Evolving extrinsic curvature and the cosmological constant problem

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Abstract

The concept of smooth deformation of Riemannian manifolds associated with the extrinsic curvature is explained and applied to the Friedmann–Lemaître–Robertson–Walker cosmology. We show that such deformation can be derived from the Einstein–Hilbert-like dynamical principle may produce an observable effect in the sense of Noether. As a result, we show how the extrinsic curvature compensates both quantitative and qualitative differences between the cosmological constant \(\Lambda\) and the vacuum energy \(\rho_{\text{vac}}\), obtaining the observed upper bound for the cosmological constant problem at electroweak scale. The topological characteristics of the extrinsic curvature are discussed showing that the produced extrinsic scalar curvature is an evolving dynamical quantity.

Keywords: Nash embedding theorem, cosmological constant, dark energy, topological changing Universe

(Some figures may appear in colour only in the online journal)

1. Introduction

The recent proof of the Poincaré conjecture by G Perelman [1–3] suggests a new paradigm for geometry and for Einstein’s gravitation, namely the possibility that space-times can be deformed in arbitrary directions, changing its shape. In a previous investigation [4], we studied a modification imposed on Friedman’s equation when the standard model of the Universe is regarded as an embedded space–time [5]. It was shown that a more fundamental explanation for the dynamics of the extrinsic curvature was given by Gupta equations [7], which is essentially a set of Einstein-like equations applied to extrinsic curvature. Different from other embedding models, the novelty of this approach is that the extrinsic curvature is thought of as a (normal) component of the spin-2 gravitational field and its dynamics can be studied. As a result, the accelerated expansion of the Universe may be explained as an effect of the extrinsic curvature [4, 8].

In this work, we investigate the cosmological constant (CC) problem that primarily consists in a seemingly unexplainable difference between the small value of the cosmological constant estimated by cosmological observations, i.e., \(\Lambda/8\pi G \sim 10^{-47}\) GeV\(^4\). Its theoretical value, given by the vacuum energy density, results from the gravitationally coupled quantum fields in space-time estimated to be of the order of \(\langle \rho_v \rangle \sim 10^{11}\) GeV\(^4\). Such a large difference cannot be eliminated by renormalization techniques in quantum field theory as it would require extreme fine tuning [9, 10]. In the last decade, it became a central issue in the context of the \(\Lambda\)CDM cosmological model regarded as the simpler model for the accelerated expansion of the Universe. In addition, another dilemma that requires attention is that of a proper explanation to the apparently coincidence between the current matter density energy and the CC (as interpreted as the vacuum energy) commonly known as the coincidence problem [11, 12]. Varieties of solutions for the CC problems have been proposed in literature such as in general relativity [13–15], Strings [16] and Branes [17–19], conformal symmetry of gravity [20] and other works [21–25].

In a different direction, we address the CC problem using essentially the fact that in an embedded space-time the gauge fields remain confined to the embedded space, but the
gravitational field propagates along the extra dimensions similar to the original brane-world proposed in [26]. On the other hand, it is important to point out that the difference between the vacuum energy and the CC is hidden in most brane-world models because the extrinsic curvature is commonly replaced by a function of the confined source fields. As commonly thought, the only accepted relation of the extrinsic curvature with matter sources is the Israel–Lanczos boundary condition, as applied to the Randall–Sundrum brane-world cosmology [27, 28]. However, this condition fixes once and for all the extrinsic curvature and does not follow the dynamics of the brane-world. Other approaches have been developed with no need for particular junction conditions [29, 30] and/or with different junction conditions which lead to several approaches of brane-world models widely studied in literature [31–36].

The main purpose of this paper is to show that the CC problem comes from a fundamental origin, not only because it involves the structure of the Einstein–Hilbert principle, but also because it reinforces a clearly distinction between gravitational and gauge fields. In what follows, we focus on the CC problem at low redshift, since it is verified in the present epoch [23, 24]. We obtain an explicit relation involving the extrinsic curvature and the absolute difference between CC and the vacuum energy density. As we shall see, we show that the dynamics of the extrinsic curvature has a more profound meaning, which a four-dimensional observer can detect a difference between Einstein’s CC and the confined vacuum energy through a conserved quantity. Another relevant aspect is about the dynamics of the extrinsic curvature and how its extrinsic scalar $Q$ evolves. In this framework we are neglecting fluctuations and/or effects of structure formation. Finally, remarks are presented in the conclusion section.

2. The FLRW embedded Universe

2.1. Modified Friedmann equations

We start with the Friedman–Lemaître–Robertson–Walker (FLRW) with line element in coordinates $(r, \theta, \phi, t)$, which is given by

$$\text{d}s^2 = -\text{d}t^2 + a^2(r)[\text{d}r^2 + f_\kappa^2(r)(\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2)],$$

where $f_\kappa(r) = \sin r$, $r$, $\sin r$ corresponding to $\kappa = 1$, $0$, $-1$, and $a = a(t)$ is the expansion parameter. This model can be regarded as a four-dimensional hypersurface dynamically evolving in a five-dimensional bulk with constant curvature. This geometry defined by four-dimensional FLRW line element is completely embedded in a five-dimensional bulk. The Riemann tensor is given by

$$\mathcal{R}_{ABCD} = K_\kappa(G_{AC}G_{DB} - G_{AD}G_{CB}),$$

where $G_{AB}$ denotes the bulk metric components in arbitrary coordinates. The constant curvature $K_\kappa$ has three possible values: it is either zero (flat bulk), a positive (de Sitter) or negative (anti-de Sitter) constant curvatures.

Since we are dealing with locally embedding of geometries, the solution was given by John Nash in 1956 [37], using only differentiable (non-analytic) properties. In short, starting with an embedded Riemannian manifold with metric $g_{\mu\nu}$ and extrinsic curvature $\bar{k}_{\mu\nu}$, Nash showed that any other embedded Riemannian geometry can be generated by differentiable perturbations, with metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, where

$$\delta g_{\mu\alpha} = -2\bar{k}_{\mu\alpha},$$

and where $\delta y^a$ is an infinitesimal displacement in one of the extra dimension $y^a$. From this new metric, we obtain a new extrinsic curvature $k_{\mu\nu}$ and the procedure can be repeated indeﬁnitely:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta y^a k_{\mu\alpha} + \delta y^b \bar{g}^{ab} k_{\mu\beta} k_{\nu\epsilon} \cdots$$

For this reason, depending on the size of the bulk, the embedding map can be well defined and, for instance, one does not need to separate the coordinates in order to perturb the line element in equation (1) with the $y$-coordinate, since Nash theorem already guarantees this property. It is worth noting that, originally, the Nash theorem was valid only for Riemannian metrics of positive signature, lately extended to pseudo-Riemannian metrics by Greene in the 70’s [38].

Another important aspect is that the seminal work on perturbation of geometry concerning Riemannian embeddings was proposed in 1926 by Campbell [39] with a posthumous work using analytic embedding functions and was revitalized by Romero and co-workers in [40] extending Campbell’s theorem to non-positive signatures and a reduction of four-dimensional general relativity to the minor dimensions, which may lead to a quantization of gravity.

2.2. Cosmology in embedded space-time

Taking the perfect fluid of the standard cosmology as composed of ordinary matter interacting with gauge fields, then it must remain confined to the four-dimensional space-time on all stages of the evolution of the Universe. Since all cosmological observations point to an accelerated expanding Universe towards a de Sitter configuration [41, 42], we choose $K_\kappa > 0$, although our results also hold for any other choice of $K_\kappa$. The bulk geometry is actually defined by the Einstein–Hilbert principle, which leads to Einstein equations for the bulk as

$$\mathcal{R}_{AB} - \frac{1}{2} R g_{AB} = \alpha_\kappa T_{AB}^\kappa,$$

The confinement condition implies that $K_\kappa = \Lambda/6$ and $T_{AB}^\kappa$ denotes the energy-momentum tensor of the known sources.

The confinement of gauge fields and ordinary matter is a standard assumption especially in what concerns the brane-
world program as a part of the solution of the hierarchy problem of the fundamental interactions: the four-dimensionality of space-time is a consequence of the invariance of Maxwell’s equations under the Poincaré group. Such a condition was later extended to all gauge fields expressed in terms of differential forms and their duals. However, in spite of many attempts, gravitation, in the sense of Einstein, does not fit in such a scheme. Thus, while all known gauge fields are confined to the four-dimensional submanifold, gravitation as defined in the whole bulk space by the Einstein–Hilbert principle, propagates in the bulk. The proposed solution of the hierarchy problem says that gravitational energy scale is somewhere within the TeV scale.

The most general expression of this confinement is that the confined components of $T_{\mu\nu}$ are proportional to the energy-momentum tensor of general relativity: $\kappa_{\alpha} T_{\mu\nu} = -8\pi G T_{\mu\nu}$. On the other hand, since only gravity propagates in the bulk we have $T_{\mu\nu} = 0$ and $T_{\alpha\beta} = 0$. Even though any gauge theory can be mathematically constructed in a higher dimensional space, we adopt the confinement as a condition. Accordingly, the four-dimensionality of space-time will suffice in our case based on experimentally high-energy tests suggest [43].

We restrict ourselves to the analysis of the local embedding of five-dimensions, which can be summarized defining an embedding map $\mathcal{Z} : V_4 \to V_5$ admitting that $\mathcal{Z}^{\mu}$ is a regular and differentiable map, with $V_4$ and $V_5$ being the embedded space-time and the bulk, respectively. The components $\mathcal{Z}^{\mu} = f^A(x^1, \ldots, x^4)$ associate with each point of $V_4$ a point in $V_5$ with coordinates $\mathcal{Z}^A$, which are the components of the tangent vectors of $V_4$. Moreover, taking the tangent, vector and scalar components of equation (5) defined in the Gaussian frame vielbein $\{ \mathcal{Z}^A, \eta^A \}$, where $\eta^A$ are the components of the normal vectors of $V_4$, one can write the set of the embedded four dimensional equations [4, 5] as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - Q_{\mu\nu} = -8\pi G T_{\mu\nu},$$

(6)

$$k_{\mu}^\rho \rho - h_{\mu} = 0,$$

(7)

where $T_{\mu\nu}$ is the four-dimensional energy-momentum tensor of the perfect fluid expressed in co-moving coordinates as

$$T_{\mu\nu} = (p + \rho) U_\mu U_\nu + \rho \ g_{\mu\nu}, \quad U_\mu = \delta_\mu^0.$$

The derivations of equations (5), (6) and (7) as well as the higher dimensional equations are shown in detailed in [5, 6]. We point out that by means of embedding, these geometries (the five-dimensional bulk and the embedded four dimensional space-time) are compatibilized by the integrability conditions for the perturbed geometry denoted by the Gauss, Codazzi and Ricci equations (e.g. see [6] and references therein). In summary, the local embedding proposed in our paper is built in a neighborhood of each point of the embedded four-dimensional space-time, defining an embedding bundle whose total space consists of all embedding spaces (which can be naturally extended to higher dimensional spaces).

It is important to point out that the quantity $Q_{\mu\nu}$ is a completely geometrical term given by

$$Q_{\mu\nu} = g^{\mu\sigma} k_{\nu\rho} k_{\sigma\nu} - k_{\mu\nu} h - \frac{1}{2} (K^2 - h^2) g_{\mu\nu},$$

(8)

where $h = g^{\mu\nu} k_{\mu\nu}$, $h^2 = h\cdot h$ and $K^2 = k^{\mu\nu} k_{\mu\nu}$. It follows that $Q_{\mu\nu}$ is conserved in the sense that

$$Q^{\mu\nu,\rho}_{\ ;\rho} = 0.$$  

(9)

According to the Noether theorem, the conservation of $Q_{\mu\nu}$ represents the observable effects from the action of the extrinsic curvature on the embedded space-time.

The general solution for equation (7) using the FLRW metric is

$$k_{ij} = \frac{b}{a^2} g_{ij}, \quad k_{44} = -\frac{1}{a} \frac{d}{dt} \left( \frac{b}{a} \right),$$

in this case $i, j = 1, 2, 3$, where we also notice that the function $b(\cdot) = k_{11}$ remains an arbitrary function of time. This follows from the confinement of the gauge fields that produces the homogeneous equation in equation (7).

The usual Hubble parameter in terms of the expansion scaling factor $a(\cdot) = a$ is denoted by $H = \dot{a}/a$ and the extrinsic parameter $B = \ddot{b}/b$. Solving the set of equation (6) and equation (7), one can obtain

$$k_{44} = -\frac{b}{a} \left( \frac{B}{H} + 1 \right) g_{44}, \quad h = \frac{b}{a} \left( \frac{B}{H} + 2 \right),$$

$$K^2 = \frac{b^2}{a^2} \left( \frac{B^2}{H^2} - 2 \frac{B}{H} + 4 \right),$$

$$Q_{ij} = \frac{b^2}{a^2} \left( \frac{2B}{H} - 1 \right) g_{ij}, \quad Q_{44} = -\frac{3b^2}{a^2},$$

$$Q = -(K^2 - h^2) = \frac{6b^2}{a^2} \frac{B}{H}. $$

(10)

(11)

(12)

(13)

In the case of equation (12), consider $i, j = 1, 2, 3$.

Replacing these results in equation (6), we obtain the Friedman equation modified by the extrinsic curvature as

$$\frac{\dot{a}}{a}^2 + \frac{k}{a^2} = \frac{4}{3} \pi G \rho + \frac{\Lambda}{3} + \frac{b^2}{a^2}.$$  

(14)

2.3. Gupta extrinsic equation and the unique solution for the function $b(t)$

The arbitrariness of $b(t)$ is a consequence of the homogeneity of equation (7), which follows from the confinement condition $T_{\mu\nu} = 0$. If these components were non zero, we would violate the intended solution of the hierarchy problem. For instance, the Randall–Sundrum brane-world models avoid such difficulty by fixing the brane-world as a boundary at $y = 0$ applying the Israel–Lanczos boundary condition. On the other hand, in order to obtain dynamical equations the Gupta equations can be used and the function $b(t)$ are determined by constructing the dynamics of extrinsic curvature $k_{\mu\nu}$ interpreted as a component of gravitational field besides the metric $g_{\mu\nu}$.
In short, the study of linear massless spin-2 fields in Minkowski space-time by Fierz and Pauli dates back to late 1930s [44]. In 1954, Gupta [7] noted that the Fierz–Pauli equation has a remarkable resemblance with the linear approximation of Einstein’s equations for the gravitational field, suggesting that such an equation could be just the linear approximation of a more general, non-linear equation for massless spin-2 fields. In reality, he found that any spin-2 field in Minkowski space-time must satisfy an equation that has the same formal structure as Einstein’s equations. This amounts to saying that, in the same way as Einstein equations can be obtained by an infinite sequence of infinitesimal perturbations of the linear gravitational equation, it is possible to obtain a non-linear equation for any spin-2 field by applying an infinite sequence of infinitesimal perturbations to the Fierz–Pauli equations. The result obtained by S Gupta is an Einstein-like system of equations [7, 45].

In the following, we use an analogy with the derivation of the Riemann tensor to write Gupta equations in a Riemannian manifold with metric geometry $g_{\mu\nu}$ embedded in a five-dimensional bulk. In this analogy we construct a ‘connection’ associated with $b_{\mu\nu}$ and then, find its corresponding Riemann tensor, but keeping in mind that the geometry of the embedded space-time has been previously defined by $g_{\mu\nu}$. To do so, define the tensors

$$f_{\mu\nu} = \frac{2}{K} k_{\mu\nu}, \quad \text{and} \quad f^{\mu\nu} = \frac{2}{K} k^{\mu\nu},$$  

so that $f^{\rho\sigma} f_{\rho\sigma} = \delta^{\mu}_{\nu}$. In the sequence we construct the ‘Levi–Civita connection’ associated with $f_{\mu\nu}$, based on the analogy with the ‘metricity condition’ $f_{\nu\mu||\rho} = 0$, where $||$ denotes the covariant derivative with respect to $f_{\mu\nu}$ (while keeping the usual $\nabla$ notation for the covariant derivative with respect to $g_{\mu\nu}$). With this condition we obtain the ‘$f$-connection’

$$\Gamma_{\rho\mu\sigma} = \frac{1}{2} (\partial_{\rho} f_{\sigma\mu} + \partial_{\sigma} f_{\rho\mu} - \partial_{\mu} f_{\rho\sigma})$$

and

$$\Gamma_{\rho\mu}^{\lambda} = f^{\rho\lambda} \Gamma_{\mu\sigma}$$

and the ‘$f$-Riemann tensor’

$$F_{\rho\sigma\lambda\mu} = \partial_{\rho} \Gamma_{\sigma\lambda\mu} - \partial_{\sigma} \Gamma_{\rho\lambda\mu} + \Gamma_{\rho\lambda\mu} \Gamma_{\sigma\lambda} - \Gamma_{\rho\lambda\sigma} \Gamma_{\sigma\mu},$$

and the ‘$f$-Ricci tensor’ and the ‘$f$-Ricci scalar’, defined with $f_{\mu\nu}$ are, respectively,

$$F_{\mu\nu} = f^{\rho\lambda} F_{\rho\sigma\lambda\mu} \quad \text{and} \quad F = f^{\mu\nu} F_{\mu\nu}.$$  

Finally, write the Gupta equations for the $f_{\mu\nu}$ field

$$F_{\mu\nu} = \frac{1}{2} F f_{\mu\nu} = \alpha_f \tau_{\mu\nu}$$

where $\tau_{\mu\nu}$ stands for the source of the $f$-field, with coupling constant $\alpha_f$. However, unlike the case of Einstein’s equations, here we do not have the equivalent to the Newtonian weak field limit, so that we cannot tell about the nature of the source term $\tau_{\mu\nu}$ based on experience, then we adopt $\tau_{\mu\nu} = 0$. With this choice Gupta’s equations for $f_{\mu\nu}$ becomes simply

$$F_{\mu\nu} = 0.$$  

Interestingly, this assumption is compatible with a possible explanation for the dark energy problem [4, 8].

As we use equation (1) and calculate equation (17), we can take equation (8) and equation (9) and obtain the unique expression for the functions $b(t)$ that is given by

$$b(t) = \alpha_0 \alpha_0 c^{-1 / 2} e^{\pm \frac{1}{2} \gamma(t)}.$$  

We denote $\alpha_0 = b_0 / a_0^{\beta_0}$, $a_0$ by the present value of the expansion scaling factor and $b_0$ is an integration constant representing the current warp of the Universe. Also, the exponential function has the exponent given by

$$\gamma(t) = \sqrt{4 b_0 a_4} - 3 - \sqrt{3} \arctan \left( \frac{3}{3} \sqrt{4 b_0 a_4} - 3 \right).$$

The two signs represent two possible signatures of the evolution of the function $b(t)$, which can be important to study how it evolves, and in the next section, we study how it can be related to the CC problem.

3. The balance through extrinsic curvature

Analyzing equation (6), a four-dimensional observer realizes that the quantum vacuum energy density $\langle \rho \rangle$ can be related to $\Lambda g_{\mu\nu} - Q_{\mu\nu}$ different from the case of general relativity that we only have the term $\Lambda g_{\mu\nu}$. Thus, taking equation (6), we have

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - Q_{\mu\nu} = -8 \pi G T_{\mu\nu},$$

and considering the vacuum contribution $T_{\mu\nu} = -\langle \rho \rangle g_{\mu\nu}$, one can write

$$8 \pi G \langle \rho \rangle g_{\mu\nu} = \Lambda g_{\mu\nu} - Q_{\mu\nu},$$

contracting with $g_{\mu\nu}$, we obtain

$$\langle \rho \rangle = - \frac{Q}{32 \pi G}$$

where $Q = g^{\mu\nu} Q_{\mu\nu}$ is the trace of $Q_{\mu\nu}$. In addition, using equation (13) we can write

$$\langle \rho \rangle = - \frac{Q}{32 \pi G} = - \frac{6 b_0^2 B}{32 \pi G a^3 H^2},$$

which indicates that the discrepancy ceases to be if the extrinsic curvature can compensate such difference.

We can make now an analysis of equation (21) starting from the extrinsic term $b(t)$. Thus, we seek a general relation that can relate the expansion parameter $a(t)$ with the difference between $\langle \rho \rangle$ and $\rho_0$. Taking the Gupta solutions in equation (18) for FLRW cosmology [4] and equation (14), we can write the modified Friedman equation in terms of the redshift $z$ and cosmological parameters as

$$H^2 = H_0^2 \left[ \Omega_m + \Omega_\Lambda e^{\pm \gamma(z)} \right],$$

where $\Omega_m$ is the matter density cosmological parameter defined as $\Omega_m = \Omega_m^0 (1 + z)^3$ and $H_0$ is the current Hubble constant. Hereafter, the upper script ‘0’ indicates the present value of certain quantity.

In order to be consistent with [4], we consider the current value for the expansion factor as $a_0 = 1$. The term $\gamma(z)$ is written in terms of the redshift $z$ and given by
Interestingly, using equations (18), (21) and (23), we find

$$|\langle \rho_v \rangle - \rho_m| = \frac{H_0^2}{2} \rho_{ex} \xi(z)(1 + z)^{4 - 2\beta_0},$$

where

$$\xi(z) = \left( \beta_0 \pm \frac{4\beta_0}{\sqrt{1 + z^2}} - 3 \right) \exp[\pm \gamma(z)].$$

From this equation one can obtain that the evolution of the difference of vacuum energy and CC is balanced through the extrinsic curvature evolving on redshift. Hence, in order to test the effectivity of this expression, we calculate that difference for today, i.e., \( z = 0 \). To this matter, we use the pair of parameters \((\beta_0, \eta_0)\) that was already constrained in \([8]\) and the values \(\beta_0 = 2\) and \(\eta_0 = 0.25\). It is important to stress that those values matched the cosmokinetics tests studied in the accelerated expansion of the Universe. It was found that \(\beta_0\) affects the value of current deceleration parameter \(q_0\) and \(\eta_0\) rules mainly on the width of the transition phase \(z_c\). Based on the fact that equation (18) can provide different solutions with the term \(\gamma\), when it holds for \(\pm \gamma(z) = 0\), one can obtain the similar pattern as obtained from phenomenological solutions as shown in \([4]\) that mimics the X-CDM model with a correspondence

$$4 - 2\beta_0 = 3(1 + w),$$

where the parameter \(w\) holds for the exotic X-fluid parameter \([46]\). Rather than only reproducing known phenomenological models we are interested also in solutions \(\pm \gamma = 0\). Thus, we adopt the current value of Hubble constant \(H_0\) as \(H_0 = 67.8 \pm 0.9\, \text{km.s}^{-1}\text{Mpc}^{-1}\) and the current matter density parameter \(\Omega_m^0 = 0.308 \pm 0.012\) based on the latest observations \([47]\). Thus, using equation (23), equation (25) turns

$$|\langle \rho_v \rangle - \rho_m| = \frac{3\alpha_0^2}{16\pi G}H_0^2\xi(z = 0).$$

With \(H_0 \sim 10^{-42}\, \text{GeV}^4\), we apply those values to equation (28) and find the difference \(|\langle \rho_v \rangle - \rho_m| \sim 10^{-47}\, \text{GeV}^4\) or lesser that this value and matches the upper bound value for the cosmological constant problem in electroweak scale. This shows that the effect of extrinsic curvature has become subtle in the present time. This interpretation seems to be very reasonable since the main process of formation of structures at cosmological scale in Universe happened a long time ago and today it appears to be reduced at local scale. Since the extrinsic curvature can warp, bend or stretch a geometry, it is expected that in early times the presence of this perturbational effect played a fundamental role.

### 3.1. The evolving extrinsic scalar

As already pointed out, the quantity \(Q\) is an independent quantity and is defined without the need of existence of \(\Lambda\). In order to get an explicit form for the evolution of this quantity, we can estimate how the extrinsic scalar \(Q\) evolves as the Universe expands. Thus, using the equation (18), we can rewrite equation (13) as a function of the expansion factor \(a\) in...
terms of the Hubble constant and the current extrinsic parameter $\Omega_{\text{ext}}^0$ as
\[ Q(a) = 6H_0^2\Omega_{\text{ext}}^0(\kappa) a^{3/2} - 4, \]
(29)
One can obtain different solutions that basically depend on the signs from equation (26), which we use to denote the absolute value of $Q(a)$ as
\[ Q^+ = \omega(a)(\beta - \sqrt{4\beta_0 a^4 - 3}) \exp[+\gamma(a)] \]
\[ Q^- = \omega(a)(\beta + \sqrt{4\beta_0 a^4 - 3}) \exp[-\gamma(a)] \]
where we denote $\omega(a) = 6H_0^2\Omega_{\text{ext}}^0 a^{3/2}$. Moreover, from equation (29) one can obtain the resulting plots as shown in the upper and lower panels in figure (1). In both panels, the dashed line represents the solution $Q^+$, the thick line represents the solution $Q^-$, the thick-dashed line represents the solution $Q^{++}$, and those solutions vary around $10^{-3} \sim 10^{-84}$ GeV$^{-2}$. As shown in the upper panel in figure (1), we obtain the evolution of the absolute value of the extrinsic scalar $Q$ ranging from $a = 0.3$ to $a = 1$. In the lower panel, we extrapolate the results and they strongly suggest that the solutions presented induce some changing in topology of an asymptotical future Universe. Interestingly, solutions $Q^+$ and $Q^-$ present an increasing of the absolute value of $Q$. The former solution provides an earlier acceleration than the latter inducing to a more accelerated regime of the expanding Universe, since we are considering the extrinsic curvature the main cause of the accelerated expansion. On the other hand, $Q^+$ and $Q^-$ suggest that after the phase transition at $a \sim 1.3$ the absolute value of $Q$ will decay and both solutions seem to converge to value less than $10^{-84}$ GeV$^2$.

These results reinforce the idea that the extrinsic scalar $Q$ is a dynamical quantity that evolves in time, which is expected for an expanding Universe and is roughly of order of the physical CC and the Ricci scalar curvature. Such results have twofold considerations. First, the quantitative issue: the current value of $Q$ is quantitatively similar to the physical cosmological constant and, rather, is a dynamical quantity that dominates the cosmological constant term. Second, the qualitative issues must be accounted carefully. The extrinsic scalar $Q$ and CC have strikingly different meanings. The extrinsic scalar $Q$ represents the solution $Q^+$, which we use to denote the extrinsic scalar $Q$ ranging from $a = 0.3$ to $a = 1$. In the lower panel, we extrapolate the results and they strongly suggest that the solutions presented induce some changing in topology of an asymptotical future Universe. Interestingly, solutions $Q^+$ and $Q^-$ present an increasing of the absolute value of $Q$. The former solution provides an earlier acceleration than the latter inducing to a more accelerated regime of the expanding Universe, since we are considering the extrinsic curvature the main cause of the accelerated expansion. On the other hand, $Q^+$ and $Q^-$ suggest that after the phase transition at $a \sim 1.3$ the absolute value of $Q$ will decay and both solutions seem to converge to value less than $10^{-84}$ GeV$^2$.

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4. Final remarks
In a previous paper [4], we have used a model independent formulation based on the Nash embedding theorem where the extrinsic curvature is an independent variable required for the definition of the embedding. However, this comes at the price that the extrinsic curvature cannot be completely determined, because Codazzi’s equations become homogeneous (incidentally, the Randall–Sundrum model avoids this problem by imposing the Israel–Lanczos condition on a fixed boundary-like brane-world). Therefore, in order to restore the definition of the extrinsic curvature an additional equation compatible with a dynamically evolving embedded space-time is required. As a rank-2 symmetric tensor, the extrinsic curvature can be seen as a spin-2 field, which satisfies an Einstein-like equations constituting so-called the Gupta equations for the extrinsic curvature.

The present paper complements that result where the solution of these equations describes not only how the Universe presents an accelerated expansion but also on how it is inner related to the CC problem. At the present cosmic scale, we have shown that the extrinsic curvature balances the vacuum energy and the CC energy density as a consequence of the embedding. Thus, since the cosmological constant problem takes into account the fact that the gauge fields contributing to the vacuum energy are confined to the embedded space-time, the gravitational field, including the cosmological term is not. Therefore, a four-dimensional observer in the embedded space-time is able to perceive this difference through a conserved quantity built with the extrinsic curvature, whose effect is to induce a warp effect in the embedded geometry. A interesting point is that the extrinsic quantity $Q$ is a geometrical entity resulting from the extrinsic curvature and no prior ansätze were necessary. Interestingly, it may lead to some changing in topology of an asymptotical future Universe for both very accelerating or decelerating regimes. On the other hand, the extrinsic quantity $Q$ may have implications for nucleosynthesis epoch and must
be analyzed since it provides topological changing as expected for the earlier Universe. This topic will be a subject of future research.

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