Avoiding the cosmological constant issue in a class of phenomenologically viable $F(R, G)$ theories

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In this paper we investigate a class of phenomenologically viable $F(R, G)$ theories that are able to avoid the cosmological constant issue. While the absence of ghosts and other kinds of instability issues is of prime importance, other reasonable requirements such as vanishing effective (low-curvature) cosmological constant, including the flat space as a stable vacuum solution, are also imposed on the viable models. These are free of the cosmological constant problem thanks to the following outstanding feature: the de Sitter space is an attractor of the asymptotic cosmological dynamics, with the resulting constant Hubble rate being unrelated both to the energy density of vacuum and to the low-curvature effective cosmological constant.

I. INTRODUCTION

The cosmological constant problem (CCP) is one of the current unsolved puzzles in fundamental physics. In the most widespread version of the issue, the challenge is to explain the origin of the large discrepancy between the theoretically predicted value of the energy density of vacuum $\rho_{\text{vac}}^{\text{theor}} \approx 10^{47} \text{GeV}^4$ and the observed value $\rho_{\text{vac}}^{\text{obs}} \approx 10^{-48} \text{GeV}^4$.

In this paper we shall not search for a solution to the CCP, that has shown to be a very complex issue with strong roots in the particle’s physics sector of the field theory. Instead, we shall look for theoretically consistent modifications of general relativity (GR) that are able to avoid the issue. The absence of the problem can be an alternative explanation to the unsolved puzzle. In order to ensure this goal, the following necessary and sufficient conditions should be satisfied.

- **Necessary condition**: de Sitter space with constant Hubble rate $H = H_0$, where $H_0$ is its present value, should be an attractor of the asymptotic dynamics of the related Friedmann-Robertson-Walker (FRW) cosmological model.

- **Sufficient condition**: $H_0$ should be unrelated both to the quantum vacuum energy density and to the low-curvature effective cosmological constant (if different from the energy density of vacuum). The necessary condition ensures that, no matter which modification of GR one is dealing with, at present it should be indistinguishable from the $\Lambda$CDM cosmological model, and that this de Sitter stage is a natural outcome of the cosmological evolution, quite independent of the chosen initial conditions. The sufficient condition assures that the present (observed) value of the Hubble rate $H_0 \approx 10^{-10} h \text{yr}^{-1}$, where $h$ is a dimensionless parameter in the range $0.62 \lesssim h \lesssim 0.82$ or, in Planck units: $H_0 \sim 10^{-42} \text{GeV}$, has nothing to do neither with the vacuum energy density $\rho_{\text{vac}} \sim \rho_{\text{Pl}} \sim 10^{47} \text{GeV}^4 \rightarrow H_{\text{vac}} \sim M_{\text{Pl}} \sim 10^{18} \text{GeV}$, nor with the effective (low-curvature) cosmological constant $\Lambda_{\text{eff}}$ (assuming that these are not coincident), which may be assumed to be vanishing if flat space is to be a solution of the equations of motion (EOM).

The phenomenologically viable models should satisfy additional reasonable consistency requirements:

1. The theoretical framework should be free of ghosts and other harmful instabilities.

2. For sufficiently small curvature the theory should be indistinguishable from GR with an effective – presumably very small or even vanishing – cosmological constant.

3. Although in curved space the energy density of the quantum vacuum must be non-vanishing, in flat space it should be zero due to some (yet undiscovered) symmetry. Hence, flat space should be a stable solution of the vacuum EOM.
The first consistency requirement above is an unavoidable theoretical criterion that any viable model of actual physical processes should respect. Ghosts that arise in modified gravity theories describe physical excitations that are drawn as external lines in Feynman diagrams. The existence of physical ghost leads, eventually, to the existence of negative norm states or to negative energy eigenstates. Hence, one is faced either with problems for the formulation of a consistent quantum theory or with catastrophic instabilities when the ghost couples to conventional matter fields. The second requirement impacts directly the phenomenological viability of the theoretical framework. It reflects our belief, deeply rooted in the existing amount of experimental evidence, that any modification of gravity in the Solar system, at leading order, must be very close to GR. In this regard, the third requirement is a consequence of our understanding that weak gravity may be viewed as a small deformation of Minkowski space or, in other words, that, for an isolated source of gravity, the space is asymptotically flat as in GR. One example of a theory that certainly does not satisfy one of the conditions stated above: the sufficient condition for avoidance of the cosmological constant issue, as well as the additional requirements for phenomenological viability. The asymptotic cosmological dynamics of the class of BI inspired models is investigated in section V for the particular case when the vacuum has vanishing energy density, while the asymptotic dynamics of the general case when the energy density of vacuum is non-vanishing, is discussed in section VI. In both cases the late time de Sitter attractor is identified and fully characterized. The way in which the cosmological constant problem is avoided in the chosen class of models, is discussed in section VII. Other interesting aspects of the model are discussed in section VIII while brief conclusions are given in section IX.

II. F(R, G) MODIFICATIONS OF GRAVITY

Here we shall focus in \( F(R, G) \) theories of the kind \( F(Lovelock) \) gravity \[27, 28\], i. e.,

\[
F(R, G) = F(\alpha R + \beta G),
\]

where \( \alpha \) and \( \beta \) are parameters with mass dimensions \( M^{-2} \) and \( M^{-4} \), respectively. Hence, the following relationships take place:

\[
F_G = \frac{\beta}{\alpha} F_R, \quad F_{GR} = F_{RG}, \quad F_{RR} = \frac{\alpha}{\beta} F_{GR}, \quad F_{GG} = \frac{\beta}{\alpha} F_{RG},
\]

where \( F_R \equiv \partial F/\partial R, F_{GR} \equiv \partial^2 F/\partial G \partial R, \) etc.

We consider an action of the form:
\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ F(R, \mathcal{G}) + 2\epsilon \kappa^2 \mu^4 \right], \quad (3) \]

where \( \mu \) and \( \kappa \) are free parameters with the dimension of mass and inverse mass, respectively, while \( \epsilon = \pm 1 \). The EOM that are derived from the above action – plus a matter piece action – read \cite{29, 30}:

\[ G_{\mu\nu} + \Sigma_{\text{curv}} = \kappa^2 \left[ T^{(m)}_{\mu\nu} + \epsilon \mu^4 g_{\mu\nu} \right], \quad (4) \]

where we have defined the effective gravitational coupling

\[ \kappa^2 \equiv \frac{\kappa^2}{F_R}, \quad (5) \]

while

\[ \Sigma_{\text{curv}} = -\frac{1 + \frac{2G}{F_R}}{F_R} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) F_R + \frac{1}{2} \left( R + \frac{\beta}{\alpha} \mathcal{G} - \frac{F}{F_R} \right) g_{\mu\nu} \]

\[ + \frac{4\beta}{\alpha F_R} (R_{\lambda\mu} \nabla^\lambda \nabla_\nu + R_{\lambda\nu} \nabla^\lambda \nabla_\mu - R_{\mu\nu} \nabla^2) F_R + \frac{4\beta}{\alpha} (R_{\mu\lambda\sigma} - g_{\mu\nu} R_{\lambda\sigma}) \nabla^\lambda \nabla^\sigma \frac{F_R}{F_R}, \quad (6) \]

comprises the contribution coming from the fourth-order curvature terms, \( \nabla^2 \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \) and \( T^{(m)}_{\mu\nu} \) is the stress-energy tensor of the matter degrees of freedom. While deriving the EOM (6) we have taken into account the relationships (2). The trace of (4):

\[ 3 \nabla^2 F_R = \frac{4\beta}{\alpha} G_{\lambda\sigma} \frac{\nabla^\lambda \nabla^\sigma F_R}{F_R} + R + \frac{2\beta}{\alpha} \mathcal{G} - \frac{2F}{F_R} = \frac{\kappa^2}{F_R} \left[ T^{(m)} + 4\epsilon \mu^4 \right], \quad (7) \]

where \( T^{(m)} = g^{\mu\nu} T^{(m)}_{\mu\nu} \) is the trace of the stress-energy tensor of matter, amounts to an additional dynamical equation on the variable \( F_R \).

### A. Small curvature limit \( (R \approx \mathcal{G} \approx 0) \)

Let us to expand the function \( F(R, \mathcal{G}) \) at small curvature up to fourth-order curvature terms \cite{25}:

\[ F(R, \mathcal{G}) = F_0 + F_0^0 R + \frac{1}{2} F_{RR} R^2 + F_{\mathcal{G}}^0 \mathcal{G}, \quad (8) \]

where \( F_0 = F(R, \mathcal{G}) = F(0,0), \ F_0^0 = \partial F/\partial R\big|_{(0,0)}, \) etc. The following effective (low curvature) action is retrieved:

\[ S_{\text{eff}} = \frac{M_{P1}^2}{2} \int d^4x \sqrt{-g} \left( R - 2\Lambda_{\text{eff}} + \frac{1}{6m_0^2} R^2 + \frac{\beta}{\alpha} \mathcal{G} \right), \quad (9) \]

where

\[ M_{P1}^2 = \frac{F_0^0}{\kappa^2}, \ \Lambda_{\text{eff}} = -\frac{F_0}{2F_R} + \frac{\epsilon \mu^4}{M_{P1}^2}, \ m_0^2 = \frac{F_0}{3F_{RR}}, \quad (10) \]

The Gauss-Bonnet term in (9) amounts to a total divergence so that it does not modify the EOM and may be safely omitted.

The exchange of the extra scalar degree of freedom with mass \( m_0 \) between two test particles with masses \( m_1 \) and \( m_2 \), modifies the Newtonian gravitational potential through an additional Yukawa interaction:
\[ V(r) = -G_N \frac{m_1 m_2}{r} \left[ 1 + \alpha \exp \left( -\frac{m_0 c}{h} r \right) \right], \]

where \( c \) is the speed of light, \( h \) is the reduced Planck constant and

\[ m_0 = \frac{h}{c \lambda} = \frac{1.967}{\lambda} \times 10^{-10} \text{ GeV}, \quad (11) \]

with the length scale \( \lambda \) in \( \mu \text{m} \). According to [31] the gravitational-strength Yukawa interactions are limited to ranges \( \lambda < 38.6 \mu \text{m} \) with 95\% confidence, so that \( m_0 > 5 \times 10^{-30} M_{\text{Pl}} \). Hence the following bound is to be satisfied:

\[ m_0^2 = \frac{F_0^R}{3 F_0^{RR}} > 2.5 \times 10^{-59} M_{\text{Pl}}^2, \quad (12) \]

It should be stressed that, for arbitrary small curvature – close enough to flat space – the above expansion is just a mirage since, as \( R \to 0 \), the related curvature quantities \( R^2 \) and \( \mathcal{G} \) vanish faster than \( R \), unless

\[ \frac{\beta}{\alpha} \mathcal{G} \sim \frac{R^2}{m_0^2} \sim R. \quad (13) \]

In other words, the effective action [3] is correct only for small curvatures down to scales of the order \( m_0^2 \sim \alpha/\beta \). For much smaller curvature the effective action just coincides with the Einstein-Hilbert action:

\[ S_{\text{EH}}^{\text{eff}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda_{\text{eff}}) . \quad (14) \]

For instance, at present cosmological scales, where

\[ R \sim H_0^2 \sim 10^{-84} \text{GeV}^2 \sim 10^{-120} M_{\text{Pl}}^2 \ll m_0^2, \quad (15) \]

since \( m_0^2 \gg H_0^2 \), the scalar degree of freedom decouples from the cosmic dynamics.

### III. GHOST FREEDOM

In order to investigate the propagating degrees of freedom, we shall study the linearization of the action [3] around maximally symmetric spaces of constant curvature \( R_0 \) [26]. In this case \( \mathcal{G}_0 = R_0^2/6 \). We shall expand the action up to terms quadratic in the curvature, so that terms like \( (R - R_0)^3 \), \( (R - R_0) (\mathcal{G} - \mathcal{G}_0) \) and higher, will be omitted. We have that:

\[ F(R, \mathcal{G}) = \mathcal{F}_0 + \mathcal{F}_R^0 (R - R_0) + \frac{1}{2} \mathcal{F}_{RR}^0 (R - R_0)^2 + \mathcal{F}_{\mathcal{G}}^0 (\mathcal{G} - \mathcal{G}_0) + \mathcal{O}(3), \]

where \( \mathcal{F}_0 = F(R_0, \mathcal{G}_0) \), \( \mathcal{F}_R^0 = F_R(R_0, \mathcal{G}_0) \), etc. If reorganize the above equation we can write it in more compact form (in the given approximation):

\[ F(R, \mathcal{G}) = \xi_0 + \zeta_0 R + v_0 R^2 + \omega_0 \mathcal{G}, \quad (16) \]

where we have introduced the following identifications:

\[ \xi_0 \equiv \mathcal{F}_0 - R_0 \mathcal{F}_R^0 \rightleftharpoons \frac{1}{2} R_0^2 \mathcal{F}_{RR}^0 - \frac{1}{6} R_0^2 \mathcal{F}_{\mathcal{G}}^0, \]

\[ \zeta_0 \equiv \mathcal{F}_R^0 - R_0 \mathcal{F}_{RR}^0, \quad v_0 \equiv \frac{\mathcal{F}_{\mathcal{G}}^0}{2}, \quad \omega_0 \equiv \mathcal{F}_{\mathcal{G}}^0. \]
If substitute the above expansion back into the action (3) we get:

\[ S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} \left( R - 2\Lambda + \frac{1}{6m^2}R^2 \right), \]  

(17)

where

\[ M_{\text{Pl}}^2 = \frac{\zeta_0}{\kappa^2}, \quad \Lambda = -\frac{\xi_0 + 2\kappa^2\mu^4}{2\zeta_0}, \quad m^2 = \frac{\zeta_0}{6\nu_0} = \frac{\tilde{F}_R^0 - R_0\tilde{F}_{RR}^0}{3\tilde{F}_{RR}^0}, \]  

(18)

and the term under the integral \( \propto G \) has been omitted since it amounts to a total derivative. It is a well-known fact that the linearization (17) is associated with three propagating degrees of freedom \([32, 33]\): the two polarizations of the (massless) graviton and a massive scalar mode with mass squared \( m^2 \). In order to avoid a tachyon instability it is then required that:

\[ m^2 \geq 0 \Rightarrow \tilde{F}_R^0 > R_0\tilde{F}_{RR}^0, \quad \tilde{F}_{RR}^0 > 0. \]  

(19)

As it was for the expansion around flat space, the present linearization is correct for small departures from de Sitter space with constant curvature \( R_0 \), down to the scale \( R \sim m^2 \). For much smaller curvature scales \( R \ll m^2 \), the action (17) reduces to the Einstein-Hilbert action.

A. Ghosts due to anisotropy of space

Even if the \( F(R,G) \) modified theory of gravity is free of ghosts when linearized around maximally symmetric spaces, when other less symmetric backgrounds such as anisotropic spaces, are considered, it is not for granted that the theory will be free of ghost in this latter case. In Ref. [34], the study of linear perturbation theory for general \( F(R,G) \) was carried out over an empty anisotropic background of the Kasner-type:

\[ ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)(dy^2 + dz^2), \]  

(20)

where \( a(t) \) and \( b(t) \) are the scale factors, in order to show that, within general \( F(R,G) \) theories, an anisotropic background has ghost degrees of freedom, which are absent on Friedmann-Robertson-Walker (FRW) backgrounds. Their study revealed that on this background the number of independent propagating degrees of freedom is four. It reduces to three on FRW backgrounds, since one mode becomes highly massive and decouples from the physical spectrum. The ghost mode is inevitable unless the following condition is fulfilled [34]:

\[ \left| \frac{\partial (\chi, \xi)}{\partial (R, G)} \right|^2 = F_{RR} F_{GG} - (F_{RG})^2 = 0, \]  

(21)

where \( \chi, \xi \) are auxiliary fields introduced in the study of [34]. If the above condition is fulfilled, then perturbations of \( \chi \) and \( \xi \) are not independent. In general backgrounds this is true if [34] \( \mathcal{L} = \chi(\phi)R + \xi(\phi)G - V(\phi) \). But this is not the only possibility left to avoid the ghosts due to anisotropy.

Actually, for theories of the kind we consider in this paper: \( F(R,G) = F(R + cG) \), where \( c = \beta/\alpha \) is a free constant, the condition (21) is fulfilled since, for this class of theories: \( F_G = cF_R \) and \( F_{GG} = cF_{RG} = cF_{RR} = c^2F_{RR} \), as seen from (2). Hence, for the latter more general class of fourth-order theories, ghosts due to anisotropy of space are absent.

B. Scalar perturbations and a modification of the dispersion relation

By the same reason as above, i.e., that the relationships (2) take place, neither a strong instability nor superluminal propagation occurs due to a modification of the dispersion relation found in [35]. In this reference the authors perform a general study of cosmological perturbations in vacuum for general \( F(R,G) \) theories. They found a modification of the dispersion relation for scalar perturbations, in comparison with previous similar studies [36, 37], that leads to
unwanted – either unstable or tachyonic – behavior. Actually, in [35] the following non-standard wave equation was obtained for the gauge invariant (Fourier) field \( \Phi \):

\[
\frac{1}{a^3 Q} \partial_t \left( a^3 Q \dot{\Phi} \right) + B_1 \frac{k^2}{a^2} \Phi + B_2 \frac{k^4}{a^4} \Phi = 0,
\]

(22)

where \( Q = Q(t) \), \( B_1 = B_1(t) \) and \( B_2 = B_2(t) \) are time-dependent parameters and \( k \) is the wavenumber of the perturbation. In this study the degrees of freedom \( \Phi \) and \( \dot{\Phi} \) are enough to describe the behavior of the metric perturbations. The above wave equation contains a term proportional to \( k^4 \), which does not vanish in generic \( F(R,G) \) theories.\(^1\) This term is responsible for non-standard behavior of the scalar metric perturbations. For instance, if \( B_2 \) were negative, then the Friedmann-Robertson-Walker (FRW) space were unstable on small scales (short wavelength limit) [35]. If \( B_2 \) were positive instead, up to the leading term the group velocity \( v_g(k) \approx 2 \sqrt{F_{RR} k/a} \). It exceeds the speed of light for modes above the critical wavenumber \( k_c = a/2 \sqrt{B_2} \). Hence, the propagation of short wavelength modes eventually (inevitably) becomes superluminal. This is true, except for the above mentioned special cases where (equation (6.18) of Ref. [35]):

\[
F_{RR} F_{GG} - F_{RG} F_{GR} = F_{RR} F_{GG} - (F_{RG})^2 = 0.
\]

(23)

The small wavelength modes inevitably either suffer from strong instability or undergo superluminal propagation.

In the kind of theories we are investigating here [11], thanks to the relationship [2], the condition for absence of instability/tachyonic behavior [23], is identically fulfilled. This means that the above discussed kind of instability is not present in the models of our interest.

C. Absence of other instabilities

Among the most dangerous instabilities, when higher-curvature corrections of gravity are considered, is the so called Dolgov-Kawasaki (matter) instability [38–41]. This instability, which is specially important in the \( f(R) \) theories since the curvature scalar \( R \) is a dynamical degree of freedom [40, 41], is of special importance in the present setup as well. The stability criterion in this case requires that

\[
\frac{d\kappa^2_{\text{eff}}}{dR} = - \frac{\kappa^2 F_{RR}}{F_R} < 0.
\]

(24)

Hence, the Dolgov-Kawasaki instability is avoided only for non-negative \( F_{RR} \geq 0 \). If the \( R \)-derivative of \( \kappa^2_{\text{eff}} \) in (24) were positive, the effective gravitational coupling increased with the curvature, so that, at larger curvature gravity becomes stronger which then implies that \( R \) itself generates a larger curvature through the trace equation (7). In other words, a positive feedback mechanism acts to destabilize the theory [26, 41].

In addition to the above stability criteria, a constraint coming from requiring positivity of the effective gravitational coupling:

\[
\kappa^2_{\text{eff}} = \frac{\kappa^2}{F_R} > 0,
\]

(25)

is also to be satisfied.

As shown in [32], for models of the class \( F(R,G) = F(R + cG) \), like the ones we are interested in here, the wave equation (22) for the scalar perturbations is given by:

\[
\frac{1}{a^3 Q} \partial_t \left( a^3 Q \dot{\Phi} \right) + \frac{c^2 k^2}{a^2} \Phi = 0.
\]

(26)

\(^1\) This term corresponds to fourth order spatial derivative in real space and is not a spurious result due to a bad choice of gauge since \( \Phi \) is gauge invariant.
where the squared sound speed is defined in the following way:

$$c_s^2 = 1 + \frac{8\beta\dot{H}/\alpha}{1 + 4\beta H^2/\alpha}. \quad (27)$$

We should require non-negative squared sound speed $c_s^2 \geq 0$ since, otherwise, a Laplacian or gradient instability develops. Meanwhile, for any $F(R, G)$ model, for the squared speed of propagation of the tensor modes one gets $c_T^2$:

$$c_T^2 = \frac{F_R + 4\beta\dot{F}_R/\alpha}{F_R + 4\beta H\dot{F}_R/\alpha}. \quad (28)$$

The absence of ghosts requires that $F_R + 4\beta H\dot{F}_R/\alpha > 0$, while, in order to avoid the Laplacian instability: $c_T^2 \geq 0$.

The absence of ghosts and instabilities such as: ghosts due to anisotropy of space or to linear perturbations around spherically symmetric static background $\text{[42]}$, tachyonic, Dolgov-Kawasaki and Laplacian instabilities, is required if the given $F(R, G)$ theories are phenomenologically viable options for the description of our universe. In particular, the choice in $\text{[1]}$ makes these theories very attractive possibilities for viable fourth-order theories of gravity since all of the mentioned instabilities may be avoided in a given subspace of the parameter’s space.

IV. POWER-LAW $F(R + cG)$ MODELS OF MODIFIED GRAVITY

Here we study a three-parametric class of models of the kind $F(R + cG)$ modified gravity and check them to stability and phenomenological viability. For the present choice of the $F(R, G)$ modification of gravity, several sources of instability such as ghosts due to anisotropy of space and non-standard behavior of the scalar metric perturbations – potentially leading to superluminal propagation of short wavelength modes – are eliminated. However, avoidance of other kinds of instability such as the Dolgov-Kawasaki and Laplacian instabilities, as well as the requirement of positivity of the effective gravitational coupling, lead to additional constraints on the parameter space.

In the present case we choose the following power-law function $F(R, G)$:

$$F(R, G) = -\lambda^2 (1 - \alpha R - \beta G)^\nu, \quad (29)$$

where $\lambda$, $\alpha$ and $\beta$ are free constants with mass dimensions $M$, $M^{-2}$ and $M^{-4}$, respectively, while $\nu$ is a dimensionless constant. Notice that, although there are four free parameters in $\text{[24]}$, the parameter $\lambda^2$ may be combined with $\kappa^2$ in $\text{[2]}$, so that the resulting $F(R, G)$ is actually a three-parametric function. We have that:

$$F_R = \alpha\lambda^2 \nu (1 - \alpha R - \beta G)^{\nu - 1}, \quad F_G = \beta\lambda^2 \nu (1 - \alpha R - \beta G)^{\nu - 1},$$

$$F_{RG} = F_G = -\alpha\lambda^2 \nu (\nu - 1) (1 - \alpha R - \beta G)^{\nu - 2},$$

$$F_{RR} = -\alpha^2\lambda^2 \nu (\nu - 1) (1 - \alpha R - \beta G)^{\nu - 2}, \quad F_{GG} = -\beta^2\lambda^2 \nu (\nu - 1) (1 - \alpha R - \beta G)^{\nu - 2}, \quad (30)$$

so that the relationships $\text{[2]}$ are satisfied. In what follows we shall assume that the following constraint on the curvature quantities is satisfied:

$$1 - \alpha R - \beta G \geq 0. \quad (31)$$

Under the above assumption, for the three-parametric class of function $\text{[24]}$, absence of the Dolgov-Kawasaki instability and positivity of the effective gravitational coupling – requirements $\text{[24]}$ and $\text{[25]}$, respectively – amount to the following conditions:

$$\frac{d\kappa^2_{\text{eff}}}{dR} = \frac{\kappa^2 (\nu - 1) (1 - \alpha R - \beta G)^{-\nu}}{\lambda^2 \nu} < 0,$$

$$\kappa^2_{\text{eff}} = \frac{\kappa^2 (1 - \alpha R - \beta G)^{1-\nu}}{\alpha \lambda^2 \nu} > 0. \quad (32)$$

Hence, phenomenologically viable theories of this type require that $\alpha > 0$ and $0 < \nu < 1$. 

The so called Born-Infeld inspired $F(R, \mathcal{G})$ models of the kind \cite{25,43}:

$$F(R, \mathcal{G}) = -\lambda^2 \sqrt{1 - \alpha R - \beta \mathcal{G}}, \quad (33)$$

fall into the above phenomenologically viable class of models of modified gravity, when we set $\nu = 1/2$. In what follows we shall focus in the investigation, specifically, of this two-parametric class of models.

V. ASYMPTOTIC DYNAMICS OF BI-INSPIRED $F(R, \mathcal{G})$ COSMOLOGY

Here we shall investigate the cosmological dynamics of the BI-inspired $F(R, \mathcal{G})$ model \cite{33} with action \cite{3}, in a FRW background space with flat spatial sections, whose line-element reads:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad i, j = 1, 2, 3.$$  \quad (34)

In this case we have that:

$$R = 6\dot{H} + 12H^2, \quad \mathcal{G} = 24H^2 \left( \dot{H} + H^2 \right),$$  \quad (35)

where $H = \dot{a}/a$ is the Hubble parameter.

For the $F(\alpha R + \beta \mathcal{G})$ class of function the FRW equations of motion \cite{41} read:

$$\frac{3H^2 + 3H\dot{F}_R}{FR} \left( 1 + \frac{\beta}{\alpha}H^2 \right) - \frac{1}{2} \left( R + \frac{\beta}{\alpha} \mathcal{G} - \frac{F}{FR} \right) = \frac{\kappa^2}{FR} (\rho_m - \epsilon \mu^4),$$ \quad (36)

$$\frac{\dot{F}_R}{FR} = -\frac{\kappa^2 (\omega_m + 1)\rho_m}{FR \left( 1 + \frac{2\beta}{\alpha}H^2 \right)} + \frac{H\dot{F}_R}{FR} - \frac{2 \left( 1 + \frac{2\beta}{\alpha}H^2 \dot{F}_R \right)}{1 + \frac{2\beta}{\alpha}H^2} \dot{H},$$ \quad (37)

where $\rho_m$ and $p_m = \omega_m \rho_m$ are the energy density and pressure of the matter fluid, while $\omega_m$ is its equation of state (EOS) parameter, respectively. In the present case the trace equation (7) is not an independent equation so that we do not write it. The above EOM-s can be written in the following alternative way:

$$\frac{\dot{H}}{H^2} = -1 + \frac{\Omega_m - \Omega_{\mu^4} - 2}{1 + \frac{2\beta}{\alpha}H^2} - \frac{\dot{F}_R}{HFR} + \frac{1}{3\alpha H^2(1 + \frac{2\beta}{\alpha}H^2)},$$ \quad (38)

$$\frac{\ddot{F}_R}{H^2FR} = -\frac{3(\omega_m + 1)\Omega_m}{1 + \frac{2\beta}{\alpha}H^2} + \frac{\dot{F}_R}{HFR} - \frac{2 \left( 1 + \frac{2\beta}{\alpha}H^2 \dot{F}_R \right)}{1 + \frac{2\beta}{\alpha}H^2} \dot{H},$$ \quad (39)

where we have introduced the dimensionless energy densities of matter and of $\mu^4$:

$$\Omega_m \equiv \frac{\kappa^2 \rho_m}{3FRH^2}, \quad \Omega_{\mu^4} \equiv \frac{\epsilon \kappa^2 \mu^4}{3FRH^2}.$$ \quad (40)

Notice that, for $\epsilon = +1$ the constant $\mu^4$ contributes a negative energy density. This is not problematic since the effective cosmological constant at low curvature is

$$\Lambda_{\text{eff}} = \frac{\lambda^2 - 2\epsilon \kappa^2 \mu^4}{\alpha \lambda^2},$$ \quad (41)

as seen from \cite{10}. Hence, as long as $\lambda^2 \geq 2\epsilon \kappa^2 \mu^4$, $\Lambda_{\text{eff}}$ is a non-negative quantity even if $\epsilon = +1$.

For the choice \cite{33}, the assumption \cite{31} is not an independent requirement but a constraint on the physical viability of the resulting cosmological model. Hence, for the specific model of interest here, this constraint amounts to a phenomenological bond which, in FRW space, can be written in the following way:

$$\frac{\dot{H}}{H^2} \leq \frac{1}{6\alpha H^2 \left( 1 + \frac{2\beta}{\alpha}H^2 \right)} - \frac{2 \left( 1 + \frac{2\beta}{\alpha}H^2 \right)}{1 + \frac{2\beta}{\alpha}H^2}.$$ \quad (42)
A. Simplified dynamical system: matter vacuum with vanishing vacuum energy

In what follows, for simplicity, we shall investigate the particular case when the density of matter vanishes \( \Omega_m = 0 \) (vacuum case) and \( \mu^4 = 0 \). This means that at small curvature:

\[
S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left( R - \frac{2}{\alpha^2} + \frac{\alpha}{4} R^2 \right),
\]

(43)

where \( M_{Pl}^2 = \alpha \lambda^2 / 2 \kappa^2 \). Although this is not the most general situation in which we may have even a vanishing effective cosmological constant, anyway the basic features of the model are preserved.\(^2\)

In order to perform the asymptotic dynamics analysis of this model, let us introduce the following dimensionless (bounded) variables of some state space:

\[
x = \frac{1}{1 + 4 \beta \alpha H^2} \Rightarrow 4 \beta \alpha H^2 = \frac{1-x}{x}, \quad 0 \leq x \leq 1,
\]

\[
y_{\pm} \equiv \frac{\dot{F}_R}{HF_R \pm FR} \Rightarrow \left[ \frac{\dot{F}_R}{HF_R} \right]_{\pm} = \frac{y_{\pm}}{1 \mp y_{\pm}}, \quad -1 \leq y_{-} \leq 0, \quad 0 \leq y_{+} \leq 1,
\]

(44)

where the whole phase space is covered by the bounded variables \( x \in [0, 1] \) and \( y = y_- \cup y_+ \in [-1, 1] \). The cosmological equations (36), (37) for the case of interest can be traded, accordingly, by the following autonomous dynamical system on these variables:

\[
x' = -2x(1-x) \left[ \frac{\dot{H}}{H^2} \right]_{\pm},
\]

\[
y'_{\pm} = (1 \mp y_{\pm})^2 \left[ \frac{\dot{F}_R}{HF_R} \right]_{\pm} - y_{\pm}(1 \mp y_{\pm}) \left[ \frac{\dot{H}}{H^2} \right]_{\pm} - y^2_{\pm},
\]

(45)

where

\[
\left[ \frac{\dot{H}}{H^2} \right]_{\pm} = -1 - 2x - \frac{y_{\pm}}{1 \mp y_{\pm}} + \frac{4 \beta x^2}{3 \alpha^2(1 - x)},
\]

\[
\left[ \frac{\dot{F}_R}{HF_R} \right]_{\pm} = \frac{y_{\pm}}{1 \mp y_{\pm}} - \frac{2[x(1 \mp y_{\pm}) + (1 - x)y_{\pm}]}{1 \mp y_{\pm}} \left[ \frac{\dot{H}}{H^2} \right]_{\pm},
\]

(46)

and the prime denotes derivative with respect to the time variable \( N = \ln a \). Notice that there are two different dynamical systems in (45); one for the choice of the '+' sign and another one for the choice '-''. However, these describe a unique phase space spanned by the variables \( x \) and \( y = y_- \cup y_+ \).

The model (43) is phenomenologically viable only if the function \( F = F(R, \mathcal{G}) \) is a real quantity, i. e., if the condition (42) is fulfilled. For the present case, in terms of the variables \( x, y \) the latter reads:

\[
\left[ \frac{\dot{H}}{H^2} \right]_{\pm} \leq -1 - x + \frac{2 \beta x^2}{3 \alpha^2(1 - x)},
\]

(47)

or \( y_{\pm} \geq y^*_{\pm} \), where

---

\(^2\) Recall that for non-vanishing \( \mu^4 \), the cosmological constant at small curvature: \( \Lambda_{\text{eff}} = (\lambda^2 - 2 \kappa^2 \mu^2) / \alpha \lambda^2 \), can be made as small as one desires by properly arranging the parameters \( \lambda^2 \) and \( \mu^2 \) if \( \epsilon = +1 \). For instance, by letting \( \lambda^2 = 2 \kappa^2 \mu^2 + \delta \lambda \), where \( \delta \lambda \) is a very small quantity.

\(^3\) For a compact introduction to the dynamical systems analysis close to this presentation see [44].
FIG. 1: Phase portraits of the dynamical system (45) for different (positive) values of the free parameters $\alpha$ and $\beta$. From left to the right ($\alpha, \beta$): (0.1, 0.1), (1, 1) and (10, 10), so that the dimensionless ratio $\beta/\alpha^2$ equals $10^{-3}$, 1 and $10^{-1}$, respectively. The 'gray' region is unphysical since the condition (42) is not satisfied. The critical points are represented by the small (red) solid circles. The thick dash-dot (blue) curve represents the condition $\dot{H}/H^2 = -1$. Hence, the critical points that are located below this curve represent accelerated expansion.

$$y^-_* = \frac{-x(1-x) + \frac{2\beta}{3\alpha^2}x^2}{(1-x)(1+x) - \frac{2\beta}{3\alpha^2}x^2}, \quad y^+_* = \frac{-x(1-x) + \frac{2\beta}{3\alpha^2}x^2}{(1-x)^2 + \frac{2\beta}{3\alpha^2}x^2}.$$  \tag{48}

Hence, the physically meaningful phase space corresponds to the following region of the plane: $\Psi_{2D}^\ominus = \Psi_{2D}^- \cup \Psi_{2D}^+$. Thus, $\Psi_{2D}^- = \{(x,y) : 0 \leq x \leq 1, \ -1 \leq y \leq y_0^-, y \geq y_0^- \}$, $\Psi_{2D}^+ = \{(x,y) : 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ y \geq y_0^+ \}$ .  \tag{49}

From the first equation in (46) it also follows that the expansion is accelerated ($q = -1 - \dot{H}/H^2 < 0$) if $y < y^-_*$ for $-1 \leq y^- \leq 0$, or if $y < y^+_*$ for $0 \leq y^+ \leq 1$, where:

$$y^-_* = \frac{-2x(1-x) + \frac{4\beta}{3\alpha}x^2}{(1-x)(1+2x) - \frac{4\beta}{3\alpha}x^2}, \quad y^+_* = \frac{-2x(1-x) + \frac{4\beta}{3\alpha}x^2}{(1-x)^2 - \frac{4\beta}{3\alpha}x^2}. \tag{50}$$

The phase portraits corresponding to the dynamical system (45) are shown in FIG. 1 for different values of the free parameters $\alpha$ and $\beta$. The different orbits appearing in these phase portraits are generated by given sets of initial conditions $(x_i(0), y_i(0))$. Each orbit may be associated with a whole cosmic history, starting (possibly) in a past attractor (origin of the given evolutionary pattern) and ending up in a future attractor (destiny of the cosmic evolution). The 'gray' region is unphysical since the condition (42) is not fulfilled. Hence, this region is excluded from the phase space $\Psi$. The thick dash-dot curves represent the condition $q = -1 - \dot{H}/H^2 = 0$. In consequence, points in $\Psi$ that are located below these curves represent accelerated expansion. In what follows we consider only non-negative $\beta$-s. The parameter $\alpha$ is also positive due to the requirements of absence of the Dolgov-Kawasaki instability and of positivity of the gravitational coupling as we have discussed before. The case with negative Gauss-Bonnet coupling ($\beta < 0$) has been studied in detail in [43].

B. Critical points of the dynamical system

Below we list those isolated critical points $P_i : (x_i, y_i)$ of the dynamical system (45) in $\Psi_{2D}$, that are located within the phenomenologically viable region, together with their main properties. These points are marked by small (red) dots in FIG. 1. In what follows, without loss of generality, we call a given point of the phase space as a "critical point" only if it is located within the phenomenologically viable region of the phase space, i. e., if it is in $\Psi_{2D}$, no matter whether it is, mathematically speaking, a critical
solid circles in FIG. 1.

1. Origin. The point $P_O : (0, 0)$ is the global past attractor in the phase space $\Psi_{2D}$ since the eigenvalues of the linearization matrix for this point: $\lambda_1 = 2$ and $\lambda_2 = 4$, are both positive. At this point:

$$x = 0 \Rightarrow H^2 \gg \alpha/\beta, \quad y = 0 \Rightarrow \frac{\dot{F}_R}{HF_R} = -\frac{\dot{F}}{HF} \to 0.$$  \hspace{1cm} (51)

Besides, the function $F$ is undefined at this equilibrium point. Since at $P_O$, the deceleration parameter $q = -1 - \frac{\dot{H}}{H^2} = 0$, then:

$$\frac{\dot{H}}{H^2} \to -1 \Rightarrow H = t^{-1} \Rightarrow a \propto t.$$  \hspace{1cm} (52)

This means that the evolution of the Universe starts in a big-bang singularity where $a(t) \to 0$ and $\dot{H} \approx -H^2 \to -\infty$.

2. Transient stages.

- Point $P_{\text{sdd}}^0 : (0, 1)$ is a saddle critical point since the eigenvalues of the corresponding linearization matrix are of different sign. Hence, this point is associated with a transient state of the cosmic evolution. It is characterized by a very high curvature with $H^2 \gg \alpha/\beta$ and $\dot{F}_R/FF_R$ undefined:

$$y \to 1 \Rightarrow \frac{\dot{F}_R}{HF_R} \to \infty \quad (H > 0),$$  \hspace{1cm} (53)

which means, in turn, that $F_R \to 0$, i.e., that $F \to \infty$. This latter limit implies that, at least,

$$\frac{\dot{H}}{H^2} < -1 \Rightarrow q > 0 \quad (H > 0),$$  \hspace{1cm} (54)

i.e., this point represents a transient stage of decelerated expansion. As a matter of fact, since at $P_{\text{sdd}}^0$, $x = 0$ and $y = 1$, then (see first equation in (46)):

$$\frac{\dot{H}}{H^2} \to -\infty \Rightarrow q \to \infty.$$  \hspace{1cm} (55)

This point should be associated with a curvature singularity.

- The point $P_{\text{sdd}}^1 : (1, 1)$, where

$$x = \frac{1}{1 + 4\frac{\alpha}{\beta}H^2} = 1, \quad y = 1 \Rightarrow \frac{\dot{F}_R}{HF_R} \to \pm \infty,$$  \hspace{1cm} (56)

represents a transient cosmic stage with low curvature $H^2 \ll \alpha/\beta$ (the numerical investigation reveals that this is a saddle equilibrium point as well). It is seen from the first equation in (46) that at $P_{\text{sdd}}^1$, due to the competition between the negative (first) and the positive (last) terms, the quantity $\dot{H}/H^2$ is undefined. As a matter of fact this equilibrium state represents a turning point in what regards to the peace of the cosmic expansion: It is seen from FIG. 1 that, as the given orbit evolves in the vicinity of $P_{\text{sdd}}^1$, the cosmic history turns from decelerating into accelerating expansion. Recall that the curve (46) corresponding to $q = 0$, joints the points $P_O$ and $P_{\text{sdd}}^1$.  

---

point of the dynamical system.
3. Destiny. The point

\[ P_{dS} : \left( \frac{1}{\sqrt{1 + 2\beta/3\alpha^2}}, \frac{1}{2} \right), \]

is a future attractor since the eigenvalues of the corresponding linearization matrix: \( \lambda_1 = -2 \) and \( \lambda_2 = -4 \), are both negative. It is the global future attractor in \( \Psi_{2D} \). This critical point represents a de Sitter solution since \( q = -1 \Rightarrow H = 0 \Rightarrow H = H_0 \). It is a de Sitter attractor with constant Hubble rate squared:

\[ H_0^2 = \frac{\alpha}{4\beta} \left( \sqrt{1 + \frac{2\beta}{3\alpha^2}} - 1 \right). \]

(VI. THREE DIMENSIONAL PHASE SPACE DYNAMICS AND THE DE SITTER SOLUTIONS)

In the above section we have investigated in detail the asymptotic dynamics of the BI inspired \( F(R,G) \) theory \((33)\) in the simplified case when \( \mu^4 = 0 \), i.e., vanishing vacuum energy density (we are investigating the vacuum case exclusively, i.e., \( \Omega_m = 0 \)). In this simplified case the asymptotic dynamics is described in a 2-dimensional (2D) phase space which means, in turn, that the mathematical handling is simpler. However, the assumption that \( \mu^4 = 0 \), means that at small curvature, where the effective theory is given by the action:

\[ S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left( R - \frac{2}{\alpha} + \frac{\alpha}{4} R^2 \right), \]

we are not able to set the cosmological constant \( 1/\alpha \) to any negligible small value – as required by the observations – without forcing an unnaturally large coupling to the higher-curvature contribution. This is why, in the present section, we shall consider a non-vanishing vacuum energy \( \mu^4 \neq 0 \). This amounts to increasing the dimension of the phase space from 2D to 3D. The corresponding mathematical handling is by far more complex. Our strategy to simplify the mathematics \( y \) to focus, exclusively, in the de Sitter solutions which are the ones that can be associated with the late-time accelerated expansion of the universe.

A. Dynamical system and the de Sitter critical points

In the present case where \( \mu^4 \neq 0 \), in addition to the phase space variables \( x, y \) in \((44)\), it is convenient to introduce the new bounded variable:

\[ u = \frac{\Omega_{\mu^4}}{\Omega_{\mu^4} + \epsilon} \Rightarrow \Omega_{\mu^4} = \frac{\epsilon u}{1 - u}, \]

(58)

where \( 0 \leq u \leq 1 \) and \( \epsilon = \pm 1 \). The resulting 3D dynamical system reads:

\[ x' = -2x(1 - x) \left[ \frac{\dot{H}}{H^2} \right] \pm, \]

\[ y'_{\pm} = (1 \mp y_{\pm})^2 \left[ \frac{\ddot{F}_R}{H^2 F_R} \right]_{\pm} - y_{\pm}^2 - y_{\pm}(1 \mp y_{\pm}) \left[ \frac{\dot{H}}{H^2} \right]_{\pm}, \]

\[ u' = -u(1 - u) \left\{ \frac{y_{\pm}}{1 \mp y_{\pm}} + 2 \left[ \frac{\dot{H}}{H^2} \right]_{\pm} \right\}, \]

(59)

where, as before, the prime denotes derivative with respect to the time variable \( N = \ln a \). Besides:
\[
\begin{align*}
\frac{\dot{H}}{H^2} \pm &= -1 - \left[ 2 + (\epsilon - 2)u \right] x - \frac{y_\pm}{1 \mp y_\pm} + \frac{4\beta x^2}{3\alpha^2(1 - x)}, \\
\frac{\ddot{F}_R}{H^2 F_R} \pm &= \frac{y_\pm}{1 \mp y_\pm} - \frac{2[x(1 \mp y_\pm) + (1 - x)y_\pm]}{1 \mp y_\pm} \left( \frac{\dot{H}}{H^2} \right)_\pm.
\end{align*}
\] (60)

In this section we shall focus in the de Sitter solutions exclusively. In order to check whether the corresponding critical points are within the phenomenologically viable region, it is required that 
\[F^2 \geq 0,\]
where \(F\) is defined in (33).

Hence,
\[
\frac{\dot{H}}{H^2} \left( 1 + 4\frac{\beta}{\alpha} H^2 \right) + 2 + 4\frac{\beta}{\alpha} H^2 \leq \frac{1}{6\alpha H^2}.
\]

but, since at the de Sitter points \(\dot{H} = 0\), then, for these critical points to be in the physically meaningful region it is required that:
\[
\frac{1}{\sqrt{1 + 2\beta/3\alpha^2}} \leq x \leq 1.
\] (61)

At the de Sitter point, for \(y \geq 0\), the equations (59) and (60) become:
\[
x' = 0, \quad y' = y(1 - 2y), \quad u' = \frac{u(1 - u)y}{y - 1},
\] (62)

while for negative \(y < 0\):
\[
x' = 0, \quad y' = y, \quad u' = \frac{u(u - 1)y}{y + 1}.
\] (63)

1. de Sitter critical manifold

For \(y = 0\) one obtains a critical manifold:
\[
\mathcal{P}_{\text{dS}} = \left\{ (x, 0, u_\ast) \left| \frac{1}{\sqrt{1 + 2\beta/3\alpha^2}} \leq x \leq 1, \quad u_\ast = u_\ast(x) \right. \right\},
\] (64)

where we have defined
\[
u_\ast(x) = \frac{(1 - x)(1 + 2x) - 4\beta x^2}{(1 - x)[1 + (2 - \epsilon)x] - \frac{12\beta}{3\alpha^2} x^2}.
\]

Depending on location (smaller or larger \(u\)-values), points in \(\mathcal{P}_{\text{dS}}\) can be either saddle critical points or local attractors instead. For points in \(\mathcal{P}_{\text{dS}}\) we have that,
\[
12\beta H^4 + 9\alpha H^2 - \frac{\dot{F}_R - c\alpha^2 H^4}{F_R} = 0.
\] (65)

On the other hand, since \(y = 0 \Rightarrow \dot{F}_R = 0\), one gets that \(F_R = \tilde{F}_R^0 \) =const., where the value of the constant \(\tilde{F}_R^0\) depends on the initial conditions.\(^5\) From (65) we obtain the following second-order algebraic equation:

\(^5\) This is a consequence of \(\mathcal{P}_{\text{dS}}\) being a manifold instead of an isolated critical point.
\[ H^4 + \frac{3\alpha}{4\beta} H^2 - \frac{\tilde{F}_R^0 - \epsilon \alpha \kappa^2 \mu^4}{12 \beta \tilde{F}_R^0} = 0, \]

whose real root is:

\[ H^2(F_0) = \frac{3\alpha}{8\beta} \left[ \sqrt{1 + \frac{16\beta}{27\alpha^2} \left( 1 - \frac{\epsilon \alpha \kappa^2 \mu^4}{\tilde{F}_R^0} \right)^2} - 1 \right]. \quad (66) \]

Given that for a set of points in \( P_{dS} \): the saddle points, the de Sitter solution is a transient stage, one may think that these can be associated with primordial inflation. However, as we shall see below, this is not the case since, if associate the late-time attractor \( P_{dS} \) (see below) with the late-time inflationary stage, there is not possible to get the required amount of e-foldings of inflation. Hence, points in the above de Sitter manifold are to be associated with intermediate to late-time cosmological evolution.

2. **Isolated de Sitter attractor**

The other possibility is for \( y = 1/2 \), where we are led with the isolated attractor:

\[ P_{dS}^\pm : \left( \frac{1}{\sqrt{1 + 2\beta/3\alpha^2}}, \frac{1}{2}, 0 \right). \quad (67) \]

At this point we have that (57):

\[ H_0^2 = \frac{\alpha}{4\beta} \left( \sqrt{1 + \frac{2\beta}{3\alpha^2}} - 1 \right). \]

This isolated attractor is to be associated with the late-time stage of accelerated expansion of the universe. It can be checked that the ratio between \( H^2(F_0) \) in (66) and \( H_0^2 \) above, at large \( \alpha \gg 1 \) amounts to:

\[ \frac{H^2(F_0)}{H_0^2} \approx \frac{4}{3} \left( 1 - \frac{\epsilon \alpha \kappa^2 \mu^4}{\tilde{F}_R^0} \right). \]

Hence, the saddle points in the manifold \( P_{dS} \) can not be associated with primordial inflation since there is not possible to get the required amount of inflation.

**B. The 3D phase portrait**

The 3D phase space where to look for phenomenologically viable behavior of the dynamical system (59) is defined in the following way:

\[ \Psi_{3D} = \{(x, y, u) : 0 \leq x \leq 1, -1 \leq y \leq 1, 0 \leq u \leq 1, u \leq u^\pm \}, \quad (68) \]

where

\[ u^\pm = \frac{\left[ x(1-x) - \frac{2\beta}{3\alpha x} x^2 \right] (1 \mp y_\pm) + (1-x)y_\pm}{(1-\epsilon)x(1-x) - \frac{2\beta}{3\alpha x} x^2} \left( 1 \mp y_\pm \right) + (1-x)y_\pm, \]
FIG. 2: Phase portrait of the dynamical system (59) for \( \epsilon = +1 \) and the following values of the free parameters: \( \alpha = 1, \beta = 1 \). Two sets of orbits that are generated by chosen initial data are shown in different colors. The boundary of the physically meaningful phase space \( F^2 = 0 \), where \( F \) is defined in (33) – surface with contours – is included. In the left and center figures the orientation is given by \( [\theta, \phi, \psi] = [-130^o, 70^o, 10^o] \) and \( [\theta, \phi, \psi] = [130^o, 70^o, -10^o] \), respectively, where \( \theta, \phi \) and \( \psi \) are the Euler angles. Meanwhile, in the right figure the \( x, y \)-projection is shown: \( [\theta, \phi, \psi] = [-90^o, 0^o, 0^o] \). The isolated point represents the de Sitter attractor \( P_{dS} \), while the dotted curved segment represents the de Sitter critical manifold \( P_{dS} \).

and, as before, \( y = y^- \cup y^+ \).

The 3D phase portrait of the dynamical system (59) is shown in FIG. 2 for \( \epsilon = 1 \) and the following values of the free parameters: \( \alpha = 1, \beta = 1 \). Two sets of orbits that are generated by chosen initial data sets, are shown in different colors. A surface with contours representing the boundary of the physically meaningful phase space: \( F^2 = 0 \), where \( F \) is defined in (33), has been included in the figure. In the left and center figures the orientation is given by \( [\theta, \phi, \psi] = [-130^o, 70^o, 10^o] \) and \( [\theta, \phi, \psi] = [130^o, 70^o, -10^o] \), respectively, where \( \theta, \phi \) and \( \psi \) are the Euler angles. Meanwhile, in the right figure the \( x, y \)-projection is shown: \( [\theta, \phi, \psi] = [-90^o, 0^o, 0^o] \). The isolated point in each figure represents the de Sitter attractor \( P_{dS} \), while the dotted curved segment represents the de Sitter critical manifold \( P_{dS} \).

Notice that the phase space orbits end up either at the isolated de Sitter attractor \( P_{dS} \), or at local attractor de Sitter points in the manifold \( P_{dS} \) (uppermost points in the manifold). The green orbits end up, precisely, at the uppermost points of \( P_{dS} \), while navy orbits end up at the isolated de Sitter attractor. In any case the state towards the universe is attracted which corresponds to de Sitter expansion \( a(t) \propto \exp H_0 t \), where \( H_0^2 \) is given either by (66) or by (67).

VII. AVOIDANCE OF THE COSMOLOGICAL CONSTANT ISSUE IN THE PRESENT SETUP

Perhaps the most interesting solution in the phenomenologically viable subspace of the phase space is the de Sitter attractor \( P_{dS} \) (here we should add the de Sitter points in the critical manifold \( P_{dS} \) which exist only in the case where \( \mu^4 \) is non-vanishing). The de Sitter attractor solutions are interesting in the present model because these entail that, at late time, the FRW universe described by the theory (3), (33), is almost indistinguishable from the \( \Lambda \)CDM cosmological model. One should not be surprised by this result since, at low curvature, a non-vanishing cosmological constant \( \Lambda_{\text{eff}} = (\lambda^2 - 2\kappa^2 \mu^4)/\alpha \lambda^2 \), arises in this model. The surprising result is that, at the de Sitter attractor \( H_0^2 \neq \Lambda_{\text{eff}}/3 \), which challenges our intuition.

In order to fix ideas, momentarily, we shall choose \( \epsilon = +1 \). Let us to set \( \lambda^2 = 2\kappa^2 \mu^4 \) in (3). Under this choice our model coincides with the one previously investigated in [22], where an exact cancellation mechanism of the cosmological constant has been applied.\(^6\) Actually, under the above choice the effective cosmological constant at low curvature, \( \Lambda_{\text{eff}} = 0 \), exactly vanishes. This includes, as a particular case, the flat space (\( R = G = 0 \)), for which:

\(^6\) The asymptotic dynamics of this model was studied in [43] for negative Gauss-Bonnet coupling.
\[ F(R, \mathcal{G}) = -\lambda^2 \sqrt{1 - \alpha R - \beta \mathcal{G}}, \quad F_R = -\frac{\alpha \lambda^4}{2F}, \quad \Rightarrow \quad F(0,0) = -\lambda^2, \quad F_R(0,0) = \frac{\alpha \lambda^2}{2}. \]

In this particular case, assuming vacuum background, we have that \( G_{\mu\nu} = 0 \ (R_{\mu\nu} = 0) \), while the fourth-order curvature contributions \( \Sigma_{\mu\nu}^\text{curv} \) in \( \text{(6)} \), amount to:

\[ \Sigma_{\mu\nu}^\text{curv}(0,0) = -\frac{F(0,0)}{2F_R(0,0)} g_{\mu\nu} = \frac{1}{\alpha} g_{\mu\nu}, \]

so that the EOM \( \text{(4)} \) for vacuum:

\[ \Sigma_{\mu\nu}^\text{curv}(0,0) = \frac{\kappa^2 \mu^4}{F_R(0,0)} g_{\mu\nu} = \frac{2\kappa^2 \mu^4}{\alpha \lambda^2} g_{\mu\nu}, \]  

become an identity after our choice \( 2\kappa^2 \mu^4 = \lambda^2 \). Hence, as long as both \( F_R(0,0) = \alpha \lambda^2 / 2 > 0 \) and \( F_{RR}(0,0) = \alpha^2 \lambda^2 / 4 > 0 \) are positive quantities, flat space is a stable solution of the EOM of our setup. This is to be contrasted with the \( F(R) \) model investigated in \( \text{(7)} \), where the flat space was a unstable solution of the equations of motion.

From equation \( \text{(14)} \) it is seen that, for our above choice \( \epsilon = +1 \), the vacuum energy density is a negative quantity: \( \rho_{\text{vac}} = -\mu^4 \). But this is not problematic since, as mentioned above, the effective energy density of vacuum at small curvature vanishes. The fact we want to underline here is that the energy density of vacuum \( \rho_{\text{vac}} \), the effective cosmological constant \( \Lambda_{\text{eff}} \) in the low-curvature regime and the present value of the Hubble rate \( H_0 \), are unrelated quantities in our setup. Actually, for the de Sitter attractor \( P_{\text{dS}} \) \( \text{(67)} \), that arises in the phase space corresponding to our cosmological model, the constant expansion rate reads:

\[ H_0^2 = \frac{\alpha}{4\beta} \left( \sqrt{1 + \frac{2\beta}{3\alpha^2}} - 1 \right). \]  

(70)

It has nothing to do neither with \( \rho_{\text{vac}} \) nor with \( \Lambda_{\text{eff}} \). Actually, given that \( \Lambda_{\text{eff}} = 0 \) thanks to our choice \( (\lambda^2 = 2\kappa^2 \mu^4) \), and that the de Sitter attractor \( P_{\text{dS}} \) arises no matter whether \( \mu^4 = 0 \), as in Sec. \( \text{VII} \) or \( \mu^4 \neq 0 \), as in Sec. \( \text{VI} \), the constant Hubble rate \( \text{(70)} \) is independent of the vacuum energy density \( \rho_{\text{vac}} = -\mu^4 \), as well as of the effective (low-curvature) cosmological constant \( \Lambda_{\text{eff}} = 0 \). This non-trivial fact is at the core of the avoidance of the cosmological constant issue in the present setup.

It is apparent from above that the theory \( \text{(3)} \) with \( F(R, \mathcal{G}) \) given by \( \text{(33)} \) and \( \lambda^2 = 2\kappa^2 \mu^4 \ (\epsilon = +1) \), satisfies the necessary and sufficient conditions discussed in the introduction (Sec. \( \text{I} \)), as well as the additional reasonable requirements, that are to be satisfied in order to have a phenomenologically satisfactory theory of gravity where the cosmological constant problem does not arise. There are, however, certain observational constraints that should be satisfied as well. Take, for instance, the bond imposed on the mass of the scalar perturbation \( \text{(31)} \) around flat space \( \text{(12)} \):

\[ m_0^2 = \frac{F_R(0,0)}{3F_{RR}(0,0)} = \frac{2}{3\alpha} > 2.5 \times 10^{-59} M_{\text{Pl}}^2 \Rightarrow \alpha < 10^{60} M_{\text{Pl}}^{-2}. \]  

(71)

Let us consider two limiting situations. Assume, first, that the dimensionless quantity \( \beta / \alpha^2 \ll 1 \) is very small. In this case from \( \text{(70)} \) it follows that:

\[ H_0^2 \approx \frac{1}{12\alpha} > 10^{-60} M_{\text{Pl}}^2, \]

which means that the observational constraint \( H_0^2 \sim 10^{-120} M_{\text{Pl}}^2 \) on the present value of the Hubble rate, can not be satisfied. The other limiting situation \( \beta / \alpha^2 \gg 1 \) yields a better physical scenario. In this case, from \( \text{(70)} \) it follows that:

\[ H_0^2 \approx \frac{1}{2\sqrt{6\beta}}, \]
so that the experimental bond (71) is avoided in this case. The price to pay for the beautiful behavior is the strong Gauss-Bonnet coupling required: $\beta \sim 10^{-240} \ M_{\text{Pl}}^{-4}$. This sets the scale of smallness of the Gauss-Bonnet term $|\mathcal{G}|$ much below $1/\beta \sim 10^{-240} \ M_{\text{Pl}}^4$, i.e., the scale of small curvature $\sqrt{|\mathcal{G}|}$ is far below the present value of the curvature of the Universe $\sim 10^{-120} \ M_{\text{Pl}}^2$. Hence, the present cosmological dynamics is still (partly) dictated by the Gauss-Bonnet interaction. At much smaller curvature scales, however, the Gauss-Bonnet term amounts to a total derivative and may be safely removed, so that the coupling $\beta$ does not play any role.

The resulting physical picture is one in which the energy density of vacuum is of the order of the Planck mass to the 4th power, with vanishing effective (low-curvature) cosmological constant — the flat space is a stable solution of the vacuum EOM — and a FRW de Sitter attractor with the required (present day) Hubble rate $H_0$.

VIII. DISCUSSION

The absence of ghosts and instabilities such as: ghosts due to anisotropy of space or to linear perturbations around spherically symmetric static background [42], tachyonic, Dolgov-Kawasaki and graviton instabilities, in the present model is a consequence of the choice of $F(R, \mathcal{G})$ in (33). This makes of the theory (3), with $F(R, \mathcal{G})$ given by (33), a very attractive possibility for a viable fourth-order theory of gravity. Nevertheless, in general, the model is not free of other kinds of problems such as the Laplacian or gradient instability. Their absence would impose additional requirements on the physical phase space as well as on the space of parameters of the theory.

An interesting aspect of fourth-order gravity theories, like the one subject of the present study, is that the measured gravitational constant $8\pi G_*$, in general, differs from the effective gravitational coupling $8\pi G_{\text{eff}} = \kappa^2 / F_R$, defined in (3) (recall that positivity of the gravitational coupling requires that $F_R > 0$). Actually, on sub-horizon scales the evolution equation for the matter density perturbation $\delta_m = \delta \rho_m / \rho_m$ reads [34, 43]:

$$\ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi G_* \rho_m \delta_m = 0,$$

(72)

where, for the present Born-Infeld inspired $F(R, \mathcal{G})$ theory, the gravitational constant measured in Cavendish type experiment is $G_* = G_N Q = \kappa^2 Q$, with:

$$Q \equiv Q(k, t) = \frac{A_1 + A_2 (\frac{k}{\eta})^2}{B_1 + B_2 (\frac{k}{\eta})^2}.$$

(73)

Here we follow the notation of [43] where, for the present case:

$$A_1 = F_R + 4 \frac{\beta}{\alpha} \tilde{F}_R, \quad A_2 = \frac{8F_R^3 H^2}{\alpha \lambda^4} \left(1 - 4 \frac{\beta}{\alpha} H^2 q\right)^2, \quad B_1 = \left(F_R + 4 \frac{\beta}{\alpha} H \tilde{F}_R\right)^2,$$

$$B_2 = \frac{2F_R^2 H^2}{\alpha \lambda^4} \left(1 + 4 \frac{\beta}{\alpha} H^2\right) \left\{ \left(1 - 4 \frac{\beta}{\alpha} H^2 q\right) \left[3 + 4 \frac{\beta}{\alpha} \left(4H \tilde{F}_R - \frac{\tilde{F}_R}{F_R}\right)\right] + 4 \frac{\beta}{\alpha} H \left(1 + 4 \frac{\beta}{\alpha} \frac{\tilde{F}_R}{F_R}\right) \right\}.$$

(74)

where $q = -1 - \dot{H} / H^2$. At long wavelength $k \to 0$, the measured gravitational constant reads:

$$G_*^0 \equiv G_N Q(0, t) = G_N \frac{A_1}{B_1} = \frac{G_N \left(1 + 4 \frac{\beta}{\alpha} \frac{\tilde{F}_R}{F_R}\right)}{F_R \left(1 + 4 \frac{\beta}{\alpha} \frac{\tilde{F}_R}{F_R}\right)^2},$$

while, at short wavelength $k \to \infty$:

$$G_*^\infty \equiv G_N Q(\infty, t) = \frac{4G_N \left(1 - 4 \frac{\beta}{\alpha} H^2 q\right)}{F_R \left(1 + 4 \frac{\beta}{\alpha} H^2 q\right) \left[3 + 4 \frac{\beta}{\alpha} \left(4H \tilde{F}_R - \frac{\tilde{F}_R}{F_R}\right)\right] + 4 \frac{\beta}{\alpha} \tilde{H} \left(1 + 4 \frac{\beta}{\alpha} \frac{\tilde{F}_R}{F_R}\right)}.$$

Notice that, in the limit when the Gauss-Bonnet coupling vanishes $\beta \to 0$, we get that $G_*^\infty = 4G_*^0 / 3$ as it should be for the $F(R)$ theories [7]. If the quantity $Q(\infty, t)$ is larger or smaller than $1/F_R$, then an enhancement or a suppression
of the matter power-spectrum is obtained. Hence, the class of theories [33] may leave an imprint that makes possible to differentiate these from other modifications of general relativity through the cosmological observations.

As we have mentioned before, the full asymptotic dynamics of the present model has been investigated in [43]. In that reference, however, only the case with negative Gauss-Bonnet coupling ($\beta < 0$) was considered. Although this latter case was not investigated here, the results of our present study also apply to this case. It is necessary to mention that, the particular model [33] with $\beta = 0$, i.e., if neglect the Gauss-Bonnet term, will no satisfy the observational constraints coming from cosmology, in particular that the present value of the Hubble rate $H_0 \sim 10^{-60} \text{M}_{\odot}$. Hence, the Gauss-Bonnet interaction plays an important role in the way the cosmological constant problem is avoided in the present setup. The class of theory [33] meets the necessary and sufficient conditions to avoid the cosmological constant issue, however, here we have not investigated in detail the constraints the absence of several instability issues, such as the Laplacian or gradient instability, impose on the phase space as well as on the space of parameters of the theory. In the last instance one may think of the present setup as a class of toy models that serve to show that, an alternative to explain the huge discrepancy between the theoretically predicted and the observed values of the cosmological constant, exist. It consists just in evading the problem.

Although our model complies the requirements mentioned in the introduction, nevertheless, there should be other models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requirements as well. The proposal investigated in [7] seems to be an example of that. In that reference the author studies models of modified gravity that may fulfill these requi

\[ F(R) = R + \lambda R_0 \left( 1 + \frac{R^2}{R_0^2} \right)^{-n} - 1, \]

where $n, \lambda > 0$ and $R_0$ of the order of the presently observed effective cosmological constant, are the free parameters. Then, $F(0) = 0$ (the cosmological constant “disappears” in flat space) and $R_{\mu\nu} = 0$ is always a solution of the EOM in the absence of matter. For $R > R_0$, $F(R) = R - 2\Lambda\gamma$, where $\Lambda = \lambda R_0/2$. The model has de Sitter solutions with $R = x_1 R_0$, where $x_1$ is the maximal root of a given algebraic equation (equation (6) of [7]). This model is free of the Dolgov-Kawasaki instability, unfortunately, given that $F_{RR} < 0$, flat space is unstable. For $n \geq 2$ the model passes laboratory and Solar System tests of gravity. In order to check that this proposal meets the necessary and sufficient conditions stated in the introduction it is mandatory to consider an action of the kind [33] with $F(R) \rightarrow F(R, \mathcal{G})$. I.e., from the start a non-vanishing vacuum energy density $\propto \mu^4$ should be considered. Besides, it should be demonstrated that the de Sitter solution with $R = x_1 R_0$ is a future attractor in the phase space of the model. To our knowledge these items have not been investigated yet.

IX. CONCLUSION

In this paper we have explored a class of Born-Infeld inspired $F(R, \mathcal{G})$ models of modified gravity of type $F(R, \mathcal{G}) \propto \sqrt{L_{\text{Lovelock}}}$ (see equation [33]):

\[ F(R, \mathcal{G}) = -\lambda^2 \sqrt{1 - \alpha R - \beta \mathcal{G}}, \]

This two-parametric class of theory is free of most of the unwanted ghosts and instabilities. Yet, we have not checked it for the absence of the Laplacian instability, but surely, the requirement of absence of this kind of instability will further constraint the phenomenologically viable phase space as well as the space of parameters. Models of this kind are very interesting alternatives to GR, not only because the inclusion of higher-order curvature operators is dictated by renormalization [32, 33] and by the formulation of GR as an effective theory [14, 15], but because the CCP is avoided in these models.

We do not expect that the present setup may account for a realistic description of our Universe, instead, it may be viewed as a toy model showing that the cosmological constant issue, if not solved, at least may be evaded.

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