Out-of-Time-Order correlators in driven conformal field theories

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Abstract: We compute Out-of-Time-Order correlators (OTOCs) for conformal field theories (CFTs) subjected to either continuous or discrete periodic drive protocols. This is achieved by an appropriate analytic continuation of the stroboscopic time. After detailing the general structure, we perform explicit calculations in large-c CFTs where we find that OTOCs display an exponential, an oscillatory and a power-law behaviour in the heating phase, the non-heating phase and on the phase boundary, respectively. In contrast to this, for the Ising CFT representing an integrable model, OTOCs never display such exponential growth. This observation hints towards how OTOCs can demarcate between integrable and chaotic CFT models subjected to a periodic drive. We further explore properties of the light-cone which is characterized by the corresponding butterfly velocity as well as the Lyapunov exponent. Interestingly, as a consequence of the spatial inhomogeneity introduced by the drive, the butterfly velocity, in these systems, has an explicit dependence on the initial location of the operators. We chart out the dependence of the Lyapunov exponent and the butterfly velocities on the frequency and amplitude of the drive for both protocols and discuss the fixed point structure which differentiates such driven CFTs from their undriven counterparts.

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1 Introduction

Stroboscopic dynamics of periodically driven closed quantum systems has been intensely studied in recent years [1–9]. The reason for such intense activity in the field is two-fold. First, at least some theoretical predictions emerging from such studies have received support from experiments carried out on ultracold atoms in optical lattices [10–19]. Second, many of such studies lead to understanding of several phenomena that have no analogue in equilibrium quantum systems.

The properties of any driven quantum system is controlled by its unitary evolution operator $U(t, 0)$ given by $U(t, 0) = T_t \{ \exp \left[ -i \int_0^t H(t') dt' / \hbar \right] \}$, where $H(t)$ is the (time-dependent) Hamiltonian of the system, $T_t$ denotes time ordering, and $\hbar$ is the Planck’s constant. For periodically driven systems, characterized by a time period $T$, this evolution operator, at times $t = nT$, where $n \in Z$, satisfies [7, 9]

$$U(nT, 0) = T_t \left\{ \exp \left[ -i \int_0^T H(t') dt' / \hbar \right] \right\} = \exp \left[ -i n T H_F(T) / \hbar \right]$$  \hspace{1cm} (1.1)

where $H_F(T) \equiv H_F$ is the Floquet Hamiltonian of the system. Thus the Floquet Hamiltonian of the systems controls stroboscopic dynamics of any periodically driven system. This allows for several features in the dynamics of such a driven system at stroboscopic times $t = nT$.
which have no analog for other times \((t \neq nT)\); such features appear for both discrete and continuous drive protocols and have been extensively studied in the literature in several contexts \([7, 9]\).

The exact computation of \(H_F\) for a generic non-integrable system is challenging due to time ordering involved in the definition of \(U\); consequently, it is customary to resort to perturbative methods for its computation. For small \(T\), the Magnus expansion constitutes one such perturbative scheme \([9, 20]\); however, its prediction starts to deviate from the exact result as one approaches the intermediate frequency regime. In contrast, the Floquet perturbation theory \([9, 21, 22]\), provides a more accurate description of such driven systems in the intermediate and small \(T\) regime provided that the drive amplitude is large \([9]\). However, at large \(T\), there are no known reliable analytic scheme for computing \(H_F\); in this regime, one has to usually rely on exact numerics.

Out of the several possible protocols used to drive a system out of equilibrium, periodic protocols which can be understood in terms of their Floquet Hamiltonians have been studied most intensively in recent years. The main reason for this focus is the presence of several phenomena in such systems that are usually not found in aperiodically driven systems. These include generating quantum states with non-trivial topology \([23–30]\), realization of Floquet time crystals \([31–33]\), demonstration of dynamical localization \([34–40]\) and dynamical freezing \([41–44]\), induction of dynamical phase transitions \([45–51]\), and tuning ergodicity of a quantum many-body system using frequency of the drive \([52, 53]\). More recently, systems with quasiperiodic and aperiodic drives \([54–60]\) has also been studied in this context. See also \([61]\) for a discussion on measures of quantum chaos in Floquet systems.

A class of such driven systems involves conformal field theories (CFT) subjected to periodic drives. It is usually expected that driving a CFT would lead to generation of a timescale which will in turn spoil its conformal invariance and drive it away from its conformal fixed point \([62]\). However, recently, it was shown \([63–68]\) that this is not necessarily the case; indeed it is possible to subject a CFT to a periodic drive without spoiling the conformal symmetry of the problem. See \([69–71]\) for more studies in Floquet CFT systems.

As explained in section 2, this can be most easily done using a CFT model with sine square deformation whose Hamiltonian is valued in \(\text{su}(1, 1)\) \([72, 73]\). The action of the drive in this case leads to an evolution operator

\[
U(T, 0) = \mathcal{T}_i \left\{ \exp \left[ -i \int_0^T dtH(t)/\hbar \right] \right\} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

where \(a d - b c = 1\) and the coefficients \(a, b, c,\) and \(d\) depend on the drive protocol. We note that \(U\) is valued in \(\text{SU}(1, 1)\). Since this group is isomorphic to \(\text{SL}(2, \mathbb{R})\), this allows us to write \(U\) in imaginary time as

\[
U(T = i\tau, 0) \equiv U_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}
\]

where \(a_i d_i - b_i c_i = 1\). The action of \(U_i\) in the complex plane generates a Möbius transformation \(z \to z_n = (a_i z + b_i)/(c_i z + d_i)\). The evolution of the quantum state under \(U_i\) gets translated into the dynamics of operators of the CFT, when we move to the Heisenberg
picture. Thus the stroboscopic time-dependence of any primary operator can be obtained using the relation

\[ U^\dagger(T,0)O(z,\bar{z})U(T,0) = \left( \frac{\partial z_n}{\partial z} \right)^h \left( \frac{\partial \bar{z}_n}{\partial \bar{z}} \right)^\bar{h} O(z_n,\bar{z}_n) \] (1.4)

where one analytically continues to real time at the end of the calculation and \((h,\bar{h})\) denotes the conformal dimension of \(O\). Using this prescription, energy density, equal and unequal-time correlation functions, and entanglement entropies of driven CFTs have been computed in both cylindrical and strip geometries \[63–68\]. We note in this context that for the cylindrical geometry such a drive does not affect the ground state of the CFT; consequently all equal-time correlation functions computed starting from an initial ground state retain their initial values. The effect of dynamics shows up in the unequal-time correlations computed using initial ground state or in equal-time correlators computed using initial asymptotic excited states which evolve under the dynamics \[68\]. Another alternative to circumvent this issue is to use the strip geometry where all correlation functions computed using the initial ground state exhibit non-trivial dynamics \[63, 64, 66\]. An interesting property found in all these quantities is the emergence of spatial inhomogeneity due to the drive which is usually not found in typical condensed matter systems. However, out-of-time-order correlation function for such driven CFTs has not been studied so far.

The Out-of-Time-Order correlators (OTOC) are known to provide a diagnostic for scrambling which precedes thermalization in a typical non-integrable quantum many-body system \[74–79\]. Examples of such systems include large \(N\) spin and bosonic models \[80–82\]. For such models, which have well-understood semi-classical limits, the early time behavior of the OTOC constitutes an exponential growth and can be written as

\[ C(x,t) = \text{Tr} \left[ e^{-\beta H/2} W(x,t)V(0,0) e^{-\beta H/2} W(x,t)V(0,0) \right] \sim e^{\lambda_L(t-x/v_B)} \] (1.5)

where \(\beta = (k_B T_0)^{-1}\) is the inverse of the temperature \(T_0\) and \(k_B\) is the Boltzmann constant.

Note that at zero temperature, the trace in eq. (1.5) is replaced by expectation in the ground state and this will be the limit which we shall be interested in here. The quantity \(\lambda_L\) is the equivalent of the Lyapunov exponent in a classical chaotic system; its inverse \(t_* = 1/\lambda_L\) provides the timescale for information scrambling. It is well known that in thermal system \(\lambda_L\) satisfies the bound \(\lambda_L \leq 2\pi/(\beta \hbar) \sim T_0\) \[83\]. In contrast, \(v_B\), the butterfly velocity, is bounded only by the speed of light \(c\) and measures the speed with which local perturbations grow. It constitutes an analogue of the Lieb-Robinson velocity \[84\] for information spreading in the present case. Such OTOCs have been studied in several contexts both in condensed matter physics (e.g. \[85, 86\]), quantum field theories (e.g. \[87, 88\]), and conformal field theories with AdS duals (e.g. \[89–91\]).

In this work, we extend the studies of such OTOCs to periodically driven CFTs. One of the motivations of our present study is to understand the efficacy of the OTOC as a diagnostic of chaos, beyond thermal equilibrium.\(^1\) Floquet CFTs provide us with a set-up

\(^1\)This is part of a larger goal to use OTOCs to study quantum chaos in systems outside its original purview, i.e.- in QFTs without boundaries and in thermal equilibrium. OTOCs in QFTs in the presence of a boundary has been studied in \[74\]. See also \[78, 79\], where OTOCs are studied in another out of equilibrium setting-the CC state in a quantum quench.
where we have analytic control to answer this question. We answer this question in the affirmative and show that unlike lower point function probes such as the entanglement entropy and expectation value of the energy, the 4-pt OTOC can demarcate integrable and chaotic CFT’s in the heating phase.

One key difference of these systems from the undeformed CFT is that they break both spatial translational as well as time translational symmetry. Another key difference is the existence of fixed points of the flow under the deformed Hamiltonian, unlike the undeformed one, where there are no fixed points. These differences leads to novel consequences, which have no analogue in the undeformed thermal CFT case. Our main findings are as follows:

1. For the continuous drive case, we show that in a large $c$ CFT, the OTOC as defined in (1.5), shows an exponential growth for sufficiently large $n$, smaller than the scrambling time in the heating phase, but only for a range of values of $x$ bounded by the fixed points of the flow. These fixed points are characteristic of driven CFTs; they have no analogue in CFTs in equilibrium. The Lyapunov exponent, in this example, is a function of the drive parameters. In the non-heating phase, there is no exponential growth for any value of $x$.

2. In the same example, we show that the butterfly velocity is position dependent. This is a consequence of the lack of translational invariance in these systems. Our definition of the butterfly velocity is a natural generalization of the same in the undeformed case. We show that the butterfly velocity in such driven systems can be tuned using drive frequency.

3. We further generalize our study of the OTOCs to the case where the two operators ($V$ and $W$) are kept at arbitrary points ($x_1$) and ($x_2$), instead of ($x$) and (0). In the undeformed case, due to spatial homogeneity, one can always choose one of the operators to be at the origin, without any loss of generality. This is not true in the driven CFT case, and as a result the OTOC is naturally a function of the two positions of the operators and not only a function of the relative distance between the two. Interestingly, for sufficiently large $n$, the expression for the OTOC factorizes as a result of which the explicit expression of the butterfly velocity remains same as the case when the operators are placed at $x$ and 0.

4. Finally, we study OTOCs in the case of the discrete drive where again the two operators ($V$ and $W$) are kept at arbitrary point ($x_1$) and ($x_2$). The qualitative features remain the same as before, even though the expressions for the butterfly velocity and the Lyapunov exponent are different.

5. For driven Ising CFT, we find that the OTOC does not show exponential behaviour even in the heating phase. We expect our result to be true for more general integrable CFTs. Thus the OTOC, unlike lower point functions, distinguishes between integrable and chaotic CFTs, under the periodic drive.

The plan of the rest of the work is as follows. After a quick review of the dynamics of the driven CFT systems in the next section, we present our analysis of the OTOCs in
the section 3. We first discuss the OTOC in the case of the continuous drive in section 3.1. We present the explicit results in the case of a large c CFT first for the case where the operators are placed at \( x \) and 0, and then later for the more general case where the positions are taken to be \( x_1 \) and \( x_2 \). In section 3.2, we study the case of a discretely driven CFT, with two periods \( T_1 \) and \( T_2 \). In the appendix, we discuss the OTOC computation for the driven Ising CFT. We conclude with a discussion of our results and some comments on the possible holographic realization of these results in section 4.

2 Driven CFTs: the set-up

In this section we present a brief review of driven CFTs. We discuss both the continuous as well as the discrete drive protocols. Our goal here is to introduce the various phases which arise under such drive protocols as well as to present the explicit form for the time evolution of operators under these protocols. These results will be used in section 3 to compute OTOC’s in these systems.

Driven many-body systems provide a useful set-up for studying non-equilibrium dynamics. Among the many driving protocols, a very interesting class is realized by the so-called Floquet dynamics of CFT, see e.g. [63–65, 67, 68]. This class of systems, despite a non-trivial driving, preserves conformal symmetry and allows for an analytical control over the dynamics, see e.g. [72, 73]. In this protocol, the drive is executed by independently controlling, the left (\( T(x - t) \)) and right moving components \( \bar{T}(x + t) \) of the energy-momentum density (\( T_{00}(x, t) \)) of the CFT. The Hamiltonian, \( H \), of the driven CFT, at \( t = 0 \) is given by

\[
H = \int dx \left( f(x)T(x) + g(x)\bar{T}(x) \right). 
\]  

(2.1)

Here \( T_{00} = \frac{1}{\pi^2}(T + \bar{T}) \), while \( f(x) \) and \( g(x) \) represent the externally controlled functions, which are both unity for an undriven CFT. The drive is accomplished by changing these functions periodically, quasi-periodically or randomly, either in discrete steps or continuously. In this note, we will be concerned only with periodically driven CFTs on a ring under both the continuous as well as the discrete drive. Under the periodic drive protocol, evolution of the system is periodic with a period \( T \), so that the total evolution happens in steps(\( n \)) of \( T \), i.e. \( t = nT \). In the discrete drive case, the evolution within this time period \( T \) is updated discretely (i.e. \( T = T_1 + T_2 + T_3 + \ldots + T_k \)). During each of these time intervals \( T_j \), the Hamiltonian is changed externally, for example, by updating the functions \( f(x) \) and \( g(x) \). Consequently, the Hamiltonian for the time \( T_j \) may be parameterized by an integer (\( H(j) \)). In contrast, for the continuous drive case, it is updated continuously throughout the time period \( T \). We will now review, in more detail, the basic setup for these two cases.

On a ring, the functions \( f(x) \) and \( g(x) \) have to obey periodic boundary conditions, and thus admit a Fourier series expansion in terms of its modes. This fact then allows one to express the Hamiltonian as a linear sum of the Virasoro generators. Thus for generic functions \( f \) and \( g \) and a discrete drive (say), the Hamiltonian at the \( j^{th} \) iteration, takes the form

\[
H^{(j)} = \sum_n \left( a_n^{(j)} L_n + b_n^{(j)} \bar{L}_n \right). 
\]  

(2.2)
Here $L_n$ and $\bar{L}_n$ generate two independent Virasoro algebras

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m+n,0},$$

$$[\bar{L}_m, L_n] = (m - n) \bar{L}_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m+n,0}$$

and are related to the chiral and anti-chiral stress tensor components in the usual way.

$$L_n = \frac{L}{2\pi} \int_0^{2\pi} dx e^{i\pi nx} T_{00}(x), \quad \bar{L}_n = \frac{L}{2\pi} \int_0^{2\pi} dx e^{-i\pi nx} T_{00}(x)$$

with $n \in \mathbb{Z}$, $x \in [0, 2\pi]$ is an angular coordinate on the $S^1$ and $L$ is the circumference of the ring.

For general $f$ and $g$, the sum is over all the modes and hence the Hamiltonian dynamics is complicated, due to the presence of these infinite number of generators. One can simplify the dynamics, by restricting the form of the functions $f$ and $g$, so that the Hamiltonian has contributions only from generators of a sub-algebra of the Virasoro algebra. In this work, we will restrict ourselves to the case when $f = g$ and the Hamiltonian is built out of the generators of the diagonal $sl(2,\mathbb{R})$ sub-algebra of the global $sl(2,\mathbb{C})$, generated by $\{L_0 + \bar{L}_0, L_{\pm 1} + \bar{L}_{\pm 1}\}$.

In this case, the evolution operator at each step is the usual $sl(2,\mathbb{R})$ transformation, and consequently the full time evolution is given by combining the conformal transformation at each step, which is again an $sl(2,\mathbb{R})$ transformation. To work out the full dynamics, the strategy used is as follows. We first Wick rotate the Lorentzian time to Euclidean time and then map the cylinder to the plane, using: $w = \frac{L}{2\pi} \log z$. On the plane, once we obtain the expressions of the combined conformal transformation, we Wick rotate back to Lorentzian time, to obtain the corresponding transformation in real time.

### 2.1 Time evolution in continuously driven CFTs

The Hamiltonian of the driven system in this case, is given by

$$H(t') = \frac{2\pi}{L} \left[ f(t') L_0 + \frac{1}{2} f_1(t') (L_1 + L_{-1}) \right] + \text{anti-holomorphic},$$

where $\{f(t'), f_1(t')\}$ are real-valued functions that encodes the drive protocol and $t'$ is an auxiliary parameter that describes the function-space. Setting $f_1(t') = 0$ and $f(t') = 1$ yields us the standard (undriven) CFT Hamiltonian on the cylinder. Given (2.5), the corresponding evolution operator can be calculated. This yields:

$$U(T, 0) = \exp \left[ -i \int_0^T H(t) dt \right] \quad \Rightarrow \quad U(T, 0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
protocol, which is real-valued. The evolution is step-wise: we imagine driving the system with a Hamiltonian $H(t)$ for an interval of $T$, for $n \in \mathbb{Z}$ number of cycles. This $n$ plays the role of stroboscopic time. For a given $T$, and an integer $n$, $nT$ measures the time. Here, the Hamiltonian $H(t)$ is determined by choosing a drive protocol data $\{f(t), f_1(t)\}$. Correspondingly, in the Heisenberg picture, operator evolutions for stroboscopic times $t = nT$, are simply given by

$$\mathcal{O}(x,t) = U_1^n(t,0)\mathcal{O}(x,0)U(t,0), \quad U(t) = U^n(T,0), \quad U(T,0) = e^{-i\int_0^T H(t) dt}. \tag{2.7}$$

Above, we have introduced a notion of time, denoted by $t$. Clearly, $n \in \mathbb{Z}$, but $t \in \mathbb{R}$.

The map from the cylinder coordinates $(w)$ to the plane coordinates $(z)$, is given by $w = \frac{1}{2\pi} \log z$. Thus on the plane, we have (eq. (1.3)):

$$U_i^\dagger \mathcal{O}(z,\bar{z})U_i = \left(\partial_{z'}^n\right)^h \left(\partial_{\bar{z}'}\right)^\hbar \mathcal{O}(z',\bar{z}'). \tag{2.8}$$

As explained earlier, $U_i \in \text{SL}(2,\mathbb{R})$. On the plane, this acts as:

$$z' = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad \text{with} \quad \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in \text{SL}(2,\mathbb{R}). \tag{2.9}$$

Correspondingly, the complete evolution of the Heisenberg operator is determined by a combined conformal transformation:

$$\left(\prod_i U_i^\dagger\right) \mathcal{O}(z,\bar{z}) \left(\prod_i U_i\right) = \left(\partial_z^n\right)^h \left(\partial_{\bar{z}}\right)^\hbar \mathcal{O}(z_n,\bar{z}_n) \tag{2.10}$$

with $z_n = \frac{a_n z_{n-1} + b_n}{c_n z_{n-1} + d_n}, \ldots, z_1 = \frac{a_1 z + b_1}{c_1 z + d_1}. \tag{2.11}$

The last line above defines a recursion relation that determines $z_n$ at the $n$th-step, starting from the initial assignment of $z$. A similar transformation holds for the anti-holomorphic part as well.

For explicit calculations, we collect some important results of [68]. The evolution operator in (2.6) can be written as $U(T,0) = e^{-iH_F T}$, where $H_F = p(T)\sigma_z + i q(T)\sigma_y$. Here $\sigma$’s are the Pauli matrices and $p(T)$ and $q(T)$ can either be numerically exactly determined or computed analytically within FPT [68]. The former is obtained by dividing $T$ into $N = T/\delta T$ steps, within each of which the Hamiltonian in (2.5) remains approximately constant. The total $U(T,0)$ is constructed by multiplying the resulting step-wise evolution operators.

Now, following [68], the drive protocol data is given by $f_1(t) = 1$ and

$$f(t) = f_0 \cos (\omega_D t) + \delta f. \tag{2.12}$$

Here the drive amplitude $f_0 \gg \delta f$, where $\delta f$ is the static (DC) component of the drive. In the regime $\omega_D \geq \delta f, 1$, the following parameter characterizes the dynamics:

$$\alpha = \sum_{n=-\infty}^{\infty} J_n \left(\frac{2f_0 \pi}{L \omega_D}\right) \frac{T}{n \pi + \pi \frac{\delta f T}{T}}. \tag{2.13}$$
where $J_n$ are the Bessel functions of the first kind. In particular, $\alpha^2 > 1$ ($\alpha^2 < 1$) corresponds to the heating (non-heating) phase while $\alpha^2 = 1$ describes the phase boundary. We note that in the limit $\omega_D L \gg 2 f_0 \pi$, one simply obtains: $\alpha \simeq L/(\pi \delta f)$,\footnote{One makes use of the asymptotic behaviour of Bessel function of the first kind:}

\begin{equation}
J_n(x) \approx \frac{1}{\Gamma(n+1)} \left(\frac{x}{2}\right)^n, \quad \text{for} \quad 0 < x \ll \sqrt{n+1},
\end{equation}

along with $J_{-n}(x) = (-1)^n J_n(x)$.

We note that in the limit $\omega D L \gg 2 f_0 \pi$, one simply obtains:

\begin{equation}
\alpha \simeq \frac{L}{\pi \delta f},
\end{equation}

which is independent of the drive amplitude $f_0$ and the drive frequency $\omega_D$. Setting further $L \gg \delta f$, we get $\alpha \gg 1$. For a large drive frequency, we obtain a heating phase for small $\delta f$; the transition to a non-heating phase occurs when $\delta f = \pi/L$. In the opposite limit of small frequencies, no such simplification occurs and $\alpha$ depends on all parameters in the drive protocol.

The last ingredient that we need is how the drive protocol data are related to the SL(2, $\mathbb{R}$) transformations in (2.11). For $\alpha^2 > 1$, the analytical expressions of these quantities follow from those of $p(T)$ and $q(T)$ computed using FPT and are given by [68]

\begin{equation}
a_n = \cosh (n\theta) - i \frac{1}{\sqrt{\alpha^2 - 1}} \sinh (n\theta), \quad d_n = a_n^*, \quad \theta = s \sqrt{\alpha^2 - 1},
\end{equation}

\begin{equation}
b_n = \frac{i \alpha}{\sqrt{\alpha^2 - 1}} \sinh (n\theta), \quad c_n = b_n^*, \quad s = \cos^{-1} \left[ \cos \left( \frac{\delta f \pi}{L} \right) \right].
\end{equation}

We note that the unitarity of the evolution matrix in the heating phase is preserved for $\delta f T / L = -n$ (for $n < 0$) for which $\alpha$ diverges.

For $\alpha^2 < 1$, similar relations hold, except that $\cosh \to \cos$ and $\sinh \to \sin$. The $\alpha^2 = 1$ boundary needs a separate analysis. In this case, we obtain [68]:

\begin{equation}
a_n = 1 - ins, \quad d_n = a_n^*,
\end{equation}

\begin{equation}
b_n = -ins, \quad c_n = b_n^*.
\end{equation}

We shall use these results for computation of OTOC in section 3.

### 2.2 Time evolution in discretely driven CFTs

We now consider a discrete drive, similar to the ones considered in [64]. We shall restrict ourselves here to a two-step protocol for which the Hamiltonian is given by:

\begin{align}
H(t) &= H_1 \quad \text{for} \quad T_1, \\
&= H_2 \quad \text{for} \quad T_2,
\end{align}

where $T_{1,2}$ are the corresponding time intervals with $T_1 + T_2 = T$. Using the notations of [64], the explicit Hamiltonians are given by

\begin{equation}
H_\phi = \int_0^L T_{\phi=0} dx \left( 1 - \tanh(2\phi) \cos \left( \frac{2\pi x}{L} \right) \right),
\end{equation}

\begin{equation}
H_1 = H_\phi=0, \quad H_2 = H_\phi \neq 0.
\end{equation}
The full Hamiltonian can be written in terms of \( \{L_0, L_\pm 1\} \) that generate the \( \text{SL}(2, \mathbb{R}) \) symmetry \cite{64}. Thus, the corresponding time evolution is governed by a set of conformal transformations, corresponding to \( H_1 \) and \( H_2 \) with time-intervals \( T_1 \) and \( T_2 \), \textit{ad infinitum}.

The explicit Möbius-transformation, for a single step, is given by

\[
M_\phi (H_\phi, T_\phi) = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}, \tag{2.22}
\]

with

\[
a = \cos \left( \frac{\pi T_\phi}{L_{\text{eff}}} \right) + i \cosh(2\phi) \sin \left( \frac{\pi T_\phi}{L_{\text{eff}}} \right), \tag{2.23}
\]

\[
b = -i \sinh(2\phi) \sin \left( \frac{\pi T_\phi}{L_{\text{eff}}} \right), \tag{2.24}
\]

We intend to explore the heating-phase, which occurs for: \( (T_1, T_2) = (L/2, L_{\text{eff}}/2) \). The \( n^{\text{th}} \)-evolution is obtained by multiplying products of \( (M_0 M_1) \), \( n \) times. This yields

\[
U(nT, 0) = (M_0 M_1)^n = (-1)^n \begin{pmatrix} \cosh(2n\phi) & -\sinh(2n\phi) \\ -\sinh(2n\phi) & \cosh(2n\phi) \end{pmatrix}. \tag{2.25}
\]

We shall use this result in section 3.

### 3 Correlation functions in driven CFTs

Correlation functions are the primary observables in any QFT. In the context of driven CFTs, one is interested in the growth of the energy density and entanglement entropy of a region with time. In particular, different phases found in Floquet CFTs are classified by the qualitatively different temporal dependence of these quantities. For instance, in the “heating phase”, the energy grows exponentially with \( n \)- the stroboscopic time, while the entanglement entropy grows linearly. While the energy density is given by the one-point function of the stress tensor in the appropriate state, the computation of the entanglement entropy involves an equal time two-point function of a specific “twist operator”, which is a primary operator in the CFT. In this note, we study four point functions in these theories. In particular, we are interested in the computation of the 4-point OTOC function in the vacuum state. These correlation functions are known to be good probes of quantum chaos at early times (times much smaller than the scrambling time).

In this section, we will present our main results on the computation of OTOC’s both for the continuous drive (section 3.1) as well as for the discrete drive protocols (section 3.2). A summary of the key findings of this section has also been given in the introduction as well as in the discussion sections.

We begin with a review of OTOC in the undeformed 2D CFT. The normalized 4-point Euclidean correlator, has the following form:

\[
\frac{\langle W(z_1, \bar{z}_1)W(z_2, \bar{z}_2)V(z_3, \bar{z}_3)V(z_4, \bar{z}_4) \rangle}{\langle W(z_1, \bar{z}_1)W(z_2, \bar{z}_2) \rangle \langle V(z_3, \bar{z}_3)V(z_4, \bar{z}_4) \rangle} = F(\eta, \bar{\eta}), \tag{3.1}
\]

\[
\eta = \frac{z_1 z_2 z_3 z_4}{\bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4}, \quad \bar{\eta} = \frac{\bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4}{z_1 z_2 z_3 z_4}. \tag{3.2}
\]
Here $F(\eta, \bar{\eta})$ is an undetermined function which depends on the dynamics of the corresponding CFT. The dynamical function $F$ can be expanded in the basis of global conformal blocks [75], which are given by Hypergeometric functions, with coefficients that are determined by the OPE-coefficients of the corresponding CFT. It is the latter that encodes details of the CFT-dynamics. The Lorentzian correlators are obtained by an analytic continuation in the complex time plane using the $i\epsilon$ prescription. To be specific, if we define $z_i = x_i + i\tau_i$, where $\tau_i$ is the complex time (i.e. after continuing the time to the complex domain), then the $i\epsilon$ prescription dictates that we take $\tau_i = it_i + \epsilon_i$ in the limit $\epsilon_i \to 0$, where $t_i$ is the Lorentzian time. The different possible ways in which the limits can be taken correspond, in the Lorentzian theory, to the different possible ordering of operators in the correlation functions. As an example, taking the following ordering of limits ($\epsilon_1 > \epsilon_3 > \epsilon_2 > \epsilon_4 \to 0$), we get the following Lorentzian correlation function $\left[\langle W(z_1, \bar{z}_1)W(z_3, \bar{z}_3)W(z_2, \bar{z}_2)V(z_4, \bar{z}_4)\rangle\right]/\left[\langle W(z_1, \bar{z}_1)W(z_2, \bar{z}_2)\rangle\langle V(z_3, \bar{z}_3)V(z_4, \bar{z}_4)\rangle\right]$. Since the $z_i$s appear in the correlation function through a single complex number, the cross ratio $\eta$, the analytic continuation happens effectively in the complex $\eta$ plane. It is well-known that for a 2D CFT on the plane, the function $F(\eta)$ has branch cuts along the real line from $\eta = 1$ to $\eta = \infty$. The behavior of the real-time correlation function depends crucially on whether $\eta$ crosses this branch cut during the analytic continuation to real time.

In the study of quantum chaos, the commutator square of two observables $-\langle[V(x_1, t), W(x, t')|^2\rangle_\beta$ in the thermal state has been proposed to be a good measure of early time chaos. Physically this measures the effect of a perturbation $W$ at time $t'$ on the measurement of an observable $V$ at time $t$. The thermal correlation functions naturally live on a cylinder, with $\beta$ being the radius of the circle. Using space time translation invariance, one can set, without loss of generality, $(x_1 = 0)$, $(t > 0)$ and $(t' = 0)$. In 2D CFTs, one can map this geometry to the plane via an appropriate conformal map. Thus effectively, with this map, we are computing vacuum correlation functions on the plane of the type described above. When studying quantum chaos, we are interested in the behaviour of the commutator square, or equivalently, the 4-pt correlation functions at late times ($t \gg x$). It can be shown that as we take $t$ from small values to late times, the cross ratio $\eta$ transverses a closed path trajectory in the complex time plane. It starts from very close to the origin, crosses the real line at $t = x$, and finally again takes a very small value at late times. In particular, with this configuration, in 2D CFT in the thermal state, $\eta$ crosses the real line at precisely $\eta = \epsilon_1\epsilon_3/\epsilon_2\epsilon_4$ (where $\epsilon_{ij} = \epsilon_i - \epsilon_j$) in the $\epsilon_i \to 0$ limit. It is easy to see that this is greater than one for the out-of-time-ordered (OTR) configurations and less than one for time-ordered (TO) configurations. Thus for TO configurations, $\eta$ crosses the branch cut during the analytic continuation, while it does not do so for TO configurations. In a quantum chaotic CFT, this fact results in an exponential growth of the OTOC at late times. However, to see this explicitly one has to work with theories where the explicit form of the function $F(\eta)$ is known. A prime example of this is the OTO in a “large-c” CFT, of two scalar Hermitian operators—“the heavy operator” $W$ and “the light operator” $V$.

---

4Here heavy and light refer to the fact that the conformal dimension of $W$ scales linearly with $c$ in the large-$c$ limit, i.e. $\frac{\Delta W}{c}$ is finite, while $\frac{\Delta V}{c} \to 0$ in the same limit. In these theories and for this choice of operators, one can explicitly carry out the analytic continuation in $\eta$ to get the exponential temporal growth in the OTO in the thermal state at late times.
We will now set up the analogous computation for these normalized 4-pt correlation functions, in the periodically driven CFTs, in the vacuum state. Given the fact that the driven Hamiltonian we consider here really generates an \( sl(2,\mathbb{R}) \) transformation under which the vacuum state is actually invariant, one might not expect any non-trivial time evolution of these correlation functions in the vacuum. For instance the energy density and entanglement entropy, will be time independent in the vacuum state. However, this is true only for equal time correlation functions. Unequal time correlation functions, either time-ordered or out-of-time ordered such as the one considered in this work, display non-trivial dynamics. This feature follows from the fact that they carry information about the time-dependence of the Hamiltonian through the dynamics of the operators in the Heisenberg picture.

### 3.1 4pt OTOC in the continuously driven CFTs

#### 3.1.1 A simple 4-point OTOC

We begin by considering a correlator of the form \( \langle WWVV \rangle \), in which both \( W \)'s are placed at \( x = 0 \) and the \( V \)'s are placed at \( x \). The initial time, for all the operators is chosen to be zero. The corresponding arrangement, on the plane, is given by:

\[
    z_1 = 1, \quad z_2 = 1, \quad z_3 = e^{2\pi i x/L} = z_4,
\]

which also determines the corresponding complex-conjugates. As described in the previous section, time evolution will be realized as the stroboscopic time \( n \) in conformal transformations on the co-ordinates and therefore the operators located on them. For our purpose, the two \( W \) operators placed initially at the points \( \{z_1, z_2\} \) are evolved to time \( n \) while the two \( V \) operators placed initially at the points \( \{z_3, z_4\} \) do not undergo any time evolution. To impose operator ordering, we will shift in euclidean time \( n\theta + i\epsilon_i \), where the subscript refers to the \( i \)th position \( z_i \). This has to be done for all operator positions. For this purpose, it is useful to first consider time evolution of the \( V \) operator by \( m \) steps, do the \( \epsilon \) shift in \( n\theta \) and then put \( m = 0 \). In addition to ensuring operator ordering, the \( i\epsilon \) prescription also regulates the “contact divergence” of the correlation functions which arises from the fact that the two \( V \)'s (as also the two \( W \)'s) were initially located at the same point.

After \( n \)-iteration of conformal transformations on \( \{z_1, z_2\} \) and \( m \)-iteration of the same on \( \{z_3, z_4\} \), one obtains:

\[
    z_{1n} = \frac{a_n z_1 + b_n}{c_n z_1 + d_n}, \quad z_{2n} = \frac{a_n z_2 + b_n}{c_n z_2 + d_n},
\]

\[
    z_{3m} = \frac{a_m z_3 + b_m}{c_m z_3 + d_m}, \quad z_{4m} = \frac{a_m z_4 + b_m}{c_m z_4 + d_m},
\]

\[
    \eta = \frac{(z_{1n} - z_{2n})(z_{3m} - z_{4m})}{(z_{1n} - z_{3m})(z_{2n} - z_{4m})}.
\]

But for the \( \epsilon \) regulator, the numerator in the expression for \( \eta \) would be zero always and hence we expect \( \eta \) to be proportional to \( O(\epsilon^2) \). So \( \eta \) would be finite iff the denominator also is of the same order in \( \epsilon \). This happens when \( z_{1n} - z_3 \sim O(\epsilon) \) and \( z_{2n} - z_4 \sim O(\epsilon) \).

The situation is similar to the underformed CFT case, where again the expression for \( \eta \) is finite when \( (z_1(t, 0) \sim z_3(0, x)) \). In that case, this happens when \( (t \sim x) \). It is also
precisely at \((t = x)\), that the analytically continued curve \(\eta(t)\) crosses the real axis in the \(\eta\) complex plane. This will turn out to be true for our case as well.

We first analyze the case of continuous protocols for which the coefficients \((a_n, b_n, c_n, d_n)\) are given by eq. (2.15) for \(\alpha^2 > 1\). In this case, one can work out the condition for \(z_{1n} = z_3\). In the heating phase, this turns out to be

\[
\tan \left( \frac{\pi x}{L} \right) = \sqrt{\frac{\alpha - 1}{\alpha + 1}} \tanh(n \theta). \tag{3.7}
\]

Substituting this back into the expression of \(\eta\), then yields:

\[
\eta = \frac{\epsilon_{12} \epsilon_{34}}{\epsilon_{13} \epsilon_{24}} \implies \eta < 1 \text{ with } \epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4, \quad \eta > 1 \text{ with } \epsilon_1 > \epsilon_3 > \epsilon_2 > \epsilon_4. \tag{3.8}
\]

The above holds for values of \(\{n, x\}\) satisfying (3.7). We note here that initially \((n = 0)\), \(\eta\) vanishes as \(\epsilon \to 0\). Similarly, in the limit of large \(n\), the denominator is dominated by \(e^{n \theta}\) and therefore \(\eta \sim e^{-2n \theta} \sim O(\epsilon^2) \to 0\). Thus we can conclude that for this arrangement of \(\epsilon_i\)'s, which corresponds to the OTO configuration, \(\eta\) starts of from near the origin at \(n = 0\) and then crosses the real axis for a value greater than one, and therefore crosses the branch cut in the complex plane, before going to zero from the second sheet. It’s interesting to note that despite the difference in the dynamics, the nature of the analytic continuation is very similar to the case of the undeformed CFT in thermal equilibrium, where too the \(\eta(t)\) curve crosses the real axis at precisely the value given in (3.8) and goes to zero for large and small times.

There are however some important differences as well, which arise due to the fact that the drive explicitly breaks time-translation invariance as well as spatial homogeneity. This breakdown leaves an imprint on lower point functions as can be explicitly seen in the expectation values of one-point functions. For example, the relation

\[
\frac{2\pi x}{L} = \pi \pm \cos^{-1} \left( \frac{1}{\alpha} \right), \tag{3.9}
\]

determines the positions of the energy peaks (i.e. positions where the one-point function of the stress tensor maximizes) that is obtained from the one-point function of the stress-tensor [68].

These spatial and time translation non-invariance also leads to key differences at the level of the four point functions. First, the stroboscopic time \(n\) is discrete, unlike the real continuous time in the thermal CFT example. Thus the analytical continuation produces a discrete set of points instead of a continuous curve. Therefore, there may not be any integer \(n\) solution to eq. (3.7). This means that under the Möbius evolution, for a particular value of \(n\), \(\eta(n)\) is a point above the real axis (i.e. in the upper half of the complex \(\eta\) plane) while for the next value \((n + 1)\), it’s below the real axis (in the lower half plane), without touching the real axis for any integer \(n\). In such a case, it becomes difficult to conclude whether after crossing over, \(\eta\) lies in the first or second Riemann sheet. We will use eq. (3.7) as a criteria to decide that. Second, unlike the equation \(t = x\), which always has a solution for any \(x\), eq. (3.7) has solutions only for a range of values of the initial position \(x\). This range is: \(0 \leq \pi x/L \leq \tan^{-1} \left( \sqrt{\frac{\alpha - 1}{\alpha + 1}} \right)\).
To complete the discussion on kinematics, let us confirm that if $\eta$ crosses the branch-cut, then $\bar{\eta}$ does not, and vice-versa. The crucial point is that both of them should not simultaneously cross the branch-cut, to yield a non-trivial dynamics in the analytically continued correlator. The corresponding condition for the $\bar{\eta}(n)$ to cross the real axis is given by

$$\tan\left(\frac{\pi x}{L}\right) = -\sqrt{\frac{\alpha - 1}{\alpha + 1}} \tanh(n\theta). \quad (3.10)$$

Similar to (3.7), eq. (3.10) admit solutions in the range, $0 \leq \pi x/L \leq -\tan^{-1}\left(\frac{\alpha - 1}{\alpha + 1}\right)$. Taken together, the condition that either $\eta$ or $\bar{\eta}$ crosses the branch cut can only be satisfied for

$$\left[-\tan^{-1}\left(\sqrt{\frac{\alpha - 1}{\alpha + 1}}\right) \leq \frac{\pi x}{L} \leq \tan^{-1}\left(\sqrt{\frac{\alpha - 1}{\alpha + 1}}\right)\right]. \quad (3.11)$$

This range is a monotonically increasing function of $\alpha$, which closes off near the transition line $\alpha = 1$ and approaches $-L/4 \leq x \leq +L/4$ in the limit $\alpha \to \infty$. Clearly, for all $x$-values outside this range, the corresponding OTOC will display no non-trivial behaviour.

Unlike the scaling transformations, generated by the undeformed CFT Hamiltonian on the plane, the Möbius transformation has fixed points. The fixed points are by definition, points which do not evolve under the Möbius transformation i.e. $z_n = z$. From its definition, it follows that, at these points $\eta$ vanishes. Interestingly, the two end points of the range $x_{fp \pm} = \pm(L/\pi)\tan^{-1}\left(\sqrt{(\alpha - 1)/(\alpha + 1)}\right)$ are actually the fixed points of the Möbius flow. Thus, as shown in figure 1 below, when the two operators ($V$ and $W$) are on the same side of the fixed points, we see nontrivial $\eta(n)$ flow, while when the two operators are on opposite side of the fixed points, there is no such nontrivial behaviour. Thus, unsurprisingly, the existence of fixed points seem to control the nature of the OTOC behaviour in these systems.

We now study the explicit form of the cross ratio $\eta$. This can be done by substituting eq. (2.15) in eq. (3.6), and setting $m = 0$. A straightforward calculation yields

$$\eta = \frac{2\sin(\epsilon_{12})\sin(\epsilon_{34})}{\sqrt{\alpha + 1} \left(1 - e^{\frac{2\pi x}{L}}\right)} \frac{\left(\alpha - 2e^{\frac{2\pi x}{L}} + \alpha e^{\frac{4\pi x}{L}}\right)}{\cosh(n\theta + i\epsilon_{13}) + i\sqrt{\alpha - 1} \left(1 + e^{\frac{2\pi x}{L}}\right) \sinh(n\theta + i\epsilon_{13})} [1 \to 2, 3 \to 4]. \quad (3.12)$$

To determine the large time behaviour, let us note that the cross-ratio in (3.12), in the limit $n\theta \to \infty$, behaves as: $\eta = \epsilon_{12}\epsilon_{34}A(x)e^{-2n\theta}$

$$A(x) = -\frac{4 \left(\alpha \cos\left(\frac{2\pi x}{L}\right) - 1\right)}{\left(\sqrt{\alpha - 1}\cos\left(\frac{\pi x}{L}\right) - \sqrt{\alpha + 1}\sin\left(\frac{\pi x}{L}\right)\right)^2}. \quad (3.13)$$

We note that $A(x)$ is purely negative for all $|\alpha| > 1$. The above can subsequently be substituted to $F(\eta)$, when such a dynamical function is known. Let us use the simplest example: large-$c$ CFT, with $h_w/c$ held small and fixed and $h_v$ fixed. In this case, using the

---

Note that this expression is valid only when $x$ is not one of the fixed points. As mentioned previously at the fixed point, $\eta$ is identically zero for all $n$.
We see that the butterfly velocities are

\[ v_{B}(x) = \frac{A(x)}{A'(x)} = \frac{\lambda_{L} L}{\alpha \sqrt{\alpha^{2} - 1}} \left( \alpha \cos \left( \frac{2\pi x}{L} \right) - 1 \right), \quad t^{*}(x) = \frac{1}{\lambda_{L}} \left( \log c + \log(-A(x)) \right). \]  

By definition, we have

\[ F(\eta) = \left( \frac{1}{1 - \frac{24\pi h_{w}}{c\eta}} \right)^{2h_{v}}. \]  

In the limit where \( \frac{24\pi h_{w}}{c\eta} \ll 1 \), this becomes:

\[ F(t, x) = 1 + \frac{48i\pi h_{w} h_{v}}{c\epsilon_{12}\epsilon_{34}} \exp \left[ \frac{\lambda_{L}}{\lambda_{L}} \left( t - \frac{1}{\lambda_{L}} \log(-A(x)) \right) \right] + O(c^{-2}), \]  

where \( t = nT \) and the Lyapunov exponent \( \lambda_{L} \) is given by \( \lambda_{L} = 2\theta/T \). We note that this expression makes sense only if \( c\eta \gg 24\pi h_{w} \). It is customary to think that this is guaranteed by large \( c \); however, this is not the case if \( \alpha \gg 1 \), i.e. near \( \delta f T/L \simeq -n \). For the rest of this section, we shall focus on the regime where \( \alpha \) is such that \( c\eta \gg 24\pi h_{w} \); we note that this problem does not arise in the limit of high drive frequency where \( \alpha \) do not diverge for non-zero \( \delta f \).

In this limit, one can define a position dependent butterfly velocity \( v_{B}(x) \) and scrambling time \( t^{*}(x) \) as follows. We first define

\[ \phi(x, t) \equiv t - \frac{1}{\lambda_{L}} \log(-A(x)) - \frac{1}{\lambda_{L}} \log(c), \]  

so that the butterfly velocity is defined as the velocity at which \( \phi(x, t) \) is an extremum i.e. \( \frac{d\phi}{dt} = 0 \). This reduces to the usual definition when \( \phi \) is linear in \( x \), as is the case for 2D thermal CFT at large \( c \). The scrambling time is defined as the time at which \( \phi(t^{*}, x) \sim 0 \). This reduces to what is referred to as the “relevant scrambling time” in the 2D CFT case [77]. With these definitions, we get

\[ v_{B}(x) = \frac{\lambda_{L} A(x)}{A'(x)} = \frac{\lambda_{L} L}{2\pi \sqrt{\alpha^{2} - 1}} \left( \alpha \cos \left( \frac{2\pi x}{L} \right) - 1 \right), \quad t^{*}(x) = \frac{1}{\lambda_{L}} \left( \log c + \log(-A(x)) \right). \]  

Figure 1. Schematic representation of fixed points on the circle. The two red dots correspond to the two fixed points, while the blue dots label the position of the two operators. For non-trivial behaviour, both the operators have to be on the same side of the fixed points.
We note that the conformal symmetry of the driven CFT forces a global drive protocol to explicitly break spatial translational invariance. Such a translational symmetry breaking manifests itself as inhomogeneous structures in the correlation functions. This is a generic feature in such driven systems. In the context of OTOC, as we find here, this leads to a spatial dependence of the butterfly velocity. In contrast, the scrambling time has weak spatial dependence. This is due to the fact that \( \log(-A(x)) \) is usually small compared to \( \log c \) for a large-\( c \) system leading to a small spatial variation of the scrambling time.

The spatial variation of the butterfly velocity is shown in the top panels of figure 2. We find that depending on the drive frequency (for a fixed \( \delta f \)) or \( \delta f \) (for a fixed drive frequency), \( v_B(x) \) may either increase or decrease as a function of \( x \). The end points of these curves indicate the fixed points defined in eq. (3.11). The plot of \( v_B \) for a fixed \( x \) is shown in the bottom panels as a function of \( \delta f \) (left) and \( \omega_D \) (right). The horizontal dashed line indicate the schematic marks for \( c\eta \simeq 24\pi h_w \); between these lines, the values of \( |\alpha| \) are large enough to invalidate the expansion of \( F(x) \). We find that the sign of \( \alpha \) and
hence \( v_B(x) \) (eq. (3.18)) changes across these regions; thus the direction of the butterfly velocity at a given spatial point can be tuned by tuning either \( \omega_D \) or \( \delta f \). To the best of our knowledge, this feature has no analogue in homogeneous CFTs.

### 3.1.2 4-point OTOC for generic configurations

The initial positions of the operators taken in the above discussion, \( (x \text{ and } 0) \) are not the most general. In the undeformed CFT case, due to translational invariance, one can without loss of generality choose one operator to be at the origin. However, the drive is not spatially homogeneous and so the most general configuration would be when the two operators are placed at two arbitrary points \( (x_1 \text{ and } x_2) \), with the corresponding configurations on the plane.

\[
z_1 = e^{\frac{2\pi i x_1}{L}} = z_2, \quad z_3 = e^{\frac{2\pi i x_2}{L}} = z_4. \tag{3.19}
\]

Proceeding with the definition in (3.6), we obtain the following expression for the cross-ratio \( \eta \):

\[
\eta = \frac{\epsilon_{12}\epsilon_{34}(\alpha-2D_1+\alpha D_2^2)}{(D_1-D_2)\sqrt{\alpha^2-1}\cosh(n\theta+i\epsilon_{13})+i(\alpha+\alpha D_1 D_2-D_1-D_2)\sinh(n\theta+i\epsilon_{13})}[\epsilon_{13} \rightarrow \epsilon_{24}], \tag{3.20}
\]

where \( D_{1,2} = \exp[2\pi i x_{1,2}/L] \). The corresponding kinematics is no longer a function of \( (x_1 - x_2) \), as is also obvious from the expression of the cross-ratio above. As before, the \( \eta \) would vanish if the initial position of either of the operators is at the fixed point of the Möbius flow. Also, like before, \( \eta \) crosses the real axis at the point when \( (z_{1n} = z_3) \). The corresponding condition is

\[
\tanh(n\theta) = i \frac{(D_1 - D_2)\sqrt{\alpha^2 - 1}}{\alpha + \alpha D_1 D_2 - D_1 - D_2}. \tag{3.21}
\]

The corresponding condition for \( \bar{\eta} \) is:

\[
\tanh(n\theta) = -i \frac{(D_1 - D_2)\sqrt{\alpha^2 - 1}}{\alpha + \alpha D_1 D_2 - D_1 - D_2}. \tag{3.22}
\]

It is easy to check that setting \( D_1 = 0 \) in (3.21) and (3.22) yields (3.7) and (3.10). It is also straightforward to check that, when (3.21) is satisfied, one obtains: \( \eta = (\epsilon_{12}\epsilon_{34})/(\epsilon_{13}\epsilon_{24}) \). As before, one can solve (3.21) to find a range of values for \( x_1 \) and \( x_2 \). For this purpose, it’s convenient to define the variables \( X = (x_1 + x_2)/2 \) and \( x = (x_1 - x_2)/2 \). The range of these variables is \( (-X \leq x \leq X) \) and \( 0 \leq X < 2\pi \). In terms of these variables, eqs. (3.21) and (3.22) maybe rewritten as:

\[
\tanh(n\theta) = \pm \frac{\sin(\frac{2\pi x}{L})\sqrt{\alpha^2 - 1}}{\cos(\frac{2\pi x}{L}) - \alpha \cos(\frac{2\pi X}{L})}, \tag{3.23}
\]

where the \(+(-)\) signs correspond to the conditions from \( \eta \) and \( \bar{\eta} \) respectively. The range of the \( x_1 \) and \( x_2 \) for which this is valid is given by:

\[
1 \leq \frac{\sin(\frac{2\pi x}{L})\sqrt{\alpha^2 - 1}}{\cos(\frac{2\pi x}{L}) - \alpha \cos(\frac{2\pi X}{L})} \leq -1. \tag{3.24}
\]
Figure 3. Schematic representation of two regions marked in the region plot as A and B. The green part of the circle represents B while the blue part represents A. Fixed points on the circle are marked with two red dots. The brown dots label the positions of the two operators at $x_1$ and $x_2$. For non trivial behaviour, both the operators have to be on the same side of the fixed points (figure 5). In this figure both points are in region A.

Figure 4. Same as figure 3 but with both points in region B. This corresponds to the shaded center square of figure 5.

The explicit solution space of the above inequality is shown for a particular value of $\alpha$ in figure 5. Here $\gamma_1$ and $\gamma_2$ are the two fixed points. The solutions are given by the shaded region in figure 5. This corresponds to both points being on the same side of the fixed points. This is schematically represented in figures 3 and 4. Indeed this is as it should be, since $z_1(n) = z_3$ only if initially $z_1$ and $z_3$ were placed on the same side of the fixed points, else $z_1$ would end up at the fixed point and eq. (3.23) could not have been satisfied.

One can also obtain the large time behaviour, we take $n \to \infty$, which yields:

$$\eta = \epsilon_{12} \epsilon_{34} e^{-2\eta \theta} A(x_1, x_2)$$

where,

$$A(x_1, x_2) = 4 \frac{(\alpha - 2D_1 + \alpha D_1^2) (D_1 \to D_2)}{[(D_1 - D_2) \sqrt{\alpha^2 - 1} + i (\alpha + \alpha D_1 D_2 - D_1 - D_2)]^2}$$

$$= -4 \frac{(\alpha \cos(\frac{2\pi x_1}{L}) - 1)(\alpha \cos(\frac{2\pi x_2}{L}) - 1)}{(\sqrt{\alpha - 1} \cos(\frac{\pi x_1}{L}) - \sqrt{\alpha + 1} \sin(\frac{\pi x_1}{L}))^2(\frac{\pi x_1}{L}) + \sqrt{\alpha + 1} \sin(\frac{\pi x_1}{L}))^2}$$
Figure 5. Plot showing the allowed regions in $x_1$ and $x_2$. The green regions indicate the allowed ranges for $\eta$ while the brown regions indicate that for $\bar{\eta}$, $\gamma_1$ and $\gamma_2$ label the fixed points of the flow (These are the same points that are labelled as $x_{fp\pm}$ on the circle in figures 1, 3 and 4). All distances are scaled in units of $\frac{L}{\pi}$.

With this information, one can again find the OTOC for a large c CFT. The Lyapunov exponent is again given by (3.18). There is however an ambiguity in the definition of the butterfly velocity, since now there are two positions $x_1$ and $x_2$. Intuitively, it should be a measure of how fast information spreads between the two points. There are two possible definitions, consistent with this intuition. One way to define it is as: $v_B^- = 2 \frac{dx}{dt} |_{x_1=\text{constant}}$. The second possible definition would be as: $v_B = \frac{dx_2}{dt} |_{x_1=\text{constant}}$. This second definition also has the advantage that it reduces to the definition of the previous section.

$$
\frac{d}{dt} \left( t - \log \frac{A(x_1, x_2)}{\lambda_L} \right) \bigg|_{x_1=\text{constant}} = 0 \quad \implies \quad 1 - \frac{1}{\lambda_L} \left( \frac{\partial_2 A}{A} v_B \right) = 0
$$

$$
v_B = \lambda_L \left( \frac{\partial_2 A}{A} \right)^{-1} = \lambda_L \frac{L}{2\pi} \left( \alpha \cos \left( \frac{2\pi x_2}{L} \right) - 1 \right) \sqrt{\alpha^2 - 1} \quad (3.26)
$$

Remarkably, with the second definition, the butterfly velocity turns out to be independent of the position $x_1$ and exactly matches with the previous case $x_1 = 0$. This is a direct consequence of the fact $A(x_1, x_2)$ can be written as products of functions of $x_1$ and $x_2$ at late times as can be seen from eq. (3.25).

3.1.3 On the transition line: $\alpha = 1$

The analysis above assumes $|\alpha| > 1$. On the transition line, i.e. when $\alpha = 1$, we can write [68]:

$$
\frac{1}{p} \approx \alpha, \quad p \approx \frac{s}{T} \quad (3.27)
$$
and subsequently, the coefficients are:

\[ a_n = 1 - i p n T = 1 - i n s , \quad d_n = 1 + i n s , \quad b_n = - i n s = - c_n . \quad (3.28) \]

With these, one now obtains:

\[ \eta = \frac{4 s^2 e^{\frac{2 \pi i x}{L}} \epsilon_{12} \epsilon_{34}}{[1 - 2 i n s + 2 s \epsilon_{13} - e^{\frac{2 \pi i x}{L}} (1 + 2 i n s - 2 s \epsilon_{13})] [1 \to 2, 3 \to 4]} . \quad (3.29) \]

Proceeding as before, setting the \( z_1 n = z_3 \), we get:

\[ \cos \left( \frac{2 \pi i x}{L} \right) = \left( \frac{1 - 4 n^2 s^2}{1 + 4 n^2 s^2} \right) . \quad (3.30) \]

Using (3.30) in (3.29), we again obtain:

\[ \eta = \frac{\epsilon_{12} \epsilon_{34}}{\epsilon_{13} \epsilon_{24}} . \quad (3.31) \]

Evidently, in the limit \( n \to 0 \), \( \eta \sim \mathcal{O}(e^2) \) and in the limit of large \( n \), \( \eta \sim A(x) n^{-2} \epsilon_{12} \epsilon_{34} \).

Here \( A(x) = - \frac{e^{\frac{2 \pi i x}{L}}}{(1 + e^{\frac{2 \pi i x}{L}})^2} \). Hence the OTOC in this phase transition regime shows power law growth of \( \mathcal{O}(n^2) \). The absence of Lyapunov growth is also a feature of non-heating phase \( (\alpha^2 < 1) \), where one could obtain oscillatory behavior of OTOC in a similar manner by using appropriate \( a_n, b_n, c_n, d_n \) as discussed in section 2.1.

### 3.2 4pt OTOC in the discrete driven CFTs

We now consider the case of the discrete drive similar to the ones considered in [64]. The corresponding Hamiltonian is given in eq. (2.2). Let us begin with the initial operator locations: \( z_1 = z_2 = e^{\frac{2 \pi i x_1}{L}} \equiv D_1 \) and \( z_3 = z_4 = e^{\frac{2 \pi i x_2}{L}} \equiv D_2 \). The corresponding cross-ratio is given by

\[ \eta = - \frac{(D_1^2 - 1) (D_2^2 - 1) \sin(2 \phi \epsilon_{12}) \sin(2 \phi \epsilon_{34})}{[(D_1 D_2 - 1) \sinh(2 \phi (n + i \epsilon_{13}) + (D_1 - D_2) \cosh(2 \phi (n + i \epsilon_{13})) [\epsilon_{13} \to \epsilon_{24}]]} \quad (3.32) \]

Interestingly, the cross-ratio in this case vanishes if we take either \( x_1 = 0 \) or \( x_2 = 0 \). This is because \( x = 0 \) is a fixed point of the flow. This situation is schematically shown in the left panel of figure 6.

Repeating the analysis from the previous cases, we find that the condition for \( z_1 n = z_3 \) gives us:

\[ \tanh (2 n \phi) = \frac{D_1 - D_2}{1 - D_1 D_2} = - \frac{\sin \left( \frac{\pi (x_1 - x_2)}{L} \right)}{\sin \left( \frac{\pi (x_1 + x_2)}{L} \right)} . \quad (3.33) \]

As in the continuous drive example, for eq. (3.33) to have any solution, \( x_1 \) and \( x_2 \) have to be on the same side of the fixed points. Once again, as was the case previously, when this

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\(^6\) Note that, by definition \( n \in \mathbb{Z}_+ \cup \{0\} \). However, \( n T \), where \( T \) is the period of the drive, is a real-valued parameter. Thus, all smooth limits e.g. \( n \to 0 \) or large \( n \) is to be understood as \( n T \to 0 \) and \( n T \to \infty \), respectively.
Figure 6. Left: plot showing the allowed regions in $x_1$ and $x_2$ in discrete drive case. The green regions indicate the allowed ranges for $\eta$ while the grey regions indicate that for $\bar{\eta}$. Right: change in $v_B$ with $x_2$ when $x_1$ is kept fixed. All distances are scaled in units of $L/2\pi$. 

condition is met we see that the cross ratio just crosses the real axis at the point $\eta = \epsilon_{12}\epsilon_{34}/\epsilon_{13}\epsilon_{24}$. Thus, $\eta$ crosses the branch cut for OTOCs. Including the condition from $\bar{\eta}$, as before, we get the full range of values of $x_i$ for which we have non-trivial OTOCs. This is given by:

$$-1 \leq -\frac{\sin(\pi(x_1-x_2)/L)}{\sin(\pi(x_1+x_2)/L)} \leq 1.$$  

(3.34)

In the large $n$ limit,

$$\eta = \epsilon_{12}\epsilon_{34}e^{-4n\phi}A(x_1,x_2),$$  

(3.35)

where:

$$A(x_1,x_2) = -4\frac{(D_1^2-1)(D_2^2-1)(2\phi)^2}{[(D_1D_2-1)+(D_1-D_2)]^2}$$  

(3.36)

$$= -16\phi^2\frac{\tan(\pi x_2/L)}{\tan(\pi x_1/L)}.$$  

(3.37)

The large $c$ CFT analysis is the same, except for a change in the form of $A(x_1,x_2)$. In this case we get $\lambda_L = \frac{4\phi}{T_1+T_2}$ and (using the second definition for the butterfly velocity ($v_B = \frac{\lambda_L}{\partial_{x_2} A}$))

$$v_B = \lambda_L\frac{L}{2\pi}\sin\left(\frac{2\pi x_2}{L}\right).$$  

(3.38)

A plot of the butterfly velocity for the discrete drive protocol is shown in the right panel of figure 6; as before, this is independent of $x_1$.

4 Discussion

In this section we discuss the dual holographic perspective of our results. This is followed by a summary of our main results and conclusion in section 4.2.
4.1 Comments on the dual holographic perspective

An OTOC computation in a thermal CFT translates to a geodesic computation in a shock-wave geometry created by a massive particle (dual to one of the boundary operators), moving in a black hole background [77]. Crucial to obtaining the exponential time dependence is the existence of the black-hole horizon, in the presence of which there is an exponentially large blue shift in the energy of the massive particle.

The key difference of the above-mentioned scenario with ours is that in our case, OTOC is computed in the vacuum state. Under the $\text{SL}(2,\mathbb{R})$ drive, the vacuum state does not change. Thus the dual geometry, would be AdS, instead of a black-hole. It would be interesting to understand, how the exponential and oscillatory growths of the OTOC’s which we find in the heating and non-heating phases of the driven CFT emerge from the bulk in this case. A natural way to proceed would be to work out the trajectories that the deformed Hamiltonian generates in the bulk and use it to define a new coordinate system in which we can rewrite the AdS metric. One expects that in this new frame, an effective horizon, similar to the Rindler frame, emerges in the heating phase, which is the source of the exponential growth of the OTOC from the bulk.

Interestingly, preliminary investigations [103] seems to confirm this picture. When the parameters of the drive correspond to the heating phase, we find that the corresponding bulk frame is that of a $\text{AdS}_2$ black hole slicing of $\text{AdS}_3$, while the non-heating phase corresponds to a 2D global $\text{AdS}_2$ slicing of $\text{AdS}_3$. For parameter ranges corresponding to the transition line, it corresponds to a 2d Poincare patch $\text{AdS}_2$ slicing of $\text{AdS}_3$. Moreover, atleast for the discrete drive case discussed here in section 3.2, the corresponding 2d black hole temperature matches with the Lyapunov exponent, that we find in this paper. It would be nice to understand the emergence of the exponential and oscillatory growth of the OTOC as well from such a bulk picture [103].

We now offer some remarks and comments that are qualitatively relevant in understanding the holographic picture of driven systems of more general cases than what has been considered in this paper, for instance when the initial state is an excited state of the CFT.

Let us begin with the existing literature. First, note that, given an initial thermal state in the dual CFT and a particular drive protocol in any dimensions, the Holographic description is a well-posed problem in an asymptotically AdS-space. For example, in [92–94] a deformation of the following kind has been considered:

$$S = S_{J=0} + \int d^d x \sqrt{-\gamma} J(x) \mathcal{O}(x),$$

where $\mathcal{O}$ is a relevant operator in the CFT. The coupling $J(x)$ can be promoted to a function of time which defines the drive protocol. For small amplitudes of the drive, the physics is dissipation-dominated, since the system can dissipate the drive-energy into its thermal bath. At large amplitudes, however, this is not true and one obtains a rich phase structure. Correspondingly, low-point correlation functions were also explored in [92–94].

One subtlety lies in formulating an analogous question in the AdS-geometry within a Heisenberg picture. Typically, a classical field in the bulk geometry back-reacts on...
the geometry and as a result the geometry also becomes dynamical: correspondingly the CFT state evolves. Usually, given a boundary scalar deformation, one expects that the gravitational back-reaction becomes stronger as one approaches towards the IR, which results in the formation of a black hole in the geometry. Thus, intuitively, a heating phase is generically expected for an energy injection at the conformal boundary of the AdS-geometry, which also changes the bulk geometry from vacuum AdS background to an AdS-BH geometry. The existence of a non-heating phase is less intuitive. Note that, in [95], both local states and scattering states in global AdS were constructed for (regularized) operators with large conformal dimensions. This provides us with a natural framework in which the evolution of an operator may be naturally understood within Holography. It will be interesting to explore further how this possible scenario works out in detail for various kinds of correlation functions. Related to the existence of oscillatory correlation functions (e.g. an oscillating OTOC), note that thin time-like shells can undergo an oscillatory motion in AdS, which yields a corresponding behaviour in real-time for two point functions [96]. It was further noted that such oscillating configurations rely on when the perfect fluid on the shell is described by a polytropic equation of state. This is perfectly consistent with weak energy condition in the bulk geometry. It is therefore conceivable that by suitably adjusting the equation of state of the fluid on the collapsing shell, one interpolates between a heating phase and a non-heating phase. Introducing an inhomogeneity in the system can further be modelled by injecting the thin shell locally at the conformal boundary. This description is potentially more general than what we have considered in this paper, as the corresponding states are not the CFT vacua.

4.2 Summary of our results

In summary, we have studied four point out of time ordered correlation functions in large \( c \) driven CFTs in this paper. We find that these functions show qualitatively different behaviour in the heating and non-heating phases. In the heating phase, the OTOCs show an exponential behaviour for times sufficiently smaller than a suitably defined “scrambling time”, while in the non-heating phase we see a oscillatory behaviour. Both the “scrambling time” and a “butterfly velocity” in our case are position dependent. Moreover, unlike the undeformed CFT Hamiltonian, the exponential growth of the OTOC in the heating phase happens only when the operators are placed within a special range of spatial coordinate values determined by the fixed points of the Möbius flow. The existence of this spatial profile in the OTOC and the associated spatial dependence of the butterfly velocity are a consequence of the spatial inhomogeneity introduced by the drive.

The Lyapunov exponent that we obtain from our analysis for the large \( c \) CFT matches exactly with the group theoretic Lyapunov exponent obtained in [64] when applied for the continuous drive case, and differs by a factor of 4 from the discrete drive case. Of course these two notions are very different. The Lyapunov exponent obtained in this note is intrinsically quantum mechanical, as it is defined through a four-point function, while the group-theoretical Lyapunov may be interpreted as a classical notion measuring the sensitivity of the coordinate \( z_f \) of a quasi-particle to its initial position \( z_i \) on the complex plane after \( n \) iterations of the Möbius evolution. Therefore a priori there is no reason
to expect them to match. Indeed the matching seems to be primarily a consequence of dimensional analysis and the fact that $n$ always enters into all expressions as $n\theta$.

The existence of novel phases like the heating and non-heating phases is universal and exists for all Floquet CFTs [64, 65]. However, as we show in the appendix, for driven Ising CFT, the OTOC shows no exponential behaviour at large values of the stroboscopic time, even in the heating phase. We expect this to hold for other integrable CFTs as well. For example, one could explicitly check that in driven minimal models, at very late time the OTOC differs from TOC by a constant (which is completely determined by modular S-matrix) up to an additive exponential decay term. Thus our analysis shows that the OTO correlation function continues to be a good diagnostic of chaos, demarcating driven chaotic CFT Hamiltonians, from integrable Hamiltonians even in these out of equilibrium situations.

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**A OTOC in driven Ising CFT**

Here we briefly review the computation of OTOC in driven Ising CFTs. We find that the large stroboscopic time behavior of OTOC in this class of integrable systems remains similar to the large time behaviour of the thermal OTOC [77]. Let us consider 2d Ising CFT with Virasoro Identity operator $I$, spin operator $\sigma$ and energy operator $\epsilon$. The four point functions of these operators have explicit closed form expression as follows:

$$\frac{\langle \sigma\sigma\sigma\sigma \rangle}{\langle \sigma\sigma \rangle^2} = \frac{1}{2} \left( \frac{1}{(1 - \eta)(1 - \bar{\eta})} \right)^{1/8} \left[ \left( \frac{1 + \sqrt{1 - \eta}}{\sqrt{1 + \sqrt{1 - \eta}}} \right) \left( \frac{1 + \sqrt{1 - \bar{\eta}}}{\sqrt{1 + \sqrt{1 - \bar{\eta}}} \sqrt{1 - \sqrt{1 - \bar{\eta}}} \right) \right], \quad (A.1)$$

$$\frac{\langle \sigma\epsilon\sigma\epsilon \rangle}{\langle \sigma\sigma \rangle \langle \epsilon\epsilon \rangle} = \frac{2 - \eta}{2\sqrt{1 - \eta}} \frac{2 - \bar{\eta}}{2\sqrt{1 - \bar{\eta}}}, \quad (A.2)$$

$$\frac{\langle \epsilon\epsilon\epsilon\epsilon \rangle}{\langle \epsilon\epsilon \rangle^2} = \frac{1 - \eta + \eta^2}{1 - \eta} \frac{1 - \bar{\eta} + \bar{\eta}^2}{1 - \bar{\eta}}. \quad (A.3)$$

From the above expressions, we can see all these correlators have branch cuts in the $\eta : [1, \infty)$ regime, except $\langle \epsilon\epsilon\epsilon\epsilon \rangle$. We have seen that for large values of the stroboscopic time:

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8This is in the same spirit of computing thermal OTOC in minimal models [99–101]. The late time behavior of thermal cross ratio is very similar to driven cfts except the existence of novel spatial profile. Thus the analytic continuation of Euclidean four point functions provides a leading constant OTOC in the heating phase of driven cfts which is similar to thermal OTOC.
(\eta \sim \epsilon_{1234} A(x) e^{-2n\theta}).^{9} \text{ Now, using } (1 - \eta) \rightarrow (1 - \eta) e^{-2\pi i} \text{ and taking } \eta, \bar{\eta} \ll 1 \text{ at large time, one gets:}

\frac{\langle \sigma \sigma \sigma \rangle}{\langle \sigma \sigma \rangle^2} \sim \frac{e^{\pi/4}}{2} (\sqrt{\eta} + \sqrt{\bar{\eta}}) + O(\eta \bar{\eta}). \quad (A.4)

Hence at large } n \theta \text{ it goes to zero. Similarly, we obtain for the other correlation functions:

\frac{\langle \sigma \epsilon \sigma \epsilon \rangle}{\langle \sigma \sigma \rangle \langle \epsilon \epsilon \rangle} \sim -\left(1 - \frac{\eta}{2}\right)\left(1 - \frac{\bar{\eta}}{2}\right)\left(1 + \frac{\eta}{2}\right)\left(1 + \frac{\bar{\eta}}{2}\right)

\sim -1 + O(\eta \bar{\eta}, \ldots). \quad (A.5)

And for the } \langle \epsilon \epsilon \epsilon \epsilon \rangle \text{ correlator we get:

\frac{\langle \epsilon \epsilon \epsilon \epsilon \rangle}{\langle \epsilon \epsilon \rangle^2} = 1. \quad (A.6)

Thus we get the usual behaviour of Ising OTOC where at late time these saturate to constant values (0, -1 and 1 respectively). Thus even in driven } CFT_2, \text{ OTOC distinguishes integrable systems from chaotic systems like large } c \text{ } CFT_2.

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