DERHAM–BETTI CLASSES ON PRODUCTS OF ELLIPTIC CURVES OVER
A NUMBER FIELD ARE ALGEBRAIC

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Abstract. A de Rham–Betti class on a smooth projective variety \( X \) over an algebraic extension \( K \) of the rational numbers is a rational class in the Betti cohomology of the analytification of \( X \) that descends to a class in the algebraic de Rham cohomology of \( X \) via the period comparison isomorphism. The period conjecture of Grothendieck implies that de Rham–Betti classes should be algebraic. We prove that any de Rham–Betti class on a product of elliptic curves is algebraic. The case of the power of a CM elliptic curve goes back to Chudnovsky. A key step in our proof consists in showing that any de Rham–Betti isometry between the second cohomology groups of hyper-Kähler varieties, with second Betti number not 3, is Hodge. This builds on the analytic subgroup theorem of Wüstholz, its consequence observed by André and Bost regarding the algebraicity of codimension-1 de Rham–Betti classes on abelian varieties, and results of Deligne and André regarding the Kuga–Satake correspondence. As further result, we obtain a global de Rham–Betti Torelli theorem for K3 surfaces over \( \mathbb{Q} \).

Introduction

De Rham–Betti classes. Let \( X \) be a smooth projective variety defined over a field \( K \subset \mathbb{C} \). Serre’s GAGA and the analytic Poincaré lemma provide a canonical isomorphism

\[
c^n_X : H^n_{dR}(X/K) \otimes_K \mathbb{C} \xrightarrow{\sim} H^n_B(X^\text{an}_C, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},
\]

called the period comparison isomorphism, between the algebraic de Rham cohomology of \( X \) and the Betti cohomology with \( \mathbb{C} \)-coefficients of the analytification \( X^\text{an}_C \) of \( X \). The de Rham–Betti cohomology of \( X \) is defined as the triple

\[
H^n_{dRB}(X, \mathbb{Q}(k)) := (H^n_{dR}(X/K), H^n_B(X^\text{an}_C, \mathbb{Q}), (2\pi i)^{-k}c^n_X).
\]

If \( Z \) denotes an algebraic cycle of codimension \( k \) on \( X \), then its classes in de Rham cohomology and in Betti cohomology are related, up to a sign, by

\[
\frac{1}{(2\pi i)^k} c_X^k (c_{dR}(Z) \otimes_K 1_C) = c_B(Z) \otimes_{\mathbb{Q}} 1_C,
\]

see e.g., [BC16, Prop. 1.1]. A de Rham–Betti class in \( H^n_{dRB}(X, \mathbb{Q}(k)) \), also called Grothendieck class in loc. cit., is a pair \( (\alpha_B, \alpha_{dR}) \) with \( \alpha_B \) a class in \( H^n_B(X^\text{an}_C, \mathbb{Q}) \) and \( \alpha_{dR} \) a class in \( H^n_{dR}(X/K) \) that correspond to one another under \( (2\pi i)^{-k}c_X^k \); see Definitions 4.1 and 4.2. By (2) the Betti class and the de Rham class of an algebraic cycle of codimension \( k \) define a de Rham–Betti class in \( H_{dRB}^{2k}(X, \mathbb{Q}(k)) \). In terms of periods, a de Rham class \( \omega \in H^n_{dR}(X/K) \) can be extended to a de Rham–Betti class in \( H^n_{dRB}(X, \mathbb{Q}(k)) \) if and only if the complex periods

\[
\frac{1}{(2\pi i)^k} \int_{\gamma} \omega
\]

lie in \( \mathbb{Q} \) for every rational homology class \( \gamma \in H_n(X^\text{an}_C, \mathbb{Q}) \).

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In case $K$ is an algebraic extension of $\mathbb{Q}$ inside $\mathbb{C}$, the Grothendieck period conjecture for $X$ stipulates that any polynomial relation with $\mathbb{Q}$-coefficients among the periods should be induced by algebraic cycles on some power of $X$. If $X$ satisfies the Grothendieck period conjecture and if $X$ satisfies the standard conjectures, then every de Rham–Betti class in $H^{2k}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is algebraic, i.e., is the class of an algebraic cycle; see [And04, Prop. 7.5.2.2] and [BC16, Prop. 2.13]. The only non-trivial case for which the Grothendieck period conjecture is known is for CM elliptic curves [Chu80]; this gives in particular the algebraicity of de Rham–Betti classes on powers of CM elliptic curves. On the other hand, the algebraicity of de Rham–Betti classes has so far been established in the following cases: for codimension-$1$ classes on abelian varieties [And04, Bos13, BC16, Wüs84] and, via the Kuga–Satake correspondence of André [And96a], for codimension-$1$ classes on hyper-Kähler varieties [BC16]; see Proposition 4.10. Here a hyper-Kähler variety over $K \subseteq \mathbb{C}$ means a variety over $K$ whose base-change to $\mathbb{C}$ is projective, irreducible holomorphic symplectic and, deviating from the usual definition, is such that its second Betti number satisfies $b_2 > 3$. (This latter condition, which holds for all known deformation families of hyper-Kähler varieties, ensures that the deformation space of a hyper-Kähler variety is big enough and is crucial in André’s work [And96a].) As a particular instance of the above, any de Rham–Betti class should be a Hodge class. Even this latter expectation is wide open and we are not aware of any examples beyond the ones mentioned above for which this expectation is met.

De Rham–Betti classes on products of elliptic curves. Our main result is the following extension of the consequence of Chudnovsky’s theorem concerned with the algebraicity of de Rham–Betti classes on powers of CM elliptic curves.

**Theorem 1.** Let $X$ be an abelian variety over an algebraic extension $K$ of $\mathbb{Q}$ inside $\mathbb{C}$ and assume that $X_{\overline{\mathbb{Q}}}$ is isogenous to a product of elliptic curves over $\mathbb{Q}$. Then:

(i) any de Rham–Betti class in $H^{2k}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is algebraic;
(ii) any de Rham–Betti class in $H^j_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is zero for $j \neq 2k$.

We refer to Theorem 7.1 for a more precise version. Theorem 1 is the analogue in the de Rham–Betti setting of the following results.

(a) The Hodge conjecture holds for products of complex elliptic curves. Its proof essentially goes back to Tate (unpublished); see [Lew99, §B.3] for a proof and further references.

(b) The Tate conjecture holds for products of elliptic curves over a finitely generated extension of $\mathbb{Q}$. This follows from the validity of the Mumford–Tate conjecture for products of elliptic curves. The latter is established in [Lom16, Cor. 1.2] and builds on the validity of the Mumford–Tate conjecture for elliptic curves due to Serre [Ser68].

(c) The Tate conjecture holds for products of elliptic curves over a finite field. This is due to Spieß [Spi99].

In contrast to the case of Hodge classes and Tate classes, it is a priori not known, for lack of a theory of weights in the de Rham–Betti setting, that $H^j_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ does not support any non-zero de Rham–Betti class for $j \neq 2k$. Theorem 1(ii) confirms that this is indeed the case for products of elliptic curves.

De Rham–Betti isometries between hyper-Kähler varieties. Recall that the second cohomology group of hyper-Kähler varieties comes equipped with a canonical quadratic form called the Beauville–Bogomolov form. In the case of K3 surfaces, this form coincides with the cup-product pairing. If $X$ is a hyper-Kähler variety over $K$, we denote $T^2_{\mathrm{dRB}}(X, \mathbb{Q})$ its de Rham–Betti transcendental cohomology; it is the orthogonal complement to the subspace spanned by classes of divisors. The following theorem is an essential step towards proving Theorem 1. It will be used in the special case where $X$ is the Kummer surface associated to the square of an elliptic curve.
**Theorem 2** (special instance of Theorem 5.1). Let $X$ and $X'$ be hyper-Kähler varieties over an algebraic extension $K$ of $\mathbb{Q}$ inside $\mathbb{C}$. Then any de Rham–Betti isometry
\[ T^2_{dRB}(X, \mathbb{Q}) \sim T^2_{dRB}(X', \mathbb{Q}) \]
is Hodge.

As an application, we obtain:

**Theorem 3** (Global de Rham–Betti Torelli theorem for K3 surfaces over $\overline{\mathbb{Q}}$; see Theorem 5.4). Let $S$ and $S'$ be two K3 surfaces over $\overline{\mathbb{Q}}$. If there is an integral de Rham–Betti class in $H^4_{dRB}(S \times S', \mathbb{Z}(2))$ inducing an isometry
\[ H^3_{dRB}(S, \mathbb{Z}) \sim H^3_{dRB}(S', \mathbb{Z}), \]
then $S$ and $S'$ are isomorphic.

Our proof of Theorem 5.1 makes essential use of André’s theory of motivated cycles [And96b]: we indeed show that de Rham–Betti isometries as in Theorem 5.1 are motivated and hence (absolute) Hodge. It is inspired by André’s proof [And96a] of the Mumford–Tate conjecture for K3 surfaces, but differs from it in several aspects as we have to avoid representation-theoretic arguments and base-change of the field of coefficients; see the introduction to §5 for more details. A common and important feature is the Kuga–Satake construction and the proof due to André that the Kuga–Satake correspondence descends to $\overline{\mathbb{Q}}$ and is motivated. The main result that we use is summed up in Theorem 3.3.

**Strategy of proof of Theorem 1.** First we may assume that $K = \overline{\mathbb{Q}}$; see Lemma 4.4. Our proof consists in showing that the Tannakian torsor of periods of $A$ agrees with its torsor of motivic periods; see Definitions 4.12 and 4.14. This is a special instance of the more general period conjecture of Grothendieck (Conjecture 4.15). This Tannakian torsor of periods is a torsor under a $\mathbb{Q}$-algebraic group that we call the de Rham–Betti group and denote $G_{dRB}(A)$. In Theorem 4.9, we observe that, for an abelian variety $A$ over $\overline{\mathbb{Q}}$, the group $G_{dRB}(A)$ is reductive as a consequence of Wüstholz’ analytic subgroup theorem 4.7. Consequently, it can be defined as the $\mathbb{Q}$-subgroup of $\text{GL}(H^1(A^n, \mathbb{Q}))$ that fixes all de Rham–Betti classes in $H^j_{dRB}(A^n, \mathbb{Q}(k))$ for all $j, n \geq 0$ and all $k \in \mathbb{Z}$. This group is the analogue in the de Rham–Betti setting of the Mumford–Tate group $\text{MT}(A)$ of $A$, but in contrast is neither known to be connected nor to contain the torus $G_m$ acting via homotheties (theory of weights). Nevertheless, by Deligne’s theorem [Del82], Hodge classes on abelian varieties are absolute Hodge and hence de Rham–Betti so that there is in this case a natural inclusion $G_{dRB}(A) \subseteq \text{MT}(A)$.

If now $A$ is isogenous to a product of elliptic curves, then we know that Hodge classes on powers of $A$ are algebraic. We are therefore reduced to showing that $G_{dRB}(A) = \text{MT}(A)$. In Theorem 7.1, we reduce this to showing that $G_{dRB}(E) = \text{MT}(E)$ for a single elliptic curve $E$. The latter is established in Theorem 6.5. In both cases, the arguments are group-theoretic (reductive groups and their invariants) and notably involve knowing that de Rham–Betti isometries
\[ T^2_{dRB}(E_1 \times E_1, \mathbb{Q}) \sim T^2_{dRB}(E_2 \times E_2, \mathbb{Q}) \]
are algebraic for any two elliptic curves $E_1$ and $E_2$; this is Theorem 2 applied to the Kummer surfaces associated to $E_1 \times E_1$ and $E_2 \times E_2$. Specifically, Theorem 2 is used in the proof of Theorem 6.5 to discard the non-connected candidates for $G_{dRB}(E)$ and it is used in the proof of Theorem 7.1 to determine $G_{dRB}(E_1 \times \mathbb{Q} E_2)$ for the product of two non-isogenous elliptic curves without CM. Finally, regarding the proof of $G_{dRB}(E) = \text{MT}(E)$, we note that this follows in the case of a CM elliptic curve $E$ from Chudnovsky’s theorem [Chu80] on the periods of elliptic curves. We obtain however a proof based on Wüstholz’ analytic subgroup theorem, which only deals with linear relations among periods. In the case of a non-CM elliptic curve, we use our
Theorem 2 and proceed in several steps in order to avoid the issue that there is no theory of weights in the de Rham–Betti setting. Namely, we first compute the de Rham–Betti group of $T^2_{\text{dR}}(E \times E, \mathbb{Q}(1))$, then that of $T^2_{\text{dR}}(E \times E, \mathbb{Q})$ to finally obtain that $G_{\text{dR}}(E) = MT(E)$.

1. Generalities on Tannakian categories

Our main references concerning Tannakian categories are [DM82, SR72], and [And04, §2.3] for a quick overview.

1.1. Tannakian categories. Let $F$ be a field. Suppose that $\mathcal{T}$ is a rigid $\otimes$-category, that $\mathcal{T}$ is abelian, and that $\text{End}_\mathcal{T}(1) = F$. A fiber functor on $\mathcal{T}$ is an exact and faithful $\otimes$-functor $\omega : \mathcal{T} \to \text{Vec}_K$ to the rigid $\otimes$-category $\text{Vec}_K$ of finite-dimensional vector spaces over a finite field extension $K$ of $F$. If such a fiber functor exists, we say that $\mathcal{T}$ is a Tannakian category. Given a Tannakian category $\mathcal{T}$, equipped with a fiber functor $\omega$, one can define its Tannakian fundamental group $\text{Aut}^\otimes \omega$; it is the affine group scheme over $K$ such that for all field extensions $K'/K$ the group $(\text{Aut}^\otimes \omega)(K')$ is the automorphism group of the extended $\otimes$-functor $\omega_{K'} : \mathcal{T} \to \text{Vec}_{K'}$.

1.2. Tannakian subcategories. A Tannakian subcategory of a Tannakian category $\mathcal{T}$ is a full subcategory $\mathcal{T}'$ of $\mathcal{T}$, stable under $\otimes$ and duals, and such that any quotient object in $\mathcal{T}'$ of an object in $\mathcal{T}$ is in $\mathcal{T}'$. The restriction of a fiber functor $\omega : \mathcal{T} \to \text{Vec}_K$ to $\mathcal{T}'$ induces by pull-back a faithfully flat homomorphism of $K$-group schemes $\text{Aut}^\otimes(\omega) \to \text{Aut}^\otimes(\omega_{\mathcal{T}'})$. In case $X$ is an object of $\mathcal{T}$, we denote $\langle X \rangle$ the Tannakian subcategory of $\mathcal{T}$ generated by $X$; its objects are the quotients of “spaces of tensors” $\bigoplus_{\text{finite}} X^\otimes m \otimes (X^\vee)^{\otimes m}$.

1.3. Neutral Tannakian categories. If there exists a fiber functor $\omega$ with $F = K$, we say that $\omega$ is neutral and that $\mathcal{T}$ is a neutral Tannakian category. Let $(\mathcal{T}, \omega : \mathcal{T} \to \text{Vec}_F)$ be a neutral Tannakian category with Tannakian fundamental group $G$. Then the functor $\omega : \mathcal{T} \to \text{Rep}_F G$ defined by the fiber functor $\omega : \mathcal{T} \to \text{Vec}_F$ is an equivalence of categories. For $X$ an object of $\mathcal{T}$, the Tannakian fundamental group $G_X := \text{Aut}^\otimes(\omega|_{\langle X \rangle})$ is then a closed $K$-subgroup of $\text{GL}(\omega(X))$.

1.4. Semi-simple neutral Tannakian categories. Finally, a neutral Tannakian category $(\mathcal{T}, \omega)$ is semi-simple if and only if its Tannakian fundamental group $G = \text{Aut}^\otimes \omega$ is pro-reductive, i.e., an inverse limit of (not necessarily connected) reductive groups. In that case, for $X$ any object of $\mathcal{T}$, $G_X := \text{Aut}^\otimes(\omega|_{\langle X \rangle})$ is reductive and it is the closed subgroup of $\text{GL}(\omega(X))$ that fixes elements in $\omega\left(\text{Hom}_\mathcal{T}(1, M)\right)$ inside $\text{Hom}_F(F, \omega(M))$ for all spaces of tensors $M = \bigoplus_{\text{finite}} X^\otimes m \otimes (X^\vee)^{\otimes m}$.

2. Motivated cycles and André motives

2.1. Motivated cycles and André motives: definitions and properties. The notion of motivated cycle was introduced by Yves André in [And96b]. Let $K \subseteq \mathbb{C}$ be a subfield of the complex numbers and let $X$ be a smooth projective variety over $K$. A motivated cycle on $X$ is an element in $H^r_B(X^\text{an}, \mathbb{Q}(r))$ of the form $p_X \ast x, \alpha \cdot \ast_L \beta$, where $Y$ is an arbitrary smooth projective variety over $K$, $\alpha$ and $\beta$ are algebraic cycles on $X \times_K Y$, and $\ast_L$ is the (inverse of the) Lefschetz isomorphism attached to polarization choices on $X$ and $Y$. As detailed in [And96b, §2], the motivated cycles define a graded $\mathbb{Q}$-subalgebra of $\bigoplus_r H^r_B(X^\text{an}, \mathbb{Q}(r))$ that contains the classes of algebraic cycles and that is stable under pullbacks and pushforwards along morphisms of smooth projective varieties. As such, one can define motivated correspondences and their compositions. By replacing algebraic correspondences with motivated correspondences in the construction of pure motives (as outlined in [And04, §4]), one obtains a pseudo-abelian rigid $\otimes$-category over $\mathbb{Q}$. From the fact that inverses to the Lefschetz isomorphisms are motivated, one obtains that the Künneth projectors are motivated. This provides a grading on the above category and after changing the commutativity constraint along the Koszul rule of signs we obtain the André category of motives $M^\text{And}_K$ over $K$. Its objects have the form $M = p_h(X)(n) = (X, p, n)$, where $X$ is a...
smooth projective variety over $K$ of dimension $d_X$, $p$ is a motivated idempotent correspondence in $H_B^{2d_X}((X \times_K X)^{an}, \mathbb{Q}(d_X))$ and $n$ is an integer. The unit object is $1 := \mathfrak{h}(\text{Spec} K)$ and with the above formalism the space of motivated cycles on $M$ is $\text{Hom}_{\text{Mot}}(1, M)$.

**Theorem 2.1** (André [And96b, §4]). The category $M^\text{And}_K$ is graded Tannakian semi-simple over $\mathbb{Q}$, neutralized by the fiber functor given by the Betti cohomology realization functor

$$\omega_B : M^\text{And}_K \to \text{Vec}_{\mathbb{Q}}, \quad M := (X, p, n) \mapsto H_B^*(M) := p_*H_B^*(X^{an}, \mathbb{Q}(n)).$$

The grading of an object $M = \bigoplus_{k \in \mathbb{Z}} M^k$ is such that $H_B^*(M^k) = H_B^*(M)$. We write

$$\mathfrak{h}(X) = \bigoplus_{0 \leq k \leq 2d_X} \mathfrak{h}_k(X)$$

for the grading of the André motive of $X$. The various realization functors attached to Weil cohomology theories provide other fiber functors. Under the $\ell$-adic realization functor $\omega_\ell : M^\text{And}_K \to \text{Vec}_{\mathbb{Q}}$, motivated cycles are invariant under the action of the Galois group $\text{Gal}(\overline{K}/K)$. Under the de Rham realization functor $\omega_{dR} : M^\text{And}_K \to \text{Vec}_{\mathbb{C}}$, motivated cycles are mapped to classes in $F^0$ for the Hodge filtration. In addition these are compatible with the canonical comparison isomorphisms, so that motivated cycles are absolute Hodge in the sense of Deligne [Del82] (and in particular Hodge) and they are also de Rham–Betti (see Definition 4.2 below). Since the Hodge conjecture is known for codimension-1 cycles (Lefschetz (1,1)-theorem), we see that any motivated cycle of codimension 1 on $X$ is algebraic.

### 2.2. The motivated Galois group and the Mumford–Tate group

Using the Tannakian formalism, André makes the following

**Definition 2.2** (Motivated Galois group). Given a André motive $M$ over $K$, its motivated Galois group $G^\text{And}_M(K)$ is the Tannakian fundamental group

$$G^\text{And}_M(K) := \text{Aut}^\otimes(\omega_B|_{(M)}).$$

Since the neutral Tannakian category $M^\text{Neutral}_K$ is semi-simple, the motivated Galois group $G^\text{And}_M(K)$ is reductive. In addition, $G^\text{And}_M(K)$ is the closed subgroup of $\text{GL}(\omega_B(M))$ that fixes motivated classes inside spaces of tensors $\bigoplus_{\text{finite}} M^\otimes_{n_1} \otimes (M')^\otimes_{m_1}$.

Recall that a pure rational Hodge structure $H$ consists of a finite-dimensional $\mathbb{Q}$-vector space $H$ together with a decomposition $H \otimes \mathbb{C} = \bigoplus_{i,j} H^{i,j}$ such that $H^{i,j} = H^{j,i}$, where complex conjugation on $H \otimes \mathbb{C}$ acts on the second factor. Equivalently, $H$ is a finite-dimensional $\mathbb{Q}$-vector space equipped with a representation $\rho : \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}} \to H \otimes \mathbb{R}$ of the Deligne torus; the pieces $H^{i,j}$ then corresponding to the eigenspace for the character $z^i \overline{z}^j$. The *Mumford–Tate group* $MT(H)$ of $H$ is the smallest $\mathbb{Q}$-subgroup of $\text{GL}(H)$ that contains the image of the representation $\rho$. The category of pure Hodge structures naturally defines a neutral Tannakian category $HS$ with fiber functor the forgetful functor $\omega$ associating to $H$ the underlying $\mathbb{Q}$-vector space. A *Hodge class* in $H$ is an element of $H$ that lies in $H^{0,0}$ after base-change to $\mathbb{C}$; in other words, it is an element of $\text{Hom}_{HS}(\mathbb{Q}, H)$, where $\mathbb{Q}$ is the trivial Hodge structure with grading concentrated in bidegree $(0, 0)$. The Mumford-Tate group of $H$ can then be described as

$$MT(H) = \text{Aut}^\otimes(\omega|_{(H)}).$$

In case $H$ is polarizable, $(H)$ is semi-simple and hence $MT(H)$ is reductive and is the closed subgroup of $\text{GL}(H)$ that fixes all Hodge classes in spaces of tensors $\bigoplus_{\text{finite}} H^\otimes_{n_1} \otimes (H')^\otimes_{m_1}$.

Since motivated cycles are Hodge, the Betti realization functor $\omega_B : M^\text{And}_K \to \text{Vec}_{\mathbb{Q}}$ factors through $\omega : HS \to \text{Vec}_{\mathbb{Q}}$. Moreover the Hodge structure associated to a André motive is polarizable. It follows that for a André motive $M$ we have an inclusion $MT(M) \subseteq G^\text{And}_M(M)$ of reductive groups. For future reference, we then have
Proposition 2.3. The inclusion $\text{MT}(M) \subseteq G_{\text{And}}(M)$ is an equality if and only if Hodge classes in spaces of tensors $\bigoplus_{\text{finite}} M^\otimes n_i \otimes (M^\vee)^\otimes m_i$ are motivated.

Proof. This follows at once from the facts that $\text{MT}(M)$ is the closed subgroup of $\text{GL}(\omega_B(M))$ that fixes all Hodge classes in spaces of tensors while $G_{\text{And}}(M)$ is the closed subgroup of $\text{GL}(\omega_B(M))$ that fixes all motivated cycles in spaces of tensors.

2.3. The case of abelian varieties. Deligne [Del82] famously proved that any Hodge cycle on a complex abelian variety is absolute Hodge. André [And96b] established the following generalization:

Theorem 2.4 (André [And96b]). Let $A$ be a complex abelian variety. Any Hodge cycle in $H^{2k}(A, \mathbb{Q}(k))$ is motivated and $\text{MT}(A) := \text{MT}(h(A)) = G_{\text{And}}(h(A)) = G_{\text{And}}(A)$. □

3. The Kuga–Satake correspondence

We review the Kuga–Satake construction and André’s proof [And96a] that the Kuga–Satake correspondence is motivated and descends to $\overline{K}$. The purpose is to fix notation and to state Theorem 3.3 which will be crucial in our proof of Theorem 2; see Theorem 5.1.

3.1. The Kuga–Satake construction. Let $R$ be a commutative ring, e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Let $V_R$ be a free $R$-module of finite rank equipped with a symmetric bilinear form $Q : V_R \times V_R \rightarrow R$.

The associated Clifford algebra $C(V_R)$ is the quotient of the tensor algebra $T(V_R)$ by the two-sided ideal generated by $v \otimes v - Q(v,v)$, for all $v \in V_R$; it admits a natural $\mathbb{Z}/2\mathbb{Z}$-grading and we have

$$C(V_R) = C^+(V_R) \oplus C^-(V_R).$$

The sub-algebra $C^+(V_R)$ is called the even Clifford algebra. The even Clifford group is

$$\text{CSpin}^+(V_R) := \{ \gamma \in C^+(V_R) \times | \gamma V \gamma^{-1} = V \}.$$

Each element $\gamma \in \text{CSpin}^+(V_R)$ induces an $R$-linear homomorphism $\tau(\gamma) : V_R \rightarrow V_R$, $v \mapsto \gamma v \gamma^{-1}$. One easily checks that the homomorphism $\tau(\gamma)$ preserves the bilinear form $Q$ and hence provides a group homomorphism

$$\tau : \text{CSpin}^+(V_R) \rightarrow \text{SO}(V_R).$$

The above constructions are compatible with scalar extensions $R \rightarrow R'$. As such $\text{CSpin}^+(V_R)$ has the natural structure of an algebraic group over $R$ making $\tau$ a surjective homomorphism of $R$-groups with kernel given by $\mathbb{G}_{m,R}$.

Now we consider the special case $R = \mathbb{Z}$. Assume that $V_\mathbb{Z}$ is equipped with a Hodge structure of K3 type, that is, a Hodge structure of type $(-1,1) + (0,0) + (1,-1)$ such that $\dim V^{-1,1} = 1$. Assume that the integral bilinear form $Q : V_\mathbb{Z} \otimes V_\mathbb{Z} \rightarrow \mathbb{Z}$ is a homomorphism of Hodge structures. Then the Hodge structure of $V$ is given by

$$h : S = \text{Res}^{C/R}(C^*) \rightarrow \text{SO}(V_R)$$

and it lifts uniquely to a homomorphism

$$\tilde{h} : S \rightarrow \text{CSpin}^+(V_R)$$

such that the restriction of $\tilde{h}$ to $\mathbb{G}_m$ is the natural inclusion of $\mathbb{G}_m$ in $\text{CSpin}^+(V_R)$. The action of $\text{SO}(V_R)$ on the tensor algebra $T(V_R)$ induces an action on the quotient $C(V)$ and hence also on $C^+(V_R)$. Thus $h$ induces a Hodge structure of weight 0 on $C^+(V_\mathbb{Z})$. This Hodge structure also has type $(-1,1) + (0,0) + (1,-1)$. 
Now let $L_Z$ be a free (left) $C^+(V_Z)$-module of rank one. Then $L_Z$ is naturally a $\text{CSpin}^+(V_Z)$-module via left multiplication. After base change to $\mathbb{R}$ and composing with $\tilde{h}$, we get a group homomorphism

$$S \longrightarrow \text{GL}(L_{\mathbb{R}})$$

that defines a weight one Hodge structure on $L_Z$.

Now we assume that $Q$ has signature $(2,N)$ and the restriction of $Q$ to the real part of $V^{-1,1} \oplus V^{1,-1}$ is positive definite. In other words, $-Q$ is a polarization on $V_{\mathbb{R}}$. Under these assumptions, the weight 1 Hodge structure on $L_Z$ is polarizable (see below) and hence we get an abelian variety $A$ such that $H^1(A, \mathbb{Z}) = L_Z$. This abelian variety is called the Kuga–Satake variety attached to $V_{\mathbb{Z}}$ and is denoted $\text{KS}(V_{\mathbb{Z}})$; it has dimension $2^k v_{\mathbb{Z}} - 2$.

**Example 3.1** (The Kuga–Satake construction for complex hyper-Kähler varieties). Let $X/C$ be a hyper-Kähler variety of dimension $2n$ with ample divisor class $\eta$. The second cohomology group $H := H^2(X, \mathbb{Z}(1))$ is naturally endowed with a Hodge structure of type $(-1,1) + (0,0) + (1,-1)$ such that $\dim H^{-1,1} = 1$ and it comes equipped with two quadratic forms: the one induced by the polarization $\eta$, namely $(\alpha, \beta) \mapsto \int_X \alpha \cup \beta \cup \eta^{2n-2}$, and the Beauville–Bogomolov form

$$\mathfrak{B} : P^2(X, \mathbb{Z}(1)) \times P^2(X, \mathbb{Z}(1)) \longrightarrow \mathbb{Z}.$$

We denote $P^2(X, \mathbb{Z}(1))$ the primitive cohomology of $X$; by definition it is the orthogonal complement of $\mathfrak{B}$ in $H^2(X, \mathbb{Z}(1))$. It is a fact that $\mathfrak{B}$ and $P^2(X, \mathbb{Z}(1))$ are orthogonal with respect to the Beauville–Bogomolov form, and that the polarization form and the Beauville–Bogomolov form restricted to $\langle \mathfrak{B} \rangle$ (resp. to $P^2(X, \mathbb{Z}(1))$) differ by a positive rational multiple. Since the restriction of the polarization form on $P^2(X, \mathbb{Z}(1))$ has signature $(2,N)$, the above Kuga–Satake construction can be carried out for $V_{\mathbb{Z}} = P^2(X, \mathbb{Z}(1))$ equipped with the Beauville–Bogomolov bilinear form. The corresponding Kuga–Satake variety is denoted $\text{KS}(X)$. The Kuga–Satake variety constructed in [And96a] is the one obtained from $V_{\mathbb{R}} = P^2(X, \mathbb{Z}(1))$ equipped with the polarization form. This gives however rise to isogenous Kuga–Satake varieties. Indeed, let $Q' = cQ$, $c \in \mathbb{Q}_{>0}$. Then we can define an isomorphism $T^+V \to T^+V$, which is multiplication by $c^k$ on $T^{2k}V$. Then this induces an isomorphism between $C^+(V_{\mathbb{Q}}, Q')$ and $C^+(V_{\mathbb{Q}}, Q)$ and hence an isogeny between the associated Kuga–Satake varieties.

**Example 3.2** (The transcendental cohomology of a hyper-Kähler variety). Let $X/C$ be a hyper-Kähler variety of dimension $2n$ with ample divisor class $\eta$. Denote $\text{NS}(X)$ the image of the cycle class map $\text{CH}^1(X) \to H^2(X, \mathbb{Z}(1))$ and denote $\rho$ its rank. The second transcendental cohomology group of $X$ is by definition the orthogonal complement of $\text{NS}(X)$ inside $H^2(X, \mathbb{Z}(1))$, that is,

$$T^2(X, \mathbb{Z}(1)) := \text{NS}(X)^\perp.$$

Here the orthogonal complement is with respect to either the polarization form or the Beauville–Bogomolov form; it doesn’t matter which, according to Example 3.1. Moreover, the Beauville–Bogomolov form and the polarization form differ by a positive rational multiple when restricted to $T^2(X, \mathbb{Z}(1))$ and have signature $(2,M)$. The Kuga–Satake variety of $X$ is then isogenous to the $2^\rho$-th power of the Kuga–Satake variety associated to $T^2(X, \mathbb{Z}(1))$.

Let $C^+$ be the opposite ring $(\text{End}_{C^+(V_Z)} L_Z)^{\text{op}}$ of the ring of $C^+(V_Z)$-linear endomorphisms of $L_Z$. Note that $L_Z$ admits a right-action of $C^+$ that respects the Hodge structure of $L_Z$; in particular, $C^+$ is endowed with the trivial Hodge structure. More precisely, $C^+$ is a subring of $\text{End}(A)$ with action on $L_Z$ given by pull back of cohomology classes. The choice of a generator of $L_Z$ provides an isomorphism $L_Z \simeq C^+(V_Z)$ and also an induced isomorphism $C^+ \simeq C^+(V_Z)$ of algebras (but not of Hodge structures).

Note that there is a canonical algebra isomorphism

$$\psi : C^+(V_Z) \cong \text{End}_{C^+} L_Z$$
which is also an isomorphism of Hodge structures.

A polarization on $L_Z$ can be given as follows. Pick a generator of $L_Z$ and consider the induced identifications $C^+(V_Z) = L_Z = C^+$ of $\mathbb{Z}$-modules. Satake [Sat66, Prop. 3] gave all possible polarizations and one special case can be described as follows. Pick $f_1$ and $f_2$ in $V_Z$ with $Q(f_1, f_1) > 0$, $Q(f_2, f_2) > 0$ and $Q(f_1, f_2) = 0$ and let $a = f_1 f_2 \in C^+$. Then the skew symmetric bilinear form

$$\varphi_a : L_Z \times L_Z \rightarrow \mathbb{Z}(-1), \quad (x, y) \mapsto \pm \text{tr}_{C(V)}(ax^t y)$$

defines a polarization on $L_Z$.

Let $V_Z$ and $Q$ be as above. We can also define a new lattice

$$V^\# := V_Z \oplus \mathbb{Z}(0), \quad Q^\# := Q \oplus \langle -1 \rangle$$

and run the Kuga–Satake construction for $(V^\#, Q^\#)$. In this way, we also get an abelian variety $A^\#$ and $L^\# = H^1(A^#, Q)$. We will call this the #-construction; see [And96a, Variant 4.1.3].

### 3.2. The Kuga–Satake correspondence is motivated

In this section, we discuss the above Kuga–Satake construction in the setting of André motives. We focus on the special case of hyper-Kähler varieties.

Let $X/K$ be a hyper-Kähler variety with an ample divisor class $\eta$. Let $v := p^2(X)(1) \subset h^2(X)(1)$ (resp. $v := t^2(X)(1) \subset h^2(X)(1)$) be the primitive submotive (resp. transcendental submotive), whose Betti realization is simply $V := P^2(X, Q(1))$ (resp. $V = T^2(X, Q(1))$). The Beauville–Bogomolov bilinear form restricted to $V$ is a rational multiple of the one defined by the power of $\eta$ and hence it is motivated. In other words, we have a morphism

$$Q : v \otimes v \rightarrow 1$$

of André motives, which induces the Beauville–Bogomolov form $\mathfrak{B}$ under Betti realization.

Now we define the even Clifford algebra motive $C^+(v, Q)$. Let

$$\mu = (\text{Id} + \sigma, -2Q) : v \otimes v \rightarrow v \otimes v \oplus 1$$

be the morphism of André motives, where $\sigma$ is the automorphism of $v \otimes v$ that switches the two factors. In other words, the Betti realization of $\mu$ is given by $(u, v) \mapsto u \otimes v + v \otimes u - 2\mathfrak{B}(u, v)$. For each pair of integers $a, b \geq 0$, we define

$$\mu_{a,b} = \text{Id}_{v^a} \otimes \mu \otimes \text{Id}_{v^b} : v^a \otimes v^b \otimes v^a(v^b) \rightarrow v^{a+b+2} \otimes v^a(v^b).$$

Let $T^+(v) := \bigoplus_{n \geq 0} v^\otimes 2n$ be the even tensor algebra of $v$, which is viewed as a formal direct sum. Let $i \subset T^+(v)$ be the image of $\sum \mu_{a,b}$ where $a$ and $b$ run through all non-negative integers such that $a+b$ is even. It turns out that $C^+(v, Q) := T^+(v)/i$ is a well-defined object in $M^K_{\text{And}}$. Indeed, let $T^+_n(v) = \bigoplus_{0 \leq i \leq n} v^\otimes 2i$. Then we have a morphism $T^+_n(v) \rightarrow C^+(v, Q)$ and let $i_n$ be the kernel of this morphism. One checks that $C^+(v, Q) = T^+_n(v)/i_n$ for all $n \geq \frac{\dim V}{2}$. Let $F_n C^+(v, Q)$ be the image of $T^+_n(v)$ in $C^+(v, Q)$ and this defines an increasing filtration on $C^+(v, Q)$.

Let $A$ be the Kuga–Satake abelian variety associated to $P^2(X, Z(1))$ (resp. $T^2(X, Z(1))$). Then $A$ is defined over some finite extension $K'$ of $K$; see [And96a, Lem. 6.5.1]. Hence $L$ is the Betti realization of $h^1(A) \in M^K_{\text{And}}$.

Note that $C^+ \subset \text{End}(A_C)$. We may enlarge $K'$ and assume that all endomorphisms of $A_C$ are actually defined over $K'$. This gives rise to a subalgebra object $C^+_Q \subset \text{End}(h^1(A))$, where $C^+_Q$ is the motive associated to the algebra $C^+_Q$ and $\text{End}(h) = h^V \otimes h$ is the internal endomorphism object. Hence we get the subalgebra object

$$\text{End}_{C^+}(h^1(A)) \subset \text{End}(h^1(A)).$$
Let $M$ and $N$ be two André motives. A homomorphism $H^*_p(M) \to H^*_q(N)$ is said to be *motivated* if it is induced by a morphism $M \to N$ of motives. The following theorem is essentially due to André [And96a].

**Theorem 3.3.** Let $X$ be a hyper-Kähler variety defined over a field $K \subseteq \mathbb{C}$ with an ample class $\eta$. Let $V_Z := \mathbb{P}^2(X, \mathbb{Z}(1))$ (resp. $V_\mathbb{Q} := \mathbb{T}^2(X, \mathbb{Z}(1))$) be the primitive cohomology (resp. the transcendental cohomology) and let $A := \text{KS}(V_Z)$ be the associated Kuga–Satake variety. We write $V = V_Z \otimes \mathbb{Q}$ and $L = H^1(A, \mathbb{Q})$. Then there exists a finite extension $K'/K$ such that the following statements are true.

1. The abelian variety $A$ is defined over $K'$.
2. The canonical algebra isomorphism
   \[ \psi : C^+(V) \cong \text{End}_{C^+}(L) \]
   is motivated and descends to $K'$.
3. There is a canonical algebra isomorphism
   \[ \text{End}(L) \cong (\text{End}_{C^+} L) \otimes (C^+)^\text{op} \]
   which is motivated and descends to $K'$.
4. The Kuga–Satake correspondence $C^+(V) \hookrightarrow \text{End}(L)$, which corresponds to the inclusion
   \[ \text{End}_{C^+} L \hookrightarrow (\text{End}_{C^+} L) \otimes (C^+)^\text{op}, \quad \alpha \mapsto \alpha \otimes 1, \]
   is motivated and descends to $K'$.

**Proof.** We will prove for the case of $\text{KS}(\mathbb{P}^2(X, \mathbb{Z}(1)))$. Note that the $\text{KS}(\mathbb{P}^2(X, \mathbb{Z}(1)))$ is isogenous to a self-product of $\text{KS}(\mathbb{T}^2(X, \mathbb{Z}(1)))$ and hence the statements also holds for $\text{KS}(\mathbb{T}^2(X, \mathbb{Z}(1)))$.

Statement (1) for $\text{KS}(\mathbb{P}^2(X, \mathbb{Z}(1)))$ follows from [And96a, Lem. 6.5.1]. We choose $K'$ such that all endomorphisms of $A_C$ are defined over $K'$. Statement (2) follows from [And96a, Prop. 6.2.1 and Lem. 6.5.1]. For the remaining part of the proof, all motives are considered as defined over $K'$. It follows from (2) that we have an algebra isomorphism $\tilde{\psi} : C^+(\mathfrak{h}, \mathbb{Q}) \to \text{End}_{C^+}(\mathfrak{h}^1(A))$ which induces $\psi$. Note that both $C^+_Q$ and $\text{End}_{C^+}(\mathfrak{h}^1(A))$ are subalgebra objects of $\text{End}(\mathfrak{h}^1(A))$. Hence we have a canonical algebra homomorphism

\[ (\text{End}_{C^+}(\mathfrak{h}^1(A))) \otimes (C^+_Q)^\text{op} \to \text{End}(\mathfrak{h}^1(A)) \]

which induces an isomorphism of the Betti realizations. As a consequence, this is an isomorphism of André motives. Here we use the general fact that $\text{End}_Q(R)$ is canonically isomorphic to $R \otimes_Q R^\text{op}$ for any finite dimensional $\mathbb{Q}$-algebra $R$. This proves statement (3). The element $1 \in C^+_Q$ induces a morphism $1 \to (C^+_Q)^\text{op}$ yielding a morphism

\[ \text{End}_{C^+}(\mathfrak{h}^1(A)) \to \text{End}_{C^+}(\mathfrak{h}^1(A)) \otimes 1 \to \text{End}_{C^+}(\mathfrak{h}^1(A)) \otimes (C^+_Q)^\text{op} \]

whose Betti realization is given by $\alpha \mapsto \alpha \otimes 1$. This shows statement (4). \hfill $\Box$

**Remark 3.4.** The above theorem also holds for the $\#$-construction. The corresponding key statement (2) is established in [And96a, Cor. 6.4.4].

From Theorem 3.3, André deduces:

**Theorem 3.5** (André [And96a, Prop. 6.4.3 & Cor. 1.5.3]).

(i) Let $X$ be a hyper-Kähler variety defined over a field $K \subseteq \mathbb{C}$. Then there is a finite field extension $K'/K$ and an abelian variety $A$ over $K'$ such that the André motive $\mathfrak{h}^2(X_{K'})$ is a direct summand of $\mathfrak{h}(A)$;

(ii) Any Hodge class in a direct summand $M$ of $(\otimes \mathfrak{h}^2(X_i)) \otimes (\otimes \mathfrak{h}(A_j))$, where the $X_i$ are complex hyper-Kähler varieties and the $A_j$ are complex abelian varieties, is motivated. In particular, the Mumford–Tate group $\text{MT}(M)$ of $M$ coincides with its motivated Galois group $G_{\text{And}}(M)$. 
Proof. If \( \dim V = 2n + 1 \) is odd, then its primitive motive \( v \) satisfies
\[
v \cong v^\vee \cong v^\vee \otimes \det v \cong \wedge^{2n} v \cong \text{Gr}_n C^+(v, Q)
\]
where the last term is the graded piece \( F_n/F_{n-1} \). It follows that \( v \) is a subquotient of \( C^+(v, Q) \) and hence a sub motive of \( C^+(v, Q) \) by semi-simplicity. Theorem 3.3 thus provides a split inclusion
\[
v \hookrightarrow C^+(v, Q) \cong \text{End}_{C^+}(\mathfrak{h}^1(A)) = \mathfrak{h}^1(A) \otimes (A^\vee)^\vee \hookrightarrow \mathfrak{h}(A \times A^\vee)(1).
\]
If \( \dim V = 2n \) is even, then we consider \( V^\# = V \oplus \mathbb{Z}(0) \) and Remark 3.4 yields
\[
v \hookrightarrow v^\# \hookrightarrow \mathfrak{h}(A^\# \times (A^\#)^\vee)(1).
\]
Since \( \mathfrak{h}^2(X)(1) = 1 \oplus v \), this proves (i). Finally, (ii) follows from (i) and the fact (Theorem 2.4) proved by \( \text{André} \) [And96b] that Hodge cycles on complex abelian varieties are motivated (the statement \( MT(M) = G_{\text{And}}(M) \) is Proposition 2.3).

\[ \square \]

4. De Rham–Betti objects and the Grothendieck period conjecture

In this section we fix an algebraic closure \( \overline{Q} \) of \( Q \) inside \( C \) and we let \( K \) be a subfield of \( \overline{Q} \).

4.1. De Rham–Betti objects and de Rham–Betti classes. The following definition will be mostly used in the special case where \( L = Q \); but see §4.6 for remarks on \( Q \)-deRham–Betti classes.

**Definition 4.1** (De Rham–Betti objects [And04, §7.1.6]). Let \( L \) be another subfield of \( \overline{Q} \). The category of \( L \)-de Rham–Betti objects over \( K \) is the \( (K \cap L) \)-linear category \( \mathcal{C}_{dRB,KdR,L_B} \) whose objects \( M \) are triples of the form
\[
M = (M_{dR}, M_B, c_M),
\]
where \( M_{dR} \) is a finite-dimensional \( K \)-vector space, \( M_B \) is a finite-dimensional \( L \)-vector space and \( c_M : M_{dR} \otimes_K C \rightarrow M_B \otimes_L C \) is a \( C \)-linear isomorphism. A \textit{de Rham–Betti homomorphism} \( f \in \text{Hom}_{dRB}(M, N) \) between \( L \)-de Rham–Betti objects over \( K \) consists of a \( K \)-linear map \( f_{dR} : M_{dR} \rightarrow N_{dR} \) together with an \( L \)-linear map \( f_B : M_B \rightarrow N_B \) such that \( c_N \circ (f_{dR} \otimes_K \text{id}_C) = (f_B \otimes_L \text{id}_C) \circ c_M \).

For any \( k \in \mathbb{Z} \), we denote \( Q(k) \) the object of \( \mathcal{C}_{dRB,KdR,L_B} \) defined by
\[
Q(k)_{dR} := K, \quad Q(k)_{B} := L, \quad \text{and} \quad c_{Q(k)} : C \rightarrow C, z \mapsto (2\pi i)^{-k}z.
\]
The category \( \mathcal{C}_{dRB,KdR,L_B} \) is then naturally a rigid \( \otimes \)-category, with unit object \( \mathbb{1} := Q(0) \).

In case \( L = Q \), we write \( \mathcal{C}_{dRB,K} \) for \( \mathcal{C}_{dRB,KdR,Q_B} \) and simply say \textit{de Rham–Betti object} (the field \( K \) will usually be clear from the context). In case \( K = \overline{Q} \) and \( L = Q \), we omit the subscripts \( K_{dR} \) and \( L_B \) and simply write \( \mathcal{C}_{dRB} \) for \( \mathcal{C}_{dRB,KdR,Q_B} \).

From now on, and except in §4.6, we will only consider the case \( L = Q \). To a smooth projective variety \( X \) defined over \( K \), one associates its \textit{de Rham–Betti cohomology groups}
\[
H^n_{dRB}(X, Q(k)) := (H^n_{dR}(X/K), H^n_{B}(X_C^\an, Q), (2\pi i)^{-k}c_X),
\]
where \( c_X : H^n_{dR}(X/K) \otimes_K C \rightarrow H^n_{B}(X_C^\an, Q) \otimes Q C \) is Grothendieck’s period comparison isomorphism (1).

**Definition 4.2** (De Rham–Betti classes [And04, §7.5.1]). Let \( M \) be a de Rham–Betti object. A \textit{de Rham–Betti class} on \( M \) is an element of \( \text{Hom}_{\mathcal{C}_{dRB,K}}(\mathbb{1}, M) \).
Recall from §2 that the Betti and de Rham realizations of motivated cycle classes are compatible with the comparison isomorphisms and therefore that motivated cycle classes are de Rham–Betti. As such, there is a well-defined faithful realization functor

\[ \rho_{dRB} : M_{K}^{\text{And}} \to \mathcal{C}_{dRB,K}, \quad M = (X, p, n) \mapsto (p_\ast \mathbb{H}_{dR}^i(X/K), p_\ast \mathbb{H}_{B}^i(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}), (2\pi i)^{-n} p_\ast \circ c_X \circ p_\ast), \]

and we may speak of de Rham–Betti classes on an André motive \( M \). Conversely, de Rham–Betti classes are conjectured to be algebraic [And04, Conj. 7.5.1.1]. A weaker form of this expectation is the following

**Conjecture 4.3.** The realization functor \( \rho_{dRB} : M_{K}^{\text{And}} \to \mathcal{C}_{dRB,K} \) is full. In other words, de Rham–Betti classes are motivated.

We note in particular that the conjecture implies that de Rham–Betti classes should be Hodge classes, and even absolute Hodge classes (in the sense of Deligne [Del82]). The following easy lemma reduces Conjecture 4.3 to the case \( K = \overline{K} \):

**Lemma 4.4.** Let \( M \) be a André motive over \( K \subseteq \overline{K} \) and let \( M_{\overline{K}} \) be its base-change to \( \overline{K} \). If every de Rham–Betti class on \( M_{\overline{K}} \) is motivated, then every de Rham–Betti class on \( M \) is motivated.

**Proof.** Recall from [And96b, Scolie p.17] that the action of the Galois group \( \text{Gal}(\overline{K}/K) \) on the space of motivated cycles on \( M_{\overline{K}} \) factors through a finite quotient and the space of motivated cycles on \( M \) is exactly the space of \( \text{Gal}(\overline{K}/K) \)-invariant motivated cycles on \( M_{\overline{K}} \).

By assumption, we have an isomorphism

\[ \text{Hom}_{\text{And}}(\mathbb{I}, M_{\overline{K}} \to \text{Hom}_{dRB}(\mathbb{I}, \mathbb{H}_{dRB}(M_{\overline{K}})). \] (3)

This isomorphism gives rise to a Galois action on the right-hand side which is induced from that on the left. Explicitly, the Galois action on the de Rham component of \( \mathbb{H}_{dRB}(M_{\overline{K}}) \) is the usual Galois action on the algebraic de Rham cohomology. Hence the Galois invariants on the right-hand side are exactly the de Rham–Betti classes in \( \mathbb{H}_{dRB}(M_{\overline{K}}) \) defined over \( K \). The result then follows by taking Galois invariants on both sides of the isomorphism (3). \( \square \)

### 4.2. The Tannakian category of de Rham–Betti objects

The category \( \mathcal{C}_{dRB,K} \) is neutral Tannakian and the two natural forgetful functors

\[ \omega_{B} : \mathcal{C}_{dRB,K} \to \text{Vec}_{\mathbb{Q}} \quad \text{and} \quad \omega_{dR} : \mathcal{C}_{dRB,K} \to \text{Vec}_{K} \]

provide fiber functors, whose compositions with the de Rham–Betti realization functor \( \rho_{dRB} : M_{K}^{\text{And}} \to \mathcal{C}_{dRB,K} \) from the category of André motives over \( K \) provide the fiber functors abusively also denoted \( \omega_{B} \) and \( \omega_{dR} \) defined in Section 2.

**Definition 4.5** (De Rham–Betti group). Let \( M \) be an object in \( \mathcal{C}_{dRB,K} \). Its de Rham–Betti group \( G_{dRB}(M) \) is the Tannakian fundamental group

\[ G_{dRB}(M) := \text{Aut}^{\otimes}(\omega_{B}|_{(M)}). \]

In particular, a class \( \alpha \) in \( M_{B}^{\otimes n} \otimes (M_{B}^{\text{and}})^{\otimes m} \) is de Rham–Betti if and only if it is fixed by \( G_{dRB}(M) \).

Let now \( M \) be a André motive over \( K \). By abuse, we denote \( G_{dRB}(M) \) the de Rham–Betti group of \( \rho_{dRB}(M) \). Since motivated classes are de Rham–Betti and since \( G_{\text{And}}(M) \) is the closed subgroup of \( \text{GL}(\omega_{B}(M)) \) fixing motivated classes in tensor spaces of \( M \) (see Section 2), we have the inclusion \( G_{dRB}(M) \subseteq G_{\text{And}}(M) \). In this situation, we have the analogue of Proposition 2.3:

**Proposition 4.6.** Let \( M \) be a André motive over \( K \). The following statements are equivalent:

(i) The inclusion \( G_{dRB}(M) \subseteq G_{\text{And}}(M) \) is an equality;

(ii) \( G_{dRB}(M) \) is reductive and \( (\rho_{dRB})|_{(M)} \) is full.

**Proof.** This is a direct consequence of the basic facts concerning Tannakian categories (see Section 1) and the fact that \( G_{\text{And}}(M) \) is reductive (see Section 2). \( \square \)
4.3. The case of abelian varieties. Let $K$ and $L$ be two subfields of $\mathbb{C}$ and let us denote $\text{Vec}_{K,L}$ the category whose objects are triples of the form $M = (W, V, c)$, where $W$ (resp. $V$) is a finite-dimensional $K$-vector space (resp. $L$-vector space) and $c : W \otimes_K \mathbb{C} \to V \otimes_L \mathbb{C}$ is a $\mathbb{C}$-linear map. In particular $\mathcal{C}_{\text{dRB}} := \text{Vec}^\mathbb{Q}_{\overline{\mathbb{Q}}}$ is the category whose objects are triples of the form $M = (M_{\text{dR}}, M_{B}, c_{M})$ as for $\mathcal{C}_{\text{dRB}}$ except that we no longer require $c_{M}$ to be bijective. Let $\text{CCG}(\overline{\mathbb{Q}})_{\mathbb{Q}}$ be the category of connected commutative group schemes over $\overline{\mathbb{Q}}$ up to isogeny. The following formulation is borrowed from [And04, §7.5.3], but see also [HW22, Chap. 6].

**Theorem 4.7** (Wüstholz’ analytic subgroup theorem [Wüs84]). The functor

$$\omega : \text{CCG}(\overline{\mathbb{Q}})_{\mathbb{Q}} \to \text{Vec}^\mathbb{Q}_{\overline{\mathbb{Q}}}, \quad G \mapsto ((\ker \text{exp}_{G}) \otimes_{\mathbb{Z}} \mathbb{Q}, \text{Lie}(G), \iota_{G}),$$

where $\iota_{G} : (\ker \text{exp}_{G}) \otimes_{\mathbb{Z}} \mathbb{C} \to \text{Lie}(G) \otimes_{\mathbb{Q}} \mathbb{C}$ is the natural map, is fully faithful and every subobject of $\omega(G)$ is of the form $\omega(H)$ for some subobject $H$ of $G$. A fortiori, the same holds for the functor

$$\hat{\omega} : \text{CCG}(\overline{\mathbb{Q}})^{\text{op}}_{\mathbb{Q}} \to \mathcal{C}_{\text{dRB}}, \quad G \mapsto (\text{Lie}(G)^{\vee}, \text{Hom}(\ker \text{exp}_{G}), \mathbb{Q}, (\iota_{G})^{\vee}).$$

From this theorem and the fact that, for an abelian variety $A$ over $K \subseteq \mathbb{C}$, $H^{1}_{\text{dRB}}(A, \mathbb{Q})$ identifies with $\text{Lie}(E(A^{\vee}))$, where $E(A^{\vee})$ is the universal vector extension of the dual $A^{\vee}$ of $A$, one can derive the following two theorems:

**Theorem 4.8.** Let $A$ and $B$ be abelian varieties over $K \subseteq \mathbb{C}$. The natural map

$$\text{Hom}(A, B) \otimes \mathbb{Q} \xrightarrow{\sim} \text{Hom}_{\text{dRB}}(H^{1}_{\text{dRB}}(B, \mathbb{Q}), H^{1}_{\text{dRB}}(A, \mathbb{Q}))$$

is an isomorphism. In particular, any codimension-1 de Rham–Betti class on an abelian variety defined over $K \subseteq \overline{\mathbb{Q}}$ is algebraic.

**Proof.** This is stated over $K = \overline{\mathbb{Q}}$ in [And04, §7.5.3] and established as a consequence of Wüstholz’ analytic subgroup theorem 4.7. One uses in addition the fact [Bos13, Lem. 5.2] that the map

$$\text{Hom}(A, B) \longrightarrow \text{Hom}(E(A), E(B))$$

obtained using the universal property of the universal vector extension is bijective. We also note that Bost [Bos13, Thm. 5.1 & 5.3] provides a proof based on the older transcendence theorems of Schneider and Lang. The result over $K \subseteq \overline{\mathbb{Q}}$ follows from Lemma 4.4. \(\square\)

Let $A$ be an abelian variety over $\overline{\mathbb{Q}}$ and let $G_{\text{dRB}}(A) := G_{\text{dRB}}(H^{1}_{\text{dRB}}(A, \mathbb{Q}))$ be its de Rham–Betti group. Since $H^{1}_{\text{dRB}}(A, \mathbb{Q})$ is the exterior algebra on $H^{1}_{\text{dR}}(A, \mathbb{Q})$, we have $G_{\text{dRB}}(A) = G_{\text{dR}}(H^{1}_{\text{dR}}(A))$. We have the following observation (for which we could not find a reference):

**Theorem 4.9.** The de Rham–Betti group $G_{\text{dRB}}(A)$ of an abelian variety $A$ over $\overline{\mathbb{Q}}$ is reductive.

**Proof.** As a consequence of Wüstholz’ analytic subgroup theorem 4.7, the de Rham–Betti cohomology group $H^{1}_{\text{dRB}}(A, \mathbb{Q})$ is semi-simple as an object in $\mathcal{C}_{\text{dRB}}$. This is stated in [And04, §7.5.3]. Let us flesh out the details; this is analogous to the proof of [Tan18, Thm. 2.3.5]. We may assume that $A$ is a simple abelian variety over $\overline{\mathbb{Q}}$ and it suffices to show that the de Rham–Betti object $H^{1}_{\text{dRB}}(A, \mathbb{Q})$ is simple. Let $V$ be a de Rham–Betti subobject of $H^{1}_{\text{dRB}}(A, \mathbb{Q})$. Up to replacing $V$ with $(H^{1}_{\text{dRB}}(A, \mathbb{Q}))/V)^{\vee}(1)$ inside $H^{1}_{\text{dRB}}(A, \mathbb{Q})^{\vee}(1) \cong H^{1}_{\text{dR}}(A, \mathbb{Q})$, we may assume $V \geq \dim A$. We aim to show that $V = H^{1}_{\text{dR}}(A/\mathbb{Q})$. By Wüstholz’ analytic subgroup theorem 4.7, $V$ is the Lie algebra of some $\overline{\mathbb{Q}}$-algebraic subgroup $H$ of $E(A^{\vee})$. Recall that we have a short exact sequence

$$0 \longrightarrow \text{Lie}(A)^{\vee} \longrightarrow E(A^{\vee}) \longrightarrow A^{\vee} \longrightarrow 0,$$

where $\text{Lie}(A)^{\vee}$ canonically identifies with $F^{1}H^{1}_{\text{dR}}(A/\overline{\mathbb{Q}})$. Since $A$ is simple, the image of $H$ under the projection $E(A^{\vee}) \to A^{\vee}$ is either $A^{\vee}$ or $0$. The latter does not occur: otherwise the $\overline{\mathbb{Q}}$-subspace $V = F^{1}H^{1}_{\text{dR}}(A/\overline{\mathbb{Q}})$ inside $\text{Lie}(E(A^{\vee})) = H^{1}_{\text{dR}}(A/\overline{\mathbb{Q}})$ would descend to $\mathbb{Q}$ on the Betti
side, i.e., the Hodge filtration on $H^2(A_{C}^{an}, \mathbb{Q}) \otimes \mathbb{C}$ would be defined over $\mathbb{Q}$. Therefore $H$ maps onto $A^\vee$ and we aim to show that $H = E(A^\vee)$. Let $W$ be the connected component of $H \cap \text{Lie}(A)^\vee$. We have a short exact sequence

$$
0 \longrightarrow \text{Lie}(A)^\vee/W \longrightarrow E(A^\vee)/W \longrightarrow A^\vee \longrightarrow 0 .
$$

The subgroup $H \subseteq E(A^\vee)$ induces $H/W \rightarrow E(A^\vee)/W$. Since $H/W \rightarrow A^\vee$ is an isogeny, this provides a splitting $E(A^\vee)/W \rightarrow \text{Lie}(A)^\vee/W$ of the short exact sequence above. However, $E(A^\vee)$ is anti-affine (e.g., [Bri09, Prop 2.3]), so that $W = \text{Lie}(A)^\vee$ and hence $H = E(A^\vee)$.

It follows from the facts recalled in §1 that $G_{\text{dR}}(A)$ acts faithfully and semi-simply on $H^1_{\text{dR}}(A, \mathbb{Q})$. Since we are working in characteristic zero, we conclude that $G_{\text{dR}}(A)$ is reductive. \hfill □

4.4. First consequences for hyper-Kähler varieties. Via André’s Theorem 3.5, Theorems 4.8 and 4.9 admit the following two consequences:

**Proposition 4.10** (Bost–Charles [BC16, Thm. 5.6]). Let $X$ be a hyper-Kähler variety over $K \subseteq \overline{\mathbb{Q}}$. Then any de Rham–Betti class in $H^2_{\text{dR}}(X, \mathbb{Q}(1))$ is algebraic.

**Proof.** By Theorem 3.5, there exists an abelian variety $A$ over $\overline{\mathbb{Q}}$ such that the André motive $h^2(X_{\overline{\mathbb{Q}}})$ is a direct summand of $h^2(A)$. The proposition in case $K = \overline{\mathbb{Q}}$, which is [BC16, Thm. 5.6], is then a direct consequence of Theorem 4.8. The case $K \subseteq \overline{\mathbb{Q}}$ follows from Lemma 4.4. \hfill □

**Proposition 4.11.** Let $X$ be a hyper-Kähler variety over $\overline{\mathbb{Q}}$. Then $G_{\text{dR}}(h^2(X))$ is reductive.

**Proof.** By Theorem 3.5, there exists an abelian variety $A$ over $\overline{\mathbb{Q}}$ such that $h^2(X)$ is a direct summand in the category $\mathcal{C}_{\text{dR}}$ of $h(A)$. Hence $G_{\text{dR}}(h^2(X))$ is a quotient of $G_{\text{dR}}(A)$ and is thus reductive by Theorem 4.9. \hfill □

4.5. The Grothendieck period conjecture and the de Rham–Betti conjecture. We start with an overview of [And04, §7.5] and [BC16, §2.2.2] regarding the Grothendieck period conjecture.

**Definition 4.12** (Tannakian torsor of periods). The Tannakian torsor of periods $\Omega^T_M$ of a de Rham–Betti object $M \in \mathcal{C}_{\text{dR}, K}$ is

$$
\Omega^T_M := \text{Iso}^\otimes(\omega_{\text{dR}}\vert_{\langle M \rangle} \otimes_K \overline{\mathbb{Q}}, \omega_B\vert_{\langle M \rangle} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) ;
$$

it is a torsor under under $\text{Aut}^\otimes(\omega_B\vert_{\langle M \rangle} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, which coincides with $G_{\text{dR}}(M)_{\overline{\mathbb{Q}}}$ by [DM82, Rmk. 3.12].

By definition, the Tannakian torsor of periods $\Omega^T_M$ is included in the intersection of the $\Omega_\alpha$, where $\Omega_{\alpha}$ is the torsor whose $\overline{\mathbb{Q}}$-points are given by

$$
\Omega_\alpha(\overline{\mathbb{Q}}) = \{ f \in \text{Iso}_{\overline{\mathbb{Q}}}(M_{\text{dR}} \otimes_K \overline{\mathbb{Q}}, M_B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \mid f(\alpha_{\text{dR}} \otimes_K 1_{\overline{\mathbb{Q}}}) = \alpha_B \otimes_{\mathbb{Q}} 1_{\overline{\mathbb{Q}}} \},
$$

for $\alpha$ running through the de Rham–Betti classes in the various spaces of tensors $M^\otimes n \otimes (M^\vee)^\otimes m$. In case the Tannakian subcategory $\langle M \rangle$ is semi-simple (or, equivalently, if $G_{\text{dR}}(M)$ is reductive), then the above inclusion is an equality and one recovers in this case the definition of the Tannakian torsor of periods $\Omega^T_M$ in terms of invariants given in [BC16, Def. 2.4].

Let now $M$ be an André motive defined over $K$. By definition, its Tannakian torsor of periods is the Tannakian torsor of periods of the de Rham–Betti realization $\rho_{\text{dR}}(M)$; By abuse, we denote it by $\Omega^T_M$. On the other hand, we have
Definition 4.13 (Torsor of motivated periods). The torsor of motivated periods $\Omega_{\text{And}}^M$ of a André motive $M \in M^\text{And}_K$ is

$$\Omega_{\text{And}}^M := \text{Iso} \otimes (\omega_{\text{dR}}|_{\langle M \rangle} \otimes_K \mathcal{Q}, \omega_{\text{B}}|_{\langle M \rangle} \otimes_K \mathcal{Q})$$

it is a torsor under $G_{\text{And}}(M)\mathcal{Q}$.

Since the neutral Tannakian category of André motives is semi-simple, the torsor of motivated periods has the following description in terms of invariants: it is the intersection of the $\Omega_\alpha$ as above, where $\alpha$ runs through the motivated classes on spaces of tensors $M^\otimes n \otimes (M^\vee)^\otimes m$. This description coincides with [BC16, Def. 2.9(3)].

Restricting the intersection to those $\Omega_\alpha$ with $\alpha$ algebraic classes yields the notion of torsor of motivic periods. A homological motive $M$ over $K$ is an object of the form $M = (X, p, n)$ with $X$ smooth projective over $K$ of dimension $d_X$, $p$ an idempotent in $\text{im}(\text{CH}^d_X(X \times X) \to H_B((X \times X)^{an}, \mathcal{Q}))$ and $n$ an integer.

Definition 4.14 (Torsor of motivic periods). The torsor of motivic periods $\Omega_{\text{mot}}^M$ of a homological motive $M$ is defined as the intersection of the $\Omega_\alpha$ as above, where $\alpha$ runs through the algebraic classes on spaces of tensors $M^\otimes n \otimes (M^\vee)^\otimes m$.

We note that this torsor has a Tannakian description in case $X$ satisfies Grothendieck’s standard conjectures; see [And04, §7.5.2].

From the descriptions above, we have clear inclusions $\Omega_T^M \subseteq \Omega_{\text{And}}^M$ for $M$ a André motive and $\Omega_{\text{And}}^M \subseteq \Omega_{\text{mot}}^M$ for $M$ a homological motive. It is also clear that, for a de Rham–Betti object $M$, the period comparison isomorphism $c_M : M_{\text{dR}} \otimes_K \mathcal{C} \to M_{\text{B}} \otimes \mathcal{Q} \mathcal{C}$ defines a complex point of $\Omega_T^M$, so that the Zariski closure $Z_M$ of $c_M$ inside the $\mathcal{Q}$-scheme $\text{Iso}(M_{\text{dR}} \otimes_K \mathcal{Q}, M_{\text{B}} \otimes \mathcal{Q})$ is contained in $\Omega_T^M$.

Conjecture 4.15 (Grothendieck Period Conjecture [Gro66]).

(i) Let $M$ be a André motive over $\overline{\mathcal{Q}}$. We say $M$ satisfies the motivated version of Grothendieck’s period conjecture if the inclusions

$$Z_M \subseteq \Omega_T^M \subseteq \Omega_{\text{And}}^M$$

are equalities.

(ii) Let $M$ be a homological motive over $\overline{\mathcal{Q}}$. We say $M$ satisfies Grothendieck’s period conjecture if the inclusions

$$Z_M \subseteq \Omega_T^M \subseteq \Omega_{\text{And}}^M \subseteq \Omega_{\text{mot}}^M$$

are equalities.

Note that Conjecture 4.15 predicts that $G_{\text{And}}(M)$ is connected for a André motive $M$ over $\overline{\mathcal{Q}}$; this is also predicted by the Hodge conjecture (or more simply from the conjecture that Hodge classes are motivated) since then $G_{\text{And}}(M)$ coincides with the Mumford–Tate group $\text{MT}(M)$, but is unknown in general. We have formulated Conjecture 4.15 for $K = \overline{\mathcal{Q}}$. In general, for $K \subseteq \overline{\mathcal{Q}}$, one still expects the various torsors to coincide; however one cannot expect them to be connected and $Z_M$ is only expected to be a connected component of $\Omega_T^M$. In fact, we have the following easy observation (and a similar statement holds for homological motives and torsors of motivic periods provided the standard conjectures hold):

Lemma 4.16. Let $M$ be a André motive over $K$. Then

$$Z_{M_{\mathcal{Q}}} = \Omega_T^M_{M_{\mathcal{Q}}} = \Omega_{\text{And}}^M_{M_{\mathcal{Q}}} \implies \Omega_T^M = \Omega_{\text{And}}^M.$$
Proof. In general, we have the chain of inclusions
\[ Z_M \longrightarrow \Omega_M^T \longrightarrow \Omega_M^{And} \]
\[ \Omega_M^T \longrightarrow \Omega_M^{T'} \longrightarrow \Omega_M^{And} \]
If the top row inclusion is an equality, then we have inclusions \( \Omega_M^{And} \subseteq \Omega_M^T \subseteq \Omega_M^{And} \). This gives inclusions \( G_{And}(\mathcal{M}_Q) \subseteq G_{dRB}(\mathcal{M}_Q) \subseteq G_{And}(\mathcal{M}_Q) \). Since the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on tensor spaces of \( \mathcal{M}_Q \) preserves motivated cycles, \( G_{And}(\mathcal{M}_Q) \) has finite index in \( G_{And}(\mathcal{M}_Q) \); indeed, \( G_{And}(\mathcal{M}_Q) \) can be defined as the closed subgroup of \( \text{GL}(\omega_B(\mathcal{M}_Q)) \) fixing a finite number of motivated classes in tensors spaces of \( \mathcal{M}_Q \) and the quotient set \( G_{And}(\mathcal{M}_Q)/G_{And}(\mathcal{M}_Q) \) can be shown, e.g., as in the proof of [Tan18, Lem. 2.2.7], to preserve the Galois orbit of each of these classes and therefore is finite. As a consequence, \( G_{dRB}(\mathcal{M}_Q) \) is a finite extension of the reductive group \( G_{And}(\mathcal{M}_Q) \). Hence \( G_{dRB}(\mathcal{M}) \) is reductive. From Lemma 4.4 and Proposition 4.6, we conclude that \( G_{dRB}(\mathcal{M}) = G_{And}(\mathcal{M}) \) and hence that \( \Omega_M^T = \Omega_M^{And} \). \( \square \)

The Grothendieck Period Conjecture 4.15 has so far only been fully established in the following cases:

(a) \( M = I(k) \) : this is trivial for \( k = 0 \) and this amounts to the transcendence of \( \pi \) for \( k \neq 0 \).

(b) \( M = \mathfrak{h}(E) \) for \( E \) a CM elliptic curve : this is due to Chudnovsky [Chu80] who proves more generally that, for any elliptic curve \( E \), the degree of transcendence of the residue field of the comparison \( c_E \) in \( \Omega_M^T \) is \( \geq 2 \).

In this work, we will address the following special instance of the Grothendieck Period Conjecture 4.15:

Conjecture 4.17 (De Rham–Betti Conjecture).

(i) Let \( M \) be a André motive over \( \overline{\mathbb{Q}} \). We say \( M \) satisfies the motivated de Rham–Betti conjecture if the inclusion
\[ \Omega_M^T \subseteq \Omega_M^{And} \]
is an equality.

(ii) Let \( M \) be a homological motive over \( \overline{\mathbb{Q}} \). We say \( M \) satisfies de Rham–Betti conjecture if the inclusions
\[ \Omega_M^T \subseteq \Omega_M^{And} \subseteq \Omega_M^{mot} \]
are equalities.

The following proposition shows that Conjecture 4.17(i) for a André motive \( M \) over \( \overline{\mathbb{Q}} \) is a strengthening of Conjecture 4.3 restricted to \( \langle M \rangle \); see also [BC16, Prop. 2.14].

Proposition 4.18. Let \( M \) be a André motive over \( K \). The following statements are equivalent:

(i) \( M \) satisfies the motivated de Rham–Betti conjecture, i.e., \( \Omega_M^T = \Omega_M^{And} \);

(ii) The functor \( (\rho_{dRB})_{\langle M \rangle} \) is full and \( G_{dRB}(\mathcal{M}) \) is reductive.

In particular, if \( M_Q \) satisfies the motivated de Rham–Betti conjecture, then any de Rham–Betti class on a space of tensors \( \bigoplus_{\text{finite}} M_{Q_m}^{\otimes m} \otimes (M_{Q}^{\otimes m_i}) \) is motivated, and if \( M = \mathfrak{h}(X) \), then any de Rham–Betti class in \( H_{dRB}^j(X^n, \mathbb{Q}(k)) \) is motivated and in particular zero if \( j \neq 2k \).

Proof. The equivalence of (i) and (ii) is a reformulation of Proposition 4.6. The assumption \( \Omega_M^T = \Omega_M^{And} \) implies that \( G_{dRB}(\mathcal{M}_Q) = G_{And}(\mathcal{M}_Q) \) and hence that \( (\rho_{dRB})_{\langle M \rangle} \) is full. From Lemma 4.4 it follows that \( (\rho_{dRB})_{\langle M \rangle} \) is full. The latter exactly means that de Rham–Betti classes
on spaces of tensors $\bigoplus_{\text{finite}} M^\otimes n \otimes (M^\vee)^\otimes m$ are motivated. If $M = h(X)$, we note that $1(-1)$ is a direct summand of $h(X)$, so that $h(X^n)(k)$ is a direct summand of $h(X)^\otimes r \otimes (h(X)^\vee)^\otimes s$ for some $r, s \geq 0$. Hence any de Rham–Betti class in $H^j_{\text{dR}}(X^n, Q(k))$ is motivated. That de Rham–Betti classes in $H^j_{\text{dR}}(X^n, Q(k))$ are zero for $j \neq 2k$ follows at once from the fact that a André motive with no grade zero component does not support any non-zero motivated class.

4.6. On $\overline{Q}$-de Rham–Betti classes. To a smooth projective variety $X$ over $\overline{Q}$, one may associate its de Rham–Betti cohomology groups with $\overline{Q}$-coefficients

$$H^n_{\text{dR}}(X, \overline{Q}(k)) := (H^n_{\text{dR}}(X/K), H^n_B(X^{an}, Q) \otimes_{\overline{Q}} (2\pi i)^{-k}c_X).$$

This defines an object in the $\overline{Q}$-linear category $\mathcal{C}_{\text{dR}, \overline{Q}}$ consisting of $\overline{Q}$-de Rham–Betti objects over $\overline{Q}$. In that setting, a $\overline{Q}$-de Rham–Betti class is a class $\alpha_{\text{dR}} \in H^n_{\text{dR}}(X^{an}, Q) \otimes_{\overline{Q}} Q$ that corresponds to a class $\alpha_{\text{dR}} \in H^n_{\text{dR}}(X/\overline{Q})$ under $(2\pi i)^{-k}c_X$. It is a priori not at all clear that $\overline{Q}$-de Rham–Betti classes are $\overline{Q}$-linear combinations of de Rham–Betti classes. This is however predicted by parts of the Grothendieck period conjecture:

**Proposition 4.19.** Let $M$ be a André motive over $\overline{Q}$. Assume that the motivated version of Grothendieck’s period conjecture 4.15 holds for $M$, i.e., that the inclusions

$$Z_M \subseteq \Omega^T_M \subseteq \Omega^\text{And}_M$$

are equalities. Then, for any André motive $N \in \langle M \rangle$, any $\overline{Q}$-de Rham–Betti class $H^n_{\text{dR}}(N, \overline{Q})$ is a $\overline{Q}$-linear combinations of motivated, and hence de Rham–Betti, classes in $H^n_{\text{dR}}(N, Q)$.

**Proof.** Let $M \otimes \overline{Q}$ be the image of the motive $M$ in the category $M^\text{And}_K \otimes \overline{Q}$ of André motives with $\overline{Q}$-coefficients. Its de Rham–Betti realization provides an object in the category $\mathcal{C}_{\text{dR}, \overline{Q}}$ of $\overline{Q}$-de Rham–Betti objects over $\overline{Q}$. The latter comes equipped with two neutral fiber functors

$$\omega_{B, \overline{Q}} : \mathcal{C}_{\text{dR}, \overline{Q}} \to \text{Vec}_{\overline{Q}} \quad \text{and} \quad \omega_{\text{dR}, \overline{Q}} : \mathcal{C}_{\text{dR}, \overline{Q}} \to \text{Vec}_{\overline{Q}}.$$

The corresponding Tannakian torsor is

$$\Omega^T_{M \otimes \overline{Q}} := \text{Iso}^\otimes(\omega_{\text{dR}, \overline{Q}}(M \otimes \overline{Q}), \omega_{B, \overline{Q}}(M \otimes \overline{Q})).$$

It is included in the intersection of the torsors $\Omega_\alpha$ whose $\overline{Q}$-points are given by

$$\Omega_\alpha(\overline{Q}) = \{ f \in \text{Iso}_{\overline{Q}}(M_{\text{dR}}, M_B \otimes_{\overline{Q}} \overline{Q}) \mid f(\alpha_{\text{dR}}) = \alpha_{B} \},$$

for $\alpha$ running through the $\overline{Q}$-de Rham–Betti classes in the various spaces of tensors $M^\otimes n \otimes (M^\vee)^\otimes m$. Hence, since motivated classes become $\overline{Q}$-de Rham–Betti classes after base-change of coefficients to $\overline{Q}$, we have a natural inclusion of $\overline{Q}$-torsors

$$\Omega^T_{M \otimes \overline{Q}} \subseteq \Omega^\text{And}_M.$$

We conclude by noting that the comparison isomorphism $c_M$ defines a $\mathcal{C}$-point of $\Omega^T_{M \otimes \overline{Q}}$. □

In particular, by Chudnovsky [Chu80], the conclusion of Proposition 4.19 holds for powers of CM elliptic curves and more generally for any André motive belonging to the Tannakian subcategory $\langle h(E) \rangle$ generated by the motive of a CM elliptic curve $E$. We are not aware of any example of a variety $X$ over $\overline{Q}$ whose motive does not belong to $\langle h(E) \rangle$ for any CM elliptic curve $E$, for which it is known that $\overline{Q}$-de Rham–Betti classes are $\overline{Q}$-linear combinations of de Rham–Betti classes.
5. De Rham–Betti isometries and hyper-Kähler varieties

As before, $\overline{Q}$ is the algebraic closure of $Q$ inside $C$ and $K$ is a subfield of $\overline{Q}$. In this section we prove Theorem 2, i.e., that de Rham–Betti isometries between degree-2 cohomology groups of hyper-Kähler varieties are motivated. This is the analogue of [And96a, Lem. 7.3.1], where it is shown that Galois-equivariant isometries between degree-2 $\ell$-adic cohomology groups are $\mathbb{Q}_\ell$-linear combinations of motivated cycles. The approach of [And96a, Lem. 7.3.1] however is representation-theoretic and proceeds through base-change to an algebraically closed field of coefficients and the use of spinorial representations. In our situation, beyond the fact that we deal with torsors rather than groups, we cannot base-change to $\overline{Q}$ as it is unknown in general whether $\overline{Q}$-de Rham–Betti classes of codimension 1 on abelian varieties are $\overline{Q}$-linear combinations of divisors; see §4.6. As another caveat we do not know whether the de Rham–Betti group $G_{\text{dR}}$ is connected (the connectedness of both the motivic Galois group and the $\ell$-adic Galois group are used in Andrè’s proof). Instead, we use the canonical algebra isomorphisms of Andrè motives provided by Theorem 3.3, which allows us to simultaneously deal with both the de Rham and the Betti sides, and then use a polarization on the Kuga–Satake variety to connect the two sides. As a result of independent interest, we derive from Theorem 5.1 a “de Rham–Betti Torelli theorem” for K3 surfaces over $\overline{Q}$; see Theorem 5.4.

5.1. De Rham–Betti isometries between hyper-Kähler varieties are motivated. For a hyper-Kähler variety $X$ over $K$, we denote $T^2_{\text{dR}}(X, Q(1))$ its degree-two de Rham–Betti transcendental cohomology. By definition it is the orthogonal complement with respect to the Beauville–Bogomolov form of the subspace of $H^2_{\text{dR}}(X, Q(1))$ spanned by the classes of divisors. (Note from Example 3.1 that the Beauville–Bogomolov form with rational coefficients is indeed defined over $K$.)

**Theorem 5.1.** Let $X$ and $X'$ be hyper-Kähler varieties over $K$. Then

(i) any de Rham–Betti isometry $P^2_{\text{dR}}(X, Q) \xrightarrow{\sim} P^2_{\text{dR}}(X', Q)$ is motivated.

(ii) any de Rham–Betti isometry $T^2_{\text{dR}}(X, Q) \xrightarrow{\sim} T^2_{\text{dR}}(X', Q)$ is motivated.

(iii) any de Rham–Betti isometry $H^2_{\text{dR}}(X, Q) \xrightarrow{\sim} H^2_{\text{dR}}(X', Q)$ is motivated.

Here, the isometries are with respect to either the form induced by $\eta$ or the Beauville–Bogomolov form (see Example 3.1).

**Proof.** By Proposition 4.10, if $X$ is a hyper-Kähler variety, then there are no non-trivial de Rham–Betti homomorphisms between $\mathcal{I}$ and $T^2_{\text{dR}}(X, Q(1))$ (both ways). Hence $(ii) \implies (iii)$ and it thus suffices to show items (i) and (ii). Let $X$ and $X'$ be two hyper-Kähler varieties over $K \subseteq \overline{Q}$ with respective polarizations $\eta$ and $\eta'$. Let $V := P^2_{\text{dR}}(X, Q(1))$ (resp. $V := T^2_{\text{dR}}(X, Q(1))$) and $V' := P^2_{\text{dR}}(X', Q(1))$ (resp. $V' := T^2_{\text{dR}}(X', Q(1))$) be the primitive de Rham–Betti cohomology (resp. transcendental cohomology). Assume that we have a de Rham–Betti isometry

$$i : V \xrightarrow{\sim} V'.$$

We will show that $i$ is motivated. By [And96b, Scolie p.17], it suffices to show this after base-change of $X$ and $X'$ to $\overline{Q}$.

Let $V_{B,Z}$ and $V'_{B,Z}$ be the natural $\mathbb{Z}$-lattices inside $V_B$ and $V'_B$, respectively. Let $A := \text{KS}(V_{B,Z})$ and $A' := \text{KS}(V'_{B,Z})$ be the associated Kuga–Satake varieties over $\overline{Q}$ provided by Theorem 3.3. Let $L := H^1_{\text{dR}}(A, Q)$ and $L' := H^1_{\text{dR}}(A', Q)$, and denote $L_{B,Z} := H^1_B(A, Z)$ and $L'_{B,Z} := H^1_B(A', Z)$. We have the two algebras $C^+ := (\text{End}_C(V_{B,Z}))^{op}$ and $C'^+ := (\text{End}_C(V'_{B,Z}))^{op}$. We fix a generator $e_B$ of $L_{B,Z}$ and a generator $e'_B$ of $L'_{B,Z}$. This induces identifications

$$C^+ = C^+(V_{B,Z}) \quad \text{and} \quad C'^+ = C^+(V'_{B,Z}).$$
The isometry \( i_B : V_B \to V'_B \) induces algebra isomorphisms

\[
\begin{array}{c}
C^+_Q \xrightarrow{\phi} C'^+_Q \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
C^+(V_B) \xrightarrow{C^+(i_B)} C^+(V'_B),
\end{array}
\]

while the de Rham–Betti isometry \( i \) induces de Rham–Betti algebra isomorphisms

\[
\begin{array}{c}
\text{End}_{C^+} L \xrightarrow{j} \text{End}_{C'^+} L' \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
C^+(V) \xrightarrow{C^+(i)} C^+(V').
\end{array}
\]

By Theorem 3.3, we get a de Rham–Betti isomorphism

\[
\begin{array}{c}
\text{End}(L) \xrightarrow{j} \text{End}(L') \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
(\text{End}_{C^+} L) \otimes (C^+_{Q})^{\text{op}} \xrightarrow{j \otimes j_0} (\text{End}_{C'^+} L') \otimes (C'^+_{Q})^{\text{op}}.
\end{array}
\]

of algebras. Here \( C^+_Q \) is viewed as a constant de Rham–Betti object \((C^+ \otimes \overline{Q}, C^+ \otimes Q, \text{id})\). By Lemma 5.2 below, there exists isomorphisms \( \nu_{\text{dr}} : L_{\text{dr}} \to L'_{\text{dr}} \) and \( \nu_B : L_B \to L'_B \) such that

\[
J_{\text{dr}}(f_{\text{dr}}) = \nu_{\text{dr}} \circ f_{\text{dr}} \circ \nu_{\text{dr}}^{-1} \quad \text{and} \quad J_B(f_B) = \nu_B \circ f_B \circ \nu_B^{-1}
\]

for all \( f_{\text{dr}} \in \text{End}_{Q}(L_{\text{dr}}) \) and all \( f_B \in \text{End}_{Q}(L_B) \). The isomorphism \( \nu_{\text{dr}} \) (resp. \( \nu_B \)) is unique up to scaling by a scalar in \( k^* \).

**Lemma 5.2.** Let \( W \) and \( W' \) be two vector spaces over a field \( k \) and let \( \Phi : \text{End}_k(W) \to \text{End}_k(W') \) be an isomorphism of \( k \)-algebras. Then there exists a \( k \)-linear isomorphism \( \phi : W \to W' \) such that \( \Phi(f) = \phi \circ f \circ \phi^{-1} \) for all \( f \in \text{End}_k(W) \). The isomorphism \( \phi \) is unique up to a scalar in \( k^* \).

**Proof.** Pick a basis \( \{w_1, w_2, \ldots, w_n\} \) of \( W \). Let \( pr_1 \in \text{End}_k(W) \) be the projector onto the subspace generated by \( w_1 \). Let \( pr'_1 := \Phi(pr_1) \in \text{End}_k(W') \), which is nonzero. Pick some \( w' \in W' \) such that \( w'_1 := pr'_1(w') \) is nonzero. Let \( f_i \in \text{End}_k(W), i = 2, 3, \ldots, n \), be defined by \( f_i(w_1) = w_i \) and \( f_i(w_j) = 0 \) for all \( j > 1 \). Let \( f'_i := \Phi(f_i) \) and set \( w'_j := f'_i(w'_1) \). In this way, we get a basis \( \{w'_1, w'_2, \ldots, w'_n\} \) of \( W' \) and the isomorphism \( \phi : W \to W', \phi(w_i) = w'_i \), satisfies the required condition. If \( \phi' \) is another such isomorphism, then \( \phi^{-1} \circ \phi' \) sits in the center of \( \text{End}_k(W) \) and hence \( \phi' = c\phi \) for some \( c \in k^* \).

The condition that \( J \) is a de Rham–Betti isomorphism of algebras means \( J_B \circ c_{\text{End}(L)} = c_{\text{End}(L')} \circ J_{\text{dr}} \) over \( C \), where the comparison \( c_{\text{End}(L)} \) is given by \( f_{\text{dr}} \mapsto c_L \circ f_{\text{dr}} \circ c_L^{-1} \). Hence we have

\[
\nu_B \circ c_L \circ f_{\text{dr}} \circ c_L^{-1} \circ \nu_B^{-1} = c_L' \circ \nu_{\text{dr}} \circ f_{\text{dr}} \circ \nu_{\text{dr}}^{-1} \circ c_L'^{-1}
\]

over \( C \) for all \( f_{\text{dr}} \in \text{End}_C(L_{\text{dr}}, C) \). As a consequence, \( \nu_{\text{dr}}^{-1} \circ c_L'^{-1} \circ \nu_B \circ c_L \) lies in the center of \( \text{End}_C(L_{\text{dr}}, C) \) and hence is a constant complex number. In other words,

\[
\nu_B \circ c_L = \lambda \ c_L' \circ \nu_{\text{dr}}, \quad \lambda \in C^*.
\]

(4)

Our next goal is to show that \( \nu_{\text{dr}} \) and \( \nu_B \) can be chosen so that they are compatible with the comparison isomorphism \( c_L \), i.e., that \( \lambda \in \overline{Q}^* \). This is done by choosing polarizations on \( A \).
Similarly, we also have \( \text{Gr} \cdot \) motivated filtration \( F \) isomorphism \( C \) semi-simplicity of the category of André motives), this further implies that \( \nu \) and hence \( \lambda \) such that the comparison isomorphism \( c \) for all \( x, y \in C^+(V_B) \).

The above choice of generators also induces an isomorphism \( \nu_{ib} : L_B \to L'_B \), \( \nu_{ib}(xe_B) = j(x)e'_B \).

For any \( f_B \in \text{End}(L_B) \), there exists \( \alpha \in C^+(V_B) \) and \( a \in (C_\overline{Q})^{op} \) such that \( f_B(xe_B) = (\alpha x a)e'_B \)
for all \( x \in C^+(V_B) \). Let \( f'_B := J_B(f_B) \). By the definition of \( J \), we have \( f'_B(x'e'_B) = J_B(\alpha) x'j_0(a)e'_B \)
for all \( x' \in C^+(V'_B) \). Meanwhile, we also have

\[
\nu_{ib} \circ f_B \circ \nu_{ib}^{-1}(x'e'_B) = \nu_{ib}(\alpha j_B^{-1}(x')ae_B) = j_B(\alpha) x'j_0(a)e'_B.
\]

Thus \( J_B(f_B) = \nu_{ib} \circ f_B \circ \nu_{ib}^{-1} \), for all \( f_B \in \text{End}(L_B) \). It follows that \( \nu_{ib} \) is a rational multiple of \( \nu_B \). By rescaling \( \nu_B \) we may assume that \( \nu_B = \nu_{ib} \). With this choice, we have

\[
\varphi' \nu_B(xe_B), \nu_B(ye_B) = \varphi'(j(x)e'_B, j(y)e'_B), \quad \text{for all } x, y \in C^+(V_B)
\]

\[
= \text{tr}(v'_i v'_j x^* y^*)
= \text{tr}(j(v_1 v_2 x^* y^*))
= \text{tr}(v_1 v_2 x^* y)
= \varphi(xe_B, ye_B),
\]

that is, \( \nu_B \) respects the polarizations.

Since the polarization \( \varphi \) is algebraic, there exists a unique bilinear pairing \( \varphi_{dR} : \text{L}_{dR} \times \text{L}_{dR} \to \overline{Q} \) such that the comparison isomorphism \( c_L \) respects the bilinear pairings. This means

\[
\varphi(c_L(u), c_L(v)) = (2\pi i) \varphi_{dR}(u, v), \quad \text{for all } u, v \in \text{L}_{dR,c}.
\]

and a similar equality for \( L' \). From equation (4), we get \( \lambda \nu_{dR} = c_L^{-1} \circ \nu_B \circ c_L \). As a consequence, \( \lambda \nu_{dR} \) respects the bilinear pairings. In other words,

\[
\varphi_{dR}(u, v) = \varphi'_{dR}(\lambda \nu_{dR}(u), \lambda \nu_{dR}(v)) = \lambda^2 \varphi'_{dR}(\nu_{dR}(u), \nu_{dR}(v)),
\]

for all \( u, v \in \text{L}_{dR,c} \). Since \( \varphi_{dR} \), \( \varphi'_{dR} \) and \( \nu_{dR} \) are all defined over \( \overline{Q} \), we conclude that \( \lambda^2 \in \overline{Q}^* \)
and hence \( \lambda \in \overline{Q}^* \). After rescaling \( \nu_{dR} \) by \( \lambda \), we may assume that \( \lambda = 1 \). This means that \( \nu = (\nu_{dR}, \nu_B) : L \to L' \) is a de Rham–Betti isomorphism.

It follows from Theorem 4.8 that \( \nu \) is algebraic. As a consequence, \( J \) is motivated and hence \( j \otimes j_0 \) is motivated. By restricting \( J \) to the motivated sub-algebra \( \text{End}_{C^+} \), \( L \subseteq \text{End}(L) \) (and the semi-simplicity of the category of André motives), this further implies that \( j \) is motivated and hence that \( C^+(i) : C^+(V) \to C^+(V') \) is also motivated. By the discussion in Section 3.2, we have a motivated filtration \( F_{\bullet}C^+(V) \) on \( C^+(V) \) and a motivated filtration \( F_{\bullet}C^+(V') \). By construction, \( C^+(i) \) preserves these filtrations.

We first assume that \( \dim V = 2n + 1 \) is odd. Then we have canonical isomorphisms

\[
\text{Gr}_n C^+(V) = F_n/F_{n-1} \cong \lambda^{2n} V \cong V^\vee \cong V.
\]

Similarly, we also have \( \text{Gr}_n C^+(V') \cong V' \). Under these canonical identifications, the induced isomorphism

\[
\text{Gr}_n C^+(i) : \text{Gr}_n C^+(V) \to \text{Gr}_n C^+(V')
\]

reverses the isometry \( i : V \to V' \). Thus \( i \) is motivated.
If \( \dim V = 2n \) is even, then we consider \( i^\# = i \oplus \text{id} : V^\# \to (V')^\# \), which is a de Rham–Betti isometry by definition. Since \( \dim V^\# = 2n + 1 \) is odd, we conclude that \( i^\# \) is motivated. Thus \( i \) is also motivated. \( \square \)

**Corollary 5.3.** If \( X \) and \( X' \) are hyper-Kähler varieties over \( K \) of \( K\mathbb{B}[n] \)-deformation type, then any de Rham–Betti isometry \( H^2_{\text{dRB}}(X, \mathbb{Q}) \to H^2_{\text{dRB}}(X', \mathbb{Q}) \) is algebraic.

**Proof.** By Markman [Mar22], any Hodge isometry \( H^2_B(X^\text{an}, \mathbb{Q}) \to H^2_B((X')^\text{an}, \mathbb{Q}) \) is algebraic. (The case where \( X \) and \( X' \) are K3 surfaces is due to Buskin [Bus19]). The corollary then follows directly from Theorem 5.1(iii). \( \square \)

### 5.2. A global de Rham–Betti Torelli theorem for K3 surfaces over \( \overline{\mathbb{Q}} \)

Let \( S \) and \( S' \) be two K3 surfaces over \( \overline{\mathbb{Q}} \). If there is an integral de Rham–Betti isometry

\[
i : H^2_{\text{dRB}}(S, \mathbb{Z}) \to H^2_{\text{dRB}}(S', \mathbb{Z}),
\]

i.e., an isometry \( i : H^2_B(S^\text{an}, \mathbb{Z}) \to H^2_B(S'^\text{an}, \mathbb{Z}) \) that becomes de Rham–Betti after base-change to \( \mathbb{Q} \), then \( S \) and \( S' \) are isomorphic.

**Proof.** Let \( \eta \) be an ample divisor class on \( S \). Then \( \eta' = i(\eta) \) is a de Rham–Betti class and hence algebraic by [BC16, Thm. 5.6]; see also Proposition 4.10. Thus either \( \eta' \) or \( -\eta' \) is a positive class on \( S' \). Without loss of generality, we assume that \( \eta' \) is a positive class. Then \( \eta' \) can be moved to an ample class by a series of reflections along \((-2)\)-classes; see [Huy16, Ch. 8, Cor. 2.9]. Such reflections are all algebraic isometries of \( H^2_{\text{dRB}}(S', \mathbb{Z}) \) since each \((-2)\)-class is represented by a rational curve on \( S' \). By composing \( i \) with these reflections, we may assume that \( \eta' \) is an ample class. Thus \( i \) restricts to a de Rham–Betti isometry

\[
i' : P^2_{\text{dRB}}(S, \mathbb{Z}) \to P^2_{\text{dRB}}(S', \mathbb{Z}),
\]

where \( P^2_{\text{dRB}}(S, \mathbb{Z}) \) (resp. \( P^2_{\text{dRB}}(S', \mathbb{Z}) \)) is the orthogonal complement of \( \eta \) (resp. \( \eta' \)). It follows from Theorem 5.1 that \( i' \otimes 1_{\mathbb{Q}} \) is motivated and hence respects Hodge structures. As a consequence, \( i \) is a Hodge isometry and the usual Torelli theorem for K3 surfaces provides an isomorphism \( S_C \simeq S'_C \), from which we obtain an isomorphism \( S \simeq S' \). \( \square \)

### 6. The de Rham–Betti conjecture for elliptic curves

We address the de Rham–Betti conjecture (Conjecture 4.17) for elliptic curves. The main result of this section is Theorem 6.5. We note that the proof in the CM case only involves knowing that codimension-1 de Rham–Betti classes are motivated (Theorem 4.8), while the proof in the non-CM case involves our Theorem 5.1. Before we proceed to the proof, a few preliminary results are needed.

We start by recalling that the Hodge conjecture is known for products of complex elliptic curves. Recall that the *Hodge group* \( \text{Hdg}(H) \) of a rational Hodge structure \( H \) is the smallest algebraic \( \mathbb{Q} \)-subgroup of \( \text{GL}(H) \) that contains the image of \( U_{\mathbb{C}/\mathbb{R}} \to \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \text{GL}(H)_{\mathbb{R}} \), where the right arrow is the morphism defining the Hodge structure on \( H \). The group \( \text{Hdg}(H) \) can be characterized as being the largest subgroup of \( \text{GL}(H) \) that fixes all Hodge classes in tensor spaces associated to \( H \). In case \( H \) is of pure weight 0, then the Hodge group of \( H \) agrees with its Mumford–Tate group, while in case \( H \) is of pure weight \( n \neq 0 \), its Mumford–Tate group \( \text{MT}(H) \) is the image of the multiplication map \( \mathbb{G}_m \times \text{Hdg}(H) \to \text{GL}(H) \). If \( A \) is a complex abelian variety, the Hodge group \( \text{Hdg}(A) \) (resp. the Mumford–Tate group \( \text{MT}(A) \)) of \( A \) is the Hodge group (resp. the Mumford–Tate group) of the Hodge structure \( H^1(A, \mathbb{Q}) \).
Proposition 6.1. Let $E_1, \ldots, E_s$ be pairwise non-isogenous complex elliptic curves and let $A$ be an abelian variety isogenous to $E_1^{m_1} \times \cdots \times E_s^{m_s}$. Denote $V_i := H^1(E_i, \mathbb{Q})$ and $K_i := \text{End}(E_i) \otimes \mathbb{Q}$. Then the Hodge group of $A$ is

$$\text{Hdg}(A) = \text{Hdg}(E_1) \times \cdots \times \text{Hdg}(E_s),$$

where $\text{Hdg}(E_i) = \begin{cases} \bigcup K_i & \text{if } E_i \text{ has CM;} \\ \text{SL}(V_i) & \text{if } E_i \text{ is without CM}. \end{cases}$

In particular, the subspace of Hodge classes on $A$ is generated by Hodge classes in $H^2(A, \mathbb{Q})$ and the Hodge conjecture holds for $A$.

Proof. This is classical and we refer to [Lew99, App. B] for a proof. □

For the convenience of the reader we recall Goursat’s lemma:

Lemma 6.2 (Goursat’s lemma). Let $G$ and $G'$ be two algebraic groups over $\mathbb{Q}$ and let $H \subseteq G \times G'$ be an algebraic subgroup such that both projections $H \to G$ and $H \to G'$ are surjective. Let

$$N := \ker(H \to G') = H \cap (G \times \{e'\}) \quad \text{and} \quad N' := \ker(H \to G) = H \cap (\{e\} \times G').$$

Then $N$ (resp. $N'$) is a normal subgroup of $G$ (resp. $G'$) and the image of $H$ in $G/N \times G'/N'$ is the graph of a group isomorphism.

The following construction will be used.

Construction 6.3. Let $V$ be a two-dimensional $\mathbb{Q}$-vector space equipped with a symplectic form $\langle -, - \rangle : V \otimes V \to \mathbb{Q}$. The symplectic group $\text{Sp}(V)$ agrees with the special linear group $\text{SL}(V)$. The symmetric square $S^2 V$ is then a three-dimensional $\mathbb{Q}$-vector space equipped with a quadratic form

$$q : S^2 V \otimes S^2 V \to \mathbb{Q}, \quad (v_1 v_2) \otimes (w_1 w_2) \mapsto (v_1, w_1) (v_2, w_2) + (v_1, w_2) (v_2, w_1).$$

The action of $\text{SL}(V)$ on $S^2 V$ preserves $q$ and thereby induces a group homomorphism

$$\rho : \text{SL}(V) \to \text{SO}(S^2 V, q);$$

it is surjective with kernel $\{\pm 1\}$.

Moreover, given $V$ and $W$ two rational symplectic spaces of dimension 2, we have $\langle \varphi(v), \varphi(w) \rangle = (\det \varphi) \cdot \langle v, w \rangle$ for any isomorphism $\varphi : V \sim W$. Here the determinant of $\varphi$ is with respect to the choice of any symplectic bases of $V$ and $W$. In particular, the induced isomorphism on the symmetric squares

$$\frac{1}{\det \varphi} S^2 \varphi : S^2 V \sim S^2 W, \quad v_1 v_2 \mapsto \frac{1}{\det \varphi} \varphi(v_1) \varphi(v_2)$$

is an isometry.

In case $E$ is an elliptic curve, we may consider $V = H^1_B(E_{\text{an}}^\text{c}, \mathbb{Q})$ equipped with the symplectic form given by $\langle v_1, v_2 \rangle = \int_E v_1 \cup v_2$. In that case, viewing $S^2 V$ as a direct summand of $H^2_B((E \times E)^{an}_{\text{c}}, \mathbb{Q})$, the cup-product pairing on $E \times E$ agrees with $-2$ times the quadratic form $q$, i.e., $\int_{E \times E} \alpha \cup \beta = -2q(\alpha, \beta)$. The Hodge structure $S^2 V \otimes \mathbb{Q}(1)$ is of K3-type and coincides with the transcendental cohomology $\text{T}^2_B((E \times E)^{an}_{\text{c}}, \mathbb{Q}(1))$ when $E$ is without CM.

Finally, we have the following description of reductive $\mathbb{Q}$-subgroups of $\text{SL}_2$.

Lemma 6.4. The reductive $\mathbb{Q}$-algebraic subgroups of $\text{SL}_2$ are, up to conjugation, given by:

(i) The finite subgroups.

(ii) The maximal tori; for some $D \in \mathbb{Q}^\times$, these are of the form

$$\left\{ \begin{pmatrix} a & b \\ bD & a \end{pmatrix} \bigg| a^2 - b^2 D = 1 \right\}.$$
(iii) Subgroups admitting a maximal torus as an index-2 subgroup; for some $D \in \mathbb{Q}^\times$, these are of the form

$$\left\{ \begin{pmatrix} a & b \\ bD & a \end{pmatrix} \mid a^2 - b^2 D = 1 \right\} \cup \left\{ \begin{pmatrix} x & y \\ -yD & -x \end{pmatrix} \mid x^2 - y^2 D = -1 \right\}.$$

(iv) $\text{SL}_2$.

Proof. The reductive subgroups of $\text{SL}_2, \mathbb{C}$ are, up to conjugation, given by (see e.g. [NvdPT08])

(i) The finite subgroups.

(ii) The maximal torus $G_m = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \mid c \in \mathbb{C}^\times \right\}$.

(iii) The infinite dihedral group $D_\infty = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \mid c \in \mathbb{C}^\times \right\} \cup \left\{ \begin{pmatrix} 0 & -d \\ d^{-1} & 0 \end{pmatrix} \mid d \in \mathbb{C}^\times \right\}$.

(iv) $\text{SL}_2$.

It thus suffices to determine the $\mathbb{Q}$-forms of maximal tori, and of subgroups admitting a maximal torus as index-2 subgroup, inside $\text{SL}_2$. For that purpose let $\rho : G \hookrightarrow \text{SL}_2$ be a reductive subgroup defined over $\mathbb{Q}$.

We first assume that $G_\mathbb{C} \simeq G_{m, \mathbb{C}}$ is a maximal torus in $\text{SL}_2, \mathbb{C}$. If $G \simeq G_{m, \mathbb{Q}}$, then the inclusion $G \subseteq \text{SL}_2$ is given, up to conjugation, by

$$\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{G}_m.$$

This situation is covered by (ii) with $D = 1$. Hence we assume that $G$ does not split. In this case, there is a non-trivial quadratic field extension $K = \mathbb{Q}(\sqrt{D})$ that splits $G$ and

$$G = U_K := \{a + b\sqrt{D} \mid a^2 - b^2 D = 1\} \subseteq \text{Res}_{K/\mathbb{Q}} G_{m, K}.$$

There is a natural surjective group homomorphism

$$\mathbb{G}_m, \mathbb{Q} \times U_K \longrightarrow \text{Res}_{K/\mathbb{Q}} G_{m, K}, \quad (\lambda, x) \mapsto \lambda x,$$

whose kernel is $\{(1, 1), (-1, -1)\}$. Let $V$ be the standard representation of $\text{SL}_2$. Then the inclusion $\rho : G \simeq U_K \subseteq \text{SL}(V)$ can be extended to a group homomorphism

$$\tilde{\rho} : \mathbb{G}_m \times U_K \longrightarrow \text{GL}(V), \quad (\lambda, x) \mapsto \lambda \rho(x)$$

whose kernel is $\{(1, 1), (-1, -1)\}$. Thus there is an induced inclusion

$$\tilde{\rho} : \text{Res}_{K/\mathbb{Q}} G_{m, K} \longrightarrow \text{GL}(V).$$

After taking $\mathbb{Q}$-points in the above inclusion, we see that $K^\times$ acts on $V$ and hence $V \simeq K$. With respect to the $\mathbb{Q}$-basis $\{1, \sqrt{D}\}$ of $V$, we have

$$\rho(a + b\sqrt{D}) = \begin{pmatrix} a & b \\ bD & a \end{pmatrix}, \quad a + b\sqrt{D} \in U_K.$$

This proves statement (ii).

Now we assume that $G_\mathbb{C} \subseteq \text{SL}_2, \mathbb{C}$ is conjugate to the infinite dihedral group. Let $G^\circ \subseteq G$ be the identity component. Hence $G^\circ$ is a maximal torus and, by the above argument, we have

$$G^\circ = \left\{ \begin{pmatrix} a & b \\ bD & a \end{pmatrix} \mid a^2 - b^2 D = 1 \right\}.$$
Let \( G^- \subset G \) be the other component. Let

\[
g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in G^-
\]

be an element. Then the following three conditions are satisfied

\[
xw - yz = 1; \quad (5)
\]
\[
g^{-1} h g \in G^0, \text{ for all } h \in G^0; \quad (6)
\]
\[
g^2 \in G^0. \quad (7)
\]

Writing \( h = \begin{pmatrix} a & b \\ bD & a \end{pmatrix} \) with \( a^2 - bD^2 = 1 \), we get

\[
g^{-1} h g = \frac{1}{xw - yz} \begin{pmatrix} a(xw - yz) + b(wz - Dxy) & b(w^2 - Dy^2) \\ b(-z^2 + Dx^2) & a(xw - yz) + b(Dxy - zw) \end{pmatrix}
\]

Then the condition (6) becomes

\[
-z^2 + Dx^2 = Dw^2 - D^2y^2 \quad \text{and} \quad wz - Dxy = 0. \quad (8)
\]

One spells out condition (7) as

\[
(z - Dy)(x + w) = 0.
\]

If \( x + w \neq 0 \), then \( z = Dy \), which combined with equation (8) gives \( x = w \) and thus \( g \in G^0 \), which is a contradiction. Hence we have \( w = -x \) and consequently \( z = -Dy \). Thus

\[
g = \begin{pmatrix} x & y \\ -yD & -x \end{pmatrix}, \quad -x^2 + y^2D = 1.
\]

One easily checks that the above condition does define a component whose union with \( G^0 \) forms an algebraic group whose base change to \( \mathbb{C} \) is conjugate to \( D_\infty \).

We can now proceed to the proof of the main result of this section.

**Theorem 6.5.** Let \( E \) be an elliptic curve over \( K \subseteq \overline{\mathbb{Q}} \). Then the de Rham–Betti conjecture (Conjecture 4.17(ii)) holds for \( E_{\overline{\mathbb{Q}}} \), i.e.,

\[
\Omega^T_{E_{\overline{\mathbb{Q}}}} = \Omega^{\text{mot}}_{E_{\overline{\mathbb{Q}}}^\text{And}}.
\]

In particular, by Proposition 4.18, for any \( n \geq 0 \) and any \( k \in \mathbb{Z} \), any de Rham–Betti class on \( \mathfrak{h}(E^n)(k) \) is algebraic.

**Proof.** We may and do assume that \( K = \overline{\mathbb{Q}} \). Since curves satisfy the standard conjectures and since motivated cycles on powers of elliptic curves are algebraic, it suffice to show the motivated version of the de Rham–Betti conjecture, i.e., \( \Omega^T_E = \Omega^{\text{And}}_E \). In order to avoid the lack of theory of weights as well as the difficulty of dealing with (not necessarily connected) dimension-1 reductive subgroups of \( \text{GL}_2 \), we proceed in several steps.

Let \( X = E \times_{\overline{\mathbb{Q}}} E \). We first show that

\[
\Omega^T_{\mathfrak{h}^2(X)(1)} = \Omega^{\text{And}}_{\mathfrak{h}^2(X)(1)}.
\]

Let \( t^2(X) \) be the transcendental part of the André motive \( \mathfrak{h}^2(X) \), i.e., the orthogonal complement with respect to the polarization on \( X \) of \( 1(1)^{\otimes \rho} \) inside \( \mathfrak{h}^2(X) \), and denote \( T_{\text{dR}}^2(X/\overline{\mathbb{Q}}) \) and \( T_{\text{B}}^2(X^{an}, \mathbb{Q}) \) its de Rham and Betti realizations. We denote further \( T_{\text{dRB}}^2(X, \mathbb{Q}(1)) \) the de Rham–Betti realization of \( t^2(X)(1) \). Since \( \mathfrak{h}^2(X)(1) \simeq t^2(X)(1) \oplus 1^{\otimes \rho} \), the various Tannakian fundamental groups associated to \( (\mathfrak{h}^2(X)(1)) \) and \( (t^2(X)(1)) \) coincide and it suffices to show that \( \Omega^T_{\mathfrak{h}^2(X)(1)} = \Omega^{\text{And}}_{\mathfrak{h}^2(X)(1)} \). Recall that \( \Omega^{\text{And}}_{\mathfrak{h}^2(X)(1)} \) is a torsor under \( G_{\text{And}}(t^2(X)(1)) \) and that the latter
coincides by Theorem 3.5(ii) with the Mumford–Tate group \( \text{MT}(t^2(X)(1)) \). Zarhin [Zar83] determined the Hodge group of irreducible Hodge structures \( H \) of K3-type: in the special case where \( H \) is 2-dimensional, then \( F := \text{End}_{\text{idg}}(H) \) is a quadratic CM-field [Zar83, Rmk. 1.5.3], and in the special case where \( H \) is 3-dimensional, then \( F = \mathbb{Q} \) [Huy16, Rmk. 3.14(iii)], and we have

\[
\text{MT}(t^2(X)(1)) \simeq \begin{cases} 
U_F & \text{if } E \text{ has CM;} \\
\text{SO}_4 & \text{if } E \text{ is without CM}, 
\end{cases}
\]

as standard representations of \( T_B^2(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \), where \( U_F \) denotes the one-dimensional unitary group. (We have used that the Mumford–Tate group of a Hodge structure of weight zero coincides with its Hodge group.) Now we observe that the Zariski closure in the \( \mathbb{Q} \)-scheme \( \text{Iso}_T := \text{Iso}(T_{dR}(X/\overline{\mathbb{Q}}), T_B^2(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \otimes \overline{\mathbb{Q}}) \) of the comparison isomorphism

\[
(2\pi i)^{-1}c_{X} : \text{X}_{dR}(X/\overline{\mathbb{Q}}) \otimes \overline{\mathbb{Q}} \mathbb{C} \rightarrow T_B^2(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \otimes \mathbb{C}
\]

has positive dimension. (In other words, one of the periods of \( t^2(X) \) is transcendental. If not, then \( (2\pi i)^{-1}c_{X} \) defines a \( \overline{\mathbb{Q}} \)-point in the \( \overline{\mathbb{Q}} \)-scheme \( \text{Iso}_T \) and it follows that any class in \( T_{dR}^2(X, \mathbb{Q}(1)) \) is de Rham–Betti. By Proposition 4.10 we conclude that \( T_{dR}^2(X, \mathbb{Q}(1)) \) is spanned by algebraic classes, contradicting the very definition of \( t^2(X)(1) \). From the natural inclusion \( G_{\text{dR}}(t^2(X)(1)) \subseteq G_{\text{And}}(t^2(X)(1)) \) it follows that \( G_{\text{dR}}(t^2(X)(1)) \) is a positive-dimensional \( \mathbb{Q} \)-subgroup of \( \text{MT}(t^2(X)(1)) \). We now claim that we have equality

\[
G_{\text{dR}}(t^2(X)(1)) = G_{\text{And}}(t^2(X)(1)).
\]

In the CM case, this is clear since \( U_F \) viewed as a \( \mathbb{Q} \)-group scheme is connected and 1-dimensional. Suppose now that \( E \) is without CM. By the above and by Proposition 4.11, \( G := G_{\text{dR}}(t^2(X)(1)) \) is a positive-dimensional reductive \( \mathbb{Q} \)-subgroup of \( \text{SO}_3 = \text{SO}(T_B^2(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q})) \). By Construction 6.3, we view the quadratic space \( T_B^2(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \) as the symmetric square \( S^2V \) of the symplectic plane \( V := H_B^1(E_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \). Under the surjective homomorphism \( \rho : \text{SL}(V) \to \text{SO}(S^2V) \) whose kernel is \( \{ \pm 1 \} \), we then have \( G = \rho(H) \) for the positive-dimensional reductive \( \mathbb{Q} \)-subgroup \( H := \rho^{-1}(G) \) of \( \text{SL}(V) \). We have to show that \( H = \text{SL}(V) \). By Lemma 6.4, it is either \( \text{SL}(V) \), a \( \mathbb{Q} \)-form of the finite dihedral group as in Lemma 6.4(iii), or a maximal torus. Since a maximal torus is included in a \( \mathbb{Q} \)-form of the infinite dihedral group, it is enough to show that \( H \) is not a subgroup of a \( \mathbb{Q} \)-form of the infinite dihedral group. Assume for contradiction that it is. We are going to produce an \( H \)-invariant isometry

\[
f : T_B^2(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \sim \rightarrow T_B^2(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q})
\]

which is not \( \text{SL}(V) \)-invariant. Being \( H \)-invariant the isometry \( f \) defines a de Rham–Betti isometry and is hence thanks to Theorem 5.1 motivated, in particular Hodge. However, since \( f \) is not \( \text{SL}(V) \)-invariant, it cannot define by Proposition 6.1 a Hodge class, which provides a contradiction. For that purpose, we may in fact assume that \( H \) is a \( \mathbb{Q} \)-form of the dihedral group. Let \( (e_1, e_2) \) be a \( \mathbb{Q} \)-basis of \( V \) such that \( H \) takes the form given in Lemma 6.4(iii). In the \( \mathbb{Q}(\sqrt{D}) \)-basis \( (\varepsilon_1 = e_1 + \sqrt{D}e_2, \varepsilon_2 = e_1 - \sqrt{D}e_2) \), the group \( H \) takes the simple form (with \( c = a + b\sqrt{D} \) and \( d = -x + y\sqrt{D} \)):

\[
H = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & -d \\ d^{-1} & 0 \end{pmatrix} \right\}.
\]

Consider then the \( \mathbb{Q}(\sqrt{D}) \)-basis \( (\varepsilon_1\varepsilon_1, \varepsilon_2\varepsilon_2, \varepsilon_1\varepsilon_2) \) of \( S^2V \). Note that \( \varepsilon_1\varepsilon_2 \) is defined over \( \mathbb{Q} \) and that its orthogonal complement is the \( \mathbb{Q} \)-plane \( P \) spanned by \( \varepsilon_1\varepsilon_1 \) and \( \varepsilon_2\varepsilon_2 \). We then define the \( \mathbb{Q} \)-isometry \( f : S^2V \to S^2V \) as the orthogonal reflection across the \( \mathbb{Q} \)-plane \( P \). Since the action of \( H \) on \( S^2V \) diagonalizes in the basis \( (\varepsilon_1\varepsilon_1, \varepsilon_2\varepsilon_2, \varepsilon_1\varepsilon_2) \), it is clear that \( f \) is \( H \)-invariant. Since \( \text{SO}_3 \) is not abelian, it is also clear that \( f \) is not \( \text{SL}(V) \)-invariant. This finishes the proof that \( \Omega_{t^2(X)(1)}^{H} \) = \( \Omega_{t^2(X)(1)}^{\text{And}} \). (We note that the infinite dihedral group has no invariants in \( S^2V \) so that...
our Theorem 5.1 is essential to our proof. On the other hand, Theorem 4.8 suffices to discard the possibility that $G$ is a maximal torus; indeed, such a maximal torus leaves $\varepsilon_1 \varepsilon_2$ invariant and Theorem 4.8 would provide a non-trivial algebraic class in $\mathfrak{t}^2(X)(1)$ which is a contradiction.)

We now proceed to showing $\Omega^T_{h^2(X)} = \Omega^\text{And}_{h^2(X)}$. The Tannakian subcategory of $M^\text{And}_{Q}$ generated by $h^2(X)$ coincides with the Tannakian subcategory generated by $h^2(X)(1) \oplus 1(1)$. We therefore have natural surjective homomorphisms $G_{\text{dRB}}(h^2(X)) \to G_{\text{dRB}}(h^2(X)(1))$ and $G_{\text{dRB}}(h^2(X)) \to G_{\text{dRB}}(1(1))$. We also have similar natural surjective homomorphisms at the level of Mumford–Tate groups and, by definition of the Mumford–Tate group as the $Q$-Zariski closure of the image of the Deligne torus, these induce an inclusion $\text{MT}(h^2(X)) \to \text{MT}(h^2(X)(1)) \times \text{MT}(1(1))$. Note that we have $G_{\text{dRB}}(1(1)) = G_{\text{And}}(1(1)) = G_m$. By Theorem 3.5(ii) and the above, we obtain a commutative diagram

\[
\begin{array}{ccc}
G_{\text{dRB}}(h^2(X)) & \longrightarrow & G_{\text{dRB}}(h^2(X)(1)) \times G_{\text{dRB}}(1(1)) \\
\downarrow & & \downarrow \\
G_{\text{And}}(h^2(X)) & \longrightarrow & G_{\text{And}}(h^2(X)(1)) \times G_{\text{And}}(1(1)) \\
\downarrow & & \downarrow \\
\text{MT}(h^2(X)) & \longrightarrow & \text{MT}(h^2(X)(1)) \times Q_G_m.
\end{array}
\]

We apply Goursat’s lemma 6.2 to the subgroup $G_{\text{dRB}}(h^2(X)) \subseteq \text{MT}(h^2(X)(1)) \times Q_G_m$: in the CM case, the relative dimension of $G_{\text{dRB}}(h^2(X)) \to G_m$ cannot be zero since otherwise we would obtain an isogeny over $Q$ between $U_F$ and $G_m$; in the non-CM case, the kernel of $G_{\text{dRB}}(h^2(X)) \to G_m$ has to be isomorphic to $\text{MT}(h^2(X)(1)) = SO_3$ since the latter is a simple algebraic group. Therefore the chain of inclusions $G_{\text{dRB}}(h^2(X)) \to G_{\text{And}}(h^2(X)) = \text{MT}(h^2(X)) \to \text{MT}(h^2(X)(1)) \times Q_G_m$ is an equality, and we conclude that $\Omega^T_{h^2(X)} = \Omega^\text{And}_{h^2(X)}$.

Finally, we conclude that $\Omega^T_E = \Omega^\text{And}_E$. We have a commutative diagram

\[
\begin{array}{ccc}
G_{\text{dRB}}(E) & \longrightarrow & G_{\text{dRB}}(h^2(E \times E)) \\
\downarrow & & \downarrow \\
G_{\text{And}}(E) & \longrightarrow & G_{\text{And}}(h^2(E \times E)),
\end{array}
\]

where the right vertical equality was established above. Now both $G_{\text{And}}(E) = \text{MT}(E)$ and $G_{\text{And}}(h^2(E \times E))$ are 2-dimensional if $E$ is CM, and both are 4-dimensional if $E$ is without CM. Since $G_{\text{And}}(E)$ is connected, we conclude that the natural inclusion $G_{\text{dRB}}(E) \subseteq G_{\text{And}}(E)$ is an equality. \hfill $\square$

Remark 6.6. Note that Theorem 6.5 for CM elliptic curves follows at once from Chudnovsky’s theorem [Chu80] on the transcendence degree of periods of elliptic curves. (In fact Chudnovsky’s theorem implies more strongly the Grothendieck period conjecture for $E$ in that case.) Note however that our proof does not involve Chudnovsky’s theorem but rather Wüstholz’ analytic subgroup theorem [Wüs84].

7. The de Rham–Betti conjecture for products of elliptic curves

Using Theorem 5.1 and Theorem 6.5 we can prove the following generalization of Theorem 1:
Theorem 7.1. Let $E_1, \cdots, E_s$ be elliptic curves over $\overline{Q}$ and let $A$ be an abelian variety over $K \subseteq \overline{Q}$ such that $A_{\overline{Q}}$ is isogenous to $E_1 \times \cdots \times E_s$. Then the de Rham–Betti conjecture (Conjecture 4.17(ii)) holds for $h(A_{\overline{Q}})$, i.e.,

$$\Omega^{T}_{A_{\overline{Q}}} = \Omega_{A_{\overline{Q}}}^{mot}. \tag{9}$$

In particular, by Proposition 4.18, for any $n \geq 0$ and any $k \in \mathbb{Z}$, any de Rham–Betti class on $h(A^n)(k)$ is algebraic.

Proof. We may and do assume that $K = \overline{Q}$. Since Hodge classes on powers of $A$ are algebraic, it suffices to show that the natural inclusion $G_{dRB}(A) \subseteq G_{And}(A)$ is an equality. Moreover, since any Hodge class on an abelian variety $A$ over $\overline{Q}$ is motivated, the natural inclusion $MT(A) \subseteq G_{And}(A)$ is an equality (see André’s Theorem 2.4) and we will use interchangeably $MT(A)$ and $G_{And}(A)$.

The case where $A$ is an elliptic curve was covered in Theorem 6.5. We now consider the case where $A$ is a product of several elliptic curves.

Step 1: $A = E_1 \times E_2$, where $E_1, E_2$ are non-isogenous elliptic curves without CM.

Let $V_i := H^1(E_i, Q), i = 1, 2$. Then we know that $MT(E_i) = \text{GL}(V_i), i = 1, 2$. We have a commutative diagram

$$\begin{array}{ccc}
G_{dRB}(A) & \longrightarrow & G_{dRB}(E_1) \times G_{dRB}(E_2) \\
\downarrow & & \downarrow \text{MT}(E_1) \times \text{MT}(E_2) \\
\text{MT}(A) & \longleftarrow & \text{MT}(E_1) \times \text{MT}(E_2).
\end{array}$$

Moreover $MT(A) = G_m \cdot (\text{SL}(V_1) \times \text{SL}(V_2)) \subseteq MT(E_1) \times MT(E_2) = \text{GL}(V_1) \times \text{GL}(V_2)$ and both projections $\pi_i : G_{dRB}(A) \rightarrow MT(E_i)$ are surjective. We aim to show that the inclusion $G_{dRB}(A) \subseteq MT(A)$ is an equality. Let $N_i := \ker(\pi_i)$ and view $N_i$ as a subgroup of $MT(E_i) = \text{GL}(V_i)$. By Goursat’s lemma 6.2, $N_i$ is normal inside $\text{GL}(V_i)$ and the image of $G_{dRB}(A)$ inside $MT(E_1)/N_1 \times MT(E_2)/N_2$ is the graph of an isomorphism. Since the normal subgroups of $\text{GL}_2$ are $\text{GL}_2, \mu_m \cdot \text{SL}_2$ for $m > 0$ and subgroups of $G_m$, and since $G_{dRB}(A)$ contains both $N_1 \times \{e_2\}$ and $\{e_1\} \times N_2$, there are two possibilities, namely, $N_1 = \text{SL}(V_1)$ and $N_2 = \text{SL}(V_2)$, or $N_1 \subseteq \{\pm 1\}$ and $N_2 \subseteq \{\pm 1\}$.

We first make the following observation. Let $V$ be a two dimensional vector space over $Q$, then there is a canonical isomorphism

$$\text{GL}(V)/\{\pm 1\} \cong G_m \times \text{PGL}(V), \quad (g \mod \{\pm 1\}) \mapsto (\det(g), [g]) \tag{10}$$

where $[g] \in \text{PGL}(V)$ is the image of $g \in \text{GL}(V)$. Thus the composition

$$G_m \times \text{SL}(V) \rightarrow \text{GL}(V) \rightarrow \text{GL}(V)/\{\pm 1\}$$

realizes $\text{GL}(V)/\{\pm 1\}$ as the quotient of $G_m \times \text{SL}(V)$ by the subgroup $\{\pm 1\} \times \{\pm 1\}$, while $\text{GL}(V)$ is the quotient of $G_m \times \text{SL}(V)$ by the subgroup $\{(1,1), (-1,-1)\}$. Since $\text{SL}(V)$ is simply connected, we have $\pi_1(\text{PGL}(V)) = \mathbb{Z}/2\mathbb{Z}, \pi_1(\text{GL}(V)/\{\pm 1\}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(\text{GL}(V)) \cong \mathbb{Z}$. It follows that $\text{GL}(V)$ is not isomorphic to $\text{GL}(V)/\{\pm 1\}$. Furthermore, the degree 4 covering map in (10) corresponds to the canonical subgroup $2 \pi_1(\text{GL}(V)/\{\pm 1\})$ inside $\pi_1(\text{GL}(V)/\{\pm 1\})$.

We now treat the case where $N_1 \subseteq \{\pm 1\}$ and $N_2 \subseteq \{\pm 1\}$.

If $N_1 = \{1\}$ and $N_2 = \{1\}$, then the group $G_{dRB}(A)$ induces an isomorphism

$$\psi : \text{GL}(V_1) \sim \text{GL}(V_2).$$

If $N_1$ or $N_2$ is equal to $\{\pm 1\}$, then by the above observation they are both equal to $\{\pm 1\}$ and $G_{dRB}(A)$ induces an isomorphism

$$\psi' : \text{GL}(V_1)/\{\pm 1\} \sim \text{GL}(V_2)/\{\pm 1\}.$$
Since the degree 4 covering map (10) is canonical, we see that the isomorphism \( \psi' \) lifts to an isomorphism

\[
\tilde{\psi} : G_m \times SL(V_1) \xrightarrow{\sim} G_m \times SL(V_2).
\]

Since there is no nontrivial homomorphism \( SL(V_i) \to G_m \), the above isomorphism \( \tilde{\psi} \) restricts to an isomorphism \( \psi_2 : SL(V_1) \xrightarrow{\sim} SL(V_2) \) and hence also induces an isomorphism \( \psi_1 : G_m \xrightarrow{\sim} G_m \) such that

\[
\tilde{\psi}(\lambda, a) = (\psi_1(\lambda), \tau(\lambda)\psi_2(a)), \quad \forall \lambda \in G_m, a \in SL(V_1),
\]

where \( \tau : G_m \to SL(V_2) \) is a homomorphism. Note that \((\lambda, a) = (\lambda, 1) \cdot (1, a) = (1, a) \cdot (\lambda, 1)\). Therefore \( \tau(\lambda) \) lies in the center of \( SL(V_2) \) and hence \( \tau(\lambda) = 1 \) for all \( \lambda \in G_m \). Since automorphisms of \( SL \) are inner [ABD+66, Exp. XXIV, 1.3 & 3.6], there is a \( \overline{\mathbb{Q}} \)-linear isomorphism \( \varphi_2 : V_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \xrightarrow{\sim} V_2 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \) such that \( \psi_2(a) = \varphi_2a\varphi_2^{-1} \). Since \( \psi_1 \) is an automorphism of \( G_m \), we have \( \psi_1(\lambda) = \lambda \) or \( \lambda^{-1} \). In both cases, one checks that the isomorphism descends to an isomorphism \( \psi : GL(V_1) \xrightarrow{\sim} GL(V_2) \) which lifts the isomorphism \( \psi' \). Hence \( G_{dRB}(A) \) induces an isomorphism

\[
\psi : GL(V_1) \xrightarrow{\sim} GL(V_2).
\]

The group of automorphisms of \( GL_2 \) is generated by the inner automorphisms together with the transpose-inverse automorphism. Therefore either \( \Gamma_\psi = \{(g, \varphi g \varphi^{-1})\} \) or \( \Gamma_\psi = \{(g, \varphi(\varphi'g)^{-1} \varphi^{-1})\} \) for some \( \overline{\mathbb{Q}} \)-linear isomorphism

\[
\varphi : V_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \xrightarrow{\sim} V_2 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}.
\]

However, as \( G_{dRB}(A) \) sits in \( G_m \cdot (SL(V_1) \times SL(V_2)) \), the latter cannot occur. It follows that, in case \( N_1 \subseteq \{\pm 1\} \) and \( N_2 \subseteq \{\pm 1\} \), the group \( G_{dRB}(A) \) is included in the subgroup generated by

\[
G_m, \quad \{(X, \varphi X \varphi^{-1}) \mid X \in SL(V_1)\} \quad \text{and} \quad \{(X, -\varphi X \varphi^{-1}) \mid X \in SL(V_1)\}.
\]

Now we observe that the isomorphism on symmetric squares

\[
\frac{1}{\det \varphi} S^2 \varphi : S^2(V_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \xrightarrow{\sim} S^2(V_2 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})
\]

is defined over \( \mathbb{Q} \). To see this, fix \( \mathbb{Q} \)-bases of \( V_1 \) and \( V_2 \) and let

\[
\Phi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{ij} \in \overline{\mathbb{Q}},
\]

be the matrix of \( \varphi \) with respect to those bases. The isomorphism \( \psi \), given by conjugation by \( \varphi \), is defined over \( \mathbb{Q} \), so that for all invertible rational matrices \( X = (x_{ij})_{1 \leq i,j \leq 2} \in GL_2(\mathbb{Q}) \), the matrix \( \Phi X \Phi^{-1} \) has rational coefficients. By considering invertible matrices with \( x_{12} = x_{21} = 0 \) and \( x_{11} = -x_{22} \) and matrices with \( x_{11} = x_{22} = 0 \) and \( x_{12} = \pm x_{21} \), we find that

\[
\frac{a_{ij}a_{kl}}{\det \Phi} \in \mathbb{Q}, \quad \forall 1 \leq i,j,k,l \leq 2,
\]

and it follows that \( \frac{1}{\det \varphi} S^2 \varphi \) is defined over \( \mathbb{Q} \).

In view of the above and Construction 6.3, the \( \overline{\mathbb{Q}} \)-isomorphism \( \varphi \) induces a \( \mathbb{Q} \)-isometry

\[
\frac{1}{\det \varphi} S^2 \varphi : S^2 V_1 \xrightarrow{\sim} S^2 V_2.
\]

Moreover this isometry is \( G_{dRB}(A) \)-invariant. It thus defines a de Rham–Betti isometry between \( T^2_{dRB}(E_1 \times E_1, Q) \) and \( T^2_{dRB}(E_2 \times E_2, Q) \). By Theorem 5.1, it defines a non-trivial Hodge class in \( V_1 \otimes_{E_1} V_1 \otimes_{E_2} V_2 \). This however contradicts Proposition 6.1 since this class is not \( SL(V_1) \times SL(V_2) \)-invariant.

Finally, if \( N_1 = SL(V_1) \) and \( N_2 = SL(V_2) \), then \( G_{dRB}(A) \) induces an isomorphism \( G_m \to G_m \), which must be either \( \lambda \mapsto \lambda \) or \( \lambda \mapsto \lambda^{-1} \). Thus, after choosing an isomorphism \( V_1 \simeq V_2, G_{dRB}(A) \)
is the subgroup of $GL_2 \times GL_2$ generated by $SL_2 \times \{e\}$, $\{e\} \times SL_2$, and $\text{Graph}(\text{id} : M \mapsto M^{-1})$. It follows that either

$$G_{\text{dRB}}(A) = \{(M, M') \mid \det M = \det M'\} \text{ or } G_{\text{dRB}}(A) = \{(M, M') \mid \det M = (\det M')^{-1}\}.$$  

However, $G_{\text{dRB}}(A)$ sits inside $G_m \cdot (SL_2 \times SL_2)$, so that the latter does not occur and we do get $G_{\text{dRB}}(A) = G_m \cdot (SL_2 \times SL_2) = MT(A)$.

**Step 2:** $A = E_1^{r_1} \times \cdots \times E_r^{r_r}$, $r \geq 2$, with the $E_i$ without CM and pairwise non-isogenous.

Let $V_i = H^1(E_i, \mathbb{Q})$, $1 \leq i \leq r$; we know that $MT(E_i) = GL(V_i)$ and $MT(A) = G_m \cdot (SL(V_1) \times \cdots \times SL(V_r))$ viewed as a subgroup of $GL(V_1 \oplus \cdots \oplus V_r)$. Consider the following commutative diagram

$$
\begin{array}{ccc}
G_{\text{dRB}}(A) & \longrightarrow & MT(E_1) \times \cdots \times MT(E_r) \\
\downarrow & & \downarrow \\
MT(A) & & MT(A)
\end{array}
$$

where all the arrows are inclusions. In particular, the vertical inclusion provides the following observation: Let $g = (g_1, \ldots, g_r) \in G_{\text{dRB}}(A)$ and suppose $g_i \in SL(V_i)$ for some $1 \leq i \leq r$; then $g_j \in SL(V_j)$ for all $1 \leq j \leq r$.

Let $S := G_{\text{dRB}}(A) \cap (SL(V_1) \times \cdots \times SL(V_r))$. Consider the following diagram

$$
\begin{array}{ccc}
G_{\text{dRB}}(A) & \longrightarrow & G_{\text{dRB}}(E_i) \times G_{\text{dRB}}(E_j) \\
\downarrow & & \downarrow \\
MT(E_1) \times \cdots \times MT(E_r) & \longrightarrow & MT(E_i) \times MT(E_j)
\end{array}
$$

where $1 \leq i < j \leq r$. Let $S_{ij} := G_{\text{dRB}}(E_i \times E_j) \cap (SL(V_i) \times SL(V_j))$. The above observation implies that $S$ surjects onto $S_{ij}$ for all $1 \leq i < j \leq r$. By Step 1, we have $G_{\text{dRB}}(E_i \times E_j) = G_m \cdot (SL(V_i) \times SL(V_j))$ and hence $S_{ij} = SL(V_i) \times SL(V_j)$. Thus $S \subseteq SL(V_1) \times \cdots \times SL(V_r)$ is a subgroup that surjects onto any two-factor product $SL(V_i) \times SL(V_j)$. By Goursat’s lemma 6.2, we conclude $S = SL(V_1) \times \cdots \times SL(V_r)$. Thus we get the inclusions

$$SL(V_1) \times \cdots \times SL(V_r) \subseteq G_{\text{dRB}}(A) \subseteq G_m \cdot (SL(V_1) \times \cdots \times SL(V_r)).$$

Since $G_{\text{dRB}}(A)$ surjects onto $G_{\text{dRB}}(E_i) = GL(V_i)$ for all $1 \leq i \leq r$, we conclude that

$$G_{\text{dRB}}(A) = G_m \cdot (SL(V_1) \times \cdots \times SL(V_r)) = MT(A).$$

**Step 3:** The general case of $A$ isogenous to a product of elliptic curves.

Since isogeny does not affect the validity of the statement, we may assume $A$ is a product of elliptic curves and we write $A = E_1^{r_1} \times \cdots \times E_r^{r_r}$ with the $E_i$ pairwise non-isogenous. From Step 2, we know that the theorem holds for $A$ and $G_{\text{dRB}}(A) = MT(A) = G_m \cdot \text{Hdg}(A)$, if all the $E_i$’s are non-CM. Now we assume that the theorem holds for $A$ and

$$G_{\text{dRB}}(A) = G_m \cdot \text{Hdg}(A)$$

Recall from Proposition 6.1 that the Hodge group $\text{Hdg}(A)$ is equal to $\text{Hdg}(E_1) \times \cdots \times \text{Hdg}(E_r)$. Let $E_{r+1}$ be a CM elliptic curve not isogenous to any of the $E_i$, $1 \leq i \leq r$. Since $\text{Hdg}(E_i)$ is either $SL_2$ (non-CM case) or a non-split torus (CM case), for all $1 \leq i \leq r$, there is no non-trivial homomorphism $\text{Hdg}(A) \to G_m$. Similarly there is also no non-trivial homomorphism $\text{Hdg}(E_{r+1}) \to G_m$. By Lemma 7.2, if two CM elliptic curves are not isogenous to each other, then their Hodge groups do not have a non-trivial common quotient. Since $SL_2$ does not admit any torus quotient, $\text{Hdg}(A)$ and $\text{Hdg}(E_{r+1})$ do not share a non-trivial common quotient. We
can then apply Lemma 7.3 to the situation where \( G = G_{\text{dRB}}(A \times E_{r+1}^{n+1}) \), \( G_1 = G_{\text{dRB}}(A) \), \( G_2 = G_{\text{dRB}}(E_{r+1}) \), \( S_1 = \text{Hdg}(A) \) and \( S_2 = \text{Hdg}(E_{r+1}) \). We conclude

\[
G_{\text{dRB}}(A \times E_{r+1}^{n+1}) = G_m \cdot (\text{Hdg}(A) \times \text{Hdg}(E_{r+1})) = G_m \cdot \text{Hdg}(A \times E_{r+1}^{n+1}) = \text{MT}(A \times E_{r+1}^{n+1}).
\]

This finishes the induction step and the theorem follows. \( \square \)

**Lemma 7.2.** Let \( T/\mathbb{Q} \) be a torus of rank 1 and \( T \to T' \) a non-trivial quotient. Then \( T' \simeq T \).

**Proof.** If \( T \) is split, i.e., if \( T \simeq \mathbb{G}_m \), the result is clear. Assume that \( T \) is non-split. The torus \( T' \) is then also non-split. Otherwise we have \( T' \simeq \mathbb{G}_m \) and hence \( T \), as a quotient of \( T' \), is also isomorphic to \( \mathbb{G}_m \). The rank one non-split tori over \( \mathbb{Q} \) are classified by degree 2 field extensions of \( \mathbb{Q} \). The torus \( T \) is determined by its splitting field. Let \( K \supset \mathbb{Q} \) be the degree-2 extension that splits \( T \). Then \( T'_K \) is a quotient of \( T_K \simeq \mathbb{G}_{m,K} \). Hence \( T'_K \simeq \mathbb{G}_{m,K} \). In other words, \( T \) and \( T' \) have the same splitting field \( K \). Hence they are isomorphic. \( \square \)

**Lemma 7.3.** Let \( V_1 \) and \( V_2 \) be two \( \mathbb{Q} \)-vector spaces. Let \( Z_i \cong \mathbb{G}_m \subset \text{GL}(V_i) \), \( i = 1, 2 \) and \( Z \cong \mathbb{G}_m \subset \text{GL}(V_1 \oplus V_2) \) be the center of the corresponding group. Let \( Z_1 \subseteq G_1 \subseteq \text{GL}(V_1) \) and \( Z_2 \subseteq G_2 \subseteq \text{GL}(V_2) \) be two reductive subgroups. Let \( S_1 = G_1 \cap \text{SL}(V_1) \) and \( S_2 = G_2 \cap \text{SL}(V_2) \). Let \( G \subseteq G_1 \times G_2 \) be a subgroup that fits into the following commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\iota} & G_1 \times G_2 \\
\downarrow & & \downarrow \\
Z \cdot (S_1 \times S_2) & \xrightarrow{\iota} & \text{GL}(V_1) \times \text{GL}(V_2) \xrightarrow{\iota} \text{GL}(V_1 \oplus V_2)
\end{array}
\]

Assume that the following conditions hold.

- The two projections \( \pi_i : G \to G_i \), \( i = 1, 2 \), are surjective.
- The groups \( S_i \), \( i = 1, 2 \), admit no non-trivial homomorphisms \( S_i \to \mathbb{G}_m \).
- \( S_1 \) and \( S_2 \) have no non-trivial isomorphic quotients.

Then \( G = Z \cdot (S_1 \times S_2) \).

**Proof.** The natural homomorphisms

\[
Z_1 \times S_1 \to G_1, \quad Z_2 \times S_2 \to G_2
\]

are surjective with finite kernels \( \mu_{r_1} \) and \( \mu_{r_2} \). Let \( N_1 := \ker(\pi_2 : G \to G_2) \) and \( N_2 := \ker(\pi_1 : G \to G_1) \). Then \( N_1 \subseteq G_1 \times e_2 = G_1 \) and \( N_2 \subseteq e_1 \times G_2 = G_2 \) are normal subgroups, where \( e_i \in G_i(\mathbb{Q}) \), \( i = 1, 2 \), is the identity element. By Goursat’s lemma 6.2, the image of \( G \) in \( G_1/N_1 \times G_2/N_2 \) is the graph of an isomorphism \( \varphi : G_1/N_1 \xrightarrow{\sim} G_2/N_2 \). Consider the following commutative diagram

\[
\begin{array}{cccc}
\{1\} & \xrightarrow{\{1\}} & \{1\} \\
\uparrow & & \uparrow \\
Z_i \times S_i / \tilde{N}_i & \xrightarrow{\sim} & G_i / N_i \\
\uparrow & & \uparrow \\
\{1\} & \xrightarrow{\mu_{r_1}} & \{1\} \\
\uparrow & & \uparrow \\
\{1\} & \xrightarrow{\mu_{r_2}} & \{1\} \\
\uparrow & & \uparrow \\
\{1\} & \xrightarrow{\{1\}} & \{1\}
\end{array}
\]
where $\tilde{N}_i$ is the inverse image of $N_i$ in $Z_i \times S_i$ and $i = 1, 2$. Thus we get an exact sequence

$$
\{1\} \longrightarrow \tilde{N}_i \longrightarrow Z_i \times S_i \longrightarrow G_i/N_i \longrightarrow \{1\}
$$

for $i = 1, 2$. Let $S_i' \subseteq G_i/N_i$ be the image of $\{1\} \times S_i$ in $G_i/N_i$. We further let $T_i$ be the quotient of $G_i/N_i$ by the subgroup $S_i'$. One readily checks that $T_i$ can also be realized as a quotient of $Z_i = Z_i \times S_i/\{1\} \times S_i$. Since $Z_i \cong \mathbb{G}_m$, we conclude that either $T_i \cong \mathbb{G}_m$ or $T_i = \{1\}$, $i = 1, 2$.

Case 1: $T_1 = \{1\}$.

In this case we have $S_1' \cong G_1/N_1 \subseteq G_2/N_2 \\rightarrow T_2$. Since $S_1'$ is a quotient of $S_1$, we see that $T_2$ is a quotient of $S_1$. Thus by assumption $T_2$ can not be $\mathbb{G}_m$ and hence $T_2 = \{1\}$. Then we get

$$
S_1' \cong G_1/N_1 \cong G_2/N_2 \cong S_2'.
$$

Note that $S_1'$ is a quotient of $S_1$ and $S_2'$ is a quotient of $S_2$. By assumption we have $S_1' = \{1\}$ and $S_2' = \{1\}$. Thus we get $N_i = G_i$, $i = 1, 2$, and it follows that $G = G_1 \times G_2$, which is not possible.

Case 2: $T_1 \cong \mathbb{G}_m$.

In this case, we also have $T_2 \cong \mathbb{G}_m$. Consider the diagram

$$
\begin{array}{ccc}
\{1\} & \longrightarrow & S_1' \\
\downarrow & & \downarrow \varphi \\
\{1\} & \longrightarrow & S_2'
\end{array}
$$

$$
\begin{array}{ccc}
\{1\} & \longrightarrow & G_1/N_1 \\
\downarrow & \cong & \downarrow \\
\{1\} & \longrightarrow & G_2/N_2 \\
\downarrow & & \downarrow \\
\{1\} & & \mathbb{G}_m
\end{array}
$$

Note that the composition $S_1' \rightarrow G_1/N_1 \rightarrow G_2/N_2 \rightarrow \mathbb{G}_m$ is trivial since $S_1'$ is a quotient of $S_1$ and there is no non-trivial homomorphism $S_1 \rightarrow \mathbb{G}_m$. As a consequence $\varphi$ restricts to a homomorphism $\varphi|_{S_1'} : S_1' \rightarrow S_2'$, which is an isomorphism since the same argument can be carried out for $\varphi^{-1}$. From the assumption that $S_1$ and $S_2$ do not have any non-trivial common quotient, we see that $S_1' \cong S_2' = \{1\}$. Thus we get

$$
G_i/N_i \cong \mathbb{G}_m, \quad \tilde{N}_i \supseteq \{1\} \times S_i,
$$

for $i = 1, 2$. As a consequence $N_i \supseteq S_i$ and we get

$$
S_1 \times S_2 \subseteq G \subseteq Z \cdot (S_1 \times S_2).
$$

Note that $Z \cong \mathbb{G}_m$ and the above inclusion implies either $G = Z \cdot (S_1 \times S_2)$ or $G = \mu_1 \cdot (S_1 \times S_2)$ where $\mu_1 \subset Z$ is a finite subgroup. The condition that $\pi : G \rightarrow G_i = Z_i \cdot S_i = \mathbb{G}_m \cdot S_i$ is surjective forces $G = Z \cdot (S_1 \times S_2)$.

\[\square\]

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