Fraction of isospectral states exhibiting quantum correlations

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For several types of correlations: mixed-state entanglement in systems of distinguishable particles, particle entanglement in systems of indistinguishable bosons and fermions and non-Gaussian correlations in fermionic systems we estimate the fraction of non-correlated states among the density matrices with the same spectra. We prove that for the purity exceeding some critical value (depending on the considered problem) fraction of non-correlated states tends to zero exponentially fast with the dimension of the relevant Hilbert space. As a consequence a state randomly chosen from the set of density matrices possessing the same spectra is asymptotically a correlated one. To prove this we developed a systematic framework for detection of correlations via nonlinear witnesses.

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The notion of quantum correlations in physical systems is a concept that depends both on the system as well as on the physical property in question. Taking as a paradigmatic example a familiar notion of entanglement in systems of distinguishable particles [1], we may construct a general scheme of defining quantum correlations. We start with a class of pure states \( \mathcal{M} \) lacking the desired correlation property (in the case of entanglement these are all pure product states). The non-correlated (non-entangled) mixed states are further defined as statistical mixtures of pure states taken from the chosen class \( \mathcal{M} \). All other states are then called correlated (entangled).

The same scheme can be extended to other interesting cases by modifying the choice of the class \( \mathcal{M} \) of ‘non-correlated’ pure states. For indistinguishable particles the indispensable (anti)symmetrization of the wavefunction under permutation of subsystems introduces strong quantum correlations. Nevertheless one can pose a legitimate question about nature of correlations which go beyond the mere fact that the states are anti(symmetric). In fact, as recently shown in [2], in the case of bosons such correlations can be extracted into an entangled state of distinguishable subsystems represented by independent modes. To analyze a role of double-occupancy errors for operation of quantum gates composed of two quantum dots, the authors of [3] introduced a measure of correlations in fermionic two-particle systems. It ascribes the vanishing entanglement (or, in other words, vanishing correlations) only to pure states which are expressible in terms of a single Slater determinant. The pure non-correlated states are again probabilistic mixtures of non-correlated pure ones. This construction was generalized in [4] to fermionic and bosonic systems of arbitrary fixed number of particles occupying a finite number of one-particle states. Here the underlying Hilbert space is no longer a product of Hilbert spaces of individual subsystems, but rather an antisymmetric or symmetric part of it. The class \( \mathcal{M} \) of non-correlated pure states consists of, respectively, the states in the form of a single Slater determinant and the product bosonic states.

Quantum information theory with bosons [5], and fermions [6–8] in the Gaussian settings, where the number of particles can vary, is another area where we can apply the above scheme to discriminate non-correlated and correlated states. The underlying Hilbert spaces are the bosonic and fermionic Fock spaces, whereas the classes of non-correlated pure states are obtained from the vacuum state by actions of Hamiltonians quadratic in, respectively, bosonic and fermionic creation and annihilation operators. In the later case states that are not-correlated are related to computation protocols performed with Majorana fermions that can be classically simulated [3]. Moreover, states that cannot be written as a statistical combination of fermionic Gaussian states are states not described by the Bogolyubov mean field theory [9].

Although in the following we concentrate on correlations in the systems mentioned above (distinguishable and non-distinguishable particles, fermionic Gaussian states) it is worth mentioning that the outlined general scheme of defining (non)-correlated states can be further extended to encompass, e.g., \( k \)-separable states [10]. Another important type of correlations that can be analyzed within the same frame are ‘non-classical’ properties of light [11]. Here the class \( \mathcal{M} \) consist of Glauber coherent states and ‘classical’ (‘non-correlated’) mixed states are precisely those having a positive \( P \)-representation. This notion of classicality was extended to spin states [12], where again the same construction applies [1].

In all considered cases the correlation properties are invariant with respect to specific classes of transformations performed on the system in question. Thus, e.g., entanglement of distinguishable particles is unchanged under local unitary transformations. In all cases such correlation-preserving operations form a proper subset of all (global) unitary transformation that can be applied to the whole system. Global unitary transformation preserve the spectrum of a density matrix but, at the same time, change its correlation properties. As a consequence the correlation properties can not be decided upon examining the spectrum of a state. Among density matrices with the same spectra we find correlated as well as non-correlated states. An answer to a natural question about
the fraction of non-correlated states is the main result of the paper.

To achieve the goal we present a unified scheme for detection of correlations defined in the above-described way. It has its own merits that will be elaborated in forthcoming publications, here we describe only the simplest form of it, suitable for the present purposes.

The first step is a proper description of the uncorrelated pure states $\mathcal{M}$. Observe that there is no observable having vanishing expectation value only on non-correlated states [14]. Instead, we assume that the class of non-correlated pure states $\mathcal{M}$ is defined by a condition involving two copies of a state,

$$|\psi\rangle \text{ is non-correlated } \iff \langle\psi| (\rho \otimes \rho)|\psi\rangle = 0, \quad (1)$$

where $\rho$ is a suitably chosen projection operator acting in the double tensor product, $\mathcal{H} \otimes \mathcal{H}$, of the underlying Hilbert space $\mathcal{H}$. In the following we show that indeed, this is a correct definition of $\mathcal{M}$ in all considered cases. Our criterion for detection of correlations in the mixed states takes a particularly simple form:

$$\text{tr} \,( (\rho_1 \otimes \rho_2 ) \cdot V ) > 0 \implies \rho_1 \text{ and } \rho_2 \text{ are correlated}, \quad (2)$$

where $V = A - P_{\text{asym}}$ and $P_{\text{asym}}$ denotes the orthogonal projection onto the two fold antisymmetrization, $\Lambda^2 \mathcal{H}$, of $\mathcal{H}$. A particular choice $\rho_1 = \rho_2 = \rho$ leads to a quadratic witness of correlations

$$f(\rho) = \text{tr} \,( (\rho \otimes \rho ) \cdot V ) > 0 \implies \rho \text{ is correlated}. \quad (3)$$

It is important to note that the criterion is independent on the dimension of $\mathcal{H}$ and uses only algebraic structure of the set $\mathcal{M}$.

In the following it will be expedient to identify pure states with rank-one density matrices, i.e., $|\psi\rangle \sim |\psi\rangle\langle\psi|$ for a normalized $|\psi\rangle$. Under such an identification the set of uncorrelated pure states $\mathcal{M}$ can be treated as subset of $\mathcal{H}$, as well as a subset of the set of rank-one density matrices denoted in the following by $\mathcal{D}_1(\mathcal{H})$. To keep the notation compact we will alternate between both interpretations of $\mathcal{M}$, as it usually does not cause confusion. We will use $\mathcal{D}(\mathcal{H})$ do denote the set of all states (non-negatively definite, trace-one operators on $\mathcal{H}$).

The set of mixed correlated states can be now identified with the convex hull $\mathcal{M}^c$ of $\mathcal{M}$,

$$\mathcal{M}^c = \left\{ \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| : p_i > 0, \sum_i p_i = 1, |\psi_i\rangle \in \mathcal{M} \right\}. \quad (4)$$

We are now ready to state two theorems from which we deduce the criterion (2). We present their proofs in the Supplemental Material [15].

**Theorem 1.** Assume that there exists a Hermitian operator acting on $\mathcal{H} \otimes \mathcal{H}$ such that $\langle v | (w \otimes V |v\rangle |w\rangle \leq 0$ for all $|v\rangle \in \mathcal{M}$ and for arbitrary $|w\rangle \in \mathcal{H}$. Then, for any state $\rho \in \mathcal{M}^c$ and for arbitrary non-negatively defined operator $B$ acting on $\mathcal{H}$, we have

$$\text{tr} \,( (\rho \otimes B ) \cdot V ) \leq 0. \quad (5)$$

**Theorem 2.** Consider the class of pure states $\mathcal{M}$ defined by the condition $\text{tr} \,( (\rho \otimes \rho ) \cdot V ) > 0$. The operator $V = A - P_{\text{asym}}$ fulfills assumptions of Theorem 1. Note that $\tau V \tau = V$, where $\tau$ is the operator swapping between two factors of the tensor product $\mathcal{H} \otimes \mathcal{H}$. Using Theorems 1 and 2 we arrive at the result given by (2).

Below we give formulas for the operator $A$ for four considered classes of correlations and accompany them with some exemplary applications for Slater determinants. For Slater determinants we consider the depolarisation of an arbitrary pure state of two fermions [4].

**Separable states.** For a system of $L$ distinguishable particles the Hilbert space is $\mathcal{H}_d = \bigotimes_{i=1}^L \mathcal{H}_i$. For simplicity we assume that all $\mathcal{H}_i$ are identical, $\mathcal{H}_i \cong \mathbb{C}^d$. Pure separable states are given by

$$\mathcal{M}_\text{sep} = \{|\phi_1\rangle \otimes \ldots \otimes |\phi_L\rangle | |\phi_i\rangle \in \mathcal{H}_i \}. \quad (6)$$

We introduce the notation

$$\mathcal{H}_d \otimes \mathcal{H}_d = \left( \bigotimes_{i=1}^L \mathcal{H}_i \right) \otimes \left( \bigotimes_{i'=1}^{L'} \mathcal{H}_{i'} \right) \quad (7)$$

where where $L = L'$ and spaces from the second copy of the total space are labeled with primes. It was proven in [17] that the set $\mathcal{M}_\text{sep}$ is characterized by the condition $\mathcal{M}_\text{sep}$ where operator $A$ is given by

$$A_d = P_{\text{asym}} - \sum_{i=1}^L P_{i1'}^+ P_{22'}^+ \ldots P_{L L'}^+ \quad (8)$$

where $P_{i'}$ projects onto $\text{Sym}^2(\mathcal{H}_i)$ and operators $P_{i'}^+ : \mathcal{H}_d \otimes \mathcal{H}_d \rightarrow \mathcal{H}_d \otimes \mathcal{H}_d$ that projects onto the subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$ completely symmetric under interchange spaces $i$ and $i'$. Applying the above above result to criterion (2) we recover “quadratic entanglement witness” considered before by, among others, P. Horodecki [13], F. Mintert, A. Buchleitner [17]. For a general discussion of non-linear entanglement witnesses see also [19]. Interesting variation of this method can be found in [20].

**Separable bosonic states** [4]. The relevant Hilbert space describing the system consisting of $L$ bosonic particles is the $L$-fold symmetrization of a single-particle $d$-dimensional space, $\mathcal{H}_d = \text{Sym}^L(\mathbb{C}^d)$. The set of pure bosonic separable states are defined by

$$\mathcal{M}_b = \{|\phi\rangle \otimes |\phi\rangle \otimes \ldots \otimes |\phi\rangle | |\phi\rangle \in \mathbb{C}^d \}. \quad (9)$$
We can treat $\mathcal{H}_b$ and $\mathcal{H}_b \otimes \mathcal{H}_a$ as subspaces of respectively $\mathcal{H}_c$ and $\mathcal{H}_d \otimes \mathcal{H}_d$ defined in the previous part. It was shown [3] that operator $A$ can be expressed by

$$A_b = \mathbb{P}_b^{\text{sym}} - \left( \mathbb{P}_{1,11}^+ \circ \mathbb{P}_{2,22}^+ \circ \cdots \circ \mathbb{P}_{L,L'}^+ \right) \left( \mathbb{P}_c \mathbb{P}_c^\dagger \mathbb{P}_c \mathbb{P}_c^\dagger \cdots \mathbb{P}_d \mathbb{P}_d^\dagger \right) \mathbb{P}_d \mathbb{P}_d^\dagger$$

(10)

where $\mathbb{P}_b^{\text{sym}}$ projects onto $\text{Sym}^2 (\mathcal{H}_b)$ and $\mathbb{P}_c \mathbb{P}_c^\dagger \mathbb{P}_c \mathbb{P}_c^\dagger$ are projectors onto subspaces of $\mathcal{H}_d \otimes \mathcal{H}_d$ completely symmetric under interchange of spaces labeled by indices from the set $\{1,2,\ldots,L\}$ and $\{1',2',\ldots,L'\}$ respectively.

**Slater determinants** [4]. The Hilbert space describing $L$ fermions is the $L$-fold antisymmetrization of the single-particle $d$-dimensional space, $\mathcal{H}_f = \mathbb{A}^L (\mathbb{C}^d)$. We distinguish the class of Slater determinants

$$\mathcal{M}_f = \left\{ |\phi_1 \wedge |\phi_2 \wedge \cdots \wedge |\phi_L \rangle | \langle \phi_1 | \phi_2 \rangle = \delta_{ij} \right\}$$

(11)

As before, we treat $\mathcal{H}_f$ and $\mathcal{H}_f \otimes \mathcal{H}_f$ as subspaces of, respectively, $\mathcal{H}_a$ and $\mathcal{H}_d \otimes \mathcal{H}_d$. It was proven [3] that in this case the operator $A$ is given by

$$A_f = \mathbb{P}_f^{\text{sym}} - \frac{2L}{L + 1} \left( \mathbb{P}_{1,11}^+ \circ \mathbb{P}_{2,22}^+ \circ \cdots \circ \mathbb{P}_{L,L'}^+ \right) \left( \mathbb{P}_c \mathbb{P}_c^\dagger \mathbb{P}_c \mathbb{P}_c^\dagger \cdots \mathbb{P}_d \mathbb{P}_d^\dagger \right)$$

(12)

where $\mathbb{P}_f^{\text{sym}}$ projects onto $\text{Sym}^2 (\mathcal{H}_f)$ and $\mathbb{P}_c \mathbb{P}_c^\dagger \mathbb{P}_c \mathbb{P}_c^\dagger$ are projectors onto subspaces of $\mathcal{H}_d \otimes \mathcal{H}_d$ completely asymmetric under interchange of spaces labeled by indices from the set $\{1,2,\ldots,L\}$ and $\{1',2',\ldots,L'\}$, respectively. Let us now study arbitrary depolarized pure states of two fermions. Any state $|\psi\rangle \in \mathcal{H}_f$ can be written [4] as

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |\phi_{2i-1}\rangle \wedge |\phi_{2i}\rangle,$$

(13)

where $\lambda_i \geq 0$, $\sum_{i=1}^{d} \lambda_i^2 = 1$, and the vectors $|\phi_i\rangle$ are pairwise orthogonal. As an example we consider an arbitrary depolarization of the state $|\psi\rangle$,

$$\rho_\psi (p) = (1 - p)|\psi\rangle \langle \psi| + p \frac{2\mathbb{I}}{d(d-1)},$$

(14)

where $p \in [0,1]$ and $\mathbb{I}$ is the identity operator on $\mathcal{H}_f$. Direct usage of the criterion [3] shows that the state $\rho_\psi (p)$ is correlated if

$$(1 - p)^2 \left( 5 - 2 \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \lambda_i^4 \right) + 2p(1-p)\chi_1(d) + p^2\chi_2(d) > 3,$$

(15)

where $\chi_1(d) = 3 + \frac{2(d-2)(d-3)}{d(d-1)}$ and $\chi_2(d) = \frac{2(d+1)}{d-1} + \frac{6}{d(d-1)}$.

**Fermionic Gaussian states.** Hilbert space describing fermions with unconstrained number of particles is the Fock space, $\mathcal{H}_{\text{Fock}} (\mathbb{C}^d) = \bigoplus_{L=0}^{\infty} \mathbb{A}^L (\mathbb{C}^d)$, where $\mathbb{A}^L (\mathbb{C}^d)$ is the one dimensional linear subspace spanned by the Fock vacuum $|0\rangle$. Standard fermionic creation and annihilation operators: $a_i^\dagger$, $a_i$, $i = 1,\ldots,d$, satisfying canonical anti-commutation relations, act in $\mathcal{H}_{\text{Fock}}$. In order to define pure fermionic Gaussian states it is convenient to introduce Majorana fermion operators [6,8]:

$$c_{2k-1} = a_k + a_k^\dagger, \quad c_{2k} = i \left( a_k - a_k^\dagger \right), \quad k = 1,\ldots,d.$$

(16)

Every two density matrices from $\mathcal{H}_{\text{Fock}}$ projects onto $\text{Sym}^2 (\mathcal{H}_{\text{Fock}})$. In a recent paper [22] the set of convex-Gaussian fermionic states was characterized analytically in the first non-trivial case of $d = 4$ modes. The method presented here cannot reproduce this result but can be used to detect states that are not convex-Gaussian for arbitrary number of modes.

Having proved the criterion [3] and demonstrating its usefulness it is natural to ask how often it is satisfied and what does it say about correlation properties of the system in question. We answer these questions by studying typical properties of function $f$ for the set of density matrices having the same spectrum (isospectral density matrices). We denote by $\Omega_{\{p_1,\ldots,p_N\}}$ the set of all density matrices having an ordered spectrum $\{p_1,\ldots,p_N\}$ where $p_i$ are real numbers satisfying $0 \leq p_1 \leq \cdots \leq p_N$, $\sum_i p_i = 1$. Obviously, we have $\Omega_{\{1,0,\ldots,0\}} = D_1 (\mathcal{H})$. On the set $\Omega_{\{p_1,\ldots,p_N\}}$ the special unitary group $SU (\mathcal{H})$ acts naturally via the conjugation: $U \rho U^\dagger$. Every two density matrices from $\Omega_{\{p_1,\ldots,p_N\}}$ are conjugate in this manner by some element of $SU (\mathcal{H})$. In what follows we will write for short $\Omega_{\{p_1,\ldots,p_N\}} \equiv \Omega$. The geometry of the considered setting is presented on Figure [4].

The set $\Omega$ is equipped with a natural unitarily invariant probability measure $\mu_0$ that stems from the (normalized) Haar measure $\mu$ on $SU (\mathcal{H})$ and the transitive action of this group on $\Omega$. Our strategy is as follows: for each $\Omega$ we employ the concentration of measure inequality [2] for the function $f_\Omega$ which is the restriction of $f$ to $\Omega$. Having done so we have the information about typical properties of $f$ on $\Omega$. This gives us, provided the average
of $f_{\Omega}$ is non-negative, the lower bound for the measure of correlated states on $\Omega$. This insight is different from the previous approaches to similar problems, usually arising from the entanglement theory, in which typical properties of the quantity in question (some entanglement measure or the particular property of a quantum state) were studied on the whole space $D(\mathcal{H})$ with a particular choice of the probability measure $\{25–27\}$. Our reasoning is more general because it gives the information about typical behaviour of correlations for each choice of the spectrum. Our final result is the following.

\textbf{Theorem 3.} \textit{Let the class $\mathcal{M}$ be defined by (1) and let $V$ be defined as in Theorem 2. Let $X = \frac{\text{dim}(\text{Im}(A))}{\text{dim}(\text{Sym}^{\frac{d}{2}}(\mathcal{H}))}$, $P_{\text{cr}} = \frac{1}{X}$ and let $P(\Omega) = \sum_{i} P_{i}$, denote the purity of states belonging to $\Omega$. Assume that $P(\Omega) = P_{\text{cr}} + \delta$ ($\delta > 0$). Then, the following inequality holds,}

$$
\mu_{\Omega}(\{\rho \in \Omega | \rho \text{ is correlated}\}) \geq 1 - \exp\left(-\frac{N \delta^2 (X + 1)^2}{64}\right).
$$

(18)

Here $N$ denotes the dimension of the Hilbert space and $\text{Im}(A)$ the image of the operator $A$ relevant for the problem in question. The proof of Theorem 3 is presented in the Supplemental Material. Values of the relevant parameters appearing in (18) for the four discussed classes of states are presented in Table I. Value of $X$ for separable states follows directly from (8). Value of $X$ for fermionic Gaussian states [28] follows easily from the discussion contained in [8]. The origin of the remaining two values is discussed in the Supplemental Material [14]. Notice that for separable states, separable bosonic states and Slater determinants $P_{\text{cr}} \rightarrow 0$ and $N \rightarrow \infty$ as $L \rightarrow \infty$. To our knowledge results contained in Theorem 3 and Table I were not obtained elsewhere. Closely related problems in the context of entanglement theory were considered [26] but mostly with the use of numerical methods.

| Class of pure states $\mathcal{M}$       | $N$   | $1 - X$                                      |
|-----------------------------------------|-------|---------------------------------------------|
| Separable states                        | $d^L$ | $2^{1-L} \left(\frac{(d+1)^2}{2^{d+1}}\right)$ |
| Separable bosonic states                | $(d^2 L^{-1})$ | $1 - \frac{2^{2(d+L)}}{(d+L+1)(d+L+1)}$ |
| Slater determinants                     | $(d L)$ | $\frac{2}{(d+1)(d+1-L)}$                    |
| Fermionic Gaussian states               | $2^{d-1}$ | $\frac{1}{(2^{d-1}+1)2^{d-1}}$             |

\textbf{Table I.} Parameters characterising typical behaviour of correlated states that appear in Theorem 3.

To summarize, we have presented a criterion for detection of correlated mixed quantum states i.e. states that cannot be expressed as a convex combination of uncorrelated pure states belonging to the class $\mathcal{M}$ given by the operator $A$ (see Eq. (1)). We have demonstrated our criterion for four physically relevant classes of pure states: separable states, separable bosonic states, Slater determinants and fermionic Gaussian states. Moreover, we have shown that our criterion leads to the characterisation of typical properties of set of correlated states belonging to the set of isospectral density matrices. Let us end with comments concerning the obtained results. First, we would like to remark that it is not a coincidence that the projector $A$ exists in all four considered classes of pure states. It is a general result in the representation theory of semisimple Lie groups [4, 7, 29] that such operator exists for so-called Perelomov coherent states [30], i.e. states that form the orbit of the relevant symmetry group through the highest weight vector of an irreducible representation. This observation covers first three cases as it was discussed in [5]. On the other hand, fermionic Gaussian states can be also treated as the orbit through the Fock vacuum of the group of fermionic Bogoliubov transformations or, equivalently, the group $\text{Pin}(2d)$ [9]. It is tempting to ask whether the operator $A$ exists for other physically interesting classes of states (like Glauber coherent states or bosonic Gaussian states). There are other ways in which one can generalize the presented approach. For instance, on can try to subtract in Eq. (2) not $P_{\text{sym}}$ but some operator that would be more suitable for a given problem. In the future we plan to extend our framework to cases when $\mathcal{M}$ is given by the operator acting on many copies of the physical Hilbert space.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865–942 (2009).
[2] N. Killoran, M. Cramer, M. B. Plenio, arxiv:1312.4311
[3] J. Schliemann, D. Loss, and A. H. MacDonald, Phys. Rev. B 63, 085311 (2001).
[4] K. Eckert, J. Schliemann, D. Bruss, and M. Lewenstein, Ann. Phys. N. Y. 299, 88–127 (2002).
[5] C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. 84, 621–669 (2012).
[6] S. Bravyi, Quantum Inf. Comp 5, 216–238 (2005).
[7] E. Grepliova, Master Thesis, Quantum Information with Fermionic Gaussian States, 2013.
[8] F. de Melo, P. Ćwikliński, and B. M. Terhal, New J. Phys. 15, 013015 (2013).
[9] J. Dereziński, M. Napiórkowski, and J.P. Solovej, arxiv:1102.2931.
[10] A. Gabriel, B. C. Hiesmayr, and M. Huber, Quantum Inf. Comput. 10, 829–836 (2010).
[11] R. J. Glauber, Phys. Rev. 131, 2766–2788 (1963).
[12] O. Giraud, P. Braun, and D. Braun, Phys. Rev. A 78, 042112 (2008).
[13] M. Kuś and I. Bengtsson, Phys. Rev. A 80, 022319 (2009).
[14] P. Badziag, P. Deuar, M. Horodecki, P. Horodecki, and R. Horodecki, J. Mod. Opt. 49, 1289 (2002).
[15] See Supplemental Material at [URL will be inserted by publisher] for proofs of Theorems 1, 2 and 3 and for the justification of values of $X$.
[16] S. Bravyi, Phys. Rev. A 73, 042313 (2006).
[17] F. Mintert and A. Buchleitner, Phys. Rev. Lett. 98, 140505 (2007).
[18] P. Horodecki, Phys. Rev. A 68, 052101 (2003).
[19] R. Augusiak and M. Lewenstein, Quant. Inf. Proc. 8, 493-521 (2009).
[20] P. Badziag, P. Horodecki, R. Horodecki, and R. Augusiak, Phys. Rev. A 88, 010301(R) (2013).
[21] M. Oszmaniec and M. Kuś, Phys. Rev. A 88, 052328 (2013).
[22] M. Oszmaniec, J. Gutt and M. Kuś, arXiv:1406.1577
[23] M. Ledoux, The concentration of measure phenomenon (American Mathematical Society, Providence, 2001).
[24] G.W. Anderson, A. Guionnet, and O. Zeitouni, An Introduction to Random Matrices (Cambridge University Press, Cambridge, 2010).
[25] K. Życzkowski, P. Horodecki, A. Sanpera, and, M. Lewenstein, Phys. Rev. A 58, 883 (1998)
[26] K. Życzkowski, Phys. Rev. A 60, 3496 (1999)
[27] G. Aubrun, S. J. Szarek, and D. Ye, Phys. Rev. A 85, 030302(R) (2012)
[28] From [5] it follows that all convex-Gaussian states in Fock ($\mathbb{C}^d$) have a block structure with respect to the decomposition $\text{Fock}(\mathbb{C}^d) = \text{Fock}_+ (\mathbb{C}^d) \oplus \text{Fock}_- (\mathbb{C}^d)$, where $\text{Fock}_\pm (\mathbb{C}^d)$ denote the subspaces of the Fock space spanned by even and respectively odd number of excitations. For this reason fraction of convex-Gaussian states in $\Omega$ equals 0. Therefore, in order not to consider a trivial situation we consider states and 'global unitary operations' defined solely on $\text{Fock}_+ (\mathbb{C}^d)$.
[29] W. Lichtenstein, Proc. Am. Math. Soc. 84, 605–608 (1982).
[30] M. Oszmaniec and M. Kuś, J. Phys. A: Math. Theor. 45, 244034 (2012).
[31] A. Perelomov, Generalized Coherent States and Their Applications (Springer-Verlag, New York, 1986).
SUPPLEMENTAL MATERIAL

We provide here technical details that were omitted in the main part of the manuscript. We give proofs of Theorems 1, 2, and 3. Moreover, we give the reasoning from which we deduce the forms of $X = \frac{\dim(\text{Lie}(A))}{\dim(\text{Sym}^2(H))}$ in the case of bosonic separable states and Slater determinants.

Proof of Theorem 1

Proof. Since the expression (5) is linear in $B$ and every non-negative operator is of the form $B = \sum_i |w_i⟩⟨w_i|$, the condition $\rho \in M^e$ is equivalent to $\rho = \sum_j p_j |v_j⟩⟨v_j|$, where $p_j \geq 0$, $\sum p_j = 1$. Using that and the assumption (5) about the operator $V$ we get

$$\text{tr}((\rho \otimes |w⟩⟨w|) V) = \sum_j p_j ⟨v_j| V |w⟩|v_j⟩ \leq 0.$$ 

This concludes the proof.

Proof of Theorem 2

Proof. Let $|v⟩ \in M$ and let $|w⟩ \in H$. Let us write $|w⟩ = |v_1⟩ + |v_⊥⟩$, where $|v_1⟩ \propto |v⟩$ and $|v_⊥⟩ \perp |v⟩$. We have the following equalities

$$\langle v|w⟩ A |v⟩ = \langle v|v_1⟩ A |v_1⟩,$$

$$\langle v|w⟩ |\text{P}^\text{asy}⟩ |v⟩ = \langle v|v_⊥⟩ |\text{P}^\text{asy}⟩ |v_⊥⟩ = \frac{1}{2} |v_⊥⟩ |v_⊥⟩.$$ 

We have used the fact that operator $A$ is an orthonormal projector and therefore we have $A|v⟩ = A|v| |v⟩$. Consequently, we get the desired inequality

$$\langle v|w⟩ V |v⟩ = \langle v|v_1⟩ A |v⟩ |v_1⟩ - \frac{1}{2} |v_⊥⟩ |v_⊥⟩ \leq 0,$$

where the estimate stems from the fact that $\langle v|v_⊥⟩ A |v⟩ |v_⊥⟩ \leq \langle v|v_⊥⟩ |\text{P}^\text{asy}⟩ |v_⊥⟩ = \frac{1}{2} |v_⊥⟩ |v_⊥⟩$, where $\text{P}^\text{asy}$ is the projector onto $\text{Sym}^2(H)$.

Proof of Theorem 3

Proof. In what follows we use the notation of Theorem 3. By $E_\Omega(\cdot)$ we denote the expectation value with respect to the probability measure $\mu_\Omega$. In order to prove (18) we first show that

$$E_\Omega(f_\Omega) = \frac{P_0(\Omega)}{2}(X + 1) + \frac{X - 1}{2}. \quad (S.1)$$

We first construct an auxiliary function on the unitary group $SU(H)$:

$$f_\Omega(U) = f(U \rho_0 U^T),$$

where $\rho_0 = \text{diag}(p_1, \ldots, p_N)$ denotes the diagonal matrix with the spectrum $\{p_1, \ldots, p_N\}$. Let $E(\cdot)$ denote the expectation value with respect to the Haar measure $\mu$. From the definition of the measure $\mu_\Omega$ we have that $E_\mu(f_\Omega) = E(f_\Omega)$.

Using the definition of the function $f$ and the linearity of the trace we have

$$E(f_\Omega) = \int_{SU(H)} dU \text{tr}(U \otimes U (\rho_0 \otimes \rho_0) U^T \otimes U^T V) = \text{tr}\left( \int_{SU(H)} dU \left(U \otimes U (\rho_0 \otimes \rho_0) U^T \otimes U^T \right) V \right). \quad (S.2)$$

The integral inside the trace can be computed via the known formula valid for an arbitrary operator $X$ acting on $H \otimes H$,

$$\int_{SU(H)} dU \left(U \otimes U X U^T \otimes U^T \right) = a \text{P}^\text{asy} + b \text{P}^\text{sym}, \quad (S.3)$$

where $a = 2 \frac{\text{tr}(X \text{P}^\text{sym})}{N(N+1)}$ and $b = 2 \frac{\text{tr}(X \text{P}^\text{sym})}{N(N-1)}$. Applying (S.3) to (S.2) and using definitions of operators $V$ and $\rho_0$ we obtain (S.19). We now use the inequality

$$\mu_\Omega(\{|\rho \in \Omega| f_\Omega(\rho) > 0\}) \geq \mu_\Omega(\{|\rho \in \Omega| f_\Omega(\rho) > 0\}) \geq \mu(\{|U \in SU(H)\} f_\Omega(U) > 0\}) \quad (S.4)$$

If $\mu(f_\Omega)$ is positive we can use a variant of the concentration of measure inequality for the group $SU(H)$ to estimate right hand side of (S.4). Indeed, for every smooth function $F$ defined on $SU(H)$ we have the inequality $\mu(F)$

$$\mu(\{U \in SU(H)\} | F(U) > E(F) - \epsilon \}) \geq 1 - \exp\left(-\frac{N \epsilon^2}{4C^2}\right), \quad (S.5)$$

where $\epsilon > 0$ and $L = \max_{U \in SU(H)} |\nabla F|$. The gradient is taken with respect to the natural metric induced on $SU(H)$ when it is viewed as a subset of the algebra of linear operators, $\text{Lin}(H)$, equipped with the Hilbert-Schmidt inner product $\langle A, B \rangle = \text{tr}(A^T B)$. Note that $E(f_\Omega) = \delta(X + 1)$ for $P(\Omega) = \rho_\epsilon + \delta$. We take $F = f_\Omega$ and apply (S.5) with $\epsilon = E(f_\Omega)$,

$$\mu(\{U \in SU(H)\} f_\Omega(U) > 0\}) \geq 1 - \exp\left(-\frac{N \delta^2 (X + 1)^2}{4C^2}\right), \quad (S.6)$$

where $C = \max_{U \in SU(H)} |\nabla f_\Omega|$. Inequalities (S.6) and (S.4) almost give (18). We complete the proof by observing that $C \leq 4$. Indeed, from definition of the gradient we have:

$$\nabla f_\Omega \bigg|_{U^T Y} = \frac{d}{dt} \bigg|_{t=0} f_\Omega(U), \quad (S.7)$$

where $U \in SU(H)$, $U(t) = e^{t Y} U$, $Y = \rho \otimes \rho$, and $Y \in T_U SU(H)$ (the tangent space treated as a (real) subspace of $\text{Lin}(H)$). It follows that

$$\nabla f_\Omega \bigg|_{U^T Y} = \text{tr}(\Omega \otimes I + I \otimes Y, \rho \otimes \rho | A), \quad (S.8)$$

where $\rho = U \rho_0 U^T$ and $I$ is the identity operator. When passing from (S.7) to (S.8) we have used the fact that $\text{P}^\text{asy} = \text{P}^\text{sym}$.
Expanding $\rho$ in the eigenbasis and using the fact that $A$ is an orthonormal projector we get

$$
\left\|\left< \nabla \tilde{f}_1 \big|_U, YU \right> \right\| \leq 4\sqrt{\langle YU, YU \rangle}.
$$

(S.9)

Plugging to the above $YU = \nabla \tilde{f}_1 \big|_U$ gets the desired result.

Values of $X$ for bosonic separable states and Slater determinants

We give here the reasoning justifying formulas for $X = \frac{\dim(\text{Im}(A))}{\dim(\text{Sym}^2(\mathcal{H}))}$ appearing in (18). We use the fact that the operator $A$ has a clear group theoretical interpretation in these two cases. One can identify $\mathcal{H}_b$ and $\mathcal{H}_f$ with the carrier spaces of irreducible representations of $SU(d)$ [4, 5]. All irreducible representations of $SU(d)$ are parameterized by so-called highest weights [1], i.e. non increasingly ordered sequences of length $d-1$ consisting of non-negative integers:

$$
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d-1}) ,
$$

(S.10)

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{d-1}$, and $\lambda_i \in \{0, 1, \ldots\}$. One can also represent $\lambda$ by a Young diagram [6] - a collection of boxes arranged in left-justified rows, with non increasing lengths when looked from the top to the bottom. In what follows we will use the notation $2\lambda = (2\lambda_1, 2\lambda_2, \ldots, 2\lambda_{d-1})$. Clearly, $2\lambda$ is also a highest weight so it corresponds to some irreducible representation of $SU(d)$.

The highest weights corresponding to $\mathcal{H} = \mathcal{H}_b$ and $\mathcal{H} = \mathcal{H}_f$ describing respectively $L$ bosons and $L$ fermions are

$$
\lambda_b = (L, 0, \ldots, 0) \text{ and } \lambda_f = (1, \ldots, 1, 0, \ldots, 0) ,
$$

(S.11)

where there are precisely $L$ ones appearing in the formula for $\lambda_f$. In [7] it was proven that the operator $A \in \text{Herm} \left( \text{Sym}^2(\mathcal{H}) \right)$ defining families of separable bosonic states and Slater determinants is given by

$$
A = \mathbb{P}^{\text{sym}} - \mathbb{P}^{2\lambda} ,
$$

(S.12)

where $\mathbb{P}^{2\lambda}$ is the projector onto the unique irreducible representation of the type $2\lambda$ that appear in $\text{Sym}^2(\mathcal{H})$ (treated as a carrier space of a representation of $SU(d)$). Therefore, the problem of computing $X$ for bosonic separable states and Slater determinants reduces to computation of dimensions of representations of $SU(d)$ described by highest weights $2\lambda_b$ and $2\lambda_f$ respectively. We perform these computations explicitly with the usage of methods described in [8].

References

[1] A. Barut and R. Rączka, Theory of group representations and applications (World Scientific, 1986).
[2] M. Ledoux, The concentration of measure phenomenon (American Mathematical Society, Providence, 2001).
[3] G.W. Anderson, A. Guionnet, and O. Zeitouni, An Introduction to Random Matrices (Cambridge University Press, Cambridge, 2010).
[4] M. Kuś and I. Bengtsson, Phys. Rev. A 80, 022319 (2009).
[5] M. Oszmaniec and M. Kuś, Phys. Rev. A 88, 052328 (2013).
[6] W. Fulton, Young Tableaux, with Applications to Representation Theory and Geometry (Cambridge University Press, 1997).
[7] M. Oszmaniec and M. Kuś, J. Phys. A: Math. Theor. 45, 244034 (2012).
[8] H. Elvang, P. Cvitanovic, and A.D. Kennedy, J. Math. Phys. 46, 043501 (2005).