SHARP REGULARITY THEORY OF SECOND ORDER
HYPERBOLIC EQUATIONS WITH NEUMANN BOUNDARY
CONTROL NON-SMOOTH IN SPACE

ROBERTO TRIGGIANI

Department of Mathematical Sciences
University of Memphis
Memphis, TN, 38152, USA

Abstract. The purpose of this paper is to complement available literature on sharp regularity theory of second order mixed hyperbolic problem of Neumann type [13, 15, 26] with a series of new results in the case–so far rather unexplored–where the Neumann boundary term (input, control) possesses a regularity below $L^2(\Gamma)$ in space on the boundary $\Gamma$. We concentrate on the cases $H^{-\frac{1}{2}}(\Gamma)$, $H^{-\beta}(\Gamma)$, $H^{-1}(\Gamma)$, $\beta$ being a distinguished parameter of the problem. Our present results are consistent with the sharp result of [13, 15, 26] (obtained through a pseudo-differential/micro-local analysis approach), whose philosophy is expressed by a gain of $\beta$ in space regularity in going from the boundary control to the position in the interior. A number of physically relevant illustrations are given.

1. Problem formulation, literature. Orientation

The purpose of this paper is to complement available literature on sharp regularity theory of second order mixed hyperbolic problem of Neumann type [13, 15, 26] (see also [17, p 739]) with a series of new results in the case–so far rather unexplored–where the Neumann boundary term (input, control) possesses a regularity below $L^2$ in space on the boundary. We let throughout $\dim \Omega \geq 2$. Though the starting point of our present analysis will be the sharp regularity theory available in [13, 15, 26], the results given in this paper are, except for Theorem 4.1, not explicitly contained in these references. Our present effort is prompted and stimulated by the recent papers [1], [2]. More precisely, [2] studies the shape differentiability and sensitivity analysis for the solution of the wave equation on a bounded domain with an active Neumann boundary term with respect to the shape of the geometric domain on which the wave equation is defined. In doing so, [2] needs a regularity result that guarantees that the wave equation will have $H^1$-space interior regularity in the position. One such result–with highly asymmetric assumptions on the Neumann boundary datum—is given in [1], with an ad-hoc proof which uses Galerkin techniques. Such regularity result of [1]–to be recalled as Theorem 1.3 below–is not sharp. It is however in line with the philosophy of the original treatment in [21, p 120] which yields a gain of regularity in space from the Neumann boundary term to the solution (position) in the interior of about $\frac{1}{2}$-space derivative. See the precise statement in (1.3) below. Why the result of [1] is within the philosophical setting of [21] can be explained if one factors-in that, for the wave equation, one derivative in time corresponds to one derivative
in space; see Remark 1.4 below Theorem 1.3. A completely different, soft proof of the (non-optimal) regularity result of [1] is given in the Appendix: starting from the original abstract model of the wave equation with Neumann boundary control—which was introduced originally in [27] and was extensively used by the authors of [9] to culminate in the treatment of the monograph [17]—it only employs basic elliptic theory and sine/cosine theory, in a purely functional analytic approach. It has been known since [13] that technical tools other than pseudo-differential microlocal analysis techniques are not suitable, and are not expected, to provide sharp regularity results for wave equation mixed problems in dimension greater than or equal to 2, with Neumann boundary control, a case where the Lopatinski condition is not satisfied. In contrast, in the case of second order hyperbolic mixed problems with Dirichlet boundary control—where the Lopatinski condition is satisfied—differential multipliers suffice to obtain sharp/optimal interior and boundary regularity [8]. [Such reference provided a cleaner proof by duality of the basic level regularity result of [9], originally obtained for an $L^2(\Sigma)$-Dirichlet boundary control through a more complicated analysis of the non-homogeneous problem.]

**Problem formulation.** Let $\Omega$ be an open bounded domain in $\mathbb{R}^n, n \geq 2$, with sufficiently smooth boundary $\Gamma$. We consider a second order hyperbolic problem defined on $\Omega$ with Neumann boundary control $g$ acting on $\Gamma$:

\[
\begin{align*}
y_{tt} &= -A(\xi, \partial)y + h \quad &\text{in } (0, T] \times \Omega \equiv Q \\
y_{t=0} &= y_0, \quad y_{t=0} = y_1 \quad &\text{in } \Omega \\
\frac{\partial y}{\partial \nu} \bigg|_{\Sigma} &= g \quad &\text{in } (0, T] \times \Gamma \equiv \Sigma
\end{align*}
\]

In (1.1a), $A(\xi, \partial)$ is a (time-independent) partial differential operator of order two in $\Omega$, with smooth real coefficients:

\[
A(\xi, \partial) = \sum_{i,j=1}^{n} a_{ij}(\xi) \frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{j=1}^{n} b_j(\xi) \frac{\partial}{\partial \xi_j} + c_0(\xi)
\]

and with principal part uniformly elliptic in $\Omega$: \[\sum_{i,j=1}^{n} a_{ij}(\xi) \eta_i \eta_j \geq c \sum_{j=1}^{n} \eta_j^2, \quad a_{ij} = a_{ji}, \quad c > 0,\]

whereby then $g$ is applied to the corresponding co-normal derivative, denoted by $\frac{\partial}{\partial \nu}$. As noted, this is the generality of references [13, 15, 26], whose results will be the starting point of our present analysis.

**Goal** The purpose of this paper is to provide sharp regularity theory for problem (1.1a)–(1.1c)—fully consistent with the philosophical setting of references [13, 15, 26] noted below in Theorem 1.1 in the new and presently considered case where the Neumann boundary control is assumed to possess a space regularity (on $\Gamma$) below the $L^2(\Gamma)$-level; for instance, $H^{-\frac{3}{2}}(\Gamma)$, or $H^{-1}(\Gamma)$ being relevant cases. These are treated in Section 4 and 5, respectively. This is the key feature and novelty of the present paper over available literature.

**Literature** To the best of our knowledge, the relevant references on interior and boundary sharp regularity of second order hyperbolic equations with Neumann boundary control $g$ of at least $L^2(\Sigma)$-regularity are [13, 15, 26]. [For the purpose
of the present discussion, we may take throughout $y_0 = 0, y_1 = 0, h \equiv 0$, for the standard data.]

The $\frac{1}{2}$-gain in space \[21, 22\] References \[13, 15, 26\] (in chronological order) followed by at least twenty years the original contribution of \[21\]. More precisely, \[21\], Vol. II, p 120\] shows the following result on the wave equation problem corresponding to \((1.1a)–(1.1c)\) \[-A(\xi, \partial) = \Delta/:\]

$$g \in L^2(0, T; L^2(\Gamma)) \quad y = Lg \in C([0, T]; H^{\frac{1}{2}-\epsilon}(\Omega) \cap L^2(0, T; H^{\frac{1}{2}}(\Omega)), \quad \epsilon > 0 \text{ arbitrary} \quad (1.3)$$

continuously, with two unites less in space for $yt = \Delta y$. Result \((1.3)\) embodies the ‘about $\frac{1}{2}$-gain in space’ philosophy from Neumann boundary term $g$ to solution $y$ in the interior, which was referred to above.

Reference \[22\] studies likewise the regularity of second order hyperbolic problem \((1.1a)–(1.1c)\) with Neumann boundary control, where however the coefficients of the operator $A(\cdot, \cdot)$ in \((1.2a)\) are also time-dependent. Fourier/pseudo-differential operator techniques, in the style of \[23\] for the Dirichlet-problem, are used in \[22\].

The main result of \[22\] shares with \[21\] the philosophy of an improvement of “$1\frac{1}{2}$” in Sobolev space regularity, from $H^{\frac{1}{2}}(\Gamma)$ for $g$ to $H^1(\Omega)$ for the interior solution (position) $y$.

Lifting of time regularity With regard to result \((1.3)\) for $y \in L^2(0, T; H^{\frac{1}{2}}(\Omega))$, we note that it was proved in \[10, 11\] (originally, in the context of second order hyperbolic equations with Dirichlet control) that the $L^2$-regularity in time such as the one in \((1.3)\) can be lifted to $C([0, T]; \cdot)$ regularity in time, while preserving the regularity in space. More precisely, and in particular in the present case, with reference to problem \((1.1a)–(1.1c)\), as soon as one has

$$L : g \longrightarrow Lg = y : \text{continuous } L^2(0, T; L^2(\Omega)) \longrightarrow L^2(0, T; H^{\frac{1}{2}}(\Omega)) \quad (1.4)$$

as in \((1.3)\), it then follows that

$$L : g \longrightarrow Lg = y : \text{continuous } L^2(0, T; L^2(\Gamma)) \longrightarrow C([0, T]; H^{\frac{1}{2}}(\Omega)). \quad (1.5)$$

This result was originally proved in \[10\], lifting to $C$-the original $L^2$-time regularity of \[9\], in the context of the wave equation with Dirichlet boundary control, where $L^2(\Gamma)$ is the boundary space, and $L^2(\Omega)$ is the interior space for the position. Its proof relied on the property that, given the abstract model for a wave equation with either Dirichlet or Neumann boundary control which was introduced in \[27\], the corresponding variation of parameter formula for the map: boundary control $\hat{g} \rightarrow$ solution $\{L\hat{g}, \frac{\partial}{\partial t}(L\hat{g})\}$ can be expressed in terms of a strongly continuous group, precisely the group generated by the free-dynamics operator. An abstract version of the time-lifting property was next given in \[11, 17, \text{Chapter 8, p 651}\], assuming more generally initial $L_p(0, T; \cdot)$-regularity, for any $1 \leq p < \infty$, not only $p = 2$. It is reproduced below for further use. The implication \((1.4)–(1.5)\), combined with the result in \((1.3)\), makes now precise the notion of $\frac{1}{2}$-gain in space regularity for the mixed problem \((1.1a)–(1.1c)\) referred to before in going from the Neumann boundary datum $g$ to the corresponding hyperbolic solution $Lg = y$ (when $y_0 = y_1 = 0, h \equiv 0$). It thus improves by “$\epsilon$” the original result of \[21\] in \((1.3)\).

Abstract result on time-lifting regularity \[17, \text{Theorem 7.3.1, p 651}\] Let $X$ and $U$ be reflexive Banach spaces with $X^*$ and $U^*$ their adjoints. For given $0 < T < \infty$
fixed, we study the operator
\[ (Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau, \quad 0 \leq t \leq T \] (1.6)
corresponding to the mild solution
\[ x(t) = e^{At} x_0 + (Lu)(t) \] (1.7)
of the abstract equation
\[ \dot{x} = Ax + Bu \in [\mathcal{D}(A^*)]' \quad x(0) = x_0, \] (1.8)
subject to the following standing assumptions:
(i) \( A : X \supset \mathcal{D}(A) \to X \) is a linear operator, which is the infinitesimal generator of a strongly continuous (s.c.) group \( e^{At} \) on \( X \);
(ii) \( B \) is a linear, continuous operator \( U \to [\mathcal{D}(A^*)]' \), where \( A^* \) is the \( X \)-adjoint of \( A \), and \( [\mathcal{D}(A^*)]' \) is the dual of \( \mathcal{D}(A^*) \) with respect to the pivot space \( X \), or equivalently
\[ A^{-1}B \in \mathcal{L}(U : X). \] (1.9)
[W.l.o.g. for consideration of the operator \( L \) below over a finite interval, we may assume \( A^{-1} \in \mathcal{L}(X) \) for otherwise we replace \( A \) with a suitable translation].

**Theorem 1.0.** [17, p 651] Assume (i) and (ii). Moreover, suppose that the operator \( L \) in (1.6) satisfies
\[ L : \text{continuous } L_p(0,T;U) \to L_p(0,T;X), \quad 1 \leq p < \infty. \] (1.10)
Then, in fact, in follows that
\[ \mathbb{L} : \text{continuous } L_p(0,T;U) \to C([0,T];X); \] (1.11)
so that, equivalently by duality, the following (abstract trace regularity) property holds true
\[ \int_0^T \| B^* e^{At} x^* \|^q_{U^*} \, dt \leq C_T \| x^* \|^q_{X^*}, \quad x^* \in \mathcal{D}(A^*), 1 \leq q < \infty, \frac{1}{p} + \frac{1}{q} = 1. \] (1.12)
Thus, the closable operator \( B^* e^{At} \) admits a continuous extension (denoted by the same symbol) satisfying
\[ B^* e^{At} : \text{continuous } X^* \to L_q(0,T;U^*). \] (1.13)

**Remark 1.1.** As already noted, this result was originally given in [10] in the context of second order hyperbolic equations with \( L^2(0,T;L^2(\Gamma)) \)-Dirichlet boundary control. More precisely, in this reference, a soft argument was given to lift the regularity \( \{y, y_t \} \in L^2(0,T;L^2(\Omega) \times H^{-1}(\Omega)) \) in [9] to \( C([0,T];L^2(\Omega) \times H^{-1}(\Omega)) \), for the interior solution. This argument was then abstracted in [11] and [17, Chapter 7, p 651].

From a \( \frac{1}{2} \)-gain to a \( \alpha=\beta \)-gain, where \( \alpha = \beta = \frac{2}{3} \) for a general domain \( \Omega \), and \( \alpha = \beta = \frac{3}{4} \) for parallelepipeds: from the boundary datum to the interior solution \( y \). For ready connection with the results in the references [13, 15, 26] to be critically invoked throughout this paper, we shall henceforth set
\[ \begin{cases} \alpha = \beta = \frac{2}{3} & \text{for a general sufficiently smooth domain } \Omega \text{ in } \mathbb{R}^n \quad (1.14a) \\ \alpha = \beta = \frac{3}{4} & \text{for a parallelepiped } \Omega \text{ in } \mathbb{R}^n. \quad (1.14b) \end{cases} \]
Theorem 1.1. [13] [15, Theorem 2.0, p 123; Theorem 2.1, p 124] With reference to problem (1.1a)–(1.1c), let \( y_0 = y_1 = 0, h = 0 \), and \( n \geq 2 \). Then we have the interior regularity

\[
g \in L^2(0, T; L^2(\Omega)) \equiv L^2(\Sigma) \implies \begin{cases} \frac{dg}{dt}(y) \in C([0, T]; H^{\alpha}) \equiv D(A^{\alpha/2}) \quad (1.15a) \\
\end{cases}
\]

as well as the independent boundary (trace) regularity

\[
\frac{dg}{dt}(y)|_\Sigma = y|_\Sigma \in H^{2\alpha-1}(\Sigma). \quad (1.16)
\]

In (1.15a)–(1.15b), \( A \) is the operator defined in (3.1) below.

Theorem 1.1 shows that, for a general smooth domain \( \Omega \subset \mathbb{R}^n \) (respectively, for a parallelepiped in \( \mathbb{R}^n \), \( n \geq 2 \), the actual gain in space regularity from \( g \) to \( y \) is \( \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \) (respectively, \( \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \)) higher than the original result (1.3) in [21], as improved by (1.4)–(1.5). Corresponding improved regularity results for the adjoint problem (i.e. the operator \( L^* \) to be defined below) apply.

Consider the problem

\[
\begin{cases}
\phi_{tt} - A^*(\xi, \partial)\phi + f & \text{in } (0, T] \times \Omega \equiv Q \\
\phi(T, \cdot) = 0, \quad \phi_t(T, \cdot) = 0 & \text{in } \Omega \\
\frac{\partial\phi}{\partial\nu}|_{\Sigma} = 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma
\end{cases} \quad (1.17a)
\]

where \( -A^*(\xi, \partial) \) is the formal adjoint, which together with the zero corresponding co-normal derivative realize the adjoint operator \( A^* \). The adjoint operator \( L^* \) with respect to \( Lg = y \) (for \( y_0 = y_1 = 0, h = 0 \) in (1.1a)–(1.1c)) is the trace operator

\[
(L^* f)(t) = \phi(t)|_\Sigma, \quad \text{where } (Lg, f)_{L^2(Q)} = (g, L^* f)_{L^2(\Sigma)} \quad (1.18)
\]

Theorem 1.2. (a) [13] [15, Theorem 2.0, Eq (2.10), p 124] With reference to the adjoint problem (1.17a)–(1.17c) and the corresponding operator \( L^* \) in (1.18), we have

\[
f \in L^2(0, T; L^2(\Omega)) \equiv L^2(\Sigma) \implies L^* f = \phi|_\Sigma \in H^{3}(\Sigma) = L^2(0, T; D^3(\Gamma)) \cap H^2(0, T; L^2(\Gamma)) \quad (1.19)
\]

\[\text{(b) [15, Corollary 6.3, p 158] More generally, for } 0 \leq \theta \leq 1:
\]

\[
f \in H^\theta(Q) \\
f(T) = 0 \implies L^* f = \phi|_\Sigma \in H^{3+\theta}(\Sigma) \quad (1.20)
\]

continuously [the compatibility condition (C.C.) \( f(T) = 0 \) is not recognized for \( \theta \leq \frac{1}{2} \)].

Remark 1.2. Throughout the paper we are maintaining the distinction between the parameter \( \alpha \) (which pertains to interior regularity) and the parameter \( \beta \) (which pertains to boundary regularity), even though at the end of the investigations in [13, 15, 26] it turned out that they are numerically equal, \( \alpha = \beta \), as noted in (1.14). This strategy will make it easier to follow the arguments of the present paper when they make direct reference to, typically, results of [15], where they were stated in terms of \( \alpha \) or \( \beta \) separately, as it was not yet determined at that time that in all cases \( \alpha = \beta \).
The result of [1] We close this section by stating the regularity result of [1], which, under asymmetric assumptions on the Neumann datum \( g \), yields the \( H^1(\Omega) \)-space regularity in the interior for \( Lg = y \), which is needed in the shape differentiability and sensitivity analysis of [2]. By asymmetric assumptions, we mean that the Neumann control \( g \) is assumed smooth in time (in fact, with two time derivatives) and non-smooth in space (in fact, with space regularity \( H^{-\frac{1}{2}}(\Gamma) \)).

**Theorem 1.3.** [1] With reference to problem (1.1a)–(1.1c), let \( y_0 = y_1 = 0, h \equiv 0 \), and

\[
g \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma)), \quad \dot{g} \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma)); \quad g(0) = 0.
\]

Then, continuously,

\[
Lg = y \in L^\infty(0, T; H^1(\Omega)), \quad \frac{d}{dt}(Lg) = y_t \in L^\infty(0, T; L^2(\Omega)).
\]  

**Remark 1.3.** We note that the regularity assumptions on \( g \) and \( \dot{g} \) in (1.21) imply [20, Theorem 3.1 for \( j = 0 \), page 19] that \( g \in C([0, T]; H^{-\frac{1}{2}}(\Gamma)) \), whereby then the extra condition \( g(0) = 0 \) makes sense in \( H^{-\frac{1}{2}}(\Gamma) \). The proof in [1] with \( C^2 \)-domains is obtained by Galerkin technique. It is further noted in [2] that the same proof can be extended to Lipschitz domains as well, by a density argument. In the Appendix, we shall provide an altogether different proof yielding a slightly stronger result than Theorem 1.3 by means, essentially, of ‘soft’ arguments.

**Remark 1.4.** Theorem 1.3 is not sharp. Qualitatively, for second order hyperbolic equations, one time derivative corresponds to one space derivative. Thus, philosophically, Theorem 1.3 corresponds to taking an extra space derivative for \( g \); i.e. \( g \in H^\frac{1}{2}(\Gamma) \). It then delivers a solution \( Lg \) in \( H^1(\Omega) \), with a gain of \( \frac{1}{2} \) from the Neumann boundary datum \( g \) to the interior solution \( Lg \). In this case, Theorem 1.3 (as well as its counterpart in Theorem A.1, in the Appendix) is in the \( \frac{1}{2} \)-gain setting of [21], as explained in (1.4). The definite improvement of Theorem 1.3 will be provided instead in Theorem 4.5 in Section 4 (same as Theorem B in Section 2). See also Theorem 4.3 in Section 4 (same as Theorem A in Section 2). The same conclusion (1.22) with time regularity actually boosted to \( C([0, T]; \cdot) \) will be obtained essentially only with \( g \in H^\frac{1}{2}(0, T; H^{-\beta}(\Gamma)), g(0) = 0 \). This improvement from \( \frac{1}{2} \) to \( \beta \) in the gain of regularity will rest on the sharp regularity theory of [13, 15, 26], of which Theorem 1.1 (where \( \alpha = \beta \)) is a canonical illustration.

**Remark 1.5.** After the sharp/optimal regularity theory in [9, 10, 8] was obtained for the Dirichlet boundary control case of wave equations, it was conjectured that in going from Dirichlet to Neumann boundary datum, the regularity will improve by one unit, as in the case of elliptic or parabolic equations. However, a counterexample for \( \text{dim } \Omega \geq 2 \) was given in [13], showing that with an \( L^2(0, T; L^2(\Gamma)) \)-Neumann boundary control, the interior regularity of the position \( y = Lg \) cannot exceed \( H^\frac{1}{2}(\Omega) \); that is, a regularity of \( H^{\frac{1}{2}+\epsilon}(\Omega) \), \( \epsilon > 0 \), is not possible for \( \text{dim } \Omega \geq 2 \). Moreover, the regularity \( H^{\frac{1}{2}}(\Omega) \) is achieved in the case of \( \Omega \) being a parallelepiped. On the other hand, it was also shown that the conjectured 1-gain of regularity from Dirichlet to Neumann is possible if the data are compactly supported [14, 25]. Finally, of course, in the one dimensional case, an elementary treatment gives the desired improvement of one unit [17, p 758, or 859, or 882, or 962].
2. Some main results. While we refer to the subsequent Sections 4 and 5 for a list of results, and their proofs, in this section we single out a selection of them, with particular reference to the motivating Theorem 1.3.

**Theorem A.** (Recall $\beta$ in (1.14a)–(1.14b).) Let
\[ g \in H^1(0, T; H^{-\frac{1}{2}}(\Gamma)) \cap C([0, T]; H^{\beta-1}(\Gamma)); \quad g(0) = 0. \] (2.1)

Then, continuously,
\[ Lg \in C([0, T]; H^{\beta+\frac{1}{2}}(\Omega)), \quad \frac{d}{dt}(Lg) \in C([0, T]; H^{-\frac{1}{2}}(\Omega)). \] (2.2)

**Remark 2.1.** This is Theorem 4.3 in Section 4. Under a slightly stronger hypothesis than in Theorem 1.3, it yields a solution $Lg$ which is slightly smoother in time regularity, and more importantly is smoother in space regularity over Theorem 1.3, Eq. (1.22), by \((\beta + \frac{1}{2}) - 1 = \beta - \frac{1}{2}\), which is equal to \(\frac{1}{6}\) for a general domain $\Omega$, or is equal to \(\frac{1}{4}\) for parallelepipeds.

**Theorem B.** Let
\[ g \in H^1(0, T; H^{-\beta}(\Gamma)) \cap C([0, T]; H^{-\frac{1}{2}}(\Gamma)); \quad g(0) = 0. \] (2.3)

Then, continuously,
\[ Lg \in C([0, T]; H^1(\Omega)), \quad \frac{d}{dt}(Lg) \in C([0, T]; L^2(\Omega)). \] (2.4)

**Remark 2.2.** This is Theorem 4.5. It is the definite improvement of Theorem 1.3.

**Theorem C.** For $0 \leq \theta \leq 1$, let
\[ g \in L^2(0, T; H^{-\beta(1-\theta)}(\Gamma)). \] (2.5)

Then, continuously,
\[ Lg \in C([0, T]; H^{\beta\theta}(\Omega)). \] (2.6)

In particular, for $\theta = \frac{\beta - \frac{1}{2}}{\beta} < 1$, we have continuously
\[ g \in L^2(0, T; H^{-\frac{1}{2}}(\Omega)) \rightarrow Lg \in C([0, T]; H^{(\beta - \frac{1}{2})}(\Omega)). \] (2.7)

**Remark 2.3.** This result is a-fortiori in Corollary 4.2 (see also Equation (4.13)). Consistently with (1.15a) of Theorem 1.1, where $\alpha = \beta$, it shows a gain of space regularity equal to $\beta$ from $g$ to $Lg$.

**Theorem D.** Let
\[ g \in L^2(0, T; H^{-1}(\Gamma)). \] (2.8)

Then, continuously,
\[ Lg \in C([0, T]; H^{\beta-1}(\Omega)). \] (2.9)

**Remark 2.4.** This is Theorem 5.4. It shows a gain of $\beta$ (from $-1$ to $(\beta - 1)$) in the space regularity, in going from $g$ to $Lg$, again consistently with (1.15a) of Theorem 1.1, where $\alpha = \beta$. 
3. Preliminary background. We return to problem (1.1a)–(1.1c) and its adjoint (1.17a)–(1.17c) and introduce a number of relevant objects:

1. the operator 
\[ A : L^2(\Omega) \supset D(A) \rightarrow L^2(\Omega) \]
defined by
\[ A h = A(\xi, \partial) h, h \in D(A) = \left\{ h \in H^2(\Omega) : \frac{\partial h}{\partial \nu} \bigg|_{\Gamma} = 0 \right\} \] (3.1)
[canonically, \(-Ah = \Delta h\).]

2. The operator \(-A\) is the generator of a s.c. (strongly continuous) ‘cosine’ operator \(C(t)\) on \(L^2(\Omega), t \in \mathbb{R}\), with (sometimes called ‘sine’ operator) \(S(t) = \int_0^t C(\tau) d\tau\). Similarly, \(-A^*\) generates a s.c. cosine operator on \(L^2(\Omega)\), given precisely by \(C^*(t)\). For general background on cosine operators, we refer to [24, 3], and bibliography cited therein. If \(A\) is self-adjoint, so are \(C(t)\) and \(S(t)\), and the standard self-adjoint calculus applies.

3. Without loss of generality for the regularity problem here considered, we may assume that the null space of \(A\) is trivial: \(N(A) = \{0\}\), for replacement of the original \(A\) with a suitable translation \((A + k^2 I)\) does not change the regularity over \([0, T], T < \infty\). Thus, we may take fractional powers \(A^\theta\) of \(A\), \(0 < \theta < 1\), to be well defined [Recall that \(-A\) (canonically the Laplacian \(\Delta\) with Neumann homogeneous B.C.) a fortiori generates a s.c. analytic semigroup on \(L^2(\Omega)\), \(t > 0\)].

4. As in [17, 28], define the Neumann map \(N\) (elliptic extension of a Neumann boundary datum) by
\[ h = Nu \iff \left\{ -A(\xi, \partial) h = 0 \text{ in } \Omega; \frac{\partial u}{\partial \nu} = u \text{ on } \Gamma \right\} \] (3.2)
We have [20] as well as [4, 5, 7]

\[ \text{range of } N = NL^2(\Gamma) \subset H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2} - 2\rho}(\Omega) = D(A^{\frac{3}{2} - \rho}) = D((A^*)^{\frac{3}{2} - \rho}), \quad 0 < \rho < \frac{3}{4} \] (3.3)

\[ N : \text{continuous } H^{\sigma}(\Gamma) \rightarrow H^{\frac{3}{2} + \sigma}(\Omega), \quad \sigma \text{ real}, \] (3.4)
the identification in (3.3) being set-theoretical and topological with
\[ \|x\|_{H^{\frac{3}{2} - 2\rho}(\Omega)}, \quad \left\| A^{\frac{3}{2} - \rho} x \right\|_{L^2(\Omega)}, \quad \left\| (A^*)^{\frac{3}{2} - \rho} x \right\|_{L^2(\Omega)} \text{ equivalent norms} \] (3.5)
\[ A^{\frac{3}{2} - \rho} N, \quad (A^*)^{\frac{3}{2} - \rho} N : \text{continuous operator } L^2(\Gamma) \rightarrow L^2(\Omega) \] (3.6)

5. As introduced in [27, 9], the abstract model of the operator \(L : g \rightarrow y\) in (1.1a)–(1.1c) with \(y_0 = y_1 = 0, h \equiv 0\), is
\[ (Lg)(t) = \int_0^t A S(t - \tau) N g(\tau) d\tau \] (3.7)
in appropriate topologies based on \(L^2(\Omega)\) to be described below, where \(A\) in (3.7) is the isomorphic extension: \(L^2(\Omega) \rightarrow [D(A^*)]' = \text{dual of } D(A^*)\) w.r.t. the \(L^2(\Omega)\)-topology, of the original operator \(A\) in (3.1). Thus, the solution of
Then, (recall $\alpha$ contained in Theorem 1.1 (1.15a) and the statement

\[ (Kh)(t) = \int_0^t S(t - \tau)h(\tau)d\tau \]  

(3.9)

6. The following are standard properties of the cosine/sine operators: $C(t)$ is

even on $L^2(\Omega)$, $C(0) = I$, thus $S(t)$ is odd, $t \in \mathbb{R}, S(0) = 0$. Moreover

\[ \frac{d^2C(t)x}{dt^2} = -AC(t)x, \quad x \in \mathcal{D}(A); \quad \frac{dC(t)x}{dt} = -AS(t)x, \quad x \in \mathcal{D}(A\frac{1}{2}); \]  

(3.10)

\[ C(t), A^{\frac{1}{2}}S(t) : \text{continuous } L^2(\Omega) \longrightarrow C([0,T]; L^2(\Omega)) \]  

(3.11)

Moreover with reference to (1.17a)–(1.17c) and (1.19)

\[ (L^*f)(t) = \phi(t)|_{\Sigma} = N^* A \int_t^T S^*(\tau - t)f(\tau)d\tau. \]  

(3.12)

4. Sharp regularity of $Lg$ for $g$ in $H^{-\beta}(\Gamma)$ or $H^{-\frac{1}{2}}(\Gamma)$ in space. Proofs. Throughout this section, without further mention, we shall provide results for the

operator $L : g \rightarrow y$ for problem (1.1a)–(1.1c) with $y_0 = y_1 = 0, h \equiv 0$; as well as for its adjoint $L^* : f \rightarrow \phi_{\Sigma}$ for the adjoint problem (1.17a)–(1.17c). The parameter

$\alpha = \beta$ are characterized in (1.14a)–(1.14b).

We begin with a result which is actually contained in [15, Theorem 8.1, for $\theta = 0$, p 161].

**Theorem 4.1.** Let

\[ g \in [H^{\beta}(\Sigma)]' \quad (\text{in particular}, \quad g \in L^2(0,T; H^{-\beta}(\Gamma))). \]  

(4.1)

Then, continuously,

\[ Lg \in L^2(Q), \quad \text{hence } Lg \in C([0,T]; L^2(\Omega)). \]  

(4.2)

**Proof.** The proof is by duality over Theorem 1.2 on $L^*$, see (1.19). Take $g \in [H^{\beta}(\Sigma)]'$ and $f \in L^2(Q)$. Then, in terms of duality pairings, we have

\[ (Lg, f)_{L^2(Q)} = (g, L^* f)_{L^2(\Sigma)} = \text{well defined}, \]  

(4.3)

where the RHS duality pairing in (4.3) is well defined with $g \in [H^{\beta}(\Sigma)]'$ and $L^* f \in H^{\beta}(\Sigma)$ by Theorem 1.2, (1.19). Thus, the LHS duality pairing of (4.3) is well-defined and then $Lg \in L^2(Q)$, since $f \in L^2(Q)$. Then the lifting Theorem 1.0 applies and yields $Lg \in C([0,T]; L^2(\Omega))$. \hfill $\Box$

**Corollary 4.2.** Let $0 \leq \theta \leq 1$ and

\[ g \in [H^{\beta(1-\theta)}(\Sigma)]' \quad (\text{in particular}, \quad g \in L^2(0,T; H^{-\beta(1-\theta)}(\Gamma))). \]  

(4.4)

Then, (recall $\alpha = \beta$ from (1.14a)–(1.14b)) continuously,

\[ Lg \in L^2(\Omega; H^{\alpha\theta}(\Omega)) = L^2(0,T; H^{\beta\theta}(\Omega)), \quad \text{hence } Lg \in C([0,T]; H^{\alpha\theta}(\Omega)). \]  

(4.5)

**Proof.** We interpolate between the statement

\[ g \in L^2(\Sigma) \equiv L^2(0,T; L^2(\Gamma)) = H^0(\Sigma) \longrightarrow Lg \in L^2(0,T; H^{\alpha}(\Omega)) \]  

(4.6)

contained in Theorem 1.1 (1.15a) and the statement

\[ g \in [H^{\beta}(\Sigma)]' \longrightarrow Lg \in L^2(0,T; L^2(\Omega)) = H^0(Q) \]  

(4.7)
of Theorem 4.1, Eq (4.2), to get \[20, \text{Thm } 5.1, \text{p } 27\], \[21, \text{Eq (5.5), p } 109\]

\[g \in [H^0(\Sigma), [H^\beta(\Sigma)]_1]_{1-\theta} \longrightarrow L^2 \left(0, T; [H^\alpha(\Omega), L^2(\Omega)]_{1-\theta}\right). \tag{4.8}\]

But \[20, \text{Theorem 6.2, p } 29\]

\[[H^0(\Sigma), [H^\beta(\Sigma)]_1]_{1-\theta} = [[H^0(\Sigma)]', [H^\beta(\Sigma)]']_{1-\theta} = [H^\beta(\Sigma), H^0(\Sigma)]_\theta = [H^{\beta(1-\theta)}(\Sigma)]'. \tag{4.9}\]

Moreover,

\[L^2 \left(0, T; [H^\alpha(\Omega), L^2(\Omega)]_{1-\theta}\right) \equiv L^2 \left(0, T; H^\alpha(\Omega)\right). \tag{4.10}\]

Thus, (4.9) and (4.10), used in (4.8), yield Corollary 4.2, after invoking Theorem 1.0 to lift the time regularity to \[C([0, T]). \]

With an eye toward finding a sharper counterpart of Theorem 1.3 of [1], we begin by providing a sharp result under an assumptions on \(g\) slightly stronger than in Theorem 1.3.

**Theorem 4.3.** Let

\[g \in H^{1/2}(0, T; H^{-\frac{1}{2}}(\Gamma)) \cap C([0, T]; H^{\beta-1}(\Gamma)); \quad g(0) = 0. \tag{4.11}\]

Then, continuously,

\[Lg \in C([0, T]; H^{\beta+\frac{1}{2}}(\Omega)), \quad \frac{d}{dt}(Lg) \in C([0, T]; H^{\beta-\frac{1}{2}}(\Omega)). \tag{4.12}\]

**Proof.**

**Step 1.** Set \(\theta = \frac{\beta - 1}{2} < 1\), so that \(\beta(1-\theta) = \frac{1}{2}\). Then the implication (4.4) \(\rightarrow\) (4.5) reads now with such \(\theta:\)

\[g \in [H^{\frac{1}{2}}(\Sigma)]' \longrightarrow Lg \in C([0, T]; H^{(\beta-\frac{1}{2})}(\Omega)) \subset L^2(0, T; H^{(\beta-\frac{1}{2})}(\Omega)). \tag{4.13}\]

continuously since \(\alpha = \beta\).

**Step 2.** Likewise

\[\dot{g} \in [H^{\frac{1}{2}}(\Sigma)]' \longrightarrow \dot{L}g \in C([0, T]; H^{(\beta-\frac{1}{2})}(\Omega)) \subset L^2(0, T; H^{(\beta-\frac{1}{2})}(\Omega)). \tag{4.14}\]

**Step 3.** It was already used in \[15, \text{Eq (3.18), p } 131\] that differentiating (3.7) yields

\[
\begin{cases}
\frac{d}{dt}(Lg)(t) = AS(t)Ng(0) + (L\dot{g})(t). \tag{4.15a} \\
\text{hence } \frac{d}{dt}(Lg)(t) = (L\dot{g})(t) \quad \text{for } g(0) = 0. \tag{4.15b}
\end{cases}
\]

In fact, returning to (3.7), we differentiate in time

\[(Lg)(t) = \int_0^t AS(t-\tau)Ng(\tau)d\tau = \int_0^t AS(r)Ng(t-r)dr \tag{4.16}\]

to obtain

\[
\frac{d}{dt}(Lg)(t) = AS(t)Ng(0) + \int_0^t AS(r)Ng(t-r)dr \\
= AS(t)Ng(0) + \int_0^t AS(t-\tau)Ng(\tau)d\tau \tag{4.17}\]

and (4.15a) follows.
Step 4. Our present assumption $\dot{g} \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma))$ in (4.11) a fortiori implies the assumption on $\dot{g}$ in (4.14). So (4.14) shows a fortiori conclusion (4.12) on $\frac{d}{dt}(Lg) = \dot{L}g$ in (4.12).

Step 5. We now use (4.14) to boost the regularity of $Lg$ over the preliminary result (4.13). By (3.7), recalling also the RHS of (3.10) and integrating by parts, we obtain

$$
(Lg)(t) = \int_0^t A S(t - \tau) N g(\tau)d\tau = \int_0^t \frac{d}{d\tau} C(t - \tau) N g(\tau)d\tau
$$

$$
= [C(t - \tau) N g(\tau)]_{\tau=0}^{\tau=t} - \int_0^t C(t - \tau) \dot{N} g(\tau)d\tau
$$

$$
= N g(t) - \int_0^t C(t - \tau) \dot{N} g(\tau)d\tau \quad (4.18)
$$

By (4.14), we have

$$
\dot{L} g \in C([0, T]; H^{(\beta - \frac{1}{2})}(\Omega)) \equiv D(A^{\frac{\beta}{2} - \frac{1}{2}})
$$

(4.20) as $\beta - \frac{1}{2} = \frac{1}{2}$ for $\beta = \frac{3}{4}$ and $\beta - \frac{1}{2} = \frac{1}{4}$ for $\beta = \frac{5}{4}$. Then, with $1 + \frac{1}{2}(\beta - \frac{1}{2}) = \frac{1}{2}(\beta + \frac{3}{2})$, $A^{-1}(L\dot{g}) \in C([0, T]; D(A^{\frac{\beta}{2} + \frac{1}{2}})) \subset C([0, T]; H^{(\beta + \frac{1}{2})}(\Omega))$ (4.21)

and hence via (4.19) and (4.21), recalling (1.15a)-(1.15b)

$$
\int_0^t C(t - \tau) \dot{N} g(\tau)d\tau = \frac{d}{dt} (A^{-1}(L\dot{g})) \in C([0, T]; H^{(\beta + \frac{1}{2})}(\Omega)) \quad (4.22)
$$

Step 7. By elliptic regularity of the Neumann map $N$ in (3.4) with $\sigma = \beta - 1$, we obtain using now the continuity regularity of $g$ in $H^{(\beta - 1)}(\Gamma)$ stated in (4.11)

$$
Ng \in C([0, T]; H^{(\beta + \frac{1}{2})}(\Omega)) \quad (4.23)
$$

Step 8. Then using (4.22) and (4.23) in (4.18) yields (4.12) also for $Lg$. The proof of Theorem 4.3 is complete.

Remark 4.1. Without the additional assumption $g \in C([0, T]; H^{(\beta - 1)}(\Gamma))$ we may proceed as follows

$$
Ng \in C([0, T]; H^1(\Omega)) \quad (4.24)
$$

since $g, \dot{g} \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma))$ implies $g \in C([0, T]; H^{-\frac{1}{2}}(\Gamma))$, [20, Theorem 3.1 for $j = 0$, p 19], as in Remark 1.3. Then, in this case, by (4.22), (4.24) used in (4.18), one only obtains $Lg \in C([0, T]; H^1(\Omega))$.

Corollary 4.4. (of Theorem 4.1) Assume

$$
g \in H^1(0, T; H^{-\beta}(\Gamma)), \quad g(0) = 0 \quad (4.25)
$$

Then, continuously

$$
Lg \in H^1(0, T; L^2(\Omega)) \quad (4.26)
$$
Proof. Assumption (4.25) implies a-fortiori \( g \in [H^\beta(\Sigma)]' \) and \( \dot{g} \in [H^\beta(\Sigma)]' \), so that Theorem 4.1 implies \( Lg \in L^2(Q) \) and \( L\dot{g} \in L^2(Q) \). Moreover the full assumption (4.25)-whereby then \( g \in C([0,T]; H^{-\beta}(\Gamma)) \) by [20, Theorem 3.1 for \( j = 0 \), p 19]-also yields \( \frac{d}{dt}(Lg) = L\dot{g} \) via (4.15b). Hence (4.26) follows.

**Remark 4.2.** Corollary 4.4 yields a result for \( Lg \) being \( H^1 \)-in time. We would next like to trade derivatives in time into derivatives in space. For second order hyperbolic equations, one derivative in time corresponds to one derivative in space. As noted explicitly in [10, bottom of page 121], in the micro-local proofs of [9], complemented by [26], yielding the basic results Theorem 1.1 for \( L \) (and Theorem 1.2 for \( L^* \)) (the starting point of our present analysis), the critical cone in the dual variables is the cone where the dual variable corresponding to time is comparable to the dual variable corresponding to space. Thus, in such critical case for Theorem 1.1 regularity in time and regularity in space are interchangeable.

The next result transfers one derivative in time to one derivative in space over the conclusion of Corollary 4.4, under an additional smoothness assumption \( g \), in the style of [15, Theorem 3.1(ii), p 129]. In fact, the present proof is similar in spirit to that of this result, mutatis mutandis.

**Theorem 4.5.** (Compare with Corollary 4.4) Let

\[
g \in H^1(0,T; H^{-\beta}(\Gamma)) \cap C([0,T]; H^{-\frac{1}{2}}(\Gamma)), \quad g(0) = 0
\]  

(4.27)

Then, continuously \( (y = Lg \text{ for } y_0 = y_1 = 0, h \equiv 0 \text{ in } (1.1a)-(1.1c)) \)

\[
y = Lg \in C([0,T]; H^1(\Omega)), \quad y_t = \frac{d}{dt}(Lg) \in C([0,T]; L^2(\Omega))
\]  

(4.28)

**Proof.** We proceed as in the proof of Theorem 4.3.

**Step 1.** We return to (4.18),(4.19)

\[
(Lg)(t) = Ng(t) - \int_0^t C(t - \tau)N\dot{g}(\tau)d\tau
\]  

(4.29)

\[
\int_0^t C(t - \tau)N\dot{g}(\tau)d\tau = \frac{d}{dt}(A^{-1}(L\dot{g})(t))
\]  

(4.30)

**Step 2.** By conclusion (4.26) of Corollary 4.4 supplemented by Theorem 1.0, we have with \( \dot{g}(0) = 0 \)

\[
\frac{d}{dt}(Lg) = L\dot{g} \in C([0,T]; L^2(\Omega))
\]  

(4.31)

thus

\[
A^{-1}(L\dot{g})(t) \in C([0,T]; D(A)) \subset C([0,T]; H^2(\Omega))
\]  

(4.32)

hence recalling (1.15a)-(1.15b)

\[
\frac{d}{dt}(A^{-1}(L\dot{g})(t)) \in C([0,T]; H^{2-1}(\Omega) \equiv H^1(\Omega)).
\]  

(4.33)

Using (4.33) in (4.30) yields

\[
\int_0^t C(t - \tau)N\dot{g}(\tau)d\tau \in C([0,T]; H^1(\Omega)).
\]  

(4.34)
Step 3. By the elliptic regularity of the Neumann map $N$ in (3.4) with $\sigma = -\frac{1}{2}$, we obtain using now the continuity regularity in $H^{-\frac{1}{2}}(\Gamma)$ of $g$ in (4.27):

$$Ng \in C([0, T]; H^1(\Omega))$$  (4.35)

Step 4. Using (4.34) and (4.35) in (4.29) yields then $Lg \in C([0, T]; H^1(\Omega))$, from which $\frac{d}{dt}(Lg) \in C([0, T]; L^2(\Omega))$ follows. Theorem 4.5 is proved.

Remark 4.3. Theorem 4.5 offers a set of assumptions, weaker than those in Theorem 1.3 by $\beta - \frac{1}{2} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ in space regularity of $g$ (for a general domain), which yield a slightly stronger conclusion for $Lg$ in time, however with the $H^1$-regularity in space of $Lg$, which is what is needed in the analysis of [2]

5. Sharp regularity of $Lg$ with $g$ even less regular in space, say in $H^{-1}(\Gamma)$.

Proofs. In this section, we shall provide additional results on the sharp regularity of $Lg$, when $g$ is even less regular in space, the case $H^{-1}(\Gamma)$ being our present objective. Section 4 rests, as a starting point, on Theorem 1.2(a), on the adjoint $L^*$, which is nothing but [15, Theorem B, p 118, or Theorem 2.0, Eq (2.10), p 124]. The present section is likewise based on Theorem 1.2(b) which is nothing but [15, Corollary 6.3, p 158]. With these results at hand, we shall prove

Theorem 5.1. We have ($1 - \beta = \frac{1}{3}$ for a general domain; $1 - \beta = \frac{1}{4}$ for parallelepipeds)

$$g \in [H^1(\Sigma)]' \implies Lg \in [H^{1-\beta}(Q)]'$$ continuously.  (5.1)

Proof. By Theorem 1.2(b) we have, for $0 \leq \theta \leq 1$

$$f \in H^\theta(Q), f(T) = 0 \implies L^*f = \phi|_\Sigma \in H^{\beta+\theta}(\Sigma)$$ continuously.  (5.2)

Specializing (5.2) with $0 < \theta = 1 - \beta$, so that $0 < \theta < \frac{1}{2}$ and thus the compatibility condition does not apply, we obtain :

$$f \in H^{1-\beta}(Q) \implies L^*f = \phi|_\Sigma \in H^1(\Sigma)$$ continuously.  (5.3)

With $f$ as in (5.3), let $g \in [H^1(\Sigma)]'$ and consider the duality pairings

$$(Lg, f)_{L^2(Q)} = (g, L^*f)_{L^2(\Sigma)} = \text{well defined},$$  (5.4)

where the RHS duality pairing in (5.4) is well-defined with $g \in [H^1(\Sigma)]'$ and $L^*f \in H^1(\Sigma)$ by (5.3) (ultimately by Theorem 1.2(b)). Thus, the LHS duality pairing of (5.4) is likewise well-defined and then $Lg \in [H^{1-\beta}(Q)]'$, since $f \in H^{1-\beta}(Q)$.  □

Remark 5.1. Theorem 5.1 is sharp in both time and space. Nevertheless, for our purposes, we prefer a result that start with $g$ smoother in time, i.e. assumed in $L^2(0, T; \cdot)$ in time. This is provided by the analysis below.

Theorem 5.2. Let

$$f \in L^2(0, T; H^1(\Omega)).$$  (5.5)

Then, continuously

$$L^*f = \phi|_\Sigma \in H^{1+\beta}(\Sigma) \equiv L^2(0, T; H^{1+\beta}(\Gamma)) \cap H^\beta(0, T; L^2(\Gamma)).$$  (5.6)
Proof. The proof follows in part the argument of [15, Theorem 6.1, p.153], except for two main changes: a different proof of its Step 2, which now exploits assumption (5.5) while in [15, p.154] exploited the assumption \( v \in H^1(0,T;L^2(\Omega)) \); and a differential proof also in our present Step 4 over [15].

**Step 1.** As in [15, Step 1, p.154], we note that proving (5.6) is equivalent to introducing the new variable \( w = B_1 \phi \) with respect to the \( \phi \)-problem in (1.17a)–(1.17c) and proving that

\[
w |_\Sigma = B_1 \phi |_\Sigma \in H^\beta(\Sigma) \equiv L^2(0,T;H^\beta(\Gamma)) \cap H^\beta(0,T;L^2(\Gamma)) \tag{5.7}
\]

where \( B_1 \) is a time-independent first order differential operator

\[
B_1 = \sum_i b_i(\xi) \frac{\partial}{\partial \xi_i}, \text{ with smooth coefficients } b_i \text{ in } \overline{\Omega} \tag{5.8a}
\]

\[
\text{tangential to } \Gamma; \text{ i.e. } \sum_i b_i \nu_i = 0 \text{ on } \Gamma. \tag{5.8b}
\]

In term of the new variable \( w = B_1 \phi \), the adjoint \( \phi \)-problem in (1.17a)–(1.17b) becomes

\[
\begin{aligned}
&\begin{cases}
  w_{tt} = -A^*(\xi,\partial)w + R_2 \phi + B_1 f \\
  w |_{t=T} = w_t |_{t=T} = 0
\end{cases} \quad \text{in } Q \tag{5.9a} \\
&\begin{cases}
  \frac{\partial w}{\partial \nu} |_\Sigma = F \phi |_\Sigma
\end{cases} \quad \text{in } \Sigma \tag{5.9b}
\end{aligned}
\]

where the relevant commutators are

\[
B_1 (-A^*(\xi,\partial)) - (-A^*(\xi,\partial)) B_1 = R_2 \tag{5.10a}
\]

\[
R_2 = \text{second order operator } (2 + 1 - 1) \text{ in } \Omega \text{ with smooth coefficients in } \overline{\Omega} \tag{5.10b}
\]

\[
\nabla(B_1 \cdot) \cdot \nu = F = \text{first order commutator operator } (1 + 1 - 1) \text{ in } \Omega \text{ with smooth coefficients in } \overline{\Omega}; \text{ the normal } \nu \text{ extended smoothly into the interior, } \tag{5.10c}
\]

as (1.17c) with \( B_1 \) tangential implies \( B_1 \frac{\partial \phi}{\partial \nu} |_\Gamma = 0 = \frac{\partial}{\partial \nu}(B_1 \phi) + [B_1, \frac{\partial}{\partial \nu}] \phi \). With \( w = w_1 + w_2 \), we split problem (5.9a)–(5.9c) into two problems

\[
\begin{aligned}
&\begin{cases}
  w_{1tt} = -A^*(\xi,\partial)w_1 + R_2 \phi + B_1 f \\
  w_1 |_{t=T} = w_1 |_{t=T} = 0
\end{cases} \quad \text{in } Q \tag{5.11a} \\
&\begin{cases}
  w_{2tt} = -A^*(\xi,\partial)w_2
\end{cases} \quad \text{in } Q \tag{5.11b}
\end{aligned}
\]

\[
\begin{aligned}
&\begin{cases}
  \frac{\partial w_1}{\partial \nu} |_\Sigma = 0 \quad \text{in } \Sigma \\
  \frac{\partial w_2}{\partial \nu} |_\Sigma = F \phi |_\Sigma \quad \text{in } \Sigma
\end{cases}
\end{aligned} \tag{5.11c}
\]

**Step 2.** We need a-priori regularity of \( \phi \), solution of problem (1.17a)–(1.17c). This step is different from [10, Step 2, p.154]. Now we shall use assumption (5.5) in \( f \) in [10, Step 2, p.154] we used the assumption \( f \in H^1(0,T;L^2(\Omega)), \text{ instead} \). By (3.9) we obtain for problem (1.17a)–(1.17c)

\[
\phi(t) = (Kf)(t) = \int_0^t S(t-\tau)f(\tau) d\tau \tag{5.12}
\]

hence

\[
A\phi(t) = \int_0^t A^{\frac{3}{2}} S(t-\tau) A^{\frac{3}{2}} f(\tau) d\tau \in C([0,T];L^2(\Omega)) \tag{5.13}
\]
by convolution recalling (3.11) on $A^{\frac{1}{2}}S(\cdot)$ and $f \in L^2(0,T;H^1(\Omega) = D(A^{\frac{1}{2}}))$ by (5.5). Hence by (5.13) via (5.10b) [20, Proposition 12.1, p 85],

$$\phi \in C([0,T];H^2(\Omega)); \quad R_2\phi \in C([0,T];L^2(\Omega))$$  (5.14)

$$F\phi \in C([0,T];H^1(\Omega)); \quad \text{hence } F\phi|_{\Sigma} \in C([0,T];H^\frac{1}{2}(\Gamma))$$  (5.15)

by (5.10c) and trace theory. Thus, recalling assumption (5.5) on $f$, we obtain

$$RHS \text{ of (5.11a) (left)} = R_2\phi + B_1f \in L^2(Q).$$  (5.16)

Then, by virtue of (5.16), we can invoke Theorem 1.2, Eq (1.19) to the $w_1$-problem and obtain

$$w_1|_{\Sigma} = L^*(R_2\phi + B_1f) \in H^\beta(\Sigma).$$  (5.17)

**Step 3.** We return to the $\phi$-problem (1.17a)–(1.17c). Complementing (5.15), we have with $f \in L^2(0,T;D(A^{\frac{1}{2}}) = H^1(\Omega))$ as in (5.5)

$$A^{\frac{1}{2}}\phi_t = \int_0^t C(t-\tau)A^{\frac{1}{2}}f(\tau)d\tau \in C([0,T];L^2(\Omega)).$$  (5.18)

Thus, a-fortiori from (5.14) and (5.18), we have

$$\phi \in H^{2,1}(Q) = L^2(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)).$$  (5.19)

Hence, as $F$ in (5.10c) in a space-dependent first order operator

$$F\phi \in H^{1,1}(\Omega); \quad \text{hence } F\phi|_{\Sigma} \in H^{\frac{1}{2},\frac{1}{2}}(\Sigma) \equiv H^{\frac{1}{2}}(\Sigma)$$  (5.20)

continuously, where the trace result follows from [21, Vol II, Theorem 2.1, p 9 with $r = s = 1, j = 0$; thus $\mu_j = \nu_j = \frac{1}{2}$].

**Step 4.** This step is also different from [15, Step 3, p 155]. With the Neumann boundary term $F\phi|_{\Sigma} \in H^{\frac{1}{2}}(\Sigma)$ by (5.20) for the $w_2$-problem in (5.11a)–(5.11c), where in addition $F\phi(T)|_{\Sigma} = 0$ by (1.17b), we invoke [15, Corollary 4.3, Eq (4.32) with $\theta = \frac{1}{2}$] (which compliments (1.16)) and obtain

$$w_2|_{\Sigma} \in H^{2\alpha-1+\theta}(\Sigma) = H^{2\alpha-\frac{1}{2}}(\Sigma) \subset H^\beta(\Sigma)$$  (5.21)

since $\alpha = \beta$ (see (1.14a)–(1.14b)) and $\beta > \frac{1}{2}$.

**Step 5.** Combining (5.17) for $w_1|_{\Sigma}$ and (5.21) for $w_2|_{\Sigma}$, we obtain

$$w|_{\Sigma} = w_1|_{\Sigma} + w_2|_{\Sigma} \in H^\beta(\Sigma)$$  (5.22)

and (5.7) is proved. Equivalently, with $w = B_1\phi$, we conclude that (5.6) is proved, as noted in Step 1.

**Remark 5.2.** Theorem 5.2 shows an improvement of the sharp gain $\beta$ in time (from 0 to $\beta$) as well as in space (from 1 to $1+\beta$).

**Corollary 5.3.** Let $0 \leq \theta \leq 1$ and

$$f \in L^2(0,T;H^\theta(\Omega)).$$  (5.23)

Then, continuously

$$L^*f = \phi|_{\Sigma} \in H^{\beta+\theta,\beta}(\Sigma) = L^2(0,T;H^{\beta+\theta}(\Gamma)) \cap H^\beta(0,T;L^2(\Gamma)).$$  (5.24)
Proof. We interpolate between implication (1.19) in Theorem 1.2(a) and implication (5.5) → (5.6) in Theorem 5.2. We then obtain, as desired

\[ L^* : \text{continuous } L^2 \left(0, T; [H^1(\Omega), L^2(\Omega)]_{1-\theta} \right) = L^2(0, T; H^\theta(\Omega)) \]

\[ \rightarrow [H^{1+\beta,\beta}(\Sigma), H^{\beta,\beta}(\Sigma)]_{1-\theta} = H^{\beta+\theta,\beta}(\Sigma) \] (5.25)

as \((1+\beta)\theta + (1-\theta)\beta = \beta + \theta\), by invoking [21, Vol II, Proposition 2.1, Eq (2.5)]. \(\square\)

In the next result we deliberately accept a loss of time regularity in order to have an attractive result expressed in terms of \(L^2(0, T; \cdot)\) regularity, consistently with a gain \(\beta\) in space regularity from the boundary to the interior.

**Theorem 5.4.** We have, continuously

\[ g \in L^2(0, T; H^{-1}(\Gamma)) \rightarrow Lg \in C([0, T]; H^{\beta-1}(\Omega)). \] (5.26)

**Proof.** Specialize implication (5.23) → (5.24) with \(\theta = 1 - \beta\), so that

\[ f \in L^2(0, T; H^{1-\beta}(\Omega)) \rightarrow L^* f = \phi|_\Sigma \in L^2(0, T; H^1(\Gamma) \cap H^\beta(0, T; L^2(\Gamma))). \] (5.27)

With such \(f\) and with \(g \in L^2(0, T; H^{-1}(\Gamma))\), consider the duality pairings

\[ (Lg, f)_{L^2(\Omega)} = (g, L^* f)_{L^2(\Sigma)} = \text{well defined.} \] (5.28)

where the RHS duality pairing in (5.28) is well defined with \(g \in L^2(0, T; H^{-1}(\Gamma))\) and \(L^* f \in L^2(0, T; H^1(\Gamma))\) by (5.27). Thus, the LHS duality pairing of (5.28) is also well-defined and then \(Lg \in L^2(0, T; H^{\beta-1}(\Omega))\) due to \(f\) in (5.27), where \(1 - \beta < \frac{1}{2}\), see (1.14a)–(1.14b), and so [20, Theorem 11.1, p 55]

\[ [H^{1-\beta}(\Omega)]' = [H^{1-\beta}_0(\Omega)]' = H^{\beta-1}(\Omega) \] (5.29)

The obtained regularity \(Lg \in L^2(0, T; H^{\beta-1}(\Omega))\) is then boosted to \(Lg \in C([0, T]; H^{\beta-1}(\Omega))\), by Theorem 1.0. \(\square\)

**Theorem 5.5.** Let

\[ g \in H^1(0, T; H^{-1}(\Gamma)) \cap C([0, T]; H^{\beta-\frac{3}{2}}(\Gamma)), \quad g(0) = 0. \] (5.30)

Then, continuously,

\[ Lg \in C([0, T]; H^\beta(\Omega)). \] (5.31)

**Proof.** **Step 1.** Along with (5.26) we also have

\[ \dot{g} \in L^2(0, T; H^{-1}(\Gamma)) \rightarrow \frac{d}{dt} (Lg) = L\dot{g} \in C([0, T]; H^{\beta-1}(\Omega)). \] (5.32)

since \(g(0) = 0\), as in (4.15b).

**Step 2.** We return once more to the expressions (4.29) and (4.30)

\[ (Lg)(t) = Ng(t) - \int_0^t C(t-\tau) N\dot{g}(\tau) d\tau = Ng(t) - \frac{d}{dt} \left( A^{-1}(L\dot{g}) \right)(t) \] (5.33)

**Step 3.** With \(0 < 1 - \beta < \frac{1}{2}\), we return to, and compliment, (5.29) by writing

\[ H^{\beta-1}(\Omega) = [H^{1-\beta}_0(\Omega)]' = [H^{1-\beta}(\Omega)]' = \left[ D \left( A^{\frac{1}{2}(1-\beta)} \right) \right]' \] (5.34)

so from (5.32) and (5.34)

\[ L\dot{g} \in C \left( [0, T]; \left[ D \left( A^{\frac{1}{2}(1-\beta)} \right) \right]' \right) \] (5.35)
hence with $\frac{1}{2}(\beta - 1) + 1 = \frac{1}{2}(\beta + 1)$:

$$A^{-1} L g \in C \left([0,T];D \left(A^{\frac{\beta + 1}{2}}\right)\right) \subset C([0,T];H^{\beta+1}(\Omega)). \quad (5.36)$$

Therefore, as in (1.15a)–(1.15b),

$$\frac{d}{dt} \left(A^{-1} L g\right)(t) \in C([0,T];H^{\beta}(\Omega)). \quad (5.37)$$

**Step 4.** By elliptic regularity of the Neumann map $N$ in (3.4) with $\sigma = \beta - \frac{3}{2}$, we obtain by using the regularity of $g$ in (5.30)

$$Ng \in C([0,T];H^{\beta}(\Omega)). \quad (5.38)$$

**Step 5.** Equations (5.37) and (5.38), used in (5.33) yield

$$Lg \in C([0,T];H^{\beta}(\Omega)). \quad (5.39)$$

and Theorem 5.5 is proved.

6. **Illustrations.** Illustration #1: A Dirichlet-controlled wave equation feeding in a serial connection another wave equation via the Neumann trace.

Consider a sufficiently smooth internal domain $\Omega_i$ in $\mathbb{R}^n$, $n \geq 2$, being immersed in a sufficiently smooth external domain $\Omega_e$ in $\mathbb{R}^n$, $n \geq 2$. Call $\Gamma_i = \partial \Omega_i$, the boundary of $\Omega_i$ (interface between $\Omega_i$ and $\Omega_e$). Call $\Gamma_e$ the external boundary of $\Omega_e$ and $\partial \Omega_e = \Gamma_e \cup \Gamma_i$ the boundary of $\Omega_e$. See Figure 1.

![Figure 1](image-url)

**Theorem 6.1.** Consider the following $y$-wave mixed problem defined on the external domain $\Omega_e$:
\[ \begin{cases} y_{tt} = \Delta y + F & \text{in } Q_e = (0, T) \times \Omega_e \\ y(0, \cdot) = y_0, \ y_t(0, \cdot) = y_1 & \text{in } \Omega_e \\ y|_{\Sigma_e} = u & \text{in } \Sigma_e = (0, T) \times \partial \Omega_e \end{cases} \quad (6.1a) \]

feeding through its normal trace \( \frac{\partial y}{\partial \nu} \) on \( \Gamma_i = \partial \Omega_i \) a second \( w \)-wave mixed problem defined on the internal domain \( \Omega_i \):

\[ \begin{cases} w_{tt} = \Delta w & \text{in } Q_i = (0, T) \times \Omega_i \\ w(0, \cdot) = 0, \ w_t(0, \cdot) = 0 & \text{in } \Omega_i \\ \frac{\partial w}{\partial \nu}|_{\Sigma_i} = \frac{\partial y}{\partial \nu}|_{\Sigma_i} & \text{in } \Sigma_i = (0, T) \times \Gamma_i \end{cases} \quad (6.2c) \]

Assume:

\[ \begin{cases} F \in L^1(0, T; H^{-1}(\Omega_e)); \quad \{y_0, y_1\} \in L^2(\Omega_e) \times L^{-1}(\Omega_e) \\ u \in L^2(\Sigma_e) \equiv L^2(0, T; L^2(\Gamma_e)); \quad u \equiv 0 \text{ on } \Sigma_i \end{cases} \quad (6.3) \]

Then, continuously

\[ w = L \left( \frac{\partial y}{\partial \nu}|_{\Sigma_i} \right) \in \left[ H^{1-\beta}(Q_i) \right]' \quad (6.4) \]

**Proof. Step 1.** By Theorem 1.2(b), Eq (1.20), with \( \theta = 1 - \beta \) we obtain

\[ f \in H^{1-\beta}(Q_i) \implies L^* f = \phi|_{\Sigma} \in H^1(\Sigma_i) \text{ continuously} \quad (6.5) \]

as \( 1 - \beta < \frac{1}{2} \), so that the compatibility condition \( f(T) = 0 \) is void. Eq. (6.5) is nothing but (5.3). We now consider the duality pairings in (5.4) with \( g = \frac{\partial y}{\partial \nu}|_{\Sigma_i} \) from (6.2c); that is:

\[ (w, f)_{L^2(Q_i)} = \left( L \left( \frac{\partial y}{\partial \nu}|_{\Sigma_i} \right), f \right)_{L^2(Q_i)} = \left( \frac{\partial y}{\partial \nu}|_{\Sigma_i}, L^* f \right)_{L^2(\Sigma_i)}. \quad (6.6) \]

We now notice that in addition to the regularity noted in (6.5), it also holds that by (3.12)

\[ (L^* g)(t) = N^* A \int_t^T S^*(\tau - t) f(\tau) d\tau, \text{ so that } (L^* f)(T) = 0. \quad (6.7) \]

Moreover, under the hypotheses in (6.3), we have that the solution of the \( y \)-problem (6.1a)–(6.1c) satisfies

\[ y \in C([0, T]; L^2(\Omega)), \quad y_t \in C([0, T]; H^{-1}(\Omega)). \quad (6.8) \]

In addition, the proof in [8, Eq. (2.66), p 163] yields then that

\[ \left| \int_{\Sigma} \left( \frac{\partial y}{\partial \nu} \right) h d\Sigma \right| \leq c \| h \|_{H^1(\Sigma)} \text{ for all } h \in H^1(\Sigma) \text{ s.t. } h(\cdot, T) = 0. \quad (6.9) \]

Define

\[ H^1_0(\Sigma) = \{ h \in H^1(\Sigma) : h(\cdot, T) = 0 \} \quad (6.10) \]

so that we can take \( h = L^* f \in H^1_0(\Sigma) \) by (6.5) and (6.7), and define

\[ \left[ H^1_0(\Sigma) \right]' = \text{dual of } H^1_0(\Sigma). \quad (6.11) \]
Then (6.9) with \( h = L^* f \) yields

\[
\frac{\partial y}{\partial \nu}\bigg|_{\Sigma_i} \in \left[H^1_0(\Sigma)\right]'.
\] (6.12)

This refinement is explicitly noted in [8, (2.64)–(2.66) and the conclusion of the proof] over the actual statement of [8, Theorem 2.3, Eq. (2.14) p 153]. Consequently, we have that the RHS duality pairings in (6.6) is well-defined. Thus the LHS duality pairings in (6.6) is likewise well-defined and then we then conclude that

\[
w \in \left[H^{1-\beta}(Q_i)\right]', \text{ since } f \in H^{1-\beta}(Q_i).
\] (6.13)

Theorem 6.1 is proved.

**Remark 6.1.** The above illustration deals with an irregular set of data in (6.3) which produce a highly irregular normal derivative on \( \Gamma_i \) given by (6.11). This very irregular ‘input’ then produces the corresponding solution \( w \) in \( Q_i \), as in (6.13).

**Illustration #2:** An uncontrolled Dirichlet-wave equation feeding in a serial connection another wave equation via the tangential gradient of the Neumann trace.

The setting of the internal domain \( \Omega_i \) immersed in an external domain \( \Omega_e \) is exactly the same as in Illustration #1, with same symbols \( \Gamma_i = \partial \Omega_i \), \( \Gamma_e \), and \( \partial \Omega_e = \Gamma_e \cup \Gamma_i \).

**Theorem 6.2.** Consider the following \( \psi \)-wave mixed problem defined in the external domain \( \Omega_e \):

\[
\begin{cases}
p_{tt} = \Delta p + F & \text{in } Q_e = (0, T) \times \Omega_e \\
p(0, \cdot) = \psi_0, \quad p_t(0, \cdot) = \psi_1 & \text{in } \Omega_e \\
|\Sigma_e = 0 & \text{in } \Sigma_e = (0, T) \times \partial \Omega_e
\end{cases}
\] (6.14a-6.14c)

feeding through the tangential gradient of the normal trace \( \left[ \nabla_{\tan} \left( \frac{\partial \psi}{\partial \nu} \right) \right]_{\Sigma_i} \) on \( \Gamma_i = \partial \Omega_i \) a second \( w \)-wave mixed problem defined on the internal domain \( \Omega_i \):

\[
\begin{cases}
p_{tt} = \Delta p & \text{in } Q_i = (0, T) \times \Omega_i \\
p(0, \cdot) = \psi_0, \quad p_t(0, \cdot) = \psi_1 & \text{in } \Omega_i \\
\left[ \frac{\partial w}{\partial \nu} \right]_{\Sigma_i} = \left[ \nabla_{\tan} \left( \frac{\partial \psi}{\partial \nu} \right) \right]_{\Sigma_i} & \text{in } \Sigma_i = (0, T) \times \Gamma_i
\end{cases}
\] (6.15a-6.15c)

Assume:

\[
\{\psi_0, \psi_1\} \in H^1_0(\Omega_e) \times L^2(\Omega_e); \quad F \in L^1(0, T; L^2(\Gamma_e)).
\] (6.16)

Then, continuously

\[
w \in C([0, T]; H^{\beta-1}(\Omega_e)), \quad w_t \in C([0, T]; H^{\beta-2}(\Omega_e)).
\] (6.17)

**Proof.** We have [8, Theorem 2.1, p 151], [9, 10] that assumption (6.16) implies that the solution \( \psi \) of problem (6.14a)–(6.14c) satisfies

\[
\frac{\partial \psi}{\partial \nu}\bigg|_{\Sigma_e} \in L^2(0, T; L^2(\Omega_e)), \quad \nabla_{\tan} \left( \frac{\partial \psi}{\partial \nu} \right) \in L^2(0, T; H^{-1}(\partial \Omega_e)).
\] (6.18)
With such Neumann input as in (6.18) applied in (6.15b), Theorem 5.4 yields

\[ w = L \left( \nabla_{\mathrm{tan}} \left( \frac{\partial \psi}{\partial \nu} \right) \right)_{\Sigma_i} \in C([0,T]; H^{\beta-1}(\Omega_i)) \] (6.19)

\[ w_t = \frac{d}{dt} L \left( \nabla_{\mathrm{tan}} \left( \frac{\partial \psi}{\partial \nu} \right) \right)_{\Sigma_i} \in C([0,T]; H^{\beta-2}(\Omega_i)) \] (6.20)

and (6.17) is shown.

Illustration #3: A point-control wave equation feeding in a serial connection another wave equation via time integration of the Neumann trace for \( n = 3 \); via Neumann trace for \( n = 2 \).

The setting of the internal domain \( \Omega_i \) immersed in an external domain \( \Omega_e \) is exactly the same as in Illustration #1 and #2, with same symbols \( \Gamma_i = \partial \Omega_i \), \( \Gamma_e \), and \( \partial \Omega_e = \Gamma_e \cup \Gamma_i \).

**Theorem 6.3.** (a) Case \( n = 3 \). Consider the following \( y \)-wave problem with point control defined in the internal 3-dimensional domain \( \Omega_i \), containing the origin in its interior (w.l.o.g.):

\[
\begin{align*}
{y_{tt} = \Delta y + \delta(x)u(t)} & \quad \text{in } Q_i = (0,T] \times \Omega_i \quad (6.21a) \\
y(0,\cdot) = 0, \quad y_t(0,\cdot) = 0 & \quad \text{in } \Omega_i \quad (6.21b) \\
y \big|_{\Sigma_i} = 0 & \quad \text{in } \Sigma_i = (0,T] \times \partial \Omega_i \quad (6.21c)
\end{align*}
\]

feeding through time integration of the Neumann trace on \( \Gamma_i = \partial \Omega_i \) a second \( w \)-wave mixed problem defined on the external 3-dimensional domain \( \Omega_e \):

\[
\begin{align*}
{w_{tt} = \Delta w} & \quad \text{in } Q_e = (0,T] \times \Omega_e \quad (6.22a) \\
w(0,\cdot) = 0, \quad w_t(0,\cdot) = 0 & \quad \text{in } \Omega_e \quad (6.22b) \\
\frac{\partial w}{\partial \nu} \big|_{\Sigma_i} = \int_0^t \frac{\partial y}{\partial \nu} \big|_{\Sigma_i} \, d\tau & \quad \text{in } \Sigma_i = (0,T] \times \Gamma_i \quad (6.22c) \\
\frac{\partial w}{\partial \nu} \big|_{\Sigma_e} = 0 & \quad \text{in } \Sigma_e = (0,T] \times \Gamma_e \quad (6.22d)
\end{align*}
\]

Assume:

\[ u \in L^2(0,T). \] (6.23)

Then, continuously

\[ w = L \left( \frac{\partial w}{\partial \nu} \big|_{\Sigma_i} \right) \in C([0,T]; H^{\beta-1}(\Omega_e)) \] (6.24)

\[ w_t = \frac{d}{dt} L \left( \frac{\partial w}{\partial \nu} \big|_{\Sigma_i} \right) \in C([0,T]; H^{\beta-2}(\Omega_e)) \] (6.25)

(b) Case \( n = 2 \). When \( \Omega_i \) and \( \Omega_e \) are 2-dimensional, consider the same \( y \)-problem (6.21a-b-c) and \( w \)-problem (6.22), except that now

\[ \frac{\partial w}{\partial \nu} \big|_{\Sigma_i} = \frac{\partial y}{\partial \nu} \big|_{\Sigma_i} \quad \text{in } \Sigma_i = (0,T] \times \Gamma_i \] (6.26)
replaces (6.22c). Assume again (6.23) for the scalar control \( u \). Then, continuously,

\[
w = L \left( \frac{\partial w}{\partial \nu} \bigg|_{\Sigma_i} \right) \in C([0, T]; H^{\beta-\frac{1}{2}}(\Omega))
\]

\[
w_t = \frac{d}{dt} L \left( \frac{\partial w}{\partial \nu} \bigg|_{\Sigma_i} \right) \in C([0, T]; H^{\beta-\frac{3}{2}}(\Omega))
\] (6.27)

(6.28)

**Proof.** (a) **Case** \( n = 3 \). Under assumption (6.23) of the point control, we have, continuously, the following regularity of the \( y \)-problem on \( \Omega_i \) [17, Theorem 9.8.1.1, p 844], [28]

\[
\{ y, y_t \} \in C([0, T]; L^2(\Omega_i) \times H^{-1}(\Omega_i))
\]

\[
\frac{\partial y}{\partial \nu} \bigg|_{\Sigma_i} \in H^{-1}(\Sigma_i), \text{ hence } \int_0^t \frac{\partial y}{\partial \nu} \bigg|_{\Sigma_i} \ d\tau \in L^2(0, T; H^{-1}(\Gamma_i)))
\] (6.29)

Then with \( g(t) = \int_0^t \frac{\partial y}{\partial \nu} \bigg|_{\Sigma_i} \ d\tau \), the Neumann control acting in (6.22c), we invoke Theorem 5.4 to the \( w \)-problem and obtain (6.24), (6.25).

(b) **Case** \( n = 2 \). Here, under assumption (6.23) of the point control we have, continuously, the following regularity of the \( y \)-problem on \( \Omega_i \) [17, Theorem 9.8.1.1, p 844], [28]

\[
\{ y, y_t \} \in C([0, T]; \dot{H}^{1/2}_{00}(\Omega_i) \times [H^{1/2}_{00}(\Omega_i)])
\] (6.31)

\[
\frac{\partial y}{\partial \nu} \bigg|_{\Sigma_i} \in H^{-1/2}(\Sigma_i) = [H^{1/2}(\Sigma_i)]',
\]

Then, with \( g(t) = \frac{\partial y}{\partial \nu} \bigg|_{\Sigma_i} \), the Neumann control acting in (6.22c), we invoke (4.13) and obtain (6.29), (6.30).

**Remark 6.2.** Actually, the proof of this result ultimately relies on [8, Eq. (2.66), p 163] (see [17, p 870] that quotes [8, Eq. (2.66), p 163] with reference to the \( h \)-problem [17, (9.9.1.49a-b-c), p 868]). Thus, we are in exactly the same situation as below (6.9). The tight result of [8] is accordingly

\[
\frac{\partial y}{\partial \nu} \bigg|_{\Sigma_i} \in [H^1(\Sigma_i)]' = \text{dual of } H^1(\Sigma_i).
\]

rather than (6.29), precisely as (6.12), since \( (L^*f)(T) = 0 \).

**Illustration #4:** A point-control thermoelastic system feeding in a serial connection a wave equation via the tangential gradient of the trace of the “moment” of the elastic displacement; \( n = 2 \).

The setting of the internal domain \( \Omega_i \) immersed in an external domain \( \Omega_e \) is exactly the same as in Illustration #1, #2, and #3, with same symbols \( \Gamma_i = \partial \Omega_i \) and \( \partial \Omega_e = \Gamma_e \cup \Gamma_i \).

**Theorem 6.4.** With constant \( \gamma > 0 \), consider the following 2-dimensional clamped /Dirichlet thermoelastic system with point control defined in the internal domain \( \Omega_i \),
containing (w.l.o.g.) the origin in its interior:

\[
\begin{cases}
y_{tt} - \gamma \Delta y_{tt} + \Delta^2 y + \Delta \theta = \delta u & \text{in } Q_i \equiv (0, T] \times \Omega_i \\
\theta_t - \Delta \theta - \Delta y_t = 0 & \text{in } Q_i \\
y(0, \cdot) = 0, \quad y_t(0, \cdot) = 0, \quad \theta(0, \cdot) = 0 & \text{in } \Omega_i \\
y|_{\Sigma_i} = 0; \quad \frac{\partial y}{\partial \nu}|_{\Sigma_i} = 0; \quad \theta|_{\Sigma_i} = 0 & \text{in } \Sigma_i = (0, T] \times \Gamma_i
\end{cases}
\] (6.34a-c)

feeding through the tangential gradient \[\nabla_{\text{tan}}(\Delta y|_{\Sigma_i})\] of its elastic moment on \[\Gamma_i = \partial \Omega_i\], a \(w\)-wave mixed problem defined on an external 2-dimensional domain \(\Omega_e\):

\[
\begin{cases}
w_{tt} = \Delta w & \text{in } \Sigma_e = (0, T] \times \Omega_e \\
w(0, \cdot) = 0, \quad w_t(0, \cdot) = 0 & \text{in } \Omega_e \\
\frac{\partial w}{\partial \nu}|_{\Sigma_i} = \nabla_{\text{tan}}(\Delta y|_{\Sigma_i}) & \text{on } (0, T] \times \Gamma_i \\
\frac{\partial w}{\partial \nu}|_{\Sigma_e} = 0 & \text{on } \Sigma_e = (0, T] \times \Gamma_e
\end{cases}
\] (6.35a-d)

Assume:

\[u \in L^2(0, T).\] (6.36)

Then, continuously

\[w \in C([0, T]; H^{\beta-1}(\Omega_e)), \quad w_t \in C([0, T]; H^{\beta-2}(\Omega_e))\] (6.37)

Proof. From [29] (this result is also quoted in [18, Theorem 1.2, p 247]) we know that assumption (6.36) implies the following boundary regularity of the elastic component

\[\Delta y|_{\Sigma_i} \in L^2(0, T; L^2(\Gamma_i)) \equiv L^2(\Sigma_i).\] (6.38)

Hence, the input

\[\nabla_{\text{tan}}(\Delta y|_{\Sigma_i}) \in L^2(0, T; H^{-1}(\Gamma_i))\] (6.39)

fed in the Neumann B.C. (6.35c) yields conclusion (6.37) for \(\{w, w_t\}\), as desired, by invoking Theorem 5.4 to the resulting \(w\)-problem.

Illustration #5: An uncontrolled Kirchoff Equation feeding in a serial connection a wave equation.

The setting of the internal domain \(\Omega_i\) immersed in an external domain \(\Omega_e\) is exactly the same as in Illustration #1, #2, #3, and #4, with same symbols \(\Gamma_i = \partial \Omega_i\) and \(\partial \Omega_e = \Gamma_e \cup \Gamma_i\).

**Theorem 6.5.** With constant \(\gamma > 0\), consider the following \(\phi\)-Kirchoff mixed problem defined on the external domain \(\Omega_e\):

\[
\begin{cases}
\phi_{tt} - \gamma \Delta \phi_{tt} + \Delta^2 \phi = f & \text{in } Q_e \equiv (0, T] \times \Omega_e \\
\phi(0, \cdot) = \phi_0, \quad \phi_t(0, \cdot) = \phi_1 & \text{in } \Omega_e \\
\phi|_{\Sigma_e} = 0; \quad \Delta \phi|_{\Sigma_e} = 0 & \text{in } \Sigma_e \equiv (0, T] \times \partial \Omega_e
\end{cases}
\] (6.40a-c)
mixed problem defined on the internal domain $\Omega_i$: \begin{align*}
  \left\{ \begin{array}{ll}
  w_{tt} = \Delta w & \text{in } Q_i \equiv (0, T) \times \Omega_i \\
  w(0, \cdot) = 0, & \text{in } \Omega_i \\
  w_t(0, \cdot) = 0 & \text{in } \Omega_i \\
  \left. \frac{\partial w}{\partial \nu} \right|_{\Sigma_i} = \left[ \nabla \tan \left( \frac{\partial \Delta \phi}{\partial \nu} \right) \right]_{\Sigma_i} & \text{on } \Sigma_i \equiv (0, T) \times \Gamma_i
  \end{array} \right.
  \tag{6.41a}
\end{align*}

Assume:

\begin{align*}
  \left\{ \begin{array}{l}
  \phi_0, \phi_1 \in V \times [H^2(\Omega_e) \cap H^1_0(\Omega_e)], \quad f \in L^1(0, T; L^2(\Omega_e)) \\
  V = \{ h \in H^3(\Omega_e) : h|_{\partial \Omega_e} = \Delta h|_{\partial \Omega_e} = 0 \}
  \end{array} \right.
  \tag{6.42a}
\end{align*}

Then, \begin{align*}
  w \in C([0, T]; H^{3-1}(\Omega_i)), \quad w_t \in C([0, T]; H^{2-2}(\Omega_i))
  \tag{6.43}
\end{align*}

Proof. Under assumption (6.42), it is known \cite{16}, \cite[Theorem 10.7.3.1, p 993]{17} that

\begin{align*}
  \left. \frac{\partial \Delta \phi}{\partial \nu} \right|_{\partial \Omega_e} \in L^2(0, T; L^2(\partial \Omega_e)).
  \tag{6.44}
\end{align*}

Hence, the input

\begin{align*}
  \left[ \nabla \tan \left( \frac{\partial \Delta \phi}{\partial \nu} \right) \right]_{\Sigma_i} \in L^2(0, T; H^{-1}(\Gamma_i))
  \tag{6.45}
\end{align*}

fed in the Neumann B.C. (6.41c) yields conclusion (6.43) for $\{w, w_t\}$, as desired, by invoking Theorem 5.4 to the resulting $w$-problem.

Illustration #6: An uncontrolled Euler-Bernoulli feeding in a serial connection a wave equation.

The setting of the internal domain $\Omega_i$ immersed in an external domain $\Omega_e$ is exactly the same as in Illustration #1–#5 with same symbols $\Gamma_i = \partial \Omega_i$ and $\partial \Omega_e = \Gamma_e \cup \Gamma_i$.

\begin{theorem}
  Consider the following $\phi$-Euler Bernoulli mixed problem ($\gamma = 0$ in (6.40a)) defined on the external domain $\Omega_e$: \begin{align*}
  \left\{ \begin{array}{ll}
  \phi_{tt} + \Delta^2 \phi = f & \text{in } Q_e \equiv (0, T) \times \Omega_e \\
  \phi(0, \cdot) = \phi_0, & \text{in } \Omega_e \\
  \phi_t(0, \cdot) = \phi_1 & \text{in } \Omega_e \\
  \phi|_{\Sigma_e} = 0, \quad \Delta \phi|_{\Sigma_e} = 0 & \text{in } \Sigma_e \equiv (0, T) \times \partial \Omega_e
  \end{array} \right.
  \tag{6.46a}
\end{align*}

feeding through the tangential gradient \begin{align*}
  \left[ \nabla \tan \left( \frac{\partial \phi_t}{\partial \nu} \right) \right]_{\Sigma_i} \text{ on } \Gamma_i = \partial \Omega_i \text{ a } w\text{-wave}
  \tag{6.46c}
\end{align*}

mixed problem defined on the internal domain $\Omega_i$: \begin{align*}
  \left\{ \begin{array}{ll}
  w_{tt} = \Delta w & \text{in } Q_i \equiv (0, T) \times \Omega_i \\
  w(0, \cdot) = 0, & \text{in } \Omega_i \\
  w_t(0, \cdot) = 0 & \text{in } \Omega_i \\
  \left. \frac{\partial w}{\partial \nu} \right|_{\Sigma_i} = \left[ \nabla \tan \left( \frac{\partial \phi_t}{\partial \nu} \right) \right]_{\Sigma_i} & \text{on } \Sigma_i \equiv (0, T) \times \Gamma_i
  \end{array} \right.
  \tag{6.47a}
\end{align*}

Assume:

\begin{align*}
  \left\{ \begin{array}{l}
  \phi_0, \phi_1 \in V \times H^1_0(\Omega_e), \quad f \in L^1(0, T; H^1_0(\Omega_e))
  \end{array} \right.
  \tag{6.48}
\end{align*}
where $V$ is defined in (6.42b). Then, continuously

$$w \in C([0, T]; H^{\beta-1}(\Omega)), \quad w_t \in C([0, T]; H^{\beta-2}(\Omega))$$  \hfill (6.49)

**Proof.** Under assumption (6.48), it is known [19], [17, Theorem 10.8.10.1, p 1022] that

$$\Delta \phi \cdot \phi_t \in L^2(0, T; L^2(\partial \Omega_e)).$$  \hfill (6.50)

Hence, the input

$$\left[ \nabla \tan \left( \frac{\partial \phi}{\partial \nu} \bigg|_{\Sigma_i} \right) \right]_{\Sigma_i} \in L^2(0, T; H^{-1}(\Gamma_i))$$  \hfill (6.51)

fed in the Neumann B.C. (6.47c) yields conclusion (6.49) for \{w, w_t\}, as desired, by invoking Theorem 5.4 to the resulting \(w\)-problem.

---

**Appendix: A soft proof of the non-optimal Theorem 1.3 ("\(1/2\)-gain in space").** In this Appendix we provide a result which is a slightly stronger version of Theorem 1.3. Refer also to Remark 1.4. The proof is in spirit of the radically different re-proof, given in [15, p 122], of result (1.3) in [21, Vol II, p 120].

**Theorem A.1.** Let

$$g \in C([0, T]; H^{-\frac{1}{2}}(\Gamma)) \quad g(0) = 0 \quad \text{(A.1)}$$

$$\dot{g} \in L^1(0, T; H^{-\frac{1}{2}}(\Gamma)) \quad \text{(A.2)}$$

Then, continuously

$$Lg \in C([0, T]; H^1(\Omega)) \quad \text{(A.3)}$$

$$\frac{d}{dt}(Lg) \in C([0, T]; L^2(\Omega)). \quad \text{(A.4)}$$

**Proof.** (A.3). We return to the explicit expression of \(Lg\) given in (3.7), which we re-write as in (4.22)–(4.24) by integrating by parts with \(g(0) = 0\):

$$\begin{align*}
(Lg)(t) &= \int_0^t A_S(t - \tau) N g(\tau) d\tau = \int_0^t \frac{d}{d\tau} C(t - \tau) N g(\tau) d\tau \\
&= N g(t) - \int_0^t C(t - \tau) N \dot{g}(\tau) d\tau \\
&= N g(t) - A^{-\frac{1}{2}} \int_0^t C(t - \tau) A^{\frac{1}{2}} N \dot{g}(\tau) d\tau.
\end{align*}$$  \hfill (A.5)

By (A.2) and (3.4) on the Neumann map with \(\sigma = -\frac{1}{2}\), we have

$$N : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^1(\Omega) = D(A^{\frac{1}{2}}); \quad A^{\frac{1}{2}} N \dot{g}(\cdot) \in L^1(0, T; L^2(\Omega)).$$  \hfill (A.6)

hence by convolution

$$\int_0^t C(t - \tau) A^{\frac{1}{2}} N \dot{g}(\tau) d\tau \in C([0, T]; L^2(\Omega))$$  \hfill (A.7)

By (A.2) and (3.4) on the Neumann map with \(\sigma = -\frac{1}{2}\), we have

$$N : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^1(\Omega) = D(A^{\frac{1}{2}}); \quad A^{\frac{1}{2}} N \dot{g}(\cdot) \in L^1(0, T; L^2(\Omega)).$$  \hfill (A.8)

hence by convolution

$$\int_0^t C(t - \tau) A^{\frac{1}{2}} N \dot{g}(\tau) d\tau \in C([0, T]; L^2(\Omega))$$  \hfill (A.9)

$$A^{-\frac{1}{2}} \int_0^t C(t - \tau) A^{\frac{1}{2}} N \dot{g}(\tau) d\tau \in C([0, T]; D(A^{\frac{1}{2}}) \equiv H^1(\Omega)).$$  \hfill (A.10)

Moreover, by (A.1) and the LHS of (A.8)

$$Ng \in C([0, T]; D(A^{\frac{1}{2}}) \equiv H^1(\Omega)).$$  \hfill (A.11)

Using (A.10) and (A.11) in (A.7) yields (A.3).
As in (4.16)–(4.17), differentiate
\[(Lg)(t) = \int_0^t A\mathcal{S}(t-\tau)Ng(\tau)d\tau = \int_0^t A\mathcal{S}(r)Ng(t-r)dr\] (A.12)
to obtain with \(g(0) = 0\) in (A.1)
\[\frac{d}{dt}(Lg)(t) = \int_0^t A\mathcal{S}(t-\tau)N\dot{g}(\tau)d\tau\] (A.13)
\[= \int_0^t A^{\frac{1}{2}}\mathcal{S}(t-\tau)A^{\frac{1}{2}}N\dot{g}(\tau)d\tau \in C([0,T];L^2(\Omega))\] (A.14)
by convolution, recalling the RHS of (A.8) and the regularity of \(A^{\frac{1}{2}}\mathcal{S}(\cdot)\) in (3.11). (A.14) proves (A.4).

Acknowledgments. The author wishes to thank the referee. Research partially supported by the National Science Foundation under grant DMS-1434941. This research was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation.

REFERENCES

[1] L. Bociu and J. P. Zolesio, A pseudo-extractor approach to hidden boundary regularity for the wave equation with Neumann boundary condition, Journal of Differential Equations, 259 (2015), 5688–5708.
[2] L. Bociu and J. P. Zolesio, Hyperbolic equations with mixed boundary conditions: Shape differentiability analysis, Applied Mathematics & Optimization, (2016), 1–24.
[3] H. O. Fattorini, The Cauchy Problem, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1983.
[4] D. Fujiwawa, Concrete characterization of the domain of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad, 43 (1967), 82–86.
[5] P. Grisvard, Caracterization de quelques espaces d’interpolation, Arch. Rational Mech. Anal., 25 (1967), 40–63.
[6] T. Kato, Fractional powers of dissipative operators, J. Math. Soc. Japan, 13 (1961), 246–274.
[7] I. Lasiecka, Unified theory for abstract parabolic boundary problems—a semigroup approach, Appl. Math. & Optimiz., 6 (1980), 287–333.
[8] I. Lasiecka, J. L. Lions and R. Triggiani, Non homogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures Appl., 65 (1986), 149–192.
[9] I. Lasiecka and R. Triggiani, A cosine operator approach to modelling \(L_2(0,T;L_2(\Omega))\) boundary input hyperbolic equations, Appl. Math. Optimiz., 7 (1981), 35–93.
[10] I. Lasiecka and R. Triggiani, Regularity of hyperbolic equations under \(L_2(0,T;L_2(\Gamma))\)-Dirichlet boundary terms, Appl. Math. Optimiz., 10 (1983), 275–286.
[11] I. Lasiecka and R. Triggiani, A lifting theorem for the time regularity of solutions to abstract equations with unbounded operators and applications to hyperbolic equations, Proceedings American Mathematical Society, 104 (1988), 745–755.
[12] I. Lasiecka and R. Triggiani, Exact boundary controllability for the wave equation with Neumann boundary control, Appl. Math. Optimiz., 19 (1989), 243–290. (Also, preliminary version in Springer-Verlag Lecture Notes, 100 (1987), 316–371.)
[13] I. Lasiecka and R. Triggiani, Sharp regularity for mixed second order hyperbolic equations of Neumann type Part I: The \(L_2\)-boundary case, Annali Matem. Pura Appl., (IV) CLVII (1990), 285–367. (Announcements in Accad. Lincei, 85 (1989), 109–113, Classe di Scienze Matematiche, Rome, Italy, and Springer-Verlag Lecture Notes, 114.)
[14] I. Lasiecka and R. Triggiani, Trace regularity of the solutions of the wave equation with homogeneous Neumann boundary conditions and compactly supported data, J. Math. Anal. Appl., 141 (1989), 49–71.
[15] I. Lasiecka and R. Triggiani, Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions, Part II: General boundary data, J. Diff. Eqns., 94 (1991), 112–164.
[16] I. Lasiecka and R. Triggiani, Exact Controllability and uniform stabilization of Kirchoff plates with boundary controls only on $\Delta w|_{\Sigma}$, J. Diff. Eqts., 93 (1991), 62–101.

[17] I. Lasiecka and R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories II, Abstract Hyperbolic Systems over a Finite Time Horizon, Encyclopedia of Mathematics and Its Applications Series, Cambridge University Press, January 2000.

[18] C. Lebiedzik and R. Triggiani, The optimal interior regularity for the critical case of a clamped thermoelastic system with point control revisited, Modern Aspects of the Theory of Partial Differential Equations, Operator Theory Advances and Applications, M. Ruzhansky, Jens Wirth Editors, Birkhäuser, 216 (2010), 243–259.

[19] J. L. Lions, Contrôlabilité Exacte et Stabilisation de Systemes Distribués, vol 1, Masson, 1988.

[20] J. L. Lions and E. Magenes, Nonhomogeneous Boundary Value Problems and Applications, Vol. I, Springer-Verlag, 1972.

[21] J. L. Lions and E. Magenes, Nonhomogeneous Boundary Value Problems and Applications, Vol. II, Springer-Verlag, 1972.

[22] S. Myatake, Mixed problems for hyperbolic equations of second order, J. Math. Kyoto Univ, 130 (1973), 435–487.

[23] R. Sakamoto, Hyperbolic Boundary Value Problems, Cambridge University Press, London/New York, 1982.

[24] M. Sova, Cosine operator functions, Rozprawy Mat, 49 (1966), 1–47.

[25] W. Symes, A trace theorem for solutions of the wave equation, and the remote determination of acoustic sources, Mathematical Methods in the Applied Sciences, 5 (1983), 131–152.

[26] D. Tataru, On the regularity of boundary traces for the wave equation, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 26 (1998), 185–206.

[27] R. Triggiani, A cosine operator approach to modeling boundary input problems for hyperbolic systems, Springer-Verlag Lecture Notes in Control and Information Sciences, 6 (1978), 380–390.

[28] R. Triggiani, Interior and boundary regularity of the wave equation of interior point control, J. Diff. Eqts, 103 (1993), 394–420.

[29] R. Triggiani, The critical case of clamped thermoelastic systems with interior point control: optimal interior and boundary regularity results, J. Diff. Eqts., 245 (2008), 3764–3805.

Received August 2016; revised August 2016.

E-mail address: rtrggani@memphis.edu