Equilibrium electric current of massive electrons with anomalous magnetic moments induced by a magnetic field and the electroweak interaction with matter

Maxim Dvornikov\textsuperscript{a,b}\textsuperscript{*}

\textsuperscript{a} Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation (IZMIRAN), 108840 Troitsk, Moscow, Russia;
\textsuperscript{b} Physics Faculty, National Research Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia

Abstract

We study the possibility of the existence of the electric current, formed by massive electrons and positrons, flowing along an external magnetic field. The charged fermions are supposed to have nonzero anomalous magnetic moments and electroweakly interact with background matter. The expression for the current is obtained on the basis of the exact solution of the Dirac equation in the corresponding external fields. We demonstrate that, in the state of equilibrium, such a current is vanishing for any characteristics of the electron-positron plasma as well as the external fields. Our results are compared with the recent findings of other authors.

1 Introduction

The dynamo mechanism is widely used in cosmology and astrophysics for the generation of strong large-scale magnetic fields \cite{1}. This mechanism is based on the excitation of the electric current along the external magnetic field $B$: $J = \Pi B$. Indeed, if a current $J \parallel B$ is accounted for in the Maxwell equations, the magnetic field becomes unstable and can be dynamo amplified. In QED plasma, where the parity is conserved, the parameter $\Pi$ should be a pseudoscalar since $J$ is a vector and $B$ is an axial-vector. For example, in classical MHD, the parameter $\Pi \sim \langle \mathbf{v} \cdot (\nabla \times \mathbf{v}) \rangle$ \cite{2}, where $\mathbf{v}$ is the random component of the fluid velocity, $\langle \mathbf{v} \rangle = 0$. Thus, one can see that the situation when $\Pi \neq 0$ is implemented effectively in a medium.

Recently, the dynamo mechanism based on the chiral magnetic effect (the CME) \cite{3,4} becomes popular. The CME consists in the appearance of the current $J_{\text{CME}} = \alpha_{\text{em}}(\mu_R - \mu_L)B/\pi$ of massless charged particles, as the consequence of the Adler-Bell-Jackiw anomaly in QED \cite{5}. Here $\mu_{R,L}$ are the chemical potentials of right and left chiral fermions and $\alpha_{\text{em}} \approx 1/137$ is the fine structure constant. We remind that the chiral imbalance $\mu_5 = (\mu_R - \mu_L)/2$

\textsuperscript{*}maxdvo@izmiran.ru
is the pseudoscalar under the spatial inversion, $\mu_R \leftrightarrow \mu_L$, i.e. the parity is conserved in QED for $J_{\text{CME}}$. The manifestations of the CME are extensively studied in astrophysics and cosmology [6], accelerator experiments [7], and solid state physics [8]. The correction to the CME from the parity violating electroweak interaction was considered for the first time in Ref. [9].

The main feature of the CME is the unbroken chiral symmetry of charged particles. It means that any nonzero mass makes $J_{\text{CME}}$ to vanish [3]. The majority of known elementary particles acquire masses through the electroweak mechanism. However, it is likely to be an electroweak crossover rather than a first order phase transition [10]. Therefore, charged particles will remain massive at any pressure and chemical potential unless a new physics beyond the standard model is accounted for (see, e.g., Ref. [11]). There are indications that a chiral phase transition can happen in dense matter owing to the QCD effects [12]. Some astrophysical applications for the magnetic fields generation in compact stars due to the CME and the electroweak interaction between quarks in dense matter are considered in Ref. [13].

In this connection, there is a particular interest in searching for the possibility of the generation of the current $J \parallel B$ in the system of massive charged particles. In this situation, one would have an instability of the magnetic field without necessity to demand the restoration of the chiral symmetry. One of the examples of such a system was studied in Ref. [14], where the dynamo amplification of magnetic fields in an inhomogeneous electroweak matter was discussed.

Recently, in Refs. [15, 16], the existence of an electric current $J \parallel B$ of massive fermions, having anomalous magnetic moments and interacting with an external axial-vector field, was claimed. The axial-vector field can be represented in the form of the electroweak background matter [15] or in a hypothetical CPT-odd extension of the standard model [16]. In both cases, a nonzero $J \parallel B$ was shown to exist in the considered system.

In the present work, we revise the results of Refs. [15, 16]. In Sec. 2, we demonstrate that the current $J \parallel B$ of massive particles with anomalous magnetic moments, interacting with the parity violating axial-vector field, is vanishing. This result is based on the computation of this current using the exact solution of the Dirac equation in the corresponding external fields performed in Appendix [A] (see also Refs. [17, 18]). Our results are summarized in Sec. 3.

2 Cancellation of the anomalous electric current

Let us consider a plasma of electrons and positrons, electroweakly interacting with background matter under the influence of the external magnetic field $B$. Charged particles are considered to be massive. Their nonzero anomalous magnetic moments are taken into account. The Lagrangian for an electron, described by the bispinor $\psi_e$, has the form,

$$\mathcal{L} = \bar{\psi}_e \left[ \gamma_\mu (i\partial^\mu + eA^\mu) - m + \frac{\mu}{2} \sigma^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \sigma^{\mu\nu} \gamma_\mu \gamma_\nu \right] \psi_e,$$

where $A^\mu = (0, 0, Bx, 0)$ is the vector potential corresponding to the constant and homogeneous magnetic field, directed along the $z$-axis, $e > 0$ is the elementary charge, $P_{L,R} = (1 \mp \gamma^5)/2$ are the chiral projectors, $\gamma^\mu = (\gamma^0, \gamma^i)$, $\sigma^{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu]$, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ are the Dirac matrices, $m$ is the electron mass, $\mu$ is the anomalous magnetic moment [19], $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (E, B)$ is the electromagnetic field tensor (with $E = 0$), and $V^{\mu}_{L,R} = (V^{0}_{L,R}, V_{L,R})$ are the effective potentials of the electroweak interaction of the electron chiral
projections with background matter. We shall suppose that the background matter is macro-
scopically at rest and unpolarized. In this situation, \( V_{L,R} = 0 \) and \( V_{L,R}^0 \equiv V_{L,R} \neq 0 \). The explicit form of \( V_{L,R} \) is given in Ref. [20] for the case of background matter consisting of neutrons and protons.

The most general expression for the induced current of electrons \( J_e \) was obtained in Ref. [15] (see also Eq. (34)). The electric current of positrons \( J_\bar{e} \) can be derived on the basis of the exact solution of the Dirac equation for positrons \( \psi_\bar{e} \), accounting for the external fields, which, in its turn, can be obtained from \( \psi_\bar{e} \) by applying the charge conjugation. The details of the derivation are provided in Appendix A.

Summing up the induced currents of electrons (see Eq. (34) and Ref. [15]), and positrons (see Eqs. (38)), one gets the following expression for the total current

\[
J = J_e + J_\bar{e},
\]

(2)

where \( \alpha_{em} = e^2/4\pi \) is the fine structure constant, \( V_5 = (V_L - V_R)/2 \), \( \Delta f = f(\mathcal{E} - \chi_{\text{eff}}) - f(\mathcal{E} + \chi_{\text{eff}}) \), \( f(\mathcal{E}) = [\exp(\beta\mathcal{E}) + 1]^{-1} \) is the Fermi-Dirac distribution function, \( \chi_{\text{eff}} = \chi - \bar{V}, \chi \) is the chemical potential of the electron-positron plasma, \( \beta = 1/T \) is the reciprocal temperature, \( \bar{V} = (V_L + V_R)/2, n = 1, 2, \ldots \) and \( s = \pm 1 \) are discrete quantum numbers, which the energy levels depend on (see Eqs. (20) and (21)), and

\[
\mathcal{E} = \sqrt{p_z^2 + 2eBn + m^2 + (\mu B)^2 + V_5^2 + 2sR^2},
\]

\[
R^2 = \sqrt{(p_zV_5 - \mu Bm)^2 + 2eBn [(\mu B)^2 + V_5^2]}.
\]

(3)

It is interesting to mention that the integrand in Eq. (2) can be represented as

\[
\frac{1}{\mathcal{E}} \left[ p_z \left( 1 + s \frac{V_5^2}{R^2} \right) - s \frac{\mu BmV_5}{R^2} \right] = \frac{\partial \mathcal{E}}{\partial p_z}.
\]

(4)

Hence the total current is proportional to the averaged group velocity of a charged particle along the magnetic field, \( v_z = \partial \mathcal{E}/\partial p_z : J = -\alpha_{em} \langle v_z \rangle \mathbf{B}/\pi \).

Considering Eq. (2) in case of a degenerate electron gas, it was claimed in Ref. [15] that \( \Pi \neq 0 \). Analogous result was obtained in Ref. [16] on the basis of the analysis of the effective Lagrangians in the one-loop approximation. The instability of the magnetic field, driven by the anomalous current \( J = \Pi \mathbf{B} \), and some astrophysical applications were studied in Ref. [15]. The claim of Refs. [15] that there is \( J = \Pi \mathbf{B} \neq 0 \) in the considered system is based on the fact that the energy levels at \( n > 0 \) in Eq. (3) are neither symmetric nor antisymmetric functions of \( p_z \), which is the momentum projection along the magnetic field. Thus, the integration over \( p_z \) in the symmetric limits in Eq. (2) could give a nonzero result. The asymmetry coefficient of the energy levels in Eq. (3) is \( \mu BmV_5 \). It is this term, which \( J \parallel \mathbf{B} \) in Refs. [15] is proportional to.

Nevertheless, a careful analysis reveals that \( \Pi = 0 \) in Eq. (2). This fact is not quite obvious. To demonstrate it, we introduce the notation in Eq. (2),

\[
\frac{\Delta f}{\mathcal{E}} = F(Q^2 + 2sR^2), \quad Q^2 = p_z^2 + 2eBn + m^2 + (\mu B)^2 + V_5^2.
\]

(5)
Then we decompose $F(Q^2 + 2sR^2)$,

$$F(Q^2 + 2sR^2) = \sum_{k=0}^{\infty} 2^{2k} R^{4k} \left[ \frac{F^{(2k)}(Q^2)}{(2k)!} + 2sR^2 \frac{F^{(2k+1)}(Q^2)}{(2k+1)!} \right],$$

$$F^{(k)}(Q^2) = \frac{d^k F(Q^2)}{d(Q^2)^k},$$

in a formal series.

The sum over $s$ of the integrand in Eq. (2) gives

$$I = \sum_{s=\pm 1} \left[ p_z + s(p_zV_5 - \mu Bm) \frac{V_5}{R^2} \right] F(Q^2 + 2sR^2)$$

$$= 2 \sum_{k=0}^{\infty} 2^{2k} R^{4k} \left[ p_z \frac{F^{(2k)}(Q^2)}{(2k)!} + 2V_5(p_zV_5 - \mu Bm) \frac{F^{(2k+1)}(Q^2)}{(2k+1)!} \right]$$

$$= 2 \sum_{k=0}^{\infty} 2^{2k} \left\{ p_z R^{4k} \frac{F^{(2k)}(Q^2)}{(2k)!} \right\}$$

$$+ \frac{1}{k+1} \frac{\partial}{\partial p_z} \left[ R^{4(k+1)} \frac{F^{(2k+1)}(Q^2)}{(2k+1)!} \right],$$

where we use the identities,

$$\frac{\partial R^4}{\partial p_z} = 2V_5(p_zV_5 - \mu Bm), \quad \frac{1}{k+1} \frac{\partial}{\partial p_z} \left[ R^{4(k+1)} \right] = R^{4k} \frac{\partial R^4}{\partial p_z},$$

and take into account Eq. (3).

Integrating Eq. (7) over $p_z$ and then by parts, one gets

$$\int_{-\infty}^{+\infty} dp_z = 2 \sum_{k=0}^{\infty} 2^{2k} \left[ \int_{-\infty}^{+\infty} dp_z \left\{ R^{4k} p_z \frac{F^{(2k)}(Q^2)}{(2k)!} \right\} \right.$$

$$\left. - \frac{1}{k+1} \frac{\partial}{\partial p_z} \left[ F^{(2k+1)}(Q^2) \right] R^{4(k+1)} \frac{(2k+1)!}{(k+1)(2k)!} \right|_{-\infty}^{+\infty} \right].$$

The function $F(Q^2)$ is proportional to the Fermi-Dirac distribution functions, which are vanishing at great values of the argument. The same property has any derivative of $F(Q^2)$ in Eq. (3). Thus, the last term in Eq. (9) disappears at $p_z \to \pm \infty$.

Taking into account that

$$\frac{\partial}{\partial p_z} \left[ F^{(2k+1)}(Q^2) \right] = 2p_z \frac{d}{dQ^2} \left[ F^{(2k+1)}(Q^2) \right] = 2p_z F^{(2k+2)}(Q^2),$$

and changing the summation index $k \to k - 1$ in the second term in Eq. (9), one obtains

$$\int_{-\infty}^{+\infty} dp_z = 2 \sum_{k=0}^{\infty} 2^{2k} \left[ \int_{-\infty}^{+\infty} dp_z R^{4k} p_z \frac{F^{(2k)}(Q^2)}{(2k)!} \right] - 2 \sum_{k=1}^{\infty} 2^{2k} \int_{-\infty}^{+\infty} dp_z R^{4k} p_z \frac{F^{(2k)}(Q^2)}{(2k)!}$$

$$= 2 \int_{-\infty}^{+\infty} dp_z p_z F(Q^2) = 0,$$
where we use the fact that $F^{(0)}(Q^2) = F(Q^2)$. To get the vanishing result of the integration in the symmetric limits in Eq. (11), we account for that $Q^2$ is the even function of $p_z$ (see Eq. (5)), i.e. the integrand in Eq. (11) is the odd function.

Thus, we have demonstrated that in Eq. (2)

$$J = \Pi B = 0, \quad \text{since} \quad \Pi = 0,$$

(12)

for arbitrary characteristics of the external fields and charged particles. Accounting for the fact that the lowest energy level with $n = 0$ does not contribute to the anomalous current either (see Eq. (35) and Ref. [15]), we get that there is no electric current $J \parallel B$ in the system of massive electrons with anomalous magnetic moments, electroweakly interacting with background matter, in the state of equilibrium.

Let us comment on the cancellation of the contribution of the lowest energy level to the current $J \parallel B$ in Eq. (35), where we use the equilibrium Fermi-Dirac distribution function $f_{eq}(E) = \left[\exp(\beta E) + 1\right]^{-1}$ to eliminate the ultraviolet divergence of the integral. Another regularization scheme was applied in Refs. [5, 16]. This scheme is based on imposing the momentum cut-off: $|p_z| < P_{\text{max}}$, where $P_{\text{max}} \gg \max(m, \mu_B V_5)$, and setting $f \rightarrow 1$. This regularization is equivalent to the replacement of the equilibrium distribution function $f_{eq}$ by the nonequilibrium one,

$$f_{\text{non-eq}}(p_z) = \begin{cases} 1, & \text{if } |p_z| < P_{\text{max}}, \\ 0, & \text{if } |p_z| > P_{\text{max}}. \end{cases}$$

(13)

The computation of the integral in Eq. (35) with $f_{\text{non-eq}}(p_z)$ in Eq. (13) gives $J_{e}^{(n=0)} = -e^2 V_5 B / 2n^2$, which formally coincides with the prediction of the CME if we replace $\mu_5 \rightarrow -V_5$ in $J_{\text{CME}}$. It is interesting to mention, that, in case of $V_5 = 0$, the regularization scheme used in Ref. [5] and that applied in Eq. (35) give coinciding results, $J_{e}^{(n=0)} = 0$, for massive particles.

It is, however, known that, if the distribution function of a system happens to differ from $f_{eq}(E)$, the system becomes unstable and will tend to the equilibrium state, i.e. $f_{\text{non-eq}}(p_z) \rightarrow f_{eq}(E)$. If we consider an ultrarelativistic plasma, the relaxation time for the process $f_{\text{non-eq}}(p_z) \rightarrow f_{eq}(E)$ was estimated in Ref. [21] as $\tau \sim 10T^2/n$, where $n$ is the number density of charged particles in plasma. For instance, in the case of a degenerate electron gas in a neutron star with $T = 10^9$ K and $n = 10^{36}$ cm$^{-3}$, we get that $\tau \sim 10^{-25}$ s. Such a small relaxation time is the indication that one should use $f_{eq}$ while calculating $J \parallel B$ in astrophysical media. Since the typical lifetime of a neutron star is much longer than the estimated $\tau$, the gas of massive electrons should be in the equilibrium and, thus, the anomalous current is vanishing, as shown above.

We can consider a situation when the phases with broken and unbroken chiral symmetry coexist. It can happen if the first order chiral phase transition takes place and a bubble with an unbroken phase appears in a neutron star. Then, in the vicinity of the bubble wall, the chiral symmetry can be considered as approximately broken, i.e. $J_{\text{CME}} \sim \mu_5 B$ exists, however, $\mu_5$ is no longer constant. The evolution of $\mu_5$ in this case is driven by the Adler anomaly completed by the chirality flip term [22],

$$\frac{d\mu_5}{dt} = \cdots - \Gamma_f \mu_5,$$

(14)

where $\Gamma_f$ is the chirality flip rate, which was calculated in Refs. [23, 24] in the degenerate matter of a neutron star as $\Gamma_f \sim 10^{11}$ s$^{-1} \times (T/10^8$ K). In Eq. (14), we omitted the terms
containing the magnetic fields. Taking $T = 10^8$ K, one gets that the chiral imbalance relaxation time $\tau = 1/\Gamma_f \sim 10^{-11}$ s, which is again much shorter than the neutron star life-time. It means that, even if the chiral symmetry is considered approximately broken, the contribution of the chiral imbalance to the CME is negligible in astrophysical media.

The opposite situation is implemented if we discuss a quark plasma formed in a collision of heavy ions. In this case, we can take $T \sim 10^2$ MeV and $n \sim 10^{38}$ cm$^{-3}$. The relaxation time for the distribution function $f_{\text{eq}} \rightarrow f_{\text{non-eq}}$ is $\tau \sim 10^{-12}$ cm, which is much greater than a nucleus radius $r_N \sim 10^{-13}$ cm. It means that a plasma emerging in such a collision is strongly nonequilibrium and there is a chance to generate $J = \Pi B$ with $\Pi \neq 0$. This fact is in agreement with the possibility of the appearance of strong magnetic fields in heavy ion collisions \cite{7}.

The main reason for the disappearance of the current $J \parallel B$ in the state of equilibrium is the range of the $p_z$ variation for massive particles: $-\infty < p_z < +\infty$. This fact distinguishes the considered situation from the CME, where, for massless particles, one has either $-\infty < p_z < 0$ or $0 < p_z < +\infty$, depending on the particle chirality and its charge.

Now let us compare the obtained result that $J \parallel B$ is vanishing with the previous findings. In Ref. \cite{15}, the general expression for the electron current in Eq. (2) was obtained correctly (see Eq. (9) in Ref. \cite{15}). However, while considering the particular case of the degenerate electron gas, the error in integration over $p_z$ was made, that lead to a nonzero $J \parallel B$.

It is also interesting to compare our results with those in Ref. \cite{16}, where a massive electron with the anomalous magnetic moment, interacting with an external magnetic field and the axial-vector field $b^\mu = (b^0, b^\gamma)$, was considered. The Lagrangian for the electron interaction with the field $b^\mu$ was taken as $L_b = -\bar{\psi} \gamma^5 \gamma^\mu b_\mu \psi$. Comparing $L_b$ with Eq. (1), we get that $b^0 = V_5$. Nonzero spatial components $b$ can be present if polarized or moving background matter is considered.

It was claimed in Ref. \cite{16} that, in the situations (i) $(\mu = 0, b^0 \neq 0, b^\gamma = 0)$; and (ii) $(\mu \neq 0, b^0 \neq 0, b^\gamma = 0)$, there is a nonzero electric current $J \parallel B$ in the system of massive electrons. The case (i) was previously considered in our work \cite{25}, where the induced current was shown to vanish for massive charged particles. Note that neither the lowest nor higher energy levels contribute to the current. The situation (ii) is considered in the present work. Although the cancellation of the current is not so straightforward as in the case (i), the current $J \parallel B$ turns out to be vanishing for massive electrons nonetheless. It should be also noted that any energy level does not contribute to the current. This finding is again in the contradiction with the results of Ref. \cite{16}.

The reason of the discrepancy of our results with those of Ref. \cite{16} is the following. The anomalous current in Ref. \cite{16} was calculated in vacuum. The regularization was used to eliminate the ultraviolet divergence in the integrals. We have already mentioned above that this regularization is equivalent to the consideration of a nonequilibrium state of the system described by the distribution function in Eq. (13). If one makes electrons, forming this current, to thermalize, then $f_{\text{non-eq}} \rightarrow f_{\text{eq}}$ very rapidly.\footnote{Unless one considers the situation analogous to a collision of heavy ions.} Calculations in the present work are free of the ultraviolet divergencies since we consider the system of electrons in the thermodynamic equilibrium with the nonzero temperature $T$ and the chemical potential $\chi$. The distribution function, used in Eq. (2), eliminates the appearance of the ultraviolet divergencies. In fact, this distribution function serves as a natural regularization.
3 Conclusion

In this work, we have analyzed the possibility of the existence of the electric current induced along the external magnetic field in the system of massive charged electrons, having anomalous magnetic moments and electroweakly interacting with background matter, which was supposed to be nonmoving and unpolarized. Using the exact solution of the Dirac equation in the corresponding external fields, which was obtained in Refs. [17,18] (see also Appendix A), the most general expression of the current, which accounts for the contributions of both electrons and positrons, has been derived.

The energy of an electron in the external fields in question is quantized (see Eqs. (20) and (27)) and depends on the discrete quantum number \( n = 0, 1, \ldots \). The cancellation of the contribution of the lowest energy level with \( n = 0 \) to the induced current can be established directly from the Dirac equation (see Eqs. (28) and (35)) even without analyzing the spin integral of the Dirac equation in Eq. (22). It is important to suppose that the system is in the equilibrium state while considering the cancellation of \( J^{(n=0)} \) (see the discussion in Sec. 2).

The analysis of the contribution of the higher energy levels with \( n > 0 \) to the induced current is not trivial. Nevertheless, in Sec. 2, we have revealed that this contribution is vanishing; cf. Eq. (12). This result is valid at any characteristics of the electron-positron field, such as \( m, \mu \), etc., and any parameters of the external fields, like \( B \) and \( V_5 \).

The cancellation of the induced current \( J \parallel B \) for \( n > 0 \) in the considered system, which was supposed to be in the equilibrium, corrects the recent claims in Refs. [15,16] that such a current can be nonzero. The incorrect nonzero expression for the current, obtained in Ref. [15], was because of the error in the integration over the longitudinal momentum in the case of the degenerate electron gas. The discrepancy between our results and the findings of Ref. [16] can be explained by the consideration of a nonequilibrium state of the system in Ref. [16]. Therefore the current \( J \parallel B \neq 0 \), derived in Ref. [16], will tend to zero very rapidly in a realistic medium. The typical relaxation time for the current to vanish in the astrophysical medium was estimated in Sec. 2.

The main reason for the cancellation of the current consists in the fact that the longitudinal momentum \( p_z \) can vary from \(-\infty\) to \(+\infty\) for a particle with a nonzero mass. Even the feature of the energy spectrum for \( n > 0 \) that it is not symmetric with respect to the transformation \( p_z \rightarrow -p_z \) (see Eq. (3)), which Ref. [15] appealed to in order to justify the existence of the nonzero induced current, does not help to generate \( J \parallel B \neq 0 \). Thus the cases of \( m \neq 0 \) and \( m = 0 \) are different generically. In the latter situation, the induced current can exist owing to the CME (see Sec. 1), which is based on the asymmetric motion of charged massless particles at the lowest energy level with respect to the external magnetic field. The difference between the systems of massive and massless particles consists in the chiral symmetry: it is broken in the former case and restored in the latter one.

Therefore, one can expect the existence of the induced current \( J \parallel B \), and, thus, the instability of the magnetic field in a system in equilibrium only if fermions, present in this system, have zero masses. Of course, in this situation, one should somehow restore the chiral symmetry. This task is not trivial as explained in Sec. 1.

The nonsmooth behavior of the current \( J \parallel B \) as a function of the particle mass can be treated as follows. In some circumstances, the induced current itself \( J \parallel B \) is not the object

\[ \text{Note that the general expression for the current, derived in Ref. [15], turns out to be correct; cf. Eq. (34) and Eq. (9) in Ref. [15].} \]
of a study. For instance, if one considers the generation of magnetic fields, one is interested in the evolution of the magnetic field driven by the current $\mathbf{J} \parallel \mathbf{B}$. For instance, let us discuss the electron-positron plasma with a nonzero temperature. In this case, the mass of a particle becomes the function of the temperature $m = m(T)$. Suppose that a first order chiral phase transition can happen in this plasma at a certain temperature $T_c$. If $T < T_c$, particles are massive and there is no CME in the system.

Let the temperature to increase. As soon as $T = T_c$, bubbles of the new phase with restored chiral symmetry appear in plasma. Since masses of fermions are equal to zero inside a bubble, $\mathbf{J}_{\text{CME}} \parallel \mathbf{B}$ can flow there, causing the magnetic field instability, which, in its turn, leads to the enhancement of a seed field. Just after the phase transition, the size of these bubbles is small and, hence, the magnetic field is small-scale. However, in the course of time, the scale of the magnetic field will smoothly increase together with the size of bubbles. This qualitative analysis shows that a smooth change of an external parameter, like the plasma temperature, will result in the smooth variation of a measurable quantity, such as the magnetic field length scale, which is related to the CME. In this consideration, the particle mass is an auxiliary parameter.

Note that the disappearance of the current $\mathbf{J} \parallel \mathbf{B}$ of massive fermions, participating in parity violating interactions, was mentioned for the first time in Ref. [3]. That result was obtained perturbatively by considering the loop contribution to the photon polarization tensor. In the present work we have generalized the finding of Ref. [3] to a more complex physics system, which also accounts for anomalous magnetic moments of massive charged fermions. Moreover, our approach to demonstrate that $\mathbf{J} \parallel \mathbf{B} = 0$ is nonperturbative since it is based on the exact solution of the Dirac equation in the presence of the external fields.

**Acknowledgments**

I am thankful to V. B. Semikoz for useful comments, to the Tomsk State University Competitiveness Improvement Program, RFBR (research project No. 18-02-00149a), and the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS” for a partial support.

**A Calculation of the induced current**

In this appendix, we compute the electric currents of both electrons and positrons. The charged particles are supposed to be massive, have anomalous magnetic moments, and electroweakly interact with background matter under the influence of the constant and homogeneous magnetic field. The computation of the currents is based on the exact solution of the Dirac equation in the external fields. Earlier this equation for an electron was solved in Refs. [17,18] using the standard representation of the Dirac matrices.

Let us suppose that an electron interacts with the unpolarized and nonmoving background matter consisting of neutrons and protons and with the magnetic field, which is along the $z$-axis: $\mathbf{B} = B\mathbf{e}_z$. Then, the Dirac equation, which can be obtained from the Lagrangian in Eq. (11), for an electron wave function $\psi_e$ has the form,

$$i\gamma^\mu \partial_\mu \psi_e = \mathbf{H}\psi_e, \quad \mathbf{H} = (\alpha \mathbf{P}) + \beta m + \mu B\beta \Sigma_3 + V_L P_L + V_R P_R,$$

(15)
where \( P = -i \nabla - eA \) is the canonical momentum operator, \( A = (0, Bx, 0) \) is vector potential in the Landau gauge, \( \alpha = \gamma^0 \gamma, \beta = \gamma^0 \) and \( \Sigma = \gamma^0 \gamma^5 \) are the Dirac matrices. The effective potentials \( V_{L,R} \) of the electroweak interaction with matter are defined in Sec. [2].

Let us look for the solution of Eq. (15) in the form,

\[
\psi_x = \exp (-iEt + ip_y y + ip_z z) \psi_x, \tag{16}
\]

where \( \psi_x = \psi(x) \) is the bispinor which depends on \( x \) and \(-\infty < p_{y,z} < +\infty \). We shall choose the chiral representation of the Dirac matrices [26],

\[
\gamma^\mu = \begin{pmatrix} 0 & -\sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (\sigma_0, -\sigma), \quad \bar{\sigma}^\mu = (\sigma_0, \sigma), \tag{17}
\]

where \( \sigma_0 \) is the unit \( 2 \times 2 \) matrix and \( \sigma \) are the Pauli matrices. Using Eq. (17), we can represent \( \psi_x \) in the form,

\[
\psi_x^T = (C_1 u_{n-1}, i C_2 u_n, C_3 u_{n-1}, i C_4 u_n), \tag{18}
\]

where \( C_i, i = 1, \ldots, 4 \), are the spin coefficients,

\[
u_n(\eta) = \left( \frac{eB}{\pi} \right)^{1/4} \exp \left( -\frac{\eta^2}{2} \right) \frac{H_n(\eta)}{\sqrt{2^n n!}}, \quad n = 0, 1, \ldots, \tag{19}
\]

are the Hermite functions, \( H_n(\eta) \) are the Hermite polynomials, and \( \eta = \sqrt{eB} x + p_y/\sqrt{eB} \).

The energy levels for \( n > 0 \) were found in Refs. [17][18] as

\[
E = \tilde{V} + \lambda \xi, \quad \xi = \sqrt{p_z^2 + 2eBn + m^2 + (\mu B)^2 + V_5^2 - 2sm|S|}, \tag{20}
\]

where

\[
|S| = \frac{1}{m} \sqrt{(p_z V_5 - \mu B m)^2 + 2eBn \left[(\mu B)^2 + V_5^2\right]}, \tag{21}
\]

is the absolute value of the eigenvalue of the spin operator [17][18],

\[
\hat{S} = V_5 \hat{S}_t - \mu B \hat{S}_s, \quad \hat{S}_t = \frac{(\Sigma \cdot P)}{m}, \quad \hat{S}_s = \Sigma_3 - \frac{i}{m} (\gamma \times P)_3, \tag{22}
\]

which commutes with the Hamiltonian \( \hat{H} \) in Eq. (15). In Eq. (20), the discrete quantum number \( s = \pm 1 \) is the sign of \( S \), \( \tilde{V} = (V_L + V_R)/2, V_5 = (V_L - V_R)/2 \), and \( \lambda = \pm 1 \) is the sign of the energy, i.e. the electron energy reads \( E_\epsilon = E(\lambda = +1) = \xi + \tilde{V} \), and the positron energy has the form, \( E_\bar{\epsilon} = -E(\lambda = -1) = \xi - \tilde{V} \).

The spin coefficients for \( n > 0 \) also were found in Refs. [17][18], using the standard representation of the Dirac matrices. Since we choose the chiral representation, which is more convenient for our purposes, we just briefly list the main results. The spin coefficients can be represented in the form,

\[
\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \left( 1 + \frac{p_z V_5 - \mu B m}{m S} \right)^{1/2} \begin{pmatrix} Z & -\mu B/Z \\ \mu B/Z & Z \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},
\]

\[
\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \frac{s}{\sqrt{2}} \left( 1 - \frac{p_z V_5 - \mu B m}{m S} \right)^{1/2} \begin{pmatrix} Z & \mu B/Z \\ -\mu B/Z & Z \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{23}
\]
where 

\[ Z = \left( V_5 + \sqrt{V_5^2 + (\mu B)^2} \right)^{1/2} \]

The new auxiliary coefficients \( A_i, i = 1, 2 \), are completely defined by the following expressions:

\[
A_1^2 = \left( 1 + \frac{mS - V_5^2 - (\mu B)^2}{\mathcal{E}\sqrt{V_5^2 + (\mu B)^2}} \right) C^2, \quad A_2^2 = \left( 1 - \frac{mS - V_5^2 - (\mu B)^2}{\mathcal{E}\sqrt{V_5^2 + (\mu B)^2}} \right) C^2, 
\]

\[
A_1A_2 = -\frac{\mu B p_z + mV_5}{\mathcal{E}\sqrt{V_5^2 + (\mu B)^2}} C^2, 
\]

where \( C \) is the normalization coefficient having the form

\[
C^2 = \frac{1}{4(2\pi)^2 \sqrt{V_5^2 + (\mu B)^2}},
\]

which can be found if we normalize the total wave function as

\[
\int d^3x \psi^\dagger (p_{y,z} \psi_{y,z}^e \gamma \psi_e f(E^e - \chi)) = 0,
\]

where \( f(E) = [\exp(\beta E) + 1]^{-1} \) is the Fermi-Dirac distribution function, \( \chi \) is the chemical potential, and \( \beta = 1/T \) is the reciprocal temperature. If the electron wave functions are treated as secondly quantized objects, we should define the current in Eq. (29) using the normal ordering : \( J_\text{e} \) : to remove the infinite contribution of the vacuum energy. The normal ordering of creation and annihilation operators should be performed accounting for the external fields, i.e. Fock states are defined in the presence of the magnetic field and the electroweak interaction with background matter.

First, we notice that

\[
J_{x,y} \sim \langle \psi_e^\dagger \alpha_{1,2} \psi_e \rangle = 0 \quad \text{since} \quad \alpha_{1,2} = \text{diag}(\sigma_{1,2}, -\sigma_{1,2}) \quad \text{and} \quad \sigma_{1,2} \quad \text{are the nondiagonal Pauli matrices. Indeed, the integration over } x \text{ in Eq. (30) is vanishing for } J_{x,y} \text{ owing to the orthogonality of Hermite functions with different indexes,}
\]

\[
\int_{-\infty}^{+\infty} dx u_n(x) u_{n'}(x) = \delta_{nn'}.
\]
Therefore only the component of the current along the magnetic field $J_z$ should be analyzed.

Then, let us consider the contribution to $J_z$ from the higher energy levels with $n > 0$. In the chiral representation of the Dirac matrices, with help of Eqs. (30) and (31), we have

$$
\int_{-\infty}^{+\infty} dp_y \bar{\psi}_e \gamma^3 \psi_e = eB \left( C_1^2 + C_4^2 - C_2^2 - C_3^2 \right). 
$$

(32)

Using Eqs. (23)-(25), we obtain that

$$
C_1^2 + C_4^2 - C_2^2 - C_3^2 = -4\mu B A_1 A_2 + 2V_5 \frac{\mu B m - V_5 p_z}{mS} \left( A_1^2 - A_2^2 \right)
= \frac{1}{(2\pi)^2} \left[ p_z \left( 1 - \frac{V_5^2}{mS} \right) + V_5 \frac{\mu B}{S} \right].
$$

(33)

Introducing the quantity $R^2 = m|S|$, changing $s \rightarrow -s$, and returning to the vector notations, we get the contribution of the higher energy levels to the current as

$$
J_{e}^{(n>0)} = -\frac{e^2 B}{(2\pi)^2} \sum_{n=1}^{\infty} \sum_{s=\pm 1} \int_{-\infty}^{+\infty} dp_z f\left( E_e - \chi \right) \left( C_2^2 - C_4^2 \right)
= -\frac{e^2 B}{(2\pi)^2} \int_{-\infty}^{+\infty} dp_z f\left( E_e - \chi \right) \frac{p_z + V_5}{\sqrt{(p_z + V_5)^2 + (m - \mu B)^2}} = 0,
$$

(34)

since the integrand is the odd function. One can see in Eq. (35) that the contribution to the electric current from the lowest energy level vanishes. This result is valid for any characteristics of the external fields and charged particles. This is a finding that extends the result of Ref. [25] to the situation when the anomalous magnetic moment of charged particles is taken into account.

The positron wave function can be obtained on the basis of Eq. (16) by applying the charge conjugation operation, $\psi_e = i\gamma^2 \psi^*_e$, and setting $\lambda = -1$ in the energy spectrum. Finally, using Eqs. (16) and (18) we get

$$
\psi^T_e = \exp \left( -iE_e t - ip_y y - ip_z z \right)
\times (-iC_4 u_n, -C_3 v_{n-1}, iC_2 u_n, C_1 u_{n-1}),
$$

(36)

where the coefficients $C_i$ are defined by Eqs. (23)-(25).

The expression for the current of positrons has the form,

$$
J_e = e \sum_{n=0}^{\infty} \sum_{s} \int_{-\infty}^{+\infty} dp_y dp_z \bar{\psi}_e \gamma \psi_e f(\bar{E}_e + \chi).
$$

(37)
Analogously to the electron case, we obtain that the transversal (with respect to $B$) components of $\mathbf{J}$ are equal to zero. Using Eqs. (32), (33), and (36), we get on the basis of Eq. (37) that

$$J_{\parallel}^{(n>0)} = e^2 B \left( \frac{2\pi}{2} \right)^2 \sum_{n=1}^{\infty} \sum_{s=\pm1} \int_{-\infty}^{+\infty} \frac{dz}{z} \left[ p_z \left( 1 + \frac{V_z^2 R^2}{z^2} \right) - \frac{s \mu_B MV_5}{z^2} \right] f(E + \chi_{\text{eff}}).$$

Comparing Eqs. (34) and (38), we can see that the positrons current flows in the opposite direction and has the opposite sign of $\chi_{\text{eff}}$ in the distribution function. Eq. (38) is used in Eq. (2).

The contribution of the lowest energy level with $n = 0$ to $\mathbf{J}$ can be obtained analogously to $\mathbf{J}$ by setting $C_1 = C_3 = 0$ in Eq. (36). One obtains that $J_{\parallel}^{(n=0)} = -J_{\parallel}^{(n=0)} = 0$; cf. Eq. (35).

There is a special case when $m = \mu B$. In this situation, particles at the lowest energy level with $n = 0$ become effectively massless; cf. Eq. (27). The wave function of a “left” electron, which satisfies the normalization condition in Eq. (26), reads

$$\psi_{eL}^{T} = \frac{i u_0}{2\pi} \exp (-i E_{eL} t + ip_y y + ip_z z) \times (0, 0, 0, 1),$$

where the energy level has the form,

$$E_{eL} = p_z + V_L, \quad -V_5 < p_z < +\infty.$$

Analogously for “right” electrons one has,

$$\psi_{eR}^{T} = \frac{i u_0}{2\pi} \exp (-i E_{eR} t + ip_y y + ip_z z) \times (0, 1, 0, 0),$$

and

$$E_{eR} = -p_z + V_R, \quad -\infty < p_z < -V_5.$$

One can see in Eqs. (40) and (42) that the range of the $p_z$ variation becomes not symmetric like in the case of the CME [3–5].

The electric current of electrons can be computed using Eqs. (29)-(31) and (39)-(42) as

$$\mathbf{J} = \frac{e^2 B}{(2\pi)^2} \left[ \int_{-V_5}^{+V_5} dp_z f (p_z + V_L - \chi) - \int_{-\infty}^{+V_5} dp_z f (-p_z + V_R - \chi) \right].$$

Changing the variables $p_z \rightarrow p = p_z - V_5$ and $p_z \rightarrow p = -p_z - V_5$ in the first and the second integrals in Eq. (43) respectively, we can see that $\mathbf{J} = 0$. Analogously we can show that $\mathbf{J} = 0$. Thus, even in this special situation, when $B = m/\mu$, the induced current $\mathbf{J} \parallel B$ is absent.

References

[1] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, Magnetic Fields in Astrophysics (Gordon & Breach Science Publishers, New York, 1983).

[2] M. Stix, The Sun (Springer, Berlin, 2004), 2nd ed., p. 370.

[3] A. Vilenkin, Equilibrium parity-violating current in a magnetic field, Phys. Rev. D 22, 3080–3084 (1980).
[4] H. B. Nielsen and M. Ninomiya, The Adler-Bell-Jackiw anomaly and Weyl fermions in a crystal, Phys. Lett. B 130, 389–396 (1983).

[5] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, The chiral magnetic effect, Phys. Rev. D 78, 074033 (2008), arXiv:0808.3382.

[6] G. Sigl, Astroparticle Physics: Theory and Phenomenology (Atlantis Press, Paris, 2017).

[7] V. Koch, S. Schlichting, V. Skokov, P. Sorensen, J. Thomas, S. Voloshin, G. Wang, and H.-U. Yee, Status of the chiral magnetic effect and collisions of isobars, Chin. Phys. C 41, 072001 (2017), arXiv:1608.00982.

[8] N. P. Armitage, E. J. Mele, and A. Vishwanath, Weyl and Dirac semimetals in three dimensional solids, Rev. Mod. Phys. 90, 15001 (2018), arXiv:1705.01111.

[9] A. Boyarsky, O. Ruchayskiy, and M. Shaposhnikov, Long-range magnetic fields in the ground state of the Standard Model plasma, Phys. Rev. Lett. 109, 111602 (2012), arXiv:1204.3604.

[10] M. Laine and M. Meyer, Standard Model thermodynamics across the electroweak crossover, J. Cosmol. Astropart. Phys. 07 (2015) 035, arXiv:1503.04935.

[11] J. M. Cline, K. Kainulainen, and D. Tucker-Smith, Electroweak baryogenesis from a dark sector, Phys. Rev. D 95, 115006 (2017), arXiv:1702.08909.

[12] M. G. Alford, A. Schmitt, K. Rajagopal, and T. Schäfer, Color superconductivity in dense quark matter, Rev. Mod. Phys. 80, 1455–1515 (2008), arXiv:0709.4635.

[13] M. Dvornikov, Magnetic fields in turbulent quark matter and magnetar bursts, Int. J. Mod. Phys. D 27, 1750184 (2018), arXiv:1612.06540.

[14] V. B. Semikoz and D. D. Sokoloff, Large-scale magnetic field generation by $\alpha$-effect driven by collective neutrino-plasma interaction, Phys. Rev. Lett. 92, 131301 (2004), astro-ph/0312567.

[15] M. Dvornikov, Magnetic field instability driven by anomalous magnetic moments of massive fermions and electroweak interaction with background matter, JETP Lett. 106, 775–779 (2017), arXiv:1704.03403.

[16] A. F. Bubnov, N. V. Gubina, and V. Ch. Zhukovsky, Vacuum current induced by an axial-vector condensate and electron anomalous magnetic moment in a magnetic field, Phys. Rev. D 96, 016011 (2017).

[17] I. A. Balantsev, A. I. Studenikin, and I. V. Tokarev, New solutions to the Dirac equation for particles in a magnetic field and a medium, Phys. Part. Nucl. 43, 727–741 (2012).

[18] I. A. Balantsev, A. I. Studenikin, and I. V. Tokarev, Motion of a charged fermion with an anomalous magnetic moment in magnetized media, Phys. Atom. Nucl. 76, 489–503 (2013).

[19] B. C. Odom, D. Hanneke, B. D’Urso, and G. Gabrielse, New measurement of the electron magnetic moment using a one-electron quantum cyclotron, Phys. Rev. Lett. 97, 030801 (2006); Erratum: Phys. Rev. Lett. 99, 039902 (2007).
[20] M. Dvornikov and V. B. Semikoz, Magnetic field instability in a neutron star driven by the electroweak electron-nucleon interaction versus the chiral magnetic effect, Phys. Rev. D 91, 061301 (2015), arXiv:1410.6676.

[21] A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, Plasma Electrodynamics. Vol. 1: Linear Theory (Pergamon Press, Oxford, 1975).

[22] A. Boyarsky, J. Fröhlich, and O. Ruchayskiy, Self-consistent evolution of magnetic fields and chiral asymmetry in the early universe, Phys. Rev. Lett. 108, 031301 (2012), arXiv:1109.3350.

[23] D. Grabowska, D. B. Kaplan, and S. Reddy, Role of the electron mass in damping chiral plasma instability in supernovae and neutron stars, Phys. Rev. D 91, 085035 (2015), arXiv:1409.3602.

[24] M. Dvornikov, Relaxation of the chiral imbalance and the generation of magnetic fields in magnetars, JETP 123, 967–978 (2016), arXiv:1510.06228.

[25] M. Dvornikov, Role of particle masses in the magnetic field generation driven by the parity violating interaction, Phys. Lett. B 760, 406–410 (2016), arXiv:1608.04940.

[26] C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980), pp. 691–696.

[27] E. S. Fradkin, D. M. Gitman, and Sh. M. Shvartsman, Quantum Electrodynamics with Unstable Vacuum (Springer, Berlin, 1991).