text{Isom}^+ (\mathbb{H}^2)$, originally due to Huber [11].

4. Surface groups

Our primary tool for working with surface groups will be various moduli spaces of hyperbolic metrics. For an oriented surface $\Sigma$ of genus $g$ with $p$ punctures and $b$ boundary circles we denote its *moduli space* of complete, finite-area hyperbolic metrics with totally geodesic boundary $M_{g,p,b}$ or $M(\Sigma)$; when $b = 0$ or $p = 0$ we omit the index. We need the following consequence [17] §12.4.3 of Mumford’s compactness criterion [19] in order to prove that certain collections of surfaces land in a compact part of a fixed moduli space.

**Fact 4.1.** Suppose $\Sigma$ is a finite-type surface with $\chi(\Sigma) < 0$. For all $\epsilon > 0$ the subset $M^*(\Sigma) \subset M(\Sigma)$ consisting of hyperbolic structures where all essential arcs and simple closed curves have length at least $\epsilon$ is compact.

Fact 4.1 allows us to bound $\tau$, because $\tau$ is a continuous function on $M(\Sigma)$. We will appeal to this in the sequel without further comment.

**Theorem 4.2.** Let $S$ be a good compact $2$-orbifold and let $r \in \mathbb{N}$. Every ascending chain of free groups of constant rank in $\pi_1(S)$ stabilizes.

Proof. The orbifold $S$ has a finite cover $S'$ which is an orientable surface. By taking intersections, an ascending chain of free groups of constant rank in $\pi_1(S)$ will give rise to an ascending chain of free groups of bounded rank in $\pi_1(S')$, which by the pigeonhole principle contains an ascending subsequence of free groups of constant rank. This reduces the proof to the case that $S$ is an orientable surface. Furthermore, if $S$ has boundary, then $\pi_1(S)$ is free, and the result follows from Takahasi or Higman [10][27]. If $S$ has genus zero or one there is nothing to prove. Thus, we may assume that $S$ is a closed orientable hyperbolic surface.

Let $G = \pi_1(S)$, and let $H_1 \leq H_2 \leq \ldots$ be an ascending chain of free subgroups of rank $r$ in $G$. By Lemma 2.1 assume further that each $H_i$ is not contained in a
free factor of $H_{i+1}$. If $r = 1$ let $H_i = \langle \alpha_i \rangle$, since $\tau(\alpha_{i+1}) \leq \tau(\alpha_i)$ and the translation lengths are bounded below by the girth of $S$ the chain must stabilize.

Now suppose $r \geq 2$. Note that each $H_i$ is a discrete group of isometries of $\mathbb{H}^2$. Let $S_i = CC(H_i)$. Since the rank of $H_i$ is fixed, by the pigeonhole principle, after passing to a subsequence we may assume that all $S_i$ are homeomorphic to a fixed surface $\Sigma$. Since $H_1 \leq H_i$ we have $\Lambda H_1 \leq \Lambda H_i$; after passing to the quotient this produces an isometric immersion $f_i : S_1 \hookrightarrow S_i$. Let $\alpha(S_i)$ be the length of the shortest essential arc in $S_i$. By Lemma 2.2 $H_1$ is not contained in a free factor of $H_i$, so the immersed surface $f_i(S_1)$ crosses every essential arc transversely. Therefore, $\alpha(S_i) \geq \alpha(S_1) > 0$.

Moreover, the shortest non-parabolic curve on each $S_i$ is bounded below by that of $S$. We conclude that the hyperbolic surfaces $S_i$ are contained in a compact subset of $\mathcal{M}(\Sigma)$, so there is a uniform bound on $\tau(S_i)$. Let $A_i \subset H_i$ be a generating set realizing $\tau(S_i)$. By Lemma 4.1 and the pigeonhole principle there exists an infinite set of indices $J$ such that for all $i \leq j \in J$ the sets $A_i$ and $A_j$ are conjugate in $\pi_1(S)$, and therefore $H_i$ is a conjugate of $H_j$ and a subgroup of $H_j$. Hence $H_i = H_j$ for all $i, j \in J$ by Theorem 3.2, and the chain stabilizes. □

**Remark 4.3.** Every subgroup of a surface group is either free or a closed surface group of strictly greater rank. Therefore, Theorem 4.2 implies that every ascending chain of constant rank subgroups in a fixed orbifold group stabilizes.

### 5. Seifert-fibered 3-manifold groups

Recall that a group is **slender** if every subgroup is finitely generated. Equivalently, every ascending chain of subgroups stabilizes.

**Lemma 5.1.** Let $1 \to N \to G \to S \to 1$ be a short exact sequence of groups. If $N$ is slender and any ascending chain of constant rank subgroups of $S$ stabilizes then every ascending chain of constant rank subgroups of $G$ stabilizes.

**Proof.** Let $H_1 \leq H_2 \leq \ldots \leq G$ be an ascending chain of constant rank subgroups of $G$. The chain $H_i \cap N$ stabilizes because $N$ is slender, and so does the image chain $\bar{H}_i \leq \bar{S}$ by hypothesis. By the short five lemma, the chain $\bar{H}_i$ also stabilizes. □

**Corollary 5.2.** Let $M$ be a Seifert-fibered 3-manifold. Every ascending chain constant rank subgroups in $\pi_1(M)$ stabilizes.

**Proof.** Let $M$ be a Seifert-fibered 3-manifold, and let $G = \pi_1(M)$. Thus, $G$ fits into an exact sequence

$$1 \to N \to G \to S \to 1$$

where $N \cong \mathbb{Z}$ is the fiber group and $S$ is the fundamental group of a good 2-orbifold. Note that $N \cong \mathbb{Z}$ is slender, and every ascending chain of constant rank subgroups of $S$ stabilizes by Theorem 4.2 and Remark 4.3. The corollary follows from Lemma 5.1. □

### 6. Hyperbolic 3-manifold groups

**Proposition 6.1.** Let $M$ be a finite-volume hyperbolic 3-manifold. Every ascending chain of free groups of constant rank in $\pi_1(M)$ stabilizes.
Proof. As in the surface setting by passing to the orientation double cover we may assume without loss that $M$ is orientable. Let $G = \pi_1(M)$, and let $H_1 \leq H_2 \leq \ldots \leq G$ be an ascending chain of free groups of rank $r$. By Lemma 2.1 assume further that each $H_i$ is not contained in a free factor of $H_{i+1}$.

Since each $H_i$ is finitely generated it is a consequence of the tameness theorem for hyperbolic 3-manifolds that $H_i$ is either geometrically finite or a virtual fiber [1][5].

If some $H_i$ is a virtual fiber, the chain stabilizes at $H_i$. This follows from the rank restriction. Suppose $H_i$ is a virtual fiber and let $G_i$ be the finite-index subgroup of $G$ such that the corresponding cover of $M$ fibers over the circle with fiber subgroup $H_i$. Algebraically we have the short exact sequence

$$1 \to H_i \to G_i \to \mathbb{Z} \to 1.$$ 

Consider $H_j$ for $j \geq i$. The intersection $H_j \cap G_i$ is finite-index in $H_j$. Observe $H_i$ is normal in $H_j \cap G_i$, hence $[H_j \cap G_i : H_i] < \infty$. Thus $[H_j : H_i] < \infty$. However, $\text{rk}(H_i) = \text{rk}(H_j)$, so we conclude $H_i = H_j$.

Now suppose that every $H_i$ is geometrically finite. If $\tau(H_i) = 0$ for all $i$, then all $H_i$ are contained in a single parabolic subgroup $P$ of $\pi_1(M)$. Since $M$ is finite-volume and orientable, $P \cong \mathbb{Z}^2$ [28, Theorem 4.5] and the result is obvious. So we assume further that each $H_i$ is not a parabolic subgroup. We will show that there is a uniform upper bound for $\tau(H_i)$. The theorem follows from this bound via the pigeonhole principle, Lemma 3.1 and Theorem 3.2 as in the surface case.

In order to bound $\tau(H_i)$ we consider two cases depending on the limit sets of $H_i$.

If there is a round circle $C \subset \partial \mathbb{H}^3$ such that the limit sets $\Lambda H_i \subseteq C$ for all $i$, then each $H_i$ acts on a hyperbolic plane $\mathbb{H}^2 = \text{Hull}(C)$ and each convex core $S_i = \text{CC}(H_i)$ is a hyperbolic surface. By the pigeonhole principle we may pass to a subsequence and assume all $S_i$ are homeomorphic to a fixed surface $\Sigma$. As in the proof of Theorem 4.2 the length $\alpha(S_i)$ of the shortest essential arc in each $S_i$ is bounded below by $\alpha(S_1) > 0$. Since there is a lower bound on the length of each non-parabolic element of $G$, there is a lower bound on the girth of each $S_i$. Therefore the $S_i$ land in a compact subset of the moduli space of $\Sigma$, which implies there is a uniform upper bound bound on $\tau(H_i)$.

Otherwise, after passing to a tail sequence, no limit set $\Lambda H_i$ is contained in a round circle. Let $M_i = \text{CC}(H_i)$ and consider the surfaces $\partial M_i$. Each $\partial M_i$ is a finite-area connected hyperbolic surface in the intrinsic metric [6, Theorem II.1.12.1]. Each $M_i$ is a homeomorphic to a handlebody of genus $r$ with $0 \leq k \leq r$ disjoint simple closed curves removed from the boundary (each curve represents a primitive parabolic conjugacy class of $M_i$). After passing to a subsequence we suppose $k$ is constant, so each hyperbolic surface $\partial M_i$ determines a point in the moduli space $\mathcal{M}_{r-k,2k}$. First we will show that the sequence $\{\partial M_i\}$ is contained in a compact subset of $\mathcal{M}_{r-k,2k}$.

For a contradiction suppose there exists a sequence $\gamma_i \subset \partial M_i$ of non-peripheral simple closed curves, geodesic in the intrinsic metric, such that the lengths $\ell(\gamma_i) \to 0$. The curves $\gamma_i$ must be eventually be nullhomotopic in $M$. Indeed, for all $\gamma_i$, we have $\ell(\gamma_i) \geq \tau([\gamma_i])$. If $\gamma_i$ is non nullhomotopic then because it is non-peripheral in $\partial M_i$ the conjugacy class $[\gamma_i]$ is non-parabolic, but there is a lower bound on the

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3When $M$ is closed all virtual fiber subgroups are closed surface groups. The following argument is only necessary in the torus-boundary case.
translation length of non-parabolic elements of $G$. So the curves $\gamma_i$ are eventually nullhomotopic. By Dehn’s lemma [8 Lemma 4.1] each $\gamma_i$ is therefore a compression curve of $M_i$. Since $M_i$ are convex, the isoperimetric inequality of $H^2$ holds also for $M_i$, hence the minimal area compression disks $D_i \subset M_i$ with $\gamma_i = \partial D_i$, have $\text{Area}(D_i) \to 0$.

To arrive at a contradiction, we will show that $\text{Area}(D_i)$ is bounded from zero. Since $H_1 \leq H_i$ we have $\rho_i : M_1 \cong M_i$. By Lemma 2.2 $H_1$ is not contained in a free factor of $H_i$, so $\rho_i(M_1)$ intersects $D_i$ essentially. Pull back the intersection to get a compression disk $D' \subset M_1$. By considering nested lifts $D' \subset D_i$ in the universal cover we deduce $\text{Area}(D') \leq \text{Area}(D_i)$. The minimum area of a compression disk for $M_i$ is positive. This is a contradiction, so we conclude $\partial M_i$ is contained in a compact subset of $M_{r-k,2k}$.

The inclusion $\partial M_i \to M_i$ induces a surjection $f_i : \pi_1(\partial M_i) \to H_i$. For each $\alpha \in \pi_1(\partial M_i)$, and $x \in \partial C(\Lambda H_i) = \partial M_i$, convexity implies $d_{\partial M_i}(f_i(\alpha).x, x) \leq d_{\partial M_i}(\alpha.x, x)$. Thus $\tau(H_i) \leq \tau(\partial M_i)$. The latter is uniformly bounded above for all $i$ since $\partial M_i$ is contained in a compact subset of $M_{r-k,2k}$.

7. Proof of the main theorem

Proof of Theorem 1.1 Suppose $M$ is a closed 3-manifold. As in the orbifold portion of the surface case we may pass to a finite-sheeted cover, so assume $M$ is orientable. By the prime decomposition [8 Theorem 3.15], JSJ decomposition [12, 13], and geometrization [21, 22] theorems, $M$ decomposes along essential spheres and incompressible tori into finite-volume hyperbolic and Seifert-fibered pieces. This decomposition splits $\pi_1(M)$ as a graph of groups with abelian edge groups. The theorem now follows from Theorem 1.2 Proposition 6.1 and Corollary 5.2. □

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4See also Bessi`eres et al. [9], Kleiner and Lott [15], and Morgan and Tian [18].
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