Dissipation process of binary gas mixtures in thermally relativistic flow

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Abstract. In this paper, dissipation process of binary gas mixtures in thermally relativistic flows is discussed with focus on characteristics of diffusion flux. As an analytical object, we consider the relativistic rarefied-shock layer around a triangular prism. Numerical results for the diffusion flux are compared with the Navier–Stokes–Fourier (NSF) order approximation of the diffusion flux, which is calculated using the diffusion and thermal-diffusion coefficients by Kox et al (1976 Physica A 84 165–74). In the case of uniform flow with small Lorentz contraction, the diffusion flux, which is obtained by calculating the relativistic Boltzmann equation, is roughly approximated by the NSF order approximation inside the shock wave, whereas the diffusion flux in the vicinity of a wall is markedly different from the NSF order approximation. The magnitude of the diffusion flux, which is obtained by calculating the relativistic Boltzmann equation, is similar to that of the NSF order approximation inside the shock wave, unlike the pressure deviator, dynamic pressure and heat flux, even when the Lorentz contraction in the uniform flow becomes large, because the diffusion flux does not depend on the generic Knudsen number from its definition in Eckart’s frame. Finally, the author concludes that for accuracy diffusion flux must be calculated using the particle four-flow and averaged four velocity, which are formulated using the four velocity defined by each species of hard spherical particles.

Keywords: Boltzmann equation, diffusion, rarefied gases dynamics, shock waves
1. Introduction

Relativistic hydrodynamics has been a significant issue for understanding of the quark gluon plasma (QGP) [1] in the Relativistic Heavy Ion Collider (RHIC) [2] and Large Hadron Collider (LHC) [3] or astrophysical phenomena such as space jets [4]. In particular, the dissipation process of relativistic matter has been discussed in the framework of relativistic kinetic theory, in which various types of relativistic hydrodynamics equation have been discussed from the viewpoint of rational mechanics [5, 6]. Provided that multi-body-interactions beyond three bodies [7] are negligible under asymptotic freedom [8], the scattering of partons is expected to be demonstrated by the relativistic Boltzmann equation (RBE). The Boltzmann approach to multiparton scatterings (BAMPS) [9] has surely been widely prevalent in the calculation of QGP [10, 11], in which the collisional cross section between two colliding partons is calculated using the leading order in perturbative quantum chromodynamics (pQCD) [12]. The dependency of the collisional cross section on momentums of two colliding partons in the basis of...
the pQCD, however, yields an intricate collisional term in the RBE, from which the analytical calculation of the transport coefficients involves mathematical difficulties. Of course, we can calculate the transport coefficients from numerical results of the BAMPS using Green–Kubo formula [13], numerically, whereas such numerical datum of the transport coefficients on the basis of Green–Kubo formula must be distinguished from the analytical calculation of the transport coefficients from the RBE on the basis of the Chapman–Enskog method [14]. Consequently, the simple scattering of two colliding particles such as hard spherical particles or Israel–Stewart (Maxwellian) particles [14, 15] enables us to calculate the transport coefficients from the RBE on the basis of Chapman–Enskog method [14, 15] as a function of the temperature.

In previous studies by the author [16, 17], the dissipation process in relativistic gasses composed of single component hard spherical particles has been investigated. The primary reason why the author has considered hard spherical particles, whose collisional cross section seems to be far from the realistic collisional cross section of the QGP [1, 8, 12, 18], is that the author aimed to clarify relativistic effects on the dissipation process by comparing numerical results for dissipating terms such as the dynamic pressure, pressure deviator and heat flux, which are calculated by solving the RBE, with analytical results for dissipating terms, which are approximated by the Navier–Stokes–Fourier (NSF) law using the transport coefficients of the hard spherical particles [14, 15]. As discussed above, we recall that analytical results for the transport coefficients are obtained for hard spherical particles or Israel–Stewart particles as a function of the temperature [14, 15]. Thus, Yano et al [16] calculated the relativistic rarefied-shock layer by calculating RBE on the basis of the direct simulation Monte Carlo (DSMC) method [19] to investigate two types of relativistic effects, that is thermally relativistic effect and Lorentz contraction effect, on the dissipation process, where thermally relativistic matter is characterized by the thermally relativistic measure \( \chi = mc^2/\theta kT \) (\( m \): mass of a particle, \( c \): speed of light, \( k \): Boltzmann constant, \( \theta \): temperature) such as \( \chi \leq 100 \). The relativistic shock layer problem is suitable to investigate dissipation processes in thermally relativistic matter, because the steady flow-field enables us to calculate dissipating terms with less numerical (thermal) fluctuation as a result of ensemble averages in the DSMC method. Additionally, the dependency of dissipating terms on their time derivatives [15] can be excluded by the steady flow-field, when dissipating terms are approximated using the NSF law.

In recent studies of QGP, the characteristics of thermally relativistic matter mixtures were discussed through the formulation of the viscosity coefficients of binary mixtures of partons by Itakura et al [20] or El et al [13]. Of course, the most classical formulation of transport coefficients of the thermally relativistic gas mixture was obtained by van Leeuwen et al [21]. In the recent study by Wiranata et al [22], the viscosity coefficient of thermally relativistic binary mixtures of hard spherical particles was calculated on the basis of its classical formulation by van Leeuwen et al [21]. Indeed, the diffusion flux is a markedly significant physical quantity for understanding

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1 In [20], Itakura et al calculate viscosity coefficients of pions and nucleons using the form of equation (5). On the other hand, the author considers that they should have used not equation (5) but equation (2) to calculate viscosity coefficients, because the symmetric property in equation (5) is not obtained for each species of pions and nucleons.
of the characteristics of the fluid mixtures, as described in past studies of nonrelativistic fluid mixtures by Onsager [23], Meixner [24], Truesdell [25] and Müller–Ruggeri [5].

We cannot assert, however, that the full characteristics of the dissipation process in thermally relativistic gas mixtures have been understood, because the flow velocity and temperature were not defined for each species of the multi-component gas in all the previous studies [21, 26, 27]. In other words, the flow velocity of the species ‘a’ is defined as the averaged flow velocity over all the species and diffusion flux, which is approximated by gradients of the fraction of the number density and temperature with the diffusion and thermal diffusion coefficients [27]. Consequently, the particle four-flow of the species ‘a’ [21, 26] was defined using the sum of the product of the number density and the averaged four velocity over all the species and diffusion flux of the species ‘a’, instead of the product of the number density and the four velocity, which are defined for the species ‘a’. Hence, the diffusion flux in previous studies [21, 26, 27] was included in the particle four-flow of the species ‘a’ as a dissipating term like the dynamic pressure, pressure deviator and heat flux [26].

Finally, a significant aim in this paper is an investigation of the characteristics of the dissipation process in thermally relativistic binary gas mixtures by solving the RBE, which postulates projected moments (number density, flow velocity, temperature, etc), which are defined by each species of hard spherical particles. In particular, we focus on the characteristics of the diffusion flux in the thermally relativistic binary gas mixture by calculating the diffusion flux of the species ‘a’ from the particle four-flow, which is calculated by the distribution function of the species ‘a’. To attain our aim, the rarefied shock layer of the thermally relativistic binary gas mixture, which is composed of hard spherical particles, is investigated by solving the RBE on the basis of the DSMC method. In outline, this paper addresses the following three items:

- The diffusion flux of a species ‘a’ is defined by the particle four-flow, which is calculated by the distribution function of the species ‘a’, in section 2.1.
- The diffusion flux, which is calculated by the RBE, is compared with the NSF order approximation of the diffusion flux, which was calculated using the diffusion and thermal diffusion coefficients by Kox et al [27], in section 3.1, when two species of hard spherical particles have equal masses and different diameters. Here, readers will recall that the NSF order approximation of the diffusion flux by Kox et al [27] used the common temperature and flow velocity of the species ‘a’, which is expressed as the averaged flow velocity over all the species and diffusion flux of the species ‘a’, which is approximated by gradients of the fraction of number density and temperature with diffusion and thermal diffusion coefficients [27]. Finally, we confirm that diffusion flux depends on differences in flow velocities, which are calculated for each species of hard spherical particle individually.
- The reduced NSF order approximations of the dynamic pressure, pressure deviator and heat flux are calculated by the first Maxwellian iteration of Grad’s 28 (2 (number of species) × 14) moment equations [15] by neglecting diffusive terms in Grad’s 28 moment equations, which are caused by differences in Grad’s 14 moments [28] between two species of hard spherical particles with equal masses and different diameters, in section 2.2. The reduced NSF order approximations
of the dynamic pressure, pressure deviator and heat flux are compared with the dynamic pressure, pressure deviator and heat flux as calculated by solving the RBE, in section 2.1.

In addition to the above three items, the effects of mass ratio on overshoots of the temperature of the heavier hard spherical particles inside the shock wave, are addressed in section 3.2, briefly.

2. Definition of diffusion flux and reduced NSF order approximations

The diffusion flux of a species ‘a’ is defined by its particle four-flow, which is calculated by the distribution function of the species ‘a’, in section 2.1, and the reduced NSF law for the dynamic pressure, pressure deviator and heat flux is formulated in section 2.2.

2.1. RBE for gas mixture and definition of diffusion flux

Firstly, we assume that thermal equilibrium distribution follows the Maxwell–Jüttner function [15], since the analytical calculation of the transport coefficients for hard spherical particles [14, 15] was performed under such an assumption. Consequently, the RBE for the gas mixture is written in the following form, when effects via the quantum spin [29] are negligible, as [14, 15]

\[ p_\alpha^0 \partial_\alpha^0 \mathbf{f}_a(p_\alpha) = \sum_b \int_{\mathbb{R}^3} \int_{\Omega} \left[ \mathbf{f}_a'(p_\alpha') \mathbf{f}_b(p_\beta) - \mathbf{f}_a(p_\alpha) \mathbf{f}_b'(p_\beta') \right] F_{ab} \sigma_{ab} d\Omega \frac{d^3p_\beta}{p_\beta^0}, \tag{1} \]

where subscripts ‘a’ and ‘b’ correspond to the species of particles respectively, \( p_\alpha^0 = (p_\alpha^0, p_\alpha) := m_\alpha \gamma(v_\alpha)(c, v_\alpha^0) (p \in \mathbb{R}^3): \) momentum vector of a particle, \( v \in \mathbb{C}^3 \{ x \in \mathbb{C}^3 | |x| \leq c \cap x \in \mathbb{R}^3 \}: \) velocity of a particle) and \( p_\beta^0 = (p_\beta^0, p_\beta) := m_\beta \gamma(v_\beta)(c, v_\beta^0) \) are four momentums of two colliding species ‘a’ and ‘b’, in which \( \gamma(v_\alpha) := 1/\sqrt{1 - v_\alpha^2/c^2} \) and \( \gamma(v_\beta) := 1/\sqrt{1 - v_\beta^2/c^2} \) are Lorentz factors of two colliding species ‘a’ and ‘b’. In equation (1), \( p_a' \) and \( p_b' \) are momentum vectors after the binary collision. In equation (1), \( \mathbf{f}_a(p_\alpha) := f_a(t, x' , p_\alpha) \) \( (\mathbf{f}_b(p_\beta) := f_b(t, x' , p_\beta)) \) in \( \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \) is the distribution function of the species ‘a’ (‘b’). \( \sigma_{ab} \) is the differential cross section between two colliding species ‘a’ and ‘b’. In equation (1), \( F_{ab} := \sqrt{(p_\alpha^0 p_\beta^0)^2 - m_\alpha^2 m_\beta^2 c^4} \) is the Lorentz invariant flux and \( \Omega \) is the solid angle on the spherical surface with radius 1. Provided that all the particles are hard spherical particles, we obtain \( \sigma_{ab} = d_{ab}^2/4 \), in which \( d_{ab} := (d_a + d_b)/2 \) (d: diameter of a particle), whereas \( \sigma_{ab} \) must be a function of \( p_a^0 \) and \( p_b^0 \) to demonstrate the collisional cross section of the QGP [9, 18].

Grad’s 14 moments [15] of species ‘a’ are calculated using Eckart’s decomposition [30] of \( N_a := \int_{\mathbb{R}^3} p_a^0 d^3p_a/p_a^0 \) (particle four-flow of species ‘a’) and \( T_a^{\alpha\beta} := \int_{\mathbb{R}^3} p_a^0 p_{\alpha\beta} d^3p_a/p_a^0 \) (energy-momentum tensor of the species ‘a’) [30]. Consequently, \( n_a \) (number density), \( \Pi_a^{(\alpha\beta)} \) (pressure deviator), \( p_a + \Pi_a \) (static pressure + dynamic pressure), \( q_a^\alpha \) (heat flux) and \( e_a \) (energy density) of species ‘a’ are calculated as [30]

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\[ n_a = N_a U_{a,\alpha}, \]

\[ \Pi_a^{(\alpha\beta)} = \left( \Delta^\alpha_\beta \Delta^\beta_\delta - \frac{1}{3} \Delta^\alpha_\beta \Delta^\beta_\gamma \right) T^\gamma_\delta, \]

\[ p_a + \Pi_a = - \frac{1}{3} \Delta^\alpha_\beta T^\alpha_\beta, \]

\[ q_a^\alpha = \Delta^\gamma_\beta U_{a,\beta} T^\gamma_\beta, \]

\[ e_a = \frac{1}{n_a} U_{a,\alpha} T^\alpha_\beta U_{a,\beta}, \]

where \( U_a^\alpha := \gamma(u)(c, u_i^a) (U_{0,a} := \gamma(u)(c, -u_i^a)) \) (i = 1, 2, 3) is the four velocity of the species \('a'\), and \( \Delta^\alpha_\beta = \Delta^\beta_\gamma \Delta^\gamma_\alpha \) and \( \Delta^\alpha_\beta \equiv \eta^\alpha_\beta - U_a^\alpha U_a^\beta/c^2 \), in which \( \eta^\alpha_\beta := \text{diag}(+1, -1, -1, -1) \).

The temperature of the species \('a'\), designated \( \theta_a \), is calculated from the relation

\[ e_a := m_a c^2 (G_a - 1/\chi_a). \]

Here, \( G_a \equiv K_0(\chi_a)/K_2(\chi_a) \), in which \( K_n \) is the \( n \)th order modified Bessel function of the second kind, \( \chi_a := m_a c^2/(k\theta_a) \) is the thermally relativistic measure of the species \('a'\) and \( p_a = n_a k\theta_a \). We recall, however, that the relation \( e_a := m_a c^2 (G_a - 1/\chi_a) \), is obtained, successfully, when \( f_a \) follows the thermal equilibrium distribution, that is, the Maxwell–Jüttner function. Therefore, the accuracy of the calculation of \( \theta_a \) from \( e_a := m_a c^2 (G_a - 1/\chi_a) \) is a mathematically open problem under strongly nonequilibrium states.

In Eckart’s frame [30], the particle four-flow of species \('a'\), that is, \( N_a \), is written as

\[ N_a^\alpha = n_a U_a^\alpha, \]

\[ = n_a U_a^\alpha + J_a^\alpha, \]

where \( J_a^\alpha \) is the diffusion flux of species \('a'\), and \( \bar{U}^\alpha \) is the averaged four velocity over all species. In our discussion, we consider projected moments in Eckart’s frames, which are defined for each species respectively. Meanwhile, we remind readers that previous studies [26, 27] used a common Eckart’s frame for all species using not \( U_a^\alpha \) but \( \bar{U}^\alpha \), where \( J_a^\alpha \) was regarded as a dissipating term, formulated using the five-field variables (number density, flow velocity and temperature) as well as the dynamic pressure, pressure deviator and heat flux, which are approximated in the framework of the Chapman–Enskog method. In later numerical analyses of the RBE on the basis of the DSMC method, we calculate \( N_a^\alpha \) for the species \('a'\), from which \( J_a^\alpha \) is calculated using \( \bar{U}^\alpha \). Therefore, \( \bar{U}^\alpha \) must be defined using \( U_a^\alpha \), because the definition of \( \bar{U}^\alpha \) with \( U_a^\alpha \) has not been discussed in the previous studies, as far as the author knows.

Summing over all species in both sides of equation (3), we obtain

\[ \sum_a n_a U_a^\alpha = \sum_a n_a \bar{U}^\alpha = n \bar{U}^\alpha, \]

where \( n = \sum_a n_a \) and \( \sum_a J_a^\alpha = (0, 0) \) are used.

From equation (4), the averaged flow velocity (\( \bar{u}^i \)) and four velocity are obtained using the relation, \( n \bar{U}^0 = \sum_a n_a U_a^0 \), in equation (4), as follows
In later numerical analyses of the RBE on the basis of the DSMC method, we can also use the simple relation $\tilde{U}^\alpha = \sum_a N_a^\alpha n_a$ from equation (4).

Using $\tilde{U}^\alpha$ in equation (5), the diffusion flux ($J_a^\alpha$) is obtained as [26]

$$J_a^\alpha = \bar{\Delta}_\alpha^\beta N_a^\beta,$$

where $\bar{\Delta}_\alpha^\beta = \bar{\Delta}_{\alpha\gamma} \bar{\Delta}^{\alpha\gamma}$ and $\bar{\Delta}^{\alpha\beta} \equiv \bar{U}^{\alpha\beta} U^{\beta\gamma} c^2$. From equation (6), the diffusion flux of the species ‘$a$’ is defined by the particle four-flow, $N_a^\alpha$, which is calculated by the distribution function of the species ‘$a$’ such as $N_a^\alpha := c \int \delta^3 \mathbf{p}_a \mathbf{p}_a^0$.

Of course, $J_a^\alpha = (0, 0)$ in equation (6), when the relation $\Delta^{\alpha\beta} \equiv \eta^{\alpha\beta} - U_a^{\alpha\beta} U^{\beta\gamma} c^2$ instead of $\bar{\Delta}^{\alpha\beta}$ is used. Thus, $J_a^\alpha$ never emerges in balance equations in equations (A.2)–(A.4), when the four velocity $U_a^\alpha$ is used to define $N_a^\alpha$, $\Gamma_{\alpha\beta} := c \int \delta^3 \mathbf{p}_a \mathbf{p}_a \mathbf{p}_a^0$ (energy-momentum tensor) and $T_{\alpha\beta\gamma} := c \int \delta^3 \mathbf{p}_a \mathbf{p}_a \mathbf{p}_a \delta^3 \mathbf{p}_a / p_a^0$ in Eckart’s frame, as shown in equations (A.9)–(A.11). Consequently, $f_a$ can be expanded using Grad’s 14 moment equation as [15]

$$f_a \sim [f_a]_{14} = f_a^{MJ}(n_a, \theta_a, \mathbf{u}_a) + \left(1 + \frac{\Pi_a}{p_a} A_a + \frac{q_{a\alpha} B_a^\alpha}{p_a^0} + \frac{\Pi_a^{(a\beta)} C_a^{a\beta}}{p_a^0}\right),$$

where $f_a^{MJ}(n_a, \theta_a, \mathbf{u}_a) = n_a / (4\pi m_a^2 c k_\theta a k_\theta a) \exp \left[-U_a^\alpha p_a^0 \alpha (k\theta_a)\right]$ is the Maxwell–Jüttner function of the species ‘$a$’ [15]. $A_a$, $B_a^\alpha$ and $C_a^{a\beta}$ in equation (7) are defined as [15]

$$A_a = \frac{1}{20 G_a^2} \left[ \frac{15 G_a^2 + 2 \chi_a - 6 G_a^2 \chi_a + 5 G_a \chi_a^2 + \chi_a^3 - G_a \chi_a^3}{1 - 5 C_a G_a \chi_a - \chi_a^2 + G_a^2 \chi_a^2} \right. \left. + \frac{3 \chi_a}{m_a c^2} \frac{6 G_a + \chi_a - G_a^2 \chi_a}{1 - 5 C_a G_a \chi_a - \chi_a^2 + G_a^2 \chi_a^2} U_{aa} p_a^\alpha + \frac{\chi_a}{m_a c^4} U_{aa} U_{a\beta} p_a^\alpha p_a^\beta \right],$$

$$B_a^\alpha = \frac{\chi_a}{\chi_a + 5 G_a - G_a^2 \chi_a} \left[ \frac{G_a}{m_a c^2} p_a^\alpha - \frac{1}{m_a c^4} U_{a\alpha} p_a^\alpha \right],$$

$$C_a^{a\beta} = \frac{\chi_a}{2 G_a m_a^2 c^2} p_a^\alpha p_a^\beta.$$
function, that is, \( f_a^{(0)} = f_a^{M_1}(n_a, \theta_a, u_a) \). As a result, \( f_a^{(0)} \) is substituted into the left-hand side of equation (1), and the form of the first order approximation of \( f_a \), that is, \( f_a^{(1)} \), is predicted using undetermined parameters. Substituting \( f_a^{(1)} \), which is predicted from the left-hand side of equation (1), into the right-hand side of equation (1), undetermined parameters in \( f_a^{(1)} \) are determined from the equality between both sides of equation (1) in the case of a single component gas. Meanwhile, \( f_a^{(0)} \) is not always equivalent to \( f_a^{M_1}(n_a, \theta_a, u_a) \) in a multi-component gas. Such an inequality, that is, \( f_a^{(0)} \neq f_a^{M_1}(n_a, \theta_a, u_a) \), in a multi-component gas yields mathematical difficulties, when the transport coefficients are calculated from \( f_a^{(1)} \). In Grad’s method, \( f_a \) can be expanded around \( f_a^{M_1}(n_a, \theta_a, u_a) \), because Grad’s method never postulates an expansion of \( f_a \) around \( f_a^{(0)} \) and the left-hand side of equation (1) must be dependent of the state of the species ‘\( a \)’ and independent of states of other species, that is, \( f_b \ (a \neq b) \). Consequently, Grad’s 14 moment equation in equation (7) is considered to introduce the NSF law in section 2.2.

Substituting \( f_a = [f_a]_4 \) into equations (A.2)–(A.4), balance equations of \( N_a^\alpha, T_a^{\alpha\beta} \) and \( T_a^{\alpha\gamma} \) can be evaluated using Grad’s 14 moments, as discussed in appendix B. Here, we find that the left-hand sides of balance equations in equations (A.2)–(A.4) are almost the same as those for the single component gas. Therefore, effects of the gas mixture appear in right-hand sides of equations (A.3) and (A.4), exclusively. The mathematical difficulties exist in the calculation of such right-hand sides of equations (A.3) and (A.4), so that previous studies are limited to hard spherical particles with equal masses and different diameters are calculated in Eckart’s frame [30], as shown in equations (A.13), (A.18) and (A.19), from which the NSF law is introduced by taking the first Maxwellian iteration [26].

The first Maxwellian iteration of equations (A.13), (A.18) and (A.19) yields the NSF law by setting \( \Pi_a = \Pi_a^{(a/b)} = q_a^\alpha = 0 \) in the left-hand sides of equations (A.13), (A.18) and (A.19) and \( \Pi_a = [\Pi_a]_N, \Pi_a^{(a/b)} = [\Pi_a^{(a/b)}]_N \) and \( q_a^\alpha = [q_a^\alpha]_N \) in the right-hand sides of equations (A.13), (A.18) and (A.19), when we assume that \( ||\Pi_a||_{NSF} - ||\Pi_a||_N \ll ||\Pi_a||_{NSF} \) in \( \delta \Pi_a, [\Pi_a^{(a/b)}]_{NSF} - [\Pi_a^{(a/b)}]_N \ll [\Pi_a^{(a/b)}]_{NSF} \) in \( \delta \Pi_a^{(a/b)} \) and \( [q_a^\alpha]_{NSF} - [q_a^\alpha]_N \ll [q_a^\alpha]_{NSF} \) in \( \delta q_a^\alpha \), as discussed in appendix B, as follows

\[
[\Pi_a]_{NSF} = -\eta_a \nabla_a U_a^\alpha + \mathcal{L}_a, \quad (9)
\]

\[
[\Pi_a^{(a/b)}]_{NSF} = 2\mu_a \nabla^\alpha U_a^{(a/b)} + \mathcal{M}_a^{(a/b)}, \quad (10)
\]

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\[ \rho_a \frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a u_a p_a) = \frac{\partial}{\partial t} \left( \rho_a \left( \frac{\Theta_a}{n_a h_a} \nabla \times \mathbf{v}_a \right) \right) + \mathcal{N}_a^\alpha, \]

where \( \nabla^\alpha := \nabla^\alpha \equiv (\eta^\alpha - U_a^\alpha U_b^\beta c^2) \partial_{\gamma}. \)

\( \eta_a \) (bulk viscosity), \( \mu_a \) (viscosity coefficient) and \( \lambda_a \) (thermal conductivity) in equations (9)–(11) are written as

\[
\eta_a = \frac{c^2 n_a m^2}{3} \frac{20 G_a + 2 \chi_a^2 G_a^3 + 3 \chi_a - 13 \chi_a G_a - 2 \chi_a^2 G_a}{\chi_a^3 G_a + \chi_a - \chi_a^3 - 5 \chi_a^2 G_a},
\]

\[
\mu_a = \frac{n_a m^2 c^2 G_a}{\sum_b \mathcal{E}_{ab} \chi_a},
\]

\[
\lambda_a = \frac{n_a m^2 c^4 G_a'}{\sum_b \mathcal{D}_{ab} \theta_a},
\]

where \( G_a' = dG_a/d\chi_a. \)

Additionally, \( \mathcal{L}_a, \mathcal{M}_a^{(\alpha \beta)} \) and \( \mathcal{N}_a^\alpha \) in equations (9)–(11) are written as

\[
\mathcal{L}_a = \frac{1}{\sum_b \mathcal{B}_{ab}} \left( \sum_b \mathcal{B}_{ab} \delta \Pi_{ab} + \frac{1}{6m \chi_a ^2 + 5 \chi_a G_a - \chi_a G_a^2 - 1} U_{a \alpha} [\Psi_a^\alpha]^{(0)} \right),
\]

\[
\mathcal{M}_a^{(\alpha \beta)} = \frac{\sum_b \mathcal{E}_{ab} \delta \Pi_{ab}^{(\alpha \beta)}}{\sum_b \mathcal{E}_{ab}},
\]

\[
\mathcal{N}_a^\alpha = \frac{1}{\sum_b \mathcal{D}_{ab}} \left[ \sum_b \mathcal{D}_{ab} \delta q_{ab}^\alpha + mc^2 \left( 1 + \frac{5G_a}{\chi_a} \right) \Delta_{\alpha}^\beta [\Psi_a^\beta]^{(0)} \right],
\]

where \( [\Psi_a^\alpha]^{(0)} = (0, 0) \) is assumed in the right-hand sides of equations (15) and (17), and \( \mathcal{L}_a, \mathcal{M}_a^{(\alpha \beta)} \) and \( \mathcal{N}_a^\alpha \) are finite terms in the NSF law in equations (9)–(11), which are derived from differences in Grad’s 14 moments between species ‘a’ and ‘b’. In the nonrelativistic gas, concrete forms of \( \mathcal{L}_a, \mathcal{M}_a^{(\alpha \beta)} \) and \( \mathcal{N}_a^\alpha \) were calculated for the inelastic Maxwell model using averaged flow velocity and temperature over all the species and diffusion flux by Garzo and Astillero [31] (i.e. see equation (B22) in [31]).

\(^2\)In the single component relativistic gas, \( \Psi_a^\alpha = 0 \) is obtained. Thus, we can conclude that \( \Psi_a^\alpha \) reflects differences in Grad’s 14 moments between two species of hard spherical particles with equal masses and different diameters, exclusively. In this paper, the contribution of such a diffusive term, \( \Psi_a^\alpha \), which is caused by differences in Grad’s 14 moments between two species of hard spherical particles with equal masses and different diameters, on the NSF law for the dynamic pressure, pressure deviator and heat flux is assumed to be negligible. Consequently, we set \( \Psi_a^\alpha = (0, 0) \) in \( \mathcal{L}_a \) in equation (15) and \( \mathcal{N}_a^\alpha \) in equation (17).

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for $\mathcal{N}_a^\alpha$\(^3\), whereas the NSF law for the relativistic binary gas mixture is obtained using Israel–Stewart particles, when $\theta_a = \theta_b = \tilde{\theta}$, and $m_a \approx m_b$\[20\]. Provided that $\delta_0 = \delta_0$ and $\delta_0 = d_0$, transport coefficients in equations (12)--(14) coincide with those for a single component gas, and the NSF law for a single component gas is reproduced, because $\mathcal{L}_a = \mathcal{M}_a^{(\alpha\beta)} = \mathcal{N}_a^{\alpha} = 0$ is obtained in equations (9)--(11). Concrete forms of $\mathcal{L}_a$, $\mathcal{M}_a^{(\alpha\beta)}$ and $\mathcal{N}_a^{\alpha}$ in equations (15)--(17) are obtained using five-field variables ($n_a$, $\mathbf{u}_a$ and $\theta_a$, $i = a$, $b$), because $\delta\mathbf{Pi}_a$, $\delta\Pi^{(\alpha\beta)}_a$ and $\delta q_a^{\alpha}$ reflect differences in five-field variables between two species owing to $\Psi_a^{\alpha} = (0, 0)$ in equations (15)--(17). Such calculations, however, involve mathematical difficulties. Similarly, calculations of $\mathcal{B}_a$, $\mathcal{C}_a$ and $\mathcal{D}_a$ also involve mathematical difficulties. Thus, two assumptions are made to compare later numerical results of $\Pi_a$, $\Pi^{(\alpha\beta)}_a$ and $q_a^{\alpha}$ with their analytical results. One is $|\mathcal{L}_a| \ll \left| -n_a \nabla_a U_a^{\alpha} \right|$, $|\mathcal{M}_a^{(\alpha\beta)}| \ll \left| 2\mu_a \nabla^{<\alpha} U_a^{\beta>} \right|$ and $|\mathcal{N}_a^{\alpha}| \ll \left| \lambda_a (\nabla^{\alpha} \theta_a - \theta_a / (n_a h_a) \nabla^{\alpha} p_a) \right|$ in equations (9)--(11) because of $|\sum_b \mathcal{B}_{ab}^{\alpha} \Pi_{ab} / \sum_b \mathcal{B}_{ab}^{\alpha}| \ll \left| \nabla_a U_a^{\alpha} \right|$, $|\delta\mathbf{Pi}_a|$, $|\sum_b \mathcal{C}_{ab}^{(\alpha\beta)} / \sum_b \mathcal{C}_{ab}^{\alpha}| \ll \left| 2\mu_a \nabla^{<\alpha} U_a^{\beta>} \right|$ and $|\sum_b \mathcal{D}_{ab} \delta q_{ab}^{\alpha} / \sum_b \mathcal{D}_{ab} | \ll \left| \lambda_a (\nabla^{\alpha} \theta_a - \theta_a / (n_a h_a) \nabla^{\alpha} p_a) \right|$. In other words, equations (9)--(11) are reduced to

$$[\Pi_a]_{\text{NSF}} \simeq \nabla_a U_a^{\alpha},$$

$$[\Pi^{(\alpha\beta)}_a]_{\text{NSF}} \simeq 2\mu_a \nabla^{<\alpha} U_a^{\beta>},$$

$$[q_a^{\alpha}]_{\text{NSF}} \simeq \lambda_a \left( \nabla^{\alpha} \theta_a - \theta_a / (n_a h_a) \nabla^{\alpha} p_a \right).$$

The other is that $\mathcal{B}_{ab}$, $\mathcal{C}_{ab}$ and $\mathcal{D}_{ab}$ in equations (12)--(14) are obtained by replacing $n_a$, $\chi_a$ and $d_a$ in $\mathcal{B}_{aa}$, $\mathcal{C}_{aa}$ and $\mathcal{D}_{aa}$[15] with $n_b$, $\chi_{ab} (\theta_{ab})$ and $d_{ab}$ as follows:

$$\mathcal{B}_{ab} = \frac{64\pi n_b \theta_a \sigma_{ab}}{3cK_2(\chi_a)^2 \chi_b^2} \left( 1 - \chi_a^2 - 5 \chi_a G_{ab} + \chi_a^2 G_{ab}^2 \right) \left[ 2K_2(2\chi_{ab}) + \chi_{ab} K_2(2\chi_{ab}) \right],$$

$$\mathcal{C}_{ab} = \frac{64\pi n_b \theta_a \sigma_{ab}}{15cK_2(\chi_a)^2 \chi_b^2} \left[ \frac{2}{G_{ab}} + \frac{3}{G_{ab}} - \frac{13}{G_{ab}} \chi_{ab} - \frac{2}{G_{ab}} \chi_{ab}^2 + 2 \chi_{ab} \right],$$

$$\mathcal{D}_{ab} = \frac{64\pi n_b \theta_a \sigma_{ab}}{3cK_2(\chi_a)^2 \chi_b^2} \left( \frac{2}{G_{ab}} + \frac{5}{G_{ab}} \chi_{ab} K_2(2\chi_{ab}) \right),$$

where $G_{ab} = K_2(\chi_{ab}) + K_2(\chi_{ab}) \chi_{ab} = k \theta_{ab} / (m \epsilon^2)$ will be mentioned in section 3.

Finally, we recall that equations (18)--(20) are obtained by neglecting diffusive terms, which are caused by differences in Grad’s 14 moments between two species of hard spherical particles with equal masses and different diameters, in equations (A.13), (A.18) and (A.19). Then, equations (18)--(20) are called as the reduced NSF law to discriminate equations (18)--(20) from the full NSF law, which is straightforwardly calculated by the

\[\text{As far as the author knows, the best study of Grad’s moment equations for the multi-component gas, which are obtained using averaged flow velocity and temperature and diffusion flux, was performed by Garzo and Astillero [31]. Of course, Grad’s moment equations for the granular gas mixture can be readily reduced to those for the elastic gas by setting the restitution coefficient as unity in Grad’s moment equations for the granular gas mixture.}\]
first Maxwellian iteration of (A.13), (A.18) and (A.19) without any assumptions. As mentioned in appendix B, the NSF law of the species ‘a’ surely depends on gradients of five-field variables of other species. Meanwhile, effects via diffusive terms in the NSF law can be confirmed, when the reduced NSF law in equations (18)–(20) for the binary gas mixture of hard spherical particles with equal masses and different diameters approximates dissipating terms with worse accuracies than the NSF law for the single component gas does, because so much worse an approximation by the reduced NSF law in equations (18)–(20) is caused by neglect of the diffusive terms in the NSF law in equations (18)–(20). Actually, later numerical results confirm that the reduced Fourier law in equation (20) for the binary gas mixture of hard spherical particles with equal masses and different diameters approximates $\alpha q_a$ ($a = A, B$) with worse accuracies than the NSF law for a single component gas does.

3. Numerical analysis of thermally relativistic rarefied-shock layer of binary gas mixture

In this section, we investigate the characteristics of the dissipation process in the thermally relativistic flow of a binary gas mixture, which is composed of two species of hard spherical particles with equal masses and different diameters, by solving the RBE on the basis of the DSMC method. The NSF order approximation of the diffusion flux, which was calculated by Kox et al [27] for the binary gas mixture of hard spherical particles with equal masses and different diameters, is defined by equation (C.3), whereas definitions of the diffusive coefficients and thermal diffusive coefficient are defined by equations (C.1) and (C.2). The DSMC method for the RBE is described in appendix D, in detail. As an object of the numerical analysis, we investigate the thermally relativistic rarefied-shock layer around a triangular prism, as shown in figure 1, because such a problem has been discussed in our previous studies [16, 17] using a single component gas. The vertical angle of the triangle prism is set as 120 degrees, whereas the upper-half of the triangle prism ($Y = 0$) is analyzed owing to symmetries of numerical results at both sides of $Y = 0$, as shown in figure 1. In our later discussion, the quantity with a bracket [] indicates the approximated value obtained using the analytical result in section 2, whereas the quantity without a bracket indicates the numerical value, which is obtained by solving the RBE on the basis of the DSMC method.

3.1. Dissipation process of binary gas mixture of hard spherical particles with equal masses and different diameters

Firstly, we investigate the dissipation process of the thermally relativistic flow of a binary gas mixture (species $A$ and $B$), when the masses of the two species, $A$ and $B$, are equal, that is, $m_A = m_B$ and diameters of $A$ and $B$ are different, that is, $d_A \neq d_B$. Then, we consider two types of uniform flows in tests (I) and (II). In test (I), the uniform flow corresponds to mildly thermally relativistic flow with small Lorentz contraction, in which the temperature of the uniform flow is set as $\chi_{A\infty} = \chi_{B\infty} = \chi_\infty = 47$, and the flow velocity of the uniform flow is set as $u_{A\infty}^r = u_{B\infty}^r = u_\infty^r = 0.6c$, where symbols with
the subscription ∞ indicate quantities in the uniform flow. In test (II), the uniform flow corresponds to mildly thermally relativistic flow with large Lorentz contraction, in which the temperature of the uniform flow is set as \( \chi_A = \chi_B = \chi_\infty = 30.6 \), and the velocity of the uniform flow is set as \( u_{A,\infty} = u_{B,\infty} = u_\infty = 0.999465c \). The schematic of the flow-field in test (II) is shown in figure 1. In both tests (I) and (II), \( d_A/d_\infty = 0.5 \), \( d_B/d_\infty = 1 \), \( m_A = m_B = m \) and \( n_{A,\infty} = n_{B,\infty} = n_\infty \), whereas the scale parameter, \( n_\infty \pi d_B^2 L_\infty \), is set as 2.5. In later descriptions, symbols with \( \tilde{} \) indicate nondimensionalized quantities, such as \( \tilde{m}_i = m_i/m \), \( \tilde{d}_i = d_i/d \) (\( i = A, B \)), \( \tilde{x}^i = x^i/L \), \( \tilde{v}^i = v^i/c \), \( \tilde{\omega}^i = \omega^i/c \), \( \tilde{\theta} = \theta/\theta_\infty \), \( \tilde{J}^\alpha = J^\alpha/(n_\infty mc) \), \( \tilde{\Pi} = \Pi/(n_\infty mc^2) \), \( \tilde{\Pi}^{(\alpha\beta)} = \Pi^{(\alpha\beta)}/(n_\infty mc^2) \) and \( \tilde{q}^\alpha = q^\alpha/(n_\infty mc^3) \). As shown in figure 1, the Cartesian coordinate \((x^1, x^2) \rightarrow (X, Y)\) in the laboratory frame is considered. In particular, physical quantities along the stagnation stream line (SSL) \((Y = 0 \wedge X \leq -L)\) are discussed. \((X, Y) = (48, 80)\) girds are equally spaced in \(-4L \leq X \leq -L\), and \(0 \leq Y \leq 8L\) and \((X, Y) = (12, 80)\) girds are equally spaced in \(-L \leq X \leq 0\) and \(-\sqrt{3}/2X \leq Y \leq 8L - \sqrt{3}/2X\). In both tests (I) and (II), about 130 sample particles are set per unit cell in the uniform flow. The hard spherical particles, which collide with the wall, are reflected from the wall with the thermally equilibrium state, which is defined by Maxwell–Jüttner function, whose flow velocity and temperature are set as zero and \( \theta_\infty \) (wall temperature), respectively. In test (I), the wall temperature is set as \( \chi_w = mc^2/(k\theta_\infty) = 30 \). In test (II), the wall temperature is set as \( \chi_w = 0.8 \).

Figure 2 shows profiles of the number density, flow velocity and temperature along the SSL in test (I) (upper frame) and test (II) (lower frame). The shock wave separation between two species is confirmed in profiles of the number density, velocity and temperature in both tests (I) and (II), as shown in both frames of figure 2. Additionally, the thickness of the shock wave obtained for species \( A \) is thicker than that for species \( B \) in both tests (I) and (II), because the mean free path for species \( A \) is longer than that for species \( B \). Meanwhile, marked differences between \( \tilde{n}_A \) and \( \tilde{n}_B \), or \( \tilde{\sigma}^A_2 \) and \( \tilde{\sigma}^B_2 \) are confirmed in the thermal boundary layer behind the shock wave, that is, \(-X/L \leq 1.7\) in test (II). As a result, species \( A \) and \( B \) are under the nonequilibrium state in the thermal
boundary layer in test (II), whereas we conjecture that species $A$ and $B$ are similar to the equilibrium state in the thermal boundary layer ($-X/L \leq 2.5$) in test (I). Thus, increase of the Lorentz contraction in the uniform flow yields increase of the nonequilibrium in the thermal boundary layer. Such an increase of the nonequilibrium in the thermal boundary layer in accordance with increase of the Lorentz contraction in the uniform flow was surely confirmed in our previous study of the single component gas [17], whereas the present study indicates that the nonequilibrium between two species also increases via the increase of the Lorentz contraction in the uniform flow\footnote{Readers should note that two types of nonequilibrium states exist in this numerical test. One is the nonequilibrium in each species $A$ and $B$, in which $f_A = f_M^A$ and $f_B = f_M^B$ are allowable, even when $f_A = f_B$. The other is the nonequilibrium between species $A$ and $B$, in which $f_A = f_M^A$, $f_B = f_M^B$ and $f_A \neq f_B$ are allowable. Therefore, the perfect equilibrium state is realized by $f_A = f_B = f_M^A = f_M^B$.}

Figure 2. Profiles of $\tilde{n}_i$, $\tilde{u}_i^x$ and $\tilde{\theta}_i$ ($i = A, B$) along the SSL in tests (I) (upper frame) and (II) (lower frame).
Figure 3 shows profiles of the diffusion flux along the SSL in tests (I) (upper frame) and (II) (lower frame). Here, $\tilde{J}^x_{AB}$, $\tilde{J}^x_{BA}$, $[\tilde{J}^x_{AB}]_1$, $[\tilde{J}^x_{AB}]_2$ and $[\tilde{J}^x_{AB}]_T$ are defined to investigate the accuracy of the approximation with Laguerre polynomials in equation (C.3) and effects via the thermal-diffusion as

$$
\begin{align*}
[\tilde{J}^x_{AB}]_1 &= -\rho c_A c_B [D_T]_1 \tilde{\nabla}^n \tilde{\theta} - \rho [D]_1 \tilde{\nabla}^n c_A, \\
[\tilde{J}^x_{AB}]_2 &= -\rho c_A c_B [D_T]_1 \tilde{\nabla}^n \tilde{\theta} - \rho [D]_2 \tilde{\nabla}^n c_A, \\
[\tilde{J}^x_{AB}]_T &= -\rho c_A c_B [D_T]_1 \tilde{\nabla}^n \tilde{\theta},
\end{align*}
$$

where $\tilde{\theta}$ is calculated using the relation $n_A e_A + n_B e_B = n e$, although the flow velocity of the species $A$ is different from that of the species $B$.

In test (I), $[\tilde{J}^x_{AB}]_1$ is dominant in $[\tilde{J}^x_{AB}]_1$ or $[\tilde{J}^x_{AB}]_2$ inside the shock wave ($2.5 \leq -X/L \leq 3.3$), whereas $[\tilde{J}^x_{AB}]_2$ is obtained in all the domain. Additionally, $[\tilde{J}^x_{AB}]_T$ approximates to zero in the range of $1.4 \leq -X/L \leq 2.5$, whereas $0 \leq [\tilde{J}^x_{AB}]_2$ is obtained behind the shock wave, that is, $2.2 \leq -X/L \leq 2.5$. The global tendency of the profile of $\tilde{J}^x_{AB}$ is similar to $[\tilde{J}^x_{AB}]_2$ in the range of $1.4 \leq -X/L$, whereas the negative peak
value of \( \tilde{J}_{AB} \) is smaller than \( \tilde{J}_{AB}^2 \) inside the shock wave \((-X/L \approx 2.6) \) and the positive peak of \( \tilde{J}_{AB}^2 \) is larger than \( \tilde{J}_{AB}^2 \) behind the shock wave \((-X/L \approx 2.5) \). Numerical results surely confirm the relation \( \tilde{J}_{AB} = -\tilde{J}_{BA}^2 \), as shown in both frames of figure 3. On the other hand, the positive signature of \( \tilde{J}_{AB}^2 \) is opposite to the negative signature of \( \tilde{J}_{AB}^2 \) in the thermal boundary layer \((-X/L \leq 1.38) \).

In test (II), \( \tilde{J}_{AB}^2 \) has a positive peak around \(-X/L = 2.9) \), which corresponds to the forward regime of the shock wave, as shown in the lower frame of figure 2. Meanwhile, \( \tilde{J}_{AB}^2 \) is dominant in \( \tilde{J}_{AB}^2 \) inside the shock wave \( (2.3 \leq -X/L \leq 2.6) \). \( \tilde{J}_{AB}^2 \) approximates to zero behind the shock wave \( (1 \leq -X/L \leq 2.3) \). \( \tilde{J}_{AB}^2 \) is approximately equal to zero in the forward regime of the shock wave \( (3 \leq -X/L \leq 2) \), whereas \( \tilde{J}_{AB}^2 \) has a negative peak around \(-X/L \approx 2.5) \) and a positive peak around \(-X/L \approx 2.1) \). \( \tilde{J}_{AB}^2 \) decreases toward the wall \( (1 \leq -X/L \leq 2.1) \). As a result, \( \tilde{J}_{AB}^2 \) never has a positive peak in the forward regime of the shock wave like \( \tilde{J}_{AB}^2 \), whereas \( \tilde{J}_{AB}^2 \) never decreases toward the wall \( (1 \leq -X/L \leq 2.1) \) unlike \( \tilde{J}_{AB}^2 \).

In summary, the profile of the diffusion flux is roughly approximated by the NSF order approximation inside the shock wave in test (I), as shown in the upper frame of figure 3, when the Lorentz contraction in the uniform flow is small. On the other hand, the magnitude of the diffusion flux obtained using the DSMC method is similar to that obtained using the NSF order approximation inside the shock wave in test (II), when the Lorentz contraction in the uniform flow is large, as shown in the lower frame of figure 3. In other words, the magnitude of the diffusion flux is not sensitive to nonequilibrium terms beyond the NSF order approximation inside the shock wave, when the Lorentz contraction is large. The magnitude of the diffusion flux in the vicinity of the wall is, however, markedly larger than that obtained using the NSF order approximation in test (II), as shown in the lower frame of figure 3. In other words, the magnitude of the diffusion flux is sensitive to nonequilibrium terms beyond the NSF order approximation in the vicinity of the wall, although effects via the Lorentz contraction are markedly small in the vicinity of the wall. Such a strong nonequilibrium in the vicinity of the wall is demonstrated by the fact that the nonequilibrium states between species A and B are not relaxed behind the shock wave owing to the thermally relativistic effects (see the lower frame of figure 6) together with nonequilibrium states owing to discontinuous distribution functions of species A and B on the wall [33].

In the above discussion on the diffusion flux, we postulated that the diffusion flux can be approximated by the Chapman–Enskog method, which has also been applied to nonrelativistic gas mixtures [34, 35]. On the other hand, we know that diffusion flux does not depend on the generic Knudsen number [17] from equation (6). As discussed in appendix C, diffusion flux depends on the difference between \( \tilde{U}_A^x = \gamma(\tilde{u}_A^x)\tilde{r}_A^x \) and \( \tilde{U}_B^x = \gamma(\tilde{u}_B^x)\tilde{r}_B^x \). Then, the approximate diffusive flux, \( \tilde{J}_{AB}^x \), is defined as

\[
\tilde{J}_{AB}^x = \frac{1}{2} \tilde{n}(\tilde{U}_A^x - \tilde{U}_B^x) = -\tilde{J}_{BA}^x, \quad (23)
\]

where \( \tilde{n} = (\tilde{n}_A + \tilde{n}_B)/2 \).

Figure 4 shows profiles of \( \tilde{J}_{AB}^x \) along the SSL together with those of \( \tilde{J}_{AB}^x \) in tests I (upper frame) and II (lower frame). As shown in the upper frame of figure 4, \( \tilde{J}_{AB}^x \)
is markedly similar to $\tilde{J}_{AB}^x$. Such a similarity indicates that the diffusion flux can be found using $\tilde{J}_{AB}^x$ with a good accuracy. The upper frame of figure 4 shows that the difference between $\tilde{u}_A^x$ and $\tilde{u}_B^x$ contributes to $\tilde{J}_{AB}^x$ in the vicinity of the wall. The lower frame of figure 4 indicates that there are some differences between $\tilde{J}_{AB}$ and $\tilde{J}_{AB}^x$ in the forward regime of the shock wave ($2.44 \leq -X/L \leq 3.27$), whereas $\tilde{J}_{AB}^x$ is markedly similar to $\tilde{J}_{AB}^x$ in the range of $1 \leq -X/L \leq 2.44$. Therefore, $\tilde{J}_{AB}^x = \tilde{\alpha}(\tilde{U}_A^x - \tilde{U}_B^x)/2$ is a rough approximation of $\tilde{J}_{AB}$, when the local Lorentz contraction becomes large. As shown in the lower frame of figure 4, the difference between $\tilde{u}_A^x$ and $\tilde{u}_B^x$ contributes to $\tilde{J}_{AB}^x$ in the vicinity of the wall, exclusively. Finally, we have a mathematically open problem of how the higher order approximation of $J_{AB}^a$ beyond the NSF approximation converges to $J_{AB}^a$, which is independent of the generic Knudsen number, when the generic Knudsen number increases in accordance with the increase of the Lorentz contraction in the uniform flow [17].
Figures 5–7 show profiles of $\tilde{\Pi}_i^{(xx)}$ and $[\Pi_i^{(xx)}]_{\text{NSF}}$ ($i = A, B$) along the SSL in tests (I) (upper frame) and (II) (lower frame).

Finally, we refer to comparisons of profiles of heat fluxes in test (I) with that for a single component gas, which was obtained using $m_A = m_B = 1$ and $\chi_\infty = 45$ (see the left frame of figure 2 in [17]). As shown in the left frame of figure 2 in [17], profile of the heat flux for a single component gas is better approximated by the Fourier law in the thermal boundary layer than two heat fluxes in test (I). Consequently, we conclude that neglect...
of all the diffusive terms in the reduced Fourier law in equation (20) degrades accuracies of approximations of two heat fluxes, whereas we must investigate whether the reduced Navier–Stokes (NS) law for the pressure deviator in equation (19) approximates pressure deviators ($\Pi_A^{(xx)}$ and $\Pi_B^{(xx)}$) with worse accuracies than the NS law for the single component gas does, in our future study. As shown in the upper frame of figure 3, $J_{AB}$ has nonzero value in the thermal boundary layer. Therefore, we conjecture that such worse NSF order approximations of two heat fluxes in the thermal boundary layer in test (I) are caused by neglecting diffusive effects in the reduced Fourier law in equation (20) (i.e. $\mathcal{N}_u^\alpha = 0$ in equation (11) yields such worse NSF order approximations of heat fluxes). In test (II), $[\Pi_i^{(xx)}]_{\text{NSF}} \ll [\Pi_i^{(xx)}]_{\text{NSF}} \ll [\Pi_i^{(xx)}]_{\text{NSF}}$ and $[\Pi_i^{(xx)}]_{\text{NSF}} \ll [\Pi_i^{(xx)}]_{\text{NSF}}$ inside the shock wave and thermal boundary layer around the wall, as shown in lower frames of figures 5–7. Our previous studies [17] described that such marked differences are caused by the marked increase of the generic Knudsen number [17] owing to the marked increase of the Lorentz contraction. Consequently, terms beyond Burnett order
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As mentioned above, the nonequilibrium between species $A$ and $B$ appears in the thermal boundary layer in test (II), as shown in the lower frame of figure 2, whereas the nonequilibrium between species $A$ and $B$ seems to disappear behind the shock wave in test (I), as shown in the upper frame of figure 2. Figure 8 shows distribution functions, $f_i(v_i^r) = \int \gamma(v_i) \tilde{f}(v_i) dv_i d\tilde{v}_i n_i$ versus $\tilde{v}_i^r (i = A, B)$ on the SSL in tests (I) (upper frame) and (II) (lower frame). Differences between $f_A(\tilde{v}_A^r)$ and $f_B(\tilde{v}_B^r)$ indicate the nonequilibrium between species $A$ and $B$. The upper frame of figure 8 shows that the nonequilibrium between species $A$ and $B$ exists at point (A) $-X/L = 2.96$, which corresponds to the forward regime of the shock wave, as shown in the upper frame of figure 2, whereas the nonequilibrium between species $A$ and $B$ disappears at point (B) $-X/L = 2.54$, which corresponds to the backward regime of the shock wave, as shown in the upper frame of figure 2. On the other hand, the lower frame of figure 8 shows that the nonequilibrium between species $A$ and $B$ exists at point (A) $-X/L = 1.38$, which corresponds to the thermal boundary layer, as shown in the lower frame of figure 2, whereas the

Figure 7. Profiles of $\tilde{\Pi}_i$ and $[\tilde{\Pi}_i]_{NSF} (i = A, B)$ along the SSL in tests (I) (upper frame) and (II) (lower frame).
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nonequilibrium between species $A$ and $B$ still exists at point (B) $-X/L = 1.04$, which corresponds to the vicinity of the wall, as shown in the lower frame of figure 2. As a result, we can conclude that the nonequilibrium between species $A$ and $B$ is not dismissed behind the shock wave in test (II).

### 3.2. Effect of mass ratio $m_A/m_B$ on dissipation process of binary gas mixture

Finally, we numerically investigate effects of the mass ratio ($m_A/m_B$) on the dissipation process using $m_A/m_B = 0.25$ and $m_A/m_B = 0.5$ together with $m_A/m_B = 1$ in test (II), when $\tilde{m}_B = 1$, $\tilde{d}_A = 0.5$ and $\tilde{d}_B = 1$. Other conditions of the uniform flow are the same as those in test (II).

Figure 9 shows profiles of the flow velocity (upper frame) and temperature (lower frame) along the SSL, when $\tilde{m}_A = 0.25$, 0.5 and 1. Figure 9 shows that the location of the shock wave moves toward the wall as $\tilde{m}_A$ increases. The shock wave separation increases as $\tilde{m}_A$ decreases. The increase of the shock wave separation yields the increase of the difference between $\tilde{U}_A^\epsilon$ and $\tilde{U}_B^\epsilon$ in equation (23). As a result, we can easily predict that the diffusion flux increases as the shock wave separation increases. The overshoot


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of $\theta_A$ becomes larger when $\tilde{m}_A$ decreases from $\tilde{m}_A = 0.5$. $\theta_A$ is similar to $\theta_B$ behind the shock wave when $m_A = 0.25$ and 0.5.

Figure 10 shows profiles of diffusion fluxes, $\tilde{J}_{AB}$, $\tilde{J}_{BA}$ and $\tilde{J}_{AB}$ along the SSL, when $\tilde{m}_A = 0.5$ (upper frame) and 0.25 (lower frame). Firstly, $\tilde{J}_{AB} = -\tilde{J}_{BA}$ is confirmed. The absolute value of the negative peak of $\tilde{J}_{AB}$ inside the shock wave increases as $\tilde{m}_A$ decreases, owing to the increase of the shock separation. $\tilde{J}_{AB}$ is markedly similar to $\tilde{J}_{AB}$ in the range of $1 \leq -X/L \leq 2.5$, when $\tilde{m}_A = 0.5$, whereas $\tilde{J}_{AB}$ is markedly similar to $\tilde{J}_{AB}$ in the range of $1 \leq -X/L \leq 2.4$, when $\tilde{m}_A = 0.25$. Consequently, the diffusion flux does not depend on the mass ratio $m_A/m_B$.

Finally, we put forward further comments on the overshoot of $\theta_B$, when $\tilde{m}_A = 0.25$ and $m_A = 0.5$. The increase of the shock wave separation means the increase of the deceleration of species $B$ in accordance with the deceleration of species $A$ inside the shock wave. The increase of the deceleration of $\tilde{u}_B$ yields an increase of the thermal energy of species $B$, which is converted from its kinetic energy. Such an increase of the thermal energy leads to overshoots of $\tilde{\theta}_B$. Figure 11 shows profiles of $\tilde{\theta}_A$, $\tilde{\theta}_B$, and the averaged temperature $\tilde{\theta}$ along the SSL, when $\tilde{m}_A = 0.5$ (upper frame) and $\tilde{m}_A = 0.25$ (lower frame). $\tilde{\theta}_B$ decreases toward $\tilde{\theta}$ behind the point of its overshoot, when $\tilde{m}_A = 0.25$ and 0.5.
\[ \tilde{\theta}_A \] increases toward \( \bar{\theta} \) in the range of \( 1.77 \leq -X/L \leq 2.36 \), when \( \tilde{m}_A = 0.5 \), whereas \( \tilde{\theta}_A \) increases toward \( \bar{\theta} \) in the range of \( 1.87 \leq -X/L \leq 2.47 \), when \( \tilde{m}_A = 0.25 \). Of course, such relaxations of \( \tilde{\theta}_A \) and \( \tilde{\theta}_B \) to \( \bar{\theta} \) are expressed by the term \( U_{a\beta} \Psi_a^\beta \) in equation (A.16).

4. Concluding remarks

In this paper, the dissipation process in a thermally relativistic binary gas mixture, composed of hard spherical particles, was investigated. In particular, the characteristics of the diffusion flux in the thermally relativistic binary gas mixture were considered by solving the RBE, numerically. The diffusion and thermal-diffusion coefficients calculated by Kox et al were used to calculate the NSF order approximation of the diffusion flux, when two species of hard spherical particles have equal masses and different diameters, whereas the diffusion flux of species ‘a’ was defined by the particle four-flow, which is calculated by the distribution function of species ‘a’. Such a definition of the diffusion
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As an object of the numerical analysis, the thermally relativistic rarefied-shock layer of the binary gas mixture around a triangular prism was investigated by solving the RBE on the basis of the DSMC method. Focusing on the profile of the diffusion flux along the SSL, the diffusion flux via the thermal-diffusion coefficient is dominant over the diffusion flux via the thermal-diffusion coefficient inside the shock wave, whereas the difference between the first order approximation of the diffusion coefficient and the second order approximation of the diffusion coefficient by Kox et al is markedly small. The profile of the diffusion flux inside the shock wave, which is obtained using the DSMC method, is roughly approximated by the NSF order approximation on the basis of transport coefficients by Kox et al, when the Lorentz contraction in the uniform flow is small. Meanwhile, the NSF order approximation of the diffusion flux has a positive peak in the forward regime of the shock wave, where the diffusion flux, which is obtained using the DSMC method, is approximately zero, when the Lorentz contraction in the uniform flow is large. The negative peak of the diffusion flux inside the shock wave, which is obtained using the DSMC method, however, is roughly approximated by the NSF order approximation on the basis of transport coefficients by Kox et al, although magnitudes of peak values of the dynamic pressure, pressure deviator and heat flux inside the shock

Figure 11. Profiles of $\tilde{\theta}_A$, $\tilde{\theta}_B$ and $\tilde{\theta}$ along the SSL, when $\tilde{m}_A = 0.5$ (upper frame) and $\tilde{m}_A = 0.25$ (lower frame).
wave, which are obtained using the DSMC method, are markedly larger than those approximated using the NSF law because the generic Knudsen number becomes large in accordance with the increase of the Lorentz contraction. Thus, the approximate diffusion flux \( \tilde{J}_a \) is proposed using the product of the difference between four velocities of two species of hard spherical particles with a half of averaged number density. \( \tilde{J}_a \) is markedly similar to the diffusion flux obtained using the DSMC method, when the local Lorentz contraction is not so large. Such similarity confirms that the diffusion flux is surely independent of the generic Knudsen number from its definition. The nonequilibrium between two species of hard spherical particles remains behind the shock wave, when the Lorentz contraction in the uniform flow is large, whereas the nonequilibrium between two species of hard spherical particles disappears behind the shock wave, when the Lorentz contraction in the uniform flow is small. Such a nonequilibrium state remaining in the vicinity of the wall yields differences between flow velocities of two species of hard spherical particles, which yields a diffusion flux whose magnitude is markedly larger than the magnitude of the NSF order approximation, when the Lorentz contraction in the uniform flow is large. Thus, we consider that the relaxation process of the nonequilibrium between two species strongly depends on the local Lorentz contraction together with the local temperature. Finally, effects of the mass ratio between two species on the dissipation process were investigated. \( \tilde{J}_a \) is still markedly similar to the diffusion flux obtained using the DSMC method, even when the mass ratio of two species of hard spherical particles is changed. The decrease of the mass ratio emphasizes the overshoot of the temperature of the heavier hard spherical particles, whereas the peak absolute value of the diffusion flux inside the shock wave increases in accordance with the decrease of the mass ratio. Such an overshoot of the temperature of the heavier hard spherical particles is explained by the increase of the shock wave separation in accordance with the decrease of the mass ratio, which leads to increase of the magnitude of the diffusion flux inside the shock wave. In summary, the diffusion flux must be calculated using the particle four flow and averaged four velocity (see equation (6)), which are formulated using the four velocity defined by each species of hard spherical particles, whereas the diffusion flux never appears in Grad’s moment equations, which are formulated using the four velocity defined by each species of hard spherical particles. Finally, numerical results confirm that the reduced Fourier law for the binary gas mixture, which neglects all the diffusive terms, approximates heat fluxes with worse accuracies than the Fourier law for the single component gas does, owing to neglect of all the diffusive terms in the reduced Fourier law.

Appendix A. Grad’s 28 moment equations for binary gas mixture with equal masses

Multiplying \( \psi(p_o) = p_o^b p_o^c p_o^d \ldots \) by both sides of equation (1) and integrating over \( d^3p_o/p_o^0 \), we obtain

\[
\int_{\mathbb{R}^3} \psi(p_o) p_o^a \partial_a f_o \frac{d^3p_o}{p_o^0} = \sum_b \int_{\Omega} \int_{\mathbb{R}^3} \left[ \psi(p_o') - \psi(p_o) \right] f_o(p_o) f_b(p_o') \delta_{ab} \delta_{\alpha\beta\gamma\delta} d\Omega \frac{d^3p_b}{p_b^0} \frac{d^3p_a}{p_a^0} \\
= \sum_b \psi_{ab} \delta_{\alpha\beta\gamma\delta} \ldots \tag{A.1}
\]

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Substituting $\psi(p_a) = c$, $c p_a^\beta$ and $c p_a^\beta p_a^\gamma$ into equation (A.1), we obtain

$$\partial_\alpha N_a^\alpha = 0, \quad \text{(mass conservation in species } 'a'),$$

(A.2)

$$\partial_\alpha T_a^{\alpha \beta} = \sum_b \Psi_{ab}^{\beta} = \Psi_a^{\beta}, \quad \text{(momentum-energy equation in species } 'a'),$$

(A.3)

$$\partial_\alpha T_a^{\alpha \beta \gamma} = \sum_b \Psi_{ab}^{\beta \gamma} = \Psi_a^{\beta \gamma}, \quad \text{(balance equation of the third order moment in species } 'a').$$

(A.4)

$N_a^\alpha$ is the particle four-flow of species ‘$a$’, $T_a^{\alpha \beta}$ is the energy-momentum tensor of species ‘$a$’ and $T_a^{\alpha \beta \gamma}$ is the third order moment. In a single component gas, $\Psi_a^{\beta} = (0, 0)$, whereas $\Psi_a^{\beta}$ in equation (A.3) is finite owing to the momentum and energy transfer between different species ‘$a$’ and ‘$b$’ ($\Psi_{ab}^{\beta} \neq (0, 0)$ for $a \neq b$).

From the symmetry between species ‘$a$’ and ‘$b$’ in equation (A.1), we readily obtain

$$\sum_a \int_{\mathbb{R}^3} \psi(p_a) p_a^\alpha \partial_\beta f_a \frac{d^3 p_a}{p_a^0} = \frac{1}{2} \sum_a \sum_b \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left[ \psi(p'_a) + \psi(p'_b) - \psi(p_a) - \psi(p_b) \right]$$

$$\times f_a(p_a) f_b(p_b) E_{ab} \sigma_{ab} d\Omega \frac{d^3 p_a}{p_a^0} \frac{d^3 p_b}{p_b^0}.$$  

(A.5)

From the mass and momentum-energy conservation, that is, $p_a^{\alpha} + p_b^{\alpha} = p_a^{\alpha} + p_b^{\alpha}$, we obtain the following relation by substituting $\psi = c$, $c p_a^\beta$ and $c p_a^\beta p_a^\gamma$ into equation (A.5)

$$\partial_\alpha \sum_a N_a^\alpha = \partial_\alpha N^\alpha = 0, \quad \text{(mass conservation in total species)},$$

(A.6)

$$\partial_\alpha \sum_a T_a^{\alpha \beta} = \partial_\alpha T^\alpha T^\beta = 0, \quad \text{(momentum-energy conservations in total species)},$$

(A.7)

$$\partial_\alpha \sum_a T_a^{\alpha \beta \gamma} = \partial_\alpha T^\alpha T^\beta T^\gamma = \Psi^\beta \gamma, \quad \text{(balance equation of the third order moment in total species)},$$

(A.8)

where $\sum_a N_a^\alpha = N^\alpha$, $\sum_a T_a^{\alpha \beta} = T^\alpha T^\beta$, $\sum_a \Psi_a^{\beta} = 0$ and $\sum_a \Psi_a^{\beta \gamma} = \Psi^{\beta \gamma}$.

In this appendix, moment equations of $\Pi_a$, $\Pi_a^{\alpha \beta}$ and $q_a^\alpha$ are calculated for two species (‘$a$’ = A and B) of hard spherical particles with $m_A = m_B = m$ and $d_A = d_B$ ($d_a$, diameter of species ‘$a$’). Firstly, Grad’s 28 moment equations for a binary gas mixture are calculated using equation (7) in Eckart’s frame [30], from which the NSF law is calculated by taking the first Maxwellian iteration [26].

In Eckart’s frame, $N_a^\alpha$, $T_a^{\alpha \beta}$ and $T_a^{\alpha \beta \gamma}$ in left-hand sides of equations (A.2)–(A.4) are obtained using Grad’s 14 moment equation in equation (7), that is, substituting $f = [f_0]_{14}$ into their definitions as [15]

$$N_a^\alpha = n_a U_a^\alpha,$$

(A.9)

$$T_a^{\alpha \beta} = \Pi_a^{(\alpha \beta)} - (p_a + \Pi_a) \Delta^{\alpha \beta} + \frac{1}{c^2} (U_a^\alpha q_a^\beta + U_a^\beta q_a^\alpha) + \frac{\alpha a n_a}{c^2} (U_a^\alpha U_a^\beta + U_a^\beta U_a^\alpha),$$

(A.10)
Dissipation process of binary gas mixtures in thermally relativistic flow

\[ T^{(\alpha \beta \gamma)} = (n_a C_1 + C_2 \Pi_a) \frac{c^2}{6} \frac{1}{(1 - n_a C_1 - C_2 \Pi_a)} (\eta^{(\alpha \beta \gamma)} U^{(\gamma \alpha \beta \gamma)} + \eta^{(\alpha \beta \gamma)} U^{(\gamma \alpha \beta \gamma)} + \eta^{(\alpha \beta \gamma)} U^{(\gamma \alpha \beta \gamma)}) \]

\[ + C_3 (\eta^{(\alpha \beta \gamma)} q^{(\alpha \beta \gamma)} + \eta^{(\alpha \beta \gamma)} q^{(\alpha \beta \gamma)} + \eta^{(\alpha \beta \gamma)} q^{(\alpha \beta \gamma)}) - \frac{6}{c^2} C_3 (U_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)} + U_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)} + U_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)}) \]

\[ + C_4 (\Pi_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)} + \Pi_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)} + \Pi_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)}) \]

where \( C_1 = \frac{m_a^2}{\chi_a} (\chi_a + 6 G_a) \),

\[ C_2 = -\frac{6 m_a}{c^2 \chi_a} [2 \chi_a^3 - 5 \chi_a + (19 \chi_a^2 - 30) G_a - (2 \chi_a^3 - 45 \chi_a) G_a^2 - 9 \chi_a G_a^2] \]

\[ \times (20 G_a + 3 \chi_a - 13 G_a^2 \chi_a - 2 \chi_a^2 G_a + 2 \chi_a^2 G_a^3) \]

\[ C_3 = -\frac{m_a}{\chi_a} (\chi_a + 6 G_a - G_a^2 \chi_a) (\chi_a + 5 G_a - G_a^2 \chi_a)^{-1} \]

\[ \text{and } C_4 = m_a (G_a \chi_a)^{-1} (\chi_a + 6 G_a). \]

Provided that we restrict ourselves to \( m_A = m_B = m \) in the binary gas mixture, differences of the right-hand side of equation (A.1) from that in the single component gas are forms of \( \sigma \) and \( f \) because of \( \neq d_{ab} \) and \( \neq f_{ab} \). In other words, the difference between \( f_a \) and \( f_b \) can be demonstrated by differences in Grad’s 14 moments between two species, when \( f_a \) and \( f_b \) are approximated by \( f_a^{(ab)} \) and \( f_b^{(ab)} \), respectively. As a result of equations (A.2)–(A.4) and (A.9)–(A.11), we obtain moment equations of \( \Pi_a^{(\alpha \beta \gamma)} \), \( \Pi_a^{(\alpha \beta \gamma)} \), and \( \Pi_a^{(\alpha \beta \gamma)} \) by multiplying \( U_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)} \), \( U_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)} \), and \( U_a^{(\alpha \beta \gamma)} U_a^{(\gamma \alpha \beta \gamma)} \), respectively. The nonlinear terms, which are neglected in the left-hand side of equations (A.12)–(A.14), never emerge in the NSF law, because such nonlinear terms are expressed with dissipating terms such as \( \Pi_a^{(\alpha \beta \gamma)} \) and \( q_a^{(\alpha \beta \gamma)} \), which are set as zero in the left-hand side of Grad’s 28 moment equations via the first Maxwellian iteration [26].

Of course, terms \( \delta \Pi_{ab}^{(\alpha \beta \gamma)} = \delta \Pi_{ab}^{(\alpha \beta \gamma)} = \delta q_{ab}^{(\alpha \beta \gamma)} = 0 \) in equations (A.12)–(A.14), because diffusive effects never emerge in binary collision between two hard spherical particles belonging to the same species.
Multiplying $\Delta^\alpha_\beta$ or $U_{a\beta}$ by both sides of equation (A.3) with equation (A10), the following relations are obtained by neglecting nonlinear terms

$$\frac{n_a h_a}{c^2} DU_a^\alpha = \nabla^\alpha (p_a + \Pi_a) - \nabla^\beta \Pi_a^{\alpha\beta} - \frac{1}{c^2} Dq_a^\alpha + \Delta^\alpha_\beta \Psi^\beta,$$

(A.15)

$$n_a Du_a = -p_a \nabla^\beta U_a^\beta - \nabla^\beta q_a^\beta + U_{a\beta} \Psi^\beta,$$

(A.16)

where $h_a = mc^2 G_a$ ($G_a = K(\chi_a)/K_2(\chi_a)$) is the enthalpy density and nonlinear terms, which are neglected in equations (A.15) and (A.16), never emerge in the NSF law, because they are expressed with dissipating terms.

From equation (A.2), the following relation is readily obtained using equation (A.9)

$$Dn_a + n_a \nabla^\alpha U_a = 0.$$  

(A.17)

Substituting equations (A.15)–(A.17) into equations (A.12) and (A.14), equations (A.12) and (A.14) are rewritten as

$$\begin{align*}
\frac{C_2}{2} D\Pi_a - \frac{1}{2} n_a (m^2 + C_1) \nabla^\alpha U_a + \frac{\chi_a}{2 \theta_a} \frac{C_1}{\chi_a^2 + 5 \chi_a G_a - \chi_a^2 G_a^2 - 1} \\
\times (\nabla^\alpha q_a + p_a \nabla^\alpha U_a - U_{a\beta} \Psi^\beta)
\end{align*}$$

$$- \frac{5}{c^2} C_3 \nabla^\alpha q_a + \frac{1}{6} (n_a m^2 + 5 n_a C_1) \nabla^\alpha U_a = - \frac{3}{c^2} \sum b \omega_{ab}(\Pi_a + \delta \Pi_a),$$

(A.18)

$$\begin{align*}
5 C_3 Dq_a^\alpha - \frac{c^4}{6} \left( (m^2 - C_1) \nabla^\alpha n_a + \frac{\chi_a}{\theta_a} n_a C_1 \nabla^\alpha \theta_a - C_2 \nabla^\alpha \Pi_a \right) - c^2 C_4 \nabla^\beta \Pi_a^{\alpha\beta} \\
- \frac{c^4}{6 n_a h_a} (n_a m^2 + 5 n_a C_1) \left[ \nabla^\alpha (p_a + \Pi_a) - \nabla^\beta \Pi_a^{\alpha\beta} - \frac{1}{c^2} Dq_a^\alpha + \Delta^\alpha_\beta q_a^\beta \right]
\end{align*}$$

$$= - \sum b \omega_{ab}(q_a^\alpha + \delta q_a^\alpha).$$

(A.19)

### Appendix B. Comments on right-hand sides of equations (A.12)–(A.14)

We mention the right-hand sides of equations (A.12)–(A.14). As discussed in appendix A, $f_a$ and $f_b$, which are approximated using Grad’s 14 moment equations in equation (7), are used to evaluate $\Psi^\beta_a$ and $\Psi^{\beta\gamma}_a$ in equations (A.3) and (A.4). We can easily conjecture that $\Psi^\beta_a$ and $\Psi^{\beta\gamma}_a$ are functions of Grad’s 28 ($=2$(number of species) $\times$ 14) moments, when the binary gas mixture, which is composed of two species (‘a’ and ‘b’) of hard spherical particles with equal masses and different diameters, is considered.

Actually, collisional moments of $\Pi_a$, $\Pi_a^{\alpha\beta}$ and $q_a^\alpha$ were calculated for the binary gas mixture of Israel–Stewart particles with similar masses by Kremer and Marquis Jr such as [26]

$$\frac{C_2}{2} D\Pi_a + ... = - \chi_{1,a} \Pi_a - \chi_{2,a} (\Pi_b - \Pi_a) + \delta \chi_a,$$

(B.1)
\[ C_4 D \Pi^{(\alpha\beta)}_a + \ldots = -X_{3, a} \Pi^{(\alpha\beta)}_a - X_{4, a}(\Pi^{(\alpha\beta)}_b - \Pi^{(\alpha\beta)}_a), \]

\[ 5 C_2 D q^a_a + \ldots = -X_{5, i} q^a_a - X_{6, a}(q^b_a - q^a_a) + \mathcal{G}_a, \]

where \( X_{\ell, a} \) (\( \ell = 1, 2, 3, 4, 5, 6 \)) are dissipation rates, and \( \mathcal{G}_a \) and \( \mathcal{G}_b \) are expressed using five-field variables [26]. From equations (B.1)–(B.3), the NSF law can be obtained by solving three sets of simultaneous equations, that is, simultaneous equations of \( \Pi_a \) and \( \Pi_b \), \( \Pi^{(\alpha\beta)}_a \) and \( \Pi^{(\alpha\beta)}_b \), and \( q^a_a \) and \( q^b_a \) using the first Maxwellian iteration. As a result, we can readily predict that right-hand sides of equations (A.12)–(A.14) can be written in similar forms to those in equations (B.1)–(B.3). Provided that the right-hand side of equation (A.12) can be expressed with the linear combination of \( \Pi_a \) and \( \Pi_b \), the Navier–Stokes law for \( \Pi_a \) must be written with the linear combination of \( \nabla a U^a \) and \( \nabla a U^a \) as a result of the first Maxwellian iteration of moment equations of \( \Pi_a \) and \( \Pi_b \). However, we are unable to calculate \( X_{\ell, a} \) in the case of hard spherical particles with equal masses and different diameters owing to mathematical difficulties. Then, \( X_{\ell, a} = \sum_b \mathcal{B}_{ab} \) and \( X_{5, a} = \sum_b \mathcal{D}_{ab} \) are assumed in the right-hand sides of equations (A.12)–(A.14). In equation (B.1), the term with \( \Pi_b - \Pi_a \) is equal to zero when a single component gas is considered. Therefore, the term with \( \Pi_b - \Pi_a \) is the diffusive term between \( \Pi_a \) and \( \Pi_b \), which is included in the diffusive term \( \delta \Pi_{ab} \) in the right-hand side of equation (A.12). In the first Maxwellian iteration of equations (A.13), (A.18) and (A.19), \( ||\Pi_b||_{\text{NSF}} - ||\Pi_a||_{\text{NSF}} \ll ||\Pi_a||_{\text{NSF}} \) is assumed, because we cannot evaluate \( X_{\ell, a} \) (\( \ell = 1, 2 \)) in the right-hand side of equation (A.18). Similarly, \( ||\Pi^{(\alpha\beta)}_b||_{\text{NSF}} - ||\Pi^{(\alpha\beta)}_a||_{\text{NSF}} \ll ||\Pi^{(\alpha\beta)}_a||_{\text{NSF}} \) and \( ||q^a_a||_{\text{NSF}} - ||q^b_a||_{\text{NSF}} \ll ||q^b_a||_{\text{NSF}} \) are assumed in the right-hand sides of equations (A.13) and (A.19). As a result of such assumptions, equations (9)–(11) are obtained by setting terms with \( ||\Pi^{(\alpha\beta)}||_{\text{NSF}} - ||\Pi^{(\alpha\beta)}||_{\text{NSF}} \), \( ||\Pi^{(\alpha\beta)}||_{\text{NSF}} - ||\Pi^{(\alpha\beta)}||_{\text{NSF}} \) and \( ||q^a_a||_{\text{NSF}} - ||q^b_a||_{\text{NSF}} \) as zero in \( \delta \Pi_{ab}, \delta \Pi^{(\alpha\beta)}_{ab} \) and \( \delta q^{(\alpha\beta)}_{ab} \) in equations (A.13), (A.18) and (A.19), when the first Maxwellian iterations of equations (A.13), (A.18) and (A.19) are performed. Here, we recall that diffusive terms in \( \Pi_a, \Pi^{(\alpha\beta)}_a \) and \( q^a_a \) (\( i = a, b \)) are set as zero in \( \Psi^a_a \) in the left-hand sides of equations (A.18) and (A.19), as discussed in footnote5.

Appendix C. NSF order approximation of diffusion flux for binary gas mixture of hard spherical particles with equal masses and different diameters [27]

The diffusion and thermal diffusion coefficients for the binary gas mixture with equal masses, which were calculated by Kox et al [27], are addressed. Therefore, the author recommends readers to follow the mathematical procedures to calculate the diffusion and thermal diffusion coefficients in their original paper [27].

Setting \( m_a = m_b = m \) in the binary gas mixture (species ‘\( a \)’ and ‘\( b \)’), the diffusion and thermal-diffusion coefficients were calculated by Kox et al [27] using the approximation on the basis of Laguerre polynomials as

\[ \text{Appendix C. NSF order approximation of diffusion flux for binary gas mixture of hard spherical particles with equal masses and different diameters [27]} \]

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\[ \text{Appendix C. NSF order approximation of diffusion flux for binary gas mixture of hard spherical particles with equal masses and different diameters [27]} \]
and $K$, even when in equation (C.1), in which $\alpha$ and $\chi$ are approximated, $d_T\alpha$ is assumed to be calculated using the energy conservation, that is, in equation (C.1), $T_1(\chi) = (10\chi^{-1} + 4\chi^{-3})K_2(2\chi) + (2 + 34\chi^{-2})K_3(2\chi)$, $T_2(\chi) = (2 + \chi^{-1})K_2(2\chi) + 7\chi^{-1}K_3(2\chi)$, $N_1(\chi) = (4\chi^{-2} + 10\chi^{-4} + 4\chi^{-6})K_2^2(2\chi) + (34\chi^{-3} + 38\chi^{-5})K_2(2\chi)K_3(2\chi) + 70\chi^{-4}K_3^2(2\chi)$, $N_2(\chi) = (2 + 11\chi^{-1} + 14\chi^{-4} + 4\chi^{-6})K_2^2(2\chi) + (5\chi^{-1} + 75\chi^{-3} + 50\chi^{-5})K_2(2\chi)K_3(2\chi) + (-2 - 5\chi^{-2} + 136\chi^{-4})K_3^2(2\chi)$, $N_3(\chi) = (4 + 18\chi^{-2} + 18\chi^{-4} + 4\chi^{-6})K_2^2(2\chi) + (10\chi^{-1} + 116\chi^{-3} + 62\chi^{-5})K_2(2\chi)K_3(2\chi) + (-4 - 10\chi^{-2} + 202\chi^{-4})K_3^2(2\chi)$, where $\chi$ is assumed to be calculated using the energy conservation, that is, $n_a e_a + n_b e_b = (n_a + n_b)\epsilon$, even when $U_a^a = U_a^b$, where $\epsilon = mc^2(G - 1/\chi)$ ($G \equiv K_0(\chi)/K_2(\chi)$) is the averaged energy density.

Finally, the diffusion flux $(J_{ab}^a)$ between species ‘$a$’ and ‘$b$’ is obtained as

$$J_{ab}^a = -J_{ba}^b = -\rho c_a c_b D_T \hat{\nabla}^a \bar{\theta} - \rho D \hat{\nabla}^a c_a,$$

where $\hat{\nabla}^a = \tilde{\Delta}^{a\beta} \partial_\beta$, $\rho = \rho_a + \rho_b$ and $c_a = n_a/(n_a + n_b)$, and $\bar{\theta}$ is the averaged temperature over species ‘$a$’ and ‘$b$’.

From the definition of the diffusion flux $J_a^a$ in equation (6), we can readily deduce that $J_a^a$ is not a dissipating term, which depends on the generic Knudsen number, because $N_a^a$ in equation (A.9) does not include dissipating terms, which depend on the generic Knudsen number. As a result, $J_a^a$ depends on the difference between $\tilde{u}^a$ ($U_a^a$) and $u_a^a$ ($U_a^a$), exclusively. In previous studies on the diffusion flux [26, 27], $J_a^a$ is approximated with gradients of the fraction of the number density ($c_a$) and temperature, as shown in equation (C.3), because the diffusion flux was regarded as a dissipating term, which depends on the generic Knudsen number [17]. In later discussions on numerical results, we certainly confirm that $J_a^a$ is not sensitive to the increase of the generic Knudsen number unlike other dissipating terms such as $\Pi_a$, $\Pi_a^{(\alpha\beta)}$ and $q_a^a$. However, we recall that
the diffusion flux is surely dissipated by binary collisions between species \textquoteleft a\textquoteleft \ and \textquoteleft b\textquoteleft , because the momentum transfer between two different species of particles via binary collisions decreases the difference between \( U_a^\alpha \) and \( U_b^\alpha \), (namely, \( U_a^\alpha, U_b^\alpha \rightarrow U^\alpha \) via binary collisions between species \textquoteleft a\textquoteleft \ and \textquoteleft b\textquoteleft ).

Appendix D. DSMC method of solving relativistic Boltzmann equation

Equation (1) can be rewritten as [15]

\[
\partial_t f_a(p_a) + v_a^i \partial_i f_a(p_a) = \sum_b \int_{p_a \in \mathbb{R}^3} \int_{\Omega} \{ f_a(p_a')f_b(p_b') - f_a(p_a)f_b(p_b) \} (g_{ab})_{\Omega} \sigma_{ab} \Omega \, d^3p_b,
\]

where \((g_{ab})_{ab} := \sqrt{(v_a - v_b)^2 - (v_a \times v_b)^2/c^2}\) is Möller’s relative velocity.

The direct simulation Monte Carlo (DSMC) method [19], which was developed for nonrelativistic gasses, can be extended to relativistic gasses. The Courant–Friedlich–Lewy (CFL) condition [36], which is required on the left-hand side of equation (1), requires that the time step \( \Delta t \) approximates to zero when \( p^\alpha \rightarrow \infty \), that is \( \nu \rightarrow c \). Consequently, the left-hand side of equation (1) leads to the numerical stiffness via \( \Delta t \rightarrow 0 \), when the thermally relativistic flow is considered. Thus, equation (D.1) is solved instead of equation (1). As a numerical scheme to solve the collisional term in equation (D.1), the majorant frequency scheme by Ivanov [37] is used, whereas the BAMPS [9] uses Bird’s scheme [19], which calculates the collision-probability for all binary collisional pairs in the numerical cell. Consequently, the computational time required by Bird’s scheme is markedly longer than that required by the majorant frequency scheme, because the majorant frequency scheme calculates the collision pair \( ab \) the maximum collision number \((\nu_{ab})_{\text{max}}\) times. In the majorant frequency scheme, the maximum collision number during \( \Delta t \) is obtained for a hard-sphere particle from equation (D.1) as

\[
(\nu_{ab})_{\text{max}} = \frac{1}{2}(\mathcal{N}_a - \delta_{ab})n_a \gamma(u_b)(\sigma_{ab})_T [(g_{ab})_{\text{max}}] \Delta t = (\mathcal{N}_a - \delta_{ab})n_a \gamma(u_b)(\sigma_{ab})_T \Delta t \cdot [(g_{ab})_{\text{max}} = 2c),
\]

where \( \mathcal{N}_a \) is the number of sample particles of the species \textquoteleft a\textquoteleft \ in the cell, and \((\sigma_{ab})_T := \pi(d_a + d_b)^2/4\) is the total collisional cross section. A collision pair is selected \((\nu_{ab})_{\text{max}}\) times. The two particles selected induce a binary collision when the random number \( \mathcal{W} \) \( (0 < \mathcal{W} \leq 1) \) satisfies

\[
\frac{(g_{ab})_{ab}}{[(g_{ab})_{ab}]_{\text{max}}} = \frac{(g_{ab})_{ab}}{2c} < \mathcal{W}.
\]

Before and after the binary collision between the species \textquoteleft a\textquoteleft \ and \textquoteleft b\textquoteleft , the total energy and total momentum of the binary collisional particles must be conserved. The conservations of energy \( (E_a + E_b) \) and momentum \( (p_a = (p_a^1, p_a^2, p_a^3) + p_b = (p_b^1, p_b^2, p_b^3)) \) before and after a binary collision are written as

\[
E_a + E_b = E_a' + E_b' = (E_{ab})_{\text{tot}},
\]
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\[ \mathbf{p}_a + \mathbf{p}_b = \mathbf{p}'_a + \mathbf{p}'_b = (\mathbf{p}_{ab})_{\text{tot}}, \]  

(D.5)

where \( E_a = m_a \gamma(v_a)c^2 \), \( E_b = m_b \gamma(v_b)c^2 \), \( \mathbf{p}_a = m_a \gamma(v_a)\mathbf{v}_a \) and \( \mathbf{p}_b = m_b \gamma(v_b)\mathbf{v}_b \). \( E_a \) and \( \mathbf{p}_a \) or \( E_b \) and \( \mathbf{p}_b \) are related as follows

\[ E_a = \sqrt{c^2|\mathbf{p}_a|^2 + m_a^2c^4}, \quad E_b = \sqrt{c^2|\mathbf{p}_b|^2 + m_b^2c^4}. \]  

(D.6)

In this paper, a binary collision is calculated using the following algorithm:

(a) Calculate the total energy \((E_{ab})_{\text{tot}}\) and total momentum \(((\mathbf{p}_{ab})_{\text{tot}} = ((\mathbf{p}_{ab})_{\text{tot}}^1, (\mathbf{p}_{ab})_{\text{tot}}^2, (\mathbf{p}_{ab})_{\text{tot}}^3)) \) of binary particles from the left-hand sides of equations (D.4) and (D.5).

(b) Redistribute the energy to binary particles, that is, \( E'_a \) and \( E'_b \), using a random number on the right-hand side of equation (D.4). From equation (D.6), the norms of momenta, that is, \(|\mathbf{p}'_a|\) and \(|\mathbf{p}'_b|\) are fixed.

(c) Decide the direction of the momentum by the law of cosines so that the total momentum is conserved in equation (D.5) as

\[
\begin{pmatrix}
\mathbf{p}'_a \\
\mathbf{p}'_b \\
\mathbf{p}'_a
\end{pmatrix} = M(\phi, \varphi)N(r) \begin{pmatrix} 0 \\ -|\mathbf{p}'_a|\sin\phi_1 \\ |\mathbf{p}'_a|\cos\phi_1 \end{pmatrix}, \quad \begin{pmatrix}
\mathbf{p}'_a \\
\mathbf{p}'_b \\
\mathbf{p}'_b
\end{pmatrix} = M(\phi, \varphi)N(r) \begin{pmatrix} 0 \\ -|\mathbf{p}'_b|\sin\phi_2 \\ |\mathbf{p}'_b|\cos\phi_2 \end{pmatrix},
\]

(D.7)

where \( \phi_1, \phi_2, M(\phi, \varphi) \) and \( N(r) \) are given by

\[
\phi_1 = \arccos \left( \frac{|\mathbf{p}'_a|^2 + (\mathbf{p}_{ab})_{\text{tot}}^2 - |\mathbf{p}'_a|^2}{2|\mathbf{p}'_a|(|\mathbf{p}_{ab})_{\text{tot}}|} \right), \quad \phi_2 = \arccos \left( \frac{|\mathbf{p}'_b|^2 + (\mathbf{p}_{ab})_{\text{tot}}^2 - |\mathbf{p}'_b|^2}{2|\mathbf{p}'_b|(|\mathbf{p}_{ab})_{\text{tot}}|} \right)
\]

(D.8)

\[
M(\phi, \varphi) = \begin{pmatrix}
\cos\phi \cos\varphi & -\sin\varphi & \sin\phi \cos\varphi \\
\cos\phi \sin\varphi & \cos\varphi & \sin\phi \sin\varphi \\
-\sin\phi & 0 & \cos\phi
\end{pmatrix}, \quad N(r) = \begin{pmatrix}
\cos r & -\sin r & 0 \\
\sin r & \cos r & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(D.9)

where \( r \) is a random number in the range of \( 0 \leq r \leq 2\pi \), and \( \phi \) and \( \varphi \) are given by

\[
\phi = \arccos \left( \frac{(\mathbf{p}_{ab})_{\text{tot}}^3}{(|\mathbf{p}_{ab})_{\text{tot}}|} \right).
\]

(D.10)
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\[
\begin{align*}
\text{if } 0 \leq (p_{ab})_{\text{tot}} &\quad \varphi = \arccos\left( \frac{(p_{ab})_{\text{tot}}^1}{|(p_{ab})_{\text{tot}}| \sin \phi} \right) \\
\text{else } &\quad 2\pi - \arccos\left( \frac{(p_{ab})_{\text{tot}}^1}{|(p_{ab})_{\text{tot}}| \sin \phi} \right).
\end{align*}
\] (D.11)

The validity of above DSMC algorithm has been confirmed in the author’s previous studies [16, 17, 33, 38].

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