Use of Material-Shape Factors in Mechanical Design

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During design process materials and their shapes selection is usually performed unsystematically, selection is mostly based on previous experience, although this is the most influencing part during mechanical design. This paper is devoted to the application of the material-shape factors and indexes to the standard bicycle frame. Paper to show the method with material-shape factors and indexes and how to use it for a simply and fast redesign for a different material and shapes. The whole method is described in detail for two examples and the results are verified by FEM analysis.

Keywords: shape factor; material index; optimization; mechanical design

1 Introduction

When we want to design or redesign new structure and find the best ratio between lightness and stiffness we can use two ways how to reach it. First way is topological optimization. This optimization can be used by some of computer designing program to calculate ideal shape of a construction. Second way is using the “trial and error” method, when after first design and calculation the structure is modified and calculated again. All this process is repeated so long, until the appropriate solution is found. But what if we want to make this designing part faster but do not have the possibility to use topological optimization? In this situation can be used material-shape factors as a simple and useful tool.

2 Material-SHAPE factors

2.1 Material index

| Design requirements | Design requirements of cantilever beam |
|---------------------|----------------------------------------|
| Function            | What does component do?                |
|                     | A cantilever beam with loaded by force on free end, circular cross-section |
| Constrains          | What conditions must be met?           |
|                     | Requires stiffness S and length L.     |
| Objective           | What is to be optimized?               |
|                     | Minimize mass of the beam              |
| Free variable       | Which parameters are free to change?   |
|                     | Radius r (half of diameter)            |

Objective equation of the beam loaded by a force is the mass equation of the beam

\[
m = AL\rho
\]  (1)

Where cross-section area \( A \) depend on radius \( r \)
\[
A = \pi r^2
\]  (2)

In this case, the objective function contains free variable \( r \), it must to be eliminated. Stiffness influence also missing, it can be used for substitution

\[
S = \frac{3EI}{FL^3} \Rightarrow I = \frac{FL^3S}{3E}
\]  (3)

The last step is to create relationship between cross-section area \( A \) and cross-section moment of inertia \( I \)
\[
I = \frac{\pi}{4} \cdot r^4 = \frac{\pi^2 r^4}{4\pi} = \frac{4^2}{4\pi} \Rightarrow A = (4\pi I)^{\frac{1}{2}}
\]  (4)

Then by substituting of two previous equations into objective function

\[
m = L\rho \left(4\pi \cdot \frac{FL^3S}{3E}\right)^{\frac{1}{2}} = \frac{\rho}{\varepsilon^2} \left(4\pi \cdot \frac{FL^3S}{3E}\right)^{\frac{1}{2}}
\]  (5)

Where \( \frac{\rho}{\varepsilon^2} \) formula describes the influence of material properties on beam stiffness. Material index value should be higher for better material so material index equation is in form

\[
M^\sigma = \left(\frac{\rho}{\varepsilon^2}\right)^{\frac{1}{2}} = \frac{\varepsilon^2}{\rho}
\]  (6)

According to the equation 6 can be created the material index for strength, where \( \sigma_f \) is yield strength. [1]
\[
M^f = \frac{\sigma_f^2}{\rho}
\]  (7)

Another option is to create the material index using FEM. This approach is very effective in the case of a complex mathematical description of a physical phenomenon. However, advanced numerical analysis of FEM output is required to derive the material index.
Material index can be displayed in a material map to better explanation of the suitability of materials based on its properties. Figure 2 compare the suitability of the basic material against different material indices. In addition to material selection, material maps with material indices are excellent learning aids to understand the relationship between material properties and design requirements. [1]

2.2 Shape factor

Shape factor is a tool for determining the suitability of a shape for specific application. It is developed for the case of uneven distribution of stresses in the cross-section. For this reason, shape factors are use for bending or torsion of a structure. It is further assumed that the shape factor is not dependent on the cross-section area. The principle of the function of the shape factors is to compare some shape to any cross section to square cross section. The shape factors can be created using analytical or numerical solution. It is important that there are two types of shape factors for any shape, elastic and plastic. Elastic shape factor is use for deformation, plastic shape factor is use for the evaluating of the stress influence or the evaluating of failure.

For the understanding, it is best to make a sample elastic shape factor derivation for a circular cross-section in the case of bending. [1]

Cross-section area of square \( A_0 \):

\[
A_0 = \alpha^2
\]  

And its moment of inertia \( I_0 \):

\[
I_0 = \frac{\alpha^4}{12} = \frac{A^2}{12}
\]

According to the equation 12 can be created the shape factor for strength where \( Z \) is the bending cross section modulus.

\[
\phi_B^e = \frac{I}{I_0} = \frac{\frac{\alpha^4}{4\pi}}{\frac{\alpha^4}{4\pi}} = \frac{3}{\pi}
\]  

2.3 Combination of material indices and shape factors

Material indices or shape factors can be used separately, however, to assess the level of maturity of a design there is a possibility of their interconnection. The essence of connection is the transformation of material properties by shape factor. Evaluation of suitability is evaluated by comparison of material index values of transformed materials.

The material index combined with the elastic shape factor is created similarly to pure material index, but elimination of free variables must be based on selected shape factor. In this case, it is necessary to replace \( I_0 \) by the equation of its area and create the equation of moment of inertia based on cross section area. [1]

\[
\phi_B^e = \frac{I}{I_0} = \frac{\frac{\alpha^4}{4\pi}}{\frac{\alpha^4}{4\pi}} = \frac{3}{2\pi}
\]

Following procedure is the same as the creation of the material index.

\[
S = \frac{3E}{FL^3} = \frac{3E}{FL^3} \frac{\phi_B^e A^2}{12} \Rightarrow A = \left(\frac{4FL^3S}{E\phi_B^e}\right)^\frac{1}{2}
\]

\[
m = L\rho \left(\frac{4FL^3S}{E\phi_B^e}\right)^\frac{1}{2} = \rho \left(\frac{4\pi \cdot FL^2S}{\phi_B^e}\right)^\frac{1}{2}
\]

At the end, the material-shape factor for bending consist of material parameters and shape factor.

\[
M_B^e = \frac{(\phi_B^e)^\frac{1}{2}}{\rho}
\]

According to the equation 17 can be created the material-shape factor for strength.

\[
M_B^f = \frac{(\phi_B^f)^\frac{1}{2}}{\rho}
\]
2.4 Material-Shape indexes

Material-shape factors can show which combination is better in which category (stiffness, strength) but cannot show nothing more. For this case we create indexes which are based on the shape factors.

The first index is for mass. Lightness of a profile depends on volume and density. When we will compare two profiles of the different materials but the same length, then we need only cross-sections of the profiles and densities of both materials.

Equation of the index for light profile has form:

$$i_m = \frac{m_1}{m_2} = \frac{A_1\rho_1}{A_2\rho_2} \geq 1$$  \hspace{1cm} (19)

Where result of the equation bigger than 1 means lighter profile than original one.

Similarly we can get an index for rigid profile in bending which is based on the equation 3. This index depends on the ratio of Young's modulus and the second moment of area of both materials.

$$i_x = \frac{E_x}{E_x} = \frac{E_x}{E_x} \geq 1$$  \hspace{1cm} (20)

Where result of the equation bigger than 1 means stiffer profile than original one.

Both equations we can combine into one and get combined index for stiff and light profile.

$$i_c = \frac{i_m + i_x - 2}{\sqrt{(i_m - 1)^2 + (i_x - 1)^2}} \geq 1$$  \hspace{1cm} (21)

If the result is higher than 1, it means that the new profile is lighter and stiffer than the original one. If the result is lower than -1, it means that the new profile is heavier and flexible than the original one. If the result is between (-1; 1), it means that the new profile is either lighter and flexible, or heavier and stiffer than the original one.

3 Example of use

With this method can be compared materials for every construction which is made from profiles like is construction of hang-glider, extension ladder or cabin for agriculture machine. [2][3] For our needs we use as an example aluminium alloy bicycle frame.

This frame is a common bicycle frame which has nowadays almost every bicycle. Advantages of this frame is low weigh, good stiffness and cheap price for profiles. But the cost of work, as welding and normalization, is much more expensive, then for steel. In the past, the steels do not have enough high yield strength to create stiff and light construction, but with the high strength steels (HSS) whole situation changed. [4] In this example we will try to design a steel bicycle frame with the same output dimensions, approximately stiffness and lightness as the aluminum frame.

The aluminium frame consist of 7 different profiles as shown in Figure 4. Profiles 1, 4 and 5 are depends on the inner diameter, as these profiles must be prepare for connecting standard parts of the bicycle as a saddle tube, handlebars and shaft hub. Profiles 6 and 7 are the same and profile no. 3 is profile of the elliptical shape.

| Number of profile | Dimension [mm] |
|-------------------|----------------|
| 1                 | Ø32x2          |
| 2                 | Ø32x3          |
| 3                 | 40x30x3        |
| 4                 | Ø32x3          |
| 5                 | Ø52x3          |
| 6                 | Ø20x2          |
| 7                 | Ø20x2          |

Fig. 4 Numbers of profiles
The original material is the aluminium alloy AW 6061 T6 and as the comparative material will be used HSS Reynolds 531. Input values for calculation are density, Young’s modulus and yield strength of both materials and dimension of the profiles of the original profile. Profiles are loaded by bending.

**3.1 Designing of profile 1**

The dimensions of the first profile are limited by the requirement of the inner diameter which must to be equal to \( d_2 = 28 \text{ mm} \). For this condition we need to upgrade equation for \( i_m \):

\[
A_{12,\text{max}} \leq \frac{p_1}{p_2} \cdot A_{11} \tag{22}
\]

Where \( A_1 \) and \( A_{2,\text{max}} \):

\[
A_{11} = \frac{\pi}{4} (D_{11}^2 - d_{11}^2) \tag{23}
\]

\[
A_{12,\text{max}} = \frac{\pi}{4} (D_{12}^2 - d_{12}^2) \tag{24}
\]

Then:

\[
\frac{\pi}{4} (D_{12}^2 - d_{12}^2) \leq \frac{p_1}{p_2} \cdot \frac{\pi}{4} (D_{11} t_{11} - t_{11}^2) \tag{25}
\]

\[
D_{12,\text{max}} \leq \frac{p_1}{p_2} \cdot \frac{\pi}{4} (D_{11} t_{11} - t_{11}^2) - d_{12}^2 \tag{26}
\]

After values have been set we have relations:

\[
D_{12} \leq 29.437 \text{ mm} \tag{27}
\]

If we want this profile lighter than aluminium, then we need to choose an outer diameter smaller than 29.437 mm. If we choose \( D_{12} = 29 \text{ mm} \), then \( t_{12} = 0.5 \text{ mm} \) and combine index will be:

\[
i_{c1} = 0.154 \tag{28}
\]

That means that the HSS profile is lighter but flexible than the original one. If we calculate other indices, we can see how much it is.

\[
i_{m1} = 1.448 \Rightarrow 44.8\% \tag{29}
\]

\[
i_{s1} = 0.640 \Rightarrow -36\% \tag{30}
\]

The profile is almost 45% lighter but stiffness is lower about 36%. We saved a lot of weight but we also lost a lot of stiffness. If \( D_{12} = 30 \text{ mm} \), than \( t_{12} = 1 \text{ mm} \) and combine index will be:

\[
i_{c1} = 0.137 \tag{31}
\]

\[
i_{m1} = 0.712 \Rightarrow -28.8\% \tag{32}
\]

\[
i_{s1} = 1.351 \Rightarrow 35.1\% \tag{33}
\]

Last try is select \( D_2 \) as closer as possible to 29.437 mm, then for \( D_{12} = 29.5 \text{ mm} \) will be \( t_{12} = 0.75 \text{ mm} \) and combine index will be:

\[
i_{c1} = -1.255 \tag{34}
\]

\[
i_{m1} = 0.957 \Rightarrow -4.3\% \tag{35}
\]

\[
i_{s1} = 0.986 \Rightarrow -1.4\% \tag{36}
\]

These result are pretty close to the original profile, but still a little bit worse. As we can see, in these small dimensions with this material we cannot reach better result in both categories.

The length of this profile is only 125 mm so selecting variation with bigger \( i_{m1} \) still does not save enough weight. Therefore, it will be better to use variation with larger \( D_2 \) which will increase stiffness of the first profile and also help to save weight to profiles 2 and 3 because these profiles are much longer. For this the best dimensions of first profile are \( D_{12} = 30 \text{ mm} \), \( t_{12} = 1 \text{ mm} \).

Before we can use this profile we need to check it if is not inclinable to buckling. For this we use Ashby’s equation for tube:

\[
\phi_{B1}^{f} = \frac{3}{\sqrt{2\pi}} \cdot \frac{D_{12}}{2t_{12}} \leq \phi_{B,\text{max}}^{f} \tag{37}
\]

**3.2 Designing of profile 2**

The profile 2 is round tube in the original frame and does not have any requirements to shape. Only limit is width 30 mm because it must be connected to the profile 1.

The original dimensions of the profile 2 are \( D_{21} = 32 \text{ mm} \), \( t_{21} = 3 \text{ mm} \). For round tube we know that \( D_{22} = 30 \text{ mm} \) and only thickness of the tube is missing. According to equation 23:

\[
t_{22,\text{max}} \leq \frac{D_{22} - \frac{p_1}{p_2} \cdot 4(D_{21} t_{21} - t_{21}^2)}{2} \tag{38}
\]

For \( D_{22} = 30 \text{ mm} \) and \( t_{21} = 1 \text{ mm} \) shape factors will be:

\[
i_{c2} = 0.676 \tag{40}
\]

\[
i_{m2} = 1.032 \Rightarrow 3.2\% \tag{41}
\]

\[
i_{s2} = 0.991 \Rightarrow -0.9\% \tag{42}
\]

This HSS profile is almost equal to aluminium and we cannot get higher values for one index without we lose in second one. But if we use the elliptical shape of the tube, we can get better results.

For this case, the width must be equal to 30 mm and thickness less than 1 mm, so if \( a_{22} = 30 \text{ mm} \) and \( t_{22} = 0.8 \text{ mm} \) then \( b_{22,\text{max}} \) will be:

\[
A_{22} = \frac{\pi}{2} \cdot t_{22} (a_{22} + b_{22} - 2t_{22}) \tag{43}
\]

\[
b_{22,\text{max}} \leq \frac{8b_{22}^2 (a_{22} t_{21} - t_{21}^2)}{\pi t_{22}} - a_{22} + 2t_{22} \tag{44}
\]

\[
b_{22,\text{max}} \leq 46.409 \text{ mm} \tag{45}
\]

If \( b_{22} = 46 \text{ mm} \), then:

\[
i_{c2a} = 1.038 \geq 1 \tag{46}
\]

\[
i_{c2b} = 1.005 \geq 1 \tag{47}
\]

\[
i_{m2} = 1.005 \Rightarrow 0.5\% \tag{48}
\]
Both axes are stiffer than the original profile, but as we can see, there is still potential to reduce weight if we reduce stiffness in the axis “a” to difference close to 0%.

For checking profile for buckling we need to use Ashby’s equation for ellipse shaped tube:

\[
\phi_{B_2} = \frac{1}{\sqrt{\pi}} \sqrt{\frac{\pi}{\frac{e^2}{2} \left[ \frac{e^2}{2} \frac{b_{22}}{e^2} \right] \left[ \frac{e^2}{2} \frac{a_{22}}{e^2} \right]}} \leq \phi_{B_{max}}
\]

(59)

\[
\phi_{B_2} = 5.141 \leq 5.961
\]

(60)

As we can see, this profile is 9.4% lighter than the original one, stiffer in axis “b” by 58.2% and in axis “a” is stiffness almost identical. Checkup for buckling is all right though we are close to the maximum limit of material-shape factor.

### Tab. 4 Results of HSS profiles

| Number of profile | Dimension [mm] | \(i_{5a}\) [%] | \(i_{5b}\) [%] | \(i_{m}\) [%] | Weight saving [g] |
|-------------------|----------------|----------------|----------------|---------------|------------------|
| 1                 | 030x1          | 35.1           | 35.1           | -28.8         | -25.78           |
| 2                 | 30x40x0.8      | 1.6            | 58.2           | 9.4           | 36.42            |
| 3                 | 30x50x0.8      | 18.7           | 0.1            | 5.3           | 18.82            |
| 4                 | 030x1          | -0.9           | -0.9           | 3.2           | 10.93            |
| 5                 | 048x1          | -12            | -12            | 7.6           | 10.54            |
| 6                 | 20x24x0.5      | 8.7            | 43.9           | 15.2          | 2x 19.32         |
| 7                 | 20x24x0.5      | 8.7            | 43.9           | 15.2          | 2x 17.31         |

### Tab. 5 Comparison of shape factors for AW and HSS

| Number of profile | \(\phi_B\) AW [1] | \(\phi_B\) HSS [1] | \(\phi_B\) AW [1] | \(\phi_B\) HSS [1] |
|-------------------|------------------|------------------|------------------|------------------|
| 1                 | 7.639            | 14.324           | 3.385            | 4.635            |
| 2                 | 5.093            | 16.443           | 2.764            | 5.141            |
| 3                 | 4.385            | 15.107           | 2.655            | 5.049            |
| 4                 | 5.093            | 14.324           | 2.764            | 4.635            |
| 5                 | 8.276            | 22.918           | 3.523            | 5.863            |
| 6                 | 4.775            | 18.152           | 2.676            | 5.335            |
| 7                 | 4.775            | 18.152           | 2.676            | 5.335            |

### Tab. 6 Comparison of material-shape factors for AW and HSS

| Number of profile | \(M_y^a\) AW [1] | \(M_y^a\) HSS [1] | \(M_y^f\) AW [1] | \(M_y^f\) HSS [1] |
|-------------------|------------------|------------------|------------------|------------------|
| 1                 | 270.84           | 220.94           | 33.14            | 32.52            |
| 2                 | 221.14           | 236.72           | 28.95            | 34.84            |
| 3                 | 205.19           | 226.90           | 28.18            | 24.43            |
| 4                 | 221.14           | 220.94           | 28.95            | 32.52            |
| 5                 | 281.90           | 279.47           | 34.03            | 38.04            |
| 6                 | 214.12           | 248.71           | 28.33            | 35.72            |
| 7                 | 214.12           | 248.71           | 28.33            | 35.72            |
In Table 4 we can see that all HSS profiles have higher or same stiffness as the original ones expect profile No. 5 and total weight saving is equal to 124.184 g. If we compare values in Table 5 with Table 2 we find out that profile No. 5 is also very close to the maximum limit of material-shape factor for buckling. This could be dangerous, but because the shaft hub for pedals will be placed inside, the whole profile will be reinforced. By comparison of these two tabs we can also see that the aluminium alloy has a higher potential of use than the HSS, because factors of the HSS for buckling are very close to the limit.

From Table 6 is obvious that almost all HSS profiles have better stiffness and strength than aluminium profiles. This means that results of FEM analysis should say that HSS frame will be stiffer and stronger than the aluminium frame.

3.4 Computer model and FEM

With dimensions of profiles from the Table 4 was created a 3D computer model of bicycle frame and it was compared to the original one. The premise says that HSS frame should be slightly stiffer and lighter than the original one.

Total weight of aluminium frame is:

\[ m_{AW} = 2.196 \text{ kg} \]  

Total weight of HSS frame is:

\[ m_{HSS} = 2.322 \text{ kg} \]

As we see, the HSS frame is slightly heavier than the aluminium, how is it possible? There are two reasons. First, the frame is composed of profiles and holders of the rear wheel. These holders are made of solid material of the same volume, so the HSS holders are heavier than the aluminium ones. Second, the frame is welded so that it is necessarily count with extra weight of the welds, where the HSS welds have a higher weight again.

![Deformation and stress of aluminum frame](image1)

![Deformation and stress of HSS frame](image2)

Frames were loaded by two forces. First equal to 2000N was placed in the middle of profile No. 5. The second one was placed on the top of profile No. 1. equal to 1000N. This distribution of forces corresponds to the dynamic load of the cyclist during impact after jump.

Deformation of aluminium frame is 4.41 mm and maximum stress is 100,39 MPa. Deformation of HSS frame is 3.05 mm and maximum stress is 267,53 MPa.

If we compare the maximum values of stress both frames with the yield modulus of both materials, we reach the safety factor of the structure.

For aluminium frame:

\[ k_{AW} = \frac{\sigma_f}{\sigma_{AW}} = \frac{250}{100.39} = 2.49 \]
For HSS frame:
\[
k_{sfr} = \frac{\sigma_f}{\sigma_{st}} = \frac{880}{267.53} = 3.29
\] (64)

The safety factor of the HSS frame is 1.3x higher than aluminium one. This means that HSS is stronger than aluminium alloy and we can pick up steel with lower yield modulus. If we would want the same value of safety, we need to find a steel with a yield modulus close to 666 MPa.

By comparing the results from Table 6 with the results from FEM analysis it is clear that HSS frame is really stronger and stiffer than the aluminium and the prediction made by material-shape factors was correct.

4 Conclusion

The goal of this paper was to show the possibilities of using material-shape factors for the design of profile structures. With a combination of Microsoft Excel we can get quick, easy and powerful tool for structure optimization who can be used by everyone.

Acknowledgments

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