Transverse Ward-Takahashi Relation for the Fermion-Boson Vertex Function in 4-dimensional QED

Han-xin He\textsuperscript{a,b,*}

\textsuperscript{a} China Institute of Atomic Energy, P.O.Box 275(18), Beijing 102413, P.R.China

\textsuperscript{b} Institute of Theoretical Physics, the Chinese Academy of Science, Beijing 100080, P.R.China

Abstract

I present a general expression of the transverse Ward-Takahashi relation for the fermion-boson vertex function in momentum space in 4-dimensional QED, from which the corresponding one-loop expression is derived straightforwardly. Then I deduce carefully this transverse Ward-Takahashi relation to one-loop order in d-dimensions, with $d = 4 + \epsilon$. The result shows that this relation in d-dimensions has the same form as one given in 4-dimensions and there is no need for an additional piece proportional to $(d - 4)$ to include for this relation to hold in 4-dimensions. This result is confirmed by an explicit computation of terms in this transverse WT relation to one-loop order. I also make some comments on the paper given by Pennington and Williams who checked the transverse Ward-Takahashi relation at one loop order. PACS number(s): 11.30.-j; 11.15.Tk; 12.20.Ds.

Keywords: Transverse Ward-Takahashi relation; One-loop calculations

* E-mail address: hxhe@iris.ciae.ac.cn
I. INTRODUCTION

The gauge symmetry imposes powerful constraints on the basic interaction vertices, leading to a variety of exact relations among Green’s functions—referred to as Ward-Takahashi (WT) relations[1]. They play an important role in the study of gauge theories through the use of Dyson-Schwinger equations[2-5]. The well-known and simplest WT relation relates the fermion-gauge-boson vertex $\Gamma^\mu_V$ to the fermion propagator $S_F$:

$$q_\mu \Gamma^\mu_V(p_1,p_2) = S_F^{-1}(p_1) - S_F^{-1}(p_2),$$  (1)

where $q = p_1 - p_2$. The normal WT identity (1) is satisfied both perturbatively and nonperturbatively, but it specifies only the longitudinal part of the vertex, leaving the transverse part unconstrained. It has long been known that the transverse part of the vertex plays the crucial role in ensuring multiplicative renormalizability and so in determining the propagator [2-5]. How to determine the transverse part of the vertex then becomes a crucial problem. Although much effort has been devoted to constructing the transverse part of the vertex in terms of an ansatz which satisfies some constraints but all such attempts remain ad hoc[2-6]. Such a constructed vertex is not unique since it is not fixed by the symmetry of the system. Takahashi first discussed the constraint relation for the transverse part of the vertex from symmetry, which is called the transverse WT relation[7]. So far a basic formula of the transverse WT relation for the fermion-boson vertex in coordinate space has been given as[8]

$$\partial^\mu_x \left\langle 0 \left| T j_\nu(x) \tilde{\psi}(x_1) \tilde{\psi}(x_2) \right| 0 \right\rangle - \partial^\nu_x \left\langle 0 \left| T j_\mu(x) \tilde{\psi}(x_1) \tilde{\psi}(x_2) \right| 0 \right\rangle = i\sigma^{\mu\nu} \left\langle 0 \left| T \tilde{\psi}(x_1) \tilde{\psi}(x_2) \right| 0 \right\rangle \delta^4(x_1 - x) + i \left\langle 0 \left| T \tilde{\psi}(x_1) \tilde{\psi}(x_2) \right| 0 \right\rangle \sigma^{\mu\nu} \delta^4(x_2 - x) + 2m \left\langle 0 \left| T \tilde{\psi}(x) \sigma^{\mu\nu} \psi(x_1) \tilde{\psi}(x_2) \right| 0 \right\rangle + \lim_{x' \to x} i(\partial^\xi_x - \partial^\xi_{x'}) \epsilon^{\lambda\mu\nu\rho} \left\langle 0 \left| T \tilde{\psi}(x') \gamma_\rho \gamma_5 U_P(x', x) \psi(x_1) \psi(x_1) \tilde{\psi}(x_2) \right| 0 \right\rangle,$$  (2)

where $j_\mu(x) = \tilde{\psi}(x) \gamma^\mu \psi(x)$, $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ and $m$ is the bare fermion mass. The Wilson line $U_P(x', x) = P \exp(-ig \int_{x'}^{x} dy A_\rho(y))$ is introduced in order that the operator is locally
gauge invariant, where $A_\rho$ are gauge fields and $g = e$ in QED case. The last matrix element in Eq.(2) expresses a non-local axial-vector vertex function in coordinate space.

The momentum space representation of this transverse WT relation is obtained by computing the Fourier transformation of Eq.(2), which gives[10]:

$$
i q^\mu \Gamma_\nu(p_1, p_2) - i q^\nu \Gamma_\mu(p_1, p_2) = S_F^{-1}(p_1) \sigma^{\mu\nu} + \sigma^{\mu\nu} S_F^{-1}(p_2) + 2 m \Gamma_\mu^\nu(p_1, p_2)$$

$$+ (p_1 \lambda + p_2 \lambda) \varepsilon^{\lambda\mu
u\rho} \Gamma_A\rho(p_1, p_2) - \int \frac{d^4k}{(2\pi)^4} 2k_\chi \varepsilon^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k),$$

(3)

where $\Gamma_{A\rho}$ and $\Gamma_\mu^\nu$ denote the axial-vector and tensor vertex functions, respectively, and the integral-term involves the non-local axial-vector vertex function $\Gamma_{A\rho}(p_1, p_2; k)$, with the internal momentum $k$ of the gauge boson appearing in the Wilson line. This integral-term was missing in the earlier works [8,9]. In perturbation theory the integral-term at one-loop order is easy to write

$$\int \frac{d^4k}{(2\pi)^4} 2k_\chi \varepsilon^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k) = g^2 \int \frac{d^4k}{(2\pi)^4} 2k_\chi \varepsilon^{\lambda\mu\nu\rho}{\alpha} \epsilon_{\mu\nu,\rho,\sigma} \gamma_\rho \gamma_5 \frac{1}{p_1 - k - m} \frac{1}{p_2 - k - m} \gamma_5 \frac{1}{k^2} \frac{\gamma^\lambda}{k^2} [g_{\alpha\beta} + (\xi - 1) \frac{k_\alpha k_\beta}{k^2}]$$

$$+ g^2 \int \frac{d^4k}{(2\pi)^4} 2\varepsilon^{\alpha\mu\nu\rho} \gamma^{\rho\gamma_5} \frac{1}{p_1 - k - m} \gamma_\rho \gamma_5 + \gamma_\rho \gamma_5 \frac{1}{p_2 - k - m} \gamma_5 \frac{1}{k^2} \frac{\gamma^\alpha}{k^2} [g_{\alpha\beta} + (\xi - 1) \frac{k_\alpha k_\beta}{k^2}],$$

(4)

where $k' = \gamma_\mu k^\mu$, $\xi$ is the covariant gauge parameter. Here the integral-term involves two parts: the first part is the one-loop axial-vector vertex contribution, and the second part is the one-loop self-energy contribution accompanying the vertex correction.

The transverse WT relation to one-loop order given by Eq.(3) with Eq.(4) was derived in 4-dimensions[10] and was demonstrated to be satisfied to one-loop order by Refs.[11,12] in the Feynman gauge but without performing the check in d-dimensions, with $d = 4 + \epsilon$. Especially, so far the complete expression of the integral-term has not been given.

Recently, Pennington and Williams made a good comment on the potential of the transverse WT relation to determine the full fermion-boson vertex and then checked the transverse WT relation to one loop order in d-dimensions[13]. However, they claimed that an additional piece, say $(d - 4) N^{\mu\nu}$, must be included in the evaluation of the integral-term for this
transverse WT relation to hold in 4-dimensions. This problem is crucial and so is worth to clarify for the further study and the application of this transverse WT relation. Obviously, the central subject is attributed to the study of the integral-term given in Eq.(3).

In this paper, at first, I present a complete expression of the integral-term involving the non-local axial-vector vertex function and hence give the general formula of the transverse WT relation for the fermion-boson vertex in momentum space in 4-dimensional QED, from which the corresponding one-loop expression can be derived straightforwardly. To see if an additional piece proportional to $(d - 4)$ should be included in Eq.(3), I then deduce carefully the transverse WT relation to one-loop order in d-dimensions and compute explicitly the terms of this transverse WT relation. The result shows that this transverse WT relation in d-dimensions has the same form as one, Eq.(3) with Eq.(4), given in 4-dimensions and there is no need for an additional piece $\sim (d - 4)$ to include for this relation to hold in 4-dimensions. The complete expression of the integral-term and the detailed deducing in d-dimensions are given in Sec.II. The one-loop result is checked, by an explicit computation of terms in this relation, in Sec.III. In Sec.IV, I show how the authors of Ref.[13] separate out a so-called additional piece $(d - 4)N^{\mu \nu}$ from the integral-term given by Eq.(4) by defining a modifying integral-term, which in fact does not change the original formula of the transverse WT relation to one-loop order. The conclusion and remark are given in Sec.V.

II. TRANSVERSE WARD-Takahashi RELATION IN D-DIMENSIONS

At first, let me write the complete expression of the integral-term involving $\Gamma_{A\rho}(p_1, p_2; k)$ in the transverse WT relation (3), where $\Gamma_{A\rho}(p_1, p_2; k)$ is given by the Fourier transformation of the last matrix element in Eq.(2):

$$\begin{align*}
\int d^4x d^4x' d^4x_1 d^4x_2 e^{i(p_1 \cdot x_1 - p_2 \cdot x_2 + (p_2 - k) \cdot x' - (p_1 - k) \cdot x')} \langle 0 | T\bar{\psi}(x') \gamma_\rho \gamma_5 U_F(x', x) \psi(x) \psi(x_1) \bar{\psi}(x_2) | 0 \rangle \\
= (2\pi)^4 \delta^4(p_1 - p_2 - q)iS_F(p_1)\Gamma_{A\rho}(p_1, p_2; k)iS_F(p_2),
\end{align*}$$

(5)

where $q = (p_1 - k) - (p_2 - k)$. Eq.(3), together with Eq.(5), gives the general expression of the transverse WT relation for the fermion-boson vertex function in momentum space in 4-
dimensional QED, which should be satisfied both perturbatively and non-perturbatively like the normal WT identity (1). In fact, perturbative calculations of Eq.(5) can be performed order by order in the interaction representation, which, at one-loop order, leads straight to the expression given by Eq.(4).

Eq.(5) shows that $\Gamma_{A\rho}(p_1, p_2; k)$ is a non-local axial-vector vertex function, which and hence the integral-term is the four-point-like function. This integral-term is essential for this transverse WT relation to be satisfied both perturbatively and non-perturbatively. Indeed, as shown by Refs.[11,12], this integral-term is crucial to prove the transverse WT relation being satisfied to one-loop order in perturbation theory.

In the following, let me check if the transverse WT relation to one-loop order, given by Eq.(3) with Eq.(4), holds in 4-dimensions by deducing this transverse WT relation to one-loop order in $d$-dimensions with a similar procedure outlined in Ref.[11].

The fermion-boson vertex to one-loop order in perturbation theory is known as:

$$\Gamma^{\mu}_{V}(p_1, p_2) = \gamma^{\mu} + \Lambda^{\mu}_{(2)}(p_1, p_2)$$

(6)

with

$$\Lambda^{\mu}_{(2)}(p_1, p_2) = \int \frac{d^d k}{(2\pi)^d} \gamma^{\alpha} q^{\gamma} q^{\gamma}_{\alpha},$$

(7)

where the shorthand denotes

$$\int \frac{d^d k}{(2\pi)^d} = g^2 \int \frac{d^d k}{k^2 a^2 b^2}, \quad a = p_1 - k, b = p_2 - k,$$

(8)

and the case of massless fermion and the Feynman gauge are taken for simplifying the discussion. Eq.(6) gives

$$i q^{\mu} \Gamma^{\nu}_{V}(p_1, p_2) - i q^{\nu} \Gamma^{\mu}_{V}(p_1, p_2) = i(q^{\mu}\gamma^{\nu} - q^{\nu}\gamma^{\mu}) + i(q^{\mu}\Lambda^{\nu}_{(2)}(p_1, p_2) - q^{\nu}\Lambda^{\mu}_{(2)}(p_1, p_2)).$$

(9)

Here $i(q^{\mu}\gamma^{\nu} - q^{\nu}\gamma^{\mu})$ satisfies the transverse WT relation (3) at tree level, which is reduced to a trivial identity of $\gamma$-matrices:

$$i q^{\mu}\gamma^{\nu} - i q^{\nu}\gamma^{\mu} = \not{\sigma}^{\mu\nu} + \sigma^{\mu\nu} = (p_1 + p_2) \epsilon^{\lambda\mu\rho\nu} \gamma^{\rho}\gamma^{5}.$$
Using Eq.(7) and Eq.(10) and the identity of $\gamma$-matrices,

$$i\Gamma^{\mu\nu} = \{\gamma^\lambda, \sigma^{\mu\nu}\} = -2\varepsilon^{\mu\nu\rho\gamma_\rho\gamma_5} = i(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\lambda),$$  \hspace{1cm} (11)

the last term in the right-hand side of Eq.(9) can be written as

$$iq^\mu\Lambda_\nu^\mu(p_1, p_2) - iq^\nu\Lambda_\mu^\mu(p_1, p_2)$$

$$= \int d^4k\gamma^\alpha q^\mu\gamma_\alpha^\nu + \sigma^{\mu\nu}\gamma^\alpha_\alpha - \frac{1}{2}\int d^4k\gamma^\alpha q^\nu\gamma_\alpha^\mu + \sigma^{\mu\nu}\gamma_\alpha^\alpha, \hspace{1cm} (12)$$

To one-loop order, the inverse of the fermion propagator reads

$$S_F^{-1}(p_i) = \not{p}_{\not{i}} - \Sigma_{(2)}(p_i), \hspace{1cm} i = 1, 2,$$  \hspace{1cm} (13)

where $\Sigma_{(2)}(p_i)$ are one loop self-energy:

$$\Sigma_{(2)}(p_1) = \int d^4k\gamma^\alpha q^\mu\gamma_\alpha, \hspace{1cm} \Sigma_{(2)}(p_2) = \int d^4k\gamma^\alpha a^2\gamma_\alpha. \hspace{1cm} (14)$$

Using Eq.(14) and with some $\gamma$- algebraic calculations, Eq.(12) leads to

$$iq^\mu\Lambda_\nu^\mu(p_1, p_2) - iq^\nu\Lambda_\mu^\mu(p_1, p_2)$$

$$= -\Sigma_{(2)}(p_1)\sigma^{\mu\nu} - \sigma^{\mu\nu}\Sigma_{(2)}(p_2) + (p_{1\lambda} + p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Lambda_{\rho(2)}(p_1, p_2)$$

$$+ \int d^4k\gamma^\alpha q^\mu\gamma_\alpha^\nu + \sigma^{\mu\nu}\gamma^\alpha_\alpha$$

$$+ 2\int d^4k\{\gamma^\alpha q^\mu\gamma_\alpha^\nu + \sigma^{\mu\nu}\gamma_\alpha^\alpha - (a^2b_\lambda + b^2a_\lambda)(\gamma^\lambda\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\lambda)\}, \hspace{1cm} (15)$$

where $\Lambda_{\rho(2)}(p_1, p_2) = \int d^4k\gamma^\alpha q^\rho\gamma_\alpha^\rho f^\alpha$. Note that Eq.(15) is same as Eq.(23) of Ref.[13].

Using the identity $\gamma^\lambda\sigma^{\mu\nu} = \sigma^{\mu\nu}\gamma^\lambda + 2i(g^{\lambda\mu}\gamma^\nu - g^{\lambda\nu}\gamma^\mu)$, and performing some $\gamma$- algebraic calculations, I obtain from Eq.(15):

$$iq^\mu\Lambda_\nu^\mu(p_1, p_2) - iq^\nu\Lambda_\mu^\mu(p_1, p_2)$$

$$= -\Sigma_{(2)}(p_1)\sigma^{\mu\nu} - \sigma^{\mu\nu}\Sigma_{(2)}(p_2) + (p_{1\lambda} + p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Lambda_{\rho(2)}(p_1, p_2)$$

$$+ \int d^4k\gamma^\alpha q^\mu\gamma_\alpha^\nu + \sigma^{\mu\nu}\gamma_\alpha^\alpha$$

$$+ \int d^4k\{b^2\gamma_\alpha^\rho(\gamma^\rho\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\rho) + a^2(\gamma^\alpha\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\alpha)\}, \hspace{1cm} (16)$$
which shows that there is no additional piece $\sim (d - 4)$. To confirm this result, let me make further check by an explicit calculation in $d$-dimensions. The possible additional piece $(d - 4)\tilde{N}^{\mu\nu}$, if exits, should be given by the difference between Eq.(15) and Eq.(16):

$$
(d - 4)\tilde{N}^{\mu\nu} = 2\int \frac{dk}{(2\pi)^d} \{a^2 b_\lambda + b^2 a_\lambda(\gamma^\lambda \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\lambda)\}
$$

$$- \int \frac{dk}{(2\pi)^d} \{b^2 \gamma^\lambda(\gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha) + a^2 (\gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha)\}. \quad (17)
$$

Using the Dirac algebra in $d$-dimensions

$$\gamma^\alpha \gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\alpha = -2\gamma^\nu \gamma^\lambda \gamma^\mu - (d - 4)\gamma^\mu \gamma^\lambda \gamma^\nu, \quad \gamma^\alpha \gamma^\mu \gamma^\alpha = -(d - 2)\gamma^\mu, \quad (18)
$$

and performing the $\gamma$-algebraic calculations, I find that $(d - 4)\tilde{N}^{\mu\nu} = 0$, which indicates again that there is no additional piece $\sim (d - 4)$ in Eq.(16).

Combining Eqs.(9)-(13) with (16) gives the transverse WT relation for the fermion-boson vertex to one-loop order in $d$-dimensions (for massless case):

$$iq^\mu \Gamma^\nu(p_1, p_2) - iq^\nu \Gamma^\mu(p_1, p_2)
$$

$$= S_f^{-1}(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu} S_f^{-1}(p_2) + (p_{1\lambda} + p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho} \Gamma_A^\rho(p_1, p_2)
$$

$$- \int \frac{d^d k}{(2\pi)^d} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho} \Gamma_A^\rho(p_1, p_2; k) \quad (19)
$$

with

$$\int \frac{d^d k}{(2\pi)^d} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho} \Gamma_A^\rho(p_1, p_2; k)
$$

$$= g^2 \int \frac{d^d k}{(2\pi)^d} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho} [\gamma^\rho \gamma^5 \frac{1}{H_1 - k^\rho} \frac{1}{H_2 - k^\rho} \gamma^\rho - i \frac{1}{k^2} (g_{\alpha\beta} + (\xi - 1) \frac{k_\alpha k_\beta}{k^2})]
$$

$$+ g^2 \int \frac{d^d k}{(2\pi)^d} 2\varepsilon^{\alpha\mu\nu\rho} [\gamma^\rho \gamma^5 \frac{1}{H_1 - k^\rho} \frac{1}{H_2 - k^\rho} \gamma^\rho - i \frac{1}{k^2} (g_{\alpha\beta} + (\xi - 1) \frac{k_\alpha k_\beta}{k^2})], \quad (20)
$$

where the covariant gauge is used to replace the Feynman gauge. The result shows that Eq.(19) with Eq.(20) given in $d$-dimensions has the same form as Eq.(3) with Eq.(4) given in 4-dimensions and so there is no need for an additional piece $\sim (d - 4)$ to include for the transverse WT relation to hold in 4-dimensions. I would like to emphasis that the relation (19) together with (20) are exact to one-loop order because they have been deduced exactly without any ambiguity.
III. COMPUTING TERMS OF THE TRANSVERSE WARD-TAKAHASHI
RELATION TO ONE-LOOP ORDER

Let me check the transverse WT relation (19) with (20) by an explicit computation of terms in Eq.(19). For simplicity, I consider the massless fermion case and the Feynman gauge. The main task is to compute the integrals given by Eq.(20). These integrals were computed directly in Ref.[13](I will return to discuss the problem in the computation of Ref.[13] in next section ). Here I use another way to compute the integrals, that is, at first, using the identity of γ matrices, Eq.(11), and then do the integral calculations. Thus I find

\[- \int \frac{d^d k}{(2\pi)^d} \frac{1}{2k_\lambda \varepsilon^{\lambda \mu \nu \rho} \Gamma_{\Lambda \rho}(p_1, p_2; k)} \]

\[= g^2 \int \frac{d^d k}{(2\pi)^d} \gamma^\alpha \frac{1}{p_1^2 - k^2} (2\gamma_1 \sigma^{\mu \nu} + \sigma^{\mu \nu} \gamma_1) \frac{1}{p_2^2 - k^2} \gamma^\alpha \frac{i}{k^2} \]

\[+ g^2 \int \frac{d^d k}{(2\pi)^d} \gamma^\alpha \frac{1}{p_1^2 - k^2} (2\gamma_1 \sigma^{\mu \nu} + \sigma^{\mu \nu} \gamma_1) \frac{1}{p_2^2 - k^2} \gamma^\alpha \frac{i}{k^2}, \quad (21)\]

which may be denoted as $P^\mu_4$. The computation gives

\[P^\mu_4 = - \frac{i\alpha}{2\pi^3} [\gamma_2 (\gamma_2 \sigma^{\mu \nu} + \sigma^{\mu \nu} \gamma_2) J^{(0)}(p) - \gamma_2 \gamma_1 \gamma_1 J^{(1)}(p) + \gamma_2 \gamma_1 \gamma_1 J^{(2)}(p)] \]

\[+ \gamma_2 \gamma_1 \gamma_2 \sigma^{\mu \nu} + \sigma^{\mu \nu} \gamma_2 \gamma_2 J^{(0)}(p) - \gamma_2 \gamma_1 \gamma_1 J^{(1)}(p) + \gamma_2 \gamma_1 \gamma_1 J^{(2)}(p)] \]

\[+ \Sigma_{(2)}(p_1) \sigma^{\mu \nu} + \sigma^{\mu \nu} \Sigma_{(2)}(p_2), \quad (22)\]

where

\[\Sigma_{(2)}(p_i) = - \frac{i\alpha}{4\pi^3} (d - 2) [\gamma_1 K^{(0)}(p) - \gamma_1 K^{(1)}(p)], \quad i = 1, 2. \quad (23)\]

Here $\alpha = g^2/4\pi$, $J^{(0)}$, $J^{(1)}_\lambda$, $J^{(2)}_{\lambda \eta}$, $K^{(0)}(p_i)$ and $K^{(1)}_\lambda(p_i)$ are some integrals listed in Appendix.

Now let me compute other terms in the transverse WT relation (19) by following notations: $P^\mu_1 = iq^\mu \Gamma^\nu_{V(2)}(p_1, p_2) - iq^\mu \Gamma^\nu_{V(2)}(p_1, p_2)$, $P^\mu_2 = S^{-1}_{F(2)}(p_1) \sigma^{\mu \nu} + \sigma^{\mu \nu} S^{-1}_{F(2)}(p_2)$, $P^\mu_3 = (p_{1\lambda} + p_{2\lambda}) \varepsilon^{\lambda \mu \nu \rho} \Gamma_{\Lambda \rho}(p_1, p_2)$. 

8
The fermion-boson vertex function at one-loop order is familiar:

\[ \Lambda_{(2)}^\mu(p_1, p_2) = -\frac{i\alpha}{4\pi^3} \{ \gamma^\alpha p_1^\mu \gamma^\mu p_2 \gamma_\alpha J^{(0)} \]

\[ - (\gamma^\alpha p_1^\mu \gamma^\mu p_2 \gamma_\alpha) J^{(1)} + \gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha J^{(2)} \}. \tag{24} \]

Using Eq.(24) and Eq.(18), it is straightforward to get

\[ P_1^{\mu\nu} = iq^\mu \gamma^\nu - iq^\nu \gamma^\mu - \frac{\alpha}{2\pi^3} [\gamma^\mu p_1^\nu - q^\nu \gamma^\mu p_1 J^{(0)}]
\]

\[ - [\gamma^\mu p_1^\nu - q^\nu \gamma^\mu] J^{(1)} + \gamma^\mu (q^\nu \gamma^\nu - q^\nu \gamma^\mu) J^{(2)} \]

\[ -(d - 4) \frac{\alpha}{4\pi^3} [\gamma^\mu p_1^\nu - q^\nu \gamma^\mu J^{(0)} - \gamma^\mu (q^\nu \gamma^\nu - q^\nu \gamma^\mu) J^{(1)}
\]

\[ + \gamma^\mu (q^\nu \gamma^\nu - q^\nu \gamma^\mu) J^{(2)} \}. \tag{25} \]

The axial-vector vertex function at one-loop order can be obtained from Eq.(24) by replacing \( \gamma^\mu \) with \( \gamma^\mu \gamma_5 \), thus it gives

\[ P_3^{\mu\nu} = (p_{1\lambda} + p_{2\lambda}) \varepsilon^{\lambda\mu\nu\rho} \gamma_\rho \gamma_5
\]

\[ + \frac{i\alpha}{4\pi^3} [\gamma^\mu p_1^\nu - q^\nu \gamma^\mu] J^{(0)} - \gamma^\mu (q^\nu \gamma^\nu - q^\nu \gamma^\mu) J^{(1)}
\]

\[ + \gamma^\mu (q^\nu \gamma^\nu - q^\nu \gamma^\mu) J^{(2)} \}

\[ -(d - 4) \frac{i\alpha}{8\pi^3} [\gamma^\mu p_1^\nu - q^\nu \gamma^\mu J^{(0)} - \gamma^\mu (q^\nu \gamma^\nu - q^\nu \gamma^\mu) J^{(1)}
\]

\[ + \gamma^\mu (q^\nu \gamma^\nu - q^\nu \gamma^\mu) J^{(2)} \}. \tag{26} \]

At last, it is easy to get

\[ P_2^{\mu\nu} = h_1 \sigma^{\mu\nu} + \sigma^{\mu\nu} h_2 - \Sigma_{(2)}(p_1) \sigma^{\mu\nu} - \sigma^{\mu\nu} \Sigma_{(2)}(p_2). \tag{27} \]

Now using the identity (11) and Eq.(9), I obtain

\[ P_1^{\mu\nu} - P_2^{\mu\nu} - P_3^{\mu\nu} - P_4^{\mu\nu} = 0. \tag{28} \]

This shows that the transverse WT relation for the fermion-boson vertex to one-loop order is satisfied indeed and there is no need for an additional piece proportional to \( (d - 4) \) to include, which confirms the conclusion obtained in last section.
IV. HOW A SO-CALLED ADDITIONAL PIECE MIGHT BE SEPARATED OUT?

In a recent paper, Pennington and Williams[13] claimed that an additional piece must be included in the evaluation of the Wilson line component (i.e. the integral-term) for the transverse WT relation to hold in 4-dimensions as (see Eq.(25) of Ref.[13]):

\[ iq^\mu \Gamma^\nu_V (p_1, p_2) - iq^\nu \Gamma^\mu_V (p_1, p_2) \]

\[ = S_{F}^{-1}(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu} S_{F}^{-1}(p_2) + (p_{1\lambda} + p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho} \Lambda_\rho(p_1, p_2) \]

\[ - \int \frac{d^dk}{(2\pi)^d} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho} \Gamma_\rho(p_1, p_2; k) P_{-M} \]

\[ -4(d - 4)g^2 \int \frac{d^dk}{(2\pi)^d} \frac{i(p_1 - k)_\lambda \varepsilon^{\lambda\mu\nu\rho} \gamma_\rho \gamma_5}{(p_1 - k)^2}, \quad (29) \]

where the last term is the so-called additional piece denoted as \((d - 4)N^{\mu\nu}\) in Ref.[13]. To analyze how they write this form in Ref.[13], let me follow their relative derivation. The key step is the derivation from Eq.(23) to Eq.(24) in Ref.[13]. Eq.(23) of Ref.[13] is same as Eq.(15) of present work. Rearranging Eq.(15) can gave

\[ iq^\mu \Lambda^\nu_{(2)}(p_1, p_2) - iq^\nu \Lambda^\mu_{(2)}(p_1, p_2) \]

\[ = -\Sigma_{(2)}(p_1)\sigma^{\mu\nu} - \sigma^{\mu\nu} \Sigma_{(2)}(p_2) + (p_{1\lambda} + p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho} \Lambda_\rho(2)(p_1, p_2) \]

\[ + \int \overline{dk} \gamma^\alpha \{ [\sigma^{\mu\nu} + \sigma^{\mu\nu}]_\lambda \} \gamma_\alpha \]

\[ - \int \overline{dk} (\gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha) (a^2 b^2 - q^2 b^2) \gamma_\alpha \]

\[ -2(d - 4) \int \overline{dk} (q^2 \sigma^{\mu\nu} + \sigma^{\mu\nu} q^2) \gamma_\alpha \]

\[ = \int \frac{d^dk}{(2\pi)^d} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho} \Gamma_\rho(p_1, p_2; k) P_{-M} \]

\[ = - \int \overline{dk} \gamma^\alpha q^2 \{ [\sigma^{\mu\nu} + \sigma^{\mu\nu}]_\lambda \} \gamma_\alpha \]

\[ + \int \overline{dk} (\gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha) (a^2 b^2 - q^2 b^2) \gamma_\alpha \]

\[ = g^2 \int \frac{d^dk}{(2\pi)^d} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho} \gamma_\alpha \frac{1}{b^2_l - k^2_l} \gamma_\rho \gamma_5 \frac{1}{b^2_\rho - k^2_\rho} \gamma_\alpha - \frac{i}{k^2} \quad (30) \]

This is the same equation as Eq.(24) of Ref.[13]. Combining Eq.(30) with Eqs.(9) and (10), then leads to the expression (29), where a modifying integral-term is defined as
\[ +g^2 \int \frac{d^d k}{(2\pi)^d} 2\varepsilon^\alpha\mu\nu\rho \gamma_5 \bigg[ \frac{1}{p_1 - k'} - \frac{1}{p_2 - k'} \bigg] \gamma^\alpha - i \frac{k}{k^2}, \] (31)

which is obviously different from the integral-term given by Eq.(20) if comparing the second part of the right-hand side of Eq.(31) with that of Eq.(20) (in the Feynman gauge). Furthermore, comparing Eqs.(16) and (19) with Eqs.(30) and (29), I then obtain following relation:

\[ \int \frac{d^d k}{(2\pi)^d} 2k_\lambda \varepsilon^\lambda\mu\nu\rho \Gamma_{A\rho}(p_1, p_2; k) = \int \frac{d^d k}{(2\pi)^d} 2k_\lambda \varepsilon^\lambda\mu\nu\rho \tilde{\Gamma}_{A\rho}(p_1, p_2; k)_{p-M} \]
\[ +4(d-4)g^2 \int \frac{d^d k}{(2\pi)^d} k^2 \varepsilon^\alpha (p_1 - k)\lambda \varepsilon^\lambda\mu\nu\rho \gamma_5. \] (32)

Thus, it shows clearly that the authors of Ref.[13] separated out a so-called additional piece \((d-4)N^{\mu\nu}\) from the integral-term by defining a modifying integral-term, as shown by Eqs.(31)-(32), which, of course, does not change the formula of the transverse WT relation to one-loop order as given by Eq.(19) with Eq.(20).

However, in the explicit one-loop computation of terms in the transverse WT relation given in Sec.3 of Ref.[13], the authors of Ref.[13] replaced the modified integral-term with the original integral-term (see Eq.(42) of Ref.[13]) but still included the additional piece \((d-4)N^{\mu\nu}\). Such a computation is obviously inconsistent. In fact, there was a problem in their computation of the second part of the integral-term given in Ref.[13]. To see what is the problem, let me compute this part in the Feynman gauge in the following:

\[ -g^2 \int \frac{d^d k}{(2\pi)^d} 2k_\lambda \varepsilon^\lambda\mu\nu\rho \Gamma_{A\rho}(p_1, p_2; k) [\text{second part}] \]
\[ = -g^2 \int \frac{d^d k}{(2\pi)^d} 2\varepsilon^\alpha\mu\nu\rho \bigg[ \gamma^\alpha \frac{1}{p_1 - k'} \gamma_5 + \gamma_\rho \gamma_5 \frac{1}{p_2 - k'} \gamma^\alpha \bigg] - i \frac{k}{k^2} \]
\[ = \frac{\alpha}{4\pi^3} \gamma^\alpha \gamma^\lambda \gamma^\alpha \mu \nu \int \frac{d^d k}{(p_1 - k)^2 k^2} + \gamma^\alpha \mu \nu \gamma^\lambda \gamma_\alpha \int \frac{d^d k}{(p_2 - k)^2 k^2}. \] (33)

where the notation \(\gamma^\alpha\mu\nu\) is given in Eq.(11). Noticing that

\[ \gamma_\alpha \gamma^\lambda \gamma^\alpha \mu \nu = -\gamma^\alpha \mu \nu \gamma^\lambda \gamma_\alpha + 2i(d-4)(\gamma^\lambda \sigma^\mu \nu + \sigma^\mu \nu \gamma^\lambda), \] (34)
and using this relation into Eq.(33), I find

\[- \int \frac{d^dk}{(2\pi)^d} 2k\varepsilon^{\lambda\mu\nu}\Gamma_{A\rho}(p_1,p_2;k) [\text{second part}]\]

\[= -\frac{\alpha}{2\pi^3}[p_{2\lambda}K^{(0)}(0,p_2) - p_{1\lambda}K^{(0)}(p_1,0) - K^{(1)}_\lambda(0,p_2) + K^{(1)}_\lambda(p_1,0)]\]

\[\times (2\gamma\mu g^{\lambda\nu} - 2\gamma^\mu g^{\lambda\nu} + (d-4)(\gamma^\mu g^{\lambda\nu} - g^{\mu\nu})\gamma^\lambda)\]

\[+ \frac{i\alpha}{2\pi^3}(d-4)[p_{1\lambda}K^{(0)}(p_1,0) - K^{(1)}_\lambda(p_1,0)][(\gamma^\lambda\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\lambda)]\],

(35)

where the first piece of the right-hand side of Eq.(35) gives the corresponding result of Ref.[13], while the contribution of second piece \(\sim (d-4)\) was missing in the corresponding computation of Ref.[13](see Eq.(44) of Ref.[13]). This missing piece by Ref.[13] in the integral form is

\[-4(d-4)g^2 \int \frac{d^dk}{(2\pi)^d} \frac{-i(p_1 - k)\varepsilon^{\lambda\mu\rho}\gamma_\rho \gamma_5}{k^2(p_1 - k)^2}\]

(36)

which is just equal to the so-called additional piece given in Eq.(29). Thus this computation indicates clearly that a piece equal to the so-called additional piece was missing in the computation of the integral-term in Ref.[13]. Including such a missing piece into their computation of the integral-term, then an additional piece does not need to be introduced.

V. CONCLUSION AND REMARK

In conclusion, I present the complete expression of the integral-term involving the non-local axial-vector vertex function and hence give the general expression of the transverse Ward-Takahashi (WT) relation for the fermion-boson vertex function in momentum space in 4-dimensional QED, which should be satisfied both perturbatively and non-perturbatively like the normal WT identity. I have deduced that this transverse WT relation to one-loop order in d-dimensions, with \(d = 4 + \epsilon\), has the same form as one given in 4-dimensions and there is no need for an additional piece proportional to \((d-4)\) to include in the evaluation of the integral-term for this relation to hold in 4-dimensions. This result has been confirmed by an explicit computation of terms in this transverse WT relation to one-loop order in the Feynman gauge and the massless fermion case.
It has been shown that the authors of Ref. [13] in their paper separated out a so-called additional piece, \((d - 4)N^{\mu\nu}\), from the integral-term involving the non-local axial-vector vertex by defining a modified integral-term, which in fact does not change the original formula of the transverse WT relation to one loop order. Thus it needs not have introduced such an additional piece.

The transverse WT relation for the fermion-boson vertex, Eq. (3), shows that the transverse part of the fermion-boson (vector) vertex is related to the axial-vector and tensor vertices. Thus in order to constrain completely the fermion-boson vertex, it is needed to build the transverse WT relations for the axial-vector and tensor vertices as well. Their expressions in coordinate space have been derived in Ref. [9], where, however, the corresponding expressions in momentum space neglected the contributions from integral-terms. The complete expressions of the transverse WT relations for the axial-vector and tensor vertices in momentum space and their expressions to one-loop order can be obtained by a similar procedure given in this paper. Then the full fermion-boson vertex can be derived consistently in terms of a set of the normal and transverse WT relations for the fermion vertex functions, which will be discussed elsewhere [14].

**APPENDIX**

\[
J^{(0)}(p_1, p_2) = \int_M d^4k \frac{1}{k^2(p_1 - k)^2(p_2 - k)^2}, \tag{37}
\]

\[
J^{(1)}_{\mu}(p_1, p_2) = \int_M d^4k \frac{k_{\mu}}{k^2(p_1 - k)^2(p_2 - k)^2}, \tag{38}
\]

\[
J^{(2)}_{\mu\nu}(p_1, p_2) = \int_M d^4k \frac{k_{\mu}k_{\nu}}{k^2(p_1 - k)^2(p_2 - k)^2}, \tag{39}
\]

\[
K^{(0)}(p_i) = \int_M d^4k \frac{1}{(p_i - k)^2k^2}, \quad i = 1, 2, \tag{40}
\]

\[
K^{(1)}_{\mu}(p_i) = \int_M d^4k \frac{k_{\mu}}{(p_i - k)^2k^2}, \quad i = 1, 2, \tag{41}
\]
where $K^{(0)}(p_1) = K^{(0)}(p_1, 0)$, $K^{(0)}(p_2) = K^{(0)}(0, p_2)$, $K^{(1)}_{\mu}(p_1) = K^{(1)}_{\mu}(p_1, 0)$, $K^{(1)}_{\mu}(p_2) = K^{(1)}_{\mu}(0, p_2)$. These integrals can be carried out in the cutoff regularization scheme or in the dimensional regularization scheme[4,5,6,13]. Here I do not list the results of these integrals since I do not need to use these detailed results in this work.

**ACKNOWLEDGMENTS**

I would like to thank H.S.Zong for telling me Ref.[13] and useful conversations during a recent CCAST workshop (March 6-10,2006, Beijing, China). This work is supported by the National Natural Science Foundation of China under grant No.90303006.
REFERENCES

[1] J.Ward, Phys.Rev.78, 182 (1950); Y.Takahashi, Nuovo Cimento 6, 370 (1957).

[2] C.D.Roberts and A.G.Williams, Prog.Part.Nucl.Phys.33, 477 (1994), and references therein.

[3] R.Alkofer and L.von Smekal, Phys.Rep.353,281 (2001), and references therein.

[4] J.S.Ball and T.W.Chiu, Phys.Rev.D22, 2542 (1980).

[5] D.C.Curtis and M.R.Pennington, Phys.Rev.D42, 4165 (1990).

[6] A.Kizilersu, M.Reenders and M.R.Pennington, Phys.Rev.D52, 1242 (1995).

[7] Y.Takahashi, Phys.Rev.D15, 1589 (1977); Nuovo Cim.A47, 392 (1978).

[8] H.X.He, F.C.Khanna and Y.Takahashi, Phys.Lett.B 480, 222 (2000).

[9] Hanxin He, Phys.Rev.C63, 025207(2001).

[10] H.X.He, Nonperturbative fermion-boson vertex function in gauge theories, hep-th/0202013.

[11] H.X.He and H.W.Yu, Commun.Theor.Phys.(Beijing,China) 39, 559 (2003).

[12] H.X.He, Commun.Theor.Phys.(Beijing,China) 44, 103(2005).

[13] M.R.Pennington and R.Williams, Checking the transverse Ward-Takahashi relation at one loop order in 4-dimensions, hep-ph/0511254.

[14] Han-xin He, Full fermion-boson vertex function derived in terms of the Ward-Takahashi relations, to be published.