On Strongly Nonlinear Eigenvalue Problems in the Framework of Nonreflexive Orlicz-Sobolev Spaces

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Abstract

It is established existence and multiplicity of solutions for strongly nonlinear problems driven by the Φ-Laplacian operator on bounded domains. Our main results are stated without the so called Δ2 condition at infinity which means that the underlying Orlicz-Sobolev spaces are not reflexive.

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1 Introduction

In this work, we study the nonlinear eigenvalue problem

\[ \begin{cases} -\text{div}(\phi(|\nabla u|)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1} \]

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter and $\phi : (0, \infty) \to (0, \infty)$ is a $C^1$-function satisfying

$(\phi_1)$ (i) $t\phi(t) \to 0$ as $t \to 0$,

(ii) $t\phi(t) \to \infty$ as $t \to \infty$,

$(\phi_2)$ $t\phi(t)$ is strictly increasing in $(0, \infty)$.
Throughout this work, \( f : [0, \infty) \to [0, \infty) \) is a continuous function satisfying

\[ f(0) \geq 0, \]

\( (f_1) \) \( f(0) \geq 0, \)

\( (f_2) \) there exist positive numbers \( a_k, b_k, \ k = 1, \cdots, m - 1 \) such that

\[ 0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m, \]

\[ f(s) \leq 0 \text{ if } s \in (a_k, b_k), \]

\[ f(s) \geq 0 \text{ if } s \in (b_k, a_{k+1}), \]

\( (f_3) \) \( \int_{a_k}^{a_{k+1}} f(s) \, ds > 0, \ k = 1, \cdots, m - 1. \)

**Remark 1.1** We extend \( t \mapsto t \phi(t) \) to the whole of \( \mathbb{R} \) as an odd function.

We shall consider the N-function

\[ \Phi(t) = \int_0^t s \phi(s) \, ds, \quad t \in \mathbb{R}, \]

and we shall use the notation

\[ \Delta_\Phi u := \text{div} (\phi(|\nabla u|) \nabla u) \]

for the \( \Phi \)-Laplacian operator. Due to the use of this more general operator we shall work in the framework of Orlicz-Sobolev spaces such as \( W_0^{1} L_\Phi(\Omega) \) which under our assumptions on \( \phi \) is not reflexive.

The main novelty in this work is to ensure existence and multiplicity of solutions for problem (1.1) without the so called \( \Delta_2 \) condition which amounts that \( W_0^{1} L_\Phi(\Omega) \) is not reflexive.

Examples of functions \( \Phi \) covered by the main theorem in the present work:

\[ \Phi(t) = e^t - t + 1, \quad (1.2) \]

\[ \Phi(t) = (1 + t^2)^{\gamma} - 1 \quad \text{where} \quad \gamma > \frac{1}{2}, \quad (1.3) \]

\[ \Phi(t) = t^p \log(1 + t) \quad \text{where} \quad p \geq 1. \quad (1.4) \]

Our main result stated below extends Theorem 1.1 by Loc & Schmidt in [16] to the more general operator \( \Delta_\Phi \) in the case that \( W_0^{1} L_\Phi(\Omega) \) is not reflexive.
Theorem 1.1 Assume \((\phi_1) - (\phi_2)\). Then

(i) if \((f_1) - (f_3)\) hold, there is \(\lambda > 0\) such that for each \(\lambda > \lambda\), \((1.1)\) admits at least \(m - 1\) non negative weak solutions \(u_1, \ldots, u_{m-1} \in W^1_0 L_\Phi(\Omega) \cap L^\infty(\Omega)\) satisfying

\[a_1 < \|u_1\|_\infty \leq a_2 < \|u_2\|_\infty \leq \cdots \leq a_{m-1} < \|u_{m-1}\|_\infty \leq a_m,\]

(ii) conversely, if \(u \in W^1_0 L_\Phi(\Omega) \cap L^\infty(\Omega)\) is a nonnegative weak solution of problem \((1.1)\) such that \(a_k < \|u\|_\infty \leq a_{k+1}\) and \((f_1) - (f_2)\) holds then \((f_3)\) also holds true.

Remark 1.2 If \(f\) is extended to the whole of \(\mathbb{R}\) as an odd function then with minor modifications on the arguments of the present work, problem \((1.1)\) admits at least \(2(m - 1)\) weak solutions \(u_1, \ldots, u_{m-1}\) and \(v_1, \ldots, v_{m-1}\) such that \(u_i > 0, v_i < 0\) in \(\Omega, i = 1, 2, \ldots, m - 1\) and

\[a_1 < \|u_1\|_\infty \leq a_2 < \|u_2\|_\infty \leq \cdots \leq a_{m-1} < \|u_{m-1}\|_\infty \leq a_m,\]

\[a_1 < \|v_1\|_\infty \leq a_2 < \|v_2\|_\infty \leq \cdots \leq a_{m-1} < \|v_{m-1}\|_\infty \leq a_m.\]

We recall that Hess in [13] employed variational and topological methods and arguments with lower and upper solutions to prove a result on existence of multiple positive solutions for the problem

\[-\Delta u = \lambda f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

which is problem \((1.1)\) with \(\phi(t) \equiv 1\). As noticed by Hess, the results in [13] were motivated by Brown & Budin [2, 3] which in turn were motivated by the literature on nonlinear heat generation.

In [16], Loc & Schmitt extended the result by Hess to the \(p\)-Laplacian operator by taking \(\phi(t) = t^{p-2}\) with \(1 < p < \infty\) in \((1.1)\). Actually, in [16] the authors showed that \((f_1) - (f_3)\) are sufficient conditions for the existence of \(m - 1\) positive solutions of

\[-\Delta_p u = \lambda f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

for \(\lambda\) large, while if \((f_1) - (f_2)\) hold, and \(u\) is a positive solution of the problem above with \(a_k < \|u\|_\infty \leq a_{k+1}\) then \((f_3)\) holds.

Regarding the rich literature on [13, 16] we further refer the reader to [4, 6, 7, 8], where several techniques were employed.
In particular, in [7], the authors proved a version of Theorem 1.1 in the case that both $\Phi$ and its conjugate function $\tilde{\Phi}$ satisfy the $\Delta_2$ condition, or equivalently, there exist $\ell, m$ with $1 < \ell < m < \infty$ such that

$$\ell \leq \frac{t\Phi'(t)}{\Phi(t)} \leq m, \ t > 0,$$

see e.g. Section 2 for clarification on notation and terms above.

Examples of functions $\Phi$ for which the $\Delta_2$ condition does not hold are (1.2) and (1.4) with $p = 1$. In these two cases $\Phi$ grows too slow or too fast, respectively. In the first case we have that $\ell = 1$ while in the second case $m = \infty$.

In the present work, we do not require the $\Delta_2$ condition. Due to the lack of the $\Delta_2$ condition, we have to overcome many difficulties which do not appear in the $\Delta_2$ case. Indeed, without that condition, the Orlicz spaces are not reflexive and the energy functional $J$ associated to problem (1.1) is not $C^1$, in fact, it is not even well defined in the whole Orlicz-Sobolev space. This difficulty is overcome by working in some appropriate subspaces in the Orlicz-Sobolev space. On this subject we refer the reader to Gossez [11] and García-Huidobro et al [9]. For further results without $\Delta_2$, we refer the reader to Le Vy Khoi [14], Loc & Schmitt [16], V. Mustonen & M. Tienari [17] and M. Tienari [18], Clément et al [5].

Other classes of functions $\phi$ which satisfy $(\phi_1) - (\phi_2)$ are:

(i) $\phi(t) = t^{p-2} + t^{q-2}$ with $1 < p < q < N$. In this case with $\ell = p$ and $m = q$, the corresponding operator is the $(p,q)$-Laplacian and (1.1) becomes

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(u) \text{ in } \Omega, \\
u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega, \end{cases}$$

(ii) $\phi(t) = \sum_{i=1}^N t^{p_i-2}$ where $1 < p_1 < p_2 < \ldots < p_N$, $\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}$ with $\overline{p} < N$. In this case the corresponding problem

$$\begin{cases} -\sum_{i=1}^N \Delta_{p_i} u = \lambda f(u) \text{ in } \Omega, \\
u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \end{cases}$$

is known in the literature as an anisotropic elliptic problem,
(iii) \( \phi(t) = a(tp)t^{p-2} \) where \( 2 \leq p < N \) and \( a : (0, \infty) \to (0, \infty) \) is a suitable \( C^1(\mathbb{R}^+) \)-function. In this case the corresponding problem reads as

\[
\begin{cases}
-\text{div}(a(|\nabla u|^p)|u|^{p-2}\nabla u) = \lambda f(u) \text{ in } \Omega, \\
u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\end{cases}
\] (1.5)

This work is organized as follows: in Section 2 we present some tools and references on Orlicz-Sobolev spaces. Section 3 is devoted to some auxiliary problems and the variational setting. Section 4 is devoted to some technical lemmata. In Section 5 we give the proof of Theorem 1.1. In the Appendix we give, for completeness, some technical results used in this work.

2 Basics on Orlicz-Sobolev Spaces

The main references for this Section are Gossez [11, 12], Adams [1], Kufner [15] and Tienari [18]. Further references will be given timely.

Let \( \Phi : \mathbb{R} \to \mathbb{R} \) be an \( N \)-Function. We say that \( \Phi \) satisfies the \( \Delta_2 \) condition if there exist \( t_0 > 0 \) and \( K > 0 \) such that

\[
\Phi(2t) \leq K\Phi(t), \quad |t| \geq t_0.
\] (2.1)

It is well known that this condition is equivalent to

\[
\frac{t\Phi'(t)}{\Phi(t)} \leq m, \quad |t| \geq t_0 \text{ for some } m \in (1, \infty).
\]

The \( \Delta_2 \) condition is crucial to ensure that Orlicz and Orlicz-Sobolev spaces are reflexive Banach spaces.

The Orlicz class associated with \( \Phi \) is

\[
\mathcal{L}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u \text{ measurable and } \int_\Omega \Phi(u)dx < \infty \right\}.
\]

The Orlicz space \( L_\Phi(\Omega) \) is the linear hull of \( \mathcal{L}_\Phi(\Omega) \), that is,

\[
L_\Phi(\Omega) = \bigcap_{\mathcal{L}(\Omega) \subset V} \{ V \mid V \text{ is a vector space } \}.
\]

The norm of a function \( u \in L_\Phi(\Omega) \), (Luxemburg norm), is defined by

\[
\|u\|_\Phi = \inf \left\{ \lambda > 0 \mid \int_\Omega \frac{\Phi(u)}{\lambda} \leq 1 \right\}
\]
and $L_\Phi(\Omega)$ endowed with the norm $\| \cdot \|_\Phi$ is a Banach space. On the other hand, the space $E_\Phi(\Omega)$ is defined by

$$E_\Phi(\Omega) = \text{closure of } L^\infty(\Omega) \text{ in } L_\Phi(\Omega) \text{ with respect to } \| \cdot \|_\Phi.$$  

We recall that

$$L_\Phi(\Omega) = \left\{ u \mid \int_\Omega \Phi(\lambda u) < \infty \text{ for some } \lambda > 0 \right\}.$$  

and

$$E_\Phi(\Omega) = \left\{ u \mid \int_\Omega \Phi(\lambda u) < \infty \text{ for each } \lambda > 0 \right\}.$$ 

It is well known that $L_\Psi(\Omega) \hookrightarrow L^1(\Omega)$ (cf. Adams [1]). The Orlicz-Sobolev space is defined by

$$W^{1}L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega) \mid \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \ i = 1, \ldots, N \right\},$$

and similarly

$$W^{1}E_\Phi(\Omega) = \left\{ u \in E_\Phi(\Omega) \mid \frac{\partial u}{\partial x_i} \in E_\Phi(\Omega), \ i = 1, \ldots, N \right\}.$$ 

It is known that $W^{1}L_\Phi(\Omega)$ endowed with the norm

$$\| u \|_{1, \Phi} = \| u \|_\Phi + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_\Phi$$

is a Banach space. We point out that by the Poincaré Inequality (see [11, Lemma 5.7 p 202]),

$$\| u \| := \| \nabla u \|_\Phi, \ u \in W^{1}L_\Phi(\Omega)$$

defines a norm in $W^{1,\Phi}_0(\Omega)$ equivalent to $\| \cdot \|_{1, \Phi}$.

The conjugate function of $\Phi$ is defined by

$$\tilde{\Phi}(t) = \sup\{ ts - \Phi(s) \mid s \in \mathbb{R} \}.$$ 

It is known that $\tilde{\Phi}$ is also an $N$-function and actually

$$\tilde{\Phi}(t) = \int_{0}^{|t|} t\phi(t)dt$$

for a suitable function $\phi$. 


where \( \bar{\varphi} \) satisfies the same basic conditions as \( \varphi \). The Young inequality holds,
\[
ts s \leq \Phi(t) + \Phi(s), \quad t, s \in \mathbb{R},
\]
and the equality is true if and only if \( t = s\bar{\varphi}(s) \) or \( s = t\varphi(t) \).

It is well known that \( L_\Phi(\Omega) \) is the dual space of \( E_\bar{\Phi}(\Omega) \), that is
\[
L_\Phi(\Omega) = E_\bar{\Phi}(\Omega)'.
\]
The Hölder inequality is true, that is
\[
\int_\Omega |uv| \leq 2\|u\|_\varphi \|v\|_{\bar{\Phi}}, \quad u \in L_\Phi(\Omega), \quad v \in L_\bar{\Phi}(\Omega).
\]

Yet following Gossez \cite{11,12} we recall that the spaces \( W^{1}L_\Phi(\Omega) \) and \( W^{1}E_\Phi(\Omega) \) are identified with subspaces of the product spaces \( \prod L_\Phi(\Omega), \ \prod E_\Phi(\Omega) \), respectively.

We define
\[
W^{1}_0L_\Phi(\Omega) = C_0^\infty(\Omega)^{E_\Phi(\Omega)} \quad \text{and} \quad W^{1}_0E_\Phi(\Omega) = C_0^\infty(\Omega)^{\Phi}.\]

3 Auxiliary Problems and Variational Setting

Consider the family of problems associated to (1.1)
\[
\begin{cases}
-\Delta \varphi u = \lambda f_k(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( f_k : \mathbb{R} \to \mathbb{R} \) is a continuous function for each \( k = 2, \ldots, m \) which is given by
\[
f_k(s) = \begin{cases}
f(0) & \text{if } s \leq 0, \\
f(s) & \text{if } 0 \leq s \leq a_k, \\
0 & \text{if } s > a_k.
\end{cases}
\]

Here we emphasize that \( f_k \) is a bounded function which has at least \( m \) bumps. The energy functional associated to the problem (3.1) is given by
\[
I_k(\lambda, u) = \int_\Omega \Phi(|\nabla u|)dx - \lambda \int_\Omega F_k(u)dx, \quad u \in W^{1}_0L_\Phi(\Omega),
\]
where
\[
F_k(s) = \int_0^s f_k(t)dt, \quad k = 1, 2, \ldots, m - 1.
\]
Now we observe that \( I_k(\lambda, \cdot) : W^1_0 L_\Phi(\Omega) \to \mathbb{R} \cup \{\infty\} \). In fact, the domain of \( I_k \), that is the set for which \( I_k \) is finite, is
\[
\{ u \in W^1_0 L_\Phi(\Omega) \mid |\nabla u| \in \mathcal{L}(\Omega) \}.
\]
Moreover, there are points in \( W^1_0 L_\Phi(\Omega) \) where \( I_k \) is not differentiable. However, we mention that \( I_k \) is differentiable in \( W^1_0 L_\Phi(\Omega) \), (cf. [9, Lemma 3.4]). In this way, we say that \( u \in W^1_0 L_\Phi(\Omega) \) is a weak solution for problem (3.1) if
\[
\int_\Omega \phi(|\nabla u|) \nabla u \cdot \nabla v dx = \lambda \int_\Omega f_k(u) v dx, \quad v \in W^1_0 L_\Phi(\Omega).
\]

**Remark 3.1** When working with the \( \Delta_2 \) condition in both \( \Phi \) and \( \tilde{\Phi} \), the energy functional \( I_k \) is \( C^1 \) (which is easy to prove), and so, every critical point of \( I_k \) satisfies the Euler equation above. Without \( \Delta_2 \) condition, the lack of differentiability is a problem which we have to overcome in order to show that the minimum of \( I_k \) satisfies the above Euler equation.

## 4 Technical Lemmas

The result below is crucial in this work and it was proved originally by Hess [13] and then extended by Loc & Schmitt in [16] to more general Sobolev spaces. In the present paper due to the non-reflexivity of the Orlicz-Sobolev space \( W^1_0 L_\Phi(\Omega) \) it was necessary to employ a version of Stampacchia’s theorem for that space. See Proposition 6.1 in the Appendix.

Consider the problem
\[
\begin{align*}
-\Delta_\Phi u &= g(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
(4.1)
The result is:

**Lemma 4.1** Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( g(s) \geq 0 \) for \( s \in (-\infty, 0) \) and assume that there is some \( s_0 \geq 0 \) such that \( g(s) \leq 0 \) for \( s \geq s_0 \). Let \( u \in W^1_0 L_\Phi(\Omega) \) be a weak solution for the problem (4.1). Then \( 0 \leq u \leq s_0 \) a.e. in \( \Omega \).

**Proof.** Let \( \Omega_0 = \{ x \in \Omega : u(x) < 0 \} \). Using Proposition 6.1 in the Appendix, with the function \( \min\{t, 0\} \), we can take \( u^- \) as a test function.
As a consequence we know that
\[ \int_{\Omega_0} \phi(|\nabla u|)|\nabla u|^2 dx = \int_{\Omega_0} g(u) u dx. \]

In particular, the last assertion says that
\[ \int_{\Omega_0} \phi(|\nabla u|)|\nabla u|^2 dx \leq 0, \]
which implies that $\Omega_0$ has zero measure. Again, define $\Omega_{s_0} = \{ x \in \Omega : u(x) > s_0 \}$. Using Proposition 6.1 with the function $\max\{t - s_0, 0\}$, we take $(u - s_0)^+$ as a test function. As a byproduct we get
\[ \int_{\Omega_{s_0}} \phi(|\nabla u|)|\nabla u|^2 dx = \int_{\Omega_{s_0}} g(u)(u - s_0) dx, \]
showing that
\[ \int_{\Omega_{s_0}} \phi(|\nabla u|)|\nabla u|^2 dx \leq 0. \]
Therefore the set $\Omega_{s_0}$ has zero measure.

The result below is crucial. In the case that the $\Delta_2$ condition holds its proof is rather straightforward. In the setting of the present paper it is much more difficult. We shall detail it by using arguments employed in [9] and [17].

**Lemma 4.2** Let $\lambda > 0$. Then there is $v_k \equiv v_k(\lambda) \in W^{1}_0 L_\Phi(\Omega)$ such that
\[ I_k(\lambda, v_k) = \min_{u \in W^{1}_0 L_\Phi(\Omega)} I_k(\lambda, u). \]

**Proof.** It is enough to show that $I_k(\lambda, \cdot)$ is both coercive and weak* sequentially lower semicontinuous, ($w^*, s.l.s.c.$ for short).

In order to show the coercivity property for $I_k(\lambda, \cdot)$ take $u \in W^{1}_0 L_\Phi(\Omega)$ such that $\|u\| \geq 1 + \epsilon$ with $\epsilon > 0$. Once $\Phi$ is convex and satisfies $\Phi(0) = 0$ we have that
\[ \frac{1 + \epsilon}{\|u\|} \int_{\Omega} \Phi(|\nabla u|) dx \geq \int_{\Omega} \Phi \left( \frac{(1 + \epsilon)|\nabla u|}{\|u\|} \right) dx > 1. \]
In the last inequality it was used the definition for Luxemburg norm. Here we point out that $f_k$ is a bounded function for any $k = 1, 2, \ldots, m - 1$. The coerciveness follows by a straightforward argument.
In order to show that $I_k(\lambda, \cdot)$ is $w^*\text{s.l.s.c.}$, we first prove that 

$$u \in W_0^1 L(\Omega) \mapsto \int_{\Omega} \Phi(|\nabla u|)dx$$

is $w^*\text{s.l.s.c.}$ Indeed, it follows by Young’s inequality (2.2) that

$$\int_{\Omega} \Phi(|\nabla u|)dx = \sup \left\{ \int_{\Omega} |\nabla u|wdx - \int_{\Omega} \tilde{\Phi}(w)dx \mid w \in \tilde{E}(\Omega) \right\}.$$

Assume that $u_n \overset{w^*}{\rightharpoonup} u$. Given $\epsilon > 0$ it follows from the previous identity that there is $w \in \tilde{E}(\Omega)$ such that

$$\int_{\Omega} \Phi(|\nabla u_n|)dx \geq \int_{\Omega} |\nabla u_n|wdx - \int_{\Omega} \tilde{\Phi}(w)dx - \epsilon.$$

Due the fact that $|\nabla u|, |\nabla u_n| \geq 0$, we may assume without loss of generality that $w \geq 0$. As a consequence

$$\int_{\Omega} \Phi(|\nabla u_n|)dx - \int_{\Omega} \Phi(|\nabla u|)dx \geq \int_{\Omega} |\nabla u_n|wdx - \int_{\Omega} |\nabla u|wdx - \epsilon \geq \int_{\Omega} |\nabla u_n w|dx - \int_{\Omega} |\nabla uw|dx - (4.2)$$

Since

$$\int_{\Omega} \frac{\partial u_n}{\partial x_i}wdx \rightarrow \int_{\Omega} \frac{\partial u}{\partial x_i}wdx, \quad v \in \tilde{E}(\Omega),$$

and $\tilde{E}(\Omega)L^\infty(\Omega) = E(\tilde{\Phi})$ (see [18]), we have that $\nabla u_n w \rightarrow \nabla uw$ for $\phi(\|L^1(\Omega), \|L^\infty(\Omega)).$ Hence, by the weak lower semicontinuity of norms, we conclude that

$$\int_{\Omega} |\nabla u|wdx \leq \lim inf \int_{\Omega} |\nabla u_n|wdx.$$

This inequality and (4.2) imply that

$$\lim inf \int_{\Omega} \Phi(|\nabla u_n|)dx \geq \int_{\Omega} \Phi(|\nabla u|)dx - \epsilon.$$

Since $\epsilon$ was taken arbitrarily, we obtain that

$$u \mapsto \int_{\Omega} \Phi(|\nabla u|)dx, u \in W_0^1 L(\Omega)$$
is a $w^*,s.l.s.c.$ function. Now, we shall prove that

$$\int_{\Omega} F(u_n) dx \to \int_{\Omega} F(u) dx$$

First of all, using the compact embedding $W_0^1 L_\Phi(\Omega) \overset{cpt}{\hookrightarrow} L_\Psi(\Omega)$ for $\Psi \ll \Phi^*$ and the continuous embedding $L_\Psi(\Omega) \hookrightarrow L^1(\Omega)$ we conclude that

$$W_0^1 L_\Phi(\Omega) \overset{cpt}{\hookrightarrow} L^1(\Omega).$$

Consequently we can take a function $h \in L^1(\Omega)$ such that

$$u_n \to u \text{ and } |u_n| \leq h \text{ a.e. in } \Omega.$$ 

Using the dominated convergence theorem we have

$$\int_{\Omega} F_k(u_n) dx \to \int_{\Omega} F_k(u) dx.$$

It follows from the previous arguments that

$$I_k(\lambda, u) \leq \lim \inf I_k(\lambda, u_n).$$

As a consequence, there is a minimum $v_k \equiv v_k(\lambda)$ of $I_k(\lambda, \cdot)$. \hfill \Box

Now we shall prove that $v_k$ is in fact a weak solution of problem (3.1). In order to do that, we will show that the functional $I_k$ is differentiable at its minimum point, found in Lemma 4.2. We first give some auxiliary results and definitions which can be found in [17].

**Definition 4.1** Let

$$\operatorname{dom}(\phi(t)t) = \{u \in L_\Phi(\Omega) : \phi(|u|)|u| \in L_{\tilde{\Phi}}(\Omega)\}.$$ 

For the next result we infer the reader to [17] Lemma 4.1.

**Lemma 4.3** For any $\epsilon \in (0,1]$ we have

(i) $(1 - \epsilon)L_\Phi(\Omega) \subset \operatorname{dom}(\phi(t)t)$,

(ii) $(1 - \epsilon)L_\Phi(\Omega) + E_\Phi(\Omega) \subset (1 - \epsilon/2)L_\Phi(\Omega) \subset \operatorname{dom}(\phi(t)t)$. 

Proof. At first we show (i): take $u \in L(\Omega)$ and $\varepsilon \in (0, 1)$. By the Young inequality (2.2), we easily see that (recall that by taking $s = t\phi(t)$ we obtain the equality in the Young inequality) $\tilde{\Phi}(t\phi(t)) \leq t(\phi(t))$. We also see that (due the fact that $t\phi(t)$ is increasing) that $t(\phi(t)) \leq \Phi(2t)$.

Therefore

$$\frac{\varepsilon}{1 - \varepsilon} \tilde{\Phi}((1 - \varepsilon)u\phi((1 - \varepsilon)u)) \leq \varepsilon(1 - \varepsilon)u\phi((1 - \varepsilon)u). \quad (4.3)$$

Now due the fact that $t\phi(t)$ is increasing, we obtain that

$$\varepsilon(1 - \varepsilon)u\phi((1 - \varepsilon)u) \leq \int_{(1 - \varepsilon)u}^{u} s\phi(s)ds \leq \Phi(u). \quad (4.4)$$

Using (4.3) and (4.4) together, we conclude the proof of item i).

For the item ii) we put $v \in E_\Phi(\Omega)$. Note that

$$\frac{v}{1 - \varepsilon/2} = \frac{2}{\varepsilon} \left(1 - \frac{1 - \varepsilon}{1 - \varepsilon/2}\right) v.$$  

Therefore, using the convexity of $\Phi$, we obtain that

$$\Phi \left(\frac{1}{1 - \varepsilon/2}((1 - \varepsilon)u + v)\right) \leq \frac{1 - \varepsilon}{1 - \varepsilon/2} \Phi(u) + \left(1 - \frac{1 - \varepsilon}{1 - \varepsilon/2}\right) \Phi \left(\frac{2v}{\varepsilon}\right). \quad (4.5)$$

This assertion implies that

$$(1 - \varepsilon)L_\Phi(\Omega) + E_\Phi(\Omega) \subset (1 - \varepsilon/2)L_\Phi(\Omega).$$

Now using item i) the proof for item ii) is now finished.

Now we shall prove, with some adapted ideas from Tienari [17], that $|\nabla v_k| \in \text{dom}(\phi(t))$.

Lemma 4.4 The function $|\nabla v_k|$ satisfies $\phi(|\nabla v_k|)|\nabla v_k| \in L_\tilde{\Phi}(\Omega)$.

Proof. Consider the following sequence $f_k(\varepsilon) = I_k(\lambda, (1 - \varepsilon)v_k)$ for any $\varepsilon \in [0, 1]$. Recall that $v_k$ is the minimizer of $I_k(\lambda, \cdot)$. This ensures that $f_k(\varepsilon) < \infty$ is finite for any $\varepsilon \in [0, 1]$. Moreover, we observe that $|\nabla v_k| \in L_\Phi(\Omega)$. Therefore, using the item i) of Lemma 4.3 we already conclude that $f_k$ is differentiable in the interval $(0, 1]$. Furthermore, we know that

$$f'_k(\varepsilon) = -\int_\Omega \phi((1 - \varepsilon)|\nabla v_k|)(1 - \varepsilon)|\nabla v_k|^2dx + \lambda \int_\Omega f_k((1 - \varepsilon)v_k)v_kdx. \quad (4.5)$$
Suppose, on the contrary that \( \phi(|\nabla v_k|)|\nabla v_k| \notin \mathcal{L} \Phi(\Omega) \). Recall from Young inequality (2.2) that
\[
\int_{\Omega} \phi(|\nabla v_k|)|\nabla v_k|^2 dx = \int_{\Omega} \Phi(|\nabla v_k|) dx + \int_{\Omega} \tilde{\Phi}(\phi|\nabla v_k|)|\nabla v_k| dx = \infty.
\]
From the last equality and (4.5), we mention that \( f'(\epsilon) \to -\infty \) if \( \epsilon \to 0 \). Hence, there exists \( \epsilon_0 \in (0,1] \) such that \( f(\epsilon_0) < f(0) \) which is an contradiction because of \( v_\tau \) is a minimizer for \( I_k(\lambda, \cdot) \).

With the help of the last Lemma, we can now prove that the minimum we found in Lemma 4.2 does satisfies the Euler equation:

**Lemma 4.5** For \( v_k \) as defined in lemma (4.2) we have
\[
\int_{\Omega} \phi(|\nabla v_k|)|\nabla v_k| \nabla w dx = \lambda \int_{\Omega} f(v_k) w dx, \ w \in W^1_0 L_\Phi(\Omega).
\]

**Proof.** Let \( v \in W^1_0 E_\Phi(\Omega) \) and define
\[
f_v(\epsilon) = I_k(\lambda, v_\epsilon), \ \epsilon \in [0,1],
\]
where \( v_\epsilon = 1/(1-\epsilon/2)[(1-\epsilon)v_k + \epsilon v] \). The item (ii) of lemma 4.3 and Young inequality (2.2) imply that \( f_v(\epsilon) < \infty \) for all \( \epsilon \in [0,1] \). Once \( v_k \) is a minimizer of \( I_k(\lambda, \cdot) \), we have that
\[
0 \leq \frac{f_v(\epsilon) - f_v(0)}{\epsilon}, \ v \in W^1_0 E_\Phi(\Omega).
\]

Note that for \( \epsilon < 2/3 \), the monotonicity of \( \phi(t) t \) and the triangle inequality, the following is true
\[
\left| \frac{\Phi(|\nabla v_\epsilon|) - \Phi(|\nabla v_k|)}{\epsilon} \right| \leq \phi(|\nabla v_\epsilon|) |\nabla v_\epsilon| + \phi(|\nabla u_k|) |\nabla u_k| \frac{|\nabla v_k - \nabla v_k|}{\epsilon},
\]
\[
\leq (2\phi(|\nabla u_k||\nabla v_k| + \phi(|\nabla v||\nabla v_k| + |\nabla v|).)
\]

As \( |\nabla v| \in \text{dom}(\phi(t)t) \) for every \( v \in W^1_0 E_\Phi(\Omega) \) (this is true because of the inequality \( \tilde{\Phi}(\phi(t)t) \leq \Phi(2t) \)), we conclude that the right hand side of the last inequality, is a function in \( L^1(\Omega) \). Now using the fact that
\[
\Phi(|\nabla v_\epsilon|) - \Phi(|\nabla v_k|) \to \phi(|\nabla v_k|)|\nabla v_k| (v - \nabla v_k/2), \ a.e., x \in \Omega, \ when \ \epsilon \to 0,
\]
we infer from inequality (4.6) and Lebesgue theorem that
\[
0 \leq \int_{\Omega} \phi(|\nabla v_k|) |\nabla v_k| (v - \nabla v_k/2) dx - \lambda \int_{\Omega} f(v_k)(v - v_k/2) dx, \ v \in W^1_0 E_\Phi(\Omega).
\]
As a consequence
\[
\int_{\Omega} \phi(|\nabla v_k|) \nabla v_k \nabla v = \lambda \int_{\Omega} f(v_k)vd, \quad lv \in W^1_0 E\Phi(\Omega).
\]

At this moment, using the weak star density of \( W^1_0 E\Phi(\Omega) \) in \( W^1_0 L\Phi(\Omega) \), we mention that
\[
\int_{\Omega} \phi(|\nabla v_k|) \nabla v_k \nabla v dx = \lambda \int_{\Omega} f(v_k)vd, \quad v \in W^1_0 L\Phi(\Omega).
\]
Hence \( v_k \) is a weak solution to the problem \( (1.1) \). This finishes the proof for this lemma.

To continue, we shall prove that \( I_k(\lambda, \cdot) \) admits at least one weak solution \( v_k \) satisfying \( a_k < \|v_k\|_{\infty} \leq a_{k+1}, k = 1, \ldots, m - 1 \). This is crucial in order to get our main result.

**Lemma 4.6** There is \( \lambda_k > 0 \) such that
\[
a_{k-1} < \|v_k\|_{\infty} \leq a_k
\]
for each minimum \( v_k \equiv v_k(\lambda) \) of \( I_k(\lambda, \cdot) \) with \( \lambda > \lambda_k \).

**Proof.** The proof is similar to those given in [13] and [16] and the idea is the following: Take \( \delta > 0 \) and consider the open set
\[
\Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \delta\}.
\]

Set
\[
\bar{\alpha}_k := F(a_k) - \max\{F(s) \mid 0 \leq s \leq a_{k-1}\}
\]
and note that by \( (f_3) \), \( \bar{\alpha}_k > 0 \). Choose \( w_\delta \in C^\infty_0(\Omega) \) such that
\[
0 \leq w_\delta \leq a_k \quad \text{and} \quad w_\delta(x) = a_k, \quad x \in \Omega \setminus \Omega_\delta.
\]
Writing \( \Omega = \Omega_\delta \cup (\Omega \setminus \Omega_\delta) \) and setting \( C_k = \max\{|F(s)| \mid 0 \leq s \leq a_k\} \) we get to,
\[
\int_{\Omega} F(w_\delta)dx \geq \int_{\Omega} F(a_k)dx - 2C_k|\Omega_\delta|.
\]

Let \( u \in W^1_0 L\Phi(\Omega) \) such that \( 0 \leq u \leq a_{k-1} \). By the inequality above we have
\[
\int_{\Omega} F(w_\delta)dx - \int_{\Omega} F(u)dx \geq \bar{\alpha}_k|\Omega| - 2C_k|\Omega_\delta|.
\]
Since $|\Omega_\delta| \to 0$ as $\delta \to 0$ there is $\delta > 0$ such that
\[ \eta_k := \tilde{\alpha}_k|\Omega| - 2C_k|\Omega_\delta| > 0. \]

Set $w = w_\delta$ and pick $u \in W^1_0 L_\Phi(\Omega)$ with $0 \leq u \leq a_{k-1}$. Choosing $\lambda_k > 0$ large enough, taking $\lambda \geq \lambda_k$ and making use of the expressions of $I_k(\lambda, w_\delta), I_{k-1}(\lambda, u)$ and the inequality just above we infer that
\[ I_k(\lambda, w_\delta) - I_{k-1}(\lambda, u) \leq \int_\Omega \Phi(|\nabla w_\delta|)dx - \lambda \eta_k < 0 \quad (4.7) \]
and hence
\[ I_k(\lambda, w_\delta) < I_{k-1}(\lambda, u) \quad \text{for} \quad \lambda \geq \lambda_k. \quad (4.8) \]

To finish, assume, on the contrary, that there is a minimum $v_k(\lambda)$ of $I_k(\lambda, \cdot)$ such that $v_k(\lambda) \leq a_{k-1}$. It follows by (4.8) and lemma 4.1 that
\[ I_k(\lambda, w_\delta) < I_{k-1}(\lambda, v_k(\lambda)). \]

On the other hand, because $I_{k-1}(\lambda, v_k(\lambda)) = I_k(\lambda, v_k(\lambda))$ and since $v_k(\lambda)$ is a minimum of $I_k(\lambda, \cdot)$ we have
\[ I_{k-1}(\lambda, v_k(\lambda)) = I_k(\lambda, v_k(\lambda)) \leq I_k(\lambda, w_\delta). \]

The inequalities just above lead to a contradiction.

5 Proof of Theorem 1.1

The proof is based on Loc & Schmitt [16]. However, we will get into details taking into account the Orlicz-Sobolev spaces framework. In this sense we will make use of a general result on lower and upper solutions by Le [14, theorem 3.2].

For the proof of (ii) we will need the lemma below. In order to state it, take an open ball $B$ centered at 0 with radius $R$ containing $\Omega$. Consider the functions $\alpha, \beta : \overline{B} \to \mathbb{R}$ defined as follows:
\[ \alpha(x) = \begin{cases} u(x), & x \in \overline{\Omega} \\ 0, & x \in \overline{B} \setminus \Omega, \end{cases} \quad \beta(x) = a_{k+1}, \ x \in \overline{B}. \]

Since $u \in W^1_0 L_\Phi(\Omega)$, we know by the Lemma (6.1) in the Appendix, that the extension by zero $\overline{u}$ of $u$ belongs to $W^1_0 L_\Phi(\mathbb{R}^N)$. Hence, due to the fact that the extension by zero of the function $\alpha$ to $\mathbb{R}^N$, coincides with $\overline{u}$, we also conclude by using Lemma (6.1) (appendix) again that $\alpha \in W^1_0 L_\Phi(B)$. 
Lemma 5.1 The functions $\beta$ and $\alpha$ are respectively upper and lower solutions to the elliptic problem

\[
\begin{cases}
-\Delta \Phi u = \lambda f(u) \text{ in } B, \\
u \in W^1_0 L_\Phi(B).
\end{cases}
\]  

(5.1)

Proof. That $\beta$ is an upper solution is immediately. Now we shall prove that $\alpha$ is a subsolution for the problem (5.1). Let $v_n(x) = n \min\{u(x), 1/n\}$ be a fixed function. Note that $u_n(x) \to 1$ for each $x$ where $u(x) \neq 0$ and $u(x) \to 0$ for each $x$ where $u(x) = 0$. Moreover, on the set $\{x \in \Omega : u(x) = 0\}$, we have that $\nabla u(x) = 0$ a.e.. Therefore, for each $v \in W^1_0 L_\Phi(\Omega)$ with $v \geq 0$, the following inequalities are true

\[
\int_B \phi(|\nabla \alpha|) \nabla \alpha \nabla v dx = \int_\Omega \phi(|\nabla u|) \nabla u \nabla v dx = \lim_{n \to \infty} \int_\Omega v_n \phi(|\nabla u|) \nabla u \nabla v dx = \lim_{n \to \infty} \left( \int_\Omega \phi(|\nabla u|) \nabla u \nabla (v_n v) dx - \int_\Omega v \phi(|\nabla u|) \nabla u \nabla v_n dx \right) \leq \lim_{n \to \infty} \int_\Omega f(u) v_n dx \leq \int_B f(\alpha) v dx.
\]

This proves the lemma.

Proof of (i) of theorem 1.1 Take $\lambda > 0$. By lemma 4.2, for each $k = 2, \cdots, m$ there is a minimum $v_k \equiv v_k(\lambda)$ of $I_k(\lambda)$, which is actually a weak solution of problem (3.1). By lemma 4.1, we know that $0 \leq v_k \leq a_k$ a.e. in $\Omega$.

Now we mention that, using Lemma 4.6, there is $\lambda \geq \max_{2 \leq k \leq m} \{\lambda_k\}$ such that $v_2, \cdots, v_m$ are solutions of problem (1.1) for any $\lambda > \lambda$. Moreover, these solutions satisfy

\[a_1 < \|v_2\|_{\infty} \leq a_2 < \|v_3\|_{\infty} \leq \cdots \leq a_{m-1} < \|v_m\|_{\infty} \leq a_m\]

Now we take $u_{k-1} \equiv v_k(\lambda), \ k = 2, \cdots, m$. This ends the proof of the first part of theorem 1.1.

Proof of (ii) of theorem 1.1 We distinguish between two cases.
Case 1 $f(0) > 0$.

This case is more difficult. In order to address it we state and prove the lemma below.

**Lemma 5.2** Assume $(\phi_1) - (\phi_4)$, $(f_1) - (f_2)$ and $f(0) > 0$. If $u$ is a non-negative weak solution of (1.1) such that $a_{k-1} < \|u\|_{\infty} \leq a_k$ then

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0.$$ 

**Proof** Let us consider the case that $k = 2$, the other cases may be treated in a similar manner. Take the lower and upper solutions respectively $\alpha$ and $a_2$ for (5.1).

Now, applying Theorems 3.2, 4.1 and 5.1 of [14], we find a maximal solution say $u$ of (5.1) such that $\alpha(x) \leq u(x) \leq a_2$ for $x \in B$.

The verification of the **Claim** below follows as in [16].

**Claim 5.1** $\overline{u}$ is radially symmetric, i.e. $\overline{u}(x_1) = \overline{u}(x_2)$, $x_i \in B$, $|x_1| = |x_2|$. Now we set $u(r) = \overline{u}(x)$ where $r = |x|$ and $x \in B$;

and because the extension by zero outside of $\Omega$, of the function $\overline{u}$, is an absolutely continuous function, with respect to a.e. segment of line in the direction of a vector $\eta \in \{y \in \mathbb{R}^N : \|y\| = 1\}$, we conclude that $u$ is continuous in $(0, R]$ and

$$u \in W^{1,1}(0, R), \ u(1) = 0.$$ 

Let $r \in (0, R)$ and pick $\epsilon > 0$ small such that $r + \epsilon < R$. Note that

$$\int_B \phi(|\nabla \overline{u}|) \nabla \overline{u} \nabla vdx = \lambda \int_B f(\overline{u})vdx, \ v \in W_0^{1,\Phi}(B). \quad (5.2)$$ 

Consider the radially symmetric cut-off function $v_{r,\epsilon}(x) = v_{r,\epsilon}(r)$, where

$$v_{r,\epsilon}(t) := \begin{cases} 
1 & \text{if } 0 \leq t \leq r, \\
linear & \text{if } r \leq t \leq r + \epsilon, \\
0 & \text{if } r + \epsilon \leq t \leq R.
\end{cases}$$

and notice that $v_{r,\epsilon} \in W_0^{1,\Phi}(B) \cap \text{Lip}(\overline{B})$. Setting $v = v_{r,\epsilon}$ in (5.2) and using the radial symmetry we get to

$$\frac{-1}{\epsilon} \int_r^{r+\epsilon} t^{N-1} \phi(|u'|)u' \ dt = \int_0^r t^{N-1} \lambda f(u) \ dt + \int_r^{r+\epsilon} t^{N-1} \lambda f(u)vdt.$$
Once $t^{N-1}\phi(|u'|)u' \in L^1(0, R)$, we conclude by letting $\epsilon \to 0$, that for a.e. $r \in (0, R)$

$$- r^{N-1}\phi(|u(r)|)u'(r) = \int_0^r \lambda f(u)t^{N-1} \, dt, \quad 0 < r < R. \quad (5.3)$$

From equation (5.3) and by the boundedness of $u$, we obtain that $u'$ is continuous and $u'(0) = 0$. Set

$$\|u\|_{\infty} = \max\{u(r) \mid r \in [0, R]\};$$

and choose numbers $r_0, r_1 \in [0, R)$ with $r_1 \in (r_0, R)$ such that

$$u(r_0) = \|u\|_{\infty} \text{ and } u(r_1) = a_1.$$

Note that $u(r_0) > u(r_1)$ and $0 \leq r_0 < r_1 < R$.

**Claim 5.2** $\|u\|_{\infty} > b_1$.

Indeed, assume on the contrary that, $u(r_0) \leq b_1$. Take $\delta > 0$ small such that

$$a_1 < u(r) \leq u(r_0), \quad r_0 \leq r \leq r_0 + \delta.$$

We have by (5.3)

$$- r_0^{N-1}\phi(|u'(r_0)|)u'(r_0) = \int_0^{r_0} \lambda f(u)t^{N-1} \, dt, \quad (5.4)$$

$$- r^{N-1}\phi(|u'(r)|)u'(r) = \int_0^r \lambda f(u)t^{N-1} \, dt. \quad (5.5)$$

Subtracting (5.5) from (5.4) and recalling that $u'(r_0) = 0$ one obtains,

$$- r^{N-1}\phi(|u'(r)|)u'(r) = \int_{r_0}^r \lambda f(u)t^{N-1} \, dt, \quad r_0 \leq r \leq r_0 + \delta.$$

Since $f \leq 0$ on $[a_1, b_1]$,

$$r^{N-1}\phi(|u'(r)|)u'(r) \geq 0, \quad r_0 \leq r \leq r_0 + \delta.$$

It follows that $u'(r) \geq 0$ for $r_0 \leq r \leq r_0 + \delta$. But, since $u(r_0)$ is a global maximum on $[0, R]$, it follows that $u' = 0$ on $[r_0, r_0 + \delta]$. By a continuation argument we get $u' = 0$ on $[r_0, r_1)$ so that $u = \|u\|_{\infty}$ on $[r_0, r_1]$, contradicting $u(r_0) > a_1$. As a consequence, $\|u\|_{\infty} > b_1$, proving Claim 5.2.
Claim 5.3 \( u \in C^2(\mathcal{O}) \) where \( \mathcal{O} := \{ r \in (0, R) \mid u'(r) \neq 0 \} \).

Of course \( \mathcal{O} \) is an open set. Motivated by the left hand side of (5.3) consider

\[
G(z) = \phi(z)z, \quad z \in \mathbb{R},
\]

where \( z \) is set to play the role of \( u' \). Recall that \( G \) is odd, \( G'(z) = (\phi'(z)z)' > 0 \) for \( z > 0 \)

and

\[
G(z) = \phi(|u'(r)|)u'(r) = -\frac{1}{r^{N-1}} \int_0^r \lambda f(u)t^{N-1} \, dt.
\]

Since \( \phi(z)z \in C^1 \) and \( (\phi(z)z)' \neq 0 \) for \( z \neq 0 \), we get by applying the Inverse Function Theorem in \( \mathcal{O} \) that \( z = z(r, u) \) is a \( C^1 \)-function of \( r \). Since \( z = u' \), the claim is proved.

Claim 5.4 \( \int_{a_1}^{||u||_{\infty}} f(s)ds > 0 \).

Differentiating in (5.3) and multiplying by \( u' \) we get

\[
(t^{N-1}\phi(|u'(t)|)u'(t))'u'(t) = -\lambda f(u(t))u'(t)t^{N-1},
\]

and hence

\[
\frac{(N-1)}{t} \phi(|u'|)(u')^2 + (\phi(|u'|)u')'u' = -\lambda f(u)u'.
\]

Integrating from \( r_0 \) to \( r_1 \) we have

\[
-\left[ \int_{r_0}^{r_1} \frac{(N-1)}{t} \phi(|u'|)(u')^2 \, dt + \int_{r_0}^{r_1} [\phi(|u'|)u']'u' \, dt \right] = \int_{r_0}^{r_1} \lambda f(u)u' \, dt.
\] (5.6)

Making the change of variables \( s = u'(t) \) in the second and third integrals in (5.6) and applying the arguments in [16] leads to Claim 5.4.

Since \( f \geq 0 \) on \((b_1, a_2)\) and \( ||u||_{\infty} > b_1 \) it follows that

\[
\int_{a_1}^{a_2} f(s)ds > 0,
\]

ending the proof of lemma 5.2.

Case 2 \( f(0) = 0 \).

This case is handled as in [16]. The Theorem 1.1 is now proved.
6 Appendix

In this Section we state and prove a version of the Stampacchia Theorem (generalized Chain Rule) for the case of nonreflexive Orlicz-Sobolev spaces.

Proposition 6.1 Let \( g : \mathbb{R} \to \mathbb{R} \) be a Lipschitz continuous function such that \( \|g'\|_\infty < M \) and \( g(0) = 0 \). Let \( u \in W^1_0L_\Phi(\Omega) \). Then \( g(u) \in W^1_0L_\Phi(\Omega) \) and
\[
\frac{\partial g(u)}{\partial x_i} = g'(u) \frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega.
\]

At first we recall, for the reader’s convenience, the definition and basic properties of the trace on \( \partial \Omega \) of an element of \( W^1L_\Phi(\Omega) \). We refer the reader to Gossez [12].

Let \( \gamma : C^\infty(\Omega) \to C(\partial \Omega) \) be defined by the linear map \( \gamma(u) = u|_{\partial \Omega} \). Then \( \gamma \) is continuous with respect to the topologies
\[
\sigma\left( \prod L_\Phi(\Omega), \prod E_{\tilde{\Phi}}(\Omega) \right) \text{ and } \sigma(L_\Phi(\partial \Omega), E_{\tilde{\Phi}}(\partial \Omega)).
\]

Using the facts that \( C^\infty(\overline{\Omega}) \) is dense in \( W^1L_\Phi(\Omega) \) with respect to the topology \( \sigma(\prod L_\Phi(\Omega), \prod E_{\tilde{\Phi}}(\Omega)) \) and \( C(\partial \Omega) \) is dense in \( L_\Phi(\partial \Omega) \) with respect to the topology \( \sigma(L_\Phi(\partial \Omega), E_{\tilde{\Phi}}(\partial \Omega)) \), \( \gamma \) admits an only continuous extension to a linear map namely \( \gamma : W^1_0L_\Phi(\Omega) \to L_\Phi(\partial \Omega) \).
It can be shown that
\[
W^1_0L_\Phi(\Omega) = \{ u \in W^1L_\Phi(\Omega) \mid \gamma(u) = 0 \}.
\]

Let \( u : \Omega \to \mathbb{R} \). We define \( \overline{u} : \mathbb{R}^N \to \mathbb{R} \) by
\[
\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c. \end{cases}
\]

Let \( \nu = (\nu_1, \cdots, \nu_N) \) be the outward unit normal vector field of \( \partial \Omega \). The Green’s formula reads as,
\[
\int_\Omega u \frac{\partial v}{\partial x_i} \, dx + \int_\Omega v \frac{\partial u}{\partial x_i} \, dx = \int_{\partial \Omega} \gamma(u)\gamma(v)\nu_i \, dx, \quad i = 1, \cdots, N, \tag{6.1}
\]
where \( u \in W^1L_\Phi(\Omega), \, v \in W^1L_{\tilde{\Phi}}(\Omega) \).

We will give the proof of the result below:
Lemma 6.1

\[ W_0^1 L_\Phi(\Omega) = \{ u \in W^1 L_\Phi(\Omega) \mid \pi \in W^1 L_\Phi(\mathbb{R}^N) \}. \]

**Proof** Set

\[ E = \{ u \in W^1 L_\Phi(\Omega) \mid \pi \in W^1 L_\Phi(\mathbb{R}^N) \}. \]

Take \( u \in W^1_0 L_\Phi(\Omega) \) and let \( g_i : \mathbb{R}^N \to \mathbb{R} \) be defined by

\[ g_i(x) = \begin{cases} (\partial u/\partial x_i)(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c. \end{cases} \]

Computing derivatives in the distribution sense and using the generalized Green’s formula we have

\[
\int_{\mathbb{R}^N} u \frac{\partial \varphi}{\partial x_i} dx = \int_{\Omega} u \frac{\partial \varphi}{\partial x_i} dx,
\]

\[
= -\int_{\Omega} \frac{\partial u}{\partial x_i} \varphi dx,
\]

\[
= -\int_{\mathbb{R}^N} g_i \varphi dx, \quad \varphi \in C^\infty_0(\mathbb{R}^N).
\]

Using the fact that \( \pi, g_i \in L_\Phi(\mathbb{R}^N) \), we conclude that \( u \in E \). On the other hand, take \( u \in E \). We claim that \( \gamma(u) = 0 \). Indeed, by the definition of weak derivative and Green’s identity it follows that for all \( v \in W^1 \Phi(\Omega) \) and \( i = 1, \cdots, N \), we have

\[
\int_{\partial\Omega} \gamma(u) \gamma(v) \nu_i dx = 0.
\]

and so

\[
\int_{\partial\Omega} \gamma(u) \nu_i w dx = 0, \quad w \in C(\partial\Omega).
\]

Because \( \gamma(u) \nu_i \in L_\Phi(\partial\Omega) \) and \( C(\partial\Omega) \) is dense (with respect to the norm topology) in \( E^\Phi_\Phi(\partial\Omega) \), we conclude that

\[
\int_{\partial\Omega} \gamma(u) \nu_i w dx = 0, \quad w \in E^\Phi_\Phi(\partial\Omega).
\]

By taking \( w(x) = 1 \) if \( \gamma(u) \nu_i \) is positive, \( w(x) = 0 \) if \( \gamma(u) \nu_i \) is zero and \( w(x) = -1 \) if \( \gamma(u) \nu_i \) is negative, we conclude that \( w \in L^\infty(\partial\Omega) \subset E^\Phi_\Phi(\partial\Omega) \) and

\[
\int_{\partial\Omega} |\gamma(u) \nu_i| d\sigma = 0.
\]
So, $\gamma(u) = 0$ a.e. on the support of $\nu_i$ and because

$$\bigcup_{i=1}^{N} \text{supp}(\nu_i) = \partial \Omega,$$

we conclude that $\gamma(u) = 0$ and hence $u \in W_0^1 L_\Phi(\Omega)$. This finishes the proof. \hfill \Box

**Proof of Proposition 6.1.** Indeed, by lemma 6.1, $\overline{u} \in W^1 L_\Phi(\mathbb{R}^N)$. Reminding that

$$W^1 L_\Phi(\Omega) \hookrightarrow W^{1,1}(\Omega), \tag{6.2}$$

we have $\overline{u} \in W^{1,1}(\mathbb{R}^N)$ and by the Chain rule for $W^{1,1}$, (cf. Gilbarg & Trudinger [10]), we infer that $g(\overline{u}) \in W^{1,1}(\mathbb{R}^N)$ and in addition,

$$\int_{\Omega} g(u) \frac{\partial \varphi}{\partial x_i} dx = \int_{\Omega} g'(u) \frac{\partial u}{\partial x_i} \varphi dx, \varphi \in C_0^\infty(\Omega).$$

We will show next that $\overline{g(u)} \in W^1 L_\Phi(\mathbb{R}^N)$. Indeed, on one hand, we have that $g(\overline{u}) = g(u)$. On the other hand, setting

$$h_i(x) = \begin{cases} \frac{\partial g(u)}{\partial x_i}(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c, \end{cases}$$

we have $\overline{g(u)}, h_i \in L^\Phi(\mathbb{R}^N)$. Therefore, $\overline{g(u)} \in W^1 L_\Phi(\mathbb{R}^N)$. Now, using lemma 6.1 again, we obtain that $g(u) \in W_0^1 L_\Phi(\Omega)$ and

$$\frac{\partial g(u)}{\partial x_i} = g'(u) \frac{\partial u}{\partial x_i}, \text{ a.e. in } \Omega.$$

This concludes the proof of Proposition 6.1. \hfill \Box

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