Riemann problem for the photon fluid: self-steepening effects

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We consider the Riemann problem of evolution of initial discontinuities for the photon fluid propagating in a normal dispersion fiber with account of self-steepening effects. The dynamics of light field is described by the nonlinear Schrödinger (NLS) equation with self-steepening term appearing due to retardation of the fiber material response to variations of the electromagnetic signal. It is shown that evolution dynamics in this case is much richer than that for the NLS equation. Complete classification of possible wave structures is given for all possible jump conditions at the discontinuity.

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I. INTRODUCTION

Dispersive shock waves (DSWs, or undular bores), that are oscillatory wave structures emerging in evolution of wave pulses after wave breaking, are ubiquitous, being observed in various physical systems including water waves, Bose-Einstein condensates, waves in magnetics and in nonlinear optics (see, e.g., review article [1] and references therein). In nonlinear optics, the formation of temporal dispersive shock waves was observed in single-mode optical fibers for the wavelength corresponding to a normal group velocity dispersion upon steepening of powerful picosecond optical pulses acquiring almost rectangular shapes and linear frequency chirp due to combined action of the self-phase-modulation and dispersion effects [2]. Formation of dispersive shock wave in [2] was identified by means of analysis of the spectrum of transmitted pulses, while in the subsequent work [3] the evidence of the shock wave formation was obtained already in the time domain. Optical shock waves were observed not only in light pulses, but also in light beams. For example, the propagation of high-intensity localized beams superimposed on low-intensity plane-wave background led to formation of both one- and two-dimensional spatial shock waves in photo-refractive crystals with defocusing nonlinearity [4] and allowed observation of interactions between several shocks. A shock fan filled with non-interacting one-dimensional gray solitons that emanate from a gradient catastrophe developing around the notch of powerful dark beam in defocusing optical medium was observed in Ref. [5]. Recently, a fiber-optics analogue of the dam-breaking phenomenon was studied experimentally in Ref. [6].

Theoretically, the DSWs are represented as modulated nonlinear periodic waves and then the process of their formation and evolution is described by the Whitham theory of modulations (for a review see Ref. [11]). In the fiber optics applications, the dynamics of pulses is described usually by the nonlinear Schrödinger (NLS) equation that accounts for two main effects—quadratic normal dispersion and Kerr nonlinearity. For this case, the theory of DSWs is already well developed and the main parameters of the arising wave structures can be calculated for typical idealized situations in simple analytical form. In particular, consideration of many realistic problems can be reduced to analysis of the so-called Riemann problem of evolution of discontinuity in the initial data. Such a discontinuity can appear, for example, as a jump in the time dependence of the light intensity, what is most typical in physics of light pulses in fibers, or evolve from a “collision” of two pulses in which case not only intensity has a discontinuity but also the time and space derivatives of the phase. Classification of possible wave structures in the NLS equation theory was given in Refs. [7, 8], and it provides the theoretical basis for calculation of characteristic parameters of such experiments as that of Ref. [6].

However, in nonlinear optics, besides quadratic dispersion and Kerr nonlinearity, many other effects can play important role in propagation of pulses. For example, in experiment [4] with photo-refractive material the saturation of nonlinearity is quite essential and the corresponding theory of DSWs was developed in Ref. [9]. In fiber optics, one needs to take into account such effects as dissipation, higher-order dispersion, intra-pulse Raman scattering and self-steepening (see, e.g., [10]). These effects can drastically change evolution of DSWs leading sometimes to violation of the supposition that such an evolution is adiabatically slow, as apparently it happens in the case of considerable higher order dispersion [11]. On the other hand, small dissipation can stop spreading out of the oscillatory region so that its width is stabilized with the size being inverse proportional to the dissipation coefficient. These effects have been studied in different physical contexts and their role in nonlinear optics seems to be quite clear. The self-steepening effects are usually described by the last term in the modified NLS (mNLS) equation which can be written in non-dimensional form as

$$i q_{t} + \frac{1}{2} q_{tt} + |q|^2 q - i \alpha (|q|^2 q) \chi = 0. \quad (1)$$

In early publication [12] it was shown in dispersionless approximation that during an evolution the pulse acquires an asymmetric form instead of gradual symmetric deformation of its form in the NLS equation theory. This ob-
The aim of this paper is to give a full solution to the mKdV equation (1) by the finite-gap integration method which yields these solutions in the form convenient for applications in the Whitham theory of modulations and the Whitham equations are also derived in section IV. In section V we describe the elementary wave structures that appear as building blocks in the general wave patterns. In section VI we apply the developed theory to derivation of the full classification of wave structures arising in evolution of the initial discontinuities. The last section VII is devoted to conclusions.

II. LINEAR WAVES

We shall start with the study of linear waves in a waveguide along a uniform wave background with the amplitude \( \sqrt{I_0} = |q_0| = \text{const} \). It is more convenient to make a substitution \( q(t,x) = \tilde{q}(t,x) \exp(-i\alpha t) \) and then the modified NLS equation (1) with “minus” sign in the Kerr nonlinearity (normal dispersion) and \( \alpha > 0 \) transforms to

\[
i \frac{\partial \tilde{q}}{\partial t} + \frac{1}{2} \frac{\partial^2 \tilde{q}}{\partial x^2} - i \frac{p}{\alpha I_0} \frac{\partial \tilde{q}}{\partial x} + |\tilde{q}|^2 \tilde{q} + i\alpha |\tilde{q}|^2 \frac{\partial \tilde{q}}{\partial x} = 0.
\]

We suppose that at the undisturbed state the phase is everywhere equal to zero and linearize the equation with respect to small disturbance

\[
\delta q = \sqrt{I_0} + \delta q, \quad |\delta q| \ll \sqrt{I_0}
\]

to obtain equation for \( \delta q \):

\[
i \delta q_x + \frac{1}{2} \delta q_{tt} - I_0(\delta q + \delta q^*) - i\alpha I_0(2\delta q_t + \delta q_t^*) = 0.
\]

This equation should be solved with the initial condition \( \delta q|_{t=0} = \delta q_0(t) \). After separation of the real and imaginary parts \( \delta q = A + iB \), we obtain the system from which we can exclude \( B \) and get the linear equation

\[
A_{xx} + I_0 \left( 3\alpha^2 I_0 - 1 \right) A_{tt} + \frac{1}{4} A_{tttt} - 4\alpha I_0 A_{xt} = 0.
\]

It can be readily solved by the Fourier method. To this end, we note that linear harmonic waves \( A \propto \exp[i(kx - \omega t)] \) satisfy to the dispersion law

\[
k_{1,2}(\omega) = \omega \left( -2\alpha I_0 \pm \sqrt{\frac{\omega^2}{4} + I_0(2\alpha^2 I_0 + 1)} \right). \]

After standard calculations we arrive at the solution expressed in terms of the Fourier transform \( I_0(\omega) \) of the initial (input) intensity disturbance \( I'(x,t) = 2\sqrt{I_0}A(x,t), \)

\[
I'(t,x) = J_1(x,t) - J_2(x,t),
\]

\[
J_1 = \int_{-\infty}^{\infty} \delta I(\omega) K_1(\omega) e^{ixf_1(\omega)} \frac{d\omega}{2\pi},
\]

\[
J_2 = \int_{-\infty}^{\infty} \delta I(\omega) K_2(\omega) e^{ixf_2(\omega)} \frac{d\omega}{2\pi},
\]
where
\[ K_1(\omega) = \frac{k_1(\omega) + 3\alpha I_0\omega}{k_2(\omega) - k_1(\omega)}, \quad K_2(\omega) = \frac{k_2(\omega) + 3\alpha I_0\omega}{k_2(\omega) - k_1(\omega)}, \]
(9)
and
\[ f_1(\omega) = k_1(\omega) - \omega \frac{t}{x}, \quad f_2(\omega) = k_2(\omega) - \omega \frac{t}{x}. \]
(10)
These integrals can be estimated for large distance of propagation \( x \) by the method of stationary phase resulting in
\[ J_1 \simeq \frac{2\delta I_0(\omega^{(1)}) K_2(\omega^{(1)})}{\sqrt{2\pi x} |\frac{d^2}{dx^2}|^{(1)}} \cos \left( x f_1(\omega^{(1)}) + \frac{\pi}{4} \right), \]
\[ J_2 \simeq \frac{2\delta I_0(\omega^{(2)}) K_1(\omega^{(2)})}{\sqrt{2\pi x} |\frac{d^2}{dx^2}|^{(2)}} \cos \left( x f_2(\omega^{(2)}) + \frac{\pi}{4} \right), \]
(11)
where \( \omega^{(1)} \) and \( \omega^{(2)} \) are the values of \( \omega \) at the points of the stationary phase that are defined by the equations
\[ \frac{df_1}{d\omega} = 0, \quad \frac{df_2}{d\omega} = 0. \]
(12)
In Fig. 1, we compare the numerical calculation of the integral (8) with its approximate estimation (11) for the initial perturbation
\[ I'(t) = \frac{1}{\sqrt{\pi a}} \exp \left( -\frac{t^2}{a^2} \right), \quad I'(\omega) = \exp \left( -\frac{\omega^2 a^2}{4} \right). \]
(13)
As we see, the pulse splits into two smaller pulses, however, on the contrary to the NLS case, they are not symmetrical pulses propagating in opposite directions. Now these two pulses have different profiles and propagate with different group velocities. This is manifestation of lack of the time inversion invariance mentioned in the introduction, which is caused by the last term in the mNLS equation (14). It should be noted that the asymptotic solution (11) describes well the wave packet even for not very large \( x \).

The two modes of propagation in the linear approximation to the equation (14) differ from each other not only by the dispersive properties of their propagation; in fact, their nonlinear properties are also drastically different, as we shall see in the next section.

III. SMALL DISPERSION AND WEAK NONLINEAR LIMITS OF THE MODIFIED NLS EQUATION

We are interested in the leading dispersive and nonlinear corrections to the dispersionless linear propagation of disturbances along the background pulse. Therefore they can be considered separately and after that their contributions should be added to give the resulting approximate equation.

In the small dispersion limit the series expansion of the expressions (7) in degrees of \( \omega \) yields
\[ k(\omega) \equiv \omega \left\{ -2\alpha I_0 \right. \]
\[ \pm \left[ \sqrt{I_0(1 + \alpha^2 I_0)} + \frac{\omega^2}{8\sqrt{I_0(1 + \alpha^2 I_0)}} \right] \left. \right\}. \]
This approximation of dispersion laws corresponds to linear equations for propagation of, say, small disturbances of intensity, \( I = I_0 + I' \),
\[ \frac{\partial I'}{\partial x} + \left[ \pm \sqrt{I_0(1 + \alpha^2 I_0)} - 2\alpha I_0 \right] \frac{\partial I'}{\partial t} \]
\[ \mp \frac{1}{8\sqrt{I_0(1 + \alpha^2 I_0)}} \frac{\partial^3 I'}{\partial t^3} = 0. \]
(14)
To find small nonlinear corrections, we turn to the dispersionless limit which can be obtained by means of well-known Madelung transformation
\[ q(t, x) = \sqrt{I(t, x)} \exp \left( i \int t u(t', x) dt' \right) \]
which after substitution into the mNLS equation (1) and separation of real and imaginary parts yields the system
\[ I_x + \left( u I - \frac{3}{2} \alpha I^2 \right)_t = 0, \]
\[ u_x + uu_t + I_t - \alpha (u I)_t + \left( \frac{I^2}{8I^2} - \frac{I_t}{4I} \right)_t = 0. \]
(15)
The last term in the second equation describes the dispersion effects and the full system will be considered later. Now we shall discuss the dispersionless limit when this term is omitted and we arrive at the hydrodynamic system

\[ I_x + \left( uI - \frac{3}{2} \alpha I^2 \right)_t = 0, \quad u_x + uu_I + I_I - \alpha(uI)_t = 0, \]  

where the first equation can be interpreted as the continuity equation for the intensity \( I \) and the second one as the Euler equation for the “flow velocity” \( u \). This system can be cast in standard way to the Riemann diagonal form

\[ \frac{\partial r_\pm}{\partial x} + \frac{1}{v_\pm} \frac{\partial r_\pm}{\partial t} = 0 \]  

for the Riemann invariants

\[ r_\pm = \frac{u}{2} - \alpha I \pm \sqrt{I(1 + \alpha^2 I - \alpha u)} \]

with inverse velocities

\[ \frac{1}{v_\pm} = u - 2\alpha I \pm \sqrt{I(1 + \alpha^2 I - \alpha u)}. \]

As one can see, if we put chirp \( u \) equal to zero, then we reproduce the low frequency limit \( \omega \to 0 \) of the inverse phase velocities \( k/\omega \) of linear waves given by Eq. \[7\], as it should be. This means that the two linear modes correspond to the linear approximation of the so-called simple waves with one of the Riemann constant. Hence, the weakly nonlinear waves correspond to the next order approximation of these simple waves with respect to amplitude of propagating disturbance. Since properties of these two modes are very different, they should be considered separately.

A. Korteweg-de Vries mode

In dispersionless approximation, the KdV equation is obtained in the case of the weakly nonlinear simple wave evolution with constant Riemann invariant \( r_+ = \text{const} \). Assuming that a pulse propagates along the same background \( I = I_0, u = 0 \), we have the relation between \( I \) and \( u \),

\[ \frac{u}{2} - \alpha I + \sqrt{I(1 + \alpha^2 I - \alpha u)} = -\alpha I_0 + \sqrt{I_0(1 + \alpha^2 I_0)}, \]

which defines \( u \) as a function of \( I, u = u(I) \), along this simple wave. This function can be substituted into the varying Riemann invariant \( r_- \) and the corresponding inverse velocity \( 1/v_- \). Hence, series expansion of equation \[14\] for \( r_- \) with respect to small disturbance \( I' \) of the intensity \( I = I_0 + I' \) yields the weakly nonlinear approximation

\[ \frac{\partial I'}{\partial x} = (\sqrt{I_0(1 + \alpha^2 I_0)} + 2\alpha I_0) \frac{\partial I'}{\partial t} \]

\[ - 3\left( \sqrt{1 + \alpha^2 I_0} + \sqrt{\alpha^2 I_0} \right) I'_I \frac{\partial I'}{\partial t} + 0. \]

Combining the nonlinear term from this equation and the small dispersion effects described by equation \[14\], we arrive at the KdV equation

\[ \frac{\partial I'}{\partial x} - (\sqrt{I_0(1 + \alpha^2 I_0)} + 2\alpha I_0) \frac{\partial I'}{\partial t} \]

\[ - 3\left( \sqrt{1 + \alpha^2 I_0} + \sqrt{\alpha^2 I_0} \right) I'_I \frac{\partial I'}{\partial t} + \frac{1}{8\sqrt{I_0(1 + \alpha^2 I_0)^3}} \frac{\partial^3 I'}{\partial t^3} = 0. \]

It is worth noticing that in both limits \( \alpha^2 I_0 \gg 1 \) and \( \alpha^2 I_0 \ll 1 \) the nonlinear term has finite value and we need not to include higher order corrections for taking into account higher order nonlinear effects. The situation is different for another simple wave in the small amplitude approximation.

B. Gardner mode

Derivation of evolution equation for another weakly nonlinear simple wave is similar, however, at it will be clear from the result, now we have to take into account the dispersion effects described by equation \[14\], in the opposite limit \( \alpha^2 I_0 \gg 1 \) this equation reduces to the mKdV equation

\[ \frac{\partial I'}{\partial x} + (\sqrt{I_0(1 + \alpha^2 I_0) + 2\alpha I_0}) \frac{\partial I'}{\partial t} \]

\[ + 3\left( \sqrt{1 + \alpha^2 I_0} + \sqrt{\alpha^2 I_0} \right) I'_I \frac{\partial I'}{\partial t} \]

\[ - \frac{3}{8\sqrt{I_0(1 + \alpha^2 I_0)^3}} \frac{\partial^3 I'}{\partial t^3} = 0. \]

In the limit \( \alpha^2 I_0 \ll 1 \) this equation reduces to the mKdV equation

\[ \frac{\partial I'}{\partial x} - \alpha I_0 \frac{\partial I'}{\partial t} - \frac{3\alpha I_0 I'_I}{4\sqrt{I_0(1 + \alpha^2 I_0)}} \frac{\partial I'}{\partial t} - 8\sqrt{I_0(1 + \alpha^2 I_0)^3} \frac{\partial^3 I'}{\partial t^3} = 0. \]

Disappearance of quadratic nonlinearity from this equation was the reason why the cubic nonlinearity was included into equation \[21\]. In the opposite limit \( \alpha^2 I_0 \ll 1 \), neglecting higher order correction, we return to the KdV equation

\[ \frac{\partial I'}{\partial x} + \sqrt{I_0} \frac{\partial I'}{\partial t} + \frac{3}{2\sqrt{I_0}} I'_I \frac{\partial I'}{\partial t} - \frac{1}{8I_0} \frac{\partial^3 I'}{\partial t^3} = 0. \]
It differs from analogous limit of equation (20) by the replacement $t \to -t$, that is in this limit the symmetry between the left and right propagating waves is restored.

Formation of DSWs from initial discontinuities in the KdV equation theory is well known since the pioneering paper Ref. [20] — the initial discontinuity evolves into either rarefaction wave or cnoidal DSW. However, situation for the mKdV equation is much more complicated [21] and in this case we can get eight different structures including, besides the rarefaction waves and cnoidal DSWs, also trigonometric DSWs, combined shocks and their combinations separated by plateau. Therefore one should expect that in the case of Riemann problem for the equation (1) we have also to get much richer structure than in the NLS case. To solve this problem, at first we have to find periodic solutions of the equation (1) in convenient for us form, that is in the form parameterized by the parameters related with the Riemann invariants of the corresponding Whitham modulation equations by simple formulae. This is achieved by the restricted finite-gap integration method developed in Ref. [15], and in the next section we shall obtain the periodic solutions by this method and derive the Whitham equations.

IV. PERIODIC SOLUTIONS AND WHITHAM EQUATIONS

The finite-gap integration method (see, e.g., [24]) is based on possibility of representing of the mNLS equation (1) as a compatibility condition of two systems of linear equations with a spectral parameter $\lambda$

$$\frac{\partial}{\partial t} (\psi_1, \psi_2) = \begin{pmatrix} F & G \\ H & -F \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\frac{\partial}{\partial x} (\psi_1, \psi_2) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where

$$F = -2i \left( \lambda^2 - \frac{1}{4\alpha} \right), \quad G = 2\sqrt{\alpha} \lambda q, \quad H = 2\sqrt{\alpha} \lambda q^*, \quad A = -i \left\{ 4 \left( \lambda^2 - \frac{1}{4\alpha} \right)^2 + 2\alpha \lambda^2 |q|^2 \right\},$$

$$B = \sqrt{\alpha} \left\{ 4\lambda \left( \lambda^2 - \frac{1}{4\alpha} \right) q + \lambda (iq_1 - 2\alpha |q|^2 q) \right\},$$

$$C = \sqrt{\alpha} \left\{ 4\lambda \left( \lambda^2 - \frac{1}{4\alpha} \right) q^* - \lambda (iq_1^* - 2\alpha |q|^2 q^*) \right\}.$$

This Lax pair can be obtained by simple transformation from the known Lax pair for the DNLS equation (2) (see Ref. [25]). Here $2 \times 2$ linear problems [24] and [25] have two linearly independent basis solutions which we denote as $(\psi_1, \psi_2)^T$ and $(\varphi_1, \varphi_2)^T$. We define “squared basis function” by the formulae

$$f = -\frac{i}{2}(\psi_1 \varphi_2 + \psi_2 \varphi_1), \quad g = \psi_1 \varphi_1, \quad h = -\psi_2 \varphi_2.$$

They obey the linear equations

$$f_t = i Gh - i Hg, \quad g_t = 2F g + 2i G f, \quad h_t = -2F h - 2i H f,$$

and

$$f_x = i Bh - i C g, \quad g_x = 2Ag + 2i B f, \quad h_x = -2Ah - 2i C f.$$

We look for the solutions of these equations in the form

$$f = \left( \lambda^2 - \frac{1}{4\alpha} \right)^2 - f_1 \left( \lambda^2 - \frac{1}{4\alpha} \right) + f_2,$$

$$g = \sqrt{\alpha} \lambda \left( \lambda^2 - \frac{1}{4\alpha} - \frac{\mu}{2} \right) q,$$

$$h = \sqrt{\alpha} \lambda \left( \lambda^2 - \frac{1}{4\alpha} - \frac{\mu^*}{2} \right) q^*.$$

Here the functions $f_1(x,t), f_2(x,t), \mu(x,t)$ and $\mu^*(x,t)$ are unknown; $\mu(x,t)$ and $\mu^*(x,t)$ are not interrelated a priori, but we shall find soon that they are complex conjugate, whence the notation.

It is easy to check that the expression $f^2 - gh = P(\lambda)$ is independent of $x$ and $t$, and periodic solutions are distinguished by the condition that $P(\lambda)$ be a polynomial in $\lambda$ in accordance with the ansatz (30).

$$f^2 - gh = P(\lambda) = \prod_{i=1}^{4} (\lambda^2 - \lambda_i^2) = \lambda^8 - s_1 \lambda^6 + s_2 \lambda^4 - s_3 \lambda^2 + s_4.$$

Equating the coefficients of like powers of $\lambda$ at two sides of this identity, we get

$$s_1 = \frac{1}{\alpha} + 2f_1 + \alpha |q|^2,$$

$$s_2 = f_1^2 + \frac{3}{2\alpha} f_1 + 2f_2 + \frac{3}{8\alpha^2} + \frac{1}{2} |q|^2 + \frac{1}{2\alpha} |q|^2 (\mu + \mu^*),$$

$$s_3 = \frac{f_2^2}{2\alpha} + \frac{3f_1}{8\alpha^2} + 2f_1 f_2 + \frac{f_2}{\alpha} + \frac{1}{16\alpha^3} + \frac{|q|^2}{16\alpha},$$

$$s_4 = \left( \frac{f_1}{4\alpha} + f_2 \right)^2 + \frac{1}{8\alpha^2} \left( \frac{f_1}{4\alpha} + f_2 \right) + \frac{1}{256\alpha^4}.$$

(27)
Here $s_i$ are standard symmetric functions of the four zeros $\lambda_i^2$ of the polynomial,

\[
\begin{align*}
  s_1 &= \sum_{i} \lambda_i^2, & s_2 &= \sum_{i<j} \lambda_i^2\lambda_j^2, & s_3 &= \sum_{i<j<k} \lambda_i^2\lambda_j^2\lambda_k^2, \\
  s_4 &= \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2.
\end{align*}
\]

Equations (32) allow us to express $\mu, \mu^*$ as functions of $I = |q|^2$. The last equation (32d) gives

\[
\begin{align*}
  f_1 &= \frac{1}{2} \left( s_1 - \frac{1}{\alpha} - \alpha I \right), \\
  f_2 &= \frac{1}{8\alpha^2} \left( 1 + \alpha^2 I - \alpha s_1 \pm 8\alpha^2 s_4 \right).
\end{align*}
\]

We substitute that into (32b) and (32c) and obtain the system for $\mu$ and $\mu^*$, which can be easily solved to give

\[
\mu = \frac{1}{4\alpha I} \left[ 4s_2 - (s_1 - \alpha I)^2 - 2I \\
+ 8\sqrt{s_4} \pm i\sqrt{-R(\alpha I)} \right],
\]

where

\[
R(\nu) = \nu^4 - 4s_1\nu^3 + (6s_1^2 - 8s_2 \pm 48\sqrt{s_4}) \nu^2 - (4s_1^3 - 16s_1s_2 + 64s_3 \pm 32s_1\sqrt{s_4}) \nu + (-s_2^2 + 4s_2 \pm 8\sqrt{s_4}).
\]

The introduced here function $R$ is a fourth-degree polynomial in $\nu$ and it is called an algebraic resolvent of the polynomial $P(\lambda)$, because zeros of $R(\nu)$ are related to zeros of $P(\lambda)$ by the following simple symmetric expressions: the upper sign (+) in (30) corresponds to the zeros

\[
\begin{align*}
  \nu_1 &= (-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2, \\
  \nu_2 &= (\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4)^2, \\
  \nu_3 &= (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2, \\
  \nu_4 &= (\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)^2,
\end{align*}
\]

and the lower sign (−) in equation (30) corresponds to the zeros

\[
\begin{align*}
  \nu_1 &= (-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)^2, \\
  \nu_2 &= (\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2, \\
  \nu_3 &= (\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4)^2, \\
  \nu_4 &= (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2.
\end{align*}
\]

This can be proved by a simple check of the Vieté formulæ.

Substitution of Eqs. (30) into Eqs. (29) gives after equating the coefficients of like powers of $\lambda$ expressions for the time derivatives of $f_1$ and $f_2$

\[
\begin{align*}
  f_{1,t} &= i\alpha|q|^2 (\mu - \mu^*), & f_{2,t} &= -\frac{1}{4\alpha} f_{1,t},
\end{align*}
\]
cn and \( sn \) being Jacobi elliptic functions \(^{24}\). The period of the oscillating with change of \( t \) function \(^{46}\) is

\[
T = \frac{4K(m)}{\sqrt{(\nu_3 - \nu_1)(\nu_4 - \nu_2)}} = \frac{K(m)}{\sqrt{(\lambda_3^2 - \lambda_1^2)(\lambda_4^2 - \lambda_2^2)}},
\]

where \( K(m) \) is the complete elliptic integral of the first kind \(^{24}\).

In the limit \( \nu_3 \to \nu_2 \) (\( m \to 1 \)) the period tends to infinity and the solution \(^{46}\) acquires the soliton form

\[
\alpha I = \nu_2 - \frac{\nu_2 - \nu_1}{\cosh^2 \theta + \frac{\nu_2 - \nu_1}{\nu_4 - \nu_2} \sinh^2 \theta}.
\]

(50)

This is a “dark soliton” for the variable \( I \).

The limit \( m \to 0 \) can be reached in two ways.

(i) If \( \nu_2 \to \nu_1 \), then the solution transforms into a linear harmonic wave

\[
\alpha I \cong \nu_2 - \frac{1}{2} (\nu_2 - \nu_1) \cos(\omega \xi),
\]

\[
\omega = \sqrt{(\nu_3 - \nu_1)(\nu_4 - \nu_2)}.
\]

(51)

(ii) If \( \nu_4 = \nu_3 \) but \( \nu_1 \neq \nu_2 \), then we arrive at the nonlinear trigonometric solution:

\[
\alpha I = \nu_2 - \frac{(\nu_2 - \nu_1) \cos^2 \theta}{1 + \frac{\nu_2 - \nu_1}{\nu_4 - \nu_2} \sin^2 \theta},
\]

\[
\theta = \sqrt{(\nu_3 - \nu_1)(\nu_4 - \nu_2)} \xi/2.
\]

(52)

If we take the limit \( \nu_2 - \nu_1 \ll \nu_3 - \nu_1 \) in this solution, then we return to the small-amplitude limit \(^{51}\) with \( \nu_4 = \nu_3 \). On the other hand, if we take here the limit \( \nu_2 \to \nu_3 = \nu_4 \), then the argument of the trigonometric functions becomes small and we can approximate them by the first terms of their series expansions. This corresponds to an algebraic soliton of the form

\[
\alpha I = \nu_2 - \frac{\nu_2 - \nu_1}{1 + (\nu_2 - \nu_1)\xi^2/4}.
\]

(53)

(B) In the second case, the variable \( \alpha I \) oscillates in the interval

\[
\nu_3 \leq \alpha I \leq \nu_4.
\]

(54)

Here again, a standard calculation yields

\[
\alpha I = \nu_3 + \frac{(\nu_4 - \nu_3)\sin^2(\theta, m)}{1 + \frac{\nu_4 - \nu_3}{\nu_4 - \nu_1} \sin^2(\theta, m)},
\]

(55)

with the same definitions \(^{47}\), \(^{48}\), and \(^{49}\) for \( \theta, m, \) and \( T \), correspondingly. In this case we have \( \alpha I(0) = \nu_4 \).

In the soliton limit \( \nu_3 \to \nu_2 \) (\( m \to 1 \)) we get

\[
\alpha I = \nu_2 + \frac{\nu_4 - \nu_2}{\cosh^2 \theta + \frac{\nu_4 - \nu_2}{\nu_4 - \nu_1} \sinh^2 \theta}.
\]

(56)

This is a “bright soliton” for the variable \( I \). Again, the limit \( m \to 0 \) can be reached in two ways.

(i) If \( \nu_4 \to \nu_3 \), then we obtain a small-amplitude harmonic wave

\[
\alpha I \cong \nu_3 + \frac{1}{2} (\nu_4 - \nu_3) \cos(\omega \xi),
\]

\[
\omega = \sqrt{(\nu_3 - \nu_1)(\nu_4 - \nu_2)}.
\]

(57)

(ii) If \( \nu_2 = \nu_1 \), then we obtain another nonlinear trigonometric solution,

\[
\alpha I = \nu_3 + \frac{(\nu_4 - \nu_3) \cos^2 \theta}{1 + \frac{\nu_4 - \nu_3}{\nu_4 - \nu_1} \sin^2 \theta},
\]

\[
\theta = \sqrt{(\nu_3 - \nu_1)(\nu_4 - \nu_2)} \xi/2.
\]

(58)

If we assume that \( \nu_4 - \nu_3 \ll \nu_1 - \nu_1 \), then this reproduce the small-amplitude limit \(^{51}\) with \( \nu_2 = \nu_1 \). On the other hand, in the limit \( \nu_3 \to \nu_2 = \nu_1 \) we obtain the algebraic soliton solution:

\[
\alpha I = \nu_1 + \frac{\nu_4 - \nu_1}{1 + (\nu_4 - \nu_1)^2 \xi^2/4}.
\]

(59)

The convenience of this form of periodic solutions of our equation is related with the fact that the parameters \( \lambda_i \), connected with \( \nu_i \) by the formulae \(^{37}\), \(^{38}\), play the role of Riemann invariants in Whitham theory of modulations. For both cases \(^{37}\), \(^{38}\) we have the identities

\[
m = \frac{(\nu_4 - \nu_3)(\nu_2 - \nu_1)}{(\nu_4 - \nu_2)(\nu_4 - \nu_1)} = \frac{(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)}{(\lambda_3^2 - \lambda_2^2)(\lambda_4^2 - \lambda_2^2)}.
\]

(60)

Now we shall consider slowly modulated waves. In this case, the parameters \( \lambda_i \) (\( i = 1, 2, 3, 4 \)) become slowly varying functions of \( x \) and \( t \) changing little in one period and, as was found in Ref. \(^{15}\), they can serve as Riemann invariants. Evolution of \( \lambda_i \) is governed by the Whitham modulation equations

\[
\frac{\partial \lambda_i}{\partial x} + \frac{1}{v_i} \frac{\partial \lambda_i}{\partial t} = 0.
\]

(61)

The inverse Whitham velocities appearing in these equations can be computed by means of the formulæ

\[
\frac{1}{v_i} = \left(1 - \frac{T}{\partial_i I}\right) \frac{1}{V_i}, \text{ where } \partial_i \equiv \frac{\partial}{\partial \lambda_i^2},
\]

(62)

with the use of equations \(^{44}\), \(^{49}\). Hence, a simple calculation yields the explicit expressions

\[
\frac{1}{v_3} = \frac{1}{\alpha} - \frac{1}{2} \sum_{i=1}^{4} \lambda_i^2 - \frac{(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_4^2)K(m)}{\lambda_i - \lambda_4^2 E(m)}
\]

\[
\frac{1}{v_2} = \frac{1}{\alpha} - \frac{1}{2} \sum_{i=1}^{4} \lambda_i^2 + \frac{(\lambda_3^2 - \lambda_2^2)(\lambda_4^2 - \lambda_2^2)K(m)}{\lambda_i - \lambda_2^2 E(m)}
\]

\[
\frac{1}{v_3} = \frac{1}{\alpha} - \frac{1}{2} \sum_{i=1}^{4} \lambda_i^2 - \frac{(\lambda_4^2 - \lambda_3^2)(\lambda_3^2 - \lambda_4^2)K(m)}{\lambda_i - \lambda_4^2 E(m)}
\]

\[
\frac{1}{v_4} = \frac{1}{\alpha} - \frac{1}{2} \sum_{i=1}^{4} \lambda_i^2 + \frac{(\lambda_4^2 - \lambda_3^2)(\lambda_3^2 - \lambda_4^2)K(m)}{\lambda_i - \lambda_4^2 E(m)}
\]

(63)
In a modulated wave representing a dispersive shock wave, the Riemann invariants change with $t$ and $x$. The dispersive shock wave occupies a time interval at whose edges two of the Riemann invariants coincide. The solition edge corresponds to $\lambda_3 = \lambda_2$ ($m = 1$) and at this edge the Whitham velocities are given by

$$
\frac{1}{v_1} = \frac{1}{\alpha} - (3\lambda_1^2 + \lambda_2^2), \quad \frac{1}{v_2} = \frac{1}{\alpha} - (\lambda_1^2 + 3\lambda_2^2),
$$

and if $\lambda_2 = \lambda_1$, then

$$
\frac{1}{v_1} = \frac{1}{\alpha} -(3\lambda_1^2 + \lambda_2^2), \quad \frac{1}{v_2} = \frac{1}{\alpha} - 4\lambda_1^2 - \frac{(\lambda_2^2 - \lambda_1^2)^2}{\lambda_1^2 - \lambda_2^2 - 2\lambda_1^2}, \quad \frac{1}{v_3} = \frac{1}{\alpha} - (3\lambda_1^2 + \lambda_2^2), \quad \frac{1}{v_4} = \frac{1}{\alpha} - (\lambda_1^2 + 3\lambda_2^2).
$$

The small amplitude limit $m = 0$ can be obtained in two ways. If $\lambda_3 = \lambda_4$, then we get

$$
\frac{1}{v_1} = \frac{1}{\alpha} - (3\lambda_1^2 + \lambda_2^2), \quad \frac{1}{v_2} = \frac{1}{\alpha} - (\lambda_1^2 + 3\lambda_2^2),
$$

and if $\lambda_2 = \lambda_1$, then

$$
\frac{1}{v_1} = \frac{1}{\alpha} -(3\lambda_1^2 + \lambda_2^2), \quad \frac{1}{v_2} = \frac{1}{\alpha} - 4\lambda_1^2 - \frac{(\lambda_2^2 - \lambda_1^2)^2}{\lambda_1^2 - \lambda_2^2 - 2\lambda_1^2},
$$

Now we are ready to discuss the key elements from which any wave structure evolving from an initial discontinuity consists.

V. ELEMENTARY WAVE STRUCTURES

Let the initial (input) conditions have a step-like form,

$$
I(x = 0) = \begin{cases} 
I^L, & t < 0, \\
I^R, & t > 0,
\end{cases}
$$

$$
u(x = 0) = \begin{cases} 
u^L, & t < 0, \\
u^R, & t > 0.
\end{cases}
$$

Evolution of this step-like pulse leads to formation of quite complex structures consisting of simpler elements. We shall describe these elements in the present section.

A. Rarefaction waves

For smooth enough wave patterns we can neglect the last dispersion term in the second equation of the system (15) and arrive at the so-called dispersionless equations (16). First of all, this system admits a trivial solution for which $I = \text{const}$ and $u = \text{const}$. We shall call such a solution a “plateau”. Introducing the Riemann invariants (18), we transform the hydrodynamic equations (16) to the diagonal form

$$
\frac{\partial r_{\pm}}{\partial x} + \frac{1}{v_{\pm}} \frac{\partial r_{\pm}}{\partial t} = 0,
$$

where the Riemann velocities are expressed via the Riemann invariants by the relations

$$
\frac{1}{v_+} = \frac{3}{2} r_+ + \frac{1}{2} r_- - \frac{1}{r_-} = \frac{1}{2} r_+ + \frac{3}{2} r_-.
$$

In terms of $I$ and $u$ these velocities are given by Eqs. (19). It is clear that the system is modulationally unstable if

$$
u > \frac{1}{\alpha} + \alpha I.
$$

A rarefaction wave belongs to the class of simple wave solutions discussed in section III and it is characterized by the condition that one of the Riemann invariants has a constant value along the flow, $r_+ = \text{const}$ or $r_- = \text{const}$. Consequently, according to the definition (18), these simple wave solutions are represented in the $(u, I)$-plane by the parabolas

$$
I = \frac{(u - 2r_{\pm})^2}{4(1 - \alpha r_{\pm})}.
$$

FIG. 2: Relation between $u$ and $I$ for simple wave solutions in the dispersionless regime. One line corresponds to $r_- = \text{const}$, and another one to $r_+ = \text{const}$. Dashed gray area shows the modulationally unstable region.

To have the intensity positive, it is necessary to fulfill the condition $r_- \leq r_+ \leq 1/(2\alpha)$. By virtue of obvious inequality $r_+ \geq r_-$ the parabola corresponding to $r_- = \text{const}$ has greater curvature than the parabola for $r_+ = \text{const}$ (see Fig. 2). Both parabolas touch the boundary line $I = u/\alpha - 1/\alpha^2$ of the instability region (in fact, this line is an envelope of a pencil of parabolas $I = (u - 2r)^2/(4(1 - 2\alpha^2r))$ with $r$ as a parameter). In the Fig. 2 the modulationally unstable region (70) is dashed. Along the line $u = 1/\alpha$ both derivatives $\partial r_+ / \partial u = 0$, $\partial r_+ / \partial I = 0$ vanish and $r_+$ reaches here its maximal value equal to $r_+ = 1/(2\alpha)$. We say that the line $u = 1/\alpha$ separates two monotonicity regions $u < 1/\alpha$ and $u > 1/\alpha$ in the half-plane $I \geq 0$. The two intersection points of parabolas correspond to uniform flows with constant parameters $I = \text{const}$ and $u = \text{const}$, that is to the plateau solutions. It is easy to express the physical variables $I$ and $u$ in...
terms of \( r_-, r_+ \),

\[
I = \frac{1}{2\alpha^2} \left[ 1 - \alpha(r_+ + r_-) \pm \sqrt{(1 - 2\alpha r_+)(1 - 2\alpha r_-)} \right],
\]

\[
u = \frac{1}{\alpha} \left[ 1 \pm \sqrt{(1 - 2\alpha r_+)(1 - 2\alpha r_-)} \right].
\]

(72)

The initial conditions (67) do not contain any parameters with dimension of time or length. Therefore solutions of equations (68) can depend on the self-similar variable \( \tau = t/x \) only, that is \( r_\pm = r_\pm(\tau) \), and then this system reduces to

\[
\left( \frac{1}{\nu_-} - \tau \right) \frac{dr_-}{d\tau} = 0, \quad \left( \frac{1}{\nu_+} - \tau \right) \frac{dr_+}{d\tau} = 0.
\]

(73)

We note again that these equations have a simple solution \( r_- = \text{const}, \ r_+ = \text{const} \) with constant \( u \) and \( I \) which corresponds to the mentioned above plateau region.

Turning to self-similar simple wave solutions, let us consider for definiteness the case when \( r_- = \text{const} \). Then we have

\[
\frac{1}{\nu_+} = u - 2\alpha I + \sqrt{I(\alpha^2 I - \alpha u + 1)} = \frac{t}{x},
\]

\[
r_- = \frac{u}{2} - \alpha I - \sqrt{I(\alpha^2 I - \alpha u + 1)} = \text{const}.
\]

(74)

Solving this system with respect to \( I \) and \( u \) yields

\[
I(\tau) = \frac{1}{2\alpha^2} - \frac{1}{3\alpha}(r_- + \tau)
\]

\[
\pm \frac{1}{2\alpha^2} \sqrt{(1 - 2\alpha r_-) \left[ 1 + \frac{2}{3}(r_- - 2\tau) \right]},
\]

\[
u(\tau) = \frac{2}{3} [\tau + r_- + 3\alpha I(\tau)].
\]

(75)

Plots for the intensity for both choices of the sign are shown in the Fig. 3. We see that in the self-similar solutions the variable \( \tau \) must be below its maximum value

\[
\tau \leq \frac{3}{4\alpha} + \frac{r_-}{2}.
\]

(76)

FIG. 3: Dependence of simple-wave solution \( I(\tau) \) on \( \tau = t/x \). The upper sign in Eq. (75) corresponds to the upper branch curve and the lower one to the lower branch.

FIG. 4: Example of the simple-wave solution for \( I^L = 1, u^L = -1, I^R = 0.1, u^R = 0, r_+ = 0.232 \). Numerical solution of the mNLS equation (1) is shown in red, analytical approximation is shown in blue.

FIG. 5: Diagrams representing the evolution of the Riemann invariants as functions of \( \tau = t/x \) in the rarefaction wave solutions of the hydrodynamic equations: (a) \( r_- = \text{const} \), \( r_+^L < r_+^R \); (b) \( r_+ = \text{const} \), \( r_-^L < r_-^R \).

at which the solutions coincide with each other and the intensity assumes the common value equal to

\[
I = \frac{1 - 2\alpha r_-}{4\alpha^2}.
\]

(77)

This means that both types of solutions (lower or upper branches) must match some other element of the whole structure at \( \tau \) smaller than its maximal possible value. Similar formulas and plots can be obtained for the solution \( r_+ = \text{const}, \nu_-(r_-, r_+) = x/t \equiv 1/\tau \). This wave configuration represents a rarefaction wave. In the general case this type of wave can connect uniform flows with equal values of the corresponding Riemann invariants \( r_- = r_+^L \) or \( r_+ = r_+^L \). Example of corresponding distribution is shown in Fig. 4. The analytical simple wave approximation agrees with the exact numerical solution very well.

Both branches in Fig. 3 correspond to the same solution of the equations (68) written for the Riemann invariants,

\[
r_- = r_-^0 = \text{const}, \quad \frac{1}{\nu_+} = \frac{3}{2} r_0^0 + \frac{1}{2} r_+ = \frac{t}{x}.
\]

(78)
and two solutions appear due to the two-valued character of the formulae (72) and (73). Thus, in these self-similar solutions one of the Riemann invariants must be constant and another one must increase with $\tau$ according to Eqs. (75). The dependence of the Riemann invariants on the physical parameters must also be monotonic in order to keep the solution single-valued. Hence both edge points of the rarefaction wave must lie either on the left or on the right side of the line $u = 1/\alpha$, along which the Riemann invariants reach their extremal values. As was mentioned above, we shall call the two regions on the left and right sides of this line as monotonicity regions. Dependence of the Riemann invariants on $\tau$ is sketched in Fig. 5 for two possible situations with $r_-$ or $r_+$ constant. The edge velocities of these rarefaction waves are equal to

$$(a) \quad s^{-1} = \frac{1}{2} \sqrt{r_L} + \frac{3}{2} \sqrt{r_+}, \quad s^+ = \frac{1}{2} r_R + \frac{3}{2} \sqrt{r_+};$$

$$(b) \quad s^{-1} = \frac{3}{2} \sqrt{r_L} + \frac{1}{2} r_+, \quad s^+ = \frac{3}{2} r_R + \frac{1}{2} \sqrt{r_+}. \quad (79)$$

Obviously, the corresponding wave structures must satisfy the conditions (a) $r_+ < r_R$, $r_L = r^R$ or (b) $r_+ = r_R$, $r_L < r^L$. It is natural to ask, what happens if we have the initial conditions satisfying opposite inequalities, and to answer this question we have to consider the DSW structures.

**B. Cnoidal dispersive shock waves**

The other two possible solutions of Eqs. (68) are sketched in Fig. 6 where for future convenience we have made the change $r \rightarrow \lambda$ ($r_{\pm}$ will be functions of $\lambda_{\pm}$ defined below), and they satisfy the boundary conditions (a) $\lambda_+ = \lambda_L$, $\lambda_+ > \lambda_R$ or (b) $\lambda_+ > \lambda_L$, $\lambda_+ = \lambda_R$. In the dispersionless approximation these multi-valued solutions are nonphysical. However, we can give them clear physical sense by understanding $\lambda_i$ as four Riemann invariants of the Whitham system that describe evolution of a modulated nonlinear periodic wave. We interpret this as formation of cnoidal dispersive shock wave from the initial discontinuity with such a type of the boundary conditions.

To find the solution of equations (61), we use again the argument that the wavelength of the DSW is negligibly small compared with the large scale of the whole structure and at this scale the initial conditions for the Whitham equations (61), which are similar to the Riemann equations (68), do not contain parameters with dimension of length, so the modulation parameters depend on the self-similar variable $\tau = t/x$ only. Therefore, equations (61) reduce to

$$\left( \frac{1}{v_i} - \tau \right) \frac{d\lambda_i}{d\tau} = 0. \quad (80)$$

Hence we find again that only one Riemann invariant varies along the DSW, while the other three are constant, that is the corresponding diagram reproduces the picture shown in Fig. 6. The limiting expressions (64) for the Whitham velocities must coincide with expressions (69) for dispersionless Riemann velocities and therefore we can relate the corresponding dispersionless and dispersive Riemann invariants by the formulae

$$(a) \quad \lambda^L = -\frac{1}{4\alpha} - \frac{r_L}{2}; \quad \lambda^R = -\frac{1}{4\alpha} - \frac{r^R}{2}, \quad (81)$$

$$(b) \quad \lambda^L = -\frac{1}{4\alpha} - \frac{r^L}{2}; \quad \lambda^R = -\frac{1}{4\alpha} - \frac{r^R}{2}.$$ 

At the soliton edges of the DSW, Here $r^L$, $r^R$ are the Riemann invariants of the Whitham system that describe evolution of a modulated nonlinear periodic wave. In a similar way, at the small-amplitude edges we find similar relations

$$(a) \quad \lambda^L = -\frac{1}{4\alpha} - \frac{r^L}{2}; \quad \lambda^R = -\frac{1}{4\alpha} - \frac{r^R}{2}, \quad (82)$$

and

$$(b) \quad \lambda^L = -\frac{1}{4\alpha} - \frac{r^L}{2}; \quad \lambda^R = -\frac{1}{4\alpha} - \frac{r^L}{2}. \quad (83)$$

Again the limiting expressions (65) and (66) coincide with the dispersionless expressions (69). Then the self-similar solutions of the Whitham equations (60) are given by

$$(a) \quad v^{-1}_3(\lambda^L, \lambda^R, \lambda_3(\tau), \lambda^L_2) = \tau; \quad (84)$$

or

$$(b) \quad v^{-1}_2(\lambda^R, \lambda_2(\tau), \lambda^L, \lambda^L_2) = \tau; \quad (84)$$

which define the dependence of the Riemann invariants (modulation parameters) $\lambda_3$ or $\lambda_2$ on $\tau$ in implicit form.
The edges of the DSW propagate with velocities

\begin{align*}
(a) \quad s_1^{-1} &= \frac{1}{\alpha} - \frac{(\lambda^L_0)^2 + 2(\lambda^R_0)^2 + (\lambda^L_0)^2}{\alpha}, \\
&= \frac{1}{\alpha} - 4(\lambda^L_0)^2 - \frac{(\lambda^R_0)^2 - (\lambda^R_0)^2}{\alpha}, \\
&= \frac{1}{\alpha} - 4(\lambda^L_0)^2 - \frac{(\lambda^L_0)^2 - (\lambda^L_0)^2}{\alpha}, \\
(b) \quad s_2^{-1} &= \frac{1}{\alpha} - \frac{(\lambda^R_0)^2}{\alpha} + 2(\lambda^L_0)^2 - \frac{2(\lambda^L_0)^2}{\alpha}, \\
&= \frac{1}{\alpha} - (\lambda^R_0)^2 + 2(\lambda^L_0)^2 - (\lambda^R_0)^2.
\end{align*}

\begin{equation}
(85)
\end{equation}

It should be stressed that each \( \lambda \)-diagram in Fig. 6 corresponds to two different dispersive shock waves, because we have two mappings \([37]\) and \([38]\) from Riemann invariants to the physical parameters. This point will be important in classification of the wave structures evolving from the initial discontinuities. For example, let us consider the case \( b \) \( \lambda^L_0 = \lambda^R_0, \lambda^L_\alpha > \lambda^R_\alpha \) (the diagram Fig. 6b). In Fig. 7 the parabolas of constant Riemann invariants in the \((u,I)\)-plane are shown. We see that there are two paths \( L_1 \to R_1 \) and \( L_2 \to R_2 \) which connect pairs of points with the same values of both Riemann invariants. The points \( L_1 \) and \( L_2 \) correspond to the left boundary condition with the Riemann invariants equal to \( \lambda^L_0 \) and \( \lambda^L_\alpha \), and the points \( R_1 \) and \( R_2 \) correspond to the right boundary condition with the Riemann invariants equal to \( \lambda^R_0 \) and \( \lambda^R_\alpha \) \( (\lambda^L_0 = \lambda^R_\alpha) \). These paths are described correspondingly by the maps \([37]\) or \([38]\) of Riemann invariants to the parameters \( \nu \) that parameterize the periodic solutions. Substitution of the solutions \([34]\) for the Riemann invariants in the formulae \([37]\) and \([38]\) with the use of \([40]\) yields the \( \tau \)-dependence of the parameters in the modulated periodic solutions resulting in the DSW structure. In Fig. 8 we compare the numerical and analytical approximate solution for the DSW with the constant Riemann invariant \( \lambda_\alpha \). For the path \( L_1 \to R_1 \) \( \nu \) at \( \nu = \frac{1}{\alpha} \) and \( \nu = \frac{2}{\alpha} \) and, consequently, \( \lambda^L_0 = \lambda^R_0 \), \( \lambda^L_\alpha = \lambda^R_\alpha \). In this case we obtain a new type of structures: contact dispersive shock wave. For this situation, the parabolas corresponding to \( \nu = \frac{1}{\alpha} \) and \( \nu = \frac{2}{\alpha} \) in Fig. 7 coincide with each other and a cnoidal DSWs disappear. Instead, there appears the path connecting the identical left and right states labeled by the crossing points of two parabolas as is shown in Fig. 9. Such waves can arise only if the boundary points are located on the opposite sides of the line \( u = 1/\alpha \), i.e. in different regions of monotonicity.

In this situation shown in Fig. 10 the invariants \( \lambda_1 \) and \( \lambda_2 \) are constant within the shock region and they match the boundary conditions \( \lambda_1 = \lambda^L_\alpha = \lambda^R_\alpha, \lambda_2 = \lambda^L_\alpha = \lambda^R_\alpha \), whereas the two other Riemann invariants remain equal to each other along the shock \( (\lambda_3 = \lambda_4) \) and satisfy the same Whitham equation \( v_3^{-1}(\lambda^L_\nu, \lambda^L_\alpha, \lambda_4(\tau), \lambda_4(\tau)) = v_3^{-1}(\lambda^L_\nu, \lambda^L_\alpha, \lambda_4(\tau), \lambda_4(\tau)) = \tau \). Thus we obtain

\begin{equation}
\lambda_1 = \lambda^L_\alpha = \lambda^R_\alpha, \quad \lambda_2 = \lambda^L_\alpha = \lambda^R_\alpha, \quad \frac{1}{v_4} = \frac{1}{\alpha} - 4\lambda_\nu^2 - \frac{(\lambda^L_\alpha)^2 - (\lambda^L_\nu)^2}{2\lambda_\alpha^2} = \tau,
\end{equation}

where the last formula determines the dependence of \( \lambda_4 \) on the opening angle \( \nu \).
FIG. 9: Example of path for a contact DSW ($r^L_+ = r^R_+$ and $r^L_+ = r^R_-$). The $r_-$-parabola crosses the vertical $u = 1/\alpha$ separating the regions of monotonicity. Two directions $P_1 \leftrightarrow P_2$ are described by the mappings (37) and (38). Corresponding $\lambda$-diagrams are shown in Fig. 11 and the wave structures are shown in Fig. 11.

on $\tau$, which can be presented in the explicit form

$$\lambda_1^2(\tau) = \frac{1}{8} \left\{ 2 (\lambda^R_-)^2 + (\lambda^R_+)^2 + \frac{1}{\alpha} - \tau \right. \\
- \left[ 2 ((\lambda^R_-)^2 + (\lambda^R_+)^2) - \frac{1}{\alpha} + \tau \right]^2 + 8 ((\lambda^R_-)^2 - (\lambda^R_+)^2)^2 \right\}^{1/2}, \quad (87)$$

Here $\tau$ varies within the interval $s_1^{-1} \leq \tau \leq s_1^{-1}$ with

$$s_1^{-1} = \frac{1}{\alpha} - (\lambda^R_-)^2 - 3(\lambda^R_+)^2,$$

$$s_1^{-1} = \frac{1}{\alpha} - \frac{(\lambda^R_-)^2 - (\lambda^R_+)^2)^2}{(\lambda^R_-)^2 + (\lambda^R_+)^2}. \quad (88)$$

The period in this case is given by the formula

$$T = \frac{\pi}{2 \sqrt{(\lambda^R_-)^2 - (\lambda^L_-)^2(\lambda^R_+)^2 - (\lambda^L_+)^2}}. \quad (89)$$

As in the case of cnoidal DSWs, the single contact DSW diagram Fig. 12 corresponds to two structures due to different mappings (37) or (38). Example of such a structure is shown in Fig. 11 where in the first case (a), the path $P_1 \rightarrow P_2$ is realized and formulae (38) are used, and in the second case (b) the opposite path $P_1 \leftarrow P_2$ takes place with corresponding formulae (37).

D. Combined shocks

Now we turn to the last elementary structure connecting two plateau states and therefore it can be symbolized by a single path between two points in the $(u, I)$-plane. This type of paths is illustrated in Fig. 12 and obviously it is a generalization of the preceding structure. In this case the boundary points are also located in different monotonicity regions. One of the Riemann invariants still remains constant ($r^L_+ = r^R_-$ or $\lambda^L_+ = \lambda^R_-$), however the boundary values of the other Riemann invariant are different: we have $r^L_+ < r^R_-$ in case (a) and $r^L_+ > r^R_-$ in case (b). The corresponding $\lambda$-diagrams are shown in Fig. 13.

In the case corresponding to Fig. 13(a) the contact dispersive shock wave is attached at its soliton edge to the rarefaction wave which matches at its left edge with the left boundary plateau. The velocities of the characteristic points identified in Fig. 13(a) are expressed in terms of

FIG. 10: Diagram represents evolution of the Riemann invariants as functions of $\tau = t/x$ in the contact DSW solution of the Whitham equations: $r^L_+ = r^R_-$ ($\lambda^L_+ = \lambda^R_-$), $r^L_+ = r^R_+$ ($\lambda^L_+ = \lambda^R_-$).

FIG. 11: Comparison of analytical and numerical solutions of the mNLS equation (1) with contact DSW for two possible choices of directions $P_1 \leftrightarrow P_2$ and corresponding mappings (37) and (38). Here $\alpha = 1$ and (a) $I^L = 0.5$, $u^L = -0.5$, $r^L = 2$, $u^L = -0.5$; (b) $I^L = 2$, $u^L = 2.5$, $I^R = 0.5$, $u^R = -0.5$. The Riemann invariants are equal to $r^L_+ = r^R_+ = 0.25$ and $r^L_+ = r^R_+ = -1.75$. Thin line corresponds to the analytic solution, thick gray line to numerics, dashed lines show analytical envelopes.
the boundary Riemann invariants by the formulae

\[
\begin{align*}
 s_{1,-}^{-1} &= \frac{1}{\alpha} - \frac{(\lambda_L^L)^2 + 3(\lambda_L^L)^2}{\lambda_L^L}, \\
 s_{2,-}^{-1} &= \frac{1}{\alpha} - \frac{3(\lambda_R^R)^2 + (\lambda_R^R)^2}{\lambda_R^R}, \\
 s_{+}^{-1} &= \frac{1}{\alpha} - \frac{(\lambda_R^R)^2 - (\lambda_L^L)^2}{(\lambda_R^R)^2 + (\lambda_L^L)^2}. 
\end{align*}
\]

The resulting composite wave structure is shown in Fig. 14(a) (thin black line) where it is compared with the numerical solution of the mNLS equation (thick gray (red) line).

In the case corresponding to Fig. 13(b) the trigonometric DSW is attached at its left edge to the cnoidal dispersion shock wave. At the left soliton edge the cnoidal wave matches with the left boundary plateau. The velocities of the characteristic points identified in Fig. 13(b)

FIG. 12: Paths in the \((u, I)\)-plane associated with two types of combined shocks. The left and right boundary conditions correspond to points \(L\) and \(R\) respectively; they lie on the parabolas along which the dispersionless Riemann invariant \(r_\pm = r_\pm^L = r_\pm^R (\lambda_L = \lambda_L^L = \lambda_L^R)\) is constant. One has \(r_\pm^L < r_\pm^R (\lambda_L^L < \lambda_L^R)\) in case (a) and \(r_\pm^L > r_\pm^R (\lambda_L^L > \lambda_L^R)\) in case (b).

FIG. 13: Diagram representing the evolution of the Riemann invariants as functions of \(\tau = t/x\) for combined shocks corresponding to the paths in the \((u, I)\)-plane shown in Fig. 12. Thin line corresponds to the analytic solution, thick gray line to numerics, dashed lines show analytical envelopes.

FIG. 14: Comparison of analytical and numerical solutions of the mNLS equation (1) for combined shocks corresponding to the paths in the \((u, I)\)-plane (Fig. 12) and to the diagrams of Riemann invariants, Fig. 13. Here \(\alpha = 1\) and (a) \(I^L = 0.5, u^L = 0.2, I^R = 1.768, u^R = 2.5\); (b) \(I^L = 0.94, u^L = -0.46, I^R = 2.12, u^R = 2\).

are given by

\[
\begin{align*}
 s_{1,-}^{-1} &= \frac{1}{\alpha} - \frac{(\lambda_L^-)^2 + 2(\lambda_R^L)^2 + (\lambda_L^L)^2}{\lambda_L^L}, \\
 s_{2,-}^{-1} &= \frac{1}{\alpha} - \frac{4(\lambda_R^L)^2 - (\lambda_R^L)^2}{(\lambda_R^L)^2 + (\lambda_R^L)^2 - 2(\lambda_L^L)^2}, \\
 s_{+}^{-1} &= \frac{1}{\alpha} - \frac{(\lambda_R^R)^2 - (\lambda_L^L)^2}{(\lambda_R^R)^2 + (\lambda_L^L)^2}. 
\end{align*}
\]

The resulting composite wave structures are shown in Fig. 14(b) (blue lines) where they are compared with the numerical solution of the mNLS equation (red lines).

Now, after description of all elementary wave structures arising in evolution of discontinuities in the mNLS equation theory, we are in position to formulate the main principles of classification of all possible wave structures.

VI. CLASSIFICATION OF WAVE PATTERNS

Classification of possible structures is very simple in the KdV equation case when any discontinuity evolves into either rarefaction wave, or cnoidal DSW [20]. It becomes more complicated in the NLS equation case [27, 28] and similar situations as, e.g., for the Kaup-Boussinesq equation [27, 28], where the list consists of eight or ten structures which can be seen after simple enough inspection of available possibilities and studied one by one. However, the situation changes drastically when we turn to non-convex dispersive hydrodynamics: even in the case of unidirectional Gardner (mKdV) equation we get eight different patterns (instead of two in KdV case) due to appearance of new elements (kinks or trigonometric and combined dispersive shocks), but these patterns can be labeled by two parameters only and therefore these possibilities can be charted on a two-dimensional diagram. In our present case the initial discontinuity (67) is parameterized by four parameters \(u^L, I^L, u^R, I^R\), hence the number of possible wave patterns considerably increases and it is impossible to present them in a two-dimensional chart. Therefore it seems more effective to
formulate the principles according to which one can predict the wave pattern evolving from a discontinuity with given parameters. Similar method is used [29] in classification of wave patterns evolving from initial discontinuities according to the Landau-Lifshitz equation for easy-plane magnetics or polarization waves in two-component Bose-Einstein condensate.

It is convenient to begin with the consideration of the classification problem from the case when both boundary points lie on one side of the line $u = 1/\alpha$ separating two monotonicity regions in the $(u, I)$-plane. At first we shall consider situation when the boundary points lie in the left monotonicity region. We show in Fig. 15 the two parabolas corresponding to the constant dispersionless Riemann invariants $r_L^\pm$ related with the left boundary state. Evidently, they cross at some point $L(u_L^L, I_L^L)$ where $r_L^- = r_L^+$. These two parabolas cut the left monotonicity region into six domains labeled by the symbols $A, B, \ldots, F$. Depending on the domain, in which the point $R$ with coordinates $(u_R, I_R)$, representing the right boundary condition, is located, one gets one of the six following possible orderings of the left and right Riemann invariants:

- **A**: $\lambda_R^L < \lambda_R^R < \lambda_L^L < \lambda_L^R$,
- **B**: $\lambda_R^L < \lambda_L^L < \lambda_R^R < \lambda_L^R$,
- **C**: $\lambda_L^L < \lambda_R^R < \lambda_L^R < \lambda_L^L$,
- **D**: $\lambda_R^L < \lambda_L^R < \lambda_L^L < \lambda_R^R$,
- **E**: $\lambda_L^L < \lambda_R^R < \lambda_L^L < \lambda_R^R$,
- **F**: $\lambda_L^L < \lambda_R^L < \lambda_L^R < \lambda_R^R$.

(92)

All these six domains and corresponding orderings yield six possible wave structures evolving from initial discontinuities. Let us consider briefly each of them.

- **In case (A)** two rarefaction waves are combined into a single wave structure where they are separated by an empty region. This means that two light fluids flow in opposite directions with velocities so large that the rarefaction waves are not able to fill in an empty region between them. Evolution of Riemann invariants and sketch of wave structure are shown in Fig. 16(A).
- **In case (B)** two rarefaction waves are connected by a plateau whose parameters are determined by the dispersionless Riemann invariants $r_P^\pm$ equal to $r_P^- = r_P^R$ and $r_P^+ = r_P^L$. Here rarefaction waves are able now to provide enough flux of the light fluid to create a plateau in the region between them (see Fig. 16(B)).
- **In case (C)** we obtain a dispersive shock wave on the left, a rarefaction wave on the right and a plateau in between are produced (see Fig. 16(C)).
- **In case (D)** we get the same situation as in the case
(C), but now the dispersive shock wave and rarefaction wave exchange their places (see Fig. [16](D)).

- In case (E) two DSWs are produced with a plateau between them. Here we have a collision of two light fluids (see Fig. [16](E)).
- In case (F) the plateau observed in the case (E) disappears. It is replaced by a nonlinear wave which can be presented as a non-modulated cnoidal wave (see Fig. [16](F)).

The possible structures for this part of the \((u, I)\)-plane coincide qualitatively with the patterns found in similar classification problem for the nonlinear Schrödinger equation [8]. It is clear that as \(\alpha\) tends to zero, the mNLS equation transforms to the nonlinear Schrödinger equation. Then the line \(u = 1/\alpha\) goes to infinity and therefore there remains only the left monotonicity region.

Now we turn to consideration of the classification problem for the case when both boundary points lie to the right of the line \(u = 1/\alpha\). This situation is shown in Fig. [16](B). We see that the parabolas divide again this right monotonicity region into six domains. For this case the Riemann invariants can have the same orderings (92) as in the previous case. Depending on the location of the right boundary point in a certain domain, the corresponding wave structure will be formed. For all cases these structures coincide with those for the previous case.

At last, we have to investigate the situation when the boundary points lie on different sides of the line \(u = 1/\alpha\), that is in different monotonicity regions. As we have seen in the previous section, in this case new complex structures consisting of contact dispersive shock waves or combined shocks appear. Since the total number of possible wave patterns is very large, we shall not list all of them here but rather illustrate the general principles of their classification.

For given boundary parameters, we can construct the parabolas corresponding to constant Riemann invariants \(r^{L,R}_i\): each left or right pair of these parabolas crosses at the point \(L\) or \(R\) representing the left or right boundary state’s plateau. Our task is to construct the path joining these two points, then this path will represent the arising wave structure. We already know the answer for the case when the left and right points lie on the same parabola, see, e.g., Fig. [12] If this is not the case and the right point \(R\) lies, say, below the parabola \(r^L = \text{const}\), see Fig. [18](a), then we can reach \(R\) by means of more complicated path consisting of two arcs of parabolas joined at the point \(P\). Evidently, this point \(P\) represents the plateau between two waves represented by the arcs. At the same time, each arc corresponds to a wave structure discussed in the preceding section. In fact, there are two paths with a single intersection point that join the left and right boundary points, and one can easily see another path made of dashed lines in Fig. [18](a). We choose the physically relevant path by imposing the condition that velocities of edges of all regions must increase from left to right. Having constructed a path from the left boundary point to the right one, it is easy to draw the corresponding \(\lambda\)-diagram. To construct the wave structure, we use the formulae connecting the zeros \(\nu_i\) of the resolvent with the Riemann invariants \(\lambda_i\) and expressions for the solutions parameterized by \(\nu_i\). This solves the problem of construction of the wave structure evolving from the initial discontinuity with given boundary conditions.

For example, let us consider the case \(I^L = I^R = 1\), \(u^L = -u^R = -1.5\) which corresponds to Fig. [18](a). We see that the branch of the parabola with \(r^L = \text{const}\) crosses the line \(u = 1/\alpha\). Taking into account that the left wave corresponds to the continuation of \(r^L = \text{const}\) and the right wave to the continuation of \(r^R = \text{const}\), we arrive at the diagram shown in Fig. [18](b). Consequently, at the left edge we have a rarefaction wave and at the right edge the combination of a trigonometric shock with a rarefaction wave. Between these waves we get a plateau characterized by the Riemann invariants \(r^P = r^R\) and \(r^P = r^L\). This plateau is represented by a single point \(P\) in Fig. [18](a). The wave structure can be obtained by substitution of solution of the Whitham equation into expressions for wave oscillations. As we see, our analytical

FIG. 17: Domains in the \((u, I)\)-plane on the right side of the line \(u = 1/\alpha\) corresponding to different structures.

FIG. 18: (a) The branches of the parabola corresponding to the path between the left and right points. (b) The corresponding diagram for the Riemann invariants.
results agree very well with numerical calculations shown in Fig. 19.

Another instructive example describes the situation when we get “expansion into vacuum” wave pattern with formation of a contact shock wave. Such a situation is impossible in the NLS theory where expansion into vacuum leads always to formation of a rarefaction wave. However, the mNLS equation differs drastically in this respect from the NLS equation case. In the problem of evolution of the initial discontinuity, if velocity and intensity on the left boundary are equal to zero ($u^L = 0$, $I^L = 0$), then the Riemann invariants also vanish at this boundary ($r^L = r^L = 0$). In spite of that, they can form a contact shock in transition to the right boundary if the dispersionless Riemann invariants on the right boundary are also equal zero ($r^R = r^R = 0$). From equations we find that there are two possibilities for that: either $u^R = 0$, $I^R = 0$ or $u^R = 2$, $I^R = 1$. The first trivial option refers to the absence of light in the waveguide and therefore it is not of any interest. The second option corresponds precisely to the case of the formation of the contact shock wave. Fig. 20 shows such a structure with comparison of the numerical solution with the analytical one. This comparison shows that the analytic Whitham theory agrees with numerics very well. The corresponding diagram of the Riemann invariants qualitatively coincides with the diagram in Fig. 10 with one difference: the Riemann invariant $r_-$ coincides with $r_+$. Therefore the dispersive Riemann invariants $\lambda_1$ and $\lambda_2$ also coincide with each other.

VII. CONCLUSION

In this paper, we have developed the Whitham method of modulations for propagation of long enough pulses in fibers with account of steepening effects. The theory is applied to the problem of classification of wave patterns evolving from given discontinuity in the initial data. Because of non-convex behavior of nonlinear velocities in this case, previously known methods of solving such kind of problems should be modified with inclusion of new types of elementary wave structures, such as ‘contact dispersive shocks’. Evolution of these structures is described by the degenerate limits of the Whitham modulation equations. In the resulting scheme, one solution of the Whitham equations corresponds to two different wave patterns, and this correspondence is provided by a two-valued mapping of Riemann invariants to physical modulation parameters. In this respect, situation is similar to that of modified KdV case already discussed in Ref. [21], but here the system with two-directional propagation of waves is considered, and one can compare this with transition from the KdV equation case [20] to NLS equation case [8]. The resulting set of possible wave patterns is very rich and we have developed a graphical method for determining which structure will evolve from given initial data. The method is quite flexible and it was also applied to another system with non-convex hydrodynamics—Landau-Lifshitz equation for dynamics of magnetics with uniaxial easy-plane anisotropy [29].

In principle, one may hope that the results found here can be observed experimentally in systems similar to that used in the recent experiment [6]. However, one should keep in mind that in standard fibers the Raman effect is typically much stronger than the self-steepening effect (see, e.g., Ref. [10]). Fortunately, the manifestations of these two effects are quite different and therefore they can be identified separately. As was shown in this paper, the main new effect of the self-steepening term is
formation of combined shocks caused by the non-convex properties of the nonlinearity, whereas the Raman effect leads to formation of stationary shocks with finite length (see, e.g., [30–33]). In the limit of long-time evolution, the combined action of both effects must lead to formation of combined stationary shocks different from shocks predicted by the theory which takes into account the Raman effect only. Qualitatively, these shocks must look similar to those described here. The quantitative theory of this new type of combined shocks can be developed in framework of the presented here approach, however this task is definitely beyond the present paper.

Another possibility of observation of predicted here effects is related with the use of photonic crystal waveguides which are free from the Raman scattering, as it was observed experimentally in Ref. [34], and the waveguides can be engineered in such a way that the self-steepening parameter is considerably increased [35].

Thus, the presented here theory, on one side, predicts some new phenomena which can be observed experimentally and, on the other side, it forms the basis for development of more complete theories which take into account other effects.

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