Coupling coefficients for SO(5) with applications to nuclear physics

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Abstract. The Lie algebra SO(5) has several applications in nuclear theory, involving its various subalgebra chains. In this contribution, the calculation of SO(5) coupling coefficients, reduced with respect to these different subalgebras, is discussed. It is shown that “Racah’s method” for extracting coupling coefficients can be formulated generically for a subalgebra chain \( G \supset H \), provided the reduced matrix elements of the generators of \( G \) and the recoupling coefficients of \( H \) are known. The calculation of SO(5) \( \supset \) SO(4) reduced coupling coefficients, for generic irreps, is considered as an example.

1. Introduction

Continuous symmetries and their associated Lie algebras facilitate the description of many-body systems both directly and indirectly. When a symmetry occurs as a dynamical symmetry of the system, the corresponding algebra immediately gives the spectroscopic properties of the system. However, even when a symmetry is strongly broken, the algebraic structure nonetheless provides a calculational tool, classifying the basis states used in a full computational treatment of the many-body problem and greatly simplifying the underlying calculational machinery. Lie algebras have a long history of application, in both these capacities, to nuclear spectroscopy and related problems [1–3]. The fundamental quantities underlying calculations within a Lie algebraic framework are the coupling coefficients of the algebra, also known as generalized Clebsch-Gordan coefficients or Wigner coefficients. These are needed in order to couple states (or operators) of good symmetry to yield new states (or operators) of good symmetry, and they are required for the calculation of matrix elements through the generalized Wigner-Eckart theorem of the algebra.

The Lie algebra SO(5) has several distinct applications in nuclear theory, involving different physical realizations of the operators, and in which different subalgebra chains are relevant to the symmetry properties. Application as the proton-neutron pairing quasispin algebra [4–8] requires reduction with respect to the U(1) \( \otimes \) SO(3) algebra of isospin and occupation number operators. For the dynamics of spin-2 bosons (as in the interacting boson model [9, 10]) and for the Bohr collective model [11–15], the appropriate reduction is instead with respect to the physical angular momentum SO(3) subalgebra. Finally, the canonical SO(4) \( \sim \) SO(3) \( \otimes \) SO(3) subalgebra is the most mathematically natural for computational use.

In this contribution, it is shown that the method of infinitesimal generators (“Racah’s method”) for extracting coupling coefficients can be formulated generically for a subalgebra chain \( G \supset H \), provided the reduced matrix elements of the generators of \( G \) and the recoupling coefficients of \( H \) are known. After a brief overview of the SO(5) algebra (section 2), the general
problem of calculating reduced coupling coefficients is discussed, and Racah’s method is broadly and systematically formulated as a method applicable to the calculation of reduced coupling coefficients for any subalgebra chain (section 3). More specifically, the problem of calculating coupling coefficients for generic irreps of SO(5), reduced with respect to any of the subalgebra chains, is completely resolved by this approach. The calculation of reduced coupling coefficients for the SO(5) ⊃ SO(4) canonical chain is considered in detail (section 4). Coupling coefficients reduced with respect to the noncanonical subalgebra chains of SO(5) may be obtained by a similar application of Racah’s method, or they can simply be deduced from the canonical chain coupling coefficients by a straightforward unitary transformation.

2. The SO(5) algebra
The rotation algebra SO(5) in 5-dimensional space is spanned by the infinitesimal generators of rotation,

\[ L_{rs} \equiv -i(x_r \partial_s - x_s \partial_r), \]

with \( r \) and \( s = 1, \ldots, 5 \). These operators are Hermitian and antisymmetric in the indices. The structure of the algebra is determined by the commutators

\[ [L_{pq}, L_{rs}] = -i(\delta_{qr} L_{ps} + \delta_{ps} L_{qr} + \delta_{sq} L_{rp} + \delta_{rp} L_{sq}). \]

The canonical construction of SO(5) begins with SO(3), with the usual generators

\[ \text{SO(3)} : \quad J_1 \equiv L_{23}, \quad J_2 \equiv L_{31}, \quad J_3 \equiv L_{12}. \]

Then, SO(4) is obtained with the addition of generators involving the fourth coordinate,

\[ \text{SO(4)} : \quad N_1 \equiv L_{14}, \quad N_2 \equiv L_{24}, \quad N_3 \equiv L_{34}. \]

These generators have commutation relations \([J_r, J_s] = i \varepsilon_{rst} J_t, \quad [N_r, N_s] = i \varepsilon_{rst} N_t\), and \([J_r, N_s] = i \varepsilon_{rst} N_t\). However, SO(4) can be recast as the direct sum of two SO(3) algebras, with the definitions

\[ \text{SO}_X(3) \otimes \text{SO}_Y(3) : \quad X_k \equiv \frac{1}{2}(J_k + N_k), \quad Y_k \equiv \frac{1}{2}(J_k - N_k), \]
so \([X_r, X_s] = i\varepsilon_{rst}X_t, \ [Y_r, Y_s] = i\varepsilon_{rst}Y_t\), and \([X_r, Y_s] = 0\). The ladder operator realizations of these generators \((X_\pm \equiv X_1 \pm iX_2, X_0 \equiv X_3, Y_\pm \equiv Y_1 \pm iY_2, \ym 0 \equiv Y_3)\) are indicated in the root vector diagram figure 1, where each operator is placed according to the \(SO(3)\) weights \(M_X\) and \(M_Y\) carried by the operator. The Clebsch-Gordan series and Wigner calculus (coupling and recoupling coefficients) for \(SO(4)\) follow immediately from the \(SO(3)\) structure \([16]\). Finally, \(SO(5)\) is obtained by introducing the four generators \(L_{r5} \ (r = 1, \ldots, 4)\), which are most naturally incorporated as the linear combinations

\[
SO(5): \quad \begin{align*}
T_{++} &= -\frac{1}{2}(L_{15} + iL_{25}) \\
T_{-+} &= \frac{1}{2}(L_{35} + iL_{45}) \\
T_{+-} &= \frac{1}{2}(L_{35} - iL_{45}) \\
T_{--} &= \frac{1}{2}(L_{15} - iL_{25}).
\end{align*}
\]

These together constitute a double spherical tensor \(T^{(1\frac{1}{2}, 1\frac{1}{2})}_{M_X M_Y}\) \((M_X = \pm \frac{1}{2} \text{ and } M_Y = \pm \frac{1}{2})\) with respect to the two \(SO(3)\) algebras, completing the root vector diagram of figure 1.

The generators of \(SO(5)\) as realized in (1) are differential operators on \(\mathbb{R}^5\), which can only support symmetric irreps. However, the commutation relations (2) provide the fundamental definition of the algebra, and these support irreps of a much more general nature. The weights within a generic \(SO(4)\) irrep \((XY)\) follow from the usual angular momentum rules \(M_X = -X, \ldots, X - 1, X \text{ and } M_Y = -Y, \ldots, Y - 1, Y\), as shown graphically in figure 2(a). The decomposition of an outer product of \(SO(4)\) irreps (Clebsch-Gordan series) as \((X_1 Y_1) \otimes (X_2 Y_2) \to (XY)\) follows by simple application of the triangle inequality to each \(SO(3)\) algebra, \(i.e., X = |X_1 - X_2|, \ldots, X_1 + X_2 - 1, X_1 + X_2 \text{ and } Y = |Y_1 - Y_2|, \ldots, Y_1 + Y_2 - 1, Y_1 + Y_2\). The \(SO(4)\) irreps contained within an \(SO(5)\) irrep \((RS)\) follow from the branching rule \([17, 18]\)

\[
X = R - \frac{1}{2}m - \frac{1}{2}n \\
Y = S + \frac{1}{2}m - \frac{1}{2}n,
\]

with \(m\) and \(n\) integers such that \(0 \leq n \leq 2(R - S)\) and \(0 \leq m \leq 2S\). The \(SO(4)\) irrep highest weights form a lattice bounded by a tilted rectangle, as illustrated in figure 2(b). Note that

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**Figure 2.** Weight and branching diagrams for \(SO(4)\) and \(SO(5)\) irreps: (a) Weight points \((M_X M_Y)\) contained within an \(SO(4) \sim SO(3) \otimes SO(3)\) irrep \((XY)\). The irrep \((2\frac{3}{2})\) is used for illustration. (b) Branching of an \(SO(5)\) irrep \((RS)\). The irrep \((\frac{3}{2})\) is used for illustration. The open circle indicates the highest weight for the \(SO(5)\) irrep, and the solid dots indicate \(SO(4)\) highest weights, according to branching rule (7). The shaded rectangles are the boundaries of the weight sets for these \(SO(4)\) irreps.
the SO\(_X(3) \otimes SO_Y(3)\) labels for an SO(4) irrep are simply related to the canonical labels by \([pq] = [X + Y, X - Y]\), and the SO\(_X(3) \otimes SO_Y(3)\) labels for an SO(5) irrep are related to the canonical labels by \([l_1 l_2] = [R + S, R - S]\).

In addition to the natural subalgebra chains arising from the canonical construction above, involving the SO(4) \(\sim SO_X(3) \otimes SO_Y(3)\) subalgebra, two noncanonical subalgebra chains also arise. These involve SO(3) subalgebras [or, equivalently, SU(2) subalgebras] distinct from those already considered. It is the noncanonical chains which appear in physical applications. The chains may be summarized as follows:

\[
\begin{align*}
\text{SO}(5) & \supset \text{SO}(4) \supset \text{SO}_X(3) \supset \text{SO}_Y(3) \supset \text{SO}_X(2) \supset \text{SO}_Y(2) \quad (\text{I}) \\
\supset U_N(1) \otimes \text{SO}_T(3) & \supset \text{SO}_T(2) \quad (\text{II}) \\
\supset \text{SO}_L(3) & \supset \text{SO}_L(2) \quad (\text{III}).
\end{align*}
\]

The irrep labels have been noted beneath each subalgebra, and branching multiplicity labels are required for the noncanonical chains as well.

The isospin algebra SO\(_T(3)\) of chain (II) is the relevant subalgebra for the description of proton-neutron pairing [4–8, 19–21]. In this context, the SO(5) generators arise as quasispin operators for pairing of protons and neutrons occupying the same \(j\)-shell. The operators

\[
X_+ \propto (a_p^\dagger a_n^+)^{(0)} \quad X_0 \propto (a_p^\dagger \tilde{a}_p + \tilde{a}_p^\dagger a_n^1)^{(0)} \quad X_- \propto (\tilde{a}_p a_p)^{(0)}
\]

constitute an SO(3) quasispin algebra for proton pairs (of angular momentum zero), and

\[
Y_+ \propto (a_n^\dagger a_n^+)^{(0)} \quad Y_0 \propto (a_n^\dagger \tilde{a}_n + \tilde{a}_n^\dagger a_n^1)^{(0)} \quad Y_- \propto (\tilde{a}_n a_n)^{(0)}
\]

constitute an SO(3) quasispin algebra for neutron pairs. The remaining generators (6) consist of the proton-neutron pair creation and annihilation operators

\[
S_+ \propto (a_p^\dagger a_n^1 + a_n^\dagger a_p^1)^{(0)} \quad S_- \propto (\tilde{a}_n a_p + \tilde{a}_p a_n)^{(0)}
\]

and isospin ladder operators

\[
T_+ \propto -(a_p^\dagger \tilde{a}_n)^{(0)} \quad T_- \propto -(\tilde{a}_n^\dagger a_p)^{(0)}.
\]

identified as \(S_+ = -2T_{++}, \ S_- = 2T_{--}, \ T_+ = 2T_{+-}, \ \text{and} \ T_- = 2T_{-+}\). The natural weight operators, providing the labels \(M_S = \frac{1}{2}[N - (2j + 1)]\) and \(M_T = \frac{1}{2}(N_p - N_n)\) of (II), are \(S_0 = X_0 + Y_0\) and \(T_0 = X_0 - Y_0\).

On the other hand, the “physical” or “geometric” angular momentum subalgebra SO\(_L(3)\) of chain (III) is the relevant subalgebra for application to systems of spin-2 bosons [9, 10] or the nuclear collective model [11–13, 15]. For the bosonic realization, the angular momentum generators are naturally given by the products \(L_M = \sqrt{10}(b_1^\dagger \tilde{b}_1)^{(1)}_M\), which close as the subalgebra SO\(_L(3)\). The remaining seven generators are realized as \(O_M = \sqrt{10}(b_1^\dagger \tilde{b}_1)^{(3)}_M\), constituting a spherical tensor of octupole character with respect to SO\(_L(3)\). The commutation relations are again found to be those of SO(5), with the \(L_M\) and \(O_M\) identified as certain linear combinations of the canonical generators, e.g., \(L_{+1} = -\sqrt{2}X_+ - \sqrt{6}T_{-+}\).
3. General formulation of Racah’s method in reduced form

First, let us establish what is meant by a coupling coefficient for $G$ reduced with respect to a subalgebra $H$. For a Lie algebra $G$, with subalgebra $H$, containing the Cartan subalgebra of $G$, states which reduce this subalgebra chain may be classified by the corresponding irreps labels $\Gamma$ and $\Lambda$ and Cartan weights $\lambda$,

$$G \supset_a H \supset \lambda \quad \text{(Cartan subalgebra),} \quad \text{(13)}$$

where multiplicity indices may be required as well. The coupling coefficients for $G$ relate the uncoupled product states of two irreps of $G$ to the coupled states, as

$$\left\langle \Gamma_1 \Gamma_2 \rho \Lambda \left| a_1 \Lambda_1 a_2 \Lambda_2 \lambda \right\rangle = \sum_{a_1 \Lambda_1 a_2 \Lambda_2} \left( \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \rho \Lambda \\ a_1 \Lambda_1 & a_2 \Lambda_2 & \lambda \end{array} \right) \left( \begin{array}{ccc} \Gamma_1 & \Gamma_2 \rho \Lambda \\ a_1 \Lambda_1 & a_2 \Lambda_2 & \lambda \end{array} \right) \left\langle \Gamma_1 \Gamma_2 \rho \Lambda \left| a_1 \Lambda_1 a_2 \Lambda_2 \lambda \right\rangle \right\rangle, \quad \text{(14)}$$

where $\rho$ is the outer multiplicity index for $G \otimes G$. By Racah’s factorization lemma, the coupling coefficient appearing can be decomposed as the product

$$\left( \begin{array}{ccc} \Gamma_1 & \Gamma_2 \rho \Lambda \\ a_1 \Lambda_1 & a_2 \Lambda_2 & \lambda \end{array} \right) = \left( \begin{array}{ccc} \Lambda_1 & \Lambda_2 \rho \Lambda \\ a_1 \Lambda_1 & a_2 \Lambda_2 & \lambda \end{array} \right) \left( \begin{array}{ccc} \Gamma_1 & \Gamma_2 \rho \Lambda \\ a_1 \Lambda_1 & a_2 \Lambda_2 & \lambda \end{array} \right) \left( \begin{array}{ccc} \Lambda_1 & \Lambda_2 \rho \Lambda \\ a_1 \Lambda_1 & a_2 \Lambda_2 & \lambda \end{array} \right) \left( \begin{array}{ccc} \Gamma_1 & \Gamma_2 \rho \Lambda \\ a_1 \Lambda_1 & a_2 \Lambda_2 & \lambda \end{array} \right)$$

of a coupling coefficient of $H$ with a reduced coupling coefficient for $G \supset H$. (Here we have taken the subalgebra $H$ and its outer product to be multiplicity free. See [22] for the appropriate generalizations.)

For the symmetric irreps of SO(5), one may work with an explicit construction, deriving the five-dimensional spherical harmonics as functions on the four-sphere. Their triple overlap integrals are then proportional to SO(5) coupling coefficients. Such an approach was recently developed for chain (III), where the spherical harmonics $\Psi_{\nu \lambda \Lambda, \Lambda, \lambda}^L \gamma, \Omega$ are of specific interest as basis functions for nuclear collective model calculations. Further details may be found in refs. [13, 23].

However, calculation of coupling coefficients involving generic irreps requires a broader approach. Several methods may be considered, in general, for constructing the reduced coupling coefficients of Lie algebras:

1. Recurrence relations among coupling coefficients may be obtained by considering the action of an infinitesimal generator $G = G^{(1)} + G^{(2)}$ on uncoupled and coupled states. This approach is broadly termed “Racah’s method” [22] and generalizes the classic recurrence method for evaluating SO(3) Clebsch-Gordan coefficients [24].

2. Recurrence relations may be obtained by considering the action of a “shift tensor”, lying outside the algebra [25]. For SO(5), shift tensors can be constructed by vector coherent state methods [26].

3. Consistency relations among coupling and recoupling coefficients serve as the basis for a “building up” process [8, 27], in which unknown coupling coefficients can be deduced from a few known coefficients.

4. Explicit realizations of an algebra can be obtained in terms of bosonic or fermionic creation and annihilation operators. Relations among coupling coefficients follow from considering the matrix elements of tensor operators operating on bosonic or fermionic states (e.g., [28]). This approach yields a limited but useful class of coupling coefficients, generally restricted to symmetric irreps, antisymmetric irreps, or irreps which can be obtained as simple combinations thereof.
Indeed, all of these approaches have been applied, in various forms, to the calculation of specific classes of SO(5) coupling coefficients [1, 8, 17, 22, 28–32].

Let us now consider how the first approach, i.e., Racah’s method based on the action of infinitesimal generators, can be generally and systematically formulated as a method applicable to the calculation of reduced coupling coefficients involving generic irreps of an arbitrary subalgebra chain. (A comprehensive development of the method will be given in [33].)

Consider the action of a generator $T_{\lambda_T}^{\Lambda_T} = T_{\lambda_T}^{\Lambda_T(1)} + T_{\lambda_T}^{\Lambda_T(2)}$ of $G$ (written as a tensor operator with respect to $H$) between coupled and uncoupled states,

$$\left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T} \bigg| a_{A}^{\Lambda} \right\rangle = \left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T(1)} \bigg| a_{A}^{\Lambda} \right\rangle + \left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T(2)} \bigg| a_{A}^{\Lambda} \right\rangle.$$

Its action (to the right) on the coupled state and (to the left) on each factor of the uncoupled state is then known, and the requirement of equality between the two sides of (16) provides the necessary condition for determination of the coupling coefficients in (14).

In particular, if the right hand side is expanded entirely in terms of uncoupled states, and if the left hand side is expanded entirely in terms of coupled states, by (14) and (15), then (16) constitutes a relation among (unknown) coupling coefficients and (known) matrix elements of the generator $T_{\lambda_T}^{\Lambda_T}$ between basis states. The Wigner-Eckart theorem for $H$ allows the action of $T_{\lambda_T}^{\Lambda_T}$ to be expressed in terms of a coupling coefficient of $H$ and a reduced matrix element as

$$\left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T} \bigg| a_{A}^{\Lambda} \right\rangle = \left( \frac{\Lambda}{\lambda} \bigg| \frac{\Lambda_T}{\lambda_T} \bigg| \frac{\Lambda'}{\Lambda'} \right) \left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T(1)} \bigg| a_{A}^{\Lambda} \right\rangle \left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T(2)} \bigg| a_{A}^{\Lambda} \right\rangle.$$

Thus, the basic condition (16) expands to

$$\sum_{\lambda_{\lambda'}} \left( \frac{\Lambda}{\lambda} \bigg| \frac{\Lambda_T}{\lambda_T} \bigg| \frac{\Lambda'}{\lambda'} \right) \left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T} \bigg| a_{A}^{\Lambda} \right\rangle = \sum_{\lambda'_{\lambda'}} \left( \frac{\Lambda'}{\lambda'} \bigg| \frac{\Lambda_T}{\lambda_T} \bigg| \frac{\Lambda}{\lambda} \right) \left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T(1)} \bigg| a_{A}^{\Lambda} \right\rangle \left\langle a_{\lambda}^{A} \bigg| T_{\lambda_T}^{\Lambda_T(2)} \bigg| a_{A}^{\Lambda} \right\rangle.$$

However, this relation may be rearranged, by application of the orthonormality conditions on coupling coefficients of $H$, so that all these coupling coefficients of the lower algebra are bundled into recoupling coefficients, the unitary “6-Λ symbols”

$$\left[ \begin{array}{ccc} \Lambda_1 & \Lambda_2 & \Lambda_{12} \\ \Lambda_3 & \Lambda_2 & \Lambda_{23} \end{array} \right]_{\lambda_1 \lambda_2 \lambda_3} = \sum_{\lambda_1 \lambda_2 \lambda_3} \left( \frac{\Lambda_1}{\lambda_1} \bigg| \frac{\Lambda_2}{\lambda_2} \bigg| \frac{\Lambda_3}{\lambda_3} \right) \left( \frac{\Lambda_{12}}{\lambda_{12}} \bigg| \frac{\Lambda_{13}}{\lambda_{13}} \right) \left( \frac{\Lambda_{23}}{\lambda_{23}} \right).$$

These are the transformation brackets connecting coupling schemes $[(\Lambda_1 \Lambda_2) \Lambda_{12} \Lambda_3]_{\lambda}$ and $[\Lambda_1 (\Lambda_2 \Lambda_3) \Lambda_{23}]_{\lambda}$. We can thereby eliminate the coupling coefficients of $H$, and thus all reference...
to weights, from the condition (18). Let $\sigma(\Lambda_1\Lambda_2\Lambda)$ be the phase factor incurred when the first and second irreps are interchanged in a coupling coefficient of $H$. Then (18) simplifies to

$$
\sum_{a'} \left< a' \Lambda' \right| T^{\Lambda_T} \left| a \Lambda \right> \left( \frac{\Gamma_1}{a_1 \Lambda_1} \frac{\Gamma_2}{a_2 \Lambda_2} \rho^\Gamma \right)
$$

$$= \sum_{a'_1 \Lambda'_1} \sigma(\Lambda_1\Lambda_2\Lambda')\sigma(\Lambda'_1\Lambda_2\Lambda) \left[ \frac{\Lambda_2}{\Lambda_T} \frac{\Lambda'_1}{\Lambda} \frac{\Lambda}{\Lambda_1} \right] \left< a_1 \Lambda_1 \right| T^{\Lambda_T} \left| a'_1 \Lambda'_1 \right> \left( \frac{\Gamma_1}{a'_1 \Lambda'_1} \frac{\Gamma_2}{a_2 \Lambda_2} \rho^\Gamma \right)
$$

$$+ \sum_{a'_2 \Lambda'_2} \left[ \frac{\Lambda_1}{\Lambda_T} \frac{\Lambda'_2}{\Lambda} \frac{\Lambda}{\Lambda_2} \right] \left< a'_2 \Lambda'_2 \right| T^{\Lambda_T} \left| a_2 \Lambda_2 \right> \left( \frac{\Gamma_2}{a'_2 \Lambda'_2} \frac{\Gamma_1}{a_1 \Lambda_1} \rho^\Gamma \right),
$$

(20)

expressed entirely in terms of $G \supset H$ reduced coupling coefficients, $H$-reduced matrix elements of the generator $T^{\Lambda_T}$, and recoupling coefficients of the lower algebra $H$.

The question remains as to how to apply these relations in order to systematically obtain the values of coupling coefficients. Consider the coupling coefficients for $\Gamma_1 \otimes \Gamma_2 \rightarrow \rho \Gamma$. Then (20) yields a different relation among specific reduced coupling coefficients for each quadruplet of values $(a_1 \Lambda_1 a_2 \Lambda_2 a \Lambda' \Lambda')$ for the irrep labels, such that the conditions

$$
\Lambda \otimes \Lambda_T \rightarrow \Lambda' \quad \Lambda_1 \otimes \Lambda_2 \rightarrow \Lambda'
$$

$$\Gamma_1 \rightarrow \Lambda_1 \quad \Gamma_2 \rightarrow \Lambda_2 \quad \Gamma \rightarrow \Lambda \quad \Gamma \rightarrow \Lambda'
$$

(21)

are satisfied.

The most familiar and traditional approach to extracting coupling coefficients, after obtaining some set of relations among them, is to proceed by recurrence, both in the familiar case of SO(3) Clebsch-Gordan coefficients [24] as well as in the various approaches noted above for higher algebras. A seed value is given for one coupling coefficient, and further coefficients are deduced inductively from those already obtained.

Since calculation of SO(3) Clebsch-Gordan coefficients provides the archetypical example, let us recall how recurrence relations are applied in that problem [24]. Considering the matrix elements of the generators $J_\pm$ between coupled an uncoupled states, as in (16), i.e.,

$$
\left< J_1 M_1 J_2 M_2 \mid J_\pm \mid J M \right> = \left< J_1 M_1 J_2 M_2 \mid J_\pm^{(1)} \mid J M \right> + \left< J_1 M_1 J_2 M_2 \mid J_\pm^{(2)} \mid J M \right>,
$$

(22)

yields a relation among Clebsch-Gordan coefficients only if $M = M_1 + M_2 \mp 1$. Then,

$$
K_\pm(J M) \left( \frac{J_1}{M_1} \frac{J_2}{M_2} \mid J \mid M \pm 1 \right) = K_\pm(J_1 M_1 \mp 1) \left( \frac{J_1}{M_1 \mp 1} \frac{J_2}{M_2} \mid J \mid M \right) + K_\pm(J_1 M_2 \mp 1) \left( \frac{J_1}{M_1} \frac{J_2}{M_2 \mp 1} \mid J \mid M \right),
$$

(23)

where $K_\pm(J M) \equiv \left< JM \pm 1 \mid J_\pm \mid J M \right> = [(J \mp M)(J \pm M + 1)]^{1/2}$ is the generator matrix element. If the Clebsch-Gordan coefficients for $J_1 \otimes J_2 \rightarrow J$ are represented as a lattice of points $(M_1, M_2)$, as in figure 3(a), these relations connect neighboring points at the vertices of an upward-oriented triangle (for $J_+$) or downward-oriented triangle (for $J_-$). A natural order for traversing the coefficients can therefore easily be chosen, such that only one unknown arises at each step. It is necessary to involve certain known-zero or “forbidden” Clebsch-Gordan coefficients, represented by the triangle vertices without dots in figure 3(a), in the relations. However, from figure 3(b), it is apparent that all nonvanishing coefficients may also be directly connected by the relations (23), without involving any forbidden coefficients. This yields a system of equations which fully determines the coefficients, to within an overall phase and normalization, without the need for choosing a specific seed coefficient and recurrence order.

Returning to the general problem for $G \supset H$ reduced coupling coefficients, if there are $N$ coupling coefficients for the coupling $\Gamma_1 \otimes \Gamma_2 \rightarrow \rho \Gamma$, then the relations obtained from (20)
Figure 3. The classic problem of constructing the SO(3) Clebsch-Gordan coefficients \((J_1 M_1 J_2 M_2 | J, M_1 + M_2)\) by use of the relations (23). Dots indicate allowed non-zero coefficients. Coefficients at the vertices of a dashed triangle are connected by (23). (a) The conventional recurrence approach, in which coefficients are calculated inductively from a seed coefficient, making use of relations which in some cases invoke known-zero Clebsch-Gordan coefficients. (b) A full set of relations among allowed Clebsch-Gordan coefficients, yielding a linear, homogeneous system of equations in the Clebsch-Gordan coefficients.

constitute a system of equations in \(N\) unknowns for these coupling coefficients. In general, many irreps of \(H\) will be connected by the generator \(T^{(\Lambda \Lambda')}\), and therefore each relation obtained from (20) will involve many unknown coupling coefficients. A simple recurrence pattern, as in figure 3(a), may be impractical to devise.

Instead, it is possible to solve the full system of equations derived from (20), as in figure 3(b), simultaneously. This is a linear, homogeneous system of equations in the unknown coupling coefficients. Let us label the \(N\) unknown coupling coefficients for \(\Gamma_1 \otimes \Gamma_2 \rightarrow \rho\) with a single counting index, as \(C_i \equiv (a_1 \Lambda_1 a_2 \Lambda_2 | \rho\Gamma\), where \(i = 1, \ldots, N\). Then, for each generator \(T^{(\Lambda \Lambda')}\) in \(G\) but not in \(H\), and for each quadruplet of values \((a_1 \Lambda_1 a_2 \Lambda_2 a' \Lambda a'')\) allowed by (21), the relation (20) yields an equation, which we label by an index \(k\), of the form \(\sum_{i=1}^{N} a_{ki} C_i = 0\). In matrix form, the system of equations for the unknown \(C_i\) is then

\[
\begin{pmatrix}
C_1 \\
C_2 \\
\vdots \\
C_N
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\ddots \\
0
\end{pmatrix}
\]

The problem of solving this linear, homogeneous system of equations (24) is equivalent to finding the null vector (or vectors)

\[
\begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1N} \\
C_{21} & C_{22} & \cdots & C_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
C_{D1} & C_{D2} & \cdots & C_{DN}
\end{pmatrix}
\begin{pmatrix}
\rho = 1 \\
\rho = D
\end{pmatrix}
\]
of the matrix on the left hand side. These may be obtained most simply by Euler row reduction.

The entries of the null vectors \([C_{\rho 1} C_{\rho 2} \cdots C_{\rho N}]\) are the reduced coupling coefficients, to within (ortho)normalization and a possible overall phase, to be determined by convention. The appropriate orthonormality condition for reduced coupling coefficients is

\[
\sum_{a_1 A_1 a_2 A_2} \left( \Gamma_1 \big| a_1 A_1 \big| \Gamma_2 \big| a_2 A_2 \big| \rho' \big| a' A' \big) = \delta(\rho(\rho') \delta_{aa'}.
\]

In the case where \(\Gamma_1 \otimes \Gamma_2 \rightarrow \Gamma\) is free of outer multiplicity, the matrix in (24) is of rank \(N - 1\), yielding a unique null vector. More generally, if the coupling \(\Gamma_1 \otimes \Gamma_2 \rightarrow \Gamma\) has outer multiplicity \(D\) (\(\rho = 1, \ldots, D\)), the matrix is of rank \(N - D\), yielding \(D\) linearly independent null vectors. The coupling coefficients in the presence of multiplicity are only defined to within a unitary transformation, which in the present approach is manifested as the freedom in choosing the orthonormal basis vectors spanning the \(D\)-dimensional null space.

4. Calculation of \(SO(5) \supset SO(4)\) coupling coefficients

Let us now consider the application of Racah’s method in “reduced” form, as described in section 3, to the canonical chain of \(SO(5)\), i.e., for the case in which \(G \supset H\) is \(SO(5) \supset SO(4)\). The necessary ingredients for constructing relations of the type (20) are:

1. matrix elements for generators of \(G\), reduced with respect to \(H\), and
2. the recoupling coefficients of \(H\).

Both components are readily available, as simple closed-form expressions, for \(SO(5) \supset SO(4)\). The operators \(T^J(3)\) of (6) are generators of \(SO(5)\) but not \(SO(4)\). They constitute a double spinor with respect to \(SO(4)\), i.e., for \(SO(4)\) the outer product conditions \((XY) \otimes (\pm \frac{1}{2}) \rightarrow (X'Y')\) and \((X_1 Y_1) \otimes (X_2 Y_2) \rightarrow (X'Y')\) are satisfied (the relevant branching and Clebsch-Gordan series relations are given in section 2). Since the matrix elements (27) and recoupling coefficients (28) defining the system of equations are known exactly, as (signed) square roots of rational numbers,
coefficients. The full matrix has dimension 36 (a) Calculation of SO(5)

The orthonormality condition (26) interrelates groups of coupling coefficients sharing the same
of dimension 2. (c) Orthonormal vectors, containing the desired coupling coefficients as entries.

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Figure 4. Calculation of SO(5) ⊃ SO(4) reduced coupling coefficients for (10) ⊗ (1\frac{1}{2}) → (1\frac{1}{2}): (a) Coefficient matrix for the linear, homogenous system of equations determining the coupling coefficients. The full matrix has dimension 36 × 18. (b) Basis vectors for the null space, which is of dimension 2. (c) Orthonormal vectors, containing the desired coupling coefficients as entries. The orthonormality condition (26) interrelates groups of coupling coefficients sharing the same (XY) label, which are delimited by the vertical lines in parts (b) and (c).
the SO(5) ⊃ SO(4) reduced coupling coefficients can be obtained exactly by Euler row reduction, again as (signed) square roots of rational numbers.

A concrete numerical example is provided by the SO(5) coupling (10) ⊗ (1½) → (1½). (In the canonical labeling scheme, this coupling is [11] ⊗ [3½] → [3½].) There are 18 SO(4)-reduced coupling coefficients to be determined, related by a system of 36 equations. The first few rows of the coefficient matrix are shown in figure 4(a). The coupling has outer multiplicity \( D = 2 \). Correspondingly, the coefficient matrix is of rank 16, admitting two null vectors, given in figure 4(b). Orthonormal null vectors, with respect to the condition (26), are obtained by the Gram-Schmidt procedure, with the result given in figure 4(c). The entries of the upper and lower rows may thus be taken as the SO(5) ⊃ SO(4) reduced coupling coefficients for \( \rho = 1 \) and 2, respectively.

5. Conclusions

It has been shown that Racah’s method of infinitesimal generators can be systematically generalized to the calculation of reduced coupling coefficients for an arbitrary subalgebra chain, provided the matrix elements of the generators (reduced with respect to the lower algebra) and the recoupling coefficients of the lower algebra are known. For the algebra SO(5), the problem of calculating coupling coefficients for generic irreps, reduced with respect to the canonical or noncanonical chains, is thereby completely resolved.

The necessary ingredients take on a particularly simple form for the canonical chain of SO(5). However, the generator matrix elements reduced with respect to the noncanonical isospin subalgebra [chain (II)] and physical angular momentum algebra [chain (III)] are also known, from vector coherent state theory [34, 35]. Hence, Racah’s method may be applied directly to the calculation of chain (II) and chain (III) reduced coupling coefficients.

Alternatively, once coupling coefficients for the canonical chain have been obtained, the coupling coefficients for the noncanonical chains can readily be deduced by a unitary transformation. Transformation brackets between basis states reducing the canonical and noncanonical chains are obtained by diagonalizing the appropriate Casimir operator, i.e., \( T^2 \) or \( L^2 \), in the canonical basis (as in [14]). Then the transformed reduced coupling coefficients follow, e.g., for chain (II), as

\[
\begin{align*}
\text{SO}(5) \supset U_N(1) \supset \text{SO}(3) &= \sum_{(X_1 Y_1)(X_2 Y_2)(XY)} \left< T_1 M_{T1} T_2 M_{T2} T M \right| \left< X_1 Y_1 M_{X1} M_{Y1} \right| \left< X_2 Y_2 M_{X2} M_{Y2} \right| \left< XY M_X M_Y \right> \times \left< (R_1 S_1) M_{S1} M_{T1} \right| \left< (R_1 S_1) M_{S1} M_{T1} \right| \left< (R_2 S_2) M_{S2} M_{T2} \right| \left< (R_2 S_2) M_{S2} M_{T2} \right| \left< (RS) M_{S} M_{T} \right| \left< (RS) M_{S} M_{T} \right| \left< (XY M_X M_Y) \right|
\end{align*}
\]

where \( M_X = \frac{1}{2}(M_S + M_T) \), \( M_Y = \frac{1}{2}(M_S - M_T) \), and similarly for \( M_{X1} \), \( M_{Y1} \), \( M_{X2} \), and \( M_{Y2} \).

The specific example of SO(5) coupling coefficients may be considered as a prototype for the systematic calculation of coupling coefficients for other higher algebras. For instance, the

\[
\begin{align*}
\text{SO}(5) \supset U_N(1) \supset \text{SO}(3) &= \sum_{(X_1 Y_1)(X_2 Y_2)(XY)} \left< T_1 M_{T1} T_2 M_{T2} T M \right| \left< X_1 Y_1 M_{X1} M_{Y1} \right| \left< X_2 Y_2 M_{X2} M_{Y2} \right| \left< XY M_X M_Y \right> \times \left< (R_1 S_1) M_{S1} M_{T1} \right| \left< (R_1 S_1) M_{S1} M_{T1} \right| \left< (R_2 S_2) M_{S2} M_{T2} \right| \left< (R_2 S_2) M_{S2} M_{T2} \right| \left< (RS) M_{S} M_{T} \right| \left< (RS) M_{S} M_{T} \right| \left< (XY M_X M_Y) \right|
\end{align*}
\]

where \( M_X = \frac{1}{2}(M_S + M_T) \), \( M_Y = \frac{1}{2}(M_S - M_T) \), and similarly for \( M_{X1} \), \( M_{Y1} \), \( M_{X2} \), and \( M_{Y2} \).
computational machinery for SU(3) is well established [36, 37] and may therefore be used as the starting point for calculation of Sp(6) ⊂ U(3) reduced coupling coefficients, for the fermion dynamical symmetry model [38], or Sp(6, R) ⊂ U(3) reduced coupling coefficients, for the symplectic shell model [39]. The requisite generator matrix elements for Sp(6) and Sp(6, R) may again be calculated from vector coherent state realizations [40, 41]. The Sp(6, R) ⊂ U(3) coupling coefficients are required, for instance, if large-scale calculations are to be carried out in the ab initio symplectic scheme of Dytrych et al. [42, 43].

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