A NOTE ON THE RANDOMIZED KACZMARZ METHOD WITH
A PARTIALLY WEIGHTED SELECTION STEP

JÜRGEN GROSS

Institute for Mathematics and Applied Informatics, University of Hildesheim,
Germany

Abstract. In this note we reconsider two known algorithms which both usually converge faster than the randomized Kaczmarz method introduced by Strohmer and Vershynin [27], but require the additional computation of all residuals of an iteration at each step. As already indicated in the literature, e.g. Steinerberger [26], Jiang et al. [16], it is shown that the non-randomized version of the two algorithms converges at least as fast as the randomized version, while still requiring computation of all residuals. Based on that observation, a new simple random sample selection scheme has been introduced by Jiang et al. [16] to reduce the required total of residuals. In the same light we propose an alternative random selection scheme which can easily be included as a 'partially weighted selection step' into the classical randomized Kaczmarz algorithm without much ado. Numerical examples show that the randomly determined number of required residuals can be quite moderate.

1. Introduction

Consider the system of linear equations denoted by

\[ Ax = b \]

where \( A \in \mathbb{C}^{m \times n} \) for \( m \geq n \) is a matrix of rank \( n \).

Let \( a_i \) denotes the \( i \)-th row of the matrix \( A \) considered as a column vector. Starting with an initial guess \( x_0 \), the simple Kaczmarz [17] method provides an iterative algorithm for the solution to (1) with respect to \( x \). The \((k+1)\)-th iteration is given by

\[ x_{k+1} = x_k + \frac{b_i - \langle a_i, x_k \rangle}{\|a_i\|^2} a_i, \]

where \( i = (k \mod m) + 1 \), and \( \| \cdot \|_2 \) denotes the Euclidean norm of a vector. The algorithm generates cycles of \( m \) iterations by sweeping repeatedly through the rows of the matrix \( A \) with the last iteration from one cycle as the starting value for the first iteration of the next cycle. This procedure is also known as the algebraic reconstruction technique (ART), e.g. Gordon et al. [9].

The randomized Kaczmarz method by Strohmer and Vershynin [27] improves upon the simple Kaczmarz method by choosing row \( i \) in (2) not in a successive...
manner but randomly from the set of row numbers \( \{1, \ldots, m\} \), where row \( i \) is given probability \( \|a_i\|_2^2/\|A\|_F^2 \), and \( \|A\|_F \) denotes the Frobenius norm of \( A \).

Convergence rates of Kaczmarz related methods and procedures for improvements were investigated by a number of authors, see e.g. [4, 5, 8, 29, 7, 20, 11, 19, 21, 13, 15, 1, 2, 24, 10, 28, 13, 16, 26].

In the following we will assume that the matrix \( A \) has Euclidean row norm
\[
\|a_i\|_2 = 1
\]
for \( i = 1, \ldots, m \), in which case the matrix \( A \) is also called standardized, see Needell and Tropp [20]. In that case \( \|A\|_F^2 = m \) and row \( i \) is selected by the randomized Kaczmarz method with uniform probability \( 1/m \).

2. Two Algorithms

Recently, algorithms had been proposed which randomly select a row for computing \( x_{k+1} \) by assigning a probability to row \( i \) depending on the \( i \)-th element
\[
r_k(i) := b_i - \langle a_i, x_k \rangle
\]
of the residual vector \( r_k = b - Ax_k \) of the \( k \)-th iteration \( x_k \). Algorithm 3 in Jiang et al. [16] and the Algorithm in Steinerberger [26] slightly differ with respect to the precise implementation, while Algorithm 2.1 in [13] can be seen as a more restricted version. By referring to Steinerberger [26], row \( i \) is selected with probability
\[
p_k(i) = \frac{|r_k(i)|^p}{\sum_{i=1}^m |r_k(i)|^p}
\]
for some given integer \( p \), see Algorithm 1.

**Algorithm 1** Randomized Weighted Kaczmarz

**Input:** matrix \( A \in \mathbb{C}^{m,n} \) with \( m \geq n \) and \( \|a_i\|_2 = 1 \) for all \( i \in \{1, \ldots, m\} \), \( b \in \mathbb{C}^n \) satisfying \( b = Ax \) for some \( x \), initial guess \( x_0 \), integer \( p \).

1. Set \( k = 0 \). Compute \( r_0 = b - Ax_0 \).
2. Select an element \( i \) from \( \{1, \ldots, m\} \) with probability \( p_k(i) \) from (5).
3. Compute \( x_{k+1} = x_k + r_k(i) a_i \) with \( r_k(i) \) from (4).
4. Compute \( r_{k+1} = b - Ax_k \). Set \( k = k + 1 \). Go to 2.

As shown by Steinerberger [26], Algorithm 1 is at least as efficient as the classical randomized Kaczmarz method from Strohmer and Vershynin [27] irrespective of the choice of \( p \), see also Algorithm 3 in Jiang et al. [16]. The case \( p \to \infty \) corresponds to Algorithm 2 below, which actually is the maximum residual (MR) rule, coinciding with the maximum distance (MD) rule for a standardized matrix \( A \), see [6, 12, 22].

**Algorithm 2** Non-Randomized Greedy Kaczmarz

**Input:** matrix \( A \in \mathbb{C}^{m,n} \) with \( m \geq n \) and \( \|a_i\|_2 = 1 \) for all \( i \in \{1, \ldots, m\} \), \( b \in \mathbb{C}^n \) satisfying \( b = Ax \) for some \( x \), initial guess \( x_0 \).

1. Set \( k = 0 \). Compute \( r_0 = b - Ax_0 \).
2. Select an element \( i \) from \( \{1, \ldots, m\} \) such that \( i = \arg\max_j |r_k(j)| \).
3. Compute \( x_{k+1} = x_k + r_k(i) a_i \).
4. Compute \( r_{k+1} = b - Ax_k \). Set \( k = k + 1 \). Go to 2.
Partially Weighted Randomized Kaczmarz

The non-randomized Algorithm 2 is called ‘partially randomized’ by Jiang et al. [16] and stated as their Algorithm 4. The authors show a convergence result that also relates Algorithm 4 to the greedy randomized Kaczmarz method from Bai and Wu [1, 2]. It is noted, however, that there is nothing random about the actual selection of a row in Algorithm 2, apart from the fact that a deterministic selection can be modelled as a probability one (almost sure) decision, see also the proof of the following Proposition in Appendix A.

**Proposition.** Let $x$ denote the solution of (1). Let $E_{Alg1}$ and $E_{Alg2}$ be the respective conditional expectation given $x_k$ corresponding to the probability distribution implied by the choice of row $i$ in step 2 of Algorithm 1 and 2. Then

$$E_{Alg2} \| x_{k+1} - x \|^2_2 \leq E_{Alg1} \| x_{k+1} - x \|^2_2 .$$

**Remark.** The above Proposition together with the Theorem in Steinerberger [26] implies that Algorithm 2 converges with at least the rate of Algorithm 1.

### 3. Partially Weighted Variant of Randomized Kaczmarz

Both Algorithms 1 and 2 require the computation of the complete residual vector $r_k$ prior to the selection step 2 which can be quite time-consuming. In order to reduce the number of required residuals $r_k(i)$, it has been proposed to select a random sample of rows in advance and then greedily select a row within the sample, see [3, 14, 16]. Our Algorithm 3 below describes a proposal in the same light, where, however, the sample size by itself is random. It may also be seen as a variant of the classical randomized Kaczmarz method which makes use of individual residuals $r_k(i)$ only when needed, and does not require the computation of the complete residual vector $r_k$, except in rare cases.

The main idea is to look out for $\arg\max_j |r_k(j)|$, but with considerable reservation. In a first step a candidate row $i_1$ is randomly chosen from the set $\{1, \ldots, m\}$ according to the uniform distribution. Then a second competitor row $i_2$ is chosen from the remaining set $\{1, \ldots, m\} \setminus \{i_1\}$ due to the uniform distribution on this set. If $|r_k(i_1)| > |r_k(i_2)|$ row $i_1$ is selected, otherwise $i_2$ becomes the new candidate and a new competitor row $i_3$ is randomly selected from the set $\{1, \ldots, m\} \setminus \{i_1, i_2\}$. The absolute residuals of the two rows in question are again compared in the same manner. This is repeated until a candidate row is actually selected, where one may always select the last possible row when all other residual comparisons did not lead to a candidate selection before.

The procedure requires at least the computation of 2 residuals for each iteration. At most, $m$ residuals are required, but this will rarely be the case. Of course, an obvious simpler variant is to compare only 2 residuals in each iteration and select the row admitting the larger residual. This may then be seen as employing a simple random sample of size 2. The performance of Algorithm 3 and its two residuals variant is considered in Section 4.

Skipping steps 2.2 and 2.3 and employing $i = i_1$ from step 2.1 gives the classical randomized Kaczmarz by Strohmer and Vershynin [27] for the case of a standardized matrix $A$.

**Remark.** If a specific row $i$ is selected using elements of $r_k$ by step 2 of one of the Algorithms 1, 2, and 3 then $r_{k+1}(i) = 0$. Hence, for all three algorithms, a selected row in the subsequent iteration is (almost surely) different from the selected row in the actual iteration as long as the exact solution has not been found.
Algorithm 3 Randomized Kaczmarz With Partially Weighted Selection Step

Input: matrix $A \in \mathbb{C}^{m \times n}$ with $m \geq n$ and $\|a_i\|_2 = 1$ for all $i \in \{1, \ldots, m\}$, $b \in \mathbb{C}^n$ satisfying $b = Ax$ for some $x$, initial guess $x_0$.

1. Set $k = 0$.
2. Set $U = \{1, \ldots, m\}$. Select an element $i$ from $U$ according to the following selection scheme:
   2.1 Select an element $i_1$ from $U$ with uniform probability.
   2.2 Set $U = U \setminus \{i_1\}$. If $U = \emptyset$ set $i = i_1$ and go to 3. Otherwise, select an element $i_2$ from the set $U$ with uniform probability.
   2.3 If $|r_k(i_1)| > |r_k(i_2)|$, set $i = i_1$ and go to 3. Otherwise, set $i_1 = i_2$ and go to 2.2.
3. Compute $x_{k+1} = x_k + r_k(i) a_i$.
4. Set $k = k + 1$. Go to 2.

The convergence of Algorithm 3 can be concluded from the convergence of the classical randomized Kaczmarz algorithm. For each iteration the full set $\{1, \ldots, m\}$ of row numbers is again available for both, the classical and the partially weighted row selection step. Indeed, if there were a sequence of selected row numbers for which the partial weighted variant had failed to converge, then the classical algorithm too could not have converged for this very sequence. See in addition Patel et al. [23, Sect. 5.5], also confirming convergence of algorithms of this type.

Moreover, it is to be expected that Algorithm 3 converges faster than the classical randomized Kaczmarz, since the selection of rows is directed towards Algorithm 2. Numerical examples discussed in the following section support this conclusion.

4. Numerical Examples

In this section we reconsider the settings described by Steinerberger [26]. Computations are carried out with the statistical software R, see R Core Team [25].

Figure 1. Error $\|x_k - x\|_2$ for the classical randomized Kaczmarz (blue), randomized Kaczmarz with partially weighted selection step (green), two residuals variant of the former (orange), and non-randomized greedy Kaczmarz (red)
4.1. **Nice Matrix.** A $1000 \times 1000$ matrix $A$ is created by sampling the elements independently from the standard normal distribution. Then the matrix $100I_{1000}$ is added to $A$ and the result is standardized. The vector $b$ is the $1000 \times 1$ vector of $0$s and the initial guess $x_0$ is the $1000 \times 1$ vector of $1$s. The outcome from the considered algorithms is displayed in Figure 1. The classical randomized Kaczmarz (blue line) performs in a very similar manner to what can be seen from Figure 2 in Steinerberger [26]. Also, the non-randomized Kaczmarz from Algorithm 2 (red line) performs similar to what can be seen from Figure 2 in Steinerberger [26]. Visibly, the randomized Kaczmarz with partially weighted selection step (green line) performs better than the classical randomized Kaczmarz and also slightly better than the two residuals variant. In order to assess the amount of complexity in Algorithm 3, the number of required residuals is recorded in Table 1 for each of the first 10000 iterations.

| # residuals | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|
| freq        | 4947| 3334| 1292| 355 | 59  | 10  | 2   | 1   |

As it is seen, about half of the considered iterations require only the minimal number of two residuals, while no iteration requires more than 9 residuals. Several repetitions have shown similar results.

![Figure 2](image1.png)

**Figure 2.** Error $\|x_k - x\|_2$ for the classical randomized Kaczmarz (blue), randomized Kaczmarz with partially weighted selection step (green), two residuals variant of the former (orange), and non-randomized greedy Kaczmarz (red).

4.2. **Challenging Matrix.** For the more challenging setting, a $1000 \times 1000$ matrix $A$ is created by sampling the elements independently from the standard normal distribution. The vector $b$ and the initial choice $x_0$ are the same as before. Figure 2 displays the results. As before, results are comparable to those in Figure 3 from Steinerberger [26], the overall convergence being quite slower. Table 2 shows the number of required residuals in Algorithm 3 for each of the first 20000 iterations.
As in the case of the nice matrix, about half of the iterations require only two residuals and no iterations requires more than 9 residuals.

Table 2. Number of required residuals per iteration.

| # residuals | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|
| freq        | 9938| 6665| 2532| 687 | 153 | 19  | 4   | 2   |

The results from the numerical examples support the conclusion that Algorithm 3 requires a rather moderate number of residuals, but has the capability to outperform the classical randomized Kaczmarz method.

Appendix A. Proof of the Proposition

Let \( a \) be a random vector with possible values \( a_1, \ldots, a_m \), being the rows of the standardized matrix \( A \) considered as column vectors. Let row \( i \) be selected with \( \text{prob}(i) \) for the computation of \( x_{k+1} \). Let \( d_k = x_k - x \), where \( x \) denotes the solution to (1).

In the following, it is assumed that \( x_k \) (and hence \( d_k \)) is given and thus non-random, while \( x_{k+1} \) (and hence \( d_{k+1} \)) is random, depending on the random vector \( a \). The probability distribution \( \text{prob}(i) \), \( i = 1, \ldots, m \), may be chosen as depending on \( x_k \), and all considered expectations are regarded as conditional with respect to given \( x_k \).

Now, the conditional expectation of the random variable \( |\langle a, d_k \rangle|^2 \) is given by

\[
E|\langle a, d_k \rangle|^2 = \sum_{i=1}^{m} |\langle a_i, d_k \rangle|^2 \text{prob}(i) .
\]

Then, since the expectation of a discrete random variable cannot exceed its largest possible value,

\[
E|\langle a, d_k \rangle|^2 \leq |\langle a_i, d_k \rangle|^2
\]

when \( |\langle a_i, d_k \rangle|^2 \leq |\langle a_i, d_k \rangle|^2 \) for \( i = 1, \ldots, m \).

Suppose that \( E_* \) denotes the conditional expectation of \( |\langle a, d_k \rangle|^2 \) with respect to the specific probability distribution that assigns probability 1 to row \( i_* \) satisfying

\[
|b_i - \langle a_i, x_k \rangle| \leq |b_{i_*} - \langle a_{i_*}, x_k \rangle|
\]

for \( i = 1, \ldots, m \). In view of the identity,

\[
|b_i - \langle a_i, x_k \rangle| = |\langle a_i, d_k \rangle|
\]

it follows that

\[
E_*|\langle a, d_k \rangle|^2 = |\langle a_{i_*}, d_k \rangle|^2 ,
\]

and thus

\[
E|\langle a, d_k \rangle|^2 \leq E_*|\langle a, d_k \rangle|^2 .
\]

By following the lines in Steinerberger [26 Sect. 4.1], it is concluded that

\[
\|d_k\|^2 - |\langle a_i, d_k \rangle|^2 , \quad i = 1, \ldots, m ,
\]

are the possible values of the random variable \( \|d_{k+1}\|^2 \), so that

\[
E\|d_{k+1}\|^2 = \|d_k\|^2 - E|\langle a, d_k \rangle|^2
\]

is the conditional expectation of \( \|d_{k+1}\|^2 \) given \( x_k \). Thus

\[
E_*\|d_{k+1}\|^2 \leq E\|d_{k+1}\|^2 ,
\]
implying the Proposition.

Remark. The derived inequality $\mathbb{E}_e \|x_{k+1} - x\|_2^2 \leq \mathbb{E}\|x_{k+1} - x\|_2^2$ holds for any probability distribution specified for the selection of a row in computing $x_{k+1}$ from (2) when $A$ is standardized.

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