Continuity of Powerspaces Structures in Directed Spaces

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Abstract

Powerspaces of directed spaces play an important role in modeling the semantics of nondeterministic functional programming languages. The notions of upper, lower and convex powerspace of a directed space are defined by the way of free algebras[25]. In this paper, We study the continuity of power structures of directed spaces and show that the directed lower powerspaces, directed upper powerspaces and directed convex powerspaces of continuous spaces are continuous spaces.

Keywords: directed space; continuous-space; directed lower powerspace; directed upper powerspace; directed convex powerspace

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1. Introduction

Domain theory was first introduced by Dana Scott in the early 1970s, and the main purpose is to provide a mathematical tool for the semantics of functional programming languages, see[1, 2, 3, 20]. The most distinctive feature of domain theory is that it integrates order structures, topology structures and computer science[4, 6, 7, 15, 16, 17]. The main objects of domain theory are posets and domains, see[4, 5, 8]. In [13, 16], directed space is defined. It is easy to see that directed spaces are equivalent to monotone determined spaces, which is defined in [17]. DTop, the category of all directed spaces and continuous functions was shown to be cartesian closed category.

Powerdomain is one of the most important parts of domain theory. Its purpose is to provide a mathematical model for the semantics of nondeterministic functional programming languages. There are the Hoarer powerdomain, the Smyth powerdomain and the Plotkin powerdomain. Moreover, each power structure has a standard topological representation. In recent years, papers [26, 27, 28] have made a lot of generalizations on these power structures. In [25], the power structures of directed spaces are introduced by Xie, Chen and Kou. They defined the notions of directed lower, upper and convex powerspace of a directed space by the way of free algebras and showed that the directed lower, upper and convex powerspace over any directed space exist and give their concrete structures. Generally, the directed lower, upper and convex powerspace of directed space are different from the lower, upper and convex powerdomain of dcpo endowed with the Scott topology.

In [13], Xie and Kou introduced the approximation relation on T0 topological spaces through directed subsets and then defined continuous spaces. They showed that a directed space is continuous iff it is a c-space.

In this paper, we are going to study the continuity of power structures of directed spaces and show that the directed lower powerspaces, directed upper powerspaces and directed convex powerspace of continuous spaces are continuous spaces.

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2. Preliminaries

Now, we introduce the concepts needed in this paper. The readers can also consult [8, 11, 12, 14]. A nonempty set $L$ endowed with a partial order $\leq$ is called poset. A subset $D \subseteq L$ is called directed set if for any $x, y \in D$, there exists $d \in D$ such that $x, y \leq d$. A poset $L$ is called a directed complete poset (dcpo, for short) if any directed subset of $L$ has a sup in $L$. For any $x, y \in L$, we say that $x$ is way below $y$ (denoted by $x \ll y$) if for any directed set $D$ of $L$, $y \leq \lor D$ implies that there is some $d \in D$ with $x \leq d$. A poset $L$ is called continuous if for any $x \in L$, $\{a \in L : a \ll x\}$ is directed set and has $x$ as its supremum. For a subset $A$ of $L$, let $\uparrow A = \{x \in L : 3a \in A, a \leq x\}$, $\downarrow A = \{x \in L : 3a \in A, x \leq a\}$. We use $\uparrow a$ (resp. $\downarrow a$) instead of $\uparrow \{a\}$ (resp. $\downarrow \{a\}$) when $A = \{a\}$. $A$ is called an upper (resp. a lower) set if $A = \uparrow A$ (resp. $A = \downarrow A$).

Let $L$ be a poset and $U \subseteq L$. Then $U$ is called Scott open iff it satisfies: (1) $U = \uparrow U$; (2) For any directed sets $D \subseteq L$, $\lor D \subseteq U$ implies $D \cap U \neq \emptyset$. The collection of all Scott open subsets of $L$ is called the Scott topology of $L$ and denoted by $\sigma(L)$.

In this paper, topological spaces will always be supposed to be $T_0$ spaces. For a topological space $X$, its topology is denoted by $\tau$. The partial order $\leq$ defined on $X$ by $x \leq y \iff x \in \text{cl}_\tau\{y\}$ is called the specialization order [11, 18, 19], where $\text{cl}_\tau\{y\}$ is the closure of $\{y\}$. From now on, all order-theoretical statements about $T_0$ spaces, such as upper sets, lower sets, directed sets, and so on, always refer to the specialization order.

A net of a topological space is a map $\xi : J \to X$, where $J$ is a directed set. Usually, we denote a net by $(x_j)_{j \in J}$. Let $x \in X$, saying $(x_j)_{j \in J}$ converges to $x$, denoted by $(x_j)_{j \in J} \to x$, if $(x_j)_{j \in J}$ is eventually in every open neighborhood of $X$, that is, for any given open neighborhood $U$ of $x$, there exists $j_0 \in J$ such that for any $j \in J$, $j \geq j_0 \Rightarrow x_j \in U$.

Let $X$ be a $T_0$ topological space, then any directed subset of $X$ can be regarded as a net, and its index set is itself. We use $D \to x \to X$ to represent $D$ converges to $x$. Define notation $D(X) = \{(D, x) : x \in X, D \text{ is a directed subset of } X \text{ and } D \to x\}$. It is easy to verify that, for any $x, y \in X$, $x \leq y$ implies $\{y\}$ converges to $x$, and for directed set $D$, if $\exists d \in D, x \leq d$, then $D \to x$. Therefore, if $x \leq y$, then $\{(y), x\} \in D(X)$. Next, we give the concept of directed space. A subset $U$ of $X$ is called a directed open set if $\forall (D, x) \in D(X)$, $x \in U \Rightarrow D \cap U \neq \emptyset$. Denote all directed open sets of $X$ by $d(X)$. Obviously, every open set of $X$ is directed open, that is, $\tau \subseteq d(X)$.

Let $X$ be a $T_0$ topological space. $X$ is called directed space if every directed open set of $X$ is an open set, that is, $d(X) = \tau$ [13, 16].

Remark 2.1. [13, 16] Let $X$ be a $T_0$ topological space.

(1) The definition of directed space here is equivalent to the monotone determined space defined in [17].

(2) Every poset equipped with the Scott topology is a directed space [16, 18], besides, each Alexandroff space is a directed space. Thus, the directed space extends the concept of the Scott topology.

(3) If $U \in d(X)$, $U = \uparrow U$.

(4) $X$ equipped with $d(X)$ is a $T_0$ topological space such that $\leq_d = \leq$, where $\leq_d$ is the specialization order relative to $d(X)$.

(5) For a directed subset $D$ of $X$, $D \to x \iff D \to d(X) x$ for all $x \in X$, where $D \to d(X) x$ means that $D$ converges to $x$ with respect to the topology $d(X)$.

(6) For each $x \in X$, $\downarrow x$ is directed closed.

(7) In [22], it is showed that the subspaces of directed spaces are directed spaces.

Definition 2.2. [13, 19] Let $X$ be a directed space and $x, y \in X$. We say that $x$ is way below $y$, denoted by $x \ll y$, if for any directed subset $D$ of $X$, $D \to y$ implies $x \leq d$ for some $d \in D$. If $x \ll y$, then $x$ is called compact element of $X$.

For any directed space $X$ and $x \in X$, we denote $K(X) = \{x \in X : x \ll \downarrow x\}$, $\downarrow x = \{y \in X : y \ll x\}$, and $\uparrow x = \{y \in X : x \ll y\}$. The way below relation is a natural extension of $\ll_s$ on poset. Similarly, the notion of continuous space can be defined.

Definition 2.3. [13, 19] Let $X$ be a directed space. $X$ is called continuous if $\downarrow x$ is directed and $\downarrow x \to x$ for any $x \in X$. 

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A topological space $X$ is called a c-space, if it is $T_0$ space and every point $y \in X$ has a neighborhood basis of sets of the form $\uparrow x$. Many scholars have done a lot of researchs on c-space, and have given many meaningful conclusions about c-space. For example, a poset equipped with Scott topology is c-space; The concept of c-cone is given by combining c-space with topological cone in [20].

**Theorem 2.4.** [13, 19] Let $X$ be a directed space. Then the following three conditions are equivalent to each other.

1. $X$ is a continuous space;
2. $X$ is a c-space;
3. There exists a directed set $D \subseteq \downarrow x$ such that $D \to x$ for any $x \in X$.

**Theorem 2.5.** [13, 19] Let $X$ be a continuous space. Then we have the following statements.

1. For all $x, y \in X$, $x \ll y$ implies $x \ll z \ll y$ for some $z \in X$.
2. $\{\uparrow x : x \in X\}$ is a basis of $\tau$.
3. For all $x, y \in X$, the following are equivalent:
   i. $x \ll y$;
   ii. $y \in (\uparrow x)^\circ$;
   iii. For any net $(x_j)_{j \in J} \subseteq X$, $(x_j)_{j \in J} \to y$ implies $x \leq x_{j_0}$ for some $j_0 \in J$.

Let $X$ be a $T_0$ topological space and $\mathcal{P}^w(X)$ be the family of all nonempty finite subsets of $X$. Given any two subsets $G, H$ of $X$, we define $G \leq H$ if $H \subseteq \uparrow G$. A family of finite sets $\mathcal{F} \subseteq \mathcal{P}^w(X)$ is said to be directed if given $F_1, F_2$ in the family, there exists $F \in \mathcal{F}$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$. Let $\mathcal{F} \subseteq \mathcal{P}^w(X)$ be a directed family. We say that $\mathcal{F} \to x$ if for any open neighbourhood $U$ of $x$, there exists some $F \in \mathcal{F}$ such that $F \subseteq U$.

Let $X$ be a directed space and $G, H \subseteq X$. We say that $G$ approximates $H$, denoted by $G \ll H$, if for any directed subset $D$ of $X$, $D \to h$ for some $h \in H$ implies $D \cap \uparrow G \neq \emptyset$. We write $G \ll x$ for $G \ll \{x\}$. $G$ is said to be compact if $G \ll G$. For any $T_0$ topological space $X$ and $F \in \mathcal{P}^w(X)$, we denote $\uparrow F = \{x \in X : F \ll x\}$. A topological space $X$ is called quasicontinuous if it is a directed space such that for any $x \in X$, the family $\text{fin}(x) = \{F : F \in \mathcal{P}^w(X), F \ll x\}$ is a directed family and converges to $x$.

**Theorem 2.6.** [21] Let $X$ be a d-quasicontinuous space. The following statements hold.

1. Given any $H \in \mathcal{P}^w(X)$ and $y \in X$, $H \ll y$ implies $H \ll F \ll y$ for some finite subset $F \in \mathcal{P}^w(X)$.
2. Given any $F \in \mathcal{P}^w(X)$, $\uparrow F = (\uparrow F)^\circ$. Moreover, $\{\uparrow F : F \in \mathcal{P}^w(X)\}$ is a base of $\tau$.

A $T_0$ topological space $X$ is called locally hypercompact, if for any open subsets $U$ of $X$ and $x \in U$, there exists some $F \in \mathcal{P}^w(X)$ such that $x \in (\uparrow F)^\circ \subseteq \uparrow F \subseteq U$.

Continuous dcpos and quasicontinuous dcpos endowed with the Scott topology can be viewed as special continuous spaces and quasicontinuous spaces.

**Definition 2.7.** [13, 21] Suppose $X, Y$ are two $T_0$ topological spaces. A function $f : X \to Y$ is called directed continuous if it is monotone and preserves all limits of directed subset of $X$; that is, $(D, x) \in D(X) \Rightarrow (f(D), f(x)) \in D(Y)$.

Here are some characterizations of the directed continuous functions.

**Proposition 2.8.** [13, 21] Suppose $X, Y$ are two $T_0$ topological spaces. $f : X \to Y$ is a function between $X$ and $Y$. Then

1. $f$ is directed continuous if and only if $\forall U \in d(Y), f^{-1}(U) \in d(X)$.
2. If $X, Y$ are directed spaces, then $f$ is continuous if and only if it is directed continuous.

**Definition 2.9.** [23] Let $X$ be a directed space and $B \subseteq X$.

1. $B \subseteq X$ is called a base of $X$ if $B \cap \downarrow x$ is directed and converges to $x$ for any $x \in X$.
2. $X$ is called algebraic space if $K(X)$ is a base of $X$.

Obviously, if $B \subseteq X$ is a base of $X$ and $B \subseteq A \subseteq X$, then $A$ is a base of $X$.  


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Proposition 2.10. Let $X$ be a directed space, $x \in X$, $B \subseteq X$.

(1) $B$ is a base of $X$.

(2) $X$ is continuous space and for any $x \ll y$, there exists $b \in B$ such that $x \ll b \ll y$.

(3) For any $x \in X$, there exists directed set $D \subseteq B \cap \downarrow x$ such that $D \rightarrow x$.

Proof. (1)$\Rightarrow$(2) Let $B$ is a base of $X$ and $x \ll y$. Then $X$ is continuous space by Theorem 2.4. Since $x \ll y$, $x \ll z \ll y$ for some $z \in X$. Since $B \cap \downarrow y \rightarrow y$, there exists a $b \in B \cap \downarrow x$ such that $z \ll b$. Hence $x \ll b \ll y$.

(2)$\Rightarrow$(1) For any $m, n \in B \cap \downarrow x$, there exists $z \in X$ such that $m, n \leq z \ll x$. Then $z \ll b \ll x$ for some $b \in B$ and $m, n \leq b$. Thus $B \cap \downarrow x$ is a directed set. Let $x \in U \subseteq d(X)$, then $x \in \downarrow z \subseteq U$ for some $z \in X$. Then $x \in \downarrow z \subseteq U$ for some $z \in X$. Hence $B \cap \downarrow x \rightarrow x$, $B$ is a base of $X$.

(1)$\Rightarrow$(3) Obviously.

(3)$\Rightarrow$(1) Let $x \in U \subseteq d(X)$. For any $m, n \in B \cap \downarrow x$, Since $D \rightarrow x$, $m, n \leq d$ for some $d \in D \subseteq B \cap \downarrow x$. Then $B \cap \downarrow x$ is a directed set. Since $x \in \downarrow d \subseteq U$ for some $d \in D \subseteq B \cap \downarrow x$. Hence $B \cap \downarrow x \rightarrow x$. $B$ is a base of $X$. □

3. Continuity of Lower Powerspaces of directed spaces

As mentioned above, continuous space and quasicontinuous space are extended framework of domain theory. In [10, 25], Xie, Chen and Kou constructed the directed lower powerspace of the directed space, which is a free algebra generated by the inflationary operation of the directed space. In this section, we will show that the directed lower powerspace of the continuous space is a continuous space.

Definition 3.1. [25] Let $X$ be a directed space.

(1) A binary operation $\oplus : X \oplus X \rightarrow X$ on $X$ is called an inflationary operation if it is continuous and satisfies the following four conditions: $\forall x, y, z \in X$,

(a) $x \oplus x = x$;

(b) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

(c) $x \oplus y = y \oplus x$;

(d) $x \oplus y \geq x$.

(2) If $\oplus$ is an inflationary operation on $X$, then $(X, \oplus)$ is called a directed inflationary semilattice, that is, directed inflationary semilattices are those directed spaces with inflationary operations.

(3) Suppose $(X, \oplus), (Y, \sqcup)$ are two directed inflationary semilattices, $f : (X, \oplus) \rightarrow (Y, \sqcup)$ is called an inflationary homomorphism between $X$ and $Y$, if $f$ is continuous (respect to $d(X)$) and $f(x \oplus y) = f(x) \sqcup f(y)$ holds, $\forall x, y \in X$.

(4) A directed space $Z$ is called the directed lower powerspace of $X$ if and only if the following two conditions are satisfied:

(a) $Z$ is a directed inflationary semilattice, that is the sup operation $\vee$ on $Z$ exists and which is continuous;

(b) There is a continuous function $i : X \rightarrow Z$ satisfying: for arbitrary directed inflationary semilattice $(Y, \vee)$ and continuous function $f : X \rightarrow Y$, there exists an unique inflationary homomorphism $\overline{f} : (Z, \vee) \rightarrow (Y, \vee)$ such that $\overline{f} = f \circ i$.

Up to order isomorphism and topological homomorphism, the directed lower powerspace of a directed space is unique. Particularly, we denote the directed lower powerspace of each directed space $X$ by $P_L(X)[25]$.

Definition 3.2. [25] Let $X$ be a directed space.

(1) Denote $\mathcal{L}_{fin}(X) = \{ \downarrow F : F \in \mathcal{P}^w(X) \}$. Define an order $\leq_L$ on $\mathcal{L}_{fin}(X)$ as follows: $\downarrow F_1 \leq_L \downarrow F_2 \iff F_1 \subseteq \downarrow F_2$.

(2) Let $\mathcal{D} = \{ \downarrow F : F \in \mathcal{P}^w(X) \}$ be a directed set(respect to order $\leq_L$) and $\downarrow H \in \mathcal{L}_{fin}(X)$. Define a convergence notation: $\mathcal{D} \Rightarrow \downarrow L \iff$ there exists directed set $D_h \subseteq \bigcup \mathcal{D}$ such that $D_h \rightarrow h$ for any $h \in H$.

(3) A subset $\mathcal{H} \subseteq \mathcal{L}_{fin}(X)$ is called a $\Rightarrow_L$ convergence open set of $\mathcal{L}_{fin}(X)$ if and only if for any directed subset $\mathcal{D}$ of $\mathcal{L}_{fin}(X)$ and $\downarrow H \in \mathcal{L}_{fin}(X)$, $\mathcal{D} \Rightarrow_L \downarrow H \in \mathcal{H}$ implies $\mathcal{D} \cap \mathcal{H} \neq \emptyset$. Denote all $\Rightarrow_L$ convergence open set of $\mathcal{L}_{fin}(X)$ by $\mathcal{O}_{w_L}(\mathcal{L}_{fin}(X))$. 

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Proposition 3.3. [25] Let 𝑋 be a directed space, then

1. \((\mathcal{L}_{\text{fin}}(X), \mathcal{O}_{\leq}(\mathcal{L}_{\text{fin}}(X)))\) is a directed space, that is, \(\mathcal{O}_{\leq}(\mathcal{L}_{\text{fin}}(X)) = d(\mathcal{L}_{\text{fin}}(X))\) and the specialization order \(\leq\) on \((\mathcal{L}_{\text{fin}}(X), \mathcal{O}_{\leq}(\mathcal{L}_{\text{fin}}(X)))\) equals to \(\leq_{\leq}\);
2. \((\mathcal{L}_{\text{fin}}(X), \mathcal{O}_{\leq}(\mathcal{L}_{\text{fin}}(X)))\) respect to the set union operation \(\cup\) is a directed inflationary semilattice;
3. Suppose \(X\) is a directed space, then \((\mathcal{L}_{\text{fin}}(X), \mathcal{O}_{\leq}(\mathcal{L}_{\text{fin}}(X)))\) is the directed lower powerspace of \(X\), that is, endowed with topology \(\mathcal{O}_{\leq}(\mathcal{L}_{\text{fin}}(X)), (\mathcal{L}_{\text{fin}}(X), \cup) \cong P_L(X)\).

Let \(X\) be a directed space and \(↓ G, ↓ H \in \mathcal{L}_{\text{fin}}(X)\), then \(↓ G \ll \ll_{\leq} H \iff \) for any directed set \(D \subseteq \mathcal{L}_{\text{fin}}(X)\) (respect to order \(\leq\)), \(D \Rightarrow_{\leq} H, ↓ G \subseteq F\) for some \(F \in D\) by the definition of way below relation. In this paper, the way below relation with respect to \(\leq\) and \(\leq_{\leq}\) is denoted by \(\ll_{\leq}\). Let \(↓ G \ll_{\leq} H\). Since \(↓ x \subseteq G\) for any \(x \in H\), \(↓ x \ll_{\leq} H\). Obviously, \(↓ G \ll_{\leq} H \iff \) for any \(g \in G\).

The following proposition gives a sufficient condition of \(↓ G \ll_{\leq} H\).

Proposition 3.4. Let \(X\) be a directed space and \(↑ G, ↑ H \in \mathcal{L}_{\text{fin}}(X)\). If \(\forall g \in G\), there exists \(h \in H\) such that \(g \ll h\), then \(↓ G \ll_{\leq} H\).

Proof. Let \(D \subseteq \mathcal{L}_{\text{fin}}(X)\) be a directed set and \(D \Rightarrow_{\leq} H\). Let \(G = \{g_1, \cdots, g_n\}\). For any \(g_i\), there exists \(h_i \in H\) such that \(g_i \ll h_i\). Since \(D \Rightarrow_{\leq} H\), there exists directed set \(D_i \subseteq \bigcup D\) such that \(D_i \rightarrow h_i\). Hence \(D_i \cap \uparrow g_i \neq \emptyset\). Let \(f_i \in D_i \cap \uparrow g_i\). Since \(D_i \subseteq \bigcup D\), \(f_i \in F_i\) for some \(F_i \in D\). Then \(g_i \ll F_i\) and \(G \subseteq \bigcup_{i=1}^n F_i\). Since \(D\) is directed set, there exists \(\downarrow F \in D\) such that \(\bigcup_{i=1}^n F_i \subseteq F\). Hence \(↓ G \subseteq \downarrow F\) and \(↓ G \ll_{\leq} H\).

Corollary 3.5. Let \(X\) be a directed space, \(x, X \subseteq \mathcal{L}_{\text{fin}}(X)\) and \(↑ G \in \mathcal{L}_{\text{fin}}(X)\). Then \(↓ G \ll_{\leq} x \iff g \ll x\) for any \(g \in G\). In particular, \(↓ x \ll_{\leq} y \iff x \ll y\) for any \(x, y \in X\).

Proof. If \(g \ll x\) for any \(g \in G\), then \(↓ G \ll_{\leq} x\) by Proposition 3.4. Let \(↓ G \ll_{\leq} x\), \(D \subseteq \mathcal{L}_{\text{fin}}(X)\) be a directed set and \(D \Rightarrow x\). Let \(D = \{\downarrow d : d \in D\}\), then \(D\) is a directed set of \(\mathcal{L}_{\text{fin}}(X)\) and \(D \Rightarrow_{\leq} x\). Hence there exists \(a \in D\) such that \(↓ G \subseteq \downarrow d\). That is \(\uparrow g \cap D \neq \emptyset\) for any \(g \in G\), thus \(g \ll x\) for any \(g \in G\).

Example 3.6. (1) Let \(X\) be a poset with Scott topology, then \(X\) is a directed space. If \(x \ll y\) for some \(x, y \in X\), then \(↓ x \ll_{\leq} F\) for any \(y \in F\) and \(↓ F \in \mathcal{L}_{\text{fin}}(X)\) by Proposition 3.4. So, it’s possible that if \(↓ G \ll_{\leq} H\), then \(\exists h \in H\) such that \(g \ll h\) is not true for any \(g \in G\).

2. Let \(X = N \cup \{a, \infty\}\). Define an order on \(X\): \(\forall x, y \in X, x \leq y \equiv y = \infty\) or \(x, y \in N, x \leq y\). Then \((X, \sigma(X))\) is a directed space and \(\leq_{\sigma} \leq_{\leq}\), where \(\leq_{\sigma}\) is specialization order with respect to \(\sigma(X)\). Obviously, \(X\) is quasicontinuous space but is not continuous space. It is easy to see that \(a \ll_{\leq} a\), then \(\downarrow a \ll_{\leq} \{a, n\}\) for any \(n\) by Proposition 3.4.

In the other hand, if \(D \Rightarrow_{\leq} \{a, n\}\), then \(a \in \bigcup D\) or \(\infty \in \bigcup D\). So, \(a \in F\) or \(\infty \in F\) for some \(F \subseteq D\) and \(↓ a \subset F\), that is \(↓ a \ll_{\leq} \{a, n\}\).

Proposition 3.7. Let \(X\) be continuous space, then \(\mathcal{L}_{\text{fin}}(X)\) is continuous space.

Proof. Suppose \(X\) is continuous space, then \(D = \{\{\downarrow g : g \ll h\} : \exists h \in H\}\), then \(D\) is directed family and \(\subseteq \{↓ G : G \ll_{\leq} H\}\) by Proposition 3.4. For any \(h \in H\) and \(x \in X\), \(\downarrow x \in D\) and \(\downarrow h \subseteq \bigcup D\). Since \(X\) is continuous space, \(\downarrow h \rightarrow h\). Let \(D_h = \downarrow h\), then \(D_h \Rightarrow_{\leq} H\) by Definition 3.2. Thus \(\mathcal{L}_{\text{fin}}(X)\) is continuous space by Theorem 2.4.

Proposition 3.8. Let \(X\) be a continuous space and \(B \subseteq X\) be a base of \(X\), then (1) \(B = \{↓ F : F \in \mathcal{P}^w(X), F \subseteq B\}\) is a base of \(\mathcal{L}_{\text{fin}}(X)\); (2) If \(X\) has a countably base, then so is \(\mathcal{L}_{\text{fin}}(X)\); (3) If \(X\) is a algebraic space, then so is \(\mathcal{L}_{\text{fin}}(X)\).

Proof. (1) Let \(B \subseteq X\) be a base of \(X\) and \(B = \{↓ F : F \in \mathcal{P}^w(X), F \subseteq B\}\). For any \(\downarrow h \in \mathcal{L}_{\text{fin}}(X)\), let \(D^* = B \cap \{\{\downarrow b : g \ll h\} : \exists h \in H\}\), \(\{\downarrow b : b \ll h\} \subseteq \bigcup D^*\). Then \(\downarrow h \cap B \subseteq \bigcup D^*\). Since \(B \subseteq X\) is a base of \(X\), \(\downarrow h \cap B \rightarrow h\). Let \(D_h = \downarrow h \cap B\), then \(D_h \Rightarrow_{\leq} H\) by Definition 3.2. Thus \(B = \{↓ F : F \in \mathcal{P}^w(X), F \subseteq B\}\) is a base of \(\mathcal{L}_{\text{fin}}(X)\) by Proposition 2.10.
Define a convergence notation $\mathcal{L}_{\text{fin}}(X)$ for $X$ a directed space is unique. Particularly, we denote the directed upper powerspace of each directed space.

**Proposition 4.3.** By (1), $\mathcal{B} = \{ \downarrow F : F \in \mathcal{P}_{\text{up}}(X), F \subseteq B \}$, then $\mathcal{B}$ is a countably base of $\mathcal{L}_{\text{fin}}(X)$.

**Definition 4.1.** [25] Let $X$ be a directed space.

1. A binary operation $\oplus : X \oplus X \to X$ on $X$ is called an deflationary operation if it is continuous and satisfies the following four conditions: $\forall x, y, z \in X$,
   - (a) $x \oplus x = x$;
   - (b) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
   - (c) $x \oplus y = y \oplus x$;
   - (d) $x \oplus y \leq x$.

2. If $\oplus$ is a deflationary operation on $X$, then $(X, \oplus)$ is called a directed deflationary semilattice, that is, directed deflationary semilattices are those directed spaces with deflationary operations.

3. Suppose $(X, \oplus)$, $(Y, \sqcup)$ are two directed deflationary semilattices, $f : (X, \oplus) \to (Y, \sqcup)$ is called an deflationary homomorphism between $X$ and $Y$, if $f$ is continuous and $f(x \oplus y) = f(x) \sqcup f(y)$ holds, $\forall x, y \in X$.

4. A directed space $Z$ is called the directed upper powerspace of $X$ if and only if the following two conditions are satisfied:
   - (a) $Z$ is a directed deflationary semilattice, that is the meet operation $\wedge$ on $Z$ exists and which is continuous.
   - (b) There is a continuous function $i : X \to Z$ satisfying: for an arbitrary directed deflationary semilattice $(Y, \vee)$ and continuous function $f : X \to Y$, there exists an unique deflationary homomorphism $\overline{f} : (Z, \vee) \to (Y, \vee)$ such that $f = \overline{f} \circ i$.

To up of order isomorphism and topological homomorphism, the directed lower powerspace of a directed space is unique. Particularly, we denote the directed upper powerspace of each directed space $X$ by $P_U(X)$ [25].

**Definition 4.2.** [25] Let $X$ be a directed space.

1. Denote $Q_{\text{fin}}(X) = \{ \uparrow F : F \in \mathcal{P}_{\text{up}}(X) \}$. Define an order $\leq_{\mathcal{Q}}$ on $Q_{\text{fin}}(X)$: $\uparrow F_1 \leq_{\mathcal{Q}} \uparrow F_2 \iff \uparrow F_2 \subseteq \uparrow F_1$.
2. Let $\mathcal{F} = \{ \uparrow F : F \in \mathcal{P}_{\text{up}}(X) \} \subseteq Q_{\text{fin}}(X)$ be a directed set (respect to order $\leq_{\mathcal{Q}}$) and $\uparrow H \subseteq Q_{\text{fin}}(X)$. Define a convergence notation $\mathcal{F} \Rightarrow_{\mathcal{Q}} \uparrow H$ for there exists finite directed sets $D_1, \ldots, D_n \subseteq X$ such that
   - 1. $H \cap \{ x : D_i \to x \} \neq \emptyset$ for any $i = 1, \ldots, n$;
   - 2. $H \subseteq \{ x : D_i \to x, i = 1, \ldots, n \}$;
   - 3. $\forall (d_1, \ldots, d_n) \in \prod^n_i D_i$, there exists some $\uparrow F \in \mathcal{F}$ such that $\uparrow F \subseteq \uparrow H$.

3. A subset $\mathcal{U} \subseteq Q_{\text{fin}}(X)$ is called a $\Rightarrow_{\mathcal{Q}}$ convergence open set of $Q_{\text{fin}}(X)$ if and only if for any directed subset $\mathcal{F}$ of $Q_{\text{fin}}(X)$ and $\uparrow H \subseteq Q_{\text{fin}}(X)$, $\mathcal{F} \Rightarrow_{\mathcal{Q}} \uparrow H \in \mathcal{U}$ implies $\mathcal{F} \cap \mathcal{U} \neq \emptyset$. Denote all $\Rightarrow_{\mathcal{Q}}$ convergence open set of $Q_{\text{fin}}(X)$ by $O_{\Rightarrow_{\mathcal{Q}}}(Q_{\text{fin}}(X))$.

**Proposition 4.3.** [25] Let $X$ be a directed space, then

1. $(Q_{\text{fin}}(X), O_{\Rightarrow_{\mathcal{Q}}}(Q_{\text{fin}}(X)))$ is a directed space, that is, $O_{\Rightarrow_{\mathcal{Q}}}(Q_{\text{fin}}(X)) = d(Q_{\text{fin}}(X))$ and the specialization order $\leq$ of $O_{\Rightarrow_{\mathcal{Q}}}(Q_{\text{fin}}(X))$ equals to $\leq_{\mathcal{Q}}$;
(2) \((Qfin(X), \mathcal{O}_{\mathcal{Q}}(Qfin(X)))\) respect to the set union operation \(\cup\) is a directed deflationary semilattice;

(3) Suppose \(X\) is a directed space, then \((Qfin(X), \mathcal{O}_{\mathcal{Q}}(Qfin(X)))\) is the upper powerspace of \(X\), that is, endowed with topology \(\mathcal{O}_{\mathcal{Q}}(Qfin(X))\), \((Qfin(X), \cup) \cong \mathbb{P}_U(X)\).

Let \(X\) be a directed space and \(\uparrow G, \uparrow H \in Qfin(X)\), then \(\uparrow G \ll \uparrow H \iff \forall F \Rightarrow \mathcal{Q} \uparrow H, \uparrow F \subseteq \uparrow G\) for some \(\uparrow F \in \mathcal{F}\) by the definition of way below. In this paper, the way below relation with respect to \(\leq\) and \(\Rightarrow\) is denoted by \(\ll_{\mathcal{Q}}\). Let \(\uparrow G \ll_{\mathcal{Q}} \uparrow H\). Since \(\uparrow H \leq \uparrow x\) for any \(x \in H\), \(\uparrow G \ll_{\mathcal{Q}} \uparrow x\).

The following lemma shows that the way below relation on the poset \(Qfin(X)\) agrees with the way-below relation defined for finite subsets of directed space.

**Proposition 4.4.** Let \(X\) be a directed space and \(\uparrow G, \uparrow H \in Qfin(X)\), then \(\uparrow G \ll_{\mathcal{Q}} \uparrow H\) iff \(G \ll H\).

**Proof.** Obviously Proposition 4.4.

**Proposition 4.6.** Let \(X\) be a algebraic space, \(F, H \in \mathcal{P}_w(X)\) and \(F \subseteq H\) for any \(\uparrow F = \uparrow H\). Then \(F \ll F\) iff \(f \ll f\) for any \(f \in F\).

**Proof.** For any \(f \ll f\) for any \(f \in F\). Let \(D \subseteq X\) be a directed set and \(D \rightarrow f \in F\), then \(D \cap \uparrow f \neq \emptyset\) by \(f \ll f\). Since \(\uparrow f \subseteq F, D \cap \uparrow f \neq \emptyset\). Hence \(F \ll F\).

Let \(F \ll F\) and \(f \ll f\) be not true for some \(f \in F\). Since \(X\) is algebraic space, \(\downarrow f \cap K(X) \rightarrow f; then \downarrow f \cap K(L) \cap \uparrow F \neq \emptyset\) by \(f \ll f\). So, there exists \(d \in \downarrow f \cap K(L) \cap \uparrow F\) such that \(a \leq d \ll d \ll f\) for some \(a \in F\). If \(a = f, f \ll f, a\) contradiction. If \(a \neq f\), then \(\uparrow F = \uparrow F\setminus \{f\}\) is not ture, a contradiction. Hence \(f \ll f\) for any \(f \in F\).

**Proposition 4.7.** Let \(X\) be a continuous space, \(F \subseteq Qfin(X)\) and \(H \in \mathcal{P}_w(X)\). Then \(F \Rightarrow_{\mathcal{Q}} H\) iff \(H \subseteq U\) implies there exists \(F \in \mathcal{F}\) such that \(F \subseteq F \subseteq U\) for any \(U \in d(X)\).

**Proof.** Let \(F \Rightarrow_{\mathcal{Q}} H\) and \(H \subseteq U \in d(X)\), then there exists finite directed sets \(D_1, \cdots, D_n \subseteq X\) such that
\[
\begin{align*}
1. &H \cap \{x : D_i \rightarrow x\} \neq \emptyset \text{ for any } i = 1, \cdots, n; \\
2. &H \subseteq \{x : D_i \rightarrow x, i = 1, \cdots, n\}; \\
3. &\forall(d_1, \cdots, d_n) \in \prod^n_i D_i, \text{ there exists some } F \in \mathcal{F} \text{ such that } \uparrow F \subseteq U^n \uparrow d_i. \\
\end{align*}
\]
Hence for any \(i, D_i \rightarrow h\) for some \(h \in H\). Since \(h \in U, D_i \cap U \neq \emptyset\). Let \(d_i \subseteq D_i \cap U\), then there exists some \(\uparrow F \in \mathcal{F}\) such that \(\uparrow F \subseteq U^n \uparrow d_i \subseteq U^n \uparrow d_i\), that is, \(F \subseteq F \subseteq U\).

If \(H \subseteq U\) implies there exists \(\uparrow F \in \mathcal{F}\) such that \(F \subseteq F \subseteq U\) for any \(U \in d(X)\). For any \(h \in H\), since \(X\) is continuous space, \(\downarrow h\) is directed set and converges to \(x\). Let \(H = \{h_1, \cdots, h_n\}\) and \(D_i = \downarrow h_i\). Obviously, \(H \cap \{x : D_i \rightarrow x\} \neq \emptyset\) for any \(i = 1, \cdots, n\) and \(H \subseteq \{x : D_i \rightarrow x, i = 1, \cdots, n\}\). \(\forall(d_1, \cdots, d_n) \in \prod^n_i D_i, d_i \ll h_i\). Since \(H \subseteq U^n \uparrow d_i \subseteq d(x)\), then there exists some \(F \in \mathcal{F}\) such that \(\uparrow F \subseteq U^n \uparrow d_i \subseteq U^n \uparrow d_i\). Thus \(F \Rightarrow_{\mathcal{Q}} H\) be Definition 4.2.

**Definition 4.8.** A directed space \(X\) is called \(\mathcal{Q}\)-quasicontinuous if the directed space \(Qfin(X)\) of nonempty finitely generated upper sets ordered by reverse inclusion \(\leq_{\mathcal{Q}}\) is a continuous space.
Proposition 4.9. Let $X$ be a $Q$-quasicontinuous and $\uparrow G, \uparrow H \in Q_{fin}(X)$, $\uparrow G \ll_Q \uparrow H$. Then $\uparrow G \ll_Q \uparrow F \ll_Q \uparrow H$ for some $\uparrow F \in Q_{fin}(X)$.

Proof. Obviously Proposition 4.4 and Definition 4.8.

Proposition 4.10. Let $X$ be a directed space, then (1) If $X$ is a $Q$-quasicontinuous space, then $X$ is quasicontinuous space; (2) If $X$ is a continuous space, then $X$ is $Q$-quasicontinuous space.

Proof. If $X$ is $Q$-quasicontinuous space, then $Q_{fin}(X)$ is a continuous space. Thus $\text{fin}_d(x) =$ \{ $F : \uparrow F \ll_Q \uparrow x$ \} is a directed family. Since and \{ $\uparrow F : \uparrow F \ll_Q \uparrow x$ \} $\forall x$, there exists $\uparrow F \ll_Q x$ such that $\uparrow F \subseteq U$ for any $U \in d(X)$ by Proposition 4.7. Hence $X$ is quasicontinuous space.

Corollary 4.11. Let $X$ be a directed space, then $X$ is a continuous space ⇒ $Q_{fin}(X)$ is continuous space ⇒ $X$ is quasicontinuous space.

Proof. Obviously by Proposition 4.10.

Proposition 4.12. Let $X$ be a continuous space and $B \subseteq X$ be a base of $X$, then (1) $B = \{ \uparrow F : F \in P^w(x), F \subseteq B \}$ is a base of $Q_{fin}(X)$; (2) If $X$ has a countably base, then so is $Q_{fin}(X)$; (2) If $X$ is an algebraic space, then so is $Q_{fin}(X)$.

Proof. (1) Let $B \subseteq X$ be a base of $X$ and $B = \{ \uparrow F : F \in P^w(x), F \subseteq B \}$. For any $H \subseteq U \in d(X)$ and $h \in H$, there exists $d_h \in U \cap B$ such that $d_h \ll h$. Then $\{ \uparrow d_h : h \in H \} \subseteq U \cap B$. Since $\{ d_h : h \in H \} \ll h$ for any $h \in H$, $\{ \uparrow d_h : h \in H \} \ll H$. Since $\{ \uparrow d_h : h \in H \} \subseteq U$, $F = \{ \uparrow F : F \in P^w(x), F \subseteq B, F \ll H \} \Rightarrow \ll_Q H$ by Proposition 4.7. Hence, $B = \{ \uparrow F : F \in P^w(x), F \subseteq B \}$ is a base of $Q_{fin}(X)$.

(2) Obviously, if $B \subseteq X$ is a countably base of $X$. Let $B = \{ \uparrow F : F \in P^w(x), F \subseteq B \}$, then $B$ is a countably base of $Q_{fin}(X)$.

(3) If $X$ is an algebraic space, then $K(X)$ is a base of $X$. Hence $B = \{ \uparrow F : F \ll_Q F, F \in P^w(x), F \subseteq K(X) \}$ a base of $Q_{fin}(X)$. Since $B \subseteq K(Q_{fin}(X))$, $Q_{fin}(X)$ is an algebraic space.

Definition 4.13. [25] (1) A $\lor\land$-directed space $X$ is a directed space that is both a $\lor$-directed space and $\land$-directed space that the distributive law $a \land (b \lor c) = (a \land b) \lor (a \land c), \forall a, b, c \in X$.

(2) A $\lor\land$-morphism between two $\lor\land$-directed spaces that is both a $\lor$-morphism and a $\land$-morphism.

Proposition 4.14. [25] (1) If $X$ is a $\land$-directed space, then $P_L(X)$ is a $\lor\land$-directed space, and $\eta_L : X \rightarrow P_L(X)$ is a $\lor$-morphism, here $\eta_L(x) = \{ x \} \subseteq X$.

(2) If $X$ is a $\lor$-directed space, then $P_U(X)$ is a $\lor\land$-directed space, and $\eta_L : X \rightarrow P_U(X)$ is a $\lor$-morphism, here $\eta_L(x) = \{ x \} \subseteq X$.

(3) For each directed space $X$, $P_L(P_U(X))$ and $P_U(P_L(X))$ are isomorphic by a $\lor\land$-morphism, which maps $\uparrow (\downarrow x)$ to $\downarrow (\uparrow x)$.

Corollary 4.15. For each continuous space $X$, $P_L(P_U(X))$ and $P_U(P_L(X))$ are continuous space and isomorphic by a $\lor\land$-morphism, which maps $\uparrow (\downarrow x)$ to $\downarrow (\uparrow x)$.

Proof. Obviously by Proposition 3.7, 4.10 and 4.13.
5. Continuity of Convex Powerspaces of directed spaces

The directed convex powerspace of the continuous space is a free algebra generated by the directed semilattice operation of the directed space. In this section, We show that the directed convex powerspace of the continuous space is a continuous space.

**Definition 5.1.** [25] Let $X$ be a directed space.

(1) A binary operation $+: X \times X \to X$ on $X$ is called a semilattice operation if it is continuous and satisfies the following three conditions: $\forall x, y, z \in X$,
- (a) $x + x = x$;
- (b) $x + y + z = x + (y + z)$;
- (c) $x + y = y + x$.

(2) If $+$ is a semilattice operation on $X$, then $(X, +)$ is called a directed semilattice, that is, directed semilattices are those directed spaces with semilattice operations.

(3) Suppose $(X, \oplus)$, $(Y, \uplus)$ are two directed semilattices, $f: (X, \oplus) \to (Y, \uplus)$ is called an semilattice homomorphism between $X$ and $Y$, if $f$ is continuous and $f(x \oplus y) = f(x) \uplus f(y)$ holds, $\forall x, y \in X$.

(4) A directed space $Z$ is called the directed convex powerspace of $X$ if and only if the following two conditions are satisfied:
- (a) $Z$ is a directed semilattice, that is the semilattice operation $+$ on $Z$ exists and which is continuous;
- (b) There is a continuous function $i: X \to Z$ satisfying: for an arbitrary directed semilattice $(Y, +)$ and continuous function $f: X \to Y$, there exists an unique semilattice homomorphism $\overline{f}: (Z, +) \to (Y, +)$ such that $f = \overline{f} \circ i$.

Up to order isomorphism and topological homomorphism, the directed convex powerspace of a directed space is unique. Particularly, we denote the directed convex powerspace of each directed space $X$ by $P_P(X)$ [25].

**Definition 5.2.** [25] Let $X$ be a directed space.

(1) Denote $\mathcal{P}_{\text{fin}}(X) = \{ F: F \in \mathcal{P}^w(X) \}$, where $\hat{F} = (\downarrow F, \uparrow F)$. Define an order $\leq_P$ on $\mathcal{P}_{\text{fin}}(X): \hat{F}_1 \leq_P \hat{F}_2 \iff \downarrow \hat{F}_1 \subseteq \downarrow \hat{F}_2$ and $\uparrow \hat{F}_2 \subseteq \uparrow \hat{F}_1$.

(2) Let $\mathcal{D} = \{ \hat{F}: F \in \mathcal{P}^w(X) \} \subseteq \mathcal{P}_{\text{fin}}(X)$ be a directed set(respect to order $\leq_P$) and $\hat{H} \subseteq \mathcal{P}_{\text{fin}}(X)$. Define a convergence notation $\mathcal{D} \Rightarrow_P \hat{H}$ if there exists finite directed sets $D_1, \ldots, D_k \subseteq X$ such that

1. $D_k \subseteq \bigcup_{F \in \mathcal{D}} \downarrow F$.
2. $H \cap \{ x: D_i \to x \} \neq \emptyset$ for any $i = 1, \ldots, k$;
3. $H \subseteq \{ x: D_i \to x, i = 1, \ldots, k \}$;
4. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^k D_i$, there exists some $\hat{F} \in \mathcal{D}$ such that $\uparrow F \in \bigcup_{i=1}^k \uparrow d_i$.

(3) A subset $\mathcal{U} \subseteq \mathcal{P}_{\text{fin}}(X)$ is called a $\Rightarrow_P$ convergence open set of $\mathcal{P}_{\text{fin}}(X)$ if and only if for any directed subset $\mathcal{D}$ of $\mathcal{P}_{\text{fin}}(X)$ and $\hat{H} \in \mathcal{P}_{\text{fin}}(X)$, $\mathcal{D} \Rightarrow_P \hat{H}$ implies $\mathcal{D} \cap \mathcal{U} = \emptyset$. Denote all $\Rightarrow_P$ convergence open set of $\mathcal{P}_{\text{fin}}(X)$ by $\mathcal{O}_{\Rightarrow_P}(\mathcal{P}_{\text{fin}}(X))$.

Define a binary operation $\oplus: \mathcal{P}_{\text{fin}}(X) \times \mathcal{P}_{\text{fin}}(X) \to \mathcal{P}_{\text{fin}}(X)$ on $\mathcal{P}_{\text{fin}}(X)$: $\forall \hat{G}, \hat{H} \in \mathcal{P}_{\text{fin}}(X), \hat{G} \oplus \hat{H} = \hat{G} \cup \hat{H}$. It is easy to see that $\oplus$ is well-defined [25].

**Proposition 5.3.** [25] Let $X$ be a directed space, then

(1) $(\mathcal{P}_{\text{fin}}(X), \mathcal{O}_{\Rightarrow_P}(\mathcal{P}_{\text{fin}}(X)))$ is a directed space, that is, $\mathcal{O}_{\Rightarrow_P}(\mathcal{P}_{\text{fin}}(X)) = d(\mathcal{P}_{\text{fin}}(X))$ and the specialization order $\leq$ of $\mathcal{O}_{\Rightarrow_P}(\mathcal{P}_{\text{fin}}(X))$ equals to $\leq_P$;

(2) $(\mathcal{P}_{\text{fin}}(X), \oplus)$ is a directed semilattice;

(3) Suppose $X$ is a directed space, then $(\mathcal{P}_{\text{fin}}(X), \mathcal{O}_{\Rightarrow_P}(\mathcal{P}_{\text{fin}}(X)))$ is the convex powerspace of $X$, that is, endowed with topology $\mathcal{O}_{\Rightarrow_P}(\mathcal{P}_{\text{fin}}(X), (\mathcal{P}_{\text{fin}}(X), \cup)) \cong P_P(X)$.

Let $X$ be a directed space and $\hat{G}, \hat{H} \in \mathcal{P}_{\text{fin}}(X)$, then $\hat{G} \ll \hat{H} \iff \forall \mathcal{D} \Rightarrow_P \hat{H}, \hat{G} \ll_P \hat{F}$ for some $\hat{F} \in \mathcal{D}$ by the definition of way below. In this paper, the way below relation with respect to $\leq_P$ and $\Rightarrow_P$ is denoted by $\ll_P$.  

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Proposition 5.4. Let $X$ be a directed space and $\hat{G}, \hat{H} \in \mathcal{P}_{\text{fin}}(X)$, then $\forall y \in G$, $g \ll h$ for some $h \in H$, and $G \ll H$ implies $\hat{G} \ll \hat{H}$.

**Proof.** Let $\mathcal{D} \subseteq \mathcal{P}_{\text{fin}}(X)$ be a directed set and $\mathcal{D} \Rightarrow \hat{H}$. Then there exists finite directed sets $D_1, \cdots, D_k$ of $X$ satisfy the following four conditions:

1. $D_i \subseteq \bigcup_{F \in \mathcal{D}} \downarrow F$, $i = 1, \cdots, k$;
2. $\forall i = 1, \cdots, k$, $H \cap \{x \in X : D_i \rightarrow x\} \neq \emptyset$;
3. $H \subseteq \bigcup_{i=1}^{k} \{x \in X : D_i \rightarrow x\}$;
4. $\forall (d_1, \cdots, d_k) \in \prod_{i=1}^{k} \exists F \in \mathcal{D}$ such that $\uparrow F \subseteq \bigcup_{i=1}^{k} \downarrow d_i$.

Let $G = \{g_1, \cdots, g_n\}$, then $g_i \ll h_i$ for some $h_i \in H$. There exists $D_i$ such that $D_i \rightarrow h_i$. Since $g_i \ll h_i$, $\uparrow g_i \cap D_i \neq \emptyset$. Let $d_i \in \uparrow g_i \cap D_i$, then $d_i \in \downarrow F_i$ for some $\hat{F}_i \in \mathcal{D}$ and $\downarrow G \subseteq \bigcup \downarrow d_i \subseteq \bigcup \downarrow F_i$. By directness of $\mathcal{D}$, there exists $\hat{F} \in \mathcal{D}$ such that $\hat{F}_i \leq \hat{F}$ for any $\forall g_i \in G$. Hence $\downarrow G \subseteq \bigcup \downarrow F_i \subseteq \downarrow \hat{F}^*$.

For any $i$, let $x_i \in H \cap \{x : D_i \rightarrow x\}$, then $D_i \rightarrow x_i \in \hat{H}$. Thus $D_i \cap \uparrow G \neq \emptyset$ by $G \ll x$. Let $d_i \in D_i \cap \uparrow G$, then there exists $\hat{F} \in \mathcal{F}$ such that $\uparrow F \subseteq \bigcup \uparrow d_i \cap \uparrow G$, that is $\uparrow F \subseteq \uparrow G$.

Since $\mathcal{D}$ is a directed set, there exists $\hat{F}$ such that $\hat{F}_i, \hat{F} \leq \hat{F}$. Then $\hat{G} \leq \hat{F}, \hat{G} \ll \hat{H}$. □

Proposition 5.5. Let $X$ be a directed space and $\hat{G}, \hat{H} \in \mathcal{P}_{\text{fin}}(X)$, then $\downarrow G \ll_{\downarrow} H$ and $\uparrow G \ll_{\uparrow} \hat{H}$ imply $\hat{G} \leq_{\downarrow} \hat{H}$.

**Proof.** If $\downarrow G \ll_{\downarrow} H$ and $\uparrow G \ll_{\uparrow} \hat{H}$ and $\mathcal{D} \Rightarrow \hat{H}$. Then there exists finite directed sets $D_1, \cdots, D_k$ of $X$ satisfy the following four conditions:

1. $D_i \subseteq \bigcup_{F \in \mathcal{D}} \downarrow F$, $i = 1, \cdots, k$;
2. $\forall i = 1, \cdots, k$, $H \cap \{x \in X : D_i \rightarrow x\} \neq \emptyset$;
3. $H \subseteq \bigcup_{i=1}^{k} \{x \in X : D_i \rightarrow x\}$;
4. $\forall (d_1, \cdots, d_k) \in \prod_{i=1}^{k} \exists F \in \mathcal{D}$ such that $\uparrow F \subseteq \bigcup_{i=1}^{k} \downarrow d_i$.

By Definition 3.2, $D_1 = \{\downarrow F : \hat{F} \in \mathcal{D}\} \Rightarrow_{\downarrow} H$, then $\downarrow G \subseteq \downarrow F$ for some $\hat{F} \in F$. By Definition 4.2, $\mathcal{D}_2 = \{\uparrow F : \hat{F} \in \mathcal{D}\} \Rightarrow_{\uparrow} \hat{H}$, then $\uparrow F' \subseteq \uparrow G$ for some $\hat{F}' \in \mathcal{D}$. Since $\mathcal{D}$ is a directed set, there exists $\hat{F}$ such that $\hat{F}', \hat{F} \leq \hat{F}$. Then $\hat{G} \leq \hat{F}, \hat{G} \ll \hat{H}$. □

Corollary 5.6. Let $X$ be a directed space, then $\hat{G} \ll_{\downarrow} \hat{H} \Rightarrow g \ll_{\downarrow} x \forall g \in G$.

**Proof.** By Proposition 5.4, $g \ll_{\downarrow} x \forall g \in G \Rightarrow \hat{G} \ll_{\downarrow} \hat{H}$. If $\hat{G} \ll_{\downarrow} \hat{H}$ and directed set $D \rightarrow x$, then $\mathcal{D} = \{\hat{d} : \hat{d} \in D\} \Rightarrow_{\downarrow} \hat{H}$. There exists $\hat{d} \in D$ such that $\hat{G} \leq_{\downarrow} \hat{d}$. Hence $\uparrow G \subseteq_{\downarrow} \hat{d}, g \ll_{\downarrow} x \forall g \in G$. □

Proposition 5.7. Let $X$ be a continuous space, then $\mathcal{P}_{\text{fin}}(X)$ is continuous space.

**Proof.** Suppose $X$ is continuous space. Let $H = \{h_i : i \in \{1, \cdots, n\}\}$, and $\mathcal{D} = \{g_i : i \in \{1, \cdots, n\}\}$, where $g_i \ll_{\downarrow} h_i$ for any $i$. Then $\mathcal{D} \subseteq \{\hat{G} : \hat{G} \ll \hat{H}\}$ by Proposition 5.4.

For any $h_i \in H$ and $x_i \ll_{\downarrow} h_i$, $\{x_i : i \in \{1, \cdots, n\}\} \in \mathcal{D}$. Then $\downarrow h_i \subseteq \bigcup_{F \in \mathcal{D}} \downarrow F$ and $\downarrow h_i \rightarrow h_i$. Let $D_i = \downarrow h_i$, then $D_i$ and $D_i$ satisfy the conditions (1)-(3) of 5.2. $\forall x_i \in D_i$, we have $x_i \ll_{\downarrow} h_i$ for some $z_i \in X$. Let $F = \{z_i : h_i \in H\}$, then $\hat{F} \in \mathcal{D}$ and $\uparrow F \subseteq \uparrow x_i$. Hence $\mathcal{D} \Rightarrow_{\downarrow} H$ by Definition 5.2. Thus $\mathcal{L}_{\text{fin}}(X)$ is continuous space by Theorem 2.4.

Proposition 5.8. Let $X$ be a continuous space and $B \subseteq X$ be a base of $X$, then (1) $B = \{\hat{F} : F \in \mathcal{P}_{\text{fin}}(X), F \subseteq B\}$ is a base of $\mathcal{P}_{\text{fin}}(X)$; (2) If $X$ has a countably base, then so is $\mathcal{P}_{\text{fin}}(X)$; (2) If $X$ is an algebraic space, then so is $\mathcal{P}_{\text{fin}}(X)$.

**Proof.** (1) Let $B \subseteq X$ be a base of $X$, $H = \{h_1, \cdots, h_n\}$ and $\mathcal{D} = \{g_i : i \in \{1, \cdots, n\}\}$, where $g_i \ll_{\downarrow} h_i$ for any $i$. Then $\mathcal{D} \subseteq \{\hat{G} : \hat{G} \ll \hat{H}\}$ by Proposition 5.4. Let $\mathcal{D}^* = B \cap \{g_i : i \in \{1, \cdots, n\}\} = \{b_i : i \in \{1, \cdots, n\}\}$, where $b_i \in B$. Then $\mathcal{D}^*$ is directed family. For any $h_i \in H$ and $b_i \in \downarrow h_i \cap B$, $\{b_i : i \in \{1, \cdots, n\}\} \in \mathcal{D}^*$. Let $D_i = B \cap \downarrow h_i$, then $\mathcal{D}^*$ and $D_i$ satisfy the conditions (1)-(3) of 5.2. If $x_i \in D_i$, then $x_i \ll_{\downarrow} h_i$ for some
Let $z_i \in B$. Let $F = \{ z_i : h_i \in H \}$, then $\hat{F} \in \mathcal{D}^*$ and $\uparrow F = \bigcup \uparrow x_i$. Hence $\mathcal{D}^* \Rightarrow_{\mathcal{P}} \downarrow H$ by Definition 5.2. Thus $\mathcal{B} = \{ \hat{F} : F \in \mathcal{P}^w(X), F \subseteq B \}$ is a base of $\mathcal{P}_{fin}(X)$ by Proposition ??.

(2) Obviously, if $B \subseteq X$ is a countably base of $X$, let $\mathcal{B} = \{ \hat{F} : F \in \mathcal{P}^w(X), F \subseteq B \}$. Then $\mathcal{B}$ is a countably base of $\mathcal{P}_{fin}(X)$.

(3) If $X$ is an algebraic space, then $K(X)$ is a base of $X$. By (1), $\mathcal{B} = \{ \hat{F} : F \in \mathcal{P}^w(X), F \subseteq B \}$ is a base of $\mathcal{P}_{fin}(X)$. For any finite set $F \subseteq K(X)$, $\hat{F} \ll_{\mathcal{P}} \hat{F}$ by Proposition 5.4. Since $\mathcal{B} = \{ \hat{F} : F \in \mathcal{P}^w(X), F \subseteq K(X) \}$ is a base of $\mathcal{P}_{fin}(X)$. Hence $\mathcal{P}_{fin}(X)$ is an algebraic space. \qed

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