On a geometrical notion of dimension for partially ordered sets

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Abstract

The well-known notion of dimension for partial orders by Dushnik and Miller allows to quantify the degree of incomparability and, thus, is regarded as a measure of complexity for partial orders. However, despite its usefulness, its definition is somewhat disconnected from the geometrical idea of dimension, where, essentially, the number of dimensions indicates how many real lines are required to represent the underlying partially ordered set.

Here, we introduce a variation of the Dushnik-Miller notion of dimension that is closer to geometry, the Debreu dimension, and show the following main results: (i) how to construct its building blocks under some countability restrictions, (ii) its relation to other notions of dimension in the literature, and (iii), as an application of the above, we improve on the classification of preordered spaces through real-valued monotones.

1 Introduction

The notion of dimension has a geometrical origin and can be defined, roughly, as the number of copies of the real line needed to describe a certain object. This is not only true in the most common sense of dimension as the amount of (basis) vectors required to represent elements of linear spaces by linear combinations of these vectors, but also arises naturally in physics, which impulsed the study of high dimensional geometry greatly, where a system is said to have \( n \geq 1 \) dimensions or degrees of freedom when \( n \) real-valued functions are needed to describe its behaviour. More precisely, for each point of an \( n \)-dimensional manifold, there exists an \( \mathbb{R}^n \)-valued structure preserving map representing the manifold in a neighbourhood of that point. In a similar fashion, one may say a partial order \((X, \leq)\) has dimension \( n \geq 1 \) when \( n \) is the minimal natural number such that there exists an order embedding with values in \( \mathbb{R}^n \), that is, some \( f : (X, \leq) \to (\mathbb{R}^n, \leq_n) \) such that \( x \leq y \iff f(x) \leq_n f(y) \) for all \( x, y \in X \), where \( \leq_n \) is the standard ordering on \( \mathbb{R}^n \), that is, if \( x, y \in \mathbb{R}^n \), \( x \leq_n y \) if and only if \( x_i \leq y_i \) for \( i = 1, \ldots, n \). Notice, this is precisely the definition of a multi-utility of cardinality \( n \).\(^1\) Thus, this notion of order dimension relates a partial ordering

\(^1\)If \((X, \leq)\) is a partial order, then a multi-utility with the cardinality of some set \( I \) is a family of functions \((u_i)_{i \in I}\), where \( u_i : X \to \mathbb{R} \) for all \( i \in I \), such that we have for all \( x, y \in X \)
Note that a pair of partial orders holds. Finally, the cardinality of the smallest set \( I \) of \( \{x\} \), and, thus, the geometrical dimension is equal to the Dushnik-Miller dimension and, thus, also equal to the Ore dimension (see [15, Theorem 10.4.2]). However, this is not the standard definition of dimension for partial orders, which requires some concepts to be introduced first. If \( (X, \leq) \) is a partial order, we say the partial order \( (X, \leq') \) is an extension of \( (X, \leq) \) if \( x \leq y \) implies \( x \leq' y \) for all \( x, y \in X \) [11]. We may denote this by \( \leq \leq' \). Moreover, \( (X, \leq') \) is a linear extension of \( (X, \leq) \) if it is an extension and a total partial order [11]. A family of linear extensions \( (\leq_i)_{i \in I} \) is called a realizer of \( \leq \) if

\[
x \leq y \iff x \leq_i y \quad \forall i \in I
\]

holds. Finally, the cardinality of the smallest set \( I \) for which a realizer \( (\leq_i)_{i \in I} \) of \( (X, \leq) \) exists is called the Dushnik-Miller dimension of \( (X, \leq) \) [5].

Alternatively, the Ore dimension of \( (X, \leq) \) [15, 19] is the cardinality of the smallest set \( I \) such that \( (X, \leq) \) is order-isomorphic to \( (S, \leq') \), where \( S \subseteq \times_{i \in I} C_i \) is a subset of the Cartesian product of \( I \) chains \( (C_i)_{i \in I} \) and \( \leq' \) is the product-induced order with respect to the family of the corresponding partial orders of the chains \( C_i \).\footnote{The \textit{product-induced} order \( \leq' \) with respect to a family of preordered spaces \( ((X_i, \leq_i))_{i \in I} \) is the preorder defined on the product space \( \times_{i \in I} X_i \) given by \( x \leq' y \) if and only if \( x_i \leq_i y_i \) for all \( i \in I \).}

Note that a pair of partial orders \( (X, \leq) \) and \( (Y, \leq') \) are order isomorphic if there exists a bijective order embedding \( f : X \to Y \) [15]. Notice, also, the Dushnik-Miller dimension and the Ore dimension are equivalent (see [15, Theorem 10.4.2]).

As argued in [15], the Ore dimension was introduced in order to bring the Dushnik-Miller dimension closer to geometry. Indeed, it does generalize the definition of multi-utility, which corresponds to the case where every chain \( C_i \) is the real line \( \mathbb{R} \) equipped with its usual ordering \( \leq \), and, thus, the geometrical dimension. Furthermore, whenever the ground set \( X \) is countable, then the geometrical dimension is equal to the Dushnik-Miller dimension and, thus, also equal to the Ore dimension (see [14, Proposition 1] and, for the case where the ground set is finite, [7, Proposition 1]). However, in case the ground set is uncountable, they may be significantly different. Take, for example, the lexicographic plane \( (\mathbb{R}^2, \leq_L) \) where for all \( x, y, z, t \in \mathbb{R} \) we have

\[
(x, y) \leq_L (z, t) \iff \begin{cases} x < z \\ x = z \text{ and } y \leq t \end{cases}
\]

[4]. Since it is a total order, its Dushnik-Miller dimension is \( n = 1 \). However, given it has no utility function, that is, no multi-utility consisting of a...
single function (see [3, Example 1.4.1]), it has no countable multi-utility by [1, Proposition 4.1]. Hence, its geometrical dimension is uncountable.

Here, in order to avoid this disconnection between the Dushnik-Miller and Ore dimensions and the geometrical dimension, we introduce the Debreu dimension, a variation of Dushnik and Miller’s definition of dimension where some specific linear extensions, Debreu separable linear extensions, are required in the definition of realizer. In particular, we begin, in Section 2, defining the Debreu dimension and showing, under certain countability restrictions on \((X, \leq)\), Debreu separable linear extensions can be constructed as the limit of a sequence of partial orders that consecutively extend \(\leq\). In Section 3, we study the relation between countability of the geometrical, Debreu and Dushnik-Miller dimension. More specifically, we show the equivalence between countability of the Debreu dimension and that of the geometrical dimension. We obtain, thus, that the Debreu dimension is more intimately related to geometry than the Dushnik-Miller dimension. Despite this, we show the equivalence is broken in the case of finite dimensions and provided hypotheses under which it is recovered. Lastly, we show linear extensions can be also be obtained as the limit of a sequence of partial orders with the first one being defined through the strict part of any monotone. We recover, hence, that the Dushnik-Miller dimension is finite whenever the geometrical is (see [14]) through a different method. We conclude in Section 4, where we apply Section 3 to improve on the classification of pre-orders through real-valued monotones by showing that there are simple partial orders where finite multi-utilities exist and finite strict monotones do not (with majorization [12] being such an example if \(|\Omega| \geq 3\). However, we show the equivalence is recovered under a mild condition.

2 The Debreu dimension

The disagreement between the geometrical dimension and the Dushnik-Miller dimension originates from the fact that the first one is intimately related to the real numbers, while the second one is not. In order to avoid this disconnection in the second case, we can consider, instead of just linear extensions, Debreu separable linear extensions, that is, linear extensions which are Debreu separable. We do so since a classical result by Debreu [4] (see also [3, Theorem 1.4.8]) states that Debreu separable total orders are precisely the linear orders which can be completely characterized by a single real-valued function, that is, the ones which have a utility function. We have, thus, the connection to the real numbers we were seeking. Analogously, we say a realizer \((\leq_i)_{i \in I}\) is Debreu separable if each linear order \(\leq_i\) is Debreu separable. We can now introduce a new notion of dimension, the Debreu dimension.

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3A function \(f : X \rightarrow \mathbb{R}\) is a monotone on \((X, \leq)\) if \(x \leq y\) implies \(f(x) \leq f(y)\) for all \(x, y \in X\) [6]. Moreover, \(f\) is a strict monotone or a Richter-Peleg function if \(x < y\) implies \(f(x) < f(y)\) for all \(x, y \in X\); where \(x < y\) stands for \(x \leq y\) and \(\neg (y \leq x)\) [18, 16].

4Recall a partial order \((X, \leq)\) is Debreu separable whenever there exists a countable subset \(D \subseteq X\) such that, if \(x < y\) holds for \(x, y \in X\), then there exists some \(d \in D\) such that \(x < d < y\) [3]. Furthermore, we call a subset \(D\) with this property a (countable) Debreu dense subset.
Definition 2 (Debreu dimension of a partial order). If \((X, \leq)\) is a partial order, then its Debreu dimension is the cardinality of the smallest set \(I\) for which \((X, \leq)\) has a Debreu separable realizer \((\leq_1)_{i \in I}\).

Notice, in contrast to both the Dushnik-Miller dimension and the Ore dimension, it is easy to see the Debreu dimension is only defined for partial orders with \(|X| \leq c\), where \(c\) is the cardinality of the continuum.

Since the existence of a Debreu separable linear extension \(\leq'\) of a partial order defined on some space \(X\) is a countable denseness restriction on \((X, \leq')\), it is natural to start by relating the existence of such linear extensions with that of countable denseness properties. In particular, we relate their existence to Debreu upper separability. We say a partial order \((X, \leq)\) is *Debreu upper separable* if there exists a countable subset \(D \subseteq X\) such that \(X\) is Debreu separable because of \(D\) and for all \(x, y \in X\), if \(x \not\approx y\), then there exists some \(d \in D\) such that \(x \not\approx d \leq y\) \([10]\). Moreover, a subset \(D \subseteq X\) that fulfills the second property is called *Debreu upper dense*. Here, \(x \not\approx y\) stands for \(\neg(x \leq y)\) and \(\neg(y \leq x)\), and we refer to such a pair of elements as *incomparable*. Notice, Debreu separability appears in the context of real-valued representations of total orders. Debreu upper separability is an extension to partial orders with incomparable elements. In particular, it is easy to see that the geometrical dimension of Debreu upper separable partial orders is countable \([10, \text{Proposition 9}]\), that is, countable multi-utilities exist for such partial orders. A summary of relevant order denseness properties can be found in \([10, \text{Table 1}]\).

As a first result, we show in Proposition 1 that any Debreu upper separable partial order \(\leq\) has a Debreu separable linear extension that can be obtained as the limit of a sequence of consecutive extensions of \(\leq\). Thus, in order to state Proposition 1, we need to define the notion of *limit* of a sequence of binary relations, which plays an important role in the following.

**Definition 3 (Limit of a sequence of binary relations).** If \((\leq_n)_{n \geq 0}\) is a sequence of binary relations on a set \(X\), the limit of \((\leq_n)_{n \geq 0}\) is the binary relation \(\leq\) on \(X\), that for all \(x, y \in X\) fulfills

\[
x \leq y \iff \exists n_0 \geq 0 \text{ s.t. } x \leq_n y \forall n \geq n_0.
\]

We denote this limit also by \(\lim_{n \to \infty} \leq_n\).

Note that Definition 3 corresponds to the set-theoretic *limit infimum* \([17]\) of a certain sequence of sets in the Cartesian product \(X \times X\), since we can think of any binary relation on a set \(X\) as a subset of \(X \times X\). In particular, if we define \(A_n := \{(x, y) \in X \times X | x \leq_n y\}\) for all \(n \geq 0\), then Definition 1 corresponds to

\[
\liminf_{n \to \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,
\]

as defined in \([17, \text{Section 1.3}]\).

The instance of Definition 3 that concerns us here is the one where all binary relations are partial orders. As we state in the following lemma, whenever
\((\leq_n)_{n\geq 0}\) is a sequence of partial orders, then its limit is also a partial order. Moreover, whenever \(\leq_{n+1}\) extends \(\leq_n\) for all \(n \geq 0\), then the limit of \((\leq_n)_{n\geq 0}\) extends \(\leq_n\) for all \(n \geq 0\).

**Lemma 1.** If \((\leq_n)_{n\geq 0}\) is a sequence of partial orders on a set \(X\), then \(\lim_{n\to\infty} \leq_n\) is a partial order. Moreover, if \(\leq_{n+1}\) extends \(\leq_n\) for all \(n \geq 0\), then \(\lim_{n\to\infty} \leq_n\) extends \(\leq_n\) for all \(n \geq 0\).

**Proof.** For the first statement, we simply have to show \(\leq\) is reflexive, antisymmetric, and transitive. Given \(x \in X\), we have \(x \leq x\) for all \(n \geq 0\) since each \(\leq_n\) is a partial order. Thus, by (1), we can take \(n_0 = 0\) and get \(x \leq y\). To see \(\leq\) is antisymmetric, take \(x, y \in X\) such that \(x \leq y\) and \(y \leq x\). By definition, there exist \(n_0, n_0' \geq 0\) such that \(x \leq_n y\) for all \(n \geq n_0\) and \(x \leq_n y\) for all \(n \geq n_0'\). Taking \(m_0 := \max\{n_0, n_0'\}\), we get \(x \leq_{m_0} y\) and \(y \leq_{m_0} x\). By antisymmetry of \(\leq_{m_0}\), we conclude \(x = y\). To see \(\leq\) is transitive, we can proceed as for antisymmetry, taking advantage of the fact \(\leq_n\) is transitive for all \(n \geq 0\).

For the second statement, take \(x, y \in X\) such that \(x \leq_n y\) for some \(n \geq 0\). Since \(\leq_{n+1}\) extends \(\leq_n\) for all \(n \geq 0\), we have \(x \leq_m y\) for all \(m \geq n\). Thus, by definition (1), we have \(x \leq y\) and, hence, \(\leq\) extends \(\leq_n\).

Notice, in the second statement of Lemma 1, the fact that \(\leq_{n+1}\) extends \(\leq_n\) for all \(n \geq 0\) is key for the limit to be related to the sequence of partial orders through which it is defined. Take, for example, \(X = \mathbb{N}\), and consider the sequence of partial orders \((\leq_n)_{n\geq 0}\) where for all \(m, p \in \mathbb{N}\)

\[
m \leq_n p \iff \begin{cases} m = p, & \text{or} \\ m = n \text{ and } p = n + 1. \end{cases}
\]

Notice, in this pathological case, \(\lim_{n\to\infty} \leq_n\) is the trivial ordering, which carries no information about \((\leq_n)_{n\geq 0}\) since, given that \(\lim_{n\to\infty} \leq_n\) is a partial order by Lemma 1, the trivial ordering is always contained in the limit.\(^5\) Note, also, whenever \(\leq_{n+1}\) extends \(\leq_n\) for all \(n \geq 0\), then Definition 3 coincides with the set-theoretic limit supremum of \((A_n)_{n\geq 0}\)

\[
\lim_{n\to\infty} \sup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,
\]

as defined in [17, Section 1.3]. We can now show Proposition 1.

**Proposition 1.** If \((X, \leq)\) is Debreu upper separable and \(x, y \in X\) are two incomparable elements \(x \not\asymp y\), then there exists a sequence \((\leq_n)_{n\geq 0}\) of extensions of \(\leq\), such that \(\leq':= \lim_{n\to\infty} \leq_n\) is a Debreu separable extension of \(\leq\) with \(x \leq' y\).

\(^5\)We say a binary relation \(\leq\) on a set \(X\) is a trivial ordering if \(x \leq y \iff x = y\) for all \(x, y \in X\).
Proof. Let \((d_n)_{n \geq 1}\) be an enumeration of a countable subset \(D \subseteq X\) by which \((X, \leq)\) is Debreu upper separable and such that we have \(x \bowtie d_1 \leq y\). Starting with \(\leq_0 \coloneqq \leq\), we define recursively a family of partial orders \((\leq_n)_{n \geq 0}\) by

\[
x \leq_n y \iff \begin{cases} x \leq_{n-1} y, & \text{or} \\ x \bowtie_{n-1} y, & x \bowtie_0 d_n, \text{ and } d_n \leq_0 y \end{cases}
\]  

(2)

for all \(x, y \in X\) (see Figure 1 for an example of \((\leq_n)_{n \geq 0}\) with finite \(X\)). Notice, by definition, it is clear that \(x \leq_n y\) implies \(x \leq_{n+1} y\) for all \(n \geq 0\). Furthermore, as we show in Lemma 2 in the appendix, \(\leq_n\) is a partial order for all \(n \geq 0\). Thus, by Lemma 1, \(\leq'\) is a partial order that extends \(\leq = \leq_0\).

In order to see that \(\leq'\) is total, we only need to consider the pairs \(x, y \in X\) such that \(x \bowtie y\). By Debreu upper separability of \(\leq\) and (2), there exists a minimal \(n_{xy} \geq 1\) such that we either have \(x \leq_n y\) or \(y \leq_n x\) for all \(n \geq n_{xy}\). In particular, we have \(\neg (x \bowtie' y)\) for every pair \(x, y \in X\), that is, \(\leq'\) is total. We conclude \(\leq'\) is a linear extension of \(\leq\). Lastly, we show \(\leq'\) is Debreu separable. Assume \(x \bowtie' y\) for some \(x, y \in X\). If \(x < y\), then there exists some \(d_n \in D\) such that \(x \leq d_n \leq y\) by Debreu separability of \(\leq\) and, thus, \(x \bowtie' d_n \leq y\). In case \(x \bowtie y\), by Debreu upper separability of \(\leq\) and (2), there exists a minimal \(n_0 \geq 1\) such that \(x \bowtie_{n_0} y\). In particular, we have \(x \bowtie_{n_0} d_{n_0} \leq_{n_0} y\) and, hence, \(x \bowtie' d_{n_0} \leq' y\). In summary, \(\leq'\) is a Debreu separable linear extension of \(\leq\).

Notice, since by definition we have \(x \leq_1 y\) for the pair \(x, y \in X\) in the statement of the result, we have \(x \leq' y\) as desired. \(\square\)

The converse of Proposition 1 does not hold since, for example, the trivial ordering on the real numbers \((\mathbb{R}, =)\) is a partial order which has Debreu separable linear extensions, like \((\mathbb{R}, \leq)\), although it is not Debreu upper separable. As this example shows, if \((X, \leq)\) is a partial order and \((X, \leq')\) is a linear extension which is Debreu separable by some countable subset \(D \subseteq X\), then \(D\) may not distinguish between different elements of \((X, \leq)\) which are not in \(D\). That is, we may have \(x \bowtie d \bowtie y\) for all \(d \in D\) and for all \(x, y \in X/D\) where \(x \neq y\). Notice, also, the axiom of choice is not needed in Proposition 1, which contrasts with Szpilrajn’s extension theorem [20] (see also [11, Section 2.3]).

The specific procedure we follow in Proposition 1 to construct the Debreu separable linear extension (see Figure 1 for an example) is fundamental since we can even strengthen the order denseness requirement to separable partial orders and still find linear extensions which are not Debreu separable, as we show in Proposition 2. We say a partial order \((X, \leq)\) is separable whenever there exists a countable subset \(D \subseteq X\) such that for all \(x, y \in X\) there exists \(d \in D\) such that \(x < d < y\) if \(x < y\) and there exist \(d_1, d_2 \in Z\) such that \(x \bowtie y < d_1\) and \(x \bowtie d_2 < y\) if \(x \bowtie y\) [14].

Proposition 2. There exist separable partial orders with linear extensions which are not Debreu separable.

Proof. Consider the set \(X \coloneqq \mathbb{R}\setminus\{0\}\) equipped with the partial order \(\leq\) defined
Figure 1: Hasse diagram of a sequence of partial orders \((\leq_i)_{i=1}^3\) extending \(\leq_0\) following Proposition 1. We consider the finite partial order \((X, \leq_0)\) with \(X := \{A, B, C, D, E, F\}\) and \(A, B \leq_0 C; D \leq_0 E, F; A \leq_0 F; B \leq_0 E\). Moreover, we color in red the subset \((d_i)_{i=1}^3 \subseteq X\) used to define \((\leq_i)_{i=1}^3\) following (2). Note \((X, \leq_3)\) is a Debreu separable linear extension of \((X, \leq_0)\).

by

\[
 x \leq y \iff \begin{cases} 
 x, y > 0 \text{ and } x \leq y \\
 x, y < 0 \text{ and } y \leq x 
\end{cases}
\]  

(3)
Figure 2: Representation of the partial order defined in Proposition 2, which is a linear extension of the separable space \((\mathbb{R}\setminus \{0\}, \leq)\), where \(x \leq y\) if and only if either \(x, y > 0\) and \(x \leq y\) or \(x, y < 0\) and \(y \leq x\), and is not Debreu separable. In particular, we include three positive real numbers \(0 < x < y < z\) in the top line and their negatives \(-x, -y, -z\) in the bottom line. Note that an arrow from one element \(x\) to another \(y\) represents \(x < y\) and that we only include the arrows among the nearest neighbours, although their compositions also belong to the ordered set by transitivity.

for all \(x, y \in X\). Notice, because of \(\mathbb{Q}\setminus \{0\}, (X, \leq)\) is separable. We introduce now a linear extension \(\leq'\) of \(\leq\), defined by

\[
x \leq' y \iff \begin{cases} x \leq y \\ xy < 0 \text{ and } |x| < |y| \\ x < 0, y > 0 \text{ and } |x| = |y|. \end{cases}
\]

for all \(x, y \in X\) (see Figure 2 for a representation of \((X, \leq')\)). Notice \(\leq'\) is linear and extends \(\leq\) by definition. To finish, we will show any Debreu separable subset of \((X, \leq')\) is uncountable. Consider a pair \(x, y \in X\) where \(x < 0, y > 0\) and \(|x| = |y|\). Notice, by definition, we have \(x < y\). Consider some \(d \in X\) such that \(x \leq' d \leq' y\). If \(|d| < |x|\), then we have \(d < x\) by definition and we get a contradiction with the definition of \(d\). Similarly, we get a contradiction when assuming \(|x| < |d|\). Thus, we have \(d \in \{x, y\}\). As a result, any Debreu separable subset of \((X, \leq')\) contains either \(x\) or \(-x\) for all \(x > 0\) and, hence, it is uncountable. 

Notice, clearly, a countable ground set \(X\) could not have been used to prove Proposition 2, since any total order on such a set is Debreu separable.

The result in Proposition 1 can actually be improved upon substituting is Debreu upper separable by has a countable multi-utility. Notice, we know how
these assumptions are related by both [10] and [8]. In particular, while Debreu upper separable partial orders have countable multi-utilities [10, Proposition 9], the converse is false, as illustrated in both [10, Lemma 5 (ii)] and [8, Proposition 9]. We show the improved result in the following proposition. We say a family of sets \((A_n)_{n \geq 1}\) separates \(x\) from \(y\) if there exists some \(n_0 \geq 1\) such that \(x \notin A_{n_0}\) and \(y \in A_{n_0}\) [10].

**Theorem 1.** If \((X, \preceq)\) is a partial order with a countable multi-utility and \(x, y \in X\) are two incomparable elements, \(x \preceq y\), then there exists a sequence \((\preceq_n)_{n \geq 0}\) of extensions of \(\preceq\), such that \(\preceq' := \lim_{n \to \infty} \preceq_n\) is a Debreu separable extension of \(\preceq\) where \(x \preceq' y\).

**Proof.** Consider an enumeration \((A_n)_{n \geq 1}\) of a countable family of increasing sets which for all \(x, y \in X\) with \(x < y\) separates \(x\) from \(y\) and for all \(x, y \in X\) with \(x \preceq y\) separates both \(x\) from \(y\) and \(y\) from \(x\) (see Remark 1). Notice such a family exists (see [10] and the references therein), since there is a countable multi-utility by hypothesis. We assume w.l.o.g. that the enumeration fulfills \(x \notin A_1\) and \(y \in A_1\) for the pair \(x, y \in X\) of the statement. Starting with \(\preceq_0 := \preceq\), we recursively define \((\preceq_n)_{n \geq 0}\) by

\[
x \preceq_n y \iff \begin{cases} x \preceq_{n-1} y, & \text{or} \\ x \preceq_{n-1} y, & x \notin A_n, \text{ and } y \in A_n \end{cases}
\]

for all \(x, y \in X\) (see Figure 3 for an example of \((\preceq_n)_{n \geq 0}\) with finite \(X\)). By definition, for all \(x, y \in X\), \(x \preceq_n y\) implies \(x \preceq_{n+1} y\) for all \(n \geq 0\). By Lemma 3 in the appendix, \(\preceq_n\) is a partial order for all \(n \geq 0\). Furthermore, by Lemma 1, \(\preceq'\) is an extension of \(\preceq = \lim_{n \to \infty} \preceq_n\). In order to see that \(\preceq'\) is total, consider a pair \(x, y \in X\) such that \(x \not\succ y\). By definition of multi-utility, there exists a minimal integer \(n_0 \geq 1\) such that either \(A_{n_0}\) separates \(x\) from \(y\) or \(y\) from \(x\). Thus, we have \(\neg(x \not\succ_{n_0} y)\) and, hence, \(\neg(x \not\succ' y)\). Therefore, \(\preceq'\) is a linear extension of \(\preceq\).

It remains to show that \(\preceq'\) is Debreu separable. In order to do so, we construct a utility function \(u\) for \(\preceq'\) and, since any preorder with a utility function is Debreu separable by [3, Theorem 1.4.8], this concludes the proof. Take

\[
u(x) := \sum_{n \geq 0} r^n \chi_{A_{n+1}}(x)
\]

with \(r \in (0, \frac{1}{2})\). As we showed in [10, Lemma 1], we have \(u(x) < u(y)\) if and only if, for the first \(m \geq 1\) with \(\chi_{A_m}(x) \neq \chi_{A_m}(y)\), we have \(\chi_{A_m}(x) < \chi_{A_m}(y)\). To finish, we show for all \(x, y \in X\)

\[
x \preceq' y \iff u(x) \leq u(y).
\]

Assume \(x \preceq' y\). If \(x = y\), then \(u(x) = u(y)\). Otherwise, by antisymmetry, we have \(x <' y\). Then, the first \(m \geq 1\) such that \(\chi_{A_m}(x) \neq \chi_{A_m}(y)\) fulfills \(\chi_{A_m}(x) < \chi_{A_m}(y)\), either by construction if \(x \not\succ y\) or by definition of \((A_n)_{n \geq 1}\)
if $x < y$, and we get $u(x) < u(y)$. Conversely, if $\neg (x \leq' y)$, then $y <' x$ since $\leq'$ is total and, as we just showed, $u(x) > u(y)$.

Notice, since by definition we have $x \leq 1 y$ for the pair $x, y \in X$ in the statement of the result, we have $x \leq' y$ as desired.

\[ \textbf{Remark 1.} \) If $\{u_n\}_{n \geq 0}$ is a countable multi-utility, then we can take $\{A_{n,q}\}_{n \geq 0, q \in \mathbb{Q}}$ as the family of increasing sets we used in the proof of \textit{Theorem 1} (see \cite{10} and the references therein), where $A_{n,q} := u_n^{-1}([q, \infty))$ for all $n \geq 0, q \in \mathbb{Q}$.

Although a technique to construct linear extensions for partial orders $(X, \leq)$ where countable multi-utilities $U := \{u_n\}_{n \geq 0}$ exist is already known (see \cite{14, Proposition 1}), it fails in general to yield Debreu separable linear extensions, as we show in the reminder of this paragraph. Since the technique in \cite{14} constructs linear extensions starting from a monotone procedure in \cite{3}, it fails in general to yield Debreu separable linear extensions, or, equivalently, \textit{Proposition 1}, and, since the converse is immediate, \textit{Propositions 2}) are equal in this case.

\[ x \leq' y \iff \begin{cases} x = y, \\
\quad u_1(x) < u_1(y), \\
\quad u_i(x) = u_i(y) \quad i = 1, \ldots, s - 1 \quad \text{and} \quad u_s(x) < u_s(y)\end{cases} \]  

(6)

for all $x, y \in X$. To exemplify that this method can lead to linear extensions which are not Debreu separable, we can take $(X, \leq)$ with $X := \mathbb{R} \setminus \{0\}$ and $\leq$ defined as in (3). Note that $U := \{u_1\}_{i=1}^n$ is a multi-utility, where $u_1(x) := |x|$ is the absolute value function, $u_2(x) := 1$ if $x > 0$ and $u_2(x) := 0$ if $x < 0$, and $u_3(x) := 1$ if $x < 0$ and $u_3(x) := 0$ if $x > 0$. Notice, if we follow the procedure in (6), we obtain the same linear extension of $(X, \leq)$ we constructed in Proposition 2, which is not Debreu separable, as we showed there.

Notice, in case there exists some $D = \{d_n\}_{n \geq 1} \subseteq X$ by which $(X, \leq)$ is Debreu upper separable, the definition of $\leq_n$ for $n \geq 1$ in \textit{Theorem 1} and \textit{Proposition 1} is the same provided we take $A_n := i_0(d_n)$ for all $n \geq 1$, where $i_0(x) := \{y \in X | x \leq 0, y \}$ for all $x \in X$. This is the case since, if we assume we have $x \leq_n y$, $x \notin i_0(d_n)$ and $y \in i_0(d_n)$, then $x < 0 d_n$ implies, by transitivity of $\leq_n$, that $x < 0 y$, contradicting, thus, the fact $x \in D$. Thus, (4) implies (2) and, since the converse is immediate, (4) and (2) are equal in this case.

The technique in both \textit{Proposition 1} and \textit{Theorem 1} relies on the sequential definition of partial orders which extend each other and, thus, it cannot be applied to partial orders where only uncountable multi-utilities exist. Moreover, notice starting $(\leq_n)_{n \geq 0}$ with the original partial order, $\leq_0 = \leq$, is not necessary to conclude a Debreu separable linear extension of $\leq$ exists neither in \textit{Proposition 1} nor in \textit{Theorem 1}. In fact, for example, we could have started \textit{Theorem 1}, or, equivalently, \textit{Proposition 1}, with $\leq_0$ being the trivial ordering (or taking any $\leq_0 \leq \leq$), defined $\leq_n$ for $n \geq 1$ using (4) and still have concluded $\lim_{n \to \infty} \leq_n$.

\[ \text{Note the geometrical dimension of } (X, \leq) \text{ is } 2 \text{ and } U \text{ is a non-minimal multi-utility, since } (X, \leq) \text{ is non-total and } V := \{v_i\}_{i=1}^\infty \text{ is a multi-utility, where } v_1(x) := x \text{ if } x > 0 \text{ and } v_1(x) := 0 \text{ if } x < 0, \text{ and } v_2(x) := |x| \text{ if } x < 0 \text{ and } v_2(x) := 0 \text{ if } x > 0. \]
Figure 3: Hasse diagram of a sequence of partial orders \((\preceq_n)_{n=1}^3\) extending \(\preceq_0\) following Theorem 1. We consider the same finite partial order \((X, \preceq_0)\) in Figure 1. Moreover, we color in red the family of subsets \((A_n)_{n=1}^3 \subseteq X\) we used to define \((\preceq_n)_{n=1}^3\) following (4). Furthermore, notice, although it is clearly not necessary, \((X, \preceq_3)\) is the same Debreu separable linear extension of \((X, \preceq_0)\) we obtained in Figure 1.

was a Debreu separable linear extension of \(\preceq\) (see Figure 4 for an example of \((\preceq_n)_{n\geq 0}\) with finite \(X\)). In fact, although they are not extensions of \(\preceq\) as in
Proposition 1 and Theorem 1, showing that \((\leq_n)_{n \geq 0}\) is composed of partial orders is easier in this scenario.

The procedures by which we concluded that the linear extensions we constructed in each proposition are Debreu separable are not the same. In Proposition 1, it is based directly on the construction we presented, while, in Theorem 1, we relied on the argument in [3, Theorem 1.4.8]. Actually, it remains a challenge to show Debreu separability of the linear extension in Theorem 1 by a more direct method.

The Debreu separability of the extension \(\leq\) constructed in Theorem 1 is similar to a construction we introduced recently in [10, Proposition 5] when showing injective monotones exist whenever countable multi-utilities do (we comment on the difference right after Proposition 3). Notice, if \((X, \leq)\) is a partial order, then \(u : X \rightarrow \mathbb{R}\) is an injective monotone provided it is a monotone and it fulfills for all \(x, y \in X\) that \(u(x) = u(y)\) implies \(x = y\) [10]. As a matter of fact, we can improve upon Theorem 1 and achieve equivalence, weakening the condition from the existence of countable multi-utilities to the existence of injective monotones, as we show in Proposition 3.\(^7\) Furthermore, this equivalence allows us to conclude the existence of a Debreu separable linear extension \(\leq'\) is, actually, equivalent to the existence of a sequence of extensions of \(\leq\), \((\leq_n)_{n \geq 0}\), such that \(\leq' = \lim_{n \rightarrow \infty} \leq_n\).

**Proposition 3.** If \((X, \leq)\) is a partial order and \(x, y \in X\) are two incomparable elements, \(x \not\succ y\), then the following are equivalent:

(i) There exists a Debreu separable linear extension \(\leq'\) such that \(x \leq' y\).

(ii) There exists an injective monotone \(u\) such that \(u(x) \leq u(y)\).

(iii) There exists a sequence \((\leq_n)_{n \geq 0}\) of extensions of \(\leq\), such that its limit \(\leq' = \lim_{n \rightarrow \infty} \leq_n\) is a Debreu separable extension of \(\leq\) with \(x \leq' y\).

**Proof.** In order to show (i) implies (ii), note that, by [3, Theorem 1.3.2], there exists a utility function \(u : X \rightarrow \mathbb{R}\), that is,

\[
x \leq' y \iff u(x) \leq u(y) \quad \forall x, y \in X,
\]

because \((X, \leq')\) is a Debreu separable total order. To conclude, it is easy to show that \(u\) is an injective monotone for \((X, \leq)\).

The proof that (ii) implies (iii) is analogous to the proof of Theorem 1. That is, we can take \(\leq_0 = \leq\) and define \(\leq_n\) for \(n \geq 1\) using (4) but, instead of the \((A_n)_{n \geq 1}\) there, using \((A_n)_{n \geq 1}\) the family of increasing sets that for all \(x, y \in X\) with \(x < y\) separates \(x\) from \(y\) and for all \(x, y \in X\) with \(x \sim y\) separates either \(x\) from \(y\) or \(y\) from \(x\) (see Remark 2). Notice such a family exists by [10, Proposition 7 (ii)], since \((X, \leq)\) has injective monotones by assumption. Notice, also, the key property to conclude \(\leq_n\) is a partial order for all \(n \geq 0\) in Theorem 1 is that \(A_n\) is increasing for all \(n \geq 1\), which is also true here. In contrast with

\(^7\)Notice, we know how countable multi-utilities are related to injective monotones by [10, Propositions 5 and 8].
Theorem 1, it should be noted that, instead of (4), we can simply define $\leq_n$ for each $n \geq 1$ by

$$x \leq_n y \iff \begin{cases} x \leq_{n-1} y \\ x \notin A_n \text{ and } y \in A_n \end{cases}$$

(8)

for all $x, y \in X$. This is the case since, whenever $(A_n)_{n \geq 0}$ comes from an injective monotone, then, if there exists some $n_0 \geq 1$ such that $y \in A_{n_0}$ and $x \notin A_{n_0}$, we have that $y \in A_n$ if $x \in A_n$ for all $n \geq 1$. Thus, since it is not possible to have $x \leq_n y$ and $y \leq_m x$ for $x \neq y$ and $n \neq m$, which would contradict the fact $\leq'$ is required to be a partial order, we can avoid including $x \triangleright_{n-1} y$ in the second line of (8). Note we cannot erase $x \triangleright_{n-1} y$ in (4).

To finish the proof, note that (iii) implies (i) by definition.

**Remark 2.** If $u$ is an injective monotone and $(q_n)_{n \geq 1}$ is a numeration of the rational numbers, then, by [10, Proposition 7 (ii)], we can take $(A_n)_{n \geq 1}$, where $A_n := u^{-1}([q_n, \infty))$ for all $n \geq 1$, as the family of increasing sets we used in the proof of Proposition 3.

Notice, by Proposition 3, Theorem 1 shows how to construct injective monotones from countable multi-utilities. However, it improves upon [10, Proposition 5] since, instead of simply constructing a Debreu separable linear extension as the limit of a sequence of binary relations that do not necessarily extend the original partial order, it shows how such an extension can be achieved as the limit of a sequence of partial orders where each of them extends the original one. In fact, much of the effort in Theorem 1 is devoted to showing the binary relations in the sequence are, in fact, partial orders.

Joining Proposition 3 and [10, Proposition 8], we obtain the converse of Theorem 1 does not hold or, equivalently, that the existence of Debreu separable linear extensions is insufficient for that of countable multi-utilities. This is the case since there are partial orders where injective monotones exist and countable multi-utilities do not (see [10, Proposition 8] for a counterexample) while, by Proposition 3, Debreu separable linear extensions exist whenever injective monotones do. Notice, also, we can show (i) implies (iii) directly, that is, without referring to injective monotones, in analogy to the proof of Proposition 1. In particular, we could take $\leq_0 := \leq$ and define $\leq_n$ for each $n \geq 1$ like

$$x \leq_n y \iff \begin{cases} x \leq_{n-1} y \\ x \leq' d_n \leq' y \end{cases}$$

(9)

for all $x, y \in X$, where $(d_n)_{n \geq 1}$ is an enumeration of a countable subset $D \subseteq X$ by which $(X, \leq')$ is Debreu separable. Note (9) simplifies (2) (we can argue like we did to justify (8) simplifies (4)) and it can be used instead of (2) in the proof of Theorem 1.

It is straightforward, as we show in Proposition 4, to extend the result in Proposition 3 to an equivalence between Debreu separable realizers and *injective monotone multi-utilities*, that is, multi-utilities where all involved functions are injective monotones [10].
Proposition 4. If \((X, \leq)\) is a partial order, then it has a Debreu separable realizer \((\leq_i)_{i \in I}\) if and only if it has an injective monotone multi-utility \((u_i)_{i \in I}\).

Proof. Assume first \((X, \leq)\) has a Debreu separable realizer \((\leq_i)_{i \in I}\). As we noted in Proposition 3, there exists for each \(i \in I\) a utility function \(u_i\) such that for all \(x, y \in X\) \(x \leq_i y \iff u_i(x) \leq u_i(y)\). Thus, since \((\leq_i)_{i \in I}\) is a realizer of \(\leq\), we have for all \(x, y \in X\)

\[
x \leq y \iff x \leq_i y \ \forall i \in I \iff u_i(x) \leq u_i(y) \ \forall i \in I.
\]

Following Proposition 3, \(u_i\) is an injective monotone for all \(i \in I\) and, thus, \((u_i)_{i \in I}\) is an injective monotone multi-utility by (10).

Conversely, given an injective monotone multi-utility \((u_i)_{i \in I}\) for \((X, \leq)\), we define \((\leq_i)_{i \in I}\) using \((u_i)_{i \in I}\), \(x \leq_i y \iff u_i(x) \leq u_i(y)\) for all \(x, y \in X\). Since \(u_i\) is a monotone for all \(i \in I\), we have that \(\leq_i\) is an extension of \(\leq\) that is reflexive and transitive for all \(i \in I\). Given that \(u_i\) is injective, \(\leq_i\) is also antisymmetric. Since \(\leq_i\) is clearly total and defined by a utility function, it is Debreu separable by [3, Theorem 1.3.2]. We conclude \(\leq_i\) is a Debreu separable extension of \(\leq\) for all \(i \in I\). Finally, since we have

\[
x \leq y \iff u_i(x) \leq u_i(y) \ \forall i \in I \iff x \leq_i y \ \forall i \in I,
\]

the sequence \((\leq_i)_{i \in I}\) is a realizer. \(\square\)

To illustrate the advantage of the characterization of Debreu separable extensions by injective monotones, we show, in Proposition 5, a stronger version of Proposition 3. In particular, we recover [11, Theorem 3.3] for the restricted case of partial orders with injective monotones. This restriction, however, allows us to avoid using the axiom of choice in its proof, which is the main novelty in the proof of Proposition 5.

Proposition 5. Let \((X, \leq)\) be a partial order where injective monotones exist and \(A \subseteq X\) be an antichain. If \(\leq_A\) is a Debreu separable total order on \(A\), then there exists a Debreu separable total order which extends both \((X, \leq)\) and \((A, \leq_A)\).

Proof. Take \(B := \{x \in X / A | x \equiv a \ \forall a \in A\}\). Since \(B \subseteq X\), then \(B\) has an injective monotone \(c_B : B \to (0, 1)\) and, since \(\leq_A\) is Debreu separable, it also has an injective monotone \(c_A : A \to (1, 2)\) (see Proposition 3). Thus, \(c_{AB} : A \cup B \to \mathbb{R},\) where \(c_{AB}(x) := c_B(x)\) if \(x \in B\) and \(c_{AB}(x) := c_A(x)\) if \(x \in A\), is an injective monotone for \((A \cup B, \leq_A \cup \leq_A)\). Define now

\[
Y := \{x \in X | \exists a \in A \text{ such that } x < a\},
\]

\[
Z := \{x \in X | \exists a \in A \text{ such that } a < x\}
\]

and note \(X = Y \cup B \cup A \cup Z\) is a decomposition of \(X\) into pairwise disjoint sets. Notice both \(Y\) and \(Z\) inherit, respectively, injective monotones \(c_Y : Y \to (-1, 0)\) and \(c_Z : Z \to (2, 3)\) from \(X\). Consider, thus, the total order \(x \leq_u y \iff u(x) \leq u(y)\)
Figure 4: Hasse diagram of a sequence of partial orders \((\leq_{n})_{n=1}^{3}\) extending \(\leq'_0\), a proper subset of \(\leq_0\) from Figure 3, according to the procedure in Theorem 1. The notation is equal to the one in Figure 3.

\(u(y)\) where \(u(x) := c_Y(x)\) if \(x \in Y\), \(u(x) := c_{AB}(x)\) if \(x \in A \cup B\) and \(u(x) := c_Z(x)\) if \(x \in Z\). Notice, since it has a utility function, \(\leq_u\) is a Debreu separable partial order (see Proposition 3).

In order to finish the proof, we ought to see that \((X, \leq_u)\) is an extension of both \((X, \leq)\) and \((A, \leq_A)\). Note \(x \leq_A y\) implies \(x \leq_u y\) by construction for all
Figure 5: Representation of $\leq_u$, the total order defined on $X = Y \cup B \cup A \cup Z$ by $u$ in Proposition 5. In particular, $\leq_u$ is the transitive closure of the arrows in the figure, where an arrow from a set $C \subseteq X$ to a set $D \subseteq X$ implies $c \leq_u d$ for all $c \in C$, $d \in D$. The dotted lines represent the specific relations, which lead to a contradiction, that we check in Proposition 5 to conclude $\leq_u$ is an extension of the original partial order $\leq$.

$x, y \in A$. Since the rest are straightforward, the only three cases with $x \leq y$ where we need to check whether $\leq_u$ extends $\leq$ are $x \in A \cup B$ and $y \in Y$, $x \in Z$ and $y \in Y$ and, lastly, $x \in Z$ and $y \in A \cup B$ (see Figure 5). As we will show, all three cases lead to contradiction. In the first case, there exists some $a \in A$ such that $y \prec a$ by definition. Thus, if $x \in A$ ($x \in B$), we have $x \prec a$ by transitivity of $\leq$, contradicting the definition of $A$ ($B$). In the second case, there exist $a, a' \in A$ such that $a \prec x \leq y \prec a'$ by definition. Thus, we have $a \prec a'$ by transitivity of $\leq$, contradicting the definition of $A$. Lastly, in the third case, there exists some $a \in A$ such that $a \prec x$ by definition. Thus, $a \prec y$, contradicting the definition of $A$ if $y \in A$ and that of $B$ if $y \in B$.

\section{Relation between the different notions of order dimension}

In this section, we address the relation between the three different notions of order dimension we defined above, namely, the Debreu, geometrical and Dushnik-Miller dimension. We focus, in particular, in their relation when either of them is countable.

\subsection{Countable geometrical and Debreu dimension}

We begin, in Corollary 1, showing the equivalence between countability of the geometrical dimension and that of the Debreu dimension. To do so, we use the equivalence in Proposition 4 between the existence of injective monotone multi-utilities of certain cardinality and that of Debreu separable realizers of the same cardinality, and the equivalence between the existence of countable
Corollary 1. If $(X, \leq)$ is a partial order, then its geometrical dimension is countable if and only if its Debreu dimension is countable.

Proof. The only if implication follows directly from Proposition 4. For the if implication, it is sufficient, by Proposition 4, to show there exists a countable injective monotone multi-utility. Using the increasing family $(A_n)_{n \geq 1}$ from Remark 1, we can construct an injective monotone $u$ as in (5) (see [10, Proposition 5]). Thus, for $m \geq 1$, we can define $u_m$ like $u$ but using $(A_n^m)_{n \geq 1}$, which is equal to $(A_n)_{n \geq 1}$ except for the permutation of the sets $A_1$ and $A_m$. To conclude, we need to show $(u_m)_{m \geq 1}$ is a multi-utility. By definition, we just have to show, if $x \preceq y$ for $x, y \in X$, then there exists some $m_0 \geq 1$ such that $u_{m_0}(x) > u_{m_0}(y)$. To finish, notice that is the case if we take some $m_0 \geq 1$ such that $x \in A_1^{m_0}$ and $y \notin A_1^{m_0}$ (see [10, Lemma 1]), which exists by Remark 1 and definition of multi-utility.

Corollary 1 illustrates the close relation between the Debreu dimension and the geometrical dimension. As we discussed in the introduction, this is not the case when one considers, instead of the Debreu dimension, either the Dushnik-Miller dimension [5] or the Ore dimension [15] of a partial order. Note, as a direct consequence of Corollary 1, we get that the Dushnik-Miller dimension is countable whenever the geometrical dimension is. However, the converse is not true, as we discussed in the introduction.

3.2 Finite dimension

In the reminder of this section, we discuss the relation between these notions of dimension when they are finite. We begin relating the geometrical dimension and the Dushnik-Miller relation. In particular, we show, in Corollary 2, the Dushnik-Miller dimension is finite whenever the geometrical dimension also is. While this is already known (see [14, Proposition 1]), we show it can be proven using the technique from Theorem 1, which we extend in Proposition 6.

Proposition 6. If $(X, \leq)$ is a partial order with a countable multi-utility and $u : X \to \mathbb{R}$ is a monotone, then there exists a sequence of extensions of $\leq$, $(\leq_n)_{n \geq 0}$, such that $\leq_u := \lim_{n \to \infty} \leq_n$ is a linear extension of $\leq$ where $x \preceq u y$ for all $x, y \in X$ such that both $x \preceq y$ and $u(x) < u(y)$ hold.

Proof. Given the monotone $u : X \to \mathbb{R}$, we will construct a linear extension of $\leq$ which preserves the properties of $u$ in the statement, $\leq_u$, via a sequence of partial orders $(\leq_n)_{n \geq 0}$ which we will define using both $u$ and $(A_n)_{n \geq 1}$, an enumeration of a countable family of increasing sets defined as in the proof of Theorem 1 (see Remark 1). We will conclude by showing

$$\leq_u := \lim_{n \to \infty} \leq_n$$
is a linear extension of $\leq$ such that, if $x \bowtie y$ and $u(x) < u(y)$ hold, then $x <_u y$.

As a starting point, we define the binary relation $\leq_0$ like

$$x \leq_0 y \iff \begin{cases} x \leq y \\ x \bowtie y \text{ and } u(x) < u(y) \end{cases}$$ (11)

for all $x, y \in X$. Note $\leq_0$ is reflexive since $\leq$ is and $\leq \leq_0$. $\leq_0$ is also antisymmetric since, whenever we have $x \leq_0 y$ and $y \leq_0 x$ for some $x, y \in X$, then $x \leq y$ and $y \leq x$ hold. Thus, by antisymmetry of $\leq$, we have $x = y$. Furthermore, $\leq_0$ is transitive. To show this, take $x, y, z \in X$, assume $x \leq_0 y$ and $y \leq_0 z$ hold and consider four cases. If we have $x \leq y \leq z$, then $x \leq z$ by transitivity of $\leq$ and, thus, $x \leq_0 z$. If $x \leq y \approx z$ holds, then $u(y) < u(z)$. In case $x \approx z$, then $u(x) < u(y) < u(z)$, by monotonicity of $u$, and $x \leq_0 z$. If $x \leq z$, then $x \leq_0 z$ by definition. Lastly, $z \leq x$ implies $z \leq y$ by transitivity of $\leq$, which contradicts the fact $y \bowtie y$. In case $x \bowtie y \leq z$ holds, transitivity follows analogously. In particular, if $x \bowtie z$, then $u(x) < u(y) \leq u(z)$, by monotonicity of $u$, and $x \leq_0 z$. If $x \leq z$, then $x \leq_0 z$ by definition. Lastly, $z \leq x$ implies $y \leq x$ by transitivity of $\leq$, which contradicts the fact $x \bowtie y$. The final case is that where $x \bowtie y \bowtie z$ and $u(x) < u(y) < u(z)$ hold. If $x \bowtie z$ holds, then $x \leq_0 z$ by definition. $z \leq x$ contradicts the monotonicity of $u$, as we have $u(z) > u(x)$. Lastly, $x \leq z$ implies $x \leq_0 z$, as desired. Hence, $\leq_0$ is transitive and, thus, a partial order. We define now the binary relations $\leq_n$ for all $n \geq 1$ like

$$x \leq_n y \iff \begin{cases} x \leq_{n-1} y \\ x \bowtie_{n-1} y, \ x \notin A_n \text{ and } y \in A_n \end{cases}$$ (12)

for all $x, y \in X$ (note, although (12) is equal to (4), $(\leq_n)_{n \geq 0}$ varies between them, as they use a different $\leq_0$). In order to show $\leq_n$ is a partial order for all $n \geq 0$, we use induction, analogously to the proof of Theorem 1. In particular, note $\leq_0$ is a partial order, as we proved above, and we can follow the proof of Theorem 1 to conclude, for all $n \geq 0$, $\leq_{n+1}$ is reflexive and antisymmetric whenever $\leq_n$ is. To show the same holds for transitivity, that is, that $x \leq_{n+1} y$ and $y \leq_{n+1} z$ imply $x \leq_{n+1} z$ for all $x, y, z \in X$ whenever $\leq_n$ is transitive, we can also follow Theorem 1 except for a couple of cases which we need to consider separately. In particular, the cases where we either have $x \leq_n y$ and $y \bowtie_n z$ with $x \bowtie y$ and $u(x) < u(y)$ or $x \bowtie_n y$ and $y \leq_n z$ with $y \bowtie z$ and $u(y) < u(z)$. In the first case, note the only situation of interest is $x \bowtie_n z$ (see the proof of Theorem 1), which implies $x \bowtie z$ and $u(x) = u(z)$, since, otherwise, we would have $\neg(x \bowtie z)$, contradicting the fact $x \bowtie_n z$. Similarly, we conclude we have $u(y) = u(z)$. We arrive, thus, to a contradiction, since both $u(x) < u(y)$ and $u(x) = u(z) = u(y)$ cannot hold. In the second case, we can similarly argue the only case of interest is that where $x \bowtie_n z$ and obtain a contradiction, since this scenario implies both $u(y) < u(z)$ and $u(y) = u(x) = u(z)$ are fulfilled. Following Theorem 1, thus, we obtain $\leq_n$ is a partial order for all $n \geq 0$ and, since we also have $\leq_{n+1}$ extends $\leq_n$ for all $n \geq 0$ by definition, we conclude, by Lemma 1, $\leq_u = \lim_{n \to \infty} \leq_n$ is a partial order that extends $\leq$. Moreover, we
have \( \leq_u \) is a total order, which can be proven like we showed \( \leq' \) was a total order in Theorem 1. In particular, \( \leq_u \) is a linear extension of \( \leq \). To conclude, notice, if \( x \succ y \) and \( u(x) < u(y) \) hold for some \( x, y \in X \), then we have \( x \prec_0 y \) by definition of \( \leq_0 \) and, by the definition of \( \leq_n \) for \( n \geq 1 \) and the fact \( \leq_u \) extends \( \leq_0 \), we obtain \( x \prec_u y \), as desired. □

Note \( (\leq_n)_{n \geq 0} \) is defined in the same manner in both Theorem 1 and Proposition 6. However, the way we define \( \leq_0 \) in Proposition 6 prevents us from proving the extensions we construct are Debreu separable, which contrasts with Theorem 1. To illustrate this, we present, in the following paragraph, a partial order and a monotone such that, after following the procedure in Proposition 6, we obtain a linear extension \( \leq' \) which is not Debreu separable.

Take \( (X, \leq) \) with \( X := \mathbb{R} \setminus \{0\} \) and \( \leq \) defined as in (3). Note \((\chi_{B_n}, \chi_{C_n})_{n \geq 1}\) is a countable multi-utility, where \( \chi_A \) is the characteristic function of a set \( A \), \((q_n)_{n \geq 1}\) is a numeration of the strictly positive rational numbers, \( B_n := \{x \in X | q_n < x\} \) and \( C_n := \{x \in X | x < -q_n\} \). If we take \( u \) to be the absolute value function and follow the definition of \( \leq_0 \) in (11), then we have \( x \succ_0 y \iff x = -y \). We define, furthermore, for all \( n \geq 1 \)

\[
A_n := \begin{cases} 
B_{\frac{n+1}{2}} & \text{if } n \text{ is odd}, \\
C_{\frac{n}{2}} & \text{if } n \text{ is even},
\end{cases}
\]

and take \( (A_n)_{n \geq 1} \) as the family of increasing sets in Proposition 6. Note we have \( \leq_{n+1} = \leq_n \) whenever \( n \) is odd, since \( x \succ_n y \) with \( y \in A_{n+1} \) and \( x \notin A_{n+1} \) implies \( y < -q_{\frac{n+1}{2}} \) and \( x \geq -q_{\frac{n+1}{2}} \), which leads to \( x = -y > q_{\frac{n+1}{2}} \) and \( y = -x < -q_{\frac{n+1}{2}} \) (\( x = -y \) since \( x \succ_n y \) implies \( x \succ_0 y \)) and results in \( x \in A_n \), \( y \notin A_n \) and \( y \succ_{n-1} x \) being true. Hence, we contradict the fact \( x \succ_n y \). As a result, we obtain \( x \prec' y \) whenever \( x = -y \) with \( x < 0 \) and \( y > 0 \). Thus, \((X, \leq')\) is the same linear extension of \((X, \leq)\) we constructed in Proposition 2. Hence, as we showed there, it is not Debreu separable. We conclude the procedure we introduced in Proposition 6 to construct linear extensions from monotones can lead to extensions which are not Debreu separable.

As a final remark regarding Proposition 6, notice, in contrast with Proposition 1 and Theorem 1 (as illustrated in Figure 4), substituting \( \leq \) by the trivial ordering in the definition of \( \leq_0 \) in (11) could result in \( \leq_0 \) not being transitive. We use now Proposition 6 to prove Corollary 2.

**Corollary 2** ([14, Proposition 1]). If \( (X, \leq) \) is a partial order with a finite multi-utility \((u_i)_{i=1}^N\), then it has a finite realizer \((\leq_i)_{i=1}^N\). In particular, any partial order whose geometrical dimension is finite has a finite Dushnik-Miller dimension.

**Proof.** Take \((u_i)_{i=1}^N\) a finite multi-utility for \((X, \leq)\), which exists by hypothesis. Analogously to how we defined \( \leq_u \) for a monotone \( u \) in the proof of Proposition 6, we define a linear extension of \( \leq \), \( \leq_i \), for each \( u_i \in (u_i)_{i=1}^N \). To conclude, we ought to notice that \((\leq_i)_{i=1}^N\) is a realizer of \( \leq \), that is, that we have

\[
x \leq y \iff x \leq_i y \forall i = 1, ..., N,
\]

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which is easy to see.

Note Corollary 2 is a straightforward application of Proposition 6 and was shown in [14, Proposition 1] through a different constructive method. In fact, taking a countably infinite multi-utility $(u_n)_{n \geq 0}$, we can also follow the approach in [14] to construct a sequence of extensions of the original partial order $\leq$ that, in the limit, yield a linear extension of $\preceq$. We can take, in particular,

$$x \leq_0 y \iff \begin{cases} x \leq y \\ x \preceq y \text{ and } u_0(x) < u_0(y) \end{cases}$$

for all $x, y \in X$ and define for all $n \geq 1$

$$x \leq_n y \iff \begin{cases} x \leq_{n-1} y \\ x \preceq_{n-1} y, \ u_n(x) < u_n(y) \end{cases}$$

for all $x, y \in X$. Moreover, Corollary 2 was shown for partial order with finite ground sets in [7, Proposition 1]. However, the direct extension to infinite sets of the proof in [7] using standard techniques would follow a non-constructive approach relying on Szpilrajn’s extension theorem [20]. This contrasts with the constructive approach in both Proposition 6 and [14, Proposition 1].

We address now the relation between having a finite geometrical dimension and a finite Debreu dimension. As a first remark, notice the technique in Proposition 6 is not applicable in general to construct a Debreu realizer $(\leq)_{i=1}^N$ from a multi-utility $(u_i)_{i=1}^N$ when $N \leq \infty$. In order to illustrate this, take $(X, \leq)$ with $X := \mathbb{R}\setminus\{0\}$ and $\leq$ defined as in $(3)$. Following the technique in Corollary 2, we would take an arbitrary finite multi-utility $(u_i)_{i=1}^N$ of $(X, \leq)$ and construct a Debreu separable realizer $(\leq_i)_{i=1}^N$ where, for $i = 1, \ldots, N$, $\leq_i$ fulfills $x \prec_i y$ whenever $x \preceq y$ and $u_i(x) < u_i(y)$ hold for a pair $x, y \in X$. In particular, we could apply this technique to the multi-utility $U := (u_i)_{i=1}^3$, where $u_1(x) := |x|$, $u_2(x) := 1$ if $x > 0$ and $u_2(x) := 0$ if $x < 0$, and $u_3(x) := 1$ if $x < 0$ and $u_3(x) := 0$ if $x > 0$. However, like $\leq'$ in the proof of Proposition 2, any $\leq_1$ with the properties above will fail to be Debreu separable, since it will either have $x \prec_1 y$ with no $z \in X$ such that $x \prec_1 z \prec_1 y$ or $y \prec_1 x$ with no $z \in X$ such that $y \prec_1 z \prec_1 x$ for any pair $x, y \in X$ where $x = -y$. As a result, although the Debreu dimension of $(X, \leq)$ is 2, the technique in Corollary 2 will fail to construct a Debreu separable realizer $(\leq_i)_{i=1}^2$ from $U$. Note the the Debreu dimension of $(X, \leq)$ is 2 since it is not total and $(v_i)_{i=1}^2$ is an injective monotonically multi-utility, where $v_1(x) := x$ if $x > 0$ and $v_1(x) := -e^x$ if $x < 0$, and $v_2(x) := -x$ if $x < 0$ and $v_2(x) := -e^{-x}$ if $x > 0$. Thus, by Proposition 4, there exists a Debreu separable realizer $(\leq_i)_{i=1}^2$. Note, also, any procedure similar to the one in Proposition 6 would result in a linear extension that is not Debreu separable when applied to $u_1$, not only the one that follows Proposition 6 using $(A_n)_{n \geq 1}$ as defined in $(13)$.

As the previous paragraph suggests, there exist, in fact, partial orders where the geometrical dimension is finite and the Debreu dimension is not. We show
this in Theorem 2, where we introduce a partial order on \( \mathbb{R} \setminus \{0\} \) that extends the one defined in (3).

**Theorem 2.** There exist partial orders with finite geometrical dimension and countably infinite Debreu dimension. In particular, such partial orders whose geometrical dimension is 2 exist.

*Proof.* Consider the set \( X := \mathbb{R} \setminus \{0\} \) equipped with the partial order \( \leq \) defined by

\[
x \leq y \iff \begin{cases} |x| \leq |y|, & \text{and} \\ \text{sgn}(x) \leq \text{sgn}(y) \end{cases}
\]

for all \( x, y \in X \), where \(| \cdot |\) is the absolute value function and \( \text{sgn}(\cdot) \) is the sign function, that is, \( \text{sgn}(x) := 1 \) if \( x > 0 \) and \( \text{sgn}(x) := -1 \) if \( x < 0 \). (A representation of \((X, \leq)\) can be found in Figure 6.) Note the absolute value function together with the sign function form a multi-utility of minimal cardinality, since \((X, \leq)\) is not total. Thus, the geometrical dimension of \((X, \leq)\) is 2.

As shown in Corollary 1, the Debreu dimension is, at most, countably infinite. To conclude, we will show, by reduction to the absurd, \((X, \leq)\) has no finite Debreu separable realizer. As a result, the Debreu dimension of \((X, \leq)\) is countably infinite. Assume, hence, there exists a Debreu separable realizer \((\leq_i)_{i=1}^N\) for some \( N < \infty \).

To begin with, notice, for all \( x \in X \) where \( 0 < x \), there exists some \( i \), \( 1 \leq i \leq N \), such that \( x \leq_i -y \) for all \( y \in X \) such that \( x < y \). To show this, fix such an \( x > 0 \) and assume it is not true. Then, define for each \( i \), \( 1 \leq i \leq N \), \( A_i := \{ y \in X | x < y \text{ and } x \leq_i -y \} \) and \( x_i := \inf A_i \), if \( A_i \neq \emptyset \), and \( x_i := \infty \) otherwise. Note \( \inf \) denotes the infimum in \( \leq \) and, moreover, that we can assume \( x < x_i \) for all \( i \), \( 1 \leq i \leq N \), since, otherwise, there exists some \( i_0 \), \( 1 \leq i_0 \leq N \), such that \( x \leq_{i_0} -y \) for all \( y \in X \) such that \( x < y \) and, hence, we have finished. Thus, there exists some \( z \in X \) such that \( x < z \leq \min\{x_1, \ldots, x_N\} \).

Hence, we have \( -z \leq_i x \) for all \( i \), \( 1 \leq i \leq N \). By definition of realizer, we obtain \( -z \leq x \), which contradicts the fact \( x = -z \) (note this holds since \( x < z \)). In conclusion, for each \( x \in X \) such that \( x > 0 \), there exists some \( i \), \( 1 \leq i \leq N \), such that \( x \leq_i y \) for all \( y \in X \) for which \( x < y \). Given that \( N \) is finite and there exists some \( i \), \( 1 \leq i \leq N \), for each \( x \in X \) such that \( 0 < x \), then there exists some \( i_0 \), \( 1 \leq i_0 \leq N \), which possesses this property for an uncountable set of strictly positive elements \( X_0 \subseteq X \).

We finish showing \( \leq_{i_0} \) is not Debreu separable. Assume \( D \subseteq X \) is a Debreu dense subset of \( X \), take \( x, y \in X_0 \) and consider w.l.o.g. \( x < y \). There exist, then, \( d_x, d_y \in D \) such that \( -x \leq_{i_0} d_x \leq_{i_0} x \leq_{i_0} -y \leq_{i_0} d_y \leq_{i_0} y \), where \( x \leq_{i_0} -y \) since \( x \in X_0 \) and \( x < y \). Given that \( \leq_{i_0} \) is a partial order, we obtain that \( d_x \neq d_y \). Hence, \( D \subseteq X \) is uncountable and \( \leq_{i_0} \) is not Debreu separable, which contradicts our assumption. In summary, \((X, \leq)\) has no Debreu separable realizer of finite cardinality. \( \square \)

Note having a finite geometrical dimension and a finite Debreu dimension are indeed equivalent for certain partial orders (see [8, Proposition 10]), including
Figure 6: Representation of the partial order defined in Theorem 2, whose geometrical dimension is finite (in fact, 2) and whose Debreu dimension is countably infinite. In particular, we include three positive real numbers $0 < x < y < z$ in the top line and their negatives $-x, -y, -z$ in the bottom line. Note that an arrow from one element $x$ to another $y$ represents $x < y$ and that we only include the arrows among the nearest neighbours, although their compositions also belong to the ordered set by transitivity. As we note in Theorem 3, this partial order also illustrates that the existence of a finite multi-utility does not imply that of finite strict monotone multi-utilities.

those with a countable ground set. Note, also, the argument in Theorem 2 fails for countably infinite Debreu separable realizers $(\leq_i)_{i \geq 0}$, since we cannot define $z$ as in its proof. In particular, it is not necessarily true that, for each $x \in X$ such that $x > 0$, there exists some $i \geq 0$ such that $x \leq_i -y$ for all $y \in X$ such that $x < y$.

Corollary 1 and Theorem 2 address the question whether a well-known theorem by Debreu [3, Theorem 1.4.8] (namely, that there exists a utility function for a total preorder if and only if it is Debreu separable) can be extended. As we have shown, its generalization works for countable multi-utilities, that is, that countable multi-utilities exist if and only if countable Debreu realizers do, although it fails in general when only finite multi-utilities with cardinality larger than one exist.

4 Application: the classification of preorders

In this section, we show two applications of the framework we have developed here, in particular, of Theorem 2. The first one concerns the general classification of preordered spaces through monotones [8] and the second one the placement of majorization, a preorder with several applications [2, 12], inside
improves upon the classification of preordered spaces through real-valued monotones, showing, in particular, there are such spaces where finite multi-utilities exist and finite injective monotone multi-utilities do not. As we argue in Theorem 3, we can use the same counterexample in Theorem 2 to show the stronger statement that there are partial orders where finite multi-utilities exist and finite strict monotone multi-utilities do not. Moreover, applying [8, Proposition 10], we recover the equivalence when considering Debreu separable preorders.

**Theorem 3.** The following statements hold:

(i) If \((X, \leq)\) is a Debreu separable preorder, then it has a finite multi-utility \((u_i)_{i \in I}\) if and only if it has a finite strict monotone multi-utility \((v_i)_{i \in I}\).

(ii) There exist partial orders with finite multi-utilities and whose smallest strict monotone multi-utility is countably infinite. In particular, such partial orders with multi-utilities of cardinality 2 exist.

**Proof.** (i) Since the converse holds by definition without requiring Debreu separability on \((X, \leq)\), we simply show that a finite strict monotone multi-utility \((v_i)_{i \in I}\) can be constructed from a finite multi-utility \((u_i)_{i \in I}\) provided \((X, \leq)\) is Debreu separable. In order to do so, in analogy with \(8, \text{Proposition 10}\), it suffices to show that \(I_{u_i} := \{ r \in \mathbb{R} | \exists x, y \in X \text{ such that } x \leq y \text{ and } u_i(r) < u_i(x) \} \) is countable for all \(i \in I\). Fix, thus, some \(i \in I\) and a countable Debreu dense subset \(D \subseteq X\) and note that, for all \(r \in I_{u_i}\), there exists some \(d \in D\) such that \(d \in u_i^{-1}(r)\). This is the case since, by definition, there exist \(x, y \in X\) such that \(x \leq y\) and \(x, y \in u_i^{-1}(r)\). Hence, since \(D\) is Debreu dense, there exists some \(d \in D\) such that \(x \leq d \leq y\) and, by monotonicity of \(u_i\), \(r = u_i(x) \leq u_i(d) \leq u_i(y) = r\). As a result, for all \(r \in I_{u_i}\), we have that \(D \cap u_i^{-1}(r) \neq \emptyset\) and, hence, we can fix some \(d_r \in D \cap u_i^{-1}(r)\). To conclude, we simply notice the map \(f : I_{u_i} \to D, r \mapsto d_r\) is injective.

(ii) We can follow the proof of Theorem 2, with \((\leq_i)_{i=1}^N\) being, instead of a Debreu separable realizer, defined by \(x \leq_i y \iff v_i(x) \leq v_i(y)\), where \((v_i)_{i=1}^N\) is a strict monotone multi-utility and \(N < \infty\). We can follow the proof of Theorem 2 to show that, for all \(x \in X\) where \(0 < x\), there exists some \(i, 1 \leq i \leq N\), such that \(v_i(x) < v_i(y)\) for all \(y \in X\) such that \(x < y\). We arrive to a contradiction following the remainder of the proof. We simply ought to notice that \(\leq_i\) is a total preorder with a utility function \(v_i\), which implies it is Debreu separable by [3, Theorem 1.4.8].

It should be noted that we cannot improve on Theorem 3 (i) by including Debreu separability into the first condition of the equivalence. That is, there exist preorders that posses finite strict monotone multi-utilities and are not Debreu separable. We include such a preorder in the following proposition, which shows a slightly stronger fact and improves on both [8, Proposition 9] and, as we will state in Corollary 4, [10, Lemma 5 (ii)].
Proposition 7. There exist preorders with finite strict monotone multi-utilities such that every subset that is either Debreu dense or Debreu upper dense is uncountable.

Proof. Take $X := \mathbb{R}\{0\}$ equipped with $\leq$, where

$$x \leq y \iff u_1(x) \leq u_1(y) \text{ and } u_2(x) \leq u_2(y),$$

with $u_1(x) := x$ for all $x \in X$, and $u_2(x) := 1/x$ if $x > 0$ and $u_2(x) := -1/|x|$ if $x < 0$ (see a representation of $(X, \leq)$ in Figure 7). Note that $\{u_1, u_2\}$ is a finite strict monotone multi-utility. For the assertion regarding Debreu density, take a Debreu dense subset $D \subseteq X$. Since $-x < x$ holds for all $x > 0$, there exists some $d_x$ such that $-x \leq d_x \leq x$. We will show, if $x \neq y$ for a pair $x, y > 0$, then $d_x \neq d_y$ and, hence, $D$ is uncountable. Assume, first, that $d_x > 0$. Then $d_x = x$ and $d_x \not\approx y$. Thus, $d_x \neq d_y$. Otherwise, $d_x < 0$ and $d_x \not\approx -y$. For the part of the statement that involves Debreu upper density, we can argue as for Debreu density. In particular, if $D \subseteq X$ is a Debreu upper dense subset, it is straightforward to see that it is uncountable. In order to do so, we simply notice that $\not\approx y$ for all $x, y > 0$ and that, if $d \in X$ fulfills that $\not\approx d \leq y$, then $d = y$. □

As a consequence of Theorem 3, we obtain that the equivalence between the geometrical and Debreu dimensions can be recovered under some hypotheses, as we state in the following proposition.

Corollary 3. If $(X, \leq)$ is a Debreu separable partial order and $N < \infty$, then the following statements hold:

(i) The geometrical dimension of $(X, \leq)$ equals two if and only if its Debreu dimension equals two.

(ii) If there exists a countable subset $B \subseteq X$ such that, for all $x, y \in X$ where $\not\approx y$, we either have $x \in B$ or $y \in B$, then the geometrical dimension of $(X, \leq)$ equals $N$ if and only if its Debreu dimension equals $N$.

Proof. (i) Since the converse is straightforward, we simply ought to show that having a geometrical dimension equal to two implies the Debreu dimension is two. In order to do so, take a multi-utility $\{u_1, u_2\}$. Since $(X, \leq)$ is Debreu separable, we can apply Theorem 3 and obtain a strict monotone multi-utility $\{v_1, v_2\}$. Assume then w.l.o.g. that $v_1$ is not an injective monotone, that is, that there exist $x, y \in X$ such that $\not\approx y$ and $v_1(x) = v_1(y)$. Then, if $v_2(x) \leq v_2(y)$ ($v_2(y) \leq v_2(x)$), we obtain $x \leq y$ ($y \leq x$) by definition of multi-utility, which contradicts the fact $\not\approx y$. Hence, $\{v_1, v_2\}$ constitutes an injective monotone multi-utility and, by Proposition 4, it defines a Debreu separable realizer of dimension two.

(ii) This follows, like Theorem 3, as a consequence of [8, Proposition 10]. As stated there, we simply ought to see that, for every $u_i$ in a multi-utility, we have that $I_{u_i} := \{r \in \mathbb{R} | \exists x, y \in X \text{ such that } x, y \in u_i^{-1}(r) \text{ and } -(x \sim y)\}$
Figure 7: Graphical representation of a preordered space, defined in Proposition 7, that is not Debreu separable despite having a finite strict monotone multi-utility. In particular, we include three positive real numbers \(0 < x < y < z\) in the top line and their negatives \(-x, -y, -z\) in the bottom line. Notice, an arrow from an element \(w\) to an element \(t\) represents \(w \succ t\).

is countable. By Debreu separability (see Theorem 3), the only case left to consider is that of \(r \in I_u\) such that \(x \succ y\) for all \(x, y \in u^{-1}_i(r)\). In this scenario, countability follows directly by assumption.

Note that the hypothesis in (i) is indeed mandatory, as Theorem 2 proves. Moreover, it is easy to see, under the hypotheses in (ii), that a result equivalent to Proposition 6 holds (although with a Debreu separable linear extension in this case). It remains a task for the future to continue refining the classification of preorders through real-valued monotones (see Figure 8 and [8, Figure 1] for the current classification). In particular, the relation between the existence of finite strict monotone multi-utilities and that of finite injective monotone multi-utilities should be clarified (specially, by Corollary 3, the cases where only finite strict monotone multi-utilities with \(N \geq 3\) exist).

4.1 Majorization

Regarding applications of partial orders in economics, quantum physics and other areas [2, 13], majorization constitutes a relevant example. That is, the partial order which is defined on \(\mathbb{P}^1_\Omega\), the set of probability distributions on a finite set \(\Omega\) whose components are non-increasingly ordered, by

\[ p \leq_m q \iff u_i(p) \leq u_i(q) \quad \forall i \in \{1, \ldots, |\Omega| - 1\}, \quad (14) \]

where \(u_i(p) := \sum_{n=1}^i p_n\) [2].
Preordered spaces

Debreu separable

Finite multi-utility

Finite strict monotone multi-utility

Utility function

Preordered spaces

Figure 8: Subset of the classification of preordered spaces. The other classes as well as their relation to the ones presented here can be found in [8, Figure 1]. Our contribution to the classification is included in Theorem 3, where we showed that there are preorders where finite multi-utilities exist while finite strict monotone multi-utilities do not (with majorization being such an example, as stated in Corollary 4) and that they are equivalent for Debreu separable preorders. To conclude, note that Proposition 7 includes a preorder that is not Debreu separable where finite strict monotone multi-utilities exists while utilities do not.

Both the geometrical and Dushnik-Miller dimension of \( (\mathcal{P}_\Omega^1, \preceq_m) \) are equal to \(|\Omega| - 1\) (see Proposition 8 in the Appendix A for a proof).\(^8\) However, as we state in Corollary 4, majorization has no finite strict monotone multi-utility whenever \(|\Omega| \geq 3\). In particular, its Debreu dimension is countably infinite for \(|\Omega| \geq 3\). We include the details of the proof in [9] and only sketch the idea here.

**Corollary 4 ([9, Theorem 1]).** If \(|\Omega| \geq 3\), then, although the geometrical dimension of \( (\mathcal{P}_\Omega^1, \preceq_m) \) is \(|\Omega| - 1\), it has no finite strict monotone multi-utility. In particular, the Debreu dimension of \( (\mathcal{P}_\Omega^1, \preceq_m) \) is countably infinite whenever \(|\Omega| \geq 3\).

**Proof.** Since the geometrical dimension of \( (\mathcal{P}_\Omega^1, \preceq_m) \) is \(|\Omega| - 1\) by Proposition 8 and, hence, countably infinite Debreu separable realizers exist by Corollary 1, we

\(^8\)Note this is an interesting application of Corollary 2 since its ground set has the cardinality of the continuum and \(|X| \leq \omega\) whenever the geometrical dimension is finite.
only ought to show is that there are no finite strict monotone multi-utilities. In order to do so, we simply define a subset \( S \subseteq P \Omega \) that, when equipped with the restriction of \( \leq_m \) on it, is order isomorphic to the partial order we constructed in Theorem 2 (see [9, Theorem 1] for the details). Lastly, we obtain the result by applying Theorem 3.

Note Corollary 4 is false when \( |\Omega| = 2 \), since \( u_1 \) is a utility function and, hence, both a strict monotone and an injective monotone. Note, also, what we have called majorization is the quotient space of the usual definition (see [12]), which consists of a preorder that is not a partial order. Nonetheless, Corollary 4 applies to the usual definition as well.

5 Conclusion

In this paper, we are mainly concerned with finding a useful notion of dimension for partial orders which reflects the geometrical intuition underlying it. As we discuss in Section 1, one could consider the minimal cardinality for which a multi-utility exists, a concept borrowed from mathematical economics [6], as a proper notion of dimension. This would essentially correspond to the number of lines required to represent the partial order. As noted, this geometrical notion of dimension can be drastically different from the usual definitions of dimension for partial orders, which were introduced by Dushnik and Miller [5] and Ore [15]. In fact, we include a partial order where, although the geometrical dimension is uncountable, both the Dushnik-Miller dimension and the Ore dimension are finite. The Debreu dimension, our variation of the Dushnik-Miller dimension, partially overcomes this gap by following Debreu’s classical result in mathematical economics [4], which, essentially, characterizes the total orders that behave exactly like a subset of the real line \( S \subseteq \mathbb{R} \) with its usual order \( \leq \).

Our main contributions can be summarized as follows:

(i) We have introduced the notion of Debreu dimension on a partial order \((X, \leq)\), which is based on Debreu separable linear extensions and is limited to partial orders where the ground set has at most the cardinality of the continuum, unlike the Dushnik-Miller dimension. Moreover, we have shown, under certain countability restrictions on \((X, \leq)\), Debreu linear extensions \((X, \leq')\) can be obtained as the limit of a sequence of partial orders \((\leq_n)_{n \geq 0}\) that starts with the original partial order \(\leq_0 = \leq\) and consecutively extends each of them \(\leq_n \subseteq \leq_{n+1}\) for all \(n \geq 0\) until, in the limit, a Debreu linear extension of \(\leq\) is obtained, \(\leq' = \lim_{n \to \infty} \leq_n\) (see Theorem 1 and Propositions 1 and 3). Furthermore, we have shown that the linear extensions that result from a previous technique [14] fail, in general, to be Debreu separable. As a consequence of the above, we have gained insight into the construction of injective monotones using countable multi-utilities (see [10, Proposition 5]) by showing injective monotones can be constructed as the limit of a sequence of binary relations where each extends the original one and is a partial order.
We have related the Debreu dimension with the geometrical dimension when they are countable. In particular, we have shown the one is countable whenever the other is (see Corollary 1). Thus, unlike the Dushnik-Miller dimension, our definition is intimately related to the real line and its usual order and, hence, to geometry. However, we have introduced a partial order which shows that, although the converse is false, there exist partial orders with finite geometrical dimension and countably infinite Debreu dimension (see Theorem 2). Moreover, we have provided hypotheses under which the equivalence between finite dimensions is recovered. Lastly, for partial orders with a countable geometrical dimension, we have constructed linear extensions for each monotone using, as in (i), a sequence of partial orders that yield the desired result in the limit (see Proposition 6). As a result, we have recovered, through a different method, that having a finite geometrical dimension implies the Dushnik-Miller dimension is finite as well (see Corollary 2 and [14, Proposition 1]).

As an application of our study of the Debreu dimension, we have improved on the classification of preordered spaces through real-valued monotones. In particular, we have constructed a preorder where finite multi-utilities exist and finite strict monotone multi-utilities do not (see Theorem 3). As a consequence of this, a preorder with several applications in economics and physics, majorization, shares these characteristics (see Corollary 4 for a precise statement and [9, Theorem 1] for the details). However, the equivalence is recovered for Debreu separable preorders.

Regarding the techniques we used, we should emphasize the fact that, since the partial orders of interest here fulfill some sort of countability restriction, we have avoided the axiom of choice throughout this work. This is particularly important in Propositions 1 and 5, given that their generalizations where no countability restriction is assumed, Szpilrajn extension theorem [20] and [11, Theorem 3.3], respectively, rely on the axiom of choice. Proposition 6 and its consequence, Corollary 2, also exemplify this improvement, although a different constructive approach for them was already known [14].

Since several scientific disciplines (see [10, Section 7] for an overview of them) rely on ordered structures and their representation via real-valued monotones and, as we have shown, these representation are closely related to the Debreu dimension, several areas of applied mathematics could benefit from a better understanding of the notion of dimension for partial orders. Specific formal questions also remain open. For example, regarding the classification of preorders through real-valued monotones, it is unclear whether preorders with finite strict monotone multi-utilities and no finite injective monotone multi-utilities exist.

A Appendix

Lemma 2. The binary relation $\leq_n$ defined in (2) is a partial order for all $n \in \mathbb{N}$. 

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Proof. We will see $\leq_n$ is a partial order for all $n \geq 0$ using induction. Notice, $\leq_0$ is a partial order by definition. Assume now $\leq_n$ is a partial order. Using this assumption, we will conclude $\leq_{n+1}$ is reflexive, transitive and antisymmetric, i.e. a partial order, and, by induction, that $\leq_n$ is a partial order for all $n \geq 0$.

To see $\leq_{n+1}$ is reflexive, it suffices to notice $\leq_0$ is reflexive by definition and that $\leq_{n+1}$ extends $\leq_n$.

We will first show that $\leq_{n+1}$ is antisymmetric. Assume we have $x \leq_{n+1} y$ and $y \leq_{n+1} x$ for $x, y \in X$. In case we have $x \leq_n y$ and $y \leq_n x$, then we have $x = y$ by the induction hypothesis. If $x \gg_n y$, then, by definition of $\leq_{n+1}$, we cannot have both $x \leq_{n+1} y$ and $y \leq_{n+1} x$. Lastly, whenever we have $x \leq_n y$ and $\neg(y \leq_n x)$ (equivalently for the symmetric case where the roles of $x$ and $y$ are interchanged), then, by definition, we have $x \leq_{n+1} y$ and $\neg(y \leq_{n+1} x)$, a contradiction. We conclude $\leq_{n+1}$ is antisymmetric.

For the transitivity of $\leq_{n+1}$, assume we have $x \leq_{n+1} y$ and $y \leq_{n+1} z$ for $x, y, z \in X$ and consider four possible situations. In case we have $x \leq_n y$ and $y \leq_n z$, then $x \leq_n z$ by transitivity of $\leq_n$ and, by definition, $x \leq_{n+1} z$. The case where $x \gg_n y$ and $y \gg_n z$ hold leads to contradiction, as it implies both $y \gg_0 d_{n+1}$ and $d_{n+1} \leq_0 y$ are true. If $x \leq_n y$ and $y \gg_n z$, note that $z \leq_n x$ implies $y \leq_n z$ by transitivity of $\leq_n$ and, thus, contradicts the fact $y \gg_0 n$. Furthermore, $x \leq_n z$ implies $x \leq_{n+1} z$ by definition. Hence, the only case left to consider is $x \gg_n z$. We deal with this in two subcases. If we have $x \leq_0 y$, then, since $d_{n+1} \leq_0 x$ contradicts that $y \gg_0 d_{n+1}$ by transitivity of $\leq_0$, we either have $x \leq_0 d_{n+1}$ or $x \gg_0 d_{n+1}$. Since $x \leq_0 d_{n+1}$ implies $x \leq_{n+1} z$ by transitivity of $\leq_0$ and definition of $\leq_{n+1}$ and $x \gg_0 d_{n+1}$ implies $x \leq_{n+1} z$ by definition of $\leq_{n+1}$, we conclude $x \leq_{n+1} z$, as desired. On the contrary, if we have $\neg(x \leq_0 y)$, then there exists a minimal $m \in \mathbb{N}$, $1 \leq m \leq n$, such that $x \gg_0 d_m$ and $d_m \leq_0 y$. If $d_m \leq_0 z$, then, since $x \gg_{m-1} z$ holds (anything else would contradict $x \gg_0 z$), we have $x \leq_m z$ and, by definition, $x \leq_{n+1} z$, as we wanted. If $z \leq_0 d_m$ holds, then, by transitivity of $\leq_0$, we have $z \leq_0 y$, contradicting the fact $y \gg_0 z$. Lastly, if $z \gg_0 d_m$, then, since $z \gg_{m-1} y$ holds, we have $z \leq_m y$ by definition of $\leq_m$, which contradicts the fact $y \gg_0 z$. The case where $x \gg_n y$ and $y \leq_n z$ holds follows similarly. We include the details for completeness. If $x \gg_0 y$ and $y \leq_n z$, note that $z \leq_n x$ implies $y \leq_n x$ by transitivity of $\leq_n$ and, thus, contradicts the fact $y \gg_0 x$. Furthermore, $x \leq_n z$ implies $x \leq_{n+1} z$ by definition. Hence, the only case left to consider is $x \gg_n z$. We deal with this in two subcases. If we have $y \leq_0 z$, then, since $\leq_0$ is transitive, we have $d_{n+1} \leq_0 z$ and, given that $x \gg_0 d_{n+1}$ holds because both $x \leq_{n+1} y$ and $x \gg_0 y$ are true, we have, as desired, $x \leq_{n+1} z$, since $x \gg_0 z$ holds. On the contrary, if we have $\neg(y \leq_0 z)$, then there exists a minimal $m \in \mathbb{N}$, $1 \leq m \leq n$, such that $y \gg_0 d_m$ and $d_m \leq_0 z$. If $x \leq_0 d_m$, then $x \leq_0 z$ by transitivity of $\leq_0$ and, thus, $x \leq_{n+1} z$. If $x \gg_0 d_m$, then, since $x \gg_{m-1} z$ holds (anything else would contradict $x \gg_0 y$), we have $x \leq_m z$ and, by definition, $x \leq_{n+1} z$, as we wanted. Lastly, if $d_m \leq_0 x$, then, since $x \gg_{m-1} y$ holds, we would have $y \leq_m x$, which contradicts the fact $x \gg_0 y$.

In summary, $\leq_{n+1}$ is transitive and, in fact, a partial order. By induction, we conclude $\leq_n$ is a partial order for all $n \geq 0$. □
Lemma 3. The binary relation \( \preceq_n \) defined in (4) is a partial order for all \( n \in \mathbb{N} \).

Proof. We will see \( \preceq_n \) is a partial order for all \( n \geq 0 \) using induction (analogously to how we did it in Lemma 2). Notice \( \preceq_0 \) is a partial order by definition. Assume now \( \preceq_n \) is a partial order. Using this assumption, we will conclude \( \preceq_{n+1} \) is reflexive, transitive and antisymmetric, i.e. a partial order, and, by induction, that \( \preceq_n \) is a partial order for all \( n \geq 0 \). Since we can show reflexivity and antisymmetry of \( \preceq_{n+1} \) analogously to how we did it in Proposition 1, we only include, for convenience, the details leading to the transitivity of \( \preceq_{n+1} \).

Assume we have \( x \preceq_{n+1} y \) and \( y \preceq_{n+1} z \) for \( x, y, z \in X \) and consider four possible situations. In case we have \( x \preceq_n y \) and \( y \preceq_n z \), then \( x \preceq_n z \) by transitivity of \( \preceq_n \) and, by definition, \( x \preceq_{n+1} z \). The case where \( x \npreceq_n y \) and \( y \npreceq_n z \) hold leads to contradiction, as it implies both \( y \in A_{n+1} \) and \( y \notin A_{n+1} \) are true. If \( x \preceq_n y \) and \( y \npreceq_n z \), note that \( z \preceq_n x \) implies \( z \preceq_n y \) by transitivity of \( \preceq_n \) and, thus, contradicts the fact \( y \npreceq_n z \). Furthermore, \( x \preceq_n z \) implies \( x \preceq_{n+1} z \) by definition. Hence, the only case left to consider is \( x \npreceq_n z \). We deal with this in two subcases. If we have \( x \preceq_0 y \), then, since \( A_{n+1} \) is increasing in \( \preceq_0 \) and \( y \notin A_{n+1} \), then \( x \notin A_{n+1} \) (otherwise, \( y \in A_{n+1} \), a contradiction) and, since \( x \npreceq_n z \) and \( z \in A_{n+1} \) hold, we have \( x \preceq_{n+1} z \) as desired. On the contrary, if we have \( \neg(x \preceq_0 y) \), then there exists a minimal \( m \in \mathbb{N}, 1 \leq m \leq n \), such that \( x \notin A_m \) and \( y \in A_m \). If \( z \in A_m \), then, since \( x \npreceq_{m-1} z \) holds (anything else would contradict \( x \npreceq_{m} z \)), we have \( x \preceq_{m} z \) and, by definition, \( x \preceq_{n+1} z \), as we wanted. Conversely, if \( z \notin A_m \), then, since \( z \npreceq_{m-1} y \) holds, we have \( z \preceq_m y \), which contradicts the fact \( y \npreceq_{n} z \). The case where \( x \npreceq_n y \) and \( y \preceq_n z \) hold follows similarly. We include the details for completeness. If \( x \npreceq_n y \) and \( y \preceq_n z \), note that \( z \preceq_n x \) implies \( y \preceq_n x \) by transitivity of \( \preceq_n \) and, thus, contradicts the fact \( y \npreceq_n x \). Furthermore, \( x \preceq_n z \) implies \( x \preceq_{n+1} z \) by definition. Hence, the only case left to consider is \( x \npreceq_n z \). We deal with this in two subcases. If we have \( y \preceq_0 z \), then, since \( A_{n+1} \) is increasing in \( \preceq_0 \), \( y \in A_{n+1} \), \( x \notin A_{n+1} \) and \( x \npreceq_n z \), we have \( x \preceq_{n+1} z \), as desired. On the contrary, if we have \( \neg(y \preceq_0 z) \), then there exists a minimal \( m \in \mathbb{N}, 1 \leq m \leq n \), such that \( y \notin A_m \) and \( z \in A_m \). If \( x \notin A_m \), then, since \( x \npreceq_{m-1} z \) holds (anything else would contradict \( x \npreceq_{m} z \)), we have \( x \preceq_{m} z \) and, by definition, \( x \preceq_{n+1} z \), as we wanted. Conversely, if \( x \in A_m \), then, since \( x \npreceq_{m-1} y \) holds, we have \( y \preceq_m x \), which contradicts the fact \( x \npreceq_n y \). In summary, \( \preceq_{n+1} \) is transitive and, in fact, a partial order for all \( n \geq 0 \). By induction, we conclude \( \preceq_n \) is a partial order for all \( n \geq 0 \). \( \square \)

Proposition 8. If \( |\Omega| \geq 2 \), then the geometrical and Dushnik-Miller dimensions of \( (P_{\Omega}^1, \preceq_m) \) are \( |\Omega| - 1 \).

Proof. If \( |\Omega| = 2 \), then, since \( (P_{\Omega}^1, \preceq_m) \) is total, the results follows from (14). If \( |\Omega| = 3 \), then it follows since (14) holds and \( (P_{\Omega}^1, \preceq_m) \) is not total. If \( |\Omega| \geq 4 \), then, by definition (14), the geometrical dimension of \( (P_{\Omega}^1, \preceq_m) \) is, at most, \( |\Omega| - 1 \). Hence, by Corollary 2, its Dushnik-Miller dimension is also, at most, \( |\Omega| - 1 \). To conclude, we ought to show that \( (P_{\Omega}^1, \preceq_m) \) has no realizer \( (\leq)^{|\Omega|-2} \), which implies that both its geometrical and Dushnik-Miller dimension are equal.
to $|\Omega| - 1$. In order to do so, it suffices to construct a subset $S \subset P^1_{\Omega}$ that is order isomorphic to the subset $S_0 := \{1 - |\Omega|, \ldots, -1, 1, \ldots, |\Omega| - 1\}$ of the partial order in [8, Proposition 7]. We can then follow [8, Proposition 7] to conclude that no realizer with $|\Omega| - 2$ components exists.

For simplicity, we conclude indicating the general idea that needs to be followed in order to construct $S$. We begin taking some distribution $p_1 \in P^1_{\Omega}$ such that $(p_1)_i > (p_1)_{i+1} > 0$ for all $i$ such that $1 \leq i < |\Omega|$. $p_1$ is then used to sequentially define the set of distributions $S := (p_i, q_i)_{i=1}^{[\Omega]-1}$, where each distribution is defined from the previous one by simple adding or subtracting some probability mass at the appropriate component in order to fulfill the following relations:

$$u_1(p_1) > \cdots > u_1(p_{n-1}) > u_1(q_n) > u_1(p_n) > u_1(q_1) > \cdots > u_1(q_{n-1}),$$

$$u_2(p_n) > u_2(p_1) > \cdots > u_2(p_{n-2}) > u_2(q_{n-1}) > u_2(p_{n-1}) > u_2(q_n) >$$

$$> u_2(q_1) > \cdots > u_2(q_{n-2}),$$

$$\vdots$$

$$u_n(p_2) > \cdots > u_n(p_n) > u_n(q_1) > u_n(p_1) > u_n(q_2) > \cdots > u_n(q_n),$$

where each restriction is obtained from the previous one by permuting the subindexes and we fix $n := |\Omega| - 1$ for commodity. From the above relations, it is direct to check the order isomorphism between $S$ and $S_0$ and, hence, to obtain the desired result. \qed
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