A DECOMPOSITION OF A MEASURABLE FUNCTION $f$ BY A
ONE-SIDED LOCAL SHARP MAXIMAL FUNCTION AND
APPLICATIONS TO ONE-SIDED OPERATORS

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Abstract. Following the ideas of Andrei Lerner in [ “A pointwise estimate for the
local sharp maximal function with applications to singular integrals” Bull. London
Math. Soc. 42 (2010) 843–856], we obtain another decomposition of an arbitrary
measurable function $f$ in terms of local mean oscillations. This allows us to get new
estimates involving one-sided singular integrals and one-sided maximal operator. As
an application to this result we obtain two weighted inequality for one-sided
singular integrals and a $L^1(w)$ inequality relating a measurable function $f$ and sharp one-
sided operator. These estimates are more precise in sense that they are valid for a
greater class of weights.

1. Introduction

In this paper we give a version of the Lerner formula obtain in [6]. The motivation
to study this result was a $L^1$-weighted inequality involving a function $f$ and the
one-sided sharp-$\delta$ maximal function. This type of inequality was needed to obtain
the best constant while dealing with weighted $A_p^+$ norms of the commutator of the
one-side singular integral given by a symbol $b \in BMO$. This results will appear in
[20].

Given a measurable function $f$ on $\mathbb{R}^n$ and a cube $Q$, we define
$$\tilde{\omega}_\lambda(f, Q) = \inf_{c \in \mathbb{R}} ((f - c) \chi_Q)^*(\lambda |Q|), \quad 0 < \lambda < 1,$$
where $f^*$ denotes the non-increasing rearrangement of $f$. The local sharp maximal
function relative to $Q$ is defined by
$$M_{\lambda Q}^# f(x) = \sup_{x \in Q' \subset Q} \tilde{\omega}_\lambda(f, Q').$$

In [6] A. Lerner, obtained the following result:

Given a cube $Q^0$, denote by $D(Q^0)$ the set of all dyadic cubes with respect to $Q^0$.
If $Q \in D(Q^0)$ and $Q \neq Q^0$, then we denote by $\hat{Q}$ its dyadic parent, that is, the unique
cube from $D(Q^0)$ containing $Q$ and such that $|\hat{Q}| = 2^n |Q|$.

Theorem 1.1 ([6]). Let $f$ be a measurable function on $\mathbb{R}^n$ and let $Q^0$ be a fixed cube.
Then there exists a (possibly empty) collection of cubes $Q^k_j \in D(Q^0)$ such that:

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\end{itemize}
where $m_f(Q)$ is a median value of $f$ over $Q$, i.e., there is a possibly non-unique real number such that

$$\{|x \in Q : f(x) > m_f(Q)\} \leq |Q|/2$$

and

$$\{|x \in Q : f(x) < m_f(Q)\} \leq |Q|/2.$$ 

In this paper we obtained a similar formula to the one obtain in Theorem 1.1 for a measurable function $f$ defined on $\mathbb{R}$, that will be useful to apply in the case of one-sided operators. We also give several applications of this formula that appear in section 3. Through out this paper we will use the following notation:

Given the interval $I = (b, c)$, we denote by $I^- = (a, b)$ and by $I^+ = (c, d)$ the intervals where $b - a = c - b = d - c$. Let $f$ be a measurable function on $\mathbb{R}$ and let $I$ be an interval, the local mean oscillation of $f$ on $I$ is defined by

$$\omega_\lambda(f, I) = ((f - m_f(I))\chi_I)^*|\lambda|I) \quad 0 < \lambda < 1.$$ 

Given a fix interval $I^0$, for $x \in (I^0)^-$ we define $B_{x,I^0} = \{I : x \in I^- \subset (I^0)^-\}$. Observe that if $I \in B_{x,I^0}$ then $I^+ \subset (I^0)^- \cup I^0 \cup (I^0)^+.$

**Definition 1.2.** Given a measurable function $f$ on $\mathbb{R}$ and an interval $I^0$, the one-sided local sharp maximal function relative to $(I^0)^-$, is defined by

$$M_{(I^0)^-}^+(f,x) = \sup \{\omega_\lambda(f, I^+), I \in B_{x,I^0}\}.$$ 

**Theorem 1.3.** Let $f$ be a measurable function on $\mathbb{R}$ and let $I^0$ be a fixed interval. Then there exists a (possibly empty) collection of intervals $I^+_{j,r}, \ (I^+_{j,r^-}) \subset (I^0)^-$ such that:

- for a.e. $x \in (I^0)^-$,

$$|f(x) - m_f((I^0)^+)| \leq 2M_{(I^0)^-}^+(f)(x) + \sum_{k=1}^\infty \sum_{j=1}^\infty \omega_{1/4}(f, (I^+_{j,r})^-) \chi_{(I^+_{j,r})^-}(x);$$

- for each fixed $k$ the intervals $(I^+_{j,r})^-$ are pairwise disjoint;

- if $\Omega_k = \bigcup_{j,r} (I^+_{j,r})^-$, then $\Omega_{k+1} \subset \Omega_k$;

- $|\Omega_{k+1} \cap (I^+_{j,r})^-| \leq 1/2|I^+_{j,r}|$;

- for each fixed $k$ and each fixed $j$, $(I^+_{j,r})^- \subset I^+_{j,r}$ and $\frac{3}{2}|(I^+_{j,r})^-| = |I^+_{j,r}|$;

- if $E^+_{j,r} = (I^+_{j,r})^- \setminus \Omega_{k+1}$ then $E^+_{j,r}$ are pairwise disjoint (for all $k$, $j$, $r$) and

$$\frac{1}{2}|(I^+_{j,r})^-| \leq |E^+_{j,r}|.$$

The paper is organized as follows: in section 2 we give some preliminaries, in section 3 we give applications of Theorem 1.3 and in section 4 we prove all the results.
2. Preliminaries

In this section we give some definitions and well known results.

2.1. One-sided singular integral operators and Sawyer’s weights.

Definition 2.1. Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). The one-sided maximal operators are defined as
\[
M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)| \, dt, \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \, dt.
\]

The good weights for these operators are those of the Sawyer’s classes. We recall this definition.

Definition 2.2. Let \( w \) be a non-negative locally integrable function and \( 1 \leq p < \infty \). We say that \( w \in A_p^+ \) if there exists \( C_p < \infty \) such that for every \( a < c < b \)
\[
\left( \frac{1}{(b-a)^p} \left( \int_c^b w \right) \left( \int_a^c w \right)^{p-1} \right) \leq C_p,
\]
when \( 1 < p < \infty \), and for \( p = 1 \),
\[
M^- w(x) \leq C_1 w(x), \quad \text{for a.e. } x \in \mathbb{R},
\]
finally we set \( A_\infty^+ = \cup_{p \geq 1} A_p^+ \).

The smallest possible \( C_1 \) in (2.2) here is denoted by \( \|w\|_{A_1^+} \) and the smallest possible \( C_p \) in (2.1) here is denoted by \( \|w\|_{A_p^+} \).

It is well known that the Sawyer classes characterize the boundedness of the one-sided maximal function on weighted Lebesgue spaces. Namely, \( w \in A_p^+ \), \( 1 < p < \infty \), if and only if \( M^+ \) is bounded on \( L^p(w) \) and \( w \in A_1^+ \) if and only if \( M^+ \) maps \( L^1(w) \) into \( L^{1,\infty}(w) \). See \([21],[10],[11]\) for more details. The classes \( A_p^- \) for \( 1 \leq p < \infty \) are defined analogously.

F. J. Martín-Reyes and A. de la Torre in \([13]\) introduced the one-sided sharp function.

Definition 2.3. Let \( f \) be a locally integrable function. The one-sided sharp maximal function is defined by
\[
M^+ (# f)(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} \left( f(y) - \frac{1}{h} \int_{x}^{x+h} f \right)^+ \, dy.
\]

Now we give some definitions and several results about Young functions. A Young function, is a function \( \mathcal{A} : [0, \infty) \rightarrow [0, \infty) \) continuous, convex and increasing such that \( \mathcal{A}(0) = 0 \) and \( \mathcal{A}(t) \rightarrow \infty \) as \( t \rightarrow \infty \). The Luxemburg norm of a function \( f \), given by \( \mathcal{A} \) is
\[
\|f\|_{\mathcal{A}} = \inf \left\{ \lambda > 0 : \int \mathcal{A} \left( \frac{|f|}{\lambda} \right) \leq 1 \right\},
\]
and the \( \mathcal{A} \)-average of \( f \) over an interval \( I \) is
\[
\|f\|_{\mathcal{A},I} = \inf \left\{ \lambda > 0 : \frac{1}{|I|} \int_I \mathcal{A} \left( \frac{|f|}{\lambda} \right) \leq 1 \right\}.
\]
We will denote by $\overline{A}$ the complementary function associated to $A$ (see [2]). Then the
generalized Hölder’s inequality
\[ \frac{1}{|T|} \int_T |fg| \leq ||f||_{A,I}||g||_{\overline{A},I}, \]
holds. There is a further generalization that turns to be out useful for our purposes
(see[14]). If $A, B, C$ are Young functions such that
\[ A^{-1}(t)B^{-1}(t) \leq C^{-1}(t), \]
then
\[ ||fg||_{C,I} \leq 2||f||_{A,I}||g||_{B,I}. \]

**Definition 2.4.** For each locally integrable function $f$, the maximal and one-sided
maximal operators associated to the Young function $A$ are defined by
\[ M_A f(x) = \sup_{x \in I} ||f||_{A,I}, \quad M_A^+ f(x) = \sup_{x < b} ||f||_{A,(x,b)}, \quad \text{and} \quad M_A^- f(x) = \sup_{a < x} ||f||_{A,(a,x)}. \]

Observe that for $A(t) = t^r$, $M_A^+ f(x) = M_A^+ f(x) = (M^+ |f|^r(x))^{1/r}$, for all $r \geq 1$.

**Definition 2.5.** For $1 < p < \infty$, a Young function $A$ is said to belong to $B_p$ if there
exists $c > 0$ such that
\[ \int_c^\infty A(t) \frac{dt}{tp - t} < \infty. \]

This condition appears first in [17] and it was shown that $A \in B_p$ if and only if $M_A$
is bounded on $L^p(\mathbb{R}^n)$. Observe that as $M_A^+ f \leq M_A f$, $A \in B_p$ implies $M_A^+$ is bounded
on $L^p(\mathbb{R})$.

In [7] (see Theorem 3.1), the authors proved that if $B$ is a Young function such
that $B \in B_p$, $p > 1$, and
\[ ||u^{1/p}||_{p,(a,b)} ||v^{-1/p}||_{B,(b,c)} \leq \infty, \quad (2.4) \]
for all $a < b < c$ and $b - a < c - b$, then
\[ ||M^+ f||_{L^p(a)} \leq C ||f||_{L^p(b)}. \quad (2.5) \]

**Definition 2.6.** We shall say that a function $K$ in $L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is a Calderón-
Zygmund kernel if the following properties are satisfied:

1. $||\hat{K}||_{\infty} < c_1,$
2. $|K(x)| \leq \frac{c_2}{|x|^{n+1}},$
3. $|K(x) - K(x - y)| < \frac{c_3 |y|}{|x|^{n+1}}, \quad \text{where} \ |y| < \frac{|x|}{2}.$

The Calderón-Zygmund singular integral operator associated to $K$ is defined
\[ T(f) = \text{p.v.}(K * f)(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B(x, \epsilon)} K(x - y)f(y) \ dy. \]

A one-sided singular integral $T^+$ is a singular integral associated to a Calderón–
Zygmund kernel with support in $(−\infty, 0)$; therefore, in that case,
\[ T^+ f(x) = \lim_{\epsilon \to 0^+} \int_{x+\epsilon}^\infty K(x - y)f(y) \ dy. \]

Examples of such kernels are given in [1]. In an analogous way we defined $T^-$. 
Remark 2.7. H. Aimar, L. Forzani and F.J. Martín-Reyes proved in [1] that the one-sided singular integral $T^+$ is controlled by the one-sided maximal functions $M^+$ in the $L^p(w)$ norm if $w \in A_w^\infty$.

Remark 2.8. It is well known to that the classes $A_p$ are included in $A_p^+$ and $A_p^-$; namely $A_p = A_p^- \cap A_p^+$.

Remark 2.9. The one-sided classes of weights satisfy the following factorization, $w \in A_p^+$ if only if $w = w_1w_2^{-p}$ with $w_1 \in A_1^+$ and $w_2 \in A_1^-$, and $\|w\|_{A_p^+} \leq \|w_1\|_{A_1^+}\|w_2\|_{A_1^-}^{p-1}$.

Remark 2.10. It is easy to check that $(M^- f)^\delta \in A_1^+$ for all $0 < \delta < 1$ with \[\|(M^- f)^\delta\|_{A_1^+} \leq \frac{C}{1-\delta}.\]

Remark 2.11. Usually while working with $T^+$ it is used the following one-sided sharp maximal function

\[M^+\# f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x+h} \left| f(t) - \frac{1}{h} \int_{x-h}^{x+h} f \right| dt.\]

This maximal function is bigger that the one in Definition 2.3. The one in Definition 2.3 is used to study the one-sided fractional integral.

We will also use the following maximal sharp function

\[M_\delta^+\# f(x) = \sup_{h>0} \left( \frac{1}{h} \int_{x-h}^{x+h} \left| f(t) - \frac{1}{h} \int_{x-h}^{x+h} f \right| \delta \right)^{\frac{1}{\delta}}.\]

In [8], M. Lorente and M.S. Riveros, give the following pointwise estimate for the sharp maximal function of the one-sided singular integral. Let $0 < \delta < 1$. Then there exists $C = C_\delta > 0$ such that

\[M_\delta^+\# (T^+ f)(x) \leq CM^+ f(x).\]

Recently F.J. Martín-Reyes and A. de la Torre in [12] proved the one-sided version of the well known result that S. M. Buckley, proved in [3], for the Hardy-Littlewood maximal function.

Lemma 2.12 (Theorem 1.4 in [12]). Let $w \in A_p^-$, then

\[\|M^-\|_{L^p(w)} \leq C p' 2^p \|w\|_{A_p}^{-\frac{1}{p-1}}.\]

Finally,

Lemma 2.13 (Kolmogorov’s inequality). Let $T$ be an operator in $L^1(\mathbb{R}^n)$ with $T$ is weak $(1,1)$, $Q$ is a cube, and $0 < \epsilon < 1$. Then

\[\left( \frac{1}{|Q|} \int_Q |Tf|^\epsilon \right)^{\frac{1}{\epsilon}} \leq \frac{C}{2|Q|} \int_{2Q} |f|\]

where $2Q$ is a cube with the same centre as $Q$ and having side length two times larger, and $\text{supp}(f) \subset 2Q$. 


2.2. The non-increasing rearrangement of a measurable function \( f \).

**Definition 2.14.** Let \( f \) be a measurable function on \( \mathbb{R}^n \), we define the non-increasing rearrangement of \( f \) by

\[
f^*(t) = \inf \{ \alpha > 0 : |\{ x \in \mathbb{R}^n : |f(x)| > \alpha \}| \leq t \} \quad (0 < t < \infty).
\]

If \( E \) is any measurable set, an important fact is that

\[
\int_E |f|^p \, dx = \int_0^{\|E\|^p} f^*(t)^p \, dt.
\]

If \( f \) is only a measurable function and if \( Q \) is a cube then we define the following quantity:

\[
(f\chi_Q)^*(\lambda|Q|) \leq \left( \frac{1}{\lambda|Q|} \int_Q |f|^\delta \right)^{\frac{1}{\delta}},
\]

for all \( 0 < \delta \) and \( 0 < \lambda < 1 \).

It is easy to check, from the definition of median value, that

\[
|m_f(Q)| \leq (f\chi_Q)^*(|Q|/2) \tag{2.8}
\]

and if \( f > 0 \)

\[
m_f(Q) = (f\chi_Q)^*(|Q|/2).
\]

It was proved in \cite{5} (see Lemma 2.2), that

\[
\lim_{|Q| \to 0, x \in Q} m_f(Q) = f(x) \tag{2.9}
\]

and for any constant \( c \),

\[
m_f(Q) - c = m_{f-c}(Q) \tag{2.10}
\]

**Remark 2.15.** If \( |f(x)| < |g(x)| \) then \( f^*(t) > g^*(t) \) for all \( t > 0 \).

3. APPLICATIONS

In this section we give several application to the “one-sided Lerner formula” (Theorem 1.3).

3.1. Weighted \( L^1 \)-norms for a Coifman-Fefferman inequality.

First we start with the following application: a weighted \( L^1 \)-norms inequality relating a \( f \in L^1(w) \) and a sharp maximal operator, when the weight \( w \in A^+_p \). We also give a local version of this one (see Lemma 4.6). In \cite{16} the authors obtain similar results as Theorems 3.1 and 3.2 for a sharp operator, the Hardy-Littlewood Maximal function, Calderón-Zygmund operators and Muckenhoupt weights. We obtain the following results:

**Theorem 3.1.** Let \( w \in A^+_p \) and \( 0 < \delta < 1 \). Then there is a constant \( C > 0 \), \( C = C_\delta \) such that

\[
\int_{\mathbb{R}} |f(x)|w(x) \, dx \leq C 6^p\|w\|_{A^+_p} \int_{\mathbb{R}} M^+_{\delta \#} f(x)w(x) \, dx.
\]

The next theorem is direct consequence of Theorem 3.1. This result was already proved in a different way in \cite{19}. 

**Theorem 3.2.** Let $T^+$ be an one-sided singular integral. Given $w \in A^+_p$, there is a constant $C > 0$ such that

$$\int_{\mathbb{R}} |T^+ f(x)| w(x) \, dx \leq C 2^p ||w||_{A^+_p} \int_{\mathbb{R}} M^+ f(x) w(x) \, dx.$$  

3.2. **Two-weight norm inequalities for one-sided singular integrals.**

Recently A. Lerner in [6] proved a conjecture stated by D. Cruz-Uribe and C. Pérez in [4] namely,

Let $T$ be a Calderón-Zygmund singular integral, $p > n$ and let $A$ and $B$ be two Young functions such that $A \in B_{p'}$ and $B \in B_p$. If

$$||u^{1/p}||_{A,Q} ||v^{-1/p}||_{B,Q} \leq \infty,$$

for all cube $Q \subset \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |T f(x)|^p u(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx.$$  

This result generalized the one obtain by Neugebauer in [9], where he proved inequality (3.2) in the case that $u, v$ satisfy the following bump condition,

$$||u^{1/p}||_{r_p,Q} ||v^{-1/p}||_{r_p,Q} \leq \infty.$$  

for some $r > 1$. The next theorem give a more general result, when we consider $T^+$ an one-sided singular integral. Using **Theorem 1.3** we will obtain a greater class of weights for witch inequality (3.2) is also true. Similar results with Orlicz bumps in one of the two-weights, for the one-sided case, were obtained in [18] and [7].

**Theorem 3.3.** Let $T^+$ be an one-sided singular integral. Let $A$ and $B$ be two Young functions such that

$$\overline{A} \in B_{p'} \quad \text{and} \quad \overline{B} \in B_p.$$  

If

$$||u^{1/p}||_{A,(a,b)} ||v^{-1/p}||_{B,(b,c)} \leq \infty,$$

for all $a < b < c$ with $b - a < c - b$. Then

$$\int_{\mathbb{R}} |T^+ f(x)|^p u(x) \, dx \leq C \int_{\mathbb{R}} |f(x)|^p v(x) \, dx.$$  

3.3. **Sharp $A^+_1$ inequality.**

In [19], it was studied that, for any $w \in A^+_1$,

$$||T^+ f||_{L^p(w)} \leq C p ||w||_{A^+_1} ||f||_{L^p(w)}.$$  

The fundamental result to prove (3.4) is the following inequality

$$||T^+ f||_{L^p((M^+_w)^{(p-1)})} \leq C p ||M^+ f||_{L^p((M^+_w)^{(p-1)})}.$$  

This last inequality is regarded as a Coifman-type inequality. Let us observe the sharp dependence on $p$ on the right-hand side. Using **Theorem 1.3** we will prove the following more general result.
Theorem 3.4. Let $T^+$ be an one-sided singular integral. For any appropriate function $f$ and for any locally integrable function $\varphi$, we have

$$||T^+ f||_{L^p((M^+\varphi)^{-\mu})} \leq C_T \max\{p2^\mu, \mu2^\mu\}||M^+ f||_{L^p((M^+\varphi)^{-\mu})}$$

where $1 < p < \infty$ and $\mu > 0$.

By Remark 2.10 and Remark 2.9 we get $(M^+\varphi)^{-\mu} \in A^+_p$.

Remark 3.5. Previously, we mention the sharp dependence on $p$. Let us consider the $A^+_p$ constant with the following definition

$$[w]_{A^+_p} = \sup_{a<b, c = a+b/2} \left\{ \left( \frac{1}{c-a} \int_a^c w \right) \left( \frac{1}{b-c} \int_c^b w^{p-1} \right)^{p-1} \right\}.$$ 

In [12] it was proved that

$$2^{-p'} [w]_{A^+_p}^{1/p-1} \leq ||w||_{A^+_p}^{1/p-1} \leq [w]_{A^+_p}^{1/p-1}.$$

Then Lemma 2.12 (see [12]) gives

$$||M^-||_{L^p(w)} \leq C_p[w]_{A^+_p}^{1/p-1}.$$ 

Using this definition in the constant $A^+_p$ the inequality of the last Theorem can be rewritten in the following way:

$$||T^+ f||_{L^p((M^+\varphi)^{-\mu})} \leq C_T \max\{p, \mu\}||M^+ f||_{L^p((M^+\varphi)^{-\mu})}.$$ 

To obtain Theorem 3.4 we need the following results that yield from Theorem 1.3.

Theorem 3.6. For any measurable function $f$ with $f^*(t) \to 0$ where $t \to \infty$ and for any weight $w$ we have

$$\int_{\mathbb{R}} |f|w(x) \, dx \leq C \int_{\mathbb{R}} (M^+_{1/4} f(x))^{\delta} M^- [(M^+_{1/4} f)^{1-\delta} w](x) \, dx,$$

where the constant $C$ not depends of $w$, and $0 < \delta < 1$.

Now, using that $M^+_{\delta} (T^+ f)(x) \leq CM^+ f(x)$ and Theorem 3.6 we get

Theorem 3.7. Let $T^+$ be an one-sided singular integral. For any appropriate function $f$ and for any weight $w$, we have

$$\int_{\mathbb{R}} |T^+ f| w(x) \, dx \leq C_T \int_{\mathbb{R}} (M^+ f(x))^{\delta} M^- [(M^+ f)^{1-\delta} w](x) \, dx,$$

where $0 < \delta < 1$. 

4. Proof of the main result.

Proof of **Theorem 1.3** For $I$ be a fixed interval and $f$ be a measurable function and $x \in I$. We define

$$m_I f(x) = \sup_{h > 0} \{m_f(x - h, x) : (x - h, x) \subset I^-\}.$$ 

Set $f_1(x) = f(x) - m_f((I^0)^+)$ and

$$E_1 = \{x \in (I^0)^- : |f_1(x)| > \omega_1/4(f, (I^0)^+)\}.$$ 

If $|E_1| = 0$ we trivially have

$$|f - m_f((I^0)^+)| \leq \omega_1/4(f, (I^0)^+) \leq M_{1/4}^+(f)(x).$$

Assume therefore that $|E_1| > 0$, and consider

$$\Omega_1 = \{x \in (I^0)^- : m_{f_1}(x) > \omega_1/4(f, (I^0)^+)\}.$$ 

By (2.9), $|\Omega_1| \geq |E_1| > 0$, then $\Omega_1 \neq \emptyset$. We write $\Omega_1 = \bigcup J^j_1$, where $J_1^j = (a_1^j, b_1^j)$ are pairwise disjoint maximal interval such that

$$m_{f_1}(x, b_1^j) \leq \omega_1/4(f, (I^0)^+) \leq m_{f_1}(a_1^j, x), \quad (4.1)$$

for all $x \in J_1^j$ (see proof of Lemma 1 in [13]).

Now fix $j$ and we define the sequences $(x_{j,r})$ and $(y_{j,r})$ by

$$b_1^j - x_{j,r}^j = 2(b_{j-r}^j - y_{j-r}^j) = (2/3)^k |J_1^j|,$$

and the intervals $(I_1^j) = (x_{j,r}^j, y_{j,r}^j)$, (see **Proposition 3.6 in [15]**). Therefore the intervals $(I_1^j)^-\subset J_1^j$, and by (4.1)

$$m_{J_1^j}((I_1^j)^+) \leq \omega_1/4(f, I^0). \quad (4.2)$$

Let us show

$$\sum_j \sum_r |(I_1^j)^-| \leq 1/2 |(I^0)^-|. \quad (4.3)$$

By (2.8) we get

$$(f_1 \chi_{J_1^j})^*((|J_1^j|/2) \geq m_{f_1}(J_1^j) = \omega_1/4(f, (I^0)^+)$$

$$= ((f - m_f((I^0)^+) \chi_{(I^0)^+})^*((|I^0)^+|/4) = (f_1 \chi_{(I^0)^+})^*((|I^0)^+|/4).$$

Hence,

$$|\{x \in J_1^j : |f_1(x)| > (f_1 \chi_{(I^0)^+})^*((|I^0)^+|/4)\}| \geq |\{x \in J_1^j : |f_1(x)| > (f_1 \chi_{J_1^j})^*((|J_1^j|/2)\}|$$

$$\geq |J_1^j|/2,$$

and thus,

$$1/2 \sum_j \sum_r |(I_1^j)^-| = 1/2 \sum_j |J_1^j| \leq \sum_j |\{x \in J_1^j : |f_1(x)| > (f_1 \chi_{(I^0)^+})^*((|I^0)^+|/4)\}|$$

$$\leq |\{x \in \bigcup J_1^j : |f_1(x)| > (f_1 \chi_{(I^0)^+})^*((|I^0)^+|/4)\}|$$

$$\leq \{x \in (I^0)^- \cup I^0 \cup (I^0)^+ : |f_1(x)| > (f_1 \chi_{(I^0)^+})^*((|I^0)^+|/4)\}| \leq |I^0|/4,$$
where the last equation follows from Remark 2.15.

Now we define \( g_1 = f_1 \chi(I_0^-) - \Omega_1 \), then for all \( x \in (I_0^-) \), using that \( m_f((I_1^-)_{j,r}) = m_f((I_0^-)) \), we have

\[
f(x) - m_f((I_0^-)) = g_1(x) + f \chi_{\Omega_1}(x) - m_f((I_0^-)) \chi_{\Omega_1}(x)
\leq g_1(x) + \sum_{j,r} f(\chi_{(I_1^-)_{j,r}} - (x) + \sum_{j,r} \left( m_f((I_1^-)_{j,r}) - m_f((I_1^-)) \right) \chi_{(I_1^-)_{j,r}}(x)
\leq g_1(x) + \sum_{j,r} m_f((I_1^-)_{j,r}) \chi_{(I_1^-)_{j,r}}(x) + \sum_{j,r} \left( f(x) - m_f((I_1^-)_{j,r}) \right) \chi_{(I_1^-)_{j,r}}(x).
\]

We observe that

\[
|g_1(x)| \leq \omega_{1/4}(f, (I_0^-)) \leq M_{1/4, 0}(f), \quad \text{for a.e. } x \in (I_0^-) \setminus \Omega_1.
\]

The function \( f - m_f((I_1^-)_{j,r}) \) has the same behavior on \((I_1^-)_{j,r}\) as \( f - m_f((I_0^-)) \) has on \((I_0^-)\). Therefore, we can repeat the process for each \((I_1^-)_{j,r}\), and continue by induction.

Denote by \( I_{k+1}^{j,r} \) the intervals obtained at the \( k \)th stage. Let \( \Omega_k = \bigcup_{j,r}(I_k^{j,r})^- \) and \( f_{i,l,k}(x) = f(x) - m_f((I_{k+1}^{j,r})^-) \). Denote

\[
R_{1,k} = \{(i,l) : \Omega_k \cap (I_{k+1}^{j,r})^- = \emptyset\}, \quad R_{2,k} = \{(i,l) : \Omega_k \cap (I_{k+1}^{j,r})^- \neq \emptyset\}.
\]

Assume that \((i,l) \in R_{2,k}\). Setting \( S_{i,l,k} = \{(j,r) : (I_{l+1}^{j,r})^- \subset (I_{k+1}^{j,r})^-\} \), we have

\[
\Omega_{i,l,k} = \{x \in (I_{k+1}^{j,r})^- : m_{f_{i,l,k}}((I_{l+1}^{j,r})^-)(x) > \omega_{1/4}(f, (I_{k+1}^{j,r})^-)\} = \bigcup_{(j,r) \in S_{i,l,k}} (I_{k+1}^{j,r})^-.
\]

Observe that

\[
m_{f_{i,l,k}}((I_{k+1}^{j,r})^-) \leq \omega_{1/4}(f, (I_{k+1}^{j,r})^-), \quad \text{(where } (j,r) \in S_{i,l,k}). \tag{4.4}
\]

Further, similarly to (4.3),

\[
|\Omega_{i,l,k}| = |\Omega_k \cap (I_{k+1}^{j,r})^-| = \sum_{(j,r) \in S_{i,l,k}} |(I_{k+1}^{j,r})^-| \leq 1/2 |(I_{k+1}^{j,r})^-|. \tag{4.5}
\]

Now we define

\[
g_k(x) = \sum_{(i,l) \in R_{1,k}} f_{i,l,k} \chi_{(I_{k+1}^{j,r})^-}(x) + \sum_{(i,l) \in R_{2,k}} f_{i,l,k} \chi_{(I_{k+1}^{j,r})^- \setminus \Omega_{i,l,k}}(x). \tag{4.6}
\]

Then,

\[
f(x) - m_f((I_0^-)) \leq \sum_{\nu=1}^{k} g_{\nu}(x) + \sum_{\nu=1}^{k} \sum_{(i,l) \in R_{2,\nu}} \sum_{(j,r) \in S_{i,l,\nu}} m_{f_{i,l,\nu}}((I_{j,r}^{\nu})^+) \chi_{(I_{j,r}^{\nu})^-}(x) + \varphi_k(x),
\]

where

\[
\varphi_k(x) = \sum_{(i,l) \in R_{2,\nu}} \sum_{(j,r) \in S_{i,l,\nu}} \left( f(x) - m_f((I_{j,r}^{\nu})^+) \right) \chi_{(I_{j,r}^{\nu})^-}(x),
\]

and for the case \( \nu = 1 \)

\[
\sum_{(i,l) \in R_{2,1}} \sum_{(j,r) \in S_{i,l,1}} m_{f_{i}}((I_{j,r}^{1})^+) \chi_{(I_{j,r}^{1})^-}(x) \equiv \sum_{j} \sum_{r} m_{f_{1}}((I_{j,r}^{1})^+) \chi_{(I_{j,r}^{1})^-}(x).
\]
By (1.5), $|\Omega_k| \leq |\Omega_{k-1}|/2$ then $|\Omega_k| \leq |(I^0)^-|/(2^k)$. Since the support of $\varphi_k$ is $\Omega_k$ we have that $\varphi_k \to 0$ a.e. $x \in (I^0)^-$ when $k \to \infty$. Therefore a.e. $x \in (I^0)^-$,

$$f(x) - m_f(I^0) \leq \sum_{\nu=1}^{\infty} g_{\nu}(x) + \sum_{\nu=1}^{\infty} \sum_{(i,j) \in R_{2,\nu}} \sum_{(j_x,r) \in S_{i,t,\nu}} m_{f_{i,t,\nu}}((I^0_{j_x,r})^+) \chi_{(I^0_{j_x,r})^-}(x) \equiv \xi_1(x) + \xi_2(x).$$

It is easy to see that the supports of $g_{\nu}$ are pairwise disjoint and for a.e. $x \in (I^0)^-$,

$$|g_{\nu}(x)| \leq M^{1+\#}_{1/4,f_0}(f)(x) \chi_{\text{supp}(g_{\nu})},$$

hence $|\xi_1(x)| \leq M^{1+\#}_{1/4,f_0}(f)(x)$.

Next, we write

$$\xi_2(x) = \sum_{j} \sum_{r} m_{f_{j,r}}((I^0_{j,r})^+) \chi_{(I^0_{j,r})^-}(x) + \sum_{\nu=2}^{\infty} \sum_{(i,l) \in R_{2,\nu}} \sum_{(j_x,r) \in S_{i,l,\nu}} m_{f_{i,l,\nu}}((I^0_{j_x,r})^+) \chi_{(I^0_{j_x,r})^-}(x),$$

by (1.2),

$$\sum_{j} \sum_{r} |m_{f_{j,r}}((I^0_{j,r})^+) \chi_{(I^0_{j,r})^-}(x) \leq \sum_{j} \sum_{r} (\omega^+_{1/4}(f, (I^0)^+)) \chi_{(I^0_{j,r})^-}(x) \leq M^{1+\#}_{1/4,f_0}(f)(x).$$

Applying (4.4), we get that the second term on the right-hand sided is bounded by

$$\sum_{\nu=2}^{\infty} \sum_{(i,l) \in R_{2,\nu}} \sum_{(j_x,r) \in S_{i,l,\nu}} |m_{f_{i,l,\nu}}((I^0_{j_x,r})^+) \chi_{(I^0_{j_x,r})^-}(x) \leq \sum_{\nu=2}^{\infty} \sum_{(i,l) \in R_{2,\nu}} \sum_{(j_x,r) \in S_{i,l,\nu}} (\omega^+_{1/4}(f, (I^0_{i,l})^{-1})) \chi_{(I^0_{j_x,r})^-}(x) \leq \sum_{\nu=2}^{\infty} \sum_{i} \sum_{l} \omega^+_{1/4}(f, (I^0_{i,l})^{-1})) \chi_{(I^0_{i,l})^-}(x).$$

Combining this with the previous estimate yields

$$|\xi_2(x)| \leq M^{1+\#}_{1/4,(f_0)^+}(f)(x) + \sum_{\nu=1}^{\infty} \sum_{i} \sum_{l} (\omega^+_{1/4}(f, (I^0_{i,l})^+)) \chi_{(I^0_{i,l})^-}(x).$$

Unifying this with the estimate for $\xi_1$ completes the proof.

\[ \square \]

**Corollary 4.1.** Let $w \in A^+_p$, then

$$w((I^k_{j,r})^-) \leq 6^p ||w||_{A^+_p} w(E^k_{j,r+1}).$$

**Proof.** Since $w \in A^+_p$ and $E^k_{j,r+1} \subset I^k_{j,r}$, then

$$\frac{|E^k_{j,r+1}|^p}{|I^k_{j,r}|^p} \leq 2^p ||w||_{A^+_p} \frac{w(E^k_{j,r+1})}{w((I^k_{j,r})^-)},$$

and recalling that $\frac{1}{2} |(I^k_{j,r})^-| \leq |E^k_{j,r}|$ and $\frac{2}{3} |(I^k_{j,r})^-| = |(I^k_{j,r+1})^-|$, $w((I^k_{j,r})^-) \leq 6^p ||w||_{A^+_p} w(E^k_{j,r+1}).$
4.2. Proof of the results of the weighted $L^1$-norm of a Coifman-Fefferman inequality.

In order to obtain these results first we give some previous lemmas.

**Lemma 4.2.** Let $f > 0$ be a measurable function in $\mathbb{R}$ and $a < b < c$ with $b - a = 2(c - b)$. For all $x \in (a, b)$ we have

\[
\left( \frac{1}{c - b} \int_b^c |f(y) - f(b,c)|^\delta \, dy \right)^{\frac{1}{\delta}} \leq C_\delta M_\delta^+\# f(x). \tag{4.7}
\]

**Proof.** Fix $x \in (a, b)$ we define $h = c - x$, observe that $\frac{1}{c - b} \leq \frac{3}{h}$, then

\[
\left( \frac{1}{c - b} \int_b^c |f(y) - f(b,c)|^\delta \, dy \right)^{\frac{1}{\delta}} \leq \left( \frac{3}{h} \int_x^{x+h} |f(y) - f(x+h,c)|^\delta \, dy \right)^{\frac{1}{\delta}} + 3|f(x+h,c) - f(b,c)| \\
\leq \left( \frac{3}{h} \int_x^{x+h} |f(y) - f(x+h,c)|^\delta \, dy \right)^{\frac{1}{\delta}} + 3 \left( \frac{3}{h} \int_x^{x+h} |f(y) - f(x+h,c)|^\delta \, dy \right)^{\frac{1}{\delta}} \\
\leq C_\delta \left( \frac{1}{h} \int_x^{x+h} |f(y) - f(x+h,c)|^\delta \, dy \right)^{\frac{1}{\delta}} \leq C_\delta M_\delta^+\# f(x).
\]

□

**Lemma 4.3.** Let $f > 0$ be a measurable function in $\mathbb{R}$, $\lambda \in (0, 1)$ and $\delta > 0$ then for all $x \in I^+ \cup I^-$,

\[
\omega_\lambda(f, I^+) \leq C_{\lambda,\delta} M_\delta^+\# f(x),
\]

therefore,

\[
\omega_\lambda(f, I^+) \leq C_{\lambda,\delta} \inf_{y \in I^+ \cup I^-} M_\delta^+\# f(y).
\]

**Proof.** By the rearrangement properties and Lemma 4.2

\[
\omega_\lambda(f, I^+) \leq \left( \frac{2}{\lambda |I^+|} \int_{I^+} |f - m_{I^+}(I^+)|^\delta \right)^{\frac{1}{\delta}} \\
\leq \left( \frac{2}{\lambda |I^+|} \int_{I^+} |f - f_{I^+}|^\delta \right)^{\frac{1}{\delta}} + C_{\lambda,\delta} |f_{I^+} - m_{I^+}(I^+)| \\
\leq \left( \frac{2}{\lambda |I^+|} \int_{I^+} |f - f_{I^+}|^\delta \right)^{\frac{1}{\delta}} + C_{\lambda,\delta} \left( \frac{1}{|I^+|} \int_{I^+} |f - f_{I^+}|^\delta \right)^{\frac{1}{\delta}} \\
\leq C_{\lambda,\delta} M_\delta^+\# f(x).
\]

□

**Corollary 4.4.** Let $f > 0$ be a measurable function on $\mathbb{R}$, $I^0$ an interval, $x \in (I^0)^-$, $\lambda \in (0, 1)$ and $\delta > 0$. Then

\[
M_{\lambda,I^0}^+\# f(x) \leq C_{\lambda,\delta} M_\delta^+\# f(x). \tag{4.8}
\]
Remark 4.5. Let \( T^+ \) be an one-sided singular integral and \( \text{supp}(f) \subset J^- \cup J \cup J^+ \). If \( J^+ = (a, b) \) then \( T^+ f(x) = T^+ f(x(a,\infty))(x) \) for all \( x \in J^+ \). Let us observe the following estimate for the median value of \( T^+ f \) over interval \( J^+ \). By Kolmogorov’s inequality we have that

\[
m_{T^+ f}(J^+) \leq \left( \frac{2}{|J^+|} \left| f(x) \right|^{1/\delta} \right)^{1/\delta} \leq \left( \frac{2}{|J^+|} \left( \mu a, b \right)^{1/\delta} \right)^{1/\delta} = \frac{C_\delta}{|J^- \cup J \cup J^+|} \int_{J^- \cup J \cup J^+} |f(x)| \, dx \leq C_\delta \int_{J^+} |f| \, dx,
\]

therefore for all \( x \in J^- \cup J \) we get

\[
m_{T^+ f}(J^+) \leq CM^+ f(x). \tag{4.9}
\]

Lemma 4.6. Let \( J \) be an interval and \( f \in L^\infty_c(\mathbb{R}) \) with \( \text{supp}(f) \subset J^- \cup J \cup J^+ \). Given \( w \in A_p^+ \) and \( 0 < \delta < 1 \) there exists a constant \( C > 0 \), \( C = C_\delta \) such that

\[
\| f - m_f(J^+) \|_{L^1(w,J-)} \leq C 6^p \| w \|_{A_p^+} \| M^+ f \|_{L^1(w,J-)}.
\]

Proof. By the Theorem 1.3 we get

\[
\int_{J^-} |f(x) - m_f(J^+)| w(x) \, dx \leq 2 \int_{J^-} M^+_{1,4,J}(f)(x) w(x) \, dx + \int_{J^-} \sum_{k=1}^{\infty} \sum_j \sum_{r=1}^{\infty} \omega_{1/4}(f, (I_{j,r}^k)^+) \chi_{(I_{j,r}^k)^-} w(x) \, dx = I + II.
\]

Let us start with \( I \). Observe that by Lemma 4.3 for \( \lambda = \frac{1}{4} \), we get

\[
I = 2 \int_{J^-} M^+_{1/4,J}(f)(x) w(x) \, dx \leq C_\delta \int_{J^-} M^+_{\delta}(f)(x) w(x) \, dx.
\]

Let estimate \( II \). Observe that by Lemma 4.3

\[
\omega_{1/4}(f, (I_{j,r}^k)^+) \leq C_\delta \inf_{x \in I_{j,r}^k} M^+_{\delta}(f)(x),
\]

and by Corollary 4.1

\[
w((I_{j,r}^k)^-) \leq 6^p \| w \|_{A_p^+} w(E_{j,r+1}^k).
\]

Then,

\[
II = \sum_{k,j,r} \omega_{1/4}(f, (I_{j,r}^k)^+) \int_{J^-} \chi_{(I_{j,r}^k)^-} w(x) \, dx \leq C_\delta 6^p \| w \|_{A_p^+} \sum_{k,j,r} \inf_{x \in I_{j,r}^k} M^+_{\delta}(f)(x) \int_{I_{j,r}^k} \chi_{E_{j,r+1}^k}(x) w(x) \, dx.
\]
Finally using that \( \{E_{j,r}^k\} \) is a pairwise disjoint family such that \( E_{j,r+1}^k \subset I_{j,r}^k \),

\[
II \leq C_6 \delta^p \|w\|_{A_p^+} \int_{J^-} M_{\delta}^{+\#}(f)(x) w(x)dx.
\]

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Suppose \( f \in L_c^\infty(\mathbb{R}) \) with \( \text{supp}(f) \subset (-r,r) \), and \( w_N(x) = \text{sup}\{w(x), N\} \), \( w_N \in A_p^+ \). Given \( J^- = (-n,n) \), Lemma 4.6 and equation (2.8) implies

\[
\|f\|_{L^1(w_N, J^-)} \leq \|f - m_f(J^+)\|_{L^1(w_N, J^-)} + \|m_f(J^+)\|_{L^1(w_N, J^-)} \\
\leq C_6 \delta^p \|w\|_{A_p^+} \|M^{+\#}_\delta f\|_{L^1(w_N, J^-)} + \|f\|_{\infty}(\frac{2r}{n})^\frac{1}{2} 2nN,
\]

as the second term of the equation tends to zero as \( n \to \infty \), we get what stated.

Lemma 4.7. Let \( T^+ \) be an one-sided singular integral. Let \( J \) an interval and \( f \in L_c^\infty(\mathbb{R}) \) such that \( \text{supp}(f) \subset J^- \cup J \cup J^+. \) Given \( w \in A_p^+ \) there is a constant \( C > 0 \) such that

\[
\|T^+ f\|_{L^1(w, J^-)} \leq C_6 \delta^p \|w\|_{A_p^+} \|M^{+\#} f\|_{L^1(w, J^-)}.
\]

Proof. By Lemma 4.6 we get

\[
\int_{J^-} |T^+ f(x)| w(x)dx \leq \int_{J^-} |T^+ f(x) - m_{T^+ f}(J^+)| w(x)dx + \int_{J^-} m_{T^+ f}(J^+) w(x)dx \\
\leq C_6 \delta^p \|w\|_{A_p^+} \int_{J^-} M^{+\#}_\delta (T^+ f)(x) w(x)dx + \int_{J^-} m_{T^+ f}(J^+) w(x)dx = I + II.
\]

To estimate \( II \), we use Remark 4.5 then \( m_{T^+ f}(J^+) \leq CM^+ f(x) \) for all \( x \in J^- \). For \( I \) we use (2.6) to get that \( M^{+\#}_\delta (T^+ f)(x) \leq CM^+ f(x) \).

As a consequence of the previous lemma we get Theorem 3.2.

4.3. Proof of the two-weight inequality result.

Before proving Theorem 3.3 we need the following lemma.

Lemma 4.8. Let \( T^+ \) be an one-sided singular integral, then

\[
\omega_\lambda(T^+ f, I) \leq C_6 \sum_{l=1}^\infty \frac{1}{2^l} \frac{1}{2^l |I|} \int_{2^{l+1} I} |f(t)| dt, \tag{4.10}
\]

where if \( I = (x, x+h) \), we write \( 2^l I = (x, x+2^l h) \) for all \( l \in \mathbb{Z} \).

Proof. Observing the proof of Lemma 4.3,

\[
\omega_\lambda(T^+ f, I) \leq C_{\lambda, \delta} \left( \frac{1}{|I|} \int_I |T^+ f - T^+ f_I|^\delta \right)^\frac{1}{\delta} \leq C_{\lambda, \delta} \left( \frac{1}{|I|} \int_I |T^+ f - a|^\delta \right)^\frac{1}{\delta},
\]

for all \( a \in \mathbb{R} \). If \( I = (x, x+h) \), we write \( 2^l I = (x, x+2^l h) \) and \( 2^{\tilde{l}} I = (x+2^\tilde{l} h, x+2^{\tilde{l}+1} h) \) for all \( l \in \mathbb{N} \). We define \( f = f_1 + f_2 \), where \( f_1 = f_{\lambda 4I} \), then

\[
\omega_\lambda(T^+ f, I) \leq C_{\lambda, \delta} \left[ \left( \frac{1}{|I|} \int_I |T^+ f_1|^\delta \right)^\frac{1}{\delta} + \left( \frac{1}{|I|} \int_I |T^+ f_2 - a|^\delta \right)^\frac{1}{\delta} \right] = C_{\lambda, \delta} [I + II].
\]
Let us consider $I$. As $T^+$ is of weak type $(1, 1)$ using Kolmogorov’s inequality,

$$I = \left( \frac{1}{|I|} \int_I |T^+ f_1| \right)^\frac{1}{2} \leq C_\delta \left( \frac{1}{4|I|} \int_{4I} |f| \right).$$

For $II$ we take $a = T^+ f_2(x + h)$, then

$$II = \left( \frac{1}{|I|} \int_I |T^+ f_2(y) - T^+ f_2(x + h)|^\delta \, dy \right)^\frac{1}{\delta}.$$

Using property (3) of the kernel $K$, for every $y \in I$,

$$|T^+ f_2(y) - T^+ f_2(x + h)| \leq \int_{x+h}^\infty (K(y - t) - K(x + h - t)) f(t) \, dt \leq C \int_{x+h}^\infty \frac{x + h - y}{(t - x - h)^2} f(t) \, dt \leq C \sum_{l=2}^\infty h \int_{2^l I} \frac{|f(t)|}{(t - x - h)^2} \, dt \leq C \sum_{l=2}^\infty \frac{2^l}{(2^l - 1)^2} \frac{1}{2^l |I|} \int_{2^l I} |f(t)| \, dt,$$

then

$$II \leq C \sum_{l=2}^\infty \frac{1}{2^l |I|} \int_{2^l I} |f(t)| \, dt.$$

Therefore

$$\omega_\lambda(T^+ f, I) \leq C_{\lambda, \delta} \sum_{l=1}^\infty \frac{1}{2^l |I|} \int_{2^l I} |f(t)| \, dt. \quad \square$$

**Proof of Theorem 3.3** If $1 < p < \infty$, by the standard density argument, it is enough to prove

$$\int_\mathbb{R} |T^+ f(x)|^p u(x) \, dx \leq C \int_\mathbb{R} |f(x)|^p v(x) \, dx.$$

for $f \in C_c^\infty(\mathbb{R})$. For such an $f$, $T^+ f$ is well defined and $(T^+ f)^*(\infty) = 0$. Observe that $f^*(\infty) = 0$, if and only if $|\{x : |f(x)| > \alpha\}| < \infty$ for any $\alpha > 0$. Also observe that by Remark 4.5

$$\lim_{|I^+| \to \infty} m_{T^+ f}(I^+) = 0. \quad (4.11)$$

Let $g$ be a positive function $g \in L^p(\mathbb{R})$ such that $\|g\|_{L^p(\mathbb{R})} = 1$ and

$$\left( \int_\mathbb{R} |T^+ f(x)|^p u(x) \, dx \right)^{\frac{1}{p}} = \int_\mathbb{R} |T^+ f(x)| \, u^{\frac{1}{p}}(x) g(x) \, dx.$$

Then for $n \in \mathbb{N}$, let $I^- = (-n, n)$. Applying Theorem 1.3,

$$\int_{-n}^n |T^+ f(x) - m_{T^+ f}(I^+)| u^{1/p} g \, dx \leq 2 \int_{-n}^n M_{I/I}^{1/4, f}(T^+ f)(x) u^{1/p} g \, dx$$
\[ + \sum_{k=1}^{\infty} \sum_{j} \sum_{r=1}^{\infty} \omega_{1/4}(T^+ f, (I^k_{j,r})^+) \int_{(I^k_{j,r})^-} u^{1/p} g \, dx. \]

It is easy to check that, if \( \mathcal{A} \subset B_p \) then \( \mathcal{A}(t) \geq C t^p \). Hence the assumption (3.3) on \( u, v \) imply the condition (2.4), and therefore \( M^+ : L^p(v) \to L^p(u) \) is bounded. Hence by (4.4), (2.6) and Hölder’s inequality

\[ \int_{-n}^{n} M^+_{1/4}(T^+ f)(x) u^{1/p} g \, dx \leq \int_{\mathbb{R}} M^+_{1/4}(T^+ f)(x) u^{1/p} g \, dx \leq C \int_{\mathbb{R}} M^+(f)(x) u^{1/p} g \, dx \]

\[ \leq C||M^+ f||_{L^p(v)}||g||_{L^p} \leq C||f||_{L^p(v)}. \]

Now by (4.10) we have

\[ \sum_{k,j,r} \omega_{1/4}(T^+ f, (I^k_{j,r})^+) \int_{(I^k_{j,r})^-} u^{1/p} g \, dx \leq \sum_{k,j,r} \sum_{l=1}^{\infty} \frac{1}{2} \frac{1}{2^l} \int_{2^{l+1}(I^k_{j,r})^-} |f(t)| \, dt \int_{(I^k_{j,r})^-} u^{1/p} g \, dx. \]

Observe that for \( \mathcal{A} \) a young function and \( a < b < c \), exists \( C > 0 \), such that \( ||w||_{\mathcal{A}(b,c)} \leq C ||w||_{\mathcal{A}(a,c)} \). Applying generalized Hölder’s inequality for \( \mathcal{A}, \overline{\mathcal{A}} \) and \( \mathcal{B}, \overline{\mathcal{B}} \) and using (3.3), we get that the previous sum is bounded by

\[ \sum_{k,j,r} \sum_{l=1}^{\infty} \frac{1}{2} \frac{1}{2^l} \int_{2^{l+1}(I^k_{j,r})^-} |f(t)| v^{1/p} |v|^{-1/p} \, dt \int_{(I^k_{j,r})^-} u^{1/p} g \chi(I^k_{j,r}) \, dx \]

\[ \leq \sum_{k,j,r} \sum_{l=1}^{\infty} \frac{1}{2} \frac{1}{2^l} \int_{2^{l+1}(I^k_{j,r})^-} |f(t)| v^{1/p} |v|^{-1/p} \, dt \int_{(I^k_{j,r})^-} u^{1/p} g \chi(I^k_{j,r}) \, dx \]

\[ \leq C \sum_{k,j,r} \sum_{l=1}^{\infty} \frac{1}{2} \frac{1}{2^l} \int_{2^{l+1}(I^k_{j,r})^-} |f(t)| v^{1/p} |v|^{-1/p} \, dt \int_{(I^k_{j,r})^-} u^{1/p} g \chi(I^k_{j,r}) \, dx \]

\[ \leq C \sum_{k,j,r} \sum_{l=1}^{\infty} \frac{1}{2} \frac{1}{2^l} \int_{2^{l+1}(I^k_{j,r})^-} |f(t)| v^{1/p} |v|^{-1/p} \, dt \int_{(I^k_{j,r})^-} u^{1/p} g \chi(I^k_{j,r}) \, dx \]

\[ \leq C \sum_{l=1}^{\infty} \frac{1}{2} \sum_{k,j,r} \int_{E_{j,r}^k} M^+(f v^{1/p}) M^-_{\mathcal{A}}(g) \, dx \]

\[ \leq C \sum_{l=1}^{\infty} \frac{1}{2} \int_{\mathbb{R}} M^+(f v^{1/p}) M^-_{\mathcal{A}}(g) \, dx \]

\[ \leq C \sum_{l=1}^{\infty} \frac{1}{2} \int_{\mathbb{R}} M^+(f v^{1/p}) \, dx \]

\[ \leq C \sum_{l=2}^{\infty} \frac{1}{2^l} ||f||_{L^p(v)} \leq C \sum_{l=2}^{\infty} \frac{1}{2^l} ||f||_{L^p(v)} \leq C ||f||_{L^p(v)}. \]

Combining the obtained estimates

\[ \int_{-n}^{n} |T^+ f(x) - m_{T^+ f}(I^+)| u^{1/p} g \, dx \leq C ||f||_{L^p(v)}. \]
Now, taking limit when $n \to \infty$ in (4.13) and using (4.11) along with Fatou’s convergence theorem, we get

$$
\left( \int_{\mathbb{R}} |T^+ f(x)|^p u(x) dx \right)^{\frac{1}{p}} = \int_{\mathbb{R}} |T^+ f(x)|u^{1/p}g dx \leq C||f||_{L^p(v)},
$$

which completes the proof. \hfill \Box

4.4. Proof of the Sharp $A^+_I$ inequalities results.

Proof of Theorem 3.6 Let $(I^0)^- = (-n,n)$. Again as inequality (4.13) we set

$$
\int_{-n}^{n} |f(x)| w(x) dx \leq \int_{-n}^{n} |f(x) - m_f((I^0)^+)| w(x) dx + \int_{-n}^{n} m_f((I^0)^+) w(x) dx. \tag{4.14}
$$

Let study the first summand in (4.14). By Theorem 1.3,

$$
\int_{-n}^{n} |f(x) - m_f((I^0)^+)| w(x) dx \leq 2 \int_{-n}^{n} M^{+,\#}_{1/4,I^0}(f)(x) w(x) dx,
$$

and

$$
\int_{-n}^{n} \sum_{k=1}^{n} \sum_{j,r} \omega^{+,\#}_{1/4}(f, (I^k_{j,r})^+) \chi_{(I^k_{j,r})^-}(x) w(x) dx = I + II.
$$

By Corollary 4.4 we get $M^{+,\#}_{1/4,I^0} f(x) \leq C_{\delta} M^{+,\#}_{\delta} f(x)$ for $x \in (I^0)^-$, then

$$
I = 2 \int_{-n}^{n} M^{+,\#}_{1/4,I^0}(f)(x) w(x) dx \leq 2 C \int_{-n}^{n} M^{+,\#}_{1/4}(f)(x) w(x) dx \leq C \int_{-n}^{n} [M^{+,\#}_{1/4}(f)]^\delta M^{-([M^{+,\#}_{1/4}(f)]^{1-\delta} w)](x) dx.
$$

Now $II$. Recall that $\{E^k_{j,r}\}$ is a pairwise disjoint family with $E^k_{j,r+1} \subset I^k_{j,r}$ and $\frac{1}{2}|(I^k_{j,r})^-| \leq |E^k_{j,r}|$. By Lemma 4.3 we get $\omega^{+,\#}_{1/4}(f, (I^k_{j,r})^+) \leq C_{\delta} \inf_{x \in (I^k_{j,r})^-} M^{+,\#}_{\delta} f(x)$. Then

$$
II = \int_{-n}^{n} \sum_{k=1}^{n} \sum_{j} \sum_{r=1}^{\infty} \omega^{+,\#}_{1/4}(f, (I^k_{j,r})^+) \chi_{(I^k_{j,r})^-}(x) w(x) dx
$$

$$
\leq \sum_{k=1}^{n} \sum_{j} \sum_{r=1}^{\infty} C \inf_{x \in (I^k_{j,r})^-} M^{+,\#}_{1/4} f(x) w(I^k_{j,r})^- 
$$

$$
\leq C \sum_{k=1}^{n} \sum_{j} \sum_{r=1}^{\infty} \int_{E^k_{j,r+1}} [M^{+,\#}_{1/4} f(x)]^\delta M^{-[M^{+,\#}_{1/4} f(x)]^{1-\delta} w(x)] dx
$$

$$
\leq C \int_{-n}^{n} [M^{+,\#}_{1/4} f(x)]^\delta M^{-[M^{+,\#}_{1/4} f(x)]^{1-\delta} w(x)] dx.
$$

Then putting together the estimates obtained for $I$ and $II$, we get

$$
\int_{-n}^{n} |f(x) - m_f((I^0)^+)| w(x) dx \leq C \int_{-n}^{n} [M^{+,\#}_{1/4} f(x)]^\delta M^{-[M^{+,\#}_{1/4} f(x)]^{1-\delta} w(x)] dx. \tag{4.15}
$$
Combining inequalities (4.15) and (4.14),
\[
\int_{-n}^{n} |f|(x)w(x) \leq C \left( \int_{-n}^{n} [M_{1/4}^+(f)]^\delta M^-(M_{1/4}^+(f))^{1-\delta} w(x) \right) dx + \int_{-n}^{n} m_f((I^0)^+) w(x) dx \tag{4.16}
\]
Finally letting \( n \to \infty \) and using the same argument as in the proof of Theorem 3.1 for the second summand in (4.16), completes the proof.

**Proof of Theorem 3.4** Using duality,
\[
||T^+ f||_{L^p(M^+\varphi^{-\mu})} = \sup_{\|h\|_{L^q(M^+\varphi^{(p-\mu)/p-1})} = 1} \int_{\mathbb{R}} |T^+ f(x)||h(x)| \, dx.
\]
Let \( h \) with \( ||h||_{L^q(M^+\varphi^{(p-\mu)/p-1})} = 1 \). By Theorem 3.7 with \( \delta = \min\{p/(2\mu + 1), 1\} \), using H"older’s inequality with \( q = p/\delta \) and \( q' = p/(p-\delta) \),
\[
\int_{\mathbb{R}} |T^+ f(x)||h(x)| \, dx \leq C_T \int_{\mathbb{R}} (M^+ f(x))^\delta M^-(M^+ f)^{1-\delta}|h|| \, dx \\
\leq C_T \int_{\mathbb{R}} (M^+ f(x))^\delta M^-(M^+ f)^{1-\delta}|h||(x)(M^+\varphi(x))^{-\mu\delta/p}(M^+\varphi(x))^{\mu\delta/p} \, dx \\
\leq C_T A \|M^+(f)\|_{L^p(M^+\varphi^{-\mu})}^{\delta},
\]
where
\[
A = \left( \int_{\mathbb{R}} (M^-[(M^+ f)^{1-\delta}|h|| \, dx)^{p/(p-\delta)}(M^+\varphi(x))^{\mu\delta/(p-\delta)} \, dx \right)^{(p-\delta)/p}.
\]
Suppose \( \mu \leq (p-1)/2 \), then in this case \( \delta = 1 \). By Lemma 2.12 and Remark 2.10 we get \( (M^+\varphi)_{p/\delta} \in A_p^- \), therefore
\[
A = \|M^- h\|_{L^p(M^+\varphi^{(p-\mu)/p-1})} \leq C p^{2\mu} ||h||_{L^q(M^+\varphi^{(p-\mu)/p-1})} = C p^{2\mu}.
\]
Assume now that \( \mu > (p-1)/2 \), then \( \delta = p/(2\mu + 1) \) and hence \( \mu\delta/(p-\delta) = 1/2 \). Applying again Lemma 2.12, we obtain
\[
A = \|M^-[(M^+ f)^{1-\delta}|h|| \, L^p/(p-\delta)((M^+\varphi)^{1/2}) \leq C \mu^{2\mu} \|M^+ f\|^{1-\delta} ||h||_{L^p/(p-\delta)((M^+\varphi)^{1/2})}.
\]
By H"older’s inequality with \( q = (p-\delta)/(1-\delta) \) and \( q' = (p-\delta)/(p-1) \),
\[
\|(M^+ f)^{1-\delta}|h||\|_{L^p/(p-\delta)((M^+\varphi)^{1/2})} \leq \int_{\mathbb{R}} [(M^+ f)^{1-\delta}|h||(M^+\varphi)^{(\delta-1)/2\delta}] \|h||_{L^p/(p-\delta)((M^+\varphi)^{1/2})} \\
\leq \|(M^+ f)^{1-\delta}|h||_{L^p/(p-\delta)((M^+\varphi)^{-\mu})} \|h||_{L^p/(p-\delta)((M^+\varphi)^{-\mu})},
\]
then
\[
A \leq C \mu^{2\mu} ||M^+ f||_{L^p(M^+\varphi^{-\mu})}^{1-\delta}.
\]
Combining these estimates the proof is completed.
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