Quantum-memory-assisted entropic uncertainty relation with a single nitrogen-vacancy center in diamond

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The limitation of simultaneous measurements of noncommuting observables can be eliminated when the measured particle is maximally entangled with a quantum memory. We present a proposal for testing this quantum-memory-assisted entropic uncertainty relation in a single nitrogen-vacancy (N-V) center in diamond only by local electronic measurements. As an application, this entropic uncertainty relation is used to witness entanglement between the electron and nuclear spins of the N-V center, which is close to reach the currently available technology.

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The Heisenberg’s uncertainty principle is the cornerstone of quantum mechanics, which provides a limitation of simultaneous measurements of canonically conjugate observables1. In quantum information theory, the uncertainty relation is usually characterized by entropic measures rather than standard deviations2-3. However, the traditional entropic uncertainty relation may be violated if particle A to be measured is initially entangled with a quantum memory. We present a proposal for testing this quantum-memory-assisted entropic uncertainty relation in a single N-V center in diamond only by local electronic measurements. As an application, this entropic uncertainty relation is used to witness entanglement between the electron and nuclear spins of the N-V center, which is close to reach the currently available technology.

S (Q|B) + S (R|B) ≥ log2 1 ε + S (A|B). (1)

Here, S (A|B) with Λ ∈ (Q, R) is the conditional von Neumann entropy of the post-measurement state ρAB = m (ψm) ⟨ψm| ⊗ 1⟩ ρAB ⟨ψm| ⟨ψm| ⊗ 1⟩, where 1 is the identity operator and {ψm} are the eigenstates of the observable Λ. The parameter log2 1 ε quantifies the incompatibility of two measurements, where ε = maxα,β |⟨φα| φβ⟩|^2, |⟨φα⟩| and |φβ⟩ are the eigenstates of two noncommuting observables Q and R, respectively. Equation (1) can be understood in a straightforward way: the entanglement between A and B may create a negative conditional entropy S (A|B) which will beat the incompatibility log2 1 ε. In other words, the quantum information held by the quantum memory B will help us to eliminate the uncertainty of the measurements performed on particle A. This entropic uncertainty relation together with other forms8-10 has practical applications such as for witnessing entanglement6 and crytography11 and has been explored under noise12 and in optical photon systems13,14.

Over the past few years, the negatively charged nitrogen-vacancy (N-V) center in diamond has been considered as one of the most promising building block for solid-state quantum information processing15. Indeed, due to weak magnetic interaction with the environment, the proximal nuclear spins in a single N-V center become a good quantum memory for information storage16-18. In addition, the paramagnetic electron spin in a single N-V center can be optically polarized and readout with high fidelity at room temperature19 and can also be used to polarize and readout the nuclear spins20 and realize conditional gating21,22. Based on above advantages, the N-V center provides an excellent test bed to explore this quantum-memory-assisted entropic uncertainty relation.

In this Letter, we present a practical proposal to verify this entropic uncertainty relation in a single N-V center in diamond by local electronic measurements. As a by-product, this entropic uncertainty relation is employed to witness entanglement of electron and nuclear spins in diamond. The experimental feasibility is also justified with current laboratory parameters.

Before introducing the experimental proposal, we first display a theoretical framework of the quantum-memory-assisted entropic uncertainty relation for two-qubit system, in which an arbitrary two-qubit state can be of the form23

ρAB = 1/4 [1A ⊗ 1B + ∑i=1 3 (xiσi ⊗ 1B + 1A ⊗ yiσiB) + ∑i,j=1,3 Tijσi ⊗ σjB],

where σi(j) with i(j) ∈ {1, 2, 3} correspond to standard Pauli matrices. We define two 3 × 3 vectors x with real components xi = trAB (ρABσi ⊗ 1B) and y with real components yi = trAB (ρABσi ⊗ σjB), and a 3 × 3 correlation tensor T with real components Tij = trAB (ρABσi ⊗ σjB).

If we choose two Pauli observables Q = σ1 and R = σ3 to be measured, the uncertainty, i.e., the left hand side of Eq. (1), can be expressed as

U = − ∑μ,ν=0,1 ημν log2 (ημν) − 2Hbin (1 − ||y||/2), (2)

where Hbin (p) = −p log2 p − (1 − p) log2 (1 − p) denotes the binary entropy, ημν = [1 + (−1)^μxλ + (−1)^ν ± (−1)^μyλ] / 1. Since the complementarity c of the observables σ1 and σ3 is always equal to 1/2, the lower bound of uncertainty, i.e., the right hand side of Eq. (1), takes the form

Ub = S (ρAB) + 1 − Hbin (1 − ||y||/2). (3)

In experiment, if we choose the same measurement Λ on particles A and B, we may get H (Λ) ≥ S (Λ) with H (·)
the Shannon entropy. According to Fano’s inequality \(^7\), we have \(H(\Lambda|\Lambda) \leq H_{bin}(\kappa_A)\) with \(\kappa_A\) the probability that the outcomes of the same measurement \(\Lambda\) on particles \(A\) and \(B\) are different. Therefore, \(H_{bin}(\kappa_A) \geq S(\Lambda|B)\), which will in general lead to a higher measurement estimation \(ME\) of the uncertainty, and can be used to conveniently test \(U\) with experimental counts\(^{13,14}\). For two-qubit states and observables \(\sigma_1\) and \(\sigma_3\),

\[
ME = H_{bin} \left(\frac{1 - T_{11}}{2}\right) + H_{bin} \left(\frac{1 - T_{33}}{2}\right).
\]

In Fig. 1(c), the structure of a pure N-V center in diamond is depicted, where we treat the electronic spin \((S = 1)\) as particle \(A\) and a proximal \(^{13}\)C nuclear spin \((I = 1/2)\) as quantum memory \(B\). In the following, we will employ the electronic states \(\{|0\rangle, |-1\rangle\}\) and the nuclear states \(\{|\downarrow\rangle, |\uparrow\rangle\}\) as two-qubit system [Fig. 1(b)].

An experimental schematic sequence for testing this quantum-memory-assisted entropic uncertainty relation is depicted in Fig. 1(c). The first dashed box represents the entangled state (for example, we consider the Schmidt state \(|\Phi\rangle = \cos \chi |0\downarrow\rangle + \sin \chi |1\uparrow\rangle\) with \(\chi \in [0, \pi/2]\) preparation with the following key steps\(^{16}\): (i) The electronic ground state is first polarized at state \(|0\rangle\) by a 532 nm green laser pulse and the two-qubit state now reads \(|0\rangle \otimes |\rho_n\rangle\) with \(\rho_n\) an unknown nuclear mixed state; (ii) Then we can transfer the nuclear state to the electronic state \(|0\rangle \otimes |\rho_n\rangle \rightarrow \rho_e \otimes |\downarrow\rangle \langle \downarrow|\) with a conditional MW1-\(\pi\) pulse followed by a conditional RF1-\(\pi\) pulse; (iii) The electronic state can then be polarized again by the 532 nm laser pulse, which reduces the two-qubit state to \(|0\rangle \langle 0|\); (iv) A MW1-2\(\chi\) pulse is then performed on the electronic state, which yields the product state \((\cos \chi |0\rangle + \sin \chi |1\rangle) \otimes |\downarrow\rangle\). Then a conditional RF2-\(\pi\) pulse will create the final entangled state \(|\Phi\rangle\).

After preparation of the entangled state, we then perform measurement \(Q = \sigma_1\) \(\text{(or } R = \sigma_3\text{)}\) on the electron spin. In our case, the \(\sigma_3\) operation can be directly manipulated by electron shelving projecting onto \(|0\rangle\) or \(|-1\rangle\) with 532 nm laser pulse, and the \(\sigma_1\) operation can be reduced to detecting \(\sigma_3\) by applying a stronger MW-\(\pi/2\) pulse before and after \(\sigma_3\) operation (i.e., \(\sigma_1 = H \sigma_3 H\) with \(H\) the Hadamard gate).

To acquire the measurement estimation of the uncertainty in experiment, the same observable \(Q = \sigma_1\) \(\text{(or } R = \sigma_3\text{)}\) should be performed on the proximal \(^{13}\)C nuclear spin. However, since electron spin is good for efficient processing and readout, while the nuclear spin is more suitable for long-term storage\(^{24}\), we will transfer the nuclear state to the electronic state and then perform the same operation \(\sigma_1\) \(\text{(or } \sigma_3\text{)}\) on the electron spin to quantify the uncertainty. This task can be accomplished by first polarizing the electron spin to the state \(|0\rangle\) again [the 3rd box in Fig. 1(c)], and then employing the conditional MW2-\(\pi\) pulse followed by a conditional RF2-\(\pi\) pulse to transfer the nuclear state to the electronic state [the 4th box in Fig. 1(c)]. Finally, we may perform the \(\sigma_1\) \(\text{(or } \sigma_3\text{)}\) operation on the electron spin again [the 5th box in Fig. 1(c)]. In comparison with the first and the second outcomes of \(\sigma_1\) \(\text{(or } \sigma_3\text{)}\) operation [the 2nd and 5th boxes in Fig. 1(c)], we may acquire the measurement estimation of the uncertainty.

For state \(|\Phi\rangle\) with \(x = y = (0, 0, \cos 2\chi)^T\) and \(\bar{T} = \text{diag}(\sin 2\chi, -\sin 2\chi, 1)\), it is easy to check that the uncertainty \(U = U_b = 1 - H_{bin}(\sin^2 2\chi)\) and the measurement estimation \(ME = H_{bin}(\frac{1 - \sin^2 2\chi}{2})\). In Fig. 2(a), the dependence of the uncertainty \(U\) and the measurement estimation \(ME\) on the Schmidt state angle \(\chi\) is plotted. The blue solid line represents the uncertainty \(U\) and the lower bound \(U_b\) (the equality is achieved in this case). The
orange dashed line represents the measurement estimation, which is higher than the uncertainty. To illustrate the role of entanglement on this quantum-memory-assisted entropic uncertainty relation, we employ the concurrence with $C = 2 \max \{0, \sin \chi \cos \chi \}$ [black dotted line in Fig. 2(a)] to characterize the entanglement. It is clearly illustrated that when the electron and nuclear spins are in a maximum entangled state ($\chi = \pi/4$), the uncertainty of two noncommuting observables can be totally eliminated.

One of the most remarkable applications of this quantum-memory-assisted entropic uncertainty relation is to witness entanglement of electron and nuclear spins in the N-V center. Our scheme with only local electronic measurements and no need for quantum state tomography is helpful for experimental implementation. The procedures are similar to the former testing case except for the initial state preparation, i.e., the preparation of initial entangled state is actually not required, so the first box in Fig. 1(c) could be removed. However, to illustrate the application on the entanglement witness, we assume that the electron and nuclear spins are in state $|\Xi\rangle = \frac{1}{\sqrt{2}} \{|0\downarrow\rangle \otimes |+\uparrow\rangle + |q\Phi^+\rangle \otimes |\Phi^+\rangle\}$, with $|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|0\downarrow\rangle + |−1\uparrow\rangle)$. For state $|\Xi\rangle$ with $x = y = (0, 0, 0)^t$ and $T = \text{diag}(q, −q, q)$, we have $U = ME = 2H_{bin}(\frac{1}{2q})$, $U_b = H_{bin}(\frac{1+q}{2}) + \log_2 3$, and $C = 2 \max (0, \frac{3q-1}{2})$. As shown in Fig. 2(b), the electron and nuclear spins must be entangled if the uncertainty is less than 1.

Finally, we survey the relevant experimental parameters. As reported in recent N-V experiments, the longitudinal relaxation time $T_{1n}$ of $^{13}C$ nuclear spin is around 1.7 s$^{18}$, and the coherence time $T_{2n}$ is about 20 ms$^{16}$ (We note that $T_{2n}$ of $^{13}C$ can reach on the order of one second with elegant dissipative decoupling technique (DDT)$^{18}$). However, the straightforward employment of DDT in our scheme will also induce complications with ionization and deionization of the N-V center). While for N-V electron spin at room temperature, the relaxation time $T_{1e}$ is about 6 ms and coherence time $T_{2e}$ is about from 350 $\mu$s$^{23}$ to 1.8 ms$^{28}$. Since the key step of our proposal is the state mapping from nucleus to electron, which will take on the scale of 100 $\mu$s$^{32}$, $T_{2e}$ becomes the key factor in our proposal ($T_{2n} = 20$ ms$^{16}$ is long enough for implementing our proposal). Here, we briefly study the electronic dephasing effect on this entropic uncertainty relation. As an example, the initial state $|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|0\downarrow\rangle + |−1\uparrow\rangle)$ under electronic dephasing effect yields $U = U_b = ME = H_{bin}(\frac{1−e^{-\frac{t}{\tau_e}}}{2})$, which is depicted in Fig. 3. Clearly, longer electronic dephasing time $T_{2e}$ will retain less uncertainty of two incompatible measurements.

In conclusion, the quantum-memory-assisted entropic uncertainty relation has been explored in a single N-V center in diamond. By investigating relevant experimental parameters, our proposal can be immediately verified under current N-V-based experimental conditions.

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