Work and energy in rotating systems

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Abstract

Literature analyzes the way in which Newton’s second law can be used when non-inertial rotating systems are used. However, the treatment of the work and energy theorem in rotating systems is not considered in textbooks. In this paper, we show that the work and energy theorem can still be applied to a closed system of particles in a rotating system, as long as the work of fictitious forces is properly included in the formalism. The coriolis force does not contribute to the work coming from fictitious forces. It worths remarking that real forces that do not do work in an inertial reference frame can do work in the rotating reference frame and vice versa. The combined effects of acceleration of the origin and rotation of the non-inertial system are also studied.

Keywords: Fundamental theorem of work and energy, fictitious forces, rotating reference frames.

PACS: 01.55.+b, 45.20.D-; 45.20.D-, 45.20.dg, 45.50.-j.

1 Introduction.

Most of our laboratory reference frames are non-inertial, for instance, the dynamics of air or water clusters is usually studied from a reference frame attached to the earth, this dynamics is strongly determined by the presence of the coriolis term. Meteorological and oceanographic phenomena are also influenced by fictitious forces generated by considering the earth as a rotating system. Further, the work and energy formalism could have some advantages in rotating systems, similar to the case of inertial reference frames. Consequently, it is important to study the transformation properties of work and energy quantities from an inertial reference frame to a frame in relative rotation with respect to the inertial one.

The transformation properties of work and energy between reference frames in relative translation have been studied. On the other hand, the transformation of forces between an inertial and a rotating reference frames is quite well studied in most of the literature (e.g. Ref. [1] [2]). Nevertheless, the transformation properties of work and energy quantities between reference frames in relative rotation have not been considered. The latter is important since fictitious forces arising from rotating systems could have potential energies associated, and real forces that do not do work in a given inertial frame can do work in the rotating frame.

In Ref. [5], it was shown that the work and energy theorem is covariant between inertial reference frames, and that the theorem still holds in non-inertial translational frames if the works done by fictitious forces are included appropriately. Further, it was shown that fictitious forces can do work, and that even in transformations between inertial frames forces that do not do work in an inertial frame can do work in another inertial frame obtaining a non-trivial potential energy (see also Ref. [6]). In this work we extend the study of the work and energy formalism for non-inertial systems that combine accelerated translation with rotation with respect to an inertial frame. The main results are illustrated with a pedagogical example, where we solve the problem in both an inertial and a non-inertial frames, and we show a force that does do work in the inertial frame, but does not do work in the non-inertial frame. The latter fact simplifies the problem considerably when treated in the non-inertial frame.

2 Formulation of the problem.

The system under study consists of \( N \) interacting particles. The system is closed, i.e. there is not any flux of particles from or toward the system and the number of particles is conserved so that not processes of creation or destruction of particles are considered. However, the system could be under the influence of time dependent (or time independent) external forces. Our description is non-relativistic such that time and the mass of the particles are independent of the reference frame.

Let us define an inertial system \( \Sigma \), and a non-inertial rotating system \( \Sigma'' \) with constant angular velocity \( \Omega \) with respect to \( \Sigma \) and a common origin between them. With respect to \( \Sigma \), a given \( j - \text{th} \) particle has a position, velocity and acceleration \( r_j', v_j', a_j' \). These variables measured by \( \Sigma'' \) are denoted by \( r_j'', v_j'', a_j'' \). Since \( \Sigma \) and \( \Sigma'' \) have a common origin, we see that

\[
r_j'' = r_j.
\]

Figure [1] illustrates these statements. We shall analyze...
The relationship between velocities and accelerations in Σ from the inertial system Σ, to the non-inertial system Σ′′ is well known from the literature [1].

\[ \mathbf{v}_j'' = \mathbf{v}_j - \dot{\Omega} \times \mathbf{r}_j, \quad (1) \]
\[ \mathbf{a}_j'' = \mathbf{a}_j - 2 \dot{\Omega} \times \mathbf{v}_j'' - \Omega \times (\Omega \times \mathbf{r}_j), \quad (2) \]

where \( \mathbf{F}_j \) represents the total real force on the \( j \)-th particle (summation of internal and external forces on \( j \)), \( \mathbf{F}_{\text{cor}} \) and \( \mathbf{F}_{\text{cent}} \) are the well-known coriolis and centrifugal forces. Taking the dot product on both sides of Eq. (3) by \( \mathbf{v}_j'' dt \) on the left and by \( d\mathbf{r}_j'' \) on the right, and summing over all particles of the system we have

\[
\sum_j d \left( \frac{1}{2} m_j \mathbf{v}_j''^2 \right) = \sum_j (\mathbf{F}_j + \mathbf{F}_{\text{fict}}) \cdot d\mathbf{r}_j'', \quad dK'' = dW''.
\]

Where \( dK'' \) and \( dW'' \) are the differentials of kinetic energy and work when an infinitesimal path \( d\mathbf{r}_j'' \) is taken for each particle. This equation shows the covariance of the fundamental theorem of work and energy between Σ and Σ′′. In relating work and energy observables between Σ and Σ′′ it is important to write \( dK_j'' \) and \( dW_j'' \) (differential of kinetic energy and work associated with the \( j \)-th particle in the system Σ′′) in terms of quantities measured by Σ. For \( dK_j'' \), we use Eq. (1) to get

\[
dK_j'' = m_j \mathbf{v}_j'' \cdot d\mathbf{v}_j'' = m_j (\mathbf{v}_j - \dot{\Omega} \times \mathbf{r}_j) \cdot \{d\mathbf{v}_j - \Omega \times d\mathbf{r}_j\}
\]
\[
dK_j'' = dK_j + dZ_j,
\]

with

\[
dZ_j = - (\dot{\Omega} \times \mathbf{r}_j) \cdot d\mathbf{P}_j - m_j (\dot{\Omega} \times (\dot{\Omega} \times \mathbf{r}_j)) \cdot d\mathbf{r}_j,
\]

where \( d\mathbf{P}_j \) denotes the differential of linear momentum associated with the \( j \)-th particle measured by Σ. The coriolis force given by Eq. (5) does not do work with respect to Σ′′. To obtain \( dW_j'' \) in terms of variables measured by Σ, we use Eqs. (11, 10).

\[
\begin{align*}
\mathbf{F}_j'' &= \mathbf{F}_j - m_j [2 \dot{\Omega} \times \mathbf{v}_j'' + \dot{\Omega} \times (\dot{\Omega} \times \mathbf{r}_j)], \\
d\mathbf{r}_j'' &= \mathbf{v}_j'' dt = \dot{\mathbf{r}}_j - (\dot{\Omega} \times \mathbf{r}_j) dt, \\
dW_j'' &= \mathbf{F}_j'' \cdot d\mathbf{r}_j'' = (\mathbf{F}_j + \mathbf{F}_{\text{fict}}) \cdot d\mathbf{r}_j'', \\
&= \{\mathbf{F}_j - m_j \dot{\Omega} \times (\dot{\Omega} \times \mathbf{r}_j)\} \cdot \{d\mathbf{r}_j - \dot{\Omega} \times \mathbf{r}_j dt\},
\end{align*}
\]

so the covariance of the fundamental theorem of work and energy is expressed by Eq. (9), or equivalently by (8, 10). For pedagogical reasons the covariance of the work and energy theorem for the pure rotation case is realized first, for the reader to assimilate the formalism in a simple way.

The additional subtleties that involve the combination of translation and rotation are introduced in appendix A in which we consider a non-inertial system Σ′′, that possesses a relative rotation with time-dependent angular velocity and translation with respect to Σ.

### 3 Fictitious work

We shall consider the general case in which Σ′′ rotates and translates with respect to Σ (see appendix A). The work observed by Σ′′ can be separated in the work coming from real forces and those coming from fictitious forces

\[
\begin{align*}
dW'' &= dW_{\text{real}} + dW_{\text{fict}}, \\
dW_{\text{fict}} &= - \sum_j m_j [\dot{\Omega} \times (\dot{\Omega} \times \mathbf{r}_j) + \dot{\Omega} \times \mathbf{r}_j'' + \mathbf{A}(t)] \cdot d\mathbf{r}_j'', \\
dW_{\text{real}} &= \sum_j [\mathbf{F}_j'' + m_j (\dot{\Omega} \times (\dot{\Omega} \times \mathbf{r}_j) + \dot{\Omega} \times \mathbf{r}_j'') + \mathbf{A}(t)] \cdot d\mathbf{r}_j''.
\end{align*}
\]

Eqs. (11, 13) are written is terms of observables measured by Σ′′ (except for \( \dot{\Omega} \) and \( \ddot{\Omega} \) which are measured with respect to Σ). It is because in most of the problems involving non-inertial systems, experiments are done on
the non-inertial system and measured with respect to it. In particular, real forces that do not do work in $\Sigma$ can do work in $\Sigma''$, or real forces that do work in $\Sigma$ could do no work in $\Sigma''$.

For the particular case $\Omega = 0$ it can be proved that if $\Sigma''$ is attached to the center of mass of the system of particles (CM), the total fictitious work is null [5],

$$dW_{\text{fict}}^{CM} = -MA_C(t) \cdot d \left( \sum_j \frac{m_j r_j''}{M} \right) = 0,$$

where $A_C$ is the acceleration of the CM. In the most general case with $\Omega \neq 0$, the total fictitious work measured by $\Sigma''$ can be different from zero even if its origin is attached to the CM.

These results are in close analogy with the case of torques analyzed from systems attached to the CM. For non-rotating systems (with respect to $\Sigma$) attached to the CM, the total fictitious torque and work are null [5]. On the other hand, for rotating systems attached to the CM both the total fictitious torque and the total fictitious work are in general non-vanishing [7]. It worths saying that even in non-rotating systems, the fictitious torque and work on each particle are in general different from zero.

In addition, it is also possible to find non-inertial systems for which fictitious work is null and fictitious torque is non-null or vice versa. However, these features depend no only on the reference frame, but also on the system of particles involved.

### 4 Pedagogical example.

![Figure 2: A block of mass $m$ slides on a table along a groove, which rotates with constant angular velocity $\omega$. The block is attached to other block of mass $M$. The Normal forces $N_1$ and $N_2$ on the block, are illustrated.](image)

Let us consider a block of mass $m$ sliding without friction on a horizontal table which rotates with constant angular velocity $\omega$. The block is constrained to move on a groove that is radially directed. The block is attached through a rope of negligible mass to other block of mass $M$, which hangs from the table through its center (see Fig. 2). Our physical system of interest are the two blocks, and their description will be made from the point of view of an inertial reference frame $\Sigma$ whose origin is located at the center of the table, and other non-inertial system $\Sigma''$ fixed to the table, with a common origin with $\Sigma$, and with constant angular velocity $\omega$. The dimensions of the two blocks are neglected, so that we regard the two blocks as point-like particles.

Our aim is to apply the fundamental work-energy theorem, therefore the work done by the resultant force due to all applied forces on the system is calculated. We take in mind that the block of mass $m$, is constrained to move on a groove. $N_2$ is a normal force from the wall of the groove on the mass $m$, which corresponds to the reaction of the table over the block due to rotation. This force is tangential, and it is different from the Normal force $N_1$ which is the force on $m$ due to the vertical contact with the table. It is clear that $N_1$ does not do work with respect to $\Sigma$ or $\Sigma''$.

We remark the fact that from the point of view of $\Sigma$, the normal force $N_2$ does work, since there are few examples in the literature in which normal forces do work [5]. For an observer in $\Sigma$, the work done by $N_2$ is given by

$$W_N = \int_{\theta_0}^{\theta_f} N_2 r d\theta,$$

from Newton’s second law the normal force is given by

$$N_2 = m(\ddot{r} + 2\dot{r} \dot{\theta}) u_\theta = 2m\omega^2 u_\theta,$$

so that,

$$W_N = m\omega \int_{\theta_0}^{\theta_f} 2r \frac{dr}{dt} d\theta = m\omega \int_{r_0}^{r_f} 2r \frac{dr}{dt} d\theta,$$

$$W_N = m\omega^2 \int_{r_0}^{r_f} 2r \, dr$$

hence, the work done by $N_2$ and the total external work (with respect to $\Sigma$) are given by

$$W_N = m\omega^2(r_f^2 - r_0^2), \quad W = m\omega^2(r_f^2 - r_0^2) - Mg(z_f - z_0).$$

From the point of view of $\Sigma''$, the centrifugal force on the block of mass $M$ is zero, since $\Omega$ is parallel to the position of $M$. Furthermore, the coriolis force does not do work. Therefore, the total fictitious force on $M$ does not do work. In $\Sigma''$ the displacement of the block of mass $m$ is radial so that we have

$$\omega = \omega \, u_z, \quad r'' = r = ru_r, \quad dr'' = dr''u_r = dr u_r.$$
hence, the work done by \( \mathbf{N}_2 \) is zero in \( \Sigma'' \). Therefore, \( \mathbf{N}_2 \) does work in \( \Sigma \) but does not do work in \( \Sigma'' \). On the other hand, the work done by the fictitious forces in \( \Sigma'' \) is

\[
dW'' = -m\omega \times (\mathbf{v} \times \mathbf{r}'') \cdot d\mathbf{r}'' = m\omega^2 r'' dW''.
\]

The total fictitious work seen by \( \Sigma'' \) is

\[
W'' = \frac{m\omega^2}{2} (r''_f^2 - r''_0^2) = \frac{m\omega^2}{2} (r^2_f - r^2_0).
\]

The work-energy theorem applied on both \( \Sigma \) and \( \Sigma'' \), yields

\[
W = m\omega^2 (r^2_f - r^2_0) - Mg(z_f - z_0)
\]

\[
= \frac{m\omega^2}{2} (r^2_f - r^2_0) + \frac{m + M}{2} (v^2_{r,f} - v^2_{r,0}),
\]

\[
W'' = \frac{m\omega^2}{2} (r''_f^2 - r''_0^2) - Mg(z_f - z_0)
\]

\[
= \frac{m + M}{2} (v^2_{r,f} - v^2_{r,0}).
\]

We have taken into account that \( v^2_M = v^2_r \) where \( v_M, v_r \) are the velocity of \( M \) and the radial velocity of \( m \) respectively. The change of kinetic energy of the system has been separated in tangential and radial parts. For convenience, we define the radial kinetic energy of the system as the sum of radial kinetic energy of \( m \) plus the kinetic energy of \( M \). We can see from \[17\] \[18\], that the change in radial kinetic energy is the same in both reference frames. As a consequence, it could be useful to define an effective work in \( \Sigma \), so that the transversal component of the kinetic energy is included in the work, and therefore an effective work-energy theorem can be established.

\[
W = \frac{m\omega^2}{2} (r^2_f - r^2_0) + W_{ef} = \Delta K_{\theta} + W_{ef}
\]

\[
W_{ef} = \frac{m\omega^2}{2} (v^2_{r,f} - v^2_{r,0}) - Mg(z_f - z_0)
\]

\[
= \frac{m + M}{2} (v^2_{r,f} - v^2_{r,0})
\]

where \( \Delta K_{\theta} \) denotes the change in transversal kinetic energy. The effective work \( W_{ef} \) defined in \( \Sigma \) in this way, is equal to the work \( W'' \) seen by \( \Sigma'' \) and both are equal to the change in radial kinetic energy seen by either frame. From \( W_{ef} \) we can define a effective potential as it is done in the problem of a central force \[1\].

\[
V_{ef} = -\frac{m\omega^2}{2} r^2 + Mg z
\]

Equations \[18\] \[19\] can be rewritten by taking into account that \( z_f - z_0 = r_f - r_0 \)

\[
W_{ef} = W'' = m\omega^2 \left[ \frac{(\omega / \omega_c)^2}{\tau - r_0} \right] (z_f - z_0),
\]

\[
\omega_c = \left( \frac{Mg}{mr_0} \right)^{1/2}, \quad \tau = \frac{r_f + r_0}{2}.
\]

From Eq. \[20\] we can give a physical interpretation of the problem where \( \omega_c \) is the critical frequency that determines the sense of motion. For the particular case in which the initial radial velocity of the block \( m \) vanishes (\( v_{r,0} = 0 \)), there are three possible situations: i) For \( \omega < \omega_c \), the block \( m \) moves toward the center of the table and the block \( M \) descends, hence we get \( \tau < r_0 \) and \( z_f - z_0 < 0 \), so that a positive work \( W'' \) is done on the system; ii) For \( \omega > \omega_c \), the block \( m \) moves away from the center of the table and the block \( M \) ascends, therefore we have \( \tau > r_0 \) and \( z_f - z_0 > 0 \), and a positive work \( W'' \) is done on the system; iii) If \( \omega = \omega_c \), the block \( M \) remains at rest and the radial velocity of \( m \) vanishes, then \( \tau = r_0 \) and \( z_f - z_0 = 0 \), and the work done is null. We suggest for the reader to interpret the equation \[20\] for the case in which the block of mass \( m \) possesses an initial radial velocity different from zero.

5 Conclusions.

We have shown the covariance of the work and energy theorem for a non-uniform rotational frame, where the effects of acceleration of the origin and rotation of the non-inertial system are included. This covariance is complied when the work done for the fictitious forces is properly included. The coriolis force does not contribute to the work coming from fictitious forces. We generalize the properties when the reference frame is attached to the center of mass, where the fictitious work on each particle is in general different from zero but the total fictitious work is zero for non-rotating non-inertial systems.

Acknowledgments: This work was supported by División de Investigaciones de la Universidad Nacional de Colombia (DIB), sede Bogotá.

A General case.

Let us define an inertial system \( \Sigma \), a non-inertial translational and non-rotating system \( \Sigma' \) (with respect to \( \Sigma \)), \[\footnote{The solution from the dynamical point of view to obtain the critical frequency is obtained in appendix \[1\].}\]
and a rotating system $\Sigma''$ with origin common with $\Sigma'$ and with angular velocity $\Omega$. The position, velocity and acceleration of the origin of $\Sigma'$ and $\Sigma''$ with respect to $\Sigma$ are called $R(t)$, $V(t)$, $A(t)$. With respect to $\Sigma$, a given $j$-th particle has a position, velocity and acceleration $r_j$, $v_j$, $a_j$. These variables measured by $\Sigma'$ are denoted by $r_j'$, $v_j'$, $a_j'$ and measured by $\Sigma''$ are denoted by $r_j''$, $v_j''$, $a_j''$. Since $\Sigma'$ and $\Sigma''$ have a common origin, we see that $r_j' = r_j''$. The axes of $\Sigma$ and $\Sigma'$ are parallel each other at all times. Figure 3 illustrates these statements, this figure also shows that

$$r_j'' = r_j' = r_j - R(t),$$  \hspace{1cm} (21)  

$$v_j'' = v_j' = v_j - V(t),$$  \hspace{1cm} (22)

the relationship between velocities and accelerations in $\Sigma'$ and $\Sigma''$ is well known from the literature

$$v_j'' = v_j - \Omega \times r_j',$$  \hspace{1cm} (23)  

$$a_j'' = a_j' - 2\Omega \times v_j'' - \Omega \times (\Omega \times r_j') - \dot{\Omega} \times r_j',$$  \hspace{1cm} (24)

where we have included the term corresponding to the variation in time of $\Omega$. Combining (22) (23) we have

$$v_j'' = v_j - \Omega \times r_j' - V(t),$$  \hspace{1cm} (25)  

deriving Eq. (22) with respect to time we find

$$a_j' = a_j - A(t),$$  \hspace{1cm} (26)  

substituting Eq. (26) in Eq. (24), we obtain

$$a_j'' = a_j - 2\Omega \times v_j'' - \Omega \times (\Omega \times r_j') - \dot{\Omega} \times r_j' - A(t),$$  \hspace{1cm} (27)

multiplying Eq. (27) by the mass $m_j$ we find

$$m_j \frac{dv_j''}{dt} = F_j + F_{j,\text{fict}},$$  \hspace{1cm} (28)  

$$F_{j,\text{fict}} = F_{j,\text{cor}} + F_{j,\text{cent}} + F_{j,\text{azim}} + F_{j,\text{tras}},$$  \hspace{1cm} (29)  

$$F_{j,\text{cor}} = -2m_j\Omega \times v_j'',$$  \hspace{1cm} (30)  

$$F_{j,\text{cent}} = -m_j\Omega \times (\Omega \times r_j'),$$  \hspace{1cm} (31)  

$$F_{j,\text{azim}} = -m_j\Omega \times r_j',$$  \hspace{1cm} (32)  

$$F_{j,\text{tras}} = -m_jA(t).$$  \hspace{1cm} (33)

In comparison with Eqs. (30), two additional terms appear, $F_{azim}$ is the fictitious force coming from the angular acceleration of $\Sigma''$ with respect to $\Sigma$, and $F_{tras}$ is the term coming from the linear acceleration of the origin of $\Sigma''$ with respect to $\Sigma$. Taking the dot product on both sides of Eq. (28) by $v_j''dt$ on the left and by $dr_j''$ on the right, and summing over all particles of the system, we obtain the covariance of the fundamental work energy theorem expressed by Eq. (7), in the general case.

The Coriolis force given by equation (30) does not do work with respect to $\Sigma''$. The differentials of kinetic energy and work, can be written in terms of quantities measured by $\Sigma$, so

$$dK_j'' = m_j \{v_j - \Omega \times [r_j - R(t)] - V(t)\} \cdot \{dv_j - \Omega \times [dr_j - V(t) dt] - d\Omega \times [r_j - R(t)] - A(t) dt\},$$

$$dW_j'' = F_j'' \cdot dr_j'' = (F_j + F_{j,\text{fict}}) \cdot v_j''dt,$$

$$= \{F_j - m_j(\Omega \times (\Omega \times (r_j - R(t))) + \Omega \times (r_j - R(t))) + A(t)\} : \{dr_j - \Omega \times (r_j - R(t)) + V(t) dt\}.$$

The covariance of the fundamental theorem of work and energy can be expressed as

$$dK_j'' = dK_j + dZ_j,$$  \hspace{1cm} (34)  

$$dW_j'' = dW_j + dZ_j,$$  \hspace{1cm} (35)

$$dZ_j \equiv -[\Omega \times (r_j - R(t)) + V(t)] \cdot dP_j$$

$$-m_j \{\Omega \times (\Omega \times (r_j - R(t))) + \Omega \times (r_j - R(t)) + A(t)\}$$

$$\cdot \{dr_j - \Omega \times (r_j - R(t)) + V(t) dt\}.$$

### B Dynamics.

In the example exposed in Sec. 4 we can determine the condition for the angular velocity so that if the block $m$ starts with null radial velocity ($v_{r_o} = 0$), its radial velocity remains null at all times. From the point of view of $\Sigma$, we
analyze the dynamics of the system composed by the two blocks. In polar coordinates the equations of motion for the block of mass \(m\) are

\[
-T = m(\ddot{r} - r\omega^2), \tag{36}
\]

\[
N_2 = 2m\omega\dot{r}. \tag{37}
\]

For the block of mass \(M\) we have

\[
T - Mg = M\ddot{z}. \tag{38}
\]

The motion of the system is constrained by

\[
r - z = \text{constant} \tag{39}
\]

We combine (36, 38), and employ (39), which leads to

\[
\ddot{r} - \frac{m\omega^2}{M + m} r = -\frac{Mg}{M + m} \tag{40}
\]

The solution of Eq. (40), is obtained from the solutions of the homogeneous equation, hence

\[
r = C + (r_0 - C) \cosh (\kappa t) + \frac{v_{r,0}}{\kappa} \sinh (\kappa t), \tag{41}
\]

\[
\kappa = \frac{\omega}{\sqrt{\frac{m}{m + M}}}, \quad C = \frac{Mg}{m\omega^2}. \tag{42}
\]

For \(v_{r,0} = 0\), we can see that if \(\omega = \omega_c = \left(\frac{Mg}{mr_0}\right)^{1/2}\), then the block remains at rest.

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