Generalized Uncertainty Principle and Klein Paradox

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Abstract

We study the Klein paradox in presence of a new generalized uncertainty principle. We define a new momentum operator and derive the modified dispersion relation for a Dirac particle. Apart from the forbidden band within the range $\pm m$, $m$ being the mass of the particle, we find the existence of additional forbidden bands at the both ends of the energy spectrum. It causes the allowed energy bands to be localised within finite energy range. This band structure forbids a Dirac particle from penetrating a potential barrier of sufficient height which is estimated to be of the order of Planck energy ($E_P$). This is also true for a massless particle. It reflects from the wall of a sufficiently high ($\sim E_P$) potential just like a massive particle does in nonrelativistic quantum mechanics. We propose this to be a signature of the existence of a minimum measurable length and a maximum measurable energy.

1 Introduction

Introduction of a minimum length scale [1]-[11] has revealed several salient features of gravity and quantum field theory. The essence of a minimum length can be captured by various means - by a generalized uncertainty principle (GUP), by a modified dispersion relation (MDR), by a deformed (doubly) special relativity (DSR) etc. One can initially start from a modified momentum operator and successively obtain a GUP or an MDR [7]. The manifestation of the presence of a minimal length in different fields, such as black hole thermodynamics [12]-[15], relativistic and non-relativistic quantum mechanics [16]-[26] etc, has stimulated a large number of extensive studies.

In nonrelativistic quantum mechanics, a particle with energy $E$ bounces from potential region of height $V$ if $V > E$. For a relativistic particle if one keep increasing the potential, the particle can once again propagate in the potential region. This is known as Klein paradox [27]. In this paper we study the Klein paradox in presence of a new GUP which imply a minimum length as well as a maximum energy. We start from a modified momentum operator and derive the MDR for a Dirac particle. We show that apart from the forbidden energy band in between $\pm m$ ($m$ is the mass of the particle) as predicted by standard relativistic quantum mechanics, there are other forbidden regions at some high (both positive and negative) energies which are of the order of Planck energy ($E_P$). The wave vector has an upper bound which is consistent with the existence of a minimum length scale. The allowed energy band is localised within finite energy range which gives rise to interesting features of the Klein paradox. We further study the case of a massless particle and find some new insight. According to non-GUP relativistic quantum mechanics, a massless Dirac particle can penetrate a potential barrier of any arbitrary height. In presence of our GUP the situation is entirely different from that. We find that a massless particle cannot penetrate a potential barrier of the order of $E_P$ which we propose to be a signature of the existence of minimum measurable length and maximum measurable energy.

The organisation of paper is as follows. In section 2, We introduce a modified momentum operator and obtain the corresponding Dirac equation with a discussion on the structure modified commutation relation and GUP. In section 3, we find the solution for the modified Dirac equation and derive the MDR which is used to find the band structure. This band structure is used to explain the nature of the Klein paradox in section 4. In section 5 we investigate the scenario for a massless particle. Section 6 contains the discussions.

2 GUP and the Dirac equation

In this section we will start from a modified momentum operator and derive a new GUP and corresponding modified Dirac equation.

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A GUP corresponds a modified commutation relation between position and momentum. One way to achieve this is to define a modified momentum operator whose structure is determined by the GUP. In a recent paper [15] we have proposed a GUP based on some heuristic arguments. The GUP contains only even terms on the right hand side. This is needed for our prescription, but not explicitly necessary for a GUP. Here we use a more general form which contains both odd and even terms. This kind of GUP is recently proposed in [19]. It not only accommodate a minimum length, but also a maximum measurable momentum (and hence a maximum energy).

We start by defining a new momentum operator \( \vec{P} \), keeping first two correction terms

\[
\vec{P}_i = p_i \left( 1 - a_0 \frac{p}{E_P/c} + b_0 \frac{p^2}{(E_P/c)^2} \right)
\]

where the suffix \( i \) \((i = 1, 2, 3)\) corresponds the \( i \) -th component. \( E_P \) is the Planck energy. \( a_0 \) and \( b_0 \) are real positive numbers. \( p \) is the magnitude of the operator \( \Sigma P_i^2 \). \( p_i \) obeys the standard commutation relation

\[
[x_j, p_i] = i \hbar \delta_{ji}
\]

where \( x_j \) being the \( j \) -th component of position operator.

It would be convenient for future calculations to introduce the scaled coefficients

\[
a = \frac{a_0}{E_P/c} \quad b = \frac{b_0}{(E_P/c)^2}
\]

and express (1) in terms of them. The modified momentum operator can be recast as

\[
\vec{P}_i = p_i (1 - ap + bp^2)
\]

We define the new position operator as

\[
\vec{X} = \vec{x}
\]

These modified momentum \((P)\) and position operator \((X)\) satisfy the commutation relation

\[
[X_i, P_j] = i \hbar \delta_{ij} - a \left( \delta_{ij}p + \frac{pbp_j}{p} \right) + b \left( \delta_{ij}p^2 + 2pp_j \right)
\]

We shall now obtain the modified Dirac equation. The Dirac equation in presence of the modified momentum operator (1) is given by [20][21]

\[
H\psi = \left[ c(\vec{\alpha} \cdot \vec{p})(1 - a(\vec{\alpha} \cdot \vec{p}) + b(\vec{\alpha} \cdot \vec{p})(\vec{\alpha} \cdot \vec{p})) + \beta mc^2 \right] \psi
\]

where we have used Dirac’s prescription for the operator \( p \) (i.e \( p \rightarrow \vec{\alpha} \cdot \vec{p} \)). \( \alpha \) and \( \beta \) are the Dirac matrices defined as

\[
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

where \( \sigma_i \) is the \( i \)th Pauli matrix \((i = 1, 2, 3)\).

For simplicity we will consider one dimensional motion (say along \( z \) axis). In one dimension the modified momentum operator (1) and the commutation relation (6) becomes

\[
\begin{align*}
P &= p(1 + ap + bp^2) \\
[X, P] &= i \hbar \left( 1 - 2aP + (3b - 2a^2)P^2 \right)
\end{align*}
\]

The GUP in one dimension in terms of the new operators \( P \) and \( X \) is

\[
\Delta X \Delta P \geq \frac{1}{2} |\langle X, P \rangle| \\
\geq \frac{\hbar}{2} \left( 1 - 2a \langle P \rangle + (3b - 2a^2) \langle P^2 \rangle \right)
\]

Note that by putting \( b = 2a^2 \) we can obtain the GUP proposed in [19].

For minimum position uncertainty, we can replace \( \langle P \rangle \) and \( \langle P^2 \rangle \) by \( \Delta P \) and \( \langle \Delta P \rangle^2 \) [19] and hence the GUP becomes

\[
\Delta X \Delta P \geq \frac{\hbar}{2} \left( 1 - 2a \Delta P + (3b - 2a^2)(\Delta P)^2 \right)
\]
The existence of minimum position uncertainty (i.e. $\Delta X$ to have a minima) requires the coefficient of $(\Delta P)^2$ to be positive. i.e

\[ 3b - 2a^2 > 0 \]  

(13)

In [19], they have used $b = 2a^2$ which is also consistent with the above condition (13).

For a particle moving along $z$ direction, (7) simplifies as

\[ [\alpha p - c\alpha p^2 + \beta mc^2] \psi = E_G \psi \]  

(14)

where $E_G$ is the GUP corrected energy of the system. For future purpose we will replace the momentum operator $p$ by its differential form $-i\hbar \frac{d}{dz}$ and recast the modified Dirac equation as

\[ \left[-i\hbar\alpha z \frac{d}{dz} + \hbar^2 c^2 \frac{d^2}{dz^2} - i\hbar\beta mc \frac{d^3}{dz^3} + \beta mc^2 \right] \psi = E_G \psi \]  

(15)

Note that the modified Dirac equation (15) contains third order spatial derivative and a complete solution requires the continuity of higher derivatives also. Since the GUP correction is very small one can also take a perturbative approach [18]-[24] as well.

### 3 Solution of modified Dirac equation for a free particle and its band structure

In this section we will find a solution of the modified Dirac equation (15) and subsequently derive the MDR. This will be used to find the energy band structure.

The standard one dimensional Dirac equation for a free particle is given by

\[ \left(-i\hbar\alpha z \frac{d}{dz} + \beta mc^2 \right) \psi(z) = E\psi(z) \]  

(16)

Note that one can also obtain this equation by putting $a = b = 0$ in (15). Its positive energy ($+E$) solution is given by

\[ \psi(z) = Ne^{ikz} \begin{pmatrix} \chi \\ \sigma_z \chi \end{pmatrix} \]  

(17)

Here $\chi$ is a $(2 \times 1)$ column matrix which satisfies orthonormality condition $\chi^\dagger \chi = 1$, $N$ is the normalization factor and $\mathcal{U}$ is a dimensionless quantity defined as

\[ \mathcal{U} = \frac{\hbar k}{E + mc^2} \]  

(18)

The wave number ($k$) obeys the dispersion relation

\[ E = \pm \sqrt{(\hbar k)^2 + m^2c^4} \]  

(19)

For the modified Dirac equation (15), the solution would not be so simple. Whatever the solution be, it must converge to (17) in absence of GUP. Let us consider the following ansatz

\[ \psi_G(z) = Ne^{ik_G z} \begin{pmatrix} \chi \\ \sigma_z \chi \end{pmatrix} \]  

(20)

to be a solution of (15).

The assumption

\[ \lim_{(a,b)\to0} \psi_G \to \psi \]  

(21)

where $\psi$ is given by (17), requires

\[ \lim_{(a,b)\to0} \mathcal{U}_G \to \mathcal{U} \]  

(22)

where $\mathcal{U}$ is given by (18). Besides, the modified dispersion relation should also converge to (19). i.e.

\[ \lim_{(a,b)\to0} E_G \to \pm \sqrt{\hbar k_G^2 + m^2c^4} \]  

(23)
We now substitute ansatz (20) in (15) and obtain the GUP corrected expression for energy \(E_G\) or the MDR and \(U_G\) which are given by

\[
E_G = \pm \sqrt{c^2\hbar^2 k_G^2 (1 + bh^2 k_G^2)^2 + m^2 c^4 - c\hbar^2 k_G^2}
\]

\[
U_G = \frac{c\hbar k_G (1 + bh^2 k_G^2)}{c\hbar^2 k_G^2 + mc^2 + E_G}
\]

One can readily see that both \(E_G\) and \(U_G\) satisfy (23) and (22).

The condition for the existence of a positive energy solution is

\[
\sqrt{c^2\hbar^2 k_G^2 (1 + bh^2 k_G^2)^2 + m^2 c^4} > c\hbar^2 k_G^2
\]

which can be simplified to

\[
(a^2 - 2b) < \left( \frac{1}{(\hbar k_G)^2} + \frac{m^2 c^2}{(\hbar k_G)^4} + b^2 (\hbar k_G)^2 \right)
\]

If we consider particles for which \(\hbar k >> m\) and \(0 < b(\hbar k_G)^2 << 1\) then the condition reduces to

\[
(a^2 - 2b) < 0
\]

Notice that in (19), \(b = 2a\) which also satisfies this condition. We can refine this condition by using (13), from which one can easily obtain

\[
a^2 - 2b < -b/2
\]

Since the coefficient \(b\) is positive, from (29) we get

\[
(a^2 - 2b) < 0
\]

Therefore the existence of a minimum length scale also asserts the existence of a positive energy solution.

The expression for the wave vector \(k_G\) can be obtained from (24) by ignoring \(O(b^2)\) term as

\[
\hbar^2 k_G^2 = \frac{(c - 2a E_G)}{2(a^2 - 2b)c} \left[ 1 \pm \sqrt{1 - \frac{4(a^2 - 2b)(E_G^2 - m^2 c^4)}{(c - 2a E_G)^2}} \right]
\]

Since we are interested up to the terms \(O(a^2, b)\), the above relation could be further simplified as,

\[
\hbar^2 k_G^2 = \frac{E_G^2 - m^2 c^4}{c^2 - 2a E_G c} \left[ 1 + \frac{(a^2 - 2b)(E_G^2 - m^2 c^4)}{4(c - 2a E_G)^2} \right]
\]

Note that here we have considered only negative (-) part of the square root term, because it is consistent with the fact that in the limit \((a, b) \to 0\) it gives

\[
\lim_{(a, b) \to 0} \hbar^2 k_G^2 = E^2 - m^2 c^4
\]

where we used the fact that \(\lim_{(a, b) \to 0} E_G = E\).

Let us discuss some properties of (32) and let see what they say about the GUP. Let us first consider the denominator \((c - 2a E_G)\). Using the basic definition (3) we can rewrite this as \((1 - \frac{2a_0 E_G}{E_P})\). It clearly shows that as the particle energy \(E_P\) approaches \(E_P\) the denominator approaches zero which will cause the wave vector \(k_G\) to diverge. This contradicts with the basic assumption of the existence of minimum length. Hence it gives a upper limit for the measurable energy which is given by

\[
E_G|_{max} = \frac{E_P}{2a_0}
\]

In other words this is a direct indication of the existence of maximum energy and hence a maximum momentum. Note that the order of maximum energy matches well with that predicted in (19).

We will now study the band structure. The simplest way to know the occurrence of a propagative or non-propagative mode is to find the nature of \(k_G^2\) with respect to energy \((E_G)\). A negative value of \(k_G^2\) suggests a imaginary \(k_G\) which is a damped mode. Corresponding energy values form a forbidden band. In (fig.1) we plot \(k_G^2\) as a function of \(E_G\) as given by equation (32), which shows the band structure for the modified Dirac
equation. The forbidden band within the range ±m is well known in Dirac theory without GUP. The interesting feature is the presence of two more forbidden bands which are the sole effect of the GUP. The edges of the allowed and forbidden bands, which are actually zeros of (32), are given by

\[ W_{G1} = \sqrt{c^2(a^2 - 2b)(c^2m^2(17a^2 - 2b) - 4)} + 8ac \quad (35) \]

\[ W_{G2} = -m \quad (36) \]

\[ W_{G3} = m \quad (37) \]

\[ W_{G4} = \sqrt{c^2(a^2 - 2b)(c^2m^2(17a^2 - 2b) - 4)} - 8ac \quad (38) \]

The edge \( W_{G1} \) is the bottom of the Dirac sea, whereas \( W_{G4} \) gives the maximum positive energy for a free particle. Based on this values we can divide the entire energy spectrum in five different regions which are described in (Table 1).

| FB1  | 1st forbidden band | \( E_G < W_{G1} \) |
|------|-------------------|------------------|
| AB1  | 1st allowed band  | \( W_{G2} < E_G < W_{G1} \) |
| FB2  | 2nd forbidden band| \( W_{G3} < E_G < W_{G2} \) |
| AB2  | 2nd Allowed band  | \( W_{G4} < E_G < W_{G3} \) |
| AB3  | 3rd forbidden band| \( W_{G4} < E_G \) |

Table 1: Nomenclature and positions of different bands

Figure 1: \( k_{G}^2 \) as a function of energy \( E_G \) for \((a = 0.05, b = 1, m = 0.25)\) with \( c = \hbar = E_P = 1 \). The dashed line shows non GUP dispersion relation \( k^2 = E^2 - m^2 \). The light-red regions (FB1, FB2, FB3) are forbidden bands and the light-blue regions (AB1, AB2) are the allowed bands.

The dashed line in (fig 1) corresponds the standard relativistic dispersion relation and the solid line gives the GUP corrected MDR. We see that the solid line for low energy matches well with the relativistic dispersion relation. As \( E_G \) is increased, \( k_{G}^2 \) gradually attains a maximum value and then start decreasing. The asymmetry of the curve is due to term \( a \). If we put \( a = 0 \) then the curve is symmetric about the \( k_{G}^2 \) axis (fig 2). The upper bound of the \( k_{G}^2 \) suggests the fact no matter how energetic particle we use we can’t reduce its wave length below a certain limit and hence we can’t measure below a certain length.

With this discussion let us now proceed to the next section where we will discuss the Klein paradox in this modified picture.

4 GUP and Klein paradox

In this section we will study the effect of GUP on Klein paradox. Klein paradox is a well known phenomenon for a relativistic particle. The motion of a plane wave inside a step potential of height \( V_0 \) with energy (E) less than the height of the barrier is forbidden according to the classical mechanics. However, the wave function of the particle can penetrate the barrier and emerge from the other side, which is against the classical prediction. The GUP provides a modified dispersion relation \( k^2 = E^2 - m^2 \), which suggests that the GUP can resolve the Klein paradox. The modified dispersion relation allows the wave function to penetrate the barrier and emerge from the other side, which is consistent with the quantum mechanical prediction.

3In that case the GUP (12) is \( \Delta p \Delta x \geq \frac{\hbar}{2} \), which is similar to that predicted in string theory (13). It supports the existence of a minimum length scale but doesn’t suggest a maximum energy. It agrees with the fact that for \( a = 0 \), \( E_G|_{\text{max}} = \infty \).
than $V_0$ is well known in nonrelativistic quantum mechanics and we know that the wave will decay exponentially with an exponent $\kappa = \sqrt{2m(V_0 - E)}$. The expression suggests that if we increase $V_0$ the wave will die off faster and so long $E < V_0$ there will not be any propagative mode. In case of a Dirac particle one can easily check that if the barrier height $V_0 > 2m$ then there can be a propagative mode for $E < V_0$ and the wave can penetrate through the barrier. Let us see how this effect is modified in presence of a GUP.

Consider the situation when a particle with positive energy and up spin is incident on the potential step from the left. The solution for the incident wave in region (I) is (apart from the normalization factor)

$$\psi_{in}^G(z) = e^{ik_G z} \left( \frac{\chi}{U_G \sigma_z \chi} \right)$$

(39)

where $U_G$ and $k_G$ are given by (25) and (32) respectively.

Let us look for the solution in region II (fig. 3). The transmitted part will only contain the spin up component and the solution can be given by the ansatz

$$\psi_{tr}^G = e^{iq_G z} \left( \frac{\chi}{U_G' \sigma_z \chi} \right)$$

(40)
where $q_G$ and $U'_G$ can be obtained from (24) and (25) replacing $E_G$ by $E_G - V_0$

\[
\begin{align*}
\hbar^2 q_G^2 &= \frac{(E_G - V_0)^2 - m^2 c^4}{c^2 - 2a(E_G - V_0)c} \left[ 1 + \frac{(a^2 - 2b)(E_G - V_0)^2 - m^2 c^4}{4(c - 2a(E_G - V_0))^2} \right] \\
U'_G &= \frac{chq_G(1 + bhq_G^2)}{cah^2q_G^2 + mc^2 + (E_G - V_0)}
\end{align*}
\]  

(41) (42)

Let consider a particle with such an energy that in region (II) it falls in a forbidden energy band (i.e. its momentum will be imaginary and the particle cannot propagate). By gradually increasing the potential energy we can lift up the negative energy bands such a way that the particle will find itself in an allowed band. Its momentum will be a real quantity and it can propagate inside the potential step. This is the physical scenario behind the Klein paradox. In a non-GUP Dirac theory, if we keep increasing the barrier height arbitrarily, there will always be room for propagating modes in region (II). In presence of a GUP the picture is entirely different. Due to GUP correction the allowed energy bands, i.e. $AB1$ and $AB2$ (fig. 4) are situated within a finite region. If a particle from region (I) encounter a forbidden band (say FB2) in region (II) (fig. 4, 5), then we can bring it to an allowed energy band (AB1) by raising the potential (fig. 4, 5). If we further increase the potential we will find the particle again to be in a forbidden energy band (FB1) (fig. 4, 5) which does not occur in Dirac theory without GUP. The particle would not be able to penetrate the barrier on further increment of potential.

Figure 4: (a) A free particle (black dot) in an allowed band (AB2) ($E_G = 0.5E_P$, $V_0 = 0$). (b) In a potential step the energy spectra will shift by an amount equal to the step height and the particle will find itself in a forbidden band (FB2)($V_0 = 0.5E_P$). (c) By raising the potential energy, the spectrum can be shifted further to the left which can bring the particle again in an allowed band (AB1) ($V_0 = 1.7E_P$). (d) If the potential is further increased then at a point the particle will again fall in a forbidden band (FB1) ($V_0 = 2.3E_P$).

5  GUP, massless particle and (reverse) Klein paradox

In this section we study the behaviour of a massless particle inside a potential step. The solution for the modified Dirac equation will be similar to the ansatz (40). We can write it as

\[
\psi_{fr}^{G0} = e^{iq_G z} \chi (U'_G, \sigma_z \chi) 
\]  

(43)

An additional constant positive potential will shift the entire energy spectrum to the left and thus can bring the particle to a forbidden band from an allowed band.
where \( q_{G_0} \) can be find by putting \( m = 0 \) in (41) and is given by

\[
\hbar^2 q_{G_0}^2 = \frac{(E_G - V_0)^2}{c^2 - 2a(E_G - V_0)c}\left[ 1 + \frac{(a^2 - 2b)(E_G - V_0)^2}{4(c^2 - 2a(E_G - V_0)c)} \right]
\]

Expression for the wave vector in free space \( (k_{G_0}) \) can easily be found from (44) by putting \( V_0 = 0 \).

**Figure 5:** \( k_{G_0}^2 \) as a function of energy for a massless Dirac particle for \( a = 0.5 \) and \( b = 1 \). There is no forbidden energy band around the zero energy, but the forbidden energy bands on the both ends still exist (FB1 and FB3). The dashed line shows the dispersion relation in absence of GUP.

In a non GUP Dirac theory, a massless particle can penetrate a potential step of any height due to the absence of any forbidden band. In presence of a GUP we see that (fig.5) there exist another couple of forbidden bands (FB1 and FB3) on the both sides of an allowed band \((AB(1+2))\). By applying a sufficiently large potential \((\sim E_P)\) we can bring this forbidden region to any energy level which will cause a massless Dirac particle with that energy to bounce from the potential wall. This phenomenon is not like standard Klein paradox, rather the reverse of it where due to an increase of the potential a particle which could propagate earlier is now reflected from the potential wall. This is the consequence of the existence of a minimum length and a maximum energy which is manifested through the modified momentum operator \( \hat{p} \).

6 Discussion

In this paper we discussed the modification of the dispersion relation of a Dirac particle and how it affects the band structure of a Dirac particle in presence of a minimum length and maximum energy. Then we studied the Klein paradox with the help of this new band structure. If a Dirac particle with some fixed energy face a forbidden energy band in a potential region, it will rebound from the wall of the potential. By increasing the potential strength we can lift up the negative energy levels such a way that there will be an allowed energy band corresponding to the particle energy and the particle will propagate within the potential region (fig.3). This is the essence of Klein paradox. In presence of a GUP the the scenario is different. We see that there are additional forbidden bands apart from the \( \pm m \) region, denoted by a negative \( k_{G_0}^2 \) in (fig.1) which appear at the both ends of the energy spectra. The allowed energy bands are not extended over \( \pm \infty \), rather localized within a finite region. This kind of band structure is the reason behind the difference in Klein paradox at high energy from that in case of a non GUP theory where the allowed bands are not bounded. We can initially bring a particle to an allowed band from a forbidden band by raising the potential (like what happen in a non GUP Klein paradox) (fig.1a,b,c); but if we keep the potential increasing we will reach the end of the allowed band and the particle will find itself again in a forbidden band (fig.1d). As a result it will reflect from the wall of the potential.

We further studied the phenomenon for a massless Dirac particle. In case of a massless Dirac particle there is nothing like a forbidden energy band in relativistic quantum mechanics and hence a massless particle with any energy can penetrate and propagate within a potential region of arbitrary height. In presence of a GUP the situation is different. For zero mass the forbidden region in the middle (FB2 in fig.1) disappears but those at the both ends (FB1,FB3) still survive (fig.3). So in presence of a sufficiently large potential \((\sim E_P)\) it is always possible for a massless particle with any energy to encounter a forbidden band. So with increase of height of the potential, a massless Dirac particle which earlier can penetrate inside the potential region will bounce off the wall like a massive particle does in nonrelativistic quantum mechanics. This is solely due the boundedness of the allowed energy bands which is a consequence of the existence of a minimum length and maximum energy.
(momentum). Hence the reflection of a massless Dirac particle from a potential barrier of height of the order of Planck energy is direct a signature of existence of a minimum length and maximum energy.

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