TRAJECTORY OPTIMIZATION OF UAV BASED ON HP-ADAPTIVE RADAU PSEUDOSPECTRAL METHOD

YI CUI
School of Electro-Optical Engineering
Changchun University of Science and Technology
Changchun 130012, Jilin, China

XINTONG FANG
College of Control Science and Engineering
Zhejiang University
Hangzhou 310027, Zhejiang, China

GAOQI LIU∗
School of Electrical Engineering
Sichuan University
Chengdu 610065, Sichuan, China

BIN LI
School of Aeronautics and Astronautics
Sichuan University
Chengdu 610065, Sichuan, China

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ABSTRACT. Unmanned Aerial Vehicles (UAVs) have been extensively studied to complete the missions in recent years. The UAV trajectory planning is an important area. Different from the commonly used methods based on path search, which are difficult to consider the UAV state and dynamics constraints, so that the planned trajectory cannot be tracked completely. The UAV trajectory planning problem is considered as an optimization problem for research, considering the dynamics constraints of the UAV and the terrain obstacle constraints during flight. An hp-adaptive Radau pseudospectral method based UAV trajectory planning scheme is proposed by taking the UAV dynamics into account. Numerical experiments are carried out to show the effectiveness and superior of the proposed method. Simulation results show that the proposed method outperform the well-known RRT* and A* algorithm in terms of tracking error.

1. Introduction. Unmanned aerial vehicle (UAV) is a hot topic of research in recent years, which have a wide range of applications in both of the commercial and the military fields [3, 25]. For example, battlefield reconnaissance, drones trikes and emergency rescue [23, 6, 1]. In various application scenarios, UAVs must be able to plan trajectories, and the planned path has to be able to avoid obstacles...
Therefore, the trajectory planning with obstacles avoidance for the UAV is becoming increasingly important.

The aim of the trajectory planning algorithm is determining a collision-free trajectory between the starting position and the target for the UAV through the cluttered obstacles [22]. Due to the high computational efficiency, sampling-based path planning methods are widely used to plan a trajectory of UAV in complex environments with obstacles. In [27], the dynamic domain rapidly-exploring random tree (DDRRT) is combined with linear quadratic Gaussian motion planning (LQG-MP) method to search for local paths under threat and uncertainty conditions, and simulation results show that the algorithm behaves well in planning safe paths online in uncertain and hostile environments. Although the RRT-based methods can generate a feasible anti-collision paths, the optimality of the path cannot be guaranteed. To overcome the optimality problem of RRT-based methods, the optimal rapidly exploring random tree (RRT*) [12] is proposed. It is demonstrated that this RRT* method can keep searching for better trajectory solutions in real-time. For example, the RRT* algorithm is applied to find a near-optimal trajectory for aerial vehicles traveling in 3D space while avoiding obstacles [20], and the artificial potential fields is integrated to accelerate the convergence speed to the optimal trajectory.

In addition to the sampling-based methods, path planning algorithms based on heuristic algorithms, such as particle swarm optimization (PSO) algorithm [9, 10] and genetic algorithm (GA) [21], are also widely used due to the global optimization capabilities. For example, a PSO-based method [31] is developed to improve the target location accuracy through imaging it from an UAV, an expression of optimal trajectory can be derived. For global heuristic algorithms, a genetic algorithm for global path planning is proposed in UAV cooperative systems [13].

However, for the algorithms based on path planning mentioned above, which can find a feasible flight trajectory with high computational efficiency and acceptable optimality, the system modeling of UAV can include kinematics characteristics, but cannot easily handle complex dynamics characteristics, which will cause a bad performance on actual trajectory tracking. And this shortcoming can be overcome by the perspective of optimization, that is, a trajectory planning problem of UAV is considered as an constrained optimal control problem. However, solving such a numerical problem is challenging because it is a NP-hard problem. There are two main types of numerical methods for trajectory optimization problems: the direct method and the indirect method [5]. Because the indirect method is difficult to deal with optimization problems with complex nonlinear state constraints, the direct method is more widely used in solving trajectory planning problem of UAV, which discretizes both of the system states and costates or only the system controls, and then transforming the original trajectory planning problem into an nonlinear programming (NLP) problem. As a direct method, the control parameterization method [32, 15] is applied to handle the UAVs formation reconfiguration trajectory planning problem, and a hybrid scheme that combines global simulated annealing algorithm and gradient method is proposed to find the global optical trajectory [14]. Among several direct methods, the hp-adaptive Radau pseudospectral method [18, 26] is efficient in dealing with the state-constrained optimal control problem, where both of the state and control variables are parametrized by the weighted Lagrange polynomials along the finite LGR collocation points within each mesh interval. Motivated by the shortcoming of path planning algorithms and inspired
by the hp-adaptive Radau pseudospectral method, a novel UAV trajectory planning scheme is developed in this paper.

By imposing the UAV dynamics into the formulation and considering several performance measure, the trajectory planning problem is formulated as an optimal control problem subject to continuous state inequality constraints. To solve this kind of problem, the Radau pseudospectral method is combined with the hp adaptive finite element method for transforming the state-constrained optimal control problem into a nonlinear programming problem, which can be efficiently solved by the sequential quadratic program (SQP) method. To verify the effectiveness of the proposed method, several numerical examples are carried out by comparing the proposed method with the well-known A* and RRT* algorithm. The generated trajectories by the three methods are tracked by a PID controller which is tuned the particle swarm optimization (PSO) algorithm. Simulation results show that the proposed method outperform A* [30] and RRT* [20] algorithm in terms of tracking error.

The rest of this paper is organized as follows. The UAV trajectory planning problem is stated in Section 2. In Section 3, the Radau pseudospectral method and the hp adaptive finite element method are introduced, respectively. The hp-adaptive Radau pseudospectral method for UAV trajectory planning is developed in Section 3. In Section 4, simulation experiments are carried out to test the proposed method. Finally, we conclude this paper in Section 5.

2. Problem statement.

2.1. The dynamics model of the UAV. The quad-rotor UAV with the Earth-frame and the fixed-body frame is shown in Fig. 1. The thrusts generated by the four rotors are denoted by \( F_i, i = 1, 2, 3, 4 \) respectively. Let \( E = [X_e, Y_e, Z_e] \) represents a right hand inertia frame with \( Z_e \) being the vertical direction towards the sky. The body fixed frame, denoted by \( B = [X_b, Y_b, Z_b] \) is located at the center of gravity of the aircraft. The Euclidean position and Euler angle of the UAV with respect to the frame are represented by \( Q(t) = [x(t), y(t), z(t)]^T \) and \( W(t) = [\phi(t), \theta(t), \psi(t)]^T \), respectively. The dynamics model of the UAV [33, 29]
can be described by the following nonlinear differential equations:

\begin{align*}
\ddot{x} &= u_1 (\cos(\varphi) \sin(\theta) \cos(\psi) + \sin(\varphi) \sin(\psi)) - \frac{K_1 \dot{x}}{m} \\
\ddot{y} &= u_1 (\sin(\varphi) \sin(\theta) \cos(\psi) - \cos(\varphi) \sin(\psi)) - \frac{K_2 \dot{y}}{m} \\
\ddot{z} &= u_1 \cos(\theta) \cos(\psi) - g - \frac{K_3 \dot{z}}{m} \\
\dot{\theta} &= u_2 - \frac{L K_4 \dot{\theta}}{I_y} \\
\dot{\psi} &= u_3 - \frac{L K_5 \dot{\psi}}{I_x} \\
\dot{\varphi} &= u_4 - \frac{L K_6 \dot{\varphi}}{I_z}
\end{align*}

(1)

where $O_b$ and $O_e$ denote the fixed-body and the Earth frame, respectively; $x, y,$ and $z$ represent inertial horizontal, lateral, and vertical positions in the Earth-frame; $\theta, \psi, \varphi$ are the pitch angle, roll angle and yaw angle, respectively; $g$ is the acceleration of gravity; $L$ represents the arm length from the motor to the center of mass; $m$ is the total mass of the quad-rotor; $I_i, i = x, y, z$ stands for the moment of inertia about the coordinate axis; $K_i, i = 1, 2, \ldots, 6$ denotes the coefficient of air resistance; The virtual control inputs $u_i, i = 1, 2, 3, 4$ are defined as follows:

\begin{align*}
    u_1 &= \frac{F_1 + F_2 + F_3 + F_4}{m} \\
    u_2 &= \frac{L (F_2 - F_4)}{I_y} \\
    u_3 &= \frac{L (F_3 - F_1)}{I_x} \\
    u_4 &= \frac{C (F_1 - F_2 + F_3 - F_4)}{I_z}
\end{align*}

(2)

where $C$ denotes the thrust-moment scale factor and the thrust generated by the $i$th rotor $F_i, i = 1, 2, 3, 4$, is defined by [17]

\begin{equation}
    F_i = K_i \omega_i^2
\end{equation}

(3)

where $K_i$ is the propeller thrust coefficient, $\omega_i, i = 1, 2, 3, 4$ is the rotational angular velocity of the $i$th rotor, which can be obtained from the virtual input $u_1, u_2, u_3$ and $u_4$ through the following conversion:

\begin{equation}
\begin{bmatrix}
\omega_1^2 \\
\omega_2^2 \\
\omega_3^2 \\
\omega_4^2
\end{bmatrix} = \frac{1}{K_v} \begin{bmatrix}
\frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\
0 & \frac{L}{I_y} & 0 & -\frac{L}{I_y} \\
-\frac{L}{I_x} & 0 & \frac{L}{I_x} & 0 \\
\frac{C}{I_z} & -\frac{C}{I_z} & \frac{C}{I_z} & -\frac{C}{I_z}
\end{bmatrix}^{-1} \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
\end{equation}

(4)

To obtain the state-space model, we define the state vector and the control vector as

\begin{equation}
X = \begin{bmatrix}
\theta & \psi & \varphi & \dot{\theta} & \dot{\psi} & \dot{\varphi} & x & y & z & \dot{x} & \dot{y} & \dot{z}
\end{bmatrix}^T
\end{equation}

(5)
and

\[ U = [u_1 \ u_2 \ u_3 \ u_4]^T \]  \hspace{1cm} \text{(6)}

respectively. Considering (5) and (6), differential equations (1) can be redefined as

\[
\begin{align*}
\dot{x}_1(t) &= x_4(t) \\
\dot{x}_2(t) &= x_5(t) \\
\dot{x}_3(t) &= x_6(t) \\
\dot{x}_4(t) &= u_2(t) - \frac{L K_1 x_4(t)}{I_y} \\
\dot{x}_5(t) &= u_3(t) - \frac{L K_2 x_5(t)}{I_x} \\
\dot{x}_6(t) &= u_4(t) - \frac{L K_3 x_6(t)}{I_z} \\
\dot{x}_7(t) &= x_{10}(t) \\
\dot{x}_8(t) &= x_{11}(t) \\
\dot{x}_9(t) &= x_{12}(t) \\
\dot{x}_{10}(t) &= u_1(t) \left( \cos(x_3(t)) \sin(x_1(t)) \cos(x_2(t)) + \sin(x_2(t)) \sin(x_3(t)) \right) - \frac{K_4 x_{10}(t)}{M} \\
\dot{x}_{11}(t) &= u_1(t) \left( \sin(x_3(t)) \sin(x_1(t)) \cos(x_2(t)) - \sin(x_2(t)) \cos(x_3(t)) \right) - \frac{K_5 x_{11}(t)}{M} \\
\dot{x}_{12}(t) &= u_1(t) \cos(x_3(t)) \cos(x_2(t)) - g - \frac{K_6 x_{12}(t)}{M} \\
\end{align*}
\]  \hspace{1cm} \text{(7)}

For the notation simplicity, above equation (7) is denoted as

\[ \dot{X}(t) = f(X(t), U(t)) \]  \hspace{1cm} \text{(8)}

2.2. Obstacle model. Two classical barriers are considered in this paper, and the obstacle avoidance can be formulated as continuous state inequality constraints.

The first type of obstacle is a multiple columnar shape barrier which is similar to the architectural area in the city. The obstacle avoidance constraint can be defined as follows [8]:

\[
\left( \frac{|x(t) - L|}{a} \right)^{P_x} + \left( \frac{|y(t) - L|}{b} \right)^{P_y} + \left( \frac{|z(t) - z_c|}{c} \right)^{P_z} \geq 1 \]  \hspace{1cm} \text{(9)}

where \((x_c, y_c, z_c)\) is the center of the obstacle, and the radius of the obstacle in the direction \(x, y, z\) are defined by \(a, b, c\), respectively. \(P_x, P_y, P_z\) determine the shape of the obstacle. For example, \(P_x = P_y = P_z = 2\) defines an ellipsoid and \(P_x = P_y = P_z = 10\) defines a box-shaped obstacle.

The second type of obstacle is a mountain-shaped barrier which is similar to the mountains in the rural area. The obstacle avoidance constraint can be defined as follows [16]:

\[
z(t) - \sum_{i=1}^{n} h_i \exp \left[ \left( \frac{|x(t) - L| - x_i}{x_{si}} \right)^2 + \left( \frac{|y(t) - L| - y_i}{y_{si}} \right)^2 \right] \geq 0 \]  \hspace{1cm} \text{(10)}

where \((x_i, y_i)\) is the coordinates of the center for the \(i\)th mountain, and \(h_i\) is the terrain parameter, which determines the height of the mountain. \(x_{si}\) and \(y_{si}\) are used for controlling the slope of the mountain, which are the attenuation of the
ith mountain along the $x$-axis and $y$-axis, respectively. $n$ is the total number of mountains.

2.3. **Objective function and constraints.** In this paper the flying time $t_f$ is chosen as the performance measure, and the objective function of UAV optimization can be defined by

$$J = t_f$$

(11)

The initial condition and constraints are listed below [4].

1. Initial state constraints and terminal constraints are defined by

$$
X(0) = X_0.
$$

(12)

$$
\begin{bmatrix}
x(t_f) \\
y(t_f) \\
z(t_f)
\end{bmatrix} =
\begin{bmatrix}
x_{t_f} \\
y_{t_f} \\
z_{t_f}
\end{bmatrix}
$$

(13)

where $X_0$ is the initial state of the quadrotor UAV and $[x_{t_f}, y_{t_f}, z_{t_f}]^T$ is the target position.

2. Due to the characteristics of the UAV, the flying altitude should not be too high. The height constraint of the UAV can be written as:

$$0 \leq z(t) \leq z_{\text{max}}$$

(14)

where $z_{\text{max}}$ is the highest altitude of the quadrotor. Considering the physical limitations of the motor, the speed of the UAV cannot be too high. In order to limit the speed of the UAV, the following constraints are imposed:

$$
|\dot{x}(t)| \leq V_{x_{\text{max}}} \\
|\dot{y}(t)| \leq V_{y_{\text{max}}} \\
|\dot{z}(t)| \leq V_{z_{\text{max}}}
$$

(15)

where $V_{x_{\text{max}}}$, $V_{y_{\text{max}}}$ and $V_{z_{\text{max}}}$ are the components of the maximum flight speed of the quadrotor. Similarly, since the UAV is flying at a small angle in this experiment, considering the limitation of angle, we introduce the following constraints:

$$
\theta_{\text{min}} \leq \theta(t) \leq \theta_{\text{max}} \\
\psi_{\text{min}} \leq \psi(t) \leq \psi_{\text{max}} \\
\varphi_{\text{min}} \leq \varphi(t) \leq \varphi_{\text{max}}
$$

(16)

where $\theta_{\text{min}}$, $\theta_{\text{max}}$, $\psi_{\text{min}}$, $\psi_{\text{max}}$, $\varphi_{\text{min}}$ and $\varphi_{\text{max}}$ are the minimum and maximum value of pitch angle, roll angle and yaw angle, respectively.

3. There are physical limitations on the rotational angular velocity $\omega$ and its acceleration $\ddot{\omega}$ of the quadrotor motors, which can be achieved by the following physical constraints on $u$ and its derivatives:

$$u_{i_{\text{min}}} \leq u_i(t) \leq u_{i_{\text{max}}}, \ i = 1, 2, 3, 4$$

(17)

$$|\dot{u}_i(t)| \leq \dot{u}_{i_{\text{max}}}, \ i = 1, 2, 3, 4$$

(18)

where $u_{i_{\text{min}}}$ and $u_{i_{\text{max}}}$ are the minimum and maximum input of each control variable, respectively, $\dot{u}_{i_{\text{max}}}$ is the maximum value of the derivative of each control variable.

It can be seen that this is an optimal control problem restricted by multiple states and controls variables constraints.
3. **Hp adaptive radau pseudospectral method.** Based on the UAV trajectory planning optimization problem presented in Section 2, we use the hp adaptive Radau pseudospectral method to solve the optimal control problem.

3.1. **Radau pseudospectral method.** The main steps of the Radau pseudospectral method can be described as follows:

Transforming the optimization problem from the time interval $t \in [t_0, t_f]$ to $\tau \in [-1, 1]$ by the following affine transformation [24]:

$$t = \frac{t_f - t_0}{2} \tau + \frac{t_f + t_0}{2}$$ (19)

Then, we consider $N - 1$ LGR collocation points $\tau_1, \tau_2, \ldots, \tau_{N-1}$ in region $\tau \in [-1, 1)$, which can be obtained from the root of $P_{N-1}(\tau) + P_N(\tau) = 0$, where the $P_N(\tau)$ is the Nth-order Legendre polynomial [11]. And an additional noncollocated point $\tau_N = 1$ is introduced to describe the terminal approximation of the state variable.

After transforming the time interval and selecting LGR collocation points, we can approximate the state and control variables of quadrotor as

$$X(t) = X(\tau) \approx \sum_{i=1}^{N} X_i L_i(\tau), \quad i = 1, 2, \ldots, N$$

$$U(t) = U(\tau) \approx \sum_{i=1}^{N-1} U_i \tilde{L}_i(\tau), \quad i = 1, 2, \ldots, N-1$$ (20)

where $X_i$ and $U_i$ are the state approximation and control approximation at $\tau = \tau_i$, respectively. $L_i$ and $\tilde{L}_i$ are the basis of Lagrange polynomials, which can be expressed as

$$L_i(\tau) = \prod_{j=1, j \neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$

$$\tilde{L}_i(\tau) = \prod_{j=1, j \neq i}^{N-1} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$ (21)

Through the derivation with respect to $\tau$ on both sides of the first equation in (20), we can obtain:

$$X(\tau_k) \approx \sum_{i=1}^{N} X_i \dot{L}_i(\tau_k), \quad k = 1, 2, \ldots, N-1$$ (22)

then, the state-space dynamics constraint (8) can be approximated by following algebraic nonlinear equality constraints according to (19), (20) and (22):

$$\sum_{i=1}^{N} X_i \dot{L}_i(\tau_k) - \frac{t_f - t_0}{2} f(X_k, U_k) = 0$$ (23)

where $X_k \equiv X(\tau_k), \quad U_k \equiv U(\tau_k), \quad k = 1, 2, \ldots, N-1$.

After approximating the state variables, control variables and the state-space dynamics constraint, we can then approximate the boundary and path constraints (12)-(17) of the UAV trajectory optimization problem mentioned in Section 2.3 (Since the cost function (11) does not involve the state and control variables, it remains unchanged).
The boundary and path constraints are approximated at the boundary points and each LGR points respectively. More specifically, the boundary constraints (12) and (13) can be described as:

\[
X(\tau_1) = X_0
\]  
(24)

\[
\begin{bmatrix}
x(\tau_N) \\
y(\tau_N) \\
z(\tau_N)
\end{bmatrix} =
\begin{bmatrix}
x_t \\
y_t \\
z_t
\end{bmatrix}
\]  
(25)

For simplicity, the path constraints (14)-(17) can be uniformly expressed in the following form,

\[
C_i(X(t), U(t)) \leq 0, \quad i = 1, 2, \ldots, 22
\]  
(26)

and each constraint can be approximated as:

\[
C_i(X_k, U_k) \leq 0, \quad i = 1, 2, \ldots, 22, \quad k = 1, 2, \ldots, N - 1
\]  
(27)

Finally, the UAV trajectory optimization problem can be transformed into the nonlinear programming problem by above RPM, that is, minimize the cost function (11) under the algebraic constraints (23)-(25) and (27).

3.2. **Hp-adaptive segment adjustment strategy.**

3.2.1. **Definition of scaled midpoint residual vector.** After solving the above nonlinear programming problem, the approximated states, controls and time intervals can be obtained. To estimate the error in the differential-algebraic equations of (23), we insert the following midpoints \(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{N_s}\) on the \(s\)th time segment \([t_{s-1}, t_s]\) [11]:

\[
\bar{t}_i = \frac{t_i + t_{i+1}}{2}, \quad i = 1, 2, \ldots, N_s - 1
\]  
(28)

where \(t_1, t_2, \ldots, t_{N_s} \in [t_{s-1}, t_s]\) are the original time points.

Then, let the following \(\bar{X}\) and \(\bar{U}\) to represent the approximation of the state and control at each midpoint, which are obtained using the Lagrange polynomial approximation and the cubic interpolation [7], respectively.

\[
\bar{X} = \begin{bmatrix} X(\bar{t}_1) \\ \vdots \\ X(\bar{t}_{N_s-1}) \end{bmatrix}, \quad \bar{U} = \begin{bmatrix} U(\bar{t}_1) \\ \vdots \\ U(\bar{t}_{N_s-1}) \end{bmatrix}
\]  
(29)

Next, we have

\[
E = \left| D\bar{X} - \frac{t_s - t_{s-1}}{2} J(X, U) \right| \in \mathbb{R}^{(N_s-1) \times 12}
\]  
(30)

where \(D\) denotes the differentiation matrix [18]. Then, the following \(\alpha\) is written to contain the largest element in each row of \(E\) which is used to determine if the segment should be further subdivided or the number of collocation points in the segment should be increased.

\[
\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{N_s-1} \end{bmatrix}
\]  
(31)

Then, we let \(\bar{\alpha}\) be the arithmetic mean of the components of \(\alpha\):

\[
\bar{\alpha} = \frac{\sum_{i=1}^{N_s-1} \alpha_i}{N_s - 1}
\]  
(32)
Next, define the scaled midpoint residual vector \[ \beta \]

\[
\beta = \begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_{N_s-1}
\end{bmatrix} = \begin{bmatrix}
\alpha_1/\bar{\alpha} \\
\vdots \\
\alpha_{N_s-1}/\bar{\alpha}
\end{bmatrix}
\] (33)

3.2.2. Criteria for segment adjustment. Let \( \epsilon \) be a tolerance for the maximum entry of \( E \), \( E_{\text{max}} \). If \( E_{\text{max}} \leq \epsilon \), the approximation error of (23) meets the requirements, the segment does not need to be changed. If \( E_{\text{max}} > \epsilon \), the segments needs to be divided into more segments or more LGR collocation points needs to be added according to \( \beta \).

Let \( \eta \) be a threshold for each element of \( \beta \), we have the following two ways to change the segment: (1) If there is \( \beta_i > \eta \), the corresponding \( s \)th segment is divided into more segments, which can be determined by the method in [18]. (2) If there is no elements of \( \beta \) exceed \( \eta \), the number of collocation points in the \( s \)th segment needs to be increased, that is

\[
N_s = N_s + K
\] (34)

where \( K > 0 \) is a specified number.

The algorithm of hp-Adaptive mesh refinement method can be summarized as follows:

**STEP1.** Initialize the problem choosing \( M \) collocation points.

**STEP2.** Solve the corresponding nonlinear programming problem transformed by the Radau pseudospectral method.

**STEP3.** Calculate the error matrix \( E \) and the scaled midpoint residual vector \( \beta \) for each segment, continue to **STEP4**.

**STEP4.** Determine whether all the segments satisfy that \( E_{\text{max}} \leq \epsilon \), if not, continue to **STEP5**, or the optimization process is stopped, the optimal solution of UAV trajectory planning problem is outputed.

**STEP5.** the corresponding segment needs to be changed into more segments or the more LGR collocation points needs to be added according to the hp-adaptive segment adjustment strategy, then continue to **STEP2**.

**STEP6. END.**

4. Numerical simulation. To verify the effectiveness of the hp adaptive Radau pseudospectral algorithm and compare it with the AStar algorithm and the RRT* algorithm, this paper conducted three different numerical experiments. In Case I, we consider the UAV flying in columnar obstacles to simulate an urban area environment; In Case II, we consider the UAV flying in a mountain-shaped obstacle to simulate a mountainous environment, and achieve trajectory optimization by using AStar, RRT* and hp adaptive Radau pseudospectral method, respectively. In Case III, the trajectory generated by the algorithm is used as the reference input of the closed-loop control system to test whether it can be tracked in practical applications and complete the closed-loop trajectory tracking experiment. The quad-rotor UAV control constraints are shown in Table 1. And the parameters of the quad-rotor UAV are listed in Table 2.

The simulation implemented in this paper are under Windows 10 and Intel (R) i7-9750H CPU, 2.60 GHZ, with 16.00 GB RAM. The experiment simulation software uses MATLAB 2019a [34]. And it uses the GPOPS-II [19] to solve the trajectory
Table 1. State and control constraints of UAV

| Parameters                          | Value                  |
|-------------------------------------|------------------------|
| Altitude \( z \) (m)               | \([0, 500]\)           |
| Velocity of x direction \( V_x \) (m/s) | \([0, 25]\)         |
| Velocity of y direction \( V_y \) (m/s) | \([0, 25]\)         |
| Velocity of z direction \( V_z \) (m/s) | \([0, 20]\)         |
| pitch angle \( \theta \) (°)      | \([-\frac{\pi}{6}, \frac{\pi}{6}]\) |
| roll angle \( \psi \) (°)         | \([-\frac{\pi}{6}, \frac{\pi}{6}]\) |
| yaw angle \( \varphi \) (°)       | \([-\frac{\pi}{6}, \frac{\pi}{6}]\) |
| Virtual control input 1 \( u_1 \) (N) | \([10, 12]\)       |
| Virtual control input 2 \( u_2 \) (N · M) | \([-0.005, 0.005]\) |
| Virtual control input 3 \( u_3 \) (N · M) | \([-0.005, 0.005]\) |
| Virtual control input 4 \( u_4 \) (N · M) | \([-0.005, 0.005]\) |
| Derivative of virtual control input 1 \( \dot{u}_1 \) (N/s) | \([-1.5, 1.5]\)   |
| Derivative of virtual control input 2 \( \dot{u}_2 \) (N · M/s) | \([-1, 1]\)        |
| Derivative of virtual control input 3 \( \dot{u}_3 \) (N · M/s) | \([-1.5, 1.5]\)   |
| Derivative of virtual control input 4 \( \dot{u}_4 \) (N · M/s) | \([-1.5, 1.5]\)   |

Table 2. Parameters of UAV

| Parameters                                | Value |
|-------------------------------------------|-------|
| Length from motor to center of mass \( L \) (m) | 0.2   |
| Total mass of UAV \( M \) (kg)            | 1.5   |
| Acceleration of gravity \( g \) (m/s²)    | 9.8   |
| Moment of inertia about the x axis \( I_x \) (kg · m²) | 0.0075 |
| Moment of inertia about the y axis \( I_y \) (kg · m²) | 0.0075 |
| Moment of inertia about the z axis \( I_z \) (kg · m²) | 0.013 |
| Coefficient of air resistance \( K_1 \) (N/m/s) | 0.06  |
| Coefficient of air resistance \( K_2 \) (N/m/s) | 0.06  |
| Coefficient of air resistance \( K_3 \) (N/m/s) | 0.09  |
| Coefficient of air resistance \( K_4 \) (N/m/s) | 0.002 |
| Coefficient of air resistance \( K_5 \) (N/m/s) | 0.002 |
| Coefficient of air resistance \( K_6 \) (N/m/s) | 0.1   |

planning problem in the MATLAB environment, \( \epsilon = 1.0 \times 10^{-6}, \eta = 1.0 \times 10^{-4}, K = 2 \). Three different numerical experiments and analyses are as follows:

4.1. **Case I: UAV trajectory optimization in columnar obstacles.** In the numerical simulation of case I, the algorithm we proposed is implemented in the environment of the first type of obstacle \( (9) \). We set the starting point as \((100, 100, 100)\),
the initial conditions as $x_0 = [0; 0; 0; 0; 0; 100; 100; 100; 5; 5; 1]^T$, and the target point as $(400, 400, 400)$. We set $P_x = P_y = P_z = 8$, that is, the obstacle is a box-like shape to simulate a rectangular parallelepiped building group. The optimal trajectory of the UAV generated by the numerical simulation experiment is shown in Fig. 2(a), Fig. 2(b) is the top view, and the optimal state and optimal control input are shown in Fig. 3(a) and Fig. 3(b), respectively. It can be seen that the UAV do not encounter any obstacles in the process from the starting point to the target point, and the trajectory has no sharp corners, and it is also a smooth curve, therefore, this method is suitable for UAV flight trajectory optimization in environment of urban buildings. As shown in Fig. 3(a) and Fig. 3(b), all constraints are satisfied, and the minimum flight time is 25.64 seconds.
4.2. Case II: UAV trajectory optimization in mountain-shaped obstacles.

In the numerical simulation of case II, the algorithm we propose is implemented in the environment of the second type of obstacle (10). Through experimental simulation, we separately studied the UAV trajectory optimization problem based on the three algorithms of AStar, RRT*, and hp adaptive Radau pseudospectral method. The specific process is as follows:

1. AStar Algorithm

Achieve path planning by using AStar algorithm. The optimal trajectory of the UAV generated by the numerical simulation experiment is shown in Fig. 4(a), and Fig. 4(b) is the top view. Because the AStar algorithm ignores the dynamics characteristics of the UAV, we can see that two sharp corners can
be observed on the planned path, the planned flight path is a polyline section by section, and the flight path is long, so that it will consume more time and energy for UAV flighting.

Figure 4. Trajectory optimization by using AStar algorithm.

2. RRT* Algorithm

Using RRT* algorithm achieve path planning. The optimal trajectory of the UAV generated by the numerical simulation experiment is shown in Fig. 5(a), and Fig. 5(b) is the top view. We can also find that there are two sharp corners on the optimized path. Although the generated corners on the optimal flight path is smoother than the path of the AStar algorithm, the planned path is obviously not the best and not smooth enough to be tracked because of the neglect of UAV dynamics.
3. Hp Adaptive Radau Pseudospectral Algorithm

We set the starting point as (5, 70, 10), the target point as (80, 20, 15), and the initial conditions as $x_0 = [0; 0; 0; 0; 0; 0; 5; 70; 10; 3; -3; 1.2]^T$. The optimal trajectory of the UAV generated by the numerical simulation experiment is shown in Fig. 6(a), and Fig. 6(b) is the top view. The optimal state and optimal control input are shown in Fig. 7(a) and Fig. 7(b), respectively. We can see that the UAV reaches the target without touching obstacles, and no sharp corners are found in the generated optimal trajectory, which is a very smooth curve and the path length is significantly shorter than the path generated by the above two algorithms. As shown in Fig. 7(a) and Fig. 7(b), all the constraints of the UAV are satisfied.

**Figure 5.** Trajectory optimization by using RRT* algorithm.
4.3. **Case III: Trajectory tracking of UAV.** When the required optimal control trajectory is obtained as a command, a limited time tracking controller of the UAV is designed to achieve high-precision tracking. In the numerical simulation of case III, we use a closed-loop control system to track the trajectory generated in the previous section. The closed-loop control system structure is shown in Fig. 8.

The trajectory tracking closed-loop simulation experiment is implemented using particle swarm optimization algorithm. The general process is to set each parameter value, initialize the particle swarm, set individual extreme values and group extreme values, including individual best and global best, individual best fitness value and the optimal fitness value of the group, after iterative optimization, stops the iterative
operation after reaching the requirement or the specified number of times, and finally obtains the gain value.
Figure 9. Trajectory tracking by using AStar algorithm.

Figure 10. Trajectory tracking by using RRT* algorithm.

The optimal trajectory (reference trajectory) and tracking trajectory generated by AStar, RRT* and pseudospectral methods are shown in Fig. 9, Fig. 10, and Fig. 11, respectively. Among them, the green solid line represents the optimal trajectory (reference trajectory), and the tracking trajectory is represented by a red solid line. It can be seen from Fig. 11 that in the hp adaptive Radau pseudospectral method, the tracking trajectory almost follows the optimal trajectory. In contrast, the error between the tracking trajectory and the optimal trajectory in Fig. 9 and Fig. 10 is relatively large, because the AStar algorithm and the RRT* algorithm do not consider the dynamics characteristics of the UAV.
5. Conclusion. In this paper, a method combining hp adaptive finite element method and Radau pseudospectral method to solve the trajectory optimization problem of quad-rotor UAV in a barrier 3D environment is proposed. Simulation results showed that the hp adaptive Radau pseudospectral algorithm outperforms the AStar algorithm, RRT* algorithm in terms of optimality and feasibility in the generated trajectory, and the tracking error of this algorithm is smaller in closed-loop feedback trajectory tracking. In summary, we can conclude that the hp adaptive Radau pseudospectral method is suitable to solve the problem of quad-rotor UAV trajectory optimization, smooth and well-trackable trajectories can be obtained. In the future research, other pseudospectral methods will be discussed, such as Gauss pseudospectral method and Chebyshev pseudospectral method. At the same time, there is not much in-depth research and discussion on other methods for solving optimal control problems, which lays a foundation for further exploration of UAV trajectory optimization methods, and we will try to apply the proposed method in actual UAV flight.

REFERENCES

[1] N. Ahn and S. Kim, Optimal and heuristic algorithms for the multi-objective vehicle routing problem with drones for military surveillance operations, J. Industrial and Management Optimization, 2021.
[2] A. F. Alkaya and D. Oz, An optimal algorithm for the obstacle neutralization problem, J. Ind. Manag. Optim., 13 (2017), 835–856.
[3] R. Austin, Unmanned aircraft systems: UAVs design, development, and deployment, Journal Publications Chestnet.org, 50 (2010), 31–36.
[4] A. Chamseddine, Y. Zhang, C. A. Rabbath, C. Join and D. Theilliol, Flatness-based trajectory planning/replanning for a quadrotor unmanned aerial vehicle, IEEE Transactions on Aerospace and Electronic Systems, 48 (2012), 2832–2848.
[5] J. Chen, Y. Cao, N. Du, X. Liu and Y. Han, Gaussian pseudospectral longitudinal trajectory optimization algorithm of a solar powered communication/remote-sensing UAV, 2019 IEEE International Conference on Unmanned Systems and Artificial Intelligence (ICUSAI), (2019), 303–308.
[6] W. P. Coutinho, M. Battarra and J. Fliege, The unmanned aerial vehicle routing and trajectory optimisation problem, a taxonomic review, *Computers and Industrial Engineering*, **120** (2018), 116–128.

[7] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, 1996.

[8] B. T. Gatzke, Trajectory optimization for helicopter unmanned aerial vehicles (UAVs), *NPS Thesis*, 2012.

[9] A. Goli, H. K. Zare, R. Tavakkoli-Moghaddam and A. Sadeghieh, A comprehensive model of demand prediction based on hybrid artificial intelligence and metaheuristic algorithms: A case study in dairy industry, *MPRA Paper*, **11** (2018), 190–203.

[10] A. Goli, H. Khademi-Zare, R. Tavakkoli-Moghaddam, A. Sadeghieh, M. Sasanian and R. M. Kordestanizadeh, An integrated approach based on artificial intelligence and novel metaheuristic algorithms to predict demand for dairy products: A case study, *Network Computation in Neural Systems*, **1** (2021), 1–35.

[11] M. Y. Hussaini and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer Series in Computational Physics. Springer-Verlag, New York, 1988.

[12] S. Karaman and E. Frazzoli, Optimal kinodynamic motion planning using incremental sampling-based methods, *49th IEEE Conference on Decision and Control (CDC)*, (2010), 7681–7687.

[13] J. Li, G. Deng, C. Luo, Q. Lin, Q. Yan and Z. Ming, A hybrid path planning method in unmanned Air/Ground vehicle (UAV/UGV) cooperative systems, *IEEE Transactions on Vehicular Technology*, **65** (2016), 9585–9596.

[14] B. Li, J. Zhang, L. Dai, K. L. Teo and S. Wang, A hybrid offline optimization method for reconfiguration of multi-UAV formations, *IEEE Transactions on Aerospace and Electronic Systems*, **57** (2021), 506–520.

[15] C. Y. Liu, Z. H. Gong, K. L. Teo, J. Sun and L. Caccetta, Robust multi-objective optimal switching control arising in 1,3-propanediol microbial fed-batch process, *Nonlinear Anal. Hybrid Syst.*, **25** (2017), 1–20.

[16] H. Liu, Q. Chen, N. Pan, Y. Sun and Y. Yang, Three-dimensional mountain complex terrain and heterogeneous multi-UAV cooperative combat mission planning, *IEEE Access*, **8** (2020), 197407–197419.

[17] R. Mahony, V. Kumar and P. Corke, Multirotor aerial vehicles: Modeling, estimation, and control of quadrotor, *IEEE Robotics and Automation Magazine*, **19** (2012), 20–32.

[18] M. A. Patterson, W. W. Hager and A. V. Rao, A ph mesh refinement method for optimal control, *Optimal Control Appl. Methods*, **36** (2015), 398–421.

[19] M. A. Patterson and A. V. Rao, GPOPS-II: A MATLAB software for solving multiple-phase optimal control problems using hp-adaptive gaussian quadrature collocation methods and sparse nonlinear programming, *ACM Trans. Math. Software*, **41** (2014), Art. 1, 37 pp.

[20] P. Pharpatara, B. Hérisse and Y. Bestaoui, 3-D trajectory planning of aerial vehicles using RRT*, *IEEE Transactions on Control Systems Technology*, **25** (2017), 1116–1123.

[21] B. Salamat and A. M. Tonello, A modelling approach to generate representative UAV trajectories using PSO, *2019 27th European Signal Processing Conference (EUSIPCO)*, (2019), 1–5.

[22] N. Sariff and N. Buniyamin, An overview of autonomous mobile robot path planning algorithms, *2006 4th Student Conference on Research and Development*, (2006), 183–188.

[23] Y. Shi, R. Li and H. Xu, Control augmentation design of UAVs based on deviation modification of aerodynamic focus, *J. Industrial and Management Optimization*, **11** (2015), 231–240.

[24] Z. S. Shui, J. Zhou and Z. L. Ge, On-line predictor-corrector reentry guidance law based on Gauss pseudospectral method, *J. Astronautics*, **6** (2011), 1249–1255.

[25] K. P. Valavanis and G. J. Vachtsevanos, *Handbook of Unmanned Aerial Vehicles*, 1st edition, Springer Netherlands, 2014.

[26] S. Vera, J. A. Cobano, G. Heredia and A. Ollero, An hp-adaptive pseudospectral method for collision avoidance with multiple UAVs in real-time applications, *2014 IEEE International Conference on Robotics and Automation (ICRA)*, (2014), 4717–4722.

[27] N. Wen, X. Su, P. Ma, L. Zhao and Y. Zhang, Online UAV path planning in uncertain and hostile environments, *International J. Machine Learning and Cybernetics*, **8** (2017), 469–487.

[28] F. Yan, Y. Liu and J. Xiao, Path planning in complex 3D environments using a probabilistic roadmap method, *International J. Automation and Computing*, **6** (2013), 525–533.
[29] S. Yang and Z. Wang, Quad-rotor UAV control method based on PID control law, *2017 International Conference on Computer Network, Electronic and Automation (ICCNEA)*, (2017), 418–421.

[30] W. Zeng and R. L. Church, Finding shortest paths on real road networks: The case for A*, *International J. Geographical Information Science*, 23 (2009), 531–543.

[31] L. Zhang, F. Deng, J. Chen, Y. Bi, S. K. Pang and X. Chen, Trajectory planning for improving vision-based target geolocation performance using a quad-rotor UAV, *IEEE Transactions on Aerospace and Electronic Systems*, 55 (2019), 2382–2394.

[32] Y. Zhang, C. Yu and Y. Xu, Minimizing almost smooth control variation in nonlinear optimal control problems, *J. Ind. Manag. Optim.*, 16 (2020), 1663–1683.

[33] B. Zhao, B. Xian, Y. Zhang and X. Zhang, Nonlinear robust adaptive tracking control of a quadrotor UAV via immersion and invariance methodology, *IEEE Transactions on Industrial Electronics*, 62 (2015), 2891–2902.

[34] MATLAB, 9.7.0.1190202 (R2019a), Natick, Massachusetts: The MathWorks Inc, 2019.