A priori bounds for geodesic diameter. Part II.
Fine connectedness properties of varifolds

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Abstract

For varifolds whose first variation is representable by integration, we introduce the notion of indecomposability with respect to locally Lipschitzian real valued functions. Unlike indecomposability, this weaker connectedness property is inherited by varifolds associated with solutions to geometric variational problems phrased in terms of sets, $G$ chains, and immersions; yet it is strong enough for the subsequent deduction of substantial geometric consequences therefrom. Our present study is based on several further concepts for varifolds put forward in this paper: real valued functions of generalised bounded variation thereon, partitions thereof in general, partition thereof along a real valued generalised weakly differentiable function in particular, and local finiteness of decompositions.

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1 Introduction

Throughout the introduction, we suppose the following assumptions to be valid.

**General hypotheses.** Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \) is an \( m \) dimensional varifold in \( U \), and its first variation, \( \delta V \), is representable by integration.

After reviewing the notion of sets of locally finite perimeter on varifolds, we describe our contributions—five concepts, four examples, eight theorems, and three corollaries—in their logical order.

**Sets of locally finite perimeter**

A fundamental concept for the present study is that of *distributional* \( V \) boundary of a \( \|V\| + \|\delta V\| \) measurable set \( E \) (equivalently, a set \( E \) which is measurable with respect to both measures, \( \|V\| \) and \( \|\delta V\| \)) defined by

\[
V \partial E = (\delta V) \llcorner E - \delta(V \llcorner E \times G(n, m)) \in \mathcal{D}'(U, \mathbb{R}^n)
\]

in [Men16a, 5.1]. With respect to the geometry of \( V \), the distribution \( V \partial E \) and its Borel regular variation measure, \( \|V \partial E\| \), could be understood as the distributional gradient of the characteristic function of \( E \) and the measure-theoretic perimeter of \( E \), respectively. Thus, one may say that the perimeter of \( E \) with respect to \( V \) is locally finite if and only if \( V \partial E \) is representable by integration or, equivalently, \( \|V \partial E\| \) is a Radon measure.

**Real valued functions of generalised bounded variation**

For a real valued \( \|V\| + \|\delta V\| \) measurable function, we shall apply the preceding concept of locally finite perimeter to its subgraph to obtain a definition of a function possessing generalised bounded variation with respect to \( V \). Alternatively, as suggested in [Men16a, 8.1, 8.10], we could base a definition of this notion on a distribution in \( U \times \mathbb{R} \) of type \( \mathbb{R}^n \) associated with the distributional boundaries of superlevel sets of the function in question. Both approaches are equivalent by the following theorem, Theorem \( A \) which includes a characterisation—by means of a natural absolute continuity condition—of the subclass \( T(V) \) of generalised \( V \) weakly differentiable functions introduced in [Men16a, 8.3].

**Theorem A** (see 4.2 and 4.5). Suppose \( V \) fulfils the **general hypotheses**, \( f \) is a real valued \( \|V\| + \|\delta V\| \) measurable function,

\[
E = \{(x, y) : f(x) > y \},
\]
$W$ is the Cartesian product of $V$ with the one-dimensional varifold in $\mathbb{R}$ associated with $\mathcal{L}^1$, the distribution $T \in \mathcal{D}'(U \times \mathbb{R}, \mathbb{R}^n)$ satisfies

$$T(\phi) = \int_V \partial_x(x,y) \cdot E(\phi(x,y)) \, d\mathcal{L}^1 y \quad \text{for } \phi \in \mathcal{D}(U \times \mathbb{R}, \mathbb{R}^n),$$

and $F : \text{dnn} f \to U \times \mathbb{R}$ is defined by $F(x) = (x, f(x))$ for $x \in \text{dnn} f$.

Then, $W \partial E$ is representable by integration if and only if $T$ is representable by integration. Moreover, $f \in T(V)$ if and only if $\|W \partial E\|$, or equivalently $\|T\|$, is absolutely continuous with respect to $F_\# V$.

Coarea formulae for functions $f$ of generalised $V$ bounded variation in terms of $T$—which acts as distributional derivative of $f$—follow, see [4.3] in particular, for characteristic functions, we recover the concept of locally finite $V$ perimeter of the corresponding set. These developments are in line with those originating from [DGA88] and [BBG 1965] for the special case $m = n$ and $\|V\|(A) = \mathcal{L}^n(A)$ for $A \subset U$, see [Men16a, 8.8, 8.19] and [4.4].

**Partitions**

We employ sets of zero distributional $V$ boundary to introduce the notion of partition of $V$ which generalises that of decomposition of $V$.

**Definition** (see 5.5). Under the general hypotheses a subfamily $\Pi$ of

$$\{V \perp E \times G(n,m) : E \text{ is } [V] + \|\delta V\| \text{ measurable, } V \partial E = 0, \|V\|(E) > 0\}$$

is termed a partition of $V$ if and only if

$$V(k) = \sum_{W \in \Pi} W(k) \quad \text{and} \quad \|\delta V\|(f) = \sum_{W \in \Pi} \|\delta W\|(f)$$

for $k \in \mathcal{H}^n(U \times G(n,m))$ and $f \in \mathcal{H}^n(U)$.

In view of 5.5, the previous notions of indecomposability, component, and decomposition from [Men16a, 6.2, 6.6, 6.9] can be rephrased: $V$ is indecomposable if and only if either $V = 0$ or $\{V\}$ is the only partition of $V$; $W$ is a component of $V$ if and only if $W$ is an indecomposable member of some partition of $V$; and, $\Xi$ is a decomposition of $V$ if and only if $\Xi$ is a partition of $V$ consisting of components of $V$. Decompositions were introduced for the study of $T(V)$ to characterise those members $f$ of that space which have vanishing generalised $V$ weak derivative, $V \partial D f$. In fact, in the case $V$ is rectifiable, $f \in T(V)$ satisfies $V \partial D f = 0$ if and only if there exist some decomposition $\Xi$ of $V$ and $v : \Xi \to \mathbb{R}$ such that

$$f(x) = v(W) \quad \text{for } \|W\| + \|\delta W\| \text{ almost all } x,$$

whenever $W \in \Xi$, see [Men16a, 10.10, 8.24, 8.34]. Decompositions are in general non-unique; in fact, one may consider three lines intersecting in a common point at equal angles, see [Men16a, 6.13]. For nonrectifiable $V$, elementary examples show that decompositions may fail to exist even if $\delta V = 0$, see [MS18, 4.12(3)].

The structure of functions in $T(V)$ with non-vanishing generalised $V$ weak derivative may be more complex. Whereas it is possible to define a member $f$ of $T(V)$ by firstly selecting a partition $\Pi$ of $V$ and then, for each $W \in \Pi$, a member of $T(W)$ subject only to a natural summability condition, see [MS18, 4.14] and [5.9], our first example shows that those $f$ may not exhaust $T(V)$.
Example 1 (see [5.10]). There exists a nonzero two-dimensional varifold $X$ in $\mathbb{R}^3$ with $\|\delta X\| \leq \kappa \|X\|$ for some $0 \leq \kappa < \infty$ and $\Theta^m(\|X\|, x) = 1$ for $\|X\|$ almost all $x$ such that, for some $f \in T(X)$, the function $f$ does not belong to $T(W)$ for any component $W$ of $X$.

This varifold $X$ has four components, each associated with a properly embedded two-dimensional submanifold of class $\infty$ of $\mathbb{R}^3$, and two distinct decompositions; there is a straight line which is the intersection of the submanifolds corresponding to the two components of each decomposition: $f$ possesses generalised bounded variation with respect to $W$ for each component $W$ of $X$; and the singular parts of the distributional derivative of $f$ with respect to $W$—that is, those not representable with respect to $F_\# \|W\|$—are concentrated along this line so as to cancel after summing over the two members $W$ of any decomposition of $X$. We have included Theorem A in our present paper because it allows us to employ the geometric viewpoint of boundaries of subgraphs in this construction.

Examples of decomposable varifolds

We shall demonstrate that—when viewed by means of the existing notion of indecomposability—the varifolds associated with compact $(\mathcal{H}^m, m)$ rectifiable sets, indecomposable $G$ chains (see [6.6]), and immersions may fail to inherit the connectedness properties of these objects. We firstly introduce a related concept.

Definition (see [6.2]). A family $\Xi$ of $m$ dimensional varifolds in $U$ is termed locally finite if and only if

$$\text{card}(\Xi \cap \{W: K \cap \text{spt} \|W\| \neq \emptyset\}) < \infty$$

whenever $K$ is a compact subset of $U$.

Example 2 (Two touching spheres, see [6.4] and [6.5]). Suppose $m = n - 1$, $A = \{a_1, a_2\} \subset \mathbb{R}^n$, where $a_1 = 0$ and $a_2 = (0, \ldots, 0, 2)$,

$$M = \mathbb{R}^n \cap \{x: \text{dist}(x, A) = 1\},$$

and $V$ is the $m$ dimensional varifold in $\mathbb{R}^n$ associated with $M$.

Then, $V$ satisfies the general hypotheses $\|\delta V\| \leq m\|V\|$, $M = \text{spt} \|V\|$ is a connected compact $(\mathcal{H}^m, m)$ rectifiable set, and $V$ is decomposable.

In fact, $V$ admits a unique decomposition whose elements correspond to the varifolds associated with the two spheres whose union equals $M$. In the case $m = 1$, slightly modifying the shape of the set $M$, one may require that $V$ is associated with some figure-eight immersion $F: S^1 \to \mathbb{R}^2$ of class $\infty$.

Example 3 (Three line segments, see [6.8]). Suppose $B = \{b_1, b_2, b_3\} \subset \mathbb{R}^2$, where $b_1 = (1, 0), b_2 = (0, 1),$ and $b_3 = (-1, 0)$,

$$M = \{tb: 0 \leq t \leq 1, b \in B\},$$

and $V$ is the one-dimensional varifold in $\mathbb{R}^2$ associated with $M$.

Then, $V$ is a varifold satisfying the general hypotheses $V$ is associated with an indecomposable integral $\mathbb{Z}/3\mathbb{Z}$ chain, and $V$ is decomposable.
The unique decomposition of this varifold \( V \) consists of the varifolds associated with the two line segments \( \{tb_1 + (1 - t)b_3 : 0 \leq t \leq 1 \} \) and \( \{tb_2 : 0 \leq t \leq 1 \} \).

Similar situations do occur with integer coefficients; in fact, using a variant of the map \( f \) in [Fed69, p.426], one may construct two real projective planes with \( m = 2 \) embedded into \( \mathbb{R}^6 \) touching along a common bounding projective line. In Examples 2 and 3, \( \text{spt} \| V \| \) is connected only through a single point. This illustrates the challenge involved in creating a \textit{measure-theoretic} notion of connectedness for varifolds which treats these varifolds as connected and is yet powerful enough to entail meaningful geometric consequences; this will be accomplished by the present paper and the final paper, see [MS23].

We now exhibit an example of a varifold associated with a properly \textit{immersed} submanifold-with-boundary in which decompositions are not locally finite. Even for general varifolds not necessarily associated with an immersion, this behaviour can be precluded by either of the natural Regularity hypotheses \([1]\) and \([2]\) below involving properly \textit{embedded} boundary data of class 2 of Dirichlet or Neumann type, respectively, see Theorems \([3]\) and \([E]\) below.

**Example 4** (see 6.15). There exist a two-dimensional manifold-with-boundary \( M \) and a proper immersion \( F : M \rightarrow \mathbb{R}^3 \) of class \( \infty \) such that no decomposition of the two-dimensional varifold in \( \mathbb{R}^3 \) associated with \( F \) is locally finite.

One may partially visualise the construction as follows: \( M \) equals the disjoint union of four half-planes and one plane; \( F \) restricted to any of these components of \( M \) is an embedding; the images of the four half-planes meet the isometrically embedded plane at an angle of \( 60^\circ \) in two curved lines (two half-planes per line); the two lines tangentially meet in a prescribed closed set; the two lines enclose an infinite number of accumulating open topological discs between each other; and, each of these discs occurs as component in the unique decomposition of \( V \).

**Properties of indecomposability with respect to a family of generalised weakly differentiable real valued functions**

The most central concept for our present study is the following notion of indecomposability of \( V \) with respect to a subfamily \( \Psi \) of \( T(V) \).

**Definition** (see 7.1). If \( V \) satisfies the \textit{general hypotheses} and \( \Psi \subset T(V) \), then we term \( V \) \textit{indecomposable of type} \( \Psi \) if and only if, whenever \( f \in \Psi \), the set of \( y \in \mathbb{R} \) such that \( E(y) = \{ x : f(x) > y \} \) satisfies

\[
\| V \| (E(y)) > 0, \quad \| V \| (U \sim E(y)) > 0, \quad V \partial E(y) = 0
\]

has \( \mathcal{L}^1 \) measure zero.

Equivalently, only for \( y \) in an exceptional set of \( \mathcal{L}^1 \) measure zero, we allow

\[
\{ V \setminus E(y) \times G(n,m), V \setminus (U \sim E(y)) \times G(n,m) \}
\]

to be a partition of \( V \). The smaller \( \Psi \), the less restrictive is the corresponding notion of indecomposability. We shall consider the following six families as \( \Psi \).
\(T(V)\) Generalised \(V\) weakly differentiable real valued functions.

\(\Gamma\) Continuous functions \(f \in T(V)\) with domain \(\text{spt} \|V\|\).

\(\Lambda\) Locally Lipschitzian functions \(f : U \to \mathbb{R}\).

\(E(U, \mathbb{R})\) Functions \(f : U \to \mathbb{R}\) of class \(\infty\).

\(D(U, \mathbb{R})\) Functions \(f : U \to \mathbb{R}\) of class \(\infty\) with compact support in \(U\).

\(\{f\}\) A particular function \(f \in T(V)\).

The strongest condition—indecomposability of type \(T(V)\)—coincides with the existing notion of indecomposability, see 7.2. In the case that \(V\) is rectifiable and every decomposition of \(V\) is locally finite, connectedness of \(\text{spt} \|V\|\) implies indecomposability of type \(\Gamma\), see Theorem H below; in particular, the decomposable varifolds of Examples 2 and 3 are indecomposable of type \(\Gamma\). Indecomposability of type \(\Lambda\) is expedient for varifolds associated with immersions and integral \(G\) chains, see Theorems F and G below, respectively. Indecomposability of type \(E(U, \mathbb{R})\) suffices to guarantee connectedness of \(\text{spt} \|V\|\), see (7.7). Indecomposability of type \(D(U, \mathbb{R})\) will be employed in the final paper of our series (see [MS23]) to establish—for varifolds satisfying a uniform lower density bound, local \(p\)-th power summability of their mean curvature, and a boundary condition—two types of varifold-geometric results: lower density bounds \(\mathcal{H}^{m-p}\) almost everywhere, provided \(m-1 \leq p < m\), and an a priori bound on the geodesic diameter of \(\text{spt} \|V\|\). Theorem C and Corollaries 2 and 3 below then entail that these results are applicable to various geometric variational problems including a variety of Plateau problems. The latter refers to the classical formulations of Reifenberg and Federer-Fleming as well as to those solutions obtained using integral \(G\) chains, min-max methods, and Brakke flow. Finally, the notion of indecomposability of type \(\{f\}\) serves as a tool to study the behaviour of \(V\) with regard to a fixed \(f \in T(V)\).

For varifolds \(V\) associated with \(m\) dimensional properly embedded submanifolds \(M\) of class 2 of \(U\), see 7.9 indecomposability of type \(E(U, \mathbb{R})\) is equivalent to both previously existing notions—connectedness of \(M\) and indecomposability of \(V\); moreover, such \(V\) is indecomposable of type \(D(U, \mathbb{R})\) if and only if either \(M\) is connected or no connected component of \(M\) is compact.

Whereas indecomposability of types \(T(V), E(U, \mathbb{R}), D(U, \mathbb{R})\), and \(\{f\}\) all differ by the examples already discussed (and the fact that \(V\) is always indecomposable of type \(\{0\}\)), we did not study whether the families \(\Gamma, \Lambda,\) and \(E(U, \mathbb{R})\) yield different types of indecomposability; in the case that \(V\) is rectifiable and every decomposition of \(V\) is locally finite, these three types of indecomposability are all equivalent to connectedness of \(\text{spt} \|V\|\) by Corollary F below.

Indecomposability of type \(\Psi\) may be exploited by coarea type arguments. The following theorem serves as an example of this method and is the foundation for the varifold-geometric results in the final paper of our series [MS23].

**Theorem B** (see 7.11[14] and 7.12). Suppose \(f \in T(V), V\) is indecomposable of type \(\{f\}\), \(Y \subset \mathbb{R}\), and \(f(x) \in Y\) for \(\|V\|\) almost all \(x\).

Then, \(\text{spt} f\|V\|\) is a subinterval of \(\mathbb{R}\) whose diameter is bounded by \(\mathcal{L}^1(Y)\). Moreover, if \(V \text{ D } f = 0\), then \(f\) is \(\|V\| + \|\delta V\|\) almost constant.

By way of contrast, in the case \(V\) admits a countably infinite partition and \(Y\) is a countable dense subset of \(\mathbb{R}\), there exists \(f \in T(V)\) with \(V \text{ D } f = 0\), \(\text{im } f = Y\), and \(\text{spt} f\|V\| = \mathbb{R}\) by [MS18, 4.14]. This shows the substance of the assumed indecomposability of type \(\{f\}\) and the diameter bound established.
Refining indecomposability in the opposite direction, a more stringent notion for integral varifolds $V$ was formulated in [Mon14, 2.15] and subsequently studied in [Cho23]. Such $V$ is termed *integrally indecomposable* if and only if there exists no $m$ dimensional integral varifold $W$ in $U$ such that $0 \neq W \neq V$,

$$\|V\| = \|W\| + \|V - W\|, \quad \|\delta V\| = \|\delta W\| + \|\delta(V - W)\|.$$  

The purpose of [Cho23]—a foundation to merge key elements of [Alm00, Chapters 1 and 2] and [Men16a]—is quite different from ours.

Unique partition along a generalised weakly differentiable real valued function

We next introduce two concepts expedient in deriving criteria for indecomposability of type $\Psi$. Firstly, the situation described in Example 1 led us to develop the notion of partition along a member of $T(V)$ as substitute.

**Definition** (see 8.6). For $V$ satisfying the general hypotheses and $f \in T(V)$, a subfamily $\Pi$ of

$$\{V \downarrow f^{-1}[I] \times G(n,m) : I \text{ subinterval of } \mathbb{R}, V \partial f^{-1}[I] = 0\}$$

is termed *partition of $V$ along $f$* if and only if $\Pi$ is a partition of $V$ and, whenever $W \in \Pi$, there exists no partition of $\mathbb{R}$ into subintervals $J_1$ and $J_2$ such that

$$\{W \downarrow f^{-1}[J_1] \times G(n,m), W \downarrow f^{-1}[J_2] \times G(n,m)\}$$

is a partition of $W$.

In contrast to the behaviour of general partitions exhibited in Example 1, we always have $f \in T(W)$ for members $W$ of a partition of $V$ along $f$ by 8.7. Moreover, each member of a partition along $f$ is indecomposable of type $\{f\}$, see 8.7, but a varifold $V$ indecomposable of type $\{f\}$ may still admit a nontrivial partition along $f$, see 8.9. The different behaviour corresponds to the absence of an exceptional set in the definition of partition along $f$.

**Theorem C** (see 8.10 and 8.13). *Under the general hypotheses, let $f \in T(V)$. Then, there exists at most one partition of $V$ along $f$; if $V$ is rectifiable, then a partition of $V$ along $f$ exists.*

The uniqueness established distinguishes the behaviour of partitions of $V$ along $f$ from that of decompositions, whereas the existence proof elaborates on the machinery yielding existence of decompositions. For non rectifiable varifolds, similar to decompositions, existence may fail in simple examples, see 8.11.

Criteria for local finiteness of decompositions

Secondly, we discuss two sets of regularity hypotheses—tailored to cover Dirichlet and Neumann boundary data, respectively—which allow us to study $\text{spt} \|V\|$ by providing criteria for local finiteness of decompositions of $V$, see Theorems D and E below. For this purpose, we recall the following definition.

7
Definition (see [Fed69, 2.4.12]). Suppose $\phi$ measures $X$, $Y$ is a Banach space, $1 \leq p \leq \infty$, and $f$ is a $\phi$ measurable $Y$ valued function.

Then, the quantity $\phi(p)(f)$ is defined by

$$
\phi(p)(f) = \left( \int |f|^p \, d\phi \right)^{1/p} \text{ if } p < \infty,
$$

$$
\phi(p)(f) = \inf \left\{ s : s \geq 0, \phi(\{ x : |f(x)| > s \}) = 0 \right\} \text{ if } p = \infty.
$$

Regularity hypotheses 1 (see 9.1). Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $B$ is a properly embedded $m - 1$ dimensional submanifold of class 2 of $U$, $X$ is an $m$ dimensional varifold in $U$,

$$
\beta = \infty \text{ if } m = 1, \quad \beta = m/(m - 1) \text{ if } m > 1,
$$

$$
sup \left\{ (\delta X)(\theta) : \theta \in \mathcal{D}(U, \mathbb{R}^n), spt \theta \subset K \sim B, \|X\|_1(\theta) \leq 1 \right\} < \infty
$$

whenever $K$ is a compact subset of $U$, $\|X\|(B) = 0$, and

$$
\Theta^m(\|X\|, x) \geq 1 \text{ for } \|X\| \text{ almost all } x; \quad \text{hence, } \|\delta X\| \text{ is a Radon measure over } U \text{ and } X \text{ is rectifiable.}
$$

Using the inclusion map $i : U \sim B \rightarrow U$, we may equivalently require that, for some $m$ dimensional rectifiable varifold $W$ in $U \sim B$ such that

$$
\Theta^m(\|W\|, x) \geq 1 \text{ for } \|W\| \text{ almost all } x,
$$

$$
i_\#(\|W\| + \|\delta W\|) \text{ is a Radon measure over } U,
$$

$$
\|\delta W\| \text{ is absolutely continuous with respect to } \|W\| \text{ if } m > 1,
$$

$$
i_\#(\|W\| \land |h(W, \cdot)|^m) \text{ is a Radon measure over } U \text{ if } m > 1,
$$

where $h(W, \cdot)$ denotes the mean curvature vector of $W$, there holds $i_\# W = X$.

Theorem D (see 9.2). Suppose $X$ satisfies the Regularity hypotheses 1.

Then, every decomposition of $X$ is locally finite.

For $B = \emptyset$, this was obtained in [Men16a, 6.11] where also the sharpness of the summability condition on the mean curvature vector was noted. Thus, the difficulty stems from the boundary $B$. For this purpose, we additionally make use of the concept of reach and isoperimetric considerations near the boundary, see [Fed59, Section 4] and [All75, Section 3], respectively.

Regularity hypotheses 2 (see 9.14 and 9.16 with $p = m$). Suppose $m$ and $n$ are integers, $2 \leq m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $M$ is a relatively closed subset of $U$ and an $n$ dimensional submanifold-with-boundary of class 2 of $\mathbb{R}^n$, $B = \partial M$, $X \in V^m(U)$, $\operatorname{spt} \|X\| \subset M$, $\Theta$ consists of all $\theta : U \rightarrow \mathbb{R}^n$ of class 1 with compact support and $\theta(b) \in \text{Tan}(B, b)$ for $b \in B$, $\beta = m/(m - 1),$

$$
sup \left\{ \int S_2 \cdot D \theta(x) \, dX(x, S) : \theta \in \Theta, spt \theta \subset K, \|X\|_1(\theta) \leq 1 \right\} < \infty
$$

whenever $K$ is a compact subset of $U$, and

$$
\Theta^m(\|X\|, x) \geq 1 \text{ for } \|X\| \text{ almost all } x; \quad \text{hence, } \|\delta X\| \text{ is a Radon measure over } U, \langle h(X, \cdot), \tau \rangle \in L^m_\#(\|X\|, \mathbb{R}^n)$, and $X$ is rectifiable, where $\tau : U \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is defined by

$$
\tau(x) = 1_{\mathbb{R}^n} \text{ for } x \in U \sim B, \quad \tau(x) = \text{Tan}(B, x)_2 \text{ for } x \in B.$$

8
This set of hypotheses for instance occurs in the study of constrained free boundary problems (see 9.17), of minimal sets with sliding boundary conditions (see 9.18), of the Willmore functional with free boundary (see 9.19), and of Brakke, or level set mean curvature, flow with free boundary (see 9.20).

**Theorem E** (see 9.24). Suppose $X$ satisfies the Regularity hypotheses. Then, every decomposition of $X$ is locally finite.

Similar to Theorem E, the preceding theorem relies on the concept of reach and isoperimetric considerations near the boundary. However, in the present setting, an isoperimetric lower density ratio bound analogous to [All75, 3.4(1)] was not available; therefore, we establish it in 9.22.

**Criteria for indecomposability with respect to a family of generalised weakly differentiable real valued functions**

The following two theorems concern varifolds associated with connected immersions and indecomposable $G$ chains, respectively. For this purpose, we recall that $\Lambda$ denotes the class of all locally Lipschitzian functions $f : U \to \mathbb{R}$.

**Theorem F** (see 10.7). Suppose $M$ is a connected $m$ dimensional manifold-with-boundary of class 2, $F : M \to U$ is a proper immersion of class 2, and $X$ is the $m$ dimensional varifold in $U$ associated with $F$.

Then, $X$ is indecomposable of type $\Lambda$.

**Theorem G** (see 10.9). Suppose $G$ is a complete normed commutative group, $S$ is an indecomposable $m$ dimensional integral $G$ chain in $U$, $X$ is the $m$ dimensional varifold in $U$ associated with $S$, and $\delta X$ is representable by integration.

Then, $X$ is indecomposable of type $\Lambda$.

Both theorems ultimately rely on the general coarea formulae obtained in [Men16a, 12.1] and [MS22, 3.33, 3.34], which entail that, whenever $f \in \Lambda$, we have

$$\|V \partial \{x : f(x) > y\}\| = (\mathcal{H}^{m-1} \cup \{x : f(x) = y\}) \cup \Theta^m(\|V\|, \cdot)$$

for $\mathcal{L}^1$ almost all $y$. For Theorem F, this is combined with a constancy theorem based on [Fed69, 4.5.11], see 10.5. For Theorem G, we instead employ the basic theory of slicing for integral $G$ chains as recorded in [MS22, 4.8, 5.13(8)].

Capitalising on the topological notion of connectedness of $\text{spt} \|V\|$ is more subtle. For this purpose, we employ the family

$$\Gamma = T(V) \cap \{f : \text{dimg} f = \text{spt} \|V\|, f \text{ continuous}\}$$

and assume all decompositions of the varifold in question to be locally finite. This additional hypothesis distinguishes the purely varifold-geometric setting of the next theorem and its corollaries from the more specific Theorems F and G.

**Theorem H** (see 10.16). Suppose that $X$ is an $m$ dimensional rectifiable varifold in $U$, that $\delta X$ is representable by integration, that every decomposition of $X$ is locally finite, and that $\text{spt} \|X\|$ is connected.

Then, $X$ is indecomposable of type $\Gamma$. 9
By [10.15] the local finiteness of the decompositions of $X$ ensures that
\[ \text{spt} \| X \| = \bigcup \{ \text{spt} \| W \| : W \in \Pi \} \quad \text{whenever } \Pi \text{ is a partition of } X; \]
otherwise, $\text{spt} \| X \|$ could be larger than this union. For $f \in \Gamma$, we then consider the partition $\Pi$ of $X$ along $f$ furnished by Theorem [C] and the subintervals $J(W) = \text{spt} f_{\mu} \| W \|$ of $\mathbb{R}$ corresponding to $W \in \Pi$, see Theorem [B]. Exploiting the equation for the supports and the continuity of $f$, we finally prove that the set
\[ B = \left\{ y : \| X \| (E(y)) > 0, \| X \| (U \sim E(y)) > 0, X \partial E(y) = 0 \right\}, \]
where $E(y) = \{ x : f(x) > y \}$, is in fact contained in the countable set of boundary points of the intervals $J(W)$ corresponding to $W \in \Pi$; hence, $\mathcal{L}^1(B) = 0$. Thus, the proof combines local finiteness of decompositions with partitions along $f$.

Combining 7.7 and Theorem H yields the following equivalence.

**Corollary 1** (see 10.19). Suppose that $V$ satisfies the general hypotheses, $V$ is rectifiable, and every decomposition of $V$ is locally finite.

Then, connectedness of $\text{spt} \| V \|$ is equivalent to indecomposability of $V$ of type $\Gamma$ and to indecomposability of $V$ of type $\mathcal{E}(U, R)$.

Combining Theorems [D] and [H] implies the following corollary.

**Corollary 2** (see 10.20). Suppose $X$ satisfies the Regularity hypotheses, the set $\text{spt} \| X \|$ is connected, and $\Gamma = T(X) \cap \{ f : \text{dmn} f = \text{spt} \| X \|, f \text{ continuous} \}$.

Then, $X$ is indecomposable of type $\Gamma$.

The hypotheses of the preceding corollary for instance apply to the study of the Willmore energy, see [10.21]. In the final paper of our series [MS23], we will demonstrate the same for solutions of Plateau’s problem stemming from Reifenberg’s formulation, Brakke flow, and min-max methods. For solutions based on integral currents or—more generally—integral $G$ chains, we shall employ Theorem [G] instead.

Finally, combining Theorems [E] and [H] entails the following corollary.

**Corollary 3** (see 10.22). Suppose $X$ satisfies the Regularity hypotheses, the set $\text{spt} \| X \|$ is connected, and $\Gamma = T(X) \cap \{ f : \text{dmn} f = \text{spt} \| X \|, f \text{ continuous} \}$.

Then, $X$ is indecomposable of type $\Gamma$.

### Possible lines of further study

A natural goal would be the development of a theory of real valued functions possessing generalised bounded variation with respect to $V$ analogous to that of generalised $V$ weakly differentiable functions laid out in [Men16a, Sections 8–13], [Men16b, Section 4], and [MS18, Sections 4 and 5]; ideally, such a theory would include a definition of generalised $V$ bounded variation for functions with values in a finite-dimensional normed space. In this process, the concept of partitions of $V$ along $f$ and indecomposability of type $\{ f \}$ could be defined—in the obvious way—and investigated for real valued $\| V \| + \| \delta V \|$ measurable functions $f$ having generalised bounded variation with respect to $V$. 

10
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2 Notation

Basic sources As in the first paper of our series [MS22], our notation follows [Men16a] and is thus largely consistent with H. Federer’s terminology for geometric measure theory listed in [Fed69, pp. 669–676] and W. Allard’s notation for varifolds introduced in [All72]. This includes, for certain distributions $S$ and sets $A$, the definition of the variation measure $\|S\|$ and of the restriction $S \upharpoonright A$ of $S$ to $A$, see [Men16a, 2.18, 2.20], respectively; and, for certain varifolds $V$ and sets $E$, the notions of distributional boundary, $V \partial E$, of $E$ with respect to $V$, indecomposability, component, and decomposition, see [Men16a] 5.1, 6.2, 6.6, 6.9, respectively.

Review Here, we list symbols not already reviewed in [Men16a Introduction, Section 1]: The determinant, det $f$, of an endomorphism $f$, see [Fed69] 1.3.4; the Grassmann manifold, $\mathbf{G}(n, m)$, of all $m$ dimensional subspaces of $\mathbb{R}^n$, see [Fed69] 1.6.2; the norms associated with inner products, $|\cdot|$, see [Fed69] 1.7.1; the adjoint, $f^*$, of a linear map $f$ between inner product spaces, see [Fed69] 1.7.4; the seminorm, $\|\cdot\|$, on $\text{Hom}(V, W)$ associated with normed spaces $V$ and $W$, see [Fed69] 1.10.5; the infimum and supremum, inf $S$ and sup $S$, of a subset $S$ of the extended real numbers and the sign function, denoted sign, see [Fed69] 2.1.1; the class $2^X$ of all subsets of $X$, see [Fed69] 2.1.2; the restriction, $\phi \upharpoonright A$, of a measure $\phi$ to a set $A$, see [Fed69] 2.1.2; the support, abbreviated spt $\phi$, of a measure $\phi$, see [Fed69] 2.2.1; the Lebesgue spaces, $L_p(\phi, Y)$ and $L_p(\phi)$, and the dual, $Y^*$, of a Banach space $Y$, see [Fed69] 2.4.12; the Dirac measure, $\delta_a$, at $a$, see [Fed69] 2.5.19; the $n$ dimensional Lebesgue measure, $\mathcal{L}^n$, see [Fed69] 2.6.5; the $\mathcal{L}^n$ measure of the unit ball, $\alpha(n)$, see [Fed69] 2.7.16;
the diameter of $S$, $\text{diam} \, S$, see [Fed69, 2.8.8]; the limit, $(V) \lim_{x \to z} f(S)$, of $f$ at $x$ with respect to a Vitali relation $V$, see [Fed69, 2.8.16]; the absolutely continuous part $\psi_\phi$ of a measure $\psi$ with respect to $\phi$, see [Fed69, 2.9.1]; the derivative, $g'$, of a function on the real line, see [Fed69, 2.9.19]; the $m$ dimensional Hausdorff measure, $\mathcal{H}^m$, see [Fed69, 2.10.2]; the integralgeometric measure with exponent 1, denoted $\mathcal{I}_1^m$, see [Fed69, 2.10.5]; the multiplicity function, $N(f, \cdot)$, see [Fed69, 2.10.9]; the $m$ dimensional densities, $\Theta^m(\phi, a)$ and $\Theta^m(\phi, a)$, see [Fed69, 2.10.19]; the differential, $Df$, and the gradient, $\text{grad} \, f$, see [Fed69, 3.1.1]; the closed cones of tangent and normal vectors, $\text{Tan}(S, a)$ and $\text{Nor}(S, a)$, see [Fed69, 3.1.21]; the normal bundle, $\text{Nor}(\mathcal{B})$, of a submanifold $\mathcal{B}$ of class 1 of $\mathbb{R}^n$, see [Fed69, 3.1.21]; the unit sphere, $S^{n-1}$, in $\mathbb{R}^n$, see [Fed69, 3.2.13]; the space, $\mathcal{E}(U, Y)$, of functions of class $\infty$, the support, $\text{spt} \, \gamma$, of a member $\gamma$ of $\mathcal{E}(U, Y)$, see [Fed69, 4.1.1]; the boundary, denoted $\partial T$, for currents $T$ and the Euclidean currents, $\mathcal{E}_e$, see [Fed69, 4.1.7]; the one-dimensional current $[u, v]$ in $\mathbb{R}^n$ associated with the line segment from $u$ to $v$, see [Fed69, 4.1.8]; the group, $L_m(\mathbb{R}^n)$, of $m$ dimensional integral currents in $\mathbb{R}^n$, see [Fed69, 4.1.24]; the exterior normal, $\text{n}(A, \cdot)$, of $A$, see [Fed69, 4.5.5]; the weight, $\|V\|$, of a varifold $V$, see [All72, 3.1]; and, the first variation, $\partial V$, of a varifold $V$, see [All72, 4.2].

Modifications For the push forward, $f_# \phi$, of a measure $\phi$ by a function $f$, we use the definition [All72, 3.9] which extends [Fed69, 2.1.2]. Whenever $M$ is an $m$ dimensional submanifold of class 1 of $\mathbb{R}^n$,

$$G_m(M) = (M \times G(n, m)) \cap \{(a, S) : S \subset \text{Tan}(M, a)\};$$

this extends [All72, 2.5]. Whenever $M$ is a submanifold of class 2 of $\mathbb{R}^n$, the second fundamental form of $M$ at $a$, for $a \in M$, and the mean curvature vector of $M$ at $(a, S)$, for $(a, S) \in G_m(M)$, are defined to be the unique (symmetric) bilinear form $b(M, a) : \text{Tan}(M, a) \times \text{Tan}(M, a) \to \text{Nor}(M, a)$ and the unique vector $h(M, a, S) \in \text{Nor}(M, a)$ such that

$$v \cdot (w, D \, g(a)) = -b(M, a)(v, w) \cdot g(a) \quad \text{for} \quad v, w \in \text{Tan}(M, a),$$

$$S_k \cdot (D \, g(a) \circ \text{Tan}(M, a)_k) = -h(M, a, S) \cdot g(a)$$

whenever $g : M \to \mathbb{R}^n$ is of class 1 and satisfies $g(x) \in \text{Nor}(M, x)$ for $x \in M$; these notions extend [All72, 2.5]. Whenever $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, and $E$ is a $\mathcal{H}^m$ measurable set which meets every compact subset of $U$ in a $(\mathcal{H}^m, m)$ rectifiable set, we define the $m$ dimensional varifold associated with $E$, denoted $v_m(E)$, as the unique member of $\mathcal{RV}_m(U)$ satisfying $\|v_m(E)\| = \mathcal{H}^m \cdot E$; this modifies the notation of [All72, 3.5]. The concepts relating to generalised weak differentiability on varifolds—that is, generalised $V$ weak derivatives, $Df$, of generalised $V$ weakly differentiable functions $f$ and the resulting function spaces, $\mathcal{T}(V, Y)$ and $\mathcal{T}(V)$,—are employed in accordance with the generalisation [Men16a, 4.2] of [Men16a, 8.3].

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1 As was stipulated in [Men16a] for similar notions, $\psi_\phi$ will also be employed in the case where one of the measures $\psi$ or $\phi$ on $X$ fails to be finite on bounded sets but there exist open sets $U_1, U_2, U_3, \ldots$ covering $X$ at which $\phi$ and $\psi$ are finite.

2 Our topology on $\mathcal{D}(U, Y)$ differs from that employed in [Fed69, 4.1.1] but yields the same dual $\mathcal{D}'(U, Y)$ of continuous linear functionals $S : \mathcal{D}(U, Y) \to \mathbb{R}$, see [Men16a, 2.13, 2.17 (1)].
Amendments Whenever $n \geq 2$, the trace of an $\mathbb{R}^n$ valued bilinear form $B$ on $\mathbb{R}^n$ is defined so that $\alpha(\text{trace } B) = \text{trace}(\alpha \circ B)$ whenever $\alpha \in \text{Hom}(\mathbb{R}^n, \mathbb{R})$. Modelled on [Fed69, 4.1.7], we employ the restriction notation, $\phi \lvert f$, for the measure $\phi$ weighted by $f$ introduced in [MS22, 3.6]; this weighted measure was discussed—without name—in [Fed69, 2.4.10]. For subsets $A$ of Euclidean space, we adopt the concepts of unique nearest point and reach, and the symbols $\text{Unp}(A)$, $\text{reach}(A, \cdot)$, and $\text{reach}(A)$ from [Fed59, 4.1] as well as those of pointwise and approximate differentiability of order $k$ with the corresponding pointwise and approximate differentials of order $k$, denoted by $\text{pt } D^k A$ and $\text{ap } D^k A$, respectively, from [Men19] and [San19]. The terms immersion and embedding are employed in accordance with [Hir94, p. 21]. Whenever $k$ is a positive integer or $k = \infty$, we mean by a [sub]manifold-with-boundary of class $k$ a Hausdorff topological space with a countable base of its topology that is, in the terminology of [Hir94] pp. 29–30, a $C^k$ [sub]manifold. For manifolds-with-boundary $M$ of class $k$, we similarly adapt the notion of chart of class $k$ and Riemannian metric of class $k - 1$ from [Hir94] p. 29, p. 95 and denote by $\partial M$ its boundary as in [Hir94] p. 30]. The image, $f \circ V$, of a varifold $V$ under suitable locally Lipschitzian maps $f$ is used in accordance with [MS22, 3.28]. Finally, whenever $G$ is a complete normed commutative group as defined in [MS22, 3.1], we employ the following notation regarding the group $\mathcal{H}^m_c(U, G)$ of $m$ dimensional locally rectifiable $G$ chains $S$ in an open subset $U$ of $\mathbb{R}^n$, see [MS22, 4.5]; the notions of weight measure, $\|S\|$, and restriction, $S \lvert A$, see [MS22, 4.5]; the slice, $\langle S, f, \cdot \rangle$, of $S$ by $f$, see [MS22, 4.8]; the isomorphism $i_{U, m} : \mathcal{H}^m_{loc}(U) \rightarrow \mathcal{H}^m_{loc}(U, Z)$, see [MS22, 5.1]; the bilinear operation $\cdot : \mathcal{H}^m_{loc}(U, Z) \times G \rightarrow \mathcal{H}^m_{loc}(U, G)$, see [MS22, 5.5]; and, the chain complex of integral $G$ chains in $U$ with $m$-th chain group $L_m(U, G)$—identified with a subgroup of $\mathcal{H}^m_{loc}(U, G)$—and boundary operator $\partial G$, see [MS22, 5.11, 5.13, 5.17].

Definitions in the text The $Y$ topology on the dual space $Y^*$ of a Banach space $Y$ is specified in [3.1]. The flow associated with a vector field occurs in [3.18]. For varifolds, the function $\eta(V, \cdot)$ representing the first variation $\delta V$ with respect to $\|\delta V\|$ is defined in [3.20] the concepts of partition, local finiteness of families thereof, indecomposability of type $\Psi$, and partition along $f$ are introduced in [5.5] [7.2] and [8.6] respectively. For integral $G$ chains, the definition of indecomposability is recorded in [6.6]. Based on [Men19] and [San19], the notions of pointwise and approximate mean curvature vector $\text{pt } h(A, \cdot)$ and $\text{ap } h(A, \cdot)$ of subsets $A$ of Euclidean space are introduced in [6.9]. For immersions $F$, we lay down the concepts of tangent cone $\text{Tan}(F, \cdot)$, normal cone $\text{Nor}(F, \cdot)$, mean curvature vector $h(F, \cdot)$, and exterior normal $n(F, \cdot)$ in [6.10] associated varifold in [6.13] and Riemannian distance in [10.1].

3 Preliminaries

We compile properties of distributions representable by integration in [3.1] 3.6 approximate a retraction in [4.7] study the reach of the graph of functions and submanifolds of class 2 in [5.8] 3.13 and [5.14] 3.17 respectively, consider the flow associated to certain vector fields in [5.18] and discuss the intersection of sets of locally finite perimeter in [5.19]. Proceeding to varifolds, we treat notions related to the first variation in [5.20] 5.21 and observe some new basic properties of the
distributional boundary of superlevel sets of generalised weakly differentiable functions in [3.22 3.24] 3.1 Definition (see [DS58 p. 420]). Suppose $Y$ is a Banach space. Then, the topology on $Y^*$ inherited from $R^Y$ is termed the $Y$ topology.

3.2 Theorem. Suppose $U$ is an open subset of $R^n$, $Y$ is a separable Banach space, $\phi$ is a Radon measure over $U$, $T \in D'(U,Y)$ is representable by integration, $g$ is a $Y^*$ valued function that is $\|T\|$ measurable with respect to the $Y$ topology,\\n\\n\[\|g(x)\| = 1 \quad \text{for } \|T\| \text{ almost all } x,\\nT(\theta) = \int (\theta, g) \, d\|T\| \quad \text{for } \theta \in D(U,Y),\]
\\nthe function $h$ mapping a subset of $U$ into $R^Y$ is defined by\\n\\n\[\langle y, h(a) \rangle = (V) \lim_{S \to a} T(b_S \cdot y)/\phi(S) = T(b_S \cdot y)/\phi(S) \in R \quad \text{for } y \in Y, \quad \text{ whenever } a \in U,\]
\\nwhere $V = \{(a, B(a, r)) : a \in U, 0 < r < \infty, B(a, r) \subset U\}$
\\nand $b_S$ is the characteristic function of $S$ on $U$.
\\nThen, $\lim h \subset Y^*$, $h$ is a Borel function with respect to the $Y$ topology on $Y^*$, $\text{dnn} h$ is a Borel set, and, for $\phi$ almost all $a$, there holds, for $y \in Y$,
\\n\[\langle y, h(a) \rangle = D([|T|], \phi, V, a)\langle y, g(a) \rangle \quad \text{if } a \in \text{dnn} g,\]
\\n\[\langle y, h(a) \rangle = 0 \quad \text{if } a \notin \text{dnn} g,\]
\\nin particular, $\|h(a)\| = D([|T|], \phi, V, a)$ for $\phi$ almost all $a$, $\|T\|_\phi = \phi \|h\|$, and
\\n\[T(\theta) = \int (\theta, h) \, d\phi + \int (\theta, g) \, d(||T|| - \|T\|_\phi) \quad \text{for } \theta \in L_1([|T|], Y).\]
\\nProof. The first three conclusions may be verified by employing the uniform boundedness principle (see [DS58 II.1.11]) and a countable dense subset of $Y$. In view of [Fed69 2.8.18], $V$ is a Vitali relation with respect to $\phi$ and $\|T\|$. If either $D([|T|], \phi, V, a) = 0$ or $0 < D([|T|], \phi, V, a) < \infty$ and, whenever $y \in Y$, the point $a$ belongs to the $(|T|, V)$ Lebesgue set of the function mapping $x \in \text{dnn} g$ onto $(y, g(x))$, then the fourth conclusion holds at $a$. Employing a countable dense subset of $Y$ and [Fed69 2.9.5, 2.9.7, 2.9.9], we see that $\phi$ almost all $a$ satisfy these conditions. Since $\|T\|_\phi = \phi \cdot D([|T|], \phi, V, \cdot)$ by [Fed69 2.9.7] and [MS22 3.7] and $T(\theta) = \int (\theta, g) \, d\|T||$ for $\theta \in L_1([|T|], Y)$, the postscript follows from [Fed69 2.4.10, 2.9.2, 2.9.6].

3.3 Example. Suppose $U$ is an open subset of $R^n$, $Y$ is a separable Banach space, $\phi$ is a Radon measure over $U$, $k$ is a $Y^*$ valued function that is $\phi$ measurable with respect to the $Y$ topology, $\|k\| \in L^{1\ast}(\phi)$, and $T \in D'(U,Y)$ satisfies
\\n\[T(\theta) = \int (\theta, k) \, d\phi \quad \text{for } \theta \in D(U,Y).\]
\\nThen, $\|T\|$ is a Radon measure absolutely continuous with respect to $\phi$ and
\\n\[\langle y, k(a) \rangle = (V) \lim_{S \to a} T(b_S \cdot y)/\phi(S) \quad \text{for } y \in Y, \quad \text{ for } \phi \text{ almost all } a,\]

by [Fed69 2.8.18, 2.9.9], where the notation of [3.2] is employed; hence, [Fed69 2.9.2, 4.1.5] and [3.2] yield $\|T\| = \phi \|k\|$ and
\\n\[T(\theta) = \int (\theta, k) \, d\phi \quad \text{for } \theta \in L_1([|T|], Y).\]
\\nIn particular, whenever $S \in D'(U,Y)$ is representable by integration and $A$ is $\|S\|$ measurable, we have $\|S \cdot A\| = \|S\| \cdot A$; in fact, we may take $T = S \cdot A$. 14
3.4 Theorem. Suppose \(1 \leq p < \infty\), \(q = \infty\) if \(p = 1\) and \(q = p/(p-1)\) if \(p > 1\). Let \(U\) be an open subset of \(\mathbb{R}^n\), \(\phi\) is a Radon measure over \(U\), \(Y\) is a separable Banach space, \(T \in \mathcal{D}'(U,Y)\), and \(M = \sup\{ \| T(\theta) \| : \theta \in \mathcal{D}(U,Y), \phi_{(p)}(\theta) \leq 1 \} < \infty\). Then, there exists a \(\phi\) almost unique function \(k : U \to Y^*\) that is \(\phi\) measurable with respect to the \(Y^*\) topology such that \(\| k \| \in L^{1}_{\text{loc}}(\phi)\) and
\[
T(\theta) = \int \phi(\theta,k) \, d\phi \quad \text{for } \theta \in \mathcal{D}(U,Y).
\]
Moreover, we have \(M = \phi_{(q)}(\| k \|)\).

Proof. In view of [Men16b, 2.1], we may define the linear map
\[
S : L_p(\phi,Y) \cap \{ \theta : \text{dmm } \theta = U \} \to R
\]
to be the unique \(\phi_{(p)}\) continuous extension of \(T\); hence,
\[
M = \sup\{ S(\theta) : \theta \in L_p(\phi,Y), \text{dmm } \theta = U, \phi_{(p)}(\theta) \leq 1 \}.
\]
Using [ITIT69, Chapter 7, Sections 4 and 5], we obtain a function \(k : U \to Y^*\) that is \(\phi\) measurable with respect to the \(Y^*\) topology such that
\[
S(\theta) = \int \phi(\theta,k) \, d\phi \quad \text{whenever } \theta \in L_p(\phi,Y) \text{ and } \text{dmm } \theta = U,
\]
and \(M = \phi_{(q)}(\| k \|)\). Finally, the uniqueness statement follows from 3.3.

3.5 Remark. We recall that, for general \(Y\), the functions \(g, h, \phi, k\) occurring in 3.2 and 3.4 may be \(\phi\) nonmeasurable with respect to the norm topology on \(Y^*\), see [Men21, 2.32]. In the present paper, we only consider the case \(Y = \mathbb{R}^n\).

3.6 Example. Suppose \(1 \leq p < \infty\), \(q = \infty\) if \(p = 1\) and \(q = p/(p-1)\) if \(p > 1\), \(k\) is a positive integer or \(k = \infty\), \(U\) is an open subset of \(\mathbb{R}^n\), \(\phi\) is a Radon measure over \(U\), \(B\) is a relatively closed subset and a submanifold of class \(k\) of \(U\), the vector space \(\Theta\) consists of all vector fields \(\theta : U \to \mathbb{R}^n\) of class \(k-1\) with compact support satisfying \(\theta(b) \in \text{Tan}(B,b)\) for \(b \in B\), and
\[
X_r = L_r(\phi, \mathbb{R}^n) \cap \{ \theta : \text{dmm } \theta = U, \theta(b) \in \text{Tan}(B,b) \text{ for } b \in B \}
\]
for \(1 \leq r \leq \infty\); hence, \(\Theta = \phi_{(p)}\) dense in \(X_p\) as may be verified combining [Fed69, 2.17, 2.4.18 (1), 3.1.19 (1)] and [Men16b, 2.1]. Next, suppose \(R \in \text{Hom}(\Theta, \mathbb{R})\) and
\[
M = \sup \{ R(\theta) : \theta \in \Theta, \phi_{(p)}(\theta) \leq 1 \} < \infty.
\]
Let \(S \in \text{Hom}(X_p, \mathbb{R})\) be the unique \(\phi_{(p)}\) continuous extension of \(R\). Defining the endomorphism \(E\) of \(L_p(\phi, \mathbb{R}^n) \cap \{ \theta : \text{dmm } \theta = U \}\) with \(E \circ E = E\) by
\[
E(\theta)(x) = \theta(x) \text{ if } x \in U \sim B, \quad E(\theta)(x) = (\theta(x), \text{Tan}(B,x)_\perp) \text{ if } x \in B
\]
for \(\theta \in L_p(\phi, \mathbb{R}^n)\) with \(\text{dmm } \theta = U\) and noting \(\phi_{(p)}(E(\theta)) \leq \phi_{(p)}(\theta)\) for such \(\theta\), we apply 3.4 with \(T = S \circ E|\mathcal{D}(U, \mathbb{R}^n)\) to furnish \(k \in X_q\) satisfying
\[
T(\theta) = \int k \cdot \theta(\cdot) \, d\phi \quad \text{for } \theta \in L_p(\phi, \mathbb{R}^n)
\]

More elementary, one may pass from the case \(Y = \mathbb{R}\) treated in [Fed69, 2.5.7 (i) (ii)] to the general case adapting the method of [Fed69, 2.5.12]; this is carried out in [Men14, 4.6.3 (1) (2)].
and $\phi_{(q)}(k) = M$. Finally, we deduce from \[ \text{3.3} \] that, for $\phi$ almost all $x$, there holds
\[
k(x) \cdot \theta(x) = \lim_{r \to 0+} \frac{S(b_{x,r} \cdot \theta)}{\phi B(x,r)} \quad \text{whenever } \theta \in \Theta,
\]
where $b_{x,r}$ is the characteristic function of $B(x,r)$ on $U$ since $\theta$ is continuous and $\lim_{r \to 0+} \int_{B(x,r)} |k| d\phi / \phi B(x,r) < \infty$ for $\phi$ almost all $x$ by \[ \text{Fed69} \] 2.8.18, 2.9.5.

\[ \text{3.7 Lemma.} \] Suppose $n$ is a positive integer and $\epsilon > 0$.
Then, there exists a function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ of class $\infty$ such that $\text{Lip } \rho = 1$, $\im \rho \subset B(0,1 + \epsilon)$, $\rho(b) = b$ for $b \in \mathbb{R}^n \cap B(0,1)$, and $\rho(b) \in \{tb : 0 \leq t \leq 1\}$ whenever $b \in \mathbb{R}^n$.

\[ \text{Proof.} \] Choosing a concave nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ of class $\infty$ such that $f(t) = t$ for $-\infty < t \leq 1$ and $f(t) = 1 + \epsilon$ for $t \in \mathbb{R}$, we define $\rho : \mathbb{R}^n \to \mathbb{R}^n$ of class $\infty$ by requiring $\rho(y) = (f(|y|)/|y|)y$ for $y \in \mathbb{R}^n \sim \{0\}$. Since $D \rho(0) = 1_{\mathbb{R}^n}$ and
\[
\|D \rho(y)\| = \sup\{|f'(|y|), f(|y|)/|y|\} \leq 1 \quad \text{for } y \in \mathbb{R}^n \sim \{0\},
\]
we conclude $\text{Lip } \rho \leq 1$ from \[ \text{Fed69} \] 2.2.7, 3.1.1.

\[ \text{3.8.} \] Suppose $m$ and $n$ are positive integers and $m < n$. Following \[ \text{Fed69} \] 5.1.9, we define orthogonal projections $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q : \mathbb{R}^n \to \mathbb{R}^{n-m}$ by
\[
p(z) = (z_1, \ldots, z_m), \quad q(z) = (z_{m+1}, \ldots, z_n)
\]
whenever $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$.

\[ \text{3.9 Theorem.} \] In the situation of \[ \text{3.8} \] suppose $n - m = 1$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is of class $2$, $F = p^* f + q^* \circ f$, $C = \im F$, $0 < r \leq \infty$, and
\[
\|b(C,c)\| \leq r^{-1} \quad \text{for } c \in C.
\]
Then, whenever $\text{dist}(z,C) < r$, there exists a unique point $c$ in $C$ satisfying $|z - c| = \text{dist}(z,C)$; moreover, we have $\text{sign } q(z - c) = \text{sign } (q(z) - f(p(z)))$.

\[ \text{Proof.} \] Abbreviating $A = \{s : -r < s < r\}$, we define functions $\nu : C \to \mathbb{S}^{n-1}$, $\phi_s : C \to \mathbb{R}^n$, and $\psi : A \times C \to \mathbb{R}^n$ of class $1$ by
\[
\nu(c) = (1 + |D f(p(c))|)^{-1/2}(q^*(1 - p^*(\text{grad } f(p(c))))), \quad \phi_s(c) = \psi(s,c) = c + s \nu(c)
\]
whenever $(s,c) \in A \times C$; hence, we have
\[
|\phi_s(c) - c| \leq |s|, \quad \phi_s \text{ is proper}, \quad D \phi_s(c) = 1_{\text{Tan}(C,c)} + s D \nu(c), \quad D \phi_s(c) \in \text{Hom}(\text{Tan}(C,c), \text{Tan}(C,c)), \quad \|D \phi_s(c) - 1_{\text{Tan}(C,c)}\| \leq |s|\|b(C,c)\| \leq |s|r^{-1} < 1, \quad \det D \phi_s(c) > 0.
\]
Defining functions $h = p \circ \psi \circ (1_A \times F)$ and $g_s = p \circ \phi_s \circ F$ of class $1$, we see that, whenever $(s,c) \in A \times \mathbb{R}^{n-1}$, we have
\[
p(F(x)) = x, \quad g_s(x) = x + s p(\nu(F(x))), \quad |g_s(x) - x| \leq |s|, \quad p \circ D F(x) = 1_{\mathbb{R}^{n-1}}, \quad D g_s(x) = p \circ D \phi_s(F(x)) \circ D F(x), \quad \det D g_s(x) > 0.
\]
Thus, \( h(I \times \mathbb{R}^{n-1}) \) is proper whenever \( I \) is a compact subinterval of \( A \) and, for \( s \in A \), \( N(g_{s}, \cdot) \) is a constant \( Z \) valued function because \( g_{s} \) is a proper map which locally is a diffeomorphism of class 1 by [Fed69 3.1.1]. In view of [MS22 5.1], we conclude

\[
g_{s} \# E^{n-1} = g_{0} \# E^{n-1} \quad \text{for} \quad s \in A
\]

from [MS22 5.16] applied with \( m, n, U, G, \nu, V, t, \) and \( S \) replaced by \( n-1, n-1, \mathbb{R}^{n-1}, Z, n-1, \mathbb{R}^{n-1}, s, \) and \( e_{R^{n-1},n-1}(E^{n-1}) \). Noting \( g_{0} = 1_{R^{n-1}} \) and computing \( g_{s} \# E^{n-1} \) by means of [MS22 4.6, 5.1], we obtain that

\[
N(g_{s}, \chi) = 1 \quad \text{for} \quad (s, \chi) \in A \times \mathbb{R}^{n-1}.
\]

Therefore, \( \phi_{s} \) is univalent for \( s \in A \). Moreover, we have

\[
\text{sign} q(\phi_{s}(c) - c) = \text{sign} s \quad \text{for} \quad (s, c) \in A \times C.
\]

Next, suppose \( 0 < d = \text{dist}(z, C) < r \) and \( \epsilon = \text{sign}(q(z) - f(p(z))) \). Observing

\[
q(z) \notin q[C \cap p^{-1}[B(p(z), d)]
\]

because \( p[Tan(C, c)] = R^{n-1} \) for \( c \in C \), we conclude

\[
\text{sign}(q(z - F(x))) = \epsilon \quad \text{for} \quad x \in B(p(z), d).
\]

Finally, for \( c \in C \cap B(z, d) \), we shall verify \( z = \phi_{c}(c) \) and \( q(z - c) = \epsilon \); in fact, noting \( z - c \in \text{Nor}(C, c) \), there exists \( s \in A \) such that \( |s| = d \) and \( \phi_{s}(c) = z \), whence we infer \( \text{sign} s = \text{sign} q(z - c) = \epsilon \) as \( C \cap B(z, d) \subset F[B(p(z), d)] \). \( \square \)

3.10 Remark. The principal conclusion is equivalent to \( \text{reach}(C) \geq r \); for this lower bound to hold, the hypothesis \( n - m = 1 \) is essential, see [3.11] below.

3.11 Example. Suppose \( m \) and \( n \) are positive integers satisfying \( n - m \geq 2 \), \( \pi = \mathcal{A}^{\ast}(S^{1})/2, 0 < \epsilon \leq 1, f : \mathbb{R}^{m} \to \mathbb{R}^{n-m} \) is defined by

\[
f(x) = (\cos(\pi x_{1}/\epsilon), \sin(\pi x_{1}/\epsilon), 0, \ldots, 0) \in \mathbb{R}^{n-m} \quad \text{for} \quad x = (x_{1}, \ldots, x_{m}) \in \mathbb{R}^{m},
\]

and \( C \) is associated with \( f \) as in [3.9] Then, there holds

\[
|b(c, C)| \leq 1 \quad \text{for} \quad c \in C \quad \text{but} \quad \text{reach}(C) \leq \epsilon;
\]

in fact, reducing to \( (m, n) = (1, 3) \) and defining \( F \) as in [3.9] we compute

\[
|F'(x)|^{2} = 1 + \epsilon^{-2} \pi^{2}, \quad |F''(x)| = \epsilon^{-2} \pi^{2}, \quad \|b(C, F(x))\| = \frac{\epsilon^{-2} \pi^{2}}{1 + \epsilon^{-2} \pi^{2}} \leq 1
\]

for \( x \in \mathbb{R} \), take \( z = (0, -1, 0) \), \( d = \text{dist}(z, C) \), and note

\[
|z - F(x)| = |z - F(-x)| \quad \text{for} \quad x \in \mathbb{R},
\]

so that \( |z - F(0)| = 2 > \epsilon = |z - F(\epsilon)| \geq d \) implies \( \text{reach}(C) \leq \epsilon \).

3.12 Corollary. Suppose additionally \( U = \{ z : \text{dist}(z, C) < r \} \) and the functions \( \xi : U \to C \) and \( \delta : U \to \mathbb{R} \) satisfy

\[
\{\xi(z)\} = C \cap \{ c : |z - c| = \text{dist}(z, C) \}, \quad \delta(z) = \text{dist}(z, C) \text{ sign } q(z - \xi(z))
\]

for \( z \in U, f \) is of class \( k \geq 2 \), and \( E = \{ z : f(p(z)) > q(z) \} \). Then, \( \xi \) is of class \( k - 1 \), \( \delta \) is of class \( k \), and

\[
\text{grad} \delta(z) = n(E, \xi(z)) \quad \text{for} \quad z \in U.
\]
From [Fed59, 4.8(12)] we deduce that whenever $x$, we deduce that $\xi(s,c) = s$, $\delta(s,c) = s$.

From [Fed59, 4.8(3)(5)] and the fact that $\frac{\psi(x)-c}{x}$, in particular, $(s,c) \in A \times C$ and $s \neq 0$; hence, $\text{grad } \delta = \nu \circ \xi$ and $\delta$ is of class $k$.

**3.13 Remark.** In the context of bounded domains, a similar deduction of properties of the distance function is carried out in [GT01, Lemma 14.6].

**3.14 Theorem.** Suppose $B$ is a submanifold of class 2 of $\mathbb{R}^n$, $A = \text{Clos } B$,

$$
\xi = (\mathbb{R}^n \times A) \cap \{(x,a) : \|a\| = A \cap B(x, |x-a|)\},
X = \xi^{-1}[B] \cap \{x : \text{reach}(A, \xi(x)) > \text{dist}(x, A)\},
$$

and $\psi : X \to \mathbb{R}^n \times \mathbb{R}^n$ satisfies $\psi(x) = (\xi(x), x - \xi(x))$ for $x \in X$.

Then, $X$ is an open set containing $B$, $\psi$ is a univalent function of class 1 whose image is contained in $\text{Nor}(B)$, and $D \psi(x)$ is univalent for $x \in X$.

**Proof.** Assume $A \neq \varnothing$. For $b \in B$, there holds $\text{reach}(A, b) > 0$; in fact, [Fed59, 4.12] and [Fed69, 3.1.19] yield a neighbourhood $U$ of $b$ and a set $C$ satisfying $\text{reach}(C, b) > 0$ and $U \cap A = U \cap C$. It follows that $B \subset X$. From [Fed59, 4.8(4)], we recall that $\text{reach}(A, \cdot)$ and $\xi$ are continuous functions; in particular, observing that $X \subset \text{Int } \text{Unp}(A)$, we infer that $X$ is open, as $B$ is open relative to $A$. Moreover, combining the continuity with [Fed59, 4.8(13)], we conclude that $\psi$ is locally Lipschitzian with $\psi \subset \text{Nor}(B)$, that $\psi$ is univalent, and that $\psi^{-1}(b, v) = b + v$ for $(b, v) \in \psi$. Recalling [Fed69, 3.1.19] and the fact that $\text{Nor}(B)$ is an n-dimensional submanifold of class 1, we deduce firstly that $D \psi(x)$ is univalent for $x \in \text{dom } D \psi$ by [Fed69, 3.1.1], secondly, that $\psi$ is open relative to $\text{Nor}(B)$ by [Fed69, 4.1.26], and finally that $\psi$ is of class 1 by [Fed69, 3.1.1].

**3.15 Remark.** Instead of applying degree theory for Lipschitzian maps via [Fed69, 4.1.26], the general principle of invariance of domain could have been employed.

**3.16 Remark.** From [Fed59, 4.8(4)(12)], we deduce that $B = X \cap \{x : x = \xi(x)\}$ is closed relative to $X$ and that we have $\xi(b + v) = b$ whenever $(b, v) \in \text{Nor}(B)$ and $|v| < \text{reach}(A, b)$. Defining $Q : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ by

$$v \cdot \langle u, Q(x) \rangle = v \cdot (\text{Tan}(B, \xi(x))_2(u), D \text{Nor}(B, \cdot)\theta(\xi(x))(x - \xi(x)))
- \text{b}(B, \xi(x))((u, \text{Tan}(B, \xi(x))_2, (v, \text{Tan}(B, \xi(x))_2)) \cdot (x - \xi(x))
$$

whenever $x \in X$ and $u, v \in \mathbb{R}^n$ with the help of [KAM17, 3.1(2)], we thus obtain

$$\|\text{b}(B, \xi(x))\| \leq \text{reach}(A, \xi(x))^{-1}, \quad D \xi(x) = \text{Tan}(B, \xi(x))_2 + \sum_{i=1}^{\infty} (-1)^i Q(x)^i$$
3.17 Remark. If \( E \) is of class \( n_{\text{adj}} \) whenever \( x, r < \infty \). Abbreviating \( \delta = \text{dist}(\cdot, B) \), we have
\[
\delta(x) = \delta(x)^{-1}(x - \xi(x)) \quad \text{for} \quad x \in X \sim B \quad \text{by} \quad \|Q(x)\| \leq \frac{||Q(x)||}{1 - ||Q(x)||} \quad \text{for} \quad x \in X \sim B.
\]

3.19 Example. Suppose \( R^n \to R^n \) is of class 1, \( \theta(b) \in \text{Tan}(B, b) \) for \( b \in B, I \subset R \), \( f : I \to R^n \) is of class 1, and \( f' = \theta \circ f \), then \( f^{-1}[B] \) is open; in fact, whenever \( U \) is an open subset of \( R^n, \kappa = \text{Lip}(\theta(U)) < \infty \), and \( B \neq \emptyset \), we recall \( \text{Fed69} \) and \( \text{All72} \) abbreviate \( J = f^{-1}[U \cap X \cap \xi^{-1}[U]] \), let \( g = \delta \circ f | J \), define \( h_\pm : J \to R \cap \{y : y \geq 0\} \) by \( h_\pm(t) = g(t) \exp(\pm t) \) for \( t \in J \), and estimate \( |g'(t)| \leq \kappa g(t), \quad h'_+(t) \geq 0, \quad h'_-(t) \leq 0 \) for \( L^1 \) almost all \( t \in J \), because we have \( (\delta \circ f)'(t) = \delta(f(t)) + (f(t)) \circ \{\theta(f(t)) - \xi(f(t))\} \) for \( t \in f^{-1}[U \cap X \sim B] \), hence, using \( \text{Fed69} \), we infer \( \{t : g(t) = 0\} \) is open. Thus, if \( I \) is an interval, \( B \cap \text{spt} \theta \) is closed, and \( B \cap \text{im} f \neq \emptyset \), then \( \text{im} f \subset B \) as either \( B \cap \text{im} f \subset \text{spt} \theta \) or \( f \) is constant by \( \text{Rei71} \). Chapter 1, Theorem 5.2.3.18 Example. Suppose \( U \) is an open subset of \( R^n, \theta : U \to R^n \) is of class 1, and \( \text{spt} \theta \) is compact. Then, by \( \text{Rei71} \) Chapter 1, Theorems 3.4, 5.2, 5.7, 10.5, there exists \( \phi : R \times U \to U \) of class 1 with \( \phi(0, \cdot) = 1_U \) such that
\[
\phi(\cdot, x)'(t) = \theta(\phi(t, x)) \quad \text{for} \quad (t, x) \in R \times U
\]
and such that, if \( I \) is an interval, \( 0 \in I, f : I \to U \) is of class 1, and \( f' = \theta \circ f \), then \( f = \phi(\cdot, f(0)) \cap I \); hence, \( \phi(t, \cdot)^{-1} = \phi(-t, \cdot) \) and \( \text{spt} \phi(t, \cdot) = 1_U \) for \( t \in R \). The unique such \( \phi \) is termed the \textit{flow associated with} \( \theta \). By the method of \( \text{All72} \) it follows that, for \( V \in V_m(U) \), the derivative, at 0, of the function mapping \( t \in R \) onto \( \|\phi(t, \cdot)\|_{W^p(V)} \|\text{spt} \theta\| = \int S_t \cdot \nabla \theta(x) \, dV(x, S) \) equals \( \int S_t \cdot \nabla \theta(x) \, dV(x, S) \). Finally, if \( M \) is a relatively closed subset of \( U \) and an \( n \) dimensional submanifold-with-boundary of class 2, \( B = \partial M \), and \( \theta(b) \in \text{Tan}(B, b) \) for \( b \in B \), then \( 3.17 \) implies
\[
\phi(t, \cdot)|B] \subset B, \quad \phi(t, \cdot)[M] \subset M, \quad \text{for} \quad t \in R.
\]

3.19 Example. Suppose \( E \) and \( F \) are subsets of \( R^n \) with locally finite perimeter, and
\[
P = \{x : \Theta^n(\mathcal{L}^n \cup E, x) = 1\}, \quad Q = \{x : \Theta^n(\mathcal{L}^n \cup F, x) = 1\},
A = \{x : n(E, x) \in S^{n-1}\}, \quad B = \{x : n(F, x) \in S^{n-1}\}.
\]
Then, \( E \cap F \) is a set with locally finite perimeter by \( \text{Fed69} \) and, recalling \( \text{Fed69} \), one readily verifies that
\[
n(E \cap F, x) = n(E, x) \quad \text{if} \quad x \in A \cap Q, \quad n(E \cap F, x) = n(F, x) \quad \text{if} \quad x \in B \cap P,
\]
for almost all \( x \in R^n \) by means of \( \text{Fed69} \); a related fact is recorded in \( \text{ACMM01} \).
3.20 Definition. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathbf{V}_m(U)$, and $\|\delta V\|$ is a Radon measure.

Then, the $\mathbb{R}^n$ valued function $\eta(V, \cdot)$ is defined on a subset of $U$, by the requirement that, for $x \in U$,

$$\eta(V, x) \cdot u = \lim_{r \to 0^+} \frac{(\delta V)(b_{x,r} \cdot u)}{\|\delta V\| B(x, r)} \quad \text{whenever } u \in \mathbb{R}^n,$$

where $b_{x,r}$ denotes the characteristic function of $B(x, r)$ on $U$; hence, $x \in U$ belongs to the domain of $\eta(V, \cdot)$ if and only if the above limit exists for $u \in \mathbb{R}^n$.

3.21 Remark. Our definitions of $h(V, \cdot)$, see [Men16a, p.992], and $\eta(V, \cdot)$ adapt [All72, 4.3] in the spirit of [Fed69, 4.1.5]; in particular, 3.2 and 3.3 yield

$$h(V, \cdot) \in L^1_\loc(\|V\|, \mathbb{R}^n), \quad \|\delta V\|_{\|V\|} = \|V\| \cdot |h(V, \cdot)|,$$

$h(V, \cdot)$ and $\eta(V, \cdot)$ are Borel functions, $\text{d}m\eta(V, \cdot)$ and $\text{d}m\eta(V, \cdot)$ are Borel sets, $|\eta(V, x)| = 1$ for $\|\delta V\|$ almost all $x$,

$$h(V, x) = |h(V, x)| \eta(V, x) \quad \text{if } x \in \text{d}m\eta(V, \cdot)$$

$$h(V, x) = 0 \quad \text{if } x \notin \text{d}m\eta(V, \cdot)$$

for $\|V\|$ almost all $x$, and

$$(\delta V)(\theta) = \int \eta(V, x) \cdot \theta(x) \, d\|\delta V\| x$$

$$= -\int h(V, x) \cdot \theta(x) \, d\|V\| x + \int \eta(V, x) \cdot \theta(x) \, d(\|\delta V\| - \|\delta V\|_{\|V\|}) x$$

for $\theta \in L_1(\|\delta V\|, \mathbb{R}^n)$. Applying [MS18, 4.1, 4.5, 4.6] and convolution shows

$$\theta \in T(V, \mathbb{R}^n) \quad \text{and} \quad (\delta V)(\theta) = \int \text{trace}(V \partial E) \, d\|V\|$$

whenever $\theta : U \to \mathbb{R}^n$ is Lipschitzian with compact support.

3.22 Theorem. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $f \in T(V)$, and

$$E(y) = \{ x : f(x) > y \} \quad \text{for } y \in \mathbb{R}.$$

Then, for $\mathcal{L}^1$ almost all $y$, $V \partial E(y)$ is representable by integration and

$$V \partial E(y)(\theta) = \int \langle \theta(x), [V \partial f(x)]^{-1} V \partial f(x) \rangle \, d\|V \partial E(y)\| x.$$

for $\theta \in L_1(\|V \partial E(y)\|, \mathbb{R}^n)$.

Proof. By [Men16a, 8.30] and [MS18, 4.11], we have

$$\int_s^\infty \|V \partial E(y)\|(K) \, d\mathcal{L}^1 y < \infty$$

whenever $K$ is a compact subset of $U$ and $0 \leq s < \infty$. In view of [Men16a, 2.2, 2.24] and 3.3 it thus suffices to prove

$$\int \omega(y) V \partial E(y)(\theta) \, d\mathcal{L}^1 y$$

$$= \int \omega(y) \int \langle \theta(x), [V \partial f(x)]^{-1} V \partial f(x) \rangle \, d\|V \partial E(y)\| x \, d\mathcal{L}^1 y$$

for $\theta \in \mathcal{D}(U, \mathbb{R}^n)$ and $\omega \in \mathcal{D}(\mathbb{R}, \mathbb{R})$. The latter equation follows by applying [Men16a, 8.5, 8.30] and [MS18, 4.11] with $\phi$ and $y$ satisfying $\phi(x, y) = \omega(y) \theta(x)$ and $g(x, y) = \omega(y) \theta(x)$, $[V \partial f(x)]^{-1} V \partial f(x)$ for $x \in U$ and $y \in \mathbb{R}$. □
3.23 Remark. In the special case that \( \Theta^m(\|V\|, x) \geq 1 \) for \( \|V\| \) almost all \( x \), the conclusion is immediate from [Men16a, 12.2] and [3.3].

3.24 Remark. Under the hypotheses of 3.22, we will show that, if \( y \in \mathbb{R}, V \partial E(y) \) is representable by integration, \( \theta : U \to \mathbb{R}^n \) is locally Lipschitzian, and

\[
K = \text{spt} \theta \cap \text{spt} \|V\| \cap \text{Clos} E(y)
\]

is compact, then \( \theta \in T(V, \mathbb{R}^n) \) and there holds

\[
V \partial E(y)(\theta) = \int_{E(y)} \eta(V, x) \cdot \theta(x) \, d\|V\| \, x - \int_{E(y)} \text{trace}(V \, D \theta(x)) \, d\|V\| \, x.
\]

By [MS18, 4.6 (1)], \( \theta \) is generalised \( V \) weakly differentiable. For the equation, we reduce the problem: firstly, to the case that \( \text{spt} \theta \) is compact by expressing \( \theta \) as sum of \( \zeta \theta \) and \( (1 - \zeta) \theta \) for some \( \zeta \in \mathcal{D}(U, \mathbb{R}) \) such that \( K \subset \text{Int} \{ x : \zeta(x) = 1 \} \) and noting [Men16a, 8.20 (1) (3)] and [MS18, 4.11]; secondly, to the case that \( \theta \in \mathcal{D}(U, \mathbb{R}^n) \) by approximating \( \theta \) using convolution and noting [MS18, 4.6 (2)]. The latter case follows from [MS18, 4.1, 4.5] and [3.21].

4 Real valued functions of generalised bounded variation

We prove Theorem A of the introductory section in 4.2 and 4.5 coarea formulae in [3.3] and relations to the case of Lebesgue measure in [4.1] and [4.3].

4.1 Example. Suppose \( V \in \mathbf{V}_m(\mathbb{R}^n) \) is characterised by \( \|V\| = \mathcal{L}^n \) and \( A \) is an \( \mathcal{L}^n \) measurable set. Then, \( \|V \partial A\| \) is a Radon measure if and only if \( A \) is a set of locally finite perimeter by [All72, 4.5, Remark] applied to \( V \cap A \setminus G(n, n) \).

4.2 Theorem. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \) is an \( m \) dimensional varifold in \( U \), \( \|DV\| \) is a Radon measure, \( f \) is a real valued \( \|V\| + \|DV\| \) measurable function whose domain is contained in \( U \), \( E = \{(x, y) : f(x) > y \}, T \in \mathcal{D}(U \times \mathbb{R}, \mathbb{R}^n) \) satisfies

\[
T(\phi) = \int_{E(y)}(\phi(\cdot, y)) \, d\mathcal{L}^1 y \quad \text{for} \quad \phi \in \mathcal{D}(U \times \mathbb{R}, \mathbb{R}^n),
\]

\( p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( q : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) are defined by

\[
p(x, y) = x \quad \text{and} \quad q(x, y) = y \quad \text{for} \quad (x, y) \in \mathbb{R}^n \times \mathbb{R},
\]

and \( W \in \mathbf{V}_{m+1}(U \times \mathbb{R}) \) is characterised by

\[
W(k) = \int k((x, y), S \times \mathbb{R}) \, d(V \times \mathcal{L}^1)((x, S), y)
\]

whenever \( k \in \mathcal{K}(\{(U \times \mathbb{R}) \times G(\mathbb{R}^n \times \mathbb{R}, m + 1)\}) \).

Then, there holds

\[
W \partial E(\theta) = T(p \circ \theta) - \int (q \circ \theta)(x, f(x)) \, d\|V\| \, x \quad \text{for} \quad \theta \in \mathcal{D}(U \times \mathbb{R}, \mathbb{R}^n \times \mathbb{R}).
\]

Thus, \( \|W \partial E\| \) is a Radon measure if and only if \( \|T\| \) has this property; in this case, \( f \) possesses generalised bounded variation with respect to \( V \).
Proof. Firstly, defining $\iota : (U \times G(n,m)) \times R \to (U \times R) \times G(R^n \times R, m+1)$ by $\iota((x, S), y) = ((x, y), S \times R)$ whenever $x \in U$, $y \in R$, and $S \in G(n,m)$, we conclude that $W = \iota(V \times L^1)$ from [Fed69, 2.2.17, 2.4.18(1)]. Moreover, we deduce $\|W\| = \|V\| \times L^1$ and $\|\delta W\| = \|\delta V\| \times L^1$ from [KM17, 3.6(1)(5)] in conjunction with [Men16a, 3.4(1)(2)] applied with $T$ replaced by $\delta W$ and [Fed69, 2.6.2(4)]. The last two equations imply that $\|\delta W\|$ is a Radon measure and that $E$ is a $\|W\| + \|\delta W\|$ measurable set by [Fed69, 2.6.2(2)].

Next, suppose $\theta \in G(U \times R, R^n \times R)$ and let $\phi = \theta \circ \psi$ and $\psi = q \circ \theta$. Noting

$$(S \times R)_\phi \cdot D\theta(x, y) = S_\psi \cdot D(\phi(\cdot, y))(x) + (\psi(x, \cdot))'(y)$$

whenever $(x, y) \in U \times R$ and $S \in G(n,m)$ by [KM17, 3.6(4)], we employ the preceding paragraph, [Fed69, 2.4.18(1)(2), 2.6.2(2)(4), 2.9.20(1)], and [MS18, 4.1] to obtain

$$\delta(W \cup E \times G(R^n \times R, m+1))(\theta)$$

$$= \int_{E \times G(R^n \times R, m+1)} R_{\psi} \cdot D\theta(x, y) \, d\mu(V \times L^1)((x, y), R)$$

$$= \int \{((x, S), y) \in E, S \in G(n,m)\} S_\psi \cdot D(\phi(\cdot, y))(x) \, dV(x, S) \, dL^1 y$$

$$+ \int \{((x, y) \in E\} \psi(x, \cdot)'(y) \, dL^1 y \, d\|V\| x$$

$$= \delta(V \cup \{x : (x, y) \in E\} \times G(n,m)) (\phi(\cdot, y)) \, dL^1 y + \int \psi(x, f(x)) \, d\|V\| x.$$  

Combining this with the equation (see [KM17, 3.6(5)(6)])

$$(\delta W) \cup E(\phi) = \int (\delta V) \cup \{x : (x, y) \in E\} (\phi(\cdot, y)) \, dL^1 y,$$

the conclusion follows.  

4.3 Remark. If $f$ has generalised bounded variation with respect to $V$, then abbreviating $E(y) = \{x : (x, y) \in E\}$ for $y \in R$ and taking $J = R$ and $Z = R^n$ in [Men16a, 3.2.3.4(2)] shows the coarea formulae

$$T(\phi) = \int V \partial E(y)(\phi(\cdot, y)) \, dL^1 y,$$

$$\int g \, d\|V\| = \int g(x, y) \, d\|V \partial E(y)\| \, x \, dL^1 y$$

whenever $\phi \in L^1(\|T\|, R^n)$ and $g$ is a $\|T\|$ integrable function. Therefore, $f$ has generalised bounded variation with respect to $V$ if and only if

$$\int_0^s \|V \partial E(y)\|(K) \, dL^1 y < \infty$$

whenever $K$ is a compact subset of $U$ and $0 \leq s < \infty$; in the case that $f$ is the characteristic function of some $\|V\| + \|\delta V\|$ measurable set $A$, the latter condition is equivalent to $\|V \partial A\|$ being a Radon measure.

4.4 Remark. In the case $n = m$ and $\|V\| = L^\infty 2^U$, the function $f$ possesses generalised bounded variation with respect to $V$ if and only if $f$ is a generalized function of bounded variation in the sense of [AFP00, 4.26]; in fact, in view of 4.3 this readily follows from [AFP00, 4.27] and [ACMM01, Lemma 1].

4.5 Corollary. Suppose additionally $f$ possesses generalised bounded variation with respect to $V$ and $F : dmn f \to U \times R$ satisfies $F(x) = (x, f(x))$ for $x \in dmn f$.

Then, $f$ is generalised $V$ weakly differentiable if and only if $\|W \partial E\|$, or equivalently $\|V\|$, is absolutely continuous with respect to $F_{ab}\|V\|$.  

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Proof. There exists a Borel function \( k : U \times \mathbb{R} \to \text{Hom}(\mathbb{R}^n, \mathbb{R}) \cap \{ h : |h| = 1 \} \) such that \( T(\phi) = \int (\phi, k) d\|T\| \) for \( \phi \in \mathcal{D}(U \times \mathbb{R}, \mathbb{R}^n) \) by [Fed69 2.3.6, 4.1.5]. Since \( F_{\#}\|V\| \) is a Radon measure by [MS22 3.11] and we have \( \|T\| \leq \|W \partial E\| \leq \|T\| + F_{\#}\|V\| \) by 4.2, the two absolute continuity conditions are equivalent. If \( f \) is generalised \( V \) weakly differentiable, they are satisfied by [Men16a 8.5] and [MS18 4.11]. If \( \|T\| \) is absolutely continuous with respect to \( F_{\#}\|V\| \), then, by [Fed69 2.8.18, 2.9.2, 2.9.7] and [MS22 3.8], there exists a nonnegative Borel function \( g \in L^1_{ac}(F_{\#}\|V\|) \) with \( \|T\| = (F_{\#}\|V\|) \downarrow g \). Employing [Fed69 2.4.10, 2.4.18(1)] and [Men16a 8.1, 8.2], we therefore verify that \( f \) is generalised \( V \) weakly differentiable with \( V Df(x) = (gk)(F(x)) \) for \( \|V\| \) almost all \( x \).

5 Partitions

Properties of sets with vanishing distributional boundary follow in [5.1]. The distributional boundary of an intersection of certain sets then appears in [5.2] and [3.3] in [5.3] this leads in particular to an example relating indecomposability of varifolds to the notion of [ACMM01, p.52] for sets of finite perimeter bearing the same name. With these preparations at hand, we obtain basic properties of partitions in [5.5] including Example 1 of the introductory section in [5.10].

5.1 Lemma. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \in \mathcal{V}_m(U) \), \( \|\delta V\| \) is a Radon measure, \( E \) is \( \|V\| + \|\delta V\| \) measurable, \( V \partial E = 0 \), and \( W = V \cup E \times G(n, m) \).

Then, there holds \( \|\delta W\| = \|\delta V\| \cup E \), \( \|\delta W\|_W = \|\delta V\|_{V \cup E} \), and

\[
\eta(W, x) = \eta(V, x) \quad \text{for } \|\delta W\| \text{ almost all } x, \\
\mathbf{h}(W, x) = \mathbf{h}(V, x) \quad \text{for } \|W\| \text{ almost all } x.
\]

Proof. We have \( \|W\| = \|V\| \cup E \) and \( \delta W = (\delta V) \cup E \). Taking \( S = \delta W \in [3.3] \) yields \( \|\delta W\| = \|\delta V\| \cup E \). Next, we verify the last conclusion; in fact, for \( x \in U \), the condition

\[
\lim_{r \to 0+} \frac{\|V\| + \|\delta V\|)(B(x, r) \sim E)}{\|V\|B(x, r)} = 0,
\]

which is valid for \( |V| \) almost all \( x \in E \) as follows from [Fed69 2.8.18, 2.9.7] applied with \( \psi = (\|V\| + \|\delta V\|) \cup E \), \( \phi = \|V\| \), and \( A = E \), implies

\[
\mathbf{h}(W, x) \ast v = -\lim_{r \to 0+} \frac{(\delta V)(b_{x,r} \cdot v)}{\|W\|B(x, r)} = -\lim_{r \to 0+} \frac{(\delta V)(b_{x,r} \cdot v)}{\|V\|B(x, r)} = \mathbf{h}(V, x) \ast v
\]

whenever \( v \in \mathbb{R}^n \). Thus, we have \( \|\delta W\|_W = \|\delta V\|_V \cup E \) by [3.21]. Finally, as

\[
(\delta V)(\theta) = ((\delta V) \cup E)(\theta) = \int_G \eta(V, x) \ast \theta(x) d\|\delta V\| x = \int \eta(V, x) \ast \theta(x) d\|\delta W\| x
\]

for \( \theta \in \mathcal{D}(U, \mathbb{R}^n) \) by [3.21], the remaining conclusion follows from [3.3].

5.2 Lemma. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \in \mathcal{V}_m(U) \), \( \|\delta V\| \) is a Radon measure, \( E \) is a \( \|V\| + \|\delta V\| \)
measurable set, \(\|V \Delta E\|\) is a Radon measure, \(W = V \setminus E \times G(n,m)\), and \(F\) is a \(\|V\| + \|\delta V\| + \|V \Delta E\|\) measurable set.

Then, there holds

\[V \partial (E \cap F) = W \partial F + (V \partial E) \setminus F.\]

**Proof.** Approximation of \(F\) reduces the assertion to [Men16a, 5.4]. \[\Box\]

5.3 Remark. If \(m = n, U = \mathbb{R}^n, \|V\| = \mathcal{L}^n\), and \(E, F, P, Q, A,\) and \(B\) are as in 3.19, then we compute, for \(\theta \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)\), that

\[W \partial (Q \cup B)(\theta) = \int \theta(x) \cdot n(F, x) \, d\mathcal{H}^{n-1}x + \int_{A \cap B} \theta(x) \cdot \frac{n(F, x) - n(E, x)}{2} \, d\mathcal{H}^{n-1}x\]

by [Fed69, 4.5.6(5)]; if additionally \(F \subset \mathbb{R}^n\), then the second integral is zero.

5.4 Example. Suppose \(V \in V_n(\mathbb{R}^n)\) such that \(\|V\| = \mathcal{L}^n\), the set \(E\) is \(\|V\|\) measurable, and \(\|V \Delta E\|((\mathbb{R}^n)) < \infty\). Then, \(V \setminus E \times G(n, n)\) is indecomposable if and only if \(E\) is indecomposable in the sense of [ACMM01, p. 52] as may be verified using 5.2 and 5.3. (This comparison was planned for inclusion in [Men16a] but was unintentionally omitted in that paper.)

5.5 Definition. Suppose \(m\) and \(n\) are positive integers, \(m \leq n\), \(U\) is an open subset of \(\mathbb{R}^n\), \(V \in V(m)(U)\), \(\|\delta V\|\) is a Radon measure, and \(\Pi \subset V_m(U)\).

Then, \(\Pi\) is termed a *partition of \(V\)* if and only if the following three conditions are satisfied.

1. Whenever \(W \in \Pi\), there exists a \(\|V\| + \|\delta V\|\) measurable set \(E\) such that \(W = V \setminus E \times G(n, m), \|V\|(E) > 0\), and \(V \partial E = 0\).
2. If \(k \in \mathcal{H}(U \times G(n, m))\), then \(V(k) = \sum_{W \in \Pi} W(k)\).
3. If \(f \in \mathcal{H}(U)\), then \(\|\delta V\|(f) = \sum_{W \in \Pi} \|\delta W\|(f)\).

5.6 Remark. The family \(\Pi\) is countable and the equations in (2) and (3) also hold for \(V\) integrable functions \(k\) and \(\|\delta V\|\) integrable functions \(f\), respectively. Thus, whenever \(F\) is a \(\|V\| + \|\delta V\|\) measurable set, 3.21 and 5.1 yield

\[V \partial F(\theta) = \sum_{W \in \Xi} W \partial F(\theta) \quad \text{for } \theta \in \mathcal{D}(U, \mathbb{R}^n)\]

5.7 Remark. Every decomposition of \(V\) forms a partition of \(V\). Conversely, if \(\xi : \Pi \to 2^{V_m(U)}\) such that \(\xi(W)\) is a decomposition of \(W\) for \(W \in \Pi\), then the map \(\xi\) is univalent, \(\text{im} \xi\) is disjointed, and \(\bigcup \text{im} \xi\) is a decomposition of \(V\); in fact, for \(X \subset \bigcup \text{im} \xi\), we construct a \(\|V\| + \|\delta V\|\) measurable set \(E\) with \(\|V\|(E) > 0\), \(X = V \setminus E \times G(n, m)\), and \(V \partial E = 0\) using [Men16a, 5.4] or 5.2. In case \(V\) is rectifiable, such \(\xi\) exists by [Men16a, 6.12].

5.8 Remark. Employing 5.1 and 5.6 we may verify that a subfamily \(\Pi\) of \(V_m(U)\) is a partition of \(V\) if and only if there exists a map \(\pi : \Pi \to 2^U\) mapping distinct members of \(\Pi\) to disjoint Borel subsets of \(U\) of positive \(\|V\|\) measure satisfying

\[(\|V\| + \|\delta V\|)(U \cup \text{im} \pi) = 0\]

and \(W = V \setminus \pi(W) \times G(n, m)\) with \(V \partial \pi(W) = 0\) for \(W \in \Pi\).
5.9 Remark. If $m$, $n$, $U$, $V$, and $\Xi$ satisfy the hypotheses of [MS18 4.14], then $\Xi \sim \{0\}$ is a partition of $V$ by $5.1$ and $5.8$.

5.10 Example. There exist $L$, $L_1, \ldots, L_4$, $M$, $V$, $\kappa$, and $f$ such that

$L$ is a submanifold of class $\infty$ of $R^2$, \( \dim L = 1 \), \( (\text{Clos} \, L) \sim L = \{0\}, \)

$L_1, \ldots, L_4$ enumerate the connected components of $L$,

\[
\begin{align*}
\text{Tan}(L_1, 0) &= \text{Tan}(L_2, 0) = \{(t, 0) : t \geq 0\}, \\
\text{Tan}(L_3, 0) &= \text{Tan}(L_4, 0) = \{(t, 0) : t \leq 0\}, \\
\end{align*}
\]

$\text{Clos}(L_i \cup L_j)$ is a submanifold of class $\infty$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$,

\[
\begin{align*}
M &= R^3 \cap \{(x_1, x_2, x_3) : (x_1, x_2) \in L\}, \\
V &= RV_2(R^3), \\
\|V\| &= \mathcal{H}^2 \cup M, \\
0 &\leq \kappa < \infty, \\
\|\delta V\| &\leq \kappa \|V\|, \\
f : M \to \{y : 0 \leq y \leq 1\} &\text{ is of class } \infty \text{ relative to } M, \\
f \in T(V), \quad \text{but} \quad f \notin T(W) \text{ whenever } W \text{ is a component of } V.
\end{align*}
\]

Proof. We choose $\gamma : R \to \{s : 0 \leq s \leq 1\}$ of class $\infty$ such that

\[
\{r : \gamma(r) = 0\} = \{r : -\infty < r \leq 0\} \quad \text{and} \quad \{r : \gamma(r) = 1\} = \{r : 1 \leq r < \infty\}.
\]

For $i \in \{1, 2, 3, 4\}$, we define functions $h_i : R \to R$ of class $\infty$ by

\[
h_1(r) = 0, \quad h_2(r) = \gamma(r), \quad h_3(r) = \gamma(|r|), \quad h_4(r) = \gamma(-r)
\]

whenever $r \in R$ and associate with them $M_i$ and $V_i \in RV_2(R^3)$ such that

\[
M_i = R^3 \cap \{(x_1, x_2, x_3) : x_1 \neq 0, x_2 = h_i(x_1)\}, \quad \|V_i\| = \mathcal{H}^2 \cup M_i.
\]

Defining $V = V_1 + V_3 = V_2 + V_4$, there exists $0 \leq \kappa < \infty$ with $\|\delta V\| \leq \kappa \|V\|$ and $V_1, \ldots, V_4$ enumerate the components of $V$. Abbreviating

\[
C = R^2 \cap \{(u_1, u_2) : 0 < u_1 < u_2\},
\]

we define $g : R^2 \cap \{(u_1, u_2) : u_1 \neq 0\} \to \{y : 0 < y \leq 1\}$ by

\[
g(u) = \gamma(u_1/u_2) \text{ if } u_1 > 0 < u_2, \quad g(u) = 1 \text{ else}
\]

whenever $u = (u_1, u_2) \in R^2$ and $u_1 \neq 0$. We see that $g$ is of class $\infty$ with

\[
\int_{B(0, r)} |Dg| dL^2 < \infty \quad \text{for } 0 \leq r < \infty,
\]

since $C = \{u : g(u) < 1\}$ and $|Dg(u)| \leq (2 \sup \text{sup} |\gamma'|)/u_2$ for $u = (u_1, u_2) \in C$.

We let $M = M_1 \cup M_3 = M_2 \cup M_4$, notice $\|V\| = \mathcal{H}^2 \cup M$, and define the function $f : M \to \{y : 0 \leq y \leq 1\}$ of class $\infty$ relative to $M$ by

\[
f(x) = g(x_1, x_3) \text{ if } x_2 = 0, \quad f(x) = 1 - g(x_1, x_3) \text{ if } x_2 > 0
\]

whenever $x = (x_1, x_2, x_3) \in M$. We take

\[
E = \{(x, y) : f(x) > y\}, \quad G = \{(x, y) : f(x) = y\},
\]

and verify $\mathcal{H}^2(G \cap K) < \infty$ whenever $K$ is a compact subset of $R^3 \times R$ by means of [Fed69 3.2.20]. Associating $W_i$ with $V_i$ as in [4.2] and noting $\|\delta W_i\| \leq \kappa \|W_i\|$. 

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by \([\text{KLM17}]\) 3.6 (1) (6), we define \(T = (\mathbb{R}^3 \times \mathbb{R}) \cap \{(x_1, x_2, x_3) : x_1 = 0\}\) and apply \([\text{Men16a}]\) 5.9 (3) with \(V\) replaced by \(W_i\) to compute

\[
\|W_i \partial E\|_{\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})} = T = \mathcal{H}^2 \cup \mathcal{L}(M_1 \times \mathbb{R})
\]

for \(i \in \{1, 2, 3, 4\}\),

\[
\|W_1 \partial E\|_{\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})} = T = \mathcal{H}^2 \cup T \cap \{(x_2 = 0, x_3 \geq 0, 0 \leq y \leq 1)\},
\]

\[
\|W_2 \partial E\|_{\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})} = T = \mathcal{H}^2 \cup T \cap \{(x_2 = 0, x_3 \leq 0, 0 \leq y \leq 1)\},
\]

\[
(W_1 \partial E + W_3 \partial E) \cup T = 0 = (W_2 \partial E + W_4 \partial E) \cup T.
\]

Defining \(W = W_1 + W_3 = W_2 + W_4\), we infer

\[
\|W \partial E\| = \mathcal{H}^2 \cup \mathcal{L}
\]

by \([\text{Men16a}]\) 5.2. In view of 4.5, the conclusion is now evident. \(\Box\)

5.11 Remark. Whereas, whenever \(V\) is an \(m\) dimensional rectifiable varifold in an open subset of \(\mathbb{R}^n\) such that \(\|SV\|\) a Radon measure, a generalised \(V\) weakly differentiable function may be defined (see \([\text{Men16a}]\) 6.10, 6.12, 8.24) by selecting a decomposition \(\Xi\) of \(V\) and, subject to the natural summability condition, a member of \(\mathcal{T}(W)\) for each \(W \in \Xi\), the preceding example shows that, in general, not all members of \(\mathcal{T}(V)\) arise in this way.

6 Examples of decomposable varifolds

After some preparations in 6.1–6.2, we construct Example 2 of the introductory section in 6.15. In 6.3 and 6.5, we introduce the terminology of indecomposability for integral chains with coefficients in a complete normed commutative group to construct Example 3 of the introductory section in 6.8. In 6.6–6.7, we discuss basic properties of immersions and their associated varifolds to construct Example 4 of the introductory section in 6.4 and 6.5. In 6.9–6.14, we introduce the terminology of indecomposability for integral chains with coefficients in a complete normed commutative group to construct Example 4 of the introductory section in 6.15.

6.1 Lemma. Suppose \(m\) and \(n\) are positive integers, \(m \leq n\), \(U\) is an open subset of \(\mathbb{R}^n\), \(M\) is a connected \(m\) dimensional submanifold of \(U\) of class 2 meeting every compact subset of \(U\) in a set of finite \(\mathcal{H}^m\) measure, \(V \in \mathcal{V}_m(U)\), \(\|SV\|\) is a Radon measure, \(0 < c < \infty\),

\[
\text{spt}(V - cv_m(M)) \subset (U \sim M) \times G(n, m),
\]

\(E\) is a \(\|V\| + \|SV\|\) measurable set, and \(V \partial E = 0\).

Then, there holds \(\|V||(M \cap E) = 0\) or \(\|V||(M \sim E) = 0\); in particular, if \(V = cv_m(M)\), then \(V\) is indecomposable.

Proof. We use \([\text{All72}]\) 4.4, 4.6 (3) to conclude that each \(z \in M\) admits a neighbourhood \(X\) in \(M\) such that \(E \cap X\) is \(\mathcal{H}^m\) almost equal to \(\partial\) or \(E \cap X\), whence the principal conclusion follows because \(M\) is connected. \(\Box\)

6.2 Definition. Suppose \(m\) and \(n\) are positive integers, \(m \leq n\), \(U\) is an open subset of \(\mathbb{R}^n\), and \(\Xi \subset \mathcal{V}_m(U)\).

Then, \(\Xi\) is termed locally finite if and only if

\[
\text{card}(\Xi \cap \{W : K \cap \text{spt} \|W\| \neq \emptyset\}) < \infty
\]

whenever \(K\) is a compact subset of \(U\).
6.3 Remark. If $V \in V_m(U)$ satisfies that $\|\delta V\|$ is a Radon measure and that, for each $a \in U$, there exists a positive radius $r$ with $B(a, 2r) \subset U$ and

$$\inf\{\|W\|B(a, 2r) : W \text{ is a component of } V, B(a, r) \cap \text{spt } \|W\| \neq \emptyset\} > 0,$$

then every decomposition of $V$ is locally finite.

6.4 Example. Suppose $m$ is a positive integer, $n = m + 1$, $q$ is as in 6.3 and $A = q^*[2\mathbb{Z}]$, $M = \mathbb{R}^n \cap \{x : \text{dist}(x, A) = 1\}$, $V = v_m(M) \in V_m(\mathbb{R}^n)$.

Then, we have $\Theta^n(\|V\|, x) = 1$ for $\|V\|$ almost all $x$, $\text{spt } \|V\|$ is connected, $\|\delta V\| \leq m\|V\|$, and the following three statements hold.

(1) Whenever $I$ is a subinterval of $\mathbb{R}$, there holds $V \partial q^{-1}[I] = 0$ if and only if $\text{Bdry } I$ is contained in the set of odd integers.

(2) The family $\{V \cup q^{-1}[U(2i, 1)] \times G(n, m) : i \in \mathbb{Z}\}$ is a decomposition of $V$; in particular, $V$ is decomposable.

(3) Every decomposition of $V$ is locally finite.

Noting that $N = M \sim q^{-1}([z : z \text{ odd integer}])$ is an $m$ dimensional submanifold of class $\infty$ of $\mathbb{R}^n$, it suffices to apply 6.1 to each connected component of $N$.

6.5 Remark. Similar statements hold if $A$ is replaced by its subset $q^*[\{0, 2\}]$ in the preceding example; in that case, the decomposition of $V$ is unique.

6.6 Definition. Suppose $m$ and $n$ are nonnegative integers, $n \geq 1$, $U$ is an open subset of $\mathbb{R}^n$, $G$ is a complete normed commutative group, and $S \in I_m(U, G)$.

Then, $S$ is called indecomposable if and only if there exists no $T \in I_m(U, G)$ with $T \neq 0 \neq S - T$, $\|S\| = \|T\| + \|S - T\|$, either $m = 0$ or $\partial_G S = \partial_G T + \partial_G(S - T)$.

6.7 Remark. By [MS22 5.1], an integral current $R \in I_m(\mathbb{R}^n)$ is indecomposable if and only if $S_{R, m}(R) \in I_m(\mathbb{R}^n, Z)$ is indecomposable.

6.8 Example. This example is based on [MS22 5.1, 5.5] with $G = Z/3Z$. Abbreviating $b_1 = (1, 0)$, $b_2 = (0, 1)$, and $b_3 = (-1, 0)$, we define $B = \{b_1, b_2, b_3\}$ and

$$Q = \sum_{i=1}^{3} R_i,$$

where $R_i = [0, b_i] \in I_1(\mathbb{R}^2)$

so that we have $\partial Q = (\sum_{b \in B} \delta_b) - 3\delta_0 \in I_0(\mathbb{R}^2)$ and $N = (\text{spt } Q) \sim \text{spt } \partial Q$ is a one-dimensional submanifold of class 1 of $\mathbb{R}^2$ satisfying

$$\text{(Clos } N) \sim N = B \cup \{0\}.$$ 

Choosing $g \in G$ with $|g| = 1$, we see $g$ generates $G$. We define $S$ and $V$ such that $S = q_{R^2, 1}(Q) \cdot g \in I_1(\mathbb{R}^2, G), \quad V \in RV_1(\mathbb{R}^2), \quad \|V\| = \|S\|,$

hence, $\|V\| = H^1 \cup N$ and $\|\delta V\| = H^0 \cup B \cup \{0\}$. Moreover, $V$ is decomposable because

$$\{v_1(\text{spt } R_1 \cup \text{spt } R_2), v_1(\text{spt } R_3)\}$$

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is the only decomposition of $V$ by [6.1]. Since $\partial_G S = \{ \sum_{k \in B} h_k R^z_k (S_k) \} \cdot g$, we have $\text{spt} \| \partial_G S \| = B$. Finally, we show that $S$ is indecomposable. Whenever $T \in L^1(R^2, G)$ with $\text{spt} \| T \| \subset \text{Clos} N$ and $N \cap \text{spt} \| \partial_G T \| = \emptyset$, we observe that [MS22, 7.1] may be applied to each connected component $M$ of $N$ to represent $T = \sum_{i=1}^3 t R^z_i (R_i) \cdot h_i$ for some $h_i \in G$; accordingly, the condition $\| T \| + \| S - T \| \leq \| S \|$ would imply $|h_i| + |g - h_i| \leq 1$, hence $h_i \in \{0, g\}$, for $i \in \{1, 2, 3\}$ and the additional requirement $0 \notin \text{spt} \partial_G T$ would then allow us to conclude $\sum_{i=1}^3 h_i = 0$, hence $h_1 = h_2 = h_3$, and thus either $T = 0$ or $T = S$.

### 6.9 Definition

Suppose $A \subset R^n$, $a \in R^n$, and $A$ is pointwise [approximately] differentiable of order $2$ at $a$. Then, the pointwise [approximate] mean curvature vector of $A$ at $a$ is defined by

$$\text{pt } h(A, a) = \text{trace } \text{pt } D^2 A(a, \text{Tan}(A, a)) \in R^n$$

$$\text{ap } h(A, a) = \text{trace } \text{ap } D^2 A(a) \in R^n.$$  

### 6.10 Definition

Suppose $M$ is a manifold-with-boundary of class $k$, the map $F : M \to R^k$ is an immersion of class $k$, and $k \geq 1$. Then, for $c \in M$, the tangent cone $\text{Tan}(F, c)$, the normal cone $\text{Nor}(F, c)$, and, in case $k \geq 2$ and $c \notin \partial M$, also the mean curvature vector $h(F, c)$ along $F$ at $c$ are characterised by

$$\text{Tan}(F, c) = \text{Tan}(F[W], F(c)), \quad \text{Nor}(F, c) = \text{Nor}(F[W], F(c)),$$

$$h(F, c) = h(F[W], F(c))$$

whenever $W$ is an open neighbourhood of $c$ in $M$ and $F[W]$ is an embedding. Moreover, we define the exterior normal $n(F, c)$ along $F$ at $c$ by

$$\{ -n(F, c) \} = S^{n-1} \cap \text{Tan}(F, c) \cap \text{Nor}(F|\partial M, c) \quad \text{for } c \in \partial M.$$

### 6.11 Remark

Whenever $\phi$ is a chart of $M$ of class $2$, $c \in (\text{dmn } \phi) \sim \partial M$, $\psi = \phi^{-1}$, and $e_1, \ldots, e_m \in R^m$ are such that $\langle e_i, D(F \circ \psi)(\phi(c)) \rangle, \ldots, \langle e_m, D(F \circ \psi)(\phi(c)) \rangle$ form an orthonormal basis of $\text{Tan}(F, c)$, we have (cf. [All72, pp. 423–424])

$$h(F, c) = \sum_{i=1}^m \langle (e_i, e_i), \text{Nor}(F, c) \rangle \circ D^2 (F \circ \psi)(\phi(c)).$$

### 6.12 Example

Suppose $k, m,$ and $n$ are positive integers, $m \leq n$, $M$ is an $m$-dimensional manifold-with-boundary of class $k$, $U$ is an open subset of $R^n$, $F : M \to U$ is a proper immersion of class $k$, and $A = \text{im } F$. Then, there holds

$$\sup N(F, \cdot)[K] < \infty \text{ whenever } K \text{ is a compact subset of } U,$$

$A$ is pointwise and approximately differentiable of order $k$ at $a$,

$$\text{Tan}(A, a) = \text{Tan}^m(\mathcal{H}^m \downharpoonright A, a) = \text{Tan}(F, c),$$

$$\text{pt } D^k A(a, \text{Tan}(A, a)) = \text{ap } D^k A(a),$$

if $k \geq 2$, then $\text{pt } h(A, a) = \text{ap } h(A, a) = h(F, c),$ whenever $F(c) = a$, for $\mathcal{H}^m$ almost all $a \in A$; in fact, recalling [San19, 3.1, 3.3, 3.16, 3.22, 4.1, 4.3, 4.10], [Men19, 3.7], and [Fed69, 2.10.19 (4)], we apply first [Men19, Footnote 19] and then [Men19, 3.11] regarding approximate and pointwise differentiation, respectively.

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6.13 Definition. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( M \) is an \( m \) dimensional manifold-with-boundary of class 1, \( U \) is an open subset of \( \mathbb{R}^n \), and \( F : M \to U \) is a proper immersion of class 1.

Then, we define the *varifold* \( V \) associated with \((F, U)\) by

\[
V(k) = \int k(x, \text{Tan}(\text{im} F, x)) N(F, x) \, d\mathcal{H}^m x \quad \text{for } k \in \mathcal{H}(U \times G(n, m)).
\]

6.14 Remark. We notice that

Accordingly, we may construct a function \( \theta \) for \( \mathcal{H}^m \setminus N(F, \cdot) \), and using \( [\text{All72}, 4.4, 4.7] \) and \( 3.21 \) and \( 6.12 \) to verify firstly that \( \theta \) satisfies

\[
\Theta^m(\|V\|, x) = \text{Tan}(\text{im} F, x), \quad \Theta^m(\|V\|, x) = N(F, x)
\]

for \( \|V\| \) almost all \( x \). If \( M \) and \( F \) are of class 2, then we employ \( [\text{All72}, 4.4, 4.7] \) and \( 6.12 \) to verify firstly that \( \|\delta V\| \) is a Radon measure satisfying

\[
(\delta V)(\theta) = -\int_{\partial M} \left( \sum_{x \in \partial M} h(F, c) \right) \theta(x) \, d\mathcal{H}^m x + \int_{\partial M} \left( \sum_{x \in \partial M} n(F, c) \right) \theta(x) \, d\mathcal{H}^{m-1} x
\]

for \( \theta \in \mathcal{P}(U, \mathbb{R}^n) \) and secondly that, for \( \|V\| \) almost all \( x \), there holds

\[
h(V, x) = h(F, c) \quad \text{whenever } F(c) = x;
\]

in particular, \( \|\delta V\| \leq \|V\| \cdot |h(V, \cdot)| + \mathcal{H}^{m-1} \setminus N(F|\partial M, \cdot) \) with equality in case \( F|\partial M \) is an embedding.

6.15 Example. There exists a two-dimensional manifold-with-boundary \( M \) of class \( \infty \) and a proper immersion \( F : M \to \mathbb{R}^3 \) of class \( \infty \) such that the varifold associated with \((F, \mathbb{R}^3)\) has a unique decomposition \( \Xi \) and \( \Xi \) is not locally finite.

**Proof.** We define a compact subset of \( \mathbb{R} \) by

\[
X = \mathbb{R} \cap \{ x : x \leq 0, or \ x \geq 1, or \ x = 1/i \ for \ some \ positive \ integer \ i \}
\]

and use [MS22, 3.14] to construct a nonpositive function \( f : \mathbb{R} \to \mathbb{R} \) of class \( \infty \) satisfying

\[
X = \{ x : f(x) = 0 \} \quad \text{and} \quad \sup \text{im } |f''| \leq r^{-1},
\]

where \( r = 4 \sup \text{im } |f| \). We set \( T = \mathbb{R}^2 \cap \{ (x, y) : |y| < r/2 \} \) and define

\[
A = \{ (x, y) : f(x) < y \}, \quad G = \{ (x, y) : f(x) = y \}.
\]

Since we have \( \sup \text{im } |b(G, \cdot)| \leq \sup \text{im } |f''| \), we conclude \( \text{reach}(G) \geq r/2 \) from [3.9] and [3.10] hence, taking \( U \) and \( \delta \) as in [3.9] and [3.12] we have

\[
G \subset T \subset U, \quad A \cup T = \mathbb{R}^2 \cap \{ (x, y) : y > -r/2 \}.
\]

Accordingly, we may construct a function \( g : A \cup T \to \mathbb{R} \) of class \( \infty \) such that

\[
\begin{align*}
g(x, y) &= 3^{1/2} \delta(x, y) \quad \text{if } -r/2 < y \leq r/4, \\
g(x, y) &= 0 \quad \text{if } r/4 < y < r/2, \\
g(x, y) &= 0 \quad \text{if } y \geq r/2
\end{align*}
\]

whenever \( (x, y) \in \mathbb{R}^2 \) and \( y > -r/2 \); hence, \( T \cap \{ (x, y) : g(x, y) = 0 \} = G \) and

\[
D_y g(x, y) > 0 \quad \text{whenever } x \in \mathbb{R} \text{ and } -r/2 < y \leq r/4.
\]
Defining $M = \mathbb{R}^3 \cap \{(x, y, z) : g(x, y) = z\}$ as well as closed sets

$$E = M \cap \{(x, y, z) : (x, y) \in A \cup G\}, \quad F = M \cap \{(x, y, z) : y \geq 0\},$$

$E$ and $F$ are two-dimensional submanifolds-with-boundary of class $\infty$ of $\mathbb{R}^3$, whenever $(x, y, z) \in \mathbb{R}^3$, where we have identified $\mathbb{R}^3 \simeq \mathbb{R}^2 \times \mathbb{R}$, and

$$n(M, E, (x, y, z)) = 2^{-1} \left( n(A, (x, y)), -3^{1/2} \right) \quad \text{if} \ (x, y) \in G \text{ and } z = 0$$

$$n(M, E, (x, y, z)) = 0 \quad \text{else}$$

$$n(M, F, (x, y, z)) \in \{(u, v, w) : v < 0, w < 0\} \quad \text{if} \ y = 0, g(x, y) = z,$$

$$n(M, F, (x, y, z)) = 0 \quad \text{else}$$

for $(x, y, z) \in \mathbb{R}^3$. Defining

$$B_1 = \mathbb{R}^3 \cap \{(x, y, z) : (x, y) \in A \cup G, z = g(x, y)\},$$

$$B_2 = \mathbb{R}^3 \cap \{(x, y, z) : (x, y) \in A \cup G, z = g(x, y)\},$$

$$B_3 = \mathbb{R}^3 \cap \{(x, y, z) : (x, y) \in A \cup G, z = -g(x, y)\},$$

$$B_4 = \mathbb{R}^3 \cap \{(x, y, z) : (x, y) \in A \cup G, z = -g(x, y)\},$$

$$B_5 = \mathbb{R}^3 \cap \{(x, y, z) : z = 0\},$$

we take $V = \sum_{j=1}^{5} v_2(B_j) \in \mathcal{V}_2(\mathbb{R}^3)$ and compute

$$(\delta V)(\theta) = -\left( \sum_{j=1}^{4} \int_{B_j} \theta \bullet h(B_j, \cdot) \, d\mathcal{H}^2 \right) + \int \theta \bullet n(B_5, K, \cdot) \, d\mathcal{H}^1$$

for $\theta \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, where $K = \mathbb{R}^3 \cap \{(x, y, z) : (x, y) \in A, (x, -y) \in A, z = 0\}$. The connected components of $K$ are given by

$$D_i = \mathbb{R}^3 \cap \{(x, y, z) : \frac{r_i}{1 + r_i} < x < \frac{r_i}{1 + r_i}, |y| < |f(x)|, z = 0\}$$

corresponding to every positive integer $i$.

Next, we consider the closed sets and one-dimensional submanifolds $L_1, \ldots, L_6$ of class $\infty$ of $\mathbb{R}^3$ defined by

$$L_1 = \mathbb{R}^3 \cap \{(x, y, z) : y = r/2, z = 0\},$$

$$L_2 = \mathbb{R}^3 \cap \{(x, y, z) : (x, y) \in G, z = 0\},$$

$$L_3 = \mathbb{R}^3 \cap \{(x, y, z) : y = 0, g(x, y) = z\},$$

$$L_4 = \mathbb{R}^3 \cap \{(x, y, z) : y = 0, -g(x, y) = z\},$$

$$L_5 = \mathbb{R}^3 \cap \{(x, y, z) : (x, y) \in G, z = 0\},$$

$$L_6 = \mathbb{R}^3 \cap \{(x, y, z) : y = -r/2, z = 0\},$$

note $L_j \subset \text{spt} \|V\|$ for $j = 1, \ldots, 6$, set $S = \bigcup_{j=1}^{6} L_j$ and $R = (\text{spt} \|V\|) \sim S$, let

$$P_{i,1} = \{(x, y, z) : \frac{r_i}{1 + r_i} < x < \frac{r_i}{1 + r_i}, y < 0, (x, y) \in A, g(x, y) = z\},$$

$$P_{i,2} = \{(x, y, z) : \frac{r_i}{1 + r_i} < x < \frac{r_i}{1 + r_i}, y > 0, (x, -y) \in A, g(x, -y) = z\},$$

$$P_{i,3} = \{(x, y, z) : \frac{r_i}{1 + r_i} < x < \frac{r_i}{1 + r_i}, y < 0, (x, y) \in A, -g(x, y) = z\},$$

$$P_{i,4} = \{(x, y, z) : \frac{r_i}{1 + r_i} < x < \frac{r_i}{1 + r_i}, y > 0, (x, -y) \in A, -g(x, -y) = z\},$$

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for every positive integer $i$ and $Q_j = \bigcup_{i=1}^{\infty} P_{i,j}$ for $j \in \{1, 2, 3, 4\}$ and abbreviate

$N_1 = B_1 \cap \{(x, y, z) : y > 0, z > 0\}$, $N_5 = R^3 \cap \{(x, y, z) : y > r/2, z = 0\}$,

$N_2 = B_2 \cap \{(x, y, z) : y < 0, z > 0\}$, $N_6 = \{(x, y, z) : -f(x) < y < r/2, z = 0\}$,

$N_3 = B_3 \cap \{(x, y, z) : y > 0, z < 0\}$, $N_7 = \{(x, y, z) : -r/2 < y < f(x), z = 0\}$,

$N_4 = B_4 \cap \{(x, y, z) : y < 0, z < 0\}$, $N_8 = R^3 \cap \{(x, y, z) : y < -r/2, z = 0\}$;

hence, $R$ is a two-dimensional submanifold of class $\infty$ of $R^3$, the family $\Phi$ of its connected components consists of the sets $D_i$ and $P_{i,j}$ corresponding to positive integers $i$ and $j \in \{1, 2, 3, 4\}$ together with the sets $N_1, \ldots, N_8$, the function $\Theta(\|V\|, \cdot)C$ is constant for $C \in \Phi$, we have $Q_j \subseteq B_j$ for $j \in \{1, 2, 3, 4\}$,

$B_1 \cap B_2 = L_3$, $B_1 \cap B_3 = L_1 \cup L_5 \cup N_5$

$B_1 \cap B_4 = L_4$, $B_2 \cap B_3 = L_2 \cup L_6 \cup N_8$,

and we let $X = (\bigcup \Phi) \sim K$. Whenever $W$ is a member of a partition of $V$, we obtain

$\|W\| \ll C = \|V\| \ll C$ whenever $\|W\|(C) > 0$ and $C \in \Phi$

by applying \[ \text{for a suitable set } E, \text{ with } m, n, U, M, \text{ and } \{e\} \text{ replaced by } 2, 3, R^3, C, \text{ and } \Theta(\|V\|, \cdot)[C]. \text{ Thus, whenever } K \text{ is a partition of } V, \text{ we may partition } \Phi \text{ into nonempty subfamilies } Y_2(W) \text{ corresponding to } W \in K \text{ such that } W = V \ll (\bigcup Y (W)) \times G(3,2) \text{ for } W \in K. \text{ We also record that }$

$\mathcal{H}^2((\text{Clos } D_i) \sim D_i) = 0$ and $V \partial \text{Clos } D_i = 0$

whenever $i$ is a positive integer, as $n(B_5, K, \cdot)$ and $n(B_5, D_i, \cdot)$ agree on $\text{Clos } D_i$ and applying \[ \text{with } M \text{ and } E \text{ replaced by } B_5 \text{ and } D_i \text{ yields } \]

$\delta(V \ll D_i \times G(3,2))(\theta) = \int \theta \bullet n(B_5, D_i, \cdot) d\mathcal{H}^1$ for $\theta \in \mathcal{D}(R^3, R^3)$.

Accordingly, $\text{Clos } D_1, \text{Clos } D_2, \text{Clos } D_3, \ldots$ form a $\|V\| + \|\delta V\|$ almost disjoint sequence of sets with vanishing distributional $V$ boundary whose union $\text{Clos } K$ is $\|V\| + \|\delta V\|$ equal to $R^3 \sim X$; in particular, we have $V \partial X = 0$ and, defining

$Z_1 = V \ll X \times G(3,2)$, $Z_2 = V \ll (\text{Clos } K) \times G(3,2)$,

and $\Pi = \{V \ll D_i \times G(3,2) : i = 1, 2, 3, \ldots\}$,

we see that $\{Z_1, Z_2\}$ and $\{Z_i\} \cup \Pi$ are partitions of $V$ by \[ \text{Employing } 5.2 \text{ we also verify that } \Pi \text{ is the unique decomposition of } Z_2. \]

Since $V$ admits a decomposition by \[ \text{the proof may be concluded by showing that } Z_1 \text{ belongs to every decomposition } K \text{ of } V \text{ because in this case } K \sim \{Z_1\} \text{ is a decomposition of } Z_2 \text{ by } \text{6.12}. \]

For this purpose, computing $\delta(V \ll C \times G(3,2))$ whenever $C \in \Phi$, we shall verify:

If $\|SW\|(L_1) = 0$ and $N_5 \in Y$, then $\{N_1, N_3, N_6\} \subseteq Y$;

if $\|SW\|(L_6) = 0$ and $\{N_2, N_4, N_7\} \subseteq Y$, then $N_8 \in Y$;

if $\|SW\|(L_2 \cap L_5) = 0$ and $\{N_1, N_3, N_6\} \subseteq Y$, then $\{N_2, N_4, N_7\} \subseteq Y$;

if $\|SW\|(L_3) = 0$ and $N_1 \in Y$, then $Q_1 \subseteq Y$;

if $\|SW\|(L_4) = 0$ and $N_4 \in Y$, then $Q_2 \subseteq Y$;

if $\|SW\|(L_3) = 0$ and $N_3 \in Y$, then $Q_3 \subseteq Y$ and, if $\|SW\|(L_4) = 0$ and $N_4 \in Y$, then $Q_4 \subseteq Y$. 

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whenever \( Y \subseteq \Phi \) and \( W = V \cup (U \cup Y) \times G(3, 2) \). In fact, the first two implications are elementary; the third implication follows from the equations for \( n(M.E.) \); and the last four implications follow from the inclusions for \( n(M.F.) \). Therefore, if \( Y \subseteq \Phi \) and \( W = V \cup (U \cup Y) \times G(3, 2) \) is a member of a partition of \( V \), then

\[
either X \subseteq U or X \cap U = \emptyset;
\]

hence, there exists \( W \in \Xi \) with \( X \subseteq \bigcup Y_{\Xi}(W) \). Noting \( \|V\| \leq X \) is absolutely continuous with respect to \( \|V\| \), we apply 5.2 for a suitable set \( E \), with \( m, n, U, \) and \( F \) replaced by \( 2, 3, \mathbb{R}^3 \), and \( X \) to conclude \( W \delta X = 0 \), whence we infer \( \bigcup Y_{\Xi}(W) \subseteq X \). Thus, \( Z_1 \subseteq \Xi \).

\( \square \)

7 Properties of indecomposability with respect to a family of generalised weakly differentiable real valued functions

We introduce the main new concept of the present paper—indecomposability of type \( \Psi \)—in \( \text{7.1 Definitions} \). Its basic relations to topological connectedness are given in \( \text{7.2 Remark} \), before establishing first geometric consequences in \( \text{7.3 Remark} \). The latter includes Theorem B of the introductory section in \( \text{7.4 Remark} \) and \( \text{7.5 Remark} \).

7.1 Definition. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \in \mathbf{V}_m(U) \), \( \|V\| \) is a Radon measure, and \( \Psi \subseteq T(V) \).

Then, \( V \) is called indecomposable of type \( \Psi \) if and only if, whenever \( f \in \Psi \), the set of \( y \in \mathbb{R} \), such that \( E(y) = \{ x : f(x) > y \} \) satisfies

\[
\|V\|(E(y)) > 0, \quad \|V\|(U \sim E(y)) > 0, \quad V \delta E(y) = 0,
\]

has \( \mathcal{L}^1 \) measure zero.

7.2 Remark. If \( V \) is indecomposable, then \( V \) is also indecomposable of type \( \Psi \) whenever \( \Psi \subseteq T(V) \). For \( \Psi = T(V) \), the converse implication holds; in fact, \( \text{MS13} \) readily yields that, whenever \( E \) is a \( \|V\| + \|\delta V\| \) measurable set satisfying \( V \delta E = 0 \), its characteristic function belongs to \( T(V) \).

7.3 Remark. If \( \text{spt} \|V\| \) is compact, then indecomposability of types \( \mathcal{E}(U, \mathbb{R}) \) and \( \mathcal{D}(U, \mathbb{R}) \) agree. In general, these concepts differ as will be shown in \( \text{7.9} \).

7.4 Remark. If \( V \) is indecomposable of type \( \mathcal{D}(U, \mathbb{R}) \), \( a \in U, \ 0 < r < \infty \), \( B(a, r) \subseteq U \), and \( f : U \to \mathbb{R} \) satisfies \( f(x) = \sup \{ x - |x - a|, 0 \} \) for \( x \in U \), then \( V \) is indecomposable of type \( \{f\} \), as may be verified by approximation.

7.5 Remark. If \( V \) is indecomposable of type \( \mathcal{D}(U, \mathbb{R}) \) and \( A \) is a relatively closed subset of \( U \), then \( V|\mathcal{D}(U \sim A, \mathbb{R}) \) is indecomposable of type \( \mathcal{D}(U \sim A, \mathbb{R}) \), as may be verified using the canonical extension map of \( \mathcal{D}(U \sim A, \mathbb{R}) \) into \( \mathcal{D}(U, \mathbb{R}) \).

7.6 Example. Taking \( V \) and \( q \) as in \( \text{6.4} \) and defining \( E(y) = \{ x : q(x) > y \} \) whenever \( y \in \mathbb{R} \), the set \( \{ y : \|V\|(E(y)) > 0, \|V\|(U \sim E(y)) > 0, V \delta E(y) = 0 \} \) is countably infinite and \( V \) is indecomposable of type \( \{q\} \).

7.7 Theorem. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \in \mathbf{V}_m(U) \), \( \|\delta V\| \) is a Radon measure, and \( V \) is indecomposable of type \( \mathcal{E}(U, \mathbb{R}) \).

Then, \( \text{spt} \|V\| \) is connected.
Proof. If spt $\|v\|$ were not connected, then there would exist nonempty disjoint relatively closed subsets $E_0$ and $E_1$ of $U$ with $\text{spt } \|v\| = E_0 \cup E_1$, and [MS22 3.16] would yield $f$ satisfying $\|v\| \cap \{x : f(x) > y\} = E_1$ for $0 \leq y < 1$, in contradiction to $\|v\|(E_1) > 0$, $\|v\|(U \sim E_1) > 0$, and $V \partial E_1 = 0$ by [Men16a 6.5].

7.8 Remark. In view of 7.2, the preceding theorem extends [Men16a 6.5].

7.9 Example. Suppose $m$ and $n$ are integers, $1 \leq m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $M$ is an $m$ dimensional submanifold-with-boundary of class 2 of $U$, the inclusion map $F : M \to U$ is proper, $V$ is associated with $(F,U)$, and $\Phi$ is the family of connected components of $M$. Then, $\|\delta V\|$ is a Radon measure by 6.14 and one verifies the equivalence of the following four conditions using 6.1 and 6.7.

1. The submanifold-with-boundary $M$ is connected.
2. The submanifold $M \sim \partial M$ is connected.
3. The varifold $V$ is indecomposable.
4. The varifold $V$ is indecomposable of type $\mathcal{E}(U,\mathbb{R})$.

Hence, $V \perp C \times G(n,m)$ is indecomposable and $V \partial C = 0$ for $C \in \Phi$, as $C$ is relatively open in $M$. Next, we notice that [Men16a 5.2] may be used to obtain that
\[
V \partial E(\theta) = \sum_{C \in \Phi} (V \perp C \times G(n,m)) \partial E(\theta),
\]
\[
\|V \partial E\|(f) = \sum_{C \in \Phi} \|(V \perp C \times G(n,m)) \partial E\|(f)
\]
whenever $E$ is $\|v\| + \|\delta V\|$ measurable, $\theta \in \mathcal{D}(U,\mathbb{R}^n)$, and $f \in \mathcal{X}(U)$. These equations are readily used to verify that $\{V \perp C \times G(n,m) : C \in \Phi\}$ is the unique decomposition of $V$ and that the following two conditions are equivalent:

5. If $C$ is compact for some $C \in \Phi$, then $M$ is connected.
6. The varifold $V$ is indecomposable of type $\mathcal{D}(U,\mathbb{R})$.

7.10 Lemma. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in V_m(U)$, $\|\delta V\|$ is a Radon measure, $Y$ is a finite dimensional normed space, and $f \in T(V,Y)$.

Then, there holds
\[
f(x) \in \text{spt } f_\# \|v\| \quad \text{for } \|v\| + \|\delta V\| \text{ almost all } x.
\]

Proof. Noting [MS18 4.11], we see that $\gamma \circ f \in T(V)$ by [Men16a 8.15] and obtain
\[
f(x) \in \{y : \gamma(y) = 0\} \quad \text{for } \|\delta V\| \text{ almost all } x,
\]
since we have $\|\delta V\|_\infty (\gamma \circ f) \leq \|v\|_\infty (\gamma \circ f) = 0$ by [Men16a 8.33], whenever $\gamma \in \mathcal{D}(Y,\mathbb{R})$ satisfies $\text{spt } \gamma \cap \text{spt } f_\# \|v\| = \emptyset$. 

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Remark. The utility of an estimate of an open subset of is a Radon measure, is indecomposable of type \( \{f\} \), and \( J = \text{spt } f_\# \|V\| \).

Then, the following four statements hold.

(1) The set \( J \) is a subinterval of \( \mathbb{R} \).

(2) For \( L^1 \) almost all \( y \in J \), we have \( V \partial f(x) > y \) \( \neq 0 \).

(3) If \( Y \subset \mathbb{R} \) and \( V D f(x) = 0 \) for \( \|V\| \) almost all \( x \in f^{-1}[Y] \), then we have \( L^1(Y \cap J) = 0 \).

(4) If \( V D f = 0 \), then \( f \) is \( \|V\| + \|\delta V\| \) almost constant.

Proof. Abbreviate \( I = \mathbb{R} \cap \{y: \inf J \leq y \leq \sup J\} \), choose compact sets \( K_i \) with \( K_1 \subset \text{Int } K_{i+1} \) and \( \bigcup_{i=1}^\infty K_i = U \), and pick \( \varepsilon_i > 0 \) such that (see [MS22, 3.8, 3.11])

\[
\nu = \sum_{i=1}^\infty \varepsilon_i f_\#((\|V\| \cap K_i \cap \{x: f(x) \leq i\}) \cup |V D f|)
\]
satisfies \( \nu(\mathbb{R}) < \infty \). Define \( B = \mathbb{R} \cap \{y: V \partial f(x) > y \} = 0 \}. Clearly, we have \( L^1(B \cap I) = 0 \); in particular (2) holds. [MS18, 4.11] and [Men16a, 8.29] yield

\[
\{y: \Theta^1(\nu, y) = 0\} \subset B.
\]

If \( Y \) satisfies the hypotheses of (3), then we have \( \nu(Y) = 0 \) which entails firstly \( L^1(Y \sim \{y: \Theta^1(\nu, y) = 0\}) = 0 \) by [Fed69, 2.10.19 (4)], and then \( L^1(Y \cap I) = 0 \). Now, (1) and (3) follow; in particular (2) holds. In the hypotheses of (2), we have \( \nu = 0 \) which implies \( B = \mathbb{R} \), \( L^1(I) = 0 \), and that \( f \) is \( \|V\| \) almost constant; thence, (7.10) implies (4).

Theorem. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \in \mathcal{V}_m(U) \), \( \|\delta V\| \) is a Radon measure, \( f \in T(V) \), \( V \) is indecomposable of type \( \{f\} \), \( Y \subset \mathbb{R} \), and \( f(x) \in Y \) for \( \|V\| \) almost all \( x \).

Then, there holds

\[
\text{diam } \text{spt } f_\# \|V\| \leq L^1(Y).
\]

Proof. From (7.11) and (7.11) applied with \( Y \) replaced by \( \mathbb{R} \sim Y \), we conclude that \( \text{spt } f_\# \|V\| \) is an interval which is \( L^1 \) almost contained in \( Y \).

Remark. The utility of an estimate of \( \text{diam } \text{spt } f_\# \|V\| \) is illustrated by the following two facts concerning Radon measures \( \phi \) over \( U \), see [MS22, 3.11].

(1) Whenever \( g \) is a nonnegative real valued \( \phi \) measurable function satisfying \( \phi \{x: g(x) \leq y\} \geq 0 \) for \( y > 0 \), we have \( \phi(\infty)(g) = \text{diam } \text{spt } g_\# \phi \).

(2) If \( g : \text{spt } \phi \to \mathbb{R} \) is continuous, then \( \text{im } g \) is a dense subset of \( \text{spt } g_\# \phi \).

In the final paper of our series [MS23], for certain \( f \), such a set \( Y \) satisfying a geometric estimate for \( L^1(Y) \) will be constructed to obtain a novel type of Sobolev Poincaré inequality.
8 Unique partition along a generalised weakly differentiable real valued function

Following some preparations in 8.1–8.5 we introduce the concept of partition along $f$ in 8.6–8.8. Its uniqueness and existence properties are proven in 8.10–8.16. This includes Theorems C of the introductory section in 8.10 and 8.13.

8.1 Lemma. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathcal{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $f \in \mathbf{T}(V)$, $y \in \mathbb{R}$, and

$$E = \{x : f(x) > y\}, \quad V \partial E = 0, \quad W = V \cup E \times \mathbf{G}(n, m).$$

Then, there holds $f \in \mathbf{T}(W)$ and

$$W \mathbf{D} f(x) = V \mathbf{D} f(x) \quad \text{for } \|W\| \text{ almost all } x.$$

Proof. Employing [MIS13, 4.11] and [Men16a, 8.12, 8.13(4)], we define the function $g = \sup \{f, y\} \in \mathbf{T}(V)$ and notice

$$V \mathbf{D} g(x) = V \mathbf{D} f(x) \quad \text{for } \|V\| \text{ almost all } x \in E,$$

$$V \mathbf{D} g(x) = 0 \quad \text{for } \|V\| \text{ almost all } x \in U \sim E.$$

Splitting the left integrals along the partition $\{E, U \sim E\}$, we obtain

$$\begin{align*}
\langle 0 & \rangle \gamma(y) = (\delta V)((\gamma \circ f) \theta) + \gamma(y)((\delta V) \cup (U \sim E))(\theta), \\
\int_{\gamma(y)}(f(x)S_{V} \bullet D \gamma \theta(x)) dV(x, S)
\end{align*}$$

$$= \int \gamma(f(x))S_{V} \bullet D \theta(x) dW(x, S) + \gamma(y)\delta V \cup (U \sim E) \times \mathbf{G}(n, m))\langle \theta \rangle(x),$$

$$\int \|\theta(x)\| D \gamma(g(x)) \circ V \mathbf{D} g(x)) d\|V\| \quad \text{almost all } x = \int \|\theta(x)\| D \gamma(f(x)) \circ V \mathbf{D} f(x)) d\|W\|$$

whenever $\theta \in \mathcal{D}(U, \mathbb{R}^n)$, $\gamma \in \mathcal{E}(\mathbb{R}, \mathbb{R})$, and $\text{spt } D \gamma$ is compact. Subtracting the last two equations from the first, the conclusion follows, as $V \partial (U \sim E) = 0$. □

8.2 Remark. In view of 8.10 or [Men16a, 8.25], the condition on $E$ to equal $\{x : f(x) > y\}$ for some $y \in \mathbb{R}$ may not weakened to $\|V\| + \|\delta V\|$ measurability.

8.3 Lemma. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathcal{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $f \in \mathbf{T}(V)$, and $R$ is the family of all $\|V\| + \|\delta V\|$ measurable sets $E$ such that $V \partial E = 0$ and such that $W = V \cup E \times \mathbf{G}(n, m)$ satisfies $f \in \mathbf{T}(W)$ and

$$W \mathbf{D} f(x) = V \mathbf{D} f(x) \quad \text{for } \|W\| \text{ almost all } x.$$

Then, the following four statements hold:

1. The set $U$ belongs to $R$.
2. If $E \in R$, $F \in R$, and $F \subset E$, then $E \sim F \in R$.
3. If $E_1, E_2, E_3, \ldots$ form a disjoint sequence in $R$, then $\bigcup_{i=1}^{\infty} E_i \in R$.
4. If $E_1 \supset E_2 \supset E_3 \supset \cdots$ form a sequence in $R$, then $\bigcap_{i=1}^{\infty} E_i \in R$.

In particular, if $I$ is a subinterval of $\mathbb{R}$, $E = f^{-1}[I]$, and $V \partial E = 0$, then $E \in R$. 

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Proof. Recalling [Men16a, 5.3], we verify the principal conclusion using that $E \in R$ if and only if $E$ is $\|V\| + \|\delta V\|$ measurable, $V \partial E = 0$, and

$$(\delta V) \cdot E)((\gamma \circ f) \theta) = \int_{E \times G(n,m)} (\gamma \circ f)(x)S_\theta \cdot \mathbf{D} \theta(x) + (\theta(x), \mathbf{D} \gamma(f(x)) \circ V \mathbf{D} f(x)) \, dV(x, S)$$

whenever $\theta \in \mathcal{D}(U, \mathbb{R}^n)$, $\gamma \in \mathcal{E}(\mathbb{R}, \mathbb{R})$, and $\text{spt} \, \gamma$ is compact. Finally, the postscript follows from the principal conclusion and 8.1. □

8.4 Remark. The family $R$ need not to be a Borel family; in fact, it may happen that $E \in R$ and $F \in R$ but neither $E \cup F \in R$ nor $E \cap F \in R$ as is evident from considering the varifold constructed in [Men16a, 6.13] and taking $f = 0$.

8.5 Remark. If $\Theta^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all $x$, then the condition that $W \mathbf{D} f(x) = V \mathbf{D} f(x)$ for $\|W\|$ almost all $x$ in the definition of $R$ is redundant, as may be verified by means of [Men16a, 11.2] in conjunction with [Fed69, 2.8.18, 2.9.11], [All72, 3.5(1b)], and [MS22, 3.26].

8.6 Definition. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in V_m(U)$, $\|\delta V\|$ is a Radon measure, and $f \in T(V)$.

Then, $\Pi$ is called a partition of $V$ along $f$ if and only if $V$ is a partition of $V$ and, whenever $W \in \Pi$, the following two conditions hold.

1. There exists a subinterval $I$ of $\mathbb{R}$ such that $W = V \cap f^{-1}[I] \times G(n, m)$ and $V \partial f^{-1}[I] = 0$.

2. There exists no partition of $\mathbb{R}$ into subintervals $J_1$ and $J_2$ such that

$$\|W\|(f^{-1}[J_i]) > 0 \text{ for } i \in \{1, 2\}, \quad W \partial f^{-1}[J_i] = 0.$$

8.7 Remark. For $W \in \Pi$, we see $\|\delta W\|$ is a Radon measure by (1), $f \in T(W)$ with $W \mathbf{D} f(x) = V \mathbf{D} f(x)$ for $\|W\|$ almost all $x$ by (3). $W$ is indecomposable of type $\{f\}$ by (2), $J = \text{spt} \, f_\# \|V\|$ is a subinterval of $\mathbb{R}$ by (7.11)(1), and $W = V \cap f^{-1}[I] \times G(n, m)$ for some subinterval $I$ of $f$ with $V \partial f^{-1}[I] = 0$ by (7.10) and (1). In this case, $J$ is dense in $J$, whence we infer

$$\text{Int } J \subset I \subset J, \quad \text{Bdry } I = \text{Bdry } J.$$

8.8 Remark. By (5.1) and (8.7) there exists a function $\pi: \Pi \to 2^\mathbb{R}$ whose value at $W \in \Pi$ is characterised to be the dense subinterval $\pi(W)$ of $f_\# \|W\|$ such that $W = V \cap f^{-1}[\pi(W)] \times G(n, m)$ with $V \partial f^{-1}[\pi(W)] = 0$ and $b \in \pi(W)$ if and only if $\{(\|W\| + \|\delta W\|) \cdot x : f(x) = b\} > 0$ whenever $b \in \text{Bdry } \pi(W)$; hence, $\|\delta W\| = \|\delta V\| \cup f^{-1}[\pi(W)]$ for $W \in \Pi$. This entails that $\pi$ maps distinct members of $\Pi$ to disjoint subintervals of $\mathbb{R}$ and

$$\{(\|V\| + \|\delta V\|) \cdot U \sim f^{-1}(\bigcup \text{im } \pi)\} = 0$$

because $\text{card}\{W : x \in f^{-1}[\pi(W)]\} = 1$ for $\|V\| + \|\delta V\|$ almost all $x$ by (5.6).

8.9 Example. Taking $V$ and $q$ as in (6.4) we see that $V$ admits a nontrivial partition along $q$ but is nevertheless indecomposable of type $\{q\}$.
8.10 Theorem. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathcal{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and $f \in T(V)$.

Then, there exists at most one partition of $V$ along $f$.

Proof. We suppose $\Pi$ and $P$ are partitions of $V$ along $f$ and associate functions $\pi$ and $\rho$ to $\Pi$ and $P$, respectively, as in 8.8. Whenever $W \in \Pi$, there exists $X \in P$ satisfying

$$f_\# \|V\|((\pi(W) \cap \rho(X)) > 0$$

by 5.6. Accordingly, the proof may be conducted by showing that, whenever $W \in \Pi$ and $X \in P$ are related by the preceding condition, there holds

$$f_\# \|V\|((\rho(X) \sim \pi(W)) = 0.

Clearly, we have $\pi(W) \cap \rho(X) \neq \emptyset$. Moreover, whenever $J$ is a subinterval of $\mathbb{R}$, $\mathbb{R} \sim J$ is a subinterval of $\mathbb{R}$, every member of $(\text{im} \pi \cup \text{im} \rho) \sim \{\rho(X)\}$ is contained either in $J$ or in $\mathbb{R} \sim J$, and $\pi(W) \subset \mathbb{R} \sim J$, we apply 5.6 twice to conclude

$$X \partial f^{-1}[J] = V \partial f^{-1}[J] = 0$$

so that $\|X\|(f^{-1}[\mathbb{R} \sim J]) \geq f_\# \|V\|((\pi(W) \cap \rho(X)) > 0$ implies

$$f_\# \|V\|((\rho(X) \cap J) = \|X\|(f^{-1}[J]) = 0.

This allows us to deduce

$$f_\# \|V\|((\rho(X) \cap \{y : \sup \pi(W) \leq y\} \sim \pi(W)) = 0;

in fact, assuming $b = \sup \pi(W) \in \rho(X)$, we take $J = \{y : b \leq y < \infty\} \sim \pi(W)$. Similarly, we obtain

$$f_\# \|V\|((\rho(X) \cap \{y : y \leq \inf \pi(W)\} \sim \pi(W)) = 0

by assuming $b = \inf \pi(W) \in \rho(X)$ and taking $J = \{y : -\infty < y \leq b\} \sim \pi(W)$.

8.11 Example. Suppose $m$ and $n$ are positive integers, $m < n$, $T \in G(n, m)$, $V = \mathcal{L}^n \times \delta T \in \mathcal{V}_m(\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{R}$ is a nonzero linear map, and $T \subset \ker f$. Then, $\delta V = 0$ by [All72, 4.8(2)] and there exists no partition of $V$ along $f$; in fact, whenever $I$ is a subinterval of $\mathbb{R}$, we verify $V \partial f^{-1}[I] = 0$ by means of [MS13, 4.12(1)], 8.1, and 8.3 whence, taking $W = V \cup f^{-1}[I] \times G(n, m)$, we infer $W \partial f^{-1}[J] = 0$ whenever $J$ is a subinterval of $\mathbb{R}$ by 5.2.

8.12 Lemma. Suppose $m$ and $n$ are positive integers with $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathcal{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $f \in T(V)$, $P$ is the family of all subintervals $I$ of $\mathbb{R}$ such that $V \partial f^{-1}[I] = 0$ and $\|V\|(f^{-1}[I]) > 0$, and $C$ is a countable disjointed subfamily of $P$ satisfying

$$\|V\|(U \sim f^{-1}[\bigcup C]) = 0$$

such that, for $I \in C$, there exists no $J$ with

$$J \subset I, \quad J \in P, \quad I \sim J \in P.

Then, $\{V \cup f^{-1}[I] \times G(n, m) : I \in C\}$ is a partition of $V$ along $f$. 

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Proof. Abbreviating $W_I = V \cup f^{-1}[I] \times G(n, m)$, we define $\Pi = \{W_I : I \in C\}$. Firstly, we see that $\Pi$ is a partition of $V$ because
\[
\|\delta V\|(f) \leq \sum_{I \in C} \|\delta W_I\|(f) = (\|\delta V\| \cup C)(f) \quad \text{for } 0 \leq f \in \mathcal{X}(U)
\]
by 5.1. Then, in view of [Men16a, 5.3], it suffices to note $V \cap f^{-1}[I \cap J] = 0$ whenever $I \in C$, $J$ is a subinterval of $\mathbb{R}$, and $W_I \cap f^{-1}[J] = 0$ by 5.2. □

8.13 Theorem. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^m$, $V \in \mathcal{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and $f \in \mathcal{T}(V)$.

Then, there exists a partition of $V$ along $f$.

Proof. Assuming $V \neq 0$, we will verify the conditions of 8.12.

First, for every positive integer $i$, we define
\[
\delta_i = \alpha(m)2^{-m-1}i^{-1-2m}, \quad \epsilon_i = 2^{-1-i^2}
\]
and let $A_i$ denote the Borel set of $a \in \mathbb{R}^n$ satisfying
\[
|a| \leq i \iff U(a, 2\epsilon_i) \subset U, \quad \Theta^m(\|V\|, a) \geq 1/i, \quad \|\delta V\| B(a, r) \leq \alpha(m)i \epsilon_i \quad \text{for } 0 < r < \epsilon_i.
\]

Clearly, we have $A_i \subset A_{i+1}$ for every positive integer $i$ and $\|V\|\big(U \cup \bigcup_{i=1}^\infty A_i\big) = 0$ by [All72, 3.5 (1a)] and [Fed69, 2.8.18, 2.9.5]. Moreover, we define
\[
P_i = P \cap \{I : f_\#(\|V\| \cup A_i)(I) > 0\}
\]
and notice that $P_i \subset P_{i+1}$ for every positive integer $i$ and $P = \bigcup_{i=1}^\infty P_i$. Next, we observe the lower bound given by
\[
f_\#(\|V\| \cup B(a, \epsilon_i))(I) \geq \delta_i
\]
whenever $i$ is a positive integer, $I \in P$, $a \in A_i$, and $\Theta^m(\|V\| \cup f^{-1}[I], a) \geq 1/i$; in fact, noting
\[
f_0^\epsilon r^{-m}\|\delta(V \cup f^{-1}[I] \times G(n, m))\| B(a, r) d\mathcal{L}^1 r \leq \Theta(m)\epsilon_i,
\]
the inequality follows from [Men16a, 4.5, 4.6]. Let $Q_i$ denote the set of $I \in P$ such that there is no $J$ satisfying
\[
J \subset I, \quad J \in P_i, \quad I \sim J \in P_i.
\]

We denote by $\Omega$ the class of all disjointed subfamilies $H$ of $P$ with $\bigcup H = \mathbb{R}$ and let $G_0 = \{R\} \in \Omega$. The previously observed lower bound implies
\[
\delta_i \text{card}(H \cap P_i) \leq \|V\|(U \cap \{x : \text{dist}(x, A_i) \leq \epsilon_i\}) < \infty
\]
whenever $H$ is a disjointed subfamily of $P$ and $i$ is a positive integer; in fact, for each $I \in H \cap P_i$, there exists $a \in A_i$ with
\[
\Theta^m(\|V\| \cup f^{-1}[I], a) = \Theta^m(\|V\|, a) \geq 1/i
\]
Therefore, there exists 

\[ \|V\| \left( \{ J^{-1}[I] \cap \{ x : \text{dist}(x, A_i) \leq \epsilon_i \} \} \right)(I) \geq f_{\#}(\|V\| \cdot B(a, \epsilon_i))(I) \geq \delta_i. \]

In particular, such \( H \) is countable.

Next, we inductively (for every positive integer \( i \)) define \( \Omega_i \) to be the class of all \( H \in \Omega \) such that every \( E \in G_{i-1} \) is the union of some subfamily of \( H \), and choose \( G_i \in \Omega_i \) such that

\[ \text{card}(G_i \cap P_i) \geq \text{card}(H \cap P_i) \quad \text{whenever } H \in \Omega_i. \]

The maximality of \( G_i \) implies \( G_i \subset Q_i \); in fact, if there existed \( I \in G_i \sim Q_i \), there would exist \( J \) satisfying

\[ J \subset I, \quad J \in P_i, \quad I \sim J \in P_i, \]

and \( H = (G_i \sim \{ J \}) \cup \{ J, I \sim J \} \) would belong to \( \Omega_i \), with

\[ \text{card}(H \cap P_i) > \text{card}(G_i \cap P_i). \]

Moreover, it is evident that, to each \( x \in \text{dmn} f \), there corresponds a sequence \( I_1, I_2, I_3, \ldots \) characterised by the conditions \( f(x) \in I_i \in G_i \), hence \( I_{i+1} \subset I_i \), for every positive integer \( i \).

We define \( G = \bigcup_{i=1}^{\infty} G_i \) and notice that \( G \) is countable. We let \( C \) denote the collection of sets \( \bigcap_{i=1}^{\infty} I_i \) with positive \( f_{\#}\|V\| \) measure corresponding to some sequence \( I_1, I_2, I_3, \ldots \) with \( I_{i+1} \subset I_i \in G_i \) for every positive integer \( i \). Clearly, \( C \) is a disjointed subfamily of \( P \), and hence \( C \) is countable. We will show that

\[ f_{\#}\|V\|(R \sim \bigcup C) = 0. \]

In view of (Fed69) 2.8.18, 2.9.11, it is sufficient to prove

\[ A_i \cap (\text{dmn} f) \sim \bigcup C \]

\[ \subset \bigcup \{ f^{-1}[I] \cap \{ x : \Theta^m(\|V\| \cdot f^{-1}[I], x)  < \Theta^m(\|V\|, x) \} : I \in C \} \]

for every positive integer \( i \). For this purpose, we consider \( a \in A_i \cap (\text{dmn} f) \sim \bigcup C \) with corresponding sequence \( I_1, I_2, I_3, \ldots \) as above. It follows that

\[ f_{\#}\|V\| \left( \bigcap_{j=1}^{\infty} I_j \right) = 0. \]

Therefore, there exists \( j \) with \( f_{\#}(\|V\| \cdot B(a, \epsilon_i))(I_j) < \delta_i \), and the lower bound implies

\[ \Theta^m(\|V\| \cdot f^{-1}[I_j], a) < 1/i \leq \Theta^m(\|V\|, a). \]

If, for some \( I \in C \), there existed \( J \) with

\[ J \subset I, \quad J \in P_i, \quad I \sim J \in P_i, \]

then there would exist intervals \( I_i \) with \( I = \bigcap_{i=1}^{\infty} I_i \) and \( I_i \in G_i \) for every positive integer \( i \). We could choose \( i \) such that \( J \in P_i \) and \( I \sim J \in P_i \). Defining \( E = f^{-1}[I_i \sim I] \) and \( W = V \cup E \times G(n, m) \), we would notice

\[ V \cap E = 0, \quad f \in T(W) \]
by \[8.3\] As \(J\) and \(I \sim J\) would be nonempty, \(I\) would have nonempty interior. Accordingly, picking an interval \(L\) in \(\mathbb{R}\) such that \(\mathbb{R} \sim L\) would also be an interval in \(\mathbb{R}\) and
\[ J = I \cap L, \]
we could employ \[\text{Men16a} 8.5, 8.30\] to deduce \(W \partial f^{-1}[L] = 0\), whence it would follow
\[ V \partial f^{-1}[I_i \cap L \sim I] = 0 \]
from \[5.2\] with \(F = f^{-1}[L]\). Finally, noting that \(I_i \cap L\) would equal the disjoint union of \(J\) and \(I_i \cap L \sim I\), we could establish
\[ V \partial f^{-1}[I_i \cap L] = 0, \quad V \partial f^{-1}[I_i \sim L] = 0, \]
\[ J \subset I_i \cap L \in P_i, \quad I \sim J \subset I_i \sim L \in P_i \]
by \[\text{Men16a} 5.3\], in contradiction to \(I_i \in Q_i\).

\[\text{8.14 Remark.}\] The novelty of the preceding proof is its last paragraph; otherwise, it rests on the machinery developed in \[\text{Men16a} 6.12\]. A refinement of that machinery in a different direction appears in \[\text{Cho23} 5.12\].

\[\text{8.15 Remark.}\] The rectifiability hypothesis may not be omitted by \[8.11\].

\[\text{8.16 Remark.}\] In view of \[5.6\] and \[5.7\], one may obtain the case \(Y = \mathbb{R}\) of the constancy theorem \[\text{Men16a} 8.34\] from \[8.13\] by means of \[7.11(4)\] and \[8.7\].

9 Criteria for local finiteness of decompositions

A first criterion is obtained in \[9.1\]–\[9.2\]; in particular, Theorem \[D\] of the introductory section is provided in \[9.2\]. A second criterion is derived in \[9.3\]–\[9.24\]: Firstly, the material of \[\text{DM21} pp.2614–2625\] is localised and streamlined in \[9.3\]–\[9.15\]; secondly, an isoperimetric lower density ratio bound is established in \[9.16\]–\[9.23\]; and, finally, Theorem \[E\] of the introductory section is provided in \[9.24\].

\[\text{9.1 Example.}\] Suppose \(m\) and \(n\) are positive integers, \(m \leq n\), \(U\) is an open subset of \(\mathbb{R}^n\), \(B\) is an \(m - 1\) dimensional submanifold of class \(2\) of \(U\) satisfying
\[ U \cap (\text{Clos } B) \sim B = \emptyset, \quad V \in V_m(U), \quad \beta = \infty \text{ if } m = 1, \quad \beta = m/(m - 1) \text{ if } m > 1, \]
\[ \sup \left\{ \langle \delta V \rangle(\theta) : \theta \in \mathcal{D}(U, \mathbb{R}^n), \text{spt } \theta \subset K \sim B, \|V\|_{\beta}(\theta) \leq 1 \right\} < \infty \]
whenever \(K\) is a compact subset of \(U\), \(\|V\|(B) = 0\), and
\[ \Theta_m(\|V\|, x) \geq 1 \text{ for } \|V\| \text{ almost all } x. \]

Then, \(V\) is rectifiable, \(\|\delta V\|\) is a Radon measure, \(h(V, \cdot) \in L^\text{loc}_m(\|V\|, \mathbb{R}^n)\), and, if \(m > 1\), then \(\text{spt}(\|\delta V\| - \|\delta V\|_{\|V\|}) \subset B\); in fact, noting that \(\|\delta V\| \cap (U \sim B)\) is a Radon measure and that \[\text{All72} 5.5(1)\] remains valid for submanifolds of class \(2\), we conclude that \(\|\delta V\|\) is a Radon measure, hence \(V\) is rectifiable by \[\text{All72} 5.5(1)\], and the remaining assertions follow from \[3.4\] and \[3.21\].

\[\text{9.2 Theorem.}\] Suppose \(V\) is as in \[9.1\].

Then, every decomposition of \(V\) is locally finite.
Proof. We take \( m, n, U, \) and \( B \) as in 9.1.

If \( m = 1 \), then the assertion is a special case of \[\text{[Men16a 6.11]} \]. To prove the conclusion in case \( m > 1 \), we will verify the hypotheses of 6.3. If \( a \in U \sim B \), then, in view of [Men09 2.5] and 5.1, we may take \( r > 0 \) with \( B(a, 2r) \subset U \sim B \) and \[\int_{B(a, 2r)} |\mathbf{h}(\mathbf{V}, \cdot)|^m d\|\mathbf{V}\| \leq 2^{-m} \gamma(m)^{-m} \].

If \( a \in B \), we firstly notice that, for instance by 3.19 there exists \( R > 0 \) with \( B(a, R) \subset U \) and

\[ |\text{Nor}(B, b, y_c(y - b))| \leq R^{-1} |y - b|^2 / 2 \]

whenever \( b, y \in B \cap U(a, R) \). Then, we pick \( 0 < r \leq R / 2 \) such that

\[ \int_{B(a, 2r)} |\mathbf{h}(\mathbf{V}, \cdot)|^m d\|\mathbf{V}\| \leq 8^{-m} \gamma(m)^{-m} \].

We will conclude the proof by showing that

\[ \|\mathbf{W}\| B(a, 2r) \geq (16m \gamma(m))^{-m} r^m. \]

whenever \( \mathbf{W} \) is a component of \( V \) and \( B(a, r) \cap \text{spt} \|\mathbf{W}\| \neq \emptyset \). For this purpose, we abbreviate \( A = \text{spt}(\|\delta \mathbf{W}\| - \|\delta \mathbf{W}\| \|\mathbf{W}\|) \) and assume

\[ \|\mathbf{W}\| B(a, 2r) < (4m \gamma(m))^{-m} r^m. \]

Noting 5.1 we infer \( A \cap B(a, 3r / 2) \neq \emptyset ; \) in fact, taking \( x \in B(a, r) \cap \text{spt} \|\mathbf{W}\| \), we have \( A \cap B(x, r / 2) \neq \emptyset \) by [Men09 2.5]. As \( A \subset B \cap \text{spt} \|\mathbf{W}\| \) by 5.1 we may hence select

\[ b \in B \cap B(a, 3r / 2) \cap \text{spt} \|\mathbf{W}\|. \]

Noting that [All75 3.4 (1)] remains valid\(^4\) when \( c_1 \) is replaced by \( \gamma(k) \) and \( B \) is required to be of class 2, instead of class \( \infty \), we apply [All75 3.4 (1)] with

\[ B, k, s, V, \alpha, \text{and } r \text{ replaced by} \]

\[ B \cap U(a, R), m, r / 4, W |2^{U(b, r / 4) \times G(n, m)}, (8 \gamma(m))^{-1}, \text{and } r / 4 \]

to obtain

\[ \|\mathbf{W}\| B(b, r / 4) \geq (16m \gamma(m))^{-m} r^m, \]

as \( R / (R - r / 4) \leq 2 \) and \( m \gamma(m)\|\mathbf{W}\| (B(b, r / 4))^{1 / m} / (R - r / 4) \leq 1 / 4. \]

\[ \square \]

9.3 Lemma. Suppose \( m \) and \( n \) are positive integers, \( m \leq n, U \) is an open subset of \( \mathbb{R}^n \), \( B \) is a relatively closed subset of \( U \), and \( H \) is the set of Lipschitzian functions \( \eta : U \to \mathbb{R}^n \) with compact support satisfying \( \eta(b) = 0 \) for \( b \in B \).

Then, the following three statements hold.

1. If \( \eta \in H \) and \( \epsilon > 0 \), then there exists \( \theta \in H \) such that

\[ \sup |\eta - \theta| \leq \epsilon, \quad \text{Lip} \theta \leq \text{Lip} \eta, \quad \text{spt} \eta \subset (\text{spt} \eta) \sim B, \]

and, if \( \eta \) is of class 1, then so is \( \theta \).

2. If \( W \in V_m(U \sim B) \) satisfies \( (\|W\| + \|\delta W\|| K \sim B) < \infty \) whenever \( K \) is a compact subset of \( U \), then, for \( \eta \in H \), we have \( h = \eta(U \sim B) \in T(W, \mathbb{R}^n) \) and

\[ \int \text{trace}(W \mathbf{D} h) d\|\mathbf{W}\| = \int \eta(W, \cdot) \cdot h d\|\delta \mathbf{W}\|. \]

\(^4\)Notice that “\( m(s)^{1/k} \)” should be replaced by “\( km(s)^{1/k} \)” in the cited statement and that “\( +m(t) \)” in line six of its proof should read “\( +km(t) \)”. 

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Remark. We record the following related basic fact: If $K$ is proper, subset of $B$ with $\sup im \{\int S_2 \bullet D \eta(x) \, dV(x, S): \eta \in H\}$ of class $1$, spt $\eta \subset K$, $|\eta| \leq 1 \} < \infty$ whenever $K$ is a compact subset of $U$, then
\[ \int_{B \times G(n, m)} S_2 \bullet D \eta(x) \, dV(x, S) = 0 \quad \text{whenever } \eta \in H \text{ is of class 1}. \]

Proof. Choosing a function $\gamma: \mathbb{R} \to \mathbb{R}$ of class 1 such that
\[ \sup \{ \int S_2 \bullet D \eta(x) \, dV(x, S): \eta \in H\} \leq \epsilon, \quad \text{Lip } \gamma \leq 1, \quad 0 \notin \text{spt } \gamma, \]
we may take $\theta = \gamma \circ \eta$ in (1) because $\text{spt } \theta \subset \eta^{-1}[\text{spt } \gamma] \subset (\text{spt } \eta) \sim B$. Combining [MS18 4.6] with (1), we reduce the proof of (2) to the case that $\text{spt } \eta \subset U \sim B$ which follows from (3.21). Next, under the hypotheses of (3), we suppose $\eta \in H$ is of class 1. The special case $\text{spt } \eta \subset U \sim B$ is trivial. To treat the general case, we define $W = V[2(U \sim B) \times G(n, m)] \in \mathcal{V}_m(U \sim B)$, note
\[ \int_{B \times G(n, m)} S_2 \bullet D \eta(x) \, dV(x, S) = \int S_2 \bullet D \eta(x) \, dV(x, S) - \int \eta(W_i) \times \eta d||\delta W|| \]
by (2) in conjunction with [MS18 4.1, 4.5], and use the fact that, by (1), there exist $\theta_1, \theta_2, \theta_3, \ldots$ in $H$ of class 1 such that $\text{spt } \eta \subset (\text{spt } \eta) \sim B$ for every positive integer $i$ and $\lim_{i \to \infty} \sup \text{im } \eta - \theta_i = 0$.

9.4 Remark. If $||V||/(B) = 0$ and $W = V[2(U \sim B) \times G(n, m)]$, then the hypotheses of (2) and (3) are equivalent by (3.21). In general, they differ as examples with card $B = 1$ and card spt $V = 1$ show.

9.5 Theorem. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $B$ is a submanifold of class 2 in $U$, the inclusion map $i: B \to U$ is proper, $V \in \mathcal{V}_m(U)$, and
\[ \sup \{ \int S_2 \bullet D \eta(x) \, dV(x, S): \eta \in H\} \leq 1 \} < \infty \]
whenever $K$ is a compact subset of $U$, where $H$ is as in (9.3).

Then, there holds $V \cup B \times G(n, m) \in i_\#[\mathcal{V}_m(B)]$.

Proof. In view of (9.3, 9), we may proceed as in [All72 4.6 (1) (2)].

9.6 Remark. The case dim $B = n - 1$ was treated, under slightly more restrictive hypotheses, in [DM21 Lemma 3.1]; the case dim $B \neq n - 1$ is new.

9.7 Remark. If $||V||$ is a Radon measure (see (9.13 below) and $m = n - 1$, then $V \cup B \times G(n, m)$ is rectifiable by [All72 5.1 (4)] and [MS22 3.25].

9.8 Remark. We record the following related basic fact: If $\theta: U \to \mathbb{R}^n$ is a function of class 1 with compact support and $\theta(b) \in \text{Nor}(B, b)$ for $b \in B$, then
\[ S_2 \bullet D \theta(b) = -h(B, b, S) \bullet \theta(b) \quad \text{for } (b, S) \in G_m(B). \]

9.9 Example. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $M$ is a relatively closed subset of $U$ and an $n$ dimensional submanifold-with-boundary of class 2 in $U$, $B = \partial M$, the open subset $X$ of $\mathbb{R}^n$ is associated with $B$ as in (3.14) $\delta: X \to \mathbb{R}$ satisfies
\[ \delta(x) = \text{dist}(x, B) \text{ if } x \in M, \quad \delta(x) = -\text{dist}(x, B) \text{ if } x \notin M. \]
whenever $x \in X$; hence, $B \subset X$, $\delta$ is of class 2, $|\operatorname{grad} \delta(x)| = 1$ for $x \in X$, and
\[
\operatorname{grad} \delta(b) = -\mathbf{n}(M, b) \quad \text{for } b \in B
\]
by [Fed59 4.8 (3)], [Fed69 3.1.19 (5)], [Fed69 3.1.12] and [Fed69 3.1.14].

Suppose $W \in V_m(U \cap X \sim B)$, $\operatorname{spt} \|W\| \subset M$, and
\[
(\|W\| + \|\delta W\|)(K \sim B) < \infty \quad \text{whenever } K \text{ is a compact subset of } U \cap X.
\]

Let $f = \delta((U \cap X \sim B)$, note $f \in T(W)$ and $\|W D f\| \leq 1$ by [MS18 4.6 (1)], and abbreviate
\[
E(y) = \{x : f(x) > y\} \quad \text{for } y \in \mathbb{R}.
\]

Whenever $\zeta : U \cap X \to \mathbb{R}$ is a Lipschitzian function with compact support, we define a right-continuous function $g_\zeta : \{y : 0 \leq y < \infty\} \to \mathbb{R}$ by
\[
g_\zeta(y) = \int_{E(y)}(\eta(W, \cdot) \cdot \operatorname{grad} f)\zeta \|\delta W\| - \int_{E(y)}\operatorname{trace}(W D(\zeta \operatorname{grad} f))\|W\|
\]
for $0 \leq y < \infty$ with the help of [MS18 4.6 (1)], hence
\[
g_\zeta(0) = \int(\eta(W, \cdot) \cdot \operatorname{grad} f)\zeta \|\delta W\| - \int \operatorname{trace}(W D(\zeta \operatorname{grad} f))\|W\|
\]
we also let $Q_\zeta \in \mathcal{D}'(\mathbb{R}, \mathbb{R})$ be defined by $Q_\zeta(\omega) = \int_0^\infty g_\zeta \omega dL^1$ for $\omega \in \mathcal{D}'(\mathbb{R}, \mathbb{R})$. Finally, we define $T \in \mathcal{D}'(U \cap X, \mathbb{R})$ by $T(\zeta) = g_\zeta(0)$ for $\zeta \in \mathcal{D}(U \cap X, \mathbb{R})$.

9.10 Remark. We record that
\[
|\operatorname{trace}(h \circ S)| \leq m \|h\| \quad \text{for } h \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \text{ and } S \in \text{G}(n, m).
\]

9.11 Lemma. Suppose $m$, $U$, $B$, $X$, $W$, $f$, $g_\zeta$, $Q_\zeta$, and $T$ are as in 9.7.

Then, the following two statements hold.

(1) The distribution $T$ is representable by integration, $\operatorname{spt} T \subset B$,
\[
T(\zeta) = \int \zeta d\|T\| \quad \text{for } \zeta \in \mathbf{L}_1(\|T\|).
\]

(2) If $\zeta : U \cap X \to \mathbb{R}$ is a nonnegative Lipschitzian function with compact support, then $\alpha = \zeta((U \cap X \sim B) \in T(W)$ and
\[
\int \alpha d\|T\| \leq \int \alpha d\|\delta W\| + \int (\|W D \alpha\| + m\alpha\|D^2 f\|) d\|W\|.
\]

(3) Suppose $\zeta : U \cap X \to \mathbb{R}$ is Lipschitzian with compact support. Then,
\[
Q_\zeta(\omega) = \int \zeta(\operatorname{grad} f, W D f)(\omega \circ f) d\|W\|,
\]
\[
D_1 Q_\zeta(\omega) = g_\zeta(0) \omega(0) - \int \zeta(\eta(W, \cdot) \cdot \operatorname{grad} f)(\omega \circ f) d\|\delta W\|
\]
\[
+ \int (\operatorname{grad} f, W D \zeta)(\omega \circ f) d\|W\|
\]
\[
+ \int \zeta \operatorname{trace}(W D(\operatorname{grad} f))(\omega \circ f) d\|W\|
\]

whenever $\omega \in \mathcal{D}'(\mathbb{R}, \mathbb{R})$; in particular,
\[
\|Q_\zeta\| \leq f_#(\|W\| \circ |\zeta W D f|),
\]
\[
\|D_1 Q_\zeta\| \leq |g_\zeta(0)| \delta_0 + f_#(\|\delta W\| \circ |\zeta| + \|W\| \circ (|W D \zeta| + m\|D^2 f\|)),
\]
and, if $\zeta \geq 0$, then $g_\zeta \geq 0$. Moreover, there holds
\[
\int \zeta d\|T\| = g_\zeta(0).
\]
Proof. We suppose $\zeta : U \cap X \to \mathbb{R}$ is Lipschitzian with compact support.

Noting $B \cap \text{Clos} E(y) = \emptyset$ for $0 < y < \infty$, we apply [3.22] and [3.24] with $U$, $V$, and $\theta$ replaced by $U \cap X \sim B$, $W$, and $\zeta \text{ grad } f$ to conclude
\[
g_{\zeta}(y) = W \partial E(y)(\zeta \text{ grad } f)
= \int \zeta(x)(\text{grad } f(x), |W D f(x)|^{-1}W D f(x)) \, d\|W \partial E(y)\|_x
\]
for $L^1$ almost all $0 \leq y < \infty$; in particular,
\[
g_{\zeta}(0) = 0 \quad \text{in case } B \cap \text{spt } \zeta = \emptyset
\]
because $\text{spt } W \partial E(y) \subset \{x : f(x) = y\}$ for $y \in \mathbb{R}$. Noting [MS18 4.11], we apply [Men16a 8.5, 8.30] for each $\omega \in \mathcal{D}(\mathbb{R}, \mathbb{R})$ with $V$ and $g(x, y)$ replaced by $W$ and $i(y)\omega(y)\zeta(x)(\text{grad } f(x), |W D f(x)|^{-1}W D f(x))$, where $i$ is the characteristic function of $\{y : 0 \leq y < \infty\}$ on $\mathbb{R}$, to infer the first equation in (3). In case $\zeta \geq 0$, we deduce $Q_{\zeta}(\omega) \geq 0$ whenever $0 \leq \omega \in \mathcal{D}(\mathbb{R}, \mathbb{R})$, as
\[
\langle \text{grad } f(x), W D f(x) \rangle = \int |S_\zeta(\text{grad } f(x))|^2 \, dW(x) \, S \geq 0
\]
for $\|W\|$ almost all $x$ by [All72 2.3 (1)] and [MS18 4.1, 4.5]. By [3.3] with $U$, $Y$, $\phi$, $k$, and $T$ replaced by $\mathbb{R}$, $\mathbb{R}$, $L^1$, $g$, and $Q_{\zeta}$, where $\mathbb{R}^* \approx \mathbb{R}$, it follows that
\[
g_{\zeta} \geq 0 \quad \text{in case } \zeta \geq 0.
\]

By means of [Fed69 4.1.5], the preceding considerations readily yield (1) and the final equation in (3) which in turn imply (2) by [Men16a 8.20 (4)], [MS18 4.6 (1), 4.11], and [9.10].

To prove the second equation of (3), we additionally suppose $\omega \in \mathcal{D}(\mathbb{R}, \mathbb{R})$ and take $\theta = \zeta(\omega \circ f) \text{ grad } f$. Noting [MS18 4.6 (1), 4.11], we compute
\[
\begin{align*}
\text{trace}(W D \theta(x)) &= \langle \text{grad } f(x), W D \zeta(x)\omega(f(x)) \rangle \\
&= \langle \zeta(x)(\text{grad } f(x), W D f(x))\omega'(f(x)) \rangle \\
&= \langle \zeta(x) \text{ trace}(W D(\text{grad } f(x)))\omega(f(x)) \rangle
\end{align*}
\]
for $\|W\|$ almost all $x$ by [Men16a 8.20 (4)]. Thus, the first equation of (3) yields
\[
\begin{align*}
- Q_{\zeta}(\omega') &= - \int \text{trace}(W D \theta) \, d\|W\| + \int \langle \text{grad } f, W D \zeta(\omega \circ f) \rangle \, d\|W\| \\
&= \int \langle \zeta \text{ trace}(W D(\text{grad } f))\omega \circ f) \rangle \, d\|W\|.
\end{align*}
\]
Taking $\delta$ as in [9.9], hence $\delta[\text{spt } T] \subset \{0\}$ by (1), the final equation in (3) shows
\[
\int \eta(W, \cdot) \cdot \theta \, d\|\delta W\| - \int \text{trace}(W D \theta) \, d\|W\| = \eta(\omega \circ \delta)(0) = g_{\zeta}(0)\omega(0)
\]
and the second equation of (3) follows. Taking [MS18 4.1, 4.5] and [9.10] into account, the remaining assertions of (3) may now be verified by means of [Fed69 2.4.18 (1)] and [MS22 3.8, 3.11].

9.12 Theorem. Suppose that $n$, $U$, $M$, $B$, $X$, $W$, and $T$ are as in [9.9] that $\theta : U \cap X \to \mathbb{R}^n$ is a Lipschitzian function with compact support satisfying $\theta(b) \in \text{Nor}(B, b)$ for $b \in B$, and that $h = \theta(U \sim B)$.

Then, $h \in T(W, \mathbb{R}^n)$ and there holds
\[
\int \text{trace}(W D h) \, d\|W\| = \int \eta(W, \cdot) \cdot \theta \, d\|\delta W\| + \int \eta(M, \cdot) \cdot \theta \, d\|T\|.
\]
Proof. We take $\delta$ and $g_{\varepsilon}$ as in [9.9 and recall [MS18 4.6(1)]. By additivity, see [Men16a 8.20(3)] and [MS18 4.11], it suffices to consider two cases, namely
\[
\theta = (\varepsilon \cdot \text{grad } \delta) \text{ grad } \delta \text{ and } \theta \cdot \text{grad } \delta = 0.
\]
In the first case, we take $\zeta = \theta \cdot \text{grad } \delta$ and note that
\[
g_{\varepsilon}(0) = \int \zeta d\|T\| = -\int n(M, \cdot) \cdot \theta d\|T\|
\]
by [9.11] which yields the conclusion in that case. The second case implies $\theta(b) = 0$
for $b \in B$ and thus follows from [9.3.2] applied with $\eta$ replaced by $U \cap X$
and $\theta$ in conjunction with [9.11][4].

9.13 Example. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an
open subset of $R^n$, $M$ is a relatively closed subset of $U$ and an $n$
dimensional submanifold-with-boundary of class 2, $B = \partial M$, $X$ is associated with $B$ as in
[3.14] and $V \in V_{m}(U)$ satisfies $spt \|V\| \subset M$ and
\[
\sup\left\{ \int S_{2} \cdot D \theta(x) dV(x, S) : \theta \in \Theta, spt \theta \subset K, |\theta| \leq 1 \right\} < \infty
\]
whenever $K$ is a compact subset of $U$, where $\Theta$ is the vector space of functions
\[
\theta : U \rightarrow R^n \text{ of class 1 with compact support and } \theta(b) \in \text{tan}(B, b) \text{ for } b \in B.
\]
Then,
\[
W = V|2^{U \cap X \sim B} \times G(n, m) \in V_{m}(U \cap X \sim B)
\]
satisfies the conditions of [9.9] and, defining $T \in \mathcal{D}(U \cap X, R)$ by
\[
T(\zeta) = \int (\eta(W, \cdot) \cdot \text{grad } f) \zeta d\|W\| - \int \text{trace}(W D(\text{grad } f)) d\|W\|
\]
for $\zeta \in \mathcal{D}(U \cap X, R)$, where $f : U \cap X \sim B \rightarrow R$ is given by
\[
f(x) = \text{dist}(x, B) \text{ if } x \in M, \quad f(x) = -\text{dist}(x, B) \text{ if } x \notin M
\]
for $x \in U \cap X \sim B$, we conclude that $\|\delta V\|$ is a Radon measure and
\[
\int_{B} \eta(V, b) \cdot \text{Nor}(B, b) \|\delta V\| b
\]
\[
= \int n(M, x) \cdot \theta(x) d\|T\| - \int_{G_{m}(b)} h(B, b, S) \cdot \theta(b) dV(b, S)
\]
whenever $\theta : U \rightarrow R^n$ is of class 1 with compact support; in fact, recalling [9.11][4],
it suffices to note that if such $\theta$ satisfies $spt \theta \subset U \cap X$ and $\theta(b) \in \text{nor}(B, b)$
for $b \in B$, then combining [9.5, 9.3, 9.12] and [MS18 4.1, 4.5] yields
\[
\int S_{2} \cdot D \theta(x) dV(x, S) = -\int_{G_{m}(b)} h(B, b, S) \cdot \theta(b) dV(b, S)
\]
\[
+ \int \eta(W, x) \cdot \theta(x) d\|\delta W\| x + \int n(M, x) \cdot \theta(x) d\|T\| x.
\]

With the techniques of [Fed69 2.5.14], we construct a Radon measure $\psi$ over
$U$ such that
\[
\psi(G) = \sup \{|\delta V| \theta : \theta \in \Theta, spt \theta \subset G, |\theta| \leq 1\}
\]
whenever $G$ is an open subset of $U$. Since the vector subspace $\Theta$ is $\|\delta V\|_{(L^1)}$
dense in $L_{1}(\|\delta V\|, R^n) \cap \{\theta : \text{dmm } \theta = U, \theta(b) \in \text{tan}(B, b) \text{ for } b \in B\}$ by [3.6] we conclude
\[
\psi = \|\delta V \|\|\eta(V, \cdot, T)|
\]
by means of [MS22 3.8], [3.7] and [5.21] where $\tau : U \rightarrow \text{Hom}(R^n, R^n)$ is defined by
\[
\tau(x) = 1_{R^n} \text{ for } x \in U \sim B, \quad \tau(x) = \text{tan}(B, x) \text{ for } x \in B.
\]
Whenever \(a \in \mathbb{R}^n\), \(0 < r < \infty\), and \(B(a, 2r) \subset U \cap X\), we will estimate

\[
\|\delta V\| \mathcal{B}(a, r) \leq (\psi + \|T\| + r^{-1} m\lambda \|V\|) \mathcal{B}(a, r)
\leq (2\psi + r^{-1}(1 + 2m\lambda)\|V\|) \mathcal{B}(a, 2r),
\]

where \(\lambda = r \sup \|D^2 f\|\mathcal{B}(a, 2r)\); in fact, for the first inequality, it suffices to note \(\|b(B, b)\| = \|D^2 f(b)\|\) for \(b \in B\) and recall \(9.11(1)\). whereas, for the second inequality, in view of \([\text{All72}] 4.6(1)\], we may apply \(9.11(2)\) with

\[
\alpha(x) = \sup\{0, 1 - \text{dist}(x, \mathcal{B}(a, r))/r\} \quad \text{for} \quad x \in U \cap X \sim B.
\]

Finally, if \(E\) is \(\|V\| + \|\delta V\|\) measurable, \(V \partial E = 0\), and \(V' = V \cup E \times G(n, m)\), then \(\delta V' = (\delta V) \cup E\) yields that, whenever \(K\) is a compact subset of \(U\), we have

\[
\sup\{\int_{S} S_{\mathcal{B}} \bullet \tau(x) dV'(x, S) : \theta \in \Theta, \text{spt} \theta \subset K, |\theta| \leq 1\} < \infty
\]

and \(\psi' = \psi \land E\) by \(3.21\) and \(5.1\) where \(\psi'\) is associated with \(V'\) as \(\psi\) with \(V\).

9.14 Remark. If \(\Theta^{\mathcal{B}}(\|V\|. x) \geq 1\) for \(\|V\|\) almost all \(x\), then \(V\) is rectifiable by \([\text{All72}] 5.5(1)\).

9.15 Remark. Based on \(3.14\) and \(3.16\) the development in \(9.3[9.14]\) localises the results of \([\text{DM21}]\) Section 3, Subsections 4.1–4.3. By fully employing the machinery of \([\text{Fed59}]\), \([\text{All72}]\), \([\text{All75}]\), \([\text{Men16a}]\), and \([\text{MS18}]\)—in particular, \([\text{Fed59}] 4.8\), \([\text{All72}] 2.5, 4.6\), \([\text{All75}] 2.2\), \([\text{Men16a}] 8.5, 8.30\), and \([\text{MS18}] 4.6, 4.11\)—and re-organising the material, we intend to provide a deeper understanding of the long computations in \([\text{DM21}]\) pp. 2614–2625: Namely, \([\text{DM21}]\) Lemma 3.1 is a special case of \(9.5\). the content of \([\text{DM21}]\) Theorem 1.1, Theorem 4.1, and Corollary 4.2] was split and, recalling \([\text{Fed59}] 4.8(3)\), it is implied by \(9.5, 9.8, 9.11(1)\), and \(9.12(1)\) for \([\text{DM21}]\) Corollary 4.3], the same holds when \(9.7\) and \(9.13\) are taken into account; for \([\text{DM21}]\) Corollaries 4.4 and 4.6], one similarly takes note of \(9.4\); finally, \([\text{DM21}]\) Corollary 4.7 is contained in \(9.11(1)\) and \(9.13\). The statements on the distributional derivative of \(Q_{\xi}\) in \(9.11(\xi)\) are new.

9.16 Example. Suppose \(m\) and \(n\) are integers, \(1 \leq m \leq n\), \(U\) is an open subset of \(\mathbb{R}^n\), \(M\) is a relatively closed subset of \(U\) and \(n\) dimensional submanifold-with-boundary of class 2, \(B = \partial M, V \in \mathcal{V}_m(U)\), \(\text{spt} \|V\| \subset M\), \(\Theta\) consists of all \(\theta : U \to \mathbb{R}^n\) of class 1 with compact support and \(\theta(b) \in \text{Tan}(B, b)\) for \(b \in B\), \(1 < p \leq \infty\), \(q = 1\) if \(p = \infty\), \(q = p/(p-1)\) if \(p < \infty\). Then, we will prove that the following two conditions are equivalent and imply those of \(9.13\):

1. If \(K\) is a compact subset of \(U\), then

\[
\sup\{\int_{S} S_{\mathcal{B}} \bullet \theta(x) dV'(x, S) : \theta \in \Theta, \text{spt} \theta \subset K, \|V\|_{(q)}(\theta) \leq 1\} < \infty.
\]

2. The measure \(\|\delta V\|\) is a Radon measure, \(\langle h(V, \cdot), \tau \rangle \in \mathcal{L}^{\text{loc}}(\|V\|, \mathbb{R}^n)\), and

\[
\int_{S} S_{\mathcal{B}} \bullet \tau(x) dV'(x, S) = -\int h(V, x) \bullet \theta(x) d\|V\| x \quad \text{whenever} \ \theta \in \Theta,
\]

where \(\tau : U \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)\) is defined by

\[
\tau(x) = 1_{B^c} \quad \text{for} \ x \in U \sim B, \quad \tau(x) = \text{Tan}(B, x)_3 \quad \text{for} \ x \in B.
\]

\(\text{In} \ [\text{DM21}] \text{Theorems 1.1 and 4.1], to obtain assertions supported by the proofs provided, second fundamental form should be replaced by reach regarding the dependence of } c(\#). \)
Clearly, (2) implies (1) and (1) implies the conditions of 9.13. Thus, if (1) holds, then \( \|\delta V\| \) is a Radon measure by 9.13 and (2) follows by 3.6 as the function \( \langle h(V, \cdot), \tau \rangle \) is \( \|V\| \) almost characterised by the fact that, for \( \|V\| \) almost all \( x \),

\[
-\frac{h(V,x) \bullet \theta(x)}{\|V\| B(x,r)} = \lim_{r \to 0^+} \frac{\langle \delta V, (b_x, \theta) \rangle}{\|V\| B(x,r)} \quad \text{whenever } \theta \in \Theta,
\]

where \( b_x \) is the characteristic function of \( B(x,r) \) on \( U \), which is true since \( \theta \) is continuous and \( \lim_{r \to 0^+} \frac{\langle \delta V, (b_x, \theta) \rangle}{\|V\| B(x,r)} < \infty \) for \( \|V\| \) almost all \( x \) by \[Fed69\] 2.8.18, 2.9.5. Taking \( \psi \) as in 9.13 the preceding conditions imply

\[
(\|\delta V\| - \|\delta V\|_{V\|}) \cdot |\langle \nabla(V, \cdot), \tau \rangle| = 0, \quad \psi = \|V\|_{\langle h(V, \cdot), \tau \rangle};
\]

in fact, the first equation follows from \[Fed69\] 2.9.2, as \( \psi \) is absolutely continuous with respect to \( \|V\| \), and entails the second equation by \[MS22\] 3.8 and 3.21.

Finally, if \( E \) is \( \|V\| + \|\delta V\| \) measurable, \( V \ominus E = 0 \), and \( W = V \ominus E \times G(n, m) \), then we employ \( \delta W = (\delta V) \ominus E \) in conjunction with 3.21 and 5.1 to verify

\[
\sup \{ \int_{S} \Theta(x) \cdot d\mathcal{W}(x,S) : \Theta \in \Theta, \ spt \Theta \subset K, \ \|W\|_{\partial E}(\Theta) \leq 1 \} < \infty
\]

whenever \( K \) is a compact subset of \( U \).

9.17 Remark. By 6.14 whenever \( N \) is an \( m \)-dimensional manifold-with-boundary of class 2, \( F : N \to U \) is a proper immersion of class 2, \( F[N] \subset M \), and \( V \) is associated with \( (F,U) \), the varifold \( V \) satisfies the conditions of 9.16 with \( p = \infty \) if

\[
n(F,c) \in \operatorname{Nor}(M, F(c)) \quad \text{for } c \in \partial N;
\]

in case \( F[\partial N] \) is an embedding, the condition on the exterior normal is necessary whenever \( 1 < p \leq \infty \). The special case that \( U = \mathbb{R}^n \), \( M \) compact, \( N \subset M \), \( F = 1_N \), and \( h(F, \cdot) = 0 \), pertaining to the constrained free boundary problem, is of particular importance; a general existence theorem of such \( N \) was obtained in case \( 2 \leq m \leq 6 \), \( n = m + 1 \), and \( M \) of class \( \infty \) in \[LZ21\] Theorem 1.1. In the classical subcase that \( n = 3 \) and \( M = B(0,1) \), existence of \( N \) with prescribed number of boundary components and genus 0 or with prescribed genus and connected boundary was treated in \[FS16\] Theorem 1.6 and \[CFS22\] Theorem 1.1, respectively.

9.18 Remark. If \( E \) is an \( \mathcal{H}^m \) measurable subset of \( M \) which meets every compact subset of \( U \) in an \( \mathcal{H}^m \), \( m \) rectifiable set, and

\[
\mathcal{H}^m(E \cap \partial \Theta) \leq \mathcal{H}^m(\phi(t, \cdot)|E) \cap \partial \Theta) \quad \text{for } t \in \mathbb{R} \text{ and } \theta \in \Theta,
\]

where \( \phi \) is the flow associated with \( \theta \), then \( V = v_m(E) \in \mathcal{R}v_m(U) \) satisfies the conditions of 9.13 and of 9.16 with \( \psi = 0 \) by \[MS22\] 3.27. According to \[Dav19\] Theorem 5.16, our treatment therefore locally applies to minimal sets in \( U \) with sliding boundary conditions given by \( M \) and \( B \) in the sense of \[Dav19\]. In the special case \( m = 2 \), \( n = 3 \), \( U = \mathbb{R}^n \), and \( M \) compact, an existence result for such sets is established in \[Fan21\] Theorem 8.1. Adapting the terminology of \[FK18\] Section 12 and noting \[Lab22\] Lemma 2.1.2, we obtain the following proposition: If \( M \) is a compact \( n \)-dimensional submanifold-with-boundary of class 2 of \( \mathbb{R}^n \), \( B = \partial M \), \( G \) is a commutative group, \( L \) is a subgroup of the
$$(m - 1)$-th Čech homology group of $B$, $\mathcal{C}(B, L, G)$ denotes the family of closed subsets of $\mathbb{R}^n$ spanning $L$, $M \supset E \in \mathcal{C}(B, L, G)$,

$$\mathcal{H}^m(E \sim B) = \inf\{\mathcal{H}^m(F \sim B) : M \supset F \in \mathcal{C}(B, L, G)\},$$

and $E \sim B$ is $(\mathcal{H}^m, m)$ rectifiable, then $E \sim B$ satisfies the preceding conditions; for given $(M, n, G, L, m)$ such that the infimum is finite, the existence of such a set $E$ follows from [FK18] Theorem 3.20 with $U$ replaced by $\mathbb{R}^n \sim B$. The preceding proposition remains valid when “$\sim B$” is omitted both in its hypotheses and its conclusion; however, there is no known existence resulting in this case. Finally, it is not clear whether such sets $E$ must be minimal in $\mathbb{R}^n$ with sliding boundary conditions given by $M$ and $B$, see [Dev14] Remark 7.7.

9.19 Remark. Two related classes of curvature varifolds with boundary and $p$-th power summable weak second fundamental form in the sense of [Man96] occur in [KM22] Theorems 4 and 5; the first one satisfies the conditions of 9.13 and, if $p > 1$, of 9.16 whereas the second one satisfies 9.14 and 9.16 with $p = 2$.

9.20 Remark. For $\mathcal{L}^1$ almost all times, rectifiable varifolds satisfying the conditions of 9.16 with $p = 2$ occur in Brakke flow with free boundary, see [MT15] Theorems 2.1 and 2.5 jointly with [MT16], [Kag19] Theorems 3.3, 3.5, and 3.6 (A1)], and [Ldc20] Theorem 9.1. Such varifolds also appear in level set mean curvature flow with Neumann boundary conditions, see [Aim23] Theorem 2.6.

9.21 Lemma. Suppose $m$, $r$, and $\gamma$ are positive real numbers, $0 < \delta \leq (8m\gamma)^{-1}$, $m \geq 1$, $g : \{s : r/4 \leq s \leq r\} \rightarrow \mathbb{R}$ is a nondecreasing function, $g(r/4) \geq (\delta r/4)^m$, and

$$g(s)^{1-1/m} \leq 2^{-2m-2}\delta^{-1}s^{-1}g(2s) + 2^{-2m}g(2s)^{1-1/m} + \gamma g'(s)$$

for $\mathcal{L}^1$ almost all $s$ with $r/4 \leq s \leq r/2$.

Then, $g(r) \geq \delta^m r^m$.

Proof. If the lemma were false, noting that, for $r/4 \leq s \leq r/2$, we would have

$$g(2s) \leq \delta^m r^m \leq 2^{2m}\inf\{g(s), \delta^m s^m\},$$

$$2^{-2m-2}\delta^{-1}s^{-1}g(2s) + 2^{-2m}g(2s)^{1-1/m} \leq 2^{-2m}g(2s)^{1-1/m} \leq 2^{-1}g(s)^{1-1/m},$$

we would conclude $(2m\gamma)^{-1} \leq (g'^{1/m})'(s)$ for $\mathcal{L}^1$ almost all $s$ with $r/4 \leq s \leq r/2$ so that

$$(8m\gamma)^{-1}r \leq \int_{r/4}^{r/2} (g'^{1/m})' d\mathcal{L}^1 \leq g(r)^{1/m} < \delta r$$

by [Fed99] 2.9.19, in contradiction to $\delta \leq (8m\gamma)^{-1}$. $\square$

9.22 Theorem. Suppose $m$, $U$, $X$, and $f$ are as in 9.13, $G$ is an open subset of $U \cap X$, $\kappa = \sup \|D^2 f\| |G| < \infty$,

1. if $m = 1$, then $V$ and $\tau$ are as in 9.13 and

$$\int_G |(\eta(V, \cdot), \tau)| d\|V\| \leq 2^{-3}\gamma(1)^{-1},$$

2. if $m \geq 2$, then $V$ and $\tau$ are as in 9.16 and

$$\left( \int_G |(h(V, \cdot), \tau)|^m d\|V\| \right)^{1/m} \leq 2^{-2m-1}\gamma(m)^{-1},$$
and $Θ^m(\|V\|, x) ≥ 1$ for $\|V\|$ almost all $x \in G$.

Then, there holds

$$\|V\|B(a, r) ≥ δ^m r^m,$$

where $δ = 1/(2^{m+2} γ(m)(1 + 2mk))$, whenever $a \in \text{spt} \|V\|$, $0 < r < \infty$, and $B(a, r) \subset G$.

**Proof.** Taking $n$ as in 9.16 $V \subset G \times \mathcal{G}(n, m) \in \mathcal{R}_m(U)$ by 9.14. It suffices to consider $a$ with $Θ^m(\|V\|, a) ≥ 1$. We define $g : \{s : 0 < s ≤ r\} → \mathbb{R}$ by

$$g(s) = \|V\|B(a, s) \quad \text{for } 0 < s ≤ r.$$  

The isoperimetric inequality applied to $ι_#(V \subset B(a, s) \times \mathcal{G}(n, m)) \in \mathcal{R}_m(\mathbb{R}^n)$, where $ι : U → \mathbb{R}^n$ is the inclusion map, in conjunction with [Men16a, 8.7, 8.29], yields

$$g(s)^{1−1/m} ≤ γ(m)(\|δV\|B(a, s) + g(s))$$

for $\mathcal{L}^1$ almost all $s$ with $0 < s ≤ r$. Since

$$\|δV\|B(a, s) ≤ 2^{−2m} γ(m)^{1−1/m} g(2s)^{1−1/m} + s^{−1}(1 + 2mk) g(2s),$$

for $0 < s ≤ r/2$ by 9.13 and Hölder’s inequality, we infer

$$g(s)^{1−1/m} ≤ 2^{−2m−2} δ^{−1} s^{−1} g(2s) + 2^{−2m} g(2s)^{1−1/m} + γ(m) g(s)$$

for $\mathcal{L}^1$ almost all $s$ with $0 < s ≤ r/2$. Recalling $γ(m) ≥ α(m)^{−1/m}/m$ from [Men99, 2.4], we observe that $δ^m < α(m)$. Applying 9.21 with $γ = γ(m)$ and $r$ replaced by $2^{−r}$ for nonnegative integers $i$ then yields the conclusion. 

**9.23 Remark.** The preceding theorem adapts [All72, 8.3] to the present setting. A possible strengthening analogous to [Men09, 2.5] remains open though.

**9.24 Corollary.** Suppose $m$ and $V$ satisfy the equivalent conditions of 9.16 with $p = m ≥ 2$ and $Θ^m(\|V\|, x) ≥ 1$ for $\|V\|$ almost all $x$.

Then, every decomposition of $V$ is locally finite.

**Proof.** Recalling $\|δV\|$ is a Radon measure by 9.16, that $V$ is rectifiable by 9.14 we verify the condition of 5.4. Noting 5.1 and taking $U$ and $B$ as in 9.16 this follows from [Men99, 2.5] for $a ∈ U \sim B$ and from 9.22 for $a ∈ B$.

**9.25 Remark.** By [Men16a, 6.11], the conclusion also holds if $m = 1$, $V$ satisfies the conditions of 9.13 and $Θ^1(\|V\|, x) ≥ 1$ for $\|V\|$ almost all $x$.

10 Criteria for indecomposability with respect to a family of generalised weakly differentiable real valued functions

Firstly, in 10.1–10.8 we treat varifolds associated with immersions. This includes a constancy theorem in 10.5 and Theorem [F] of the introductory section in 10.7.

Secondly, in 10.9–10.11 we proceed to varifolds associated with integral chains with coefficients in a complete normed commutative group; in particular, we provide Theorem [G] of the introductory section in 10.9. Finally, we study several purely varifold-geometric settings in 10.12–10.23 and derive Theorem [I] and Corollaries [1, 2] and [3] of the introductory section in 10.16–10.20 and 10.22 respectively.
10.1 Definition. Suppose the pair \((M, g)\) consists of a connected manifold-with-boundary \(M\) of class 1 and a Riemannian metric \(g\) on \(M\) of class 0.

Then, the \textit{Riemannian distance} \(\sigma\) of \((M, g)\) is the function on \(M \times M\) whose value at \((c, z) \in M \times M\) equals the infimum of the set of numbers

\[
\int_{K}^{}((C'(y), C'(y)), g(C(y)))^{1/2} \, dL^1\,y
\]

corresponding to all locally Lipschitzian\footnote{That is, \(C\) is continuous and \(\phi \circ C\) is locally Lipschitzian whenever \(\phi\) is a chart of \(M\) of class 1. A posteriori, this is equivalent to \(C\) being locally Lipschitzian with respect to \(\sigma\).} functions \(C\) mapping some compact interval \(K\) into \(M\) with \(C(\inf K) = c\) and \(C(\sup K) = z\).

10.2 Remark. In our development, \(g\) is always induced by an immersion into \(\mathbb{R}^n\).

10.3 Lemma. Suppose \(m\) and \(n\) are positive integers, \(m \leq n\), \(M\) is a connected \(m\)-dimensional manifold-with-boundary of class 1, \(F: M \to \mathbb{R}^n\) is an immersion of class 1, \(g\) is the Riemannian metric on \(M\) induced by \(F\) from \(\mathbb{R}^n\), and \(\sigma\) is the Riemannian distance associated with \((M, g)\).

Then, the function \(\sigma\) is a metric on \(M\) inducing the given topology on \(M\) and \(\mathcal{F}_\# \mathcal{H}_g = \mathcal{H}_k \cap N(F, \cdot)\) whenever \(0 \leq k < \infty\).

\textit{Proof.}\quad We first verify that one may reduce the statement to the case that \(F\) is an embedding, hence to case that \(M \subset \mathbb{R}^n\) and \(F = 1_M\). Clearly, \(|z - c| \leq \sigma(z, \zeta)\) for \(z, \zeta \in M\), hence \(\mathcal{H}_k^m(S) \leq \mathcal{H}_k^m(S)\) for \(S \subset M\). On the other hand, given \(1 < \lambda < \infty\) and \(c \in M\), there exists \(\delta > 0\) such that

\[
\sigma(z, \zeta) \leq \lambda|z - \zeta|\quad \text{whenever } z, \zeta \in M \cap B(c, \delta);
\]

in fact, we observe that it is sufficient to note that the chart \(\psi\) of \(\mathbb{R}^n\) of class 1 occurring in the definition of submanifold-with-boundary of \(\mathbb{R}^n\) of class 1 (see [Hir94, p.30]) may be required to satisfy \(D(\psi(c) = 1_{\mathbb{R}^n}\). This in particular implies \(\mathcal{H}_k^m(S) \leq \lambda^k \mathcal{H}_k^m(S)\) for \(S \subset M \cap B(c, \delta)\) and the conclusion follows.\footnote{That is, \(C\) is continuous and \(\phi \circ C\) is locally Lipschitzian whenever \(\phi\) is a chart of \(M\) of class 1. A posteriori, this is equivalent to \(C\) being locally Lipschitzian with respect to \(\sigma\).}

10.4 Remark. Denoting by \(h\) the Riemannian metric on \(\partial M\) induced by \(F|\partial M\) from \(\mathbb{R}^n\) and by \(\mu\) the Riemannian distance associated with \((\partial M, h)\), the preceding lemma yields in particular \(\mathcal{H}_k^m -h^{-1}(S) = \mathcal{H}_k^m -h^{-1}(S)\) for \(S \subset \partial M\). Moreover, the measures \(\mathcal{H}_k^m\) and \(\mathcal{H}_k^m -h^{-1}\) agree with the usual Riemannian measures (see [Sak90, Section 2.5]) associated with \((M, g)\) and \((\partial M, h)\), respectively, by \([Fed69, 3.2.46]\).

10.5 Theorem. Suppose \(m, M,\) and \(\sigma\) are as in 10.3. \(C\) is a subset of \(M\), and \(\mathcal{H}_\sigma^{m-1}((\text{Bdry} C) \sim \partial M) = 0\).

Then, either \(\mathcal{H}_\sigma^{m}(C \sim \partial M) = 0\) or \(\mathcal{H}_\sigma^{m}(M \sim C) = 0\).

\textit{Proof.}\quad Noting that \(M \sim \partial M\) is connected and \(\mathcal{H}_\sigma^{m}(\partial M) = 0\) by 10.4, we assume \(\partial M = \emptyset\). Next, we observe that it is sufficient to prove that there holds either \(\mathcal{H}_\sigma^m(C \cap \text{dmm} \phi) = 0\) or \(\mathcal{H}_\sigma^{m}((\text{dmm} \phi) \sim C) = 0\) whenever \(\phi\) is a chart of \(M\) of class 1 satisfying \(\text{im} \phi = \mathbb{R}^m\), as \(M\) is covered by the domains of a countable collection of such charts. To verify this dichotomy, we let \(A = \phi[C]\) and infer \(\mathcal{H}_{m-1}(\text{Bdry} A) = 0\) from \([Fed69, 3.2.46]\), hence \(\text{Bdry} A = \emptyset\) if \(m = 1\) and \(\mathcal{H}_{m-1}(\text{Bdry} A) = 0\) if \(m > 1\) by \([Fed69, 2.10.15]\). This implies \(A\) is of locally finite perimeter by \([Fed69, 4.5.11]\) and that \(\partial(\mathbb{R}^m \cup A) = 0\) by \([Fed69, 4.5.6(1)]\). Consequently, there holds \(\mathcal{L}_m(A) = 0\) or \(\mathcal{L}_m(\mathbb{R}^m - A) = 0\) by the constancy theorem \([Fed69, 4.1.7]\), whence we deduce the assertion by \([Fed69, 3.2.46]\).\footnote{That is, \(C\) is continuous and \(\phi \circ C\) is locally Lipschitzian whenever \(\phi\) is a chart of \(M\) of class 1. A posteriori, this is equivalent to \(C\) being locally Lipschitzian with respect to \(\sigma\).}
10.6 Remark. Alternatively to using [Fed69, 4.5.6, 4.5.11], one could employ an argument based on capacity (see, e.g., [Men16a, 5.7]) to infer $\partial (E^m \cup A) = 0$ from $\mathcal{H}^{m-1}(\partial V) = 0$.

10.7 Theorem. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $\Lambda$ is the class of all locally Lipschitzian real valued functions with domain $U$, $M$ is a connected $n$ dimensional manifold-with-boundary of class 2, $F: M \rightarrow U$ is a proper immersion of class 2, and $V$ is the varifold associated with $(F,U)$.

Then, $V$ is indecomposable of type $\Lambda$.

Proof. Clearly, $V$ is rectifiable and $\|\delta V\|$ is a Radon measure by 6.14. Suppose $f: U \rightarrow \mathbb{R}$ is locally Lipschitzian, let

$$B(y) = \{x: f(x) = y\}, \quad E(y) = \{x: f(x) > y\}$$

for $y \in \mathbb{R}$, note $\text{Bdry} F^{-1}[E(y)] \subset F^{-1}[B(y)]$, and define $\sigma$ as in 10.3. Since $\Theta^m(\|V\|, x) = N(F, x)$ for $\mathcal{H}^m$ almost all $x \in U$ by [Al72, 3.5 (1b)] and 6.14, we employ [MS22, 3.33, 3.34], [Fed69, 2.10.25], and 10.3 to infer

$$\|V \partial E(y)\|(U) = \int_{B(y)} \Theta^m(\|V\|, x) \, d\mathcal{H}^{m-1}x$$

for $\mathcal{L}^1$ almost all $y$. Whenever $V \partial E(y) = 0$ for such $y$, we apply 10.5 with $C = F^{-1}[E(y)]$ to conclude that

either $F_#\mathcal{H}_\sigma^m(E(y)) = 0$ or $F_#\mathcal{H}_\sigma^m(U \sim E(y)) = 0$.

Since $F_#\mathcal{H}_\sigma^m = \|V\|$ by 10.3 and 6.14, the conclusion follows.

10.8 Remark. Simple examples with $N(F, x) = 2$ for $x \in \text{im} F$, yield that conversely even indecomposability of $V$ need not imply connectedness of $M$.

10.9 Theorem. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $G$ is a complete normed commutative group, $S \in I_m(U,G)$ is indecomposable, $V \in RV_m(U)$ is characterised by $\|V\| = \|S\|$, $\|\delta V\|$ is a Radon measure, and $\Lambda$ denotes the class of all locally Lipschitzian real valued functions with domain $U$.

Then, $V$ is indecomposable of type $\Lambda$.

Proof. Suppose $f: U \rightarrow \mathbb{R}$ is locally Lipschitzian, define $E(y) = \{x: f(x) > y\}$ for $y \in \mathbb{R}$, and let $Y = \mathbb{R} \cap \{y: V \partial E(y) = 0\}$. Firstly, [MS22, 3.33, 3.34] yield

$$(\mathcal{H}^{m-1} \cup \{x: f(x) = y\}) \subseteq \Theta^m(\|V\|, x) = 0$$

for $\mathcal{L}^1$ almost all $y \in Y$.

We then infer $(S, f, y) = 0$ for $\mathcal{L}^1$ almost all $y \in Y$ by [MS22, 4.8] and hence

$$(S \cup E(y)) \in I_m(U,G) \quad \text{and} \quad \partial_G(S \cup E(y)) = (\partial_G S) \cup E(y)$$

for $\mathcal{L}^1$ almost all $y \in Y$ by [MS22, 5.13 (8)]. In view of [MS22, 4.5], the indecomposability of $S$ finally yields that either $\|S\|(E(y)) = 0$ or $\|S\|(U \sim E(y)) = 0$ for such $y$. \hfill $\Box$

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10.10 Remark. The indecomposability hypothesis on $S$ may be weakened to the requirement that there exists no Borel subset $E$ of $U$ satisfying

\[ \|S\|(E) > 0, \quad \|S\|(U \sim E) > 0, \quad \text{and} \quad \partial_G(S \cap E) = (\partial_G S) \cap E. \]

In [GM23, Definition 1.1(2)], the strictly stronger condition of set-indecomposability—still strictly weaker than indecomposability—is introduced in the context of the notion of flat $G$ chains defined in [Whi99].

10.11 Remark. It may happen that $V$ is decomposable, see 6.8.

10.12 Lemma. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in V_m(U)$, $\|\delta V\|$ is a Radon measure, $\Phi$ is the family of connected components of $\text{spt} \|V\|$, and $\Xi$ is a locally finite decomposition of $V$.

Then, the following four statements hold.

1. If $C \in \Phi$, then $C = \bigcup \{\text{spt} \|W\| : W \in \Xi, C \cap \text{spt} \|W\| \neq \emptyset\}$.
2. If $K$ is a compact subset of $U$, then $\text{card}(\Phi \cap \{C : C \cap K \neq \emptyset\}) < \infty$.
3. If $C \in \Phi$, then $C$ is open relative to $\text{spt} \|V\|$.
4. If $C \in \Phi$, then $\text{spt}(\|V\| \cap C) = C$ and $V \cap C = 0$.

Proof. Our hypotheses yield $\text{spt} \|V\| = \bigcup \{\text{spt} \|W\| : W \in \Xi\}$; hence, [Men16a, 6.8] implies (1). Moreover, (1) implies (2) and (2) implies (3). Finally, (3) and [Men16a, 6.5] yield (4). \qed

10.13 Remark. The proof is almost identical to [Men16a, 6.11, 6.14].

10.14 Remark. For nonrectifiable varifolds $V$, no decomposition needs to exist even if $\delta V = 0$, see [MS18, 4.12(3)].

10.15 Lemma. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in RV_m(U)$, $\|\delta V\|$ is a Radon measure, and every decomposition of $V$ is locally finite.

Then, there holds

\[ \text{spt} \|V\| = \bigcup \{\text{spt} \|W\| : W \in \Pi\} \quad \text{whenever} \quad \Pi \text{ is a partition of} \quad V. \]

Proof. With a decomposition $\Xi$ related to $\Pi$ as in 5.7, we apply 10.12(1). \qed

10.16 Theorem. Suppose $m$ and $n$ are positive integers, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in RV_m(U)$, $\|\delta V\|$ is a Radon measure, every decomposition of $V$ is locally finite, $\text{spt} \|V\|$ is connected, and

\[ \Gamma = T(V) \cap \{f : \text{dmn} f = \text{spt} \|V\|, \; f \text{ is continuous}\}. \]

Then, whenever $f \in \Gamma$ and $E(y) = \{y : f(x) > y\}$ for $y \in \mathbb{R}$, the set

\[ \{y : \|V\|(E(y)) > 0, \quad \|V\|(U \sim E(y)) > 0, \quad V \cap E(y) = 0\} \]

is countable; in particular, $V$ is indecomposable of type $\Gamma$.

\[ \footnote{We may consider $m = n = 1$, $U = \mathbb{R}$, $G = \mathbb{Z}$, and $Q = [0,1] + [0,-1] \in I_1(\mathbb{R})$.} \]
Proof. Suppose \( f \in \Gamma \) and \( \Pi \) denotes a partition of \( V \) along \( f \), see [8.13]. Since \( f \) is continuous and \( \text{dmn} \, f \) is connected, \( \text{im} \, f \) is an interval. Moreover, we have

\[
\text{spt} \| V \| = \bigcup \{ \text{spt} \| W \| : W \in \Pi \}
\]

by [10.15]. It follows

\[
\text{im} \, f \subset \bigcup \{ J(W) : W \in \Pi \}
\]

by [7.13(2)], where \( J(W) = \text{spt} \| f \| \). Abbreviating

\[
B = \{ y : \| V \| (E(y)) > 0, \| V \| (U \sim E(y)) > 0, \text{and } V \partial E(y) = 0 \},
\]

we conclude

\[
B \cap \bigcup_{W \in \Pi} \{ y : \inf J(W) < y < \sup J(W) \} = \emptyset.
\]

because we have \( V \partial E(y) = W \partial E(y) \neq 0 \) whenever \( \inf J(W) < y < \sup J(W) \) and \( W \in \Pi \) by [5.6] and [8.8]. Noting \( B \subset \text{im} \, f \), the set \( B \) is accordingly contained in the countable set \( \bigcup \{ \text{Bdry} \, J(W) : W \in \Pi \} \); in particular, \( \mathcal{L}^1(B) = 0 \).

10.17 Remark. By [7.6], our estimate is sharp and \( V \) may be decomposable. This also shows that indecomposability of type \( \Gamma \) and indecomposability differ which, by [Men16a, 8.7], answers the second question posed in [Sch16, Section A].

10.18 Remark. If the hypotheses of [9.2] are satisfied with \( B = \emptyset \) and \( \text{spt} \| V \| \) is connected, then geometrically significant members of \( \Gamma \) which may fail to be Lipschitzian are the geodesic distance functions on \( \text{spt} \| V \| \) to points in \( \text{spt} \| V \| \), see [Men16b, 6.8(2), 6.11]; more generally, this applies to functions in the image of the embedding of the local Sobolev space with respect to \( V \) and exponent \( q \) satisfying \( q > m \) if \( m > 1 \), into the space of real valued continuous functions on \( \text{spt} \| V \| \), see [Men16b, 7.12].

10.19 Corollary. Suppose \( m \) and \( n \) are positive integers, \( m \leq n \), \( U \) is an open subset of \( \mathbb{R}^n \), \( V \in \text{RV}_m(U) \), \( \| \delta V \| \) is a Radon measure, every decomposition of \( V \) is locally finite, and \( \Gamma = \mathcal{T}(V) \cap \{ f : \text{dmn} \, f = \text{spt} \| V \|, f \text{ is continuous} \} \).

Then, the following three statements are equivalent.

1. The set \( \text{spt} \| V \| \) is connected.
2. The varifold \( V \) is indecomposable of type \( \Gamma \).
3. The varifold \( V \) is indecomposable of type \( \mathcal{E}(U, \mathbb{R}) \).

Proof. We combine [7.7] and [10.16].

10.20 Corollary. Suppose \( V \) is as in [9.1] the set \( \text{spt} \| V \| \) is connected, and \( \Gamma = \mathcal{T}(V) \cap \{ f : \text{dmn} \, f = \text{spt} \| V \|, f \text{ is continuous} \} \).

Then, \( V \) is indecomposable of type \( \Gamma \).

Proof. We combine [9.2] and [10.16].

10.21 Remark. Integral varifolds satisfying the hypotheses with \( m = 2 \), \( n = 3 \), and \( U = \mathbb{R}^3 \) occur in the minimisation of the Willmore energy with \textit{clamped boundary condition} amongst connected surfaces; see [NP20, Theorem 4.1].
10.22 Corollary. Suppose \( m \) and \( V \) satisfy the equivalent conditions of 9.16, \( \Theta^m(\|V\|, x) \geq 1 \) for \( \|V\| \) almost all \( x \), \( \text{spt} \|V\| \) is connected, and
\[
\Gamma = T(V) \cap \{ f : \text{dnn} f = \text{spt} \|V\|, \text{f is continuous} \}.
\]

Then, \( V \) is indecomposable of type \( \Gamma \).

Proof. Recalling 9.14 and 9.16(2), we combine 9.24 and 10.16. \( \square \)

10.23 Remark. By 9.14, 9.25, and 10.16, the conclusion also holds if \( m = 1 \), \( V \) satisfies the conditions of 9.13, \( \Theta^1(\|V\|, x) \geq 1 \) for \( \|V\| \) almost all \( x \), and \( \text{spt} \|V\| \) is connected.

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