TWO-WEIGHTED INEQUALITIES FOR HARDY-LITTLEWOOD MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS IN $L^{p(\cdot)}$ SPACES

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Abstract. Two-weight criteria of various type for the Hardy–Littlewood maximal operator and singular integrals in variable exponent Lebesgue spaces defined on the real line are established.

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Introduction

We study the two-weight problem for for Hardy–Littlewood maximal functions and singular integrals in variable exponent Lebesgue spaces $L^{p(\cdot)}$. In particular, we derive various type two-weight criteria for the maximal functions and the Hilbert transforms on the line. For a bounded interval we assume that the exponent $p$ satisfies the local log-Hölder continuity condition and for the real line we require that $p$ is constant outside some interval. In the framework of variable exponent analysis such a condition first appeared in the paper [4], where the author established the boundedness of the Hardy–Littlewood maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$. Unfortunately we do not know whether the established criteria remain valid or not when $p$ satisfies log-Hölder decay condition at infinity (see [3] for this condition). It is known that the local log-Hölder continuity condition for the exponent $p$ together with the log-Hölder decay condition guarantees the boundedness of operators of harmonic analysis in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces (see [3], [26], [1], [2]).

The boundedness of the maximal, potential and singular operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces was derived in the papers [4], [5], [7], [3], [26], [2], [1]. Weighted inequalities for classical operators in $L^{p(\cdot)}_w$ spaces, were $w$ is a power–type weight, were established in the papers [18]–[21], [30], [27], [8] etc, while the same problems with general weights for Hardy, maximal and fractional integral operators were studied in [10]–[12], [16], [20], [22], [21], [9]. Moreover, in [8] a complete solution of the one-weight problem for maximal functions defined on Euclidean spaces are given in terms of Muckenhoupt–type conditions. Finally we notice that in the paper [12] modular–type sufficient conditions governing the two-weight inequality for maximal and singular operators were established.

Throughout the paper $J$ denotes an interval (bounded or unbounded) in $\mathbb{R}$.

Let $p$ be a non–negative function on $\mathbb{R}$. Suppose that $E$ is a measurable subset of $\mathbb{R}$. We use the following notation:

$$p_-(E) := \inf_{E} p; \quad p_+(E) := \sup_{E} p; \quad p_- := p_-(\mathbb{R}); \quad p_+ := p_+(\mathbb{R}).$$
Assume that $1 \leq p_{-}(J) \leq p_{+}(J) < \infty$. The variable exponent Lebesgue space $L^{p_{\cdot}}(J)$ (sometimes it is denoted by $L^{p_{\cdot}(J)}$) is the class of all $\mu$-measurable functions $f$ on $X$ for which $S_{p}(f) := \int_{J} |f(x)|^{p(x)}dx < \infty$. The norm in $L^{p_{\cdot}}(J)$ is defined as follows:

$$\|f\|_{L^{p_{\cdot}}(J)} = \inf\{\lambda > 0 : S_{p}(f/\lambda) \leq 1\}.$$ 

It is known (see e.g. [23], [28], [18]) that $L^{p_{\cdot}}(J)$ is a Banach space. For other properties of $L^{p_{\cdot}}$ spaces we refer, e.g., to [33], [23], [28].

Finally we point out that constants (often different constants in the same series of inequalities) will generally be denoted by $c$ or $C$. The symbol $f(x) \approx g(x)$ means that there are positive constants $c_{1}$ and $c_{2}$ independent of $x$ such that the inequality $f(x) \leq c_{1}g(x) \leq c_{2}f(x)$ holds. Throughout the paper by the symbol $p'(x)$ is denoted the function $p(x)/(p(x) - 1)$.

## 1 Sawyer-type Condition for Maximal Operators in $L^{p_{\cdot}(J)}$ Spaces.

### 1.1 The case of bounded interval

Let $J$ be a bounded interval in $\mathbb{R}$ and let

$$(M^{(J)}_{\alpha}f)(x) = \sup_{I \subset J} \frac{1}{|I|^{1-\alpha}} \int_{I} |f(y)| dy, \quad x \in J,$$

where $x \in J$ and $\alpha$ is a constant satisfying the condition $0 \leq \alpha < 1$.

For a weight function $u$ we denote

$$u(E) := \int_{E} u(x) dx.$$

**Definition 1.1.** Let $J$ be a bounded interval in $\mathbb{R}$. We say that a non–negative function $u$ satisfies the doubling condition on $J$ $(u \in DC(J))$ if there is a positive constant $b$ such that for all $x \in J$ and all $r$, $0 < r < |J|$, the inequality

$$u(I(x - 2r, x + 2r) \cap J) \leq bu(I(x - r, x + r) \cap J)$$

holds.

**Definition 1.2.** We say that $p \in LH_{\cdot}(J)$ ($p$ satisfies the local log-Hölder condition) if there is a positive constant $c$ such that

$$|p(x) - p(y)| \leq \frac{c}{\log |x - y|}$$

for all $x, y \in J$ satisfying the condition $|x - y| \leq 1/2$.

**Theorem 1.1.** Let $1 < p_{-} \leq p(x) \leq p_{+} < \infty$ and let the measure $dv(x) = w(x)^{-p'(x)}dx$ belongs to $DC(J)$. Suppose that $0 \leq \alpha < 1$ and that $p \in LH_{\cdot}(J)$ . Then the inequality

$$\|v(\cdot)M^{(J)}_{\alpha}f\|_{L^{p_{\cdot}}(J)} \leq c\|w(\cdot)f(\cdot)\|_{L^{p_{\cdot}}(J)}$$

holds, if and only if there exist a positive constant $c$ such that for all interval $I$, $I \subset J$,

$$\int_{I} (v(x))^{p(x)}(M^{(J)}_{\alpha}(w(\cdot)^{-p'(x)}X_{I(\cdot)}))^{p(x)}dx \leq c \int_{I} w^{-p'(x)}dx < \infty.$$
To prove Theorem 1.1 we need some auxiliary statements.

**Proposition A.** ([32], Lemma 3.20) Let $s$ be a constant satisfying the condition $1 < s < \infty$ and let $u \geq 0$ on $\mathbb{R}$. Suppose that $\{Q_i\}_{i \in A}$ is a countable collection of dyadic intervals in $\mathbb{R}$ and that $\{a_i\}_{i \in A}, \{b_i\}_{i \in A}$ are sequences of positive numbers satisfying the conditions:

(i) $\int_{Q_i} u \leq a_i$ for all $i \in A$;

(ii) $\sum_{\{j \in A : Q_j \subset Q_i\}} b_j \leq ca_i$ for all $i \in A$.

Then there is a positive constant $c_s$ depended on $s$ such that the inequality

$$\left( \sum_{i \in A} b_i \left( \frac{1}{a_i} \int_{Q_i} gu \right)^s \right)^{1/s} \leq c_s \left( \int_{\mathbb{R}} g^s u \right)^{1/s}$$

holds for all non-negative functions $g$.

**Corollary A.** Let $1 < s < \infty$ and let $u$ be a non-negative measurable function on $\mathbb{R}$. Suppose that $\{Q_i\}_{i \in A}$ is a sequence of dyadic cubes in $\mathbb{R}^n$ and that $\{b_i\}_{i \in A}$ is a sequence of positive numbers satisfying the condition

$$\sum_{\{j \in A : Q_j \subset Q_i\}} b_j \leq cu(Q_i).$$

Then there is a positive constant $c$ such that for all non-negative functions $g$ the inequality

$$\sum_{i \in A} b_i \left( \frac{1}{u(Q_i)} \int_{Q_i} gu \right)^s \leq c \left( \int_{\mathbb{R}} g^s u \right)^{1/s}$$

holds.

**Lemma A.** Let $J$ be a bounded interval and let $1 \leq r_-(J) \leq r_+(J) < \infty$. Suppose that $r \in LH(J)$ and that the measure $\mu$ satisfies the condition $\mu \in DC(J)$. Then there is a positive constant $c$ such that for all $f$, $\|f\|_{L^r(J, \mu)} \leq 1$, intervals $I \subseteq J$ and $x \in I$ the inequality

$$\left( \frac{1}{\mu(I)} \int_I |f(y)| d\mu(y) \right)^{r(x)} \leq c \left[ \left( \frac{1}{\mu(I)} \int_I |f(y)|^{r(y)} d\mu(y) \right) + 1 \right]$$

holds.

**Proof.** We follow the idea of L. Diening [3] (see also [14] for the similar statement in the case of metric measure spaces with doubling measure). We give the proof for completeness.

First recall that (see, e.g., [14]) since $J$ with the Euclidean distance and the measure $\mu$ is a bounded doubling space with the finite measure $\mu$ the condition $r \in LH(J)$ implies the following inequality:

$$(\mu(I))^{r_-(I) - r_+(I)} \leq C$$  \hspace{1cm} (1.1)

for all subintervals $I$ of $J$.

Assume that $\nu B \leq 1/2$. By Hölder’s inequality we have that

$$\left( \frac{1}{\mu(I)} \int_I |f(y)| d\mu(y) \right)^{r(x)} \leq \left( \frac{1}{\mu(I)} \int_I |f(y)|^{r-(I)} d\mu(y) \right)^{r(x)/r_-(I)}$$

for all non-negative functions $g$. We use this inequality to estimate the integrals over intervals $I \subseteq J$.
\[ c\mu(I)^{-r(x)/r_-(I)} \left[ \frac{1}{2} \int_I |f(y)|^{r(y)}d\mu(y) + \frac{1}{2}\mu(I) \right]^{r(x)/r_-(I)}. \]

Observe now that the expression in brackets is less than or equal to 1. Consequently, by (1.1) we find that
\[ \left( \frac{1}{\mu(I)} \int_I |f(y)|d\mu(y) \right)^{r(x)} \leq c\mu(I)^{1-r(x)/r_-(I)} \left( \frac{1}{\mu(I)} \int_I |f(y)|^{r(y)}d\mu(y) + 1 \right) \]
\[ \leq c\mu(I)^{(r_-(I)-r_+(I))/r_-(I)} \left( \frac{1}{\mu(I)} \int_I |f(y)|^{r(y)}d\mu(y) + 1 \right) \]
\[ \leq c \left( \frac{1}{\mu(I)} \int_I |f(y)|^{r(y)}d\mu(y) + 1 \right). \]

The case \( \mu(I) > 1/2 \) is trivial. \( \square \)

Suppose that \( S \) is an interval in \( \mathbb{R} \) and let us introduce the dyadic maximal operator
\[ (M^{d,S}_\alpha f)(x) = \sup_{I \in D(S)} |I|^{-\alpha} \int_I |f(y)|dy, \]
where \( 0 \leq \alpha < 1 \) and \( D(S) \) is a dyadic lattice in \( S \).

To prove Theorem 1.1 we need the following statement:

**Lemma 1.1.** Let \( S \) be a bounded interval on \( \mathbb{R} \) and let \( J \) be a subinterval of \( S \). Suppose that \( \sigma(x) := w^{-p(x)} \) belongs to the class \( DC(J) \) and that \( p \in LH(J) \), where \( 1 < p_-(J) \leq p(x) \leq p_+(J) < \infty \). Let \( 0 \leq \alpha < 1 \). If there is a positive constant \( c \) such that for all interval \( I, I \subset J \),
\[ \int_I (v(x))^{p(x)} \left( M^{(d,S)}_{\alpha}( \chi_{I}(\cdot)\sigma(\cdot) \right)^{p(x)} dx \leq c \int_I \sigma(x)dx < \infty, \]
then the estimate
\[ \|v(\cdot)M^{(d,S)}_{\alpha}(f(\cdot)\chi_{J}(\cdot))\|_{L^{p(\cdot)}(J)} \leq c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)} \]
holds.

**Proof.** Suppose that \( \|f\|_{L^{p(\cdot)}(J)} \leq 1 \). Assume that \( f_1 := \chi J f \). Let us introduce the set
\[ J_k = \{ x \in S : 2^k < (M^{(d,S)}_{\alpha} f_1)(x) \leq 2^{k+1} \}, \quad k \in \mathbb{Z}. \]
Suppose that for \( k, J_k \neq \emptyset \), \( \{I^k_j\} \) is a maximal dyadic interval, \( I^k_j \subset D(S) \), such that
\[ \frac{1}{|I^k_j|^{1-\alpha}} \int_{I^k_j} |f_1(y)|dy > 2^k. \] (1.2)

It is obvious that such a maximal interval always exists. Now observe that
(i) \( \{I^k_j\} \) are disjoint for fixed \( k \);
(ii) \( \mathcal{J}_k := \{ x \in S : (M^{(d,S)}_{\alpha} f_1)(x) > 2^k \} = \cup_j I^k_j. \)

Indeed, (i) holds because if \( I^k_i \cap I^k_j \neq \emptyset \), then \( I^k_i \subset I^k_j \) or \( I^k_j \subset I^k_i \). Consequently, if \( I^k_i \subset I^k_j \), then \( I^k_j \) is maximal interval for which (1.2) holds.
To see that $(ii)$ holds, observe that if $x \in J_k$, then $M^{(d),S} f_1(x) \geq 2^k$. Hence, there is a maximal dyadic interval $I^k_j$ containing $x$ such that (1.2) hold for $I^k_j$. Let now $x \in \bigcup_j I^k_j$. Then $x \in I^k_{j_0}$ for some $j_0$. Hence, $M^{(d),S} f_1(x) > 2^k$ because (1.2) holds for $I^k_{j_0}$.

Denote:

$$E^k_j := I^k_j \setminus \{x \in S : M^{(d),S} f_1(x) > 2^{k+1}\}.$$  

Then $E^k_j = I^k_j \cap J_k$. Indeed, if $x \in E^k_j$, then $x \in I^k_j$ and $M^{(d),S} f_1(x) \leq 2^{k+1}$. Hence, by (1.2) we find that

$$2^k < |I^k_j|^{-1} \int_{I^k_j} |f_1(y)|dy \leq M^{(d),S} f_1(x) \leq 2^{k+1}.$$  

This means that $x \in I^k_j \cap J_k$. Let now $x \in I^k_j \cap J_k$. Then obviously $M^{(d),S} f_1(x) \leq 2^{k+1}$. Consequently, $x \in E^k_j$.

Observe that $\{E^k_j\}$ are disjoint for every $j, k$ because, as we have seen,

$$E^k_j = \{x \in I^k_j : 2^k < M^{(d),S} f_1(x) \leq 2^{k+1}\}.$$  

Also, $E^k_j \subset I^k_j$. Assume that $\|w(\cdot) f_1(\cdot)\|_{L^p(\cdot)(S)} \leq 1$. Denote:

$$v_1 := v\chi_J, \quad \sigma_1 := \sigma\chi_J.$$  

By the arguments observed above and using Lemma A with $r(\cdot) = p(\cdot)/p_-$ and the measure
\[ d\mu(x) = \sigma(x)dx \] we have that

\[
\int_J (v(x))^{p(x)} \left( M^{(d), S}_{\alpha} f_1 \right)^{p(x)} (x)dx = \int_S (v_1(x))^{p(x)} \left( M^{(d), S}_{\alpha} f_1 \right)^{p(x)} (x)dx \\
\leq \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} 2^{(k+1)p(x)} dx \\
\leq c \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \left( \frac{1}{|I_j^k|^{1-\alpha}} \int_{I_j^k} |f_1(y)|dy \right)^{p(x)} dx \\
= c \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \left( \frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \right)^{p(x)} \left( \frac{1}{\sigma(I_j^k \cap J)} \int_{I_j^k} |f_1(y)| \sigma(y) dy \right)^{p(x)} dx \\
= c \sum_{j,k} \left( \int_{E_j^k} (v_1(x))^{p(x)} \left( \frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \right)^{p(x)} dx \right) \left( \frac{1}{\sigma(I_j^k \cap J)} \int_{I_j^k} |f_1(y)|^{p(x)} \sigma(y) dy \right)^{p(x)} \\
+ c \sum_{j,k} \left( \int_{E_j^k} (v_1(x))^{p(x)} \left( \frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \right)^{p(x)} dx \right) \\
\equiv c \left( \sum_{j,k} A_j^k + \sum_{j,k} B_j^k \right).
\]

Notice that the sign of sum is taken over all those \( j \) ad \( k \) for which \( \sigma(I_j^k \cap J) > 0 \).

To use Corollary A observe that

\[
\sum_{I_j^k \subset I_i} \int_{E_j^k} (v_1(x))^{p(x)} \left( \frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \right)^{p(x)} dx \\
\leq \sum_{I_j^k \subset I_i} \int_{E_j^k} (v_1(x))^{p(x)} \left( M^{(d), S}_{\alpha} (\chi_{I_i \cap J}) \right)^{p(x)}(x)dx \\
\leq \int_{I_i} (v_1(x))^{p(x)} \left( M^{(d), S}_{\alpha} (\chi_{I_i \cap J}) \right)^{p(x)}(x)dx \\
\leq c \int_{I_i} \sigma(x)dx = c \int_{I_i} \sigma_1(x)dx.
\]
Now Corollary A implies that
\[
\sum_{j,k} A_k^j = \sum_{j,k} \left( \int (v_1(x))^p \left( \frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \right)^{p(x)} \right) \left( \frac{1}{\sigma_1(I_j^k)} \int |f_1(y)|^{p(x)} \sigma_1(y) dy \right)^{p-1} 
\leq c \int v_1(x)^p \sigma(x)^{-p} \sigma_1(x) dx = c \int |f_1(x)|^p w(x) dx \leq c.
\]

For the second term we have that
\[
\sum_{j,k} B_j^k = \sum_{j,k} \int (v_1(x))^p \left( \frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \right)^{p(x)} dx 
\leq \sum_{j,k} \int (v_1(x))^p \left( M_{\alpha}^{(d),S}(\chi_J \sigma) \right)^{p(x)} dx 
= \int (v(x))^p \left( M_{\alpha}^{(d),S}(\chi_J \sigma) \right)^{p(x)} dx 
\leq c \int \sigma(x) dx < \infty.
\]

Finally we conclude that
\[
\|v(\cdot)(M_{\alpha}^{(d),S}f_1)(\cdot)\|_{L^p(J)} \leq c
\]
for \(\|w(\cdot)f(\cdot)\|_{L^{p}(J)} \leq 1\). □

**Proof of Theorem 1.1. Sufficiency.** Let us take an interval \(S\) containing \(J\). Without loss of generality we can assume that \(S\) is a maximal dyadic interval and that \(|J| \leq \frac{|S|}{8}\). Further, suppose also that \(J\) and \(S\) have one and the same center. Without loss of generality assume that \(|S| = 2^{m_0}\) for some integer \(m_0\). Then every interval \(I \subset J\) has the length \(|I|\) less than or equal to \(2^{m_0-3}\). Assume that \(|I| \in [2^j, 2^{j+1})\) for some \(j, j \leq m_0 - 4\). Let us introduce the set
\[
F = \{ t \in (-2^{m_0-4}, 2^{m_0-4}) : \text{there is } I_1 \in D(S) - t, I \subset I_1 \subset S, |I_1| = 2^{j+1}\}.
\]

The simple geometric observation (see also [13], p. 431) shows that \(|F| \geq 2^{m_0-4}\).

Further, let
\[
(K_t f)(x) := \sup_{S \supset I_1 \ni x, I_1 \subset D(S) - t} \frac{1}{|I_1|^{1-\alpha}} \int_{I_1} |f_1|, \ t \in F,
\]
where \(f_1 = \chi_J f\). Then for \(x \in J\) there exist \(I \ni x, I \subset J\) such that
\[
|I|^{-\alpha} \int_I |f_1| > \frac{1}{2} (M_{\alpha}^{(J)} f_1)(x).
\]

For the interval \(I\), we have that \(|I| \in [2^j, 2^{j+1}), j \leq m_0 - 4\). Therefore for \(t \in F\), there is an interval \(I_1, I_1 \in D(S) - t, I \subset I_1 \subset S, |I_1| = 2^{j+1}\), such that
\[
|I|^{-\alpha} \int_I |f_1| \leq \frac{c}{|I_1|^{1-\alpha}} \int_{I_1} |f_1|.
\]
Hence,
\[(M_{\alpha}^{(J)} f)(x) \leq c(K_{t} f_{1})(x), \text{ for every } t \in F, x \in J,\]
with the positive constant $c$ depending only on $\alpha$. Consequently,
\[(M_{\alpha}^{(J)} f)(x) \leq \frac{1}{|F|} \int_{F} (K_{t} f_{1})(x)dt \leq \frac{c}{|I(0,2^{m_{0}-4})|} \int_{I(0,2^{m_{0}-4})} (K_{t} f_{1})(x)dt.\]

Suppose that $\|w(\cdot)f(\cdot)\|_{L^{p}(\cdot)} \leq 1$. Then by Lemma 1.1 we have that
\[S_{t} := \int_{J} (v(x))^{p(x)}((K_{t} f_{1})(x))^{p(x)}dx \]
\[= \int_{J} (v(x))^{p(x)} \left( \sup_{S \supset I_{1} \ni x} \frac{1}{|I_{1}|} \int_{I_{1}} |f_{1}| dx \right)^{p(x)} dx \]
\[= \int_{J+t} (v_{t}(x))^{p(x-t)} \left( \sup_{I_{1} \ni x} \frac{|I_{1}|}{|I_{1}|} \int_{I_{1}} \chi_{J}(s-t)f_{1}(s-t)ds \right)^{p(x-t)} dx \]
\[= \int_{J+t} (v_{t}(x))^{p(x-t)} \left( \sup_{I_{1} \ni x} \frac{|I_{1}|}{|I_{1}|} \int_{I_{1}} \chi_{J+t}(s)f_{1}(s-t)ds \right)^{p(x-t)} dx \]
\[= \int_{J+t} (v_{t}(x))^{p(x-t)} \left( M_{\alpha}^{(J+t)}(\cdot f_{1}(\cdot-t)) \right)^{p(x-t)} dx \leq c \]
provided that
\[\int_{J+t} (v_{t}(x))^{p(x-t)}(f_{1}(x-t))^{p(x-t)}dx = \int_{J} w(x)|f(x)|^{p(x)}dx \leq 1,\]
where $v_{t}(x) = v(x-t), w_{t}(x) = w(x-t), p_{t}(x) = p(x-t)$. To justify this conclusion we need to check that for every $I, I \subset J + t$,
\[\int_{I} (v_{t}(x))^{p(x)} \left( M_{\alpha}^{(J+t)}(\sigma_{t}\chi_{I})(x) \right)^{p_{t}(x)} dx \leq c \int_{I} \sigma_{t}(x)dx < \infty,\]
where the positive constant $c$ is independent of $I$ and $t$. Indeed, observe that

$$
\int_I (v_t(x))^{p(x)} \left( M^{(d)}_{\alpha} (\sigma I)(x) \right)^{\frac{p(x)}{p_t(x)}} dx
$$

$$
= \int_I (v_t(x))^{p(x)} \left( \sup_{I_1 \supset x, I_1 \in D(S)} |I_1|^{-1} \int_I \chi_{I_1}(s) \sigma(s - t) ds \right)^{\frac{p(x)}{p_t(x)}} dx
$$

$$
= \int_I (v_t(x))^{p(x)} \left( \sup_{I_1 \supset x, I_1 \in D(S)} |I_1|^{-1} \int_{I_1 - t} \chi_{I_1 - t}(s) \sigma(s) ds \right)^{\frac{p(x)}{p_t(x)}} dx
$$

$$
\leq \int_I (v(x))^{p(x)} \left( M^{(J)}_{\alpha} (\chi_{I_t} \sigma) \right)^{p(x)} (x) dx \leq \int_I \sigma(x) dx
$$

$$
= \int_I \sigma_t(x) dx < \infty.
$$

Further, let $g \in L^{p(\cdot)}(J)$ with $\|g\|_{L^{p(\cdot)}(J)} \leq 1$. Then we find that

$$
\int_J (M^{(J)}_{\alpha} f)(x) v(x) g(x) dx
$$

$$
\leq \int_J \left( \frac{1}{|I(0, 2^{m_0})|} \int_{I(0, 2^{m_0})} (K_t f_1)(x) dt \right) v(x) g(x) dx
$$

$$
\leq \frac{1}{|I(0, 2^{m_0})|} \int_{I(0, 2^{m_0})} \left( \int_J (K_t f_1)(x) g(x) v(x) dx \right) dt
$$

$$
\leq \frac{1}{|I(0, 2^{m_0})|} \int_{I(0, 2^{m_0})} \| (K_t f_1) v \|_{L^{p(\cdot)}(J)} \| g \|_{L^{p(\cdot)}(J)} dt
$$

$$
\leq c,
$$

provided that $\|f\|_{L^{p(\cdot)}(J)} \leq 1$.

Finally we conclude that $\| (M^{(J)}_{\alpha} f) v \|_{L^{p(\cdot)}(J)} \leq c$ if $\|f w\|_{L^{p(\cdot)}(J)} \leq 1$.

Sufficiency is proved.

Necessity. Let $f_1(t) = \chi_I(t) w^{-p'(t)}(t)$. Suppose that $\beta = \|w^{-1}(\cdot)\|_{L^{p(\cdot)}(J)} \leq 1$. We have that

$$
\| v(\cdot) (M^{(J)}_{\alpha} f)^{p(\cdot)} (\cdot) \|_{L^{p(\cdot)}(J)} \geq \| \chi_I(\cdot) v(\cdot) (M^{(J)}_{\alpha} (w^{-p'(\cdot)}(\cdot) \chi_I(\cdot))) (\cdot) \|_{L^{p(\cdot)}(J)} =: A.
$$

Hence, by the boundedness of $M^{(J)}_{\alpha}$, Lemma B (recall that the measure $dv(x) = w(x)^{-p'(x)}dx$ satisfies the doubling condition) and the fact that $1/p \in LH(J)$ we find that
Then, on the other hand,

\[ A = \left\| \chi_I(\cdot) v(\cdot) M_\alpha^{(J)} (w^{-p'(\cdot)} \chi_I(\cdot)) (\cdot) \right\|_{L^p(\cdot)(J)} \]

\[ \leq c \left\| w(\cdot) w^{-p'(\cdot)} \chi_I(\cdot) \right\|_{L^p(\cdot)(J)} \]

\[ \leq c \left( \int_I w^{-p'(x)} p(x) (x) u_p(x)(x) dx \right)^{1/p(\cdot)} \]

\[ \leq \bar{c} \left( \int_I w^{-p'(x)} (x) dx \right)^{1/p(\cdot)} \leq \bar{c}. \]

Summarizing these inequalities we conclude that

\[ \int_I (v(x))^{p(x)} \left( M_\alpha^{(J)} (w^{-p'(\cdot)} \chi_I(\cdot)) (x) \right)^{p(x)} dx \leq c \int_I w^{-p'(x)} (x) dx < \infty. \]

Suppose now that \( \beta \geq 1 \). Let us take

\[ f(t) = \frac{w^{-p'(t)}(t) \chi_I(t)}{\beta}. \]

Then

\[ \left\| f(\cdot) w(\cdot) \right\|_{L^p(\cdot)(J)} = \left\| \frac{w^{-p'(\cdot)}(\cdot) \chi_I(\cdot)}{\beta} \right\|_{L^p(\cdot)(J)} \leq 1. \]

Arguing as above we have desire result. It remains to show that

\[ A := \int_J w^{-p'(x)}(x) dx < \infty. \]

Suppose that \( A = \infty \). Then \( \left\| w^{-1}(\cdot) \right\|_{L^p(\cdot)(J)} = \infty \). Hence, there exist a function \( g, \| g \|_{L^p(\cdot)(J)} \geq 0 \) such that

\[ \int_J g(x) w^{-1}(x) dx = \infty. \]

Let \( f(x) = g(x) w^{-1}(x) \). Then

\[ \left\| v(\cdot) \left( M_\alpha^{(J)} f(\cdot) \right) \right\|_{L^p(\cdot)(J)} \geq \left( \int_J w^{-1}(x) g(x) \right) \left\| v(\cdot) |J|^{a-1} \right\|_{L^p(\cdot)(J)} = \infty, \]
while
\[ \|fw\|_{L^p(J)} = \|g\|_{L^p(J)} < \infty. \]
\[ \Box \]

**Corollary 1.1.** Let \( J \) be a bounded interval and let \( 1 < p_-(J) \leq p(x) \leq p_+(J) < \infty \) and let \( 0 \leq \alpha < 1 \). Assume that \( p \in LH(J) \) then the inequity
\[ \|v(\cdot)(M^a_J f)(\cdot)\|_{L^p(J)} \leq c\|f\|_{L^p(J)} \quad (\text{Trace inequality}) \]
holds if and only if
\[ \sup_{I, I \subseteq J} \frac{1}{|I|} \int_I (v(x))^{p(x)} |I|^{\alpha p(x)} dx < \infty. \]

**Proof. Sufficiency.** By Theorem 1.1 it is enough to see that
\[ (M^a_J \chi_I)(x) \leq |I|^\alpha \quad \text{for} \quad x \in I. \]
This is true because of the following estimates:
\[ \sup_{S, S \subseteq J} |S|^{\alpha - 1} \int_S \chi_I \leq \sup_{S \cap I \subseteq x} |S \cap I|^{\alpha - 1} \int_S dx = \sup_{S \cap I \subseteq J} |S \cap I| = |I|^\alpha. \]

**Necessity** follows by choosing the appropriate test functions in the trace inequality. \( \Box \)

### 1.2 The case of unbounded interval

Now we derive criteria for the two–weight inequality for the following maximal operators:
\[ \left( M^{(\mathbb{R}^+)}_{\alpha} f \right)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{(x-h,x+h) \cap \mathbb{R}^+} |f(y)| dy \]
and
\[ \left( M^{(\mathbb{R})}_{\alpha} f \right)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x-h}^{x+h} |f(y)| dy, \]
where \( 0 \leq \alpha < 1 \).

In the sequel we will assume that \( v^{p(\cdot)}(\cdot) \) and \( w^{-p'(\cdot)}(\cdot) \) are a.e. positive locally integrable function.

**Theorem 1.2.** Let \( 0 \leq \alpha < 1, \ 1 < p_-(\mathbb{R}^+) \leq p \leq p_+(\mathbb{R}^+) < \infty \) and let \( p \in LH(\mathbb{R}^+). \) Suppose that there is a bounded interval \([0, a]\) such that \( w^{-p'(\cdot)}(\cdot) \in DC([0, a]) \) and \( p \equiv p_c \equiv \text{const} \) outside \([0, a]\). Then the inequity
\[ \|vM^{(\mathbb{R}^+)}_{\alpha} f\|_{L^p(\mathbb{R}^+)} \leq \|w f\|_{L^p(\mathbb{R}^+)}, \]
holds if and only if there is a positive constant \( b \) such that for all bounded intervals \( I \subseteq \mathbb{R}^+ \),
\[ \|vM^{(\mathbb{R}^+)}_{\alpha}(w^{-p'(\cdot)} \chi_I)\|_{L^p(I)} \leq c\|w^{1-p'(\cdot)}\|_{L^p(I)} < \infty. \] (1.3)
Proof. Sufficiency. Suppose that \(\|wf\|_{L^p(\mathbb{R}_+)} < \infty\). We will show that \(\|vM^{(R_+)}_\alpha\|_{L^p(\mathbb{R}_+)} < \infty\).

Represent \(M^{(R_+)}_\alpha f(x)\) as follows:

\[
M^{(R_+)}_\alpha f(x) = \chi_{[0,a]}(x)M^{(R_+)}_\alpha(f \cdot \chi_{[0,a]})(x) \\
+ \chi_{[0,a]}(x)M^{(R_+)}_\alpha(f \cdot \chi_{(a,\infty)})(x) + \chi_{(a,\infty)}(x)M^{(R_+)}_\alpha(f \cdot \chi_{[0,a]})(x) \\
+ \chi_{(a,\infty)}(x)M^{(R_+)}_\alpha(f \cdot \chi_{(a,\infty)})(x) \\
=:\ M^{(1)}_\alpha f(x) + M^{(2)}_\alpha f(x) + M^{(3)}_\alpha f(x) + M^{(4)}_\alpha f(x).
\]

Since \(\|wf\|_{L^p(\mathbb{R}_+)} < \infty\) we have that \(\|wf\|_{L^p([0,a])} < \infty\). Applying now Theorem 1.1 we find that \(\|vM^{(1)}_\alpha f\|_{L^p(\mathbb{R}_+)} < \infty\). Further, observe that

\[
M^{(2)}_\alpha f(x) \leq \sup_{h > a-x} \frac{1}{h} \int_a^{x+h} |f(y)|dy \leq (M^{(R_+)}_\alpha f)(a) < \infty.
\]

Hence,

\[
\|vM^{(2)}_\alpha f\|_{L^p(\mathbb{R}_+)} \leq (M^{(R_+)}_\alpha f)(a) \cdot \|v\|_{L^p([0,a])} < \infty.
\]

Let us use the following representation for \(M^{(3)}_\alpha f(x)\):

\[
(M^{(3)}_\alpha f)(x) = \chi_{(a,2a]}(x)M^{(R_+)}_\alpha(f \cdot \chi_{[0,a]})(x) + \chi_{(2a,\infty)}(x)M^{(R_+)}_\alpha(f \cdot \chi_{[0,a]})(x) \\
=:\ (\overline{M}^{(3)}_\alpha f)(x) + (\overline{M}^{(3)}_\alpha f)(x).
\]

It is easy to check that for \(x \in (a,2a]\),

\[
(\overline{M}^{(3)}_\alpha f)(x) \leq \sup_{h > a-x} \frac{1}{(a-x+h)^{1-\alpha}} \int_{x-h}^a |f(y)|dy \leq (M^{(R_+)}_\alpha f)(a).
\]

Consequently,

\[
\|v\overline{M}^{(3)}_\alpha f\|_{L^p(\mathbb{R}_+)} \leq \|f\|_{L^p((a,2a])} (M^{(R_+)}_\alpha f)(a) < \infty,
\]

because \(v^{p(c)}(\cdot)\) is locally integrable on \(\mathbb{R}_+\). Further we have that for \(x > 2a\),

\[
(\overline{M}^{(3)}_\alpha f)(x) \leq \frac{1}{(x-a)^{1-\alpha}} \int_0^a |f(y)|dy.
\]

Hence, by using Hölder’s inequality in \(L^p(\cdot)\) spaces, we find that

\[
\left\|v\overline{M}^{(3)}_\alpha f\right\|_{L^p(\mathbb{R}_+)} \leq \left\|v(x)\right\|_{L^p((a,2a])} \left(\int_0^a |f(y)|dy\right) \\
\leq \left\|v(x)\right\|_{L^p((a,2a])} \left\|f\right\|_{L^p((0,a])} \left\|w^{-1}\right\|_{L^p((0,a])} \\
= I_1 \cdot I_2 \cdot I_3.
\]
Since \( I_2 < \infty \) and \( I_3 < \infty \), we need to show that \( I_1 < \infty \). This follows from the fact that condition (1.3) yields
\[
\| vM_\alpha (w^{- (p_c)'}) \chi_I \|_{L^{p_c}((2a, \infty))} \leq \| w^{1 - (p_c)'} (\cdot) \chi_I (\cdot) \|_{L^{p_c}((2a, \infty))}, \quad I \subset (2a, \infty),
\]
where \( M_\alpha \) is the maximal operator defined on \((2a, \infty)\) as follows:
\[
(M_\alpha f)(x) = \sup_{h > 0} \frac{1}{h^{1-\alpha}} \int_{(2a, \infty) \cap (x-h, x+h)} |f(y)| dy.
\]

Using the result by E. Sawyer see [31] (see also [13], Ch. 4) for Lebesgue spaces with constant parameter, we see that (1.4) implies the inequality
\[
\| vM_\alpha f \|_{L^{p_c}((2a, \infty))} \leq c \| w \|_{L^{p_c}((2a, \infty))}.
\]

Since
\[
M_\alpha f(x) \geq \frac{1}{(x-a)^{1-\alpha}} \int_{2a}^{x} |f(y)| dy \quad \text{for} \quad x > 2a,
\]
we have that for the Hardy operator
\[
(H_\alpha f)(x) = \int_{2a}^{x} f(t) dt, \quad x > 2a,
\]
the two-weight inequality
\[
\| v(x)(x-a)^{-1} H_\alpha f \|_{L^{p_c}((2a, \infty))} \leq \| w f \|_{L^{p_c}((2a, \infty))}
\]
holds. Let us recall that (see e.g. [25], Section 1.3) necessary condition for (1.5) is that
\[
\sup_{t > 2a} \left( \int_{t}^{\infty} \left[ \frac{v(x)}{(x-a)^{1-\alpha}} \right]^{p_c} dx \right)^{\frac{1}{p_c}} \left( \int_{2a}^{t} \left( w^{1 - (p_c)'}(x) \right)^{\frac{1}{(p_c)'}} dx \right)^{\frac{1}{(p_c)'}} < \infty.
\]
Hence,
\[
\int_{2a}^{\infty} \left[ \frac{v(x)}{(x-a)^{1-\alpha}} \right]^{p_c} dx = \int_{2a}^{3a} (\cdots) + \int_{3a}^{\infty} (\cdots)
\]
\[
\leq a^{\alpha-1} \int_{2a}^{3a} (v(y))^{p_c} + \int_{3a}^{\infty} \left[ \frac{v(x)}{(x-a)^{1-\alpha}} \right]^{p_c} dx < \infty.
\]

It remains to estimate \( I := \| vM_\alpha^{(4)} f \|_{L^{p_c}(\mathbb{R}^+)} \). But \( I < \infty \) because of the two-weight result by E. Sawyer [31] (see also [13], Ch. 4) for the maximal operator defined on \((a, \infty)\) in Lebesgue spaces with constant exponent. Sufficiency is proved.

Necessity follows easily by taking the test functions \( f(\cdot) = \chi_I (\cdot) w^{-p_c}'(\cdot) \) in the two-weight inequality. \( \Box \)
The next statement follows in the same way as the previous one; therefore we omit the proof.

**Theorem 1.3.** Let $0 \leq \alpha < 1$, $1 < p_- \leq p \leq p_+ < \infty$, and let $p \in LH(\mathbb{R})$. Suppose that there is a positive number $a$ such that $w^{-p'(\cdot)} \in DC([-a,a])$ and $p \equiv p_c \equiv \text{const}$ outside $[-a,a]$. Then the inequity

$$
\|vM_{\alpha}^a f\|_{L^p(\mathbb{R})} \leq \|w f\|_{L^p(\mathbb{R})},
$$

holds if and only if there is a positive constant $b$ such that for all bounded intervals $I \subset \mathbb{R}$,

$$
\|vM_{\alpha}^a (w^{-p'(\cdot)} \chi_I)\|_{L^p(\mathbb{R})} \leq c \|w^{1-p'(\cdot)}\|_{L^p(I)} < \infty.
$$

2 Integral operators on $\mathbb{R}^+$

In this section we derive two–weight criteria of other type for the operators

$$(\mathcal{H}f)(x) = (\text{p.v.}) \int_0^\infty \frac{f(t)}{x-t}dt, \quad x \in \mathbb{R}^+,$$

$$(\mathcal{M}f)(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|dt, \quad x \in \mathbb{R}^+,$$

provided that weights are monotonic, where the supremum is taken over all finite intervals $I \subset \mathbb{R}^+$ containing $x$.

In this section we shall use the notation

$$g_- := g_-(\mathbb{R}_+); \quad g_+ := g_+(\mathbb{R}_+),$$

for a measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

First we present the following statement regarding the weighted Hardy transform

$$(H_{v,w}f)(x) = v(x) \int_0^x f(t)w(t)dt$$

and its dual

$$(H'_{v,w}f)(x) = v(x) \int_x^\infty f(t)w(t)dt$$

defined on $\mathbb{R}_+$.

**Theorem A.** Let $1 < p_- \leq p(x) \leq q(x) \leq q_- < \infty$ and let $p,q \in LH(\mathbb{R}_+)$. Suppose that $p = p_c \equiv \text{const}$, $q = q_c \equiv \text{const}$ outside some interval $(0,a)$. Then

(i) the operator $H_{v,w}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ if and only if

$$D := \sup_{t>0} D(t) := \sup_{t>0} \|v\|_{L^p(\cdot)(t,\infty)} \|w\|_{L^{p'}(\cdot)(0,t)} < \infty;$$

(ii) the operator $H'_{v,w}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+) \to L^{q(\cdot)}(\mathbb{R}_+)$ if and only if

$$D' := \sup_{t>0} D'(t) := \sup_{t>0} \|v\|_{L^p(\cdot)(0,t)} \|w\|_{L^{p'}(\cdot)(t,\infty)} < \infty.$$
Proof. We prove part (i). Part (ii) follows from the duality arguments. Let \( \|f\|_{L^q([a,\infty))} \leq 1 \). We represent \( H_{v,w}f \) as follows:

\[
H_{v,w}f(x) = \chi_{[0,a]}(x) \int_0^x f(t)w(t)dt + \chi_{(a,\infty)}(x) \int_0^x f(t)w(t)dt := H_{v,w}^{(1)}f(x) + H_{v,w}^{(2)}f(x).
\]

Observe that the condition \( D < \infty \) implies that

\[
D^{(a)} := \sup_{0 < t < a} \|v\|_{L^q((t,a))} \|w\|_{L^{p'}((0,t))} < \infty.
\]

Consequently (see \[22\]),

\[
\|H_{v,w}^{(1)}f\|_{L^q([a,\infty))} \leq c\|f\|_{L^p([0,a])} \leq c.
\]

It remains to estimate \( \|H_{v,w}^{(2)}f\|_{L^q([a,\infty))} \). Let \( \|g\|_{L^{p'}([a,\infty])} \leq 1 \). We have that

\[
\int_0^\infty (H_{v,w}^{(2)}f)(x)g(x)dx = \int_0^\infty (H_{v,w}^{(2)}f)(x)g(x)dx
\]

\[
\leq \int_a^\infty v(x) \left( \int_a^x f(t)w(t)dt \right) g(x)dx + \int_a^\infty v(x)g(x)dx \left( \int_a^\infty f(t)w(t)dt \right) := S_1 + S_2.
\]

We can now apply the boundedness of the Hardy transform \( T_{v,w}^{(a)}f(x) = v(x) \int_a^x f(t)w(t)dt \) from \( L^p([a,\infty)) \) to \( L^q([a,\infty]) \) (see e.g. \[25\], Section 1.3) because

\[
\sup_{t > a} \|v\|_{L^q((t,\infty))} \|w\|_{L^{p'}((a,t))} \leq D < \infty.
\]

Consequently, by this fact and Hölder’s inequality we derive that

\[
S_1 \leq \|T_{v,w}^{(a)}f\|_{L^p([a,\infty))} \|g\|_{L^q([a,\infty])} \leq c\|f\|_{L^{p'}([a,\infty])} \leq C.
\]

Applying Hölder’s inequality for \( L^p(\cdot) \) spaces we find that

\[
S_2 \leq \left( \int_a^\infty v(x)g(x)dx \right) \|f\|_{L^{p'}([0,a])} \|w\|_{L^{p'}([0,a])} \leq C.
\]

Necessity follows by the standard way choosing the appropriate test functions. \( \square \)

Theorem B (\[12\]). \( 1 < p_+ \leq p < \infty \). Suppose that \( p \in LH(\mathbb{R}^+) \) and that \( p = p_c = \text{const} \) outside some interval. Then the inequality

\[
\|vTf\|_{L^q([a,\infty))} \leq c\|w\|_{L^{p'}([a,\infty])};
\]

(2.1)

where \( T \) is \( M \) or \( H \), holds if

(i) \( H_{\overline{v},\overline{w}} \) is bounded in \( L^p(\cdot)(\mathbb{R}) \), where \( \overline{v}(x) := \frac{v(x)}{x}, \overline{w}(x) := \frac{1}{w(x)} \);

(ii) \( H_{v_1,\overline{w}} \) is bounded in \( L^p(\cdot)(\mathbb{R}) \), where \( \overline{w}_1(x) := \frac{1}{w(x)x} \);

(iii) \( v_+([x/4, 4x]) \leq cw \) a.e. or \( v(x) \leq cw_+([x/4, 4x]) \) a.e.

(2.2)
Theorems A and B imply the following statement:

**Theorem 2.1.** Let $1 < p_- \leq p_+ < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that $p = p_c \equiv \text{const}$ outside some interval $[0, a]$. Suppose also that $v$ and $w$ are weights on $\mathbb{R}_+$. Then the inequality (2.1), where $T$ is $\mathcal{M}$ or $\mathcal{H}$, holds if

(i) $$E_1 := \sup_{t > 0} E_1(t) := \sup_{t > 0} \| v(x)x^{-1} \|_{L^{p_c}(\{t, \infty\})} \| w^{-1} \|_{L^{p'}(\{0, t\})} < \infty;$$

(ii) $$E_2 := \sup_{t > 0} E_2(t) := \sup_{t > 0} \| v \|_{L^{p_c}(\{0, t\})} \| w^{-1}(x)x^{-1} \|_{L^{p'}(\{0, t\})} < \infty;$$

(iii) condition (2.2) is satisfied.

Now we prove the next statement.

**Theorem 2.2.** Let $1 < p_- \leq p_+ < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that $p = p_c \equiv \text{const}$ outside some interval $[0, a]$. Suppose also that $v$ and $w$ are positive increasing functions on $\mathbb{R}_+$. Then inequality (2.1), where $T$ is $\mathcal{M}$ or $\mathcal{H}$, holds if and only if (2.3) is satisfied.

**Proof. Sufficiency.** Taking Theorem 2.1 into account it is enough to see that condition (2.3) implies conditions (2.4) and (2.2). For (2.2) we will show that there is a positive constant $c$ such that for all $t > 0$ inequality

$$v(4t) \leq cw(t), \quad t > 0.$$  

holds. Indeed, inequality (1.1) with respect to the Lebesgue measure $d\mu(x) = dx$ and the exponent $r = p'$ which belongs to $LH([0, a])$, for small $t$, yields that

$$E_1(t) \geq \| \chi_{[t, 4t]}(\cdot) | \cdot |^{-1} \|_{L^{p_c}(\mathbb{R}_+)} \| \chi_{[0, t/4]}(\cdot) w^{-1}(\cdot) \|_{L^{p'}(\mathbb{R}_+)}$$

$$\geq c \frac{v(t)}{t^{p_c - \frac{1}{\overline{p'} - \frac{1}{4}}} w^{-1}(t/4) t^{\frac{1}{\overline{p'} - \frac{1}{4}}}} \geq c \frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_c - \frac{1}{4}}} t^{\frac{1}{\overline{p'} - \frac{1}{4}}} = c \frac{v(t)}{w(t/4)}.$$  

Further, for large $t$, we have that

$$E_1(t) \geq \| v(x)x^{-1} \chi_{(t, 2t)}(x) \|_{L^{p_c}(\mathbb{R}_+) \| \chi_{[t/8, t/4]}(\cdot) w^{-1}(\cdot) \|_{L^{p'}(\mathbb{R}_+)} \geq c \frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_c} t^{\frac{1}{p_c}}} = c \frac{v(t)}{w(t/4)}$$

Thus, condition (2.2) is satisfied.

Taking into account the fact that $v$ and $w$ are increasing and inequality (2.5) we can easily conclude that condition (2.4) is satisfied.

**Necessity.** First observe that inequality (2.1) implies that $\| w^{-1} \|_{L^{p'}(\{0, t\})} < \infty$ for all $t > 0$.

Let $T = \mathcal{M}$. Then using the obvious inequality

$$\mathcal{M}f(x) \geq \frac{c}{x} \int_0^x f(t)dt, \quad x > 0,$$

and taking into account Theorem A we have necessity for $\mathcal{M}$. Let now $T = \mathcal{H}$. We take $f \geq 0$ so that $\| f \|_{L^{p_c}(\mathbb{R}_+)} \leq 1$. Then we have that

$$\| v \mathcal{H} f \|_{L^{p}(\mathbb{R}_+)} \leq C.$$  

(2.6)
Obviously, (2.6) yields that

\[ C \geq \|v \mathcal{H} f\|_{L^p(\mathbb{R}^n)} \geq \|\chi_{(t,\infty)}(\cdot) v \mathcal{H} f\|_{L^p(\mathbb{R}^n)}. \]

If \( f \) has support on \((0, t), t > 0\), then this inequality implies that

\[ C \geq \left\| \chi_{(t,\infty)}(\cdot) v(\cdot) \left( \int_0^t \frac{f(y)}{-y} dy \right) \right\|_{L^p(\mathbb{R}^n)} \geq c \left\| \chi_{(t,\infty)}(x) v(x) x^{-1} \right\|_{L^p(\mathbb{R}^n)} \left( \int_0^t f(y) dy \right). \]

By taking now supremum with respect to \( f \) and using the inequality

\[ \|g\|_{L^p(\cdot)} \leq \sup_{\|h\|_{L^p(\cdot)} \leq 1} \left| \int gh \right|, \]

(see e.g. [28]) we have necessity. □.

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