Robust parameter estimation of regression model with weaker moment

Kangqiang Li† Songqiao Tang‡ Lixin Zhang§

School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China

December 9, 2021

Abstract

This paper provides some extended results on estimating the parameter matrix of high-dimensional regression model when the covariate or response possess weaker moment condition. We investigate the $M$-estimator of Fan et al. (Ann Stat 49(3):1239–1266, 2021) for matrix completion model with $(1 + \epsilon)$-th moments. The corresponding phase transition phenomenon is observed. When $\epsilon \geq 1$, the robust estimator possesses the same convergence rate as previous literature. While $1 > \epsilon > 0$, the rate will be slower. For high dimensional multiple index coefficient model, we also apply the element-wise truncation method to construct a robust estimator which handle missing and heavy-tailed data with finite fourth moment.

Keywords: Linear and nonlinear-type models; Heavy-tailed data; Missing observation; Element-wise truncation; Robust estimation.

†This work was supported by grants from the NSF of China (Grant No.11731012), Ten Thousands Talents Plan of Zhejiang Province (Grant No. 2018R52042) and the Fundamental Research Funds for the Central Universities.

Corresponding author E-mail address: 11935023@zju.edu.cn (Kangqiang Li)
E-mail address: 11835013@zju.edu.cn (Songqiao Tang)
E-mail address: stazlx@zju.edu.cn (Lixin Zhang)
1 Introduction

Under the traditional settings, sub-Gaussian assumption is often required for noises and designs, regardless of whether the regression model is linear or nonlinear-type. Due to the heavy-tailed phenomena of real-world data, in recent years, there has been a growing body of literature on the robust regression estimation when the design and response are heavy-tailed. For robust parameter estimation of linear-type model, Fan et al. (2017)\cite{11} applied the robust approximate quadratic loss to the sparse linear model and showed that under \((2 + \epsilon)\)-th moment assumption on the noise, the proposed estimator exhibits the same statistical error rate as that of the light-tail noise case. Further, Sun et al. (2020)\cite{32} proposed the adaptive Huber regression and extended the results of Fan et al. (2017)\cite{11} to the case of \((1 + \epsilon)\)-th moment condition on the noise. A tight phase transition for the estimation error of the robust regression parameter estimator was established which paralleled those first discovered by Bubeck et al. (2013)\cite{3} and Devroye et al. (2016)\cite{8} for robust mean estimation without finite variance. Motivated by Sun et al. (2020)\cite{32}, Tan et al. (2018)\cite{33} established the similar phase transition results on sparse reduced rank regression. Fan et al. (2021)\cite{10} focused on robust estimation for the trace regression and their \(M\)-estimator achieves the minimax statistical error rate under only bounded \((2 + \epsilon)\)-th moment response or both bounded fourth moment response and design. Zhu and Zhou (2020)\cite{39} studied the corrupted general linear model with heavy-tailed data under finite fourth moment assumption.

For robust estimation of sparse nonlinear-type regression problem, Yang et al. (2017)\cite{36} and Yang et al. (2017)\cite{37} considered robust estimators of a high-dimensional single index model (SIM) when the covariate and response only have the bounded fourth moment. The proposed estimator achieves the optimal error bound via the truncation method for the design and response. Furthermore, Goldstein et al. (2018)\cite{13} analyzed high-dimensional SIM with elliptical distribution. Fan et al. (2020)\cite{9} studied implicit regularization in SIM with heavy-tailed data. As a generalization of SIM and trace regression model, Na et al. (2019)\cite{26} considered high dimensional varying coefficient index model introduced by Ma and Song (2015)\cite{22} and applied to economics and medical science by Fan and Zhang (2008)\cite{12}. For estimating sparse parameter matrix, they required the existence of bounded 6-th moment of the covariate and response in order to obtain the optimal rate,
but whether the moment constraint can be further relaxed is unknown. Meanwhile, it is worth noting that Fan et al. (2021)[10] tested the superiority of their estimator for trace regression via selecting the scaled Cauchy noise, beyond the corresponding theoretical condition. Motivated by those, a natural question arises:

*Can we further generalize their results and obtain the optimal estimation rate?*

To address this problem, on the basis of Fan et al. (2021)[10]’s work, we further study matrix completion model in which the noise distribution has no finite variance. The applicable condition of their $M$-estimator is promoted. Simultaneously, the sharp phase transition of the convergence rate is observed. As a generalization of matrix completion model, we consider robust parameter estimation of high-dimensional varying index coefficient model with missing observations. To handle missing and heavy-tailed data with only finite fourth moment together, we propose a robust element-wise truncated estimator (see (6)) with the help of Theorem 1 and derive its convergence rate. It turns out that under finite fourth moment assumption, our method shows the robustness against the low order moments such that our estimator can achieve the same statistical error rate as that of Na et al. (2019)[26] with complete observations.

For complete observations, there is a body of literature on robust covariance estimation with heavy-tailed data. Catoni (2012)[6] proposed a robust mean estimator for heavy-tailed sample via an innovative way. Minsker (2018)[23] generalized Catoni’s idea to the multivariate self-adjoint random matrix case. Furthermore, Minsker and Wei (2017)[24], Ke et al.(2019)[15], Minsker and Wei (2020)[25] and Fan et al. (2021)[10] respectively constructed different robust covariance estimators when the samples have bounded fourth moment. Avella-Medina et al. (2018)[1] applied Huber (1964)[14]’s loss to construct robust covariance and precision matrix estimators without finite kurtosis of the samples. Afterwards, Zhang (2021)[38] investigated Huber estimators for high-dimensional time series.

Another line of research considered covariance estimation with missing observations where the missingness is independent with the data. In order to construct unbiased estimators of covariance matrices, most of the recent literature assumed sub-Gaussian distribution for samples. For example, Lounici (2014)[21], Loh and Wainwright (2012)[20], Kolar and Xing (2012)[16] and Cai and Zhang (2016)[5] all put forward procedures for estimating high dimensional near low-rank covariance and sparse precision matrices under sub-Gaussian assumption. Park, S. and Lim, J. (2019)[29] and Park et al. (2021)[30] generalized the
result of Lounici (2014) [21] via inverse probability weighting. In particular, Pavez and Ortega (2019) [28] further considered that the missing observation distribution depends on original data. Although there are some work on non-Gaussian distribution such as Wang et al. (2014) [35], Cui et al. (2017) [7] for non-paranormal distribution and Little (1988) [18], Liu and Palomar (2019) [19] for multivariate t distribution, few research on generic heavy-tailed data with missing observations has been reported. To this end, in this paper, we construct a plug-in estimator for generic heavy-tailed and missing data under the missing completely at random mechanism, and establish its nonasymptotic deviation bound in terms of the max norm (Theorem 1). When samples have finite kurtosis, maintain the same convergence rate with that of sub-Gaussian tail in Park et al. (2021) [30].

The remainder of our paper is organized as follows. In Section 2, we propose a plug-in covariance matrix estimator for missing and heavy-tailed data to lay the groundwork for Section 3. In Section 3, we analyze two specific linear and nonlinear-type models, and derive the statistical error rates of the corresponding M-estimators under weaker moment assumptions. All the proofs are presented in the Appendix A.

Notations

For any positive integer \( n \), we denote the set \( \{1, 2, \ldots, n\} \) by \([n]\). For two vectors \( X, Y \in \mathbb{R}^n \), we write \( \langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i \) to be the inner product and if \( X, Y \in \mathbb{R}^{d_1 \times d_2} \), \( \langle X, Y \rangle := \text{tr}(X^T Y) \) and \( X \odot Y \) denotes element-wise (Hadamard) division. Given a subset \( S \) of index set \([d]\), the projection \( V_S \) is defined by \( (V_S)_i = 0 \), if \( i \notin S \) and \( (V_S)_i = (V)_i \), if \( i \in S \). For a matrix \( A = (a_{ij}) \in \mathbb{R}^{d_1 \times d_2} \), the max norm and operator norm of \( A \) are defined as \( \|A\|_{\text{max}} = \max_{i \in [d_1], j \in [d_2]} |a_{i,j}| \) and \( \|A\|_F = \sqrt{\sum_{i \in [d_1], j \in [d_2]} a_{i,j}^2} \) respectively. \( \|A\|_* = \text{tr} \left( \sqrt{AA^T} \right) \), \( \|A\|_{1,1} = \sum_{i \in [d_1]} \sum_{j \in [d_2]} |a_{i,j}| \), \( \|A\|_{\infty} = \max_{i \in [d_1]} \sum_{j \in [d_2]} |a_{i,j}| \) and \( \|A\|_{L_1} = \max_{j \in [d_2]} \sum_{i \in [d_1]} |a_{i,j}| \). Given two sequences \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \), we use the notation \( a_n \asymp b_n \) or \( a_n = O(b_n) \), if there exist two positive constants \( C_1 \) and \( C_2 \) such that \( C_1 b_n \leq a_n \leq C_2 b_n \) for all \( n \).
2 Plug-in estimator for missing and heavy-tailed data

Suppose that \( X = (x_1, \ldots, x_d)^T \) is a random vector from a \( d \)-dimensional distribution with zero mean and \( \mathbb{E}[XX^T] = \Sigma \). We say \( Y = (y_1, \ldots, y_d)^T \) is missing completely at random which comes from \( X \), if

\[
y_i = \xi_i x_i \quad \text{for } \forall i \in [d]
\]

where \( \xi_i \) is independent with \( X_i \) and follows Bernoulli distribution with success probability \( \rho_i \in (0, 1] \) and \( \rho_{i,j} := \mathbb{P}(\xi_i = \xi_j = 1) \). Note that \( \Sigma \) can be written as

\[
\Sigma = \Sigma^\rho \otimes \Xi = \text{diag}(\Sigma^\rho)\text{diag}(\Xi)^{-1} + (\Sigma^\rho - \text{diag}(\Sigma^\rho)) \otimes \Xi
\]

where \( \Sigma^\rho = \mathbb{E}[Y_1 Y_1^T] \) and \( (\Xi)_{j,k} = \rho_{j,k} \). Therefore, when \( \{\xi_i\}_{i=1}^d \) are i.i.d. with the known parameter \( \rho \), and \( X \) is sub-Gaussian, Lounici (2014) \[21\] proposed the following unbiased estimator of \( \Sigma \) based on the sample covariance matrix of the missing data \( \{Y_i\}_{i=1}^n \):

\[
\widehat{\Sigma}_n = \rho^{-1}\text{diag}(\widehat{\Sigma}_n^\rho) + \rho^{-2}(\widehat{\Sigma}_n^\rho - \text{diag}(\widehat{\Sigma}_n^\rho))
\]

where \( \widehat{\Sigma}_n^\rho = \frac{1}{n} \sum_{i=1}^n Y_i Y_i^T \). Nevertheless, the above three strong assumptions: the known parameter \( \rho \), i.i.d. \( \{\xi_i\}_{i=1}^d \) and sub-Gaussian sample are very restrictive in practice. Therefore, we construct the following plug-in estimator of \( \Sigma \) when the data have heavy-tails and missing values with unknown parameters \( \{\rho_i\}_{i=1}^d \).

\[
\widehat{\Sigma}_{n,\text{plug}} = \text{diag}(\widehat{\Sigma}_n^\rho)\text{diag}(\widehat{\Xi})^{-1} + (\widehat{\Sigma}_n^\rho - \text{diag}(\widehat{\Sigma}_n^\rho)) \otimes \widehat{\Xi}
\]

where \( (\widehat{\Sigma}_n^\rho)_{j,k} = \frac{1}{n} \sum_{i=1}^n \psi_{\tau_{j,k}}(y_j^{(i)} y_k^{(i)}) \) and\( (\widehat{\Xi})_{j,k} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{y_j^{(i)} y_k^{(i)} \neq 0\}} \). The truncation parameters \( \{\tau_{j,k}\}_{j,k \in [d]} \) strike a balance between the induced bias and tail robustness.

The following lemma gives the convergence rate of the proposed robust estimator \( \widehat{\Sigma}_{n,\text{plug}} \) under max norm.

**Theorem 1.** Let \( Y_1, \ldots, Y_n \in \mathbb{R}^d \) be \( n \) i.i.d. random vectors drawn from \([1]\) with unknown parameters \( \{\rho_j\}_{j \in [d]} \) and \( M_\alpha := \max_{k,s \in [d]} (\mathbb{E} |X_{i(k)} X_{i(s)}|^{\alpha})^{\frac{1}{\alpha}} < \infty \) for some \( \alpha > 1 \). Then there exists a universal constant \( C > 0 \) such that as long as

\[
2 \log d + \log \delta^{-1} \leq \min \left\{ \left( \frac{3}{2} - \sqrt{2} \right) \min_{j \neq k} \rho_{j,k}, \frac{\rho_{\min}^2}{4} \right\} n,
\]

\[1\]In this paper, we only consider continuous distributions with \( \mathbb{P}(X = 0_{d \times 1}) = 0 \).
we have with probability at least $1 - \left( \frac{3}{2} + \frac{5}{2d} \right) \delta$,

$$
\| \tilde{\Sigma}_n^{\text{plug}} - \Sigma \|_{\text{max}} \leq CM_{\alpha} \left( \max_{j \neq k} \frac{\rho_{j,k}^{\text{max}}\left( \frac{1}{\alpha + \frac{1}{2}} \right)}{\min_{j \neq k} \rho_{j,k}} + \rho_{\text{max}}^{\frac{1}{\alpha + \frac{1}{2}}} / \rho_{\text{min}} \right) \left( \frac{2 \log d + \log \delta - 1}{n} \right)^{\min\left\{ \frac{\alpha}{2}, \frac{1}{2} \right\}} + C \left( \frac{\max_{j \in [d]} (\rho_{j,j} \sigma_{j,j})}{\rho_{\text{min}}^2} + \frac{\max_{j \neq k} (\rho_{j,k} \sigma_{j,k})}{\min_{j \neq k} (\rho_{j,k})^{\frac{1}{2}}} \right) \sqrt{\frac{2 \log d + \log \delta - 1}{n}}
$$

where $\tau_{k,s} = M_{\alpha} (\rho_{j,k} / (2 \log d - \log \delta))^{\max\left( \frac{1}{\alpha}, \frac{1}{2} \right)}$, $\rho_{\text{max}} := \max_{j \in [d]} \rho_j$ and $\rho_{\text{min}} := \min_{j \in [d]} \rho_j$.

Remark 1. If $\{ \xi_i \}_{i \in [d]}$ are i.i.d. and $\rho_1 := \rho$, Theorem 1 reveals that $\| \Sigma_n^{\text{plug}} - \Sigma \|_{\text{max}} \asymp M_{\alpha} \left( \frac{\log d}{\rho_{\text{min}}^2} \right)^{\min\left\{ \frac{\alpha}{2}, \frac{1}{2} \right\}}$ with high probability, if the scaling condition (2) is satisfied. When $\alpha \geq 2$, our proposed estimator achieves the same statistical rate of convergence with those of Park et al. (2021) [30] for sub-Gaussian distribution and Wang et al. (2014) [35] for non-paranormal distribution.

The following corollary is of avail for subsection 3.2 and can be easily obtained by Theorem 1. The corresponding proof is omitted.

When $\Omega^* := \Sigma^{-1}$ belongs to the following family of precision matrices $U(\varpi, q, s_0(d)) := \left\{ \Omega : \Omega \succ 0, \| \Omega \|_{L_1} \leq \varpi, \max_{1 \leq i \leq d} \sum_{j=1}^{d} |(\Omega)_{i,j}|^q \leq s_0(d) \right\}$ for $q \in [0, 1)$, we plug $\tilde{\Sigma}_n^{\text{plug}}$ into the following Cai et al. (2011) [4]'s CLIME procedure:

$$
\widehat{\Omega} = \arg\min \| \Omega \|_{L_1, r} \text{ s.t. } \| \tilde{\Sigma}_n^{\text{plug}} \Omega - I_d \|_{\text{max}} \leq \gamma.
$$

Corollary 1. Under the condition of Theorem 1, choosing the RHS of (3) as $\gamma$, we have $\| \widehat{\Omega} - \Omega^* \|_{\text{max}} \leq 4 \varpi \gamma$.

3 Parameter matrix estimation of linear and nonlinear-type models

In this section, we analyze the following two type of regression models and present the optimal statistical rates of the corresponding regularized estimators.

3.1 Matrix completion model with weaker moment

We first consider the following matrix completion model:

$$
y = \langle X, \Theta^* \rangle + \varepsilon
$$
where $X$ is uniformly sampled from $\{\sqrt{d_1 d_2} \cdot e_j e_k^T \}_{j \in [d_1], k \in [d_2]}$ and $\mathbb{E}(\varepsilon | X) = 0$. To recover the coefficient matrix $\Theta^*$ under near low-rank assumption, Fan et al. (2021) studied the following $M$-estimator of $\Theta^*$:

$$\hat{\Theta} \in \arg\min_{\|\Theta\|_{\text{max}} \leq R/\sqrt{d_1 d_2}} \left\{ \text{vec}(\Theta)^T \hat{\Sigma}_{XX} \text{vec}(\Theta) - 2 \left\langle \hat{\Sigma}_{yX}, \Theta \right\rangle + \lambda \|\Theta\|_* \right\}$$

where $\hat{\Sigma}_{XX} = \frac{1}{n} \sum_{i=1}^n \text{vec}(X_i) \text{vec}(X_i)^T$ and $\hat{\Sigma}_{yX} = \frac{1}{n} \sum_{i=1}^n \text{sign}(y_i) (|y_i| \wedge \tau) X_i$ with a truncation parameter $\tau$. Under finite $(2 + \epsilon)$-th moment condition on the response, their robust estimator has the same statistical error rate as that of Negahban and Wainwright (2012) for sub-exponential noise. In order to fill the gap for the robust estimator’s scope of use, the following theorem further relaxes the distributional conditions from the bounded $(2 + \epsilon)$-th moment to the $(1 + \epsilon)$-th moment assumption.

**Theorem 2.** Suppose the following conditions hold:

1. $\|\Theta\|_F \leq 1$, $\|\Theta\|_{\text{max}} \leq R/\sqrt{d_1 d_2}$, $\|\Theta^*\|_{\text{max}} / \|\Theta^*\|_F \leq R/\sqrt{d_1 d_2}$ and $\text{rank}(\Theta^*) \leq r$;
2. $\{X_i\}_{i=1}^n$ are i.i.d. uniformly sampled from $\{\sqrt{d_1 d_2} \cdot e_j e_k^T \}_{j \in [d_1], k \in [d_2]}$ and $\mathbb{E}(\epsilon_i^a | X_i) \leq M < \infty$ a.s. for some $a \in (1, 2]$.

Then for any $\delta > 1$, choose $\tau = \left(\frac{\ln}{(d_1 \lor d_2) \log(d_1 + d_2)}\right)^{\frac{1}{\delta}}$ and for some constant $C > 0$,

$$\lambda = 4C \left(\frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n}\right)^{\frac{\alpha - 1}{\alpha}} \left( L^\frac{\alpha}{\alpha \delta} \delta^{\frac{\alpha - 1}{\alpha}} + R \sqrt{\delta} + L^\frac{1}{\alpha}\right).$$

There exist constants $\{C_i\}_{i=1}^4$ such that as long as $n \geq (d_1 \lor d_2) \log(d_1 + d_2)$, we have with the probability at least $1 - (d_1 + d_2)^{1-\delta} - (d_1 + d_2)^{1-\frac{\alpha - 2}{\alpha}} - C_1 \exp(-C_2(d_1 + d_2))$,

$$\left\| \hat{\Theta} - \Theta^* \right\|_F \leq C_3 \max \left\{ \sqrt{r} \left(\frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n}\right)^{\frac{\alpha - 1}{\alpha}} \left( L^\frac{\alpha}{\alpha \delta} \delta^{\frac{\alpha - 1}{\alpha}} + R \sqrt{\delta} + L^\frac{1}{\alpha}\right), \frac{R}{\sqrt{n}} \right\},$$

$$\left\| \hat{\Theta} - \Theta^* \right\|_* \leq C_4 \max \left\{ r \left(\frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n}\right)^{\frac{\alpha - 1}{\alpha}} \left( L^\frac{\alpha}{\alpha \delta} \delta^{\frac{\alpha - 1}{\alpha}} + R \sqrt{\delta} + L^\frac{1}{\alpha}\right), R \sqrt{\frac{r}{n}} \right\},$$

where $L = 2^{\alpha - 1}(R^\alpha + M)$.

**Remark 2.** According to Theorem 2, we obtain that

$$\left\| \hat{\Theta} - \Theta^* \right\|_F \asymp \sqrt{r} L^\frac{1}{\alpha} \left(\frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n}\right)^{\min\left(\frac{\alpha - 1}{\alpha}, \frac{1}{2}\right)},$$

with high probability.

Compared with the result of Fan et al. (2021), when $\alpha < 2$, there exists a tight phase transition phenomenon for the statistical error rate and the truncation parameter $\tau$ should
adapt to the moment of the noise which is in line with Theorem 1 as well as those of linear regression in Sun et al. (2020)\[32\] and mean estimation in Bubeck et al. (2013)\[3\]. However, this transition is observed in the low-rank matrix completion model via the shrinkage technique, which is a visible difference with previous literature.

Although Theorem 2 still doesn’t account for the case of the scaled Cauchy noise in the simulation of Fan et al. (2021)\[10\], it widens the applying condition for all noise distributions with finite mean.

3.2 High-dimensional varying index coefficient model with missing data

As a generalization of model (5), in this subsection, we concentrate on robustly estimating the direction of parameters estimation of the following varying index coefficient model with missing observations:

\[ y = \sum_{i=1}^{d_2} z_i \cdot f_i ((X, \theta_i^*)) + \varepsilon \]

where \( X \in \mathbb{R}^{d_1} \) and \( Z = (z_1, z_2, \ldots, z_{d_2})^T \in \mathbb{R}^{d_2} \) are independent covariates, and \( \varepsilon \) is the stochastic error with \( \mathbb{E}[\varepsilon \mid X, Z] = 0 \). We assume that \( \|\theta_i^*\|_2 = 1 \) for model identifiability and \( X \) has the known probability density function \( p(X) \).

Based on the following two assumptions and first-order Stein’s identity (Stein et al. (2004)\[31\]), a feasible method to robustly estimate the direction of \( \{\theta_i^*\}_{i=1}^{d_1} \) without the knowledge of the link functions \( \{f_i(\cdot)\}_{i \in [d_2]} \) is pointed out.

**Assumption 1.** Assume that the covariate \( X \) has the differentiable density function \( p(X) : \mathbb{R}^{d_1} \to \mathbb{R} \) and the link functions \( \{f_i(\cdot) \mid i \in [d_2]\} \) are differentiable such that \( \mu_i^* := \mathbb{E}[f_i'((X, \theta_i^*))] \neq 0 \) for \( \forall i \in [d_2] \), and \( \mathbb{E}[Z] = 0_{d_2 \times 1} \). Denote \( \Sigma^* := \mathbb{E}[ZZ^T] \) and \( \Omega^* := (\Sigma^*)^{-1} \). For some \( \varpi \) and \( q \in [0, 1) \), we assume \( \Omega^* \in \mathcal{U}(\varpi, q, s_0(d_2)) \).

**Assumption 2.** there exists an absolute constant \( M > 0 \) such that

\[ \mathbb{E}[y^4] \vee \mathbb{E}[(S(X)]^4 \vee \mathbb{E}[z_k^4] \leq M, \quad \forall j \in [d_1], k \in [d_2] \]

where the first-order score function \( S : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1} \) is defined as \( S(X) := -\nabla p(X)/p(X) \).

Given \( n \) i.i.d. samples \( \{y_i, X_i, Z_i\}_{i=1}^{n} \) satisfying Assumption 1 and 2, we now consider that the entries of the coefficient vector \( Z_i \) are missing at random. Denote \( \tilde{Z}_i := \ldots \).
\((z_1^{(i)}, z_2^{(i)}, \ldots, z_d_z^{(i)})^T = (\xi_1^{(i)} z_1^{(i)}, \xi_2^{(i)} z_2^{(i)}, \ldots, \xi_d_z^{(i)} z_d_z^{(i)})^T\) as a missing observation defined in (1) from \(Z_i\). Therefore, by first-order Stein’s identity,

\[\mathbb{E}[y \cdot S(X) \bar{Z}^T] \Omega^* = \sum_{j=1}^{d_z} \mathbb{E} [f_j \langle \theta_j^*, X \rangle] S(X) \mathbb{E}[z_j \cdot \bar{Z}^T] \Omega^* = \sum_{j=1}^{d_z} \rho_j \mu_j^* \theta_j^* e_j^T \Sigma^* \Omega^* := (\tilde{\theta}_1, \ldots, \tilde{\theta}_{d_z}) = \tilde{\Theta}.\]

For further relaxing the moment condition of the covariates and response, instead of separately truncating the data \(\{y_i, S(X_i), Z_i\}_{i=1}^n\) via the truncation function \(\hat{x} = x 1_{\{|x| \leq \rho\}}\) proposed by Na et al. (2019) [20], we consider \(y_i S(X_i) Z_i^T\) as a matrix-valued data and then use \(\hat{x} := \psi_\tau(x) = (|x| \wedge \tau) \text{sign}(x)\) to truncate the \((j, k)\)-th entry of the matrix-variate data. Specifically, the robust element-wise truncated matrix estimator is defined as

\[
\hat{\Theta} := \arg\min_{\Theta \in \mathbb{R}^{d_1 \times d_2}} \left\{ \|\Theta\|_F^2 - 2 \left( \frac{1}{n} \sum_{i=1}^n \psi_{T_1} \left( y_i S(X_i) Z_i^T \hat{\Theta}, \Theta \right) \right) + \lambda \|\Theta\|_{1,1} \right\}
\]

where \(\Gamma_1 = \left( \tau_{j,k}^{(1)} \right)_{j \in [d_1], k \in [d_2]}\) is a truncation parameter matrix and \(\hat{\Omega}\) is defined by (4). The following theorem gives the statistical error rate of the robust estimator above.

**Theorem 3.** Consider that \(\{\xi_i\}_{i=1}^{d_z}\) are dependent with unknown parameters \(\{\rho_i\}_{i=1}^{d_z}\). Suppose Assumption 4 and 2 hold with \(\|\theta_j^*\|_0 = s\) for all \(j \in [d_z]\). For \(k, s \in [d_2]\), choose \(\tau_{j,k}^{(1)} \asymp (\rho_k M^3)^{\frac{1}{2}} \sqrt{n \log(d_1 d_2)}, \tau_{k,s}^{(2)} \asymp M^{\frac{1}{2}} \sqrt{\frac{n \rho_k s}{\log d_2}}, \gamma \asymp M^{\frac{1}{2}} \sqrt{\frac{\log d_2}{n}}\) and

\[\lambda \asymp (\rho_{\max} M^3)^{\frac{1}{2}} \|\Omega^*\|_{1,1} \sqrt{\frac{3 \log(d_1 d_2)}{n}} + \max_{j \in [d_2]} \|\Theta^* \Sigma^*\|_{\infty} M^{\frac{1}{2}} \sqrt{\frac{3 \log d_2}{n}}.\]

Then as long as \(\min \left\{ \left( \frac{3}{2} - \sqrt{2} \right) \min_{j \neq k} (\rho_{j,k}), \frac{\rho_{\min}}{4} \right\} n \geq 3 \log d_2\), we have with the probability at least \(1 - \frac{2}{(d_1 d_2)^2} - \frac{3}{2d_2} - \frac{5}{2d_2}\),

\[\|\hat{\Theta} - \Theta\|_F \leq 2\lambda \sqrt{s d_2} \text{ and } \|\hat{\Theta} - \Theta\|_{1,1} \leq 8\lambda s d_2\]

where \(\kappa = C \left( \frac{\rho_{\max}^2}{\rho_{\min}^2} \frac{\max_{j \neq k} \rho_{j,k}}{\min_{j \neq k} \rho_{j,k}} + \sqrt{\frac{\rho_{\max}}{\rho_{\min}}} + \frac{\max_{j \neq k} \rho_{j,k}}{\min_{j \neq k} \rho_{j,k}} \right).

**Remark 3.** When \(\rho_{\min} = \rho_{\max} = 1\), we have with high probability,

\[\|\hat{\Theta} - \Theta\|_F \asymp \sqrt{s d_2 \left( \frac{\log(d_1 d_2)}{n} \right)^{\frac{1}{2}}} \text{ and } \|\hat{\Theta} - \Theta\|_{1,1} \asymp s d_2 \left( \frac{\log(d_1 d_2)}{n} \right)^{\frac{1}{4}}\]

which shows that the proposed estimator possesses the same statistical error rate as that of Na et al. (2019) [20] with bounded 6-th moment assumption.
A Appendix

A.1 Proof of Theorem 1

Proof. Denote $\hat{\Sigma} = \text{diag}(\Sigma^\rho) \text{diag}(\hat{\Xi})^{-1} + (\Sigma^\rho - \text{diag}(\Sigma^\rho)) \otimes \hat{\Xi}$. Then, we have

$$
\|\hat{\Sigma}^\text{plug} - \Sigma\|_{\text{max}} \leq \|\hat{\Sigma}^\text{plug} - \hat{\Sigma}\|_{\text{max}} + \|\hat{\Sigma} - \Sigma\|_{\text{max}} \\
\leq \|\left(\hat{\Sigma}^\rho_n - \Sigma^\rho + \text{diag}(\Sigma^\rho - \hat{\Sigma}^\rho_n)\right) \otimes \hat{\Xi}\|_{\text{max}} + \|\text{diag}(\hat{\Sigma}^\rho_n - \Sigma^\rho)\text{diag}(\hat{\Xi})^{-1}\|_{\text{max}} \\
+ \|\text{diag}(\Sigma^\rho)\text{diag}(\hat{\Xi})^{-1} - \text{diag}(\Xi)^{-1}\|_{\text{max}} + \|\left(\Sigma^\rho - \text{diag}(\Sigma^\rho)\right) \otimes \hat{\Xi} - (\Sigma^\rho - \text{diag}(\Sigma^\rho)) \otimes \Xi\|_{\text{max}} \\
\leq \max_{j \neq k} \|\hat{\Xi}^{-1}_{j,k}\| \|\hat{\Sigma}^\rho_n - \Sigma^\rho + \text{diag}(\Sigma^\rho - \hat{\Sigma}^\rho_n)\|_{\text{max}} + \max_{j \in [d]} \|\hat{\Xi}_{j,j}^{-1}\| \|\text{diag}(\hat{\Sigma}^\rho_n - \Sigma^\rho)\|_{\text{max}} \\
+ \max_{j \in [d]} (\rho_j \sigma_{j,j}) \|\text{diag}(\hat{\Xi})^{-1} - \text{diag}(\Xi)^{-1}\|_{\text{max}} + \max_{j \neq k} (\rho_j \sigma_{j,j}) \cdot \max_{j \neq k} \|\hat{\Xi}_{j,j}^{-1} - (\Xi)_{j,j}^{-1}\|.
$$

By Theorem 1 of Li et al. (2021)[17], we get that

$$
P_1 \left( \frac{\|\hat{\Sigma}^\rho_n - \Sigma^\rho + \text{diag}(\Sigma^\rho - \hat{\Sigma}^\rho_n)\|_{\text{max}}}{\max \left\{ \min \left\{ \frac{\alpha \cdot \rho_j}{\alpha - \frac{1}{2}} \right\} \right\}} \leq 2M_1 \left( \frac{2 \log d + \log \delta^{-1}}{n} \right) \geq 1 - \left(1 - d^{-1}\right)\delta,
$$

$$
P_2 \left( \frac{\|\text{diag}(\hat{\Sigma}^\rho_n) - \text{diag}(\Sigma^\rho)\|_{\text{max}}}{\max \left\{ \min \left\{ \frac{\alpha \cdot \rho_j}{\alpha - \frac{1}{2}} \right\} \right\}} \leq 2M_2 \left( \frac{2 \log d + \log \delta^{-1}}{n} \right) \geq 1 - \frac{2\delta}{d}.
$$

where $M_1 = M_{\alpha} \max_{j \neq k} \rho_j \sigma_{j,j} \left( \frac{1}{\alpha - \frac{1}{2}} \right)$ and $M_2 = M_{\alpha} \max_{j \in [d]} \rho_j \left( \frac{1}{\alpha - \frac{1}{2}} \right)$.

By Hoeffding inequality, we have $P \left( \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{y_i^{(i)} \neq 0\}} - \rho_j \right) \right) \leq \sqrt{\frac{\log d + \log \delta^{-1}}{2n}} \geq 1 - \frac{2\delta}{d}$.

Based on the above event, it yields that $\mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{y_i^{(i)} \neq 0\}} - \rho_j \right) \right] \leq \sqrt{\frac{\log d + \log \delta^{-1}}{2n}}$. Therefore, when $\sqrt{\frac{2 \log d + \log \delta^{-1}}{2n}} < \frac{1}{2} \rho_{\text{min}}$, we obtain that

$$
\left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{y_i^{(i)} \neq 0\}} \right)^{-1} \leq \left( \rho_j - \sqrt{\frac{2 \log d + \log \delta^{-1}}{2n}} \right)^{-1} \leq \frac{2}{\rho_j}.
$$

Since $\|\text{diag}(\hat{\Xi})^{-1} - \text{diag}(\Xi)^{-1}\|_{\text{max}} = \max_{j \in [d]} \left( \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{y_i^{(i)} \neq 0\}} - \rho_j \right) \cdot \left( \frac{\rho_j}{\nu} \sum_{i=1}^{n} \mathbb{1}_{\{y_i^{(i)} \neq 0\}} \right)^{-1} \right)^{-1}$, by union bound,

$$
P \left( \frac{\|\text{diag}(\hat{\Xi})^{-1} - \text{diag}(\Xi)^{-1}\|_{\text{max}}}{\rho_{\text{min}}} \leq \frac{2}{\rho_{\text{min}}} \sqrt{\frac{2 \log d + \log \delta^{-1}}{2n}} \right) \geq 1 - \frac{2\delta}{d}.
$$

Since $\mathbb{E} \left[ \mathbb{1}_{\{y_j^{(i)} \neq 0\}} \right]^2 = P(\xi_j = \xi_k = 1) = \rho_{j,k}$, according to Bernstein’s inequality in
Boucheron et al. (2013)\cite{2}, we have for \(j \neq k\),
\[
P\left( \frac{1}{n} \sum_{i=1}^{n} 1\{y_{j}^{(i)}-y_{k}^{(i)} \neq 0 \} - \rho_{j,k} \right) \leq \sqrt{2 \rho_{j,k} (2 \log d + \log \delta^{-1} / n) + \frac{2 \log d + \log \delta^{-1}}{n}} \geq 1 - \frac{1}{d^2}.
\]
Thus, when \(s_{j,k} := \sqrt{2 \rho_{j,k} (2 \log d + \log \delta^{-1} / n) + \frac{2 \log d + \log \delta^{-1}}{n}} < \frac{1}{2} \rho_{j,k}\) for \(\forall j \neq k\) (i.e. \(\frac{2 \log d + \log \delta^{-1}}{n} \leq \frac{3-2\sqrt{2}}{2} \min_{j \neq k} \rho_{j,k}\)), we have
\[
\bar{\varepsilon}_{j,k}^{-1} \leq (\rho_{j,k} - s_{j,k})^{-1} \leq 2 / \rho_{j,k}, \tag{11}
\]
\[
\left| \bar{\varepsilon}_{j,k}^{-1} - (\Xi)^{-1}_{j,k} \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} 1\{y_{j}^{(i)}-y_{k}^{(i)} \neq 0 \} - \rho_{j,k} \right| \cdot \frac{\rho_{j,k}}{n} \sum_{i=1}^{n} 1\{y_{j}^{(i)}-y_{k}^{(i)} \neq 0 \}^{-1} \leq \frac{2 s_{j,k}}{\rho_{j,k}}.
\]
Since \(\frac{2 \log d + \log \delta^{-1}}{n} \leq (\frac{3}{2} - \sqrt{2}) \min_{j \neq k} \rho_{j,k} \leq 2 \min_{j \neq k} \rho_{j,k}\), we have \(\frac{2 \log d + \log \delta^{-1}}{n} \leq \sqrt{2 \rho_{j,k} (2 \log d + \log \delta^{-1} / n)}\). By union bound,
\[
P\left( \max_{j \neq k} \left| \bar{\varepsilon}_{j,k}^{-1} - (\Xi)^{-1}_{j,k} \right| \leq \frac{2 \rho_{j,k}}{\min_{j \neq k} (\rho_{j,k})^{2}} \sqrt{\frac{2(2 \log d + \log \delta^{-1})}{n}} \right) \geq 1 - \left( 1 - \frac{1}{2} - \frac{1}{2 d} \right) \delta. \tag{12}
\]
Combining (11)-(12), there exists a constant \(C > 0\) such that the conclusion holds. \(\square\)

### A.2 Proof of Theorem 2

**Proof.** The proof follows the lines of Lemma 4 in Fan et al. (2021)\cite{10}. The difference is that we bound the first two terms more carefully such that the corresponding convergence rate is tight for \(\alpha\)-th moment of the stochastic error \(\varepsilon\). By the triangle inequality, we have

\[
\left\| \Sigma_{yX} - \text{mat} \left( \Sigma_{XX} \text{vec} (\Theta^{*}) \right) \right\|_{2} \leq \left\| \Sigma_{yX} - E[\hat{\Sigma}_{yX}] \right\|_{2} + \left\| E[\hat{\Sigma}_{yX}] - \Sigma_{yX} \right\|_{2}.
\]

For the first term, since
\[
\left\| E[\psi_{\tau}(y_{i})^{2}X_{i}^{T}X_{i}] \right\|_{2} = d_{1}d_{2} \left\| E[\psi_{\tau}(y_{i})^{2}e_{k_{(i)}}e_{k_{(i)}}^{T}] \right\|_{2} \leq d_{1}d_{2} \tau^{2-\alpha} \left\| E[|y_{i}|^{\alpha}X_{i}^{T}]e_{k_{(i)}}e_{k_{(i)}}^{T} \right\|_{2}
\]
\[
\leq 2^{\alpha+1}d_{1}d_{2} \tau^{2-\alpha} \left\| E[|y_{i}|^{\alpha}X_{i}^{T}]e_{k_{(i)}}e_{k_{(i)}}^{T} \right\|_{2}
\]
\[
\leq 2^{\alpha+1}d_{1}d_{2} \tau^{2-\alpha}(R^{\alpha} + M) \left\| E[e_{k_{(i)}}e_{k_{(i)}}^{T}] \right\|_{2}
\]
\[
= 2^{\alpha+1}d_{1} \tau^{2-\alpha}(R^{\alpha} + M) \left\| \sum_{k_{0}=1}^{d_{2}} e_{k_{0}}e_{k_{0}}^{T} \right\|_{2} = 2^{\alpha+1}d_{1} \tau^{2-\alpha}(R^{\alpha} + M)
\]
and \( \|E[y_i^2 X_i X_i^T]\|_2 \leq 2^{\alpha-1} d_2 \tau^{2-\alpha}(R^\alpha + M) \), we have

\[
\max \{ \|E[\psi_r(y_i)^2 X_i X_i^T]\|_2, \|E[\psi_r(y_i)^2 X_i X_i^T]\|_2 \} \leq 2^{\alpha-1} (d_1 \lor d_2) \tau^{2-\alpha}(R^\alpha + M).
\]

Moreover, \( \|\psi_r(y_i) X_i - E[\psi_r(y_i) X_i]\|_2 \leq \|\psi_r(y_i) X_i\|_2 + E \|\psi_r(y_i) X_i\|_2 \leq 2\sqrt{d_1 d_2} \tau \). By the Matrix Bernstein inequality in Tropp (2015)[34], we obtain that

\[
P\left( \left\| \bar{\Sigma}_{yX} - E[\bar{\Sigma}_{yX}] \right\|_2 \geq t \right) \leq (d_1 + d_2) \exp \left( \frac{-nt^2/2}{L(d_1 \lor d_2) \tau^{2-\alpha} + 2\sqrt{d_1 d_2} \tau t/3} \right)
\]

By choosing \( \tau = \left( \frac{L\alpha}{(d_1 \lor d_2) \log(d_1 + d_2)} \right)^{\frac{1}{2}} \) and \( t = L^{\frac{1}{2}} \left( \frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n} \right)^{\frac{\alpha-1}{\alpha}} \), we have for a constant \( C > 0 \), with the probability at least \( 1 - (d_1 + d_2)^{-\delta} \frac{2n^{2-n}}{n} \),

\[
\left\| \bar{\Sigma}_{yX} - E[\bar{\Sigma}_{yX}] \right\|_2 \leq CL \left( \frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n} \right)^{\frac{\alpha-1}{\alpha}}.
\]

For the second term, denote \( Z_i := u^T X_i v \) where \( u \in S^{d_1-1} \) and \( v \in S^{d_2-1} \). By Hölder inequality,

\[
\left\| E[\psi_r(y_i) Z_i] - E[y_i Z_i] \right\| = \sqrt{d_1 d_2} \left\| E[y_i 1\{|y_i| \geq \tau\} u_{j(i)} v_{k(i)}] \right\|
\]

\[
= \frac{1}{\sqrt{d_1 d_2}} \left\| \sum_{j_0=1}^{d_1} \sum_{k_0=1}^{d_2} E[y_i 1\{|y_i| \geq \tau\} | j(i) = j_0, k(i) = k_0] u_{j_0} v_{k_0} \right\|
\]

\[
\leq \frac{1}{\sqrt{d_1 d_2}} \left\| \sum_{j_0=1}^{d_1} \sum_{k_0=1}^{d_2} (E[y_i^\alpha]^{\frac{1}{\alpha}} (P(|y_i| \geq \tau))^{1-\frac{1}{\alpha}} u_{j_0} v_{k_0} \right\|
\]

\[
\leq (L)^{\frac{1}{\alpha}} (L/\tau^\alpha)^{1-\frac{1}{\alpha}} \frac{1}{\sqrt{d_1 d_2}} \sum_{j_0=1}^{d_1} \sum_{k_0=1}^{d_2} |u_{j_0} v_{k_0}| \leq L \tau^{1-\alpha}
\]

\[
\leq L^{\frac{1}{\alpha}} \left( \frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n} \right)^{\frac{\alpha-1}{\alpha}}.
\]

The treatment of the last term is the same as that of Fan et al. (2021)[10]. Therefore, with the probability at least \( 1 - (d_1 + d_2)^{-\delta} \),

\[
\left\| \Sigma_{yX} - \text{mat} \left( \bar{\Sigma}_{X X} \text{vec}(\Theta^*) \right) \right\|_2 \leq R \sqrt{\frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n} \delta}.
\]

Furthermore, when \( (d_1 \lor d_2) \log(d_1 + d_2) \leq n \), we obtain that

\[
\left\| \Sigma_{yX} - \text{mat} \left( \bar{\Sigma}_{X X} \text{vec}(\Theta^*) \right) \right\|_2 \leq R \left( \frac{(d_1 \lor d_2) \log(d_1 + d_2)}{n} \right)^{\frac{\alpha-1}{\alpha}} \sqrt{\delta}.
\]

12
By union bound, with the probability at least \(1 - (d_1 + d_2)^{1-\delta} - (d_1 + d_2)^{1-\delta} \frac{2\alpha - 2}{n}\),
\[
\left\| \hat{\Sigma}_{yX} - \text{mat} \left( \hat{\Sigma}_{XX} \text{vec} (\Theta^*) \right) \right\|_2 \leq C \left( \frac{(d_1 \vee d_2) \log (d_1 + d_2)}{n} \right)^{\frac{\alpha-1}{\alpha}} \left( L \frac{1}{\delta} \frac{\alpha-1}{\alpha} + R \sqrt{\delta} + L \frac{1}{\alpha} \right). 
\]

\[ \Box \]

### A.3 Proof of Theorem 3

**Proof.** Denote \( \hat{L}(\Theta) = \| \Theta \|_F^2 - \frac{2}{n} \sum_{i=1}^{n} \left\langle \psi_{T_1} \left( y_i S(X_i) \tilde{Z}_{iT} \right), \hat{\Omega} \right\rangle \). Then we have
\[
\nabla \hat{L}(\Theta) = 2 \hat{\Theta} - \frac{2}{n} \sum_{i=1}^{n} \psi_{T_1} \left( y_i S(X_i) \tilde{Z}_{iT} \right) \hat{\Omega} = 2 \mathbb{E} \left[ y \cdot S(X) \tilde{Z}^T \right] \Omega^* - \frac{2}{n} \sum_{i=1}^{n} \psi_{T_1} \left( y_i S(X_i) \tilde{Z}_{iT} \right) \hat{\Omega} 
\]
\[
= 2 \mathbb{E} \left[ y \cdot S(X) \tilde{Z}^T \right] \left( \Omega^* - \hat{\Omega} \right) + 2 \left( \mathbb{E} \left[ y \cdot S(X) \tilde{Z}^T \right] - \frac{1}{n} \sum_{i=1}^{n} \psi_{T_1} \left( y_i S(X_i) \tilde{Z}_{iT} \right) \right) \left( \hat{\Omega} - \Omega^* \right) 
\]
\[
+ 2 \left( \mathbb{E} \left[ y \cdot S(X) \tilde{Z}^T \right] - \frac{1}{n} \sum_{i=1}^{n} \psi_{T_1} \left( y_i S(X_i) \tilde{Z}_{iT} \right) \right) \Omega^*.
\]
Let \( T := \frac{1}{n} \sum_{i=1}^{n} \psi_{T_1} \left( y_i S(X_i) \tilde{Z}_{iT} \right) - \mathbb{E} \left[ y \cdot S(X) \tilde{Z}^T \right] \). Applying the similar treatment of Theorem 1 of Li et al. (2021) [17] with \( \alpha = 2 \) to \( T \) yields that
\[
P \left( \| T \|_{\text{max}} \leq 2 \max_{j \in [d_2], k \in [d_2]} \sqrt{\mathbb{E} \left( y \cdot [S(X)]_j \cdot \tilde{z}_k \right)^2} \sqrt{\frac{3 \log(d_1 d_2)}{n}} \right) \geq 1 - \frac{2}{(d_1 d_2)^2} \tag{13}\]
where \( r_{j,k}^{(1)} = \sqrt{\mathbb{E} \left( y \cdot [S(X)]_j \cdot \tilde{z}_k \right)^2 \sqrt{\frac{n}{3 \log(d_1 d_2)}}} \). By Cauchy-Schwarz inequality, \( \mathbb{E} \left( y \cdot [S(X)]_j \cdot \tilde{z}_k \right)^2 \leq \sqrt{\mathbb{E}[y^2] \cdot \mathbb{E}([S(X)]_j \cdot \tilde{z}_k)^2} \leq \sqrt{\rho_k M^2} < \infty \). Because \( \hat{\Omega} \) is obtained by \( \text{(4)} \), according to Corollary \( \text{(4)} \) we obtain
\[
P \left( \left\| \hat{\Omega} - \Omega^* \right\|_{\text{max}} \leq M^{\frac{1}{2}} \delta \kappa \sqrt{\frac{3 \log d_2}{n}} \right) \geq 1 - \frac{3}{2d_2} - \frac{5}{2d_2^2},
\]
where \( \gamma \geq M^{\frac{1}{2}} \kappa \sqrt{\frac{3 \log d_2}{n}} \). Since
\[
\left\| \nabla \hat{L}(\Theta) \right\|_{\text{max}} \leq 2 \| T \|_{\text{max}} \left\| \Omega^* - \hat{\Omega} \right\|_{1,1} + 2 \| T \|_{\text{max}} \left\| \Omega^* \right\|_{1,1} + 2 \left\| \Omega^* - \hat{\Omega} \right\|_{\text{max}} \mathbb{E} \left[ y \cdot S(X) \tilde{Z}^T \right] \| \Omega^* \|_{\text{max}} \| \Theta^* \|_{\text{max}} \| \Sigma^* \|_{\text{max}},
\]
we have with probability at least \(1 - \frac{2}{(d_1 d_2)^2} - \frac{3}{2d_2} - \frac{5}{2d_2^2}\),
\[
\left\| \nabla \hat{L}(\Theta) \right\|_{\text{max}} \leq 8 \left( \rho_{\text{max}} M^3 \right)^{\frac{1}{2}} \left\| \Omega^* \right\|_{1,1} \sqrt{\frac{3 \log(d_1 d_2)}{n}} + 8 \max_{j \in [d_2]} \| \mu^*_j \rho_j \| \cdot \left\| \Theta^* \Sigma^* \right\|_{\text{max}} M^{\frac{1}{2}} \delta \kappa \sqrt{\frac{3 \log d_2}{n}}. \tag{14}\]
By the optimality of $\hat{\Theta}$, we have $\hat{L}(\hat{\Theta}) + \lambda \|\hat{\Theta}\|_{1,1} \leq \tilde{L}(\hat{\Theta}) + \lambda \|\hat{\Theta}\|_{1,1}$. Since

$$\|\hat{\Theta}\|_{1,1} = \|\hat{\Theta} + (\hat{\Theta} - \hat{\Theta})_{sc} + (\hat{\Theta} - \hat{\Theta})_{s}\|_{1,1}$$

$$\geq \|\hat{\Theta} + (\hat{\Theta} - \hat{\Theta})_{sc}\|_{1,1} - \|((\hat{\Theta} - \hat{\Theta})_{s})\|_{1,1}$$

$$= \|((\hat{\Theta} - \hat{\Theta})_{sc} - (\hat{\Theta} - \hat{\Theta})_{s})\|_{1,1} + \|\hat{\Theta}\|_{1,1}$$

and $\hat{L}(\hat{\Theta}) - \tilde{L}(\hat{\Theta}) = \langle \nabla \hat{L}(\hat{\Theta}), \hat{\Theta} - \hat{\Theta}\rangle + \|\hat{\Theta} - \hat{\Theta}\|_{F}^2$, thus, we have

$$\|\hat{\Theta} - \hat{\Theta}\|_{F}^2 \leq - \langle \nabla \hat{L}(\hat{\Theta}), \hat{\Theta} - \hat{\Theta}\rangle + \lambda \|\hat{\Theta}\|_{1,1} - \lambda \|\hat{\Theta}\|_{1,1}$$

$$\leq \|\nabla \hat{L}(\hat{\Theta})\|_{\max} \|\hat{\Theta} - \hat{\Theta}\|_{1,1} - \lambda \|((\hat{\Theta} - \hat{\Theta})_{sc}\|_{1,1} + \lambda \|((\hat{\Theta} - \hat{\Theta})_{s}\|_{1,1}$$

Choosing the RHS of (14) as $\lambda$, we have with probability at least $1 - \frac{2}{s_1^2 s_2^2} - \frac{1}{2s_2} - \frac{1}{2s_2}$,

$$\|\hat{\Theta} - \hat{\Theta}\|_{F}^2 \leq 2\lambda \|((\hat{\Theta} - \hat{\Theta})_{s}\|_{1,1} \leq 2\lambda \sqrt{s d_2} \|\hat{\Theta} - \hat{\Theta}\|_{F}$$

(15)

Similarly, we can obtain that $\|\hat{\Theta} - \hat{\Theta}\|_{1,1} \leq 8\lambda s d_2$. \hfill \Box

References

[1] Avella-Medina, M., Battey, H. S., Fan, J. and Li, Q. (2018). Robust estimation of high-dimensional covariance and precision matrices. *Biometrika*, 105(2):271–284.

[2] Boucheron, S., Lugosi, G. and Massart, P. (2013). Concentration inequalities: A nonasymptotic theory of independence. OUP Oxford.

[3] Bubeck, S., Cesa-Bianchi, N and Lugosi, G. (2013). Bandits with heavy tail. *Information Theory, IEEE Transactions on*, 59(11):7711–7717.

[4] Cai, T., Liu, W. and Luo, X. (2011). A constrained $\ell_1$ minimization approach to sparse precision matrix estimation. *Journal of the American Statistical Association*, 106(494):594–607.

[5] Cai, T. and Zhang, A. (2016). Minimax rate-optimal estimation of high-dimensional covariance matrices with incomplete data. *Journal of Multivariate Analysis*, 150:55-74.
[6] Catoni, O. (2012). Challenging the empirical mean and empirical variance: A deviation study. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 48(4):1148–1185. MR3052407

[7] Cui, R., Groot, P., and Heskes, T. (2017). Robust estimation of gaussian copula causal structure from mixed data with missing values. In *IEEE International Conference on Data Mining*, 835–840.

[8] Devroye, L., Lerasle, M., Lugosi, G., and Oliveira, R. I. (2016). Sub-Gaussian mean estimators. *Annals of Statistics*, 44:2695–2725.

[9] Fan, J., Yang, Z., and Yu, M. (2020). Understanding implicit regularization in over-parameterized nonlinear statistical model. Preprint. Available at arXiv:2007.08322.

[10] Fan, J., Wang, W., and Zhu, Z. (2021). A shrinkage principle for heavy-tailed data: High-dimensional robust low-rank matrix recovery. *Annals of Statistics*, 49(3):1239–1266.

[11] Fan, J., Li, Q., and Wang, Y. (2017). Estimation of high dimensional mean regression in the absence of symmetry and light tail assumptions. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79(1):247–265.

[12] Fan, J., and Zhang, W. (2008). Statistical methods with varying coefficient models. *Statistics and its Interface*, 1(1):179–195.

[13] Goldstein, L., Minsker, S., and Wei, X. (2018). Structured signal recovery from nonlinear and heavy-tailed measurements. *IEEE Transactions on Information Theory*, 64(8), 5513–5530.

[14] Huber, P. J. (1964). Robust estimation of a location parameter. *Annals of Mathematical Statistics*, 35:73–101. MR0161415

[15] Ke, Y., Minsker, S., Ren, Z., Sun, Q., and Zhou, W.-X. (2019). User-friendly covariance estimation for heavy-tailed distributions. *Statistical Science*, 34(3):454–471. MR4017523

[16] Kolar, M. and Xing, E. (2012) Estimating sparse precision matrices from data with missing values. In *International Conference on Machine Learning*, 635–642.
[17] Li, K., Bao, H. and Zhang, L. (2021). Robust covariance estimation for distributed principal component analysis. *Metrika*, 1–26.

[18] Little, R. J. (1988). Robust estimation of the mean and covariance matrix from data with missing values. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 37(1):23–38.

[19] Liu, L. and Palomar, D. (2019). Regularized robust estimation of mean and covariance matrix for incomplete data. *Signal Processing*, 165:278–291.

[20] Loh, P.-L. and Wainwright, M. J. (2012). High-dimensional regression with noisy and missing data: Provable guarantees with nonconvexity. *Annals of Statistics*, 40(3):1637–1664.

[21] Lounici, K. (2014). High-dimensional covariance matrix estimation with missing observations. *Bernoulli*, 20(3):1029–1058.

[22] Ma, S. and Song, P. (2015). Varying index coefficient models. *Journal of the American Statistical Association*, 110(509):341–356.

[23] Minsker, S. (2018). Sub-gaussian estimators of the mean of a random matrix with heavy-tailed entries. *Annals of Statistics*, 46(6A):2871–2903. [MR3851758](#)

[24] Minsker, S. and Wei, X. (2017). Estimation of the covariance structure of heavy-tailed distributions. In *Advances in Neural Information Processing Systems*, 2855–2864.

[25] Minsker, S. and Wei, X. (2020). Robust modifications of U-statistics and applications to covariance estimation problems. *Bernoulli*, 26(1):694–727. [MR4036049](#)

[26] Na, S., Yang, Z., Wang, Z. and Kolar, M. (2019). High-dimensional varying index coefficient models via Stein’s identity. *Journal of Machine Learning Research*, 20:1–44, 2019.

[27] Negahban, S. and Wainwright, M. J. (2012). Restricted strong convexity and weighted matrix completion: Optimal bounds with noise. *Journal of Machine Learning Research*, 13:1665–1697.
[28] Pavez, E. and Ortega, A. (2020). Covariance matrix estimation with non uniform and data dependent missing observations. *IEEE Transactions on Information Theory, 67*(2):1201–1215.

[29] Park, S. and Lim, J. (2019). Non-asymptotic rate for high-dimensional covariance estimation with non-independent missing observations. *Statistics & Probability Letters, 153*:113–123.

[30] Park, S., Wang, X. and Lim, J. (2021). Estimating high-dimensional covariance and precision matrices under general missing dependence. *Electronic Journal of Statistics, 15*(2):4868–4915.

[31] Stein, C., Diaconis, P., Holmes, S. and Reinert, G. (2004) Use of exchangeable pairs in the analysis of simulations. In *Stein’s Method*, Institute of Mathematical Statistics.

[32] Sun, Q., Zhou, W. X. and Fan, J. (2020). Adaptive huber regression. *Journal of the American Statistical Association, 115*(529):254–265.

[33] Tan, K. M., Sun, Q. and Witten, D. M. (2018). Robust sparse reduced rank regression in high dimensions. Preprint. Available at [arXiv:1810.07913](https://arxiv.org/abs/1810.07913).

[34] Tropp, J. A. (2015). An introduction to matrix concentration inequalities. Preprint. Available at [arXiv:1501.01571](https://arxiv.org/abs/1501.01571).

[35] Wang, H., Fazayeli, F., Chatterjee, S. and Banerjee, A. (2014). Gaussian copula precision estimation with missing values. In *Artificial Intelligence and Statistics*, 978–986.

[36] Yang, Z., Balasubramanian, K. and Liu, H. (2017). High-dimensional non-Gaussian single index models via thresholded score function estimation. In *Proceedings of the 34th International Conference on Machine Learning, volume 70* of *Proceedings of Machine Learning Research*, 3851–3860.

[37] Yang, Z., Balasubramanian, K., Wang, Z. and Liu, H. (2017). Learning non-gaussian multi-index model via second-order stein’s method. *Advances in Neural Information Processing Systems, 30*:6097–6106.

[38] Zhang, D. (2021). Robust estimation of the mean and covariance matrix for high dimensional time series. *Statistica Sinica, 31*(2):797–820.
[39] Zhu, Z. and Zhou, W. (2020). Taming heavy-tailed features by shrinkage. In *International Conference on Artificial Intelligence and Statistics*.