CONVEXITY OF SINGULAR AFFINE STRUCTURES AND
TORIC-FOCUS INTEGRABLE HAMILTONIAN SYSTEMS

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Abstract. This work is devoted to a systematic study of symplectic convexity for integrable Hamiltonian systems with elliptic and focus-focus singularities. A distinctive feature of these systems is that their base spaces are still smooth manifolds (with boundary and corners), similarly to the toric case, but their associated integral affine structures are singular, with non-trivial monodromy, due to focus singularities. We obtain a series of convexity results, both positive and negative, for such singular integral affine base spaces. In particular, near a focus singular point, they are locally convex and the local-global convexity principle still applies. They are also globally convex under some natural additional conditions. However, when the monodromy is sufficiently big then the local-global convexity principle breaks down, and the base spaces can be globally non-convex even for compact manifolds. As one of surprising examples, we construct a 2-dimensional “integral affine black hole”, which is locally convex but for which a straight ray from the center can never escape.

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1. Introduction

This paper is devoted to the study of convexity properties of integrable Hamiltonian systems in the presence of focus-focus singularities. These singularities are one of the three kinds of elementary non-degenerate singularities for integrable systems (the other
two being elliptic and hyperbolic). They are of special interest in mathematics and physics for at least the following two reasons:

- They can be found everywhere, even in the most simple physical systems, e.g., the spherical pendulum, the symmetric spinning top, the internal movement of molecules, the nonlinear Schrödinger equation, the Jaynes-Cummings-Gaudin model, and so on. (See, e.g., [9, 20, 30, 31, 40, 135] and references therein.) They also appear in the theory of special Lagrangian fibrations on Calabi-Yau manifolds related to mirror symmetry. (See, e.g., [26, 49, 50, 80] and references therein.)

- They give rise to the non-trivial monodromy phenomenon, both classical and quantum, for both the Lagrangian fibrations on symplectic manifolds and the singular integral affine structure on the base spaces. This monodromy is an obstruction to the existence of global action-angle variables, and is also an obstruction to the existence of global quantum numbers in a very large class of integrable quantum systems, including such simple systems as the $H_2O$ molecule.

For our study of convexity, this monodromy also creates a challenge and leads to some very surprising results.

We restrict our attention to toric-focus integrable Hamiltonian systems, i.e., systems which admit only non-degenerate elliptic and focus-focus singularities and no hyperbolic singularity. The reason is that the base space (i.e., the space of connected components of the level sets of the momentum map) branches out at hyperbolic singular points, so it does not make much sense to talk about convexity at those points. On the other hand, if the system is toric-focus, then the base space is still a smooth manifold, together with an associated integral affine structure, and one can talk about the convexity of that structure.

When there are no focus-focus singularities, then the system is toric (i.e, it admits a Hamiltonian torus action of half the dimension), and its convexity properties are well-known, due to the famous Atiyah–Guillemin–Sternberg theorem [6, 54] and the Delzant classification theorem [32]. For a subclass of toric-focus systems in dimension 4 called semitoric systems, Văcuţă [127] developed a theory of multi-valued momentum polytopes which are similar to Delzant polytopes. Nevertheless, the problem of convexity of semi-toric systems has never been studied anywhere in the literature, as far as we know.

Let us now present the main concepts and results of this paper.

In order to study convexity of singular affine structures, we first need to define clearly what does convexity mean for them. So we introduce the notion of straight lines, especially those passing through singular points, and study their existence and their multiple branchings. Then we say that a set in a singular affine space is convex if any two points can be joined by at least one (and maybe several) straight line segment in that set. We also need several refinements of convexity: local and global convexity, strong convexity, $\Sigma$-convexity, and the usual notion of convex hull. We also need the notion of multiple-valued integral affine charts around focus singular points in our study.
The key technical assumption in almost all theorems is either compactness or a properness condition for non-compact spaces. As expected from the theory of non-linear Lie group actions on manifolds, without some hypothesis of properness convexity may fail. The so-called local-global convexity principle remains one of the main technical tools which links local to global convexity; it is presented in detail, in the framework needed in this paper, in Section 7, even in the presence of focus singularities.

The paper presents two kinds of convexity results, positive and negative.

**Positive convexity results.**

- (Theorem 7.1, Theorem 7.3, Proposition 7.1) **Local convexity near focus points:** Any box around a focus singular point is convex and, moreover, the local-global convexity principle holds in such a box.
- (Theorem 8.14, Theorem 8.21, Theorem 8.24) **Global convexity in dimension 2:** Let $\mathcal{B}$ be the 2-dimensional base space of a toric-focus integrable Hamiltonian system on a connected, 4-dimensional, symplectic manifold (with or without boundary). Assume that the base $\mathcal{B}$ is compact or satisfies some properness conditions. If $\mathcal{B}$ is non-empty and locally convex, then $\mathcal{B}$ is convex (in its own affine structure) and is topologically either a disk, an annulus, or a Möbius band. There are globally convex examples if $\mathcal{B}$ is a sphere.
- (Theorem 8.26, Theorem 8.28). **Global convexity if there exists a system-preserving $\mathbb{T}^{n-1}$-action or, equivalently, $(n-1)$ globally well-defined affine functions on the base space:** Let $\mathcal{B}$ be the base space of a toric-focus integrable Hamiltonian system with $n$ degrees of freedom on a connected symplectic manifold $\mathcal{M}$. Assume that the system admits a global Hamiltonian $\mathbb{T}^{n-1}$-action and that $\mathcal{B}$ is either compact or satisfies some properness conditions. Then $\mathcal{B}$ is convex.

**Negative convexity results.**

All negative convexity results we have in this paper are due to monodromy, which is the main potential obstruction to convexity. We find integrable Hamiltonian systems whose base spaces of the associated Lagrangian foliations possess sufficiently complicated monodromy which then enforces non-convexity of these base spaces.

- (Theorem 8.22) **Existence of “affine black holes” in dimension 2 (or higher), leading to locally convex but globally non-convex integral affine structures on the 2-sphere $S^2$** (with 24 focus singular points, counted with multiplicities).
- (Theorem 8.25) **Existence of a non-convex box of dimension 3 (or higher) around two focus curves passing nearby each other.** The local monodromy group in this case is a representation of the free group with 2 generators.
- (Theorem 7.6) **Existence of a non-convex focus$^m$ box for any $m \geq 2$**. The local monodromy group in this case is the free Abelian group $\mathbb{Z}^m$.

**Organization of the paper.**

The paper is organized as follows.
In Section 2 and 3 we give a brief overview of convexity results in symplectic geometry and in integrable Hamiltonian systems. Strictly speaking, these sections are not necessary for the understanding of the rest of the paper. However, we believe that this overview is interesting in its own right, especially to the non-experts, and give a clear picture of where our main results are situated in the field of symplectic convexity.

In Section 4 we recall the notion of integrable Hamiltonian systems and present their interpretation as singular Lagrangian fibrations. We recall the main results about their singularities, give the associated normal forms, and present the topology and affine structure on the base space of the fibration. This naturally leads to the definition of toric-focus systems, the object of study in this paper, whose known properties are reviewed.

Section 5 is devoted to the study of affine manifolds with focus singularities. We first show how monodromy induces affine coordinates near focus points in arbitrary dimensions and study the behavior of the affine structure in the presence of focus points in increasing order of complexity.

In Section 6 we define the notion of straight lines, with special emphasis on the ones passing through singular points. This allows us to introduce the notions of convexity and local convexity with respect to the underlying singular affine structure. We show the existence and branching of extensions of straight lines hitting singular points. In view of global convexity, we also introduce the notions of strongly convex subsets of a singular affine manifold.

In Section 7 we prove local convexity theorems for the singular affine structure around a singular point with just one focus component. We also explain that the presence of a singular point with many focus components may lead to non-convexity, even for compact manifolds, by showing why it is not convex on an example.

Section 8 is devoted to both positive and negative results on global convexity. We show that if the base space of the singular Lagrangian fibration of a two degrees of freedom integrable system with no hyperbolic singularities has non-empty boundary and is either compact or proper, then it is convex. If this base space is a two dimensional sphere, in contrast to the case when it has boundary, it is not necessarily convex. We present both a convex and non-convex example in this situation. If the dimension of this base space is at least three, then there are non-convex examples, even when the manifold is compact, has a boundary, and is topologically a cube. All the non-convex examples are related to the fact that the monodromy group is big. When there exist \( n - 1 \) global affine functions on the base space, which means that the monodromy group is not too complicated, we prove that the base space is convex in any dimension \( n \), under the assumption that the base space is compact or its affine structure is proper. If the affine structure is not proper, then it is already known [111] that convexity fails, even if there is a Hamiltonian \( \mathbb{T}^{n-1} \)-action.
2. A brief overview of convexity in symplectic geometry

The present work is part of a big program aimed at understanding the relationship between symplectic and Poisson geometry to convexity of the momentum maps of Hamiltonian actions and several of its generalizations. Readers who are familiar with this subject can skip this overview section.

2.1. Kostant’s Linear Convexity Theorem.

The link between convexity and coadjoint orbits appears for the first time in a 1923 paper of Schur [120] who showed that the set of diagonals of an isospectral set of Hermitian $n \times n$ matrices, viewed as a subset of $\mathbb{R}^n$, lies in the convex hull whose vertices are the vectors formed by the $n!$ permutations of its eigenvalues. Horn [65] proved that this inclusion is an equality. The fundamental breakthrough pointing the way towards the basic relationship between convexity and certain aspects of Lie theory and symplectic geometry is due to Kostant [81] who realized that the Schur-Horn Convexity Theorem is just a special case of a much more general statement: the projection of a coadjoint orbit of a connected compact Lie group relative to a bi-invariant inner product onto the dual of a Cartan subalgebra is the convex hull of the Weyl group orbit of one (hence any) intersection points of the orbit with the Cartan algebra. Even this result is a special case of a convexity theorem for symmetric spaces.

The influence of this theorem, called Kostant’s Linear Convexity Theorem in the development of the relationship between geometric aspects of Lie theory, symplectic and Poisson geometry, infinite dimensional differential geometry, affine geometry, and integrable systems cannot be overstated. We give below a very quick and incomplete synopsis of the enormous work generated by this result in order to put our paper in this larger context. For an excellent review of the convexity results in Lie theory and symplectic geometry see Guillemin and Sjamaar’s book [53] and references therein. We isolate five main areas where Kostant’s Linear Convexity Theorem has important extensions and has led to remarkable results.

2.2. Infinite dimensional Lie theory.

The first infinite dimensional convexity result, due to Atiyah and Pressley [7], extends Kostant’s Convexity Theorem to loop groups of compact connected and simply connected Lie groups. A more general convexity theorem, valid for all coadjoint orbits of arbitrary Kac-Moody Lie groups associated to a symmetrizable generalized Cartan matrix, except for some degenerate orbits, is due to Kac and Peterson [68]. The direct generalization of the Schur-Horn Convexity Theorem to the Banach Lie group of unitary operators on a separable Hilbert space is due to Neumann [104]. The main issue in the formulation of the theorem is the topology used to close the convex hull and the author discusses both Schatten classes as well as the operator topology. This work is continued in [105] by considering the infinite dimensional orthogonal and symplectic groups. Again, the essential issue is the topology used for the closure of the convex hull.

A different type of Schur-Horn Convexity Theorem for an appropriate completion of the group of area preserving diffeomorphisms of the annulus $[0, 1] \times S^1$ is due to Bloch,
Flaschka, and Ratiu [17]. The completion relative to various topologies is shown to equal the semigroup of not necessarily invertible measure preserving maps of the annulus. The role of the maximal torus is played the Hilbert space of $L^2$-functions on the interval $[0, 1]$ and that of the Weyl group by semigroup of maps $[0, 1] \times S^1 \ni (z, \theta) \mapsto (a(z), j(x)\theta)$, where $a(z)$ is measure preserving and $j(z) = \pm 1$ a.e. It is shown that the Schur-Horn Convexity Theorem holds: the orthogonal projection of the set of functions with the same moments, given by integration over the circle, equals a weakly compact convex set in the Hilbert space of $L^2$-functions on the interval $[0, 1]$ whose extreme points coincide with the Weyl semigroup orbit. The link of this convexity theorem with Toeplitz quantization, measure theory, the dispersionless Toda PDE, and the PDE version of Brockett’s double bracket equation is discussed in [18, 15]. The convexity result has been extended to the subgroup of equivariant symplectomorphisms of a symplectic toric manifold in the Ph.D. thesis of Mousavi [102] (unpublished). Our paper does not explore such infinite dimensional generalizations.

2.3. “Linear” symplectic formulations.

The so-called “linear” convexity theorems have their origin in the study of the behavior of eigenvalues of matrices under linear operations; the Schur-Horn Convexity Theorem and Konstant’s Linear Convexity Theorem are representatives of this point of view. The foundational work extending these theorems to symplectic geometry is due to Atiyah [6] and Guillemin and Sternberg [54, 56]. Let $(M^{2n}, \omega)$ be a $2n$-dimensional symplectic manifold endowed with a Hamiltonian $\mathbb{T}^k$-action, with invariant momentum map $J : M \to \mathbb{R}^k$. Then the fibers of $J$ are connected and $J(M)$ is a compact convex polytope, namely the convex hull of the image of the fixed point set of the $\mathbb{T}^k$-action; $J(M)$ is called the momentum polytope. If the $\mathbb{T}^k$-action is effective (the intersection of all isotropy subgroups of each point in $M^{2n}$ is the identity), then there must be at least $k + 1$ fixed points of the action and $k \leq n$.

The generalization of the Atiyah-Guillemin-Sternberg theorem to compact Lie group actions on compact connected symplectic manifolds is due to Kirwan [75]: the fibers of the momentum map are connected and the image of the momentum map intersected with a Weyl chamber is a connected polytope. For Hamiltonian actions of compact Lie groups, this intersection is called momentum polytope. The momentum polytope of the restriction of the Hamiltonian action of a compact Lie group to its maximal torus is the convex hull of the Weyl group orbit of the momentum polytope of the Hamiltonian compact Lie group action.

Brion [23] sharpened Kirwan’s Convexity Theorem in the framework of projective algebraic varieties by finding a more detailed description of this momentum polytope and linked it to the Kirwan stratification [76]. For an up to date account of stratified spaces, see Pflaum’s book [114]. Sjamaar [121] extended Brion’s methods to symplectic manifolds and gave a description of the polytope in a neighborhood of any of its points $p$ in terms of the action of the stabilizer of the group action on the manifold at a point $m$ satisfying $J(m) = p$; this also yields a necessary condition for $p$ to be a vertex. In addition, Kirwan’s Convexity Theorem was extended to actions on affine varieties and cotangent bundles.
Hilgert, Neeb, and Plank [62] extended Kirwan’s Convexity Theorem to non-compact symplectic manifolds with a proper momentum map and obtained other convexity results. Lerman [83] proved the same result for linear compact Lie group actions on symplectic vector spaces whose associated momentum map is not necessarily proper employing his symplectic cuts technique. Lerman, Meinrenken, Tolman and Woodward [84, 85] also extended Atiyah–Guillemin–Sternberg–Kirwan’s convexity theorems to the case of symplectic orbifolds, gave a local description of the momentum polytope, and proved the openness of the map from the orbit space to the momentum polytope.

The momentum polytope for non-compact manifolds has discrete vertices and is, in general, an unbounded, convex, locally finite intersection of polyhedral cones, each of which is determined by local data on the manifold, conveniently expressed in terms of the Marle-Guillemin-Sternberg Normal Form [92, 93, 57] (a slice theorem for Hamiltonian actions; see also [58], [109] for a presentation of the normal form and various extensions thereof).

Heinzner and Huckleberry [60] replaced the compact symplectic manifold and compact Lie group in Kirwan’s theorem by an irreducible complex Kähler, not necessarily compact, manifold with an action of the complexification of the compact Lie group and showed that the image by the momentum map of the open subset consisting of the Lie group orbits of maximal dimension, intersected with a Weyl chamber, is convex.

Knop [78] introduced the concept of convex Hamiltonian manifold in the following way. Suppose that a symplectic manifold \( M \) admits a Hamiltonian Lie group action. Choose a Cartan subalgebra \( \mathfrak{t} \) in the Lie algebra of the compact symmetry Lie group and a positive Weyl chamber \( \mathfrak{t}^*_+ \). For each point in the manifold, the image of the orbit by the momentum map intersects the positive Weyl chamber in exactly one point. This defines a map \( \psi: M \to \mathfrak{t}^*_+ \). A convex Hamiltonian manifold is defined then as a Hamiltonian manifold such that the inverse image by \( \psi \) of any straight line segment in \( \mathfrak{t}^*_+ \) is connected in \( M \). Then it is proved that convexity of \( M \) is equivalent to the convexity of \( \psi(M) \) together with the connectedness of the fibers of \( \psi \) and openness of \( \psi: M \to \psi(M) \), the range being endowed with the subspace topology. It is shown that several convexity results are a consequence of the fact that the hypotheses of these theorems guarantee that the symplectic manifold is convex. It is also proved that \( \psi(M) \) is locally a polyhedral cone and that results that required in the hypothesis properness of the momentum map, generalize to convex symplectic manifolds.

Kostant [81] already showed that his Linear Convexity Theorem holds also for real flag manifolds and Atiyah [6] generalized it to the symplectic setting. The general version of this theorem is due to Duistermaat [38] who proved that the image of the fixed point set of an antisymplectic involution by the momentum map of a toral Hamiltonian action on a compact connected symplectic manifold, coincides with the momentum polytope. This result has striking consequences, for example, in the study of the gradient character of the finite Toda lattice system for compact semisimple Lie algebras, the convexity properties of the associated momentum map and their relationship the Brockett double bracket equation (a gradient system in the normal metric), as well as certain “non-linear” convexity theorems (see [16]), which will be discussed in the next subsection.
If the symplectic action of a compact Lie group does not admit a momentum map, there are two generalizations of the convexity theorems. The first one is based on a construction of Condevaux, Dazord, and Molino [29]. They introduce a flat connection on the trivial bundle whose structure group is the underlying additive group of the dual of the Lie algebra of symmetries over the symplectic manifold, consider the holonomy bundle in this product containing a given point (whose structure group is the holonomy group based at that point), and the projection to the dual of the Lie algebra. The cylinder valued momentum map is the quotient of this projection from the original symplectic manifold to the quotient of the dual of the Lie algebra of symmetries by the closure of the holonomy group; for a detailed presentation of this momentum map and its properties see [109, Sections 5.2-5.4, 7.6].

Birtea, Ortega, and Ratiu [12] proved a convexity theorem for cylinder valued momentum maps of compact symplectic group actions in the spirit of Kirwan’s Convexity Theorem supposing that the holonomy group mentioned above is closed and that the cylinder valued momentum map is a closed map and is tube-wise Hamiltonian (each point admits an open group invariant neighborhood such that the restriction of the action to this neighborhood admits a standard momentum map; for connected Abelian Lie group actions, this hypothesis is not necessary): the intersection of the range of the momentum map in the quotient of the dual of the symmetry Lie algebra by the closed holonomy group (i.e., the “cylinder”) intersected with the quotient of a Weyl chamber by this holonomy group is weakly convex. This statement uses the natural length metric of the cylinder (so it is a geodesic metric space) and weak convexity of a set means that any two points can be joined by a geodesic contained in this set, however, not necessarily the shortest one.

The second approach, due to Benoist [11] and Giacobbe [46, 47] is to work with the momentum map naturally associated to an appropriate covering of the symplectic manifold. Benoist works with the universal covering of the symmetry group acting symplectically on the universal covering of the symplectic manifold. Suppose that this momentum map is proper modulo the holonomy group, i.e., the inverse image of any compact set is included in the holonomy group orbit of a compact set in the symplectic manifold. Then the image of the momentum map intersected with a Weyl chamber is convex (and has the other usual properties in convexity theorems), also extending the Kirwan Convexity Theorem.

Giacobbe [46, 47] showed first that if the group is a torus, there is a minimal covering on which the given symplectic action admits a momentum map and that the image of this covering by the momentum map equals a convex compact polytope times a vector space, both being explicitly described. Then he showed that, for an non-Abelian compact group $H$ which acts symplectically on $M$, there exists an $H$-equivariant momentum map $J : \tilde{M} \to h^* \cong g^* \times t^*$, where $h$ is the Lie algebra of $H$, $g$ is the Lie algebra of the commutator $G$ of $H$ which is semisimple, $t$ is the Lie algebra of the center of $H$ which is a torus $T = T_e \times T_c$, where $T_e$ is the maximal subtorus of $T$ acting in a Hamiltonian fashion on $M$, $T_c$ is a torus complement to $T_e$ in $T$, and $\tilde{M}$ is the minimal covering of $M$ on which the $T$-action can be lifted to a Hamiltonian action. Then
\( J(\tilde{M}) \cap (s_+^* \times t^*) = P \times t^*_c \), where \( t_c = \text{Lie } T_c, t_e = \text{Lie } T_e \), \( s_+^* \) is the positive Weyl chamber of \( g \), and \( P \subseteq s_+^* \times t^*_c \) is the product of the momentum polytope in \( t^*_c \) given by the Hamiltonian \( T_c \)-action on \( M \) and the \( G \)-Kirwan polytope in \( s_+^* \). Giacobbe also proved any effective symplectic action of a \( n \)-dimensional torus on a closed connected symplectic 2n-manifold with a fixed point must be Hamiltonian.

There are also convexity theorems for presymplectic manifolds. For torus actions, this result is due to Ratiu and Zung [117]. Let \( J : M^{2d+q} \to \mathbb{R}^{q+d} \) be a flat momentum map of a Hamiltonian torus \( \mathbb{T}^{q+d} \)-action on a connected compact presymplectic manifold whose presymplectic form has constant corank \( d \). The flatness condition means that the image \( J(M^{2d+q}) \) lies in the intersection of \( d \) hyperplanes in \( \mathbb{R}^{q+d} \). Then \( J(M^{2d+q}) \) is a convex \( q \)-dimensional polytope (rational or non-rational) lying in the \( q \)-dimensional affine subspace of \( \mathbb{R}^{q+d} \) given by the flatness condition. In addition, any such presymplectic manifold admits a unique equivariant symplectization. For general compact Lie groups, the convexity result is due to Lin and Sjamaar [87], extending to presymplectic manifolds Kirwan’s Convexity Theorem and, of course, containing the presymplectic convexity theorem just described.

2.4. “Non-linear” symplectic formulations.

“Non-linear” convexity theorems appear as the result of the analysis of the behavior of eigenvalues or singular values of matrices under nonlinear operations (usually, multiplication). There are far less theorems of this kind and they are considerably more involved, although many known convexity theorems (for example, those presented in Marshall, Olkin, and Arnold’s book [95]) await a symplectic interpretation, albeit mostly not in the classical sense, as will be apparent from the presentation below. Reformulating them in a symplectic, Poisson, Dirac, or groupoid context, remains a challenge. The matrix case of such theorems goes back to Weyl [131] and Horn [66]. Let \( P \) be the set of positive definite Hermitian matrices whose determinant equals 1 and \( \Sigma_A \) the isospectral subset of matrices in \( P \) defined by the given eigenvalues \( (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \). The image of the map \( P \ni p \mapsto (\log(\text{det } p_1), \ldots, \log(\text{det } p_n)) \in \mathbb{R}^n \), where \( p_k = (p_{ij}), i, j = 1, \ldots, k \) is the \( k \times k \) principal sub-matrix of \( p \), is a convex polytope.

The Lie theoretical generalization of the above Weyl–Horn theorem is again due to Kostant [81] and is called Kostant’s Non-linear Convexity Theorem which has the following formulation. Let \( G \) be a connected semisimple Lie group and \( G = K \text{AN} = PK \) its Iwasawa and Cartan decomposition, respectively, with \( A \subseteq P \). Let \( K \) be the \( K \)-orbit of \( a \in A \) in \( P \) (by conjugation) and \( \rho_A : G \to A \) the Iwasawa projection \( \rho_A(\text{kan}) = a \). Identify \( A \) with its Lie algebra \( a \) by the exponential map. Then \( \rho_A(\mathcal{O}_a) \) is the convex hull of the Weyl group orbit through \( a \). The Weyl-Horn Convexity theorem is a special case of Kostant’s Non-linear Convexity Theorem by making the following choices: \( G = SL(n, \mathbb{C}), K = SU(n), P \) positive definite Hermitian matrices of determinant 1 (i.e., the Cartan complement of \( K \) in \( G \), \( A \) positive diagonal matrices of determinant equal to 1, \( \Sigma_A \) the orbit (by conjugation) of \( K = SU(n) \) in \( P \) through \( (\lambda_1, \ldots, \lambda_n) \in A \), \( p = \text{kan} \) the Iwasawa decomposition of \( p \in P, n \) upper triangular with ones on the
diagonal. Then the map \( P \ni p \mapsto (\log(\det p_1), \ldots, \log(\det p_n)) \in \mathbb{R}^n \) is the logarithm of \( \rho_A \).

A symplectic proof of Kostant’s theorem for a large class of Lie groups was given by Lu and Ratiu [89]; in the special case of the Weyl-Horn Convexity Theorem, \( \Sigma_\lambda \) is a symplectic manifold, the map \( P \to \mathbb{R}^n \) given above is the momentum map for a Hamiltonian torus action, and convexity follows from the Atiyah-Guillemin-Sternberg Convexity Theorem. However, there is a twist: the symplectic form on the orbit \( \Sigma_\lambda \) is not due to a Lie-Poisson structure on the dual of a Lie algebra, but it is induced from a Poisson structure compatible with the Lie group structure on the group \( SL(n, \mathbb{C}) \). This has far reaching consequences, as discussed below. The symplectic proof of Kostant’s Non-linear Convexity Theorem given in [89] is based on the following idea. Consider the complexification \( G^C \) of \( G \) and write the Iwasawa decomposition \( G^C = UB \), where \( U \) is a maximal compact subgroup and \( B \) is the solvable complement; \( B \) is a Poisson-Lie group (i.e., a Lie group and a Poisson manifold for which multiplication is a Poisson map) endowed with the so-called Lu-Weinstein Poisson structure [90] whose symplectic leaves are the dressing orbits (left or right) of \( U \) on \( B \). The restriction of this action to a maximal torus of \( U \) is Hamiltonian and its associated momentum map is the Iwasawa projection of \( G^C \), which then proves Kostant’s Convexity Theorem for \( G^C \) viewed as a real Lie group by invoking the Atiyah-Guillemin-Sternberg Convexity Theorem.

To get the general case, one would like to apply Duistermaat’s Convexity Theorem for the antisymplectic involution induced by the Cartan involution, viewing \( AN \) in the Iwasawa decomposition \( G = KAN \) as the fixed point set in \( B \). However, \( B \) is not invariant (as claimed in [38] and then used in [89]) and the argument holds only if the centralizer of \( a \) in \( \mathfrak{k} \) is Abelian, as Hilgert and Neeb [61] pointed out. This paper contains several interesting results on nonlinear convexity theorems. The map \( B \ni b \mapsto \log(b^*b) \in \mathfrak{p} \) is a diffeomorphism sending each each symplectic leaf of \( B \) (a dressing orbit) to a coadjoint orbit of \( K \) in \( \mathfrak{p} \); here \( u \) is the \(+1\) eigenspace of the Cartan involution \( \tau \) on \( G^C \) viewed as a real Lie group, so \( u \) is the Lie algebra of \( U \), \( \mathfrak{p} \) is the \(-1\) eigenspace of \( \tau \), \( b^* = \tau(g^{-1}) \) (so for \( G = SL(n, \mathbb{C}) \) it is just the usual transpose conjugate), \( u^* \) is identified with \( \mathfrak{p} \) via the imaginary part of the Killing form. So, a coadjoint orbit in \( \mathfrak{p} \) carries two symplectic forms: the usual orbit symplectic form and the push forward of the dressing symplectic form by the map \( B \ni b \mapsto \log(b^*b) \in \mathfrak{p} \).

Ginzburg and Weinstein [48] proved that there is a global diffeomorphism on \( \mathfrak{p} \) taking the Lie-Poisson tensor on \( \mathfrak{p} \) to the push forward of the Poisson Lie tensor on \( B \) to \( \mathfrak{p} \) by the map defined above. The symplectic proof of the Kostant Non-linear Convexity Theorem for all connected real semisimple Lie groups was given in Sleewaegen’s 1999 Ph.D. thesis [122] (unpublished) and later by Krötz and Otto [82] each extending the Duistermaat Convexity Theorem in such a way that the symplectic proof outlined above works in full generality.

The previously described proof of the Kostant Non-linear Convexity Theorem has sparked interest in other types of non-linear convexity results. The first such theorem was given by Flaschka and Ratiu [44] and is very general. Let \( K \) be the compact real form of a connected complex semisimple Lie group \( G \) (hence \( K \) is connected) and denote by \( G^\mathbb{R} \) the real underlying Lie group. Let \( G^\mathbb{R} = KAN \) be the Iwasawa decomposition, \( \mathfrak{g} \),
the Lie algebras of $G$, $G^R$, $K$, $A$, $N$, respectively, and $T \subset K$ the connected maximal torus in $K$ whose Lie algebra is $t = ia$. Then $b = a + n$ and $t$ are dual to each other via the imaginary part of the Killing form; thus $t$ is isomorphic to $b^*$. The Cartan decomposition $G^R = PK$ defines the Cartan involution $\tau : G^R \to G^R$, $t$ is the $+1$ eigenspace of $\tau$ and let $p$ be the $-1$ eigenspace of $\tau$. Denote by $a_+$ a positive Weyl chamber in $a$. Now suppose that $K$ acts on a compact connected symplectic manifold $(M,\omega)$. Assume that the restriction of this action to the maximal torus $T \subset K$ is Hamiltonian with associated invariant momentum map $\phi : M \to a \cong t^*$. Suppose there exists a map $j : M \to p$ with the following four properties: $j$ is equivariant relative to the adjoint action of $K$ on $p$; $T_m j(T_m M)$ is the annihilator of the stabilizer subalgebra $t_m$ for every $m \in M$; $\ker T_m j$ coincides with the $\omega$-orthogonal complement of the tangent space to the $K$-orbit at $m$ for every $m \in M$; the restriction of $j$ to $j^{-1}(a_+)$ is proportional to $\phi$. (Note that the second and third conditions are the content of the Reduction Lemma; see, e.g., [1, Lemma 4.3.4], [109, Propositions 4.5.12 and 4.5.14].) Then the fibers of $j$ are connected and $j(M) \cap a_+$ is a convex polytope. The hypotheses of this theorem are non-trivially verified if $j$ is the Lu momentum map ([88, 90]) of a Poisson Lie group structure on $K$ [44].

All Poisson Lie group structures on compact Lie groups have been classified and the dual groups computed ([91, 44]), the most important one being the Lu-Weinstein Poisson Lie structure [90]. If the compact Lie group $K$ in the Flaschka-Ratiu Convexity Theorem is a Poisson Lie group, Alekseev [2] has given a different proof of this theorem, by modifying the symplectic structure on $M$ using the Poisson Lie group structure on $K$, which then reduces this theorem for Poisson actions of compact Poisson Lie groups on symplectic manifolds to the usual Kirwan Convexity Theorem (see also [53] for a discussion of this method).

A non-linear convexity theorem for quasi-Hamiltonian actions is due to Alekseev, Malkin, and Meinrenken [3]. The momentum map for quasi-Hamiltonian actions is group valued, so convexity means that the projection of the image of the group valued momentum map to the space of conjugacy classes of the group, identified with the fundamental Weyl alcove in a choice of a positive Weyl chamber, is convex; this projection is the momentum polytope for connected quasi-Hamiltonian spaces of compact, connected, simply connected Lie group actions. It should be emphasized that the Lu momentum map for Poisson Lie group actions (whose values lie, by definition, in the dual Poisson Lie group of the Poisson Lie group whose action generates it, if it exists) is not an example of this group valued momentum map; the two momentum maps coincide only for Abelian groups; see [109, Section 5.4] for a discussion and its relationship to the cylinder valued momentum map introduced in [29].

The existing momentum map convexity theorems strongly suggest a convexity theorem for the optimal momentum map introduced by Ortega and Ratiu [107] (see [109, Sections 5.5 and 5.6] for a study of its properties), which always exists for any canonical Lie group action on a Poisson manifold. The target of the optimal momentum map is the quotient topological space of the Poisson manifold on which the group acts by the polar pseudo-group of the group of diffeomorphisms given by the action. The problem is to
find an intrinsic definition of the concept of convexity for this topological space. The conjecture is that this topological space is intrinsically convex.

Zung [142] showed that every proper quasi-symplectic groupoid \((\Gamma \Rightarrow P, \omega + \Omega)\) in the sense of Xu [134] (also known as twisted presymplectic groupoid [25]) is symplectically linearizable, and studied the intrinsic affine structures of appropriate quotient spaces of proper quasi-symplectic groupoids and of their quasi-Hamiltonian manifolds. He showed that, under some mild conditions, these intrinsic affine structures are intrinsically convex. Most existing symplectic convexity results, both linear and nonlinear, and also for group-valued momentum maps, can be recovered from this convexity result for quasi-Hamiltonian manifolds of proper quasi-symplectic groupoids as important particular cases. There are remarkable similarities between Ortega–Ratiu’s construction of the optimal momentum map and Zung’s construction of the intrinsic transverse structures, so it is likely that these objects are closely related together, though no one worked it out yet.

2.5. Local-Global Convexity Principle.

All proofs of the symplectic convexity theorems rely on the following strategy. First one proves some local convexity properties of the map in question, using some kind of local normal forms, e.g., the Marle-Guillemin-Sternberg normal form [92, 93, 57], which has its roots in the quadratic nature of the singularities of the momentum map, first observed by Arms, Marsden, and Moncrief [5]. Then one uses, under certain hypotheses, a passage from local convexity to global convexity.

Originally, this passage from local to global was done using Morse theory, but it became later apparent that these Morse techniques do not always apply. A better tool is the so-called Local-Global Convexity Principle. The origins of this principle go back to Tietze [124] and Nakajima [103]: if a connected closed subset \( S \subset \mathbb{R}^n \) is locally convex, then it is convex. Since then, there are many extensions and developments of this result, due to Schoenberg [119], Klee [77], Sacksteder, Straus, and Valentine [118], Blumenthal and Freese [19], Kay [74], Cel [27], and so on.

Condevaux, Dazord, and Molino [29] were the first authors to use the local-global convexity principle to give a different and much simpler proof of the Atiyah-Guillemin-Sternberg and Kirwan Convexity Theorems. Then Hilgert, Neeb, and Plank [62] isolated this principle as a fundamental tool, and gave the following version of it, which is well adapted for symplectic convexity.

Let \( f : S \to V \) be a continuous map from a connected Hausdorff topological space \( S \) to a vector space \( V \). The map \( f \) is said to be locally fiber connected if for each \( s \in S \), there is an open neighborhood \( U \subset S \) of \( s \) such that \( f^{-1}(f(u)) \cap U \) is connected for all \( u \in U \). Suppose that there is an assignment of a closed convex cone \( C_s \subset V \) with vertex \( f(s) \) to each \( s \in S \). Such an assignment is called local convexity data for \( f \) if for each \( s \in S \) there is an open neighborhood \( U_s \subset S \) of \( s \) such that \( f|_{U_s} : U_s \to C_s \) is an open map and \( f^{-1}(f(u)) \cap U_s \) is connected for all \( u \in U_s \). The Local-Global Convexity Principle in [62] states that if \( f : S \to V \) is a proper locally fiber connected map with local convexity data \( \{ C_s \mid s \in S \} \), then \( f(S) \subset V \) is a closed convex locally
polyhedral set, the fibers of \( f \) are connected, \( f : S \to f(S) \) is an open map (with respect to the subspace topology of \( f(S) \subset V \)), and \( C_s = f(s) + L_{f(s)}(f(S)) \), where \( L_{f(s)}(f(S)) \) denotes the closure of the cone \( \mathbb{R}_+(f(S) - f(s)) \).

The above Local-Global Convexity theorem of Hilgert, Neeb, and Plank [62] was crucial in the proof of the Flaschka-Ratiu Convexity Theorem [44], in particular for all compact Poisson Lie groups, because the usual Morse theoretic arguments failed to work for the Poisson Lie group structures different from the Lu-Weinstein one.

Prato [115] pointed out that the loss of compactness of the manifold also implies, in general, the loss of the convexity of the momentum map, even for torus actions. She also showed that if there is an integral element in the Lie algebra of the symmetry torus such that the momentum map component for this element is proper and has its minimum as its unique critical value, then the image of the momentum map of the Hamiltonian torus action is the convex hull of a finite number of affine rays starting in the fixed point set of the action. This convexity theorem has been vastly generalized in several directions. Birtea, Ortega, and Ratiu [13] have shown that only closedness of the momentum map is sufficient to obtain a convexity result, thereby setting the stage for other convexity theorems in infinite dimensions than the ones mentioned earlier. These results have been further generalized in Birtea, Ortega, and Ratiu [12] where the target map is a length metric space; this has as consequence the convexity theorem for cylinder valued momentum maps, mentioned earlier. Bjorndahl and Karshon [14] have provided another Local-Global Convexity theorem for proper continuous maps from a connected Hausdorff topological space to a convex set in Euclidean space.

The paper [142] by Zung contains another simple version of the local-global convexity principle, which is the one that we will use this paper to show some positive global convexity results for toric-focus integrable Hamiltonian systems (see Section 8).

3. Convexity in integrable Hamiltonian systems

Integrable Hamiltonian systems in the classical sense of Liouville are a special case of Hamiltonian group actions on symplectic manifolds, where the group is the non-compact Abelian group \( \mathbb{R}^n \), with \( n \) being half of the dimension of the manifold.

Each integrable Hamiltonian system with a proper momentum map has an associated base space, whose points correspond to connected components of the level sets of the momentum map. Due to the existence of local action coordinates, this base space is equipped with an intrinsic integral affine structure which is singular, in general, and plays a key role in the classification of integrable systems; see, e.g., [37, 140].

If the integrable system admits hyperbolic singularities, then the base space has branching points, and it does not make much sense to even talk about local convexity at those points. However, there are large classes of integrable Hamiltonian systems without hyperbolic singularities, for which we can study convexity properties of the base spaces. We present a brief overview of such systems in this section.

3.1. Toric systems and their momentum polytopes.
A symplectic toric manifold is a compact connected symplectic $2n$-dimensional manifold endowed with an effective Hamiltonian $\mathbb{T}^n$-action. From the point of view of integrable Hamiltonian systems, they are nothing else but integrable systems whose singular points are all elliptic non-degenerate. See, e.g., Audin’s book [8] for an introduction to this topic of toric integrable systems.

As a special case of the Atiyah–Guillemin–Sternberg theorem, the base space of a symplectic toric manifold is a convex polytope embedded in $\mathbb{R}^n$ via the momentum map, which satisfies three additional properties: rationality (each facet is given by a linear equation whose linear coefficients are integers), simplicity (each vertex has exactly $n$ edges), and regularity (near each vertex, the polytope is locally integral-affinely isomorphic to the orthant $(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0, \ldots, x_n \geq 0$ in $\mathbb{R}^n$). Such a convex polytope is called a Delzant polytope, because Delzant [32] proved a 1-to-1 correspondence between these polytopes and connected compact symplectic toric manifolds (up to isomorphisms). For non-compact symplectic toric manifolds, the situation is considerably more involved; Karshon and Lerman [70] analyzed this situation.

If we consider convex polytopes which are rational simple but not regular, then they correspond to compact symplectic toric orbifolds, instead of manifolds. According to a result of Lerman and Tolman [84], there is a 1-to-1 correspondence between connected compact symplectic toric orbifolds and weighted simple rational convex polytopes: each facet of the polytope is given an arbitrary natural number $w$, called its weight, which corresponds to the orbifold type $D^{2(n-1)} \times D^2/\mathbb{Z}_w$ at its preimage under the momentum map.

If the polytope is irrational then the situation is more complicated, because there is no symplectic Hausdorff space corresponding to it. Prato [116] invented the notion of symplectic quasifolds to deal with this case. Some other authors [73] talk about non-commutative toric varieties in this situation.

Ratiu and Zung [117] extended Delzant’s classification theorem to compact presymplectic toric manifolds. It states that connected compact presymplectic manifolds are classified, up to equivariant presymplectic diffeomorphisms, by their associated framed momentum polytopes, which are convex and may be rational or non-rational. Unlike the symplectic case, the polytope is not enough to characterize Hamiltonian presymplectic manifolds and additional information, in this case the framing of the polytope, is needed. Furthermore, they defined a Morita equivalence relation on the set of framed polytopes and, using it, the classification of connected presymplectic toric manifolds is alternatively given by the Morita equivalence classes of their framed momentum polytopes. Toric orbifolds [84], quasifolds [10], and non-commutative toric varieties [73] can be viewed as quotients of presymplectic toric manifolds by the kernel isotropy foliation of the presymplectic form; thus, they are classified by the Morita equivalence classes of their framed momentum polytopes. The Lerman-Tolman [84] classification of symplectic orbifolds by weighted simple rational convex polytopes can be recovered from this Morita equivalence. (See also [87]).

3.2. Toric degenerations.
There are many interesting integrable Hamiltonian systems on compact symplectic manifolds with degenerate singularities, whose base spaces (with the associated integral affine structures) are still convex polytopes. The most famous examples are probably the so-called Gelfand-Cetlin system, introduced by Guillemin and Sternberg [55], and the bending flows of polygons in $\mathbb{R}^3$, introduced by Kapovich and Millson [69].

It turns out that these two famous examples, and other systems with convex momentum polytopes but degenerate singularities, have some very similar topological and geometrical characteristics, namely:
- Their symplectic manifolds are not toric, but admit toric degenerations. See, e.g., [4, 79, 59] for toric degenerations related to integrable systems. It’s interesting to note that the momentum polytope corresponds to that of a (singular) toric variety at the degeneration, see, e.g., [24].
- Even though their smooth momentum maps have degenerate singularities, the inverse image of every point of the momentum polytope under the momentum map is not a singular variety but rather a smooth isotropic manifold or orbifold. See, e.g., [21, 22] for some results in this direction.

3.3. Log-symplectic convexity.

The Delzant correspondence extends to certain classes of toric Poisson manifolds which are generically symplectic but their Poisson structure admits a degeneracy locus. If this degeneracy locus is a smooth hypersurface, Guillemin, Miranda, Pires, and Scott [52] proved a generalization of the Delzant correspondence. If, however, this degeneracy locus has singularities of the type of normal crossing configurations of smooth hypersurfaces, the manifolds are called log symplectic, to emphasize the nature of the singularities. For the formulation of the analogue of the Delzant correspondence, several new ideas need to be introduced. This was done by Gualtieri, Li, Pelayo, and Ratiu [51] and required the appeal to the Mazzeo-Melrose [96] decomposition for manifolds with corners, free divisors appearing in algebraic geometry, and certain ideas of tropicalization. The notion of the momentum map is extended to the global tropical momentum map whose range is constructed from partial compactifications of affine spaces intimately linked to extended tropicalizations of toric varieties.

The analogue of the Delzant theorem to this log symplectic case states that there is a bijective correspondence between equivariant isomorphism classes of oriented compact connected toric Hamiltonian log symplectic $2n$-manifolds and equivalence classes of pairs $(\Delta, M)$, where $\Delta$ is a compact convex log affine polytope of dimension $n$, satisfying the Delzant condition, and $M$ is a principal $n$-torus bundle over $\Delta$ with vanishing obstruction class. From the point of view of integrable systems, which is the subject of this paper, this convexity result yields a classification of a large family of toric integrable systems having a base whose integral affine structure degenerates in a very precise manner along a stratification.

3.4. Non-Abelian integrability.
Non-Abelian integrability appears for the first time, in a very general context, in Abraham and Marsden’s book [1, Exercise 5.2I]: a Hamiltonian action of a symmetry group is completely integrable if, generically, all reduced spaces are zero dimensional, or, if the Hamiltonian is not a functional of the momentum map, they should be two dimensional. This idea is developed further in Marsden’s lectures [94], where even a model of action-angle coordinates are proposed, based on a local product structure of the symplectic manifold that takes into account the reduced space. These ideas were vague at the time because the Marle-Guillemin-Sternberg Normal Form did not exist yet.

An identical definition of non-Abelian integrability as that in Abraham and Marsden’s book [1], both appearing in the same year, but aimed at linking non-Abelian to Abelian, i.e., standard Liouville, integrability, is due to Mishchenko and Fomenko [101]: the action of a Hamiltonian symmetry group $G$ on a symplectic manifold $M$ is completely integrable if the dimension of the manifold is the sum of the dimension and the rank of $G$. By definition, the rank of $G$ is the rank of $g^*$, which, in turn is defined to be the dimension of the minimal coadjoint isotropy subalgebra in $g$; by the Duflo-Vergne Theorem [34] the set of points whose coadjoint isotropy algebra is minimal is Zariski open and the coadjoint isotropy algebras at all of these points are Abelian. Note that the Mishchenko-Fomenko condition means that the reduced spaces associated to every point in this Zariski open set are zero dimensional, which is exactly the condition imposed in [1].

The link of non-Abelian integrability with Kirwan’s convexity theorem is the following: the inverse image of a coadjoint orbit by the momentum map is a $G$-orbit in $M$, i.e., the Kirwan polytope can be identified, in this case, with $M/G$.

In [101] it is shown that if a system is integrable in the non-Abelian sense defined above, then the system is also integrable in the Abelian sense (i.e., there is another Abelian Lie algebra of functions relative to the Poisson bracket whose dimension is half of the dimension of the manifold) and its trajectories are straight lines on a torus whose dimension is the rank of the group (which is strictly smaller than half of the dimension of the manifold); see also [45, Subsections 3.1–3.4] for a detailed discussion.

Due to the fact that the solutions of a non-Abelian integrable system take place on tori of dimension strictly smaller than half of the dimension of the manifold, these systems are also often called super-integrable systems.

The natural question then arises to classify all compact connected symplectic manifolds admitting a completely integrable action of a compact connected Lie group. Iglesias [67] carried out the classification for rank one groups. Delzant [33] did it for rank two groups: if $G$ is a compact connected Lie group of rank two acting on a compact connected symplectic manifold in a completely integrable manner, then the image of the momentum map and the principal isotropy group classify the manifold up to equivariant symplectic diffeomorphisms.

3.5. Convexity in the presence of focus-focus singularities.
As we mentioned earlier, if an integrable Hamiltonian system admits only elliptic singularities, then it is a toric system and enjoys convexity properties. If it also has hyperbolic singularities then it does not make much sense to speak about convexity of the base space at hyperbolic points because of the branching at these points.

If the system admits no hyperbolic or degenerate singularities, only non-degenerate elliptic and focus-focus singularities, then it is said to be of toric-focus type. For a subclass of toric-focus systems called semitoric systems with 2 degrees of freedom, Vũ Ngọc [127] developed a theory of multi-valued momentum polytopes which are similar to Delzant polytopes in order to classify them. However, as far as we know, convexity properties of toric-focus integrable systems have never been treated systematically in the literature before, and our present paper is the first one to deal with it.

4. Toric-focus integrable Hamiltonian systems

4.1. Integrable systems and Lagrangian torus fibrations.

Recall that an integrable Hamiltonian system with $n$ degrees of freedom in the classical sense means a Hamiltonian function $H = H_1$ together with a so-called momentum map $\mathbf{H} = (H_1, \ldots, H_n) : M^{2n} \to \mathbb{R}^n$ consisting of $n$ functions on a symplectic manifold $M^{2n}$ which are functionally independent (i.e., $dH_1 \wedge \ldots \wedge dH_n \neq 0$ almost everywhere) and pairwise commuting ($\{H_i, H_j\} = 0$) under the Poisson bracket defined by the symplectic form. Throughout this paper it is assumed that $M^{2n}$ is paracompact and that $\mathbf{H}$ is a proper map.

At each point $x \in M^{2n}$, the number $\text{rank}_x = \text{rank} \ dH(x)$, which is the dimension of the vector subspace of the cotangent space $T^*_x M$ generated by $(dH_1(x), \ldots, dH_n(x))$, is called the rank of the system at $x$. If $\text{rank}_x < n$ then we say that $x$ is a singular point of the system, and $\text{corank}_x = n - \text{rank}_x$ is called the corank of $x$.

Each integrable system gives rise to a so-called (singular) associated Lagrangian torus fibration

$$\mathcal{F} := \{ \text{A connected component of } \mathbf{H}^{-1}(c), c \in \mathbf{H}(M) \},$$

whose fibers are the connected components of the momentum map. Singular fibers are those which contain at least one singular point of the system and regular fibers are those which contain only regular (i.e., non-singular) points.

The classical action-angle variables theorem (due to Liouville and Mineur [98], but often called Arnold-Liouville theorem; see, e.g., [63, Appendix A2] for a proof and [143] for a generalization to many different situations, based on the invariance of the underlying geometric structures with respect to natural torus actions associated to dynamical systems) says that every regular fiber of the associated Lagrangian torus fibration is indeed a Lagrangian torus which admits a neighborhood

$$\mathcal{U}(N) \cong D^n \times \mathbb{T}^n$$

with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$, where $(p_1, \ldots, p_n)$ are coordinates on a small disk $D^n$, and $(q_1 \pmod{1}, \ldots, q_n \pmod{1})$ are periodic coordinates on the torus $\mathbb{T}^n$, in
which the symplectic form is canonical,
\[ \omega = \sum dp_i \wedge dq_i, \]
and the components of the momentum map depend only on the variables \((p_1, \ldots, p_n)\),
\[ H_i = h_i(p_1, \ldots, p_n). \]
The variables \((p_1, \ldots, p_n)\) are called \textit{actions} while \((q_1, \ldots, q_n)\) are called \textit{angles}.

From a geometric point of view, we can forget about the momentum map, and only look at the associated Lagrangian fibration. So we will adopt the following notion of geometric integrable Hamiltonian systems: \textit{a geometric integrable Hamiltonian system} is a singular Lagrangian torus fibration such that each fiber admits a saturated neighborhood (i.e., consisting of whole fibers) and an integrable Hamiltonian system in this neighborhood whose associated Lagrangian fibration coincides with the given fibration.

4.2. Local normal form of non-degenerate singularities.

In this subsection, we recall the necessary definitions and results about non-degenerate singularities of integrable Hamiltonian systems, which will be used in the paper.

A \textit{fixed point}, or a singular point of rank 0, of the momentum map \(H = (H_1, \ldots, H_n) : M^{2n} \to \mathbb{R}^n\), is a point \(p \in M^{2n}\) such that \(dH_1(p) = \ldots = dH_n(p) = 0\).

At a singular point \(p\), the quadratic parts of the functions \(H_1, \ldots, H_n\) are well-defined and span an Abelian subalgebra \(A_p\) of the Lie algebra \(Q(T_p M)\) of homogeneous quadratic functions on the tangent space \((T_p M)\): the Lie bracket is the Poisson bracket given by the symplectic form at \(p\). \(Q(T_p M)\) is isomorphic to the simple Lie algebra \(\mathfrak{sp}(2n, \mathbb{R})\).

The fixed point is called \textit{non-degenerate} if \(A_p\) is a Cartan subalgebra of \(Q(T_p M)\), i.e., it has dimension \(n\) and all its elements are semi-simple.

If the singular point \(q\) has corank \(\kappa < n\), we may assume, without loss of generality, that \(dH_1(q) = \ldots = dH_{\kappa}(q) = 0\) and \(dH_{\kappa+1} \wedge \ldots \wedge dH_n(q) \neq 0\). The Hamiltonian vector fields of the functions \(H_{\kappa+1}, \ldots, H_n\) commute and give rise to a free local \(\mathbb{R}^{n-\kappa}\)-action. The quotient of the local coisotropic codimension-\((n - \kappa)\) submanifold \(\{H_{\kappa+1} = \text{const}, \ldots, H_n = \text{const}\}\) containing \(q\) by the orbits of this free local \(\mathbb{R}^{n-\kappa}\)-action is the \textit{local Marsden-Weinstein reduction} of the system (with respect to \((H_{\kappa+1}, \ldots, H_n))\). This local reduced space is a \(2\kappa\)-dimensional symplectic manifold with an integrable system given by the momentum map \((H_1, \ldots, H_\kappa)\), and the image of \(q\) in this local reduced space is a fixed point \(p\) for the reduced system. We say that \(q\) is \textit{non-degenerate of corank} \(\kappa\) if and only if its image \(p\) is a non-degenerate fixed point for the local reduced integrable Hamiltonian system.

According to a classical theorem of Williamson [133], at each non-degenerate fixed point \(p\), there exist a triple of nonnegative integers \(k(p) = (k_e, k_h, k_f)\), called the \textit{Williamson type} of \(p\), \(k_e + k_h + 2k_f = n\) is the corank of \(p\), such that, after the local Marsden-Weinstein reduction, the quadratic part of each first integral function
whose differential vanishes at \( p \) is a linear combination of the quadratic functions
\[ e_i(1 \leq i \leq k_e), h_i(1 \leq i \leq k_h), f_i^1, f_i^2(1 \leq i \leq k_f) \]
given by
\[
\begin{align*}
&\bullet e_i = x_i^2 + \xi_i^2, \quad \forall 1 \leq i \leq k_e, \\
&\bullet h_i = x_i+k_e \xi_i+k_i, \quad \forall 1 \leq i \leq k_h \\
&\bullet \begin{cases} 
  f_i^2 = x_{2i-1+k_c+k_h} \xi_{2i-1+k_c+k_h} + x_{2i+k_c+k_h} \xi_{2i+k_c+k_h} \\
  f_i^1 = x_{2i-1+k_c+k_h} \xi_{2i-1+k_c+k_h} + x_{2i+k_c+k_h} \xi_{2i+k_c+k_h}
\end{cases} \quad \forall 1 \leq i \leq k_f.
\end{align*}
\]
In addition, \((x_1, \xi_1, \ldots, x_n, \xi_n)\) is a canonical linear coordinate system on the tangent space at \( p \). The numbers \( k_e, k_h, \) and \( k_f \) are called the numbers of \textit{elliptic, hyperbolic,} and \textit{focus-focus} components of \((\text{the system at})\ p\), respectively.

If \( q \) is a non-degenerate singular point of corank \( \kappa < n \) and \( n - \kappa > 0 \), we define its \textit{Williamson type} \( k(p) = (k_e, k_h, k_f) \) to be the numbers of elliptic, hyperbolic, and focus-focus components of the image of \( p \) in the local Marsden-Weinstein reduction at \( p \); in this case we have \( k_e + k_h + 2k_f = \kappa \).

We will need the following local symplectic linearization \((\text{i.e., normal form})\) theorem for non-degenerate singular points, which is due to Vey \cite{vey} in the analytic case and to Eliasson \cite{eliasson, eliasson2} in the smooth case \(\text{(see also } [28, 35, 99, 128, 139, 141]).}\)

**Theorem 4.1** \((\text{Local linearization } [125, 41, 42])\). If \( p \in M^{2n} \) is a non-degenerate singular point of rank \( n - \kappa \) and Williamson type \( k(p) = (k_e, k_h, k_f) \) of an integrable Hamiltonian system \( H = (H_1, \ldots, H_n) : M \to \mathbb{R}^n \), then there exist local symplectic coordinates \((x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)\) about \( p \), in which \( m \) is represented as \((0, \ldots, 0)\), such that \(\{f_i, q_j\} = 0\), for all \( i, j \), where
\[
\begin{align*}
&\bullet q_i = e_i = x_i^2 + \xi_i^2 \quad (1 \leq i \leq k_e) \text{ are elliptic components}, \\
&\bullet q_{k_e+i} = h_i = x_{i+k_e \xi_i+k_i} \quad (1 \leq i \leq k_h) \text{ are hyperbolic components}, \\
&\bullet \begin{cases} q_{2i-1+k_c+k_h} = f_i^1 = x_{2i-1+k_c+k_h} \xi_{2i-1+k_c+k_h} - x_{2i+k_c+k_h} \xi_{2i-1+k_c+k_h} \\
q_{2i+k_c+k_h} = f_i^2 = x_{2i-1+k_c+k_h} \xi_{2i-1+k_c+k_h} + x_{2i+k_c+k_h} \xi_{2i+k_c+k_h}\end{cases} \leq i \leq k_f\text{ are focus-focus components}, \\
&\bullet q_{k+c+i} = x_i \quad (1 \leq i \leq n - \kappa) \text{ are regular components}.
\end{align*}
\]
Moreover, if \( p \) does not have any hyperbolic component, then the system of commuting equations \(\{f_i, q_j\} = 0\), for all indices \( i, j \), may be replaced by the single equation
\[
(F - F(m)) \circ \varphi = g \circ (q_1, \ldots, q_n),
\]
where \(\varphi = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)^{-1} \), \( g \) is a diffeomorphism from a small neighborhood of the origin in \( \mathbb{R}^n \) into another such neighborhood, and \( g(0, \ldots, 0) = (0, \ldots, 0) \).

For example, when the symplectic manifold \( M \) is of dimension \( 4 \) and the non-degenerate singular point \( p \) has no hyperbolic component, then one of the following three cases occur:
\[
\begin{align*}
&\text{(i) (elliptic-elliptic singularity: } k_e = 2, k_h = k_f = 0 \text{) } q_1 = (x_1^2 + \xi_1^2)/2 \text{ and } q_2 = (x_2^2 + \xi_2^2)/2; \\
&\text{(ii) (focus-focus singularity: } k_e = k_h = 0, k_f = 1 \text{) } q_1 = x_1 \xi_2 - x_2 \xi_1 \text{ and } q_2 = x_1 \xi_1 + x_2 \xi_2; \\
\end{align*}
\]
(iii) (transversally-elliptic singularity: \( k_e = 1, k_h = k_f = 0 \)) \( q_1 = (x_1^2 + \xi_1^2)/2 \) and \( q_2 = \xi_2 \).

If \( p \) is a “most singular point” (i.e., of highest corank) in a singular fiber of an integrable system, then the orbit \( O(p) \) of the Poisson \( \mathbb{R}^n \)-action generated by the momentum map through \( p \) must be compact (otherwise it will contain in its boundary singular points of even higher corank), and hence is diffeomorphic to the \((n - \kappa)\)-dimensional torus \( \mathbb{T}^{n-\kappa} \), where \( \kappa \) is the corank of \( p \). According to a theorem by Miranda and Zung [100], under the non-degeneracy condition of \( p \), the system can be linearized not only at \( p \) but also in a neighborhood of the orbit \( O(p) \). To formulate this theorem precisely, we need first to construct the linear model around such an orbit, which is what we do next.

Take a symplectic manifold of direct product type

\[ V = D^{2\kappa} \times D^{n-\kappa} \times \mathbb{T}^{n-\kappa} \]

with a canonical coordinate system

\[ (x_1, \ldots, x_\kappa, \xi_1, \ldots, \xi_\kappa, x_{\kappa+1}, \ldots, x_n, \xi_{\kappa+1}, \ldots, \xi_n), \]

where \((x_1, \ldots, x_\kappa, \xi_1, \ldots, \xi_\kappa)\) are canonical coordinates on the symplectic disk \( D^{2\kappa} \) and \((x_{\kappa+1}, \ldots, x_n, \xi_{\kappa+1}, \ldots, \xi_n)\) are action-angle coordinates on \((D^{n-\kappa} \times \mathbb{T}^{n-\kappa})\). They look the same as the coordinate system in the statement of Theorem 4.1, except for the fact that the coordinates \( \xi_{\kappa+1} \pmod{1}, \ldots, \xi_n \pmod{1} \) are not local but periodic coordinates of period 1 on \( \mathbb{T}^{n-\kappa} \). Take the functions \( q_1, \ldots, q_n \) with the same expression as in Theorem 4.1. We get a direct product linear model around a compact orbit \( O(p) \) of Williamson type \((k_e, k_h, k_f)\).

Let \( \Gamma \) be a finite group with a free symplectic action \( \rho : \Gamma \times V \to V \), preserving the momentum map \((q_1, \ldots, q_n)\), and is linear in the following sense: \( \Gamma \) acts on the product \( V = D^{2\kappa} \times D^{n-\kappa} \times \mathbb{T}^{n-\kappa} \) component wise; the action of \( \Gamma \) on \( D^{n-\kappa} \) is trivial, its action on \( \mathbb{T}^{n-\kappa} \) is by translations relative to the coordinate system \((\xi_1, \ldots, \xi_{n-\kappa})\), and its action on \( D^{2\kappa} \) is linear with respect to the coordinate system \((x_1, \xi_1, \ldots, x_\kappa, \xi_\kappa)\). Then we can form the quotient symplectic manifold \( V/\Gamma \), with an integrable Hamiltonian system on it given by the same momentum map \((q_1, \ldots, q_n)\) as before. We call it an almost direct product linear model. It is a direct product model if \( \Gamma \) is trivial.

**Theorem 4.2 ([100]).** The associated Lagrangian fibration of an integrable Hamiltonian system near a compact orbit \( O(p) \) through a non-degenerate singular point \( p \) of Williamson type \((k_e, k_h, k_f)\) is symplectically isomorphic to one of the above almost direct product linear models.

### 4.3. Semi-local structure of singularities.

A singular fiber \( N \) of an integrable Hamiltonian system with proper momentum map is called a non-degenerate singular fiber, or non-degenerate singularity for short, if its singular points are non-degenerate, and moreover it satisfies a mild additional condition called the non-splitting condition: the bifurcation diagram (i.e., the set of singular values of the momentum map) for the system in a neighborhood of \( N \) does not split at \( N \), i.e., this bifurcation diagram coincides (locally) with the
bifurcation diagram of the system in a neighborhood of a singular point of maximal corank in $N$. This condition was first introduced by Zung [137] under a different name; later Bolsinov and Fomenko [20] called it “non-splitting,” a term which is more intuitive and will be adopted in this paper. Analytic integrable systems (and most natural examples of integrable systems from mechanics and physics that we know are analytic) automatically satisfy this non-splitting condition.

As was shown in [137], all non-degenerate singular points of maximal corank in a singular fiber $N$ of an integrable system (even without the splitting condition) have the same Williamson type $k = (k_e, k_h, k_f)$, which is called the **Williamson type of** $N$. The corank $\kappa = k_e + k_h + 2k_f$ of the most singular points on $N$ is also called the **corank** of $N$.

**Elementary** non-degenerate singularities of integrable systems are those with $k_e + k_h + k_f = 1$ and $n - \kappa = 0$, i.e., they have rank 0 and only one component, either elliptic, hyperbolic, or focus-focus. They live on 2-dimensional (the elliptic and hyperbolic case) and 4-dimensional (the focus-focus case) symplectic manifolds. In the following theorem, we denote a neighborhood of an elementary non-degenerate singular fiber $N$ (with some upper index, in dimension 2 or 4), together with the associated Lagrangian foliation, by $(P^2(N^e), L^e)$ in the elliptic case, $(P^2(N^h), L^h)$ in the hyperbolic case, and $(P^4(N^f), L^f)$ in the focus-focus case. $(D^{n-\kappa} \times \mathbb{T}^{n-\kappa}, L^e)$ denotes a regular Lagrangian torus fibration in dimension $2(n-\kappa)$ in the neighborhood of a regular fiber.

**Theorem 4.3** (Semi-local structure of non-degenerate singularities, [137]). Let $N$ be a non-degenerate singular fiber of corank $\kappa$ and Williamson type $k = (k_e, k_h, k_f)$ in an integrable Hamiltonian system given by a proper momentum map $H : M^{2n} \to \mathbb{R}^n$ on a symplectic manifold $(M^{2n}, \omega)$. Then there is a neighborhood $U(N)$ on $N$ in $M^{2n}$, saturated by the fibers of the system, with the following properties:

(i) There exists an effective Hamiltonian action of $\mathbb{T}^{k_e+k_h+(n-\kappa)}$ on $U(N)$ which preserves the system. This number $k_e + k_h + (n - \kappa)$ is the maximal possible. There is a (non-unique, in general) locally free $\mathbb{T}^{n-\kappa}$-subaction of this torus action.

(ii) $U(N)$ together with the associated Lagrangian torus fibration is homeomorphic (though not symplectomorphic, in general) to the quotient of a direct product of elementary non-degenerate singularities (2-dimensional elliptic, 2-dimensional hyperbolic, and/or 4-dimensional focus-focus) and a regular Lagrangian torus foliation of the type

\[
(1) \quad (U(\mathbb{T}^{n-\kappa}), L^e) \times (P^2(N^e_1), L^e_1) \times \cdots \times (P^2(N^e_{k_e}, L^e_{k_e})) \times \\
\times (P^2(N^h_1, L^h_1) \times \cdots \times (P^2(N^h_{k_h}, L^h_{k_h})) \times (P^4(N^f_1), L^f_1) \times \cdots \times (P^4(N^f_{k_f}, L^f_{k_f}))
\]

---

1A smooth map $f : M \to N$ is said to be locally trivial at $n_0 \in f(M)$, if there is an open neighborhood $U \subset N$ of $n_0$ such that $f^{-1}(n)$ is a smooth submanifold of $M$ for each $n \in U$ and there is a smooth map $h : f^{-1}(U) \to f^{-1}(n_0)$ such that $f \times h : f^{-1}(U) \to U \times f^{-1}(n_0)$ is a diffeomorphism. The **bifurcation set** $\Sigma_f$ consists of all the points of $N$ where $f$ is not locally trivial. The set of critical values of $f$ is included in the bifurcation set; if $f$ is proper, this inclusion is an equality (see [1, Proposition 4.5.1] and the comments following it).
by a free action of a finite group $\Gamma$ with the following property: $\Gamma$ acts on the above product component-wise (i.e., it commutes with the projections onto the components), it preserves the system, and, moreover, it acts trivially on elliptic components.

### 4.4. Topology of the base space.

The set of fibers of the associated Lagrangian fibration of an integrable system is called the base space of the system and denoted by $\mathcal{B}$. There is a natural projection from $M^{2n}$ to $\mathcal{B}$ and the momentum map projects to a map from $\mathcal{B} \to \mathbb{R}$. The topology on $\mathcal{B}$ is the quotient topology from $M^{2n}$ via the projection map.

There is also a natural notion of (local or global) smooth function on $\mathcal{B}$: such a function is the push forward of smooth first integrals of the system, i.e., it is a function whose pull back via the projection map is smooth on $M^{2n}$. Equipped with these smooth functions, $\mathcal{B}$ becomes a differential space. It means that if $f_1, \ldots, f_m$ is a finite family of smooth functions on an open subset $U \subset \mathcal{B}$ and $g : \mathbb{R}^m \to \mathbb{R}$ is a smooth function then $g \circ (f_1, \ldots, f_m)$ is again a smooth function on $U$.

According to Theorem 4.3, under the non-degeneracy assumption, locally $\mathcal{B}$ is homeomorphic to an almost direct product of elementary components, which are either regular, elliptic, hyperbolic, or focus-focus: each regular component is just an open interval, each elliptic component is a half-closed interval with one end point (which corresponds to the elliptic singular fiber), each hyperbolic component is a bouquet of half-closed intervals (a star-shaped graph, with the vertex in the center corresponding to the hyperbolic singular fiber), each focus-focus component is a 2-dimensional disk whose center corresponds to the focus-focus fiber.

The phrase “almost direct” means that we may have to take the quotient of a direct product, as described above, of 1-dimensional and 2-dimensional components by a diagonal action of a finite group $\Gamma$, which acts non-trivially only on hyperbolic components. (The action of $\Gamma$ on local regular, elliptic, and focus-focus components of the base space is trivial).

In particular, if there is no hyperbolic component, then the base space is locally homeomorphic to a direct product of intervals (maybe half-closed) and disks, and so it is an $n$-dimensional manifold (maybe with boundary and corners). The points on $\mathcal{B}$ which correspond to singularities of the system with at least one elliptic component ($k_e \geq 1$) are the boundary points of $\mathcal{B}$, and those points with at least two elliptic components ($k_e \geq 2$) lie on the corners of $\mathcal{B}$.

When there are hyperbolic singularities, the base space is not a manifold, but rather a space which is locally an almost-direct product of graphs.

### 4.5. Integral affine structure on the base space.

We say that a local function on $\mathcal{B}$ is a local action function if its pull-back to the symplectic manifold generates a Hamiltonian flow whose time-one map is the identity, i.e., if it can be viewed as the momentum map of a Hamiltonian $\mathbb{T}^1$-action which preserves the system.
By the Arnold-Mineur-Liouville theorem about action-angle variables, near each regular point \( x \in \mathcal{B} \) (which corresponds to a regular Lagrangian torus of the system), there is a smooth local coordinate system consisting of \( n \) actions functions \( (p_1, \ldots, p_n) \) in a neighborhood of \( x \). If \( p' \) is another local action function near \( x \), then, up to a constant, it is a linear combination of the action functions \( p_1, \ldots, p_n \) with integer coefficients:

\[
p' = \sum_{i=1}^{n} a_i p_i + c, \quad a_i \in \mathbb{Z}, \ c \in \mathbb{R}.
\]

It follows that the sum or difference of two local action functions is again a local action function, even near a singular fiber. Indeed, near each regular fiber the statement is true because of (2). Near a singular fiber it is also true because of the continuity of the time-one flow map of a vector field: if the map is the identity outside the singular fibers, then it is also the identity at a singular fiber. Thus the sheaf \( \mathcal{A} \) of local action functions on \( \mathcal{B} \) is an Abelian sheaf which contains the constant functions. (Constant functions correspond to the trivial torus action). The quotient \( \mathcal{A}/\mathbb{R} \) of \( \mathcal{A} \) by constant functions is a free Abelian sheaf called the sheaf of local \textbf{action 1-forms}. Theorem 4.3 yields the following formula for the rank of the stalk \( (\mathcal{A}/\mathbb{R})(x) \) of \( \mathcal{A}/\mathbb{R} \) at every point \( x \in \mathcal{B} \) of corank \( \kappa \) and Williamson type \((k_e, k_h, k_f)\): \[
\text{rank } (\mathcal{A}/\mathbb{R})(x) = (n - \kappa) + k_e + k_f = n - k_h - k_f, \text{ i.e. }
\]

\[
(\mathcal{A}/\mathbb{R})(x) \cong \mathbb{Z}^{n-k_h-k_f}.
\]

The sheaf \( \mathcal{A} \) is a (singular) \textbf{integral affine structure} on \( \mathcal{B} \). Indeed, near regular points of \( \mathcal{B} \) we have local coordinate systems consisting of action functions, and the transformation maps between different action coordinate systems are elements of \( GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n \), i.e., they are affine maps whose linear part is in the group \( GL(n, \mathbb{Z}) \) of linear isomorphisms of the lattice \( \mathbb{Z}^n \). Near points of \( \mathcal{B} \) which correspond to singular fibers we still have (smooth) affine functions, though not sufficiently many to make a local affine coordinate system, except for elliptic singularities.

4.6. \textbf{Systems of toric-focus type.}

The base space of an integrable system \textit{with} hyperbolic singularities is not a manifold but rather an \( n \)-dimensional generalization of graphs: locally it is topologically equivalent to an almost direct product of graphs. Such spaces are \textbf{branched manifolds} in the sense of Williams; see [132] It is not clear what does it mean for a branched manifold to be convex. Hence, in this paper we will not consider hyperbolic singularities when talking about convexity. (However, in the hyperbolic case, one can still talk about (quasi) convexity of the closure of each regular region of the base space and the (quasi) convexity of a function on \( \mathcal{B} \); see [136]). So we adopt the following definition.

**Definition 4.4.** An integrable Hamiltonian system with non-degenerate singularities is of \textbf{toric-focus} type if its singular points have no hyperbolic components, only elliptic and/or focus-focus components.

As we mentioned above, the base space \( \mathcal{B} \) of a toric-focus integrable system is a manifold, possibly with boundary and corners. As a matter of fact, it is not only a topological manifold, but is also a \textbf{smooth manifold}, with a natural smooth structure
given by the smooth first integrals of the system: it follows from the Eliasson-Miranda-Zung normal form theorem [41, 42, 100], that at every point $x \in \mathcal{B}$ there are $n$ smooth coordinate functions (those linear or quadratic first integrals in the normal form) such that any other local smooth function in a neighborhood of $x$ can be written as a smooth function in these coordinates.

Systems of toric-focus type also admit local smooth sections of the associated Lagrangian fibrations, not only at regular fibers but also at singular fibers. (See [140] for the role of local sections in the definition of characteristic classes of systems with singularities.) The projection maps (i.e., restrictions of the momentum map) from these local smooth sections to the base space are local diffeomorphisms.

We will not consider the image of an elliptic singular fiber on $\mathcal{B}$ of corank $\kappa$ as a singular point on $\mathcal{B}$, but rather a boundary or corner point, in a neighborhood of which $\mathcal{B}$ looks locally like a standard corner \{\(p_1 \geq 0, \ldots, p_\kappa \geq 0\}\} in an appropriate local action coordinate system \((p_1, \ldots, p_n)\).

On the other hand, the image of a singular fiber of Williamson type \((k_e, 0, k_f)\), with \(k_f > 0\), is called a singular point of type focus power \(k_f\), or, more concisely, focus\(^k_f\) point, on $\mathcal{B}$. Even though the base space $\mathcal{B}$ of a toric-focus integrable system is a smooth manifold (with boundary and corners), and its integral affine structure is also smooth outside of the focus points, the focus points are singular points for the integral affine structure; one of the reasons for it is the monodromy, discussed in the next subsection.

**Remark 4.5.** In dimension 4, toric-focus systems are also called almost-toric and they have been studied by several authors (see [123, 140, 86, 110]). In particular, compact 4-dimensional toric-focus symplectic manifolds have been classified up to diffeomorphisms in [86, Table 1]. Fiber connectivity and bifurcation diagrams were studied in [110]. We prefer the terminology toric-focus because it is more descriptive.

**Remark 4.6.** If all the singularities are elliptic, then we say that the integrable system is of toric type. Toric systems are closely related to symplectic toric manifolds; they have been studied by Delzant [32] and others; their convexity properties are known, under some properness conditions.

Another important special class of toric-focus systems are the semitoric and proper semitoric systems. **Proper semitoric systems** $F = (J, H) : M^4 \to \mathbb{R}^2$ are toric-focus systems for which

- $J : M^4 \to \mathbb{R}$ is the momentum map of an effective Hamiltonian $T^1$-action
- the fibers of $J$ are connected
- the bifurcation set of $J$ is discrete
- for any critical value $x$ of $J$, there exists a neighborhood $V \ni x$ such that the number of connected components of the critical set of $J$ in $J^{-1}(V)$ is finite
- $F$ is a proper map

If the momentum map $J$ of the proper semitoric system is itself proper, the system is called semitoric. Thus semitoric systems are a subclass of proper semitoric ones (so the terminology is a bit confusing). Hamiltonian $T^1$-actions and semitoric systems
on symplectic 4-manifolds have been studied by many authors, including Karshon and Tolman [71, 72], Leung and Symington [86], Pelayo and Vũ Ngọc [113], Sepe, Hohloch and Sabatini [64]. If the semitoric system is defined on a connected compact four-manifold, Vũ Ngọc [127] has shown that $F(M)$ is simply connected, $F$ has connected fibers, and that the set critical values of $F$ consists of the points on the boundary of $F(M)$ and a finite number of points, corresponding to the focus-focus fibers. In addition, it is shown that one can naturally map $F(M)$ to a convex polygon $P$ which is almost everywhere the range of a local momentum map for a torus action, defining the same foliation by tori as $F$. This is achieved by cutting along each of the focus-focus lines in $F(M)$ (they are vertical half-lines containing the focus-focus singularity), thereby obtaining vertices of the convex polygon. This polygon is not unique, due to monodromy, in stark contrast to momentum map ranges of $T^2$-actions. An invariant for semitoric systems is obtained by attaching labels to the polygon $P$ associated to the affine structure induced by the integrable system and the direction of the cuts. The complete invariant for semitoric systems (of course, containing this labeled polygon $P$) has been described in work of Pelayo and Vũ Ngọc [112, 113]. In all of this investigation of semitoric systems, properness of the momentum map $J$ is essential, since it permits the use of Morse-Bott theory and leads to connectedness of the fibers. Wacheux [129, 130] extended some of the results of Vũ Ngọc and Pelayo to higher dimensions.

We will come back to proper semitoric systems in Subsection 8.5 and describe a result from [111] showing that the loss of properness of $J$ has dramatic consequences, not only from a technical point of view, but also because $F(M)$ can be non-convex.

5. Base spaces and affine manifolds with focus singularities

5.1. Monodromy and affine coordinates near elementary focus points. The monodromy (of the Gauss-Manin flat connection on the bundle of homology groups of the fibers) of a Lagrangian torus fibration was first introduced in 1980 by Duistermaat [37] as an obstruction to the existence of global action-angle variables. This discovery has generated much work, both among mathematicians and physicists; see, e.g., [30] and references therein.

It is now well-known (see [138, 139]) that the monodromy around an elementary focus-focus singular fiber in dimension 4 is non-trivial and is given by the matrix
\[
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix},
\]
where $k \geq 1$ is the number of singular focus-focus points on the singular fiber. Concretely, this means the following. Take a simple closed path on the 2-dimensional base space $B$ with the focus point lying in its interior. For a chosen point $c$ on this path, take two appropriate 1-cycles $\gamma_1$ and $\gamma_2$ on the corresponding Lagrangian fiber $N_c \cong \mathbb{T}^2$, which form a basis of $H_1(N, \mathbb{Z})$. Move $c$ along this closed path. Then $N_c$ and $\gamma_1, \gamma_2$ move together with it. When $c$ has moved once along this closed path, coming back to its initial position in $B$, $N_c$ also comes back to itself, but the homology classes of $\gamma_1, \gamma_2$ change by the following formula:
\[
\begin{pmatrix}
[\gamma_1^{new}] \\
[\gamma_2^{new}]
\end{pmatrix} =
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
[\gamma_1] \\
[\gamma_2]
\end{pmatrix}.
\]
The number $k$ in the above monodromy formula is called the *index* of the focus singularity.

For the above formula to hold, one must choose $\gamma_1$ to be the cycle represented by the orbits of the system-preserving Hamiltonian $\mathbb{T}^1$-action given by Theorem 4.3 and choose appropriate orientations for the path on the base space and the cycles.

In order to avoid confusion about the orientations, it is be better to look at the monodromy on action functions or action 1-forms, instead of 1-cycles on the tori; these are essentially the same, because the action 1-forms generate 1-cycles on the tori. In terms of action functions, the monodromy around an elementary focus-focus singularity can be described as follows.

For each point $y \in \mathcal{B}$, denote by $\gamma_i(y)$ ($i = 1, 2$) the result of parallel transport of $\gamma_i$ from $c$ to $d$ by the Gauss-Manin connection along a simple path on $\mathcal{B}$ avoiding the focus point. Let

$$F(y) = \int_{\gamma_1(y)} \alpha, \quad G(y) = \int_{\gamma_2(y)} \alpha,$$

where $\alpha$ is a primitive 1-form of the symplectic 2-form $d\alpha = \omega$. Such a primitive 1-form exists near the focus-focus singular fiber, because the cohomology class of the symplectic form in a neighborhood of the focus-focus fiber is 0, due to the fact that the fiber is Lagrangian.
Then $F$ and $G$ are two action functions. $F$ is single-valued and is, in fact, a smooth action function on $\mathcal{B}$ given by Theorem 4.3. The function $G$ is multi-valued: it depends on the homotopy of the path from $c$ to $y$.

The singular focus-focus fiber of the Lagrangian torus fibration is called a *pinched torus*, with the number of pinches equal to the number $k$ of singular points. (See Figure 1 for the case of 2 pinches). Its first homology group with integral coefficients is isomorphic to $\mathbb{Z}$. When $c$ tends to the focus point on $\mathcal{B}$ then $\gamma_1$ tends to 0 while $\gamma_2$ tends to a generator of this homology group (independently of the path taken), hence $G$ admits a continuous extension to the focus point $O$ on $\mathcal{B}$, and we may arrange so that $F(O) = G(O) = 0$ and choose $c$ such that $F(c) < 0$. The line $\{F = 0\}$ cuts $\mathcal{B}$ into two parts and $G$ is single-valued and strictly monotonous on that line.

We choose two different branches of $G$, denoted by $G_l$ and $G_r$, by specifying the homotopy of the path from $c$ to $y$ in $\mathcal{B}\setminus\{O\}$ as follows. For $G_l$, if $F(y) \leq 0$ then the path from $c$ to $y$ lies in the region $\{F \leq 0\}$; if $F(y) > 0$ then the path from $c$ to $y$ cuts the ray $\{F = 0, G \leq 0\}$ but does not cut the ray $\{F = 0, G \geq 0\}$. For $G_r$, if $F(y) \leq 0$ then the path from $c$ to $y$ lies in the region $\{F \leq 0\}$, if $F(y) > 0$ then the path from $c$ to $y$ cuts the ray $\{F = 0, G \geq 0\}$ but does not cut the ray $\{F = 0, G \leq 0\}$.

The monodromy formula around the focus-focus singularity, or the focus point on $\mathcal{B}$, can now be expressed as a relation between the two branches $G_l$ and $G_r$ of the multi-valued action function $G$:

$$G_r = G_l + kF \text{ when } F > 0; \quad G_r = G_l \text{ when } F \leq 0. \quad (4)$$

In turn, Formula (4) can be seen as a special case of the Duistermaat-Heckman formula [39] for the Hamiltonian $\mathbb{T}^1$-action generated by $F$; see [139].

![Figure 2. Affine structure near the focus point.](image)

Visually, the local affine structure at the focus point $O$ on $\mathcal{B}$ can be obtained from an Euclidean plane with a standard coordinate system by cutting out the angle given by the two rays $OA, OB$, where $O = (0,0), A = (k, 1), B = (0, 1)$, and then gluing the ray $OB$ to the ray $OA$ by the transformation matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. (See Figure 2).

Since we have both smooth and a singular structures on $\mathcal{B}$, it is natural to ask how does the affine structure behave in terms of the smooth structure near the singular
focus points. In particular, how does $G$ behave on the line $\{F = 0\}$ with respect to the smooth structure?

Recall that we have a smooth coordinate system $(F, H)$ near $O$, where $F$ is the action function. Their pull backs to the symplectic manifold are given by the following formulas in a local canonical coordinate system near a focus-focus point, according to Theorem 4.1:

$$F = x_1y_2 - x_2y_1; \quad H = x_1y_1 + x_2y_2.$$ 

Without loss of generality (using some rescaling if necessary), we may assume that the point $p : (x_1, y_1, x_2, y_2) = (1, 0, 0, 0)$ is in a small neighborhood of the focus-focus point where the normal form is well-defined. Take a small 2-dimensional disk $D = \{x_1 = 1, x_2 = 0\}$ which intersects the local plane $\{y_1 = y_2 = 0\}$ (which is part of the local focus-focus fiber) transversally at $p$. On $D$ we have $H = y_1, F = y_2$, which means that the local smooth structure of $B$ near $O$ can be projected from the smooth local structure of the disk $D$ which intersects all the fibers near the focus-focus fiber transversally. This implies, in particular, that the smooth structure on $B$ near $O$ does not depend on the choice of the singular point on the singular fiber where one takes the normal form.

![Figure 3. Cycle $\gamma_2$.](image)

We can assume that the 1-form $\alpha$ in formula (3) for action functions is given in the local canonical coordinates $(x_1, y_1, x_2, y_2)$ by

$$\alpha = x_1dy_1 + x_2dy_2$$

and that the part $\gamma_2^{\text{sing}}$ of the 1-cycle $\gamma$ on the Lagrangian torus $\{F = 0, H = \varepsilon\}$, which is inside the ball with the canonical coordinate system $(x_1, y_1, x_2, y_2)$, is given by the equations $\{x_2 = y_2 = 0, x_1y_1 = \varepsilon = H\}$ and the inequalities $0 \leq x_1 \leq 1, 0 \leq y_1 \leq 1$. 

If the focus-focus fiber contains only one singular point (see Figure 3), then at \( F = 0 \), \( H = \varepsilon \) we have:

\[
G = \int_{\gamma_2} \alpha = \int_{\gamma_2^{\text{sing}}} \alpha + \int_{\gamma_2^{\text{\neg sing}}} \alpha = \int_H^1 (H/y_1) dy_1 + g(H) \\
= H \log(1/H) + g(H),
\]

where \( g \) is a smooth function. The case when there are many singular points on the focus-focus fiber is absolutely similar, except for the fact that there are many “singular” pieces of \( \gamma_2 \) (each piece near one singular point) instead of just one. We have proved the following lemma.

**Lemma 5.1.** With the above notations and assumptions, \( G \) is strictly increasing with respect to \( H \) on the line \( \{ F = 0 \} \) on the base space \( \mathcal{B} \), has real positive derivative \( \partial G/\partial H > 0 \) on this line, except for the point \( O \) where \( \partial G/\partial H = +\infty \).

Both Formula (5) and Lemma 5.1 are in fact well-known to people working on symplectic invariants of integrable Hamiltonian systems, and go back at least to Dufour, Molino, and Toulet [36]; see also Võ Ngọc [126] and Wacheux [129].

### 5.2. Affine coordinates near focus points in higher dimensions.

A singularity with 1 focus-focus component in a toric-focus integrable system with \( n \) degrees of freedom (\( n \geq 3 \)) may be viewed as a parametrized version of elementary focus-focus singularities.

In this case, topologically we have a direct product of an elementary focus-focus singularity with a regular Lagrangian torus fibration (i.e., the finite group \( \Gamma \) in Theorem 4.3 can be made trivial). On the base space \( \mathcal{B} \) near the focus point \( O \), we have a local smooth coordinate system \( (F, H, L_1, \ldots, L_{n-2}) \) which consists of \( n - 1 \) action functions \( L_1 = x_3, \ldots, L_{n-2} = x_n, F = x_1 y_2 - x_2 y_1 \), and one additional first integral \( H = x_1 y_1 + x_2 y_2 \), where \((x_1, y_2, \ldots, x_n, y_n)\) is a local canonical coordinate system on the symplectic manifold. Some of the action functions \( L_1, \ldots, L_{n-2} \) may eventually correspond to the elliptic components of the singularity (if it also has elliptic components).

The set of focus points on \( \mathcal{B} \) near \( O \), or equivalently, the family of focus-focus fibers near the fiber corresponding to \( O \), is smooth \( (n - 2) \)-dimensional and given by the equations \( F = H = 0 \). The functions \( (L_1, \ldots, L_{n-2}) \) form a smooth coordinate system on this local submanifold of focus points on \( \mathcal{B} \).

Similarly to the case with 2 degrees of freedom, there is a multi-valued action function \( G \) in a neighborhood of \( O \) in \( \mathcal{B} \). We can choose, as in Subsection 5.1, two branches of \( G \), denoted by \( G_l \) and \( G_r \), by specifying the homotopy of the path from a point \( c \in \mathcal{B} \), satisfying \( F(c) < 0 \), to \( y \in \mathcal{B} \setminus \{ F = H = 0 \} \) as follows. For \( G_l \): if \( F(y) \leq 0 \), then the path from \( c \) to \( y \) lies in the region \( \{ F \leq 0 \} \), whereas if \( F(y) > 0 \), then the path from \( c \) to \( y \) cuts half-plane \( \{ F = 0, H \leq 0 \} \) but does not cut the half-plane \( \{ F = 0, H \geq 0 \} \). For \( G_r \): if \( F(y) \leq 0 \), then the path from \( c \) to \( y \) lies in the region \( \{ F \leq 0 \} \), whereas if \( F(y) > 0 \), then the path from \( c \) to \( y \) cuts the half-plane \( \{ F = 0, H \geq 0 \} \) but does not cut the half-plane \( \{ F = 0, H \leq 0 \} \).
The monodromy formula around the local codimension two submanifold of focus points in $\mathcal{B}$ can be expressed as a relation between the two branches $G_l$ and $G_r$ of the multi-valued action function $G$, just as in the case of an elementary focus-focus singularity:

$$G_r = G_l + kF \text{ when } F > 0; \quad G_r = G_l \text{ when } F \leq 0.$$ 

On $\{F = 0\}$, the multi-valued action function $G$ is, in fact, continuous, single-valued, and has the same behavior with respect to $H$ as in the case of an elementary focus-focus singularity: $\partial G/\partial H > 0$ when $H \neq 0$, and $\partial G/\partial H = +\infty$ when $H = 0$.

Local Marsden-Weinstein reduction with respect to the $\mathbb{R}^{n-2}$-action generated by $(L_1, \ldots, L_{n-2})$ yields a $(n-2)$-dimensional family of elementary focus-focus singularities, each for one level of $(L_1, \ldots, L_{n-2})$. It means that we can slice a neighborhood of $O$ in $\mathcal{B}$ by the functions $(L_1, \ldots, L_{n-2})$ into a $(n-2)$-dimensional family of 2-dimensional base spaces containing each one a focus point. The pair $(F, H)$ is a local smooth coordinate system and the pair $(F, G)$ is a multi-valued system of action coordinates on each such 2-dimensional base space. The restriction of $G$ to the local $(n-2)$-dimensional submanifold $\{F = H = 0\}$ of focus points in $\mathcal{B}$ is not constant, in general, but it is a smooth function in the variables $(L_1, \ldots, L_{n-2})$. The reason is that the family of 1-cycles $\gamma_2$, as shown in Figure 3, consists of a “singular part” $\gamma_2^{sing}$ which is identical for all values of $(L_1, \ldots, L_{n-2})$ (with respect to a canonical system of coordinates) and a “regular part” which depends smoothly on $(L_1, \ldots, L_{n-2})$. We denote this restriction of $G$ to the submanifold $\{F = H = 0\}$ by $G^{\text{critical}}(L_1, \ldots, L_{n-2})$ and call it the (function of) critical values of $G$. (This function is only unique up to a constant; one can put, for example, $G(O) = 0$ to fix it).

![Figure 4](image-url)  

**Figure 4.** Construction of a base space with focus points in higher dimensions.

To visualize the affine structure of $\mathcal{B}$ near $O$, one can proceed as follows. In a neighborhood of the origin in the Euclidean space $\mathbb{R}^n$ with a local coordinate system $(F, G, L_1, \ldots, L_{n-2})$ (if there are elliptic singular components then just take a corner of this space, e.g., $L_1, L_2 \geq 0$ if the number of elliptic components is equal to 2), draw the
“critical” \((n - 2)\)-dimensional submanifold
\[
\mathcal{S} = \{ F = 0, \ G = G_{\text{critical}}(L_1, \ldots, L_{n-2}) \}.
\]

Dig out a “ditch”
\[
\mathcal{D} = \{ F \geq 0, \ G_{\text{critical}}(L_1, \ldots, L_{n-2}) \leq G \leq G_{\text{critical}}(L_1, \ldots, L_{n-2}) + kF \},
\]
which has \(\mathcal{S}\) as its “sloping bottom” and glue the two “walls” of the ditch together, slice by slice (glue the two edges of each \(\mathcal{D} \cap \{ L_1 = \text{const.}, \ldots, L_{n-2} = \text{const.} \}\) in the same way as in the case of a 2-dimensional base space). The result is the local affine model for \(\mathcal{B}\); see Figure 4 for the case \(n = 3\). (See also [130].) Notice that our ditch is “curved”: its base \(\mathcal{S}\) is a submanifold but not an affine submanifold, in general. However, \(\mathcal{S}\) lies on the affine hypersurface \(\{ F = 0 \}\).

5.3. Behavior of the affine structure near focus\(^m\) points.

Consider a singular point \(O\) of Williamson type \(k = (k_e, 0, k_f)\) with \(2 \leq k_f = m \leq n/2\) in the base space \(\mathcal{B}\) of a toric-focus integrable Hamiltonian system with \(n\) degrees of freedom.

For simplicity, we assume, for the moment, that the singularity corresponding to \(O\) of the Lagrangian fibration is a topological direct product of elementary singularities, i.e., the finite group \(\Gamma\) in Theorem 4.3 is trivial.

Similarly to the situation with only one focus component treated in the previous subsections, Theorems 4.1, 4.2, and 4.3 provide a local smooth coordinate system \((F_1, H_1, \ldots, F_m, H_m, L_1, \ldots, L_{n-2m})\) and a local multi-valued action coordinate system \((F_1, G_1, \ldots, F_m, G_m, L_1, \ldots, L_{n-2m})\) in a neighborhood \(\mathcal{U}(O)\) of the focus\(^m\) point \(O\), with the following properties:

- All coordinate functions vanish at \(O\):
  \[
  F_i(O) = H_i(O) = G_i(O) = L_j(O) = 0 \ \forall \ i = 1, \ldots, m; j = 1, \ldots, n - 2m.
  \]

- The set of singular points in \(\mathcal{U}(O)\) is the union of \(m\) codimension two submanifolds
  \[
  \mathcal{S}_i = \{ F_i = H_i = 0 \}, \ i = 1, \ldots, m.
  \]

- The fundamental group of the local set of regular points in \(\mathcal{U}(O)\) is isomorphic to \(\mathbb{Z}^m\):
  \[
  \pi_1(\mathcal{U}(O) \setminus (\bigcup_{i=1}^m \mathcal{S}_i)) \cong \mathbb{Z}^m.
  \]

- The linear monodromy representation \(\rho : \mathbb{Z}^m \to GL(n, \mathbb{Z})\) of \(\pi_1(\mathcal{U}(O) \setminus (\bigcup_{i=1}^m \mathcal{S}_i))\) for the regular integral affine structure on \(\mathcal{U}(O) \setminus (\bigcup_{i=1}^m \mathcal{S}_i)\), or equivalently, for the Lagrangian torus fibration over \(\pi_1(\mathcal{U}(O) \setminus (\bigcup_{i=1}^m \mathcal{S}_i))\), is generated by the matrices
  \[
  M_i = I_{2i-2} \oplus \begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix} \oplus I_{n-2i}, \ \forall \ i = 1, \ldots, m,
  \]
  where \(I_d\) means the identity matrix of size \(d \times d\), and \(k_i > 0\) is the index of the focus points on \(\mathcal{S}_i \setminus \bigcup_{j \neq i} \mathcal{S}_j\).
• If the fiber over $O$ has no elliptic component, then $\mathcal{U}(O)$ together with the coordinate functions $(F_i, H_i, L_j)$ looks like an $n$-dimensional cube $] - a, a[^n$ in $\mathbb{R}^n$ for some $a > 0$. If there are $k_e > 0$ elliptic components then the functions $L_{n-2m-k_e+1}, \ldots, L_{n-2m}$ admit only nonnegative values, and $\mathcal{U}(O)$ looks like $1/2^{k_e}$ part of a cube $] - a, a[^{n-k_e} \times [0, a[^{k_e}$.

• For each $i = 1, \ldots, m$, the action function $G_i$ will change to $G_i + k F_i$ if one goes one full circle around $S_i$ in an appropriate direction. To be more precise, we can think of $G_i$ as having two branches $(G_i)_l$ and $(G_i)_r$: $(G_i)_l$ is a smooth action function on $\mathcal{U}(O) \setminus \{F_i = 0, H_i \geq 0\}$ with a continuous extension to the whole $\mathcal{U}(O)$ but not smooth on $\{F_i = 0, H_i \geq 0\}$; $(G_i)_r$ is a smooth action function on $\mathcal{U}(O) \setminus \{F_i = 0, H_i \geq 0\}$ with a continuous extension to the whole $\mathcal{U}(O)$ but not smooth on $\{F_i = 0, H_i \leq 0\}$. The monodromy can be expressed in terms of the following relations:

\[
(G_i)_r = (G_i)_l + k_i F_i \text{ when } F_i > 0 \quad \forall \ i = 1, \ldots, m.
\]

\[
(G_i)_r = (G_i)_l \text{ when } F_i \leq 0
\]

• For each $i = 1, \ldots, m$, we have

\[
\frac{\partial G_i}{\partial H_i} > 0 \quad \text{is real positive on } \{F_i = 0\},
\]

except on $S_i$ where $\frac{\partial G_i}{\partial H_i} = +\infty$.

In the case when we have only a topological almost direct product, i.e., the finite group $\Gamma$ in Theorem 4.3 is non-trivial, its action preserves every fiber of the singular Lagrangian fibration, and so the base space of the quotient fibration by $\Gamma$ is exactly the same as the base space of the direct product model. Thus $\mathcal{B}$ is locally still the same as in the direct product case. It still has a local smooth coordinate system $(F_1, H_1, \ldots, F_m, H_m, L_1, \ldots, L_{n-2m})$ consisting of $n - m$ action functions $(F_1, \ldots, F_m, L_1, \ldots, L_{n-2m})$ and $m$ additional functions $(H_1, \ldots, H_m)$; it still has $m$ complementary multi-valued action functions $G_1, \ldots, G_m$, with the same formulas and monodromy as above. The only difference is that, unlike the direct product case, the $n$-tuple $(dF_1, dG_1, \ldots, dF_m, dG_m, dL_1, \ldots, dL_{n-2m})$ does not generate the whole lattice of action 1-forms at a regular point near $O$ in general, but only a sub-lattice of finite index. For example, consider the almost direct product

\[
(\mathcal{F}_1 \times \mathcal{F}_2)/\mathbb{Z}_2
\]

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are elementary focus-focus singularities of index 2, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts freely on each of them in such a way that $\mathcal{F}_1/\mathbb{Z}_2$ and $\mathcal{F}_2/\mathbb{Z}_2$ are elementary focus-focus singularities of index 2. For this almost direct product, the function

\[
(G_1 + G_2)/2
\]

is a multi-valued action function, though such a function cannot be a multi-valued action function in the direct product situation. $(G_1 + G_2)$ is a multi-valued action function in the direct product situation.
Remark 5.2. Vũ Ngọc [126] and Wacheux [129] obtained more refined asymptotic formulas for the functions $G_i$ in the particular case when $k_i = 1 \forall i$, using the complex algorithm function. More precisely, they showed that in that case Formula (5) becomes $G = \Re(-(H + \sqrt{-1}F) \log(H + \sqrt{-1}F)) + g(H, F)$ (where $\Re$ means the real part, and $g$ is a smooth function) (or a similar formula for higher corank), which holds also for $F \neq 0$, whereas Formula (5) only holds for $F = 0$. We will not need these refined formulas in the present paper.

5.4. Definition of affine structures with focus points.

Guided by the properties of the base spaces of toric-focus integrable system, we define the following notion of singular integral affine structures with focus points on manifolds with or without boundary and corners.

Definition 5.3. A (singular) integral affine structure with focus points on a smooth manifold $B$ (possibly with boundary and corners) is a free Abelian subsheaf $A_Z$ of the sheaf of local smooth functions on $B$, called the sheaf of local integral affine functions, which satisfies the following properties:

(i) $A$ contains all constant functions. We will denote by $A_Z/\Re$ the quotient of $A_Z$ by the constant functions and call it the sheaf of local integral affine 1-forms on $B$.

(ii) For $x$ in a open dense set in $B$, called the set of regular points, $A_Z(x)/\Re \cong \mathbb{Z}^n$ (where $A_Z(x)$ denotes the stalk of $A_Z$ at $x$), and a basis of $A_Z(x)$ forms a local coordinate system on $B$, called a local integral affine coordinate system.

(iii) If $x \in B$ is singular (i.e., non-regular), then rank $A_Z(x) < n$. The set $F_m = \{x \in B \mid \text{rank } A_Z(x) = n - m\}$ is a smooth submanifold of codimension $2m$ in $B$ (whose intersection with the boundary and corners of $B$ is transversal). Each point $x \in F_m$ is called a singular focus point of corank $m$, or of type focus power $m$, or an $F^m$-point for short.

(iv) For every point $x \in F_m$, there is a small neighborhood $V$ of $x$ and $k$ codimension two submanifolds $S_1, \ldots, S_m$, with transversal intersections, such that $F_k \cap V = \cap_{i=1}^m S_i$. Moreover, there is a family of functions

$$(F_1, (G^1)_r, \ldots, F_m, (G_m)_r, L_1, \ldots, L_{n-2m})$$

in $V$, called a multi-valued system of integral affine coordinates near $x$, which satisfies the following conditions:

- $F_1, \ldots, F_k, L_1, \ldots, L_{n-2m} \in A_Z(V)$ and together with some other smooth functions $H_1, \ldots H_m$ form a smooth coordinate system of $V$.
- $S_i = \{F_i = H_i = 0\}$ for every $i = 1, \ldots, m$.
- $(G_i)_l$ is continuous on $V$ and $(G_i)_l |_{\{F_k = 0, H_i \geq 0\}} \in A_Z(\{F_i = 0, H_i \geq 0\})$. $(G_i)_r$ is continuous on $V$ and $(G_i)_r |_{\{F_k = 0, H_i \leq 0\}} \in A_Z(\{F_i = 0, H_i \leq 0\})$.
- $(G_i)_r = (G_i)_l$ when $F_i \leq 0$ and $(G_i)_r = (G_i)_l + k_i F_i$ when $F_i \geq 0$, for some positive constant $k_i \in \mathbb{Z}_+$. Moreover, $(\partial(G_i)_l/\partial H_i)(y) = (\partial(G_i)_l/\partial H_i)(y) > 0$ with respect to the coordinate system $(F_1, H_1, \ldots, F_m, H_m, L_1, \ldots, L_{n-2m})$ for every $y \notin S_i$. 

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Of course, if $\mathcal{B}$ is the base space of a toric-focus integrable Hamiltonian system, then the sheaf of local action function on $\mathcal{B}$ is an integral affine structure with focus singularities. Local integral affine functions on $\mathcal{B}$ are those functions whose pull back on $M^{2n}$ give rise to Hamiltonian vector fields whose time-one flow is the identity map, i.e., they are generators of $\mathbb{T}^1$-Hamiltonian actions which preserve the system. We do not consider the points of $\mathcal{B}$ corresponding to elliptic singularities of the system as singular points of $\mathcal{B}$, but rather as regular points lying on the boundary and corners.

We also define affine structures with focus points on a manifold $\mathcal{B}$ (without the adjective integral) by simply removing all the words “integral” and “$\mathbb{Z}$” from Definition 5.3, and by considering the sheaf $\mathcal{A}$ of all local affine functions (instead of just integral affine functions). At a regular point, the stalk of $\mathcal{A}/\mathbb{R}$ is isomorphic to $\mathbb{R}^n$ instead of $\mathbb{Z}^n$. In the monodromy formulas $(G_i)_r = (G_i)_l + k_i F_i$, the numbers $k_i$ are still positive but are not required to be integers.

Of course, an integral affine structure with focus point is a special case of affine structures with focus points: in that case, the relation between $\mathcal{A}$ and $\mathcal{A}_\mathbb{Z}$ is $\mathcal{A} = \mathcal{A}_\mathbb{Z} \otimes \mathbb{R}$.

6. Straight lines and convexity

6.1. Regular and singular straight lines.

A (parametrized) curve in $\mathcal{B}$ is a continuous map from an interval $I \subset \mathbb{R}$ (which can be bounded or unbounded, with or without end points) to $\mathcal{B}$.

Affine lines, a.k.a. straight lines, in the regular part of a singular affine manifold $\mathcal{B}$ can be defined in an obvious way: a curve $\gamma$ lying entirely in the regular part of a singular affine manifold $\mathcal{B}$ is called an affine line or straight line, if it is locally affine in every local affine chart of the regular part of $\mathcal{B}$. In other words, for any two local affine functions there is a non-trivial affine combination of them which vanishes on $\gamma$.

Any non-trivial regular straight line can be naturally (affinely) parametrized by local affine functions: the parametrization is unique up to transformations of the type $t \mapsto at + b$ ($a$ and $b$ are constants, $a \neq 0$).

We also want to study straight lines which contain singular points of $\mathcal{B}$. They will be called singular straight lines, or singular affine lines. We can define them by using limits of regular straight lines.

Definition 6.1. A non-constant parametrized continuous curve $\gamma : I \to \mathcal{B}$ in a singular affine manifold $\mathcal{B}$ (I is an interval in $\mathbb{R}$), which contains singular points of $\mathcal{B}$, is called a singular straight line or singular affine line if for every point $t_0$ in the interior of $I$ there exists a subinterval $[s, t] \subset I$, $s < t_0 < t$, and a sequence of affinely parametrized regular straight lines $\gamma_n : [t, s] \to \mathcal{B}$ which converges to $\gamma$ (in the standard open-compact topology).

Of course, the regular part of a singular straight line $\gamma$ (i.e., $\gamma$ minus singular points of $\mathcal{B}$), if not empty, consists of regular straight lines, because in the regular region, the limit of a sequence of straight lines is again a straight line.
6.2. **Singular straight lines in dimension 2 and branched extension.**

To understand the nature of singular straight lines, let us first consider the simplest case of a single focus point $O$ in a 2-dimensional singular affine manifold $\mathcal{B}^2$.

Let $F,G$ be a multi-valued affine coordinate system near $O$ in $\mathcal{B}^2$: $F(0) = G(0) = 0$, $F$ is single valued, $G$ is multivalued. $G$ has two branches $G_t$ and $G_r$ (extensions of an affine function $G$ from the region $\{F \leq 0\}$ to the region $\{F > 0\}$, from the left or from the right of the singular point $O$), with the monodromy formula

$$G_r = G_t + kF \text{ when } F > 0 \text{ and } G_r = G_t \text{ when } F \leq 0,$$

where $k > 0$ is the index of the focus point $O$.

Let $\gamma : I \rightarrow \mathcal{B}^2$ be a parametrized singular straight line in $\mathcal{B}^2$, with $\gamma(t_0) = O$ for some $t_0 \in I$ ($I$ is an interval in $\mathbb{R}$). Let $\gamma_n : I \rightarrow \mathcal{B}^2$ be parametrized regular straight lines in $\mathcal{B}^2$ which converge to $\gamma$ when $n \rightarrow \infty$. Since $F$ is single-valued and is affine on every $\gamma_n$ with respect to the parameter $t \in I$, it follows that $F$ is also affine on $\gamma$ with respect with to the parameter $t \in I$. Moreover, $F(t_0) = F(O) = 0$ on $\gamma$. We distinguish two possible situations.

1) $F$ is identically zero on $\gamma$, i.e., $\gamma(I) \subset \{F = 0\}$. On $\{F = 0\}$ the function $G$ is single-valued and becomes an affine parametrization for $\gamma$, i.e., $G$ is affine with respect to $t$ on $\gamma$. The curves $\{F = c\}$ (where $c \neq 0$ is a constant) are straight lines affinely parametrized by $G$ (or $G_t$ or $G_r$ when $c > 0$) and they converge to the singular affine curve $\{F = 0\}$ (also affinely parametrized by $G$) when $c$ tends to 0.

2) $F$ is not identically zero on $\gamma$. Then we can take $F$ as an affine (re)parametrization of $\gamma$. Let $\gamma_n$ be a family of straight lines which converges to $\gamma$ when $n$ goes to infinity. Without losing generality, we may assume that $F(\gamma_n(t)) = F(\gamma(t))$ for every $n \in \mathbb{N}$ and every $t \in I$. Put $X_n = \gamma_n(t_0)$. Then $X_n \neq O$, but the sequence of points $(X_n)$ lies on the straight line $\{F = 0\}$ and converges to the singular point $O$ when $n \rightarrow \infty$. Put $c_n = G(X_n)$, then $c_n \neq 0$ and $c_n \overset{n \rightarrow \infty}{\rightarrow} 0$. By taking a subsequence of $(\gamma_n)$, if necessary, we may assume that either $c_n < 0$ for all $n$ or $c_n > 0$ for all $n$.

![Figure 5](image)

**Figure 5.** Extension of a straight line through a focus point.

If $c_n < 0$ for all $n$, then we say that the straight lines $(\gamma_n)$ tend to $\gamma$ **from the left**, and if $c_n > 0$ for all $n$, then we say that the straight lines $(\gamma_n)$ tend to $\gamma$ **from the right** (with respect to the singular multi-valued affine coordinate system $(F,G)$).
Assume now that \( t_0 \) lies in the interior of the interval \( I \), and cut \( I \) by \( t_0 \) into two parts: \( I_1 = I \cap (-\infty, t_0) \) and \( I_2 = I \cap (t_0, +\infty) \). We will say that \( \gamma|_{I_1} \) is a straight line coming from the region \( \{ F < 0 \} \) which hits \( O \), and \( \gamma|_{I_2} \) is a straight extension of \( \gamma|_{I_1} \) after hitting \( O \).

Our main observation in this subsection is the following:

**Proposition 6.2.** In a neighborhood of a focus point \( O \), every straight line hitting \( O \) and not lying on \( \{ F = 0 \} \) has exactly two different straight extensions after hitting \( O \).

**Remark 6.3.** The straight line has only a unique straight extension if it lies on \( \{ F = 0 \} \).

*Proof.* Indeed, without loss of generality, working locally, we may assume for simplicity that \( I = [-\delta, \delta] \) for some small \( \delta > 0 \), \( t_0 = 0 \), \( F(\gamma(t)) = t \) and \( G(\gamma(t)) = at \) for some constant \( a \in \mathbb{R} \) on \( \gamma|_{I_1} \). We can define straight lines \( \gamma^l_n : I \to \mathcal{B}^2 \) and \( \gamma^r_n : I \to \mathcal{B}^2 \) by \( F(\gamma^l_n(t)) = t \), \( G_l(\gamma^l_n(t)) = at - \varepsilon_n \) and \( F(\gamma^r_n(t)) = t \), \( G_r(\gamma^r_n(t)) = at + \varepsilon_n \), where \( \varepsilon_n \) are small positive numbers which tend to 0 when \( n \) goes to infinity.

Then \( (\gamma^l_n) \) tend to a singular straight line \( \gamma^l \) defined by
\[
F(\gamma^l_n(t)) = t, \quad G_l(\gamma^l(t)) = at
\]
(for every \( t \in I = [-\delta, \delta] \)), while \( (\gamma^r_n) \) tend to a singular straight line \( \gamma^r \) defined by
\[
F(\gamma^r_n(t)) = t, \quad G_r(\gamma^r(t)) = at.
\]

Recall that when \( F \leq 0 \), then \( G_l = G_r = G \) are the same function, but when \( F > 0 \), then \( G_l \) and \( G_r \) are two different functions (two branches of \( G \)) related by the monodromy formula \( G_r = G_l + kF \). Because of that, \( \gamma^l|_{I_1} = \gamma^r|_{I_1} = \gamma \), where \( \gamma : I_1 \to \mathcal{B} \) is a straight line that hits \( O \), but \( \gamma^l|_{I_2} \neq \gamma^r|_{I_2} \).

We will say that \( \gamma^l \) is the (local) extension from the left of \( \gamma \) after hitting \( O \), while \( \gamma^r \) is the (local) extension from the right of \( \gamma \); together they form the branched, double-valued extension of \( \gamma \).

The difference between \( \gamma^l \) and \( \gamma^r \) in terms of \( G \) is as follows:
\[
G_l(\gamma^l(t)) - G_l(\gamma^r(t)) = G_r(\gamma^l(t)) - G_r(\gamma^r(t)) = kF(t),
\]
when \( F(t) > 0 \). The difference is 0 when \( F(t) \leq 0 \). Geometrically, it means that the extension from the left lies on the right of the extension from the right.

### 6.3. Straight lines in dimension \( n \) near a focus point.

Let \( O \) be a \( F^1 \) singular point (i.e., with just one focus component) in a singular \( n \)-dimensional affine manifold \( \mathcal{B} \), with \( n \geq 3 \). (\( O \) can lie on the boundary of \( \mathcal{B} \)). Near \( O \), we have a local multi-valued affine coordinate system \( (G, F, L_1, \ldots, L_{n-2}) \), where \( F, L_1, \ldots, L_{n-2} \) are single valued affine functions, \( F \) is the “angular momentum” for the focus-focus singularities, \( F(O) = 0 \), \( G \) is single valued when \( F \leq 0 \) and admits a double-valued affine extension, denoted by \( G_l \) and \( G_r \), to the region \( F > 0 \), which are related to each other by the same formula as in the 2-dimensional case:
\[
G_r = G_l + kF \text{ when } F > 0, \quad G_l = G_r = G \text{ when } F \leq 0,
\]
where $k$ is some positive constant.

The set of all singular points of $\mathcal{B}$ near $O$ is a local $(n-2)$-dimensional hypersurface $S$ containing $O$, lying in the local affine subspace $\{F = 0\}$ of $B$, on which $(L_1, \ldots, L_{n-2})$ is a regular local coordinate system. We consider $G_{\text{critical}} = G|_S$ as a function of $(L_1, \ldots, L_{n-2})$, the function of critical values of $G$ near $O$. (This function is smooth but not constant, in general.) $G_l$ (respectively, $G_r$) is obtained from $G$ by affine extension from $F \leq 0$ to $F > 0$ via the paths “on the left” (respectively, “on the right”) of the critical set $S$, i.e., paths which cut the subspace $\{F = 0\}$ at points whose value of $G$ is less than (respectively, grater than) the critical value of $G$ for the same level of $(L_1, \ldots, L_{n-2})$.

Similarly to the 2-dimensional situation, for a local singular parametrized straight line $\gamma$ in dimension $n$ which contains $O$, we can distinguish two cases (see Figure 6).

Case 1. $F$ is identically zero on the line, i.e., the straight line lies on the hypersurface $\{F = 0\}$. On this hypersurface the function $G$ is single valued and, together with $(L_1, \ldots, L_{n-2})$, form a local single-valued affine coordinate system on $\{F = 0\}$. In other words, if we restrict our attention to $\{F = 0\}$, then we can forget about the singular points; $\{F = 0\}$ admits a regular affine structure compatible with the singular affine structure on $\mathcal{B}$, and (singular) affine straight lines on $\{F = 0\}$ are simply straight lines with respect to the regular affine structure on it. Such a straight line can cut the singular set $S \subset \{F = 0\}$ at one, or many, or an infinite number of points, but there is no branching. Such a (local) straight line can, of course, be constructed as a limit of (local) regular straight lines in $\mathcal{B}$: to generate a regular straight line $\gamma_c$, where $c \neq 0$ is some small constant, just keep the same values of $G, L_1, \ldots, L_{n-2}$ but change the value of $F$ from $0$ to $c$. Then $\lim_{c \to 0} \gamma_c = \gamma$ is a singular straight line.

![Figure 6. The two cases of straight lines going through focus points.](image-url)
Case 2. $F$ is identically zero on the line. Then we can assume that $\gamma$ is parametrized by $F$, i.e., $F(\gamma(t)) = t$ for every $t \in I = [-\delta, \delta]$. By passing affine equations from regular straight line to our singular straight line via the limit, we see that for each $j = 1, 2, \ldots n - 2$, there exists two positive constants $a_j, c_j$ such that

$$a_j F + L_j = b_j$$

identically on $\gamma$. Define the following local 2-dimensional subspace $P$ of $B$:

$$P = \{ x \in B \mid a_j F(x) + L_j(x) = b_j, \forall j = 1, \ldots, n - 2 \}.$$ 

$P$ is given by $n - 2$ independent linear equations and it intersects the $(n - 2)$-submanifold $S$ of focus points transversally at $O$. $P$ inherits from $B$ the structure of a local affine manifold which contains $O$ as the only singular focus point. The singular straight line $\gamma$ lies in $P$. By doing this reduction, we fall back to the 2-dimensional case with a focus singular point. So, similarly to the 2-dimensional case, $\gamma|_{[-\delta, 0]}$ is a straight line which hits a singular point $O$ and which admits exactly two different straight line extensions $\gamma^l$ and $\gamma^r$ after hitting $O$. The difference between $\gamma^l$ and $\gamma^r$ in terms of the local multi-valued affine function $G$ is the same as in the 2-dimensional case:

$$G_l(\gamma^l(t)) - G_l(\gamma^r(t)) = G_r(\gamma^l(t)) - G_r(\gamma^r(t)) = kF(t)$$

when $F(t) > 0$. The difference is 0 when $F(t) \leq 0$.

6.4. **Straight lines near a focus** $m$ **point.**

The case when a singular straight line $\gamma : [-\delta, \delta] \to B$ contains a $F^m$ point $O$ of type (focus power $m$) is similar.

We have a local multi-valued affine coordinate system

$$(F_1, G_1, \ldots, F_m, G_m, L_1, \ldots, L_{n-2m})$$

centered at $O$ ($F_i(O) = G_i(O) = L_j(O) = 0$), such that $F_1, \ldots, F_m, L_1, \ldots, L_{n-2m}$ are single-valued, $G_i$ is single-valued when $F_i \leq 0$ and is double-valued with branches $(G_i)_l$ and $(G_i)_r$ when $F_i > 0$ for each $i = 1, 2, \ldots, m$. The relation between $(G_i)_l$ and $(G_i)_r$ is $(G_i)_r = (G_i)_l + k_i F_i$ when $F_i > 0$ for some positive constants $k_i > 0$. (These numbers $k_i$ are the monodromy indices of the singular point $O$.) We also have a local smooth coordinate system

$$(F_1, H_1, \ldots, F_m, H_m, L_1, \ldots, L_{n-2m}).$$

The local singular set on $B$ is $\bigcup_{i=1}^m S_i$ where $S_i = \{ F_i = H_i = 0 \}$.

In the generic case, when none of the functions $F_1, \ldots, F_m$ is identically zero on $\gamma$, then $\gamma$ intersects the set of singular points of $B$ only at $O$, and the straight line $\gamma|_{[-\delta, 0]}$ admits exactly $2^m$ different straight line extensions after hitting $O$ (see Figure 7).

Without loss of generality, we may assume that $\gamma(\delta) = A$ is a point such that $F_1(A) < 0, \ldots, F_m(A) < 0$. For each multi-index $d = (d_1, \ldots, d_m)$, where each $d_i$ is either $l$ (left, minus) or $r$ (right, plus), and each small positive number $c$, we can define a regular straight line $\gamma_{c,d} : [-\delta, \delta] \to B$ near $O$ which satisfy the following conditions for every $t \in [-\delta, 0]$:

$$L_i(\gamma_{c,d}(t)) = L_i(\gamma(t)), \quad \forall i = 1, \ldots, n - 2m$$
and

\[ F_i(\gamma_{c,d}(t)) = F_i(\gamma(t)), \quad G_i^{d_i}(\gamma_{c,d}(t)) = G_i(\gamma(t)) + G_i(p_{c,d}), \quad \forall i = 1, \ldots, m, \]

where \( p_{c,d} \) is the point with coordinates \( F_1(p_{c,d}) = \cdots = F_m(p_{c,d}) = L_1(p_{c,d}) = \cdots = L_{n-2m}(p_{c,d}) = 0, \quad H_i(p_{c,d}) = -c \) if \( d_i \) is \( l \) and \( H_i(p_{c,d}) = c \) if \( d_i \) is \( r \). Taking the limit of \( \gamma_{c,d} \) when \( c \) tends to 0, we get \( 2^m \) branches \( \gamma_d \) of straight extensions of \( \gamma \), one for each \( d \).

If, among the values \( F_1(A), \ldots, F_m(A) \), there are only \( s \) values different from 0, and the other \( m - s \) values are equal to 0 (\( 0 \leq s < m \)), then we only have \( 2^s \) branches of straight extensions of \( \gamma \) instead of \( 2^m \) branches. In particular, if \( F_1(A) = \cdots = F_m(A) = 0 \), then \( \gamma \) lies entirely on the local \((n - m)\)-dimensional subspace \( \{ F_1(x) = \cdots = F_m(x) = 0 \} \) of \( \mathcal{B} \) with a flat affine structure (we can ignore the singular points when we restrict our attention to this subspace because all the multi-valued affine coordinate functions are single-valued there), and so \( \gamma \) admits a unique straight extension in this case.

6.5. The notions of convexity and strong convexity.

We begin with a definition.

**Definition 6.4.** A singular affine manifold \( \mathcal{B} \) (or a subset \( \mathcal{C} \subset \mathcal{B} \)) is called (globally) **convex** if for any two points \( A, B \in \mathcal{B} \) (or \( A, B \in \mathcal{C} \)) there exists a (regular or singular) straight line from \( A \) to \( B \) in \( \mathcal{B} \) (or in \( \mathcal{C} \), respectively). We say that \( \mathcal{B} \) is **locally convex** if any neighborhood of any point \( x \in \mathcal{B} \) admits a sub-neighborhood of \( x \) which is convex. Similarly, a subset \( \mathcal{C} \subset \mathcal{B} \) is called is **locally convex** if any neighborhood of any point \( x \in \mathcal{C} \) admits a sub-neighborhood of \( x \) whose intersection with \( \mathcal{C} \) is convex.

Of course, any globally convex singular affine manifold is locally convex, but the converse is not true, in general.

The above notion of convexity using straight lines is very natural and is a generalization of the notion of **intrinsic convexity** (see [142]) from the regular case to the singular case. When \( \mathcal{B} \subset \mathbb{R}^n \), this Definition 6.4 is equivalent to the usual definition of convexity.
In the Euclidean space, every (locally) convex set is automatically (locally) path-connected, and the intersection of two (locally) convex sets is again a (locally) convex set.

![Figure 8. “Bad” intersections of convex sets near a focus point.](image)

In a singular affine manifold, a (locally) convex set is again automatically (locally) path-connected (because we can go from one point to another by a straight line). However, due to singular points and monodromy, the intersection of two convex sets is neither necessarily locally convex nor connected, as shown in Figure 8.

So, in order to make sure that the intersection of two sets with convexity properties still inherit convexity properties, we need a stronger notion of convexity for sets in singular affine manifolds.

**Definition 6.5.** (i) A subset $C$ of a singular convex affine manifold $B$ is called **strongly convex** in $B$ if $C$ is convex, i.e., any two distinct points $A, B \in C$ can be joined by a straight line which lies in $C$, and, moreover, any straight line going from $A$ to $B$ in $B$ also lies in $C$.

(ii) We say that $C \subset B$ is **strongly locally convex** in $B$ if any neighborhood $U$ of any point $x \in C$ admits a sub-neighborhood $V$ of $x$ such that $C \cap V$ is strongly convex in $V$. In other words, for any two distinct points $A, B \in C \cap V$ there is a straight line going from $A$ to $B$ in $C \cap V$, and, moreover, any straight line going from $A$ to $B$ in $V$ also lies in $C$.

**Remark 6.6.** Recall that, because of monodromy, given two points in $B$, there could be more than one straight line segment in the affine structure of $B$ linking these two points. So Definition 6.4 requires only that at least one such segment lies in $C$, whereas Definition 6.5 requires that all such segments lie in $C$.

**Proposition 6.7.** (i) The intersection of two strongly convex sets in a singular affine manifold $B$ is again a strongly convex set in $B$.

(ii) The intersection of two strongly locally convex sets in a singular affine manifold $B$ is again a strongly locally convex set in $B$.

*Proof.* The proof is straightforward.
(i) Let $C_1, C_2$ be two strongly convex subsets of $B$, and $A, B \in C_1 \cap C_2$. Let $\gamma$ be a straight line from $A$ to $B$ in $C_1$ ($\gamma$ exists because $C_1$ is convex). Then $\gamma \subset C_2$ because $C_2$ is strongly convex, so $\gamma \subset C_1 \cap C_2$, i.e., there is at least one straight line from $A$ to $B$ in $C_1 \cap C_2$. Moreover, if $\gamma'$ is any other straight line from $A$ to $B$ in $B$, then $\gamma' \subset C_i$ ($i = 1, 2$) because $C_i$ is strongly convex, and so $\gamma' \subset C_1 \cap C_2$.

(ii) Let $C_1, C_2$ be two strongly locally convex subsets of $B$, $x \in C_1 \cap C_2$, and $U$ is an arbitrary neighborhood of $x$ in $B$. By definition, there is a neighborhood $V_1 \subset U$ of $x$ such that $C_1 \cap V_1$ is strongly convex in $V_1$, and a neighborhood $V \subset V_1$ of $x$ such that $C_2 \cap V$ is strongly convex in $V$. Then $(C_1 \cap C_2) \cap V$ is strongly convex in $V$.

Indeed, let $A, B \in (C_1 \cap C_2) \cap V$ be two arbitrary distinct points. Since $C_2 \cap V$ is convex, there is a straight line $\gamma$ going from $A$ to $B$ in $C_2 \cap V$. Since $\gamma \subset V \subset V_1$, $A, B \in C_1 \cap V_1$ and $C_1 \cap V_1$ is strongly convex in $V_1$, we have that $\gamma \subset C_1 \cap V_1 \subset C_1$. Hence $\gamma \subset C_1 \cap C_2 \cap V$. Let $\gamma'$ be any other straight line going from $A$ to $B$ in $V \subset V_1$. Then, since $C_1 \cap V_1$ is strongly convex in $V_1$ and $C_2 \cap V$ is strongly convex in $V$, we have that $\gamma' \subset C_1 \cap V_1$ and $\gamma' \subset C_2 \cap V$, and hence $\gamma' \subset (C_1 \cap C_2) \cap V$. \qed

**Remark 6.8.** In for regular affine manifolds there is an notion of convexity, which requires the universal covering of the universal covering of the manifold to be convex, see, e.g.

7. **Local convexity at focus points**

7.1. **Convexity of focus boxes in dimension 2.**

Let $O$ be a focus singular point of index $k > 0$ in a singular 2-dimensional affine manifold $B^2$ with a local multi-valued affine coordinate system $F, G$: $F(0) = G(0) = 0$, $F$ is single valued, $G$ is multivalued: when $F \leq 0$ then $G$ is single valued but when $F > 0$ then $G$ has 2 branches $G_l$ and $G_r$ (extension of $G$ from the region $\{F \leq 0\}$ to the region $\{F > 0\}$ by the left or by the right of the singular point $O$), with the monodromy formula $G_r = G_l + kF$ when $F > 0$.

Let $\delta, \delta' > 0$ be two sufficiently small positive numbers. The closed neighborhood $Box = Box(\delta, \delta')$ of $O$ defined by the inequalities

$$Box = \{ x \text{ such that } -\delta \leq F(x) \leq \delta, -\delta' \leq G_l(x), G_r(x) \leq \delta' \}$$

is called a **focus box**, i.e., a box with one focus point in it.

**Remark.** The box is a quadrilateral figure (i.e., a figure whose boundary consists of four straight lines) if $2\delta' > k\delta$; for simplicity of the exposition we assume that this is the case. If $2\delta' \leq k\delta$ then the box is a triangle, but our subsequent arguments based on the quadrilateral figure are easily seen to apply to this case too.

**Theorem 7.1.** Any 2-dimensional focus box is convex.

**Proof.** We give two simple proofs.

**Proof 1.** Let $A$ be an arbitrary point in the box. We show that the box is $A$-star shaped, i.e., every other point can be connected to $A$ by a straight line in the box.
If $A$ lies on the line $\{F = 0\}$, then cut the box in two closed parts by the line $\{F = 0\}$ (see Figure 9). Each part is affinely isomorphic to a convex polygon which contains $A$ on the boundary. (The part $\{F \leq 0\}$ is a rectangle and the part $\{F \geq 0\}$ is a trapezoid.) So both parts are $A$-star shaped and their union is also $A$-star shaped, which proves the claim.

![Figure 9. Cutting the box into two convex polygons with $A$ (and $O$) on the boundary.](image)

If $F(A) \neq 0$ then the straight line $AO$ admits two different straight extensions at $O$. Just take any of these two and keep extending this line on both ends so that the obtained straight line cuts the box in two parts (see Figure 9). Again each part is a convex polygon with $A$ on the boundary, which proves the claim.

**Proof 2.** Let $A$ and $B$ be two arbitrary points in the box. We have to find a straight line in the box going from $A$ to $B$.

- If $F(A) \leq 0$ and $F(B) \leq 0$, then $A$ and $B$ lie in the rectangle $\{F \leq 0\}$ and in this case there is a unique straight line going from $A$ to $B$ consisting of points $\gamma(t), \ 0 \leq t \leq 1$, such that $F(\gamma(t)) = tF(B) + (1-t) F(A)$ and $G(\gamma(t)) = tG(B) + (1-t) G(A)$. (The uniqueness follows from the fact that $F$ is single valued and affine on the line, so $F$ is always smaller than or equal to 0 on the line, and when $F \leq 0$, then there is no monodromy, so $G$ is single valued and is affine on the line).

- If $F(A) \geq 0$ and $F(B) \leq 0$, then $A$ and $B$ lie in the trapezoid $\{F \geq 0\}$, and in this case there is a unique straight line going from $A$ to $B$, consisting of points $\gamma(t), \ 0 \leq t \leq 1$, such that $F(\gamma(t)) = tF(B) + (1-t) F(A)$ and $G_l(\gamma(t)) = tG_l(B) + (1-t) G_l(A)$ (or, equivalently, $G_r(\gamma(t)) = tG_r(B) + (1-t) G_r(A)$).

- The more tricky situation is when $F(A) < 0$, $F(B) > 0$, or vice versa. In this case, there exists at least one, but maybe two different straight lines from $A$ to $B$ in the box. We construct up to two different straight lines from $A$ to $B$, denoted by $\gamma_l : [0, 1] \to \Box$ and $\gamma_r : [0, 1] \to \Box$. The equations for the points of $\gamma_l : [0, 1] \to \Box$ are

$$F(\gamma_l(t)) = tF(B) + (1-t) F(A); \ G_l(\gamma_l(t)) = tG_l(B) + (1-t) G_l(A)$$

and the equations for the points of $\gamma_r : [0, 1] \to \Box$ are

$$F(\gamma_r(t)) = tF(B) + (1-t) F(A); \ G_r(\gamma_r(t)) = tG_r(B) + (1-t) G_r(A)$$

for all $t \in [0, 1]$; when $F \leq 0$ then $G_l = G_r = G$. 
In particular, at the point \( t_0 = \frac{-F(A)}{F(B) - F(A)} \), we have \( F(\gamma_r(t_0)) = 0 \), \( G_l(\gamma_l(t_0)) = t_0 G_l(B) + (1 - t_0) G(A) \), and \( G_r(\gamma_r(t_0)) = t_0 G_r(B) + (1 - t_0) G(A) \).

The lines \( \gamma_l \) and \( \gamma_r \) can be defined on the interval \([0, t_0]\) (for \( F \) going from \( F(A) \) to 0) without any problem. Indeed, since the values of \( G \) remain in the interval \([-\delta', \delta']\), this line cannot leave the box. However, at \( t_0 \) we may run into problems.

- \( G_l \) is the extension of \( G \) from the left of \( O \), i.e., through points on \( \{ F = 0 \} \), where \( G \) has negative values. Thus, if \( t_0 G_l(B) + (1 - t_0) G(A) > 0 \), then \( G_l \) is the wrong function to use and the equations \( F(\gamma_l(t)) = t F(B) + (1 - t) F(A) \), \( G_l(\gamma_l(t)) = t G_l(B) + (1 - t) G_l(A) \) do not give a straight line, but rather a broken line at \( t_0 \). So, in order for \( \gamma_l \) to be a straight line from \( A \) to \( B \), we need the condition \( t_0 G_l(B) + (1 - t_0) G(A) \leq 0 \) (which is a necessary and sufficient condition).

- Similarly, in order for \( \gamma_r \) to be a straight line from \( A \) to \( B \), we need the condition \( t_0 G_r(B) + (1 - t_0) G(A) \geq 0 \).

Note that \( G_r(B) = G_l(B) + k F(B) > G_l(B) \), and so \( t_0 G_r(B) + (1 - t_0) G(A) > t_0 G_l(B) + (1 - t_0) G(A) \). We have three different possibilities (see Figure 10).

![Figure 10. The three cases in a focus box.](image)

i) \( t_0 G_l(B) + (1 - t_0) G(A) > 0 \): then, automatically \( t_0 G_r(B) + (1 - t_0) G(A) > 0 \), and we have exactly one straight line from \( A \) to \( B \) in the box, which is \( \gamma_r \).

ii) \( t_0 G_r(B) + (1 - t_0) G(A) < 0 \): then automatically \( t_0 G_l(B) + (1 - t_0) G(A) < 0 \), and we have exactly one straight line from \( A \) to \( B \) in the box, which is \( \gamma_l \).

iii) \( t_0 G_l(B) + (1 - t_0) G(A) \leq 0 \) and \( t_0 G_r(B) + (1 - t_0) G(A) \geq 0 \): then we have two different straight lines from \( A \) to \( B \) in the box, which are \( \gamma_l \) and \( \gamma_r \). \( \square \)

**Remark 7.2.** The number \( k \) in the monodromy formula (\( G_r = G_l + k F \) when \( F > 0 \)) can be any positive number (not necessarily an integer) and the focus box is still convex, as the proof above shows. However, the requirement \( k > 0 \) is essential: if \( k < 0 \) then the box is non-convex, because there exist points \( A \) and \( B \) in the box such that \( F(A) < 0, F(B) > 0 \), \( t_0 G_l(B) + (1 - t_0) G(A) > 0 \), and \( t_0 G_r(B) + (1 - t_0) G(A) < 0 \), where \( t_0 = \frac{-F(A)}{F(B) - F(A)} \).

7.2. Convexity of focus boxes in higher dimensions.

Theorem 7.1 can be extended to the \( n \)-dimensional case in a straightforward manner.
Let $O$ be a focus singular point of index $k > 0$ in a singular $n$-dimensional affine manifold $\mathcal{B}$ ($n \geq 3$) with a local multi-valued affine coordinate system $F, G, L_1, \ldots, L_{n-2}$; $F(O) = G(O) = L_1(O) = \cdots = L_{n-2}(O) = 0$, $F, L_1, \ldots, L_{n-2}$ are single valued, and $G$ is multivalued: when $F \leq 0$ then $G$ is single valued but when $F > 0$ then $G$ has 2 branches $G_l$ and $G_r$ (extensions of $G$ from the region $\{F \leq 0\}$ to the region $\{F > 0\}$ on the left or on the right of the $(n-1)$-dimensional manifold $\mathcal{S} \subset \{F = 0\}$ of focus points, $O \in \mathcal{S}$), with the monodromy formula $G_r = G_l + kF$ when $F > 0$ and $G_r = G_l = G$ when $F \leq 0$.

Similarly to the 2-dimensional case, we define a **focus box** around the focus points $O$ to be the set of all points $x$ in a neighborhood of $O$ in $\mathcal{B}$ which satisfies the inequalities

$$-\delta \leq F(x) \leq \delta, \quad -\delta' \leq G_l(x), \quad G_r(x) \leq \delta', \quad -\delta_1 \leq L_1(x) \leq \delta_1, \ldots, -\delta_{n-2} \leq L_{n-2} \leq \delta_{n-2},$$

where $\delta, \delta', \delta_1, \ldots, \delta_{n-2}$ are arbitrary sufficiently small positive numbers. We do this if $O$ is an interior point of $\mathcal{B}$.

If $O$ lies on the boundary of $\mathcal{B}$, where $\mathcal{B}$ comes from an integrable Hamiltonian system whose singularities are non-degenerate and have only elliptic and/or focus-focus components, then the singularity of the integrable system at $O$ has $e$ elliptic components for some positive number $e$, and we can assume that $L_1, \ldots, L_e$ are the action functions corresponding to the $e$ elliptic components at $O$, $L_1, \ldots, L_e \geq 0$ near $O$ (and the other functions $L_{e+1}, \ldots, L_{n-2}$ can admit both negative and positive values near $O$). On this boundary (i.e., elliptic$^e$-focus-focus case), the above inequalities for $F, G, L_i$ still define a box around $O$, and for $i = 1, \ldots, e$ we can replace the inequalities $-\delta_i \leq L_i(x) \leq \delta_i$ by the inequalities $0 \leq L_i(x) \leq \delta_i$.

**Theorem 7.3.** Any $n$-dimensional focus box is convex.

**Proof.** We simply repeat the second proof of Theorem 7.1, with $L_1, \ldots, L_{n-2}$ added to the picture. Let $A$ and $B$ be two arbitrary points in the box. We have to find a straight line in the box going from $A$ to $B$. 

![Figure 11. Two straight lines (the case $k > 0$) vs. no straight line ($k < 0$) from $A$ to $B$.](image-url)
- If \( F(A) \leq 0 \) and \( F(B) \leq 0 \) then \( A \) and \( B \) lie in the (affinely flat) convex polytope \( \{x \in Box, F(x) \leq 0\} \). In this case, there is a unique straight line going from \( A \) to \( B \), consisting of points \( \gamma(t) \), \( 0 \leq t \leq 1 \), such that \( F(\gamma(t)) = tF(B) + (1 - t)F(A) \), \( G(\gamma(t)) = tG(B) + (1 - t)G(A) \), and \( L_i(\gamma(t)) = tL_i(B) + (1 - t)L_i(A) \) for all \( i = 1, \ldots, n - 2 \). (This straight line is unique for the same reasons as in the 2-dimensional case).

- Similarly, if \( F(A) \geq 0 \) and \( F(B) \leq 0 \), then \( A \) and \( B \) lie in the affinely flat convex polytope \( \{x \in Box, F(x) \geq 0\} \), and in this case, there is a unique straight line going from \( A \) to \( B \), consisting of points \( \gamma(t) \), \( 0 \leq t \leq 1 \) such that \( F(\gamma(t)) = tF(B) + (1 - t)F(A) \), \( L_i(\gamma(t)) = tL_i(B) + (1 - t)L_i(A) \) for all \( i = 1, \ldots, n - 2 \), and \( G_i(\gamma(t)) = tG_i(B) + (1 - t)G_i(A) \) (or, equivalently, \( G_r(\gamma(t)) = tG_r(B) + (1 - t)G_r(A) \)).

- The more tricky situation is when \( F(A) < 0 \) and \( F(B) > 0 \) or vice versa. In this case, there exists at least one, but maybe two different straight lines from \( A \) to \( B \) in the box. We construct up to two different straight lines from \( A \) to \( B \), denoted by \( \gamma_l : [0, 1] \to B \) and \( \gamma_r : [0, 1] \to Box \). The equations for the points of \( \gamma_l : [0, 1] \to Box \) are (for \( t \in [0, 1] \)):

\[
L_i(\gamma_l(t)) = tL_i(B) + (1 - t)L_i(A), \quad \forall i = 1, \ldots, n - 2,
\]

\[
F(\gamma_l(t)) = tF(B) + (1 - t)F(A), \quad G_l(\gamma_l(t)) = tG_l(B) + (1 - t)G_l(A),
\]

and the equations for the points of \( \gamma_r : [0, 1] \to Box \) are

\[
L_i(\gamma_r(t)) = tL_i(B) + (1 - t)L_i(A) \quad \forall i = 1, \ldots, n - 2,
\]

\[
F(\gamma_r(t)) = tF(B) + (1 - t)F(A), \quad G_r(\gamma_r(t)) = tG_r(B) + (1 - t)G_r(A).
\]

In particular, at the point \( t_0 = \frac{-F(A)}{F(B) - F(A)} \), we have \( F(\gamma_r(t_0)) = 0 \), \( G_l(\gamma_l(t_0)) = 0 \), \( G_l(\gamma_l(t_0)) = t_0G_l(B) + (1 - t_0)G_l(A) \), and \( L_i(t_0) = c_i \in [-\delta_i, \delta_i] \) (\( i = 1, \ldots, n - 2 \)) are some small numbers.

The line \( \{x \mid F(x) = 0, L_1(x) = c_1, \ldots, L_{n-2}(x) = c_{n-2}\} \) contains exactly one focus point, denoted by \( O_{A,B} \), and the value \( g_{A,B}^{\text{critical}} = G(O_{A,B}) \) is called the critical value of \( G \) (on this line, or with respect to \( A \) and \( B \)).

The lines \( \gamma_l \) and \( \gamma_r \) can be defined on the interval \([0, t_0]\) (for \( F \) going from \( F(A) \) to 0) without any problem, because the values of the functions \( F, G, L_i \) always stay in the intervals of allowed values for the box; hence these values cannot leave the box. However, at \( t_0 \) we may run into problems.

- \( G_l \) is the extension of \( G \) from the left of \( O_{A,B} \), i.e., through points on \( \{F = 0, L_1 = c_1, \ldots, L_{n-2} = c_{n-2}\} \), where \( G \) has values less than the critical value. Thus, if \( t_0G_l(B) + (1 - t_0)G(B) > g_{A,B}^{\text{critical}} \), then \( G_l \) is the wrong function to use: the equations \( F(\gamma_l(t)) = tF(B) + (1 - t)F(A) \), \( G_l(\gamma_l(t)) = tG_l(B) + (1 - t)G_l(A) \) do not give a straight line but rather a line which is broken at \( t_0 \). So in order for \( \gamma_l \) to be a straight line from \( A \) to \( B \), we need the condition \( t_0G_l(B) + (1 - t_0)G(A) \leq g_{A,B}^{\text{critical}} \) (which is necessary and sufficient).

- Similarly, in order for \( \gamma_r \) to be a straight line from \( A \) to \( B \), we need the condition \( t_0G_r(B) + (1 - t_0)G(A) \geq g_{A,B}^{\text{critical}} \).
Note that $G_r(B) = G_l(B) + kF(B) > G_l(B)$, and so $t_0G_r(B) + (1 - t_0)G(A) > t_0G_l(B) + (1 - t_0)G(A)$. We have three different possibilities:

i) $t_0G_l(B) + (1 - t_0)G(A) > g^{\text{critical}}_{A,B}$, then, automatically, $t_0G_r(B) + (1 - t_0)G(A) > g^{\text{critical}}_{A,B}$, and we have exactly one straight line from $A$ to $B$ in the box, which is $\gamma_r$.

ii) $t_0G_r(B) + (1 - t_0)G(A) < g^{\text{critical}}_{A,B}$, then, automatically, $t_0G_l(B) + (1 - t_0)G(A) < g^{\text{critical}}_{A,B}$, and we have exactly one straight line from $A$ to $B$ in the box, which is $\gamma_l$.

iii) $t_0G_l(B) + (1 - t_0)G(A) \leq g^{\text{critical}}_{A,B}$ and $t_0G_r(B) + (1 - t_0)G(A) \geq g^{\text{critical}}_{A,B}$; then we have two different straight lines from $A$ to $B$ in the box, which are $\gamma_l$ and $\gamma_r$. \hfill \Box

**Remark 7.4.** Another proof of Theorem 7.3 can be given by the use of appropriate $n - 2$ independent affine equations to define a 2-dimensional singular affine “subbox”

$$P = \{ x \in \text{Box} \mid a_jF(x) + L_j(x) = b_j \forall j = 1, \ldots, n - 2 \}$$

of $\text{Box}$, which contains $A$, $B$, and $O_{A,B}$ as the only focus point in $P$. This reduces the problem to the 2-dimensional case already treated in the previous subsection.

**Remark 7.5.** The fact that the local $(n - 2)$-submanifold of focus points near $O$ lies on $\{ F = 0 \}$ is important in the proof of Theorem 7.3. If focus points were able to move more freely, one would be able to construct counter-examples (where both choices $\gamma_l$ and $\gamma_r$ in the construction of a straight line from $A$ to $B$ are bad choices).

### 7.3. Existence of non-convex focus$^m$ boxes.

Toric-focus systems may lose convexity due to the presence of a focus$^m$ singularity.

**Theorem 7.6.** For any $m \geq 2$ and $n \geq 2m$, there exists a toric-focus integrable system with $n$ degrees of freedom whose base space $\mathcal{B}^n$ admits an interior singular point $O$ with $m$ focus components (and no elliptic component) with a corresponding multi-valued system of action coordinates $(F_1, G_1, \ldots, F_m, G_m, L_1, \ldots, L_{n-2m})$ (where $F_1, \ldots, F_m, L_1, \ldots, L_{n-2m}$ are smooth single valued and each function $G_i$ has two branches $(G_i)_l$ and $(G_i)_r$), and a number $\delta > 0$ such that the focus$^m$ box

$$\{|F_i|, |L_j| \leq \delta \forall i, j; -\delta \leq (G_i)_l \& (G_i)_r \leq \delta, \forall i\}$$

around $O$ is locally convex at its boundary points, but is not convex.

**Proof.** We prove for the case $m = 2$ and $n = 4$. The other cases are obtained from this case by simply taking a symplectic direct product.

Let $O$ be a singular point of type focus square in the four dimensional base space $\mathcal{B}$ of a toric-focus integrable system with four degrees of freedom. We assume that the singularity of the system over $O$ is topologically a direct product of two elementary focus-focus singularities of index 1. So we have a local smooth coordinate system $(F_1, H_1, F_2, H_2)$ and a local multi-valued system of action coordinates $(F_1, G_1, F_2, G_2)$ as described in Subsection 5.3.

Take a number $\delta > 0$ and consider the box

$$\text{Box} = \{ x \in \mathcal{B} \mid |F_i(x)| < \delta; -\delta \leq (G_i)_l(x); (G_i)_r(x) \leq \delta, \forall i = 1, 2 \}$$
containing \(O\) in \(B\). According to Theorem 7.3, this box is locally convex at its boundary points. We want to show that the system can be chosen in such a way that our box is not convex.

If the decomposition of the given (focus-focus)\(^2\) singularity of the integrable Hamiltonian system into a product of two elementary focus-focus singularities is not only topological but also symplectic, then the local base space near \(O\) is also a direct product of two affine 2-dimensional manifolds (each one with a focus-focus singular point), and then the convexity of \(B\) near \(O\) is obvious, because it is clear that the direct product of two convex (singular) affine manifolds is convex. However, in general, this local product decomposition of \(B\) is only topological but not affine and that’s why we may run into trouble.

Let us recall that the set of singular points in the box is the union \(S_1 \cup S_2\) of two 2-dimensional quadrilateral disks intersecting each other transversally at \(O\), each disk being given by the formula

\[
S_i = \{ x \in Box \mid F_i(x) = H_i(x) = 0 \}.
\]

For each \(i = 1, 2\), the function \(G_i\) is single-valued when \(F_i \leq 0\) but it has two branches \((G_i)_l\) and \((G_i)_r\) when \(F_i > 0\); \((G_i)_l = (G_i)_r = G_i\) when \(F_i \leq 0\). These two branches of \(G_i\) are related by the monodromy formula \((G_i)_r = (G_i)_l + F_i\) when \(F_i > 0\).

Take two different points \(A\) and \(B\) in the box, with \(F_1(A) < 0\), \(F_2(A) < 0\) and \(F_1(B) < 0\), \(F_2(B) < 0\). Then, potentially, there may be up to four different straight lines in the box going from \(A\) to \(B\), each line corresponding to one of the four choices of the value of the couple \((G_1, G_2)\) at \(B\). (This couple is single-valued at \(A\.)\) Define four (straight or broken piecewise straight) lines

\[
\gamma^{l,r}_l, \gamma^{l,r}_r, \gamma^{r,l}_l, \gamma^{r,l}_r : [0, 1] \to Box
\]

which go from \(A\) to \(B\) by the following formulas:

\[
F_1(\gamma^{l,r}_l(t)) = tF_1(B) + (1-t)F_1(A),
\]
\[
F_2(\gamma^{l,r}_r(t)) = tF_2(B) + (1-t)F_2(A),
\]
\[
(G_1)_l(\gamma^{l,r}_l(t)) = t(G_1)_l(B) + (1-t)G_1(A),
\]
\[
(G_2)_r(\gamma^{l,r}_r(t)) = t(G_2)_r(B) + (1-t)G_2(A)
\]

for \(\gamma^{l,r}_r\), and similarly for the other three lines.

It is easy to see that these four lines lie inside the box, because the inequalities defining the box are satisfied on all of them. If at least one of these four lines is a straight line (without any breaking, i.e., changing of direction in the middle of the line) then we are done. But similarly to the 2-dimensional cases, some of them may be broken. Actually, as we will see, it may happen that all of them are broken and, in that case, we cannot join \(A\) and \(B\) by a straight line.

Look at \(\gamma^{r,l}_r\), for example. It is broken at the intersection with the hyperplane \(\{F_1 = 0\}\), if it intersects \(\{F_1 = 0\}\), on the “left side” (i.e., the side given by \(\{H_1 < 0\}\), which is the “wrong side” for \(\gamma^{r,l}_r\) of the surface \(S_1\) of focus-focus points on \(\{F_1 = 0\}\) (i.e., the side with lower \(G_1\)-values than on \(S_1\)). It can also be broken at the intersection
with the hyperplane \( \{ F_2 = 0 \} \), if it intersects \( \{ F_2 = 0 \} \), on the “right side” (i.e., the side given by \( \{ H_2 > 0 \} \)), which is again the “wrong side” for \( \gamma^{r,l} \) of the surface \( S_2 \) of focus-focus points on \( \{ F_2 = 0 \} \) (i.e., the side with higher \( G_2 \)-values than on \( S_2 \)). See Figure 12.

Let us assume, for example, that \( \gamma^{l,l} \) is broken at \( \{ F_1 = 0 \} \). Then take \( \gamma^{r,l} \) instead of \( \gamma^{l,l} \); by arguments absolutely similar to the 2-dimensional situation, \( \gamma^{r,l} \) is not broken at \( \{ F_1 = 0 \} \) in this case. But \( \gamma^{r,l} \) can be broken at \( \{ F_2 = 0 \} \). If \( \gamma^{r,l} \) is broken at \( \{ F_2 = 0 \} \), then \( \gamma^{r,r} \) is not broken at \( \{ F_2 = 0 \} \), but then \( \gamma^{r,r} \) can be broken at \( \{ F_1 = 0 \} \). If \( \gamma^{r,r} \) is broken at \( \{ F_1 = 0 \} \), we take \( \gamma^{l,r} \) which is not broken at \( \{ F_1 = 0 \} \), but then \( \gamma^{l,r} \) can be broken at \( \{ F_2 = 0 \} \), in which case we try to take \( \gamma^{l,l} \) to avoid being broken at \( \{ F_2 = 0 \} \). However, we already know that \( \gamma^{l,l} \) is broken at \( \{ F_1 = 0 \} \). See again Figure 12.

So, at least theoretically, it may happen that all of our four choices go wrong: all of the lines \( \gamma^{l,l}, \gamma^{l,r}, \gamma^{r,l}, \gamma^{r,r} \) are broken somewhere. More specifically, up to a permutation of indices, \( \gamma^{l,l} \) and \( \gamma^{r,r} \) are broken at \( \{ F_1 = 0 \} \), while \( \gamma^{l,r} \) and \( \gamma^{r,l} \) are broken at \( \{ F_2 = 0 \} \). If this happens, then our box is not convex.

Such a situation (where all the possible four choices go wrong) can actually happen. For example, we may have \( F_1(A) = F_2(A) = -\delta, F_1(B) = F_2(B) = \delta \), the four lines \( \gamma^{l,l}, \gamma^{l,r}, \gamma^{r,l}, \gamma^{r,r} \) intersect the plane \( \{ F_1 = F_2 = 0 \} \) (with affine coordinates \( (G_1, G_2) \)) at four respective points \( E^{l,l}, E^{l,r}, E^{r,l}, E^{r,r} \) as shown in Figure 12, where the curves \( H_1 = 0 \) and \( H_2 = 0 \) (i.e., the curves of critical values for \( G_1 \) and \( G_2 \), respectively) are also shown. One can see from this picture that all the four choices in the situation on it are wrong choices, so there is no straight line from \( A \) to \( B \). By general results on integrable surgery (the method for constructing integrable system by glueing small pieces together, see [140]: one first glues the base spaces, and then lift the glueing to Lagrangian torus fibrations provided that the cohomological obstruction vanishes; the cohomological obstruction lies in the so called Lagrange-Chern class, which automatically vanishes.

\[ \text{Figure 12. A situation when all the choices are wrong choices.} \]
when the base space is simple enough), there is no obstruction to the creation of an integrable Hamiltonian system with such a picture. □

8. Global convexity

8.1. Local-global convexity principle.

Let us recall the following two lemmas from [142] for regular affine manifolds (with boundary, but without focus singularities), which are along the lines of the well-known local-global convexity principle (see, e.g., [29, 62]).

Lemma 8.1. Let \( X \) be a connected compact locally convex regular affine manifold (with boundary) and \( \phi : X \to \mathbb{R}^m \) a locally injective affine map, where \( \mathbb{R}^m \) is endowed with the standard affine structure. Then \( \phi \) is injective and its image \( \phi(X) \) is convex in \( \mathbb{R}^m \).

Lemma 8.2. Let \( X \) be a connected locally convex regular affine manifold (with boundary) and \( \phi : X \to \mathbb{R}^m \) a proper locally injective affine map, where \( \mathbb{R}^m \) is endowed with the standard affine structure. Then \( \phi \) is injective and its image \( \phi(X) \) is convex in \( \mathbb{R}^m \).

A simplified version of Lemma 8.1 is the following lemma, whose proof we recall because the method (of stars) is useful also for the case of singular affine structures.

Lemma 8.3. Let \( C \subset \mathbb{R}^m \) be a compact connected locally convex set. Then \( C \) is convex.

Proof. If \( x, y \in \mathbb{R}^m \) denote \( [x, y] := \{tx + (1-t)y \mid t \in [0,1] \} \) the straight line segment with extremities \( x \) and \( y \), respectively. For any \( x \in C \) define the star \( S(x) := \{y \in C \mid [x, y] \subset C\} \) of \( x \). It is clear that \( C \) is convex if and only if \( S(x) = C \) for all \( x \in C \). So, fix an arbitrary \( x \in C \). We will show that \( S(x) \) is both open and closed; connectedness of \( C \) guarantees then that \( S(x) = C \).

To show that \( S(x) \) is closed, let \( y_n \to y \), where \( \{y_n\}_{n \in \mathbb{N}} \subset S(x) \). Since \( C = \overline{C} \), it follows that \( y \in C \). This then implies that for any \( t \in [0,1] \), \( tx + (1-t)y_n \to tx + (1-t)y \). However, \( tx + (1-t)y_n \in [x, y_n] \subset C \), by the definition of \( S(x) \), and hence \( [x, y] \subset C \) since \( C = \overline{C} \).

We now show that \( S(x) \) is open, i.e., for any \( y \in S(x) \) there is some \( \varepsilon > 0 \) such that \( B(y, \varepsilon) \cap C \subset S(x) \), where \( B(y, \varepsilon) \) is the open ball in \( \mathbb{R}^m \) centered at \( y \) of radius \( \varepsilon \).

Figure 13. Piling up of small intervals.
Since $C$ is locally convex, for any $z \in C$, there is some $\delta(z) > 0$ such that $B(z, \delta(z)) \cap C$ is convex. By compactness of $C$, a finite number of such open balls covers $C$; call them $B(z_1, \delta_1), \ldots, B(z_p, \delta_p)$. Since $y \in S(x)$, we have $[x, y] \subset C$. Let $\delta = \min\{\delta_1, \ldots, \delta_p\}$ and divide the segment $[x, y]$ into parts each one of length $< \delta/2$. Call the extremities of this subdivision $y_0 := x, y_1, \ldots, y_{r-1}, y_r := y$. Let $z \in B(y, \delta')$, where $\delta' < \delta$ and $C \cap B(y, \delta')$ is convex, by local convexity of $C$. Therefore, the triangle with vertices $y_{r-1}, z, y_r$ is contained in $C$. Now use the definition of local convexity at the point $y_{r-1}$ to conclude that the triangle whose vertices are the middle segment of $y_{r-2}, y_{r-1}$ and the middle of the segment $[y_{r-1}, z]$ is contained in $C$. Now keep doing this construction using local convexity of $C$ at all vertices of the triangles, except $z$, till one reaches $y_0 = x$ (see Figure 13). Take the limit of the upper sides of these triangles; since each segment is in $C$ and $C = \overline{C}$, we conclude that the limit lies in $C$. However, this limit is $[x, z]$, which shows that $z \in S(x)$. \hfill $\square$

For integral affine manifolds with focus points, the local-global convexity principle no longer works, and indeed, as we will see, there are many examples of locally convex but globally non-convex integral affine manifolds with focus points (which are base spaces of toric-focus integrable systems), even in dimensions 2 and 3.

However, when there is only one focus point, then the local-global convexity principle still holds, as the following propositions show. Recall that a 2-dimensional focus box is an affine convex quadrilateral figure with one focus point in it.

**Proposition 8.4.** Let $B$ be a 2-dimensional focus box and $C$ a closed connected subset of $B$ which is strongly locally convex in $B$. Then $C$ is convex. If, moreover, $C$ contains the focus point then it is strongly convex in $B$.

**Proof.** Since $B$ is a focus box, it is compact. Denote by $O$ the focus point in $B$.

**Case 1:** $O \notin C$: It is not possible to invoke Lemma 8.3, because $C$ is not flat, only locally flat. However, the proof of Lemma 8.3 works. For any $x \in C$ define $S(x) = \{y \in C \mid [x, y] \subset C\}$ to be the $x$-star in $C$. Since none of the segments in the definition of $S(x)$ contains $O$, one can repeat the proof of Lemma 8.3, even though there may be more than one straight line segment joining $x$ to $y$ in the affine structure of $B$.

**Case 2:** $O \in C$: Cut $B$ into two parts $B_+ = \{F \geq 0\}$ and $B_- = \{F < 0\}$, where $F$ is the smooth single valued affine coordinate of the box, $F(O) = 0$. Then $B_+$ and $B_-$ are affinely isomorphic to convex polygons in $\mathbb{R}^2$ (if we forget about the singular point $O$), and $C_+ = C \cap B_+$ and $C_- = C \cap B_-$ are locally convex, hence they are disjoint union of convex sets.

The intersection $D = C \cap \{F = 0\} \ni O$ is also locally convex, and so is a finite union of closed intervals. If $D$ is not connected, say $D$ has connected components $D_1, D_2, \ldots$, then $D_1$ must be connected to $D_2$ by a path in $C_+$ or $C_-$ because $C$ is connected (and hence path connected, because it is also locally convex). But then $C_+$ or $C_-$ will not be convex, contradiction.

Denote by $x$ and $y$ the two end points of $D$, $x$ is on the left and $y$ is on the right of $O$ (or equal to $O$). Then, because of local convexity at $x$ and $y$ and convexity of
$C_+$ and $C_-$, there are straight lines $\ell_t$ and $\ell_r$, which pass through $x$ and $y$ respectively, such that $C$ lies on the right of $\ell_t$ and on the left of $\ell_r$. If, for example, $y = O$ then $\ell_r$ is a straight line “coming from the right”. It means that, in the multi-valued affine coordinate system $(F,G)$ of the box, where $G$ has 2 branches $G_1$ and $G_r$, $\ell_t$ is given by an affine equation of the type $G_1 + a_t F = b_t$ while $\ell_t$ is given by an affine equation of the type $G_r + a_r F = b_r$.

If $p, q$ are two arbitrary different points in $C$, and they are both in $C_+$ or both in $C_-$, then there is only one straight line from $p$ to $q$ in $B$ and that straight line also lies in $C$.

Consider the opposite case, when $F(p) < 0$ and $F(q) > 0$ (or vice versa). Then for both potential straight lines $\gamma_t$ from $p$ to $q$ constructed in Subsection 7.1, we have that the intersection of this (straight or broken) line lies on the right of $x$ (because both points $p$ and $p$ lie on the right of $\ell_t$), and if this point also lies on the left of $O$, then $\gamma_t$ is a true straight line from $p$ to $q$. The same for $\gamma_r$. At least of the two lines $\gamma_t$ and $\gamma_r$ must be straight, and whichever line is straight, is a straight line in $C$.

The strong convexity of $C$ is proved.

\[ \square \]

8.2. Angle variation of a curve on an affine surface.

This subsection is an adaptation of some old ideas and results from Zung’s thesis (see [136]), which were inspired by Milnor’s paper [97] on the non-existence of (locally flat regular) affine structures on closed surfaces of genus greater than or equal to 2.

In this subsection we define and show some results about the angle variation of a closed curve on an oriented affine surface, which will be used to show topological restrictions on affine surfaces with focus points which are locally convex at the boundary.

Let $B$ be a smooth surface equipped with an affine structure with or without focus points. We assume that $B$ is oriented; if $B$ is not orientable, then we work on an oriented covering of $B$ instead of $B$ and, of course, if a double covering of $B$ is convex then $B$ is also be convex.

Let $\gamma : [0, 1] \to B$ be a continuously differentiable closed curve (i.e., $\gamma(0) = \gamma(1)$, $\dot{\gamma}(0) = \dot{\gamma}(1)$) on $B$ which does not contain focus points of $B$ and whose velocity is non-zero everywhere: $\dot{\gamma}(t) \neq 0$, $\forall t \in [0, 1]$. We say that such a curve is non-critical.

Put a smooth conformal structure on $B$ and take a nonzero tangent vector $v \in T_{\gamma(0)}B$. We define the angle variation of $\gamma$ with respect to the affine structure of $B$, the choice of $v$, and the conformal structure\(^2\) as follows.

Denote by $v(t) = v(t, \gamma)$ the parallel transport of $v$ by the affine structure of $B$ along $\gamma$ from $\gamma(0)$ to $\gamma(t)$ for each $t \in [0, 1]$. Define $A(t)$ to be the angle from $v(t)$ to $\dot{\gamma}(t)$ with respect to the chosen conformal structure. Our angle $A(t)$ is algebraic, in the sense that it can be negative and can be greater than $\pi$. With every given choice of $A(0)$ (it is only unique up to a multiple of $2\pi$) there is a unique choice of $A(t)$ for each $t \in [0, 1]$ such that $t \mapsto A(t)$ is continuous.

\(^2\)Recall that a conformal structure on a real smooth manifold $B$ is an equivalence class of Riemannian metrics, where two metrics are equivalent if one is a multiple of the other via a smooth strictly positive function.
Definition 8.5. With the above assumptions and notations, the value

\[ AV(\gamma; v) := AV(\gamma; v, C) = A(1) - A(0) \]

is called the **angle variation** of \( \gamma \) on the affine surface \( B \) with respect to the conformal structure \( C \) on \( B \) and non-zero vector \( v \in T_{\gamma}(B) \).

We collect some simple and useful facts about the angle variation.

**Lemma 8.6.** With the above notations, we have:

(i) If two \( C^1 \) non-critical curves \( \gamma \) and \( \mu \) have the same initial point \( \gamma(0) = \mu(0) \) and are homotopic by a 1-dimensional family of non-critical curves with the same initial point, then \( AV(\gamma; v, C) = AV(\mu; v, C) \) with respect to any vector \( v \in T_{\gamma}(B) \).

(ii) If \( C_1 \) and \( C_2 \) are two conformal structures on \( B \) which coincide at \( \gamma(0) \), then \( AV(\gamma; v, C_1) = AV(\gamma; v, C_2) \), i.e., we only have to specify the conformal structure at the initial point \( \gamma(0) \).

(iii) If \( AV(\gamma; v, C) = m\pi \) for some integer \( m \), then \( AV(\gamma; v, C') = m\pi \) for any other conformal structure \( C' \). If \( m\pi < AV(\gamma; v, C) < (m + 1)\pi \), then \( m\pi < AV(\gamma; v, C') < (m + 1)\pi \) for any other \( C' \).

(iv) For any two different non-zero vectors \( v \) and \( v' \) we have

\[ |AV(\gamma; v', C) - AV(\gamma; v, C)| < \pi. \]

(v) If \( \gamma \) bounds a regular disk (without focus points) in \( B \) then \( AV(\gamma; v, C) = 2\pi \) for any \( v \) and \( C \), provided that \( \gamma \) is positively oriented with respect to the disk.

(vi) If \( B \) (or a neighborhood of \( \gamma \)) can be affinely immersed into \( \mathbb{R}^2 \), then \( AV(\gamma; v, C) \) is nothing but the **winding number** of \( \gamma \) times \( 2\pi \).

(vii) If \( \gamma \) is an affine straight line, then \( AV(\gamma; \gamma(0), C) = 0 \).

(viii) If \( \gamma \) is locally convex, in the sense that the set of points near \( \gamma \) and lying on \( \gamma \) or “on the left” of \( \gamma \) is locally convex at \( \gamma \), then \( AV(\gamma; \gamma(0), C) \geq 0 \). If \( \mu \) is homotopic by a path of non-critical closed curves to a locally convex \( \gamma \), then \( AV(\mu; v) > -\pi \) for any \( v \).

**Lemma 8.7** (Collars of handles have negative angle variation). Let \( \gamma \) be a \( C^1 \) simple closed curve on \( B \) which bounds a compact orientable domain \( T \) of genus \( \geq 1 \) which does not contain any focus point and such that \( T \) “lies on the left” of \( \gamma \) with respect to the orientation of \( B \). Then we have

\[ AV(\gamma; v) < 0 \]

with respect to any vector \( v \) (and any conformal structure \( C \)).

**Proof.** We will give the proof if the genus is 1. (The higher genus case is proved in a completely similar manner.) Draw two simple closed curves \( \mu_1 \) and \( \mu_2 \) on the handle \( T \), which are tangent to \( \gamma \) at the point \( S = \gamma(0) = \gamma(1) = \mu_1(0) = \mu_1(1) = \mu_2(0) = \mu_2(1) \), as shown in the left part of Figure 14. Cutting \( T \) by \( \mu_1 \) and \( \mu_2 \), we get a disk \( T' \) with the boundary consisting of five consecutive paths \( \gamma, \mu_2^{-1}, \mu_1, \mu_2, \mu_1^{-1} \), as shown in the right part of Figure 14. Notice that, after this cutting, \( S \) becomes five points.
Figure 14. Collars of handles have negative angle variation.

S₁, S₂, S₃, S₄, S₅, with U-turns at S₁, S₂, S₃, S₅ and has C¹ continuation at S⁴ for the boundary of the disk T'. The total angle variation of the boundary of T' would be 2π, but since each U-turn accounts for π in this variation, if we count the sum of the angle variations of the five pieces of the boundary of T, it is only 2π − 4π = −2π. In other words, we have

\[ AV(\gamma; v) + AV(\mu_2^{-1}; v') + AV(\mu_1; v'') + AV(\mu_2; v''') + AV(\mu_1^{-1}; v''') = -2\pi, \]

where v' is the parallel transport of v by \( \gamma \) with respect to the affine structure, v'' is the parallel transport of v' by \( \mu_2^{-1} \) with respect to the affine structure, and so on.

According to Lemma 8.6 we have \(-\pi < AV(\mu_2^{-1}; v') + AV(\mu_2; v''')\) and \(-\pi < AV(\mu_1; v'') + AV(\mu_1^{-1}; v''')\). These two inequalities together with equality (7) imply that \( AV(\gamma; v) < 0 \).

We say that a non-critical closed curve \( \gamma \) is of **completely negative angle variation** if \( AV(\gamma; v, C) < 0 \) for any choice of \( v \) and \( C \). So Lemma 8.7 says that the collar of a handle is of completely negative angle variation. It is easy to see that the property of completely negative angle variation is invariant under homotopy.

**Lemma 8.8.** If \( \gamma_s \ (s \in [0, 1]) \) is a continuous family of non-critical closed curves, then \( \gamma_0 \) is of completely negative angle variation if and only if \( \gamma_1 \) is of completely negative angle variation.

The following lemma says what happens when the homotopy passes over a focus point.

**Lemma 8.9.** Assume that two closed non-critical curves \( \gamma \) and \( \mu \) on \( B \) form the boundary of an annulus which contains exactly one focus point \( O \) inside it, and are oriented in such a way that the annulus “lies of the left” of \( \mu \) and “lies on the right” of \( \gamma \). If \( \gamma \) is of completely negative angle variation, then \( \mu \) is also of completely negative angle variation.

**Proof.** The reason is simply that the affine monodromy around a focus point in the positive direction pushes every tangent vector to the left (except for one direction which
remains unchanged under this monodromy). When \( v(t) \) is pushed to the left then the angle between \( v(t) \) and \( \dot{\gamma}(t) \) becomes smaller, and so the angle variation becomes even more negative.

**Lemma 8.10.** Assume that two closed non-critical curves \( \gamma \) and \( \mu \) on \( B \) form the boundary of an annulus which contains a finite number of focus points inside it, and are oriented in such a way that the annulus “lies of the left” of \( \mu \) and “lies on the right” of \( \gamma \). If \( \gamma \) is of negative angle variation, then \( \mu \) is also of negative angle variation.

*Proof.* Just apply Lemma 8.9 \( m \) times, where \( m \) is the number of focus points in the annulus.

**Lemma 8.11.** If \( \gamma_1, \gamma_2, \ldots, \gamma_k (k \geq 3) \) are three closed non-critical curves which bound a regular domain \( D \) (without focus points) of genus 0 and are positively oriented with respect to \( D \), and such that \( D \) is locally convex at \( \gamma_2, \ldots, \gamma_k \), then \( \gamma_1 \) is of completely negative angle variation.

*Proof.* The proof is similar to the proof of Lemma 8.7. For example, if \( k = 3 \), we can draw curves \( \mu_2 \) and \( \mu_3 \) which are homotopic by paths of non-critical closed curves to \( \gamma_2 \) and \( \gamma_3 \) respectively, such that \( AV(\gamma_1;v_1,C) + AV(\mu_1;w_2,C) + AV(\mu_3,w_3,C) = -2\pi \), where \( v_1 \) is arbitrary and the choice of \( v', v'' \) depends on \( v \). Then use the inequalities \( AV(\mu_i;w_i,C) > -\pi \) to conclude the statement.

**Proposition 8.12.** Let \( C \) be a compact subset of an orientable affine surface \( B \) with (or without) focus points, such that \( C \) has non-empty boundary and is locally convex at its boundary. Then \( C \) has no handle (i.e., it can be embedded into \( \mathbb{R}^2 \)) and has at most two boundary components.

*Proof.* Just put together Lemma 8.7, Lemma 8.10, and Lemma 8.11.

**Remark 8.13.** In [136], it was shown that the conclusion of 8.12 still holds for each regular domain of the base space even in the case with 8.12 singularities. Proposition 8.12 was also mentioned in [140] (without a proof), and reproved by Leung and Symington in [86].

### 8.3. Convexity of compact affine surfaces with non-empty boundary.

The Local-Global Principle (see Proposition 7.1) in a focus box allows us to prove global convexity results in dimension 2, for base spaces of toric-focus integrable systems on symplectic 4-manifolds.

First we consider the compact case.

**Theorem 8.14.** Let \( B \) be the 2-dimensional base space of a toric-focus integrable Hamiltonian system on a connected, compact, 4-dimensional, symplectic manifold (with or without boundary). Assume that the boundary of \( B \) is non-empty and \( B \) is locally convex. Then \( B \) is convex. Moreover, if \( B \) is orientable, then it is topologically a disk or an annulus. If \( B \) is an annulus, there is a global single-valued non-constant affine function \( F \) on \( B \) such that \( F \) is constant on each of the two boundary components of \( B \) and, in particular, the boundary components of \( B \) are straight curves.
Remark 8.15. Recall that, according to the results presented in Section 4 and Section 7, the local convexity condition of $\mathcal{B}$ is automatically satisfied if the toric-focus integrable system has elliptic singularities and the symplectic manifold is compact without boundary.

In order to prove the above theorem, we will use the shrinking method to define and study convex hulls of subsets on affine manifolds with singular points. In the Euclidean space, the convex hull of a given subset is unique. In our case, it may be non-unique, but exists and can be defined as follows.

By compactness of $\mathcal{B}$, there is a finite set of closed focus boxes and regular boxes $\Sigma := \{B_i \subset \mathcal{B} \mid i \in I, I \text{ finite}\}$ whose interiors cover $\mathcal{B}$. We need to introduce the following definition:

$C \subset \mathcal{B}$ is $\Sigma$-convex if $C \cap B_i$ is strongly convex in $B_i$ (see Definition 6.5) for all $B_i \in \Sigma$.

Note that if $C$ is $\Sigma$-convex, then $C$ is locally convex since each $B_i$ is convex (see Theorem 7.3). For each given non-empty subset $S$ of $\mathcal{B}$, the family

$$C_S = \{C \subset \mathcal{B} \mid S \subset C, C \text{ is closed, connected, } \Sigma\text{-convex}\}$$

of subsets of $\mathcal{B}$ is not empty since $\mathcal{B} \in \mathcal{C}$.

We claim that $C_S$ has a minimal element. Indeed, $C_S$ is a partially ordered set with respect to inclusion. Let $\{C_\alpha\} \subset C_S$ be a totally ordered subset of $C$, i.e., for any two elements in $C$ one is included in the other. Let $C_\infty = \cap_\alpha C_\alpha$. Clearly $S \subset C_\infty$ and $C_\infty$ is strongly convex in each box $B_i$ by Proposition 8.4. The set $C_\infty$ is also connected (see [43, Theorem 6.1.18, page 355]) and, therefore, $C_\infty \in C_S$. By Zorn’s Lemma, the set $C_S$ has a minimal element $C_0$, which is hence connected, closed, $\Sigma$-convex, and $X, Y \in C_0$.

We will call a minimal element $C_S$ of $C_S$ a $\Sigma$-convex hull of $S$ in $\mathcal{B}$. It exists but is not necessarily unique.

Lemma 8.16. With the above notations, under the assumption that $\mathcal{B}$ has non-empty boundary, any $\Sigma$-convex hull $C_S$ of a non-empty set $S \subset \mathcal{B}$ has a non-empty boundary. Moreover, each boundary component of $C_S$ contains at least one point of $S$, and is locally straight outside the points of $S$.

Proof. Since the boundary of $\mathcal{B}$ is not empty, neither is the boundary of $C_S$. Indeed, if the boundary of $C_S$ was empty, then $C_S$ would be open and different from $\mathcal{B}$, which contradicts the closedness of $C_S$ and the connectedness of $\mathcal{B}$.

If $C_S$ is one-dimensional, then the boundary of $C_S$ is $C_S$ itself, and it contains $S$, so we are done.

If $C_S$ is two-dimensional, it follows that the interior of $C_S$ is not empty. Let $\mu$ be a boundary component of $C_S$. We know that $\mu$ must be homeomorphic to a circle and that $C_S$ is locally convex at $\mu$.

For every $z \in \mu$ such that $z \notin S$ and $z$ is not a focus point, we have that $\mu$ is locally straight at $z$. Indeed, if $\mu$ is not straight at $z$, we can cut a small piece of $C_S$ with $z \in \mu$.
Figure 15. Making $C_S$ smaller by cutting out a corner at $Z$.

being a vertex and the resulting set is still $\Sigma$-convex, closed, connected, containing $S$ and it is strictly included in $C_S$ (see Figure 15). This contradicts minimality of $C_S$.

If $z$ is a focus point, then because $C_S$ is convex at $z$, we can either cut out a small piece containing $z$ just like in the previous case, or the boundary component $\mu$ must be locally a limit of straight lines from the interior of $C_z$. Since $C_z$ is minimal, this is the only possible case. By forgetting about the complement of $C_z$, we may forget the fact that $z$ is a focus point and pretend that it is a regular point and have the same situation as in the previous case.

Thus, if $\mu$ does not contain any point of $S$, then $\mu$ is a straight circle, and we can push the boundary $\mu$ a bit into the interior of $C$ to another straight curve $\mu'$, and then cut the "collar" bounded by $\mu$ and $\mu'$. (Here we use the fact that the affine structure is integral). The resulting set is clearly in $C_S$ if the collar that has been cut out is sufficiently thin. This contradicts the minimality of $C_S$ in $C_S$. □

Proof of Theorem 8.14 if $B$ is not a disk.

We may assume that $B$ is orientable (if not just take an orientable covering of it). Then by Proposition 8.12, if $B$ is not a disk, it must be topologically an annulus.

Denote the two components of the boundary of $B$ by $\gamma$ and $\mu$. Notice that a $\Sigma$-convex hull $C_\gamma$ of $\gamma$ in $B$ must be 1-dimensional, for otherwise it must have some other boundary component $\mu'$ disjoint from $\gamma$, and we can still shrink $C_\gamma$ near $\mu$, which is a contradiction. However, if $C_\gamma$ is 1-dimensional, then it is equal to $\gamma$, i.e., $\gamma$ must be a $\Sigma$-convex set in $B$, and it follows that it is a straight line.

Apply the same argument to $\mu$ to conclude that both $\gamma$ and $\mu$ are straight lines. Now we want to show the existence of a global single-valued integral affine function $F$ such that $F$ is constant on $\gamma$ and on $\mu$. We do this by induction on the number of focus points in $B$.

If $B$ does not contain any focus point, then the universal covering of $B$ is affinely isomorphic to a part of $\mathbb{R}^2$ by two straight lines, and these lines must be parallel. From that we can construct our affine function $F$.

Assume now that the statement is true when there are less than $n$ focus points ($n \in \mathbb{N}$). Let us show that the statement is also true when there are exactly $n$ focus points.
Denote the set of focus points by $\mathcal{O} = \{O_1, \ldots, O_n\}$ and consider a $\Sigma$-convex hull of the union $\gamma \cup \mathcal{O}$ in $B$. Its boundary must contain $\gamma$ but not only $\gamma$, so it is an annulus, and has another boundary component, say $\mu'$, which contains a focus point, say $O_n$ (see Figure 16). Then $\mu'$ must be a straight line and, by induction, there is a single-valued non-constant affine function $F$ on the annulus between $\gamma$ and $\mu'$, which is constant on $\gamma$ and on $\mu'$. It follows that the strip between $\mu'$ and $\mu$ is also locally convex, and hence it is an annulus on which we have a non-constant affine function $F'$ such that $F'$ is constant on $\mu'$ and on $\mu$. It is easy to see that we can “glue” $F$ with $F'$ after some affine transformation to get a non-constant affine function on $\mathcal{B}$ which is constant on $\gamma$ and on $\mu$. The global convexity of the annulus is now easy to see, by the same method of potential straight lines as in the proof of Theorem 7.1.

Proof of Theorem 8.14 in the case when $\mathcal{B}$ is a disk.

Take $x, y \in \mathcal{B}$. We need to show the existence of a straight line segment $[x, y] \subset \mathcal{B}$.

Denote by $C_0 = C_{\{x,y\}}$ a $\Sigma$-convex hull of $x$ and $y$ in $\mathcal{B}$. If $C_0$ is one-dimensional, it follows that it is locally straight (in the affine structure of $\mathcal{B}$) so it is straight and hence there is a straight line $[x, y] \subset C_0 \subset \mathcal{B}$.

If $C_0$ is two-dimensional, it follows that the interior of $C_0$ is not empty. Let $\mu$ be a boundary component of $C_0$. We know that $\mu$ must be homeomorphic to a circle and that $C_0$ is locally convex at $\mu$. So, according to Lemma 8.16, $\mu$ must contain $x$ or $y$, or both of them, and $\mu$ is locally straight outside these points.

If both $x$ and $y$ belong to $\mu$ then both paths on $\mu$ from $x$ to $y$ are straight, and we are done.

Consider now the case when $y \in \mu$ but $x \notin \mu$ (or vice versa). Then Lemma 8.17 below states, in particular, that there is a straight line from $x$ to $y$ in $C_0$, and we are again done.

Lemma 8.17. Let $D$ be a $\Sigma$-convex closed disk on $\mathcal{B}$ with boundary $\mu$ and $x \in D$. 

(i) (See Figure 17). If \( x \) is in the interior of \( D \), then there exist a finite number of consecutive points \( A_1, \ldots, A_n \in \mu \) (\( n \geq 1 \)) and \( n \) corresponding disjoint closed segments \( I_1, \ldots, I_n \) on the circle of nonzero tangent vectors at \( x \), such that:

- Each straight ray emanating from \( x \) in direction \( d \) belonging to the interior of \( I_i \) hits a point \( y(d) \) on the “arc” \( A_iA_{i+1} \) on \( \mu \) (with the convention that \( A_{n+1} = A_1 \));

- The above straight line \([x, y(d)]\) lies in \( D \), is regular in \( D \) (i.e., it does not hit any focus point in \( D \)), and the map \( d \mapsto y(d) \) is a homeomorphism from the interior of \( I_i \) to the interior of the “arc” \( A_iA_{i+1} \) on \( \mu \);

- When \( d \) tends to the end points of \( I_i \), then \([x, y(d)]\) tends to straight lines going from \( x \) to \( A_i \) and \( A_{i+1} \) (these straight lines may contain focus points).

![Figure 17. Dividing the disk into sectors and gaps.](image)

(ii) (See Figure 18). If \( x \in \mu \), then either \( \mu \setminus \{x\} \) is a straight line, or there exist a finite number of consecutive points \( A_1, \ldots, A_n \in \mu \) (\( n \geq 1 \), it may happen that \( A_1 = x \) or \( A_n = x \), or both) and \( n - 1 \) corresponding disjoint closed segments \( I_1, \ldots, I_{n-1} \) on the circle of nonzero tangent vectors at \( x \), such that, the “arcs” \( xA_1 \) and \( xA_n \) on \( \mu \) are straight lines, and the other “arcs” \( A_iA_{i+1} \) on \( \mu \) (\( 1 \leq i \leq n - 1 \)) satisfy the same properties as in the previous case:

- Each straight ray emanating from \( x \) in direction \( d \) belonging to the interior of \( I_i \) hits a point \( y(d) \) on the “arc” \( A_iA_{i+1} \) on \( \mu \);

- The above straight line \([x, y(d)]\) lies in \( D \), is regular in \( D \), and the map \( d \mapsto y(d) \) is a homeomorphism from the interior of \( I_i \) to the interior of the “arc” \( A_iA_{i+1} \) on \( \mu \);

- When \( d \) tends to the end points of \( I_i \), then \([x, y(d)]\) tend to straight lines going from \( x \) to \( A_i \) and \( A_{i+1} \).

In other words, the above lemma says that \( D \) can be divided into \textbf{sectors} and \textbf{gaps}: each sector is a regular “convex cone” with \( x \) as the vertex. The interiors of the sectors
do not overlap and the whole boundary $\mu$ is covered by these sectors. To go from $x$ to the points on $\mu$, we can move in the sectors and forget about the gaps.

**Proof.** We prove the lemma by induction on the number of focus points in the interior of $D$. (Focus points on the boundary of $D$ can be ignored).

If $D$ does not contain any focus point, then we can apply the regular local-global convexity principle to $D$ to conclude that $D$ is affinely isomorphic to a compact convex set in $\mathbb{R}^2$, in which case the lemma becomes trivial (there is only one sector).

Assume now that there are exactly $n$ focus points $O_1, \ldots, O_n$ in the interior of $D$. Denote by $C = C_{\{x,O_1,\ldots,O_n\}}$ a $\Sigma$-convex hull of $x, O_1, \ldots, O_n$, and let $\gamma$ be the boundary of $C$. (If $C$ has a hole, then denote by $C'$ the union of the $C$ with the disk inside the hole and by $\gamma$ the boundary of $C'$). The following cases may happen:

**Case 1.** $x$ lies in the interior of $C$ (or $C'$, if $C$ has a hole) and at least one of the focus points, say $O_1$, lies on $\gamma$. Then we can apply the induction hypothesis to $C$ (if $C$ is a disk) or the annulus case of Theorem 8.14 (if $C$ has a hole) to divide $C$ into sectors and holes with the above properties.

Let $d$ be any direction at $x$ in one of the sectors of $C$ (or $C'$) and consider the corresponding straight line $\ell = \ell(d)$ passing through $x$ and $y(d)$. After hitting the point $y(d)$ on $\gamma = \partial C$ (or $\partial C'$), this straight line $\ell$ gets out of $C$ and goes in the regular region between $\gamma$ and $\mu$. It must then hit $\mu$ at a point, say $z(d)$ (it may happen that $z(d) = y(d)$, i.e., it already hits $\mu$ at $y(d)$ without having to extend), or, otherwise, one of the following 3 bad possibilities happens (see Figure 19):

a) The straight line $\ell$ falls back to $C$ at a point $y'(d)$. This situation is impossible, because the region between the segment from $y(d)$ to $y'(d)$ on the straight line and the “arc” from $y(d)$ to $y'(d)$ on $\gamma$ can be affinely immersed into $\mathbb{R}^2$. However, $\gamma$ viewed from $\ell$ is concave while $\ell$ is straight, a situation that cannot happen in $\mathbb{R}^2$.

b) The straight line $\ell$ cuts itself (after going around). Then we have locally an annulus between a loop created by the straight line and $\mu$, whose boundary is not straight
the self-intersection point of the straight line), which contradicts the annulus case of Theorem 8.14. So this situation is also impossible.

c) The straight line $\ell$ winds around an infinite number of times without cutting itself or hitting $\mu$. Denote by $\mu'$ the limit set of $\ell$. Then $\mu'$ is a closed straight line union $\ell$. Such a situation cannot happen in integral affine geometry, because in a neighborhood of $\mu'$ there would exist a non-constant affine function $F$ such that $\mu' = \{F = 0\}$, and $F$ is not constant on $\ell$, implying that $F = 0$ on some point of $\ell$, i.e., $\ell$ would have to cut $\mu'$.

So none of the above three “bad” situations can happen, which means that $\ell$ must necessarily cut $\mu$ at a point, say $z(d)$. The map $d \mapsto z(d)$ maps the segments $I_i$ of directions $d$ at $x$ (which correspond to the sectors of $C$) to “arcs” in $\mu$. Then one can easily see that these arcs overlap and cover the whole $\mu$. (The focus point on $\gamma$ is also responsible for some overlapping). In order to avoid overlapping, one simply shrinks the segments as much as necessary (the choice of shrinking is not unique). After this shrinking, one gets the required sectors and gaps for $D$.

Case 2. $x$ lies in the interior of $D$ and also on the boundary of $C$. The treatment of this case is similar to Case 1, with one segment of directions at $x$ added: the one which consists of the directions which point towards the exterior of $C$.

Case 3. $x$ lies in on the boundary $\mu$ of $D$. If $\mu \setminus \{x\}$ is a straight line, then the conclusion of the lemma is empty, we have nothing to prove. If $\mu \setminus \{x\}$ is not a straight line then $C$ is strictly included in $D$ and $C$ contains a focus point on its boundary. We can apply the induction hypotheso to $C$ and treat this case similarly to Case 2. \hfill \Box

**Remark 8.18.** In the proof of Theorem 8.14 we used the fact that the affine structure is integral, especially for the annulus case. However, if there is a global affine function on $\mathcal{B}$, then the fact that the affine structure is integral is not needed and the conclusions of the theorem still hold. In particular, if there is a global affine function on $\mathcal{B}$, the indices of the focus points are allowed to be non-integers, and $\mathcal{B}$ would still be convex.

8.4. **Convexity in the non-compact proper case.**
Let \( B \) be the 2-dimensional base space of a toric-focus integrable Hamiltonian system on a non-compact symplectic 4-manifold which has both elliptic and focus-focus singularities. Assume that the number of focus-focus singularities is finite. The boundary of \( B \) corresponds to the elliptic singularities of the system. In this subsection, we assume that the interior of \( B \) is homeomorphic to an open disk.

Fix a point \( p \) on the boundary. Take continuous paths from \( p \) to each focus point of \( B \) and modify, if necessary, the paths such that they do not intersect in \( B \). Now cut along these paths, thereby obtaining a new set \( \hat{B} \), whose boundary consists of the old boundary of \( B \) union with twice each path linking the elliptic point to the focus singularity. This new set does not contain any points with monodromy, because of the cuts that place the former focus singularities on the boundary of \( \hat{B} \). Thus we get a global affine map \( \varphi : \hat{B} \to \mathbb{R}^2 \).

**Definition 8.19.** If \( B \) is such a base space, then \( B \) is said to be proper if the affine map \( \varphi : \hat{B} \to \mathbb{R}^2 \) is proper (i.e., the inverse image by \( \varphi \) of any compact set in \( \mathbb{R}^2 \) is compact in \( \hat{B} \)).

We show that the definition does not depend on the choice of the paths linking the elliptic singularity on the boundary of \( B \) to the focus singularities inside \( B \). The following statement is used to prove this statement.

**Lemma 8.20.** Let \( X \) be a topological space, \( Y = \overline{Y}, Z = \overline{Z} \subset X \). Assume that the inclusions \( Y \hookrightarrow X, Z \hookrightarrow X \) are proper. Then a continuous map \( f : X \to T \) to some other topological space is proper if and only if both restrictions \( f|_Y : Y \to T \) and \( f|_Z : Z \to T \) are proper.

**Proof.** If \( K \subset T \), then \( f^{-1}(K) = f^{-1}(K) \cap (Y \cup Z) = (f^{-1}(K) \cap Y) \cup (f^{-1}(K) \cap Z) \).

If \( K \subset T \) is compact and \( f : X \to T \) is proper, it follows that \( f^{-1}(K) \) is compact in \( X \). Therefore, the inverse images of \( f^{-1}(K) \) by the proper inclusions \( Y \hookrightarrow X, Z \hookrightarrow X \) are compact sets in \( Y \) and \( Z \), respectively. These inverse images coincide with \( (f|_Y)^{-1}(K) \subset Y \) and \( (f|_Z)^{-1}(K) \subset Z \), which shows that both \( f|_Y : Y \to T \) and \( f|_Z : Z \to T \) are proper.

Conversely, if \( f|_Y : Y \to T \) an \( f|_Z : Z \to T \) are proper, then \( f^{-1}(K) \cap Y \) is a compact set in \( Y \) and \( f^{-1}(K) \cap Z \) is a compact set in \( Z \). Since \( Y = \overline{Y}, Z = \overline{Z} \), it follows that \( f^{-1}(K) \cap Y \) and \( f^{-1}(K) \cap Z \) are compact in \( X \) which then implies that their union, which equals \( f^{-1}(K) \) is compact in \( X \). \( \square \)

Now we can formulate a global convexity theorem in the proper non-compact 2-dimensional case.

**Theorem 8.21.** Let \( B \) be the 2-dimensional base space of a toric-focus integrable Hamiltonian system on a connected, non-compact, symplectic, 4-manifold without boundary. Assume:

(i) the system has elliptic singularities (i.e., the boundary of \( B \) is not empty);
(ii) the number of focus points in \( B \) is finite and the interior of \( B \) is homeomorphic to an open disk;
(iii) $\mathcal{B}$ is proper (see Definition 8.19).

Then $\mathcal{B}$ is convex (in its own underlying affine structure).

Proof. The main idea is to reduce Theorem 8.21 to Theorem 8.14, in a way similar to the reduction of Lemma 8.2 to Lemma 8.1.

Fix the proper integral affine map $\varphi : \tilde{\mathcal{B}} \to \mathbb{R}^2$ given in the definition of the properness of $\mathcal{B}$. Then, for $N \in \mathbb{N}$ large enough, the square $D_N = \{(s, t) \in \mathbb{R}^2 \mid |s|, |t| \leq N\}$ contains the images of all the paths used to cut $\mathcal{B}$, because these paths are compact and there are only finitely many of them. It follows that $\varphi^{-1}(D_N)$ corresponds to a locally convex compact subset $\mathcal{B}_N$ in $\mathcal{B}$ whose boundary is not empty. By theorem 8.14, we obtain that $\mathcal{B}_N$ is convex. For any $x, y \in \mathcal{B}$ there exists $N$ large enough such that $x, y \in \mathcal{B}_N$, hence there is a straight line from $x$ to $y$. Thus $\mathcal{B}$ is convex. □

8.5. Non-convex examples in the non-proper case.

In this subsection we consider only four dimensional symplectic manifolds $(M, \omega)$ endowed with a symplectic circle action. Let $H : M \to \mathbb{R}$ be the circle invariant Hamiltonian (total energy) of a dynamical system on $M$. We assume that the circle action admits a circle invariant momentum map $J : M \to \mathbb{R}$ such that $F := (J, H) : M \to \mathbb{R}^2$ is an integrable system of toric-focus type (see Definition 4.4). If the system is semitoric, i.e., $J$ is proper plus additional assumptions on the structure of the singularities (see Remark 4.6), then, as recalled in Subsection 4.6, to $F(M)$ one can associate a family of convex polygons.

The case of proper semitoric systems ($F$ is proper, $J$ is not necessarily proper) is fundamentally different, as shown in [111]. In the study of semitoric systems, properness of $J$ plays a crucial role since it permits the use of Morse-Bott theory and of techniques related to the Duistermaat-Heckman theorem, ultimately leading to the proof of the connectedness of the fibers of $F$ and of the convexity result of the polygons naturally associated to $F(M)$. These methods are not available if $J$ is not proper. The consequences of the loss of properness of $J$ are remarkable. Not only are the proofs totally different, but even the properties of $F(M)$ change radically. In [111], an invariant generalizing that for semitoric and toric systems, the cartographic projection, was introduced. A cartographic projection is the natural planar representation of the singular affine structure of the proper semitoric system. A cartographic projection is the union of subsets $\mathcal{R} \subset \mathbb{R}^2$ of four very specific types, which we now describe:

- $\mathcal{R} \subset \mathbb{R}^2$ is of type I if there is an interval $I \subseteq \mathbb{R}$ and $f, g : I \to \mathbb{R}$ such that $f$ is a piecewise linear continuous convex function, $g$ is a piecewise linear continuous concave function, and
  \[ \mathcal{R} = \{ (x, y) \in \mathbb{R}^2 \mid x \in I \text{ and } f(x) \leq y \leq g(x) \}. \]

- $\mathcal{R} \subset \mathbb{R}^2$ is of type II if there is an interval $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ such that $f$ is a piecewise linear continuous convex function, $g$ is lower semicontinuous,
and

\[ \mathcal{R} = \left\{ (x, y) \in \mathbb{R}^2 \mid x \in I \text{ and } f(x) \leq y < g(x) \right\}. \]

- \( \mathcal{R} \subset \mathbb{R}^2 \) is of type III if there is an interval \( I \subseteq \mathbb{R} \) and \( f: I \to \mathbb{R}, \ g: I \to \mathbb{R} \) such that \( f \) is upper semicontinuous, \( g \) is a piecewise linear continuous concave function, and

\[ \mathcal{R} = \left\{ (x, y) \in \mathbb{R}^2 \mid x \in I \text{ and } f(x) < y \leq g(x) \right\}. \]

- \( \mathcal{R} \subset \mathbb{R}^2 \) is of type IV if there is an interval \( I \subseteq \mathbb{R} \) and \( f, g: I \to \mathbb{R} \) such that \( f \) is upper semicontinuous, \( g \) is lower semicontinuous, and

\[ \mathcal{R} = \left\{ (x, y) \in \mathbb{R}^2 \mid x \in I \text{ and } f(x) < y < g(x) \right\}. \]

Here, \( \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\} \) is endowed with the standard topology, i.e., a basis of neighborhoods of \( \pm \infty \) consists of the set of intervals \([-\infty, r) \) and \((r, +\infty]\), respectively, \( r \in \mathbb{R} \). Notice that \( \mathcal{R} \) is a type I set if and only if there exists a convex polygon \( \mathcal{P} \subset \mathbb{R}^2 \) and an interval \( I \subseteq \mathbb{R} \) such that \( \mathcal{R} = \mathcal{P} \cap \{(x, y) \in \mathbb{R}^2 \mid x \in I\} \).

An example of a possible cartographic projection is given in Figure 20 (taken from [111]). The cartographic projection contains the information given by the singular affine structure induced by the (singular) Lagrangian fibration \( F: M \to \mathbb{R}^2 \) on the base \( F(M) \). For its construction and the study of its properties we refer to [111, Theorems B, C, and Corollary 4.3]. In [111, Theorem D], examples of proper semitoric systems are produced for which the cartographic projection is neither polygonal, nor convex. More precisely, it is shown, by construction, that there are uncountably many proper semitoric systems having the range of \( F \) unbounded and the cartographic projections not convex, neither open nor closed in \( \mathbb{R}^2 \), and containing every type of the four possible sets listed above in the union forming the cartographic projection. In addition, one can build two uncountable subfamilies such that the cartographic projections for the first family are bounded and those for the second family are unbounded. One can even construct
the semitoric systems such that they are isomorphic if and only if their parameter indices coincide. For the detailed construction of this family and the verification of these properties we refer to [111, Section 7].

8.6. An affine black hole and non-convex $S^2$.

In this subsection, we prove the following statement.

**Theorem 8.22.** There exists a non-degenerate singular Lagrangian torus fibration on a symplectic manifold diffeomorphic to $K3$, with only focus-focus singularities, and whose base space is a sphere $S^2$ with a non-convex singular integral affine structure.

The theorem is proved by constructing an explicit example of an integral affine structure on $S^2$ with focus points, which is, of course, locally convex, but which is globally non-convex. This example is an instance of the phenomenon monodromy can kill global convexity, where the local-global convexity principle no longer holds.

Look at the “8-vertex shuriken” drawn in Figure 21, together with a standard integral lattice in $\mathbb{R}^n$. We glue the edges of this shuriken together, by the arrows shown in Figure 21. For example:

$O_1 Q_8$ is glued to $O_1 P_1$ by the linear transformation which admits $O_1$ as the origin and is given by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Indeed, $\overrightarrow{O_1 Q_8} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$, $\overrightarrow{O_1 P_1} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

$O_2 Q_1$ is glued to $O_2 P_2$ by the linear transformation which admits $O_2$ as the origin and is given by the matrix

$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}$

Indeed, $\overrightarrow{O_2 Q_1} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$, $\overrightarrow{O_2 P_2} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and

$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

After this glueing process, we get an “8-petal flower”, as shown in Figure 22. Its boundary consists of the straight lines $P_1 P_2$, $P_2 P_3$, $\ldots$, $P_8 P_1$, and it is concave at the vertices $P_i$. (Note that $Q_i$ is identified with $P_{i+1}$, and $P_9 = P_1$ by our convention). The flower has 8 focus points $O_1, O_2, \ldots, O_8$ inside ($O_1, O_3, O_5, O_7$ have index 2, and $O_2, O_4, O_6, O_8$ have index 1). Each curve $AO_i P_{i+1}$ (consisting of a dashed part and a boundary part) is in fact a straight line “coming from the left” with respect to the singular affine structure on the flower, and each petal $AO_i P_{i+1} O_{i+1}$ is affinely isomorphic to a triangle with vertices $A_i, P_{i+1}, O_{i+1}$ ($P_9 = P_1$ and $O_9 = O_1$ by convention). (The fact that each petal is a triangle is more clearly shown on the shuriken picture).

The boundary of our 8-petal flower $P_1 P_2 \ldots P_8 P_1$ is a piecewise straight simple closed curve with the following integral affine invariants:

- Every edge $P_i P_{i+1}$ has **integral direction**, which means that there is a local integral affine function $F_i$ which vanishes on $P_i P_{i+1}$. 

Each vertex $P_i$ is a **simple vertex**, which means that the covectors $dF_{i-i}(P_i)$, $dF_i(P_i)$ form a basis of the lattice of integral covectors at $P_i$.

- Each edge has **integral affine length** equal to 2.

- Each edge $P_i P_{i+1}$ of the boundary has its **characteristic number** defined as follows. We can embed a small neighborhood of the 3-piece broken line $P_{i-1} P_i P_{i+1} P_{i+2}$ by an integral affine map $\Phi_i$ into the standard integral affine plane $\mathbb{R}^2$, such that under this map $\Phi(P_i) = (0,0)$, $\Phi(P_{i+1}) = (\ell, 0)$ (where $\ell = 2$ is the integral affine length of $P_i P_{i+1}$), $\Phi(P_{i-1})$ lies on the axis $(0, y)$, and $\Phi_{i+2}$ lies on the line $x = cy + \ell$. Then $c$ is called the **characteristic number** of the edge $P_i P_{i+1}$ (with respect to the two adjacent edges). The edges $P_1 P_2$, $P_3 P_4$, $P_5 P_6$, $P_7 P_8$ have index 4, while the edges $P_2 P_3$, $P_4 P_5$, $P_6 P_7$, $P_8 P_1$ have index 2, as shown on Figure 21.

Since our 8-petal flower has concave boundary, we can glue an appropriate octagon with convex boundary to it to obtain a sphere $S^2$ with a singular integral affine structure with focus points. Such an explicit octagon is given on Figure 23, with exactly the same integral lattice as in Figure 21.

We take a flat octagon, then create 8 focus points inside it (4 focus points of index 1 and 4 focus points of index 2) by cutting out 8 small triangles and glueing the edges of the created angles together, as shown on Figure 23.
Figure 22. The “shuriken” becomes an 8-petal flower after glueing.

Figure 23. Glue by the arrows to get the complementary octagon.

It is easy to check that the boundary of our octagon has exactly the same integral
affine invariants as the boundary of our flower. So the octagon can be glued to the
flower edge by edge in an integral affine way to obtain a sphere $S^2$ with an integral affine structure on it. By integrable surgery (see [140]), where $S^2$ is the base space of a singular Lagrangian torus fibration with 8 simple and 8 double focus-focus fibers (for a total of 24 singular focus-focus points in the symplectic manifold, and it is well-known that such a manifold is diffeomorphic to a complex K3 surface; see [86]). We now show that this singular affine $S^2$ is not convex.

Indeed, take any straight line going from the center $A$ of the flower. One can see that this straight line is trapped inside the flower, can never get out of it, and so for any point $B$ lying in the interior of the complementary octagon, there is no straight line going from $A$ to $B$. One may imagine the flower as a kind of “black hole” in which the “light rays” are bent so much by the “affine gravity” (i.e., monodromy) of the focus points that no light ray from $A$ can escape it.

For example, let’s say that we take a ray (i.e., straight line) starting from $A$ and lying between $AO_1$ and $AO_2$. Then it must get out of the petal $AO_1P_1P_2O_2$ by the edge $O_2P_2$. When it gets out of the first petal, it enters the second petal $AO_2P_2P_3O_3$ by the edge $AO_2P_2P_3$. Hence it must get out of this petal by one of the edges $AO_3$ and $O_3P_3$ to enter the third petal. But $AO_3$ and $O_3P_3$ lie on the same edge $AO_3P_3P_4$ of the petal $AO_3P_3P_4O_4$, and so it must get out of the third petal by $AO_4$ or $O_4P_4$, to go into the next petal, and so on. By induction, the ray from $A$ will never get out of the flower.

Theorem 8.22 is proved.

Remark 8.23. Since any function on $S^2$ has critical points, there is no integrable Hamiltonian system (with a global momentum map) with only focus-focus singularities and whose base space is $S^2$. However, if we take the direct product of our non-convex integral affine $S^2$ with a closed affine integral $D^1$ (with the product affine structure), then from a standard embedding of $S^2 \times D^1$ into $\mathbb{R}^3$ we get the momentum map of a 3-degrees-of-freedom integrable Hamiltonian system whose base space is our product $S^2 \times D^1$, which is non-convex.

8.7. A globally convex $S^2$ example.

This subsection is devoted to the proof of the following theorem.

Theorem 8.24. There exists a non-degenerate singular Lagrangian torus fibration on a symplectic manifold diffeomorphic to $K3$, with only focus-focus singularities, and whose base space is a sphere $S^2$ with a globally convex singular integral affine structure.

The original example constructed by Zung in 1993 (see [136, 140]) of a singular Lagrangian torus fibration with only focus-focus singularities on a symplectic 4-manifold diffeomorphic to K3, whose base space is the sphere $S^2$ equipped with a singular affine structure, is as follows:

Take a standard triangle $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq c\}$ in the Euclidean space $\mathbb{R}^2$ with the standard integral affine structure. Cut out from it three small triangles homothetic to it, one on each edge, as shown on the left in Figure 24, and then glue the edges of the created angles together to obtain a “rounded” singular affine triangle with 3 focus points of index 1 inside, as shown in the middle of Figure 24. We can now take 8
copies of this “rounded out” singular affine triangle, glue them together to get an affine sphere $S^2$ with 24 singular points, which is the base space of a singular Lagrangian torus fibration with 24 simple focus-focus singular fibers on a 4-dimensional symplectic manifold, which therefore is diffeomorphic to a complex K3 surface; see [86] for details.

We can simplify the construction by pushing the focus points to the edges of the “rounded” affine triangles. By doing so, the 24 focus points of index 1 on $S^2$ become 12 focus points of index 2, each lying on one quarter of one of the three “great circles” on $S^2$, as shown in Figure 25. Of course, this new example of an affine $S^2$ with 12 double focus points is also the base space of a singular Lagrangian torus fibration with 12 double focus-focus fibers. Each 1/8 of the sphere (rounded triangle) in this example is affinely isomorphic to the standard triangle $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq c\}$ in $\mathbb{R}^2$ (because the singular points have been pushed to the boundary). Let us show that this simplified example is convex.

Notice that in each triangle which is 1/8 of our sphere we have 3 trapezoids “X”, “Y”, “Z”, and any two of them already cover the whole triangle. If we take all the trapezoids “X” in all the eight triangles, then they form a strip, which is an annulus whose boundary consists of two affine circles (there are regular affine circles in the interior of the annulus which tend to these boundary circles). The same with trapezoids “Y” and trapezoids “Z”. So we have three locally convex annuli, and any two of these annuli already cover the whole $S^2$.

If $A$ and $B$ are two arbitrary points on our $S^2$, then at least one of these three annuli contains both $A$ and $B$. From Theorem 8.14 we know that each annulus is convex. So we can connect $A$ to $B$ by a straight line lying in one of the annuli on our $S^2$. Thus our $S^2$ is convex.

8.8. Convexity of toric-focus base spaces in higher dimensions.

We have seen in Subsection 7.3 that there exist non-convex compact toric-focus base space $B$ of any dimension $\geq 4$ with a focus$^2$ point, due to the “all choices go wrong” phenomenon near such points. In fact, already in dimension 3, this phenomenon can already happen, with just two curves of focus points, and so we get the following result.
Theorem 8.25. There exists a toric-focus integrable system on a compact 6-dimensional symplectic manifold, with elliptic and focus-focus singularities, whose base space is not convex.

Proof. We construct a 3-dimensional box
\[ \mathcal{B} = \{ x \mid -1 \leq F(x) \leq 1; -1 \leq (G_1)_l(x), (G_1)_r(x) \leq 1; -1 \leq (G_2)_l(x), (G_2)_r(x) \leq 1 \}, \]
where:

- There is a smooth coordinate system \((F, H_1, H_2)\) on \(\mathcal{B}\), and two curves \(S_1 = \{ x \in \mathcal{B} \mid F(x) = H_1(x) = 0 \} \) and \(S_1 = \{ x \in \mathcal{B} \mid F(x) = \delta, H_2(x) = 0 \} \) lying on the disks \(\{ F = 0 \}\) and \(\{ F = \delta \}\), respectively.
- \(G_1\) is a function of \(\mathcal{B}\) with two branches \((G_1)_l, (G_1)_r\) satisfying \((G_1)_r = (G_1)_l + F\) when \(F \geq 0\) and \((G_1)_r = (G_1)_l + (F - \delta)\) when \(F \leq 0\).
- \(G_2\) is a function of \(\mathcal{B}\) with two branches \((G_2)_l, (G_2)_r\) satisfying \((G_1)_r = (G_1)_l\) when \(F \leq \delta\) and \((G_1)_r = (G_1)_l + (F - \delta)\) when \(F \geq 0\).
- \((G_1)_l\) is smooth outside of the set \(\{ F = 0, H_1 \geq 0 \}\); \((G_1)_r\) is smooth outside of the set \(\{ F = \delta, H_1 \leq 0 \}\).
- \((G_2)_l\) is smooth outside of the set \(\{ F = \delta, H_1 \geq 0 \}\); \((G_2)_r\) is smooth outside of the set \(\{ F = 0, H_1 \leq 0 \}\).
- \((F, G_1, G_2)\) is a multi-valued integral affine coordinate system for the singular integral affine structure on \(\mathcal{B}\) with two curves of focus points \(S_1\) and \(S_2\).
- \(G_1\) is a smooth parametrization for \(S_2\) and \(G_2\) is a smooth parametrization for \(S_1\).
- The restriction of \(G_i\) to \(S_i\) is a smooth function for each \(i = 0, 1\).

Using integrable surgery, one can construct an integrable system with such a box \(\mathcal{B}\) as the base space in which \((F, G_1, G_2)\) is a multi-valued system of action coordinates and \(S_1, S_2\) are the two curves of focus points. For each \(i = 1, 2\), the restriction of \(G_i\) to \(S_i\) can be chosen to be an arbitrary smooth function with absolute value smaller than or equal to 1.
Figure 26. All four potential straight lines are broken at respective points $E^{l,l}, E^{r,l}, E^{l,r}, E^{r,r}$.

To go from a point $A$ with $F(A) = -1$ to a point $B$ with $F(B) = 1$ in $B$, we have to go through the planes $\{F = 0\}$ and $\{F = \delta\}$ containing focus curves. For each intersecting plane we have two potential choices: either go “on the left” (i.e., in the domain with $H_i \leq 0$) or “on the right” (i.e., in the domain with $H_i \geq 0$) of the focus curve. So, in total we have 4 potential choices. However, similarly to the case considered in Theorem 7.6, we can choose our data in such a way that all the 4 potential choices are wrong, so there is no straight line going from $A$ to $B$, as illustrated on Figure 26. □

Theorem 8.25 can, of course, be extended to higher dimensions, simply by taking symplectic direct products. Notice that in all non-convex Theorems 8.25, 8.22, and 7.6, the monodromy group is big (its image in $GL(n, \mathbb{Z})$ has at least 2 generators). So one may say that big monodromy allows non-convexity in all this case. The following theorems essentially state that if there are $n-1$ independent global affine functions on $B$ (so the monodromy group is not too complicated), then we always have convexity (under the compactness or properness condition) in any dimension.

**Theorem 8.26.** Let $B$ be the base space of a toric-focus integrable Hamiltonian system with $n$ degrees of freedom on a connected compact symplectic manifold $M$. Assume that the system admits a global Hamiltonian $\mathbb{T}^{n-1}$-action. Then $B$ is convex.

**Proof.** Let $L = (L_1, \ldots, L_{n-1}) : M \to \mathbb{R}^{n-1}$ be the momentum map of the $\mathbb{T}^{n-1}$-action. By the Atiyah-Guillemin-Sternberg theorem [6, 54], $L(M)$ is a convex polytope in $\mathbb{R}^{n-1}$ and the preimage in $M$ by $L$ of each point is connected. Because the torus action preserves the integrable system, $L$ descends to a map $B \to \mathbb{R}^{n-1}$, which we still call $L$, and the preimage of each point by $L$ in $B$ is still connected.
Let \( x, y \in \mathcal{B} \) be arbitrary distinct points; we need to show that there is a straight line connecting \( x \) to \( y \). If \( \mathbf{L}(x) = \mathbf{L}(y) = c \in \mathbb{R}^{n-1} \), then \( x, y \in \mathbf{L}^{-1}(c) \subset \mathcal{B} \) which is connected and is a straight line, so we are done.

Consider the case when \( \mathbf{L}(x) = c_1, \mathbf{L}(y) = c_2 \), with \( c_1 \neq c_2 \). Let \( \ell \) be the intersection of the straight line passing through \( c_1 \) and \( c_2 \) with the polytope \( \mathbf{L}(M) \). Then \( \ell \) is a closed interval. Let \( P = \mathbf{L}^{-1}(\ell) \subset \mathcal{B} \). Then \( P \) contains \( x \) and \( y \), it inherits a singular affine structure from \( \mathcal{B} \), it is compact and locally convex (since \( \mathcal{B} \) is compact and locally convex). Note that \( P \) is connected since the interval \( \ell \) is connected, the preimage of every point in \( \ell \) by \( \mathbf{L} \) is connected, and \( P \) is compact.

The singular points in \( \mathcal{B} \) are still of focus type, except for the fact their indices may not be integers and the affine structure on \( \mathcal{B} \) may not be integral. However, on \( \mathcal{B} \) we have a global affine function (one of the \( L_i \)), so by Remark 8.18, the conclusion of Theorem 8.14 still holds, which means that \( P \) is convex and hence there is a straight line from \( x \) to \( y \) in \( P \subset \mathcal{B} \). \( \square \)

**Definition 8.27.** The singular affine manifold \( \mathcal{B} \) is called **proper** if for any closed, connected, simply connected subset \( S \subset \mathcal{B} \), the local injective map \( \varphi : S \rightarrow \mathbb{R}^n \) given by an \( n \)-tuple of independent affine functions is a proper map.

**Theorem 8.28.** Let \( \mathcal{B} \) be the \( n \)-dimensional base space of a toric-focus integrable Hamiltonian system on a connected, non-compact, symplectic, \( 2n \)-manifold without boundary. Assume:

(i) The system admits a global Hamiltonian \( \mathbb{T}^{n-1} \)-action with momentum map \( \mathbf{L} : M \rightarrow \mathbb{R}^n \);

(ii) the set of focus points in \( \mathcal{B} \) is compact;

(iii) the interior of \( \mathcal{B} \) is homeomorphic to an open ball in \( \mathbb{R}^n \);

(iv) \( \mathcal{B} \) is proper (see Definition 8.27).

Then \( \mathcal{B} \) is convex (in its own underlying affine structure).

**Proof.** This is done by reducing the statement to Theorem 8.26 exactly in the same way Theorem 8.21 was proved as a consequence of a cutting construction and Theorem 8.14. \( \square \)

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