LYAPUNOV-TYPE CHARACTERISATION OF EXPONENTIAL DICHOTOMIES WITH APPLICATIONS TO THE HEAT AND KLEIN-GORDON EQUATIONS

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Abstract. We give a sufficient condition for existence of an exponential dichotomy for a general linear dynamical system (not necessarily invertible) in a Banach space, in discrete or continuous time. We provide applications to the backward heat equation with a potential varying in time and to the heat equation with a finite number of slowly moving potentials. We also consider the Klein-Gordon equation with a finite number of potentials whose centres move at sub-light speed with small accelerations.

1. Introduction

1.1. Exponential dichotomies. Consider a linear dynamical system

\[ v_{n+1} = B_n v_n, \quad n \geq 0, \quad v_n \in \mathbb{C}^d, \quad B_n \in \mathbb{C}^{d \times d}. \]

In the special case where \( B_n = B \in \mathbb{C}^{d \times d} \) is independent of \( n \) (autonomous dynamics), the dynamical behavior of the solutions of (1.1) can be described using the Jordan normal form of the matrix \( B \).

In particular, if \( B \) has no eigenvalues \( \lambda \in \mathbb{C} \) such that \( |\lambda| \in [a,b] \), where \( 0 < a < b \), then the phase space \( \mathbb{C}^d \) decomposes as a direct sum

\[ \mathbb{C}^d = X_s \oplus X_u, \]

where \( X_s \) and \( X_u \) are invariant for (1.1) and there exist constants \( c,C > 0 \) such that

- if \( v_0 \in X_s \), then \( |v_n| \leq Ca^n |v_0| \) for all \( n \geq 0 \),
- if \( v_0 \in X_u \), then \( |v_n| \geq cb^n |v_0| \) for all \( n \geq 0 \).

Such a situation is called an exponential dichotomy. We call \( X_s \) the stable subspace and \( X_u \) the unstable subspace.

The purpose of this paper is to construct exponential dichotomies for (1.1) and similar systems in the case where \( B_n \) changes with \( n \). There are many classical examples exhibiting “surprising” behavior of the system (1.1). One such example is given by

\[ B_{2m} = \begin{pmatrix} 0 & -2 \\ 1/8 & 0 \end{pmatrix}, \quad B_{2m+1} = \begin{pmatrix} 0 & -1/8 \\ 2 & 0 \end{pmatrix}. \]

It is easy to see that the eigenvalues of \( B_{2m} \) and \( B_{2m+1} \) are \( \pm \frac{i}{2} \), and thus have modulus \( < 1 \). However, the eigenvalues of \( B_1 B_0 \) are \( -\frac{1}{64} \) and \(-4\), and it turns out that if \( x_2 \neq 0 \), then the initial data \( v_0 = (x_1, x_2) \) yields to exponential growth of the sequence \( (v_n) \). This example shows that the spectra of \( B_n \) do not provide enough information to describe exponential dichotomies of (1.1). Indeed, it is necessary to control how contracting/expanding directions relate to each other as \( n \) changes.

There exists an extensive literature on exponential dichotomies for non-autonomous dynamical systems. The monograph by Coppel [7] deals with the case of linear ordinary differential equations. In particular, it provides a necessary and sufficient condition for an exponential dichotomy in terms of existence of a Lyapunov functional satisfying certain properties. Related results were obtained by Coppel [8], Muldowney [19] and, in the case of difference equations, Papaschinopoulos [20].
A different approach to exponential dichotomies is based on the evolution semigroup introduced by Howland \[13\], which means that the non-autonomous system is transformed to an operator semigroup on some space, whose properties are then studied using spectral methods. This theory, both in finite and infinite dimension, is developed in the works of Rau \[22\], Latushkin and Montgomery-Smith \[14\], Räbiger and Schnaubelt \[21\], as well as subsequent works. One can consult the monograph \[15\] for a comprehensive bibliography.

The works \[7, 8, 19, 20\] mentioned above do not seem to directly generalise to infinite dimension. However, a Lyapunov-type characterisation of exponential dichotomies in infinite-dimensional Hilbert spaces was obtained by Barreira, Dragičević and Valls \[1\], using the theory of evolution semigroups, see also \[2\].

In this paper, we adopt the Lyapunov-type approach and formulate conditions for existence of exponential dichotomies in terms of existence of Lyapunov (or energy) functionals satisfying certain properties. Instead of invoking the evolution semigroup theory, we provide an alternative and more direct method, which we believe can be useful in applications. One advantage of our method is that in many cases we can easily obtain some supplementary information about the (un)stable spaces, for example the (co)dimension or an approximate basis.

In the theory of linear cocycles, exponential dichotomies are related to the existence of the so-called Oseledets flag, see \[25\]. Our proof of existence of exponential dichotomies resembles known proofs of the Oseledets Theorem, especially the one given in \[11\].

Finally, would like to point out that one of the important properties of exponential dichotomies is that they often persist under (not necessarily linear) perturbations of the dynamical system. This general principle is called the Lyapunov-Perron method, see for instance \[3\].

1.2. Statement of the results. Because we are interested in applications to dynamics of partial differential equations, we need to work with an infinite dimensional phase space. As we are not going to rely on Spectral Theory, we take it to be a real Banach space denoted \(X\). Let \(B_n \in \mathcal{L}(X), n \in \{0, 1, \ldots\}\) be a sequence of bounded linear operators on \(X\). We consider the dynamical system
\[
(1.2) \quad v_{n+1} = B_nv_n, \quad v_0 \in X.
\]
For \(n \leq m\) we denote
\[
B(n,n) := \text{Id}, \quad B(m,n) := B_{m-1}B_{m-2} \ldots B_n.
\]
Note that we do not require boundedness of the sequence \((B_n)\) in \(\mathcal{L}(X)\).

**Definition 1.1.** We say that (1.2) has an exponential dichotomy with values \(a\) and \(b, 0 < a < b\), if for all \(n \geq n_0\) there exists a direct sum decomposition \(X = X_s(n) \oplus X_u(n)\) such that \(X_s(n), X_u(n)\) and the associated projections \(\pi_s(n) : X \to X_s(n)\) and \(\pi_u(n) : X \to X_u(n)\) have the following properties for some \(C > 0\) and all \(n \leq m\):

1. \(B(m,n) \circ \pi_u(n) = \pi_u(m) \circ B(m,n)\) and \(B(m,n) \circ \pi_s(n) = \pi_s(m) \circ B(m,n)\),
2. \(\|\pi_s(n)\|_{\mathcal{L}(X)} + \|\pi_u(n)\|_{\mathcal{L}(X)} \leq C\),
3. \(B(m,n)|_{X_u(m)} : X_u(n) \to X_u(m)\) is invertible,
4. \(\|B(m,n)v_n\| \leq Ca^{m-n}\|v_n\|\) for all \(v_n \in X_s(n)\),
5. \(\|B(m,n)^{-1}v_m\| \leq Cb^{m-n}\|v_m\|\) for all \(v_m \in X_u(m)\).

**Remark 1.2.** It is clear that \(X_s(n)\) is unique. In general, \(X_u(n)\) is not unique.

Our sufficient condition for existence of an exponential dichotomy is expressed in terms of two sequences of (nonlinear) continuous homogeneous functionals \(I^-_n, I^+_n : X \to \mathbb{R}_+\). Given \(I^-_n, I^+_n\) and a number \(c > 0\), we define the stable and the unstable cone
\[
(1.3) \quad \mathcal{V}_s(c,n) := \{v \in X : I^+_n(v) \leq cI^-_n(v)\},
\]
\[
(1.4) \quad \mathcal{V}_u(c,n) := \{v \in X : I^-_n(v) \geq cI^+_n(v)\}.
\]
We find it helpful to keep in mind that if \( c \) is small, then \( V_h(c,n) \) is “thin” and \( V_u(c,n) \) is “wide”. Conversely, if \( c \) is large, then \( V_h(c,n) \) is “wide” and \( V_u(c,n) \) is “thin”.

Firstly, we assume that there exists \( c_1 > 0 \) (independent of \( n \)) such that

\[
(1.5) \quad c_1 \|v\|_X \leq I_n^-(v) + I_n^+(v) \leq \frac{1}{c_1} \|v\|_X, \quad \text{for all } n \geq 0 \text{ and } v \in X.
\]

Note that, directly from the definitions above, we obtain

\[
(1.6) \quad v \in V_h(c,n) \Rightarrow c_1 I_n^-(v) \leq \|v\|_X \leq \frac{1+c}{c_1} I_n^-,
\]

\[
(1.7) \quad v \in V_u(c,n) \Rightarrow c_1 I_n^+(v) \leq \|v\|_X \leq \frac{1+c}{c_1 c} I_n^+,
\]

thus on the stable cone the norm is equivalent to \( I_n^- \), and on the unstable cone it is equivalent to \( I_n^+ \).

Secondly, we assume that there exist \( c_2 > 0, K \in \{0,1,2,\ldots\} \) and \( \alpha_{k,n}^+ \in X^* \) for \((k,n) \in \{1,\ldots,K\} \times \{0,1,\ldots\} \) such that

\[
(1.8) \quad c_2 \max_{1 \leq k \leq K} |\langle \alpha_{k,n}^+, v \rangle| \leq I_n^+(v) \leq \frac{1}{c_2} \max_{1 \leq k \leq K} |\langle \alpha_{k,n}^+, v \rangle|.
\]

Lastly, we assume that there exist \( c_3, c_4 > 0 \) and \( 0 < a < b < \infty \) such that

\[
(1.9) \quad V_u(c_3,n) \text{ contains a linear space of dimension } K \text{ for all } n,
\]

\[
(1.10) \quad c_4 < \frac{1}{3} (c_1 c_2)^2 \frac{c_3}{1+c_3},
\]

\[
(1.11) \quad I_{n+1}^- (B_n v_n) \leq a I_n^- (v_n) \quad \text{if } B_n v_n \in V_h(c_3,n+1),
\]

\[
(1.12) \quad I_{n+1}^+ (B_n v_n) \geq b I_n^+ (v_n) \quad \text{if } v_n \in V_u(c_4,n).
\]

**Theorem 1.** Under assumptions (1.5), (1.8), (1.12), the system (1.2) has an exponential dichotomy with values \( a \) and \( b \). For all \( n \geq 0 \) the stable subspace \( X_s(n) \) is contained in \( V_h(c_4,n) \) and has codimension \( K \).

**Remark 1.3.** Note that if (1.11) holds, then it also holds with \( c_3 \) replaced by any smaller number. Similarly, if (1.12) holds, then it also holds with \( c_4 \) replaced by any bigger number. In other words, (1.11) and (1.12) imply

\[
(1.13) \quad I_{n+1}^- (B_n v_n) \leq a I_n^- (v_n) \quad \text{if } B_n v_n \in V_h(c,n+1),
\]

\[
(1.14) \quad I_{n+1}^+ (B_n v_n) \geq b I_n^+ (v_n) \quad \text{if } v_n \in V_u(c,n)
\]

for all \( c \in [c_4,c_3] \).

**Remark 1.4.** One can show that (1.9) always holds if \( \alpha_{k,n}^+ \) are uniformly linearly independent and \( c_3 \) small enough, see Proposition 2.9.

The condition (1.10) that we impose on \( c_4 \) is far from being optimal. In the applications, it only matters that \( c_4 \) is required to be smaller than \( c_3 \) multiplied by some small positive constant depending on \( c_1,c_2,K \).

**Remark 1.5.** The appropriate energy functionals \( I_n^- \) and \( I_n^+ \) are constructed in each particular case using the specific structure of a given problem, and in particular the natural energy functionals associated with it. Intuitively, we would like \( I_n^- \) to control the “shrinking” in the stable direction. Similarly, \( I_n^+ \) has to control the “expanding” in the unstable directions. Note that the assumption of the stable/unstable component being significant is “before the step” in the direction of expansion. The condition (1.8) means that there are only finitely many expansion directions, which is true for
any application we could think of. If $c_4$ is small, then $X_s(n) \subset \mathcal{V}_s(c_4, n)$ gives a precise information about the subspace $X_s(n)$.

To complete our analysis, we will prove that existence of an exponential dichotomy implies existence of energy functionals $I_n^-$ and $I_n^+$ satisfying conditions which are apparently stronger than the conditions listed above (thus, in reality, equivalent).

**Proposition 1.6.** If \( (1.2) \) has an exponential dichotomy with values $a < b$, then there exist semi-norms $\tilde{I}_n^-$, $\tilde{I}_n^+$ satisfying \((1.3)\) and

$$
X_s(n) = \{v_n : \tilde{I}_n^+(v_n) = 0\},
$$
$$
X_u(n) = \{v_n : \tilde{I}_n^-(v_n) = 0\},
$$
$$
\tilde{I}_n^-(B_n v_n) \leq a \tilde{I}_n^-(v_n) \quad \text{for all} \quad v_n \in X,
$$
$$
\tilde{I}_n^+(B_n v_n) \geq b \tilde{I}_n^+(v_n) \quad \text{for all} \quad v_n \in X.
$$

If $X_s(n)$ has finite codimension $K$, then $\tilde{I}_n^+$ satisfies \((1.8)\).

We will also consider the case of a backward dynamical system

\( (1.15) \)

\[ v_{n-1} = A_n v_n, \quad v_0 \in X. \]

We do not require $A_n$ to be invertible or the sequence $(A_n)$ to be bounded in $\mathcal{L}(X)$. For $n \leq m$ we denote

\[ A(n, n) := \text{Id}, \quad A(n, m) := A_{n+1} A_{n+2} \ldots A_m. \]

**Definition 1.7.** We say that \( (1.2) \) has a (uniform) exponential dichotomy with values $a$ and $b$, $0 < a < b$, if for all $n \geq n_0$ there exists a direct sum decomposition $X = X_s(n) \oplus X_u(n)$ such that $X_s(n)$, $X_u(n)$ and the associated projections $\pi_s(n) : X \to X_s(n)$ and $\pi_u(n) : X \to X_u(n)$ have the following properties for all $n \leq m$:

1. $A(n, m) \circ \pi_u(m) = \pi_u(n) \circ A(n, m)$ and $A(n, m) \circ \pi_u(m) = \pi_u(n) \circ A(n, m)$,
2. there exists a constant $C$ such that $\|\pi_s(n)\|_\mathcal{L}(X) + \|\pi_u(n)\|_\mathcal{L}(X) \leq C$,
3. $A(n, m)|_{X_s(m)} : X_s(m) \to X_s(n)$ is invertible,
4. there exists a constant $C$ such that $\|A(n, m)^{-1} v_n\| \leq C a^{n-m} \|v_n\|$ for all $v_n \in X_s(n)$,
5. there exists a constant $C$ such that $\|A(n, m) v_m\| \leq C b^{m-n} \|v_m\|$ for all $v_m \in X_u(m)$.

We make the following assumptions about the functionals $I_n^\pm$. We assume that there exist $c_2 > 0$, $K \in \{0, 1, 2, \ldots\}$ and $\alpha_{k,n} \in X^*$ for $(k, n) \in \{1, \ldots, K\} \times \{0, 1, \ldots\}$ such that

\[ c_2 \max_{1 \leq k \leq K} |\langle \alpha_{k,n}, v \rangle| \leq I_n^-(v) \leq \frac{1}{c_2} \max_{1 \leq k \leq K} |\langle \alpha_{k,n}, v \rangle|. \]

We define the stable and unstable cone by the same formulas \((1.3)\) and \((1.4)\). Instead of \((1.9) - (1.12)\), we assume

\[ \mathcal{V}_s(c_3, n) \text{ contains a linear space of dimension } K \text{ for all } n, \]
\[ c_4 > 3(c_1 c_2)^{-2} (c_3 + 1), \]
\[ I_n^-(v_n) \leq a I_{n-1}^- (A_n v_n) \quad \text{if} \quad v_n \in \mathcal{V}_s(c_4, n), \]
\[ I_n^+(v_n) \geq b I_{n-1}^+ (A_n v_n) \quad \text{if} \quad A_n v_n \in \mathcal{V}_u(c_3, n - 1). \]

**Theorem 2.** Under assumptions \((1.5), (1.16) - (1.20)\), the system \((1.15)\) has an exponential dichotomy with values $a$ and $b$. For all $n \geq 0$ the stable subspace $X_s(n)$ is contained in $\mathcal{V}_s(c_3, n)$ and has dimension $K$. 

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Remark 1.8. As before, (1.19) and (1.20) imply
\[ I_n^-(v_n) \leq a I_{n-1}^-(A_n v_n) \quad \text{if} \quad v_n \in \mathcal{V}_s(c, n), \]
\[ I_n^+(v_n) \geq b I_{n-1}^+(A_n v_n) \quad \text{if} \quad A_n v_n \in \mathcal{V}_u(c, n - 1). \]
for all \( c \in [c_3, c_4] \).

Remark 1.9. Note that if \( c_3, c_4 \) are large, then \( \mathcal{V}_s(c_4, n) \) is a wide cone and \( \mathcal{V}_u(c_3, n - 1) \) is a thin cone. We will prove in Proposition 2.3 that (1.17) holds if \( \alpha_{k,n} \) are uniformly linearly independent and \( c_3 \) is large enough. However, it is often possible to use a much smaller value of \( c_3 \), which gives more information about \( X_s(n) \). Again, (1.18) is not optimal.

One can state and prove analogous results for continuous dynamical systems, see Sections 2.3 and 2.4.

1.3. Applications in PDE. As a typical application, we can think of a heat equation with a time-dependent potential in the case of a forward dynamical system and of a backward heat equation with a time-dependent potential in the case of a backward dynamical system. We explain in Section 3 how to apply our result to the heat equation in the following two situations:

- the potential is almost constant on short time intervals in a suitable \( L^p \) norm,
- a potential of a fixed shape (or a finite number of such potentials) are moving in space with a small velocity.

Our result in the first case is quite similar to previous results of Schnaubelt [23, 24].

In Section 4, we apply the general results to the Klein-Gordon equation with moving potentials. To our knowledge, this is a first non-trivial example of exponential dichotomies for a wave-type equation.

Our main motivation is the study of multi-solitons for nonlinear models. In this situation, the potential is given by linearising the equation around an approximate solution. The hyperbolic structure of the flow around this approximate solution can often by obtained by the Lyapunov-Perron method if the existence of exponential dichotomy for the linear model is proved. We believe that this approach could lead to an alternative construction of multi-solitons in the weak interaction regime, see for instance [15, 6, 10, 9, 18].

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2. Constructions of exponential dichotomies

2.1. Discrete backward dynamical systems. In this section we prove Theorem 2. We use the so-called method of invariant cones, cf. [23, Section 4.4.2].

Lemma 2.1. For all \( n \geq 1 \) and \( c \in [c_3, c_4] \) there is
\[ A_n^{-1} \mathcal{V}_u(c, n - 1) \subset \mathcal{V}_u(c, n), \]
\[ A_n \mathcal{V}_s(c, n) \subset \mathcal{V}_u(c, n - 1). \]

Proof. In order to prove the first inclusion, suppose \( v_n \in X \) is such that \( A_nv_n \in \mathcal{V}_u(c, n - 1) \) and \( v_n \notin \mathcal{V}_u(c, n) \), thus \( v_n \in \mathcal{V}_s(c, n) \). From (1.21) and (1.22) we get
\[ c I_{n-1}^-(A_nv_n) \geq \frac{c}{a} I_n^-(v_n) > \frac{1}{a} I_n^+(v_n) \geq \frac{b}{a} I_{n-1}^+(A_nv_n) \geq I_{n-1}^+(A_nv_n), \]
which contradicts \( A_nv_n \in \mathcal{V}_u(c, n - 1) \).
In order to prove the second inclusion, suppose \( v_n \in X \) is such that \( v_n \in \mathcal{V}_b(c,n) \) and \( A_nv_n \notin \mathcal{V}_b(c,n-1) \). From (1.21) and (1.22) we get
\[
cI_{n-1}^-(A_nv_n) \geq \frac{c}{a}I_n^-(v_n) \geq \frac{1}{a}I_n^-(v_n) \geq \frac{b}{a}I_{n-1}^+(A_nv_n) \geq I_{n-1}^+(A_nv_n),
\]
which contradicts \( A_nv_n \notin \mathcal{V}_b(c,n-1) \).

For \( c \in [c_3,c_4] \) we define the stable subspace by
\[
X_s(c,n) := \bigcap_{n'>n} A(n,n')\mathcal{V}_b(c,n').
\]
Clearly, \( X_s(c,n) \) is a closed set, but it is not even obvious if \( X_s(c,n) \) is a linear subspace of \( X \).

**Lemma 2.2.** For all \( n \geq 0 \) and \( c \in [c_3,c_4] \) the following conditions are equivalent:

(i) \( w \in X_s(c,n) \),

(ii) there exists \( C \geq 0 \) such that for all \( \delta \in (0,1) \) and \( n' \geq n \) there is a solution \( (v_n,\ldots,v_{n'}) \) of (1.15) satisfying \( \|v_n-w\|_X \leq \delta \) and \( \|v_{n'}\|_X \leq Ca^{n'-n} \).

(iii) there exist \( C \geq 0 \) and \( d < b \) such that for all \( \delta \in (0,1) \) and \( n' \geq n \) there is a solution \( (v_n,\ldots,v_{n'}) \) of (1.15) satisfying \( \|v_n-w\|_X \leq \delta \) and \( \|v_{n'}\|_X \leq Ca^{n'-n} \).

**Proof.**

(i) \( \Rightarrow \) (ii) Let \( w \in X_s(c,n) \), \( \delta \in (0,1) \), and \( n' > n \). By the definition of \( X_s(c,n) \), there exists \( v_{n'} \in \mathcal{V}_b(c,n') \) such that \( \|A(n,n')v_{n'}-w\|_X \leq \delta \). We will show that there exists \( C \geq 0 \), independent of \( \delta \) and \( n' \), such that \( \|v_{n'}\|_X \leq Ca^{n'-n} \).

Consider \( v_m := A(m,n')v_{n'} \) for \( m \in \{n,\ldots,n'-1\} \). Then, by Lemma 2.1, \( v_m \in \mathcal{V}_b(c,m) \) for all \( m \in \{n,\ldots,n'-1\} \). Since \( v_m \in \mathcal{V}_b(c,m) \) for \( n \leq m < n' \), (1.21) implies \( I^+_m(v_m) \leq aI^-_{m-1}(A_mv_m) \) for \( n < m < n' \), which yields \( I^+_m(v_{n'}) \leq a^{n'-n}I^+_n(v_n) \). From (1.10) we obtain
\[
\|v_{n'}\|_X \leq \frac{1+c}{c_1}I^+_n(v_{n'}) \leq \frac{1+c}{c_1}a^{n'-n}I^+_n(v_n) \leq \frac{1+c}{c_1}a^{n'-n}\|v_n\|_X,
\]
which proves (ii) with \( C = \frac{(1+c)(1+\|w\|_X)}{c_1^2} \).

(ii) \( \Rightarrow \) (iii) follows from \( a < b \).

(iii) \( \Rightarrow \) (i) Let \( v_n \in X \) be such that (iii) holds. Suppose that \( w \notin X_s(c,n) \). This means that there exists \( m > n \) such that
\[
w \notin A(n,m)\mathcal{V}_b(c,m).
\]
In other words, there exists \( \delta > 0 \) such that if \( (v_n,\ldots,v_m) \) is a solution of (1.15) such that \( \|v_n-w\|_X \leq \delta \), then \( v_m \in X \setminus \mathcal{V}_b(c,m) \subset \mathcal{V}_d(c,m) \). We fix this \( \delta \) (without loss of generality assume \( \delta < \frac{1}{2}\|w\|_X \)) and let \( (v_n,\ldots,v_m) \) be a solution of (1.15) having the properties described in (iii) with \( n' \) large. Since \( v_m \in \mathcal{V}_d(c,m) \), Lemma 2.1 yields \( v_k \in \mathcal{V}_b(c,k) \) for \( k \in \{m,\ldots,n'\} \). Thus
\[
\frac{C}{c_1}a^{n'-n} \geq \frac{1}{c_1}\|v_{n'}\| \geq I^+_n(v_{n'}) \geq b^{n'-m}I^+_m(v_m) \geq \frac{c_1C}{1+c}b^{n'-m}\|v_m\|_X.
\]
Since we assume \( d < b \), by taking \( n' \) sufficiently large we can ensure that
\[
\|v_m\|_X \leq \frac{\delta}{\|A_{n+1}\|\ldots\|A_m\|}.
\]
This implies \( \|v_n\| \leq \delta \), thus \( \|w\| \leq \|v_n\| + \delta \leq 2\delta \), contradicting the choice of \( \delta \).

**Remark 2.3.** In the proof of the last lemma, assumptions (1.16)–(1.18) were not used.

**Corollary 2.4.** For all \( n \geq 0 \), the set \( X_s(c,n) = X_s(n) \) does not depend on \( c \in [c_3,c_4] \). It is a closed linear subspace of \( X \) and \( A_n: X_s(n) \to X_s(n-1) \) is a linear embedding.
Proof. Condition (ii) in Lemma 2.2 defines a linear subspace independent of $c$ so, by Lemma 2.2, $X_s(n)$ is a linear subspace of $X$. We see directly from (2.1) that it is closed and that $A_n w \in X_s(n)$ whenever $w \in X_s(n)$. The fact that $A_n$ is an embedding on $X_s(n)$ follows from (1.21) and the fact that $\| \cdot \|$ is comparable to $I_n^-$ on $V_s(c,n)$.

Proof of Theorem 2. We set $X_u(0) := \bigcap_{k=1}^K \ker \alpha_{k,0}$ and we define inductively

$$X_u(n) := A_n^{-1}(X_u(n-1)),$$

for $n > 0$.

By the definition of $V_u(c_3, n)$ and (1.19), we have $\bigcap_{k=1}^K \ker \alpha_{k,0} \subset V_u(c_4, 0)$, thus Lemma 2.4 yields $X_u(n) \subset V_u(c_4, n)$, for all $n \geq 0$. For all $n \geq 0$, $X_u(n)$ is a linear subspace of $X$ of codimension at most $K$ (as we will see later, in fact equal to $K$).

Note that the choice of $X_u(0)$ is not canonical, in fact we could take as $X_u(0)$ any subspace of dimension $\leq K$ contained in $V_u(c_4, 0)$.

We will find a constant $c_5 > 0$ depending on $c_1, \ldots, c_4$ such that if $v \in X_s(n)$ and $w \in X_u(n)$, then

$$\|v + w\| \geq c_5 \|v\|.$$  \hfill (2.2)

If $\|w\| \geq \frac{3}{2} \|v\|$, then (2.2) follows from the triangle inequality. Assume $\|w\| \leq \frac{3}{2} \|v\|$. Since $X_s(n) \subset V_s(c_3, n)$, (1.16) yields $I_n^-(v) \geq \frac{c_1 c_2}{1 + c_3} \|v\|_X$, thus by (1.16) there is $k_0 \in \{1, \ldots, K\}$ such that

$$\|\langle \alpha_{k_0}, v \rangle\| \geq \frac{c_1 c_2}{1 + c_3} \|v\|.$$  \hfill (2.3)

Since $X_u(n) \subset V_u(c_4, n)$, we have $c_4 I_n^-(w) \leq I_n^-(w)$, so (1.15) yields $(1 + c_4) I_n^-(w) \leq \frac{3}{c_1} \|w\| \leq \frac{3}{2c_1} \|v\|$. Invoking again (1.16) we obtain

$$\|\langle \alpha_{k_0}, w \rangle\| \leq \frac{3}{2c_1 c_2 (1 + c_4)} \|v\|.$$  \hfill (2.4)

From (2.3) and (2.4) we get

$$\|v + w\| \geq c_1 I_n^-(v + w) \geq c_1 c_2 \|\langle \alpha_{k_0}, v + w \rangle\| \geq c_1 c_2 \left( \frac{c_1 c_2}{1 + c_3} - \frac{3}{2c_1 c_2 (1 + c_4)} \right) \|v\|.$$  \hfill (2.5)

Assumption (1.18) implies that the constant in front of $\|v\|$ is greater than 0, so we have proved (2.2).

Next, we prove that $X = X_s(n) \oplus X_u(n)$. Bound (2.2) directly yields $X_u(n) \cap X_s(n) = \emptyset$.

Let $v \in X$. Let $\Pi := v + X_u(n)$. By assumption (1.17), for any $n' \geq n$ the cone $V_s(c_3, n')$ contains a linear space $X_s(n')$ of dimension $K$. We see that $A(n, n')|_{X_s(n')} = A(n, n')|_{X_s(n')}$ is one-to-one. Indeed, since $\ker A(n, n') \subset V_u(c_4, n')$ and $X_s(n') \subset V_s(c_3, n')$, this follows from $V_s(c_3, n') \cap V_u(c_4, n') = \emptyset$.

Since $X_u(n) \cap A(n, n') X_s(n') = \emptyset$ and codim$(X_u(n)) \leq K$, we actually have

$$\text{codim}(X_u(n)) = K$$  \hfill (2.5)

and the intersection $\Pi \cap A(n, n') X_s(n')$ is non-empty. In particular, $\Pi \cap A(n, n') V_s(c_3, n')$ is a nested family of closed non-empty sets. It suffices to show that their diameters tend to 0 as $n' \to \infty$.

Let $w_1, w_2 \in \Pi \cap A(n, n') V_s(c_3, n')$. Then $w := w_1 - w_2 \in X_u(n)$. Let $\delta > 0$. There exist $w'_1, w'_2 \in V_s(c_3, n')$ such that $\|A(n, n') w'_1 - w\| \leq \delta$ for $k \in \{1, 2\}$. Using Lemma 2.1 (1.21) and (1.15), we have

$$I_{n'}(w'_k) \leq a^{n'-n} I_n^-(A(n, n') w'_k) \leq \frac{1}{c_1} a^{n'-n} \|A(n, n') w'_k\| \leq \frac{1}{c_1} a^{n'-n} (\|w_k\| + \delta).$$

Since $w'_k \in V_s(c_3, n')$, (1.15) yields

$$\|w'_k\| \leq \frac{1 + c_3}{c_1} I_{n'}(w'_k) \leq \frac{1 + c_3}{c_1^2} a^{n'-n} (\|w_k\| + \delta).$$  \hfill (2.6)
Let \( w' := w'_1 - w'_2 \) and \( \bar{w} := A(n, n')w' \). We have \( \|\bar{w} - w\| \leq 2\delta \). There are two cases: either \( \bar{w} \in \mathcal{V}_u(c_3, n) \), or not.

In the first case, we also have \( w' \in \mathcal{V}_u(c_3, n') \), so we obtain
\[
\|w'\| \geq c_1 I_n^+(w') \geq c_1 b^{n'-n} I_n^+(\bar{w}) \geq \frac{c_1^2 c_3}{1 + c_3} b^{n'-n} \|\bar{w}\|.
\]
Combining this with (2.5) and \( \|w'\| \leq \|w_1'\| + \|w_2'\| \) we get
\[
\|\bar{w}\| \leq \frac{(1 + c_3)^2}{c_1^2 c_3} \left( \frac{a}{b} \right)^{n'-n} (\|w_1\| + \|w_2\| + 2\delta),
\]
which implies \( \|w\| \leq \|\bar{w}\| + 2\delta \leq 4\delta \) by taking \( n' \) large enough (depending on \( \delta \)). Since \( \delta > 0 \) is arbitrary, this finishes the proof.

In the second case, since \( X_u(n) \subset \mathcal{V}_u(c_4, n) \), we have
\[
I_n^+(w) \geq c_4 I_n^+(w), \quad I_n^+(\bar{w}) < c_3 I_n^+(\bar{w}), \quad \|w - \bar{w}\| \leq 2\delta.
\]
By continuity of \( I_n^+, I_n^- \), since \( \delta > 0 \) is arbitrary, this yields \( I_n^-(w) = 0 \). Again by continuity, we have \( I_n^-(\bar{w}) \) as small as we wish, thus also \( \|\bar{w}\| \) as small as we wish. Hence \( \|w\| \) is small as well.

This finishes the proof that \( X = X_u(n) \oplus X_u(n) \) for all \( n \). The fact that
\[
(2.7) \quad \dim X(u(n)) = K
\]
follows from (2.5). We are ready to verify all the requirements in Definition 1.7.

Invertibility of \( A(n, m)|_{X_u(m)} : X_u(m) \to X_u(n) \) follows from Corollary 2.4 and (2.7).

Uniform boundedness of the projections \( \pi_u(n), \pi_u(n) \) follows from (2.2).

The fact that the projections commute with \( A(n, m) \) follows from \( X_u(n) = A_n^{-1}(X_u(n-1)) \) and \( X_u(n) = A_n^{-1}(X_u(n-1)) \).

If \( v_n \in X_u(n) \), then for any \( m \geq n \) there exists \( v_m = A(n, m)^{-1}v_n \). Moreover, \( v_m \in \mathcal{V}_u(c_3, m) \) for \( m \geq n \), so \( (1.9) \) yields \( I_n^-(v_m) \leq a^{m-n} I_n^-(v_n) \). Hence \( (1.6) \) yields \( \|v_m\| \leq C a^{m-n} \|v_n\| \) with \( C = \frac{1 + c_3}{c_1^2} \).

Similarly, one can prove that if \( v_m \in X_u(m) \), then \( \|A(n, m)v_m\| \leq C b^{n-m} \|v_m\| \), with \( C = \frac{1 + c_3}{c_1^2} \).

Next, we prove below that (1.9) holds for \( c_3 \) sufficiently large if \( \alpha_{k,n}^- \) are uniformly linearly independent, by which we mean that there exists \( c_0 > 0 \) such that
\[
(2.8) \quad \left\| \sum_{k=1}^{K} b_k \alpha_{k,n}^- \right\|_{X^*} \geq c_0 \max_{1 \leq k \leq K} |b_k|, \quad \text{for all } (b_1, \ldots, b_K) \in \mathbb{R}^K.
\]

**Proposition 2.5.** The cone \( \mathcal{V}_n(c_3, n) \) contains no linear subspace of dimension \( K + 1 \). If (2.8) holds and \( c_3 > \frac{2K}{c_1^2 c_2 c_6} \), then it contains a linear subspace of dimension \( K \).

**Proof.** If \( \Sigma \subset X \) is a linear subspace of dimension \( K + 1 \), then there exists \( 0 \neq v \in \Sigma \) such that
\[
\langle \alpha_{k,n}^-, v \rangle = 0, \quad \text{for all } k \in \{1, \ldots, K\},
\]
which implies \( v \notin \mathcal{V}_n(c, n) \), for any \( c > 0 \).

Now assume \( c_3 > \frac{2K}{c_1^2 c_2 c_6} \). Fix \( n \geq 0 \) and for \( k \in \{1, \ldots, K\} \) let \( Y_k := \bigcap_{j \neq k} \ker \alpha_{j,n}^- \). We have
\[
(2.9) \quad \sup_{v_0 \in Y_k, \|v_0\|=1} \langle \alpha_{k,n}^-, v_0 \rangle = \inf_{(b_j) \in \mathbb{R}^K, b_k=1} \left\| \sum_{j=1}^{K} b_j \alpha_{j,n}^- \right\|_{X^*}.
\]
Indeed, for all \( v_0 \in Y_k \) and \( (b_j) \in \mathbb{R}^K \) such that \( \|v_0\| = b_k = 1 \) we have

\[
\langle \alpha_{k,n}, v_0 \rangle = \left\langle \sum_{j=1}^{K} b_j \alpha_{j,n}, v_0 \right\rangle \leq \left\| \sum_{j=1}^{K} b_j \alpha_{j,n} \right\|_{X^*},
\]

which implies that the left hand side in (2.9) is smaller or equal to the right hand side. Suppose the strict inequality holds, in other words \( \alpha_{k,n} \) defines a linear functional on \( Y_k \) of norm strictly smaller than the right hand side of (2.9). Then, by the Hahn-Banach theorem, there exists \( \alpha \in X^* \) such that \( \|\alpha\|_{X^*} \) is strictly smaller than the right hand side of (2.9) and \( Y_k \subset \ker(\alpha - \alpha_{k,n}) \). But it is well-known that the last condition implies that there exist \( b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_K \in \mathbb{R} \) such that \( \alpha - \alpha_{k,n} = \sum_{j \neq k} b_j \alpha_{j,n} \), so we get a contradiction. This proves (2.9).

By (2.8), the right hand side of (2.9) is \( \geq c_6 \), thus for all \( k \in \{1, \ldots, K\} \) there exists \( z_k \in X \) such that

\[
\|z_k\| = 1, \quad \langle \alpha_{k,n}, z_k \rangle \geq \frac{1}{2} c_6, \quad \langle \alpha_{j,n}, z_k \rangle = 0 \quad \text{for} \quad j \neq k.
\]

Let \( \tilde{X}_s(n) \) be the subspace spanned by the vectors \( z_k \). Clearly, the vectors \( z_k \) are linearly independent, so \( \dim \tilde{X}_s = K \). Let \( (a_k) \in \mathbb{R}^K \). We should prove that \( v_0 := \sum_k a_k z_k \in \mathcal{V}_s(c_2, n) \). From (2.10) and (1.16), we have

\[
I_n^+(v_0) = \sup_{0 \leq m \leq n} b^{n-m} \| B(n, m)^{-1} \pi_u(n)v_n \|, \\
I_n^-(v_0) = \sup_{m \geq n} a^{n-m} \| B(m, n)\pi_u(n)v_n \|.
\]

Directly from Definition 1.1 we get

\[
\| \pi_u(n)v_n \| \leq I_n^+(v_0) \leq C \| \pi_u(n)v_n \|, \\
\| \pi_s(n)v(n) \| \leq I_n^-(v_0) \leq C \| \pi_s(n)v_n \|,
\]

which implies (1.13). It is clear that \( I_n^+ \) and \( I_n^- \) are seminorms, in particular they are continuous. Moreover, we have

\[
I_{n+1}^+(B_n v_n) = \sup_{0 \leq m \leq n+1} b^{n+1-m} \| B(n+1, m)^{-1} \pi_u(n+1)(B_n v_n) \| \geq b \sup_{0 \leq m \leq n} b^{n-m} \| B(n+1, m)^{-1} B_n \pi_u(n)v_n \| = b I_n^+(v_n),
\]

and similarly \( I_{n+1}^- (B_n v_n) \leq a I_n^-(v_n) \).

Now assume that \( X(n) \) has finite dimension \( K \). Since \( I_n^+ \) is a norm on \( X(n) \), existence of linear functionals \( \alpha_{k,n}^+ \in (X(n))^* \) such that (1.8) holds on \( X(n) \) (with the constant depending only on \( K \) is a classical fact in Convex Geometry (it can be proved for example using the John’s ellipsoid). Now it suffices to extend \( \alpha_{k,n}^+ \) on the whole \( X \) be setting \( \langle \alpha_{k,n}^+, v \rangle = 0 \) for \( v \in X(n) \).
2.2. Discrete forward dynamical systems. In this section, we prove Theorem 1.

**Lemma 2.6.** For all \( n \geq 0 \) and \( c \in [c_4, c_3] \) there is
\[
\overline{B}_n V_u(c, n) \subset V_u(c, n + 1),
\]
\[
\overline{B}_n^{-1} V_s(c, n + 1) \subset V_s(c, n).
\]

**Proof.** In order to prove the first inclusion, suppose \( v_n \in X \) is such that \( v_n \in V_u(c, n) \) and \( B_n v_n \notin V_u(c, n + 1) \), thus \( B_n v_n \in V_s(c, n + 1) \). From (1.13) and (1.14) we get
\[
I_{n+1}^+ (B_n v_n) \geq b I_n^+ (v_n) \geq b c I_n^- (v_n) \geq \frac{b c}{a} I_{n+1}^- (B_n v_n) \geq c I_{n+1}^- (B_n v_n),
\]
which contradicts \( B_n v_n \notin V_u(c, n + 1) \).

In order to prove the second inclusion, suppose \( v_n \in X \) is such that \( B_n v_n \in V_s(c, n + 1) \) and \( v_n \notin V_s(c, n) \), thus \( v_n \in V_u(c, n) \). From (1.13) and (1.14) we get
\[
I_{n+1}^+ (B_n v_n) \geq b I_n^+ (v_n) > b c I_n^- (v_n) \geq \frac{b c}{a} I_{n+1}^- (B_n v_n) \geq c I_{n+1}^- (B_n v_n),
\]
which contradicts \( B_n v_n \in V_s(c, n + 1) \). \( \square \)

For \( n \geq 0 \) and \( c \in [c_4, c_3] \) we define the stable subspace by
\[
X_s(c, n) := \bigcap_{n' \geq n} B(n', n)^{-1}(V_s(c, n')).
\]
Clearly, \( X_s(c, n) \) is a closed set.

The statement and proof of Lemma 2.7 are very similar to the statement and proof of Lemma 2.2. We provide the details for the sake of completeness.

**Lemma 2.7.** For all \( n \geq 0 \) and \( c \in [c_4, c_3] \) the following conditions are equivalent:

(i) \( w \in X_s(c, n) \),

(ii) there exists \( C \geq 0 \) such that for all \( n' \geq n \) the bound \( \| B(n', n)w \| \leq C a^{n'-n} \) holds,

(iii) there exist \( C \geq 0 \) and \( d < b \) such that for all \( n' \geq n \) the bound \( \| B(n', n)w \| \leq C d^{n'-n} \) holds.

**Proof.** \( (i) \Rightarrow (ii) \) Let \( w \in X_s(c, n) \) and set \( v_{n'} := B(n', n)w \) for \( n' > n \). By the definition of \( X_s(c, n) \), we have \( v_{n'} \in V_s(n') \) for all \( n' > n \). In particular, the bound (1.16) holds. Also, by the proof of Lemma 2.6 we have \( I_{n+1}^- (v_{m+1}) \leq \alpha I_m^+ (v_m) \) for all \( m \geq n \). Hence, we obtain (ii) with \( C = \frac{1+c}{1+a} \).

\( (i) \Rightarrow (iii) \) follows from \( a < b \).

\( (ii) \Rightarrow (iii) \) Let \( w \in X_s(c, n) \) be such that (iii) holds and set \( v_{n'} := B(n', n)w \) for \( n' > n \). Suppose that \( w \notin X_s(c, n) \). This means that there exists \( m > n \) such that \( v_m \notin V_s(c, m) \), thus \( v_m \in V_u(c, m) \). By Lemma 2.6 we have \( v_{n'} \in V_u(c, n') \) for all \( n' \geq m \). Thus
\[
\frac{C}{c_1} a^{n'-n} \geq \frac{1}{c_1} \| v_{n'} \| \geq \frac{1}{c_1} I_n^+ (v_n) \geq \frac{1}{c_1} b^{n'-m} I_m^+ (v_m) \geq \frac{c_1}{c_1} b^{n'-m} \| v_m \|.
\]
Since we assume \( d < b \), this implies \( \| v_m \| = 0 \), contradicting \( v_m \notin V_s(c, m) \). \( \square \)

**Corollary 2.8.** For all \( n \geq 0 \), \( X_s(c, n) = X_s(n) \) does not depend on \( c \in [c_4, c_3] \). It is a closed linear subspace of \( X \) and \( X_s(n) = B_n^{-1} X_s(n + 1) \).

**Proof.** Condition (ii) in Lemma 2.7 defines a linear subspace independent of \( c \in [c_4, c_3] \) so, by Lemma 2.6, \( X_s(n) \) is a linear subspace of \( X \). We see directly from (2.11) that it is closed and that \( X_s(n) = B_n^{-1} X_s(n + 1) \). \( \square \)
Proof of Theorem 1. We let \( X_u(0) \) be any \( K \)-dimensional linear subspace contained in \( \mathcal{V}_u(c_3, 0) \). Such a space exists by assumption (1.9). If \( v_0 \in X_u(0) \) and \( v_{n+1} = B_n v_n \) for \( n \geq 0 \), then by Lemma 2.6 \( v_n \in \mathcal{V}_u(c_3, n) \) for all \( n \), which by (1.12) yields \( I_n^+(v_n) \geq b_n^+ I_0^+(v_0) \). Using (1.7), we obtain
\[
\|v_n\| \geq \frac{c_1^2 c_3}{1 + c_3} b_n^+ \|v_0\|,
\]
thus \( B(n, 0)|_{X_u(0)} \) is a linear embedding. We set
\[
X_u(n) := B(n, 0)X_u(0),
\]
which is a linear subspace of \( X \) of dimension \( K \). As for backward systems, the choice of \( X_u(n) \) is not canonical.

We will find \( c_5 > 0 \) such that if \( v \in X_u(n) \) and \( w \in X_u(n) \), then
\[
(2.12) \quad \|v + w\| \geq c_5 \|w\|.
\]
We can assume \( \|v\| \leq \frac{3}{2} \|w\| \). Since \( X_u(n) \subset \mathcal{V}_u(c_3, n) \), (1.7) yields \( I_n^+(w) \geq \frac{c_1 c_2 c_3}{1 + c_3} \|w\| \), thus by (1.8) there is \( k_0 \in \{1, \ldots, K\} \) such that
\[
|\langle \alpha_{k_0}^+, w \rangle| \geq \frac{c_1 c_2 c_3}{1 + c_3} \|w\|.
\]
Since \( X_u(n) \subset \mathcal{V}_u(c_4, n) \), we have \( I_n^+(v) \leq c_4 I_n^+(v) \), so (1.5) yields \( (1 + c_4) I_n^+(v) \leq \frac{c_4}{c_1} \|v\| \leq \frac{3c_4}{2c_1} \|w\| \). Invoking again (1.8) we obtain
\[
|\langle \alpha_{k_0}^+, v \rangle| \leq \frac{3c_4}{2c_1 c_2 (1 + c_4)} \|w\|.
\]
From (2.3) and (2.4) we get
\[
\|v + w\| \geq c_4 I_n^+(v + w) \geq c_1 c_2 |\langle \alpha_{k_0}^+, v + w \rangle| \geq c_1 c_2 \left( \frac{c_1 c_2 c_3}{1 + c_3} - \frac{3c_4}{2c_1 c_2 (1 + c_4)} \right) \|w\|.
\]
Assumption (1.10) implies that the constant in front of \( \|w\| \) is \( > 0 \), so we have proved (2.12).

As in the proof of Theorem 2 we obtain \( X_u(n) \cap X_u(n) = \{0\} \) for all \( n \geq 0 \). Let \( u \in X \) and let \( \Pi := u + X_u(n) \). For any \( n' \geq n \), let \( \bar{X}_u(n') := \bigcap_{k=1}^K \ker(\alpha_{k,n'}) \). We see that \( \text{codim}(\bar{X}_u(n')) \leq K \) and \( \bar{X}_u(n') \subset \mathcal{V}_u(n') \), which implies that \( B(n', n)^{-1}(\mathcal{V}_u(n')) \) contains a space of codimension \( K \). Since \( X_u(n) \cap B(n', n)^{-1}(\mathcal{V}_u(n')) = \{0\} \), we obtain \( \Pi \cap B(n', n)^{-1}(\mathcal{V}_u(n')) \neq \emptyset \). Thus \( \Pi \cap B(n', n)^{-1}(\mathcal{V}_u(n')) \) is a nested family of closed non-empty sets and it suffices to show that their diameters tend to 0 as \( n' \to \infty \), which can be done similarly as in the proof of Theorem 2.

If \( \alpha_{k,n} \) are uniformly linearly independent, by which we mean that there exists \( c_0 > 0 \) such that
\[
(2.13) \quad \left\| \sum_{k=1}^K b_k \alpha_{k,n}^+ \right\|_{X^*} \geq c_0 \max_{1 \leq k \leq K} |b_k| \quad \text{for all } (b_1, \ldots, b_K) \in \mathbb{R}^K,
\]
then the proof of Proposition 2.5 shows that (1.17) holds if \( c_3 \) is small enough. We have

**Proposition 2.9.** The cone \( \mathcal{V}_u(c_3, n) \) contains no linear subspace of dimension \( K + 1 \). If (2.13) holds and \( c_3 < \frac{c_1 c_2 c_6}{2 K} \), then it contains a linear subspace of dimension \( K \).

The proof is exactly the same as the proof of Proposition 2.5, so we skip it.
2.3. Strongly continuous backward dynamical systems. Our proof adapts easily to the case of continuous dynamics.

**Definition 2.10.** Let $X$ be a Banach space. A family of operators $S(t, \tau) \in \mathcal{L}(X)$ for $0 \leq t \leq \tau$ is called a strongly continuous backward evolution operator if it satisfies:

1. $S(t, t) = \text{Id}$ for all $t \geq 0$,
2. for all $\tau \geq 0$ and all $v_\tau \in X$ the function $[0, \tau] \ni t \mapsto S(t, \tau)v_\tau \in X$ is continuous,
3. for all $0 \leq t \leq s \leq \tau$ there is $S(t, s) \circ S(s, \tau) = S(t, \tau)$.

Let $X$ be a Banach space and let $S(t, \tau)$ for $0 \leq t \leq \tau$ be a strongly continuous backward evolution operator. We consider the dynamical system

$\begin{align*}
\frac{d}{dt} v(t) &= S(t, \tau)v(t), \\
v(0) &= v_0 \in X.
\end{align*}$

Note that we do not require $S(t, \tau)$ to be invertible.

**Definition 2.11.** We say that (1.15) has a (uniform) exponential dichotomy with exponents $\lambda$ and $\mu$, $-\infty < \lambda < \mu < \infty$, if for all $t \geq 0$ there exists a direct sum decomposition $X = X_s(t) \oplus X_u(t)$ such that $X_s(t)$ and $X_u(t)$ and the associated projections $\pi_s(t) : X \to X_s(t)$ and $\pi_u(t) : X \to X_u(t)$ have the following properties for all $t \leq \tau$:

1. $S(t, \tau) \circ \pi_s(\tau) = \pi_s(t) \circ S(t, \tau)$ and $S(t, \tau) \circ \pi_u(\tau) = \pi_u(t) \circ S(t, \tau)$,
2. there exists a constant $C$ such that $\|\pi_s(t)\|_{\mathcal{L}(X)} + \|\pi_u(t)\|_{\mathcal{L}(X)} \leq C$,
3. $S(t, \tau)|_{X_s(\tau)} : X_s(\tau) \to X_s(t)$ is invertible,
4. there exists a constant $C$ such that $\|S(t, \tau)^{-1} v_\tau\| \leq C e^{\lambda(\tau-t)} \|v_\tau\|$ for all $v_\tau \in X_s(t)$,
5. there exists a constant $C$ such that $\|S(t, \tau)v_\tau\| \leq C e^{\mu(\tau-t)} \|v_\tau\|$ for all $v_\tau \in X_u(t)$.

Our sufficient condition for existence of an exponential dichotomy is expressed in terms of two families of (nonlinear) homogeneous functionals $I_t^-, I_t^+ : X \to \mathbb{R}_+$. We assume that $I_t^+(v)$ is continuous in $(t, v) \in \mathbb{R}_+ \times X$. Given $I_t^-, I_t^+$ and a number $c > 0$, we define the stable and the unstable cone

$\begin{align*}
\mathcal{V}_s(c, t) &:= \{v \in X : I_t^+(v) \leq c I_t^-(v)\}, \\
\mathcal{V}_u(c, t) &:= \{v \in X : I_t^+(v) \geq c I_t^-(v)\}.
\end{align*}$

Firstly, we assume that there exists $c_1 > 0$ (independent of $t$) such that

$\begin{align*}
(2.15) \quad c_1 \|v\|_X \leq I_t^-(v) + I_t^+(v) \leq \frac{1}{c_1} \|v\|_X, & \quad \text{for all } t \geq 0 \text{ and } v \in X.
\end{align*}$

Secondly, we assume that there exist $c_2 > 0$, $K \in \{0, 1, 2, \ldots\}$ and $\alpha_{k, t}^- \in X^*$ for $(k, t) \in \{1, \ldots, K\} \times \mathbb{R}_+$ such that

$\begin{align*}
(2.16) \quad c_2 \max_{1 \leq k \leq K} |\langle \alpha_{k, t}^-, v \rangle| \leq I_t^-(v) \leq \frac{1}{c_2} \max_{1 \leq k \leq K} |\langle \alpha_{k, t}^-, v \rangle|.
\end{align*}$

Continuity of $\alpha_{k, t}^-$ with respect to $t$ is not required.

Lastly, we assume that there exist $c_3, c_4 > 0$ and $-\infty < \lambda < \mu < \infty$ such that

$\begin{align*}
(2.17) \quad & \mathcal{V}_s(c_4, t) \text{ contains a linear space of dimension } K \text{ for all } t, \\
(2.18) \quad & c_4 > 3(c_1 c_2)^{-2}(c_3 + 1), \\
(2.19) \quad & I_t^-(v_\tau) \leq e^{\lambda(\tau-t)} I_t^-(S(t, \tau)v_\tau) \quad \text{if } S(t', \tau)v_\tau \in \mathcal{V}_s(c_4, t') \text{ for all } t' \in [t, \tau], \\
(2.20) \quad & I_t^+(v_\tau) \geq e^{\mu(\tau-t)} I_t^+(S(t, \tau)v_\tau) \quad \text{if } S(t', \tau)v_\tau \in \mathcal{V}_s(c_3, t') \text{ for all } t' \in [t, \tau].
\end{align*}$

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Note that, unlike in (1.21) and (1.22), here we assume that the $I^-_t$ or $I^+_t$ direction is significant on the whole time interval $[t, \tau]$. This is why the proof of the invariance of cones given below contains an additional continuity argument.

**Lemma 2.12.** For all $c \in (c_3, c_4)$ and $0 \leq t \leq \tau$ there is

$$S(t, \tau)^{-1}V_u(c, t) \subset V_u(c, \tau),$$

$$S(t, \tau)V_s(c, \tau) \subset V_s(c, t).$$

**Proof.** In order to prove the first inclusion, suppose $v_\tau \in X$ is such that $S(t, \tau)v_\tau \in V_u(c, t)$ and $v_\tau \notin V_u(c, \tau)$. Let $t_1 := \sup\{t' \leq \tau : S(t', \tau)v_\tau \in V_u(c, t')\}$.

By continuity,

(2.21) \quad $I^+_t(S(t_1, \tau)v_\tau) = cI^-_t(S(t_1, \tau)v_\tau)$,

(2.22) \quad $S(t', \tau)v_\tau \in V_s(c, t')$, for all $t' \in [t_1, \tau]$.

Assumption (2.19) together with (2.22) yields

$$I^-_\tau(v_\tau) \leq e^{\lambda(t-t_1)}I^-_t(S(t_1, \tau)v_\tau).$$

In particular, since $I^-_\tau(v_\tau) > 0$, we have $I^-_t(S(t_1, \tau)v_\tau) > 0$. Thus, again by continuity, (2.21) and $c > c_3$ imply that there exists $t_2 \in (t_1, \tau]$ such that

$$S(t', \tau)v_\tau \in V_u(c_3, t'), \quad \text{for all } t' \in [t_1, t_2].$$

Let $v_{t_2} := S(t_2, \tau)v_\tau$ and $v_{t_1} := S(t_1, t_2)v_2$. Using (2.20) with $\tau = t_2$ and $t = t_1$ we get

$$I^+_t(v_{t_2}) \geq e^{\mu(t_2-t_1)}I^+_t(v_{t_1}).$$

On the other hand, (2.19) and (2.22) yield

$$I^-_{t_2}(v_{t_2}) \leq e^{\lambda(t_2-t_1)}I^-_{t_1}(v_{t_1}).$$

Thus (2.21) and $\lambda < \mu$ yield $I^+_t(v_{t_2}) > cI^-_{t_1}(v_{t_2})$, which contradicts (2.22).

In order to prove the second inclusion, suppose $v_\tau \in X$ is such that $v_\tau \in V_s(c, \tau)$ and $S(t, \tau)v_\tau \notin V_s(c, t)$. Set

$$t_2 := \inf\{t' \geq t : S(t', \tau)v_\tau \in V_s(c, t')\}.$$

By continuity,

(2.23) \quad $I^+_t(S(t_2, \tau)v_\tau) = cI^-_t(S(t_2, \tau)v_\tau)$,

(2.24) \quad $S(t', \tau)v_\tau \in V_u(c, t')$, for all $t' \in [t, t_2]$.

Assumption (2.20) together with (2.24) yields

$$I^+_t(v_{t_1}) \geq e^{\mu(t_2-t)}I^+_t(S(t, \tau)v_\tau).$$

In particular, since $I^+_t(S(t, \tau)v_\tau) > 0$, we have $I^+_t(S(t_2, \tau)v_\tau) > 0$. Thus, again by continuity, (2.23) and $c < c_4$ imply that there exists $t_1 \in [t, t_2)$ such that

$$S(t', \tau)v_\tau \in V_s(c_4, t'), \quad \text{for all } t' \in [t_1, t_2].$$

The remaining arguments are the same as in the first part of the proof. \hfill \Box

**Theorem 3.** Under assumptions (2.15)–(2.20), the system (2.14) has an exponential dichotomy with exponents $\lambda$ and $\mu$. For all $t \geq 0$ the stable subspace $X_s(t)$ is contained in $V_u(c_3, t)$ and has dimension $K$.  

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The proof would follow the lines of the proof of Theorem 2 with $c_3$ and $c_4$ replaced everywhere by some $\tilde{c}_3 > c_3$ and $\tilde{c}_4 < c_4$ such that $\tilde{c}_4 > 3(c_1c_2)^{-2}(\tilde{c}_3 + 1)$. At the end we obtain $X_s(t) \subset V_s(\tilde{c}_3, t)$ for any $\tilde{c}_3 > c_3$, which means $X_s(t) \subset V_s(c_3, n)$.

2.4. **Strongly continuous forward dynamical systems.**

**Definition 2.13.** Let $X$ be a Banach space. A family of operators $T(\tau, t) \in \mathcal{L}(X)$ for $0 \leq t \leq \tau$ is called a **strongly continuous evolution operator** if it satisfies:

1. $T(t, t) = \text{Id}$ for all $t \geq 0$,
2. for all $t \geq 0$ and all $v_t \in X$ the function $[t, \infty) \ni \tau \mapsto T(\tau, t)v_t \in X$ is continuous,
3. for all $0 \leq t \leq s \leq \tau$ there is $T(\tau, s) \circ T(s, t) = T(\tau, t)$.

Let $T(\tau, t)$ be a strongly continuous evolution operator and consider the system

$$v_\tau = T(\tau, t)v_t, \quad v_0 \in X.$$

**Definition 2.14.** We say that (2.25) has an **exponential dichotomy** with exponents $\lambda$ and $\mu$, $-\infty < \lambda < \mu < \infty$, if for all $t \geq 0$ there exists a direct sum decomposition $X = X_s(t) \oplus X_u(t)$ such that $X_s(t)$, $X_u(t)$ and the associated projections $\pi_s(t) : X \rightarrow X_s(t)$ and $\pi_u(t) : X \rightarrow X_u(t)$ have the following properties for all $t \leq \tau$:

1. $T(\tau, t) \circ \pi_s(t) = \pi_s(\tau) \circ T(\tau, t)$ and $T(\tau, t) \circ \pi_u(t) = \pi_u(\tau) \circ T(\tau, t)$,
2. there exists a constant $C$ such that $\|\pi_s(t)\|_{\mathcal{L}(X)} + \|\pi_u(t)\|_{\mathcal{L}(X)} \leq C$,
3. $T(\tau, t)|_{X_u(t)} : X_u(t) \rightarrow X_u(\tau)$ is invertible,
4. there exists a constant $C$ such that $\|T(\tau, t)v_t\| \leq Ce^{\lambda(t-t')}\|v_t\|$ for all $v_t \in X_s(t)$,
5. there exists a constant $C$ such that $\|T(\tau, t)^{-1}v_r\| \leq Ce^{\mu(t-t')}\|v_r\|$ for all $v_r \in X_u(\tau)$.

Our sufficient conditions for existence of an exponential dichotomy are similar as in Section 2.3. Instead of (2.16), we assume

$$c_2 \max_{1 \leq k \leq K} |\langle \alpha_{k,t}^+, v \rangle| \leq I_t^+(v) \leq \frac{1}{c_2} \max_{1 \leq k \leq K} |\langle \alpha_{k,t}^+, v \rangle|.$$

Instead of (2.17)–(2.20), we assume

$$V_u(c_3, t) \text{ contains a linear space of dimension } K \text{ for all } t,$$

$$c_4 < \frac{1}{3} \frac{(c_1c_2)^2}{1 + c_3},$$

$$I_t^- (T(\tau, t)v_t) \leq e^{\lambda(t-t')}I_t^- (v_t) \quad \text{if } T(t', t)v_t \in V_s(c_3, t') \text{ for all } t' \in [t, \tau],$$

$$I_t^+ (T(\tau, t)v_t) \geq e^{\mu(t-t')}I_t^+ (v_t) \quad \text{if } T(t', t)v_t \in V_u(c_4, t') \text{ for all } t' \in [t, \tau].$$

**Theorem 4.** Under assumptions (2.15) and (2.26)–(2.30), the system (2.25) has an exponential dichotomy with exponents $\lambda$ and $\mu$. For all $t \geq 0$ the stable space $X_s(t)$ is contained in $V_s(c_4, t)$ and has codimension $K$.

3. **Some simple examples**

3.1. **Avalanche dynamics in finite dimension.** Let $B_n$ be a sequence of real matrices of size $d \in \{1, 2, \ldots\}$. We consider the linear dynamical system

$$v_{n+1} = B_nv_n, \quad v_0 \in \mathbb{R}^d.$$

Assume that there exist $0 \leq a < b$ such that for all $n$ the matrix $B_n^*B_n$ has $d_s$ eigenvalues $\leq a^2$ and $d_u = d - d_s$ eigenvalues $\geq b^2$. 

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Let $Y_s(n) \subset \mathbb{R}^d$ be the subspace spanned by eigenvectors of $B_n^*B_n$ corresponding to eigenvalues $\leq a^2$, and let $Z_s(n) \subset \mathbb{R}^d$ be spanned by eigenvectors of $B_n^*B_n$ corresponding to eigenvalues $\geq b^2$. The standard formula $\|B_nv\|^2 = \langle v, B_n^*B_nv \rangle$, together with the Spectral Theorem for symmetric matrices yields

$$v \in Y_s(n) \Rightarrow \|B_nv\| \leq a\|v\|,$$
$$v \in Y_u(n) \Rightarrow \|B_nv\| \geq b\|v\|.$$ 

Let $Z_s(n) := B_nY_s(n)$ and $Z_u(n) := B_nY_u(n)$. Note that we do not assume that $B_n$ is invertible, so it may happen that $\dim(Z_s(n)) < \dim(Y_s(n))$. It turns out that $\dim(X_s) = d_s$ if the angles between $Z_s(n)$ and $Y_s(n+1)$, as well as the angles between $Z_u(n)$ and $Y_u(n+1)$, are small. A similar assumption appears in the “Avalanche Principle” of Goldstein and Schlag [12].

If $Y, Z \subset \mathbb{R}^d$ are two linear subspaces of the same dimension, we can measure their proximity by the Hausdorff distance of their unit spheres $S_Y := \{v \in Y : \|v\| = 1\}$ and $S_Z := \{w \in Z : \|w\| = 1\}$. We set

$$\phi(Y, Z) := \max \left( \sup_{v \in S_Y} \inf_{w \in S_Z} \|v - w\|^2, \sup_{w \in S_Z} \inf_{v \in S_Y} \|v - w\|^2 \right).$$

Note that $\phi(Y, Z) = 0$ if and only if $Y = Z$.

**Lemma 3.1.** Assume $\phi(Y, Z) \leq \delta$. Let $v \in Y$ and let $\tilde{v} \in Z$ be the orthogonal projection of $v$ on $Z$. Then

$$\|\tilde{v}\|^2 \geq (1 - \delta)\|v\|^2.$$ 

**Proof.** By rescaling, we can assume $\|v\| = 1$. By compactness and the definition of $\phi(Y, Z)$, there exists $w \in Z$ such that $\|v - w\|^2 \leq \delta$. This implies $\|v - \tilde{v}\|^2 \leq \delta$, thus $\|\tilde{v}\|^2 = \|v\|^2 - \|v - \tilde{v}\|^2 \geq 1 - \delta$. 

We have $X = Y_s(n) \oplus Y_u(n)$ and $Z_s(n)$ is orthogonal to $Y_u(n)$. Since $Y_s(n)$ and $Y_u(n)$ are invariant for $B_n^*B_n$, for $v \in Y_s(n)$ and $w \in Y_u(n)$ we obtain $\langle B_nv, B_nw \rangle = \langle v, B_n^*B_nw \rangle = 0$, so $Z_s(n)$ and $Z_u(n)$ are orthogonal as well (but do not have to span $\mathbb{R}^d$).

**Proposition 3.2.** For any $0 \leq a < b$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi(Z_s(n), Y_s(n+1)) \leq \delta$ and $\phi(Z_u(n), Y_u(n+1)) \leq \delta$ for all $n$, then the system [3.1] has an exponential splitting $\mathbb{R}^d = X_s(n) \oplus X_u(n)$ with values $a + \epsilon$ and $b - \epsilon$. Moreover, $\dim(X_s(n)) = d_s$ and $\dim(X_u(n)) = d_u$.

**Proof.** We apply Theorem[1]. Let $\alpha^-_{1,n}, \ldots, \alpha^-_{d_s,n}$ be an orthonormal basis of $Y_s(n)$ and let $\alpha^+_{1,n}, \ldots, \alpha^+_{d_u,n}$ be an orthonormal basis of $Y_u(n)$. We define

$$I^-_n(v) := \sqrt{\sum_{k=1}^{d_s} \langle \alpha^-_{k,n}, v \rangle^2}, \quad I^+_n(v) := \sqrt{\sum_{k=1}^{d_u} \langle \alpha^+_{k,n}, v \rangle^2},$$

so that $I^-_n(v)$ and $I^+_n(v)$ are the lengths of the orthogonal projections of $v$ on $Y_s(n)$ and $Y_u(n)$. We have to check (1.13) and (1.14) for some $c_4 < c_3$ (all the other conditions are immediate).

Let $v = v_s + v_u$ with $v_s \in Y_s(n)$ and $v_u \in Y_u(n)$. Then $w = B_nv = w_s + w_u$, where $w_s = B_nv_s \in Z_s(n)$ and $w_u = B_nv_u \in Z_u(n)$. We have $\|w_s\| \leq a\|v\|$ and $\|w_u\| \geq b\|v\|$. We decompose further $w_s = w_{ss} + w_u$ and $w_u = w_{us} + w_{uu}$, with $w_{ss}, w_{us} \in Y_s(n+1)$ and $w_{uu}, w_{uu} \in Y_u(n+1)$. Note that $\|w_s\|^2 = \|w_{ss}\|^2 + \|w_{su}\|^2$, $\|w_u\|^2 = \|w_{us}\|^2 + \|w_{uu}\|^2$, $I^-_{n+1}(w) = \|w_{ss} + w_{su}\|$ and $I^+_{n+1}(w) = \|w_{su} + w_{uu}\|$. We need to show that

$$\|v_u\| \geq c_4\|v_s\| \Rightarrow \|w_{su} + w_{uu}\| \geq (b - \epsilon)\|v_u\|,$$

$$\|w_u\| \leq c_3\|w_{ss} + w_{us}\| \Rightarrow \|w_{ss} + w_{us}\| \leq (a + \epsilon)\|v_u\|.$$
In order to prove (3.2), we observe that Lemma 3.1 yields \( \|w_{uu}\| \geq \sqrt{1 - \delta} \|w_u\| \), thus
\[
(3.4) \quad \|w_{uu}\| \geq b\sqrt{1 - \delta} \|w_u\|.
\]
Using again Lemma 3.1, we obtain
\[
(3.5) \quad \|w_s\|^2 = \|w_s\|^2 - \|w_{ss}\|^2 \leq \delta \|w_s\|^2 \leq \delta a^2 \|v_s\|^2 \leq \frac{\delta a^2}{c_4^2} \|v_u\|^2,
\]
where in the last step we use the assumption \( \|v_u\| \geq c_4 \|v_s\| \). Combining (3.4) and (3.5) we get
\[
\|w_s + w_{uu}\| \geq \|w_{uu}\| - \|w_{ss}\| \geq \left( b\sqrt{1 - \delta} - \frac{a\sqrt{\delta}}{c_4} \right) \|v_u\|,
\]
which is (3.2) with \( \epsilon = b(1 - \sqrt{1 - \delta}) + \frac{a\sqrt{\delta}}{c_4} \).

We are left with (3.3). We have \( \|w_{ss}\| \leq \|w_s\| \leq \|v_s\| \) and, similarly as in (3.5),
\[
\|w_{ss} + w_{uu}\| \leq \left( 1 - \frac{\delta}{c_3} \right) \|w_{ss}\| \leq \left( 1 - \frac{\delta}{c_3} \right)^{-1} \|v_s\|.
\]
This proves (3.3) with \( \epsilon = a \left( 1 - \frac{\delta}{c_3} \right)^{-1} - 1 \). \( \square \)

**Remark 3.3.** We see that for any \( c_3, c_4 > 0 \) all the conditions are satisfied if \( \delta \) is small enough. In particular, taking \( c_4 \) small enough, we deduce from Theorem 1 that \( \phi(X_s(n), Y_s(n)) \rightarrow 0 \) as \( \delta \rightarrow 0 \).

### 3.2. Backward heat equation with an almost constant potential.

As our next example, we consider the backward heat equation with a time-dependent potential:
\[
(3.6) \quad \partial_t u(t, x) = -\Delta u(t, x) + V(t, x) u(t, x), \quad \text{for} \ (t, x) \in \mathbb{R}_+ \times \Omega.
\]
We assume that \( \Omega \subset \mathbb{R}^d \) is bounded with smooth boundary and that \( V \in L^\infty([0, \infty); L^p(\Omega)) \) for some \( p > \frac{d}{2} \). To simplify, we will also assume \( d \geq 4 \), but straightforward modifications allow to cover \( d = 1, 2, 3 \) as well (in this case, one should take \( p = 2 \)). Given a potential \( V \in L^p(\Omega) \), we denote \( \lambda_j(V) \) the \( j \)-th smallest eigenvalue (counted with multiplicities) of the Schrödinger operator \(-\Delta + V\) with Dirichlet boundary conditions.

We assume that there exists \( \mu > 0 \) such that for all \( t \in [0, \infty) \)
\[
\lambda_1(V(t)) \leq -\mu, \quad \lambda_2(V(t)) \geq \mu.
\]
Note that, upon adding a fixed constant to the potential, we could cover the case where \([\lambda_1(V(t)), \lambda_2(V(t))]\) contains a given interval of strictly positive length for all \( t \).

**Proposition 3.4.** For any \( \epsilon > 0 \) there exists \( \delta = \delta(\Omega, \mu, \|V\|_{L^\infty L^p}, \mu, \epsilon) > 0 \) such that if
\[
(3.7) \quad \|V(t_1) - V(t_2)\|_{L^p} \leq \delta \quad \text{for all} \ t_1, t_2 \text{with} \ |t_1 - t_2| \leq 1,
\]
then there exists a unique (up to multiplying by a constant) non-trivial solution \( u_s(t) : [0, \infty) \rightarrow H_0^1(\Omega) \) of (3.6) satisfying
\[
\sup_{t \geq 0} e^{-\epsilon(t)} \|u_s(t)\|_{H_0^1} < \infty.
\]
In addition, this solution satisfies
\[
\sup_{t \geq 0} e^{\epsilon(t)} \|u_s(t)\|_{H_0^1} < \infty.
\]

**Remark 3.5.** Condition (3.7) means that on any time interval of unit length the potential, though potentially highly oscillatory, is close in \( L^p \) to a fixed function. Note however that on large time intervals the potential can change considerably.
Before giving a proof, recall a few elementary facts from Spectral Theory. For a given potential $V \in L^p$, we denote $\phi_1(V)$ the positive eigenfunction corresponding to the smallest eigenvalue $\lambda_1(V)$, normalised so that $\|\phi_1(V)\|_{L^2} = 1$.

**Lemma 3.6.** For any $M \geq 0$ there exists $C = C(\Omega, p, M, \mu) \geq 0$ such that for all $V, W$ with $\|V\|_{L^p}, \|W\|_{L^p} \leq M$, $\lambda_1(V) \leq -\mu$, $\lambda_2(V) \geq \mu$ the following bounds hold:

\[
\begin{align*}
\lambda_1(V) & \leq C, \quad \|\phi_1(V)\|_{H^1_0} \leq C, \\
\lambda_1(V) - \lambda_1(W) & \leq C\|V - W\|_{L^p}, \\
\|\phi_1(V) - \phi_1(W)\|_{H^1_0} & \leq C\sqrt{\|V - W\|_{L^p}}, \\
\langle u, (-\Delta + V)u \rangle & \geq \frac{1}{C}\|u\|_{H^1_0}^2 - C\langle \phi_1(V), u \rangle^2, \quad \forall u \in H^1_0(\Omega), \\
\|(-\Delta + V)u\|_{L^2}^2 & \geq \frac{1}{C}\|\Delta u\|_{L^2}^2 - C\langle \phi_1(V), u \rangle^2, \quad \forall u \in H^1_0(\Omega) \cap H^2(\Omega), \\
\|(-\Delta + V)u\|_{L^2}^2 & \geq \mu\langle u, (-\Delta + V)u \rangle - C\langle \phi_1(V), u \rangle^2, \quad \forall u \in H^1_0(\Omega) \cap H^2(\Omega).
\end{align*}
\]

*Proof.* By Hölder and Sobolev, we have

\[
\int_{\Omega} V\phi_1(V)^2 \, dx \leq \|V\|_{L^p} \|\phi_1(V)\|^{2 - \frac{2}{p}}_{L^2} \|\phi_1(V)\|^{\frac{2}{p}}_{L^{\frac{2}{p - 2}}} \leq C(\Omega)M\|\phi_1(V)\|_{H^1_0}^\frac{2}{p}.
\]

Thus

\[
0 \leq -\lambda_1(V) = \int_{\Omega} (V\phi_1(V)^2 - |\nabla \phi_1(V)|^2) \, dx \leq C(\Omega)M\|\phi_1(V)\|_{H^1_0}^\frac{2}{p} - \|\phi_1(V)\|_{H^1_0}^2,
\]

which implies

\[
\|\phi_1(V)\|_{H^1_0} \leq (C(\Omega)M)^{\frac{p}{2p - 2}}, \quad |\lambda_1(V)| \leq (C(\Omega)M)^{\frac{2p}{2p - 2}}.
\]

In order to prove (3.9), we observe that

\[
\lambda_1(W) \leq \int_{\Omega} (|\nabla \phi_1(V)|^2 + W|\phi_1(V)|^2) \, dx \\
\leq \lambda_1(V) + \int_{\Omega} |V - W||\phi_1(V)|^2 \, dx \leq \lambda_1(V) + C\|V - W\|_{L^p},
\]

where the last step follows from Hölder, Sobolev and (3.8). Analogously, $\lambda_1(V) \leq \lambda_1(W) + C\|V - W\|_{L^p}$.

Next, we prove (3.11). By the Spectral Theorem we have

\[
\int_{\Omega} (|\nabla u|^2 + Vu^2) \, dx + (\mu - \lambda_1(V))\langle \phi_1(V), u \rangle^2 \geq \mu\|u\|_{L^2}^2.
\]

For any $\eta > 0$ we thus have

\[
\int_{\Omega} (|\nabla u|^2 + Vu^2) \, dx + (1 - \eta)(\mu - \lambda_1(V))\langle \phi_1(V), u \rangle^2 \geq (1 - \eta)\mu\|u\|_{L^2}^2 + \eta \int_{\Omega} (|\nabla u|^2 + Vu^2) \, dx \\
\geq \eta\|u\|_{H^1_0}^2 - \eta C\|u\|_{L^2}^{2 - \frac{2}{p}}\|\phi_1(V)\|_{H^1_0}^\frac{2}{p} + (1 - \eta)\mu\|u\|_{L^2}^2,
\]

so if we take $\eta$ small enough, then the Young’s inequality for products yields (3.11).

We prove (3.10). Using the bounds already proved, we have

\[
\int_{\Omega} (|\nabla \phi_1(W)|^2 + V\phi_1(W)^2) \, dx \leq \lambda_1(W) + C\|V - W\|_{L^p} \leq \lambda_1(V) + C\|V - W\|_{L^p}.
\]
Let \( \phi_1(W) = a\phi_1(V) + bu \), with \( a^2 + b^2 = 1 \), \( \|u\|_{L^2} = 1 \) and \( \langle \phi_1(V), u \rangle = 0 \). We then have
\[
\lambda_1(V)a^2 + \mu b^2 \leq \int_\Omega (|\nabla \phi_1(V)|^2 + V \phi_1(W)^2) \, dx \leq \lambda_1(V) + C\|V - W\|_{L^p},
\]
thus \(|b| \leq C\sqrt{\|V - W\|_{L^p}}\), which implies
\[
\|\phi_1(W)\|_{L^2} \leq C\sqrt{\|V - W\|_{L^p}} \Rightarrow \|\phi_1(W) - \phi_1(W)\|_{L^2} \leq C\sqrt{\|V - W\|_{L^p}},
\]
where the last implication follows because both functions are positive. Now (3.10) easily follows from (3.11) for \( u := \phi_1(V) - \phi_1(W) \). Indeed, we have
\[
\langle \phi_1(V) - \phi_1(W), (-\Delta + V)(\phi_1(V) - \phi_1(W)) \rangle \\
\leq \langle \phi_1(V) - \phi_1(W), \lambda_1(V)\phi_1(V) - \lambda_1(W)\phi_1(W) \rangle + C\|V - W\|_{L^p} \leq C\|V - W\|_{L^p}.
\]
Inequality (3.13) follows from (3.14) applied to \( \sqrt{L}u \) instead of \( u \), where \( Lu := (-\Delta + V)u + (\mu - \lambda_1(V))(\phi_1(V), u)\phi_1(V) \).

Finally, in order to prove (3.12), we write
\[
\|(-\Delta + V)u\|_{L^2}^2 = \eta\|(-\Delta + V)u\|_{L^2}^2 + (1 - \eta)\|(-\Delta + V)u\|_{L^2}^2 \\
\geq \frac{\eta}{2}\|\Delta u\|_{L^2}^2 - 2\eta\|Vu\|_{L^2}^2 + (1 - \eta)\|(-\Delta + V)u\|_{L^2}^2.
\]
By the Sobolev inequality, we have \( \|Vu\|_{L^2} \leq C\|\Delta u\|_{L^2}^{\alpha} \|u\|_{L^2}^{2 - \alpha} \) for some \( \alpha > 0 \). Thus, if we take \( \eta \) small enough, (3.12) follows from (3.13) and (3.11). \( \square \)

**Proposition 3.7.** If \( r \) is large enough, then for any \( V \in L_{\text{loc}}^r([0, \infty), L^p) \) equation (3.6) defines a strongly continuous backward evolution operator \( S(\tau, t) \) in \( H^1_0(\Omega) \). Moreover, for any \( [\tau, t] \subset [0, \infty) \) the mapping
\[
L^r([\tau, t], L^p) \ni V \mapsto S(\tau, t) \in \mathcal{L}(H^1_0(\Omega))
\]
is continuous.

**Proof.** Since \( \Omega \) is bounded, without loss of generality we can assume \( p < d \). Set \( q := \left( \frac{1}{p} + \frac{d - 2}{2d} \right)^{-1} \in \left( \frac{2d}{d + 2}, 2 \right) \), so that \( \|V\|_{L^q} \lesssim \|V\|_{L^p} \|u\|_{H^1_0} \). Using the regularising effect of the heat-flow and the \( L^p - L^q \) estimates, see [1] pages 42–44), we get for all \( t > 0 \)
\[
\|e^{t\Delta}u\|_{H^1_0} = \|e^{t\Delta}e^{\frac{t}{2}\Delta}u\|_{H^1_0} \lesssim t^{-\frac{d}{2}}\|e^{\frac{t}{2}\Delta}u\|_{L^2} \lesssim t^{-\frac{d}{2} - \frac{d}{q} \left( \frac{1}{q} - \frac{1}{2} \right)} \|u\|_{L^q} \lesssim t^{-\beta} \|u\|_{L^q},
\]
where \( \beta := \frac{1}{2} + \frac{d}{2} \left( \frac{1}{q} - \frac{1}{2} \right) \in \left( \frac{1}{2}, 1 \right) \) and the constant depends only on \( \Omega \) and \( p \).

Fix \( \tau \leq t \), denote \( I := [\tau, t] \) and consider the bilinear operator
\[
\Phi : L^r(I, L^p) \times C(I, H^1_0) \to C(I, H^1_0), \quad \Phi(V, w)(s) := \int_s^t e^{s'(t-s)}(V(s')w(s')) \, ds'.
\]
Take \( r := 2(1 - \beta)^{-1} \) (in fact any \( r \in ((1 - \beta)^{-1}, \infty) \) would work). From (3.15) we obtain
\[
\|\Phi(V, w)\|_{L^\infty H^1_0} \lesssim (t - \tau)^{-\frac{1}{2} - \beta} \|V\|_{L^r L^p} \|w\|_{L^\infty H^1_0},
\]
thus there exists \( c_0 > 0 \) such that if \( t - \tau \leq c_0\|V\|_{L^r L^p} \), then \( \|\Phi(V, \cdot)\|_{\mathcal{L}(C(I, H^1_0))} \leq \frac{1}{2} \). One can check that if we define
\[
S(s, t) := ((Id + \Phi(V, \cdot))^{-1}(e^{(t-s)\Delta})) (s), \quad \forall s \in [\tau, t],
\]
then
\[
S(s, t) = e^{(t-s)\Delta}u - \int_s^t e^{(s'-s)\Delta}(V(s')S(s', t)u) \, ds',
\]
where \( S(s, t) = ((Id + \Phi(V, \cdot))^{-1}(e^{(t-s)\Delta})) (s), \quad \forall s \in [\tau, t], \]
and
\[
\|\Phi(V, w)\|_{L^\infty H^1_0} \lesssim (t - \tau)^{-\frac{1}{2} - \beta} \|V\|_{L^r L^p} \|w\|_{L^\infty H^1_0},
\]
thus there exists \( c_0 > 0 \) such that if \( t - \tau \leq c_0\|V\|_{L^r L^p} \), then \( \|\Phi(V, \cdot)\|_{\mathcal{L}(C(I, H^1_0))} \leq \frac{1}{2} \). One can check that if we define
\[
S(s, t) := ((Id + \Phi(V, \cdot))^{-1}(e^{(t-s)\Delta})) (s), \quad \forall s \in [\tau, t],
\]
then
\[
S(s, t) = e^{(t-s)\Delta}u - \int_s^t e^{(s'-s)\Delta}(V(s')S(s', t)u) \, ds',
\]
where
which means that $S(s, t)u$ satisfies the integral form of $(3.6)$. We see that $S(s, t)$ depends continuously on $V$.

This finishes the proof for sufficiently short time intervals. In general, we divide any given time interval into a finite number of sufficiently short subintervals.

\[\square\]

**Proof of Proposition 3.4.** We will obtain the result as a corollary of Theorem 3. We assume $V \in L^\infty([0, \infty), L^p)$, in particular $V \in L^r_{loc}([0, \infty), L^p)$ for any $r$, thus Proposition 3.7 implies that $(3.6)$ defines a strongly continuous backward evolution operator.

Let $\chi$ be a $C^\infty$ positive function supported in $(-\frac{1}{2}, \frac{1}{2})$ such that $\int_\mathbb{R} \chi(x) \, dx = 1$. We set

$$W(t) := \int_\mathbb{R} \chi(t - \tau)V(\tau) \, d\tau, \quad W \in C^\infty((1/2, \infty), L^p).$$

Observe that $\|W'(t)\|_{L^p} \lesssim \delta$ and

$$\|V(t) - W(t)\|_{L^p} \leq \int_\mathbb{R} \chi(t - \tau)\|V(t) - V(\tau)\|_{L^p} \, d\tau \leq \delta.$$  

By Lemma 3.6, we have $\lambda_1(W(t)) \leq -\mu + \frac{1}{10} \epsilon$ and $\lambda_2(W(t)) \geq \mu - \frac{1}{10} \epsilon$ for all $t$ if $\delta$ is small enough. We set $\phi(t) := \phi_1(W(t))$ and

$$I^-_t(v) := C_0|\phi_1(W(t)), v|,$$

$$I^+_t(v) := \sqrt{\max\left(0, \int_\Omega (|\nabla v|^2 + W(t)v^2) \, dx\right)},$$

where $C = C(\Omega, p, M)$ is a large constant. Clearly, $I^-_t(v)$ and $I^+_t(v)$ are continuous with respect to $(t, v)$ (for $I^-_t$ we use Lemma 3.6).

Assumption $(2.15)$ follows from $(3.11)$, if $C_0$ is large enough. Assumption $(2.16)$ obviously holds. It is also clear that $I^+_t(\phi_1(W(t))) = 0$, which implies $(2.17)$ for any choice of $c_4$. We now prove that assumption $(2.19)$ holds. We will choose $c_4$ later (we will see that $c_4$ can be chosen as large as we want, in particular we can guarantee that $(2.18)$ holds).

Let $u$ be a solution of $(3.5)$ and let $\tau \leq t$ be such that $I^-_s(u(s)) \geq \frac{1}{c_4} I^+_s(u(s))$ for all $s \in [\tau, t]$, which implies $\|u(s)\|_{H^1_0} \lesssim I^-_s(u(s))$, see (1.6). Suppose that $(2.19)$ fails and set

$$t_0 := \inf\{t' : I^-_s(u(s)) \geq e^{(-\mu + \epsilon)(s-t)}I^-_t(u(t)) \text{ for all } s \in [t', t]\}$$

(we allow the possibility $t_0 = t$). By continuity, $I^-_{t_0}(u(t_0)) \geq e^{(-\mu + \epsilon)(t_0-t)}I^-_t(u(t))$. To reach a contradiction, it suffices to show

$$(3.16) \quad I^-_s(u(s)) \geq (1 + (-\mu + \epsilon/2)(s-t_0))I^-_{t_0}(u(t_0)) \quad \text{for } s \in [t_0 - \eta, t_0] \text{ for some } \eta > 0.$$  

Set $\phi := \phi(t_0)$. If $\eta > 0$ is small, then for $s \in [t_0 - \eta, t_0]$ we have

$$\|u(s) - u(t_0)\|_{H^1_0} \ll \|u(t_0)\|_{H^1_0},$$

$$\|W(t_0) - V(s)\|_{L^p} \lesssim \delta \ll 1,$$

$$\|\phi - \phi(s)\|_{L^2} \lesssim \delta |s - t_0| \ll |s - t_0|.$$ 

Below, we write “≈” when we mean “up to terms ≪ |s - t_0||u(t_0)||".

\[ \langle \phi(s), u(s) \rangle - \langle \phi, u(t_0) \rangle \approx \langle \phi, u(s) \rangle - \langle \phi, u(t_0) \rangle \]

\[ = \int_{s}^{t_0} \langle (\Delta - V(s'))\phi, u(s') \rangle \, ds' \]

\[ = \int_{s}^{t_0} \langle (\Delta - W(t_0))\phi, u(s') \rangle \, ds' + \int_{s}^{t_0} \langle (W(t_0) - V(s'))\phi, u(s') \rangle \, ds' \]

\[ \approx -\lambda_1(W(t_0)) \int_{s}^{t_0} \langle \phi, u(s') \rangle \approx \lambda_1(W(t_0))(s - t_0)\langle \phi, u(t_0) \rangle. \]

Since \( \lambda_1(W(t_0)) \leq -\mu + \frac{1}{40}\epsilon \) and \( s - t_0 \leq 0 \), we obtain (3.16).

We proceed similarly with \( I_t^+ \). Let \( \tau \leq t \) be such that \( I_s^+(u(s)) \geq c_3 I_s^-(u(s)) \) for all \( s \in [\tau, t] \), which implies \( \|u(s)\|\_{H_0^1} \approx I_s^+(u(s)) \), see (3.17). If \( I_s^+(u(s)) = 0 \) for some \( s \in [\tau, t] \), then the solution is identically 0, so assume \( I_s^+(u(s)) > 0 \) for all \( s \in [\tau, t] \). Suppose that (2.20) fails and set

\[ t_0 := \inf\{t' : I_s^+(u(s)) \leq e^{(\mu - \epsilon/2)(s-t)}I_t^+(u(t)) \} \text{ for all } s \in [t', t] \]

(we allow the possibility \( t_0 = t \)). By continuity, \( I_{t_0}^+(u(t_0)) \geq e^{(\mu - \epsilon/2)(t_0-t)}I_t^+(u(t)) \). To reach a contradiction, it suffices to show

\[ I_s^+(u(s)) \leq (1 + (\mu - \epsilon/2)(s-t_0))I_{t_0}^+(u(t_0)) \quad \text{for } s \in [t_0 - \eta, t_0] \text{ for some } \eta > 0. \]

If \( \eta > 0 \) is small, then for all \( s \in [t_0 - \eta, t_0] \) we have

\[ \|u(s) - u(t_0)\|\_{H_0^1} \ll \|u(t_0)\|\_{H_0^1}, \quad \|\partial_s W(s)\|L^p \ll \delta \ll 1. \]

Approximating \( V \) by a smooth potential in the norm \( L^qL^p \) and using Proposition 3.7 we can assume that \( u \) and \( V \) are smooth in both space and time. In the computation below, “≈” means “up to terms \( \approx \|u(t_0)\|\_{H_0^1} \).”

\[ \frac{1}{2} \frac{d}{ds} I_s^+(u(s))^2 = \langle \partial_s W(s)u(s), u(s) \rangle + \langle (-\Delta + W(s))u(s), (-\Delta + V(s))u(s) \rangle \]

\[ \approx \|(-\Delta + W(s))u(s)\|_{L^2}^2 + \langle (-\Delta + W(s))u(s), (V(s) - W(s))u(s) \rangle \]

\[ \geq (\mu - \epsilon/8)I_s^+(u(s))^2 + \frac{\epsilon}{C} \|\Delta u(s)\|_{L^2}^2 - \|V - W\|_{L^p} \|\Delta u(s)\|_{L^2}^2 - CI_s^-(u(s))^2. \]

Thus, if \( c_3 = c_3(\epsilon) \) is large enough and \( I_s^+(u(s)) \geq c_3 I_s^-(u(s)) \), then

\[ \frac{1}{2} \frac{d}{ds} I_s^+(u(s))^2 \geq (\nu - \epsilon/4)I_s^+(u(s))^2, \]

which implies (3.17). \( \square \)

**Remark 3.8.** For ordinary differential equations, a more general result (dealing with the non self-adjoint case) is proved in [7, Chapter 6].

**Remark 3.9.** We expect that a similar result could be obtained in \( H^k \cap H_0^1 \) for any \( k \in \{0, 1, 2, \ldots\} \).

We would use the functional \( I_t^+(v) := \sum_{j=0}^{k} a_j \langle v, (-\Delta + W(t))^j v \rangle \), for appropriate strictly positive numbers \( a_0, \ldots, a_k \).

### 3.3. Heat equation with slowly moving potentials

Let \( d \geq 3 \) and let \( V_1, \ldots, V_J \in L^\infty(\mathbb{R}^d) \cap L^{\frac{d}{d-j}}(\mathbb{R}^d) \) be potentials. Assume \(-\Delta + V_j \) has \( K_j \) strictly negative eigenvalues \(-\lambda_{j,1}, \ldots, -\lambda_{j,k_j}\), with corresponding eigenfunctions \( \mathcal{Y}_{j,1}, \ldots, \mathcal{Y}_{j,k_j} \). Note that \( K_j \) is finite by the Cwiok-Lieb-Rozenblum theorem. Let \( K := \sum_{j=1}^{J} K_j \) and \( \lambda := \min\{\lambda_{j,k} : 1 \leq j \leq J, 1 \leq k \leq k_j\} \). For \( j \in \{1, \ldots, J\} \), let \( x_j : [0, \infty) \rightarrow \mathbb{R}^d \) be a \( C^1 \) trajectory.
We consider the heat equation with moving potentials:

$$\partial_t u(t, x) = \Delta u(t, x) - \sum_{j=1}^J V_j(x - x_j(t))u(t, x).$$

By standard arguments, similar to the one given in Section 3.2 this defines a strongly continuous dynamical system $T(\tau, t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ for $\tau \geq t$.

**Proposition 3.10.** For any $\epsilon > 0$ there exists $\eta > 0$ such that if for $t$ large enough $|x_j'(t)| \leq \eta$ and $|x_j(t) - x_l(t)| \geq \frac{1}{\eta}$ for all $j \neq l$, then $T(\tau, t)$ has an exponential dichotomy with exponents $\epsilon$ and $\lambda - \epsilon$. Moreover, codim $X_\alpha = K$.

We need the following fact.

**Lemma 3.11.** Let $V \in L^\infty(\mathbb{R}^d)$ and let $-\lambda_1, \ldots, -\lambda_{K_0}$ be the strictly negative eigenvalues of $-\Delta + V$, with corresponding eigenfunctions $Y_1, \ldots, Y_{K_0}$. Then there exists $C_0 > 0$ such that for all $u \in L^2(\mathbb{R}^d)$ the following inequality is true:

$$\int_{\mathbb{R}^d} (|\nabla u|^2 + Vu^2) \, dx + C_0 \sum_{k=1}^{K} \langle Y_k, u \rangle^2 \geq 0.$$  (3.18)

**Proof.** The self-adjoint operator corresponding to the quadratic form in (3.18) is

$$L := -\Delta + V + C_0 \sum_{k=1}^{K} \langle Y_k, \cdot \rangle Y_k.$$  

The spaces $Y := \text{span}(Y_1, \ldots, Y_K)$ and its orthogonal complement (in $L^2$) $Y \perp$ are invariant subspaces of $L$. On $Y \perp$, the quadratic form is positive by the Spectral Theorem. On $Y$, it is positive if we take $C_0$ large enough. \qed

This easily implies coercivity for multiple potentials.

**Lemma 3.12.** Let $V_1, \ldots, V_J \in L^\infty(\mathbb{R}^d)$ be potentials. Assume $-\Delta + V_j$ has $K_{j}$ strictly negative eigenvalues $-\lambda_{j,1}, \ldots, -\lambda_{j,K_j}$, with corresponding eigenfunctions $Y_{j,1}, \ldots, Y_{j,K_j}$. There exists $C_0 > 0$ with the following property. For any $\epsilon > 0$ there exists $\eta > 0$ such that if $|x_j - x_l| \geq \frac{1}{\eta}$ for all $j \neq l$, then for all $u \in L^2(\mathbb{R}^d)$ the following bound holds:

$$\int_{\mathbb{R}^d} (|\nabla u|^2 + \sum_{j=1}^{J} V_j(\cdot - x_j)u^2) \, dx + C_0 \sum_{j=1}^{J} \sum_{k=1}^{K_j} \langle Y_{j,k}(\cdot - x_j), u \rangle^2 \geq -\epsilon \|u\|_{L^2}^2.$$  

**Proof.** Let $\chi$ be a smooth cut-off function, with the support contained in $B(0, \frac{1}{2})$ and equal 1 on $B(0, \frac{1}{4})$. For $j \in \{1, \ldots, J\}$, let $u_j(x) := \chi(\eta(x - x_j))u(x)$. We obtain the result by summing (3.18), applied for $u_j$ instead of $u$ for $j \in \{1, \ldots, J\}$. \qed

**Proof of Proposition 3.10.** For $t \geq t_0$ and $v \in L^2(\mathbb{R}^d)$, we define

$$I^+_t(u) := \sum_{j=1}^{J} \sum_{k=1}^{K_j} \langle Y_{j,k}(\cdot - x_j(t)), u \rangle$$

and

$$I^-_t(u) := \|u\|_{L^2}.$$
We have
\[
\frac{d}{dt} \|u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^d} \left( \nabla u(t)^2 + \sum_{j=1}^{J} V_j(\cdot - x_j)u(t)^2 \right) dx,
\]
so the required sub-exponential growth of \( I^- \) follows from Lemma 3.12 provided that \( c_3 \) is taken small enough.

In order to check (4.3), we compute
\[
\frac{d}{dt} \langle \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), u(t) \rangle = \langle \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), \Delta u(t) - V_{j_0}(\cdot - x_{j_0}(t))u(t) \rangle
- x'_{j_0}(t) \cdot \langle \nabla \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), u(t) \rangle - \sum_{j \neq j_0} \langle \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), V_j(\cdot - x_j(t))u(t) \rangle.
\]
We see that the second line is negligible when \( \eta \) is small, more precisely for any \( \bar{\varepsilon} > 0 \)
\[
\left| \frac{d}{dt} \langle \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), u(t) \rangle - \lambda_{j_0,k_0} \langle \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), u(t) \rangle \right| \leq \bar{\varepsilon} \|u(t)\|_{L^2}.
\]
if \( \eta \) is small enough. From this we deduce
\[
\left| \frac{d}{dt} \langle \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), u(t) \rangle - \lambda_{j_0,k_0} \langle \mathcal{Y}_{j_0,k_0}(\cdot - x_{j_0}(t)), u(t) \rangle \right| \leq \bar{\varepsilon} \|u(t)\|_{L^2},
\]
which in turn yields
\[
\frac{d}{dt} I^\pm_-(u(t)) \geq \lambda I^\pm_-(u(t)) - K\bar{\varepsilon} \|u(t)\|_{L^2}.
\]
It suffices to take \( c_4 \) some number satisfying (2.28) and \( \bar{\varepsilon} = \frac{c_4}{K}. \)

4. Klein-Gordon equation with potentials having almost constant velocity

4.1. One potential. The purpose of this section is to relate properties of the flow with one moving potential to the properties of the corresponding flow with a stationary potential. We will use many facts from [9].

We will need the Lorentz boosts. Let \( \beta \in \mathbb{R}^d, |\beta| < 1, \) be a velocity vector. For a function \( \phi: \mathbb{R}^d \to \mathbb{R} \) we define
\[
\phi_\beta(x) := \phi(\Lambda_\beta x), \quad \Lambda_\beta x := x + (\gamma - 1) \left( \frac{\beta \cdot x}{|\beta|^2} \right) \beta, \quad \gamma := \frac{1}{\sqrt{1 - |\beta|^2}}.
\]
With this notation, the Lorentz transformation is given by
\[
(t', x') = \left( \gamma(t - \beta \cdot x), \Lambda_\beta x - \gamma \beta t \right) = \left( \gamma(t - \beta \cdot x), \Lambda_\beta (x - \beta t) \right).
\]
For a pair of functions \( \mathbb{R}^d \to \mathbb{R}, \phi = (\phi, \Phi), \) we will also write
\[
\Phi_\beta := (\phi_\beta, \Phi_\beta).
\]
Let \( V \) be a smooth exponentially decaying potential. Let \( \beta \in \mathbb{R}^d \) with \( |\beta| < 1 \) and \( \xi \in \mathbb{R}^d \). We consider the following linear Klein-Gordon equation:
\[
\partial_t^2 u(t, x) = \Delta u(t, x) - u(t, x) - V_\beta(x - \beta t - \xi)u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.
\]
We see that \( u(t, x) \) is a solution of (4.1) if and only if \( u(t, x) = w(t', x') \), where
\[
(t', x') = \left( \gamma(t - \beta \cdot x), \Lambda_\beta (x - \beta t - \xi) \right)
\]
and \( w(t, x) \) is a solution of
\[
\partial_t^2 w = \Delta w - c_2 w - V w.
\]
The following observation will be useful. It expresses the conservation of charge (for a complex-valued solution \( w = w_1 + iw_2 \)) and energy for equation (4.3), which is a consequence of Noether’s Theorem.

**Lemma 4.1.** Let \( w, w_1, w_2 \) be smooth solutions of (4.3). The following vector fields in \( \mathbb{R}^{1+d} \) are divergence free:

\[
F(w_1, w_2) = (w_1 \partial_t w_2 - w_2 \partial_t w_1, -w_1 \nabla w_2 + w_2 \nabla w_1),
\]

\[
G(w) = \left( \frac{1}{2} ((\partial_t w)^2 + (\nabla w)^2 + (1 + V)w^2), -\partial_t w \nabla w \right).
\]

**Proof.** We have

\[
\partial_t(w_1 \partial_t w_2 - w_2 \partial_t w_1) + \text{div}(-w_1 \nabla w_2 + w_2 \nabla w_1) = w_1 \partial_t^2 w_2 - w_2 \partial_t^2 w_1 - w_1 \Delta w_2 + w_2 \Delta w_1
\]

\[
= w_1(\partial_t^2 w_2 - \Delta w_2) - w_2(\partial_t^2 w_1 - \Delta w_1) = -w_1(w_2 + Vw_2) + w_2(w_1 + Vw_1) = 0
\]

and

\[
\frac{1}{2} \partial_t ((\partial_t w)^2 + (\nabla w)^2 + (1 + V)w^2) + \text{div}(-\partial_t w \nabla w)
\]

\[
= \partial_t^2 w \partial_t w + \partial_t \nabla w \cdot \nabla w + (1 + V)w \partial_t w - \partial_t \nabla w \cdot \nabla w - \partial_t w \Delta w
\]

\[
= \partial_t w(\partial_t^2 w - \Delta w + (1 + V)w) = 0.
\]

\( \square \)

We can write (4.1) as a dynamical system:

\[
(4.4) \quad \partial_t u(t) = JH_{\beta}(\beta t + \xi)u(t),
\]

where

\[
(4.5) \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H_{\beta}(\xi) := \begin{pmatrix} -\Delta + 1 + V_{\beta}(\cdot - \xi) & 0 \\ 0 & 1 \end{pmatrix}.
\]

The Schrödinger operator \( L = -\Delta + 1 + V \) has the essential spectrum \([1, \infty)\), and might have a finite number of eigenvalues in \((-\infty, \frac{1}{2})\). Let \( K \) be the number of strictly negative eigenvalues (counted with multiplicities) and let \( M := \text{dim ker } L \). Let \(-\nu_1^2, \ldots, -\nu_K^2\) (with \(\nu_k > 0\)) be the strictly negative eigenvalues and let \((\phi_k)_{k=1,\ldots,K}\) and \((\phi_m^0)_{j=1,\ldots,M}\) be orthonormal (in \(L^2\)) families such that

\[
L\phi_k = -\nu_k^2 \phi_k,
\]

\[
L\phi_m^0 = 0.
\]

**Lemma 4.2.** For any \( \varepsilon > 0 \) and \( n \in \mathbb{N}^d \) there exists \( C > 0 \) such that

\[
|\partial^k \phi_k(x)| \leq Ce^{-(1-\varepsilon)\sqrt{1+\nu_k^2}|x|}, \quad \text{for all } k \in \{1, \ldots, K\}, \ x \in \mathbb{R}^d;
\]

\[
|\partial^n \phi_m^0(x)| \leq Ce^{-(1-\varepsilon)|x|}, \quad \text{for all } m \in \{1, \ldots, M\}, \ x \in \mathbb{R}^d.
\]

**Proof.** We only prove the second inequality, as the first one can be obtained in the same way. Take \( C \) large and suppose there exists \( x \in \mathbb{R}^d \) such that \( \phi_m^0(x) \geq Ce^{-(1-\varepsilon)|x|} \). By interior regularity we have \( \lim_{|x| \to \infty} |\phi_m^0(x)| = 0 \), so there exists \( x_0 \in \mathbb{R}^d \) such that

\[
\phi_m^0(x_0) - Ce^{-(1-\varepsilon)|x_0|} = \sup_{x \in \mathbb{R}^d} \left( \phi_m^0(x) - Ce^{-(1-\varepsilon)|x|} \right) > 0,
\]

which implies

\[
\Delta(\phi_m^0(x_0)) \leq C\Delta(e^{-(1-\varepsilon)|x_0|}).
\]
If \( C \) is large, then \(|x_0|\) is large as well. It is easy to see from the formula for the Laplacian in radial coordinates that if \(|x_0|\) is large enough, then \( \Delta (e^{-(1-\epsilon)|x_0|}) \leq (1-\epsilon)e^{-(1-\epsilon)|x_0|} \). We obtain
\[
\Delta (\phi_m^0(x_0)) \leq C(1-\epsilon)e^{-(1-\epsilon)|x_0|} < (1-\epsilon)\phi_m^0(x_0),
\]
which is impossible for \(|x_0|\) large.

The bound \( \phi_m^0(x) > -C e^{-(1-\epsilon)|x_0|} \) is obtained by considering \(-\phi_m^0\) instead of \(\phi_m^0\).

The bound on derivatives follows from interior regularity. 

By the Spectral Theorem,
\[
\langle \phi_k, \psi \rangle = \langle \phi_m^0, \psi \rangle = 0 \quad \Rightarrow \quad \langle \psi, L\psi \rangle \geq c\|\psi\|_{L^2}^2.
\]
Note that we also have
\[
\langle \phi_k, \psi \rangle = \langle \phi_m^0, \psi \rangle = 0 \quad \Rightarrow \quad \langle \psi, L\psi \rangle \geq c\|\psi\|_{H^1}^2.
\]
Indeed, (4.6) implies that \(\langle \psi, (a(-\Delta + 1) + V)\psi \rangle \geq 0\) for some \(a < 1\), which yields (4.7) with \(c = 1 - a\).

Following [10] and [9] Lemma 1, we now give explicit formulas for the stable, unstable and null components of the flow (4.6). We define
\[
\begin{align*}
\mathcal{Y}_{k,\beta}^-(x) &:= e^{\gamma v_k \beta \cdot x} (\phi_k, \gamma \beta \cdot \nabla \phi_k + \gamma v_k \phi_k)\beta(x), \\
\mathcal{Y}_{k,\beta}^+(x) &:= e^{-\gamma v_k \beta \cdot x} (\phi_k, -\gamma \beta \cdot \nabla \phi_k + \gamma v_k \phi_k)\beta(x), \\
\mathcal{Y}_{m,\beta}^0(x) &:= (\phi_m^0, -\gamma \beta \cdot \nabla \phi_m^0)\beta(x), \\
\alpha_{k,\beta}(x) &:= J\mathcal{Y}_{k,\beta}^+(x) = e^{-\gamma v_k \beta \cdot x} (\gamma \beta \cdot \nabla \phi_k - \gamma v_k \phi_k, \phi_k)\beta(x), \\
\alpha_{k,\beta}^+(x) &:= J\mathcal{Y}_{k,\beta}^-(x) = e^{\gamma v_k \beta \cdot x} (-\gamma \beta \cdot \nabla \phi_k - \gamma v_k \phi_k, \phi_k)\beta(x), \\
\alpha_{m,\beta}^0(x) &:= J\mathcal{Y}_{m,\beta}^0(x) = (\gamma \beta \cdot \nabla \phi_m^0, \phi_m^0)\beta(x).
\end{align*}
\]
Since \(|\Lambda \beta x| \geq \gamma|x|\), Lemma 4.2 implies that all these functions are smooth and exponentially decaying, uniformly in \(\beta\) of \(|\beta| \leq v < 1\). Observe also that
\[
\|\partial_\beta \alpha_{k,\beta}^-\|_{L^2} + \|\partial_\beta \alpha_{k,\beta}^+\|_{L^2} + \|\partial_\beta \alpha_{m,\beta}^0\|_{L^2} + \|\partial_\beta \mathcal{Y}_{k,\beta}^-\|_{L^2} + \|\partial_\beta \mathcal{Y}_{k,\beta}^+\|_{L^2} + \|\partial_\beta \mathcal{Y}_{m,\beta}^0\|_{L^2} \lesssim 1.
\]

**Lemma 4.3.** The following functions are solutions of (4.4):
\[
\begin{align*}
\mathbf{u}(t) &= e^{\frac{\nu_k}{\gamma} t} \mathcal{Y}_{k,\beta}^- (\cdot - \beta t - \xi), \\
\mathbf{u}(t) &= e^{\frac{\nu_k}{\gamma} t} \mathcal{Y}_{k,\beta}^+ (\cdot - \beta t - \xi), \\
\mathbf{u}(t) &= \mathcal{Y}_{m,\beta}^0 (\cdot - \beta t - \xi).
\end{align*}
\]

If \(\mathbf{u}(t)\) is any solution of (1.4), then
\[
\begin{align*}
\frac{d}{dt} (\alpha_{k,\beta}^- (\cdot - \beta t - \xi), \mathbf{u}(t)) &= -\frac{\nu_k}{\gamma} (\alpha_{k,\beta}^- (\cdot - \beta t - \xi), \mathbf{u}(t)), \\
\frac{d}{dt} (\alpha_{k,\beta}^+ (\cdot - \beta t - \xi), \mathbf{u}(t)) &= \frac{\nu_k}{\gamma} (\alpha_{k,\beta}^+ (\cdot - \beta t - \xi), \mathbf{u}(t)), \\
\frac{d}{dt} (\alpha_{m,\beta}^0 (\cdot - \beta t - \xi), \mathbf{u}(t)) &= 0.
\end{align*}
\]

**Proof.** It is easy to see that \(w(t, x) = e^{\nu_k t} \phi_k(x)\) is a solution of (4.3), which implies that
\[
\mathbf{u}(t, x) = w(t', x') = e^{\nu_k (t' - \beta x')} \phi_k(\Lambda \beta (x - \beta t - \xi))
\]
We will construct a continuous one-to-one linear map $T_{\norm}$, a standard weak convergence argument shows that it suffices to prove that
\begin{equation}
\text{(4.7)}
\end{equation}
is a solution of (4.1). Now we observe that
\begin{equation}
\text{(4.22)}
e^{-\nu_k(t-x)} = e^{\gamma \nu_k \beta \xi} e^{-\gamma \beta^2 (x-\beta t - \xi)} = e^{\gamma \nu_k \beta \xi} e^{-\gamma \nu_k \beta \phi_k (\Lambda \beta (x - \beta t - \xi))}.
\end{equation}
The first factor is constant and can be discarded. The second factor is the exponential growth factor in (4.16). Finally, $e^{-\gamma \nu_k \beta \phi_k (\Lambda \beta (x - \beta t - \xi))}$ is precisely the first component of $Y^{+}_{k, \beta}$. The second component of $Y^{+}_{k, \beta}$ is found by computing the time derivative of (4.21):
\begin{equation}
\frac{d}{dt} u(t, x) = e^{\gamma \nu_k(t-x)} \left( \gamma \nu_k \phi_k (\Lambda \beta (x - \beta t - \xi)) \right).
\end{equation}
Using again (4.22), we see that the second component of $e^{\nu_k} Y^{+}_{k, \beta}(x - \beta t - \xi)$ is indeed the time derivative of the first component. One can treat (4.15) and (4.17) similarly.

If $v(t)$ and $u(t)$ are solutions of (4.1), then, using the fact that $H(t)$ is self-adjoint, $J$ is skew-adjoint and $J^2 = -\text{Id}$, we get
\begin{equation}
\frac{d}{dt} \langle Jv(t), u(t) \rangle = \langle J^2 H(t)v(t), u(t) \rangle + \langle Jv(t), JH(t)u(t) \rangle = 0.
\end{equation}
Taking $v(t) = e^{\nu_k} Y^{+}_{k, \beta}(x - \beta t - \xi)$ we obtain (4.18). Similarly, (4.19) follows by considering $v(t) = e^{-\nu_k} Y^{-}_{k, \beta}(x - \beta t - \xi)$, whereas for (4.20) we take $v(t) = Y^{0}_{m, \beta}(x - \beta t - \xi)$.

4.2. Energy estimates. Consider the following quadratic form, also appearing in [10, 9]:
\begin{equation}
Q_{\beta}(\xi; u_0, u_0) := \frac{1}{2} \int_{\mathbb{R}^d} \left( (\dot{u}_0)^2 + 2u_0 (\beta \cdot \nabla u_0) + |\nabla u_0|^2 + (1 + V_{\beta}(\cdot - \xi)) u_0^2 \right) dx.
\end{equation}
We have the following coercivity property, proved in [9].

Proposition 4.4. [9, Proposition 3] For any $\beta \in (-1, 1)$ there exists $c > 0$ such that for all $\xi \in \mathbb{R}^d$ and $u_0 \in H^1 \times L^2$ the following bound is true:
\begin{equation}
Q_{\beta}(\xi; u_0, u_0) \geq c \|u_0\|^2_{H^1 \times L^2}
\end{equation}
(4.23)
\begin{equation}
- \frac{1}{c} \left( \sum_{k=1}^{K} \langle \alpha_{k, \beta}^-(\cdot - \xi), u_0 \rangle^2 + \sum_{k=1}^{K} \langle \alpha_{k, \beta}^+(\cdot - \xi), u_0 \rangle^2 + \sum_{m=1}^{M} \langle \alpha_{m, \beta}^0(\cdot - \xi), u_0 \rangle^2 \right).
\end{equation}

Proof. For the convenience of the reader, we will provide a proof, different from the one given in [9].

Let $Y_{\beta, \xi} \subset H^1 \times L^2$ be defined by
\begin{equation}
Y_{\beta, \xi} := \{ u_0 : (\alpha_{m, \beta}^-(\cdot - \xi), u_0) = (\alpha_{m, \beta}^+(\cdot - \xi), u_0) = (\alpha_{m, \beta}^0(\cdot - \xi), u_0) = 0 \text{ for all } m, k \}.
\end{equation}
Since $u_0 \mapsto \int_{\mathbb{R}^d} \left( (\dot{u}_0)^2 + 2u_0 (\beta \cdot \nabla u_0) + |\nabla u_0|^2 + u_0^2 \right) dx$ defines a norm equivalent to the $H^1 \times L^2$ norm, a standard weak convergence argument shows that it suffices to prove
\begin{equation}
u_0 \in Y_{\beta, \xi} \Rightarrow Q_{\beta}(\xi; u_0, u_0) > 0.
\end{equation}
We will construct a continuous one-to-one linear map $T_{\beta, \xi} : Y_{\beta, \xi} \to Y_{0, 0}$ such that
\begin{equation}
Q_{\beta}(\xi; u_0, u_0) = \gamma^{-1} Q_0(0; T_{\beta, \xi} u_0, T_{\beta, \xi} u_0), \quad \text{for all } u_0 \in Y_{\beta, \xi}.
\end{equation}
(4.24)
Now it easily follows from (4.7) that $w_0 \in Y_{0, 0} \Rightarrow Q_0(0; w_0, w_0) > 0$, so this will finish the proof.

Let $u_0 \in Y_{\beta, \xi} \cap (C_0^\infty \times C_0^\infty)$ and let $u(t, x)$ be the solution of (4.1) with the initial conditions $(u(0, \cdot), \partial_t u(0, \cdot)) = u_0$. Let $w(t, x)$ be defined by $w(t', x') = u(t, x)$, where $t', x'$ are given by (4.2). We set $T_{\beta, \xi} u_0 := (w(0, \cdot), \partial_t w(0, \cdot))$.

By the Chain Rule, we have
\begin{equation}
\partial_t u = \gamma (\partial_t u - \beta \cdot \nabla x' u), \quad \nabla x u = -\gamma \beta \partial_t u + \Lambda \beta \nabla x' u,
\end{equation}
(4.25)
and after a somewhat tedious computation we arrive at

\[(\partial_t u)^2 + 2\partial_t u (\beta \cdot \nabla_x u) + |\nabla_x u|^2 + (1 + V_\beta (\cdot - \xi)) u^2\]

(4.25)

\[(\partial_t w)^2 - 2\partial_t w (\beta \cdot \nabla_x w) + |\nabla_x w|^2 + (1 + V) w^2.\]

Let \(P\) be the hyperplane of the \((t', x')\) spacetime delimited by \(t' + \gamma \beta \cdot \xi = -\beta \cdot x'\) and let \(d\sigma\) be the measure inherited from the Lebesgue measure. In \((t, x)\) coordinates, \(P\) is the hyperplane \(t = 0\), so taking into account the change of measure and (4.25) we obtain

\[Q_\beta (\xi; u_0, u_0) = \sqrt{1 + \beta^2} \frac{1}{1 - \beta^2} \int_P ((\partial_t w)^2 - 2\partial_t w (\beta \cdot \nabla_x w) + |\nabla_x w|^2 + (1 + V) w^2) \, d\sigma.\]

We can now use the Divergence Theorem for the vector field \(G(w)\) in the region of the \((t', x')\) spacetime delimited by \(t' = 0\) and \(P\). This leads to (4.24).

Next, we need to prove that \(T_{\beta, \xi} u_0 \in Y_{0, 0}\). For this purpose, we integrate the vector field \(F(w_1, w_2)\) from Lemma 4.9 with \(w_1(t') := w(t')\) and \(w_2(t') := e^{\nu_k t'} (\phi_k, \nu_k \phi_k)\), in the region between \(\{t' = 0\}\) and \(P\). We have \(\partial_t w_2(0) = \nu_k \phi_k\), hence the boundary term corresponding to \(t' = 0\) equals

\[- \int_{\mathbb{R}^d} (w_1(0) \partial_t w_2(0) - w_2(0) \partial_t w_1(0)) \, dx' = - ((-\nu_k \phi_k, \phi_k), (w(0), \partial_t w(0))) = - \langle \alpha_{k, 0}, T_{\beta, \xi} u_0 \rangle.\]

For \((t', x') \in P\) we have

\[w_1 \partial_t w_2 - w_2 \partial_t w_1 - w_1 \beta \cdot \nabla w_2 + w_2 \beta \cdot \nabla w_1\]

\[= w(t', x') \nu_k e^{\nu_k t'} \phi_k (x') - \partial_t w(t', x') e^{\nu_k t'} \phi_k (x')\]

\[= w(t', x') e^{\nu_k t'} \beta \cdot \nabla \phi_k (x') + \beta \cdot \nabla w(t', x') e^{\nu_k t'} \phi_k (x')\]

\[= - e^{\nu_k t'} \phi_k (x') (\partial_t w(t', x') - \beta \cdot \nabla w(t', x')) + (\nu_k e^{\nu_k t'} \phi_k (x') - e^{\nu_k t'} \beta \cdot \nabla \phi_k (x')) w(t', x')\]

\[= - \frac{1}{\gamma} \alpha_{k, \beta} (x - \xi) \cdot u_0 (x).\]

If \(u_0 \in Y_{\beta, \xi}\), we deduce that the boundary term over \(P\) equals 0, thus the boundary term over \(\{t' = 0\}\) equals 0 as well. Orthogonality to \(\alpha_{k, 0}^+\) and \(\alpha_{m, 0}^0\) are checked similarly, and we obtain \(T_{\beta, \xi} u_0 \in Y_{0, 0}\).

From (4.24) and the coercivity of \(Q_0 (0; w_0, u_0)\) for \(w_0 \in Y_{0, 0}\) we deduce that \(T_{\beta, \xi}: Y_{\beta, \xi} \to Y_{0, 0}\) is continuous for the \(H^1 \times L^2\) norm. Thus, we can extend it by continuity from \(C_0^\infty \times C_0^\infty\) to \(T_{\beta, \xi}\). In order to prove that it is one-to-one, we need to check that if \(T_{\beta, \xi} u_0 \to 0\) in \(H^1 \times L^2\), then \(u_0 \to 0\) in \(H^1 \times L^2\). Let \(w_n := T_{\beta, \xi} u_n\) and let \(w_n(t', x')\) be the corresponding solution of (4.3). We apply the Divergence Theorem to the vector field \((\partial_t w_n)^2 + \|\nabla_x w_n\|^2 + w_n^2 - 2\partial_t \phi_n \nabla_x w_n),\) in the region \(\Omega\) contained between \(\{t' = 0\}\) and \(P\). The divergence equals \(-V(x') \partial_t w_n(t', x')\), and we see that the exponential decay of \(V\) implies \(V \in L^1_{L^2} (\Omega)\), thus

\[\int_{\Omega} |V(t', x')| |\partial_t w_n(t', x')| \, dx' \, dt' \leq \|\partial_t w_n\|_{L^\infty L^2} \to 0 \quad \text{as} \quad n \to \infty,\]

so we obtain

\[\int_P ((\partial_t w)^2 - 2\partial_t w (\beta \cdot \nabla_x w) + |\nabla_x w|^2 + w^2) \, d\sigma \to 0 \quad \text{as} \quad n \to \infty.\]

After a change of variables, this yields

\[\int_{\mathbb{R}^d} (\hat{u}_n)^2 + 2\hat{u}_n (\beta \cdot \nabla u_n) + |\nabla u_n|^2 + u_n^2) \, dx \to 0,\]

which finishes the proof. \(\square\)
Remark 4.5. The quantity $Q_0(0; w_0, w_0)$ is the energy of $\|1\|$, and from the above considerations it easily follows that $Q_0(\beta t + \xi; u(t), u(t))$ is constant for any solution $u(t)$ of (1.1). This can also be checked by a direct computation, which is the method we will have to adopt below in the case of multiple potentials.

4.3. Many potentials. We consider the linear Klein-Gordon equation with a finite number of moving potentials.

Let $V_j$ be a smooth exponentially decaying potential for $j \in \{1, 2, \ldots, J\}$, such that $L_j := -\Delta + V_j$ has $K_j$ strictly negative eigenvalues $-\nu^2_{j,k}$ (for $k = 1, \ldots, K_j$) and dim ker $L_j = M_j$.

Let $y_j(t)$ be positions of the potentials. We denote $\beta_j(t) := y_j'(t)$. We write $\beta(t) = (\beta_1(t), \ldots, \beta_J(t))$, $y(t) = (y_1(t), \ldots, y_J(t))$. We consider the equation

$$\partial_t^2 u = \Delta u - u - \sum_{j=1}^J (V_j)_{\beta_j(t)}(\cdot - y_j(t))u.$$ 

(4.26)

Note that the Lorentz transformation is applied to the potentials $V_j$, according to their instantaneous velocity.

Fix $j \in \{1, 2, \ldots, J\}$ and let $Y_{j,k}^-, Y_{j,k}^+, Y_{0,m}^-, \alpha_{j,k}^-, \alpha_{j,k}^+, \alpha_{0,m}^-$ be the functions defined in Paragraph 4.11 for $V_j$ instead of $V$. We denote

$$Y_{j,k}^{-}(t) := Y_{j,k}^{-}(\cdot - y_j(t)),$$

$$Y_{j,k}^{+}(t) := Y_{j,k}^{+}(\cdot - y_j(t)),$$

$$Y_{0,m}^{-}(t) := Y_{0,m}^{-}(\cdot - y_j(t)),$$

$$\alpha_{j,k}^{-}(t) := \alpha_{j,k}^{-}(\cdot - y_j(t)),$$

$$\alpha_{j,k}^{+}(t) := \alpha_{j,k}^{+}(\cdot - y_j(t)),$$

$$\alpha_{0,m}^{+}(t) := \alpha_{0,m}^{+}(\cdot - y_j(t)),$$

$$V(t) := \sum_{j=1}^J (V_j)_{\beta_j(t)}(\cdot - y_j(t)),$$

where $k \in \{1, \ldots, K_j\}$ and $m \in \{1, \ldots, M_j\}$.

If we let

$$H(t) := \begin{pmatrix} -\Delta + 1 + \sum_{j=1}^J (V_j)_{\beta_j(t)}(\cdot - y_j(t)) & 0 \\ 0 & 1 \end{pmatrix}$$

then (4.26) can be written as

$$\partial_t u(t) = JH(t)u(t).$$

(4.27)

By standard arguments based on energy estimates, this equation defines a strongly continuous evolution operator in $H^1 \times L^2$, which we denote $T(\tau, t)$.

In order to define the relevant quadratic form $Q$, we need to use cut-offs, cf. [9, Section 3.5]. We let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a $C^\infty$ function such that

$$\chi(x) = 0 \text{ for } |x| \geq \frac{1}{2}, \quad \chi(x) = 1 \text{ for } |x| \leq \frac{1}{4}, \quad 0 \leq \chi(x) \leq 1 \text{ for } x \in \mathbb{R}^d.$$

Assume $|y_l(t) - y_j(t)| \geq \frac{1}{\eta}$ for $j \neq l$ and $t \geq t_0$ and some (small) $\eta > 0$. We set

$$\chi_j(t, x) := \chi(\eta(x - y_j(t))).$$

Assume $|y_l(t) - y_j(t)| \geq \frac{1}{\eta}$ for $j \neq l$ and $t \geq t_0$ and some (small) $\eta > 0$. We set
and we define

\[ Q(t; u_0, u_0) := \frac{1}{2} \int_{\mathbb{R}^d} \left( (\dot{u}_0)^2 + 2 \sum_{j=1}^{J} \chi_j(t) \dot{u}_0(\beta_j(t) \cdot \nabla u_0) + |\nabla u_0|^2 + (1 + V(t))u_0^2 \right) dx. \]

Note that similar localised functionals were used by Martel, Merle and Tsai in \([16, 17]\).

**Lemma 4.6.** There exists \( c > 0 \) such that for all \( u_0 \in H^1 \times L^2 \) the following bound is true:

\[ Q(t; u_0, u_0) \geq c\|u_0\|_{H^1 \times L^2}^2 - \frac{1}{c} \sum_{j=1}^{J} \left( \sum_{k=1}^{K_j} \langle \alpha_{j,k}^-(t), u_0 \rangle^2 + \sum_{k=1}^{K_j} \langle \alpha_{j,k}^+(t), u_0 \rangle^2 + \sum_{m=1}^{M_j} \langle \alpha_{j,m}^0(t), u_0 \rangle^2 \right). \]

**Proof.** For \( j \in \{1, \ldots, J\} \), let \( u_j := \chi_j u_0 \). We obtain the result by summing (4.23), applied for \( u_j \) instead of \( u_0 \) for \( j \in \{1, \ldots, J\} \).

**Proposition 4.7.** Let \( \nu < 1 \), \( \nu := \min \{\nu_j, k\} \) and \( K := \sum_{j=1}^{J} K_j \). For any \( \epsilon > 0 \) there exists \( \eta > 0 \) such that if for \( t \) large enough

\[ |\beta_j(t)| \leq \nu, \quad |\beta_j'(t)| \leq \eta, \quad |y_j(t) - y_l(t)| \geq \frac{1}{\eta} \quad \text{for all } j \neq l, \]

then the semigroup \( T(\tau, t) \) has an exponential dichotomy with exponents \( \epsilon \) and \( \nu(1 - v^2) - \epsilon \). Moreover, \( \text{codim} X_\nu = K \).

Before giving a proof, we need one more lemma about a dynamical control of stable and unstable directions. Let \( u(t) \) be a solution of (4.27). The stable and unstable components are defined by

\[ a^\pm_{j,k}(t) := \langle \alpha^\pm_{j,k}(t), u(t) \rangle, \quad j \in \{1, \ldots, J\}, \quad k \in \{1, \ldots, K_j\}. \]

**Lemma 4.8.** For any \( c > 0 \) there exists \( \eta > 0 \) such that if (4.28) holds, then for all \( t \)

\[ \left| \frac{d}{dt} a^\pm_{j,k}(t) \right| + \frac{\nu_j}{\tau_j} a^\pm_{j,k}(t) \leq c\|u(t)\|_{H^1 \times L^2}, \quad \text{for all } j \text{ and } 1 \leq k \leq K_j, \]

\[ \left| \frac{d}{dt} a^0_{j,m}(t) \right| \leq c\|u(t)\|_{H^1 \times L^2}, \quad \text{for all } j \text{ and } 1 \leq m \leq M_j. \]

**Proof.** We prove the first bound for the sign “-”, the remaining cases being similar. Fix \( t_0 \) and let \( \beta := \beta_j(t_0) \), \( \xi := y_j(t_0) \). Let \( \alpha_{k,\beta}^- \) be defined by (4.11) and let \( H_{\beta} \) be defined by (4.5) with \( V = V_j \). Then (4.14) yields

\[ \left\langle \frac{d}{dt} \alpha_{j,k}(t_0), u(t_0) \right\rangle = \left\langle \frac{d}{dt} \big|_{t=0} \alpha_{k,\beta}^-(\cdot - \beta t - \xi), u(t_0) \right\rangle + O(\eta\|u(t_0)\|_{L^2}). \]

We also have

\[ \langle \alpha_{j,k}(t_0), \partial_\xi u(t_0) \rangle = \langle \alpha_{k,\beta}^-(\cdot - \xi), JH_{\beta}(\xi)u(t_0) \rangle + \sum_{j' \neq j} \langle \alpha_{k,\beta}^-(\cdot - \xi), (V_{j'})_{\beta_j(t_0)}(\cdot - y_{j'}(t_0))u(t_0) \rangle. \]

If (4.28) holds with \( \eta \ll 1 \), then the second term above is \( \ll \|u(t_0)\|_{L^2} \) when \( t_0 \gg 1 \) (similarly as in the proof of Proposition 3.10). We thus obtain

\[ \frac{d}{dt} a^\pm_{j,k}(t_0) = \left| \frac{d}{dt} \big|_{t=0} \alpha_{k,\beta}^-(\cdot - \beta t - \xi), u(t_0) \right\rangle + \langle \alpha_{k,\beta}^-(\cdot - \xi), JH_{\beta}(\xi)u(t_0) \rangle + o(\|u(t_0)\|_{L^2}), \]

and the conclusion follows from (4.18). \( \square \)
Proof of Proposition 4.7. We set

\[ I_t^+(u(t)) := \left( \sum_{j=1}^{J} \sum_{k=1}^{K_j} |a_{j,k}^+(t)|^2 \right)^{\frac{1}{2}}, \]

\[ I_t^-(u(t)) := \left( \max(0, Q(t; u(t), u(t)) + \sum_{j=1}^{J} \sum_{k=1}^{K_j} |a_{j,k}^-(t)|^2 + \sum_{j=1}^{J} \sum_{m=1}^{M_j} |a_{j,m}^0(t)|^2) \right)^{\frac{1}{2}}. \]

We need to verify the assumptions of Theorem 4.8, with \( \epsilon \) instead of \( \lambda \) and \( \nu - \epsilon \) instead of \( \mu \). As in the case of the heat equation, this boils down to showing that

\[ \frac{d}{dt} I_t^+(u(t)) \geq \nu \sqrt{1 - \nu^2 I_t^+(u(t)) - \varepsilon \|u(t)\|_{H^1 \times L^2}} \quad \text{if } I_t^+(u(t)) \geq c_4 I_t^-(u(t)), \]

\[ \frac{d}{dt} I_t^-(u(t)) \leq \varepsilon \|u(t)\|_{H^1 \times L^2} \quad \text{if } I_t^+(u(t)) \leq c_3 I_t^-(u(t)), \]

where \( \varepsilon \to 0 \) when \( \eta \to 0 \). Inequality (4.29) follows from Lemma 4.8. When proving (4.30), we can assume that \( I_t^-(u(t)) > 0 \), because otherwise \( u(t) = 0 \). From Lemma 4.8 we obtain

\[ \frac{d}{dt} \left( \sum_{j=1}^{J} \sum_{k=1}^{K_j} |a_{j,k}^-(t)|^2 + \sum_{j=1}^{J} \sum_{m=1}^{M_j} |a_{j,m}^0(t)|^2 \right) \leq \varepsilon \|u(t)\|^2, \quad \varepsilon \ll 1 \text{ when } \eta \to 0, \]

so we are left with computing \( \frac{d}{dt} Q(t; u(t), u(t)) \). By density, we can assume the solution is smooth.

Fix \( t_0 \) and let \( \beta_j := \beta_j(t_0), \xi_j := \xi_j(t_0), \ u_j := \chi_j u(t_0) \). We have

\[ \partial_t Q(t_0; u(t_0), u(t_0)) \approx \frac{1}{2} \partial_t V(t_0) u(t_0)^2 \approx \frac{1}{2} \sum_{j=1}^{J} \partial_{t=t_0} ((V_j)_{\beta_j} \xi_j(\cdot - \xi_j(t))) u(t_0)^2 \]

(4.31)

\[ \approx \frac{1}{2} \sum_{j=1}^{J} \beta_j \cdot \nabla ((V_j)_{\beta_j} \xi_j(\cdot - \xi_j)) u(t_0)^2, \]

where the passage from the first to the second line is justified by the rapid decay of the potentials.

Next, we compute

\[ 2Q(t_0; u(t_0), JH(t_0) u(t_0)) = \int_{\mathbb{R}^d} \left( \hat{u}(t_0) \left( \Delta u(t_0) - u(t_0) - \sum_{j=1}^{J} (V_j)_{\beta_j} (\cdot - \xi_j) u(t_0) \right) \right. \]

\[ + \sum_{j=1}^{J} \chi_j(t_0) \left( \Delta u(t_0) - u(t_0) - \sum_{l=1}^{J} (V_l)_{\beta_l} (\cdot - \xi_l) u(t_0) \right) (\beta_j \cdot \nabla u(t_0)) \]

\[ + \sum_{j=1}^{J} \chi_j(t_0) \hat{u}(t_0)(\beta_j \cdot \nabla \hat{u}(t_0)) + \nabla u_0 \cdot \nabla \hat{u}(t_0) \]

\[ + \left( 1 + \sum_{j=1}^{J} (V_j)_{\beta_j} (\cdot - \xi_j) \right) u(t_0) \hat{u}(t_0) \] dx.

We integrate by parts and note that whenever the differentiation falls on the cut-off function, we obtain a negligible term. We obtain

\[ 2Q(t_0; u(t_0), JH(t_0) u(t_0)) \approx - \sum_{j=1}^{J} \sum_{l=1}^{J} \chi_j(t_0)(V_l)_{\beta_l}(\cdot - \xi_l) u(t_0) \hat{u}(t_0). \]
In this sum, the terms for which \( l \neq j \) are negligible because of the fast decay of the potentials. For the same reason, for \( l = j \) we can neglect the cut-off function. We thus have

\[
2Q(t_0; u(t_0), JH(t_0)u(t_0)) \simeq - \sum_{j=1}^{J} (V_j)_{\beta_j} (\xi_j)_{\beta_j} \cdot \nabla u(t_0).
\]

Comparing with [431], we obtain

\[
\left| \frac{d}{dt} \bigg|_{t=t_0} Q(t; u(t), u(t)) \right| \ll \|u(t_0)\|_{H^1 \times L^2}^2.
\]

\[\square\]

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