ABSOLUTE INTEGRAL CLOSURE IN POSITIVE CHARACTERISTIC

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Abstract. Let $R$ be a local Noetherian domain of positive characteristic. A theorem of Hochster and Huneke (1992) states that if $R$ is excellent, then the absolute integral closure of $R$ is a big Cohen-Macaulay algebra. We prove that if $R$ is the homomorphic image of a Gorenstein local ring, then all the local cohomology (below the dimension) of such a ring maps to zero in a finite extension of the ring. There results an extension of the original result of Hochster and Huneke to the case in which $R$ is a homomorphic image of a Gorenstein local ring, and a considerably simpler proof of this result in the cases where the assumptions overlap, e.g., for complete Noetherian local domains.

1. Introduction

Let $R$ be a commutative Noetherian domain with fraction field $K$. The absolute integral closure of $R$, denoted $R^+$, is the integral closure of $R$ in a fixed algebraic closure $\overline{K}$ of $K$. This ring was studied by Artin in [3] where among other results he proved that in the case $R$ is Henselian and local, the sum of two primes ideals of $R^+$ remains prime.

In [6], Hochster and Huneke proved that if $(R, m)$ is an excellent local Noetherian domain of positive characteristic $p > 0$, then $R^+$ is a big Cohen-Macaulay algebra, i.e., every system of parameters in $R$ is a regular sequence on $R^+$. The corresponding statement in equicharacteristic 0 is false if the dimension is at least three. Smith [9] further proved that the tight closure of an ideal $I$ generated by parameters is exactly the extension and contraction of $I$ to $R^+$: $I^* = IR^+ \cap R$. It is an open question whether the latter equality is true for every ideal $I$ in an excellent Noetherian local domain of positive characteristic. See [1], [2], and [8] for additional work concerning $R^+$.

Throughout this paper, $R$ is a commutative Noetherian domain of characteristic $p > 0$ with fraction field $K$, $\overline{K}$ is a fixed algebraic closure of $K$, and $R^+$ is the integral closure of $R$ in $\overline{K}$. The theorem of Hochster and Huneke in [6] implies the following:

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Let \((R, \mathfrak{m})\) be an excellent local commutative Noetherian domain of characteristic \(p > 0\). Then the natural homomorphism \(H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^+)\) is the zero map for every \(i < \dim R\).

In fact, as we observe in this paper (see Corollary 2.3), the statement above is basically equivalent to the statement that \(R^+\) is a big Cohen-Macaulay algebra for \(R\).

The main result of this paper, Theorem 2.1 below, states that if \(R\) is a homomorphic image of a Gorenstein local ring (though not necessarily excellent), then in fact one can find a finite extension ring \(S, R \subset S \subset R^+\), such that the map from \(H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(S)\) is zero for all \(i < \dim R\). Our proof is independent of the results of [6] and in particular gives a considerably simpler proof of the main result of that paper, with a stronger conclusion, when the assumptions overlap. For example, if \(R\) is complete, then it is both excellent and a homomorphic image of a Gorenstein ring. Our proof was in part inspired by the work of Hartshorne and Speiser in [5] and Lyubeznik in [7], concerning the structure of local cohomology modules in positive characteristic.

2. Main Result

Let \(R\) be a commutative ring containing a field of characteristic \(p > 0\), let \(I \subset R\) be an ideal, and let \(R'\) be an \(R\)-algebra. The Frobenius ring homomorphism \(f : R' \rightarrow R'p \rightarrow R'\) induces a map \(f_* : H^i_I(R') \to H^i_I(R')\) on all local cohomology modules of \(R'\) called the action of the Frobenius on \(H^i_I(R')\). For an element \(\alpha \in H^i_I(R')\) we denote \(f_*(\alpha)\) by \(\alpha^p\).

We recall that for a Gorenstein local ring \(A\) of dimension \(n\), local duality says that there is an isomorphism of functors \(D(\text{Ext}^n_{A}(-, A)) \cong H^n_{\mathfrak{m}}(-)\) on the category of finite \(A\)-modules, where \(D = \text{Hom}_{A}(-, E)\) is the Matlis duality functor (here \(E\) is the injective hull of the residue field of \(A\) in the category of \(A\)-modules) [11.2.6].

**Theorem 2.1.** Let \(R\) be a commutative Noetherian local domain containing a field of characteristic \(p > 0\), let \(K\) be the fraction field of \(R\) and let \(\overline{K}\) be the algebraic closure of \(K\). Assume \(R\) is a surjective image of a Gorenstein local ring \(A\). Let \(\mathfrak{m}\) be the maximal ideal of \(R\). Let \(R'\) be an \(R\)-subalgebra of \(\overline{K}\) (i.e. \(R \subset R' \subset \overline{K}\)) that is a finite \(R\)-module. Let \(i < \dim R\) be a non-negative integer. There is an \(R'\)-subalgebra \(R''\) of \(\overline{K}\) (i.e. \(R' \subset R'' \subset \overline{K}\)) that is finite as an \(R\)-module and such that the natural map \(H^i_{\mathfrak{m}}(R') \to H^i_{\mathfrak{m}}(R'')\) is the zero map.

**Proof.** Let \(n = \dim A\) and let \(N = \text{Ext}^n_{A}(-, A)\). Since \(R'\) is a finite \(R\)-module, so is \(N\).

Let \(d = \dim R\). We use induction on \(d\). For \(d = 0\) there is nothing to prove, so we assume that \(d > 0\) and the theorem proven for all smaller dimensions. Let \(P \subset R\) be a non-maximal prime ideal. We claim there
exists an $R'$-subalgebra $R^P$ of $K$ (i.e. $R' \subset R^P \subset K$) such that $R^P$ is a finite $R$-module and for every $R^P$-subalgebra $R^*$ of $K$ (i.e. $R^P \subset R^* \subset K$) such that $R^*$ is a finite $R$-module, the image $\mathcal{I} \subset N$ of the natural map $\text{Ext}_A^{n-i}(R^*, A) \to N$ induced by the natural inclusion $R' \to R^*$ vanishes after localization at $P$, i.e. $\mathcal{I}_P = 0$. Hence, let $d_P = \dim R/P$. Since $P$ is different from the maximal ideal, $d_P > 0$. As $R$ is a surjective image of a Gorenstein local ring, it is catenary, hence the dimension of $R_P$ equals $d - d_P$, and $i < d$ implies $i - d_P < d - d_P = \dim R_P$. By the induction hypothesis applied to the local ring $R_P$ and the $R_P$-algebra $R'_P$, which is finite as an $R_P$-module, there is an $R'_P$-subalgebra $\hat{R}$ of $K$, which is finite as an $R_P$-module, such that the natural map $H^{i-d_P}_P(R'_P) \to H^{i-d_P}_P(\hat{R})$ is the zero map. Let $\hat{R} = R'[z_1, z_2, \ldots, z_i]$, where $z_1, z_2, \ldots, z_i \in K$ are integral over $R_P$. Multiplying, if necessary, each $z_j$ by some element of $R \setminus P$, we can assume that each $z_j$ is integral over $R$. We set $R^P = R'[z_1, z_2, \ldots, z_i]$. Clearly, $R^P$ is an $R'$-subalgebra of $K$ that is finite as $R$-module.

Now let $R^*$ be both an $R^P$-subalgebra of $K$ (i.e. $R^P \subset R^* \subset K$) and a finite $R$-module. The natural inclusions $R' \to R^P \to R^*$ induce natural maps $\text{Ext}_A^{n-i}(R^*, A) \to \text{Ext}_A^{n-i}(R_P, A) \to N$. This implies that $\mathcal{I} \subset J$, where $J$ is the image of the natural map $\phi : \text{Ext}_A^{n-i}(R_P, A) \to N$. Hence it is enough to prove that $J_P = 0$. Localizing this map at $P$ we conclude that $J_P$ is the image of the natural map $\phi_P : \text{Ext}_A^{n-i}(\hat{R}, A_P) \to \text{Ext}_A^{n-i}(\hat{R}_P, A_P)$ induced by the natural inclusion $R'_P \to \hat{R}$ (by a slight abuse of language we identify the prime ideal $P$ of $R$ with its full preimage in $A$). Let $D_P(-) = \text{Hom}_{A_P}(-, E_P)$ be the Matlis duality functor in the category of $R_P$-modules, where $E_P$ is the injective hull of the residue field of $R_P$ in the category of $R_P$-modules. Local duality implies that $D_P(\phi_P)$ is the natural map $H^{i-d_P}_P(R'_P) \to H^{i-d_P}_P(\hat{R})$ which is the zero map by construction (note that $i - d_P = \dim A_P - (n - i)$).

Since $\phi_P$ is a map between finite $R_P$-modules and $D_P(\phi_P) = 0$, it follows that $\phi_P = 0$. This proves the claim.

Since $N$ is a finite $R$-module, the set of the associated primes of $N$ is finite. Let $P_1, \ldots, P_s$ be the associated primes of $N$ different from $m$. For each $j$ let $R^{P_j}$ be an $R'$-subalgebra of $K$ corresponding to $P_j$, whose existence is guaranteed by the above claim. Let $R' = R'[P_1, \ldots, P_s]$ be the compositum of all the $R^{P_j}$, $1 \leq j \leq s$. Clearly, $R'$ is an $R'$-subalgebra of $K$ (i.e. $R' \subset K \subset K$). Since each $R^{P_j}$ is a finite $R$-module, so is $R'$. Clearly, $R'$ contains every $R^{P_j}$. Hence the above claim implies that $\mathcal{I}_{P_j} = 0$ for every $j$, where $\mathcal{I} \subset N$ is the image of the natural map $\text{Ext}_A^{n-i}(R', A) \to N$ induced by the natural inclusion $R' \to R'$. It follows that not a single $P_j$ is an associated prime of $\mathcal{I}$. But $\mathcal{I}$ is a submodule of $N$, and therefore every associated prime of $\mathcal{I}$ is an associated prime of $N$. Since $P_1, \ldots, P_s$ are all the associated primes of $N$ different from $m$, we conclude that if $\mathcal{I} \neq 0$, then $m$ is the only associated prime of $\mathcal{I}$. Since $\mathcal{I}$, being a submodule of a finite
$R$-module $N$, is finite, and since $\mathfrak{m}$ is the only associated prime of $I$, we conclude that $I$ is an $R$-module of finite length.

Writing the natural map $\Ext^n_A (\hat{R}, A) \to N$ as the composition of two maps $\Ext^n_A (\hat{R}, A) \to I \to N$, the first of which is surjective and the second injective, and applying the Matlis duality functor $D$, we get that the natural map $\varphi : H^i_m (R') \to H^i_m (\hat{R})$ induced by the inclusion $R' \to \hat{R}$ is the composition of two maps $H^i_m (R') \to D(I) \to H^i_m (\hat{R})$, the first of which is surjective and the second injective. This shows that the image of $\varphi$ is isomorphic to $D(I)$ which is an $R$-module of finite length since so is $I$. In particular, the image of $\varphi$ is a finitely generated $R$-module. Let $\alpha_1, \ldots, \alpha_s \in H^i_m (\hat{R})$ generate $\Im \varphi$.

The natural inclusion $R' \to \hat{R}$ is compatible with the Frobenius homomorphism, i.e. with the raising to the $p$th power on $R'$ and $\hat{R}$. This implies that $\varphi$ is compatible with the action of the Frobenius $f_s$ on $H^i_m (R')$ and $H^i_m (\hat{R})$, i.e. $\varphi (f_s (\alpha)) = f_s (\varphi (\alpha))$ for every $\alpha \in H^i_m (R')$, which, in turn, implies that $\Im \varphi$ is an $f_s$-stable $R$-submodule of $H^i_m (\hat{R})$, i.e. $f_s (\alpha) \in \Im \varphi$ for every $\alpha \in \Im \varphi$. We finish the proof by applying the following lemma to each element of a finite generating set $\alpha_1, \ldots, \alpha_s$ of $\Im \varphi$. Applying Lemma 2.2 below we obtain a $\hat{R}$-subalgebra $R'_j$ of $\hat{K}$ (i.e. $R' \subset R'_j \subset \hat{K}$) such that $R'_j$ is a finite $R'$-module and the natural map $H^i_m (\hat{R}) \to H^i_m (R'_j)$ sends $\alpha_j$ to zero. Let $R'' = R'[R_1, \ldots, R_s]$ be the compositum of all the $R'_j$. Then $R''$ is an $R'$-subalgebra of $\hat{K}$ and is a finite $R$-module since so is each $R'_j$. The natural map $H^i_m (\hat{R}) \to H^i_m (R'')$ sends every $\alpha_j$ to zero, hence it sends the entire $\Im \varphi$ to zero. Thus the natural map $H^i_m (R') \to H^i_m (R'')$ is zero.

To finish the proof we prove the following lemma, which is closely related to the “equational lemma” in [6] and its modification in [9], (5.3).

**Lemma 2.2.** Let $R$ be a commutative Noetherian domain containing a field of characteristic $p > 0$, let $K$ be the fraction field of $R$ and let $\hat{K}$ be the algebraic closure of $K$. Let $I$ be an ideal of $R$ and let $\alpha \in H^1_I (R)$ be an element such that the elements $\alpha, \alpha^p, \alpha^{p^2}, \ldots, \alpha^{p^s}, \ldots$ belong to a finitely generated $R$-submodule of $H^1_I (R)$. There exists an $R$-subalgebra $R'$ of $\hat{K}$ (i.e. $R \subset R' \subset \hat{K}$) that is finite as an $R$-module and such that the natural map $H^1_I (R) \to H^1_I (R')$ induced by the natural inclusion $R \to R'$ sends $\alpha$ to 0.

**Proof.** Let $A_t = \sum_{i=1}^{t} R \alpha^{p^i}$ be the $R$-submodule of $H^1_I (R)$ generated by $\alpha, \alpha^p, \ldots, \alpha^{p^s}$. The ascending chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ stabilizes because $R$ is Noetherian and all $A_t$ sit inside a single finitely generated $R$-submodule of $H^1_I (R)$. Hence $A_s = A_{s-1}$ for some $s$, i.e. $\alpha^{p^s} \in A_{s-1}$. Thus there exists an equation $\alpha^{p^s} = r_1 \alpha^{p^{s-1}} + r_2 \alpha^{p^{s-2}} + \cdots + r_{s-1} \alpha$ with $r_i \in R$ for all $i$. Let $T$ be a variable and let $g(T) = T^{p^s} - r_1 T^{p^{s-1}} - r_2^{p^{s-2}} - \cdots - r_{s-1} T$. Clearly, $g(T)$ is a monic polynomial in $T$ with coefficients in $R$ and $g(\alpha) = 0$. 


Let \( x_1, \ldots, x_d \in R \) generate the ideal \( I \). If \( M \) is an \( R \)-module, the Čech complex \( C^\bullet(M) \) of \( M \) with respect to the generators \( x_1, \ldots, x_d \in R \) is
\[
0 \to C^0(M) \to \cdots \to C^{i-1}(M) \overset{d_{i-1}}{\to} C^i(M) \overset{d_i}{\to} C^{i+1}(M) \to \cdots \to C^d(M) \to 0
\]
where \( C^0(M) = M \) and \( C^i(M) = \bigoplus_{1 \leq j_1 < \cdots < j_i \leq d} R_{x_{j_1} \cdots x_{j_i}} \), and \( H_i^i(M) \) is the \( i \)th cohomology module of \( C^\bullet(M) \) [5.1.19].

Let \( \tilde{\alpha} \in C^i(R) \) be a cycle (i.e. \( d_i(\tilde{\alpha}) = 0 \)) that represents \( \alpha \). The equality \( g(\alpha) = 0 \) means that \( g(\tilde{\alpha}) = d_{i-1}(\beta) \) for some \( \beta \in C^{i-1}(R) \). Since \( C^{i-1}(R) = \bigoplus_{1 \leq j_1 < \cdots < j_{i-1} \leq d} R_{x_{j_1} \cdots x_{j_{i-1}}} \), we may write \( \beta = \frac{r_{j_1 \cdots j_{i-1}}}{x_{j_1} \cdots x_{j_{i-1}}} \) where \( r_{j_1 \cdots j_{i-1}} \in R \), the integers \( e_1, \ldots, e_i-1 \) are non-negative, and \( \frac{r_{j_1 \cdots j_{i-1}}}{x_{j_1} \cdots x_{j_{i-1}}} \in R_{x_{j_1} \cdots x_{j_{i-1}}} \).

Consider the equation \( g(x_{j_1} \cdots x_{j_{i-1}}) - \frac{r_{j_1 \cdots j_{i-1}}}{x_{j_1} \cdots x_{j_{i-1}}} = 0 \) where \( Z_{j_1 \cdots j_{i-1}} \) is a variable. Multiplying this equation by \( (x_{j_1} \cdots x_{j_{i-1}})^{p^n} \) produces a monic polynomial equation in \( Z_{j_1 \cdots j_{i-1}} \) with coefficients in \( R \). Let \( z_{j_1 \cdots j_{i-1}} \in \overline{K} \) be a root of this equation and let \( R' \) be the \( R \)-subalgebra of \( \overline{K} \) generated by all the \( z_{j_1 \cdots j_{i-1}} \), i.e. by the set \( \{ z_{j_1 \cdots j_{i-1}} | 1 \leq j_1 < \cdots < j_{i-1} \leq d \} \). Since each \( z_{j_1 \cdots j_{i-1}} \) is integral over \( R \) and there are finitely many \( z_{j_1 \cdots j_{i-1}} \), the \( R \)-algebra \( R'' \) is finite as an \( R \)-module.

Let \( \tilde{\alpha} = \frac{Z_{j_1 \cdots j_{i-1}}}{x_{j_1} \cdots x_{j_{i-1}}} \in C^{i-1}(R'') \). The natural inclusion \( R \to R'' \) makes \( C^\bullet(R) \) into a subcomplex of \( C^\bullet(R'') \) in a natural way, and we identify \( \tilde{\alpha} \in C^i(R) \) and \( \beta \in C^{i-1}(R) \) with their natural images in \( C^i(R'') \) and \( C^{i-1}(R'') \) respectively. With this identification, \( \tilde{\alpha} \in C^i(R'') \) is a cycle representing the image of \( \alpha \) under the natural map \( H_i^i(R) \to H_i^i(R'') \), and so is \( \overline{\alpha} = \tilde{\alpha} - d_{i-1}(\tilde{\alpha}) \in C^i(R'') \). Since \( g(\tilde{\alpha}) = \beta \) and \( g(\tilde{\alpha}) = d_{i-1}(\beta) \), we conclude that \( g(\overline{\alpha}) = 0 \). Let \( \overline{\alpha} = (\rho_{j_1 \cdots j_{i-1}}) \) where \( \rho_{j_1 \cdots j_{i-1}} \in R'_{x_{j_1} \cdots x_{j_{i-1}}} \). Each individual \( \rho_{j_1 \cdots j_{i-1}} \) satisfies the equation \( g(\rho_{j_1 \cdots j_{i-1}}) = 0 \). Since \( g(T) \) is a monic polynomial in \( T \) with coefficients in \( R \), each \( \rho_{j_1 \cdots j_{i-1}} \) is an element of the fraction field of \( R' \) that is integral over \( R \). Let \( R' \) be obtained from \( R'' \) by adjoining all the \( \rho_{j_1 \cdots j_{i-1}} \).

Each \( \rho_{j_1 \cdots j_{i-1}} \in R' \), so the image of \( \alpha \) in \( H_i^i(R') \) is represented by the cycle \( \overline{\alpha} = (\rho_{j_1 \cdots j_{i-1}}) \in C^i(R') \) which has all its components \( \rho_{j_1 \cdots j_{i-1}} \) in \( R' \). Each \( R'_{x_{j_1} \cdots x_{j_{i-1}}} \) contains a natural copy of \( R' \), namely, the one generated by the element \( 1 \in R'_{x_{j_1} \cdots x_{j_{i-1}}} \). There is a subcomplex of \( C^\bullet(R') \) that in each degree is the direct sum of all such copies of \( R' \). This subcomplex is exact because its cohomology groups are the cohomology groups of \( R' \) with respect to the unit ideal. Since \( \overline{\alpha} \) is a cycle and belongs to this exact subcomplex, it is a boundary, hence it represents the zero element in \( H_i^i(R') \).

\[ \square \]

**Corollary 2.3.** Let \( R \) be a commutative Noetherian local domain containing a field of characteristic \( p > 0 \). Assume that \( R \) is a surjective image of a Gorenstein local ring. Then the following hold:

(a) \( H_i^i(R^+) = 0 \) for all \( i < \dim R \), where \( m \) is the maximal ideal of \( R \).
(b) Every system of parameters of $R$ is a regular sequence on $R^+$.

Proof. (a) $R^+$ is the direct limit of the finitely generated $R$-subalgebras $R'$, hence $H^i_m(R^+) = \lim H^i_m(R')$. But Theorem 2.1 implies that for each $R'$ there is $R''$ such that the map $H^i_m(R') \to H^i_m(R'')$ in the inductive system is zero. Hence the limit is zero.

(b) Let $x_1, \ldots, x_d$ be a system of parameters of $R$. We prove that $x_1, \ldots, x_j$ is a regular sequence on $R^+$ by induction on $j$. The case $j = 1$ is clear, since $R^+$ is a domain. Assume that $j > 1$ and $x_1, \ldots, x_{j-1}$ is a regular sequence on $R^+$. Set $I_t = (x_1, \ldots, x_t)$. The fact that $H^i_m(R^+) = 0$ for all $i < d$ and the short exact sequences

$$0 \to R^+ / I_{t-1}R^+ \to R^+ / I_tR^+ \to 0$$

for $t \leq j - 1$ imply by induction on $t$ that $H^q_m(R^+ / (x_1, \ldots, x_t)R^+) = 0$ for $q < d - t$. In particular, $H^0_m(R^+ / (x_1, \ldots, x_{j-1})R^+) = 0$ since $0 < d - (j - 1)$. Hence $m$ is not an associated prime of $R^+ / (x_1, \ldots, x_{j-1})R^+$. This implies that the only associated primes of $R^+ / (x_1, \ldots, x_{j-1})R^+$ are the minimal primes of $R / (x_1, \ldots, x_{j-1})R$. Indeed, if there is an embedded associated prime, say $P$, then $P$ is the maximal ideal of the ring $R_P$ whose dimension is bigger than $j - 1$ and $P$ is an associated prime of $(R^+ / (x_1, \ldots, x_{j-1})R^+)_P = (R_P)^+ / (x_1, \ldots, x_{j-1})R_P^+$ which is impossible by the above. Hence every element of $m$ not in any minimal prime of $R / (x_1, \ldots, x_{j-1})R$, for example, $x_j$, is a regular element on $R^+ / (x_1, \ldots, x_{j-1})R^+$. \hfill \qed

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