Isotropy vs anisotropy in small-scale turbulence

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Abstract

The decay of large-scale anisotropies in small-scale turbulent flow is investigated. By introducing two different kinds of estimators we discuss the relation between the presence of a hierarchy for the isotropic and the anisotropic scaling exponents and the persistence of anisotropies. Direct measurements from a channel flow numerical simulation are presented.

One of the main assumptions made by A.N. Kolmogorov in his 1941 theory is the restoring of universality and isotropy at small scales in turbulent flows. The idea is that the effects of a large-scale anisotropic forcing and/or boundary conditions are rapidly lost during the process of energy transfer toward the small scales. The overall result is that the isotropy and the universality of turbulent fluctuations should be locally restored at small enough scales and large enough Reynolds numbers. The rate of convergence toward isotropy can be quantitatively predicted within the K41 theory both as a function of the scale, e.g. for the structure functions, and as a function of the Reynolds numbers, e.g. for the single-point moments of the velocity gradients. Experiments [1] and numerical simulations [2, 3, 4] do not confirm those predictions. The skewness of the transversal gradients, $S_3 = \frac{\langle (\partial_y u_x)^3 \rangle}{\langle (\partial_y u_x)^2 \rangle^{3/2}}$ is for example found to have a very slow decay with $Re_\lambda$. The effect is even stronger for the fifth-order skewness $S_5 = \frac{\langle (\partial_y u_x)^5 \rangle}{\langle (\partial_y u_x)^2 \rangle^{5/2}}$, observed to remain $O(1)$ for all available $Re_\lambda$. Similar results were recently reported on a series of hydrodynamical problems. The most striking ones were obtained analytically in passive scalar/vector models advected by isotropic, Gaussian and white-in-time velocity fields (the so-called Kraichnan model [5]) with a large scale anisotropic forcing [6, 7]. Numerical [8, 9] and experimental (see, e.g., [10, 11]) evidences of persistence of anisotropies in real passive scalars have also been reported.

On one hand, there are then strong indications in favor of a persistent memory of the large-scale anisotropies even at the smallest scales of a turbulent flow. On the other hand, there are theoretical arguments [12] going in the opposite direction, i.e. that anisotropic fluctuations are sub-dominant with respect to the
isotropic ones (see below). This short note is meant to clarify the relation between the previous results and support the arguments by numerical simulations on channel flow turbulence.

The analysis in [12] is based on the invariance under rotations of the unforced Navier-Stokes equations. We shall specifically restrict here to the structure functions and refer to the original paper for more complicated tensorial objects. Since the Navier-Stokes equations are invariant under rotations, the correlations are conveniently decomposed in terms of the irreducible representations of the rotation group. For the $n$-th order longitudinal structure function we have for example

$$S_n(r) = \langle [(v(x) - v(x + r)) \cdot r]^n \rangle = \sum_{jm} S_j^m(|r|) Y_j^m(\hat{r})$$

where we have explicitly used the fact that the basis of the rotation group for scalar functions is the set of spherical harmonics $Y_j^m$. The coefficients $S_j^m(|r|)$ are expected to behave as power laws $r^{\xi_j}$ and the different scaling exponents to depend on the index $j$ (the exponents should not depend on $m$ since it does not appear in the equations of motion — see [12] for more details). The previous strong assumption is motivated by the idea of universality, i.e. that inertial-range scaling behaviors are independent of the large-scale boundary and forcing effects. Furthermore, it is natural to suppose a hierarchical organization of the different sectors in the inertial range, i.e. the existence of a hierarchy among the scaling exponents characterizing different sectors:

$$\xi_n^{j=0} < \xi_n^{j=1} < \xi_n^{j=2} < \ldots$$

This statement, even if not proved for the Navier-Stokes equations, is verified analytically in various Kraichnan models of passive fields [13, 14]. The existence of the hierarchy (2) implies that the anisotropic fluctuations become more and more subdominant at the small scales as their degree of anisotropy increases.

Let us now analyze in a quantitative way the relative importance of isotropic and anisotropic fluctuations. In the following we shall concentrate for simplicity on the structure functions, but the same arguments could be generalized to other correlations. Isotropic flows are characterized by having only the sector $j = 0, m = 0$ excited. One is therefore naturally lead to introduce two different tests to quantify the degree of isotropy/anisotropy. First, (case A) one can analyze fluctuations of comparable intensity, i.e. fixing the order $n$ of the structure function and measuring the scaling in different sectors. We can for example introduce the ratio between the projection on the anisotropic sector with the non-vanishing indices $j, m$ and the projection on the isotropic sector $j = m = 0$:

$$T_n^{jm}(r) = \frac{S_j^m(r)}{S_0^0(r)}$$

We thus have the possibility to disentangle different degrees of anisotropy depending on the typical intensity of the velocity fluctuations. Looking at the
structure functions of low order (small \( n \)'s) gives a test on the isotropy of the weak fluctuations, while looking at high orders (large \( n \)'s) gives a test on the statistics of strong turbulent fluctuations. A second possible estimator (case B) consists in first normalizing the field and then taking moments of it. As it is done for the skewness, the kurtosis etc etc., we can for example normalize by the isotropic component of the second order longitudinal structure function, \( S_{00}^2(|r|) = \langle \left[ (\mathbf{v}(x) - \mathbf{v}(x + r)) \cdot r \right]^2 \rangle_{j=0,m=0} \). The resulting dimensionless stochastic variable can then be studied by looking at its decomposition in different \( j, m \) sectors:

\[
\hat{S}_{jm}^n(|r|) = \frac{S_{jm}^n(|r|)}{(S_{00}^2(|r|))^{n/2}}.
\] (4)

If the hierarchy (2) holds, all the observables (A) tend to zero as the scale is decreased. The decay rates possibly differ from the dimensional predictions due to intermittency, but they are guaranteed to be positive. There is no experimental or numerical evidence that the hierarchy (2) is violated. The situation with observable (B) is quite different. The dimensionless quantities are indeed formed by comparing anisotropic and isotropic fluctuations of different intensity (in the numerator and denominator of (4) structure functions of different orders are involved). The hierarchy (2) does not give any constraint in this case and it is well possible that \( \xi_{jn} < \frac{n}{2} \xi_{2}^{jm} \). The corresponding observable (B) \( \hat{S}_{jm}^n \) defined in (4) would then diverge going toward the small scales, even in the presence of the hierarchy (2). That divergence is the effect of persistence of anisotropies reported in experiments and numerical simulations both for the passive scalars and Navier-Stokes turbulence (see [6, 11]). It is of importance to notice that the persistence of anisotropies is a combined effect of anisotropy and intermittency.

Let us now support the above arguments by presenting some results obtained in channel flow simulations. The simulations are performed on a grid of \( 128 \times 128 \times 256 \) points with periodic boundary conditions in the stream-wise and span-wise directions and no-slip boundary conditions at the top and the bottom walls. At the center of the channel we have \( \text{Re}_\lambda \sim 70 \). Due to the relatively moderate Reynolds number, no scaling laws are observed. Still, even in the absence of scaling laws, it is quite clear from the data that the two sets of observable (A) and (B) behave in a very different way. In Fig. 1 we present the quantities (A) and (B) for the structure functions of order 4 and 6 at the center of the channel for the sector \( j = 2, m = 2 \). In Fig. 2 the same is presented but for a higher sector, \( j = 4, m = 2 \). While the observable (A) always monotonically decreases with the scale, the observable (B) of the sixth order shows a clean tendency to increase. That is the manifestation of the persistence of anisotropies at the small scales and gives further support to the observations first made in [6]. Note that the scales shown in the figure go from the largest available one (the box size) to the beginning of the viscous scale. The decomposition in spherical harmonics at the very small scales (inside the viscous range) is hard to obtain numerically because of interpolation errors of the cubic grid on the
sphere. Details on the numerical procedure to compute the observable shown here can be found in [15].

As for the intermittency in the anisotropic sectors, the situation is still moot. There is only one attempt to directly measure the projections on each single sector in the same channel flow data set used here [15, 16]. As stated previously, the Reynolds number is unfortunately not high enough and scaling exponents of the anisotropic sectors can be measured only via the ESS [17]. In the anisotropic sectors it is even not quite clear what would be the dimensional prediction for the $\xi^n_j$ with $j > 0$. Different dimensionless quantities can indeed be built by using some anisotropic mean observable, e.g. the mean shear, and the usual energy dissipation. The dimensional predictions would then depend on the requirement that the anisotropic correction is (or is not) an analytical, smooth deviation from the isotropic sector. Furthermore, the comparison with the behavior observed in the Kraichnan models of scalar/vector fields [5, 18] suggests that the anisotropic sectors may show intermittent corrections induced by the homogeneous (non-linear, in the Navier-Stokes case) part of the equations for the correlation functions. If that is the case, the dimensional predictions might be very far from the observed behaviors.

In conclusion, we have discussed the decay of large-scale anisotropy memory in the small scales of turbulent flows. The analysis of numerical data from channel flow simulations indicate that the anisotropies persist at the small scales but still respecting the hierarchy (3) between the isotropic and anisotropic velocity components.

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**FIGURE CAPTIONS**

**FIGURE 1**: Analysis of the persistence of anisotropies with observable belonging to case A and B for the projection of structure functions in the sector $j = 2, m = 2$. Bottom: projections of fourth moment, $T_{4}^{2,2}(|r|) \times$ and $S_{4}^{2,2}(|r|) (+)$. Top: projection of sixth moment, $T_{6}^{2,2}(|r|)$ (Squares) and $S_{6}^{2,2}(|r|) (*)$. Notice how for the moment of order 6 we have a clear tendency toward increasing of $S_{6}^{2,2}(|r|)$ at small scales. Scales are dropped at $R \sim 10$ which corresponds to the onset of the viscous scale in the simulation. For details on how to compute numerically the projections, $S_{n}^{jm}(|r|)$ see [15].

**FIGURE 2**: The same of figure 1 but for the sector $j = 4, m = 2$. Bottom: projections of fourth moment, $T_{4}^{4,2}(|r|) \times$ and $S_{4}^{4,2}(|r|) (+)$. Top: projection of sixth moment, $T_{6}^{4,2}(|r|)$ (Squares) and $S_{6}^{4,2}(|r|) (*)$. 

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FIGURE 1, L. Biferale and M. Vergassola
FIGURE 2, L. Biferale and M. Vergassola