Hall-magnetohydrodynamic waves in flowing ideal incompressible solar-wind plasmas

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Abstract

It is well established now that the solar atmosphere, from the photosphere to the corona and the solar wind, is a highly structured medium. Satellite observations have confirmed the presence of steady flows there. Here, we investigate the propagation of magnetohydrodynamic (MHD) eigenmodes (kink and sausage surface waves) travelling along an ideal incompressible flowing plasma cylinder (flux tube) surrounded by a flowing plasma environment in the framework of the Hall magnetohydrodynamics. The propagation characteristics of the waves are studied in a reference frame moving with the mass flow outside the tube. In general, the flows change the waves’ phase velocities compared with their magnitudes in a static MHD flux tube and the Hall effect extends the number of the possible wave dispersion curves. It turns out that while the kink waves, considered in the context of the standard magnetohydrodynamics, are unstable against the Kelvin–Helmholtz instability, they become stable when the Hall term in the generalized Ohm’s law is taken into account. The sausage waves are stable in both considerations. All results concerning the waves’ propagation and their stability/instability status are obtained on the basis of the linearized Hall-magnetohydrodynamic equations and are applicable mainly to the solar wind plasmas.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The magnetohydrodynamic (MHD) waves, first theoretically predicted by Alfvén [1], have been the subject of numerous studies and publications. They are known to describe the dynamics of low-frequency electromagnetic phenomena in the terrestrial magnetosphere, on the Sun and in astrophysical plasmas. These waves interact strongly both with ions and electrons, which leads to direct acceleration and heating of particles in space and laboratory plasmas and to indirect effects on high-energy particles scattered on wave’s turbulence.
Recently, a great deal of attention was devoted to the various waves and oscillations which occur in structured solar atmosphere \[2, 3\]. Concerning the space structuring of the solar atmosphere one usually considers slab or cylindrical geometries. As was first theoretically established by Ruderman \[4\], the kink oscillations of thin straight magnetic tubes in a cold plasma can depend on the form of the cylindrical cross-section—in particular one can have two kink modes which are linearly polarized in the direction of the large and small axes of the elliptic cross-section of the tube, respectively. It turns out, however, that the damping rates of these two kink modes are of the order of the damping rate of the kink mode of a tube with a circular cross-section. A similar dependence of the dispersion characteristics of both kink and sausage modes on the tube cross-section for a compressible plasma has recently been reported by Erdélyi and Morton \[5\]. More specifically, when the cross-section is elliptical the magnetic tube supports slow and fast magnetoacoustic waves, which (the slow sausage mode and the slow and fast kink modes) in a thin tube approximation are analogous to the waves running on a circular cross-section flux tube. However, the kink modes propagate with different phase speeds depending on whether the axial displacement takes place along the major or minor axis of the ellipse. It is worth mentioning, however, that there is no observational evidence of ellipticity dependence for kink modes up to now. Goossens et al \[6\] studying the nature of the kink mode for slow, fast and Alfvén waves report that these waves have mixed properties which are often ignored. Considering the waves’ propagation in a compressible pressure-less plasma, an incompressible plasma and a compressible plasma which allows for MHD radiation the authors show that in all three cases the frequency and the damping rate are, for practical purposes, the same as they differ at most by terms proportional to \((kR)^2\) (\(R\) being the flux-tube radius). In the magnetic flux tube the kink waves are in all three cases, to a high degree of accuracy, incompressible waves with negligible pressure perturbations and with mainly horizontal motions. The main restoring force of kink waves in the magnetized flux tube is the magnetic tension force. But as commented on by Van Doorsselaere et al \[7\], the very existence of the kink mode requires a guiding non-uniformity.

The next step in investigating the wave phenomena in solar and stellar atmospheres was the consideration of steady flows there. Satellite measurements performed by \textit{SOHO}, \textit{Ulysses}, \textit{Yohkoh}, \textit{Wind}, \textit{ACE}, and more recently by \textit{STEREO} and \textit{TRACE}, of plasma characteristics of, for instance, the solar wind and coronal plumes flows, such as the magnetic field, the thermal and flow velocity and density of plasma or plasma compositions, are important to understand the various plasma wave modes which may arise. However, wave analysis requires further information and special tools to identify which set of modes is contributing to the observed wave features. In practice, one may use filters to perform the so-called \textit{pattern recognition} to detect the various kinds of modes that may propagate in a plasma and to determine their contribution to the wave energy \[8\]. In solar coronal studies, two wave pattern recognition techniques have recently been developed by Nakariakov and King \[9\] and Marsh \textit{et al} \[10\]. The so-called periodmap method \[9\] reduces a three-dimensional data cube to a two-dimensional map of the analysed field of view. The map reveals the presence and distribution of the most pronounced frequencies in the power spectrum of the time signal recorded at spatial pixels. The authors demonstrate the applicability of this method as a pre-analysis tool with the use of \textit{TRACE} EUV coronal data, which contain examples of transverse and longitudinal oscillations of coronal loops. A Bayesian probability-based approach suggested by Marsh \textit{et al} \[10\] is applied to the problem of detecting and parametrizing oscillations in the upper solar atmosphere. Due to its statistical origin, this method provides a mechanism for determining the number of oscillations present, gives precise estimates of the oscillation parameters with a self-consistent statistical error analysis and allows the oscillatory model signals to be reconstructed within these errors. A highly desirable feature of the Bayesian approach is the ability to resolve oscillations with
extremely small frequency separations. Another important issue is the waves’ stability. The magnetosonic waves in structured atmospheres with steady flows have been examined by Nakariakov and Roberts [11], Nakariakov et al [12] and Andries and Goossens [13]. Andries and Goossens also studied the conditions under which resonant flow and Kelvin–Helmholtz instability take place.

It is worth pointing out that all the aforementioned studies were performed in the framework of the standard magnetohydrodynamics. It was Lighthill [14] who pointed out half a century ago that for an adequate description of wave phenomena in fusion and astrophysical plasmas one has to include the Hall term, \( m_i (j \times B) / (e \rho) \), in the generalized Ohm's law. That approach is termed Hall magnetohydrodynamics (Hall MHD). In this way, it is possible to describe waves with frequencies up to \( \omega \approx \omega_{ci} \). Since the model still neglects the electron mass, it is limited to frequencies well below the lower hybrid frequency: \( \omega \ll \omega_{lh} \). Generally speaking, the theory of Hall MHD is relevant to plasma dynamics occurring on length scales shorter than an ion inertial length, \( L < l_{Hall} = c / \omega_{pi} \) (where \( c \) is the speed of light and \( \omega_{pi} \) is the ion plasma frequency), and time scales of the order or shorter than the ion cyclotron period, \( t < \omega_{ci}^{-1} \) [15]. Thus, the Hall MHD should affect the dispersion characteristics of the MHD waves in spatially bounded magnetized plasmas. An extensive review for the studies of wave propagation in bounded MHD plasmas (largely for slab geometry) in the context of both the standard and Hall MHD can be found in [16] and references therein.

The Hall term in the generalized Ohm’s law affects not only the wave dispersion relations but also the parametric instabilities of incoherent Alfvén waves [17], the wave turbulence [18, 19], as well as the strongest competitor of the wave turbulence heating mechanism, notably the magnetic reconnection [20,21]. One even speaks of Hall reconnection being akin to the Petschek reconnection model. It has been shown that the equilibrium state of thin Keplerian discs embedded in mixed poloidal and toroidal magnetic fields can be achieved by means of the Hall-MHD approach [22]. The so-called Tayler (pinch type) instability in strong toroidal magnetic fields in young neutron stars is also heavily influenced by the Hall effect [23]. The study of astrophysical jets in strong magnetic fields also requires consideration of the Hall term in the generalized Ohm's law [24]. As seen, the Hall MHD has an impact on many important astrophysical phenomena and objects.

Here, we investigate the influence of flow velocities on the dispersion characteristics and stability of hydromagnetic surface waves (sausage and kink modes) travelling along an infinitely conducting, magnetized jet moving past also (with a different speed) an infinitely conducting, magnetized plasma. Recently, a similar problem, but in terms of standard ideal MHD, was studied by Vasheghani Farahani et al [25] for the interpretation of the kink waves on coronal jets observed by Cirtain et al [26]. If in the solar corona the plasma \( \beta \) (the ratio of gas to magnetic pressure) is much less than unity, in the solar wind flux tubes \( \beta \approx 1 \). Since we are going to study the wave propagation in flowing solar wind plasma, we can assume that we have a 'high-\( \beta \)' magnetized plasma and treat it as an incompressible fluid. For simplicity, we consider a cylindrical jet of radius \( a \) (immersed together with the environment in a constant magnetic field \( B_0 \) directed along the \( z \)-axis), allowing for different plasma densities within and outside the jet, \( \rho_o \) and \( \rho_e \), respectively (figure 1). The most natural discontinuity which occurs at the surface binding the cylinder is the tangential one because it is the discontinuity that ensures an equilibrium total pressure balance. Moreover, it is worth noting that the jet is non-rotating and without twist—otherwise the centrifugal and the magnetic tension forces should be taken into account. For typical values of the background constant magnetic field \( B_0 = 5 \times 10^{-9} \text{T} \) and the electron number density inside the jet \( n_e = 2.43 \times 10^9 \text{m}^{-3} \) at 1 AU, the ion cyclotron frequency \( \omega_{ci} / 2\pi = 76 \text{ mHz} \), the Alfvén speed \( v_{Ao} = 70 \text{ km s}^{-1} \) and the Hall scale length \( (=v_{Ao} / \omega_{ci}, \text{ which is equivalent to } c / \omega_{pi}) \) is \( l_{Hall} \approx 150 \text{ km} \). This scale length
is small, but not negligible compared with the tube radius of a few hundred kilometres. Here, we introduce a scale parameter $\varepsilon = l_{\text{Hall}}/a$ called the Hall parameter. In the limit $\varepsilon \to 0$, the Hall-MHD system reduces to the standard MHD system. The flow speeds of the jet and its environment are generally rather irregular. To investigate the stability of the travelling MHD waves it is convenient to consider the wave propagation in a frame of reference attached to the flowing environment. Thus, we can define the relative flow velocity $U^\text{rel} = U_o - U_e$ ($U_o$ and $U_e$ being the steady flow speeds correspondingly inside and outside the flux tube) as an entry parameter whose value determines the stability/instability status of the jet. As usual, we normalize that relative flow velocity with respect to the Alfvén speed in the jet, $v_A = B_0/(\mu_0 \rho_0)^{1/2}$, and call it the Alfvénic Mach number $M_A$, omitting for simplicity the superscript ‘rel’. Another important entry parameter of the problem is $\eta = \rho_e/\rho_o$. It turns out that the waves’ dispersion characteristics and their stability critically depend on the magnitude of $\eta$. That is why, in this study, we take two different values of $\eta$, notably $\eta = 4$ and $\eta = 0.586$. The latter choice of $\eta$, in a compressible plasma approach, would correspond to $v_{Ao} = c_s = 70 \text{ km s}^{-1}$ and $v_{ Ae} = 100 \text{ km s}^{-1}$, $c_s = 70 \text{ km s}^{-1}$, respectively, where $c_s$ is the speed of sound in the corresponding medium. The bigger value of $\eta (=4)$ is still acceptable for some extreme values of the solar wind jet and its environment’s plasma parameters (densities and Alfvén speeds). Thus, the waves’ dispersion characteristics (the dependence of the wave phase velocity $v_{ph} = \omega/k_z$ on the wave number $k_z$) and their stability states are determined by the three parameters, $\eta$, $\varepsilon$, and $M_A$, two of which are fixed ($\eta$ and $\varepsilon$) and the third one, $M_A$, is variable.

2. Basic equations and dispersion relations

As seen in figure 1, the three important vectors, the embedded magnetic field $B_0$, the relative flow velocity $U$ and the wave vector $k$, lie along the $z$-axis. The basic equations which govern the propagation of Hall-MHD waves flowing with velocity $U$ in an incompressible ideal magnetized plasma are the linearized equations for the perturbed fluid velocity $v_1$ and wave magnetic field $B_1$:

$$\rho \frac{\partial v_1}{\partial t} + \rho (U \cdot \nabla) v_1 + \nabla \left( \frac{1}{\mu_0} B_0 \cdot B_1 \right) - \frac{1}{\mu_0} (B_0 \cdot \nabla) B_1 = 0,$$

$$\frac{\partial B_1}{\partial t} + (U \cdot \nabla) B_1 = (B_0 \cdot \nabla) v_1 + B_0 \nabla \cdot v_1 + \frac{v_A^2}{c_s} \hat{z} \cdot \nabla \nabla \times B_1 = 0,$$

with the constraints

$$\nabla \cdot v_1 = 0 \quad \text{and} \quad \nabla \cdot B_1 = 0,$$
where \( \hat{\xi} \) is the unit vector of the z-axis; the other notation is standard. After Fourier transforming the perturbed quantities assuming to be proportional to \( g(r) \ exp(-i\omega t+i\nu t+ik_z z) \), we get in a cylindrical coordinate system the following set of ordinary differential equations for the components of the perturbed wave magnetic field \( B_1 \) and fluid velocity \( v_1 \):

\[
-\mathbf{\hat{\xi}} \rho(\omega - k \cdot \mathbf{U}) v_{1r} + \frac{d}{dr} \left( \frac{1}{\mu_0} B_0 B_{1z} \right) - ik_z \frac{1}{\mu_0} B_0 B_{1r} = 0, \quad (4)
\]

\[
-\mathbf{\hat{r}} \rho(\omega - k \cdot \mathbf{U}) v_{1\phi} + \frac{1}{r} \frac{d}{dr} \left( \frac{1}{\mu_0} B_0 B_{1z} \right) - ik_z \frac{1}{\mu_0} B_0 B_{1\phi} = 0, \quad (5)
\]

\[
-\mathbf{\hat{z}} \rho(\omega - k \cdot \mathbf{U}) v_{1z} + i k_z \left( \frac{1}{\mu_0} B_0 B_{1\phi} \right) - i k_z^2 \frac{1}{\mu_0} B_0 B_{1z} = 0, \quad (6)
\]

\[
-\mathbf{\hat{\xi}} \rho(\omega - k \cdot \mathbf{U}) B_{1r} - ik_z B_0 v_{1r} + \frac{v_A^2}{\omega_\xi} \left( -k_z \frac{m}{r} B_{1z} + k_z^2 B_{1\phi} \right) = 0, \quad (7)
\]

\[
-\mathbf{\hat{r}} \rho(\omega - k \cdot \mathbf{U}) B_{1\phi} - ik_z B_0 v_{1\phi} - \frac{v_A^2}{\omega_\xi} \left( k_z^2 B_{1r} + i k_z \frac{d}{dr} B_{1z} \right) = 0, \quad (8)
\]

\[
-\mathbf{\hat{z}} \rho(\omega - k \cdot \mathbf{U}) B_{1z} - ik_z B_0 v_{1z} + \frac{v_A^2}{\omega_\xi} \left[ i k_z \left( \frac{d}{dr} + \frac{1}{r} \right) B_{1\phi} - k_z \frac{m}{r} B_{1\phi} \right] = 0, \quad (9)
\]

\[
\left( \frac{d}{dr} + \frac{1}{r} \right) v_{1r} + \frac{1}{r} m v_{1\phi} + ik_z v_{1z} = 0, \quad (10)
\]

\[
\left( \frac{d}{dr} + \frac{1}{r} \right) B_{1r} + \frac{1}{r} m B_{1\phi} + ik_z B_{1z} = 0, \quad (11)
\]

where \( v_A^2 = B_0^2 / (\mu_0 \rho) \). Recall that for an incompressible plasma \( \rho \) is the unperturbed (equilibrium) value of the plasma density—it has different magnitudes in both media—correspondingly \( \rho_0 \) and \( \rho_\ell \) (see figure 1). Although the derivation of the dispersion relations of surface MHD waves might look straightforward, it is not the case for Hall-MHD bounded plasmas. That is why we will briefly sketch the way of obtaining them.

Let us differentiate equation (4) with respect to \( r \) to get

\[
-\mathbf{\hat{\xi}} \rho(\omega - k \cdot \mathbf{U}) \frac{d}{dr} v_{1r} + \frac{d^2}{dr^2} \left( \frac{1}{\mu_0} B_0 B_{1z} \right) - \frac{1}{\mu_0} B_0 i k_z \frac{d}{dr} B_{1r} = 0.
\]

After replacing \( d v_{1r}/dr \) and \( d B_{1z}/dr \) with their expressions calculated from equations (10) and (11) the above equation becomes

\[
\rho(\omega - k \cdot \mathbf{U}) \left( \frac{1}{r} v_{1r} - \frac{m}{r} v_{1\phi} - k_z v_{1z} \right) + \frac{d^2}{dr^2} \left( \frac{1}{\mu_0} B_0 B_{1z} \right)
+ \frac{1}{\mu_0} B_0 i k_z \left( \frac{1}{r} B_{1r} - \frac{m}{r} B_{1\phi} - k_z B_{1z} \right) = 0.
\]

The final form of this equation, after putting in it the expressions for the component of the perturbed wave magnetic field \( B_1 \), is

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( k_z^2 + \frac{m^2}{r^2} \right) \right] p_{\text{imag}} = 0, \quad (12)
\]

where \( p_{\text{imag}} = B_0 B_{1z} / \mu_0 \) is the perturbed magnetic pressure.

From equations (7)–(9) after some lengthy but standard calculations, bearing in mind that in an incompressible plasma \( v_{1z} = 0 \), one can get the following relation between \( v_{1\phi} \) and \( v_{1r} \):

\[
\left( \frac{d}{dr} + \frac{1}{r} \right) v_{1\phi} = \frac{m}{r} v_{1r}. \quad (13)
\]
Another important relationship between $v_{1\phi}$ and $v_{1r}$ can be obtained from equation (7) with the help of equation (13) and it has the form

$$v_{1\phi} = \frac{\epsilon}{C - 1} v_{1r},$$

where

$$\epsilon = \frac{\omega - k \cdot U}{\omega_{ci}} \quad \text{and} \quad C = \left(\frac{\omega - k \cdot U}{k_z v_A}\right)^2.$$

Now we go back to (4) to get an expression for $v_{1r}$, namely

$$v_{1r} = -i \frac{\rho}{\mu_0} \frac{(\omega - k \cdot U)(C - 1)}{(C - 1)^2 - \epsilon^2} \frac{d}{dr} p_{1\text{mag}} - i \frac{\epsilon}{(\omega - k \cdot U)^2} k_z v_A v_{1\phi}.$$

This expression for $v_{1r}$ can be reduced, by replacing $v_{1\phi}$ from equation (14), to the following form:

$$v_{1r} = -i \frac{\rho}{\mu_0} \frac{(\omega - k \cdot U)(C - 1)}{(C - 1)^2 - \epsilon^2} \frac{d}{dr} p_{1\text{mag}}.$$

It is worth noting that we have different expressions for $C$ and $\epsilon$ inside and outside the jet.

Thus, we have derived a second-order ordinary differential equation (12) for the perturbed magnetic pressure and an expression (15) for the perturbed radial fluid velocity component in terms of the first derivative of $p_{1\text{mag}}$. The solutions to the differential equation are the modified Bessel functions, more specifically

$$p_{1\text{mag}}(r) = \begin{cases} p_{1o} I_m(k_z r) & \text{for } r < a, \\ p_{1e} K_m(k_z r) & \text{for } r > a. \end{cases}$$

Accordingly, the expressions for $v_{1r}$ inside and outside the jet are

$$v_{1r}(r < a) = -i \frac{\mu_0}{\mu_0} \frac{(\omega - k \cdot U)(C_o - 1)}{k_z B_0^2} \frac{d}{dr} p_{1o} I_m(k_z r)$$

and

$$v_{1r}(r > a) = -i \frac{\mu_0}{\mu_0} \frac{\omega(C_e - 1)}{k_z B_0^2} \frac{d}{dr} p_{1e} K_m(k_z r),$$

where

$$C_o = \left(\frac{\omega - k \cdot U}{k_z v_{Ao}}\right)^2, \quad C_e = \left(\frac{\omega}{k_z v_{Ae}}\right)^2, \quad \epsilon_o = \frac{\omega - k \cdot U}{\omega_{ci}}, \quad \epsilon_e = \frac{\omega}{\omega_{ci}}.$$

To solve our problem we need two boundary conditions, applied at the interface $r = a$.

These boundary conditions are as follows:

- the continuity of the perturbed interface $\frac{v_{1r}}{\omega - k \cdot U}[27]$,
- the continuity of the perturbed magnetic pressure $p_{1\text{mag}}$.

The application of the boundary conditions yields the following dispersion relations for the sausage ($m = 0$) and kink ($m = 1$) modes running along the jet’s interface:

$$\frac{C_o - 1}{(C_o - 1)^2 - \epsilon_o^2} I_m(k_z a) - \frac{C_e - 1}{(C_e - 1)^2 - \epsilon_e^2} K_m'(k_z a) = 0.$$

As seen, the wave frequency $\omega$ is Doppler-shifted inside the jet, as well as the two modes are pure surface waves.
If we ignore the Hall effect (\(\epsilon_{oe} = 0\)), the above equation reduces to

\[
\frac{\rho_e}{\rho_0} (\omega^2 - k_z^2 v_{Ac}^2) \frac{I'_m(k_z a)}{I_m(k_z a)} - [(\omega - k \cdot U)^2 - k_z^2 v_{Ac}^2] \frac{K'_m(k_z a)}{K_m(k_z a)} = 0,
\]

i.e. we recover the well-known dispersion relations [11].

As we are interested in the stability of the surface waves travelling along the jet interface, we have to assume that the wave frequency is complex, i.e. \(\omega \rightarrow \omega + i\gamma\), where \(\gamma\) is the expected instability growth rate. Since we plot dispersion diagrams as dependences of the wave phase velocity \(v_{ph}\) on the wave number \(k_z\), we normalize all quantities by defining the dimensionless wave phase velocity \(V_{ph} = \omega/k_z v_{Ao}\), wave number \(K = k_z a\) and the relative Alfvénic Mach number \(M_A = U/v_{Ao}\) to get the dimensionless form of our dispersion relations. It is worth investigating together both waves’ dispersion relations (those in the context of the standard magnetohydrodynamics and the Hall-MHD waves) in order to see how the Hall term in the generalized Ohm’s law modifies the waves’ dispersion curves and the instability growth rates when the waves become unstable. Thus, the dimensionless forms of the aforementioned dispersion relations are

\[
\left((V_{ph} - M_A)^2 - 1\right) \frac{I'_m(K)}{I_m(K)} Z_2 - (\eta V_{ph}^2 - 1) \frac{K'_m(K)}{K_m(K)} Z_1 = 0,
\]

where

\[
Z_1 = \left((V_{ph} - M_A)^2 - 1\right)^2 - \epsilon^2 K^2 (V_{ph} - M_A)^2,
\]

\[
Z_2 = (\eta V_{ph}^2 - 1)^2 - \epsilon^2 K^2 v_{ph}^2.
\]

In the above expressions \(\epsilon = l_{Hall}/a\) is the Hall parameter. The dispersion relation of the standard surface MHD waves takes the form

\[
(\eta V_{ph}^2 - 1) \frac{I'_m(K)}{I_m(K)} - [(V_{ph} - M_A)^2 - 1] \frac{K'_m(K)}{K_m(K)} = 0.
\]

It is easy to see that the dispersion relation (19) of the standard MHD surface waves is a quadratic one and its roots are

\[
V_{ph} = -\frac{M_A B \pm \sqrt{D}}{\eta A - B},
\]

where

\[
A = \frac{I'_m(K)}{I_m(K)}, \quad B = \frac{K'_m(K)}{K_m(K)},
\]

and the discriminant \(D\) is

\[
D = M_A^2 B^2 + (\eta A - B)(1 - M_A^2)B - A.
\]

Obviously, if \(D \geq 0\), then

\[
\text{Re}(V_{ph}) = -\frac{M_A B \pm \sqrt{D}}{\eta A - B}, \quad \text{Im}(V_{ph}) = 0,
\]

else

\[
\text{Re}(V_{ph}) = -\frac{M_A B}{\eta A - B}, \quad \text{Im}(V_{ph}) = \frac{\sqrt{D}}{\eta A - B}.
\]

We note that our choice for the sign of \(\sqrt{D}\) is plus although, in principle, it might be minus too—in that case, due to the arising instability, the wave’s energy is transferred to the jet.
3. Numerical results and discussion

While the dispersion equation of the standard MHD surface waves is a quadratic one and its roots can be expressed in closed forms, that of the Hall-MHD waves is a complex polynomial of sixth order and it can be solved only numerically. It is well known that finding the roots of a complex equation certainly is not a trivial task [28]. We have used the Müller method [29] to determine the complex roots of equation (18).

Before starting the numerical procedure for solving either dispersion equation, we have to specify the jet’s entry parameters \(\eta, M_A\) and the Hall parameter \(\varepsilon\) for the Hall-MHD waves. Since the dispersion relations, as we have already mentioned in section 1, are sensitive to the values of \(\eta\), we have calculated the waves’ dispersion curves (and growth rates when the waves are unstable) for \(\eta = 4\) and 0.586. The relative Alfvénic Mach number \(M_A\) is a running entry parameter whose values vary from zero to some reasonable numbers. Our choice for the Hall parameter is \(\varepsilon = 0.4\). We will first discuss the results for the kink mode, and later on for the sausage one.

3.1. Kink waves

It is clear from the explicit solutions to the dispersion equation of the standard kink waves that for each \(M_A\) we have two curves when the waves are stable and only one curve as they become unstable. That is seen in figures 2 and 3(a), where the curves are calculated with \(\eta = 4\). For relatively small Alfvénic Mach numbers one observes one forward and one backward propagating wave (see figure 2). For slightly bigger Alfvénic Mach numbers, say \(M_A \geq 1.42\), both waves become forward running ones and at \(M_A = 1.525\) (figure 3(a)) they start to merge forming close dispersion curves. At some critical value of \(M_A\), in our case at \(M_A = 1.6\), the waves become unstable and their growth rate can be seen in figure 3(b). The instability is found to be of the Kelvin–Helmholtz type and the growth rates for the other three relative Alfvénic Mach numbers are plotted in the same figure—the corresponding dispersion curves are clearly depicted in figure 3(a). We have to emphasize that according to Andries and Goossens [13] the Kelvin–Helmholtz instability onset starts at \(U > v_Ao + v_Ae\), or equivalently at

\[ M_A > 1 + 1/\eta^{1/2}. \]
In our case ($\eta = 4$) the above inequality requires $M_A > 1.5$ in order to expect an instability onset. Obviously, our numerical calculations confirm the applicability of that criterion. It is interesting to note that Holzwarth \textit{et al} [30] in their study on the flow instabilities of magnetic flux tubes (in the framework of the approximation of thin magnetic flux tubes) also stress the circumstance that the Kelvin–Helmholtz instability onset crucially depends on the parameter $\eta$. However, their criterion for the instability onset in our notation (see equation (5) in [30]) has the form

$$M_A > (1 + 1/\eta)^{1/2}$$

and yields threshold values for $M_A$ much lower than those obtained by the numerical solving of equation (19) for the kink mode. That is why we think that the Andries and Goossens inequality is a better choice for such a criterion.

One must mention that one can get dispersion curves and growth rates of kink unstable waves for negative values of the relative Alfvénic Mach number; however, those waves are backward propagating ones and not acceptable (for the solar wind) from a physical point of view. The reason for that conclusion is that the MHD surface waves are the incompressible vestige of the slow magnetosonic waves of the compressible magnetohydrodynamics which propagate at the group velocity, $v_A$, either parallel or antiparallel to $B_0$ depending upon the sign of $k_z$ [16].

As already mentioned, the dispersion relations of the Hall-MHD surface waves are polynomials of sixth order possessing all non-zero coefficients. That means we expect to get six different dispersion curves for each $M_A$. In figure 4(a) we show the dispersion curves of the kink waves propagating along a static ($M_A = 0$) flux tube with $\eta = 4$. It is seen that we have three forward and three backward propagating waves, the latter being mirror images of the forward running waves. Considering only the forward waves one can divide them into three categories, namely fast (super-Alfvénic) kink waves, Alfvénic kink waves and slow (sub-Alfvénic) kink waves. Figure 4(b) shows the evolution of the fast forward kink waves with increasing relative Alfvénic Mach number $M_A$. It turns out that all the dispersion curves correspond to stable wave propagation. The evolution of the Alfvénic kink waves can be seen in figure 4(c). It becomes clear from this figure that for $M_A > 1.5$ the shape of the dispersion curves dramatically changes. In the region of the relative Alfvénic Mach numbers, $1.575–1.6$, where one can expect an instability onset, we get dispersion curves corresponding to a stable wave propagation—for these and bigger Alfvénic Mach numbers the imaginary part of the normalized wave phase velocity is zero. One observes similar picture for the sub-Alfvénic
Hall-MHD kink waves also—all curves shown in figure 4(d) correspond to stable travelling waves. It is worth mentioning that for relatively Alfvénic Mach numbers correspondingly equal to 1.5, 1.525 and 1.55 the kink waves start as forward propagating ones, but for dimensionless wave numbers \( k_\alpha \) greater (roughly) than one they change the direction of their propagation and become backward running waves.

As already pointed out, the propagation (and stability) properties of the eigenmodes running on a flux tube depend on the value of the parameter \( \eta \). Let us now take \( \eta = 0.586 \) corresponding more or less to frequently detected sound and Alfvén speeds in the solar wind. The dispersion curves of forward propagating standard kink waves are shown in figure 5(a). There, one can observe closed dispersion curves (see curves labelled 2.35, 2.375 and 2.4) and for \( M_\lambda \geq 2.4 \) the waves become unstable—the instability is of the Kelvin–Helmholtz type. We note that according to the above-mentioned criterion of Andries and Goossens for the instability onset, for this specific value of \( \eta \) it requires \( M_\lambda > 2.3 \). It is worth pointing out that for relatively small values of \( \eta \) the critical value of the Alfvénic Mach number at which the instability starts can be rather large. That is why it is not surprising, for instance, that Vasheghani Farahani et al [25] in their study of transverse wave propagation in soft x-ray coronal jets claim that the kink surface waves are not found to be subjected to the Kelvin–Helmholtz instability. This is because their Alfvénic Mach number equal to 0.725 is very small compared with the critical value of 4.511 (more exactly 4.5100735), numerically obtained.
with their $\eta = 0.13$. The growth rates of the unstable waves for our $\eta = 0.586$ are plotted in figure 5(b). One immediately sees how different these curves are in shape compared with those depicted in figure 3(b). Here, we should remark that all obtained growth rates actually correspond to the instability onset—when the waves’ amplitudes become higher the linear theory is no longer applicable and one has to use another, non-linear, approach to study further the evolution of the waves.

The dispersion curves of the kink Hall-MHD surface waves for $\eta = 0.586$ propagating on a static flux tube ($M_A = 0$) and on a jet for various magnitudes of the relative Alfvénic Mach number are shown in figure 6. It turns out that with $\eta = 0.586$ the kink waves are also stable for each $M_A$. The most interesting figure here is figure 6(c). For $M_A = 2.5$ one can obtain a piece of curve starting with $\text{Re}(v_{ph}/v_A) \approx 1.7$ and touching the lobe of the green curve labelled 2.5—the imaginary parts of the normalized wave phase velocities associated with that piece of curve are non-zero. However, in our opinion, this would be a spurious unstable branch. It is extremely unbelievable that the change in the wave phase velocity at the cross point is an abrupt one. Similar spurious pieces of dispersion curves have been obtained in solving the dispersion equation (19) (with $m = 1$) of the standard kink surface waves for $2.35 \leq M_A \leq 2.4$—those pieces of curves have been common for both solutions of $V_{ph}$ (with plus and minus signs in front of the square root) and they touch the lobes of the corresponding closed (island-shaped) dispersion curves. Thus, we have to conclude that for $\eta = 0.586$ the kink Hall-MHD waves are not subjected to the Kelvin–Helmholtz instability irrespective of the value of the relative Alfvénic Mach number. It is really a surprising conclusion that the Hall effect stabilizes the kink mode in plasmas with parameters similar to those observed in the solar wind, for which the same mode in the context of the standard magnetohydrodynamics is unstable.

### 3.2. Sausage waves

The dispersion curves of the standard sausage MHD surface waves travelling along the jet have been calculated by using equation (19) with $m = 0$ and are displayed in figure 7. For fairly small relative Alfvénic Mach numbers, less than or equal to 1, we have both forward and backward propagating waves. With an increase in $M_A$ the shape of the dispersion curves changes and at $M_A = 2.5$ both curves merge (see figure 7(b)) narrowing the range of propagation of the waves. Moreover, for such big enough relative Alfvénic Mach numbers for a fixed normalized wave number $k_c a$ we have multiple (in our case two) solutions for the normalized wave phase
velocity. Which one of these two velocities will be registered during the wave propagation cannot be predicted by the theory. It is rather astonishing that we get no complex solutions to the wave dispersion equation, i.e. the standard sausage MHD surface waves are stable against the Kelvin–Helmholtz instability.
Figure 8. (a) Dispersion curves of the Alfvénic forward propagating Hall-MHD sausage waves for $\eta = 0.586$ and various values of $M_A$. (b) Dispersion curves of the sub-Alfvénic forward propagating Hall-MHD sausage waves for the same magnitude of $\eta$ and various values of $M_A$. (Colour online.)

Let us see whether the inclusion of the Hall effect will change the stability state of the sausage mode. In the next figures we show the dispersion curves of the Hall-MHD sausage surface waves running on a solar wind jet. As in the case of the kink waves, for $M_A = 0$ (propagation along a static flux tube) we obtain six distinctive dispersion curves for the Hall-MHD sausage waves—three of them correspond to forward wave propagation and the next three to backward wave propagation. We are not going to plot these curves because they are very similar to the curves shown in figure 6(a). The same is also true for the super-Alfvénic sausage waves. However, where the evolution of the fast Hall-MHD sausage waves with the inclusion of flow is similar to that of the kink waves, the evolution of the initially almost Alfvénic sausage waves is completely different (compare figures 6(c) and 8(a)). It is seen in figure 8(a) that those waves quickly become super-Alfvénic and only at sufficiently large relative Alfvénic Mach numbers are their ranges of propagation severely reduced—see, for example, the curve labelled 3 in figure 8(a). Figure 8(b) shows the dispersion curves of the sub-Alfvénic Hall-MHD sausage waves for various relative Alfvénic Mach numbers. It is seen, as in the case of the kink Hall-MHD waves, that the sausage waves possess semi-closed dispersion curves which significantly narrow the range of propagation of the waves—see, for instance, the curve labelled 2.25 in figure 8(b). Here, however, we observe a notable peculiarity: for $M_A = 2.5$ and 2.75 in the same range of the normalized waves’ phase velocities there appear two dispersion curves (see the curves labelled 2.5 and 2.75 on the right-hand side of the plot in figure 8(b)), which initially (for zero or small relative Alfvénic Mach numbers) belong to backward propagating Hall-MHD sausage waves. Probably it is not very surprising now to say that the sausage Hall-MHD waves, similarly to the standard ones, are stable against the Kelvin–Helmholtz instability. The Müller method which has been used for solving the complex dispersion equation (18) with $m = 0$ does not yield complex roots—the imaginary parts of the normalized wave phase velocities were always equal to zero. The stability state of the sausage Hall-MHD waves is not changed for negative Alfvénic Mach numbers, either. In that case, however, there occur fast sausage waves whose normalized phase velocities are smaller than those of the super-Alfvénic waves for positive values of the relative Alfvénic Mach numbers. A similar phenomenon takes place for the kink Hall-MHD waves also.

As mentioned in section 1, the dispersion characteristics of both standard and Hall-MHD waves depend upon the value of the entry parameter $\eta$. We have performed calculations for a wide range of magnitudes of that important parameter, namely $\eta = 0.16, 0.98, 10$ and, of
course, for $\eta = 0.586$ and 4 without finding any reliable dispersion curves corresponding to some instability wave states. The shapes of the obtained dispersion curves for the aforementioned values of the parameter $\eta$ actually looked different, but their general features for a given eigenmode are more or less rather similar.

4. Conclusion

Let us summarize the main findings of our study. In investigating the wave propagation along an incompressible plasma jet moving with respect to the environment with a constant speed $U$ we had to take into account the influence of two factors: (i) the Hall term in the generalized Ohm’s law and (ii) the flow itself. The combining effect of these two factors can be expressed as follows:

- The Hall term generally expands the range of propagation of the wave modes, not only for static tubes but also for jets with $M_A \neq 0$. While the number of dispersion curves of a standard MHD surface mode for a fixed $M_A$ is two (one for the forward propagating waves and the other for the backward propagating waves), for a Hall-MHD surface mode that number is six—we have generally three forward travelling waves and three backward running ones. One may expect that the two waves with phase velocities close to the Alfvén speed would correspond to left- and right-circularly polarized waves while the third one—the super-Alfvénic wave—might be a linearly polarized mode. The correct treatment of the waves’ polarizations, however, requires retaining terms of order $\alpha/\omega_{ci}$ in the governing equations and for a full description of the right-hand polarized wave one also has to include the electron dynamics. Thus, in order to model all effects associated with the Hall term and waves polarizations, one must use a two-fluid magnetohydrodynamics which falls beyond the scope of this paper.

- With increasing relative Alfvénic Mach number, $M_A$, at some critical value the standard kink waves become unstable and the instability is of the Kelvin–Helmholtz type. It turns out, however, that the kink Hall-MHD waves are stable for any relative Alfvénic Mach number (be it positive or negative).

- The sausage surface waves are stable against the Kelvin–Helmholtz instability for any relative Alfvénic Mach number, both in the framework of the standard magnetohydrodynamics and the Hall one.

- The Hall term in the generalized Ohm’s law, in addition to extending the number of the possible dispersion curves, also dramatically changes the shapes of the curves corresponding in the limit $M_A \to 0$ to slow (sub-Alfvénic) and almost Alfvénic waves. In particular, for sufficiently large Alfvénic Mach numbers their ranges of propagation are narrowed and for a fixed normalized wave number $K = k/a$ one may have two distinctively different normalized phase velocities.

It is interesting and very instructive to compare our results with those associated with the propagation of Hall-MHD waves in a flowing incompressible plasma slab along the constant external magnetic field $B_0$ [16]. In the slab geometry the Hall term generally limits the range of propagation of the wave modes, not only for static tubes/layers but also for jets possessing positive Alfvénic Mach numbers. The limiting normalized wave number [31]

$$K_{\text{limit}} = (1 + \eta)^{1/2}/\varepsilon$$

is specified by two plasma parameters: the ratio of densities of the two plasma media (outside and inside the jet), $\eta$, and the Hall parameter, $\varepsilon$. On approaching that wave number the wave phase velocity becomes very high. When $M_A$ is negative the real part of the phase velocity of
the eigenmodes (kink and sausage waves) is forced due to the presence of flow to go beyond that limiting wave number in a region where the wave becomes unstable (or overstable). The instability which occurs is of the Kelvin–Helmholtz type. The main conclusion in [16] is that the Hall current keeps stable the surface modes travelling in flowing solar plasmas within the dimensionless wave number range between 0 and $K_{\text{limit}}$ for each relative Alfvénic Mach number. Instability of the Kelvin–Helmholtz type for the forward propagating waves is only possible at negative Alfvénic Mach numbers in a wave number range lying beyond the $K_{\text{limit}}$. The instability can disappear as $|M_A|$ reaches some value depending on the magnitude of the parameter $\eta$ (see figures 12 and 13 and read the comments for them in [16]). One of the unexpected surprises of our study in cylindrical geometry was that we did not get, either analytically or numerically, such a $K_{\text{limit}}$. Hence, if we extrapolate the main conclusion of [16], we can claim that the Hall-MHD surface waves propagating along a cylindrical incompressible solar wind plasma jet should be stable with respect to the Kelvin–Helmholtz instability. It remains to be seen whether this statement will be valid when the plasma compressibility is taken into account. An examination of the dispersion characteristics of the Hall-MHD eigenmodes and their stability status in a compressible cylindrical solar wind jet is in progress and will be reported elsewhere. Another important extension of the Hall MHD in studying waves and turbulence in solar plasmas is to take into account the electron inertia and finite Larmor radius effects [32], i.e. to consider the aforementioned phenomena in the context of a two-fluid Hall MHD.

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