EIGENVALUES OF THE BILAYER GRAPHENE OPERATOR WITH A COMPLEX VALUED POTENTIAL

FRANCESCO FERRULLI, ARI LAPTEV AND OLEG SAFRONOV

Abstract. We study the spectrum of a system of second order differential operator $D_m$ perturbed by a non-selfadjoint matrix valued potential $V$. We prove that eigenvalues of $D_m + V$ are located near the edges of the spectrum of the unperturbed operator $D_m$.

1. Statement of the main results

Spectral properties of non-selfadjoint operators have been recently a subject of interest of many papers. A particular interest was related to the location of eigenvalues of differential operators in the complex plane $\mathbb{C}$. The corresponding results for Schrödinger operators can be found in [1], [3]-[4] and in [5]. Some other problems were studied in the papers [6]-[10] and [12].

The operator we study is related to the quantum theory of a material consisting of two layers of graphene. Namely, we consider the operator $D = D_m + V$, where

$$D_m = \left( \frac{m}{4 \partial_z^2} + \frac{4 \partial_z^2}{-m} \right), \quad \partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + \frac{i}{\partial x_2} \right), \quad \partial_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - \frac{1}{i} \frac{\partial}{\partial x_2} \right), \quad m \geq 0.$$ 

This operator acts in the Hilbert space $L^2(\mathbb{R}^2; \mathbb{C}^2)$. The domain of $D$ is the Sobolev space $H^2(\mathbb{R}^2; \mathbb{C}^2)$. The potential $V$ is a non-necessary self-adjoint matrix-valued function

$$V(x) = \begin{pmatrix} V_{1,1}(x) & V_{1,2}(x) \\ V_{2,1}(x) & V_{2,2}(x) \end{pmatrix},$$

where the matrix elements are allowed to take complex values. For the matrix $V$ we denote

$$|V(x)| = \sqrt{\sum_{i,j=1,2} |V_{i,j}(x)|^2}.$$

Assuming that $V$ decays at the infinity in some integral sense we would like to answer the question: "Where are the eigenvalues of $D$ located?"

Note that since $D_m^2 = \Delta^2 + m^2$, the spectrum $\sigma(D_m)$ of $D_m$ is the set $(-\infty, m] \cup [m, \infty)$. Our results show that the eigenvalues of $D$ are located near the edges of the absolutely continuous spectrum, i.e. near the points $\pm m$. Since the spectrum of the unperturbed operator has two edges, our results resemble some of the theorems of the paper [2] related to the Dirac operator. However, the main difference between the two papers is that we study a differential operator on a plane, while the article [2] deals with operators on a line.

**Theorem 1.1.** Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator $D$. Let $1 < p < 4/3$. Then

$$\frac{C_p}{\mu^{p-1}} \int_{\mathbb{R}^2} |V(x)|^p dx \left( \sqrt{\frac{k-m}{k+m}} + \sqrt{\frac{k+m}{k-m}} + 1 \right)^p \geq 1, \quad \mu^2 = k^2 - m^2,$$
with $C_p > 0$ independent of $V$, $k$ and $m$. In particular, if $m = 0$, then
\[ |k|^{p-1} \leq 3^p C_p \int_{\mathbb{R}^2} |V(x)|^p dx, \quad 1 < p < 4/3. \]

The next statement tells us about what happens when $p \to 1$.

**Theorem 1.2.** Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator $D$. Let $\mu^2 = k^2 - m^2$. Then
\[
C \left( |\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y|<|2|\mu|} |V(y)| dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \left( 1 + |\ln |x-y|| \right) |V(y)| dy \right) + \\
+ C \int_{\mathbb{R}^2} |V(x)| dx \left( \sqrt{|k-m|/k+m} + \sqrt{|k+m|/k-m} + 1 \right) \geq 1,
\]
where the constant $C > 0$ is independent of $V$, $k$ and $m$.

Note that this statement also holds true for $m = 0$.

**Corollary 1.1.** Let $m = 0$ and let $k \notin \mathbb{R}$ be an eigenvalue of the operator $D$. Then
\[
C \left( |\ln |k|| \sup_{x \in \mathbb{R}^2} \int_{|x-y|<|2|k|} |V(y)| dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \left( 1 + |\ln |x-y|| \right) |V(y)| dy \right) + \\
+ 3C \int_{\mathbb{R}^2} |V(x)| dx \geq 1,
\]
where the constant $C > 0$ is independent of $V$ and $k$.

In particular, we see that if $m = 0$, then for small $V$, the eigenvalues of $D$ are situated in the circle $\{ k \in \mathbb{C} : |k| < r \}$ of radius $r$ which has the following asymptotical behavior
\[ r \asymp \exp \left( - \frac{C}{\int |V| dx} \right), \quad \text{as} \quad \int |V| dx \to 0. \]

The proof of Theorems 1.1 and 1.2 are given in Section 2. In Section 3 we consider a special case where $V = iW^2$, $W = W^*$, In this case we can get a more precise information about location of the complex eigenvalues, see Theorem 3.1. It is interesting to note that if $m = 0$ (no gap in the continuous spectrum), then perturbations by such matrix-functions do not create any complex eigenvalues. Here we have similarities with the result obtained for the one dimensional Dirac operators in [2].

## 2. Proofs of the main results

In order to prove our main results we need the Birman-Schwinger principle formulated below.

**Proposition 2.1.** Let $V = W_2 W_1$, where $W_1$ and $W_2$ are two matrix-valued decaying functions. A point $k \in \mathbb{C} \setminus \sigma(D_m)$ is an eigenvalue of $D$ if and only if $-1$ is an eigenvalue of the operator
\[ X(k) := W_1(D_m - k)^{-1} W_2. \]
In particular, if $k \in \mathbb{C} \setminus \sigma(D_m)$ is an eigenvalue of $D$ then $||X(k)|| \geq 1$. 
The proof of this statement is standard and it is left to the reader as an exercise.

Below we always denote

\[ W = \sqrt{V^*V} \]

and use the Birman-Schwinger principle with \( W_1 = W \) and \( W_2 = VW^{-1/2} \).

**Proof of Theorem 1.1** Since

\[ (D_m - k)^{-1} = (D_m + k)(D_m - k)^{-1}(D_m + k)^{-1} = (D_m + k)(D_m^2 - k^2)^{-1}, \]

it is easy to see that

\[ (D_m - k)^{-1} = (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1}, \]

where

\[ D_0 = \begin{pmatrix} 0 & 4\partial_z^2 \\ 4\partial_z^2 & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

One can also note that the last term in the right hand side of (1) can be rewritten in the form

\[ (D_0 - \mu)^{-1} = (D_0 + \mu)(\Delta^2 - \mu^2)^{-1}. \]

The operator \((\Delta^2 - \mu^2)^{-1}\) is an integral operator with the kernel

\[ g_k(x, y) = \frac{i}{8\mu} \left( H(\sqrt{\mu r}) - H(i\sqrt{\mu r}) \right), \]

where \( H(z) = H_0^{(1)}(z) \) is the Hankel function of first kind and \( r = |x - y| \). It is a simple consequence of the fact that

\[ (\Delta^2 - \mu^2)^{-1} = \frac{1}{2\mu} \left( (-\Delta - \mu)^{-1} - (-\Delta + \mu)^{-1} \right). \]

The kernel of \((-\Delta - \mu)^{-1}\) is \(4^{-1}iH(\sqrt{\mu r})\). Another useful representation of \( g_k(x, y) \) follows from the fact that the kernel of \((-\Delta - \mu)^{-1}\) equals (see [11])

\[ (2\pi)^{-1}K_0(-i\sqrt{\mu}|x - y|), \]

where

\[ K_0(z) = \frac{e^{-z}}{\Gamma(1/2)} \sqrt{\frac{\pi}{2z}} \int_0^\infty e^{-t}t^{-1/2}(1 + \frac{t}{2z})^{-1/2}dt, \quad |\arg z| < \pi. \]

Let us define

\[ G(z) = H(z) - H(iz). \]

We need to know the behaviour of the function \( G \) only in the region \( 0 < \arg z < \pi/2 \), where we have

\[ |G(z)| + |G'(z)| + |G''(z)| \leq \frac{C}{\sqrt{|z|}}, \quad \text{if} \quad |z| > 1/2. \]

The behaviour of the function \( G \) near zero is determined by the expansion of the Hankel function in the neighbourhood of \( z = 0 \). It turns out that

\[ |G(z)| \leq C, \quad |G'(z)| \leq C_1|z|\ln|z|^{-1}, \quad |G''(z)| \leq C_1\ln|z|^{-1}, \quad \text{if} \quad |z| < 1/2. \]

Let \( \rho_\mu(|x - y|) \) be the kernel of the integral operator \((D_0 - \mu)^{-1}\)

\[ \rho_\mu(|x - y|) = \frac{i}{8\mu} \left( \mu G(\sqrt{\mu}|x - y|) \partial_z^2 G(\sqrt{\mu}|x - y|) \right). \]
Therefore

\[ |\rho_\mu(|x - y|)| = \frac{1}{8|\mu|} \sqrt{2|\mu|^2|G(\sqrt{|\mu|}|x - y|)|^2 + |\partial^2 G(\sqrt{|\mu|}|x - y|)|^2 + |\partial^2 G(\sqrt{|\mu|}|x - y|)|^2}. \]

As a consequence, if we denote by \( \rho_\theta(|x - y|) \) the kernel of the operator \( (D_0 - e^{i\theta})^{-1} \) then

\[ |\rho_\theta(r)| \leq C \ln r^{-1}, \quad \text{if} \quad r < 1/2, \tag{3} \]

and

\[ |\rho_\theta(r)| \leq Cr^{-1/2}, \quad \text{if} \quad r > 1/2. \tag{4} \]

In order to prove the latter relations, one has to differentiate the integral kernel of \( (\Delta^2 - \mu^2)^{-1} \), using the formulas

\[ \frac{\partial}{\partial z} = \frac{1}{2} z, \quad \frac{\partial^2}{\partial z^2} = -\frac{1}{4} z^2 \]

and

\[ \frac{\partial}{\partial z} = \frac{1}{2} z, \quad \frac{\partial^2}{\partial z^2} = -\frac{1}{4} z^2. \]

Since the integral kernel of \( (\Delta^2 - \mu^2)^{-1} \) is \( \frac{i}{8\mu} G(\sqrt{\mu}r) \), we obtain from (2) that

\[ 8|\rho_\theta(r)| \leq \left( \left| \frac{\partial^2 G(e^{i\theta}/2r)}{\partial z^2} \right|^2 + \left| \frac{\partial^2 G(e^{i\theta}/2r)}{\partial z^2} \right|^2 + 2|G(e^{i\theta}/r)|^2 \right)^{1/2} \leq C(r^{-1}|G'(e^{i\theta}/2r)| + |G''(e^{i\theta}/2r)| + |G(e^{i\theta}/r)|). \]

The positive constants in the inequalities (3) and (4) do not depend on \( \theta \in [0, \pi/2] \). In particular,

\[ M := \sup_{\theta} \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |\rho_\theta(|x - y|)|^q dy < \infty, \quad q > 4. \]

Let us estimate now the norm of the operator \( T = W(D_0 - e^{i\theta})^{-1}W \) with the kernel

\[ \tau(x, y) = W(x)\rho_\theta(|x - y|)W(y). \]

For that purpose, we estimate the sesquilinear form of this operator:

\[ (Tu, v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{v}(x)W(x)\rho_\theta(|x - y|)W(y)u(y) \, dx \, dy. \]

Obviously,

\[ |(Tu, v)|^2 = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{v}(x)W(x)\rho_\theta(|x - y|)W(y)u(y) \, dx \, dy \right|^2 \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v(x)|^2|\rho_\theta(|x - y|)||W(y)||^2 \, dx \, dy \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W(x)|^2|\rho_\theta(|x - y|)||u(y)||^2 \, dx \, dy \leq \left( \sup_x \int_{\mathbb{R}^2} |\rho_\theta(|x - y|)||W(y)||^2 \, dy \right)^2 \|u\|^2 \|v\|^2 \leq \left( \int_{\mathbb{R}^2} |\rho_\theta(|x - y|)|^q \, dy \right)^{2/q} \|V\|^2_p \|u\|^2 \|v\|^2, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad q > 4. \]

Therefore,

\[ \|T\| \leq C\|V\|_p, \quad 1 < p < 4/3. \]
We are now able to estimate the norm of the operator $T_k = W(D_0 - k)^{-1}W$ for $k \notin \sigma(D_0)$. Indeed,

$$|(T_k u, v)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x)W(x)\rho_\theta(\sqrt{|k|}|x - y|)W(y)u(y) \, dx \, dy \right| =$$

$$\frac{1}{|k|^2} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x/\sqrt{|k|})W(x/\sqrt{|k|})\rho_\theta(|x - y|)W(y/\sqrt{|k|})u(y/\sqrt{|k|}) \, dx \, dy \right| \leq$$

$$\frac{C}{|k|^2} \| V(\cdot/\sqrt{|k|}) \|_p \| u(\cdot/\sqrt{|k|}) \|_p \| v(\cdot/\sqrt{|k|}) \|_p = \frac{C\| V \|_p}{|k|^{(p-1)/p}} \| u \| \| v \|.$$

Consequently,

$$\| T_k \| \leq \frac{C\| V \|_p}{|k|^{(p-1)/p}}.$$

Observe now that the kernel of the operator $(\Delta^2 - \mu^2)^{-1}$ is the function $iG(\sqrt{\mu}|x - y|)/(8\mu)$. The function $G(\sqrt{\mu}|x - y|)$ has the same properties as $\rho_\theta(\sqrt{\mu}|x - y|)$. Moreover it is bounded. Therefore, by mimicking the above arguments, one proves that

$$\| W(\Delta^2 - \mu^2)^{-1}W \| \leq \frac{C\| V \|_p}{|\mu|^{(2p-1)/p}}, \quad 1 \leq p < 4/3. \quad (5)$$

This leads to the estimate

$$\| W(D_m - k)^{-1}W \|_p \leq \frac{C \int_{\mathbb{R}^2} |V(x)|^p \, dx}{|\mu|^{(p-1)/p}} \left( \sqrt{\frac{k + m}{k - m}} + \sqrt{\frac{k - m}{k + m}} + 1 \right)^p, \quad \mu^2 = k^2 - m^2.$$

Now the statement of our theorem follows from the fact that if $k$ is an eigenvalue of $D = D_m + V$, then $\| W(D_m - k)^{-1}W \| \geq 1$. The proof is complete.

In the picture below we describe the areas of possible location of complex eigenvalues depending on the value of $C \int_{\mathbb{R}^2} |V(x)|^p \, dx$, where $m = 1$ and $p = 1.2$. 
Proof of Theorem 1.2. As before we use the representation

\[(D_m - k)^{-1} = (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1}.\]

The operator \((\Delta^2 - \mu^2)^{-1}\) is an integral operator with the kernel

\[g_k(x, y) = \frac{i}{8\mu} \left( H(\sqrt{\mu r}) - H(i\sqrt{\mu r}) \right),\]

where \(H(z) = H_0^{(1)}(z)\) is the Hankel function of first kind. Again we denote

\[G(z) = H(z) - H(iz).\]

We need to know the behavior of the function \(G\) only in the region \(0 < \arg z < \pi/2\), where this function is bounded. The boundedness of \(G\) implies the estimate \((5)\) with \(p = 1\).

It remains to estimate the norm of the operator \(T_\mu = W(D_0 - \mu)^{-1}W\) for \(\text{Im } \mu > 0\). We already know that if \(\mu = |\mu|e^{i\theta}\) the operator \((D_0 - \mu)^{-1}\) is an integral operator with the kernel \(\rho_\theta(\sqrt{|\mu|}|x - y|)\), where \(\rho_\theta\) is a function having the properties

\[(6)\]

\[|\rho_\theta(r)| \leq C \ln r^{-1}, \quad \text{if} \quad r < 1/2,\]

and

\[(7)\]

\[|\rho_\theta(r)| \leq Cr^{-1/2}, \quad \text{if} \quad r > 1/2.\]

The positive constants in these inequalities do not depend on \(\theta \in [0, \pi]\). As before, we estimate the sesquilinear form of this operator:

\[
(T_\mu u, v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x)W(x)\rho_\theta(\sqrt{|\mu|}|x - y|)W(y)u(y) \, dx \, dy.
\]
Obviously,
\[ |(T_\mu u, v)|^2 = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x)W(x)\rho_\theta(\sqrt{\mu}|x-y|)|W(y)|u(y)\,dxdy \right|^2 \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v(x)|^2|\rho_\theta(\sqrt{\mu}|x-y|)||W(y)||^2\,dxdy \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W(x)|^2|\rho_\theta(\sqrt{\mu}|x-y|)||u(y)||^2\,dxdy \leq \left( \sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{\mu}|x-y|)||V(y)||\,dy \right)^2 ||u||^2 ||v||^2.
\]

Therefore,
\[ ||T_\mu|| \leq \sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{\mu}|x-y|)||V(y)||\,dy. \]

The bounds (6) and (7) imply
\[ \sup_x \int_{\mathbb{R}^2} \left| \rho_\theta(\sqrt{\mu}|x-y|)||V(y)||\,dy \right| \leq C \left( |\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y|<(2|\mu|)^{-1}} |V(y)||\,dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \left( 1 + |\ln |x-y|| \right) |V(y)||\,dy \right), \]

which leads to
\[ ||T_\mu|| \leq C \left( |\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y|<(2|\mu|)^{-1}} |V(y)||\,dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \left( 1 + |\ln |x-y|| \right) |V(y)||\,dy \right). \]

Since
\[ ||W(D_m - k)^{-1}W|| \leq ||W(m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1}W|| + ||T_\mu|| \]
and since (3) holds with \( p = 1 \), we obtain
\[ ||W(D_m - k)^{-1}W|| \leq C \left( |\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y|<(2|\mu|)^{-1}} |V(y)||\,dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \left( 1 + |\ln |x-y|| \right) |V(y)||\,dy \right) + C \int_{\mathbb{R}^2} |V(x)|\,dx \left( \sqrt{\frac{|k - m|}{k + m}} + \sqrt{\frac{k + m}{k - m}} + 1 \right). \]

The statement of Theorem 1.2 follows from the fact that if \( k \) is an eigenvalue of \( D = D_m + V \), then \( ||W(D_m - k)^{-1}W|| \geq 1 \).

3. A special case

Consider now a special case, when \( V(x) = iW^2(x) \), where \( W(x) = W^*(x) \) is a matrix valued function. It turns out, that in this case we can get a more precise information about the spectral properties of the operator \( D \).

**Theorem 3.1.** Let \( k \notin \sigma(D_m) \) be an eigenvalue of the operator \( D = D_m + V \), where \( V = iW^2 \). Let \( \mu \) be the number in the upper half-plane defined by \( \mu^2 = k^2 - m^2 \). Then
\[ \left( C \left( \left| \frac{k + m}{\mu} - 1 \right| + \left| \frac{k - m}{\mu} - 1 \right| \right) + 1 \right) \frac{1}{4} \int_{\mathbb{R}^2} \text{tr}|V|\,dx \geq 1, \]

where \( \text{tr} \) denotes the trace of the matrix.
where the constant $C$ is independent of $V$, $m$ and $k$.

Proof. According to the Birman-Schwinger principle, $k$ is an eigenvalue of the operator $D$ if and only if 1 is an eigenvalue of the operator $X = -iW(D_m - k)^{-1}W$. On the other hand, if 1 is an eigenvalue of $X$ then $\|\text{Re }X\| \geq 1$. Since,

$$\text{Re }X = W \text{Im}(D_m - k)^{-1}W,$$

we would like to have the explicit expression for the operator $\text{Im}(D_m - k)^{-1}$. Let us first obtain this representation for the case $m = 0$. Observe that $D_0$ is the operator with the symbol

$$\begin{pmatrix} 0 & - (\xi_1 + i\xi_2)^2 \\ - (\xi_1 + i\xi_2)^2 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\pm|\xi_1 + i\xi_2|^2$. The orthogonal projections $P_1(\xi)$ and $P_2(\xi)$, $\xi = \xi_1 + i\xi_2$, onto the eigenvectors depend only on $\arg(\xi)$. Therefore, the symbol of $D_0$ is

$$|\xi|^2P_1(\xi) - |\xi|^2P_2(\xi),$$

which implies that the integral kernel of $\text{Im}(D_0 - k)^{-1}$ is

$$(2\pi)^{-2}\int_{\mathbb{R}^2} \exp(i\xi(x - y)) \left( \frac{(\text{Im }k) P_1(\xi)}{(|\xi|^2 - \text{Re }k)^2 + (\text{Im }k)^2} + \frac{(\text{Im }k) P_2(\xi)}{(-|\xi|^2 - \text{Re }k)^2 + (\text{Im }k)^2} \right) d\xi.$$

It follows from this representation that the kernel of the operator $\text{Im}(D_0 - k)^{-1}$ is bounded by $1/4$ as using polar coordinates and changing variables $|\xi|^2 = t$ we obtain

$$\|\text{Im}(D_0 - k)^{-1}\| \leq (2\pi)^{-2} \int_{S^1} \int_{-\infty}^{\infty} \frac{\text{Im }k}{t^2 + (\text{Im }k)^2} dt = \frac{1}{4}, \quad \text{Im }k > 0.$$

Consequently,

$$\|W\text{Im}(D_0 - k)^{-1}W\| \leq \text{tr} (W\text{Im}(D_0 - k)^{-1}W) \leq \frac{1}{4} \int_{\mathbb{R}^2} \text{tr } W^2(x) dx.$$

If $m > 0$, then we have

$$\|\text{Re }X\| \leq \left\| \frac{1}{2\mu} W(m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1}W \right\| + \|W\text{Im}(D_0 - \mu)^{-1}W\|$$

and that

$$\left\| \frac{m\gamma_0 + k - \mu}{2\mu} \right\| \leq \frac{1}{2} \left( \left| \frac{k+m}{\mu} - 1 \right| + \left| \frac{k-m}{\mu} - 1 \right| \right).$$

It remains to note that according to (5) with $p = 1$,

$$\|W(\Delta^2 - \mu^2)^{-1}W\| \leq C \int_{\mathbb{R}^2} \text{tr } |V| dx.$$

The proof is completed. \qed

The next result says that the spectrum of the operator $D_0$ is stable with respect to small perturbations of the form $V = iW^2$. 
Corollary 3.1. Let \( m = 0 \) and let \( V = iW^2 \) with \( W^* = W \). Assume that
\[
\frac{1}{4} \int_{\mathbb{R}^2} \text{tr}|V|dx < 1.
\]
Then the operator \( D = D_0 + V \) does not have eigenvalues outside of the real line \( \mathbb{R} \), i.e. the spectrum of \( D \) is real.

References

[1] A. A. Abramov, A. Aslanyan, E. B. Davies, Bounds on complex eigenvalues and resonances. J. Phys. A 34 (2001), 57–72.
[2] J-C. Cuenin, A. Laptev and Ch. Tretter, Eigenvalue estimates for non-selfadjoint Dirac operators on the real line, Annales Henri Poincare 15 (2014), 707-736
[3] E. B. Davies, Non-self-adjoint differential operators. Bull. London Math. Soc. 34 (2002), no. 5, 513–532.
[4] E. B. Davies, J. Nath, Schrödinger operators with slowly decaying potentials. J. Comput. Appl. Math. 148 (2002), 1–28.
[5] R. L. Frank, Eigenvalue bounds for Schrödinger operators with complex potentials. Bull. Lond. Math. Soc. 43 (2011), no. 4, 745–750.
[6] R. L. Frank, A. Laptev, E. H. Lieb, R. Seiringer, Lieb–Thirring inequalities for Schrödinger operators with complex-valued potentials. Lett. Math. Phys. 77 (2006), 309–316.
[7] R.L. Frank, A. Laptev and O. Safronov, On the number of eigenvalues of Schrödinger operators with a complex potentials J. of LMS to appear
[8] R. L. Frank, A. Laptev, R. Seiringer, A sharp bound on eigenvalues of Schrödinger operators on the half-line with complex-valued potentials. Spectral theory and analysis, 39–44, Oper. Theory Adv. Appl. 214, Birkhäuser/Springer Basel AG, Basel, 2011.
[9] R. L. Frank, J. Sabin, Restriction theorems for orthonormal functions, Strichartz inequalities and uniform Sobolev estimates. Preprint (2014), [http://arxiv.org/pdf/1404.2817.pdf]
[10] R. L. Frank, B. Simon, Eigenvalue bounds for Schrödinger operators with complex potentials. III. J. Spectr. Theory, to appear.
[11] I. M. Gel’fand, G. E. Shilov, Generalized functions. Vol. 1. Properties and operations. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1964 [1977].
[12] A. Laptev, O. Safronov, Eigenvalue estimates for Schrödinger operators with complex potentials. Comm. Math. Phys. 292 (2009), 29–54.

Francesco Ferrulli, Department of Mathematics, Imperial College London, SW7 2AZ, London, UK
E-mail address: f.ferrulli@imperial.ac.uk

Ari Laptev, Department of Mathematics, Imperial College London, SW7 2AZ, London, UK
E-mail address: laptev@mittag-leffler.se

Oleg Safronov, Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
E-mail address: osafron@uncc.edu