Classification of multi-qubit mixed states: separability and distillability properties

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We give a complete, hierarchic classification for arbitrary multi-qubit mixed states based on the separability properties of certain partitions. We introduce a family of $N$-qubit states to which any arbitrary state can be depolarized. This family can be viewed as the generalization of Werner states to multi-qubit systems. We fully classify those states with respect to their separability and distillability properties. This provides sufficient conditions for nonseparability and distillability for arbitrary states.

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I. INTRODUCTION

Entanglement is one of the basic concepts of Quantum Mechanics and an important feature of most applications of Quantum Information. It arises when a state of a multiparticle quantum system cannot be prepared by acting on the particles individually, i.e. is non–separable. Despite the fact that we do not know yet how to classify and quantify entanglement in general, much progress was made in recent years. In particular, the concept of entanglement distillation or purification was introduced. This process, which is the creation of (few) maximally entangled states out of many not–maximally entangled ones, turned out to be one of the most important concepts in quantum information theory. When combined with teleportation, it allows to send quantum information over noisy channels and to convey secret information via quantum privacy amplification.

Particular important states of two qubits are the so called Werner states (WS), which are mixtures of a maximally entangled state, e.g. $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, with the totally depolarized state. These states are fully characterized by the fidelity $F$, which measures the overlap of the maximally entangled state $|\Phi^+\rangle$ with the WS. They play an essential role in the understanding of the entanglement and distillability properties of two qubit systems. On the one hand, it has been shown that WS are separable for $F \leq 1/2$ and non–separable (entangled) for $F > 1/2$. On the other hand, Bennett et al. showed that one can purify WS with arbitrary high fidelity out of many pairs with $F > 1/2$ by using local operations and classical communication. Furthermore, any arbitrary state can be depolarized to a WS without changing the fidelity $F$, which automatically provides a sufficient criterion for non–separability and distillability for arbitrary states.

The description of the entanglement and distillability properties of systems with more than two particles is still almost unexplored (see Refs. [11,12], however). In [3] some steps towards the understanding of 3–particle entanglement of mixed states were taken. In particular, a complete classification of arbitrary 3-qubit states was proposed and the distillability and separability properties of a family of states was obtained. In this paper we generalize the ideas introduced in [3] to multipartite quantum systems. We provide a complete classification of a family of states of $N$–qubit systems. These states are characterized in terms of $2^{N-1}$ parameters and play the role of WS in such systems, since any arbitrary state can be depolarized to this form. We fully analyze the separability and distillability properties of this family, thereby generalizing the purification procedure introduced in [3] to multipartite quantum systems. This automatically provides us - as in the bipartite case - with sufficient conditions for arbitrary multi–qubit states. Among other things, this allows us to give the necessary and sufficient separability and distillability conditions of mixtures of a maximally entangled state and the completely depolarized state. Furthermore we introduce a hierarchic classification of general $N$–qubit states with respect to their entanglement properties.

The paper is organized as follows. We start by briefly reviewing the present knowledge about distillability and entanglement of bipartite quantum systems in Sec. I. In the following we generalize the results of [3] to multi–qubit systems. We start out by giving a classification of arbitrary $N$–qubit systems in Sec. II. Then we introduce a family of states that can be obtained via depolarization from an arbitrary one in Sec. III. Here we also investigate the separability and distillability properties of this family. Sec. IV gives examples to illustrate the results obtained in the previous Sections. In particular, we analyze in detail the simplest cases of 3 and 4 qubit systems. In Sec. V we apply our results to the case where we have a maximally entangled state of $N$ qubits mixed with the totally depolarized state. Finally we conclude and summarize in Sec. VI.

II. BIPARTITE SYSTEMS AND PARTIAL TRANSPOSITION

Let us start out by briefly reviewing the separability and distillability properties of bipartite systems. A bipartite mixed state $\rho$ is called separable iff it can be prepared locally, i.e. it can be written as a convex combination of...
A state is called distillable iff one can create out of (infinitely) many copies of $\rho$ a maximally entangled state, e.g. $|\Phi^+\rangle$. In practice, it is difficult to decide whether a given state is separable or distillable respectively. As shown by Peres [7] and the Horodecki [8–10], the partial transposition of a density operator turns out to provide a simple, sufficient criterion for the classification of bipartite systems. Given an operator $X$ acting on $\mathcal{H}^{d_1} \otimes \mathcal{H}^{d_2}$, the partial transposition with respect to the first subsystem in the standard basis $\{|1\rangle, |2\rangle, \ldots, |d_1\rangle\}$, $X^{T_A}$, is defined as follows:

$$X^{T_A} = \sum_{i,j=1}^{d_1} \langle i|X|j\rangle \langle j|i\rangle.$$

Clearly, the partial transposition of the operator $X$ is basis dependent, but the eigenvalues are not. We say that a self adjoint operator has positive partial transposition ($X^{T_A} \geq 0$) - PPT - iff all eigenvalues of $X^{T_A}$ are non–negative. On the opposite, we say an operator has non–positive partial transposition (NPPT) iff at least one eigenvalue is negative. Sometimes NPPT is also called ”negative partial transposition” (NPT).

For bipartite two–level systems ($d_1 = d_2 = 2$) it was shown that positive partial transposition (PPT) is a necessary and sufficient condition for separability [14] while negative partial transposition (NPT) is a necessary and sufficient condition for distillability [10]. For higher dimensional systems, however, the partial transposition only provides necessary conditions for separability [10] and it seems that it provides only a necessary condition for distillability [14][17]. In $\mathcal{H}^2 \otimes \mathcal{H}^d (d \geq 2)$ systems we have that a sufficient condition for separability is that $\rho = \rho^{T_A}$ [18], while the negativity of the partial transposition already ensures distillability of those systems [14].

In the following we generalize the notion of separability and distillability to multi–qubit systems. It turns out that in order to characterize an important family of multi–qubit mixed states, it is useful to consider bipartite splits of multiparticle systems and their corresponding partial transpositions. Since a bipartite split of a multi–qubit system can be viewed as a state in $\mathcal{H}^{d_1} \otimes \mathcal{H}^{d_2}$, the partial transposition of the density operator $\rho$ is well defined in this case.

III. MULTI–QUBIT SYSTEMS:

We will give a classification of general $N$–qubit systems in terms of the separability properties of their partitions. In particular, we consider $k$–partite splits (that are partitions dividing a $N$–partite systems into $k \leq N$ parties), which gives rise to a whole hierarchy of classes.

A. Separability with respect to certain splits

Let us start by generalizing the notion of separability to the case of multiparticle systems. We consider $N$ parties, each holding a system with dimension $d_i$, i.e. $\mathcal{H} = \mathcal{H}^{d_1} \otimes \ldots \otimes \mathcal{H}^{d_N}$. We call $\rho$ fully separable if it can be written as a convex combination of (unnormalized) product states, i.e.

$$\rho = \sum_i |a_i\rangle_{\text{party}1} \langle a_i| \otimes |b_i\rangle_{\text{party}2} \langle b_i| \otimes \ldots \otimes |n_i\rangle_{\text{party}N} \langle n_i|.$$

In the following, we will consider a system of $N$ qubits, each hold by one of the parties $A_1, A_2, \ldots, A_N$. In this case, $d_1 = d_2 = \ldots d_N = 2$. Let us now consider a partition of the $N$–qubit system into $k \leq N$ sets, which we call a $k$–partite split of the system (see Fig. 1). That is we allow some of the parties to act together such that finally $k$ parties remain. As a special case, we have 2–partite splits which we will also call bipartite splits. A state $\rho$ is called $k$–separable with respect to this specific partition (or equivalently split) iff it is fully separable in the sense that we consider $\rho$ as a $k$–party system, i.e. as a state in $\mathcal{H} = \mathcal{H}^{d_1} \otimes \ldots \otimes \mathcal{H}^{d_k}$.

Considering all possible partitions (including all permutations) of the $N$–qubit system and determining the corresponding separability properties is sufficient to fully characterize the system in terms of its entanglement properties. However, the number of possible partitions grows rapidly with the number of parties involved (see Sec. IV) the information of one level (in particular level 2) already implies all properties at the other levels. Furthermore, there are connections between the different levels, which can be used to reduce the effort to determine the full entanglement properties of the system. However, we learned in the case of 3–qubits that these connections are sometimes not obvious or even counter–intuitive. For example we have that for a 3–qubit system separability with respect to all bipartite splits (i.e. partitions into two sets) is not sufficient to guarantee 3-separability (i.e. full separability when considering each system $A_1, A_2, A_3$ as a separate party) of the system [13][17].

B. Classification of arbitrary states $\rho$

In [13], the biseparability properties of the state $\rho$ were used to classify those states completely, and all together (apart from permutations among the parties) 5 distinct classes were found (see Sec. VA for details).
While splits of the system into two or three parties were sufficient to fully classify three qubit systems, it turns out that for $N$–qubit systems we need all possible partitions of the system for a complete classification.

1. Hierarchic classification

The basic idea of the hierarchic classification we propose here is to consider all possible $k$–partite splits of a $N$–partite system for all ($k \in \{N, N-1, \ldots, 2\}$) and determine for each split whether it is $k$–separable or not. For simplicity, we divide this procedure into levels, starting with $k = N$, continue with $k = N-1$ etc. until we reach $k = 2$. This minimizes the necessary effort for a full classification, since, as mentioned in the previous section, there are connections between the different levels which will be explained in more detail in Sec. IIIB3.

Level $k$ of the characterization consists of the complete determination of the $k$–separability properties of the state $\rho$, i.e. considering all possible partitions into exactly $k$ sets and determine whether the state is separable.

At each level $k$, we have various classes, namely all possible combinations of $k$–separability and $k$–inseparability. If the number of $k$–partite splits is $k_0$, we have $2^{k_0}$ possible configurations at this level in principle.

However, the different levels of this structure are not independent of each other and thus some of the possible configurations are forbidden by the structure at higher/lower levels. We call each allowed configuration of the whole hierarchic classification a "class", since it corresponds to different physical properties.

We have that permutations of the parties lead to different classes.

Note that all levels of this characterization are required to fully classify a state. It is not sufficient to give only the number of $k$–separable splits at each level and define classes in terms of this numbers, as done for $N = 3$ in [13]. In this case, one obtains the remaining configurations by permuting the parties, while for $N > 3$ this last property is no longer valid, i.e. one can have two physically different situations (not only up to permutations) corresponding to the same number of $k$–separable states at a certain level. This will be explained in more detail in Sec. IVB.

In principle, it may turn out that some of the classes we give here are empty. In fact, for the family of states we are going to consider in the following, we have that $k$–separability is implied by the corresponding 2–separabilities, i.e. by the biseparability properties of all bipartite splits containing the $k$–partite split $S_k$ in question. This means that these states are already fully classified by the structure at level 2 (biseparability properties).

However, for $N = 3$ examples for 3–inseparable states which are biseparable with respect to all bipartite splits are known [21], which makes it likely that similar examples (apart from the trivial generalization of those states $\rho_B$ to $N$ qubits by taking e.g. $|0\rangle (|0\rangle^{\otimes N-3} \otimes \rho_B)$ also exist for $N > 3$.

2. Partitions of $N$–qubits

A partition of $N$, $L$, is given by $L = \{1^{r_1}, 2^{r_2}, \ldots, N^{r_N}\}$ with $\sum_{j=0}^{N} j r_j = N$, and the number of sets $k = \sum r_i$. For example, $N = 4$ and $L_4 = \{1^2, 2^1\}$ denotes all possible partitions of $N$ into 3 sets such that one set consists of 2 parties, the other two sets consist of one party each. Note that we may have many partitions which correspond to the same number of sets, say $k$, which we called "$k$–partite splits".

Using the well developed theory of partitions (see e.g. [19]), one finds that the number of possible configurations (including all possible permutations among the parties) for a certain partition $L$ is given by

$$|L| = \frac{N!}{\prod_{j=1}^{r_j} (j!)^{r_j}}. \quad (4)$$

The total number of partitions $L$ is given by the partition function $p(N)$, which grows rapidly with $N$, e.g. $p(10) = 42$, $p(50) = 204226$, $p(100) = 190569292$. A closed expression for $p(N)$ is known and can be found e.g. in [19]. Using these relations, one can in principle obtain the number of possible $k$–partite splits of a $N$–qubit system.

3. Contained splits and implications for classification

Let $l < k$. We say a $k$–partite split $S_k$ belongs to (equivalently is contained in) a $l$–partite split $S_l$ iff $S_l$ can be obtained from $S_k$ by joining some of the parties of $S_k$. For 3 parties ($N = 3$) we have, for example, that the bipartite split $A_1 - (A_2 A_3)$ contains the 3–partite split $A_1 - A_2 - A_3$, since the split $A_1 - (A_2 A_3)$ can be obtained by joining the parties $(A_2 A_3)$. Note that in general we do not have a one–to–one correspondence in neither direction. On one hand, each $k$–partite split is contained in various $l$–partite splits, while on the other hand a number of different $k$–partite splits may be contained in the same $l$–partite split (see also Fig.2).

Furthermore, $k$–separability with respect to a certain $k$–partite split $S_k$ implies $l$–separability with respect to all those $l$–partite splits which contain $S_k$ ($l < k$). However we learned in the case of three qubits [13] that 2–separability with respect to all possible 2–partite splits is not sufficient to guarantee the corresponding 3–separability in general. Thus we have that $l$–separability with respect to all those $l$–partite splits which contain a certain $k$–partite split $S_k$ is a necessary, but not sufficient condition for $k$–separability with respect to $S_k$.

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1 The number of sets is equivalent to the number of parties
Let us apply this observation to our classification. We have that $k$–separability partly fixes the structure at lower levels $l < k$ ($k$–inseparability has no influence at lower levels), while $l$–inseparability fixes some properties at higher levels $k > l$ ($l$–separability only provides necessary conditions for separability at higher levels). In particular, $k$–separability with respect to a certain $k$–partite split $S_j$ already implies $l$–separability of all $l$–partite splits containing $S_k$ ($l < k$). On the other hand, $l$–inseparability with respect to a certain $l$–partite split $S_l$ implies that all $k$–partite splits which belong to $S_l$ ($k > l$) are also $k$–inseparable. This means that once one finds a $k$–partite split $S_k$ to be $k$–separable, one does not have to consider all $l$–partite splits containing $S_k$ since they are automatically $l$–separable, which reduces the necessary effort to fully classify a state.

C. Distillability properties within a specific class

One can also consider the process of distillation (entanglement purification) and relate it to this classification. Let us consider a specific class, characterized by all their $k$–separability properties. A necessary condition for the distillation of a maximally entangled pair e.g. between $A_i$ and $A_j$ is that all those splits for which $A_i$ and $A_j$ belong to different parties are $k$–inseparable. In fact, it is sufficient to consider only the bipartite splits fulfilling this property, since this already implies the inseparability of all $k$–partite splits ($k > 2$) of this kind.

In a similar way, we find a necessary condition for the creation of a $j$–GHZ state, i.e a GHZ state shared among $j$ parties, e.g. $A_i \equiv \{A_{i_0}, \ldots, A_{i_j}\}$: We consider all those bipartite splits where not all of the parties $A_j$ are joint at one site. The inseparability of all those bipartite splits is a necessary condition for the creation of a $j$–GHZ state between the parties $A_i$.

By investigating these necessary conditions for distillability, we immediately observe that there exists a huge number of classes which are inseparable at some level (and thus entangled), but cannot be distilled. Hence all this classes correspond to different kinds of bound entanglement. For example, we have that inseparability with respect to any $k$–partite split already implies that the state is entangled, but one can still have the $2$–separability properties such that the necessary conditions for distillation of a pair between any two parties are not fulfilled. Sometimes this bound entanglement may be activated by allowing some additional entanglement between some subsystems. An example for this is given in Sec IV A.

IV. FAMILY OF STATES $\rho_N$

In the following we show that any arbitrary $N$–qubit state $\rho$ can be brought to a standard form $\rho_N$ [13]. We also give a full classification of this family of states in terms of their separability and distillability properties.

A. Notation

We introduce the orthonormal GHZ-basis [20]

$$|\Psi_{\pm j}\rangle \equiv \frac{1}{\sqrt{2}} ((|j⟩|0⟩ \pm |(2^{N−1}−j−1)⟩|1⟩),$$

where $j = j_1j_2 \ldots j_{N−1}$ is understood in binary notation. We have that $|j⟩_{A_1 \ldots A_{N−1}}$ is the state of the first $(N−1)$ qubits. For example, for $N = 5$ and $j = 6$ this reads $|\Psi_{0\pm 6}\rangle \equiv \frac{1}{\sqrt{2}}(|0110⟩_{A_1 \ldots A_4}|0⟩_{A_5} ± |1001⟩_{A_1 \ldots A_4}|1⟩_{A_5})$, since $(6 \equiv 0110)$ in binary notation. Each basis state is a GHZ state, and all basis elements are connected by $N$–local unitary operations. So $|\Psi_{0+}⟩ = \frac{1}{\sqrt{2}}(|0⟩ \ldots + |1⟩ \ldots 1)$ is only an arbitrary GHZ state, which can be selected by the choice of a local basis in $A_1 \ldots A_N$. We emphasis this here, since the states $|\Psi_{0+}⟩$ seem to play a special role in what follows. In the following we consider the family of states

$$\rho_N = \sum_{\sigma=\pm} \lambda^2_\sigma |\Psi_\sigma^+⟩⟨\Psi_\sigma^+|$$

$$+ \sum_{j=1}^{2^{N−1}−1} \lambda_j (|\Psi_j^+⟩⟨\Psi_j^+| + |\Psi_j^−⟩⟨\Psi_j^−|),$$

which is the straightforward generalization of the family $\rho_3$ introduced in [13] to $N$ qubits. Due to the normalization condition tr($\rho_N$) = 1, we have that $\rho_N$ is described by $2^{N−1}$ independent real parameters. The labeling is chosen such that $\Delta \equiv \lambda^2_0 − \lambda^2_6 \geq 0$.

B. Depolarization to $\rho_N$

In this Section we are going to show that an arbitrary state $\rho$ can be depolarized to the standard form $[13]$ by a sequence of $N$–local operations while keeping the values of $\lambda^2_0 \equiv ⟨\Psi_0^+|\rho|\Psi_0^+⟩$ and $2\lambda_1 \equiv ⟨\Psi_1^+|\rho|\Psi_1^+⟩ + ⟨\Psi_1^-|\rho|\Psi_1^-⟩$ unchanged. Similarly as in the three–qubit case, this implies that the necessary and sufficient conditions for distillability and non–separability found for $\rho_N$ automatically translate into sufficient conditions for arbitrary states.

We will now explicitly construct the required sequence of $N$–local operations to obtain the desired depolarization procedure. By mixing we understand in the following that a certain operation is (randomly) performed with $p = \frac{1}{2}$, while with $p = \frac{1}{2}$ no operation is performed. The following $N$ rounds of mixing operations are sufficient to make $\rho$ diagonal in the basis [13] without changing the diagonal coefficients: In the first round we apply simultaneous spin flips at all $N$ locations; The result of this mixing operation is that all off–diagonal elements of
the form \(|\Psi_j^\pm\rangle\langle\Psi_j^\pm|\) and \(|\Psi_i^\pm\rangle\langle\Psi_i^\pm|\) are eliminated. The remaining \((N-1)\) rounds consist of applying \(\sigma_z\) to particles \(A_k\) and \(A_{\bar{k}}\) (and the identity to all other particles), where \(k\) runs from 1 to \((N-1)\). The effect of the \(k^{th}\) operation is the following: a state \(|\Psi_j^\pm\rangle\) picks up a minus sign if \(j\), written in binary notation, has a “1” at the \(k^{th}\) position and remains unchanged if it has a “0” there. Since the corresponding \(j\) and \(i\) of two different basis states \(|\Psi_j^\pm\rangle\) and \(|\Psi_i^\pm\rangle\) differ in at least on digit, this implies that in at least one mixing round one state, say \(|\Psi_j^\pm\rangle\) will pick up a minus sign while the other state, say \(|\Psi_i^\pm\rangle\) will remain unchanged. This ensures that all off–diagonal elements of the form \(|\Psi_j^\pm\rangle\langle\Psi_i^\pm|\) are eliminated in this mixing round. Finally, we have that after all \(N\) mixing rounds, \(\rho\) is diagonal in the basis \([3]\).

It remains to depolarize the subspaces spanned by \(|\Psi_j^\pm\rangle\rangle\) for each \(j > 0\). This can be accomplished by using random operations that change \(|0\rangle_\alpha\) to \(e^{i\phi_\alpha}|0\rangle_\alpha\) (\(\alpha = A_1, \ldots, A_N\)) with \(\sum_k \phi_{Ak} = 2\pi\) (this condition ensures that \(\lambda_{\bar{k}}^2\) remains unchanged). This implies that an arbitrary state \(\rho\) can be brought to the standard form \(\rho_N\) by a sequence of \(N\)–local operations.

One can readily check that the partial transpose of \(\rho_N\) with respect to the bipartite split \((A_1 \ldots A_{N-1}) - A_N\) is positive iff \(\Delta \equiv \lambda_{\bar{k}}^2 - \lambda_\bar{k}^2 \leq 2\lambda_{N-1}\). Similar conditions hold for each possible bipartite split, i.e. \(\rho_N\) has PPT with respect to a certain bipartite split \(\Delta \leq 2\lambda_k\) for a specific (unique) \(k\) corresponding to this bipartite split. To determine the corresponding \(k\), let us consider a bipartite split where \(l\) qubits \(A_k = \{A_{k_1}, \ldots, A_{k_l}\}\) are jointly located at one side, while the remaining \(N - l\) qubits are located at the other side. If we have that the partial transpose corresponding to this bipartite split is positive, i.e. \(T_{A_k} \equiv T_{A_k1} \ldots T_{A_kl} \geq 0\) then \(\rho_N\) is fully separable with respect to this bipartite split, i.e. it can be written in the form

\[
\rho_N = \sum_i |\chi_i\rangle A_k (\chi_i | \otimes |\varphi_i\rangle_{\text{rest}} \langle \varphi_i|).
\]

(iii) We consider all possible \(2^{N-1} - 1\) bipartite splits of a \((N-1)\)–qubit system. If for each of those splits the corresponding partial transposition is positive, then \(\rho_N\) is fully separable, i.e. \(\rho_N\) is \(N\)–separable.

These statements are illustrated for the simplest cases of a 3 and 4–qubit system in Sec. [3]. From (i) follows that \(k\)–separability with respect to a certain \(k\)–partite split \(S_k\) of the states \(\rho_N\) is implied by the 2–separability properties of the bipartite splits containing \(S_k\). Thus the family \(\rho_N\) is completely characterized by its 2–separability properties, which already determine the hierarchical structure proposed in [21]

In the remainder of this section, we are going to prove the statements (i)–(iii). Let us start by proving (ii). The basic idea of the proof is to define a state \(\hat{\rho}\) which we show to be 2–separable with respect to the bipartite split in question and which can be depolarized to \(\rho_N\). Since a separable state is converted into a separable one by depolarization (which is a \(N\)–local process), this automatically implies the 2–separability of \(\rho_N\). We have that \(\rho_N\) has positive partial transposition with respect to the bipartite split \(A_k\) iff \(\Delta \leq 2\lambda_k\) for \(k\) corresponding to \(\bar{k}\). We define

\[
\hat{\rho} = \rho_N + \frac{\Delta}{2} (|\Psi_k^\pm\rangle\langle\Psi_k^\pm| - |\Psi_{\bar{k}}^\pm\rangle\langle\Psi_{\bar{k}}^\pm|).
\]

The state \(\hat{\rho}\) is positive since \(\Delta \leq 2\lambda_k\) and has the property \(\hat{\rho} = \hat{\rho}_2 \hat{\rho}_{A_k}\). For a bipartite split of the form (one qubit)–rest, this already implies the separability of \(\hat{\rho}\), since it has been shown in [18] that all states in \(\mathcal{O}^2 \otimes \mathcal{O}^N\) which fulfill \(\hat{\rho}_A = \hat{\rho}\) are separable. For all the other splits we show the separability directly. We rewrite \(\hat{\rho}\) as follows

\[
\hat{\rho} = \Delta (|\Psi_0^+\rangle\langle\Psi_0^+| + |\Psi_{\bar{k}}^+\rangle\langle\Psi_{\bar{k}}^+|)
+ (\lambda_k - \frac{\Delta}{2})(|\Psi_k^-\rangle\langle\Psi_k^-| + |\Psi_{\bar{k}}^-\rangle\langle\Psi_{\bar{k}}^-|)
+ \lambda_0 (|\Psi_0^-\rangle\langle\Psi_0^-| + |\Psi_0^+\rangle\langle\Psi_0^+|)
+ \sum_{j=1,j\neq k}^{2^{N-1}-1} \lambda_j (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|).
\]

We have that all prefactors are positive (since \(\Delta \leq 2\lambda_k\)). The terms in lines 2–4 are completely separable, which can be seen by using that \((|\Psi_0^+\rangle\langle\Psi_0^+| + |\Psi_{\bar{k}}^+\rangle\langle\Psi_{\bar{k}}^+|) = |j0\rangle\langle j0| + (2^{N-1} - j - 1)1\langle(2^{N-1} - j - 1)1| \). The term
in line 1 is biseparable with respect to the bipartite split $k$. To see this, let us rewrite the basis states as follows:

$$|Ψ_0^+⟩ = \frac{1}{\sqrt{2}}((0|0)_{A_k}|0|0)_{rest} + |1|1)_{A_k}|1|1)_{rest}$$

$$|Ψ_k^+⟩ = \frac{1}{\sqrt{2}}((1|1)_{A_k}|0|0)_{rest} + |0|0)_{A_k}|1|1)_{rest}$$

We define $|Σ⟩ = \frac{1}{\sqrt{2}}((0|0)_{A_k}|±|±)_{rest}$. It is now straightforward to check that line one of the left can be written as $N(Σ)_{A_k} |+⟩+|−⟩_{A_k} |−⟩_{rest} (−)$, which is clearly biseparable with respect to the bipartite split $k$ and concludes the proof in one direction. If we consider the other hand that $ρ_N$ is biseparable with respect to the bipartite split $k$, it follows trivially that it also has PPT corresponding to this bipartite split, since positive partial transposition is a necessary condition for separability.

To prove the third statement (iii), we show that if $ρ_N$ has PPT with respect to all possible bipartite splits then $ρ_N$ is $N$–separable (note again that the opposite is trivial). This condition implies that $Δ/2 ≤ λ_j$ for all $j$. Again, the idea is to define an operator $\hat{ρ}$ which can be depolarized into the form $ρ_N$ by using local operations and that is fully separable. Let $\hat{ρ}$ be a state of the form $|\hat{Ψ}⟩$ with coefficients $λ^+_{i} = λ^+_{k}$ and $λ^-_{i} = λ_{k} ± Δ/2$ $(k = 1, . . . , 2^{N-1} - 1)$. Clearly, $\hat{ρ}$ can be depolarized into $N$. We now rewrite $\hat{ρ}$ as follows:

$$\hat{ρ} = \frac{1}{2} 2^{N-1} \sum_{k=0}^{2^{N-1} - 1} (λ^+_{k} + λ^−_{k} - Δ)(|Ψ^+_{k})|\langleΨ^+_{k}| + |Ψ^−_{k})|\langleΨ^−_{k}|)$$

$$+ Δ 2^{N-1} \sum_{k=0}^{2^{N-1} - 1} |Ψ^+_{k})|\langleΨ^−_{k}|.$$

Since all possible partial transposes are positive, we have that all coefficients in $|\hat{Ψ}⟩$ are positive. The first term in $|\hat{Ψ}⟩$ can be written as $\sum_{k=0}^{2^{N-1} - 1} (λ^+_{k} + λ^-_{k} - Δ)(k, 0)(k, 0) + (2^{N-1} - k, 0)(2^{N-1} - k, 0)$ and is thus fully separable. It remains to show that the second term in $|\hat{Ψ}⟩$ is also $N$–separable. Let us first define the states $|±⟩ = (|0⟩ ± |1⟩)/√2$. To show the separability of the second term, we write it as $\sum_{j=0}^{2^{N-1} - 1} |ϕ_j⟩|ϕ_j⟩$ where $|ϕ_j⟩$ are all the states of the form $|σ|σ|, . . . , |σ⟩⟩$ with $σ = ±$, which have an even number of minuses. All the states $|ϕ_j⟩$ are fully separable, which concludes the proof.

Now we are ready to prove the first statement (i). The basic idea of the proof is very similar to the one used in the previous proofs: We define a state $ρ'$ which can be depolarized to $ρ_N$ and we show that $ρ'$ is $k$–separable. We have that a number of bipartite splits have PPT. To a specific bipartite splits corresponds the relation $Δ ≤ 2λ_{i}$, which is the condition that this specific bipartite split has PPT. Thus we have that $Δ ≤ 2λ_{i}$, where $i \in \{i_0, . . . , i_l\} ≡ i$, and each $λ_{i}$ corresponds to a bipartite split which does not further divide systems which were joint for the $k$–partite split we consider. Let $ρ'$ be a state of the form $|\hat{Ψ}⟩$ with coefficients $λ^+_{i} = λ^+_{k}$, $λ^-_{i} = λ_{k} ± Δ/2$ for $i \in i$ and $λ_{k} ± Δ/2$ for $k \notin i$. Clearly, $ρ'$ can be depolarized to $ρ_N$. Similarly as in the previous proofs, we rewrite $ρ'$ as follows:

$$ρ' = Δ(|Ψ^+_{i})|Ψ^+_{i}| + \sum_{i \in i} |Ψ^+_{i})|Ψ^+_{i}|$$

$$+ \sum_{i \in i} (λ_i - Δ/2)(|Ψ^+_{i})|Ψ^−_{i}| + |Ψ^−_{i})|Ψ^+_{i}|)$$

$$+ \sum_{k \notin i} λ_k(|Ψ^+_{k})|Ψ^+_{k}| + |Ψ^−_{k})|Ψ^−_{k}|$$

$$+ λ^0_{0}(|Ψ^0_{0})|Ψ^0_{0}| + |Ψ^−0_{0})|Ψ^0_{0}⟩.$$

All coefficients in $|\hat{Ψ}⟩$ are positive, and lines (2–4) are fully separable. It remains to show the $k$–separability of line 1 of (12). To see this, we define the states $|±⟩_{A_i} = (|0⟩ ± |1⟩)/\sqrt{2}$ and the states $|ϕ⟩ ≡ |σ|σ|, . . . , |σ⟩⟩$ with $σ = ±$ and the number of minuses is even. It is now straightforward to check that line 1 of (12) can be written as $Δ |ϕ⟩|ϕ⟩$ and is thus $k$–separable with respect to the $k$–partite split we consider, and concludes the proof in one direction.

If we consider the other hand that $ρ_N$ is $k$–separable with respect to a specific $k$–partite split $S_k$, it follows that it is also biseparable with respect to all bipartite splits which contain $S_k$, since any of those bipartite splits corresponds to joining systems which were divided for the $k$–partite split. But the positivity of the partial transposition is a necessary condition for separability corresponding to a certain bipartite split $|\hat{Ψ}⟩$, from which follows that positivity of all bipartite splits we consider is also a necessary condition for $k$–separability. So again the conditions we found are necessary and sufficient.

### D. Distillability of $ρ_N$

We will turn now to analyze the distillability properties of $ρ_N$:

(i) We consider all possible bipartite splits of the $N$ qubits where the particles $A_i$ and $A_k$ belong to different parties. If all such splits have negative partial transposition, then a maximally entangled pair between particle $A_i$ and $A_k$ can be distilled.

(ii) We consider the $k$ parties $A_t \equiv \{A_0, . . . , A_k\} (k ≤ N)$ and consider all those bipartite splits where not all of the parties $A_t$ are joint at one side. If all those splits have negative partial transposition, then a $k$–GHZ state (i.e. a GHZ state shared between $k$ parties) between the parties $A_t$ can be distilled.

To show (i) and without loss of generality we take $i = N$ and $k = N - 1$. In that case, the condition we impose on the partial transpositions is equivalent to require that $Δ/2 > λ_j$ with $j$ odd [note that with the notation
we are using, the state of the $N-1$ qubit determines the parity of the states $|j⟩$ in (3). In order to show the distillability of a maximally entangled state between $A_{N-1}$ and $A_N$, it is sufficient to show that a pair with fidelity $F > 0.5$ (overlap with the maximally entangled state $|Ψ^+⟩$) between those two parties can be created (4). If we project all the qubits except the ones at $A_N$ and $A_{N-1}$ onto the state $|+⟩$ we see that the resulting state obtained from $ρ_N$ has $F > 0.5$ and can thus be distilled to a maximally entangled state between $A_N$ and $A_{N-1}$ iff
\[
Δ/2 > \sum_{j\text{odd}} λ_j.
\] (13)

Even though we have that $Δ/2 > λ_j$ for all $j$ odd, the condition (13) might not be fulfilled. In this case we use the following purification procedure: The idea is to combine $M$ systems in the same state $ρ_N$, perform a measurement and obtain one system with the same form (4) but in which the new $Δ$ is exponentially amplified with respect to $λ_k$, $k$ odd. In order to do that, let us define the operator
\[
P = |00...00⟩⟨00...00| + |10...00⟩⟨11...11|,
\] (14)
which acts on $M$ qubits. Now we proceed as follows: We take $M$ systems, and apply the operator $P$ in all $N$ locations. This corresponds to measuring a POVM that contains $P$ obtaining the outcome associated to $P$. The resulting state $P⊗Nρ_N⊗M(P)⊗N$ has the first system in an (unnormalized) density operator of the form (4) but with new coefficients $Δ$ and $λ_k$. In order to calculate these new coefficients, we need the following observations: First, the operator $ρ_N⊗M$ is diagonal in the basis \{ $|σ_1...σ_M⟩$\} with coefficients $λ_{k_1}...λ_{k_M}$ where
\[
|σ_1...σ_M⟩ = |Ψ^0⟩_k_0 ⊗...⊗ |Ψ^σ⟩_k_M,
\] (15)
and $k_j ∈ \{0,1,2,...,2N-1\}$, $σ_j = ±$. Second, we need the action of the operator $P⊗N$ on the basis states (15). We find
\[
P⊗N|σ_1...σ_M⟩ = δ_{k_0...k_M}|Ψ^σ⟩_{k_0/-system1}100...0_{rest},
\] (16)
where $σ = +$ if the number of minuses in \{ $σ_1...σ_M$\} is even, and $σ = -$ otherwise. Note also that we only have a contribution if $k_0 = k_1 = ... = k_M$. Using this results, it is now straightforward to check that the first system of $P⊗Nρ_N⊗M(P)⊗N$ is an (unnormalized) density operator of the form (4) with new coefficients
\[
\tilde{Δ}/2 = (Δ/2)^M; \tilde{λ}_k = λ_k^M.
\] (17)

Given that $Δ/2 > λ_k$, $k$ odd, for sufficiently large $M$ we have that condition (13) is fulfilled, i.e. that after the projection of all systems except $A_{N-1}$ and $A_N$ on the state $|+⟩$, the resulting state has $F > 0.5$ and is thus distillable, which concludes the proof in one direction. On the other hand, the condition we impose for distillability is also necessary. Having a maximally entangled pair between the parties $A_{N-1}$ and $A_N$ implies that all bipartite splits in question have NPT. Since local operations keep the positivity of the partial transposition (7), we must start with NPT of all bipartite splits in question.

To prove (ii), we just have to recognize that the condition we impose guarantees that maximally entangled pairs between any two parties within $A_7$ can be distilled. This is clearly sufficient to create a GHZ state among those parties, e.g. by means of teleportation (creating the GHZ state locally at $A_{i_0}$ and teleporting the $(k-1)$ qubits to the parties $\{A_{i_1},...,A_{i_k}\}$ using maximally entangled pairs created among $A_{i_0}$ and $A_j$ with $j ∈ \{i_1,...,i_k\}$). Note also that the condition we impose is also necessary, since a GHZ state allows us to create maximally entangled pairs among all parties involved, which implies that all bipartite splits in question have NPT and thus have to have NPT at the beginning, since one cannot convert a state from PPT to NPT by means of local operations (10).

V. EXAMPLES

A. Three-qubit systems

In this section we investigate the simplest case of three qubits, each hold by one of the parties $A, B$ or $C$.

1. Classification

In order to perform the classification proposed in (11), we consider the 3-partite split of the system as well as all possible bipartite splits, where all together three such splits exist. Each corresponds to having one system (e.g. $A$) on one side and the two other systems (e.g. $B$ and $C$) on the other side. In other words, for this specific bipartite split of our three-qubit system we allow the parties $B$ and $C$ to act together, i.e. $H = G_A^2 ⊗ G_{BC}^1$. Thus each of this bipartite splits will give us an upper limit of what can be done by three–local operations on the system. To perform the classification, we have to consider the separability properties of the 3–partite and bipartite splits. In particular, whether they can be written in one or more of the following forms:

\[
ρ = \sum_i |a_i⟩_A⟨a_i| ⊗ |b_i⟩_B⟨b_i| ⊗ |c_i⟩_C⟨c_i|,
\] (18a)
\[
ρ = \sum_i |a_i⟩_A⟨a_i| ⊗ |φ_i⟩_BC⟨φ_i|.
\] (18b)
\[
ρ = \sum_i |b_i⟩_B⟨b_i| ⊗ |φ_i⟩_AC⟨φ_i|,
\] (18c)
\[
ρ = \sum_i |c_i⟩_C⟨c_i| ⊗ |φ_i⟩_AB⟨φ_i|.
\] (18d)
Here, $|a_i\rangle$, $|b_i\rangle$ and $|c_i\rangle$ are (unnormalized) states of systems $A$, $B$ and $C$, respectively, and $|\phi_{ij}\rangle$ are states of two systems. We call a state biseparable with respect to a certain bipartite split if it is separable with respect to this split, e.g. a state is biseparable with respect to the bipartite split $A- (BC)$ iff it can be written in the form \[(8b)\]. Similarly, a state is called triseparable (3–separable) if it is separable with respect to the split $A- B- C$, i.e. can be written in the form \[(8c)\] or \[(8d)\].

At the top level of the classification (level 3), we consider the tripartite split $A-B- C$ and determine whether the state $\rho$ is 3–separable or not. At the second level (level 2), one considers all possible bipartite splits \[{A- (BC), B- (AC), C- (AB)}\] and determines whether the state $\rho$ can be written in one or more of the forms \[(18a)\], \[(18b)\], \[(18c)\], \[(18d)\]. At this level of the classification, one has $2^3 = 8$ different possibilities, each corresponding to a physically different situation. For arbitrary three–qubit systems one thus finds the following complete set of 9 disjoint classes

**Class 1 Fully inseparable states**: states that cannot be written in any of the above forms \[(12)\]. An example is the GHZ state \[(13)\] $\Psi^\pm_0$. 

**Classes 2.1, 2.2, 2.3 1-qubit biseparable states**: Class 2.1: biseparable states with respect to qubit $A$ are states that are separable in $A- (BC)$, but non–separable otherwise. That is, states that can be written in the form \[(18c)\] but not as \[(18c)\] or \[(18d)\]. An example is the state $|0\rangle_A \otimes |\Phi^+_BC\rangle$, where $|\Phi^+_BC\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a maximally entangled state of two qubits. Similarly, class 2.2 and 2.3 correspond to biseparable states with respect to qubit $B$ and $C$ respectively.

**Classes 3.1, 3.2, 3.3 2-qubit biseparable states**: Class 3.1: biseparable states with respect to qubits $A$ and $B$ are states that are separable in $A- (BC)$ and $B- (AC)$, but non–separable in $C- (AB)$. That is, states that can be written in the forms \[(18b)\] and \[(18d)\] but not as \[(18a)\]. For examples, see below. Similarly, classes 3.2 (3.3) are biseparable with respect to the qubits $A$ and $C$ ($B$ and $C$).

**Class 4 3-qubit biseparable states**: Those are states that are separable in $A- (BC)$, $B- (AC)$ and $C- (AB)$ (i.e. separable with respect to each bipartite split), but which are not completely separable, i.e. cannot be written as \[(18a)\]. For an example, see Ref. [21].

**Class 5 Fully separable states**: These are states that can be written in the form \[(18a)\] and are thus also separable with respect to each bipartite split. A trivial example is a product state $|1\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C$. 

Note that the classes 2.1, 2.2, 2.3 (respectively 3.1, 3.2, 3.3) where identified in \[(12)\], since they can be obtained from each other by permuting the parties. In this case, 5 distinct classes remain.

2. **Family $\rho_3$**

Let us now concentrate on the family of three–qubit states $\rho_3$. This family is characterized by 4 parameters, \[
\{\Delta \equiv \lambda_0^+ - \lambda_1, \lambda_1, \lambda_2, \lambda_3\},
\] and we have that any state can be depolarized to this standard form. The GHZ basis \[(13)\] reads in this case

\[
|\Psi^+_0\rangle = \frac{1}{\sqrt{2}}((j)_{AB}|0\rangle_C \pm |(3-j)\rangle_{AB}|1\rangle_C),
\]

where $|j\rangle_{AB} = |j_1\rangle_A |j_2\rangle_B$ with $j = j_1j_2$ in binary notation. For example, $|\Psi^+_0\rangle = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$ are standard GHZ states, as well as $|\Psi^\pm_3\rangle = \frac{1}{\sqrt{2}}(|110\rangle \pm |001\rangle) (3 = 11$ in binary notation). Note that all 8 basis states are connected by 3–local unitary operations, i.e. each basis state is a maximally entangled GHZ state. Due to the fact that the local bases in $A$, $B$ and $C$ can be chosen arbitrarily, none of the basis states has any preferences.

We will now investigate the separability and distillability properties of $\rho_3$ and give a full classification in terms of the classes introduced above. It turns out that using the partial transpose criterion for each bipartite split characterizes the state $\rho_3$ completely. The conditions under which the operator $\rho_3$ has positive partial transpose with respect to each qubit are as follows

\[
\begin{align*}
\rho_3^{TA} &\geq 0 \quad \text{iff } \Delta \leq 2\lambda_2 \\
\rho_3^{TB} &\geq 0 \quad \text{iff } \Delta \leq 2\lambda_1 \\
\rho_3^{TC} &\geq 0 \quad \text{iff } \Delta \leq 2\lambda_3.
\end{align*}
\]

Recall that each of this conditions correspond to a (virtually) bipartite split of the system. Investigating e.g. $\rho_3^{TA}$, we actually have in mind a bipartite split of the system into $A$ on one side and $BC$ on the other side. From these conditions we also see that in general no further depolarization which keeps the form of $\rho_3$ is possible (except the depolarization toward the completely depolarized state, which can always be done trivially). We show this by giving a counterexample. Imagine we would like to depolarize the subspaces spanned by \{ $|\Psi_3^\pm\rangle, |\Psi_2^\pm\rangle$ \} and thereby equalize the coefficients $\lambda_1$ and $\lambda_2$. For certain values of the parameters, this would imply that the state after this depolarization has negative partial transposition with respect to one party, while it started with positive partial transposition. Since one cannot change the positivity of the partial transpose by local operations \[(22)\], this further depolarization is impossible in general. (e.g. $\rho_3$ with $\lambda_0^+ = \frac{3}{4}, \lambda_2 = \frac{1}{4}$, all other parameter 0 has $\rho_3^{TA} \geq 0$, but would have negative partial transposition with respect to all bipartite splits after the (imaginary) further depolarization in question)

3. **Separability of $\rho_3$**

Let us specialize the theorems about separability obtained in \[IVC\] to $N = 3$. In this case, we do not need
the general theorem (i), but only give examples for the statements (ii) and (iii):

(ii) $\rho_3$ is separable with respect to the bipartite split $A - (BC)$, i.e. it can be written in the form \[ \rho_3 = \sum_{\alpha, \beta} \rho_{3}^{T\alpha} \rho_{3}^{T\beta} \] iff $\rho_3^{T\alpha} \geq 0$ and analogously for $\rho_3^{T\beta}$ with $\rho_3^{T\alpha} \geq 0$ and $\rho_3^{T\beta} \geq 0$, respectively.

(iii) $\rho_3$ is completely separable, i.e. it can be written as $\rho_3 = \sum_{\alpha, \beta} \rho_{3}^{T\alpha} \rho_{3}^{T\beta}$.

Note that these statements and thus provide a full characterization of $\rho_3$ in terms of the separability properties. The resulting classification is summarized in Table 5.

Furthermore, this also provides us with sufficient conditions for non–separability for arbitrary states $\rho$. Namely if there exist a basis $\{1\}$ such that the corresponding state $\rho_3$ after depolarization has e.g. the property that $\rho_3^{T\alpha}$ is negative, this implies that $\rho$ is non–separable in $A - (BC)$. Note also that no conclusion can be drawn about the separability properties of $\rho$ given PPT of the depolarized state $\rho_3$, since the depolarization process might convert a non–separable state $\rho$ into a separable state $\rho_3$.

4. Distillability of $\rho_3$

We now state the distillability properties of $\rho_3$:

(i) One can distill a maximally entangled state $|\Phi^+\rangle_{\alpha\beta}$ between $\alpha$ and $\beta$ iff both $\rho_3^{T\alpha}, \rho_3^{T\beta}$ have negative partial transposition.

(ii) If all three partial transposes are negative, we can distill a GHZ state (since we can distill an entangled state between $A$ and $B$ and another between $A$ and $C$ and then connect them to produce a GHZ state $|\Phi^+\rangle_{\alpha\beta}$ between $\alpha$ and $\beta$).

(iii) If we have that $\rho_3^{TC} = 0$ is negative but $\rho_3^{TA}, \rho_3^{TB}$ have negative partial transposition.

Note that (iii) is an example for the activation of bound entanglement. In this example, the state is inseparable with respect to the bipartite split $C - (AB)$ and thus entangled. However no entanglement between any two subsystems can be created, since we have that the PPT of $A$ and $B$ implies the separability of $A - (BC)$, i.e. have PPT and we have maximally entangled states between $A$ and $B$ at our disposal, then we can activate the entanglement between $ABC$ and create a GHZ state.

B. Four–qubit system

Here we consider the special case of a four–partite system in order to illustrate the theorems about separability and distillability obtained in the previous sections. For convenience, let us call the parties $A, B, C, D$ instead of $A_1, \ldots, A_4$.

1. Classification

We start by illustrating the classification for general 4–qubit systems. At the top level of the structure is the 4–separability, that is the question whether the state $\rho$ is separable with respect to the 4–partite split (4) $A - B - C - D$, i.e. fully separable. At the second level, we have to consider 6 different 3–partite splits of the system: (3a) $A - B - (CD)$, (3b) $A - (BC) - D$, (3c) $A - (BD) - C$, (3d) $(AB) - C - D$, (3e) $(AC) - B - D$ and (3f) $(AD) - B - C$. At the third level, all together 7 different 2–partite splits...
exist, namely (2a) \( A - (BCD) \), (2b) \( B - (ACD) \), (2c) \( C - (ABD) \), (2d) \( D - (ABC) \), (2e) \( (AB) - (CD) \), (2f) \( (AC) - (BD) \) and (2g) \( (AD) - (BC) \).

We have for instance that the 3–partite split (3a) is contained in the 2–partite splits (2a), (2b) and (2e). All other bipartite splits cannot be obtained from (3a) by joining some of the parties, since they would divide the system \((CD)\) along different parties, and thus the tripartite split (3a) does not belong to them.

The classification thus takes place as follows (see also Fig. 3): At the top level (level 4), one has to decide whether the state is 4–separable or not. In the case it is 4–separable, it automatically follows that it is also 3– and 2–separable with respect to all possible 3– or 2–partite splits. If it is 4–inseparable, one has to investigate the various kinds of 3–separability at level 3, where one can have all possible combinations of the 6 kinds (3a)-(3f) of 3–separability and and 3–inseparability. We have 2^6 different configurations at this level. At the next level of the classification (level 2), one investigates all possible bipartite splits (2a)-(2g) closer. One finds 2^7 different configurations at this level.

The structure at level 2 is partly determined by the structure at level 3 and vice versa. If, e.g., the state is 3–separable with respect to the 3–partite split (3a), it follows that the bipartite splits (2a),(2b) and (2e) at level 2 which contain (3a) are also 2–separable. In the case where a state is 3–separable with respect to the splits (3a), (3b) and (3c), it even follows that the state is 2–separable with respect to all possible bipartite splits. Although it still can be 3–inseparable with respect to the 3–partite splits (3d),(3e) or (3f) in principle, the underlying structure at level 2 is already completely determined by the structure at level 3. 3–inseparability with respect to a specific 3–partite split \( S_3 \), on the other hand, still allows all combinations of 2–separability and 2–inseparability within the bipartite splits at level 2 which contain \( S_3 \). Conversely, 2–inseparability with respect to the bipartite split (2a) implies 3–inseparability with respect to the 3–partite splits (3a), (3b) and (3c), while 2–separability still leaves all possibilities at level 3 open. From this one also sees that it is neither sufficient to consider only the 4– and 2–separability to classify the state completely, nor to consider only 4– and 3–separability.

Furthermore, it is not sufficient to classify the states by the number of \( k \)–separable state at level \( k \) of the hierarchic classification. For example, we have that 3–separability with respect to the 3–partite splits (3a),(3b) and (3c) already implies 2–separability with respect to all bipartite splits. On the other hand, 3–separability with respect to the 3–partite splits (3a),(3b) and (3d) does not determine the inseparability properties of the bipartite split (2f), which may thus still be 2–inseparable. The two kinds of 3 times 3–separability correspond to different physical situations, and one cannot obtain one configuration from the other one by permuting the parties. Thus it is not sufficient to give only the number of 3–separable states, one also needs the information which of the splits is separable and which is not.

2. Separability and distillability properties of \( \rho_4 \)

Let us now turn to the family of states \( \rho_4 \) and illustrate its separability properties. We will give an example for each theorem.

(i) Let us consider the 3–partite split \( A - B - (CD) \) (3a). Iff we have that the 2–partite splits (2a), (2b) and (2e) (which contain (3a)) have PPT, then \( \rho_4 \) is 3–separable with respect to this 3–partite split.

(ii) Iff the partial transposition with respect to the bipartite split \( (AB) - (CD) \) (2e) is positive, then \( \rho_4 \) is 2–separable in \( (AB) - (CD) \).

(iii) Iff for all possible 2–partite splits (2a)-(2g) we have that the corresponding partial transposition is positive, then \( \rho_4 \) is 4–separable.

Note that this family of states is completely characterized by its 2–separability properties, since from 2–separability follows the corresponding 3–separability as well as the corresponding 4–separability. So in this case only one level of the hierarchic structure, namely level 2, is required to fully classify the states \( \rho_4 \).

Finally we consider the distillability properties of \( \rho_4 \).

(i) Iff the partial transposition with respect to the bipartite splits (2c), (2d), (2f) and (2g) is negative, then a maximally entangled pair between \( C \) and \( D \) can be distilled. Note that these four splits are the only ones of relevance, since the parties \( C \) and \( D \) are not joint there.

(ii) Iff the partial transpositions with respect to the bipartite splits (2a), (2b), (2e), (2f) and (2g) are negative, then a GHZ state between the parties \( A - B - C \) can be distilled. Note that the split (2d) is the only one which is not of relevance here, since the parties \( ABC \) are joint in this case. If in addition also the split (2d) has NPT, then a GHZ state between all four parties can be created.

If it turns out that there exist non–distillable states in \( \mathcal{A}^4 \otimes \mathcal{A}^4 \) with NPT as conjectured in [16,17], this automatically implies that the conditions (i) for distillability obtained for the states \( \rho_4 \) are not sufficient for arbitrary states \( \rho \). To see this, let us consider the question of distillability of a maximally entangled pair between \( C \) and \( D \). Assume that the partial transposition with respect to the bipartite splits (2c), (2d), (2f) and (2g) is negative. Let us concentrate on the bipartite split (2f). According to the conjecture in [16,17], the negativity of the partial transposition with respect to this split is not sufficient to ensure distillability. So we have that there exist states which are not distillable even if we allow for joint actions at \( (AC) \) and \( (BD) \) and are thus also not distillable when allowing only local operations in \( A, B, C \) and \( D \) (As mentioned earlier, each bipartite split provides us with an upper limit of what can be done by local operations).
VI. MIXTURES OF GHZ-STATE WITH IDENTITY

We will apply now our results to the case in which we have a maximally entangled state of N particles mixed with the completely depolarized state

\[
\rho(x) = x|\Psi^+_N\rangle\langle\Psi^+_N| + \frac{1-x}{2^N}I. \tag{21}
\]

This is clearly a special case of the state \(\rho_N\) with \(\lambda^{-}_j = \frac{x}{2^N}\), \(\lambda^+_j = x + \frac{x}{2^N}\) and thus \(\Delta = x\). These states have been analyzed in the context of robustness of entanglement \[1\], NMR computation \[4\], and multiparticle purification \[1\]. In all these contexts bounds are given regarding the values of \(x\) for which \(\rho(x)\) is separable or purifiable. For example, in Refs. \[1,4\] they show that in the case \(N = 3\) if \(x \leq 1/(3 + 6\sqrt{2})\), \(1/25\) then the state is separable, respectively. In Ref. \[1\] it is shown that for \(N = 3\) if \(x > 0.32263\) then \(\rho(x)\) is distillable. Using our results we can state that \(\rho(x)\) is fully non-separable and distillable to a maximally entangled state iff \(x > 1/(1 + 2^{N-1})\), and fully separable otherwise. Specializing this for the case \(N = 3\) we obtain that for \(x > 1/5\) it is non-separable and distillable \[2\].

Note also that the purification procedure proposed in this work is pretty inefficient compared to the two-step procedure proposed in \[1\], although it allows us to determine stronger bounds for the value of \(x\). However, one can slightly modify the procedure proposed in \[1\] such that the protocols P1 and P2 are no longer performed alternately as in the original version, but rather in a specific (state dependent) order, e.g. P1-P1-P1-P2-P1 etc. When doing so, we found by numerical investigation (for \(N = 3\)) that all states \(\rho_3\) which are purifiable to a GHZ state using our procedure are also purifiable using the modified procedure of \[1\], which thus provides an efficient purification protocol for states of the form \(\rho_3\).

VII. SUMMARY

In summary, we have proposed a classification of arbitrary multi-qubit systems. For a family of states, we gave a full characterization of the separability and distillability properties. These states play the role of Werner states in these systems since any state can be reduced to such a form by depolarization. Thus, our results provide sufficient conditions for non-separability and distillability for general states.

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[1] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin and W. K. Wootters, Phys. Rev. Lett. 76, 722 (1996);
[2] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[3] C. H. Bennett, G. Brassard, C. Crpeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[4] H.-J. Briegel, W. Dür, J. I. Cirac and P. Zoller, Phys. Rev. Lett. 81, 5932 (1998);
[5] W. Dür, H.-J. Briegel, J. I. Cirac and P. Zoller, Phys. Rev. A 59, 169 (1999);
[6] S. J. van Enk, J. I. Cirac, and P. Zoller, Phys. Rev. Lett., 78, 4293(1997);
[7] S. J. van Enk, J. I. Cirac, and P. Zoller, Science, 279, 205 (1998).
[8] D. Deutsch, A. Ekert, C. Macchiavello, S. Popescu, and A. Sanpera, Phys. Rev. Lett. 77, 2818 (1996).
[9] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[10] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[11] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[12] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A223, 8 (1996).
[13] P. Horodecki, Phys. Lett. A 232, 333 (1997).
[14] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. 78, 574 (1997).
[15] M. Murao, M. B. Plenio, S. Popescu, V. Vedral, and P. L. Knight, Phys. Rev. A 57, 4075 (1998).
[16] J. Kempe, quant-ph/9902036;
[17] A. V. Thapliyal, Phys. Rev. A 59, 3336 (1998);
[18] G. Vidal, quant-ph/9807074;
[19] N. Linden and S. Popescu, Fortsch.Phys. 46, 567 (1998);
[20] N. Linden, S. Popescu and A. Sudbery, quant-ph/9801071;
[21] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, A. V. Thapliyal, quant-ph/9908073.
[22] W. Dür, J. I. Cirac, and R. Tarrach, Phys. Rev. Lett. 83, 3562 (1999).
[23] S.L. Braunstein, C.M. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, Phys. Rev. Lett. 83, 1054 (1999).
[24] G. Vidal and R. Tarrach, Phys. Rev. A 59, 141 (1999).
[25] W. Dür, J. I. Cirac, M. Lewenstein and D. Bruß, quant-ph/9910022;
[26] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, A. V. Thapliyal, quant-ph/9908073.
[27] M. Lewenstein, J. I. Cirac and S. Karnas, quant-ph/9903012.
[28] G. E. Andrews, The theory of partitions, Encyclopedia of mathematics and its applications, Addison-Wesley Publishing Company 1976.
A. Smolin, and B. M. Terhal, Phys. Rev. Lett. 82, 5385 (1999).
[22] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. 82, 1056 (1999).
[23] M. Zukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. 71, 4287 (1993).
[24] In R. Schack and C. M. Caves, quant-ph 9904109, it was independently shown using different methods that for $N = 3$, the state $\rho(x)$ is separable for $x \leq 1/5$. For larger $N$ however, only weaker bounds were obtained.

| Positive Operators | Class | Distillability          |
|--------------------|-------|-------------------------|
| None               | 1     | (GHZ) $|\Psi_3^+\rangle_{ABC}$ |
| $\rho_3^{A^3}$     | 2.1   | (Pair) $|\Phi^+\rangle_{BC}$ |
| $\rho_3^{A^3} + \rho_3^{A^2}$ | 3.1   | Activate with $|\Phi^+\rangle_{AB}$ |
| All                | 5     |                          |

TABLE I. Separability and distillability classification of $\rho_3$

FIG. 1. Example of two partitions of a 9-qubit system into three sets (full line - $S_3 = (A_1 A_6 A_9) - (A_2 A_3 A_7) - (A_4 A_8)$) and five sets (dotted line - $S_5 = (A_1 A_6) - (A_9) - (A_2 A_7) - (A_3 A_5) - (A_4 A_8)$). We have that $S_5$ is contained in $S_3$.

FIG. 2. Hierarchy structure for classification of 4-qubit systems. Note that not all connections at the lowest level are drawn.