Towards a New ODE Solver Based on Cartan’s Equivalence Method

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ABSTRACT
The aim of the present paper is to propose an algorithm for a new ODE–solver which should improve the abilities of current solvers to handle second order differential equations. The paper provides also a theoretical result revealing the relationship between the change of coordinates, that maps the generic equation to a given target equation, and the symmetry D-groupoid of this target.

Categories and Subject Descriptors
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1. INTRODUCTION
Current ODE-solvers make use of a combination of symmetry methods and classification methods. Classification methods are used when the ODE matches a recognizable pattern (that is, for which a solving method is already implemented), and symmetry methods are reserved for the non-classifiable cases — Fig. 1. Using symmetry methods, the solvers first look for the generators of 1-parameter symmetry groups of the given ODE, and then use this information to integrate it, or at least reduce its order [4, 5].

In practice, present solvers are often unable to return closed form solution. Consider for instance the following equation

\[ y'' = -y^3 y^4 - \frac{y^2}{y} - \frac{1}{2}y, \]  (1)

which admits only one 1-parameter symmetry group. Using this information, actual solvers return a complicated first order ODE and a quadrature. Clearly, such output is quite useless for practical applications. More dramatically, consider the following equation

\[ y'' = \frac{2x^4y' - 6y^2x - 1}{x^5}. \]  (2)

When applied actual solvers, output no result. This failure is due to the fact that the above equation does not match any recognizable pattern and has zerodimensional point symmetry group(oid). Thus neither symmetry method nor classification method works.

Our solver (the implementation is in progress) is designed to handle such equations. It returns an equation from Kamke’s book [9], equivalent to the equation to be solved, and the equivalence transformation \( \varphi \). Thus, for the equation (1) we obtain the Rayleigh equation \( y'' + y^4 + y = 0 \) (number 72 in [9]) and the change of coordinates \( \varphi : (x, y) \rightarrow (x, y^2/2) \). For the equation (2), we obtain the first Painlevé equation \( y'' = 6y^2 + x \) (number 3 in [9]) and the change of coordinates \( \varphi : (x, y) \rightarrow (1/x, y) \). It is worth noticing that this transfor-
...there exists a unique inverse arrow. We shall see that this transformation change of coordinates is the source and target equation. This jet is said to be invertible in which every arrow is invertible. Moreover, the degree of \( \varphi \) is equal to \( \text{card}(\mathcal{S}_f) \) which is finite. Note that, the use of the D-groupoids formalism is dictated by the non-global invertibility of the transformation \( \varphi \).

As we shall see, \( \varphi \) can be obtained using differential elimination. Unfortunately, such approach is rarely effective due to expressions swell. For this reason, we propose in section 2 a new method to precompute the transformation \( \varphi \) in terms of differential invariants, for each target equation \( \mathcal{E}_f \) in Kamke’s list. These invariants are provided by Cartan’s method. In the last section, we present our solver. This solver uses 7 possible types of transformations \( \Phi_1, \ldots, \Phi_7 \). Using Lie infinitesimal method we precalculate to each target equation a signature. That is, the dimensions of the 7 symmetry groupoids associated to the 7 groupoids \( \Phi_1, \ldots, \Phi_7 \), If two differential equations are equivalent then their signature match. This fact significantly restricts the space of research in Kamke’s list at the run-time (when the input equation \( \mathcal{E}_f \) is known).

2. EQUIVALENCE PROBLEMS

2.1 Groupoids

Definition 1 (Groupoid). A groupoid is a category in which every arrow is invertible. Let \( (G, o, s, t) \) be a category. Each arrow \( \varphi \in G \) admits a source \( s(\varphi) \) and a target \( t(\varphi) \) which are objects of this category. For each arrows \( \alpha, \beta \in G \) such that \( s(\beta) = t(\alpha) \), there exists a unique arrow \( \beta \circ \alpha \in G \) with the source \( s(\alpha) \) and the target \( t(\beta) \). If \( G \) is a groupoid, for each arrow \( \varphi \in G \), there exists a unique inverse arrow \( \varphi^{-1} \) such that \( \varphi^{-1} \circ \varphi = \text{Id}_{s(\varphi)} \) and \( \varphi \circ \varphi^{-1} = \text{Id}_{t(\varphi)} \). Let \( X \) and \( U \) be two manifolds and \( x \in X \). The Taylor series up to order \( q \) (i.e. the jet of order \( q \)) of a function \( f : X \to U \), of class \( C^q \), is denoted \( j_q^X f \). The Taylor series of \( f \) about \( x \) is denoted \( j_0^X f \) or \( j_0^U f \). We shall say that \( x \) is in \( X \) is the source and \( f(x) \in U \) is the target of the \( q \)-jet \( j_q^X f \).

Example 1. For instance, when \( X = U = \mathbb{C} \), we have 
\[
j_0^X f := \left( x, f(x), f'(x), \ldots, f^{(q)}(x) \right) \in \mathbb{C}^{q+2}.
\]
This jet is said to be invertible if \( f'(x) \neq 0 \). The jet of the function \( \text{Id} \) about the point \( x \) is \( (x, x, 1, 0, \ldots, 0) \).

For each \( q \in \mathbb{N} \) and each \( x \in X \), we set \( J_0^X f := j_q^X f \) and \( J^q f := j_q^X f \). We denote by \( J^q f \) the manifold of \( J^q f \) for the composition of Taylor series up to order \( q \) according to 
\[
j_q^X (g \circ f) = \left( j_q^X g \right) \circ (j_q^X f).
\]

By definition, a D-groupoid \( \mathbb{G} \subset J^q \) is a sub-groupoid of \( J^q \) for the Taylor series solutions (see def. 5 of an algebraic PDE system called the Lie defining equations. This system contains an inequation which expresses the invertibility of the jets. The set of smooth functions \( \varphi : X \to X \) that are local solutions of the Lie defining equations of \( \mathbb{G} \) is a pseudo-group denoted by \( \mathbb{G} \). We define the dimension \( \dim \mathbb{G} : = \dim C \) and, if \( \dim C = 0 \), \( \deg \mathbb{G} : = \deg C \) where \( C \) is a characteristic set (see sect. 3) of the Lie defining equations. We have \( \deg \mathbb{G} = \text{card}(\Gamma \mathbb{G}) \).

Example 2 (\( \Phi_3 \)). Let \( \Phi_3 \) (see table 1) be the D-groupoid of infinite jets of transformations \( (\bar{x}, \bar{y}) = \varphi(x, y) \) where 
\[
\bar{x} = x + C \text{ and } \bar{y} = y \times (x, y).
\]
The constant \( C \in \mathbb{C} \) and the function \( \eta : \mathbb{C} \to \mathbb{C} \) are arbitrary. \( \Phi_3 \subset J^q \) (\( C^2, C^2 \)) is an infinite dimensional D-groupoid where the corresponding Lie defining equations are 
\[
\bar{x}_x = 1, \bar{x}_y = 0, \bar{y}_x \neq 0.
\]

Definition 2 (\( \mathbb{G} \)-invariant). An invariant of the D-groupoid \( \mathbb{G} \subset J^q \) is a function \( I : X \to \mathbb{C} \) which is constant on the orbits of \( \mathbb{G} \).

Clearly, the sum, the product and the ratio of two invariant functions is still an invariant function. Consequently, invariant functions of \( \mathbb{G} \) define a field.

2.2 Differential equations and diffeologies

Let \( \mathcal{E}_f \) denotes the generic ODE 
\[
y^{(n+1)} = f(x, y, y', \ldots, y^{(n)}).
\]
Let \( M := J^n(\mathbb{C}, \mathbb{C}) \) be the \( n \)-th order jets space of functions from \( \mathbb{C} \) to \( \mathbb{C} \). Let \( x := (x, y, y_1, \ldots, y_n) \in \mathbb{C}^n \) be a local coordinates system over \( M \) where \( m := n + 2 = \dim M \).

Every differential equation \( \mathcal{E}_f \) defines a diffeology \( \mathcal{D}_f \). This diffeology is given by the manifold \( M \) and a set of 1-forms, called contact forms, satisfying the Frobenius condition of complete integrability. Contact forms are linear combinations of the basic contact 1-forms \( dy - y_1 dx, dy_1 - y_2 dx, \ldots, dy_n - f(x) dx \). Vector fields which are orthogonal to the contact forms are colinear to the Cartan field 
\[
D_z := \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \cdots + f(x) \frac{\partial}{\partial y_n}.
\]
They generate a distribution denoted by \( \Delta_f \). A local isomorphism \( \varphi \) between two diffeologies \( \mathcal{E}_f \) and \( \mathcal{E}_f \) is, by definition, a local diffeomorphism \( \varphi : M \to M \) such that 
\[
\Delta_f = \varphi_*(\Delta_f).
\]

2.3 Equivalence problem and symmetries

Definition 3 (Equivalence problem). An equivalence problem (EPB) is an ordered pair (\( M, \Phi \)) where \( M = J^n(\mathbb{C}, \mathbb{C}) \) and \( \Phi \subset J^q(\mathbb{C}^2, \mathbb{C}^2) \) is a D-groupoid of point transformations from \( \mathbb{C}^2 \to \mathbb{C}^2 \).
There exists a unique prolongation of $\Phi$, denoted $\Phi^{(n)}$, that acts on $M$ (see section 3.4.1). Two differential equations $E_f$ and $E_{\tilde{f}}$ are said to be equivalent if there exists a local transformation $\varphi : M \rightarrow M$ satisfying the differential system
\[
\Delta_f = \varphi_*(\Delta_{\tilde{f}}) \quad \text{and} \quad \varphi \in \Gamma \Phi^{(n)}.
\] (7)
The second condition means that $\varphi$ fulfills the Lie defining equations of the $D$-groupoid $\Phi^{(n)}$.

The system $\Phi$ is fundamental and we shall see that it can be treated by two different approaches: brute-force method based on differential algebra (section 3) and geometric approach relying on Cartan's theory of exterior differential systems (section 4). It is classically known that the existence of a solution to the system $\Delta = \{\partial_{x_1}, \ldots, \partial_{x_n}\}$ is, actually, a group with 3 elements. The system $\Phi$ is completely algorithmic whenever $f$ and $\tilde{f}$ are explicitly known [3, 10, 1]. However, there is no general algorithm for computing closed form of $\varphi$. In the sequel, we shall show that if the function $f$ is fixed such that a certain $D$-groupoid $S_f$ is zero-dimensional, then $\varphi$ is obtained without integrating any differential equation.

**Definition 4** ($S_f$). To any EPB, with fixed target equation $E_f$, we associate the $D$-groupoid $S_f \subset J^\infty(M, M)$ formed by the Taylor series solutions of the self-equivalence problem
\[
\Delta_{\tilde{f}} = \sigma_*(\Delta_f) \quad \text{and} \quad \sigma \in \Gamma \Phi^{(n)}.
\] (8)

**Example 3.** Consider the EPB $(\mathcal{J}^1(C, C), \Phi_3)$ and the Emden-Fowler equation $E_f$ (number 11 in [9])
\[
y'' = \frac{x}{y} y'.
\] (9)
The Lie defining equations of the $D$-groupoid $S_f$ are given by the characteristic set
\[
\{ \bar{\varphi} = \frac{\bar{y}p}{y}, \quad \bar{y}^3 = y^3, \quad \bar{x} = x \}.
\] (10)
This PDE system is particular. Indeed, it contains only non differential equations. We have $\dim S_f = 0$ and $\deg S_f = 3$. We deduce that its associated pseudo-group
\[
\Gamma S_f = \{(x, y, p) \mapsto (x, \lambda y, \lambda p) \mid \lambda^3 = 1\}
\] is, actually, a group with 3 elements.

**Equivalence problem and associated $D$-groupoid**
Let $X := J^\infty(M, C)$. Any EPB $(M, \Phi)$ defines a $D$-groupoid $\mathcal{G} \subset J^\infty(X, X)$ formed by the set of triplets
\[
(j_x, f, J_{x(x)} \tilde{f})
\] where $x \in M$ and the functions $(f, \varphi, \tilde{f})$ are local solutions of the differential system $\Phi$. The source of a triplet is the infinite jet $j_x f \in X$ and the target is the infinite jet $j_{\varphi(x)} \tilde{f} \in X$. The composition of two triplets $(j_{x_1} f, j_{x_1} \varphi_1, j_{x_1} f_1)$ and $(j_{x_2} f_1, j_{x_2} \varphi_2, j_{x_2} f_2)$ is the triplet $(j_{x_3} f, j_{x_3} \varphi, j_{x_3} f_2)$ where we have $\varphi := \varphi_2 \circ \varphi_1$.

**Definition 5** (Specialized invariant). For each $G$-invariant $I$ and each function $f : M \rightarrow C$, we define the specialized invariant $I[f] : M \rightarrow C$ by
\[
I[f](x) := I(j_x f), \quad x \in M.
\] (11)

### 3. USING DIFFERENTIAL ALGEBRA
The aim of this section is to use differential elimination to solve the EPB when the target function $f$ is a $Q$-rational function, explicitly known and the $D$-groupoid of symmetries $S_f$ is zerodimensional.

#### 3.1 The vocabulary
The reader is assumed to be familiar with the basic notions and notations of differential algebra. Reference books are [18] and [10]. We also refer to [2, 8, 1]. Let $U = \{u_1, \ldots, u_n\}$ be a set of differential indeterminates. $k$ is a differential field of characteristic zero endowed with the set of derivations $\Delta = \{\partial_1, \ldots, \partial_n\}$. The monoid of derivations
\[
\Theta := \{\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N}\}
\] (12)
acts freely on the alphabet $U$ and defines a new (infinite) alphabet $\Theta U$. The differential ring of the polynomials built over $\Theta U$ with coefficients in $k$ is denoted $R = k[U]$. Fix an admissible ranking over $\Theta U$. For $f \in R$, $\text{Id}(f) \in \Theta U$ denotes the leader (main variable). $I_j \in R$ denotes the initial of $f$ and $S_f \subset R$ denotes the separant of $f$. Recall that $S_f = \text{Id}(f)$ where $v = \text{Id}(f)$. Let $C \subset R$ be a finite set of differential polynomials. Denote by $[C]$ the differential ideal generated by $C$ and by $\sqrt{[C]}$ the radical of $[C]$. Let $H_C := \{I_j \mid f \in C\} \cup \{S_f \mid f \in C\}$. As usual, $\text{full}_\text{rem}$ is the Ritt full reduction algorithm [10]. If $r = \text{full}_\text{rem}(f, C)$ then $\exists h \in H_C^n, hf = r \mod [C]$. Then the normal form is defined by $\text{normal}_\text{form}(f) := r/h$.

**Definition 6** (Characteristic set). The set $C \subset R$ is said to be a characteristic set of the differential ideal $\epsilon := \sqrt{[C]} : H_C^n \subset R$ is prime.

(1) $C$ is autoreduced,
(2) $f \in \epsilon$ if and only if $\text{full}_\text{rem}(f, C) = 0$.

**Definition 7** (Quasi-linear characteristic set). The characteristic set $C \subset R$ is said to be quasi-linear if for each $f \in C$ we have $\deg (f, v) = 1$ where $v$ is the leader of $f$.

**Proposition 1.** When the characteristic set $C$ is quasi-linear, the differential ideal $\epsilon := \sqrt{[C]} : H_C^n \subset R$ is prime.

#### 3.2 Taylor series solutions space
Let $k := \mathbb{Q}(x_1, \cdots, x_p)$ be the differential field of coefficients endowed with the set of derivations $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p}\}$. Let $C$ be a characteristic set of a prime differential ideal $\epsilon \subset R$. We associate to $C$ the system
\[
(C = 0, H_C \neq 0)
\] (13)
of equations $f = 0, f \in C$ and inequations $h \neq 0, h \in H_C$.

**Definition 8** (Taylor series solution). $A$ Taylor series solution of the PDE system (13) above is a morphism $\mu : R \rightarrow C$ of (non differential) $\mathbb{Q}$-algebras such that
\[
[C] \subset \ker \mu \neq 0 \quad \text{and} \quad \epsilon \cap \ker \mu = 0.
\] The morphism $\mu$ defines an infinite jet where the source is $\mu(x_0) := (\mu(x_1), \ldots, \mu(x_n)) \in C^n$ and the target is $t(\mu) := (\mu(u_1), \ldots, \mu(u_n)) \in C^n$. Thus, a Taylor series is a $C$-point of an algebraic quasi-affine variety. Its Zarisky closure is an affine variety defined by the ideal $\epsilon$. The dimension of the solutions space of (13) is the number of arbitrary constants appearing in the Taylor series solutions $\mu$ when the
source point \( x := s(\mu) \in \mathbb{C}^p \) is determined. Let \( K \) be the fractions field \( \text{Frac}(R/c) \). Recall that the transcendence degree of a field extension \( K/k \) is the greatest number of elements in \( K \) which are k-algebraically independent. The degree \( [K : k] \) is the dimension of \( K \) as a k-vector space. When \( \text{tr deg}(K/k) = 0 \), the field \( K \) is algebraic over \( k \) and \( [K : k] < \infty \). If \( f \in C \), we denote \( \text{rank}(f) := (v, d) \) where \( v := \text{ld } f \) and \( d := \text{deg}(f, v) \).

\[
\begin{align*}
\text{rank } C & := \{ \text{rank}(f) \mid f \in C \} \\
\text{ld } C & := \{ \text{ld}(f) \mid f \in C \} \\
\text{dim } C & := |\text{card}(\Theta U \setminus \Theta(\text{ld } C))| \\
\text{deg } C & := \prod_{f \in C} \text{deg}(f, \text{ld } f).
\end{align*}
\]

**Proposition 2.** \( \text{dim } C = \text{tr deg}(K/k) \) is the dimension of the solutions space of (13). If \( \text{dim } C = 0 \) then the cardinal of the solutions space is finite and equal to \( \text{deg } C = [K : k] \).

### 3.3 Differential elimination

Let \( U = U_1 \cup U_2 \) be a partition of the alphabet \( U \). A ranking which eliminates the indeterminates of \( U_2 \) is such that

\[
\forall v_1 \in \Theta U_1, \forall v_2 \in \Theta U_2, \quad v_2 \succ v_1. \tag{14}
\]

Assume that \( C \) is a characteristic set of the prime differential ideal \( \epsilon = \sqrt{C} \subset R^p_C \) w.r.t. the elimination ranking \( \Theta U_2 \succ \Theta U_1 \). Let \( R_1 := k\{U_1\} \) be the differential polynomials \( k \)-algebra generated by the set \( U_1 \). Consider the set \( C_1 := C \cap R_1 \) and the differential ideal \( \epsilon_1 := \epsilon \cap R_1 \).

**Proposition 3.** \( C_1 \) is a characteristic set of \( \epsilon_1 \).

Consider the differential field of fractions \( K := \text{Frac}(R/c) \) and denote by \( \alpha : R \to K \) the canonical \( k \)-algebra morphism. Let \( K_1 \) be the differential subfield of \( K \) generated by the set \( \alpha(R_1) \). Then \( K_1 \) is the fraction field associated to the prime differential ideal \( \epsilon_1 := \epsilon \cap R_1 \). The partition of the characteristic set

\[
C = C_1 \cup C_2 \quad (\text{i.e. } C_2 := C \setminus C_1), \tag{15}
\]

enables us to study the field extension \( K_1/K \).

**Proposition 4.** \( \text{tr deg}(K/K_1) = \text{dim } C_2 \). If \( \text{dim } C_2 = 0 \) then \( [K : K_1] = \text{deg } C_2 \).

### 3.4 The system \((7)\) revisited

#### 3.4.1 Prolongation algorithm

Our aim, here, is to prolong the action of \( \Phi \subset J^n(C^2, C^2) \) on the manifold \( M := J^n(C, C) \). For each integer \( q \geq 0 \), define

\[
\begin{align*}
k^{(q)} & := Q(x, y, y_1, \ldots, y_q) \\
R^{(q)} & := k^{(q)}\{ \bar{x}, \bar{y}, \bar{y}_1, \ldots, \bar{y}_q \}
\end{align*}
\]

The differential field \( k^{(q)} \) is the coefficients field of the ring of differential polynomials \( R^{(q)} \) endowed with the set of derivations \( \{ \partial_1, \partial_2, \ldots, \partial_q \} \). Let us assume that the Lie defining equations of \( \Phi \) are given by a characteristic set \( C^{(0)} \subset R^{(0)} \). The \( D \)-groupoid \( \Phi^{(q)} \) acting on \( J^q \) and prolonging the action of \( \Phi \) is characterized by a characteristic set \( C^{(q)} \subset R^{(q)} \). The prolongation formulae of the point transformation \((x, y) \to (\xi(x, y), \eta(x, y))\) are of the form

\[
\bar{y}_q = \eta_q(x, \bar{x}, \bar{y}, \ldots, \bar{y}_q),
\]

where \( \bar{y}_q = \eta(x, y) \) if \( q = 0 \). The computation of the characteristic set \( C^{(q)} \) is done incrementally using the infinite Cartan field \( D_{\xi} := \partial_{\bar{y}} + y_1 \partial_{\bar{y}_1} + y_2 \partial_{\bar{y}_2} + \cdots \)

\[
\eta_q := \left( D_{\xi} \eta_{q-1} \right)^{-1}, \quad C^{(q)} = C^{(q-1)} \cup \left\{ \bar{y}_q - \text{normal form}\left( \eta_q, C^{(q-1)} \right) \right\}
\]

**Proposition 5.** If \( C^{(0)} \) is a quasi-linear characteristic set of \( \Phi \) then \( C^{(q)} \) is a quasi-linear characteristic set of the algebraic system \( \Phi^{(q)} \) w.r.t. the elimination ranking \( \Theta \bar{y}_q \succ \Theta \bar{y}_{q-1} \succ \cdots \succ \Theta \{ \bar{y}, \bar{x} \} \).

The previous proposition gives an efficient method to prolong a \( D \)-groupoid \( \Phi \) without the explicit knowledge of transformations.

#### 3.4.2 EPB with fixed target

Let us compute a characteristic set \( C[\bar{f}] \subset R^{(n)}[f] \) for the PDE system (7) (where \( \bar{f} \) is fixed \( Q \)-rational function). First, prolong \( C^{(0)} \) up to the order \( n + 1 \) as above. Then \( C[\bar{f}] \) is obtained by substituting in \( C^{(n-1)} \) the indeterminate \( y_{n+1} \) by the symbol \( f \) and the indeterminate \( \bar{y}_{n+1} \) by the rational function \( f(\bar{x}, \ldots, \bar{y}_n) \).

**Example 4.** For the EPB \((J^1(C, C), \Phi_{3})\), we have

\[
\begin{align*}
\bar{p} - \bar{y}_1 - \bar{p}y_2 & = 0, \\
\bar{y}_x + 2\bar{p}y_{x\bar{y}} + \bar{p}^2 \bar{y}_{x\bar{y}} + f \bar{y} - \bar{f}(\bar{x}, \bar{y}, \bar{p}) & = 0, \\
\bar{x}_y - 1 & = 0, \quad \bar{y}_x = 0, \quad \bar{x}_p = 0, \quad \bar{y}_p = 0, \quad \bar{y}_y \neq 0.
\end{align*}
\]

These equations constitute a quasi-linear characteristic set w.r.t. the elimination ranking \( \Theta \bar{f} \succ \Theta \bar{p} \succ \Theta \bar{y} \succ \Theta \bar{x} \).

Hence, the associated differential ideal is prime.

**Corollary 1.** The PDE system (7) (where \( \bar{f} \) is a fixed \( Q \)-rational function) is a quasi-linear characteristic set \( C[\bar{f}] \subset R^{(0)}[f] \) w.r.t the elimination ranking \( \Theta \bar{f} \succ \Theta \bar{p} \succ \Theta \bar{y} \succ \Theta \bar{x} \).

### 3.5 Brute-force method

Using ROSENFIELD-GRÖBNER we compute a new characteristic set \( C[\bar{f}] \) of the PDE system (7) w.r.t. the new ranking \( \Theta \{ y_1, \ldots, y_1, \bar{x} \} \succ \Theta \{ f \} \). We make the partition of \( C := C[\bar{f}] \) as in (16)

\[
C = C_f \cup C_{\varphi}, \tag{17}
\]

where \( C_f := C \cap k^{(n)} \{ f \} \) and \( C_{\varphi} := C \setminus C_f \).

**Proposition 6.** The transformation \( \varphi \) does exist for almost any function \( f \) satisfying the PDE system associated to the characteristic set \( C_f[\bar{f}] \). The function \( \bar{x} = \varphi(x) \) is solution of the PDE system associated to \( C_{\varphi}[\bar{f}] \).

If \( \text{dim } C_{\varphi}[\bar{f}] = 0 \), one can calculate \( \varphi \) by an algebraic process without integrating differential equations.

**Definition 9.** When \( \text{dim } C_{\varphi}[\bar{f}] = 0 \), the algebraic system associated to \( C_{\varphi}[\bar{f}] \) is called the necessary form of the change of coordinates \( \bar{x} = \varphi(x) \).
Example 5. Consider the EPB \((J^1(C,\mathbb{C}),\Phi_3)\). Suppose that the target \(E_j\) is the Airy equation
\[
y'' = \bar{y}^3.
\]
In this case, Rosenfeld-Gröbner returns \(C_{p}[\bar{f}]\) and \(C_{f}[\bar{f}]\) resp. given by (13) and (19)
\[
\begin{align*}
y_{xx} &= -f_{yy} + p_{xy}y + 1/2 p_{yy}f_{yy} + \bar{y} - 1/2 y f_{yy} - 1/2 y f_{yy} \quad (18) \\
y_{xy} &= -1/2 f_{yy} y + 1/2 p_{yy} y \\
y_{yy} &= -1/2 f_{yy} y + 1/4 y f_{yy} - 1/2 y f_{yy} \\
x &= f_{yy} - 1/2 f_{yy} f + 1/4 f_{yy} f + 1/2 f_{yy} f \\
fy & = y_f - p_{pp} f_{pp} \quad (19) \\
f_{xxp} &= 2 f_{xy} + p_{xxp} - 2 f_{pp} f_{xxp} - 2 p_{pp} f_{xxp} + p_{pp} f_{xxp} \\
f_{xyp} &= 2 f_{xy} - p_{xy} - p_{pp} f_{xyp} - 2 p_{pp} f_{xyp} + p_{pp} f_{xyp} \\
f_{yp} &= f_{pp} - p_{pp} f_{yp} \\
f_{pp} &= 0.
\end{align*}
\]
We have \(\dim C_{p}[\bar{f}] = 3\) which means that the transformation \(\bar{x} = \varphi(x)\), when \(\varphi\) satisfies \(C_{f}[\bar{f}]\), depends on 3 arbitrary constants.

Example 6. Consider the EPB \((J^1(C,\mathbb{C}),\Phi_1)\) where \(\Phi_1\) is defined in table 1. Assume that the target equation \(E_j\) is
\[
y'' = \bar{y}^3.
\]
Here, Rosenfeld-Gröbner returns \(C_{p}[\bar{f}]\) and \(C_{f}[\bar{f}]\) resp. given by (20) and (21)
\[
\begin{align*}
y^2 &= 1/2 (4 f_y - 2 f_{xy} - 2 f_{pp} f - 2 p_{pp} f + p_{pp} f), \\
x &= x, \\
f_{xxp} &= (4 f_y - 2 f_{xy} - 2 f_{pp} f - 2 p_{pp} f + p_{pp} f)^{-1} \times \\
&\vdots \\
f_{xyp} &= (4 f_y - 2 f_{xy} - 2 f_{pp} f - 2 p_{pp} f + p_{pp} f)^{-1} \times \\
&\vdots \\
f_{yp} &= f_{pp} - p_{pp} f \quad (21) \\
f_{pp} &= 0.
\end{align*}
\]
Consequently \(\dim C_{p}[\bar{f}] = 0\) and \(\deg C_{p}[\bar{f}] = 2\). Thus, \(\varphi\) is the algebraic transformation of degree 2, given by equations (20).

3.6 Discrete symmetries \(\mathcal{D}\)-groupoids

The self-equivalence problem, is in fact, the EPB when the PDE system (4) is specialized by substituting the symbol \(f\) by the value \(\bar{f}(x)\), that is
\[
f := \bar{f}(x, \bar{y}, \ldots, \bar{y}_n).
\]
After specialization, the differential system \(C_{f}[\bar{f}]\) constraining the function \(f\) is automatically satisfied (since there exists at least one solution \(\bar{x} = \sigma(x)\) of the problem, namely \(\sigma = \text{Id}\)). The symmetries \(\sigma\) are solutions of a characteristic set \(C_{p}[\bar{f}]\) obtained form \(C_{f}[\bar{f}]\) by the specialization (22).

By definition, the degree of an algebraic transformation \(\bar{x} = \varphi(x)\) is the generic number of points \(\bar{x}\) when \(x\) is determined.

Theorem 1. The following conditions are equivalent
\[
(1) \quad \dim (C_{p}[\bar{f}]) = 0, \\
(2) \quad \dim (S_{\mathcal{D}}) = 0, \\
(3) \quad \deg (S_{\mathcal{D}}) < \infty.
\]
In this case, \(\deg S_{\mathcal{D}} = \deg (C_{p}[\bar{f}]) = \deg \varphi\).

Proof. Define
\[
\mathcal{G}_{\mathcal{D}} := \{(j_{\mathcal{D}}, j_{\mathcal{D}}\varphi, j_{\mathcal{D}}\bar{f}) \in \mathcal{G} \mid \bar{f} \text{ determined}\}. 
\]
\(\mathcal{G}_{\mathcal{D}}\) is an algebraic covering of \(\mathcal{M}\) defined by the characteristic set \(C_{p}[\bar{f}]\). The \(\mathcal{D}\)-groupoid \(S_{\mathcal{D}} \subset \mathcal{G}_{\mathcal{D}}\) is defined by differential system (5) i.e. the characteristic set \(C_{p}[\bar{f}]\). Figure 2 shows that \(S_{\mathcal{D}}\) acts simply transitively on \(\mathcal{G}_{\mathcal{D}}\).

\[
\begin{array}{c}
\varphi_0 \\
\mathcal{G}_{\mathcal{D}} \\
j_{\mathcal{D}} \bar{f} \\
\mathcal{S}_{\mathcal{D}} \ni \varphi \\
\sigma \\
j_{\mathcal{D}} \bar{f}
\end{array}
\]

Figure 2: Simply transitive action of \(S_{\mathcal{D}}\) on \(\mathcal{G}_{\mathcal{D}}\) where \(\bar{x}_0 = \varphi_0(x)\) and \(\bar{x} = \varphi(x)\).

Choose a point \(\bar{x}_0\) in \(M\). For every \(\varphi_0 \in \Gamma \Phi\), define the rational transformation \(S_{\mathcal{D}} \rightarrow \mathcal{G}_{\mathcal{D}}\)
\[
j_{\mathcal{D}} \sigma \rightarrow \varphi_0 \circ j_{\mathcal{D}} \varphi, \quad (\sigma \in \Gamma \mathcal{S}_{\mathcal{D}}).
\]
In fact, according to the Taylor series composition formulae, this transformation is birational. Thus, the one-to-one correspondence between the two algebraic varieties \(\mathcal{G}_{\mathcal{D}}\) and \(S_{\mathcal{D}}\) is birational. Consequently, the two characteristic sets \(C_{p}[\bar{f}]\) and \(C_{f}[\bar{f}]\) have the same dimension and the same degree.

Lemma 1. The rank of the characteristic set \(C_{p}[\bar{f}]\) is stable under the specialization (22) i.e. rank \(C_{p}[\bar{f}] = \text{rank} C_{f}[\bar{f}]\).

Proof. The specialization (22) transforms the characteristic set \(C_{p}[\bar{f}]\) to \(C_{f}[\bar{f}]\). A fall of the rank of \(C_{p}[\bar{f}]\) during the specialization contradicts the existence of birational correspondence between \(\mathcal{G}_{\mathcal{D}}\) and \(S_{\mathcal{D}}\).

Remark 1. When the transformation \(\bar{x} := \varphi(x)\) is locally bijective but not globally, \(S_{\mathcal{D}}\) and \(S_{\mathcal{D}}\) need not to have the same degree. Indeed, consider again the groupoid \(\Phi_3\) and the equations
\[
y'' = \frac{6y^4 + x - 2y^2}{2y} \quad \text{and} \quad y''' = 6y^2 + \bar{x}
\]
which are equivalent under \((\bar{x} = x, \bar{y} = y^2)\). The corresponding symmetry group are respectively given by
\[
\Gamma S_{\mathcal{D}} = \{(x, y) \rightarrow (x, \lambda y) \mid \lambda^2 = \pm 1\} \quad \text{and} \quad \Gamma S_{\mathcal{D}} = \{\text{Id}\}.
\]
They have the same dimension but different cardinal.

3.7 Expression swell

In practice, the above brute-force method, which consists of applying Rosenfeld-Gröbner to the PDE system (4), is rarely effective due to expressions swell. Much of the examples treated here and in [6], using our algorithm Chgt-Cords, can not be treated with this approach.

It seems that the problem lies in the fact that we can not separate the computation of \(C_{p}[\bar{f}]\) from that of \(C_{f}[\bar{f}]\) which contains, very often, big expressions.
An other disadvantage of the above method is that we have to restart computation from the very beginning if the target equation is changed. In the next section, we propose our algorithm ChtgCoords to compute the transformation \( \varphi \) alone and in terms of differential invariants. These invariants are provided by Cartan method for a generic \( f \) which means that we have not re-apply Cartan method if the target equation is changed and a big part of calculations is generic. Furthermore, the computation of \( \varphi \) in terms of differential invariants reduces significantly the size of the expressions.

4. USING CARTAN’S METHOD

In this paper, differential invariants are obtained using Cartan’s equivalence method. We refer the reader to [4, 13, 16, 7] for an expanded tutorial presentation and application to second order ODE. When applied Cartan’s method furnishes a finite set of fundamental invariants and a certain number of invariant derivations generating the differential field of invariant functions.

**Example 7.** Consider the EPB \((J^1(C, C), \Phi_3)\). The PDE system \(7\) reads

\[
\begin{pmatrix}
\frac{d\bar{p} - f(\bar{x}, \bar{y}, \bar{p}) d\bar{x}}{d\bar{y} - p \bar{d} x}
\end{pmatrix}
= \begin{pmatrix}
a_1 & a_2 & 0 \\
0 & a_3 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{d\bar{p} - f(x, y, p) dx}{dy - p dx}
\end{pmatrix}
\]

with \( \det(S(a)) \neq 0 \). In accordance with Cartan, this system is lifted to the linear Pfaffian system

\[
S(a) \omega_f = S(a) \omega_f
\]

defined on the manifold of local coordinates \((x, a, \bar{x}, \bar{a})\). After two normalizations and one prolongation, Cartan’s method yields three fundamental invariants \((p = y', a = a_3)\)

\[
I_1 = -\frac{1}{4} f_p^2 - f_{xy} + \frac{1}{2} D_x f_p, \quad I_2 = f_{ppp}, \quad I_3 = \frac{f_{ppp} - D_x f_{pp}}{2a^2},
\]

and the invariant derivations

\[
X_1 = \frac{1}{a} \frac{\partial}{\partial p}, \quad X_3 = D_x - \frac{1}{2} f_p \frac{\partial}{\partial a}, \quad X_4 = a \frac{\partial}{\partial a},
\]

\[
X_2 = \frac{1}{a} \frac{\partial}{\partial y} + \frac{1}{2} f_p \frac{\partial}{\partial a} - \frac{1}{2} f_{pp} \frac{\partial}{\partial a},
\]

where \( D_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + f(x, y, p) \frac{\partial}{\partial y} \).

When \( \dim(S_f) = 0 \), the additional parameter \( a \) can be (post)normalized by fixing some invariant to some suitable value. In this manner one constructs invariants defined on \( M \) (not depending on the additional parameter).

**Theorem 2 (Olver [16]).** If \( \dim(S_f) = 0 \), then there exist exactly \( m \) functionally independent specialized invariants \( I_1[f], \ldots, I_m[f] \).

Note that the invariants \( I_1[f], \ldots, I_m[f] \) are functionally independent if and only if \( dI_1[f] \wedge \cdots \wedge dI_m[f] \neq 0 \). Note also that if the function \( f \) is rational, then the specialized invariants \( I[f] : M \to C \) are algebraic functions. In the sequel, we use the notation \( I_{i,j} \cdots k \) to denote the differential invariant \( X_k \cdots X_j(I_i) \).

4.1 Computation of \( \varphi \)

Suppose that the \( D \)-groupoid \( S_f \) is zerodimensional. Then, according to the theorem 2 there exists \( m \) functionally independent invariants \( F_k := I_k[\bar{f}], 1 \leq k \leq m \). This implies that the algebraic (non differential) system

\[
\{ F_1(\bar{x}) = I_1, \ldots, F_m(\bar{x}) = I_m \}
\]

is locally invertible and has a finite number of solutions \( \bar{x} = F^{-1}(I_1, \ldots, I_m) \).

The specialization of \( I_1, \ldots, I_m \) on the source function \( f \) yields

\[
\bar{x} = F^{-1}(I_1[f], \ldots, I_m[f]).
\]

Let \( C \) denote the (non differential) characteristic set associated to the system \([24]\) w.r.t. the elimination ranking \( \{ x, y, \ldots, y_n \} \succ \{ I_1, \ldots, I_m \} \). Thus, \( C \) describes the inversion \([20]\). The most simple situation happens when \( \deg(C) = 1 \). In this case, the necessary form of the change of coordinates \( \varphi \) is the rational transformation defined by \( C \).

**Example 8.** Consider the EPB \((J^1(C, C), \Phi_3)\) and the target equation \( \mathcal{E}_f \) introduced by G. Reid [77]

\[
\bar{y}'' = \frac{\bar{y}'}{x} + \frac{4\bar{y}^2}{x^2}.
\]

The following invariants are functionally independent

\[
\bar{I}_{1,23} = -20 \frac{1}{a_3^4}, \quad \bar{I}_{1,31} = 8\frac{1}{a_3^2}, \quad \bar{I}_1 = \frac{3}{4a_3^2} + 8\frac{\bar{y}}{a_3^3}, \quad \bar{I}_{1,3} = \frac{-3\bar{x} - 48\bar{y} + 16\bar{y}^2}{2a_3^4}.
\]

We normalize the parameter \( \bar{a} \) by setting \( \bar{I}_{1,23} = -20 \). The characteristic set \( C \) is

\[
\begin{cases}
\bar{p} = -\frac{3}{32} + \frac{3}{512} I_{1,31}^2 I_1 + \frac{1}{4096} I_{1,3}^3 I_{1,31}, \\
\bar{y} = \frac{3}{256} I_{1,31} + \frac{1}{4096} I_{1,31}, \\
\bar{x} = \frac{1}{8} I_{1,31},
\end{cases}
\]

which gives the sought necessary form of \( \varphi \). As a byproduct we deduce that the symmetry group \( \Gamma S_f = \{ \text{Id} \} \).

Let us return to the general situation, that is when \( \deg(C) \) is strictly bigger than 1. We have two cases. First, \( \deg(C) = \deg(S_f) \) and then \( \varphi \) is the algebraic transformation defined by \( C \). Second, \( \deg(C) > \deg(S_f) \). In this case, to obtain the transformation \( \varphi \), we have to look for \( m \) other functionally independent invariants such that the new characteristic set \( C \) has degree equal to \( \deg(S_f) \).

**Example 9.** Consider the EPB \((J^1(C, C), \Phi_3)\) and the target equation \( \mathcal{E}_f \) (number 8 in [29])

\[
\bar{y}'' = \bar{y}' + \bar{x} \bar{y}'
\]

which the corresponding symmetry group is

\[
\Gamma S_f = \{ (x, y) \mapsto (x, \lambda y) | \lambda^2 = 1 \}.
\]

One can verify that \( I_1, I_{1,13} \) and \( I_{1,133} \), when specialized on the considered equation, are functionally independent. In
In this case, the associated characteristic set \( C \) is
\[
\begin{align*}
\tilde{p} &= -\left(4\bar{x}^2 + 2I_1\bar{x} - 3I_{1,33} - 2I_1^3\right)\bar{y}, \\
\tilde{y}^2 &= -\frac{1}{3}\bar{x} - \frac{1}{3}I_1, \\
\tilde{x}^3 &= -\frac{3}{4}I_1\bar{x}^2 + \frac{3}{4}I_{1,32}\bar{x} - \frac{3}{4}I_{1,3} - \frac{3}{8}I_{1,3}^2 + \frac{3}{4}I_{1,33}I_1 \\
&+ \frac{1}{2}I_1 - \frac{3}{8}.
\end{align*}
\]
The degree of this set is equal to 6 which is different from the degree of the symmetry groupoid.

However, if instead of the above invariants we consider the invariants \( K_1 := I_{1,233}/I_{1,31}, \ K_2 := I_{1,234}/I_{1,31} \) and \( K_3 := I_{1,231}/I_{1,31} \), we obtain
\[
\begin{align*}
\tilde{p} &= -K_1\bar{y}, \\
\tilde{y}^2 &= \frac{1}{6}K_3, \\
\tilde{x} &= -\frac{1}{6}K_4 + K_1.
\end{align*}
\]
This characteristic gives the necessary form of \( \varphi \) since it has degree two.

### 4.1.1 Heuristic of degree reduction

In practice, one has to search the invariants giving the required degree in the algebra of invariants. However, this is not an easy task since this algebra can be very large (although it is algorithmic). For this reason we provide an important heuristic which enables us to obtain the desired invariants. This heuristic is explained in the following example.

**Example 10.** Consider the Emden–Fowler equation \(^{24}\) and the D-groupoid of transformations \( \Phi_3 \). We have already computed the corresponding symmetry groupoid. The specialization of the invariants \( I_1, I_{1,13} \) and \( I_{1,133} \) gives three functionally independent functions. As explained above, we obtain the following characteristic set computed \( w.r.t. \) the ranking \( \bar{p} > \bar{y} > \bar{x} > I_1 > I_{1,3} > I_{1,33} \)
\[
\begin{align*}
\tilde{p} &= \left(\frac{3}{8}I_1 - \frac{1}{4}I_{1,33} + \frac{1}{3}I_{1,3}^2\right)x\bar{y} - \frac{1}{6}I_1\bar{y}, \\
\tilde{y}^2 &= -\frac{9}{4}I_{1,3}^2 + \frac{3}{2}I_{1,33}\bar{x} - \frac{1}{6}I_1, \\
\tilde{x}^2 &= 4\left(9I_{1,4} - 8I_{1,3}^2 + 6I_{1,33}I_1\right)x + 8I_1I_{1,3} - 8I_{1,3}^2 + 6I_{1,33}I_1.
\end{align*}
\] (28)

Comparing with the D-groupoid of symmetries \(^{10}\) we deduce that, in contrary to \( \bar{y} \), the degree of \( \bar{x} \) must be reduced to one. This can be done in the following manner. First, observe that the Lie defining equations of \( \Phi_3 \), more exactly \( \bar{x}_p = 0 \), implies that \( X_1(\bar{x}) = 0 \) where \( X_1 = \frac{\partial}{\partial x} \) is the invariant derivation \(^{24}\). Now, differentiate the last equation of the characteristic set, which we write as \( \bar{x}^2 = A\bar{x} + B \), \( w.r.t. \) the derivation \( X_1 \). We find \( A_1\bar{x} + B_1 = 0 \). The coefficient of \( \bar{x} \) in this equation, which was invariant, could not vanish (since it is not identically zero when specializing on the Emden–Fowler equation). Thus, \( \bar{x} = -\frac{B_1}{A_1} \) or explicitly
\[
\bar{x} = -\frac{2KI_{1,11} + I_3K_1}{KI_{1,31} + I_{1,3}K_1} \quad \text{with} \quad K := \frac{I_1}{9I_1^3 - 8I_{1,3}^2 + 6I_{1,33}I_1}.
\] (29)

The necessary form of the change of coordinates \( \varphi \) is then given by \(^{29}\) and the two first equations of \(^{23}\).

The above reasoning can be summarized as follows

**Procedure ChgCoords**

**Input:** \( \mathcal{E}_f \) and \( \Phi \) such that \( \dim(\mathcal{S}_f) = 0 \)

**Output:** \( \bar{x} = \varphi(x) \) the necessary form of the change of coordinates

1. Find \( m \) functionally independent invariants \( (\eta_1[\bar{f}], \ldots, \eta_m[\bar{f}]) \) defined on \( M \).
2. Compute a char. set \( C \) of the algebraic system \(^{20}\).
3. If \( \deg(C) = 1 \) then Return \( C \).
4. Compute \( \mathcal{S}_f \) with ROSENFIELD-GRÖBNER.
5. WHILE \( \deg(C) \neq \deg(\mathcal{S}_f) \) DO
   a. Reduce the degree of \( C \).
   b. END DO
6. Return \( C \).

### 5. THE SOLVER

#### 5.1 Precalculation of \( \varphi \)

**5.1.1 The first step : the adapted D-groupoid**

Let \( \Phi_1, \ldots, \Phi_\tau \subset J^*_\varnothing(C^2, C^\infty) \) denote the D-groupoids defined in the table 1 above. It is not difficult to see that \( \Phi_1 \subset \Phi_3 \subset \Phi_5 \) and \( \Phi_2 \subset \Phi_1 \subset \Phi_8 \) and finally \( \Phi_5, \Phi_6 \subset \Phi_7 \).

Let \( d(\mathcal{E}_f, \Phi) := \dim(\mathrm{aut}(\mathcal{E}_f) \cap \Phi) \) where \( \mathrm{aut}(\mathcal{E}_f) \) is the contact symmetry D-groupoid of the second order ODE \( \mathcal{E}_f \).

Let \( d_\tau := d(\mathcal{E}_f, \Phi_\tau) \) for \( 1 \leq \tau \leq 7 \).

**Definition 10** (Signature). The signature of \( \mathcal{E}_f \) is
\[
\mathrm{sign}(\mathcal{E}_f) := (d_1, d_3, d_5, d_2, d_4, d_6, d_7).
\]

Clearly, \( d_1 \leq d_3 \leq d_5 \leq d_7 \) and \( d_2 \leq d_4 \leq d_6 \leq d_7 \).

Recall that the calculation of these dimensions does not require solving differential equations. We shall say that the signature \( \mathrm{sign}(\mathcal{E}_f) \) matches the signature \( \mathrm{sign}(\mathcal{E}_f) \) if and only if \( d_7 = d_\tau \) and \( (s_1 = s_1 \ or \ s_2 = s_2) \) where \( s_1 \) and \( s_2 \) stand for \( (d_1, d_3, d_5) \) and \( (d_2, d_4, d_6) \) resp. Two second order ODE \( \mathcal{E}_f \) and \( \mathcal{E}_f \) are said to be strongly equivalent if
\[
\forall \Phi \in \{\Phi_1, \cdots, \Phi_\tau\}, \exists \varphi \in \Phi, \varphi \mathcal{E}_f = \mathcal{E}_f, \ d(\mathcal{E}_f, \Phi) = 0.
\]

**Lemma 2.** If \( \mathcal{E}_f \) and \( \mathcal{E}_f \) are strongly equivalent then their signatures match.

**Definition 11** (Adapted D-groupoid). A D-groupoid \( \Phi \) is said to be adapted to the ODE \( \mathcal{E}_f \) if \( d(\mathcal{E}_f, \Phi) = 0 \) and \( \Phi \) is maximal among \( \Phi_1, \cdots, \Phi_\tau \) satisfying this property.

| \( \Phi_1 \) | \( x = x, \ y = \eta(x, y) \) | 1, 2, 4, 7, 10, 21, 23, 24, 30, 31, 32, 40, 42, 45, 47, 50 |
| \( \Phi_3 \) | \( x = x + C', \ y = \eta(x, y) \) | 9, 11, 17, 19, 23, 27, 30, 34, 36, 40, 42, 45, 47, 50 |
| \( \Phi_5 \) | \( x = \xi(x, y), \ y = \eta(x, y) \) | Null |
| \( \Phi_6 \) | \( x = \xi(x, y), \ y = y + C \) | 81, 89, 135, 143, 145, 235 |
| \( \Phi_7 \) | \( x = \xi(x, y), \ y = \eta(x, y) \) | 11, 24, 79, 90, 92, 93, 94, 97, 98, 105, 106, 156, 172 |
| \( \Phi_8 \) | \( x = \xi(x, y), \ y = \eta(x, y) \) | 80, 86, 156, 219, 233 |

**Table 1.**
The above table associates to each equation in the third column its adapted groupoids. For instance, the first Painlevé equation (number 3) appears in the last row which means that its adapted $\mathcal{D}$-groupoid is the point transformations $\mathcal{D}$-groupoid $\Phi_7$. To the Emden–Fowler equation, number 11, we associate the $\mathcal{D}$-groupoids $\Phi_3$ and $\Phi_4$. In the case of homogeneous linear second order ODE (e.g. Airy equation, Bessel equation, Gauß hyper-geometric equation) we prove that, generically, the adapted $\mathcal{D}$-groupoid is $\Phi_4$.

5.1.2 The second step
Once the list of adapted $\mathcal{D}$-groupoids $\Phi$ is known, we proceed by computing the necessary form of the change of coordinates $\varphi \in \Phi$ using ChgtCoord. Doing so, we construct a MAPLE table indexed by Kamke’s book equations and where entries corresponding to the index $\mathcal{E}_f$ are:
1- the signature of $\mathcal{E}_f$,
2- the list of the adapted $\mathcal{D}$-groupoids $\Phi$ of $\mathcal{E}_f$,
3- the necessary form of the change of coordinates $\varphi \in \Phi$.
For instance, the entries associated to Rayleigh equation $y'' + y' + y = 0$ are:
1- the signature $((0, 1, 1), (1, 1, 1), 1)$,
2- the $\mathcal{D}$-groupoid $\Phi_3$,
3- the necessary form of the change of coordinates
\[
\begin{align*}
\bar{\rho} &= -36 I_{2;1}^1 + 72 I_{2;1}^1 I_{1;1}^1 + I_{2;1}^2 \bar{y}, \\
\bar{x} &= x,
\end{align*}
\]
\[
\bar{y} = \begin{cases} 
-1 & 559872 I_{2;1}^1 + 216 I_{1;1}^1 + 216 I_{2;1}^2 + 373248 \cdot (I_{2;1}^1 + 1191744 I_{1;1}^1 + 31104 I_{1;1}^2 + 1119744 I_{1;1}^3 + 15552 I_{1;1}^4) \\
+1191744 I_{1;1}^1 + 373248 I_{1;1}^3 + 15552 I_{1;1}^4 + 216 I_{2;1}^2 + 216 I_{1;1}^2 + 72 I_{1;1}^1 \end{cases}
\]
with the normalization $I_3/I_{2;1} = 1$. Invariants here are those generated by (23) and (24) plus the essential invariant $\bar{x} = x$.

5.2 Algorithmic scheme of the solver
To integrate a differential equation $\mathcal{E}_f$ our solver proceeds as follows

\textbf{PROEDURE} Newdsolve
\begin{itemize}
\item \textbf{Input :} $\mathcal{E}_f$
\item \textbf{Output :} An equation $\mathcal{E}_f$ in Kamke’s book and the transformation $\varphi$ such that $\varphi_*(\mathcal{E}_f) = \mathcal{E}_f$
\end{itemize}
1- Compute the signature of $\mathcal{E}_f$.
2- Select from the table the list of equations $\mathcal{E}_f$ such that $\text{sign}(\mathcal{E}_f)$ matches $\text{sign}(\mathcal{E}_f)$.
3- FOR each equation $\mathcal{E}_f$ in the selected list DO
\begin{itemize}
\item (i) Specialize, on $\mathcal{E}_f$, the necessary form of the change of coordinates associated to $\mathcal{E}_f$. We obtain $\varphi$.
\item (ii) If $\varphi \in \Phi$ and $\varphi_*(\mathcal{E}_f) = \mathcal{E}_f$ then return $(\mathcal{E}_f, \varphi)$.
\end{itemize}
END DO.

It is worth noticing that the time required to perform steps (i)- (ii) is very small. In fact, it is about one hundredth of a second using Pentium(4) with 256 Mo. The second feature of our solver is, contrarily to the symmetry methods, neither the table construction nor the algorithm of the solver involves integration of differential equations.

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