Gossip-Based Model for Opinion Dynamics with Probabilistic Group Interactions

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Abstract

Classical models of opinion dynamics utilize deterministic methods to describe how opinions evolve within a social network. More recent models are shifting towards randomized processes to emulate real-world interactions. These include gossip-based models that use pairwise interactions to represent the opinion formation process. In this paper, we propose to extend the gossip-based models for opinion dynamics by allowing group interactions with randomly selected participants.

1 Introduction

One of the important research topics in the area of social network analysis is understanding how opinions propagate over a group of interconnected individuals. Control theory provides various methods for modeling the opinion dynamics of social networks [1]. For this research, we consider agent-based models for reaching consensus.

In agent-based models for opinion dynamics, individuals are represented as agents and their opinions evolve based on their interpersonal relationships, which define the structure of their social network, and how they interact with one another. Earlier models employ deterministic methods that update opinions synchronously using a fixed computation within a given timeframe. As an example, the DeGroot model demonstrates how consensus can be reached by iteratively updating the opinions of all agents synchronously using a weighted average of the opinions of the agents that are influential to them [3].

Recent models for opinion dynamics, however, are shifting towards the use of randomized processes [2]. This is meant to emulate the nature of real-world interactions which are asynchronous and usually unpredictable. A particular method that is commonly used for randomized opinion formation is the gossip algorithm used in different types of networks [4], [5]. Examples of gossip-based models for opinion dynamics can be found in [6] and [7]. By design, gossip-based models are restricted to pairwise interactions. The work in [8] addressed this restriction by proposing an opinion dynamics model that uses a gossip algorithm with asynchronous group interactions. However, in this model, the members of the groups remain fixed during the duration of its process.

In this paper, we propose a gossip-based extension of the DeGroot model that involves asynchronous group interactions with random participants. This approach is a closer approximation of real-world communications compared to pairwise and fixed-group gossiping. We analyze its conditions for achieving consensus and its expected dynamics. Additionally, we provide numerical examples for demonstrating its behavior.

The organization of this paper is as follows. Section 2 discusses existing models for reaching consensus. One is the classic DeGroot model, and another is a gossip-based model. In Section 3, we describe in detail our proposed model for opinion dynamics, followed by its analysis. Section 4 contains simulations that use our proposed model. Finally, we state our concluding remarks in Section 5.

Notation and Preliminaries. In this paper, a social network is represented by a directed graph $G = (V, E)$, where $V = \{1, ..., n\}$ is the set of agents and $E \subseteq V \times V$ is the set of edges such that $(i, j) \in E$ if and only if agent $i$ can interact with agent $j$. The neighbors of agent $i$ is given by $N_i = \{j \mid (i, j) \in E\}$. A node $i$ is globally reachable if all the other nodes have a path to it. We assume that $G$ is at least weakly connected, otherwise each connected component is treated as an independent network. The opinions of the agents at time $k$ are stored in the vector $x(k) \in \mathbb{R}^n$, and $x(0)$ contain the initial opinions. We use $e_i \in \mathbb{R}^n$ to denote a standard basis vector, where the $i$th element is 1 while the rest are zeroes. The Hadamard product of two matrices $P$ and $W$ is denoted by $P \circ W$.

2 Reaching Consensus

Various agent-based models exist for modeling the opinion dynamics of social networks. For this research, we are interested in models that describe how a group of interrelated individuals can achieve consensus by recurrent exchange of opinions. In this paper, we consider the classic DeGroot model and a gossip-based model for reaching consensus.
2.1 Deterministic Consensus

The DeGroot model is an iterative process for reaching a consensus. Given \( n \) agents, let \( W \in \mathbb{R}^{n \times n} \) be a nonnegative matrix, where \( w_{ij} > 0 \) if and only if \((i,j) \in E\). It is assumed that \( W \) is a row-stochastic matrix, which means \( W1 = 1 \). The \( W \) matrix can be viewed as a matrix of interpersonal influence, where each \( w_{ij} \) represents the relative importance given by agent \( i \) on the opinion of agent \( j \).

Starting with the initial opinions \( x(0) \), the opinion of each agent \( i \) is updated at each time \( k > 0 \) as

\[
x_i(k + 1) = \sum_{j \in N_i} w_{ij} x_j(k).
\]

Thus, the DeGroot model can be written as

\[
x(k + 1) = W x(k).
\]

The model (2) is deterministic because the opinions of all agents are updated synchronously every iteration using a convex combination of a fixed set of neighbors whose weights are defined by \( W \).

We recall some of the DeGroot model’s important properties that are relevant to this paper. The model (2) reaches consensus if and only if \( G \) has a globally reachable node that belongs to a strongly connected component that is aperiodic. In such cases, the opinions of the agents converge to the limit \( x^* \) given by

\[
x^* = \lim_{k \to \infty} x(k) = \pi^T x(0)
\]

where \( \pi \) is the dominant left eigenvector of \( W \). The vector \( \pi \) can also be viewed as the unique stationary distribution of a Markov chain whose transition matrix is defined by \( W \). Additionally, rate of convergence of (2) is dependent on the eigenvalue \( W \) with the second largest magnitude.

2.2 Gossip-Based Consensus

In communication networks, gossiping refers to asynchronous pairwise interactions of agents. In this section, we consider the asymmetric gossip model described in [5]. The model can be described as follows. Given a directed graph \( G \) representing a social network, let \( C \in \mathbb{R}^{n \times n} \) be a matrix where each \( c_{ij} \) is the probability that edge \((i,j)\) is activated and \( 1^T C 1 = 1 \). For any agent, the opinions of other agents are given a weight of \( \gamma \in (0,1) \). At each time \( k > 0 \), an edge \((i,j)\) is selected with probability \( c_{ij} \). This results to agent \( i \) receiving the opinion of agent \( j \). Thus, \( i \) updates its opinion as

\[
x_i(k + 1) = (1 - \gamma)x_i(k) + \gamma x_j(k).
\]

Only the opinion of agent \( i \) is updated, while the others remain the same.

Let \( B^{ij} \in \mathbb{R}^{n \times n} \) be the weight matrix based on the selected pair of agents \( i \) and \( j \) at time \( k \), and defined as

\[
B^{ij} = I - q(e_i e_i^T + q e_i e_j^T).
\]

Then, the model can be written as

\[
x(k + 1) = B(k)x(k)
\]

where \( B(k) = B^{ij} \) and \( \mathbb{P}[B(k) = B^{ij}] = c_{ij} \). Thus, (6) is a time-varying version of (2) with independent and identically distributed weight matrices.

The expected value of \( B(k) \) is given by

\[
\mathbb{E}[B(k)] = I - \gamma \operatorname{diag}(C1) + \gamma C.
\]

Noted that \( B(k) \) always has a positive diagonal. Based on this property, the model (6) achieves probabilistic consensus when the graph adapted from \( \mathbb{E}[B(k)] \) is strongly connected.

3 Gossiping with Random Groups

In this section, we introduce our proposed model for opinion dynamics, which applies pairwise gossiping to the DeGroot model, then extends it by allowing group interactions with random participants.

Let \( P \in \mathbb{R}^{n \times n} \) be a nonnegative matrix, where \( p_{ij} > 0 \) if and only if \((i,j) \in E\). In this matrix, each \( p_{ij} \) corresponds to the independent probability that agent \( i \) receives the opinion of agent \( j \) at time \( k \). Let \( W \in \mathbb{R}^{n \times n} \) be a weight matrix that has the same properties as the \( W \) matrix for the DeGroot model.

At each time \( k > 0 \), an agent \( i \) is selected with uniform probability from \( V \). The chosen agent then interacts with a random subset of its neighbors \( S_i(k) \), where each \( j \in S_i(k) \) is selected with probability \( p_{ij} \) via an independent Bernoulli trial. Formally, let \( \phi_{ij}(k) \) be a Bernoulli random variable such that

\[
\phi_{ij}(k) = \begin{cases} 1, & \text{if } j \in S_i(k) \\ 0, & \text{if } j \notin S_i(k) \end{cases}
\]

where

\[
\mathbb{P}[\phi_{ij}(k) = 1] = p_{ij}.
\]

For brevity, we refer to the participating neighbors as a group. The opinion of agent \( i \) is then updated as

\[
x_i(k + 1) = \left(1 - \sum_{j \in S_i(k)} w_{ij}\right)x_i(k) + \sum_{j \in S_i(k)} w_{ij}x_j(k)
\]

while the opinions of the other agents remain unchanged.

The weight matrix corresponding to the update performed
in (7) is given by
\[ A(k) = I - \sum_{j \in S_i(k)} w_{ij} e_i e_j^T + \sum_{j \in S_i(k)} w_{ij} e_j e_i^T \] (8)
Thus, our model can be compactly expressed as
\[ x(k + 1) = A(k)x(k) \] (9)
which is a time-varying version of the DeGroot model.

Note that if \( S_i(k) = N_i \), meaning all neighbors of \( i \) are selected at time \( k \), then the opinion of agent \( i \) is updated using (1). On the other hand, if \( |S_i(k)| = 1 \) as a result of selecting only one neighbor \( j \) at the current turn, then the opinion of \( i \) is updated using (4), where \( y = w_{ij}. Thus, our proposed model can be seen as a generalization for both the DeGroot model and the asymmetric gossip model in [5].

### 3.1 Convergence to a Consensus

Even if our gossip-based model has random participants at every iteration, it still achieves probabilistic consensus under certain conditions.

**Theorem 1.** If \( G \) contains a globally reachable node and every node has a self-loop, that is \((i, i) \in E\) for all \( i \in V \), then consensus is achieved in (9).

**Proof.** Let \( Q(k, r) = A(k + r)A(k + r - 1) \ldots A(k) \). We are going to use the theorem in [5] which states that a random network achieves probabilistic consensus if and only if
\[ \mathbb{P}[\exists k, \exists h \mid Q_{ih}(0, k)Q_{jh}(0, k) > 0] = 1 \] (10)
for all \( i, j \in V \).

Since every node in \( G \) has a self-loop, then for any \( i \) selected at any time \( k \), \( A_{ii}(k) \geq w_{ii} \). Thus \( A_{ii}(k + 1) > 0 \) implies \( Q_{ij}(1, k) > 0 \). If \( i \) has path to node \( h \), then
\[ \mathbb{P}[\exists k, \exists r \mid Q_{ih}(r, k) > 0] > 0 \]
Let
\[ \kappa_{ij}(k, r) = \{\exists k, \exists r \mid Q_{ih}(r, k)Q_{jh}(k, r) > 0\} \] (11)
for any \( i, j \in V \). If \( h \) is a globally reachable node, then \( \mathbb{P}[\kappa_{ij}(k, r)] > 0 \). As \( k \to \infty \), then \( \mathbb{P}[\kappa_{ij}(k, r)] \to 1 \). Notice that (11) implies that \( A_{ih}(k + r) > 0 \) for all \( i \). Since \( A(k) \) is row-stochastic with positive diagonal, then \( Q_{ih}(0, k + r) > 0 \) for all \( i \), which satisfies (10).

### 3.2 Expected Dynamics

The consensus value that our proposed model (9) achieves varies depending on the selected agents at each time \( k \). Here, we analysis its expected dynamics.

**Lemma 2.** Let \( \bar{A} = \mathbb{E}[A(k)] \). The expected dynamics of the model (9) is
\[ \mathbb{E}[x(k + 1)] = \bar{A}\mathbb{E}[x(k)] \] (12)
where
\[ \bar{A} = I - \frac{1}{n} (\text{diag}(P \circ W)1 - P \circ W) \] (13)

**Proof.** At any given time \( k \), \( \mathbb{P}[j \in S_i(k)] = p_{ij} \). Since the selection of \( j \in S_i(k) \) is an independent event, then
\[ \bar{A} = \frac{1}{n} \sum_i \left(I - \left(\sum_{j \in N_i} p_{ij}w_{ij}\right)e_i e_j^T + \sum_{j \in N_i} p_{ij}w_{ij}e_j e_i^T\right) \]
Note that
\[ \sum_i \sum_{j \in N_i} w_{ij}e_i e_j^T = W \]
\[ \sum_i \sum_{j \in N_i} w_{ij}e_j e_i^T = \text{diag}(W1) \]
Thus
\[ \bar{A} = I - \frac{1}{n} (\text{diag}(P \circ W1) - P \circ W) \]

**Theorem 3.** If \( G \) contains a globally reachable node, then the expected dynamics (12) achieves consensus and the consensus value is given by \( 1\pi^T x(0) \), where \( \pi \) is the dominant eigenvector of \( \bar{A} \).

Notice that (12) is the same as (2), so they share the same convergence properties. \( \bar{A} \) is a stochastic matrix with a positive diagonal, so it is aperiodic. Additionally, \( A_{ij} > 0 \) if and only if \( \bar{A}_{ij} > 0 \). Thus, the only requirement needed to guarantee its consensus is the presence of a globally reachable node.

For both (2) and (12), the convergence behaviour and consensus value are dependent on the properties of the matrices \( W \) and \( \bar{A} \), respectively. An important distinction of our proposed model, however, is the presence of the \( P \) matrix which allows the modification of the model dynamics without having to alter the social network structure. We demonstrate this idea via the succeeding corollary, which involves a special case for our model.

**Corollary 4.** Let \( p_{ij} = \rho \) for all \( i, j \in V \), which means the probability for selecting a neighbor is uniform for all agents (i.e. \( P \circ W = \rho W \)). Let \( A_0 = I - \text{diag}(W1) + W \) which is a row-stochastic matrix. Then any eigenvalue of \( \bar{A} \) can be represented as \( 1 - \rho/n + (\rho/n)\lambda \), where \( \lambda \) is an eigenvalue of \( A_0 \).
Proof. Based on the assumptions stated in Corollary 4, we can represent $\bar{A}$ as

$$
\bar{A} = I - \frac{\hat{p}}{n} (\text{diag}(W) - W)
$$

$$
= \left(1 - \frac{\hat{p}}{n}\right) I + \frac{\hat{p}}{n} \left(I - \text{diag}(W) + W\right)
$$

$$
= \left(1 - \frac{\hat{p}}{n}\right) I + \frac{\hat{p}}{n} A_{0}
$$

from which we can characterize its eigenvalues. ■

Corollary 4 does not guarantee that the magnitude of the eigenvalues of $\bar{A}$ is inversely proportional to $\hat{p}$. However, we can generalize that a $\hat{p}$ value closer to 1 means lower eigenvalue magnitudes while a $\hat{p}$ value close to 0 means higher eigenvalue magnitudes. Recall that convergence rate of (2) is based on the eigenvalue with the second largest magnitude. Thus, Corollary 4 implies that higher interaction frequencies can lead to faster convergence to consensus.

4 Numerical Examples

We consider an Erdős-Rényi graph with $n = 20$ as our social network. All nodes in the graph have a self-loop. The matrices $W$, $P$, and the vector of initial opinions $x(0)$ are randomly generated. However, the same values are used for our examples in Fig. 1 and Fig. 2.

Fig. 1 shows the behavior of our proposed model, while Fig. 2 shows its expected dynamics. In both cases, consensus is reached.

To demonstrate the effects of having low and high interaction frequencies in the expected dynamics of our model, we replace the values in $P$ with $\hat{p} = 0.25$ and $\hat{p} = 0.95$, while the other variables remain the same. Fig. 3 shows the expected dynamics when $\hat{p} = 0.25$, while Fig. 4 shows the expected dynamics when $\hat{p} = 0.95$. It can be seen that the model reached consensus much faster in Fig. 4 than in Fig. 3.

5 Conclusion

In this paper, we proposed a gossip-based model for opinion dynamics that involves group interactions with random participants. Compared to traditional models and other gossip-based models, our proposed model is a closer representation of real-world interactions. We have established that, under suitable conditions, the model achieves probabilistic consensus. Furthermore, we have shown that its expected dynamics varies depending on the probabilities that an agent interacts with its neighbors. In particular, higher interaction frequencies may lead to faster convergence to consensus.
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