Elements of causality theory\textsuperscript{1}

Piotr T. Chruściel\textsuperscript{2}
University of Vienna
http://homepage.univie.ac.at/piotr.chrusciel/

November 1, 2011

\textsuperscript{1}Preprint UWThPh-2011-32
\textsuperscript{2}Supported in part by the Polish Ministry of Science and Higher Education grant Nr N N201 372736.
## Contents

| Contents | iii |
|----------|-----|
| 1 Introduction | 1 |
| 2 Causality | 3 |
| 2.1 Time orientation | 3 |
| 2.2 Normal coordinates | 5 |
| 2.3 Causal paths | 13 |
| 2.4 Futures, pasts | 17 |
| 2.5 Extendible and inextendible paths | 28 |
| 2.5.1 Maximally extended geodesics | 31 |
| 2.6 Accumulation curves | 32 |
| 2.6.1 Achronal causal curves | 37 |
| 2.7 Causality conditions | 38 |
| 2.8 Global hyperbolicity | 41 |
| 2.9 Domains of dependence | 44 |
| 2.10 Cauchy horizons | 51 |
| 2.10.1 Semi-convexity of future horizons | 53 |
| 2.11 Cauchy surfaces | 57 |
| 2.12 Some applications | 62 |
| Bibliography | 65 |
Chapter 1

Introduction

These notes present some elements of causality theory, as useful to study general relativity. They amount to an incremental compilation (and thus are far from being well synchronized and balanced between topics) of notes for lectures I held at various summer schools over the years. While they are not as complete as other presentations of the topic [4, 24, 29, 36, 43, 45], there is some originality in that the whole treatment is based on a definition of causal curves which allows one to simplify many arguments.

Now, in light of studies of the Einstein equations with metrics of low differentiability [28, 31, 32, 53, 54], it is of interest to understand the differentiability needed for the causality part of the theory. The standard references are either vague about differentiability, or assume smoothness of the metric. In our presentation we keep track of the differentiability of the metric needed for the arguments. This leads to a coherent causality theory for $C^2$ metrics. This differentiability threshold can be traced back to Proposition 2.6.1, page 33 below, as used to prove that accumulation curves of causal curves are causal, as well as to the deformation lemma 2.4.14, page 26. The threshold for the accumulation result, and some of its consequences, is relaxed in [11] using different methods, and note that the approach there requires developing first a causality theory for smooth metrics in any case.

Given the number of alternative more complete treatments, it is not clear that the above two reasons justify making the notes public. However, the notes serve as a crossreference for the accompanying notes [11] on causality for continuous metrics, which is the main reason for posting.

The reader is warned that some of our proofs do not apply to metrics which are not $C^2$, and that a few key results (e.g., deforming not-everywhere-null-causal curves to timelike ones keeping end points fixed) are plain wrong for metrics with lower differentiability.
Chapter 2

Causality

Unless explicitly indicated otherwise, or otherwise clear from the context, we consider manifolds equipped with a $C^\infty$ atlas and a continuous metric. As already pointed out, the considerations below give a coherent causality theory for metrics which are $C^2$ and manifolds which are $C^3$. However, for many considerations a metric of $C^0$ differentiability class suffices. We will strive to indicate explicitly the differentiability of the metric needed as the presentation evolves.

2.1 Time orientation

Recall that at each point $p \in \mathcal{M}$ the set of timelike vectors in $T_p M$ has precisely two components. A time-orientation of $T_p \mathcal{M}$ is the assignment of the name “future pointing vectors” to one of those components; vectors in the remaining component are then called “past pointing”. The set of future pointing timelike, or causal, vectors, is stable under addition and multiplication by positive numbers; similarly for past pointing ones. (In particular this implies convexity.) In order to see this, suppose that $X = (X^0, \vec{X})$ and $Y = (Y^0, \vec{Y})$ are timelike future pointing, in an ON-frame this is equivalent to

$$|\vec{X}| < X^0, \quad |\vec{Y}| < Y^0,$$

and the inequality

$$|\vec{X} + \vec{Y}| \leq |\vec{X}| + |\vec{Y}| < X^0 + Y^0$$

follows. Two timelike vectors $X$ and $Y$ have the same time orientation if and only if

$$g(X, Y) < 0;$$

this is immediate in an ON frame in which $X$ is proportional to $e_0$.

A time-orientation of $T_p \mathcal{M}$ can always be propagated to a neighborhood of $p$ by choosing any continuous vector field $X$ defined around $p$ which is timelike and future pointing at $p$. By continuity of the metric and of $X$, the vector field $X$ will be timelike in a sufficiently small neighborhood $\mathcal{O}_p$ of $p$, and for $q \in \mathcal{O}_p$ one can define future pointing vectors at $q$ as those lying in the same component of the set of timelike vectors as $X(q)$: for $q \in \mathcal{O}_p$ the vector $Y \in T_q \mathcal{M}$
will be said to be timelike future pointing if and only if \( g(Y, X(q)) < 0 \). A Lorentzian manifold is said to be \textit{time-orientable} if such locally defined time-orientations can be defined globally in a consistent way; that is, we can cover \( \mathcal{M} \) by coordinate neighborhoods \( \mathcal{O}_p \), each equipped with a vector field \( X_{\mathcal{O}_p} \), such that \( g(X_{\mathcal{O}_p}, X_{\mathcal{O}_q}) < 0 \) on \( \mathcal{O}_p \cap \mathcal{O}_q \).

Some Lorentzian manifolds will not be time-orientable, as is shown by the flat metric\(^1\) on the Möbius strip. On a time-orientable manifold there are precisely two choices of time-orientation possible, and \((\mathcal{M}, g)\) will be said \textit{time oriented} when such a choice has been made. This leads us to the fundamental definition:

**Definition 2.1.1** A pair \((\mathcal{M}, g)\) will be called a \textit{space-time} if \((\mathcal{M}, g)\) is a time-oriented Lorentzian manifold. We write \((\mathcal{M}, g)_{C^k}\) to denote a space-time with a metric of \(C^k\)-differentiability class.

**Remark 2.1.2** A Lorentzian manifold \((\mathcal{M}, g)\) which is not time-orientable has a double cover which is \([18]\). The proof goes as follows: Choose any \( p_0 \in \mathcal{M} \) and set

\[
\hat{\mathcal{M}} := \{(p, \gamma) : p \text{ is a point of } \mathcal{M} \text{ and } \gamma \text{ is a continuous curve from } p \text{ to } p_0 \}/\sim,
\]

where \( \sim \) is the following equivalence relation: \((p, \gamma) \sim (p', \gamma')\) if \( p = p' \) and if there exists a continuous timelike vector field defined along the curve obtained by first following \( \gamma \) from \( p \) to \( p_0 \) and then \( \gamma' \) from \( p_0 \) to \( p' = p \). The usual arguments from the theory of covering spaces show that \( \hat{\mathcal{M}} \) can be equipped with a manifold structure, and covers \( \mathcal{M} \) twice. \( \hat{\mathcal{M}} \) is then equipped with the pull-back \( \hat{g} \) of \( g \) using the covering map; time-orientability of \((\hat{\mathcal{M}}, \hat{g})\) should be clear. Furthermore, any time-orientable cover of \( \mathcal{M} \) also covers \( \hat{\mathcal{M}} \), so \( \hat{\mathcal{M}} \) can be thought-of as the smallest time-oriented covering of \( \mathcal{M} \).

On any space-time there always exists a globally defined future directed timelike vector field — to show this, consider the locally defined timelike vector fields \( X_{\mathcal{O}_p} \) defined on neighborhoods \( \mathcal{O}_p \) as described above. One can choose a locally finite covering of \( \mathcal{M} \) by such neighborhoods \( \mathcal{O}_p, i \in \mathbb{N} \), and construct a globally defined vector field \( X \) on \( \mathcal{M} \) by setting

\[
X = \sum_i \phi_i X_{\mathcal{O}_p_i},
\]

where the functions \( \phi_i \) form a partition of unity dominated by the covering \( \{\mathcal{O}_p_i\}_{i \in \mathbb{N}} \). The resulting vector field will be timelike future pointing everywhere, as follows from the fact that the sum of an arbitrary number of future pointing timelike vectors is a future pointing timelike vector.

Now, non-compact manifolds always admit a nowhere vanishing vector field. However, compact manifolds possess a nowhere vanishing vector field if and only if they have vanishing Euler characteristic \( \chi \). More generally, if \( M \) is a compact, orientable manifold, then the Poincaré–Hopf theorem (see, e.g., [21]) implies that the index of any smooth vector field, \( X \), on \( M \) (i.e. the zeroes of \( X \) counted with signs) satisfies

\[
\text{index}(X) = \chi(M).
\]

\(^1\)In two dimensions \(-g\) is a Lorentzian metric whenever \( g \) is, and the operation \( g \to -g \) has the effect of interchanging the role of space and of time.
2.2. NORMAL COORDINATES

As such, if $M$ admits a non-vanishing vector field $X$, then $\text{index}(X) = 0$ and, hence, $\chi(M) = 0$. Conversely, if $M$ has $\chi(M) = 0$, then any smooth vector field $X$ on $M$ is of index zero. A theorem of Hopf then implies that there exists a non-vanishing vector field on $M$ homotopic to $X$.

As such, vanishing of the Euler characteristic, $\chi$, is a necessary and sufficient condition of topological nature for a compact, orientable manifold to be a time-orientable Lorentzian manifold. We actually have the following:

**Proposition 2.1.3** A manifold $\mathcal{M}$ admits a space-time structure if and only if there exists a nowhere vanishing, continuous vector field on $\mathcal{M}$.

**Proof:** The necessity of the existence of a nowhere vanishing vector field on $\mathcal{M}$ has already been established. Conversely, suppose that such a vector field $X$ exists, and let $h$ be any Riemannian metric on $\mathcal{M}$. Then the formula

$$g(Y,Z) = h(Y,Z) - 2 \frac{h(Y,X)h(Z,X)}{h(X,X)} \quad (2.1.2)$$

defines a Lorentzian metric on $\mathcal{M}$. Finally, the existence of a globally defined timelike vector field $X$ on a Lorentzian manifold $(\mathcal{M},g)$ implies time-orientability of $\mathcal{M}$ in the obvious way – choose $O_p = \mathcal{M}$ and $X_{O_p} = X$. \[\square\]

Summarising, non-compact $\mathcal{M}$’s always admit both a Lorentzian metric and a space-time structure. Now, because the Euler characteristic of a double-cover of $\mathcal{M}$ is zero if and only if that of $\mathcal{M}$ is, it follows from Remark 2.1.2 and Proposition 2.1.3 that compact $\mathcal{M}$’s admit a Lorentzian metric if and only if they have vanishing Euler characteristic. For example, no Lorentzian metrics exist on $S^2$.

2.2 Normal coordinates

Given a $C^2$ metric, for $p \in \mathcal{M}$ the exponential map

$$\exp_p : T_p \mathcal{M} \to \mathcal{M}$$

is defined as follows; if $X$ is a vector in the tangent space $T_p \mathcal{M}$, then $\exp_p(X) \in \mathcal{M}$ is the point reached by following a geodesic with initial point $p$ and initial tangent vector $X \in T_p \mathcal{M}$ for an affine distance one, provided that the geodesic in question can be continued that far. Now an affinely parameterized geodesic solves the equation

$$\nabla_{\dot{x}} \dot{x} = 0 \iff \frac{d^2 x^\mu}{d s^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}, \quad (2.2.1)$$

where the $\Gamma^\mu_{\alpha\beta}$’s are the Christoffel symbols of the metric $g$, defined as

$$\Gamma^\mu_{\alpha\beta} := \frac{1}{2} g^{\mu\sigma} \left( \frac{\partial g_{\sigma\alpha}}{\partial x^\beta} + \frac{\partial g_{\sigma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \right), \quad (2.2.2)$$

$$g^{\mu\sigma} := g^\flat(dx^\mu, dx^\sigma), \quad g_{\alpha\beta} := g(\partial_\alpha, \partial_\beta). \quad (2.2.3)$$
Here, as elsewhere, we use the symbol $g^#$ to denote the “contravariant metric”, that is, the metric on $T^*\mathcal{M}$ constructed out from $g$ in the canonical way (the matrix $g^{\alpha\beta}$ is thus the matrix inverse to $g_{\alpha\beta}$). However, it is usual in the literature to use the same symbol $g$ for the metric $g^#$, as well as for all other metrics induced by $g$ on tensor bundles over $\mathcal{M}$, and we will often do so.

Equations (2.2.2)-(2.2.3) show that when the metric is of $C^{1,1}$ differentiability class, then the Christoffel symbols are Lipschitz continuous, which guarantees local existence and uniqueness of solutions of (2.2.1). Due to the lack of uniqueness$^2$ of the Cauchy problem for (2.2.1) for metrics which are not $C^{1,1}$, various problems arise when attempting to develop causality theory on manifolds with a metric with less regularity$^3$ than $C^{1,1}$, addressed in [11] (see also [30, 49]).

The domain $\mathcal{U}_p$ of $\exp_p$ is always the largest subset of $T_p\mathcal{M}$ on which the exponential map is defined. By construction, and by homogeneity properties of solutions of (2.2.1) under a linear change of parameterization (see (2.2.11)), the set $\mathcal{U}_p$ is star-shaped with respect to the origin (this means that if $X \in \mathcal{U}_p$ then we also have $\lambda X \in \mathcal{U}_p$ for all $\lambda \in [0, 1]$). When the metric is $C^{1,1}$, continuity of solutions of ODE’s upon initial values shows that $\mathcal{U}_p$ is an open neighborhood of the origin of $T_p\mathcal{M}$.

The exponential map is neither surjective nor injective in general. For example, on $\mathbb{R} \times S^1$ with the flat metric $-dt^2 + dx^2$, the “left-directed” null geodesics $\Gamma_- (s) = (s, -s \mod 2\pi)$ and the “right-directed” null geodesics $\Gamma_+ (s) = (s, s \mod 2\pi)$ meet again after going each “half of the way around $S^1$”, and injectivity fails. Both in de-Sitter and in anti-de-Sitter space-time all timelike geodesics meet again at a point, which leads to lack of surjectivity of the exponential map.

A Lorentzian manifold is said to be geodesically complete if all geodesics can be defined for all real values of affine parameter; this is equivalent to the requirement that for all $p \in \mathcal{M}$ the domain of the exponential map is $T_p\mathcal{M}$. One also talks about timelike geodesically complete space-times, future timelike geodesically complete space-times, etc., with those notions defined in an obvious way.

It follows from the Hopf–Rinow theorem [25, 33] that compact Riemannian manifolds are geodesically complete. There is no Lorentzian analogue of this, the standard counter-example proceeds as follows:

**Example 2.2.1** Consider the following symmetric tensor field on $\mathbb{R}^2$:

$$g = \frac{2dx dy}{x^2 + y^2}.$$  \hspace{1cm} (2.2.4)

$^2$Examples of $C^{1,\alpha}$ metrics with non-unique null geodesics for $0 < \alpha < 1$ can be found in [9, Appendix F] and [11], compare [23] for spacelike geodesics in a Riemannian context. Here $C^{k,\alpha}$ is the space of $k$ times differentiable functions (or maps, or sections — whichever is the case should be clear from the context), the $k$'th derivatives of which satisfy, locally, a Hölder condition of order $\alpha$.

$^3$One can construct large classes of solutions to the Cauchy problem for the vacuum Einstein equations which are not of $C^{1,1}$ differentiability class [2, 27, 28, 48]. This leads to a mismatch in differentiability between the Cauchy problem and causality theory which has not been completely clarified yet.
2.2. NORMAL COORDINATES

We have

\[ g_{\mu\nu} = \frac{1}{x^2 + y^2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \det g_{\mu\nu} = \frac{1}{(x^2 + y^2)^2}, \]

(2.2.5)

which shows that \( g \) is indeed a Lorentzian metric. Note that for all \( \lambda \in \mathbb{R}^* \) the maps

\[ \mathbb{R}^2 \ni (x, y) \rightarrow \phi_\lambda(x, y) := (\lambda x, \lambda y) \]

are isometries of \( g \):

\[ \phi_\lambda^* g = \frac{2d(\lambda x)d(\lambda y)}{(\lambda x)^2 + (\lambda y)^2} = \frac{2dxdy}{x^2 + y^2} = g. \]

It follows that for any \( 1 \neq \lambda > 0 \) the metric \( g \) passes to the quotient space

\[ \left\{ \mathbb{R}^2 \setminus \{0\} \right\} / \phi_\lambda = \{(x, y) \sim (\lambda x, \lambda y)\} \approx S^1 \times S^1 = T^2. \]

(Clearly the quotient spaces with \( \lambda \) and \( 1/\lambda \) are the same, so without loss of generality one can assume \( \lambda > 1 \).) In order to show geodesic incompleteness of \( g \) we will use the following result:

**Proposition 2.2.2** Consider a spacetime \((\mathcal{M}, g)_{C^2}\). Let \( f \) be a function such that \( g(\nabla f, \nabla f) \) is a constant. Then the integral curves of \( \nabla f \) are affinely parameterized geodesics.

**Proof:** Let \( X := \nabla f \), we have

\[ (\nabla_X X)^j = \nabla^i f \nabla_i \nabla^j f = \nabla^i f \nabla^j \nabla_i f = \frac{1}{2} \nabla^j (\nabla^i f \nabla_i f) = \frac{1}{2} \nabla^j (g(\nabla f, \nabla f)) = 0. \]

Returning to the metric (2.2.4), let \( f = x \), by (2.2.5) we have

\[ g^{\mu\nu} = (g^{\mu\nu})^{-1} = (x^2 + y^2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]

so that

\[ \nabla f = (x^2 + y^2) \partial_y \implies g(\nabla f, \nabla f) = 0. \]

Proposition 2.2.2 shows that the integral curves of \( \nabla f \) are null affinely parameterized geodesics. Let \( \gamma(s) = (x^\mu(s)) \) be any such integral curve, thus

\[ \frac{dx^\mu}{ds} = \nabla^\mu f \implies \frac{dx}{ds} = 0, \quad \frac{dy}{ds} = (x^2 + y^2). \]

It follows that \( x(s) = x(0) \) for all \( s \). The equation for \( y \) is easily integrated; for our purposes it is sufficient to consider that integral curve which passes through \((0, y_0) \in \mathbb{R}^2 \setminus \{0\}, y_0 > 0 \) — we then have \( x(s) = 0 \) for all \( s \) and

\[ \frac{dy}{ds} = y^2 \implies y(s) = \frac{y_0}{1 - y_0 s}. \]

(2.2.6)

This shows that \( y(s) \) runs away to infinity as \( s \) approaches

\[ s_\infty := \frac{1}{y_0}. \]
It follows that \( \gamma \) is indeed incomplete on \( \mathbb{R}^2 \setminus \{0\} \). To see that it is also incomplete on the quotient torus \( \{ \mathbb{R}^2 \setminus \{0\} \} / \phi_{\lambda} \), \( \lambda > 1 \), note that the image of \( \gamma(s) = (0, y(s)) \) under the equivalence relation \( \sim \) is a circle, and there exists a sequence \( s_j \to s_{\infty} \) such that \( \gamma(s_j) \) passes again and again through its starting point:

\[
y(s_j) = \lambda^j y_0 \quad \Rightarrow \quad (0, y(s_j)) \sim (0, y_0) \quad \text{in} \quad \{ \mathbb{R}^2 \setminus \{0\} \} / \phi_{\lambda}.
\]

By (2.2.6) we have

\[
\frac{dy}{ds}(s_j) = (y(s_j))^2 = (\lambda^j y_0)^2 \to_{s_j \to s_{\infty}} \infty,
\]

which shows that the sequence of tangents \( (dy/ds)(s_j) \) at \( (0, y_0) \) blows up as \( j \) tends to infinity. This clearly implies that \( \gamma \) cannot be extended beyond \( s_{\infty} \) as a \( C^1 \) curve.

When the metric is \( C^2 \), the inverse function theorem\(^4\) shows that there exists a neighborhood \( V_p \subset \mathcal{M}_p \) of the origin in \( \mathbb{R}^{\dim \mathcal{M}} \) on which the exponential map is a diffeomorphism between \( V_p \) and its image

\[
\mathcal{O}_p := \exp_p(V_p) \subset \mathcal{M}.
\]

This allows one to define normal coordinates centred at \( p \):

**Proposition 2.2.3** Let \((\mathcal{M}, g)\) be a \( C^3 \) Lorentzian manifold with \( C^2 \) metric \( g \). For every \( p \in \mathcal{M} \) there exists an open coordinate neighborhood \( \mathcal{O}_p \) of \( p \), in which \( p \) is mapped to the origin of \( \mathbb{R}^{n+1} \), such that the coordinate rays \( s \to sx^\mu \) are affinely parameterized geodesics. If the metric \( g \) is expressed in the resulting coordinates \( (x^\mu) = (x^0, \vec{x}) \in V_p \), then

\[
g_{\mu\nu}(0) = \eta_{\mu\nu}.
\]

If \( g \) is \( C^3 \) then we also have

\[
\partial_\sigma g_{\mu\nu}(0) = 0.
\]

Further, if the function \( \sigma_p : \mathcal{O}_p \to \mathbb{R} \) is defined by the formula

\[
\sigma_p(\exp_p(x^\mu)) := \eta_{\mu\nu} x^\mu x^\nu \equiv -(x^0)^2 + (\vec{x})^2,
\]

then

\[
\nabla \sigma_p \text{ is } \begin{cases} 
timelike & \text{past directed on } \{ q \mid \sigma_p(q) < 0, \ x^0(q) < 0 \}, 
future & \text{directed on } \{ q \mid \sigma_p(q) < 0, \ x^0(q) > 0 \}, 
null & \text{past directed on } \{ q \mid \sigma_p(q) = 0, \ x^0(q) < 0 \}, 
space & \text{future directed on } \{ q \mid \sigma_p(q) = 0, \ x^0(q) > 0 \}, 
space & \text{space directed on } \{ q \mid \sigma_p(q) > 0 \}.
\end{cases}
\]

\(^4\)It follows from the invariance-of-domain theorem that one can construct normal coordinates for \( C^{1,1} \) metrics. However, those coordinates will only be continuous a priori, which is a problem for some arguments below; note that one cannot even calculate the metric functions in such coordinates. It is conceivable that Clarke’s implicit function theorem [13] could provide some more information in this context. We have not investigated this line of thought any further, as the approach in [11] provides more general results anyway.
Remark 2.2.4 The definition of normal coordinates leads to $C^{k-1}$ coordinate functions if the metric is $C^k$. Hence the metric, when expressed in normal coordinates, will be of $C^{k-2}$ differentiability class. This implies that there is a $C^2$ threshold for the introduction of normal coordinates, and that two derivatives are lost when expressing the metric in those coordinates. This is irrelevant for our purposes, as the main point here is that for $C^2$ metrics there exists a function $\sigma_p$ satisfying (2.4.7) below, together with the following three facts:

1. $q \mapsto \sigma_p(q)$ is differentiable;
2. $(q, p) \mapsto \sigma_p(q)$ is continuous; and
3. if $g_n$ converges to $g$ in $C^2$, then the corresponding functions $\sigma_p$ converge in $C^0$.

These are standard properties of solutions of ODEs (cf., e.g., [51]).

Remark 2.2.5 The coefficients of a Taylor expansion of $g_{\mu\nu}$ in normal coordinates can be expressed in terms of the Riemann tensor and its covariant derivatives (compare [38, 39]).

Proof: Let us start by justifying that the implicit function theorem can indeed be applied: Let $x^\mu$ be any coordinate system around $p$, and let $e_a = e_a^\mu \partial_\mu$ be any ON frame at $p$. Let

$$X = X^\alpha e_a = X^\alpha e_a^\mu \partial_\mu \in T_p M$$

and let $x^\mu(s, X)$ denote the affinely parameterized geodesic passing by $p$ at $s = 0$, with tangent vector

$$\dot{x}^\mu(0, X) := \frac{dx^\mu(s, X)}{ds}|_{s=0} = X^\alpha e_a^\mu.$$

Homogeneity properties of the ODE (2.2.1) under the change of parameter $s \to \lambda s$ together with uniqueness of solutions of ODE’s show that for any constant $a \neq 0$ we have

$$x^\mu(as, X/a) = x^\mu(s, X).$$

This, in turn, implies that there exist functions $\gamma^\mu$ such that

$$x^\mu(s, X) = \gamma^\mu(sX). \quad (2.2.11)$$

From (2.2.1) and (2.2.11) we have

$$x^\mu(s, X) = x_0^\mu + sX^\alpha e_a^\mu + O((s|X|)^2).$$

Here $x_0^\mu$ are the coordinates of $p$, $|X|$ denotes the norm of $X$ with respect to some auxiliary Riemannian metric on $M$, while the $O((s|X|)^2)$ term is justified
by (2.2.11). The usual considerations of the proof that solutions of ODE’s are differentiable functions of their initial conditions show that

\[
\frac{\partial x^\mu(s, X^a)}{\partial X^a} = \frac{\partial (x^\mu_0 + sX^a e_a^\mu)}{\partial X^a} + O(s^2)|X| = se_a^\mu + O(s^2)|X|.
\]

At \(s = 1\) one thus obtains

\[
\frac{\partial x^\mu(1, X^a)}{\partial X^a} = e_a^\mu + O(|X|). \quad (2.2.12)
\]

This shows that \(\partial x^\mu / \partial X^a\) will be bijective at \(X = 0\) provided that \(\det e_a^\mu \neq 0\). But this last inequality can be obtained by taking the determinant of the equation

\[
\eta(e_a, e_b) = g_{\mu\nu}e_a^\mu e_b^\nu \implies -1 = (\det g_{\mu\nu})(\det e_a^\mu)^2. \quad (2.2.13)
\]

This justifies the use of the implicit function theorem to obtain existence of the neighborhood \(\mathcal{O}_p\) announced in the statement of the proposition. Clearly \(\mathcal{O}_p\) can be chosen to be star-shaped with respect to \(p\). Equation (2.2.12) and the fact that \(e^\mu_a\) is an ON-frame show that

\[
g(\partial_a, \partial_b)|_{X^a=0} = g_{\mu\nu}e_a^\mu e_b^\nu|_{X^a=0} = \eta_{ab},
\]

which establishes in (2.2.7).

By construction the rays

\[ s \to \gamma^a(s) := sX^a \]

are affinely parameterized geodesics with tangent \(\dot{\gamma} = X^a \partial_a\), which gives

\[
0 = (\nabla_\gamma \dot{\gamma})^a = \frac{d^2(sX^a)}{ds^2} + \Gamma^a_{bc}(sX^d)X^b X^c
\]

\[
= \Gamma^a_{bc}(sX^d)X^b X^c.
\]

Setting \(s = 0\) and differentiating this equation twice with respect to \(X^d\) and \(X^e\) one obtains \(\Gamma^a_{de}(0) = 0\).

The equation

\[
0 = \nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d_{ba}g_{dc} - \Gamma^d_{ca}g_{bd}
\]

evaluated at \(X = 0\) gives (2.2.8).

Let us pass now to the proof of the main point here, namely (2.2.10). From now on we will denote by \(x^\mu\) the normal coordinates obtained so far, and which were denoted by \(X^a\) in the arguments just done. For \(x \in \mathcal{O}_p\) define

\[
f(x) := \eta_{\mu\nu}x^\mu x^\nu, \quad (2.2.14)
\]
2.2. NORMAL COORDINATES

and let $\mathcal{H}_\tau \subset \partial_p \setminus \{p\}$ be the level sets of $f$:

$$\mathcal{H}_\tau := \{ x : f(x) = \tau , \ x \neq 0 \} . \quad (2.2.15)$$

We will show that

the vector field $x^\mu \partial_\mu$ is normal to the $\mathcal{H}_\tau$’s. \quad (2.2.16)

Now, $x^\mu \partial_\mu$ is tangent to the geodesic rays $s \rightarrow \gamma^\mu(s) := sx^\mu$. As the causal character of the field of tangents to a geodesic\(^5\) is point-independent along the geodesic, we have

- $x^\mu \partial_\mu$ is timelike at $\gamma(s)$ $\iff$ $f(x) < 0$,
- $x^\mu \partial_\mu$ is null at $\gamma(s)$ $\iff$ $f(x) = 0 , \ x \neq 0$,
- $x^\mu \partial_\mu$ is spacelike at $\gamma(s)$ $\iff$ $f(x) > 0$.

This follows from the fact that the right-hand-side is precisely the condition that the geodesic be timelike, spacelike, or null, at $\gamma(0)$. Since $\nabla f$ is always normal to the level sets of $f$, when (2.2.16) holds we will have

$$x^\mu \partial_\mu \text{ is proportional to } \nabla^\mu f . \quad (2.2.18)$$

This shows that (2.2.10) will follow from (2.2.17) when (2.2.16) holds.

It remains to establish (2.2.16). In order to do that, consider any differentiable curve $\lambda \mapsto \gamma^\mu(\lambda)$ lying on $\mathcal{H}_\tau$:

$$\eta_{\mu\nu}x^\mu(\lambda)x^\nu(\lambda) = \tau \implies \eta_{\mu\nu}x^\mu(\lambda)\partial_\lambda x^\nu(\lambda) = 0 . \quad (2.2.19)$$

Let $\gamma^\mu(\lambda,s)$ be the following one-parameter family of geodesic rays:

$$\gamma^\mu(\lambda,s) := sx^\mu(\lambda) .$$

For any function $f$ set

$$T(f) = \partial_s (f \circ \gamma(s,\lambda)) , \quad X(f) = \partial_\lambda (f \circ \gamma(s,\lambda)) ,$$

so that

$$T(\lambda,s) := (\partial_s \gamma^\mu(\lambda,s))\partial_\mu = x^\mu(\lambda,s)\partial_\mu , \quad X(\lambda,s) := (\partial_\lambda \gamma^\mu(\lambda,s))\partial_\mu .$$

For any fixed value of $\lambda$ the curves $s \rightarrow \gamma^\mu(\lambda,s)$ are geodesics, which shows that

$$\nabla_T T = 0 .$$

This gives

$$\frac{d(g(T,T))}{ds} = 2g(\nabla_T T,T) = 0 ,$$

hence

$$g(T,T)(s) = g(T,T)(0) = \eta_{\mu\nu}x^\mu(\lambda)x^\nu(\lambda) = \tau$$

\(^5\)Without loss of generality an affine parameterization of a geodesic $\gamma$ can be chosen, the result follows then from the calculation $d(g(\dot{\gamma},\dot{\gamma}))/ds = 2g(\nabla_\gamma \dot{\gamma}, \dot{\gamma}) = 0.$
by (2.2.19), in particular $g(T, T)$ is $\lambda$-independent.

Next, for any twice-differentiable function $\psi$ we have

$$[T, X](\psi) := T(X(\psi)) - X(T(\psi)) = \partial_s \partial_\lambda (\psi(\gamma^\mu(s, \lambda))) - \partial_\lambda \partial_s (\psi(\gamma^\mu(s, \lambda))) = 0,$$

because of the symmetry of the matrix of second partial derivatives. It follows that

$$[T, X] = \nabla_T X - \nabla_X T = 0. $$

Finally,

$$\frac{dg(T, X)}{ds} = g(\nabla_T T, X) + g(T, \nabla_T X) = 0,$$

This yields

$$g(T, X)(s, \lambda) = g(T, X)(0, \lambda) = 2\partial_\lambda (g(T, T)) = 0.$$

again by (2.2.19). Thus $T$ is normal to the level sets of $f$, which had to be established. \hfill \Box

As already pointed out, some regularity of the metric is lost when going to normal coordinates; this can be avoided using coordinates which are only approximately normal up to a required order, which is often sufficient for several purposes.

It is sometimes useful to have a geodesic convexity property at our disposal. This is made precise by the following proposition:

**Proposition 2.2.6** Let $\mathcal{O}$ be the domain of definition of a coordinate system $\{x^\mu\}$. Let $p \in \mathcal{O}$ and let $B_p(r) \subset \mathcal{O}$ denote an open coordinate ball of radius $r$ centred at $p$. There exists $r_0 > 0$ such that every geodesic segment $\gamma : [a, b] \rightarrow \overline{B_p(r_0)} \subset \mathcal{O}$

$$\gamma(a), \gamma(b) \in B_p(r), \quad r < r_0$$

is entirely contained in $B_p(r)$.

**Proof:** Let $x^\mu(s)$ be the coordinate representation of $\gamma$, set

$$f(s) := \sum_\mu (x^\mu - x_0^\mu)^2,$$

where $x_0^\mu$ is the coordinate representation of $p$. We have

$$\frac{df}{ds} = 2 \sum_\mu (x^\mu - x_0^\mu) \frac{dx^\mu}{ds},$$

$$\frac{d^2 f}{ds^2} = 2 \sum_\mu \left( \frac{dx^\mu}{ds} \right)^2 + 2 \sum_\mu (x^\mu - x_0^\mu) \frac{d^2 x^\mu}{ds^2}$$

$$= 2 \sum_\mu \left( \frac{dx^\mu}{ds} \right)^2 - 2 \sum_\mu (x^\mu - x_0^\mu) \Gamma^\mu_{\alpha \beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}. $$
Compactness of $\overline{B_p(r_0)}$ implies that there exists a constant $C$ such that we have
\[
\left| \sum_{\mu} (x^\mu - x_0^\mu) \Gamma^\alpha_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right| \leq C r_0 \sum_{\mu} \left( \frac{dx^\mu}{ds} \right)^2.
\]
It follows that $d^2 f / ds^2 \geq 0$ for $r_0$ small enough. This shows that $f$ has no interior maximum if $r_0$ is small enough, whence the result. \qed

It is convenient to introduce the following notion:

**Definition 2.2.7** An elementary region is an open coordinate ball $\mathcal{O}$ within the domain of a normal coordinate neighborhood $\mathcal{U}_p$, such that

1. $\mathcal{O}$ has compact closure $\overline{\mathcal{O}}$ in $\mathcal{U}_p$, and
2. $\nabla t$ and $\partial_t$ are timelike on $\overline{\mathcal{U}}$.

Note that $\partial_t$ is timelike if and only if
\[
g_{tt} = g(\partial_t, \partial_t) < 0,
\]
while $\nabla t$ is timelike if and only if
\[
g^{tt} = g^\#(dt, dt) < 0.
\]
Existence of elementary regions containing some point $p \in M$ follows immediately from Proposition 2.2.3: In normal coordinates centred at $p$ one chooses $\mathcal{O}$ be an open coordinate ball of sufficiently small radius.

### 2.3 Causal paths

Let $(\mathcal{M}, g)$ be a space-time. The basic objects in causality theory are paths. We shall always use parameterized paths: by definition, these are continuous maps from some interval to $\mathcal{M}$. We will use interchangeably the terms “path”, “parameterized path”, or “curve”, but we note that some authors make the distinction. (For example, in [7] a path is a map and a curve is the image of a path, oriented by parameterisation.)

Let $\gamma : I \to \mathcal{M}$ and let $\mathcal{U} \subset \mathcal{M}$, we will write
\[
\gamma \subset \mathcal{U}
\]
whenever the image $\gamma(I)$ of $I$ by $\gamma$ is a subset of $\mathcal{U}$. We will sometimes write
\[
\gamma \cap \mathcal{U}
\]
for the path obtained by removing from $I$ those parameters $s$ for which $\gamma(s) \notin \mathcal{U}$. Strictly speaking, our definition of a path requires $\gamma$ to be defined on a connected interval, so if the last construction gives several intervals $I_i$, then $\gamma \cap \mathcal{U}$ will actually describe the collection of paths $\gamma|_{I_i}$.
Some authors define a path in $\mathcal{M}$ as what would be in our terminology the image of a parameterized path. In this approach one forgets about the parameterization of $\gamma$, and identifies two paths which have the same image and differ only by a reparameterization. This leads to various difficulties when considering end points of causal paths — cf. Section 2.5, or limits of sequences of paths — cf. Section 2.6, and therefore we do not adopt this approach.

If $\gamma : I \to \mathcal{M}$ where $I = [a, b)$ or $I = [a, b]$, then $\gamma(a)$ is called the starting point of the path $\gamma$, or of its image $\gamma(I)$. If $I = (a, b]$ or $I = [a, b]$, then $\gamma(b)$ is called the end point. (This definition will be extended in Section 2.5, but it is sufficient for the purposes here.) We shall say that $\gamma : [a, b) \to \mathcal{M}$ is a path from $p$ to $q$ if $\gamma(a) = p$ and $\gamma(b) = q$.

In previous treatments of causality theory [4, 19, 24, 45, 52] one defines future directed timelike paths as those paths $\gamma$ which are piecewise differentiable, with $\dot{\gamma}$ timelike and future directed wherever defined; at break points one further assumes that both the left-sided and right-sided derivatives are timelike. This definition turns out to be quite inconvenient for several purposes. For instance, when studying the global causal structure of space-times one needs to take limits of timelike curves, obtaining thus — by definition — causal future directed paths. Such limits will not be piecewise differentiable most of the time, which leads one to the necessity of considering paths with poorer differentiability properties. One then faces the unhandy situation in which timelike and causal paths have completely different properties. In several theorems separate proofs have then to be given. The approach we present avoids this, leading — we believe — to a considerable simplification of the conceptual structure of the theory.

It is convenient to choose once and for all some auxiliary Riemannian metric $h$ on $\mathcal{M}$, such that $(\mathcal{M}, h)$ is complete — such a metric always exists [42]; let $\text{dist}_h$ denote the associated distance function. A parameterized path $\gamma : I \to \mathcal{M}$ from an interval $I \subset \mathbb{R}$ to $M$ is called locally Lipschitzian if for every compact subset $K$ of $I$ there exists a constant $C(K)$ such that

$$\forall s_1, s_2 \in K \quad \text{dist}_h(\gamma(s_1), \gamma(s_2)) \leq C(K)|s_1 - s_2|.$$

It is natural to enquire whether the class of paths so defined depends upon the background metric $h$?

**Proposition 2.3.1** Let $h_1$ and $h_2$ be two complete Riemannian metrics on $\mathcal{M}$. Then a path $\gamma : I \to \mathcal{M}$ is locally Lipschitzian with respect to $h_1$ if and only if it is locally Lipschitzian with respect to $h_2$.

**Proof:** Let $K \subset I$ be a compact set, then $\gamma(K)$ is compact. Let $L_a$, $a = 1, 2$ denote the $h_a$-length of $\gamma$, set

$$\mathcal{K}_a := \bigcup_{s \in K} B_{h_a}(\gamma(s), L_a),$$

where $B_{h_a}(p, r)$ denotes a geodesic ball, with respect to the metric $h_a$, centred at $p$, of radius $r$. Then the $\mathcal{K}_a$’s are compact. Likewise the sets

$$\hat{\mathcal{K}}_a \subset T_M,$$

defined as the sets of $h_a$-unit vectors over $\mathcal{K}_a$, are compact. This implies that there exists a constant $C_K$ such that for all $X \in T_pM$, $p \in \mathcal{K}_a$, we have

$$C_K^{-1} h_1(X, X) \leq h_2(X, X) \leq C_K h_1(X, X).$$
2.3. CAUSAL PATHS

Let \( \gamma_{a,s_1,s_2} \) denote any minimising \( h_a \)-geodesic between \( \gamma(s_1) \) and \( \gamma(s_2) \), then

\[
\forall s_1, s_2 \in K \quad \gamma_{a,s_1,s_2} \subset \mathcal{K}_a .
\]

This implies

\[
\text{dist}_{h_2}(\gamma(s_1), \gamma(s_2)) = \int_{\sigma = \gamma_{2,s_1,s_2}} \sqrt{h_2(\dot{\sigma}, \dot{\sigma})} \\
\geq C_K^{-1} \int_{\sigma = \gamma_{2,s_1,s_2}} \sqrt{h_1(\dot{\sigma}, \dot{\sigma})} \\
\geq C_K^{-1} \inf_{\sigma} \int \sqrt{h_1(\dot{\sigma}, \dot{\sigma})} \\
= C_K^{-1} \int_{\sigma = \gamma_{1,s_1,s_2}} \sqrt{h_1(\dot{\sigma}, \dot{\sigma})} \\
= C_K^{-1} \text{dist}_{h_1}(\gamma(s_1), \gamma(s_2)) .
\]

From symmetry with respect to the interchange of \( h_1 \) with \( h_2 \) we conclude that for all \( s_1, s_2 \in K \)

\[
C_K^{-1} \text{dist}_{h_1}(\gamma(s_1), \gamma(s_2)) \leq \text{dist}_{h_2}(\gamma(s_1), \gamma(s_2)) \leq C_K \text{dist}_{h_1}(\gamma(s_1), \gamma(s_2)) ,
\]

and the result easily follows. \( \square \)

More generally, if \((N,k)\) and \((M,h)\) are Riemannian manifolds, then a map \( \phi \) is called \( \text{locally Lipschitzian} \), or \( \text{locally Lipschitz} \), if for every compact subset \( K \) of \( N \) there exists a constant \( C(K) \) such that

\[
\forall p, q \in K \quad \text{dist}_{h}(\phi(p), \phi(q)) \leq C(K) \text{dist}_{k}(p, q) .
\]

A map is called \( \text{Lipschitzian} \) if the constant \( C(K) \) above can be chosen independently of \( K \).

The following important theorem of Rademacher will play a key role in our considerations:

**Theorem 2.3.2 (Rademacher)** Let \( \phi : M \rightarrow N \) be a locally Lipschitz map from a manifold \( M \) to a manifold \( N \). Then:

1. \( \phi \) is classically differentiable almost everywhere, with “almost everywhere” understood in the sense of the Lebesgue measure in local coordinates on \( M \).

2. The distributional derivatives of \( \phi \) are in \( L^\infty_{\text{loc}} \) and are equal to the classical ones almost everywhere.

3. Suppose that \( M \) is an open subset of \( \mathbb{R} \) and \( N \) is an open subset of \( \mathbb{R}^n \). Then \( \phi \) is the integral of its distributional derivative,

\[
\phi(x) - \phi(y) = \int_y^x \frac{d\phi}{dt} \, dt . \quad (2.3.1)
\]
CHAPTER 2. CAUSALITY

Proof: Point 1. is the classical statement of Rademacher, the proof can be found in [14, Theorem 2, p. 235]. Point 2. is Theorem 5 in [14, p. 131] and Theorem 1 of [14, p. 235]. (In that last theorem the classical differentiability a.e. is actually established for all $W^{1,p}_{\text{loc}}$ functions with $p > n$). Point 3. can be established by approximating $\phi$ by $C^1$ functions as in [14, Theorem 1, p. 251], and passing to the limit.

Point 2. shows that the usual properties of the derivatives of continuously differentiable functions — such as the Leibniz rule, or the chain rule — hold almost everywhere for the derivatives of locally Lipschitzian functions. By point 3. those properties can be used freely whenever integration is involved.

We will use the symbol $\dot{\gamma}$ to denote the classical derivative of a path $\gamma$, wherever defined. A parameterized path $\gamma$ will be called causal future directed if $\gamma$ is locally Lipschitzian, with $\dot{\gamma}$ — causal and future directed almost everywhere\(^6\). Thus, $\dot{\gamma}$ is defined almost everywhere; and it is causal future directed almost everywhere on the set on which it is defined. A parameterized path $\gamma$ will be called timelike future directed if $\gamma$ is locally Lipschitzian, with $\dot{\gamma}$ — timelike future directed almost everywhere. Past directed parameterized paths are defined by changing “future” to “past” in the definitions above.

A useful property of locally Lipschitzian paths is that they can be parameterized by $h$-distance. Let $\gamma : [a, b) \to M$ be a path, and suppose that $\dot{\gamma}$ is non-zero almost everywhere — this is certainly the case for causal paths. By Rademacher’s theorem the integral

$$s(t) = \int_a^t |\dot{\gamma}|_h(u) du$$

is well defined. Clearly $s(t)$ is a continuous strictly increasing function of $t$, so that the map $t \to s(t)$ is a bijection from $[a, b)$ to its image. The new path $\hat{\gamma} := \gamma \circ s^{-1}$ differs from $\gamma$ only by a reparametrization, so it has the same image in $M$. The reader will easily check that $|\dot{\hat{\gamma}}|_h = 1$ almost everywhere. Further, $\hat{\gamma}$ is Lipschitz continuous with Lipschitz constant smaller than or equal to 1: denoting by $\text{dist}_h$ be the associated distance function, we claim that

$$\text{dist}_h(\hat{\gamma}(s), \hat{\gamma}(s')) \leq |s - s'|. \quad (2.3.2)$$

In order to prove (2.3.2), we calculate, for $s > s'$:

$$s - s' = \int_{s'}^s dt = \int_{s'}^s \sqrt{h(\dot{\hat{\gamma}}, \dot{\hat{\gamma}})(t)} dt \overset{=1 \text{ a.e.}}{=} \inf_{\gamma} \int \sqrt{h(\dot{\gamma}, \dot{\gamma})} \overset{=1 \text{ a.e.}}{=} \inf_{\gamma} \int \sqrt{h(\dot{\gamma}, \dot{\gamma})} \overset{=1 \text{ a.e.}}{=} \int \sqrt{h(\dot{\gamma}, \dot{\gamma})} = \text{dist}_h(\hat{\gamma}(s), \hat{\gamma}(s')),$$  \quad (2.3.3)\)

\(^6\)Some authors allow constant paths to be causal, in which case the sets $J^\pm(\mathcal{U}; \mathcal{O})$ defined below automatically contain $\mathcal{U}$. This leads to unnecessary discussions when concatenating causal paths, so that we find it convenient not to allow such paths in our definition.
2.4 Futures, pasts

Let $\mathcal{U} \subset \mathcal{O} \subset \mathcal{M}$. One sets

$$I^+(\mathcal{U}; \mathcal{O}) := \{ q \in \mathcal{O} : \text{there exists a timelike future directed path from } \mathcal{U} \text{ to } q \text{ contained in } \mathcal{O} \},$$

$$J^+(\mathcal{U}; \mathcal{O}) := \{ q \in \mathcal{O} : \text{there exists a causal future directed path from } \mathcal{U} \text{ to } q \text{ contained in } \mathcal{O} \} \cup \mathcal{U}.$$  

$I^-(\mathcal{U}; \mathcal{O})$ and $J^-(\mathcal{U}; \mathcal{O})$ are defined by replacing “future” by “past” in the definitions above. The set $I^+(\mathcal{U}; \mathcal{O})$ is called the timelike future of $\mathcal{U}$ in $\mathcal{O}$, while $J^+(\mathcal{U}; \mathcal{O})$ is called the causal future of $\mathcal{U}$ in $\mathcal{O}$, with similar terminology for the timelike past and the causal past. We will write $I^\pm(\mathcal{U})$ for $I^\pm(\mathcal{U}; \mathcal{M})$, similarly for $J^\pm(\mathcal{U})$, and one then omits the qualification “in $\mathcal{M}$” when talking about the causal or timelike futures and pasts of $\mathcal{U}$. We will write $I^\pm(p; \mathcal{O})$ for $I^\pm(\{p\}; \mathcal{O})$, $I^\pm(p)$ for $I^\pm(\{p\}; \mathcal{M})$, etc.

Although our definition of causal curves does not coincide with the usual ones [4, 24, 45, 52], it is equivalent to those. Indeed, it is easily seen that our definition of $J^\pm$ is identical to the standard one. On the other hand, the class of timelike curves as defined here is quite wider than the standard one; nevertheless, the resulting sets $I^\pm$ are again identical to the usual ones (compare Proposition 2.4.11).

It is legitimate to raise the question, why is it interesting to consider sets such as $J^+(\mathcal{O})$. The answer is two-fold: From a mathematical point of view, those sets appear naturally when describing the finite speed of propagation property of wave-type equations, such as Einstein’s equations. From a physical point of view, such constructs are related to the fundamental postulate of general relativity, that no signal can travel faster than the speed of light. This is equivalent to the statement that the only events of space-times that are influenced by an event $p \in \mathcal{M}$ are those which belong to $J^+(\mathcal{M})$.

Example 2.4.1 Let $\mathcal{M} = S^1 \times S^1$ with the flat metric $g = -dt^2 + d\varphi^2$. Geodesics of $g$ through $(0, 0)$ are of the form

$$\gamma(s) = (\alpha s \text{ mod } 2\pi, \beta s \text{ mod } 2\pi),$$  \hspace{1cm} (2.4.1)

where $\alpha$ and $\beta$ are constants; the remaining geodesics are obtained by a rigid translation of (2.4.1). Clearly any two points of $\mathcal{M}$ can be joined by a timelike geodesic, which shows that for all $p \in \mathcal{M}$ we have

$$I^+(p) = J^+(p) = \mathcal{M}.$$  

It is of some interest to point out that for irrational $\beta/\alpha$ in (2.4.1) the corresponding geodesic is dense in $\mathcal{M}$. 

where the infimum is taken over $\tilde{\gamma}$'s which start at $\gamma(s')$ and finish at $\gamma(s)$. 

2.4 Futures, pasts

Let $\mathcal{U} \subset \mathcal{O} \subset \mathcal{M}$. One sets

$$I^+(\mathcal{U}; \mathcal{O}) := \{ q \in \mathcal{O} : \text{there exists a timelike future directed path from } \mathcal{U} \text{ to } q \text{ contained in } \mathcal{O} \},$$

$$J^+(\mathcal{U}; \mathcal{O}) := \{ q \in \mathcal{O} : \text{there exists a causal future directed path from } \mathcal{U} \text{ to } q \text{ contained in } \mathcal{O} \} \cup \mathcal{U}.$$  

$I^-(\mathcal{U}; \mathcal{O})$ and $J^-(\mathcal{U}; \mathcal{O})$ are defined by replacing “future” by “past” in the definitions above. The set $I^+(\mathcal{U}; \mathcal{O})$ is called the timelike future of $\mathcal{U}$ in $\mathcal{O}$, while $J^+(\mathcal{U}; \mathcal{O})$ is called the causal future of $\mathcal{U}$ in $\mathcal{O}$, with similar terminology for the timelike past and the causal past. We will write $I^\pm(\mathcal{U})$ for $I^\pm(\mathcal{U}; \mathcal{M})$, similarly for $J^\pm(\mathcal{U})$, and one then omits the qualification “in $\mathcal{M}$” when talking about the causal or timelike futures and pasts of $\mathcal{U}$. We will write $I^\pm(p; \mathcal{O})$ for $I^\pm(\{p\}; \mathcal{O})$, $I^\pm(p)$ for $I^\pm(\{p\}; \mathcal{M})$, etc.

Although our definition of causal curves does not coincide with the usual ones [4, 24, 45, 52], it is equivalent to those. Indeed, it is easily seen that our definition of $J^\pm$ is identical to the standard one. On the other hand, the class of timelike curves as defined here is quite wider than the standard one; nevertheless, the resulting sets $I^\pm$ are again identical to the usual ones (compare Proposition 2.4.11).

It is legitimate to raise the question, why is it interesting to consider sets such as $J^+(\mathcal{O})$. The answer is two-fold: From a mathematical point of view, those sets appear naturally when describing the finite speed of propagation property of wave-type equations, such as Einstein’s equations. From a physical point of view, such constructs are related to the fundamental postulate of general relativity, that no signal can travel faster than the speed of light. This is equivalent to the statement that the only events of space-times that are influenced by an event $p \in \mathcal{M}$ are those which belong to $J^+(\mathcal{M})$.

Example 2.4.1 Let $\mathcal{M} = S^1 \times S^1$ with the flat metric $g = -dt^2 + d\varphi^2$. Geodesics of $g$ through $(0, 0)$ are of the form

$$\gamma(s) = (\alpha s \text{ mod } 2\pi, \beta s \text{ mod } 2\pi),$$  \hspace{1cm} (2.4.1)

where $\alpha$ and $\beta$ are constants; the remaining geodesics are obtained by a rigid translation of (2.4.1). Clearly any two points of $\mathcal{M}$ can be joined by a timelike geodesic, which shows that for all $p \in \mathcal{M}$ we have

$$I^+(p) = J^+(p) = \mathcal{M}.$$  

It is of some interest to point out that for irrational $\beta/\alpha$ in (2.4.1) the corresponding geodesic is dense in $\mathcal{M}$.
There is an obvious *meta-rule* in the theory of causality that whenever a property involving \( I^+ \) or \( J^+ \) holds, then an identical property will be true with \( I^+ \) replaced by \( I^- \), and with \( J^+ \) replaced by \( J^- \), or both. This is proved by changing the time-orientation of the manifold. Thus we will only make formal statements for the futures.

Example 2.4.1 shows that in causally pathological space-times the notions of futures and pasts need not to carry interesting information. On the other hand those objects are useful tools to study the global structure of those space-times which possess reasonable causal properties.

We start with some elementary properties of futures and pasts:

**Proposition 2.4.2** Consider a spacetime \((\mathcal{M}, g)_{\mathbb{C}^0}\). We have:

1. \( I^+(\mathcal{U}) \subset J^+(\mathcal{U}) \).
2. \( p \in I^+(q) \iff q \in I^-(p) \).
3. \( \mathcal{V} \subset I^+(\mathcal{U}) \implies I^+(\mathcal{V}) \subset I^+(\mathcal{U}) \).

Similar properties hold with \( I^+ \) replaced by \( J^+ \).

**Proof:**

1. A timelike curve is a causal curve.
2. If \([0, 1] \ni s \to \gamma(s)\) is a future directed causal curve from \( q \) to \( p \), then \([0, 1] \ni s \to \gamma(1-s)\) is a past directed causal curve from \( p \) to \( q \).
3. Let us start by introducing some notation: consider \( \gamma_a : [0, 1] \to \mathcal{M} \), \( a = 1, 2 \), two causal curves such that \( \gamma_1(1) = \gamma_2(0) \). We define the *concatenation operation* \( \gamma_1 \cup \gamma_2 \) as follows:

\[
(\gamma_1 \cup \gamma_2)(s) = \begin{cases} 
\gamma_1(s), & s \in [0, 1], \\
\gamma_2(s-1), & s \in [1, 2].
\end{cases}
\]

(2.4.2)

There is an obvious extension of this definition when the ranges of parameters of the \( \gamma_a \)’s are not \([0, 1]\), or when a finite number \( i \geq 3 \) of paths is considered, we leave the formal definition to the reader.

Let, now, \( r \in I^+(\mathcal{V}) \), then there exists \( q \in \mathcal{V} \) and a future directed timelike curve \( \gamma_2 \) from \( q \) to \( r \). Since \( \mathcal{V} \subset I^+(\mathcal{U}) \) there exists a future directed timelike curve \( \gamma_1 \) from some point \( p \in \mathcal{U} \) to \( q \). Then the curve \( \gamma_1 \cup \gamma_2 \) is a future directed timelike curve from \( \mathcal{U} \) to \( r \). \( \square \)

We have the following, intuitively obvious, description of futures and pasts of points in Minkowski space-time (see Figure 2.4.1); in Proposition 2.4.5 below we will shortly prove a similar local result in general space-times, with a considerably more complicated proof.

**Proposition 2.4.3** Let \((\mathcal{M}, g)\) be the \((n+1)\)-dimensional Minkowski space-time \( \mathbb{R}^{1,n} := (\mathbb{R}^{1+n}, \eta) \), with Minkowskian coordinates \( (x^\mu) = (x^0, \vec{x}) \) so that

\[\eta(\partial_\mu, \partial_\nu) = \text{diag}(-1, +1, \ldots, +1) .\]

Then

1. \( I^+(0) = \{ x^\mu : \eta_{\mu\nu} x^\mu x^\nu < 0, \ x^0 > 0 \} \),

\( \square \)
2.4. FUTURES, PASTS

2. \( J^+(0) = \{ x^\mu : \eta_{\mu\nu} x^\mu x^\nu \leq 0, \ x^0 \geq 0 \} \),

3. in particular the boundary \( \partial J^+(0) \) of \( J^+(0) \) is the union of \( \{0\} \) together with all null future directed geodesics with initial point at the origin.

**Proof:** Let \( \gamma(s) = (x^\mu(s)) \) be a parameterized causal path in \( \mathbb{R}^{1,n} \) with \( \gamma(0) = 0 \). At points at which \( \gamma \) is differentiable we have

\[
\eta(\dot{\gamma}, \dot{\gamma}) = \left( \frac{dx^0}{ds} \right)^2 + \left| \frac{d\vec{x}}{ds} \right|_\delta^2 \leq 0, \quad \frac{dx^0}{ds} \geq \left| \frac{d\vec{x}}{ds} \right|_\delta \geq 0.
\]

Now, similarly to a differentiable function, a locally Lipschitzian function is the integral of its distributional derivative (see Theorem 2.3.2) hence

\[
x^0(s) = \int_0^s \frac{dx^0}{ds}(u) du \quad \text{ (2.4.3a)}
\]

\[
\geq \int_0^s \left| \frac{d\vec{x}}{ds} \right|_\delta(u) du =: \ell(\gamma_s). \quad \text{ (2.4.3b)}
\]

Here \( \ell(\gamma_s) \) is the length, with respect to the flat Riemannian metric \( \delta \), of the path \( \gamma_s \), defined as

\[
[0, s] \ni u \to \vec{x}(u) \in \mathbb{R}^n.
\]

Let \( \text{dist}_\delta \) denote the distance function of the metric \( \delta \), thus

\[
\text{dist}_\delta(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|_\delta,
\]

it is well known that

\[
\ell_s \geq \text{dist}_\delta(\vec{x}(s), \vec{x}(0)) = |\vec{x}(s) - \vec{x}(0)|_\delta = |\vec{x}(s)|_\delta.
\]

Therefore

\[
x^0(s) \geq |\vec{x}(s)|_\delta,
\]

and point 2. follows. For timelike curves the same proof applies, with all inequalities becoming strict, establishing point 1. Point 3. is a straightforward consequence of point 2. \( \square \)
CHAPTER 2. CAUSALITY

There is a natural generalisation of Proposition 2.4.3 to the following class of metrics on $\mathbb{R} \times \mathcal{I}$:

$$g = -\varphi dt^2 + h, \quad \partial_t \varphi = \partial_t h = 0,$$

where $h$ is a Riemannian metric on $\mathcal{I}$, and $\varphi$ is a strictly positive function. Such metrics are sometimes called warped-products, with warping function $\varphi$.

**Proposition 2.4.4** Let $\mathcal{M} = \mathbb{R} \times \mathcal{I}$ with the metric (2.4.4), and let $p \in \mathcal{I}$. Then $J^+((0,p))$ is the graph over $\mathcal{I}$ of the distance function $\text{dist}_h(p, \cdot)$ of the optical metric

$$\hat{h} := \varphi^{-1} h,$$

while $I^+((0,p))$ is the epigraph of $\text{dist}_h(p, \cdot)$,

$$I^+((0,p)) = \{ (t,q) : t > \text{dist}_h(p,q) \}.$$

**Proof:** Since the causal character of a curve is invariant under conformal transformations, the causal and timelike futures with respect to the metric $g$ coincide with those with respect to the metric

$$\varphi^{-1} g = -dt^2 + \hat{h}.$$

Arguing as in the proof of Proposition 2.4.3, (2.4.3) becomes

$$x^0(s) = \int_0^s \frac{dx^0}{ds}(u)du \geq \int_0^s \frac{dx}{ds} \bigg|_h (u)du =: \ell_h(\gamma_s),$$

where $\ell_h(\gamma_s)$ denotes the length of $\gamma_s$ with respect to $\hat{h}$, and one concludes as before. 

The next result shows that, locally, causal behaviour is identical to that of Minkowski space-time. The proof of this “obvious” fact turns out to be surprisingly involved:

**Proposition 2.4.5** Consider a spacetime $(\mathcal{M},g)_{C^2}$. Let $\mathcal{O}_p$ be a domain of normal coordinates $x^\mu$ centered at $p \in \mathcal{M}$ as in Proposition 2.2.3. Let

$$\mathcal{O} \subset \mathcal{O}_p$$

be any normal-coordinate ball such that $\nabla x^0$ is timelike on $\mathcal{O}$. Recall (compare (2.2.9)) that the function $\sigma_p : \mathcal{O}_p \to \mathbb{R}$ has been defined by the formula

$$\sigma_p(\exp_p(x^\mu)) := \eta_{\mu\nu}x^\mu x^\nu.$$

Then

$$\mathcal{O} \ni q = \exp_p(x^\mu) \in \begin{cases} I^+(p; \mathcal{O}) \iff \sigma_p(q) < 0, \ x^0 > 0, \\ J^+(p; \mathcal{O}) \iff \sigma_p(q) \leq 0, \ x^0 \geq 0, \\ \hat{J}^+(p; \mathcal{O}) \iff \sigma_p(q) = 0, \ x^0 \geq 0, \end{cases}$$

with the obvious analogues for pasts. In particular, a point $q = \exp_p(x^\mu) \in J^+(p; \mathcal{O}_p)$ if and only if $q$ lies on the null geodesic segment $[0,1] \ni s \to \gamma(s) = \exp_p(sx^\mu) \in \hat{J}^+(p; \mathcal{O}_p)$. 
Remark 2.4.6 Example 2.4.1 shows that $I^\pm(p; \mathcal{O})$, etc., cannot be replaced by $I^\pm(p)$, because causal paths through $p$ can exit $\mathcal{O}$, and reenter it; this can actually happen again and again.

Before proving Proposition 2.4.5, we note the following straightforward implication thereof:

**Proposition 2.4.7** Let $\mathcal{O}$ be as in Proposition 2.4.5, then $I^+(p; \mathcal{O})$ is open. \(\square\)

**Proof of Proposition 2.4.5:** As the coordinate rays are geodesics, the implications “⇐” in (2.4.7) are obvious. It remains to prove “⇒”. We start with a lemma:

**Lemma 2.4.8** Let $\tau$ be a time function, i.e., a differentiable function with timelike past-pointing gradient. For any $\tau_0$, a future directed causal path $\gamma$ cannot leave the set $\{q : \tau(q) > \tau_0\}$; the same holds for sets of the form $\{q : \tau(q) \geq \tau_0\}$. In fact, $\tau$ is non-decreasing along $\gamma$, strictly increasing if $\gamma$ is timelike.

**Proof:** Let $\gamma : I \to \mathcal{M}$ be a future directed parameterized causal path, then $\tau \circ \gamma$ is a locally Lipschitzian function, hence equals the integral of its derivative on any compact subset of its domain of definition, so that

\[
\tau(\gamma(s_2)) - \tau(\gamma(s_1)) = \int_{s_1}^{s_2} \frac{d(\tau \circ \gamma)}{du}(u)du = \int_{s_1}^{s_2} (d\tau, \dot{\gamma})(u)du = \int_{s_1}^{s_2} g(\nabla \tau, \dot{\gamma})(u)du \geq 0 ,
\]

since $\nabla \tau$ is timelike past directed, while $\dot{\gamma}$ is causal future directed or zero wherever defined. The function $s \to \tau(\gamma(s))$ is strictly increasing when $\gamma$ is timelike, since then the integrand in (2.4.8) is strictly positive almost everywhere. \(\square\)

Applying Lemma 2.4.8 to the time function $x^0$ we obtain the claim about $x^0$ in (2.4.7). To justify the remaining claims of Proposition 2.4.5, we recall Equation (2.2.10)

\[
\nabla \sigma_p \text{ is } \begin{cases} \text{timelike future directed} & \text{on } \{q : \sigma_p(q) < 0 , x^0(q) > 0\} , \\ \text{null future directed} & \text{on } \{q : \sigma_p(q) = 0 , x^0(q) > 0\} . 
\end{cases}
\]

(2.4.9)

Let $\gamma = (\gamma^\mu) : I \to \mathcal{O}$ be a parameterized future directed causal path with $\gamma(0) = p$, then $\sigma_p \circ \gamma$ is a locally Lipschitzian function, hence

\[
\sigma_p \circ \gamma(t) = \int_0^t \frac{d(\sigma_p \circ \gamma)(s)}{ds}ds = \int_0^t g(\nabla \sigma_p, \dot{\gamma})(s)ds .
\]

(2.4.10)

We note the following:
LEMMA 2.4.9 A future directed causal path \( \gamma \subset \mathcal{O}_p \) cannot leave the set \( \{ q : x^0(q) > 0, \sigma_p(q) < 0 \} \).

**Proof:** The time function \( x^0 \) remains positive along \( \gamma \) by Lemma 2.4.8. If \( -\sigma_p \) were also a time function we would be done by the same argument. The problem is that \( -\sigma_p \) is a time function only on the set where \( \sigma_p \) is negative, so some care is needed; we proceed as follows: The vector field \( \nabla \sigma_p \) is causal future directed on \( \{ x^0 > 0, \eta_{\mu\nu}x^\mu x^\nu \leq 0 \} \), while \( \dot{\gamma} \) is causal future directed or zero wherever defined, hence \( g(\nabla \sigma_p, \dot{\gamma}) \leq 0 \) as long as \( \gamma \) stays in \( \{ x^0 > 0, \eta_{\mu\nu}x^\mu x^\nu \leq 0 \} \). By Equation (2.4.9) the function \( \sigma_p \) is non-increasing along \( \gamma \) as long as \( \gamma \) stays in \( \{ x^0 > 0, \eta_{\mu\nu}x^\mu x^\nu \leq 0 \} \). Suppose that \( \sigma_p(\gamma(s_1)) < 0 \) and let

\[
s_* = \sup\{ u \in I : \sigma_p(\gamma(s)) < 0 \text{ on } [s_1, u] \}.
\]

If \( s_* \in I \), then \( \sigma_p \circ \gamma(s_*) = 0 \) and \( \sigma_p \circ \gamma \) is not non-increasing on \([s_1, s_*]\), which is not possible since \( \gamma(s) \in \{ x^0 > 0, \eta_{\mu\nu}x^\mu x^\nu \leq 0 \} \) for \( s \in [s_1, s_*) \). It follows that \( \sigma_p \circ \gamma < 0 \), as desired. \( \square \)

Proposition 2.4.5 immediately follows for those future direct causal paths through \( p \) which do enter the set \( \{ \eta_{\mu\nu}x^\mu x^\nu < 0 \} \). This is the case for \( \gamma \)'s such that \( \dot{\gamma}(0) = (\dot{\gamma}_{\mu}(0)) \) exists and is timelike: We then have

\[
\gamma_{\mu}(s) = s \dot{\gamma}_{\mu}(0) + o(s),
\]

hence

\[
\eta_{\mu\nu} \gamma_{\mu}(s) \gamma_{\nu}(s) = s^2 \eta_{\mu\nu} \dot{\gamma}_{\mu}(0) \dot{\gamma}_{\nu}(0) + o(s^2) < 0
\]

for \( s \) small enough. It follows that \( \gamma \) enters the set \( \{ \eta_{\mu\nu}x^\mu x^\nu < 0 \} \equiv \{ q : \sigma_p(q) < 0 \} \), and remains there for \( |s| \) small enough. We conclude using Lemma 2.4.9.

We continue with arbitrary parameterized future directed timelike paths \( \gamma : [0, b) \to \mathcal{M} \), with \( \gamma(0) = p \), thus \( \dot{\gamma} \) exists and is timelike future directed for almost all \( s \in [0, b) \). In particular there exists a sequence \( s_i \to i \to \infty \) such that \( \dot{\gamma}(s_i) \) exists and is timelike.

Standard properties of solutions of ODE’s show that for each \( q \in \mathcal{O}_p \) there exists a neighborhood \( \mathcal{W}_{p,q} \) of \( p \) such that the function

\[
\mathcal{W}_{p,q} \ni r \to \sigma_r(q)
\]

is defined, continuous in \( r \). For \( i \) large enough we will have \( \gamma(s_i) \in \mathcal{W}_{p,\gamma(s)} \); for such \( i \)'s we have just shown that

\[
\sigma_{\gamma(s_i)}(\gamma(s)) < 0.
\]

Passing to the limit \( i \to \infty \), by continuity one obtains

\[
\sigma_p(\gamma(s)) \leq 0, \quad (2.4.11)
\]

thus

\[
\gamma \subset \{ x^0 \geq 0, \eta_{\mu\nu}x^\mu x^\nu \leq 0 \}. \quad (2.4.12)
\]
Since $\dot{\gamma}$ is timelike future directed wherever defined, and $\nabla \sigma_p$ is causal future directed on $\{x^0 > 0, \eta_{\mu\nu}x^\mu x^\nu \leq 0\}$, Equations (2.4.10) and (2.4.12) show that the inequality in (2.4.11) must be strict.

To finish the proof, we reduce the general case to the last one by considering perturbed metrics, as follows: let $e_0$ be any unit timelike vector field on $\mathcal{O}$ ($e_0$ can, e.g., be chosen as $\nabla x^0 / \sqrt{-g(\nabla x^0, \nabla x^0)}$, for $\epsilon > 0$ define a family of Lorentzian metrics $g_\epsilon$ on $\mathcal{O}$ by the formula

$$g_\epsilon(X, Y) = g(X, Y) - \epsilon g(e_0, X)g(e_0, Y).$$

Consider any vector $X$ which is causal for $g$, then

$$g_\epsilon(X, X) = g(X, X) - \epsilon(g(e_0, X))^2 \leq -\epsilon(g(e_0, X))^2 < 0,$$

so that $X$ is timelike for $g_\epsilon$. Let $\sigma(g_\epsilon)p$ be the associated functions defined as in Equation (2.4.6), where the exponential map there is the one associated to the metric $g_\epsilon$. Standard properties of solutions of ODE’s (see, e.g., [51]) imply that for any compact subset $K$ of $\mathcal{O}_p$ there exists an $\epsilon_K > 0$ and a neighborhood $\mathcal{O}_{p, K}$ of $K$ such that for all $\epsilon \in [0, \epsilon_K]$ the functions

$$\mathcal{O}_{p, K} \ni q \rightarrow \sigma(g_\epsilon)p(q)$$

are defined, and depend continuously upon $\epsilon$. We take $K$ to be $\gamma([0, s])$, where $s$ is such that $[0, s] \subset I$, and consider any $\epsilon$ in $(0, \epsilon_{\gamma([0, s])})$. Since $\gamma$ is timelike for $g_\epsilon$, the results already established show that we have

$$\sigma(g_\epsilon)p(\gamma(s)) < 0.$$

Continuity in $\epsilon$ implies

$$\sigma_p(\gamma(s)) \leq 0.$$

Since $s$ is arbitrary in $I$, Proposition 2.4.5 is established. \qed

For certain considerations it is useful to have the following:

**Corollary 2.4.10** Let $\mathcal{O}_p$ be a domain of normal coordinates $x^\mu$ centered at $p \in \mathcal{M}$ as in Proposition 2.2.3, and let $\mathcal{O} \subset \mathcal{O}_p$ be any normal–coordinates ball such that $\nabla x^0$ is timelike on $\mathcal{O}$. If $\gamma \subset \mathcal{O}$ is a causal curve from $p$ to $q = \exp_p(x^\mu) \in \mathcal{J}^+(p; \mathcal{O})$, then $\gamma$ lies entirely in $\mathcal{J}^+(p; \mathcal{O})$, and there exists a reparameterization $s \rightarrow r(s)$ of $\gamma$ so that $\gamma$ is a null-geodesic segment through $p$:

$$[0, 1] \ni s \rightarrow \gamma(r(s)) = \exp_p(sx^\mu).$$

**Proof:** Proposition 2.4.5 shows that $\sigma_p(q) = 0$. It follows that

$$0 = \sigma_p \circ \gamma(t) = \int_0^t g(\nabla \sigma_p, \dot{\gamma})(s)ds. \quad (2.4.13)$$
Since $\nabla \sigma_p$ and $\dot{\gamma}$ are causal oppositely directed we have $g(\nabla \sigma_p, \dot{\gamma}) \geq 0$ almost everywhere. It thus follows from (2.4.13) that

$$g(\nabla \sigma_p, \dot{\gamma}) = 0$$

almost everywhere. This is only possible if

$$\nabla \sigma_p \sim \dot{\gamma} \quad (2.4.14)$$

almost everywhere. Thus $\nabla \sigma_p$ is null a.e. along $\gamma$. But $\nabla \sigma_p$ is null only on $J^+(p; \emptyset)$, thus $\gamma(s) \in J^+(p; \emptyset)$ a.e.; by continuity this is true for all $s$, and we have shown that $\gamma$ lies entirely on $J^+(p; \emptyset)$.

To continue, in normal coordinates (2.4.14) reads

$$\frac{dx^\mu(s)}{ds} = f(s) x^\mu(s)$$

a.e., for some strictly positive function $f \in L^\infty$. Define $r(s)$ by

$$r(s) = \int_0^s f(t) dt,$$

then $r$ is strictly increasing, hence a bijection from the interval of definition of $\gamma$ to some interval $[0, r_0]$. Further $r$ is Lipschitz, differentiable on a set of full measure, and on that set it holds

$$\frac{dr}{ds} = f.$$

Now, this equation shows that the set where the map $r \mapsto s(r)$ might fail to be differentiable is the image by $r$ of the set $\Omega_1$ where $r$ fails to be differentiable, together with the image by $r$ of the set of points $\Omega_2$ where $f$ vanishes. Both $\Omega_1$ and $\Omega_2$ have zero measure, and the image of a negligible set by a Lipschitz map is a negligible set. We thus obtain, almost everywhere,

$$\frac{dx^\mu(s(r))}{dr} = \frac{dx^\mu(s(r))}{ds} \frac{ds}{dr} = x^\mu(s(r)),$$

(2.4.15)

so $dx^\mu/dr$ can be extended by continuity to a continuous function. It easily follows that $x^\mu(s(\lambda))$ is $C^1$, and the result is obtained by integration of (2.4.15).

Penrose’s approach [45] to the theory of causality is based on the notion of timelike or causal trips: by definition, a causal trip is a piecewise broken causal geodesic. The following result can be used to show equivalence of the definitions of $I^+$, etc., given here, to those of Penrose:

**Corollary 2.4.11** Consider a spacetime $(\mathcal{M}, g)_{C^2}$. If $q \in I^+(p)$, then there exists a future directed piecewise broken future directed timelike geodesic from $p$ to $q$. Similarly, if $q \in J^+(p)$, then there exists a future directed piecewise broken future directed causal geodesic from $p$ to $q$.

**Proof:** Let $\gamma : [0, 1] \to \mathcal{M}$ be a parameterized future directed causal path with $\gamma(0) = p$ and $\gamma(1) = q$. Continuity of $\gamma$ implies that for every $s \in [0, 1]$ there exists $\epsilon_s > 0$ such that

$$\gamma(u) \in \mathcal{O}_{\gamma(s)}$$

for all

$$u \in (s - 2\epsilon_s, s + 2\epsilon_s) \cap [0, 1] = [\max(0, s - 2\epsilon_s), \min(1, s + 2\epsilon_s)],$$

Penrose’s approach [45] to the theory of causality is based on the notion of timelike or causal trips: by definition, a causal trip is a piecewise broken causal geodesic. The following result can be used to show equivalence of the definitions of $I^+$, etc., given here, to those of Penrose:
where $\mathcal{O}_r$ is a normal-coordinates ball centred at $r$, and satisfying the requirements of Proposition 2.4.5. Compactness of $[0,1]$ implies that from the covering $\{(s - \epsilon_s, s + \epsilon_s)\}_{s \in [0,1]}$ a finite covering $\{(s_i - \epsilon_{s_i}, s_i + \epsilon_{s_i})\}_{i=0,...,N}$ can be extracted, with $s_0 = 0$, $s_N = 1$. Reordering the $s_i$’s if necessary we may assume that $s_i < s_i + 1$. By definition we have

$$\gamma|_{[s_i, s_{i+1}]} \subset \mathcal{O}_{\gamma(s_i)} ,$$

and by Proposition 2.4.5 there exists a causal future directed geodesic segment from $\gamma(s_i)$ to $\gamma(s_{i+1})$: if $\gamma(s_{i+1}) = \exp_{\gamma(s_i)}(x^\mu)$, then the required geodesic segment is given by

$$[0,1] \ni s \to \exp_{\gamma(s_i)}(sx^\mu).$$

If $\gamma$ is timelike, then all the segments are timelike. Concatenating the segments together provides the claimed piecewise broken geodesic. \hfill \Box

Proposition 2.4.3 shows that the sets $I^\pm(p)$ are open in Minkowski space-time. Similarly it follows from Proposition 2.4.5 that the sets $I^\pm(p; \mathcal{O}_p)$ are open. This turns out to be true in general:

\textbf{Proposition 2.4.12} Consider a spacetime $(\mathcal{M}, g)_{C^2}$. For all $\mathcal{U} \subset \mathcal{M}$ the sets $I^\pm(\mathcal{U})$ are open.

\textbf{Proof:} Let $q \in I^+(\mathcal{U})$, and let, as in the proof of Corollary 2.4.11, $s_{N-1}$ be such that $q \in \mathcal{O}_{\gamma(s_{N-1})}$. Then

$$\mathcal{O}_{\gamma(s_{N-1})} \cap I^+(\gamma(s_{N-1}); \mathcal{O})$$

is an open neighborhood of $q$ by Corollary 2.4.7. Clearly

$$I^+(\gamma(s_{N-1}); \mathcal{O}) \subset I^+(\gamma(s_{N-1})).$$

Since $\gamma(s_{N-1}) \in I^+(\mathcal{U})$ we have

$$I^+(\gamma(s_{N-1})) \subset I^+(\mathcal{U})$$

(see point 3. of Proposition 2.4.2). It follows that

$$\mathcal{O}_{\gamma(s_{N-1})} \cap I^+(\gamma(s_{N-1}); \mathcal{O}) \subset I^+(\mathcal{U}),$$

which implies our claim. \hfill \Box

In Minkowski space-time the sets $J^\pm(p)$ are closed, with

$$\overline{I^\pm(p)} = J^\pm(p). \quad (2.4.16)$$

We will show below (see Corollary 2.4.19) that we always have

$$\overline{I^\pm(p)} \supset J^\pm(p), \quad (2.4.17)$$

but this requires some work. Before proving (2.4.17), let us point out that (2.4.16) does not need to be true in general:
Example 2.4.13 Let \((\mathcal{M}, g)\) be the two-dimensional Minkowski space-time \(\mathbb{R}^{1,1}\) from which the set \(\{x^0 = 1, x^1 \leq -1\}\) has been removed. Then
\[
J^+(0; \mathcal{M}) = J^+(0, \mathbb{R}^{1,1}) \setminus \{x^0 = -x^1, x^1 \in (-\infty, 1]\},
\]
cf. Figure 2.4.2. hence \(J^+(0; \mathcal{M})\) is neither open nor closed, and Equation (2.4.16) does not hold.

We have the following:

Lemma 2.4.14 ("Push-up Lemma 1") Consider a spacetime \((\mathcal{M}, g)_C^2\). For any \(\Omega \subset \mathcal{M}\) we have
\[
I^+(J^+(\Omega)) = I^+(\Omega) .
\] (2.4.18)

Remark 2.4.15 In \cite{11} an example is presented which shows that the result is wrong for metrics which are merely continuous.

Proof: The obvious property
\[
\mathcal{U} \subset \mathcal{V} \implies I^+(\mathcal{U}) \subset I^+(\mathcal{V})
\]
provides inclusion of the right-hand-side of (2.4.18) into the left-hand-side. It remains to prove that
\[
I^+(J^+(\Omega)) \subset I^+(\Omega) .
\]

Let \(r \in I^+(J^+(\Omega))\), thus there exists a past-directed timelike curve \(\gamma_0\) from \(r\) to a point \(q \in J^+(\Omega)\). Since \(q \in J^+(\Omega)\), then either \(q \in \Omega\), and there is nothing to prove, or there exists a past-directed causal curve \(\gamma : I \to \mathcal{M}\) from \(q\) to some point \(p \in \Omega\). We want to show that there exists a past-directed timelike curve \(\hat{\gamma}\) starting at \(r\) and ending at \(p\). The curve \(\hat{\gamma}\) can be obtained by “pushing-up” \(\gamma\) slightly, to make it timelike, the construction proceeds as follows: Using compactness, we cover \(\gamma\) by a finite collection \(\mathcal{U}_i, i = 0, \cdots, N\), of elementary regions \(\mathcal{U}_i\) centered at \(p_i \in \gamma(I)\), with
\[
p_0 = q, \quad p_i \in \mathcal{U}_i \cap \mathcal{U}_{i+1}, \quad p_{i+1} \subset J^- (p_i), \quad p_N = p .
\]
Let \(\gamma_0 : [0, s_0] \to \mathcal{M}\) be the already mentioned causal curve from \(r\) to \(q \in \mathcal{U}_1\); let \(s_1 \neq s_0\) be close enough to \(s_0\) so that \(\gamma_0(s_1) \in \mathcal{U}_1\). By Proposition 2.2.3
2.4. FUTURES, PASTS

together with the definition of elementary regions there exists a past directed timelike curve \( \gamma_1 : [0, 1] \to \mathcal{U}_1 \) from \( \gamma_0(s_1) \) to \( p_1 \in \mathcal{U}_1 \cap \mathcal{U}_2 \). For \( s \) close enough to 1 the curve \( \gamma_1 \) enters \( \mathcal{U}_2 \), choose an \( s_2 \neq 1 \) such that \( \gamma_1(s_2) \in \mathcal{U}_2 \), again by Proposition 2.2.3 there exists a a past directed timelike curve \( \gamma_2 : [0, 1] \to \mathcal{U}_2 \) from \( \gamma_1(s_2) \) to \( p_2 \). One repeats that construction iteratively obtaining a (finite) sequence of past-directed timelike curves \( \gamma_i \subset I^+(\gamma) \cap \mathcal{O}_1 \) such that the end point \( \gamma_i(s_{i+1}) \) of \( \gamma_i|_{[0,s_{i+1}]} \) coincides with the starting point of \( \gamma_{i+1} \). Concatenating those curves together gives the desired path \( \hat{\gamma} \).  

We have the following, slightly stronger, version of Lemma 2.4.14, which gives a sufficient condition to be able to deform a causal curve to a timelike one, keeping the deformation as small as desired:

**Corollary 2.4.16** Let the metric be twice differentiable. Consider a causal future directed curve \( \gamma : [0, 1] \to \mathcal{M} \) from \( p \) to \( q \). If there exist \( s_1 < s_2 \in [0, 1] \) such that \( \gamma|_{[s_1,s_2]} \) is timelike, then in any neighborhood \( \mathcal{O} \) of \( \gamma \) there exists a timelike future directed curve \( \hat{\gamma} \) from \( p \) to \( q \).

**Remark 2.4.17** The so-called maximising null geodesics can not be deformed as above to timelike curves, whether locally or globally. We note that all null geodesics in Minkowski space-time are maximising.

**Proof:** If \( s_2 = 1 \), then Corollary 2.4.16 is essentially a special case of Lemma 2.4.14: the only difference is the statement about the neighborhood \( \mathcal{O} \). This last requirement can be satisfied by choosing the sets \( \mathcal{U}_i \) in the proof of Lemma 2.4.14 so that \( \mathcal{U}_i \subset \mathcal{O} \). If \( s_1 = 0 \) (and regardless of the value of \( s_2 \)) the result is obtained by changing time-orientation, applying the result already established to the path \( \gamma'(s) = \gamma(1 - s) \), and changing-time orientation again. The general case is reduced to the ones already covered by first deforming the curve \( \gamma|_{[0,s_2]} \) to a new timelike curve \( \hat{\gamma} \) from \( p \) to \( \gamma(s_2) \), and then applying the result again to the curve \( \hat{\gamma} \cup \gamma|_{[s_2,1]} \).

Another result in the same spirit is provided by the following:

**Proposition 2.4.18** Let \( \gamma \) be causal curve from \( p \) to \( q \) in \((\mathcal{M}, g)_{C^2}\) which is not a null geodesic. Then there exists a timelike curve from \( p \) to \( q \).

**Proof:** By Corollary 2.4.11 we can without loss of generality assume that \( \gamma \) is a piecewise broken geodesic. If one of the geodesics forming \( \gamma \) is timelike, the result follows from Corollary 2.4.16. It remains to consider curves which are piecewise broken null geodesics with at least one break point, say \( p \). Let \( q \in J^-(p) \) be close enough to \( p \) so that \( p \) belongs to a domain of normal coordinates \( \mathcal{O}_q \) centred at \( q \). Corollary 2.4.10 shows that points on \( \gamma \) lying to the causal future of \( p \) are not in \( J^+(q, \mathcal{O}_q) \), hence they are in \( I^+(q, \mathcal{O}_q) \), and so \( \gamma \) can be deformed within \( \mathcal{O}_q \) to a timelike curve. The result follows now again from Corollary 2.4.16.

As another straightforward corollary of Lemma 2.4.14 one obtains a property of \( J \), which is wrong in general for metrics which are not \( C^2 \):
CHAPTER 2. CAUSALITY

**Corollary 2.4.19** Consider a spacetime \((\mathcal{M}, g)_{C^2}\). For any \(p \in \mathcal{M}\) we have

\[
J^+(p) \subset I^+(p) .
\]

**Proof:** Let \(q \in J^+(p)\), and let \(r_i \in I^+(q)\) be any sequence of points accumulating at \(q\), then \(r_i \in I^+(p)\) by Lemma 2.4.14, hence \(q \in I^+(p)\). \(\square\)

2.5 Extendible and inextendible paths

To avoid ambiguities, recall that we only assume continuity of the metric unless explicitly indicated otherwise.

A useful concept, when studying causality, is that of a causal path with cannot be extended any further. Recall that, from a physical point of view, the image in space-time of a timelike path is supposed to represent the history of some observer, and it is sometimes useful to have at hand idealised observers which do never cease to exist. Here it is important to have the geometrical picture in mind, where all that matters is the image in space-time of the path, independently of any parameterisation: if that image “stops”, then one can sometimes continue the path by concatenating with a further one; continuing in this way one hopes to be able to obtain paths which are inextendible.

In order to make things precise, let \(\gamma : [a, b) \rightarrow \mathcal{M}\), be a parameterized, causal, future directed path. A point \(p\) is called a future end point of \(\gamma\) if \(\lim_{s \to b} \gamma(s) = p\). Past end points are defined in the obvious analogous way. An end point is a point which is either a past end point or a future end point.

Given \(\gamma\) as above, together with an end point \(p\), one is tempted to extend \(\gamma\) to a new path \(\hat{\gamma} : [a, b] \rightarrow \mathcal{M}\) defined as

\[
\hat{\gamma}(s) = \begin{cases} 
\gamma(s), & s \in [a, b), \\
p, & s = b .
\end{cases}
\]

(2.5.1)

The first problem with this procedure is that the resulting curve might fail to be locally Lipschitz in general. An example is given by the timelike future-directed path

\[
[0, 1) \ni s \mapsto \gamma_1(s) = (-s^{1/2}, 0) \in \mathbb{R}^{1,1} ,
\]

which is locally Lipschitzian on \([0, 1)\), but is not on \([0, 1]\). (This follows from the fact that the difference quotient \((f(s) - f(s'))/(s - s')\) blows up as \(s\) and \(s'\) tend to one when \(f(s) = (1 - s)^{1/2}\). Recall that in our definition of a causal curve \(\gamma\), a prerequisite condition is the locally Lipschitz character, so that the extension \(\hat{\gamma}_1\) fails to be causal even though \(\gamma_1\) is.

The problem is even worse if \(b = \infty\): consider the timelike future-directed curve

\[
[1, \infty) \ni s \mapsto \gamma_3(s) = (-1/s, 0) \in \mathbb{R}^{1,1} .
\]

Here there is no way to extend the curve to the future, as an application from a subset of \(\mathbb{R}\) to \(\mathcal{M}\), because the range of parameters already covers all \(s \geq 1\).

Now, the image of both \(\gamma_1\) and \(\gamma_2\) is simply the interval \([-1, 0) \times \mathbb{R}\), which can
be extended to a longer causal curve in \( \mathbb{R}^{1,1} \) in many ways if one thinks in terms of images rather than of maps.

Both problems above can be taken care of by requiring that the parameter \( s \) be the proper distance parameter of some auxiliary Riemannian metric \( h \). (At this stage \( h \) is not required to be complete). This might require reparameterizing the path. From the point of view of our definition this means that we are passing to a different path, but the image in space-time of the new path coincides with the previous one. If one thinks of timelike paths as describing observers, the new observer will thus have experienced identical events, even though he will be experiencing those events at different times on his time-measuring device. We note, moreover, that (locally Lipschitz) reparameterizations do not change the timelike or causal character of paths.

**Example 2.5.1** Consider a sequence of null geodesics in \( \mathbb{R}^{1,1} = (\mathbb{R}^2, g = -dt^2 + dx^2) \), with \( h = dt^2 + dx^2 \) as the Riemannian background metric, threading back and forth up to a space-distance \( 1/n \) around the \{\( x = 0 \)\} axis. The limit curve is \( \gamma(s) = (s/\sqrt{2}, 0) \) which is not \( \text{dist}_h \)-parameterized.

We have already shown in Section 2.3 that a locally Lipschitzian path can always be reparameterized by \( h \)-distance, leading to a uniformly Lipschitzian path, with Lipschitz constant one. It should be clear from the examples given above, as well as from the examples discussed at the beginning of Section 2.6, that it is sensible to use such a parameterization, and it is tempting to build this requirement into the definition of a causal path. One reason for not doing that is the existence of affine parameterization for geodesics, which is geometrically significant, and which is convenient for several purposes. Another reason is the arbitrariness related to the choice of \( h \). Last but not least, a limit curve for a sequence of \( \text{dist}_h \)-parameterized curves does not have to be \( \text{dist}_h \)-parameterized, as seen in Example 2.5.1. Therefore we will not assume \textit{a priori} an \( h \)-distance parameterization, but such a reparameterization will often be used in the proofs.

Returning to (2.5.1), we want to show that \( \check{\gamma} \) will be uniformly Lipschitz if \( \text{dist}_h \)-parameterization is used for \( \gamma \). More generally, suppose that \( \gamma \) is uniformly Lipschitz with Lipschitz constant \( L \),

\[
\text{dist}_h(\gamma(s), \gamma(s')) \leq L|s - s'|.
\]

(2.5.2)

Passing with \( s' \) to \( b \) in that equation we obtain

\[
\text{dist}_h(\gamma(s), p) \leq L|s - b|,
\]

and the Lipschitzian character of \( \check{\gamma} \) easily follows. We have therefore proved:

**Lemma 2.5.2** Let \( \gamma : [a, b) \to \mathcal{M}, b < \infty \), be a uniformly Lipschitzian path with an end point \( p \). Then \( \gamma \) can be extended to a uniformly Lipschitzian path \( \check{\gamma} : [a, b] \to \mathcal{M}, \) with \( \check{\gamma}(b) = p \).

Let \( \gamma : [a, b) \to \mathcal{M}, b \in \mathbb{R} \cup \{\infty\} \) be a path, then \( p \) is said to be an \( \omega \)-\textit{limit point} of \( \gamma \) if there exists a sequence \( s_k \to b \) such that \( \gamma(s_k) \to p \). An end point is always an \( \omega \)-limit point, but the inverse does not need to be true in general
(consider $\gamma(s) = \exp(is) \in \mathbb{C}$, then every point $\exp(ix) \in S^1 \subset \mathbb{C}^1$ is seen to be a $\omega$-limit point of $\gamma$ by setting $s_k = x + 2\pi k$). For $b < \infty$ and for uniformly Lipschitz paths the notions of end point and of $\omega$-limit point coincide:

**Lemma 2.5.3** Let $\gamma : [a,b) \to \mathcal{M}$, $b < \infty$, be a uniformly Lipschitzian path. Then every $\omega$-limit point of $\gamma$ is an end point of $\gamma$. In particular, $\gamma$ has at most one $\omega$-limit point.

**Proof:** By (2.5.2) we have

$$\text{dist}_h(\gamma(s_i), \gamma(s)) \leq L|s_i - s|,$$

and since $\text{dist}_h$ is a continuous function of its arguments we obtain, passing to the limit $i \to \infty$

$$\text{dist}_h(p, \gamma(s)) \leq L|b-s|.$$

Thus $p$ is an end point of $\gamma$. Since there can be at most one end point, the result follows. \hfill \Box

A future directed causal curve $\gamma : [a,b) \to \mathcal{M}$ will be said to be future extendible if there exists $b < c \in \mathbb{R} \cup \{\infty\}$ and a causal curve $\tilde{\gamma} : [a,c) \to \mathcal{M}$ such that

$$\tilde{\gamma}|_{[a,b)} = \gamma. \quad (2.5.3)$$

The path $\tilde{\gamma}$ is then said to be an extension of $\gamma$. The curve $\gamma$ will be said future inextendible if it is not future extendible. The notions of past extendibility, or of extendibility, are defined in the obvious way.

Extendibility in the class of causal paths forces a causal $\gamma : [a,b) \to \mathcal{M}$ to be uniformly Lipschitzian: This follows from the fact that $[a,b)$ is a compact subset of the domain of definition of any extension $\tilde{\gamma}$, so that $\tilde{\gamma}|_{[a,b)}$ is uniformly Lipschitzian there. But then $\tilde{\gamma}|_{[a,b)}$ is also uniformly Lipschitzian, and the result follows from (2.5.3).

Whenever a uniformly Lipschitzian path can be extended by adding an end point, it can also be extended as a strictly longer path:

**Lemma 2.5.4** A uniformly Lipschitzian causal path $\gamma : [a,b) \to \mathcal{M}$, $b < \infty$, is extendible if and only if it has an end point.

**Proof:** Let $\tilde{\gamma}$ be given by Proposition 2.5.2, and let $\tilde{\gamma} : [0, d)$ be any maximally extended to the future, future directed causal geodesic starting at $p$, for an appropriate $d \in (0, \infty)$. Then $\tilde{\gamma} \cup \tilde{\gamma}$ is an extension of $\gamma$. \hfill \Box

It turns out that the paths considered in Lemma 2.5.4 are always extendible:

**Theorem 2.5.5** Consider a spacetime $(\mathcal{M}, g)_{C_0}$. Let $\gamma : [a,b) \to \mathcal{M}$, $b \in \mathbb{R} \cup \{\infty\}$, be a future directed causal path parameterized by $h$-distance, where $h$ is any complete auxiliary Riemannian metric. Then $\gamma$ is future inextendible if and only if $b = \infty$. 
2.5. EXTENDIBLE AND INEXTENDIBLE PATHS

Proof: Suppose that \( b < \infty \). Let \( B_h(p, r) \) denote the open \( h \)-distance ball, with respect to the metric \( h \), of radius \( r \), centred at \( p \). Since \( \gamma \) is parameterized by \( h \)-distance we have, by (2.3.2),

\[
\gamma([a, b)) \subset B_h(\gamma(a), b - a).
\]

The Hopf-Rinow theorem [25, 33] asserts that \( B_h(\gamma(a), b - a) \) is compact, therefore there exists \( p \in B_h(\gamma(a), b - a) \) and a sequence \( s_i \) such that

\[
[a, b) \ni s_i \to i \to \infty b \quad \text{and} \quad \gamma(s_i) \to p.
\]

Thus \( p \) is an \( \omega \)-limit point of \( \gamma \). Clearly \( \gamma \) is uniformly Lipschitzian (with Lipschitz modulus one), and Lemma 2.5.3 shows that \( p \) is an end point of \( \gamma \). The result follows now from Lemma 2.5.4.

\[\square\]

2.5.1 Maximally extended geodesics

Consider the Cauchy problem for an affinely-parameterized geodesic \( \gamma \):

\[
\nabla \dot{\gamma} = 0, \quad \gamma(0) = p, \quad \dot{\gamma}(0) = X.
\]

This is a second-order ODE which, by the standard theory [23], for \( C^{1,1} \) metrics, has unique solutions defined on a maximal interval \( I = I(p, X) \ni 0 \). \( I \) is maximal in the sense that if \( I' \) is another interval containing 0 on which a solution of (2.5.4) is defined, then \( I' \subset I \). When \( I \) is maximal the geodesic will be called maximally extended. Now, it is not immediately obvious that a maximally extended geodesic is inextendible in the sense just defined: To start with, the notion of inextendibility involves only the pointwise properties of a path, while the notion of maximally extended geodesic involves the ODE (2.5.4), which involves both the first and second derivatives of \( \gamma \). Next, the inextendibility criteria given above have been formulated in terms of uniformly Lipschitzian parameterizations. While an affinely parameterized geodesic is certainly locally Lipschitzian, there is no a priori reason why it should be uniformly so, when maximally extended. All these issues turn out to be irrelevant, and we have the following:

**Proposition 2.5.6** Consider a spacetime \((\mathcal{M}, g)_{C^{1,1}}\). A geodesic \( \gamma : I \to \mathcal{M} \) is maximally extended as a geodesic if and only if \( \gamma \) is inextendible as a causal path.

Proof: Suppose, for contradiction, that \( \gamma \) is a maximally extended geodesic which is extendible as a path, thus \( \gamma \) can be extended to a path \( \hat{\gamma} \) by adding its end point \( p \) as in (2.5.1). Working in a normal coordinate neighborhood \( \mathcal{O}_p \) around \( p \), \( \hat{\gamma} \cap \mathcal{O}_p \) has a last component which is a geodesic segment which ends at \( p \). By construction of normal coordinates the component of \( \hat{\gamma} \) in question is simply a half-ray through the origin, which can be clearly be continued through \( p \) as a geodesic. This contradicts maximality of \( \gamma \) as a geodesic. It follows that a maximally extended geodesic is inextendible. Now, if \( \gamma \) is inextendible as a path, then \( \gamma \) can clearly not be extended as a geodesic, which establishes the reverse implication.

\[\square\]

A result often used in causality theory is the following:
Theorem 2.5.7 Consider a spacetime \((\mathcal{M},g)\)\(_{C^0}\). Let \(\gamma\), be a future directed causal, respectively timelike, path. Then there exists an inextendible causal, respectively timelike, extension of \(\gamma\).

Proof: We start with a proof assuming a \(C^{1,1}\) metric, as it is simpler: If \(\gamma : [a,b) \to \mathcal{M}\) is inextendible there is nothing to prove; otherwise the path \(\hat{\gamma} \cup \tilde{\gamma}\), where \(\hat{\gamma}\) is given by Proposition 2.5.2, and \(\tilde{\gamma}\) is any maximally extended future directed causal geodesic as in the proof of Proposition 2.5.4, provides an extension. This extension is inextendible by Proposition 2.5.6.

When the metric is merely assumed to be continuous, one can proceed as follows: Suppose that \(\gamma\) is extendible, in particular \(\gamma\) has an end point \(p\). Let \(\Omega_p\) denote the collection of all future directed, parameterized by \(h\)-proper distance, timelike paths starting at \(p\). Obviously \(\Omega_p\) is non-empty. \(\Omega_p\) can be directed using the property of “being an extension”: we write \(\gamma_1 < \gamma_2\) if \(\gamma_2\) is an extension of \(\gamma_1\). The existence of inextendible paths in \(\Omega_p\) easily follows from the Kuratowski-Zorn lemma.

If \(\gamma_1\) is any maximal element of \(\Omega_p\), then \(\hat{\gamma} \cup \gamma_1\), with \(\hat{\gamma}\) given by Lemma 2.5.2, is an inextendible future directed extension of \(\gamma\).

\(\square\)

2.6 Accumulation curves

A key tool in the analysis of global properties of space-times is the analysis of sequences of curves. One typically wants to obtain a limiting curve, and study its properties. The object of this section is to establish the existence of such limiting curves.

We wish, first, to find the ingredients needed for a useful notion of a limit of curves. It is enlightening to start with several examples. The first question that arises is whether to consider a sequence of curves \(\gamma_n\) defined on a common interval \(I\), or whether one should allow different domains \(I_n\) for each \(\gamma_n\). To illustrate that this last option is very unpractical, consider the family of timelike curves

\[ (-1/n, 1/n) \ni s \to \gamma_n(s) = (s, 0) \in \mathbb{R}^{1,1}. \]  

(2.6.1)

The only sensible geometric object to which the \(\gamma_n(s)\) converge is the constant map

\[ \{0\} \ni s \to \gamma_\infty(s) = 0 \in \mathbb{R}^{1,1}, \]  

(2.6.2)

which is quite reasonable, except that it takes us away from the class of causal curves. To avoid such behavior we will therefore assume that all the curves \(\gamma_n\) have a common domain of definition \(I\).

Next, there are various reasons why a sequence of curves might fail to have an “accumulation curve”. First, the whole sequence might simply run to infinity. (Consider, for example, the sequence

\[ \mathbb{R} \ni s \to \gamma_n(s) = (s, n) \in \mathbb{R}^{1,1}. \])

This is avoided when one considers curves such that \(\gamma_n(0)\) converges to some point \(p \in \mathcal{M}\).
Further, there might be a problem with the way the curves are parameterized. As an example, let $\gamma_n$ be defined as

$$(-1, 1) \ni s \mapsto \gamma_n(s) = (s/n, 0) \in \mathbb{R}^{1,1}.$$  

As in (2.6.1), the $\gamma_n(s)$ converge to the constant map

$$(-1, 1) \ni s \mapsto \gamma_\infty(s) = 0 \in \mathbb{R}^{1,1},$$

again not a causal curve. Another example of pathological parameterizations is given by the family of curves

$$\mathbb{R} \ni s \mapsto \gamma_n(s) = (ns, 0) \in \mathbb{R}^{1,1}.$$  

In this case one is tempted to say that the $\gamma_n$’s accumulate at the path, say $\gamma_1$, if parameterization is not taken into account. However, such a convergence is extremely awkward to deal with when attempting to actually prove something. This last behavior can be avoided by assuming that all the curves are uniformly Lipschitz continuous, with the same Lipschitz constant. One way of ensuring this is to parameterize all the curves by a length parameter with respect to our auxiliary complete Riemannian metric $h$.

Yet another problem arises when considering the family of Euclidean-distance-parameterized causal curves

$$\mathbb{R} \ni s \mapsto \gamma_n(s) = (s + n, 0) \in \mathbb{R}^{1,1}.$$  

This can be gotten rid of by shifting the distance parameter so that the sequence $\gamma_n(s_0)$ stays in a compact set, or converges, for some $s_0$ in the domain $I$.

The above discussion motivates the hypotheses of the following result:

**Proposition 2.6.1** Let $(\mathcal{M}, g)$ be a $C^3$ Lorentzian manifold with a $C^2$ metric. Let $\gamma_n : I \to \mathcal{M}$ be a sequence of uniformly Lipschitz future directed causal curves, and suppose that there exist $p \in \mathcal{M}$ such that

$$\gamma_n(0) \to p.$$ 

Then there exists a future directed causal curve $\gamma : I \to \mathcal{M}$ and a subsequence $\gamma_{n_i}$ converging to $\gamma$ in the topology of uniform convergence on compact subsets of $I$.

**Remark 2.6.2** The hypothesis that $g$ is $C^2$ is made to guarantee that the function $(p, q) \mapsto \sigma_q(p)$ is continuous, and depends continuously upon the metric. It is shown in [11] that the result remains true for metrics which are merely continuous. The analysis there relies on the result here and suitable smooth approximations of the metric.

Proposition 2.6.1 provides the justification for the following definition:

**Definition 2.6.3** Let $\gamma_n : I \to \mathcal{M}$ be a sequence of paths in $(\mathcal{M}, g)_{C^0}$. We shall say that $\gamma : I \to \mathcal{M}$ is an accumulation curve of the $\gamma_n$’s, or that the $\gamma_n$’s accumulate at $\gamma$, if there exists a subsequence $\gamma_{n_i}$ that converges to $\gamma$ uniformly on compact subsets of $I$.  


In their treatment of causal theory, Hawking and Ellis [24] introduce a notion of \textit{limit curve} for paths, regardless of parameterization, which we find very awkward to work with. A related but slightly more convenient notion of \textit{cluster curve} is considered in [29], where the name of “limit curve” is used for yet another notion of convergence. As discussed in [4, 29], those definitions lead to pathological behavior in some situations. We have found the above notion of “accumulation curve” the most convenient to work with from several points of view.

A sensible terminology, in the context of Definition 2.6.3, could be “$C^0$-limits of curves”, but we prefer not to use the term “limit” in this context, as limits are usually unique, while Definition 2.6.3 allows sequences that have more than one accumulation curve.

**Proof of Proposition 2.6.1:** The hypothesis that all the $\gamma_n$’s are uniformly Lipschitz reads

\[
\text{dist}_h(\gamma_n(s), \gamma_n(s')) \leq L|s - s'|, \tag{2.6.5}
\]

for some constant $L$. This shows that the family $\{\gamma_n\}$ is equicontinuous, and (2.6.4) together with the Arzela-Ascoli theorem implies that for every compact set $K \subset I$ there exists a curve $\gamma_K : K \to \mathcal{M}$ and a subsequence $\gamma_{n_k}$ which converges uniformly to $\gamma_K$ on $K$. One can obtain a $K$-independent curve $\gamma$ by the so-called diagonalisation procedure.

The diagonalisation procedure goes as follows: For ease of notation we consider $I = \mathbb{R}$, the same argument applies on any interval with obvious modifications. Let $\gamma_{n(i,1)}$ be the sequence which converges to $\gamma_{[-1,1]}$; applying Arzela-Ascoli to this sequence one can extract a subsequence $\gamma_{n(i,2)}$ of $\gamma_{n(i,1)}$ which converges uniformly to some curve $\gamma_{[-2,2]}$ on $[-2,2]$. Since $\gamma_{n(i,2)}$ is a subsequence of $\gamma_{n(i,1)}$, and since $\gamma_{n(i,1)}$ converges to $\gamma_{[-1,1]}$ on $[-1,1]$, one finds that $\gamma_{[-2,2]}$ restricted to $[-1,1]$ equals $\gamma_{[-1,1]}$. One continues iteratively: suppose that $\{\gamma_{n(i,k)}\}_{i \in \mathbb{N}}$ has been defined for some $k$, and converges to a curve $\gamma_{[-k,k]}$ on $[-k,k]$, then the sequence $\{\gamma_{n(i,k+1)}\}_{i \in \mathbb{N}}$ is defined as a subsequence of $\{\gamma_{n(i,k)}\}_{i \in \mathbb{N}}$ which converges to some curve $\gamma_{[-(k+1),k+1]}$ on $[-(k+1),k+1]$. The curve $\gamma$ is finally defined as

\[
\gamma(s) = \gamma_{[-k,k]}(s),
\]

where $k$ is any number such that $s \leq k$. The construction guarantees that $\gamma_{[-k,k]}(s)$ does not depend upon $k$ as long as $s \leq k$.

It remains to show that $\gamma$ is causal. Passing to the limit $n \to \infty$ in (2.6.5) one finds

\[
\text{dist}_h(\gamma(s), \gamma(s')) \leq L|s - s'|. \tag{2.6.6}
\]

For $q \in \mathcal{M}$ let $\mathcal{O}_q$ be an elementary neighborhood of $q$ as in Proposition 2.4.5, and let $\sigma_q$ be the associated function defined by (2.4.6). Let $s \in \mathbb{R}$ and consider any point $\gamma(s) \in \mathcal{M}$. Now, the size of the sets $\mathcal{O}_q$ can be controlled uniformly when $q$ varies over compact subsets of $\mathcal{M}$. It follows that for all $s'$ close enough to $s$ and for all $n$ large enough we have $\gamma_n(s') \in \mathcal{O}_{\gamma_n(s)}$. Since the $\gamma_n$’s are causal, Proposition 2.4.5 shows that we have

\[
\sigma_{\gamma_n(s)}(\gamma_n(s')) \leq 0. \tag{2.6.7}
\]

Passing to the limit in (2.6.7) gives

\[
\sigma_{\gamma(s)}(\gamma(s')) \leq 0. \tag{2.6.8}
\]
This is only possible if $\gamma$ is causal, which can be seen as follows: Suppose that $\gamma$ is differentiable at $s$. In normal coordinates on $\mathcal{O}_{\gamma(s)}$ we have, by definition of a derivative,

$$\gamma^\mu(s') = \gamma^\mu(s) + \dot{\gamma}^\mu(s)(s' - s) + o(s' - s),$$

hence

$$0 \geq \sigma_{\gamma(s)}(\gamma(s')) \equiv \eta_{\mu\nu} \dot{\gamma}^\mu(s') \dot{\gamma}^\nu(s') = \eta_{\mu\nu} \dot{\gamma}^\mu(s)(s' - s)^2 + o((s' - s)^2).$$

For $s' - s$ small enough this is only possible if

$$\eta_{\mu\nu} \dot{\gamma}^\mu(s) \dot{\gamma}^\nu(s) \leq 0,$$

and $\dot{\gamma}$ is causal, as we desired to show. \qed

Let us address now the question of inextendibility of accumulation curves. We note the following lemma:

**Lemma 2.6.4** Let $\gamma_n$ be a sequence of dist$_h$-parameterized inextendible causal curves converging to $\gamma$ uniformly on compact subsets of $\mathbb{R}$, then $\gamma$ is inextendible.

**Proof:** Note that the parameter range of $\gamma$ is $\mathbb{R}$, and the result would follow from Theorem 2.5.5 if $\gamma$ were dist$_h$-parameterized, but this might fail to be the case, as seen in Example 2.5.1.

So we need to show that both $\gamma|_{[0,\infty)}$ and $\gamma|_{(-\infty,0]}$ are of infinite length. As usual it suffices to consider $\gamma|_{[0,\infty)}$, we retain the name $\gamma$ for this last path. Suppose that this is not the case, then there exists $a < \infty$ so that $\gamma$ is defined on $[0,a)$, when reparameterised by dist$_h$-distance. By Theorem 2.5.5 the curve $\gamma$ can be extended to a causal curve defined on $[0,a]$, still denoted by $\gamma$.

Let $\mathcal{U}$ be an elementary neighborhood centred at $\gamma(a)$, and let $0 < b < a$ be such that $\gamma(b) \in \mathcal{U}$. By definition of accumulation curve there exists a sequence $n_i \in \mathbb{N}$, a compact interval $[-k,k] \subset \mathbb{R}$ and a sequence $s_i \in [-k,k]$ such that $\gamma_{n_i}(s_i)$ converges to $\gamma(b)$, in particular we will have $\gamma_{n_i}(s_i) \in \mathcal{U}$ for $i$ large enough. We note the following:

**Lemma 2.6.5** Let $\mathcal{U}$ be an elementary neighborhood, as defined in Definition 2.2.7. There exists a constant $\ell$ such that for any causal curve $\gamma : I \to \mathcal{U}$ the $h$-length $|\gamma|_h$ of $\gamma$ is bounded by $\ell$.

To prove Lemma 2.6.5 we need the following variation of the inverse Cauchy-Schwarz inequality:

**Lemma 2.6.6** Let $K$ be a compact set and let $X$ be a continuous timelike vector field defined there, then there exists a strictly positive constant $C$ such that for all $q \in K$ and for all causal vectors $Y \in T_q\mathcal{M}$ we have

$$|g(X,Y)| \geq C|Y|_h. \quad (2.6.9)$$
Proof: By homogeneity it is sufficient to establish (2.6.9) for causal \( Y \in T_q \mathcal{M} \) such that \( |Y|_h = 1 \); let us denote by \( U(h)_q \) this last set. The result follows then by continuity of the strictly positive function

\[
\bigcup_{q \in K} U(h)_q \ni Y \rightarrow |g(X,Y)|
\]
on the compact set \( \bigcup_{q \in K} U(h)_q \).

Returning to the proof of Lemma 2.6.5, let \( x^0 \) be the local time coordinate on \( \mathcal{U} \), since \( X := \nabla x^0 \) is timelike we can use Lemma 2.6.6 with \( K = \mathcal{U} \) to conclude that there exists a constant \( C \) such that for any causal curve \( \gamma \subset \mathcal{U} \) we have

\[
|g(X,\dot{\gamma})| \geq C > 0
\]
at all points at which \( \gamma \) is differentiable. This implies, for \( s_2 \geq s_1 \),

\[
|x^0(s_2) - x^0(s_1)| \geq \int_{s_1}^{s_2} |g(\nabla x^0, \dot{\gamma})| ds \geq C \int_{s_1}^{s_2} ds = C|s_2 - s_1|.
\]

It follows that

\[
|\gamma|_h \leq \ell := \frac{2}{C} \sup_{\mathcal{U}} |x^0| < \infty, \quad (2.6.10)
\]

as desired.

Returning to the proof of Lemma 2.6.4, it follows from Lemma 2.6.5 applied to \( \gamma_n \) that \( \gamma_n|_{[s_i,s_i+\ell]} \) must exit \( \mathcal{U} \). This implies that \( \gamma_n|_{[-k,k+\ell]} \) cannot accumulate at a curve which has an end point \( \gamma(b) \in \mathcal{U} \), and the result follows.

In summary, it follows from Lemmata 2.5.4 and 2.6.4 together with Proposition 2.6.1 that:

**Theorem 2.6.7** Let \( (\mathcal{M},g) \) be a \( C^3 \) Lorentzian manifold with a \( C^2 \) metric. Every sequence of future directed, inextendible, causal curves which accumulates at a point \( p \in \mathcal{M} \) accumulates at some future directed, inextendible, causal curve through \( p \).

One is sometimes interested in sequences of maximally extended geodesics:

**Proposition 2.6.8** Let \( \gamma_n \) be a sequence of maximally extended geodesics accumulating at \( \gamma \) in \( (\mathcal{M},g)_{C^{1,1}} \). Then \( \gamma \) is a maximally extended geodesic.

Proof: If we use a dist\(_h\)-parameterization of the \( \gamma_n \)'s and of \( \gamma \) such that \( \gamma_n(0) \rightarrow \gamma(0) \), then by the Arzela-Ascoli Theorem (passing to a subsequence if necessary) the \( \gamma_n \)'s converge to \( \gamma \), uniformly on compact subsets of \( \mathbb{R} \). Let \( \mathcal{K} \) be a compact neighborhood of \( \gamma(0) \), compactness of \( \bigcup_{p \in \mathcal{K}} U_p \mathcal{M} \), where \( U_p \mathcal{M} \subset T_p \mathcal{M} \) is the set of \( h \)-unit vectors tangent to \( \mathcal{M} \), implies that there exists a subsequence such that \( \hat{\gamma}_n(0) \) converges to some vector \( X \in U_{\gamma(0)} \mathcal{M} \subset T_{\gamma(0)} \mathcal{M} \). Let \( \sigma : (a,b) \rightarrow \mathcal{M} \), \( a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{\infty\} \), be an affinely parameterised
maximally extended geodesic through $\gamma(0)$ with initial tangent vector $X$. By continuous dependence of ODE’s upon initial values it follows that 1) for any $a < \alpha < \beta < b$ all the $\gamma_n$’s, except perhaps for a finite number, are defined on $[\alpha, \beta]$ when affinely parameterized, and 2) they converge to $\sigma_{|[\alpha, \beta]}$ in the (uniform) $C^1([\alpha, \beta], \mathcal{M})$ topology. Thus $\dot{\gamma}_n(s) \rightarrow \dot{\sigma}(s)$ uniformly on compact subsets of $(a, b)$, which implies that a dist$_h$-parameterization is preserved under taking limits. Hence the $\gamma_n$’s, when dist$_h$-parameterized, converge uniformly to a dist$_h$-reparameterization of $\sigma$ on compact subsets of $\mathbb{R}$, call it $\mu$. It follows that $\gamma = \mu$, and $\gamma$ is a maximally extended geodesic. $\blacksquare$

2.6.1 Achronal causal curves

A curve $\gamma : I \rightarrow \mathcal{M}$ is called achronal if

$$\forall s, s' \in I \quad \gamma(s) \notin I^+(\gamma(s')).$$

Any spacelike geodesic in Minkowski space-time is achronal. More interestingly, it follows from Proposition 2.4.3 that this is also true for null geodesics. However, null geodesics do not have to be achronal in general: consider, e.g., the two-dimensional space-time $\mathbb{R} \times S^1$ with the flat metric $-dt^2 + dx^2$, where $x$ is an angle-type coordinate along $S^1$ with periodicity, say, $2\pi$. Then the points $(0, 0)$ and $(2\pi, 0)$ both lie on the null geodesic

$$s \rightarrow (s, s \mod 2\pi),$$

and are clearly timelike related to each other.

In this section we will be interested in causal curves that are achronal. We start with the following:

**Proposition 2.6.9** Consider a spacetime $(\mathcal{M}, g)_{C^2}$. If $\gamma$ is an achronal causal curve, then $\gamma$ is a null geodesic.

**Proof:** Let $\mathcal{O}$ be any elementary neighborhood, then any connected component of $\gamma \cap \mathcal{O}$ is a null geodesic by Corollary 2.4.10. $\blacksquare$

**Theorem 2.6.10** Consider a spacetime $(\mathcal{M}, g)_{C^2}$. Let $\gamma_n : I \rightarrow \mathcal{M}$ be a sequence of achronal causal curves accumulating at $\gamma$, then $\gamma$ is achronal.

**Remarks 2.6.11** 1. Propositions 2.6.9 and 2.6.8 show that $\gamma$ is inextendible if the $\gamma_n$’s are.

2. The theorem uses the fact that timelike futures and pasts are open, which is not clear if the metric is not $C^2$.

**Proof:** Suppose $\gamma$ is not achronal, then there exist $s_1, s_2 \in I$ such that $\gamma(s_2) \in I^+(\gamma(s_1))$, thus there exists a timelike curve $\tilde{\gamma} : [s_1, s_2] \rightarrow \mathcal{M}$ from $\gamma(s_1)$ to $\gamma(s_2)$. Choose some $\tilde{s} \in (s_1, s_2)$. We have $\gamma(s_2) \in I^+(\tilde{\gamma}(\tilde{s}))$, and since $I^+(\tilde{\gamma}(\tilde{s}))$ is open there exists an open neighborhood $\mathcal{O}_2$ of $\gamma(s_2)$ such that $\mathcal{O}_2 \subset I^+(\tilde{\gamma}(\tilde{s}))$. Similarly there exists an open neighborhood $\mathcal{O}_1$ of $\gamma(s_1)$ such
that \( \mathcal{O}_1 \subset I^-(\hat{\gamma}(\hat{s})) \). This shows that any point \( p_2 \in \mathcal{O}_2 \) lies in the timelike future of any point \( p_1 \in \mathcal{O}_1 \): indeed, one can go from \( p_1 \) along some timelike path to \( \hat{\gamma}(\hat{s}) \), and continue along another timelike path from \( \hat{\gamma}(\hat{s}) \) to \( p_2 \).

Passing to a subsequence if necessary, there exist sequences \( s_{1,n} \) and \( s_{2,n} \) such that \( \gamma_n(s_{1,n}) \) converges to \( \gamma(s_1) \) and \( \gamma_n(s_{2,n}) \) converges to \( \gamma(s_2) \). Then \( \gamma_n(s_{1,n}) \in \mathcal{O}_1 \) and \( \gamma_n(s_{2,n}) \in \mathcal{O}_2 \) for \( n \) large enough, leading to \( \gamma_n(s_{2,n}) \in I^+(\gamma_n(s_{1,n})) \), contradicting achronality of \( \gamma_n \).

### 2.7 Causality conditions

Space-times can exhibit various causal pathologies, most of which are undesirable from a physical point of view. The simplest example of unwanted causal behaviour is the existence of closed timelike curves. A space-time is said to be chronological if no such curves exist. An example of a space-time which is not chronological is provided by \( S^1 \times \mathbb{R} \) with the flat metric \(-dt^2 + dx^2\), where \( t \) is a local coordinate defined modulo \( 2\pi \) on \( S^1 \). Then every circle \( x = \text{const} \) is a closed timelike curve.

The class of compact manifolds is a very convenient one from the point of view of Riemannian geometry. The following result of Geroch shows that such manifolds are always pathological from a Lorentzian perspective:

**Proposition 2.7.1 (Geroch [18])** Every compact space-time \((\mathcal{M}, g)\) contains a closed timelike curve.

**Proof:** Consider the covering of \( \mathcal{M} \) by the collection of open sets \( \{I^-(p)\}_{p \in \mathcal{M}} \), by compactness a finite covering \( \{I^-(p_i)\}_{i=1,...,t} \) can be chosen. The possibility \( p_1 \in I^-(p_1) \) yields immediately a closed timelike curve through \( p_1 \), otherwise there exists \( p_{i(1)} \) such that \( p_1 \in I^-(p_{i(1)}) \). Again if \( p_{i(1)} \in I^-(p_{i(1)}) \) we are done, otherwise there exists \( p_{i(2)} \) such that \( p_{i(1)} \in I^-(p_{i(2)}) \). Continuing in this way we obtain a — finite or infinite — sequence of points \( p_{i(j)} \) such that

\[
p_{i(j)} \in I^-(p_{i(j+1)}) \tag{2.7.1}
\]

If the sequence is finite we are done. Now, we have only a finite number of \( p_i \)'s at our disposal, therefore if the sequence is finite it has to contain repetitions:

\[
p_{i(j+\ell)} = p_{i(j)}
\]

for some \( j \), and some \( \ell > 0 \). It should be clear from (2.7.1) that there exists a closed timelike curve through \( p_{i(j)} \).

**Remark 2.7.2** Galloway [16] has shown that in compact space-times \((\mathcal{M}, g)\) there exist closed timelike curves through any two points \( p \) and \( q \), under the supplementary condition that the Ricci tensor \( \text{Ric} \) satisfies the following energy condition:

\[
\text{Ric}(X, X) > 0 \quad \text{for all causal vectors } X. \tag{2.7.2}
\]

The chronology condition excludes closed timelike curves, but it just fails to exclude the possibility of occurrence of closed causal curves. A space-time is said to be causal if no such curves can be found. The existence of space-times which are chronological but not causal requires a little work:
Example 2.7.3 Let $\mathcal{M} = \mathbb{R} \times S^1$ with the metric
\[ g = 2dt \, dx + f(t)dx^2 \, , \]
where $f$ is any function satisfying
\[ f \geq 0 \, , \text{ with } f(t) = 0 \text{ iff } t = 0 \, . \]
(The function $f(t) = t^2$ will do.) Here $t$ runs over the $\mathbb{R}$ factor of $\mathcal{M}$, while $x$ is a coordinate defined modulo $2\pi$ on $S^1$. In matrix notation we have
\[ [g_{\mu\nu}] = \begin{bmatrix} 0 & 1 \\ 1 & f \end{bmatrix} \, , \]
which leads to the following inverse metric
\[ [g^{\mu\nu}] = \begin{bmatrix} -f & 1 \\ 1 & 0 \end{bmatrix} \, . \]
It follows that
\[ g(\nabla t, \nabla t) = -f \leq 0 \, , \text{ with } g(\nabla t, \nabla t) = 0 \text{ iff } t = 0 \, . \quad (2.7.3) \]
Recall, now, that a function $\tau$ is called a time function if $\nabla \tau$ is timelike, past-pointing. Equation (2.7.3) shows that $t$ is a time function on the set $\{t \neq 0\}$. Since a time-function is strictly increasing on any causal curve (see Lemma 2.4.8), one easily concludes that no closed causal curve in $\mathcal{M}$ can intersect the set $\{t \neq 0\}$. In other words, closed causal curves — if they do exist — must be entirely contained in the set $\{t = 0\}$. Now, any curve $\gamma$ contained in this last set is of the form
\[ \gamma(s) = (0, x(s)) \, , \]
with tangent vector
\[ \dot{\gamma} = \dot{x} \partial_x \implies g(\dot{\gamma}, \dot{\gamma}) = (\dot{x})^2 g(\partial_x, \partial_x) = (\dot{x})^2 g_{xx} = 0 \, . \]
This shows in particular that
- $\mathcal{M}$ does contain closed causal curves: an example is given by $x(s) = s \mod 2\pi$.
- All closed causal curves are null.

It follows that $(\mathcal{M}, g)$ is indeed chronological, but not causal, as claimed.

It is desirable to have a condition of causality which is stable under small changes of the metric. By way of example, consider a space-time which contains a family of causal curves $\gamma_n$ with both $\gamma_n(0)$ and $\gamma_n(1)$ converging to $p$. Such curves can be thought of as being “almost closed”. Further, it is clear that one can produce an arbitrarily small deformation of the metric which will allow one to obtain a closed causal curve in the deformed space-time. The object of
our next causality condition is to exclude this behaviour. A space-time will be said to be strongly causal if every neighborhood \( O \) of a point \( p \in \mathcal{M} \) contains a neighborhood \( \mathcal{U} \) such that for every causal curve \( \gamma : I \to \mathcal{M} \) the set

\[
\{ s \in I : \gamma(s) \in \mathcal{U} \} \subset I
\]

is a connected subset of \( I \). In other words, \( \gamma \) does not re-enter \( \mathcal{U} \) once it has left it.

Clearly, a strongly causal space-time is necessarily causal. However, the inverse does not always hold. An example is given in Figure 2.7.1.

![Figure 2.7.1: A causal space-time which is not strongly causal. Here the metric is the flat one \(-dt^2 + dx^2\), with \( t \) a parameter along \( S^1 \), so that the light cones are at 45\(^o\), in particular \( \hat{\gamma} \) is a null geodesic. It should be clear that no matter how small the neighborhood \( \mathcal{U} \) of \( p \) is, there will exist a causal curve as drawn in the figure which will intersect this neighborhood twice. In order to show that \((\mathcal{M},g)\) is causal one can proceed as follows: suppose that \( \gamma \) is a closed causal curve in \( \mathcal{M} \), then \( \gamma \) has to intersect the hypersurfaces \( \{t = \pm 1\} \) at some points \( x_\pm \), with \( x_- > 1 \) and \( x_+ < -1 \). If we parameterize \( \gamma \) so that \( \gamma(s) = (s,x(s)) \) we obtain

\[
-2 > x_+ - x_- = \int_{-1}^{1} \frac{dx}{ds} ds,
\]

hence there must exist \( s_\ast \in [-1,1] \) such that \( dx/ds < -1 \), contradicting causality of \( \gamma \).

The definition of strong causality appears, at first sight, somewhat unwieldy to verify, so simpler conditions are desirable. The following provides a useful criterion: A space-time \((\mathcal{M},g)\) is said to be stably causal if there exists a time function \( t \) globally defined on \( \mathcal{M} \). Recall — see Lemma 2.4.8 — that time functions are strictly increasing on causal curves. It then easily follows that stable causality implies strong causality:

**Proposition 2.7.4** If \((\mathcal{M},g)_{C^0}\) is stably causal, then it is strongly causal.

**Proof:** Let \( \mathcal{O} \) be a connected open neighborhood of \( p \in \mathcal{M} \), and let \( \varphi \) be a nonegative smooth function such that \( \varphi(p) \neq 0 \) and such that the support \( \text{supp} \varphi \) of \( \varphi \) is a compact set contained in \( \mathcal{O} \). Let \( \tau \) be a time function on \( \mathcal{M} \), for \( a \in \mathbb{R} \) set

\[
\tau_a := \tau + a \varphi.
\]

As \( \nabla \tau \) is timelike, the function \( g(\nabla \tau, \nabla \tau) \) is bounded away from zero on the compact set \( \text{supp} \varphi \), which implies that there exists \( \epsilon > 0 \) small enough so that
\( \tau_{\pm \epsilon} \) are time functions on \( \text{supp} \varphi \). Now, \( \tau_{\pm \epsilon} \) coincides with \( \tau \) away from \( \text{supp} \varphi \), so that the \( \tau_{\pm \epsilon}'s \) are actually time functions on \( \mathcal{M} \) as well. We set
\[
\mathcal{U} := \{ q : \tau_{-\epsilon}(q) < \tau(p) < \tau_{+\epsilon}(q) \}.
\]

We have:
- \( p \in \mathcal{U} \), therefore \( \mathcal{U} \) is not empty;
- \( \mathcal{U} \) is open because the \( \tau\_\alpha\_s \)'s are continuous;
- \( \mathcal{U} \subset \mathcal{O} \) because \( \varphi \) vanishes outside of \( \mathcal{O} \).

Consider any causal curve \( \gamma \) the image of which intersects \( \mathcal{U} \), \( \gamma \) can enter or leave \( \mathcal{U} \) only through
\[
\partial \mathcal{U} \subset \{ q : \tau_{-\epsilon}(q) = \tau(p) \} \cup \{ q : \tau(p) = \tau_{+\epsilon}(q) \}.
\]

At a point \( s_- \) at which \( \gamma(s_-) \in \{ q : \tau_{-\epsilon}(q) = \tau(p) \} \) we have
\[
\tau(p) = \tau_{-\epsilon}(\gamma(s_-)) = \tau(\gamma(s_-)) - \epsilon \varphi(\gamma(s_-)) \implies \tau(\gamma(s_-)) > \tau(p).
\]

Similarly at a point \( s_+ \) at which \( \gamma(s_+) \in \{ q : \tau_{+\epsilon}(q) = \tau(p) \} \) we have
\[
\tau(p) = \tau_{+\epsilon}(\gamma(s_+)) = \tau(\gamma(s_+)) + \epsilon \varphi(\gamma(s_+)) \implies \tau(\gamma(s_+)) < \tau(p).
\]

As \( \tau \) is increasing along \( \gamma \), we conclude that \( \gamma \) can enter \( \mathcal{U} \) only through \( \{ q : \tau_{+\epsilon}(q) = \tau(p) \} \), and leave \( \mathcal{U} \) only through \( \{ q : \tau_{-\epsilon}(q) = \tau(p) \} \). Lemma 2.4.8 shows that \( \gamma \) can intersect each of the two sets at the right-hand-side of (2.7.4) at most once. Those facts obviously imply connectedness of the intersection of (the image of) \( \gamma \) with \( \mathcal{U} \). \( \Box \)

There exist various alternative definitions of stable causality which are equivalent for \( C^2 \) metrics, but note that equivalence is not obvious and its proof requires work. For example, Yvonne Choquet-Bruhat [7] defines stable causality as the requirement of existence of a timelike vector field \( v \) such that \( g - v \otimes v \) is chronological. Hawking and Ellis [24] define stable causality by requiring that \( C^0 \)-small perturbations of the metric preserve causality. Compare [34, 35].

The strongest causality condition is that of global hyperbolicity, considered in the next section.

### 2.8 Global hyperbolicity

A space-time \((\mathcal{M}, g)\) said to be globally hyperbolic if it is strongly causal, and if for every \( p, q \in \mathcal{M} \) the sets \( J^+(p) \cap J^-(q) \) are compact.

It is often convenient to use the equivalent requirement of stable causality together with compactness of the sets \( J^+(p) \cap J^-(q) \); compare Theorem 2.11.1, page 57. The current definition is the one that appears to be the most widely used. From a Cauchy-problem point of view, a natural definition is by requiring the existence of a Cauchy surface, compare Section 2.9 and Theorem 2.11.1, page 57. The last definition is again equivalent for \( C^2 \) metrics, but the equivalence for \( C^0 \) metrics is not clear.
It is not too difficult to show that Minkowski space-time $\mathbb{R}^{1,n}$ is globally hyperbolic: first, the Minkowski time $x^0$ provides a time-function on $\mathbb{R}^{n,1}$; this implies strong causality. Compactness of $J^+(p) \cap J^-(q)$ for all $p$'s and $q$'s is easily checked by drawing pictures; it is also easy to write a formal proof using Proposition 2.4.3, this is left as an exercise to the reader.

The notion of globally hyperbolicity provides excellent control over causal properties of $(\mathcal{M},g)$. This will be made clear at several other places in this work. Anticipating, let us list a few of those:

1. Let $(\mathcal{M},g)$ be globally hyperbolic. If $J^+(p) \cap J^-(q) \neq \emptyset$, then there exists a causal geodesic from $p$ to $q$. Similarly if $I^+(p) \cap I^-(q) \neq \emptyset$, then there exists a timelike geodesic from $p$ to $q$.

2. The Cauchy problem for linear wave equations is globally solvable on globally hyperbolic space-times.

3. A key theorem of Choquet-Bruhat and Geroch asserts that maximal globally hyperbolic solutions of the Cauchy problem for Einstein’s equations are unique up to diffeomorphism.

We start our study of globally hyperbolic space-times with the following property:

**Proposition 2.8.1** Let $(\mathcal{M},g)\in C^2$ be globally hyperbolic, and let $\gamma_n$ be a family of causal curves accumulating both at $p$ and $q$. Then there exists a causal curve $\gamma$, accumulation curve of the (perhaps reparameterized) $\gamma_n$’s which passes both through $p$ and $q$.

**Remark 2.8.2** The result is wrong if stable causality is assumed only. Indeed, let $(\mathcal{M},g)$ be the two-dimensional Minkowski space-time with the origin removed. Let $\gamma_n$ be obtained by following a timelike geodesic from $p = (-1,0)$ to $(0,1/n)$ and then another timelike geodesic to $q = (1,0)$. Then $\gamma_n$ has exactly two accumulation curves $s \rightarrow (s,0)$, with $s \in [-1,0)$ for the first one and $s \in (0,1]$ for the second, none of which passes through both $p$ and $q$.

**Proof:** Extending the $\gamma_n$’s to inextendible curves, and reparameterizing if necessary, we can assume that the $\gamma_n$’s are dist$_h$-parameterized, with common domain of definition $I = \mathbb{R}$, and with $\gamma_n(0)$ converging to $p$. If $p = q$ the result has already been established in Proposition 2.6.1, so we assume that $p \neq q$. Consider the compact set

$$\mathcal{K} := \left( J^+(p) \cap J^-(q) \right) \cup \left( J^+(q) \cap J^-(p) \right)$$

(2.8.1)

(since a globally hyperbolic space-time is causal, one of those sets is, of course, necessarily empty). $\mathcal{K}$ can be covered by a finite number of elementary domains $\mathcal{U}_i$, $i = 1, \cdots, N$. Strong causality allows us to choose the $\mathcal{U}_i$’s small enough so that for every $n$ the image of $\gamma_n$ is a connected subset in $\mathcal{U}_i$. We can choose a parameterization of the $\gamma_n$’s by $h$-length so that, passing to a subsequence of the $\gamma_n$’s if necessary, we have $\gamma_n(0) \rightarrow p$. Extending the $\gamma_n$’s if necessary we
can assume that all the $\gamma_n$’s are defined on $\mathbb{R}$. Now, Lemma 2.6.5, p. 35, shows that there exists a constant $L_i$ — independent of $n$ — such that the $h$–length $|\gamma_n \cap \mathcal{U}_i|_h$ is bounded by $L_i$. Consequently the $h$–length $|\gamma_n \cap \mathcal{K}|_h$, with $\mathcal{K}$ as in (2.8.1), is bounded by

$$|\gamma_n \cap \mathcal{K}|_h \leq L := L_1 + L_2 + \ldots + L_I. \tag{2.8.2}$$

By hypothesis the $\gamma_n$’s accumulate at $q$, therefore there exists a sequence $s_n$ (passing again to a subsequence if necessary) such that $\gamma_n(s_n) \to q$.

Equation (2.8.2) shows that the sequence $s_n$ is bounded, hence — perhaps passing to a subsequence — we have $s_n \to s_*$ for some $s_* \in \mathbb{R}$.

At this stage we could use Proposition 2.6.1, but one might as well argue directly: by our choice of parametrization we have

$$\text{dist}_h(\gamma_n(s), \gamma_n(s')) \leq |s - s'| \tag{2.8.3}$$

(see (2.3.2)-(2.3.3)). This shows that the family $\{\gamma_n\}$ is equicontinuous, and (2.8.3) together with the Arzela-Ascoli theorem (on the compact set $[-L, L]$) implies existence of a curve $\gamma : [-L, L] \to \mathcal{M}$ and a subsequence $\gamma_{n_i}$ which converges uniformly to $\gamma$ on $[-L, L]$. As $\gamma_{n_i}(s_{n_i})$ converges both to $\gamma(s_*)$ and to $q$ we have

$$\gamma(s_*) = q.$$

This shows that $\gamma$ is the desired causal curve joining $p$ with $q$. \hfill $\Box$

**Remark 2.8.3** It should be clear from the proof above that, as emphasised in [7], a space-time is globally hyperbolic if and only if the length of causal paths between two points, as measured with respect to a smooth complete Riemannian metric, is bounded by a number independent of the path.

As a straightforward corollary of Proposition 2.8.1 we obtain:

**Corollary 2.8.4** Let $(\mathcal{M}, g)_{C^2}$ be globally hyperbolic, then

$$\overline{I^\pm(p)} = J^\pm(p).$$

**Proof:** Let $q_n \in I^+(p)$ be a sequence of points accumulating at $q$, thus there exists a sequence $\gamma_n$ of causal curves from $p$ to $q$, then $q \in J^+(p)$ by Proposition 2.8.1. Hence

$$\overline{I^\pm(p)} \subset J^\pm(p).$$

The reverse inclusion is provided by Corollary 2.4.19, page 28. \hfill $\Box$

As already mentioned, global hyperbolicity gives us control over causal geodesics:

**Theorem 2.8.5** Let $(\mathcal{M}, g)_{C^2}$ be globally hyperbolic, if $q \in I^+(p)$, respectively $q \in J^+(p)$, then there exists a timelike, respectively causal, future directed geodesic from $p$ to $q$. 
2.9 Domains of dependence

A set \( U \subset M \) is said to be **achronal** if

\[
I^+(U) \cap I^-(U) = \emptyset.
\]

There is an obvious analogous definition of an **acausal** set

\[
J^+(U) \cap J^-(U) = \emptyset.
\]

Let \( S \) be an achronal topological hypersurface in a space-time \((M, g)\). (By a hypersurface we mean an embedded submanifold of codimension one.) Unless explicitly indicated otherwise we will assume that \( S \) has no boundary.

The **future domain of dependence** \( D^+(S) \) of \( S \) is defined as the set of points \( p \in M \) with the property that every past-directed past-inextendible timelike curve starting at \( p \) meets \( S \) precisely once. The **past domain of dependence** \( D^-(S) \) is defined by changing past-directed past-inextendible to future-directed future-inextendible above. Finally one sets

\[
D^I(S) := D^+(S) \cup D^-(S).
\] (2.9.1)

The “precisely” in “precisely once” above follows of course already from achronality of \( S \); the repetitiveness in our definition is deliberate, to emphasize the property. We always have

\[
S \subset D^\pm(S).
\]

We have found it useful to build in the fact that \( S \) is a topological hypersurface in the definition of \( D^+(S) \). Some authors do not impose this restriction [19], which can lead to various pathologies. From the point of view of differential equations the only interesting case is that of a hypersurface anyway.

The domain of dependence is usually denoted by \( D(S) \) in the literature, and we will sometimes write so. We have added the subscript \( I \) to emphasise that the definition is based on timelike curves. Hawking and Ellis [24] define the domain of dependence using causal curves instead of timelike ones, we will denote the resulting domains of dependence by \( D_I \), etc., if need arises. On the other hand timelike curves are used by Geroch [19] and by Penrose [45]. For \( C^3 \) metrics and spacelike acausal hypersurfaces \( S \), the resulting sets differ by a boundary. The definition with causal curves has the advantage that the resulting set \( D_I(S) \) is open when \( S \) is an acausal topological hypersurface. However, this excludes piecewise null hypersurfaces as Cauchy surfaces, and this is the reason why we use the definition based on timelike curves in the current treatment. It appears that the definition using causal curves is easier to handle when continuous metrics are considered [11].

The following examples are instructive, and are left as exercises to the reader; note that some of the results proved later in this section might be helpful in verifying our claims:

**Example 2.9.1** Let \( S = \{ x^0 = 0 \} \) in Minkowski space-time \( \mathbb{R}^{1,n} \), where \( x^0 \) is the usual time coordinate on \( \mathbb{R}^{1,n} \). Then \( D_I(S) = \mathbb{R}^{1,n} \). Thus both \( D_I^+(S) \) and \( D_I^-(S) \) are non-trivial, and their union covers the whole space-time.
Example 2.9.2 Let \( S = \{ \text{the set of points in } \mathbb{R}^n \text{ with rational coordinates} \} \subset \{ x^0 = 0 \} \) in Minkowski space-time \( \mathbb{R}^{1,n} \), where \( x^0 \) is the usual time coordinate on \( \mathbb{R}^{1,n} \). Then \( D^+_I(S) = \mathcal{I} \), in the sense that “the set of points \( p \in \mathcal{M} \) with the property that every past-directed past-inextendible timelike curve starting at \( p \) meets \( \mathcal{I} \) precisely once” coincides with \( \mathcal{I} \). Such examples are the reason why we assumed that \( S \) is a hypersurface in the definition of \( D_I(S) \).

Example 2.9.3 Let \( S = \{ x^0 - x^1 = 0 \} \) in Minkowski space-time, where the \( x^\mu \)'s are the usual Minkowskian coordinates on \( \mathbb{R}^{1,n} \). Then \( D^+_I(S) = D^-_I(S) = \{ x^0 \geq |x^1| \} \). The fact that \( D^-_I(S) \) makes \( D^-_I(S) \) rather uninteresting. On the other hand \( D^+_I(S) \) coincides with the causal future of \( S \).

Example 2.9.4 Let \( S = \{ x^0 = |x^1| \} \) be the upper component of the unit spacelike hyperboloid in Minkowski space-time. Then \( D^-_I(S) = J^+(0) \). Thus both \( D^-_I(S) \) and \( D^+_I(S) \) are non-trivial, however \( D^-_I(S) \) does not cover the whole past of \( S \).

As a warm-up, let us prove the following elementary property of domains of dependence:

**Proposition 2.9.7** Consider a spacetime \((\mathcal{M}, g)_{C^2}\). Let \( p \in D^+_I(S) \), then

\[ I^-(p) \cap J^+(S) \subset D^+_I(S). \]

**Proof:** Let \( q \in I^-(p) \cap J^+(S) \), thus there exists a past-directed timelike curve \( \gamma_0 \) from \( p \) to \( q \). Let \( \gamma_1 \) be a past-inextendible timelike curve \( \gamma_1 \) starting at \( q \). The
CHAPTER 2. CAUSALITY

curve \( \gamma := \gamma_0 \cup \gamma_1 \) is a past-inextendible past-directed timelike curve starting at \( p \), thus it meets \( \mathcal{I} \) precisely once at some point \( r \in \mathcal{I} \). Suppose that \( \gamma \) passes through \( r \) before passing through \( q \), as \( q \in J^+(\mathcal{I}) \) Lemma 2.4.14 shows that \( r \in I^+(\mathcal{I}) \), contradicting achronality of \( \mathcal{I} \). This shows that \( \gamma \) must meet \( \mathcal{I} \) after passing through \( q \), hence \( \gamma_1 \) meets \( \mathcal{I} \) precisely once. \( \square \)

Let \( \mathcal{I} \) be achronal, we shall say that a set \( \mathcal{O} \) forms a one-sided future neighborhood of \( p \in \mathcal{I} \) if there exists an open set \( U \subset \mathcal{M} \) such that \( U \) contains \( p \) and

\[
U \cap J^+(\mathcal{I}) \subset \mathcal{O}.
\]

As \( I^-(p) \) is open, Proposition 2.9.7 immediately implies:

**Corollary 2.9.8** Consider a spacetime \( (\mathcal{M},g)_{C^2} \). Suppose that \( D^+ I(\mathcal{I}) \neq \mathcal{I} \), consider any point \( p \in D^+ I(\mathcal{I}) \setminus \mathcal{I} \). For any \( q \in \mathcal{I} \cap I^-(p) \) the set \( D^+ I(\mathcal{I}) \) forms a one-sided future neighborhood of \( q \). \( \square \)

Transversality considerations near \( \mathcal{I} \) should make it clear that the hypothesis of Corollary 2.9.8 is satisfied for achronal, \( C^1 \), spacelike hypersurfaces without boundary, and therefore for such \( \mathcal{I} \) the set \( D^+ I(\mathcal{I}) \) forms a neighborhood of \( \mathcal{I} \). Example 2.9.3 shows that this will not be the case for general \( \mathcal{I} \)'s.

The next theorem shows that achronal topological hypersurfaces can be used to produce globally hyperbolic space-times:

**Theorem 2.9.9** Let \( \mathcal{I} \) be an achronal hypersurface in \( (\mathcal{M},g)_{C^2} \), and suppose that the interior \( \mathcal{D}^+_1(\mathcal{I}) \) of the domain of dependence \( \mathcal{D}^+_1(\mathcal{I}) \) of \( \mathcal{I} \) is not empty. Then \( \mathcal{D}^+_1(\mathcal{I}) \) equipped with the metric obtained by restriction from \( g \) is globally hyperbolic.

**Proof:** We need first to show that a causal curve can be pushed-up by an amount as small as desired to yield a timelike curve:

**Lemma 2.9.10 ("Push-up Lemma II")** Consider a spacetime \( (\mathcal{M},g)_{C^2} \). Let \( \gamma : \mathbb{R}^+ \to \mathcal{M} \) be a past-inextendible past-directed causal curve starting at \( p \), and let \( \mathcal{O} \) be a neighborhood of the image \( \gamma(\mathbb{R}^+) \) of \( \gamma \). Then for every \( r \in I^+(q) \cap \mathcal{O} \) there exists a past-inextendible past-directed timelike curve \( \hat{\gamma} \) starting at \( r \) such that

\[
\hat{\gamma} \subset I^+(\gamma) \cap \mathcal{O}, \quad \forall \, s \in [0,\infty) \quad I^-(\hat{\gamma}(s)) \cap \gamma(\mathbb{R}^+) \neq \emptyset. \quad (2.9.2)
\]

**Proof:** The construction is essentially identical to that of the proof of Lemma 2.4.14, p. 26, except that we will have to deal with a countable collection of curves, rather than a finite number. One also needs to make sure that the final curve is inextendible. As usual, we parameterize \( \gamma \) by \( h \)-distance as measured from \( p \). Using an exhaustion of \( [0,\infty) \) by compact intervals \( [m,m+1] \) we cover \( \gamma \) by a countable collection \( \mathcal{U}_i \subset \mathcal{O} \), \( i \in \mathbb{N} \) of elementary regions \( \mathcal{U}_i \) centered at \( p_i = \gamma(r_i) \).
2.9. DOMAINS OF DEPENDENCE

with

\[ p_1 = p, \quad p_i \in \mathcal{U}_i \cap \mathcal{U}_{i+1}, \quad p_{i+1} \subseteq J^-(p_i). \]

We further impose the following condition on the \( \mathcal{U}_i \)'s: if \( r_i \in [j, j+1) \), then the corresponding \( \mathcal{U}_i \) is contained in a \( h \)-distance ball \( B_h(p_i, 1/(j+1)) \).

Let \( \gamma_0 : [0, s_0] \to \mathcal{M} \) be a past directed causal curve from \( r \) to \( p \in \mathcal{U}_1 \cap \mathcal{U}_2 \); let \( s_1 \) be close enough to \( s_0 \) so that

\[ \gamma_0(s_1) \in \mathcal{U}_2. \]

Proposition 2.2.3, p. 8, together with the definition of elementary regions shows that there exists a past directed timelike curve \( \gamma_1 : [0, 1] \to \mathcal{U}_1 \subset \mathcal{O} \) from \( q \) to \( p_2 \). (In particular \( \gamma_1 \setminus \{p\} \subset I^+(p) \subset I^+(\gamma) \)). Similarly, for any \( s \in [0, 1] \) there exists a past directed timelike curve \( \gamma_{2,s} : [0, 1] \to \mathcal{U}_2 \subset \mathcal{O} \) from \( \gamma_1(s) \) to \( p_2 \).

We choose \( s =: s_2 \) small enough so that

\[ \gamma_1(s_2) \in \mathcal{U}_3. \]

One repeats that construction iteratively, obtaining a sequence of past-directed timelike curves \( \gamma_i \subset I^+(\gamma) \cap \mathcal{U}_i \subset I^+(\gamma) \cap \mathcal{O} \) such that the end point of \( \gamma_i \) lies in \( \mathcal{U}_{i+1} \) and coincides with the starting point of \( \gamma_{i+1} \). Concatenating those curves together gives the desired path \( \hat{\gamma} \). Since every path \( \gamma_i \) lies in \( I^+(\gamma) \cap \mathcal{O} \), so does their union.

Since \( \gamma_i \subset \mathcal{U}_i \subset B_h(p_i, 1/(j+1)) \) when \( r_i \in [j, j+1) \) we obtain, for \( r \in [j, j+1), \)

\[ \text{dist}_h(\gamma(r), \hat{\gamma}) \leq \text{dist}_h(\gamma(r), \gamma(r_i)) + \text{dist}_h(\gamma(r_i), \gamma_i) \leq \frac{2}{j+1}, \]

where we have ensured that \( \text{dist}_h(\gamma(r), \gamma(r_i)) < 1/(j+1) \) by choosing \( r_i \) appropriately. It follows that

\[ \text{dist}_h(\gamma(r), \hat{\gamma}) \leq \frac{2}{r}. \quad (2.9.4) \]

To finish the proof, suppose that \( \hat{\gamma} : [0, s_\ast] \to \mathcal{M} \) is extendible, call \( \hat{p} \) the end point of \( \hat{\gamma} \). By (2.9.4)

\[ \lim_{r \to \infty} \text{dist}_h(\gamma(r), \hat{p}) = 0. \]

Thus \( \hat{p} \) is an end point of \( \gamma \), which together with Theorem 2.5.5 contradicts inextendibility of \( \gamma \). \( \square \)

By the definition of domains of dependence, inextendible timelike curves through \( p \in \mathcal{D}_I^+(\mathcal{I}) \) intersect all the sets \( \mathcal{I}, I^+(\mathcal{I}) \), and \( I^-(\mathcal{I}) \). This is wrong in general for inextendible causal curves through points in \( \mathcal{D}_I^+(\mathcal{I}) \setminus \mathcal{D}_I^+(\mathcal{I}) \), as shown on Figure 2.9.2. Nevertheless we have:

**Lemma 2.9.11** If \( p \in \mathcal{D}_I^+(\mathcal{I}) \), then every inextendible causal curve \( \gamma \) through \( p \) intersects \( \mathcal{I}, I^-(\mathcal{I}) \) and \( I^+(\mathcal{I}) \).

**Remark 2.9.12** In contradistinction with timelike curves, for causal curves the intersection of \( \gamma \) with \( \mathcal{I} \) does not have to be a point. An example is given by the hypersurface \( \mathcal{I} \) of Figure 2.9.3.
Figure 2.9.2: Let $\mathcal{I} = \{t = 0, x \in (-1, 1)\} \subset \mathbb{R}^{1,1}$, then $\mathcal{D}_I(\mathcal{I})$ is the closed dotted diamond region without the two rightmost and leftmost points that lie on the closure of $\mathcal{I}$. The past directed null geodesic $\gamma$ starting at $(1, 1) \in \mathcal{D}_I^+(\mathcal{I})$ does not intersect $\mathcal{I}$.

Figure 2.9.3: A null geodesic $\gamma$ intersecting an achronal topological hypersurface $\mathcal{I}$ at more than one point.

Proof: Changing time-orientation if necessary we may suppose that $p \in \mathcal{D}_I^+(\mathcal{I})$. Let $\gamma : I \to \mathcal{M}$ be any past-directed inextendible causal curve through $p$. Since $p$ is an interior point of $\mathcal{D}_I^+(\mathcal{I})$ there exists $q \in I^+(p) \cap \mathcal{D}_I^+(\mathcal{I})$. By the Push-up Lemma 2.9.10 with $\mathcal{O} = \mathcal{M}$ there exists a past-inextendible past-directed timelike curve $\hat{\gamma}$ starting at $q$ which lies to the future of $\gamma$. The inextendible timelike curve $\hat{\gamma}$ enters $I^-(\mathcal{I})$, and so does $\gamma$ by (2.9.3).

If $p \in \mathcal{I}$, we can repeat the argument above with the time-orientation changed, showing that $\gamma$ enters $I^+(\mathcal{I})$ as well, and we are done.

Otherwise $p \notin \mathcal{I}$, then $p$ is necessarily in $I^+(\mathcal{I})$, hence $\gamma$ meets $I^+(\mathcal{I})$ as well. Now, each of the two disjoint sets

$$I_{\pm} := \{ s \in I : \gamma(s) \in I^\pm(\mathcal{I}) \} \subset \mathbb{R}$$

is open in the connected interval $I$. They cover $I$ if $\gamma$ does not meet $\mathcal{I}$, which implies that either $I_+$ or $I_-$ must be empty when $\gamma \cap \mathcal{I} = \emptyset$. But we have shown that both $I_+$ and $I_-$ are not empty, and so $\gamma$ meets $\mathcal{I}$, as desired. $\Box$

Returning to the proof of Theorem 2.9.9, suppose that $\mathcal{D}_I(\mathcal{I})$ is not strongly causal. Then there exists $p \in \mathcal{D}_I(\mathcal{I})$ and a sequence $\gamma_n : \mathbb{R} \to \mathcal{D}_I(\mathcal{I})$ of inextendible causal curves which exit the $h$-distance geodesic ball $B_h(p, 1/n)$.
(centred at \( p \) and of radius \( 1/n \)) and rerenter \( B_h(p,1/n) \) again. Changing time-orientation of \( \mathcal{M} \) if necessary, without loss of generality we may assume that \( p \in I^+(\mathcal{S}) \cup \mathcal{S} \). Note that the property “leaves and reenters” is invariant under the change \( \gamma_p(s) \to \gamma_p(-s) \), so that changing the orientation of (some of) the \( \gamma_p \)’s if necessary, there is no loss of generality in assuming the \( \gamma_p \)’s to be future-directed. Finally, we reparameterize the \( \gamma_n \)’s by \( h \)-distance, with \( \gamma_n(0) \in B_h(p,1/n) \). Then, there exists a sequence \( s_n > 0 \) such that \( \gamma_n(s_n) \in B_h(p,1/n) \), with \( \gamma_n(0) \) and \( \gamma_n(s_n) \) lying on different connected components of \( \gamma \cap B_h(p,1/n) \).

Let \( \mathcal{O} \) be an elementary neighborhood of \( p \), as in Definition 2.2.7, p. 13, and let \( n_0 \) be large enough so that \( B_h(p,1/n_0) \subset \mathcal{O} \). Note that the local coordinate \( x^0 \) on \( \mathcal{O} \) is monotonous along every connected component of \( \gamma_n \cap \mathcal{O} \) which implies, for \( n \geq n_0 \), that any causal curve which exits and reenters \( B_h(p,1/n) \) has also to exit and reenter \( \mathcal{O} \). This in turn guarantees the existence of an \( \varepsilon > 0 \) such that \( s_n > \varepsilon \) for all \( n \geq n_0 \).

Let \( \gamma \) be an accumulation curve through \( p \) of the \( \gamma_n \)’s, passing to a subsequence if necessary the \( \gamma_n \)’s converge uniformly to \( \gamma \) on compact subsets of \( \mathbb{R} \). The curve \( \gamma \) is causal and \( p \) is, by hypothesis, in the interior of \( D_1(\mathcal{S}) \), we can therefore invoke Lemma 2.9.11 to conclude that there exist \( s_\pm \in \mathbb{R} \) such that \( \gamma(s_-) \in I^-(\mathcal{S}) \) and \( \gamma(s_+) \in I^+(\mathcal{S}) \). Since \( \mathcal{S} \) is achronal and \( \gamma \) is future directed we must have \( s_- < s_+ \). Since the \( I^\pm(\mathcal{S}) \)’s are open, and since (passing to a subsequence if necessary) \( \gamma_n(s_\pm) \to \gamma(s_\pm) \), we have \( \gamma_n(s_\pm) \in I^\pm(\mathcal{S}) \) for \( n \) large enough.

The situation which is simplest to exclude is the one where the sequence \( \{s_n\} \) is bounded. Then there exists \( s_* \in \mathbb{R} \) such that, passing again to a subsequence if necessary, we have \( s_n \to s_* \). Note that \( \gamma_n(s_*) \to p \) and that \( s_* \geq \varepsilon \). Since \( \gamma_n|_{[0,s_*]} \) converges uniformly to \( \gamma|_{[0,s_*]} \), we obtain an inextendible periodic causal curve \( \gamma' \) through \( p \) by repetitively circling from \( p \) to \( p \) along \( \gamma|_{[0,s_*]} \). By Lemma 2.9.11 \( \gamma' \) meets all of \( \mathcal{S} \), \( I^+(\mathcal{S}) \) and \( I^-(\mathcal{S}) \), which is clearly incompatible with periodicity of \( \gamma' \) and achronality of \( \mathcal{S} \). (In detail: there exist points \( q_\pm \in \gamma' \cap I^\pm(\mathcal{S}) \). Following backwards \( \gamma' \) from \( p \) to \( q_\pm \) we obtain \( q_\pm \in I^-(p) \). But \( I^-(q_\pm) \cap \mathcal{S} \neq \emptyset \), and Lemma 2.4.14 implies that \( I^-(p) \cap \mathcal{S} \neq \emptyset \). Since \( p \in I^-(\mathcal{S}) \cup \mathcal{S} \), this contradicts achronality of \( \mathcal{S} \).

Note that if \( p \in I^-(\mathcal{S}) \) we would need to have \( s_n \leq s_* \) for \( n \) large enough: Otherwise, for \( n \) large, we could follow \( \gamma_n \) in the future direction from \( \gamma_n(s_*) \in I^+(\mathcal{S}) \) to \( \gamma_n(s_n) \in I^-(\mathcal{S}) \), which is not possible if \( \mathcal{S} \) is achronal. But then the sequence \( s_n \) would be bounded, which has already been excluded. So \( p \in I^-(\mathcal{S}) \) cannot occur.

There remains the possibility \( p \in \mathcal{S} \). We then must have \( \gamma_n|_{[0,\infty)} \cap I^-(\mathcal{S}) = \emptyset \), otherwise we would obtain a contradiction with achronality of \( \mathcal{S} \) by following \( \gamma_n \) to the future from \( p = \gamma_n(0) \in \mathcal{S} \) to a point where \( \gamma_n \) intersects \( I^-(\mathcal{S}) \). Set

\[
\hat{\gamma}_n(s) = \gamma_n(s + s_n),
\]

then \( \hat{\gamma}_n \) accumulates at \( p \) since \( \hat{\gamma}_n(0) \to p \), therefore there exists an inextendible accumulation curve \( \hat{\gamma} : \mathbb{R} \to \mathcal{M} \) of the \( \hat{\gamma}_n \)’s passing through \( p \). As

\[
\hat{\gamma}_n([-s_n, \infty)) \cap I^-(p) = \gamma_n([0, \infty)) \cap I^-(\mathcal{S}) = \emptyset
\]
we have \( \hat{\gamma}(\mathbb{R}) \cap I^-(p) = \emptyset \) as well. This is, however, not possible if \( p \in \hat{\mathcal{D}}_I(\mathcal{I}) \) by Lemma 2.9.11. We see that the possibility that \( p \in \mathcal{I} \) cannot occur either. We conclude that \( \hat{\mathcal{D}}_I(\mathcal{I}) \) is strictly causal, as desired.

To finish the proof, we need to prove compactness of the sets of the form

\[
J^+(p) \cap J^-(q) , \quad p, q \in \hat{\mathcal{D}}_I(\mathcal{I}) .
\]

If \( p \) and \( q \) are such that this set is empty or equals \( \{ p \} \) there is nothing to prove. Otherwise, consider a sequence \( r_n \in J^+(p) \cap J^-(q) \). One of the following is true:

1. we have \( r_n \in I^-(\mathcal{I}) \cup \mathcal{I} \) for all \( n \geq n_0 \), or
2. there exists a subsequence, still denoted by \( r_n \), such that \( r_n \in I^+ (\mathcal{I}) \).

In the second case we change time-orientation, pass to a subsequence, rename \( p \) and \( q \), reducing the analysis to the first case. Note that this leads to \( p \in I^-(\mathcal{I}) \cup \mathcal{I} \).

By definition, there exists a future directed causal curve \( \hat{\gamma}_n \) from \( p \) to \( q \) which passes through \( r_n \),

\[
\hat{\gamma}_n(s_n) = r_n .
\]  

(2.9.5)

Let \( \gamma_n \) be any dist\(_h\)-parameterized, inextendible future directed causal curve extending \( \hat{\gamma}_n \), with \( \gamma_n(0) = p \). Let \( \gamma \) be an inextendible accumulation curve of the \( \gamma_n \)'s, then \( \gamma \) is a future inextendible causal curve through

\[
p \in (\mathcal{D}_I^+(\mathcal{I}) \cup \mathcal{I}) \cap \hat{\mathcal{D}}_I(\mathcal{I}) .
\]

By Lemma 2.9.11 there exists \( s_+ \) such that \( \gamma(s_+) \in I^+(\mathcal{I}) \). Passing to a subsequence, the \( \gamma_n \)'s converge uniformly to \( \gamma \) on \([0, s_+]\), which implies that for \( n \) large enough the \( \gamma_n|_{[0, s_+]} \)'s enter \( I^+(\mathcal{I}) \). This, together with achronality of \( \mathcal{I} \), shows that the sequence \( s_n \) defined by (2.9.5) is bounded; in fact we must have \( 0 \leq s_n \leq s_+ \). Eventually passing to another subsequence we thus have \( s_n \to s_\infty \) for some \( s_\infty \in \mathbb{R} \). This implies

\[
r_n \to \gamma(s_\infty) \in J^+(p) \cap J^-(q) ,
\]

which had to be established. \( \square \)

We have the following characterisation of interiors of domains of dependence:

**Theorem 2.9.13** Consider a spacetime \( (\mathcal{M},g)_{C^2} \). Let \( \mathcal{I} \) be a differentiable acausal spacelike hypersurface. A point \( p \in \mathcal{M} \) is in \( \hat{\mathcal{D}}_I^+(\mathcal{I}) \) if and only if

\[
\text{the set } \mathcal{I}^- (p) \cap \mathcal{I} \text{ is non-empty, and compact as a subset of } \hat{\mathcal{I}} .
\]  

(2.9.6)

**Remark 2.9.14** The set \( \mathcal{I} = \{ t = 0, x \in [-1, 1] \} \subset \mathbb{R}^{1,1} \) (compare Figure 2.9.2, but note that a different \( \mathcal{I} \) was meant there) shows that the condition (2.9.6) cannot be replaced by the requirement that the set \( \mathcal{I}^- (p) \cap \mathcal{I} \) is non-empty, and compact as a subset of \( \mathcal{M} \).
Proof: For \( p \in \mathcal{D}_I^+(\mathcal{I}) \) compactness of \( \overline{I^-(p)} \cap \mathcal{I} \) can be established by an argument very similar to that given in the last part of the proof of Theorem 2.9.9, the details are left to the reader.

In order to prove the reverse implication assume that (2.9.6) holds, then there exists a future directed causal curve \( \gamma : [0, 1] \to \mathcal{M} \) from some point \( q \in \mathcal{I} \) to \( p \). Set

\[
I := \{ t \in (0, 1) : \gamma(s) \in \mathcal{D}_I^+(\mathcal{I}) \text{ for all } 0 < s \leq t \} \subseteq (0, 1].
\]

Now, elementary considerations show that for \( C^1 \), spacelike, acausal hypersurfaces we have \( \gamma(t) \in \mathcal{D}_I^+(\mathcal{I}) \) for \( t > 0 \) small enough, hence \( I \) is not empty. Clearly \( I \) is open in \((0, 1]\). In order to show that it equals \((0, 1]\) set

\[
t_* := \sup I.
\]

Consider any past-inextendible past-directed causal curve \( \hat{\gamma} \) starting at \( \gamma(t_*) \). For \( t < t_* \) let \( \hat{\gamma}_t \) be a family of past-inextendible causal push-downs of \( \hat{\gamma} \) which start at \( \gamma(t) \), and which have the property that

\[
\text{dist}_h(\hat{\gamma}_t(s), \gamma(s)) \leq |t - t_*| \text{ for } 0 \leq s \leq 1/|t - t_*|.
\]

Then \( \hat{\gamma}_t \) intersects \( \mathcal{I} \) at some point \( q_t \in J^-(p) \). Compactness of \( J^-(p) \cap \mathcal{I} \) implies that the curve \( t \to q_t \in \mathcal{I} \) accumulates at some point \( q_* \in \mathcal{I} \), which clearly is the point of intersection of \( \gamma \) with \( \mathcal{I} \). This shows that every causal curve \( \gamma \) through \( \gamma(t_*) \) meets \( \mathcal{I} \), in particular \( \gamma(t_*) \in \mathcal{D}_I^+(\mathcal{I}) \). So \( I \) is both open and closed in \((0, 1]\), hence \( I = (0, 1]\), and the result is proved.

\[
\square
\]

2.10 Cauchy horizons

Definition 2.10.1 Let \( \mathcal{I} \) be an achronal topological hypersurface. The future Cauchy horizon \( \mathcal{H}_I^+(\mathcal{I}) \) of \( \mathcal{I} \) is defined as

\[
\mathcal{H}_I^+(\mathcal{I}) = \mathcal{D}_I^+(\mathcal{I}) \setminus \overline{I^-(\mathcal{D}_I^+(\mathcal{I}))},
\]

with an obvious corresponding definition for the past Cauchy horizon \( \mathcal{H}_I^-(\mathcal{I}) \). One defines the Cauchy horizon as

\[
\mathcal{H}_I(\mathcal{I}) = \mathcal{H}_I^-(\mathcal{I}) \cup \mathcal{H}_I^+(\mathcal{I}).
\]

Our definition follows that of Penrose [45]. Similarly to the domains of dependence, the usual notation for Cauchy horizons is \( \mathcal{H} \) and not \( \mathcal{H}_I \), and we will sometimes write so. The analogous definition of future Cauchy horizon with \( \mathcal{D}_I^+(\mathcal{I}) \) replaced by \( \mathcal{D}_J^+(\mathcal{I}) \) leads in general to essentially different sets for continuous metrics [11].

It is instructive to consider a few examples:

Example 2.10.2 Let \( \mathcal{I} = \{ x^0 = 0 \} \) in Minkowski space-time \( \mathbb{R}^{1,n} \), where \( x^0 \) is the usual time coordinate on \( \mathbb{R}^{1,n} \). Then \( \mathcal{H}_I(\mathcal{I}) = \emptyset \).
Example 2.10.3 Let \( \mathcal{I} \) be the open unit ball in \( \mathbb{R}^n \), viewed as a subset of \( \{x^0 = 0\} \) in Minkowski space-time \( \mathbb{R}^{1,n} \), where \( x^0 \) is the usual time coordinate on \( \mathbb{R}^{1,n} \). Then \( \mathcal{H}^+_I(\mathcal{I}) \) is the intersection of the past-light cone of the point \( (x^0 = 1, \tilde{x} = 0) \) with \( \{x^0 \geq 0\} \).

Example 2.10.4 Let \( \mathcal{I} = \{\eta_{\mu\nu}x^\mu x^\nu = -1, x^0 > 0\} \) be the upper component of the unit spacelike hyperboloid in Minkowski space-time. Then \( \mathcal{H}^+_I(\mathcal{I}) = \emptyset \), while \( \mathcal{H}^-_I(\mathcal{I}) \) coincides with the future light-cone of the origin.

Example 2.10.5 The Taub-NUT space-times \([41, 50]\) provide examples of space-times with an achronal spacelike \( S^3 \) and with two corresponding past and future Cauchy horizons, each diffeomorphic to \( S^3 \) \([37]\).

For any open set set \( \Omega \) one has \( \Omega \setminus I^-(\Omega) = \emptyset \), which shows that

\[
\mathcal{D}_I(\mathcal{I}) \cap \mathcal{H}^+_I(\mathcal{I}) = \emptyset . \tag{2.10.1}
\]

It follows that \( \mathcal{H}^+_I(\mathcal{I}) \) is a subset of the topological boundary \( \partial \mathcal{D}^+_I(\mathcal{I}) \) of \( \mathcal{D}^+_I(\mathcal{I}) \):

\[
\mathcal{H}^+_I(\mathcal{I}) \subset \partial \mathcal{D}^+_I(\mathcal{I}) := \overline{\mathcal{D}^+_I(\mathcal{I})} \setminus \mathcal{D}^+_I(\mathcal{I}) . \tag{2.10.2}
\]

The important notion of *generators* of horizons stems from the following result in which we assume, for simplicity, that \( \mathcal{I} \) is differentiable and spacelike:

**Proposition 2.10.6** Let \( \mathcal{I} \) be a spacelike \( C^1 \) hypersurface in \( (\mathcal{M}, g)_{C^2} \). For any \( p \in \mathcal{H}^+_I(\mathcal{I}) \) there exists a past directed null geodesic \( \gamma_p \subset \mathcal{H}^+_I(\mathcal{I}) \) starting at \( p \) which either does not have an endpoint in \( \mathcal{M} \), or has an endpoint on \( \mathcal{F} \setminus \mathcal{I} \).

**Remark 2.10.7** There might be more than one such geodesic for some points on the horizon.

**Proof:** Let \( p \in \mathcal{H}^+_I(\mathcal{I}) \), then there exists a sequence of points \( p_n \notin \mathcal{D}^+_I(\mathcal{I}) \) which converge to \( p \), and past inextendible timelike curves \( \gamma_n \) through \( p_n \) that do not meet \( \mathcal{I} \). Let \( \gamma \) be an accumulation curve of the \( \gamma_n \) through \( p \). Then \( \gamma \) does not meet \( \mathcal{I} \): indeed, if it did, then the \( \gamma_n \)’s would be meeting \( \mathcal{I} \) as well for all \( n \) large enough. If \( \gamma \) meets \( \mathcal{F} \), we let \( \gamma_p \) be the segment of \( \gamma \) from \( p \) to the intersection point with \( \mathcal{F} \), otherwise we let \( \gamma_p = \gamma \).

The curve \( \gamma_p \) is achronal: otherwise \( \gamma_p \) would enter the interior of \( \mathcal{D}^+_I(\mathcal{I}) \), but then it would have to intersect \( \mathcal{I} \) by Lemma 2.9.11. We can thus invoke Proposition 2.6.9, p. 37, to conclude that \( \gamma_p \) is a null geodesic.

It remains to show that \( \gamma_p \subset \mathcal{H}^+_I(\mathcal{I}) \). Let \( \gamma \) be an inextendible past-directed timelike curve through a point \( q \) on \( \gamma_p \), with \( q \notin \mathcal{F} \). Let \( \mathcal{O} \) be a neighborhood of \( q \) that does not meet \( \mathcal{I} \), and let \( r \in \mathcal{O} \) be a point on \( \gamma \) lying to the timelike past of \( q \). By Corollary 2.4.16 there exists a timelike curve \( \gamma_1 \) from \( p \) to \( r \). Consider the past-inextendible timelike curve, say \( \gamma_2 \), obtained by following \( \gamma_1 \) from \( p \) to \( r \), and then following \( \gamma \) to the past. Since \( p \in \mathcal{H}^+_I(\mathcal{I}) \) the curve \( \gamma_2 \) has to meet \( \mathcal{I} \). As \( \gamma_1 \) does not meet \( \mathcal{I} \), it must be the case that \( \gamma \) meets \( \mathcal{I} \), and so \( q \in \mathcal{H}^+_I(\mathcal{I}) \). \( \square \)
2.10. CAUCHY HORIZONS

For any \( p \in \mathcal{H}_I^+(\mathcal{I}) \) let \( \hat{\gamma}_p \) denote a maximal future extension of the geodesic segment \( \gamma_p \) of Proposition 2.10.6, and set \( \tilde{\gamma}_p = \gamma_p \cap \mathcal{H}_I^+(\mathcal{I}) \). (Note that \( \tilde{\gamma}_p \) might exit \( \mathcal{H}_I^+(\mathcal{I}) \) when followed to the future, an example of this can be seen in Figure 2.9.2.) Then \( \tilde{\gamma}_p \) is called a generator of \( \mathcal{H}_I^+(\mathcal{I}) \). Using this terminology, Proposition 2.10.6 can be reworded as the property that every \( p \in \mathcal{H}_I^+(\mathcal{I}) \) is either an interior point or a future end point of a generator of \( \mathcal{H}_I^+(\mathcal{I}) \). If \( \mathcal{T} = \mathcal{I} \), then generators of \( \mathcal{H}_I^+(\mathcal{I}) \) do not have past end points, remaining forever on \( \mathcal{H}_I^+(\mathcal{I}) \) to the past.

### 2.10.1 Semi-convexity of future horizons

A hypersurface \( \mathcal{H} \subseteq \mathcal{M} \) will be said to be future null geodesically ruled if every point \( p \in \mathcal{H} \) belongs to a future inextensible null geodesic \( \Gamma \subset \mathcal{H} \); those geodesics are called the generators of \( \mathcal{H} \). We emphasize that the generators are allowed to have past endpoints on \( \mathcal{H} \), but no future endpoints. Past null geodesically ruled hypersurfaces are defined by changing the time orientation. Examples of future geodesically ruled hypersurfaces include past Cauchy horizons \( \mathcal{D}^- (\mathcal{I}) \) of achronal sets \( \mathcal{I} \) of Proposition 2.10.6 (compare [45, Theorem 5.12]) and black hole event horizons \( \mathcal{J}^- (\mathcal{I}^+) \) [24, p. 312].

Note that our definition involves explicitly geodesics, and therefore throughout this section we assume that the metric is twice-continuously differentiable. It should be kept in mind that the notion of the generator of a horizon in space-times with merely continuous metrics is not completely clear, so allowing metrics of lower differentiability might require a reformulation of the problem.

We always assume that the space-time dimension \( \dim \mathcal{M} = n + 1 \).

Suppose that \( \mathcal{O} \) is a domain in \( \mathbb{R}^n \). Recall that a continuous function \( f : \mathcal{O} \to \mathbb{R} \) is called semi–convex if there exists a \( C^2 \) function \( \phi : \mathcal{O} \to \mathbb{R} \) such that \( f + \phi \) is convex. We shall say that the graph of \( f \) is a semi–convex hypersurface if \( f \) is semi–convex. A hypersurface \( \mathcal{H} \) in a manifold \( \mathcal{M} \) will be said semi–convex if \( \mathcal{H} \) can be covered by coordinate patches \( \mathcal{U}_\alpha \) such that \( \mathcal{H} \cap \mathcal{U}_\alpha \) is a semi–convex graph for each \( \alpha \).

Consider an achronal hypersurface \( \mathcal{H} \neq \emptyset \) in a globally hyperbolic space–time \((\mathcal{M}, g)\). Let \( t \) be a time function on \( \mathcal{M} \) which induces a diffeomorphism of \( \mathcal{M} \) with \( \mathbb{R} \times \mathcal{I} \) in the standard way [19, 47], with the level sets \( \mathcal{I}_\tau \equiv \{p|t(p) = \tau\} \) of \( t \) being Cauchy surfaces. As usual we identify \( \mathcal{I}_0 \) with \( \mathcal{I} \) and, in the identification above, the curves \( \mathbb{R} \times \{q\}, q \in \mathcal{I}, \) are integral curves of \( \nabla t \).

Define
\[
\mathcal{I}_\mathcal{H} = \{q \in \mathcal{I} \mid \mathbb{R} \times \{q\} \text{ intersects } \mathcal{H}\}.
\] (2.10.3)

For \( q \in \mathcal{I}_\mathcal{H} \) the set \((I \times q) \cap \mathcal{H}\) is a point by achronality of \( \mathcal{H} \), which will be denoted by \((f(q), q)\). Thus an achronal hypersurface \( \mathcal{H} \) in a globally hyperbolic space–time is a graph over \( \mathcal{I}_\mathcal{H} \) of a function \( f \). The invariance-of-the-domain theorem shows that \( \mathcal{I}_\mathcal{H} \) is an open subset of \( \mathcal{I} \). We have the following:

**Theorem 2.10.1** Let \( \mathcal{H} \neq \emptyset \) be an achronal future null geodesically ruled hypersurface in a globally hyperbolic space–time \((\mathcal{M} = \mathbb{R} \times \mathcal{I}, g)_{C^2}\). Then \( \mathcal{H} \) is the graph of a semi–convex function \( f \) defined on an open subset \( \mathcal{I}_\mathcal{H} \) of \( \mathcal{I} \), in particular \( \mathcal{H} \) is semi–convex.
Proof: As discussed above, $H$ is the graph of a function $f$. The idea of the proof is to show that $f$ satisfies a variational principle, the semi–concavity of $f$ follows then by a standard argument. Let $p \in H$ and let $O$ be a coordinate patch in a neighborhood of $p$ such that $x^0 = t$, with $O$ of the form $I \times B(3R)$, where $B(R)$ denotes a coordinate ball centered at 0 of radius $R$ in $\mathbb{R}^3$, with $p = (t(p),0)$. Here $I$ is the range of the coordinate $x^0$, we require it to be a bounded interval the size of which will be determined later on. We further assume that the curves $I \times \{\vec{x}\}$, $\vec{x} \in B(3R)$, are integral curves of $\nabla t$. Define

$$U_0 = \{\vec{x} \in B(3R) | \text{ the causal path } I \ni t \to (t,\vec{x}) \text{ intersects } H\}.$$

We note that $U_0$ is non–empty, since $0 \in U_0$. Set $H_\sigma = H \cap \mathcal{I}_\sigma$, (2.10.4) and choose $\sigma$ large enough so that $O \subset I^-(\mathcal{I}_\sigma)$. Now $p$ lies on a future inextensible generator $\Gamma$ of $H$, and global hyperbolicity of $(M,g)$ implies that $\Gamma \cap \mathcal{I}_\sigma$ is nonempty, hence $H_\sigma$ is nonempty.

For $\vec{x} \in B(3R)$ let $\mathcal{P}(x)$ denote the collection of piecewise differentiable future directed null curves $\Gamma : [a,b] \to \mathcal{M}$ with $\Gamma(a) \in \mathbb{R} \times \{\vec{x}\}$ and $\Gamma(b) \in H_\sigma$. We define

$$\tau(\vec{x}) = \sup_{\Gamma \in \mathcal{P}(x)} t(\Gamma(a)).$$ (2.10.5)

We emphasize that we allow the domain of definition $[a,b]$ to depend upon $\Gamma$, and that the “$a$” occurring in $t(\Gamma(a))$ in (2.10.5) is the lower bound for the domain of definition of the curve $\Gamma$ under consideration.

We have the following result (compare [1, 20, 46]):

**Proposition 2.10.8 (Fermat principle)** For $\vec{x} \in U_0$ we have

$$\tau(\vec{x}) = f(\vec{x}).$$

Proof: Let $\Gamma$ be any generator of $H$ such that $\Gamma(0) = (f(\vec{x}),\vec{x})$, clearly $\Gamma \in \mathcal{P}(x)$ so that $\tau(\vec{x}) \geq f(\vec{x})$. To show that this inequality has to be an equality, suppose for contradiction that $\tau(\vec{x}) > f(\vec{x})$, thus there exists a null future directed curve $\Gamma$ such that $t(\Gamma(0)) > f(\vec{x})$ and $\Gamma(1) \in H_\sigma \subset H$. Then the curve $\tilde{\Gamma}$ obtained by following $\mathbb{R} \times \{\vec{x}\}$ from $(f(\vec{x}),\vec{x})$ to $(t(\Gamma(0)),\vec{x})$ and following $\Gamma$ from there on is a causal curve with endpoints on $H$ which is not a null geodesic. By Proposition 2.4.18 the curve $\tilde{\Gamma}$ can be deformed to a timelike curve with the same endpoints, which is impossible by achronality of $H$. $\square$

The Fermat principle, Proposition 2.10.8, shows that $f$ is a solution of the variational principle (2.10.5). Now this variational principle can be rewritten in a somewhat more convenient form as follows: The identification of $\mathcal{M}$ with $\mathbb{R} \times \mathcal{I}$ by flowing from $\mathcal{I}_0 \equiv \mathcal{I}$ along the gradient of $t$ leads to a global decomposition of the metric of the form

$$g = \alpha(-dt^2 + h_t),$$
where \( h_t \) denotes a \( t \)-dependent family of Riemannian metrics on \( \mathcal{S} \). Any future directed differentiable null curve \( \Gamma(s) = (t(s), \gamma(s)) \) satisfies
\[
\frac{dt(s)}{ds} = \sqrt{h_t(s)(\dot{\gamma}, \dot{\gamma})},
\]
where \( \dot{\gamma} \) is a shorthand for \( d\gamma(s)/ds \). It follows that for any \( \Gamma \in \mathcal{P}(x) \) it holds that
\[
t(\Gamma(a)) = t(\Gamma(b)) - \int_a^b \frac{dt}{ds}(s) ds = \sigma - \int_a^b \sqrt{h_{\tilde{\sigma}}(\dot{\gamma}, \dot{\gamma})} ds.
\]
This allows us to rewrite (2.10.5) as
\[
\tau(\vec{x}) = \sigma - \mu(\vec{x}) , \quad \mu(\vec{x}) \equiv \inf_{\Gamma \in \mathcal{P}(x)} \int_a^b \sqrt{h_{t(s)}(\dot{\gamma}, \dot{\gamma})} ds .
\]
We note that in static space–times \( \mu(\vec{x}) \) is the Riemannian distance from \( \vec{x} \) to \( H_\sigma \). In particular Equation (2.10.6) implies the well known fact, that in globally hyperbolic static space–times Cauchy horizons of open subsets of level sets of \( t \) are graphs of the distance function from the boundary of those sets.

Let \( \gamma : [a, b] \to \mathcal{S} \) be a piecewise differentiable path, for any \( p \in \mathbb{R} \times \{\gamma(b)\} \) we can find a null future directed curve \( \hat{\gamma} : [a, b] \to \mathcal{M} \) of the form \( \hat{\gamma}(s) = (\phi(s), \gamma(s)) \) with future end point \( p \) by solving the problem
\[
\begin{cases}
\phi(b) = t(p) , \\
\frac{d\phi(s)}{ds} = \sqrt{h_{\phi(s)}(\dot{\gamma}(s), \dot{\gamma}(s))}.
\end{cases}
\]
The path \( \hat{\gamma} \) will be called the **null lift of \( \gamma \) with endpoint \( p \).**

As an example of application of Proposition 2.10.8 we recover the following well known result [45]:

**Corollary 2.10.9** \( f \) is Lipschitz continuous on any compact subset of its domain of definition.

**Proof:** For \( \vec{y}, \vec{z} \in B(2R) \) let \( K \subset \mathbb{R} \times B(2R) \) be a compact set which contains all the null lifts \( \Gamma_{\vec{y}, \vec{z}} \) of the coordinate segments \( \left[ \vec{y}, \vec{z} \right] := \{ \lambda \vec{y} + (1 - \lambda) \vec{z} , \lambda \in [0,1] \} \) with endpoints \( (\tau(\vec{z}), \vec{z}) \). Define
\[
\hat{C} = \sup\{ \sqrt{h_p(n, n)} | p \in K, |n|_\delta = 1 \},
\]
where the supremum is taken over all points \( p \in K \) and over all vectors \( n \in T_p \mathcal{M} \) the coordinate components \( n^i \) of which have Euclidean length \( |n|_\delta = 1 \) equal to one. Choose \( I \) to be a bounded interval large enough so that \( K \subset I \times B(2R) \) and, as before, choose \( \sigma \) large enough so that \( I \times B(2R) \) lies to the past of \( \mathcal{I}_\sigma \).

Let \( \vec{y}, \vec{z} \in B(2R) \) and consider the causal curve \( \Gamma = (t(s), \gamma(s)) \) obtained by following the null lift \( \Gamma_{\vec{y}, \vec{z}} \) in the parameter interval \( s \in [0,1] \), and then a
generator of \( \mathcal{H} \) from \( (\tau(\bar{z}), \bar{z}) \) until \( \mathcal{H}_\sigma \) in the parameter interval \( s \in [1, 2] \). Then we have

\[
\mu(\bar{z}) = \int_1^2 \sqrt{h_\sigma(\dot{\gamma}, \Gamma)}ds .
\]

Further \( \Gamma \in \mathcal{P}(x) \) so that

\[
\mu(\bar{y}) \leq \int_0^2 \sqrt{h_\sigma(\dot{\gamma}, \dot{\gamma})}ds = \int_0^1 \sqrt{h_\sigma(\dot{\gamma}, \dot{\gamma})}ds + \int_1^2 \sqrt{h_\sigma(\dot{\gamma}, \dot{\gamma})}ds \leq C|\bar{y} - \bar{z}|_\delta + \mu(\bar{z}) ,
\]

where \( | \cdot |_\delta \) denotes the Euclidean norm of a vector, and with \( C \) defined in (2.10.8). Setting 1) first \( \bar{y} = \bar{x}, \bar{z} = \bar{x} + \bar{h} \) in (2.10.9) and 2) then \( \bar{z} = \bar{x}, \bar{y} = \bar{x} + \bar{h} \), the Lipschitz continuity of \( f \) on \( B(2R) \) follows. The general result is obtained now by a standard covering argument. \( \square \)

Returning to the proof of Theorem 2.10.1, for \( \bar{x} \in B(R) \) let \( \Gamma_{\bar{x}} \) be a generator of \( \mathcal{H} \) such that \( \Gamma_{\bar{x}}(0) = (\tau(\bar{x}), \bar{x}) \), and, if we write \( \Gamma_{\bar{x}}(s) = (\phi_{\bar{x}}(s), \gamma_{\bar{x}}(s)) \), then we require that \( \gamma_{\bar{x}}(s) \in B(2R) \) for \( s \in [0, 1] \). For \( s \in [0, 1] \) and \( h \in B(R) \) let \( \gamma_{\bar{x}, \pm}(s) \in \mathcal{P} \) be defined by

\[
\gamma_{\bar{x}, \pm}(s) = \gamma_{\bar{x}}(s) \pm (1 - s)h = s\gamma_{\bar{x}}(s) + (1 - s)(\gamma_{\bar{x}}(s) \pm h) \in B(2R) .
\]

We note that

\[
\gamma_{\bar{x}, \pm}(0) = \bar{x} \pm h , \quad \gamma_{\bar{x}, \pm}(1) = \gamma_{\bar{x}}(1) , \quad \dot{\gamma}_{\bar{x}, \pm} - \dot{\gamma}_{\bar{x}} = \mp h .
\]

Let \( \Gamma_{\bar{x}, \pm} = (\phi_{\bar{x}, \pm}(s), \gamma_{\bar{x}, \pm}) \) be the null lifts of the paths \( \gamma_{\bar{x}, \pm} \) with endpoints \( \Gamma_{\bar{x}}(1) \). Let \( K \) be a compact set containing all the \( \Gamma_{\bar{x}, \pm} \)'s, where \( \bar{x} \) and \( h \) run through \( B(R) \). Let \( I \) be any bounded interval such that \( I \times B(2R) \) contains \( K \). As before, choose \( \sigma \) so that \( I \times B(2R) \) lies to the past of \( \mathcal{J}_\sigma \), and let \( b \) be such that \( \Gamma_{\bar{x}}(b) \in \mathcal{J}_\sigma \). (The value of the parameter \( b \) will of course depend upon \( \bar{x} \)). Let \( \Gamma_{\bar{x}} \) be the null curve obtained by following \( \Gamma_{\bar{x}, \pm} \) for parameter values \( s \in [0, 1] \), and then \( \Gamma_{\bar{x}} \) for parameter values \( s \in [1, b] \). Then \( \Gamma_{\bar{x}} \in \mathcal{P}(\bar{x} \pm h) \) so that we have

\[
\mu(\bar{x} \pm h) \leq \int_0^1 \sqrt{h_{\phi_{\bar{x}}} (\dot{\gamma}_{\bar{x}, \pm}, \dot{\gamma}_{\bar{x}, \pm})}ds + \int_1^b \sqrt{h_{\phi_{\bar{x}}} (\dot{\gamma}_{\bar{x}, \pm}, \dot{\gamma}_{\bar{x}, \pm})}ds .
\]

Further

\[
\mu(\bar{x}) = \int_0^b \sqrt{h_{\phi_{\bar{x}}} (\dot{\gamma}_{\bar{x}}, \dot{\gamma}_{\bar{x}})}ds ,
\]

hence

\[
\frac{\mu(\bar{x} + h) + \mu(\bar{x} - h)}{2} - \mu(\bar{x}) \leq \int_0^1 \frac{\sqrt{h_{\phi_{\bar{x}}} (\dot{\gamma}_{\bar{x}, +}, \dot{\gamma}_{\bar{x}, +})} + \sqrt{h_{\phi_{\bar{x}}} (\dot{\gamma}_{\bar{x}, -}, \dot{\gamma}_{\bar{x}, -})}}{2} - \sqrt{h_{\phi_{\bar{x}}} (\dot{\gamma}_{\bar{x}}, \dot{\gamma}_{\bar{x}})}ds .
\]

(2.10.10)
2.11. CAUCHY SURFACES

Since solutions of ODE's with parameters are differentiable functions of those, we can write

$$\phi_\pm(s) = \phi(x) + \psi(s)h^i + r(s,h), \quad |r(s,h)| \leq C|h|^2,$$

for some functions $\psi_i$, with a constant $C$ which is independent of $\vec{x}, \vec{h} \in B(R)$ and $s \in [0,1]$. Inserting (2.10.11) in (2.10.10), second order Taylor expanding the function $\sqrt{h \phi_{\pm}(s)}(\dot{\gamma}(\vec{x},\pm\dot{\gamma}(\vec{x})))$ in all its arguments around $(\phi(x), \gamma(x), \dot{\gamma}(x))$ and using compactness of $K$ one obtains

$$\frac{\mu(x + \vec{h}) + \mu(x - \vec{h})}{2} - \mu(x) \leq C|\vec{h}|^2,$$

for some constant $C$. Set

$$\psi(x) = \mu(x) - C|x|^2.$$  

Equation (2.10.12) shows that

$$\forall \vec{x}, \vec{h} \in B(R) \quad \psi(x) \geq \frac{\psi(x + \vec{h}) + \psi(x - \vec{h})}{2}.$$  

A standard argument implies that $\psi$ is concave. It follows that

$$f(x) + C|x|^2 = \tau(x) + C|x|^2 = \sigma - \mu(x) + C|x|^2 = \sigma - \psi(x)$$

is convex, which is what had to be established.  

\[\Box\]

2.11 Cauchy surfaces

A topological hypersurface $\mathcal{S}$ is said to be a Cauchy surface if

$$\mathcal{D}_{\mathcal{J}}(\mathcal{S}) = \mathcal{M}.$$  

(Note that it does not matter whether $\mathcal{D}_{\mathcal{J}}(\mathcal{S})$ or $\mathcal{D}_{\mathcal{I}}(\mathcal{S})$ is chosen in the definition when the metric is $C^2$.) Theorem 2.9.9, p. 46, shows that a necessary condition for this equality is that $\mathcal{M}$ be globally hyperbolic. A celebrated theorem, due independently to Geroch and Seifert, shows that this condition is also sufficient:

**Theorem 2.11.1 (Geroch [19], Seifert [47])** A space-time $(\mathcal{M}, g)$ is globally hyperbolic if and only if there exists on $\mathcal{M}$ a time function $\tau$ with the property that all its level sets are Cauchy surfaces. The function $\tau$ can be chosen to be smooth if the manifold is.

**Proof:** The proof uses volume functions, defined as follows: let $\varphi_i, i \in \mathbb{N}$, be any partition of unity on $\mathcal{M}$, set

$$V_i := \int_{\mathcal{M}} \varphi_i d\mu,$$
CHAPTER 2. CAUSALITY

where \( d\mu \) is, say, the Riemannian measure associated to the auxiliary Riemannian metric \( h \) on \( \mathcal{M} \). Define

\[
\nu := \sum_{i \in \mathbb{N}} \frac{1}{2iV_i} \varphi_i .
\]

Then \( \nu \) is smooth, positive, nowhere vanishing, with

\[
\int_M \nu \, d\mu = 1 .
\]

Following Geroch, we define

\[
V_\pm(p) := \int_{I^\pm(p)} \nu \, d\mu .
\]

We clearly have

\[
\forall p \in \mathcal{M} \quad 0 < V_\pm(p) < 1 .
\]

The functions \( V_\pm \) may fail to be continuous in general, an example is given in Figure 2.11.1. It turns out that such behavior cannot occur under the current conditions:

**Lemma 2.11.2** On \( C^2 \) globally hyperbolic space-times the functions \( V_\pm \) are continuous.

**Proof:** Let \( p_i \) be any sequence converging to \( p \), and let the symbol \( \varphi_\Omega \) denote the characteristic function of a set \( \Omega \). Let \( q \) be any point such that \( q \in I^-(p) \Leftrightarrow p \in I^+(q) \), since \( I^+(q) \) forms a neighborhood of \( p \) we have \( p_i \in I^+(q) \Leftrightarrow q \in I^-(p_i) \) for \( i \) large enough. Equivalently,

\[
\forall i \geq i_0 \quad \varphi_{I^-(p_i)}(q) = 1 = \varphi_{I^-(p)}(q) .
\]

(2.11.1)

Since the right-hand-side of (2.11.1) is zero for \( q \not\in I^-(p) \) we obtain

\[
\forall q \quad \liminf_{i \to \infty} \varphi_{I^-(p_i)}(q) \geq \varphi_{I^-(p)}(q) .
\]

(2.11.2)
By Corollary 2.8.4 $J^-(p)$ differs from $I^-(p)$ by a topological hypersurface so that
\[ \liminf_{i \to \infty} \varphi_{J^-(p_i)} \geq \varphi_{J^-(p)} \text{ a.e.} \] (2.11.3)

To obtain the inverse inequality, let $q$ be such that
\[ \limsup_{i \to \infty} \varphi_{J^-(p_i)}(q) = 1 \]
hence there exists a sequence $\gamma_j$ of future directed, dist-$h$-parameterised causal curves from $q$ to $p_{i_j}$. By Proposition 2.8.1 there exists a future directed accumulation curve of the $\gamma_j$’s from $q$ to $p$. We have thus shown the implication
\[ \limsup_{i \to \infty} \varphi_{J^-(p_i)}(q) = 1 \implies \varphi_{J^-(p)}(q) = 1. \]

Since the function appearing at the left-hand-side of the implication above can only take values zero or one, it follows that
\[ \limsup_{i \to \infty} \varphi_{J^-(p_i)} \leq \varphi_{J^-(p)}. \] (2.11.4)

Equations (2.11.1)-(2.11.4) show that
\[ \lim_{i \to \infty} \varphi_{J^-(p_i)} \text{ exists a.e., and equals } \varphi_{J^-(p)} \text{ a.e.} \]

Since
\[ 0 \leq \varphi_{J^-(p)} \leq 1 \in L^1(\nu \, d\mu), \]
the Lebesgue dominated convergence theorem gives
\[ V_-(p) = \int_v \varphi_{J^-(p)} \nu \, d\mu = \lim_{i \to \infty} \int_v \varphi_{J^-(p_i)} \nu \, d\mu = \lim_{i \to \infty} V_-(p_i). \]

Changing time orientation one also obtains continuity of $V_+$. \hfill \Box

We continue with the following observation:

**Lemma 2.11.3** $V_-$ tends to zero along any past-inextendible causal curve $\gamma : [a, b) \to \mathcal{M}$.

**Proof:** Let $X_i$ be any partition of $\mathcal{M}$ by sets with compact closure, the dominated convergence theorem shows that
\[ \lim_{k \to \infty} \sum_{i \geq k} \int_{X_i} \nu \, d\mu = 0. \] (2.11.5)

Suppose that there exists $k < \infty$ such that
\[ \forall s \quad J^-(\gamma(s)) \cap \left( \bigcup_{i=1}^k X_i \right) \neq \emptyset. \]

Equivalently, there exists a sequence $s_i \to b$ such that
\[ \gamma(s_i) \in K := \bigcup_{i=1}^k X_i. \]
Compactness of $K$ implies that there exists (passing to a subsequence if necessary) a point $q_{\infty} \in K$ such that $\gamma(s_i) \to q_{\infty}$. Strong causality of $M$ implies that there exists an elementary neighborhood $\mathcal{E}$ of $q_{\infty}$ such that $\gamma \cap \mathcal{E}$ is connected, and Lemma 2.6.5 shows that $\gamma \cap \mathcal{E}$ has finite $h$-length, which contradicts inextendibility of $\gamma$ (compare Theorem 2.5.5). This implies that for any $k$ we have

$$J^- (\gamma(s)) \cap \left( \bigcup_{i=1}^{k} X_i \right) = \emptyset$$

for $s$ large enough, say $s \geq s_k$. In particular

$$s \geq s_k \implies \int_{J^- (\gamma(s)) \cap \left( \bigcup_{i=1}^{k} X_i \right)} \nu d\mu = 0 .$$

This implies

$$\forall s \geq s_k \quad V^- (\gamma(s)) = \int_{J^- (\gamma(s)) \cap \left( \bigcup_{i=1}^{k} X_i \right)} \nu d\mu \leq \sum_{i \geq k+1} \int_{X_i} \nu d\mu ,$$

which, in view of (2.11.5), can be made as small as desired by choosing $k$ sufficiently large.

We are ready now to pass to the proof of Theorem 2.11.1. Set

$$\tau := \frac{V_-}{V_+} .$$

Then $\tau$ is continuous by Lemma 2.11.2. Let $\gamma : (a, b) \to \mathcal{M}$ be any inextendible future-directed causal curve. By Lemma 2.11.3

$$\lim_{s \to b} \tau(\gamma(s)) = \infty , \quad \lim_{s \to a} \tau(\gamma(s)) = 0 .$$

Thus $\tau$ runs from 0 to $\infty$ on any such curves, in particular $\gamma$ intersects every level set of $\tau$ at least once. From the definition of the measure $\nu d\mu$ it should be clear that $\tau$ is actually strictly increasing on any causal curve, hence the level sets of $\tau$ are met by causal curves precisely once.

The differentiability properties of $\tau$ constructed above are not clear. It thus remains to show that $\tau$ can be modified, if necessary, so that it is as smooth as the atlas of $\mathcal{M}$ allows (except perhaps for analyticity). This can be done as follows (compare [3, 5, 47]): By [40] there exists a smooth metric $\hat{g}$ with cones wider than those of $g$ so that $(\mathcal{M}, \hat{g})$ is globally hyperbolic. We can thus apply the construction just carried-out to construct a $\hat{g}$–time function $\hat{\tau}$. The property that the light-cones of $\hat{g}$ are strictly wider than those of $g$ implies that the $g$-gradient of $\hat{\tau}$ is everywhere timelike. A small smoothing of $\hat{\tau}$, using convolutions in local coordinates, leads to the desired smooth time function. (Note that the smoothness of $\tau$ depends only upon the smoothness of $\mathcal{M}$, regardless of the smoothness of the metric.)

An important corollary of Theorem 2.11.1 is:

**Corollary 2.11.4** A globally hyperbolic space-time $(\mathcal{M}, g)_{C^2}$ is necessarily diffeomorphic to $\mathbb{R} \times \mathcal{S}$, with the coordinate along the $\mathbb{R}$ factor having timelike gradient.
Proof: Let $X$ by any smooth timelike vector field on $X$ — if both the metric and the time function $\tau$ of Theorem 2.11.1 are smooth then $\nabla \tau$ will do, but any other choice works equally well. Choose any number $\tau_0$ in the range of $\tau$. Define a bijection $\varphi: \mathcal{M} \to \mathbb{R} \times \mathcal{I}$ as follows: for $p \in \mathcal{M}$ let $q(p)$ be the point on the level set $\mathcal{I}_0 = \{ r \in \mathcal{M} : \tau(r) = \tau_0 \}$ which lies on the integral curve of $X$ through $p$. Such a point exists because any inextendible timelike curve in $\mathcal{M}$ meets $\mathcal{I}_0$; it is unique by achronality of $\mathcal{I}_0$. The map $\varphi$ is continuous by continuous dependence of ODE’s upon initial values. If $\tau$ is merely continuous, one can invoke the invariance of domain theorem to prove that $\varphi$ is a homeomorphism; if $\tau$ is differentiable, its level sets are differentiable, $X$ meets those level sets transversely, and the fact that $\varphi$ is a diffeomorphism follows from the implicit function theorem.

It is not easy to decide whether or not a hypersurface $\mathcal{I}$ is a Cauchy hypersurface, except in spatially compact space-times:

Theorem 2.11.5 (Budič et al. [6], Galloway [15]) Let $(\mathcal{M}, g)$ be a smooth globally hyperbolic space-time and suppose that $\mathcal{M}$ contains a smooth, compact, connected spacelike hypersurface $\mathcal{I}$. Then $\mathcal{I}$ is a Cauchy surface for $\mathcal{M}$.

Remark 2.11.6 Some further results concerning Cauchy surface criteria can be found in [15, 22].

The following shows the key role of global hyperbolicity for the wave equation:

Theorem 2.11.7 Let $\mathcal{I}$ be a smooth spacelike hypersurface in a smooth space-time $(\mathcal{M}, g)$. Then the Cauchy problem for the wave equation has a unique globally defined solution on $\dot{\mathcal{D}}_I(\mathcal{I})$ for all smooth initial data on $\mathcal{I}$.

The theorem is well known to the community, but we note that an adequate reference does not seem to be available.

It is of interest to enquire what happens with solutions of the wave equation when Cauchy horizons occur:

First, examples are known where solutions of wave equations blow up at the event horizon, cf., e.g., [12, 26].

Next, a simple example where solutions extend smoothly to solutions of the wave equation, but uniqueness fails, proceeds as follows: let $\mathcal{I}$ be the unit ball within the hypersurface $\{ t = 0 \}$ in Minkowski space-time. Let $p = (1, \vec{0})$ and let $q = (-1, \vec{0})$, then the Cauchy horizon is the union of two inverted cones with tips at $p$ and $q$:

$$\mathcal{H}_I(\mathcal{I}) = (\mathcal{J}^-(p) \cap \{ t > 0 \}) \cup \mathcal{J}^+ \cap \{ t < 0 \}.$$  

Any two distinct solutions of the wave equation on Minkowski space-time which have the same Cauchy data on $\mathcal{I}$ coincide on $\mathcal{D}_I(\mathcal{I})$, and provide examples of solutions which differ beyond the event horizon.

To conclude, we have both global existence and uniqueness of solutions of the Cauchy problem for the wave equation in globally hyperbolic spacetimes. On the other hand, uniqueness or existence are problematic when the space-time is not globally hyperbolic space-times.
2.12 Some applications

Any formalism is only useful to something if it leads to interesting applications. In this section we will list some of those.

We start by pointing out the already-mentioned Theorem 2.11.7. Its counterpart for the Einstein equations is the celebrated Choquet-Bruhat – Geroch theorem:

**Theorem 2.12.1 (Choquet-Bruhat, Geroch [8])** Consider a smooth triple \((S, \gamma, K)\), where \(S\) is an \(n\)-dimensional manifold, \(\gamma\) is a Riemannian metric on \(S\), and \(K\) is a symmetric two–tensor on \(S\), satisfying the general relativistic vacuum constraint equations. Then there exists a unique up to isometries vacuum space–time \((M, g)\), called the maximal globally hyperbolic vacuum development of \((S, \gamma, K)\), with an embedding \(i : S \to M\) such that \(i^* g = \gamma\), and such that \(K\) corresponds to the extrinsic curvature tensor (“second fundamental form”) of \(i(S)\) in \(M\). \((M, g)\) is inextendible in the class of globally hyperbolic space–times with a vacuum metric.

This theorem is the starting point of many studies in mathematical general relativity. Similarly to the wave equation, examples show that uniqueness fails beyond horizons.

To continue, we say that \((\mathcal{M}, g)\) satisfies the *timelike focussing condition*, or *timelike convergence condition*, if the Ricci tensor satisfies

\[
R_{\mu\nu}n^\mu n^\nu \geq 0 \quad \text{for all timelike vectors } n^\mu. \tag{2.12.1}
\]

By continuity, the inequality in (2.12.1) will also hold for causal vectors. Condition (2.12.1) can of course be rewritten as a condition on the matter fields using the Einstein equation, and is satisfied in many cases of interest, including vacuum general relativity, or the Einstein-Maxwell theory, or the Einstein-Yang-Mills theory. This last two examples actually have the property that the corresponding energy-momentum tensor is trace-free; whenever this happens, (2.12.1) is simply the requirement that the energy density of the matter fields is non-negative for all observers:

\[
8\pi T_{\mu\nu}n^\mu n^\nu = (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) n^\mu n^\nu = R_{\mu\nu}n^\mu n^\nu . \tag{2.12.2}
\]

We say that \((\mathcal{M}, g)\) satisfies the *null energy condition* if

\[
R_{\mu\nu}n^\mu n^\nu \geq 0 \quad \text{for all null vectors } n^\mu. \tag{2.12.3}
\]

Clearly, the timelike focussing condition implies the null energy condition. Because \(g_{\mu\nu}n^\mu n^\nu = 0\) for all null vectors, the \(R\) term in the calculation (2.12.2) drops out regardless of whether or not \(T_{\mu\nu}\) is traceless, so the null energy condition is equivalent to positivity of energy density of matter fields without any provisos.

The simplest *geodesic incompleteness theorem* is:
2.12. SOME APPLICATIONS

Theorem 2.12.2 (Geroch’s geodesic incompleteness theorem [17]) Let $(\mathcal{M}, g)$ be a smooth globally hyperbolic satisfying the timelike focusing condition, and suppose that $\mathcal{M}$ contains a compact Cauchy surface $\mathcal{I}$ with strictly negative mean curvature:

$$\text{tr}_h K < 0,$$

where $(h, K)$ are the usual Cauchy data induced on $\mathcal{I}$ by $g$. Then $(\mathcal{M}, g)$ is future timelike geodesically incomplete.

Yet another incompleteness theorem involves trapped surfaces, this requires introducing some terminology: Let $\mathcal{S}$ be a spacelike hypersurface in $(\mathcal{M}, g)$, and consider a surface $S \subset \mathcal{S}$. We shall also assume that $S$ is two-sided in $\mathcal{S}$, this means that there exists a globally defined field $m$ of unit normals to $S$ within $\mathcal{S}$. There are actually two such fields, $m$ and $-m$, we arbitrarily choose one and call it outer pointing. In situations where $S$ does actually bound a compact region, the outer-pointing one should of course be chosen to point away from the compact region. We let $H$ denote the mean extrinsic curvature of $S$ within $\mathcal{S}$:

$$H := D_i m^i,$$  \hspace{1cm} (2.12.4)

where $D$ is the covariant extrinsic of the metric $h$ induced on $\mathcal{S}$. We say that $S$ is outer-future-trapped if

$$\theta_+ := H + K_{ij} (g^{ij} - m^i m^j) \leq 0,$$  \hspace{1cm} (2.12.5)

with an obvious symmetric definition for inner-future-trapped:

$$\theta_- := -H + K_{ij} (g^{ij} - m^i m^j) \geq 0,$$  \hspace{1cm} (2.12.6)

(One also has the obvious past version thereof, where the sign in front of the $K$ term should be changed.) A celebrated theorem of Penrose\(^7\) asserts that:

Theorem 2.12.3 (Penrose’s geodesic incompleteness theorem [44]) Let $(\mathcal{M}, g)$ be a smooth globally hyperbolic space-time satisfying the null energy condition, and suppose that $\mathcal{M}$ contains a non-compact Cauchy surface $\mathcal{I}$. If there exists a compact trapped surface within $S$ which is both inner-future-trapped and outer-future-trapped, then $(\mathcal{M}, g)$ is geodesically incomplete.

The significance of this theorem stems from the fact, that the Schwarzschild solution, as well as the non-degenerate Kerr black holes, possess trapped surfaces beyond the horizon. A small perturbation of the metric will preserve this. It follows that the geodesic incompleteness of these black holes is not an accident of the large isometry group involved, but is stable under perturbations of the metric.

More generally, future-trapped surfaces signal the existence of black holes. Formal statements to this effect require the introduction of the notion of a black

\(^7\)Penrose’s theorem is slightly more general, using a definition of $\theta_\pm$ which involves a discussion of null geometry which we prefer to avoid here. This is the reason why we have stated this theorem in the current form.
hole, as well as several global regularity conditions, and will therefore not be
given here.

We close the list of applications with the area theorem, which plays a funda-
mental role in black-hole thermodynamics:

**Theorem 2.12.4 ([10, 24])** Consider a spacetime $(\mathcal{M}, g)_{C^\infty}$. Let $\mathcal{E}$ be a future
geodesically complete acausal null hypersurface, and let $\mathcal{I}_1, \mathcal{I}_2$ be two spacelike
acausal hypersurfaces. If

$$\mathcal{E} \cap \mathcal{I}_1 \subset J^-(\mathcal{E} \cap \mathcal{I}_2),$$

then

$$\text{Area}(\mathcal{E} \cap \mathcal{I}_1) \leq \text{Area}(\mathcal{E} \cap \mathcal{I}_2).$$

**Acknowledgements** The author is grateful to Gregory Galloway and James
Grant for useful discussions.
Bibliography

[1] F. Antonacci and P. Piccione, *A Fermat principle on Lorentzian manifolds and applications*, Appl. Math. Lett. 9 (1996), 91–95.

[2] H. Bahouri and J.-Y. Chemin, *Équations d’ondes quasilineaires et estimations de Strichartz*, Am. Jour. Math. 121 (1999), 1337–1377.

[3] C. Bär, N. Ginoux, and F. Pfäffle, *Wave equations on Lorentzian manifolds and quantization*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2007. MR MR2298021

[4] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian geometry*, 2 ed., Pure and Applied Mathematics, vol. 202, Marcel Dekker, New York, 1996.

[5] A.N. Bernal and M. Sánchez, *Smoothness of time functions and the metric splitting of globally hyperbolic space-times*, Commun. Math. Phys. 257 (2005), 43–50. MR MR2163568 (2006g:53105)

[6] R. Budic, J. Isenberg, L. Lindblom, and P. Yasskin, *On the determination of the Cauchy surfaces from intrinsic properties*, Commun. Math. Phys. 61 (1978), 87–95.

[7] Y. Choquet-Bruhat, *General relativity and the Einstein equations*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2009. MR MR2473363 (2010f:83001)

[8] Y. Choquet-Bruhat and R. Geroch, *Global aspects of the Cauchy problem in general relativity*, Commun. Math. Phys. 14 (1969), 329–335. MR MR0250640 (40 #3872)

[9] P.T. Chruściel, *On uniqueness in the large of solutions of Einstein equations ("Strong Cosmic Censorship")*, Australian National University Press, Canberra, 1991.

[10] P.T. Chruściel, E. Delay, G. Galloway, and R. Howard, *Regularity of horizons and the area theorem*, Annales Henri Poincaré 2 (2001), 109–178, arXiv:gr-qc/0001003. MR MR1823836 (2002e:83045)

[11] P.T. Chruściel and J. Grant, *On Lorentzian causality with continuous metrics*, (2011), arXiv:1110.xxxx [gr-qc].

65
[12] P.T. Chruściel, J. Isenberg, and V. Moncrief, *Strong cosmic censorship in polarized Gowdy space–times*, Class. Quantum Grav. 7 (1990), 1671–1680.

[13] F.H. Clarke, *Optimization and nonsmooth analysis*, second ed., Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

[14] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.

[15] G.J. Galloway, *Some results on Cauchy surface criteria in Lorentzian geometry*, Illinois Jour. Math. 29 (1985), 1–10.

[16] Gregory J. Galloway, *Some global aspects of compact space-times*, Arch. Math. 42 (1984), 168–172.

[17] R. Geroch, *Singularities in closed universes*, Phys. Rev. Lett. 17 (1966), 445–447.

[18] ——–, *Topology in general relativity*, Jour. Math. Phys. 8 (1967), 782–786. MR MR0213139 (35 #4004)

[19] ——–, *Domain of dependence*, Jour. Math. Phys. 11 (1970), 437–449.

[20] F. Giannoni, A. Masiello, and P. Piccione, *A variational theory for light rays in stably causal Lorentzian manifolds: regularity and multiplicity results*, Commun. Math. Phys. 187 (1997), 375–415.

[21] V. Guillemin and A. Pollack, *Differential topology*, Prentice–Hall, Englewood Cliffs, N.J, 1974.

[22] S.G. Harris, *What is the shape of space in a spacetime?*, Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 287–296. MR 1216546 (94e:53065)

[23] P. Hartman, *Ordinary differential equations*, J. Wiley & Sons, Baltimore, 1973.

[24] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973, Cambridge Monographs on Mathematical Physics, No. 1. MR MR0424186 (54 #12154)

[25] H. Hopf and W. Rinow, *Ueber den Begriff der vollst¨aendingen differential-geometrischen Fläche*, Comment. Math. Helv. 3 (1931), 209–225.

[26] S. Kichenassamy and A. Rendall, *Analytic description of singularities in Gowdy space–times*, Class. Quantum Grav. 15 (1998), 1339–1355.

[27] S. Klainerman and I. Rodnianski, *The causal structure of microlocalized rough Einstein metrics*, Ann. of Math. (2) 161 (2005), 1195–1243. MR MR2180401 (2007d:58052)
[28] Rough solutions of the Einstein-vacuum equations, Ann. of Math. (2) 161 (2005), 1143–1193. MR MR2180400 (2007d:58051)

[29] M. Kriele, Spacetime, Lecture Notes in Physics. New Series m: Monographs, vol. 59, Springer-Verlag, Berlin, 1999, Foundations of general relativity and differential geometry. MR 2001g:53126

[30] K. Martin, Compactness of the space of causal curves, Class. Quantum Grav. 23 (2006), 1241–1251. MR MR2205482

[31] D. Maxwell, Rough solutions of the Einstein constraint equations on compact manifolds, Jour. Hyperbolic Diff. Equ. 2 (2005), 521–546, arXiv:gr-qc/0506085. MR MR2151120 (2006d:58027)

[32] Rough solutions of the Einstein constraint equations, J. Reine Angew. Math. 590 (2006), 1–29, arXiv:gr-qc/0405088. MR MR2208126 (2006j:58044)

[33] J. Milnor, Morse theory, Annals of Mathematics Studies, vol. 51, Princeton Univ. Press, 1963.

[34] E. Minguzzi, Chronological spacetimes without lightlike lines are stably causal, Comm. Math. Phys. 288 (2009), 801–819. MR 2504855 (2010e:83089)

[35] K-causality coincides with stable causality, Comm. Math. Phys. 290 (2009), 239–248. MR 2520513 (2010i:53133)

[36] E. Minguzzi and M. Sánchez, The causal hierarchy of spacetimes, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, pp. 299–358. MR 2436235 (2010b:53128)

[37] C.W. Misner and A. Taub, A singularity-free empty universe, Soviet. Phys. JEPT 28 (1969), 122–133.

[38] V. Moncrief, An integral equation for spacetime curvature in general relativity, Surveys in differential geometry. Vol. X, Surv. Differ. Geom., vol. 10, Int. Press, Somerville, MA, 2006, pp. 109–146. MR 2408224 (2009h:53166)

[39] U. Mueller, C. Schubert, and A.E.M. van de Ven, A closed formula for the Riemann normal coordinate expansion, Gen. Rel. Gravitation 31 (1999), 1759–1781.

[40] J.J.B. Navarro and E. Minguzzi, The stability of global hyperbolicity, (2011), arXiv:1108.5210 [gr-qc].

[41] E. Newman, L. Tamburino, and T. Unti, Empty-space generalization of the Schwarzschild metric, Jour. Math. Phys. 4 (1963), 915–923. MR MR0152345 (27 #2325)

[42] K. Nomizu and H. Ozeki, The existence of complete Riemannian metrics, Proc. Amer. Math. Soc. 12 (1961), 889–891. MR MR0133785 (24 #A3610)
[43] B. O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press, New York, 1983. MR MR719023 (85f:53002)

[44] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. 14 (1965), 57–59. MR 0172678 (30 #2897)

[45] [citation removed], *Techniques of differential topology in relativity*, SIAM, Philadelphia, 1972, (Regional Conf. Series in Appl. Math., vol. 7).

[46] V. Perlick, *On Fermat’s principle in general relativity: I. The general case*, Class. Quantum Grav. 7 (1990), 1319–1331.

[47] H.J. Seifert, *Smoothing and extending cosmic time functions*, Gen. Rel. Grav. 8 (1977), 815–831. MR MR0484260 (58 #4185)

[48] H. Smith and D. Tataru, *Sharp local well-posedness results for the nonlinear wave equation*, Ann. of Math. (2) 162 (2005), 291–366. MR MR2178963 (2006k:35193)

[49] R.D. Sorkin and E. Woolgar, *A causal order for space-times with $C^0$ Lorentzian metrics: proof of compactness of the space of causal curves*, Class. Quantum Grav. 13 (1996), 1971–1993. MR MR1400951 (97e:53123)

[50] A.H. Taub, *Empty space-times admitting a three parameter group of motions*, Ann. of Math. (2) 53 (1951), 472–490. MR MR0041565 (12,865b)

[51] G. Teschl, *Ordinary differential equations and dynamical systems*, 2011, http://www.mat.univie.ac.at/~gerald/ftp/book-ode/ode.pdf.

[52] R.M. Wald, *General relativity*, University of Chicago Press, Chicago, 1984.

[53] Q. Wang, *On Ricci coefficients of null hypersurfaces with time foliation in Einstein vacuum space-time*, arXiv:1006.5963.

[54] [citation removed], *On the geometry of null cones in Einstein-vacuum spacetimes*, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), 285–328. MR 2483823 (2011b:53172)