Extended phase space thermodynamics and
$P-V$ criticality of black holes with a nonlinear source

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In this paper, we consider the solutions of Einstein gravity in the presence of a generalized
Maxwell theory, namely power Maxwell invariant. First, we investigate the analogy of non-
linear charged black hole solutions with the Van der Waals liquid–gas system in the extended
phase space where the cosmological constant appear as pressure. Then, we plot isotherm
$P-V$ diagram and study the thermodynamics of AdS black hole in the (grand canonical)
canonical ensemble in which (potential) charge is fixed at infinity. Interestingly, we find the
phase transition occurs in the both of canonical and grand canonical ensembles in contrast
to RN black hole in Maxwell theory which only admits canonical ensemble phase transition.
Moreover, we calculate the critical exponents and find their values are the same as those in
mean field theory. Besides, we find in the grand canonical ensembles universal ratio $\frac{P}{V}$ is
independent of spacetime dimensions.

I. INTRODUCTION

Theoretically one may be expect the cosmological constant term to arise from the vacuum
expectation value of a quantum field and hence can vary. Therefore, it may be considered in the first
law of thermodynamics with its conjugate $\frac{P}{V}$. By this generalization, the cosmological constant
and its conjugate can be interpreted as geometrical pressure and volume of a black object system,
respectively. Moreover, this approach leads to an interesting conjecture on reverse isoperimetric
inequality for black holes in contrast to a Euclidean version of isoperimetric inequality. Regarding
the inequality conjecture, some of the black hole processes may be restricted.

Furthermore, the extension of thermodynamic phase space has dramatic effects on the studying
of famous phase transition of black holes in AdS space and improves the analogy between

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small/large black hole with the Van der Waals liquid/gas phase transitions. Indeed, the AdS charged black holes exhibit an interesting phase transition with the same critical behavior as Van der Waals model, qualitatively.

Taking into account the above statements, we should note that the charge of the black hole plays a crucial role in this phase transition. Therefore it is important to know effects of any modification in the electromagnetic field. Indeed, some characteristic features of universality class of phase transitions such as the value of critical exponents or universal ratio \( \frac{P_c}{v_c} \) may depend on electromagnetic source or spacetime dimension. On the other hand, considering strong electromagnetic field in regions near to point-like charges, Dirac suggested that one may have to use generalized nonlinear Maxwell theory in those regions. Similar behavior may occur in the vicinity of neutron stars and black objects and so it is expected to consider nonlinear electromagnetic fields with an astrophysical motive. In addition, within the framework of quantum electrodynamics, it was shown that quantum corrections lead to nonlinear properties of vacuum which affect the photon propagation. Moreover, the effects of Born–Infeld (BI) source in the thermodynamics and phase transition of black hole have been studied. Besides, in context AdS/CFT some authors consider roles of BI source on shear viscosity and holographic superconductors.

By this observations one may find it is worthwhile to study the effects of nonlinear electrodynamics (NLEDs) on phase transition of black holes in the extended phase space. In this direction, the effects of nonlinear electromagnetic field of static and rotating AdS black holes in the extended phase space have been analyzed. It has been shown that for the BI black holes, one may obtain the same qualitative behavior as RN black holes. Indeed, BI electromagnetic field does not have any effect on the values of critical exponents, but it changes the universal ratio \( \rho_c = \frac{P_c}{v_c} \).

Although BI theory is a specific model in the context of NLEDs, the recent interest on the NLEDs theories is mainly due to their emergence in the context of low-energy limit of heterotic string theory or as an effective action for the consideration of effects loop corrections in QED where quartic corrections of Maxwell field strength appear.

In the last five years, a class of NLEDs has been introduced, the so-called power Maxwell invariant (PMI) field (for more details, see [21, 22]). The PMI field is significantly richer than that of the Maxwell field, and in the special case \( s = 1 \) it reduces to linear electromagnetic source. The black hole solutions of the Einstein-PMI theory and their interesting thermodynamics and geometric properties have been examined before. In addition, in the context of AdS/CFT correspondence, the effects of PMI source on strongly coupled dual gauge theory have been investigated.
The bulk action of Einstein-PMI gravity has the following form [22]

\[ I_b = -\frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left( R + \frac{n(n-1)}{l^2} + \mathcal{L}_{PMI} \right), \]  

where \( \mathcal{L}_{PMI} = (-\mathcal{F})^s \) and \( \mathcal{F} = F_{\mu\nu}F^{\mu\nu} \). Before we proceed, we provide some of reasonable motivation for considering this form of NLEDS.

First, between NLEDS theories, the PMI theory is a toy model to generalize Maxwell theory which reduces to it for \( s = 1 \). One of the most important properties of the PMI model in \((n + 1)\)-dimensions occurs for \( s = (n + 1)/4 \) where the PMI theory becomes conformally invariant and so the trace of energy-momentum tensor vanishes, the same as Maxwell theory in four-dimensions [21]. Considering this value for the nonlinearity parameter, \( s \), one can obtain inverse square law for the electric field of charged pointlike objects in arbitrary dimensions (the same as Coulomb’s field in four-dimensions). Furthermore, it has been shown that there is an interesting relation between the solutions of a class of pure \( F(R) \) gravity and those of conformally invariant Maxwell source \((s = (n + 1)/4)\) in Einstein gravity [24].

Second, we should note that considering the \( E_8 \times E_8 \) heterotic string theory, the \( SO(32) \) gauge group has a \( U(1) \) subgroup. It has been shown that [25] taking into account a constant dilaton, the effective Lagrangian has Gauss-Bonnet term as well as a quadratic Maxwell invariant in addition to the Einstein-Maxwell Lagrangian. Since, unlike the quadratic Maxwell invariant, the Gauss-Bonnet term becomes a topological invariant and does not give any contribution in four-dimensions, it is natural to investigate Einstein-NLEDS in four dimensions. Taking into account a PMI theory as a NLEDS Lagrangian and expanding it for \( \mathcal{F} \rightarrow \mathcal{F}_0 \) (where we considered \( \mathcal{F}_0 \) as an unknown constant which we should fix it.), we find

\[ \mathcal{L}_{PMI} \simeq -a_1\mathcal{F} + (s - 1) \left[ a_0 + a_2(-\mathcal{F})^2 + a_3(-\mathcal{F})^3 + \ldots \right]. \]  

In other words, one can consider series expansion of \( \mathcal{L}_{PMI} \) near a constant \( \mathcal{F}_0 \) and obtain Eq. [2], in which the constants \( a_i \)'s are depend on \( s \) and \( \mathcal{F}_0 \). In order to obtain \( \mathcal{F}_0 \) and also have a consistent series expansion with linear Maxwell Lagrangian \((s = 1)\), one should set \( a_1 = 1 \). Taking into account \( a_1 = 1 \) and obtaining \( \mathcal{F}_0 \), we are in a position to get a new series expansion for \( \mathcal{L}_{PMI} \)

\[ \mathcal{L}_{PMI} \simeq -\mathcal{F} + (s - 1) \left[ b_0 + b_2(-\mathcal{F})^2 + b_3(-\mathcal{F})^3 + \ldots \right], \]  

where \( b_i \)'s are only depend on \( s \). Although, \( \mathcal{L}_{PMI} = (-\mathcal{F})^s \) can lead to Eq. [3] by a series expansion, working with Eq. [3] is more complicated and we postpone the study of this scenario to another paper.
Third, taking into account the applications of the AdS/CFT correspondence to superconductivity, it has been shown that the PMI theory makes crucial effects on the condensation as well as the critical temperature of the superconductor and its energy gap [23].

Motivated by the recent results mentioned above, we consider the PMI theory to investigate the effects of nonlinearity on the extended phase space thermodynamics and $P-V$ criticality of the solutions. Moreover, to better understand the role of nonlinearity, we relax the conformally invariant constraint and take $s$ as an arbitrary constant. It helps us to have a deep perspective to study the universal behavior of large/small black hole phase transitions. In particular, we are keen on understanding sensitivity of the critical exponents, universal ratio and other thermodynamic properties to nonlinearity parameter, $s$.

Outline of this paper is as follows: In Sec. II, we consider spherically symmetric black hole solutions of Einstein gravity in the presence of the PMI source. Regarding the cosmological constant as thermodynamic pressure, we study thermodynamic properties and obtain Smarr’s mass relation. In Sec. III, we investigate the analogy of black holes with Van der Waals liquid–gas system in the grand canonical ensemble by fixing charge at infinity. In this ensemble we find the free energy and plot the coexistence curve of a small/large black hole. Then, we calculate the critical exponent and find they match to mean field value (same as a Van der Waals liquid). Moreover, we consider the special case $s = n/2$ as BTZ-like solution, study its phase transition and show the critical exponents are the same as former case. In Sec. IV, we consider the possibility of the phase transition in the grand canonical ensemble and find that in contrast to RN black holes, the phase transition occurs for $s \neq 1$. Finally, we finish this work with some concluding remarks.

II. EXTENDED PHASE-SPACE THERMODYNAMICS OF BLACK HOLES WITH PMI SOURCE

We consider a spherically symmetric spacetime as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_d^2 - 2,$$  

(4)

where $d\Omega_d^2$ stands for the standard element on $S^d$. Considering the field equations following from the variation of the bulk action with Eq. (4), one can show that the metric function $f(r)$, gauge
potential one–form $A$ and electromagnetic field two–form $F$ are given by [22]

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{m}{r^{n-2}} + \frac{(2s-1)^2}{(n-1)(n-2)(2s-1)^2} \left\{ \frac{(n-1)(2s-n)^2 q^2}{(n-2)(2s-1)^2} \right\}^s,$$

$$A = -\sqrt{\frac{n-1}{2(n-2)}} q r^{(2s-n)/(2s-1)} dt,$$

$$F = dA.$$

The power $s \neq n/2$ denotes the nonlinearity parameter of the source which is restricted to $s > 1/2$ [22], and the parameters $m$ and $q$ are, respectively, related to the ADM mass $M$ and the electric charge $Q$ of the black hole

$$M = \frac{\omega_{n-1}^2}{16\pi} (n-1)m,$$

$$Q = \frac{\sqrt{2}(2s-1)s}{8\pi} \frac{\omega_{n-1}}{n-2} \frac{(n-2) q}{(2s-1)^{2s-1}}.$$

where $\omega_{n-1}$ is given by

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

It has been shown that [22] Eqs. (4) and (5) describe a black hole with a cauchy horizon ($r_-$) and an event horizon ($r_+$). The event horizon radius of this black hole can be calculated numerically by finding the largest real positive root of $f(r = r_+) = 0$. Using the surface gravity relation, we can obtain the temperature of the black hole solutions as

$$T = \frac{f'(r_+)}{4\pi} = \frac{n-2}{4\pi r_+} \left( 1 + \frac{r^2}{n-2} \right) = \frac{(2s-1)^2}{(n-1)(n-2)} \left\{ \frac{(n-1)(2s-n)^2 q^2}{(n-2)(2s-1)^2} \right\}^s.$$

The electric potential $\Phi$, measured at infinity with respect to the horizon while the black hole entropy $S$, was determined from the area law. It is easy to show that

$$\Phi = \sqrt{\frac{n-1}{2(n-2)}} q r^{(n-2s)/(2s-1)}.$$

$$S = \frac{\omega_{n-1}^{n-1} r_+^{n-1}}{4\pi}.$$

Now, as it was considered before [6], we interpret $\Lambda$ as a thermodynamic pressure $P$,

$$P = -\frac{1}{8\pi} \frac{\Lambda}{\omega_{n-1}^{n-1} r_+^{n-1}} = \frac{n(n-1)}{16\pi l^2},$$

where its corresponding conjugate quantity is the thermodynamic volume $V$

$$V = \frac{\omega_{n-1}^{n-1} r_+^{n-1}}{n}.$$
Considering obtained quantities, one can show that they satisfy the following Smarr formula

\[ M = \frac{n-1}{n-2} TS + \frac{ns - 3s + 1}{s(2s - 1)(n - 2)} \Phi Q - \frac{2}{n-2} VP. \]  

(16)

It has been shown that Eq. (16) may be obtained by a scaling dimensional argument [4, 26]. In addition, the (extended phase-space) first law of thermodynamics can be written as

\[ dM = TdS + \Phi dQ + VdP. \]  

(17)

In what follows, we shall study the analogy of the liquid–gas phase transition of the Van der Waals fluid with the phase transition in black hole solutions in the presence of PMI source.

### III. CANONICAL ENSEMBLE

In order to study the phase transition, one can select an ensemble in which black hole charge is fixed at infinity. Considering the fixed charge as an extensive parameter, the corresponding ensemble is called a canonical ensemble.

#### A. Equation of state

Using the Eqs. (14) and (11) for a fixed charge \( Q \), one may obtain the equation of state, \( P(V, T) \)

\[ P = \frac{(n - 1)}{4r_+} T - \frac{(n - 1)(n - 2)}{16\pi r_+^2} + \frac{1}{16\pi} \frac{(2s - 1)}{r_+^{2s(n-1)/(2s-1)}} \left( \frac{(n-1)/(2s-n)^2 g^2}{(n-2)/(2s-1)^2} \right)^s, \]  

(18)

where \( r_+ \) is a function of the thermodynamic volume, \( V \) [see Eq. (15)]. Following [6], we identify the geometric quantities \( P \) and \( T \) with physical pressure and temperature of system by using dimensional analysis and \( l_P^{n-1} = G_{n+1} h/c^3 \) as

\[ \text{[Press]} = \frac{h c}{l_P^{n-1}} [P], \quad \text{[Temp]} = \frac{h c}{k} [T]. \]  

(19)

Therefore, the physical pressure and physical temperature are given by

\[ \text{Press} = \frac{h c}{l_P^{n-1}} P = \frac{h c}{l_P^{n-1}} \frac{(n - 1)T}{4r_+} + \ldots \]

\[ = \frac{k \text{Temp}(n - 1)}{4l_P^{n-1} r_+} + \ldots. \]  

(20)

Now, one could compare them with the Van der Waals equation [6], and identify the specific volume \( v \) of the fluid with the horizon radius as \( v = \frac{4r_+ l_P^{n-1}}{n-1} \), and in geometric units \( (l_P = 1, \)
The equation of state can be written in the following form

\[ P = \frac{T}{v} - \frac{(n-2)}{\pi(n-1) v^2} + \frac{\kappa q^{2s}}{16 \pi q^{2s(n-1)/(2s-1)}}, \tag{21} \]

\[ \kappa = \frac{4^{2s(n-1)/(2s-1)} (2s-1) \left( \frac{(n-1)(2s-n)^2}{(n-2)(2s-1)^2} \right)^s}{(n-1)^{2s(n-1)/(2s-1)}}, \tag{22} \]

Considering Eq. (21), we plot the \( P - V \) isotherm diagram in Fig. 1. This figure shows that, similar to Van der Waals gas, there is a critical point which is a point of inflection on the critical isotherm. The pressure and volume at the critical point are known as the critical pressure and the critical volume, respectively. Above the critical point and for large volumes and low pressures, the isotherms lose their inflection points and approach equilateral hyperbolas, the so-called the isotherms of an ideal gas. It is shown that the slope of the isotherm \( P - V \) diagram passing through the critical point is zero. Furthermore, as we mentioned before, the critical point is a point of inflection on the critical isotherm, hence

\[ \frac{\partial P}{\partial v} = 0, \tag{23} \]

\[ \frac{\partial^2 P}{\partial v^2} = 0. \tag{24} \]

Using Eqs. (23) and (24) with the equation of state (21), we will be able to calculate the critical parameters

\[ v_c = \left[ \frac{\kappa s(n-1)^2(2ns-4s+1)q^{2s}}{16(n-2)(2s-1)^2} \right]^{(2s-1)/2/(ns-3s+1)}, \tag{25} \]

\[ T_c = \frac{4(n-2)(ns-3s+1)}{\pi(n-1)(2ns-4s+1)} \left[ \frac{\kappa s(n-1)^2(2ns-4s+1)q^{2s}}{16(n-2)(2s-1)^2} \right]^{(1-2s)/2/(ns-3s+1)}, \tag{26} \]

\[ P_c = \frac{(n-2)(ns-3s+1)}{\pi s(n-1)^2} \left[ \frac{\kappa s(n-1)^2(2ns-4s+1)q^{2s}}{16(n-2)(2s-1)^2} \right]^{(2s-1)/(ns-3s+1)}. \tag{27} \]

These relations lead us to obtain the following universal ratio

\[ \rho_c = \frac{P_c v_c}{T_c} = \frac{2ns-4s+1}{4s(n-1)}. \tag{28} \]

Note that for \( s = -2/(n-5) \) with arbitrary spacetime dimensions, one can recover the ratio \( \rho_c = 3/8 \), characteristic for a Van der Waals gas.

### B. Free energy

Thermodynamic behavior of a system may be governed by the thermodynamic potentials such as the free energy. It is known that the free energy of a gravitational system may be obtained by
evaluating the Euclidean on-shell action. In order to calculate it, we use the counterterm method for cancelling of divergences. Furthermore, to make an action well-defined, one should add the Gibbons-Hawking boundary term to the bulk action. In addition, in order to fix charge on the boundary (working in canonical ensemble) we should consider a boundary term for electromagnetic field. So the total action is

$$I = I_b + I_{ct} - \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{\gamma} K - \frac{s}{4\pi} \int_{\partial M} d^n x \sqrt{\gamma} (-F)^{s-1} n_\mu F^{\mu\nu} A_\nu,$$

(29)

where $I_{ct}$ is the counterterm action, and $\gamma_{ij}$ and $K$ denote the induced metric and extrinsic curvature of the boundary. Using Eq. (29), it is straightforward to calculate the on-shell value of the total action

$$I = \frac{\beta \omega_{n-1}}{16\pi} \left( 1 - \frac{r_+^2}{l^2} + \frac{(2s - 1) (2sn - 4s + 1) \Psi r_+^2}{(n - 1) (n - 2s)} \right) r_+^{n-2},$$

(30)

where

$$\Psi = \left( \frac{n - 1}{n - 2} \right) \left( \frac{2s - n}{2s - 1} \right)^2 q^2 r_+^{\frac{2(n - 1)}{2s - 1}}.$$

and $\beta$ is the periodic Euclidean time which is related to the inverse of Hawking temperature. Using the fact that $G = I \beta^{-1}$ with Eq. [14], the (fixed charge) free energy in the extended phase space may be written as

$$G(T, P) = \frac{\omega_{n-1}}{16\pi} \left( 1 - \frac{16\pi P r_+^2}{n(n - 1)} + \frac{(2s - 1) (2sn - 4s + 1) \Psi r_+^2}{(n - 1) (n - 2s)} \right) r_+^{n-2}.$$
FIG. 2: Free energy density \( \frac{G(T, P)}{\omega n - 1} \) of charged black holes with PMI source with respect to temperature for \( q = 1 \), \( s = \frac{6}{5} \) with \( n = 4 \) (left) and \( q = 1 \), \( s = \frac{3}{4} \) with \( n = 3 \) (right). The characteristic swallowtail behavior is the signature of the first order phase transition between large-small charged black holes (analogous to the Van der Waals gas).

FIG. 3: Coexistence curves of large and small charged black holes with PMI source for \( q = 1.5 \) with \( n = 3 \) (left) and \( q = 1 \) with \( s = \frac{n+1}{n} \) (right). Critical points are denoted by the small circle at the end of the coexistence curve. Above these points, the phase transition does not occur.

The behavior of the free energy is displayed in Fig. 2. In this figure the characteristic swallowtail behavior of the free energy shows the first order phase transition happen between large and small charged black holes. Using the fact that the free energy, temperature and the pressure of the system are constant during the phase transition, one can plot the coexistence curve of two phases.
large and small charged black holes in the PMI theory (see Fig. 3). Along this curve, small and large black holes have alike temperature (horizon radii) and pressure.

1. Critical exponents

One of the most important characteristics of the phase transition is the value of its critical exponents. So, following the approach of [19], we calculate the critical exponents $\alpha$, $\beta$, $\gamma$, $\delta$ for the phase transition of $(n+1)$-dimensional charged black holes with an arbitrary $s$. In order to obtain the critical exponent $\alpha$, we consider the entropy of horizon $S$ and rewrite it in terms of $T$ and $V$. So we have

$$ S = S(T,V) = \left[ \omega_{n-1} (nV)^{n-1} \right]^{1/n}. $$

(32)

Obviously, this is independent of $T$ and then the specific heat vanishes, ($C_V = 0$), and hence $\alpha = 0$. To obtain other exponents, we study equation of state (21) in terms of reduced thermodynamic variables

$$ p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}. $$

(33)

So, Eq. (21) translates into the following reduced equation of state

$$ p = \frac{4(n+1)s\tau}{(2ns - 4s + 1)\nu} - \frac{n - 1}{(ns - 3s + 1)\nu^2} + \frac{(2s - 1)^2}{(2ns - 4s + 1)(ns - 3s + 1)\nu^{2s(n-1)}}. $$

(34)

To study the recent equation, we will slightly generalize the argument of [19] for nonlinear Maxwell theory. Indeed, we can rewrite the equation of state (34) as

$$ p = \frac{1}{\rho_c \nu} \tau + f(\nu, s), $$

(35)

where $\rho_c$ stands for the critical ratio and

$$ f(\nu, s) = \frac{1}{s(1 - 4\rho_c)} \left( \frac{1}{\nu^2} - \frac{(2s - 1)^2}{4s\rho_c \nu^{2s(n-1)}} \right). $$

The function $f(\nu, s)$ depends on $\nu$ and $s$ compared to [19], where it is independent of $s$. But as we will see the nonlinearity parameter $s$ does not play any dramatic role and does not change critical exponents. Following the method of Ref. [19], one may define two new parameters $t$ and $\omega$

$$ \tau = t + 1, \quad \nu = (\omega + 1)^{1/\epsilon}, $$

(36)
where $\epsilon$ is a positive parameter. Now we can expand near the critical point to obtain

$$p = 1 + At - Bt\omega - C\omega^3 + O(t\omega^2, \omega^4),$$

(37)

with

$$A = \frac{1}{\rho_c}, \quad B = \frac{1}{\epsilon \rho_c}, \quad C = \frac{2s(n-1)}{3\epsilon^3(2s-1)}.$$  

(38)

We consider a fixed $t < 0$ and differentiate the Eq. (37) to obtain

$$dP = -P_c(Bt + 3C\omega^2)d\omega.$$  

(39)

Now, we denote the volume of small and large black holes with $\omega_s$ and $\omega_l$, respectively, and apply the Maxwell’s equal area law. One obtains

$$p = 1 + At - Bt\omega_l - C\omega_l^3 = 1 + At - Bt\omega_s - C\omega_s^3$$

$$0 = \int_{\omega_l}^{\omega_s} \omega dP.$$  

(40)

This equation leads to a unique non-trivial solution

$$\omega_s = -\omega_l = \sqrt{-\frac{Bt}{C}},$$  

(41)

and therefore we can find

$$\eta = V_c(\omega_l - \omega_s) = 2V_c\omega_l \propto \sqrt{-t} \quad \Rightarrow \quad \beta = \frac{1}{2}.$$  

(42)

Now, we should calculate the next exponent, $\gamma$. In order to obtain it, one should consider Eq. (37). After some manipulation one can obtain

$$\kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_{T} \propto \frac{1}{P_c} \frac{1}{Bt} \quad \Rightarrow \quad \gamma = 1.$$  

(43)

Next, we calculate the final exponent, $\delta$. To do this, we should obtain the shape of the critical isotherm $t = 0$, i.e.,

$$p - 1 = -C\omega^3 \quad \Rightarrow \quad \delta = 3.$$  

(44)

We conclude that the thermodynamic exponents associated with the nonlinear charged black holes in any dimension $n \geq 3$ with arbitrary nonlinearity parameter, $s \neq n/2$, coincide with those of the Van der Waals fluid (the same as critical exponents of the linear Maxwell case).
C. Equation of state for the BTZ-like black holes

So far, we have investigated the phase transition of black holes in the presence of nonlinear PMI source with the nonlinearity $s \neq n/2$. Interestingly, for $s = n/2$, the solutions (the so-called BTZ-like black holes) have different properties. In other words, the solutions for $s = n/2$ are not the special limit of the solutions for general $s$. In fact, the solutions of $s = n/2$ are completely special and differ from the solutions of other values of $s$. As we will see, for $s = n/2$ the charge term in metric function is logarithmic and the electromagnetic field is proportional to $r^{-1}$ (logarithmic gauge potential). In other words, in spite of some differences, this special higher dimensional solution has some similarity with the charged BTZ solution and reduces to the original BTZ black hole for $n = 2$.

Considering the metric (4) and the field equations of the bulk action (1) with $s = n/2$, we can find that the metric function $f(r)$ and the gauge potential may be written as

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{m}{r^{n-2}} - \frac{2^{n/2} q^n}{r^{n-2}} \ln \left( \frac{r}{l} \right),$$  \hspace{1cm} (45)

$$A = q \ln \left( \frac{T}{l} \right) dt,$$  \hspace{1cm} (46)

Straightforward calculations show that BTZ-like spacetime has a curvature singularity located at $r = 0$, which is covered with an event horizon. The temperature of this black hole can be obtained as [28]

$$T = \frac{n - 2}{4 \pi r_+} \left( 1 + \frac{n}{n - 2} \frac{r_+^2}{l^2} - \frac{2^{n/2} q^n}{(n - 2) r_+^{n-2}} \right),$$  \hspace{1cm} (47)

In this section, we will investigate the analogy of the liquid–gas phase transition of the Van der Waals fluid with the phase transition in BTZ-like black hole solutions [28]. Following the same approach and using Eqs. (14) and (47) for a fixed charge $Q$, we obtain

$$P = \frac{(n - 1)}{4 r_+} T - \frac{(n - 1)(n - 2)}{16 \pi r_+^2} + \frac{1}{16 \pi} \frac{2^{n/2} (n - 1) q^n}{r_+^{n-2}}.$$  \hspace{1cm} (48)

Using Eqs. (19) and (20) with the fact that in geometric units $v = \frac{4 r_+}{n - 1}$, Eq. (48) may be rewritten as

$$P = \frac{T}{v} - \frac{(n - 2)}{\pi (n - 1) v^2} + \frac{1}{16 \pi} \frac{\kappa' q^{2s}}{v^n},$$  \hspace{1cm} (49)

$$\kappa' = \frac{2^{5n/2}}{(n - 1)^{n - 1}}.$$  \hspace{1cm} (50)

Now, we plot the isotherm $P - V$ diagram in Fig. 4. The behavior of these plots is the same as the Van der Waals gas. In order to find the critical quantities, one may use Eqs. (23) and (24).
FIG. 4: $P - V$ diagram of BTZ-like black holes for $n = 4$ (left) and $n = 5$ (right). The temperature of isotherms decreases from top to bottom. The bold line is the critical isotherm diagram.

These relations help us to obtain

$$v_c = \frac{\kappa' n(n-1)^2 q^n}{32(n-2)}^{1/(n-2)},$$

$$T_c = \frac{2(n-2)^2 \left[ \frac{\kappa' n(n-1)^2 q^n}{32(n-2)} \right]^{1/(n-2)}}{\pi(n-1)^2},$$

$$P_c = \frac{(n-2)^2}{\pi n(n-1) \left[ \frac{\kappa' n(n-1)^2 q^n}{32(n-2)} \right]^{2/(n-2)}}.$$

Having the critical quantities at hand, we are in a position to obtain the following universal ratio

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{n - 1}{2n}.$$

It is notable that only for $n = 4$ (5-dimensional BTZ-like black holes), one can recover the ratio $\rho_c = \frac{3}{8}$, characteristic for a Van der Waals gas, where for higher dimensional Reissner–Nordström black holes, this ratio has been recovered only for 4-dimensions [19]. In addition, considering $s = n/2$ in Eq. (28), one can obtain $\rho_c$ of the BTZ-like black holes.

1. **Critical exponents of BTZ-like black holes**

In order to obtain the critical exponents of the phase transition, we follow the same procedure of [19]. The entropy of horizon $S(T,V)$ is the same as Eq. (32), which is independent of $T$ and
hence $\alpha = 0$. In addition we can use Eq. (33) and rewrite Eq. (49) in the following form

$$p = \frac{2n\tau}{(n-1)\nu} - \frac{n}{(n-2)\nu^2} + \frac{2}{(n-1)(n-2)\nu^n}. \quad (55)$$

It is straightforward to show that the thermodynamic exponents associated with the BTZ-like black holes in arbitrary dimension, coincide with those of the Van der Waals fluid (the same as critical exponents of the PMI case).

IV. GRAND CANONICAL ENSEMBLE

In addition to the canonical ensemble, one can work with a fixed electric potential at infinity. The ensemble of this fixed intensive quantity translates into the grand canonical ensemble. It is worthwhile to note that, for linear Maxwell field, the criticality cannot happen in the grand canonical ensemble [6].

A. Equation of state

In this section, we study the critical behavior of charged black holes in the grand canonical (fixed $\Phi$) ensemble. We take $q = \Phi r^{(n-2s)/(2s-1)}$ with $v = \frac{4r}{n-1}$ to rewrite Eq. (18) in the following form

$$P = T \frac{v^2}{v} - \frac{(n-2)}{(n-1)\pi \nu^2} + \frac{2s-1}{16\pi} \left( \frac{4\sqrt{2}(n-2s)\Phi}{(2s-1)(n-1)\nu} \right)^{2s}. \quad (56)$$

In the Maxwell theory ($s = 1$) Eq. (56) reduces to

$$Pv^2 = Tv - \frac{(n-2)}{(n-1)\pi} + \frac{2}{\pi (n-1)^2} \Phi^2. \quad (57)$$

Clearly, this is a quadratic equation and does not show any criticality. Interestingly, in contrast to the Maxwell field ($s = 1$), the PMI theory admits phase transition in this ensemble and one can study the fixed potential $P - V$ phase transition of the black holes in extended phase space. However, the general behavior of the isotherm $P - V$ diagram for fixed potential is same as fixed charge ensemble as displayed in Fig. 5. Applying Eqs. (23) and (24) to the equation of state, it is
FIG. 5: $P-V$ diagram of charged AdS black holes in PMI for $s = \frac{6}{5}$ with $n = 3$ (left) and $s = \frac{5}{4}$ with $n = 4$ (right). The temperature of isotherms decreases from top to bottom. The bold line is the critical isotherm diagram.

easy to calculate the critical point in the grand canonical ensemble

$$v_c = \frac{4\sqrt{2}(n - 2s)}{(2s - 1)(n - 1)} \left[ \frac{2s(2s - n)^2}{(n - 2)(n - 1)} \right]^{\frac{1}{2(s-1)}} \Phi_{\frac{s}{s-1}}^{-1},$$  \hspace{1cm} (58)

$$T_c = \frac{(s - 1)(n - 2)}{\pi(n - 2s)} \left[ \frac{(n - 2)(n - 1)}{s2s(n - 2s)^2} \right]^{\frac{1}{2(s-1)}} \Phi_{\frac{s}{s-1}}^{-1},$$  \hspace{1cm} (59)

$$P_c = \frac{(s - 1)(2s - 1)^22^{(4-5s)/(s-1)}}{s\pi} \left[ \frac{(n - 2s)^2s^{\frac{1}{n}}}{(n - 1)(n - 2)} \right]^{\frac{s}{s-1}} \Phi_{\frac{s}{s-1}}^{-2s}.$$  \hspace{1cm} (60)

One must consider that in contrast to the canonical ensemble for $s > \frac{n}{2}$ or $s < 1$ the $v_c$, $T_c$ or $P_c$ take the negative value so there is not any physical phase transition in these cases. Using the values of $v_c$, $T_c$ and $P_c$ we will be able to obtain the following universal ratio

$$\rho_c = \frac{P_cv_c}{T_c} = 2s - 1 \frac{4s}{4s}.$$  \hspace{1cm} (61)

Note that here this universal ratio is independent of $n$ and for $s = 2$ one can recover the ratio $\rho_c = 3/8$, characteristic for a Van der Waals gas. Although Eq. (61) does not depend on the spacetime dimensions, but one can take $s = n/2$ to recover Eq. (54) of BTZ-like black holes. Furthermore, it is interesting to mention that one can set $n = 2s$ in Eq. (28) to obtain Eq. (61).
FIG. 6: Free energy density \( G(T, P) \) of charged black holes with PMI source with respect to temperature for \( \Phi = 1, s = \frac{6}{5} \) with \( n = 3 \) (left) and \( \Phi = 1, s = \frac{5}{4} \) with \( n = 4 \) (right).

B. Free energy

By ignoring the surface term of PMI and fixing the potential on the boundary \( \delta A_\mu |_{\partial M} = 0 \), one can find the on-shell action correspondence to free energy in the grand canonical ensemble. So, we take the action as follows

\[
I = I_b + I_{ct} - \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{\gamma} K. \tag{62}
\]

Now we can find the free energy \( G = I\beta^{-1} \) as

\[
G(T, P) = \frac{\omega_{n-1}}{16\pi} \left( 1 - \frac{16\pi P r^2}{n(n-1)} + \frac{2^s (2s-1)^{2-2s} \Phi^{2s} r^{2-2s}}{(n-1)(2s-n)^{1-2s}} \right) r^{n-2}. \tag{63}
\]

Now, we are looking for the phase transition. We plot figure 6 and find that, in contrast to Maxwell case, there is a first order phase transition. In other words, the nonlinearity parameter, \( s \), affects the existence of the phase transition in the grand canonical ensemble.

V. CONCLUDING REMARKS

In this paper, we have considered the cosmological constant and its conjugate quantity as thermodynamic variables and investigated the thermodynamic properties of a class of charged black hole solutions. At the first step, we have introduced the black hole solutions of the Einstein-\( \Lambda \) gravity in the presence of the PMI source.
Then, we have used the Hawking temperature as an equation of state and calculated the the critical parameters, \( T_c \), \( v_c \) and \( P_c \). We have plotted the isotherm diagram \( (P-V) \) of charged black holes in PMI theory and found that the total behavior of this diagram is the same as that of the Van der Waals gas. Also, we have obtained the free energy of a gravitational system through the use of Euclidean on-shell action to investigate its thermodynamics behavior.

Furthermore, we have calculated the critical exponents of the phase transition and concluded that the thermodynamic exponents associated with the nonlinear charged black holes in arbitrary dimension coincide with those of the Van der Waals fluid (the mean field theory).

Also, we have applied the same procedure for the BTZ-like black holes to obtain their phase transition. Calculations showed that thermodynamic behaviors of BTZ-like black holes are the same as PMI ones.

Moreover, we have studied the grand canonical ensemble in which the potential, instead of charge, should be fixed on the boundary. In contrast to the Maxwell case \([6]\), here one sees a phase transition. We have also computed the universal ratio \( \frac{P_c v_c}{T_c} = \frac{2s-1}{4s} \) and found that it does not depend on the spacetime dimensions.

Finally, we have found that, \( v_c \), \( T_c \) and \( P_c \) have different dependencies of \( n \) and \( s \) for PMI and BTZ-like in the canonical ensemble, but when \( n = 2s \) both of them reduce to the universal ratio \( \rho_c = \frac{P_c v_c}{T_c} = \frac{2s-1}{4s} \) which we have found in the grand canonical ensemble.

It is interesting to investigate underlining reasons for the mentioned universality and figure out why this ratio in the grand canonical ensembles is independent of spacetime dimensions. Moreover, in the statistical physics, it is known that a universality class of criticality is characterized by dimensions of space, order parameters and fluctuations \([28]\). However, it is not clear which features of gravitational theories and black holes determine the universality class of phase transition or modify critical exponents. Clearly as we found in this paper and have been shown in \([19]\), the critical exponents do not change by crucial modifications of matter fields (such as PMI modification) or changing the space-time dimensions. In addition, it seems the geometry of spacetime is also irrelevant to critical exponents as one can see in the slowly rotating black holes \([19]\). Interestingly, these critical exponents remain unchanged in the mean field class even when one considers some corrections to gravity action \([30]\). Besides, it is worthwhile to think about whether there is any holographic interpretation for the extended phase space thermodynamics and the universal classification. Perhaps a holographic approach helps us to have a better understanding of this problem. We leave the study of these interesting questions for future studies.
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