Energy and Momentum of a Class of Rotating Gravitational Waves

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Abstract

We calculate energy and momentum for a class of cylindrical rotating gravitational waves using Einstein and Papapetrou’s prescriptions. It is shown that the results obtained are reduced to the special case of the cylindrical gravitational waves already available in the literature.

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I. INTRODUCTION

The notion of energy has been one of the most thorny and important problems in Einstein’s theory of General Relativity (GR). There have been many attempts [1,2] to get a well defined expression for local or quasi-local energy and momentum. However, there is still no generally accepted definition known. As a result, different people have different points of view. Cooperstock [3] argued that in GR, energy and momentum are localized in regions of the non-vanishing energy and momentum tensor and consequently gravitational waves are not carriers of energy and momentum in vacuum. The gravitational waves, by definition, have zero stress-energy tensor. Thus the existence of these waves was questioned. However, the theory of GR indicates the existence of gravitational waves as solutions of Einstein’s field equations [4]. Infact this problem arises because energy is not well defined in GR.

The problem for gravitational waves was resolved by Ehlers and Kundt [5], Pirani [6] and Weber and Wheeler [7] by considering a sphere of test particles in the path of the waves. They showed that these particles acquired a constant momentum from the waves. Qadir and Sharif [8] presented an operational procedure, embodying the same principle, to show that gravitational waves impart a momentum. Rosen [9] investigated whether or not cylindrical gravitational waves have energy and momentum. He used the energy-momentum pseudo tensors of Einstein and Landau Lifshitz and carried out calculations in cylindrical polar coordinates. However, he arrived at the conclusion that the energy and momentum density components vanish. These results supported the conjecture of Scheidegger [10] that physical system cannot radiate gravitational energy. Later, he pointed out [11] that the energy and momentum densities turn out to be non-vanishing and reasonable if the calculations are performed in Cartesian coordinates. Rosen and Virbhadra [12] explicitly evaluated these quantities in the Einstein’s prescription by using Cartesian coordinates and found them finite and well defined. Virbhadra [13] then used Tolman, Landau-Lifshitz and Papapetrou’s prescriptions to evaluate the energy and momentum densities and found that the same results turn out in all these prescriptions.
In this paper we use Einstein and Papapetrou’s prescriptions to evaluate energy and momentum densities to a class of cylindrical rotating gravitational waves. As we shall see from the analysis given in the paper, when rotation is included, the problem is considerably complicated. We find that the results obtained by these two prescriptions are not exactly the same. However, it is shown that they both reduce to the same result for a special case of cylindrical gravitational waves. In the next section, we shall describe the class of rotating cylindrical gravitational waves. In sections three and four, we evaluate energy and momentum using Einstein and Papapetrou’s prescriptions respectively. Finally, we shall discuss the results.

II. A CLASS OF ROTATING CYLINDRICAL GRAVITATIONAL WAVES

A class of solutions of the gravitational field equations describing vacuum spacetimes outside rotating cylindrical sources is given by the line element of the form [14]

\[ ds^2 = e^{2\gamma - 2\psi}(dt^2 - d\rho^2) - \mu^2 e^{-2\psi}(\omega dt + d\phi)^2 - e^{2\psi}dz^2, \]  

in the cylindrical coordinates \((\rho, \phi, z)\). Here the metric functions \(\gamma, \mu, \psi\) and \(\omega\) depend on the coordinates \(t\) and \(\rho\) only. In general the spacetimes have two Killing vectors, one is associated with the invariant translations along the symmetry axis, \(\xi(z) = \partial z\), and the other is associated with the invariant rotations about the axis, \(\xi(\phi) = \partial \phi\). Obviously, these Killing vectors are orthogonal. When \(\omega = 0\), the metric represents spacetimes without rotation, in which the polarization of gravitational waves has only one degree of freedom and the direction of polarization is fixed [4]. It is to be noticed that if we take \(\omega = 0\) and \(\mu = \rho\), the above metric reduces to a special case of cylindrical gravitational waves [7]. Einstein’s vacuum field equations for the metric form (1) are given by

\[ (\mu \psi_v)_u + (\mu \psi_u)_v = 0, \]  

\[ \mu_{uv} - \frac{l^2}{8} \mu^{-3} e^{2\gamma} = 0, \]
\[ \omega_v - \omega_u = l \mu^{-3} e^{2\gamma}, \]  \tag{4} 

\[ \gamma_u = \frac{1}{2\mu_u} (\mu_{uu} + 2\mu\psi^2_u), \]  \tag{5} 

\[ \gamma_v = \frac{1}{2\mu_v} (\mu_{vv} + 2\mu\psi^2_v), \]  \tag{6} 

where \( \psi_u = \frac{\partial \psi}{\partial u} \), etc. The subscripts \( u = t - \rho \) and \( v = t + \rho \) are retarded and advanced times respectively. Here \( l \) is a constant length characteristic of the rotation of the system which is positive and is specifically attributed with rotating gravitational waves. For \( l = 0 \) we have \( \omega = \omega(t) \) from Eq.(4) and \( \mu_{tt} = \mu_{\rho\rho} \) from Eq.(3). A simple transformation to a rotating frame reduces the waves to non-rotating generalized Beck spacetimes which have been studied by many authors [15,16].

In order to have meaningful results in the prescription of Einstein and Papapetrou, it is necessary to transform the metric in Cartesian coordinates. Let us transform the metric in Cartesian coordinates by using

\[ x = \rho \cos \phi, \quad y = \rho \sin \phi. \]  \tag{7} 

The corresponding metric in these coordinates will become

\[ ds^2 = e^{2\gamma-2\psi} (dt^2 - \frac{1}{\rho^2} (xdx + ydy)^2) - \mu^2 e^{-2\psi} (\omega dt + \frac{1}{\rho^2} (xdy - ydx))^2 - e^{2\psi} dz^2. \]  \tag{8} 

### III. ENERGY AND MOMENTUM IN EINSTEIN’S PRESCRIPTION

The energy-momentum complex of Einstein [17] is given by

\[ \Theta^b_a = \frac{1}{16\pi} H^{bc}_{a,c}, \]  \tag{9} 

where

\[ H^{bc}_{a} = \frac{g_{ad}}{\sqrt{-g}} [g^{bd}g^{ce} - g^{cd}g^{be}]_{,e}, \quad a, b, c, d, e = 0, 1, 2, 3. \]  \tag{10}
\( \Theta_0^a \) is the energy density, \( \Theta_0^a \) are the momentum density components, and \( \Theta_0^a \) are the components of energy current density. The Einstein energy-momentum satisfies the local conservation laws

\[
\frac{\partial \Theta_b^a}{\partial x^b} = 0. \tag{11}
\]

The required non-vanishing components of \( H_{bc}^a \) are

\[
H_{01}^0 = \frac{1}{\mu \rho^3} (\mu^2 x - 2\mu \rho \rho x + \rho^2 \rho e^{2\gamma} - \mu^2 \omega \rho^2 x + 2\mu^2 \omega \gamma \rho y - \mu^4 \omega \rho \rho \rho x e^{-2\gamma}), \tag{12}
\]

\[
H_{02}^0 = \frac{1}{\mu \rho^3} (\mu^2 y - 2\mu \rho \rho y + \rho^2 \rho e^{2\gamma} - \mu^2 \omega \rho^2 y - 2\mu^2 \omega \gamma \rho x - \mu^4 \omega \rho \rho ye^{-2\gamma}), \tag{13}
\]

\[
H_{11}^0 = -\frac{1}{\rho^4} (2\mu \gamma \rho y^2 + 2\mu \rho x^2 - \mu^3 \omega \rho \rho ye^{-2\gamma}), \tag{14}
\]

\[
H_{12}^0 = \frac{1}{\rho^4} (\mu \omega \rho^3 + 2\mu \gamma \rho xy + \mu^3 \omega \rho y^2 e^{-2\gamma}), \tag{15}
\]

\[
H_{21}^0 = \frac{1}{\rho^4} (2\mu \gamma \rho xy - 2\mu \rho xy - \mu \omega \rho^3 - \mu^3 \omega \rho \rho xe^{-2\gamma}), \tag{16}
\]

\[
H_{22}^0 = -\frac{1}{\rho^4} (2\mu \gamma \rho x^2 + 2\mu \rho y^2 + \mu^3 \omega \rho xe^{-2\gamma}), \tag{17}
\]

\[
H_{33}^0 = -\frac{2}{\rho} (\mu \gamma - 2\mu \hat{\psi} + \hat{\mu}), \tag{18}
\]

\[
H_{02}^{12} = -\mu \omega \rho + 2\mu \omega \gamma \rho - 2\mu \rho \omega, \tag{19}
\]

\[
H_{03}^0, H_{13}^0, H_{23}^0, H_{33}^0, H_{01}^0, H_{02}^0, H_{23}^0, H_{31}^0 = 0. \tag{20}
\]

Using Eqs.(12-20) in Eq.(9), we obtain energy and momentum densities in Einstein’s prescription

\[
\Theta_0^0 = \frac{1}{16\pi \mu^2 \rho^3} [\mu^2 (-\mu + \mu \rho \rho - 2\mu \rho \rho \rho^2 - \mu \omega \rho^2 \rho^2 - 2\mu \omega \rho \rho \rho^3 - \mu \rho \omega^2 \rho^3) \tag{21}
\]

\[
+ \rho^2 (\mu + 2\mu \gamma \rho - \mu \rho \rho)e^{2\gamma} - \mu^4 \rho^2 (\mu \omega \rho^2 + \mu \omega \rho \rho)
\]

\[
-2\mu \omega \rho \gamma \rho + 3\mu \rho \omega \rho e^{-2\gamma}],
\]
\[
\Theta_0^0 = \frac{1}{16\pi \rho \delta^3}[2\mu \rho^2 \gamma x - 6\mu x^3 - 2\mu \rho^2 x - 2\mu \rho x^3 - \mu^2 \rho + 2\mu \rho\gamma \rho + 2\mu \rho \gamma \rho + 3\mu \rho \omega \rho) e^{-2\gamma}],
\]

\[
\Theta_0^2 = \frac{1}{16\pi \rho^4}[2\mu \rho\gamma y - 2\mu \rho \gamma y + \mu \omega \rho - \mu \rho \gamma \rho - 2\mu \rho \gamma \rho + 3\mu \rho \omega \rho) e^{-2\gamma}],
\]

\[
\Theta_0^1 = \frac{1}{16\pi \mu \rho^2 \pi}[\mu^2(-\mu x + 2\mu \rho x + 2\mu \omega \rho^2 x - 2\mu \omega \gamma \rho^2 y + 2\mu \omega \gamma \rho^2 y - \mu \rho \omega \rho^2 y - 2\mu \rho \omega \rho^2 y) - \mu \rho 
\]

\[
2\mu \rho \omega \rho^2 y + 2\mu \omega \gamma \rho \rho^2 y + 2\mu \omega \gamma \rho \rho^2 y + 3\mu \rho \omega \rho^2 y - 3\mu \rho \omega \rho^2 y - 2\mu \rho \omega \rho^2 y) - \mu \rho \omega \rho^2 y + 2\mu \rho \omega \rho^2 y + 3\mu \rho \omega \rho^2 y - 2\mu \rho \omega \rho^2 y + 2\mu \rho \omega \rho^2 y).
\]

\[
\Theta_0^3 = \Theta_0^2 = 0.
\]

Now for \(\omega = 0\) and \(\mu = \rho\), Eqs.(21)-(26) become

\[
\Theta_0^0 = \frac{1}{8\pi}e^{2\gamma}(\psi_\rho^2 + \psi_t^2),
\]

\[
\Theta_0^1 = \frac{1}{4\pi \rho}x\psi_\rho \psi_t,
\]

\[
\Theta_0^2 = \frac{1}{4\pi \rho}y\psi_\rho \psi_t,
\]

\[
\Theta_0^3 = \Theta_0^2 = 0.
\]

These are the energy and momentum densities of cylindrical gravitational waves given by Rosen and Virbhadra [12].
IV. ENERGY AND MOMENTUM IN PAPAPETROU’S PRESCRIPTION

The symmetric energy-momentum complex of Papapetrou [18] is given by

\[ \Omega^{ab} = \frac{1}{16\pi} N^{abcd} \cd, \]  

(33)

where

\[ N^{abcd} = \sqrt{-g} (g^{ab} \eta^{cd} - g^{ac} \eta^{bd} + g^{cd} \eta^{ab} - g^{bd} \eta^{ac}), \]  

(34)

and \( \eta^{ab} \) is the Minkowski spacetime. The energy-momentum complex satisfies the local conservation laws

\[ \frac{\partial \Omega^{ab}}{\partial x^b} = 0. \]  

(35)

The locally conserved energy-momentum complex \( \Omega^{ab} \) contains contributions from the matter, non-gravitational and gravitational fields. \( \Omega^{00} \) and \( \Omega^{0a} \) are the energy and momentum (energy current) density components. The required non-vanishing components of \( N^{abcd} \) are given as

\[ N^{0101} = \frac{1}{\mu^3 \rho^3} (\mu^2 \rho^2 + \mu^2 x^2 - \mu^2 \rho^2 \omega^2 y^2 + \rho^2 y^2 e^{2\gamma}), \]  

(36)

\[ N^{0102} = \frac{1}{\mu^3 \rho^3} (\mu^2 x y + \mu^2 \omega^2 \rho^2 x y - \rho^2 x y e^{2\gamma}), \]  

(37)

\[ N^{0202} = \frac{1}{\mu^3 \rho^3} (\mu^2 \rho^2 + \mu^2 y^2 - \mu^2 \rho^2 \omega^2 x^2 - \rho^2 x^2 e^{2\gamma}), \]  

(38)

\[ N^{0303} = \frac{\mu}{\rho} (1 + e^{2\gamma-4\psi}), \]  

(39)

\[ N^{0121} = \frac{1}{\rho} (\mu \omega x), \]  

(40)

\[ N^{0122} = \frac{1}{\rho} (\mu \omega y), \]  

(41)
Substituting Eqs. (36-41) in Eq. (33), we have energy and momentum density components in Papapetrou’s prescription

\[ \Omega_{00} = \frac{1}{16\pi\mu^2\rho^3}[\mu^2(-\mu + \mu_\rho\rho - 2\mu_\rho\rho^2 - \mu^2\rho^2 - 2\mu_\rho\omega_\rho\rho^3 - \mu_\rho^2\rho^3) + (\mu\rho^2 + 2\mu_\gamma\rho^3 - \mu_\rho^3)e^{2\gamma}], \]  

\[ \Omega_{01} = \frac{1}{16\pi\mu^2\rho^3}[\mu^2(-\dot{\mu}x + 2\dot{\mu}_\rho\rho x + \mu_\omega y - \mu_\rho\omega y - \mu_\rho\rho_\rho^2 x - 2\mu_\rho\omega_\rho\rho^2 y - \mu_\rho\omega_\rho\rho^2 x - \rho^2x(2\mu_\gamma - \dot{\mu})e^{2\gamma}], \]  

\[ \Omega_{02} = \frac{1}{16\pi\mu^2\rho^3}[\mu^2(\dot{\mu}y + 2\dot{\mu}_\rho\rho y + 2\mu_\omega\rho_\rho^2 x + \mu_\omega_\rho_\rho^2 y + \mu_\omega^2\rho^2 y - \mu_\omega x + \mu_\rho\rho x + \mu_\rho_\rho_\rho^2 x + \mu_\rho_\rho_\rho^2 x + 2\mu_\rho_\rho_\rho_\rho^2 x + \mu_\rho_\rho_\rho_\rho^2 x - \rho^2y(2\mu_\gamma - \dot{\mu})e^{2\gamma}], \]  

\[ \Omega_{03} = \Omega_{30} = 0. \]  

We see that for \( \omega = 0 \) and \( \mu = \rho \), Eqs. (42)-(45) yield

\[ \Omega_{00} = \frac{1}{8\pi}e^{2\gamma}(\psi^2_\rho + \psi^2_t), \]  

\[ \Omega_{01} = -\frac{1}{4\pi\rho}x\psi_\rho\psi_te^{2\gamma}, \]  

\[ \Omega_{02} = -\frac{1}{4\pi\rho}y\psi_\rho\psi_te^{2\gamma}, \]  

\[ \Omega_{03} = 0. \]  

These turn out to be energy and momentum density components for cylindrical gravitational waves given by Virbhadra [13].
V. DISCUSSION

We have evaluated energy and momentum density components for a class of rotating cylindrical gravitational waves by using prescriptions of Einstein and Papapetrou. It can be seen that the energy and momentum densities for a class of rotating gravitational waves are finite and well defined in both the prescriptions. It follows from Eqs.(21-26) and (42-45) that though the energy-momentum complexes of Einstein and Papapetrou are not exactly the same but are similar upto certain terms. However, it is interesting to note from Eqs.(27-32) and (46-49) that both the results reduce to the same energy and momentum densities of a special case of cylindrical gravitational waves as given in [12,13].
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