Quantum expanders and growth of group representations

by
Gilles Pisier*
Texas A&M University
College Station, TX 77843, U. S. A.
and
Université Paris VI
Inst. Math. Jussieu, Équipe d’Analyse Fonctionnelle, Case 186,
75252 Paris Cedex 05, France

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Abstract

Let $\pi$ be a finite dimensional unitary representation of a group $G$ with a generating symmetric $n$-element set $S \subset G$. Fix $\varepsilon > 0$. Assume that the spectrum of $|S|^{-1} \sum_{s \in S} \pi(s) \otimes \overline{\pi(s)}$ is included in $[-1, 1 - \varepsilon]$ (so there is a spectral gap $\geq \varepsilon$). Let $r'_N(\pi)$ be the number of distinct irreducible representations of dimension $\leq N$ that appear in $\pi$. Then let $R'_{n,\varepsilon}(N) = \sup r'_N(\pi)$ where the supremum runs over all $\pi$ with $n, \varepsilon$ fixed. We prove that there are positive constants $\delta_\varepsilon$ and $c_\varepsilon$ such that, for all sufficiently large integer $n$ (i.e. $n \geq n_0$ with $n_0$ depending on $\varepsilon$) and for all $N \geq 1$, we have $\exp \delta_\varepsilon n N^2 \leq R'_{n,\varepsilon}(N) \leq \exp c_\varepsilon n N^2$. The same bounds hold if, in $r'_N(\pi)$, we count only the number of distinct irreducible representations of dimension exactly $= N$.

1 Introduction

We wish to formulate and answer a natural extension of a question raised explicitly by Wigderson in several lectures (see e.g. [23, p.59]) and also implicitly in [18]. Although the variant that we answer seems to be much easier, it may shed some light on the original question. Wigderson’s question concerns the growth of the number $r_N(G)$ of distinct irreducible representations of dimension $\leq N$ that may appear on a finite group $G$ when the order of $G$ is arbitrarily large and all that one knows is that $G$ admits a generating set $S$ of $n$ elements for which the Cayley graph forms an expander with a fixed spectral gap $\varepsilon > 0$. The problem is to find the best bound of the form $r_N(G) \leq R(N)$ with $R(N)$ independent of the order of $G$ (but depending on $n, \varepsilon$). We consider a more general framework: the finite group $G$ is replaced by a finite dimensional representation $\pi$ (playing the role of the regular representation $\lambda_G$ for finite groups) such that the representation $\pi \otimes \overline{\pi}$ admits a spectral gap, meaning that the trivial representation is isolated with a gap $\geq \varepsilon$ from the other irreducible components of $\pi \otimes \overline{\pi}$. When $\pi = \lambda_G$ we recover the previous notion of spectral gap. Let then $r'_N(\pi)$ be the number of distinct irreducible representations of dimension $\leq N$ appearing in $\pi$ (note that $r_N(G) = r'_N(\lambda_G)$), and let $R'(N)$ denote the least upper bound $r'_N(\pi) \leq R'(N)$ when the only restriction on $\pi$ is that $n, \varepsilon$ remain fixed (but the dimension of $\pi$ is arbitrary). We observe that the previously known bound for $R(N)$ namely $R(N) = e^{o(n N^2)}$ is also valid for $R'(N)$.

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and also that $R(N) \leq R'(N)$. Our main result, which follows from the metric entropy estimate for quantum expanders in [20], is that this bound for $R'(N)$ is sharp: there is $\delta > 0$ such that for all $n$ large enough (i.e. $\forall n \geq n_0(\varepsilon)$) we have $R'(N) \geq \varepsilon n N^2$ for all $N$.

The term “quantum expander” was coined in [9, 2, 3] to which we refer for background (see also [7, 8]).

2 Main result

Let $G$ be any group with a finite generating set $S \subset G$ with $|S| = n$. For any unitary representation $\pi : G \to H_\pi$ we set

$$\lambda(\pi, S) = n^{-1} \sup \{\Re \langle \sum_{s \in S} \pi(s) \xi, \xi \rangle \mid \xi \in H_\pi^{\text{inv}}, \|\xi\|_{H_\pi} = 1\}.$$ 

where $H_\pi^{\text{inv}} \subset H_\pi$ denotes the subspace of all $\pi$-invariant vectors.

When $S$ is symmetric, $\sum_{s \in S} \pi(s)$ being selfadjoint, the real part sign $\Re$ can be omitted.

We then set

$$\varepsilon(\pi, S) = 1 - \lambda(\pi, S).$$

It will be useful to record here the elementary observation that if $\pi$ is unitarily equivalent to the direct sum $\bigoplus_{i \in I} \pi_i$ of a family of unitary representations, then $\lambda(\pi, S) = \sup_{i \in I} \lambda(\pi_i, S)$ and hence

$$\varepsilon(\pi, S) = \inf_{i \in I} \varepsilon(\pi_i, S).$$

In particular, if $\pi_1$ is contained in $\pi_2$, then $\varepsilon(\pi_1, S) \geq \varepsilon(\pi_2, S)$.

We denote

$$\varepsilon(G, S) = \inf \{\varepsilon(\pi, S)\}$$

where the infimum runs over all unitary representations $\pi : G \to H_\pi$. Thus the condition

$$\varepsilon(G, S) > 0$$

means that $G$ has Kazhdan’s “property (T)”, (or in otherwords is a Kazhdan-group), see [1] for more background.

We start by the following result somewhat implicitly due to S. Wassermann [22] and explicitly proved in detail in [6].

**Proposition 2.1** ([22, 6]). For any $\varepsilon > 0$ there is a constant $c_\varepsilon$ such that for any $n$, any group $G$ and any $S \subset G$ with $|S| = n$ such that $\varepsilon(G, S) \geq \varepsilon$, the number $r_N(G)$ of distinct irreducible unitary representations $\sigma : G \to B(H_\sigma)$ with $\dim(H_\sigma) \leq N$ is majorized as follows:

$$r_N(G) \leq \exp(c_\varepsilon n N^2).$$

Of course, here distinct means up to unitary equivalence.

**Remark 2.2.** Note that it suffices to prove a bound of the same form for the number of distinct irreducible unitary representations $\sigma : G \to B(H_\sigma)$ with $\dim(H_\sigma) = N$. Indeed, if the latter number is denoted by $s_N(G)$, we have $r_N(G) = \sum_{d=1}^{N} s_d(G)$, so that it suffices to have a bound of the form $s_d(G) \leq \exp(c'_d n d^2)$ to obtain (2.2).

See [14, 15] for some examples of estimates of the growth of $r_N(G)$.  

2
We note that it was originally proved by Wang [21] that for any Kazhdan-group $G$ this number $r_N(G)$ is finite for any $N$. There is an indication of proof of (2.2) in [22], and detailed proofs appear in [6] (see also [18]). We will prove a simple extension of this bound below.

Recall that a sequence $(G_k, S_k)$ of finite groups equipped with generating sets $S_k \subset G_k$ such that

$$\sup_k |S_k| < \infty, \quad |G_k| \to \infty \quad \text{and} \quad \inf_k \varepsilon(G_k, S_k) > 0$$

is called an expander or an expanding family. This corresponds to the usual notion among Cayley graphs to which we restrict the entire discussion.

Let $\hat{G}$ denote as usual the (finite) set of all irreducible unitary representations of a finite group $G$ (up to unitary equivalence).

We note in passing that it is well known (and this also can be derived from Proposition 2.1) that any expander satisfies

$$\lim_{k \to \infty} \max \{\dim(H_\sigma) \mid \sigma \in \hat{G}_k\} = \infty. \quad (2.3)$$

We refer the reader to the surveys [10, 17] for more information on expanders.

The question raised by Wigderson in this context can be formulated as follows:

Let

$$R_{n, \varepsilon}(N) = \sup \{r_N(G)\}$$

where the supremum runs over all finite groups $G$ admitting a subset $S$ with $|S| = n$ such that $\varepsilon(G, S) \geq \varepsilon$. Actually the question is just as interesting for arbitrary (Kazhdan) groups $G$, but it is more natural to restrict to finite groups, because there are infinite Kazhdan groups without any (nontrivial) finite dimensional representations.

Moreover, since, for a finite group $G$, all representations are weakly contained in the left regular representation $\lambda_G$, we have clearly by (2.1)

$$\varepsilon(G, S) = \varepsilon(\lambda_G, S). \quad (2.4)$$

By (2.2), we have

$$R_{n, \varepsilon}(N) \leq \exp(\varepsilon_\varepsilon nN^2). \quad (2.5)$$

and a fortiori simply $R_{n, \varepsilon}(N) = \exp O(N^2)$.

Wigderson asked whether this upper bound can be improved. More explicitly, what is the precise order of growth of $\log R_{n, \varepsilon}(N)$ when $N \to \infty$. Does it grow like $N$ rather than like $N^2$ ?

The motivation for this question can be summarized like this: In [18] Th. 1.4 an exponential bound $\exp O(N)$ is proved for a special class of groups $G$ (namely monomial groups), admitting a fixed spectral gap with generating sets of very slowly growing size (but not bounded) and it is asked whether the same exponential bound holds in general for $R_{n, \varepsilon}(N)$. Moreover, in a remark following the proof of [18] Th. 1.4, Meshulam and Wigderson observe that for any prime number $p > 2$, there is a group $G_p$ with a generating set of (unbounded) size $\log p$ admitting a fixed spectral gap and such that $r_p(G) \approx 2^p/p$.

Remark 2.3. By classical results, originating in the works of Kazhdan and Margulis (see e.g. [16] or [17] Cor. 2.4), for any fixed $m \geq 3$, the family $\{\text{SL}_m(\mathbb{Z}_p) \mid p \text{ prime}\}$ is an expander, so that we have (for suitable $\ell, \delta$)

$$R_{\ell, \delta}(N) \geq \sup_p r_N(\text{SL}_m(\mathbb{Z}_p)).$$
Similarly, let $G_k$ denote the symmetric group of all permutations of a $k$ element set. Kassabov [11] proved that the family $\{G_k \mid k \geq 1\}$ forms an expanding family with respect to subsets $S_k \subset G_k$ of a fixed size $\ell$ and a fixed spectral gap $\delta > 0$. Thus we find a lower bound

$$R_{\ell, \delta}(N) \geq \sup_k r_N(G_k).$$

Quite remarkably, it is proved in [13] that the family itself of all non-commutative finite simple groups forms an expander (for some suitable $n, \varepsilon$).

**Remark 2.4.** However, it seems the resulting lower bounds are still far from being exponential in $N$. Actually, in many important cases (see e.g. [4]), the proof that certain finite groups $G$ give rise to expanders uses the fact that the smallest dimension of a (non-trivial) irreducible representation on $G$ is \( \geq c|G|^a \) for some $a > 0$. Then since $|G| = \sum_{\pi \in \hat{G}} \dim(\pi)^2$ the cardinal of $\hat{G}$ is bounded above by $|G|^{1-2\varepsilon}/c^2$. Therefore, for any $N \geq c|G|^a$ we have $r_N(G) \leq |G|^{1-2\varepsilon}/c^2 \leq c'N^{(1/a) - 2}$, so that the resulting growth implied for $R_{n, \varepsilon}(N)$ is at most polynomial in $N$. (I am grateful to N. Ozawa for drawing my attention to this point).

Nevertheless, we have:

**Remark 2.5.** (Communicated by Martin Kassabov). For suitable $n, \varepsilon$ the numbers $R_{n, \varepsilon}(N)$ grow faster than any power of $N$. In fact, we will prove the

**Claim :** There is an expanding family of Cayley graphs $(G_k)$ of groups generated by 3 elements with a positive spectral gap $\varepsilon$ and such that for $N_k = 2^{3k} - 2$, $G_k$ admits $2^{k^2}$ distinct irreducible representations of dimension $N_k$.

From this claim follows that $R_{3, \varepsilon}(N_k) \geq 2^{k^2} \geq 2^{(\log(N_k))^2}$, say for all $k$ large enough, and hence

$$R_{n, \varepsilon}(N) \geq 2^{(\log(N))^2}$$

for infinitely many $N$’s.

To prove the claim we use the ideas from [12]. Let $R_k$ denote the (finite) ring $M_k(F_2)$ of $k \times k$ matrices with entries in the field with 2 elements. It is known that the cartesian product $\Pi_k = R_k^{k^2}$ of $|R_k| = 2^{k^2}$ copies of $R_k$ is generated by 3 elements. Indeed, $R_k$ itself is generated as a ring by two elements, e.g. $a = e_{12}$ and the shift $b = e_{12} + e_{23} + \cdots + e_{k-1,k} + e_{k1}$, then $\Pi_k$ is generated as a ring by $\{A, B, C\}$ where $A$ (resp. $B$) is the element with all coordinates equal to $a$ (resp. $b$), and $C$ is such that its coordinates are in one to one correspondence with the elements of $R_k$. To check this, let $R \subset \Pi_k$ be the ring generated by $\{A, B, C\}$. Note, by the choice of $C$, the following easy observation: for any coordinate $i$, there is $x \in R$ such that $x_i = 0$ but $x_j \neq 0$ for all $j \neq i$. For any subset $I$ of the index set let $p_I : R \to R_k^I$ be the coordinate projection. One can then prove by induction on $m = |I|$ that $p_I(R) = R_k^I$ for all $I$. Indeed, assume the fact established for $m - 1$. For any $I$ with $|I| = m$ we pick $i \in I$ and we consider the set $I = \{y \in R_k^I \mid (0,y) \in p_I(R)\}$. By the induction hypothesis, $I$ is an ideal in $R_k^I$, but, since $R_k$ is simple, the above observation implies that $I = R_k^I$, and since $a, b$ generate $R_k$ we have $p_{\{i\}}(R) = R_k$, so we obtain $p_I(R) = R_k^I$.

This implies that the free associative ring $Z(x,y,z)$ (in 3 non-commutative variables) can be mapped onto the product $\Pi_k$. Consider now the group $EL_3(Z(x,y,z))$ generated by the elementary matrices in $GL_3(Z(x,y,z))$. This is a noncommutative universal lattice in the terminology of [12, 3]. First observe that $EL_3(Z(x,y,z))$ is generated by 3 elements. Indeed, let $\alpha, \beta, \gamma$ generate $SL_3(Z)$. Then $\alpha, \beta, \gamma$ will generate $EL_3(Z(x,y,z))$ where $\gamma = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$. Moreover, by [5, Th.1.1] $EL_3(Z(x,y,z))$ has Kazhdan’s property T. It follows that the groups

$$G_k = EL_3(\Pi_k)$$
have expanding generating sets with 3 elements. But it turns out that $G_k$ can be identified with the product

$$SL_{3k}(F_2)^{2k^2}.$$ 

Indeed, firstly one easily checks the natural isomorphism $EL_3(R_k^{2k^2}) \simeq EL_3(R_k)^{2k^2}$, secondly it is well known that, since $F_2$ is a field, $EL_n(F_2) = SL_n(F_2)$ for any $n$, and hence (taking $n = 3k$) we have a natural isomorphism $EL_3(R_k) = SL_{3k}(F_2)$; this yields the identification $G_k = SL_{3k}(F_2)^{2k^2}$.

To conclude, we will use the fact that $SL_{3k}(F_2)$ admits a nontrivial irreducible representation $\pi$ with dimension $N_k = 2^{3k} - 2$. (Just consider its action by permutation on the projective space, which has $2^{3k} - 1$ elements; the action is transitive and doubly transitive, therefore the associated Koopman representation $\pi$ is irreducible and of dimension $2^{3k} - 2$). This immediately produces $2^{k^2}$ distinct irreducible representations of dimension $N_k$ on $SL_{3k}(F_2)^{2k^2}$. Indeed, it is an elementary fact that if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$ is a product group, and if $\pi_1, \cdots, \pi_m$ are arbitrary nontrivial irreducible representations on the factor groups $\Gamma_1, \cdots, \Gamma_m$, then the representations $\pi_j$ defined on $\Gamma$ by $\pi_j(g) = \pi_j(g_j)$ are distinct (meaning not unitarily equivalent), irreducible on $\Gamma$ and $\dim(\pi_j) = \dim^*(\pi_j)$ for any $j$. So taking all $\Gamma_j$’s equal to $SL_{3k}(F_2)$, with $\pi_j = \pi$ and $m = 2^{k^2}$, we obtain the announced claim.

In any case, the problem of finding the correct behaviour of $\log R_{n,\varepsilon}(N)$ (or of $R_{n,\varepsilon}(N)$ itself) when $N \to \infty$ appears to be still wide open.

In this paper we consider a modified version of this question involving “quantum expanders” and show that for this (much easier) modified version, $\pi$ naturally viewed as a linear operator on the (Hilbert space sense) tensor product $H \otimes \bar{H}$. Using the (canonical) identification $H^* \simeq \bar{H}$, the tensor product $H \otimes \bar{H}$ can be isometrically identified with the space of linear operators from $H$ to $\bar{H}$ equipped with the Hilbert-Schmidt norm denoted by $\|\|_2$ (sometimes called the Frobenius norm in the present finite dimensional context). Then, the identity operator $Id_H : H \to H$ defines a distinguished element of $H \otimes \bar{H}$ that we denote by $I$.

We set

$$\lambda(u) = n^{-1} \sup \{ \Re(\langle \sum_{j=1}^n u_j \otimes \bar{u}_j \rangle \xi, \xi) \mid \xi \in H \otimes \bar{H}, \xi \perp I, \|\xi\|_{H \otimes \bar{H}} = 1 \},$$

and

$$\varepsilon(u) = 1 - \lambda(u).$$

In other words, with the preceding identifications, the condition $\varepsilon(u) \geq \varepsilon$ means that for any $x \in M_N$ with $\text{tr}(x) = 0$ we have

$$\Re \sum \text{tr}(u_j x u_j^* x^*) \leq (1 - \varepsilon)\|x\|_2,$$

where $\|x\|_2 = (\text{tr}(x^* x))^{1/2}$. When $T = \sum_{j=1}^n u_j \otimes \bar{u}_j$ is self-adjoint (in particular when the set $\{u_1, \cdots, u_n\}$ is self-adjoint) the real part $\Re$ can be omitted in the two preceding lines.

In group theoretic language, if $\pi : F_n \to U(N)$ is the group representation on the free group $F_n$, equipped with a set of $n$ free generators $S = \{g_1, \cdots, g_n\}$, such that $\pi(g_j) = u_j$ ($1 \leq j \leq n$), then we have

$$\varepsilon(u) = \varepsilon(\pi \otimes \bar{\pi}, S).$$

5
Definition 2.6. A sequence \( \{u(k) \mid k \in \mathbb{N} \} \) with each \( u(k) \in U(N_k)^n \) such that \( N_k \to \infty \) (with \( n \) remaining fixed) and \( \inf_k \{ \varepsilon(u(k)) \} > 0 \) is called a quantum expander. We say that \( n \) is its degree and \( \inf_k \{ \varepsilon(u(k)) \} > 0 \) its spectral gap.

Remark 2.7. The existence of quantum expanders can be deduced as follows from that of expanders. Recalling (2.4), assume given a finite group \( G \) and \( S \subset G \) as before such that \( \varepsilon(G, S) = \varepsilon(\lambda_G, S) \geq \varepsilon > 0 \). Recall that each \( \sigma \in \hat{G} \) is contained in \( \lambda_G \). Let \( \pi \in \hat{G} \). Since any representation on \( G \) without invariant vectors, being a direct sum of non trivial irreps, is weakly contained in \( \lambda_G \), the representation \( \rho = \pi \otimes \overline{\pi} \) restricted to \( H^\text{inv} \) is weakly contained in the non trivial part of \( \lambda_G \). In particular, we have by (2.3)

\[
\lambda(\rho, S) \leq \lambda(\lambda_G, S).
\]

Therefore, we have

\[
\varepsilon(\pi \otimes \overline{\pi}, S) \geq \varepsilon(\lambda_G, S) \geq \varepsilon.
\]

Thus if we are given an expander \( (G_k, S_k) \) as above, say with \( S_k = \{s_1(k), \ldots, s_n(k)\} \), we can choose by (2.3) \( \sigma_k \in \hat{G}_k \) such that \( \dim(H_{\sigma_k}) \to \infty \), and if we set \( u_j(k) = \sigma_k(s_j(k)) \) \((1 \leq j \leq n)\), then \( u(k) = \{u_1(k), \ldots, u_n(k)\} \) forms a quantum expander.

The next statement is a simple generalization of Proposition 2.1

Proposition 2.8. For any \( 0 < \varepsilon < 1 \) there is a constant \( \varepsilon' > 0 \) for which the following holds. Let \( G \) be any group and let \( \pi : G \to B(H) \) be any unitary representation on a finite dimensional Hilbert space \( H \). Let us assume that there is an \( n \)-element subset \( S \subset G \) and \( \varepsilon > 0 \) such that

\[
\varepsilon(\pi \otimes \overline{\pi}, S) \geq \varepsilon.
\]

In other words, \( \pi \) satisfies the following spectral gap condition:

\[
(2.6) \quad \lambda(\pi \otimes \overline{\pi}, S) \leq 1 - \varepsilon
\]

Let \( \pi = \bigoplus_{t \in T} \pi_t \) be the decomposition into distinct irreducibles (where each \( \pi_t \) has multiplicity \( d_t \geq 1 \)), then

\[
(2.7) \quad |\{t \in T \mid \dim(\pi_t) \leq N\}| \leq \exp c_n N^2.
\]

Proof. Let \( \sigma = \bigoplus_{t \in T} \pi_t \) be the direct sum where each component is included with multiplicity equal to 1. We may clearly view \( \sigma \) as a subpresentation of \( \pi \), acting on a subspace \( K \subset H \) so that the orthogonal projection \( Q : H \to K \) is intertwining, i.e. satisfies \( Q\pi = \sigma Q \). Then we also have \((Q \otimes \overline{Q})(\pi \otimes \overline{\pi}) = (\sigma \otimes \sigma)(Q \otimes \overline{Q})\), from which it is easy to derive that if we denote \( V_\pi = H^\text{inv} \), we have \((Q \otimes \overline{Q})V_\pi = V_\sigma\) and \((Q \otimes \overline{Q})V_\pi^\perp = V_\sigma^\perp\). This implies

\[
\lambda(\sigma \otimes \overline{\sigma}, S) \leq \lambda(\pi \otimes \overline{\pi}, S) \leq 1 - \varepsilon.
\]

Thus, replacing \( \pi \) by \( \sigma \), we may as well assume that the multiplicities \( d_t \) are all equal to 1.

Let \( H = \bigoplus_{t \in T} H_t \) denote the decomposition corresponding to \( \pi = \bigoplus_{t \in T} \pi_t \). We have \( \pi \otimes \overline{\pi} = \bigoplus_{t,r \in T} \pi_t \otimes \overline{\pi_r} \), with associated decomposition \( H \otimes \overline{H} = \bigoplus_{t,r \in T} H_t \otimes \overline{H}_r \). From this follows that the subspace \( V_\pi \subset H \otimes \overline{H} \) of \( \pi \otimes \overline{\pi} \)-invariant vectors is equal to \( \bigoplus_{t,r \in T} V_{t,r} \) where \( V_{t,r} \subset H_t \otimes \overline{H}_r \) is the subspace of invariant vectors of \( \pi_t \otimes \overline{\pi_r} \). Since for any \( t \neq r \in T \), \( \pi_t \not\cong \pi_r \), by Schur’s lemma \( V_{t,r} = \{0\} \), and hence \( V_\pi \subset \bigoplus_{t \in T} V_{t,t} \). In particular, this shows that

\[
\forall t \neq r \in T \quad H_t \otimes \overline{H}_r \subset V_\pi^\perp.
\]
Let \( T' = \{ t \in T \mid \dim(\pi_t) = N \} \). It suffices to show an estimate of the form

\[
|T'| \leq \exp c_r n N^2.
\]

(2.8)

Let \( \mathcal{H} \) be the Hilbert space obtained by equipping \( M_N^R \) with the norm

\[
\|x\|_\mathcal{H}^2 = N^{-1} n^{-1} \sum_1^n \text{tr}(x_j^* x_j).
\]

Let \( S = \{s_1, \cdots, s_n\} \). For any \( t \in T' \) we define \( x(t) \in M_N^R \) by

\[
x(t)_j = \pi_t(s_j) \quad 1 \leq j \leq n.
\]

Note that, by our normalization, \( \|x(t)\|_\mathcal{H} = 1 \) for any \( t \in T' \). Moreover, since for any \( t \neq r \in T \) \( \pi_t \ncong \pi_r \), by Schur’s lemma the representation \( \pi_t \otimes \pi_r \) has no invariant vector, and hence lies inside \( (\pi \otimes \pi)|_{V_r^\perp} \). Therefore, by (2.1)

\[
\lambda(\pi_t \otimes \pi_r, S) \leq \lambda(\pi \otimes \pi, S),
\]

and hence for any unit vector \( \xi \in H_{\pi_t} \otimes \overline{H_{\pi_r}} \) we have

\[
n^{-1} \Re(\sum_{s \in S} (\pi_t \otimes \overline{\pi_r})\xi, \xi) \leq 1 - \varepsilon.
\]

In particular, if \( t \neq r \in T' \), we may realize \( \pi_t, \pi_r \) as representations on the same \( N \)-dimensional space, so that taking \( \xi = N^{-1/2} I \) we find

\[
\Re(\langle x(t), x(r) \rangle_\mathcal{H}) = (nN)^{-1} \Re \left( \sum_{s \in S} \text{tr}(\pi_t(s)^* \pi_r(s)) \right) \leq 1 - \varepsilon,
\]

which implies

\[
\|x(t) - x(r)\|_\mathcal{H} \geq \sqrt{2\varepsilon}.
\]

Thus we have \( |T'| \) points in the unit sphere of \( \mathcal{H} \) that are \( \sqrt{2\varepsilon} \)-separated. Since \( \dim(\mathcal{H}) = n N^2 \), (2.8) follows immediately by a well known elementary volume argument (see e.g. [19, p. 57]).

Remark 2.9. To derive Proposition 2.1 from the preceding statement, consider, in the situation of Proposition 2.1 a finite set \( \{\sigma_t \mid t \in T\} \) of distinct finite dimensional irreducible representations of \( G \), let \( \pi \) be their direct sum and let \( \rho = \pi \otimes \overline{\pi} \). By the assumption in Proposition 2.1 we know \( \varepsilon(\rho, S) \geq \varepsilon \), and hence (2.7) implies \( |T| \leq \exp c'_\varepsilon n N^2 \). Applying this to \( \pi = \lambda_G \), this shows that Proposition 2.8 contains Proposition 2.1.

For any finite dimensional unitary representation \( \pi : G \to B(H) \) on an arbitrary group, let us denote by \( r'_N(\pi) \) the number of distinct irreducible representations appearing in the decomposition of \( \pi \) of dimension at most \( N \). Let then

\[
R_{n,\varepsilon}^c(N) = \sup r'_N(\pi)
\]

where the sup runs over all \( \pi \)'s and \( G \)'s admitting an \( n \)-element generating set \( S \subset G \) such that

\[
\varepsilon(\pi \otimes \overline{\pi}, S) \geq \varepsilon.
\]
Note that \( r_N'(\lambda_G) = r_N(G) \) and hence
\[
R_{n,\varepsilon}(N) \leq R'_{n,\varepsilon}(N).
\]

With this notation \((2.7)\) means that
\[
R'_{n,\varepsilon}(N) \leq \exp c'_n N^2.
\]

While it seems very difficult to give a good lower bound for \( R_{n,\varepsilon}(N) \), we can answer the analogous question for \( R'_{n,\varepsilon}(N) \): Indeed, the main result of [20] (see [20, Th. 1.3]), which follows, implies the desired lower bound when reformulated in terms of representations.

**Theorem 2.10 (20).** For each \( 0 < \varepsilon < 1 \), there is a constant \( \beta_\varepsilon > 0 \) such that and for all sufficiently large integer \( n \) (i.e. \( n \geq n_0 \) with \( n_0 \) depending on \( \varepsilon \)) and for all \( N \geq 1 \), there is a subset \( T \subset U(N)^n \) with
\[
|T| \geq \exp \beta_\varepsilon n N^2
\]
such that
\[
\forall u \neq v \in T \quad \| \sum_1^n u_j \otimes v_j \| \leq n(1 - \varepsilon) \quad (\text{we call these } "\varepsilon\text{-separated}"),
\]
and \( \varepsilon(u) \geq \varepsilon \) for all \( u \in T \) (we call these "\( \varepsilon \)-quantum expanders").

More precisely, for all \( u \in T \) we have
\[
\| (\sum u_j \otimes v_j)_{|U} \| \leq n(1 - \varepsilon).
\]

**Theorem 2.11.** The estimate in Proposition \(2.8\) is best possible in the sense that for any \( 0 < \varepsilon < 1 \) there is a constant \( \beta_\varepsilon > 0 \) such that for any \( n \) large enough (i.e. \( n \geq n_0(\varepsilon) \)), for any \( N \geq 1 \) there is a group \( G \) and a finite dimensional representation \( \pi \) on \( G \) satisfying \((2.6)\) and admitting a decomposition \( \pi = \oplus_{t \in T} \pi_t \), with distinct irreducibles \( \pi_t \) each with multiplicity 1 (or any specified value \( \geq 1 \)) and acting on an \( N \)-dimensional space, with
\[
|T| \geq \exp \beta_\varepsilon n N^2.
\]

**Proof.** Fix \( N > 1 \). Let \( T \subset U(N)^n \) be the subset appearing in Theorem \(2.10\) i.e. \( T \) is such that \( |T| \geq \exp \beta_\varepsilon n N^2 \) and \( \forall t \neq r \in T \) we have
\[
(2.9) \quad \| \sum t_j \otimes r_j \| \leq n(1 - \varepsilon),
\]
and also
\[
(2.10) \quad \| (\sum t_j \otimes r_j)_{|U} \| \leq n(1 - \varepsilon).
\]

Let \( s_j = \oplus_{t \in T} t_j \in U(m) \) with \( m = |T|N \), and let \( G \subset U(m) \) be the subgroup generated by \( S = \{ s_1, \ldots, s_n \} \). Note that \( \pi(G) \subset \oplus_{t \in T} M_N \). Let \( \pi : G \to U(m) \) be the inclusion map viewed as a representation on \( G \). Let \( P_t : \oplus_{t \in T} M_N \to M_N \) be the *-homomorphism corresponding to the projection onto the coordinate of index \( t \). For any \( t \in T \), let \( \pi_t : G \to U(N) \) be the representation defined by \( \pi_t = P_t(\pi) \). Then, by definition, we have \( \pi = \oplus_{t \in T} \pi_t \). By the spectral gap condition \(2.10\) the commutant of \( \pi_t(S) \) (which is but the commutant of \( \{ t_1, \ldots, t_n \} \) is reduced to the scalars, so \( \pi_t \) is irreducible, and by \(2.9\) for any \( t \neq r \in T \) the representations \( \pi_t \) and \( \pi_r \) are not unitarily equivalent.

**Remark 2.12.** In particular, this means that \( \forall n \geq n_0(\varepsilon) \) and \( \forall N \)
\[
R'_{n,\varepsilon}(N) \geq \exp \beta_\varepsilon n N^2.
\]

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