Classification of divisible design graphs with at most 39 vertices

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Abstract
A k-regular graph is called a divisible design graph (DDG) if its vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly $\lambda_1$ common neighbours, and two vertices from different classes have exactly $\lambda_2$ common neighbours. A DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is called improper, otherwise it is called proper. We present new constructions of DDGs and, using a computer enumeration algorithm, we find all proper connected DDGs with at most 39 vertices, except for three tuples of parameters: (32, 15, 6, 7, 4, 8), (32, 17, 8, 9, 4, 8), and (36, 24, 15, 16, 4, 9).

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divisible design, divisible design graph, walk-regular graph

MATHEMATICAL SUBJECT CLASSIFICATION
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1 | INTRODUCTION

An incidence structure with constant block size k is a (group) divisible design whenever the point set can be partitioned into m classes of size n, such that two points from one class occur together in exactly $\lambda_1$ blocks, and two points from different classes occur together in exactly $\lambda_2$ blocks. A divisible design $D$ is called symmetric or to have the dual property (SDD) if the dual of $D$ (i.e., the design with the transposed incidence matrix) is again a divisible design with the same parameters as $D$. A divisible design graph (DDG) is a graph whose adjacency matrix is the incidence matrix of a symmetric divisible design. A DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is called improper (these DDGs are $(v, k, \lambda)$-graphs), otherwise it is called proper.
At first, DDGs were studied in master’s thesis by Meulenberg [7] and then studied in more detail in 2011 in the following paper by Haemers et al. [6] and in 2011–2013 in two following papers by Crnkovic and Haemers [2,3].

In 2008, Meulenberg presented the list of feasible parameters of proper DDGs up to 50 vertices. In 2011–2013, feasible parameters of proper DDGs up to 27 vertices were studied and the existence of graphs was resolved in all but one case, however, the exact number of graphs corresponding to these tuples of parameters remained unknown.

In this paper, we present new constructions of DDGs and find all proper connected DDGs with at most 39 vertices, except for three tuples of parameters: \((32, 15, 6, 7, 4, 8)\), \((32, 17, 8, 9, 4, 8)\), and \((36, 24, 15, 16, 4, 9)\).

The paper is organised as follows. In Section 2 we give some definitions, notations, and preliminaries about DDGs. In Section 3 we give an overview of known constructions of DDGs and in Section 4 we describe some new constructions. In Section 5 we describe some new sporadic constructions of DDGs. In Section 6 we describe the algorithm used for enumerating DDGs and in Section 7 we present the results.

\section{Preliminaries}

Let \(I_t\) and \(J_t\) be an identity \(t \times t\) and all-ones \(t \times t\) matrix, respectively, and \(K_{(m,n)}\) be 
\[ I_n \otimes J_n = \text{diag}(J_n, ..., J_n). \]
Then a graph \(\Gamma\) is a DDG with parameters \((v, k, \lambda_1, \lambda_2, m, n)\) if and only if \(\Gamma\) has an adjacency matrix \(A\) that satisfies
\[ A^2 = kI_v + \lambda_1(K_{(m,n)} - I_v) + \lambda_2(J_v - K_{(m,n)}). \]

In DDGs \(v = mn\), and taking row sums on both sides of the equation above gives
\[ k^2 = k + \lambda_1(n - 1) + \lambda_2n(m - 1). \]

The formula for \(A^2\) also gives us strong information about the eigenvalues of \(A\) and their multiplicities.

\textbf{Lemma 1} (Haemers et al. [6, lemma 2.1]). \(A\) has at most five distinct eigenvalues \(k, \sqrt{k - \lambda_1}, -\sqrt{k - \lambda_1}, \sqrt{k^2 - \lambda_2^2v}, -\sqrt{k^2 - \lambda_2^2v}\) with corresponding multiplicities \(f_1, f_2, g_1, g_2\), where \(f_1 + f_2 = m(n - 1)\) and \(g_1 + g_2 = m - 1\).

Some of the multiplicities may be 0 and some values may coincide. In general, the multiplicities \(f_1, f_2, g_1,\) and \(g_2\) are not determined by the parameters, but if we know one, we can compute the rest because \(f_1 + f_2 = m(n - 1), g_1 + g_2 = m - 1\) and
\[ \text{trace}(A) = 0 = k + (f_1 - f_2)\sqrt{k - \lambda_1} + (g_1 - g_2)\sqrt{k^2 - \lambda_2^2v}. \]

This equation leads to the following result.

\textbf{Lemma 2} (Haemers et al. [6, theorem 2.2]). Consider a proper DDG with parameters \((v, k, \lambda_1, \lambda_2, m, n)\) and eigenvalue multiplicities \(f_1, f_2, g_1, g_2\). Then
(1) \( k - \lambda_1 \) or \( k^2 - \lambda_2 v \) is a nonzero square.
(2) If \( k - \lambda_1 \) is not a square, then \( f_1 = f_2 = m(n - 1)/2 \).
(3) If \( k^2 - \lambda_2 v \) is not a square, then \( g_1 = g_2 = (m - 1)/2 \).

Let \( V_1 \cup V_2 \cup \ldots \cup V_t \) be the partition of the vertex set of a graph \( \Gamma \) with the property that every vertex of \( V_i \) has exactly \( r_{ij} \) neighbours in \( V_j \). Then \( V_1 \cup V_2 \cup \ldots \cup V_t \) will be an equitable \( t \)-partition of \( \Gamma \). Matrix \( R = (r_{ij})_{t \times t} \) is called the quotient matrix of the equitable partition.

**Lemma 3** (Haemers et al. [6, theorem 3.1]). The vertex partition from the definition of a DDG (the canonical partition) is equitable and the quotient matrix \( R \) satisfies

\[
R^2 = RR^T = (k^2 - \lambda_2 v)I_m + \lambda_2 n I_m.
\]

The eigenvalues of \( R \) are \( k, \sqrt{k^2 - \lambda_2 v}, -\sqrt{k^2 - \lambda_2 v} \) with corresponding multiplicities \( 1, g_1, g_2 \).

**Lemma 4** (Haemers et al. [6, proposition 3.2]). The quotient matrix \( R \) of a DDG satisfies

\[
\sum_i r_{ij} = k \text{ for } j = 1, \ldots, m,
\]

\[
\sum_{i,j} r_{ij}^2 = \text{trace}(R^2) = mk^2 - (m - 1)\lambda_2 v,
\]

\[
0 \leq \text{trace}(R) = k + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v} \leq m(n - 1).
\]

A graph is walk-regular, whenever for every \( l \geq 2 \) the number of closed walks of length \( l \) at a vertex \( x \) is independent of the choice of \( x \). Note that walk-regularity implies regularity (take \( l = 2 \)).

A DDG with four distinct eigenvalues is walk-regular, provided it is connected [2, corollary 4.2]. A DDG with five distinct eigenvalues can also be walk-regular. To decide on this the following lemma can be used.

**Lemma 5** (Crnkovic and Haemers [2, theorem 4.3]). A proper DDG is walk-regular if and only if the quotient matrix \( R \) has constant diagonal.

### 3 | KNOWN CONSTRUCTIONS

#### 3.1 | \((v, k, \lambda)\)-graphs and designs

The incidence graph of a design with incidence matrix \( N \) is a bipartite graph with adjacency matrix

\[
\begin{bmatrix}
O & N \\
N^T & O
\end{bmatrix}.
\]

**Construction 1** (Haemers et al. [6, construction 4.1]). The incidence graph of an \((n, k, \lambda_1)\)-design with \( 1 < k \leq n \) is a proper DDG with \( \lambda_2 = 0 \).

**Proposition 1** (Haemers et al. [6, Proposition 4.3]). For a proper connected DDG \( \Gamma \) the following are equivalent.
(1) $\lambda_2 = 0$.
(2) $\Gamma$ comes from Construction 1.

**Construction 2** (Haemers et al. [6, construction 4.4]). If $A'$ is the adjacency matrix of an $(m, k', \lambda')$-graph ($1 \leq k' < n$), then $A = A' \otimes J_n$ is the adjacency matrix of a proper DDG with $k = \lambda_1 = nk'$, $\lambda_1 = n\lambda'$.

**Proposition 2** (Haemers et al. [6, proposition 4.5]). For a proper DDG $\Gamma$ the following are equivalent.

(1) $\lambda_1 = k$.
(2) $\Gamma$ comes from Construction 2.

**Construction 3** (Haemers et al. [6, construction 4.6]). Let $A_1, \ldots, A_m$ ($m \geq 2$) be the adjacency matrices of $m$ $(n, k', \lambda')$-graphs with $0 \leq k \leq n - 2$. Then $A = J_n - K_{m,n} + \text{diag}(A_1, \ldots, A_m)$ is the adjacency matrix of a proper DDG with $k = k' + n(m - 1)$, $\lambda_1 = \lambda' + n(m - 1)$, $\lambda_2 = 2k - v$.

**Proposition 3** (Haemers et al. [6, proposition 4.7]). For a proper DDG $\Gamma$ the following are equivalent:

(1) $\lambda_2 = 2k - v$.
(2) $\Gamma$ comes from Construction 3.

### 3.2 DDGs with $\lambda_1 = k - 1$

The lexicographic product or graph composition $G[H]$ of graphs $G$ and $H$ is a graph such that the vertex set of $G[H]$ is $V(G) \times V(H)$ and adjacency defined by

$$(u_1, u_2) \sim (v_1, v_2) \text{ if and only if } u_1 \sim v_1 \text{ or } (u_1 = v_1 \text{ and } u_2 \sim v_2).$$

**Construction 4** (Haemers et al. [6, theorem 4.11]). If $G$ is a strongly regular graph with parameters $(v, k, \lambda, \lambda + 1)$, then $G[K_2]$ is a DDG with parameters $(2v, 2k + 1, 2k, 2\lambda + 2, v, 2)$. If $G$ is $K_{x,y}$, the complete multipartite graph containing $x$ parts of $y$ vertices, then $G[K_2]$ is a DDG with parameters $(2xy, 2y(x - 1) + 1, 2y(x - 1), 2y(x - 2) + 2, x, 2y)$.

An involutive automorphism of a graph is called *Seidel automorphism* if it interchanges only nonadjacent vertices. Permuting the rows (and not the columns) of the adjacency matrix of a graph according to Seidel automorphism is called *dual Seidel switching*.

**Construction 5** (Goryainov et al. [4, construction 2]). Let $\Gamma$ be a DDG obtained with the first case of Construction 4. Let $M$ be the adjacency matrix of $\Gamma$, and $P$ be a nonidentity permutation matrix of the same size. Then $PM$ is the adjacency matrix of a DDG if and only if $P$ represents a Seidel automorphism.
Proposition 4 (Goryainov et al. [4, theorem 2]). For a proper DDG \( \Gamma \) with \( \lambda_2 \neq 0 \) the following are equivalent.

1. \( \lambda_i = k - 1 \).
2. \( \Gamma \) comes from Construction 4 or 5.

3.3 | Distance-regular graphs

Suppose that \( G \) is a graph with diameter \( d \). For any vertex \( u \) and for any integer \( i \), where \( 0 \leq i \leq d \), let \( G_i(u) \) denote the set of vertices at distance \( i \) from \( u \). If \( u' \in G_i(u) \) and \( w \) is a neighbour of \( u' \), then \( w \) must be at distance \( i - 1, i \) or \( i + 1 \) from \( u \). Let \( c_i, a_i \) and \( b_i \) denote the number of such vertices \( w \). \( G \) is a distance-regular graph if and only if the parameters \( c_i, a_i, b_i \) depend only on the distance \( i \), and not on the choice of \( u \) and \( u' \) (i.e., \( a_i + b_i + c_i = k = b_0, c_1 = 1 \)). The array \( \{k, b_1, ..., b_{d-1}; 1, c_2, ..., c_d\} \) is called the intersection array of the distance-regular graph. A distance-regular graph of diameter \( d \) is called antipodal if being at distance \( d \) or 0 defines an equivalence relation on the vertices. For a distance-regular graph the parameters \( \lambda \) and \( \mu \) give the number of common neighbours of a pair of vertices at distance 1 and 2, respectively (i.e., \( \lambda = a_1, \mu = c_2 \)).

Distance regular graphs of diameter 2 are strongly regular graphs with parameters \((v, k, \lambda, \mu)\).

Construction 6 (Haemers et al. [6, Theorem 4.13]). Suppose \( \Gamma \) is an antipodal distance-regular graph of diameter 3. If \( \lambda = \mu \), then \( \Gamma \) is a proper DDG with parameters \((\mu n + 2), \mu n + 1, 0, \mu, \mu n + 2, n)\). If \( \lambda = \mu - 2 \), then the complement of \( \Gamma \) is a proper DDG with parameters \((\mu n^2, \mu n(n - 1), \mu n(n - 2), \mu(n - 1)^2, \mu n, n)\).

3.4 | Partial complements

The partial complement of a DDG is a graph whose adjacency matrix can be obtained as the complement of all blocks of the canonical partition except the diagonal blocks.

Proposition 5 (Haemers et al. [6, proposition 4.15]). The partial complement of a proper DDG \( \Gamma \) is again a DDG if one of the following holds:

1. The quotient matrix \( R \) equals \( t(I_m - I_m) \) for some \( t \in \{1, ..., n - 1\} \).
2. \( m = 2 \).

Let \( G \) be a \( k \)-regular graph on \( v \) vertices with the smallest eigenvalue \( \lambda_{\text{min}} \). A Hoffman colouring of \( G \) is a partition of the vertices into Hoffman-cocliques, that is, cocliques meeting the Hoffman upper bound \( c = v\lambda_{\text{min}} / (\lambda_{\text{min}} - k) \). An equitable partition of a \((v, k, \lambda)\)-graph that satisfies (1) from Proposition 5 is a Hoffman colouring.

Construction 7 (Haemers et al. [6, construction 4.16]). Let \( \Gamma \) be a \((v, k, \lambda)\)-graph. If \( \Gamma \) has a Hoffman colouring or an equitable partition into two parts of equal size, then the partial complement is a DDG.
3.5 Hadamard matrices

An \( m \times m \) matrix \( H \) is a Hadamard matrix if every entry is 1 or \(-1\) and \( HH^\top = mI_m \). A Hadamard matrix \( H \) is called graphical if \( H \) is symmetric with constant diagonal, and regular if all row and column sums are equal.

**Construction 8** (Haemers et al. [6, construction 4.8]). Consider a regular graphical Hadamard matrix \( H \) of order \( m \geq 4 \) and row sum \( l = \pm \sqrt{m} \). Let \( n \geq 2 \). Replace each entry with value \(-1\) by \( J_n - I_n \), and each +1 by \( I_n \), then we obtain the adjacency matrix of a DDG with parameters \((mn, n(m - l)/2 + l, (n - 2)(m - l)/2, n(m - 2l)/4 + l, m, n)\).

**Construction 9** (Haemers et al. [6, construction 4.9]). Consider a regular graphical Hadamard matrix \( H \) of order \( l^2 \geq 4 \) with diagonal entries \(-1\) and row sum \( l \). The graph with adjacency matrix

\[
A = \begin{bmatrix}
M & N & O \\
N & O & M \\
O & M & N
\end{bmatrix},
\]

where

\[
M = \frac{1}{2} \begin{bmatrix}
J_{l^2} + H & J_{l^2} + H \\
J_{l^2} + H & J_{l^2} + H
\end{bmatrix}
\]

\[
N = \frac{1}{2} \begin{bmatrix}
J_{l^2} + H & J_{l^2} - H \\
J_{l^2} - H & J_{l^2} + H
\end{bmatrix},
\]

is a DDG with parameters \((6l^2, 2l^2 + l, l^2 + l, (l^2 + l)/2, 3, 2l^2)\).

**Construction 10** (Crnkovic and Haemers [2, theorem 3.2]). If there exist a regular graphical Hadamard matrix of order \( 4u^2 \) with row sum \( 2u \) and a Hadamard matrix of order \( 2u^2 \), then there exists a DDG with parameters \((24u^2, 12u^2 - 2u, 4u^2 - 2u, 6u^2 - 2u, 12u^2, 2)\).

3.6 DDGs with parameters \((4n, n + 2, n - 2, 2, 4, n)\) and \((4n, 3n - 2, 3n - 6, 2n - 2, 4, n)\)

An \( m \times n \)-lattice graph is a line graph of complete bipartite graph \( K_{m,n} \).

**Construction 11** (Shalaginov [8, theorem 1]). An \( m \times n \)-lattice graph is a DDG if and only if \( n = 4 \). These graphs have parameters \((4n, n + 2, n - 2, 2, 4, n)\). If \( m = 4 \), then the graph is strongly regular with parameters \((16, 6, 2, 2)\).

**Construction 12** (Shalaginov [8, construction 4]). Let \( M \) be the adjacency matrix of a \( 4 \times n \)-lattice graph. Let \( M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} \), such that \( M_{11} \) is the adjacency matrix of the subgraph \( H \), which is isomorphic to the \( 2 \times n \)-lattice graph. Consider the permutation matrix \( P = \begin{bmatrix}
P_{11} & 0 \\
0 & I
\end{bmatrix} \), where \( P_{11} \) is the permutation matrix of the Seidel automorphism \( \varphi \).
corresponding to the central symmetry of \( H \). Then \[
\begin{bmatrix}
P_{11} & M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]
is the adjacency matrix of a DDG with parameters \((4n, n + 2, n - 2, 2, 4, n)\).

The switching of edges between two sets of vertices of a graph is the reversion of the adjacency of each pair of vertices, one from the first set and other from the second set. Thus, the edge set is changed so that an adjacent pair becomes nonadjacent and a nonadjacent pair becomes adjacent.

**Construction 13** (Shalaginov [8, construction 5]). Consider \( C_{4t}[K_2] \) \((t \geq 1)\) and the 4-cube, where \( C_{4t} \) is the 4t-cycle. Each of these graphs has an equitable partition with quotient matrix \( J_4 \). Consider some copies of \( C_{4t}[K_2] \) and some copies of the 4-cube with a fixed equitable partition. Denote by \( V_1, V_2, V_3, V_4 \) the classes of this partition. The switching of edges between \( V_1 \) and \( V_2 \) and also between \( V_3 \) and \( V_4 \) gives a DDG with parameters \((4n, n + 2, n - 2, 2, 4, n)\).

For more information about equitable partitions of the 4-cube and \( C_{4t}[K_2] \) see lemmas 5–8 from [8]. Note that graphs obtained from Construction 11, Construction 12, and Construction 13 have the same parameters, but different spectra.

**Proposition 6** (Shalaginov [8, theorem 1]). Let \( \Gamma \) be a DDG with parameters \((4n, n + 2, n - 2, 2, 4, n)\). Then:

(1) If \( n \) is odd, then \( \Gamma \) is isomorphic to \( 4 \times n \)-lattice graph.

(2) If \( n \) is even, then \( \Gamma \) comes from Construction 11, 12, or 13.

**Construction 14** (Shalaginov [8, theorem 2]). Let \( \Gamma \) be a DDG with parameters \((4n, n + 2, n - 2, 2, 4, n)\). The switching of edges between the union of two classes of the canonical partition and the remaining vertices gives a DDG with parameters \((4n, 3n - 2, 3n - 6, 2n - 2, 4, n)\).

### 4 | NEW CONSTRUCTIONS

**Construction 15.** Suppose \( \Gamma \) is an antipodal distance-regular graph of diameter 3 with antipodal classes of size \( r \). Denote by \( A_i \) the matrix of a relation ‘to be at distance \( i \)’ on the vertices of \( \Gamma \). If \( \lambda = \mu + 2 \), then \( A = A_1 + A_3 \) is the adjacency matrix of a DDG with parameters \((r(2\mu + 4), 2\mu + r + 2, r - 2, \mu + 2, 2\mu + 4, r)\).

**Proof.** The intersection array of an antipodal distance-regular graph of diameter 3 is \([k, \mu(r - 1), 1; 1, \mu, k]\) [1, p. 431]. From this the statement of the construction follows straightforwardly.

**Construction 16.** Suppose \( \Gamma \) is a strongly regular graph with parameters \((v, k, \mu + 2, \mu)\) and has a Hoffman colouring with Hoffman-cocliques of size \( n \) \((v = mn)\). Let \( A \) be an adjacency matrix of \( \Gamma \), in which Hoffman-cocliques are located on the main diagonal,
\[ K = K_{(m,n)}, \quad I = I_v. \] Then \( A + K - I \) is the adjacency matrix of a DDG with parameters \((mn, k + n - 1, n + \mu - 2, \mu + \frac{2k}{m-1}, m, n)\).

**Proof.** Let’s calculate \((A + K - I)^2\) (we denote \(J = J_v\)).

\[
(A + K - I)^2 = A^2 + K^2 + I^2 + AK + KA - 2A - 2K.
\]

Since \(A^2 = kI + \lambda A + \mu (J - I - A)\), \(K^2 = nK\) and \(AK = KA = \frac{k}{m-1} (J - K)\),

\[
(A + K - I)^2 = kI + \lambda A + \mu (J - I - A) + nK + I + \frac{2k}{m-1} (J - K) - 2A - 2K
\]

\[
= (k - \mu + 1)I + \left( \mu + \frac{2k}{m-1} \right) J + \left( n - \frac{2k}{m-1} - 2 \right) K
\]

\[
+ (\lambda - \mu - 2)A.
\]

Since \(\lambda = \mu + 2\),

\[
(A + K - I)^2 = (k - \mu + 1)I + \left( \mu + \frac{2k}{m-1} \right) J + \left( n - \frac{2k}{m-1} - 2 \right) K,
\]

which can be rewritten as

\[
(A + K - I)^2 = (k + n - 1)I + (n + \mu - 2)(K - I) + \left( \mu + \frac{2k}{m-1} \right) (J - K).
\]

So, \(A + K - I\) is the adjacency matrix of a DDG with parameters \((mn, k + n - 1, n + \mu - 2, \mu + \frac{2k}{m-1}, m, n)\). \(\square\)

**Construction 17.** Let \(\Gamma\) be a DDG with parameters \((v, k, \lambda_1, \lambda_2, m, n)\) and quotient matrix \(R = aI_m + b(J_m - I_m)\). Take \(s\) copies of \(\Gamma\) and label all blocks of the canonical partition in each copy with numbers \(1, ..., m\). Then connect all vertices from blocks with the same label (adjacency inside the block does not change). The resulting graph is a DDG with parameters \((vs, k + (s - 1)n, \lambda_1 + (s - 1)n, \lambda_2, ms, n)\), if \(\lambda_2 = 2b = 2a + (s - 2)n\).

**Proof.** Consider two vertices from the same block of the same copy of \(\Gamma\). Then they have \(\lambda_1\) common neighbours in \(\Gamma\) plus \(n\) common neighbours in each other copy of \(\Gamma\). Thus, the number of common neighbours equals \(\lambda_1 + (s - 1)n\).

Consider two vertices from different blocks of the same copy of \(\Gamma\). Then they have \(\lambda_2\) common neighbours in \(\Gamma\) and no new common neighbours in other copies of \(\Gamma\). Thus, the number of common neighbours equals \(\lambda_2\).

Consider two vertices from the same block of different copies of \(\Gamma\). Then they have \(a\) common neighbours in their blocks (\(2a\) in total) plus \(n\) common neighbours in each other copy of \(\Gamma\). Thus, the number of common neighbours equals \(2a + (s - 2)n\).
Consider two vertices from different blocks of different copies of $\Gamma$. Then they have $b$ common neighbours in their copies of $\Gamma$ ($2b$ in total) and no new common neighbours in other copies of $\Gamma$. Thus, the number of common neighbours equals $2b$.

So, the resulting graph is a DDG with parameters $(v, k + (s - 1)n, \lambda_1 + (s - 1)n, \lambda_2, ms, n)$, if $\lambda_2 = 2b = 2a + (s - 2)n$.

**Construction 18.** Let $D$ be a symmetric divisible design with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, such that every block contains $\frac{k}{m}$ points from each class. $\Gamma$ is the incidence graphs of $D$. Construct a new graph $\Gamma^*$ with the same vertex set as $\Gamma$, where vertices $x, y$ are adjacent in $\Gamma^*$ when $x, y$ are adjacent in $\Gamma$, or $x, y$ are points from different classes of $D$, or $x, y$ are points from different classes of the dual design of $D$. Then $\Gamma^*$ is a DDG with parameters $(2v, k + (m - 1)n, \lambda_1 + (m - 1)n, \lambda_2 + (m - 2)n, 2m, n)$, if $2(m - 1)\frac{k}{m} = \lambda_2 + (m - 2)n$.

**Proof.** Consider two vertices, corresponding to two points from the same class of $D$. They have $\lambda_1$ common neighbours in $G$ plus $n$ common neighbours in each other class of points. Thus, we have $\lambda_1 + (m - 1)n$ common neighbours. The proof goes similarly for the case when we consider two vertices corresponding to two blocks from the same class.

Consider two vertices, corresponding to two points from different classes of $D$. They have $\lambda_2$ common neighbours in $G$ plus $n$ common neighbours in each other class of points. Thus, we have $\lambda_2 + (m - 2)n$ common neighbours. The proof goes similarly for the case when we consider two vertices corresponding to two blocks from different classes.

Consider two vertices, the first corresponding to a point and the second corresponding to a block of $D$. They have no common neighbours in $G$. They have $\frac{k}{m}$ common neighbours in each class of points except for the class the first vertex belongs to. They also have $\frac{k}{m}$ common neighbours in each class of blocks except for the class the second vertex belongs to. Thus, we have $2(m - 1)\frac{k}{m}$ common neighbours.

So, $\Gamma^*$ is a DDG with parameters $(2v, k + (m - 1)n, \lambda_1 + (m - 1)n, \lambda_2 + (m - 2)n, 2m, n)$, if $2(m - 1)\frac{k}{m} = \lambda_2 + (m - 2)n$.

**Construction 19.** Let $D$ be a symmetric divisible design with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ with even $m$, such that every block contains $\frac{k}{m}$ points from each class. Let $\Gamma$ be the incidence graphs of $D$. Split the set of classes of blocks and the set of classes of points into pairs. Construct a new graph $\Gamma^*$ with the same vertex set as $\Gamma$, where vertices $x, y$ are adjacent in $\Gamma^*$ when $x, y$ are adjacent in $\Gamma$ or $x, y$ are blocks from different classes of the dual design of $D$. Then $\Gamma^*$ is a DDG with parameters $(2v, k + n, \lambda_1 + n, \lambda_2, 2m, n)$, if $\frac{2k}{m} = \lambda_2$.

**Proof.** The proof is similar to Construction 18.

A weighing matrix $W(n, k)$ of order $n$ and weight $k$ is an $n \times n$ $(0, 1, -1)$-matrix, such that $WW^T = kl_n$.

**Construction 20.** Let $W$ be a $(4t, 4(t - 1))$-weighing matrix, such that the main diagonal of $W$ contains blocks of zeros of size 4. Construct matrix $A'$ by replacing each 0 with $O_2$, each 1 with
$I_2$ and each $-1$ with $J_2 - I_2$. Then matrix $A = A' + I_1 \otimes ((J_4 - I_4) \otimes J_2)$ is the adjacency matrix of a DDG $\Gamma$ with parameters $(8t, 4t + 2, 6, 2t + 2, 4t, 2)$.

Proof. Consider two vertices corresponding to two rows of $A$, which were obtained from one row of $W$ after replacing. These vertices form a block of canonical partition of $\Gamma$. They have only six common neighbours in $\Gamma$, which come from the addition of $I_1 \otimes ((J_4 - I_4) \otimes J_2)$.

Consider two vertices corresponding to two rows of $A$, which were obtained from different rows of $W$ after replacing. We have two cases: these vertices correspond to the same block of zeros from $W$ or they correspond to different blocks of zeros from $W$.

Consider two vertices from the first case. They have four common neighbours from addition of $I_1 \otimes ((J_4 - I_4) \otimes J_2)$ plus $4(t - 1)/2$ common neighbours from replacing. Thus, these vertices have $2t + 2$ common neighbours.

Consider two vertices from the second case. They have three common neighbours in each block of $\Gamma$, which correspond to a block of zeros from $W$. Thus, they have six common neighbours. They also have $4(t - 2)/2$ common neighbours from replacing. Thus, these vertices have $2t + 2$ common neighbours total.

So, $\Gamma$ is a DDG with parameters $(8t, 4t + 2, 6, 2t + 2, 4t, 2)$. $\square$

Construction 21. Let $\Gamma$ be a DDG obtained from Construction 20 with adjacency matrix $A$. The main diagonal of $A$ consists of $I_1 \otimes ((J_4 - I_4) \otimes J_2)$, which gives a partition of $\Gamma$ into complete multipartite graphs with four parts of size 2. Construct a new graph $\Gamma'$ by removing the edges of the complete bipartite subgraph $K_{4,4}$ from each part of this partition. Then $\Gamma'$ is a DDG with parameters $(8t, 4t - 2, 2, 2t - 2, 4t, 2)$.

Proof. The proof is similar to Construction 20. $\square$

5 | SPORADIC CONSTRUCTIONS

Construction 22. The following matrix $M$ is the adjacency matrix of a DDG with parameters $(27, 8, 4, 2, 9, 3)$:

$$M = \begin{bmatrix} O & J - I & J - I & J - I & J - I & O & O & O & O \\ J - I & O & T_1 & T_2 & T_3 & J & O & O & O \\ J - I & T_1 & O & T_3 & T_2 & O & J & O & O \\ J - I & T_2 & T_3 & O & T_1 & O & O & J & O \\ J - I & T_3 & T_2 & T_1 & O & O & O & O & J \\ O & J & O & O & O & J - I & I & I & I \\ O & O & J & O & O & I & J - I & I & I \\ O & O & O & J & O & I & I & J - I & I \\ O & O & O & O & J & I & I & I & J - I \end{bmatrix},$$

where $J = J_3, I = I_3, T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$
Construction 23. The following matrix $M$ is the adjacency matrix of a DDG with parameters $(28, 6, 2, 1, 7, 4)$:

$$M = \begin{bmatrix}
O & A & B & C & O & O & O \\
A^T & D & O & O & E & O & O \\
B^T & O & D & O & O & E & O \\
C^T & O & D & O & O & E & O \\
O & E & O & O & A & C & O \\
O & O & E & O & A^T & O & F \\
O & O & O & E & C^T & F^T & O 
\end{bmatrix}, \text{ where}

$$A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 
\end{bmatrix}, \quad
C = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 
\end{bmatrix},

$$D = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 
\end{bmatrix}, \quad
E = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 
\end{bmatrix}, \quad
F = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 
\end{bmatrix}.$$

Proposition 7. There exists two DDG with parameters $(32, 10, 2, 3, 8, 4)$ for which the adjacency matrix has the given structure: $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, where $A$ and $C$ are the adjacency matrices of the Shrikhande graph and $B$ is the incidence matrix of a symmetric divisible design with parameters $(16, 4, 0, 1, 4, 4)$.

6 | ENUMERATION ALGORITHM

6.1 | Search for feasible parameters

At the first step, for a fixed number of vertices $v$, we calculate all feasible parameters $(v, k, \lambda_1, \lambda_2, m, n)$ of DDGs with the approach based on Meulenberg's work [7]:

1. All possible numbers of classes $m$ and sizes of classes $n$ are calculated, $m \cdot n$ must be equal to $v$.
2. $k$ runs from 3 to $v - 3$, both $\lambda_1$ and $\lambda_2$ run from $\max(0, 2k - v)$ to $k$, $\lambda_1 \neq \lambda_2$.
3. For the remaining possibilities we check the conditions for parameters and spectrum given in Section 2.

6.2 | Enumeration of quotient matrices

Given feasible parameters $(v, k, \lambda_1, \lambda_2, m, n)$ of a DDG, we initially construct all possible quotient matrices $R$ using the following method:

1. We generate all possible $r_{11}$, $0 \leq r_{11} \leq n - 1$.
2. For all obtained $r_{11}$ we generate all unordered partitions of $k - r_{11}$ as sum of $m - 1$ numbers. We use unordered partitions since the permutation of $r_{ij}$, $j \geq 2$ does not change the final result.
We check if the generated row satisfies the equality for \( R^2 = (k^2 - \lambda_2 v)I_m + \lambda_2 nJ_m \). If not, we reject this row. Also in case of odd \( n \) all \( r_{i_1} \) must be even numbers.

We repeat Steps (1)–(3) for the next rows, but we generate ordered partitions instead. After we generate all possible \( i \geq 2 \) rows of the quotient matrix, we leave only nonequal up to classes renumbering options.

After we generate all nonequal quotient matrices for given parameters, in case where both \( k^2 - \lambda_2 v \) and \( k - \lambda_i \) are squares, we also compute all possible values of \( g_i \) using the equation

\[
\text{trace}(A) = 0 = k + (f_1 - f_2)\sqrt{k - \lambda_i} + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v}.
\]

Then we find the corresponding multiplicity for the generated quotient matrix and check if the found value is among possible values of \( \delta_i \). If not, we reject this quotient matrix.

6.3 | Constructing adjacency matrices

We say that two partially filled adjacency matrices are equivalent if for the graphs determined by them there is an isomorphism, which keeps partition into classes.

For a given quotient matrix \( R \) and tuple of parameters \((v, k, \lambda_1, \lambda_2, m, n)\) we generate all possible rows of the adjacency matrix, relying on the known partition into classes. For that, we use an exhaustive search of possible rows. For each new row \( t \) of the adjacency matrix \((1 \leq t \leq v)\) and each corresponding entry \( r_{i_j} \) of the quotient matrix all possible combinations of \( r_{i_j} \) 1s in \( n \) positions are considered.

For each \( t \), all partial matrices were checked for equivalence after adding all possible \( t \)th rows of the adjacency matrix and only the nonequivalent ones were considered for adding the next row.

At the last step, all obtained graphs were checked for isomorphism, and only nonisomorphic graphs were left.

We used SageMath for the search of feasible parameters, the enumeration of quotient matrices and equivalence/isomorphism check during the enumeration of adjacency matrices. For the procedure of adding a new row to the adjacency matrix, a C program was used.

7 | Enumeration results

In Table 1 below we enumerate the nontrivial proper DDGs on at most 39 vertices (trivial DDGs are DDGs that can be obtained from Constructions 1–5). The column indicated by # gives a number of nonisomorphic DDGs with specified parameters and spectrum; \( v, k, \lambda_1, \lambda_2, m, n \) are the parameters; \( \theta_1, \theta_2, \theta_3, \theta_4 \) are the nonprincipal eigenvalues with corresponding multiplicities; ‘WR’ denotes whether all graphs with specified parameters and spectrum are walk-regular, or all graphs are not walk-regular. The column ‘constructions’ refers to the constructions from Section 4, which can be used to obtain the specified graphs.

In column # the exclamation mark ‘!’ indicates that the number is the exact number of nonisomorphic graphs, a number followed by a ‘+’ gives the number of known nonisomorphic graphs (there can be more DDGs with the given parameters), a question mark ‘?’ indicates that the existence of DDGs with the given parameters remains unresolved.
| #   | v  | k  | λ₁ | λ₂ | m  | n  | $\Theta_1 f$ | $\Theta_2 f$ | $\Theta_3 f$ | $\Theta_4 f$ | WR      | constructions                                      |
|-----|----|----|----|----|----|----|-------------|-------------|-------------|-------------|---------|---------------------------------------------------|
| 1!  | 8  | 4  | 0  | 2  | 4  | 2  | $-2^1$      | $0^1$       | -           | +           | 2 $\times$ 4-lattice [c₈, c₁₁]                  |
| 1!  | 12 | 5  | 0  | 2  | 6  | 2  | $\sqrt{5}^3$ | $-\sqrt{5}^3$ | -           | $-1^3$     | +       | icosahedron [c₆]                                 |
| 1!  | 12 | 5  | 1  | 2  | 4  | 3  | $2^2$       | $-2^6$      | $1^1$       | -           | +       | 3 $\times$ 4-lattice [c₈, c₁₁]                  |
| 1!  | 12 | 6  | 2  | 3  | 3  | 4  | $2^3$       | $-2^6$      | $0^2$       | -           | +       | L(octahedron)                                    |
| 1!  | 12 | 7  | 3  | 4  | 4  | 3  | $2^2$       | $-2^6$      | $1^2$       | $-1^1$     | +       | c₈, c₁₄.11                                       |
| 1!  | 15 | 4  | 0  | 1  | 5  | 3  | $2^5$       | $-2^5$      | -           | $-1^4$     | +       | L(Petersen graph) [c₆]                           |
| 1!  | 18 | 9  | 6  | 4  | 6  | 3  | $\sqrt{3}^6$ | $-\sqrt{3}^6$ | $3^1$       | $-3^4$     | +       | c₁₈ from (9, 3, 0, 1)-SDD                       |
| 1!  | 20 | 7  | 3  | 2  | 4  | 5  | $2^4$       | $-2^{12}$   | $3^1$       | -           | +       | 5 $\times$ 4-lattice [c₈, c₁₁]                  |
| 1!  | 20 | 9  | 0  | 4  | 10 | 2  | $3^4$       | $-3^6$      | $1^3$       | $-1^6$     | -        | DSS(J(6,3))                                      |
| 1!  | 20 | 9  | 0  | 4  | 10 | 2  | $3^4$       | $-3^6$      | -           | $-1^9$     | +        | J(6,3) [c₆]                                     |
| 1!  | 20 | 13 | 9  | 8  | 4  | 5  | $2^4$       | $-2^{12}$   | $3^2$       | $-3^1$     | +       | c₈, c₁₄.11                                       |
| 2!  | 24 | 6  | 2  | 1  | 3  | 8  | $2^9$       | $-2^{12}$   | $\sqrt{12}^1$ | $-\sqrt{12}^1$ | -       | c₉                                               |
| 2!  | 24 | 7  | 0  | 2  | 8  | 3  | $\sqrt{7}^8$ | $-\sqrt{7}^8$ | -           | $-1^7$     | +        | Klein graph [c₆]                                |
| 2!  | 24 | 8  | 4  | 2  | 4  | 6  | $2^5$       | $-2^{15}$   | $4^1$       | -           | +        | 6 $\times$ 4-lattice [c₈, c₁₁]                  |
| 2!  | 24 | 8  | 4  | 2  | 4  | 6  | $2^7$       | $-2^{13}$   | $4^2$       | $-4^1$     | -        | c₁₂                                              |
| 5!  | 24 | 10 | 2  | 4  | 12 | 2  | $\sqrt{8}^6$ | $-\sqrt{8}^6$ | $2^3$       | $-2^8$     | +        | c₂₁                                              |
| 2!  | 24 | 10 | 3  | 4  | 8  | 3  | $\sqrt{7}^8$ | $-\sqrt{7}^8$ | $2^1$       | $-2^6$     | +        | one graph is Cayley (see [5])                   |
| 1!  | 24 | 10 | 6  | 3  | 3  | 8  | $2^8$       | $-2^{13}$   | $\sqrt{28}^1$ | $-\sqrt{28}^1$ | -       | c₉                                               |
| 1!  | 24 | 14 | 6  | 8  | 12 | 2  | $\sqrt{8}^6$ | $-\sqrt{8}^6$ | $2^2$       | $-2^9$     | +        | c₁₀, c₂₀                                         |
| 1!  | 24 | 14 | 7  | 8  | 8  | 3  | $\sqrt{7}^8$ | $-\sqrt{7}^8$ | $-2^7$     | +        | PC(Klein graph) [p₅]                             |
| 1!  | 24 | 16 | 12 | 10 | 4  | 6  | $2^5$       | $-2^{15}$   | $4^2$       | $-4^1$     | +        | c₈, c₁₄.11                                       |
| 1!  | 24 | 16 | 12 | 10 | 4  | 6  | $2^7$       | $-2^{13}$   | $4^1$       | $-4^2$     | -        | c₁₄.1₂                                          |
| 4!  | 24 | 16 | 12 | 10 | 4  | 6  | $2^9$       | $-2^{11}$   | $-4^3$     | +        | c₁₄.₁₃                                          |
| 1!  | 27 | 8  | 4  | 2  | 9  | 3  | $2^7$       | $-2^{11}$   | $\sqrt{16}^4$ | $-\sqrt{16}^4$ | -       | c₂₂                                              |
| 2!  | 27 | 18 | 9  | 12 | 9  | 3  | $3^6$       | $-3^{12}$   | $0^8$       | -           | +        | c₁₆ from Schläfli graph                         |
| 1!  | 28 | 6  | 2  | 1  | 7  | 4  | $2^9$       | $-2^{12}$   | $\sqrt{8}^3$ | $-\sqrt{8}^3$ | -       | c₂₃                                              |
| 1!  | 28 | 9  | 5  | 2  | 4  | 7  | $2^6$       | $-2^{18}$   | $5^1$       | -           | +        | 7 $\times$ 4-lattice [c₈, c₁₁]                  |
| 1!  | 28 | 13 | 0  | 6  | 14 | 2  | $\sqrt{13}^7$ | $-\sqrt{13}^7$ | -           | $-1^{13}$ | +        | Taylor graph [c₆]                               |
| 16! | 28 | 13 | 4  | 6  | 7  | 4  | $3^9$       | $-3^{12}$   | $1^1$       | $-3^5$     | -        | -                                                |
| 56! | 28 | 15 | 6  | 8  | 7  | 4  | $3^7$       | $-3^{14}$   | $1^6$       | -           | +        | c₁₆ from T(8), Chang graphs                    |
| 4!  | 28 | 15 | 6  | 8  | 7  | 4  | $3^8$       | $-3^{13}$   | $1^3$       | $-1^1$     | -        | DSS of the previous entry                       |
| 1!  | 28 | 19 | 15 | 12 | 4  | 7  | $2^6$       | $-2^{18}$   | $5^2$       | $-5^1$     | +        | c₈, c₁₄.1₁                                 |
| 2!  | 32 | 10 | 2  | 3  | 8  | 4  | $\sqrt{8}^{12}$ | $-\sqrt{8}^{12}$ | $2^1$       | $-2^6$     | +        | p₇                                               |

(Continues)
TABLE 1  (Continued)

| #  | v  | k  | λ₁ | λ₂ | m  | n  | θ₁ | θ₂ | θ₃ | WR constructions |
|----|----|----|----|----|----|----|-----|-----|-----|------------------|
| 1! | 32 | 10 | 6  | 2  | 4  | 8  | 2⁷  | −2⁴¹ | 6³  | + 8×4-lattice [c₈, c₁₁] |
| 1! | 32 | 10 | 6  | 2  | 4  | 8  | 2¹⁰ | −2¹⁸ | 6²  | −6¹  | −c₁₂             |
| 15!| 32 | 10 | 6  | 2  | 4  | 8  | 2¹³ | −2¹⁵ | 6¹  | −6²  | + c₁₃             |
| 15!| 32 | 14 | 2  | 6  | 16 | 2√12⁸ | −√12⁸ | 2⁴  | −2¹¹ | + c₂₁             |
| 2+ | 32 | 15 | 6  | 7  | 4  | 8  | 3¹² | −3¹⁶ | −1²  | + Cayley graphs (see [5]) |
| 1! | 32 | 16 | 0  | 8  | 16 | 2  | 4⁶  | −4⁰  | 0¹⁵ | + c₁₅ from halved 6-cube |
| ?  | 32 | 17 | 8  | 9  | 4  | 8  | 3¹⁷ | −3¹¹ | 1²  | −1²  | −                |
| 1! | 32 | 18 | 6  | 10 | 16 | 2√12⁸ | −√12⁸ | 2³  | −2¹² | + c₂₀             |
| 1! | 32 | 22 | 18 | 14 | 4  | 8  | 2⁷  | −2¹¹ | 6²  | −6¹  | + c₈, c₁₄.₁₁     |
| 1! | 32 | 22 | 18 | 14 | 4  | 8  | 2¹⁰ | −2¹⁸ | 6¹  | −6²  | −c₁₄.₁₂          |
| 9! | 32 | 22 | 18 | 14 | 4  | 8  | 2¹³ | −2¹⁵ | −    | −6³  | + c₁₄.₁₃          |
| 2! | 35 | 12 | 3  | 4  | 7  | 5  | 3¹² | −3¹⁶ | 2³  | −2³  | −                |
| 3854!|35 | 12 | 3  | 4  | 7  | 5  | 3¹⁴ | −3¹⁴ | −    | −2⁶  | + c⁷ from (35, 18, 9)-graphs |
| 3! | 36 | 9  | 3  | 2  | 12 | 3  | √6¹² | −√6¹² | 3⁴  | −3⁷  | + two graphs from c₁₉ |
| 7! | 36 | 9  | 4  | 2  | 18 | 2  | √5⁹  | −√5⁹ | 3³  | −3¹⁰ | + c₁₇ from icosahedron |
| 1! | 36 | 11 | 7  | 2  | 4  | 9  | 2⁸  | −2⁴  | 7¹  | −    | + 8×4-lattice [c₈, c₁₁] |
| 1! | 36 | 17 | 0  | 8  | 18 | 2√17⁹ | −√17⁹ | −    | −1⁷  | + Taylor graph [c₆] |
| 3+ | 36 | 24 | 15 | 16 | 4  | 9  | 3¹² | −3²⁰ | 0³  | −    | + Cayley graphs (see [5]) |
| 1! | 36 | 25 | 21 | 16 | 4  | 9  | 2⁸  | −2⁴  | 7²  | −7¹  | + c₈, c₁₄.₁₁     |
| 1! | 36 | 27 | 21 | 20 | 12 | 3  | √6¹² | −√6¹² | 3¹  | −3¹⁰ | + c₁₈ from (18, 12, 6, 8)-SDD |
| 2! | 38 | 9  | 0  | 2  | 19 | 2  | 3⁸  | −3¹¹ | √5⁹  | −√5⁹ | −                |

For the ‘constructions’ column, ‘c’ is short for ‘construction’, ‘p’ is short for ‘proposition’, ‘DSS’ means ‘dual Seidel switching’, ‘PC’ means ‘partial complement’, ‘L(G)’ denotes the line graph of G, ‘J(m,n)’ denotes the Johnson graph, ‘T(n)’ denotes the triangular graph (J(2,n)). For Construction 14, we note with ‘.’ the number of the construction (11, 12, or 13) for which the switching of edges was implemented.

The list of proper divisible design graphs up to 39 vertices is available by http://alg.imm.uran.ru/dezagraphs/ddgtable.html. This web page provides access to adjacency matrices, quotient matrices, and other properties of the graphs we found.

8  | CONCLUSION

The enumeration was incomplete for three tuples of parameters: (32, 15, 6, 7, 4, 8), (32, 17, 8, 9, 4, 8), and (36, 24, 15, 16, 4, 9). For these sets, the adjacency matrix construction algorithm was enumerating the rows of adjacency matrices corresponding to the first row of the quotient
matrix for several weeks. Therefore, we decided to stop the enumeration for these tuples and switch to others. It is possible to enumerate graphs for these tuples with more time or more powerful equipment and it could be done in the future.

After the enumeration was finished, we compared our results with the results from [2, 3, 6]. We found that the results for the existence of graphs coincide, except for one tuple of parameters: (27, 8, 4, 2, 9, 3). In [6] this tuple was rejected because it did not meet the necessary condition given in [6, theorem 5.1], which concerns the existence of non-zero integral solutions of the Diophantine equations. It turns out that the Diophantine equations for parameters (27, 8, 4, 2, 9, 3) actually have required solutions, and our enumeration produced one graph with these parameters. Our enumeration also showed the non-existence of graphs with parameters (27, 16, 12, 9, 9, 3), the only case for which the answer was not given in [2, 3, 6].

For one graph with parameters (24, 10, 3, 4, 8, 3), 16 graphs with parameters (28, 13, 4, 6, 7, 4), two graphs with parameters (35, 12, 3, 4, 7, 5), one graph with parameters (36, 9, 3, 2, 12, 3), and two graphs with parameters (38, 9, 0, 2, 19, 2) we could not find existing theoretical constructions or present new ones. We hope that 1 day new theoretical constructions will be found for the graphs that remained undescribed.

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