A note on small cycles in the Bubble-sort graph

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Abstract

The Bubble-sort graph $BS_n$, $n \geq 2$, is a Cayley graph over the symmetric group $Sym_n$ generated from the set of transpositions $\{(12),(23),\ldots,(n-1n)\}$. These graphs are bipartite, they do not contain odd cycles but contain all even cycles of length $\ell$, where $4 \leq \ell \leq n!$. We characterize all distinct 4-, 6-cycles by their canonical forms as products of generating elements. The number of these cycles in the Bubble-sort graph is also given. We show that all 8-cycles in $BS_n$ can be represented as a sum of base 4-cycle and 6-cycles. We claim that the minimal cycle basis consists of a specific family of 4- and 6-cycles. Computational experiments for small values of $n$ confirm such hypothesis.

Keywords: Cayley graphs; Bubble-sort graph; cycle embedding; product of generating elements; minimal cycle basis; sums of cycles

1 Introduction

The Bubble-sort graph $BS_n = Cay(Sym_n, B)$, $n \geq 2$, is a Cayley graph over the symmetric group $Sym_n$ of permutations $\pi = [\pi_1\pi_2\ldots\pi_n]$, where $\pi_i = \pi(i)$, $1 \leq i \leq n$, with the generating set $B = \{b_i \in Sym_n : 1 \leq i \leq n-1\}$ of all transpositions $b_i$ transposing the $i$-st and $(i+1)$-st elements of a permutation $\pi$ when multiplied on the right, i.e. $[\pi_1\pi_2\ldots\pi_{i-1}\pi_i\pi_{i+1}\ldots\pi_n]b_i = [\pi_1\pi_2\ldots\pi_{i-1}\pi_{i+1}\pi_i\ldots\pi_n]$. It is a connected bipartite $(n-1)$-regular graph of order $n!$ and diameter $diam(S_n) = \binom{n}{2}$. Since this graph is bipartite it does not contain odd cycles but it contains all even $\ell$-cycles where $l = 4, 6, 8, \ldots, n!$ [1].

The hamiltonicity of this graph was also shown in papers on generating all permutations by transpositions [4, 5].

The graph $BS_n$, $n \geq 2$, has the hierarchical structure such that it contains $n$ copies of $BS_{n-1}(i), 1 \leq i \leq n$, where each $BS_{n-1}(i)$ has the vertex set $V_i = \{[\pi_1\ldots\pi_{n-1}]\}$, where $\pi_k \in \{1, \ldots, n\}\{i\}$, $|V_i| = (n-1)!$, and the edge set $E_i = \{[[\pi_1\ldots\pi_{n-1}],[\pi_1\ldots\pi_{n-1}]b_j] : 1 \leq j \leq n-2\}$, $|E_i| = \frac{(n-1)!}{2}$. Any two copies $BS_{n-1}(i), BS_{n-1}(j), i \neq j$, are connected by $(n-2)!$ edges presented as $[[\pi_1\ldots ij],[\pi_1\ldots ji]]$, where $[\pi_1\pi_2\ldots ij] = [\pi_1\pi_2\ldots ji]b_{n-1}$. Transpositions $b_j, 1 \leq j \leq n-2$, define internal edges in all $n$ copies $BS_{n-1}(i), 1 \leq i \leq n$, and the transposition $b_{n-1}$ defines external edges between copies. Copies $BS_{n-1}(i)$ are also called $(n-1)$-copies. A vertex in $BS_n$ is identified with the permutation corresponding to this vertex.

In this work the characterization of small cycles in the Bubble-sort graph is given and the number of distinct 4- and 6-cycles is obtained. An explicit description of cycles is

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The main results of this paper are the following two theorems giving the complete description of 4– and 6– cycles in the Bubble-sort graph.

**Theorem 1.** Each of vertices of $BS_n$, $n \geq 4$, belongs to $(n - 2)(n - 3)/2$ distinct 4–cycles of the following canonical form:

$$C_4 = (b_j b_i)^2, \quad 1 \leq i < j - 1 \leq n - 2.$$  \hspace{1cm} (2)

In total, there are $\frac{(n-2)(n-3)n!}{8}$ distinct 4–cycles in $BS_n$.

**Theorem 2.** Each of vertices of $BS_n$, $n \geq 3$, belongs to $(n - 2)$ distinct 6–cycles of the canonical form:

$$C_6^1 = (b_i b_{i+1})^3, \quad 1 \leq i \leq n - 2, \quad n \geq 3;$$  \hspace{1cm} (3)
when \( n \geq 5 \), each vertex belongs to \( 6(n - 3)(n - 4) \) distinct 6–cycles of the canonical form:
\[
C^2_6 = b_j b_{i+1} b_i b_j b_i b_{i+1}, \quad 1 \leq i < j - 2 \leq n - 3, \text{or } 1 \leq j < i - 1 \leq n - 3; \tag{4}
\]
when \( n \geq 6 \), each vertex belongs to \( \frac{(n-3)(n-4)(n-5)}{2} \) distinct 6–cycles of the canonical form:
\[
C^3_6 = (b_k b_j b_i)^2, \quad 1 \leq i < j - 1 < k - 2 \leq n - 3; \tag{5}
\]
and to \( 3(n-3)(n-4)(n-5) \) distinct 6–cycles of the following canonical forms:
\[
C^4_6 = b_k b_j b_i b_k b_j b_j, \quad 1 \leq i < j - 1 < k - 2 \leq n - 3; \tag{6}
\]
\[
C^5_6 = b_k b_j b_i b_k b_j b_i, \quad 1 \leq i < j - 1 < k - 2 \leq n - 3; \tag{7}
\]
\[
C^6_6 = b_k b_i b_k b_j b_j b_j, \quad 1 \leq i < j - 1 < k - 2 \leq n - 3. \tag{8}
\]
In total, there are \( \frac{(7n^7-72n^5+247n-280)n!}{12} \) distinct 6–cycles in \( BS_n \).

2 Technical lemmas

The general idea of the proofs of the next theorems is based on considering two antipodal vertices \( \pi \) and \( \tau \) of the \( 2\ell \)-cycle, where \( \ell = 4, 6 \) and finding two non-intersecting \( \ell \)-paths between them. The element \( \pi_i \) of a permutation \( \pi \in \text{Sym}_n \), where \( 1 \leq i \leq n - d - 1 \), is called to be shifted \( d \) steps to the right from its original position, if \( \tau_{i+d} = \pi_i \). This is achieved by applying the sequence of transpositions \( (b_{i+1} b_{i+2} \ldots b_{i+d}) \).

In case of each \( \ell \), we subdivide the proof into cases, depending on the number of shifted elements of \( \pi \) and the distance they are shifted along the \((\pi, \pi)\)-path. It is obvious that along the \((\tau, \pi)\)-path the same elements must be shifted back to the left to their original positions, and therefore we only consider shifting to the right. Since we consider cycles of small length, we evade looking into more complicated shifts along the paths.

![Figure 2: Example of \((\pi, \pi)\)- and \((\tau, \pi)\)-paths in a 6–cycle.](image)

We call two transpositions \( b_i \) and \( b_j \) independent, if \( i \neq j - 1, j + 1 \). Otherwise the transpositions are called dependent. The main properties of such transpositions used in the proofs can be formulated as follows.

**Proposition 1.** Consider two vertices \( \pi \) and \( \tau \) and let the \((\pi, \pi)\)-path of length \( d \) be given as a sequence of successively dependent transpositions \( (b_j b_{j+1} \ldots b_{j+d-1}) \), where \( 1 \leq j \leq n - d \), then there is no other non-intersecting \((\tau, \pi)\)-path of length \( d \) between \( \tau \) and \( \pi \).

**Proof.** Since the graph is vertex-transitive, let for simplicity the permutation \( \pi = \text{id} = [123 \ldots n] \). For any \( j \), where \( 1 \leq j \leq n - d \), the sequence of transpositions \( (b_j b_{j+1} \ldots b_{j+d-1}) \) acts on \( \pi \) as shifting of the element \( j \) to position \( j + d \). Since this is the only moved element, the backward \((\tau, \pi)\)-path of the same length inevitably has all the transpositions in reverse order. Hence the path is unique. \( \square \)
3 Proofs

3.1 Proof of Theorem 1

Now we prove the characterisation theorem on 4-cycles.

Proof. Let us consider $\pi = id = [12 \ldots n]$. Consider two antipodal vertices $\pi$ and $\tau$ of the 4-cycle, thus the length of the $(\pi, \tau)$-path is equals two. We prove the statement by considering all possible cases of the shifted elements of $\pi$ along the $(\pi, \tau)$-path and describing the set of non-intersecting $(\tau, \pi)$-paths. Obviously, the number of shifted elements should not be greater than two.

Shift of the one element is equivalent to having only two dependent transpositions on the $(\pi, \tau)$-path and by Proposition 1 such path is unique. If two elements are shifted, then $(\pi, \tau)$-path consists of two independent transpositions $b_i b_j$, where $1 < i < j - 1 < n - 2$. Since transpositions are independent, shuffling those will result in another path, which is a non-intersecting $(\tau, \pi)$-path defined as $b_i b_j$.

Altogether the canonical form of the only possible 4-cycles is given in (2). It is straightforward to obtain the number of distinct cycles given by this family of forms. The number of possible pairs of indices is $(n - 2)(n - 3)/2$ and each form describes one distinct 4-cycle passing through a given vertex, thus in total there are $(n - 2)(n - 3)/2$ distinct 4-cycles passing through a given vertex and the total number of distinct 4-cycles in $BS_n$ is $(n - 2)(n - 3)n!$. This finishes proof of the theorem.

3.2 Proof of Theorem 2

We present now the complete description of 6-cycles.

Proof. Let $\pi = id = [12 \ldots n]$. Consider again two antipodal vertices $\pi$ and $\tau$ of the 6-cycle, thus the length of the $(\pi, \tau)$-path is equal to three. We prove the statement by considering all possible cases of the shifted elements of $\pi$ along the $(\pi, \tau)$-path and describing the set of non-intersecting $(\tau, \pi)$-paths. The number of such elements should not be greater than three. This turns to considering the following cases.

One shifted element. Suppose the element $i$ is shifted to the position of $i + 3$. In order to do so, the sequence $b_i b_{i+1} b_{i+2}$ should be applied to $\pi$, which consists of successively dependent transpositions and by Proposition 1 such $(\pi, \tau)$-path is unique. Hence, no 6-cycle is possible in this case.

Two shifted elements. Suppose the element $i$ is shifted to the position of $i + 2$ and another element $j$ is shifted to $j + 1$. In order to do so, the sequence $b_i b_{i+1}$ and the transposition $b_j$ should be applied to $\pi$. There are three possible orders of application:

I) $b_i b_{i+1} b_j$; II) $b_i b_j b_{i+1}$; III) $b_j b_i b_{i+1}$.

There are two subcases: either a) $b_j$ is independent from $b_i$ and $b_{i+1}$ or b) not.

Case a). If $b_j$ is independent from $b_i$ and $b_{i+1}$, then in order to shift the element $i$ back to the initial position, we need to apply sequence $b_i b_{i+1}$ in inverse order on $(\tau, \pi)$-path. By analysing the possible ways of this operation (see Figure 3), we find the only possible cycle form $b_i b_{i+1} b_j b_{i+1} b_i$, which corresponds to (4) from the statement. Since $b_j$ is independent from $b_i$ and $b_{i+1}$, then $j < i - 1$ or $i < j + 2$ and thus (4) describes exactly $(n - 3)(n - 4)$ forms and each form describes six distinct cycles passing through a given vertex, in total giving $6(n - 3)(n - 4)$ distinct cycles.
Figure 3: Illustration for the case of two shifted elements. Cases I) and III) produce the same form; cycle in case II) is impossible.

Case b). If $b_j$ is dependent on $b_i$ or $b_{i+1}$, then either $j = i$, $j = i + 1$ or $j = i - 1$. Along the $(\tau, \pi)$–path we must apply sequence $b_i b_{i+1}$ in the inverse order, thus for each choice I), II) and III) of $(\pi, \tau)$–path there exists a unique description of possible $(\tau, \pi)$–path. In case $(\pi, \tau)$–path is given by I), then for $j = i$ the non-intersecting $(\tau, \pi)$–path is given as $b_{i+1} b_i b_{i+1}$, whereas $j = i - 1$ and $j = i + 1$ are impossible. In case $(\pi, \tau)$–path is given by III), then for $j = i + 1$ the non-intersecting $(\tau, \pi)$–path is given as $b_i b_{i+1} b_i$, whereas $j = i - 1$ and $j = i$ are impossible. The case of $(\pi, \tau)$–path given by II) is also impossible. Therefore, we have an only form of a 6-cycle given as $b_{i+1} b_i b_{i+1} b_i b_{i+1} b_i$, which corresponds to (3) in the statement. It is easy to see that the number of distinct forms described by (3) is $(n - 2)$, and each form describes one cycle passing through a given vertex, in total giving $(n - 2)$ distinct cycles.

Three shifted elements. Suppose the element $i$ is shifted to the position $i + 1$ by $b_j$. Then we have two other elements $j$ and $k$ shifted to the positions $j + 1$ and $k + 1$ by independent transpositions $b_j$ and $b_k$ correspondingly. Suppose the $(\pi, \tau)$-path is given by $b_i b_j b_k$. The sole restriction on the non-intersecting $(\tau, \pi)$–path is that the two incident edges to $\pi$ and $\tau$ should be different. Therefore we have two non-intersecting $(\tau, \pi)$-paths: 1) $b_j b_k b_i$ and 2) $b_k b_i b_j$. Further we note there are chords between pairs of paths $((b_i b_j b_k), (b_j b_k b_i), (b_j b_k b_i), (b_i b_j b_k))$ and $((b_i b_j b_k), (b_i b_j b_k))$ (see Figure 4). Thus, combining all the non-intersecting pairs of paths, we obtain four distinct canonical forms of cycles described by (5), (6), (7) and (8) in the statement. It is easy to see that these 6-cycles appear only when $n \geq 6$. To calculate the number $N_{C_6}(n)$ of the forms described by each family, we note that when $n = 6$ we have only one way to select three independent transpositions, thus the number of forms given by each (5)-(8) is $N_{C_6}(6) = 1$. For $n \geq 7$ it

Figure 4: Case of three shifted elements: three possible non-intersecting $(\pi, \tau)$–paths with chords between them.
is easy to note the recurrence relation
\[ N_{C_6}(n) = N_{C_6}(n - 1) + \frac{(n - 3)(n - 4)}{2}. \]

From the relation we obtain that the number of forms given by each form is
\[ N_{C_6}(n) = \frac{(n - 3)(n - 4)(n - 5)}{6}. \]
The form (5) describes three distinct cycles, the rest describe six distinct
cycles passing through a given vertex, in total giving \( \frac{(n - 3)(n - 4)(n - 5)}{2} \)
distinct cycles correspondingly.

In total the graph \( BS_n \) contains \( \left( (n - 2) + 6(n - 3)(n - 4) + 7\frac{(n - 3)(n - 4)(n - 5)}{2} \right) \frac{n!}{6} = \frac{(7n^3 - 72n^2 + 247n - 280)n!}{12} \) cycles of length 6. This finishes proof of the theorem.

\[ \square \]

4 Minimal cycle basis

A sum of two subgraphs \( G_1 \) and \( G_2 \) is called a subgraph with vertices \( V(G_1) \cup V(G_2) \) and the
edges given by the symmetric difference between \( E(G_1) \) and \( E(G_2) \). A minimal cycle
basis in the graph \( G \) is the minimal cardinality set of cycles, such that any cycle in \( G \) can
be represented as a finite sum of base cycles.

The complete characterization of 4– and 6–cycles is given in Theorems 1 and 2 corre-
spondingly. Since 4–cycles are minimum possible cycles, they are chordless. The 6–cycles,
on the contrary, may have chords, which is shown in Figure 5.

![Figure 5: Analysis of forms of 6–cycles from Theorem 2.](image)

We notice that a chord in a 6–cycle in \( BS_n \) only can divide it into the sum of 4–cycles.
Therefore, cycles of forms \( C_6^1 \) and \( C_6^3 \) are chordless. However, considering the form \( C_6^3 \)
we may notice that the cycle may be represented as a sum of three 4–cycles. The form \( C_6^1 \)
cannot be decomposed as a sum of 4–cycles, since any two neighboring edges have dependent
transpositions, whereas 4–cycles require use of independent transpositions.

The overall conclusion is that the minimum cycle basis contains 4–cycles and 6–cycles
of form \( C_6^1 \). Denote this set as \( \mathcal{C} \). We show that \( \mathcal{C} \) is a cycle basis for all 8–cycles in \( BS_n \).

**Proposition 2.** Any 8–cycle in the graph \( BS_n, n \geq 4 \), can be represented as a sum of base
cycles from \( \mathcal{C} \).

**Proof.** In order to prove the proposition we simply observe two antipodal vertices of the
8–cycle and consider all possible constructions of non-intersecting \((\pi, \tau)–\) and \((\pi, \tau)–\)paths
and present a possible construction of such 8-cycle as a sum of base cycles. By the remark above we need only to find the division into sum of any 4– and 6–cycles.

Again, let \( \pi = id = [12 \ldots n] \). It is evident that length of these paths is four and if any transposition is employed in the \((\pi, \tau)\)-path, then it should be applied again on \((\tau, \pi)\)-path. We shall not write explicitly these paths, instead we use an induction on the number of dependent transpositions on \((\pi, \tau)\)-path. We use the fact that if a sequence of dependent transpositions is applied on \((\pi, \tau)\)-path, then the transpositions from this sequence should be applied in \((\tau, \pi)\)-path in reverse order. For the sake of convenience we further denote transpositions only by their indices.

**Four independent transpositions.** Suppose the element \( i \) is shifted to the position \( i + 1 \) by \( b_i \). Then we have three other elements \( j, k \) and \( m \) shifted to the positions \( j + 1, k + 1 \) and \( m + 1 \) by independent transpositions \( b_j, b_k \) and \( b_m \) correspondingly. Since we make no assumptions on relations between indices \( i, j, k, m \), then it is only necessary to consider possible \((\tau, \pi)\)-paths obtained by shuffling the indices and keeping the end transpositions different from those in \((\pi, \tau)\)-path. All such shufflings are presented in Table 1 with respective sum representations on Figure 6.

|   | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| i | i   | m   | m   | m   | m   | m   | m   | k   | k   | j   | j   | j   | j   | j   |
| j | j   | k   | i   | j   | k   | i   | m   | i   | m   | j   | m   | m   | i   | k   |
| k | k   | j   | j   | i   | i   | k   | i   | j   | m   | m   | k   | i   | m   | m   |
| m | i   | i   | k   | j   | j   | i   | j   | i   | i   | k   | k   | i   | i   |

Table 1: All possible shufflings of indices \( i, j, k, m \) that produce a \((\tau, \pi)\)-path. A column represents a \((\tau, \pi)\)-path, presented in order from \( \pi \) to \( \tau \), read from top row to bottom.

**Three independent transpositions.** Suppose the element \( i \) is shifted to the position \( i + 2 \) by sequence \( b_i, b_{i+1} \). Then we have two other elements \( k \) and \( m \) shifted to the positions \( k + 1 \) and \( m + 1 \) by independent transpositions \( b_k \) and \( b_m \) correspondingly. Since we make no assumptions on relations between indices \( i, k, m \), then it is only necessary to consider possible \((\tau, \pi)\)-paths obtained by shuffling the indices and keeping the end transpositions different from those in \((\pi, \tau)\)-path. Additionally, we note from the Figure 6 that if at the junction of \((\pi, \tau)\)- and \((\tau, \pi)\)-paths there is a sequence of indices \( i_1, i_2, i_1 \), then we trivially divide an 8–cycle into the sum of 4–cycle \((b_i b_{i+j})^2\) and a 6–cycle, thus fulfilling the statement. Thus, we present only the non-trivial shufflings in Table 2 with respective sum representations on Figure 7.

|   | k   | m   | i   | i + 1 |
|---|-----|-----|-----|-------|
| (1) | i   | m   | i + 1 | k     |
| (2) | m   | i   | i + 1 | k     |

Table 2: Non-trivial shufflings of indices \( i, i + 1, j, k \) that produce a \((\tau, \pi)\)-path. A row with lettered numbering represents a \((\tau, \pi)\)-path, presented in order from \( \pi \) to \( \tau \).

**Two independent transpositions.** Suppose the element \( i \) is shifted to the position \( i + 3 \) by sequence \( b_i, b_{i+1}, b_{i+2} \). Then we have one other element \( m \) shifted to position \( m + 1 \) by an appropriate transposition \( b_m \). Since we make no assumptions on relations between indices \( i \) and \( m \), then it is only necessary to consider possible \((\tau, \pi)\)-paths obtained by shuffling the indices and keeping the end transpositions different from those in \((\pi, \tau)\)-path.
In this case we notice that if \((\pi, \tau)\)-path is given by a sequence \(b_i b_{i+1} b_{i+2} b_m\) then the only possible non-intersecting \((\tau, \pi)\)-path is \(b_m b_i b_{i+1} b_{i+2}\), which is trivially divided into 4-cycle and 6-cycle.

By Proposition 1 we exclude the case of four dependent transpositions and consider the following cases. This finishes the proof of the proposition.

Additionally, we performed a computational experiment of finding the minimal cycle basis for Bubble-sort graphs \(BS_n\), where \(n = 4, \ldots, 6\). We used Python networkX implementation of the algorithm of finding the minimum cycle basis, described in [7]. The results show that for observed values of \(n\) indeed it is enough to have 4-cycles and 6-cycles of form \(C_6^1\) in order to generate all cycles in the graph (see Table 3).

The observations stated in this section give us the reason to state the following hypothesis relative to the minimal cycle basis in \(BS_n\).

**Conjecture 1.** Any cycle in the graph \(BS_n, n \geq 4\), can be represented as a sum of base cycles from \(C\).
Figure 7: Case of three independent transpositions: divisions of possible 8–cycles into sums of base cycles. Only non-trivial cases are shown.

| $n = 4$ | $n = 5$ | $n = 6$ |
|---------|---------|---------|
| $(b_1 b_2)^3, (b_1 b_3)^2, (b_2 b_3)^3$ | $(b_1 b_2)^3, (b_1 b_3)^2, (b_1 b_4)^2, (b_2 b_3)^3, (b_2 b_4)^2, (b_3 b_4)^3$ | $(b_1 b_2)^3, (b_1 b_3)^2, (b_1 b_4)^2, (b_1 b_5)^2, (b_2 b_3)^3, (b_2 b_4)^2, (b_2 b_5)^2, (b_3 b_4)^3, (b_3 b_5)^2, (b_4 b_5)^3$ |

Table 3: Minimal cycle basis in $BS_n$ for $n = 4, \ldots, 6$.

Acknowledgement

This work was supported by the grant 17-51-560008 and 18-01-00353 of the Russian Foundation for Basic Research.

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