NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. This review paper treats the dynamics of non-autonomous dynamical systems. To study the forwards asymptotic behaviour of a non-autonomous differential equation we need to analyse the asymptotic configurations of the non-autonomous terms present in the equations. This fact leads to the definition of concepts such as skew-products and cocycles and their associated global, uniform, and cocycle attractors. All of them are closely related to the study of the pullback asymptotic limits of the dynamical system, from which naturally emerges the concept of a pullback attractor. In the first part of this paper we want to clarify these different dynamical scenarios and the relations between their corresponding attractors.

If the global attractor of an autonomous dynamical system is given as the union of a finite number of unstable manifolds of equilibria, a detailed understanding of the continuity of the local dynamics under perturbation leads to important results on the lower-semicontinuity and topological structural stability for the pullback attractors of evolution processes that arise from small non-autonomous perturbations, with respect to the limit regime. Finally, continuity with respect to global dynamics under non-autonomous perturbation is also studied, for which appropriate concepts for Morse decomposition of attractors and non-autonomous Morse–Smale systems are introduced. All of these results will also be considered for uniform attractors. As a consequence, this paper also makes connections between different approaches to the qualitative theory of non-autonomous differential equations, which are often treated independently.

2010 Mathematics Subject Classification. 37B55, 37L05, 37C70, 34D30, 37B35, 37D15.
Key words and phrases. Skew-product semiflow, Morse decomposition, Morse–Smale systems, cocycle attractor, pullback attractor, uniform attractor.

The first author is partially supported by CNPq 305230/2011-5, CAPES/DGU 238/2011 and FAPESP 2008/55516-3, Brazil, and FEDER and Ministerio de Economía y Competitividad grant # MTM2011-22411. The second author is partially supported by by Proyecto de Excelencia P12-FQM-1492 and Ministerio de Economía y Competitividad grant # MTM2011-22411. The third author is currently partially supported by an EPSRC Leadership Fellowship, EP/G007470/1. The three authors belong to the Brazilian-European partnership in Dynamical Systems (BREUDS) from the FP7 International International Research Staff Exchange Scheme (IRSES) grant of the European Union.
1. Introduction. The theory of dynamical systems is a powerful tool for understanding many different and important real phenomena in a variety of scientific areas. Indeed, the study of compact attracting invariant sets has proved a fruitful research area, providing essential information for an increasing number of models for phenomena from physics, biology, economics, engineering, and others. In particular, the analysis of qualitative properties of semigroups in general phase spaces (infinite-dimensional Banach spaces or general metric spaces) has received a lot of attention throughout the last four decades (see, for instance, [6], [7], [25], [37], [41], [48], [45] or [49]).

1.1. Global attractors for dynamical systems. When a system is shown to possess a global attractor, all its asymptotic behaviour can be described by a detailed analysis of the internal dynamics in this compact invariant set, which leads to a reasonable understanding of the asymptotic behaviour of the associated models, including some localisation of the attractor, knowledge of the rate at which it attracts solutions in the state space, geometrical description, continuity under perturbation, and quantification of its ‘complexity’ via estimates on its dimension.
For autonomous dynamical systems there is now a general theory that goes a long way to meeting the objectives described above. In the non-autonomous case, the theory is still under development, and a number of different (although related) approaches have led to a number of different formal frameworks in which the subject of “non-autonomous dynamics” has been studied. One of the objectives of this review paper is to highlight the relationships between these different approaches, and to show the unique contributions that each can make to the study of non-autonomous dynamics.

First we recall the definition of a global attractor for a nonlinear semigroup \( T(\cdot) \) \([41, 25, 7, 48, 37]\), and then we discuss how this concept can be generalised to the attractor associated to a non-autonomous dynamical system. Throughout this paper, unless mentioned otherwise, \((X,d)\) is a metric space. A family \( \{T(t) : t \geq 0\} \) is called a continuous semigroup if

1. \( T(0) = I_X \), with \( I_X \) being the identity in \( X \),
2. \( T(t+s) = T(t)T(s) \), for all \( t, s \in \mathbb{R}^+ \) and
3. the map \( \mathbb{R}^+ \times X \ni (t,x) \mapsto T(t)x \in X \) is continuous.

We start by giving meaning to the word attraction. In the definition, \( \text{dist}(A,B) \) denotes the Hausdorff semidistance between \( A \) and \( B \) defined as

\[
\text{dist}(A,B) = \sup_{a \in A} \inf_{b \in B} d(a,b).
\]

Note that \( \text{dist}(A,B) = 0 \) implies only that \( \overline{A} \subseteq \overline{B} \), where \( \overline{M} \) denotes the closure of \( M \) in \( X \); we only have \( \text{dist}(A,B) = 0 \) implying \( A \subseteq B \) provided that \( B \) is closed. Later we will also use the symmetric Hausdorff metric,

\[
\text{dist}_H(A,B) = \max\{\text{dist}(A,B), \text{dist}(B,A)\}.
\]

**Definition 1.1.** Let \( B \) and \( C \) be subsets of \( X \). We say that \( B \) attracts \( C \) (under \( T(\cdot) \)) if

\[
\text{dist}(T(t)C,B) \to 0 \quad \text{as} \quad t \to +\infty.
\]

Subsets of \( X \) that are fixed by the semigroup, so called ‘invariant sets’, play an important dynamical role; in particular, invariance is one of the defining properties of the global attractor.

**Definition 1.2.** A set \( A \subseteq X \) is invariant under \( T(\cdot) \) if \( T(t)A = A \) for any \( t \geq 0 \).

Note that if a set is invariant not only does any solution that starts in \( A \) remain in \( A \) (i.e. \( T(t)A \subseteq A \) for all \( t \geq 0 \), known as ‘positive invariance’), but an understanding of solutions through all initial conditions in \( A \) is ‘essential’ for understanding the asymptotic dynamics, since \( A \) does not ‘shrink’ under the evolution. In fact, any invariant set must be a union of the orbits of globally defined solutions.

Given these definitions we are in a position to define an attractor for a semigroup.

**Definition 1.3.** A set \( A \subseteq X \) is the global attractor for a semigroup \( T(\cdot) \) if

1. \( A \) is compact;
2. \( A \) is invariant; and
3. \( A \) attracts each bounded subset of \( X \).

This definition yields the minimal compact set that attracts each bounded subset of \( X \) and the maximal closed and bounded invariant set.

It is well known that global attractors for semigroups are unique and that the existence of a compact attracting set is a necessary and sufficient condition for their existence (see \([41, 48]\)).
As already pointed out, the global attractor can be characterised as the collection of all globally defined bounded solutions. If the semigroup arises from a differential equation (particularly a partial differential equation) this gives an analytic, rather than dynamical, characterisation of the global attractor. The attractor is therefore of interest from both a dynamical and an analytical point of view. We now make this precise.

A global solution for a semigroup \( \{T(t) : t \geq 0\} \) is a continuous function \( \xi : \mathbb{R} \to X \) such that \( T(t)\xi(s) = \xi(t+s) \) for all \( s \in \mathbb{R} \) and all \( t \in \mathbb{R}^+ \). We say that \( \xi : \mathbb{R} \to X \) is a global solution through \( z \in X \) if it is a global solution with \( \xi(0) = z \).

**Lemma 1.4.** If a semigroup \( T(\cdot) \) has a global attractor \( \mathcal{A} \), then
\[
\mathcal{A} = \{ y \in X : \text{there is a bounded global solution through } y \}.
\]

**Proof.** That any \( y \in \mathcal{A} \) lies on a bounded global solution is a consequence of the invariance of \( \mathcal{A} \) and the fact that \( \mathcal{A} \) is bounded. Conversely, if \( \xi(\cdot) \) is a bounded global solution then \( \mathcal{A} \) attracts \( Y = \bigcup_{t \in \mathbb{R}} \xi(t) \). Since \( \xi(0) = T(t)\xi(-t) \), it follows that \( \text{dist}(\xi(0), \mathcal{A}) \leq \text{dist}(T(t)Y, \mathcal{A}) \) for any \( t \geq 0 \), and so \( \text{dist}(\xi(0), \mathcal{A}) = 0 \). Since \( \mathcal{A} \) is closed it follows that \( \xi(0) \in \mathcal{A} \).

In the second part of the proof we have used an idea that will be central to the definition of an attractor in the non-autonomous case; we study the behaviour at a fixed time (here \( t = 0 \)) by taking a ‘pullback’ limit, namely considering
\[
\xi(0) = \lim_{t \to \infty} T(t)\xi(-t);
\]
we consider an initial condition further and further in the past, on which the semigroup is able to act for ever-longer time intervals.

In the next sections we recall the notions of a dynamically gradient semigroup (see \[15, 19\]) and the Morse decomposition (see, for instance, \[45\]) for a global attractor. We first define isolated invariant sets.

### 1.2. Dynamically gradient semigroups.

**Definition 1.5.** Let \( \{T(t) : t \geq 0\} \) be a semigroup on \( X \). We say that an invariant set \( E \subset X \) for the semigroup \( \{T(t) : t \geq 0\} \) is an isolated invariant set if there is an \( \epsilon > 0 \) such that \( E \) is the maximal invariant subset in the neighbourhood \( O_\epsilon(E) \).

A disjoint family of isolated invariant sets is a family \( \{E_1, \cdots, E_n\} \) of isolated invariant sets with the property that,
\[
O_\epsilon(E_i) \cap O_\epsilon(E_j) = \emptyset, \quad 1 \leq i < j \leq n,
\]
for some \( \epsilon > 0 \).

**Definition 1.6.** Let \( \{T(t) : t \geq 0\} \) be a semigroup that possesses a disjoint family of isolated invariant sets \( E = \{E_1, \cdots, E_n\} \). A homoclinic structure associated to \( E \) is a subset \( \{E_{k_1}, \cdots, E_{k_p}\} \) of \( E \) \((p \leq n)\) together with a set of global solutions \( \{\xi_1, \cdots, \xi_p\} \) such that
\[
E_{k_j} \xrightarrow{t \to -\infty} \xi_j(t) \xrightarrow{t \to \infty} E_{k_{j+1}}, \quad 1 \leq j \leq p,
\]
where \( E_{k_{p+1}} := E_{k_1} \).

We now define dynamically gradient semigroups (using the terminology of \[19\]; in \[15\] they are termed gradient-like semigroups).
Definition 1.7. Let \( \{T(t) : t \geq 0\} \) be a semigroup with a global attractor \( \mathcal{A} \) and a disjoint family of isolated invariant sets \( \mathbf{E} = \{E_1, \cdots, E_n\} \). We say that \( \{T(t) : t \geq 0\} \) is a \textit{generalized dynamically gradient semigroup relative to} \( \mathbf{E} \) if

\( (G1) \) for any global solution \( \xi : \mathbb{R} \to \mathcal{A} \) there are \( 1 \leq i, j \leq n \) such that
\[
E_i \xymatrix{ t \to \infty \ar@<1ex>[r] & \rightleftharpoons \ar@<1ex>[l] \xi(t) \xymatrix{ \leftarrow & t \to \infty \ar@<1ex>[r] & \leftarrow \ar@<1ex>[l] E_j }
\]

and
\( (G2) \) there is no homoclinic structure associated to \( \mathbf{E} \).

When each \( E_i \) consists only of a single stationary point, we say that the semigroup is a \textit{dynamically gradient semigroup}.

1.3. Morse decomposition of a global attractor. Next we introduce the notion of a Morse decomposition for the attractor \( \mathcal{A} \) of a semigroup \( \{T(t) : t \geq 0\} \) (see [23], [42] or [45]). We start with the notion of an attractor-repeller pair.

Definition 1.8. Let \( \{T(t) : t \geq 0\} \) be a semigroup with a global attractor \( \mathcal{A} \). We say that a non-empty subset \( A \) of \( \mathcal{A} \) is a \textit{local attractor} if there is an \( \epsilon > 0 \) such that \( \omega(O_\epsilon(A)) = A \), where \( \omega(B) \) is the \( \omega \)-limit set of \( B \), defined as
\[
\omega(B) = \{x \in X : S(t_n)x_n \to x, \text{ for some } x_n \in B, t_n \to \infty\}.
\]
The \textit{repeller} \( A^* \) associated to a local attractor \( A \) is the set defined by
\[
A^* := \{x \in \mathcal{A} : \omega(x) \cap A = \emptyset\}.
\]
The pair \( (A, A^*) \) is called an \textit{attractor-repeller pair} for \( \{T(t) : t \geq 0\} \).

Note that if \( A \) is a local attractor, then \( A^* \) is closed and invariant.

Definition 1.9. Given an increasing family \( \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A} \) of \( n + 1 \) local attractors, for \( j = 1, \cdots, n \), define \( E_j := A_j \cap A_{j-1}^* \). The ordered \( n \)-tuple \( \mathbf{E} := \{E_1, E_2, \cdots, E_n\} \) is called a \textit{Morse decomposition} for \( \mathcal{A} \).

An equivalent definition of a Morse decomposition (see [42, 1]) for the attractor \( \mathcal{A} \) of a semigroup \( \{T(t) : t \geq 0\} \) is the following.

Definition 1.10. Let \( \{T(t) : t \geq 0\} \) be a semigroup with a global attractor \( \mathcal{A} \). A \textit{Morse decomposition} of \( \mathcal{A} \) is a collection \( \mathbf{E} = \{E_1, E_2, \cdots, E_n\} \) of disjoint, compact and invariant subsets of \( \mathcal{A} \) such that for a given global solution \( \xi : \mathbb{R} \to \mathcal{A} \) of \( \{T(t) : t \geq 0\} \)
\[\begin{array}{l}
i\text{ either } \xi(t) \in E_i, \text{ for all } t \in \mathbb{R} \text{ and some } i = 1, \cdots, n; \\
i\text{ or there exist } 1 \leq i < j \leq n \text{ such that } E_j \xymatrix{ t \to \infty \ar@<1ex>[r] & \rightleftharpoons \ar@<1ex>[l] \xi(t) \xymatrix{ \leftarrow & t \to \infty \ar@<1ex>[r] & \leftarrow \ar@<1ex>[l] E_i.}
\end{array}\]

Note that the local attractors are ordered by inclusion whereas the Morse sets are disjoint. Given a Morse decomposition \( \mathbf{E} = \{E_1, E_2, \cdots, E_n\} \) we can construct a sequence of local attractors setting
\[
A_i = E_i \cup \left[ \bigcup_{j=1}^{i-1} W^u(E_j) \right].
\]

Definition 1.11. We say that a semigroup \( \{T(t) : t \geq 0\} \) with a global attractor \( \mathcal{A} \) and a disjoint family of isolated invariant sets \( \mathbf{E} = \{E_1, \cdots, E_n\} \) is a \textit{gradient semigroup} with respect to \( \mathbf{E} \), if there exists a continuous function \( V : X \to \mathbb{R} \) such that
\[\begin{array}{l}
i (0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R} \text{ is non-increasing for each } x \in X; \\
i V \text{ is constant in } E_i, \text{ for each } 1 \leq i \leq n; \text{ and}
\end{array}\]

\[\begin{array}{l}
i V(x) \leq V(y) \text{ for each } x \in \bar{E}_i, y \in \bar{E}_j, i < j;
\end{array}\]
(iii) \( V(T(t)x) = V(x) \) for all \( t \geq 0 \) if and only if \( x \in \bigcup_{i=1}^{n} E_i \).

In this case we call \( V \) a Lyapunov functional related to \( E \).

It has been proved in [1] that a semigroup \( \{T(t) : t \geq 0\} \) is gradient with respect to \( E \) if and only if it is dynamically gradient with respect to \( E \). Essentially, this important result says that, given a disjoint family of isolated invariant sets \( E = \{E_1, \cdots, E_n\} \) for a semigroup \( T(t) \), the following three properties are equivalent:

- \( T(\cdot) \) is dynamically gradient;
- there exists an associated ordered family of local attractor-repellers; and
- there exists a Lyapunov functional related to \( E \).

Since, from the results in [15], dynamically gradient nonlinear semigroups are stable under perturbation, we conclude that gradient semigroups are stable under perturbation as well; that is, the existence of a continuous Lyapunov function is robust under perturbation.

1.4. Morse–Smale semigroups. We now deal with the permanence of connections of isolated invariants sets under perturbation, and to this end we introduce the concept of Morse–Smale systems. Throughout this section \( X \) is a Banach space. We begin by defining the non-wandering set for a semigroup \( \{T(t) : t \geq 0\} \).

**Definition 1.12.** Let \( \{T(t) : t \geq 0\} \) a semigroup with a global attractor \( A \). We define the **non-wandering set** \( \Omega \) of \( \{T(t) : t \geq 0\} \) as the set of points \( u \in A \) such that given \( t_0 \geq 0 \) and a neighbourhood \( V_u \) of \( u \) in \( X \), there exists \( t \geq t_0 \) such that \( T(t)V_u \cap V_u \neq \emptyset \).

We can now define the concept of a Morse–Smale semigroup.

**Definition 1.13.** Let \( \{T(t) : t \geq 0\} \) be a semigroup with a global attractor \( A \). We say that \( \{T(t) : t \geq 0\} \) is a **Morse–Smale semigroup** if it satisfies the following conditions:

1. \( \{T(t) : t \geq 0\} \) is reversible; that is, \( T(t)|_A \) is injective, \( T(t) \) is differentiable on \( A \) and \( T'(t)(z) : X \rightarrow X \) is an injective bounded linear operator, for all \( t \geq 0 \) and \( z \in A \).

2. There exists a finite number of hyperbolic equilibrium points \( z_1, \cdots, z_n \) and a finite number of normally hyperbolic periodic orbits \( \xi_1, \cdots, \xi_m : \mathbb{R} \rightarrow X \) of \( \{T(t) : t \geq 0\} \) such that

\[
\Omega = \{\{z_1\}, \cdots, \{z_n\}, \xi_1(\mathbb{R}), \cdots, \xi_m(\mathbb{R})\}.
\]

3. If \( E \in \Omega \), then \( \dim W^u_{\text{loc}}(E) < \infty \).

4. If \( e_i, e_j \in \Omega \) then \( W^u(e_i) \) and \( W^s_{\text{loc}}(e_j) \) intersect transversally, i.e. there exists a point \( z \in W^u(e_i) \cap W^s_{\text{loc}}(e_j) \) of transversal intersection, so that \( T_z W^u(e_i) + T_z W^s_{\text{loc}}(e_j) = \mathbb{R}^d \) the tangent space at point \( z \).

Our goal will be to relate Morse–Smale semigroups and dynamically gradient semigroups; that is, to find conditions under which the two definitions coincide. We begin with the definition of \( \alpha_\phi \)-limit and some lemmas that will help us to establish this relationship.

**Definition 1.14.** Let \( \{T(t) : t \geq 0\} \) a semigroup, \( u \in X \) and \( \phi : \mathbb{R} \rightarrow X \) a bounded global solution through \( u \). The \( \alpha_\phi \)-limit of \( u \) is the set

\[
\alpha_\phi(u) = \{z \in X : \text{there exists a sequence } \{t_n\}_{n \geq 0} \subset [0, \infty) \text{ with } t_n \xrightarrow{n \to \infty} \infty \text{ such that } \phi(-t_n) \rightarrow z\}.
\]
The following lemma can be found, for instance, in [9].

**Lemma 1.15.** Let \( \{ T(t) : t \geq 0 \} \) be a semigroup with a global attractor \( A \) such that \( T(t) : A \to A \) is a one-to-one homeomorphism for each \( t \geq 0 \), and a non-wandering set \( \Omega \). Then \( \Omega \) is compact and invariant, and \( \omega(x) \) and \( \alpha_\phi(x) \) are subsets of \( \Omega \) for every \( x \in A \) and \( \phi : \mathbb{R} \to X \) bounded global solution through \( x \).

The first result shows that every Morse–Smale semigroup is a generalized dynamically gradient semigroup.

**Theorem 1.16.** (see [9]) Let \( \{ T(t) : t \geq 0 \} \) be a Morse–Smale semigroup with global attractor \( A \) and non-wandering set \( \Omega = \{ \{ z_1 \}, \ldots, \{ z_n \}, \xi_1(\mathbb{R}), \ldots, \xi_m(\mathbb{R}) \} \). Then \( \{ T(t) : t \geq 0 \} \) is a generalized dynamically gradient semigroup with \( \Omega \) as its family of isolated invariant sets.

**Proof.** Assume that there exists a homoclinic structure in \( \Omega \); from the transversality and the \( \lambda \)-lemma we may assume that there is a direct connection, i.e. there exists \( x \in \Omega \) and a global solution \( \xi : \mathbb{R} \to X \) through \( x \) such that \( \xi(t_0) \notin \Omega \), for some \( t_0 \in \mathbb{R} \), and \( \xi(t) \xrightarrow{t
rightarrow \pm \infty} x \). Let \( x^0 = \xi(t_0) \) (which we can assume belongs to \( W^u_{\text{loc}}(x) \)), a neighbourhood \( V \) of \( x^0 \) in \( X \) and \( D^0 \subset V \) a submanifold transversal to \( W^u_{\text{loc}}(x) \) in \( x^0 \). Then, the global \( \lambda \)-lemma (see [36, 9]) implies that there exist a neighbourhood \( D^0 \) of \( x^0 \) in \( D^0 \) and subsets \( S^n \) of \( T^n(D^0) \subset T^n(V) \) such that \( S^n \to T^n(W^u_{\text{loc}}(x)) \), where \( n_0 \in \mathbb{N} \) is such that \( x^0 \in T^{n_0}(W^u_{\text{loc}}(x)) \), which proves that \( x^0 \in \Omega \) and gives us a contradiction. Therefore \( \{ T(t) : t \geq 0 \} \) satisfies (G2).

Now we prove that \( \{ T(t) : t \geq 0 \} \) satisfies (G1). Let \( \xi : \mathbb{R} \to A \) be a global solution with \( \xi(\mathbb{R}) \notin \Omega \) and let \( x = \xi(0) \). From Lemma 1.15 we know that \( \omega(x), \alpha_\xi(x) \subset \Omega \). Now choose \( x_1 \subset \omega(x) \) and a sequence \( \{ t_n \}_{n \geq 0} \subset [0, \infty) \) such that \( t_n \to \infty \) and \( \xi(t_n) \to x_1 \) (remember that \( x_1 \) might be either an equilibrium point or a periodic orbit). Assume that \( \xi(t) \) does not converge to \( x_1 \) as \( t \to \infty \). Then there exists \( \epsilon_0 > 0 \) and a sequence \( \{ \tau_n \}_{n \geq 0} \subset [0, \infty) \) with \( \tau_n > t_n \) and \( d(\xi(\tau_n), x_1) < \epsilon_0 \) for all \( n \geq 0 \). Since \( A \) is compact, we can assume, taking subsequences if necessary, that \( \xi(\tau_n) \to x_2 \in \omega(x) \setminus x_1 \).

Let \( \delta > 0 \) be such that \( \mathcal{O}_\delta(y) \cap \mathcal{O}_\delta(z) = \emptyset \) for any \( y \neq z \in \Omega \). Then there exists a sequence \( \{ s_n \}_{n \geq 0} \subset [0, \infty) \) with \( t_n < s_n < \tau_n \) and \( \xi(s_n) \notin \mathcal{O}_\delta(x_1) \cap \mathcal{O}_\delta(x_2) \) for all \( n \geq 0 \). Again, by the compactness of \( A \), we can assume that \( \xi(s_n) \to x_3 \in \omega(x) \setminus \{ x_1, x_2 \} \). Applying this process inductively we reach a contradiction, since we will obtain infinite equilibrium points and/or periodic orbits. Hence \( \xi(t) \to x_1 \) as \( t \to \infty \). We can argue analogously for \( \alpha_\xi(x) \), which proves that \( \{ T(t) : t \geq 0 \} \) satisfies (G1). \( \square \)

Now we prove the converse result.

**Theorem 1.17.** (see [9]) Let \( \{ T(t) : t \geq 0 \} \) be a reversible generalized gradient semigroup associated with the disjoint family of isolated invariant sets given by

\[
E = \{ \{ z_1 \}, \ldots, \{ z_n \}, \xi_1(\cdot), \ldots, \xi_m(\cdot) \},
\]

where each \( z_i \) is a hyperbolic equilibrium point and \( \xi_j : \mathbb{R} \to X \) is a normally hyperbolic periodic orbit, for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Assume also that if \( E \in \Omega \), then \( \dim W^u_{\text{loc}}(E) < \infty \) and if \( x, y \in \Omega \) then \( W^u(x) \) and \( W^u_{\text{loc}}(y) \) intersect transversally. Then \( \{ T(t) : t \geq 0 \} \) is a Morse–Smale semigroup and its non-wandering set \( \Omega \) coincides with \( E \).
Proof. It is clear that $E \subset \Omega$. Now assume that there exists an $x \in \Omega \setminus E$ and consider a global solution $\xi_1 : \mathbb{R} \to X$ through $x$. By (G1) and (G2), there exist $\Xi_1$ and $\Xi_2$ in $E$ such that $\Xi_1 \neq \Xi_2$ and

$$\Xi_1 \xrightarrow{t \to -\infty} \xi_1(t) \xrightarrow{t \to -\infty} \Xi_2.$$ 

Let $\delta > 0$ be such that $O_\delta(x) \cap O_\delta(\Xi) = \emptyset$ for all $\Xi \in E$ and $O_\delta(\Xi) \cap O_\delta(\hat{\Xi}) = \emptyset$ for all $\Xi, \hat{\Xi} \in E$.

We will construct a global solution $\xi_2 : \mathbb{R} \to X$ in the following manner: for each $k \in \mathbb{N}$, let $t_k > 0$ be such that $\xi_1(t_k) \in O_{\frac{\delta}{k}}(\Xi_2)$. By continuity, there exists a neighbourhood $V_k$ of $x$ such that $T(t_k)V_k \subset O_{\frac{\delta}{k}}(\Xi_2)$ (without any loss of generality we can assume that $\text{diam}(V_k) \to 0$ as $k \to \infty$). Since $x \in \Omega$, there exists $s_k > t_k$ such that $T(s_k)V_k \cap V_k \neq \emptyset$ and choose $x_k \in V_k$ such that $T(s_k)x_k \in V_k$; it follows directly that there exists $r_k \in (t_k, s_k)$ such that $d(T(r_k)x_k, \Xi_2) = \delta$ (for $k$ sufficiently large, so that $1/k < \delta$), and since $T(t_k)x_k \in O_{1/k}(\Xi_2)$ it follows that $t_k - t_k \xrightarrow{k \to \infty} \infty$.

Define then $\psi_k : [t_k - \tau_k, \infty) \to X$ by

$$\psi_k(t) = T(t + \tau_k)x_k, \quad \text{for all } t \geq t_k - \tau_k.$$ 

Hence, there exists a global solution $\xi_2 : \mathbb{R} \to X$ such that (taking subsequences if necessary)

$$\psi_k(t) \xrightarrow{k \to \infty} \xi_2(t) \quad \text{for all } t \in \mathbb{R}$$

and uniformly in bounded sets. Since $d(\psi_k(t), \Xi_2) \leq \delta$ for all $t \in [t_k - \tau_k, 0]$ it follows that $d(\xi_2(t), \Xi_2) \leq \delta$ for all $t \leq 0$ and therefore $\xi_2(t) \to \Xi_2$ as $t \to -\infty$ by Lemma 2.12 of [15]. By (G1) and (G2) there exists $\Xi_3 \in E \setminus \{\Xi_1, \Xi_2\}$ such that $\xi_2(t) \to \Xi_3$ as $t \to \infty$.

Now, given $m \in \mathbb{N}$, choose $r_k > 0$ such that $\xi_2(r_k) \in O_{1/k}(\Xi_3)$, and we can assume, without loss of generality, that

$$d(\psi_k(t), \xi_2(t)) < \delta \quad \text{for all } t \in [0, r_k].$$ 

Thus $\psi_k(t) \notin V_k$ for all $t \in [t_k - \tau_k, r_k]$ (hence $T(r_k + \tau_k)x_k = \psi_k(r_k) \notin V_k$) and so $\gamma_k := r_k + \tau_k < s_k$ and $s_k - \gamma_k \to \infty$ as $k \to \infty$.

Therefore, there exists $\lambda_k \in (\gamma_k, s_k)$ such that $d(T(\lambda_k)x_k, \Xi_3) = \delta$ and it is clear that $\lambda_k - \gamma_k \xrightarrow{k \to \infty} \infty$. Define $\theta_k : [\gamma_k - \lambda_k, \infty) \to X$ by

$$\theta_k(t) = T(t + \gamma_k)x_k \quad \text{for all } t \geq \gamma_k - \lambda_k.$$ 

Thus, taking subsequences if necessary, there exists a global solution $\xi_3 : \mathbb{R} \to X$ such that

$$\theta_k(t) \xrightarrow{k \to \infty} \xi_3(t) \quad \text{for all } t \in \mathbb{R},$$

and with the same argument as before, $\xi_3(t) \to \Xi_3$ as $t \to -\infty$. Again, (G1) and (G2) imply the existence of $\Xi_4 \in E \setminus \{\Xi_1, \Xi_2, \Xi_3\}$ such that $\xi_3(t) \to \Xi_4$ as $t \to \infty$.

Proceeding with this argument inductively we get a contradiction in a finite number of steps, since there are only a finite number of elements in $E$. \qed

As a consequence, we obtain the following result.

**Corollary 1.18.** Let $\{T(t) : t \geq 0\}$ be a reversible semigroup associated with finitely many equilibria and finitely periodic orbits $\Omega = \{\{z_1\}, \ldots, \{z_n\}, \xi_1(\cdot), \ldots, \xi_m(\cdot)\}$, where each $z_i$ is a hyperbolic equilibrium point and $\xi_j : \mathbb{R} \to X$ is a normally hyperbolic periodic orbit, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Assume also that if $x \in \Omega$, then $\dim W^u_{loc}(x) < \infty$ and if $x, y \in \Omega$ then $W^u(x)$ and $W^u_{loc}(y)$ intersect
transversally. Then \( \{T(t) : t \geq 0\} \) is a Morse–Smale semigroup if and only if it is a generalized dynamically gradient semigroup with respect to \( \Omega \).

1.5. Perturbation of global attractors. The study of the effect of perturbation on terms of a dynamical is very natural when we consider such systems as models of real phenomena. Indeed, since a model is always an idealisation, without some robustness under small changes in parameters and/or linear and nonlinear terms it is of dubious worth. In particular, the various notions of attractors that we define need to enjoy some kind of ‘continuity under perturbation’. Such properties have been studied in some detail in the autonomous case, and there are three distinct strands to this work.

a1) Upper semicontinuity of attracting sets under perturbation ([26, 27].)

A global attractor is an invariant compact set describing the long time dynamics of a system. As a set in the phase space \( X \), the first step is to analyse the behaviour under perturbation of the global attractor considered solely as a subset of \( X \). Upper semicontinuity of attractors is a general property and holds when the attractors under perturbation are included, asymptotically, in the limit attractor: it is straightforward to obtain from topological properties of the semigroup. A general upper semicontinuity result confirms the global attractor as a good concept for treating the asymptotic dynamics of autonomous dynamical systems.

a2) Lower semicontinuity of attracting sets under perturbation ([28, 29, 47, 20].)

Lower semicontinuity, when the perturbed attractors do not collapse inside the limit attractor, depends on continuity properties of isolated invariant sets and their local unstable manifolds, so a better understanding of the inner structure of the attractors is required. Indeed, one of the classical arguments to prove the lower semicontinuity of attractors requires detailed information about the local dynamics around isolated invariant sets inside the attractors, such as equilibria or periodic orbits.

The combination of a1) and a2) leads to the the continuity of attractors as sets.

c) Topological structural stability ([25, 19]).

A detailed understanding of the behaviour of hyperbolic equilibria, normally hyperbolic periodic orbits and their associated unstable manifolds is one of the key facts used to prove the characterization of attractors as the union of unstable manifolds and their resulting lower semicontinuity under perturbation. Moreover, similar properties are used to prove that a gradient system (i.e. one possessing a Lyapunov function) with a finite number of hyperbolic equilibria can be completely characterized by the internal dynamics between equilibria: every global solution connects two different equilibria and there are no homoclinic structures connecting equilibria (see [19, 15]).

d) Structural stability ([30, 39]).

The third step in this programme deals with the continuity under perturbation of the dynamics that occurs within the attractor. Indeed, even if the gradient structure of a system is preserved, the connection between two equilibria can be destroyed under perturbation (see [1, 19]). A transversal intersection between the stable and unstable manifolds of isolated invariant manifolds prevents this from happening. Such structural stability of a dynamical system is related to the continuity of the phase diagram that arises from
the dynamics in a global attractor and its perturbation. Morse–Smale systems are the canonical class for which one can prove such structural stability.

For autonomous dynamical systems there is now a general theory that goes a long way to meeting the objectives described above, as shown in Theorem 1.20 below. Although the corresponding theory for non-autonomous perturbations of dynamical systems has been intensively studied in recent years, with special attention paid to changes in the associated pullback attractors (see [19, 2, 9, 8, 16, 17, 18]), the area is still under development. By studying carefully the relation between pullback and uniform attractors in Section 2, here we obtain new results on the continuity and structural stability of both pullback and uniform attractors.

**Definition 1.19.** We say that a family of semigroups \( \{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]} \) is collectively asymptotically compact at \( \eta = 0 \) if, given a sequence \( \{\eta_k\}_{k \in \mathbb{N}} \) with \( \eta_k \to 0 \) as \( k \to \infty \), a bounded sequence \( \{x_k\}_{k \in \mathbb{N}} \) in \( X \) and a sequence \( \{t_k\}_{k \in \mathbb{N}} \) in \( \mathbb{R}^+ \) with \( t_k \to \infty \) as \( k \to \infty \), then \( \{T_{\eta_k}(t_k)x_k\} \) is relatively compact.

We are now ready to state the following theorem on behaviour of global attractors under perturbation, which combines results from [20, 27, 47] for (1), [1] for (2) and [30] for (3).

**Theorem 1.20.** Let \( \{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]} \) be a collectively asymptotically compact family of semigroups. Consider the following hypotheses:

a) For all \( K \subset X \) compact and \( T > 0 \), \( T_\eta \) is continuous at \( \eta = 0 \), i.e.

\[
\sup_{t \in [0,T]} \sup_{z \in K} \|T_\eta(t)z - T_0(t)z\|_X \xrightarrow{\eta \to 0} 0.
\]

b) For each \( \eta \in [0,1] \) the semigroup \( \{T_\eta(t) : t \geq 0\} \) possesses a global attractor \( A_\eta \) and \( \cup_{\eta \in [0,1]} A_\eta \) is bounded.

c) There exists \( n \in \mathbb{N} \) such that for every \( \eta \in [0,1] \) \( A_\eta \) has \( n \) isolated invariant sets \( \Xi_\eta = \{\Xi_{1,\eta}, \ldots, \Xi_{n,\eta}\} \), and \( \sup_{1 \leq i \leq n} \text{dist}_H(\Xi_{i,\eta}, \Xi_{i,0}) \to 0 \) as \( \eta \to 0 \).

d) The local unstable manifold of \( \Xi_{i,\eta} \) behaves continuously as \( \eta \to 0 \); that is, there exists a \( \delta_j > 0 \) such that

\[
\text{dist}_H(W^{u,\delta_j}_{\eta}(\Xi_{j,\eta}), W^{u,\delta_j}_{0}(\Xi_{j})) \to 0 \quad \text{as} \quad \eta \to 0.
\]

e) there are \( \delta > 0 \) and \( \eta_0 \in (0,1) \) such that, if \( \eta < \eta_0 \), \( \xi_0 : \mathbb{R} \to X \) is a global solution in \( A_\eta \), \( t_0 \in \mathbb{R} \) and \( \text{dist}(\xi_\eta(t), \Xi_{\eta,i}) \leq \delta \) for all \( t \leq t_0 \) (\( t \geq t_0 \)), then \( \text{dist}(\xi_\eta(t), \Xi_{\eta,i}) \to 0 \) as \( t \to -\infty \) (\( t \to +\infty \)).

f) \( \{T_0(t) : t \geq 0\} \) is a dynamically gradient semigroup with respect to \( \Xi_0 \).

g) \( \{T_0(t) : t \geq 0\} \) is a Morse–Smale semigroup with respect to \( \Xi_0 \).

Then

1. **(Upper semicontinuity)** Under hypotheses a) and b) the family \( \{A_\eta, 0 \leq \eta \leq \eta_0\} \) is upper semicontinuous at \( \eta = 0 \), namely

\[
\text{dist}(A_\eta, A_0) \to 0 \quad \text{as} \quad \eta \to 0.
\]

2. **(Lower semicontinuity)** Under hypotheses a), b), c), d), the family \( \{A_\eta, 0 \leq \eta \leq \eta_0\} \) is upper and lower semicontinuous at \( \eta = 0 \), namely

\[
\text{dist}_H(A_\eta, A_0) \to 0 \quad \text{as} \quad \eta \to 0.
\]
(3) (Topological structural stability) Under hypotheses a), b), c), e) and f), there exists \( \eta_0 > 0 \) such that, for all \( \eta \leq \eta_0 \), \( \{ T_\eta(t) : t \geq 0 \} \) is a dynamically gradient semigroup and consequently
\[
A_\eta = \bigcup_{i=1}^n W^u(\Xi_{t,\eta}^*), \quad \forall \eta \in [0, \eta_0].
\]

(4) (Geometrical structural stability) Under hypotheses a), b), c), e) and g), there exists \( \eta_1 > 0 \) such that, for all \( \eta \leq \eta_1 \), \( \{ T_\eta(t) : t \geq 0 \} \) is a Morse–Smale semigroup.

2. On the concept of ‘an attractor’: Different dynamical scenarios. In what follows we discuss some of the possible notions of ‘an attractor’ in the non-autonomous setting. Essentially, the attractor for a non-autonomous dynamical system should provide some information about the asymptotic dynamics of the system. But the definition of ‘asymptotic dynamics’ has to be clarified, since, due to the explicit dependence of the system on both the initial time \( s \) and the final time \( t \) (and not only on the elapsed time \( t - s \), which is the case for autonomous systems), different dynamical scenarios emerge, in the sense that we can try to analyse the asymptotic dynamics as \( t \to \infty \) (and then uniformly or not in \( s \)) or as \( s \to -\infty \), which, in general, will be unrelated. In what follows we will show that both are necessary even if we are only concerned with the forwards asymptotic behaviour of a non-autonomous differential equation.

Indeed, consider the initial value problem in a metric space \( (X, d) \)
\[
\begin{aligned}
\dot{u} &= f(t, u), \quad t \geq s, \\
u(s) &= u_0 \in X,
\end{aligned}
\]  
(1)
where \( f : [0, \infty) \times X \to X \) is a nonlinear operator which belongs to a metric space \( \mathcal{C} \). Assume that for each \( f \in \mathcal{C}, u_0 \in X \), the solution of (1) is defined for all \( t \geq s \); that is, for each \( u_0 \in X \), there is a unique continuous function \( \mathbb{R}^+ \ni t \mapsto u(t, s, f, u_0) \in X \) satisfying (1).

Suppose that there is a sequence of times \( t_n \to \infty \) such that
\[
f(\cdot + t_n, \cdot) \to \eta(\cdot, \cdot)
\]
as an element of \( \mathcal{C} \). To each such limiting vector field we can associate a limiting evolution process \( S_\eta(\cdot, \cdot) \) given by \( S_\eta(t, s)u_0 = u(t, s, \eta, u_0) \) where \( u(t, s, \eta, u_0) \) is the solution of
\[
\begin{aligned}
\dot{u} &= \eta(t, u), \quad t \geq s \\
u(s) &= u_0 \in X,
\end{aligned}
\]  
(2)
at time \( t \). Every such evolution process \( S_\eta(\cdot, \cdot) \) will play an important role in the understanding of the limiting states for the solutions of (1). Indeed, the crucial point is that in order to understand the forwards dynamics of a non-autonomous evolution equation like (1), as a general rule we must understand the dynamics of many (possibly infinitely many) non-autonomous evolution processes, one for each global solution in the asymptotic limit for the driving semigroup associated to the non-autonomous terms in the equation. This is in contrast with the autonomous case for which we have only one limiting problem (the semigroup itself) or with the asymptotically autonomous case for which we also have only one limiting problem (the limiting semigroup). Once one appreciates this feature, it becomes clear how rich the study of the “dynamics of non-autonomous dynamical systems” can be.
Coming back to (1), suppose that the map
\[ \mathbb{R}^+ \times \mathcal{C} \times X \ni (t, f, u_0) \mapsto (u(t, f, u_0), f_t) \in X \times \mathcal{C} \]
is continuous, where \( u(t, f, u_0) = u(t, 0, f, u_0) \) and \( f_t(\cdot) = f(t + \cdot) \). Then it is easy to see that
\[ u(t + s, f, u_0) = u(t, f_s, u(s, f, u_0)), \]
and from this it follows that the map \( \Pi(t) : X \times \mathcal{C} \to X \times \mathcal{C} \) defined by
\[
\Pi(t)(u_0, f) := (u(t, f, u_0), f_t)
\]
is a semigroup on \( X \times \mathcal{C} \), since
\[
\Pi(t + s)(u_0, f) = (u(t + s, f, u_0), f_{t+s}) = (u(t, f_s, u(s, f, u_0)), f_t(f_s)) = \Pi(t)(u(s, f, u_0), f_s) = \Pi(t)\Pi(s)(u_0, f).
\]
To discuss the long time behaviour of (1), assume that \( \{\Pi(t) : t \geq 0\} \) possesses a global attractor \( \mathcal{A} \) in \( X \times \mathcal{C} \) with the product metric. While it may initially seem that we have found the proper way to study the asymptotic dynamics of (1), as the set \( \mathcal{A} \) possesses dynamics associated to \( \{\Pi(t) : t \geq 0\} \), it does not have any dynamics immediately associated to (1). An element of \( \mathcal{A} \) is an element of \( X \times \mathcal{C} \); that is, an initial condition \( u_0 \in X \) and a vector field \( g \) (which is not \( f \) in general) and \( \Pi(t)(u_0, g) = (u(t, g, u_0), g_t) \) may have no straightforward relation to (1). Let us try to unravel a little the connection between the points in \( \mathcal{A} \) and the dynamics of (1).

The first step to study the dynamics of (1) is to understand the attraction property of \( \{\Pi(t) : t \geq 0\} \) as \( t \to \infty \) relative to the solution operator of (1).

Given a bounded subset \( \mathcal{B} \) of \( X \times \mathcal{C} \), we know that \( \mathcal{A} \) attracts \( \mathcal{B} \) under the action of \( \{\Pi(t) : t \geq 0\} \) if
\[
\lim_{t \to \infty} \text{dist}(\Pi(t)\mathcal{B}, \mathcal{A}) = 0.
\]

If, for a given bounded subset \( \mathcal{B} \subset X \), we only consider a bounded subset \( \mathcal{B} \) of the form \( B \times \{f\} \), this attraction property can be written as
\[
\lim_{t \to \infty} \text{dist}(u(t, f, B) \times \{f_t\}, \mathcal{A}) \geq \lim_{t \to \infty} \text{dist}(u(t, f, B), \mathcal{A}),
\]
where \( \mathcal{A} = \{u \in X : \text{there exists } g \in \mathcal{C} \text{ such that } (u, g) \in \mathcal{A}\} \). This means that the compact set \( \mathcal{A} \subset X \) attracts bounded subsets of \( X \).

We now try to formalise these ideas. To begin with we take \( f \in C(\mathbb{R}; X) \), i.e. continuous from \( \mathbb{R} \) into \( X \), where \( X \) is a complete metric space, and recast the non-autonomous equation
\[
\dot{u} = \mathcal{F}[u; f(\cdot, t)] \quad u(s) = u_0
\]
as an autonomous system by considering the coupled equations
\[
\begin{cases}
\dot{u} = \mathcal{F}[u; f(\cdot, \sigma)](u) & u(0) = u_0 \\
\sigma = 1 & \sigma(0) = s.
\end{cases}
\]
This corresponds (when the solutions exist for all \( t \geq 0 \)) to a skew-product system in which \( \sigma \) is the variable in the base space \( \Sigma = \mathbb{R} \) that evolves under a flow \( \{\theta_t : t \in \mathbb{R}\} \) via the simple rule \( \theta_t \sigma = \sigma + t \); this base flow ‘drives’ the \( u \) dynamics via changes in \( f \). The fact that the base flow drives the dynamics on \( X \) can be emphasised by writing (1) in the more compact form
\[
\dot{u} = \mathcal{F}[u; f(\cdot, \theta_t s)](u) \quad u(0) = u_0.
\]
There is a general method that provides a reasonably canonical way to obtain a more useful base space for a given non-autonomous equation, formed from the time shifts of the function $f(\cdot, t)$ occurring in the original equation. For simplicity here we can consider $f \in C_0(\mathbb{R}, X)$, the set of bounded continuous functions from $\mathbb{R}$ into $X$. Denote by $\Sigma_0$ the set of all translates of $f$, 

$\Sigma_0(f) = \{ f(s + \cdot) : s \in \mathbb{R} \}$,

and define the shift operator $\theta_t : C_0(\mathbb{R}, X) \to C_0(\mathbb{R}, X)$ by

$\theta_t f(\cdot) = f(\cdot + t)$.

The continuity of $\theta_t$ on $\Sigma_0$ depends on properties of $f$ and the choice of metric $\rho$ on $C_0(\mathbb{R}, X)$. If one places no further restrictions on $f$ than $f \in C_0(\mathbb{R}, X)$ then one must choose $\rho$ to be uniform convergence on compact subintervals; if $f$ is in addition uniformly continuous on $\mathbb{R}$ then it is possible to take $\rho$ to be uniform convergence on the whole of $\mathbb{R}$ (see [22, 35, 43, 44] for more details).

For autonomous and periodic time dependence this construction immediately yields a closed base space. However, for more general non-autonomous terms (e.g. quasiperiodic) it is convenient to consider the closure of $\Sigma_0$ with respect to $\rho$:

$\Sigma := \Sigma_\rho(f) = \text{closure of } \Sigma_0(f) \text{ in } C_0(\mathbb{R}, X)$ with respect to $\rho$, known as the hull of the function $f$ in the space $(C_0(\mathbb{R}, X); \rho)$, see [22, 45]. Continuity of $\theta_t$ on $\Sigma_0$ then extends to continuity of $\theta_t$ on $\Sigma$.

Thus, we can try to analyse non-autonomous differential equations as the combination of a base flow $\{\theta_t\}_{t \in \mathbb{R}}$ on $\Sigma$, and a semiflow $\varphi(t, \sigma)$ on the phase space $X$ that is driven by the base flow. In general the base flow consists of the base space $\Sigma$, which we take to be a metric space with metric $\rho$, and a group of continuous transformations $\{\theta_t\}_{t \in \mathbb{R}}$ from $\Sigma$ into itself such that

- $\theta_0 = \text{Id}_\Sigma$,
- $\theta_t \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$, and
- $\theta_t \Sigma = \Sigma$ for all $t \in \mathbb{R}$.

Note that we require $\Sigma$ to be invariant under $\theta_t$, but we can relax this condition later. Much of the theory of non-autonomous dynamical systems $(\varphi, \theta_{t \times \Sigma})$ has been developed under the assumption that $\Sigma$ is compact, and this is true for a number of interesting examples (see [22]).

Thus, to study the forwards asymptotic dynamics of $(1)$ on the phase space $(X, d)$ we will consider a family of mappings

$\mathbb{R}^+ \times \Sigma \ni (t, \sigma) \mapsto \varphi(t, \sigma) \in \mathcal{C}(X)$

that satisfy

- $\varphi(0, \sigma) = \text{Id}_X$ for all $\sigma \in \Sigma$,
- $\mathbb{R}^+ \times \Sigma \times X \ni (t, \sigma, u) \mapsto \varphi(t, \sigma)u \in X$ is continuous, and
- for all $t \geq s, s \in \mathbb{R}$, and $\sigma \in \Sigma$,

$\varphi(t + s, \sigma) = \varphi(t, \theta_s \sigma) \varphi(s, \sigma)$,

the ‘cocycle property’.

We will refer to $\varphi$ as the cocycle semiflow, after this last property. Roughly speaking, one interprets $\varphi(t, \sigma)u$ as the solution starting (at time zero) with the non-autonomous driving term in ‘state $\sigma$’ at $u$, after a time $t$ has elapsed.
Given a non-autonomous dynamical system \((\varphi, \theta)\) on \((X, \Sigma)\), one can define an associated autonomous semigroup \(\Pi(\cdot)\) on \(X = X \times \Sigma\) by setting (cf. (3))

\[
\Pi(t)(u, \sigma) = (\varphi(t, \sigma)u, \theta_t \sigma), \quad t \geq 0.
\]

The group property of \(\theta\) and the cocycle property of \(\varphi\) ensure that \(\Pi(\cdot)\) satisfies the semigroup property:

\[
\Pi(t + s)(u, \sigma) = (\varphi(t + s, \sigma)u, \theta_{t+s} \sigma) = (\varphi(t, \theta_s \sigma)[\varphi(s, \sigma)u], \theta_t [\theta_s \sigma])
= \Pi(t)(\varphi(s, \sigma)u, \theta_s \sigma) = \Pi(t)\Pi(s)(u, \sigma).
\]

In summary, given a non-autonomous differential equation such as (1), we need to deal with four different but closely related dynamical systems:

(a) The driving system \(\{\theta_t : t \geq 0\}\) on \(\Sigma\) associated with the dynamics of the time-dependent terms appearing in the equation, and which is defined by \(\theta_t f(\cdot, u) = f(t + \cdot, u)\);

(b) the skew-product semiflow \(\{\Pi(t) : t \geq 0\}\) defined on the product space \(X \times \Sigma\);

(c) the associated non-autonomous dynamical system \((\varphi, \theta)(X, \Sigma)\) defined by setting \(\varphi(t, \theta_s f)u_0 = u(t + s, f, u_0)\); and

(d) the evolution process \(S(\cdot, \cdot)\) associated to \(f\) and its translates, defined by \(S(t, s)u_0 = u(t - s, \theta_s f)u_0\).

Observe that an evolution process describes the dynamics of (1) referred to one particular function \(f\). In this sense, if we want to study the forwards asymptotic behaviour of (1) we need to study the asymptotic behaviour of the driving system, the cocycle and the skew-product semiflow, since they all contain information about the future of the non-autonomous terms in the equation. However, there are still some situations, some of which are described below, in which an understanding of the evolution process alone is sufficient to describe the asymptotic behaviour of a non-autonomous dynamical system.

On the other hand, when the original differential equation is autonomous, i.e. when there is no explicit dependence on time,

\[
\dot{u} = F(u) \quad u(s) = u_s,
\]

the value of the solution at time \(t\) depends only on \(t - s\), the time elapsed. In this case we only need to define a one parameter family of solution operators \(\{T(t) : t \in \mathbb{R}\}\), and then

\[
u(t, s; u_s) = T(t - s)u_s.
\]

Note that, to study the forwards asymptotic dynamics of a semigroup \(T(t - s)\) we can make \(t \to +\infty\) or, artificially, \(s \to -\infty\). This observation will be crucial when dealing with non-autonomous equations as (1). Indeed, in this last case we get two different and unrelated notions of the ‘asymptotic dynamics’:

- the forwards dynamics, when we analyse the asymptotic limits of \(S(t, s)\) as \(t \to +\infty\), and
- the pullback dynamics, when we study the state of the system \(S(t, s)\) at a fixed time \(t\) when \(s \to -\infty\).

As we will see in the following sections, some of the fine structure of the forwards dynamics of a non-autonomous dynamical system is likely to be missed if we do not take into account the emergence of invariant structures due to a pullback procedure.

To complete the general framework on basic concepts and tools to study (1), observe that each of the four types of dynamical systems described above can possess an associated attractor:
(i) a global attractor $A$ for the skew-product semiflow $\Pi(t)$;
(ii) a global attractor $A$ for the driving system $\theta$;
(iii) a cocycle attractor $\{A(\sigma)\}_{\sigma \in \Sigma}$ for the cocycle semigroup $\varphi(t, \sigma)$;
(iv) a pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$ for the evolution process $S(t, s)$
and finally, to study $A$ in the phase space $X$, it is useful to consider
(v) the uniform attractor
\[ A = \Pi_X(A) := \{ u \in X : \text{there exists } \sigma \in \Sigma \text{ with } (u, \sigma) \in A \} \]
associated to the cocycle $\varphi$ or the evolution process $S(t, s)$.

2.1. Skew-product semiflows and the uniform attractor. Suppose that the semigroup $\Pi(t)$ possesses a global attractor $A$. We want to use the global attractor of $\Pi(t)$ to define another dynamical object for the skew-product flow, the ‘uniform attractor’ $A$. We know that an autonomous semigroup has a global attractor if and only if it has a compact attracting set (see [41]). So $\Pi(t)$ has a global attractor iff there exists a compact set $K \subset X$ such that
\[ \lim_{t \to \infty} \text{dist}(\Pi(t)B, K) = 0 \]
for any bounded subset $B$ of $X$.

We denote by $\Pi_X$ and $\Pi_\Sigma$ the projections from $X \times \Sigma$ onto $X$ and $\Sigma$ respectively. If $\Sigma$ is bounded and invariant and there exists a compact attracting set $K$ for $\Pi(t)$, then the attraction must hold in particular for bounded sets of the form $B = B \times \Sigma$, where $B$ is a bounded subset of $X$. In this case $\Pi_\Sigma[\Pi(t)B] = \Sigma$ for all $t \geq 0$, and so if $\Pi(t)$ is asymptotically compact, the base space $\Sigma$ must be compact. This assumption is often adopted in the theoretical consideration of skew-product flows, and motivates many of the particular cases of non-autonomous dependence (e.g., almost periodic) that are considered (see [22]).

For the moment we suppose that $\Sigma$ is compact. We first relate the asymptotic compactness of $\Pi(t)$ to an equivalent condition for $(\varphi, \theta)$.

**Proposition 2.1.** If $(\varphi, \theta)$ is a non-autonomous dynamical system and $\Pi(t)$ is the corresponding skew-product semiflow on $X \times \Sigma$, then the following two properties are equivalent:

(i) there exists a compact subset $K$ of $X := X \times \Sigma$ such that for every bounded subset $B$ of $X \times \Sigma$
\[ \lim_{t \to \infty} \text{dist}(\Pi(t)B, K) = 0; \]

(ii) there exists a compact subset $K$ of $X$ such that for every bounded subset $B$ of $X$
\[ \lim_{t \to \infty} \left( \sup_{\sigma \in \Sigma} \text{dist}(\varphi(t, \sigma)B, K) \right) = 0. \]

**Proof.** Given (i), take $K = K \times \Sigma$, which is compact since $K$ and $\Sigma$ are compact. Given any bounded subset $B$ of $X \times \Sigma$ is contained in a set of the form $B \times \Sigma$, where $B$ is a bounded subset of $X$. Since
\[ \Pi(t)[B \times \Sigma] = \left[ \bigcup_{\sigma \in \Sigma} \varphi(t, \sigma)B \right] \times \Sigma, \]
it follows that
\[
dist(\Pi(t)[B \times \Sigma], K \times \Sigma) \leq \sup_{\sigma \in \Sigma} \dist(\varphi(t, \sigma)B, K),
\]
whence (ii) follows. Given (ii), simply observe that \( K = \Pi_X K \) is compact, since \( K \) is compact and \( \Pi_X \) is continuous.

This forms the basis of the following definition \([21, 22, 31, 49]\). Note that this definition is meaningful even if \( \Sigma \) is not compact (and this will be useful below).

**Definition 2.2.** The cocycle semiflow \( \varphi \) is *uniformly asymptotically compact* if there exists a compact set \( K \subset X \) such that
\[
\lim_{t \to \infty} \left( \sup_{\sigma \in \Sigma} \dist(\varphi(t, \sigma)B, K) \right) = 0
\]
for every bounded subset \( B \) of \( X \).

Noting that \( \varphi(t, \sigma) \) gives the solution for a fixed time \( t \), one can see that this property of ‘uniform asymptotic compactness’ represents a generalisation of the autonomous notion of asymptotic compactness in a way that retains one of the key features of autonomous systems, namely the dependence only on the elapsed time. Moreover, due to the definition of \( \Sigma \) as the closure of all time shifts of \( f \), note that the asymptotic behaviour on (4) is not only referred to the ‘future’ of \( \theta_\tau f \) for \( \tau \) large enough, but also of all the limit points of the sequences \( \theta_\tau f \) with \( \tau_n \to \infty \).

Continuing to assume compactness of \( \Sigma \), we have just shown that uniform asymptotic compactness of \( \varphi \) implies asymptotic compactness of \( \Pi(\cdot) \) and hence that \( \Pi(\cdot) \) has a global attractor \( \mathcal{A} \). The natural question is how this can be interpreted for the cocycle semiflow. We can adopt a first approach if we wish to concentrate on the asymptotic behaviour as \( t \to +\infty \). Note that the attracting property of \( \mathcal{A} \) implies that if we set \( \mathcal{A} = \Pi_X \mathcal{A} \) then
\[
\lim_{t \to \infty} \left( \sup_{\sigma \in \Sigma} \dist(\varphi(t, \sigma)B, \mathcal{A}) \right) = 0.
\]
Again, Chepyzhov & Vishik \([22]\) use this as the basis of a definition.

**Definition 2.3.** A set \( \mathcal{A} \subset X \) is *uniformly attracting* for the non-autonomous dynamical system \( (\varphi, \theta) \) if, for all bounded subsets \( B \) of \( X \),
\[
\lim_{t \to \infty} \left( \sup_{\sigma \in \Sigma} \dist(\varphi(t, \sigma)B, \mathcal{A}) \right) = 0.
\]

While the invariance of \( \mathcal{A} \) is not directly carried over to \( \mathcal{A} \) the property of minimality is preserved: the global attractor \( \mathcal{A} \) is the minimal closed set in \( X \) that attracts all bounded sets, and its projection \( \mathcal{A} \) is the minimal closed subset of \( X \) that is uniformly attracting (in the sense of (5)) for all bounded subsets of \( X \). This is easy to see, since if \( \mathcal{A} \subset X \) is uniformly attracting, \( \mathcal{A} \times \Sigma \) is attracting for \( \Pi(\cdot) \), from whence \( \mathcal{A} \subset [\mathcal{A} \times \Sigma] \) and thus \( \Pi_X \mathcal{A} \subset \mathcal{A} \).

This yields the definition of the uniform attractor.

**Definition 2.4.** The minimal closed subset \( \mathcal{A} \) of \( X \) that is uniformly attracting for all bounded subsets \( B \) of \( X \) is called the *uniform attractor* for the cocycle \( \varphi \).

We have therefore proved the following result, in the particular case of a compact base space \( \Sigma \).
Theorem 2.5 ([21]). If ϕ is uniformly asymptotically compact then it has a uniform attractor.

Without the assumption that Σ is compact one cannot appeal to the autonomous theory as we did above, but the general proof follows a fairly standard path: one defines the uniform ω-limit set of B as

$$\omega_\Sigma(B) = \bigcap_{t\geq 0} \left[ \bigcup_{\sigma \in \Sigma} \varphi(s, \sigma) B \right],$$

and shows that it is a non-empty compact subset of X that attracts B uniformly, and in fact is the minimal set with these properties. The uniform attractor is then given by

$$\mathcal{A} = \bigcup_{n \in \mathbb{N}} \omega_\Sigma(B_n)^X,$$

where $B_n$ is the ball in X of radius n and $K^X$ denotes the closure of K in X. For details of the proof see Chapter VII of [22].

The uniform attractor is a fixed subset of the phase space that describes all possible ‘asymptotic configurations’ of the system, but one cannot talk of the dynamics ‘on the uniform attractor’ referred to a fixed cocycle semiflow.

We could have adopted a slightly different approach in order to understand the forwards asymptotic behaviour of (1). Indeed, actually we do not need to take the entire closure of $\Sigma_0$ in order to get $\Sigma$. By taking the closure of $\Sigma$ we include in the skew-product semiflow $\Pi(t)$ not only the forwards limits of $f$ but also its backwards limits, which, in general, may have no relation with the forwards dynamics, as is shown by the following example.

Let us consider the planar system $\dot{x} = F(t, x)$, where $x = (x, y)$. Assume that $F(t, (x, y)) \to F_1(x, y)$ as $t \to -\infty$ and that $F(t, (x, y)) \to F_2(x, y)$ as $t \to +\infty$, where $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy

1) $F_1(x, y) = (x - x^3, (1 - x^2)y - y^3)$; and

2) in polar coordinates $\dot{\theta} = F_2(x)$ becomes $\dot{\theta} = -r(r - 1)(r - 2)$ and $\dot{\theta} = 1$.

It is not hard to see that the Morse decomposition of the global attractor $\mathcal{A}$ of the limiting system at $t = -\infty$, $\dot{x} = F_1(x)$, is $\{M_0, M_1, M_2\}$ with $M_0 = \{(-1, 0)\}$, $M_1 = \{(1, 0)\}$, and $M_2 = \{(0, y) : y \in [-1, 1]\}$. On the other hand, the limiting system at $t = +\infty$, $\dot{x} = F_2(x)$ is a gradient system, whose invariant sets are given by $E_0 = \{0\}$, $E_1 = \{(1, \theta) : \theta \in [0, 2\pi]\} = \{(x, y) : x^2 + y^2 = 1\}$ and $E_2 = \{(2, \theta) : \theta \in [0, 2\pi]\} = \{(x, y) : x^2 + y^2 = 4\}$.

Thus, by the results in [15] for asymptotically autonomous systems (see also [19, 8, 38]), every solution $\psi : \mathbb{R} \to \mathbb{R}^2$ of the system $\dot{x} = F(t, x)$ satisfies

(a) $\psi(t) \to M_i$ for some $i = 0, 1, 2$ as $t \to -\infty$,

(b) $\psi(t) \to E_j$ for some $j = 0, 1, 2$ as $t \to \infty$. 

Figure 1: Different asymptotic behaviours for solutions $\phi, \psi$ of $\dot{x} = F(t, x)$ as $t \to \pm \infty$.

Therefore, suppose that $\Sigma$ is a metric space and its associated flow $\theta_t$ has a compact global attractor $A$, given by

$$A = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \theta_t f.$$  

Note that $A$ contains all limits of the form $\theta_{t_n} f$ with $t_n \to +\infty$. As $A$ is compact and invariant for $\theta_t$, we can define the restricted skew product semiflow $\tilde{\Pi}(t)$ on $X \times (\Sigma_0 \cup A)$, which is positively invariant for $\tilde{\Pi}(t)$. Thus, if $\Pi(t)$ has a global attractor $\hat{A}$ in $X \times \Sigma$, there exists a compact global attractor $\hat{A}$ for $\tilde{\Pi}(t)$ and, generally, $\hat{A} \subset A$. Moreover,

$$\lim_{t \to \infty} \sup_{\sigma \in (\Sigma_0 \cup A)} \text{dist}(\varphi(t, \sigma) B, \Pi_{\Sigma} \hat{A}) = 0.$$  

Furthermore, we could have just defined the global attractor $\hat{A}$ for the skew-product flow on the invariant set $X \times A$, as it is also true that

$$\lim_{t \to \infty} \sup_{\sigma \in (\Sigma_0 \cup A)} \text{dist}(\varphi(t, \sigma) B, \Pi_{\Sigma} \hat{A}) = 0. \quad (6)$$  

In summary, if our aim is to study the forwards asymptotic behaviour of an equation such as (1), we do not need to take the entire hull of $f$, but only its time translations and its associated attractors. This also applies to the case in which the base flow is just a metric space, even an unbounded one. Actually this is the general situation we find when dealing with driving systems related to ordinary or partial differential equations. Indeed, consider the system of autonomous differential equations

$$\begin{cases} 
\dot{v} = f(u, v) & t > 0 \\
\dot{u} = g(u), & t > 0 \\
u(0) = u_0 \in \mathbb{R}^n, \; v(0) = v_0 \in \mathbb{R}^n,
\end{cases} \quad (7)$$

where the $u$ component is decoupled, so that the system (7) generates a skew-product semiflow on $\mathbb{R}^n \times \mathbb{R}^n$. The $u$-component here may be considered to represent an independent system that drives the $v$-component of the system in the sense that

$$\dot{v} = f(u(t), v)$$
for any given solution $u(t)$ of $\dot{u} = g(u)$. Assume that the system $\dot{u} = g(u)$ generates a semigroup \( \{ \theta_t : t \geq 0 \} \) in \( \mathbb{R}^n \), that is, \( \theta_t u_0 = u(t, u_0) \), where \( u(\cdot, u_0) \) is the unique solution for \( t > 0 \) of the problem
\[
\begin{align*}
\dot{u} &= g(u), \quad t > 0 \\
u(0) &= u_0.
\end{align*}
\]
Assume also that \( \{ \theta_t : t \geq 0 \} \) has a global attractor \( A \) and that the generated skew product semiflow \( \{ \Pi(t) : t \geq 0 \} \) has a global attractor \( A \). Then for every pair of points \( (u_0, v_0) \in \mathbb{R}^n \times \mathbb{R}^n \), the solution \( v(t, v_0, u_0) \) of the problem
\[
\begin{align*}
\dot{v}(t) &= f(\theta_t u_0, v(t)), \quad t > 0 \\
v(0) &= v_0,
\end{align*}
\]
satisfies
\[
\lim_{t \to +\infty} \text{dist}(v(t, v_0, u_0), \Pi_{\mathbb{R}^n} A) = 0.
\]
This theoretical reduction from a non-compact, non-invariant base space to a compact invariant space has relevance to the study of asymptotically autonomous systems (see below): the above results show that the uniform attractor of an asymptotically autonomous system is the same as the global attractor of the limiting system.

Finally, note that even in the case in which there is no global attractor for the driving system we can adopt the definition of the skew-product flow. This is exactly the situation we find when dealing with random dynamical systems (see \([4, 24]\)).

2.2. **Skew-product semiflows and the cocycle attractor.** One way to try to generalise the theory of attractors to the non-autonomous case is to ensure that the autonomous semigroup \( \Pi(\cdot) \) has an attractor, and then translate these conditions back to the phase space \( X \).

It is immediate that a non-autonomous set \( D(\cdot) \) is invariant for \( \varphi \) (see Definition 2.6) if and only if the corresponding subset \( D \) of \( X \times \Sigma \)
\[
D = \bigcup_{\sigma \in \Sigma} D(\sigma) \times \{ \sigma \},
\]
is invariant for the semigroup \( \Pi(\cdot) \).

Given a subset \( E \) of \( X \times \Sigma \) we denote by \( E(\sigma) = \pi_\sigma E \) the section of \( E \) over \( \sigma \in \Sigma \), i.e.
\[
E = \bigcup_{\sigma \in \Sigma} E(\sigma) \times \{ \sigma \}. \tag{8}
\]
Given a non-autonomous set \( E(\cdot) \) we denote by \( E \) the set defined by (8), and by \( \gamma_\sigma E \) the trace of \( E(\cdot) \), i.e.
\[
\gamma_\sigma E = \bigcup_{\sigma \in \Sigma} E(\sigma).
\]
Note that
\[
\gamma_\sigma E = \Pi_X \Sigma.
\]
Following \([35]\) we can now relate the concept of cocycle attractors for a non-autonomous dynamical system \( (\varphi, \theta) \) with the attractor for the semiflow \( \Pi(\cdot) \), using the assumption that \( \Sigma \) is compact.

**Definition 2.6.** A non-autonomous compact set \( \{A(\sigma)\}_{\sigma \in \Sigma} \) (\( A(\sigma) \) is compact for all \( \sigma \in \Sigma \)) is called a **cocycle attractor** of \( (\varphi, \theta)_{(X, \Sigma)} \) if
(i) \( \{ A(\sigma) \}_{\sigma \in \Sigma} \) is \( \varphi \)-invariant, i.e.
\[
\varphi(t, \sigma)A(\sigma) = A(\theta_t \sigma),
\]
and
(ii) \( \{ A(\sigma) \}_{\sigma \in \Sigma} \) pullback attracts all bounded subsets \( B \subset X \), i.e.
\[
\lim_{t \to +\infty} \text{dist}(\varphi(t, \theta_{-t} \sigma)B, A(\sigma)) = 0.
\]

The existence of a cocycle attractor is related to the existence of a non-autonomous pullback attracting compact set \( \{ K(\sigma) \}_{\sigma \in \Sigma} \).

**Theorem 2.7.** (see [35]) There exists a pullback attracting compact set \( \{ K(\sigma) \}_{\sigma \in \Sigma} \) if and only if there exists a cocycle attractor \( \{ A(\sigma) \}_{\sigma \in \Sigma} \).

**Theorem 2.8.** Let \((\varphi, \theta)\) be a non-autonomous dynamical system on \((X, \Sigma)\), where \( \Sigma \) is compact, and let \( \Pi(\cdot) \) be the associated semigroup on \( X \times \Sigma \). Assume that \( \mathbb{A} \subset X \times \Sigma \) is the global attractor of the semigroup \( \Pi(\cdot) \). Then \( \{ A(\sigma) \}_{\sigma \in \Sigma} \) with \( A(\sigma) = \pi_\sigma \mathbb{A} \) is the cocycle attractor of \((\varphi, \theta)\).

**Proof.** Since \( \mathbb{A} \) is the global attractor for \( \Pi(\cdot) \) in \( X \times \Sigma \), it follows that \( K = \Pi_X \mathbb{A} \) is compact in \( X \) and that for any bounded subset \( B \) of \( X \)
\[
\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \text{dist}(\varphi(t, \sigma)B, \Pi_X \mathbb{A}) = 0.
\]
In particular for each \( \sigma \in \Sigma \) and each bounded \( B \subset X \)
\[
\lim_{t \to +\infty} \text{dist}(\varphi(t, \theta_{-t} \sigma)B, K) = 0, \tag{9}
\]
i.e. \( K \) is a compact (uniformly) pullback attracting set for \((\varphi, \theta)\), and so \((\varphi, \theta)\) has a cocycle attractor, which is given by \( \omega_K(\cdot) \), the pullback omega-limit set of \( K \), defined as
\[
\omega_K(\cdot) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \varphi(t, \theta_{-t} \sigma)K.
\]

We now claim that this attractor is in fact the non-autonomous set \( A(\cdot) \) with \( A(\sigma) = \pi_\sigma \mathbb{A} \). Since (9) holds, it follows that \( \omega_K(\sigma) \subset K = \Pi_X \mathbb{A} \) for all \( \sigma \in \Sigma \).

We know that \( \omega_K(\cdot) \) pullback attracts \( K \) (see [19]) and is the maximal compact non-autonomous subset of \( K \) that is invariant for \((\varphi, \theta)\).

Moreover \( \omega_K(\cdot) \) is invariant for \((\varphi, \theta)\), so that the set
\[
\bigcup_{\sigma \in \Sigma} \omega_K(\sigma) \times \{ \sigma \}
\]
is invariant for \( \Pi(\cdot) \). Since \( \mathbb{A} \) is the global attractor of \( \Pi(\cdot) \), it is the maximal compact invariant subset of \( X \times \Sigma \). On the other hand, \( \mathbb{A} \subset \mathbb{K} := K \times \Sigma \), so \( \mathbb{K} \) is also the maximal compact invariant subset of \( \mathbb{K} \) for \( \Pi(\cdot) \).

Thus, the non-autonomous set \( A(\cdot) \) associated with \( \mathbb{K} \) is the maximal non-autonomous compact set in \( K \) that is invariant with respect to \((\varphi, \theta)\). That is, \( A(\cdot) = \omega_K(\cdot) \), which we know is the pullback attractor of \((\varphi, \theta)\). \( \square \)

We have just shown that the existence of a global attractor for the semigroup \( \Pi(\cdot) \) implies the existence of a cocycle attractor for \( \varphi \). Without additional conditions the converse does not hold; indeed, we can see that the cocycle attractor need not in general be bounded, while the global attractor of \( \Pi(\cdot) \) must be compact. The following result offers sufficient conditions for a converse result.
Theorem 2.9. Suppose $\Sigma$ is compact and $A(\cdot)$ is the cocycle attractor of $(\varphi, \theta)$ and that $\Pi(\cdot)$ is the associated skew-product semiflow. Assume that $A(\cdot)$ is uniformly attracting, i.e.

$$
\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \text{dist}(\varphi(t, \theta_{-t}\sigma)D(\theta_{-t}\sigma), A(\sigma)) = 0,
$$

and that $\gamma_\sigma A$ is precompact. Then $\mathcal{A}$, the subset of $X \times \Sigma$ associated with $A(\cdot)$, is the global attractor of $\Pi(\cdot)$.

Proof. Note that $A(\cdot)$ is invariant with respect to the cocycle $\varphi$, so that $\mathcal{A}$ is invariant with respect to the associated skew-product flow $\Pi(\cdot)$. Let $\mathcal{A}'$ be the closure of $\mathcal{A}$ in $\Sigma \times X$. Since $\mathcal{A} \subset \gamma_\sigma A \times \Sigma$, $\Sigma$ is compact and $\gamma_\sigma A$ is precompact, then $\mathcal{A}'$ is compact and by the continuity of $\Pi(\cdot)$ it follows that $\mathcal{A}'$ is invariant with respect to $\Pi(\cdot)$. By the definition of $\mathcal{A}'$, it follows that $A(\sigma) \subset A'(\sigma)$ for each $\sigma \in \Sigma$, where $A'(\sigma) = \pi_\sigma \mathcal{A}'$. On the other hand, since $A'(\sigma) \subset \gamma_\sigma A$ for all $\sigma \in \Sigma$, the non-autonomous set $A'(\cdot)$ is pullback attracted by $A(\cdot)$. That is,

$$
\lim_{t \to +\infty} \text{dist}(\varphi(t, \theta_{-t}\sigma)A'(\theta_{-t}\sigma), A(\sigma)) = \text{dist}(A'(\sigma), A(\sigma)) = 0,
$$

which implies that $A'(\sigma) \subset A(\sigma)$. So $A(\sigma) = A'(\sigma)$ for each $\sigma$ and hence $\mathcal{A} = \mathcal{A}'$ is compact in $X \times \Sigma$.

If $\mathcal{E} \subset \Sigma \times X$ is a compact set that is invariant with respect to $\Pi(\cdot)$, then the associated non-autonomous set $E(\cdot)$ (with $E(\sigma) = \pi_\sigma \mathcal{E}$) is invariant with respect to the cocycle $\varphi$. Furthermore, by the pullback attraction of $A(\cdot)$ it follows that $E(\sigma) \subset A(\sigma)$ for each $\sigma$ and hence $E \subset \mathcal{A}$. That is, $\mathcal{A}$ is the maximal compact set in $\Sigma \times X$ that is invariant with respect to $\Pi(\cdot)$.

Let $\mathcal{K} := \overline{\gamma_\sigma A} \times \Sigma$. Then $\mathcal{K}$ is a compact set in $\Sigma \times X$. For any $D \subset X \times \Sigma$ bounded, by the uniform pullback attraction property of $A(\cdot)$, we have

$$
\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \text{dist}(\varphi(t, \theta_{-t}\sigma)D(\theta_{-t}\sigma), A(\sigma)) = 0,
$$

which yields

$$
\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \text{dist}(\varphi(t, \theta_{-t}\sigma)D(\theta_{-t}\sigma), \gamma_\sigma A) = 0.
$$

So it follows that

$$
\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \text{dist}(\varphi(t, \sigma)D(\sigma), \gamma_\sigma A) = 0
$$

and hence

$$
\lim_{t \to +\infty} \text{dist}(\Pi(t)D, \gamma_\sigma A \times \Sigma) = \lim_{t \to +\infty} \text{dist}(\Pi(t)D, \mathcal{K}) = 0.
$$

That is, the compact set $\mathcal{K}$ attracts all bounded subsets of $X \times \Sigma$. Thus, there exists a global attractor of $\Pi(\cdot)$ and the attractor is the maximal compact set in $X \times \Sigma$ that is invariant with respect to $\Pi(\cdot)$, i.e. $\mathcal{A}$. \hfill \Box

Note that the proof demonstrates that under the assumptions of theorem, $\gamma_\sigma A$ must in fact be compact, since we know that $\gamma_\sigma A = \Pi_X \mathcal{A}$.

2.3. The uniform and the cocycle attractors. We can now reinterpret Theorems 2.8 and 2.9 in terms of the uniform attractor.

Theorem 2.10. Suppose that $(\varphi, \theta)$ is uniformly asymptotically compact. Then it has a uniform attractor $A$ and a cocycle attractor $A(\sigma)$. In general

$$
\bigcup_{\sigma \in \Sigma} A(\sigma) \subseteq A.
$$
with equality if $\Sigma$ is compact.

Proof. The cocycle attractor is bounded: it is contained in a set that is uniformly attracted to $A$, in particular is pullback attracted to the uniform attractor. Since $A(\cdot)$ is invariant, it follows that $A(\sigma) \subset A$ for every $\sigma \in \Sigma$, which gives (10). Equality when $\Sigma$ is compact follows from Theorem 2.8. \qed

Observe that, when $\Sigma$ is not compact, our uniform attractor $\tilde{A}$, satisfying (6), has to be taken from the global attractor $A$ for the driving system $\theta_t$, in which case there also exists a cocycle attractor $\tilde{A}(\sigma)$ for which

$$\bigcup_{\sigma \in A} \tilde{A}(\sigma) = \tilde{A}. \quad (11)$$

The equality in (10) or (11) leads to an important consequence for the internal dynamics in the uniform attractor $A$ or $\tilde{A}$.

Note that, given $\sigma \in \Sigma$, we can interpret $\{A(\theta_t \sigma)\}_{t \in \mathbb{R}}$ as the pullback attractor related to the evolution process $S_\sigma(t, s)$ associated to the cocycle $\varphi(t, \sigma)$, defined as $S_\sigma(t, s) = \varphi(t - s, \theta_t \sigma)$.

**Definition 2.11.** Given $x \in X$ and $\sigma \in \Sigma$, we call $\xi_\sigma : \mathbb{R} \to X$ a bounded global solution through $x$ if, for all $t \geq s \in \mathbb{R}$, it satisfies

$$\varphi(t - s, \theta_t \sigma)\xi(s) = \xi(t) \text{ and } \xi(0) = x.$$

Now, note that (11) naturally leads to the following definition.

**Definition 2.12.** Let $(\varphi, \theta)_{(X, \Sigma)}$ be a non-autonomous dynamical system. We say that a subset $\mathcal{M} \subseteq X$ is lifted-invariant if for each $x \in \mathcal{M}$ there exist $\sigma \in \Sigma$ and a bounded global solution $\xi_\sigma$ in $\mathcal{M}$.

Moreover, if there exists $\epsilon > 0$ such that $\mathcal{M}$ is the maximal lifted-invariant set in $\mathcal{O}_c(\mathcal{M})$, then we say that $\mathcal{M}$ is an isolated lifted-invariant set.

**Proposition 2.13.** Suppose $\Sigma$ is compact. The uniform attractor $A$ of the non-autonomous dynamical system $(\varphi, \theta)_{(X, \Sigma)}$ is the maximal isolated lifted-invariant set of $X$.

Proof. Let $\{\Pi(t) : t \geq 0\}$ be the associated skew product semiflow in $X \times \Sigma$, which has a global attractor $\mathcal{A}$ and $A = \Pi_X \mathcal{A}$. Hence, if $x \in A$, there exists $\sigma \in \Sigma$ such that $(x, \sigma) \in \mathcal{A}$ and thus there exists a global solution $\Psi : \mathbb{R} \to \mathcal{A}$ of $\{\Pi(t) : t \geq 0\}$ through $(x, \sigma)$. Thus, if we define $\xi_\sigma(t) = \Pi_X \Psi(t)$, for all $t \in \mathbb{R}$, then $\xi_\sigma : \mathbb{R} \to A$ is a bounded global solution of the non-autonomous dynamical system $(\varphi, \theta)_{(X, \Sigma)}$.

Now assume that $C \subseteq X$ is a lifted-invariant set. If $x \in C$ there exists $\sigma \in \Sigma$ and a bounded global solution $\xi_\sigma : \mathbb{R} \to X$ of the non-autonomous dynamical system $(\varphi, \theta)_{(X, \Sigma)}$ such that $\xi_\sigma(t) \in C$, for all $t \in \mathbb{R}$, by the lifted-invariance of $C$. Defining $\Psi(t) = (\xi_\sigma(t), \theta_t \sigma)$, for all $t \in \mathbb{R}$, Lemma 1.4 shows that $\Psi(t) \in \mathcal{A}$, for all $t \in \mathbb{R}$, which implies that $\xi_\sigma(t) \in A$ for all $t \in \mathbb{R}$; in particular $x = \xi_\sigma(0) \in A$ and hence $C \subseteq A$. \qed

The lifted-invariance of the uniform attractor can be easily appreciated if we look at the attractor for an asymptotically autonomous system, i.e. when $f(t, \cdot) \to f_0$ as $t \to +\infty$ (see below). Indeed, in this case the uniform attractor $A$ for the non-autonomous system coincides with the global attractor $A_0$ related to the nonlinear term $f_0$ (note that in this case $\{f_0\}$ is the global attractor $A$ for the driving system). Thus, given $u \in A$ it is clear that there exists a bounded global solution (of the semigroup $T(t)$ associated to the equation $\dot{u} = f_0(u)$) inside the global attractor.
2.4. The uniform and the pullback attractors. We now want to investigate the relationship between the uniform attractor and the pullback attractor in the more familiar context of processes, which are equivalent to skew-product flows with base space \( \mathbb{R} \) and shift \( \theta_t s = t + s \).

**Definition 2.14.** A non-autonomous compact set \( \{ A(t) \}_{t \in \mathbb{R}} \) is called a **pullback attractor** of \( S(t,s) \) if

(i) \( \{ A(t) \}_{t \in \mathbb{R}} \) is \( S(t,s) \)-invariant, i.e. \( S(t,s)A(s) = A(t) \), for all \( t \geq s \).

(ii) \( \{ A(t) \}_{t \in \mathbb{R}} \) pullback attracts all bounded subsets \( B \subset X \), i.e.

\[
\lim_{s \to -\infty} \text{dist}(S(t,s)B, A(t)) = 0.
\]

Again, a general theorem for the existence and uniqueness of a minimal pullback attractor is related to the existence of a pullback attracting compact set \( K \subset X \) (see [19]).

The theory of uniform attractors can be developed for a single non-autonomous process without recourse to the skew-product framework, if one makes the obvious definitions, i.e. a set \( K \) is uniformly attracting for \( S(\cdot, \cdot) \) if for every bounded subset \( B \) of \( X \),

\[
\lim_{t \to \infty} \left( \sup_{\tau \in \mathbb{R}} \text{dist}(S(t + \tau, \tau)B, K) \right) = 0. \tag{12}
\]

The process is uniformly asymptotically compact if it has a compact uniformly attracting set, and in this case Theorem 2.5 guarantees that it has a uniform attractor \( A \), i.e. a minimal uniformly attracting set.

To effect the comparison with the pullback attractor we introduce the kernel sections (once again the terminology is due to Chepyzhov & Vishik [22]). Recall that a **global solution** of a non-autonomous process \( S(\cdot, \cdot) \), is a function \( u : \mathbb{R} \to X \) such that

\[
S(t, \tau)u(\tau) = u(t) \quad \text{for all} \quad t \geq \tau, \; \tau \in \mathbb{R};
\]

such a solution is said to be (globally) bounded if

\[
\{ u(t) : t \in \mathbb{R} \} \quad \text{is bounded}.
\]

**Definition 2.15.** The kernel \( K \) consists of all bounded global solutions. The **kernel sections** are given by

\[
K(t) = \{ u(t) : u(\cdot) \in K \}. 
\]

These kernel sections are, essentially, the fibres of the pullback attractor: it is easy to prove that if \( S(\cdot, \cdot) \) is a process that has a pullback attractor \( A(t) \), then any backwards bounded trajectory is contained in \( A(t) \), and if \( A(\cdot) \) is backwards bounded then \( A(t) = K(t) \). Note that, in general, a pullback attractor is not required to be backwards bounded, unlike the kernel sections, which are uniformly bounded by definition. This fact is successfully exploited, for instance, in the theory of attractors for random dynamical systems, see [4], [24] or [46]. Indeed, in this case we can define the skew-product semiflow on \( X \times \Omega \), with \( (\Omega, \mathcal{P}) \) a given probability space. There is no hope to get an attractor for the driving system \( \theta_t \) in \( \Omega \). However, we can still define a pullback (random) attractor associated to the cocycle \( \varphi(t, \omega) \), for \( \omega \in \Omega \) (see [24, 46]).

Observe that Theorem 2.5 implies the existence of a (fixed) compact attracting set \( K \) for \( S(\cdot, \cdot) \), so that, from (12) and Theorem 2.8 it also implies the existence of
a pullback attractor $A(\cdot)$ with $\cup_{t \in \mathbb{R}} A(t) \subset K$. Moreover, the uniform attractor $\mathcal{A}$ satisfies
\[
\lim_{t \to +\infty} \sup_{s \in \mathbb{R}} \text{dist}(S(t+s,s)B, \mathcal{A}) = 0.
\]

Thus, the relationship between pullback and uniform attractors reads as follows

**Theorem 2.16.** If $S(\cdot, \cdot)$ has a uniform attractor $\mathcal{A}$ then it has a pullback attractor \{\(A(t)\)\}_{t \in \mathbb{R}}\ which satisfies
\[
\bigcup_{t \in \mathbb{R}} A(t) \subseteq A.
\]

**Proof.** Observe that (13) follows from directly from Theorem 2.8, noting that $A(t) = A(\theta_t f)$.

**Remark 2.17.** Note that equality in (13) does, in general, not hold. Consider, for instance,\[
x' = h(t)x - x^3,
\]
where $h$ is a continuous function in $\mathbb{R}$ with $h(t) = 0$ for $t \leq 0$ and $h(t) = 1$ for $t \geq 1$.
In this case the pullback attractor $A(t) = \{0\}$, for all $t \in \mathbb{R}$, while the uniform attractor is the interval $[-1, 1]$.

In the case that $\Sigma$ is not compact and there exists a global attractor $A$ for the semigroup $\theta_t$ on $\Sigma$, we have defined in (11) the uniform attractor $\mathcal{A}$. In this case $\mathcal{A}$ is not defined by the pullback attractor $A(t)$ related to $f$ but for the union of the pullback attractors associated to any global solution in $A$, i.e. we could write (compare with (11))
\[
\bigcup_{\sigma \in A} \bigcup_{t \in \mathbb{R}} A(\theta_t \sigma) = \mathcal{A}.
\]

Observe that, under the conditions of the existence of a global attractor $A$ for $\theta_t$ on a metric space $\Sigma$, this last equality shows that the forwards asymptotic behaviour of a non-autonomous differential equation of the form
\[
\dot{u} = f(t, u)
\]
is not related to the pullback attractor for the corresponding evolution process $S(t, s)$, but to the pullback attractors related to the family of non-autonomous differential equations
\[
\dot{u} = \sigma(t, u),
\]
for all $\sigma \in A \subseteq \Sigma$. This is the situation we find, for example, in driving systems such as (7) or, in general, in non-autonomous differential equations only defined for positive times (see [8, 10]).

### 2.5. The cocycle and the pullback attractors.

We are now going to write the previous ideas in a different manner, in order to reinforce them. Indeed, given a non-autonomous dynamical system $(\varphi, \theta)_{(X, \Sigma)}$ and a set $R \subset \Sigma$ that is invariant for $\{\theta_t : t \geq 0\}$, we can consider the restriction $\theta_t|_R : R \to R$ and the restriction $\varphi|_{\mathbb{R}^+ \times R \times X} : \mathbb{R}^+ \times R \times X \to X$, so that we have a new non-autonomous dynamical system. In this case, the associated skew product semiflow is $\{\Pi(t)|_{X \times R} : t \geq 0\}$ in the phase space $X \times R$.

**Definition 2.18.** A family of subsets $\{D(t)\}_{t \in \mathbb{R}}$ of $X$ is called a non-autonomous set. If each fibre $D(t)$ is closed/compact/open, then $\{D(t)\}_{t \in \mathbb{R}}$ is called a non-autonomous closed/compact/open set.
Recall that a global solution for a semigroup \( \{ t : t \geq 0 \} \) is a continuous function \( \eta : \mathbb{R} \to \Sigma \) such that \( \theta_t \eta(s) = \eta(t + s) \) for all \( s \in \mathbb{R} \) and \( t \in \mathbb{R}^+ \). We say that \( \eta : \mathbb{R} \to \Sigma \) is a global solution through \( \sigma \in \Sigma \) if it is a global solution with \( \eta(0) = \sigma \).

**Definition 2.19.** Given a global bounded solution \( \eta : \mathbb{R} \to \Sigma \) of the driving system \( \{ \theta_t : t \geq 0 \} \), a non-autonomous set \( \{ D(t) \}_{t \in \mathbb{R}} \) is said to be \( \eta \)-forwards invariant under \( (\varphi, \theta)_{(X, \Sigma)} \) if \( \varphi(t, \eta(s)) \subset D(t + s) \) for all \( s \in \mathbb{R} \) and \( t \geq 0 \). It is said to be \( \eta \)-invariant if \( \varphi(t, \eta(s)) = D(t + s) \) for all \( s \in \mathbb{R} \) and \( t \geq 0 \).

**Definition 2.20.** Given a global bounded solution \( \eta : \mathbb{R} \to \Sigma \) of the driving system \( \{ \theta_t : t \geq 0 \} \) and two non-autonomous sets \( \{ D(t) \}_{t \in \mathbb{R}} \) and \( \{ A(t) \}_{t \in \mathbb{R}} \), we say that \( \{ A(t) \}_{t \in \mathbb{R}} \) \( \eta \)-pullback attracts \( \{ D(t) \}_{t \in \mathbb{R}} \) if
\[
\lim_{t \to \infty} \text{dist}(\varphi(t, \eta(s-t))D(s-t), A(s)) = 0, \text{ for each } s \in \mathbb{R}.
\]

**Definition 2.21.** A universe \( \mathcal{D} \) is a collection of nonempty non-autonomous sets that is closed with respect to set inclusion, i.e. if \( \{ D^1(t) \}_{t \in \mathbb{R}} \in \mathcal{D}_\eta \) and \( D^2(t) \subset D^1(t) \) for all \( t \in \mathbb{R} \), then \( \{ D^2(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \). A non-autonomous compact set \( \{ A(t) \}_{t \in \mathbb{R}} \in \mathcal{D}_\eta \) is called a \( (\mathcal{D}, \eta) \)-pullback attractor of \( (\varphi, \theta)_{(X, \Sigma)} \) if
\[
\begin{align*}
& (i) \quad \{ A(t) \}_{t \in \mathbb{R}} \text{ is } \eta \text{-invariant;} \\
& (ii) \quad \{ A(t) \}_{t \in \mathbb{R}} \text{ } \eta \text{-pullback attracts all families } \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D}.
\end{align*}
\]

The above definitions are a simple rewriting of the known definitions for the non-autonomous setting given above (see also [8, 19]) for the case of a non-injective driving system \( \{ \theta_t : t \geq 0 \} \), where there may be more than one global solution through a given point \( \sigma \in \Sigma \).

The important fact here is the relationship between the global attractor of a skew product semiflow and the pullback attractors of the evolution processes it may contain. Such a relation is expressed in our next result.

**Theorem 2.22.** Assume that the skew product semiflow \( \{ \Pi(t) : t \geq 0 \} \) possesses a global attractor \( \mathcal{A} \) (consequently, the driving system \( \{ \theta_t : t \geq 0 \} \) has a global attractor \( A \)). If \( \eta(\cdot) : \mathbb{R} \to \Sigma \) is a global bounded solution for \( \{ \theta_t : t \geq 0 \} \) then, the evolution process \( \{ S_\eta(t) : t \geq 0 \} \) given by
\[
S_\eta(t,s)u = \varphi(t-s, \eta(s))u, \; u \in X,
\]
possesses a \( (\mathcal{D}, \eta) \)-pullback attractor \( \{ A_\eta(t) \}_{t \in \mathbb{R}} \) with the property that
\[
A_\eta(t) = \{ u \in X : (u, \eta(t)) \in \mathcal{A} \},
\]
where \( \mathcal{D} \) is the collection of all non-autonomous sets \( \{ D(t) \}_{t \in \mathbb{R}} \) such that \( \bigcup_{t \in \mathbb{R}} D(t) \) is bounded in \( X \). Moreover,
\[
\mathcal{A} = \left\{ \bigcup_{t \in \mathbb{R}} A_\eta(t) \times \{ \eta(t) \}, \; \eta(\cdot) \text{ is a global bounded solution for } \{ \theta_t : t \geq 0 \} \right\}
\]

**Proof.** Define \( K = \overline{\eta(\mathbb{R})} \subset A \), which is a compact set in \( \Sigma \) and invariant under the action of \( \{ \theta_t : t \geq 0 \} \). Thus the semigroup \( \{ \Pi_K(t) : t \geq 0 \} \) given by the restriction \( \Pi_K(t) = \Pi(t)|_{X \times K} : X \times K \to X \times K \) is well defined and has a global attractor \( A_K \). By Theorem 2.8, the non-autonomous set \( \{ A_\eta(t) \}_{t \in \mathbb{R}} \), given by \( A_\eta(t) = \{ x \in X : (x, \eta(t)) \in \mathcal{A} \} \), is the pullback attractor for the evolution process \( \{ T_\eta(t) : t \geq 0 \} \). The last assertion is straightforward. \( \square \)
Theorem 3.3. \[\text{Morse decomposition of attractors for skew-products semiflows.}\]

In this section we describe the internal structure and dynamics in the sense of Morse decomposition of the global attractor for the skew product semiflow \(\Pi(t)\) and its relation with the attractors for the associated driving systems.

Indeed, we will show how a Morse Decomposition of the global attractor for \(\theta_i\) produces a Morse Decomposition for the attractors both of the skew-product semiflow and the non-autonomous dynamical system. We will pay special attention to the dynamical properties inherited by this description.

3.1. The lift of a Morse decomposition from \(\Sigma\) to \(X \times \Sigma\). Our primary interest is to obtain a Morse decomposition for the global attractor \(A\) of the skew product semiflow \(\{\Pi(t): t \geq 0\}\) in terms of a Morse decomposition of the global attractor \(A\) of the driving system \(\{\theta_i: t \geq 0\}\).

**Definition 3.1.** Given any \(R \subset \Sigma\) and \(D \subset X \times \Sigma\), we define the subset \(L^D_R \subset X \times \Sigma\) by

\[
L^D_R = \{(x, \sigma) \in D : \sigma \in R\}.
\]

The set \(L^D_R\) is called the lift of \(R\) in \(D\).

If \(\Pi|_\Sigma: X \times \Sigma \to \Sigma\) is the projection on the second coordinate, that is, \(\Pi|_\Sigma(x, \sigma) = \sigma\) for all \((x, \sigma) \in X \times \Sigma\), then we get that \(L^D_R = \Pi|_\Sigma^{-1}(R) \cap D\).

We can now state the following theorem. Note that if \(\Pi(\cdot)\) is a skew-product semiflow with a global attractor \(A\), it follows that the driving flow \(\theta_i\) has a global attractor \(A\).

**Theorem 3.2.** (Theorem 4.3 in Bortolan et al. [8]) Let \(\{\Pi(t): t \geq 0\}\) be a skew product semiflow with a global attractor \(A\), and let \(A\) be the global attractor of \(\{\theta_i: t \geq 0\}\). Suppose that \(A\) possesses a Morse decomposition \(\{M_1, \cdots, M_n\}\).

Define, for each \(i = 1, \cdots, n\), the set \(M_i := L^A_{M_i}\). Then, the family \(\{M_1, \cdots, M_n\}\) is a Morse decomposition for the global attractor \(A\) of \(\{\Pi(t): t \geq 0\}\). Moreover, the set \(M_i\) coincides with the global attractor of the semigroup \(\{\Pi(t): t \geq 0\}\) defined on \(X \times M_i\) by \(\Pi(t) = \Pi(t)|_{X \times M_i}\) for each \(i = 1, \cdots, n\).

We can now reinterpret this result with respect to the uniform attractor.

**Theorem 3.3.** (see [10]) Let \(\{\Pi(t): t \geq 0\}\) be a skew product semiflow with a global attractor \(A\) and suppose \(A\) possesses a Morse decomposition \(\{M_1, \cdots, M_n\}\).

Define, for each \(i = 1, \cdots, n\), the set \(M_i := L^A_{M_i}\). Then the family \(\{M_1, \cdots, M_n\}\) with

\[
M_i = \bigcup_{\sigma \in M_i} A(\sigma) (\subset A)
\]

is a Morse decomposition for the uniform attractor \(A\) of \((\varphi, \theta)\), in the sense that

i) \(\{M_1, \cdots, M_n\}\) is a disjoint family of isolated lifted-invariant sets, i.e. each \(M_i\) is an isolated lifted-invariant set and there exists \(\epsilon > 0\) such that \(O_i(M_i) \cap O_j(M_j) = \emptyset\) if \(1 \leq i < j \leq n\).

ii) Given \(u \in A\) and \(\sigma \in \Sigma\) there exists a global solution \(\xi_\sigma: \mathbb{R} \to X\) of the non-autonomous dynamical system \((\varphi, \theta)|_{X \times \Sigma}\), and we have that either

\[
\xi_\sigma |_{\sim} t \to \infty M_i \quad \text{for all } t \in \mathbb{R},
\]

or

\[
M_j \xrightarrow{\sim} t \to \infty \xi_\sigma |_{\sim} t \to \infty \quad \text{for } 1 \leq i < j \leq n.
\]
3.2. The projection of a Morse decomposition from $X \times \Sigma$ to $\Sigma$.

We are now interested in the opposite problem. Indeed, we investigate when a given Morse decomposition on the global attractor $A$ of the skew product semiflow $\{\Pi(t) : t \geq 0\}$ generates a Morse decomposition in the global attractor $\tilde{A}$ of the driving system $\{\theta_t : t \geq 0\}$.

**Definition 3.4.** Given any $\mathcal{D} \subset X \times \Sigma$ and $R \subset \Sigma$, we define the subset $Q^R_D$ by

$$Q^R_D = \{\sigma \in R : (u, \sigma) \in \mathcal{D} \text{ for some } x \in X\}.$$ 

The set $Q^R_D$ is called the $\Sigma$-projection of $\mathcal{D}$ over $R$.

Notice that $Q^R_D = \Pi_2(\mathcal{D}) \cap R$.

**Theorem 3.5.** (Theorem 4.6 in [8]). Assume that $\{M_1, \cdots, M_n\}$ is a Morse decomposition for the global attractor $A$ of the skew product semiflow $\Pi(t) : t \geq 0$. Let $\tilde{A}$ be the global attractor of the driving system $\{\theta_t : t \geq 0\}$.

Define $M_i := Q^A_{M_i}$ for $i = 1, \cdots, n$ and assume that the family $\{M_1, \cdots, M_n\}$ is disjoint. Then $\{M_1, \cdots, M_n\}$ is a Morse decomposition for $A$.

4. Attractors under non-autonomous perturbation. There has been much intensive research to understand the continuity of attractors and of their structure for semigroups when the perturbations are of a non-autonomous nature. In this section we will try to describe some recent results for a non-autonomous approach to this classical research program, generalizing the results in the autonomous case that were described in Section 1.5.

Let $X$ be a Banach space and $B : D(B) \subset X \to X$ the generator of a strongly continuous semigroup $\{e^{Bt} : t \geq 0\}$. Consider the semilinear problem

$$\begin{align*}
y' &= By + f_0(y), \\
y(\tau) &= y_0,
\end{align*}$$

(14)

and a non-autonomous perturbation of it

$$\begin{align*}
y' &= B y + f_\eta(t, y), \\
y(\tau) &= y_0.
\end{align*}$$

(15)

Assuming that, for $\eta \in [0, 1]$, $f_\eta : \mathbb{R} \times X \to X$ is continuous, continuously differentiable with respect to the second variable and (uniformly for $t \in \mathbb{R}$) Lipschitz continuous in bounded subsets of $X$, the problems (14) and (15) are locally well posed. Assuming further that solutions exist in $[\tau, \infty)$ for any $\tau \in \mathbb{R}$ and $y_0 \in X$ we can define the continuously differentiable evolution process $\{S_\eta(t, \tau) : t \geq \tau \in \mathbb{R}\}$ by

$$S_\eta(t, \tau)y_0 = e^{B(t-\tau)}y_0 + \int_{\tau}^{t} e^{B(t-s)}f_\eta(s, S_\eta(s, \tau)y_0) \, ds.$$ 

Suppose the nonlinearity $f_\eta$ is a small non-autonomous perturbation of $f_0$, in particular assume that, for some $r > 0$,

$$\lim_{\eta \to 0} \sup_{(t, \phi) \in \mathbb{R} \times B_X(0, r)} \{\|f_\eta(t, \phi) - f_0(\phi)\|_X + \|(f_\eta)y(t, \phi) - (f_0)y(\phi)\|_{L(X)}\} = 0. \quad (16)$$

**Remark 4.1.** Observe that, if we want to analyse (15) from the point of view of skew-product semiflows, if we define

$$\Sigma_\eta = \{f_\eta(t + \cdot, \cdot)\}$$
then any sequence $\sigma_\eta \in \Sigma_\eta$ with $\eta \to 0$ satisfies (16).

4.1. **Continuity of attractors as sets.** We start with the definitions of upper and lower semicontinuity, and ‘full continuity’ of attractors.

**Definition 4.2.** Let $\{S_\eta(\cdot, \cdot) : \eta \in [0,1]\}$ be a sequence of evolution processes in a Banach space $X$ with corresponding pullback attractors $\{A_\eta(\cdot) : \eta \in [0,1]\}$. For any bounded interval $I \subset \mathbb{R}$, we say

i) that $A_\eta(\cdot)$ is upper semicontinuous for $t \in I$ if
$$\lim_{\eta \to 0} \sup_{t \in I} \text{dist}(A_\eta(t), A_0(t)) = 0,$$

ii) that $A_\eta(\cdot)$ is lower semicontinuous for $t \in I$ if
$$\lim_{\eta \to 0} \sup_{t \in I} \text{dist}(A_0(t), A_\eta(t)) = 0,$$

iii) and $A_\eta(\cdot)$ is continuous if it is upper and lower semicontinuous for $t \in I$ as $\eta \to 0$.

4.1.1. **Upper semicontinuity of attractors.** The upper semicontinuity of attractors for autonomous dynamical systems is a relatively simple matter depending only on uniform bounds on the attractors, some collective compactness and continuity of the nonlinear semigroups (see [26], or the books [25], [41], or [48] for the autonomous case, and [12], [11] in a non-autonomous framework).

We assume that for each $t \in \mathbb{R}$, for each compact subset $K$ of $X$ and each $T > 0$,
$$\sup_{\tau \in [0,T]} \sup_{u \in K} d(S_\eta(t, t-\tau)u, S_0(t, t-\tau)u) \to 0 \quad \text{as} \quad \eta \to 0; \quad (17)$$
also, given $\tau \in \mathbb{R}$,
$$\bigcup_{\eta \in [0,1]} \bigcup_{t \leq \tau} A_\eta(t) \text{ is bounded}, \quad (18)$$
and that for each $t \in \mathbb{R}$
$$\bigcup_{\eta \in [0,1]} A_\eta(t) \text{ is compact}. \quad (19)$$

**Theorem 4.3.** (*Theorem 3.1 in [17]*) Let $\{S_\eta(\cdot, \cdot) : \eta \in [0,1]\}$ be a sequence of evolution processes with corresponding pullback attractors $\{A_\eta(\cdot) : \eta \in [0,1]\}$. Assume that (17), (18) and (19) are satisfied. Then for each $t \in \mathbb{R}$, $A_\eta(t)$ is upper semicontinuous as $\eta \to 0$, and so for any $I \subset \mathbb{R}$ bounded,
$$\sup_{t \in I} \text{dist}(A_\eta(t), A_0) \to 0 \quad \text{as} \quad \eta \to 0.$$

4.1.2. **Lower semicontinuity of attractors.** Results on the lower semicontinuity of attractors are more difficult to prove. These results, in general, rely on reproducing the structures present in the limiting attractor inside the perturbed attractors. Because of this, a dynamically gradient structure for the attractors of the limiting problem has always been taken as the starting point (for autonomous problems see [27], [47], and also [5, 20]; and for non-autonomous perturbations of autonomous systems see [16] and [38]). This is why, in order to get a result on the lower semicontinuity of pullback attractors, we first need to study the behaviour of hyperbolic equilibria and unstable manifolds under non-autonomous perturbations.
4.1.3. Non-autonomous perturbation of hyperbolic equilibria. The following result shows that, near a hyperbolic equilibrium for (14), there is a unique global hyperbolic solution of (15).

We first need to give meaning to hyperbolicity of the zero solution of a linear evolution process.

**Definition 4.4.** We say that a linear evolution process \{U(t, s) : t \geq s\} has an exponential dichotomy with exponent \( \omega \) and constant \( M \) if there exists a family of projections \{Q(t) : t \in \mathbb{R}\} \subset L(X) such that

i) \( Q(t)U(t, s) = U(t, s)Q(s) \), for all \( t \geq s \);

ii) The restriction \( U(t, s)|_{R(Q(s))} \), \( t \geq s \) is an isomorphism from \( R(Q(s)) \) into \( R(Q(t)) \); we denote its inverse by \( U(s, t) : R(Q(t)) \to R(Q(s)) \).

iii) \( U(t, s) \) satisfies the estimates

\[
\|U(t, s)(I - Q(s))\| \leq Me^{-\omega(t-s)} \quad t \geq s
\]
\[
\|U(t, s)Q(s)\| \leq Me^{\omega(t-s)}, \quad t \leq s.
\]

Suppose that \( \xi_0^* : \mathbb{R} \to X \) is a global solution for (15). If \( y(t) \) is a solution to (15) and \( z = y - \xi_0^* \), we rewrite equation (15) as

\[
\dot{z} = A_\eta(t)z + h_\eta(t, z)
\]
\[
z(0) = z_0, \tag{20}
\]

where \( A_\eta(t) = B + f_\eta'(t, \xi_0^*(t)) \) and

\[
h_\eta(t, z) = f_\eta(t, \xi_0^*(t) + z) - f_\eta(t, \xi_0^*(t)) - f_\eta'(t, \xi_0^*(t))z.
\]

Then \( A_\eta(t) \) generates a linear evolution process \{\( U_\eta(t, s) : t \geq s \}\} \subset \mathcal{L}(X) \) (if \( \xi_0^* \) is an equilibrium then for \( \eta = 0 \) this in fact yields a strongly continuous semigroup \{\( e^{A\eta t} : t \geq 0 \)\}). Moreover, 0 is an equilibrium solution for (20) and \( h_\eta(t, 0) = 0, h_\eta'(t, 0) = 0 \in L(X) \).

We say that \( \xi_0^* \) is hyperbolic if \( \{U_\eta(t, s) : t \geq s\} \) has an exponential dichotomy. When \( \eta = 0 \) and \( \xi_0^* \) is an equilibrium, it is hyperbolic if \( \{e^{A0(t-s)} : t \geq s\} \) has an exponential dichotomy. It is easy to see that the family of projections associated to the exponential dichotomy of \( \{e^{A0(t-s)} : t \geq s\} \) is constant: that is \( Q(t) = Q \) for all \( t \in \mathbb{R} \). Also, if \( \{e^{A0(t-s)} : t \geq s\} \) has an exponential dichotomy with constant \( M_1 \) and exponent \( \beta \), the spectrum of \( A_0 \) does not intersect the imaginary axis, the set \( \sigma^+ = \{\lambda \in \sigma(A_0) : \text{Re} \lambda > 0\} \) is compact and, if \( \gamma \) is a smooth closed simple curve in \( \rho(A_0) \cap \{\lambda \in \mathbb{C} : \text{Re} \lambda > 0\} \) oriented counterclockwise and enclosing \( \sigma^+ \),

\[
Q = Q(\sigma^+) = \frac{1}{2\pi i} \int_{\gamma} (\lambda I - A_0)^{-1} \lambda \, d\lambda.
\]

Furthermore, \( e^{A_0 t}Q : R(Q) \to R(Q) \) is an isomorphism and

\[
\|e^{A_0 t}Q\|_{L(X^+)} \leq M_1 e^{\beta t}, \quad t \leq 0,
\]
\[
\|e^{A_0 t}(I - Q)\|_{L(X^-)} \leq M_1 e^{-\beta t}, \quad t \geq 0.
\]

(Note that the fact that the spectrum does not intersect the imaginary axis and the compactness of \( \sigma^+ \) will not, in general, imply an exponential dichotomy for a linear semigroup.)

Then we have the following result.
Theorem 4.5. (Theorem 2.1 in [14]) Let \( \xi_0^* \) be a hyperbolic equilibrium solution for (14) and assume that (16) holds. Then there exists \( \eta_0 > 0 \) such that, for each \( 0 < \eta < \eta_0 \) there is a global bounded solution \( \xi_\eta^* : \mathbb{R} \to X \) of (15) such that
\[
\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} \| \xi_\eta^*(t) - \xi_0^* \|_X = 0.
\]

Let us argue that the global bounded solutions \( \xi_\eta^* \) obtained in the previous theorem inherit the hyperbolicity of \( \xi_0^* \). We rewrite (20) as
\[
\dot{z} = (A_0 + B_\eta(t))z + h_\eta(t, z)
\]
where \( \mathbb{R} \ni t \mapsto B_\eta(t) \in L(X) \) is strongly continuous and defined as \( B_\eta(t) = (t, f_\eta)(\xi_\eta^*(t)) - f_\eta'(\xi_0^*) \). Hence, 0 is a globally defined bounded solution for (21) and \( h_\eta(t, 0) = 0 \), \( (h_\eta)_\eta(t, 0) = 0 \in L(X) \).

Consider now the linear problem associated to (21)
\[
\dot{z} = A_0 z + B_\eta(t) z
\]
where (22) has a unique mild solution \( U_\eta(t, \tau, z_0) \) for each \( z_0 \in X \), which satisfies
\[
U_\eta(t, \tau, z_0) = e^{A_0(t-\tau)} z_0 + \int_{\tau}^{t} e^{A_0(t-s)} B_\eta(s) U_\eta(s, \tau, z_0) \, ds.
\]

The family \( U_\eta(t, \tau, z_0) \) is a linear evolution process. From (23) is easy to see that, for any \( T > 0 \),
\[
\lim_{\eta \to 0} \sup_{t \in [0, T]} \sup_{\tau \in \mathbb{R}} \| U_\eta(t + \tau, \tau) - e^{A_0 \tau} \|_{L(X)} = 0.
\]

It follows from Theorem 7.6.11. in [32] that given \( \omega < \beta \) and \( M > M_1 \), there exists \( \delta_0 > 0 \) such that, if \( \sup_{\eta \in \mathbb{R}} \| B_\eta(t) \|_{L(X)} = \delta < \delta_0 \) then (22) has an exponential dichotomy. Hence, \( \xi_\eta^* : \mathbb{R} \to X \) is a hyperbolic global solution.

4.1.4. Non-autonomous unstable manifolds. Now we show that the unstable and stable manifolds of a hyperbolic solution \( \xi_\eta^*(\cdot) \) are given by graphs.

Definition 4.6. The unstable manifold of a hyperbolic solution \( \xi_\eta^* \) to (15) is the set
\[
W^u(\xi_\eta^*) = \{ (\tau, \zeta) \in \mathbb{R} \times X : \text{there is a solution } z(\cdot, \tau, \zeta) : (-\infty, \tau] \to X \text{ of (15)} \}
\]
satisfying \( z(\tau, \tau, \zeta) = \zeta \) and such that \( \lim_{\tau \to -\infty} \| z(\tau, \tau, \zeta) - \xi_\eta^*(t) \| = 0 \} \).

The stable manifold of a hyperbolic solution \( \xi_\eta^*(\cdot) \) to (15) is the set
\[
W^s(\xi_\eta^*) = \{ (\tau, \zeta) \in \mathbb{R} \times X : \text{there is a solution } z(\cdot, \tau, \zeta) : [\tau, \infty) \to X \text{ of (15)} \}
\]
satisfying \( z(\tau, \tau, \zeta) = \zeta \) and such that \( \lim_{\tau \to +\infty} \| z(\tau, \tau, \zeta) - \xi_\eta^*(t) \| = 0 \} \).

It is proved in [14] that the unstable and stable manifolds of \( \xi_\eta^* \) are given by maps
\[
\mathbb{R} \times X \ni (t, z) \mapsto \Sigma^u(t, Q_\eta(t)z) \in X
\]
with the property that \( \Sigma^u(t, Q_\eta(t)z) \in (I - Q_\eta(t))X \) for all \( z \in X \) and \( t \in \mathbb{R} \) and
\[
\mathbb{R} \times X \ni (t, z) \mapsto \Sigma^s(t, (I - Q_\eta(t))z) \in X
\]
with the property that $\Sigma^u(t, (I - Q_\eta(t))z) \in Q_\eta(t)X$ for all $z \in X$ and $t \in \mathbb{R}$.

The points in the unstable manifold will be those of the form

$$\eta^*(t) + (t, Q_\eta(t)z + \Sigma^u(t, Q_\eta(t)z)) \in \mathbb{R} \times X$$

with $(t, z) \in \mathbb{R} \times X$ and $z$ small,

and the points on the stable manifold those of the form

$$\eta^*(t) + (t, (I - Q_\eta(t))z + \Sigma^s(t, (I - Q_\eta(t))z)) \in \mathbb{R} \times X$$

with $(t, z) \in \mathbb{R} \times X$ and $z$ small. Moreover, as $\eta$ tends to zero, the unstable and stable manifolds approach the unstable manifold and stable manifolds of the autonomous problem (14).

**Theorem 4.7.** (Theorem 6.1 in [14]) For $\eta \in [0, 1]$ suppose that $h_\eta : \mathbb{R} \times X \to X$ is differentiable, and consider the initial value problem

$$\dot{z} = A_0(t)z + h_\eta(t, z), \quad z(0) = z_0 \in X.$$  \hspace{1cm} (24)

Assume that $h_0 : \mathbb{R} \times X \to X$ is such that $h_0(t, 0) = 0$, $(h_0)_t(z, 0) = 0 \in L(X)$, and that $\dot{z} = A_0(t)z$ has an exponential dichotomy with projections $\{Q_0(t) : t \in \mathbb{R}\}$. Suppose that

$$\lim_{\eta \to 0} \sup_{t \in B(0, r)} \|h_\eta(t, z) - h_0(t, z)\|_X + \|(h_\eta)_t(z, t) - (h_0)_t(z, t)\|_{L(X)} = 0$$

for some $r > 0$. Under these assumptions:

1. For each $\eta$ sufficiently small, there exists a globally defined solution of (24) $\xi^*_\eta : \mathbb{R} \to X$ with $\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} \|\xi^*_\eta(t)\|_{L(X)} = 0$ and such that

$$\dot{z} = (A_0(t) + (h_\eta)_t(t, \xi^*_\eta(t)))z$$  \hspace{1cm} (25)

has an exponential dichotomy; that is, there is a family of projections $\{Q_\eta(t) : t \in \mathbb{R}\}$ such that the conditions in Definition 4.4 are satisfied and $U_\eta(t, \tau)$ is the solution operator associated to (25).

2. For any $T > 0$

$$\lim_{\eta \to 0} \sup_{t \in [0, T]} \sup_{\tau \in \mathbb{R}} \|U_\eta(t + \tau, \tau) - U_0(t + \tau, \tau)\|_{L(X)} \to 0.$$

3. The family of projections $\{Q_\eta(t) : t \in \mathbb{R}\}$ satisfies

$$\lim_{\eta \to 0} \|Q_\eta(t) - Q_0(t)\|_{L(X)} = 0.$$

4. There exists an $\eta_0 > 0$, a neighbourhood $V$ of $z = 0$ in $X$ (independent of $\eta$) with $\xi^*_\eta(t) \in V$, for all $\eta \in [0, \eta_0]$ and, for each $0 \leq \eta \leq \eta_0$, a function $(\tau, z) \mapsto \Sigma^s_\eta(\tau, Q_\eta(t)z) : \mathbb{R} \times V \to X$, such that the local unstable manifold $W^u_{loc, \eta}(\xi^*_\eta) = W^u_{\eta}(\xi^*_\eta) \cap V$ for (24) is given by

$$W^u_{loc, \eta}(\xi^*_\eta) = \{(\tau, w) \in V : w = \eta^*(\tau) + Q_\eta(\tau)w + \Sigma^s_\eta(\tau, Q_\eta(t)w)\}.$$

5. Finally, the unstable manifolds behave continuously at $\eta = 0$ in the sense that

$$\sup_{t \leq \tau} \sup_{z \in V} \left\{\|Q_\eta(t)z - Q_0(t)z\|_X + \|\Sigma^s_\eta(t, Q_\eta(t)z) - \Sigma^s_0(t, Q_0(t)z)\|_X\right\} \to 0.$$

It is possible to prove a similar result concerning the existence and stability of local stable manifolds (see [14]).
4.1.5. Lower semicontinuity. We are now ready to state our result on the lower semicontinuity of pullback attractors.

**Theorem 4.8.** (Theorem 3.1 in [17]) Consider the family \( \{ S_\eta(t, \tau) : t \geq \tau \in \mathbb{R} \} \), \( \eta \in [0, 1] \), of nonlinear processes and assume that

\[
\sup_{t \in [0, T]} \sup_{\tau \in \mathbb{R}} \sup_{z \in K} \| S_\eta(t, \tau) z - S_0(t, \tau) z \|_X \xrightarrow{\eta \to 0} 0.
\]

Suppose that, for each \( \eta \in [0, 1] \), the nonlinear process \( \{ S_\eta(t, \tau) : t \geq \tau \in \mathbb{R} \} \) has a global non-autonomous attractor \( A_\eta \), and that the global non-autonomous attractor for the autonomous limit system is given by the union of unstable manifolds of a collection of global hyperbolic solutions \( \{ \xi_j^* (\cdot) \}_{j=1}^\infty \)

\[
A_0(t) = \bigcup_{j=1}^\infty W^u(\xi_j^* (\cdot))(t).
\]

Assume further that, for each \( j \in \mathbb{N} \),

- given \( \epsilon > 0 \), there exists an \( \eta_{j, \epsilon} \) such that for all \( 0 < \eta < \eta_{j, \epsilon} \) there exists a global hyperbolic solution \( \xi_j^{*, \eta}(\cdot) \) for (15) and

\[
\sup_{t \in \mathbb{R}} \| \xi_j^{*, \eta}(t) - \xi_j^*(t) \|_X < \epsilon,
\]

- the local unstable manifold of \( \xi_j^{*, \eta} \) behaves continuously as \( \eta \to 0 \); that is, there exists a \( \delta_j > 0 \) such that

\[
\text{dist}_H(W_{\eta_j}^{u, \delta_j}(\xi_j^{*, \eta}), W_0^{u, \delta_j}(\xi_j^*)) \xrightarrow{\eta \to 0} 0.
\]

Then the family \( \{ A_\eta(t), 0 \leq \eta \leq \eta_0 \} \) is upper and lower semicontinuous at \( \eta = 0 \), namely

\[
\sup_{t \in \mathbb{R}} \text{dist}_H(A_\eta(t), A_0(t)) \xrightarrow{\eta \to 0} 0. \tag{26}
\]

In [14] it is proved that all the hypotheses in Theorem 4.8 are satisfied for the equation

\[
\dot{y} = By + f_\eta(t, y)
\]

(this was (15), above) under the continuity assumption (16), provided that the global attractor for the autonomous limit system is given by the union of unstable manifolds associated to a finite number of hyperbolic equilibria.

Note that this result is also valid in an autonomous framework in which the global attractor is given by the union of a finite number of unstable manifolds of global solutions. On the other hand, observe that nowhere do we need the pullback attraction property of the family \( \{ A_\eta(t) \}_{t \in \mathbb{R}} \), so that our results are also applicable to any kind of non-autonomous family \( \{ A_\eta(t) \}_{t \in \mathbb{R}} \) described as in the above theorem. Finally, if we actually have a pullback attractor, it is worth remarking that in place of (16) we could have considered the following weaker hypothesis: there exists \( \tau \in \mathbb{R} \) such that

\[
\lim_{\eta \to 0} \sup_{t \in (-\infty, \tau)} \sup_{z \in B(0, \epsilon)} \| f_\eta(t, z) - f_0(t, z) \|_X + \| (f_\eta)_z(t, z) - (f_0)_z(t, z) \|_{L(X)} = 0,
\]

for all \( \epsilon > 0 \). This would imply the existence of global solutions close to \( \xi_i(t) \) that are hyperbolic in \((-\infty, \tau]\), and would therefore allow for lower semicontinuity of a family of pullback attractors that are not bounded as \( t \to +\infty \) (i.e. \( \bigcup_{[\tau, +\infty)} A_\eta(t) \)).
unbounded in $X$). This reinforces the idea that in non-autonomous differential equations the pullback behaviour determines crucial properties of the future dynamics of the system.

Finally, observe that the results in the previous theorem can be interpreted in terms of the uniform attractors associated to $(15)$. Indeed, thanks to $(13)$ we obtain the following result.

**Theorem 4.9.** (see [10]) Under the hypotheses of Theorem 4.8, it holds that the uniform attractors $A_\eta$ for $(15)$ behave upper and lower semicontinuously at $\eta = 0$ with respect to the global attractor of $(14)$, i.e.

$$\text{dist}_H(A_\eta, A_0) \xrightarrow{\eta \to 0} 0,$$

or, equivalently,

$$\lim_{\eta \to 0} \sup_{\sigma_\eta \in \Sigma_\eta} \text{dist}_H(A_\eta(\sigma_\eta), A_0) = 0,$$

if $\Sigma_\eta$ is compact for $\eta$ small enough, or

$$\lim_{\eta \to 0} \sup_{\sigma_\eta \in A_\eta} \text{dist}_H(A_\eta(\sigma_\eta), A_0) = 0,$$

when $\Sigma_\eta$ is not compact and $A_\eta$ are the global attractors of the associated driving systems.

**Proof.** For the upper semicontinuity of the uniform attractor, note the inclusion $A_\eta \subseteq A_\eta \times \Sigma_\eta$ and thus $A_\eta \subseteq \cup_{\eta \in [0,1]} A_\eta \times \cup_{\eta \in [0,1]} \Sigma_\eta$. Since $A_\eta = \cup_{\eta \in \Sigma_\eta} A_\eta(0)$, then $\cup_{\eta \in [0,1]} A_\eta$ is precompact. Then Theorem 3.6 of [19] shows that the family $\{A_\eta\}_{\eta \in [0,1]}$ is upper semicontinuous in $\eta = 0$ and hence $\{A_\eta\}_{\eta \in [0,1]}$ is also upper semicontinuous at $\eta = 0$. Remark 4.1 and Theorem 4.8 imply that the family of attractors $\{A_\eta(0)\}_{\eta \in [0,1]}$ is lower semicontinuous at $\eta = 0$ (see Theorem 3.8 of [19]), i.e.

$$\lim_{\eta \to 0} \text{dist}(A_0, A_\eta(0)) = 0,$$

and since

$$\text{dist}(A_0, A_\eta) \leq \text{dist}(A_0, A_\eta(0))$$

the result follows. \qed

4.2. **Dynamically gradient non-autonomous dynamical systems.** If there are only a finite number of isolated invariant sets inside the global attractor, then the existence of a Lyapunov function relative to those isolated invariant sets ensures that all the bounded global solutions inside the attractor tend both backwards and forwards in time to these invariant sets, and it also prevents the presence of homoclinic structures among them. Permanence of this property under perturbation is what we call **topological structural stability**, which describes robustness of the dynamics on the global attractor from the properties of the semigroup defined on a general metric space. In this section we give conditions that ensure topological structural stability.

**Definition 4.10.** An invariant set $E(\cdot)$ is called **isolated** if there exists $\delta > 0$ such that any global solution $\xi(\cdot)$ with $\xi(t) \in O_\delta(E(t))$ for all $t \in \mathbb{R}$ must be in $E(\cdot)$, i.e. $\xi(t) \subseteq E(t)$ for all $t \in \mathbb{R}$. A collection $\mathcal{S}(\cdot)$ of isolated invariant sets is **disjoint** if there exists a $\delta^* > 0$ such that for every $t \in \mathbb{R}$

$$O_{\delta^*}(E_i(t)) \cap O_{\delta^*}(E_j(t)) = \emptyset \quad \text{for all} \quad i \neq j.$$
Let \( \{S(t, \tau) : t \geq \tau \in \mathbb{R}\} \) be a nonlinear evolution process with a pullback attractor \( \{A(t) : t \in \mathbb{R}\} \) that contains a disjoint set of isolated global solutions \( \mathcal{Z} = \{\xi_1, \cdots, \xi_n\} \). We make the following definition.

**Definition 4.11.** Let \( \delta \) be as in Definition 4.10 and fix \( \epsilon_0 \in (0, \delta) \). For \( E \in E \) and \( \epsilon \in (0, \epsilon_0) \), an \( \epsilon \)-chain from \( E \) to \( E \) is a sequence of natural numbers \( \ell_i \in \{1, \cdots, n\} \), a sequence of real numbers, \( \tau_i < \sigma_i < t_i \), and a sequence \( u_i \) in \( X \), \( 1 \leq i \leq k \), such that \( u_i \in \mathcal{O}_\epsilon(E_{\ell_i}(\mathbb{R})) \), \( T(\sigma_i, \tau_i)u_i \notin \mathcal{O}_\epsilon(\bigcup_{i=1}^k(E_{\ell_i}(\mathbb{R})) \) and \( T(t_i, \tau_i)u_i \in \mathcal{O}_\epsilon(E_{\ell_i+1}(\mathbb{R})) \), \( 1 \leq i \leq k \), with \( E = E_{\ell_{k+1}} = E_{\ell_1} \). We say that \( \xi^* \in \mathcal{Z} \) is a chain recurrent global solution if there is an \( \epsilon_0 \in (0, \delta) \) such that there is an \( \epsilon \)-chain from \( \xi^* \) to \( \xi^* \) for each \( \epsilon \in (0, \epsilon_0) \).

**Examples of \( \epsilon \)-chains**

We now define dynamically gradient evolution processes.

**Definition 4.12.** Let \( X \) be a metric space and \( \{S(t, \tau) : t \geq \tau\} \) be a nonlinear evolution process in \( X \). Let \( \{A(t)\}_{t \in \mathbb{R}} \) be the pullback attractor for \( \{S(t, \tau) : t \geq \tau\} \). We say that \( \{S(t, \tau) : t \geq \tau\} \) is a dynamically gradient process if the following two hypotheses are satisfied:

(GP1) There is a disjoint set of isolated global solutions \( \mathcal{Z} = \{\xi_1, \cdots, \xi_n\} \) contained in \( \{A(t) : t \in \mathbb{R}\} \) with the property that any global solution \( \eta : \mathbb{R} \to X \) in \( \{A(t) : t \in \mathbb{R}\} \) satisfies

\[
\lim_{t \to -\infty} \text{dist}(\eta(t), \xi_i(t)) = 0 \quad \text{and} \quad \lim_{t \to \infty} \text{dist}(\eta(t), \xi_j(t)) = 0,
\]

for some \( 1 \leq i, j \leq n \).

(GP2) \( \mathcal{Z} \) does not contain any chain recurrent isolated global solution.

4.3. **Topological structural stability under non-autonomous perturbation.**

Now we will consider dynamically gradient evolution process under perturbations. Let be a family \( \{S_\eta(t, \tau) : t \geq \tau\}_{\eta \in [0, 1]} \) of evolution processes. The following assumptions will play an important role.

a) The family \( \{S_\eta(t, \tau) : t \geq \tau\}_{\eta \in [0, 1]} \) is collectively asymptotically compact at \( \eta = 0 \); that is, given \( \eta \xrightarrow{n \to 0} 0 \), \( t_n \xrightarrow{n \to \infty} \{\tau_n\} \) in \( \mathbb{R} \), and \( \{u_n\} \subset X \) bounded \( \{S_\eta(t_n + \tau_n, \tau_n)u_n\} \), is relatively compact.
b) The family \( \{S_\eta(t, \tau) : t \geq \tau\}_{\eta \in [0,1]} \) is continuous at \( \eta = 0 \); that is, for each compact subset \( K \) of \( X \) and \( S > 0 \)
\[
\sup_{z \in K} \sup_{\tau \in [0,S]} d(S_\eta(t, t - \tau)z - T_0(\tau)z) \xrightarrow{\eta \to 0} 0,
\]
where \( T_0(\tau) = S_0(t, t - \tau) \).

Theorem 4.13. (Theorem 3.9 in [15]) Let \( \{S_\eta(t, \tau) : t \geq \tau\}_{\eta \in [0,1]} \) be a family of evolution processes that is continuous and collectively asymptotically compact at \( \eta = 0 \). Assume that
\begin{itemize}
  
  \item[a)] \( \{S_\eta(t, \tau) : t \geq \tau\} \) has a pullback attractor \( \{A_\eta(t) : t \in \mathbb{R}\} \), for each \( \eta \in [0,1] \), and \( \bigcup_{\eta \in [0,1]} \bigcup_{t \in \mathbb{R}} A_\eta(t) \) is compact;
  
  \item[b)] \( S_0(t, \tau) = T_0(t - \tau), t \geq \tau \), where \( \{T_0(t) : t \geq 0\} \) is a dynamically gradient semigroup on \( X \) with global attractor \( A \) and set of isolated stationary solutions \( Z = \{z_1^\star, \cdots, z_n^\star\} \);
  
  \item[c)] for each \( \eta \in [0,1] \), \( Z_\eta = \{\xi_{1, \eta}^\star, \cdots, \xi_{n, \eta}^\star\} \) is a disjoint set of isolated global solutions for \( \{S_\eta(t, \tau) : t \geq \tau\} \), and
\[
\lim_{\eta \to 0} \max_{1 \leq i \leq n} d_H(\xi_{i, \eta}^\star(\mathbb{R}), z_i^\star) = 0;
\]
and
  
  \item[d)] there are \( \delta > 0 \) and \( \eta_0 \in (0,1) \) such that, if \( \eta < \eta_0 \), \( \xi_\eta : \mathbb{R} \to X \) is a global solution in \( \{A_\eta(t) : t \in \mathbb{R}\}, t_0 \in \mathbb{R} \) and \( \text{dist}(\xi_\eta(t), \xi_{i, \eta}^\star(\mathbb{R})) \leq \delta \) for all \( t \leq t_0 \)
\( (t \geq t_0) \), then \( \text{dist}(\xi_\eta(t), \xi_{i, \eta}^\star(t)) \to 0 \) as \( t \to -\infty \) \( (t \to +\infty) \).
\end{itemize}

Then there exists an \( \eta_0 > 0 \) such that \( \{S_\eta(t, \tau) : t \geq \tau\} \) is a dynamically gradient evolution process for all \( \eta \in (0, \eta_0) \).

In order to reinterpret this result to treat the uniform attractor, we need to make some definitions.

**Definition 4.14.** Let \( (\varphi, \theta)_{(X, \Sigma)} \) be a non-autonomous dynamical system with a uniform attractor \( \mathcal{A} \) and a disjoint family of isolated lift-invariant sets \( \mathcal{M} = \{\mathcal{M}_1, \cdots, \mathcal{M}_n\} \) in \( \mathcal{A} \). A homoclinic structure in \( \mathcal{M} \) is a subset \( \{\mathcal{M}_{\ell_1}, \cdots, \mathcal{M}_{\ell_k}\} \), together with a set of points \( \{\sigma_{\ell_1}, \cdots, \sigma_{\ell_k}\} \) and global solutions \( \xi_{\ell_i} : \mathbb{R} \to X \) of \( (\varphi, \theta)_{(X, \Sigma)} \), such that
\[
\mathcal{M}_{\ell_1} \xrightarrow{t \to -\infty} \xi_{\ell_1}(t) \xrightarrow{t \to +\infty} \mathcal{M}_{\ell_{i+1}},
\]
where \( \mathcal{M}_{\ell_{k+1}} := \mathcal{M}_{\ell_k} \) and there exists \( \varepsilon > 0 \) such that
\[
\max_{1 \leq i \leq k} \sup_{t \in \mathbb{R}} \text{dist}(\xi_{\ell_i}(t), \cup_{1 \leq j \leq k} \mathcal{O}_c(\mathcal{M}_{\ell_j})) > 0.
\]

**Definition 4.15.** A non-autonomous dynamical system \( (\varphi, \theta)_{(X, \Sigma)} \) with a uniform attractor \( \mathcal{A} \) is said to be **dynamically gradient** with a disjoint family of isolated lift-invariant sets \( \mathcal{M} = \{\mathcal{M}_1, \cdots, \mathcal{M}_n\} \) if
\begin{itemize}
  
  \item[(GU1)] for any \( x \in \mathcal{A} \) and \( \sigma \in \Sigma \) such that there exists a global solution \( \xi_{\sigma} : \mathbb{R} \to \mathcal{A} \) of the associated evolution process \( \{S_\sigma(t, s) : t \geq s\} \) through \( x \), such that
\[
\mathcal{M}_j \xrightarrow{t \to -\infty} \xi_{\sigma}(t) \xrightarrow{t \to +\infty} \mathcal{M}_i,
\]
where \( 1 \leq i, j \leq n \), for each such \( \sigma \).
  
  \item[(GU2)] There are no homoclinic structures in \( \mathcal{M} \).
\end{itemize}

Then, we have the following result (see [10]).
Theorem 4.16. Let $(\varphi_\eta, \theta)(X, \Sigma_\eta)$ be a non-autonomous dynamical system, for each $\eta \in [0, 1]$ and assume that

(a) for each $\eta \in [0, 1]$, $(\varphi_\eta, \theta)_t$ has an uniform attractor $\mathcal{A}_\eta$, and $\bigcup_{\eta \in [0, 1]} \mathcal{A}_\eta$ is bounded;

(b) $\Sigma_0 = \{\sigma_0\}$ and $\varphi_0(t, \sigma_0) \equiv T_0(t)$, where $\{T_0(t) : t \geq 0\}$ is a dynamically gradient semigroup with a disjoint family of isolated invariants $\mathcal{E}_0 = \{E_{0,1}, \cdots, E_{0,n}\}$;

(c) for each $\eta \in [0, 1]$ there exists a disjoint family of isolated lifted-invariant sets $\mathcal{E}_\eta = \{E_{\eta,1}, \cdots, E_{\eta,n}\}$ such that

$$\max_{1 \leq i \leq n} d_H(E_{\eta,i}, E_{0,i}) \to 0, \text{ as } \eta \to 0^+;$$

(d) for each $r > 0$

$$\sup_{\|u\| \leq r} \sup_{\sigma_\eta \in \Sigma_\eta} \|\varphi_\eta(t, \sigma_\eta)u - \varphi_0(t, \sigma_0)u\|_X \to 0, \text{ as } \eta \to 0^+;$$

and

(e) there exist $\delta > 0$ and $\eta \in (0, 1]$ such that if $\mathbb{R} \ni t \mapsto \xi_{\eta, \sigma_\eta} \in \mathcal{A}_\eta$ is an entire orbit of $(\varphi_\eta, \theta)(X, \Sigma_\eta)$, $t_0 \in \mathbb{R}$ and $\text{dist}(\xi_{\eta, \sigma_\eta}, E_{\eta,i}) \leq \delta$, for all $t \leq t_0$ (if $t_0$) then $\text{dist}(\xi_{\eta, \sigma_\eta}, E_{\eta,i}) \to 0$ as $t \to -\infty$ ($t \to \infty$).

Then $(\varphi_\eta, \theta)(X, \Sigma_\eta)$ is a dynamically gradient non-autonomous dynamical system with the family of isolated lifted-invariant sets given by $\mathcal{E}_\eta = \{E_{\eta,1}, \cdots, E_{\eta,n}\}$.

4.4. Asymptotically autonomous systems. One important class of problems that can be viewed as a small perturbation of an autonomous dynamical system are asymptotically autonomous evolution processes. Loosely speaking, an evolution process is asymptotically autonomous if it is very close to an autonomous evolution process when the initial times are very large. This idea leads to the following definition (for a similar definition see [40]).

Definition 4.17. Let $\{S(t, s) : t \geq s\}$ be an evolution process and $\{S_0(t) : t \geq 0\}$ be a semigroup in a metric space $X$. We say that

- $\{S(t, s) : t \geq s\}$ is asymptotically autonomous at $-\infty$ if

$$S(t + s, s)u_0 \xrightarrow{s \to -\infty} S_0(t)u_0$$

- $\{S(t, s) : t \geq s\}$ is asymptotically autonomous at $+\infty$ if

$$S(t + s, s)u_0 \xrightarrow{s \to +\infty} S_0(t)u_0$$

uniformly for $t$ in bounded intervals of $[0, \infty)$ and for $u_0$ in compact subsets of $X$.

In order to obtain information about an asymptotically autonomous evolution process using the results of the previous sections it is convenient to introduce a new evolution process which is close, for all initial times, to an autonomous evolution process.

Let $\tau \in \mathbb{R}$, $\{S(t, s) : t \geq s\}$ be an evolution process and $\{T(t) : t \geq 0\}$ be a semigroup, and construct the following truncated evolution processes:

- forwards truncation at time $\tau$

$$S_\tau(t, s) = \begin{cases} S(t, s), & \text{if } s \leq t \leq \tau, \\ T(t - \tau)S(\tau, s), & \text{if } s \leq \tau \leq t, \\ T(t - s), & \text{if } \tau \leq s \leq t. \end{cases}$$
• Backward truncation at time $\tau$

$$S_{\tau}(t,s) = \begin{cases} 
T(t-s), & \text{if } s \leq t \leq \tau, \\
S(t,\tau)T(\tau-s), & \text{if } s \leq \tau \leq t \\
S(t,s), & \text{if } \tau \leq s \leq t.
\end{cases}$$

We have the following property for the truncations:

**Theorem 4.18.** (Theorem 6.4 in [8]) Let $\{S(t,s) : t \geq s\}$ be an asymptotically autonomous evolution process at $-\infty$ (at $+\infty$) and $\{S_0(t) : t \geq 0\}$ the associated semiflow. Assume that the semiflow satisfies a uniform continuity condition, i.e. given $\epsilon > 0$, a bounded interval $I \subset \mathbb{R}^+$ and a compact set $K \subset X$ there exists $\delta = \delta(\epsilon, I, K)$ such that $\|S_0(t)u - S_0(t)v\| < \epsilon$, if $\|u - v\| < \delta$, $u, v \in K$, for all $t \in I$. Then the forwards (backward) truncation of $\{S(t,s) : t \geq s\}$ satisfies $\|S_{\tau}(t+\tau,r)u - S_0(t+\tau,r)u\| \to 0$ as $\tau \to -\infty$ ($\tau \to +\infty$) uniformly for $r \in \mathbb{R}$ and for $(t,u)$ in compact subsets of $\mathbb{R}^+ \times X$.

We can state the following result.

**Theorem 4.19.** (Theorem 6.5 in [8]) Let $(\varphi, \theta)_{(X,\Sigma)}$ be a non-autonomous dynamical system and $\{\Pi(t) : t \geq 0\}$ the associated skew product semiflow. Assume that $\{\Pi(t) : t \geq 0\}$ has a global attractor $\mathcal{A}$ and let $A$ be the global attractor for the driving semiflow $\{\theta_t : t \geq 0\}$. Let also $p_0 \in \Sigma$ be a fixed point of $\theta$ and assume that there exists a bounded global solution $\eta : \mathbb{R} \to \Sigma$ such that $\eta(s) \to p_0$ as $s \to -\infty$ and that

(a) If $T(t) := \varphi(t,p_0)$, for all $t \geq 0$ then $\{T(t) : t \geq 0\}$ is a generalized dynamically gradient semiflow with isolated invariant sets $\{\Gamma_{1,0}, \ldots, \Gamma_{n,0}\}$.

(b) If $S(t,s) := \varphi(t-s,\eta(s))$ for all $t \geq s$, the evolution process $\{\varphi(t-s,\eta(s)) : t \geq s\}$ possesses $n \in \mathbb{N}$ isolated invariant families

$$E = \{E_1(\cdot), \ldots, E_n(\cdot)\},$$

which behave upper and lower semi-continuously as $s \to -\infty$, that is

$$\sup_{1 \leq i \leq n} \left[ \text{dist}_H(E_i(s), \Gamma_{i,0}) + d_H(\Gamma_{i,0}, E_i(s)) \right] \xrightarrow{s \to -\infty} 0.$$

(c) There exist $\mu > 0$ such that, if $\xi : \mathbb{R} \to X$ is a bounded solution of $\{\varphi(t-s,\eta(s)) : t \geq s\}$ and there are $t_0 \in \mathbb{R}$ and $i \in \{1, \ldots, n\}$ with $\sup_{t \leq t_0} \text{dist}(\xi(t), E_i(t)) \leq \mu$, then $\lim_{t \to -\infty} \text{dist}(\xi(t), E_i(t)) = 0$.

Then, there exists $\tau_0 < 0$ such that, for all $\tau \leq \tau_0$, the forwards truncated evolution process $\{S_{\tau}(t,s) : t \geq s\}$ is a generalized dynamically gradient evolution process with respect to $E$.

There is an analogous result (Theorem 6.7 in [8]) for the backward truncated evolution process.

5. Morse theory for pullback attractors. The above results on the characterization of attractors under perturbation allow us to define, with some generality, the main concepts and results leading to a Morse theory for pullback attractors of infinite-dimensional non-autonomous dynamical systems.

For a dynamically gradient evolution process associated to the set of isolated global solutions $E = \{E_1(t), \ldots, E_n(t)\}$ we show that $E$ can be reordered in such a way that it constitutes a Morse decomposition for $A(t)$. We then show that one
can give an explicit construction of a Lyapunov function, associated to this Morse decomposition, for the flow in the pullback attractor. The results in this section are taken from [3].

5.1. **Attractor-repeller pairs.** We first give the definition of an attractor-repeller pair for a process.

**Definition 5.1.** Let \( \{S(t,s): t \geq s\} \) be an evolution process with a pullback attractor \( \{A(t): t \in \mathbb{R}\} \). We say that a non-empty invariant family \( \{E(t): t \in \mathbb{R}\} \) in \( A(t) \) is a local attractor if

\[
W^u(E)(t) = E(t), \quad \text{for all } t \in \mathbb{R}.
\]

Now define its associated repeller as the set, for each \( t \in \mathbb{R} \),

\[
E^s(t) = \{ z \in A(t) : \lim_{r \to -\infty} \text{dist}(S(r,t),E) = 0 \}.
\]

The family of pairs \( \{(E(t),E^s(t)): t \in \mathbb{R}\} \) will be called attractor-repeller pair.

If \( \{S(t,s): t \geq s\} \) is a dynamically gradient evolution process with a pullback attractor \( A(t) \) and attractor-repeller pair \( \{(E(t),E^s(t)): t \in \mathbb{R}\} \), it is clear that, if \( x^d \notin E(\tau) \cup E^s(\tau) \), then a global solution \( \xi: \mathbb{R} \to X \) with \( E(\tau) = x \) satisfies

\[
\lim_{t \to -\infty} \text{dist}(\xi(t),E^s(t)) = 0 = \lim_{t \to -\infty} \text{dist}(\xi(t),E(t)).
\]

If \( \{S(t,s): t \geq s\} \) is a dynamically gradient evolution process and \( \{E(t): t \in \mathbb{R}\} \) an isolated invariant family for it which is also a local attractor, then its associated repeller \( E^s(t) \) is an invariant family.

**Definition 5.2.** Let \( \{S(t,\tau): t \geq \tau\} \) be an evolution process and \( \emptyset = A_0(t) \subset A_1(t) \subset \cdots \subset A_n(t) = A(t) \) be a sequence of local attractors. For \( j = 1, \ldots, n \), define \( E_j(t) := A_j(t) \cap A_{j-1}(t) \). The ordered \( n \)-tuple \( E := \{E_1(t), \ldots, E_n(t)\} \) is a set of invariant families called a Morse decomposition for \( \{A(t): t \in \mathbb{R}\} \).

Next we describe the construction of a Morse decomposition for the pullback attractor of a dynamically gradient process relative to the disjoint set of isolated invariant sets \( E = \{E_1, \ldots, E_n\} \). To that end we construct the associated increasing collection of local attractors starting from \( X \). The following result plays a fundamental role in this.

**Lemma 5.3.** (Lemma 2.7 in [3]) Let \( \{S(t,s): t \geq s\} \) be a dynamically gradient evolution process with associated family of isolated invariant sets \( E = \{E_1, \ldots, E_n\} \). Then, there is a \( 1 \leq k \leq n \) such that \( W^u(E_k)(t) = E_k(t) \).

Let \( \{S(t,s): t \geq s\} \) be a dynamically gradient evolution process with associated family of isolated invariant sets \( E = \{E_1(t), \ldots, E_n(t)\} \). If (after possible reordering) \( E_1 \) is such that \( W^u(E_1)(t) = E_1(t) \) and \( E_1(t) \) is its associated repeller we have that, for each \( i \geq 1 \), \( E_i(t) \) is contained in \( E_i^s(t) \) and for any \( z \notin A(t) \setminus \{E_i(t) \cup E_i^s(t)\} \) and global solution \( \xi(t) \in A(t) \) with \( \xi(0) = z \) we have

\[
\lim_{t \to -\infty} \text{dist}(\xi(t),E_i(t)) = 0 = \lim_{t \to -\infty} \text{dist}(\xi(t),E_i^s(t)).
\]

Considering the restriction \( S_1(t,s) \) of \( S(t,s) \) to \( E_1^s(s) \), for all \( s \in \mathbb{R} \), the evolution process \( S_1(t,s) \) is dynamically gradient in \( E_1^s(t) \) with isolated global solutions \( \{E_2(t), \ldots, E_n(t)\} \) and we may assume without loss of generality that \( E_2(t) \) is a local attractor for the process \( \{S_1(t,s): t \geq s\} \) in \( E_1^s(s) \). If \( E_{2,1}^s(t) \) is the repeller associated to the isolated invariant set \( E_{2}^s(t) \) for \( \{S_1(t,s): t \geq s\} \) in \( E_1^s(t) \) we may
proceed and consider the restriction \( \{ S_2(t,s) : t \geq s \} \) of the process \( \{ S_1(t,s) : t \geq s \} \) to \( E_{2,1}(t) \) and \( \{ S_2(t,s) : t \geq s \} \) is a dynamically gradient process in \( E_{2,1}(t) \) with associated isolated global solutions \( \{ E_2(t), \ldots , E_n(t) \} \).

In a finite number of steps we obtain a reordering of \( \{ E_1(t), \ldots , E_n(t) \} \) in such a way that \( E_j(t) \) is a local attractor for the restriction of \( \{ S(t,s) : t \geq s \} \) to \( E_{j-1,j-2}(t) \) (\( E_{0,1}(t) := A(t) \) and \( E_{1,0}(t) := E_1(t) \)).

With the construction above, we have the following result.

**Lemma 5.4.** (Lemma 2.8 in [3]) Let \( \{ S(t,s) : t \geq s \} \) be a dynamically gradient evolution process and \( E = \{ E_1(t), \ldots , E_n(t) \} \) the associated family of isolated invariant sets, reordered as described above. If a global solution \( \xi : \mathbb{R} \to A \) satisfies

\[
\lim_{t \to \infty} \text{dist}(\xi(t), E_k(t)) = 0 = \lim_{t \to \infty} \text{dist}(\xi(t), E_k(t))
\]

then \( \ell \geq k \).

We can show that this reordering of \( \{ E_1(t), \ldots , E_n(t) \} \) (which we denote the same way) is a Morse decomposition (in the sense of Definition 5.2) for \( \{ A(t) \}_{t \in \mathbb{R}} \) for a suitably chosen sequence \( A_0(t) \subset A_1(t) \subset A_2(t) \subset \ldots \subset A_n(t) \) of local attractors.

Define \( A_0(t) = \emptyset, A_1(t) = E_1(t) \) and for \( j = 2, 3, \ldots , n \)

\[
A_j(t) = A_{j-1}(t) \cup W^u(E_j)(t). \tag{27}
\]

Observe that \( A(t) = A_n(t) = \bigcup_{j=1}^n W^u(E_j)(t) \).

**Theorem 5.5.** (Theorem 2.9 in [3]) Let \( \{ S(t,s) : t \geq s \} \) be a dynamically gradient evolution process with an associated family of isolated global solutions \( E = \{ E_1(t), \ldots , E_n(t) \} \) reordered in such a way that \( E_j(t) \) is an attractor for the restriction of \( \{ S(t,s) : t \geq s \} \) to \( E_{j-1,j-2}(t) \). Then \( A_j(t) \) defined in (27) is a local attractor and

\[
E_j^*(t) = A_j(t) \cap A_{j-1}^*, \quad \text{for each } t \in \mathbb{R} \text{ and } 1 \leq j \leq n.
\]

As a consequence, \( E \) defines a Morse decomposition for \( A(t) \).

### 5.2. Gradient evolution processes

Next, we relate our definition of local attractor for an evolution process with the definition given for semigroups. These results will be useful for the construction of a Lyapunov function for it.

**Lemma 5.6.** (Lemma 2.14 in [3]) Let \( \{ S(t,s) : t \geq 0 \} \) be a dynamically gradient evolution process in \( X \) with pullback attractor \( \{ A(t) : t \in \mathbb{R} \} \) and \( (E(t), E^*(t)) \) an attractor-repeller pair in \( A(t) \).

Suppose there is a \( \delta > 0 \) such that

\[
\mathcal{O}_\delta(E(t)) \bigcap \mathcal{O}_\delta(E^*(t)) = \emptyset, \quad \text{for each } t \in \mathbb{R} \tag{28}
\]

If \( K \) is a compact subset of \( A(t) \) such that \( K \cap E^*(t) = \emptyset \), then

\[
\text{dist}(S(r + t, t)K, E(r + t)) \xrightarrow{r \to \infty} 0.
\]

Finally, we show that a dynamically gradient evolution processes must be a gradient process, i.e. have a Lyapunov functional. We first make a formal definition.

**Definition 5.7.** We say that an evolution process \( \{ S(t,s) : t \geq s \} \) with a pullback attractor \( \{ A(t) : t \in \mathbb{R} \} \) and a disjoint family of isolated global solutions \( E = \{ E_1(t), \ldots , E_n(t) \} \) is a gradient process with respect to \( E \) if there is function \( V : \mathbb{R} \times X \to \mathbb{R} \) such that
i) $[0, \infty) \ni r \mapsto V(r + t, S(r + t, t)u) \in \mathbb{R}$ is decreasing for each $u \in X$ and $t \in \mathbb{R}$.

ii) $V(r + t, S(r + t, t)u) = V(t, u)$ for all $t \geq 0$ if and only if $u \in \bigcup_{i=1}^{n} E_i(t)$.

iii) $V(t, \cdot) : A(t) \to \mathbb{R}$ is continuous for each $t \in \mathbb{R}$.

A function $V$ with the properties above is called a Lyapunov function for the gradient process $\{S(t, s) : t \geq s\}$ with respect to $E$.

Proposition 5.8. (Theorem 3.3 in [3]) Let $\{S(t, s) : t \geq s\}$ be a dynamically gradient evolution process in a metric space $X$ with pullback attractor $\{A(t) : t \in \mathbb{R}\}$, and let $(E(t), E^*(t))$ an attractor-repeller pair in $A(t)$. Suppose that (28) holds.

Then, there exists a function $k : \mathbb{R} \times X \to \mathbb{R}$ satisfying the following:

i) $k : \mathbb{R} \times X \to \mathbb{R}$ is non-increasing along solutions.

ii) $k^{-1}(0) = E(t)$ and $k^{-1}(1) \cap A(t) = E^*(t)$.

iii) Given $z \in A(t)$, if $k(r + t, S(r + t, t)z) = k(t, z)$ for all $r \geq 0$, then $z \in (E(t) \cup E^*(t))$.

iv) $k(t, \cdot) : A(t) \to \mathbb{R}$ is continuous in $X$ for each $t \in \mathbb{R}$.

Finally, the following result holds.

Theorem 5.9. (Theorem 3.9 in [3]) Let $\{S(t, s) : t \geq 0\}$ be a dynamically gradient evolution process with pullback attractor $A(t)$ and a disjoint set of isolated global solutions $E = \{E_1(t), \ldots, E_n(t)\}$. Suppose that (28) holds for every $E_i \in E$. Then there exists a Lyapunov function $V : \mathbb{R} \times X \to \mathbb{R}$ with the properties i)-iii) of Definition 5.7 and such that $V(t, E_k(t)) = k - 1$, for all $t \in \mathbb{R}$ and $k = 1, \ldots, n$.

5.3. Gradient non-autonomous dynamical systems. The Lyapunov function constructed in Theorem 5.9 has an important weakness. Indeed, the existence of $V$ with the above properties does not guarantee that (G1) is satisfied, so that, in contrast to the autonomous case, a gradient evolution process need not be dynamically gradient. A partial resolution of this problem has been obtained in [8, 13]. Once more, to look for a Lyapunov functional for

\[ \dot{u} = f(t, u) \]

we have to look at the skew-product and cocycle formalisms, and make the following definition.

Definition 5.10. A non-autonomous dynamical system $(\varphi, \theta)_{(X, \Sigma)}$ with an uniform attractor $A$ is said to be gradient with a disjoint family of isolated lifted-invariant sets $\mathcal{M} = \{M_1, \ldots, M_n\}$ if there exists a function $V : X \times \Sigma \to \mathbb{R}$ satisfying the following conditions:

i) Given $(x, \sigma) \in X \times \Sigma$, the map $[0, \infty) \ni t \mapsto V(\varphi(t, \sigma)x, \theta(t)\sigma)$ is decreasing.

ii) If $V(\varphi(t, \sigma)u, \theta(t)\sigma) = \text{const.}$, then $u \in M_i$, for some $1 \leq i \leq n$.

Such function $V$ is called a Lyapunov function for $(\varphi, \theta)_{(X, \Sigma)}$.

With this definition we now can write the generalization of the main result in [1] to a non-autonomous framework.

Theorem 5.11. (see [10]) Suppose that there is a family of isolated lifted-invariant sets $\mathcal{M} = \{M_1, \ldots, M_n\}$ inside the uniform attractor $A$. Then the non-autonomous dynamical system $(\varphi, \theta)_{(X, \Sigma)}$ is dynamically gradient with respect to $\mathcal{M}$ if and only if $\mathcal{M} = \{M_1, \ldots, M_n\}$ is a Morse decomposition of $A$, if and only if $(\varphi, \theta)_{(X, \Sigma)}$ is gradient with respect to $\mathcal{M}$.  

Proof. We only have to take into account that a non-autonomous dynamically gradient dynamical system generates a dynamically gradient system $\Pi(t)$ in the lifted product space $X \times \Sigma$; then we can apply the results for a semigroup as in [1]. □

6. Morse–Smale systems. In this section we take $X$ to be a Banach space, and we prove that a small non-autonomous perturbation of a Morse–Smale semigroup maintains the structure of the phase diagram, i.e. connections between isolated invariants and their orientations. Note that even the definition of a non-autonomous Morse–Smale system is not straightforward.

6.1. Morse–Smale evolution process. From the results in the autonomous case (see Corollary 1.18) we can now introduce the definition of a Morse–Smale evolution process.

Definition 6.1. Let $\{S(t,s) : t \geq s \in \mathbb{R}\}$ be a nonlinear evolution process in $X$ with a pullback attractor $\{A(t) : t \in \mathbb{R}\}$. We say that $\{S(t,s) : t \geq s \in \mathbb{R}\}$ is a Morse–Smale evolution process if

(i) $\{S(t,s) : t \geq s \in \mathbb{R}\}$ is a dynamically gradient evolution process with respect to a finite family of isolated hyperbolic global solutions $E = \{\xi_1(\cdot), \ldots, \xi_n(\cdot)\}$;

(ii) $\{S(t,s) : t \geq s \in \mathbb{R}\}$ is a reversible process, that is $S(t,s)|_{\mathcal{A}(s)}$ is injective, $S'(t,s)(z) : X \to X$ is an injective bounded linear operator, for any $t \geq s \in \mathbb{R}$;

(iii) $W^s_{\text{loc}}(\xi_i)$ is finite-dimensional for all $i = 1, \ldots, n$; and

(iv) given $\xi_i, \xi_j \in E$ such that $W^u(\xi_i)(t_0) \cap W^s_{\text{loc}}(\xi_j)(t_0) \neq \emptyset$ for some $t_0 \in \mathbb{R}$ there exists a point of transversal intersection in $W^u(\xi_i)(t_0) \cap W^s_{\text{loc}}(\xi_j)(t_0)$.

Before stating our main result, we need the definition of phase diagram between two Morse–Smale evolution processes.

Definition 6.2. Let $\{T(t,s) : t \geq s \in \mathbb{R}\}$ and $\{S(t,s) : t \geq s \in \mathbb{R}\}$ be two dynamically gradient evolution processes. We say that there exists a phase diagram isomorphism between $\{T(t,s) : t \geq s \in \mathbb{R}\}$ and $\{S(t,s) : t \geq s \in \mathbb{R}\}$ if they satisfy the following condition

(i) If $E_T = \{\xi_1, \ldots, \xi_n\}$ and $E_S = \{\psi_1, \ldots, \psi_m\}$ are the family of isolated global solutions associated with $\{T(t,s) : t \geq s \in \mathbb{R}\}$ and $\{S(t,s) : t \geq s \in \mathbb{R}\}$, respectively, then $n = m$;

(ii) There exists a bijection $\mathcal{B} : E_T \to E_S$ such that there exists a global solution $\xi : \mathbb{R} \to X$ associated with $\{T(t,s) : t \geq s\}$ satisfying

$$\lim_{t \to -\infty} \|\xi_i(t) - \xi(t)\| = 0$$

and

$$\lim_{t \to \infty} \|\xi_j(t) - \xi(t)\| = 0,$$

if and only if there exists a global solution $\psi : \mathbb{R} \to X$ of $\{S(t,s) : t \geq s\}$ such that

$$\lim_{t \to -\infty} \|\mathcal{B}(\xi_i)(t) - \psi(t)\| = 0$$

and

$$\lim_{t \to \infty} \|\mathcal{B}(\xi_j)(t) - \psi(t)\| = 0.$$

6.2. Geometric structural stability under non-autonomous perturbation. Now, with the concept of Morse–Smale processes at hand, we can go one step further and prove a geometric structural stability result for evolution processes (see [9]).

Theorem 6.3. Let $\{S_\eta(t,s) : t \geq s\}_{\eta \in [0,1]}$ be a family of evolution processes satisfying the following conditions
and an infinite-dimensional version of the

Let

Proposition 6.5. \( \xi \) and \( \eta \) be three hyperbolic fixed points of a Morse–Smale semigroup \( \{ T(t) : t \geq 0 \} \). If \( W^s(e_1) \cap W^s_{\text{loc}}(e_2) \neq \emptyset \) and \( W^s(e_2) \cap W^s_{\text{loc}}(e_3) \neq \emptyset \) then \( W^s(e_1) \cap W^s_{\text{loc}}(e_3) \neq \emptyset \).
6.3. Morse–Smale non-autonomous dynamical systems. Finally, with respect to the uniform attractor we make the following definition and state a theorem on geometric structural stability (see [10]).

**Definition 6.6.** We say that a non-autonomous dynamical system \((\varphi, \theta)_{(X, \Sigma)}\) with a uniform attractor \(\mathcal{A}\) is a *Morse–Smale* non-autonomous dynamical system \((\varphi, \theta)_{(X, \Sigma)}\) if it is a dynamically gradient dynamical system and for each \(\sigma \in \Sigma\), the associated evolution process \(\{S_\sigma(t, s) : t \geq s\}\) is a Morse–Smale evolution process with a finite number of global hyperbolic solutions \(\{\xi_{\sigma, 1}, \cdots, \xi_{\sigma, n}\}\).

**Theorem 6.7** (Geometrical Structural Stability). Suppose that for each \(\eta \in [0, 1]\) we have a non-autonomous dynamical system \((\varphi, \theta)_{(X, \Sigma, \eta)}\). Assume that

(a) \((\varphi, \theta)_{(X, \Sigma, \eta)}\) has a uniform attractor \(\mathcal{A}_\eta\), for each \(\eta \in [0, 1]\), and \(\overline{\bigcup_{\eta \in [0, 1]} \mathcal{A}_\eta}\) is bounded.

(b) \(\Sigma_0 = \{\sigma_0\}\) and \(\varphi_0(t, \sigma_0) \equiv T_0(t)\), where \(\{T_0(t) : t \geq 0\}\) is a Morse–Smale semigroup with a family of hyperbolic points \(E_0 = \{z_{0, 1}, \cdots, z_{0, n}\}\).

(d) For each \(r > 0\)

\[
\sup_{\eta} \sup_{\sigma_0} \sup_{t \in \mathbb{R}} \|\varphi_\eta(t, \sigma_0)x - \varphi_\eta(t, \sigma_0)x\|_X \to 0, \text{ as } \eta \to 0^+,
\]

and

\[
\sup_{\sigma_0} \sup_{\eta \in [0, \eta_1]} \sup_{x \in E_\eta, t \in \mathbb{R}} \|\varphi'_\eta(t, \sigma_0)x - \varphi'_\eta(t, \sigma_0)x\|_X \to 0, \text{ as } \eta \to 0^+.
\]

Then, there exists \(\eta_1 > 0\) such that

(i) there exists, for each \(\eta \in [0, \eta_1]\) a disjoint family of isolated lifted-invariant sets \(E_\eta = \{E_{\eta, 1}, \cdots, E_{\eta, n}\}\) such that

\[
\max_{1 \leq i \leq n} \text{dist}_H(E_{\eta, i}, E_0, i) \to 0, \text{ as } \eta \to 0^+;
\]

(ii) \((\varphi, \theta)_{(X, \Sigma, \eta)}\) is a Morse–Smale non-autonomous dynamical system with the family of isolated lifted-invariant sets given by \(E_\eta = \{E_{\eta, 1}, \cdots, E_{\eta, n}\}\), for all \(\eta \in [0, \eta_1]\).

**REFERENCES**

[1] E. R. Aragão-Costa, T. Caraballo, A. N. Carvalho and J. A. Langa, Stability of gradient semigroups under perturbation, *Nonlinearity*, 24 (2011), 2099–2117.
[2] E. R. Aragão-Costa, T. Caraballo, A. N. Carvalho and J. A. Langa, Continuity of Lyapunov functions and of energy level for a generalized gradient system, *Topological Methods Nonl. Anal.*, 39 (2012), 57–82.
[3] E. R. Aragão-Costa, T. Caraballo, A. N. Carvalho and J. A. Langa, Non-autonomous Morse decomposition and Lyapunov functions for dynamically gradient processes, *Trans. Amer. Math. Soc.*, 365 (2013), 5277–5312.
[4] L. Arnold, *Random Dynamical Systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
[5] J. Arrieta and A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Trans. Amer. Math. Soc.*, 352 (2000), 285–310.
[6] A. V. Babin and M. Vishik, Regular attractors of semigroups and evolution equations, *J. Math. Pures Appl.*, 62 (1983), 441–491.
[7] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, Studies in Mathematics and its Applications, 25, North-Holland Publishing Co., Amsterdam, 1992.
[8] M. C. Bortolan, T. Caraballo, A. N. Carvalho and J. A. Langa, Skew-product flows and Morse decomposition, *J. Diff. Equations*, 255 (2013), 2436–2462.
[9] M. C. Bortolan, A. N. Carvalho, J. A. Langa and G. Raugel, Non-autonomous perturbations of Morse-Smale semigroups: Stability of the phase diagram, preprint.
[10] M. C. Bortolan, A. N. Carvalho, J. A. Langa, Structural stability of skew-product semiflows, J. Diff. Equations, 257 (2014), 490–522.
[11] T. Caraballo and J. A. Langa, On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 10 (2003), 491–513.
[12] T. Caraballo, J. A. Langa and J. C. Robinson, Upper semicontinuity of attractors for small random perturbations of dynamical systems, Comm. Partial Diff. Eq., 23 (1998), 570–603.
[13] T. Caraballo, J. C. Jara, J. A. Langa and Z. Liu, Morse decomposition of attractors for non-autonomous dynamical systems, Advanced Nonlinear Studies, 13 (2013), 309–329.
[14] A. N. Carvalho and J. A. Langa, The existence and continuity of stable and unstable manifolds for semilinear problems under non-autonomous perturbation in Banach spaces, J. Diff. Eq., 233 (2007), 622–653.
[15] A. N. Carvalho and J. A. Langa, An extension of the concept of gradient semigroups which is stable under perturbation, J. Diff. Eq., 246 (2009), 2646–2668.
[16] A. N. Carvalho, J. A. Langa, J. C. Robinson and A. Suárez, Characterization of non-autonomous attractors of a perturbed gradient system, J. Diff. Eq., 236 (2007), 570–603.
[17] A. N. Carvalho, J. A. Langa, J. C. Robinson, Upper semicontinuity of attractors for non-autonomous dynamical systems, Erg. Th. Dyn. Sys., 29 (2009), 1765–1780.
[18] A. N. Carvalho, J. A. Langa and J. C. Robinson, Lower semi-continuity of attractors for evolution processes, Nonlin. Anal., 71 (2009), 1812–1824.
[19] A. N. Carvalho, J. A. Langa and J. C. Robinson, Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems, Applied Mathematical Sciences, 182, Springer, New York, 2013.
[20] A. N. Carvalho and S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems, Numer. Funct. Anal. Optim., 27 (2006), 785–829.
[21] V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, J. Math. Pures Appl., 73 (1994), 279–333.
[22] V. V. Chepyzhov and M. I. Vishik, Attractors for Equations of Mathematical Physics, American Mathematical Society Colloquium Publications, 49, American Mathematical Society, Providence, RI, 2002.
[23] C. Conley, Isolated Invariant Sets and the Morse Index, CBMS Regional Conference Series in Mathematics Vol. 38, American Mathematical Society, Providence, RI, 1978.
[24] H. Crauel and F. Flandoli, Attractors for random dynamical systems, Prob. Th. Rel. Fields, 100 (1994), 365–393.
[25] J. K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys and Monographs, Amer. Math. Soc., Providence, 1988.
[26] J. K. Hale, X. B. Lin and G. Raugel, Upper semicontinuity of attractors for approximations of semigroups and partial differential equations, Math. Comp., 50 (1988), 89–123.
[27] J. K. Hale and G. Raugel, Lower semi-continuity of attractors of gradient systems and applications, Ann. Mat. Pura Appl., 154 (1989), 281–326.
[28] J. K. Hale and G. Raugel, A damped hyperbolic equation on thin domains, Trans. Amer. Math. Soc., 329 (1992), 185–219.
[29] J. K. Hale and G. Raugel, Convergence in dynamically gradient systems with applications to PDE, Z. Angew. Math. Phys., 43 (1992), 63–124.
[30] J. K. Hale, L. T. Magalhães and W. M. Oliva, An Introduction to Infinite-Dimensional Dynamical Systems - Geometric Theory, Applied Mathematical Sciences, 47, Springer-Verlag, New York, 1984.
[31] A. Haraux, Systèmes Dynamiques Dissipatifs et Applications, Masson, Paris, 1991.
[32] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, 840, Springer-Verlag, Berlin-New York, 1981.
[33] D. Henry, Semigroups, Handwritten Notes, IME-USP, São Paulo SP, Brazil, 1981.
[34] D. Henry, Perturbation of the Boundary in Boundary-Valued Problems of Partial Differential Equations, London Mathematical Society Lecture Note Series, 318, Cambridge University Press, Cambridge, 2005.
[35] P. E. Kloeden and M. Rasmussen, Nonautonomous Dynamical Systems, Mathematical Surveys and Monographs, 176, American Mathematical Society, Providence, RI, 2011.
[36] J. Palis Jr., Introdução aos Sistemas Dinâmicos, IMPA, 1977.
[37] O. A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.

[38] J. A. Langa, J. C. Robinson, A. Suárez and A. Vidal-López, The stability of attractors for non-autonomous perturbations of gradient-like systems, *J. Diff. Eq.*, **234** (2007), 607–625.

[39] K. Lu, Structural stability for scalar parabolic equations, *J. Diff. Eq.*, **114** (1994), 253–271.

[40] K. Mischaikow, H. Smith and H. R. Thieme, Asymptotically autonomous semiflows: Chain recurrent and Lyapunov functions, *Trans. Amer. Math. Soc.*, **347** (1995), 1669–1685.

[41] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, Cambridge, 2001.

[42] K. P. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Universitext, Springer-Verlag, Berlin, 1987.

[43] G. R. Sell, Nonautonomous differential equations and dynamical systems, *Trans. Amer. Math. Soc.*, **127** (1967), 241–262.

[44] G. R. Sell, *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand Reinhold, London, 1971.

[45] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences, Vol. 143, Springer-Verlag, New York, 2002.

[46] B. Schmalfuß, Backward cocycles and attractors of stochastic differential equations, in *International Seminar on Applied Mathematics - Nonlinear Dynamics: Attractor Approximation and Global Behaviour* (eds. V. Reitmann, T. Riedrich and N. Koksch), Technische Universität, Dresden, 1992, 185–192.

[47] A. M. Stuart and A. R. Humphries, *Dynamical Systems and Numerical Analysis*, Cambridge University Press, Cambridge, England, 1996.

[48] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, 68, Springer-Verlag, New York, 1988.

[49] M. I. Vishik, *Asymptotic Behaviour of Solutions of Evolutionary Equations*, Cambridge University Press, Cambridge, 1992.

Received June 2014; revised September 2014.

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