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Interface conditions simulating influence of a thin elastic wedge with smooth contacts

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Abstract The paper presents a method for deriving interface conditions simulating the influence of a thin wedge in a multi-wedge system with smooth contacts. It consists in successive (i) employing the Mellin’s transform, (ii) separation of the symmetric and anti-symmetric parts of a solution, (iii) distinguishing terms tending to infinity, when the wedge angle tends to zero, (iv) appropriate re-arrangement of the terms to avoid degeneration, (v) using truncated power series in equations for the thin wedge and (vi) inspection of the characteristic determinant and finding models simulating the influence of the thin wedge for various combinations of parameters. The paper extends and improves the results previously obtained by the authors for a harmonic problem. The analysis leads to three physical models of contact interaction, which cover all the ratios of shear modules of a thin wedge and neighbour wedges. Numerical examples illustrate the accuracy provided by the method employed and the models derived.

Keywords Plane elasticity · Wedges · Smooth contacts · Thin wedge · Singular points · Interaction models

1 Introduction

Partial differential equations of mathematical physics, when applied to regions with wedge points, lead to fields, which either themselves or at least some of their spatial derivatives tend to infinity when the distance \( d \) to the point tends to zero. This reflects the physical fact that these points generate high field concentration; often they are sources of unfavourable effects such as fracture, corrosion, fatigue, spark over and energy losses.

In elasticity, the importance of these points has been appreciated yet in the pioneering works by Tranter [24] and Williams [25,26]. Since then, the subject has been extensively studied by many researches. Not to dwell on results obtained, note that a comprehensive review of papers, published in 2004, is given by Sinclair [22,23]. Two of its parts, including over 400 references, contain a detailed analysis of the results on the singularities generated by wedge points, on the means to properly interpret them and ways to account for them in numerical calculations and engineering practice. More recent reviews may be found in papers by Linkov and Koshelev [12] and Paggi and Carpinteri [21]. Thus, in this study, we shall briefly discuss only those of late and recent papers, which are relevant to our theme of interface conditions near a (multi-) wedge point.
To our knowledge, the first paper considering non-ideal contact was that by Dundurs and Lee [6]. The authors studied asymptotic behaviour of stresses at a frictionless contact of a wedge with a half-plane. They concluded that the singularity in stresses was never stronger than the inverse square root of the distance $d$, and it had no oscillatory multiplier corresponding to complex zeros of the characteristic determinant. The study was extended to a contact with the Coulomb law of dry friction by Gdoutos and Theocaris [7], Comninou [5] and, in more detail, by Churchman, Mugadu and Hills [4]. Still no complex roots was found, whereas the singularity in stresses appeared depending on the slip direction and could be stronger than $O(1/\sqrt{d})$. Below we shall comment on our analogous conclusion on complex roots for all the studied (multi-) wedge systems with smooth contacts. The results of [4] have been successfully applied to fretting problems (see the paper [18] and references in it).

In all the mentioned papers, the interface conditions are prescribed \textit{a priori}. Meanwhile, in the material science, it is important to properly formulate interface conditions. The researches, tending to meet this need, account for the fact that contact interaction reflects processes in a thin layer, whose properties and response to particular external actions differ from those of embedding media. Hence, physically significant interface conditions may be obtained either analytically by considering behaviour of a thin layer and turning its thickness to zero (e.g. Klabring and Movchan [10], Benveniste and Miloh [1], Hasin [8]), or by numerical modelling, in particular, by using the finite element mesh in a thin layer to simulate its influence by appropriate interface conditions connecting contact tractions and displacements (e.g. Mishuris and Ochsner [16, 17]).

The last two papers have clearly shown the limitations on the models, obtained in this way. They arise from the edge zones, where wedge-type singular points are present. Similar and even stronger deterioration of models may be expected when thin contact layers terminate at a common apex of grains or structural blocks. In these cases, when modelling interfaces by thin layers, one obtains either a number of corner (wedge) points if thin layers have finite thickness at their edges (Fig. 1) or a multi-wedge point if their thickness gradually decreases down to zero (Fig. 2). In both cases, there arise wedge points, accounting for which requires special analysis. Common apexes being sources of unfavourable field concentration in materials with grainy structure, there is need in developing tools for such an analysis. Its aim is to provide interface models, applicable at singular points; it will also give benchmarks for alternative approaches, such as those based on the BEM, to evaluate the ranges of their applicability.

Mishuris and Kuhn [15] made the first step in this direction. The authors \textit{derived} interface conditions, which simulate the influence of a thin wedge representing properties of an interfacial material. This concept reflects the fact that an interaction between adjusting wedges is actually a result of processes occurring in a thin contact layer. Therefore, to have deeper understanding of asymptotic behaviour of fields and to obtain physically significant contact conditions, it is of value to study a system with a thin wedge when its angle may be arbitrary small. Although the study [15] concerned with a particular case of anti-plane strain problem for a semi-infinite crack with a thin wedge ahead of its tip, the starting concept looks promising for micromechanical analysis of interface conditions.^{
This impelled us to extend the results of [15] to an arbitrary anti-plane problem [13] for a (multi-) wedge system. It has been done by distinguishing the physically significant symmetric and anti-symmetric parts of the solution, an appropriate re-arrangement of terms and expanding trigonometric functions into truncated power series. Meanwhile, our attempt to immediately extend the method of [13], which looked quite general, to plane strain (or plane stress) problems failed. It has appeared that there is need in a deeper analysis of the equations and the characteristic determinant obtained. Therefore, it is reasonable to clarify the difficulties and to find ways to overcome them by considering an intermediate case when a plane strain (or plane stress) problem reduces to scalar equations.

Such is the case of no-friction (smooth) contacts; it is also of interest itself. Then, the problem becomes partly similar to that for the Laplace’s equation what suggests using and possibly improving the approach of [13]. The investigation below contains the study of this problem. It shows that despite the similarity with the harmonic problem, the results for a thin wedge in an elastic system with smooth contacts notably differ, and they look promising for overcoming the difficulty arising in the vectorial case of plane strain (plane stress).

Emphasize that rigorous derivation of models of contact interaction near multi-wedge points has become possible only due to the technique suggested by Blinova and Linkov [2] and developed by Linkov and Koshelev [12]. Still it looks that the advantages of this technique has not been widely appreciated (see, e.g. the review by Paggi and Carpinteri [21] and the recent paper on the theme by these authors [3]). Being of prime significance to the present research, they deserve brief discussion.

Numerous previous and still used approaches to study asymptotic fields near a multi-wedge point employ as unknowns four arbitrary constants entering the general solution of the fourth-order ODE resulting from the Mellin’s transform of plain elasticity equations. Using these constants themselves unavoidably leads to ill-conditioned matrices of fourth-order; the ill conditioning swiftly aggravates when the matrices are multiplied to meet contact and boundary conditions. This well-known ill conditioning has served even to find the exponent, characterizing the asymptotic behaviour; for instance, the authors of the paper [3] looked for the exponent as the value of the transformation parameter, for which an ill-conditioned matrix became even worse ill-conditioned.

The reason of this unfavourable feature of popular approaches is the same as that disclosed and explained by Maier and Novatti [14], Novatti [19], Linkov and Filippov [11] for analogous problems of multi-layered systems, commonly solved by applying the Fourier transform. Specifically, using simultaneously the four constants is physically equivalent to prescribing simultaneously both tractions and displacements at one boundary of a layer (wedge) what is physically impossible and mathematically corresponds to an ill-posed problem.

Obviously, it is safer to escape ill-posed problems and to avoid using ill-conditioned matrices in numerical calculations. For chain-like systems, as layered and multi-wedge systems, it is done by an appropriate choice of unknown constants instead of the mentioned four [2,11,12,19]. It is sufficient to express the mentioned constants via boundary values of two physically significant quantities (tractions or displacements) at each
of two boundaries of a wedge. This leads to a well-posed problem with clear mechanical meaning for three successive contacts of two successive wedges, decreases the order of matrices from the fourth to second and makes evaluation of the exponent stable for an arbitrary multi-wedge system.

In the problem under consideration, the approach of [2,12] has an additional crucial advantage; it employs the very quantities, which are to be connected by interface conditions simulating the influence of a thin wedge. This strongly simplifies the analysis, because we obtain clear physical interpretation of mathematical transformations and obtained equations. It would be impossible when following the mentioned line of using four constants.

The approach developed in this paper in its present form is tailored for problems involving linear equations because the Mellin’s transform is efficient for wedges with properties described by linear equations. Meanwhile, as mentioned above, applications to non-linear media look possible on usual ways, which include piece-wise linear approximations and using linear relations on steps of an incremental procedure.

2 Evaluation of stress singularity for a system with smooth contacts and a thin wedge

Consider a system open (Fig. 3a) or closed (Fig. 3b) of \( m \) linearly elastic isotropic wedges with arbitrary angles, Poisson’s ratios and shear modules. This figure, as well as the next one, differs from the analogous figure of the paper on anti-plane strain [13] by adding the Poisson’s ratio in the list of wedge properties.

The system includes a thin wedge with the angle \( \Theta_0 \), Poisson’s ratio \( \nu_0 \) and shear modules \( \mu_0 \). The angle \( \Theta_0 \) may be arbitrary small, in limit zero. The stresses and displacements within wedges satisfy equations of the elasticity theory. For plane strain (or plane stress) problems, they are expressed via Airy function, which satisfies the biharmonic equation (see, e.g. [20]). All contacts are smooth; the shear traction \( \sigma_{r\theta}(r) \) is zero on each of them. The normal tractions \( \sigma_{\theta\theta}(r) \) and displacements \( u_{\theta}(r) \) on the contacts are either continuous or have prescribed discontinuities \( \Delta p_0(r) \), \( \Delta u_0(r) \); the discontinuities may arise in additional fields, when solving a boundary value problem with the geometry, including the considered wedges by superposition of an appropriately chosen particular solution and an additional solution, with respect to which the problem is reformulated. For an open system, we assume prescribed either the normal traction \( \sigma_{\theta\theta}(r) \) or normal displacement \( u_{\theta}(r) \) at each of two external boundaries.

The problem is solved by employing the Mellin’s transform. Denote its parameter \( s \). After the transformation, the biharmonic equation becomes a linear ordinary differential equation (ODE) in the angular coordinate with constant coefficients for a fixed value of the parameter \( s \). The contact and boundary conditions have the same form with the coordinate \( r \) changed to the transformation parameter \( s \). Hence, after the transformation, we need to solve the ODE under these conditions. The way to efficiently find the solution is given in [2]. Omitting details, we briefly present it as concerned with the case of smooth contacts.

Consider any of the wedges and denote its angle \( \Theta \). Introduce a local polar system \((r, \theta)\) with the origin at the common apex of wedges and with the polar axis directed along the bisector. The general solution of the ODE for the wedge is a linear combination of the functions \( \sin(s \theta) \), \( \cos(s \theta) \), \( \sin(s + 2) \theta \) and \( \cos(s + 2) \theta \), which are linearly independent when \( s \neq -1 \). This implies that the transformed stresses and displacements are also linear combinations of these functions with four arbitrary constants. Note that these linear combinations do not depend on the wedge angle \( \Theta \). Hence, by taking \( \theta \) successively \(-\Theta/2\) and \( \Theta/2 \), we obtain four boundary values of the transformed tractions \( \sigma_{r\theta}(s, -\Theta/2) \), \( \sigma_{\theta\theta}(s, -\Theta/2) \), \( \sigma_{r\theta}(s, \Theta/2) \), \( \sigma_{\theta\theta}(s, \Theta/2) \) as linear functions of the four constants; the inversion gives us the four constants as linear functions of the boundary values of the transformed tractions. Thus, instead of physically insignificant constants, we may use

\[ \begin{align*}
\sigma_{r\theta}(s, -\Theta/2) &= A_1 s^2 + A_2 s + A_3 \\
\sigma_{\theta\theta}(s, -\Theta/2) &= B_1 s^2 + B_2 s + B_3 \\
\sigma_{r\theta}(s, \Theta/2) &= C_1 s^2 + C_2 s + C_3 \\
\sigma_{\theta\theta}(s, \Theta/2) &= D_1 s^2 + D_2 s + D_3
\end{align*} \]

Fig. 3 A system of \( m \) wedges a open, b closed
Introduce symmetric and anti-symmetric parts of these quantities each contact. \[ \cos \Theta_1 \]

Denote \( u_t \) normal tractions at the boundaries of the thin wedge and \( u_r \) displacements of the normal tractions at these boundaries. The right hand sides of (2) tend to infinity when \( \Theta_0 \to 0 \). Hence, to avoid degeneration, it is reasonable to re-write (2) as

\[ 2\mu_0(s + 1)T^S(s)u^S = k_0a_-P^S, \]
\[ 2\mu_0(s + 1)T^A(s)u^A = k_0a_+P^A. \]

Fig. 4 Schemes of a thin wedge, b a system, open or closed, without the thin wedge.
Consider now the external system(s). For arbitrary normal tractions $P_1$, $P_b$ at their boundaries, which coincide with the boundaries of the thin wedge, after solving the system of three-point difference equations, we obtain the corresponding boundary values of the normal displacements

$$
u_1 = a_{11}(s) P_1(s) + a_{12}(s) P_b(s),$$
$$
u_1 = a_{21}(s) P_1(s) + a_{22}(s) P_b(s).$$

(4)

In terms of symmetric and anti-symmetric parts (1), the equations (4) read

$$u^S = c_{11}(s) \psi^S(s) + c_{12}(s) \psi^A(s),$$
$$u^A = c_{21}(s) \psi^A(s) + c_{22}(s) \psi^A(s),$$

(5)

where

$$c_{11} = \frac{1}{2}(a_{11} - a_{21} + a_{12} - a_{22}),$$
$$c_{12} = \frac{1}{2}(a_{11} - a_{21} - a_{12} + a_{22}),$$
$$c_{21} = \frac{1}{2}(a_{11} + a_{21} + a_{12} + a_{22}),$$
$$c_{22} = \frac{1}{2}(a_{11} + a_{21} - a_{12} - a_{22}).$$

(6)

Substitution (5) into (3) gives the system

$$2\mu_0(s+1) T^S(s)[c_{11}(s) \psi^S(s) + c_{12}(s) \psi^A(s)] = k_0 a_- \psi^S,$$
$$2\mu_0(s+1) T^A(s)[c_{21}(s) \psi^A(s) + c_{22}(s) \psi^A(s)] = k_0 a_+ \psi^A,$$

(7)

which does not degenerate when $\Theta_0 \to 0$. Its determinant

$$\Delta(s) = (2\mu_0(s+1) T^S_{c_{11}} - k_0 a_-)(2\mu_0(s+1) T^A_{c_{21}} - k_0 a_+) - (2\mu_0(s+1) T^A_{c_{22}})(2\mu_0(s+1) T^S_{c_{12}})$$

(8)

does not tend to infinity when $\Theta_0 \to 0$.

As mentioned, a system of three-point difference equations is efficiently and accurately solved by Gauss elimination for an arbitrary value of the complex parameter $s$. Hence, a numerical procedure of this method gives us the coefficients $a_{ij}$ and, consequently, the coefficients $c_{ij}$ in (6), and the determinant $\Delta(s)$ defined by (8). Then, employing Muller’s iterations in $s$, we find the roots of the equation

$$\Delta(s) = 0$$

in the strip $-2 < Res < -1$, which generate singular stresses. If $s_*$ is such a distinct root, closest to the point $s = -1$, then from the residue theorem, it follows that the corresponding asymptotic behaviour of stresses is of order $O(r^{-\lambda})$, where $\lambda = s_* + 2$. If there are no roots in the strip $-2 < Res < -1$, we extend the search to the strip $-3 < Res < -2$ and so on. We shall call the root with $Res < -1$ closest to the point $s = -1$, significant. A numerical code, developed on this basis, allows us to find the needed roots for an arbitrary number of wedges with arbitrary angles and elastic modules.

With purpose to provide benchmarks [9] for comparison with data, which may be obtained by other methods, below we present results with eight significant digits. (The actual accuracy of calculations was ten digits what is confirmed by (i) starting Muller’s iterations from various initial points and (ii) comparing with results, obtained by alternative methods, for the particular cases of two similar wedges comprising a system with the contact being the axis of symmetry.)

As an example, Tables 1 and 2 contain the found values of $\lambda = s_* + 2$ for an open system of five wedges with the angles $\Theta_1 = 150^0$, $\Theta_2 = 50^0$, $\Theta_3 = 10^0(5^0, 1^0)$, $\Theta_4 = 50^0$, $\Theta_5 = 100^0$ (Fig. 5). Shear modules are $\mu_1 = 2$, $\mu_2 = 1$, $\mu_3 = 100(10, 0.1)$, $\mu_4 = 2$, $\mu_5 = 1$. The Poisson’s ratio is taken the same for all the wedges ($v_1 = v_2 = v_3 = v_4 = v_5 = 0.3$). The third wedge is considered to be a thin wedge ($\Theta_0 = \Theta_3$, $\mu_0 = \mu_3$, $v_0 = v_3$). Table 1 refers to the case of prescribed displacements at the external boundaries of the considered open system; Table 2 presents results for prescribed tractions.

From the tables, we may see that normally there are two roots in the strip $-2 < Res < -1$. Meanwhile, in the case of prescribed tractions, there are no roots in this strip for the angle $\Theta_0 = 5^0$ or $1^0$ when $\mu_0 = 1$ or 0.1. All the roots are real.
The study of the influence of the Poisson’s ratios on the evaluated exponent shows that the results do not change significantly when the Poisson’s ratio of wedges is different while being within the range $0 \leq \nu < 0.5$. Specifically, even for the contrast values $\nu_2 = 0.05$ and $\nu_4 = 0.45$ of two wedges neighbouring the thin wedge, the values of the exponent $\lambda$ are quite close to those obtained for the same Poisson’s ratios. For instance, in the case of five wedges (Fig. 5), we have $\lambda_1 = 0.54652986$, $\lambda_2 = 0.18717708$ when $\Theta_0 = 10^\circ$, $\mu_0 = 100$ against the values $\lambda_1 = 0.54902640$, $\lambda_2 = 0.18805175$ obtained for the neighbour wedges with $\nu_2 = \nu_4 = 0.3$.

The code will serve us for analysing the accuracy of physical models simulating the influence of a thin wedge by dependences between tractions and displacements on its sides. A model should not depend on particular values of external actions. Consequently, when validating a model, one can assume boundary and contact conditions to be homogeneous. Then, from the residue theorem, applied to the inversion of the Mellin’s transform, it follows that we need to compare the significant root, corresponding to the solution for a multi-wedge system including the thin wedge, with the root corresponding to the system, in which the thin wedge is simulated by approximate relations between tractions and displacements.

### 3 Approximations and physical models for a thin wedge

Equations (2), connecting boundary values of tractions and displacements for an arbitrary wedge, besides degrees of $s + 1$, contain also the trigonometric functions $\sin(s\Theta_0)$, $\sin(s + 1)\Theta_0$, $\cos(s\Theta_0)$, and $\cos(s + 1)\Theta_0$. 

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**Table 1** Values of $\lambda = 2 + s^*$ for the system of five wedges (prescribed boundary displacements)

| $\Theta_0$ | $10^\circ$ | $5^\circ$ | $1^\circ$ |
|------------|------------|------------|------------|
| $\mu_0 = 100$ | $\lambda_1 = 0.54902640$ | $\lambda_1 = 0.54803042$ | $\lambda_1 = 0.49415729$ |
| $\mu_0 = 10$ | $\lambda_2 = 0.18807573$ | $\lambda_2 = 0.18805175$ | $\lambda_2 = 0.18173797$ |
| $\mu_0 = 1$ | $\lambda_1 = 0.54770539$ | $\lambda_1 = 0.54775797$ | $\lambda_1 = 0.50538836$ |
| $\mu_0 = 0.1$ | $\lambda_2 = 0.18188878$ | $\lambda_2 = 0.18173615$ | $\lambda_2 = 0.18173599$ |

**Table 2** Values of $\lambda = 2 + s^*$ for the system of five wedges (prescribed boundary tractions)

| $\Theta_0$ | $10^\circ$ | $5^\circ$ |
|------------|------------|------------|
| $\mu_0 = 100$ | $\lambda_1 = 0.22794921$ | $\lambda_1 = 0.22743931$ |
| $\mu_0 = 10$ | $\lambda_2 = 0.00661898$ | $\lambda_2 = 0.03938239$ |
| $\mu_0 = 1$ | $\lambda_1 = 0.22304117$ | $\lambda_2 = 0.02341036$ |
The model (11), (12) is applicable for a thin wedge with arbitrary elastic properties. Meanwhile, when the wedge obtains a small angle $\Theta_0$, we keep only the lowest degree of $\Theta_0$ in the resulting equations, obtaining

$$2u^S(s) = \frac{\Theta_0}{\mu_0}E_S P^S(s),$$

$$2P^A(s) = \Theta_0^2 \mu_0 E_A u^A(s),$$

where the multipliers $E_S = \frac{1}{2(s+1)^2} [k_0 (s+1)^2 - k_0]$ and $E_A = 2(s+1)^2 (s^2+2s) 12k_0$ do not depend on the angle $\Theta_0$ and the shear modulus $\mu_0$ of the thin wedge. Mellin’s inversion of (10) gives the general model for a thin wedge

$$2u^S(r) = a^S \left[ r P^S(r) - \int_0^r \left( \frac{1}{r} \int_0^r P^S(r) \, dr \right) \, dr \right],$$

$$2P^A(r) = I_\Theta \frac{1}{r^2} \frac{d^3}{dr^3} \left( r^2 \frac{du^A}{dr} \right),$$

where $a^S = \frac{k_0 \Theta_0}{2\mu_0}$, $I_\Theta = \frac{2\mu_0 r^3 \Theta_0^3}{k_0 12k_0}$, and

$$u^S(r) = \frac{1}{2} \left[ u_\Theta \left( r, \frac{\Theta_0}{2} \right) - u_\Theta \left( r, -\frac{\Theta_0}{2} \right) \right], \quad u^A(r) = \frac{1}{2} \left[ u_\Theta \left( r, \frac{\Theta_0}{2} \right) + u_\Theta \left( r, -\frac{\Theta_0}{2} \right) \right],$$

$$P^S(r) = \frac{1}{2} \left[ \sigma_\Theta \left( r, \frac{\Theta_0}{2} \right) + \sigma_\Theta \left( r, -\frac{\Theta_0}{2} \right) \right], \quad P^A(r) = \frac{1}{2} \left[ \sigma_\Theta \left( r, \frac{\Theta_0}{2} \right) - \sigma_\Theta \left( r, -\frac{\Theta_0}{2} \right) \right].$$

The model (11), (12) is applicable for a thin wedge with arbitrary elastic properties. Meanwhile, when the neighbour wedges, embedding the thin wedge, do not differ significantly in their properties, more simple models cover the entire range of shear modules. To see it, we re-write (5) in the form similar to (10)

$$u^S = \left( c_{11}(s) - \frac{c_{12}(s) c_{21}(s)}{c_{22}(s)} \right) P^S(s) + \frac{c_{12}(s)}{c_{22}(s)} u^A(s),$$

$$P^A = -\frac{c_{21}(s)}{c_{22}(s)} P^S(s) + \frac{1}{c_{22}} u^A(s).$$

Then, by employing (10), we obtain the approximate homogeneous system, which substitutes the exact system (7),

$$\begin{bmatrix}
\mu_a \left( c_{11}(s) - \frac{c_{12}(s) c_{21}(s)}{c_{22}(s)} \right) - \frac{1}{2} \Theta_0^2 \frac{\mu_a}{\mu_0} E_S \\
-\frac{c_{21}(s)}{c_{22}(s)} P^S(s) + \left( \frac{1}{\mu_a c_{22}} - \frac{1}{2} \Theta_0^2 \frac{\mu_0}{\mu_a} E_A \right) \mu_a u^A(s)
\end{bmatrix} = 0.$$
Note that the products $\mu c_{ij}$ are dimensionless and of order $O(1)$ in $\Theta_0$. By comparing terms in (15), we distinguish three approximate models, which are simpler than the general model (11), (12).

**Model I: the case of soft or moderately rigid thin wedge.** In this case, $\mu_s/\mu_0 >> \Theta_0^3$, and inspection of (15) suggests neglecting the term with the multiplier $\Theta_0^3 \mu_0/\mu_s$. Its omitting corresponds to zero r. h. s. in the second of equations (10). This yields the condition $P^A(s) = 0$ implying continuous normal tractions through the thin wedge:

$$\sigma_{\theta\theta} \left( r, \frac{\Theta_0}{2} \right) = \sigma_{\theta\theta} \left( r, -\frac{\Theta_0}{2} \right).$$

The second condition, in general, is given by (11). Obviously, the model I includes the case of an infinitely soft wedge independently of its angle.

**Model II: the case of rigid or moderately soft thin wedge.** In this case, $\mu_s/\mu_0 << 1/\Theta_0$, and inspection of (15) suggests neglecting the term with the multiplier $\Theta_0^3 \mu_0/\mu_s$. Its omitting corresponds to zero r. h. s. in the first of equations (10). This yields the condition $u^5(s) = 0$ implying continuous normal displacements through the thin wedge:

$$u_\theta \left( r, \frac{\Theta_0}{2} \right) = u_\theta \left( r, -\frac{\Theta_0}{2} \right).$$

The second condition, in general, is given by (12). Obviously, the model II includes the case of an infinitely stiff wedge independently of its angle.

There is a wide range of the ratio $\mu_s/\mu_0$ when simultaneously $\mu_s/\mu_0 >> \Theta_0^3$ and $\mu_s/\mu_0 << 1/\Theta_0$. Then, we may use the simplest model III.

**Model III: the intermediate case, when the thin wedge is either moderately soft, or moderately rigid, or it has the rigidity similar to that of neighbour wedges.** This case corresponds to the ratio $\mu_s/\mu_0$ in the interval $\Theta_0^3 \leq \mu_s/\mu_0 \leq 1/\Theta_0$. Then, both normal tractions and displacements are continuous through the thin wedge:

$$\sigma_{\theta\theta} \left( r, \frac{\Theta_0}{2} \right) = \sigma_{\theta\theta} \left( r, -\frac{\Theta_0}{2} \right), \quad u_\theta \left( r, \frac{\Theta_0}{2} \right) = u_\theta \left( r, -\frac{\Theta_0}{2} \right).$$

(16)

As an example, we consider an open system of three wedges with the angles $\Theta_1 = 100^0$, $\Theta_0 = 10^0$, $\Theta_2 = 170^0$ under prescribed tractions at the external boundaries (Fig. 6). Particular values of these tractions, as well as the values of prescribed discontinuities $\Delta p_0(r)$, $\Delta u_0(r)$ on contacts, do not matter. (Remind that we are interested in the roots of the determinant, which define the asymptotic behaviour independent on these values). Shear modules are $\mu_1 = 2$, $\mu_0 = 10,000$ (1.000, 100, 1, 0.1, 0.01), $\mu_2 = 1$. The Poisson’s ratio is taken the same for all the wedges ($\nu_1 = \nu_0 = \nu_2 = 0.3$). The second column of Table 3 contains the values of $\lambda = s^* + 2$, where $s^*$ is the root of the determinant (8), corresponding to the system with all three wedges. Other columns present approximate values of $\lambda$ obtained under the assumption that the wedge with the angle $\Theta_0 = 10^0$ is the thin wedge, which may be simulated by the approximate models derived above.

We see that, as could be expected, the general model (11), (12) provides very accurate results for arbitrary values of $\mu_0$. The relative error is less then 0.02% except for the case of very rigid wedge with the rigidity $\mu_0 = 10,000$, then the relative error may be grater, still it is less then 0.06%. Also, in agreement with the
estimations given above, the models I, II and III cover the entire range of rigidities. Specifically, for a soft or moderately rigid wedge ($\mu_0 \leq 10$), the model I provides quite accurate results (the relative error is less than 0.03%); for a rigid wedge ($\mu_0 \geq 100$), the model II gives the relative error, which does not exceed 0.50%. The model III is applicable in the range $10 \leq \mu_0 \leq 100$, where its relative error is less than 0.50%. Thus, the numerical results perfectly agree with the theoretical estimations.

Detailed calculations show that with decreasing angle $\theta_0$, the accuracy of the approximate models grows. In particular, for the angle $\theta_0 = 1^\circ$, the model I gives $\lambda = 0.2830477494$ against the exact value $0.2830477495$, when $\mu_0 = 0.01$. For the same angle, the model II gives $\lambda = 0.3829964093$ against the exact value $0.3829962504$, when $\mu_0 = 10000$. The simplest model III is applicable when $10 \leq \mu_0 \leq 10000$ with the relative error less than 0.1%.

We conclude that the simple models I, II and III provide results with relative error not exceeding 0.1% for a system containing a thin wedge with the angle not exceeding $1^\circ$. They are still acceptable up to the angle $\Theta_0 = 10^\circ$.

From the analysis of the determinant (15), it is evident that the conclusions on the ranges of the models applicability, formulated in terms of strong inequalities, are true for an arbitrary wedge system; they are actually independent on the Poisson’s ratios of wedges; and they are defined first and foremost by the properties of the two wedges, which are neighbours of the thin wedge. Hence, the results of Table 3 for a particular example give quite general quantitative estimations of these ranges and errors within them. The code developed allows one to study an arbitrary multi-wedge system with smooth contacts. Using this opportunity, we have studied many other systems composed of various numbers of wedges with various angles, shear modules and Poisson’s ratios. In all the studied cases, the conclusions on the models applicability completely agree with the benchmarks provided by Table 3. There is no sense to reproduce these numerous quite similar results. As an illustration, we present them only for the scheme of Fig. 6 when changing the Poisson’s ratio of the upper wedge to $v_1 = 0.45$ and that of the lower wedge to $v_2 = 0.05$. The results are summarized in Table 4. It shows, that although the exponent changes (as should be), the ranges of the models’ applicability stay the same to the same accuracy as that in Table 3. Note that in all the cases studied, including many with various Poisson’s ratios of wedges, no complex roots have been detected (cf. [6,4]).

**4 Conclusions**

The conclusions from the analysis are as follows.

(i) For a system of elastic wedges with smooth contacts and a thin wedge, separation of symmetric and anti-symmetric parts accompanied with the proper re-arrangement (3) of terms excludes degeneration of equations when the angle of the thin wedge tends to zero.
(ii) Expanding the dependences between Mellin’s transformed normal tractions and displacements on the boundaries of a thin wedge into Taylor series facilitates analytical inversion of the dependences. Truncation of the series with the first terms gives a general model (11), (12) of contact interaction simulating the influence of a thin wedge.

(iii) Inspection of terms entering the characteristic determinant (8) results in three simplified physical models simulating the influence of the thin wedge. Specifically, when the thin wedge is soft or moderately rigid ($\mu_s/\mu_0 \gg \Theta^3_0$), the model I is applicable: the normal tractions may be assumed continuous across the thin wedge, while the discontinuity of the normal displacement is connected with the normal traction by the Eq. (11);

when the thin wedge is rigid or moderately soft ($\mu_s/\mu_0 < 1/\Theta_0$), the model II is applicable: the normal displacements may be assumed continuous across the thin wedge, while the discontinuity of the normal traction is connected with the normal displacement by the Eq. (12);

when the thin wedge is either moderately soft, or moderately rigid, or it has the rigidity similar to that of neighbouring wedges ($\Theta^2_0 \leq \mu_s/\mu_0 \leq 1/\Theta_0$), the model III is applicable: both normal tractions and displacements are continuous through the thin wedge.

(iv) The three models cover the entire range of thin wedge rigidities ($0 < \mu_s/\mu_0 < \infty$). Numerical calculations confirm high accuracy of the approximations provided by the models in the ranges of their applicability established by the theoretical analysis.

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