Nahm Equations and Boundary Conditions

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Abstract

We derive certain boundary conditions in Nahm’s equations by considering a system of $N$ parallel D1-branes perpendicular to a D3-brane in type IIB string theory.

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1 Introduction

Nahm’s equations for $SU(N)$ monopoles [1] were derived using D-brane techniques (for a review see eg. [2]) for the first time in [3]. Nahm’s data however are incomplete without specifying the boundary conditions (bc) [4] and a D-brane derivation of those was lacking. Steps in this direction have been made in [5,6].

In this paper we show how Nahm’s bc arise naturally in the context of D-brane physics by considering (following [3]) a system in type IIB string theory of $N$ infinite parallel D1-branes perpendicular to a D3-brane. The present analysis is relevant only to the case of discontinuous but finite Nahm’s data.

In order to derive the advertised result we consider a certain reduction / truncation of the low-energy effective action on the world-volume of the D1-branes. Nahm’s equations and bc then follow from the requirement of a supersymmetric vacuum.

The discontinuities in Nahm’s data are encoded in the vacuum expectation values of the hypermultiplets, coming from the 1-3 sector, which appear as source terms localized on the intersection of the D1/D3 branes. This is the main result of this paper and is contained in Eq. (8) below. Similar results have recently been obtained by Kapustin and Sethi [7].

2 Analysis

Consider the configuration, shown in fig.1, of a D3-brane extended in $x^{0,1,2,3}$ intersecting a system of $N$ infinite parallel D1-branes extended in $x^{9,9}$, at the point $x^9 = 0$ of the D1-branes’ world volume. Let $Q_{L,R}$ be the supercharges associated to left, right-moving degrees of freedom of the IIB world-sheet theory. A D3-brane is invariant under supersymmetry transformations $\epsilon_L Q_L + \epsilon_R Q_R$ such that

$$\epsilon_L = \Gamma^{0123} \epsilon_R ,$$

(1)

where $\epsilon_{L,R} \sim 16_+$ of Spin(1,9). Similarly the D1-brane imposes the condition

$$\epsilon_L = \Gamma^{09} \epsilon_R .$$

(2)

The above two conditions imply

$$\epsilon_L = \Gamma^{1239} \epsilon_L .$$

(3)
Hence the supersymmetry parameter $\epsilon^A_L$ must be in the $(2^+, 4^+)$ of $\text{Spin}(4)_{1239} \otimes \text{Spin}(1,5)_{04-8}$ where $A(\alpha)$ is a $\text{Spin}(4)(\text{Spin}(1,5))$ index. In addition $\epsilon^A_L$ must satisfy the “$SU(2)$ Majorana condition”

$$\epsilon^{AB} C^\alpha_{\beta \gamma} \epsilon^\gamma_L = (\epsilon^\alpha_L)^* ,$$

where $\epsilon^{AB}$ is the rank-2 antisymmetric tensor and $C$ is the charge conjugation matrix in $\mathbb{R}^{1,5}_{04-8}$ (this comes from the Majorana condition in 10d). Therefore the configuration of fig.1 leaves 8 real supercharges unbroken. Note that the original ten-dimensional $\text{Spin}(1,9)$ invariance is broken down to $\text{Spin}(3)_{123} \otimes \text{Spin}(5)_{4-8}$.

It will be convenient to parametrize the unbroken supersymmetry by a pair of chiral $\text{Spin}(4)_{1239}$ spinors $\eta_i, i = 1, 2$, transforming as a doublet of $SU(2) R$-symmetry. $SU(2)_R$ can be thought of as $\text{Spin}(3)_{678}$ (see below).

The low-energy fields on the D1-branes’ world-volume can be found by quantizing the different string-theory sectors [2,8]: The 1-1 strings give bosonic fields $X^M_m, M = 0,...,9$, transforming as a vector of $SO(1,9)$ and a Majorana-Weyl fermion $\psi_{mn}$ in 10d. These fields are in the adjoint of $SU(N): m, n = 1,...,N$. The 1-3 sector gives a spinor of $\text{Spin}(4)_{1239}$ and a spinor of $\text{Spin}(1,5)_{04-8}$, both in the fundamental of $SU(N)$. (A GSO projection matching bosonic and fermionic degrees of freedom should be imposed). There are also fields coming from the 3-3 sector but these are external and will not be taken into account here.

We will consider the situation where the branes do not fluctuate along $x^{6,7,8}$ and we will set $X^{6,7,8} = 0$. We also gauge away the longitudinal component $X^0$ of the gauge field of the D1-branes’ world-volume. Moreover, anticipating Nahm’s equations, we will require that the fields be time-independent and that they depend only on $x^9$, which we denote by $s$. The original 10-dimensional spacetime symmetry is thus further broken down to a global $\text{Spin}(3)_{123} \otimes \text{Spin}(2)_{45} \otimes \text{Spin}(3)_{678}$.

The fields $X^4, X^5, X^\mu, \mu = 1, 2, 3, 9$, constitute the bosonic part of the Yang-Mills multiplet of the Euclideanized version of “$N = 2$ matter” theory in 4d [9] reduced to one spatial dimension, namely $x^9$. Of course upon dimensional reduction only the $\text{Spin}(3)_{123}$ subgroup of the $\text{Spin}(4)_{1239}$ symmetry of the 4d theory survives as a (global) symmetry of the low-energy effective action. The full field content is organized as follows:

**Yang-Mills multiplet:** $X^9$ the world-volume “gauge field”; $X^a_{mn} a = 1, 2, 3$, a vector of $SO(3)_{123}$; $X^4_{mn}, X^5_{mn}$ an $SO(2)_{45}$ doublet of real bosons;
\( \lambda_{imn} \), \( i = 1, 2 \), an \( SU(2)_R \) doublet of \( Spin(3)_{123} \) spinors. All the fields in the Yang-Mills multiplet are in the adjoint of \( SU(N) \): \( m, n = 1, \ldots N \).

**Hypermultiplets**: An \( SU(2)_R \) doublet of complex bosons \( h_{im} i = 1, 2; \) a spinor \( \chi_m \) of \( Spin(4)_{1239} \) (which should really be thought of as a pair of \( Spin(3)_{123} \) spinors). All the matter fields are in the fundamental of \( SU(N) \): \( m = 1, \ldots N \).

The hypermultiplets live on the intersection of the D1/D3-branes and are thus localized at the points \( s=0 \) on the D1-branes’ world-volume.

For an off-shell realization we have to include the auxiliary bosonic fields \( D_{ijmn} = D_{jimn} = (D_{ij'}mn)^* \varepsilon_{ii'} \varepsilon_{jj'} \) (a triplet of \( SU(2)_R \) in the adjoint of \( SU(N) \)) and \( F_{im} \) (an \( SU(2)_R \) doublet in the fundamental of \( SU(N) \)).

The low-energy effective Langrangian reads (after setting \( g_{YM} = 1 \)):

\[
L = L_{\text{kinetic}}(X^9, X^a, X^4, X^5, \lambda_i) + L_0 + \delta(s) \{ L_{\text{kinetic}}(h_{im}, \chi_m) + L_{\text{interaction}} \} 
\]  

(5)

where

\[
L_0 = \frac{1}{2} Tr \{ (D^2 - ([X^4, X^5])^2) \} + \text{fermions} 
\]  

(6)

and

\[
L_{\text{interaction}} = |F_{im}|^2 + h_{im}^*[(X^4)^2 + (X^5)^2]_{mn} h_{in} + D_{ijmn} h_{im}^* h_{jn} + \text{fermions} 
\]  

(7)

The Langrangian is invariant under SUSY transformations parametrized by \( \eta_i \). For a supersymmetric vacuum we must require the vanishing of the SUSY variation of the gaugino. Setting \( X^9, X^4, X^5 = 0 \) and taking into account the equations-of-motion for \( D \), this condition reduces to

\[
\frac{dX^a}{ds} + \varepsilon^{abc}[X^b, X^c] = -i \delta(s) \sigma^a h_i^* \otimes h_j \ , \ a, b, c = 1, 2, 3 
\]  

(8)

where \( \sigma^a \) are the Pauli matrices. Equation (8) is easily seen to reproduce Nahm’s bc (Nahm’s equations are just: left-hand-side of (8)=0) in the case of discontinuous but finite \( X^a(s) \). The discontinuities in the mathematical literature are given in terms of a rank-one \( N \times N \) complex matrix which is parametrized by two complex \( N \)-vectors, the \( u_0, u_1 \) of [4]. These are essentially the \( h_{1m}, h_{2m} \) of Eq.(8).

\footnote{In fact the \( \delta(0)^2 \) term in the on-shell SUSY variation of \( L \) is ambiguous. The \( \delta \) function should be thought of as receiving some kind of stringy regularization. We thank Greg Moore for pointing this out and for discussions on this issue.}
3 Conclusion and Discussion

The discontinuity conditions in Nahm’s data were derived from D-brane considerations.

It should be noted that the $D^2$ term in the Langrangian apparently gives rise (on-shell) to an infinite-energy contribution. This is due to the singular nature of the configuration depicted in fig.1. However this picture is expected to get smoothed-out by the 1-3 sector strings in a similar to [10] manner. Another way of seeing this would be to “lift” the system to $M$ theory [11].

In Ref.[5] it was suggested that stringy contributions will give rise to a $\delta$ function regularization of the form

$$\delta(s) \to \frac{1}{l_s} e^{-\frac{s^2}{l_s^2}}$$

It would be interesting to see how can such stringy effects be taken into account in a systematic expansion in the string scale $l_s$. This problem will hopefully be investigated in some future work.

Note: While this paper was in the final stage of preparation an overlapping publication [7] appeared.

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**Figure 1.** A D3-brane extended in $x^{0,1,2,3}$ intersecting a system of $N$ parallel D1-branes. The intersection is at the point $x^9 = 0$ on the D1-branes’ world-volume.