Maxwell-Chern-Simons theory in covariant and Coulomb gauges

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Abstract

We quantize Quantum Electrodynamics in 2 + 1 dimensions coupled to a Chern-Simons (CS) term and a charged spinor field, in covariant gauges and in the Coulomb gauge. The resulting Maxwell-Chern-Simons (MCS) theory describes charged fermions interacting with each other and with topologically massive propagating photons. We impose Gauss’s law and the gauge conditions and investigate their effect on the dynamics and on the statistics of n-particle states. We construct charged spinor states that obey Gauss’s law and the gauge conditions, and transform the theory to representations in which these states constitute a Fock space. We demonstrate that, in these representations, the nonlocal interactions between charges and between charges and transverse currents, as well as the interactions between currents and massive propagating photons, are identical in the different gauges we analyze in this and in earlier work. We construct the generators of the Poincaré group, show that they implement the Poincaré algebra, and explicitly demonstrate the effect of rotations and Lorentz boosts on the particle states. We show that the imposition of Gauss’s law does not produce any “exotic” fractional statistics. In the case of the covariant gauges, this demonstration makes use of unitary transformations that provide charged particles with the gauge fields required by Gauss’s law, but that leave the anticommutator algebra of the spinor fields untransformed. In the Coulomb gauge, we show that the anticommutators of the spinor fields apply to the Dirac-Bergmann constraint surfaces, on which Gauss’s law and the gauge conditions obtain. We examine MCS theory in the large CS coupling constant limit, and compare that limiting form with CS theory, in which the Maxwell kinetic energy term is not included in the Lagrangian.

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I. INTRODUCTION

In earlier work [1], we discussed the quantization of (2 + 1)-dimensional QED (QED$_3$) with a topological mass term—the so-called Maxwell-Chern-Simons (MCS) theory—in the temporal ($A_0 = 0$) gauge. In that work, we showed that when Gauss’s law is imposed on the particle excitations of a charged spinor field, the charged particle states (fermions, in that case) do not develop “exotic” fractional statistics. We defined gauge-invariant fields that create, from the vacuum, charged particle states that obey Gauss’s law, and nevertheless anticommute exactly like the gauge-dependent fields that create the “bare” fermions from the vacuum. Moreover, the imposition of Gauss’s law does not cause the charged fermion states to acquire any arbitrary “anyonic” phases in a $2\pi$ rotation. These results contradict some widely accepted conjectures about (2 + 1)-dimensional gauge theories with Chern-Simons (CS) terms, in which exotic statistics and arbitrary rotational phases are regarded as consequences of the imposition of Gauss’s law on charged states [2].

In this paper, we extend our investigation by studying the same model in the covariant and the Coulomb gauges. Formulating this model in these different gauges confirms our earlier results, and leads to new insights into its gauge-independent observables and its particle states. First, we demonstrate that the time-evolution operator for propagating particle states (i.e., the Hamiltonian adjusted to comply with Gauss’s law and the gauge condition) is identical in the temporal, covariant, and Coulomb gauges, confirming that identical predictions are obtained in every one of these gauges for all questions that are in principle subject to empirical verification. This result substantiates the consistency of our formulation of this model in the gauges we have investigated. Our work in the Coulomb gauge, in which we apply the Dirac-Bergmann (DB) procedure for imposing constraints [3,4], supports our earlier demonstration that the implementation of Gauss’s law does not transform the statistical properties of the charged states of the spinor field from the standard Fermi statistics to “exotic” fractional statistics. We are able to corroborate this conclusion—already established in the temporal gauge [1] and confirmed by an identical result for the covariant gauges [Sec. VI]—by using the DB procedure in our treatment of the Coulomb gauge to explicitly evaluate the anticommutator for the spinor fields on the constraint surface on which all the theory’s constraints—including Gauss’s law—apply. We use the covariant gauge formulation of this model to obtain further insight into the kinematics of $2\pi$ rotations for charged states in MCS theory, and to investigate the effect of Lorentz boosts on the single propagating mode of the gauge field. One element in this investigation is the demonstration that the operators used to generate rotation and boosts implement the appropriate Poincaré algebra. In this work, we also provide concrete illustrations of important abstract principles—for example, illustrations of how operator-valued dynamical variables develop gauge-independent forms in different gauges, even though they are functionals of gauge and charged fermion fields whose forms and equations of motion clearly reflect the choice of gauge. And finally, in this investigation we explore the sense in which MCS theory approaches Chern-Simons theory without a Maxwell kinetic energy term in the limit $m \to \infty$, where $m$ is the topological mass.
II. FORMULATION OF MCS THEORY IN COVARIANT GAUGES

The Lagrangian for this model in the manifestly covariant gauges can be written as

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} m \epsilon_{\mu\nu\lambda} F^{\mu\nu} A^\lambda - G \partial_\mu A^\mu + \frac{1}{2} (1 - \gamma) G^2 - j_\mu A^\mu + \mathcal{L}_{\bar{e}e} \]  

(2.1)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( G \) is a gauge-fixing field, and \( \gamma \) is a parameter that permits "tuning" to various alternative covariant gauges—for example, to the Feynman \((\gamma = 0)\) and the Landau \((\gamma = 1)\) gauges. \( \mathcal{L}_{\bar{e}e} \) is the Lagrangian for free fermions, and is given by

\[ \mathcal{L}_{\bar{e}e} = \bar{\psi} (i \gamma_\mu D^\mu - M) \psi. \]  

(2.2)

\( \psi \) is the two-component spinor field required for the \((2 + 1)\)-dimensional Dirac equation, and the three \( \gamma^\mu \) are given in terms of the Pauli spin matrices as \( \gamma^0 = -\sigma_3, \gamma^1 = i\sigma_2, \) and \( \gamma^2 = -i\sigma_1 \). The spinor currents take the form \( j^\mu = e \bar{\psi} \gamma^\mu \psi \).

Equation (2.1) leads to the following Euler-Lagrange equations:

\[ \partial^\nu F_{\mu\nu} - \frac{1}{2} m \epsilon_{\mu\nu\lambda} F^{\nu\lambda} - \partial_\mu G + j_\mu = 0, \]  

(2.3)

\[ \partial_\mu A^\mu = (1 - \gamma) G, \]  

(2.4)

and

\[ (i \gamma_\mu D^\mu - M) \psi = 0, \]  

(2.5)

where \( D^\mu \) is the covariant derivative \( D^\mu = \partial^\mu + ie A^\mu \). Current conservation, \( \partial_\mu j^\mu = 0 \), follows from Eq. (2.3) and implies that

\[ \partial_\mu \partial^\mu G = 0. \]  

(2.6)

The gauge fields are not subject to any primary constraints in this formulation of MCS theory in the covariant gauges, and all components of \( A^\mu \) have canonically conjugate momenta. These are defined by

\[ \Pi_l = F_{0l} + \frac{1}{2} m \epsilon_{ln} A_n, \]  

(2.7)

for the momenta conjugate to the spatial components of the gauge field, \( A_l \), and by

\[ \Pi_0 = -G, \]  

(2.8)

for the momentum conjugate to \( A_0 \). We are thus led to the Hamiltonian density

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1For notational simplicity, we will, from here on, generally use a noncovariant notation, in which the subscript \( l \) denotes a covariant component of a covariant quantity (like \( \partial_l \)), a contravariant component of a contravariant quantity (like \( A_l \)), or the contravariant component of the second rank tensor \( \Pi_l \).
\[ H = H_0 + H_1 \] (2.9)

where

\[
H_0 = \frac{1}{2} \Pi_I \Pi_I + \frac{1}{4} F_{ln} F_{ln} - \partial_I A_0 \Pi_I + G \partial_l A_l + \frac{1}{8} m^2 A_I A_I \\
- \frac{1}{2} (1 - \gamma) G^2 - \frac{1}{4} m \varepsilon_{ln} F_{ln} A_0 + \frac{1}{2} m \varepsilon_{ln} A_I \Pi_I + H_{\bar{e}e},
\] (2.10)

with \( H_{\bar{e}e} = \psi^{\dagger} (\gamma_0 M - i \gamma_0 \gamma_l \partial_l) \psi \), and \( H_I = j_0 A_0 - j_l A_l \). The Hamiltonian \( H \) is given by \( H = \int d\mathbf{x} \, H(\mathbf{x}) \) and can be expressed as

\[ H = H_0 + H_1 \] (2.11)

where

\[
H_0 = \int d\mathbf{x} \left[ \frac{1}{2} \Pi_I \Pi_I + \frac{1}{4} F_{ln} F_{ln} + A_0 (\partial_I \Pi_I - \frac{1}{4} m \varepsilon_{ln} F_{ln} + j_0) + G \partial_l A_l \\
+ \frac{1}{8} m^2 A_I A_I - j_l A_l - \frac{1}{2} (1 - \gamma) G^2 + \frac{1}{2} m \varepsilon_{ln} A_I \Pi_I \right] + H_{\bar{e}e},
\] (2.12)

after an integration by parts has been carried out; \( H_{\bar{e}e} = \int d\mathbf{x} \, H_{\bar{e}e}(\mathbf{x}) \), and

\[ H_1 = \int d\mathbf{x} \, (j_0 A_0 - j_l A_l). \] (2.13)

Since each gauge field has a canonical momentum, the equal-time commutation (anticommutation) rules are canonical and are given by

\[
[A_0(x), G(y)] = -i \delta(x - y),
\] (2.14)

\[
[A_l(x), \Pi_n(y)] = i \delta_{ln} \delta(x - y),
\] (2.15)

and

\[
\{ \psi(x), \psi^\dagger(y) \} = \delta(x - y).
\] (2.16)

In order to describe the particle states of this theory—in particular the charged particle states that obey Gauss’s law—we must represent the gauge and spinor fields in terms of creation and annihilation operators for particle excitations. In the case of the spinor fields in 2 + 1 dimensions, a standard representation that uses creation—\( \psi^{\dagger}(k) \) and \( \bar{e}^{\dagger}(k) \)—and annihilation—\( e(k) \) and \( \bar{e}(k) \)—operators for particle modes in definite momentum states is well known [1], and given by

\[
\psi(x) = \sum_k \sqrt{\frac{M}{\omega_k}} \left[ e(k) u(k) e^{i k \cdot x} + \bar{e}^{\dagger}(k) \psi(k) e^{-i k \cdot x} \right]
\] (2.17)

and

\[
\psi^\dagger(x) = \sum_k \sqrt{\frac{M}{\omega_k}} \left[ \bar{e}^{\dagger}(k) u^\dagger(k) e^{-i k \cdot x} + \bar{e}(k) \psi^\dagger(k) e^{i k \cdot x} \right]
\] (2.18)
where $\bar{\omega}_k = \sqrt{M^2 + k^2}$; the excitation operators for the modes of the spinor field obey the anticommutation rules $\{e(k), e^\dagger(q)\} = \delta_{kq}$ and $\{\bar{e}(k), \bar{e}^\dagger(q)\} = \delta_{kq}$ as well as $\{e(k), e(q)\} = \{e^\dagger(k), e^\dagger(q)\} = 0$. The spinors $u(k)$ and $v(k)$ are normalized so that $ar{u}(k)u(k) = 1$ and $ar{v}(k)v(k) = -1$ and are given by

$$u(k) = \frac{1}{\sqrt{2M(M + \bar{\omega}_k)}} \begin{pmatrix} -k_- \\ M + \bar{\omega}_k \end{pmatrix}$$

(2.19)

and

$$v(k) = \frac{1}{\sqrt{2M(M + \bar{\omega}_k)}} \begin{pmatrix} M + \bar{\omega}_k \\ -k_+ \end{pmatrix}$$

(2.20)

where $k_{\pm} = k_1 \pm ik_2$. In a later section of this paper, in which we examine the phase changes effected by $2\pi$ rotations, we will find it convenient to represent the spinor field using creation and annihilation operators for excitations in definite angular momentum states. Such a representation was previously used in Ref. [1].

For the gauge field, “standard” representations are not available. A suitable representation of the gauge fields must be constructed for each gauge. One such representation was constructed for the temporal gauge in Ref. [1]; others, applicable to the covariant and Coulomb gauges, will be given in this paper. To be suitable, a representation must be consistent with the equal-time commutation rules given in Eqs. (2.14) and (2.15). There are too many commutation rules for the gauge field in a covariant gauge to be satisfied with only the single excitation mode of the topologically massive “photon” that exists in MCS theory. Further modes, in the form of ghost excitations, are required. These ghost modes are identical to the ones that appear in the temporal gauge [1], and that also are required in (3 + 1)-dimensional QED (QED$_4$) in covariant and axial (except for the spatial axial) gauges [3]. The excitation operators for the massive photon are the annihilation operator $a(k)$ and its adjoint creation operator $a^\dagger(k)$, which obey the commutation rule $[a(k), a^\dagger(q)] = \delta_{kq}$. Ghost excitation operators exist in pairs; in this work, we will use the ghost annihilation operators $a_Q(k)$ and $a_R(k)$ and their respective adjoint creation operators $a_Q^*(k)$ and $a_R^*(k)$ in representations of the gauge field. Ghost states have zero norm, but the single-particle ghost states $a_Q^*(k)|0\rangle$ and $a_R^*(k)|0\rangle$ have a nonvanishing inner product; similar nonvanishing inner products also arise for $n$-particle states with equal numbers of $Q$ and $R$ ghosts. These properties of the ghost states are implemented by the commutator algebra

$$[a_Q(k), a_R^*(q)] = [a_R(k), a_Q^*(q)] = \delta_{kq}$$

(2.21)

and

$$[a_Q(k), a_Q^*(q)] = [a_R(k), a_R^*(q)] = 0,$$

(2.22)

which, in turn, imply that the unit operator in the one-particle ghost (OPG) sector is

\footnote{From now on, we will refer to the excitation modes of the spinor fields as electrons and positrons, even though we are working in a (2 + 1)-dimensional space.}
and that the obvious generalization of that form apply in \( n\)-particle sectors. The ghost excitations enable us to satisfy all the equal-time commutation relations (ETCR), Eqs. (2.14) and (2.15), even though the gauge field has only a single mode that corresponds to a propagating particle excitation that can, in principle, be detected, and that carries energy and momentum.

There is another criterion that a representation must satisfy in order to be suitable: The photon modes (propagating and ghost) must appear in the Hamiltonian for free, non-interacting propagating photons and charged particles in such a manner that dynamical time-evolution never propagates state vectors into that part of Hilbert space in which inner products between the two different types of ghost states drain probability from the sector of Hilbert space spanned by observable particle states.

A representation of the gauge fields that satisfies these requirements in covariant gauges is given by

\[
A_t(x) = \sum_k \frac{8i k \epsilon_t k n_k}{m^{5/2}} [a_Q(k) e^{ik \cdot x} - a^*_Q(k) e^{-ik \cdot x}]
+ (1 - \gamma) \sum_k \frac{2k l}{m^{3/2}} [a_Q(k) e^{ik \cdot x} + a^*_Q(k) e^{-ik \cdot x}]
- \sum_k \frac{4k^2}{m^{3/2}} [a_Q(k) e^{ik \cdot x} + a^*_Q(k) e^{-ik \cdot x}]
+ \sum_k \frac{m^{3/2} k l}{16k^3} [a_R(k) e^{ik \cdot x} + a^*_R(k) e^{-ik \cdot x}]
- \sum_k \frac{\sqrt{\omega k} k l}{\sqrt{2mk}} [a(k) e^{ik \cdot x} + a^\dagger(k) e^{-ik \cdot x}]
+ \sum_k \frac{i \epsilon_t k n_k}{k \sqrt{2\omega_k}} [a(k) e^{ik \cdot x} - a^\dagger(k) e^{-ik \cdot x}],
\]

\[
\Pi_t(x) = - \sum_k \frac{4i k l}{m^{3/2}} [a_Q(k) e^{ik \cdot x} - a^*_Q(k) e^{-ik \cdot x}]
+ (1 - \gamma) \sum_k \frac{\epsilon_t k n_k}{\sqrt{m}} [a_Q(k) e^{ik \cdot x} + a^*_Q(k) e^{-ik \cdot x}]
+ \sum_k \frac{6k^2 \epsilon_t k n_k}{m^{5/2}} [a_Q(k) e^{ik \cdot x} + a^*_Q(k) e^{-ik \cdot x}]
+ \sum_k \frac{m^{5/2} \epsilon_t k n_k}{32k^3} [a_R(k) e^{ik \cdot x} + a^*_R(k) e^{-ik \cdot x}]
+ \sum_k \frac{im k l}{2^{5/2} k \sqrt{\omega_k}} [a(k) e^{ik \cdot x} - a^\dagger(k) e^{-ik \cdot x}]
+ \sum_k \frac{\sqrt{\omega k} \epsilon_t k n_k}{2^{3/2} k} [a(k) e^{ik \cdot x} + a^\dagger(k) e^{-ik \cdot x}],
\]
\[ A_0(x) = -\sum_k \frac{4k^3}{m^{7/2}} [a_Q(k)e^{ik\cdot x} + a_Q^*(k)e^{-ik\cdot x}] \]

\[ - (1 - \gamma) \sum_k \frac{2k}{m^{3/2}} [a_Q(k)e^{ik\cdot x} + a_Q^*(k)e^{-ik\cdot x}] \]

\[ + \sum_k \frac{m^{3/2}}{16k^2} [a_R(k)e^{ik\cdot x} + a_R^*(k)e^{-ik\cdot x}] \]

\[ - \sum_k \frac{k}{m\sqrt{2}\omega_k} [a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x}] \]  \hspace{1cm} (2.26)

and

\[ G(x) = \sum_k \frac{8ik^2}{m^{3/2}} [a_Q(k)e^{ik\cdot x} - a_Q^*(k)e^{-ik\cdot x}] \]  \hspace{1cm} (2.27)

where \( \omega_k = \sqrt{m^2 + k^2} \). The electric and magnetic fields then are

\[ E_l(x) = -\sum_k \frac{ikl}{k\sqrt{2}\omega_k} [a(k)e^{ik\cdot x} - a^\dagger(k)e^{-ik\cdot x}] \]

\[ - \sum_k \frac{\sqrt{\omega_k\epsilon_{ln}}k_n}{\sqrt{2}k} [a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x}] \]

\[ - \sum_k \frac{8k^2\epsilon_{ln}k_n}{m^{3/2}} [a_Q(k)e^{ik\cdot x} + a_Q^*(k)e^{-ik\cdot x}] \]  \hspace{1cm} (2.28)

and

\[ B(x) = \sum_k \frac{k}{\sqrt{2}\omega_k} [a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x}] \]

\[ + \sum_k \frac{8k^3}{m^{5/2}} [a_Q(k)e^{ik\cdot x} + a_Q^*(k)e^{-ik\cdot x}] \]  \hspace{1cm} (2.29)

When Eqs. (2.24), (2.27) are substituted into \( H_0 \), the Hamiltonian for noninteracting gauge fields and charged spinor fields, we obtain

\[ H_0 = \sum_k \frac{\omega_k}{2} [a^\dagger(k)a(k) + a(k)a^\dagger(k)] \]

\[ + \sum_k \left[ a_Q^\dagger(k)a_R(k) + a_Q(k)a_R^\dagger(k) \right] \]

\[ - (1 - \gamma) \sum_k \frac{64k^4}{m^3} a_Q^\dagger(k)a_Q(k) + H_{ee}. \]  \hspace{1cm} (2.30)

We now turn to consider the Fock space in which the operators in this theory act, and in which \( H_0 \) time translates state vectors. We can construct a Fock space, \( \{|h\rangle\} \), suitable for this model, on the foundation of the perturbative vacuum, \(|0\rangle\), which is annihilated by all the annihilation operators, \( a(k), a_Q(k) \) and \( a_R(k) \), as well as \( e(k) \) and \( \bar{e}(k) \). This perturbative Fock space includes all multiparticle states, \(|N\rangle\), consisting of observable, propagating
that could possibly cause a state vector to leave the subspace \{ |N \rangle \}, and of all states in which a chain of \( a^*_Q(k) \) operators [but no \( a^*_R(k) \) operators] acts on \( |N \rangle \), as \{ |n \rangle \}. States in which both varieties of ghosts appear simultaneously, such as \( a^*_Q(k_1)a^*_R(k_2)|N \rangle \), also are in the Fock space \{ |h \rangle \}, but not in \{ |n \rangle \}; because these states have a nonvanishing norm and contain ghosts, they are not probabilistically interpretable. Their appearance in the course of time evolution signals a defect in the theory. Since the states \( |N \rangle \) constitute the set of states in \{ |n \rangle \} from which all zero norm states (i.e., the ones with ghost constituents) have been excised, we will sometimes speak of the set of \( |N \rangle \) as a quotient space of observable propagating states. The time-evolution operator, \( \exp(-iH_{0}t) \), which excludes the effect of the interaction Hamiltonian, has the important property that if it acts on a state vector \( |\psi\rangle \) in \{ |n \rangle \}, it can only propagate it within \{ |n \rangle \}. We observe that the only parts of \( H_{0} \) that could possibly cause a state vector to leave the subspace \{ |n \rangle \}, are those that contain either \( a^*_R(k) \) or \( a^*_H(k) \) operators. The only part of \( H_{0} \) that has that feature contains the combination of operators \( \Gamma = a^*_R(k)a_Q(k) + a^*_Q(k)a_R(k) \). When \( a_R(k) \) acts on a state vector \( |n_i\rangle \), it either annihilates the vacuum, or it annihilates one of the \( a^*_Q(k) \) operators in \{ |n \rangle \}. In the latter case, \( \Gamma \) replaces the annihilated \( a^*_Q(k) \) operator with an identical one. When \( a_Q(k) \) acts on a state vector \( |n_i\rangle \), it always annihilates it. It is therefore impossible for \( \Gamma \) to produce a state vector external to \{ |n \rangle \}, in which an \( a^*_R(k) \) operator acts on \( |n_i\rangle \). The only effect of \( \Gamma \) is to translate \( |n_i\rangle \) states within \{ |n \rangle \}.

Substitution of Eqs. (2.24)–(2.27) into the \( H_1 \) in Eq. (2.23) leads to an expression in which all gauge-field excitations appear, including creation and annihilation operators for both varieties of ghosts:

\[
H_1 = -\sum_k \frac{4k^3}{m^{7/2}} \left[ a_Q(k)j_0(-k) + a^*_Q(k)j_0(k) \right] \\
- (1 - \gamma) \sum_k \frac{2k}{m^{5/2}} \left[ a_Q(k)j_0(-k) + a^*_Q(k)j_0(k) \right] \\
+ \sum_k \frac{m^{3/2}}{16k^2} \left[ a_R(k)j_0(-k) + a^*_R(k)j_0(k) \right] \\
- \sum_k \frac{k}{m^{3/2} \omega_k} \left[ a(k)j_0(-k) + a^*(k)j_0(k) \right] \\
- \sum_k \frac{8i\epsilon_{mn}k_m}{m^{5/2}} \left[ a_Q(k)j_0(-k) - a^*_Q(k)j_0(k) \right] \\
- (1 - \gamma) \sum_k \frac{2k_l}{m^{3/2}} \left[ a_Q(k)j_0(-k) + a^*_Q(k)j_0(k) \right] \\
+ \sum_k \frac{4k^2k_l}{m^{7/2}} \left[ a_Q(k)j_0(-k) + a^*_Q(k)j_0(k) \right]
\]
\[ \sum_{k} \frac{m^{3/2} k}{16k^3} [a_R(k) j_l(-k) + a_R^*(k) j_l(k)] \\
+ \sum_{k} \frac{\sqrt{\omega} k^3}{\sqrt{2mk}} [a(k) j_l(-k) + a^+(k) j_l(k)] \\
- \sum_{k} \frac{i\epsilon_{ln} k}{k^2 \sqrt{2\omega_k}} [a(k) j_l(-k) - a^+(k) j_l(k)]. \]  

(2.31)

\[ H_I \] contains terms with creation and annihilation operators for both varieties of ghosts—\( a_Q(k) \) and \( a_R(k) \) as well as \( a_Q^*(k) \) and \( a_R^*(k) \)—and therefore threatens to drive state vectors out of the subspace \( \{|n\rangle\} \). The reason for this apparent failure to maintain consistency is that Gauss’s law and the gauge condition \( \partial_{\mu} A^\mu = 0 \) have not yet been implemented. In the next section, we will show how implementation of the constraints prevents the catastrophic appearance of state vectors in which both varieties of ghosts coincide.

### III. IMPLEMENTATION OF GAUSS’S LAW AND GAUGE CONDITION

It is easily seen in Eq. (2.3) that Gauss’s law is not an equation of motion in this theory. The operator \( \mathcal{G} \) that is used to implement Gauss’s law in this model is

\[ \mathcal{G} = -\frac{1}{2} m \epsilon_{ln} F_{ln} + \partial_{l} F_{0l} + j_0; \]  

(3.1)

Eq. (2.3) includes an equation of motion that incorporates \( \mathcal{G} \), in the form \( \mathcal{G} = \partial_{0} G \), in much the same way as in the temporal gauge [1]. Gauss’s law, \( \mathcal{G} = 0 \), still remains to be implemented. Using Eq. (2.7), \( \mathcal{G} \) can also be represented as

\[ \mathcal{G} = \partial_{l} \Pi - \frac{1}{4} m \epsilon_{ln} F_{ln} + j_0. \]  

(3.2)

Substitution of Eqs. (2.24) and (2.25) into Eq. (3.2) leads to

\[ \mathcal{G}(x) = \sum_{k} \frac{8k^3}{m^{3/2}} [a_Q(k)e^{ik\cdot x} + a_Q^*(k)e^{-ik\cdot x}] + j_0(x). \]  

(3.3)

We can write this as

\[ \mathcal{G}(x) = \sum_{k} \frac{8k^3}{m^{3/2}} [\Omega(k)e^{ik\cdot x} + \Omega^*(k)e^{-ik\cdot x}] \]  

(3.4)

where \( \Omega(k) \) is defined by

\[ \Omega(k) = a_Q(k) + \frac{m^{3/2}}{16 k^3} j_0(k); \]  

(3.5)

with \( j_0(k) = \int dx \ j_0(x)e^{-ik\cdot x} \). Similarly, we can express \( G \) as

\[ G(x) = \sum_{k} \frac{8ik^2}{m^{3/2}} [\Omega(k)e^{ik\cdot x} - \Omega^*(k)e^{-ik\cdot x}]. \]  

(3.6)
We can therefore implement Gauss’s law and the gauge condition by embedding the theory in a subspace \{ |\nu\rangle \} of another Fock space, in which all the state vectors |\nu\rangle satisfy the condition

\[ \Omega(k)|\nu\rangle = 0. \quad (3.7) \]

For all state vectors |\nu\rangle and |\nu'\rangle in the physical subspace \{ |\nu\rangle \}, it can be seen from Eqs. (3.4) and (3.6) that \langle \nu'|G|\nu\rangle = 0 and \langle \nu'|G|\nu\rangle = 0. Moreover, the condition \Omega(k)|\nu\rangle = 0, once established, continues to hold at all other times because

\[ [H, \Omega(k)] = -k\Omega(k) \quad (3.8) \]

so that \( e^{iHt}\Omega(k)e^{-iHt} = \Omega(k)e^{-ikt} \), and \( \Omega(k)e^{-ikt}|\nu\rangle = 0 \) follows from Eq. (3.7). This means that a state vector initially in the physical subspace \{ |\nu\rangle \} will always remain entirely contained in it as it develops under time evolution. These considerations show that the subspace \{ |\nu\rangle \} must be used to secure the implementation of Gauss’s law and the gauge condition. We therefore direct our attention to this subspace and note that \{ |\nu\rangle \} can be related unitarily to the subspace \{ |n\rangle \} by the unitary transformation \( U = e^D \) for which

\[ U^{-1}\Omega(k)U = a_Q(k), \quad (3.9) \]

where \( D \) is given by

\[
D = i \int dx \ dy \ \xi(|x - y|) \left[ \epsilon_{ln}\partial_l \Pi_n(x) - \frac{1}{2} m \hat{\partial}_l A_l(x) \right] j_0(y) \\
+ i \int dx \ dy \ \chi(|x - y|) G(x) j_0(y) + i \int dx \ dy \ \zeta(|x - y|) G(x) j_0(y) \\
+ i \int dx \ dy \ \eta(|x - y|) \left[ \hat{\partial}_l \Pi_l(x) + \frac{1}{2} m \epsilon_{ln}\partial_l A_n(x) \right] j_0(y)
\]

and where

\[ \xi(|x - y|) = \sum_k \frac{1}{m} \frac{e^{ik(x-y)}}{k^2}, \quad (3.11) \]

and

\[ \chi(|x - y|) = \sum_k \frac{1}{4} \left[ (1 - \gamma) + \frac{2k^2}{m^2} \right] \frac{e^{ik(x-y)}}{k^2}. \quad (3.12) \]

\( \zeta(|x - y|) \) and \( \eta(|x - y|) \) are two additional functions that can be included. They can be expressed as

\[ \zeta(|x - y|) = -\sum_k \frac{m^{3/2} \theta(k)}{8} \frac{e^{ik(x-y)}}{k^2}, \quad (3.13) \]

and

\[ \eta(|x - y|) = \sum_k \frac{m^{3/2} \phi(k)}{8} \frac{e^{ik(x-y)}}{k^3}. \quad (3.14) \]
with \( \theta(k) \) and \( \phi(k) \) as some arbitrary real and even functions of \( k \). The operator \( D \) can also be expressed as

\[
D = \sum_k \frac{m^{3/2}}{16k^3} \left[ a_R(k)j_0(-k) - a_R^*(k)j_0(k) \right] \\
+ \sum_k \theta(k) \left[ a_Q(k)j_0(-k) - a_Q^*(k)j_0(k) \right] \\
+ \sum_k i\phi(k) \left[ a_Q(k)j_0(-k) + a_Q^*(k)j_0(k) \right].
\] (3.15)

The unitary operator \( U \) can be used to establish a mapping that maps \( \Omega(k) \rightarrow a_Q(k) \) and \( \{|\nu\} \rightarrow \{|n\} \), where \( \{|n\} \) is the subspace described in the preceding section. The required state vectors \( |n\rangle = U^{-1}|\nu\rangle \) are those in which \( a_Q^*(k) \) and \( a^\dagger(k) \), as well as electron and positron creation operators, act on the perturbative vacuum state \( |0\rangle \). But no \( a_R^*(k) \) operators may appear in the states \( |n\rangle \) for which \( |\nu\rangle = U|n\rangle \) comprise the subspace \( \{|\nu\} \), since for states \( |h_R\rangle \) in which \( a_R^*(k) \) operators act on the observable multiparticle state \( |N\rangle \), \( a_Q(k)|h_R\rangle \neq 0 \). Therefore, \( \Omega(k)|\rho\rangle \neq 0 \) for the states \( |\rho\rangle = U|h_R\rangle \).

We can exploit the existence of the unitary operator \( U \) to systematically construct the subspace \( \{|\nu\} \) from \( \{|n\} \). Alternatively, we can use \( U \) to transform all the operators we have previously defined, and to use \( \{|n\} \) as the representation of the subspace that fully implements Gauss's law and the gauge condition with all interactions included in the Hamiltonian. In order to make this latter choice, we note that, in the mapping effected by \( U \), operators \( \mathcal{P} \) map into \( \mathcal{P} \), i.e., \( U^{-1}\mathcal{P}U = \mathcal{P} \). For example, \( \tilde{\Omega}(k) = a_Q(k) \) so that Eq. (3.7) takes the form \( a_Q(k)|n\rangle = 0 \); and it is this equation that implements Gauss's law and the gauge condition, \( \partial_\mu A_\mu = 0 \), in this alternate transformed representation. Using the Baker-Hausdorff-Campbell formula, we find that the transformed Hamiltonian, \( \tilde{H} = U^{-1}HU \) is given by

\[
\tilde{H} = H_0 + \tilde{H}_1;
\] (3.16)

here \( H_0 \) is the untransformed noninteracting Hamiltonian given in Eq. (2.30) and

\[
\tilde{H}_1 = \sum_k \frac{i\epsilon_n k_l}{mk^2} j_n(k)j_0(-k) + \sum_k \frac{1}{2m^2} j_0(k)j_0(-k) \\
+ \sum_k \frac{3im^{3/2}k_l\phi(k)}{16k^3} j_l(k)j_0(-k) \\
- \sum_k \frac{k}{m\sqrt{2}\omega_k} \left[ a(k)j_0(-k) + a^\dagger(k)j_0(k) \right] \\
+ \sum_k \frac{\sqrt{\omega_kk_l}}{\sqrt{2}mk} \left[ a(k)j_l(-k) + a^\dagger(k)j_l(k) \right] \\
- \sum_k \frac{i\epsilon_n k_n}{k\sqrt{2}\omega_k} \left[ a(k)j_l(-k) - a^\dagger(k)j_l(k) \right] \\
- \sum_k \frac{4k^3}{m^{7/2}} \left[ a_Q(k)j_0(-k) + a_Q^*(k)j_0(k) \right]
\]
The similarly transformed fields are

\[
\begin{align*}
\tilde{A}_l(x) &= A_l(x) - \sum_k \frac{ik\epsilon_{ln}k_n}{mk^2} j_0(k)e^{ikx} + \sum_k \frac{im^{3/2}k_l\phi(k)}{8k^3} j_0(k)e^{ikx}, \\
\tilde{\Pi}_l(x) &= \Pi_l(x) + \sum_k \frac{ik_l}{2k^2} j_0(k)e^{ikx} + \sum_k \frac{im^{5/2}\epsilon_{ln}k_n\phi(k)}{16k^3} j_0(k)e^{ikx}, \\
\tilde{A}_0(x) &= A_0(x) + (1 - \gamma) \sum_k \frac{1}{4k^2} j_0(k)e^{ikx} \\
&\quad + \sum_k \frac{1}{2m^2} j_0(k)e^{ikx} - \sum_k \frac{m^{3/2}\theta(k)}{8k^2} j_0(k)e^{ikx}, \\
\tilde{G}(x) &= G(x) - \sum_k \frac{i}{k} j_0(k)e^{ikx}, \\
\tilde{\gamma} &= \partial_l \Pi_l - \frac{i}{4} m\epsilon_{ln}F_{ln}, \\
\tilde{\psi}(x) &= e^{D(x)}\psi(x),
\end{align*}
\]

where

\[
D(x) = -ie \int dy \left\{ \xi(|x - y|) \left[ \epsilon_{ln}\partial_l\Pi_n(y) - \frac{1}{2}m\partial_l A_l(y) \right] + \chi(|x - y|)G(y) \\
+ \eta(|x - y|) \left[ \partial_l \Pi_l(x) + \frac{1}{2}m\epsilon_{ln}\partial_l A_n(x) \right] + \zeta(|x - y|)G(x)j_0(y) \right\}.
\]
We note that under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$, $\mathcal{D}(x) \to \mathcal{D}(x) - i e \chi$, so that, since $\psi \to \psi \exp(i e \chi)$, $\psi$ is gauge-invariant. The transformed electric and magnetic fields are

$$\tilde{E}_t(x) = E_t(x)$$  \hspace{1cm} (3.25)

and

$$\tilde{B}(x) = B(x) + \mathcal{B}(x)$$  \hspace{1cm} (3.26)

where $E_t(x)$ and $B(x)$ are given by Eqs. (2.28) and (2.29), respectively, and

$$\mathcal{B}(x) = - \frac{j_0(x)}{m}.$$  \hspace{1cm} (3.27)

In this equivalent, alternative representation, $\exp(-i \tilde{H} t)$ is the time-translation operator. The time-translation operator will time translate state vectors entirely within the physical subspace in the transformed representation if $\tilde{H}$ is entirely devoid of $a_R^*(k)$ and $a_R(k)$ operators, or if it contains them at most in the combination $\Gamma = a_R^*(k)a_Q(k) + a_Q^*(k)a_R(k)$. Inspection of Eqs. (2.30) and (3.17) confirms that $\tilde{H}$ is, in fact, entirely devoid of $a_R^*(k)$ and $a_R(k)$ operators except those that appear in the combination $\Gamma$, so that the time-translation operator $\exp(-i \tilde{H} t)$ correctly satisfies this requirement. Observable states in the alternative transformed representation are described by state vectors in $\{|n\rangle\}$ which we designate as $|N\rangle$. These observable states consist of massive photons, electrons and positrons only, and have a positive norm. The operator $\exp(-i \tilde{H} t)$ time translates such state vectors by generating a new state vector, at a later time $t$, which consists of further positive-norm state vectors $|N'\rangle$, as well as additional ghost states, all of which are represented by products of $a_Q^*(k)$ operators acting on positive-norm observable sets of states. At all times, the positive-norm states alone just saturate unitarity. We can define a quotient space consisting of the state vectors $|N\rangle$, which is the residue of $\{|n\rangle\}$ after all zero-norm states have been excised from it. We can also define another Hamiltonian $\tilde{H}_{\text{quot}}$, which consists of those parts of $\tilde{H}$ that remain after we have removed all the terms in which $a_Q^*(k)$ or $a_Q(k)$ is a factor. This Hamiltonian is given by

$$\tilde{H}_{\text{quot}} = H_{ee} + \sum_k \frac{\omega_k}{2} \left[ a^\dagger(k) a(k) + a(k) a^\dagger(k) \right]$$

$$+ \sum_k \frac{i e_{ln} k_l}{mk^2} j_n(k) j_0(-k) + \sum_k \frac{1}{2m^2} j_0(k) j_0(-k)$$

$$+ \sum_k \frac{3im^{3/2} \phi(k)}{16k^3} j_l(k) j_0(-k)$$

$$- \sum_k \frac{k}{m \sqrt{2\omega_k}} \left[ a(k) j_0(-k) + a^\dagger(k) j_0(k) \right]$$

$$+ \sum_k \frac{\sqrt{\omega_k} k_l}{\sqrt{2mk}} \left[ a(k) j_l(-k) + a^\dagger(k) j_l(k) \right]$$

$$- \sum_k \frac{i e_{ln} k_l}{k \sqrt{2\omega_k}} \left[ a(k) j_l(-k) - a^\dagger(k) j_l(k) \right].$$  \hspace{1cm} (3.28)
It is manifest that the state vectors \( \exp\left(-i\hat{H}t\right) |N\rangle \) and \( \exp\left(-i\hat{H}_{\text{quot}}t\right) |N\rangle \) have identical projections on the set of state vectors \(|N\rangle\) that define the quotient space. The parts of \( \hat{H} \) that contain \( a_Q^*(\mathbf{k}) \) or \( a_Q(\mathbf{k}) \) as factors therefore do not play any role in the time evolution of state vectors within the quotient space of observable states, and cannot have any effect on the physical predictions of the theory.

To facilitate the comparison of the results we obtained in this covariant gauge formulation of MCS theory with those derived in other gauges \([1]\), we make another unitary transformation to a new representation so that operators \( \tilde{P} \) are transformed to \( \hat{P} = e^{\Lambda} \hat{P} e^{-\Lambda} \) where

\[
\Lambda = \sum_{\mathbf{k}} \frac{k}{\sqrt{2m\omega_k^3}} \left[ a(\mathbf{k}) j_0(\mathbf{-k}) - a^\dagger(\mathbf{k}) j_0^\dagger(\mathbf{k}) \right]. \tag{3.29}
\]

The advantage of this representation lies in the fact that, after the \( \hat{P} \rightarrow \hat{P} \) transformation, no interactions remain in \( \hat{H} \) that couple \( j_0 \) to \( a^\dagger(\mathbf{k}) \) or \( a(\mathbf{k}) \). The “static” interactions of charged particles at rest with photons have therefore been entirely absorbed into the nonlocal interactions among charge and current densities. It is very convenient to normalize Hamiltonians in all gauges to this common form, and to use the subspace \( \{|n\rangle\} \) as the Hilbert space in which the operators in the \( \hat{P} \) representation act. In this new representation, the Hamiltonian is given by

\[
\hat{H} = H_0 + \sum_{\mathbf{k}} \frac{im\epsilon_{ln}k_l}{k^2\omega_k^3} j_n(\mathbf{k}) j_0(\mathbf{-k}) + \sum_{\mathbf{k}} \frac{1}{2\omega_k^2} j_0^\dagger(\mathbf{k}) j_0(\mathbf{-k})
\]
\[
+ \sum_{\mathbf{k}} \frac{3im^3/2k_l\phi(\mathbf{k})}{16k^4} j_l(\mathbf{k}) j_0^\dagger(\mathbf{-k})
\]
\[
+ \sum_{\mathbf{k}} \frac{mk_l}{\sqrt{2k^3\omega_k^3}} \left[ a(\mathbf{k}) j_l(\mathbf{-k}) + a^\dagger(\mathbf{k}) j_l^\dagger(\mathbf{k}) \right]
\]
\[
- \sum_{\mathbf{k}} \frac{i\epsilon_{ln}k_n}{k\sqrt{2}\omega_k} \left[ a(\mathbf{k}) j_l(\mathbf{-k}) - a^\dagger(\mathbf{k}) j_l^\dagger(\mathbf{k}) \right]
\]
\[
- \sum_{\mathbf{k}} \frac{4k^3}{m^{3/2}} \left[ a_Q(\mathbf{k}) j_0^\dagger(\mathbf{-k}) + a_Q^*(\mathbf{k}) j_0(\mathbf{k}) \right]
\]
\[
- \sum_{\mathbf{k}} \frac{8ik\epsilon_{ln}k_n}{m^{5/2}} \left[ a_Q(\mathbf{k}) j_l^\dagger(\mathbf{-k}) - a_Q^*(\mathbf{k}) j_l^\dagger(\mathbf{k}) \right]
\]
\[
+ \sum_{\mathbf{k}} \frac{4k^2k_l}{m^{7/2}} \left[ a_Q(\mathbf{k}) j_l^\dagger(\mathbf{-k}) + a_Q^*(\mathbf{k}) j_l(\mathbf{k}) \right]
\]
\[
+ (1 - \gamma) \sum_{\mathbf{k}} \frac{2k}{m^{3/2}} \left[ a_Q(\mathbf{k}) j_0^\dagger(\mathbf{-k}) + a_Q^*(\mathbf{k}) j_0(\mathbf{k}) \right]
\]
\[
- (1 - \gamma) \sum_{\mathbf{k}} \frac{2k_l}{m^{3/2}} \left[ a_Q(\mathbf{k}) j_l(\mathbf{-k}) + a_Q^*(\mathbf{k}) j_l^\dagger(\mathbf{k}) \right]
\]
\[
+ \sum_{\mathbf{k}} k\theta(\mathbf{k}) \left[ a_Q(\mathbf{k}) j_0^\dagger(\mathbf{-k}) + a_Q^*(\mathbf{k}) j_0(\mathbf{k}) \right]
\]
\[
- \sum_{\mathbf{k}} ik\phi(\mathbf{k}) \left[ a_Q(\mathbf{k}) j_0^\dagger(\mathbf{-k}) - a_Q^*(\mathbf{k}) j_0(\mathbf{k}) \right]
\]
\[ + \sum_{k} k_i \theta(k) \left[ a_Q(k) j_i(-k) + a^*_Q(k) j_i(k) \right] \]
\[ + \sum_{k} i k_i \phi(k) \left[ a_Q(k) j_i(-k) - a^*_Q(k) j_i(k) \right] . \] (3.30)

The similarly transformed fields are
\[
\hat{\Delta}(x) = A_I(x) - \sum_{k} \frac{i m \epsilon_{l n} k_n}{k^2 \omega^2_k} j_0(k) e^{i k \cdot x} , \] (3.31)
\[
\hat{\Pi}(x) = \Pi_I(x) + \sum_{k} \frac{i k_l (k^2 + \omega^2_k)}{2 k^2 \omega^2_k} j_0(k) e^{i k \cdot x} , \] (3.32)
\[
\hat{\Delta}(x) = A_0(x) + (1 - \gamma) \sum_{k} \frac{1}{4 k^2} j_0(k) e^{i k \cdot x} \]
\[ + \sum_{k} \frac{1}{2 m^2} j_0(k) e^{i k \cdot x} - \sum_{k} \frac{m^3/2 \theta(k)}{8 k^2} j_0(k) e^{i k \cdot x} , \] (3.33)
\[
\hat{G}(x) = G(x) - \sum_{k} \frac{i}{k} j_0(k) e^{i k \cdot x} , \] (3.34)

and the gauge-invariant \( \hat{\psi}(x) \)
\[
\hat{\psi}(x) = e^{D'(x)} \psi(x) \] (3.35)

where
\[
D'(x) = -i e \int d y \left[ \xi'(|x - y|) \partial_l A_l(x) + \chi'(|x - y|) \epsilon_{l m n} \partial_l A_n(x) + \zeta'(|x - y|) G(x) \right] \] (3.36)

with
\[
\xi'(|x - y|) = \sum_{k} \frac{1}{2 \omega^2_k} e^{i k \cdot (x - y)} , \] (3.37)
\[
\chi'(|x - y|) = \sum_{k} \frac{1}{m \omega^2_k} e^{i k \cdot (x - y)} , \] (3.38)

and
\[
\zeta'(|x - y|) = \sum_{k} \frac{k^2}{m^2 \omega^2_k} e^{i k \cdot (x - y)} . \] (3.39)

The transformed electric and magnetic field are
\[
\hat{E}_I(x) = E_I(x) + \hat{E}_I(x) \] (3.40)
and
\[ \hat{B}(x) = B(x) + \hat{\mathcal{B}}(x) \]  
(3.41)

where \( E_l(x) \) and \( B(x) \) are given by Eqs. (2.28) and (2.29), respectively, and
\[ \hat{E}_l(x) = -\frac{1}{2\pi} \frac{\partial}{\partial x_l} \int dy \ K_0(m|x - y|) j_0(y), \]  
(3.42)

and
\[ \hat{\mathcal{B}}(x) = -\frac{m}{2\pi} \int dy \ K_0(m|x - y|) j_0(y). \]  
(3.43)

The quotient space Hamiltonian corresponding to the Hamiltonian \( \hat{H} \) given by Eq. (3.30) can be written as
\[ \hat{H}_{\text{quot}} = H_{\text{eff}} + \sum_k \omega_k \left[ a_k^\dagger a_k + a_k a_k^\dagger \right] + \hat{H}_1 \]  
(3.44)

where
\[ \hat{H}_1 = \int dxdy \ j_0(x) \epsilon_l j_l(y)(x - y)_n \mathcal{F}(|x - y|) \]
\[ + \int dxdy \ j_0(x) j_0(y) K_0(m|x - y|) \]
\[ + \sum_k \frac{3im^{3/2} k_l \phi(k)}{16k^3} j_l(k)j_0(-k) \]
\[ + \sum_k \frac{mk_l}{\sqrt{2}\omega_k^{3/2}} \left[ a(k) j_l(-k) + a_l^\dagger(k) j_l(k) \right] \]
\[ - \sum_k \frac{ie_l k_n}{k \sqrt{2}\omega_k} \left[ a(k) j_l(-k) - a_l^\dagger(k) j_l(k) \right] \]  
(3.45)

and where \( K_0(x) \) is a modified Bessel function and\(^3\)
\[ \mathcal{F}(R) = -\frac{m}{2\pi} \int_0^\infty du \ \frac{J_1(u)}{u^2 + (mR)^2}. \]  
(3.46)

We observe that \( \mathcal{F}(R) \) approaches the limits
\[ \lim_{mR \to 0} \mathcal{F}(R) = \frac{1}{4\pi R} \]  
(3.47)

and
\[ \mathcal{F}(R) \to \frac{1}{2\pi mR^2} \]  
(3.48)

\(^3\)The definition of \( \mathcal{F}(R) \) given in Ref. [1] is lacking a factor \( 1/2\pi \).
as $mR \to \infty$. The interaction Hamiltonian $\hat{H}_I$ describes the interaction of massive photons with charged fermions. It also describes nonlocal interactions between charged fermions. These interactions include the $(2 + 1)$-dimensional analogue of the Coulomb interaction, with the inverse power of distance between charges replaced by the modified Bessel function $K_0(m|x - y|)$. Another such interaction, which has no analogue in QED$_4$, couples charges and the transverse components of currents. The expressions for the nonlocal interactions among charge and current densities that result from the elimination of “ghost” components of the gauge fields, are well behaved and free from the kind of infrared singularities that one might anticipate from massless particle exchange in a $(2 + 1)$-dimensional model. The Hamiltonian $\hat{H}_I$, which is obtained from the implementation of Gauss’s law and the gauge condition $\partial_\mu A^\mu = 0$, is identical to the Hamiltonian we obtained previously by implementing Gauss’s law and the gauge condition $A_0 = 0$. This identity extends also to the electric field, $E_i(x)$, and the magnetic field, $B(x)$, which are identical to the corresponding expressions for the electric and magnetic fields in the $A_0 = 0$ gauge. These identities make the gauge invariance of this theory very manifest, because we have eliminated physically meaningless differences in form that arise when unitary equivalence between sets of dynamical variables have not been fully recognized and used to demonstrate gauge equivalence. Later in this paper we will extend this gauge equivalence to the Coulomb gauge.

The term

$$h = \sum_k \frac{3im^{3/2}k\phi(k)}{16k^3} j_i(k)j_0(-k)$$

(3.49)

that appears in the Hamiltonian given by Eq. (3.30) is a total time-derivative which can be expressed as $h = i[H_0, \chi]$ or as $h = i[\hat{H}, \chi]$ where

$$\chi = -\sum_k \frac{3m^{3/2}\phi(k)}{32k^3} j_0(k)j_0(-k).$$

(3.50)

The fact that $h$ is a total time-derivative gives us a priori confidence that it will not affect the $S$-matrix produced by this theory. A formal argument that confirms this result has been given previously and will not be repeated here [6].

The fact that $h$ is a total time-derivative of $\chi$, and that $\chi$ and $h$ commute, establishes the relationship

$$e^{-ie\chi}\hat{H}e^{ie\chi} - \hat{H} = e^{-ie\chi}\dot{\hat{H}}e^{ie\chi} - \hat{H} = e^{-ie\chi}H_0e^{ie\chi} - H_0 = h.$$  

(3.51)

This, in turn, demonstrates that if we combine the Hamiltonian $\hat{H}$ with the Fock space $\{|n\rangle\}$, the resulting formalism will be unitarily equivalent to the Hamiltonian $\hat{H} + h$ combined with the Hilbert space $\{e^{-ie\chi}|n\rangle\}$. The choice of a Hilbert space in which a Hamiltonian and the other dynamical variables of a model are to act, is an independent assumption in the axiomatic structure of the theory. We could equally well have chosen to combine the Hamiltonian $\hat{H} + h$ with the Fock space $\{|n\rangle\}$, simply by choosing a nonvanishing $\phi(k)$. The $S$-matrix would not have been affected by that substitution, but there would have been changes in the time-evolution of state vectors at times $t$ remote from the asymptotic regions $t \to \pm\infty$; the effects of these changes in time-evolution cancel by the time the asymptotic region $t \to \infty$ is reached. It would be desirable to have a “natural” principle for
associating Hamiltonians and Hilbert spaces, but when the substitution of one Hamiltonian for another has no effect on the $S$-matrix, there are no physical reasons for preferring one combination over the other. We have used a “minimal” principle in our work, somewhat in the spirit of the “minimal coupling” rule for coupling gauge fields to matter. This minimal principle dictates that parts of Hamiltonians, like $h$, that make no contribution at all to the $S$-matrix, are excluded in representations in which the Fock space $\{|n\rangle\}$ represents the states that implement Gauss’s law and the gauge condition. This principle does not help to make a selection in every case, but it answers the need adequately in the case of Abelian gauge theories.

IV. THE PERTURBATIVE REGIME

The perturbative theory involves the vertices dictated by the interaction Hamiltonian given in Eq. (2.31) and the propagators for the interaction-picture operators $\psi(x)$, $\bar{\psi}(x)$, and $A^\mu(x)$ obtained from $\mathcal{P}(x) = \exp(iH_0t)\mathcal{P}(x)\exp(-iH_0t)$. The gauge fields in the interaction picture are found to be

\[
A_t(x) = -\sum_k \frac{\sqrt{\omega k k_l}}{\sqrt{2mk}} \left[ a(k)e^{-ik_\mu x^\mu} + a^\dagger(k)e^{ik_\mu x^\mu} \right]
\]

\[
+ \sum_k \frac{\imath \epsilon_{ln} k_n}{k\sqrt{2\omega_k}} \left[ a(k)e^{-ik_\mu x^\mu} - a^\dagger(k)e^{ik_\mu x^\mu} \right]
\]

\[
+ \frac{8\imath k \epsilon_{ln} k_n}{m^{5/2}} \left[ a_Q(k)e^{-ik'_\mu x^\mu} - a_Q^*(k)e^{ik'_\mu x^\mu} \right]
\]

\[
+ (1 - \gamma) \sum_k \frac{2k_l}{m^{3/2}} \left[ a_Q(k)e^{-ik'_\mu x^\mu} + a_Q^*(k)e^{ik'_\mu x^\mu} \right]
\]

\[
- \sum_k \frac{4k^2 k_l}{m^{7/2}} \left[ a_Q(k)e^{-ik'_\mu x^\mu} + a_Q^*(k)e^{ik'_\mu x^\mu} \right]
\]

\[
+ \frac{m^{3/2} k_l}{16k^3} \left[ a_R(k)e^{-ik'_\mu x^\mu} + a_R^*(k)e^{ik'_\mu x^\mu} \right]
\]

\[
+ (1 - \gamma) \sum_k \frac{4\imath x_0 k k_l}{m^{3/2}} \left[ a_Q(k)e^{-ik'_\mu x^\mu} - a_Q^*(k)e^{ik'_\mu x^\mu} \right]
\]

(4.1)

and

\[
A_0(x) = -\sum_k \frac{k}{m\sqrt{2\omega_k}} \left[ a(k)e^{-ik_\mu x^\mu} + a^\dagger(k)e^{ik_\mu x^\mu} \right]
\]

\[
- \sum_k \frac{4k^3}{m^{7/2}} \left[ a_Q(k)e^{-ik'_\mu x^\mu} + a_Q^*(k)e^{ik'_\mu x^\mu} \right]
\]

\[
- (1 - \gamma) \sum_k \frac{2k}{m^{5/2}} \left[ a_Q(k)e^{-ik'_\mu x^\mu} + a_Q^*(k)e^{ik'_\mu x^\mu} \right]
\]

\[
+ \sum_k \frac{m^{3/2}}{16k^2} \left[ a_R(k)e^{-ik'_\mu x^\mu} + a_R^*(k)e^{ik'_\mu x^\mu} \right]
\]
\[ D_{\mu\nu}(x, y) = (1 - \gamma) \sum_k \frac{k_l k_n}{4k^3} \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) + e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right] \\
+ (1 - \gamma) \sum_k \frac{i k_l k_n (x_0 - y_0)}{4k^2} \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) - e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right] \\
- \sum_k \frac{k_l k_n}{2m^2k} \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) + e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right] \\
+ \sum_k \frac{i \epsilon_{ln} k_n}{2m} \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) - e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right] \\
- \sum_k \frac{i \epsilon_{ln} k_n}{2m} \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) - e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right] \\
+ \sum_k \frac{1}{2\omega_k} \left[ \delta_{ln} + \frac{k_l k_n}{m^2} \right] \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) - e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right], \tag{4.4} \]

\[ D_{00}(x, y) = -(1 - \gamma) \sum_k \frac{1}{4k} \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) + e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right] \\
+ (1 - \gamma) \sum_k \frac{i (x_0 - y_0)}{4} \left[ e^{-ik'_\mu(x-y)^\mu} \Theta(x_0 - y_0) - e^{ik'_\mu(x-y)^\mu} \Theta(y_0 - x_0) \right]. \tag{4.5} \]
\[
- \sum_{k} \frac{k}{2m^2} \left[ e^{-ik'_{\mu}(x-y)^{\mu}} \Theta(x_0 - y_0) + e^{ik'_{\mu}(x-y)^{\mu}} \Theta(y_0 - x_0) \right] \\
+ \sum_{k} \frac{k^2}{2m^2 \omega_k} \left[ e^{-ik_{\mu}(x-y)^{\mu}} \Theta(x_0 - y_0) + e^{ik_{\mu}(x-y)^{\mu}} \Theta(y_0 - x_0) \right]
\]

which can be represented as

\[
D^{\mu\nu}(x, y) = -i \int \frac{d^3k}{(2\pi)^3} D^{\mu\nu}(k) e^{-ik_{\mu}(x-y)^{\mu}}
\]

where

\[
D^{\mu\nu}(k) = \begin{pmatrix}
(1 - \gamma)(k^\mu k^\nu) + k^\mu k^\nu & k^\mu k^\nu \\
(2m^2 - \gamma)k^\alpha k^\beta m^2 & (2m^2 - \gamma)k^\alpha k^\beta m^2
\end{pmatrix} + \frac{g^{\mu\nu}}{k^\alpha k^\beta - \epsilon} + \frac{im_{\epsilon k^\alpha k^\beta - \epsilon}}{k^\alpha k^\beta - \epsilon}.
\]

in Eqs. (4.7) and (4.8), \(k_0, k_1\) and \(k_2\) are independent variables.

V. POINCARÉ STRUCTURE

The consistency of the canonical formulation of this model can be given further support by constructing the canonical Poincaré generators, and using the canonical commutation rules given in Sec. II to demonstrate that they implement the required algebra. In 2 + 1 dimensions, the Poincaré group has six generators: one \((J)\) for rotation, two \((K_i)\) for boosts, and two \((P_l)\) and one \((P_0)\) for space and time translations, respectively. The translation operators \(P_l\) and \(P_0\) can be written as

\[
P_l = \int dx \, \mathcal{P}_l(x)
\]

and

\[
P_0 = \int dx \, \mathcal{P}_0(x)
\]

where \(\mathcal{P}_0(x) = \mathcal{H}(x)\), with \(\mathcal{H}(x)\) given by Eq. (2.9), and the canonical form of \(\mathcal{P}_l\) is given by

\[
\mathcal{P}_l(x) = G(x) \partial_t A_0(x) - \Pi_n(x) \partial_t A_n(x) - i \psi^\dagger(x) \partial_t \psi(x).
\]

Similarly, we follow the canonical procedure [7] and express the rotation and boost operators \(J\) and \(K_i\), respectively, as

\[
J = \int dx \, \epsilon_{\mu nm} x_n \mathcal{P}_m(x) + \int dx \, \kappa_{\text{rotation}}(x)
\]

and

\[
K_l = x_0 P_l - \int dx \, x_l \mathcal{P}_0(x) + \int dx \, \kappa_{\text{boost}}^l(x)
\]
where

\[ \kappa_{\text{rotation}} = \epsilon_{ln} A_l \Pi_n - \frac{1}{2} \psi^\dagger \gamma_0 \psi \]  

(5.6)

and

\[ \kappa_l^{\text{boost}} = - A_l G + A_0 \Pi_l + \frac{1}{2} i \psi^\dagger \gamma_0 \gamma_l \psi. \]  

(5.7)

The term \( \kappa_{\text{rotation}} \) implements the mixing of the space components of the fields during a rotation. It arises from the fact that, under an infinitesimal rotation \( \delta \theta \) about an axis perpendicular to the 2-D plane, the components of \( A^\mu \) change as follows:

\[ \delta A_l(x) = - [\epsilon_{ij} x_i \partial_j A_l(x) + \epsilon_{ln} A_n(x)] \delta \theta \]  

(5.8)

and

\[ \delta A_0(x) = - \epsilon_{ij} x_i \partial_j A_0(x) \delta \theta; \]  

(5.9)

the spinor field transforms under rotations as

\[ \delta \psi(x) = - [\epsilon_{ij} x_i \partial_j \psi(x) - \frac{1}{2} i \gamma_0 \psi(x)] \delta \theta, \]  

(5.10)

so that \( \delta (\bar{\psi} \gamma_l \psi) = - [\epsilon_{ij} x_i \partial_j (\bar{\psi} \gamma_l \psi) + \epsilon_{ln} (\bar{\psi} \gamma_n \psi)] \delta \theta \). Similarly, the term \( \kappa_l^{\text{boost}} \) in \( K_l \) mixes the space-time components of the fields under a boost. For example, under an infinitesimal boost \( \delta \beta_l \) along the \( l \)-direction, the components of \( A^\mu \) transform as follows

\[ \delta A_0(x) = - [x_0 \partial_l A_0(x) + x_l \partial_0 A_0(x) - A_l(x)] \delta \beta_l \]  

(5.11)

and

\[ \delta A_l(x) = - [x_0 \partial_l A_l(x) + x_l \partial_0 A_l(x) - \delta_{il} A_0(x)] \delta \beta_l. \]  

(5.12)

The spinor field transforms as

\[ \delta \psi(x) = - [x_0 \partial_l \psi(x) + x_l \partial_0 \psi(x) - \frac{1}{2} i \gamma_0 \gamma_l \psi(x)] \delta \beta_l. \]  

(5.13)

Using Eq. (5.5), we can also demonstrate the mixing of the electric field \( E_l \) and the magnetic field \( B \) under a Lorentz transformation:

\[ \delta E_l = - [x_0 \partial_n E_l + x_n \partial_0 E_l + \epsilon_{ln} B] \delta \beta_n \]  

(5.14)

and

\[ \delta B = - [x_0 \partial_n B + x_n \partial_0 B + \epsilon_{ln} E_l] \delta \beta_n. \]  

(5.15)

Use of the canonical commutation rules leads to the following commutation rules for the Poincaré generators:

\[ [P_l, P_n] = 0, \]  

(5.16)
\[ [H, P_i] = [H, J] = 0, \quad (5.17) \]
\[ [H, K_i] = iP_i, \quad (5.18) \]
\[ [P_i, K_n] = i\delta_{in}H, \quad (5.19) \]
\[ [P_i, J] = -i\epsilon_{in}P_n, \quad (5.20) \]
\[ [J, K_i] = i\epsilon_{in}K_n, \quad (5.21) \]

and
\[ [K_i, K_n] = -i\epsilon_{in}J. \quad (5.22) \]

We observe that these commutation rules form a closed Lie algebra, and that they are consistent with the transformations given in Eqs. (5.8)–(5.13). The angular momentum, which is an axial vector in three-dimensional space, degenerates into a scalar in two dimensions. All spatial and temporal displacements commute; momentum and angular momentum are time-displacement invariant. Equations (5.18) and (5.19) express the infinitesimal Lorentz transformation in 2 + 1 dimensions.

The consistency of the Lie algebra formed by these canonical Poincaré generators for this model supports the use we make of the angular momentum, \( J \), to implement rotations. We will discuss this topic in the next section.

VI. ANOMALOUS ROTATION AND EXOTIC STATISTICS

It has been demonstrated that in pure CS theory a charged particle of charge \( e \), interacting with a CS field in the absence of the Maxwell kinetic energy, can acquire the phase \( e^2/m \) when it is rotated through 2\( \pi \) radians [8]. The occurrence of the arbitrary phase has been attributed by some authors to the imposition of Gauss’s law in the CS theory [1]. In Ref. [10], we have shown that the rotational anomaly that arises in CS theory has nothing to do with the implementation of Gauss’s law. We have constructed charged fermion states in MCS theory in the temporal gauge which rotate normally—i.e., they change sign under a 2\( \pi \) rotation—even when these charged states implement Gauss’s law. We have also demonstrated that these charged states in MCS theory obey standard Fermi-Dirac statistics although they obey Gauss’s law [11]. By pursing the same analysis detailed in Ref. [10], we will confirm that conclusion in the formulation of MCS theory in covariant gauges. Furthermore, we will show that other states can also be constructed that satisfy Gauss’s law, and that do

\[ 4 \text{The presence of } x_i \text{ in } \int dx \ x_iP_0(x) \text{ introduces ambiguities into the expression for the commutator of two boost operators, that appear when the momentum space representations of the gauge fields are used. We will discuss these ambiguities in an appendix to this paper.} \]
acquire an arbitrary phase under $2\pi$ rotations. These states also obey standard Fermi-Dirac statistics and give rise to the same $S$-matrix elements as the states that rotate normally.

The rotation operator we will use is $R(\theta) = e^{iJ\theta}$ where $J$ is the canonical (Noether) angular momentum given by Eq. (5.4), which we have just identified as one of the six Poincaré generators that close under the commutator algebra required for the Poincaré group. We will express $J$ as

$$J = J_g + J_e$$  \hspace{1cm} (6.1)

with

$$J_g = -\int d\mathbf{x} \, \Pi_i x_i \epsilon_{ij} \partial_j A_i + \int d\mathbf{x} \, Gx_i \epsilon_{ij} \partial_j A_0 - \int d\mathbf{x} \, \epsilon_{ij} \Pi_i A_j$$  \hspace{1cm} (6.2)

and

$$J_e = -i \int d\mathbf{x} \, \psi^\dagger x_i \epsilon_{ij} \partial_j \psi - \frac{1}{2} \int d\mathbf{x} \, \psi^\dagger \gamma_0 \psi.$$  \hspace{1cm} (6.3)

As pointed out in the previous section, $J$ is time independent since $[H, J] = 0$.

The interpretation of these angular momentum operators in terms of the angular momenta of the constituent particle-mode excitations is greatly simplified when single-particle plane waves are replaced with eigenstates of angular momentum. We therefore substitute gauge-field annihilation and creation operators describing excitations with definite angular momentum, $\alpha_n(k)$ and $\alpha_n^\dagger(k)$, respectively, for the corresponding plane-wave excitations $a(k)$ and $a^\dagger(k)$. This is accomplished by using

$$\alpha_n(k) = \frac{e^{in\pi/2}}{2\pi} \int d\tau \ a(k)e^{-in\tau}$$  \hspace{1cm} (6.4)

where $\tau$ is the angle that fixes the direction of $k$ in the plane; a corresponding expression relates the Hermitian adjoints $\alpha_n^\dagger(k)$ and $a^\dagger(k)$ [4]. Similarly, we can define the following single-particle solutions of the Dirac equation in polar coordinates,

$$u_+(n, k; \rho, \phi) = \frac{1}{\sqrt{2\omega_k(M + \omega_k)}} \left[ \frac{ikJ_n(k\rho)e^{in\phi}}{(M + \omega_k)J_{n+1}(k\rho)e^{i(n+1)\phi}} \right]$$  \hspace{1cm} (6.5)

and

$$u_-(n, k; \rho, \phi) = \frac{1}{\sqrt{2\omega_k(M + \omega_k)}} \left[ \frac{(M + \omega_k)J_n(k\rho)e^{in\phi}}{ikJ_{n+1}(k\rho)e^{i(n+1)\phi}} \right],$$  \hspace{1cm} (6.6)

where $J_s(x)$ is the Bessel function of order $s$. Using $u_{\pm}$, we can represent $\psi$ in the angular momentum representation as

$$\psi(\rho, \phi) = \sum_{n,k} \left[ b_n(k)u_+(n, k; \rho, \phi) + \bar{b}_n(k)u_-(n, k; \rho, \phi) \right],$$  \hspace{1cm} (6.7)

where $\sum_{n,k} = \sum_n \int kdk/2\pi$; $b_n(k)$ and $\bar{b}_n(k)$ are the electron and positron annihilation operators, respectively, for states with definite angular momentum; $b_n^\dagger(k)$ and $\bar{b}_n^\dagger(k)$ are the corresponding creation operators. The operator $b_n(k)$ is related to $e(k)$ by
\[ b_n(k) = \frac{e^{in\pi/2}}{2\pi} \int d\tau \, e(k)e^{-in\tau}. \] (6.8)

Similar expressions relate \( \tilde{b}_n(k) \) to \( \bar{e}(k) \), the adjoints \( b_n^\dagger(k) \) to \( e^\dagger(k) \), and \( \tilde{b}_n(k) \) to \( \bar{e}^\dagger(k) \). To write the angular momentum operator \( J \) in terms of annihilation and creation operators of states with definite angular momentum, we first write the gauge fields \( A_l, \Pi_l, A_0, G \) and the spinor fields \( \psi \) and \( \psi^\dagger \) in terms of \( \alpha_{n}(k), \alpha_{Q,n}(k), \alpha_{R,n}(k), b_n(k), \tilde{b}_n(k) \) and their adjoints, and then make the appropriate substitutions in Eqs. (6.2) and (6.3). The resulting expressions for the angular momenta \( J_g \) and \( J_e \) are

\[ J_g = \sum_{n,k} n \left[ \alpha_{n}^\dagger(k)\alpha_{n}(k) + \alpha_{R,n}^\dagger\alpha_{Q,n}(k) + \alpha_{Q,n}^\dagger\alpha_{R,n}(k) \right] \] (6.9)

and

\[ J_e = \sum_{n,k} (n + \frac{1}{2}) \left[ b_n^\dagger(k)b_n(k) - \tilde{b}_n^\dagger(k)\tilde{b}_n(k) \right]. \] (6.10)

Thus, the eigenvalues of \( J \) are integral for a photon state, and half-integral for an electron or positron state. We can also show, using Eqs. (6.9) and (6.10), that the rotation operator \( \tilde{R}(\theta) = e^{iJ\theta} \) rotates particle states correctly, e.g., the electron state |\( N \rangle = b_N^\dagger|0\rangle \) is rotated into \( |N'\rangle = \exp\left[i(N + \frac{1}{2}\theta)|N\rangle\right]. \)

To investigate the rotational properties of charged states that obey the Gauss’s law constraint under \( 2\pi \) rotations, let us consider the “bare” one-electron state \( |N\rangle = e^\dagger(k)|0\rangle \). The one-electron state |\( N \rangle \) does not satisfy the constraint \( \Omega(k)|N\rangle = 0 \), and thus is not in the physical subspace \{\( |\nu\rangle \)\} defined in Sec. III. The electron state which satisfies Eq. (6.7) is given by \( e^D|N\rangle \). It is therefore the rotation of the state \( e^D|N\rangle \) that we must analyze and not |\( N \rangle \) when we use the untransformed representation of the states, fields and dynamical variables; in the untransformed representation, the subspace \{\( |n\rangle \)\} represents states for which Gauss’s law still remains to be implemented. The rotation of the state \( e^D|N\rangle \), which obey Gauss’s law in this untransformed representation, can be evaluated by writing \( \tilde{R}(\theta) = e^{iJ\theta} \) where \( \tilde{J} \) is given by

\[ \tilde{J} = J + \mathcal{J} \] (6.11)

where

\[ \mathcal{J} = -\sum_k \frac{m^3/2}{16k^3} \epsilon_{ln}k_l \frac{\partial \phi(k)}{\partial k_n} j_0(-k)j_0(k) \]

\[ -\sum_k i\epsilon_{ln}k_l \frac{\partial \theta(k)}{\partial k_n} \left[ a_Q(k)j_0(-k) - a_Q^*(k)j_0(k) \right] \]

\[ +\sum_k \epsilon_{ln}k_l \frac{\partial \phi(k)}{\partial k_n} \left[ a_Q(k)j_0(-k) + a_Q^*(k)j_0(k) \right]. \] (6.12)

The rotation operator \( \tilde{R}(\theta) \) can also be obtained by noting that in the alternate, transformed representation, |\( N \rangle \) represents the one-electron state that does implement Gauss’s law and
the gauge condition. In that representation, all dynamical variables are represented by the correspondingly transformed operators, so that the rotation operator is given by $\hat{R} = e^{-D Re^D}$. Equation (3.13) reminds us that the forms of $\phi(k)$ and $\theta(k)$ may be chosen arbitrarily without disturbing the implementation of Gauss’s law. In particular, $\phi(k)$ and/or $\theta(k)$ may be set to zero. In that case, $J = J$, and the states that implement Gauss’s law will rotate like the “bare” fermion states that don’t obey Gauss’s law, i.e., they will change sign in a $2\pi$ rotation. Other choices for $\phi(k)$ and/or $\theta(k)$ will lead to different rotational properties for the charged states that obey Gauss’s law. If we choose

$$\phi(k) = -\frac{8k^3}{m^{5/2}} \frac{\delta(k)}{k} \tan^{-1} \frac{k_2}{k_1}, \quad (6.13)$$

and if we assume that we can carry out the integration over $dk$ while $j_0(k)$ is still operator-valued, then the first term in $J$ becomes $Q^2/4\pi m$, where $Q$ is the electron charge

$$Q = e \sum_{n,k}[b_n^\dagger(k)b_n(k) - \bar{b}_n(k)\bar{b}_n(k)]. \quad (6.14)$$

Hence, under a $2\pi$ rotation, the state $e^D|N\rangle$ which obeys Gauss’s law picks up an arbitrary phase $e^{i^2/4\pi m}$, that is,

$$R(2\pi)e^D|N\rangle = -e^{i^2/4\pi m}e^D|N\rangle. \quad (6.15)$$

Another important question to examine is whether “exotic” fractional statistics develop when Gauss’s law is implemented for charged states. If the anticommutators for the spinor fields that implement Gauss’s law, $\{\tilde{\psi}(x), \tilde{\psi}(y)\}$ and $\{\tilde{\psi}(x), \tilde{\psi}(y)\}$, differ from the canonical spinor anticommutators $\{\psi(x), \psi^\dagger(y)\} = \delta(x - y)$ and $\{\psi(x), \psi(y)\} = 0$, then that difference may signal that the excitations of $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ are subject to fractional statistics. To show that the anticommutation rules for $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ are identical to the anticommutation rules for the unconstrained $\psi$ and $\psi^\dagger$, we refer to Eq. (3.23). Since the gauge and the spinor fields commute at equal times, then $\mathcal{D}(x)$ commutes with $\psi(x)$ and therefore

$$\{\tilde{\psi}(x), \tilde{\psi}(y)\} = \delta(x - y) \quad (6.16)$$

and

$$\{\tilde{\psi}(x), \tilde{\psi}(y)\} = 0. \quad (6.17)$$

The gauge-independent fields $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ obey the same anticommutation rules as the gauge-dependent $\psi$ and $\psi^\dagger$, and are not subject to any exotic graded anticommutator algebra. The same conclusion also follows from the observation that $\tilde{\psi}(x)$ and $\tilde{\psi}^\dagger(x)$ are unitary transforms of $\psi(x)$ and $\psi^\dagger(x)$ respectively, and that the anticommutator algebra for fermion fields is invariant to the unitary transformation. The electron and positron states that implement Gauss’s law therefore obey standard Fermi—not fractional—statistics. For example, the two-electron state in the alternate transformed representation, $e^i(k)e^\dagger(q)|0\rangle$, represents charged fermions accompanied by the electromagnetic fields required for them to obey Gauss’s law. Nevertheless, the anticommutation rules Eqs. (5.10) and (5.17) demonstrate that $e^i(k)e^\dagger(q)|0\rangle = -e^i(q)e^\dagger(k)|0\rangle$, and that $e^i(k)e^\dagger(k)|0\rangle = 0$. The particular form
of $\tilde{\psi}(x)$ given in Eq. (3.23) only applies to the covariant gauges and to this method of quantization. In other gauges, and with other methods of implementing constraints, the spinor fields that implement Gauss’s law will have a different representation, and questions about the statistics of electron-positron states that obey Gauss’s law arise in a different way. In Sec. VII, we will formulate this theory in the Coulomb gauge and confirm the result that the charged particle states obey standard Fermi statistics.

VII. LORENTZ TRANSFORMATIONS OF THE PHOTON STATES

In a (3+1)-dimensional space, photons have two possible polarization modes; and photons in definite helicity states transform into themselves under Lorentz transformations. Photons share this property with all other zero-mass particles [11]. In contrast, massive spin-one excitations of gauge fields in definite helicity states in one Lorentz frame, are observed as mixtures of helicity states in other Lorentz frames. The model we are examining in this work offers an interesting illustration of how the restriction to $2 + 1$ dimensions and the topological mass term affect the Lorentz transformations of particle states. The photons in this model are massive, and propagate with velocities $v < c$. Nevertheless, there is only a single polarization mode available for propagating excitations that correspond to observable particles. No second helicity mode is available with which topologically massive photons can mix under Lorentz transformations, even though the photons are excitations not of a scalar, but a vector field. It therefore becomes interesting to examine how these photon states transform under a Lorentz boost.

To facilitate this investigation, we shift to a description of excitation operators that have an invariant norm under Lorentz transformations. We observe, for example that the norm of the one-particle state $a^\dagger(k)|0\rangle$,

$$\left|a^\dagger(k)|0\rangle\right|^2 = \sum_q \langle 0|[a(q), a^\dagger(k)]|0\rangle = \int dq \delta(k - q),$$  \hspace{1cm} (7.1)$$

is not a Lorentz scalar because $dk$ is not the Lorentz invariant measure for the phase space. The invariant measure can be established by noting that the invariant delta function

$$\delta(k - q)\delta(k_0 - q_0)\delta(q_\mu q^\mu - m^2)\Theta(q_0) = \frac{\delta(k - q)\delta(k_0 - \omega_q)}{2\omega_q},$$  \hspace{1cm} (7.2)$$

so that the states $A^\dagger(k)|0\rangle$, created by operators that obey

$$[A(k), A^\dagger(q)] = 2\omega_k(2\pi)^2\delta(k - q),$$  \hspace{1cm} (7.3)$$

have unit norms in every Lorentz frame. The equivalently normalized ghost operators obey

$$[A_Q(k), A^*_R(q)] = [A_R(k), A^*_Q(q)] = 2k(2\pi)^2\delta(k - q).$$  \hspace{1cm} (7.4)$$

Hence, the boost operator $\bar{K}_l$ for the interaction-free theory is written as
\[ \bar{K}_l = \sum_k \frac{m \epsilon_{ln} k_n}{2k^2 \omega_k} A^\dagger(k) A(k) - \sum_k \frac{64k^2 \epsilon_{ln} k_n}{m^4} A_Q^*(k) A_Q(k) \]

\[ + \frac{i}{4} \sum_k \left[ \frac{\partial}{\partial k_l} A^\dagger(k) A(k) - A^\dagger(k) \frac{\partial}{\partial k_l} A(k) \right] \]

\[ + \frac{5i k_l}{4k^2} \left[ A_Q^*(k) A_R(k) - A_R^*(k) A_Q(k) \right] \]

\[ + \frac{i}{2} \sum_k \frac{\partial}{\partial k_l} A_Q^*(k) A_R(k) - A_R^*(k) \frac{\partial}{\partial k_l} A_Q(k) \]

\[ - (1 - \gamma) \sum_k \frac{16i k^3}{m^3} \left[ \frac{\partial}{\partial k_l} A_Q^*(k) A_Q(k) - A_Q^*(k) \frac{\partial}{\partial k_l} A_Q(k) \right]. \] (7.5)

Using the commutations rules given by Eqs. (7.3) and (7.4), we find that

\[ \delta A^\dagger(k) = \left[ \frac{im \epsilon_{ln} k_n}{k^2} A^\dagger(k) - \omega_k \frac{\partial}{\partial k_l} A^\dagger(k) \right] \delta \beta_l \] (7.6)

and

\[ \delta A_Q^*(k) = - \left[ \frac{5k_l}{2k} A_Q^*(k) + k \frac{\partial}{\partial k_l} A_Q^*(k) \right] \delta \beta_l, \] (7.7)

where \( \delta \xi = i[\bar{K}_l, \xi] \delta \beta_l \). Equation (7.6) demonstrates that the particle state \( A^\dagger(k)|0\rangle \) is Lorentz-transformed into itself. The phase factor \( \delta \beta_l \epsilon_{ln} k_n / k^2 \) generated by the boost operator \( \bar{K}_l \), which appears in Eq. (7.6), is a cocycle [12]. This phase factor has no physical implications. The physically observable consequence of Eq. (7.6) is that, under a Lorentz transformation, the topologically massive photon states behave like the excitations of a scalar field—each photon state transforms only into itself at a new space-time point.

**VIII. FORMULATION OF THE THEORY IN THE COULOMB GAUGE**

In the Coulomb gauge, the gauge field \( A_0 \) is not involved in the gauge condition, so that a gauge-fixing term cannot be used to generate a canonical momentum conjugate to \( A_0 \). The quantization procedure used in the covariant and temporal gauge formulations of the theory therefore is not well-suited for the Coulomb gauge. The most convenient way to quantize in the Coulomb gauge is to use the Dirac-Bergmann (DB) procedure. In this method, the canonical “Poisson” commutators (anticommutators) are replaced by their respective Dirac commutators (anticommutators), which apply to the fields that obey all the constraints of the theory. Since the Dirac and the canonical commutators (anticommutators) can, and often do, differ from each other, this method enables us to investigate whether the Dirac anticommutator for the spinor field \( \psi \) and its adjoint \( \psi^\dagger \) differ from the corresponding canonical anticommutator. A discrepancy between the Dirac and canonical anticommutators for the spinor fields could signal the development of “exotic” fractional statistics due to the imposition of Gauss’s law. On the other hand, identity of the Dirac and the canonical anticommutators for the spinor fields demonstrate that the excitations of the charged spinor...
field that obey Gauss’s law (as well as all other constraints) also obey standard Fermi statistics. The question, whether the imposition of Gauss’s law produces charged particle excitations that are subject to exotic statistics, therefore arises in a new way in the Coulomb gauge. In this section, we will carry out this quantization procedure and demonstrate explicitly that the implementation of Gauss’s law for the charged spinor field does not change the anticommutation rule for $\psi$ and $\psi^\dagger$, and does not cause the excitations of these fields to develop exotic fractional statistics.

The Lagrangian density for MCS theory in the Coulomb gauge is given by

\[
\mathcal{L} = -\frac{1}{4}F_{ln}F_{ln} + \frac{1}{2}F_{0l}F_{0l} + \frac{1}{2}m\epsilon_{ln}(F_{ln}A_0 + 2F_{0l}A_n) + j_lA_l - j_0A_0 - G\partial_lA_l + \bar{\psi}(i\gamma^\mu\partial_\mu - M)\psi.
\] (8.1)

This Lagrangian differs from Eq. (2.1) only in that the gauge-fixing term $-G\partial_\mu A^\mu$ is replaced by $-G\partial_lA_l$. We have included a gauge-fixing term for the Coulomb gauge in Eq. (8.1) to avoid first class constraints and to enable us to develop all the constraints systematically from the Lagrangian. The Euler-Lagrange equations in the Coulomb gauge are

\[
\partial_0F_{0l} + m\epsilon_{ln}F_{0n} - \partial_nF_{ln} - \partial_lG = j_l,
\] (8.2)

\[
\frac{1}{2}m\epsilon_{ln}F_{ln} - \partial_lF_{0l} + \partial_0G = j_0,
\] (8.3)

\[
\partial_lA_l = 0,
\] (8.4)

and

\[
(M - i\gamma^\mu D_\mu)\psi = 0.
\] (8.5)

The momenta conjugate to the gauge fields are $\Pi_l = F_{0l} + \frac{1}{2}m\epsilon_{ln}A_n$; $\Pi_0 = 0$ and $\Pi_G = 0$, where $\Pi_0$ and $\Pi_G$ are the momenta conjugate to $A_0$ and $G$, respectively. For the spinor fields, we have $\Pi_\psi = i\psi^\dagger$ and $\Pi_{\psi^\dagger} = 0$ as the momenta conjugate to $\psi$ and $\psi^\dagger$, respectively. We have identified the following primary constraints:

\[
C_1 = \Pi_0 \approx 0,
\] (8.6)

\[
C_2 = \Pi_G \approx 0,
\] (8.7)

\[
C_\psi = \Pi_\psi - i\psi^\dagger \approx 0,
\] (8.8)

and

\[
C_{\psi^\dagger} = \Pi_{\psi^\dagger} \approx 0.
\] (8.9)

The time-evolution operator is the total Coulomb gauge Hamiltonian, $H_T^C$, given by
\[ H_T^C = \int dx \, \psi^\dagger (\gamma_0 M - i\gamma_0 \gamma_l \partial_l) \psi + \int dx \left[ \frac{1}{2} \Pi_l \Pi_l + \frac{1}{4} F_{ln} F_{ln} + A_0 \partial_l \Pi_l \right. \\
\left. + \frac{1}{8} m^2 A_l A_l + \frac{1}{2} \epsilon_{ln} A_l \Pi_n - \frac{1}{4} \epsilon_{ln} F_{ln} A_0 + G \partial_l A_l + j_0 A_0 - j_l A_l \\
- \mathcal{U}_1 \mathcal{C}_1 - \mathcal{U}_2 \mathcal{C}_2 - \mathcal{C}_3 \mathcal{U}_\psi - \mathcal{U}_{\psi^\dagger} \mathcal{C}_{\psi^\dagger} \right] \] (8.10)

where \( \mathcal{U}_1, \ldots, \mathcal{U}_4 \) designate arbitrary functions that commute with all operators; \( \mathcal{U}_\psi \) and \( \mathcal{U}_{\psi^\dagger} \) designate arbitrary functions that are Grassmann numbers, which anticommute with all fermion fields and with Grassmann numbers, but commute with bosonic operators and with \( \mathcal{U}_1, \ldots, \mathcal{U}_4 \). In the imposition of constraints, we will use the Poisson bracket, \([ A, B] \), of two operators \( A \) and \( B \), defined as \([ A, B] = AB - (-1)^n(A)n(B)BA \), where \( n(P) \) is an index for the operators \( P \), \( n(P) = 0 \) if \( P \) is a bosonic operator, such as a gauge field or a bilinear combination of fermion fields; and \( n(P) = 1 \) if \( P \) is a Grassmann number, or a fermionic operator such as \( \psi \) or \( \psi^\dagger \). The Poisson bracket \([ A, B] \) is the commutator \([ A, B] \) when \( A \) and \( B \) are both bosonic operators, or if one is bosonic and the other fermionic. But \([ A, B] \) is the anticommutator \( \{ A, B \} \) when \( A \) and \( B \) are both fermionic operators.

We use the total Hamiltonian to generate the further constraints needed to maintain the stability of the primary constraints under time evolution. For this purpose, we evaluate time derivatives of the primary constraints by using the equation \( \partial_0 \mathcal{C}_i = i \mathcal{H}_T^C, \mathcal{C}_i \), and set \( \partial_0 \mathcal{C}_i \approx 0 \). In this way, we find that \( \partial_0 \mathcal{C}_1 \approx 0 \) leads to the secondary constraint \( \mathcal{C}_3 \approx 0 \) where

\[ \mathcal{C}_3 = \partial_l \Pi_l - \frac{1}{4} \epsilon_{ln} F_{ln} + j_0, \] (8.11)

which implements Gauss’s law. \( \partial_0 \mathcal{C}_3 \approx 0 \) leads to

\[ \partial_l \partial_l G + \partial_l j_l - e \psi^\dagger \mathcal{U}_\psi + e \mathcal{U}_{\psi^\dagger} \psi = 0 \] (8.12)

which does not generate a tertiary constraint. The stability of the constraint \( \Pi_G \approx 0 \) is obtained by setting \( \partial_0 \mathcal{C}_2 \approx 0 \) which leads to the secondary constraint \( \mathcal{C}_4 \approx 0 \) and the tertiary constraint \( \mathcal{C}_5 \approx 0 \) where

\[ \mathcal{C}_4 = \partial_l A_l \] (8.13)

and

\[ \mathcal{C}_5 = \partial_l \Pi_l - \partial_l \partial_l A_0 + \frac{1}{4} \epsilon_{ln} F_{ln} \] (8.14)

The constraint \( \mathcal{C}_4 \approx 0 \) implements the gauge condition for the Coulomb gauge, and \( \mathcal{C}_5 \approx 0 \) is required for consistency between Eq. (8.3) and the Coulomb gauge condition. The constraint equation \( \partial_0 \mathcal{C}_5 \approx 0 \) contains \( \mathcal{U}_1 \), thus does not lead to any further constraints. The constraints \( \partial_l \mathcal{C}_\psi \approx 0 \) and \( \partial_0 \mathcal{C}_{\psi^\dagger} \approx 0 \) result in the following expressions for \( \mathcal{U}_\psi \) and \( \mathcal{U}_{\psi^\dagger} \):

\[ \mathcal{U}_\psi = ie A_0 \psi - ie A_l \gamma_l \psi + i M \gamma_0 \psi + \gamma_0 \gamma_l \partial_l \psi \] (8.15)

and

\[ \mathcal{U}_{\psi^\dagger} = \text{[expression]} \]

\(^{5}\)We generally follow the conventions in Sundermeyer. The definition of Poisson bracket used here, however, differs from Sundermeyer’s definition by a factor \( i \).

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\[ U_{\psi^t} = i e A_0 \psi^t - i e A_t \psi^t \gamma_\ell + i M \psi^t \gamma_0 - \partial_t \psi^t \gamma_0 \gamma_\ell. \] 

(8.16)

Substitution of Eqs. (8.13) and (8.10) into Eq. (8.12) yields still another constraint, \( C_6 \approx 0 \), where

\[ C_6 = \partial_t \partial \gamma G \] 

(8.17)

which is necessary for consistency between Eq. (8.2) and Gauss’s law. The constraint \( \partial_0 C_6 \approx 0 \) is an equation containing \( U_2 \) but does not lead to any further constraint.

The preceding analysis leads to eight second-class constraints for this gauge theory. Imposition of the constraints requires that we form the matrix \( M(x, y) \), whose elements are \( M_{ij}(x, y) = [C_i(x), C_j(y)] \). We assign the values \( C_1, \ldots, C_8 \) to the descending horizontal rows of the matrix, as well as to the sequence of vertical columns, where \( C_1, \ldots, C_6 \) refer to the previously defined constraints; for simplicity we will designate \( C_{\psi} \) and \( C_{\psi^t} \) as \( C_7 \) and \( C_8 \), respectively. The matrix \( M(x, y) \) is given by

\[
M(x, y) = \begin{pmatrix}
0 & 0 & 0 & 0 & i \nabla^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i \nabla^2 & 0 & 0 \\
0 & 0 & i \nabla^2 & 0 & 0 & i e \psi^t(x) & -i e \psi(x) & 0 \\
0 & 0 & -i \nabla^2 & 0 & 0 & 0 & 0 & 0 \\
-i \nabla^2 & 0 & 0 & i \nabla^2 & 0 & 0 & 0 & 0 \\
i \nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i e \psi^t(x) & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & i e \psi(x) & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix} \delta(x - y). \] 

(8.18)

The matrix \( M(x, y) \) has an inverse, \( \gamma(x, y) \), given by

\[
\gamma(x, y) = \begin{pmatrix}
0 & 0 & -i \frac{1}{\nabla^2} & 0 & i \frac{1}{\nabla^2} & e \psi(y) \frac{1}{\nabla^2} & -e \psi^t(y) \frac{1}{\nabla^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i \frac{1}{\nabla^2} & 0 & 0 & i \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 \\
0 & 0 & -i \frac{1}{\nabla^2} & 0 & 0 & e \psi(y) \frac{1}{\nabla^2} & -e \psi^t(y) \frac{1}{\nabla^2} & 0 \\
-i \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
e \psi(y) \frac{1}{\nabla^2} & 0 & 0 & e \psi(y) \frac{1}{\nabla^2} & 0 & 0 & 1 & 0 \\
e \psi^t(y) \frac{1}{\nabla^2} & 0 & 0 & -e \psi^t(y) \frac{1}{\nabla^2} & 0 & 0 & 1 & 0 \\
\end{pmatrix} \delta(x - y). \] 

(8.19)
We note that
\[
\int dz \, M_{ik}(x, z) Y_{kj}(z, y) = \int dz \, Y_{ik}(x, z) M_{kj}(z, y) = \delta_{ij} \delta(x - y). \tag{8.20}
\]
We apply
\[
\left[ \xi(x), \zeta(y) \right]^{D} = \left[ \xi(x), \zeta(y) \right] - \sum_{i,j=1}^{8} \int dz \, dz' \left[ \xi(x), C_i(z) \right] Y_{ij}(z, z') \left[ C_j(z'), \zeta(y) \right] \tag{8.21}
\]
to find Dirac commutators (anticommutators) for the gauge and/or spinor fields represented by \( \xi \) and \( \zeta \), and observe that these are given by
\[
\left[ \psi(x), \psi^\dagger(y) \right]^{D} = \{ \psi(x), \psi^\dagger(y) \} = \delta(x - y), \tag{8.22}
\]
\[
\left[ \psi(x), \psi(y) \right]^{D} = \{ \psi(x), \psi(y) \} = 0, \tag{8.23}
\]
\[
[A_l(x), \Pi_n(y)]^{D} = i \left( \delta_{ln} - \frac{\partial_l \partial_n}{\nabla^2} \right) \delta(x - y), \tag{8.24}
\]
\[
[A_0(x), A_l(y)]^{D} = [A_l(x), A_n(y)]^{D} = [A_l(x), \psi(y)]^{D} = 0, \tag{8.25}
\]
and
\[
[A_0(x), \psi(y)]^{D} = e\psi(y) \frac{1}{\nabla^2} \delta(x - y). \tag{8.26}
\]
Equations \( 8.22 \) and \( 8.23 \) demonstrate that the constrained spinor field obeys standard anticommutation rules, and not a graded anticommutator algebra; and that the charged excitations of that spinor field are subject to standard Fermi statistics, and not the exotic fractional statistics that would result from a graded anticommutator algebra. In contrast to the spinor field, the Dirac commutators of the gauge fields differ substantially both from the unconstrained canonical commutators, and also from their corresponding values in the temporal gauge. The observation that the spinor anticommutation rule is unaffected by constraints, and identical in the Coulomb and temporal gauges, therefore is not trivial.

The constrained Hamiltonian in the Coulomb gauge which now incorporates all the constraints \( C_1, \ldots, C_8 \), is found to be
\[
H_C = \int dx \, \left[ \frac{1}{2} \Pi_l^T \Pi_l^T + \frac{1}{4} F_{ln} F_{ln} + \frac{1}{2} m^2 A_l^T A_l^T - m \epsilon_{in} \partial_l A_n^T \nabla^{-2} J_0 - \frac{1}{2} j_0 \nabla^2 j_0 - j_l A_l^T \right] \tag{8.27}
\]
where \( A_l^T \) and \( \Pi_l^T \) are the transverse components of \( A_l \) and \( \Pi_l \), respectively; they are given by
\[
A_l^T = \left( \delta_{ln} - \frac{\partial_l \partial_n}{\nabla^2} \right) A_n \tag{8.28}
\]
and

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\[ \Pi_i^T = \left( \delta_{ln} - \frac{\partial l}{\nabla^2} \right) \Pi_n. \quad (8.29) \]

We need to find a suitable representation for the gauge fields in terms of particle creation and annihilation operators, in the Coulomb gauge formulation, just as we did in Sec. [1], when developing this model in the covariant gauges. The criterion of suitability is similar: The Hamiltonian for the free, noninteracting system of photons and electrons must have the perturbative vacuum \( |0\rangle \), and the single-particle photon and electron-positron states, \( a^\dagger(k)|0\rangle \), \( e^\dagger(k)|0\rangle \) and \( \bar{e}^\dagger(k)|0\rangle \) respectively, as eigenstates, so that the interaction-free time-evolution operator \( e^{-iH_0t} \) propagates these and other multi-particle states without changing their form or particle content. Since in this formulation the Dirac-Bergmann procedure has been implemented, time-evolution is restricted to the constraint surface, and all constraints (including Gauss’s law) apply. In the case of the Coulomb gauge, only the transverse gauge fields have commutation rules that need to be accommodated, so that ghost excitations of the gauge field are not required. The electrons and positrons are represented precisely as in the other gauges.

A suitable representation of the gauge fields can be based on the transverse part of \( A_l(x) \) in the covariant gauge, and leads to

\[ A_i^T(x) = \sum_k \frac{i\epsilon_l k_n}{k \sqrt{2\omega_k}} [a(k)e^{ik \cdot x} - a^\dagger(k)e^{-ik \cdot x}], \quad (8.30) \]

\[ \Pi_i^T(x) = \sum_k \frac{\sqrt{\omega_k \epsilon_l k_n}}{\sqrt{2k}} [a(k)e^{ik \cdot x} + a^\dagger(k)e^{-ik \cdot x}], \quad (8.31) \]

and

\[ A_0^D(x) = \sum_k \frac{m}{k \sqrt{2\omega_k}} [a(k)e^{ik \cdot x} + a^\dagger(k)e^{-ik \cdot x}] + \sum_k \frac{1}{k^2} j_0(k)e^{ik \cdot x}. \quad (8.32) \]

The Hamiltonian \( H_C \) then becomes

\[ H_C = \sum_k \omega_k a^\dagger(k)a(k) + \frac{1}{2k^2} j_0(k)j_0(-k) \]

\[ + \sum_k \frac{m}{k \sqrt{2\omega_k}} [j_0(-k)a(k) + j_0(k)a^\dagger(k)] \]

\[ + \sum_k \frac{\epsilon_l k_l}{k \sqrt{2\omega_k}} [j_n(-k)a(k) - j_n(k)a^\dagger(k)] + H_{\bar{e}e}. \quad (8.33) \]

Although the representations of \( A_i^T(x) \), \( \Pi_i^T(x) \) and \( A_0(x) \) given in Eqs. (8.30)–(8.32) are suitable, they are not unique. Other suitable representations would satisfy the requirements specified above equally well. We therefore cannot assume that the Hilbert space in which these operators—and the representation of \( H_C \) given in Eq. (8.33)—operate, is identical to the one previously established for the corresponding operators in the covariant gauges. All the requirements we stipulated for representations to be suitable remain unaffected by similarity transformations—i.e., unitary transformations carried out on the operators as well
as states of a particular representation. It is therefore necessary to consider the possibility that the appropriate Fock space for this representation of MCS theory in the Coulomb gauge differs from the Fock space \{\|N\}\ of the covariant gauge formulation of this model, by just such a similarity transformation. In fact, this possibility becomes a virtual certainty, in view of the fact that \(H_C\) does not have the identical form as \(\hat{H}_{\text{quot}}\), the generator of time displacements in the Fock space \{\|N\}\ of the covariant gauge formulation developed in Secs. \[\text{II}\] and \[\text{III}\]. As in the case of (3 + 1)-dimensional QED in axial gauges \[5\], the reason for the discrepancy in form between the two Hamiltonians, \(H_C\) and \(\hat{H}_{\text{quot}}\), lies in the fact that the two operate in different Fock spaces. We can expect that the Fock space on which the operators in Eqs. \(8.30\)–\(8.33\) act, consists of state vectors \(|\phi_i\rangle\) and \(|\phi_t\rangle\), where \(|\phi_i\rangle\) designates the state vectors in the Fock space whose single-particle elements are the familiar \(a^\dagger(k)|0\rangle\), \(e^\dagger(k)|0\rangle\) and \(\tilde{e}(k)|0\rangle\). We can then proceed to carry out a similarity transformation in which all the operators transform according to \(\hat{P} = e^T \mathcal{P} e^{-T}\); the similarly transformed states then are \(|\phi_t\rangle = e^{-T}|\phi_i\rangle\). If the objective of this transformation has been met, then we will find that \(|\phi_i\rangle = |N_i\rangle\), so that the transformed states will be the elements of the Fock space \{\|N\}\ familiar to us as the quotient space of the covariant gauge formulation. To carry out such a program, we choose

\[
\Gamma = -\sum_k \frac{m}{\sqrt{2}k\omega_k^{3/2}} \left[ a(k)j_0(-k) - a^\dagger(k)j_0(k) \right], 
\]

and find the transformed operators

\[
\hat{H}_C = \sum_k \omega_k a^\dagger(k)a(k) + \sum_k \frac{1}{2\omega_k^2} j_0(k)j_0(-k) + \sum_k \frac{i\epsilon_{ln}k_l}{k^2\omega_k^2} j_n(k)j_0(-k)
\]

\[
+ \sum_k \frac{mk_l}{\sqrt{2}k\omega_k^{3/2}} \left[ j_i(-k)a(k) + j_i(k)a^\dagger(k) \right]
\]

\[
+ \sum_k \frac{i\epsilon_{ln}k_l}{k\sqrt{2}\omega_k} \left[ j_n(-k)a(k) - j_n(k)a^\dagger(k) \right] + H_{ee},
\]

\[
\hat{A}_t^T(x) = \sum_k \frac{i\epsilon_{ln}k_n}{k\sqrt{2}\omega_k} \left[ a(k)e^{ik\cdot x} - a^\dagger(k)e^{-ik\cdot x} \right] - \sum_k \frac{i\epsilon_{ln}k_n}{k^2\omega_k^2} j_0(k)e^{ik\cdot x},
\]

\[
\hat{\Pi}_t^T(x) = \sum_k \frac{\sqrt{\omega_k\epsilon_{ln}k_n}}{\sqrt{2}k} \left[ a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x} \right]
\]

\[
\text{and} \quad \hat{A}_0^D(x) = \sum_k \frac{m}{k\sqrt{2}\omega_k} \left[ a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x} \right] + \sum_k \frac{1}{\omega_k^2} j_0(k)e^{ik\cdot x}.
\]

The Fock space \{\|N\}\ is the appropriate Hilbert space for the Coulomb gauge formulation after the similarity transformation \(\hat{P} = e^T \mathcal{P} e^{-T}\) and \(|n_i\rangle = |\phi_i\rangle\) has been carried out. \(\hat{H}_C\) operates in the same Fock space as does \(\hat{H}_{\text{quot}}\), and the two operators have the identical form. State vectors in the Fock space \{\|N\}\, representing systems of electrons, positrons and the topologically massive, propagating excitations of the gauge field,
are time translated by the same time evolution operator in both the covariant and the Coulomb gauges. Earlier work demonstrated that the same time-evolution operator also time translates these state vectors in the temporal gauge [1].

It is apparent from the preceding discussion that, had we initially chosen Eqs. (8.35)–(8.38) to represent \(A_T^l(x), \Pi_T^l(x)\) and \(A_0(x)\), we would have immediately obtained the desired form \(\hat{H}_C\) for the Hamiltonian for MCS theory in the Coulomb gauge, and would have had no occasion to carry out any unitary transformations. However, since there is no systematic way of initially recognizing the appropriate representation of \(A_T^l(x), \Pi_T^l(x)\) and \(A_0(x)\) that leads to this desired form for the Hamiltonian, we have deliberately avoided making the most convenient choice of representation from the start. It is important to formulate the question, whether two different representations describe the same physical system, in terms of the identity of two equivalence classes, in which the operators and states that are members of a class are related by similarity transformations. It is not sufficient, in testing whether operators, constructed with randomly chosen representations of space-time fields, have the same form. This point has been discussed in greater detail elsewhere [5] but applies here as well.

**IX. IS CS THEORY THE LARGE \(m\) LIMIT OF MCS THEORY?**

The Lagrangians for CS theory and MCS theory differ only by the Maxwell kinetic energy term, which is included in the latter but absent from the former. The relative size of the CS term and the Maxwell kinetic energy term is tuned by the CS coupling constant \(m\), and in the limit \(m \rightarrow \infty\) the Maxwell kinetic energy term becomes vanishingly small relative to the CS term. The question therefore naturally arises, whether CS theory is approached as a well-defined limit of MCS theory as \(m \rightarrow \infty\). The results obtained in this work provide some insights into that question.

The comparison between CS and MCS theory can be best approached through what we have called the “quotient space” Hamiltonian, \(H_{\text{quot}}\), for MCS theory, given in Eqs. (3.44) and (3.45). \(H_{\text{quot}}\) is the form the Hamiltonian takes in the quotient space for the Fock space \(\{|n\}\}\), in the representation in which the latter implements Gauss’s law and the gauge choice; and \(H_{\text{quot}}\) has the same form in covariant, temporal and Coulomb gauges. Ambiguities in \(H_{\text{quot}}\), of the form \(\hbar = i[H_{\text{quot}}, \chi] = i[H_0, \chi]\), can arise; in Sec. [11], we have discussed such ambiguous terms, and have shown that they cannot affect the \(S\)-matrix, and that they can be transformed away by unitary transformations. We will assume here that such terms have been transformed away, and are not included in \(H_{\text{quot}}\). Apart from such ambiguities, \(H_{\text{quot}}\) consists of a “free” part that counts the kinetic energy of propagating massive photons and electrons (\(e^-\) and \(e^+\)); interaction terms

\[
H_a = \int d\mathbf{x} d\mathbf{y} \ j_0(x)j_0(y)K_0(m|\mathbf{x} - \mathbf{y}|) \tag{9.1}
\]

and

\[
H_b = \int d\mathbf{x} d\mathbf{y} \ j_0(x)\epsilon_{ln}\eta_l(y)(x - y)_nF(|\mathbf{x} - \mathbf{y}|) \tag{9.2}
\]

which describe nonlocal interactions between charges, and between charges and transverse currents, respectively; and finally, parts of \(H_{\text{quot}}\) describe interactions between the massive
propagating photons and electrons. In the limit \( m \to \infty \), the following observations can be made about the component parts of \( H_{\text{quat}} \): \( H_a \) vanishes in that limit, and its leading term in powers of \( 1/m \) is of order \( 1/m^2 \). The modified Bessel function \( K_0(\xi) \) that appears in \( H_a \) takes the asymptotic form

\[
K_0(\xi) \to \sqrt{\frac{\pi}{2\xi}} e^{-\xi}
\]

in the limit \( \xi \to \infty \). And for \( \xi = m |x - y|, \xi \to \infty \) as \( m \to \infty \) for all values of \( |x - y| \) except \( |x - y| = 0 \); \( j_0(x) \) and \( j_0(y) \) are operators whose matrix elements will be superpositions of products of nonsingular wavefunctions given in Eqs. (2.19) and (2.20) or (6.5) and (6.6). The integration \( \int dx dy \cdots \) can be transformed to \( \int dr d\rho \cdots \), where \( r = x - y \) and \( \rho = \frac{1}{2}(x + y) \), and the \( r \, dr \) in \( dr \) regularizes the logarithmic singularity of \( K_0(mr) \) at \( r = 0 \), so that the integrand in \( H_a \) vanishes at \( r = 0 \) for all well-behaved charge densities. \( H_a \) can be expressed as

\[
H_a = \int dr \, K_0(mr) f(r) \quad (9.3)
\]

with

\[
f(r) = r \int d\Omega_r \, d\rho \, j_0(\rho + \frac{1}{2}r)j_0(\rho - \frac{1}{2}r).
\]

As \( m \to \infty \), the integrand of Eq. (9.3) becomes vanishingly small except when \( r \ll 1 \), and in that region \( f(r) \) can be represented as a series, whose leading term makes a contribution to \( H_a \) given by

\[
H_a = \lambda \int r \, dr \, K_0(mr) \quad (9.5)
\]

where

\[
\lambda = 4\pi \int d\rho \, [j_0(\rho)]^2.
\]

Since

\[
\int r \, dr \, K_0(mr) = \frac{1}{m^2}, \quad (9.7)
\]

this leading term of \( H_a \) is of order \( 1/m^2 \). Expansions beyond this leading order, which reflect the nonnegligible \( r \)-dependence of \( j_0(\rho \pm \frac{1}{2}r) \) as \( m \) gets smaller, will produce additional terms of order \( (1/m)^N \) with \( N > 2 \).

The \( m \)-dependence of \( H_b \) in the \( m \to \infty \) limit is exactly \( 1/m \), so that the ratio \( H_a/H_b \to 0 \) as \( m \to \infty \). Moreover, in the \( m \to \infty \) limit, \( H_b \) approaches the expression for the interaction between charges and transverse currents in CS theory. In CS theory, the function \( F(mr) \) that appears in \( H_b \) is replaced by the integral

\[
-\frac{1}{2\pi m|x - y|^2} \int_0^\infty du \, J_1(u).
\]

Since \( \int_0^\infty du \, J_1(u) = 1 \), this agrees with the large \( m \) limit of \( F(mr) \) given in Eq. (3.48). In the \( m \to \infty \) limit, the sum of the two interactions \( H_a + H_b \) therefore can be seen to approach
the same limit as $H_b$ alone; and that limit is the nonlocal interaction between charges and transverse currents in CS theory.

The interactions between propagating massive photons and currents that arise in MCS theory have no corresponding counterpart in CS theory. The massive photons of MCS theory never disappear as $m \to \infty$. They can transmit an interaction between charges which we will examine for the case of electron-electron scattering. To lowest order in $1/m$, the part of the $S$-matrix element for $e(P) + e(Q) \to e(P') + e(Q')$ that originates from the exchange of a propagating massive photon between electrons is given by

$$S_{fi}^{(2)} = \frac{i(2\pi)^3 M^2 e^2 \delta^3(P + Q - P' - Q')}{\sqrt{\omega_{P'}\omega_P\omega_{Q'}\omega_Q}} \times$$

$$\left[ \frac{ime^2(\omega_{P'} - \omega_Q)\bar{u}(Q')\gamma_\mu u(P)\bar{u}(P')\gamma_\mu u(Q)}{\omega_{P'}^2\omega_{P}^2\omega_{Q'}^2 - (\omega_{P'} - \omega_Q)^2 - i\epsilon} \right]$$

$$- \frac{ime^2(\omega_{P'} - \omega_P)\bar{u}(Q')\gamma_\mu u(P)\bar{u}(P')\gamma_\mu u(Q)}{\omega_{P'}^2\omega_{P}^2\omega_{Q}^2 - (\omega_{P'} - \omega_P)^2 - i\epsilon} + \frac{\bar{u}(Q')\gamma_\mu u(Q)\bar{u}(P')\gamma_\mu u(P)}{\omega_{P'}^2\omega_{P}^2\omega_{Q}^2 - (\omega_{P'} - \omega_P)^2 - i\epsilon} + \frac{(\omega_{P'} - \omega_Q)^2 u^\dagger(Q')u(P)u^\dagger(P')u(Q)}{\omega_{P'}^2\omega_{P}^2\omega_{Q}^2 - (\omega_{P'} - \omega_Q)^2 - i\epsilon}$$

$$- \frac{(\omega_{P'} - \omega_P)^2 u^\dagger(Q')u(Q)u^\dagger(P')u(P)}{\omega_{P'}^2\omega_{P}^2\omega_{Q}^2 - (\omega_{P'} - \omega_P)^2 - i\epsilon}. \tag{9.8}$$

The leading term in $1/m$ of $S_{fi}^{(2)}$ can be seen to be of order $1/m^2$, so that the interaction between charged particles mediated by photon exchange vanishes as quickly as $H_a$, as $m \to \infty$, namely one power of $1/m$ more rapidly that does the dominant interaction term, $H_b$. Photon exchange therefore will not prevent the interactions between charged particles in MCS theory from approaching the corresponding interaction in CS theory.

The interactions between propagating massive photons and currents also describe electron-photon scattering, and processes in which charged particles radiate energy in the form of massive photons. Since these processes do not, and indeed cannot occur in CS theory, we must take account of the fact that they do not vanish in the $m \to \infty$ limit of MCS theory. A large photon mass does not disqualify the photon from being part of an initial state in a scattering process. Nor is the matrix element for photon production very sharply attenuated in the $m \to \infty$ limit. However, in that case, the energy required to produce even a single photon increases with $m$, so that for ordinary energy regime this process is a not a realistic option as $m \to \infty$. Nevertheless, the interaction that produces photons in MCS theory does not vanish in the large $m$ limit. In that sense MCS theory never fully approaches CS theory as $m \to \infty$.

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APPENDIX A:

In this appendix, we return to the Poincaré algebra—in particular to Eq. (5.22)—to discuss an ambiguity that arises when the same equation is examined after the momentum representation of the gauge fields—Eqs. (2.24)–(2.27)—have been substituted into the boost operators. We observe, in that case, that besides the \( x_l \) in \( \int dx \ x_l \mathcal{P}_0(x) \) (and the corresponding \( y_n \) in \( \int dy \ y_n \mathcal{P}_0(y) \)), the only other explicit dependence on configuration space variables is through the exponentials \( e^{\pm ik \cdot x} \) and \( e^{\pm ik \cdot y} \) that appear in the gauge fields. The representations of \( x_l \) and \( y_n \) required to facilitate the evaluation of \( K_l \) and \( K_n \) are

\[
\begin{align*}
x_l &= \mp \frac{\partial}{\partial k_l} e^{\pm ik \cdot x} \\
y_n &= \mp \frac{\partial}{\partial k_n} e^{\pm ik \cdot y}
\end{align*}
\]

In order to generate Dirac delta functions in momentum space variables, and to permit the integration over these delta functions, all partial derivatives of the form \( \frac{\partial}{\partial k_l} \) (in various different momentum variables) must be integrated by parts. This step produces the derivatives of photon creation and annihilation operators required for the angular momentum operator

\[
J = -\sum_k i\epsilon_l n_k \left[ \frac{\partial}{\partial k_l} a^\dagger(k)a(k) - a^\dagger(k)\frac{\partial}{\partial k_l} a(k) \right]
\]

\[
-\sum_k i\epsilon_l n_k \left[ \frac{\partial}{\partial k_l} a^\ast_Q(k)a_R(k) - a^\ast_R(k)\frac{\partial}{\partial k_l} a^Q(k) \right]
\]

(A1)

in Eq. (5.22); in addition, this process generates further partial derivatives of the coefficients of the expressions \( a(k)e^{ik \cdot x} \) and \( a^\dagger(k)e^{-ik \cdot x} \) that appear in Eqs. (2.24)–(2.27). When partial derivatives of singular coefficients arise—those for which the identities

\[
\begin{align*}
\frac{\partial}{\partial k_l} k_n &= \left[ \pi\delta(k) + \frac{1}{k^2} \right] \delta_{ln} - \frac{2k_l k_n}{k^4} \\
\frac{\partial}{\partial k_l} \frac{k_n}{k} &= \left[ \pi k\delta(k) + \frac{1}{k} \right] \delta_{ln} - \frac{k_l k_n}{k^3}
\end{align*}
\]

(A2, A3)

hold—the delta functions \( \delta(k) \) that appear in these expressions give rise to spurious contributions that produce an extra term, beyond \( -i\epsilon_l n J \) in Eq. (5.22). For the representation we have given in Eqs. (2.24)–(2.27), the commutator \( [K_l, K_n] \), evaluated in the momentum space representation, is given by

\[
[K_l, K_n] = -i\epsilon_l n J + \frac{1}{2\pi} im^2 \epsilon_l n a^\dagger(0)a(0).
\]

(A4)

This discrepancy was noticed and discussed by Deser, Jackiw and Templeton [14]. As follows from Ref. [14], the unitary transformation \( e^{i\phi} \cdots e^{-i\phi} \) with \( \phi = -\sum_k (\tan^{-1} k_2/k_1)a^\dagger(k)a(k) \) removes the singularity in the coefficients of \( a(k)e^{ik \cdot x} \) and \( a^\dagger(k)e^{-ik \cdot x} \) in Eqs. (2.24)–(2.27), and with it also this discrepancy. The fact that such a discrepancy appears with singular coefficients, points to a mathematical inconsistency that arises when we carry out the formal manipulations required to evaluate \( [K_l, K_n] \) in the momentum representation of the gauge
In the configuration space representation, \( X \) than a threat to the consistency of the Poincaré algebra, and with it to the consistency of the space representations of the gauge fields are used—the commutator \( [X, \Pi] \) functions, and not as distributions, when integrations by parts are carried out. However, as operator-valued integrands are reversed, and when Dirac delta functions are treated as fields. It is not surprising, perhaps, that such problems arise when orders of integration which vanishes trivially because of the antisymmetry of \( \epsilon \). After extensive manipulations, almost all contributions to Eq. (A9) cancel, leaving, however, a single residue stemming from the \( \delta(k) \) that originates from Eq. (A3):

\[
X_{\text{in}} = \frac{i \pi m^2 \epsilon_{\text{in}}}{16\pi} \cdot \delta(0) a(0). \tag{A10}
\]

The existence of this residue is in conflict with Eq. (A7).
REFERENCES

[1] K. Haller and E. Lim-Lombridas, Phys. Rev. D 46, 1737 (1992).
[2] M. Lüscher, Nucl. Phys. B326, 557 (1989); G. W. Semenoff and P. Sodano, Nucl. Phys. B328, 753 (1989); G. W. Semenoff and L. C. R. Wijewardhana, Phys. Lett. B 184, 397 (1987); T. Matsuyama, Phys. Rev. D 42, 3469 (1990); 44, 2616 (1991).
[3] P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University Press, New York, 1964).
[4] P. G. Bergmann and I. Goldberg, Phys. Rev. 98, 531 (1955).
[5] K. Haller and E. Lim-Lombridas, Found. of Phys. 24, 217 (1994).
[6] K. Haller, Phys. Rev. D 36, 1830 (1987).
[7] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).
[8] M. Swanson, Phys. Rev. D 42, 552 (1990).
[9] G. W. Semenoff, in Physics, Geometry, and Topology, H. C. Lee, Ed., (Plenum Press, New York, 1990); G. W. Semenoff, Phys. Rev. Lett. 61, 517 (1988); G. W. Semenoff and P. Sodano, Nucl. Phys. B328, 753 (1989); M. Lüscher, Nucl. Phys. B326, 557 (1989); R. Banerjee, Phys. Rev. Lett. 69, 17 (1992).
[10] K. Haller and E. Lim-Lombridas, Anyonic states in Chern-Simons theory, to appear in Phys. Rev. D.
[11] M. Fierz, Helv. Phys. Acta 13, 45 (1940).
[12] R. Jackiw, Comments Nucl. Part. Phys. 15, 99 (1985).
[13] K. Sundermeyer, Constrained Dynamics (Springer, New York, 1982).
[14] S. Deser, R. Jackiw, and S. Templeton, Ann. of Phys. 140, 372 (1982).