Reflected BSDEs and continuous solutions of parabolic obstacle problem for semilinear PDEs in divergence form

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Abstract

We consider the Cauchy problem for semilinear parabolic equation in divergence form with obstacle. We show that under natural conditions on the right-hand side of the equation and mild conditions on the obstacle a unique continuous solution of the problem admits a stochastic representation in terms of reflected backward stochastic differential equations. We derive also some regularity properties of solutions and prove useful approximation results.

1 Introduction

In the present paper we are interested in stochastic representation of solutions of the Cauchy problem for semilinear parabolic equation in divergence form with obstacle. Let \( a : Q_T \equiv [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) be a measurable, symmetric matrix valued function such that

\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j \leq \Lambda |\xi|^2, \quad \xi \in \mathbb{R}^d
\]  

(1.1)

for some \( 0 < \lambda \leq \Lambda \) and let \( A_t \) be a linear operator of the form

\[
A_t = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a^{ij} \frac{\partial}{\partial x_j}).
\]  

(1.2)

Roughly speaking the problem consist in finding \( u : Q_T \to \mathbb{R} \) such that for given \( \varphi : \mathbb{R}^d \to \mathbb{R} \), \( h : Q_T \to \mathbb{R} \), \( f : Q_T \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \),

\[
\begin{cases}
\min(u - h, -\frac{\partial u}{\partial t} - A_t u - f_u) = 0 & \text{in } Q_T, \\
u(T) = \varphi & \text{on } \mathbb{R}^d,
\end{cases}
\]  

(1.3)

where \( f_u = f(\cdot, \cdot, u, \sigma \nabla u) \) and \( \sigma \sigma^* = a \), i.e. \( u \) satisfies the prescribed terminal condition, takes values above a given obstacle \( h \), satisfies inequality \( \frac{\partial u}{\partial t} + A_t u \leq -f_u \) in \( Q_T \) and equation \( \frac{\partial u}{\partial t} + A_t u = -f_u \) on the set \( \{ u > h \} \).

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The obstacle problem (1.3) has been studied intensively by many authors. Subject to regularity of the data \( \varphi, f, h \) and coefficients of \( A_t \), viscosity solutions (see [11]) or solutions of variational inequalities associated with (1.3) are considered. In the latter case one can consider weak solutions (see [4, 17, 19]) or strong solutions (see [4, 8, 9, 10]).

In the present paper by a solution of (1.3) we understand a pair \((u, \mu)\) consisting of a measurable function \( u : Q_T \to \mathbb{R} \) having some regularity properties and a Radon measure \( \mu \) on \( Q_T \) such that

\[
\frac{\partial u}{\partial t} + A_t u = - f u - \mu, \quad u(T) = \varphi, \quad u \geq h, \quad \int_{Q_T} (u - h) \, d\mu = 0 \quad (1.4)
\]

(see Section 2.2 for details). We adopt the above definition for three reasons. Firstly, it may be viewed as an analogue of the definition of the obstacle problem for elliptic equations (see [14, 16]). It is worth pointing out, however, that contrary to the case of elliptic equations, it is not obvious how solution of a parabolic variational inequality associated with (1.3) is related to the solution in the sense of (1.4). Secondly, since in many cases we are able to prove some additional information on \( \mu \), using (1.4) instead of variational formulation gives more information on solutions of (1.3). Finally, definition (1.4) is well suited with our main purpose which consists in providing stochastic representation of solutions of the obstacle problem.

In the case where \( A_t \) is a non-divergent operator of the form

\[
A_t = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i},
\]

problem (1.3) has been investigated carefully in [11] by using probabilistic methods. Let \( X^{s,x} \) be a solution of the Itô equation

\[
dX^s_t = \sigma(t, X^s_t) \, dW_t + b(t, X^s_t) \, dt, \quad X^s_s = x \quad (\sigma \sigma^* = a)
\]

associated with \( A_t \). In [11] it is proved, that under suitable assumptions on \( a, b \) and the data \( \varphi, f, h \), for each \((s, x) \in Q_T\) there exists a unique solution \((Y^{s,x}, Z^{s,x}, K^{s,x})\) of reflected backward stochastic differential equation with forward driving process \( X^{s,x} \), terminal condition \( \varphi(X^{s,x}) \), coefficient \( f \) and obstacle \( h(\cdot, X^{s,x}) \) (RBSDE(\( \varphi, f, h \)) for short), and moreover, \( u \) defined by the formula

\[
u(s, x) = Y^s_{s,x}, \quad (s, x) \in Q_T\]

is a unique viscosity solution of (1.3) in the class of functions satisfying the polynomial growth condition. In the present paper we give a representation similar to that proved in [11] for weak solutions of (1.4) with \( A_t \) defined by (1.2).

In the paper we assume that

(H1) \( \varphi \in L^2_{\text{loc}}(\mathbb{R}^d), \ h \in L^2_{\text{loc}}(Q_T), \)

(H2) \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is a measurable function satisfying the following conditions:

a) there is \( L > 0 \) such that \( |f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|) \)
for all \((t, x) \in [0, T] \times \mathbb{R}^d, \ y_1, y_2 \in \mathbb{R} \) and \( z_1, z_2 \in \mathbb{R}^d, \)

b) there exist \( M > 0, g \in L^2_{\text{loc}}(Q_T) \) such that \( |f(t, x, y, z)| \leq g(t, x) + M(|y| + |z|) \)
for all \((t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d, \)
(H3) \( \varphi(x) \geq h(T,x) \) for all \( x \in \mathbb{R}^d, \ h \in C(Q_T) \)

(definitions of various function spaces used in the paper are given at the end of the section).

We prove that under (1.1) and (H1)–(H3) the obstacle problem (1.4) has at most one solution such that \( u \in C(Q_T) \cap W_{2,q}^{0,1}(Q_T) \) for some \( q \) of the form \( q(x) = (1 + |x|^2)^{-\alpha}, x \in \mathbb{R}^d, \alpha \geq 0 \). From our existence results it follows in particular that if, in addition, \( \varphi \in L_{2,q}(\mathbb{R}^d), \ g \in L_{p,q}(Q_T) \cap L_{p,q}(Q_T), \ h \in C(Q_T) \cap L_{p,q}(Q_T) \) for some \( q \) as above and \( p, q \in (2, \infty) \) such that \( (2/q) + (d/p) < 1 \), and \( h \) satisfies the polynomial growth condition, then (1.4) has a solution \( (u, \mu) \) such that \( u \in C([0,T) \times \mathbb{R}^d) \cap W_{2,q}^{0,1}(Q_T) \).

Secondly, for each \((s,x) \in [0,T) \times \mathbb{R}^d\) we have

\[
(u(t,X_t), \sigma \nabla u(t,X_t)) = (Y_t^{s,x}, Z_t^{s,x}), \quad t \in [s,T], \quad P_{s,x}-a.s.,
\]  

(1.5)

where \((X,P_{s,x})\) is a Markov process associated with \( A_t \) (see [22, 24]) and \( Y_t^{s,x}, Z_t^{s,x} \) are the first two components of a solution \((Y_t^{s,x}, Z_t^{s,x}, K_t^{s,x})\) of RBSDE\((\varphi,f,h)\) with forward driving process \( X \). In particular, it follows that

\[
u(s,x) = Y_s^{s,x}, \quad (s,x) \in Q_T,
\]  

(1.6)

which may be viewed as a generalization of the Feynman-Kac formula. We show also that

\[
E_{s,x} \int_s^T \xi(t,X_t) \, dK_t^{s,x} = E_{s,x} \int_s^T \xi(t,y) \, p(s,x,t,y) \, d\mu(t,y)
\]  

(1.7)

for all \( \xi \in C_b(Q_T) \), where \( p \) stands for the transition density function of \((X,P_{s,x})\) (or, equivalently, \( p \) is the fundamental solution for \( A_t \)), which provides an additional information on the process \( K_t^{s,x} \) and solution \((u, \mu)\) of (1.4). For instance, it follows from (1.7) that in the linear case the solution of (1.3) admits the representation

\[
u(s,x) = \int_{\mathbb{R}^d} \varphi(y) \, p(s,x,T,y) \, dy + \int_{Q_T} f(t,y) \, p(s,x,t,y) \, dy
\]

\[
+ \int_{Q_T} p(s,x,t,y) \, d\mu(t,y), \quad (s,x) \in [0,T) \times \mathbb{R}^d,
\]

which, up to our knowledge, is new (for parabolic problems). Moreover, we show that \( \mu \) is absolutely continuous with respect to the Lebesgue measure \( \lambda \) and \( d\mu = r \, d\lambda \) if and only if

\[
K_t^{s,x} = \int_s^t r(\theta,X_\theta^{s,x}) \, d\theta, \quad t \in [s,T].
\]

Let us remark also that the first component \( u \) of a solution of (1.4) coincides with the solution of (1.3) in the variational sense.

Our conditions on \( \varphi, g \) and \( h \) are similar to that used in the theory of variational inequalities and seems to be close to the best possible. As for \( g \), in fact we prove existence and uniqueness of solutions of (1.4) and the representation (1.5) under the assumption that \( g \in L_{2,q}(Q_T) \) and

\[
E_{s,x} \int_s^T |g(t,X_t)|^2 \, dt
\]  

(1.8)

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is bounded uniformly in \((s,x) \in K\) for every compact subset \(K\) of \([0,T] \times \mathbb{R}^d\). We show also that if \(\varphi \in \mathbb{L}_{2,q}(\mathbb{R}^d), g \in \mathbb{L}_{2,q}(Q_T)\) and \(h \in \mathbb{C}(Q_T)\) satisfies the polynomial growth condition, then there is a version of the minimal weak solution of (1.3) in the variational sense such that if (1.8) is finite for some fixed \((s,x) \in [0,T] \times \mathbb{R}^d\), there exists a unique solution \((Y^{s,x}, Z^{s,x}, K^{s,x})\) of RBSDE\((\varphi, f, h)\) and (1.5) holds true. Thus, since (1.8) is finite for a.e. \((s,x) \in [0,T] \times \mathbb{R}^d\) if \(g \in \mathbb{L}_{2,q}(Q_T)\), (1.5) holds for a.e. \((s,x) \in [0,T] \times \mathbb{R}^d\) if \(g \in \mathbb{L}_{2,q}(Q_T)\). What is more important, it follows from our result that for each \((s,x)\) such that (1.8) is finite we get a probabilistic formula (1.6) for the minimal weak solution of the variational inequality associated with (1.3).

In case \(q = 1\) existence of a solution of (1.4) and representation (1.5) is proved by the method of stochastic penalization used earlier in [11]. For \(q < 1\) in proofs of these results we use ideas from [23]. In both cases from our proofs it follows that if \((u, \mu)\) is a solution of (1.4), \(u_n\) is a solution of the Cauchy problem

\[
\left(\frac{\partial}{\partial t} + A_t\right)u_n = -f_n - n(u_n - h)^-, \quad u_n(T) = \varphi
\]

and \(\nu_n\) is a measure on \(Q_T\) such that \(d\nu_n = n(u_n - h)^- d\lambda\) then \(u_n \rightharpoonup u\) uniformly in compact subsets of \([0,T] \times \mathbb{R}^d\) and in \(\mathbb{W}^{0,1}_2(Q_T) \cap \mathbb{C}([0,T], \mathbb{L}_{2,q}(Q_T))\) if \(q < 1\), and locally in the latter space if \(q = 1\). In particular, differently from the theory of variational inequalities, we obtain strong convergence in \(\mathbb{L}_{2,q}(Q_T)\) of gradients of \(u_n\)’s to the gradient of \(u\). Moreover, from the proofs it follows that \(\{\mu_n\}\) converges weakly to \(\mu\) and strongly in the space dual to \(\mathbb{W}^{0,1}_2(Q_T)\), and for each \((s,x) \in [0,T] \times \mathbb{R}^d\) the measures \(\nu_n\) defined by the relation \(d\nu_n/d\mu_n = p(s,x,\cdot,\cdot)\) converge weakly to the measure \(\nu\) such that \(d\nu/d\mu = p(s,x,\cdot,\cdot)\). These results allow us to deduce some properties of \(\mu\) from properties of the sequence \(\{\mu_n\}\).

In the paper we will use the following notation.

\(Q_T = [0,T] \times \mathbb{R}^d, \hat{Q}_T = (0,T) \times \mathbb{R}^d\). For \(E \subset Q_T\) we write \(E_t = \{x \in \mathbb{R}^d : (t,x) \in E\}\). \(B(0,r) = \{x \in \mathbb{R}^d : |x| < r\}, x^+ = \max(x,0), x^- = \max(-x,0), \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)\). By \(\lambda\) we denote the Lebesgue measure.

\(\mathbb{L}_p(\mathbb{R}^d)\) is the Banach space of measurable function \(u\) on \(\mathbb{R}^d\) having the finite norm \(\|u\|_p = \left(\int_{\mathbb{R}^d} |u(x)|^p \, dx\right)^{1/p}\). \(\mathbb{L}_{p,q}(Q_T)\) is the Banach space of measurable functions on \(Q_T\) having the finite norm \(\|u\|_{p,q,T} = \left(\int_0^T \left(\int_{\mathbb{R}^d} |u(t,x)|^p \, dx\right)^{p/q} \, dt\right)^{1/q}\), \(\mathbb{L}_p(Q_T) = \mathbb{L}_{p,p,T}\).

Let \(p\) be a positive function on \(\mathbb{R}^d\). By \(\mathbb{L}_{p,q}(\mathbb{R}^d)\) (\(\mathbb{L}_{p,q}(Q_T)\)) we denote the space of functions \(u\) such that \(u\varphi \in \mathbb{L}_p(\mathbb{R}^d)\) \((u\varphi \in \mathbb{L}_{p,q}(Q_T))\) equipped with the norm \(\|u\|_{p,q} = \|u\varphi\|_p (\|u\varphi\|_{p,q,T} = \|u\varphi\|_{p,q,T})\). We write \(K \subset \subset X\) if \(K\) is compact subset of \(X\). \(\mathbb{L}_p^\infty(\mathbb{R}^d) = \bigcap_{K \subset \subset \mathbb{R}^d} \mathbb{L}_p(K)\). By \(\langle \cdot, \cdot \rangle_2\) we denote the usual inner product in \(\mathbb{L}_2(\mathbb{R}^d)\) and by \(\langle \cdot, \cdot \rangle_{2,q}\) the inner product in \(\mathbb{L}_{2,q}(\mathbb{R}^d)\).

\(\mathbb{W}^{0,1}_2(\mathbb{R}^d)\) (\(\mathbb{W}^{0,1}_2(Q_T)\)) is the Banach space consisting of all elements \(u\) of \(\mathbb{L}_{2,q}(\mathbb{R}^d)\) (\(\mathbb{L}_{2,q}(Q_T)\)) having generalized derivatives \(\frac{\partial u}{\partial x_i}, i = 1, \ldots, d\), in \(\mathbb{L}_{2,q}(\mathbb{R}^d)\) (\(\mathbb{L}_{2,q}(Q_T)\)). If \(p \equiv 1\) then we denote the spaces by \(\mathbb{W}^{1}_2(\mathbb{R}^d)\) and \(\mathbb{W}^{0,1}_2(Q_T)\). \(\mathbb{W}^{1,1}_2(\mathbb{R}^d)\) is the subspace of \(\mathbb{W}^{0,1}_2(Q_T)\) consisting of all elements \(u\) having generalized derivatives \(\frac{\partial u}{\partial t}\) in \(\mathbb{L}_{2,q}(Q_T)\). \(\mathbb{W}^{1,1}_2(Q_T)\) is the set of all function from \(\mathbb{W}^{1,1}_2(Q_T)\) with compact support in \(Q_T\). \(\mathbb{W}_q = \{u \in \mathbb{L}_2([0,T], \mathbb{W}^{1,1}_2(\mathbb{R}^d)); \frac{\partial u}{\partial t} \in \mathbb{L}_2([0,T], \mathbb{W}^{-1,1}_2(\mathbb{R}^d))\}\), where \(\mathbb{W}^{-1,1}_2(\mathbb{R}^d)\) is the dual space to \(\mathbb{W}^{1,1}_2(\mathbb{R}^d)\) (see [17, 18] for details); if \(q \equiv 1\) we write \(\mathbb{W}\) instead of \(\mathbb{W}_q\).
By $C_0(Q_T)$ ($C_0(\mathbb{R}^d)$) we denote the space of all continuous function with compact support on $Q_T$ ($\mathbb{R}^d$) and by $C_0^+$ ($C_0^+(\mathbb{R}^d)$) the set of all positive functions from $C_0(Q_T)$ ($C_0(\mathbb{R}^d)$).

In what follows, by $C$ (or $c$) we will denote a general constant which may vary from line to line but depends only on fixed parameters.

## 2 Preliminary results

### 2.1 Symmetric diffusions and BSDEs

Let $\Omega = C([0,T],\mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions on $[0,T]$ equipped with the topology of uniform convergence and let $X$ be a canonical process on $\Omega$. It is known that given an operator $A_t$ defined by (1.2) with $a$ satisfying (1.1) one can construct a weak fundamental solution $p(s,x,t,y)$ for $A_t$ and then a Markov family $\mathbb{X} = \{(X,P_{s,x}); (s,x) \in [0,T) \times \mathbb{R}^d\}$ for which $p$ is the transition density function, i.e.

$$ P_{s,x}(X_t = x; 0 \leq t \leq s) = 1, \quad P_{s,x}(X_t \in \Gamma) = \int_{\Gamma} p(s,x,t,y) \, dy, \quad t \in (s,T] $$

for any $\Gamma$ in a Borel $\sigma$-field $\mathcal{B}$ of $\mathbb{R}^d$ (see [22, 24]).

**Theorem 2.1** For each $(s,x) \in [0,T) \times \mathbb{R}^d$, if $[0,T) \times \mathbb{R}^d \ni (s_n,x_n) \to (s,x)$ then $P_{s_n,x_n} \Rightarrow P_{s,x}$ weakly in $C([0,T]; \mathbb{R}^d)$.

**Proof.** Follows from the fact that $\mathbb{X}$ generates a strongly Feller continuous Markov time-inhomogeneous semigroup on $L_2(\mathbb{R}^d)$ (see [22]). $\square$

In what follows by $W$ we denote the space of all measurable functions $\varrho : \mathbb{R}^d \to \mathbb{R}$ such that $\varrho(x) = (1 + |x|^2)^{-\alpha}$, $x \in \mathbb{R}^d$, for some $\alpha \geq 0$.

Let $E_{s,x}$ denote expectation with respect to $P_{s,x}$.

**Theorem 2.2** Let $\varrho \in W$. Then there exist $0 < c \leq C$ depending only on $\lambda, \Lambda$ and $\varrho$ such that

(i) for any $\varphi \in L_{1,\varrho}(\mathbb{R}^d)$ and $0 \leq s \leq t < T$,

$$ c \int_{\mathbb{R}^d} |\varphi(x)| \varrho(x) \, dx \leq \int_{\mathbb{R}^d} E_{s,x} |\varphi(X_t)| \varrho(x) \, dx \leq C \int_{\mathbb{R}^d} |\varphi(x)| \varrho(x) \, dx, $$

(ii) for any $\psi \in L_{1,\varrho}(Q_T)$,

$$ c \int_{t}^{T} \int_{\mathbb{R}^d} |\psi(\theta,x)| \varrho(x) \, d\theta \, dx \leq \int_{t}^{T} \int_{\mathbb{R}^d} E_{s,x} |\psi(\theta,X_\theta)| \varrho(x) \, d\theta \, dx \leq C \int_{t}^{T} \int_{\mathbb{R}^d} |\psi(\theta,x)| \varrho(x) \, d\theta \, dx, \quad t \in [s,T]. $$

**Proof.** Both statements follow from [2, Proposition 5.1, Appendix], because by Aronson’s estimates there exist $0 < c_1 \leq c_2$ depending only on $\lambda, \Lambda$ such that

$$ c_1 \int_{\mathbb{R}^d} E |\varphi(x + X_{c_1(t-s)})| \varrho(x) \, dx \leq \int_{\mathbb{R}^d} E_{s,x} |\varphi(X_t)| \varrho(x) \, dx \leq c_2 \int_{\mathbb{R}^d} E |\varphi(x + X_{c_2(t-s)})| \varrho(x) \, dx $$. 

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where $E$ denotes expectation with respect to the standard Wiener measure on $\Omega$. □

Set $\mathcal{F}_t^x = \sigma(X_u, u \in [s, t])$ and define $\mathcal{G}$ as the completion of $\mathcal{F}_T^x$ with respect to the family $\mathcal{P} = \{P_{s, \mu} : \mu$ is a probability measure on $\mathcal{B}\}$, where $P_{s, \mu}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \mu(dx)$, and define $\mathcal{G}_t^x$ as the completion of $\mathcal{F}_t^x$ in $\mathcal{G}$ with respect to $\mathcal{P}$.

From [23, Theorem 2.1] it follows that there exist a martingale additive functional locally of finite energy $M = \{M_{s,t} : 0 \leq s \leq t \leq T\}$ of $\mathbb{X}$ and a continuous additive functional locally of zero energy $A = \{A_{s,t} : 0 \leq s \leq t \leq T\}$ of $\mathbb{X}$ such that

$$X_t - X_s = M_{s,t} + A_{s,t}, \quad t \in [s, T], \quad P_{s,x}-a.s. \quad (2.1)$$

for each $(s, x) \in [0, T) \times \mathbb{R}^d$. Moreover, the above decomposition is unique and for each $(s, x) \in [0, T) \times \mathbb{R}^d$ the process $M_{s,t}$ is a Wiener process. Noticing that $M_{s,t} = \mathbb{E}(M_{s,t})$, we now formulate definitions of backward stochastic differential equation (BSDE) and reflected BSDE (RBSDE) associated with $\mathbb{X}$.

Definition A pair $(Y^{s,x}, Z^{s,x})$ of processes on $[s, T]$ is a solution of BSDE($\varphi, f$) (associated with $(X, P_{s,x})$) if

(i) $Y^{s,x}, Z^{s,x}$ are $\{\mathcal{G}_t^x\}$-adapted,

(ii) $Y_{t}^{s,x} = \varphi(X_T) + \int_{t}^{T} f(\theta, X_{\theta}, Y_{\theta}^{s,x}, Z_{\theta}^{s,x}) d\theta - \int_{t}^{T} Z_{\theta}^{s,x} dB_{s,\theta}, t \in [s, T], \ P_{s,x}-a.s.,$

(iii) $E_{s,x} \int_{s}^{T} |Z_{t}^{s,x}|^2 \, dt < \infty, \ E_{s,x} \sup_{s \leq t \leq T} |Y_{t}^{s,x}|^2 < \infty.$

Definition A triple $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of processes on $[s, T]$ is a solution of RBSDE($\varphi, f, h$) (associated with $(X, P_{s,x})$) if

(i) $Y^{s,x}, Z^{s,x}, K^{s,x}$ are $\{\mathcal{G}_t^x\}$-adapted,

(ii) $Y_{t}^{s,x} \geq h(t, X_{t}), t \in [s, T], \ P_{s,x}-a.s.,$

(iii) $Y_{t}^{s,x} = \varphi(X_T) + \int_{t}^{T} f(\theta, X_{\theta}, Y_{\theta}^{s,x}, Z_{\theta}^{s,x}) d\theta + K_{t}^{s,x} - K_{s,x} - \int_{t}^{T} Z_{\theta}^{s,x} dB_{s,\theta}, t \in [s, T], \ P_{s,x}-a.s.,$

(iv) $E_{s,x} \int_{s}^{T} |Z_{t}^{s,x}|^2 \, dt < \infty, \ E_{s,x} \sup_{s \leq t \leq T} |Y_{t}^{s,x}|^2 < \infty,$

(v) $K^{s,x}$ is a continuous increasing process such that $K_{s,x}^{s,x} = 0, \ E_{s,x} |K_{T}^{s,x}|^2 < \infty$ and $\int_{s}^{T} (Y_{t}^{s,x} - h(t, X_{t})) dK_{t}^{s,x} = 0, \ P_{s,x}-a.s.
Observe that \( \{G_t^s\} \) need not coincide with the natural filtration generated by the Wiener process \( B_s \). Consequently, due to lack of the representation theorem for \( B_s \), existence of solutions of BSDE(\( \varphi, f \)) does not follow from known results for „usual” BSDEs.

Existence and uniqueness of solutions of BSDE(\( \varphi, f \)) for each starting point \((s, x) \in [0, T) \times \mathbb{R}^d \) was proved in [23] under the assumption that \( \varphi \in \mathbb{L}_2(\mathbb{R}^d) \) and \( f \) satisfies (H2) with \( g \in L_{p,q}(Q_T) \) for some \( p, q \) such that

\[
p, q \in (2, \infty], \quad \frac{2}{q} + \frac{d}{p} < 1.
\] (2.2)

(see also [3] for existence results for quasi-every starting point \( x \) proved in the case where the forward diffusion corresponds to symmetric divergence form operator with time-independent coefficients but not necessarily uniformly elliptic).

Let us recall that \( u \) is said to be a weak solutions of the Cauchy problem

\[
\frac{\partial u}{\partial t} + A_t u = - f_u, \quad u(T) = \varphi \quad \text{(PDE(\( \varphi, f \)) for short)}
\] (2.3)

if \( u \in W_{2,1}^0(Q_T) \cap C([0, T], L_{2,1}^1(\mathbb{R}^d)) \) and for any \( \eta \in W_{2,0}^1(Q_T) \),

\[
\int_t^T \langle u(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 ds + \frac{1}{2} \int_t^T \langle a(s) \nabla u(s), \nabla \eta(s) \rangle_2 ds = \int_t^T \langle f_u(s), \eta(s) \rangle_2 ds + \langle \varphi, \eta(T) \rangle_2 - \langle u(t), \eta(t) \rangle_2, \quad t \in [0, T].
\]

It is well known that if \( \varphi \in \mathbb{L}_2(\mathbb{R}^d) \), \( g \in \mathbb{L}_2(Q_T) \) then there exists a unique weak solution of PDE(\( \varphi, f \)) (see, e.g. [15]).

The next theorem strengthens slightly results proved in [23].

**Proposition 2.3** Assume that (H1)-(H3) are satisfied with \( \varphi \in \mathbb{L}_2(\mathbb{R}^d) \), \( g \in \mathbb{L}_2(Q_T) \).

(i) If

\[
\forall K \subset \subset [0, T) \times \mathbb{R}^d \quad \sup_{(s, x) \in K} E_{s,x} \int_s^T |g(t, X_t)|^2 dt < \infty \quad \text{(2.4)}
\]

then there exists a unique weak solution \( u \in W_{2,1}^0(Q_T) \cap C([0, T) \times \mathbb{R}^d) \) of PDE(\( \varphi, f \)) and for each \((s, x) \in [0, T) \times \mathbb{R}^d \) the pair

\[
(Y_{t}^{s,x}, Z_{t}^{s,x}) = (u(t, X_t), \sigma \nabla u(t, X_t)), \quad t \in [s, T]
\] (2.5)

is a unique solution of BSDE(\( \varphi, f \)).

(ii) There exists a version \( u \) of a weak solution of PDE(\( \varphi, f \)) such that if

\[
E_{s,x} \int_s^T |g(t, X_t)|^2 dt < \infty \quad \text{(2.6)}
\]

for some \((s, x) \in [0, T) \times \mathbb{R}^d \) then the pair (2.5) is a unique solution of BSDE(\( \varphi, f \)).
Proof. Let \( \bar{u} \in W^{0,1}_2(Q_T) \) be a weak solution of the problem (2.3) and let

\[
\|\bar{u}\|_{W^2(x,s,T)} = E_{s,x} \int_s^T \left( |\bar{u}(t, X_t)|^2 + |\nabla \bar{u}(t, X_t)|^2 \right) dt.
\]

From the proof of [23, Theorem 6.1] it follows that under (2.4) for every \( K \subset \subset [0, T) \times \mathbb{R}^d \),

\[
\sup_{(s,x) \in K} \|\bar{u}\|_{W^2(x,s,T)} < \infty. \tag{2.7}
\]

For \( n, m \in \mathbb{N} \) let \( u_{nm} \in W^{0,1}_2(Q_T) \cap C([0, T) \times \mathbb{R}^d) \) be a weak solution of the Cauchy problem

\[
\left( \frac{\partial}{\partial t} + A_t \right) u_{nm} = f_+^n \wedge m - f_-^n \wedge n, \quad u_{nm}(T) = \varphi.
\]

By [23, Proposition 5.1] the pair \( (u_{nm}(t, X_t), \sigma \nabla u_{nm}(t, X_t)), t \in [s, T], \) is a solution of BSDE(\( \varphi, f_+^n \wedge m - f_-^n \wedge n \)). Using Itô’s formula and performing standard calculations we conclude that there is \( C > 0 \) not depending on \( n, m \) such that

\[
E_{s,x} \sup_{s \leq t \leq T} |u_{nm}(t, X_t)|^2 + E_{s,x} \int_s^T |\sigma \nabla u_{nm}(t, X_t)|^2 dt
\]

\[
\leq C \left( E_{s,x} |\varphi(X_T)|^2 + E_{s,x} \int_s^T |g(t, X_t)|^2 dt + \|\bar{u}\|^2_{W^2(x,s,T)} \right). \tag{2.8}
\]

From comparison results (see [4, Theorem 4.1.4]) and the fact that \( u_{nm} \) are continuous it follows that for any fixed \( n \) the sequence \( \{u_{nm}\}_{m \in \mathbb{N}} \) is increasing. Hence, for each \( n \in \mathbb{N} \) there is \( u_n \) such that \( u_{nm} \uparrow u_n \) as \( m \to \infty \). Moreover, by well known convergence theorems (see [15, Theorem 3.4.5]), \( u_{nm} \to u_n \) in \( W^{0,1}_2(Q_T) \) and \( u_n \) is a weak solution of the problem

\[
\left( \frac{\partial}{\partial t} + A_t \right) u_n = f_+^n - f_-^n \wedge n, \quad u_n(T) = \varphi.
\]

If (2.4) is satisfied, then from (2.7), (2.8) and Nash’s continuity theorem (see [1]) it follows that \( \{u_{nm}\}_{m \in \mathbb{N}} \) is equicontinuous in every compact subset of \( [0, T) \times \mathbb{R}^d \). Therefore the functions \( u_n \) are continuous on \( [s, T) \times \mathbb{R}^d \). Using once again Itô’s formula we deduce that for any \( k, l, n \in \mathbb{N} \),

\[
E_{s,x} |(u_{nk} - u_{nl})(t, X_t)|^2 + E_{s,x} \int_s^T |\sigma \nabla (u_{nk} - u_{nl})(t, X_t)|^2 dt
\]

\[
\leq C \left( E_{s,x} \int_s^T |(f_+^k \wedge k - f_+^l \wedge l)(t, X_t)|^2 dt \right)^{1/2}
\]

\[
\times \left( E_{s,x} \int_s^T |(u_{nk} - u_{nl})(t, X_t)|^2 dt \right)^{1/2} \tag{2.9}
\]

for all \( t \in [s, T] \). By (H2) and (2.7), (2.8) the first term on the right-hand side of (2.9) is bounded uniformly in \( k, l \). Due to (2.7), (2.8) and the estimate \( |u_{nk}| \leq |u_{n1}| + |u_n| \) we may apply the Lebesgue dominated convergence theorem to conclude that the second term converges to zero as \( k, l \to 0 \). By the above,

\[
E_{s,x} |(u_{nm} - u_n)(t, X_t)|^2 + E_{s,x} \int_s^T |\sigma \nabla (u_{nm} - u_n)(t, X_t)|^2 dt \to 0
\]
as \( m \to \infty \). Using this it is easy to see that the pair \((u_n(t, X_t), \sigma \nabla u_n(t, X_t))\), \(t \in [s, T]\) is a solution of BSDE\((\varphi, f_n^+ - f_n^- \wedge n)\). Therefore, (2.8) holds for \(u_{nm}\) replaced by \(u_n\) and (2.9) holds for \(u_{nk}, u_{nl}\) replaced by \(u_k, u_l\) and \(f_k^+\) replaced by \(f_l^+\). Using once again (2.7) and Nash’s continuity theorem we conclude that \(u_n\) is equicontinuous in every compact subset of \([0, T] \times \mathbb{R}^d\). Therefore, by comparison results, \(u_n\) is decreasing there is \(u \in C([0, T] \times \mathbb{R}^d)\) such that \(u_n \downarrow u\). Since \(f_k^+ - f_l^+ \wedge n \to f_u \) \(d\mathbb{P}\)-a.s. in \(L^2(Q_T)\), it follows that \(u\) is a weak solution of the Cauchy problem \((\frac{d}{dt} + A_t)u = f_u, \ u(T) = \varphi\).

By uniqueness, \(u\) is a version of \(\bar{u}\). Finally, using the mentioned above analogues of (2.8), (2.9) we prove in much the same way as above that the pair (2.5) is a solution of BSDE\((\varphi, f)\), which completes the proof of (i).

To prove (ii), we first observe that using continuity of \(u_{nm}\) and the fact that \(\{u_{nm}\}\) is decreasing for every fixed \(n\) and increasing for every fixed \(m\) we can still show that \(\{u_n\}\) is decreasing. Therefore \(\{u_n\}\) converges pointwise to some version \(u\) of \(\bar{u}\). If (2.6) is satisfied for some \((s, x) \in [0, T] \times \mathbb{R}^d\) then \(\|\bar{u}\|_{W_2(x,s,T)} < \infty\). Therefore we can use (2.8), (2.9) to conclude as before that \((u(t, X_t), \sigma \nabla u(t, X_t)), t \in [s, T]\), is a solution of BSDE\((\varphi, f)\) associated with \((X, P_{s,x})\).

**Theorem 2.4** Assume that \((H1)-(H3)\) are satisfied with \(\varphi \in L_2(\mathbb{R}^d), h, g \in L_2(Q_T)\) and

\[
E_{s,x} \sup_{s \leq t \leq T} |h^+(t, X_t)|^2 + E_{s,x} \int_s^T |g(t, X_t)|^2 \, dt < \infty \tag{2.10}
\]

for some \((s, x) \in [0, T] \times \mathbb{R}^d\). Then the RBSDE\((\varphi, f, h)\) associated with \((X, P_{s,x})\) has a unique solution \((Y_{s,x}^-, Z_{s,x}^-, K_{s,x}^-)\). Moreover, if the pair \((Y_{s,x}^+, Z_{s,x}^+, K_{s,x}^+)\), \(n \in \mathbb{N}\), is a solution of BSDE\((\varphi, f + n(y - h)^+)\), then

\[
E_{s,x} \sup_{s \leq t \leq T} |Y_{s,x}^+ - Y_{s,x}^-|^2 + E_{s,x} \int_s^T |Z_{s,x}^+ - Z_{s,x}^-|^2 \, dt
+ E_{s,x} \sup_{s \leq t \leq T} |K_{s,x}^+ - K_{s,x}^-|^2 \to 0, \tag{2.11}
\]

where

\[
K_{s,x}^+ = \int_s^t n(Y_{s,x}^+ - h(\theta, X_\theta)) \, d\theta, \quad t \in [s, T], \quad P_{s,x}-\text{a.s.}
\]

Finally, there is \(C > 0\) depending neither on \(n, m \in \mathbb{N}\) nor on \(s, x\) such that

\[
E_{s,x} \sup_{s \leq t \leq T} |Y_{s,x}^+|^2 + E_{s,x} \int_s^T |Z_{s,x}^+|^2 \, dt + E_{s,x} |K_{s,x}^+|^2
\leq C \left( E_{s,x} |\varphi(X_T)|^2 + E_{s,x} \sup_{s \leq t \leq T} |h^+(t, X_t)|^2 + E_{s,x} \int_s^T |g(t, X_t)|^2 \, dt \right) \tag{2.12}
\]

and

\[
E_{s,x} \sup_{s \leq t \leq T-2\delta} |Y_{s,x}^+ - Y_{s,x}^-|^2
\leq C \left( E_{s,x} |Y_{T-\delta}^+ - Y_{T-\delta}^-|^2 + E_{s,x} \int_s^{T-\delta} (Y_{s,x}^+ - h(t, X_t))^- \, dK_{s,x}^- + E_{s,x} \int_s^{T-\delta} (Y_{s,x}^+ - h(t, X_t))^- \, dK_{s,x}^+ \right) \tag{2.13}
\]
for every $\delta \in [0, T - s]$.

**Proof.** From Proposition 2.3 we know that for each $n \in \mathbb{N}$ there exists a unique solution of $\text{BSDE}(\varphi, f + n(y - h^+))$. To prove (2.11)–(2.13) it suffices to repeat step by step arguments from the proofs of corresponding results in [11]. 

Let us remark that both terms in (2.10) are bounded uniformly in $(s, x) \in K$ for every $K \subset \subset [0, T) \times \mathbb{R}^d$ if $h, g$ satisfy the polynomial growth condition or $h$ satisfies the polynomial growth condition and $g \in L_{p,q,\rho}(Q_T)$ with $p, q$ satisfying (2.2) and $\rho \in W$.

The first statement is an immediate consequence of Proposition 3.2 proved in Section 3. Sufficiency of the second condition on $g$ follows from Hölder’s inequality and upper Aronson’s estimate on the transition density $p$ (see [1]).

Observe also that if $g \in L_{2,\rho}(Q_T)$ then (2.6) holds for a.e. $(s, x) \in [0, T) \times \mathbb{R}^d$ because by Theorem 2.2,

\[
\int_0^T \left( \int_{\mathbb{R}^d} (E_{s,x} \int_s^T |g(t, X_t)|^2 dt) \right) ds \leq C\|g\|_{L_{2,\rho}}^2.
\]

**Lemma 2.5** If $(Y_{t}^{s,x,i}, Z_{t}^{s,x,i}, K_{t}^{s,x,i}), i = 1, 2$, is a solution of $\text{RBSDE}(\xi, f, h^i)$ then for every $\delta \in [0, T - s]$,

\[
E_{s,x} \sup_{s \leq t \leq T - \delta} |Y_{t}^{s,x,1} - Y_{t}^{s,x,2}|^2 + E_{s,x} \int_s^{T - \delta} |Z_{t}^{s,x,1} - Z_{t}^{s,x,2}|^2 dt \\
+ E_{s,x} \sup_{s \leq t \leq T - \delta} |K_{t}^{s,x,1} - K_{t}^{s,x,2}|^2 \\
\leq C \left( E_{s,x} \sup_{s \leq t \leq T - \delta} |h^1(t, X_t) - h^2(t, X_t)|^2 + E_{s,x} \|Y_{T - \delta}^{s,x,1} - Y_{T - \delta}^{s,x,2}|^2 \right).
\]

**Proof.** See [11]. □

### 2.2 Obstacle problem

In this subsection we formulate precisely our definition of solutions of the obstacle problem and compare it to the well known definitions of solutions in the sense of variational inequalities. We prove also a priori estimates for solutions and some additional technical results which will be needed in the next section.

In the paper we will use the following notion of the capacity of $E \subset \subset \bar{Q}_T$:

\[
cap_{\bar{Q}_T}(E) = \inf \left\{ \int_{\bar{Q}_T} \left( |\frac{\partial \eta}{\partial t}(t, x)|^2 + |\nabla \eta(t, x)|^2 \right) dt \ dx : \eta \in C_{\infty}^0(\bar{Q}_T), \eta \geq 1_E \right\}.
\]

In the standard way we can extend the above capacity to external capacity for arbitrary subset $E \subset \bar{Q}_T$. It is known that $\cap_{\bar{Q}_T}$ is the Choquet capacity (see Chapter 2 in [13]).

In the remainder of the paper the abbreviation “q.e.” means “except for a set of capacity zero”.

Throughout the subsection we assume that $g \in W$ and (H1)–(H3) are satisfied.

**Definition** We say that a pair $(u, \mu)$, where $\mu$ is a Radon measure on $Q_T$ and $u : Q_T \to \mathbb{R}$ is a measurable function defined up to the sets of $\mu$-measure zero, is a weak solution of the obstacle problem (1.3) with data $\varphi, f, h$ (OP$(\varphi, f, h)$ for short) if
(a) \( u \in W^{0,1}_{2,\text{loc}}(Q_T) \cap C([0, T], L^{\text{loc}}_2(\mathbb{R}^d)) \) and for any \( \eta \in W^{1,1}_{2,0}(Q_T) \),

\[
\int_t^T \langle u(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 ds + \frac{1}{2} \int_t^T \langle a(s) \nabla u(s), \nabla \eta(s) \rangle_2 ds = \int_t^T \langle f_u(s), \eta(s) \rangle_2 ds \\
+ \int_t^T \int_{\mathbb{R}^d} \eta \, d\mu + \langle \varphi, \eta(T) \rangle_2 - \langle u(t), \eta(t) \rangle_2, \quad t \in [0, T], \tag{2.14}
\]

(b) \( u \geq h \) on \( Q_T \),

(c) \( \int_{Q_T} (u-h) \, d\mu = 0 \) for all \( \xi \in C^+_0(Q_T) \),

(d) \( \mu(\{t\} \times \mathbb{R}^d) = 0 \) for every \( t \in [0, T] \).

Some comments on the above definition are in order. In the next lemma we will show that (a) forces \( \mu_{|Q_T} \ll \text{cap}_{Q_T} \), which together with (d) and the well known fact that elements of \( W^{1,1}_{2,\text{loc}}(Q_T) \) are defined up to subsets of \( \bar{Q}_T \) of zero capacity (see, e.g., [6, 12, 21]) ensures that the integral \( \int_{\mathbb{R}^d} \eta \, d\mu \) is correctly defined. We shall see that (a) implies that \( \mu(\{s\} \times \mathbb{R}^d) = 0 \) for \( s \in (0, T) \), so instead of (d) we could impose the condition \( \mu(\{0, T\} \times \mathbb{R}^d) = 0 \). The condition \( \mu(\{T\} \times \mathbb{R}^d) = 0 \) is also necessary for the terminal condition \( u(T) = \varphi(T) \) to hold, and a fortiori, for uniqueness of the solution of the obstacle problem. Notice also that the integral in condition (c) is well defined because \( u - h \geq 0 \).

Let us remark that our definition of the obstacle problem is similar to that in stochastic case (condition (d) may be viewed as an analytical counterpart to continuity of the process \( K^{s,x} \)). Notice also that if the obstacle \( h \) is constant, then the above definition coincides with the one adopted in [20] (in [20] exclusively constant obstacles are considered; this implies that \( \mu \) is absolutely continuous with respect to the Lebesgue measure, so no problems arises with the definition of an obstacle problem).

**Lemma 2.6** If \( u \in W^{0,1}_{2,\text{loc}}(Q_T) \cap C([0, T], L^{\text{loc}}_2(\mathbb{R}^d)) \) and the pair \((u, \mu)\) satisfies

\[
\int_0^T \langle u(t), \frac{\partial \eta}{\partial t}(t) \rangle_2 dt + \frac{1}{2} \int_0^T \langle a(t) \nabla u(t), \nabla \eta(t) \rangle_2 dt \\
= \int_0^T \langle f_u(t), \eta(t) \rangle_2 dt + \int_{Q_T} \eta \, d\mu + \langle \varphi, \eta(T) \rangle_2 \tag{2.15}
\]

for every \( \eta \in W^{1,1}_{2,\text{loc}}(Q_T) \cap C_0(Q_T) \) such that \( \eta(0) \equiv 0 \), then

(i) \( \mu_{|Q_T} \ll \text{cap}_{Q_T} \),

(ii) \( \mu(\{t\} \times \mathbb{R}^d) = 0 \) for every \( t \in (0, T) \),

(iii) \( u(T) = \varphi \) if and only if \( \mu(\{T\} \times \mathbb{R}^d) = 0 \),

(iv) if \( \mu(\{0, T\} \times \mathbb{R}^d) = 0 \) then (2.14) holds for \( \eta \in W^{1,1}_{2,0}(Q_T) \).
Proof. Fix $E \subset \subset \tilde{Q}_T$ and choose positive $\eta \in C^\infty_0(\tilde{Q}_T)$ such that $\eta \geq 1_E$. Then, by (2.15),
\[
\mu(E) \leq \int_0^T \langle u(t), \frac{\partial \eta}{\partial t}(t) \rangle_2 dt + \frac{1}{2} \int_0^T \langle a(t)\nabla u(t), \nabla \eta(t) \rangle_2 dt - \int_0^T \langle f_u(t), \eta(t) \rangle_2 dt,
\]
and hence, by Gagliardo-Nirenberg-Sobolev inequality,
\[
\mu(E) \leq C(\text{cap}_{\tilde{Q}_T}(E))^{1/2}(\|u\|_{2,T} + \|\nabla u\|_{2,T} + \|f_u\|_{2,T}),
\]
which shows (i). Now, fix $s \in (0, T)$ and consider the sequence of functions $\{\eta^{n,s}\}$ defined by
\[
\eta^{n,s}(t, x) = \begin{cases} 0, & t \in [0, s_n], \\
\frac{\eta(s, x)}{s - s_n}(t - s_n), & t \in (s_n, s), \\
\eta(t, x), & t \in [s, T],
\end{cases}
\]
where $\{s_n\} \subset (0, s)$ is a sequence such that $s_n \uparrow s$. Observe that $\eta^{n,s} \to 1_{[s,T] \times \mathbb{R}^d} \eta$, $\nabla \eta^{n,s} \to 1_{[s,T] \times \mathbb{R}^d} \nabla \eta$ and
\[
\frac{\partial \eta^{n,s}}{\partial t}(t, x) = \begin{cases} 0, & t \in [0, s_n], \\
\frac{\eta(s, x)}{s - s_n}, & t \in (s_n, s), \\
\frac{\partial \eta}{\partial t}(t, x), & t \in [s, T].
\end{cases}
\]
From (2.15) with $\eta$ replaced by $\eta^{n,s}$ we have
\[
\frac{1}{s - s_n} \int_{s_n}^s \langle u(t), \eta(t) \rangle_2 dt + \int_s^T \langle u(t), \frac{\partial \eta}{\partial t}(t) \rangle_2 dt + \frac{1}{2} \int_0^T \langle a(t)\nabla u(t), \nabla \eta^{n,s}(t) \rangle dt
\]
\[
= \int_0^T \langle f_u(t), \eta^{n,s}(t) \rangle_2 dt + \int_{Q_T} \eta^{n,s} d\mu + \langle \varphi, \eta^{n,s}(T) \rangle_2.
\]
Letting $n \to \infty$ and using the fact that $u \in C([0, T], L^2_{\text{loc}}(\mathbb{R}^d))$ we get (2.14) for every $\eta \in W^{1,1}_2(\mathbb{Q}_T) \cap C_0(\mathbb{Q}_T), t \in (0, T)$. In particular, for any positive $\eta \in W^{1,1}_2(\mathbb{Q}_T) \cap C_0(\mathbb{Q}_T)$ and any $0 < h \leq T$ we have
\[
\int_{s-h}^s \int_{\mathbb{R}^d} \eta d\mu = \int_{s-h}^s \langle u(t), \frac{\partial \eta}{\partial t}(t) \rangle_2 dt + \frac{1}{2} \int_{s-h}^s \langle a(t)\nabla u(t), \nabla \eta(t) \rangle_2 dt
\]
\[
- \int_{s-h}^s \langle f_u(t), \eta(t) \rangle_2 dt - \langle u(s), \eta(s) \rangle_2 + \langle u(s-h), \eta(s-h) \rangle_2,
\]
so letting $h \downarrow 0$ and using continuity of $t \mapsto u(t)$ in $L_2(\mathbb{R}^d)$ we get (ii) and (iii). To show (iv) we assume that $\eta \in W^{1,1}_{2,0}(\mathbb{Q}_T)$ and consider a sequence $\{\eta_n\} \subset W^{1,1}_2(\mathbb{Q}_T) \cap C_0(\mathbb{Q}_T)$ such that $\eta_n \to \eta$ in $W^{1,1}_{2,0}(\mathbb{Q}_T)$ and quasi-everywhere in $\tilde{Q}_T$. From (i) and the assumption in (iv) it follows that $\{\eta_n\}$ converges $\mu$-a.e. in $Q_T$ as well. From (2.14) applied to $|\eta_n - \eta_m|$ we conclude that $\{\eta_n\}$ is a Cauchy sequence in $L_1([t, T] \times \mathbb{R}^d, \mu)$ for every $t \in (0, T]$. Therefore (2.14) is satisfied for any $\eta \in W^{1,1}_{2,0}(\mathbb{Q}_T)$ and $t \in (0, T]$. Clearly, if $\mu({\{0\} \times \mathbb{R}^d}) = 0$, then it is satisfied also for $t = 0$. \hfill \Box
In what follows, given some function \( u : Q_T \to \mathbb{R}^d \) we will extend it in a natural way to the function on \([-T, 2T] \times \mathbb{R}^d\), still denoted by \( u \), by putting

\[
u(t, x) = \begin{cases} 
  u(-t, x), & t \in [-T, 0], \\
  u(t, x), & t \in [0, T], \\
  u(2T - t, x), & t \in [T, 2T].
\end{cases}
\]

For \( \varepsilon > 0 \) set

\[
u_\varepsilon(t, x) = \frac{1}{\varepsilon} \int_0^\varepsilon u(t - s, x) \, ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d
\]

and note that if \( u \in C([0, T], \mathbb{L}_2^{loc}(\mathbb{R}^d)) \cap W^{0, 1}_{2, loc}(Q_T) \) then \( u_\varepsilon \in W^{1, 1}_{2, loc}(Q_T) \), \( t \mapsto u_\varepsilon(t) \in \mathbb{L}_2^{loc}(\mathbb{R}^d) \) is differentiable, \( \nabla u_\varepsilon \to \nabla u \) in \( \mathbb{L}_2^{loc}(Q_T) \) and \( u_\varepsilon(t) \to u(t) \) in \( \mathbb{L}_2^{loc}(\mathbb{R}^d) \) for every \( t \in [0, T] \).

**Lemma 2.7** If \((u, \mu)\) satisfies (a), (d), then for any \( \eta \in W^{1, 1}_{2, 0}(Q_T) \) and \( t \in (0, T) \),

\[
\int_t^T \langle u_\varepsilon(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 \, ds + \frac{1}{2} \int_t^T \langle a(s) \nabla u_\varepsilon(s), \nabla \eta(s) \rangle_2 \, ds
= \int_t^T \langle f_{u, \varepsilon}(s), \eta(s) \rangle_2 \, ds + \frac{1}{\varepsilon} \int_0^\varepsilon \left( \int_{t-\theta}^{T-\theta} \eta(s + \theta, x) \, d\mu(s, x) \right) \, d\theta
+ \langle u_\varepsilon(T), \eta(T) \rangle_2 - \langle u_\varepsilon(t), \eta(t) \rangle_2
\]

(2.16)

for all sufficiently small \( \varepsilon > 0 \).

**Proof.** Using Fubini’s theorem and (2.14) we obtain

\[
\int_t^T \langle u_\varepsilon(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 \, ds = \frac{1}{\varepsilon} \int_0^\varepsilon \left( \int_{t-\theta}^{T-\theta} \langle u(s), \frac{\partial \eta}{\partial s}(s + \theta) \rangle_2 \, ds \right) \, d\theta
\]

\[
= -\frac{1}{2\varepsilon} \int_0^\varepsilon \left( \int_{t-\theta}^{T-\theta} \langle a(s) \nabla u(s), \nabla \eta(s + \theta) \rangle_2 \, ds \right) \, d\theta
+ \frac{1}{\varepsilon} \int_0^\varepsilon \left( \int_{t-\theta}^{T-\theta} \langle f_{u}(s), \eta(s + \theta) \rangle_2 \, ds + \int_{t-\theta}^{T-\theta} \eta(s + \theta, x) \, d\mu(s, x) \right) \, d\theta
+ \frac{1}{\varepsilon} \int_0^\varepsilon \langle u(T - \theta), \eta(T) \rangle_2 - \langle u(t - \theta), \eta(t) \rangle_2 \, d\theta,
\]

from which (2.16) follows. \( \square \)

**Proposition 2.8** If \((u, \mu)\) satisfies (a), (d) and \( u \in C(\bar{Q}_T) \) then \( \int_{\bar{Q}_T} |\xi|^2 |u| \, d\mu < \infty \) for any \( \xi \in C_0^1(\bar{Q}_T) \). Moreover,

\[
\|u(t)\xi\|^2 L^2 + \int_t^T \langle a(s) \nabla u(s), \nabla (u\xi^2)(s) \rangle_2 \, ds
= \|u(t)\xi\|^2 L^2 + 2\int_t^T \langle f_u(s), u(s)\xi^2 \rangle_2 \, ds + 2\int_{\mathbb{R}^d} \xi^2 u \, d\mu
\]

(2.17)

for all \( t \in [0, T] \).
Proof. Let $\tau \in (0,T)$. Write $u_\epsilon^+ = (u_\epsilon)^+$. By (2.16) with $\eta = \xi^2 u_\epsilon^+$ we have

\[
\int_t^T \langle u_\epsilon(s), \xi^2 \frac{\partial u_\epsilon^+}{\partial s}(s) \rangle_2 \, ds + \frac{1}{2} \int_t^T \langle a(s) \nabla u_\epsilon^+(s), \nabla (\xi^2 u_\epsilon^+)(s) \rangle_2 \, ds
\]

\[
= \int_t^T \langle f_{u,\epsilon}(s), \xi^2 u_\epsilon^+(s) \rangle_2 \, ds + \langle u_\epsilon(\tau), \xi^2 u_\epsilon^+(\tau) \rangle_2 - \langle u_\epsilon(t), \xi^2 u_\epsilon^+(t) \rangle_2
\]

\[
+ \frac{1}{\epsilon} \int_0^\epsilon \left( \int_{t+s-1}^{t+1} \int_{\mathbb{R}^d} \xi^2 u_\epsilon^+(s + s_1, x) \, d\mu(s, x) \right) \, ds_1
\]

\[
= \int_t^T \langle f_{u,\epsilon}(s), \xi^2 u_\epsilon^+(s) \rangle_2 \, ds + \|\xi u_\epsilon^+(\tau)\|_2^2 - \|\xi u_\epsilon^+(t)\|_2^2 + \int_{Q_T} g_\xi \, d\mu, \quad (2.18)
\]

where

\[
g_\xi(s, x) = \frac{1}{\epsilon^2} \int_0^\epsilon 1_{[t-s_1, t-s_1]}(s) \xi^2 \left( \int_0^\epsilon u(s + s_1 - s_2, x) \, ds_1 \right) \, ds_2
\]

for $s \in [t, \tau)$ and $g_\xi(\tau, x) = 0$. Observe that for every $(s, x) \in [t, \tau) \times \mathbb{R}^d$,

\[
g_\xi(s, x) = \frac{1}{\epsilon^2} \int_0^\epsilon \xi^2 \left( \int_0^\epsilon u(s + s_1 - s_2, x) \, ds_1 \right) \, ds_2
\]

for sufficiently small $\epsilon > 0$. Since $|a^+ - b^+| \leq |a - b|$ for every $a, b \in \mathbb{R}$, we have

\[
\left| \frac{1}{\epsilon^2} \int_0^\epsilon \xi^2 \left( \int_0^\epsilon u(s + s_1 - s_2, x) \, ds_1 \right) \, ds_2 - \frac{1}{\epsilon^2} \int_0^\epsilon \xi^2 \left( \int_0^\epsilon u(s, x) \, ds_1 \right) \, ds_2 \right|
\]

\[
\leq \frac{1}{\epsilon^2} \int_0^\epsilon \xi^2 \left| \left( \int_0^\epsilon u(s + s_1 - s_2, x) \, ds_1 \right) \right| \, ds_2
\]

\[
\leq \frac{1}{\epsilon^2} \int_0^\epsilon \xi^2 \left| \int_0^\epsilon u(s + s_1 - s_2, x) \, ds_1 - \int_0^\epsilon u(s, x) \, ds_1 \right| \, ds_2
\]

and consequently, $g_\xi(s, x) \to \xi^2 u_\epsilon^+(s, x)$ for every $(s, x) \in [t, \tau) \times \mathbb{R}^d$ as $\epsilon \to 0$. Therefore from (2.18) we obtain

\[
\frac{1}{2} \|\xi \varphi^+\|_2^2 - \frac{1}{2} \|\xi u^+(t)\|_2^2 + \frac{1}{2} \int_t^T \langle a(s) \nabla u^+(s), \nabla (\xi^2 u^+)(s) \rangle_2 \, ds
\]

\[
= \int_t^T \langle f_u(s), \xi^2 u^+(s) \rangle_2 \, ds + \|\xi \varphi^+\|_2^2 - \|\xi u^+(t)\|_2^2
\]

\[
+ \liminf_{\epsilon \to 0} \int_t^T \int_{\mathbb{R}^d} g_{\xi, \epsilon}(s, x) \, d\mu. \quad (2.19)
\]

Hence, by Fatou’s lemma,

\[
\int_t^T \int_{\mathbb{R}^d} \xi^2 u^+ \, d\mu 
\leq \frac{1}{2} \|\xi u(t)\|_2^2 + \frac{1}{2} \int_t^T |\langle a(s) \nabla u(s), \nabla (\xi^2 u)(s) \rangle_2| \, ds
\]

\[
+ \int_t^T |\langle f_u(s), \xi^2 u(s) \rangle_2| \, ds + \frac{1}{2} \|\xi \varphi\|_2^2.
\]
Letting \( t \downarrow 0 \) and \( \tau \uparrow T \) we see from the above that \( \int_{Q_T} \xi^2 u^+ \, d\mu < \infty \). Analogously, putting \( \eta = \xi^2 u^- \) we show that \( \int_{Q_T} \xi^2 u^- \, d\mu < \infty \), which completes the proof of the first part of the lemma. Since \( |g_{\xi,\varepsilon}(s, x)| \leq C \xi^2 \) on \( [t, \tau] \times \mathbb{R}^d \) for some \( C > 0 \), using the Lebesgue dominated convergence theorem we conclude from (2.19) that (2.17) is satisfied with \( T \) replaced by \( \tau \) and \( t \in (0, \tau) \). Because we know already that \( \int_{Q_T} \xi^2 |u| \, d\mu < \infty \), letting \( \tau \uparrow T \) and \( t \downarrow 0 \) we complete the proof. \( \Box \)

We now are ready to prove useful a priori estimates for solutions of an obstacle problem.

**Proposition 2.9** Assume (H1)–(H3) with \( \varphi \in L_{2,p}(\mathbb{R}^d) \), \( g \in L_{2,p}(Q_T) \). If \((u, \mu)\) satisfies (a) and (d), \( u \in C(Q_T) \), \( \nabla u \in L_{2,p}(Q_T) \) and there is \( p : Q_T \to \mathbb{R} \) such that \( p^+ \in W_{2,p}^1(Q_T) \) and \( \int_{Q_T} (u - p) \xi \, d\mu \leq 0 \) for all \( \xi \in C_0^\infty(Q_T) \) then there is \( C > 0 \) such that

\[
\sup_{t \in [0, T]} \|u(t)\|_{2,p}^2 + \int_0^T \|\nabla u(s)\|_{2,p}^2 \, ds + \int_{Q_T} |u|^2 \, d\mu + \|\mu\|_{(W_{2,p}^1(Q_T))^*} \leq C \left( \|\varphi\|_{2,p}^2 + \sup_{t \in [0, T]} \|p^+(t)\|_{2,p}^2 + \int_0^T (\|\nabla p^+(s)\|_{2,p}^2 + \|g(s)\|_{2,p}^2) \, ds \right). 
\]

(2.20)

**Proof.** Let \( \xi_n \in C_0^\infty(\mathbb{R}^d) \) be a function such that \( \xi_n = 1 \) on \( B(0, n) \). By proposition 2.8, (H2) and (1.1)

\[
\|u(t)\xi_n\|_{2,p}^2 + \int_t^T \langle a(s) \nabla u(s), \nabla (u\xi_n^2 \varphi^2)(s) \rangle_2 \, ds \\
= \|\varphi\xi_n\|_{2,p}^2 + 2 \int_t^T \langle f_u(s), u(s)\xi_n^2 \varphi^2 \rangle_2 \, ds + 2 \int_t^T \int_{\mathbb{R}^d} u\xi_n^2 \varphi^2 \, d\mu \\
\leq \|\varphi\xi_n\|_{2,p}^2 + \int_t^T (\|g(s)\xi_n\|_{2,p}^2 + C\|u(s)\xi_n\|_{2,p}^2 + \lambda \|\nabla u(s)\xi_n\|_{2,p}^2) \, ds \\
+ \int_t^T \int_{\mathbb{R}^d} p^+\xi_n^2 \varphi^2 \, d\mu,
\]

Moreover, by (2.14) with \( \eta = p^+\xi_n^2 \varphi^2 \) we have

\[
\int_t^T \int_{\mathbb{R}^d} p^+\xi_n^2 \varphi^2 \, d\mu \leq \frac{1}{2}(\|\varphi\xi_n\|_{2,p}^2 + \|p^+(T)\xi_n\|_{2,p}^2 + \|u(t)\xi_n\|_{2,p}^2 + \|p^+(t)\xi_n\|_{2,p}^2) \\
+ \int_t^T (\|g(s)\xi_n\|_{2,p}^2 + \|u(s)\xi_n\|_{2,p}^2 + C\|p^+(s)\xi_n\|_{2,p}^2) \, ds \\
+ \int_t^T (\|\nabla p^+(s)\xi_n\|_{2,p}^2 + |\langle a(s) \nabla u(s), \nabla (p^+\xi_n^2 \varphi^2)(s) \rangle|) \, ds
\]

By the above estimates and the fact that \( |\nabla \varphi| \leq 2\alpha \varphi \) there is \( C \) such that

\[
\|u(t)\xi_n\|_{2,p}^2 + \int_t^T \|\nabla u(s)\xi_n\|_{2,p}^2 \leq C \left( \|\varphi\xi_n\|_{2,p}^2 + \sup_{t \in [0, T]} \|p^+(t)\xi_n\|_{2,p}^2 \right).
\]

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\[ + \int_0^T (|\frac{∂p^+}{∂s}(s)ξ_n|^2 + |∇p^+(s)ξ_n|^2 + |g(s)ξ_n|^2) \, ds \]
\[ + \int_0^T |u(s)ξ_n|^2 \, ds + \varepsilon_1^2 \varepsilon_n^2 \],
where
\[ \varepsilon_1^2 = \int_t^T |(a(s)\nabla u(s), u(s)ξ^2ξ_n)| ds, \quad \varepsilon_2^2 = \int_t^T |(a(s)\nabla u(s), p^+(s)ξ^2ξ_n)| ds. \]

Since \( \varepsilon_1^2 \to 0, \varepsilon_2^2 \to 0 \) as \( n \to \infty \), applying Gronwall’s lemma we see from the above estimates that \( \sup_{t∈[0,T]} |u(t)|^2_2 + \int_0^T |∇u(t)|^2_2 \, dt \) is bounded by the right-hand side of (2.20), which when combined with (2.17) and (2.14) gives (2.20).

For convenience of the reader we now recall definitions of solutions of an obstacle problem in the sense of variational inequalities (see, e.g., [4, 8, 17]).

**Definition** We say that \( u \) is a weak solution of \( \text{OP}(φ, f, h) \) in the variational sense if \( u \in W^{0,1}_2(Q_T) \) and for any \( v \in W^{1,1}_2(Q_T) \) such that \( v ≥ h \),
\[ \int_0^T \langle \frac{∂v}{∂t}(t), (v-u)(t) \rangle_{2,φ} \, dt + \int_0^T \langle A_i u(t), (v-u)(t) \rangle_{2,φ} \, dt \]
\[ + \int_0^T \langle f(t), (v-u)(t) \rangle_{2,φ} \, dt \leq \frac{1}{2} |φ-v(T)|^2_{2,φ}, \] (2.21)
where
\[ \langle A_i u(t), (v-u)(t) \rangle_{2,φ} = -\frac{1}{2} \langle a(t)\nabla u(t), Δ((v-u)(t)ξ^2) \rangle_{2,φ}. \]

**Definition** We say that \( u \) is a strong solution of \( \text{OP}(φ, f, h) \) in the variational sense if \( u \in W_2, u(T) = φ \) and for any \( v \in W^{0,1}_2(Q_T) \) such that \( v ≥ h \),
\[ \int_0^T \langle \frac{∂v}{∂t}(t), (v-u)(t) \rangle \, dt + \int_0^T \langle A_i u(t), (v-u)(t) \rangle_{2,φ} \, dt \]
\[ + \int_0^T \langle f(t), (v-u)(t) \rangle_{2,φ} \, dt \leq 0, \] (2.22)
where \( \langle \cdot, \cdot \rangle \) denote the duality pairing between \( W^{1,2}_2(\mathbb{R}^d) \) and \( W^{-1,2}_2(\mathbb{R}^d) \).

The following proposition shows that continuous solutions of the problem (1.4) coincide with solutions of (1.3) in the variational sense.

**Proposition 2.10** If \( (u, µ) \) is a solution of \( \text{OP}(φ, f, h) \) such that \( u ∈ W^{0,1}_2(Q_T) \) \( C(\bar{Q}_T) \) then \( u \) is a weak solution of the problem in the variational sense. If, in addition, \( u ∈ W_2 \), then \( u \) is a strong solution of \( \text{OP}(φ, f, h) \) in the variational sense.

**Proof.** Let \( u \in W^{0,1}_2(Q_T) \) \( C(\bar{Q}_T) \) and let \( (u, µ) \) be a solution of \( \text{OP}(φ, f, h) \). By proposition 2.8,
\[ \frac{1}{2} |u(0)|^2_{2,φ} - \int_0^T \langle A_i u(t), u(t) \rangle_{2,φ} \, dt \]
\[ = \frac{1}{2} |φ|^2_{2,φ} + \int_0^T \langle f(t), u(t) \rangle_{2,φ} \, dt + \int_{Q_T} uφ^2 \, dµ. \] (2.23)
On the other hand, from definition of solution of of \( \text{OP}(\varphi, f, h) \) it follows that for any \( v \in W_{2,\varrho}^{1,1}(Q_T) \) we have
\[
\int_0^T \langle u(t), \frac{\partial v}{\partial t}(t) \rangle_{2,\varrho} dt - \int_0^T \langle A_t u(t), v(t) \rangle_{2,\varrho} dt 
\]
\[
= \int_0^T \langle f_u(t), v(t) \rangle_{2,\varrho} dt + \int_{Q_T} v^2 d\mu + \langle \varphi, v(T) \rangle_{2,\varrho} - \langle u_0, v(0) \rangle_{2,\varrho}. \quad (2.24)
\]
Combining (2.23) with (2.24) we get
\[
\int_0^T \langle \frac{\partial v}{\partial t}(t), (v - u)(t) \rangle_{2,\varrho} dt + \int_0^T \langle A_t u(t), (v - u)(t) \rangle_{2,\varrho} dt 
\]
\[
+ \int_0^T \langle f_u(t), (v - u)(t) \rangle_{2,\varrho} dt 
\]
\[
= -\frac{1}{2} \|u(0)\|_{2,\varrho}^2 + \frac{1}{2} \|\varphi\|_{2,\varrho}^2 + \frac{1}{2} \|v(T)\|_{2,\varrho}^2 - \frac{1}{2} \|v(0)\|_{2,\varrho}^2 - \langle \varphi, v(T) \rangle_{2,\varrho} 
\]
\[
+ \langle u(0), v(0) \rangle_{2,\varrho} + \int_{Q_T} (u - v) g^2 d\mu 
\]
\[
= -\frac{1}{2} \|u(0) - v(0)\|_{2,\varrho}^2 + \frac{1}{2} \|\varphi - v(T)\|_{2,\varrho}^2 + \int_{Q_T} (u - v) g^2 d\mu. \quad (2.25)
\]
Since \( v \geq h \), \( \int_{Q_T} (u - v) g^2 d\mu \leq 0 \), so (2.21) follows. Now, assume additionally that \( u \in \mathcal{W}_\varrho \). Then by (2.24) we have
\[
- \int_0^T \langle \frac{\partial u}{\partial t}(t), v(t) \rangle dt - \int_0^T \langle A_t u(t), v(t) \rangle_{2,\varrho} dt 
\]
\[
= \int_0^T \langle f_u(t), v(t) \rangle_{2,\varrho} dt + \int_{Q_T} v^2 d\mu \quad (2.26)
\]
for every \( v \in W_{2,\varrho}^{1,1}(Q_T) \). Let \( E \subset \subset \tilde{Q}_T \). From (2.26) with a positive \( v \in C_0^\infty(\tilde{Q}_T) \) such that \( v \geq 1_E \) we conclude that
\[
\mu(E) \leq C(\overline{\text{cap}}_{\tilde{Q}_T}(E))^{1/2} (\|\frac{\partial u}{\partial t}\|_{L^2(0,T;W_{\varrho}^{-1}(\mathbb{R}^d))} + \|f_u\|_{L^2(Q_T)} + \|\nabla u\|_{L^2(Q_T)}),
\]
where
\[
\overline{\text{cap}}_{\tilde{Q}_T}(E) = \inf \left\{ \int_{Q_T} |\nabla \eta(t, x)|^2 dt dx : \eta \in C_0^\infty(\tilde{Q}_T), \eta \geq 1_E \right\}.
\]
On the other hand, it is known (see [5]) that \( \overline{\text{cap}}_{\tilde{Q}_T}(E) = \int_0^T \text{cap}_{\mathbb{R}^d}(E_t) dt \). Therefore, if \( v \in W_{2,\varrho}^{0,1}(Q_T) \), then there is a version of it which is defined \( \tilde{u} \). Since we know already that \( \mu \ll \overline{\text{cap}}_{\tilde{Q}_T} \), the integral \( \int_{Q_T} v d\mu \) is well defined for \( v \in W_{2,\varrho}^{0,1}(Q_T) \). Therefore, by approximation argument, we may take as a test function in (2.26) any \( v \in W_{2,\varrho}^{0,1}(Q_T) \). Now from (2.26) we conclude that for any \( v \in W_{2,\varrho}^{0,1}(Q_T) \) such that \( v \geq h \),
\[
\int_0^T \langle \frac{\partial u}{\partial t}(t), (v - u)(t) \rangle_{2,\varrho} dt + \int_0^T \langle A_t u(t), (v - u)(t) \rangle_{2,\varrho} dt 
\]
\[
+ \int_0^T \langle f_u(t), (v - u)(t) \rangle_{2,\varrho} dt = \int_{Q_T} (u - v) g^2 d\mu \leq 0,
\]
and the proof is complete.

Let us note here that in Theorem 3.14 we will prove that if \((u, \mu)\) is a solution of an obstacle problem, then \(u\) is the minimal solution of the same problem in the variational sense.

3 Existence, uniqueness and stochastic representation of solutions of an obstacle problem

We begin with a general uniqueness result for continuous solutions of (1.4) satisfying some weak integrability assumptions.

**Theorem 3.1** Assume (H1)–(H3). Then \(OP(\varphi, f, h)\) has at most one solution \((u, \mu)\) such that \(u \in C(\bar{Q}_T) \cap W^{0,1}_2(Q_T)\).

**Proof.** Suppose that \((u_1, \mu_1)\), \((u_2, \mu_2)\) are solutions of \(OP(\varphi, f, h)\) such that \(u_1, u_2 \in C(\bar{Q}_T) \cap W^{0,1}_2(Q_T)\) and let \(u = u_1 - u_2\), \(\mu = \mu_1 - \mu_2\). Let \(\xi_n : \mathbb{R}^d \to [0, 1], n \in \mathbb{N}\) be a smooth function such that \(\xi_n(x) = 1\) if \(|x| \leq n\) and \(\xi_n(x) = 0\) if \(|x| \geq n + 1\). By the definition of solution of \(OP(\varphi, f, h)\), for any \(\eta \in W_{2,0}^{1,1}(Q_T)\) we have

\[
\langle u(t), \eta(t) \rangle_2 + \int_t^T \langle u(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 ds + \frac{1}{2} \int_t^T \langle a(s) \nabla u(s), \nabla \eta(s) \rangle_2 ds
\]

\[
= \int_t^T \int_{\mathbb{R}^d} \eta \, d\mu + \int_t^T \langle f_{u_1}(s) - f_{u_2}(s), \eta(s) \rangle_2 ds, \quad t \in [0, T]. \tag{3.1}
\]

From proposition 2.8 we conclude that

\[
\|u(t) \varphi \xi_n\|_2^2 + \int_t^T \langle a(s) \nabla u(s), \nabla (u \varphi^2 \xi_n^2)(s) \rangle_2 ds
\]

\[
= 2 \int_t^T \int_{\mathbb{R}^d} u \varphi^2 \xi_n^2 \, d\mu + 2 \int_t^T \langle f_{u_1}(s) - f_{u_2}(s), u(s) \varphi^2 \xi_n^2 \rangle_2 ds
\]

\[
\leq 2 \int_t^T \langle f_{u_1}(s) - f_{u_2}(s), u(s) \varphi^2 \xi_n^2 \rangle_2 ds, \tag{3.2}
\]

the last inequality being a consequence of the fact that

\[
\int_t^T \int_{\mathbb{R}^d} u \varphi^2 \xi_n^2 \, d\mu = \int_t^T \int_{\mathbb{R}^d} u_1 \varphi^2 \xi_n^2 \, d\mu_1 - \int_t^T \int_{\mathbb{R}^d} u_2 \varphi^2 \xi_n^2 \, d\mu_2 \tag{3.3}
\]

\[
- \int_t^T \int_{\mathbb{R}^d} u_2 \varphi^2 \xi_n^2 \, d\mu_1 + \int_t^T \int_{\mathbb{R}^d} u_2 \varphi^2 \xi_n^2 \, d\mu_2
\]

\[
= \int_t^T \int_{\mathbb{R}^d} \varphi^2 \xi_n^2 (u_1 - h) \, d\mu_1 + \int_t^T \int_{\mathbb{R}^d} \varphi^2 \xi_n^2 (u_2 - h) \, d\mu_2
\]

\[
+ \int_t^T \int_{\mathbb{R}^d} \varphi^2 \xi_n^2 (h - u_2) \, d\mu_1 + \int_t^T \int_{\mathbb{R}^d} \varphi^2 \xi_n^2 (h - u_1) \, d\mu_2 \leq 0.
\]

By (3.2) and (H2),

\[
\|u(t) \varphi \xi_n\|_2^2 + \lambda \int_t^T \|\nabla u(s) \varphi \xi_n\|_2^2 ds
\]

\[18\]
\[\leq - \int_t^T \langle a(s) \nabla u(s), u(s) \xi_n \nabla \vartheta^2 \rangle ds - \int_t^T \langle a(s) \nabla u(s), u(s) \vartheta^2 \nabla \xi_n^2 \rangle ds + \frac{\lambda}{2} \int_t^T \| \nabla u(s) \vartheta \xi_n \|_2^2 ds + 2(L + \lambda^{-1} L^2) \int_t^T \| u(s) \vartheta \xi_n \|_2^2 ds.\]

Since \(|\nabla \vartheta^2| \leq 2\alpha \vartheta^2\), we have
\[
\int_t^T |\langle a(s) \nabla u(s), u(s) \xi_n \nabla \vartheta^2 \rangle| ds \leq \lambda \int_t^T \| \nabla u(s) \vartheta \xi_n \|_2^2 ds + \frac{\Lambda \alpha^2}{2\lambda} \int_t^T \| u(s) \vartheta \xi_n \|_2^2 ds.
\]

Consequently, there is \(C > 0\) not depending on \(n\) such that
\[
\| u(t) \vartheta \xi_n \|_2^2 \leq C \int_t^T \| u(s) \vartheta \xi_n \|_2^2 ds + \int_t^T |\langle a(s) \nabla u(s), u(s) \xi_n \nabla \vartheta^2 \rangle| ds
\]
for \(t \in [0, T]\). Letting \(n \to \infty\) we get
\[
\| u(t) \|_{2, \vartheta}^2 \leq C \int_t^T \| u(s) \|_{2, \vartheta}^2 ds, \quad t \in [0, T]
\]
and hence, by Gronwall’s lemma, \(u = 0\), i.e. \(u_1 = u_2\). Using this and (3.1) we see that
\[
\int_{Q_T} \eta(s, x) d\mu_1 = \int_{Q_T} \eta(s, x) d\mu_2 \text{ for any } \eta \in W^{1,1}_{2,0}(Q_T),
\]
which shows that \(\mu_1 = \mu_2\). \(\square\)

To prove existence of a solution of the problem (1.4) and its stochastic representation we have to impose additional integrability assumptions on \(g\) and \(h\) to ensure existence of a solution of RBSDE(\(\varphi, f, h\)). The assumptions must guarantee also continuity of \(u\) because we are able to prove uniqueness and a priori estimates only for continuous weak solutions of OP(\(\varphi, f, h\)). Proposition 2.3 and Theorem 2.4 therefore suggest that if we want the representation (1.5) to hold we should assume at least that
\[
\forall K \subset \subset [0, T) \times \mathbb{R}^d \sup_{(s, x) \in K} \langle E_{s, x} \sup_{s \leq t \leq T} |h^+(t, X_t)|^2 + E_{s, x} \int_s^T |g(t, X_t)|^2 dt \rangle < \infty.
\]

Our assumptions on \(h\) are slightly stronger but nevertheless seems to be close to the best possible.

Now we provide a useful inequality for moments of the diffusion \((X, P_{s, x})\). It is perhaps known but we could not find a proper reference. The inequality is given only for moments greater or equal to 4, because such a form is sufficient for our purposes.

**Proposition 3.2** If \((X, P_{s, x})\) is a Markov process associated with \(A_t\) then for every \(p \geq 4\),
\[
E_{s, x} \sup_{s \leq t \leq T} |X_t|^p \leq C E_{s, x}|X_T|^p,
\]
where \(C\) depends only on \(\lambda, \Lambda, d\) and \(T\).

**Proof.** Let \(u_n\) be a solution of PDE(\(\varphi_n, 0\)) with \(\varphi_n(x) = |x|^{p/2} 1_{B(0,n)}(x)\). From [23] we know that the pair \((u_n(t, X_t), \sigma \nabla u_n(t, X_t))\), \(t \in [s, T]\), is a solution of BSDE(\(\varphi_n, 0\)), i.e.
\[
u_n(t, X_t) = \varphi_n(X_T) - \int_t^T \sigma \nabla u_n(\theta, X_\theta) dB_{s, \theta}, \quad P_{s, x}\text{-a.s.}
\]
from which we obtain in particular that $u_n(s, x) = E_{s,x} \varphi_n(X_T)$. It is known that $u_n \to u$ uniformly in compact subsets of $Q_T$. By Aronson’s lower estimate, for all sufficiently large $n \in \mathbb{N}$ we have

$$|u_n(s, x)| = E_{s,x}|X_T|^{p/2}1_{B(0,n)}(X_T) \geq C \int_{B(0,n)} |y|^{p/2}(T-s)^{-d/2} \exp\left(-\frac{|y-x|^2}{C(T-s)}\right) dy = C^{1+(d/2)} E_{1}B(0,n)(X_{C(T-s)}+x)|X_{C(T-s)}+x|^{p/2} \geq C^{1+(d/2)} \left( E_{1}B(0,n)(X_{C(T-s)}+x) \left(|x|^2 + 2 \sum_{i=1}^d x_iX_{iC(T-s)}^2 \right) \right)^{p/4}$$

($E$ denotes expectation with respect to the standard Wiener measure on $\Omega$). Letting $n \to \infty$ we see that $|u(s, x)| \geq C^{1+(d/2)}|x|^{p/2}$. By the above and known a priori estimates for BSDE we get

$$E_{s,x} \sup_{s \leq t \leq T} |X_t|^p \leq CE_{s,x} \sup_{s \leq t \leq T} |u(t, X_t)|^2 \leq C \lim inf_{n \to \infty} E_{s,x} \sup_{s \leq t \leq T} |u_n(t, X_t)|^2 \leq C \lim inf_{n \to \infty} E_{s,x} \varphi_n(X_T)|^2 \leq C E_{s,x}|X_T|^p,$$

which completes the proof.

Here and subsequently, we write $\mu_n \Rightarrow_{s,x} \mu$ if for fixed $(s, x) \in [0, T) \times \mathbb{R}^d$,

$$\int_{Q_T} \xi(t, y)p(s, x, t, y) d\mu_n(t, y) \to \int_{Q_T} \xi(t, y)p(s, x, t, y) d\mu(t, y)$$

for every $\xi \in C_b(Q_T)$. We use the symbol “$\Rightarrow$” to denote weak convergence of measures.

**Lemma 3.3** Let $S$ be a Polish space and let $\mu, \mu_n, n \in \mathbb{N}$, be probability measures on $S$ such that $\mu_n \Rightarrow \mu$. If $f, f_n : S \to \mathbb{R}$ are continuous functions such that $f_n \to f$ uniformly in compact subsets of $S$ and

$$\sup_{n \geq 1} \int_S |f_n| d\mu_n < \infty, \quad \lim \sup_{\alpha \to \infty} \int_S |f_n|1_{\{|f_n| \geq \alpha\}} d\mu_n = 0$$

then

$$\int_S f_n d\mu_n \to \int_S f d\mu.$$

**Proof.** It is sufficient to modify slightly the proof of [7, Lemma 8.4.3]. We omit the details.

We now prove our main existence and representation results. For reasons to be explained later on, we decided to consider separately the case of square-integrable data $\varphi, g, h$ and the case where the data are square-integrable with some weight $p \in \mathcal{W}$ such that $p < 1$. 

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Theorem 3.4 Let assumptions (H1)–(H3) hold with \( \varphi \in L_2(\mathbb{R}^d) \), \( g \in L_2(Q_T) \) and moreover, assume that \( g \) satisfies (2.4), \( h \in L_2(Q_T) \cap C(Q_T), h \leq \psi \) for some \( \psi \) such that \( \psi \in W^{1,1}_2(Q_T) \) and \( h \leq c\bar{g}^{-1} \) for some \( c > 0, \bar{g} \in W \). Then there exists a unique weak solution \((u, \mu)\) of \( OP(\varphi, f, h) \) such that

(i) \( u \in C([0, T] \times \mathbb{R}^d) \),

(ii) \( u_n \uparrow u, u_n \to u \) in \( W^{0,1}_{2, \text{loc}}(Q_T) \cap C([0, T], L^1_{\text{loc}}(Q_T)) \), \( u_n \to u \) in \( L_2(Q_T) \) and \( \mu_n \Rightarrow \mu, \mu_n \Rightarrow \mu \) in \( (W^{1,1}_{2, \text{loc}}(Q_T))^* \), \( \mu_n \Rightarrow s,x \mu \) for every \((s, x) \in [0, T] \times \mathbb{R}^d \), where \( d\mu_n = n(u_n - h)^{-} d\lambda \) and \( u_n \) is a unique weak solution of the Cauchy problem

\[
(\frac{\partial}{\partial t} + A_t)u_n = -f_{u_n} - n(u_n - h)^-, \quad u_n(T) = \varphi. \tag{3.4}
\]

Moreover, for each \((s, x) \in [0, T] \times \mathbb{R}^d \),

\[
(u(t, x_t), \sigma \nabla u(t, x_t)) = (Y^{s,x}_t, Z^{s,x}_t), \quad t \in [s, T], \quad P_{s,x}-a.s. \tag{3.5}
\]

and

\[
E_{s,x} \int_s^T \xi(t, x_t) dK^{s,x}_t = \int_s^T \int_{\mathbb{R}^d} \xi(t, y)p(s, x, t, y) d\mu(t, y) \tag{3.6}
\]

for every \( \xi \in C_b(Q_T) \), where \((Y^{s,x}, Z^{s,x}, K^{s,x})\) is a solution of \( RBSDE(\varphi, f, h) \).

Proof. Step 1. We first show existence of \( u \in W^{0,1}_{2}(Q_T) \) and a Radon measure \( \mu \) on \( Q_T \) such that

\[
\int_0^T \langle u(t), \frac{\partial \eta}{\partial t}(t) \rangle_2 dt + \frac{1}{2} \int_0^T \langle a(t) \nabla u(t), \nabla \eta(t) \rangle_2 dt
= \int_0^T \langle f_{u(t)}(t), \eta(t) \rangle_2 dt + \int_{Q_T} \eta d\mu + \langle \varphi, \eta(T) \rangle_2 \tag{3.7}
\]

for every \( \eta \in W^{1,1}_{2}(Q_T) \cap C(Q_T) \) such that \( \eta(0) \equiv 0 \). From Proposition 2.3 we know that there exists a unique weak solution \( u_n \) of (3.4) such that \( u_n \in C([0, T], L^1(\mathbb{R}^d)) \) and \( u_n \in C([0, T] \times \mathbb{R}^d) \). Set \( r_n = n(u_n - h)^- \) and let \( d\mu_n = r_n d\lambda \). Then for any \( \eta \in W^{1,1}_{2}(Q_T) \),

\[
\int_t^T \langle u_n(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 ds + \frac{1}{2} \int_t^T \langle a(s) \nabla u_n(s), \nabla \eta(s) \rangle_2 ds = \int_t^T \langle f_{u_n(s)}(s), \eta(s) \rangle_2 ds
+ \int_t^T \int_{\mathbb{R}^d} \eta d\mu_n + \langle \varphi, \eta(T) \rangle_2 - \langle u_n(t) \eta(t) \rangle_2, \quad t \in [0, T]. \tag{3.8}
\]

By Proposition 2.9 with \( p = \psi \) there is \( C > 0 \) such that

\[
\sup_{t \in [0, T]} \|u_n(t)\|_2^2 + \int_0^T |\langle a(t) \nabla u_n(t), \nabla u_n(t) \rangle_2| dt \leq C \tag{3.9}
\]

for every \( n \in \mathbb{N} \). Since, by continuity of \( u_n \) and comparison results (see [4, Theorem 4.1.7]), \( u_n(t, x) \leq u_{n+1}(t, x) \) for every \((t, x) \in [0, T] \times \mathbb{R}^d \), there is \( u \) such that \( u_n \uparrow u \).
By Fatou’s lemma and (3.9), $u \in L_2(Q_T)$. In fact, since $u_1 \leq u_n \leq u$, it follows that $u_n^2 \leq u^2 + u^2$ and hence, by the Lebesgue dominated convergence theorem, that $u_n \rightarrow u$ in $L_2(Q_T)$. Let $f_n(t, x, y) = f_{u_n}(t, x) + n(y - h(t, x))$ and let $(Y_t^{s,x,n}, Z_t^{s,x,n})$ be a unique solution of BSDE$(\varphi, f_n)$. By results of [23], $(u_n(t, X_t), \sigma \nabla u_n(t, X_t)) = (Y_t^{s,x,n}, Z_t^{s,x,n})$ $P_{s,x}$-a.s. and hence, by (2.12),

$$|u_n(s, x)|^2 \leq C \mathbb{E}_{s,x} \left( |\varphi(X_T)|^2 + \int_s^T |g(t, X_t)|^2 \, dt + \sup_{s \leq t \leq T} |h^+(t, X_t)|^2 \right)$$

for all $(s, x) \in [0, T) \times \mathbb{R}^d$. Thus, $u(s, x) = \sup_{n \geq 1} u_n(s, x) < \infty$ and consequently, $u$ is lower semi-continuous on $[0, T) \times \mathbb{R}^d$. Let $K \subset Q_T$ and let $\eta \in C_0^\infty(\bar{Q}_T)$ be a positive function such that $\eta = 1$ on $K$. Since

$$\mu_n(K) \leq \int_{Q_T} \eta \, d\mu_n = \int_0^T \int_0^T \left< \nabla u_n(s, \eta) \right> \, ds + \int_0^T \int_0^T \left< a(s) \nabla u_n(s), \nabla \eta(s) \right> \, ds \leq \int_0^T \int_0^T \left< f_n(s, \eta) \right> \, ds,$$

we conclude from (3.9) that $\sup_{n \geq 1} \mu_n(K) < \infty$. Thus, by the weak compactness theorem for measures (see Section 1.9 in [12]), $\{\mu_n\}$ is tight. Therefore there is a subsequence, still denoted by $\{n\}$, such that $\int_{Q_T} f \, d\mu_n \rightharpoonup \int_{Q_T} f \, d\mu$ for every $f \in C_0(Q_T)$. Let $g \in W$ be such that $\int_{\mathbb{R}^d} (g \tilde{\phi}^{-1}(x))^2 \, dx < \infty$ and let $K \subset Q_T$. Then by Theorem 2.2,

$$\int_{\mathbb{R}^d} \left( E_{0,x} \int_0^T |\nabla (u_n - u_m)(t, X_t)|^2 \, dt \right) g^2(x) \, dx \geq C \|\nabla (u_n - u_m)g\|_2^2. \tag{3.10}$$

By (2.11), for every $x \in \mathbb{R}^d$, $\xi_{n,m}(x) \equiv E_{0,x} \int_0^T |\nabla (u_n - u_m)(t, X_t)|^2 \, dt \rightarrow 0$ as $n, m \rightarrow \infty$. Moreover, by (2.12) and Proposition 3.2,

$$|\xi_{n,m}(x)| \leq C(E_{0,x} |\varphi(X_T)|^2 + E_{0,x} \int_0^T |g(t, X_t)|^2 \, dt + \tilde{\phi}^{-1}(x))$$

for some $C$ not depending on $n, m$. Therefore it follows from Theorem 2.2 and the Lebesgue dominated convergence theorem that the left-hand side of (3.10) converges to zero as $n, m \rightarrow \infty$ and hence that $\|1_K \nabla (u_n - u_m)\|_2^2 \rightarrow 0$ for any $K \subset Q_T$. Using properties of $\{u_n\}$ and $\{\mu_n\}$ we have already proved we conclude from (3.8) that (3.7) holds for every $\eta \in W_{1,1}^{1,1}(Q_T) \cap C_0(Q_T)$ such that $\eta(0) = 0$.

**Step 2.** $u \in C([0, T) \times \mathbb{R}^d) \cap C([0, T], L_2^{loc}(\mathbb{R}^d))$. To see this, we first observe that $u(s, x) = Y_{s,x}^s$ for $(s, x) \in [0, T) \times \mathbb{R}^d$, since

$$|u(s, x) - Y_{s,x}^s|^2 \leq 2 \lim_{n \rightarrow \infty} \|(u - u_n)(s, x)\|^2 + E_{s,x} |Y_{s,x}^{s,n} - Y_{s,x}^s|^2 = 0.$$

Hence $u(s, x) = Y_{s,x}^s \leq h(s, X_s) = h(s, x)$, i.e. $(u - h)^- = 0$ and, by (2.13), for any $n, m \in \mathbb{N}$, $\delta > 0$ and any $K \subset [0, T - 2\delta] \times \mathbb{R}^d$ we have

$$|u_n(s, x) - u_m(s, x)|^2 \leq C(E_{s,x} |Y_{T-\delta}^{s,x,n} - Y_{T-\delta}^{s,x,m}|^2 + I_{n,m}^{s,x} + I_{m,n}^{s,x}), \tag{3.11}$$
where
\[ I_{s,n,m}^t = E_{s,x} \int_s^{T-\delta} (Y_{t}^{s,x,n} - h(t, X_t))^{-} \, dK_t^{s,x,m} \]
and \( K^{s,x,m} \) is defined as in Theorem 2.4. By Aronson's upper estimate,
\[
E_{s,x}(Y_{T-\delta}^{s,x,n} - Y_{T-\delta}^{s,x,m})^2 = E_{s,x}|(u_n - u_m)(T - \delta, X_{T-\delta})|^2 \\
\leq \int_{\mathbb{R}^d} |(u_n - u_m)(T - \delta, y)|^2 p(s, x, T - \delta, y) \, dy \\
\leq C\delta^{-d/2} \|u_n - u_m\|_2^2
\]
with some \( C \) depending neither on \((s, x) \in K\) nor on \(n, m \in \mathbb{N}\). Moreover,
\[
|I_{s,n,m}^t|^2 \leq E_{s,x} \sup_{s \leq t \leq T-\delta} |(u_n(t, X_t) - h(t, X_t))^{-}|^2 \cdot E|K^{s,x,m}_T|^2.
\]
In view of (2.12), \( \sup_{n \geq 1} \sup_{(s, x) \in K} E|K^{s,x,m}_T|^2 < \infty \). By Dini’s theorem, \((u_n - h)^{-} \to (u-h)^{-} = 0\) uniformly in any compact subset of \([0, T) \times \mathbb{R}^d\). Therefore, since \(|h(t, X_t)| \leq |\bar{u}(t, X_t)| + \tilde{g}^{-1}(X_t), t \in [0, T]\), where \(\bar{u}\) is a solution of PDE\((\phi, f)\), it follows from Theorem 2.1, Proposition 3.2, Lemma 3.3 and the Lebesgue dominated convergence theorem that \(\sup_{(s, x) \in K} (\mathcal{I}^{n,x}_s + \mathcal{I}^{n,x}_m) \to 0\) as \(n, m \to \infty\). From the above estimates and (3.11) it follows that \(u_n \to u\) uniformly in any compact subset of \([0, T) \times \mathbb{R}^d\), which implies continuity of \(u\) on \([0, T) \times \mathbb{R}^d\). By Theorem 2.2,
\[
\int_{\mathbb{R}^d} E_{0,x} \sup_{t \in [0, T]} |Y_{t}^{0,x,n} - Y_{t}^{0,x,m}|^2 g^2(x) \, dx \\
\geq \sup_{t \in [0, T]} \int_{\mathbb{R}^d} E_{0,x} |Y_{t}^{0,x,n} - Y_{t}^{0,x,m}|^2 g^2(x) \, dx \\
= \sup_{t \in [0, T]} \int_{\mathbb{R}^d} E_{0,x} |(u_n - u_m)(t, X_t)|^2 g^2(x) \, dx \geq c \sup_{t \in [0, T]} \|(u_n - u_m)(t)\|_{2,g}^2.
\]
Therefore, choosing \(g \in W\) such that \(\int_{\mathbb{R}^d} (g \tilde{g}^{-1}(x))^2 \, dx < \infty\) and arguing as in the proof of convergence of the left-hand side of (3.10) we deduce from the above that \(\sup_{t \in [0, T]} \|(u_n - u_m)(t)\|_{2,g}^2 \to 0\) as \(n, m \to \infty\), and hence that \(u \in C([0, T], L^2_{2,loc}(\mathbb{R}^d))\).

**Step 3.** \(u\) is the unique weak solution of the problem OP\((\phi, f, h)\). We know that
\[
E_{0,x} \int_0^T \xi(t, X_t) \, dK_{0,x}^{t} = E_{0,x} \int_0^T \xi_n(u_n - h)^{-} (t, X_t) \, dt \\
= \int_{Q_T} \xi(t,y)p(s,x,t,y) \, d\mu_n(t,y)
\]
for all \(\xi \in C_0(Q_T)\). Let \(K \subset \subset \mathbb{R}^d\) and \(\{\xi_n\} \subset C_0^+(Q_T)\) be such that \(\xi_n \downarrow 1_{\{0\} \times K}\). Since \(\mu_n \Rightarrow \mu\), it follows from (3.12) and (2.11) that
\[
\int_{\mathbb{R}^d} (E_{0,x} \int_0^T \xi_n(t, X_t) \, dK_{0,x}^{t}) \, dx \geq \int_{Q_T} \xi_n(t,y) \, d\mu(t,y)
\]
for \( n \in \mathbb{N} \). Letting \( n \to \infty \) and taking into account that \( K^{0,x} \) is continuous we deduce from the above inequality that \( \int_{Q_T} 1_{\{0\} \times K}(\theta,y) \, d\mu(\theta,y) = 0 \). Therefore \( \mu(\{0\} \times K) = 0 \) for any \( K \subset \subset \mathbb{R}^d \) and hence \( \mu(\{0\} \times \mathbb{R}^d) = 0 \). Now from Lemma 2.6 and Step 2 it follows that \( \mu(\{t\} \times \mathbb{R}^d) = 0 \) for all \( t \in [0,T] \). Using this and Lemma 2.6 we see that (2.14) is satisfied for any \( \eta \in W^{1,1}_{2,0}(Q_T) \) and \( t \in [0,T] \). Since \( u_n - h \to u - h \) uniformly in compact subsets of \([0,T] \times \mathbb{R}^d\),

\[
\int_s^{T-\delta} \int_{\mathbb{R}^d} \xi(u_n - h)(t,x) \, d\mu_n(t,x) \to \int_s^{T-\delta} \int_{\mathbb{R}^d} \xi(u - h)(t,x) \, d\mu(t,x)
\]

for any \( \xi \in C^+_0(\mathbb{R}^d) \). Hence, since

\[
\int_s^{T-\delta} \int_{\mathbb{R}^d} \xi(u_n - h)(t,x) \, d\mu_n(t,x) = n \int_s^{T-\delta} \int_{\mathbb{R}^d} \xi(u_n - h) \cdot (u_n - h)^{-}(t,x) \, dt \, dx \leq 0
\]

and \( u \geq h \), it follows that \( \int_{Q_T} \xi(u - h) \, d\mu = 0 \), which shows that \( u \) solves \( \text{OP}(\varphi,f,h) \). Uniqueness follows from Theorem 3.1.

**Step 4.** We show (3.5). From [23] we know that

\[
u_n(t,X_t) = \varphi(X_T) + \int_t^T f(\theta,X_\theta,u_n(\theta,X_\theta),\sigma \nabla u_n(\theta,X_\theta)) \, d\theta + K^{s,x,n}_T - K^{s,x,n}_t - \int_t^T \langle \sigma \nabla u_n(\theta,X_\theta), dB_{s,\theta} \rangle, \quad P_{s,x}-a.s. \quad (3.13)
\]

for all \( n \in \mathbb{N} \). Since \( u_n \to u \) uniformly on compact sets in \([0,T] \times \mathbb{R}^d\), it follows from (2.11) that \( u(t,X_t) = Y^{s,x}_t, \quad t \in [s,T], \quad P_{s,x}-a.s. \). To prove that \( \sigma \nabla u(t,X_t) = Z^{s,x}_t, \lambda \otimes P_{s,x}-a.s. \) observe that for any \( K \subset \subset \mathbb{R}^d \) and any \( \delta \in (0,T-s] \),

\[
E_{s,x} \int_{s+\delta}^T 1_{\{X_t \in K\}} |\sigma \nabla u(t,X_t) - Z^{s,x}_t| \, dt
\]

\[
\leq 2E_{s,x} \int_{s+\delta}^T 1_{\{X_t \in K\}} (|\sigma \nabla (u - u_n)(t,X_t)|^2 + |Z^{s,x,n}_t - Z^{s,x}_t|^2) \, dt
\]

\[
\leq C \delta^{-d/2} \int_{s+\delta}^T \int_T \int_K |\nabla (u - u_n)(t,y)| \, dy \, dt \, dy + 2E_{s,x} \int_{s+\delta}^T |Z^{s,x,n}_t - Z^{s,x}_t|^2 \, dt,
\]

which converges to zero as \( n \to \infty \) since \( \nabla u_n \to \nabla u \) in \( L^2_{loc}(Q_T) \). Hence, by Fatou’s lemma, \( E_{s,x} \int_s^T |\sigma \nabla u(t,X_t) - Z^{s,x}_t|^2 \, dt = 0 \), as required.

**Step 5.** We show (3.6) and that \( \mu_n \Rightarrow s,x \mu, \mu_n \to \mu \) in \( (W^{1,1}_{2,0}(Q_T))^* \) for every \( (s,x) \in [0,T] \times \mathbb{R}^d \). Let \( \xi = \xi^+ - \xi^- \in C_0(Q_T) \). By Theorem 2.4,

\[
E_{s,x} \left| \int_t^T n \xi(u_n - h)^-(\theta,X_\theta) \, d\theta - \int_t^T \xi(\theta,X_\theta) dK^{s,x}_\theta \right|^2 \to 0 \quad (3.14)
\]
for every \( t \in [s, T] \). Since \( \mu_n \Rightarrow \mu \) on \( Q_T \) and \( \mu(\{t\} \times \mathbb{R}^d) = 0 \) for every \( t \in [0, T] \), we see that \( \mu_n|_{[t_1, t_2] \times \mathbb{R}^d} \Rightarrow \mu|_{[t_1, t_2] \times \mathbb{R}^d} \) for every \( t_1 \leq t_2 \leq T \). Hence we have

\[
E_{s,x} \int_{s+\delta}^{T} \xi^+(t, X_t) \, dK^{s,x}_{t} = \lim_{n \to \infty} E_{s,x} \int_{s+\delta}^{T} \xi^+(t, X_t) \, dK^{s,x,n}_{t}
\]

\[
= \lim_{n \to \infty} \int_{s+\delta}^{T} \int_{\mathbb{R}^d} \xi^+(t, y)p(s, x, t, y) \, d\mu_n(t, y)
\]

\[
= \int_{s+\delta}^{T} \int_{\mathbb{R}^d} \xi^+(t, y)p(s, x, t, y) \, d\mu(t, y)
\]

for \( 0 \leq s < T, \delta > 0 \). Applying the monotone convergence theorem we see that

\[
E_{s,x} \int_{s}^{T} \xi^+(\theta, X_{\theta}) \, dK^{s,x}_{\theta} = \int_{s}^{T} \int_{\mathbb{R}^d} \xi^+(\theta, y)p(s, x, \theta, y) \, d\mu(\theta, y)
\] (3.15)

for all \( s \in [0, T] \). In the same manner we can see that (3.15) holds for \( \xi^- \) in place of \( \xi^+ \) and hence for \( \xi \) in place of \( \xi^+ \). Consequently, (3.6) holds for \( s \in [0, T] \). That \( \mu_n \Rightarrow s,x \mu \) now follows from (2.11), (3.14). Strong convergence of \( \{\mu_n\} \) to \( \mu \) in \( (W^{1,1}_{2,\text{loc}}(Q_T))^* \) follows from (3.8), (2.14) and the fact that \( u_n \to u \) in \( W^{0,1}_{2,\text{loc}}(Q_T) \). □

**Corollary 3.5** Under the assumption of Theorem 3.4, for any \( 0 \leq t_1 < t_2 \leq T \) and any closed subset \( F \) of \( \mathbb{R}^d \) we have

\[
\mu([t_1, t_2] \times F) = \int_{\mathbb{R}^d} E_{t_1,x} \int_{t_1}^{t_2} 1_F(X_t) \, dK^{t_1,t_2}_{t} \, dx.
\] (3.16)

**Proof.** Let us choose a sequence \( \{\xi_n\} \subset C_b(Q_T) \) of positive functions such that \( \xi_n \downarrow 1_{[t_1, t_2] \times F} \). Since (3.14) holds for \( \xi_n \) in place of \( \xi^+ \), we get (3.6) letting \( n \to \infty \) and then integrating with respect to the space variable. □

**Corollary 3.6** Let assumptions of Theorem 3.4 hold. Then \( \mu \) is absolutely continuous with respect to the Lebesgue measure with density \( r \) iff

\[
K^{s,x}_{t} = \int_{s}^{t} r(\theta, X_{\theta}) \, d\theta, \quad t \in [s, T], \quad P_{s,x} \text{-a.s.}
\] (3.17)

**Proof.** Sufficiency follows immediately from (3.6). To prove necessity, suppose that \((u, rd\lambda)\) is a weak solution of the OP\((\varphi, f, h)\) i.e. \((u - h)d\mu = 0, u \geq h\) and

\[
\frac{\partial u}{\partial t} + A_t u = -f_u - r, \quad u(T) = \varphi.
\] (3.18)

Set \( r^\varepsilon = (r \wedge \varepsilon^{-1})1_{B(0,\varepsilon^{-1})} \) and let \( u_\varepsilon \) be a weak solution of PDE\((\varphi, f_u + r^\varepsilon)\), i.e.

\[
\frac{\partial u_\varepsilon}{\partial t} + A_t u_\varepsilon = -f_u - r^\varepsilon, \quad u_\varepsilon(T) = \varphi.
\]
Then
\[
\|u_\varepsilon(t)\|^2_T + \|\sigma \nabla u_\varepsilon\|^2_{2,T} = 2 \int_t^T \langle f_u(s), u_\varepsilon(s) \rangle_2 \, ds + 2 \int_t^T \langle r^\varepsilon(s), u_\varepsilon(s) \rangle_2 \, ds
\]
\[
\leq \int_t^T \|u_\varepsilon(s)\|^2_2 \, ds + \|f_u\|^2_{2,T} + 2 \int_t^T \langle r(s), |u(s)|_2 \rangle_2 \, ds
\]
and hence, by Gronwall’s lemma,
\[
\|u_\varepsilon(t)\|^2_T + \|\sigma \nabla u_\varepsilon\|^2_{2,T} \leq C(\|f_u\|^2_{2,T} + \int_0^T \langle r(t) , |u(t)|_2 \rangle_2 \, dt),
\]
which is bounded, because \(\int_0^T \langle r(t), |u(t)|_2 \rangle_2 \, dt < \infty\) by Proposition 2.8. Since \(\{u_\varepsilon\}\) is increasing, there is \(\bar{u}\) such that \(u_\varepsilon \uparrow \bar{u}\). Since we know that \(\{u_\varepsilon\}\) is bounded in \(W^{1,2}(Q_T)\), \(u_\varepsilon \to \bar{u}\) in \(L^2(Q_T)\) and \(\nabla u_\varepsilon \to \nabla \bar{u}\) weakly in \(L^2(Q_T)\) from which it may be concluded that \(\bar{u}\) is a weak solution of (3.18). Therefore, \(u = \bar{u}\), by uniqueness of solution of PDE \(\varphi, f_u + r\). Now, define \(r_n, \mu_n\) as in Theorem 3.4. Let \(\xi \in C_b(Q_T)\). Since \(\mu_n \Rightarrow s,x \mu\) for every \((s,x) \in [0,T) \times \mathbb{R}^d\) and \(d\mu = r \, d\lambda\),
\[
E_{s,x} \int_s^T \xi(t, X_t) \, dK^{s,x}_t = E_{s,x} \int_s^T \xi(t, X_t) r(t, X_t) \, dt.
\]
Indeed, for every \(n \in \mathbb{N}\) we have
\[
E_{s,x} \int_s^T \xi(t, X_t) \, dK^{s,x,n}_t = \int_{\mathbb{R}^d} \int_s^T \xi(t, y) p(s, x, t, y) \, d\mu_n(t, y)
\]
\[
= \int_{\mathbb{R}^d} \int_s^T \xi(t, y) p(s, x, t, y) r_n(t, y) \, dt \, dy,
\]
so letting \(n \to \infty\) leads to (3.20). By approximation argument, (3.20) holds for any \(\xi \in C(Q_T)\) such that \(E_{s,x} \int_s^T |\xi(t, X_t)| \, dK^{s,x}_t < \infty\). In particular, it holds for \(\xi = u - h\). Hence, letting \(\varepsilon \downarrow 0\) and applying the Lebesgue dominated convergence theorem we obtain
\[
I^1_\varepsilon \equiv E_{s,x} \int_s^T (u_\varepsilon - h)(t, X_t) r^\varepsilon(t, X_t) \, dt
\]
\[
\to E_{s,x} \int_s^T (u - h)(t, X_t) r(t, X_t) \, dt = E_{s,x} \int_s^T (u - h)(t, X_t) \, dK^{s,x}_t = 0.
\]
By representation results proved in [23],
\[
u_\varepsilon(t, X_t) = \varphi(X_T) + \int_t^T (f_u + r^\varepsilon)(\theta, X_\theta) \, d\theta - \int_t^T \sigma \nabla u_\varepsilon(\theta, X_\theta) \, dB_{s,\theta}.
\]
Applying Itô’s formula we obtain
\[
I^2_\varepsilon \equiv E_{s,x} |u_\varepsilon(t, X_t)|^2 + E_{s,x} \int_t^T |\sigma \nabla u_\varepsilon(\theta, X_\theta^{s,x})|^2 \, d\theta
\]
for every $t \in [s, T]$. Hence, by Gronwall’s lemma,
\[ I_\varepsilon^2 \leq CE_{s,x} \left( |\varphi(X_{T}^{s,x})|^2 + \int_{s}^{T} |f_u(t, X_t)|^2 dt + |I_\varepsilon^1| + \int_{s}^{T} (\varepsilon^2 h^+(t, X_t)) dt \right). \tag{3.23} \]
For any $\alpha > 0$,
\[ E_{s,x} \int_{s}^{T} r^2(t, X_t) dt \leq E_{s,x} \left( \alpha \sup_{s \leq t \leq T} |h^+(t, X_t)|^2 + \alpha^{-1} |\int_{s}^{T} r^2(t, X_t) dt|^2 \right) \]
and, by (3.22),
\[ E_{s,x} |\int_{s}^{T} r^2(t, X_t) dt|^2 \leq E_{s,x} |\varphi(X_{T}^{s,x})|^2 + E_{s,x} \int_{s}^{T} |f_u(t, X_t)|^2 dt + I_\varepsilon^2. \tag{3.24} \]
Hence, choosing a sufficiently large $\alpha$ we see from (3.23) that
\[ I_\varepsilon^2 \leq CE_{s,x} \left( |\varphi(X_{T}^{s,x})|^2 + \int_{s}^{T} |f_u(t, X_t)|^2 dt + |I_\varepsilon^1| + \int_{s}^{T} |h^+(t, X_t)|^2 \right). \tag{3.25} \]
Therefore, combining (3.24) with (3.21), (3.25) and using Fatou’s lemma we conclude that $E_{s,x}(\int_{s}^{T} r(t, X_t) dt)^2 < \infty$. Finally, by (3.22) and Itô’s formula, for any $\varepsilon_1, \varepsilon_2 > 0$ and $\alpha > 0$ we have
\[ E_{s,x} |(u_{\varepsilon_1} - u_{\varepsilon_2})(t, X_t)|^2 + E_{s,x} \int_{s}^{T} |\sigma \nabla (u_{\varepsilon_1} - u_{\varepsilon_2})(t, X_t)|^2 dt \]
\[ \leq CE_{s,x} \int_{s}^{T} (\varepsilon_1^2 - \varepsilon_2^2)(u_{\varepsilon_1} - u_{\varepsilon_2})(t, X_t) dt \]
\[ \leq CE_{s,x} \sup_{s \leq t \leq T} |(u_{\varepsilon_1} - u_{\varepsilon_2})(t, X_t)| \int_{s}^{T} (\varepsilon_1^2 - \varepsilon_2^2)(t, X_t) dt \]
\[ \leq \alpha^{-1} CE_{s,x} \sup_{s \leq t \leq T} |(u_{\varepsilon_1} - u_{\varepsilon_2})(t, X_t)|^2 + \alpha CE_{s,x} \left( \int_{s}^{T} |(\varepsilon_1^2 - \varepsilon_2^2)(t, X_t)| dt \right)^2. \]
Hence, using the Burkholder-Davis-Gundy inequality we obtain the estimate
\[ E_{s,x} \sup_{s \leq t \leq T} |(u_{\varepsilon_1} - u_{\varepsilon_2})(t, X_t)|^2 + E_{s,x} \int_{s}^{T} |\sigma \nabla (u_{\varepsilon_1} - u_{\varepsilon_2})(t, X_t)|^2 dt \]
\[ \leq CE_{s,x} \left( \int_{s}^{T} |(\varepsilon_1^2 - \varepsilon_2^2)(t, X_t)| dt \right)^2 \]
with $C$ not depending on $\varepsilon_1, \varepsilon_2$. Therefore letting $\varepsilon \downarrow 0$ in (3.22) we see that the triple $(u(t, X_t), \sigma \nabla u(t, X_t), \int_{0}^{t} r(t, X_{t}^{s,x}) dt)$, $t \in [s, T]$, is a solution of RBSDE$(\varphi, f, h)$ which in view of uniqueness completes the proof. \qed
Lemma 3.7 If \( \tilde{u} \) is a solution of PDE(\( \varphi, f \)) and \( (u, \mu) \) is a solution of OP(\( \varphi, f, h \lor \tilde{u} \)) then \( (u, \mu) \) is a solution of OP(\( \varphi, f, h \)).

Proof. Let \( (u, \mu) \) be a solution of OP(\( \varphi, f, h \lor \tilde{u} \)). Then \( u \geq h \lor \tilde{u} \geq h \). Moreover, by comparison results, for any solution \( (u_1, \mu_1) \) of OP(\( \varphi, f, h_1 \)) with some \( h_1 \) we have \( u_1 \geq \tilde{u} \). Hence \( \mu_1 \mathbb{1}_{\{h_1 \leq \tilde{u}\}} = 0 \), and consequently,

\[
\int_{Q_T} (u - h) \, d\mu = \int_{\{h \leq \tilde{u}\}} (u - h) \, d\mu + \int_{\{h > \tilde{u}\}} (u - h) \, d\mu = \int_{\{h > \tilde{u}\}} (u - (h \lor \tilde{u})) \, d\mu = 0,
\]

which proves the lemma. \( \square \)

Lemma 3.8 Let \( \varphi \in L_{2,q}(Q_T) \), \( g \in L_{p,q,e}(Q_T) \). Then

\[
E_{s,x} |\varphi(X_T)|^2 \leq C q^{-2}(x) (T - s)^{-d/2} \|\varphi\|_{2,q}^2
\]

and

\[
E_{s,x} \int_s^T |g(t, X_t)|^2 \, dt \leq C q^{-2}(x) \|g\|_{p,q,e}^2.
\]

Proof. Both inequalities follows form Aronson’s estimates, because

\[
\int_{\mathbb{R}^d} |\varphi(y)|^2 p(s, x, T, y) \, dy \leq C q^{-2}(x) \int_{\mathbb{R}^d} |\varphi(y)|^2 |\varphi(y)| \frac{p(s, x, T, y)}{|\varphi(y - x)|^2} \, dy \leq C q^{-2}(x) (T - s)^{-d/2} \|\varphi\|_{2,q}^2
\]

and, by Hölder’s inequality,

\[
\int_s^T \int_{\mathbb{R}^d} |g(t, y)|^2 p(s, x, t, y) \, dt \, dy \\
\leq C q^{-2}(x) \int_s^T \int_{\mathbb{R}^d} |g(t, y)|^2 q^2(y)p(s, x, t, y)q^{-2}(y - x) \, dt \, dy \\
\leq C q^{-2}(x) \|g\|_{p,q,e}^2 \|p(0, 0, \cdots)q^{-2}||_{(p/2)^*,(q/2)^*},
\]

which is finite by Aronson’s estimate. \( \square \)

Lemma 3.9 If \( \varphi \in L_{2,q}(\mathbb{R}^d) \), \( g \in L_{2,q}(Q_T) \) for some \( q \in W \) and (2.6) is satisfied for every \( (s, x) \in [0, T) \times \mathbb{R}^d \), then for every \( K \subset \subset [0, T) \times \mathbb{R}^d \)

\[
\sup_{(s,x) \in K} E_{s,x} |(\varphi - \varphi_n)(X_T)|^2 \to 0
\]

and

\[
\sup_{(s,x) \in K} E_{s,x} \int_s^T |(g - g_n)(t, X_t)|^2 \, dt \to 0
\]

as \( n \to \infty \), where \( \varphi_n = \varphi \mathbb{1}_{B(0,n)} \), \( g_n = g \mathbb{1}_{B(0,n)} \).
Proof. The first assertion follows from Lemma 3.8. To prove the second, let us choose $R > 0$ such that $K \subset [0, T) \times B(0, R)$ and $x \in B(0, R)$. Then for $n \geq 2R$ we have

$$
\int_s^T \int_{\mathbb{R}^d} |(g_n - g)(t, y)|^2 p(s, x, t, y) \, dt \, dy = \int_s^T \int_{B^n(0, 2R)} |(g_n - g)(t, y)|^2 \varrho^2(y) p(s, x, t, y) \varrho^{-2}(y) \, dy \, dt
$$

$$
\leq C \varrho^{-2}(x) \int_s^T \int_{B^n(0, 2R)} |(g_n - g)(t, y)|^2 \varrho(y) \psi(s, x, t, y) \, dy \, dt,
$$

where $\psi(s, x, t, y) = (t - s)^{-d/2} \exp(-\frac{|y-x|^2}{(t-s)}) (1 + |y-x|^{2\alpha})$. Since $\psi$ is bounded for $0 \leq s < t \leq T$, $|y-x| > R$ we see that

$$
E_{s,x} \int_s^T |(g_n - g)(t, X_t)|^2 \, dt = \int_s^T \int_{\mathbb{R}^d} |(g_n - g)(t, y)|^2 p(s, x, t, y) \, dt \, dy
$$

$$
\leq C \varrho^{-2}(x) \|g_n - g\|^2_{2, \varrho, T}
$$

(3.26)

for $(s, x) \in K$, $n \geq 2R$, which completes the proof. \hfill \box

**Theorem 3.10** Let assumptions (H1)-(H3) hold with $\varphi \in L_{2, \varrho}(\mathbb{R}^d)$, $g \in L_{2, \varrho}(Q_T)$, where $\varrho \in \mathcal{W}$ and $\varrho < 1$. Moreover, assume that $g$ satisfies (2.4), $h \in C(Q_T)$ and $h \leq c \bar{\varrho}^{-1}$ for some $c > 0$ and $\bar{\varrho} \in \mathcal{W}$ such that $\bar{\varrho}^{-1} \in L_{2, \varrho}(\mathbb{R}^d)$. Then there exists a unique solution $(u, \mu)$ of $OP(\varphi, f, h)$ such that $u \in C([0, T] \times \mathbb{R}^d) \cap W^{2, 1}_{2, \varrho}(Q_T)$ and (3.5), (3.6) hold for each $(s, x) \in [0, T] \times \mathbb{R}^d$.

**Proof.** We divide the proof into two steps: the case of linear and semilinear equation. 

**Step 1.** We first assume that $f = f(t, x), (t, x) \in Q_T$ satisfies (2.4) with $g$ replaced by $f$. Suppose that $h(x) \leq c \bar{\varrho}^{-1}$, where $\bar{\varrho}(x) = (1 + |x|^2)^{-\beta}$ for some $c, \beta > 0$. Set $\varphi_n = 1_{B(0, n)} \varphi$, $f_n = 1_{B(0, n)} f$ and consider a sequence $\{h_n\} \subset W^{2, 1}_{2, \varrho}(Q_T)$ such that $h_n \leq 2c \bar{\varrho}^{-1}$, $n \in \mathbb{N}$, and $h_n \to h$ uniformly in compact subsets of $Q_T$. By Theorem 3.4, for each $n \in \mathbb{N}$ there is a unique solution $(u_n, \mu_n)$ of $OP(\varphi_n, f_n, h_n)$, and moreover, 

$$(u_n(t, X_t), \sigma \nabla u_n(t, X_t)) = (Y_t^{s, x, n}, Z_t^{s, x, n}), \quad P_{s,x}-a.s.$$ 

and 

$$
\int_s^T \int_{\mathbb{R}^d} \xi(t, y) p(s, x, t, y) \, d\mu_n(t, y) = E_{s,x} \int_s^T \xi(t, X_t) dK_t^{s, x, n}
$$

for all $\xi \in C_b(Q_T)$, where $(Y_t^{s, x, n}, Z_t^{s, x, n}, K_t^{s, x, n})$ is a solution of $RBSDE(\varphi_n, f_n, h_n)$.

By Lemma 2.5,

$$
E_{s,x} \sup_{s \leq t \leq T} |(u_n - u_m)(t, X_t)|^2 + E_{s,x} \sup_{s \leq t \leq T} |K_t^{s, x, n} - K_t^{s, x, m}|^2
$$

$$
+ E_{s,x} \int_s^T |\sigma \nabla (u_n - u_m)(t, X_t)|^2 \, dt
$$

$$
\leq E_{s,x} \sup_{s \leq t \leq T} |(h_n - h_m)(t, X_t)|^2 + E_{s,x} |(\varphi_n - \varphi)(X_T)|^2
$$

$$
+ E_{s,x} \int_s^T |(f_n - f_m)(\theta, X_\theta)|^2 \, d\theta.
$$

(3.27)
From this and Theorem 2.2 we deduce that

\[
\|(u_n - u_m)(s)\|_{2,q}^2 + \|\nabla (u_n - u_m)\|_{2,q,T}^2 \\
\leq C \left( \int_{\mathbb{R}^d} E_{s,x} \sup_{s \leq t \leq T} |(h_n - h_m)(t,X_t)|^2 \varphi^2(x) \, dx + \|\varphi_n - \varphi_m\|_{2,q}^2 \\
+ \|f_n - f_m\|_{2,q,T}^2 \right). \tag{3.28}
\]

Using Theorem 2.2 we also get

\[
\sup_{0 \leq t \leq T} \|(u_n - u_m)(t)\|_{2,q}^2 \leq C \int_{\mathbb{R}^d} \sup_{0 \leq t \leq T} E_{0,x} |(u_n - u_m)(t,X_t)|^2 \varphi^2(x) \, dx. \tag{3.29}
\]

Due to Lemma 3.7, without loss of generality we may assume that \(h_n \geq \tilde{u}_n\), where \(\tilde{u}_n\) is a solution of PDE(\(\varphi_n, f_n\)). From comparison theorem (see [11]) we know that \(\underline{u} \leq \tilde{u}_n\), where \(\underline{u}\) is a continuous solution of PDE(\(-|\varphi|, -|f|\)), and that \(\underline{u}_n \searrow \underline{u}\), where \(\underline{u}_n\) is a continuous solution of PDE(\(-|\varphi_n|, -|f_n|\)). Since

\[
E_{s,x} \sup_{s \leq t \leq T} |(\underline{u}(t,X_t))^+|^2 \leq E_{s,x} \sup_{s \leq t \leq T} |(\underline{u}_n(t,X_t))^+|^2
\]

and

\[
E_{s,x} \sup_{s \leq t \leq T} |(\underline{u}(t,X_t))^-|^2 = E_{s,x} \lim_{n \to \infty} \sup_{s \leq t \leq T} |(\underline{u}_n(t,X_t))^-|^2 \leq \liminf_{n \to \infty} E_{s,x} \sup_{s \leq t \leq T} |(\underline{u}_n(t,X_t))^-|^2,
\]

from a priori estimates for solutions of BSDE(\(-|\varphi_n|, -|f_n|\)) (see [23]) we get

\[
E_{s,x} \sup_{s \leq t \leq T} |(\underline{u}(t,X_t))|^2 \leq C E_{s,x} \left( |\varphi(X_T)|^2 + \int_{t}^{T} |g(t,X_t)|^2 \, dt \right). \tag{3.30}
\]

Since \(|h_n(t,X_t)| \leq |\underline{u}(t,X_t)| + (1 + |X_t|^2)\beta\) and \(\{h_n\}\) converges uniformly in compact subsets of \(Q_T\), using (3.30), Proposition 3.2 and Lemma 3.9 we conclude that the right-hand side of (3.27) converges to zero as \(n,m \to \infty\). From this it follows that there is \(u\) such that \(u_n \to u\) pointwise in \([0,T) \times \mathbb{R}^d\). Moreover, using (3.28), (3.29) and arguing as in the proof of convergence of the right-hand side of (3.10) we conclude that \(u_n \to u\) in \(W^{0,1}_{2,q}(Q_T)\) and \(u_n \to u\) in \(C([0,T], L_{2,q}(\mathbb{R}^d))\). By the definition of solution of \(\text{OP}(\varphi_n, f_n, h_n)\),

\[
\int_{t}^{T} \langle u_n(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 \, ds + \frac{1}{2} \int_{t}^{T} \langle a(s) \nabla u_n(s), \nabla \eta(s) \rangle_2 \, ds \\
= \int_{t}^{T} \langle f_n(s), \eta(s) \rangle_2 \, ds + \int_{t}^{T} \int_{\mathbb{R}^d} \eta \, d\mu_n + \langle \varphi_n, \eta(T) \rangle_2 - \langle u_n(t), \eta(t) \rangle_2 \tag{3.31}
\]

for any \(\eta \in W^{1,1}_{2,q}(Q_T)\). Therefore, if \(K \subset \subset [0,T) \times \mathbb{R}^d\), then choosing \(\eta \in W^{1,1}_{2,q}(Q_T) \cap C_0(Q_T)\) such that \(\eta \equiv 1\) on \(K\) and \(0 \leq \eta \leq 1\) we deduce from (3.31) and Proposition 2.9 applied to \((u_n, \mu_n)\) and \(p = 2\beta^{-1}\) that \(\sup_{n \geq 1} \mu_n(K) < \infty\). Thus, \(\{\mu_n\}\) is tight. Taking a subsequence if necessary we may assume that \(\mu_n \Rightarrow \mu\), where \(\mu\) is a Radon measure.
Step 1

the definition of $\Phi$ is correct. We are going to show that $\Phi$ is contractive on $M$. Since

$n$ by putting $\Phi(u)$

Finally, let $\Phi = \Phi(u)$.

Step 2.

We consider the general semilinear case. For $\gamma > 0$ to be determined later let $V(\gamma)$ denote the Banach space consisting of elements $u$ of $W^{0,1}_{2,\varrho}(Q_T) \cap C([0, T], L_{2,\varrho}(\mathbb{R}^d))$ equipped with the norm

$$
\|u\|_{V(\gamma)}^2 = \sup_{0 \leq s \leq T} \|u_\gamma(s)\|_{2,\varrho}^2 + \|u_\gamma\|_{2,2,T}^2 + \frac{\lambda}{2} \|\nabla u_\gamma\|_{2,\varrho,T}^2,
$$

where $u_\gamma(s, x) = e^{\gamma s/2} u(s, x)$. Write $K_n = [0, T - 1/n] \times B(0, n)$. By $W(\gamma)$ we denote the Fréchet space of elements of $W^{0,1}_{2}(Q_T)$ such that $\sup_{(s, x) \in K_n} \|u\|_{\gamma, s, x} < \infty$ for all $n \in \mathbb{N}$ equipped with the $F$-norm

$$
\|u\|_{W(\gamma)}^2 = \sum_{n=0}^{\infty} \frac{1}{2^{n}} \sup_{(s, x) \in K_n} \|u\|_{\gamma, s, x}^2,
$$

where $\|u\|_{\gamma, s, x}^2 = \int_s^T \int_{\mathbb{R}^d} e^{\gamma t} (|u(t, y)|^2 + \lambda |\nabla u(t, y)|^2) p(s, x, t, y) dt dy$, and by $B$ the Fréchet space of elements of $C([0, T] \times \mathbb{R}^d)$ with the $F$-norm

$$
\|u\|_B = \sum_{n=1}^{\infty} \frac{1}{2^{n}} \|u1_{K_n}\|_{\infty}.
$$

Finally, let $M_{\gamma}$ denote the Fréchet space $B \cap V(\gamma) \cap W(\gamma)$ equipped with the $F$-norm $\|u\|_{\gamma} = \|u\|_{V(\gamma)} + \|u\|_{W(\gamma)} + \|u\|_B$. Now, define the mapping $\Phi : M_{\gamma} \rightarrow M_{\gamma}$ by putting $\Phi(v)$ to be the first component of the solution $(u, \mu)$ of OP($\varphi, f, h$). By $Step 1$ the definition of $\Phi$ is correct. We are going to show that $\Phi$ is contractive on $M_{\gamma}$. Let $v_1, v_2 \in M_{\gamma}$ and let $(u_i, \mu_i), i = 1, 2$, be solutions of OP($\varphi, f, h$). Set
Letting \( n = u_1 - u_2 = \Phi(v_1) - \Phi(v_2), \mu = \mu_1 - \mu_2 \). By the definition of a solution of the obstacle problem,

\[
\langle u(t), \eta(t) \rangle_2 + \int_t^T \langle u(s), \frac{\partial \eta}{\partial s}(s) \rangle_2 ds + \frac{1}{2} \int_t^T \langle a(s) \nabla u(s), \nabla \eta(s) \rangle_2 ds
\]

\[
= \int_t^T \int_{\mathbb{R}^d} \eta d\mu + \int_t^T \langle f_{v_1}(s) - f_{v_2}(s), \eta(s) \rangle_2 ds.
\]

Consequently, the right-hand side of the above inequality may be estimated by \( \eta \) and performing computations similar to that in the proof of Theorem 3.1, we obtain

\[
e^{\gamma t} \langle u(t), u(t) \rangle_2 + \gamma \int_t^T e^{\gamma s} \langle u(s), u(s) \rangle_2 ds + \frac{1}{2} \int_t^T e^{\gamma s} \frac{d}{ds} \|u(s)\|_2^2 ds
\]

\[
+ \frac{1}{2} \int_t^T e^{\gamma s} \langle a(s) \nabla u(s), \nabla (u \xi_n^2) \rangle_2 ds \leq \int_t^T e^{\gamma s} \langle f_{v_1}(s) - f_{v_2}(s), u(s) \xi_n^2 \rangle_2 ds.
\]

Consequently,

\[
e^{\gamma t} \|u(t)\|_2^2 + \gamma \int_t^T e^{\gamma s} \|u(s)\|_2^2 ds + \int_t^T e^{\gamma s} \langle a(s) \nabla u(s), \nabla (u \xi_n^2) \rangle_2 ds
\]

\[
\leq 2 \int_t^T e^{\gamma s} \langle f_{v_1}(s) - f_{v_2}(s), u(s) \xi_n^2 \rangle_2 ds.
\]

Letting \( n \to \infty \) and performing computations similar to that in the proof of Theorem 3.1 we get

\[
e^{\gamma t} \|u(t)\|_2^2 + \gamma \int_t^T e^{\gamma s} \|u(s)\|_2^2 ds + \frac{\lambda}{2} \int_t^T e^{\gamma s} \|\nabla u(s)\|_2^2 ds
\]

\[
\leq 2 \int_t^T e^{\gamma s} \langle f_{v_1}(s) - f_{v_2}(s), u(s) \xi_n^2 \rangle_2 ds + \frac{\lambda}{2\lambda} \int_t^T e^{\gamma s} \|u(s)\|_2^2 ds.
\]

The right-hand side of the above inequality may be estimated by

\[
2L \int_t^T e^{\gamma s} \|(v_1 - v_2)(s)\|_2^2 \|u(s)\|_2^2 ds
\]

\[
+ 2LA \int_t^T e^{\gamma s} \|\nabla (v_1 - v_2)(s)\|_2^2 \|u(s)\|_2^2 ds + \frac{\lambda}{2\lambda} \int_t^T e^{\gamma s} \|u(s)\|_2^2 ds
\]

\[
\leq (4L^2 + \frac{8\Lambda^2 L^2}{\lambda} + \frac{\lambda}{2\lambda}) \int_t^T e^{\gamma s} \|u(s)\|_2^2 ds + \frac{1}{4} \int_t^T e^{\gamma s} \|(v_1 - v_2)(s)\|_2^2 ds
\]

\[
+ \frac{1}{2} \int_t^T e^{\gamma s} \|\nabla (v_1 - v_2)(s)\|_2^2 ds.
\]
Putting \( \gamma = 1 + 4L^2 + 8\lambda^{-1}\Lambda^2L^2 + (2\lambda)^{-1}\Lambda \) we see that
\[
\| \Phi(v_1) - \Phi(v_2) \|_{V(\gamma)} \leq 2^{-1} \| v_1 - v_2 \|_{V(\gamma)}.
\] (3.32)

Let \((Y^{s,x,i}, Z^{s,x,i}, K^{s,x,i}), i = 1, 2, \) denote a solution of \(\text{RBSDE}(\varphi, f_u, h)\) and let \(v = v_1 - v_2\). We already know that \((Y^{s,x,i}, Z^{s,x,i}) = (u_i(t, X_t), \sigma \nabla u_i(t, X_t)), t \in [s, T]\). Therefore, since \(E_{s,x} \int_t^T e^{\gamma \theta} v(\theta, X_\theta) d(K_\theta^{s,x,1} - K_\theta^{s,x,2}) \leq 0 \) for every \(t \in [s, T]\), using Itô's formula we have
\[
E_{s,x} e^{\gamma t} |u(t, X_t)|^2 + E_{s,x} \int_t^T e^{\gamma \theta} (\gamma |u(\theta, X_\theta)|^2 + |\sigma \nabla u(\theta, X_\theta)|^2) d\theta
\]
\[
\leq 2E_{s,x} \int_t^T e^{\gamma \theta} u(v_1 - f_{v_2})(\theta, X_\theta) d\theta
\]
\[
\leq 2LE_{s,x} \int_t^T e^{\gamma \theta} u(\theta, X_\theta)(|v_1| + |\sigma \nabla v_1|)(\theta, X_\theta) d\theta
\]
\[
\leq 8\lambda^{-1}\Lambda L^2 \varepsilon E_{s,x} \int_t^T e^{\gamma \theta} |u(\theta, X_\theta)|^2 d\theta
\]
\[
+ \varepsilon^{-1} E_{s,x} \int_t^T e^{\gamma \theta} (|v_1|^2 + \lambda |\nabla v_1|^2)(\theta, X_\theta) d\theta.
\]

Putting \( \gamma = 1 + 8\Lambda\lambda^{-1}L^2\varepsilon \) with suitably chosen \(\varepsilon > 0\) in a standard manner we deduce from the above and the Burkholder-Davis-Gundy inequality that
\[
E_{s,x} \sup_{s \leq t \leq T} e^{\gamma t} |u(t, X_t)|^2 + E_{s,x} \int_s^T \lambda e^{\gamma t} (|u(t, X_t)|^2 + |\nabla u(t, X_t)|^2) dt
\]
\[
\leq 4^{-1} E_{s,x} \int_s^T e^{\gamma t} (|v_1|^2 + \lambda |\nabla v_1|^2)(t, X_t) dt.
\] (3.33)

From this we obtain
\[
\| \Phi(v_1) - \Phi(v_2) \|_B + \| \Phi(v_1) - \Phi(v_2) \|_{V(\gamma)} \leq 2^{-1} (\| v_1 - v_2 \|_B + \| v_1 - v_2 \|_{V(\gamma)}),
\]
which when combined with (3.32) shows that \(\Phi\) is contractive on \(M_\gamma\). By Banach’s principle, \(\Phi\) has a unique fixed point \(u\). Clearly, the solution \((u, \mu)\) of \(\text{OP}(\varphi, f_u, h)\) has the asserted properties.

One can prove Theorem 3.10 by the method of stochastic penalization used in the proof of Theorem 3.4. To apply that method one should first generalize results of [23] on representation of solutions of the Cauchy problem proved for \(\varphi \in L_2(\mathbb{R}^d), g \in L_2(Q_T)\) to the case \(\varphi \in L_{2,\theta}(\mathbb{R}^d), g \in L_{2,\theta}(Q_T)\) for some \(\theta \in W\). Since detailed proof of such a generalization does not bring new ideas and at the same time requires some efforts, we decided to present a different approach. Note, however, that the adopted approach uses some ideas from [23].

**Corollary 3.11** Let assumptions of Theorem 3.10 hold. Define \((u_n, \mu_n)\) as in Theorem 3.4. Then

(i) \(u_n \uparrow u\) uniformly in compact subsets of \([0, T) \times \mathbb{R}^d\), \(u_n \rightharpoonup u\) in \(W_{2,\theta}^{0,1}(Q_T) \cap C([0, T], L_{2,\theta}(Q_T))\),

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(ii) \( \mu_n \Rightarrow \mu, \mu_n \rightharpoonup \mu \) in \( (W_{2,q}^{1,1}(Q_T))^*, \mu_n \rightharpoonup_{s,x} \mu \) for every \( (s, x) \in [0, T) \times \mathbb{R}^d \).

Proof. Follows from Theorems 2.2, 2.4 and 3.10. \( \square \)

Let us remark that Corollaries 3.5, 3.6 hold also under the assumptions of Theorem 3.10. The proof of Corollary 3.5 runs as before. In the proof of Corollary 3.6 the main difference consists in the fact that instead of boundedness of \( \{u_n\} \) in \( W_{2, q}^{0,1}(Q_T) \) (see (3.19)) we have to prove its boundedness in \( W_{2, q}^{1,1}(Q_T) \). The last assertion one can show using arguments from the proof of Proposition 2.9.

**Corollary 3.12** Let assumptions of Theorem 3.4 or Theorem 3.10 hold and let \( (u, \mu) \) be a solution of \( \text{OP}(\varphi, f, h) \).

(i) If \( g \in L_{p,q,\varphi}(Q_T) \) then

\[
|u(s, x)| + \|u\|_{W_2(s,x,T)} \leq C_\varphi^{-1}(x)(1 + (T - s)^{-d/2}\|\varphi\|_{L_\varphi}^2 + \|f\|_{L_{p,q,\varphi}}^2)^{1/2}.
\]

(ii) If \( |\varphi| \leq cg^{-1} \) for some \( c > 0, \varphi \in W \) (i.e. \( \varphi \) satisfies the polynomial growth condition) then

\[
|u(s, x)| + \|u\|_{W_2(s,x,T)} \leq C_\varphi^{-1}(x)(1 + \|f\|_{L_{p,q,\varphi}}^2)^{1/2}.
\]

(iii) If \( |\varphi| + |g| \leq cg^{-1} \) for some \( c > 0, \varphi \in W \) then

\[
|u(s, x)| + \|u\|_{W_2(s,x,T)} \leq C_\varphi^{-1}(x).
\]

Proof. It follows from (2.12), Theorem 3.10, Proposition 3.2 and Lemma 3.8. \( \square \)

It is known that the obstacle problem (1.3) with non-divergent operator \( A_t \) appear as the Hamilton-Jacobi-Bellman equation for an optimal stopping time problem (see [4]) and that value functions of that stopping problem is given by the first component of a solution of an RBSDEs with forward driving processes associated with \( A_t \) (see [11]). It is worth noting that similar relations hold for divergence form operators.

**Corollary 3.13** Let assumptions of Theorem 3.4 or Theorem 3.10 hold and let \( (u, \mu) \) be a solution of \( \text{OP}(\varphi, f, h) \). Then for each \( t \in [s, T] \),

\[
u(s, x) = \sup_{\tau \in T^s_t} E_{s,x}(\int_t^\tau f(\theta, X_\theta, u(\theta, X_\theta), \sigma \nabla u(\theta, X_\theta)) d\theta + h(\tau, X_\tau) 1_{\tau < T} + \varphi(X_T) 1_{\tau = T}|G_t^\tau),
\]

where \( T^s_t = \{ \tau \in T^s : t \leq \tau \leq T \} \) and \( T^s \) denote the set of all \( \{G_t^\tau\}- \)stopping times.

Proof. Let \( \tau \in T^s_t \). By (3.5) and the definition and a priori estimate for a solution of RBSDE(\( \varphi, f, h \)) we have

\[
u(t, X_t) = E_{s,x}(\int_t^\tau f(\theta, X_\theta, u(\theta, X_\theta), \sigma \nabla u(\theta, X_\theta)) d\theta + u(\tau, X_\tau) + K^s_{\tau} - K^s_{t}|G_t^\tau)
\] \[
\geq E_{s,x}(\int_t^\tau f(\theta, X_\theta, u(\theta, X_\theta), \sigma \nabla u(\theta, X_\theta)) d\theta + h(\tau, X_\tau) 1_{\tau < T} + \varphi(X_T) 1_{\tau = T}|G_t^\tau).
\]

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Let us define the optimal control by putting \( D_t = \inf \{ t \leq \theta \leq T : u(\theta, X_\theta) = h(\theta, X_\theta) \} \). Since \( K_{s,x}^{s,x} \) is continuous and \( \int_0^T (u(t, X_t) - h(t, X_t)) dK_{s,x}^t = 0 \) \( P_{s,x} \)-a.s., it follows that \( K_{D_t}^{s,x} - K_{s,x}^{s,x} = 0 \) \( P_{s,x} \)-a.s., which proves the corollary.

The next theorem provides a probabilistic formula for the minimal weak solution of the variational inequality associated with (1.3).

**Theorem 3.14** Assume that (H1)–(H3) hold with \( \varphi \in L^2_2(\mathbb{R}^d) \), \( g \in L^2_{2,\rho}(Q_T) \) for some \( \rho \in W \) and with \( h \in C(Q_T) \) satisfying the polynomial growth condition. Then there exists a version \( u \) of minimal weak solution of \( OP(\varphi, f, h) \) in the variational sense such that if (2.6) is satisfied for some \( (s, x) \in [0, T] \times \mathbb{R}^d \) then

\[
(Y_{t}^{s,x}, Z_{t}^{s,x}) = (u(t, X_t), \sigma \nabla u(t, X_t)), \quad t \in [s, T], \quad P_{s,x} \text{-a.s.}
\]  

**Proof.** By [4, Theorem 4.1.6] there exists the minimal weak solution \( \bar{u} \) of \( OP(\varphi, f, h) \) in the variational sense. Repeating arguments from the proof of Proposition 2.3 we show that there is a version \( u \) of a weak solution of the linear \( OP(\varphi, f_{\bar{u}}, h) \) in the variational sense such that (3.34) holds if (2.6) is satisfied. Since \( ||g||_{2,\rho} < \infty \), it follows that (2.6) is satisfied for a.e. \( (s, x) \in [0, T] \times \mathbb{R}^d \) (see remark following the proof of Theorem 2.4). Therefore, by Theorems 2.2 and 2.4, \( u \) is a limit in \( W^{0,1}_{2,\rho}(Q_T) \) of the penalizing sequence defined by (3.4), and hence (see the proof of [4, Theorem 4.1.6]), \( u \) is a minimal weak solution of \( OP(\varphi, f_{\bar{u}}, h) \) in the variational sense. Since the minimal solution is unique, \( u = \bar{u} \), and the proof is complete.

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