Quasiparticle spectrum inside the vortex core: 
crossover from dirty to clean limit

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The quasiparticle spectrum inside the vortex core in the mixed state of a strongly type-II superconductors are studied. The s-wave symmetry for the gap parameter is assumed. The crossover behavior from dirty to clean limit is shown by numerical calculation based on the random matrix theory.

The study of electric structure of vortices is very important for understanding the behavior of the superconductors in a magnetic field. The quasiparticle energy levels inside the vortex core is firstly studied by Caroli, de-Gennes, Matricon [3]. In the two dimensional case, the discretized energy levels with equal spacings are obtained and the quasiparticles are localized in these levels. The existence of these localized states is confirmed experimentally on NbSe2 [6-8]. The scattering between the levels due to impurities inside the vortex broadens the discretized energy levels and the system becomes extended. In the clean case, where the number of impurities are moderately small, the periodic spectrum is obtained in [2]. In the dirty limit the level statistics inside the vortex core is studied using the random matrix theory [7,8].

In the dirty limit the level statistics inside the vortex core is treated as a normal disk of the radius $\xi$ random source matrix [10]. The random symplectic matrix coupled to an external non-theory [9] where the impurity potential is handled by a random matrix approach for [10]. The density of states near the Fermi energy is obtained.

In the dirty case, where the number of impurities are fairly large $N_i \gg 1$ and the random matrix approach for the impurity scattering is appropriately applied. In this work, we present the numerical calculation for the crossover behavior of the density of states (DOS) inside a vortex core from the dirty to the clean limit in the presence of impurity scattering between the levels. The two limiting cases, superclean and dirty case, are determined by the scattering time $\tau$ between the Caroli-deGennes-Matricon (CDM) excitation states inside the vortex core. In the superclean case $\omega_0 \gg 1/\tau$ where $\omega_0$ is the level spacing of order $\omega_0 \sim \Delta^2/E_F$. In the dirty case $1/\tau \gg \Delta$. As shown by Koukakov and Larkin [4], the spectrum becomes an $\omega_0$ periodic function of energy in the wide region of the intermediately clean case. We define the dirty limit where the number of impurities are fairly large $N_i \gg 1$ and the random matrix approach for the impurity scattering is appropriately applied. In the dirty case, the impurity potential is handled by a random symplectic matrix coupled to an external non-random source matrix [10].

In the extremely type-II layered superconductors, the vortex core is treated as a normal disk of the radius $\xi_0$ inside a superconductor where $\xi_0$ is the coherence length. The angular dependencies can be eliminated and the Bogolubov-deGennes equation is solved with the basis which is written by the Bessel function $J_\mu(r)$. In the dirty case, where the number of impurities are moderately small, the periodic spectrum is obtained in [2]. In the dirty limit the level statistics inside the vortex core is studied using the random matrix theory [7,8]. In the latter limit, the impurity potential is given by the random symplectic matrix and the function form for the density of states near the Fermi energy is obtained. The crossover behavior between these two limits is recently investigated analytically with the random matrix theory [3] where the impurity potential is handled by a random symplectic matrix coupled to an external non-random source matrix [10].

The Bogolubov-deGennes equations for the two component excitation wave functions $\hat{\psi} = (u, v)$ inside the vortex of the s-wave superconductor is written as

$$\begin{align*}
\left[\sigma_z \left(-\frac{1}{2m} \frac{\partial^2}{\partial r^2} - E_F + V(r)\right) + \sigma_x \text{Re} \Delta(r) + \sigma_y \text{Im} \Delta(r) \right] \hat{\psi} = E \hat{\psi}
\end{align*}$$

(1)

where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices and $\Delta(r)$ is the order parameter. $V(r)$ is the disorder potential which is produced by the short-range impurities

$$V(r) = \sum_i V_i \delta(r-r_i)$$

(2)

where $V_i$, $r_i$ are the strength and the position of $i$-th impurity, respectively. The summation is taken over all impurities. We assume that the magnetic field is weak ($B \ll H_{c2}$) and the order parameter is given by

$$\Delta(r) = \Delta(r)e^{i\theta}$$

(3)

where $r = |r|$ and $\theta = \text{Arg}(r)$. The CDM excitation spectrum is obtained for $V(r) = 0$ case as

$$\hat{\psi}_\mu(r) = Ce^{-K(r)} \left( e^{i\mu\theta} J_\mu(k_Fr) - e^{i(\mu-1)\theta} J_{\mu-1}(k_Fr) \right)$$

(4)
where $C$ is the normalization constant, $K(r) = \Delta r/v_F = r/\xi\pi$ and $\mu = 0, \pm 1, \pm 2, \cdots$. The excitation energies are equidistant as

$$E^0_\mu = -\omega_0(\mu - \frac{1}{2}).$$

(5)

(We take $\hbar = 1$ hereafter.) We assume that the gap $\Delta$ is large enough and there are a large number of excitation levels $N \sim \Delta/\omega_0 \gg 1$. When the scattering between the CDM levels due to the impurity potential is present, the wave function is given by a linear combination of each levels

$$\hat{\psi} = \sum c_n \hat{\psi}_n$$

(6)

and the excitation spectrum $E$ is obtained by following equations

$$\det[\text{diag}(E^0_n - E) + \sum_i \hat{A}^i] = 0$$

(7)

$$A^i_{nm} = \tilde{C}e^{i\theta_i(m-n)}[J_n(k_F r_i)J_m(k_F r_i)$$

$$-J_{n-1}(k_F r_i)J_{m-1}(k_F r_i)]$$

(8)

where $r_i = (r_i, \theta_i)$ is the position of the $i$-th impurity. Since $k_F r_i \gg 1$ in our clean limit, we can use an asymptotic expansion for the Bessel functions in the matrix elements $A^i_{nm}$,

$$A^i_{nm} \sim \tilde{C}e^{i(m-n)\theta_i} \sin \left(2k_F r_i - \frac{n+m}{2}\pi\right)$$

(9)

The density of states is obtained by solving Eq. (7) numerically and averaging over the positions of impurities which are randomly generated. Fig.1 shows the result for the case where the number of impurities $N_i = 4$, the number of CDM basis $N = 50$ and the averaging has been done over 6000 generations of the impurity positions. The particle-hole symmetry of the spectrum is evident as seen from the plot, and it shows $\omega_0$ periodic structure which can be seen from the enlarged plot near the Fermi level ($E = 0$). The analytical result by Koulakov and Larkin is written as

$$\rho(E) = \frac{2}{\omega_0^2} \sin^2 \left(\frac{\pi E}{\omega_0}\right).$$

(10)

Our result of Fig.1(d) coincides with this analytical result. We obtain relatively sharp peaks when the matrix elements are given by the Bessel functions as Eq.$^6$.

In the dirty limit, it is obvious that the scattering Hamiltonian becomes a complex symplectic random matrix from the explicit form of $A^i_{nm}$ in Eq.$^5$. We take $2N$-CDM basis near the Fermi level ($-N + 1 \leq \mu \leq N$) and rearrange the order of rows and columns of the matrix $A$ in order to divide $A$ into four blocks, in each of which the indices are even-even, even-odd, odd-even and odd-odd, respectively. Since $J_{-n}(z) = (-1)^n J_n(z)$, we have

$$A^T J + JA = 0$$

(12)

where $J$ is written with the $N \times N$ identity matrix $I$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

(13)

In Fig.2 the DOS for the dirty limit is shown which is evaluated by the calculation of the eigenvalues of the Gaussian symplectic random matrices of size 100. It is consistent with the conjecture of analytical study of the level statistics for class C of the random matrix theory. When the Hamiltonian is given by a random matrix taken from a symplectic ensemble $Sp(N)$, the DOS near the Fermi level is given by

$$\rho(E) = 1 - \frac{\sin(2\pi E/\omega_0)}{2\pi E/\omega_0}. $$

(14)
FIG. 2. The density of states near the Fermi level for the dirty limit. The number of CDM modes is 50. (a) The full range plot. (b) The enlarged plot near the Fermi level. The solid line is the analytical result Eq. (14).

This result is also shown by the solid line in Fig.2(b). Although the random matrix theory is fully phenomenological and its applicability to the real system is left undetermined, the investigation of the crossover behavior between the above two limits by means of the matrix theory is quite important since it may provide a lot of interesting behavior of the level statistics inside the vortex core which can be measured by the STM experiments directly. This level statistics also determines the transport properties of superconductors in a magnetic field.

In order to investigate the crossover behavior of the spectrum between the above two limiting cases, we introduce the effective Hamiltonian in the form

$$\mathcal{H} = \mathcal{H}_0 + V$$

where $\mathcal{H}_0$ is the Hamiltonian for the clean limit whose eigen energies are written as

$$E_n = -\omega_0(n - \frac{1}{2}) + (-1)^n \bar{z}$$

where $\bar{z}$ is some function of the position of impurities. The spectrum is obtained by averaging over $\bar{z}$. $V$ is the random symplectic matrix whose probability distribution is assumed to be Gaussian. We also introduce a parameter $R$ which determines the strength of the external source $\mathcal{H}_0$ respective to the random potential $V$. The matrix elements of $\mathcal{H}_0$ is written by

$$[\mathcal{H}_0]_{nm} = \sum_i A_{nm}^i$$

$$A_{nm}^i = R e^{i\theta_i(n-m)} [J_n(k_F r_i) J_m(k_F r_i) - J_{n-1}(k_F r_i) J_{m-1}(k_F r_i)]$$

In the numerical calculations, the rearrangement of the indices of the matrix element $A_{nm}^i$, which generate the four blocks, even-even, even-odd, odd-even and odd-odd is applied. We fix the amplitude of the random matrix in the calculation. When $R \to \infty$, the clean limit is obtained and taking $R = 0$ gives the dirty limit. We obtain the DOS for several values of $R$ in Fig.3. The vertical axis is the scaled energy $E/(R\omega_0)$. We see that the peaks grow as the value of $R$ is increased.

In Ref. [10], the crossover from the clean limit to the dirty limit is investigated with the case where the deterministic matrix with the eigenvalues Eq.(16) is coupled
to the random matrix. By means of the generalization of the unitary ensemble matrix model, they discussed the symplectic ensemble case. The explicit formulae for the density of states is given by

\[
\rho(E) = \frac{N}{C} \left[ 1 - \frac{C}{4NE} \sin \left( \frac{4NE}{C} \right) \right] + \frac{C}{2N} \sin^2 \left( \frac{2NE}{C} \right)
\]

\[-\frac{CE^2}{4N(E^2 + N^2/C^2)} \left[ 1 - e^{-4N^2/C^2} \cos \left( \frac{4NE}{C} \right) \right]
\]

\[+ \frac{E}{4(E^2 + N^2/C^2)} e^{-4N^2/C^2} \sin \left( \frac{4NE}{C} \right) \] (18)

where the parameter \( C \) determines the strength of the external source \( \mathcal{H}_0 \) respective to the random potential \( V \) and \( N \) is the matrix size. Our numerical result is qualitatively in good agreement with this analytical result near the Fermi level.

On the other hand, the crossover behavior can be observed by increasing the number of impurities \( N_i \) directly. In this case, we take the Hamiltonian as

\[
\hat{H} = \sum_{i} N_i A^i + \left( \begin{array}{cc} E^0_{-N} & O \\ O & E^0_N \end{array} \right)
\] (19)

where \( A^i \) is the scattering matrix Eq.(18) generated by one impurity at site \( \mathbf{r}_i = (r_i, \theta_i) \). Fig.4 shows the result of the numerical calculation for various values of \( N_i \). As seen from Fig.4, the density of states near the Fermi level becomes non-vanishing except at the origin when the number of impurities is increased. The spectrum near the Fermi energy for the large value of \( N_i \) is similar to the dirty limit case where the Hamiltonian is given by the symplectic Gaussian random matrix. This crossover behavior appears only in the narrow region of the energy levels around the Fermi level.

To summarize, we have studied the quasiparticle spectrum inside the vortex core numerically for the clean, dirty and crossover case of the conventional (s-wave) superconductors in a magnetic field. By means of the random matrix theory, the crossover behavior of the vortex density of states is obtained. In the relatively dirty crossover region, the density of states becomes zero only at the Fermi level and the spectra near the Fermi level is obtained numerically which is consistent with the analytical result. The spectrum for the dirty limit is reproduced by increasing the number of impurities in the scattering matrix in the energy region near the Fermi level. The investigation of the vortex quasiparticle spectrum in the case of the unconventional superconductors such as d-wave superconductors would be very interesting which is in preparation.

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FIG. 4. The crossover behavior of the density of states near the Fermi level obtained by increasing the number of impurities \( N_i \) from the super clean limit \( N_i = 1 \) to the dirty case \( N_i = 100 \). The number of CDM modes is \( N = 50 \).

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