1. Introduction

Let \( \phi : X \hookrightarrow \mathbb{P}^r \) be a nonsingular connected projective variety of dimension \( n \geq 1 \). Let \( U_X \) denote its universal covering. Recall that the fundamental group \( \pi_1(X) \) is large if and only if \( U_X \) contains no proper holomorphic subsets of positive dimension [Kol].

Throughout the note we assume that \( \pi_1(X) \) is large and residually finite, and the genus \( g(C) \) of a general curvilinear section \( C \subset X \) is at least 2. Let \( \mathcal{L} = \mathcal{L}_X \) denote the very ample bundle defining the map \( \phi \).

In Section 3, we construct the metric \( \Lambda_{\mathcal{L}} \) on \( U_X \). As an application, in the Appendix, we reproduce a prove of a conjecture of Shafarevich on holomorphic convexity when \( \pi_1(X) \) is residually finite (see [T2]).

In Section 4, we assume that \( \pi_1(X) \) is, in addition, nonamenable (see, e.g., [LS, p. 300]). We construct a Bergman-type metric with a weight, denoted by \( \Sigma_{\mathcal{L}} \), employing \( L^2 \) holomorphic functions on \( U_X \) (the volume form is \( dv_X \)). As a corollary, we obtain that the canonical bundle on \( X \), denoted by \( \mathcal{K}_X \), is ample.

In Section 5, assuming \( \pi_1(X) \) is nonamenable, we construct another Bergman-type metric with a weight, denoted by \( \beta \), employing \( L^2 \) sections of \( \mathcal{K}_X^q \) where \( q \) is a fixed large positive integer. We obtain a natural embedding into an infinite-dimensional projective space. This means \( \pi_1(X) \) is very large (see [T1]).

In 2010, Frédéric Campana asked the author whether one can establish the uniformization theorem in [T1] with assumptions on the fundamental group only. Since \( \pi_1(X) \) is very large provided it is nonamenable, we can employ the proof of our earlier uniformization [T1, Propositions 3.9, 4.2] to obtain the following

**Theorem (Uniformization).** Let \( X \hookrightarrow \mathbb{P}^r \) be a nonsingular connected projective variety of dimension \( n \) with large and residually finite fundamental group \( \pi_1(X) \). If \( \pi_1(X) \) is nonamenable then \( U_X \) is a bounded Stein domain in \( \mathbb{C}^n \).

One of the main ingredients in our proof of the uniformization [T1] was the well-known Griffiths’ uniformization theorem [G] that says that every algebraic manifold has plenty of Zarisky open sets which are quotients of bounded domains.

The uniformization problem is about uniformizing complex analytic functions by means of automorphic functions (Poincaré [Po], Weierstrass [W, pp. 95, 232, 304] and Hilbert’s 10th problem on the short list (see, e.g., [Gra])). In particular, the problem is to find the assumptions on \( X \) such that its universal covering is a bounded domain. For modern discussions of the problem, see Siegel [S2, Chap. 6,
2. Preliminaries

(2.1) A countable group $\Gamma$ is called *amenable* if there is on $\Gamma$ a finite additive, translation invariant nonnegative probability measure (defined for all subsets of $\Gamma$). Otherwise, $\Gamma$ is called *nonamenable.*

(2.2) *Diastasis.* The diastasis was introduced by Calabi [C, Chap. 2]. Let $M$ denote a complex manifold with a real analytic Kahler metric. Let $\Phi$ denote a real analytic potential of the metric defined in a small neighborhood $\mathcal{V} \subset M$. Let $z = (z_1, \ldots, z_n)$ be a coordinate system in $\mathcal{V}$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$ a coordinate system in its conjugate neighborhood $\bar{\mathcal{V}} \subset \bar{M}$. Let $(p,p)$ be a point on the diagonal of $M \times M$ such that the neighborhood $\mathcal{V} \times \bar{\mathcal{V}} \subset M \times M$ contains the point.

There exists a unique *holomorphic* function $F$ on an open neighborhood of $(p,p)$ such that $F(p,p) = \Phi_p$. Here $\Phi_p$ is the germ at $p \in M$ of our real analytic function, and $F_{(p,p)}$ is the germ of the corresponding holomorphic function (complexification of $\Phi_p$ [C, Chap. 2], [U, Appendix]).

One considers the sheaf $\mathcal{A}_{M \times M, (p,q)}$ of germs of real analytic functions on $M$, and the sheaf $\mathcal{A}_{M \times M}^C$ of germs of complex holomorphic functions on $M \times M$. For each $p \in M$, we get a natural inclusion $\mathcal{A}_{M \times M, (p,q)}^R \rightarrow \mathcal{A}_{M \times M}^C$, called a complexification. The above equality is understood in this sense.

Now, let $p$ and $q$ be two *arbitrary* points of $\mathcal{V}$ with coordinates $z(p)$ and $z(q)$. Let $F(z(p), \bar{z}(q))$ denote the complex holomorphic function on $\mathcal{V} \times \bar{\mathcal{V}}$ obtained from $\Phi$. The *functional element of diastasis* is defined as follows [C, (5)]:

\[(2.2.1) \quad D_M(p,q) := F(z(p), \bar{z}(p)) + F(z(q), \bar{z}(q)) - F(z(p), \bar{z}(q)) - F(z(q), \bar{z}(p)).\]

We get the germ $D_M(p,q) \in \mathcal{A}_{M \times M, (p,q)}^C$, and $D_M(p,q)$ is uniquely determined by the Kahler metric, symmetric in $p$ and $q$ and real valued [C, Prop. 1, 2]. The real analytic function generated by the above functional element is called the diastasis [C, p. 3]. The diastasis approximates the square of the geodesic distance in the *small* [C, p. 4]. For $\mathbb{C}^r$ with its unitary coordinates, $D_{\mathbb{C}^r}(p,q) = \sum_{i=1}^{r} |z_i(q) - z_i(p)|^2$.

The *fundamental property of the diastasis* is that it is inductive on complex submanifolds [C, Chap. 2, Prop. 6].

Now, let $Q \in M$ be a *fixed* point, and $z = (z_1, \ldots, z_n)$ a local coordinate system in a small neighborhood $\mathcal{V}_Q \subset M$ with origin at $Q$. The real analytic function $\tilde{\Phi}_Q(z(p), \bar{z}(p)) := D_M(Q,p)$ on $\mathcal{V}_Q$ is called the *diastatic potential at $Q$* of the Kahler metric. It is strictly plurisubharmonic function in $p$ [C, Chap. 2, Prop. 4].

The *prolongation over $M$* of a germ of diastatic potential $\tilde{\Phi}_Q(z(p), \bar{z}(p))$ at $Q$ is a function $P_M := P_{M,Q} \in H^0(\mathcal{A}_{M}^R, M)$ such that, for every $u \in M$, $P_M(u)$ coincides with $D_M(Q,u)$ meaning $D_M(Q,u)$, initially defined in a neighborhood of $Q$, can be extended to the *whole* $M$. Moreover, the germ of $P_M$ at $u$, in the coordinate system with origin at $u$, is the diastatic potential of our metric at $u$.

The prolongation over $M$ of the germ of diastatic potential is not always possible, e.g., there are no strictly plurisubharmonic functions on $\mathbb{P}^1$. Now, let $P^N$ be a
projective space with the Fubini-Study metric \((1 \leq N \leq \infty)\). For \(Q \in P^N\), we consider Bochner canonical coordinates \(z_1, \ldots, z_N\) with origin at \(Q\) on the complement of a hyperplane at infinity. By Calabi [Chap. 4, (27)],

\[
D_{P^r}(Q, p) = \log\left(1 + \sum_{\sigma=1}^{N} |z_\sigma(p)|^2\right).
\]

In the homogeneous coordinates \(\xi_0, \ldots, \xi_N\), where \(z_\sigma := \xi_\sigma/\xi_0\), we get

\[
D_{P^N}(Q, p) = \log \frac{\sum_{\sigma=0}^{N} |\xi_\sigma(p)|^2}{|\xi_0(p)|^2}.
\]

On the other hand, let \(U \subset C^n\) be a bounded domain. Let \(z_1, \ldots, z_n\) be a local system of coordinates with origin at a point \(Q \in U\). By the characteristic property of the diastasic potential (vanishing of some partial derivatives; see [Bo, pp. 180-181], [C, p. 3, p. 14], and [U, Appendix] where this property is explicitly stated):

\[
\partial^{[I]} P_{U,Q}(Q)/\partial z_I = \partial^{[I]} P_{U,Q}(Q)/\partial \bar{z}_I = 0 \quad (I := \{i_1, \ldots, i_n\} \text{ where } i_1, \ldots, i_n \geq 0),
\]

we get \(B(z, \bar{z})\) is the diastasic potential at \(Q\) of the Bergman metric on \(U\). Thus \(P_{U,Q} = \log B(z, \bar{z})\), \(P_{U,Q}\) is defined over the whole \(U\) and \(P_{U,Q}(u) = D_U(Q, u)\).

(2.3.1) Tower of coverings. We consider a tower of Galois coverings with each \(\text{Gal}(X_i/X)\) a finite group:

\[
X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow U, \quad \bigcap_i \text{Gal}(U/X_i) = \{1\} \quad (0 \leq i < \infty).
\]

We do not assume \(U\) is simply connected. Let \(\tau_i\) denote the projection \(U \to X_i\), \(\tau := \tau_0\), and \(\tau_{jk}\) denote the projection \(X_j \to X_k\) \((j \geq k)\).

The hyperplane bundle on \(P^r\) restricts to \(\mathcal{L}_X\), called a polarization on \(X\) (see, e.g., [Ti, p. 99]). Given our polarized Kahler metric \(g\) on \(X\), one can find a Hermitian metric \(h\) on \(\mathcal{L}_X\) with its Ricci curvature form equal to the corresponding Kahler form \(\omega_g\).

We consider the volume form of the Kahler form \(\omega_g\). In local coordinates \(z_1, \ldots, z_n\) on \(X\),

\[
dv_g = V_g \prod_{\alpha=1}^{n} \left(\frac{\sqrt{-1}}{2} \cdot dz_\alpha \wedge d\bar{z}_\alpha\right)
\]

where \(V_g\) is a locally defined positive function. We will employ the same volume form on all the coverings of \(X\). Also, we will employ the same Hermitian metric on all \(\tau_{j0}(\mathcal{L}_X)\) and \(\tau^*(\mathcal{L}_X)\).

(2.3.2) Positive reproducing kernels and Bergman pseudometrics. The fundamental property of any Berman-type pseudometric is the existence of a natural continuous map to a suitable projective space \(P(H^*)\) where the corresponding Hilbert space \(H\) has a reproducing kernel.

(2.3.2.1) Let \(M\) denote an arbitrary complex manifold. Let \(B(z, w)\) be a Hermitian positive definite complex-valued function on \(M \times M\) which means:

(i) \(B(z, w) = B(w, z)\), \(B(z, z) \geq 0\);

(ii) \(\forall z_1, \ldots, z_N \in M, \quad \forall a_1, \ldots, a_N \in C \Rightarrow \sum_{j,k} B(z_k, z_j)a_j\bar{a}_k \geq 0\).
If \( B(z, w) \) is, in addition, holomorphic in the first variable then \( B \) is the reproducing kernel of a unique Hilbert space \( H \) of holomorphic functions on \( U \) (see Aronszajn [A, p. 344, (4)] and the articles by Faraut and Korányi [FK, pp. 5-14, pp. 187-191]). The evaluation at a point \( Q \in M \), \( e_Q : f \mapsto f(Q) \), is a continuous linear functional on \( H \).

(2.3.2.2) Conversely, given a Hilbert space \( H (H \neq 0) \) of holomorphic functions on \( M \) with all evaluation maps continuous linear functionals then, by the Riesz representation theorem, for every \( w \in M \) there exists a unique function \( B_w \in H \) such that \( f(w) = \langle f, B_w \rangle (\forall f \in H) \) and \( B(z, w) := B_w(z) \) is the reproducing kernel of \( H \) (which is Hermitian positive definite).

If we assume, in addition, that \( B(z, z) > 0 \) for every \( z \) then we can define \( \log B(z, z) \) and a positive semidefinite Hermitian form, called the Bergman pseudo-metric

\[
ds_M^2 = 2 \sum g_{jk} dz_j d\bar{z}_k, \quad g_{jk} := \frac{\partial^2 \log B(z, z)}{\partial z_j \partial \bar{z}_k}.
\]

We get a natural map \( \Upsilon : M \rightarrow \mathbf{P}(H^*) \) whose image does not belong to a proper subspace of \( \mathbf{P}(H^*) \) as in [Kob2, Chap. 4.10, pp. 224-228].

As in [Kol, Chap. 7, pp. 81-84, Lemma-Definition 7.2], the function \( B(z, w) \) can be replaced by a section of a relevant bundle.

(2.3.3) Assuming \( \mathcal{K}_X \) is ample, we fix a large integer \( q \) such that for every \( i, \mathcal{K}^q_{X_i} \) is very ample (see [Kol, 16.5], [De]). Given the volume form \( dv_g \), the bundle \( \mathcal{K}^q_{X_i} \) is equipped with a Hermitian metric \( h_{\mathcal{K}^q_{X_i}} := h_{\mathcal{K}^q_{X_i}}^q \) where \( h_{\mathcal{K}^q_{X_i}} \) is a Hermitian metric on \( \mathcal{K}_{X_i} \) [Kol, 5.12, 5.13, 7.1.1]. Also, we get a Hermitian metric \( h_{\mathcal{K}^q_U} := h_{\mathcal{K}^q_U}^q \).

Further, let \( \psi_0, \ldots, \psi_N \) be an orthonormal basis of \( H^0(X, \mathcal{K}^q) \) with respect to \( dv_g \) and \( h_{\mathcal{K}^q_X} \) (see also (2.3.3.1) below). Locally \( \psi_\beta = g_\beta(z)(dz_1 \wedge \cdots \wedge dz_n)^q \). We get an embedding \( \sigma : X \hookrightarrow \mathbf{P}(H^0(X, \mathcal{K}^q)^*) \). We set

\[
dv_{X,\mathcal{K}^q} := \left( \sum_{\beta=0}^N |g_\beta|^2 \right)^\frac{1}{q} (\sqrt{-1})^n dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.
\]

The associated Ricci form \( \text{Ric}(dv_{X,\mathcal{K}^q}) \) (see [Kob2, Chap. 2.4.4]) is negative. It is known that if we pull back on \( X \) the Fubini-Study metric on \( \mathbf{P}(H^0(X, \mathcal{K}^q)^*) \) then its Kahler form differs only by the sign from \( \text{Ric}(dv_{X,\mathcal{K}^q}) \) [Kob2, Chap. 7.3, p. 363].

(2.3.3.1) Let \( \Omega^{(n,n)}_U \) denote the bundle of \((n, n)\)-forms. As in [Kol, Chap. 7.1.1.2], we fix a real homomorphism of bundles:

\[
\mathcal{H}_{U,q} : \mathcal{K}^q_U \otimes \bar{\mathcal{K}}^q_U \rightarrow \Omega^{(n,n)}_U \simeq \mathcal{K}^q_U \otimes \bar{\mathcal{K}}^q_U.
\]

If \( dv \) is a volume form then \( h(\ ,\ ) := \mathcal{H}_{U,q}/dv \) is a Hermitian metric on \( \mathcal{K}^q_U \). Given \( \mathcal{H}_{U,q} \), we consider the Hilbert space \( H = H_U \) of all square-integrable holomorphic weight \( q \) differential forms \( \omega \) on \( U \). By square-integrable (or \( L^2 \)), we mean

\[
\int_U \mathcal{H}_{U,q}(\omega \otimes \bar{\omega}) < \infty.
\]
We assume $H \neq 0$. If all evaluation maps are bounded then $H$ has a reproducing kernel as in the case of classical Bergman metric [FK, pp. 8-10, pp. 187-188]. Further, if the natural map

$$U \rightarrow \mathbf{P}(H^*)$$

is a holomorphic embedding then the metric on $U$, induced from $\mathbf{P}(H^*)$, is called its $q$-Bergman metric. It is denoted by $b_{U,q}$ and the corresponding tensor is denoted by $g_{U,q}$. Of course, they depend on the choice of $\mathcal{H}_{U,q}$.

If $W \subset \mathbf{P}(H^*)$ is a proper subspace of $\mathbf{P}(H^*)$ then the image of $U$ does not lie in $W$ in view of the definition of $H$ in (2.3.3.1), as in the case of classical Bergman metric ([Kob1, Chap. 7] or [Kob2, Chap. 4.10, p. 228]).

Similarly, one defines the Euclidean space $V_i$ of square-integrable holomorphic weight $q$ differential forms on $X_i$. In the sequel, we assume that each $V_i$ is equipped with a normalized inner product and a norm

$$\| \omega \| := \left( \frac{1}{Vol(X_i)} \int_{X_i} \mathcal{H}_{X_i,q}(\omega \otimes \overline{\omega}) \right)^{1/2}$$

(so that we will have the embedding $V_j \rightarrow V_k$ ($j < k$) of Euclidean spaces induced by pullbacks of differential forms provided $\mathcal{H}_{.,q}$ are compatible as in (2.3.3.3) below). Let $\mathbf{P}(V_i^*)$ denote the corresponding projective space with its Fubini-Study metric. If the natural map

$$X_i \rightarrow \mathbf{P}(V_i^*)$$

is a holomorphic embedding then the induced metric on $X_i$ is called its $q$-Bergman metric.

(2.3.3.2) Fundamental domain. We keep the assumptions of (2.3.1) and (2.3.3)-(2.3.3.1). We set $X_\infty := U$. We assume each $X_i$ ($0 \leq i \leq \infty$) has a metric giving an embedding in an appropriate projective space. For $j > i$, let $\Gamma_{ji} := Gal(X_j/X_i)$ denote the corresponding Galois group.

For $j > i$, a subset $\mathcal{D} \subset X_j$ is called a fundamental domain of $\Gamma_{ji}$ (see, e.g., [Kol, Chap. 5.6.2-5.8]) if $X_j = \cup \gamma \mathcal{D}$ and $\gamma \mathcal{D}$ is disjoint from the interior of $\mathcal{D}$ for $\gamma \neq 1$ ($\gamma \in \Gamma_{ji}$).

Now, we pick an arbitrary point $Q_j \in X_j$. We can consider the Dirichlet fundamental domain centered at $Q_j$ (we do not exclude the case $D_{X_j}(p, Q_j) = \infty$ below):

$$\mathcal{D}_{Q_j}(\Gamma_{ji}) := \{ p \in X_j | D_{X_j}(p, Q_j) \leq D_{X_j}(\gamma(p), Q_j), \forall \gamma \in \Gamma_{ji} \}.$$ 

Indeed, $D_{X_j}(q_1, q_2) = D_{X_j}(\gamma q_1, \gamma q_2)$ for $\forall \gamma \in \Gamma_{ji}$ because of the natural embedding (see (2.3.3)) of $X_j$ into the corresponding finite-dimensional or infinite-dimensional projective space where $\Gamma_{ji}$ acts by collineations.

The boundary of $\mathcal{D}_{Q_j}(\Gamma_{ji})$ is a subset of $\mathcal{D}_{Q_j}(\Gamma_{ji})$ where “$\leq$” is replaced by “$=$”. The boundary has measure zero (with respect to $dv_{g_j}$) because it is at most a countable union of measurable sets of measure zero. For example, the latter can be seen by passing to the complexifications.

(2.3.3.3) We fix a point $Q \in U$. Let $z = (z_1, \ldots, z_n)$ be a coordinate system in a small neighborhood $\mathcal{V}$ with origin at $Q_i := \tau_i(Q) \in X_i$ ($\forall i$). Let $\zeta_q$ be a positive bounded measurable function on $U$. Also, we assume $\zeta_q$ is bounded away from 0 on every compact subset of $U$. 

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For every $i$, we restrict $\zeta_q$ on the Dirichlet fundamental domain of $\Gamma_{\infty i}$ centered at $Q$. We, then, get a bounded positive measurable function on $X_i$. By abuse of notation, we denote the latter function on $X_i$ by the same symbol $\zeta_q$.

In the present note, we define a (so-called) compatible with the tower (2.3.1.1) sequence of homomorphisms:

$$\mathcal{H}_{X,q}, \ldots, \mathcal{H}_{X_i,q}, \ldots, \mathcal{H}_{U,q}$$

as follows. In local coordinates, let $\omega_e = \rho_e(dz_1 \wedge \cdots \wedge dz_n)^q$ be two weight $q$ forms on $U$ or on the Dirichlet fundamental domain of $\Gamma_{\infty i}$ ($i = 1, 2$). Then we define

$$\mathcal{H}_{e,q}(\omega_1, \omega_2) := (-2\sqrt{-1})^{-n}(-1)^{\frac{q(n-1)}{2}}\zeta_q g_1 \bar{g}_2 dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

provided $\mathcal{H}_{e,q}(\omega_1, \omega_2)$ are well-defined (see, e.g., the classical example (2.3.3.4) below). Set $c(n) := (-2\sqrt{-1})^{-n}(-1)^{\frac{q(n-1)}{2}}$. The homomorphisms are real because

$$\sqrt{-1} \cdot 2^{-1} dz_\alpha \wedge d\bar{z}_\alpha = dx_\alpha \wedge dy_\alpha \quad \text{where} \quad z_\alpha = x_\alpha + \sqrt{-1} y_\alpha.$$

We consider $\mathcal{H}_{e,q}$ defined as above with a suitable $\zeta_q$. Recall that we assume all Hilbert spaces are nontrivial. The corresponding Hilbert spaces have reproducing kernels as in the case of classical weighted Bergman spaces because $\zeta_q$ is bounded away from 0 on every compact subset of $U$ [FK, p. 10; p. 188].

We pick a point $u \in U$ and consider its image $\tau(u) \in X \subset \mathbb{P}^n$. Let $H_\infty \subset \mathbb{P}^n$ be a hyperplane at infinity ($\tau(u) \not\in H_\infty$). We choose coordinates $z_1, \ldots, z_n$ in $X \setminus H_\infty$ with origin at $\tau(u)$. We obtain the same coordinates in a small neighborhood of $u$ in $U$. Let $\xi$ be a measurable section of $\mathcal{K}_X^q$ such that

$$\xi|_{X \setminus H_\infty} = (dz_1 \wedge \cdots \wedge dz_n)^q.$$  

We denote the inverse image of $\xi$ on $U$ by the same symbol.

We consider the volume form

$$dv_{X,\mathcal{K}_X} = d\mu := \mu \cdot (\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

where $\mu$ is a positive (locally defined) $C^\infty$ function on $X$. Its inverse image on $U$, also denoted by $d\mu$, will be a $\pi_1(X)$-invariant volume form on $U$. Let $\eta_1$ be an arbitrary Hermitian metric on $\mathcal{K}_X$ ([Kol, 5.12, 5.13, 7.1.1] or [Kob2, p. 363]). Let $\eta_q := \eta_1^q$ be the corresponding Hermitian metric on $\mathcal{K}_X^q$. By abuse of notation, we denote by $\eta_q$ the lifting of $\eta_q$ on $\mathcal{K}_U^q$. In our local coordinates, let

$$\omega = g(dz_1 \wedge \cdots \wedge dz_n)^q$$

be an arbitrary weight $q$ form on $U$. We will compare $\eta_q(\omega, \bar{\omega})d\mu$ and $\mathcal{H}_{U,q}(\omega \otimes \bar{\omega})$.

Let $\rho$ be a positive measurable (with respect to $d\mu$) function on $U$ that is bounded away from 0 on every compact subset of $U$. One can consider the following two $L^2$-norms on sections $\omega \in H^0(U, \mathcal{K}^q)$:

$$\|\omega\|' := \sqrt{\int_U \rho \cdot \eta_q(\omega, \bar{\omega})d\mu} \quad \text{and} \quad \|\omega\|'' := \sqrt{\int_U \rho \cdot \mathcal{H}_{U,q}(\omega \otimes \bar{\omega})}.$$
As in [Kol, Chap. 5.13, Chap. 5.6-5.8] with obvious modifications, one can compare the corresponding norms. We have

\[ \eta_q(\omega, \bar{\omega})d\mu = c(n)\eta_q(\xi, \bar{\xi})|g|^2\mu dz_1 \wedge \cdots \wedge z_n \wedge d\bar{z}_1 \wedge \cdots \wedge \bar{z}_n \]

and

\[ H_{U,q}(\omega \otimes \bar{\omega}) = c(n) \varsigma_q |g|^2dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n. \]

Hence

\[ \frac{H_{U,q}(\omega \otimes \bar{\omega})}{\eta_q(\omega, \bar{\omega})d\mu} = \frac{\varsigma_q}{\eta_q(\xi, \bar{\xi})\mu} \quad \text{(on } U) \]

where \(\epsilon < \eta_q(\xi, \bar{\xi})\mu < C\) on \(U\) (with constants \(\epsilon, C > 0\)) because \(X\) is compact. We observe that the right-hand side of the last equality is independent of \(\omega\). Thus, if \(||\omega||' < \infty\) then \(||\omega||'' < \infty\) because \(d\mu\) and \(\eta_q\) are \(\pi_1(X)\)-invariant and \(\varsigma_q\) is bounded.

(2.3.3.4) Classical example (see, e.g., [Kr, Chap. III, Sect. 2-4]). Now, let \(X = C\) be a compact Riemann surface (algebraic curve) of \(g(C) \geq 2\). Let \(U := \Delta \subset C\) be a disk which is the universal covering of \(C\). We set \(\lambda_\Delta := (1 - |z|^2)^{-1}\). As in (2.3.1.1), we consider a tower of Riemann surfaces with the universal covering \(\Delta\).

In the local coordinate \(z\), let \(\omega_1 := g_1(z)(dz)^q\) and \(\omega_2 := g_2(z)(dz)^q\) be weight \(q\) holomorphic forms on \(U\) or on the interior of \(D_0(\Gamma_{\infty})\). We set

\[ H_{\Delta,q} : \mathcal{K}_\Delta^q \otimes \bar{\mathcal{K}}_\Delta^q \longrightarrow \Omega^{(1,1)}_{\Delta} \cong \mathcal{K}_\Delta \otimes \bar{\mathcal{K}}_\Delta, \quad \omega_1 \otimes \bar{\omega}_2 \mapsto \frac{\sqrt{-1}}{2} \lambda_\Delta^{2q-2} g_1(z)g_2(z)dz \wedge d\bar{z}. \]

In view of the correspondence between automorphic forms on \(\Delta\) and the differential forms on the Riemann surfaces, we get a compatible sequence of homomorphisms \(H_{.,q}\) with \(\varsigma_q = \lambda^{2-2q}\). Recall that \(\lambda^2 dx \wedge dy\) is a volume form on \(\Delta\) invariant under all holomorphic automorphisms of \(\Delta\).

(2.3.3.5) We will establish a version of a statement attributed to Kazhdan by Yau [Y1, p. 139]. For a survey of known results and historical remarks, see a recent article by Ohsawa [O, Sect. 5]. In case \(U_X\) is the disk \(\Delta\), the first proof was given by Rhodes [R]. Recently, McMullen has given a short proof for the disk [M, Appendix].

**Proposition 1.** With the above notation, we assume that \(U\) and all \(X_i\)’s have the \(q\)-Bergman metrics for an integer \(q\), and the \(H_{.,q}\)’s, defined with a help of the above \(\varsigma_q\), are compatible with the tower (2.3.1.1). Then the \(q\)-Bergman metric on \(U\) equals the limit of pullbacks of the \(q\)-Bergman metrics from \(X_i\)’s. Furthermore, let \(V\) be the completion of the Euclidean space \(E := \cup V_i\). Then there is a natural isomorphism of projective spaces \(P(H^*) \cong P(V^*)\).

**Proof.** Let \(b_{U,q}\) denote the \(q\)-Bergman metric on \(U\). Set \(\bar{b}_{U,q} := \lim sup b_i\), where \(b_i\) is the pullback on \(U\) of the \(q\)-Bergman metric on \(X_i\).

First, we establish the inequality \(\bar{b}_{U,q} \leq b_{U,q}\). We consider an open precompact exhaustion of \(U\), namely: \(\{U_\nu \subset U, \ \nu = 0, 1, 2, \ldots\}\), where \(U = \cup U_\nu\), each \(U_\nu\) is compact and \(\bar{U}_\nu \subset U_{\nu+1}\). We can take \(U_\nu\) to be the interior of the Dirichlet fundamental domain of \(\Gamma_{\infty}\) centered at a point \(Q\). Let \(b(\nu)\) denote the \(q\)-Bergman metric on \(U_\nu\). It is well known that \(b_{U,q} = \lim_{\nu \to \infty} b(\nu)\).

Given \(U_\nu\), the restriction of \(\tau_i\) on \(U_\nu\) is one-to-one for \(i \gg 0\), since \(\pi_1(X)\) is residually finite. Further, \(b(\nu) > b_i|U_\nu\) for all \(i > i(\nu)\). This establishes that \(b_{U,q} \geq \bar{b}_{U,q}\). So, we have \(\bar{b}_{U,q} = \lim b_i\).
Given the metric $\tilde{b}_{U,q}$, one can define a natural map

$$\Upsilon_V : U \rightarrow \mathbf{P}(V^*)$$

by assigning to each point $u \in U_X$ the hyperplane in $V$ consisting of elements in $V$ vanishing at $u$ (compare [Kob2, Chap. 4.10, p. 228]). This map arises from the map of the fundamental domain $U_\nu$ into the corresponding $\mathbf{P}(V_\nu^*)$ (for $\nu = 0, 1, \ldots$). We consider $\mathbf{P}(V^*)$ with its Fubini-Study metric. Since $\mathcal{K}^q_{X_\nu}$ is very ample for every $\nu$, $\Upsilon_V$ is an embedding.

By assumption, we have an embedding $\Upsilon_H : U \hookrightarrow \mathbf{P}(V^*)$. Furthermore, we have a natural embedding $\Upsilon_{V,H} : \mathbf{P}(V^*) \hookrightarrow \mathbf{P}(H^*)$ and $\Upsilon_H = \Upsilon_{V,H} \Upsilon_V$. Since the image of $\Upsilon_H$ does not belong to a proper subspace of $\mathbf{P}(H^*)$, we obtain the proposition.

In (2.4)-(2.6) below, we assume that $U_X$ is equipped with a $\pi_1(X)$-invariant real analytic Kahler metric $\Lambda_X$. We consider the induced Riemannian metric on $U_X$ with the volume form $dv_\Lambda$. In Section 3, we shall recall a construction of $\Lambda_X$ from [T2].

(2.4) **Heat kernel** (see, e.g., [G]). We consider the heat kernel $p(s, x, y)$ on $U_X$ where $(s, x, y) \in (0, \infty) \times U_X \times U_X$. As a function of $s$ and $x$, the function $p(s, x, y)$ is the smallest positive fundamental solution of the heat equation $\partial p/\partial s = \Delta_x p$ where $\Delta_x$ is the Laplace-Beltrami operator on $U_X$ with its Riemannian metric. It is known that on $U_X$ we have (see, e.g., [G] and references therein):

1. $p(s, x, y)$ is a $C^\infty$ function of all three variables $(s, x, y) \in (0, \infty) \times U_X \times U_X$
2. $p(s, x, y) = p(s, y, x)$ symmetry
3. $p(s, x, y) > 0$ positivity
4. $\int_{U_X} p(s, x, y) dv_\Lambda(x) = 1$ stochastic completeness (for general manifolds, we have only $\int_M p(s, x, y) dv_\Lambda(x) \leq 1$)
5. let $\{U_\nu\}_{\nu \in \mathbb{N}} \subset U_X$ be a precompact open exhaustion with smooth boundaries, i.e., $\tilde{U}_\nu \backslash U_\nu$ are smooth and each $\tilde{U}_\nu$ is compact; let $p_{U_\nu}$ be the smallest positive fundamental solution of the heat equation on $U_\nu$ and set $p_{U_\nu}|U_X \backslash \tilde{U}_\nu := 0$; then

$$p_{U_\nu} \leq p_{U_{\nu+1}} \quad \text{and} \quad p(s, x, y) = \lim_{\nu \to \infty} p_{U_\nu}.$$  

When $\tilde{U}_\nu \subset U_X$ is a fundamental domain of a covering of $X$ then $\tilde{U}_\nu \backslash U_\nu$ is not smooth. However, one can approximate the fundamental domains by precompact open regions with smooth boundaries.

The presence of the Gaussian exponential term in the heat kernel estimates is one of the properties of the heat kernel which little depends on the structure of the manifold in question and reflects the structure of the heat equation.

(2.5) **Plurisubharmonic** (psh) and **pluriharmonic** (ph) functions on $U_X$. A function $f$ defined in a neighborhood of a point $p \in U_X$ will be psh (respectively ph) if and only if the following holds. For an arbitrary tangent vector $v$ to $U_X$ at $p$, we consider $\tau_i(p)$ and $\tau_i(v)$ on $X_i \subset \mathbb{P}^{r_i}$ ($i \gg 0$). We assume the latter embedding is nondegenerate and $r_i \gg 0$. We take a general curvilinear section $C_i \subset X_i$ tangent to $\tau_i(v)$ at $\tau_i(p)$.
By our assumption, \( C_i \) will be a nonsingular connected curve of genus at least 2. Indeed, the corresponding linear system has no fixed components because \( \pi_1(X) \) is large, in particular, \( X \) contains no rational curves. Moreover, the linear system is not composite with a pencil. Hence we can apply Bertini’s theorem to obtain a nonsingular connected curve. We, then, consider a connected open Riemann surface \( \tau^{-1}(C_i) \subset U_X \) (Campana-Deligne theorem [Kol, Theorem 2.14]). Thus, the function \( f \) is psh (ph) at \( p \) if and only if its restriction on \( \tau^{-1}(C_i) \) is subharmonic (harmonic) at \( p \).

Let \( u \) be a pluriharmonic function on \( U_X \). Since \( U_X \) is simply connected, there is a holomorphic function \( f = u + \sqrt{-1} \tilde{u} \) on \( U_X \) where \( \tilde{u} \) is also pluriharmonic [FG, Chap. VI, p. 318].

(2.6) An easy generalization of some results of Lyons-Sullivan [LS, Theorem 3′] and Toledo [To, Lemma 1]. It follows from [To, Lemma 1] that the space of bounded pluriharmonic functions on \( U_X \) is infinite dimensional. In his argument, we replace harmonic functions by pluriharmonic functions. As in [To, Lemma 1], the important step is the construction of the map

\[
\tilde{\varphi} : L^\infty(U_X) \longrightarrow \text{bounded harmonic functions on } U_X
\]

by Lyons-Sullivan [LS, Theorem 3′, p. 311]. In view of (2.5), \( \tilde{\varphi} \) produces pluriharmonic functions on \( U_X \). Also, the proof of Toledo’s lemma [To, Lemma 1] shows that that the linear span of positive pluriharmonic functions is infinite dimensional.

3. Metric \( \Lambda_L \)

The metric \( \Lambda_L \) was suggested by a problem of Yau [Y1, Sect. 6, p. 139] who proposed to study \( \lim_{t \to \infty} \frac{1}{t} g_{X,K^t} \) when \( K \) is the ample canonical bundle on \( X \). We do not exclude the case when \( \dim X \geq 2 \) and \( \pi_1(X) \) is Abelian.

The metric \( \Lambda_L \) on \( U_X \) is a generalization of the classical Poincaré metric though it is not necessary a Bergman-type metric if \( \dim X \geq 2 \). It will depend on the fixed very ample bundle \( L_X \) defining the embedding \( \phi : X \hookrightarrow \mathbb{P}^r \). We will define a real analytic potential at every point of \( U_X \).

(3.1) First, we will consider the case: \( C = X \hookrightarrow \mathbb{P}^r \) were \( C \) is a connected nonsingular projective curve of genus \( g(C) \geq 2 \). We will assume the embedding is given by a very ample line bundle \( L_C \) such that

\[
L_C \subset K_C^m,
\]

where \( K_C \) is the canonical bundle and \( m \) is a suitable integer. We get Bergman-type metrics on \( C \) corresponding to \( L_C \) and \( K_C^m \) (see [Y1, Sect. 6, p. 138] and [Ti, p. 99]) and the Poincaré metric on \( \Delta \). Since \( \Delta \) is homogeneous,

\[
B_{\Delta,K^t}(z, \zeta) = c(t) B_{\Delta,K^t}(z, \zeta), \\
B_{\Delta,K}(z, \zeta) = \pi^{-1}(1 - z\bar{\zeta})^{-2},
\]

where \( t \gg 0 \) is an integer, \( c(t) \) is a known constant depending on \( t \) only, and \( B_{\Delta,K^t} \) denotes the \( t \)-Bergman kernel (see, e.g., [FK, p. 9], [Kol, (7.7.1)]). It follows

\[
\lim_{t \to \infty} \frac{1}{t} g_{\Delta,K^t} = \left( \lim_{t \to \infty} \frac{1}{t} \frac{\partial^2 \log B_{\Delta,K^t}(z, \zeta)}{\partial z \partial \bar{z}} \right) dz d\bar{z} = g_{\Delta,K}.
\]
(3.2) Let $C$ be a sufficiently general nonsingular connected curvilinear section of $X$ ($g(C) \geq 2$). We consider the inverse image of $C$ on $U_X$. By the Campana-Deligne theorem [Kol, Theorem 2.14], we obtain a connected open Riemann surface $R = R_C \subset U_X$ in place of the disk $\Delta$. We would like to construct a metric on $U_X$ whose restriction on $R$ is well understood. Set $\Gamma := Gal(\Delta/R)$. Let

$$\mathcal{F} := \{ z \in \Delta \mid |Jac_\gamma(z)| \leq 1, \gamma \in \Gamma \}, \quad \{ \mathcal{F} := \{ z \in \bar{\mathcal{F}} \mid |Jac_\gamma(z)| < 1, \forall \gamma \neq 1 \}$$

be a fundamental domain of $R$ and the interior of the fundamental domain.

(3.2.1) High powers of $L_C := L_X|_C$ are squeezed between powers of the canonical bundle on $C$. For $t \geq 1$, let $b_{R,t}$ denote the $t$-Bergman metric on $R$ with $\mathcal{H}_{\cdot,t}$ as in the classical example (2.3.3.4). Since $R$ is an open Riemann surface, $L_R$ and $\mathcal{K}_R$ are free bundles. We do not assume $\mathcal{K}_C$ is very ample.

Let $D_{b,R,t}$ denote the functional element of diastasis of $b_{R,t}$ at an arbitrary point of $R$. We lift $b_{R,t}$ and $D_{b,R,t}$ on $\Delta$ and get $D_{b,R,t} \leq D_{b,\Delta,t}$ (locally at an arbitrary point of $\Delta$) by Proposition 1. Furthermore, it follows the convergence of the corresponding holomorphic functions on $R \times \bar{R}$. Hence on $\Delta$:

$$\lim_{t \to \infty} \frac{1}{t} D_{b,R,t} \leq D_{b,\Delta,1} = D_{b,\Delta}.$$

Set $b_R := \lim_{t \to \infty} \frac{1}{t} b_{R,t}$. It will be a real analytic $Gal(R/C)$-invariant metric on $R$. We get the metric $b_R$ whose diastastic potential $P_{b,R} := P_{b,R,a}$, where $a \in R$ is the image of the origin $0 \in \Delta$, has the prolongation over $R$, i.e., $P_{b,R}$ is a function on $R$.

One can replace $b_{R,1}$ by $b_{R,m}$, where $m$ is a sufficiently large fixed number, and repeat the previous argument with $b_{R,m,t}$ in place of $b_{R,t}$. As before, we set

$$b_{R,m} := \lim_{t \to \infty} \frac{1}{t} b_{R,m,t}.$$

We observe that $L_{b,R}^{tm} \subseteq \mathcal{K}_R^{m tm}$, where $\{m_i\}_{i \in \mathbb{N}}$ is a nondecreasing sequence of positive integers. Further,

$$b_{R,m} = \lim_{t,i \to \infty} \frac{1}{tm_i} b_{R,m,tm_i} \quad \text{and} \quad b_R = \lim_{t,i,m \to \infty} \frac{1}{mtm_i} b_{R,m,tm_i}.$$

We will denote a functional element of the diastasis of $b_{R,m,t}$ by $D_{b,R,m,t}$. As before, the diastastic potentials of $b_{R,m,t}$ and $b_{R,m}$ are functions on $R$.

(3.2.2) Now, let $L_R$ denote the inverse image of $L_C$ on $R$. For all $t \gg 0$, we have $\mathcal{K}_R \subseteq L_R^t$. Let $H^0_{(2)}(R, \mathcal{K})$ be the 1-Bergman space on $R$ giving the embedding

$$\nu_\mathcal{K} : R \hookrightarrow P([H^0_{(2)}(R, \mathcal{K})]^*) \quad (\varsigma_1 = 1).$$

Let $H_t := H^0_{(2)}(R, \mathcal{L}^t)$ ($t \gg 0$) be the Hilbert space of $L^2$-integrable (with respect to $h^t$ and the restriction of $dv_g$ on $R$) holomorphic global sections. Every $\omega$ in $H^0_{(2)}(R, \mathcal{K})$ belongs to $H_t$ by (2.3.3.3) ($\varsigma_1 = 1$ in the classical example (2.3.3.4)). Hence $H_t \neq 0$. We observe that $\omega$ is a function on $R$ because $\mathcal{K}_R$ is trivial.
As with the Bergman kernel, we obtain a natural map employing the reproducing kernel of $H_t$

$$\iota_{\mathcal{L}^t} : R \to \mathbb{P}(H_t^*) .$$

The map $\iota_{\mathcal{L}^t}$ separates points and tangents on $R$. The latter means that, for every point $p \in R$ and the tangent vector $v$ on $R$ at $p$, there exists a section $\omega \in H^0_{(2)}(R, \mathcal{K})$ such that $\omega(p) = 0$ and $\frac{\partial \omega}{\partial v}(p) \neq 0$. Hence $\iota_{\mathcal{L}^t}$ also separates points and tangents, i.e., $\iota_{\mathcal{L}^t}$ is an embedding as well.

We denote by $g_{R,t}$ the metric on $R$ induced from the Fubini-Study metric on $\mathbb{P}(H_t^*)$ via the map $\iota_{\mathcal{L}^t}$. We denote by $\frac{1}{t}g_{R,t}$ the inverse image of $\frac{1}{t}$-multiple of the Fubini-Study metric on $\mathbb{P}(H_t^*)$ ($t \gg 0$). Formally, we define

$$\Lambda_R := \lim_{t \to \infty} \frac{1}{t}g_{R,t}.$$ 

We will show the limit exists and $\Lambda_R$ is a real analytic $Gal(R/C)$-invariant Kahler metric on $R$. Let $D_{R,t}$ denote the functional element of diastasis of $g_{R,t}$ at an arbitrary point of $R$ ($t \gg 0$).

Since $\mathcal{L}^t_R \subset \mathcal{K}^t_R$ for all $t \gg 0$ and the fixed large integer $m$, we will be able to show that $D_{R,t}$ is bounded by the corresponding $D_{b,R,m,t}$. We will compare the reproducing kernel $B_t$ of $H_t$ and the $tm$-Bergman kernel $B_{m,t}$ of $H^0_{(2)}(R, \mathcal{K}^{mt})$.

Let $0 \in \Delta_1 \subset \Delta$, where $\Delta_1$ is a closed disk with center 0 whose radius is close to 1. Let $\kappa : \mathcal{O}_R \hookrightarrow \mathcal{K}_R$ be a natural inclusion, $\xi := \kappa(1)$. On $\Delta \setminus \Delta_1$, we have an estimate

$$(*) \quad \lambda^{2-2mt} < h^t(\xi, \xi) \cdot V_g$$

provided our $m \gg 0$ (see (2.3.1) and (2.3.3.3)-(2.3.3.4)). Indeed, $\lambda^{-2m}$ can be made arbitrary small on $\Delta \setminus \Delta_1$ by choosing a sufficiently large $m = m(\mathcal{L}_C)$ (depending on the radius of $\Delta_1$); recall that $h$ and $V_g$ depend on $\mathcal{L}_X$ only. Furthermore, $h$ and $V_g$ are bounded from below and from above by positive constants because the curve $C$ is compact (see (2.3.3.3)).

By (2.3.3.3), if $\omega \in H_t$ then $\omega \in H^0_{(2)}(R, \mathcal{K}^{mt})$. Hence we have a natural inclusion of linear spaces:

$$H_t \subset H^0_{(2)}(R, \mathcal{K}^{mt}).$$

Recall that $\mathcal{K}^{mt}_R$ is a free bundle. Hence $B_{m,t}$ and $B_t$ are functions on $R$.

The Poincaré series $\sum_{\gamma \in \Gamma} |Jac_{\gamma}(z)|^2$ is uniformly convergent. As well known, this yields that every compact subset of $\Delta$, in particular $\Delta_1$, is covered by a finite number of $\gamma \tilde{F}$ ($\gamma \in \Gamma$), where $\tilde{F}$ is the fundamental domains defined in (3.2).

Similarly, for each $i$, set $C_i := \tau_0^i(C)$ where $C \subset X \subset \mathbb{P}^r$ is a general curvilinear section. Let $Gal(R/C_i)$ be the corresponding Galois group, and let $\tilde{F}_i \subset R$ be the fundamental domain of $Gal(R/C_i)$. Relative Poincaré series are discussed, e.g., in [Dr]. As above, for each $i$, every compact subset of $R$ is covered by a finite number of $\gamma \tilde{F}_i \subset R$ ($\gamma \in Gal(R/C_i)$). Also, the estimate ($*$) holds on $R \setminus \Psi$ where $\Psi \subset R$ is a compact subset.

Now, we take an arbitrary $\omega \in H_t$. Let $\| \cdot \|$ and $\| \cdot \|^\prime$ denote the norms in $H_t$ and $H^0_{(2)}(R, \mathcal{K}^{mt})$, respectively. If $\omega$ arises from a form on $C_i$ then $\| \omega \|^\prime \leq \| \omega \|$ by (2.3.3.3) and the above estimate ($*$). Indeed, the inequality ($*$) can be violated on at most a finite number of fundamental domains of $C$, and $\Delta$ (and $R$) are
covered by infinitely many corresponding fundamental domains. By Proposition 1, \( \| \omega \|'' \leq \| \omega \|' \) for all \( \omega \in H_t \). Hence \( B_t \leq B_{m,t} \) on \( R \) (see, e.g., [FK, p. 6], [Sha, Sect. 51]). Thus

\[
D_{R,t} \leq D_{b,R,m,t} \ (t \gg 0) \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{t} D_{R,t} \leq \lim_{t \to \infty} \frac{1}{t} D_{b,R,m,t}
\]

where the diastasises are the logarithms of the corresponding kernels.

It follows the uniform convergence of the corresponding holomorphic functions on \( R \times \bar{R} \). Furthermore, \( \frac{1}{t} D_{R,t} \) generates a global function on \( R \) (see, e.g., [FK, p. 6], [Sha, Sect. 51]). Thus

\[
D_{R,t} \leq D_{b,R,m,t} \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{t} D_{R,t} \leq \lim_{t \to \infty} \frac{1}{t} D_{b,R,m,t}
\]

Proposition-Definition 2. With the notation of (3.2), let \( \Lambda_R := \lim_{t \to \infty} \frac{1}{t} g_{R,t} \). Then \( \Lambda_R \) is a real analytic \( \text{Gal}(R/C) \)-invariant Kahler metric. The diastasic potential \( P_R := P_{R,a} \), where \( a \in R \) is the image of the origin \( 0 \in \Delta \), has the prolongation over \( R \).

(3.3) Now, we return to the situation in (3.1) with \( U = U_X \). For each positive integer \( t \), the Hermitian metric \( h \) on \( L_X \) induces a Hermitian metric \( h^t \) on \( L^t_X \) as well as on all inverse images of \( L^t_X \) on the coverings of \( X \).

(3.3.1) We choose an orthonormal basis \( (s^t_0, \ldots, s^t_r) \) of \( H^0(X, L^t_X) \) with respect to \( dv_g \) and \( h^t \). We have an inner product and a natural embedding:

\[
\langle s^t_\alpha, s^t_\beta \rangle := \int_X h^t(s^t_\alpha, s^t_\beta) dv_g; \quad \phi_{X,t} : X \hookrightarrow \mathbb{P}^{r_t} := \mathbb{P}(H^0(X, L^t_X))^*.
\]

Let \( g_{FS} \) denote the corresponding standard Fubini-Study metric on the projective space. As in Yau [Y1, Sect. 6, p. 139] (see also Tian [Ti]), the \( \frac{1}{t} \)-multiple of \( g_{FS} \) on \( \mathbb{P}^{r_t} \) restricts to a Kahler metric on \( X \):

\[
g_{X,t} := \frac{1}{t} \phi_{X,t}^* g_{FS}.
\]

(3.3.2) We consider the finite coverings of \( X \). The bundles \( \tau^*_i L_X \ (1 \leq i < \infty) \) are ample by the Nakai-Moishezon ampleness criterion. However, \( \tau^*_i L_X \)'s are not necessary very ample bundles. Let \( \{ m_i \} \ (i \geq 1) \) be a nondecreasing sequence of positive integer such that the bundle \( (\tau^*_i L_X)^{m_i} \) is very ample. Then the bundle \( (\tau^*_i L_X)^{tm_i} \) defines a natural embedding

\[
\phi_{X_i,tm_i} : X_i \hookrightarrow \mathbb{P}^{r_{tm_i}}
\]

into an appropriate projective space. We get a metric \( g_{X_i,tm_i} := \frac{1}{tm_i} \phi_{X_i,tm_i}^* g_{FS} \) on \( X_i \) and the corresponding diastasic diastasises. We will establish that the functional elements of the diastasises converge at a point \( p \in U_X \), and we will obtain a real analytic strictly plurisubharmonic functional element at \( p \). For points of \( U_X \), such functional elements will define the desired Kahler metric \( \Lambda_{L_X} \) on \( U_X \).
Let $H_\infty$ denote the hyperplane at infinity in $\mathbb{P}^r$. We can and will assume that the functional elements of diastasises generate functions on the preimages of $X \setminus H_\infty$ on $X_i$'s (see [C, Chap. 4]).

We assume the point $p$ does not lie at infinity. We consider a small compact neighborhood $G \subset U_X$ of $p$. The pullbacks on $U_X$ of the above functions produce functions on $G$.

First, we will establish the pointwise convergence of the pullbacks of this functions in a small neighborhood of $p$. We will make use of the fundamental property of diastasis.

We take a general curvilinear section $C \subset X \subset \mathbb{P}^r$ whose inverse image on $U_X$ contains $p$. This inverse image will be a connected open Riemann surface by the Campana-Deligne theorem [Kol, Theorem 2.14]. We, then, apply Proposition-Definition 2.

The pointwise convergence is independent of the curve $C$ because we have the metric $g_{X_i,tm_i}$ on each $X_i$ ($0 \leq i < \infty$). By the fundamental property of diastasis, we get the same functional element of diastasis at $\tau_i(p) \in \tau_i^{-1}(C)$, independent of the choice of the Riemann surface $\tau_i^{-1}(C)$, that is, of the curve $C$.

We obtain a function on $G \times G$. By Hartogs’ theorem (separate analyticity implies joint analyticity), this function will be holomorphic. In particular, it follows the functional elements of diastasises converge uniformly on $G$ by Dini’s monotone convergence theorem, and we get the uniform convergence on $G \times G$ of the corresponding holomorphic functional elements (the complexifications) to a holomorphic functional element.

We obtain a real analytic functional element at the fixed point $p$, denoted by $D_{U_X}(p,u)$.

(3.3.4) It is easy to see that $D_{U_X}(p,z(u),\bar{z}(u))$ is strictly plurisubharmonic, where $z = (z_1, \ldots, z_n)$ are coordinates in a neighborhood with origin at $p$. Indeed, we take an arbitrary tangent vector $v$ to $U_X$ at $p$. We consider $\tau_i(p)$ and $\tau_i(v)$ on $X_i \subset \mathbb{P}^{m_i}$ ($i \geq 0$). We take a general curvilinear section $C_i \subset X_i$ tangent to $\tau_i(v)$ at $\tau_i(p)$.

By our assumption, $C_i$ will be a nonsingular connected curve of genus at least 2. Indeed, the corresponding linear system has no fixed components because $\pi_1(X)$ is large, in particular, $X$ contains no rational curves. Moreover, the linear system is not composite with a pencil. Hence we can apply Bertini’s theorem to obtain a nonsingular connected curve. We, then, consider a connected open Riemann surface $\tau_i^{-1}(C_i) \subset U_X$. We get $D_{U_X}(p,z(u),\bar{z}(u))$ is strictly plurisubharmonic at $p$.

Finally, the $n \times n$ matrix $(h_{\alpha\beta})$, where

$$h_{\alpha\beta}(z(u),\bar{z}(u)) := \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} D_{U_X}(p,z(u),\bar{z}(u)),$$

defines the desired real analytic Kahler metric $\Lambda_X$. By (2.2.1) and the fundamental property of diastasis, $D_{U_X}(p,u)$ is, in fact, the diastatic potential of $\Lambda_X$ at $p$.

Thus, we have established the following

**Proposition-Definition 3.** We assume $\pi_1(X)$ is large and residually finite, and a general curvilinear section $C \subset X$ has $g(C) \geq 2$. Then $U_X$ is equipped with a real analytic $\pi_1(X)$-invariant Kahler metric, denoted by $\Lambda_X$. The restriction of $\Lambda_X$ on $R_X$, the inverse image on $U_X$ of a general curvilinear section $C \subset X$, is the metric $\Lambda_R$ on $R_X$.
4. Metric $\Sigma$

(4.1) We assume, in addition, $\pi_1(X)$ is nonamenable and we will prove $\mathcal{K}_X$ is ample. Let $dv_\Lambda = V_\Lambda \prod_{\alpha=1}^n (\sqrt{-1}dz_\alpha \wedge d\bar{z}_\alpha)$ denote the volume form of $\Lambda \mathcal{L}$, where $V_\Lambda$ is a locally defined positive function. Let $p(s, x, y)$ be the heat kernel on $U_X$. We consider the following non-invariant volume form on $U_X$:

$$dv_\Sigma(x) := p(s, x, y)dv_\Lambda(x).$$

A harmonic function $u$ on the disk $\Delta$ is the real part of one and only one holomorphic function $f_u = u + \sqrt{-1}\bar{u} \in \text{Hol}(\Delta)$ such that $f_u(0) = u(0)$. By a theorem of M. Riesz (communicated to the author by Demailly; see [Ru, 17.24-17.26]) if $u$ is $L^2$ with respect to the Lebesque measure then

$$\|f_u\| \leq A\|u\|,$$

where $\|\cdot\|$ is the norm in $L^2$ and $A$ is a constant.

(4.2) Now, we consider two general subspaces $\mathbf{P}^{n-1} \subset \mathbf{P}^r$ and $\mathbf{P}^{r-n} \subset \mathbf{P}^r$, $\mathbf{P}^{n-1} \cap \mathbf{P}^{r-n} = \emptyset$. We obtain a fibration $F_X$ of curves over an open subset of the $\mathbf{P}^{n-1}$ whose general member is a nonsingular curve on $X$ of genus at least 2. The inverse image of $F_X$ on $U_X$ produces a family $F_U$ whose general member is a connected open Riemann surface on $U_X$.

A bounded pluriharmonic function $u$ on $U_X$ will be square-integrable with respect to the measure $dv_\Sigma$. Indeed, $p(s, x, y)$ contains a Gaussian exponential term. Furthermore, the restriction of $u$ on the general member $R \in F_U$ will be $L^2$ with respect to the Lebesque measure on $R$. By (4.1.1) and the Fubini-Tonelly theorem, the corresponding holomorphic function $f = u + \sqrt{-1}\bar{u}$ on $U_X$ will be square-integrable with respect to the measure $dv_\Sigma$.

(4.3) Let $H_\Sigma$ be the Hilbert space of square-integrable on $U_X$ with respect to $dv_\Sigma$ holomorphic functions on $U_X$:

$$H_\Sigma := \left\{ \varphi \in \text{Hol}(U_X) \left| \|\varphi\|^2 := \int_{U_X} |\varphi(x)|^2 dv_\Sigma(x) < \infty \right. \right\}.$$  

This Hilbert space is not trivial and has a reproducing kernel. Let $B_\Sigma$ denote its reproducing kernel; $B_\Sigma$ is a function on $U_X$. Set

$$g_{\alpha\beta} := \frac{\partial^2 \log B_\Sigma}{\partial z_\alpha \partial \bar{z}_\beta}.$$  

(4.4) The differential form $ds_\Sigma^2 := \sum_{\alpha,\beta=1}^n g_{\alpha\beta}dz_\alpha d\bar{z}_\beta$ is called a Bergman form. Clearly $ds_\Sigma^2$ is Hermitian. We claim it is positive definite (meaning $\log B_\Sigma$ is strictly plurisubharmonic), i.e., for any vector $w \in T_p,U_X$, $w \neq 0$, at an arbitrary point $p \in U_X$:

$$\sum_{\alpha,\beta=1}^n g_{\alpha\beta}w_\alpha \bar{w}_\beta > 0 \quad (w = (w_1, \ldots, w_n)).$$

Now, we fix $p$ and $w \in T_p,U_X$. To prove the positivity, we consider the set

$$\mathcal{E} := \{ \varphi \in H_\Sigma \left| \varphi(p) = 0, \langle \nabla \varphi, w \rangle = 1 \right. \}.$$  

A priori, it is not obvious that $\mathcal{E} \neq \emptyset$. First, we assume $\mathcal{E} \neq \emptyset$ and show that

$$\min_{\varphi \in \mathcal{E}} \|\varphi\|^2 = \frac{1}{B_{\Sigma} \sum_{\alpha, \beta} g_{\alpha \beta} w_{\alpha} \bar{w}_{\beta}},$$

where $B_{\Sigma}$ and $g_{\alpha \beta}$ are computed at $p$, hence $\sum_{\alpha, \beta} g_{\alpha \beta} w_{\alpha} \bar{w}_{\beta} > 0$. Our argument is similar to the one in [Sha, Sect. 52]. We will briefly recall his argument.

Let $\{\varphi_\sigma\} \subset H_{\Sigma}$ denote the complete orthonormal system in $H_{\Sigma}$. Let $\varphi = \sum_\sigma a_\sigma \varphi_\sigma$. Then our problem is to find $\min \sum_\sigma |a_\sigma|^2$ under the conditions at $p$:

$$\sum_\sigma a_\sigma \varphi_\sigma = 0, \quad \sum_\sigma a_\sigma \langle \nabla \varphi_\sigma, w \rangle = 1.$$

We employ the method of Lagrange multipliers. The uniqueness is easy, provided we have a solution (see [Sha, Sect. 51]).

For the extremal values of $a_\sigma$, we obtain as in [Sha, Sect. 52]:

$$\sum_\sigma a_\sigma a_\sigma = \frac{1}{B_{\Sigma} \sum_{\alpha, \beta} g_{\alpha \beta} w_{\alpha} \bar{w}_{\beta}}.$$

By the above, we get a Bergman-type metric $ds_{\Sigma}^2$ and holomorphic immersion into an infinite-dimensional projective space as in [Kob, Chap. 4.10] provided $\mathcal{E} \neq \emptyset$.

(4.5) It remains to verify $\mathcal{E} \neq \emptyset$. We consider a general curvilinear section $C_i$ of $X_i \subset \mathbb{P}^r$ ($i \gg 0$). Let $R := R_{C_i} \subset U_X$ be the corresponding open connected Riemann surface ($\Delta \rightarrow R \rightarrow C_i$). Now, let $u$ be a bounded positive pluriharmonic function on $U_X$, and $f$ the corresponding holomorphic function on $U_X$.

Let $z_1$ be a coordinate in $\Delta \subset \mathbb{C}$ and $0 \in \mathbb{C}$ goes to the point $p \in R$. Let $f_R$ denote the restriction of $f$ on $R$. We can assume $f_R$ is not a constant and $f_R(p) = 0$. Let

$$f_R(z_1) = a_1 z_1 + a_2 z_1^2 + \cdots$$

be the Taylor expansion around $p$. Assume $a_1 = \cdots = a_{k-1} = 0$ and $a_k \neq 0$. We differentiate $f$ in the direction $w$ at $p$ exactly $k$ times. We obtain a holomorphic function $f^{(k)}$ and the corresponding function $f_R^{(k)}$.

The function $f^{(k)}$ belongs to $H_{\Sigma}$ by a generalization of the Schwarz-Pick inequality in Dai-Pan [DP, Theorem 1.2]:

$$|f_R^{(k)}| \leq \frac{2n! \Re f_R(z_1)}{(1 - |z_1|^2)^k} (1 + |z_1|)^{k-1}.$$

Indeed, $p(s, x, y)$ contains a Gaussian exponential term.

(4.6) We consider the following volume form on $U_X$:

$$B_{\Sigma} \prod_{\alpha=1}^{n} (\frac{\sqrt{-1}}{2} \cdot dz_\alpha \wedge d\bar{z}_\alpha).$$

This volume form is $\pi_1(X)$-invariant as, e.g., in [FK, pp. 10-11 or pp. 188-191]. Hence $X$ admits a volume form with negative Ricci form (see [Kob2, Chap. 2.4.4]). It follows $\mathcal{K}_X$ is ample by Kodaira.
5. Metric $\beta$ and Uniformization

We assume $\pi_1(X)$ is nonamenable. In this section, we fix an integer $q \gg 0$ and assume $\mathcal{K}^q$ is very ample on every finite covering of $X$ (see (2.3.3)). Now, we consider the metric $\Lambda_{\mathcal{K}^q}$ on $U_X$ and construct a Bergman-type metric with a weight, denoted by $\beta := \beta_{\mathcal{K}^q}$, similar to the metric $\Sigma_{\mathcal{L}}$.

(5.1) Let $H_\beta$ be the Hilbert space of square-integrable (with respect to $dv_\Sigma$ and the Hermitian metric $\eta_q$ as in (2.3.3)) holomorphic sections of $\mathcal{K}_q$:  

\begin{equation}
H_\beta := \left\{ \omega \in H^0(\mathcal{K}_q^q) \mid \|\omega\|^2 := \int_{U_X} \eta_q(\omega, \bar{\omega}) dv_\Sigma(x) < \infty \right\}.
\end{equation}

By Section 4, this Hilbert space is not trivial because of the natural inclusion $O_{U_X} \hookrightarrow \mathcal{K}_q^q$. It has a reproducing kernel, denoted by $B$. As in (4.4)-(4.5), we obtain a positive definite Bergman form $ds_\beta^2$ and a metric on $U_X$, denoted by $\beta$.

This metric produces a natural isometric immersion into the projective space $\mathbb{P}(H^*_\beta)$ [Kob2, Chap. 4.10]. We will show this immersion is actually an embedding. This will follow at once from Proposition 1′ below whose proof is a trivial generalization of Proposition 1.

(5.2) Let $Q \in U$ be an arbitrary point, and $Q_i := \tau_i(Q) \in X_i$. Let $p_i(s, x, Q_i)$ be the heat kernel on $X_i$. We consider the following volume form on each $X_i$:

$$dv_i(x) := p_i(s, x, Q_i) dv_\Lambda(x).$$

As above, we consider the Hermitian metric $\eta_q$. For every $i$, we consider the Hilbert space $H_{i,\beta}$ defined similarly to (5.1.1). We obtain a metric on $X_i$, denoted by $\beta_i := \beta_{i,\mathcal{K}^q}$.

This metric produces a natural embedding into a finite-dimensional projective space $\mathbb{P}(H^*_{i,\beta})$.

**Proposition 1′.** With the above notation, the metric $\beta$ equals the limit of pullbacks on $U_X$ of the metrics $\beta_i$’s.

**Appendix**

We will prove the following theorem conjectured by Shafarevich (1972).

**Theorem A.** Let $X \hookrightarrow \mathbb{P}^r$ be a nonsingular connected projective variety of dimension $n \geq 1$ with large and residually finite fundamental group $\pi_1(X)$. Then $U_X$ is a Stein manifold.

**Proof.** We will prove the theorem under an additional assumption that a sufficiently general nonsingular connected curvilinear section $C \subset X$ has the genus $g(C) \geq 2$. Otherwise, $\pi_1(X)$ is Abelian by the Campana-Deligne theorem [Kol, Theorem 2.14], and the conjecture is well known when $\pi_1(X)$ is Abelian or, even, nilpotent (Katzarkov [Ka]).

(A.0) The idea of proof of the theorem is similar to the one by Siegel [S1]. He established that if $U$ is a connected bounded domain in $\mathbb{C}^n$ covering a compact complex manifold $Y$ then $U$ is a domain of holomorphy. We will sketch his argument.
He considers the Bergman metric on $U$ (see, e.g., [Kob1], [Kob2, Chap. 4]). It is complete since $Y$ is compact. Recall the fundamental property of the Bergman metric, namely, it determines a natural isometric embedding of $U$ into an infinite-dimensional projective space with a Fubini-Study metric. Infinite-dimensional projective spaces were considered by Bochner [Bo, p. 193], Calabi [C, Chap. 4] and Kobayashi [Kob1, Sect. 7]. According to Kobayashi [Kob1, p. 268], the idea of using square-integrable forms on arbitrary manifolds can be found in Washnitzer [W].

Let $B(z, \bar{z})$ denote the Bergman kernel of $U$. Siegel proves that $\log B(z, \bar{z})$ goes to infinity on any infinite discrete subset $T \subset U$. Hence $U$ is a domain of holomorphy (equivalently, holomorphically compete or Stein domain) by Oka’s solution of the Levi problem.

We observe that $Y$ is a projective variety by the Poincaré ampleness theorem [Kol, Theorem 5.22]. In 1950s, Bremermann proved that an arbitrary bounded domain in $\mathbb{C}^n$ with complete Bergman metric is Stein [Kob2, Theorem 4.10.21].

Now, Oka’s solution of the Levi problem for domains in $\mathbb{C}^n$ admits a generalization due to Grauert (manifolds) and Narasimhan (complex spaces) (see [N]). Thus, our aim is to define a metric on the manifold $U_X$ and a strictly plurisubharmonic function on $U_X$ that goes to infinity on an arbitrary infinite discrete subset of $U_X$. The function generated by the diastasic potentials of the metric will be such a function.

We assume $U_X$ and all $X_i$’s are equipped with the metric $\Lambda_L$. Let $I, R \subset U_X$ be two subsets with $I$ compact. Let $d(u, p)$ denote the distance function on $U_X$ with its Riemannian structure induced by $\Lambda_L$. We set

$$\xi(I, R) := \sup_{u \in I} \inf_{p \in R} d(u, p).$$

We say that a sequence of subsets $\{R_\gamma\}_{\gamma \in \mathbb{N}}$, where $R_\gamma \subset U_X$, approximates the set $I$ if $\lim_{\gamma \to \infty} \xi(I, R_\gamma) = 0$.

(A.1) Prolongation. Let $z = (z_1, \ldots, z_n)$ be a local coordinate system in a small neighborhood $V$ with origin at a fixed point $a \in V \subset U_X$. Let $\Phi_a(z(p), \bar{z}(p))$ be the diastasic potential at $a$. Let $b \in U_X$ be an arbitrary point. Let $I : u = u(s) \ (0 \leq s \leq 1, \ u(0) = a, \ u(1) = b)$ be a path joining $a$ and $b$.

(A.1.1) Prolongation along the path $I$. We say $\tilde{\Phi}_a(z(p), \bar{z}(p))$ has a prolongation along $I$ if the following two conditions are satisfied:

(i) To every $s \in [0, 1]$ there corresponds a functional element of diastasic potential $\tilde{\Phi}_{u(s)}(z(p), \bar{z}(p))$ at $u(s)$ ($u(s)$ is also called the center).

(ii) For every $s_0 \in [0, 1]$, we can take a suitable subarc $u := u(s)$ $(|s - s_0| \leq \epsilon, \ \epsilon > 0)$ of $I$ contained in the domain of convergence of $\tilde{\Phi}_{u(s_0)}(z(p), \bar{z}(p))$ such that every functional element $\tilde{\Phi}_{u(s)}(z(p), \bar{z}(p))$ with $|s - s_0| \leq \epsilon$ is a direct prolongation of $\tilde{\Phi}_{u(s_0)}(z(p), \bar{z}(p))$.

The direct prolongation means the following. Suppose $\tilde{\Phi}_{a_1}(z(p), \bar{z}(p))$ is defined on $V_1$ and $\tilde{\Phi}_{a_2}(z(p), \bar{z}(p))$ is defined on $V_2$ ($V_1 \cap V_2 \neq \emptyset$). Then $\tilde{\Phi}_{a_2}(z(p), \bar{z}(p))$ is the direct prolongation of $\tilde{\Phi}_{a_1}(z(p), \bar{z}(p))$ if they coincide on $V_1 \cap V_2$. Recall that
the complexification allows us to consider the corresponding holomorphic function in place of the diastasis potential. It follows a prolongation along \( I \) is unique provided it exists.

Let \( A = \{ a_\nu \} \ (a, b \in A, I = \bar{A}) \) be a countable ordered dense subset of \( I \). We would like to prolong \( \tilde{\Phi} \) along \( I \) obtaining the diastasis potential \( \tilde{\Phi}_{a_\nu}(z(p), z(\bar{p})) \) of \( \Lambda_L \) for each \( a_\nu \). We claim the prolongation along \( I \) is possible.

(A.1.2) Now, we will make use of the assumption \( \pi_1(X) \) is large. We can assume \( I \) is embedded in \( X \) via \( \tau \); otherwise, we could have replaced \( X \) by \( X_i \) for \( i \gg 0 \). The set \( A \) is a union of an increasing sequence of finite ordered subsets:

\[
A_1 \subset A_2 \subset \cdots \subset A_\gamma \subset \cdots \subset A, \quad a, b \in A_\gamma \ (\forall \gamma).
\]

We consider an arbitrary \( A_\gamma \) and the corresponding set

\[
\tau_i(A_\gamma) \subset \tau_i(I) \subset X_i \subset \mathbb{P}^{r_i},
\]

where \( i \) is a sufficiently large integer and \( r_i \) is an appropriate integer. We apply Bertini’s theorems to the linear system of curvilinear sections passing through \( \tau_i(A_\gamma) \), i.e., the moving part of the system is a one-dimensional subscheme in \( X_i \). We claim this linear system (and its inverse images) have no fixed components on \( X_i \) for all \( i \gg 0 \).

Suppose, to the contrary, \( W \subset X_i \) is a fixed component. Then \( W \) belongs to the linear span of \( \tau_i(A_\gamma) \subset \mathbb{P}^{r_i} \). We move up along the tower (2.3.1.1). For \( j \gg i \gg 0 \), the linear span of \( \tau_j(A_\gamma) \subset \mathbb{P}^{r_j} \) will not contain \( \tau_{j1}^{-1}(W) \). Hence the corresponding linear system on \( X_j \) does not contain \( \tau_{ji}^{-1}(W) \). Therefore the linear system has no fixed components.

A priori, a general member of the system may have singularities at the base points of the system. However, we can always assume \( \mathbb{P}^{r_i} \) is sufficiently large, and our system is sufficiently large as well. Thus, the general member of the linear system on \( X_j \) will be a connected nonsingular curve. Its inverse image on \( U_X \) will be a connected open Riemann surface \( R_\gamma \) by the Campana-Deligne theorem [Kol, Theorem 2.14]. These Riemann surfaces will approximate \( I \) as \( \gamma \) goes to infinity.

(A.1.3) For every \( \gamma \), the diastasis potential of the induced metric on \( R := R_\gamma \) is the restriction of the corresponding diastasis potential of \( U_X \), and \( \mathbf{P}_R := \mathbf{P}_{R,a} \) is a function on \( R \).

We replace the path \( I \) and an arbitrary \( A_\gamma \ (\gamma \gg 0) \) by a broken geodesic \( \sigma_\gamma \) between the points \( a \) and \( b \). Namely, we replace the subpath of \( I \) between two adjacent points of \( A_\gamma \) by a geodesic on \( U_X \). We also consider the corresponding broken geodesic \( \rho_\gamma \) on \( R_\gamma \). Recall (Section 2.2) that the diastasis approximates the square of the geodesic distance in the small. Hence \( \rho_\gamma \) will be close to \( \sigma_\gamma \) provided each \( a_\nu \) is close to \( a_{\nu+1} \), and we get

\[
\lim_{\gamma \to \infty} \xi(I, \sigma_\gamma) = \lim_{\gamma \to \infty} \xi(I, \rho_\gamma) = 0 \quad (\sigma_\gamma \subset U_X, \rho_\gamma \subset R_\gamma).
\]

(A.1.4) Now, we will establish the prolongation along \( I \). Assume we can prolong along \( I \setminus b \). We take a sufficiently small subarc \( E \subset I \) in the domain \( \mathcal{V}_b \) of

\[
\tilde{\Phi}_b^\gamma(z(p), \bar{z}(p)) := \tilde{\Phi}_b(z(p), \bar{z}(p)).
\]
Take a point \( w \in E \setminus b \) and its small neighborhood \( \mathcal{V}_w \subset \mathcal{V}_b \) in \( U_X \). We set
\[
\tilde{\Phi}_b^w(z(p), \overline{z(p)}) := \tilde{\Phi}_b^w(z(p), \overline{z(p)})|_{\mathcal{V}_w},
\]
more precisely, \( \tilde{\Phi}_b^w(z(p), \overline{z(p)}) \) is a real analytic function on \( \mathcal{V}_w \) with center \( w \) (functional element) obtained from the real analytic function \( \tilde{\Phi}_b^w(z(p), \overline{z(p)}) \) on \( \mathcal{V}_b \) with center \( b \). For every \( p \in \mathcal{V}_w \), we claim
\[
\Phi_{wb}(p) := \tilde{\Phi}_b^w(z(p), \overline{z(p)}) - \tilde{\Phi}_w^v(z(p), \overline{z(p)}) = D_{U_X}^v(b, p) - D_{U_X}^v(w, p) = 0,
\]
where \( D_{U_X}^v(b, p) \) is the real analytic function in \( z(p), \overline{z(p)} \) on \( \mathcal{V}_w \) with center \( w \) obtained from the real analytic function \( D_{U_X}^v(b, p) \) in \( z(p), \overline{z(p)} \) on \( \mathcal{V}_b \) with center \( b \), and \( D_{U_X}^v(w, p) \) is a real analytic function in \( z(p), \overline{z(p)} \) on \( \mathcal{V}_w \) with center \( w \). By the definition, \( D_{U_X}^v(b, p) \) is a direct prolongation of \( D_{U_X}^v(b, p) \). A priori, \( D_{U_X}^v(w, p) \) is not a direct prolongation of \( D_{U_X}^v(b, p) \).

Let \( e \in \mathcal{V}_w \) be an arbitrary point. We choose \( \{R_{\gamma}\} \), as in (A.1.2), with an additional condition: \( e, w \in R_{\gamma} (\forall \gamma) \). Then \( (\Phi_{wb})|_{R_{\gamma}}(e) = 0 \) for all \( \gamma \gg 0 \) because
\[
D_{R_{\gamma}}^v(b, e) - D_{R_{\gamma}}^v(w, e) = 0 \quad (\forall \gamma \gg 0)
\]
and the fundamental property of diastasis (Section 2.2). Here
\[
D_{R_{\gamma}}^v(b, e) = D_{U_X}^v(b, e)|_{R_{\gamma}} \quad \text{and} \quad D_{R_{\gamma}}^v(w, e) = D_{U_X}^v(w, e)|_{R_{\gamma}}.
\]
It follows we can prolong \( \Phi_a(z(p), \overline{z(p)}) \) along \( I \). Since \( U_X \) is simply connected, we obtain the desired function \( P_U := P_{U, a} \) on \( U_X \).

(A.2) In view of the Oka-Grauert-Narasimhan theorem (Grauert’s version), it remains to verify that, for any real \( \alpha \), the following set is relatively compact in \( U_X \):
\[
E_\alpha := \{u \in U_X \mid P_U(u) < \alpha\}.
\]

Suppose \( S \subset E_\alpha \) is an infinite discrete subset without limit points in \( U_X \). Then we will derive a contradiction by showing that \( P_U \) is unbounded on \( S \). Since \( X \) is compact, \( \tau(S) \) will be either a finite set or it will have a limit point. It suffices to replace \( S \) by an infinite set \( T_\alpha \) in the fiber of \( \tau \) over a point \( Q \in X \) and show that \( P_U \) is unbounded on \( T_\alpha \). If \( \tau(S) \) has a limit point then \( Q \) is such a point.

We consider a general curvilinear section \( C \subset X \) through \( Q \). Set \( R_C := \tau^{-1}(C) \). We obtain a connected open Riemann surface by the Campana-Deligne theorem [Kol, Theorem 2.14].

By the fundamental property of diastasis (Section 2.2), \( P_R = P_U|_{R_C} \) where \( P_R \) is the corresponding diastatic potentials on \( R_C \). One can find an infinite discrete subset \( \tilde{T}_\alpha \subset C \subset \Delta \) (see (3.2)) whose image on \( R_C \) will be close to the corresponding points of \( T_\alpha \). Moreover, \( \tilde{T}_\alpha \) is approaching the boundary of \( \Delta \).

We pick a point \( Q \in \mathcal{F} \) such that its image on \( X \) is close to \( Q \). We will identify \( D_{AR}(Q, \cdot) \) with its inverse image on \( \mathcal{F} \). We see that \( D_{AR}(Q, \cdot) \) goes to infinity on \( T_\alpha \) by considering the tower of coverings:
\[
C \leftarrow \cdots \leftarrow C_i \leftarrow \cdots \leftarrow R_C,
\]
where \( C_i \subset X_i \) (see (2.3.1.1)), and the diatsases of the corresponding Bergman-type metrics of members of the tower restricted to the complement of hyperplane at infinity where they generate functions as in (3.3.3).

By Proposition 1, the diatsases increase as we move up in the tower. So, \( P_R \) is unbounded on \( \tilde{T}_\alpha \) and \( T_\alpha \), and \( P_U \) will be unbounded on \( S \).

The contradiction proves the theorem.

\((A.3)\) Remarks. Bogomolov and Katzarkov suggested that the corresponding conjecture might fail in the case of nonresidually finite fundamental groups [BK].

A similar argument will establish the Shafarevich conjecture when \( X \) is singular and \( \pi_1(X) \) is residually finite and non-Abelian. Namely, let \( X \hookrightarrow \mathbb{P}^r \) be a connected normal projective variety of dimension \( n > 0 \). Assume \( \pi_1(X) \) is large, residually finite, and non-Abelian. Then its universal covering is a Stein space. The details will appear elsewhere.

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Princeton, NJ 08540

E-mail address: roberttreger117@gmail.com