The Hutchinson-Barnsley theory for generalized iterated function systems by means of infinite iterated function systems

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Abstract

The study of generalized iterated function systems (GIFS) was introduced by Mihail and Miculescu in 2008. We provide a new approach to study those systems as the limit of the Hutchinson-Barnsley setting for infinite iterated function systems (IIFS) which has been developed by many authors in the last years. We show that any attractor of a contractive generalized iterated function system is the limit with respect to Hausdorff-Pompeiu metric of attractors of contractive infinite iterated function systems. We also prove that any Hutchinson measure for a contractive generalized iterated function system with probabilities is the limit with respect to the Monge-Kantorovich metric of the Hutchinson measures for contractive infinite iterated function systems with probabilities.

Key words: iterated function systems, generalized iterated function systems, attractors, infinite iterated function systems, fractal, Hutchinson measures

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1 Introduction

The study of generalized iterated function systems, introduced in 2008 by Mihail and Miculescu [MM08], provided an interesting perspective of how to produce fractal sets that extends the classical one related to contractive iterated function systems. In 2015 Strobin [Str15] has proved the existence of attractors of generalized iterated function systems of order $m$ which are not attractor of any generalized iterated function system of order $m-1$, in particular of any contractive finite iterated function system (which is also a generalized iterated function systems of order 1).

Many work has been done to directly attack the problem of developing the Hutchinson-Barnsley theory for generalized iterated function systems by direct means such as fixed point theorems and code spaces. It is worth to notice two very recent works published in the last two years. The first one is due to Strobin [Str20], proving that a generalized iterated function system with probabilities consisting of generalized contractive maps generates the unique generalized Hutchinson measure extending the recent results due to Miculescu and Mihail. The analogous problem for the place dependent case has been left open (see [Str20, Problem 5.3]). The second one has been Guzik and Kapica [GK21] where a criteria on the existence of a unique attracting probability measure for stochastic process induced by generalized iterated function systems with (fixed) probabilities is developed. Both works require intricate and profound machinery to achieve these results.

Seeking to obtain some dynamical meaning, and hopefully an elementary way to address these problems, we employ another approach, proving that any attractor of a contractive generalized iterated function system is the limit, with respect to Hausdorff-Pompeiu metric, of attractors of contractive infinite iterated function systems and that any Hutchinson measure for a contractive generalized iterated function system with probabilities is the limit with respect to the Monge-Kantorovich metric of the Hutchinson measures for contractive infinite iterated function systems with probabilities. Regarding to fractal generation, we show that it is enough to study (possibly infinite, uncountable) iterated function systems instead

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generalized iterated function systems. As a byproduct we briefly present some new iterative procedures to approximate the attractor and the Hutchinson measure of a generalized iterated function system via evaluation maps. In the paper we deal with Banach contractive type function systems, but most cited results can be extended to function systems consisting of maps satisfying weaker contractive conditions.

The paper is organized as follows:
In Section 2 we recall some basic facts on infinite iterated function systems, in particular the sufficient conditions for the existence of an attractor from [Dum13]. In Section 3 we show how to introduce an infinite iterated function system associated to a given generalized iterated function system and a closed and bounded set. Then we prove that it satisfies sufficient conditions for generating an attractor and we define an evaluation map which assign this attractor to the chosen set. Finally, we show that the evaluation map is a contraction on the space of nonempty bounded and closed sets and that the attractor of the given generalized iterated function system is a fixed point of this contraction map. This proves our first main result, Theorem 3.23. Then, we introduce the joint evaluation map for sets and probabilities proving that it is also a contraction and has the attractor and the Hutchinson measure of the generalized iterated function system as its fixed point, which is the content of our main result, Theorem 3.22. For the last, in the Section 4 we explain some additional facts and discuss some possibilities for future work.

As this work is a connection between different areas of research on iterated function systems, for the reader convenience we provide a detailed set of known results on the Hutchinson-Barnsley theory along with a bibliographical review.

2 Preliminaries and Infinite IFS

First we recall some basic facts on sets and metric spaces, necessary to study infinite iterated function systems. Our main reference is [Dum13], however many authors have studied this subject in the last few years, see [GL10], [HCW05], [NDCS08], [MI12], [Sec02], [Sec13a], [Sec11], [Sec01], [Sec14b] and [CJM14]. From now on, \((X,d)\) is always a metric space. We will denote by \(\mathcal{U}^*(X)\) the set of nonempty subsets of \(X\), \(K^*(X)\) the set of nonempty compact subsets of \(X\), \(B^*(X)\) the set of nonempty bounded closed subsets of \(X\). As usual, we denote by \(\bar{A}\) the closure of \(A\), with respect to the topology of \((X,d)\).

The generalized Hausdorff-Pompeiu semi-distance is the function

\[
h : \mathcal{U}^*(X) \times \mathcal{U}^*(X) \to [0, +\infty]
\]

defined by

\[
h(A, B) = \max\{d(A, B), d(B, A)\},
\]

where

\[
d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \left( \inf_{y \in B} d(x, y) \right).
\]

We will be mostly interested in spaces \((K^*(X), h)\) and \((B^*(X), h)\). For a function \(\psi : X \to X\) we denote by \(\text{Lip}(\psi) \subseteq [0, +\infty]\) the Lipschitz constant associated to \(\psi\), which is given by

\[
\text{Lip}(\psi) = \sup_{x,y \in X; x \neq y} \frac{d(\psi(x), \psi(y))}{d(x, y)}.
\]

We say that \(\psi\) is a Lipschitz function if \(\text{Lip}(\psi) < +\infty\) and a contraction if \(\text{Lip}(\psi) < 1\).

**Theorem 2.1 ([Bar93]).** In the above frame, the following conditions hold

1. \((B^*(X), h)\) and \((K^*(X), h)\) are metric spaces and \((K^*(X), h)\) is closed in \((B^*(X), h)\).
2. If \((X,d)\) is complete, then \((B^*(X), h)\) and \((K^*(X), h)\) are complete metric spaces.
3. If \((X,d)\) is compact, then \((K^*(X), h)\) is compact and in this case \(B^*(X) = K^*(X)\).
4. If \((X,d)\) is separable, then \((K^*(X), h)\) is separable.
5. If \(H\) and \(K\) are two nonempty subsets of \(X\) then \(h(H, K) = h(\bar{H}, \bar{K})\).
6. If \((H_\theta)_{\theta \in \Theta}\) and \((K_\theta)_{\theta \in \Theta}\) are two families of nonempty subsets of \(X\) then

\[
h \left( \bigcup_{\theta \in \Theta} H_\theta, \bigcup_{\theta \in \Theta} K_\theta \right) = h \left( \bigcup_{\theta \in \Theta} H_\theta, \bigcup_{\theta \in \Theta} K_\theta \right) \leq \sup_{\theta \in \Theta} h(H_\theta, K_\theta).
\]
7. If $H$ and $K$ are two nonempty subsets of $X$ and $\psi : X \to X$ is a Lipschitz function then $h(\psi(K), \psi(H)) \leq \text{Lip}(\psi) \cdot h(K, H)$.

**Definition 2.2.** A family of continuous functions $(\psi_\theta)_{\theta \in \Theta}, \psi_\theta : X \to X$ is said to be bounded if for every bounded set $A \subseteq X$, the set $\bigcup_{\theta \in \Theta} \psi_\theta(A)$ is bounded.

**Definition 2.3.** An infinite iterated function system (IIFS in short) on $X$ consists of a bounded family of continuous functions $(\psi_\theta)_{\theta \in \Theta}$ on $X$, and it is denoted by $\mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta})$. When $\Theta$ is finite we obtain the classical notion of an iterated function system (IFS).

**Definition 2.4.** For an IIFS $\mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta})$, the fractal operator (or Hutchinson-Barnsley operator) $F_\mathcal{R} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$ is the function defined by $F_\mathcal{R}(A) = \bigcup_{\theta \in \Theta} \psi_\theta(A)$ for every $A \in \mathcal{B}^*(X)$.

The closure in the above definition is necessary because an arbitrary union of closed sets may not be closed.

**Remark 2.5.** It is a classical result in the Hutchinson-Barnsley theory that if the functions $\psi_\theta$ are contractions, for every $\theta \in \Theta$ with $\sup_{\theta \in \Theta} \text{Lip}(\psi_\theta) < 1$, then the function $F_\mathcal{R}$ is a contraction and verifies $\text{Lip}(F_\mathcal{R}) \leq \sup_{\theta \in \Theta} \text{Lip}(\psi_\theta) < 1$.

From the contractivity of $F_\mathcal{R}$ we obtain, via fixed point theorem for contractions, the following existence result (see [Dum13] Theorem 1.2] or [Lew93] Theorem 3.2 and Theorem 4.1 for details):

**Theorem 2.6.** Let $(X, d)$ be a complete metric space and $\mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta})$ an IIFS, such that $\alpha = \sup_{\theta \in \Theta} \text{Lip}(\psi_\theta) < 1$. Then there exists a unique set $A_\mathcal{R} \in \mathcal{B}^*(X)$ such that $F_\mathcal{R}(A_\mathcal{R}) = A_\mathcal{R}$ and for any $A_0 \in \mathcal{B}^*(X)$ the sequence $(A_k)_{k \in \mathbb{N}}$ defined by $A_{k+1} = F_\mathcal{R}(A_k)$ is convergent to $A_\mathcal{R}$ with respect to the metric $h$. Moreover, for every $k \in \mathbb{N}$, we have, $h(A_k, A_\mathcal{R}) \leq \frac{1}{\alpha^k} h(H_0, h_1)$.

**Definition 2.7.** The unique set $A_\mathcal{R} \in \mathcal{B}^*(X)$, given by Theorem 2.6, is called the attractor of the IIFS $\mathcal{R}$.

### 3 Hutchinson-Barnsley theory for GIFS using the induced IIFS

#### 3.1 Attractors of GIFS

We consider the maximum distance in $X^m$, that is, given $x, y \in X^m$ we have $d_m(x, y) = \max_{1 \leq i \leq m} d(x_i, y_i)$. Given a function $\phi : X^m \to X$, we define

$$\text{Lip}(\phi) = \inf \{k : d(F(x), F(y)) \leq k d_m(x, y) \text{ for all } x, y \in X^m\}.$$ 

When $\text{Lip}(\phi) < \infty$ say that that $\phi$ is a Lipschitz function and $\text{Lip}(\phi)$ is the Lipschitz constant of $\phi$. If $\text{Lip}(\phi) < 1$, then $\phi$ is called a Lipschitz contraction.

Generalized iterated function systems were introduced in [MM08] as follows:

**Definition 3.1.** Let $m \in \mathbb{N}^*$. A generalized iterated function system (GIFS, for short) on $X$, of order $m$, denoted by $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}})$, consists of a finite family of Lipschitz maps $\phi_j : X^m \to X$. The generalized Hutchinson operator $F_S : (\mathcal{B}^*(X))^m \to \mathcal{B}^*(X)$ is given by

$$F_S(A_1, \ldots, A_m) = \bigcup_{j \in \{1, \ldots, n\}} \phi_j(A_1 \times \ldots \times A_m),$$

for every $A_1, \ldots, A_m \in \mathcal{B}^*(X)$.

**Theorem 3.2** ([MM08] or [SS13]). Let $X$ be a complete metric space and

$$S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}})$$

be a GIFS of order $m$ consisting of Lipschitz contractions. Then there exists a unique $A_S \in \mathcal{K}^*(X)$ such that

$$F_S(A_S, \ldots, A_S) = \bigcup_{j \in \{1, \ldots, n\}}^{m \text{ times}} \phi_j(A_S \times \ldots \times A_S) = A_S.$$
Moreover, for any $A_0, \ldots, A_{m-1} \in K^*(X)$, the sequence $(A_k)_{k \in \mathbb{N}}$ defined by

$$A_{k+m} := F_S(A_k, \ldots, A_{k+m-1}),$$

$k \in \mathbb{N}$ converges to $A_S$.

The unique compact set $A_S$, given by Theorem 4.2, is called the fractal attractor of the GIFS $S$.

As pointed in the proof of Theorem 3.4 in [MM08], we could consider in the above theorem a slightly simpler version of the fractal operator $F_S$ as

$$F_S(A) = \bigcup_{j \in \{1, \ldots, n\}} \phi_j(A \times \cdots \times A),$$

for every $A \in B^*(X)$. We notice that in the Lipschitz case, $\text{Lip}(F_S) \leq \sup_{j \in \{1, \ldots, n\}} \text{Lip}(\phi_j) < 1$.

**Definition 3.3.** Let $S = \left(\{\phi_j\}_{j \in \{1, \ldots, n\}}\right)$ be a GIFS (of order $m \geq 2$). Given a set $B \in B^*(X)$ we define the IIFS induced by $B$ with respect to $S$, as the IIFS $R_B = (X, (\psi_\theta)_{\theta \in \Theta})$, where $\Theta = B^{m-1} \times \{1, \ldots, n\}$ and $\psi_\theta(x) = \phi_j(x, b_2, \ldots, b_m)$, for $\theta = (b_2, \ldots, b_m, j) \in \Theta$.

We want to study $S$ from an iteration point of view by approximating it by the induced IIFS $R_B$. In [Oli17] we made an attempt of study a finite skill IFS whose attractor describes some part of the behaviour of the original GIFS but its attractor can be strictly contained in the GIFS’s attractor, ending with a new kind of attractor associated to a GIFS.

**Lemma 3.4.** Let $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}})$ be a GIFS and $R_B = (X, (\psi_\theta)_{\theta \in \Theta})$, be the IIFS induced by the set $B \in B^*(X)$. Then

1. The IIFS $R_B$ is bounded;
2. $\sup_{\theta \in \Theta} \text{Lip}(\psi_\theta) < 1$;
3. If $B = A_S$, then $F_{R_B}(A_S) = A_S$. In particular, $A_S$ is the attractor of $R_{A_S}$;
4. $F_{R_B}(A) = \bigcup_{j \in \{1, \ldots, n\}} \phi_j(A \times B \times \cdots \times B) = F_S(A, B, \ldots, B)$, for every $A \in B^*(X)$.

**Proof.** The proof will be for $m = 2$ in order to avoid unnecessarily complex notation. Note that, from Definition 2.2 we get $F_{R_B}(A) = \bigcup_{\theta \in \Theta} \psi_\theta(A)$, for every $A \in B^*(X)$.

(1) From Definition 2.2 we must show that for every bounded set $A \subset X$ the set $\bigcup_{\theta \in \Theta} \psi_\theta(A)$ is bounded. As $A$ is bounded we can find $a_0 \in A$ and $M_A > 0$ such that $d(a, a_0) \leq M_A$, for any $a \in A$. Anologously, we can find $b_0 \in B$ and $M_B > 0$ such that $d(b, b_0) \leq M_B$, for any $b \in B$. For each $j \in \{1, \ldots, n\}$ we define $c_j = \psi_{(b_0, j)}(a_0) = \phi_j(a_0, b_0)$. For an arbitrary $\theta \in \Theta$ and $z \in \psi_\theta(A)$ we have $z = \phi_j(a, b)$, for some $a, b$ in the respective sets. It is easy to see that

$$d(z, c_j) = d(\phi_j(a, b), \phi_j(a_0, b_0)) \leq \text{Lip}(\phi_j) \max(d(a, a_0), d(b, b_0)) \leq \text{Lip}(F_S) \max(M_A, M_B).$$

Let $M = \max_{j \in \{1, \ldots, n\}} d(c_1, c_j)$. Then,

$$d(z, c_1) \leq d(z, c_j) + d(c_1, c_j) \leq \text{Lip}(F_S) \max(M_A, M_B) + M.$$

(2) Consider an arbitrary $\theta = (b, j) \in \Theta$ and $x, y \in X$. Then

$$d(\psi_\theta(x), \psi_\theta(y)) = d(\phi_j(x, b), \phi_j(y, b)) \leq \text{Lip}(\phi_j) \max(d(x, y), d(b, b)) \leq \text{Lip}(F_S) d(x, y),$$
that is, Lip ($\psi_\theta$) ≤ Lip ($F_S$) for all $\theta \in \Theta$, thus sup$_{\theta \in \Theta}$ Lip ($\psi_\theta$) ≤ Lip ($F_S$) < 1, as we claimed.

(3) Given $B = A_S \in K^*(X) \subset B^*(X)$ we know that the contraction theorem claims that there exists $\epsilon > 0$ such that Lip ($\phi_j$) ≤ $\epsilon$ for every $j \in \{1, \ldots, n\}$. From this, we conclude that Lip ($\phi_j$) ≤ $\epsilon$.

Let $F_{R_B}(A_S) = \bigcup_{\theta \in \Theta} \psi_\theta(A_S) = \bigcup_{b \in A_S, j \in \{1, \ldots, n\}} \phi_j(A_S \times \{b\}) = \bigcup_{j \in \{1, \ldots, n\}} \phi_j(A_S \times A_S).

Since $A_S$ is the attractor of a GIFS we know that $\bigcup_{j \in \{1, \ldots, n\}} \phi_j(A_S \times A_S) = F_S(A_S) = A_S \in K^*(X)$.

Substituting that in the previous computation and using the fact that $A_S = A_S$ we obtain

$$F_{R_B}(A_S) = A_S.$$

(4) Given $A \in B^*(X)$ we know that

$$F_{R_B}(A) = \bigcup_{\theta \in \Theta} \psi_\theta(A) = \bigcup_{b \in B, j \in \{1, \ldots, n\}} \phi_j(A \times \{b\}) = \bigcup_{j \in \{1, \ldots, n\}} \phi_j(A \times B) = F_S(A, B),$$

where we can get rid of the closure because we have a finite union of closed sets.

From Lemma 3.4 (1)-(2) and Theorem 2.3 we conclude that, for an arbitrary $B \in B^*(X)$, the induced IIFS always has an attractor, denoted $A_{R_B}$. From this property we can define the evaluation map with respect to a given GIFS $S$:

**Definition 3.5.** Let $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}})$ be a GIFS. Define the evaluation map $ev_S : B^*(X) \to B^*(X)$ by

$$ev_S(B) = A_{R_B} \in B^*(X),$$

for every $B \in B^*(X)$, where $A_{R_B}$ is the attractor of the induced IIFS $R_B$.

**Theorem 3.6.** Let $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}})$ be a GIFS consisting of Lipschitz contractive maps. The evaluation map is a Lipschitz contraction with Lip ($ev_S$) ≤ Lip ($F_S$) < 1, and for any $B_0 \in B^*(X)$, the sequence $(B_k)$ defined by $B_{k+1} = ev_S(B_k)$ for $k \geq 0$, converges to $A_S$ in $(B^*(X), h)$.

**Proof.** Let $B, B' \in B^*(X)$ be such that $A_{R_B} \neq A_{R_B'}$. From Lemma 3.4, item (4), we have:

$$A_{R_B} = F_{R_B}(A_{R_B}) = F_S(A_{R_B}, B, \ldots, B)$$

and the same is true for $A_{R_B'}$. Then,

$$h(ev_S(B), ev_S(B')) = h(F_S(A_{R_B}, B, \ldots, B), F_S(A_{R_B'}, B', \ldots, B')) \leq$$

$$\leq \text{Lip} (F_S) \max(h(A_{R_B}, A_{R_B'}), h(B, B'), \ldots, h(B, B')) =$$

$$= \text{Lip} (F_S) \max(h(A_{R_B}, A_{R_B'}), h(B, B')).$$

The last inequality follows from the fact $F_S$ is Lipschitz with respect to the maximum distance in $B^*(X)^m$.

If $h(A_{R_B}, A_{R_B'}) \geq h(B, B')$ we obtain $h(A_{R_B}, A_{R_B'}) \leq \text{Lip} (F_S) h(A_{R_B}, A_{R_B'})$, an absurd because Lip ($F_S$) < 1. Thus

$$\text{Lip} (F_S) \max(h(A_{R_B}, A_{R_B'}), h(B, B')) = \text{Lip} (F_S) h(B, B').$$

From this, we conclude that $h(ev_S(B), ev_S(B')) \leq \text{Lip} (F_S) h(B, B')$, in other words,

$$\text{Lip} (ev_S) \leq \text{Lip} (F_S) < 1.$$

Since $ev_S$ is a Lipschitz contraction and from Theorem 2.3 (2), $(B^*(X), h)$ is complete the Banach contraction theorem claims that $B_{k+1} = ev_S(B_k)$ converges to the unique fixed point of $ev_S$. On the other hand, from Lemma 3.4 (3) we know that $ev_S(A_S) = A_S$ proving that $(B_k)$ converges to $A_S$ with respect to the Hausdorff-Pompeiu metric $h$. ■
**Remark 3.7.** From a theoretical point of view, the iteration procedure \( B_{k+1} = ev_S(B_k) \) from Theorem 3.6 can be seen as a kind of iteration for a usual GIFS, but using the attractor of the induced IIFSs. To see that, we recall that, from Lemma 2.3, given \( B', B'' \in \mathcal{B}(X) \) we know that \( F_{\mathcal{R}_B}(B') = F_{\mathcal{R}_B}(B', B'', \ldots, B) \). In particular, if \( B' = A_{\mathcal{R}_B} \) then \( F_{\mathcal{R}_B}(A_{\mathcal{R}_B}) = A_{\mathcal{R}_B} \) and \( A_{\mathcal{R}_B} = F_{\mathcal{R}_B}(A_{\mathcal{R}_B}, B, \ldots, B) \). In this way, given \( B_0 \in \mathcal{B}(X) \), we have:

\[
B_1 = ev_S(B_0) = F_S(A_{\mathcal{R}_B_0}, B_0, \ldots, B_0);
B_2 = ev_S(B_1) = F_S(A_{\mathcal{R}_B_1}, A_{\mathcal{R}_B_0}, \ldots, A_{\mathcal{R}_B_0});
\ldots
B_{k+1} = ev_S(B_k) = F_S(A_{\mathcal{R}_B_k}, A_{\mathcal{R}_B_{k-1}}, \ldots, A_{\mathcal{R}_B_0}).
\]

On the other hand, by Theorem 3.2 we know that for any \( H_0, \ldots, H_{n-1} \in \mathcal{K}(X) \), the sequence \( (H_k)_{k \in \mathbb{N}} \) defined by \( H_{k+m} := F_S(H_k, \ldots, H_{k+m-1}) \), \( k \in \mathbb{N} \), converges to \( A_S \).

### 3.2 Hutchinson (invariant) measures of GIFS

From now on we assume that \((X,d)\) is compact. The set of all Borel positive finite measures \(\mu\) over the Borel sigma algebra of the metric space \((X,d)\) is denoted by \(\mathcal{M}(X)\). Recall that the support of a measure \(\mu\) is given by

\[
supp(\mu) = \{x \in X | \mu(U) > 0 \text{ for any open neighborhood of } x\},
\]

and that it is a closed subset of \(X\). Let \(\mathcal{M}_1(X)\) be the elements of \(\mathcal{M}(X)\) that are normalized \((\mu(X) = 1)\), that is, the set of all Borel probability measures over \((X,d)\).

We introduce the Monge-Kantorovich metric \(d_{MK}\) in \(\mathcal{M}_1(X)\) in the following way: for every \(\mu, \nu \in \mathcal{M}_1(X)\), define

\[
d_{MK}(\mu, \nu) = \sup \left\{ \left| \int_X f d\mu - \int_X f d\nu \right| : f \in Lip_1(X, \mathbb{R}) \right\},
\]

where \(Lip_1(X, \mathbb{R})\) is the set of maps \(f : X \rightarrow \mathbb{R}\) with \(\text{Lip}(f) \leq 1\). In this case, the Monge-Kantorovich metric induces the topology of weak convergence of measures on \(\mathcal{M}_1(X)\) (see [Bog07] for details). From now on, we consider the complete metric spaces \((\mathcal{M}_1(X), d_{MK})\) or \((\mathcal{M}_1(X)^m, d_{MK}^m)\), where

\[
d_{MK}^m(\langle \mu_0, \ldots, \mu_{m-1} \rangle, \langle \mu_0, \ldots, \mu_{m-1} \rangle) = \max_{0 \leq i \leq m-1} d_{MK}(\mu_i, \nu_i), \ m \geq 1.
\]

A key improvement from the classical study of IFS with probabilities (IFSp for short) was given by Stenflo (see [Sten92] Remark 3), where random iterations are used to represent the iterations of a so called IFS with probabilities, \(\mathcal{R} = (X, \psi_\theta, p)_{\theta \in \Theta}\), for an arbitrary measurable space \(\Theta\). The approach here is slightly different.

**Definition 3.8.** Let \(\Theta\) be a compact set. An iterated function system with probabilities (IFSp for short) \(\mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta}, p)\), is an IIFS endowed with a probability \(p\) on \(\Theta\), such that the map \((\theta, x) \rightarrow \psi_\theta(x)\) is continuous in both \(\theta\) and \(x\).

We denote by \(C(X, \mathbb{R})\), the set of all continuous functions from \(X\) to \(\mathbb{R}\).

**Definition 3.9.** [Men98 Section 2] Let \(\mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta}, p)\) be an IFSp. The transfer operator \(L_\mathcal{R} : C(X, \mathbb{R}) \rightarrow C(X, \mathbb{R})\) is given by

\[
L_\mathcal{R}(f)(x) = \int_\Theta f(\psi_\theta(x)) d\theta,
\]

for any \(f \in C(X, \mathbb{R})\).

**Definition 3.10.** [Men98 Section 2] Let \(\mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta}, p)\) be an IFSp. The Markov operator associated to \(\mathcal{R}\) is the operator \(M_\mathcal{R} : \mathcal{M}_1(X) \rightarrow \mathcal{M}_1(X)\) defined by:

\[
\int_X f(x) dM_\mathcal{R}(\nu) = \int_X L_\mathcal{R}(f)(x) d\nu,
\]

for any \(f \in C(X, \mathbb{R})\), \(\nu \in \mathcal{M}_1(X)\).
Theorem 3.11. \textcolor{red}{[Men98]} \textit{Theorem 1}  Let $X$ and $\Theta$ be compact metric spaces and $\mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta}, p)$, an IIFSps, which is contractive on average i.e., for all $x, y \in X$

$$\int_{\Theta} d(\psi_\theta(x), \psi_\theta(y)) \, dp(\theta) \leq \lambda \, d(x, y)$$

with $\lambda < 1$, then the Markov operator $M_{\mathcal{R}} : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ defined by

$$\int_X f(x) \, dM_{\mathcal{R}}(\mu)(x) = \int_X \int_{\Theta} f(\psi_\theta(x)) \, d\mu(\theta) \, dp(\theta),$$

for any $f \in C(X, \mathcal{R})$, is contractive in the Monge-Kantorovich metric with $\text{Lip}(M_{\mathcal{R}}) \leq \lambda < 1$. In particular, for any initial measure $\nu_0 \in \mathcal{M}_1(X)$ the sequence $(M_{\mathcal{R}}^k(\nu_0))$ converges to $\mu_{\mathcal{R}}$ as $k \to \infty$.

The unique measure $\mu_{\mathcal{R}}$, given by Theorem 3.11, is called as the Hutchinson measure for the IIFSps $\mathcal{R}$.

Theorem 3.12. \textcolor{red}{[Men98]} \textit{Theorem 3} Under the hypothesis from Theorem 3.11 if the family $\psi_\theta$ is uniformly Lipschitz contractive, that is, $\text{Lip}(\psi_\theta) \leq \lambda < 1$ for every $\theta \in \Theta$, then $\supp(\mu_{\mathcal{R}}) = A_{\mathcal{R}}$, where $A_{\mathcal{R}}$ is the attractor of $\mathcal{R}$, given by Theorem \textcolor{red}{2.11}.

In [Mih09] and [MM09], Miculescu and Mihail studied the counterpart of the Hutchinson measure for GIFS.

Definition 3.13. By a GIFS with probabilities (GIFS in short) we mean a triplet

$$\mathcal{S} = \left(\left(X^m, (\phi_j)_{j \in \{1, \ldots, n\}}, (q_j)_{j \in \{1, \ldots, n\}}\right)\right),$$

where $(X^m, (\phi_j)_{j \in \{1, \ldots, n\}})$ is a GIFS and $q_1, \ldots, q_n > 0$ with $\sum_{j=1}^n q_j = 1$. Each GIFS generates a map $M_{\mathcal{S}} : \mathcal{M}_1(X^m) \to \mathcal{M}_1(X)$, called the generalized Markov operator, which associates to any $\mu_0, \ldots, \mu_{m-1} \in \mathcal{M}_1(X)$, the measure $M_{\mathcal{S}}(\mu_0, \ldots, \mu_{m-1})$ defined by,

$$\int_X f \, dM_{\mathcal{S}}(\mu_0, \ldots, \mu_{m-1}) = \sum_{j=1}^n q_j \int_{X^m} f \circ \phi_j \, d(\mu_0 \times \ldots \times \mu_{m-1}), \quad (5)$$

for every continuous map $f : X \to \mathbb{R}$.

By the generalized Hutchinson measure of a GIFS $\mathcal{S}$ we mean the unique measure $\mu_{\mathcal{S}} \in \mathcal{M}_1(X)$ which satisfies $\mu_{\mathcal{S}} = M_{\mathcal{S}}(\mu_0, \ldots, \mu_{m-1})$ and such that for every $\mu_0, \ldots, \mu_{m-1} \in \mathcal{M}_1(X)$, the sequence $(\mu_k)$ defined by $\mu_{m+k} = M_{\mathcal{S}}(\mu_k, \ldots, \mu_{k+m-1})$, converges to $\mu_{\mathcal{S}}$ with respect to the Monge-Kantorovich metric.

A map $\phi : X^m \to X$ is an $(a_1, \ldots, a_m)$-contraction, if

$$d(\phi(x_0, \ldots, x_{m-1}), \phi(y_0, \ldots, y_{m-1})) \leq \sum_{i=1}^m a_i d(x_{i-1}, y_{i-1})$$

for all $(x_0, \ldots, x_{m-1}), (y_0, \ldots, y_{m-1}) \in X^m$, where $\sum_{i=1}^m a_i < 1$. In particular, $\phi$ is a Lipschitz contraction with $\text{Lip}(\phi) \leq \sum_{i=1}^m a_i < 1$.

As proved in [Mih09] and [MM09], if a GIFS $\mathcal{S}$ consists of $(a_1, \ldots, a_m)$-contractions, then $M_{\mathcal{S}}$ is also an $(a_1, \ldots, a_m)$-contraction.

As previously, we could redefine for proof purposes,

$$\overline{M}_{\mathcal{S}}(\mu) := M_{\mathcal{S}}(\mu, \ldots, \mu)$$

for each $\mu \in \mathcal{M}_1(X)$ and consider $\overline{M}_{\mathcal{S}}$ instead of $M_{\mathcal{S}}$ because they have the same fixed point and Lipschitz constant. Under the above hypothesis $\overline{M}_{\mathcal{S}}$ and $M_{\mathcal{S}}$ are Banach contractions with the Lipschitz constant $\text{Lip}(M_{\mathcal{S}}) \leq \sum_{i=1}^m a_i < 1$.

In consequence, Miculescu and Mihail proved the following theorem (see also [dCOS20] for additional details):

Theorem 3.14. \textcolor{red}{[Mih09, MM09].} Assume that $\mathcal{S}$ is a GIFS on a complete metric space consisting of $(a_1, \ldots, a_m)$-contractions, where $\sum_{i=1}^m a_i < 1$. Then, $\mathcal{S}$ admits the Hutchinson measure $\mu_{\mathcal{S}}$ and $\supp(\mu_{\mathcal{S}}) = A_{\mathcal{S}}$. 

It is worth to mention that Theorem 3.14 was fairly improved in [Str20, Theorem 4.3], proving that $S$ admits the Hutchinson measure $\mu_S$ and $\text{supp}(\mu_S) = A_S$ under the hypothesis that each map of the GIFS is a generalized Matkowski contraction, using new techniques and code spaces.

**Definition 3.15.** Let $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}}, (q_j)_{j \in \{1, \ldots, n\}})$ be a GIFSp (of order $m$) consisting of $(a_1, \ldots, a_m)$-contractions. Given a set $B \in \mathcal{B}(X)$ and a Borel probability $\nu \in \mathcal{M}_1(X)$, such that, $\text{supp}(\nu) \subseteq B$, we define the IIFSp induced by $(B, \nu)$ with respect to $S$ as $R_{B, \nu} = (X, (\psi_\theta)_{\theta \in \Theta}, \nu)$, where $\Theta = B^{m-1} \times \{1, \ldots, n\} \cup \psi_\theta(x, \nu(x)) = \phi_j(x, b_2, \ldots, b_m)$, for $\theta = (b_2, \ldots, b_m, j) \in \Theta$ and $\nu$ is the Borel probability on $\Theta$ given by $dp = \sum_{j=1}^n q_j \delta_j \times d\nu(m-1) \times (b)$, where $d\nu(m-1) := \nu \times \cdots \times \nu$, $m - 1$ times.

Notice that
$$\int_{\Theta} f(\theta) \, d\theta = \int_B \cdots \int_B \sum_{j=1}^n q_j f(b_2, \ldots, b_m, j) \, d\nu(b_2) \cdots d\nu(b_m),$$

for any continuous function $f : \Theta \to \mathbb{R}$.

We claim that $R_{B, \nu}$ is actually an IIFSp, according to Definition 3.8. In order to see that, we must show that the map $(\theta, x) \to \psi_\theta(x)$, given by $\psi_\theta(x) = \phi_j(x, b_2, \ldots, b_m)$, is continuous in both $\theta$ and $x$. Indeed, the topology of $\Theta = B^{m-1} \times \{1, \ldots, n\}$ is the product topology induced by $(X, d)$ on the closed set $B$ and the discrete topology on $\{1, \ldots, n\}$. As the map $(x_1, x_2, \ldots, x_m) \to \phi_j(x_1, x_2, \ldots, x_m)$ is Lipschitz continuous, for each $j$, we obtain the continuity of $(\theta, x) \to \psi_\theta(x)$ with respect to both variables.

**Remark 3.16.** We notice that the problem of finding the unique Hutchinson measure for a place dependent GIFS, obtained by Miclea et al. [Mc14], for a particular class of Lipschitz contractions and left as an open problem in [Str20, Problem 5.3], for generalized contractions, seems to be a big challenge even using our approach. If we consider a place dependent GIFS $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}}, (q_j)_{j \in \{1, \ldots, n\}})$ then the associated IIFSp will have a place dependent measure
$$dp_\nu = \sum_{j=1}^n q_j \delta_j \times d\nu(m-1) \times (b),$$
leading to the study of iterated function systems with measures IIFSp (see [BOS22, Section 2] for details)
$$R_{B, \nu} = (X, (\psi_\theta)_{\theta \in \Theta}, \nu),$$
with transfer operator $L(f)(x) = \int_{\Theta} f(\psi_\theta(x)) \, d\nu_\theta(\theta)$ [BOS22, Definition 2.1]. As far as we know, that class of process has not been well studied yet. Although, as we proved in [BOS22] it is possible to develop some classical results, such as the thermodynamical formalism, for those systems.

The next proposition shows that a GIFSp consisting of $(a_0, \ldots, a_{m-1})$-contractions always induces an IIFSp having a Hutchinson measure.

**Proposition 3.17.** Let $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}}, (q_j)_{j \in \{1, \ldots, n\}})$ be a GIFSp satisfying the hypothesis of Theorem 3.14 and $R_{B, \nu}$ be the IIFSp induced by $(B, \nu)$, according to Definition 3.15. Then, the Markov operator $M_{R_{B, \nu}}$ is contractive with respect to the Monge-Kantorovich metric with $\text{Lip}(M_{R_{B, \nu}}) \leq a_1 < 1$ and there exists $\mu_{R_{B, \nu}}$, the Hutchinson measure for $R_{B, \nu}$. Additionally, $\text{supp}(\mu_{R_{B, \nu}}) = A_{R_{B, \nu}}$ where $A_{R_{B, \nu}}$ is the attractor of $R_{B, \nu}$.

**Proof.** In view of Theorem 3.11, we need to show that the induced IIFSp $R_{B, \nu}$ is contractive on average. Indeed,
$$\int_{\Theta} d(\psi_\theta(x), \psi_\theta(y)) \, d\theta = \int_{\Theta} d(\phi_j(x, b_2, \ldots, b_m), \phi_j(y, b_2, \ldots, b_m)) \, d\theta \leq \int_{\Theta} \sum_{i=2}^m a_i d(b_i, b_i) \, d\theta = \int_{\Theta} a_i d(x, y) \, d\theta = a_i d(x, y),$$
shows that we can take $\lambda = a_1 < 1$. Thus Theorem 3.11 does apply because $X$ and $\Theta = B^{m-1} \times \{1, \ldots, n\}$ are both compact metric spaces. As each $\phi_j$ is an $(a_1, \ldots, a_m)$-contraction, we obtain
$$d(\phi_j(x, b_2, \ldots, b_m), \phi_j(y, b_2, \ldots, b_m)) \leq a_i d(x, y)$$
meaning that the induced IIFS is uniformly contractive. Then, from Theorem 3.12 we get the equality \( \text{supp}(\mu_{R_{B,v}}) = A_{R_{B,v}} \).

The next lemma shows that when the IIFSp is induced by the attractor \( A_S \) and the Hutchinson measure \( \mu_S \) of a GIFSp \( S \), its Markov operator has \( \mu_S \) as a fixed point.

**Lemma 3.18.** Under the hypothesis of Theorem 3.14 if \( B = A_S, \nu = \mu_S \) and \( R_{A_S,\mu_S} \) is the IIFSp induced by \( (A_S,\mu_S) \), we have that \( M_{R_{A_S,\mu_S}}(\mu_S) = \mu_S \). In particular, \( \mu_{R_{A_S,\mu_S}} = \mu_S \).

**Proof.** The proof relay on the formula for \( M_{R_{B,v}} \) replacing \( B = A_S \) and \( \nu = \mu_S \). Indeed, given \( f \in C(X) \) we get

\[
M_{R_{A_S,\mu_S}}(\mu_S)(f) = \int_X \int_{A_S^{m-1}} \sum_{j=1}^{n} q_j f(\phi_j(x,b)) \, d\mu_S^{(m-1)}(b) \, d\mu_S(x) = \\
= \int_X \int_{X^{m-1}} \sum_{j=1}^{n} q_j f(\phi_j(x,x_2,\ldots,x_m)) \, d\mu_S(x_2)\ldots d\mu_S(x_m) \, d\mu_S(x) = \\
= M_S(\mu_S\ldots \mu_S)(f) = \mu_S(f),
\]

thus \( M_{R_{A_S,\mu_S}}(\mu_S) = \mu_S \). By Proposition 3.17 the induced IIFSp has a Hutchinson measure \( \mu_{R_{A_S,\mu_S}} \), which is the unique measure satisfying \( M_{R_{A_S,\mu_S}}(\mu_{R_{A_S,\mu_S}}) = \mu_{R_{A_S,\mu_S}} \), thus \( \mu_{R_{A_S,\mu_S}} = \mu_S \).

Since we are dealing with compact metric spaces the topology induced by the metric \( d_{MK} \) in \( M_1(X) \) is equivalent to the one induced by the weak convergence of measures, whose main properties are given by

**Theorem 3.19.** ([Bil71, Theorem 2.1]) The following conditions are equivalent for a sequence of probabilities \( (\nu_n) \subset M_1(X) \):

a) \( \nu_n \) is weak convergent to \( \nu \in M_1(X) \), that is, \( \int_X f \nu_n \to \int_X f \nu \) for every closed set \( F \subset X \);

b) \( \limsup_{n \to \infty} \nu_n(F) \leq \nu(F) \), for every closed set \( F \subset X \);

c) \( \liminf_{n \to \infty} \nu_n(G) \geq \nu(G) \), for every open set \( G \subset X \);

d) \( \lim_{n \to \infty} \nu_n(A) = \nu(A) \), for every \( \nu \)-continuity set \( A \subset X \), that is, \( \nu(\partial A) = 0 \).

**Definition 3.20.** Let \( \Gamma \) be the subset of \( K^*(X) \times M_1(X) \) defined by:

\[
\Gamma := \{(B,\nu) | B \in K^*(X), \nu \in M_1(X) \text{ and } \text{supp}(\nu) \subset B \}.
\]

**Lemma 3.21.** The metric space \( (\Gamma, d_{\text{max}}) \) is complete.

**Proof.** To see that \( \Gamma \) is closed we consider a sequence \( (B_n,\nu_n))_{n \geq 1} \subset \Gamma \) such that \( (B_n,\nu_n) \to (B_0,\nu_0) \) with respect to the distance \( d_{\text{max}} \). By the definition of \( d_{\text{max}} \) we obtain that \( B_n \to B_0 \) with respect to the Hausdorff distance \( h \). The same is true for the second coordinate, that is, \( \nu_n \to \nu_0 \) with respect to the \( d_{MK} \) distance (and so with respect to the weak convergence). It remains to show that \( (B_0,\nu_0) \in \Gamma \), that is, \( \nu_0(B_0) = 1 \) or equivalently, \( \text{supp}(\nu_0) \subset B_0 \). Suppose, by contradiction, that it is not the case. Then, there exists \( x \not\in B_0 \) such that for any open neighborhood \( U_x \) of \( x \) we get \( \nu_0(U_x) > 0 \). Consider \( \epsilon > 0 \) such that \( U_x \cap B_0^\epsilon = \emptyset \), where \( B_0^\epsilon = \{ z \in X | d(z, B_0) < \epsilon \} \). From the convergence \( B_n \to B_0 \) with respect to \( h \) we obtain \( N_\epsilon \in \mathbb{N} \) such that for any \( n \geq N_\epsilon \) we have \( B_n \subset B_0^\epsilon \). Since \( (B_n,\nu_n) \in \Gamma \) we know that \( \text{supp}(\nu_n) \subset B_n \), so \( \nu_n(U_x) = 0 \) for any \( n \geq N_\epsilon \), thus \( \liminf_{n \to \infty} \nu_n(U_x) = 0 \). By Theorem 3.19 (c), we have \( \liminf_{n \to \infty} \nu_n(U_x) \geq \nu_0(U_x) \), thus \( \nu_0(U_x) = 0 \), a contradiction.

To complete the proof we observe that the completeness of \( (\Gamma, d_{\text{max}}) \) is a trivial consequence of the fact that \( \Gamma \) is a closed subset of a complete metric space. ■
Definition 3.22. Let \( S = (X^n, (\phi_j)_{j \in \{1, \ldots, n\}}, (q_j)_{j \in \{1, \ldots, n\}}) \) be a GIFSp consisting of \((a_0, \ldots, a_{m-1})\)-contractions. We define the joint evaluation map \( EV_S : \Gamma \to \Gamma \) by
\[
EV_S(B, \nu) = (A_{R_{B, \nu}, \mu_{R_{B, \nu}}}),
\]
for every \((B, \nu) \in \Gamma\), where \( A_{R_{B, \nu}} \) is the attractor and \( \mu_{R_{B, \nu}} \) is the Hutchinson measure of the induced IFS \( R_{B, \nu} \) (given by Proposition 3.17), so \((A_{R_{B, \nu}}, \mu_{R_{B, \nu}}) \in \Gamma\).

The first coordinate of \( EV_S \) is just \( ev_S \), which we already know is Lipschitz, by Theorem 3.6 with \( \text{Lip}(ev_S) \leq \text{Lip}(S) < 1 \). Moreover, the next theorem shows that \( EV_S \) is also Lipschitz contractive with respect to the second coordinate.

Theorem 3.23. Under the hypothesis of Theorem 3.14 the joint evaluation map \( EV_S \) is Lipschitz contractive in \( \Gamma \), with \( \text{Lip}(EV_S) \leq \max \left( \text{Lip}(S), \sum_{j=1}^{n} a_j a_{m-j} \right) < 1 \). In particular, for any \((B_0, \nu_0) \in \Gamma\), the sequence \((EV_S^k(B_0, \nu_0))_{k \geq 10} \) converges to \((A_{S, \mu_S}) \in \Gamma\).

Proof. The proof will be for \( m = 2 \) in order to avoid complex notation. From Lemma 3.21 the subset \( \Gamma \) is a complete metric space with respect to the metric \( d_{\text{max}} \), given by \( d_{\text{max}}((B, \nu), (B', \nu')) = \max(h(B, B'), d_{MK}(\nu, \nu')) \) inherited from the complete metric space \((K^*(X) \times M_1(X), d_{\text{max}})\). Our aim is to use the Banach fixed point for Lipschitz contractions in \( \Gamma \).

Denoting \( R := R_{B, \nu} \) and \( R' := R_{B', \nu} \) we need to estimate only \( d_{MK}(\mu_R, \mu_{R'}) \). We recall that, for each Lipschitz function \( f \), with \( \text{Lip}(f) \leq 1 \), we have that
\[
\mu_R(f) = \int_X \int_X \sum_{j=1}^{n} q_j f(\phi_j(x, b)) \, d\nu(b) \, d\mu_R(x)
\]
and
\[
\mu_{R'}(f) = \int_X \int_X \sum_{j=1}^{n} q_j f(\phi_j(x, b)) \, d\nu'(b) \, d\mu_{R'}(x),
\]
where the integration over \( B \) (resp. \( B' \)) is replaced by \( X \) because the support of \( \nu \) is \( B \) (resp. \( \nu' \) is \( B' \)).

Let \( f : X \to \mathbb{R} \) be such that \( \text{Lip}(f) \leq 1 \), and define functions \( g, g' : X \to \mathbb{R} \) by
\[
g(x) = \int_X \sum_{j=1}^{n} q_j f(\phi_j(x, b)) \, d\nu(b)
\]
and \( g'(x) = \int_X \sum_{j=1}^{n} q_j f(\phi_j(x, b)) \, d\nu'(b) \). We also define, for each \( x \in X \), the function \( r_x : X \to \mathbb{R} \) by
\[
r_x(b) = \sum_{j=1}^{n} q_j f(\phi_j(x, b)).
\]
We claim that \( \text{Lip}(r_x) \leq a_2 \), uniformly with respect to \( x \), and \( \text{Lip}(g') \leq a_1 \) (resp. \( \text{Lip}(g) \leq a_1 \)). Indeed,
\[
|r_x(b) - r_x(b')| \leq \sum_{j=1}^{n} q_j |f(\phi_j(x, b)) - f(\phi_j(x, b'))| \leq \sum_{j=1}^{n} q_j \text{Lip}(f)(a_1 \, d(x, x) + a_2 \, d(b, b')) \leq a_2 \, d(b, b'),
\]
meaning that \( \text{Lip}(r_x) \leq a_2 \) for all \( x \in X \).

Analogously,
\[
|g'(x) - g'(y)| \leq \int_X \sum_{j=1}^{n} q_j |f(\phi_j(x, b)) - f(\phi_j(y, b))| \, d\nu'(b) \leq \int_X \sum_{j=1}^{n} q_j \text{Lip}(f)(a_1 \, d(x, y) + a_2 \, d(b, b)) \, d\nu'(b) \leq a_1 \, d(x, y),
\]
meaning that \( \text{Lip}(g') \leq a_1 \). For each \( x \in X \) we obtain
\[
|g(x) - g'(x)| = \left| \int_X r_x(b) \, d\nu(b) - \int_X r_x(b) \, d\nu'(b) \right| =
\]
because \(\text{Lip}(\frac{r_b}{a_2}) \leq 1\). Evaluating \(|\mu_R(f) - \mu_{R'}(f)|\) we obtain

\[
|\mu_R(f) - \mu_{R'}(f)| = \left| \int_X g(x)d\mu_R(x) - \int_X g'(x)d\mu_{R'}(x) \right| 
\]

\[
\leq \left| \int_X g(x)d\mu_R(x) - \int_X g'(x)d\mu_{R}(x) \right| + \left| \int_X g'(x)d\mu_{R}(x) - \int_X g'(x)d\mu_{R'}(x) \right| 
\]

\[
\leq \int_X |g(x) - g'(x)|d\mu_R(x) + a_1 \int_X \frac{g'(x)}{a_1}d\mu_{R}(x) - \int_X \frac{g'(x)}{a_1}d\mu_{R'}(x) 
\]

\[
\leq a_2 \ d_{MK}(\nu, \nu') + a_1 \ d_{MK}(\mu_R, \mu_{R'}),
\]

because \(\text{Lip}(\frac{r_b}{a_2}) \leq 1\). Thus

\[
d_{MK}(\mu_R, \mu_{R'}) = \max_{\text{Lip}(f) \leq 1} |\mu_R(f) - \mu_{R'}(f)| \leq a_2 \ d_{MK}(\nu, \nu') + a_1 \ d_{MK}(\mu_R, \mu_{R'}),
\]

meaning that \(d_{MK}(\mu_R, \mu_{R'}) \leq \frac{a_2}{1 - a_1} d_{MK}(\nu, \nu')\).

We now return to the map \(EV_S:\)

\[
d_{\max}(EV_S(B, \nu), EV_S(B', \nu')) = d_{\max}((A_R, \mu_R), (A_{R'}, \mu_{R'})) = 
\]

\[
= \max(h(A_R, A_{R'}), d_{MK}(\mu_R, \mu_{R'})) \leq 
\]

\[
\leq \max \left(\text{Lip}(S)h(B, B'), \frac{a_2}{1 - a_1} d_{MK}(\nu, \nu')\right) = 
\]

\[
= \max \left(\text{Lip}(S), \frac{a_2}{1 - a_1}\right) \ max(h(B, B'), d_{MK}(\nu, \nu')) = 
\]

\[
= \max \left(\text{Lip}(S), \frac{a_2}{1 - a_1}\right) d_{\max}((B, \nu), (B', \nu')), 
\]

meaning that \(\text{Lip}(EV_S) \leq \max \left(\text{Lip}(S), \frac{a_2}{1 - a_1}\right) < 1\) (recall that \(\sum_{i=1}^{2} a_i < 1\)). In order to conclude our proof, we just notice that since \(EV_S\) is contractive for any \((B_0, \nu_0) \in \Gamma\), the sequence \((EV_S(B_0, \nu_0))_{k \geq 0}\) converges to \((\overline{A}, \overline{\nu})\) in \(\Gamma\) as \(k \to \infty\), where \((\overline{A}, \overline{\nu})\) is the unique fixed point of \(EV_S\). From Lemma 3.13 we obtain \(EV_S(A_S, \mu_S) = (A_S, \overline{\nu})\), thus \((\overline{A}, \overline{\nu}) = (A_S, \mu_S)\), concluding our proof. \(\square\)

4 Concluding remarks and future work

4.1 Revisiting the approximation procedure

Theorem 3.6 provides an approximation procedure to obtain the attractor of a GIFS via attractors of IIFSs. Indeed, given \(B_0 = \{x_0\}\) we obtain \(B_1 = ev_S(B_0) = A_{R_0} \) via iteration of an initial set \(H \subset K^*(X)\) as in Theorem 2.4. Then we pick \(B_2 = ev_S(B_1) = A_{R_1}\) again via iteration of an initial set \(H \subset K^*(X)\), as in Theorem 2.4, and so on, obtaining that \(h(B_k, A_S) \to 0\). We have proved that \(B_k \simeq A_S\), meaning that \(h(B_k, A_S) \to 0\) as \(k \to \infty\). For a GIFSp, from Theorem 3.23 if we start with \((B_0 = \{x_0\}, \nu_0 = \delta_{x_0}) \in K^*(X) \times M_1(X)\) we produce \((B_1, \nu_1) = EV_S(B_0, \nu_0) = (A_{R_0}, \mu_{R_0})\) where \(R_0 := R_{B_0, \nu_0}\) is the IIFS induced by the pair \((B_0, \nu_0)\). Naturally, \(\text{supp}(\nu_1) \subseteq B_1\) from Theorem 3.11. Then successively we choose \(R_1 := R_{B_1, \nu_1}\) the IIFS induced by the pair \((B_1, \nu_1)\) and produce \((B_2, \nu_2) = EV_S(B_1, \nu_1) = (A_{R_1}, \mu_{R_1})\) and so on. The sequence \((B_k, \nu_k) := EV_S^k(B_0, \nu_0) \to (A_S, \mu_S)\) as \(k \to \infty\).
4.2 Approximation algorithms

One could ask if there exist effective algorithms to approximate the attractor \( A_{\mathcal{R}_A} \) of the IIFS \( \mathcal{R}_A \) and if we can use it to approximate \( A_S \). For countable IIFSs [Sec01] Theorem 5 shows that for a given compact subset \( A \) of a metric space, is possible to construct a countable iterated function system having \( A \) as its attractor. The construction involves sequence of iterated function systems whose attractors approximate \( A \). For non countable IIFSs [Man10] proved some results on the approximation of measures generated by uncountably many one-dimensional affine maps.

The theoretical procedure described in Remark 3.7 is quite difficult in the practical use, since it requires the computation of many attractors of infinite IIFSs in order to approximate \( A_S \). A possible scheme to perform this, in a reasonable way, is the following: we could start with an algorithm:

**Theorem 4.1 (Ostrowski).** Let \( (X,d) \) be a complete metric space and \( T : X \to X \) be a contraction with \( \text{Lip}(T) \leq \alpha < 1 \). Let \( (\varepsilon_n) \) be a sequence of positive real numbers and \( (y_n) \subset X \) be such that

\[
y_0 = x_0 \quad \text{and} \quad d(y_k, T(y_{k-1})) \leq \varepsilon_k \quad \text{for} \quad k \in \mathbb{N}.
\]

Then

\[
d(y_k, p) \leq \frac{\alpha^k}{1 - \alpha} d(x_0, T(x_0)) + \sum_{i=1}^{k} \alpha^{k-i} \varepsilon_k, \quad \text{for} \quad k \in \mathbb{N},
\]

where \( p \) is the fixed point of \( T \). In particular, if \( \varepsilon_k \to 0 \), then \( y_k \to p \).

**Theorem 4.2.** The Algorithm approximates the attractor \( A_S \) of the GIFS

\[
S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}}),
\]

that is, the obtained sequence \( (B_k) \) converges to \( A_S \) as \( k \to \infty \).
Proof. The idea is to use Theorem 3.11. In order to do that we choose, $T = ev_S$, $p = A_S$, $x_0 = B_0$, $y_k = B_k$, $\alpha = \text{Lip}(ev_S) \leq \text{Lip}(F_S) < 1$, $\varepsilon_k = \alpha \beta_k + \sigma_k \to 0$. We only need to show that $h(ev_S(B_k), B_k) \leq \varepsilon_k$. Indeed,

$$h(ev_S(B_k), B_k) \leq h(ev_S(B_k), ev_S(B_{\beta_k}^{B_k-1})) + h(ev_S(B_{\beta_k}^{B_k-1}), B_k) \leq \alpha h(B_k, B_{\beta_k}^{B_k-1}) + h(A_{R_{\beta_k}^{B_k-1}}, B_k) \leq \alpha \beta_k + \sigma_k = \varepsilon_k.$$ 

We notice that the step (3) of the loop in the Algorithm 1 consists in to approximate the attractor of a finite IFS, and there are many efficient ways to do that. The only computational restriction is the computational purposes one can choose $\beta_k = \sigma_k = \frac{1}{k}$ and then $\varepsilon_k \to 0$. We presented a pseudocode here, but the implementation of an actual algorithm would be the subject of a future work employing the discrete algorithm from [dCOS21] for the step (3) of the loop and making a comparison with the classical iteration for GIFS.

4.3 Approximate Chaos Game Theorem and Ergodic theorem for GIFS

Another natural question is if there exists some natural chaos game theorem or ergodic theorem for the induced IIFS which could approximate, in a reasonable way, the attractor and integrals with respect to the Hutchinson measure of a given GIFS. We notice that a GIFS $S = (X^m, (\phi_j)_{j \in \{1, \ldots, n\}})$ is not a dynamical object, meaning that, from an initial $m$-tuple $(x_0, \ldots, x_{m-1}) \in X^m$ and a $j_0 \in \{1, \ldots, n\}$ we obtain a single value $x_m = \phi_{j_0}(x_0, \ldots, x_{m-1})$, but there is no obvious recipe to continue the iteration process. In [Oli17] we proposed a process where $x_{m+1} = \phi_{j_1}(x_1, \ldots, x_m)$, for $j_1 \in \{1, \ldots, n\}$, and so on. But this process is not capable to describe the actual attractor $A_S$, only a smaller set (see [Oli17] Example 11]). Unlike GIFSs, IIFSs are dynamically defined, meaning that they can be iterated from an initial point forming an orbit. Given a set $B \in B^*(X)$, let $R_B = (X, \psi_0 \circ \theta \in \Theta)$, where $\Theta = B^{m-1} \times \{1, \ldots, n\}$ and $\psi_0(x) = \phi_j(x, b_2, \ldots, b_m)$, for $\theta = (b_2, \ldots, b_m, j) \in \Theta$, be the induced IIFS. Given $x_0 \in X$ and $\theta_0 = (b_0^1, \ldots, b_0^m, j_0)$, define:

$$x_1 = \psi_0(x_0) = \phi_{j_0}(x_0, b_0^1, \ldots, b_0^m).$$

Then, choose $\theta_1 = (b_1^1, \ldots, b_1^m, j_1)$ and define:

$$x_2 = \psi_{j_1}(x_1) = \phi_{j_1}(x_1, b_1^1, \ldots, b_1^m),$$

and so on. In general, define

$$x_{k+1} = \psi_{j_k}(x_k) = \phi_{j_k}(x_k, b_k^1, \ldots, b_k^m), \ k \geq 0.$$ 

This sequence has a lot more freedom to spread than the one used in [Oli17], because at each iteration $\theta_k = (b_k^1, \ldots, b_k^m, j_k)$ is chosen accordingly a probability $p$.

We recall that an IFS has the chaos game property, if under some suitable hypothesis (see [BV11] Theorem 1, also [BL14] for a topological point of view), given a random orbit $(x_k)_{k=0}^{\infty}$ of $x_0$ under an IFS $F$, then, with probability one,

$$A_F = \lim_{K \to \infty} \{x_k\}_{k=K}^{\infty} = A_S?$$

where the limit is with respect to the Hausdorff metric and $A_F$ is the attractor of $F$. The first question is, if the process $\{(x_k^i)_{i \geq 0}\}$ has the chaos game property for $R_{B_k}$ then

$$\lim_{k \to \infty} \{x_k^i\}_{i \geq k} = A_S?$$

The second question is, since $\lim_{k \to \infty} A_{R_{B_k}} = A_S$ and $\mu_{R_{B_k}} = \mu_S$, if the process the process $(x_k^i)_{i \geq 0}$ is ergodic for $R_{B_k}$, that is, for almost (in the sense of Theorem 4.3) all address sequences $\gamma = (\theta_0, \theta_1, \ldots) \in \Theta^N$ we have $\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(x_k^i)$, then

$$\lim_{K \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(x_k^i) = \int f(y) d\mu_S(y).$$

A Chaos Game result for IIFS was proved by Leśniak in [Les15], but only for countable IIFS and an ergodic theorem for IIFS is given in [HYhWg05].
Theorem 4.3. \textbf{[HCY05]} Ergodicity of IIFS\] Let \( \mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta}, p) \) be an IIFS, where \((X,d)\) is a compact metric space, \( \Theta \) is compact, \( p \) is a Borel probability on \( \Theta \) and \( P \) is the product measure induced by \( p \) in \( \Theta^N \). If \( \mathcal{R} \) is bounded, uniformly contractive \( \left( \int \text{Lip}(\psi_\theta) \, dp(\theta) < 1 \right) \), and \( \mu \) is the Hutchinson measure of the IIFS, then, for any \( f \in C(X, \mathbb{R}) \) and \( \forall x \in X \), for \( P \) almost all address sequences \( \gamma = (\theta_0, \theta_1, \ldots) \in \Theta^\mathbb{N} \), we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k \leq m \leq n} f \left( \psi_{\theta_{m-1}} \circ \cdots \circ \psi_{\theta_0} \right) (x) = \int_X f(y) \, d\mu(y).
\]

Another computational ergodic theorem was proved for an IIFS in \textbf{[NDCS08]}, but only for \( \Theta = \mathbb{N} \). To answer these questions using the above results, or others like those, would be the subject of a future work.

4.4 Further generalizations and the respective induced IIFS

The family of sets which are attractors of GIFSs is wider than the one formed by attractors of finite IFSs. However, we proved that all this fractal attractors are also attractors of IIFS and they are also well approximated by them. Observe that if \( B \) is closed and bounded, then it is the attractor of the IIFS \( \mathcal{R} = (X, (\psi_\theta)_{\theta \in \Theta}) \), where \( \Theta = B \) and \( \psi_\theta(x) = \theta \) for all \( x \in X \). Hence the matter is to define IIFS with certain properties, which generates a given set \( B \). An effort towards finding more general fractals which are not attractors of any known IIFSs (and so of GIFSs) is to define and study infinite GIFSs. This was done in \textbf{[DISS15]} for a topological version of possibly infinite GIFS as a Matkowski function system, that is, a compact-to-compact family of mappings which are uniformly generalized Matkowski contractions.

Theorem 4.4. \textbf{[DISS15]} Theorem 3.5] Assume that \((X,d)\) is a complete metric space and
\[
S = \left( X^m, (\phi_j)_{j \in F} \right)
\]
is a GIFS that satisfies the following conditions:
(i) \( \sup_{j \in F} \text{Lip}(\phi_j) < 1 \), and
(ii) The fractal operator, associated to \( S \) is compact-to-compact, that is, preserves compact sets: for any \( A_1, \ldots, A_m \in \mathcal{K}^p(X) \)
\[
F_S(A_1, \ldots, A_m) = \bigcup_{j \in F} \phi_j(A_1 \times \cdots \times A_m) \in \mathcal{K}^p(X).
\]

Then, \( S \) generates a unique attractor.

For such systems we can also employ our approach producing IIFSs induced by each compact set. More precisely, for a possibly infinite GIFS on \( X \), \( S = \left( X^m, (\phi_j)_{j \in F} \right) \), of order \( m \geq 2 \) and a set \( B \in \mathcal{K}^p(X) \), the induced IIFS will be \( \mathcal{R}_B = (X, (\psi_\theta)_{\theta \in \Theta}) \), where \( \Theta = B^{m-1} \times F \) and \( \psi_\theta(x) = \phi_j(x, b_2, \ldots, b_m) \), for \( \theta = (b_2, \ldots, b_m, j) \in \Theta \).

Finally, following the program of expansion and generalization of the families of sets which are fractals generated by IIFSs one could consider (finite) GIFSs with infinite order (denoted GIFS\(_\infty\)). Such construction appeared in \textbf{[Mas20]} inspired by Seceleanu’s approach \textbf{[Sec14a]}, showing that a typical compact set in a Polish metric space is a generalized fractal. This result shows that by considering GIFS\(_\infty\) we can describe significantly more sets than using classical IFS theory. Actually, \textbf{[Sec01]} presented, for each compact subset \( K \) of a metric space, the construction of a countable iterated function system (CIFS) having \( K \) as a fractal attractor. Seceleanu in \textbf{[Sec14a]} considered mappings defined on the space \( (\ell_\infty(X), d_\infty) \) of all bounded sequences of elements from \( X \) with values in \( X \), endowed with the supremum metric \( d_\infty \), where \((X,d)\) is a metric space.

**Definition 4.5.** A generalized iterated function system of infinite order \((\text{GIFS}_\infty\text{ in short})\)
\[
S = \left( \ell_\infty(X), (\phi_j)_{j \in \{1, \ldots, n\}} \right),
\]
consists of a finite family of continuous functions \( \phi_j : \ell_\infty(X) \to X \).

We say that \( S \) satisfy the compact closure property, if for every \((K_k) \in \ell_\infty(\mathcal{K}^p(X))\), the closure of the image of the product \( \phi \left( \prod_{k=1}^\infty K_k \right) \in \mathcal{K}^p(X) \).
Theorem 4.6. ([Sec14a, Theorem 3.7] or [Mas20, Theorem 2.5]) Let \((X,d)\) be a complete metric space and \(S\) be a \(\text{GIFS}_\infty\) consisting of Banach contracting maps \((\text{Lip}(\phi_j) < 1, 1 \leq j \leq n)\). If \(S\) satisfy the compact closure property, then there is a unique \(A_S \in K^*(X)\) such that

\[ A_S = \bigcup_{j=1}^{n} \phi_j \left( \prod_{k=1}^{\infty} A_S \right). \]

The set \(A_S\) is called the fractal or the attractor of the \(\text{GIFS}_\infty\) \(S\).

To make a complete and up to date reference on recent developments regarding \(\text{GIFS}_\infty\) we notice that Maślanka and Strobin [MS18], made a significative advance on this subject, studying some further aspects of Secelean’s setting. More precisely, the attractor of a \(\text{GIFS}_\infty\) is approximated by attractors of \(\text{GIFS}\) of order \(m\) when it increases, [MS18, Theorem 4.11], assuming only that the \(\text{GIFS}\) maps are generalized Banach contractions. For the last, they present, in Section 7 of [MS18], a Cantor set on the plane which is an attractor of some \(\text{GIFS}_\infty\), but cannot be generated by any \(\text{GIFS}\), reinforcing the wider range of this theory regarding new fractals creation capability.

One more time, for a given \(\text{GIFS}_\infty\), satisfying reasonable assumptions, we could investigate the induced IIFS relating its attractors with \(A_S\). More precisely, for a \(\text{GIFS}_\infty\) on \(X\), given by \(S = (\ell_\infty(X), (\phi_j)_{j \in \{1,\ldots,n\}})\), the induced IIFS would be \(\mathcal{R}_B = (X, (\psi_{\theta})_{\theta \in \Theta})\), where \(\Theta = \ell_\infty(B) \times \{1,\ldots,n\}\) and \(\psi_{\theta} : X \to X\) is given by \(\psi_{\theta}(x) = \phi_j(x, b_2, b_3, \ldots)\), for \(\theta = ((b_2, b_3, \ldots), j) \in \Theta\).

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