Realising fusion systems

Ian J. Leary∗  Radu Stancu

June 21, 2021

Abstract
We show that every fusion system on a $p$-group $S$ is equal to the fusion system associated to a discrete group $G$ with the property that every $p$-subgroup of $G$ is conjugate to a subgroup of $S$.

1 Introduction

Let $p$ be a prime number. By a $p$-group we shall mean a finite group whose order is a power of $p$. A fusion system on a $p$-group $S$ is a category $\mathcal{F}$ whose objects are the subgroups of $S$, and whose morphisms are injective group homomorphisms, subject to certain axioms. The notion of a fusion system is intended to axiomatize the $p$-local structure of a discrete group $G \geq S$ in which every $p$-subgroup is conjugate to a subgroup of $S$. Every such $G$ gives rise to a fusion system $\mathcal{F}_S(G)$ on $S$, and we say that $G$ realises $\mathcal{F}$ if $\mathcal{F}_S(G) = \mathcal{F}$.

The notion of a saturated fusion system is intended to axiomatize the $p$-local structure of a finite group in which $S$ is a Sylow $p$-subgroup. It is known that there are saturated fusion systems $\mathcal{F}$ which are not realised by any finite group $G$, although showing that this is the case is very delicate. In the case when $p = 2$, the only known examples are certain systems discovered by Ron Solomon [4, 10, 15].

In contrast, we show that every fusion system on any $p$-group $S$ is realised by some discrete group $G \geq S$ in which every maximal $p$-subgroup is conjugate to $S$. The groups $G$ that are used in our proofs are constructed

∗Partially supported by NSF grant DMS-0505471
as graphs of finite groups. In particular each of our groups $G$ contains a free subgroup of finite index. In an appendix we give a brief account of those parts of the theory of graphs of groups that we use.

While preparing this paper, we learned that Geoff Robinson has proved a similar, but not identical result [13]. Since [13] was already submitted when we started to write this paper, we have taken it upon ourselves to compare and contrast the two results. Robinson’s construction realises a large class of fusion systems, including all saturated fusion systems, but does not realise all fusion systems. The groups that Robinson constructs are iterated free products with amalgamation, whereas the groups that we construct are iterated HNN extensions. In both cases the groups may be viewed as graphs of finite groups.

We state and outline the proof of a version of Robinson’s theorem, along the lines of the proof of our main result. We also give examples of fusion systems that cannot be realised by Robinson’s method, we give examples of non-saturated fusion systems that are realised by Robinson’s method, and we prove an analogue of Cayley’s theorem for fusion systems.

The work in this paper grew from the authors’ participation in the Banff conference ‘Homotopy theory and group actions’ and from a VIGRE reading seminar at Ohio State which studied the Aschbacher-Chermak approach to the Solomon fusion systems [3]. The authors thank Andy Chermak and Geoff Robinson for showing them early versions of [3] and [13].

2 Definitions and results

Let $p$ be a prime, and let $G$ be a discrete group. The $p$-Frobenius category $\Phi_p(G)$ of the group $G$ is a category whose objects are the $p$-subgroups of $G$. If $P$ and $Q$ are $p$-subgroups of $G$, or equivalently objects of $\Phi_p(G)$, the morphisms from $P$ to $Q$ are the group homomorphisms $f : P \to Q$ that are equal to conjugation by some element of $G$. Thus $f : P \to Q$ is in $\Phi_p(G)$ if and only if there exists $g \in G$ with $f(u) = g^{-1}ug$ for all $u \in P$. (Note that the element $g$ is not part of the morphism. If $g' = zg$ for some element $z$ in the centralizer of $P$, then $g$ and $g'$ define the same morphism.)

Now suppose that $S$ is a $p$-subgroup of $G$ that is maximal, and further suppose that every $p$-subgroup of $G$ is conjugate to a subgroup of $S$. In this case, every object of $\Phi_p(G)$ is isomorphic within the category $\Phi_p(G)$ to a subgroup of $S$. It follows that the full subcategory $\mathcal{F}_S(G)$ with objects the
subgroups of $S$ is equivalent to $\Phi_p(G)$. This example motivates Puig’s definition of a fusion system on $S$ [12]. A fusion system on a $p$-group $S$ is a category $\mathcal{F}$. The objects of $\mathcal{F}$ are the subgroups of $S$, and the morphisms from $P$ to $Q$ form a subset of the set $\text{Inj}(P, Q)$ of injective group homomorphisms from $P$ to $Q$. These are subject to the following axioms:

1. For any $s \in S$, and any $P, Q \leq S$ with $s^{-1}Ps \leq Q$, the morphism $\phi : P \to Q$ defined by $\phi : u \mapsto s^{-1}us$ is in $\mathcal{F}$;

2. If $f : P \to Q$ is in $\mathcal{F}$, with $R = f(P) \leq Q$, then so are $f : P \to R$ and $f^{-1} : R \to P$.

It is easily checked that these axioms are satisfied in the case when $\mathcal{F} = F_S(G)$ as defined above. Note that the first axiom could be rewritten as the statement $F_S(S) \subseteq \mathcal{F}$.

Remark 1 Fusion systems arise in other ways. For example, if $H$ is any group and $S$ is any $p$-subgroup of $H$, then the full subcategory of $\Phi_p(H)$ with objects the subgroups of $S$ is a fusion system on $S$. Another source of fusion systems on a $p$-group $S$ is the Brauer category of a $p$-block $b$ [2 11]. Here $H$ is a finite group, $S$ is the defect group of the $p$-block $b$, and the morphisms in the category are those conjugations by elements of $H$ that preserve some extra structure associated to $b$. In the case when $b$ is the principal block, $S$ is the Sylow $p$-subgroup of $H$ and this fusion system is just $F_S(H)$. One corollary of our Theorem 2 is that every such fusion system is realised by some group $G$.

There is a fusion system $F_S^\text{max}$ on $S$, in which the morphisms from $P$ to $Q$ consists of all injective group homomorphisms from $P$ to $Q$. Any fusion system on $S$ is a subcategory of $F_S^\text{max}$, and the intersection of a family of fusion systems on $S$ is itself a fusion system. If $\Phi = \{\phi_1, \ldots, \phi_r\}$ is a collection of morphisms in $F_S^\text{max}$, where $\phi_i : P_i \to Q_i$, the fusion system generated by $\Phi$ is defined to be the smallest fusion system that contains each $\phi_i$.

Theorem 2 Suppose that $\mathcal{F}$ is the fusion system on $S$ generated by $\Phi = \{\phi_1, \ldots, \phi_r\}$. Let $T$ be a free group with free generators $t_1, \ldots, t_r$, and define $G$ as the quotient of the free product $S \ast T$ by the relations $t_i^{-1}ut_i = \phi_i(u)$ for all $i$ and for all $u \in P_i$. Then $S$ embeds as a subgroup of $G$, every $p$-subgroup of $G$ is conjugate to a subgroup of $S$, and $F_S(G) = \mathcal{F}$. Moreover, every
finite subgroup of $G$ is conjugate to a subgroup of $S$, and $G$ has a free normal subgroup of index dividing $|S|^!$.

If $f : S' \to S$ is an injective group homomorphism between $p$-groups, and $\mathcal{F}'$ is a fusion system on $S'$, then there is a functor $f_*$ from $\mathcal{F}'$ to $\mathcal{F}_S^\text{max}$, which sends $P' \leq S'$ to $f(P')$ and $\phi' : P' \to Q'$ to

$$f \circ \phi' \circ f^{-1} : f(P') \to f(Q').$$

**Theorem 3 (Robinson [13])** Suppose that $\mathcal{F}$ is the fusion system on $S$ generated by the images $(f_i)_*(\mathcal{F}_{S_i}(G_i))$ for injective group homomorphisms $f_i : S_i' \to S$ for $1 \leq i \leq r$, where $G_i$ is a finite group with $S_i'$ as a Sylow $p$-subgroup. Define $G$ as the quotient of the free product $S \ast G_1 \ast \cdots \ast G_r$ by the relations $s = f_i(s)$ for all $i$ and for all $s \in S_i'$. Then $S$ embeds as a subgroup of $G$, every $p$-subgroup of $G$ is conjugate to a subgroup of $S$, and $\mathcal{F}_S(G) = \mathcal{F}$. Moreover, every finite subgroup of $G$ is conjugate to a subgroup of one of the $G_i$, or to a subgroup of $S$, and $G$ has a free normal subgroup of index dividing $N!$, where $N$ is the least common multiple of $|S|$ and the $|G_i|$.

**Remark 4** The above theorem can be obtained from theorem 1 of [13] by induction. The main result of [13] is theorem 2, which is similar to the above statement except that extra conditions are put on the $G_i$.

**Theorem 5** Let $\Sigma$ denote the group of all permutations of the elements of a $p$-group $S$, and identify $S$ with a subgroup of $\Sigma$ via the Cayley embedding. Every fusion system on $S$ is equal to a subcategory of the Frobenius category $\Phi_p(\Sigma)$ of $\Sigma$.

### 3 Saturated fusion systems

In this section we present the definition of a saturated fusion system, due to Puig [12], although we shall describe an equivalent definition due to Broto, Levi and Oliver [6]. There are two additional axioms as well as the axioms for a fusion system. These axioms necessitate some preliminary definitions.

As usual, if $G$ is a group and $H$ is a subgroup of $G$, we write $C_G(H)$ for the centralizer of $H$ in $G$ and $N_G(H)$ for the normalizer of $H$ in $G$.

Suppose that $\mathcal{F}$ is a fusion system on $S$. Say that $P \leq S$ is fully $\mathcal{F}$-centralized if

$$|C_S(P)| \geq |C_S(P')|$$
for every $P'$ which is isomorphic to $P$ as an object of $\mathcal{F}$. Suppose that $\mathcal{F} = \mathcal{F}_S(G)$ for some discrete group $G$ in which every $p$-subgroup is conjugate to a subgroup of $S$. In this case, if $P$ is fully $\mathcal{F}$-centralized, one sees that $C_S(P)$ is a $p$-subgroup of $C_G(P)$ of maximal order.

Similarly, say that $P$ is fully $\mathcal{F}$-normalized if

$$|N_S(P)| \geq |N_S(P')|$$

for every $P'$ which is isomorphic to $P$ as an object of $\mathcal{F}$. If $\mathcal{F} = \mathcal{F}_S(G)$ as above and $P$ is fully $\mathcal{F}$-normalized, one sees that $N_S(P)$ is a $p$-subgroup of $N_G(P)$ of maximal order.

Now suppose that $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$, and that $P \leq S$ is fully $\mathcal{F}$-normalized. In this case, $N_S(P)$ must be a Sylow $p$-subgroup of the finite group $N_G(P)$. Moreover, $C_G(P) \cap N_S(P) = C_S(P)$ must be a Sylow $p$-subgroup of $C_G(P)$, and $\text{Aut}_S(P) = N_S(P)/C_S(P)$ must be a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(P) = N_G(P)/C_G(P)$. This gives the first of two extra axioms for a saturated fusion system:

3. If $P$ is fully $\mathcal{F}$-normalized, then $P$ is also fully $\mathcal{F}$-centralized, and $\text{Aut}_S(P)$ is a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(P)$.

Next, suppose that $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$ and that $f : P \to Q \leq S$ is an isomorphism in $\mathcal{F}$ such that $Q$ is fully $\mathcal{F}$-centralized. This implies that $C_S(Q)$ is a Sylow $p$-subgroup of $C_G(Q)$. Pick an element $h \in G$ so that $f$ is equal to conjugation by $h$, i.e., so that $f(u) = c_h(u) = h^{-1}uh$ for all $u \in P$. The image $c_h(C_S(P))$ is a $p$-subgroup of $C_G(c_h(P)) = C_G(Q)$, and so there exists $h' \in C_G(Q)$ so that $c_{h'} \circ c_h(C_S(P)) \leq C_S(Q)$. Since $c_{h'}$ acts as the identity on $Q$, if we define $k = hh'$, we see that $c_k$ extends $f$ and $c_k(C_S(P)) \leq C_S(Q)$.

The map $c_k$ clearly extends to a map from $N_f = N_S(P) \cap c_k^{-1}(N_S(Q))$ to $N_S(Q)$. But since $C_S(P)$ is a subgroup of $c_k^{-1}(N_S(Q))$, we may rewrite this as

$$N_f = \{ g \in N_S(P) : c_k \circ c_g \circ c_k^{-1} \in \text{Aut}_S(Q) \}$$

$$= \{ g \in N_S(P) : f \circ c_g \circ f^{-1} \in \text{Aut}_S(Q) \},$$

which does not depend on choice of $k$. This leads to the second extra axiom:

4. If $f : P \to Q$ is an isomorphism in $\mathcal{F}$ and $Q$ is fully $\mathcal{F}$-centralized, then $f$ extends in $\mathcal{F}$ to a map from $N_f$ to $N_S(Q)$, where $N_f = \{ g \in N_S(P) : f \circ c_g \circ f^{-1} \in \text{Aut}_S(Q) \}$. 

5
**Remark 6** It has been shown [8] that the axioms for a saturated fusion system can be simplified to:

3’. $\text{Aut}_S(S)$ is a Sylow $p$-subgroup of $\text{Aut}_F(S)$.

4’. If $f : P \rightarrow Q$ is an isomorphism in $\mathcal{F}$ and $Q$ is fully $\mathcal{F}$-normalized, then $f$ extends in $\mathcal{F}$ to a map from $N_f$ to $N_S(Q)$, where $N_f$ is as defined in axiom 4.

**Remark 7** In the case when $S$ is abelian, axioms 3 and 4 simplify. In this case, every subgroup of $S$ is fully $\mathcal{F}$-centralized and fully $\mathcal{F}$-normalized for any fusion system $\mathcal{F}$, and for any $f \in \mathcal{F}$, $N_f = S$. Hence a fusion system $\mathcal{F}$ on an abelian $p$-subgroup $S$ is saturated if and only if $\text{Aut}_F(S)$ is a $p'$-group and every morphism $f : P \rightarrow S$ in $\mathcal{F}$ extends to an automorphism of $S$.

**Remark 8** As mentioned in the introduction, there are saturated fusion systems which are not realised by any finite group. One source of saturated fusion systems is the fusion systems associated to $p$-blocks of finite groups [2, 11]. The question of whether every such fusion system can be realised by a finite group is a long-standing open problem.

### 4 Examples

Let $E$ be an elementary abelian $p$-group of rank at least three, i.e., a direct product of at least three copies of the cyclic group of order $p$. Let $A = \text{Aut}(E)$ be the full group of automorphisms of $E$, which is of course isomorphic to a general linear group over the field of $p$ elements. Let $B$ be a subgroup of $A$ of order a power of $p$, and let $C$ be a non-trivial subgroup of $A$ of order coprime to $p$. Note that $A$ is generated by its subgroups of order coprime to $p$.

Each of $A$, $B$ and $C$ may be viewed as a collection of morphisms in the fusion system $\mathcal{F}_E^{\max}$. For $X = A$, $B$ or $C$, let $\mathcal{F}_E(X)$ denote the fusion system generated by all the morphisms in $X$.

**Example 9** The fusion system $\mathcal{F}_E(C)$ is saturated, and is equal to the fusion system $\mathcal{F}_E(G)$, where $G$ is the semi-direct product $G = E \rtimes C$. 

\[\]
Example 10 The fusion system \( F_E(A) \) is not saturated, since in \( F_E(A) \) the automorphism group of the object \( E \) does not have \( E/Z(E) \) as a Sylow \( p \)-subgroup. However, \( F_E(A) \) can be realised by the procedure of Theorem 3. Let \( C_1, \ldots, C_r \) be \( p' \)-subgroups of \( A \) that together generate \( A \). If we put \( G_i = E \rtimes C_i \) with \( f_i \) the identity map of \( E \), then the fusion system generated by all of the \( (f_i)_*(F_E(G_i)) \) is equal to \( F_E(A) \).

Example 11 The fusion system \( F_E(B) \) cannot be realised by the procedure used in Theorem 3. For suppose that \( G_1, \ldots, G_r \) are finite groups with Sylow \( p \)-subgroups \( E_1, \ldots, E_r \), each of which is isomorphic to a subgroup of \( E \), and suppose that \( F_E(B) \) is generated by the fusion systems \( (f_i)_*F_E(G_i) \). Those \( G_i \) for which \( f_i : E_i \to E \) is not an isomorphism do not contribute any morphisms to \( \operatorname{Aut}_F(E) \). If \( f_i : E_i \to E \) is an isomorphism, then either \( \operatorname{Aut}_{G_i}(E_i) \) contains non-identity elements of \( p' \) order, implying that \( F \neq F_E(B) \), or \( E_i \) is central in \( G_i \) and \( G_i \) does not contribute any morphisms to \( \operatorname{Aut}_F(E) \).

Next we consider some examples of fusion systems \( F \) on an abelian \( p \)-group \( E \) in which \( \operatorname{Aut}_F(E) \) is a \( p' \)-group, but for which some isomorphisms between proper subgroups of \( E \) do not extend to elements of \( \operatorname{Aut}_F(E) \).

Example 12 Let \( F \) and \( F' \) be distinct order \( p \) subgroups of \( E \), and let \( \phi : F \to F' \) be an isomorphism. Let \( F_E(\phi) \) be the fusion system generated by \( \phi \). Every morphism in \( F_E(\phi) \) is equal to either an inclusion map or the composite of either \( \phi \) or \( \phi^{-1} \) with an inclusion map. In particular, in \( F_E(\phi) \), the automorphism group of each object \( E' \leq E \) is trivial. The fusion system \( F_E(\phi) \) cannot be realised by the procedure of Theorem 3 as will be explained below.

In view of Remark 7, \( F_E(\phi) \) is not a saturated fusion system, since the morphism \( \phi : F \to F' \) does not extend to an automorphism in \( F_E(\phi) \) of the group \( E \).

Now suppose that \( F \) is a fusion system on \( E \) generated by the images \( (f_i)_*F_{E_i}(G_i) \) of some fusion systems for finite groups. If \( \phi : F \to F' \) is a morphism in \( F \), then there exists \( i \) so that \( F, F' \leq f_i(E_i) \) and \( \phi \in (f_i)_*F_{E_i}(G_i) \). But then (by the same argument as used above) there is a morphism \( \tilde{\phi} : f_i(E_i) \to f_i(E_i) \) extending \( \phi : F \to F' \). Thus \( F \) cannot be equal to the fusion system \( F_E(\phi) \), since this fusion system contains no such \( \phi \).
Example 13 Let $F$ be a proper subgroup of $E$, and suppose that $D$ is a non-trivial $p'$-group of automorphisms of $F$. Let $F \rtimes D$ denote the semi-direct product of $F$ and $D$, let $G$ be the free product with amalgamation $G = E * _F (F \rtimes D)$, and let $\mathcal{F}$ be the fusion system $\mathcal{F}_E (G)$. From this definition one sees that $\mathcal{F}$ can be obtained by the procedure of Theorem 3. On the other hand, since $\text{Aut}_F (E)$ is trivial, one sees that the non-trivial automorphisms of $F$ do not extend to automorphisms of $E$, and hence $\mathcal{F}$ is not saturated.

As remarked earlier, Robinson does not consider all fusion systems that can be built by the procedure of Theorem 3 but only those fusion systems that he calls Alperin fusion systems [13]. With the notation of Theorem 3 a fusion system is Alperin if the following conditions hold:

1. Inside each $G_i$ there is a subgroup $E_i$ which is the largest normal $p$-subgroup of $G_i$, and the centralizer of this subgroup is as small as possible, in the sense that $C_{G_i} (E_i) = Z(E_i)$;

2. The quotient $G_i / E_i$ is isomorphic to $\text{Out}_F (E_i) := \text{Aut}_F (E_i) / \text{Aut}_{E_i} (E_i)$;

3. Inside $S$, the image of the subgroup $S'_i$ (the Sylow $p$-subgroup of $G_i$ which is to be identified with a subgroup of $S$) is equal to the normalizer of the image of $E_i$, i.e., $f_i (S'_i) = N_S (f_i (E_i))$.

In terms of this definition, the content of Alperin’s fusion theorem with some later embellishments [11, 7] is that the fusion system for any finite group is Alperin. Robinson remarks [13] that work of Broto, Castellana, Grodal, Levi and Oliver implies that every saturated fusion system is Alperin [5]. It is easy to see that a fusion system on an abelian $p$-group is Alperin if and only if it is saturated. We finish this section by giving an example of a fusion system that is Alperin but not saturated.

Example 14 Let $p$ be an odd prime, let $A = (C_p)^3$, and let $B$ be a subgroup of $\text{Aut}(A)$ of order $p$ such that $A$ is indecomposable as a $B$-module. (Equivalently, the action of a generator for $B$ on $A$ should be a single Jordan block.) Let $S$ be the semi-direct product $S = A \rtimes B$. The centre $Z$ of $S$ has order $p$. Let $E = Z \times B \leq S$, a subgroup isomorphic to $C_p \times C_p$. It is readily seen that $C_S (E) = E$ and that $P = N_S (E)$ is isomorphic to a semi-direct product $(C_p)^2 \rtimes C_p$, the unique non-abelian group of order $p^3$ and exponent $p$. Let $G_1$ be the semi-direct product $G_1 = E \rtimes \text{Aut}(E)$. Since the Sylow $p$-subgroups
of $\text{Aut}(E)$ are cyclic of order $p$, there is an isomorphism between $P$ and a Sylow $p$-subgroup of $G_1$ that extends the inclusion of $E$.

By construction, the fusion system $\mathcal{F}$ for the free product with amalgamation $S \ast_p G_1$ is Alperin in the sense of Robinson [13], but this fusion system is not saturated. For example, there are non-identity self-maps of $Z$ inside $\mathcal{F}$, and if $\mathcal{F}$ were saturated, any self-map of $Z$ inside $\mathcal{F}$ would extend to a self-map of $S$. But in $\mathcal{F}$, $S$ has only inner automorphisms, and these restrict to $Z$ as the identity.

5 Proofs

Proof. (of Theorem 5.) As in the statement, let $\Sigma$ be the group of all permutations of $S$, and identify $S$ with a subgroup of $\Sigma$. Let $P$ and $Q$ be subgroups of $S \leq \Sigma$, and let $\phi : P \to Q$ be any injective group homomorphism. It suffices to show that there is some $\sigma \in S$ such that for all $u \in P$, $\sigma^{-1}u\sigma = \phi(u)$. Let $\Omega$ denote the group $S$ viewed as a set with a left $S$-action. There are two ways to view $\Omega$ as a set with a left $P$-action, via $P \leq S$ and via $\phi : P \to Q \leq S$. Denote these two $P$-sets by $\Omega$ and $\phi\Omega$ respectively. Each of $\Omega$ and $\phi\Omega$ is isomorphic as a $P$-set to the disjoint union of $|S : P|$ copies of $P$. In particular, there is an isomorphism of $P$-sets $\phi : \phi\Omega \to \Omega$. Viewing $\phi$ as an element of $\Sigma$, one has that $\sigma\phi(u)\omega = u\sigma\omega$ for all $u \in P$ and $\omega \in \Omega$. Hence $\sigma^{-1}u\sigma = \phi(u)$ for all $u$ as required.

Remark 15 A version of Theorem 5 appeared in [9], although fusion systems were not mentioned there.

Before proving Theorem 2 we give a result concerning extending group homomorphisms, and two corollaries, one of which will be used in the proof.

Lemma 16 Let $S$ and $G$ be as in the statement of Theorem 2, let $j : S \to G$ be the natural map from $S$ to $G$, let $H$ be a group and let $f : S \to H$ be a group homomorphism. There is a group homomorphism $\tilde{f} : G \to H$ with $f = \tilde{f} \circ j$ if and only if for each $i$, the homomorphisms $f : P_i \to H$ and $f \circ \phi_i : P_i \to H$ differ by an inner automorphism of $H$.

Proof. Given a homomorphism $\tilde{f}$ as in the statement, one has that for each $i$ and for each $u \in P_i$, $f\phi_i(u) = h_i^{-1}f(u)h_i$, where $h_i = \tilde{f}(t_i)$. For the converse, suppose that there exists, for each $i$, an element $h_i$ satisfying the equation
\[ f \phi_i(u) = h_i^{-1}f(u)h_i \text{ for all } u \in P_i. \] In this case one may define \( \tilde{f} \) on the generators of \( G \) by \( \tilde{f}(s) = f(s) \) for all \( s \in S \) and \( \tilde{f}(t_i) = h_i. \) \hfill \Box

**Corollary 17** With notation as in the statement of Theorem 2, there is a homomorphism from \( G \) to \( \Sigma \), the group of all permutations of the set \( S \), extending the Cayley representation of \( S \).

**Proof.** The argument used in the proof of Theorem 5 shows that the conditions of Lemma 16 hold. \hfill \Box

**Remark 18** Corollary 17 gives an alternative way to prove Corollary 24, at least in the special case of a rose-shaped graph.

**Corollary 19** With notation as in the statement of Theorem 2, a complex representation of \( S \) with character \( \chi \) extends to a complex representation of \( G \) if and only if for each \( i \) and for each \( u \in P_i \), \( \chi(u) = \chi(\phi_i(u)) \).

**Remark 20** Of course, a representation of \( S \) will extend to \( G \) in many different ways if it extends at all.

**Proof.** (of Theorem 2) As in Appendix 6.2 one sees that the group \( G \) presented in the statement is the fundamental group of a graph of groups with one vertex group, \( S \), and one edge group \( P_i \) for each \( \phi_i \), \( 1 \leq i \leq r \). From Corollary 24 it follows that \( S \) is a subgroup of \( G \). From Corollary 28 it follows that any finite subgroup of \( G \), and in particular any \( p \)-subgroup of \( G \), is conjugate to a subgroup of \( S \). By Theorem 26 there is a cellular action of \( G \) on a tree \( T \), with one orbit of vertices and \( r \) orbits of edges. By suitable choice of orbit representatives, we may choose a vertex \( v \) whose stabilizer is \( S \), and edges \( e_1, \ldots, e_r \) so that the stabilizer of \( e_i \) is \( P_i \), and so that the initial vertex of \( e_i \) is \( v \) while the final vertex is \( t_i \cdot v \).

Since every \( p \)-subgroup of \( G \) is conjugate to a subgroup of \( S \), there is a fusion system \( \mathcal{F}_S(G) \) associated to \( G \). By construction \( \mathcal{F}_S(G) \) contains each \( \phi_i \), which corresponds to conjugation by \( t_i \).

Conversely, suppose that \( g \in G \) has the property that \( g^{-1}Pg \leq Q \) for some subgroups \( P, Q \) of \( S \). It suffices to show that conjugation by \( g \), as a map from \( P \) to \( Q \), is equal to a composite of (restrictions of) the maps \( \phi_j \) and their inverses with conjugation maps by elements of \( S \).

Consider the action of \( P \) on the tree \( T \). By hypothesis, the action of \( P \) fixes both the vertex \( v \) and the vertex \( g \cdot v \). Since \( T \) is a tree, \( P \) must fix all
the vertices and edges on the unique shortest path from \( v \) to \( g \cdot v \). Let this path have length \( n \). Define \( g_0 = 1_G, g_n = g \), and for \( 1 \leq i \leq n - 1 \), choose \( g_i \in G \) so that \( g_0 \cdot v, g_1 \cdot v, \ldots, g_n \cdot v \) is the shortest path in \( T \) from \( v \) to \( g \cdot v \). For each \( i \), \( P \) is contained in the stabilizer of the vertex \( g_i \cdot v \), and so \( P \leq g_i S g_i^{-1} \), or equivalently \( g_i^{-1} P g_i \leq S \).

The edge joining \( g_i \cdot v \) and \( g_{i+1} \cdot v \) is an edge of the form \( g_i \cdot e_j \) or \( g_{i+1} \cdot e_j \) for some \( j \) depending on \( i \). Consider the two cases separately, first supposing that the edge is of the form \( g_i \cdot e_j \). In this case it follows that \( P \leq g_i P g_i^{-1} \), since \( P \) stabilizes the edge \( g_i \cdot e_j \). Also one sees that \( g_{i+1} \cdot v = g_i t_j \cdot v \), and hence \( g_{i+1}^{-1} g_i t_j \in S \). Hence conjugation by \( g_{i+1}^{-1} g_i \), viewed as a map from \( g_i^{-1} P g_i \) to \( g_{i+1}^{-1} P g_{i+1} \) is equal to the composite of the map \( \phi_j \) (restricted to \( g_i^{-1} P g_i = P_i \)) followed by conjugation by an element of \( S \).

The other case is similar. Here it follows that \( P \leq g_{i+1} P_{i+1} g_{i+1}^{-1} \), and one has that \( g_i \cdot v = g_{i+1} t_j \cdot v \), from which \( g_i^{-1} g_{i+1} t_j = s \in S \). In this case conjugation by \( g_{i+1}^{-1} g_i \), as a map from \( g_i^{-1} P g_i \) to \( g_{i+1}^{-1} P g_{i+1} \), is equal to the composite map given by conjugation by \( s \) followed by the map \( \phi_j^{-1} \) (restricted to \( s^{-1} g_i^{-1} P g_i s \leq \phi_j(P_{i+1}) \)).

Thus conjugation by \( g = g_n \) as a map from \( P \) to \( Q \) can be expressed as a composite of maps inside the fusion system generated by the \( \phi_i \), and so \( \mathcal{F}_S(G) \) is equal to this fusion system.

It remains to show that the group \( G \) contains a free normal subgroup of index at most \( |S|! \). Let \( \Sigma \) denote the symmetric group on the set \( S \). By Corollary \( \text{17} \) there is a homomorphism \( G \to \Sigma \) which extends the natural injection \( S \to \Sigma \). By Corollary \( \text{29} \) the kernel of this homomorphism is a free normal subgroup of \( G \), and its index is a factor of \( |\Sigma| = |S|! \).

\( \square \)

\textit{Proof.} (of Theorem \textit{3}—sketch.) In this case, the group \( G \) is the fundamental group of a star-shaped graph of groups, with one central vertex labelled \( S \) and \( r \) outer vertices labelled \( G_1, \ldots, G_r \). The edge from \( G_i \) to \( S \) is labelled by the group \( S'_i \). By Theorem \textit{20} there is a cellular action of \( G \) on a tree \( T \), with \( r + 1 \) orbit of vertices and \( r \) orbits of edges. We may choose orbit representatives \( v_0, v_1, \ldots, v_r \) of vertices and \( e_1, \ldots, e_r \) of edges so that the stabilizer of \( v_0 \) is \( S \), and for \( 1 \leq i \leq r \), the stabilizer of \( v_i \) is \( G_i \) (resp. of \( e_i \) is \( S'_i \)). Moreover, we may assume that \( e_i \) has initial vertex \( v_i \) and terminal vertex \( v_0 \).

In this case, one sees that any finite subgroup of \( G \) is conjugate to either a subgroup of \( S \) or to a subgroup of \( G_i \) for some \( i \). Since \( S'_i \) is a Sylow \( p \)-subgroup of \( G_i \), it follows that any \( p \)-subgroup of \( G \) is conjugate to a subgroup
of $S$ as required.

As in the previous proof, it is clear that the fusion system $\mathcal{F}_S(G)$ contains the image of each $\mathcal{F}_{S_i}(G_i)$, but an argument is needed to show that these images generate $\mathcal{F}_S(G)$. Given $g \in G$ and $P, Q \leq S$ so that $g^{-1}Pg \leq Q$, one argues that the action of $P$ fixes the vertices $v_0$ and $g \cdot v_0$ in the tree $T$, and hence fixes the shortest path (necessarily of even length, say $2n$) that joins these vertices.

Let $g_0 = 1_G, g_{2n} = g$, and pick group elements so that the vertices on the shortest path from $v_0$ to $g \cdot v_0$ are:

$$g_0 \cdot v_0, \; g_1 \cdot v_{j(1)}, \; g_2 \cdot v_0, \; g_3 \cdot v_{j(2)}, \; \ldots, \; g_{2n-1} \cdot v_{j(n)}, \; g_{2n} \cdot v_0$$

for some function $j : \{1, \ldots, n\} \to \{1, \ldots, r\}$. If $i$ is even, then $g_i^{-1}Pg_i \leq S$, and if $i$ is odd then $g_i^{-1}Pg_i \leq G_{j((i+1)/2)}$. Since $P$ stabilizes each edge, one sees that $P \leq g_i^{-1}S_kg_i$, where $S_k$ denotes the image of $S'_k$ inside $S$, and $k = j((i + 1)/2)$ if $i$ is odd and $k = j(i/2)$ if $i$ is even. In particular, each $g_i^{-1}Pg_i$ is a subgroup of $S$.

One may show that in the case when $i$ is odd, $g_i^{-1}g_{i+1} \in G_{j((i+1)/2)}$ and that in the case when $i$ is even, $g_i^{-1}g_{i+1} \in S$. Thus the map from $g_i^{-1}Pg_i$ to $g_{i+1}^{-1}Pg_{i+1}$ given by conjugation by $g_i^{-1}g_{i+1}$ is a map inside the fusion system generated by the images of the $\mathcal{F}_{S'_i}(G_i)$, and conjugation by $g = g_{2n}$ as a map from $P$ to $Q \leq S$ is expressed as a composite of maps of the required form.

Finally, if $\Omega$ is a finite set so that $|\Omega|$ is divisible by $|S|$ and by each $|G_i|$, one may define free actions of $S$ and each $G_i$ on $\Omega$ which give rise to the same (free) action of $S_i = f_i(S'_i)$. This gives rise to a group homomorphism from $G$ to $\Sigma$, the symmetric group on $\Omega$, whose kernel is free by Corollary 29. \hfill $\square$

### 6 Appendix: graphs of groups

In this section we give proofs of those results about graphs of groups that we use. Our treatment of graphs of groups follows that of Scott and Wall [14].

For the purposes of this paper, a graph $\Gamma$ consists of two sets, the vertices $V$ and the directed edges $E$, together with two functions $\iota, \tau : E \to V$. For $e \in E$, $\iota(e)$ is called the initial vertex of $e$ and $\tau(e)$ is the terminal vertex of $e$. Multiple edges and loops are allowed in this definition. $\Gamma$ is connected if the only equivalence relation on $V$ that contains all pairs $(\iota(e), \tau(e))$ is the relation with just one class.
A graph $\Gamma$ may be viewed as a category, with objects the disjoint union of $V$ and $E$ and two non-identity morphisms with domain $e$ for each $e \in E$, one morphism $e \to \imath(e)$ and one morphism $e \to \tau(e)$.

A graph of groups is a connected graph $\Gamma$ together with groups $G_v, G_e$ for each vertex and edge, and injective group homomorphisms $f_{e,\imath} : G_e \to G_{\imath(e)}$ and $f_{e,\tau} : G_e \to G_{\tau(e)}$ for each edge $e$. If $\Gamma$ is viewed as a category, this is just a functor from $\Gamma$ to the category of groups and injective group homomorphisms. Without loss of generality, one may assume that each map $f_{e,\imath} : G_e \to G_{\imath(e)}$ is the inclusion of a subgroup.

### 6.1 The fundamental group of a graph of groups

For a topologist, and arguably for anybody, the easiest way to define the fundamental group of a graph of groups is via the notion of a graph of spaces.

A graph of spaces is a connected graph $\Gamma$ together with topological spaces $X_v, X_e$ for each vertex and edge, and continuous maps $f_{e,\imath} : X_e \to X_{\imath(e)}$ and $f_{e,\tau} : X_e \to X_{\tau(e)}$. A graph of spaces is just a functor from the category $\Gamma$ to the category of topological spaces and continuous functions. A graph of based spaces is defined similarly: each $X_e$ and $X_v$ is equipped with a base point, and the maps must preserve base points. Let $I$ denote the closed unit interval $[0, 1]$. The total space of a graph of spaces is the space $X$ made from the disjoint union

$$\coprod_{v \in V} X_v \coprod \coprod_{e \in E} X_e \times I$$

by identifying $(x, 0) \in X_e \times I$ with $f_{e,\imath}(x) \in X_{\imath(e)}$ and identifying $(x, 1) \in X_e \times I$ with $f_{e,\tau}(x) \in X_{\tau(e)}$. As an example, consider the graph of spaces in which each $X_e$ and $X_v$ is a single point. For this graph of spaces the total space is the usual topological realization of the graph as a 1-dimensional CW-complex. The reader who is familiar with the homotopy colimit construction will note that if one views a graph of spaces as a functor $X_{(-)}$ on the category $\Gamma$, then the total space $X$ is naturally homeomorphic to the homotopy colimit of the functor $X_{(-)}$, or in symbols, $X = \hocolim_{\Gamma} X_{(-)}$.

Given a graph of groups, one may define a graph of connected based spaces by taking classifying spaces as the spaces $X_e$ and $X_v$:

$$X_e = BG_e = K(G_e, 1) \quad X_v = BG_v = K(G_v, 1).$$

For the continuous map $f_{e,\imath} : X_e \to X_{\imath(e)}$ (resp. $f_{e,\tau} : X_e \to X_{\tau(e)}$) one may take any continuous map that induces the given map $G_e \to G_{\imath(e)}$ (resp.
$G_e \to G_{r(e)}$ on fundamental groups. Define a total space $X$ as the realization of this graph of spaces.

For discrete groups $K$ and $H$, the space $BK$ is unique up to based homotopy, and homotopy class of based maps from $BK$ to $BH$ are in bijective correspondence with group homomorphisms from $K$ to $H$. It follows that the homotopy type of the space $X$ defined above depends only on the graph of groups, rather than on the particular choices of classifying spaces and maps between them. The fundamental group $G$ of the graph of groups can now be defined as the fundamental group of $X$. This describes the fundamental group of the graph of groups up to isomorphism. The inclusion of each $X_v$ in $X$ defines a conjugacy class of homomorphism $G_v \to G$ (which will be shown to be injective, below). For many purposes one wants a more precise description of $G$, together with a single choice of homomorphism $G_v \to G$.

This can be done by choosing a basepoint for the space $X$, and for each $v$, a path in $X$ from the basepoint for $X$ to the basepoint for $X_v \subseteq X$.

6.2 Presentations for graphs of groups

We shall only consider presentations for graphs of groups where the underlying graph is either a ‘rose’ or a ‘star’. By a rose we mean a graph with only one vertex, so that every edge has the same initial and terminal vertices. By a star we mean a connected graph with $n+1$ vertices and $n$ edges, for some $n > 0$, with one central vertex, such that all the edges have this vertex as their terminal vertex and so that every other vertex is the initial vertex of exactly one edge.

Suppose one is given a $p$-group $S$, subgroups $P_i, Q_i \leq S$, and injective group homomorphisms $\phi_i : P_i \to Q_i$ for $1 \leq i \leq r$, as in the statement of Theorem 2. Use this data to make a rose-shaped graph of groups with $r$ edges. Let $S$ be the vertex group, let $P_i$ be the $i$th edge group, with the inclusion map $P_i \leq S$ (resp. the composite $\phi_i : P_i \to Q_i \leq S$) as the $i$th initial (resp. terminal) homomorphism. There is a model for $BP_i$ having just one 0-cell and one 1-cell for each element of $P_i$. Take a model for $BS$ having just one 0-cell and take this 0-cell as the base point. To make a CW-complex of the homotopy type of the total space of the graph of groups, it suffices to add to $BS$ one 1-cell $t_i$ for each $i$ (with both ends at the unique 0-cell), one 2-cell $D_{i,u}$ for $1 \leq i \leq r$ and for each $u \in P_i$, and higher dimensional cells (which will not affect the fundamental group). The attaching map for the 2-cell $D_{i,u}$ spells out the word $u t_i \phi_i(u) t_i^{-1}$, and so the presentation coming from this
Next suppose that one is given a $p$-group $S$, groups $G_i$ for $1 \leq i \leq r$ with Sylow $p$-subgroups $S_i$, and injective group homomorphisms $f_i : S_i \to S$, i.e., the data found in the statement of Theorem 3. In this case, define a star of groups with central vertex group $P$, other vertex groups $G_1, \ldots, G_r$, and edge groups $S_1, \ldots, S_r$. The map of each edge group into its initial vertex group is the inclusion $S_i \to G_i$, and the map of each edge group into its terminal vertex group is $f_i : S_i \to S$. An argument similar to that given in the previous paragraph shows that the fundamental group of this graph of groups has the presentation given in the statement of Theorem 3. Note that here one can make a space homotopy equivalent to the total space of the graph of spaces by starting from the one-point union of $BS$ and the $BG_i$, without adding any extra 1-cells. This is reflected in the fact that the vertex groups generate the fundamental group of the graph of groups.

### 6.3 Properties of graphs of groups

**Proposition 21** Let $G$ be the fundamental group of a graph of groups based on a graph $\Gamma$. Every subgroup $H \leq G$ is itself the fundamental group of a graph of groups, indexed by a graph $\Delta$ equipped with a map $f : \Delta \to \Gamma$ which does not collapse any edges. For each $v$ and $e \in \Delta$, the group $H_v$ (resp. $H_e$) is a subgroup of $G_{f(v)}$ (resp. $G_{f(e)}$).

*Proof.* Use the bijection between connected covering spaces of a connected CW-complex (with a choice of base point) and subgroups of its fundamental group. Let $X$ be the total space of the graph of spaces used in the definition of $G$, so that there is a covering space of $X$ whose fundamental group is $H$. Any connected covering space of $X$ can be expressed as the total space of a graph of spaces indexed by some $\Delta$ as in the statement. This gives an expression for the fundamental group of any connected covering space of $X$ as the fundamental group of a graph of groups as claimed. $\square$

**Theorem 22** Let $X$ be the total space of the graph of spaces used in the definition of the fundamental group $G$ of a graph of groups. The universal covering space of $X$ is contractible, and hence $X$ is homotopy equivalent to $BG$.

*Proof.* We shall build a space $Y$, in such a way that it is clear that $Y$ is contractible, and that $Y$ is a covering space of $X$. 15
For \( v \) a vertex, define the subspace \( X'_v \) of \( X \) by

\[
X'_v = X_v \cup \bigcup_{\iota(e)=v} X_e \times [0,0.5) \cup \bigcup_{\tau(e)=v} X_e \times (0.5,1].
\]

Similarly, define for \( e \) an edge, \( X'_e = X_e \times (0,1) \). The inclusions \( X_v \to X'_v \) and \( X_e \cong X_e \times \{0.5\} \to X'_e \) are homotopy equivalences, and it may be useful to think of \( X'_e \) as a nice open neighbourhood of \( X_v \) in \( X \). Let \( Y_v, Y'_v, Y_e, \) and \( Y'_e \) be the universal covering spaces of \( X_v, X'_v, X_e \) and \( X'_e \) respectively. Each \( Y'_v \) (resp. \( Y'_e \)) is contractible since it is the universal covering space of the classifying space \( BG_v \) (resp. \( BG_e \)).

The definition of the space \( X'_v \) lifts to a description of the space \( Y'_v \). The complement \( Y'_v - Y_v \) is identified with a collection of disjoint copies of \( Y_e \times (0,0.5) \), and \( Y_e \times (0.5,1) \), for different edges \( e \). There are copies of \( Y_e \times (0,0.5) \) if and only if \( \iota(e) = v \). In this case the copies are in bijective correspondence with the cosets of \( f_{e,\iota}(G_e) \) in \( G_v \). Similarly, there are copies of \( Y_e \times (0.5,1) \) for each \( e \) with \( \tau(e) = v \), and these copies are indexed by cosets of \( f_{e,\tau}(G_e) \) in \( G_v \).

By induction, we shall construct a sequence \( Y_0 \subseteq Y_1 \subseteq Y_2 \cdots \) of spaces so that: each \( Y_n \) is contractible; there is a map \( \pi : Y_n \to X \) which is locally a covering map except at some points of \( X \); for any \( x \in X \) and any \( n \geq 0 \), at least one of \( \pi : Y_n \to X \) and \( \pi : Y_{n+1} \to X \) is locally a covering map at \( x \).

Pick a vertex \( v \) of the graph \( \Gamma \), and define \( Y_0 \) to be the space \( Y'_v \). Define a map \( \pi : Y'_v \to X \) as the composite of the map \( Y'_v \to X'_v \) and the inclusion \( X'_v \subseteq X \). As remarked earlier, \( Y'_v - Y_v \) consists of lots of subspaces of the form \( Y_e \times (0,0.5) \) for \( \iota(e) = v \) and lots of subspaces of the form \( Y_e \times (0.5,1) \) for \( \tau(e) = v \). Define \( Y_1 \) by attaching to each such subspace a copy of \( Y'_e \). The map \( \pi : Y_0 \to X \) extends uniquely to \( \pi : Y_1 \to X \) by insisting that on each newly-added \( Y'_e \) subspace, \( \pi \) is equal to the composite map \( Y'_e \to X'_e \subseteq X \). From the construction of \( Y_1 \), it is apparent that \( Y_1 \) is contractible.

In constructing \( Y_1 \), one attached to \( Y_0 \) many spaces of the form \( Y'_e \) by identifying one end of \( Y'_e \) with part of \( Y_0 \). For each copy of \( Y'_e \) that was attached via its initial end, take a copy of \( Y'_{\iota(e)} \), and attach this at the other end of \( Y'_e \). Similarly, for each copy of \( Y'_e \) that was attached to \( Y_0 \) by its terminal end, take a copy of \( Y'_{\tau(e)} \) and attach this at the other end of \( Y'_e \). This defines a space \( Y_2 \), which is clearly contractible, and the map \( \pi \) extends uniquely to a map \( Y_2 \to X \) which agrees with the covering map \( Y'_v \to X'_v \) or \( Y'_e \to X'_e \) on each such subspace.
Now suppose that \( n \) is even, and that \( Y_n \) has been constructed from \( Y_{n-1} \) by attaching subspaces \( Y'_v \) in such a way that the intersection of \( Y_{n-1} \) and each new \( Y'_v \) is equal to one of the components of \( Y'_v - Y_v \). Furthermore, suppose that the map \( \pi \) on each new \( Y'_v \) is equal to the map \( Y'_v \to X' \subseteq X \). Form \( Y_{n+1} \) by attaching a copy of \( Y'_e \) to each other component of \( Y'_v - Y_v \) for each of the copies of \( Y'_v \). Extend the map \( \pi \) as before.

In the case when \( n \) is odd, suppose that \( Y_n \) has been constructed from \( Y_{n-1} \) by attaching subspaces \( Y'_e \) in such a way that the intersection of \( Y_{n-1} \) and each new \( Y'_e \) is equal to one of the two components of \( Y'_e - Y_e \times \{0.5\} \). Suppose also that the map \( \pi \) on each of the new \( Y'_e \) is equal to the map \( Y'_e \to X'_e \subseteq X \). Form \( Y_{n+1} \) by attaching a copy of \( Y'_v \) to the other component of each \( Y'_e - Y_e \times \{0.5\} \), where \( v \) is either \( \iota(e) \) or \( \tau(e) \) depending which component of \( Y'_e - Y_e \times \{0.5\} \) was used. Extend the map \( \pi \) in the same way as before.

By construction, each \( Y_n \) is contractible, and comes equipped with a map \( \pi : Y_n \to X \). If \( n \) is even, this map is locally a covering except possibly at points of \( X \) contained in the union of the images of the \( X_v \). If \( n \) is odd, this map is a covering except possibly at point of \( X \) contained in the union of the images of the \( X_e \times \{0.5\} \). Now define \( Y \) by \( Y = \bigcup_n Y_n \). This space \( Y \) is contractible, and the map \( \pi : Y \to X \) is a covering map, since it is locally a covering map at every point of \( X \). It follows that \( Y \) is the universal covering space of \( X \). Since the universal covering space of \( X \) has been shown to be contractible, it follows that \( X \) is a model for \( BG \).

**Remark 23** The above proof relies on the fact that the edge groups map injectively to the vertex groups.

**Corollary 24** Each vertex group \( G_v \) maps injectively into the fundamental group of a graph of groups.

**Proof.** Given a vertex \( v \), construct the universal covering space as in the proof of Theorem 22 with \( Y_0 = Y'_v \). The group of all deck transformations of \( Y \) is naturally isomorphic to \( G \), the fundamental group of \( X \). Under this isomorphism, the subgroup of those deck transformations that preserve \( Y_0 \) is identified with \( G_v \). \( \square \)

**Remark 25** There is also an algebraic proof that each \( G_v \) embeds in \( G \). In the case when the graph is a rose, this argument is given in Corollary 17.

17
6.4 The action on a tree

Say that an action of a group on a tree is cellular if no element of the group exchanges the ends of any edge.

**Theorem 26** Let \( G \) be the fundamental group of a graph of groups indexed by the graph \( \Gamma \). There is a tree \( T \) with a cellular \( G \)-action and an isomorphism \( f : T/G \cong \Gamma \). If \( \tilde{x} \) is either a vertex or edge of \( T \), and \( x = f(\tilde{x}) \) is the image of \( G \cdot \tilde{x} \) under \( f \), then the stabilizer of \( \tilde{x} \) is conjugate to \( G_x \).

**Proof.** Let \( X \) be the total space of the graph of spaces used in defining \( G \). As remarked earlier, the underlying topological space of the graph \( \Gamma \) can be identified with the total space of the constant graph of 1-point spaces indexed by \( \Gamma \). The unique map from each \( X_v \) and \( X_e \) to a point induces a map from \( X \) to \( \Gamma \).

Now let \( Y \) be the universal covering space of \( X \), as constructed in the proof of Theorem 22. This \( Y \) can be viewed as a graph of spaces over some graph \( \Delta \), with vertex spaces copies of the spaces \( Y_v \) and edges spaces copies of the spaces \( Y_e \). The group \( G \) acts on \( Y \) in such a way that the setwise stabilizer of each copy of \( Y_v \) is a conjugate of \( G_v \), and similarly the setwise stabilizer of each copy of \( Y_e \times (0,1) \) is a conjugate of \( G_e \). Define \( T \) to be the total space of the graph of 1-point spaces over the graph \( \Delta \). By construction, \( T \) is a graph equipped with a \( G \)-action, an equivariant map \( \phi : Y \to T \), and an isomorphism \( f : T/G \to \Gamma \). To check that \( T \) is a tree, let \( T_n = \phi(Y_n) \). As in the proof of Theorem 22, one shows inductively that \( T_n \) is contractible, and \( T = \bigcup_n T_n \). \( \square \)

**Lemma 27** Any cellular action of a finite group \( H \) on a tree \( T \) fixes a vertex.

**Proof.** Take any vertex \( t \in T \), and define a finite subtree \( T' \) to be the union of all the shortest paths between elements of the orbit \( H \cdot t \). If \( T' \) is not itself fixed by \( H \), remove an \( H \)-orbit of ‘leaves’ (i.e., vertices of valency one) from \( T' \), and continue this process until a subtree fixed by \( H \) is all that remains. \( \square \)

**Corollary 28** Every finite subgroup of the fundamental group of a graph of groups is conjugate to a subgroup of a vertex group.
Proof. Let $G$ be the fundamental group of the graph of groups and let $T$ be the corresponding tree. If $H$ is a finite subgroup of $G$ then $H$ fixes some vertex of $T$. The stabilizer of each vertex of $T$ is a conjugate of one of the vertex groups $G_v$. □

Corollary 29 Let $H$ be a subgroup of a graph of groups whose intersection with each conjugate of each vertex group is trivial. Then $H$ is a free group.

Proof. The hypotheses imply that $H$ acts freely on the tree $T$, and so the quotient space $T/H$ is a 1-dimensional classifying space for $H$. □

References

[1] J. L. Alperin, Sylow intersections and fusion, J. Algebra 6 (1967), 222–241.

[2] J. L. Alperin and M. Broué. Local methods in block theory, Ann. of Math 110 (1979), 143–157.

[3] M. Aschbacher and A. Chermak, A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver, preprint (2005).

[4] D. J. Benson, Conway’s group $Co_3$ and the Dickson invariants, Manuscripta Math. 85 (1994), 177–193.

[5] C. Broto, N. Castellana, J. Grodal, R. Levi and R. Oliver, Subgroup families controlling $p$-local finite groups, Proc. London Math. Soc. 91 (2005), 325–354.

[6] C. Broto, R. Levi and B. Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003), no. 4, 779–856.

[7] D. M. Goldschmidt, A conjugation family for finite groups, J. Algebra 16 (1970), 138–142.

[8] R. Kessar and R. Stancu, A reduction theorem for fusion systems of blocks, J. Algebra, to appear.

[9] I. J. Leary, B. Schuster and N. Yagita, On universally stable elements, Quart. J. Math. Oxford 48 (1997) 493–498.
[10] R. Levi and R. Oliver, Construction of 2-local finite groups of a type studied by Solomon and Benson, Geometry and Topology \textbf{6} (2002), 917–990.

[11] M. Linckelmann, Simple fusion systems and the Solomon 2-local groups, J. Algebra, to appear.

[12] Ll. Puig, Full Frobenius systems and their localizing categories, preprint (2001).

[13] G. Robinson, Amalgams, blocks, weights, fusion systems and finite simple groups, preprint (2006).

[14] P. Scott and C. T. C. Wall, Topological methods in group theory. Homological group theory (Proc. Sympos., Durham, 1977), 137–203, London Math. Soc. Lecture Note Ser.,\textbf{36}, Cambridge Univ. Press, Cambridge-New York, 1979.

[15] R. Solomon, Finite groups with Sylow 2-subgroups of type .3, J. Algebra \textbf{28} (1974) 182–198.

Authors’ addresses:

Ian J. Leary: leary@math.ohio-state.edu
Radu Stancu: stancu@math.ohio-state.edu

Department of Mathematics,
The Ohio State University,
231 West 18th Avenue,
Columbus,
Ohio 43210