TWO-PARAMETER QUANTUM GROUPS AND RINGEL-HALL ALGEBRAS OF $A_\infty$–TYPE

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Abstract. In this paper, we study the two-parameter quantum group $U_{r,s}(sl_\infty)$ associated to the Lie algebra $sl_\infty$. We shall prove that the two-parameter quantum group $U_{r,s}(sl_\infty)$ admits both a Hopf algebra structure and a triangular decomposition. In particular, it can be realized as the Drinfeld double of its certain Hopf subalgebras. We will also study a two-parameter twisted Ringel-Hall algebra $H_{r,s}(A_\infty)$ associated to the category of all finite dimensional representations of the infinite linear quiver $A_\infty$. In particular, we will establish an iterated skew polynomial presentation of $H_{r,s}(A_\infty)$ and construct a PBW basis for $H_{r,s}(A_\infty)$. Via the theory of generic extensions in the category of finite dimensional representations of $A_\infty$, we shall construct several monomial bases and a bar-invariant basis for $U_{r,s}(sl_\infty)$.

Introduction

As generalizations or variations of the notation of quantum groups [13], several multi-parameter quantum groups have appeared in the literatures [1, 8, 11, 12, 16, 18, 25, 31, 32]. Let $g$ be a finite dimensional complex simple Lie algebra. Let us choose $r, s \in \mathbb{C}^*$ in such a way that $r, s$ are transcendental over $\mathbb{Q}$. The study of the two-parameter quantum group $U_{r,s}(g)$ has been revitalized in [3, 4, 5, 6, 7] and the references therein. Note that the one-parameter quantum groups associated to Lie algebras $gl_\infty, sl_\infty$ of infinite ranks [17] have been studied in the literatures [10, 19, 22, 23]. Similar to the case of one-parameter quantum groups, one might be interested in the constructions of the corresponding two-parameter quantum groups.
It is the purpose of this paper to study the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$, where the Lie algebra $\mathfrak{sl}_\infty$ consists of all infinite trace-zero square matrices with only finitely many non-zero entries. We shall first define such a two-parameter quantum group and then study some of its basic properties. As usual, we will prove that the algebra $U_{r,s}(\mathfrak{sl}_\infty)$ admits a Hopf algebra structure and it is the Drinfeld double of its certain Hopf subalgebras.

To further investigate the structure of $U_{r,s}(\mathfrak{sl}_\infty)$, we shall study its subalgebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ employing the approach of Ringel-Hall algebras. Note that the Ringel-Hall algebra approach has found many important applications in the study of one-parameter quantum groups \cite{15,20,21,24,26,27,28,30,34} and the references therein. We shall first define a two-parameter twisted Ringel-Hall algebra $H_{r,s}(A_\infty)$ associated to the category of all finite dimensional representations of the infinite linear quiver $A_\infty$. Then we shall prove that the algebra $H_{r,s}(A_\infty)$ can be presented as an iterated skew polynomial ring, and construct a PBW basis for $H_{r,s}(A_\infty)$. Furthermore, we shall prove that $H_{r,s}(A_\infty)$ is a direct limit of the two-parameter twisted Ringel-Hall algebras $H_{r,s}(A_n), n \geq 1$ associated to the finite linear quivers $A_n, n \geq 1$ \cite{24,33}.

We will establish an algebra isomorphism from $U^+_{r,s}(\mathfrak{sl}_\infty)$ onto $H_{r,s}(A_\infty)$. On the one hand, such an algebra isomorphism provides a generator-relation presentation of the two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$, which has been defined over a prescribed basis. On the other hand, via this isomorphism, we can prove that the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ can be presented as an iterated skew polynomial ring and it is a direct limit of $U^+_{r,s}(\mathfrak{sl}_{n+1}), n \geq 1$. As a result, we are able to construct a PBW basis for $U^+_{r,s}(\mathfrak{sl}_\infty)$.

To study the Borel subalgebras $U_{r,s}(\mathfrak{sl}_\infty)$ of $U_{r,s}(\mathfrak{sl}_\infty)$, we will study the extended two-parameter twisted Ringel-Hall algebra $H_{r,s}(A_\infty)$ and establish an Hopf algebra isomorphism from $U_{r,s}(\mathfrak{sl}_\infty)$ onto $H_{r,s}(A_\infty)$. We will follow the lines in \cite{15,34}. This result shall provide a realization of the whole two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ via the double of two-parameter extended Ringel-Hall algebras.

Note that there exists a $\mathbb{Q}$–algebra automorphism (which will be called the bar-automorphism) on the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$, which exchanges $r^{\pm 1}$ and $s^{\pm 1}$ and fixes the generators $e_i$. Using the theory of generic extensions in the category of finite dimensional representations of $A_\infty$, we will construct several monomial bases for the two-parameter quantum groups following the idea used in \cite{9,24}. As an application, we will also construct a bar-invariant basis for the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ following \cite{24}. 


The paper is organized as follows. In Section 1, we give the definition of $U_{r,s}(\mathfrak{sl}_\infty)$ and study some of its basic properties. In Section 2, we define and study two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$ and establish the algebra isomorphism from $U^+_r(s\mathfrak{l}_\infty)$ onto $H_{r,s}(A_\infty)$. In Section 3, we define and study the extended two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$ and establish the Hopf algebra isomorphism from $U_{\geq 0}r,s(g)$ onto $H_{r,s}(A_\infty)$. In Section 4, we use generic extension theory to construct some monomial bases and a bar-invariant basis for $U^+_r(s\mathfrak{l}_\infty)$. 

1. Definition and basic properties of the two-parameter quantum groups $U_{r,s}(\mathfrak{sl}_\infty)$

Let $r, s$ be two-parameters chosen from $\mathbb{C}^*$, such that $r, s$ are transcendental over the field $\mathbb{Q}$ and $r^m s^n = 1$ implies $m = n = 0$. Let us set $\mathcal{Z} = \mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$ and $\mathcal{A} = \mathbb{Q}[r, s](r-1, s-1)$, which is the localization of $\mathbb{Q}[r, s]$ at the maximal ideal $(r - 1, s - 1)$.

Let $\mathfrak{sl}_\infty$ denote the infinite dimensional complex Lie algebra which consists of all trace-zero square matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with only finitely many non-zero entries. The one-parameter quantum groups $U_q(\mathfrak{sl}_\infty)$ associated to $\mathfrak{sl}_\infty$ were studied by various people in the references \[10, 19, 22, 23\]. Following a similar idea in \[5, 32\], we will introduce a class of two-parameter quantum groups $U_{r,s}(\mathfrak{sl}_\infty)$ associated to the Lie algebra $\mathfrak{sl}_\infty$.

It is well known that one can also define roots for the Lie algebra $\mathfrak{sl}_\infty$ as in the finite dimensional case of $g = \mathfrak{sl}_{n+1}$. In particular, all the simple roots of $\mathfrak{sl}_\infty$ can be denoted as $\alpha_i, i \in I = \mathbb{N}$. Accordingly, all the positive roots of $\mathfrak{sl}_\infty$ are exactly given as $\alpha_{ij} = \sum_{k=i}^j \alpha_k$ for $i \leq j \in \mathbb{N}$.

Let $C = (c_{ij})_{i,j\in\mathbb{N}}$ denote the infinite Cartan matrix corresponding to the Lie algebra $\mathfrak{sl}_\infty$. Then, we have the following

$$c_{ii} = 2, \quad c_{ij} = -1 \text{ for } |i - j| = 1, \quad c_{ij} = 0 \text{ for } |i - j| > 1.$$ 

Let $\mathbb{Q}(r, s)$ denote the function field in two variables $r, s$ over the field $\mathbb{Q}$ of all rational numbers. Let $\mathcal{Q}$ denote the root lattice generated by $\alpha_i, i \in \mathbb{N}$. Then we can define a bilinear form $\langle -, - \rangle$ on the root lattice $\mathcal{Q} \cong \mathbb{Z}^{\mathbb{N}}$ as follows

$$\langle i, j \rangle = \langle \alpha_i, \alpha_j \rangle = \begin{cases} a_{ij}, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j. \end{cases}$$

**Definition 1.1.** The two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ is defined to be the $\mathbb{Q}(r, s)$-algebra generated by $e_i, f_i, w_i^{\pm 1}, w_i'^{\pm 1}, i \in \mathbb{N}$.
subject to the following relations

\[
\begin{align*}
   w_i^{\pm 1}w_j^{\pm 1} &= w_j^{\pm 1}w_i^{\pm 1}, \\
   w_i^{\pm 1}w_j^{\pm 1} &= w_j^{\pm 1}w_i^{\pm 1}, \\
   w_i^{\pm 1}w_j^{\pm 1} &= w_j^{\pm 1}w_i^{\pm 1}, \\
   w_i^{\pm 1}w_j^{\pm 1} &= w_j^{\pm 1}w_i^{\pm 1}, \\
   w_ie_j &= r^{(i,j)}s^{-<i,j>}e_jw_i, \\
   w_if_j &= r^{-(i,j)}s^{<i,j>}f_jw_i, \\
   w_if_j &= r^{(i,j)}s^{-<i,j>}f_jw_i, \\
   e_if_j - f_jw_i &= \delta_{ij}w_i - w_i', \\
   e_ie_j - e_je_i &= f_if_j - f_jf_i = 0 \text{ for } |i-j| > 1, \\
   e_i^2e_{i+1} - (r+s)e_ie_{i+1} + rs e_i^2 &= 0, \\
   e_i^2f_{i+1} - (r+s)e_{i+1}f_i + rs e_{i+1}^2 &= 0, \\
   f_i^2f_{i+1} - (r^{-1} + s^{-1})f_if_{i+1}f_i + (rs)^{-1}f_{i+1}f_i^2 &= 0, \\
   f_i^2f_{i+1} - (r^{-1} + s^{-1})f_{i+1}f_if_i + (rs)^{-1}f_{i+1}f_i^2 &= 0.
\end{align*}
\]

First of all, we have the following obvious proposition concerning a Hopf algebra structure of the algebra $U_{r,s}(\mathfrak{sl}_\infty)$.

**Proposition 1.1.** The algebra $U_{r,s}(\mathfrak{g})$ is a Hopf algebra with the co-multiplication, counit and antipode defined as follows

\[
\begin{align*}
   \Delta(w_i^{\pm 1}) &= w_i^{\pm 1} \otimes w_i^{\pm 1}, \\
   \Delta(w_i') &= w_i' \otimes w_i', \\
   \Delta(e_i) &= e_i \otimes 1 + w_i \otimes e_i, \\
   \Delta(f_i) &= 1 \otimes f_i + f_i \otimes w_i', \\
   \epsilon(w_i^{\pm 1}) &= \epsilon(w_i') = 1, \\
   \epsilon(e_i) &= \epsilon(f_i) = 0, \\
   S(w_i^{\pm 1}) &= w_i'^{\mp 1}, \\
   S(w_i') &= w_i'^{\mp 1}, \\
   S(e_i) &= -w_i^{-1}e_i, \\
   S(f_i) &= -f_iw_i^{-1}.
\end{align*}
\]

**Proof:** The proof is reduced to the finite case where $\mathfrak{g} = \mathfrak{sl}_{n+1}$, whose proof can be found in [5]. And we will not repeat the details here. \(\square\)

Let $U_{r,s}^+(\mathfrak{sl}_\infty)$ (resp. $U_{r,s}^-(\mathfrak{sl}_\infty)$) denote the subalgebra of $U_{r,s}(\mathfrak{sl}_\infty)$ generated by $e_i, i \in \mathbb{N}$ (resp. by $f_i, i \in \mathbb{N}$). Let $U_{r,s}^0(\mathfrak{sl}_\infty)$ denote the subalgebra of $U_{r,s}(\mathfrak{sl}_\infty)$ generated by $w_i^{\pm 1}, w_i'^{\mp 1}, i \in \mathbb{N}$. Then we shall have the following triangular decomposition of $U_{r,s}(\mathfrak{sl}_\infty)$.

**Proposition 1.2.** The algebra $U_{r,s}(\mathfrak{sl}_\infty)$ has a triangular decomposition

\[
U_{r,s}(\mathfrak{sl}_\infty) \cong U_{r,s}^-(\mathfrak{sl}_\infty) \otimes U_{r,s}^0(\mathfrak{sl}_\infty) \otimes U_{r,s}^+(\mathfrak{sl}_\infty).
\]

**Proof:** Once again, we can repeat the proof used in the case of $U_{r,s}(\mathfrak{sl}_{n+1})$. We refer the reader to [5] for more details. \(\square\)
Let us denote by \(\mathbb{Z}^{\oplus N}\) the free abelian group of rank \(|N|\) with a basis denoted by \(z_1, z_2, \ldots, z_n, \ldots\). Given any element \(a \in \mathbb{Z}^{\oplus N}\), say \(a = \sum a_i z_i\), we set \(|a| = \sum a_i\). Note that algebra \(U_{r,s}^+ (\mathfrak{sl}_{\infty})\) (resp. \(U_{r,s}^- (\mathfrak{sl}_{\infty})\)) is a \(\mathbb{Z}^{\oplus N}\)-graded algebra by assigning to the generator \(e_i\) (resp. \(f_i\)) the degree \(z_i\). Given \(a \in \mathbb{Z}^{\oplus N}\), we denote by \(U_{r,s}^\pm (\mathfrak{sl}_{\infty})^a\) the set of homogeneous elements of degree \(a\) in \(U_{r,s}^\pm (\mathfrak{sl}_{\infty})\).

Proposition 1.3. We have the following decomposition
\[
U_{r,s}^+ (\mathfrak{sl}_{\infty}) = \bigoplus_a U_{r,s}^+ (\mathfrak{sl}_{\infty})^a, \quad U_{r,s}^- (\mathfrak{sl}_{\infty}) = \bigoplus_a U_{r,s}^- (\mathfrak{sl}_{\infty})^a.
\]

Let us define \(U_{v,v^{-1}} (\mathfrak{sl}_{\infty})\) to be the specialization of \(U_{r,s} (\mathfrak{sl}_{\infty})\) for \(r = v = s^{-1}\). Then we shall have the following similar result as [5].

Proposition 1.4. Assume there exists an isomorphism of Hopf algebras
\[
\phi: U_{r,s} (\mathfrak{sl}_{\infty}) \longrightarrow U_{v,v^{-1}} (\mathfrak{sl}_{\infty})
\]
for some \(v\). Then \(r = v\) and \(s = v^{-1}\).

1.1. A Drinfeld double realization of \(U_{r,s} (\mathfrak{sl}_{\infty})\). In this subsection, we show that the two-parameter quantum group \(U_{r,s} (\mathfrak{sl}_{\infty})\) can be realized as the Drinfeld double of its certain Hopf subalgebras. To proceed, we need to recall a couple of standard definitions for the Hopf pairing and the Drinfeld double of Hopf algebras. For more details about these concepts, we refer the reader to the references [3, 12].

Definition 1.2. A Hopf pairing of two Hopf algebras \(H'\) and \(H\) is a bilinear form \((,): H' \times H \longrightarrow \mathbb{K}\) such that
\[
(1) \quad (1, h) = \epsilon_H (h),
\]
\[
(2) \quad (h', 1) = \epsilon_{H'} (h'),
\]
\[
(3) \quad (h', hk) = (\Delta_{H'} (h'), h \otimes k) = \sum (h'_{(1)}, h)(h'_{(2)}, k),
\]
\[
(4) \quad (h'k', h) = (h' \otimes k', \Delta_H (h)) = \sum (h', h_{(1)})(k', h_{(2)}),
\]
for all \(h, k \in H', h', k' \in H'\), where \(\epsilon_H, \epsilon_{H'}\) denote the counits of \(H, H'\) respectively, and \(\Delta_H, \Delta_{H'}\) denote their comultiplications.

It is obvious that
\[
(S_{H'} (h'), h) = (h', S_H (h))
\]
for all $h \in H$ and $h' \in H'$, where $S_{H'}$ and $S_H$ denote the respective antipodes of $H$ and $H'$.

Let $U_{r,s}^{>0}(\mathfrak{sl}_\infty)$ (resp. $U_{r,s}^{\leq0}(\mathfrak{sl}_\infty)$) be the Hopf subalgebra of $U_{r,s}(\mathfrak{sl}_\infty)$ generated by $e_i, w_i^{\pm1}$ (resp. $f_i, w_i'^{\pm1}$). Assume that $B = U_{r,s}^{>0}(\mathfrak{sl}_\infty)$ and $(B'^{\text{coop}})$ is the Hopf algebra generated by $f_j, (w_j')^{\pm1}$ with the opposite coproduct to $U_{r,s}^{\leq0}(\mathfrak{sl}_\infty)$. Using the same proof in the case of $\mathfrak{sl}_{n+1}$ [5], we shall have the following result

**Lemma 1.1.** There exists a unique Hopf pairing $B$ and $B'$ such that

$$(f_i, e_j) = \frac{\delta_{i,j}}{s-r}\quad (w_i', w_j) = r^{<e_i,e_j>}s^{<e_j,e_i>},$$

and the pairing takes the zero value on all other pairs of generators. Moreover, we have $(S(a), S(b)) = (a, b)$ for $a \in B', b \in B$.

Therefore, we have the following similar result as in [5].

**Theorem 1.1.** $U_{r,s}(\mathfrak{sl}_\infty)$ can be realized as a Drinfeld double of Hopf subalgebras $B = U_{r,s}^{>0}(\mathfrak{sl}_\infty)$ and $(B'^{\text{coop}}) = U_{r,s}^{\leq0}(\mathfrak{sl}_\infty)$, that is,

$$U_{r,s}(\mathfrak{sl}_\infty) \cong D(B, (B'^{\text{coop}})).$$

**Proof:** First of all, let us define a linear map: $\phi: D(B, (B'^{\text{coop}})) \to U_{r,s}(\mathfrak{sl}_\infty)$ as follows

$$\phi(\hat{\omega}_i^{\pm1}) = \omega_i^{\pm1}, \quad \phi((\hat{\omega}_i')^{\pm1}) = (\omega_i')^{\pm1}$$

$$\phi(\hat{e}_i) = e_i, \quad \phi_i(\hat{f}_i) = f_i.$$ 

We need to show that this mapping is a Hopf algebra automorphism. Obviously, we can still employ the proof used in [5] for the finite case $g = \mathfrak{sl}_{n+1}$ and we will not repeat the detail here.

**1.2. An integral form of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty).** In addition, we can consider an integral form of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ and its subalgebras following [20].

For any $l \geq 1$, let us set the following

$$[l] = \frac{r^l - s^l}{r - s}, \quad [l]' = [1][2] \cdots [l].$$

Let us define $e_i^{(l)} = \frac{e_i}{[l]}, f_i^{(l)} = \frac{f_i}{[l]}$. We define a $\mathbb{Z}$–subalgebra $U_{r,s}(\mathfrak{sl}_\infty)_\mathbb{Z}$ of $U_{r,s}(\mathfrak{sl}_\infty)$ which is generated by the elements $e_i^{(l)}, f_i^{(l)}, w_i^{\pm1}, w_i'^{\pm1}$ for
Similarly, we can define the integral form of $U_{r,s}^+(\mathfrak{sl}_\infty)$ and $U_{r,s}^-(\mathfrak{sl}_\infty)$. It is easy to see that we have the following

$$U_{r,s}(\mathfrak{sl}_\infty) \cong U_{r,s}(\mathfrak{sl}_\infty)_\mathbb{Z} \otimes \mathbb{Q}(r,s)$$

and

$$U^\pm_{r,s}(\mathfrak{sl}_\infty) \cong U_{r,s}^\pm(\mathfrak{sl}_\infty)_\mathbb{Z} \otimes _\mathbb{Z} \mathbb{Q}(r,s).$$

In particular, $U_{r,s}(\mathfrak{sl}_\infty)$ (resp. $U^\pm_{r,s}(\mathfrak{sl}_\infty)$) is a free $\mathbb{Z}$-algebra.

2. Two-parameter Ringel-Hall algebras $H_{r,s}(A_\infty)$

To study the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$, it is helpful to study its subalgebra $U_{r,s}^+(\mathfrak{sl}_\infty)$. We shall study this algebra in terms of two-parameter Ringel-Hall algebra associated to the infinite linear quiver. We will define and study a two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$ associated to the category of finite dimensional representations of the infinite quiver $A_\infty$:

$$A_\infty: \bullet \to \bullet \to \bullet \to \ldots \to \bullet \to \ldots.$$ 

For $n \geq 1$, let $A_n$ denote the finite quiver

$$A_n: \bullet \to \bullet \to \bullet \to \ldots \to \bullet \to \ldots,$$

with $n$ vertices. Let us fix $k$ to be a finite field and let $\Lambda_n$ denote the path algebra of the finite linear quiver $A_n$ over $k$. Then $\Lambda_n$ is a finite dimensional hereditary algebra of finite-representation type. Note that the category of finite dimensional representations of the quiver $A_n$ is equivalent to the category of finite dimensional $\Lambda_n$-modules. We will denote this category by $A_n\text{-mod}$. Let us set $q = |k|$ the cardinality of $k$, and choose $v$ to be a number such that $v^2 = q$. We know that $\Lambda_n$ is finitary in the sense that the cardinality of the extension group $Ext^1(S,S')$ is finite for any two simple $\Lambda_n$-modules $S, S'$. Let us denote by $A_\infty\text{-mod}$, the category of all finite dimensional representations of the quiver $A_\infty$. Note that the category $A_\infty\text{-mod}$ has been investigated by Hou and Ye in [14], where they have trivially described all finite dimensional indecomposable representations of $A_\infty$ and studied the one-parameter non-twisted generic Ringel-Hall algebra $H_q(A_\infty)$. Let $S_i$ be the simple representation associated to the vertex $i$ of the quiver $A_\infty$ and let $M_{ij}$ denote the indecomposable representation of $A_\infty$ with a top $S_i$ and length $j - i + 1$. It is easy to see that there is a one-one correspondence between the set of isoclasses of finite dimensional indecomposable representations $M_{ij}$ of the quiver $A_\infty$ and the set of positive roots $\alpha_{ij}$ for the Lie algebra $\mathfrak{sl}_\infty$. 

$$i \in I.$$
Concerning the relationship between the categories $A_n$-mod and $A_\infty$-mod, we now recall the following result from [14].

**Theorem 2.1.** (Theorem 1.1 in [14]) The category $A_n$-mod can be regarded as a fully faithful and extension closed subcategory of $A_\infty$-mod and $A_m$-mod for $m \geq n$.

Based on the above theorem, we know that the extension group between any two finite dimensional representations $M, N$ of $A_\infty$ can be calculated via regarding $M, N$ as the representations of a certain finite quiver $A_n$. Therefore, the number of extensions between $M, N$ is still depicted by the evaluation of the Hall polynomial at $q$, the cardinality of the base field. Recall that the two-parameter Ringel-Hall algebra $H_{r,s}(A_n), n \geq 1$ associated to the category $A_n$-mod has been studied in [24, 33]. In particular, one knows that $H_{r,s}(A_n)$ can be presented as an iterated skew polynomial ring and its prime ideals are completely prime. A PBW basis has also been constructed for $H_{r,s}(A_n)$ in [33] as well. Note that this approach is plausible because of the existence of Hall polynomials in the category $A_\infty$-mod. Indeed, we will be looking at a limit version $H_{r,s}(A_\infty)$ of the two-parameter Ringel-Hall algebras $H_{r,s}(A_n), n \geq 1$.

### 2.1. Two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$

We will denote by $\mathcal{P}$ the set of isomorphism classes of finite dimensional representations of the infinite quiver $A_\infty$. Let us define the subset

$$\mathcal{P}_1 = \mathcal{P} - 0$$

where 0 denotes the subset of $\mathcal{P}$ consisting of the only isomorphism class of the zero representation. For any $\alpha \in \mathcal{P}$, we choose a representation $u_\alpha$ corresponding to $\alpha$. We denote by $a_\alpha$ the order of the automorphism group $\text{Aut}(u_\alpha)$. It is easy to see that the number $a_\alpha$ is independent of the choices of the representatives $u_\alpha$ for any $\alpha \in \mathcal{P}$.

For any given three representatives $u_\alpha, u_\beta, u_\gamma$ of the elements $\alpha, \beta, \gamma \in \mathcal{P}$ respectively, we denote by $g_{\alpha\beta}^\gamma$ the number of submodules $N$ of $u_\gamma$ satisfying the conditions: $N \cong u_\beta$ and $u_\gamma/N \cong u_\alpha$.

Note that it does not make sense to define $\text{Ext}^1(M, N)$ for any two given representations $M, N$ of the infinite quiver $A_\infty$. Let us denote by $E_{A_\infty}(M, N)$ the set of all short exact sequences $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$. We say two such short exact sequences $0 \rightarrow N \rightarrow E_1 \rightarrow M \rightarrow 0$ and $0 \rightarrow N \rightarrow E_2 \rightarrow M \rightarrow 0$ are equivalent if there exists a homomorphism $\phi: E_1 \rightarrow E_2$ making the diagram commute. We denote by $E_{A_\infty}(M, N)$ the set of all equivalence classes of $E_{A_\infty}$ with
respect to this equivalence relation. For any given $M, N \in A_\infty - \text{mod}$, according to Theorem 1.2 in [14], we can choose some $m \geq 1$ such that there exists a bijection between $E_{A_\infty}(M, N)$ and $\text{Ext}^1_{A_m-\text{mod}}(M, N)$. If no confusion arises, we will still write $E_{A_\infty}(M, N)$ as $\text{Ext}^1(M, N)$.

For any given $M, N \in A_\infty - \text{mod}$, we define the following notation

$$\langle M, N \rangle = \dim_k \text{Hom}(M, N) - \dim_k \text{Ext}^1(M, N).$$

Once the representations $M, N$ are chosen, we can always restrict to a subcategory $A_n - \text{mod}$. Since the algebra $\Lambda_n$ is hereditary for any $n \in \mathbb{N}$, it is easy to see that for any representations $M, N \in A_n - \text{mod}$, the value of $\langle M, N \rangle$ solely depends on the dimension vectors $\dim M, \dim N$ of the $A_n - \text{modules} M$ and $N$.

Now for any given elements $\alpha, \beta \in \mathcal{P}$, we can define the following notation

$$\langle \alpha, \beta \rangle = \langle u_\alpha, u_\beta \rangle$$

where $u_\alpha, u_\beta$ are any chosen representatives of $\alpha, \beta$ respectively. It is easy to see that $\langle -, - \rangle$ is a bilinear form.

It is well known that in the category $A_n - \text{mod}$, there exists a symmetry between the objects of $A_n - \text{mod}$. This symmetry is described by Green’s formula [15]. In fact, one can also prove that Green’s formula holds for the objects in the category $A_\infty - \text{mod}$. Namely, we have the following result.

**Theorem 2.2.** Let $\alpha, \beta, \alpha', \beta' \in \mathcal{P}$, then we have

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_{\lambda \in \mathcal{P}} g_\lambda^{\alpha} g_\lambda^{\beta} a_\lambda^{-1} = \sum_{\rho, \sigma, \sigma', \tau \in \mathcal{P}} \frac{|\text{Ext}^1(u_\rho, u_\tau)|}{|\text{Hom}(u_\rho, u_\tau)|} g_\rho^{\alpha} g_\rho^{\alpha'} g_\sigma^{\beta} g_\sigma^{\beta'} a_\rho a_\sigma a_{\sigma'} a_{\tau'}.$$

**Proof:** Since all representations involved in the formula are finite dimensional representations of $A_\infty$, we can choose some positive integer $m$ such that $\alpha, \beta, \alpha', \beta'$ and $\lambda$ can actually be regarded as objects in the subcategory $A_m - \text{mod}$ instead. Note that Green’s formula holds within the subcategory $A_m - \text{mod}$. Since the category $A_m - \text{mod}$ is a fully faithful and extension closed subcategory of $A_\infty - \text{mod}$, we know that Green’s formula holds in $A_\infty - \text{mod}$. □

Let $H_{r,s}(A_n)$ denote the two-parameter Ringel-Hall algebra associated to the category $A_n - \text{mod}$ as defined in [24]. In [24], Reineke has proved that the two-parameter Ringel-Hall algebra $H_{r,s}(A_n)$ is isomorphic to the algebra $U_{r,s}^+(\mathfrak{sl}n + 1)$. In the rest of this section, we will show that a limit version of this statement is still true.

Note that there exist Hall polynomials $F_{M,N}^L(x)$ for $M, N, L \in A_n - \text{mod}$ such that $g_{M,N}^L = F_{M,N}^L(q)$, where $q$ is the cardinality of the base
field \( k \). For the existence and calculation of Hall polynomials in \( A_n - \text{mod} \), we refer the reader to the references [27, 28]. Since each \( A_n - \text{mod} \) is a fully faithful and extension closed subcategory of \( A_\infty - \text{mod} \), the Hall polynomials exists for objects in \( A_\infty - \text{mod} \), which leads to the definition of two-parameter Ringel-Hall algebra \( H_{r,s}(A_\infty) \) below.

Now let us define \( H_{r,s}(A_\infty) \) to be the free \( \mathbb{Q}(r,s) \)-module generated by the set \( \{ u_\alpha | \alpha \in \mathcal{P} \} \). Moreover, we define a multiplication on the free \( \mathbb{Q}(r,s) \)-module \( H_{r,s}(A_\infty) \) as follows

\[
u_\alpha \nu_\beta = \sum_{\lambda \in \mathcal{P}} s^{-\langle \alpha, \beta \rangle} F_{u_\alpha u_\beta} (rs^{-1})u_\lambda, \quad \text{for any } \alpha, \beta \in \mathcal{P}.
\]

It is easy to see that we have the following result.

**Theorem 2.3.** The free \( \mathbb{Q}(r,s) \)-module \( H_{r,s}(A_\infty) \) is an associative \( \mathbb{Q}(r,s) \)-algebra under the above defined multiplication. In particular, the algebra \( H_{r,s}(A_n) \) can be regarded as a subalgebra of \( H_{r,s}(A_\infty) \) and \( H_{r,s}(A_m) \) for \( m \geq n \). In particular, we have

\[
H_{r,s}(A_\infty) = \lim_{n \to \infty} H_{r,s}(A_n).
\]

**Proof:** It is straightforward to verify that \( H_{r,s}(A_\infty) \) is an associative algebra under the above defined multiplication. Once again, we can reduce the proof to the finite case thanks to Theorem 1.1 in [14]. Since each \( A_n - \text{mod} \) can be regarded as a fully faithful and extension closed subcategory of \( A_\infty - \text{mod} \) and \( A_m - \text{mod} \) when \( m \geq n \), the algebra \( H_{r,s}(A_n) \) can be regarded as a subgroup of the algebras \( H_{r,s}(A_\infty) \) and \( H_{r,s}(A_m) \). Furthermore, one notices that the multiplication of \( H_{r,s}(A_n) \) is the restriction of the multiplications of \( H_{r,s}(A_\infty) \) and \( H_{r,s}(A_m) \). Therefore, the algebra \( H_{r,s}(A_n) \) can be regarded as a subalgebra of \( H_{r,s}(A_\infty) \) and \( H_{r,s}(A_m) \) for \( m \geq n \) as desired. Furthermore, each element of \( H_{r,s}(A_\infty) \) can be regarded as an element of a certain subalgebra \( H_{r,s}(A_m) \). Thus we shall have \( H_{r,s}(A_\infty) = \lim_{n \to \infty} H_{r,s}(A_n) \) as desired. \( \Box \)

**2.2. Basic properties of \( H_{r,s}(A_\infty) \).** Since the category \( A_\infty - \text{mod} \) can be regarded the direct limit of its fully faithful and extension closed subcategories \( A_n - \text{mod} \) with \( n \geq 1 \), any two objects \( M, N \in A_\infty - \text{mod} \) can be regarded as objects in a certain subcategory \( A_m - \text{mod} \). Thus the extension between any such two objects can be handled in this subcategory \( A_n - \text{mod} \) as well. As a result, it is no surprise that the algebra \( H_{r,s}(A_\infty) \) shares many similar ring-theoretic properties with its subalgebras \( H_{r,s}(A_n) \). In this subsection, we will establish some similar results for \( H_{r,s}(A_\infty) \) without giving detailed proofs. The reader shall be reminded that all the proofs can be reconstructed the same way as.
in the case of a certain subalgebra $H_{r,s}(A_m)$. And we refer the curious reader to [33] for the details.

First of all, let us fix more notations. For any given $\alpha \in \mathcal{P}$, we will choose an element $u_\alpha \in H_{r,s}(\Lambda)$. We denote by $\epsilon(\alpha)$ the $k$-dimension of the endomorphism ring of the representative $u_\alpha$ associated to $\alpha$. For any given finite dimensional representation $M$ of the infinite quiver $A_{\infty}$, we will denote the isomorphism class of $M$ by $[M]$ and the dimension vector of $M$ by $\dim M$, which is an element of the Grothendieck group $K_0(A_{\infty})$ of the category $A_{\infty}$-mod.

Recall that there is a one-to-one correspondence between the set of all positive roots for the Lie algebra $\mathfrak{sl}_\infty$ and the set of isoclasses of finite dimensional indecomposable representations of $A_{\infty}$. Let $a \in \Phi^+$ be any positive root, we shall denote by $M(a)$ the indecomposable representation corresponding to $a$. For any given map $\alpha: \Phi^+ \to \mathbb{N}_0$ with finite support, let us set the following

$$M(\alpha) = M_\Lambda(\alpha) = \bigoplus_{a \in \Phi^+} \alpha(a)M(a).$$

Then it is easy to see there is a one-to-one correspondence between the set $\mathcal{P}$ of isomorphism classes of all finite dimensional representations of the infinite quiver $A_{\infty}$ and the set of all maps $\alpha: \Phi^+ \to \mathbb{N}_0$ with finite supports. From now on, we will not distinguish an element $\alpha \in \mathcal{P}$ from the corresponding map associated to $\alpha$, and we may denote both of them by $\alpha$ if no confusion arises.

For any given $\alpha \in \mathcal{P}$, let us set $\dim \alpha = \sum_{a \in \Phi^+} \alpha(a)a$. Then we shall have following

$$\dim M(\alpha) = \dim \alpha.$$

For any given $\alpha \in \mathcal{P}$, we will denote by $\dim(\alpha) = \dim(u_\alpha)$ the dimension of the representation $u_\alpha$ as a $k$-vector space. Furthermore, let us set

$$\langle u_\alpha \rangle = s^{\dim(u_\alpha) - \epsilon(\alpha)}u_\alpha.$$

For conveniences, we may sometimes simply denote the element $u_\alpha$ by $\alpha$ for any $\alpha \in \mathcal{P}$, and denote $F_{u_\alpha u_\beta}(rs^{-1})$ by $g_{\alpha\beta}^\Lambda$, if no confusion arises. In the rest of this subsection, we will carry out all the computations in terms of $\alpha$. It is obvious that the set $\{\langle \alpha \rangle \mid \alpha \in \mathcal{P}\}$ is also a $\mathbb{Q}(r, s)$-basis for the algebra $H_{r,s}(A_{\infty})$. Note that we have $\langle \alpha_i \rangle = \alpha_i$ for any given element $\alpha_i \in \mathcal{P}$ corresponding to the simple root $\alpha_i$, $i \geq 1$. As a result, we can rewrite the multiplication of $H_{r,s}(A_{\infty})$ in terms of this new basis as follows

$$\langle \alpha \rangle \langle \beta \rangle = s^{-\epsilon(\alpha) - \epsilon(\beta) - \dim(\alpha, \dim(\beta))} \sum_{\lambda \in \mathcal{P}} s^{\epsilon(\lambda)}g_{\alpha\beta}^\Lambda \langle \lambda \rangle$$
for any $\alpha, \beta \in \mathcal{P}$.

In addition, let us denote by
\[ e(\alpha, \beta) = \dim_k \text{Hom}_{A_\infty \text{-mod}}(M(\alpha), M(\beta)) \]
and
\[ \zeta(\alpha, \beta) = \dim_k \text{Ext}^1_{A_\infty \text{-mod}}(M(\alpha), M(\beta)). \]

Recall that Hou and Ye have given an explicit total ordering on the set of all isoclasses of finite dimensional indecomposable representations of the infinite linear quiver $A_\infty$ and used it to construct a PBW base for the generic one-parameter Ringel-Hall algebra $H_q(A_\infty)$. Following [14], we will order all the positive roots as follows:
\[ a_{11} < a_{12} < \cdots < a_{22} < a_{23} < \cdots. \]

Obviously, we can see that $\text{Hom}(M(a_{ij}), M(a_{kl})) \neq 0$ implies $a_{ij} > a_{kl}$, where $M(a_{ij}), M(a_{kl})$ are the indecomposable representations corresponding to the positive roots $a_{ij}, a_{kl}$ respectively. For more details about the ordering, we refer the reader to [14, 27]. We should mention that we may write the positive roots as $a_1, a_2, a_3, \cdots$ instead.

First of all, we have the following proposition.

**Proposition 2.1.** Let $\alpha_1, \cdots, \alpha_t \in \mathcal{P}$ such that for $i < j$, we have both $\epsilon(\alpha_j, \alpha_i) = 0$ and $\zeta(\alpha_i, \alpha_j) = 0$. Then
\[ \langle \bigoplus_{i=1}^t \alpha_i \rangle = \langle \alpha_1 \rangle \cdots \langle \alpha_t \rangle. \]

\[ \square \]

**Theorem 2.4.** Let $\alpha, \beta \in \mathcal{P}$ such that $e(\beta, \alpha) = 0, \zeta(\alpha, \beta) = 0$. Then we have the following
\[ \langle \beta \rangle \langle \alpha \rangle = r^{(\alpha, \beta)} s^{-(\beta, \alpha)} \langle \alpha \rangle \langle \beta \rangle + \sum_{\gamma \in J(\alpha, \beta)} c_\gamma \langle \gamma \rangle \]

with coefficients $c_\gamma$ in $\mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$ and $J(\alpha, \beta)$ is the set of all elements $\lambda \in \mathcal{P}$ which are different from $\alpha \oplus \beta$ and $g^\lambda_{\alpha \beta} \neq 0$.

\[ \square \]

**Proposition 2.2.** For any given $\alpha \in \mathcal{P}$, we have
\[ \langle \alpha \rangle = \langle \alpha(a_1)a_1 \rangle \cdots \langle \alpha(a_m)a_m \rangle. \]

\[ \square \]

Now let us consider the divided powers of $\langle a \rangle$ by setting
\[ \langle a \rangle^{(t)} = \frac{1}{[t]_{e(a)}} \langle a \rangle^t \]
where $[t]_{t(a)}^{\lambda} = \prod_{i=1}^{r(a)} \frac{\nu_i(a) - s_i(a)\lambda}{\nu^2(a) - s^2(a)}$.

Then we have the following lemma.

**Lemma 2.1.** Let $a$ be a positive root and $t \geq 0$ be an integer. Then we have the following

$$\langle ta \rangle = \langle a \rangle^{(t)}.$$ 

\[\blacksquare\]

For each positive root $a_i$, let us define the following symbol

$$X_i = \langle a_i \rangle.$$ 

Then we have the following proposition:

**Proposition 2.3.** Let $\alpha \in \mathcal{P}$ and regard $\alpha$ as a map $\alpha : \Phi^+ \to \mathbb{N}$ with finite support. Let us set $\alpha(a_i) = \alpha(a_i)\alpha(a_i)$, then we have the following

$$\langle \alpha \rangle = X_1^{\alpha(1)} \cdots X_m^{\alpha(m)} = \left( \prod_{i=1}^{m} \frac{1}{[\alpha(i)]_{t(a_i)}^{\lambda}} \right) X_1^{\alpha(1)} \cdots X_m^{\alpha(m)}.$$ 

\[\blacksquare\]

**Theorem 2.5.** The monomials $X_1^{\alpha(1)} \cdots X_m^{\alpha(m)}$ with $\alpha(1), \ldots, \alpha(m) \in \mathbb{N}$ form a $\mathbb{Q}(r, s)$-basis of $H_{r,s}(\Lambda)$; and for $i < j$, we have

$$X_j X_i = \sum_{I(i,j)} c(a_{i+1}, \ldots, a_{j-1}) X_1^{a_{i+1}} \cdots X_m^{a_{j-1}}$$

with coefficients $c(a_{i+1}, \ldots, a_{j-1})$ in $\mathbb{Q}(r, s)$. Here the index set $I(i,j)$ is the set of sequences $(a_{i+1}, \ldots, a_{j-1})$ of natural numbers such that $\sum_{i=1}^{j-1} a_{i+1} a_i = a_i + a_j$.

\[\blacksquare\]

Now we define some algebra automorphisms and skew derivations on $H_{r,s}(A_\infty)$. For any $d \in \mathbb{Z}^\mathbb{Z}$, we define an algebra automorphism $l_d$ of $H_{r,s}(A_\infty)$ as follows

$$l_d(w) = r^{<\dim w, d>} s^{-<d, \dim w>} w$$

where $w$ is any homogeneous element of $H_{r,s}(A_\infty)$.

Let $H_j$ denote the $\mathbb{Q}(r, s)$-subalgebra of $H_{r,s}(A_\infty)$ generated by the generators $X_1, \ldots, X_j$. Thus we have $H_0 = \mathbb{Q}(r, s)$ and for any $0 \leq j \leq m$, we have following

$$H_j = H_{j-1}[X_j, l_j, \delta_j]$$

with the automorphism $l_j$ and the $l_j$-derivation $\delta_j$ of $H_{j-1}$. Note that the automorphism $l_j$ can be explicitly defined as follows

$$l_j(X_i) = r^{<\dim X_i, \dim X_j>} s^{-<\dim X_j, \dim X_i>} X_i$$
for \( i < j \). And the skew derivation \( \delta_j \) can be defined as follows:

\[
\delta_j(X_i) = X_jX_i - l_j(X_i)X_j = \sum_{l(i,j)} c(a_{i+1}, \ldots, a_{j-1}) X_{i+1}^{a_{i+1}} \cdots X_{j-1}^{a_{j-1}}.
\]

It is easy to check that we have the following result.

**Proposition 2.4.** The automorphism \( l_j \) and the skew derivation \( \delta_j \) satisfy the following relation

\[
l_j\delta_j = r(a_j, a_j) s^{-1}(a_j, a_j) \delta_j l_j.
\]

\[\Box\]

**Theorem 2.6.** The two-parameter Ringel-Hall algebra \( H_{r,s}(A_\infty) \) can be presented as an iterated skew polynomial ring.

\[\Box\]

2.3. An algebra isomorphism from \( U^+_r(sl_\infty) \) onto \( H_{r,s}(A_\infty) \). In this subsection, we are going to establish a graded algebra isomorphism from the two-parameter quantized enveloping algebra \( U^+_r(sl_\infty) \) onto the two-parameter Ringel-Hall algebra \( H_{r,s}(A_\infty) \). Via this isomorphism, all results established in the previous subsection on \( H_{r,s}(A_\infty) \) can be transformed to the two-parameter quantized enveloping algebra \( U^+_r(sl_\infty) \). Indeed, the isomorphism from \( U^+_r(sl_\infty) \) onto \( H_{r,s}(A_\infty) \) is the direct limit of the isomorphisms from \( U^+_r(sl_{n+1}) \) onto \( H_{r,s}(A_n) \).

First of all, one can prove the following result, which induces a homomorphism from \( U^+_r(sl_\infty) \) into \( H_{r,s}(A_\infty) \).

**Lemma 2.2.** Let \( \alpha_i \in \mathcal{P} \) correspond to the simple module \( S_i \), then we have the following identities in \( H_{r,s}(A_\infty) \).

\[
\alpha_i^2 \alpha_{i+1}^2 - (r + s) \alpha_i \alpha_{i+1} \alpha_i + r s \alpha_i \alpha_{i+1}^2 = 0,
\]

\[
\alpha_i^2 \alpha_{i+1}^2 - (r + s) \alpha_i \alpha_{i+1} \alpha_i + r s \alpha_i \alpha_{i+1}^2 = 0,
\]

for \( i = 1, 2, 3, \ldots \).

**Proof:** Note that we can regard \( \alpha_i, \alpha_{i+1} \) as elements of the two-parameter Ringel-Hall algebra \( H_{r,s}(A_{i+1}) \), which is a subalgebra of \( H_{r,s}(A_\infty) \). By the result in [33], we know that these identities hold in the algebra \( H_{r,s}(A_{i+1}) \). Therefore, we have proved the result as desired.

Now we have the following result which relates Ringel-Hall \( H_{r,s}(A_\infty) \) to the algebra \( U^+_r(sl_\infty) \).

**Theorem 2.7.** The map

\[
\eta: e_i \longrightarrow \alpha_i
\]
extends to a $\mathbb{Q}(r, s)$–algebra isomorphism

$$\eta: U_{r,s}(\mathfrak{sl}_\infty) \rightarrow H_{r,s}(A_\infty).$$

**Proof:** (The proof is essentially borrowed from [24] and we include it for completeness. See also [33]). First of all, note that the quantum Serre relations of $U_{r,s}(\mathfrak{sl}_\infty)$ are preserved by the map $\eta$. Thus the map $\eta$ does define an algebra homomorphism from the two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{sl}_\infty)$ into the two-parameter twisted Ringel-Hall algebra $H_{r,s}(A_\infty)$. Now it suffices to show that the map $\eta$ is indeed a bijection.

We first show that the map $\eta$ is surjective by verifying that the algebra $H_{r,s}(A_\infty)$ is generated by the elements $u_i$ which correspond to the irreducible representation $S_i$ of the infinite quiver $A_\infty$. Let $u_\alpha$ be any element in $H_{r,s}(A_\infty)$, then we can regard $u_\alpha$ as an element of a certain subalgebra $H_{r,s}(A_n)$. Thus we can restrict our proof to the subalgebra $H_{r,s}(A_n)$. As a result, we have the following:

$$u_\alpha = (\prod_{i=1}^m \frac{1}{(\alpha(i))^{\epsilon_i(a_i)}}) u_{a_1}^{\alpha(a_1)} \ldots u_{a_m}^{\alpha(a_m)}.$$

Now we need to prove that $u_\alpha$ is generated by $u_i$ for any $\alpha$ corresponding to an indecomposable representations. We prove this claim by using induction. Note that $\zeta(\alpha, \alpha) = 0$, thus we prove the following:

$$u_\alpha = u_1^{d_1} \ldots u_n^{d_n} - \sum_{\beta \neq \alpha, \dim \beta = \dim \alpha} s^{\langle \beta, \beta \rangle} u_\beta.$$

However, one sees that the dimension of the module $u_\beta$ is less than the dimension of the module $u_\alpha$. Thus by induction on the dimension, we can reduce to the case where $\dim(u_\alpha) = 1$. In this case, the only possibility is that $u_\alpha = u_i$ for some $i$. Thus we have proved the statement that every $u_\alpha$ is generated by $u_i$, which further implies that the map $\eta$ is a surjective map. We also note that the map $\eta$ is a graded map.

Finally, we show that the map $\eta$ is also injective. Recall that $\mathcal{A} = \mathbb{Q}[r, s]_{(r-1, s-1)}$ denote the localization of the polynomial ring $\mathbb{Q}[r, s]$ at the maximal ideal $(r-1, s-1)$. Then we know that $\mathcal{A} = \mathbb{Q}[r, s]_{(r-1, s-1)}$ is a local ring with the residue field $\mathbb{Q}$ and the fractional field $\mathbb{Q}(r, s)$. Let $U_+^\mathcal{A}$ denote the free $\mathcal{A}$–algebra generated by the generators $e_i$ subject to the quantum Serre relations holding in $U_{r,s}(\mathfrak{sl}_\infty)$. Also let $U_+^\mathcal{Q}(\mathfrak{sl}_\infty)$ denote the universal enveloping algebra of the corresponding nilpotent Lie subalgebra $n^+$ of $\mathfrak{sl}_\infty$ defined over the base field $\mathbb{Q}$. Then
we have the following
\[ U_{r,s}^+(\mathfrak{sl}_\infty) = \mathbb{Q}(r, s) \otimes_{A} U_{A}^+, \quad U_{Q}^+(\mathfrak{sl}_\infty) = \mathbb{Q} \otimes_{A} U_{A}^+. \]

For any \( \beta \in \mathbb{Z}^{\mathbb{N}} \), we have the following result via Nakayama’s Lemma
\[
dim_{\mathbb{Q}} U_{Q}^+(\mathfrak{sl}_\infty)_\beta = \dim_{\mathbb{Q}} (\mathbb{Q} \otimes_{A} U_{A}^+)_\beta \\
\geq \dim_{\mathbb{Q}} (r, s) (\mathbb{Q}(r, s) \otimes_{A} U_{A}^+)_\beta \\
= \dim_{\mathbb{Q}} (r, s) U_{r,s}^+(\mathfrak{sl}_\infty)_\beta \\
\geq \dim_{\mathbb{Q}} (r, s) H_{r,s}(A_\infty)_\beta.
\]

Note that we also have the following result:
\[
dim_{\mathbb{Q}} U_{Q}^+(\mathfrak{sl}_\infty)_\beta = \dim_{\mathbb{Q}} (r, s) H_{r,s}(A_\infty)_\beta.
\]

Thus we have proved that the map \( \eta \) is injective. Therefore, the map \( \eta \) is an algebra isomorphism from \( U_{r,s}^+(\mathfrak{sl}_\infty) \) onto \( H_{r,s}(A_\infty) \) as desired.

Based on the previous theorem, the following corollary is in order.

**Corollary 2.1.** The algebra \( U_{r,s}^+(\mathfrak{sl}_\infty) \) has a \( \mathbb{Q}(r, s) \)-basis parameterized by the isomorphism classes of all finite dimensional representations of the infinite quiver \( A_\infty \). In particular, we have
\[
U_{r,s}^+(\mathfrak{sl}_\infty) = \lim_{n \to \infty} U_{r,s}^+(\mathfrak{sl}_{n+1}).
\]

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### 3. The extended two-parameter Ringel-Hall algebras

For the purpose of realizing the Borel subalgebra \( U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty) \) of the two-parameter quantum group \( U_{r,s}(\mathfrak{sl}_\infty) \), we define the extended Ringel-Hall algebra \( H_{r,s}(A_\infty) \) by adding the torus part. In particular, we show that there is a Hopf algebra structure on this extended two-parameter Ringel-Hall algebra \( H_{r,s}(A_\infty) \); as a result we prove that \( U_{r,s}^+(\mathfrak{sl}_\infty) \) is isomorphic to the extended two-parameter Ringel-Hall algebra \( H_{r,s}(A_\infty) \) as a Hopf algebra. Similarly, we can use an extended two-parameter Ringel-Hall algebra to realize the Borel subalgebra \( U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty) \). Therefore, we will obtain a PBW-basis of two-parameter quantum group \( U_{r,s}(\mathfrak{sl}_\infty) \).
3.1. Extended Ringel-Hall algebras $H_{r,s}(A_{\infty})$. Let us define $H_{r,s}(A_{\infty})$ to be a free $\mathbb{Q}(r, s)$–module with the following basis
\[ \{ k_{\alpha}u_{\lambda} \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P} \}. \]
Moreover one will define an algebra structure on the module $H_{r,s}(A_{\infty})$ as follows.
\[ u_{\alpha}u_{\beta} = \sum_{\lambda \in \mathcal{P}} s^{-(\alpha, \beta)}F_{u_{\alpha}, u_{\beta}}^{u_{\lambda}} (rs^{-1})u_{\lambda}, \text{ for any } \alpha, \beta \in \mathcal{P}, \]
\[ k_{\alpha}u_{\beta} = r^{(\beta, \alpha)}s^{-(\alpha, \beta)}u_{\beta}k_{\alpha} \text{ for any } \alpha \in \mathbb{Z}[I], \beta \in \mathcal{P}, \]
\[ k_{\alpha}k_{\beta} = k_{\beta}k_{\alpha} \text{ for any } \alpha, \beta \in \mathbb{Z}[I]. \]
Indeed, we have the following

**Proposition 3.1.** For any elements $x, y, z \in \mathbb{Z}[I]$ and $\alpha, \beta, \gamma \in \mathcal{P}$, we have the following
\[ [(k_{x}u_{\alpha})(k_{y}u_{\beta})(k_{z}u_{\gamma})] = (k_{x}u_{\alpha})([(k_{y}u_{\beta})(k_{z}u_{\gamma})]. \]
In particular, with the above defined multiplication, $H_{r,s}(A_{\infty})$ is an associative $\mathbb{Q}(r, s)$–algebra.

**Proof:** Once we choose $x, y, z$, and $\alpha, \beta, \gamma$, we can restrict to the subgroup $H_{r,s}(A_{m})$ of $H_{r,s}(A_{\infty})$ for some $m$. Since $H_{r,s}(A_{m})$ is an associative algebra with the restricted multiplication, thus we have proved all the statements. \hfill \Box

Furthermore, we have the following result.

**Theorem 3.1.** The map $\eta$ extends to a $\mathbb{Q}(r, s)$–algebra isomorphism from $U_{r,s}^{\geq 0}(\mathfrak{sl}_{\infty})$ onto $H_{r,s}(A_{\infty})$ via the map $\eta(w_{i}) = k_{i}$ and $\eta(e_{i}) = u_{\alpha_{i}}$.

**Proof:** The proof is straightforward. \hfill \Box

As a result, we have the following description about a basis for the algebra $U_{r,s}^{+}(\mathfrak{sl}_{\infty})$

**Corollary 3.1.** The set $B^{+} = \{ w_{\alpha}^{-1}(u_{\lambda}) \mid \alpha \in \mathbb{Z}[\mathbb{N}], \lambda \in \mathcal{P} \}$ is a $\mathbb{Q}(r, s)$–basis of $U_{r,s}^{\geq 0}(\mathfrak{sl}_{\infty})$. \hfill \Box

3.2. A Hopf algebra structure on $H_{r,s}(A_{\infty})$. Now we are going to introduce a Hopf algebra structure on the extended two-parameter Ringel-Hall algebra $H_{r,s}(A_{\infty})$. In particular, we have the following result.

**Theorem 3.2.** The algebra $H_{r,s}(A_{\infty})$ is a Hopf algebra with the Hopf algebra structure defined as follows.
(1) Multiplication:

\[ u_\alpha u_\beta = \sum_{\lambda \in P} s^{-\langle \alpha, \beta \rangle} g_\alpha^\lambda u_\lambda \text{ for any } \alpha, \beta \in B, \]

\[ k_\alpha u_\beta = r^{\langle \beta, \alpha \rangle} s^{-\langle \alpha, \beta \rangle} u_\beta k_\alpha \text{ for any } \alpha \in \mathbb{Z}[I], \beta \in P, \]

\[ k_\alpha k_\beta = k_\beta k_\alpha \text{ for any } \alpha, \beta \in \mathbb{Z}[I]. \]

(2) Comultiplication:

\[ \Delta(u_\lambda) = \sum_{\alpha, \beta \in P} r^{\langle \alpha, \beta \rangle} a_\alpha a_\beta g_\alpha^\lambda u_\alpha k_\beta \otimes u_\beta \text{ for any } \lambda \in P, \]

\[ \Delta(k_\alpha) = k_\alpha \otimes k_\alpha \text{ for any } \alpha \in \mathbb{Z}[I]. \]

(3) Counit:

\[ \epsilon(u_\lambda) = 0 \text{ for all } \lambda \neq 0 \text{ and } \epsilon(k_\alpha) = 1 \text{ for any } \alpha \in P. \]

(4) Antipode:

\[ \sigma(u_\lambda) = \delta_{\lambda,0} + \sum_{m \geq 1} (-1)^m \sum_{\pi, \lambda_1, \lambda_2, \ldots, \lambda_m \in P} r s^{-1} \sum_{i < j} \langle \lambda_i, \lambda_j \rangle \]

\[ \frac{a_\lambda \cdots a_{\lambda_m}}{a_\lambda} g_{\lambda_1}^\lambda \cdots g_{\lambda_m}^\lambda u_\pi \]

for any element \( \lambda \in P \) and

\[ \sigma(k_\alpha) = k_{-\alpha} \text{ for any } \alpha \in \mathbb{Z}[I]. \]

In particular, we have the following

\[ \overline{H_{r,s}(A_\infty)} = \lim_{n \to \infty} \overline{H_{r,s}(A_n)} \]

as a direct limit of Hopf subalgebras.

The proof of the above theorem consists of a couple of lemmas which can be proved as the finite dimensional case. And we refer the reader to [33, 34] for more details. Namely, we have the following lemmas.

**Lemma 3.1.** The comultiplication \( \Delta \) is an algebra endomorphism of \( H_{r,s}(A_\infty) \).

**Lemma 3.2.** For any \( \lambda \in P \), we have the following

\[ \mu(\sigma \otimes 1) \Delta(u_\lambda) = \delta_{\lambda,0} \]

and

\[ \mu(1 \otimes \sigma) \Delta(u_\lambda) = \delta_{\lambda,0}. \]
3.3. A Hopf algebra isomorphism from $U_{r,s}^0(\mathfrak{sl}_\infty)$ onto $\overline{H_{r,s}(A_\infty)}$.

In this subsection, we will prove that the Borel subalgebras $U_{r,s}^0(\mathfrak{sl}_\infty)$ and $U_{r,s}^0(\mathfrak{sl}_\infty)$ of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ can be realized as the extended two-parameter Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$ and $\overline{H_{s-1,r-1}(A_\infty)}$ as Hopf algebras. As a result, we shall derive a PBW-basis for the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$.

**Theorem 3.3.** We have that

$$U_{r,s}^0(\mathfrak{sl}_\infty) \cong \overline{H_{r,s}(A_\infty)}$$

and

$$U_{r,s}^0(\mathfrak{sl}_\infty) \cong \overline{H_{s-1,r-1}(A_\infty)}$$

as Hopf algebras.

Let $B^-$ denote the $\mathbb{Q}(r,s)-$basis constructed for the algebra $U_{r,s}^0(\mathfrak{sl}_\infty)$ via the Ringel-Hall algebra $\overline{H_{s-1,r-1}(A_\infty)}$, then we have the following:

**Corollary 3.2.** The set $B^+ \times B^-$ is a $\mathbb{Q}(r,s)-$basis for the two-parameter quantum groups $U_{r,s}(\mathfrak{sl}_\infty)$.

Furthermore, we have the following result, which provides a bridge from the finite dimensional case to the infinite case.

**Theorem 3.4.** The two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ is the direct limit of the direct system \( \{ U_{r,s}(\mathfrak{sl}_{n+1}) \mid n \in \mathbb{N} \} \) of the Hopf subalgebras $U_{r,s}(\mathfrak{sl}_{n+1})$ of $U_{r,s}(\mathfrak{sl}_\infty)$.

That is

$$U_{r,s}(\mathfrak{sl}_\infty) = \lim_{n \to \infty} U_{r,s}(\mathfrak{sl}_{n+1}).$$

In particular, we have

$$U_{r,s}^0(\mathfrak{sl}_\infty) = \lim_{n \to \infty} U_{r,s}^0(\mathfrak{sl}_{n+1}),$$

$$U_{r,s}^1(\mathfrak{sl}_\infty) = \lim_{n \to \infty} U_{r,s}^1(\mathfrak{sl}_{n+1}),$$

$$U_{r,s}^0(\mathfrak{sl}_\infty) = \lim_{n \to \infty} U_{r,s}^0(\mathfrak{sl}_{n+1}),$$

$$U_{r,s}^0(\mathfrak{sl}_\infty) = \lim_{n \to \infty} U_{r,s}^0(\mathfrak{sl}_{n+1}).$$

**Proof:** It is obvious that $U_{r,s}(\mathfrak{sl}_{n+1})$ are Hopf subalgebras of $U_{r,s}(\mathfrak{sl}_\infty)$ and $U_{r,s}(\mathfrak{sl}_{m+1})$ for $m \geq n$. In addition, any element of $U_{r,s}(\mathfrak{sl}_\infty)$ is an element of a certain $U_{r,s}(\mathfrak{sl}_{n+1})$. Thus we are done with the proof.
4. Monomial bases and bar-invariant bases of $U_{r,s}^+(sl_\infty)$

In this section, we study various bases of $U_{r,s}^+(sl_\infty)$ via the theory of generic extensions. Note that the construction of monomial bases using generic extension theory for the Ringel-Hall algebras of type $A, D, E$ has been done in [9]. The idea of the construction is to use the monoidal structure on the set $\mathcal{M}$ of isoclasses of finite dimensional representations of the corresponding quiver $Q$ and the Bruhat-Chevalley type partial ordering in $\mathcal{M}$. Note that the arguments used in [9] can be completely transformed to the case of $sl_\infty$. Therefore, we will state most of the results for monomial bases without much detail. For the details, we refer the reader to [9, 24].

For the reader’s convenience, we will recall the necessary details about the the generic extensions from [9, 24]. Note that there exists a bijective correspondence between the set of positive roots $\Phi^+$ of the root system $\Phi$ associated to $sl_\infty$ and the set of isoclasses of finite dimensional indecomposable representations of $A_\infty$. For any $\beta \in \Phi^+$, we will denote by $M(\beta) = M_k(\beta)$ the corresponding indecomposable representation of $A_\infty$. By the Krull–Remak-Schmidt theorem, we shall have the following

$$M(\lambda) = M_k(\lambda) = \bigoplus_{\beta \in \Phi^+} \lambda(\beta) M_k(\beta)$$

for some function $\lambda: \Phi^+ \to \mathbb{N}_0$ with a finite support. Therefore, the isoclasses of finite dimensional representations of $A_\infty$ are indexed by the following set

$$\Lambda = \{\lambda: \Phi^+ \to \mathbb{N} \text{ with a finite support}\} \cong \mathbb{N}_0^{\oplus \Phi^+}.$$ 

From now on, we will use the set $\Lambda$ to index the objects of the category $A_\infty \text{-mod}$. 

Next, we are going to recall some information about generic extensions of representations of Dynkin quivers. We should mention that all the arguments used in the finite dimensional cases of type $A, D, E$ can be transformed to the $sl_\infty$. We refer the interested reader to the references [9, 24] for details.

Let us fix $k$ to algebraically closed. Let us denote by $Q = (Q_0, Q_1)$ the quiver $A_\infty$. Fix a $d = (d_i) \in \mathbb{N}_0^{\oplus \Phi^+}$ and we may choose $n$ large enough so that $d$ can be regarded as an element in $\mathbb{N}_0^n$. For any given $d$, we can define an affine space as follows

$$R(d) = R(Q, d) = \prod_{\alpha \in Q_1} \text{Hom}_k(k^{d_{1(\alpha)}}, k^{d_{h\alpha}}) \cong \prod_{\alpha \in Q_1} k^{d_{1\alpha} \times d_{h\alpha}}.$$
Thus, a point \( x = (x_\alpha)_\alpha \) of \( R(d) \) determines a finite dimensional representation \( V(x) \) of \( Q = A_\infty \). The algebraic group \( GL(d) = \prod \Gamma_{i=1}^{n} GL_{d_i}(k) \) acts on the space \( R(d) \) by the conjugation

\[
(g_i)_i \ldots (x_\alpha)_\alpha = (g_{h(\alpha)})x_\alpha(g_{h(\alpha)}^{-1})_\alpha.
\]

and the \( GL(d) \)-orbits \( O_x \) in \( R(d) \) correspond bijectively to the isoclasses \([V(x)]\) of finite dimensional representations of \( Q \) with the dimension vector \( d \). The stabilizer \( GL(d)_x = \{ g \in GL(d) \mid gx = x \} \) of \( x \) is the group of automorphisms of \( M := V(x) \) which is zariski-open in \( \text{End}_{A_\infty}\text{-mod}(M) \) and has a dimension equal to the \( \text{dim}_{A_\infty}\text{-mod}(M) \). It follows that the orbit \( O_M := O_x \) of \( M \) has a dimension

\[
\text{dim}O_M = \text{dim}GL(d) - \text{dim}\text{End}_{A_\infty}\text{-mod}(M).
\]

Now we have the following result, whose proof is the same as the one in [24].

**Lemma 4.1.** For \( x \in R(d_1) \) and any \( y \in R(d_2) \), let \( E(O_x,O_y) \) be the set of all \( z \in R(d) \) where \( d = d_1 + d_2 \) such that \( V(z) \) is an extension of some \( M \in O_x \) by some \( N \in O_y \). Then \( E(O_x,O_y) \) is irreducible.

Given any two finite-dimensional representations of \( M, N \) of the infinite linear quiver \( A_\infty \), let us consider the extensions

\[
0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0
\]

of \( M \) by \( N \). By the lemma, there is a unique (up to isomorphism) such extension \( G \) with \( \text{dim}O_G \) being maximal. We call \( G \) the generic extension of \( M \) by \( N \), and denoted by \( M \ast N \). For any two representations \( M, N \), we say \( M \) degenerates to \( N \), or that \( N \) is a degeneration of \( M \), and write \( [N] \leq [M] \) (or simply \( N \leq M \)) if \( O_N \subseteq \overline{O_M} \) which is the closure of \( O_M \). Note that \( N < M \) if and only if \( O_N \subset \overline{O_M} \setminus O_M \).

Similar to the result in [9, 24], one knows that the relation \( \leq \) is independent of the base field \( k \) and it provides a partial order on the set \( \Lambda \) via setting \( \lambda \leq \mu \) if and only if \( M_k(\lambda) \leq M_k(\mu) \) for any given algebraically closed field \( k \).

Using the same arguments as in [9, 24], we shall have the following result.

**Theorem 4.1.** (1). If \( 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0 \) is exact and non-split, then \( M \oplus N < E \).

(2). Let \( M, N, X \) be finite dimensional representations of the quiver \( A_\infty \). Then \( X \leq M \ast N \) if and only if there exist \( M' \leq M, N' \leq N \) such that \( X \) is an extension of \( M' \) by \( N' \). In particular, we have \( M' \leq M, N' \leq N \implies M' \ast N' \leq M \ast N \).
(3). Let $\mathcal{M}$ be the set of isoclasses of finite dimensional representations of $A_\infty$ and define a multiplication $\ast$ on $\mathcal{M}$ by $[M] \ast [N] = [M \ast N]$ for any $[M], [N] \in \mathcal{M}$. Then $\mathcal{M}$ is a monoid with identity $1 = [0]$ and the multiplication $\ast$ preserves the induced partial ordering on $\mathcal{M}$.

(4). $\mathcal{M}$ is generated by irreducible representations $[S_i], i \in I$ subject to the following relations

1. $[E_i] \ast [E_j] = [E_j] [E_i]$ if $i, j$ are not connected by an arrow;
2. $[E_i] \ast [E_j] \ast [E_i] = [E_i] \ast [E_i] \ast [E_j]$ and $[E_j] \ast [E_i] \ast [E_j] = [E_i] \ast [E_j] \ast [E_j]$ if there exists an arrow from $i$ to $j$.

Let us denote by $\Omega$ the set of all words formed by letters in $I$. It is easy to see that for any given word $w = w_1 \cdots w_m \in \Omega$, we can set the following finite dimensional representations of $A_\infty$

$$M(w) = S_{w_1} \ast S_{w_2} \ast \cdots \ast S_{w_m}.$$ 

Note that there is a unique $M(w) \in A_\infty - \text{mod}$ such that $M(w) \cong M(p(w))$, which enables us to define a function as follows

$$p : \Omega \rightarrow A_\infty - \text{mod}, w \mapsto M(p(w)).$$

Furthermore, we shall have the following result on this function.

**Theorem 4.2.** The map $p$ induces a surjection

$$p : \Omega \rightarrow A_\infty - \text{mod}.$$ 

**Proof:** Once again, we can restrict the function to a certain subcategory $A_m - \text{mod}$, where the property holds. Therefore, $p$ induces a partition of the set $\Omega = \text{mod}$ such that $p^{-1}(\lambda)$ will call each $\Omega_\lambda$ a fiber of the map $p$.

Now we are going to recall some information on the partial ordering $\leq$. Let $w = i_1 \cdots i_m$ be a word in $\Omega$. Then $w$ can be uniquely expressed in the tight form $w = j_1^{e_1} \cdots j_t^{e_t}$ where $e_r \geq 1, 1 \leq r \leq t$, and $j_r \neq j_{r+1}$ for $1 \leq r \leq t - 1$. A filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

of a module $M$ is called a reduced filtration of type $w$ if $M_{r-1}/M_r \cong e_rS_r$, for all $1 \leq r \leq t$. Note that any reduced filtration of $M$ of type $w$ can be refined to a composition of $M$ of type $w$. Conversely, given a composition series of $M$, there is a unique reduced filtration of $M$. Let us denote by $\varphi^\lambda w(x)$ the Hall polynomial $\varphi^\lambda_{x_1 \cdots x_t}(x)$ where $M(\mu_r) = e_rS_r$. Let us denote by $\gamma^\lambda_w(g_k)$ the number of the reduced filtrations of $M_k(\lambda)$ over the base field $k$ when $k$ is a finite field. A word $w$ is called distinguished if $\gamma^\lambda_w(1) = 1$. Note that $w$ is distinguished if
and only if, for some algebraically closed field $k$, $M_k(p(w))$ has a unique reduced filtration of type $w$. Similar to [9], we have the following results.

**Lemma 4.2.** *(See also Lemma 4.1 in [9])* Let $\omega \in \Omega$ and $\mu \geq \lambda$ in $\Lambda$. Then $\varphi_\omega^\mu \neq 0$ implies $\varphi_\omega^\lambda \neq 0$.

**Theorem 4.3.** *(See also Theorem 4.2 in [9])* Let $\lambda, \mu \in \Lambda$. Then $\lambda \leq \mu$ if and only if there exists a word $\omega \in p^{-1}((\mu))$ such that $\varphi_\lambda^\mu \neq 0$.

**Lemma 4.3.** *(See also Lemma 5.2 in [9])* Every fiber of $p$ contains a distinguished word.

Let us define $[[e_a]]' = [[1]] \cdots [[e_a]]$ with $[[m]] = \frac{1-(rs^{-1})^m}{1-rs^{-1}}$. Then we shall have the following result.

**Lemma 4.4.** *(See also Lemma 6.1 in [9])* Let $w \in \Omega$ be a word with the tight form $j_1^{e_1} \cdots j_t^{e_t}$. Then, for each $\lambda \in \Lambda$,

$$\varphi_\lambda^w(rs^{-1}) = \gamma_\lambda^w(rs^{-1}) \prod_{r=1}^t [[e_r]]'.
$$

In particular, $\varphi_\lambda^{p(w)} = \prod_{r=1}^t [[e_r]]'$ if $w$ is distinguished.

For any given word $w = i_1 \cdots c_m \in \Omega$, we can associate a monomial

$$u_w = u_{i_1} \cdots u_{i_m} \in H_{r,s}(A_\infty).$$

**Proposition 4.1.** For any $w \in \Omega$ with the tight form $j_1^{e_1} \cdots j_t^{e_t}$, we have

$$u_w = \sum_{\lambda \leq p(w)} \varphi_\lambda^w(rs^{-1})u_\lambda = \prod_{r=1}^t [[e_r]]' \sum_{\lambda \leq p(w)} \gamma_\lambda^w(rs^{-1})u_\lambda.$$

Moreover, the coefficients appearing in the sum are all nonzero.

As a result, we shall have the following theorem.

**Theorem 4.4.** For each given $\lambda \in \Lambda$, let us choose an arbitrary word $w_\lambda \in p^{-1}(w)$. Then the set $\{u_{w_\lambda} \mid \lambda \in \Lambda\}$ is a $\mathbb{Q}(r,s)$–basis of $H_{r,s}(A_\infty)$. Moreover, if all the words are chosen to be distinguished, then this set is a $\mathbb{Z}[r,s]_{(r-1,s-1)}$–basis of $H_{r,s}(A_\infty)$.
4.1. A bar-invariant basis of $U^+_{r,s}(\mathfrak{sl}_\infty)$. It is easy to see that the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ admits a $\mathbb{Q}$–linear involution defined as follows
\[ \bar{r} = s, \quad \bar{s} = r, \quad \bar{e}_i = e_i \text{ for all } i \in I. \]
And we will refer this involution as the bar-involution. In this subsection, we will construct a bar-invariant basis for $U^+_{r,s}(\mathfrak{sl}_\infty)$.

Denote by $\dim_k \text{Hom}(M, N)$ and $\dim_k \text{Ext}^1(M, N)$. Let us set $c^X_{M,N} = s[X, X] - [M,N] + [M,N] - [M,M] - [N,N] F^X_{M,N}(rs^{-1})$. It is obvious that the same proof in [24] shall yield the following result.

**Proposition 4.2.** Let us write $u_\alpha = \sum_\beta \omega^\alpha_\beta u_\beta$, then we have
\begin{enumerate}
    \item $\omega^\alpha_\beta = 0$ unless $\beta \leq \alpha$, and $\omega^\alpha_\alpha = 1$,
    \item if $u_\alpha = M \oplus N$ for finite dimensional representations $M, N$ with $[N, M] = 0 = [M, N]^1$, then
      \[ \omega^\alpha_\beta = \sum_{M' \leq M, N' \leq N} \omega^M_{M'} \omega^N_{N'} c^\alpha_{M' N'}, \]
    \item if $u_\alpha$ is an exponent of a finite dimensional indecomposable representation, then
      \[ \omega^\alpha_\beta = s^{[u_\beta, u_\beta]} - \sum_{\beta \leq \gamma < \alpha} r^{[u_\gamma, u_\gamma]} \omega^\gamma_\beta, \]
    \item $\omega^\alpha_\beta \in s^{[u_\beta, u_\beta]} - [u_\alpha, u_\alpha] \mathbb{Z}[rs^{-1}]$.
\end{enumerate}

Furthermore, using the arguments in [21, 24], we shall have the following result on a bar-invariant basis of the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$.

**Theorem 4.5.** For each isoclass $\alpha$, there exists a unique element
\[ C_\alpha \in u_\alpha + s^{-1} \mathbb{Z}[rs, r^{-1}s^{-1}, s][B\setminus \{u_\alpha\}] \]
such that $\bar{C}_\alpha = C_\alpha$. Write $C_\alpha = \sum_\beta \zeta^\alpha_\beta u_\beta$, we have
\begin{enumerate}
    \item $\zeta^\alpha_\beta = 0$ unless $\beta \leq \alpha$, and $\zeta^\alpha_\alpha = 1$,
    \item $\zeta^\alpha_\beta \in s^{[u_\beta, u_\beta]} - [u_\alpha, u_\alpha] \mathbb{Z}[rs^{-1}]$,
    \item Denote by $\hat{\zeta}^\alpha_\beta(v) \in \mathbb{Z}[v, v^{-1}]$ the specialization of $\zeta^\alpha_\beta$ to $\alpha = v = s^{-1}$, we have
      \[ \zeta^\alpha_\beta = (\sqrt{r s})^{[u_\beta, u_\beta]} - [u_\alpha, u_\alpha] \hat{\zeta}^X Y (\sqrt{r s^{-1}}). \]
\end{enumerate}

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