Shear Dynamics in Higher Dimensional FLRW Cosmology

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We study the shear dynamics of higher dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology by considering a non-perfect fluid which exerts different pressure in the normal and extra dimensions. We generalise the definition of shear tensor for higher dimensional space-time and prove it to be consistent with the evolution equation for shear tensor obtained from the Ricci identities. The evolution of shear tensor is investigated numerically. The role of extra dimensions and other parameters involved in shear dynamics is discussed in detail. We find that with increase in anisotropy parameter, time of decay of shear increases while with increase in number of extra dimensions, shear tends to decay early.

Keywords: Extra dimensions, FLRW cosmology, shear and non-perfect fluids

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1. Introduction

Starting with their introduction by Kaluza\footnote{Kaluza} and Klein,\footnote{Klein} various models in gravity and cosmology in the context of higher dimensions have been used in recent times, where the matter fields reside in all dimensions including the compact extra dimensions. Such models have received a considerable support from string theory which provides a strong mathematical basis for the description of the universe in terms of higher dimensions. Further, there is a possibility that these models in the context of particle phenomenology and cosmology could provide a natural explanation to the well-known issues like hierarchy problem. Another motivation which is more funda-
mental to cosmology is that extra dimensions may play an important role in early universe. In the early phase, the universe could have been described by the ‘brane world models’ in which the standard model fields are trapped on a three dimensional hypersurface (i.e. brane), embedded in a higher dimensional spacetime. There are two such popular models, viz., the Arkani-Dimopolous-Dvali (ADD) model and Randall-Sundrum (RS) model, which have testable predictions at present day colliders.

In the early universe, the energy of the universe was typically high enough to make the existence of very small extra dimensions perceptible. The dynamics of the universe could have been different as compared to the normal (1 + 3) dimensional case due to the presence of the scale factor of extra dimensions. The scale factor of extra dimensions will in general be different from that of the normal dimensions. This motivates us to study the evolution of the universe with the extra dimension in the context of cosmology. For this purpose, one may study the kinematical quantities such as expansion scalar, shear and rotation (or vorticity) in order to see how the dynamics of the universe evolves with different conditions on scale factors as well as on the number of extra dimensions. The expansion scalar determines the rate of change of distance of neighbouring particles in the fluid (which is related to the volume expansion), while the shear tensor determines the distortions arising in the fluid flow keeping the volume preserved. The vorticity tensor however determines a rigid rotation of clusters of galaxies with respect to a local interial rest frame. It would be interesting to investigate the evolution of such quantities with extra dimensions. Our main objective here is to discuss the shear dynamics of the universe in higher dimensions with the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology.

The other motivation to do this analysis comes from an unresolved issue which is the effect of the structure formation on the cosmological observation beyond perturbative analysis of the FLRW models. The effect on the cosmic structures may be estimated via a backreaction term that arises by averaging inhomogeneous scalar quantities on spatial hypersurfaces. Thus, the backreaction term is calculated by subtracting the non-negative average shear from expansion rate. It motivates us to look at scenarios other than FRW models.

The paper is organised as follows. In the next section, we will present the necessary formalism to introduce the definition of the concerned kinematical quantities namely the expansion scalar, shear and rotation tensors to describe the behaviour of the geodesic congruence along with the Raychaudhuri equation for a congruence of timelike geodesics. In Section 3, we develop the shear dynamics of FLRW cosmology with extra spatial dimensions by calculating the non-vanishing components of shear tensor in view of the solutions of scale factors as obtained from the Einstein equations. We present the evolution of the expansion scalar and components of shear numerically in Section 4 for different number of extra dimensions. In the last section, we summarise our results with future possibilities.
2. $1 + n$ Decomposition and Kinematical Quantities

We introduce a family of observers with non-intersecting world lines. The timelike 4–velocity vector which is tangent to these world lines is $u^\alpha = \frac{dx^\alpha}{d\tau}$ and satisfies $u_\alpha u^\alpha = -1$, where $\tau$ is proper time measured along the world lines. The spacetime metric can be expressed the longitudinal $(-u_\alpha u_\beta)$ and transverse parts ($h_{\alpha\beta}$) as,

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta. \quad (1)$$

The latter projects the spacetime orthogonal to the 4–velocity into the observers’ instantaneous rest space at each event. The velocity of a comoving particle is given as $u_\alpha = (1, 0, 0, 0)$.

The vector field $u_\alpha$ and its associated tensor counterpart $h_{\alpha\beta}$ allows for a unique decomposition of every spacetime quantity into its irreducible timelike and spacelike parts. In order to characterise the evolution of the world lines, the covariant derivative of the velocity field may be split into its irreducible parts, defined by their symmetric properties:

$$u_{\beta;\alpha} = -u_\alpha u_\beta + \frac{1}{n} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}. \quad (2)$$

Here, the trace part $\Theta = u_\alpha^\alpha$ denotes the rate of expansion of the fluid, the symmetric-tracefree part $\sigma_{\alpha\beta} = (1/2)(u_{\beta;\alpha} + u_{\alpha;\beta})$ is the shear tensor and the anti-symmetric part $\omega_{\alpha\beta} = (1/2)(u_{\alpha;\beta} - u_{\beta;\alpha})$ is the rotation tensor. Here, $n$ is the total number of spatial dimensions. The governing equations of these quantities are the propagation equations which give the overall evolution (along the flow) of these quantities, i.e., $\Theta, \sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ in a given background spacetime. In particular, the evolution of $\Theta$ is governed by the Raychaudhuri equation given by:

$$\frac{d\Theta}{d\tau} = -\frac{1}{n} \Theta^2 - \sigma_{\alpha\beta}^\alpha \sigma_{\alpha\beta} + \omega_{\alpha\beta}^\alpha \omega_{\alpha\beta} - R_{\alpha\beta} u^\alpha u^\beta. \quad (3)$$

We assume the universe to be filled with a fluid which is irrotational as otherwise the velocity will die down rapidly with expansion of universe. The tangent planes of the comoving observers together form spacelike hypersurfaces which are normal to the world lines of the observers. More precisely, we consider the geodesic congruences which have rotation tensor, $\omega_{\alpha\beta} = 0$ as asserted by the Frobenius’ Theorem. The projected Riemann tensor on the hypersurface is defined as

$$R_{abcd} = h_a^q h_b^r h_c^l h_d^p R_{qrsfp} - u_{a; c} u_{b; d} + u_{a; d} u_{b; c} \quad (4)$$

The local Ricci tensor $R_{ab}$ and Ricci scalar $R$ on the hypersurface orthogonal to $u_a$ are defined by

$$R_{ab} = h^{cd} R_{cadb} \quad R = h^{ab} R_{ab} \quad (5)$$
The contraction between first and third indices of local Riemann tensor leads to Gauss-Codacci equation and a further contraction between the indices of Ricci tensor leads to,

\[ R = 2(\kappa \rho + \sigma^2 - \omega^2) - \Theta^2 \frac{(n-1)}{n} \]  

(6)

where \( \kappa = 8\pi G \), \( 2\sigma^2 = \sigma_{\alpha \beta} \sigma^{\alpha \beta} \) and \( 2\omega^2 = \omega_{\alpha \beta} \omega^{\alpha \beta} \). In the next section we consider the impact of extra dimensions on the evolution of shear and hence on the expansion scalar.

3. Evolution Equations

We consider the spacetime metric which has three normal spatial dimensions and \( D \) extra spatial dimensions, in addition to one time dimension.\(^{13}\) The corresponding line element is given by

\[ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j + b(t)^2 \delta_{IJ} dX^I dX^J \]  

(7)

where \( i, j \) denotes 1, 2, 3 and \( I, J \) represents 4, 5, ..., \((D+3)\), extra spatial dimensions. Here, \( D \) is a parameter which represents the number of extra dimensions and takes integral values and hence, \( n = 3 + D \). In the above metric, \( a(t) \) denotes the scale factor in the normal dimensions and \( b(t) \) represents the scale factor in the extra dimensions. We consider the whole \( 1 + 3 + D \) dimensional Universe to be homogeneous and hence, \( a(t) \) and \( b(t) \) are functions only of time. The visible Universe is satisfactorily described by flat space i.e. spatial curvature is zero. For simplicity, we also assume extra-dimensional subspace to be flat. For the line element given in equation (7), spatial Ricci tensor \( R = 0 \) and \( \omega = 0 \). Thus, equation (6) reduces to

\[ \kappa \rho + \sigma^2 = \Theta^2 \frac{(n-1)}{2n} \]  

(8)

The expansion scalar corresponding to the metric (7) is given as

\[ \Theta = 3 \frac{\dot{a}}{a} + D \frac{\dot{b}}{b} \]  

(9)

We follow the approach used in the paper\(^{14}\) to solve the governing equations and to calculate the evolution of \( \Theta \). The measure of total volume of the whole higher dimensional Universe is given as \( S(t)^n = a(t)^3 b(t)^D \).

The solutions for the scale factors may be obtained directly from the Einstein equations as below:\(^8\)

\[ a(t) = S(t) \exp(\Sigma_1 W(t)), \]
\[ b(t) = S(t) \exp(\Sigma_2 W(t)) \]  

(10)

where \( W(t) \) is defined as

\[ W(t) = \int \frac{dt}{S^n} \]  

(11)
and the constants $\Sigma_1$ and $\Sigma_2$ satisfy the relation $3\Sigma_1 + D\Sigma_2 = 0$.

Using the definition of shear, its components can be calculated to be

$$\sigma_{ij} = \frac{D a^2}{n S^n} (\Sigma_1 - \Sigma_2) \delta_{ij}$$

and using the relation $3\Sigma_1 = -D\Sigma_2$, we get

$$\sigma_{ij} = \frac{\Sigma_1 a^2}{S^n} \delta_{ij}$$

where $i, j = 1, 2, 3$. Similarly, we obtain

$$\sigma_{IJ} = \frac{\Sigma_2 b^2}{S^n} \delta_{IJ}$$

where $I, J = 4, 5, \ldots, D + 3$. All off-diagonal components of shear tensor are zero. The scalar square magnitude of the shear, $\sigma^2$ defined by $\sigma^2 = \sigma_{\alpha\beta} \sigma^{\alpha\beta}$ then becomes

$$\sigma^2 = \frac{\Sigma^2}{S^{2n}}$$

where,

$$\Sigma^2 = \frac{3\Sigma_1^2 + D\Sigma_2^2}{2}$$

here $\Sigma$ is a constant. Further, the expansion scalar $\Theta$ given by equation (9) can be written in terms of $S(t)$ as

$$\Theta = n \frac{\dot{S}}{S}$$

Using equations (8, 15, 17), one can obtain

$$\frac{\ddot{S}^2 n (n-1)}{2} = \kappa \rho + \frac{\Sigma^2}{S^{2n}}$$

This equation constraints the flow of the fluid. We have to solve equation (18) together with equation (11) for a specific value of $n$ and specific type of density ($\rho$). By substituting the solutions back in equation (10), we can obtain $a(t)$ and $b(t)$. One then can study the evolution of shear components from equations (13) and (14). Using the above mentioned expressions for the expansion scalar and the shear components, the Raychaudhuri equation (3) is manifestly satisfied.

The energy momentum tensor of a general imperfect fluid can be expressed as

$$T_{\alpha\beta} = \rho u_{\alpha} u_{\beta} + p_a \delta_{\alpha}^{\delta} \delta_{\beta}^{\gamma} h_{ij} + p_b \delta_{\alpha}^{\delta} \delta_{\beta}^{\gamma} h_{IJ} + \pi_{\alpha\beta}$$

where $\rho$ is the total matter energy density, $p_a = w_a \rho$ is the pressure in normal dimensions and $p_b = w_b \rho$ is the pressure in extra dimensions. The tensor, $\pi_{\alpha\beta}$ is the symmetric and trace free anisotropic stress tensor as given by

$$\pi_{\alpha\beta} = h_{\alpha}^{\gamma} h_{\beta}^{\delta} T_{\gamma\delta} - \frac{1}{n} h^{\gamma\delta} T_{\gamma\delta} h_{\alpha\beta}$$
The conservation equation for total energy momentum tensor, expressed as $T_{\alpha\beta} = 0$ reduces to

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p_a) - D\frac{\dot{b}}{b}(\rho + p_b) - \sigma^{\alpha\beta}\pi_{\alpha\beta}$$

(21)

which may further be re-written in terms of new variables as

$$\dot{\rho} = -\rho n \frac{\dot{S}}{S} + 3 \left( \frac{\dot{S}}{S} + \frac{\Sigma_1}{S^n} \right) \left( -p_a + \frac{(p_b - p_a)D}{n} \right) + D \left( \frac{\dot{S}}{S} + \frac{\Sigma_2}{S^n} \right) \left( -p_b + \frac{3(p_a - p_b)}{n} \right).$$

(22)

For simplicity, we assume that our universe is filled with radiation in visible subspace and with dust in extra-dimensional subspace. Thus, we have $p_a = \frac{\rho}{3}$ and $p_b = 0$.

4. Evolution of Expansion Scalar and Shear Components

We may now numerically solve the three coupled first order differential equations (11), (18) and (22) to see the evolution of shear components in time and also the evolution of expansion scalar as given by the equation (17). There may exist a variety of solutions depending on the values of the independent model parameters, i.e., $D$ and $\Sigma_1$. Before we do so, it is convenient to rescale the variables in terms of dimensionless quantities, namely

$$t = \frac{\tau}{M}, \quad \bar{\rho} = \frac{\rho}{M^4}, \quad \Sigma_1 \rightarrow \frac{\Sigma_1}{M^4},$$

(23)

where $M$ is the scale of quantum gravity in the entire bulk, and is related to $M_{\text{Planck}}$ (a derived quantity defined only for the theory in the 4-dimensional subspace) through $V_D M^{D+2} = M_{\text{Planck}}^4$. Here $V_D$ is volume of the extra-dimensional subspace. One would expect that $V_D \gtrsim M^{-D}$ and, thus $M \lesssim M_{\text{Planck}}$.

The equations are solved in terms of rescaled time. The evolution of scale factors $a(\tau)$ and $b(\tau)$ in the rescaled time for different values of $\Sigma_1$ and $D$. The equations are solved in terms of rescaled time. The evolution of scale factors $a(\tau)$ and $b(\tau)$ in the rescaled time for different values of $\Sigma_1$ and $D$ are shown in the figure 1. It is interesting to see that the expansion in scale factor $a(\tau)$ is more rapid.
if we increase the amount of anisotropy parameter, $\Sigma_1$ in visible subspace. This implies that shear helps in expansion. For the same $\Sigma_1$, the amount of expansion of scale factor $a(\tau)$ is less by increasing the number of extra dimensions. It implies that increase in number of extra dimensions has a role such that it reduces the effect of shear. For the case shown in figure 1, the scale factor in extra dimensional subspace flips its behavior, i.e., it is going from contracting to expanding phase if we decrease $\Sigma_1$ but it becomes constant asymptotically.

![Fig. 2. Evolution of the $\Theta$ with respect to the rescaled time $\tau$ for $\Sigma_1 = 0.1$ (left panel) and $\Sigma_1 = 0.5$ (right panel).](image)

In figure 2, we present the evolution of $\Theta$ with respect to rescaled time for different number of extra dimensions, viz., $D = 3, 6, 10$. Behaviour of $\Theta$ shows variation in the initial phase with the change in number of dimensions but as time evolves, it becomes independent of $D$ with a slight dependence on the value of $\Sigma_1$. Figure 3

![Fig. 3. Behaviour of the shear components, $\sigma_{11}$ and $\sigma_{44}$ with the rescaled time $\tau$ for $\Sigma_1 = 0.1$ (left panel) and $\Sigma_1 = 0.5$ (right panel).](image)

shows the variation of shear components as a function of time, where $\sigma_{11}$ and $\sigma_{44}$ represent shear in visible subspace and extra-dimensional subspace respectively. It is seen that shear components in both subspace, i.e., visible and extra-dimensional subspaces go to zero asymptotically. Higher the value of anisotropy parameter, larger
the time it takes to go to zero. With increase in extra dimensions, the value of shear components decreases and it tends to zero comparatively early.

5. Conclusion
In this work, we have studied the dynamical evolutions of expansion scalar and shear in the higher dimensional FLRW Universe by considering that it is filled with the non-perfect fluid, which exerts different pressure in the normal and extra spatial dimensions. We have developed the dynamical equations governing evolutions of expansion scalar and shear components involving the normal and extra spatial dimensions. Our numerical study for different number of extra spatial dimensions with different values of independent parameter $\Sigma_1$ show that the extra dimensions have a very small effect on the expansion scalar during some initial period of time and as time evolves, the expansion scalar become independent of number of extra spatial dimensions.

A number of interesting questions can arise from this mechanism which we plan to investigate in future. We plan to look at the effect of shear on nucleosynthesis in this model as there will be a change in the expansion depending on $D$ and shear. One may get quite different results than the FLRW models. Consequently, one can use the nucleosynthesis observations to limit the shear constant, i.e. the anisotropy parameter $\Sigma_1$ and the number of extra dimensions, $D$. We intend to report on this issue in near future.

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