EXISTENCE RESULTS AND STABILITY ANALYSIS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEM ON A CIRCULAR RING WITH AN ATTACHED EDGE: A STUDY OF FRACTIONAL CALCULUS ON METRIC GRAPH

VAIBHAV MEHANDIRATTA AND MANI MEHRA*

Department of Mathematics, Indian Institute of Technology Delhi
110016, Delhi, India

GÜNTER LEUGERING

Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
Lehrstuhl Angewandte Mathematik II
Cauerstr. 11, 91058 Erlangen, Germany

(Communicated by Christian Rohde)

Abstract. In this paper, we study a nonlinear fractional boundary value problem on a particular metric graph, namely, a circular ring with an attached edge. First, we prove existence and uniqueness of solutions using the Banach contraction principle and Krasnoselskii's fixed point theorem. Further, we investigate different kinds of Ulam-type stability for the proposed problem. Finally, an example is given in order to demonstrate the application of the obtained theoretical results.

1. Introduction. The study of differential equation on metric graphs dates back to the 1980s with Lumer’s work [32], where he investigated the evolution equations on so-called ramification spaces. Later in [46], Nicaise studied the propagation of nerve impulses in dendrites. This early work triggered a substantial amount of research for boundary value problems, in particular, Sturm-Liouville type problems on metric graphs, see the work by von Below [7] and [48]. In [17], Gordeziani et al. considered classical boundary value problems on metric graphs. They established existence and uniqueness of the solution and also presented a numerical approach for solving such type of equations using a double-sweep method. Leugering [31] considered an optimal control problem on a gas network modeled by the nonlinear isothermal Euler gas equations. Using a non-overlapping domain decomposition method, he converted the considered problem on a general graph into local problems on a small part of the graph, in particular into subgraphs considered here. We refer to [30, 8, 50, 52, 27, 26, 11] for more problems and applications of differential equations on metric graphs including flexible structures consisting of strings, beams and plates, quantum graphs as well as textiles and irrigation systems.

2020 Mathematics Subject Classification. Primary: 34A08, 34B15, 26A33; Secondary: 34D20.
Key words and phrases. Fractional boundary value problem, fixed point theorems, Ulam-type stability, metric graph.

* Corresponding author: Mani Mehra.
On the other hand, fractional calculus has attracted increasing interest in different fields of science and engineering [20, 33, 47, 53, 43, 28, 29] over the years. Unlike the classical derivatives, fractional derivatives are non-local in nature and thus, fractional operators are a very natural tool to model e.g. memory-dependent phenomena used by engineers and physicists. Moreover, in various phenomena, fractional derivatives have proved to be more flexible in the modeling than classical derivatives. We refer to Almeida et al. [3], where the authors considered real-life applications such as population growth model and blood alcohol level problem and proved (based on experimental data) that fractional differential equations model more efficiently certain problems than ordinary differential equations. The fractional operators that are most frequently used in the literature are the Caputo and Riemann-Liouville fractional derivatives and both have the same kernel, namely, 
\[ k(s, t) = (t - s)^{n-\alpha-1}, \quad n - 1 < \alpha < n. \]
They have been constructed using a power law (if viewed as convolution) and exhibit a singularity. However, it is not always possible to find power law behaviour in nature. Therefore, in recent years some authors have also defined new fractional operators which have non-singular kernels. The Caputo-Fabrizio fractional derivative [9] which is constructed based on an exponential decay law is an example. Later in 2016, Atangana and Baleanu suggested yet another fractional operator [4] based on the Mittag–Leffler function which is a generalization of the exponential function. The Atangana-Baleanu fractional derivative is again non-local and non-singular.

We may, therefore, classify fractional operators into those with singular and non-singular kernels. Fractional derivatives with singular kernels including Caputo and Riemann-Liouville have been applied in different applications such as the modelling of viscoelastic materials, image processing, control theory etc. On the other hand, fractional derivatives with non-singular kernels (Caputo-Fabrizio and Atangana-Balenau derivatives) have been used e.g. in thermal science, material sciences, modelling of the mass-spring-damper system, gas dynamics equation etc. For completeness, we also mention another recently defined fractional operator, namely, the M-fractional derivative and Atangana’s conformable derivative. The M-fractional derivative, in turn, has been used to define some physical process such as longitudinal wave equation in a magnetoelectro-elastic circular rod [34, 6] and Schrödinger-Hirota equation [36], see also [37, 13], while the fractional conformable derivatives have also attracted the researchers in the recent years, see in particular the work of Aguilar et al. [10, 39, 35, 14, 38] on the underlying fractional derivatives.

There has been considerable interest in developing the theory of existence of solution for fractional differential equations (FDEs) and it could be considered as one of the most preferable areas in the field of FDEs. Various authors have established existence and uniqueness of solution for fractional boundary value problems (FBVPs) on intervals, by applying different fixed point theorems, the variational method, the upper and lower solution method, etc. For detailed analysis, see [1, 16, 5, 58, 12] and the references therein. Another emerging area that has recently attracted a lot of attention is the investigation of Ulam-type stability of FDEs. Ulam-type stability, posed by Ulam [55] in 1940, has been of great interest to various researchers since the last two decades. The problem introduced by Ulam was the following: “Under what conditions does there exist an additive mapping near an approximately additive mapping?” The problem was solved by Hyers [21] in the subsequent year (1941). Since then, this type of stability has come to be known as Ulam-Hyers stability. Later in 1978, Rassias [51] provided a generalization for the Ulam-Hyers
stability of mappings. The mentioned stability analysis has various applications in optimization, physics, biology, etc. where it is difficult to find an analytic solution of the nonlinear problem and it guarantees that there is a close exact solution of the problem. Recently, considerable amount of work has been done on the Ulam-Hyers and Ulam-Hyers-Rassias stability for FBVPs on one-dimensional equations and coupled systems, see in particular [22, 23, 2, 57].

However, in the case of fractional differential equations on metric graphs, not much work has been published until now. It has been initiated by Graef et al. in [18], where the authors established the existence and uniqueness of solution for FBVP on a star graph with only two edges and the fractional derivative was considered in Riemann-Liouville sense. Later in [41], the authors proved the existence and uniqueness of solution for a FBVP on a general star graph (see Figure 1) having \( k \) edges and \( k + 1 \) vertices. In contrast to [18], the authors considered the fractional derivative in the Caputo sense and a more general nonlinear term which depends on the unknown function and its fractional derivative. Recently in [59], Liu and Zhang extended the work of Graef et al. on a general star graph and proved the existence and uniqueness of solution using the Banach contraction principle and Schaefer's fixed point theorem. Moreover, they have also investigated the different kinds of Ulam-type stability for the considered problem. We refer to [42, 45, 40] for recent articles of FDEs on metric graphs.

As is obvious from real world applications, significant networks include cyclic subgraphs. At a first glance, it seems, therefore, astounding that the majority of publications dwell on tree-like networks. A closer look at networks containing cycles, however, reveals that a number of difficulties arise when cycles are involved, in particular when questions of control and observation as well as numerical simulations are in the focus. As already mentioned, there are different types of differential equations used to model physical phenomenon on the networks (or metric graphs), namely, isothermal Euler-gas or Saint Venant equations (for gas and water networks, respectively), Kelvin-Voigt-type models (e.g. for electric circuits), hyperbolic wave equations (for the network of strings and Timoshenko beams), differential-difference equations (network of cells) [44]. In many of these applications, fractional derivatives become more and more important in extending constitutive equations for instance, damping mechanisms to more flexible models. To the best knowledge of the
authors, no results are available, however, for the corresponding nonlinear fractional boundary value problems on metric graphs containing cycles. Moreover, there are various chemical compounds such as glucose, ethane, cyclohexane that also have the cyclic graphical structure. In particular, quantum graphs, i.e. Schrödinger equations on graphs with cycles, have been considered in [27, 26, 11]. More recently, Gnutzmann et al. [15] considered nonlinear quantum graphs including cycles, while Klein et al. in [25] used a fractional nonlinear Schrödinger equation and developed numerical tools for it. The model considered in this article can, thus, be seen as a first study to approach stationary fractional quantum graphs including cycles.

In order to reduce the complexity of the presentation, we restrict ourselves to the simplest nontrivial nonlinear fractional boundary value problem on a metric graph with a cycle. In particular, we consider the nonlinear FBVP on a circular ring with an attached edge. The circular graph used in this article is, thus, paradigmatic for networks containing cycles. In order to describe the problem, we consider a graph \( G = (V, E) \) consisting of a set of nodes \( V = \{v_0, v_1, v_2\} \) and a set of edges (or arcs) \( E = \{e_1 = \overrightarrow{v_0v_1}, e_2 = \overrightarrow{v_1v_2}, e_3 = \overrightarrow{v_2v_0}\} \) connecting these nodes. Since both the edges \( e_1 \) and \( e_2 \) are connected to the same nodes \( v_0 \) and \( v_1 \), \( G \) is a connected graph with multiple edges. However, the graph considered in this paper is a metric graph, therefore each edge of the graph is parametrised by some interval. More precisely, edges \( e_1 \) and \( e_2 \) are parametrised by the interval \((0, \pi)\) and the edge \( e_3 \) by \((0, l)\) with \(0 < l < \infty\). Therefore, edges \( e_1 \) and \( e_2 \) form a cycle (circular ring) of length \(2\pi\) with an attached edge \( e_3 \) of length \(l\) (see Figure 2). Hence, by considering \(v_0\) as origin and \(x \in (0, \pi)\) as the coordinate for \(e_1, e_2\) and \(z \in (0, l)\) for \(e_3\), we model the nonlinear fractional boundary value problem on the considered metric graph as

\[
C D^\alpha_{0,z} u_i(x) = f_i \left( x, u_i(x), C D^\beta_{0,x} u_i(x) \right), \quad i = 1, 2, \quad 0 < x < \pi,
\]

\[
C D^\alpha_{0,z} u_3(z) = f_3 \left( z, u_3(z), C D^\beta_{0,z} u_3(z) \right), \quad 0 < z < l,
\]

\[
u_1(0) = u_2(0) = u_3(0), \quad u_1(\pi) = u_2(\pi),
\]

\[
u_3'(0) + u_2'(0) + u'_3(0) = 0, \quad u_1'(\pi) + u_2'(\pi) = 0,
\]

\[
u_3(l) = 0.
\]

Here, \( C D^\alpha_{0,z} \) denotes the Caputo fractional derivative of order \( \alpha \) with respect to \( z \), conditions (3) and (4) are called transmission conditions, namely, continuity and Kirchhoff conditions, respectively at the junction nodes \( v_0 \) and \( v_1 \), (5) denotes a Dirichlet condition at the boundary node \( v_2 \), \( 1 < \alpha \leq 2 \) and \( 0 < \beta \leq \alpha - 1 \). The nonlinear functions \( f_i, i = 1, 2 \) and \( f_3 \) are continuous on \([0, \pi] \times \mathbb{R} \times \mathbb{R}\) and \([0, l] \times \mathbb{R} \times \mathbb{R}\), respectively.

As indicated above, this article could be considered as the extension of the above-mentioned results, in particular [41] (with addition to Ulam-type stability results) to metric graphs containing cycles. In contrast to the star graph, where only one junction node \( v_0 \) is present (see Figure 1), we have two junction nodes (due to cyclic structure), namely, \( v_0 \) and \( v_1 \) as shown in Figure 2. Therefore, one has to incorporate the transmission conditions at both nodes \( v_0 \) \((x = 0)\) and \( v_1 \) \((x = \pi)\) given by (3) and (4). These circumstances increase the complexity for the investigation of Ulam-type stability considerably.

To the best knowledge of the authors, no results have been published so far for FBVPs on metric graphs containing cycles. This paper aims to develop the existence and uniqueness of solution and different kinds of Ulam-type stability for the FBVP...
Figure 2. A circular ring with an attached edge (1)-(5). For the sake of simplicity and in order to avoid cumbersome notations, in this paper, we consider a particular metric graph. However, the analysis carried out in the present work could be easily extended to general metric graphs containing cycles.

The rest of the paper is organised as follows: In Sec. 2, we recall basic definitions of fractional calculus, introduce lemma (Lemma 2.4) from [41] that transforms the BVP system (1)-(5) to a BVP system on [0, 1] and prove another important lemma (Lemma 2.7) which forms the basis of understanding for solving the considered nonlinear problem. In Sec. 3, first, we prove the existence and uniqueness of the solution for FBVP (1)-(5) using Banach contraction principle for Lipschitz-continuous functions \( f_i \) (defined in Sec. 2) and a certain assumption involving on fractional orders, \( \alpha, \beta \) and Lipschitz constants \( L_i \). Then, for bounded and Lipschitz continuous \( f_i' \)s, we apply Krasnoselskii’s fixed point theorem to obtain the existence of at least one solution. In Sec. 4, we establish applicable results under which the solution of the considered FBVP (1)-(5) satisfies the conditions of Ulam’s stability. The established results in sections 3 and 4 are illustrated by an example in Sec. 5.

2. Preliminaries. In this section, we first provide basic definitions of fractional calculus and then give some known results which will be used throughout this paper.

Definition 2.1. The fractional integral of order \( \alpha > 0 \) for a continuous function \( f \) on \( [a, b] \) is defined as

\[
D_{a, t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \left( \int_a^t (t - s)^{-\alpha - 1} f(s) ds \right),
\]

where \( \Gamma(\cdot) \) is the Euler gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) for a function \( f \) on \( [a, b] \) is defined as

\[
_{RL}D_{a, t}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \left( \int_a^t (t - s)^{n-\alpha-1} f(s) ds \right),
\]

where the function \( f(t) \) has absolutely continuous derivatives up to order \( (n - 1) \), \( n - 1 \leq \alpha < n, n \in \mathbb{N} \).
Definition 2.3. The Caputo fractional derivative of order \( \alpha > 0 \) for a function \( f \) on \([a, b]\) is defined as

\[
C D_{a,t}^\alpha f(t) = RL D_{a,t}^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t - a)^k \right],
\]

where \( n-1 < \alpha \leq n, n \in \mathbb{N} \).

Remark 1. If \( f \in C^n[a, b] \), then the Caputo fractional derivative of order \( \alpha > 0 \) is given by

\[
C D_{a,t}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \right)
\]

where \( n \) is same as defined in equation (8).

Now, we give the following lemma (for the proof see [41]) which results in converting the fractional BVP (1)-(5) to a BVP system on \([0, 1]\).

Lemma 2.4. Let \( u \) be a function defined on \([0, d]\) such that \( C D_{0, t}^\alpha u \) exists on \([0, d]\) with \( \alpha > 0 \). Let \( x \in [0, d], t = x/d \in [0, 1] \) and \( y(t) = u(dt) \). Then

\[
(C D_{0, t}^\alpha u)(x) = d^{-\alpha}(C D_{0, t}^\alpha y(t)).
\]

In view of the above lemma, BVP (1)-(5) can be transformed into a BVP system on \([0, 1]\) given by

\[
C D_{0, t}^\alpha y_i(t) = \pi^\alpha f_i \left( t, y_i(t), \pi^{-\beta} C D_{0, t}^\beta y_i(t) \right), \quad i = 1, 2, \quad 0 < t < 1,
\]

\[
C D_{0, t}^\alpha y_3(t) = l^\alpha f_3 \left( t, y_3(t), l^{-\beta} C D_{0, t}^\beta y_3(t) \right), \quad 0 < t < 1,
\]

\[
y_1(0) = y_2(0) = y_3(0), \quad y_1(1) = y_2(1),
\]

\[
\pi^{-1} y'_1(0) + \pi^{-1} y'_2(0) + l^{-1} y'_3(0) = 0, \quad y'_1(1) + y'_2(1) = 0,
\]

\[
y_3(1) = 0,
\]

where \( y_i(t) = u_i(pt), f_i(t, x, y) = f_i(pt, x, y), i = 1, 2, y_3(t) = u_3(lt) \) and \( f_3(t, z, y) = f_3(lt, z, y) \).

Lemma 2.5. (see [24]) Let \( \alpha > 0 \), then the solution of the fractional differential equation \( C D_{0, t}^\alpha y(t) = 0 \) is given by

\[
y(t) = c_1 + c_2 t + c_3 t^2 + \ldots + c_n t^{n-1},
\]

where \( c_i \in \mathbb{R}, i = 1, 2, \ldots, n \) and \( n \) is the smallest integer greater than or equal to \( \alpha \).

From Lemma 2.5, it follows that

\[
D_{0, t}^{-\alpha} C D_{0, t}^\alpha y(t) = y(t) + c_1 + c_2 t + c_3 t^2 + \ldots + c_n t^{n-1},
\]

for some \( c_i \in \mathbb{R}, i = 1, 2, \ldots, n \).

Theorem 2.6. [56] Let \( A \) be a square matrix of order \( n \) having positive real entries, i.e. \( A \in M_{nn}(\mathbb{R}^+) \). Then, the following statements are equivalent:

(i) The eigenvalues of matrix \( A \), denoted by \( \lambda \), are in the open unit disc, i.e. \( |\lambda| < 1 \).

(ii) The matrix \((I - A)^{-1}\) is nonsingular and \((I - A)^{-1}\) has nonnegative elements.

Moreover

\[
(I - A)^{-1} = I + A + \ldots + A^m + \ldots.
\]
Lemma 2.7. Let $\sigma_i \in C[0,1]$, then the solution of the fractional differential equations

$$c D_0^\alpha y_i(t) = \sigma_i(t), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \quad i = 1, 2, 3,$$

(16)

together with the transmission conditions (12)-(13) and the boundary condition (14) is given by

$$y_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma_1(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \sigma_3(s) ds$$

$$- \frac{l\pi^{-1}}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}(\sigma_1(s) + \sigma_2(s)) ds$$

$$- t \left[ \frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}(\sigma_1(s) + \sigma_2(s)) ds \right] + (1-t) \frac{(-1)^i+1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\sigma_1(s) - \sigma_2(s)) ds,$$

(17)

and

$$y_3(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma_3(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \sigma_3(s) ds$$

$$+ (t-1) \left[ \frac{l\pi^{-1}}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}(\sigma_1(s) + \sigma_2(s)) ds \right].$$

(18)

Proof. In view of (15), we have

$$y_1(t) = D_0^\alpha \sigma_i(t) - c_i^{(1)} - c_i^{(2)} t = \frac{1}{\Gamma(\alpha)} \left( \int_0^t (t-s)^{\alpha-1} \sigma_i(s) ds \right) - c_i^{(1)} - c_i^{(2)} t,$$

(19)

where $c_i^{(j)}$, $i = 1, 2, 3$, $j = 1, 2$, are some constants. The continuity conditions (12) gives

$$c_1^{(1)} = c_2^{(1)} = c_3^{(1)} \quad \text{and} \quad \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\sigma_1(s) - \sigma_2(s)) ds = c_1^{(2)} - c_2^{(2)}.$$ 

(20)

Moreover

$$y_1'(t) = \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \sigma_i(s) ds - c_i^{(2)} = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \sigma_i(s) ds - c_i^{(2)}.$$

Now, the Kirchoff condition (13) implies that $c_i^{(2)}$, $i = 1, 2, 3$, must satisfy

$$\pi^{-1} c_1^{(2)} + \pi^{-1} c_2^{(2)} + l^{-1} c_3^{(2)} = 0,$$

$$\frac{1}{\Gamma(\alpha-1)} \left( \int_0^1 (1-s)^{\alpha-2}(\sigma_1(s) + \sigma_2(s)) ds \right) = c_1^{(2)} + c_2^{(2)}.$$ 

(21)

Finally, the Dirichlet boundary condition (14) gives,

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \sigma_3(s) ds - \lambda - c_3^{(2)} = 0, \quad \text{where} \quad \lambda = c_1^{(1)} = c_2^{(1)} = c_3^{(1)}.$$ 

(22)
After solving (20)-(22), we get
\[
c_i^{(2)} = \frac{1}{2\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2}(\sigma_1(s) + \sigma_2(s))ds \\
+ \frac{(-1)^{i+1}}{2\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}(\sigma_1(s) - \sigma_2(s))ds, \quad i = 1, 2,
\]
\[
c_3^{(2)} = \frac{l\pi^{-1}}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2}(\sigma_1(s) + \sigma_2(s))ds
\]
and
\[
c_1^{(1)} = c_2^{(1)} = c_3^{(1)} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}\sigma_3(s)ds + \frac{l\pi^{-1}}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2}(\sigma_1(s) + \sigma_2(s))ds.
\]

On substituting the values of $c_i^{(j)}$, $i = 1, 2, 3$, $j = 1, 2$ in (19), we obtain (17)-(18).

**Remark 2.** Using the Definition 2.3 of the Caputo derivative and by direct computations, it can be easily shown that if $(y_i)_{1 \leq i \leq 3}$ are given by (17) and (18), then $(y_i)_{1 \leq i \leq 3}$ also satisfy (16) along with transmission conditions (12)-(13) and boundary condition (14).

3. **Existence and uniqueness results.** In this section, Banach’s contraction principle and Krasnoselskii’s fixed point theorem will be used to establish the existence of solution of BVP (10)-(14). For $1 < \alpha \leq 2$, $0 < \beta \leq \alpha - 1$, we define the space $X = \{y : y \in C([0, 1]),\quad cD_0^\beta y \in C([0, 1])\}$ equipped with the norm
\[
\|y\|_X = \|y\| + \|cD_0^\beta y\|; \quad \|y\| = \sup_{t \in [0, 1]} |y(t)|, \quad \|cD_0^\beta y\| = \sup_{t \in [0, 1]} |cD_0^\beta y(t)|.
\]
Then, $(X, \|\cdot\|_X)$ is a Banach space (see [54]) and, consequently, the product space $(X^3 = X \times X \times X, \|\cdot\|_{X^3})$ is a Banach space with norm
\[
\|(y_1, y_2, y_3)\|_{X^3} = \sum_{i=1}^3 \|y_i\|_X \quad \text{for} \quad (y_1, y_2, y_3) \in X^3.
\]

In view of Lemma 2.7, we define the operator $T : X^3 \to X^3$, associated with the BVP (10)-(14), by
\[
T(y_1, y_2, y_3)(t) := (T_1(y_1, y_2, y_3)(t), T_2(y_1, y_2, y_3)(t), T_3(y_1, y_2, y_3)(t)),
\]
where
\[
T_i(y_1, y_2, y_3)(t)
\]
\[
= \frac{\pi^\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1 \left(s, y_1(s), \pi^{-\beta} cD_0^\beta y_1(s)\right)ds
\]
\[
- \frac{l\pi^{-1}}{\Gamma(\alpha)} \int_0^t (1-s)^{\alpha-1} f_3 \left(s, y_3(s), l^{-\beta} cD_0^\beta y_3(s)\right)ds
\]
\[
- \left( \frac{t\pi^{-1}}{\Gamma(\alpha - 1)} + \frac{t\pi^{-1}}{2\Gamma(\alpha - 1)} \right) \int_0^1 (1-s)^{\alpha-2} \left[ f_1 \left(s, y_1(s), \pi^{-\beta} cD_0^\beta y_1(s)\right)
\right.
\]
\[
\left. + f_2 \left(s, y_2(s), \pi^{-\beta} cD_0^\beta y_2(s)\right) \right]ds.
\]
Theorem 3.1. Let BVP (10)-(14) using Banach’s contraction principle. In the following theorem, we prove the existence and uniqueness of solution of the conditions and for \( T \in (0, 1) \), let

Proof. \( R \) is a contraction mapping. To this end, let \( y = (y_1, y_2, y_3) \), \( w = (w_1, w_2, w_3) \in X^3 \) and \( t \in [0, 1] \). Then, using equation (25),

\begin{align*}
R_{\alpha, \beta} &= \left[ \frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right] \\
&\times \left( \pi^\alpha + \pi^{\alpha - \beta} \right) + \left( t\pi^{\alpha - 1} + t\pi^{\alpha - \beta - 1} \right) \left[ \frac{4}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} \right] \\
&+ \left( t\pi^\alpha + t\pi^{\alpha - \beta} \right) \left[ \frac{4}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \right].
\end{align*}

(25)

\begin{align*}
P_{\alpha, \beta} &= \left( \pi^\alpha + \pi^{\alpha - \beta} \right) \left[ \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right] \\
&\times \left( t\pi^{\alpha - 1} + t\pi^{\alpha - \beta - 1} \right) \left[ \frac{4}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} \right] + \frac{3}{\Gamma(\alpha + 1)} \left( t\pi^\alpha + t\pi^{\alpha - \beta} \right).
\end{align*}

(26)

In the following theorem, we prove the existence and uniqueness of solution of the BVP (10)-(14) using Banach’s contraction principle.

Theorem 3.2. Let \( f_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( i = 1, 2, 3 \) be continuous functions satisfying the conditions

\[ |f_i(t, x, z) - f_i(t, x_1, z_1)| \leq L_i(|x - x_1| + |z - z_1|), \quad L_i > 0, \quad t \in [0, 1], \]

(27)

then the BVP (10)-(14) has a unique solution on \([0, 1]\) if \( R_{\alpha, \beta} \left( \sum_{i=1}^{3} L_i \right) < 1 \), where \( R_{\alpha, \beta} \) is given by equation (25).

Proof. We prove the result by showing that \( T \) is a contraction mapping. To this end, let \( y = (y_1, y_2, y_3) \), \( w = (w_1, w_2, w_3) \in X^3 \) and \( t \in [0, 1] \). Then, using equation (25),

\begin{align*}
T_3(y_1, y_2, y_3)(t) &= \frac{l^\alpha}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f_3 \left( s, y_3(s), l^{-\beta}C D_0^\beta y_3(s) \right) ds \\
&- \frac{l^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} f_3 \left( s, y_3(s), l^{-\beta}C D_0^\beta y_3(s) \right) ds \\
&+ (t - 1) \frac{l^\alpha}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} \left[ f_1 \left( s, y_1(s), l^{-\beta}C D_0^\beta y_1(s) \right) \\
&\quad + f_2 \left( s, y_2(s), l^{-\beta}C D_0^\beta y_2(s) \right) \right] ds.
\end{align*}

(24)

Remark 3. In view of Remark 2, it is clear that the BVP (10)-(14) has a solution if and only if \( T \) has a fixed point in \( X^3 \).
(23), we have

\[
|T_y(t) - T_w(t)|
\]
\[
\leq \frac{\pi^\alpha}{\Gamma(\alpha + 1)} L_i \|y_i - w_i\| + \frac{\pi^\alpha}{\Gamma(\alpha + 1)} L_i \|cD^\beta_{0,s}y_i - cD^\beta_{0,s}w_i\|
\]
\[
+ \frac{l^\alpha}{\Gamma(\alpha + 1)} L_3 \|y_3 - w_3\| + \frac{l^{n-\beta}}{\Gamma(\alpha + 1)} L_3 \|cD^\beta_{0,s}y_3 - cD^\beta_{0,s}w_3\|
\]
\[
+ \sum_{j=1}^{2} \left[ \left( \frac{l^{n-\beta}}{\Gamma(\alpha + 1)} + \frac{\pi^\alpha}{2\Gamma(\alpha)} \right) L_j \|y_j - w_j\|
\]
\[
+ \left( \frac{l^{n-\beta}}{\Gamma(\alpha + 1)} + \frac{\pi^\alpha}{2\Gamma(\alpha)} \right) L_j \|cD^\beta_{0,s}y_j - cD^\beta_{0,s}w_j\| \right]
\]
\[
+ \sum_{j=1}^{2} \left( \frac{\pi^\alpha}{2\Gamma(\alpha + 1)} L_j \|y_j - w_j\| + \frac{\pi^\alpha}{2\Gamma(\alpha + 1)} L_j \|cD^\beta_{0,s}y_j - cD^\beta_{0,s}w_j\| \right)
\]
\[
\leq \frac{1}{\Gamma(\alpha + 1)} \left( \pi^\alpha + \pi^{n-\beta} \right) L_i \left( \|y_i - w_i\| + \|cD^\beta_{0,s}y_i - cD^\beta_{0,s}w_i\| \right)
\]
\[
+ \frac{1}{\Gamma(\alpha + 1)} \left( l^n + l^{n-\beta} \right) L_3 \left( \|y_3 - w_3\| + \|cD^\beta_{0,s}y_3 - cD^\beta_{0,s}w_3\| \right)
\]
\[
+ \sum_{j=1}^{2} \left[ \left( \frac{l^{n-\beta}}{\Gamma(\alpha + 1)} + \frac{\pi^\alpha}{2\Gamma(\alpha)} + \frac{\pi^\alpha}{2\Gamma(\alpha + 1)} \right) L_j \|y_j - w_j\|
\]
\[
+ \left( \frac{l^{n-\beta}}{\Gamma(\alpha + 1)} + \frac{\pi^\alpha}{2\Gamma(\alpha)} + \frac{\pi^\alpha}{2\Gamma(\alpha + 1)} \right) L_j \|cD^\beta_{0,s}y_j - cD^\beta_{0,s}w_j\| \right]
\]
\[
\leq \left( \pi^\alpha + \pi^{n-\beta} \right) \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{2\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} \right) + \frac{1}{\Gamma(\alpha)} \left( l^n + l^{n-\beta} \right)
\]
\[
\times L_i \left( \|y_i - w_i\| + \|cD^\beta_{0,s}y_i - cD^\beta_{0,s}w_i\| \right)
\]
On the other hand, by using the relation \[49\]

Therefore, we get

Following a similar analysis as above, one gets from (24)

we obtain,

\[i, j = 1, 2 \text{ and } i \neq j. \]

Hence, taking the supremum over \(t \in [0, 1]\) in above inequality, we obtain,

\[
\|T_i y - T_i w\| \leq \left[ \left( \pi^n + \pi^{n-\beta} \right) \left( \frac{1}{2\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} \right) + \frac{1}{\Gamma(\alpha + 1)} \right] L_3 \|y_3 - w_3\| + \frac{2l^{n-\beta}}{\Gamma(\alpha + 1)} \left( \|y_3 - C D_{0,s}^\beta y_3 - C D_{0,s}^\beta w_3\| \right)
\]

\[
+ \sum_{j=1}^{2} \left( \frac{2l^{n-1}}{\Gamma(\alpha)} L_j \|y_j - w_j\| + \frac{2l^{n-\beta-1}}{\Gamma(\alpha)} L_j \|C D_{0,s}^\delta y_j - C D_{0,s}^\delta w_j\| \right)
\]

\[
\leq \frac{2l^{n}}{\Gamma(\alpha + 1)} \left( \|y_3 - w_3\| + \|C D_{0,s}^\beta y_3 - C D_{0,s}^\beta w_3\| \right)
\]

\[
+ \frac{2}{\Gamma(\alpha)} \left( l^{n-1} + l^{n-\beta-1} \right) L_3\left( \|y_3 - w_3\| + \|C D_{0,s}^\beta y_j - C D_{0,s}^\beta w_j\| \right).
\]

Therefore, we get

\[
\|T_3 y - T_3 w\| \leq \frac{2l^{n}}{\Gamma(\alpha + 1)} \left( l^{n} + l^{n-\beta} \right) L_3\|y_3 - w_3\| x
\]

\[
+ \sum_{j=1}^{2} \frac{2}{\Gamma(\alpha)} \left( l^{n-1} + l^{n-\beta-1} \right) L_j\|y_j - w_j\| x.
\]

(29)

On the other hand, by using the relation \[49\]

\[
c D_{0,t}^\beta(t^\gamma) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \beta)} t^{\gamma - \beta}, \quad 0 < \beta < 1
\]

(30)
and the fact that the Caputo derivative of a constant function is zero, we get
\[
|cD_{0,i}^{\alpha}T_i y(t) - cD_{0,i}^{\beta}T_i w(t)| \\
\leq \frac{\pi^\alpha}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} \left| f_i \left( s, y_i(s), \pi^{-\beta} cD_{0,s}^{\beta} y_i(s) \right) - f_i \left( s, w_i(s), \pi^{-\beta} cD_{0,s}^{\beta} w_i(s) \right) \right| ds \\
+ \frac{\pi^\alpha t^{1-\beta}}{2\Gamma(\alpha - 1)\Gamma(2 - \beta)} \sum_{j=1}^2 \int_0^1 (1 - s)^{\alpha - 2} \left[ |f_j \left( s, y_j(s), \pi^{-\beta} cD_{0,s}^{\beta} y_j(s) \right) - f_j \left( s, w_j(s), \pi^{-\beta} cD_{0,s}^{\beta} w_j(s) \right) | ds \right]
\]
\]
Again, by using (27) and \( t^{1-\beta} \leq 1 \) (as \( 1 - \beta \geq 0 \), we obtain
\[
|cD_{0,i}^{\beta}T_i y(t) - cD_{0,i}^{\beta}T_i w(t)| \\
\leq \frac{\pi^\alpha}{\Gamma(\alpha - \beta + 1)} L_i \|y_i - w_i\| + \frac{\pi^\alpha}{\Gamma(\alpha - \beta + 1)} L_i \|cD_{0,s}^{\beta} y_i - cD_{0,s}^{\beta} w_i\| \\
+ \sum_{j=1}^2 \left( \frac{\pi^\alpha}{2\Gamma(\alpha)\Gamma(2 - \beta)} L_j \|y_j - w_j\| + \frac{\pi^\alpha}{2\Gamma(\alpha)\Gamma(2 - \beta)} L_j \|cD_{0,s}^{\beta} y_j - cD_{0,s}^{\beta} w_j\| \right) \\
+ \sum_{j=1}^2 \left( \frac{\pi^\alpha}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)} L_j \|y_j - w_j\| \right)
\]
\[
\leq \frac{\pi^\alpha + \pi^\alpha - \beta}{\Gamma(\alpha - \beta + 1) + \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta) + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)}} \times L_i \left( \|y_i - w_i\| + \|cD_{0,s}^{\beta} y_i - cD_{0,s}^{\beta} w_i\| \right) \right) \\
+ \left( \frac{\pi^\alpha + \pi^\alpha - \beta}{2\Gamma(\alpha)\Gamma(2 - \beta) + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)}} \times L_j \left( \|y_j - w_j\| + \|cD_{0,s}^{\beta} y_j - cD_{0,s}^{\beta} w_j\| \right) \right) \\
\]
Hence, we get
\[
\|cD_{0,i}^{\beta}T_i y - cD_{0,i}^{\beta}T_i w\| \\
\leq \left( \frac{\pi^\alpha + \pi^\alpha - \beta}{\Gamma(\alpha - \beta + 1) + \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta) + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)}} \times L_i \|y_i - w_i\|_x \right) \\
+ \left( \frac{\pi^\alpha + \pi^\alpha - \beta}{2\Gamma(\alpha)\Gamma(2 - \beta) + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)}} \times L_j \|y_j - w_j\|_x , \text{  } \right) \quad (31) \\
i, j = 1, 2 \text{ and } i \neq j.
Finally, using (28)-(32), we get

\[ \left| c D_{0,t}^\beta T_3 y(t) - c D_{0,t}^\beta T_3 w(t) \right| \]
\[ \leq \frac{l^\alpha}{\Gamma(\alpha - \beta + 1)} \int_0^t (t - s)^{\alpha - \beta - 1} \left| f_3 \left( s, y_3(s), \pi^{-\beta} c D_{0,s}^\beta y_3(s) \right) \right. \]
\[ \left. - f_3 \left( s, w_3(s), \pi^{-\beta} c D_{0,s}^\beta w_3(s) \right) \right| ds \]
\[ + \frac{l^{\alpha-1} t^{1-\beta}}{\Gamma(\alpha-1) \Gamma(2-\beta)} \sum_{j=1}^2 \int_0^1 (1 - s)^{\alpha-2} \left| f_j \left( s, y_j(s), \pi^{-\beta} c D_{0,s}^\beta y_j(s) \right) \right. \]
\[ \left. - f_j \left( s, w_j(s), \pi^{-\beta} c D_{0,s}^\beta w_j(s) \right) \right| ds \]
\[ \leq \frac{l^\alpha}{\Gamma(\alpha - \beta + 1)} L_3 \| y_3 - w_3 \| + \frac{l^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} L_3 \| c D_{0,s}^\beta y_3 - c D_{0,s}^\beta w_3 \| \]
\[ + \sum_{j=1}^2 \left( \frac{l^{\alpha-1} t^{1-\beta}}{\Gamma(\alpha) \Gamma(2-\beta)} L_j \| y_j - w_j \| + \frac{l^{\alpha-\beta-1}}{\Gamma(\alpha) \Gamma(2-\beta)} L_j \| c D_{0,s}^\beta y_j - c D_{0,s}^\beta w_j \| \right) \]
\[ \leq \frac{1}{\Gamma(\alpha - \beta + 1)} \left( l^\alpha + l^{\alpha-\beta} \right) L_3 \left( \| y_3 - w_3 \| + \| c D_{0,s}^\beta y_3 - c D_{0,s}^\beta w_3 \| \right) \]
\[ + \sum_{j=1}^2 \frac{1}{\Gamma(\alpha) \Gamma(2-\beta)} \left( l^{\alpha-1} + l^{\alpha-\beta-1} \right) L_j \left( \| y_j - w_j \| + \| c D_{0,s}^\beta y_j - c D_{0,s}^\beta w_j \| \right). \]

Hence,

\[ \| c D_{0,t}^\beta T_3 y - c D_{0,t}^\beta T_3 w \| \leq \frac{1}{\Gamma(\alpha - \beta + 1)} \left( l^\alpha + l^{\alpha-\beta} \right) L_3 \| y_3 - w_3 \| + \sum_{j=1}^2 \frac{1}{\Gamma(\alpha) \Gamma(2-\beta)} \left( l^{\alpha-1} + l^{\alpha-\beta-1} \right) L_j \| y_j - w_j \|. \]

Finally, using (28)-(32), we get

\[ \sum_{i=1}^3 \| T_i y - T_i w \|_X \]
\[ = \sum_{i=1}^2 \left( \| T_i y - T_i w \| + \| c D_{0,i}^\beta T_i y - c D_{0,i}^\beta T_i w \| \right) \]
\[ + \left( \| T_3 y - T_3 w \| + \| c D_{0,3}^\beta T_3 y - c D_{0,3}^\beta T_3 w \| \right) \]
\[ \leq \left( \pi^\alpha + \pi^{\alpha-\beta} \right) \left( \frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{\Gamma(\alpha) \Gamma(2-\beta)} \right) \]
\[ + \frac{1}{\Gamma(\alpha + 1) \Gamma(2-\beta)} + \frac{2}{\Gamma(\alpha)} \left( l^{\alpha-1} + l^{\alpha-\beta-1} \right) \sum_{i=1}^2 L_i \| y_i - w_i \|_X \]
\[ + \frac{2}{\Gamma(\alpha + 1)} \left( l^\alpha + l^{\alpha-\beta} \right) L_3 \| y_3 - w_3 \|_X. \]
such that the equation $Az$ is completely continuous; satisfying the condition

Let $\alpha, \beta < 1$, then $B$ is a contraction mapping. Then there exists

$$L_3 \| y_3 - w_3 \|_X$$

Thus,

$$\| Ty - Tw \|_X^3$$

$$= \sum_{i=1}^{3} \| T_i y - T_i w \|_X$$

$$\leq \left[ (\pi^\alpha + \pi^{\alpha-\beta}) \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{2}{\Gamma(\alpha + 1) \Gamma(2 - \beta)} \right) + \sum_{i=1}^{3} L_i \| y_i - w_i \|_X \right]$$

Since $R_{\alpha, \beta} \left( \sum_{i=1}^{3} L_i \right) < 1$, it follows that $T$ is a contraction map and, thus, in view of Banach’s contraction principle, BVP (10)-(14) has a unique solution on $[0, 1]$. Therefore, BVP (1)-(5) has a unique solution. \( \square \)

Now, we prove the existence of solutions of BVP (10)-(14) using Krasnoselskii’s fixed point theorem. First, we state the following result due to Krasnoselskii [19].

**Lemma 3.2.** Let $\Omega$ be a bounded closed convex subset of a Banach space $X$. Let $A, B$ be mappings of $\Omega$ into $X$ such that (i) $Ay + Bw \in \Omega$ for every $y, w \in \Omega$; (ii) $A$ is completely continuous; (iii) $B$ is a contraction mapping. Then there exists $z \in \Omega$ such that the equation $Az + Bz = z$ has a solution in $\Omega$.

**Theorem 3.3.** Let $f_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2, 3$ be continuous functions satisfying the condition (27) and suppose there exists $M_i > 0$ such that

$$|f_i(t, x, z)| \leq M_i, \quad t \in [0, 1], \quad x, z \in \mathbb{R}, \quad i = 1, 2, 3,$$

then BVP (10)-(14) has at least one solution on $[0, 1]$ if $P_{\alpha, \beta} \left( \sum_{i=1}^{3} L_i \right) < 1$, where $P_{\alpha, \beta}$ is defined by equation (26).
**Proof.** Let \( \Omega = \{ y = (y_1, y_2, y_3) \in X^3 : \|y\| \leq r \} \), where \( r \) is chosen such that
\[
 r \geq \left[ \pi^\alpha \left( \frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right) 
+ t\pi^{\alpha - 1} \left( \frac{4}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} \right) + l^{\alpha} \left( \frac{4}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \right) \right] \left( \sum_{i=1}^{3} M_i \right) ,
\]
then \( \Omega \) is a bounded and closed convex subset of the Banach space \( X^3 \). We define the operators \( A \) and \( B \) on \( \Omega \) as
\[
 A_y(t) = (A_1y(t), A_2y(t), A_3y(t)) \quad \text{and} \quad B_y(t) = (B_1y(t), B_2y(t), B_3y(t)) ,
\]
where
\[
 A_{1}y(t) = \frac{\pi^\alpha}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g_1(s) ds, \quad A_{2}y(t) = \frac{\pi^\alpha}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g_2(s) ds,
\]
\[
 B_{1}y(t) = \frac{l^{\alpha}}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} g_3(s) ds - \left( \frac{\pi^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{t\pi^\alpha}{2\Gamma(\alpha-1)} \right) \int_{0}^{1} (1-s)^{\alpha-2} (g_1(s) + g_2(s)) ds 
+ \frac{(-1)^{i+1}}{2\Gamma(\alpha)} t\pi^\alpha \int_{0}^{1} (1-s)^{\alpha-1} (g_2(s) - g_1(s)) ds,
\]
\[
 B_{3}y(t) = -\frac{l^{\alpha}}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} g_3(s) ds 
+ (t-1) \frac{l^{\alpha-1}}{\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} (g_1(s) + g_2(s)) ds,
\]
and here,
\[
 g_1(s) := f_1 \left( s, y_1(s), \pi^{-\beta} C D_{0+}^{\beta} y_1(s) \right), \quad i = 1, 2, \quad g_3(s) := f_3 \left( s, y_3(s), l^{-\beta} C D_{0+}^{\beta} y_3(s) \right) .
\]

Now using (33), for any \( y \in \Omega \), we find that
\[
 |A_{1}y(t)| \leq \frac{\pi^\alpha}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} |f_1 \left( s, y_1(s), \pi^{-\beta} C D_{0+}^{\beta} y_1(s) \right) | ds \leq \frac{\pi^\alpha}{\Gamma(\alpha+1)} M_i , \quad i = 1, 2,
\]
\[
 |A_{3}y(t)| \leq \frac{l^{\alpha}}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} |f_3 \left( s, y_3(s), l^{-\beta} C D_{0+}^{\beta} y_3(s) \right) | ds \leq \frac{l^{\alpha}}{\Gamma(\alpha+1)} M_3 .
\]
Thus,
\[
 \|A_{1}y\| \leq \frac{1}{\Gamma(\alpha+1)} (\pi^\alpha + l^{\alpha}) M_i , \quad i = 1, 2, 3.
\]
Moreover, using (30), we get
\[
 |C D_{0+}^{\beta} A_{1}y(t)| \leq \frac{\pi^\alpha}{\Gamma(\alpha - \beta)} \int_{0}^{t} (t-s)^{\alpha-\beta-1} |f_1 \left( s, y_1(s), \pi^{-\beta} C D_{0+}^{\beta} y_1(s) \right) | ds 
\leq \frac{\pi^\alpha}{\Gamma(\alpha - \beta + 1)} M_i , \quad i = 1, 2,
\]
\[
 |C D_{0+}^{\beta} A_{3}y(t)| \leq \frac{l^{\alpha}}{\Gamma(\alpha - \beta)} \int_{0}^{1} (1-s)^{\alpha-\beta-1} |f_3 \left( s, y_3(s), \pi^{-\beta} C D_{0+}^{\beta} y_3(s) \right) | ds 
\leq \frac{l^{\alpha}}{\Gamma(\alpha - \beta + 1)} M_3 .
\]
Hence
\[ \| cD_{0,t}^\beta A_i y \| \leq \frac{1}{\Gamma(\alpha - \beta + 1)} (\pi^\alpha + l^\alpha) M_i, \quad i = 1, 2, 3. \] (36)

Therefore, using (35) and (36), we obtain
\[
\| Ay \|_{X^3} = \sum_{i=1}^{3} \left( \| A_i y \| + \| cD_{0,t}^\beta A_i y \| \right) 
\leq \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \right] (\pi^\alpha + l^\alpha) \left( \sum_{i=1}^{3} M_i \right). \] (37)

Again using (33), for any \( w \in \Omega \), one gets
\[
|B_1 w(t)| = |B_2 w(t)| \leq \frac{I^\alpha}{\Gamma(\alpha + 1)} M_3 + \left( \frac{l \pi^\alpha - 1}{\Gamma(\alpha)} + \frac{\pi^\alpha}{2\Gamma(\alpha)} + \frac{\pi^\alpha}{2\Gamma(\alpha + 1)} \right) (M_1 + M_2),
\]
\[
|B_3 w(t)| \leq \frac{I^\alpha}{\Gamma(\alpha + 1)} M_3 + \frac{2l \pi^\alpha - 1}{\Gamma(\alpha)} (M_1 + M_2).
\]

Hence
\[
\| B_i w \| \leq \left[ \pi^\alpha \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha + 1)} \right) + \frac{4l \pi^\alpha - 1}{\Gamma(\alpha)} + \frac{2l^\alpha}{\Gamma(\alpha + 1)} \right] M_i, \quad i = 1, 2, 3. \] (38)

On the other hand
\[
|cD_{0,t}^\beta B_1 w(t)| = |cD_{0,t}^\beta B_2 w(t)| \leq \pi^\alpha \left[ \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right] (M_1 + M_2),
\]
\[
|cD_{0,t}^\beta B_3 w(t)| \leq \frac{l \pi^\alpha - 1}{\Gamma(\alpha)\Gamma(2 - \beta)} (M_1 + M_2).
\]

Thus, for \( i = 1, 2, 3 \), we get
\[
\| cD_{0,t}^\beta B_i w \| \leq \left[ \pi^\alpha \left( \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right) + \frac{l \pi^\alpha - 1}{\Gamma(\alpha)\Gamma(2 - \beta)} \right] M_i. \] (39)

Finally, using (38) and (39), we obtain
\[
\| B w \|_{X^3} = \sum_{i=1}^{3} \left( \| B_i w \| + \| cD_{0,t}^\beta B_i w \| \right) 
\leq \left[ \pi^\alpha \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right) \right. \]
\[
+ \left. \frac{3l^\alpha}{\Gamma(\alpha + 1)} + l \pi^\alpha - 1 \left( \frac{4}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} \right) \right] \times \left( \sum_{i=1}^{3} M_i \right). \] (40)

Hence, (37) and (40) gives
\[
\| Ay + B w \|_{X^3} \leq \| Ay \|_{X^3} + \| B w \|_{X^3} 
\leq \left[ \pi^\alpha \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right) \right. \]
\[
+ \left. \frac{3l^\alpha}{\Gamma(\alpha + 1)} + l \pi^\alpha - 1 \left( \frac{4}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} \right) \right] \times \left( \sum_{i=1}^{3} M_i \right) \leq r.
\]
Moreover, in view of (37), we deduce that

\[ Ay \]

Therefore, we have used (33). Similarly,

\[ ay(t_2) - Ay(t_1) \leq \frac{\pi^\alpha}{\Gamma(\alpha + 1)} M_i(t_2^n - t_1^n), \quad i = 1, 2. \]

where we have used (33). Similarly,

\[ |A_i y(t_2) - A_i y(t_1)| \leq \frac{t_2^n - t_1^n}{\Gamma(\alpha + 1)} M_i(t_2^n - t_1^n), \quad i = 1, 2, 3. \] (41)

Moreover, we have

\[ |c D_{0,t}^\beta A_i y(t_2) - c D_{0,t}^\beta A_i y(t_1)| \]

\[ \leq \frac{\pi^\alpha}{\Gamma(\alpha - \beta)} \int_0^{t_1} (t_2 - s)^{\alpha - \beta} - (t_1 - s)^{\alpha - \beta} |f_i(s, y_i(s), \pi^{-\beta} C D_{0,s}^\beta y_i(s))| ds \]

\[ + \frac{\pi^\alpha}{\Gamma(\alpha - \beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - \beta} - (t_1 - s)^{\alpha - \beta} |f_i(s, y_i(s), \pi^{-\beta} C D_{0,s}^\beta y_i(s))| ds \]

\[ \leq \frac{\pi^\alpha}{\Gamma(\alpha - \beta + 1)} M_i(t_2^n - t_1^n - t_1^{\alpha - \beta}), \quad i = 1, 2 \]

and

\[ |c D_{0,t}^\beta A_3(t_2) - c D_{0,t}^\beta A_3(t_1)| \leq \frac{t_2^n - t_1^n}{\Gamma(\alpha - \beta + 1)} M_i(t_2^n - t_1^n - t_1^{\alpha - \beta}). \]

Hence

\[ |c D_{0,t}^\beta A_i y(t_2) - c D_{0,t}^\beta A_i y(t_1)| \]

\[ \leq \frac{1}{\Gamma(\alpha - \beta + 1)} \left( \pi^\alpha + t^\alpha \right) M_i(t_2^n - t_1^n - t_1^{\alpha - \beta}), \quad i = 1, 2, 3. \] (42)

Thus, from (41) and (42), we get

\[ |A_i y(t_2) - A_i y(t_1)| + |c D_{0,t}^\beta A_i y(t_2) - c D_{0,t}^\beta A_i y(t_1)| \]

\[ \leq \left( \pi^\alpha + t^\alpha \right) M_i \left[ \frac{1}{\Gamma(\alpha + 1)} (t_2^n - t_1^n) + \frac{1}{\Gamma(\alpha - \beta + 1)} (t_2^n - t_1^n - t_1^{\alpha - \beta}) \right] \to 0 \text{ as } t_2 \to t_1, \]

\[ i = 1, 2, 3. \]
Hence, we deduce that the operators $A_i, i = 1, 2, 3$ are equicontinuous, which implies that $A$ is equicontinuous and by the Arzela-Ascoli theorem, it follows that $A$ is completely continuous operator. Therefore, all the conditions of Lemma 3.2 are satisfied and consequently, we conclude that BVP (10)-(14) has at least one solution on $[0, 1]$. Therefore, BVP (1)-(5) has at least one solution. □

4. Ulam Stability analysis. In this section, we investigate a different kind of Ulam-type stability analysis for BVP system (10)-(14). Let $\epsilon_i > 0, f_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, 2, 3$ be continuous functions and $\vartheta_i(t) : [0, 1] \to \mathbb{R}^+, i = 1, 2, 3$ are nondecreasing continuous functions. Consider the following inequalities:

\[ |cD_{0+}^{\alpha_i}w_i(t) - a_i^\alpha f_i\left(t, w_i(t), a_i^{-\beta}cD_{0+}^{\beta}w_i(t)\right)| \leq \epsilon_i, \quad t \in [0, 1], \quad i = 1, 2, 3, \quad (43) \]

\[ |cD_{0+}^{\alpha_i}w_i(t) - a_i^\alpha f_i\left(t, w_i(t), a_i^{-\beta}cD_{0+}^{\beta}w_i(t)\right)| \leq \vartheta_i(t)\epsilon_i, \quad t \in [0, 1], \quad i = 1, 2, 3, \quad (44) \]

\[ |cD_{0+}^{\alpha_i}w_i(t) - a_i^\alpha f_i\left(t, w_i(t), a_i^{-\beta}cD_{0+}^{\beta}w_i(t)\right)| \leq \vartheta_i(t), \quad t \in [0, 1], \quad i = 1, 2, 3, \quad (45) \]

where $a_1 = a_2 = \pi$ and $a_3 = l$.

**Definition 4.1.** The BVP system (10)-(14) is said to be Ulam-Hyers stable, if there exists a constant $C = C(\alpha, \beta) > 0$ such that for each $\epsilon = \epsilon(\epsilon_1, \epsilon_2, \epsilon_3) > 0$ and for each solution $w = (w_i)_{1 \leq i \leq 3} \in X^3$ of the inequalities (43), there exists a solution $y = (y_i)_{1 \leq i \leq 3} \in X^3$ of (10)-(14) with

\[ \|w - y\|_{X^3} \leq Ce, \quad t \in [0, 1]. \]

Moreover, if there exists a function $\Psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Psi(0) = 0$ such that

\[ \|w - y\|_{X^3} \leq \Psi(\epsilon), \quad t \in [0, 1], \]

then BVP system (10)-(14) is called generalized Ulam-Hyers stable.

**Definition 4.2.** The BVP system (10)-(14) is said to be Ulam-Hyers-Rassias stable with respect to $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \in C([0, 1], \mathbb{R}^+)$, if there exists a constant $C_\vartheta > 0$ such that for each $\epsilon = \epsilon(\epsilon_1, \epsilon_2, \epsilon_3) > 0$ and for each solution $w = (w_i)_{1 \leq i \leq 3} \in X^3$ of the inequalities (44), there exists a solution $y = (y_i)_{1 \leq i \leq 3} \in X^3$ of (10)-(14) with

\[ \|w - y\|_{X^3} \leq C_\vartheta \vartheta(t)\epsilon, \quad t \in [0, 1]. \]

**Definition 4.3.** The BVP system (10)-(14) is said to be generalized Ulam-Hyers-Rassias stable with respect to $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \in C([0, 1], \mathbb{R}^+)$, if there exists a constant $C_\vartheta > 0$ such that for each $\epsilon = \epsilon(\epsilon_1, \epsilon_2, \epsilon_3) > 0$ and for each solution $w = (w_i)_{1 \leq i \leq 3} \in X^3$ of the inequalities (45), there exists a solution $y = (y_i)_{1 \leq i \leq 3} \in X^3$ of (10)-(14) with

\[ \|w - y\|_{X^3} \leq C_\vartheta \vartheta(t), \quad t \in [0, 1]. \]

**Remark 4.** A function $w = (w_1, w_2, w_3) \in X^3$ is said to be the solution of (43), if there exist functions $\phi_i \in C([0, 1], \mathbb{R})$, respectively, such that

\[ (i) \quad |\phi_i(t)| \leq \epsilon_i, \quad t \in [0, 1], \quad i = 1, 2, 3; \]

\[ (ii) \quad cD_{0+}^{\alpha_i}w_i(t) = a_i^\alpha f_i\left(t, w_i(t), a_i^{-\beta}cD_{0+}^{\beta}w_i(t)\right) + \phi_i(t), \quad t \in [0, 1], \quad i = 1, 2, 3, \]

where $a_1 = a_2 = \pi$ and $a_3 = l$.

**Remark 5.** A function $w = (w_1, w_2, w_3) \in X^3$ is said to be the solution of (44), if there exist functions $\phi_i \in C([0, 1], \mathbb{R})$, respectively, such that
where \( a_1 = a_2 = \pi \) and \( a_3 = l \).

**Lemma 4.4.** Let \( w = (w_i)_{1 \leq i \leq 3} \in X^3 \) be the solution of the inequalities (43). Then, the following inequalities hold:

\[
|w_i(t) - m_i(t)| \leq \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{l^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} + \frac{1}{2\Gamma(\alpha + 1)} \right) \epsilon_i
\]

\[
+ \left( \frac{l^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} \right) \epsilon_j + \frac{1}{\Gamma(\alpha + 1)} \epsilon_3, \quad i, j = 1, 2, \quad i \neq j.
\]

\[
|w_3(t) - m_3(t)| \leq \frac{2l^{\alpha-1}}{\Gamma(\alpha)} (\epsilon_1 + \epsilon_2) + \frac{2}{\Gamma(\alpha + 1)} \epsilon_3,
\]

and

\[
|C D_0^\beta w_i(t) - C D_0^\beta m_i(t)|
\]

\[
\leq \left( \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{2\Gamma(\alpha)} \Gamma(2 - \beta) + \frac{1}{2\Gamma(\alpha + 1)} \Gamma(2 - \beta) \right) \epsilon_i
\]

\[
+ \left( \frac{1}{2\Gamma(\alpha)} \Gamma(2 - \beta) + \frac{1}{2\Gamma(\alpha + 1)} \Gamma(2 - \beta) \right) \epsilon_j, \quad i, j = 1, 2, \quad i \neq j,
\]

\[
|C D_0^\beta w_3(t) - C D_0^\beta m_3(t)| \leq \frac{1}{\Gamma(\alpha - \beta + 1)} \epsilon_3 + \frac{l^{\alpha-1}}{\Gamma(\alpha + 1)} (\epsilon_1 + \epsilon_2),
\]

where

\[
m_i(t) = \frac{\pi^n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_i(s) ds - \frac{\pi^n}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p_3(s) ds
\]

\[
- \left( \frac{t^{\alpha-1}}{\Gamma(\alpha - 1)} + \frac{t^{\alpha}}{2\Gamma(\alpha - 1)} \right) \int_0^1 (1-s)^{\alpha-2} (p_1(s) + p_2(s)) ds
\]

\[
+ \frac{(-1)^i+1}{2\Gamma(\alpha)} t^{\alpha} \int_0^1 (1-s)^{\alpha-1} (p_1(s) - p_2(s)) ds, \quad i = 1, 2,
\]

\[
m_3(t) = \frac{\pi^n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_3(s) ds - \frac{\pi^n}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p_3(s) ds
\]

\[
+ (t-1) \left[ \frac{l^{\alpha-1}}{\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} (p_1(s) + p_2(s)) ds \right]
\]

and here,

\[ p_i(s) := f_i \left( s, w_i(s), \pi^{-\beta} C D_{0,s}^\beta w_i(s) \right), \quad i = 1, 2, \]

\[ p_3(s) := f_3 \left( s, w_3(s), l^{-\beta} C D_{0,s}^\beta w_3(s) \right). \]
Proof. Since \((w_i)_{1 \leq i \leq 3}\) are the solution of the inequalities (43), then by Remark 4, \(w_i\) will be the solution of following BVP:

\[
\begin{align*}
C \mathcal{D}_{0,t}^{\alpha} w_i(t) &= \pi^\alpha f_i \left( t, w_i(t), \pi^{-\beta} C \mathcal{D}_{0,t}^{\beta} w_i(t) \right) + \phi_i(t), \quad 0 < t < 1, \\
C \mathcal{D}_{0,t}^{\alpha} w_3(t) &= t^\alpha \mathcal{F}_3 \left( t, w_3(t), l^{-\beta} C \mathcal{D}_{0,t}^{\beta} w_3(t) \right) + \phi_3(t), \quad 0 < t < 1, \\
& \quad w_1(0) = w_2(0) = w_3(0), \quad w_1(1) = w_2(1), \\
& \quad \pi^{-1} w_i'(0) + \pi^{-1} w_2'(0) + l^{-1} w_3'(0) = 0, \quad w_i'(1) + w_2'(1) = 0, \\
& \quad w_3(1) = 0.
\end{align*}
\]

In view of Lemma 2.7, the solution of BVP system (47) is given by

\[
\begin{align*}
w_i(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} (\pi^\alpha p_i(s) + \phi_i(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} (l^\alpha p_3(s) + \phi_3(s)) ds \\
& \quad - \left( \frac{l \pi^{-1}}{\Gamma(\alpha - 1)} + \frac{t}{2 \Gamma(\alpha)} \right) \int_0^1 (1 - s)^{\alpha - 2} (\pi^\alpha p_1(s) + \phi_1(s) + \pi^\alpha p_2(s) + \phi_2(s)) ds \\
& \quad + \frac{(-1)^{i+1}}{2 \Gamma(\alpha)} t \int_0^1 (1 - s)^{\alpha - 1} (\pi^\alpha p_1(s) + \phi_1(s) - \pi^\alpha p_2(s) - \phi_2(s)) ds, \quad i = 1, 2,
\end{align*}
\]

(47)

\[
\begin{align*}
w_3(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} (l^\alpha p_3(s) + \phi_3(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} (l^\alpha p_3(s) + \phi_3(s)) ds \\
& \quad + (t - 1) \left[ \frac{l \pi^{-1}}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} (\pi^\alpha p_1(s) + \phi_1(s) + \pi^\alpha p_2(s) + \phi_2(s)) ds \right].
\end{align*}
\]

(48)

From (48), we find that

\[
|w_i(t) - m_i(t)| \leq \frac{1}{\Gamma(\alpha + 1)} (\epsilon_i + \epsilon_3) + \left[ \frac{l \pi^{-1}}{\Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha + 1)} \right] (\epsilon_1 + \epsilon_2)
\]

\[
= \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{l \pi^{-1}}{\Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha + 1)} \right) \epsilon_i \\
+ \left( \frac{l \pi^{-1}}{\Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha + 1)} \right) \epsilon_j + \frac{1}{\Gamma(\alpha + 1)} \epsilon_3, \quad i, j = 1, 2, \ i \neq j.
\]

(49)

Similarly, using (49), we get

\[
|w_3(t) - m_3(t)| \leq \frac{2l \pi^{-1}}{\Gamma(\alpha)} (\epsilon_1 + \epsilon_2) + \frac{2}{\Gamma(\alpha + 1)} \epsilon_3.
\]

On the other hand, applying the operator \(C \mathcal{D}_{0,t}^{\beta}\) on (48) and then using (30), we get

\[
C \mathcal{D}_{0,t}^{\beta} w_i(t)
\]

\[
= \frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} (\pi^\alpha p_i(s) + \phi_i(s)) ds \\
- \frac{l^{1-\beta}}{2 \Gamma(\alpha - 1) \Gamma(2 - \beta)} \int_0^1 (1 - s)^{\alpha - 2} (\pi^\alpha p_1(s) + \phi_1(s) + \pi^\alpha p_2(s) + \phi_2(s)) ds \\
+ \frac{(-1)^{i+1}}{2 \Gamma(\alpha - \beta)} t^{1-\beta} \int_0^1 (1 - s)^{\alpha - 1} (\pi^\alpha p_1(s) + \phi_1(s) - \pi^\alpha p_2(s) - \phi_2(s)) ds.
\]
Then, the following inequalities hold:

\[
|cD_{0,t}^\beta w_i(t) - cD_{0,t}^\beta m_i(t)| \\
\leq \frac{1}{\Gamma(\alpha - \beta + 1)} \epsilon_i + \left[ \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right] (\epsilon_1 + \epsilon_2) \\
= \left( \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right) \epsilon_i \\
+ \left( \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right) \epsilon_j, \quad i, j = 1, 2, \ i \neq j.
\]

Moreover, from (49), we have

\[
|cD_{0,t}^\beta w_3(t) - cD_{0,t}^\beta m_3(t)| \leq \frac{1}{\Gamma(\alpha - \beta + 1)} \epsilon_3 + \frac{l\pi^{-1}}{\Gamma(\alpha)\Gamma(2 - \beta)} (\epsilon_1 + \epsilon_2).
\]

Therefore, the proof is completed.

**Lemma 4.5.** Let \( w = (w_i)_{1 \leq i \leq 3} \in X^3 \) be the solution of the inequalities (44). Then, the following inequalities hold:

\[
|w_i(t) - m_i(t)| \leq \frac{1}{\Gamma(\alpha + 1)} (\epsilon_1 \vartheta_1(t) + \epsilon_3 \vartheta_3(1)) \\
+ \left[ \frac{l\pi^{-1}}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} \right] (\epsilon_1 \vartheta_1(1) + \epsilon_2 \vartheta_2(1)), \quad i = 1, 2,
\]

\[
|w_3(t) - m_3(t)| \leq \frac{1}{\Gamma(\alpha + 1)} (\epsilon_3 \vartheta_3(t) + \epsilon_3 \vartheta_3(1)) + \frac{2l\pi^{-1}}{\Gamma(\alpha)} (\epsilon_1 \vartheta_1(1) + \epsilon_2 \vartheta_2(1))
\]

and

\[
|cD_{0,t}^\beta w_i(t) - cD_{0,t}^\beta m_i(t)| \\
\leq \frac{1}{\Gamma(\alpha - \beta + 1)} \epsilon_i \vartheta_i(t) \\
+ \left[ \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)} \right] (\epsilon_1 \vartheta_1(1) + \epsilon_2 \vartheta_2(1)), \quad i = 1, 2,
\]

\[
|cD_{0,t}^\beta w_3(t) - cD_{0,t}^\beta m_3(t)| \\
\leq \frac{1}{\Gamma(\alpha - \beta + 1)} \epsilon_3 \vartheta_3(t) + \frac{l\pi^{-1}}{\Gamma(\alpha)\Gamma(2 - \beta)} (\epsilon_1 \vartheta_1(1) + \epsilon_2 \vartheta_2(1)).
\]

**Proof.** The proof can be obtained using a similar analysis as in Lemma 4.4 and the fact that \( \vartheta_i, \ i = 1, 2, 3 \) are nondecreasing functions. Therefore, the proof is omitted.

Now, we prove the Ulam-Hyers stability of the BVP system (10)-(14). Again, for convenience, we set the following quantities:

\[
\gamma_1 = \frac{1}{2\Gamma(\alpha)} + \frac{3}{2\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)},
\]

\[
\gamma_2 = \frac{1}{2\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} + \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)},
\]

\[
\rho_1 = \frac{1}{\Gamma(\alpha)} (l\pi^{\alpha - 1} + l\pi^{\alpha - \beta - 1}), \quad \rho_2 = \left( \frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha)\Gamma(2 - \beta)} \right) (l\pi^{\alpha - 1} + l\pi^{\alpha - \beta - 1})
\]
\[ \rho_3 = t^{\pi^{-1}} \left( \frac{2}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha) \Gamma(2 - \beta)} \right) . \]

**Theorem 4.6.** Suppose that (27) holds and \( R_{\alpha, \beta} \left( \sum_{i=1}^{3} L_i \right) < 1 \), then the unique solution of BVP system (10)-(14) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable if the eigenvalues of matrix \( A \), denoted by \( \lambda \), are in the open unit disc, i.e., \( |\lambda| < 1 \), where

\[
A = \begin{cases} \{ (\pi^\alpha + \pi^\alpha) \gamma_1 + \rho_1 \} L_1 & (\pi^\alpha + \pi^\alpha) \gamma_2 + \rho_1 L_2 \quad \frac{1}{\Gamma(\alpha + 1)} (l^\alpha + l^{\alpha - \beta}) L_3 \\ \{ (\pi^\alpha + \pi^\alpha) \gamma_2 + \rho_1 \} L_1 & (\pi^\alpha + \pi^\alpha) \gamma_1 + \rho_1 L_2 \quad \frac{1}{\Gamma(\alpha + 1)} (l^\alpha + l^{\alpha - \beta}) L_3 \end{cases}
\]

and \( R_{\alpha, \beta} \) is given by (25).

**Proof.** Let \( w = (w_i)_{1 \leq i \leq 3} \) be the solution of the inequalities (43) and \( y = (y_i)_{1 \leq i \leq 3} \) be the solution to the following BVP system:

\[
\begin{align*}
&cD_{0,1}^\alpha y_i(t) = \pi^\alpha f_i \left( t, y_i(t), \pi^{-\beta} cD_{0,1}^\beta y_i(t) \right), \quad i = 1, 2, \quad 0 < t < 1, \\
&cD_{0,1}^\alpha y_3(t) = l^\alpha f_3 \left( t, y_3(t), l^{-\beta} cD_{0,1}^\beta y_3(t) \right), \quad 0 < t < 1, \\
y_1(0) = y_2(0) = y_3(0), \quad y_1(1) = y_2(1), \\
\pi^{-1} y'_1(0) + \pi^{-1} y_2'(0) + l^{-1} y'_3(0) = 0, \quad y'_1(1) + y'_2(1) = 0, \\
y_3(1) = 0.
\end{align*}
\]

Then, in view of Lemma 2.7 and Theorem 3.1, system (50) has a unique solution that can be written as

\[
\begin{align*}
y_i(t) &= \frac{\pi^\alpha}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} g_i(s) ds - \frac{l^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} g_3(s) ds \\
&\quad - \left( \frac{l^{\pi - 1}}{\Gamma(\alpha + 1)} + \frac{t^{\pi}}{2\Gamma(\alpha - 1)} \right) \int_0^1 (1 - s)^{\alpha - 2} (g_1(s) + g_2(s)) ds \\
&\quad + \frac{(-1)^{i+1} t^{\pi}}{2\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} (g_2(s) - g_1(s)) ds,
\end{align*}
\]

\[
y_3(t) = \frac{l^\alpha}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} g_3(s) ds - \frac{l^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} g_3(s) ds \\
&\quad + (t - 1) \frac{l^{\pi - 1}}{\Gamma(\alpha + 1)} \int_0^1 (1 - s)^{\alpha - 2} (g_1(s) + g_2(s)) ds,
\]

where \( g_i(s), i = 1, 2, 3 \) are given by (34). Now, using Lemma 4.4, for \( t \in [0, 1] \), we get

\[
|w_i(t) - y_i(t)| \leq |w_i(t) - m_i(t)| + |m_i(t) - y_i(t)|
\]

\[
\begin{align*}
&\leq \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{l^{\pi - 1}}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} + \frac{1}{2\Gamma(\alpha + 1)} \right) \epsilon_i \\
&\quad + \left( \frac{l^{\pi - 1}}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha + 1)} \right) \epsilon_j + \frac{1}{\Gamma(\alpha + 1)} \epsilon_3 \\
&\quad + \frac{\pi^\alpha}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |\zeta_i(s)| ds + \frac{l^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} |\zeta_3(s)| ds
\end{align*}
\]
EXISTENCE AND STABILITY FOR A FBVP ON A METRIC GRAPH HAVING CYCLE 177

\[ + \left( \frac{t \pi^{\alpha - 1}}{\Gamma(\alpha - 1)} + \frac{t \pi^{\alpha}}{2 \Gamma(\alpha - 1)} \right) \int_0^1 (1 - s)^{\alpha - 2} (|\zeta_1(s)| + |\zeta_2(s)|) \, ds \\
+ \frac{(-1)^{i+1}}{2 \Gamma(\alpha)} \pi^{\alpha} \int_0^1 (1 - s)^{\alpha - 2} (|\zeta_1(s)| + |\zeta_2(s)|) \, ds, \quad i, j = 1, 2, i \neq j, \]

where

\[ \zeta_i(s) := f_i \left( s, w_i(s), \pi^{-\beta} c D^{\beta}_{t,s} w_i(s) \right) \]

\[ - f_i \left( s, y_i(s), \pi^{-\beta} c D^{\beta}_{t,s} y_i(s) \right) = p_i(s) - g_i(s), \quad i = 1, 2, \]

\[ \zeta_3(s) := f_3 \left( s, w_3(s), l^{-\beta} c D^{\beta}_{t,s} w_3(s) \right) - f_3 \left( s, y_3(s), l^{-\beta} c D^{\beta}_{t,s} y_3(s) \right) = p_3(s) - g_3(s). \]

Hence, by (27), we get

\[
|w_i(t) - y_i(t)| \\
\leq \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{t \pi^{-1}}{\Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha + 1)} \right) \epsilon_i \\
+ \left( \frac{1}{\Gamma(\alpha)} + \frac{t \pi^{-1}}{2 \Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha + 1)} \right) \epsilon_j + \frac{1}{\Gamma(\alpha + 1)} \epsilon_3 \\
+ \left[ \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{2 \Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha + 1)} \right) + \frac{1}{\Gamma(\alpha)} \right] (t \pi^{-1} + l \pi^{-\beta - 1}) L_i \| w_i - y_i \|_X \\
+ \frac{1}{\Gamma(\alpha + 1)} (t \pi^{-\beta} L_3 \| y_3 - w_3 \|_X, \quad i, j = 1, 2 \text{ and } i \neq j. \]

(51)

Further, we have

\[
|c D^{\beta}_{t,i} w_i(t) - c D^{\beta}_{t,i} y_i(t)| \\
\leq |c D^{\beta}_{t,i} w_i(t) - c D^{\beta}_{t,i} m_i(t)| + |c D^{\beta}_{t,i} m_i(t) - c D^{\beta}_{t,i} y_i(t)| \\
\leq \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{2 \Gamma(\alpha)} + \frac{1}{2 \Gamma(\alpha + 1)} \right) \epsilon_i \\
+ \left( \frac{1}{2 \Gamma(\alpha) \Gamma(2 - \beta)} + \frac{1}{2 \Gamma(\alpha + 1) \Gamma(2 - \beta)} \right) \epsilon_j \\
+ \frac{\pi^{\alpha}}{\Gamma(\alpha - \beta)} \int_0^1 (t - s)^{\alpha - 1} |\zeta_i(s)| \, ds \\
+ \frac{\pi^{\alpha} t^{1-\beta}}{2 \Gamma(\alpha - 1) \Gamma(2 - \beta)} \int_0^1 (1 - s)^{\alpha - 2} (|\zeta_1(s) + \zeta_2(s)|) \, ds \\
+ \frac{\pi^{\alpha} t^{1-\beta}}{2 \Gamma(\alpha) \Gamma(2 - \beta)} \int_0^1 (1 - s)^{\alpha - 1} (|\zeta_1(s) + \zeta_2(s)|) \, ds \\
\leq \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{2 \Gamma(\alpha) \Gamma(2 - \beta)} + \frac{1}{2 \Gamma(\alpha + 1) \Gamma(2 - \beta)} \right) \epsilon_i \\
+ \left( \frac{1}{2 \Gamma(\alpha) \Gamma(2 - \beta)} + \frac{1}{2 \Gamma(\alpha + 1) \Gamma(2 - \beta)} \right) \epsilon_j 
\]

(52)
Further, if we set

\[
\begin{aligned}
&\left(\pi^\alpha + \pi^{\alpha-\beta}\right)
\left(\frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)}\right) L_i\|w_i - y_i\|_X \\
+ \left(\pi^\alpha + \pi^{\alpha-\beta}\right)
\left(\frac{1}{2\Gamma(\alpha)\Gamma(2 - \beta)} + \frac{1}{2\Gamma(\alpha + 1)\Gamma(2 - \beta)}\right) L_j\|w_j - y_j\|_X,
\end{aligned}
\]

\(i, j = 1, 2\) and \(i \neq j\).

Therefore, from (51) and (52), we obtain

\[
\|w_i - y_i\|_X = \left(\|w_i - y_i\| + \|cD_{0,t}^\beta w_i - cD_{0,t}^\beta y_i\|\right)
\leq \left(\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_1\right) \epsilon_1 + \left(\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2\right) \epsilon_2 + \frac{1}{\Gamma(\alpha + 1)} \epsilon_3
\]
\[+ \left[(\pi^\alpha + \pi^{\alpha-\beta}) \gamma_1 + \rho_1\right] L_i\|w_i - y_i\|_X + \left[(\pi^\alpha + \pi^{\alpha-\beta}) \gamma_2 + \rho_1\right] L_j\|w_i - y_i\|_X
\]
\[+ \frac{1}{\Gamma(\alpha + 1)} (l^\alpha + l^{\alpha-\beta}) L_3\|w_3 - y_3\|_X, \quad i, j = 1, 2\] and \(i \neq j\).

Using similar analysis, one also obtains

\[
\|w_3 - y_3\|_X = \left(\|w_3 - y_3\| + \|cD_{0,t}^\beta w_3 - cD_{0,t}^\beta y_3\|\right)
\leq \rho_3(\epsilon_1 + \epsilon_2) + \left(\frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}\right) \epsilon_3
\]
\[+ \left(\frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}\right) (l^\alpha + l^{\alpha-\beta}) L_3\|w_3 - y_3\|_X
\]
\[+ \rho_2\left(L_1\|w_1 - y_1\|_X + L_2\|w_2 - y_2\|_X\right).
\]

Now, inequalities (53) and (54) together can be rewritten in the matrix form as follows:

\[
\begin{bmatrix}
\|w_1 - y_1\|_X \\
\|w_2 - y_2\|_X \\
\|w_3 - y_3\|_X
\end{bmatrix}
\leq \begin{bmatrix}
\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_1 & \frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 & \frac{1}{\Gamma(\alpha + 1)} \\
\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 & \frac{1}{\Gamma(\alpha + 1)} & \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \\
\rho_3 & \rho_3 & \frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{bmatrix}
+ A
\begin{bmatrix}
\|w_1 - y_1\|_X \\
\|w_2 - y_2\|_X \\
\|w_3 - y_3\|_X
\end{bmatrix}.
\]

Using the fact that eigenvalues of \(A\) are in the open unit disc together with Theorem 2.6, we get

\[
\begin{bmatrix}
\|w_1 - y_1\|_X \\
\|w_2 - y_2\|_X \\
\|w_3 - y_3\|_X
\end{bmatrix}
\leq (I - A)^{-1} \begin{bmatrix}
\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_1 & \frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 & \frac{1}{\Gamma(\alpha + 1)} \\
\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 & \frac{1}{\Gamma(\alpha + 1)} & \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \\
\rho_3 & \rho_3 & \frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{bmatrix}.
\]

Further, if we set

\[
B = (I - A)^{-1} \begin{bmatrix}
\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_1 & \frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 & \frac{1}{\Gamma(\alpha + 1)} \\
\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 & \frac{1}{\Gamma(\alpha + 1)} & \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \\
\rho_3 & \rho_3 & \frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}
\end{bmatrix} = \begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{bmatrix}
\]

and \(\epsilon = \max(\epsilon_1, \epsilon_2, \epsilon_3)\), then we obtain from (55) that

\[
\|w - y\|_X^3 \leq \sum_{j=1}^{3} \sum_{i=1}^{3} b_{i,j} \epsilon := C\epsilon.
\]
Since \( b_{ij} \geq 0 \) for \( 1 \leq i, j \leq 3 \), we have \( C > 0 \). Therefore, we deduce that BVP system (10)-(14) is Ulam-Hyers stable.

Finally, taking \( \Psi(c) = Cc \) in (56), we have \( \Psi(0) = 0 \) and thus, by Definition 4.1, we conclude that BVP system (10)-(14) is generalized Ulam-Hyers stable.

**Theorem 4.7.** Let \( \vartheta_i(t) \in C([0,1], \mathbb{R}^+) \), \( i = 1, 2, 3 \) be nondecreasing functions. Suppose that (27) holds and \( R_{\alpha,\beta} \left( \sum_{i=1}^{3} L_i \right) < 1 \), then the unique solution of BVP system (10)-(14) is Ulam-Hyers-Rassias stable with respect to \( \vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \) if the eigenvalues of matrix \( A \), denoted by \( \lambda \), are in the open unit disc, i.e., \( |\lambda| < 1 \), where \( A \) is same as defined in Theorem 4.6 and function \( \vartheta \) is defined by

\[
\vartheta(t) = \max\{h_1(t), h_2(t), \vartheta_3(t)\},
\]

\[
h_i(t) = \left( \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha- \beta + 1)} \right) \vartheta_i(t) + \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha+1)} \right) \vartheta_i(1), \quad i = 1, 2.
\]

**Proof.** Let \( w = (w_i)_{1 \leq i \leq 3} \) be the solution of the inequalities (44) and \( y = (y_i)_{1 \leq i \leq 3} \) be the solution to the BVP system (10)-(14). Now, using Lemma 4.5 and following the similar analysis as in the proof of Theorem 4.6, we have

\[
|w_i(t) - y_i(t)| \\
\leq |w_i(t) - m_i(t)| + |m_i(t) - y_i(t)|
\]

\[
\leq \frac{1}{\Gamma(\alpha+1)} (\epsilon_1 \vartheta_i(t) + \epsilon_3 \vartheta_3(1)) + \left[ \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} \right] (\epsilon_1 \vartheta_1(1) + \epsilon_2 \vartheta_2(1))
\]

\[
+ \left( \frac{\pi^\alpha + \pi^\beta}{\Gamma(\alpha+1)} \right) \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha+1)} \right) L_i \|w_i - y_i\|_X
\]

\[
+ \frac{1}{\Gamma(\alpha)} \left( t^{\alpha-1} + t^{\beta-1} \right) \times L_j \|w_j - y_j\|_X
\]

\[
+ \frac{1}{\Gamma(\alpha+1)} (\alpha + \beta) L_3 \|y_3 - w_3\|_X, \quad i,j = 1, 2 \text{ and } i \neq j
\]

and

\[
|cD_{0,t}^\beta w_i(t) - cD_{0,t}^\beta y_i(t)|
\]

\[
\leq |cD_{0,t}^\beta w_i(t) - cD_{0,t}^\beta m_i(t)| + |cD_{0,t}^\beta m_i(t) - cD_{0,t}^\beta y_i(t)|
\]

\[
\leq \frac{1}{\Gamma(\alpha- \beta + 1)} (\epsilon_1 \vartheta_i(t) + \epsilon_3 \vartheta_3(1)) + \left[ \frac{1}{2\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} \right] (\epsilon_1 \vartheta_1(1) + \epsilon_2 \vartheta_2(1))
\]

\[
+ \left( \frac{\pi^\alpha + \pi^\beta}{\Gamma(\alpha- \beta + 1)} \right) \left( \frac{1}{2\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} \right) L_i \|w_i - y_i\|_X
\]

\[
+ \frac{1}{\Gamma(\alpha- \beta + 1)} (\alpha + \beta) L_3 \|y_3 - w_3\|_X, \quad i,j = 1, 2 \text{ and } i \neq j
\]

(57) and (58)
Now, rewriting the inequalities (59)-(60) together in the matrix form as

\[ \|w_i - y_i\|_X \]
\[ = \left(\|w_i - y_i\| + \|cD_{0,i}^\beta w_i - cD_{0,i}^\beta y_i\|\right)\]
\[ \leq \left(\frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}\right)\epsilon_i\vartheta_i(t) \]
\[ + \left(\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2\right) (\epsilon_1\vartheta_1(1) + \epsilon_2\vartheta_2(1)) + \frac{1}{\Gamma(\alpha + 1)}\epsilon_3\vartheta_3(1) \]
\[ + \left(\frac{(\pi^\alpha + \pi^\alpha-\beta) \gamma_1 + \rho_1}{\Gamma(\alpha + 1)}\right) L_i\|w_i - y_i\|_X + \left(\frac{(\pi^\alpha + \pi^\alpha-\beta) \gamma_2 + \rho_1}{\Gamma(\alpha + 1)}\right) L_j\|w_i - y_i\|_X \]
\[ + \frac{1}{\Gamma(\alpha + 1)}(l^\alpha + l^{\alpha-\beta}) L_3\|w_i - y_i\|_X, \quad i, j = 1, 2 \text{ and } i \neq j. \]

Similarly, one can obtain

\[ \|w_3 - y_3\|_X \]
\[ = \left(\|w_3 - y_3\| + \|cD_{0,i}^\beta w_3 - cD_{0,i}^\beta y_3\|\right)\]
\[ \leq \left(\frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}\right)\epsilon_3\vartheta_3(t) + \rho_3 (\epsilon_1\vartheta_1(1) + \epsilon_2\vartheta_2(1)) + \frac{1}{\Gamma(\alpha + 1)}\epsilon_3\vartheta_3(1) \]
\[ + \left(\frac{2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}\right) (l^\alpha + l^{\alpha-\beta}) L_3\|w_3 - y_3\|_X \]
\[ + \rho_2 (L_1\|w_1 - y_1\|_X + L_2\|w_2 - y_2\|_X). \]

Now, rewriting the inequalities (59)-(60) together in the matrix form as

\[
\begin{bmatrix}
\|w_1 - y_1\|_X \\
\|w_2 - y_2\|_X \\
\|w_3 - y_3\|_X
\end{bmatrix}
\leq
\begin{bmatrix}
h_1(t) & \left(\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2\right) & \frac{1}{\Gamma(\alpha + 1)}\vartheta_3(1) \\
\left(\frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2\right) & h_2(t) & \frac{1}{\Gamma(\alpha + 1)}\vartheta_3(1) \\
\rho_2\vartheta_2(1) & \rho_3\vartheta_2(1) & \left(\frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)}\right)\vartheta_3(t) + \frac{1}{\Gamma(\alpha + 1)}\vartheta_3(1)
\end{bmatrix}
\times
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{bmatrix}
+ A
\begin{bmatrix}
\|w_1 - y_1\|_X \\
\|w_2 - y_2\|_X \\
\|w_3 - y_3\|_X
\end{bmatrix}.
\]

\[ := B(t) \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} + A \begin{bmatrix} \|w_1 - y_1\|_X \\ \|w_2 - y_2\|_X \\ \|w_3 - y_3\|_X \end{bmatrix}. \]

Using the fact that eigenvalues of A are in the open unit disc together with Theorem 2.6, we obtain

\[
\begin{bmatrix}
\|w_1 - y_1\|_X \\
\|w_2 - y_2\|_X \\
\|w_3 - y_3\|_X
\end{bmatrix}
\leq (I - A)^{-1} B(t) \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}. 
\]
Further, we define
\[
(I - A)^{-1}B(t) = \begin{bmatrix} c_{11}(t) & c_{12}(t) & c_{13}(t) \\ c_{21}(t) & c_{22}(t) & c_{23}(t) \\ c_{31}(t) & c_{32}(t) & c_{33}(t) \end{bmatrix} \quad \text{and} \quad (I - A)^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.
\]

It is straightforward to see that \( a_{ij} \geq 0, \ c_{ij}(t) \geq 0, \ i, j = 1, 2, 3, \)
\[
c_{ij}(t) = a_{ij}h_j(t) + \left( \frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 \right) \vartheta_j(1) \sum_{r=1, r \neq j}^{2} a_{ir} + \rho_3 a_{i3} \vartheta_j(1) \\
\leq \left( a_{ij} + \left( \frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 \right) \frac{\vartheta_j(1)}{h_j(0)} \right) \sum_{r=1, r \neq j}^{2} a_{ir} + \rho_3 a_{i3} \frac{\vartheta_j(1)}{h_j(0)} h_j(t),
\]
\( i = 1, 2, 3, \quad j = 1, 2 \)
and
\[
c_{33}(t) = a_{33} \left[ \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \right) \vartheta_3(t) + \frac{1}{\Gamma(\alpha + 1)} \vartheta_3(1) \right] \\
+ \frac{1}{\Gamma(\alpha + 1)} (a_{i1} + a_{i2}) \vartheta_3(1) \\
\leq a_{33} \left[ \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \right) + \frac{1}{\Gamma(\alpha + 1)} (1 + a_{i1} + a_{i2}) \frac{\vartheta_3(1)}{\vartheta_3(0)} \right] \vartheta_3(t),
\]
\( i = 1, 2, 3. \)

Now, by taking \( \epsilon = \max(\epsilon_1, \epsilon_2, \epsilon_3) \), we get
\[
\|w - y\|_{X^3} \leq \sum_{i=1}^{3} \left( \sum_{j=1}^{2} M_{ij} + N_i \right) \vartheta(t) \epsilon, \quad (61)
\]
where
\[
M_{ij} = a_{ij} + \left( \frac{l\pi^{-1}}{\Gamma(\alpha)} + \gamma_2 \right) \frac{\vartheta_j(1)}{h_j(0)} \sum_{r=1, r \neq j}^{2} a_{ir} + \rho_3 a_{i3} \frac{\vartheta_j(1)}{h_j(0)}
\]
and
\[
N_i = \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \right) + \frac{1}{\Gamma(\alpha + 1)} (1 + a_{i1} + a_{i2}) \frac{\vartheta_3(1)}{\vartheta_3(0)}, \quad i = 1, 2, 3.
\]

Finally, we assume
\[
C = \sum_{i=1}^{3} \left( \sum_{j=1}^{2} M_{ij} + N_i \right)
\]
and thus, by Definition 4.2, we conclude that the BVP system (10)-(14) is Ulam-Hyers-Rassias stable with respect to \( \vartheta \).

Moreover, taking \( \epsilon = 1 \) in (61), we conclude using Definition 4.3 that the BVP system (10)-(14) is generalized Ulam-Hyers-Rassias stable with respect to \( \vartheta \).
5. **Example.** Consider the BVP (1)-(5) with $\alpha = 3/2$, $\beta = 3/4$, $t = 1/4$ and

$$
\begin{align*}
&f_1(t, x, z) = \frac{t}{\sin(t^2)} + \frac{1}{(t + 3)^4} \left( \sin x + \frac{|z|}{1 + |z|} \right), \quad (t, x, z) \in [0, \pi] \times \mathbb{R} \times \mathbb{R}, \\
&f_2(t, x, z) = \frac{1}{4(t^2 + 8)^2} \left( \frac{|x|}{1 + |x|} + \frac{|z|}{1 + |z|} \right), \quad (t, x, z) \in [0, \pi] \times \mathbb{R} \times \mathbb{R}, \\
&f_3(t, x, z) = \frac{1}{20(t + 4)^2} (x + |z|), \quad (t, x, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}.
\end{align*}
$$

Using Lemma 2.4, we get the following equivalent system

$$
\begin{align*}
&c D_{0,t}^{3/2} y_1(t) = \pi^{3/2} \left[ \frac{t}{\sin(t^2)} + \frac{1}{(t + 3)^4} \left( \sin(y_1(t)) + \frac{\pi^{\frac{3}{4}}}{1 + \pi^{\frac{3}{4}}} c D_{0,t}^{3/4} y_1(t) \right) \right], \\
&c D_{0,t}^{3/2} y_2(t) = \pi^{3/2} \left[ \frac{1}{4(t^2 + 8)^2} \left( \frac{|y_2(t)|}{1 + |y_2(t)|} + \frac{(\pi)^{-\frac{3}{4}}}{1 + (\pi)^{-\frac{3}{4}}} c D_{0,t}^{3/4} y_2(t) \right) \right], \\
&c D_{0,t}^{3/2} y_3(t) = \left( \frac{1}{4} \right)^{3/2} \left[ \frac{1}{20(t + 4)^2} \left( y_3(t) + \sin(1/4) \right)^{\frac{3}{4}} c D_{0,t}^{3/4} y_3(t) \right],
\end{align*}
$$

subject to the transmission conditions (12)-(13) and the boundary condition (14) with $0 < t < 1$. For $t \in [0, 1]$ and $x, z, x_1, z_1 \in \mathbb{R}$, it is clear that

$$
|f_1(t, x, z) - f_1(t, x_1, z_1)| \leq \frac{1}{(t + 3)^4} (|x - x_1| + |z - z_1|),
$$

$$
|f_2(t, x, z) - f_2(t, x_1, z_1)| \leq \frac{1}{4(t^2 + 8)^2} (|x - x_1| + |z - z_1|),
$$

and

$$
|f_3(t, x, z) - f_3(t, x_1, z_1)| \leq \frac{1}{20(t + 4)^2} (|x - x_1| + |z - z_1|).
$$

Therefore, we have $L_1 = \frac{1}{81}$, $L_2 = \frac{1}{256}$, $L_3 = \frac{1}{320}$ and

$$
R_{\alpha, \beta} = \left( \frac{\pi \sqrt{\pi} + \pi^{3/4}}{4 \sqrt{\pi}} \right) \left[ \frac{2}{2} + \frac{8}{3 \sqrt{\pi}} + \frac{4}{3 \Gamma(3/4)} + \frac{8}{\sqrt{\pi} \Gamma(1/4)} + \frac{16}{\sqrt{\pi} \Gamma(3/4)} \right] + \left( \frac{\sqrt{\pi}}{4} + \frac{1}{4 \pi^{1/4}} \right) \left[ \frac{8}{\sqrt{\pi}} + \frac{8}{\sqrt{\pi} \Gamma(1/4)} \right] + \left( \frac{1}{8} + \frac{1}{2 \sqrt{2}} \right) \left[ \frac{16}{\sqrt{\pi}} + \frac{4}{3 \Gamma(3/4)} \right].
$$

Now, using numerical values $\Gamma(1/4) \approx 3.625$, $\Gamma(3/4) \approx 1.2254$, we get $R_{\alpha, \beta} \approx 49.5476$ and consequently

$$
R_{\alpha, \beta}(L_1 + L_2 + L_3) \approx .9578 < 1.
$$

Therefore, by Theorem 3.1, the BVP system (63) has a unique solution on $[0, 1]$.

Further, using the given values, we have

$$
\gamma_1 \approx 3.81, \quad \gamma_2 \approx 1.977, \quad \rho_1 \approx .71187, \quad \rho_2 \approx 1.734
$$

and

$$
A = \begin{bmatrix}
{3.81(\pi \sqrt{\pi} + \pi^{3/4}) + .71187} & {1.977(\pi \sqrt{\pi} + \pi^{3/4}) + .71187} & {1.977(\pi \sqrt{\pi} + \pi^{3/4}) + .71187} \\
{1.977(\pi \sqrt{\pi} + \pi^{3/4}) + .71187} & {3.81(\pi \sqrt{\pi} + \pi^{3/4}) + .71187} & {0.001277} \\
{0.00677} & {0.00387} & {0.00387}
\end{bmatrix}.
$$

The eigenvalues of matrix $A$ are given by

$$
\lambda_1 = .4453 < 1, \quad \lambda_2 = .3174 < 1 \quad \text{and} \quad \lambda_3 = .0038 < 1.
$$
Hence, we conclude from Theorem 4.6 that BVP system (63) is Ulam-Hyers stable, and consequently generalized Ulam-Hyers stable. Moreover, using Theorem 4.7, one can easily deduce that BVP system (63) is Ulam-Hyers-Rassias stable and generalized Ulam-Hyers-Rassias stable.

Acknowledgments. The authors acknowledge the support of this work by the Indo-German exchange program “Multiscale Modelling, Simulation and optimization for energy, Advanced Materials and Manufacturing” funded by UGC (India) and DAAD (Germany) (grant number 1-3/2016 (IC)).

REFERENCES

[1] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, Applied Mathematics Letters, 23 (2010), 390–394.
[2] Z. Ali, A. Zada and K. Shah, On Ulam’s stability for a coupled systems of nonlinear implicit fractional differential equations, Bulletin of the Malaysian Mathematical Sciences Society, 42 (2019), 2681–2699.
[3] R. Almeida, N. R. O. Bastos, M. Teresa and T. Monteiro, Modelling some real phenomena by fractional differential equations, Math. Meth. Appl. Sci., 39 (2016), 4846–4855.
[4] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, preprint, arXiv:1602.03408.
[5] D. Baleanu, S. Rezapour and H. Mohammadi, Some existence results on nonlinear fractional differential equations, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 371 (2013), 20120144.
[6] H. M. Baskonus and J. F. G. Aguilar, New singular soliton solutions to the longitudinal wave equation in a magneto-electro-elastic circular rod with M-derivative, Modern Physics Letters B, 33 (2019), 1950251.
[7] J. V. Below, Sturm-Liouville eigenvalue problems on networks, Math. Meth. Appl. Sci., 10 (1988), 383–395.
[8] U. Brauer and G. Leugering, On boundary observability estimates for semi-discretizations of a dynamic network of elastic strings, Control and Cybernetics, 28 (1999), 421–447.
[9] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl., 1 (2015), 1–13.
[10] V. F. M. Delgado, J. F. G. Aguilar and M. A. T. Hernandez, Analytical solutions of electrical circuits described by fractional conformable derivatives in Liouville-Caputo sense, AEU-International Journal of Electronics and Communications, 85 (2018), 108–117.
[11] P. Exner, P. Kuchment and B. Winn, On the location of spectral edges in in-periodic media, Journal of Physics A: Mathematical and Theoretical, 43 (2010), 474022.
[12] H. Fazli and J. J. Nieto, Fractional Langevin equation with anti-periodic boundary conditions, Chaos, Solitons & Fractals, 114 (2018), 332–337.
[13] B. Ghanbari and J. F. G. Aguilar, New exact optical soliton solutions for nonlinear Schrödinger equation with second-order spatio-temporal dispersion involving M-derivative, Modern Physics Letters B, 33 (2019), 1950235.
[14] B. Ghanbari and J. F. G. Aguilar, Optical soliton solutions of the Ginzburg-Landau equation with conformable derivative and Kerr law nonlinearity, Revista Mexicana de Física, 65 (2019), 73–81.
[15] S. Gnutzmann and D. Waltner, Stationary waves on nonlinear quantum graphs: General framework and canonical perturbation theory, Phys. Rev. E, 93 (2016).
[16] C. Goodrich, Existence of a positive solution to a class of fractional differential equations, Appl. Math. Lett., 23 (2010), 1050–1055.
[17] D. G. Gordeziani, M. Kupreishvli, H. V. Meladze and T. D. Davitashvili, On the solution of boundary value problem for fractional equations given in graphs, Appl. Math. Lett., 13 (2008), 80–91.
[18] J. R. Graef, L. Kong and M. Wang, Existence and uniqueness of solutions for a fractional boundary value problem on a graph, Fractional Calculus and Applied Analysis, 17 (2014), 499–510.
[19] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, 2003.
[20] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[21] D. H. Hyers, On the stability of the linear functional equation, Proceedings of the National Academy of Sciences of the United States of America, 27 (1941), 222–224.

[22] A. Khan, H. Khan, J. F. G. Aguilar and T. Abdeljawad, Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel, Chaos, Solitons & Fractals, 127 (2019), 422–427.

[23] H. Khan, W. Chen and H. Sun, Analysis of positive solution and Hyers-Ulam stability for a class of singular fractional differential equations with p-Laplacian in Banach space, Mathematical Methods in the Applied Sciences, 41 (2018), 3430–3440.

[24] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006.

[25] C. Klein, C. Sparber and P. Markowich, Numerical study of fractional nonlinear Schrödinger equations, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 470 (2014), 20140364.

[26] P. Kuchment, Quantum graphs: An introduction and a brief survey, preprint, arXiv:0802.3442.

[27] P. Kuchment and L. Kunyansky, Differential operators on graphs and photonic crystals, Advances in Computational Mathematics, 16 (2002), 263–290.

[28] N. Kumar and M. Mehra, Collocation method for solving nonlinear fractional optimal control problems by using Hermite scaling function with error estimates, Optimal Control Applications and Methods, (2020).

[29] N. Kumar and M. Mehra, Legendre wavelet collocation method for fractional optimal control problems with fractional Bolza cost, Numerical Methods for Partial Differential Equations, (2020).

[30] J. E. Lagnese, G. Leugering and E. J. P. G. Schmidt, Control of planar networks of Timoshenko beams, SIAM J. Control Optim., 31 (1993), 780–811.

[31] G. Leugering, Domain decomposition of an optimal control problem for semi-linear elliptic equations on metric graphs with application to gas networks, Applied Mathematics, 8 (2017), 1074–1099.

[32] G. Lumer, Connecting of local operators and evolution equaitions on a network, Lect. Notes Math., 787 (1980), 219–234.

[33] R. L. Magin and M. Ovadia, Modeling the cardiac tissue electrode interface using fractional calculus, Journal of Vibration and Control, 14 (2008), 1431–1442.

[34] H. M. Martínez and J. F. G. Aguilar, Local M-derivative of order $\alpha$ and the modified expansion function method applied to the longitudinal wave equation in a magneto-electro-elastic circular rod, Optical and Quantum Electronics, 50 (2018), 375.

[35] H. Y. Martínez and J. F. G. Aguilar, Fractional sub-equation method for Hirota–Satsuma-coupled KdV equation and coupled mKdV equation using the Atangana’s conformable derivative, Waves in Random and Complex Media, 29 (2019), 678–693.

[36] H. Y. Martínez and J. F. G. Aguilar, M-derivative applied to the dispersive optical solitons for the Schrödinger-Hirota equation, The European Physical Journal Plus, 134 (2019), 1–11.

[37] H. Y. Martínez and J. F. G. Aguilar, M-derivative applied to the soliton solutions for the Lakshmanan–Porsezian–Daniel equation with dual-dispersion for optical fibers, Optical and Quantum Electronics, 51 (2019), 31.

[38] H. Y. Martínez and J. F. G. Aguilar, Optical solitons solution of resonance nonlinear Schrödinger type equation with Atangana’s-conformable derivative using sub-equation method, Waves in Random and Complex Media, (2019), 1–24.

[39] H. Y. Martínez, J. F. G. Aguilar and A. Atangana, First integral method for non-linear differential equations with conformable derivative, Mathematical Modelling of Natural Phenomena, 13 (2018), 14.

[40] V. Mehandiratta and M. Mehra, A difference scheme for the time-fractional diffusion equation on a metric star graph, Applied Numerical Mathematics, 158 (2020), 152–163.

[41] V. Mehandiratta, M. Mehra and G. Leugering, Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graph, Journal of Mathematical Analysis and Applications, 477 (2019), 1243–1264.

[42] V. Mehandiratta, M. Mehra and G. Leugering, Fractional optimal control problems on a star graph: Optimality system and numerical solution, Mathematical Control and Related Fields, 11 (2021), 189–209.

[43] V. Mehandiratta, M. Mehra and G. Leugering, An approach based on Haar wavelet for the approximation of fractional calculus with application to initial and boundary value problems, Math. Meth. Appl. Sci., (2020).
[44] M. Mehra and R. K. Malik, Solutions of differential–difference equations arising from mathematical models of granulocytopoiesis, *Differential Equations and Dynamical Systems*, 22 (2014), 33–49.

[45] G. Mophou, G. Leugering and P. S. Fotsing, Optimal control of a fractional Sturm-Liouville problem on a star graph, *Optimization*, (2020), 1–29.

[46] S. Nicaise, Some results on spectral theory over networks, applied to nerve impulses transmission, *Lect. Notes Math.*, 1771 (1985), 532–541.

[47] K. S. Patel and M. Mehra, Fourth order compact scheme for space fractional advection-diffusion reaction equations with variable coefficients, *Journal of Computational and Applied Mathematics*, 380 (2020), 112963.

[48] B. S. Pavlov and M. Faddeev, Model of free electrons and the scattering problem, *Teor. Mat. Fiz.*, 55 (1983), 257–269.

[49] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[50] Y. V. Pokornyi and A. V. Borovskikh, Differential equations on networks (geometric graphs), *Journal of Mathematical Sciences*, 119 (2004), 691–718.

[51] T. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society*, 72 (1978), 297–300.

[52] A. Shukla, M. Mehra and G. Leugering, A fast adaptive spectral graph wavelet method for the viscous Burgers’ equation on a star-shaped connected graph, *Mathematical Methods in the Applied Sciences*, 43, (2020), 7595–7614.

[53] A. K. Singh and M. Mehra, Uncertainty quantification in fractional stochastic integro-differential equations using Legendre wavelet collocation method, *Lect. Notes Comput. Sci.*, 12138, (2020), 58–71.

[54] X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, *Appl. Math. Lett.*, 22, (2009), 64–69.

[55] S. M. Ulam, *A Collection of Mathematical Problems*, New York, 1960.

[56] C. Urs, Coupled fixed point theorems and applications to periodic boundary value problems, *Miskolc Mathematical Notes*, 14, (2013), 323–333.

[57] J. R. Wang, A. Zada and H. Waheed, Stability analysis of a coupled system of nonlinear implicit fractional anti-periodic boundary value problem, *Mathematical Methods in the Applied Sciences*, 42, (2019), 6706–6732.

[58] L. Xiping, J. Mei and G. Weiago, The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator, *Applied Mathematics Letters*, 65, (2017), 56–62.

[59] W. Zhang and W. Liu, Existence and Ulam’s type stability results for a class of fractional boundary value problems on a star graph, *Mathematical Methods in the Applied Sciences*, (2020).

Received July 2020; revised November 2020.

*E-mail address: vaibhavmehandiratta@gmail.com*

*E-mail address: mmehra@maths.iitd.ac.in*

*E-mail address: guenter.leugering@fau.de*