DETERMINING FUZZY DISTANCE THROUGH NON-SELF FUZZY CONTRACTIONS

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Abstract: In the present work we solve the problem of finding the fuzzy distance between two subsets of a fuzzy metric space for which we use a non-self fuzzy contraction mapping from one set to the other. It is a fuzzy extension of the proximity point problem which is by its nature a problem of global optimization. The contraction is defined here by two control functions. We define a geometric property of the fuzzy metric space. The main result is illustrated with an example. Our result extends a fuzzy version of the Banach contraction mapping principle.

Keywords: Fuzzy Metric Spaces, Global Optimization, Proximity Point, Non-Self \((\phi - \psi)\)-Proximal Contraction, Optimal Approximate Solution, Fuzzy P-property.

MSC: 47H10, 54H25.
1. INTRODUCTION

In this paper we establish a proximity point result in a fuzzy metric space so to find the fuzzy distance between two subsets. The problem originated from the work of Eldred et al. [9] and has been well studied during the decade through works like [2, 5, 9, 15, 14, 16, 21, 22]. For our purpose we use a non-self contraction mapping which is defined by two control functions. The fuzzy metric space on which we deduce our results is as in George et al. [10]. Due to its special features, it has become the platform of several extensions of metric related studies [1, 3, 4, 6, 7, 11, 12, 13, 19]. The problem sought to be considered here is essentially a global optimization problem which is solved by transforming it to a problem of finding the optimal approximate solution to a fixed point equation for a non-self contraction defined by use of two control functions. Control functions have been used in several fixed point problems in metric spaces [20]. Here, as the contraction function is non-self mapping, there is no exact solution of the fixed point equation. The following are two special features of the present work.

1. We define and use a non-self contraction with two control functions.
2. We define and use a geometric property in the fuzzy metric space.

2. MATHEMATICAL PRELIMINARIES

George and Veeramani in their paper [10] introduced the following definition of fuzzy metric space. Throughout this paper, we use this definition of fuzzy metric space.

Definition 1. [10] The 3-tuple \((X, M, *)\) is called a fuzzy metric space if \(X\) is an arbitrary non-empty set, \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following conditions for each \(x,y,z \in X\) and \(t,s > 0\):

\[
\begin{align*}
& (j) \ M(x, y, t) > 0, \\
& (j) \ M(x, y, t) = 1 \text{ if and only if } x = y, \\
& (j) \ M(x, y, t) = M(y, x, t), \\
& (j) \ M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \text{ and} \\
& (j) \ M(x, y, :) : (0, \infty) \rightarrow (0, 1] \text{ is continuous},
\end{align*}
\]

where \(\ast\) is a continuous \(t\)-norm, that is, a continuous function \(\ast : [0, 1]^2 \rightarrow [0, 1]\) such that

\[
\begin{align*}
& (i) \ a \ast b = b \ast a \text{ for all } a, b \in [0, 1], \\
& (ii) \ a \ast (b \ast c) = (a \ast b) \ast c \text{ for all } a, b, c \in [0, 1], \\
& (iii) \ a \ast 1 = a \text{ for all } a \in [0, 1], \\
& (iv) \ a \ast b \leq c \ast d \text{ whenever } a \leq c \text{ and } b \leq d, \text{ for each } a, b, c, d \in [0, 1].
\end{align*}
\]
Let $(X, M, \ast)$ be a fuzzy metric space. For $t > 0$ and $r$ with $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by
\[ B(x, t, r) = \{ y \in X : M(x, y, t) > 1 - r \}. \]

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $r$ with $0 < r < 1$ such that $B(x, t, r) \subset A$. Let $\tau$ denote the family of all open subsets of $X$. Then $\tau$ is a topology and is called the topology on $X$ induced by the fuzzy metric $M$. The topology $\tau$ is a Hausdorff topology [10]. In fact, the definition 2.1 is a modification of the definition given in [17] for ensuring Hausdorff topology of the space.

**Definition 2.** [10] Let $(X, M, \ast)$ be a fuzzy metric space. A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$.

**Definition 3.** [10] Let $(X, M, \ast)$ be a fuzzy metric space. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if for each $\varepsilon$ with $0 < \varepsilon < 1$ and $t > 0$, there exists a positive integer $n_0$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

A fuzzy metric space is said to be complete if every Cauchy sequence is convergent in it.

The following lemma was proved by Grabiec [11] for fuzzy metric spaces defined by Kramosil et al [17]. The proof is also applicable to the fuzzy metric space given in definition 2.1.

**Lemma 4.** [11] Let $(X, M, \ast)$ be a fuzzy metric space. Then $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

**Lemma 5.** [18] $M$ is a continuous function on $X^2 \times (0, \infty)$.

We will require for use in our results the following two functions.

**Definition 6.** ($\Psi$-function)[23] A function $\psi : [0, \infty) \to [0, \infty)$ is a $\Psi$-function if

1. $\psi$ is nondecreasing and continuous,
2. $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^{n+1}(t) = \psi(\psi^n(t))$, $n \geq 1$.

It is clear that $\psi(t) < t$ for all $t > 0$ whenever $\psi$ is a $\Psi$-function.

The following function is an example of a $\psi$-function:
\[
\psi(t) = \begin{cases} 
  t - \frac{t^2}{2}, & \text{if } t \in [0, 1], \\
  \frac{t}{2}, & \text{if } t > 1.
\end{cases}
\]
Definition 7. [20] A function $\phi : [0, \infty) \to [0, \infty)$ is a $\Phi$-function if
(i) $\phi$ is nondecreasing and continuous,
(ii) $\phi(0) = 0$.

Lemma 8. [23] If $*$ is a continuous $t$-norm, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha$, $\gamma_n \to \gamma$ as $n \to \infty$, then $\lim_{k \to \infty} (\alpha_k * \beta_k * \gamma_k) = \alpha * \lim_{k \to \infty} \beta_k * \gamma$ and
$\lim_{k \to \infty} (\alpha_k * \beta_k * \gamma_n) = \alpha * \lim_{k \to \infty} \beta_k * \gamma$.

Lemma 9. [23] Let $\{f(k,.) : [0, \infty) \to [0,1]\}$ be a sequence of functions such that $f(k,.)$ is continuous and monotone increasing for each $k \geq 0$. Then $\lim_{k \to \infty} f(k,t)$ is a left continuous function in $t$ and $\lim_{k \to \infty} f(k,t)$ is a right continuous function in $t$.

3. MAIN RESULTS

Definition 10. [24] Let $(X,M,*)$ be a fuzzy metric space. The fuzzy distance of a point $x \in X$ from a nonempty subset $A$ of $X$ is
$M(x,A,t) = \sup_{a \in A} M(x,a,t)$ for all $t > 0$
and the fuzzy distance between two nonempty subsets $A$ and $B$ of $X$ is
$M(A,B,t) = \sup \{M(a,b,t) : a \in A, b \in B\}$ for all $t > 0$.

Let $A$ and $B$ be two nonempty disjoint subsets of a fuzzy metric space $(X,M,*)$. We write
$A_0 = \{x \in A : \exists y \in B \text{ such that } M(x,y,t) = M(A,B,t) \text{ for all } t > 0\}$,
$B_0 = \{y \in B : \exists x \in A \text{ such that } M(x,y,t) = M(A,B,t) \text{ for all } t > 0\}$.

Definition 11. Let $(X,M,*)$ be a fuzzy metric space and $A$, $B$ are two non-empty subsets of $X$. An element $x^* \in A$ is defined as a fuzzy best proximity point of the mapping $f : A \to B$ if it satisfies the condition that for all $t > 0$
$M(x^*,fx^*,t) = M(A,B,t)$.

In the following we define a property of a pair of subsets in a fuzzy metric space. It is essentially a geometric property.

Definition 12. Let $(A,B)$ be a pair of nonempty disjoint subsets of a fuzzy metric space $(X,M,*)$. Then the pair $(A,B)$ is said to satisfy the fuzzy P-property if for all $t > 0$ and $x_1,x_2 \in A$, $y_1,y_2 \in B$,
$M(x_1,y_1,t) = M(A,B,t)$ and $M(x_2,y_2,t) = M(A,B,t)$
jointly implies that
$M(x_1,x_2,t) = M(y_1,y_2,t)$.
The $P$-property is a geometric property which is automatically valid in Hilbert spaces for non-empty closed and convex pairs of sets [21], but does not hold in arbitrary Banach spaces. In metric spaces such property for pairs of subsets is separately assumed for specific purposes. The above definition is a fuzzy extension of that.

**Definition 13.** Let $(X, M, \ast)$ be a fuzzy metric space and $f : A \to B$ be a mapping. The mapping $f$ is non-self $(\phi - \psi)$- contraction mapping if there exist $\Psi$-function (definition 6) $\psi$, a $\Phi$-function (definition 7) $\phi$ and $0 < c < 1$ such that for all $t > 0$ and $x, y \in A$ we have

$$
\frac{1}{M(fx, fy, \phi(ct))} - 1 \leq \psi\left(\frac{1}{M(x, y, \phi(t))} - 1\right).
$$

(3.1)

**Note.** The above contraction condition with some variations in the condition on $\psi$ has already appeared in the context of fixed point studies in probabilistic metric spaces [8].

**Theorem 14.** Let $(X, M, \ast)$ be a complete fuzzy metric space. Let $A$ and $B$ be two closed subsets of $X$ and $f : A \to B$ be an $(\phi - \psi)$- contractive mapping such that the following conditions are satisfied.

(i) $(A, B)$ satisfies the fuzzy $P$-property,
(ii) $f(A_0) \subseteq B_0$,
(iii) $A_0$ is nonempty.

Then there exists an element $x^* \in A$ which is a fuzzy best proximity point of $f$.

**Proof.** By assumption (iii), $A_0$ is nonempty. Let $x_0 \in A_0$. Since $f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$
M(x_1, fx_0, t) = M(A, B, t) \text{ for all } t > 0.
$$

Again since $f(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$
M(x_2, fx_1, t) = M(A, B, t) \text{ for all } t > 0.
$$

Continuing this process, we construct a sequence $\{x_n\}$ in $A_0$ such that for all $n \geq 1$, for all $t > 0$,

$$
M(x_n, fx_{n-1}, t) = M(A, B, t).
$$

(3.2)

Also, we can write the above as

$$
M(x_{n+1}, fx_n, t) = M(A, B, t) \text{ for all } n \geq 1, \text{ for all } t > 0.
$$

(3.3)

Since $(A, B)$ satisfies the fuzzy $P$-property, we get from (3.2) and (3.3), for all $t > 0$

$$
M(x_n, x_{n+1}, t) = M(fx_{n-1}, fx_n, t) \text{ for all } n > 1.
$$

(3.4)

From the property of $\phi$ it is clear that for each $t > 0$ there exists $t_0 > 0$ such that $\phi(t_0) = t$. 

Since $f$ is $(\phi - \psi)$- contraction and from the property of $\phi$, we have for all $n \geq 1$, for all $t > 0$ there exist $t_0 > 0$ such that
\[
\frac{1}{M(fx_{n-1}, fx_n, t)} - 1 = \left( \frac{1}{M(fx_{n-1}, fx_n, t_0)} - 1 \right) \\
\leq \left( \frac{1}{M(fx_{n-1}, fx_n, \phi(t_0))} - 1 \right) \\
\leq \psi \left( \frac{1}{M(x_{n-1}, x_n, t_0)} - 1 \right) = \psi \left( \frac{1}{M(x_{n-1}, x_n, t)} - 1 \right)
\]
Therefore, we have for all $n \geq 1$, for all $t > 0$,
\[
\left( \frac{1}{M(fx_{n-1}, fx_n, t)} - 1 \right) \leq \psi \left( \frac{1}{M(x_{n-1}, x_n, t)} - 1 \right).
\]
Combining (3.4) and (3.5), we have for all $n \geq 1$, for all $t > 0$,
\[
\left( \frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 \right) \leq \psi \left( \frac{1}{M(x_{n-1}, x_n, t)} - 1 \right).
\]
(3.6)

If for some $k > 0$, $x_k = x_{k+1}$, then $x_k$ is a best proximity point of $f$.

Assuming $x_{n-1} \neq x_n$ for all $n \geq 1$, and making repeated applications of (3.6), we have for all $n \geq 1$, for all $t > 0$, 
\[
\left( \frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 \right) \leq \psi^n \left( \frac{1}{M(x_0, x_1, t)} - 1 \right).
\]
(3.7)

Taking $n \to \infty$ in the above inequality (3.7), for all $t > 0$, we obtain
\[
\lim_{n \to \infty} \left( \frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 \right) \leq \lim_{n \to \infty} \psi^n \left( \frac{1}{M(x_0, x_1, t)} - 1 \right) \to 0 \text{ as } n \to \infty, \text{ (by a property of } \psi).
\]
That is, \[
\lim_{n \to \infty} \left( \frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 \right) = 0, \text{ which implies that for all } t > 0,
\]
\[
\lim_{n \to \infty} M(x_n, x_{n+1}, t) = 1.
\]
(3.8)

Next, we show that $\{x_n\}$ is a Cauchy sequence in $A$. We suppose, if possible, that $\{x_n\}$ is not a Cauchy sequence in $A$. Then definition \[ is not satisfied by the sequence $\{x_n\}$ and, therefore, there exist some $\epsilon > 0$ and some $\lambda$ with $0 < \lambda < 1$, for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that
\[
M(x_{m(k)}, x_{n(k)}, \epsilon) \leq (1 - \lambda),
\]
for all positive integer $k$.
We may choose the $n(k)$ as the smallest integer exceeding $m(k)$ for which \[ holds. Then, for all positive integer $k$,
\[
M(x_{m(k)}, x_{n(k)-1}, \epsilon) > (1 - \lambda)
\]
(3.10)
Then, for all \( k \geq 1, 0 < s < \frac{1}{2} \), we obtain,
\[
(1 - \lambda) \geq M(x_{m(k)}, x_{n(k)}, \epsilon) \\
\geq M(x_{m(k)}, x_{m(k) - 1}, s) \ast M(x_{m(k) - 1}, x_{n(k) - 1}, \epsilon - 2s) \\
\ast M(x_{n(k) - 1}, x_{n(k)}, s).
\] (3.11)

For all \( t > 0 \), we denote
\[
h_1(t) = \lim_{k \to \infty} M(x_{m(k) - 1}, x_{n(k) - 1}, t).
\] (3.12)

Taking limit supremum on both sides of (3.11), using (3.8), the properties of \( M \) and \( \ast \), by lemma 9, we obtain
\[
(1 - \lambda) \geq 1 \ast \lim_{k \to \infty} M(x_{m(k) - 1}, x_{n(k) - 1}, \epsilon - 2s) \ast 1 = h_1(\epsilon - 2s)
\] (3.13)

Since \( M \) is bounded within the range in \([0,1]\), continuous and, by lemma 4, monotone increasing in the third variable \( t \), it follows by an application of lemma 9 that \( h_1 \), as given in (3.12), is continuous from the left. From the above, letting \( s \to 0 \) in (3.13), it then follows that
\[
\lim_{k \to \infty} M(x_{m(k) - 1}, x_{n(k) - 1}, \epsilon) \leq (1 - \lambda).
\] (3.14)

Let,
\[
h_2(t) = \lim_{k \to \infty} M(x_{m(k) - 1}, x_{n(k) - 1}, t), t > 0.
\] (3.15)

Again, for all \( k \geq 1, s > 0 \),
\[
M(x_{m(k) - 1}, x_{n(k) - 1}, \epsilon + s) \geq M(x_{m(k) - 1}, x_{m(k)}, s) \ast M(x_{m(k)}, x_{n(k) - 1}, \epsilon) \\
\geq M(x_{m(k) - 1}, x_{m(k)}, s) \ast (1 - \lambda), \text{ by (3.16)}
\] (3.16)

Taking limit infimum as \( k \to \infty \) in (3.16), by virtue of (3.8), we obtain
\[
h_2(\epsilon + s) = \lim_{k \to \infty} M(x_{m(k) - 1}, x_{n(k) - 1}, \epsilon + s) \geq \lim_{k \to \infty} M(x_{m(k) - 1}, x_{m(k)}, s) \ast (1 - \lambda) \\
= 1 \ast (1 - \lambda) = (1 - \lambda).
\] (3.17)

Since \( M \) is bounded within the range in \([0,1]\), continuous and by lemma 4, it is monotone increasing in the third variable \( t \), it follows by an application of lemma 9 that \( h_2 \), as given in (3.15) is continuous from the right. From the above, letting \( s \to 0 \) in (3.17), it then follows that
\[
\lim_{k \to \infty} M(x_{m(k) - 1}, x_{n(k) - 1}, \epsilon) \geq (1 - \lambda)
\] (3.18)

The inequalities (3.14) and (3.18) jointly imply that
\[
\lim_{k \to \infty} M(x_{m(k) - 1}, x_{n(k) - 1}, \epsilon) = (1 - \lambda).
\] (3.19)
Again by (3.9),
\[
\lim_{k \to \infty} M(x_m(k), x_n(k), \epsilon) \leq (1 - \lambda)
\]  
(3.20)

Also for all \( k \geq 1, s > 0 \), we obtain
\[
M(x_m(k), x_n(k), \epsilon + 2s) \geq M(x_m(k), x_m(k) - 1, s) \ast M(x_m(k) - 1, x_n(k) - 1, \epsilon) \ast M(x_n(k) - 1, x_n(k), s)
\]
Taking limit infimum as \( k \to \infty \) in the above inequality, using (3.8), (3.19) and the properties of \( M \) and \( \ast \), by lemma 8, we obtain
\[
\lim_{k \to \infty} M(x_m(k), x_n(k), \epsilon + 2s) \geq 1 + \lim_{k \to \infty} M(x_m(k) - 1, x_n(k) - 1, \epsilon) \ast 1 = 1 - \lambda.
\]
Since \( M \) is bounded within the range in \([0,1]\), is continuous and, by lemma 4, monotone increasing in the third variable \( t \), it follows by an application of lemma 9 that \( \lim_{k \to \infty} M(x_m(k), x_n(k), t) \) is continuous function of \( t \) from the right.

Taking \( s \to 0 \) in the above inequality, and using lemma 9, we obtain
\[
\lim_{k \to \infty} M(x_m(k), x_n(k), \epsilon) \geq (1 - \lambda),
\]  
(3.21)

Combining (3.20) and (3.21), we obtain
\[
\lim_{k \to \infty} M(x_m(k), x_n(k), \epsilon) = (1 - \lambda)
\]  
(3.22)

From (3.3), we have
\[
M(x_m(k), f x_m(k) - 1, t) = M(A, B, t)
\]  
(3.23)

\[
M(x_n(k), f x_n(k) - 1, t) = M(A, B, t)
\]  
(3.24)

Since \((A, B)\) satisfies the fuzzy \( P \)-property, we get from (3.23) and (3.24), for all \( t > 0 \),
\[
M(x_m(k), x_n(k), t) = M(f x_m(k) - 1, f x_n(k) - 1, t).
\]  
(3.25)

Now by the property of \( \phi \), there exists \( \epsilon_0 > 0 \) such that \( \phi(\epsilon_0) = \epsilon \).

Therefore, from the above and by (3.25),
\[
\frac{1}{M(x_m(k), x_n(k), \epsilon)} - 1 = \frac{1}{M(f x_m(k) - 1, f x_n(k) - 1, \epsilon)} - 1
\]

\[
= \left(\frac{1}{M(f x_m(k) - 1, f x_n(k) - 1, \phi(\epsilon_0))}\right) - 1
\]

\[
\leq \left(\frac{1}{M(f x_m(k) - 1, f x_n(k) - 1, \phi(\epsilon_0))}\right) - 1
\]

\[
= \psi(\frac{1}{M(f x_m(k) - 1, f x_n(k) - 1, \phi(\epsilon_0))} - 1)
\]

(3.25)
Taking $k \to \infty$ in the above inequality, we have
\[
\left( \lim_{k \to \infty} \frac{1}{M(x_{m(k)}, x_{n(k)}, \epsilon)} - 1 \right) \leq \psi \left( \lim_{k \to \infty} \frac{1}{M(x_{m(k)-1}, x_{n(k)-1}, \epsilon)} - 1 \right). \quad \text{(since $\psi$ is continuous)}
\]
Using (3.19) and (3.22), we have
\[
\left( \frac{1}{\lambda - 1} - 1 \right) \leq \psi \left( \frac{1}{\lambda - 1} - 1 \right) < \left( \frac{1}{\lambda - 1} - 1 \right),
\]
which is a contradiction.

Thus, it is established that $\{x_n\}$ is a Cauchy sequence. Since $(X, M, \ast)$ is complete, there exists $x^* \in A$ such that
\[
\lim_{n \to \infty} x_n = x^*.
\]
Since $f$ is $(\phi - \psi)$-proximal contractive mapping, by using (3.1), we have for all $n \geq 0, t > 0$
\[
\left( \frac{1}{M(f x_n, f x^*, t)} - 1 \right) = \left( \frac{1}{M(f x_n, f x^*, \phi(t_0))} - 1 \right)
\leq \left( \frac{1}{M(f x_n, f x^*, \phi(ct_0))} - 1 \right)
\leq \psi \left( \frac{1}{M(x_n, x^*, \phi(t_0))} - 1 \right)
\leq \psi \left( \frac{1}{M(x_n, x^*, t)} - 1 \right)
\]
Taking limit $n \to \infty$ on both sides of the above inequality, using the fact that $\psi(0) = 0$, we have
\[fx_n \to fx^* \quad \text{as} \quad n \to \infty.\]

From (3.3) and the above limit, for all $t > 0$
\[M(A, B, t) = M(x_{n+1}, f x_n, t) = M(x^*, f x^*, t) \quad \text{as} \quad n \to \infty.\]

Therefore, for all $t > 0$, $M(x^*, f x^*, t) = M(A, B, t)$. This completes the proof.

4. ILLUSTRATION

Example 15. Suppose that $X = \mathbb{R}^2$ with fuzzy metric space
\[M((x, y), (x', y'), t) = \frac{t}{t + |x-x'| + |y-y'|} \quad \text{and minimum $t$-norm} \ast.\]
Consider the closed subsets $A$ and $B$ in the topology induced by the fuzzy metric as
\[A = \{(0, x): x \in \mathbb{R}\},\]
\[B = \{(1, x): x \in \mathbb{R}\}.
\]
Let $\psi(t) = ct$ and $\phi(t) = t^2$, where $0 < c < 1$. Let $f : A \to B$ be the mapping defined by
\[f((0, x)) = (1, 1 - e^{-c} x).\]
Here $M(A, B, t) = \frac{t}{t+1}$ for all $t > 0$. 
Here \( A_0 = A \) and \( B_0 = B \) and \( f(A_0) \subseteq B_0 \).

Now we show that \( f \) satisfies fuzzy \( P^* \)-property.

Let \( u_1 = (0, x_1), u_2 = (0, x_2) \in A \) and \( v_1 = (1, y_1), v_2 = (1, y_2) \in B \) with

\[
M(u_1, v_1, t) = M(A, B, t) \quad \text{for all } t > 0 \tag{4.1}
\]

and

\[
M(u_2, v_2, t) = M(A, B, t) \quad \text{for all } t > 0 \tag{4.2}
\]

From (4.1), we get for all \( t > 0 \)

\[
\frac{t}{t+|x_1-y_1|} = \frac{t}{t+1},
\]

which implies that \( x_1 = y_1 \).

Similarly, from (4.2), we get for all \( t > 0 \)

\[x_2 = y_2.
\]

Now for all \( t > 0 \)

\[
M(u_1, u_2, t) = \frac{t}{t+|x_1-x_2|} = \frac{t+|y_1-y_2|}{t} = M(v_1, v_2, t).
\]

Hence \( f \) satisfies fuzzy \( P^* \)-property.

Let \( u = (0, x), v = (0, y) \in A \). Without loss of generality, we may assume that \( x < y \).

Now for all \( t > 0 \),

\[
\frac{1}{M(fu, fv, \phi(ct))} - 1 = \frac{|e^{-c^2x} - e^{-c^2y}|}{c^2t^2} = \frac{c^2e^{-c^2|x+\theta(y-x)|}|x-y|}{c^2t^2} \quad \text{(Using MVT, where } 0 < \theta < 1) \leq \frac{c|x-y|}{t^2}
\]

\[
= c\left(\frac{1}{M(u, v, \phi(t))} - 1\right) = \psi\left(\frac{1}{M(u, v, \phi(t))} - 1\right).
\]

Hence \( f \) satisfies \((\phi - \psi)\)-proximal contraction.

Here \( (0,0) \in A \) is the best proximity point of \( f \).

Note: The above illustration indicates that our result is an effective generalization of the fuzzy Banach contraction mapping principle given by Gregori and Sapena [13] in complete fuzzy metric space since the latter is not applicable to the above example.

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