JOINT DENSITY OF THE STABLE PROCESS AND ITS SUPREMUM: REGULARITY AND UPPER BOUNDS

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Abstract. This article uses a combination of three ideas from simulation to establish a nearly optimal polynomial upper bound for the joint density of the stable process and its associated supremum at a fixed time on the entire support of the joint law. The representation of the concave majorant of the stable process and the Chambers-Mallows-Stuck representation for stable laws are used to define an approximation of the random vector of interest. An interpolation technique using multilevel Monte Carlo is applied to accelerate the approximation, allowing us to establish the infinite differentiability of the joint density as well as nearly optimal polynomial upper bounds for the joint mixed derivatives of any order.

1. Introduction

Let \((X_t)_{t \geq 0}\) be a non-monotonic \(\alpha\)-stable process with \(\alpha \in (0, 2)\) and positivity parameter

\[
\rho := \mathbb{P}(X_1 > 0) \in [1 - 1/\alpha, 1/\alpha] \cap (0, 1).
\]

(1.1)

For any fixed \(T > 0\), denote by \(\overline{X}_T := \sup_{s \in [0, T]} X_s\) its supremum over the time interval \([0, T]\). Our main result, Theorem 1 below, provides the regularity and upper bounds for the joint density of \((X_T, \overline{X}_T)\) and its derivatives of any order. These explicit polynomial bounds, valid on the entire support set of the joint law, are nearly optimal. For a detailed explanation, see the discussion following Theorem 1.

The joint law of \((X_T, \overline{X}_T)\) arises in the scaling limit of many stochastic models, including queues with heavy-tailed workloads (see [DM15, §5.2] and the references therein). In such cases, the bounds in Theorem 1 are necessary for the construction of the asymptotic confidence intervals. Moreover, in some prediction problems (e.g. [BDP11]), regularity of the density of \(X_T - \overline{X}_T\), established in Theorem 1, is important.

Our approach is rooted in recent advances in simulation used to build an efficient approximation of the law of \((X_T, \overline{X}_T)\). More precisely, we use the representation of the concave majorant of a stable processes, recently applied in [GCMUB22] to construct a geometrically convergent simulation algorithm for sampling from the law of \((X_T, \overline{X}_T)\). In order to analyze the regularity of this joint law, we express the stable random variables arising in the concave majorant representation of the supremum \(\overline{X}_T\) via the classical Chambers-Mallows-Stuck representation. To the best of our knowledge, this approach to study the regularity and upper bounds of the densities of the joint law differs from the probabilistic and analytical techniques applied in this context in the literature so far (see YouTube [GCKHM22] for a short presentation of our results and techniques).

In general, it is well-known that the properties of the approximation do not necessarily persist in the limit (see [BR86] for a comprehensive study in the case of the central limit theorem). In our case, in order to establish regularity and achieve nearly optimal upper bounds of the limit law, we accelerate the convergence of the approximation procedure using ideas behind the multilevel Monte Carlo method. This method has been successfully applied in Monte Carlo estimation (see [Gil08] and the references in the webpage) to reduce the computational complexity of the algorithm for a pre-specified level of accuracy. In theoretical terms, we
apply the multilevel idea as an interpolation methodology (we have not been able to find multilevel Monte Carlo methods used for this purpose in the literature). Other interpolation techniques applied to stochastic equations are found in [BC16], see also the references therein. In fact, the authors in [BC16] use a different interpolation technique to obtain qualitative properties using approximation methods. In the examples they treat, it is hard to tell if they achieve optimal results. In our case, the near optimality is due to the geometrical convergence of the approximation of the joint law based on the concave majorant, see [GCMUB22].

To the best of our knowledge, only the regularity of the density of the marginals of \((X_T, \overline{X}_T)\) has been considered so far. The first component \(X_T\) follows a stable law, which is very well understood (see e.g. [Sat13] and the references therein). In fact, it is known that the density of \(X_T\) has the following asymptotic behavior

\[
\mathbb{P}(X_T \in dx) \xrightarrow{x \to \infty} T^x \rho x^{-\alpha} dx; \quad \mathbb{P}(X_T \in dx) \xrightarrow{x \to 0} T^{-1/\alpha} dx.
\]

Even though its law has been the focus of a number of papers over the past seven decades (starting with Darling [Dar56], Heyde [Hey69] and Bingham [Bin73]), far less information is available about the density of the second component, \(\overline{X}_T\), which is a functional of the path of a stable process. Most of the results about the law of the supremum of a Lévy process rely on the Wiener-Hopf factorization and/or the equivalence with laws related to excursions of reflected processes [CM16, CM21]. For example, in [Cha13], the author obtains explicit formulae for the supremum in the spectrally negative stable and symmetric Cauchy cases. The smoothness of the density of the supremum \(\overline{X}_T\) is known, see e.g. [PS18, Thm 2.4 & Rem. 2.14].

The papers [Don08, DS10] study the asymptotic behaviour of the density of the supremum at infinity and at zero. In [DS10], the authors rely on local times and excursion theory, the Wiener-Hopf factorisation and a distributional connection between stable suprema and stable meanders. Power series expansions of the density of \(\overline{X}_T\) have been established in [Kuz11, Kuz13] in some particular situations. Since stable processes are self-similar and Markov, results in [PS18] can be used to deduce the asymptotic behaviour of the density (and its derivatives) of \(\overline{X}_T\), see the paragraph following Corollary 2 below.

In short, the proofs of the results obtained so far in the literature rely on excursion theory or the Wiener-Hopf factorisation. These methods exploit the independence of \(\overline{X}_e\) and \(\overline{X}_e - X_e\) over an independent exponential time horizon \(e\). The dependence of all of the above methods on a number of specific analytical identities for the law of \(\overline{X}_T\) makes them hard to generalise to the law of \((X_T, \overline{X}_T)\).

The result closer to our study are the asymptotics established in [DS10, PS18]:

\[
\mathbb{P}(\overline{X}_T \in dy) \xrightarrow{y \to \infty} T y^{-\alpha - 1} dy; \quad \mathbb{P}(\overline{X}_T \in dy) \xrightarrow{y \to 0} T^{-\rho} y^{\alpha \rho - 1} dy.
\]

Taking into consideration the asymptotics in (1.2)–(1.3), it is natural that the asymptotics for the law of \((X_T, \overline{X}_T)\) are determined by four sub-domains in the support \(O := \{(x,y) \in \mathbb{R}^2 : y > \max(x,0)\}\). Our upper bound on the joint density and its derivatives, illustrated in Figure 2.1 below, is close to optimal in the sense that we obtain such a result for any \(\alpha'\) arbitrarily close to \(\alpha\) featuring in (1.2)–(1.3). The reason why we are unable to obtain the result for the choice \(\alpha' = \alpha\) is technical and due to the use of moments to bound tail behaviours in the spirit of Markov’s and Chebyshev’s inequalities.

Malliavin calculus is a long developed subject in the area of stochastic analysis of jump processes. The ultimate goal of the general theory is to obtain an infinite dimensional calculus with the view of investigating random quantities generated by the jump process and, in particular, the regularity of the law of path functionals of the process (see e.g. [BGJ87, NN18] for a general reference). Notably, these theoretical developments in Malliavin calculus have fallen short of the problem of the regularity of the density of \(\overline{X}_T\),
because the supremum of a jump process (as a random variable) appears not to depend smoothly on the underlying jumps. An exception is the result in [BD09], where the authors rely on the Lipschitz property of the supremum functional to prove the existence of a density for the supremum of a jump process in a general class, using the so-called lent-particle method. However, since $X_T$ is not a smooth functional of the path, it is unclear how to apply these methods to analyse the regularity and behavior of the density near the boundary of its support.

The approach used in this article does not fall in any of the above categories of Malliavin Calculus, nor does it rely on any results from Malliavin Calculus of jump processes. More precisely, we do not use infinite dimensional objects but only study limits of finite collections of random variables, arising in the noise used in our representation of the law of $(X_T, X_T)$. Our main underlying idea is to exploit the geometrically convergent approximation of the random vector of interest, establish the required properties of the densities for the approximate vectors and prove that these properties persist in the limit. In this sense, our approach is both self-contained and elementary.

More specifically, we establish a probabilistic representation for the joint density of $(X_T, X_T)$ and its derivatives in Theorem 7 below, based on a telescoping sum of successive approximations analogous to the multilevel method (cf. [Gil08]). The telescoping sum formula for the density and its derivatives is based on an elementary integration-by-parts formula for successive finite dimensional approximations of $(X_T, X_T)$. These approximations are not using the path of the stable process $(X_t)_{t \in [0,T]}$ directly as would be the case in Malliavin Calculus for processes with jumps. Instead, the concave majorant of $(X_t)_{t \in [0,T]}$, given in [PUB12, Thm 1], is used to represent $(X_T, X_T)$ as an infinite series [GCMUB22, GCMUB19]. The terms in this series are the increments of the stable process over macroscopic (but geometrically small) time steps given by an independent stick-breaking process on $[0, T]$ (for more details, see Section 3.1). We then build our elementary finite-dimensional integration-by-parts formulae for the partial sum approximations of $(X_T, X_T)$ using the scaling property of stable increments and their Chambers-Mallows-Stuck representation [Wer96], which in the non-Cauchy case $\alpha \neq 1$, amounts to a semi-linear function of independent uniform and exponential variables, Section 3.1.

1.1. Organisation. The remainder of the paper is organised as follows. In Section 2 we present Theorem 1, the main result of the paper, and some applications of these results. Subsection 3.1 introduces the technical notation for the proofs and Subsection 3.2 establishes the Ibpf. At the end of this section, we also give an important technical Proposition 8 which gives all the bounds needed in order to be applied in the Ibpf formula obtained. In Section 4, we give the proof of our main result, Theorem 1, using the ingredients developed in previous sections. This proof uses the interpolation method in the sense that the approximation method based on the convex majorant converges geometrically fast while the density bounds explode polynomially. Combining these two characteristics one obtains the almost optimal bounds.

We close the article with some technical appendices which prove the important technical Proposition 8. The proof of this proposition is composed of algebraic inequalities which are obtained in Subsection 5.2. The upper bounds are products of powers of basic random variables. After the proof we give also a heuristic interpretation of a basic interpolation technique used in the estimation of the moments. Finally, the moment estimates are obtained in Section 6. Throughout the article we concentrate on the case $\alpha \neq 1$ leaving the special Cauchy case, $\alpha = 1$ to the Appendix C.
Section 7 concludes the paper, remarking on our techniques and methodology as well as possible extensions. Appendices collect relevant bounds on the moments of a stick-breaking process and the moment generating function for powers of exponentially distributed random variables.

2. Main result and applications

As explained in the Introduction, we give first our main result:

**Theorem 1.** Assume that \( \alpha \in (0, 2) \). Let \( F(x, y) := \mathbb{P}(X_T \leq x, \overline{X}_T \leq y) \) be the distribution function of \((X_T, \overline{X}_T)\). The joint density of \( F \) exists and is infinitely differentiable on the open set \( \mathcal{O} \). Moreover, for any fixed \( n, m \geq 1 \) and \( \alpha' \in [0, \alpha) \) there is some \( C > 0 \) such that for all \( x, y > 0 \) and \( T > 0 \), we have

\[
|\partial_x^n \partial_y^m F(x, y)| \leq C y^{-m} (y-x)^{1-n-m} (2y-x)^{m-1} \\
\qquad \times \min \{ f^{00}_{\alpha'}(x, y), f^{01}_{\alpha'}(x, y), f^{10}_{\alpha'}(x, y), f^{11}_{\alpha'}(x, y) \},
\]

where \( f^{ij}_{\alpha'}(x, y) := T^{\alpha'(i(2-\rho)+j(1+\rho)-1)} (y-x)^{\alpha'(1-\rho)-i\alpha'(2-\rho)} y^{\alpha'-i\alpha'(1+\rho)} \) for \( i, j \in \{0, 1\} \).

Theorem 1 presents a bound on the mixed derivatives of the joint density of \((X_T, \overline{X}_T)\). The decay of the bound as \( y \) tends to either infinity or zero is almost sharp in the following sense: if one sets \( n = 1 \) and \( \alpha' = \alpha \) in (2.1) (cf. Figure 2.1 below) and integrates out \( x \) over \( \mathbb{R} \), the decay of the obtained bound matches the actual asymptotic behaviour of the density of \( \overline{X}_T \) known from the literature [DS10, Kuz11, Kuz13]. That is, marginals of the above bounds match the estimates in (1.2) and (1.3). In fact, the bound in Corollary 2 below is established in this way. The constant \( C \) in (2.1) can be made explicit. Instead of giving a formula for \( C \), which would be lengthy and suboptimal (cf. Remark 3(i) below), we point out that \((\alpha - \alpha')C\) remains bounded as \( \alpha' \uparrow \alpha \). An alternative way to understand the optimality property is through a change of variables in equation (3.1) which will be proven in Section 4.

**Figure 2.1.** The set \( \mathcal{O} = \{(x, y) \in \mathbb{R}^2 : y > \max\{x, 0\}\} \) (shaded in the figure) is the support of the joint density of \((X_T, \overline{X}_T)\). According to Theorem 1, the support can be partitioned into 4 sub-regions according to which of the functions \( f^{ij}_{\alpha'} \), \( i, j \in \{0, 1\} \), is the smallest in the (optimal) case \( \alpha' = \alpha \).

Theorem 1 above suggests that the asymptotic behaviour of the joint density at \((x, y)\) of \((X_T, \overline{X}_T)\) as \( T \to 0 \) is proportional to \( T^{2(\gamma - x)} - \alpha y^{-\alpha} \), see Figure 2.1. This is corroborated by the results in [Cha13, CM21] as we now explain. Recall from [Cha13, Thm 6] that the density of \((\overline{X}_T, \overline{X}_T - X_T)\) satisfies

\[
\mathbb{P}(\overline{X}_T \in dx, \overline{X}_T - X_T \in dy) = dx dy \int_0^1 q^*_T(x) q_{(1-s)T}(y) T ds,
\]

where

\[
q^*_T(x) = \frac{1}{T} \int_0^T f^{11}_{\alpha'}(x, y) dy.
\]
where \( q^*_t \) (resp. \( q_t \)) is the entrance density of the excursion measure of the reflected process of \( X \) (resp. \( -X \)).

By [CM21, Thm 3.1] and [Cha13, Ex. 3], we deduce that, as \( T \to 0 \), the quantities \( q^*_t(x)/(T^{\rho} s^{1/\rho} y^{-\alpha - 1}) \) and \( q^*_t(1-s)y(x)/(T^{1-\rho} s^{1-\rho} y^{-\alpha - 1}) \) have positive finite limits that depend neither on \( s \) nor \( (x, y) \). Thus the integral on the right-hand side of (2.2) is proportional to \( T^2 x^{-\alpha - 1} y^{-\alpha - 1} \) as predicted the bound in Theorem 1 (see also (3.1) below).

Setting \( n = 1 \) and explicitly integrating in \( y \) over \((0, \infty)\) yields the following bounds.

**Corollary 2.** Assume that \( \alpha \in (0, 2) \). Then the distribution function \( F(y) := \mathbb{P}(X_T \leq y) \) is infinitely smooth on \((0, \infty)\) and, for every \( \alpha' \in [0, \alpha) \) and \( n \geq 1 \), there exists some constant \( C > 0 \) such that for all \( y > 0 \) and \( T > 0 \), we have

\[
|\partial^n_y F(y)| \leq C y^{-n} \min \left\{ T^{\frac{n}{\alpha}} y^{-\alpha'}, T^{-\frac{n}{\alpha'}} y^{\alpha'} \right\}.
\]

Define \( \tau_{y_0} := \inf \{ t > 0 : X_t > y_0 \} \), \( y_0 > 0 \). Then the distribution function of \( \tau_{y_0} \) is infinitely smooth on \((0, \infty)\) and the following estimate is satisfied for \( n \geq 1 \):

\[
|\partial^n_T \mathbb{P}(\tau_{y_0} \leq T)| \leq C T^{-\frac{n}{\alpha}} \times \min \{ T^{\frac{n}{\alpha'}} y_0^{-\alpha'}, 1 \}.
\]

It has been pointed out to us [Sav20] that the bound in Corollary 2 for \( \alpha' = \alpha \) can be obtained from the literature. By studying the Mellin transform of \( \mathbb{X}_T \) [PS18, Thm 2.4] (via a distributional identity linking \( \mathbb{X}_T \) to an exponential integral arising in the Lamperti representation of self-similar Markov processes [PS18, Rem. 2.14]), one obtains the asymptotic behaviour in (1.3). Similar bounds can be obtained for the derivatives of the density, implying Corollary 2.

Other consequences of our main Theorem 1 can also be derived such as the following result which reveals a complex interplay between the final value of the stable process and its supremum in the interval \([0, T]\).

**Corollary 3.** Assume that \( \alpha \in (0, 2) \) and let \( y_0 \geq T^{1/\alpha} \), \( x_0 \leq 0 \). Then for any \( \alpha' \in (0, \alpha) \)

\[
\mathbb{P}(X_T \leq x_0, \tau_{y_0} < T) \leq C T^{2\frac{n}{\alpha}} y_0^{-\alpha'} \times \min \{ y_0^{-\alpha'}, (-x_0)^{-\alpha'} \}
\]

**Proof.** The inequalities are obtained by direct integration of the bound in Theorem 1. That is,

\[
\mathbb{P}(X_T \leq x_0, \mathbb{X}_T > y_0) \leq C T^{2\frac{n}{\alpha}} \int_{\frac{1}{y_0}}^{\infty} w^{-1-\alpha'} (1 + w)^{-\alpha'} dw.
\]

From here, the result follows. \( \square \)

We conclude the section by remarking on the excluded cases: our methods apply to the Brownian motion case \( \alpha = 2 \), but the result is not relevant since the density of \( (X_T, \mathbb{X}_T) \) is known explicitly; in (1.1) we exclude \( \rho \in \{0, 1\} \) as in those cases the monotonicity of paths implies \( \mathbb{X}_T = X_T \) (resp. \( \mathbb{X}_T = X_0 \)) a.s. if \( \rho = 1 \) (resp. \( \rho = 0 \)).

3. **Tools: Approximation method and sequential IBPF**

In order to avoid cumbersome multiple case studies, we will assume \( \alpha \neq 1 \) from now on until the last section in the Appendix where the appropriate changes for the case \( \alpha = 1 \) will be explained.
3.1. Approximation method for \((X_T, \overline{X}_T)\). Throughout the article, we fix \(T > 0\) and we will use the following decomposition of the random variable \(X_+ := \overline{X}_T\) and \(X_-\) which denote the supremum of \((X_t)_{t \in [0, T]}\) and its reflected process \(X_- := X_+ - X_T\). Therefore instead of working with \((X_T, \overline{X}_T)\), we will use \((X_+, X_-)\) whose law is supported in \(\mathbb{R}_+^2\). This \(\pm\) notation will be useful in order to write dual formulas that are valid for both random variables \(X_\pm\).

In fact, the proof of Theorem 1 studies the equivalent pair \((X_+, X_-)\), instead of \((X_T, \overline{X}_T)\), and shows the following: let \(\bar{F}(x, y) := \mathbb{P}(X_+ \leq x, X_- \leq y)\), then for any \(\alpha' \in [0, \alpha)\) and \(n, m \geq 1\) there exists some constant \(C > 0\) such that for any \(T, x, y > 0\) we have
\[
(3.1) \quad |\partial_x^n \partial_y^m \bar{F}(x, y)| \leq C x^{-n} y^{-m} \min \left\{ T \frac{\alpha}{\alpha'} x^{-\alpha'}, T^{-\alpha} x^{\alpha'} \right\} \min \left\{ T \frac{\alpha}{\alpha'} y^{-\alpha'}, T^{-\alpha} y^{\alpha'} \right\}.
\]

For this reason, we will use in many formulas multiple \(\pm\) and \(\mp\) signs. It is assumed that the signs match, i.e., all \(\pm\) are \(+\) (resp. \(-\)) and all \(\mp\) are \(-\) (resp. \(\mp\)) simultaneously. For example, \(A_{\pm} = \mp B_{\mp}\) if and only if \(A_+ = -B_-\) and \(A_- = +B_+\). Additionally, we use the notation \([x]^\pm = \max\{x, 0\}\) and \([x]^\mp = \max\{-x, 0\}\). We stress that if the brackets are not present, then the notation refers to a different object. For example, \(X_{\pm, n}\) denote the approximations for \(X_\pm\) respectively and \(D_n^\pm\) are the associated operators to be defined below. Finally, we denote \(x \wedge y = \min\{x, y\}\) and \(x \vee y = \max\{x, y\}\).

We will use an approximation method for the pair \((X_T, \overline{X}_T)\) used in [GCMUB22, §4.1] (see also [GCMUB19, Eq. (2.2)] and [PUB12, Thm 1]) which is based on the concave majorant of \(X\), see Figure 3.1.

**Figure 3.1.** Randomly selecting the first three faces of the concave majorant \(C\) of \(X\) (the smallest concave function dominating the path of \(X\)) in a size-biased way. The total length of the thick blue segment(s) on the abscissa equal the stick remainders \(L_0 = T, L_1 = T - \ell_1\) and \(L_2 = T - \ell_1 - \ell_2\), respectively, where \(\ell_1 = d_1 - g_1\) and \(\ell_2 = d_2 - g_2\). The independent random variables \(V_1, V_2, V_3\) are uniform on the sets \([0, T], [0, T] \setminus (g_1, d_1), [0, T] \setminus \bigcup_{i=1}^2 (g_i, d_i)\), respectively. The interval \((g_i, d_i)\), is determined by the edge of the concave majorant which includes \(V_i\). By [GCMUB22, §4.1], this procedure yields a stick-breaking process \(\ell\) and, conditionally given \(\ell\), the increments \(C(d_i) - C(g_i)\) are independent with the same law as \(X_t\) at \(t = \ell_i\), i.e., \(C(d_i) - C(g_i) \overset{d}{=} \ell_i^{1/\alpha} S_i\).

The procedure starts by constructing a random sequence of disjoint sub-intervals of the time interval \([0, T]\) which will cover it geometrically fast. This is usually called a stick-breaking process: \(\ell = (\ell_i)_{i \geq 1}\) on the interval \([0, T]\). That is, based on the i.i.d. standard uniform random variables \(U_i \sim U(0, 1)\), define \(L_0 := T\) and for each \(i \in \mathbb{N}\), \(L_i := L_{i-1} U_k\) and \(\ell_i = L_{i-1} - L_i = L_{i-1} (1 - U_i) = T (1 - U_i) \prod_{j=1}^{i-1} U_j\). It is not difficult
to see that \( \sum_{i=1}^{\infty} \ell_i = T \) and that for any \( p > 0 \), \( E[\ell_i^p] = T^p (1 + p)^{-i} \). That is, the convergence of the total length of the sequence of disjoint intervals \( \bigcup_{j=1}^{\infty} [L_{j-1}, L_j] \) to \( T \) is geometrically fast.

Now, we define the Chambers-Mallows-Stuck approximation for stable laws. We need to define a sequence of them in order to approximate \( X_+ \). For an independent i.i.d. sequence of stable random variables \( (S_i)_{i \geq 1} \) with parameters \( (\alpha, \rho) \) (i.e. \( S_i \sim X_1 \)). When \( \alpha \neq 1 \), these stable random variables can be represented as (see [Wer96])

\[
S_i = E_i^{1-1/\alpha} G_i \quad \text{and} \quad G_i = g(V_i), \quad i \in \mathbb{N},
\]

for i.i.d. exponential random variables \( (E_i)_{i \geq 1} \) with unit mean independent of the i.i.d. \( U(-\frac{\pi}{2}, \frac{\pi}{2}) \) random variables \( (V_i)_{i \geq 1} \) and function

\[
g(x) := \frac{\sin \left( \alpha (x + \omega) \right)}{ \cos^{1/\alpha}(x) \cos^{1-1/\alpha} \left( (1 - \alpha)x - \alpha \omega \right)}, \quad x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right),
\]

where \( \omega := \pi(\rho - \frac{1}{2}) \). Note that indeed \( \mathbb{P}(S_i > 0) = \rho \). We assume that all the above random variables are defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). These random elements and the coupling in [GCMUB22, §4.1] provide an almost sure representation for \((X_T, \overline{X}_T)\):

\[
(3.3) \quad \overline{X}_T = X_+ \quad \text{and} \quad X_T = X_+ - X_-, \quad \text{where} \quad X_\pm := \sum_{i=1}^{\infty} \ell_i^{1/\alpha}[S_i]^{\pm}.
\]

The series in the definitions of \( X_+ \) and \( X_- \) have non-negative terms and converge almost surely by the equalities in (3.3). Note again, that the convergence in the above infinite sum is “geometrically fast” due to the behavior of the stick breaking process.

As stated in the Introduction, we will base our finite dimensional integration by parts formulas using the exponential random variables \( E_i \) which characterize heuristically the “length” of the stable random variable \( S_i \) while the “oscillating” part \( G_i \) will not be used in order to determine the regularity of the law and therefore all calculations will be conditioned on this random sequence. This key observation makes possible our analysis.

In order to build approximations of the above random variables on finite dimensional spaces with smooth laws, we will truncate the infinite sums up to the \( n \)-th term. With this in mind and in order to preserve the existence of densities, we replace the remainder with \( a_n \eta_{\pm} \) as follows: let \( (a_n)_{n \in \mathbb{N}} \) be a positive and strictly decreasing sequence defined as \( a_n := T^{1/\alpha} \kappa^n \) with \( \kappa \in (0,1) \). Therefore \( a_n \downarrow 0 \) as \( n \to \infty \). The random variables \( \eta_{\pm} \) are exponentially distributed with unit mean independent of each other and of every other random variable. With these elements we define the \( n \)-th approximation to \( \chi = (X_+, X_-) \) as \( \chi_n = (X_{+,n}, X_{-,n}) \), \( n \in \mathbb{N} \) given by

\[
(3.4) \quad X_{\pm,n} := \sum_{i=1}^{n} \ell_i^{1/\alpha}[S_i]^{\pm} + a_n \eta_{\pm}^{1-1/\alpha} = \sum_{i=1}^{n} \ell_i^{1/\alpha} E_i^{1-1/\alpha}[G_i]^{\pm} + a_n \eta_{\pm}^{1-1/\alpha}.
\]

In the case \( n = 0 \), we define \( X_{\pm,0} := 0 \).

We introduce the following assumption which will be valid throughout the paper.

**Assumption (A-\( \kappa \)).** The constant \( \kappa \in (0,1) \) in \( a_n = T^{1/\alpha} \kappa^n \) satisfies \( \kappa^\alpha \geq \rho \lor (1 - \rho) \).

This assumption is crucial in order to obtain good positive and negative moment estimates for \( X_{\pm,n} \) within the bounds allowed by stable laws (see Lemma 11).

For any \( m \in \mathbb{N} \), \( n \in \mathbb{N} \cup \{\infty\} \) and \( A \subset \mathbb{R}^m \), let \( C^m_n(A) \) be the set of bounded and \( n \)-times continuously differentiable functions \( f : \mathbb{R}^m \to \mathbb{R} \) on the open set \( A \) and whose derivatives of order at most \( n \) are all
bounded. Furthermore for \( f \in C_b^1(\mathbb{R}^2) \) we denote the partial derivatives with respect to the first and second component by \( \partial_+ f \) and \( \partial_- f \), respectively.

### 3.2. Sequential integration by parts formulae via a multilevel method

In order to state the finite dimensional Ibpf based on exponential random variables, we will use a derivative operator notation with respect to this set of random variables. Thus, for any random variable \( F = f(\vartheta, \mathcal{K}) \), where \( f \) is differentiable in the first component and the random variable \( \vartheta \) is independent of the random element \( \mathcal{K} \), the derivative \( \partial_\vartheta f \) is well-defined and given by the formula \( \partial_\vartheta [F] = \partial_\vartheta f(\vartheta, \mathcal{K}) \). As stated above, the random variables \( \{E_i, U_i, V_i, \eta_\pm ; i \in \mathbb{N}\} \) are independent (i.e. the joint law is a product measure), making the derivatives in the following lemma well-defined. We start stating some basic properties of the differential operator which will be used in our arguments.

#### Lemma 4

For any \( m \in \mathbb{N} \), define the differential operators

\[
D_m^\pm := \eta_\pm \partial_\eta_\pm + \sum_{i=1}^{m} E_i 1_{\{[G_i]^{\geq 0}\}} \partial_{E_i}.
\]

Then for any function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and \( p \in \mathbb{R} \setminus \{0\} \) we have

\[
E_i \partial_{E_i}[X_{\pm,n}] = (1 - 1/\beta_i)^{1/\beta_i} E_i^{1-1/\beta_i} [G_i]^{\pm} 1_{\{i \leq n\}}, \quad k \in \mathbb{N},
\]

\[
D_m^\pm [(X_{\pm,n}^p, f(X_{\pm,n}))] = (1 - 1/\beta_i)(pX_{\pm,n}^p, 0), \quad m \geq n \geq 1.
\]

**Proof.** The first two identities follow easily. For the third identity, note that \( X_{\pm,n} > 0 \) a.s. and thus, its reciprocal and any of its powers are always well defined real numbers. The other identities follow from the first one and the corresponding formula for \( \eta_\pm \partial_\eta_\pm [X_{\pm,n}] \). \( \square \)

**Remark.**

1. The identity \( D_m^\pm X_{\pm,n}^p = (1 - 1/\beta_i)pX_{\pm,n}^p, m \geq n \geq 1 \), in (3.6) reveals a crucial regenerative property of \( X_{\pm,n} \) with respect to the operator \( D_m^\pm \) (like the fact that in classical calculus the derivative of the exponential function is itself). In fact, this is the main motivation behind the definition of \( D_m^\pm \).

This regenerative structure relies heavily on the particular dependence of \( X_{\pm,n} \) with respect to \( S_i \) and \( E_i, i \in \{1, \ldots, n\} \).

2. The indicators \( 1_{\{[G_i]^{\geq 0}\}} \) in the definition of \( D_m^\pm \) ensure that when applied to \( f(\chi_n) \), only one of the partial derivatives of \( f \) appear due to (3.6) (see (3.8) below).

Now, we introduce the space of smooth random variables. Given any metric space \( S \), define the space of real-valued bounded and continuous on \((0, \infty)^m \times S\) that are \( C_b^\infty \) in its first \( m \) components

\[
S_\infty((0, \infty)^m, S) := \{ \phi : (0, \infty)^m \times S \rightarrow \mathbb{R}; \phi \text{ is continuous}, \phi(\cdot, s) \in C_b^\infty((0, \infty)^m; \mathbb{R}), \forall s \in S \}.
\]

Then we define

\[
S_m(\Omega) := \{ \Phi \in L^p(\Omega): \exists \phi(\cdot, \vartheta) \in S_\infty((0, \infty)^{3m+2}, S), \Phi = \phi(E_m, U_m, V_m, \eta_+, \eta_-), \vartheta \},
\]

where \( E_m := (E_1, \ldots, E_m), U_m := (U_1, \ldots, U_m), V_m := (V_1, \ldots, V_m) \) and \( \vartheta \) is any random element in some metric space \( S \) independent of \( (E_m, U_m, V_m, \eta_+, \eta_-) \). For instance, if the random variable \( \Phi \) is a function of \( (E_\infty, U_\infty, V_\infty) \), we say that \( \Phi \in S_m(\Omega) \) if the property defining this set is satisfied with \( \theta = ((E_{m+1}, E_{m+2}, \ldots), (U_{m+1}, U_{m+2}, \ldots), (V_{m+1}, V_{m+2}, \ldots)) \) representing all the random variables with indices larger than \( m \). We describe now the following finite dimensional Ibpf for a fixed approximation parameter \( n \).

Recall that \( \chi_n = (X_{+,n}, X_{-,n}) \) for \( n \in \mathbb{N} \).
Proposition 5. Fix \( n, m \in \mathbb{N} \) with \( m \geq n \). Then for any \( \Phi \in \mathcal{S}_m(\Omega) \) and \( f \in C_b^1(\mathbb{R}) \),
\[
E[\partial_{\pm} f(\chi_n) \Phi] = E[f(\chi_n)H_{n,m}^\pm(\Phi)], \quad \text{where}
\]
\[
H_{n,m}^\pm(\Phi) := \frac{1}{X_{\pm,n}^-\alpha - 1} \left( \eta_\pm - \frac{1}{\alpha} \sum_{i=1}^m (E_i - 1) \mathbb{1}_{(\{G_i\}^\pm > 0)} \Phi - D_m^\pm(\Phi) \right) \in \mathcal{S}_m(\Omega).
\]

Proof. Note that \( |x|^\pm > 0 \) if and only if \( \pm x > 0 \). The chain rule for derivatives and (3.6) yield
(3.8)
\[
D_m^\pm[f(\chi_n)] = \partial_{\pm} f(\chi_n) D_m^\pm[X_{\pm,n}] = (1 - 1/\alpha) \partial_{\pm} f(\chi_n) X_{\pm,n}.
\]
Denote \( \tilde{D}_0[Y] := Y - \partial_0[Y] \). Let \( \eta \) be an exponential random variable with unit mean. Observe that if \( \Lambda_i := h_i(\eta) \) for some \( h_i \in C_\infty((0, \infty); \mathbb{R}) \), \( i \in \{1, 2\} \), then the classical Ibpf (with respect to the density of \( \eta \)) gives
(3.9)
\[
E[\Lambda_1 \eta \partial_\eta[\Lambda_2]] = E[\partial_\eta[\Lambda_1 \Lambda_2 \eta] - \Lambda_2 \partial_\eta[\Lambda_1 \eta]] = E[\Lambda_1 \Lambda_2 \eta - \Lambda_2 \partial_\eta[\Lambda_1 \eta]] = E[\Lambda_2 \tilde{\partial}_\eta[\Lambda_1 \eta]].
\]
Integration by parts with respect to \( \eta_\pm \) and \( E_k \) for each \( i \leq n \) gives, by (3.6), (3.8) and (3.9),
\[
E[\partial_{\pm} f(\chi_n) \Phi | \mathcal{F}_E] = \frac{\alpha}{\alpha - 1} E \left[ \frac{\Phi}{X_{\pm,n}} D_m^\pm[f(\chi_n)] | \mathcal{F}_E \right]
\]
\[
= \frac{\alpha}{\alpha - 1} E \left[ f(\chi_n) \left( \partial_{\eta_\pm} \left[ \frac{\Phi \eta_\pm}{X_{\pm,n}} \right] + \sum_{i=1}^m \partial_{E_i} \left[ \frac{\Phi E_i \mathbb{1}_{\{G_i\}^\pm > 0}}{X_{\pm,n}} \right] \right) \right] | \mathcal{F}_E
\]
(3.10)
\[
= E[f(\chi_n)H_{n,m}^\pm(\Phi) | \mathcal{F}_E].
\]
Here we have denoted by \( \mathcal{F}_E \) the \( \sigma \)-algebra generated by all but the exponential random variables \( \eta_+, \eta_- \) and \( E_i \), \( i \in \mathbb{N} \). Taking expectations in (3.10) completes the proof. \( \square \)

Remark 2. (1) Observe that the role of \( \varepsilon \) in the previous result is to ensure that the expectation on the right-hand side in (3.7) is finite (by making the quotient \( f(\chi_n)/X_{\pm,n} \) bounded).
(2) Recall that exponential laws are discontinuous at zero. Still, in the above Ibpf, these boundary terms do not appear. This is due to the factors \( E_i \partial_{E_i} \) and \( \eta_\pm \partial_{\eta_\pm} \) which appear in the definition of \( D_m^\pm \) in (3.5). In exchange, one has \( X_{\pm,n} \) in the denominator of the expression for \( H_{n,m}^\pm(\Phi) \).

As \( H_{n,m}^\pm(\Phi) \in \mathcal{S}_m(\Omega) \) for any \( \Phi \in \mathcal{S}_m(\Omega) \), \( m \geq n \), we inductively define the sequence of operators \( \{H_{n,m}^{\pm,k}(\cdot)\}_{k \in \mathbb{N}} \) for every \( n, m \in \mathbb{N} \) such that \( m \geq n \) as
\[
H_{n,m}^{\pm,k+1}(\Phi) := H_{n,m}^{\pm,k}(H_{n,m}^\pm(\Phi)) \quad \text{for } k \geq 0, \quad \text{where} \quad H_{n,m}^{\pm,0}(\Phi) := \Phi.
\]
Let us state some basic properties of the weights \( H_{n,m}^\pm(\Phi) \).

Lemma 6. (1) If \( \alpha \neq 1 \) and \( \Phi \) do not depend on \( E_m \) or \( \eta_\pm \), then we have \( D_m^\pm[\Phi] = 0 \) and hence \( H_{n,m}^\pm(\Phi) = H_{n,m}^{\pm,1}(\Phi) \).
(2) The operators \( H_{n,m}^{\pm,k}(\cdot) \) and \( H_{n,m}^{\pm,j}(\cdot) \) commute.

These iterated operators are useful in order to define the multiple Ibpf formulas for the limit random variables in combination with the so-called Multi level Monte Carlo method which can be interpreted as an interpolation formula which uses approximations in order to describe the behavior of the limit. This is done in the next result.
Theorem 7. Let $\Phi \in S_n(\Omega)$ for all $n \in \mathbb{N}$. For any $n \geq 1$, $k_+, k_- \geq 0$ and $f \in C^ {k_+ + k_-}_ b([\varepsilon, \infty)^2)$ we have

\begin{equation}
\mathbb{E}[\partial_+^{k_+} \partial_-^{k_-} f(\chi) \Phi] = \mathbb{E}[\langle f, \Phi \rangle_{n+k_+ + k_-}]
\end{equation}

\begin{equation}
\langle f, \Phi \rangle_{n+k_+ + k_-} := f(\chi_n)H_{n,n}^{+,k_+}(H_{n,n}^{-,k_-}(\Phi)) + \sum_{i=n}^{\infty} (f(\chi_{i+1})H_{i,i+1}^{+,k_+}(H_{i,i+1}^{-,k_-}(\Phi)) - f(\chi_i)H_{i,i+1}^{+,k_+}(H_{i,i+1}^{-,k_-}(\Phi)))
\end{equation}

Proof. Note that $\mathbb{E}[\tilde{f}(\chi_n)] \to \mathbb{E}[\tilde{f}(\chi)]$ as $n \to \infty$ for any bounded and continuous function $\tilde{f}$ since $\chi_n \to \chi$ a.s. Recall that $\partial_+^{k_+} \partial_-^{k_-} f$ is continuous and bounded. By telescoping we find

$$
\mathbb{E}[\partial_+^{k_+} \partial_-^{k_-} f(\chi) \Phi] = \mathbb{E}[\partial_+^{k_+} \partial_-^{k_-} f(\chi_n) \Phi] + \mathbb{E}\left[\sum_{i=n}^{\infty} (\partial_+^{k_+} \partial_-^{k_-} f(\chi_{i+1}) - \partial_+^{k_+} \partial_-^{k_-} f(\chi_i)) \Phi\right].
$$

The first term equals $\mathbb{E}[f(\chi_n)H_{n,n}^{+,k_+}(H_{n,n}^{-,k_-}(\Phi))]$ by Proposition 5. Applying Proposition 5 again shows that each term in the above sum equals its corresponding term in (3.12), yielding (3.11). \qed

It is clear that iterations of $H_{n,m}^{+,k_+}$ have long and complex explicit expressions. In particular, the remaining goal is to find proper bounds for the iterated operators which appear in formula (3.11).

In order to complete the arguments for our main proofs we will need that the infinite sum appearing in (3.11) converges absolutely. Furthermore bounding this sum becomes important in obtaining upper bounds for the joint density and its derivatives. This is all done at once in the next lemma. Its proof is technical but only uses basic algebra and moments of the random variables involved in Section 3.1.

We are interested in the explicit decay rate of the terms in the sum of Theorem 7 for a special class of functions $f$ related to the distribution of $\chi$. This description will then be used to finally prove Theorem 1. More precisely, given some measurable and bounded $h : \mathbb{R}_+^2 \to \mathbb{R}$ and $x_+, x_- > 0$, we will consider the function $f$ given by

\begin{equation}
f(x,y) := \int_0^x \int_0^y h(x', y') 1_{\{x' > x_+, y' > x_-\}} dy' dx', \quad x, y \in \mathbb{R}_+.
\end{equation}

We are interested in such class of functions since the particular choice $h = 1$ yields $\mathbb{E}[\partial_+ \partial_- f(\chi)] = \mathbb{P}(X_+ > x_+, X_- > x_-)$. Note also that for a general $h$, the inequality $|f(x,y)| \leq ||h||_{\infty}xy$ holds for any $x, y \in \mathbb{R}_+$, where $||h||_{\infty} := \sup_{x, y \in \mathbb{R}_+} |h(x,y)|$. We denote by $\mathcal{A}(K, x_+, x_-)$, $K > 0$, the class of functions $f$ satisfying (3.13) for some measurable function $h : \mathbb{R}_+^2 \to \mathbb{R}$ with $||h||_{\infty} \leq K$.

We denote the random variables arising in $\langle f, \Phi \rangle_{n+k_+ + k_-}^l$ of Theorem 7 by

\begin{equation}
\begin{aligned}
\Theta_{n,m}^l &\equiv \Theta_{n,m}^l(k_+, k_-) := f(\chi_n)H_{n,m}^{+,k_+}(H_{n,m}^{-,k_-}(\Phi)), \quad \text{for } m \geq n, \\
\tilde{\Theta}_n^l &\equiv \tilde{\Theta}_n^l(k_+, k_-) := \Theta_{n+1,n+1}^l(k_+, k_-) - \Theta_{n,n+1}^l(k_+, k_-).
\end{aligned}
\end{equation}

We will drop $\Phi$ and or $(k_+, k_-)$ from the notation if it is well understood from the context. The following key result provides bounds on moments.

**Proposition 8.** Let $\kappa \in (0,1)$ be as in Assumption (A-$\kappa$). Fix any $p \geq 1$, $k_+ \geq 2$ and $\alpha' \in [0, \alpha)$. Given some $\phi \in C^ {k_+ + k_-}_ b(\mathbb{R}^2)$, define $\Phi := \phi(\chi)$. Let the variables $\Theta_{n,m}^l$ and $\tilde{\Theta}_n^l$ be given by (3.14), then the following statements hold.
(a) For $s := p \wedge \alpha'$ there is a constant $C > 0$ such that for any $K, T, x_+, x_- > 0$ and $m \geq n$:

$$
(3.15) \quad \mathbb{E} \left[ \sup_{f \in A(K, x_+, x_-)} |\Theta_{n, m}^f|^p \right] \leq CK^p \frac{T^{2n/p} (1 + \frac{n}{m})^{-n} + \kappa^{n/m}}{x_+^{p(k_+ - 1) + \alpha'} x_-^{p(k_- - 1) + \alpha'}}.
$$

(b) Consider any $u \in (0, (\alpha - \alpha')(\rho \wedge (1 - \rho))/p)$ and let $p' = p(k_+ + k_-)$, then for some $C > 0$ and all $K, T, x_+, x_- > 0$ and $m \geq n$, the following inequalities hold:

$$
(3.17) \quad \mathbb{E} \left[ \sup_{f \in A(K, x_+, x_-)} |\Theta_{n, m}^f|^p \right] \leq CK^p \frac{T^{-\frac{n}{m}} (1 + \frac{n}{m})^{-n} + \kappa^{n/p} m^{p'}}{x_+^{p(k_+ - 1) - \alpha' \rho} x_-^{p(k_- - 1) - \alpha'(1 - \rho)}},
$$

$$
(3.18) \quad \mathbb{E} \left[ \sup_{f \in A(K, x_+, x_-)} |\Theta_{n, m}^f|^p \right] \leq CK^p \frac{T^{-\frac{n}{m}} m^{p'}}{x_+^{p(k_+ - 1) - \alpha' \rho} x_-^{p(k_- - 1) - \alpha'(1 - \rho)}}.
$$

(c) Consider any $u \in (0, (\alpha - \alpha')(\rho \wedge (1 - \rho))/p)$ and let $p' = p(k_+ + k_-)$, then for some $C > 0$ and all $K, T, x_+, x_- > 0$ and $m \geq n$, the following inequalities hold:

$$
(3.19) \quad \mathbb{E} \left[ \sup_{f \in A(K, x_+, x_-)} |\Theta_{n, m}^f|^p \right] \leq CK^p \left( 1 + \frac{n}{m} \right)^{-n} + \kappa^{n/p} m^{p'} \min \left\{ \frac{T^{-\frac{n}{m}} (1 - \rho)}{x_+^{\alpha' \rho} x_-^{\alpha'(1 - \rho)}}, \frac{T^{-\frac{n}{m}}}{x_+^{\alpha' \rho} x_-^{\alpha'(1 - \rho)}} \right\},
$$

$$
(3.20) \quad \mathbb{E} \left[ \sup_{f \in A(K, x_+, x_-)} |\Theta_{n, m}^f|^p \right] \leq CK^p \min \left\{ \frac{T^{-\frac{n}{m}} (1 - \rho)}{x_+^{\alpha' \rho} x_-^{\alpha'(1 - \rho)}}, \frac{T^{-\frac{n}{m}}}{x_+^{\alpha' \rho} x_-^{\alpha'(1 - \rho)}} \right\}.
$$

Remark 3. (i) Clearly the above inequalities imply the absolute convergence of the infinite sum in (3.11).

(ii) The reason for the different cases is that we will use (a) when $x_+$ and $x_-$ both take large values, part (b) when they are both small and part (c) for the mixed case in which $x_+$ is small and $x_-$ is large or vice versa, cf. Figure 2.1.

The proof of this key technical result is given in Section 6. With these preparations, now we are ready to give the proof of our main result.

4. Proof of Theorem 1

In the present subsection we will prove Theorem 1. We will follow the structure presented in the proof of Theorem 2.1.4 in [Nua06]. In fact, consider a test function $f \in C^\infty_b(\mathbb{R}^2)$ then a similar representation as (3.13) gives for $F(x, y) := [x - x_+]^+ [y - x_-]^{-}$

$$
f(x_+, x_-) = \int_{\mathbb{R}_+^2} F(x', y') \partial_+ \partial_- f(x', y') dy' dx'.
$$

Next, using Theorem 7 and Fubini theorem with $\hat{F}(x', y') = [x' - X_+]^+ [y' - X_-]^{-}$, we obtain

$$
\mathbb{E} [\partial_+ \partial_- f(\chi)] = \mathbb{E} [(f, 1)_{n, 1}^{1, 1}] = \int_{\mathbb{R}_+^2} \partial_+ \partial_- f(x', y') \mathbb{E} \left[ \langle \hat{F}(x', y'), 1 \rangle_{n, 1}^{1, 1} \right] dy' dx'.
$$

This readily implies that the density of $\chi$ at $(x', y')$ exists and can be expressed as $\mathbb{E} \left[ \langle \hat{F}(x', y'), 1 \rangle_{n, 1}^{1, 1} \right]$.

In a similar fashion, one considers for $k_+, k_- \geq 1$

$$
\mathbb{E} \left[ \partial_{k_+} \partial_{k_-}^{-} f(\chi) \right] = \mathbb{E} [(f, 1)_{n, k_+}^{k_+, k_-}] = \int_{\mathbb{R}_+^2} \partial_{k_+} \partial_- f(x', y') \mathbb{E} \left[ \langle \hat{F}(x', y'), 1 \rangle_{n, k_+}^{k_+, k_-} \right] dy' dx'.
$$
From here, one obtains the regularity of the law of $\chi$. The next step, is to obtain the upper bound for $E \left[ (\hat{F}(x_+, x_-), 1)^{k_+, k_-}_n \right]$. That is, our goal is to prove

$$
\left| E \left[ (\hat{F}(x_+, x_-), 1)^{k_+, k_-}_n \right] \right| \leq C x_+^{-k_-} x_-^{k_+} \times \min \left\{ T^\alpha x_+^{\alpha} x_-^{-\alpha'}, T^\alpha x_+^{\alpha'} x_-^{-\alpha}, T^\alpha x_+^{\alpha} x_-^{-\alpha'}, T^\alpha x_+^{\alpha'} x_-^{-\alpha'} \right\}.
$$

(4.1)

In fact, the bounds follow from Proposition 8 (a)–(c) (with $p = 1$). We use part (a) when $x_+$ and $x_-$ both take large values, part (b) when they are both small and part (c) for the mixed case in which $x_+$ is small and $x_-$ is large or vice versa, cf. Figure 2.1. Each application of Proposition 8 yields summable upper bounds on the summands of the series defined by $\langle \hat{F}(x_+, x_-), 1 \rangle^{k_+, k_-}_n$. The minimum in (4.1) is the smallest sum of these upper bounds as a function of $(x_+, x_-)$ and $T$.

Observe that the derivatives of $F$ in Theorem 1 can be expressed in terms of the derivatives of $G(x_+, x_-):= \mathbb{P}(X_+ > x_+, X_- > x_-)$ as follows: the linear transformation $(X_T, \overline{X}_T) \mapsto (X_T, \overline{X}_T - X_T)$ yields $\partial_x \partial_y F(x, y) = \partial_x \partial_y G(y, y - x)$ for $y > x \lor 0$ and thus

$$
\partial_x^m \partial_y^m F(x, y) = (-1)^{m-1} \sum_{i=0}^{m-1} \binom{m-1}{i} \partial_x^{m-i} \partial_y^{n+i} G(y, y - x).
$$

Therefore, (4.1) gives (2.1) as follows:

$$
|\partial_x^m \partial_y^m F(x, y)| \leq \sum_{i=0}^{m-1} \binom{m-1}{i} |\partial_x^{m-i} \partial_y^{n+i} G(y, y - x)|
$$

$$
\leq \sum_{i=0}^{m-1} \binom{m-1}{i} C y^{-m} (y - x)^{-n-i} \min \left\{ f_{\alpha_1}^{00}(x, y), f_{\alpha_1}^{01}(x, y), f_{\alpha_1}^{10}(x, y), f_{\alpha_1}^{11}(x, y) \right\}
$$

$$
= C y^{-m} (y - x)^{1-n-m} (2y - 2x)^{-m-1} \min \left\{ f_{\alpha_1}^{00}(x, y), f_{\alpha_1}^{01}(x, y), f_{\alpha_1}^{10}(x, y), f_{\alpha_1}^{11}(x, y) \right\}.
$$

□

5. Technical Results

5.1. Upper bounds on the Ibpf. In this section, we study the upper bounds in the technical Proposition 8. It is the key result in order to obtain Theorem 1. We start with some basic properties for the operator $H$ which are useful for bounding $\Theta^f_n, m$ and $\Theta^g_n, m$.

For any $m \in \mathbb{N}$, define

$$
\Sigma^\pm_m := \eta^\pm + \sum_{i=1}^{m} E_i I_{\{\xi_i \geq 0\}} \quad \text{and} \quad \sigma^\pm_m := 1 + \sum_{i=1}^{m} I_{\{\xi_i > 0\}}.
$$

With this notation and in this case, we may rewrite for $\Phi \in \mathbb{S}_m(\Omega)$,

$$
H^\pm_{n, m}(\Phi) = \frac{\alpha/(\alpha - 1)}{X_{\pm, n}} \left( \frac{\Sigma^\pm_m - \sigma^\pm_m + 1 - \frac{1}{\alpha}}{\Phi - D^\pm_m[\Phi]} \right),
$$

(5.1)

$$
D^\pm_m[\Sigma^\pm_m] = \Sigma^\pm_m, \quad D^\pm_m[\sigma^\pm_m] = 0.
$$

Lemma 9. Fix any $k_\pm \geq 0$ and suppose $\Phi := \phi(\chi)$ for some $\phi \in C^{k_+, k_-}_b(A)$ with $A \subset \mathbb{R}^2$. Then for any $m > n$, we have

$$
H^+, k_+ (H^-, k_- (\Phi)) X^{k_+} + n X^{k_-} = H^+, k_+ (H^-, k_- (\Phi)) X^{k_+} + n X^{k_-} + 1.
$$

(5.2)
Moreover, if we set
\[ Z_m := \Sigma_{+,m} + \Sigma_{-,m} = \eta_+ + \eta_- + \sum_{i=1}^m E_i, \quad m \in \mathbb{N}, \]
then there is a bivariate polynomial \( p_{k_+, k_-}^\phi(\cdot, \cdot) \) of degree at most \( k_+ + k_- \) whose coefficients do not depend on \( n \) or \( m \), such that
\[ |H_{n,m}^{+,k_+}(H_{n,m}^{-,k_-}(\Phi))X_{+,n}^{k_+}X_{-,n}^{k_-}| \leq \mathbb{1}_{(\chi \in A)} p_{k_+, k_-}^\phi(Z_m, m), \quad \text{for all } m \geq n. \]

Proof. The proof is simple: we only need to expand the formula for \( H_{n,m}^{+,k_+}(H_{n,m}^{-,k_-}(\Phi)) \) and then uniformly bound all the derivatives of \( \phi \) by the same constant.

Recalling that \( D_{m}^\pm[(\Sigma_{m, n}^\pm, X_{\pm, n}^\pm)] = (\Sigma_{m, n}^\pm, (1/\alpha - 1)pX_{\pm, n}^{-p}) \) and \( D_{m}^\pm[(\Sigma_{m, n}^\pm, \sigma_{m, n}^\pm, X_{\pm, n}^\pm)] = 0 \) for \( p > 0 \), we deduce that an iteration of (5.1) yields \( X_{+,n}^{-k_+}X_{-,n}^{-k_-} \) multiplied by a polynomial of degree \( k_+ \) in \( \Sigma_{m}^\pm \) and \( \sigma_{m}^\pm \). Its coefficients are themselves polynomials of degree \( k_- \) in \( \Sigma_{m}^- \) and \( \sigma_{m}^- \) multiplied by a linear combination of the derivatives \( \partial_{x^i} \partial_{x^j} \phi(\chi) \) for \( j \leq k_\pm \). This directly implies (5.2). Since those derivatives are bounded and we have the a.s. bounds \( \Sigma_{m}^\pm \leq Z_m \) and \( \sigma_{m}^\pm \leq m \), we may bound the entire expression by a constant (independent of \( n \) and \( m \)) multiplied by a polynomial of degree \( k_+ + k_- \) in \( Z_m \) and \( m \). This completes the proof in this case.

\[ \square \]

5.2. Proof of Proposition 8, Part 1: Interpolation inequalities. As we stated previously the proof of the technical Proposition 8 is self-contained and it is divided in two parts. In a first part, we mainly use basic inequalities which will depend on powers of \( X_{\pm, n}, Z_m, \ell_n, \eta_{\pm} \) and \( \Delta_{\pm, n} := X_{\pm, n} - X_{\pm, n-1} \). These moments properties are studied later in Section 6. We assume those results and give the proof of this Proposition here.

Proof of Proposition 8. In the estimates that follow, we will make repeated use of the following inequalities:
\[ \left| \sum_{i=1}^k x_i^q \right| \leq k^{[q-1]} \left| \sum_{i=1}^k |x_i|^q \right|, \quad \text{for any } q > 0 \text{ and } x_i \in \mathbb{R}, \]
which follows from the concavity of \( x \mapsto x^q \) if \( q \leq 1 \) and Jensen’s inequality if \( q > 1 \). Moreover, we frequently apply the following basic interpolating inequalities:
\[ 1_{\{y > x\}} \leq y^r x^{-r}, \quad \text{for all } v \geq 0 \text{ where we interpret the upper bound as } 1 \text{ if } v = 0. \]
Also, if \( y, z \geq 0 \) then for all \( r \in [0, 1] \)
\[ y \wedge z \leq y^r z^{1-r} \]
\[ y \vee z \geq y^r z^{1-r}. \]
Define \( (m_{\pm, n}, M_{\pm, n}) := (X_{\pm, n} \wedge X_{\pm, n+1}, X_{\pm, n} \vee X_{\pm, n+1}) \) then \( m_{\pm, n} = X_{\pm, n+1} \wedge X_{\pm, n} \geq \kappa X_{\pm, n} \) since \( X_{\pm, n+1} \geq \kappa X_{\pm, n} \). Similarly, \( M_{\pm, n} \leq \kappa^{-1} X_{\pm, n+1} \).

Part (a). We will proceed in three steps. Step I) is also used in the proofs of (b) and (c).

I) Recall the definition \( Z_m \) in (5.3) and consider the polynomial \( p_{k_+, k_-}^\phi \) from Lemma 9. According to Lemma 9 with \( A = \mathbb{R}_+^2 \), we have for \( \tilde{f}(x, y) := f(x, y)/(x^{k_+} y^{k_-}) \)
\[ |\tilde{f}(x, y)|^p = |f(\chi_{n+1}) H_{n+1,n+1}^{+,k_+} H_{n+1,n+1}^{-,k_-}(\Phi) - f(\chi_n) H_{n,n+1}^{+,k_+} H_{n,n+1}^{-,k_-}(\Phi)|^p \]
\[ = \left| \frac{f(\chi_{n+1})}{X_{n+1,n}^{k_+} X_{n+1,n}^{-k_-}} - \frac{f(\chi_n)}{X_{n,n}^{k_+} X_{n,n}^{-k_-}} \right|^p \left| H_{n,n+1}^{+,k_+} H_{n,n+1}^{-,k_-}(\Phi) X_{n+1,n}^{k_+} X_{n+1,n}^{-k_-} \right|^p \]
\[ \leq |\tilde{f}(\chi_{n+1}) - \tilde{f}(\chi_n)|^p \left| p_{k_+ k_-}^\phi (Z_{n+1,n+1}) \right|^p. \]
The goal for the rest of the proof is to provide algebraic inequalities for the above expression which depend explicitly on powers of $\Delta_{\pm,n}$, $X_{\pm,n}$ and $Z_{n+1}$. Through these expressions, we will later show that, in expectation, the first factor in the last line decays geometrically in $n$ while the second factor has polynomial growth in $n$.

II) Next, we obtain an upper bound for the modulus of continuity of the map $\tilde{f}$ which appears in (5.8) and where $f$ is given in (3.13). This map is absolutely continuous with respect to Lebesgue measure and thus a.e. differentiable with

$$
|\partial_+ \tilde{f}(x,y)| = \mathbb{I}_{\{x>x_+,y>y_\pm\}} \left| \frac{\partial_+ f(x,y)}{y} - \frac{k_f(x,y)}{y} \right| \leq \mathbb{I}_{\{x>x_+,y>y_\pm\}} c_1 x^{k_+} y^{-k_-},
$$

$$
|\partial_- \tilde{f}(x,y)| = \mathbb{I}_{\{x>x_+,y>y_\pm\}} \left| \frac{\partial_- f(x,y)}{y} - \frac{k_f(x,y)}{y} \right| \leq \mathbb{I}_{\{x>x_+,y>y_\pm\}} c_1 x^{1-k_+} y^{-k_-},
$$

where $c_1 := (k_+ + 1)(k_- + 1)||h||_{\infty}$. Then, for any $x, x', y, y' \in \mathbb{R}_+$, denote $(m_x, M_x) := (x \wedge x', x \vee x')$ and $(m_y, M_y) := (y \wedge y', y \vee y')$ and observe:

$$
|\tilde{f}(x,y) - \tilde{f}(x',y')| = \left| \int_{x'}^x \partial_+ \tilde{f}(z,y)dz + \int_y^{y'} \partial_+ \tilde{f}(x',z)dz \right|
$$

(5.9)

$$
\leq \frac{\mathbb{I}_{\{M_x>M_y\}} c_1 \|x-x'|}{(m_x \wedge x_+)^{k_+} (m_y \wedge x_-)^{k_-}} + \frac{\mathbb{I}_{\{M_x>M_y\}} c_1 \|y-y'|}{(m_y \wedge x_+)^{k_+} (m_x \wedge x_-)^{k_-}}
$$

(5.10)

$$
\leq \frac{\mathbb{I}_{\{M_x>M_y\}} c_1}{(x_+ \wedge x_-)^{k_+} (y_+ \wedge y_-)^{k_-}} (|x-x'| x_+ + |y-y'| y_+).
$$

Note that in the inequality in (5.9) we used that $k_+, k_- \geq 2$ and that the support of $g$ is contained in $[x_+, \infty) \times [x_-, \infty)$. Moreover, since $f$ in (3.13) satisfies $|f(x,y)| \leq \|h\|_{\infty} x y$, we have $|\tilde{f}(x,y)| \leq \|h\|_{\infty} x^{1-k_+} y^{1-k_-}$.

Hence, for any $x, x', y, y' \in \mathbb{R}_+$ we have

$$
|\tilde{f}(x,y) - \tilde{f}(x',y')| \leq \mathbb{I}_{\{M_x>M_y\}} \sup_{z>m_x, w>m_y} |\tilde{f}(z,w)|
$$

(5.11)

$$
\leq \mathbb{I}_{\{M_x>M_y\}} c_2 (m_x \wedge x_+)^{1-k_+} (m_y \wedge x_-)^{1-k_-},
$$

where $c_2 := 2\|h\|_{\infty}$. Typically, for each maximum in the denominator, we use a geometric mixing of its arguments.

III) Now, with the above bound we will show that the upper bound for $\tilde{\Theta}_\beta(t)$ depends on moments of basic random variables. Recall that $s = p \wedge \alpha'$. Applying (5.5) (with $q = s/p$) and (5.6) (with $r = s/p$) to the minimum of the two bounds obtained in (5.10) and (5.11) in the form $(5.10)^{s/p} (5.11)^{1-s/p}$ and using $x_+ \leq m_x \wedge x_+$ and $x_- \leq m_y \wedge x_-$ yields: for any $x, x', y, y' \in \mathbb{R}_+$ the following inequality holds,

$$
|\tilde{f}(x,y) - \tilde{f}(x',y')| \leq \mathbb{I}_{\{M_x>M_y\}} c_3 (x_+)^{1/p} (x_-)^{1-1/s} + (x_-)^{1/p} (x'_+)^{1-1/s} + (y_+)^{1/p} (y_-)^{1-1/s} / x_+^s
$$

where $c_3 := c_1^{p-1} c_2^{-s}$. This interpolation method is used in all cases with different combinations.

Then (5.5) gives

$$
|\tilde{f}(\chi_{n+1}) - \tilde{f}(\chi_n)|^p \leq \mathbb{I}_{\{M_x>M_y\}} c_3 (x_+)^{2p-1} (x_-)^{2p-1} / x_+^{p(k_+ - 1) + s} (|\Delta_{+, n+1}| x_+^s + |\Delta_{-, n+1}| x_-^s).
$$
Applying the inequality $\mathbb{1}_{\{M_{x,n} > x_+\}} \leq x_+^v M_{x,n}^v$, for $v = \alpha' - s \geq 0$ and (5.8) we obtain

$$\left| \mathcal{G}_n^f \right|^p \leq \frac{2^{p-1}c_3\phi_{k_3,k_3}(Z_{n+1}+n+1)^p}{\kappa^2\alpha' x_+^{(k_3-1)+\alpha'\rho(k_3-1)+\alpha'}} \left( |\Delta_{n+1}^+| p M_{n+1}^\alpha M_{n,n}^\alpha + |\Delta_{n+1}^-| p M_{n,n}^\alpha M_{n+1}^- \right)$$

where the second inequality follows from the fact that $M_{\pm,n} \leq \kappa^{-1} X_{\pm,n+1}$. Finally, as $\alpha' < \alpha$, Lemma 10 gives (3.15).

To prove the second statement in (a), we proceed as before. We start by using the inequality $|\mathcal{F}(x)|^p \leq \mathbb{1}_{\{x, x+, x_-, x_\geq x_+\}} \|h\|^p_{\infty} x_+^p (1-k_{-1}+x_+ \rho_{k_{-1}+\alpha'}(1-\rho))$ and the bound $\mathbb{1}_{\{x, x+, x_-, x_\geq x_+\}} \leq x_+^{\alpha'} x_+^{-\alpha'}$. An application of Lemma 10 then yields (3.16).

**Part (b).** Let $c_4 := \frac{2^{1-1/k}e c_2^{-1}}{u}$ where $u \in [0,1]$ is given in the statement. Applying (5.6) (with $r = u$) and (5.5) (with $q = p$) to the minimum of (5.9) and (5.11) in the form (5.9) and (5.11) in the form (5.9) yields

$$|\mathcal{G}_n^f| \leq \frac{2^{p-1}c_4\phi_{k_4,k_4}(Z_{n+1}+n+1)^p}{\kappa^2\alpha' x_+^{(k_4-1)+\alpha'\rho(k_4-1)+\alpha'}} \left( |\Delta_{n+1}^+| p x_+^{\rho_{k_4-1}+\alpha'}(1-\rho) + |\Delta_{n+1}^-| p x_+^{-\alpha'} \right).$$

By (5.6) we have $m_y \geq m_+ x_+^r$ and $m_y \leq m_y x_+^r$ for any $r, r' \in [0,1]$. Since $\alpha' < \alpha \leq 1/(\rho \vee (1-\rho))$, we choose $r = \alpha'/[p(k_{-1}+1)]$ and $r' = (1-\rho)/[p(k_{-1}+1)]$. Applying these interpolating inequalities to (5.12) and combining them with (5.8) gives

$$|\mathcal{G}_n^f| \leq \frac{2^{p-1}c_4\phi_{k_4,k_4}(Z_{n+1}+n+1)^p}{\kappa^2\alpha' x_+^{(k_4-1)+\alpha'\rho(k_4-1)+\alpha'}} \left( |\Delta_{n+1}^+| p x_+^{\rho_{k_4-1}+\alpha'}(1-\rho) + |\Delta_{n+1}^-| p x_+^{-\alpha'} \right),$$

where we used the restriction that $M_{\pm,n} \geq \kappa X_{\pm,n}$. Moreover, as $u \in (0, (\alpha - \alpha') (\rho \wedge (1-\rho))/p, we have \alpha' + pu < \alpha$ and $\alpha' + pu < \alpha(1-\rho)$. Hence, applying Lemma 12 gives (3.17).

The proof of (3.18) is analogous to that of (3.17). Indeed, using (5.6) and the inequality $|\mathcal{F}(x)| \leq \mathbb{1}_{\{x, x+, x_-, x_\geq x_+\}} x_+^{1-k_{-1}} x_+^{-1-k_{-1}}$ we obtain

$$|\mathcal{F}(x)| \leq \mathbb{1}_{\{x, x+, x_-, x_\geq x_+\}} x_+^{p(1-k_{-1})+\alpha' \rho_{k_{-1}+\alpha'}(1-\rho)} X_{\alpha' \rho_{k_{-1}+\alpha'}}^{-\alpha' \rho_{k_{-1}}}.$$
where we used the fact that \( M_{\pm,n} \leq \kappa^{-1}X_{\pm,n+1} \) and \( m_{\pm,n} \geq \kappa X_{\pm,n} \). An application of Lemma 12 then gives (3.19).

Using the inequality \( |\tilde{f}(x_n)| \leq \|h\|_{\infty} \left( X_{+,n} \lor x_+ \right)^{1-k_+} \left( X_{-,n} \lor x_- \right)^{1-k_-} \), and the bound \( \mathbb{1}_{\{X_{+,n} > x_+, X_{-,n} > x_-\}} \|h\|_{\infty} \left( X_{+,n} \lor x_+ \right)^{1-k_+} \left( X_{-,n} \lor x_- \right)^{1-k_-} \), we obtain

\[
|\Theta^f_{n,m}|^p \leq \frac{\|h\|_{\infty} p_{k_+,k_-}^p (Z_m,m)^p}{x_+^{\rho(k_+ - 1) + \alpha}(1 - \alpha - \alpha')(1 - \rho)} X_{\pm,n}^{\alpha'(1 - \rho)} X_{\pm,n}^\alpha \cdot
\]

which yields (3.20) by Lemma 12, completing the proof of the proposition.

\( \square \)

**Remark 4.** Analyzing the above proof, we can see the interpolation method at work here. In fact, the estimates of Proposition 8, one may say that all polynomial terms in \( n \) arise due to the polynomial growth of \( H_{n,m}^{k_+,k_-} \) (see (5.4) in Lemma 9), through the term \( p_{k_+,k_-}^p (Z_m,m)^p \) which appears in the upper bounds. On the other hand, the geometrically decreasing terms are produced by the exponentially fast decay of the differences \( \Delta_{\pm,n} := X_{\pm,n} - X_{\pm,n-1} \) in \( \tilde{f}'_n \). We stress that another achievement of the interpolation method is that the moment estimates of Proposition 8 hold for any \( p \geq 1 \).

### 6. Proof of Proposition 8, Part II: The moment bounds

In this section, we state the explicit moment estimates for the quantities that appear in the weights of the multiple Lpbf of Theorem 7. These bounds were the last step in the proof of Proposition 8 above. The proofs of these lemmas in this section, are independent of everything that have preceded them. In order to obtain near optimal bounds in Theorem 1, we first study the growth of the moments of \( X_{\pm,n}^p \) for \( p \) arbitrarily close to \( \alpha \) in Lemmas 10, 11 and 12. Since the \( \alpha \)-moment of the stable law does not exist, the bounds in these lemmas cannot be obtained e.g. via Hölder’s inequality. Their proofs consist of a direct, but very careful, analysis of the corresponding expectations.

There are two types of bounds according to whether they involve positive or negative moments of \( X_{\pm,n} \). They correspond to the behavior at infinity or at zero in the estimates that we obtain in Theorem 1 as can be deduced from the proof of Proposition 8. Throughout the present section we use the notation from Subsection 3.1. In particular, recall the definition of \( Z_m \) in (5.3) and Assumption (A-\( \kappa \)): \( \kappa^\alpha \in [\rho \lor (1 - \rho), 1) \). Explicit constants in the results in this section can be recovered from the proofs.

We begin by recalling the Mellin transform of a stable random variable (see [Zol86, Thm 2.6.3])

\[
\mathbb{E}[S_1^p \mathbb{1}_{\{S_1 > 0\}}] = \frac{\Gamma(1 + p) \Gamma(1 - p/\alpha)}{\Gamma(1 + p\rho) \Gamma(1 - p\rho)}, \quad p \in (-1, \alpha).
\]

When \( \alpha \neq 1 \), by the independence \( E_i \perp G_i \) we deduce that, for any \( p \in [0, \alpha) \),

\[
\mathbb{E}[G_i^p \mathbb{1}_{\{G_i > 0\}}] = \frac{\mathbb{E}[S_1^p \mathbb{1}_{\{S_1 > 0\}}]}{\mathbb{E}[E_i^{p(1 - 1/\alpha)}]} = \frac{\rho \Gamma(1 + p) \Gamma(1 - p/\alpha)}{\Gamma(1 + p\rho) \Gamma(1 - p\rho) \Gamma(p(1 - 1/\alpha) + 1)}.
\]

Finally, we recall that \( \mathbb{E}[E_i^p] = \Gamma(1 + p) \) is finite if and only if \( p > -1 \).

#### 6.1. Positive moments.

**Lemma 10.** Let \( p, q, s \geq 0 \) satisfy \( q \leq p < \alpha \). Then, there exists a constant \( C > 0 \) such that for any \( m \geq n \) and \( T > 0 \) we have

\[
\mathbb{E}\left[ X_{\pm,n}^{p-q} X_{\pm,n}^q \Delta_{\pm,n}| q Z_m^s \right] \leq CT^{2n} \left( \frac{1 + q}{\alpha} \right)^{-n} \kappa^{qn} m^{[p-1]^+ + [p-q-1]^+ + s}.
\]

Before giving the proof of the lemma, we give some remarks:
Remark 5. (i) Note that the exponent in $E^{1-1/\alpha}$ appearing in $X_{\pm,n}$ changes sign when $\alpha \in (0,1)$ and $\alpha \in (1,2)$. For this reason, most of the proofs for estimating the above bounds of moments will have to be done in three separate cases: $\alpha \in (0,1)$, $\alpha = 1$ and $\alpha \in (1,2)$. This makes the proofs slight long because some inequalities change depending on the above cases.

(ii) Note that due to the scaling property of the stick breaking process and $a_n$ the factor of $T^{2\alpha}$ is easily obtained. In fact, we will assume, without loss of generality, in all proofs in this section that $T = 1$. In the Lemma statements, we have left the dependence on $T$ and in some major points of the proof too. In a first reading, one may assume always that $T = 1$.

(iii) We will consider in all proofs only one combination of $\pm$ signs. The other case follows mutatis mutandis.

Proof of Lemma 10. We first make a number of reductions that simplify the proof. We will assume $p, q > 0$.

The remaining cases (when at least one of the two parameters is zero) follow similarly by ignoring the corresponding terms in the calculations.

Let $c = 2^{(p-1)^++[p-q-1]^++[q-1]^+}$ and use (5.5) to obtain

$$X_{\pm,n}^p X_{\pm,n}^q |\Delta_{\pm,n}|^q \leq c \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \left( \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} n \right)^{p-q} + a_n^{q} q_n^{p} \right) \times \left( \left( \ell_n^{1/\alpha} c_n^{+} \right)^{q} + a_n^{q} q_n^{p} \right).$$

(6.3)

Our goal is now to provide an upper bound for the expectation of the right-hand side of the above inequality multiplied by $Z_{m,n}$. This leads to eight terms which must be treated individually to show that their expectations decay exponentially at least as a polynomial (in $n$) multiple of $a_{n-1}$ or $E^{\ell_n^{1/\alpha}} = (1 + q/\alpha)^{-n}$. We treat the hardest term in (6.3); which contains the product of sums of $[S_n]^{\pm}$. The other terms are easier to treat as we remark at the end of the proof. Therefore we will consider, for $r \in \{0, q\}$

$$A := E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \left( \ell_n^{1/\alpha} c_n^{+} n \right)^{p-q} \right] Z_{m,n}, \quad r \in \{0, q\},$$

where $c_i = E_i^{1-1/\alpha}$ and $\mathcal{F} = \sigma(\ell, G; i \in \mathbb{N})$. We estimate (6.4) in steps:

1) In this step, we will separate the expectation in (6.4) using the independent components $G, E$ and $\ell$. Let $r \in \{0, q\}$ and $p':=[p-1]^++[p-q-1]^+$ and consider any positive constants $(c_i)_{i \in \mathbb{N}}$. Applying (5.5) yields

$$E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \left( \ell_n^{1/\alpha} c_n^{+} n \right)^{p-q} \right] \leq n^{p'} E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \left( \ell_n^{1/\alpha} c_n^{+} n \right)^{p-q} \right] E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \right] E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \right] E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \right].$$

Note that the cases $j \in \{i, n\}$ do not appear because $[x]^+[x]^-$ = 0. The above expression is a linear combination of monomials in $c_i, c_n$ and $c_j$. We will analyze and bound the coefficients.

The last two expectations within the sum on the right side of the above inequality can be computed using (6.1) and the value of their product only depends on whether $i = n$ or not. In fact, for $r \in \{0, q\}$

$$E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \left( \sum_{j=1}^n \ell_j^{1/\alpha} c_j^{-} \right)^{p} \right] \leq \max\{E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p} \right], E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \right] E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \right], E^\left[ \left( \sum_{i=1}^n \ell_i^{1/\alpha} c_i^{+} \right)^{p-q} \right]\},$$

for $r \in \{0, q\}$. 

which can be bounded by an explicit constant using (6.1).

II) Now, we obtain an important part of the bound in (6.2) which is due to the stick breaking process. That is, an application of Lemma 13(b) yields the existence of some \( c' > 0 \) independent of \( j, i \) and \( n \) such that for \( \theta = \frac{\alpha + p/r + r}{\alpha + 2p + r} < 1 \), we have

\[
E[e^{(p-q)/\alpha E_i/p/\alpha E_j/r}] \leq c' \theta^{i+j}(1 + r/\alpha)^{-n}.
\]

III) Now, we estimate the moments of the remaining random variables \( E_i \) which appear in the coefficients \( c_i \). By the previous steps and (6.5), we deduce that for some constant \( c'' > 0 \) independent of \( j, i \) and \( n \), we have

\[
E[A] \leq c'' n^{n'} (1 + \frac{r}{\alpha})^{-n} 
\]

Next, we will show that the expectation on the right side in the above inequality is bounded by a multiple of \( m^s \). As the term \( \theta^{i+j} \) vanishes geometrically fast, we would then obtain

\[
(6.6) \quad E[A] \leq c'' n^{n'} m^s (1 + \frac{r}{\alpha})^{-n}.
\]

To prove (6.6), observe that since \( Z_n \) in (5.3) is a Gamma distributed random variable then

\[
E[Z_n^s] = \int_0^\infty x^s x^{n+1} (n+1)! e^{-x} \ dx = \frac{\Gamma(n+s+2)}{(n+1)!}.
\]

Using the two-sided bounds in Stirling’s formula we see that this expression is bounded by a multiple of \( m^s \). In fact, a similar upper bound holds for \( E[Z_m^s E_i^{r_1} E_j^{r_2} E_k^{r_3}] \) with \( r_1 = (1 - 1/\alpha)(p-q) \), \( r_2 = (1 - 1/\alpha)p \) and \( r_3 = (1 - 1/\alpha)r \). Note that \( r_1, r_2, r_3 > -1 \) in the case \( i < n \) and \( r_1 + r_2 > -1 \), \( r_2 > -1 \) in the case that \( i = n \). Furthermore, even in the case \( \alpha \in (0,1) \), our hypotheses on \( p \) and \( q \) ensure that \( r_1, r_2 \) and \( r_3 \) satisfy these conditions. Indeed, for instance, when the indices \( i, j, k \) are different and \( n \geq 4 \), we can decompose \( Z_m \) into 4 terms according to the index of \( E \) within \( Z_m \) which may equal one of the indices \( n, i, j \) so that, by (5.5),

\[
E[Z_m^s E_i^{r_1} E_j^{r_2} E_k^{r_3}] = 4^{(s-1)! \ell} \left( E[E_i^{s+r_1}] E[E_i^{s+r_2}] E[E_j^{r_3}] + E[E_i^{s+r_1}] E[E_j^{s+r_2}] E[E_k^{r_3}] + E[E_i^{s+r_2}] E[E_j^{s+r_2}] E[E_k^{r_3}] \right)
\]

The quantity in (6.7) grows as a constant multiple of \( m^s \) (through the \( s \)-moment of \( Z_{m-3} \)). Thus, we can deduce that (6.6).

Finally, to bound other terms in (6.3), it is just a repetition of the above arguments but slightly easier because:

1. The variables \( \eta_+ \) and \( \eta_- \) are independent of the sequence \( \ell, S_i \).
2. Hence, when taking expectations, the variables \( \eta_+ \) and \( \eta_- \) will factorise by independence. These variables are multiplied by powers of \( a_n = \kappa^n \) and satisfy \( E[\eta_+^q] = \Gamma(1 + r) \) for \( r > -1 \) so their estimation is easier.
3. The final bound also uses the inequality \( a_n \leq a_{n-1} \), a consequence of Assumption (A-κ).

Putting the above arguments together completes the proof of Lemma 10, since all eight terms decay as fast as \( a_n^q \) or \( (1 + q/\alpha)^{-n} \) and \( n \leq m \).
6.2. Negative and mixed moments: Proofs of Lemmas 11 and 12. The ideas of the proof of Lemma 10 can be used again for negative moments but an additional idea is required in order to use similar techniques. This is provided by the following equality:

\[
\lambda^{-p} = \Gamma(p)^{-1} \int_0^\infty x^{p-1} e^{-\lambda x} \, dx, \quad \text{for } p > 0,
\]

which expresses the negative power \(\lambda^{-p}\) using an exponential expression which in our case leads to the study of the Laplace transform of the respective random variable whose inverse moment we want to estimate. The use of this technique leads to the following results.

**Lemma 11.** Recall that we always assume that \(\alpha \neq 1\). Below, we assume that \(r \geq 0\) and \(u, v, w \geq 0\) be arbitrary.

(a) Fix any \(p \in (0, \alpha), q \in [0, \alpha(1 - \rho))\). There exists a positive constant \(C\) such that for any \(j, n \in \mathbb{N}\) and \(T > 0\), the following bound holds:

\[
\mathbb{E}\left[\frac{\ell_{n+1} E_j^u \eta_j^v \eta_j^w}{X_{j, n}^q} X_{j, n}^q\right] \leq CT^{-\frac{p}{\alpha}} (1 + r)^{-n}.
\]

(b) Fix any \(p \in (0, \alpha), q \in (0, \alpha)\). There exists a positive constant \(C\) such that for any \(j, n \in \mathbb{N}\) and \(T > 0\), the following bound holds:

\[
\mathbb{E}\left[\frac{X_{j, n} q \ell_{n+1} E_j^u \eta_j^v \eta_j^w}{X_{j, n}^q} X_{j, n}^q\right] \leq CT^{-\frac{p+1}{\alpha}} (1 + r)^{-n}.
\]

**Lemma 12.** Let \(p, q, r, s \geq 0\) satisfy \(p \in [0, \alpha p), q \in [0, \alpha(1 - \rho))\) and \(r \in [0, \alpha)\). There exists a constant \(C > 0\) such that for any \(m \geq n\) and \(T > 0\) we have

\[
\mathbb{E}\left[\frac{\Delta_{j, n+1}^p |Z_m^s|}{X_{j, n}^p X_{j, n}^q}\right] \leq CT^{(2r-p-q)/\alpha} \left(1 + \frac{c}{\alpha}\right)^{-n} + \kappa^{m s'},
\]

where \(s' = \mathbb{1}_{s > 0}(s \vee 1)\). Similarly, for any \(p \in [0, \alpha), r \in [0, p]\) and \(q \in [0, \alpha(1 - \rho))\), there is some \(C > 0\) such that for all \(m \geq n\) and \(T > 0\)

\[
\mathbb{E}\left[\frac{\Delta_{j, n+1}^p |X_{j, n+1}^p - r| Z_m^s}{X_{j, n}^p X_{j, n}^q}\right] \leq CT^{(p-q)/\alpha} \left(1 + \frac{c}{\alpha}\right)^{-n} + \kappa^{m s'}.
\]

**Proof of Lemma 11.** Recall that we assume without loss of generality that \(T = 1\). The identity (6.8) will be used with \(\lambda = X_{j, n}^+\) (and later with \(\lambda = X_{j, n}^-\)). The resulting expression will be bounded by separately integrating the variables \(G_1, \ldots, G_n\) and \(\eta_+, \eta_-\), then and \(E_1, \ldots, E_n\) and finally \(\ell_1, \ldots, \ell_{n+1}\) as in the proof of Lemma 10. These bounds require preliminary calculations for the expressions arising in the inequalities developed below, so we begin with those. Let \(\zeta = 1 - 1/\alpha, c, \delta\) and \(\gamma\) be as defined in Lemma 15 of Appendix B. These are constants that appear in the bounds for the Laplace transform of random variables appearing in the Chambers-Mallows-Stuck representation of stable laws.

**Proof of (a), part 1: The case \(q = 0\).** We first provide an explicit upper bound for \(\mathbb{E}[\ell_{n+1} E_j^u \eta_j^w X_{j, n}^- p]\) (the case \(q = 0\) and \(w = 0\) in Lemma 11(a)) using the following constants and functions.
Define \( b_p := 1/(\gamma \alpha p) \geq 1 \) and
\[
d_s := 2^s \max \{ 1, s^s e^{-s} \Gamma(s+1) \}, \quad \text{for } s \geq 0,
\]
\[
P(x, p, q) := \frac{((x \wedge 1)^{q-p} - 1)}{p} + \frac{x^{-q}((x \wedge 1)^{q-p} - 1)}{q-p}, \quad \text{for } x, p, q > 0, q > p,
\]
(6.10)
\[
Q_p(r, u) := \frac{\alpha u (1 + r) - up}{p(1 - p)(\alpha u (1 + r) - p)(1 - p/\alpha)}, \quad \text{for } u \in (0, 1), \quad p \in (0, \min\{ \alpha u, 1 \}), \quad r \geq 0.
\]
d'\_u := \max \{ \Gamma(1 + u), \Gamma(1 + u)/\Gamma(1/\alpha), \Gamma(u + 1/\alpha) \} = \max \{ \mathbb{E}[E^n_j], \mathbb{E}[E^n_j \mathbb{E}[E^{\xi_n}_k]], \mathbb{E}[E^{\xi_n}_k] \}.
Using these definitions, it is enough to prove that for \( p \in (0, \alpha p), \ r, u, v \geq 0 \) and \( j, n \in \mathbb{N} \) it holds
\[
\mathbb{E} \left[ \frac{\ell_{n+1} E^n_j \eta^n_+ e^{-\alpha_n \eta^n_+} \Gamma(p)}{X^{n+1}_+} \right] \leq \frac{T^{\gamma-p} b_p c d_u d_v}{(p)(1 + r)^n} \left( \frac{T^\gamma}{p} + Q_p(r, \rho) + (1 - \rho)^n P(T^{-\frac{\gamma}{p}} a_n, p, \delta) \right).
\]
The special case of (6.9) with \( q = 0 \), \( w > 0 \) follows from (6.11), the independence \( \eta_+ \perp (\eta_+, E_j, \ell_{n+1}, X_{n+1}, \ldots) \), (1.1) and Assumption (A-\( \kappa \)). In fact, all the above terms within the parentheses are readily bounded by a constant except \( (1 - \rho)^n \kappa^{-kn-p} \) which is bounded due to Assumption (A-\( \kappa \)).
For the proof of (6.11), recall that \( X_{n+1} = \sum_{k=1}^n \ell^{1/\alpha}_k E^{\xi_n}_k[|G_i|^+] + a_n \eta^n_+ \) with \( \xi = 1 - 1/\alpha \).
Fix \( p \in (0, \alpha p) \). A change of variables applied to the definition of the Gamma function gives
\[
\Gamma(p) X^{n+1}_+ = \int_0^\infty x^{p-1} e^{-x X_{n+1}} dx \leq 1/p + J_{\pm p}, \quad \text{where } J_{\pm p} := \int_1^\infty x^{p-1} e^{-x X_{n+1}} dx.
\]
Next, we bound the conditional expectation \( \mathbb{E}[\eta^n_+ J_{\pm p}|\mathcal{G}] \), where \( \mathcal{G} := \sigma(\ell_k, E_k; k \in \mathbb{N}) \). By (B.3) and (B.4) (with parameter \( x \ell^{1/\alpha}_k E^{\xi_n}_k \)), that conditional expectation is smaller than
\[
\mathbb{E}[\eta^n_+ J_{\pm p}|\mathcal{G}] = \int_1^\infty x^{p-1} \mathbb{E}[\eta^n_+ e^{-x a_n \eta^n_+} \mathbb{E}[e^{-x \ell^{1/\alpha}_k E^{\xi_n}_k[|G_i|^+]}|\mathcal{G}]] dx
\]
\[
\leq c d_v \int_1^\infty x^{p-1} \min\{1, (a_n x)^{-\delta} \} \prod_{k=1}^n \left( 1 - \rho + \rho \min\{1, b_p E^{\xi_n}_k \ell^{1/\alpha}_k X^{n+1}_+ \} \right) dx.
\]
Using that \( \ell_{n+1} = (1 - U_{n+1}) U_{n+1} \cdots U_{k+1} L_k \) and \( L_k \leq L_{k-1} \) (see Subsection 3.1), then for any measurable function \( g \geq 0 \) and \( k \leq n \), we have
\[
\mathbb{E}[\ell_{n+1} g(\ell_k)] = (1 + r)^{k-1} \mathbb{E}[L_k r g(\ell_k)] \leq (1 + r)^{k-1} \mathbb{E}[L_{k-1} g(\ell_k)].
\]
Moreover, we have \( \mathbb{E}[E^n_j \min\{1, E^{\xi_n}_k y\}] \leq d'\_u \min\{1, y\} \) by definition (6.10). In fact, if \( y > 1 \), then \( d'\_u y \geq \mathbb{E}[E^n_j E^{\xi_n}_k] \) and if \( y < 1 \), then \( d'\_u y \geq 1/\mathbb{E}[E^n_j E^{\xi_n}_k] \).
Since the factors in the product of (6.13) are in \( [0, 1] \), the inequality in (B.1), (6.14) and \( b_p \geq 1 \) yields
\[
\mathbb{E} \left[ \ell_{n+1} E^n_j \prod_{k=1}^n \left( 1 - \rho + \rho \min\{1, b_p E^{\xi_n}_k \ell^{1/\alpha}_k X^{n+1}_+ \} \right) \right]
\]
\[
\leq (1 - \rho)^n \mathbb{E}[\ell_{n+1} E^n_j] + \sum_{k=1}^n \rho (1 - \rho)^{k-1} \mathbb{E}[\ell_{n+1} E^n_j \min\{1, b_p E^{\xi_n}_k \ell^{1/\alpha}_k X^{n+1}_+ \}]
\]
\[
\leq (1 - \rho)^n d'\_u (1 + r)^{-n} \sum_{k=1}^n \rho (1 - \rho)^{k-1} (1 + r)^{k-1} \mathbb{E}[L_{k-1} \min\{1, b_p \ell^{1/\alpha}_k X^{n+1}_+ \}]
\]
\[
\leq d'\_u (1 + r)^{-n} (1 + r)^{-n} \mathbb{E}[A_\rho(x) x] \equiv 0.
\]
Hence, the inequality $E[\ell_{n+1}E^{u}Y_{+}^{a}X_{+}^{-\rho}P]\Gamma(p) \leq d_{n}\Gamma(1+r)^{-n}(1/p + cK)$ holds for

$$K := \int_{1}^{\infty} x^{p-1} \min\{1, (a_n x)^{-\delta}\}((1 - \rho)^n + b_{p}A_{\rho}(x))dx.$$ 

Next, we apply Lemma 15(c) to find a formula for $\int_{1}^{\infty} x^{p-1}A_{\rho}(x)dx$. Note that $p < \alpha \rho$ implies $\frac{(1-\rho)(1+r)}{1+r-p/\alpha} < 1$, so Fubini's theorem and Lemmas 15(c) and 13(c) yield

$$\int_{1}^{\infty} x^{p-1}A_{\rho}(x)dx = \sum_{k=1}^{\infty} \rho(1-\rho)^{k-1}(1+r)^{k-1}E[L_{k-1}^{p}(\ell_{k}^{1/\alpha}x)^{-1}dx]$$

$$= \sum_{k=1}^{\infty} \rho(1-\rho)^{k-1}(1+r)^{k-1}E[L_{k-1}^{p}(x^{1/\alpha} - 1/p)]$$

$$= \sum_{k=1}^{\infty} \rho(1-\rho)^{k-1}(1+r)^{k-1}\left(\frac{(1+r-p/\alpha)1^{-k}}{p(1-p)(1-\rho)} - \frac{(1+r)^{1-k}}{p}\right) = Q_{\rho}(r, \rho).$$

Thus (6.11) follows from (6.15) and Lemma 15(c) since for any $p < \alpha \rho$ and $n \in \mathbb{N}$ we have

$$K \leq \int_{1}^{\infty} x^{p-1}((1-\rho)^{n}\min\{1, (a_n x)^{-\delta}\} + b_{p}A_{\rho}(x))dx \leq b_{\rho}[(1-\rho)^{n}P(a_{n}, p, \delta) + Q_{\rho}(r, \rho)].$$

Proof of (a), part 2. The case $q \in (0, \alpha(1-\rho))$. The general case of (6.9) for $q > 0$ follows similarly but with lengthier expressions. Recall that $B(\cdot, \cdot)$ denotes the beta function and define for any $u \in (0,1)$, $p \in (0, \alpha u \land 1)$, $q \in (0, \alpha \land 1)$ and $r \geq 0$,

$$R_{p,q}(r, u) := \frac{(\Gamma(1/\alpha) \lor 1)B(1+r-p/\alpha, 1-q/\alpha)(1+u)(1+r-p/\alpha)}{pq(1-p)(1-q)(1-\alpha)(u(1+r) - p/\alpha)}.$$ 

Fix $p \in (0, \alpha \rho)$, $q \in (0, \alpha(1-\rho))$ and $r, u, v, w \geq 0$. We will prove that for all $j, n \in \mathbb{N}$, we have

$$E[\ell_{n+1}X_{+}^{u}X_{+}^{w}] \leq \frac{T^{-\frac{n-\alpha}{\alpha}}b_{\rho}b_{1-p/\alpha}d_{u}d_{w}}{(\Gamma(q)\Gamma(1+r))^{n}}[(1-\rho)^{n}P(T^{-\frac{1}{\alpha}}a_{n}, p, \delta) + \rho^{n}P(T^{-\frac{1}{\alpha}}a_{n}, q, \delta)/p]$$

$$+ ((1-\rho)^{n} + \rho^{n})P(T^{-\frac{1}{\alpha}}a_{n}, p, \delta)P(T^{-\frac{1}{\alpha}}a_{n}, q, \delta)$$

$$+ 1/(pq) + Q_{\rho}(r, \rho)/q + Q_{q}(r, 1-r)/p + R_{p,q}(r, \rho) + R_{q,p}(r, 1-r),$$

Once this bound is proven the final result follows as above, by (1.1) and Assumption (A-$\kappa$). Indeed, $((1-\rho)^{n} + \rho^{n})P(T^{-1/\alpha}a_{n}, p, \delta)P(T^{-1/\alpha}a_{n}, q, \delta) \leq (1/p + 1/(\delta - p))(1/q + 1/(\delta - q))2K^{\alpha-n}(\rho + q)$, by Assumption (A-$\kappa$), which is bounded for $n \in \mathbb{N}$ when $p + q < \alpha$.

Applying (6.12) twice, we obtain

$$\Gamma(p)\Gamma(q)X_{+}^{p}X_{+}^{q} \leq 1/(pq) + J_{-}q/p + J_{+}q/q + J_{+}pJ_{-}q.$$ 

It remains to multiply (6.16) by $\ell_{n+1}X_{+}^{u}X_{+}^{w}$ and take expectations. The first term in (6.16) yields the inequality $E[\ell_{n+1}\ell_{n+2}X_{+}^{a}X_{+}^{w}]/(pq) \leq (1+r)^{-n}d_{u}d_{w}/(pq)$. The second and third terms are bounded as in the special case $q = 0$, since $\eta_{+}$ (resp. $\eta_{-}$) is independent of $X_{-n}$ (resp. $X_{+n}$).

It remains to bound $E[\ell_{n+1}E^{u}Y_{+}^{a}J_{+}pJ_{-}q]$. Note that applying Lemma 15(b) twice gives

$$E[e^{-x[G_{i}^{-} - y[G_{i}]^{+}}}] \leq \Upsilon(x, y) := \rho \min\{1, b_{\rho}/x\} + (1-\rho)\min\{1, b_{1-\rho}/y\} \leq 1.$$
Recall $\mathcal{G} = \sigma(\ell_k, E_k; k \in \mathbb{N})$ and apply (B.3) to $\mathbb{E}[\ell_{n+1}^r \eta_+^u \eta_-^w J_{+,p} J_{-,q}|\mathcal{G}] = \int_1^{\infty} \int_1^{\infty} x^{p-1} y^{q-1} S(x, y) dy dx$, where

$$S(x, y) := \ell_{n+1}^r \mathbb{E}[\eta_+^u \eta_-^w e^{-x X_{+,n} - y X_{-,n}}|\mathcal{G}]$$

$$= \ell_{n+1}^r \mathbb{E}\left[\eta_+^u e^{-x a_n \eta_-^w} \mathbb{E}[\eta_-^w e^{-y a_n \eta_-^w}] \prod_{k=1}^n \mathbb{E}[e^{-\ell_k^{1/\alpha} E_k \mathbb{E}(x|G_k)^+ - y G_k^-}]|\mathcal{G}\right]$$

$$\leq c^2 d_v d_w \min\{1, (a_n x)^{-\delta}\} \min\{1, (a_n y)^{-\delta}\} \ell_{n+1}^r \prod_{k=1}^n \mathbb{E}(\ell_k^{1/\alpha} x, E_k^{1/\alpha} y).$$

The inequality $\mathbb{E}[\min\{1, E_k^{-\xi} x\} \min\{1, E_k^{-\xi} y\}] \leq d'_v (\Gamma(1/\alpha) \vee 1) \min\{1, x\} \min\{1, y\}$ for $k \geq 2$, along with (B.2) and (6.14) yield

$$\mathbb{E}\left[\ell_{n+1}^r \eta_+^u \eta_-^w \prod_{k=1}^n \mathbb{E}(\ell_k^{1/\alpha} x, E_k^{1/\alpha} y)\right]$$

$$\leq ((1 - \rho)^n + \rho^n) \mathbb{E}[\ell_{n+1}^r |\mathbb{E}[E_n^u]] + \sum_{k=2}^n \rho(1 - \rho)^{k-1} \mathbb{E}\left[\ell_{n+1}^r \eta_+^u \eta_-^w \min\left\{1, \frac{b_y E_k^{-\xi}}{\ell_k^{1/\alpha} x}\right\} \min\left\{1, \frac{b_y E_k^{-\xi}}{\ell_k^{1/\alpha} y}\right\}\right]$$

$$+ \sum_{k=2}^n (1 - \rho) \rho^{k-1} \mathbb{E}\left[\ell_{n+1}^r \eta_+^u \eta_-^w \min\left\{1, \frac{b_y E_k^{-\xi}}{\ell_k^{1/\alpha} x}\right\} \min\left\{1, \frac{b_y E_k^{-\xi}}{\ell_k^{1/\alpha} y}\right\}\right]$$

$$\leq b \rho b \rho d'_u (1 + r)^{-n} (1 - \rho)^n + \rho^n + (\Gamma(1/\alpha) \vee 1)(B_\rho(x, y) + B_1 - \rho(y, x)),$$

where $B_\rho(x, y) := \sum_{k=2}^n s(1 - s)^{-k} (1 + r)^k \mathbb{E}[L_k^{-1} \min\{1, \ell_k^{-1/\alpha} / x\} \min\{1, \ell_k^{-1/\alpha} / y\}]$ for $x, y > 0$.

Next we give a simple bound on some integrals of $B_\rho$. Recall that we have $p + q < \alpha$ and $\mathbb{E}[U^r (1 - U)^q] = B(r + 1, s + 1)$. Thus an application of Fubini’s theorem, (B.5) and Lemma 13(c) yields

$$\int_1^{\infty} \int_1^{\infty} x^{p-1} y^{q-1} B_\rho(x, y) dy dx$$

$$= \sum_{k=2}^n \rho(1 - \rho)^{k-1} (1 + r)^{-k} \mathbb{E}\left[L_k^{-1} \int_1^{\infty} \int_1^{\infty} x^{p-1} y^{q-1} \min\{1, \ell_k^{-1/\alpha} / x\} \min\{1, \ell_k^{-1/\alpha} / y\} dy dx\right]$$

$$= \sum_{k=2}^n \rho(1 - \rho)^{k-1} (1 + r)^{-k} \mathbb{E}\left[L_k^{-1} \left(\frac{\ell_k^{p/\alpha}}{p (1 - p)} - 1\right) \left(\frac{\ell_k^{-q/\alpha}}{q (1 - q)} - 1\right)\right]$$

$$\leq \sum_{k=2}^n \rho(1 - \rho)^{k-1} (1 + r)^{-k} \mathbb{E}\left[L_k^{-1} \frac{\ell_k^{p/\alpha} \ell_k^{-q/\alpha}}{pq (1 - p) (1 - q)} = R_{p,q}(r, \rho) / (\Gamma(1/\alpha) \vee 1)\right].$$

Putting all the above arguments together, the following inequalities imply part (a):

$$\mathbb{E}[\ell_{n+1}^r \eta_+^u \eta_-^w J_{+,p} J_{-,q}]$$

$$\leq b \rho b \rho d'_u d_v d_w (1 + r)^n \int_1^{\infty} \int_1^{\infty} x^{p-1} y^{q-1} \min\{1, (a_n x)^{-\delta}\} \min\{1, (a_n y)^{-\delta}\}$$

$$\times ((1 - \rho)^n + \rho^n + (\Gamma(1/\alpha) \vee 1)(B_\rho(x, y) + B_1 - \rho(y, x))) dy dx$$

$$\leq b \rho b \rho d'_u d_v d_w \left[\left((1 - \rho)^n + \rho^n\right) P(a_n, p, \delta) P(a_n, q, \delta) + R_{p,q}(r, \rho) + R_{q,p}(r, 1 - \rho)\right].$$

Proof of (b). Again, we use a slightly different combination of some of the previously explained ideas. We begin using (5.5) to obtain

$$\frac{\Gamma(p) X_{+,n}^q}{2(q-1) X_{+,n}^p} \leq \frac{\Gamma(p) a_n^\psi \eta_-^\psi}{X_{-,n}^q} + (X_{-,n} - a_n \eta_-^\psi) \left(\frac{1}{p} + \int_1^{\infty} x^{p-1} e^{-x X_{-,n}} dx\right).$$
It remains to multiply the above expression by $\ell_{n+1}^r E_{j}^\# \eta_+^w \eta_-^w$ and take expectations.

The first term in (6.17) can be bounded as in part (a). The second term $(X_{-n} - a_n \eta_{-1/\alpha})^q / p$ in (6.17) can be handled as in Lemma 10 (see (6.3) and (6.6)). Indeed, we have

$$
(X_{-n} - a_n \eta_{-1/\alpha})^q \ell_{n+1}^r E_{j}^\# \eta_+^w \eta_-^w = \left( \sum_{k=1}^{n} \ell_{k}^x E_{k}[G_{k}] \ell_{n+1}^x \right)^q \eta_+^w \eta_-^w.
$$

The expected value of (6.18) may be bounded via $L^q$-seminorms: denote by $||\vartheta||_q = E[\vartheta^q]^{1/q}$ the $L^q$-seminorm of $\vartheta$ where $q' = q \vee 1$ (which is a true norm if $q \geq 1$). Let $g_q = E[(|G_{k}|)^q]$ and $h_a = \max\{\Gamma(1 + u + q \zeta), \Gamma(1 + q \zeta) \Gamma(1 + u)\}$; observe that when $\alpha < 1$, we have $q \zeta > \alpha - 1 > -1$. Then the triangle inequality and the independence gives

$$
\left| \left( \sum_{k=1}^{n} \ell_{k}^x E_{k}[G_{k}] \ell_{n+1}^x \right)^q \right| 
\leq \left( \sum_{k=1}^{n} \left| \ell_{k}^x \ell_{n+1}^x \right| E_{k}[G_{k}] \right) q^{q'} 
\leq \frac{h_a g_q B (1 + \frac{q}{\alpha}) (1 - k) / q'}{(1 + r)^{q'}} 
\leq \frac{h_a g_q B (1 + \frac{q}{\alpha}) (1 + r + q / \alpha)}{(1 + r)^{q'}} 
\leq \frac{h_a g_q B (1 + \frac{q}{\alpha}) (1 + r + q / \alpha)}{(1 + r)^{q'}}
$$

which completes the bound on the second term in (6.17) once one notes that $\eta_+$ and $\eta_-$ are independent from the other variables and $E[\eta_+^w \eta_-^w] = \Gamma(v + 1) \Gamma(w + 1)$.

The third term in (6.17) may be bounded as follows. Set $s = q / 2 < \alpha / 2 \leq 1$, then we may use (5.5) to obtain

$$
E \left[ \left( \sum_{k=1}^{n} \ell_{k}^x E_{k}^{\#} (G_{k}) \right)^q \ell_{n+1}^x \eta_+^w \eta_-^w \right] 
\leq \int_{1}^{\infty} x^{p-1} E \left[ \left( \sum_{k=1}^{n} \ell_{k}^x E_{k}^{\#} (G_{k}) \right)^q \ell_{n+1}^x \eta_+^w \eta_-^w \right] dx 
= 2 \int_{1}^{\infty} x^{p-1} \sum_{k=1}^{n} \sum_{i=k+1}^{n} E \left[ \ell_{k}^x \ell_{i}^x (G_{k}) \eta_+^w \eta_-^w \right] dx 
+ \int_{1}^{\infty} x^{p-1} \sum_{k=1}^{n} E \left[ \ell_{k}^x \eta_+^w \eta_-^w \right] dx.
$$

The previous expression can be dealt with as in (a) and (b). That is, first we average with respect to $(G_n)_{n \in \mathbb{N}}$, using that $E[(|G_{k}|)^s e^{-x|G_{k}|}] = E[(|G_{k}|)^s]$. In particular, one uses Lemma 15 (b) for the terms containing exponentials of $G$ and (6.1) for the terms which do not contain exponentials of $G$ in order to obtain a similar estimate as in (6.13). For the terms which contain $\eta_{\pm}$, one uses Lemma 15 (a). Next, one takes expectations with respect to $(E_n)_{n \in \mathbb{N}}$. As in the proof of Lemma 11 (a), one defines the appropriate $d_n^x$ which will bound all the required powers of $E$. Finally, as in steps II) and III) of the proof of Lemma 10, we take the expectations for $(\ell_n)_{n \in \mathbb{N}}$ using Lemma 13 (b). Each term in the first sum can be bounded by $C' \theta_1^{i+k} (1 + r)^{-n}$ for some $C' > 0$, $\theta_1 \in (0, 1)$ (independent of $i, k, n$) and all $k < i \leq n$, whereas each term in the second sum can be bounded by some $C'' \theta_2^i (1 + r)^{-n} (1 + (1 - \rho)^{n} P(a_n, p, \delta))$ for some $C'' > 0$, $\theta_2 \in (0, 1)$ (independent of $i, k, n$) and all $k \leq n$. The claim of part (b) then follows, completing the proof. \[\square\]
Proof of Lemma 12. We will prove the case \( s > 0 \), as the case \( s = 0 \) is very similar. The result is a consequence of Lemma 11(a). Since \( [S_k]^+[S_k]^- = 0 \), observe that using (5.5)

\[
Z_m^s \leq (m + 2)^{[s - 1]^+} \left( \eta_k^+ + \eta_k^- + \sum_{k=1}^m E_k^\alpha \right) \quad \text{and} \quad \Delta_{n+1}^+ = \xi_{n+1}^{1/\alpha} [S_{n+1}]^+ + (a_{n+1} - a_n) \eta_k^+.
\]

Recall that \( S_{n+1} = E_{n+1}^\alpha G_{n+1} \), where \( G_{n+1} \) and \( E_{n+1}^\alpha \) are independent of each other and of every other random variable in the expectations of the statement. Similarly, \((E_{n+2}, \ldots, E_{m})\) is independent of every other random variable in the expectations of the statement. An application of Lemma 11(a) (and (6.1)) gives the claim if one uses hypothesis (A-\( \kappa \)). The second claim follows similarly using Lemma 11(b). In particular, note that the restriction on \( r \) in the case (a) is due to the \( r \)-th moment of \( G_{n+1} \) while in the case (b), the restriction on \( r \) ensures that the power \( p - r \) of \( X_{+,n+1} \) is non-negative. \( \square \)

Remark 6. Note that in the above results the parameters for the negative moments can not achieve their upper limit. This is the main reason for not being able to achieve \( \alpha' = \alpha \) in Theorem 1.

7. Final remarks

In this section, we gathered some extra technical comments that may be useful for other developments.

(i) Our claim for nearly-optimal bound is not proven in two particular situations. That is, in the special case where the stable process is of infinite variation and has only negative jumps (i.e. \( \alpha \rho = 1 \)), \( X_T \) has exponential moments and therefore our bound is suboptimal for large \( y \). However, the optimality of the bound is retained in a neighborhood of 0. Although we do not provide the details here, our methods could be applied to obtain the corresponding exponential bound for the density as \( y \to \infty \) in this special case, one may use the techniques in the proof of Proposition 8 (a) and (c) to obtain exponential bounds in \( x_+ \). In those cases, we would show that the densities and all their derivatives decay faster than any polynomial \( x_+^{-p} \), \( p > 0 \), as \( x_+ \to \infty \). In the other extreme, when the infinite variation process has only positive jumps (i.e. \( \alpha(1 - \rho) = 1 \)), analogous remarks apply.

(ii) We stress that the constant \( C \) in Proposition 8 is independent of \( n \) and \( x_+ > 0 \). In fact, it can be shown that \((\alpha - \alpha')C\) is bounded as \( \alpha' \to \alpha \).

Appendix A. Moments of the stick-breaking process

Recall from Subsection 3.1 the definition of stick-breaking process \( \ell \) on \([0, T]\) and its remainders \((L_{k-1})_{k \in \mathbb{N}}\).

Lemma 13. (a) Let \( n \in \mathbb{N} \) and \( p_1, \ldots, p_n > -1 \) satisfy \( q_k := \sum_{i=k+1}^n p_i > -1 \) for \( k \in \{1, \ldots, n\} \) (with \( q_n := 0 \)). Let \( q_0 := \sum_{k=1}^n p_k \), then we have

\[
\mathbb{E} \left[ \prod_{k=1}^n \ell_k^{p_k} \right] = T^{q_0} \prod_{k=1}^n B(1+p_k, 1+q_k),
\]

where \( B(\cdot, \cdot) \) denotes the beta function. In particular \( \mathbb{E} [\ell_k^{p_k}] = T^{p(1+p)} \) for \( k \geq 1 \) and \( p > -1 \).

(b) Let \( p, q, r \geq 0 \) and define \( \theta = \frac{1+r+p+q}{1+r+p+q} \leq 1 \). Then there is some \( C > 0 \) such that for any \( 1 \leq j \leq k \leq n \) and \( T > 0 \), we have \( \mathbb{E} [\ell_k^{p_k} \ell_j^{p_j}] \leq C T^{p+q+r} \theta^{p+q} (1+r)^{-n} \).

(c) If \( p + q + r > -1 \) and \( k \geq 2 \) then \( \mathbb{E} [L_{k-1}^{p_k} \ell_k^{p_k}] \leq T^{p+q+r} B(1+p+q, 1+r+q) (1+q)^{-1} (1+p+q)^{2-k} \).

Proof. (a) Recall \( \ell_k = T(1-U_k) \prod_{i=k}^{k-1} U_i \) for \( k \geq 1 \), implying \( \prod_{k=1}^n \ell_k^{p_k} = T^{q_0} \prod_{k=1}^n U_k^{p_k} (1-U_k)^{p_k} \). Equation (A.1) follows from the identity \( \mathbb{E} [U_k^\theta (1-U_1)^q] = B(1+p, 1+q) \) and the independence of the uniformly distributed random variables \( U_1, \ldots, U_n \).
(b) Applying (A.1) yields (note that some factors in the product become 1 in this case)

\[ \frac{\mathbb{E}[\ell^p q^{y^n}]}{\ell^{p+q+y^n}} = (1+r)^{-n} \times \begin{cases} B(1+p, 1+q+r)(1+q+r)^{j-k-1} & j < k < n, \\ \times B(1+q, 1+r)(1+p+q+r)^{1-j}, & j = k < n, \\ (1+p+q+r)^{j-k} B(1+p+q+1+r)(1+q+r)^{j-k} & j = k = n. \end{cases} \]

In order to avoid considering four different cases to obtain the claimed bound, observe that \((1+b+c)/(1+a+c) \leq (1+b)/(1+a)\) for 0 \(\leq a \leq b\) and \(c \geq 0\). The claim then follows easily.

c) The proof is analogous to that of part (a).

\[ \square \]

**Appendix B. Technical lemmas for moment estimates**

**Lemma 14.** Let \(x_1, x_2, \ldots, y_1, y_2, \ldots \in [0,1]\). Then for any \(r \in [0,1]\) and \(n \in \mathbb{N}\), it holds that

\[ \prod_{k=1}^{n} ((1-r) + r x_k) \leq (1-r)^n + \sum_{k=1}^{n} r(1-r)^{k-1} x_k \tag{B.1} \]

\[ \prod_{k=1}^{n} ((1-r) y_k + r x_k) \leq r^n + (1-r)^n + \sum_{k=2}^{n} r(1-r)^{k-1} x_k y_1 + \sum_{k=2}^{n} (1-r)^{k-1} x_1 y_k. \tag{B.2} \]

**Proof.** Identity (B.1) follows by developing the product term by term and using the fact that every term in the product is bounded by 1. Indeed, we have

\[ \prod_{k=1}^{n} ((1-r) + r x_k) \leq r x_1 + (1-r) \prod_{k=2}^{n} ((1-r) + r x_k) \leq \cdots \leq (1-r)^n + \sum_{k=1}^{n} r(1-r)^{k-1} x_k. \]

The same ideas yield (B.2):

\[ \prod_{k=1}^{n} ((1-r) y_k + r x_k) = (1-r) y_1 \prod_{k=2}^{n} ((1-r) y_k + r x_k) + r x_1 \prod_{k=2}^{n} ((1-r) y_k + r x_k) \leq y_1 \left( (1-r)^n + \sum_{k=2}^{n} r(1-r)^{k-1} x_k \right) + x_1 \left( r^n + \sum_{k=2}^{n} (1-r)^{k-1} y_k \right) \leq r^n + (1-r)^n + \sum_{k=2}^{n} r(1-r)^{k-1} x_k y_1 + \sum_{k=2}^{n} (1-r)^{k-1} x_1 y_k. \]

\[ \square \]

In the next lemma, we give bounds for the Laplace transforms of random variables related to the Chambers-Mallows-Stuck representation of stable laws.

**Lemma 15.** (a) Suppose \(\alpha \neq 1\). For any \(s \geq 0\) define \(d_s = 2^s \max\{1, s^\alpha e^{-s}, \Gamma(s+1)\}\) (with the convention \(0^0 = 1\)) and let \(\zeta = 1 - 1/\alpha\). Define

\[ (c, \delta) := \begin{cases} \{\max\{1, \int_{0}^{\infty} \exp(-y^\alpha)dy\}, 1/\zeta\}, & \text{if } \alpha \in (1,2), \\ \{(2 + 1/\zeta) \max\{1, (2e^{-1}/\alpha)^{1/\alpha}\}, 1\}, & \text{if } \alpha \in (0,1). \end{cases} \]

If \(Y\) is a exponential variable with unit mean, then for \(s, x \geq 0\) we have

\[ \mathbb{E}[Y^s e^{-xY^\alpha}] \leq cd_s \min\{1, x^{-\delta}\}. \tag{B.3} \]

(b) Recall that \(G = g(V)\) where \(V\) follows a \(U(-\frac{x}{2}, \frac{x}{2})\) law. Suppose \(\alpha \neq 1\) then for any \(x > 0\), it holds that

\[ \mathbb{E}[e^{-xG}\mathbb{1}_{G > 0}] \leq \min\{1, (\gamma \alpha \rho x)^{-1}\}, \text{ and } \mathbb{E}[e^{-xG^+}] \leq 1 + \rho \min\{1, (\gamma \alpha \rho x)^{-1}\}, \text{ where} \tag{B.4} \]

\[ x \]
\[ \gamma := \begin{cases} 1, & \text{if } \alpha \in (1, 2), \\ \min \{ \cos (\pi (\frac{1}{2} - \rho)), \cos (\pi (\frac{1}{2} - \alpha \rho)) \}^{1/\alpha - 1}, & \text{if } \alpha \in (0, 1). \end{cases} \]

(c) For any \( p \in \mathbb{R}, q > p \) and \( b > 0 \) we have

\[ \int_1^\infty x^{p-1} \min \{1, (bx)^{-q}\} \, dx = P(b, p, q), \quad \text{where } P(b, p, q) := \frac{(b \wedge 1)^{-p} - 1}{p} + \frac{b^{-q}(b \wedge 1)^{q-p}}{q-p}. \]

Moreover, if \( q = 1 > b \), then the above integral equals \( \frac{1}{p(1-p)} b^{-p} - \frac{1}{p} \).

**Proof.** (a) Consider first the case \( \alpha > 1 \). For \( s \geq 0 \) it holds that \( \mathbb{E}[Y^s e^{-xY^\zeta}] \leq \mathbb{E}[Y^s] = \Gamma(s+1) \leq d_s \leq cd_s \).

Moreover, since \( y^s e^{-y} \leq d_s \) for all \( y \in \mathbb{R}_+ \), we have

\[ \mathbb{E}[Y^s e^{-xY^\zeta}] = \int_0^\infty y^s e^{-xy^{\zeta}} \, dy \leq d_s \int_0^\infty e^{-xy^{\zeta}} \, dy \leq cd_s x^{-1/\zeta} = cd_s x^{-\delta}, \]

implying (B.3) for \( \alpha > 1 \).

Suppose \( \alpha < 1 \), so that \( \zeta < 0 \). As before, we have \( \mathbb{E}[Y^s e^{-xY^\zeta}] \leq \mathbb{E}[Y^s] = \Gamma(s+1) \leq cd_s \). Without loss of generality assume that \( x \geq 1 \). Recall that \( y^s e^{-y} \leq d_s \) for all \( y \in \mathbb{R}_+ \). Hence

\[ \mathbb{E}[Y^s e^{-xY^\zeta}] = \int_0^\infty y^s e^{-xy^{\zeta}} \, dy \leq d_s \int_0^\infty e^{-xy^{\zeta}} \, dy. \]

Decomposing this integral into two parts yields

\[ \int_0^\infty e^{-xy^{\zeta}} \, dy = \int_0^{x^\alpha} e^{-xy^{\zeta}} \, dy + \int_{x^\alpha}^\infty e^{-xy^{\zeta}} \, dy \leq 2(2e^{-1/\alpha})^{1/\alpha} x^{-1/\alpha} \int_0^{x^\alpha} e^{-xy^{\zeta}} \, dy, \]

since \( (2e^{-1/\alpha})^{1/\alpha} \geq y e^{-y^{2/2}} \) for any \( y \geq 0 \). For the remaining integral in (B.6), note that for any \( s \geq 0 \) and \( z \geq 1 \) we have \( \int_z^\infty y^{-\alpha} e^{-y} \, dy \leq z^{-\alpha} e^{-z} \) and change of variables \( u = y^{\zeta} \) (recall that \( 1 + \alpha \zeta = \alpha \)):

\[ \int_0^{x^\alpha} e^{-xy^{\zeta}} \, dy = \frac{x^{-1/\zeta}}{\mathbb{K}} \int_0^{x^\alpha} u^{1-1/\zeta} e^{-u} \, du \leq \frac{1}{\mathbb{K}} x^{-(1-\alpha)/\zeta} e^{-x^{-\alpha}} = \frac{1}{\mathbb{K}} e^{-x^{-\alpha} \leq \frac{1}{\mathbb{K}} (e^{-1/\alpha})^{1/\alpha} x^{-1}. \]

This concludes the proof of (B.3).

(b) The conditional law of \( G \) given \( G > 0 \) is that of \( g(V) \), where \( V \) is uniformly distributed on the interval \( (\pi (\frac{1}{2} - \rho), \frac{\pi}{2}) \). Define \( z \) by \( v = z \pi \rho - \pi (\rho - \frac{1}{2}) \) and note that \( v \in (\pi (\frac{1}{2} - \rho), \frac{\pi}{2}) \) if and only if \( z \in (0, 1) \). Then, for \( v \in (\pi (\frac{1}{2} - \rho), \frac{\pi}{2}) \), we claim that the function \( g \) in (3.2) satisfies

\[ g(v) \geq \gamma \sin (z \pi \alpha \rho > 0. \]

Indeed, in the case that \( \alpha > 1 \), the product of cosines in the denominator of \( g \) is bounded above by \( 1 \), the exponents are positive and \( \alpha \rho \in [\alpha - 1, 1] \) with \( \gamma = 1 \). Similarly, in the case that \( \alpha < 1 \), then \( v - z \pi \alpha \rho \in \{\pi (\frac{1}{2} - \rho), \pi (\frac{1}{2} - \alpha \rho)\} \). Since \( \rho, \alpha \rho \in (0, 1) \), we have \( \cos^{1/\alpha - 1}(v - z \pi \alpha \rho) \geq \gamma > 0 \) and the inequality holds.

The concavity of the sine function on \( (0, \frac{\pi}{2}) \), implies \( \sin (u \frac{\pi}{2}) \geq u \) for any \( u \in (0, 1) \). Furthermore, \( \sin (z \pi \alpha \rho) \) is symmetric on \( z \in [0, \frac{1}{\alpha \rho}] \subset [0, 1] \) with respect to the point \( z = \frac{1}{\alpha \rho} \). Hence, using these properties, we have

\[ \mathbb{E}[e^{-xG} | G > 0] = \int_0^1 e^{-z \pi (\rho - \pi (\rho - 1/2)))} \, dz \leq \int_0^1 e^{-\gamma z \sin (z \pi \alpha \rho) d} \leq \int_0^{\pi / \rho} e^{-\gamma z \sin (z \pi \alpha \rho) d} \]

\[ = 2 \int_0^{\pi / \rho} e^{-\gamma z \sin (z \pi \alpha \rho) d} \leq 2 \int_0^{\pi / \rho} e^{-2 \gamma z \alpha \rho z} \, dz \leq 2 \int_0^{\infty} e^{-2 \gamma z \alpha \rho z} \, dz = \frac{1}{\gamma \alpha \rho z}. \]

The conclusion of part (b) then follows from the fact that \( \mathbb{E}[e^{-xG} | G > 0] \leq 1 \). For the second statement, it is enough to note that \( \mathbb{P}(|G|^+ = 0) = 1 - \rho \).

(c) The proof follows from elementary calculations. \qed
In this section, we will briefly remark the changes needed in all the arguments for the case $\alpha = 1$ in our proofs, we proceed in the order that the arguments are presented in the main text.

The Chambers-Mallows-Stuck method is not required in this case because when $\alpha = 1$, the stable random variables $(S_k)_{k \geq 1}$ have the explicit density

$$p(x) = \frac{\cos(\omega)/\pi}{\cos^2(\omega) + (x - \sin(\omega))^2}, \quad x \in \mathbb{R}.$$  

For the approximation $X_{\pm,n}$ we use $X_{\pm,n} := \sum_{k=1}^n \ell_k^{1/\alpha} [S_k]^{\pm} + a_n \eta_{\pm}$, where $\eta_{\pm}$ are respectively distributed as $\pm S_1$ conditioned on the events $\{S_1 > 0\}$ and $\{S_1 < 0\}$. Note that this already hints at the fact that we will not use exponential random variables as “length” in this case. Instead we will use the full Cauchy random variables to do the analysis.

The derivative operator is defined as

$$D_m^\pm \equiv \eta_{\pm} \partial \eta_{\pm} \pm \sum_{k=1}^m [S_k]^{\pm} \partial S_k.$$  

This operator satisfies

$$[S_k]^{\pm} \partial S_k [X_{\pm,n}] = \pm \ell_k [S_k]^{\pm} \mathbb{I}_{\{k \leq n\}}, \quad k \in \mathbb{N}, \text{ if } \alpha = 1,$$

$$D_m^\pm [(X_{\pm,n}^\alpha, f(X_{\pm,n}))] = (pX_{\pm,n},0), \quad m \geq n \geq 1.$$  

The space of smooth random variables is

$$S_m(\Omega) := \{ \Phi \in L^0(\Omega) : \exists \phi(\cdot, \vartheta) \in \mathbb{S}_\infty (\mathbb{R}^{2m+2}, S), \quad \Phi = \phi(S_m, U_m, \eta_{+}, \eta_{-}, \vartheta), \}.$$  

where $S_m := (S_1, \ldots, S_m)$. The Ibpf in finite dimension gives

$$H_{m,n}^\pm(\Phi) := \frac{1}{X_{\pm,n}} \left( \frac{2\eta_{\pm}(\pm \eta_{\pm} - \sin(\omega))}{\cos^2(\omega) + (\eta_{\pm} \mp \sin(\omega))^2} \pm \sum_{k=1}^m \frac{2[S_k]^{\pm}(S_k - \sin(\omega))}{\cos^2(\omega) + (S_k - \sin(\omega))^2} \right) \Phi - D_m^\pm(\Phi).$$  

Note that the formula is essentially different from the one in (3.7) as this formula is based on Cauchy random variables. In the proof of Proposition 5, one uses

$$\partial \vartheta[\omega] = \omega \frac{2(\vartheta - \sin(\omega))}{\cos^2(\omega) + (\vartheta - \sin(\omega))^2} - \partial \vartheta[\omega].$$  

In this case recall that if $\eta$ is a Cauchy random variable with parameter $\rho$. Also one has in this case

$$\mathbb{E}[\Lambda_1 \eta \partial \eta[\Lambda_2]] = \mathbb{E} \left[ \Lambda_1 \Lambda_2 \eta^{\pm} \frac{2(\eta - \sin(\omega))}{\cos^2(\omega) + (\eta - \sin(\omega))^2} - \Lambda_2 \partial \eta[\Lambda_1 \eta^{\pm}] \right] = \mathbb{E}[\Lambda_2 \partial \eta[\Lambda_1 \eta^{\pm}]].$$  

From here the rest of the proof of Proposition 5 follows similarly. The statement in Theorem 7 and its proof are independent of $\alpha$.

The statement of the main Theorem and its proof remain unchanged if $\alpha = 1$.

Starting in Lemma 9 and for the rest of the proof in this case, we let $Z_m = 1$. The proof of this lemma in this case is as follows:

Recall that $D_m^\pm [X_{\pm,n}^\alpha] = pX_{\pm,n}^\alpha$ and $D_m^\pm [X_{\pm,n}^\alpha] = 0$ for any $p \in \mathbb{R}$. Define the bounded functions $q_{\pm} : x \mapsto 2[x]^{\pm}(x - \mu)/(\gamma^2 + (x - \mu)^2)$. Recursively define the bounded functions $q_{\pm}^{(k+1)}(x) := x \partial x q_{\pm}^{(k)}(x)$ and the operators $D_m^{\pm,k+1} := D_m^{\pm,k} D_m^{\pm,1}$ for $k \geq 1$, where $q_{\pm}^{(1)} = q_{\pm}$ and $D_m^{\pm,1} = D_m^{\pm}$. Let $Z_{k,m}^{\pm} = q_{\pm}^{(k)}(S_{\pm}) + \sum_{i=1}^m q_{\pm}^{(k)}([S_i]^{\pm})$, we deduce that an iteration of (3.7) yields $X_{\pm,n}^\alpha X_{\pm,n}^{-\alpha}$ multiplied by

$$p_+(Z_{1,m}^+, \ldots, Z_{k,m}^+, D_m^{1+} [\Phi], \ldots, D_m^{+k+1} [\Phi]) p_-(Z_{1,m}^-, \ldots, Z_{k,m}^-, D_m^{-1} [\Phi], \ldots, D_m^{-k} [\Phi]).$$
where $p_{\pm}$ are multivariate polynomials of degree $k_{\pm}$ whose coefficients are linear in $\Phi$ and do not depend on $n$ or $m$. The arguments of $p_{\pm}$ are uniformly bounded by $Km$ for some $K > 0$ independent of $n$ and $m$ (recall $\|q_{\pm}(n)\|_\infty < \infty$ and $\|\partial_{+}^{j_{\pm}} \partial_{-}^{L_{\pm}} \phi\|_\infty < \infty$ for $j_{\pm} \leq k_{\pm}$). so the claim follows easily.

Not surprisingly the proof of the technical Proposition 8 and Lemmas in Section 5.1 do not depend on the fact that $\alpha = 1$ or not. In fact, we only used algebraic properties in order to obtain these results supposing the correct moment estimates. The required moment estimates are obtained in the next subsection.

C.1. Moment bounds for the Cauchy case $\alpha = 1$. For the proofs of Lemmas 10, 11 and 12, note that in comparison with the Chambers-Mallows-Stuck decomposition method we are not using any decomposition of the Cauchy random variables. This means that the proofs will be reduced to bounding the conditional expectations with respect to $(\ell_i)_{i \in \{1, \ldots, n\}}$, and then taking expectations there.

For the proof of Lemma 11 (and hence of Lemma 12), note that in order to compute inverse moments, we use a change of variable trick to handle negative moments via Laplace transforms. In the present case, this implies the computation of $E[e^{-x[S_i]^\pm}]$. This is done directly using the fact that its density is known. The rest of the calculations are very similar. Indeed, the version of Lemma 15 for the case $\alpha = 1$ contains the only noticeable change. That is,

**Lemma 16.** Suppose $\alpha = 1$, then for any $s \in (0, 1)$ and $x > 0$ we have

$$E[\eta_+^s e^{-\eta+}] \leq \frac{\cos(\omega)}{\pi \rho} \min \left\{ \int_0^{\infty} \frac{y^s}{\cos^2(\omega) + (y - \sin(\omega))^2} dy, \frac{\Gamma(s + 1)}{\cos^2(\omega)x^{s+1}} \right\}.$$  

In particular, $E[e^{-\eta+}] \leq \min\{1, (\pi \cos(\omega)\rho x)^{-1}\}$.

**Proof.** Observe that for any $s \in (0, 1)$, it holds that

$$E[\eta_+^s e^{-\eta+}] = \frac{\cos(\omega)}{\pi \rho} \int_0^{\infty} \frac{y^s}{\cos^2(\omega) + (y - \sin(\omega))^2} e^{-xy} dy \leq \frac{\cos(\omega)}{\pi \rho} \min \left\{ \int_0^{\infty} \frac{y^s}{\cos^2(\omega) + (y - \sin(\omega))^2} dy, \frac{\Gamma(s + 1)}{\cos^2(\omega)x^{s+1}} \right\}. \tag*{$\square$}$$

Given the above change, the statements in Lemma 11 are valid with the following changes: For $u \in [0, 1)$ and all other parameters as in in Lemma 11

(a) $E \left[ \frac{\ell_{r+n+1}^{\pm} \eta_{+}^u}{X_{+, n}^{r} X_{-, n}^{r}} \right] \leq CT^{r-\frac{s}{\alpha}}(1 + r)^{-n}$

(b) $E \left[ \frac{X_{-, n}^{q} \ell_{r+n+1}^{\pm} \eta_{+}^u}{X_{+, n}^{r}} \right] \leq CT^{q+\frac{s}{\alpha}}(1 + r)^{-n}$.

The Lemmas 10 and 12 remain unchanged.

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