The spectrum minimum for random Schrödinger operators with indefinite sign potentials

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Abstract

This paper sets out to study the spectral minimum for operator belonging to the family of random Schrödinger operators of the form $H_{\lambda,\omega} = -\Delta + W_{\text{per}} + \lambda V_\omega$, where we suppose that $V_\omega$ is of Anderson type and the single site is assumed to be with an indefinite sign. Under some assumptions we prove that there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$, the minimum of the spectrum of $H_{\lambda,\omega}$ is obtained by a given realization of the random variables.

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1 Introduction

Among the most investigated and dealt with operators in the field of mathematical physics problems are random Schrödinger operators of the form

$$H_\omega = -\Delta + W_{\text{per}} + V_\omega = H_0 + V_\omega,$$  \hspace{1cm} (1.1)

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where $W_{\text{per}}$ is a $\mathbb{Z}^d$-periodic function and $V_\omega$ is a random potential having the Anderson form, i.e $V_\omega(\cdot) = \sum_{n \in \mathbb{Z}^d} \omega_n f(\cdot - \gamma)$. See [4, 12], for the physical motivations.

The study of the spectral theory of operators of the form (1.3) have drawn the attention of many researchers for the importance of the related results. In fact, it is linked to the systems evolutions for which the Hamiltonian is described by (1.3). The goal of this paper is to discuss one of the problems that remain unsolved: the spectrum location of $H_\omega$, precisely the spectrum infimum. This will be carried out in the case when the single site $f$ does not have a definite sign.

As the main object is to study the location of the spectrum, let us recall the following basic results already known on this subject and stated by Kirsch and Martinelli [2, 3]:

**Theorem 1.1**

$$
\Sigma(H,\omega) = \bigcup_{\omega_\gamma \in \mathcal{P}} \Sigma(-\Delta + W_{\text{per}} + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma f(x - \gamma)).
$$

(1.2)

Here $\mathcal{P}$ is the set of all periodic sequences $\{\omega_n\}_{n \in \mathbb{Z}^d}$, with an arbitrary period such that $\omega_n$ is in the support of $\mu$ for all $n$ and $\Sigma(H)$ is the spectrum of $H$.

As has been said above the proof of Theorem 1.1 exists in [2, 3] and is based on Weyl sequences and probabilistic arguments. Notice that this theorem reduces the determination of the spectra of random Schrödinger operators for the case of periodic Schrödinger operators. As it is well known [3] that the spectrum of periodic operators have a band structure, this will be the case for $\Sigma(H_\omega)$ with the possibility to close gaps.

Under additional assumptions on $f$ more is known:

**Theorem 1.2** If $f$ has a fixed sign, i.e $f \leq 0$ or $f \geq 0$ and if $\mu$ is supported $[\omega^-, \omega^+]$, then

$$
\Sigma(H_\omega) = \bigcup_{\omega \in [\omega^-, \omega^+]} \Sigma(-\Delta + W_{\text{per}} + \omega \sum_{\gamma \in \mathbb{Z}^d} f(x - \gamma)).
$$

(1.3)

In particular,

$$
\inf \Sigma(H_\omega) = \begin{cases} 
\inf \Sigma(-\Delta + W_{\text{per}} + \omega^+ \sum_{\gamma \in \mathbb{Z}^d} f(x - \gamma)) & \text{if } f \leq 0, \\
\inf \Sigma(-\Delta + W_{\text{per}} + \omega^- \sum_{\gamma \in \mathbb{Z}^d} f(x - \gamma)) & \text{if } f \geq 0. 
\end{cases}
$$

(1.4)
Theorem 1.2 is a simple consequence of Theorem 1.1. Indeed, using (1.2) and the fact that constant sequences \( \{ \omega \gamma = \omega \in \text{supp } \mu \} \in \mathcal{P} \), we get that the r.h.s of (1.3) is naturally contained in \( \Sigma(H, \omega) \). For the inverse inclusion, let \((\omega \gamma)_{\gamma \in \mathbb{Z}^d} \in \mathcal{P}\) be \(k\)-periodic and let \([a, b]\) be the \(n\)-th band of the \(k\) periodic operator \(H_{\omega, k} = -\Delta + W_{\text{per}} + \sum_{\gamma \in \mathbb{Z}^d} \omega \gamma f(x - \gamma)\). Let \([a_1, b_2]\) and \([a_2, b_2]\) be the \(n\)-th bands of respectively \(-\Delta + W_{\text{per}} + \omega - \sum_{\gamma \in \mathbb{Z}^d} f(x - \gamma)\) and \(-\Delta + W_{\text{per}} + \omega + \sum_{\gamma \in \mathbb{Z}^d} f(x - \gamma)\) (both seen as \(k\)-periodic operators). By the min-max principle we have \(a_1 \leq a \leq a_2\) and \(b_1 \leq b \leq b_2\). As the bands of \(-\Delta + W_{\text{per}} + \omega \sum_{\gamma \in \mathbb{Z}^d} f(x - \gamma)\) depend continuously on \(\omega\), we deduce that \(\Sigma(-\Delta + W_{\text{per}} + \sum_{\gamma \in \mathbb{Z}^d} \omega f(x - \gamma))\) is contained in the r.h.s of (1.3). The proof of (1.3) is ended by taking into account the fact that these sets are closed.

For (1.4) it is a simple consequence of monotonicity of the model, it is increasing when \(f \geq 0\) and decreasing when \(f \leq 0\). Indeed if \(f \leq 0\), and \(\tilde{\omega} \gamma \leq \hat{\omega} \gamma\) for any \(\gamma \in \mathbb{Z}^d\) then in the sense form we have

\[-\Delta + W_{\text{per}} + \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega} \gamma f(x - \gamma) \leq -\Delta + W_{\text{per}} + \sum_{\gamma \in \mathbb{Z}^d} \hat{\omega} \gamma f(x - \gamma)\].

The situation is more complicated and different when the single site \(f\) changes the sign, as the monotonicity property is not true in this case. We notice that recently Lott and Stolz have conjectured [11] that in dimension one, the spectral minimum of random displacement models is realized through the pair formation of the single site.

1.1 The Model

Our basic object of study is the so-called Anderson model, a random Schrödinger operator of the form

\[H_{\lambda, \omega} = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + W_{\text{per}} + \lambda \sum_{\gamma \in \mathbb{Z}^d} \omega \gamma f(x - \gamma), \quad (1.5)\]

where,

- \(W_{\text{per}}\) is a \(\mathbb{Z}^d\)-periodic and bounded function.
• \( \lambda \) is a positive parameter

• \((\omega_\gamma)_{\gamma \in \mathbb{Z}^d}\) is a family of independent, identically-distributed random variables taking values in \([\omega^-, \omega^+].\) We denote by \(\mu\) the probability distribution.

• For \(C_0 = [-\frac{1}{2}, \frac{1}{2}]\), the single site potential \(f \in C_0^\infty(C_0)\) such that \(f \in l^1(L^p(\mathbb{R}^d))\), with \(p = 2\) if \(d \leq 3\), \(p > 2\) if \(d = 4\) and \(p = d/2\) if \(d \geq 5\) and \(f = f^+ + f^-,\) with \(f^+ \geq 0, f^- \leq 0\) and \(f^+ \cdot f^- = 0.\)

By [2, 8], we know that \(H_{\lambda, \omega}\) is an essentially self-adjoint operator on \(L^2(\mathbb{R}^d)\) with the domain \(C_0^\infty(\mathbb{R}^d),\) we denote by \(H_{\lambda, \omega}\) its self adjoint extension. It is an ergodic operator so, according to [2, 8], we know that there exist \(\Sigma_\lambda, \Sigma_{\lambda, pp}, \Sigma_{\lambda, ac}\) and \(\Sigma_{\lambda, sc}\) closed and non-random sets of \(\mathbb{R}\) such that \(\Sigma_\lambda\) is the spectrum of \(H_{\lambda, \omega}\) with probability one and such that if \(\sigma_{pp}\) (respectively \(\sigma_{ac}\) and \(\sigma_{sc}\)) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of \(H_{\lambda, \omega},\) then \(\Sigma_{pp} = \sigma_{pp}, \Sigma_{\lambda, ac} = \sigma_{\lambda, ac}\) and \(\Sigma_{\lambda, sc} = \sigma_{\lambda, sc}\) with probability one.

### 1.2 The result

As we will see (subsection 2.1) \(H_{\lambda, \omega}\) can be considered as a perturbation of some periodic operator \(H_{\lambda, \omega^-}\). Let \(\varphi_{\lambda, 1}(x, \theta(\lambda))\) be the Floquet eigenfunction associated to the first Floquet eigenvalue \(E_1(\lambda, \theta)\) of \(H_{\lambda, \omega^-}\). Let \((\theta_k(\lambda))_{1 \leq k \leq m}\) be the points where \(E_1(\lambda, \theta)\) attains its minimum. We set

\[
A(0) = \left( (f \varphi_{0, 1}(\cdot, \theta_k(0)), \varphi_{0, 1}(\cdot, \theta_{k'}(0)))_{L^2(C_0)} \right)_{1 \leq k, k' \leq m}.
\]

We prove that

**Theorem 1.3** Let \(H_{\lambda, \omega}\) be the operator defined by (1.5). If the matrix \(A(0)\) is positive-definite. Then there exists \(\lambda_0 > 0\) such that for any \(\lambda \in [0, \lambda_0]\) we have:

\[
\inf(\Sigma_\lambda) = \inf(\Sigma_{\lambda, \omega^-})
\]

If the matrix \(A(0)\) is negative-definite. Then there exists \(\lambda_0 > 0\) such that for any \(\lambda \in [0, \lambda_0]\) we have:

\[
\inf(\Sigma_\lambda) = \inf(\Sigma_{\lambda, \omega^+}).
\]

\[
\begin{align*}
\text{(1.5) \quad H_{\lambda, \omega} := (H_{\lambda, \omega}^+ - H_{\lambda, \omega}^-) + \lambda \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \cdot \delta_{\gamma^0}).
\end{align*}
\]
Remark 1.4 Theorem 1.3 is stated for the infimum of the spectrum. Under some additional assumptions the same result is still true for the internal edges of the spectrum.

The analogous problem for the random magnetic Schrödinger operator is considered and studied in [3].

Theorem 1.3 can be considered as a first step toward the physically-motivated applications. One of them is the study of the so-called Lifshitz tails of the integrated density of states. This could be done under some additional assumptions on the behavior of the random variables in the vicinity of $\omega^-$ or $\omega^+$, [4, 7, 8]. Another one is the spectral localization [7].

The proof of Theorem 1.3, is given in section 3. It is based on the reduction procedure. This powerful technique was predicted by Klopp [4] and used in several works, [1, 6, 7].

As stated the proof of Theorem 1.3 can be divided naturally divides into two parts, we shall discuss them separately.

Indeed, if $A(0)$ is positive-definite we will conjugate $H_\omega$ with $\Pi_{\lambda,0}$, the spectral projection for $H_{\lambda,\omega^-}$ on the first band. Then we prove that \[ \Pi_{\lambda,0} H_{\lambda,\omega^-} \Pi_{\lambda,0} \geq E_{\lambda,\omega^-} \Pi_{\lambda,0}. \]

Here $E_{\lambda,\omega^-}$ is the bottom of the spectrum of the periodic operator $H_{\lambda,\omega^-}$.

If $A(0)$ is negative-definite we will conjugate $H_\omega$ with $\Pi_{\lambda,0}$, the spectral projection for $H_{\lambda,\omega^+}$ on the first band. Then we prove \[ \Pi_{\lambda,0} H_{\lambda,\omega^+} \Pi_{\lambda,0} \geq E_{\lambda,\omega^+} \Pi_{\lambda,0}. \]

Here $E_{\lambda,\omega^+}$ is the bottom of the spectrum of the periodic operator $H_{\lambda,\omega^+}$.

2 Preliminary

Let us consider the following periodic operator

\[ H_{\lambda,\omega^-} = -\Delta + W_{\text{per}} + \lambda \sum_{\gamma \in \mathbb{Z}^d} \omega^- f(\cdot - \gamma). \]  

(2.6)

For this, it is convenient to consider $H_{\lambda,\omega}$ as a perturbation of $H_{\lambda,\omega^-}$. Indeed, we have:

\[ H_{\lambda,\omega} = H_{\lambda,\omega^-} + \lambda \sum_{\gamma \in \mathbb{Z}^d} (\omega_\gamma - \omega^-) f(\cdot - \gamma). \]
For this for $\gamma \in \mathbb{Z}^d$, we set $\tilde{\omega}_\gamma = \omega_\gamma - \omega^-$ and $V_\omega(\cdot) = \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma f(\cdot - \gamma)$. We notice that according to the definition of $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$ we get that, $(\tilde{\omega}_\gamma)_{\gamma \in \mathbb{Z}^d}$ is a family of random positive and bounded variables.

### 2.1 Some Floquet Theory

For $\gamma \in \mathbb{Z}^d$, we denote by $\tau_\gamma$ the translation by $\gamma$ operator i.e $(\tau_\gamma \varphi)(x) = \varphi(x - \gamma)$. We have, for any $\gamma \in \mathbb{Z}^d$

$$\tau_\gamma H_{\lambda, \omega^-} = \tau_\gamma^* H_{\lambda, \omega^-} = \tau_\gamma H_{\lambda, \omega^-}.$$

Then the so called, Floquet Theory, can be used to study $H_{\lambda, \omega^-}$. For this, we review some standard facts from the Floquet theory for periodic operators. Basic references for this material are [5, 9, 10]. Let $T^* = \mathbb{R}^d/(2\pi \mathbb{Z}^d)$. We define $H$ by

$$H = \{ u(x, \theta) \in L^2_{\text{loc}}(\mathbb{R}^d) \otimes L^2(T^*) ; \forall (x, \theta, \gamma) \in \mathbb{R}^d \times T^* \times \mathbb{Z}^d ; u(x + \gamma, \theta) = e^{i\gamma \theta} u(x, \theta) \}.$$

$H$ is equipped with the norm

$$\frac{1}{\text{vol}(T^*)} \int_{T^*} \| u(x, \theta) \|^2_{L^2(C_0)} d\theta.$$

For $\theta \in \mathbb{R}^d$ and $u \in \mathcal{S}(\mathbb{R}^d)$; the Schwartz space of rapidly decreasing functions we define

$$(U u)(x, \theta) = \sum_{\gamma \in \mathbb{Z}^d} e^{i\gamma \theta} u(x - \gamma).$$

$U$ can be extended as a unitary isometry from $L^2(\mathbb{R}^d)$ to $H$. Its inverse is given by the formula,

$$\text{for } u \in H, \quad (U^* u)(x) = \frac{1}{\text{vol}(T^*)} \int_{T^*} u(x, \theta) d\theta.$$

$U$ is a unitary isometry from $L^2(\mathbb{R}^d)$ to $H$ and $H_{\lambda, \omega}$ admits the Floquet decomposition [3, 10]

$$U H_{\lambda, \omega^-} U^* = \frac{1}{\text{vol}(T^*)} \int_{T^*} H_{\lambda, \omega^-}(\theta) d\theta.$$

Here $H_{\lambda, \omega^-}(\theta)$ is the operator $H_{\lambda, \omega^-}$ acting on $H_\theta$, defined by

$$\mathcal{H}_\theta = \{ u \in L^2_{\text{loc}}(\mathbb{R}^d) ; \forall \gamma \in \mathbb{Z}^d , u(x + \gamma) = e^{i\gamma \theta} u(x) \}.$$
As \( H_{\lambda, \omega}^- \) is elliptic, we know that, \( H_{\lambda, \omega}^- (\theta) \) has a compact resolvent; hence its spectrum is discrete \([1]\). We denote its eigenvalues, called Floquet eigenvalues of \( H_{\lambda, \omega}^- \), by

\[ E_1(\lambda, \theta) \leq E_2(\lambda, \theta) \leq \cdots \leq E_n(\lambda, \theta) \leq \cdots. \]

The corresponding Floquet eigenfunctions are denoted by \( (\varphi_{\lambda,j}(x, \cdot))_{j \in \mathbb{N}^*} \). The functions \( \theta \mapsto E_n(\lambda, \theta) \) are Lipshitz-continuous, and we have

\[ E_n(\lambda, \theta) \to +\infty \quad \text{as} \quad n \to +\infty \quad \text{uniformly in} \quad \theta. \]

The spectrum \( \Sigma_{\lambda, \omega}^- \) of \( H_{\lambda, \omega}^- \) is made of bands (i.e \( \Sigma_{\lambda, \omega}^- = \bigcup_{n \in \mathbb{N}^*} E_n(\lambda, \mathbb{T}^*) \)). Let us note by \( E_{\lambda, \omega}^- \) the bottom of the spectrum of \( \Sigma_{\lambda, \omega}^- \), i.e \( E_{\lambda, \omega}^- = \inf_{\theta \in \mathbb{T}^*} E_1(\lambda, \theta) \).

It is a well-known fact that, in any dimension the bottom (the first band) of the spectrum of a periodic Schrödinger operators is given by a simple Floquet eigenvalue and that the minimum of this Floquet eigenvalue is non-degenerate and quadratic. More precisely let \( \theta(\lambda) \) be an element of

\[ Z_\lambda = \{ \theta \in \mathbb{T}^* ; E_1(\lambda, \theta) = E_{\lambda, \omega}^- \}. \]

Then there exist \( C > 0 \) and \( \delta > 0 \) such that

\[ \forall |\theta - \theta(\lambda)| \leq \delta, \quad \frac{1}{C} |\theta - \theta(\lambda)|^2 \leq E_1(\lambda, \theta) - E_{\lambda, \omega}^- \leq C|\theta - \theta(\lambda)|^2. \]

Hence, the points where \( E_1(\lambda, \theta) \) reaches \( E_{\lambda, \omega}^- \) are isolated and as \( \mathbb{T}^* \) is compact, one concludes that \( Z_\lambda \) contains only finitely many of elements. Let \( m \) be the cardinal of \( Z_\lambda \) and let us denote them by \( (\theta_k(\lambda))_{1 \leq k \leq m} \). One can check that \( \theta_k(\lambda) \) depends continuously on \( \lambda \). For the sake of brevity, we use the notation \( \theta_k = \theta_k(\lambda) \).

For \( 1 \leq k \leq m \) and \( \theta \in \mathbb{T}^* \), we set

\[ \zeta_{k, \lambda}(\theta) = \sum_{1 \leq i \leq d} (\theta_i - \theta_{k,i})^2. \tag{2.7} \]

We notice that for any \( 1 \leq k \leq m \),

\[ \theta \mapsto \varphi_{\lambda,1}(x, \theta), \]

is analytic in a neighborhood of \( \theta_k \).
2.2 Wannier basis

We recall concepts used in [4]. Let $E \subset L^2(\mathbb{R}^d)$ be a closed subspace invariant by the $\mathbb{Z}^d$-translations, i.e. such that $\Pi^E$, the orthogonal projection on $E$, satisfies

$$\forall \gamma \in \mathbb{Z}^d, \quad \Pi^E = \tau_{-\gamma} \Pi^E \tau_{\gamma}.$$ 

Following the computations done in section 1.2 of [4], we see that there exists an orthonormal system of vectors $(\tilde{\varphi}_{j,0})_{j \in \mathbb{N}}$ such that for $\tilde{\varphi}_{j,\gamma} = \tau_{\gamma}(\tilde{\varphi}_{j,0})$; $(\tilde{\varphi}_{j,\gamma})_{(j \in \mathbb{N}, \gamma \in \mathbb{Z}^d)}$ is an orthonormal basis of $E$. Such system is called Wannier basis of $E$. The vectors $(\tilde{\varphi}_{n,0})_{n \in \mathbb{N}}$ are called Wannier generators of $E$.

Let $E \subset L^2(\mathbb{R}^d)$ be a space which is translation-invariant. $E$ is said to be of finite energy for $H_{\lambda,\omega}$ if $\Pi^E H_{\lambda,\omega} \Pi^E$ is a bounded operator. In this case, $E$ admits a finite set of Wannier generators.

Let $\Pi_{\lambda,0}(\theta)$ (respectively $\Pi_{\lambda,+(\theta)}$) be the orthogonal projection in $\mathcal{H}_\theta$ on the vector space generated by $\varphi_{\lambda,1}(\cdot, \theta)$ (respectively by $(\varphi_{\lambda,j}(\cdot, \theta))_{j \geq 2}$). These projections are two-by-two orthogonal and their sum is the identity for all $\theta \in T^*$. One defines

$$\Pi_{\lambda,\alpha} = U^{-1} \left( \int_{T^*} \Pi_{\lambda,\alpha}(\theta)d\theta \right) U : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d),$$

where $\alpha \in \{0, +\}$. $\Pi_{\lambda,\alpha}$ is an orthogonal projection on $L^2(\mathbb{R}^d)$ and for all $\gamma \in \mathbb{Z}^d$, we have $\tau^*_\gamma \Pi_{\lambda,\alpha} \tau_\gamma = \Pi_{\lambda,\alpha}$.

For $\alpha \in \{0, +\}$, we set $\mathcal{E}_{\lambda,\alpha} = \Pi_{\lambda,\alpha}(L^2(\mathbb{R}^d))$. These spaces are translation-invariant. Moreover $\mathcal{E}_{\lambda,0}$ is of finished energies for $H_{\lambda,\omega^-}$. The reduction procedure consists in decomposing the operator $H_{\lambda,\omega^-}$ according to various translation-invariants subspaces. The random operators thus obtained are reference operators.

3 The proof of Theorem 1.3

As we have indicated, our aim in this section is to prove Theorem 1.3, but first, let us introduce some notations and useful lemma.

For $u \in L^2(T^*)$, let

$$T_{\varphi_{\lambda,1}}(u) = U^*(u \varphi_{\lambda,1}(x, \theta)) = \int_{T^*} u(\theta) \varphi_{\lambda,1}(x, \theta)d\theta. \quad (3.8)$$
So, \( T_{\varphi_{\lambda,1}} \) define a unitary transformation from \( L^2(T^*) \) to \( E_{\lambda,0} \) and for \( v \in E_{\lambda,0} \) we have
\[
T_{\varphi_{\lambda,1}}^*(v) = \langle (Uv)(\cdot, \theta), \varphi_{\lambda,1}(\cdot, \theta) \rangle.
\]
For \( 1 \leq k \leq m \) and \( (x, \theta) \in \mathbb{R}^d \times T^* \), let
\[
\tilde{\varphi}_{\lambda,1,k}(x, \theta) = \varphi_{\lambda,1}(\theta_k(\lambda), x)e^{(\theta - \theta_k)x}.
\]
We set,
\[
\delta\varphi_{\lambda,1,k}(x, \theta) = \frac{1}{\sqrt{\zeta_{k,\lambda}(\theta)}}(\varphi_{\lambda,1} - \varphi_{\lambda,1,k}(x, \theta)).
\]
By this, for any \( u \in L^2(T^*) \), we have
\[
T_{\varphi_{\lambda,1}}(u) = T_{\tilde{\varphi}_{\lambda,1,k}}(u) + T_{\delta\varphi_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda}}u). \tag{3.9}
\]
For \( v \in H^2_\theta = \{ v \in H^2_{loc}(\mathbb{R}^d); v(\cdot - \gamma) = e^{-i\gamma \cdot \theta}u(\cdot) \} \) one defines the following norms:
\[
\sup_{\theta \in T^*}\{ \|v(\cdot, \theta)\|_{L^2(C_0)}^2 \} = \|v\|_{1,\infty}^2.
\]
and
\[
\sup_{\theta \in T^*} \left( \|H_{\lambda,\omega}(\theta)u(\cdot, \theta)\|_{L^2(C_0)}^2 + \|v(\cdot, \theta)\|_{L^2(C_0)}^2 \right) = \|v\|_{H_{\lambda,\omega^*}}.
\]

**Remark 3.1** The functions \( \tilde{\varphi}_{\lambda,1,k} \) and \( \delta\varphi_{\lambda,1,k} \) are well defined and
\[
\|\tilde{\varphi}_{\lambda,1,k}\|_{1,\infty} \sim \|\tilde{\varphi}_{\lambda,1,k}\|_{H_{\lambda,\omega^*}} \sim \|\delta\varphi_{\lambda,1,k}\|_{H_{\lambda,\omega^*; \infty}} \sim \|\delta\varphi_{\lambda,1,k}\|_{H_{\lambda,\omega^*; \infty}}
\]
and are finished (See [4]).

The following Lemma is of use. It will be proven at the end of this section.

**Lemma 3.2** For \( \theta_k, \theta_k' \in T^* \) and \( \varphi \in L^2(T^*, H^2) \) let \( \varphi_k = e^{i(\theta - \theta_k)x}\varphi(x, \theta_k) \), \( \varphi_{k'} = e^{i(\theta - \theta_{k'})x}\varphi(x, \theta_{k'}) \) and \( a_{\varphi, k,k'}(x) = f(x)\varphi(x, \theta_k)\varphi(x, \theta_{k'}) \). If \( \|\varphi\|_{1,\infty} < \infty \) (resp. \( \|\varphi\|_{H_{\lambda,\omega^*; \infty}} < \infty \) ) then \( T_{\varphi} \in \mathcal{L}(L^2(T^*), L^2(\mathbb{R}^d)) \) (resp. \( V_{\omega} \cdot T_{\varphi} \in \mathcal{L}(L^2(T^*), L^2(\mathbb{R}^d)) \) ) and there exist \( C, \beta > 0 \) such that for all \( u, v \in L^2(T^*) \), we have
\[
\left| \langle V_{\omega}T_{\varphi_k}(u), T_{\varphi_{k'}}(v) \rangle - \left( \int_{C_0} a_{\varphi, k,k'}(x)dx \right) \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \hat{u}(\gamma)\hat{v}(\gamma) \right| \leq C\beta \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma (|\hat{u}(\gamma)|^2 + |\hat{v}(\gamma)|^2) + C(1+1/\beta)(\langle \xi_{k,\lambda}u, u \rangle_{L^2(T^*)} + \langle \xi_{k',\lambda}v, v \rangle_{L^2(T^*)}). \tag{3.10}
\]
We set
\[ H_{0,\lambda,\omega} = \Pi_{\lambda,0} H_{\lambda,\omega} \Pi_{\lambda,0} = \Pi_{\lambda,0} H_{\lambda,\omega^-} \Pi_{\lambda,0} + \Pi_{\lambda,0} V_{\omega} \Pi_{\lambda,0}. \]

**Theorem 3.3** Assume that the matrix \( A(0) \) is positive-definite. Then there exists \( \lambda_0 > 0 \) such that for any \( \lambda \in [0, \lambda_0] \) we have:
\[ \Pi_{\lambda,0} \left( H_{\lambda,\omega} - E_{\lambda,\omega^-} \right) \Pi_{\lambda,0} \text{ is a positive operator}. \]

The proof of Theorem 3.3 is the object of the following section.

### 3.1.1 The proof of Theorem 3.3

Using (3.8), we get that \( H_{0,\lambda,\omega} \) is unitarily equivalent to the operator
\[ h_{0,\lambda,\omega} = T_{\varphi_{\lambda,1}}^* H_{0,\lambda,\omega} T_{\varphi_{\lambda,1}}, \]
acting on \( L^2(\mathbb{T}^*) \) and written as
\[ h_{0,\lambda,\omega} = h_{0,\lambda,\omega^-} + \lambda V_{\lambda,\omega}^0. \]

With \( h_{0,\lambda,\omega^-} \) is the multiplication operator by \( E_{1}(\lambda, \theta) \) and \( V_{\lambda,\omega}^0 \) is an integral operator with the kernel
\[ V_{\lambda,\omega}(\theta, \theta') = \langle V_{\omega} \varphi_{\lambda,1}(\cdot, \theta), \varphi_{\lambda,1}(\cdot, \theta') \rangle. \]

Let \( V_k \) be a neighborhood of \( \theta_k \), such that if \( \theta_{k'} \in Z \) and \( k \neq k' \) then \( \theta \notin V_k \) and \( V_k \cap V_{k'} = \emptyset \). As \( \mathbb{T}^* \) is compact, one can cover it by \( (V_k)_{1 \leq k \leq m} \) (i.e. \( \cup_{1 \leq k \leq m} V_k = \mathbb{T}^* \)). For \( 1 \leq k \leq m \) let \( \chi_k \) be the characteristic function of \( V_k \).

For simplicity for \( u \in L^2(\mathbb{T}^*) \), we will denote \( \chi_k u \) as \( u_k \) in the following. We consider \( u \) as a system of \( m \) columns denoted by \( (u_k)_{1 \leq k \leq m} \). We endow \( L^2(\mathbb{T}^*) \otimes \mathbb{C}^m \) with the scalar product generating the following Euclidean norm:
\[ \|u\|_{L^2(\mathbb{T}^*) \otimes \mathbb{C}^m} = \sum_{k=1}^m \|u_k\|_{L^2(\mathbb{T}^*)}^2. \]
3.1.2 The lower bound of $h_{0,\omega}^\lambda$

**Proposition 3.4** There exists $C > 0$ such that for any $u \in L^2(T^*)$, we have

$$\langle h_{0,\omega}^\lambda - u, u \rangle \geq \sum_{1 \leq k \leq m} E_1(\lambda, \theta_k(\lambda)) \|u_k\|_{L^2(T^*)}^2 + \frac{1}{C} \sum_{1 \leq k \leq m} \langle \zeta_k u_k, u_k \rangle_{L^2(T^*)}. \tag{3.12}$$

**Proof:** For $u \in L^2(T^*)$, one computes

$$\langle h_{0,\omega}^\lambda - u, u \rangle = \int_{T^*} E(\lambda, \theta)|u(\theta)|^2 d\theta = \sum_{1 \leq k \leq m} \int_{T^*} E(\lambda, \theta)\chi_k(\theta)|u(\theta)|^2 d\theta,$$

As for any $\theta$ in $V_k$ the support of $\chi_k$, there exists $C > 0$ such that we have

$$E(\lambda, \theta) + \frac{1}{C} \zeta(\theta) \leq E(\lambda, \theta),$$

we get the result. $\square$

3.1.3 The lower bound of $V_{\lambda,\tilde{\omega}}^0$

**Proposition 3.5** There exists $C_1, C_2 > 0$ and $\lambda_0$ such that for all $\lambda \in [0, \lambda_0]$ and $u \in L^2(T^*)$ we have

$$\langle V_{\lambda,\tilde{\omega}}^0 u, u \rangle \geq C_1 \sum_{1 \leq k \leq m, \gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2 - C_2 \sum_{1 \leq k \leq m} \langle \zeta_k u_k, u_k \rangle_{L^2(T^*)}. \tag{3.13}$$

**The proof of Theorem 3.3**

Let us notice that, by combining the results of Proposition 3.4 and 3.5, one gets that there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$ and for any $u \in L^2(T^*)$ we have,

$$\langle h_{0,\omega}^\lambda u, u \rangle \geq \sum_{1 \leq k \leq m} E(\lambda, \theta_k(\lambda)) \|u_k\|_{L^2(T^*)}^2 + \frac{1}{C} \left( \sum_{1 \leq k \leq m} \langle \zeta_k u_k, u_k \rangle_{L^2(T^*)} + \lambda \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2 \right)$$

$$\geq E_{\lambda,\omega}^\lambda \|u\|_{L^2(T^*)}^2 + \frac{1}{C} \left( \sum_{1 \leq k \leq m} \langle \zeta_k u_k, u_k \rangle_{L^2(T^*)} + \lambda \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2 \right).$$
The proof of proposition 3.5:

get that for any \( 1 \leq k \leq m \),

\[
\tag{3.14}
\langle (h_{\lambda,\omega}^0 - E_{\lambda,\omega}^-)u, u \rangle \geq \frac{1}{C} \left( \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(\mathbb{T}^d)} + \lambda \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} \bar{\omega}_{\gamma} |u_k(\gamma)|^2 \right).
\]

This gives that,

This ends the proof of Theorem 3.3. \( \square \)

**Remark 3.6** We notice that even if we know that the bottom of the spectrum of \( H_\omega \) coincides with the bottom of the spectrum of \( H_{\lambda,\omega}^- \) we cannot consider \( \Pi_{\lambda,0}(H_{\lambda,\omega}^-) \Pi_{\lambda,0} \) as a positive operator.

**The proof of proposition 3.5:**

Let us start by expanding \( \langle V_{\lambda,\omega}^0 u, u \rangle \),

\[
\tag{3.15}
\langle V_{\lambda,\omega}^0 u, u \rangle = \sum_{1 \leq k, k' \leq m} \langle V_{\lambda,\omega}^0 T_{\tilde{\varphi}_{\lambda,1,k}}(u_k), T_{\tilde{\varphi}_{\lambda,1,k'}}(u_k') \rangle_{L^2(\mathbb{R}^d)} + \sum_{1 \leq k, k' \leq m} \langle V_{\lambda,\omega}^0 T_{\delta \varphi_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda} u_k}), T_{\delta \varphi_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda} u_k'}) \rangle_{L^2(\mathbb{R}^d)} + 2 \sum_{1 \leq k, k' \leq m} \Re \left( \langle V_{\lambda,\omega}^0 T_{\tilde{\varphi}_{\lambda,1,k}}(u_k), T_{\delta \varphi_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda} u_k'}) \rangle_{L^2(\mathbb{R}^d)} \right).
\]

We start by estimating the three sums of the last equation. For the second sum, using Cauchy Schwartz inequality and Lemma 3.2, we get that for any \( 1 \leq k, k' \leq m \), there exists \( C > 0 \) such that we have,

\[
\left| \langle V_{\lambda,\omega}^0 T_{\delta \varphi_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda} u_k}), T_{\delta \varphi_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda} u_k'}) \rangle_{L^2(\mathbb{R}^d)} \right| \leq \frac{1}{2} \left( \| V_{\lambda,\omega}^0 T_{\delta \varphi_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda} u_k}) \|_{L^2(\mathbb{R}^d)}^2 + \| T_{\delta \varphi_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda} u_k'}) \|_{L^2(\mathbb{R}^d)}^2 \right) \leq C \cdot \left( \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(\mathbb{T}^d)} + \langle \zeta_{k',\lambda} u_{k'}, u_{k'} \rangle_{L^2(\mathbb{T}^d)} \right).
\]

So there exists \( C > 0 \) such that we have

\[
\sum_{1 \leq k, k' \leq m} \left| \langle V_{\lambda,\omega}^0 T_{\delta \varphi_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda} u_k}), T_{\delta \varphi_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda} u_k'}) \rangle_{L^2(\mathbb{R}^d)} \right| \leq C \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle. \quad (3.15)
\]
For the third sum, using the Cauchy-Schwartz inequality once more, we get that for $1 \leq k \leq m$, there exists $\beta > 0$ such that we have

$$
\left| \langle V_{\lambda,0}^0 T_{\tilde{\varphi},1,k}(u_k), T_{\tilde{\delta},\lambda,1,k'}(\sqrt{\zeta_{k',\lambda} u_{k'}}) \rangle \right|_2 \leq \beta \| V_{\lambda,0}^0 T_{\tilde{\varphi},1,k}(u_k) \|_2^2 \langle \zeta_{k,\lambda} u_{k'} \rangle_2^2 (3.16)
$$

Using equation (3.11), one gets that there exist $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$
\sum_{1 \leq k, k' \leq m} \left| \langle V_{\lambda,0}^0 T_{\tilde{\varphi},1,k}(u_k), T_{\tilde{\delta},\lambda,1,k'}(\sqrt{\zeta_{k',\lambda} u_{k'}}) \rangle \right|_2 \leq \tilde{C}_1 \beta \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} |\hat{\omega}_{\gamma}(\hat{u}_k(\gamma))|^2 + \tilde{C}_2 (1 + 1/\beta) \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(T^d)} (3.17)
$$

For the first sum, using (3.10), we get that there exist $C_1', C_2' > 0$ such that

$$
\left| \sum_{1 \leq k, k' \leq m} \langle V_{\lambda,0}^0 T_{\tilde{\varphi},1,k}(u_k), T_{\tilde{\delta},\lambda,1,k'}(u_{k'}) \rangle \right|_2 \leq C_1' \beta \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} |\hat{\omega}_{\gamma}(\hat{u}_k(\gamma))|^2 + C_2' (1 + 1/\beta) \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(T^d)} (3.18)
$$

Now equations (3.15), (3.17) and (3.18) give that there exist $K_1, K_2 > 0$ such that

$$
\left| \langle V_{\lambda,0}^0 u, u \rangle - \sum_{1 \leq k, k' \leq m} \sum_{\gamma \in \mathbb{Z}^d} \hat{\omega}_{\gamma} \left( \int_{C_0} a_{\varphi_{\lambda,1,k,k'}}(x) dx \right) (\hat{u}_k(\gamma))(\hat{u}_{k'}(\gamma)) \right| \leq K_1 \beta \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} |\hat{\omega}_{\gamma}(\hat{u}_k(g))^2 + K_2 (1 + 1/\beta) \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(T^d)} (3.19)
$$

Now, if the matrix

$$
A(0) = \left( \int_{C_0} a_{\varphi_{0,1,k,k'}}(x) dx \right)_{1 \leq k, k' \leq m},
$$

is positive-definite, one gets that $\inf \sigma(A(0)) = C > 0$ satisfies

$$
CI_m \leq A.
$$
Let $A(\lambda)$ be the matrix,

$$
\left( \int_{C_0} a_{\varphi_{1,\lambda},k,k'}(x)dx \right)_{1 \leq k, k' \leq m}.
$$

Notice that for any $1 \leq k, k' \leq m$, the functions

$$f_{k,k'} : \lambda \rightarrow \int_{C_0} a_{\varphi_{1,\lambda},\theta_k(\lambda),\theta_{k'}(\lambda)}(x)dx,$$

are continuous in $\lambda$. So there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$

$$
\frac{C}{2} I_m \leq A(\lambda).
$$

This gives that for any $u \in L^2(\mathbb{T}^*)$

$$
\frac{C}{2} \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2
\leq \sum_{1 \leq k, k' \leq m} \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \left( \int_{C_0} a_{\varphi_{1,\lambda},k,k'}(x)dx \right) (\hat{u}_k)(\gamma)(\hat{u}_{k'})(\gamma). \quad (3.20)
$$

Now using the expansion of $\langle V_{\lambda,\omega}^0 u, u \rangle$ and equation (3.19), we get that there exist $K_1, K_2 > 0$ and $\beta > 0$ such that

$$
\langle V_{\lambda,\omega}^0 u, u \rangle \geq \left( \frac{C'}{2} - K_1 \beta \right) \sum_{1 \leq k \leq m; \gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2 - K_2 (1 + 1/\beta) \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(\mathbb{T}^*)}, \quad (3.21)
$$

So, for $\beta > 0$, well chosen we get that there exist constants $C_1, C_2 > 0$ such that

$$
\langle V_{\lambda,\omega}^0 u, u \rangle \geq \frac{C_1}{3} \sum_{1 \leq k \leq m; \gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2 - C_2 \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(\mathbb{T}^*)}. \quad (3.22)
$$

This ends the proof of Proposition 3.5. \qed
3.2 If \( A(0) \) is negative-definite

Let \( H_{\lambda,\omega^+} \) be the following operator,

\[
H_{\lambda,\omega^+} = -\Delta + W_{\text{per}} + \lambda \sum_{\gamma \in \mathbb{Z}^d} \omega^+ f(\cdot - \gamma).
\]

As \( H_{\lambda,\omega^+} \) is a \( \mathbb{Z}^d \)-periodic operator, the analysis given in subsection 2.1 for \( H_{\lambda,\omega^+} \) is still true in the present case. For \( (E_j(\lambda, \theta))_{j \in \mathbb{N}^*} \), the Floquet eigenvalue of \( H_{\lambda,\omega^+} \) let us set \( E_{\lambda,\omega^+} = \inf_{\theta \in \mathbb{T}^*} E_1(\lambda, \theta) \).

**Theorem 3.7** Assume that the matrix \( A(0) \) is negative-definite. Then there exists \( \lambda_0 > 0 \) such that for any \( \lambda \in [0, \lambda_0] \) we have:

\[
\Pi_{\lambda,0} \left( H_{\lambda,\omega} - E_{\lambda,\omega^+} \right) \Pi_{\lambda,0} \text{ is a positive operator.}
\]

The result of Theorem 3.7 can be proved in the same way as we did for Theorem 1.3 in the previous subsection. Indeed, \( H_{\lambda,\omega} \) can be seen as a perturbation of \( H_{\lambda,\omega^+} \) as follow,

\[
H_{\lambda,\omega} = H_{\lambda,\omega^+} + V_{\omega^+}.
\]

With \( V_{\omega^+}(\cdot) = \sum_{\gamma \in \mathbb{Z}^d} \omega^+ f(\cdot - \gamma) \) and for any \( \gamma \in \mathbb{Z}^d \) \( \omega^+ = \omega^+ - \omega^- \). Notice that in this case \( (\omega^+)_\gamma = \omega^+ - \omega^- \) is a family of bounded and negative random variables.

Using the analogous unitary transformation, one gets that \( H_{\lambda,\omega}^0 \) is unitarily equivalent to

\[
h_{\lambda,\omega}^0 = h_{\lambda,\omega^+} + V_{\omega^+}^0.
\]

The lower bound of \( h_{\lambda,\omega^+} \) can be derived easily. As all arguments used to lower bound \( V_{\omega^+}^0 \) remain valid; we lower bound \( V_{\omega^+}^0 \) using the same computation done in subsection 3.1.

So we get that there exist \( K_1, K_2 > 0 \) such that

\[
\left| \langle V^0_{\lambda,\omega^+} u, u \rangle - \sum_{1 \leq k, k' \leq m} \sum_{\gamma \in \mathbb{Z}^d} \omega\gamma \left( \int_{C_0} a_{\varphi_{x,1,k,k'}(x)} dx \right) (\hat{u}_k)(\gamma) (\hat{u}_{k'})(\gamma) \right| \leq K_1 \beta (\omega^+ - \omega^-) \sum_{1 \leq k \leq m, \gamma \in \mathbb{Z}^d} |(\hat{u}_k)(\gamma)|^2 + K_2 (1 + 1/\beta) \sum_{1 \leq k \leq m} \langle \zeta_{k,\lambda} u_k, u_k \rangle_{L^2(\mathbb{T}^d)}.
\]

(3.23)
When $A(0)$ is negative-definite, there exists $C < 0$ and $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$, we have
$$A(\lambda) \leq CI_m.$$  
As the random variables $(\tilde{\omega}_\gamma)_{\gamma \in \mathbb{Z}^d}$ are negative, we get that
$$C \sum_{1 \leq k \leq m} \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2$$
$$\leq \sum_{1 \leq k, k' \leq m} \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \left( \int_{C_0} a_{\varphi_{\lambda,1,k,k'}(x)} dx \right) (\hat{u}_k)(\gamma)(\hat{u}_{k'})(\gamma).$$  \hfill (3.24)
This and equation \hfill (3.23) give the sought result on the lower bound of $V_\omega$. This ends the proof of Theorem 3.7. \hfill $\Box$

**The proof of Lemma 3.2:**
As $V_\omega$ is $H_{\lambda,\omega,-}$-relatively bound uniformly on $\tilde{\omega}_\gamma$, there exists $c > 0$ such that for any $u \in L^2(\mathbb{T}^*)$ we have
$$\|V_\omega T_\varphi(u)\| \leq c \left( \|H_{\lambda,\omega,-} T_\varphi(u)\|^2 + \|T_\varphi(u)\|^2 \right)$$
$$\leq c \int_{\mathbb{T}^*} \left( \|H_{\lambda,\omega,-}(\theta)\varphi(\cdot, \theta)\|_{L^2(C_0)}^2 + \|\varphi(\cdot, \theta)\|_{L^2(C_0)}^2 \right) |u(\theta)|^2 d\theta.$$  
$$\leq c \|\varphi\|^2_{H_{\lambda,\omega,-,\infty}} \cdot \|u\|^2_{L^2(\mathbb{T}^*)}.$$  
One computes
$$\langle V_\omega^0 T_{\varphi_k}(u), T_{\varphi_{k'}}(v) \rangle_{L^2(\mathbb{R}^d)}$$
$$= \int_{\mathbb{R}^d} \tilde{\omega}_\gamma \int_{C_0} f(x) \varphi_k(x, \theta_k) \overline{\varphi_{k'}(x, \theta_{k'})} \cdot \left( \int_{\mathbb{T}^*} e^{i(\theta-k)x} u(\theta) d\theta \right) \overline{\left( \int_{\mathbb{T}^*} e^{i(\theta-k')x} u(\theta) d\theta \right)} dx.$$
$$= \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \int_{C_0} f(x) \varphi_k(x, \theta_k) \overline{\varphi_{k'}(x, \theta_{k'})} \cdot \left( \int_{\mathbb{T}^*} e^{i(\theta-k)x} e^{i\gamma \cdot \theta} u(\theta) d\theta \right) \overline{\left( \int_{\mathbb{T}^*} e^{i(\theta-k')x} e^{-i\gamma \cdot \theta} u(\theta) d\theta \right)} dx.$$  
Let
$$\hat{u}(\gamma) = \int_{\mathbb{T}^*} e^{i\gamma \cdot \theta} u(\theta) d\theta.$$

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For any \((x, \theta) \in \mathbb{R}^d \times T^*\) and \(1 \leq k \leq m\), we set
\[
g_k(x, \theta) = \frac{e^{i(\theta - \theta_k) \cdot x} - 1}{\sqrt{\zeta_{k, \lambda}(\theta)}}.
\] (3.25)

As \(\theta_k(\lambda)\) is the only zero of \(\zeta_{k, \lambda}\) and as it is nondegenerate, there exist \(C > 0\) such that, for \((x, \theta) \in \mathbb{R} \times \mathbb{R}^d\), and \(1 \leq k \leq m\), we have
\[
|g_k(x, \theta)| \leq C(1 + |x|).
\] (3.26)

We have \(e^{i(\theta - \theta_k) \cdot x} = \sqrt{\zeta_{k, \lambda}} g_k(x, \theta) + 1\). So using this and expanding \(\langle V_{\tilde{\omega}} T_{\varphi_k}(u), T_{\varphi_{k'}}(v) \rangle_{L^2(\mathbb{R}^d)}\), we get
\[
\langle V_{\tilde{\omega}} T_{\varphi_k}(u), T_{\varphi_{k'}}(v) \rangle_{L^2(\mathbb{R}^d)} - \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \left( \int_{C_0} a_{\varphi, k, k'}(x) dx \right) \cdot |u(\gamma)|^2
\]
\[
= \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \left( \int_{C_0} a_{\varphi, k, k'}(x) \cdot \left( \int_{T^*} e^{i\gamma \cdot \theta} g_k(x, \theta) \sqrt{\zeta_{k, \lambda}(\theta)} u(\theta) d\theta \cdot \right) \right)
\]
\[
\int_{T^*} e^{i\gamma \cdot \theta} g_k'(x, \theta) \sqrt{\zeta_{k', \lambda}(\theta)} v(\theta) d\theta \right) dx
\]
\[
+ \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \hat{u}(\gamma) \int_{C_0} a_{\varphi, k, k'}(x) \cdot \left( \int_{T^*} e^{i\gamma \cdot \theta} g_k(x, \theta) \sqrt{\zeta_{k', \lambda}(\theta)} v(\theta) d\theta d\theta \right) dx
\]
\[
+ \sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \hat{v}(\gamma) \int_{C_0} a_{\varphi, k, k'}(x) \cdot \left( \int_{T^*} e^{i\gamma \cdot \theta} g_k(x, \theta) \sqrt{\zeta_{k, \lambda}(\theta)} u(\theta) d\theta d\theta \right) dx.
\] (3.27)

Now using the fact that the family \((\tilde{\omega}_\gamma)_{\gamma \in \mathbb{Z}^d}\) is bounded, Cauchy-Schwartz inequality and Perseval identity and equation (3.26), we get that there exists \(C > 0\) such that
\[
\sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \left( \int_{C_0} a_{\varphi, k, k'}(x) \cdot \left( \int_{T^*} e^{i\gamma \cdot \theta} g_k(x, \theta) \sqrt{\zeta_{k, \lambda}(\theta)} u(\theta) d\theta \cdot \right) \right)
\]
\[
\int_{T^*} e^{i\gamma \cdot \theta} g_k'(x, \theta) \sqrt{\zeta_{k', \lambda}(\theta)} v(\theta) d\theta \right) dx
\]
\[
\leq C \langle \zeta_{k, \lambda} u, u \rangle_{L^2(T^*)} + \langle \zeta_{k', \lambda} v, v \rangle_{L^2(T^*)}.
\] (3.28)
And $\beta > 0$,

$$\sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \hat{u}(\gamma) \int_{C_0} a_{\varphi,k,k'}(x) \cdot \int_{\mathbb{T}^*} e^{i\gamma \cdot \theta} g_{k'}(x, \theta) \sqrt{\zeta_{k',\lambda}(\theta)} v(\theta) d\theta dx \leq \beta \sum_{\gamma \in \mathbb{Z}^d} |\hat{u}(\gamma)|^2 + \frac{1}{4\beta} \langle \zeta_{k',\lambda} v, v \rangle_{L^2(\mathbb{T}^*)}. \quad (3.29)$$

The same argument gives

$$\sum_{\gamma \in \mathbb{Z}^d} \tilde{\omega}_\gamma \hat{v}(\gamma) \int_{C_0} a_{\varphi,k,k'}(x) \cdot \int_{\mathbb{T}^*} e^{i\gamma \cdot \theta} g_k(x, \theta) \sqrt{\zeta_{k,\lambda}(\theta)} u(\theta) d\theta dx \leq \beta \sum_{\gamma \in \mathbb{Z}^d} |\hat{v}(\gamma)|^2 + \frac{1}{4\beta} \langle \zeta_{k,\lambda} u, u \rangle_{L^2(\mathbb{T}^*)}. \quad (3.30)$$

So from (3.27), (3.28), (3.29) and (3.30) we get (3.10).

The proof of (3.11) follows by changing $T_{\varphi,\lambda} (u)$ using (3.9) and following the same steps as (3.10).

This ends the proof of Lemme 3.2. \qed

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