Center conditions: pull-back of differential equations

Yadollah Zare

Abstract

The space of polynomial differential equations of a fixed degree with a center singularity has many irreducible components. We prove that pull-back differential equations form an irreducible component of such a space. The method used in this article is inspired by Ilyashenko and Movasati’s method. The main concepts are the Picard-Lefschetz theory of a polynomial in two variables with complex coefficients, the Dynkin diagram of the polynomial and the iterated integral.

0 Introduction

Let \( \mathbb{C}[x, y]_{\leq d} \) be the set of polynomials in the two variables \( x, y \), and coefficients in \( \mathbb{C} \) of degree less than or equal to \( d \in \mathbb{N}_0 \). The space of algebraic foliations

\[
\mathcal{F} = \mathcal{F}(\omega) , \quad \omega \in \Omega_d^1,
\]

where

\[
\Omega_d^1 := \{ P(x, y)dy - Q(x, y)dx \mid P, Q \in \mathbb{C}[x, y]_{\leq d} \},
\]

is the projectivization of the vector space \( \Omega_d^1 \), and it is denoted by \( \mathcal{F}(d) \). The maximum degree of the polynomials \( P \) and \( Q \) is known as the (affine) degree of \( \mathcal{F} \). The space \( \mathcal{F}(d) \) is a rational variety by taking the coefficients of polynomials as the coordinates of affine variety \( A^N \) for some \( N \). The set of singularities of the foliation \( \mathcal{F} \) is \( V(P) \cap V(Q) \). If \( (P_y Q_y - P_y Q_x)(p) \neq 0 \), for an isolated singularity \( p \) of \( \mathcal{F} \), then \( p \) is called reduced singularity. If there is a holomorphic coordinate system \( (\tilde{x}, \tilde{y}) \) in a neighborhood of a reduced singularity \( p \) with \( \tilde{x}(p) = 0, \tilde{y}(p) = 0 \) such that in this coordinate system

\[
\omega \wedge d(\tilde{x}^2 + \tilde{y}^2) = 0,
\]

then the point \( p \) is called a center singularity. The closure of the set of algebraic foliations of fixed degree \( d \) with at least one center in \( \mathcal{F}(d) \), which is denoted by \( \mathcal{M}(d) \), is an algebraic subset of \( \mathcal{F}(d) \) (see for instance, [12] and [10]). Identifying irreducible components of \( \mathcal{M}(d) \) is the center condition problem in the context of polynomial differential equations on the real plane. The complete classification of irreducible components of \( \mathcal{M}(2) \) is done by H. Dulac in [4] (see also [2] p.601). This classification gives applications on the number of limit cycles in the context of polynomial differential equations on the real plane. Ilyashenko in [9], by computing tangent space at some smooth points of the space of Hamiltonian foliations \( \mathcal{F}(df), f \in \mathbb{C}[x, y]_{\leq d+1} \), proved the following:

**Theorem 0.1.** The space of Hamiltonian foliations of degree \( d \) forms an irreducible component of \( \mathcal{M}(d) \).

H. Movasati in [14], by computing the tangent cone \( \mathcal{M}(d) \) at a special point proved the following:

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Theorem 0.2. The space of $\mathcal{L}(d_1,d_2,\cdots,d_s)$ of logarithmic foliations

$$\mathcal{F}(\omega) \quad \omega = f_1 \cdots f_s \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}$$

$$f_i \in \mathbb{C}[x,y]_{\leq d_i} \quad \lambda_i \in \mathbb{C} \quad i = 1,2,\cdots,s, \quad d = \sum_{i=1}^{s} d_i - 1$$

is an irreducible component of $\mathcal{M}(d)$.

Let $\mathcal{P}(a,n)$ be the set of foliation

$$(1) \quad \mathcal{F}(F^*(\omega)) \quad \text{where} \quad \omega \in \Omega^1_a,$$

$F : \mathbb{C}^2 \to \mathbb{C}^2$ is defined by $(x,y) \to (R,S)$ and $R,S \in \mathbb{C}[x,y]_{\leq n}, n \geq 2$.

For a generic morphism $F$ and foliation $\mathcal{F}$, there exist a leaf of $\mathcal{F}$ such that it has an intersection with $\mathcal{F}(D)$ at some points with multiplicity 2, where $D$ is the curve $V(R_xS_y - R_yS_x)$. Therefore, $F^*(\mathcal{F})$ has a center singularity.

Theorem 0.3. The space $\mathcal{P}(a,n)$ of pull-back differential equations

$$\mathcal{F}(\omega), \quad \omega = P(R,S)dS - Q(R,S)dR$$

where

$$R, S \in \mathbb{C}[x,y]_{\leq n}, \quad P, Q \in \mathbb{C}[x,y]_{\leq a}, \quad d = an + n - 1, \quad n \geq 2$$

forms an irreducible component of $\mathcal{M}(d)$.

This paper is inspired by Ilyashenko’s paper [9] and H. Movasati’s paper [14] and a sketch of our proof is the following:

Consider a generic $F$ and a generic polynomial $f \in \mathbb{C}[x,y]$ of degree $a + 1$. It is clear that the point $\mathcal{F}(d(f \circ F))$ is in the intersection of $\mathcal{H}(an + n - 1)$ and $\mathcal{P}(a,n)$ of the algebraic set $\mathcal{M}(an + n - 1)$. It is needed to show that the tangent cone of $\mathcal{M}(an + n - 1)$ at the point $\mathcal{F}$ is equal to $T_\mathcal{F}\mathcal{H}(an + n - 1) \cup T_\mathcal{F}\mathcal{P}(a,n)$, in order to prove Theorem 0.3. The proof will be explained in sections 1,2 and 4.

In §1, by taking the deformation $d(f \circ F) + \epsilon^k \omega_k + \epsilon^{k+1}\omega_{k+1} + \cdots + \epsilon^{2k}\omega_{2k} + h.o.t$ where $\omega_k \neq 0$ of $d(f \circ F)$, and using Petrov module concept, we show that there is a polynomial 1-form $\alpha \in \Omega^1$ with degree $a$ and a polynomial $K \in \mathbb{C}[x,y]$ such that $\omega_k$ is of the form $F^*(\alpha) + dK$. This paper is organized as follows:

In §2, we are going to calculate the explicit form $dK$, by using the iterated integral and Melnikov function $M_{2k}$. This gives us the proof of Theorem 0.3.

In §3, we see some applications of theorem 0.3. We found a maximum lower bound for the cyclicity of a tangency vanishing cycle in a deformation $\mathcal{F}$ inside $\mathcal{F}(d)$ which is dependent on a factorization of $d$ to two natural numbers.

In §4, we study the action of the monodromy group on a tangency vanishing cycle in a regular fiber $f \circ F$. 
1 Pull-back of differential equations

Inspired by H. Movasati’s method (see [14]), we will calculate the tangent cone of $\mathcal{M}(n(a+1)-1)$ at the point in the intersection of Hamiltonian and pull-back algebraic differential equations. Similar to [14] and [9] our methods are based on Picard-Lefschetz theory for the foliations with a first integral.

Let $\mathcal{F} := \mathcal{F}(\omega) \in \mathcal{F}(a)$ be a foliation of degree $a$, and $F = (R, S) : \mathbb{C}^2 \to \mathbb{C}^2$ be a morphism, where $R, S \in \mathbb{C}[x, y] \leq n$ and $n \geq 2$. If a point $q$ is the tangent point of $F(D)$ and a leaf of the foliation $\mathcal{F}$ then a point in $F^{-1}(q)$ is called a tangency critical point of the foliation $F^*(\mathcal{F})$.

Theorem 1.1. Consider the deformation

$$F_\epsilon : \mathcal{F}^*(\omega) + \epsilon \omega_1 + \cdots$$

of the foliation $F^*(\mathcal{F})$. Let $p$ be one of the tangency critical points of foliation $F^*(\mathcal{F}(\omega))$. For a generic choice of $\omega$ and $F$, if the deformed foliation $F_\epsilon$ for all small $\epsilon$ has center singularity near $p$, then $F_\epsilon$ is also a pull-back foliation. More precisely, there is a foliation $\tilde{F}_\epsilon \in \mathcal{F}(a)$ and a polynomial map $F_\epsilon = (R_\epsilon, S_\epsilon) : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$F^*_{\epsilon} \tilde{F}_\epsilon = F_\epsilon, F_0 = F_0 = \mathcal{F}.$$

Note that Theorem 1.1 is equivalent to Theorem 0.3.

1.1 Tangent space

The set $\mathcal{P}(a, n)$ is an irreducible algebraic subset of $\mathcal{M}(an+n-1)$ (by taking the coefficient of the polynomials as coordinates of the map from the space of polynomials with degree $an+n-1$ to the projective space). We are going to show that $\mathcal{P}(a, n)$ is also a component of $\mathcal{M}(an+n-1)$. Let us take a point $\mathcal{F}$ of $\mathcal{P}(a, n)$, then make a deformation $F_\epsilon \in \mathcal{P}(a, n)$ and calculate the tangent space of $\mathcal{P}(a, n)$ at $\mathcal{F}$:

$$F^*_{\epsilon}(\omega_\epsilon) = (F + \epsilon F_1)^*_{\epsilon}(\omega + \epsilon \alpha_1) + O(\epsilon^2)$$

$$= F^*(\omega) + \epsilon W + O(\epsilon^2),$$

where

$$W = P(R, S)dS_1 - Q(R, S)dR_1 + R_1(\frac{\partial P}{\partial x}(R, S)dS - \frac{\partial Q}{\partial x}(R, S)dR) + S_1(\frac{\partial P}{\partial y}(R, S)dS - \frac{\partial Q}{\partial y}(R, S)dR) + F^*(\alpha_1).$$

For a smooth point $\mathcal{F}$ of $\mathcal{P}(a, n)$, the tangent space of $\mathcal{P}(a, n)$ at $\mathcal{F}$ is just the set of all vectors $W$, which is contained in the tangent space of $\mathcal{M}(d)$ at $\mathcal{F}$, and in order to prove our main theorem it is enough to prove that the equality happens. Now, we are going to compute the tangent cone of $\mathcal{M}(an+n-1)$ at the point in the intersection of Hamiltonian component and the set $\mathcal{P}(a, n)$.

By generic we mean always a non-empty Zariski open subset of the ambient space.
1.2 A foliation in the intersection of two algebraic sets

Let $F := \mathbb{C}^2 \to \mathbb{C}^2$ be defined by

$$\begin{align*}
(x, y) \to (R(x), S(y)) := \left( \prod_{i=1}^{n} (x - t_i), \prod_{j=1}^{n} (y - t_j') \right)
\end{align*}$$

where $t_i, t'_j \in \mathbb{R}_{\geq 0}$, $R$ and $S$ are Morse functions. Let $g, h$ be two polynomials of degree $a+1$ defined by

$$\begin{align*}
g(x) := \prod_{i=1}^{a+1} (x - s_i), \quad \text{and} \quad h(y) := \prod_{i=1}^{a+1} (y - s'_j),
\end{align*}$$

and meet the following conditions:

1. All $s_i, s'_j$ are positive real numbers,
2. Both equations $R(x) = s_i$ and $S(y) = s'_j$ have $n$ real roots,
3. The functions $g, h, g \circ R$ and $h \circ S$ are Morse, which is a holomorphic function with no degenerate critical points.
4. If $p$ is a critical point of $R$ (resp. $S$) and $q \in R^{-1}(q_1)$ (resp. $q \in S^{-1}(q_1)$) where $q_1$ is a critical point of $g$ (resp. $h$), then $|g \circ R(p)| > |g \circ R(q)|$ (resp. $|h \circ S(p)| > |h \circ S(q)|$).

In fact, by moving the roots of $g$ and $h$ on the real line this is the assumable definition.

Let $f \in \mathbb{C}[x, y]_{\leq a+1}$ be defined by

$$\begin{align*}
f(x, y) := g(x) + h(y).
\end{align*}$$

We can suppose that the intersection of the set of the critical values of $g \circ R$ and $h \circ S$ is empty. The foliation $F_0$ has three kinds of singularities:

1. Pull-back of centers of $F(df)$,
2. Tangency critical points of the foliation $F^*(df)$,
3. The points in $V(R_x) \cap V(S_y)$.

Let $X(a, n)$ be the irreducible component of $M(an + n - 1)$ containing $\mathcal{P}(a, n)$.

Consider the deformation

$$\begin{align*}
F_\epsilon : d(f \circ F) + \omega_k \epsilon^k + \omega_{k+1} \epsilon^{k+1} + \cdots, \ deg(\omega_i) \leq d,
\end{align*}$$

of $F_0 = F(d(f \circ F))$.

Assume that $F_\epsilon$ belongs to $X(a, n)$. This implies that $F_\epsilon$ always has a center singularity near a fixed tangency center $p$ of $F_0$. The set of all differential forms $\omega_k$ is the tangent cone of $M(an + n - 1)$ at $F$. Note that taking $k = 1$ is not sufficient for calculating the tangent cone.

Let $\delta_t$ be a continuous family of the vanishing cycles around a tangency critical $p$ and $\Sigma$ be a transverse section to $F$ at some point of $\delta_t$. We are able to write the Taylor expansion of the deformed holonomy $h_\epsilon(t)$

$$\begin{align*}
h_\epsilon(t) - t = M_1(t) \epsilon + M_2 \epsilon^2 + \cdots + M_i(t) \epsilon^i + \cdots.
\end{align*}$$
Here $M_i(t)$ is the $i$-th Melnikov function of the deformation. Since $\omega_i = 0$ for $0 \leq i \leq k-1$, then
$$M_1 = M_2 = \cdots = M_{k-1} = 0.$$If $\Sigma$ is parametrized by the image of $f$, i.e., $t = f(z), z \in \Sigma$ then
$$M_k(t) = -\int_{\delta_t} \omega_k.$$See for instance [5].

**Theorem 1.2.** The morphism $F_* : H_1((f \circ F)^{-1}(b), \mathbb{Z}) \to H_1(f^{-1}(b), \mathbb{Z})$ is surjective and $\ker(F_*)$ is a group generated by the action monodromy group $\pi_1(\mathbb{C} \setminus \mathbb{C}, b)$ on a vanishing cycle around a tangency point.

We will prove this theorem at the end of §4, see Theorem 4.10.

### 1.3 Brieskorn lattice/Petrov Modules

Consider the Brieskorn lattice/Petrov module
$$H_f := \frac{\Omega^1_{\mathbb{C}^2}}{df \wedge \Omega^0_{\mathbb{C}^2} + d\Omega^0_{\mathbb{C}^2}} \quad \text{and} \quad H_{f \circ F} := \frac{\Omega^1_{\mathbb{C}^2}}{d(f \circ F) \wedge \Omega^0_{\mathbb{C}^2} + d\Omega^0_{\mathbb{C}^2}},$$
where $H_f$ and $H_{f \circ F}$ are $\mathbb{C}[s]$-module and $\mathbb{C}[s']$-module respectively, (here $s = f$ and $s' = f \circ F$, and also $\Omega^i_{\mathbb{C}^2}, i = 0, 1$ are the set of polynomial differential forms in $\mathbb{C}^2$). 

**Definition 1.1.** A polynomial $l \in \mathbb{C}[x, y]$ of degree $d$ with homogeneous leading part $l_d$ is called transversal to infinity, if $l_d$ factors out as the product of $d$ pairwise different linear forms.

Consider the Milnor module
$$V_d := \frac{\mathbb{C}[x, y]}{< (l_d)x, (l_d)y >},$$
with the basis $\{x^i y^j | 0 \leq i, j \leq d - 2\}$ (see e.g. ([13] chapter 10)). We define
$$A_{ij} := \frac{i + 1}{d} + \frac{j + 1}{d},$$
$$\eta := xdy - ydx,$$
$$\eta_{ij} := x^i y^j \eta,$$
where $0 \leq i, j \leq d - 2$.

**Theorem 1.3.** Let $l(x, y) \in \mathbb{C}[x, y]$ be a polynomial transversal to infinity of degree $d$. The $\mathbb{C}[l]$-module $H_l$ is free and $\eta_{ij}$, where $0 \leq i, j \leq d - 2$, forms a basis of $H_l$. Furthermore, every $\omega \in \Omega^1_{\mathbb{C}^2}$ can be written
$$\omega = \sum_{0 \leq i, j \leq d - 2} h_{ij}(l) \eta_{ij} + dl \wedge \zeta_1 + d \zeta_2,$$
where $h_{ij} \in \mathbb{C}[l], \zeta_1, \zeta_2 \in \mathbb{C}[x, y], \deg(h_{ij}) \leq \frac{\deg(\omega)}{d} - A_{ij}$. 

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See e.g. [13] Theorem 10.9.1. and [8].

Proposition 1.1. Pull-backs of $\eta_{ij}$ for all $i$ and $j$ are independent in $H_{f \circ F}$ under the map $F^*$ and can be extended to a basis for $H_{f \circ F}$.

Proof. The map $F^* : H_f \rightarrow H_{f \circ F}$ is injective, and $F^*(\eta_{ij})$ are linear independent. We have $F^*(\eta_{ij} = x^i y^j(ydx - xdy)) = R^i S^j (RdS - SdR)$, and the coefficients of $R^i S^j$ are in $\mathbb{C}$. By Theorem 1.3 we can write

$$RdS - SdR = \sum_{0 \leq i,j \leq an + n - 1} P_{ij}(s')\tilde{\eta}_{ij}.$$ 

We are going to show that the functions $P_{ij}(s') \in \mathbb{C}[s' = f \circ F]$ are constant. We know that $xdy = \frac{1}{2}(x(ydx - ydx)) = \frac{1}{2}(\tilde{\eta})$ and $ydx = -\frac{1}{2}(\tilde{\eta})$ in $H_{f \circ F}$. By changing the coordinate assume that $R(0) = S(0) = 0$, therefore

$$RdS - SdR = \left(\prod_{i=2}^{n}(x - t_i)S_y\right)x dy - \left(\prod_{i=2}^{n}(y - t'_i)R_x\right)y dx$$

$$= \left(\prod_{i=2}^{n}(x - t_i)S_y\right)(\frac{1}{2}\tilde{\eta}) + \left(\prod_{i=2}^{n}(y - t'_i)R_x\right)(\frac{1}{2}\tilde{\eta}),$$

since $\text{deg}(RdS - SdR) < an + n - 1$ then the functions $P_{ij}$’s are constant. It means that if we write $F^*(\eta_\beta) = \sum_{\gamma'} P_{\gamma'}(s')\tilde{\eta}_{\gamma'}$, then all of the coefficients $P_{\gamma'}(s')$ are in $\mathbb{C}$. In other words, $F^*(\eta_\beta)$ for all $\beta = ij$ can extend to a basis of $H_{f \circ F}$. \qed

1.4 Relatively Exact 1-form

Definition 1.2. Let $\mathcal{F}$ be a foliation in $\mathbb{C}^2$. If the restriction of a meromorphic 1-form $\omega$ on $\mathbb{C}^2$ to each leaf $L$ of $\mathcal{F}$ is exact, then it is called relatively exact modulo $\mathcal{F}$, i.e. there is a meromorphic function $f$ on $L$ so that $\omega\big|_L = df$.

Note that a meromorphic 1-form $\omega$ is relatively exact modulo $\mathcal{F}$ if and only if

$$\int_{\delta} \omega = 0,$$

for all closed cycles in the leaves of $\mathcal{F}$.

Proposition 1.2. Every relatively exact polynomial 1-form $\omega$ in $\mathbb{C}^2$ of degree $d$ modulo a Hamiltonian foliation $\mathcal{F}(df)$ has the form

$$\omega = dg + p df$$

where $g$ and $p$ are polynomials so that $\text{deg}(p)$ divides $d$, and $\text{deg}(g) \leq d$.

See e.g. ([12] Theorem 4.1.).
1.5 Computing the Tangent Cone

Let $F$ be a morphism from $\mathbb{C}^2$ into itself and $f$ be a polynomial of degree $a + 1$ that are defined in (2) and (4) respectively. Let also consider the deformation $\mathcal{F}_\epsilon = \omega_\epsilon$ of $\mathcal{F}(d(f \circ F))$, where

$$\omega_\epsilon = d(f \circ F) + \epsilon^k \omega_k + \epsilon^{k+1} \omega_{k+1} + \ldots, \quad \text{deg}(\omega_j) \leq an + n - 1.$$ 

It is not necessary to start $k$ from one. Then from the equality (6) we have the following:

**Theorem 1.4.** There is a polynomial differential 1-form $\alpha_1$ with $\text{deg}(\alpha_1) \leq \text{deg}(\omega_k)$ and a polynomial $K \in \mathbb{C}[x,y]_{\leq a+1}$ such that

$$\omega_k = F^*(\alpha_1) + dK,$$

where $F : \mathbb{C}^2 \to \mathbb{C}^2$ is defined by $(x,y) \to (R,S)$ as in (2).

**Proof.** For a regular value $b$ of the function $f \circ F$, it is clear that the linear map

$$F_* : H_1((f \circ F)^{-1}(b), \mathbb{Z}) \to H_1((f)^{-1}(b), \mathbb{Z}),$$

is surjective. Then

$$F_b^* = H_1^{dR}(f^{-1}(b)) \to H_1^{dR}((f \circ F)^{-1}(b))$$

is injective. According to Theorem 1.2, $\int_\delta \omega_k = 0$ for all $\delta \in \ker(F_*)$, this implies that the linear map

$$H_1(f^{-1}(b), \mathbb{Z}) \to \mathbb{C} \text{ defined by } \delta \to \int_\gamma \omega_k,$$

for an element $\gamma \in F^{-1}(\delta)$, is well defined. By duality of de Rham cohomology and singular homology there is a differential form $\alpha_b$ in $f^{-1}(b)$ such that

$$\int_\gamma \omega_k = \int_\delta \alpha_b \text{ for a } \gamma \in F^{-1}(\delta).$$

By using Atiyah-Hodge theorem (see e.g. [13]) the form $\alpha_b$ can be taken algebraically. All these $\alpha_b$'s give us a holomorphic global section $\alpha$ of cohomology bundle of $f$ outside the critical values of $f$;

$$\alpha_t \in H_1^{dR}(f^{-1}(t)) \text{ where } t \in \mathbb{C} \setminus \{c_1, \ldots, c_{a^2}\}.$$

We are going to show that it is a holomorphic global section in the whole $\mathbb{C}$.

By the Theorem 1.3 we can write

$$\alpha = \sum_{\beta} h_{\beta} \eta_\beta \text{ where } h_{\beta} \text{ s are holomorphic in } \mathbb{C} \setminus \{c_1, \ldots, c_{a^2}\}, \beta = (i,j).$$

The Periodic matrix

$$\left[ \int_{\delta \epsilon_k} \eta_\beta \right]_{\mu \times \mu}$$
is invertible, where $\mu$ is the rank of $H_1((f \circ F)^{-1}(b), \mathbb{Z})$, (see e.g. [9] Proposition 26.44).

Therefore, the $h_\beta$’s coefficients are meromorphic functions on $t$, because

$$
\begin{bmatrix}
    h_{\beta_1} \\
    \vdots \\
    h_{\beta_n}
\end{bmatrix}
= 
\left[\int_{\delta_k} \eta_\beta\right]^{-1} 
\begin{bmatrix}
    \int_{\delta_1} \alpha \\
    \vdots \\
    \int_{\delta_n} \alpha
\end{bmatrix},
$$

and by Theorem 10.7 in Chapter 10 of [1] each integral $||\int_{\delta_k} \eta_\beta|| \leq \text{const} ||t - c_k||^{-N}$ for a natural number $N$ and $t$ close to singular value $c_k$. Thus, all the elements of the matrices on the right side of the equality have finite growth at critical values. This implies that, there is a polynomial $P(s) \in \mathbb{C}[s]$ such that $P\alpha$ is a holomorphic form. We can write $P\alpha = \sum \beta h'_{\beta} \eta_\beta$, then $F^*(P)\omega_k - F^*(P\alpha) = 0$ in $H_{f \circ F}$. According to Proposition 1.1 the set of $F^*(\eta_\beta)$ for all $\beta$ can be extended to a basis of $H_{f \circ F}$. Therefore, we have

$$
F^*(P)\omega_k = \sum _{\beta} F^*(P)h_{\beta}(s)\eta_\beta + \sum _{\sigma} F^*(P)a_{\sigma}\eta_\sigma.
$$

Since each element of $H_{f \circ F}$ can be written uniquely as a linear combination of the elements on this basis, then $a_{\sigma} = 0$ for all $\sigma$. In other words, $F^*(P)h_{\beta} = F^*(h'_{\beta})$, hence $P|h'_{\beta}$. This implies that $\alpha$ is a holomorphic 1-form. By Theorem 1.5, the degree of $h_\beta$ in the equation (8) is less than or equal to $\deg(\omega_k) - a_{\beta} < 1$, hence $h_\beta$ are constant for all $\beta$. To find the form of $\omega_k$ we use the Proposition 1.2 and we conclude that $\int_{F^*(\beta)} \alpha = \int_{\delta} \omega_k$ for all cycles $\delta$ in the fibers of $f \circ F$. This implies that $\omega - F^*(\alpha)$ is relatively exact modulo $F(d(f \circ F))$, then by Proposition 1.2 there are polynomials $K$ and $A$ such that $\omega - F^*(\alpha) = dK + Ad(f \circ F)$. The fact that $\deg(\omega - F^*(\alpha)) \leq \deg(f \circ F) - 1$ implies $A \equiv 0$, so we get our desired equality.

The proof of the main theorem is still not finished. We have to prove that the polynomial $K$ in the Theorem 1.4 is of the form (9). For this goal, we need to compute higher order Melnikov functions. This will be done in the next section.

## 2 Higher order Melnikov function

L.Gavrilov in [7] has shown that the higher order Melnikov functions can be expressed in terms of iterated integrals. Basic properties of iterated integrals are established by A. N. Parsin in 1969 and a systematic approach for de Rham cohomology type theorems for iterated integrals was made by K. T. Chen around 1977. H. Movasati and I. Nakai in [16] used the concept of higher order Melnikov functions by iterated integrals.

Let $\gamma : [0,1] \to \mathbb{C}^{2}$ be a piecewise smooth path on $\mathbb{C}^{2}$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be smooth 1-forms on $\mathbb{C}^{2}$, $\gamma^*(\omega_i) = f_i(t)dt$ for the pulled-back of the forms $\omega_i$ to the interval $[0,1]$. Recall that the ordinary line integral given by

$$
\int_{\gamma} \omega_1 = \int_{[0,1]} \gamma^*(\omega_1) = \int_{0}^{1} f_1(t_1)dt
$$

does not depend on the choice of parametrization of $\gamma$.

**Definition 2.1.** Iterated integral of $\omega_1, \omega_2, \ldots, \omega_n$ along the path $\gamma$ is defined by

$$
\int_{\gamma} \omega_1 \omega_2 \ldots \omega_n = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} (f_1(t_1)dt_1 \ldots f_n(t_n)dt_n).
$$
Let us consider the deformation

\[ \mathcal{F}_\epsilon : d(f \circ F) + \omega_k \epsilon^k + \omega_{k+1} \epsilon^{k+1} + \ldots \quad \text{deg}(\omega_i) \leq n(a + 1) - 1. \]

of \( d(f \circ F) \). The deformed holonomy along the path \( \delta_t \) in \( \Sigma \) is

\[ h_\epsilon(t) - t = M_1(t) \epsilon + \cdots + M_k(t) \epsilon^k + \cdots + M_{2k}(t) \epsilon^{2k} + \ldots. \]

Since \( \omega_i = 0 \) where \( 0 < i < k - 1 \), then \( M_1 = M_2 = \cdots = M_{k-1} = 0 \). By using Theorem 3.2 in [15] (Higher order approximation), we conclude that \( M_i(t) = -\int_{\delta_t} \omega_i \) where \( k \leq i < 2k \), and also

\[ M_{2k}(t) := -\int_{\delta_t} (\omega_k \cdot \frac{d\omega_k}{d(f \circ F)}) + \omega_{2k}. \]

Note that the vector \( W \) in the case \( \omega = d(f \circ F) \) is of the form

\[ W = d(R_1 \frac{\partial f}{\partial x}(R, S) + S_1 \frac{\partial f}{\partial y}(R, S)) + F^*(\alpha_1). \]

**Lemma 2.1.** The polynomial \( K \) in the Theorem [1,4] is of the form

\[ K = R_1 \frac{\partial f}{\partial x}(R, S) + S_1 \frac{\partial f}{\partial y}(R, S), \]

where \( R_1, S_1 \in \mathbb{C}[x, y]_{\leq n}. \)

**Proof.**

\[
\int_{\delta_t} \omega_k \cdot \left( \frac{d\omega_k}{d(f \circ F)} \right) = \int_{\delta_t} (F^* \alpha_1 + dK) \cdot F^*(\frac{d\alpha_1}{df})
= \int_{\delta_t} F^*(\alpha_1 \cdot \frac{d\alpha_1}{df}) + \int_{\delta_t} (dK) \cdot F^*(\frac{d\alpha_1}{df})
= \int_{F(\delta_t)} \alpha_1 \frac{d\alpha_1}{df} + \int_{t_1 \leq t_2} (K(t_1) - K(p_{t_1})) \cdot F^*(\frac{d\alpha_1}{df})
= \int_{F(\delta_t)} \alpha_1 \frac{d\alpha_1}{df} + \int_{t_1 \leq t_2} KF^*(\frac{d\alpha_1}{df}) - K(p_1) \int_{t_1} F^*(\frac{d\alpha_1}{df})
= \int_{\delta_t} KF^*(\frac{d\alpha_1}{df}).
\]

Here \( p_t \) is a point in the cycle \( \delta_t \). Now the equality \( M_{2k}(t) = 0 \) and a similar argument as in the last lemma implies that

\[ \omega_{2k} + K.F^*(\frac{d\alpha_1}{df}) = F^*(\alpha_2) + dK_2 + A_j d(f \circ F), \]

and therefore,

\[ K.F^*(d\alpha_1) = -\omega_{2k} \wedge d(f \circ F) + F^*(\alpha_2) \wedge d(f \circ F) + dK_2 \wedge d(f \circ F). \]

Since \( d\alpha_1 \) is a 2-form like \( h(x, y)dx \wedge dy \), then

\[ F^*(h(x, y)dx \wedge dy) = F^*(h).F^*(dx \wedge dy) = (h \circ F).((R_xS_y - R_yS_x)dx \wedge dy). \]
For the proof of our main theorem, it is enough to show that
\[ I = \langle R_x f_x(R, S) + R_y f_y(R, S), S_x f_x(R, S) + S_y f_y(R, S) \rangle. \]
Now consider the radical ideal \( I_1 = \langle f_x(R, S), f_y(R, S) \rangle \) and \( J = \langle h(F). (R_x S_y - R_y S_x) \rangle \), then it is clear that \( I \subset I_1 \) and \( I_1 \cdot J \subset I \), so
\[ I_1 \cdot J \subseteq I \cap J \subseteq I_1 \cap J. \]

We can assume that the curve \( V(h) \) does not pass any critical point of \( f \). By our hypothesis \( F \) and \( f \) are generic so we have \( V(J) \cap V(I_1) = \emptyset \). This means that \( J + I_1 = \mathbb{C}[x, y] \) thus we have
\[ I_1 \cdot J = I_1 \cap J \Rightarrow I \cap J = I_1 \cdot J, \]
which states that \( K \subseteq f_x(R, S), f_y(R, S) \) therefore, we get the result
\[ K = R_1 \frac{\partial f}{\partial x}(R, S) + S_1 \frac{\partial f}{\partial y}(R, S) \quad \text{where} \quad R_1, S_1 \in \mathbb{C}[x, y]_{\le n}. \]

**Corollary 2.1.** The point \( F_0 := F(d(f \circ F)) \) is in \( M(\text{as} + s - 1) \), so tangent cone of \( M(\text{as} + s - 1) \) at the point \( F_0 \) is
\[ TC_{F_0}M(\text{as} + s - 1) = T_{F_0}P(a, s) \cup T_{F_0}H(\text{as} + s - 1). \]

### 2.1 Proof of Main theorem

Consider a germ of an analytic variety \((X, 0)\) in \((\mathbb{C}^n, 0)\). The analytic path \( \gamma : (\mathbb{C}, 0) \to (X, 0) \) has the Taylor expansion \( \gamma = \omega \varepsilon^l + \omega' \varepsilon^{l+1} + \ldots, \omega, \omega', \ldots \in \mathbb{C} \). Let \( T_l \) be the set of all \( \omega \). The tangent cone \( TC_0X \) of \( X \) at 0 is \( TC_0X = \bigcup_{l=1}^{\infty} T_l \).

The tangent cone \( TC_0X \) is an algebraic set with pure dimension \( \text{dim}(X) \), i.e. each irreducible component of \( TC_0X \) is of dimension \( \text{dim}(X) \). If 0 is a smooth point of \( X \) then \( TC_0X \) is the usual tangent space of \( X \) at 0.

The variety \( P(n, a) \) is parametrized by
\[ \tau : P_n \times P_n \times P_a \times P_a \to F(d), \quad d = na + n - 1 \]
\[ \tau(R, S, P, Q) = P(R, S)dS - Q(R, S)dR, n \ge 2, \]
and so it is irreducible.

**Proof of Theorem 0.3:** Let \( F_0 := F(d(f \circ F)) \) where \( f, F \) defined in [2], [4] respectively. For the proof of our main theorem, it is enough to show that \( X := (P(n, a), F_0) \) is an irreducible component of \((M(d), F_0)\). According to Corollary 2.1 we have:

\[ TC_{F_0}M(d) = TC_{F_0}P(n, a) \cup TC_{F_0}H(d). \]

Let \( X' \) be an irreducible component of \((M(d), F_0)\) such that \( X \subset X' \). If \( TC_{F_0}X \subset Y \), where \( Y \) is the irreducible component of \( TC_{F_0}X' \), then must be a subset of \( TC_{F_0}X \), because the equality \((10)\) is union decomposition of \( TC_{F_0}M(d) \) to irreducible component.

This implies that \( Y = TC_{F_0}X \). Dimension of \( Y \subset TC_{F_0}X' \) is equal to dimension \( X \), so \( \text{dim}(X') = \text{dim}(X) \). Therefore, \( X = X' \) because \( X \subset X' \) and \( X, X' \) are irreducible algebraic sets and they have the same dimension. \( \square \)
3 Limit cycles

Consider a real planer 1-form \( \omega = P(x, y)dy - Q(x, y)dx \) where \( P \) and \( Q \) are polynomials of degree less than or equal to \( d \). Let the foliation \( F \) induced by the 1-form \( \omega \).

**Definition 3.1.** A closed trajectory which is limit set of some trajectories of a real foliation \( F \) is called limit cycle.

The Hilbert number, which denotes by \( H_d \), is the maximum possible number of limit cycles of a real foliation \( F(\omega) \). It is still unsolved whether \( H_d \) is finite, even for the simple case \( d = 2 \). It is known that \( H_d \geq k.d^2 \) for some constant \( k \), but in 1995, C.J Christopher and N.G. Lloyd found a strong lower bound \( d^2 \log d \) for the Hilbert numbers, see [3].

Let \( X \) be an irreducible component of \( \mathcal{M}(d) \). Let \( p \) be a real center singularity of a real foliation \( F \in X - \text{sing}(\mathcal{M}(d)) \). By real foliation we mean the equation of the foliation has real coefficient. Let \( \delta_t, t \in (\mathbb{R}, 0) \) be a family of real vanishing cycles around \( p \). Roughly speaking, the cyclicity of \( \delta_0 \) is the maximum number of limit cycles appearing near \( \delta_0 \) after a deformation of \( F \) in \( \mathcal{M}(d) \). The cyclicity of \( \delta_0 \) in a deformation of \( F \) inside \( \mathcal{F}(d) \) is greater than \( \text{codim}_{\mathcal{F}(d)}(X) - 1 \). The reader can find the exact definition of cyclicity and the proof of this fact in [15], Yu. Ilyashenko in [9] shows that \( \text{codim}_{\mathcal{F}(d)}(\mathcal{H}(d)) - 1 = \frac{(d+2)(d-1)}{2} - 1 \).

The best upper bound for the cyclicity of a vanishing cycle of a Hamiltonian equation is the P.Mardesic’s result \( \frac{d^3 + d^2}{2} \) in [11]. H. Movasat in [14], shows that the cyclicity of \( \delta_0 \) of a logarithmic foliation \( F(\int \sum_{i=1}^{s} \lambda_i \frac{d\xi}{\xi}) \in \mathcal{L}(d_1, \ldots, d_s) \) is not less than

\[
(d + 1)(d + 2) - \sum_{i=1}^{d} (\frac{d_i + 1}{2})^2 - 1.
\]

This lower bound reaches to maximum when \( d_i = 1, s = d + 1, i = 1, \ldots, s \). In this case the cyclicity of \( \delta \) is not less than \( d^2 - 1 \).

**Proposition 3.1.** Suppose that \( n > 1 \) and \( d := an + n - 1 \). The cyclicity of \( \delta_0 \) in a deformation of \( F \) in \( \mathcal{F}(d) \) is not less than

\[
C := (d + 1)(d + 2) - ((n + 1)(n + 2) + (\frac{d + 1}{n})(\frac{d + 1}{n} + 1)) - 1.
\]

By considering \( d + 1 = (a + 1)n \) as a fixed value, when \( n \leq a + 1 \) and in addition the distance of \( a + 1 \) and \( n \) is minimum, then \( (n + 1)(n + 2) + (\frac{d + 1}{n})(\frac{d + 1}{n} + 1) \) will be minimum. This minimization will be led to maximizing of the cyclicity. If \( n \) and \( (a + 1) \) are near to \( \sqrt{d + 1} \) then the cyclicity \( C \) close to \( d^2 - d - 4\sqrt{d + 1} - 3 \). If \( a + 1 = p \) and \( n = q \), where \( p, q \) are primes and \( p > q \), then \( C = (pq)^2 + pq - q^2 - 3q - p^2 - p - 3 \), for instance, when \( q = 2 \) we have \( 3p^2 + p - 13 \).

4 Picard-Lefschetz Theory

In this section, we intend to study the topology of a regular fiber of a polynomial function with one and two variables. The main idea of this section is to understand the intersection number between two vanishing cycles and the action of monodromy group on a vanishing cycle in the case of pull-back of cycles under a morphism.

Let \( f \) be a Morse function with the finite set of critical values \( C \) labeled by \( c_1, c_2, \ldots, c_s \).
**Definition 4.1.** A distinguished system of paths related to $f$ is the system of smooth paths in $\mathbb{C}$, starting at the regular point $b \in \mathbb{C} \setminus C$ and ending at a point in $C$ such that

1. The paths have no self-intersections;
2. Different paths meet only at their common point $b$.

Consider a small ball $U_p$ in $\mathbb{C}^m$ with center at the Morse critical point $p$. Let the value $b$ be very close to $c := f(p)$, but not equal to it. Let $\alpha := [0, 1] \rightarrow f(U_p)$ be a path that starts at $b$, ends at $c$ and does not pass through any other critical value of $f$. By the Morse lemma, there is a local coordinate system $x_1, \ldots, x_m$ in a neighborhood of $p$ such that the function $f$ can be written in the form $f(x_1, \ldots, x_m) = c + \sum_j x_j^2$. Consider the sphere $S_t := \{ x \in f^{-1}(\alpha(t)) | Im(x_j(x)) = 0 \} \cap U_p$. Whenever $t$ tends to 1, then $S_t$ tends to $p$.

**Definition 4.2.** If $S_0 := \delta$ in the nonsingular fiber $f^{-1}(b)$, then the homology class $\delta \in H_{m-1}(f^{-1}(b), \mathbb{Z})$ is called a vanishing cycle along the path $\alpha$ or just a vanishing cycle.

**Theorem 4.1.** (see e.g [1]) The collection of the vanishing cycles along all paths of a distinguished system of paths forms a basis of the group $H_{m-1}(f^{-1}(b), \mathbb{Z})$.

**Definition 4.3.** Let $\lambda_c$ be a path of a distinguished system, and $\lambda$ be a loop in $\pi_1(\mathbb{C} \setminus C, b)$ such that 
(a) it turns once anti-clockwise around the critical point $c$, and 
(b) the closure of the interior of $\lambda$ contains the path $\lambda_c$ and does not contain any other point of $C$. Then the loop $\lambda$ is called a simple loop, corresponding to $\lambda_c$.

The paths that are homotopic to a simple loop $\lambda$ give a class of homotopic homeomorphism maps $\{ h_{\lambda} : f^{-1}(b) \rightarrow f^{-1}(b) \}$. This class defines a unique well-defined map

$$ h_{\lambda} : H_{m-1}(f^{-1}(b), \mathbb{Z}) \rightarrow H_{m-1}(f^{-1}(b), \mathbb{Z}). $$

**Definition 4.4.** For a regular value $b$ of $f$, we have

$$ h : \pi_1(\mathbb{C} \setminus C, b) \times H_{m-1}(f^{-1}(b), \mathbb{Z}) \rightarrow H_{m-1}(f^{-1}(b), \mathbb{Z}) $$

$$ h(\lambda, \cdot) = h_{\lambda}(\cdot). $$

The image of $\pi_1(\mathbb{C} \setminus C, b)$ in $\text{Aut}(H_{m-1}(f^{-1}(b), \mathbb{Z}))$ is called the monodromy group and its action $h$ is called the action of the monodromy group on the homology group of $f^{-1}(b)$.

**Picard-Lefschetz formula:** Let $\lambda$ be a monodromy (simple loop) around the critical value $c$, the action of monodromy $h_{\lambda}$ on a cycle $\delta \in H_{m-1}(f^{-1}(b), \mathbb{Z})$ is given by

$$ h_{\lambda}(\delta) = \delta + \sum_j (-1)^{m(m-1)/2} < \delta, \delta_j > \delta_j, $$

where $j$ runs through all the vanishing cycles around the singularities with value $c$, and $< \cdot, \cdot >$ denotes the intersection number of two cycles in $f^{-1}(b)$. 

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4.1 Picard-Lefschetz theory in dimension zero

Let \( f(x) \in \mathbb{C}[x] \) be a polynomial of degree \( d \) with real roots \( t_i \), where \( i = 1, \ldots, d \), and \( d - 1 \) critical values.

**Theorem 4.2.** For the regular value \( b = 0 \), we have the following:

- \( H_0(f^{-1}(b), \mathbb{Z}) \) is generated by \( \delta_i = [t_i] - [t_{i+1}] \) for \( i = 1, \ldots, d - 1 \);
- Intersection matrix for \( H_0(f^{-1}(b), \mathbb{Z}) \) with respect to this basis is

\[
< \delta_i, \delta_j > = \begin{cases} 
2 & \text{if } i = j \\
-1 & \text{if } i = j + 1 \\
0 & \text{if } |i - j| > 1.
\end{cases}
\]

(See e.g. [13] or [6].)

**Lemma 4.1.** For any two vanishing cycles \( \delta_i, \delta_j \in H_0(f^{-1}(b), \mathbb{Z}) \) there is a monodromy \( \lambda \) such that \( \lambda(\delta_i) = \delta_j \).

Consider

\[
R(x) = \prod_{i=1}^{n}(x - t_i)
\]

where \( (t_i \in \mathbb{R}^+ , t_i < t_{i+1}) \) such that \( R \) has \( s - 1 \) different critical values. Let us define

\[
g(x) = \prod_{i=1}^{a+1}(x - s_i) \quad \text{where} \quad a > 1
\]

and meet the following conditions

1. \( s_i \)'s are positive real numbers and \( s_i \neq s_j \),
2. The function \( g \) has different critical values and also \( R(x) = s_i \) has \( n \) real roots. \( s_i \)'s are in an interval \( I \) such that \( g^{-1}(I) \) is a union of \( n \) intervals.
3. The function \( g \circ R(x) \) is a Morse function.

**Notation 4.1.** Let us denote by \( C \cup \tilde{C} \) the set of critical values of \( g \circ R \) where \( C \) is the set of the critical values of \( g \), and \( \tilde{C} \) is the image of the set of the critical points of \( R \) under \( g \circ R \). All the critical points of \( g \) and \( R \) are real. Therefore, \( C = \{c_i = g(p_i) | p_i \in V(g_x), p_1 < p_2 < \cdots < p_n \} \) when \( n \) is odd, and \( C = \{c_{a+1-i} = g(p_i) | p_i \in V(g_x), p_1 < p_2 < \cdots < p_n \} \) when \( n \) is even. Also the order of \( \tilde{C} \) is as usual \( \{\tilde{c}_{a+i} = g \circ R(q_j) | q_j \in V(R_x), q_1 < q_2 < \cdots < q_{n-1} \} \).

Take the distinguished system of paths related to the function \( g \circ R \) such that all the paths are in the upper half plane. Let \( \gamma_c \) be the vanishing cycle along the path \( \lambda_c \) of the fiber \( g^{-1}(0) \). Therefore, \( R^{-1}(\gamma_c) = \{\delta^i_c | i = 1, \ldots, n \} \) is the set of vanishing cycles along the path \( \lambda_c \) of the fiber \( (g \circ R)^{-1}(0) \).

**Theorem 4.3.** If \( b = 0 \), then the zero homology group of \((g \circ R)^{-1}(b)\) is generated by

\[
\delta^i_c, \delta^j_{\tilde{c}} \quad \text{where} \quad c \in C, \tilde{c} \in \tilde{C}, i = 1, \ldots, n.
\]
For the vanishing cycle roots of we have therefore,
without loss of generality we can assume that
Proof.\[ R \]
\[ R \]
The Dynkin diagram of a Morse polynomial with different critical values
Theorem 4.4. For the regular value point \( b=0 \) of \( g \circ R \), the intersection matrix of \( H_0((g \circ R)^{-1}(b), \mathbb{Z}) \) with respect to the basis in Theorem 4.3 is

\[
< \delta_i^j, \delta_l^j' > = \begin{cases} 
2 & \text{if } (i = i', j = j') \\
1 & \text{if } (j = j', i = i + 1, i < a) \\
-1 & \text{if } \forall (j' = 0, i = a, i' = a + 2k - 1, j = k + (\frac{1+(-1)^n}{2})) \\
\text{if } (j' = 0, i = 1, i' = a + 2k, j = 2k + (\frac{1+(-1)^n+1}{2})) \\
0 & \text{if } (j = j', |i' - i| > 1) \lor (j = j', i, i' > a, i \neq i') \\
\end{cases}
\]

by \( \delta_i^j \) and \( \delta_{a+k}^j \) we mean \( \delta_i^j \) and \( \delta_{a+k}^j \) respectively, and \( k \in \mathbb{N} \).

For the vanishing cycle \( \delta_{a+2k-1} := [r] - [r'] \) where \( r, r' \) are two consecutive roots of \( g \circ R \). Thus \( R(r) = R(r') = s_{a+1} \) is a root of the function \( g \). Let \( s_a, s_{a+1} \) be two consecutive roots of \( g \) and also let \( R^{-1}(s_a) = \{l_1 | l_1 < l_2 < \cdots < l_n \} \) such that \( l_k < r' < r < l_{k+1} \) be four consecutive roots of \( g \circ R \). Two vanishing cycles on that basis are \( \delta_{a}^{2k-1} = [r'] - [l_k] \) and \( \delta_{a}^{2k} = [r] - [l_{k+1}] \);

\[
< \delta_{a+2k-1}, \delta_{a}^{2k-1} >= - < \delta_{a+2k-1}, \delta_{a}^{2k} >= -1 < [r] - [r'], [r'] - [l_k] >= - < [r] - [r'], [r] - [l_{k+1}] >= -1.
\]

For the vanishing cycle \( \delta_{a+2k} := [r_2] - [r_1] \) where \( r_1, r_2 \) are two consecutive roots of \( g \circ R \) we have \( R(s_{a+2k}) = 0 \) and \( R(r_1) = R(r_2) = s_1 \) where \( s_1 \) is a root of \( g \). For the root \( s_2 \) of \( g \) let \( R^{-1}(s_2) = \{l_1 | l_1 < l_2 < \cdots < l_n \} \), so we have \( l_2 < r_1 < r_2 < l_{2k+1} \) are consecutive roots of \( g \circ R \). Therefore, \( \delta_{a}^{2k} = [r_1] - [l_2k] \) and \( \delta_{a}^{2k+1} = [l_{2k+1}] - [r_2] \) are vanishing cycles and they are in the basis so we have

\[
< \delta_{a+2k}, \delta_{a}^{2k} >= - < \delta_{a+2k}, \delta_{a}^{2k+1} >= 1 < [r_2] - [r_1], [l_{2k}] - [r_1] >= - < [r_2] - [r_1], [l_{2k+1}] - [r_2] >= 1.
\]

Also, the above procedure can work for an even number \( n \) but only by changing the order of \( C \). The function \( R \) induces the surjective morphism

\[
R_* : H_0((g \circ R)^{-1}(b), \mathbb{Z}) \rightarrow H_0(g^{-1}(b), \mathbb{Z}).
\]

If \( \gamma_{c_i}, \gamma_{c_{i+1}} \in H_0(g^{-1}(b), \mathbb{Z}) \) are two vanishing cycles where \( c_i, c_{i+1} \in C \), then each set \( R_*^{-1}(\gamma_{c_m}) = \{\delta_{c_m}^j\}, j = 1, 2, \ldots, n \) contains \( n \) separated vanishing cycles where \( m = i, i+1 \). For each cycle \( \delta_{c_i}^j \) there is exactly one cycle \( \delta_{c_{i+1}}^j \) such that \( < \delta_{c_i}^j, \delta_{c_{i+1}}^j >= -1 \). By definition of the functions \( R \) and \( g \) and also Theorem 4.2, we can have the other equalities.

\begin{definition}
The Dynkin diagram of a Morse polynomial with different critical values is a graph defined in the following way: Its vertices are in one-to-one correspondence with a distinguished basis of vanishing cycles \( \delta_i, i = 1, 2, \ldots, \mu = d-1 \). The \( i \)-th and \( j \)-th vertices of the graph are joined with an edge of multiplicity \( < \delta_i, \delta_j > \). The intersection indexes \((-1)^n\) are depicted by dash lines, where \( n \) is the dimension of vanishing cycles.
\end{definition}
For an illustration of the Dynkin diagram $g \circ R$ with respect to this basis (when $s$ is odd) see the figure 1.

**Lemma 4.2.** The action of monodromy group $\pi_1(C \setminus C \cup \tilde{C}, b)$ on a tangency vanishing cycle generates all

$$\delta_{\tilde{c}}, \tilde{c} \in \tilde{C} \text{ and } \delta_i^j - \delta_i^j \text{ where } c \in C, i, j = 1, \ldots, n.$$  

*Proof.* Each tangency vanishing cycle $\delta_{\tilde{c}}^{n+2i-1}$ (resp. $\delta_{\tilde{c}}^{n+2i}$) has an intersection with two vanishing cycles $\delta_{c_a}^{2i-1}$, $\delta_{c_a}^{2i}$ (resp. $\delta_{c_1}^{2i+1}$, $\delta_{c_1}^{2i+1}$) with different signs. By using Picard-Lefschetz formula, the action of monodromy $\lambda_{c_a}$ (resp. $\lambda_{c_1}$) on $\delta_{\tilde{c}}^{n+2i-1}$ (resp. $\delta_{\tilde{c}}^{n+2i}$) generates $\delta_{c_a}^{2i-1} - \delta_{c_a}^{2i}$ (resp. $\delta_{c_1}^{2i+1} - \delta_{c_1}^{2i+1}$). According to Lemma 4.1, the action of the monodromy group $\pi := \langle \lambda_{c} \mid c \in C \rangle \subset \pi_1((C \cup \tilde{C}), b)$ on $R_*(\delta_{\tilde{c}})$ (for all $c \in C$ and $j$) generates zero homology group $H_0((g^{-1}(b), \mathbb{Z})$. Therefore, for a fixed $j$, the action of $\pi$ on $\delta_{c_a}^j$ or $\delta_{c_1}^j$ can generate all $\delta_{\tilde{c}}^j$. In other words, the action $\pi$ on $\delta_{c_a}^{2i} - \delta_{c_a}^{2i-1}$ (resp. $\delta_{c_1}^{2i+1} - \delta_{c_1}^{2i}$) generates all $\delta_{\tilde{c}}^{2i} - \delta_{\tilde{c}}^{2i-1}$ (resp. $\delta_{\tilde{c}}^{2i+1} - \delta_{\tilde{c}}^{2i}$). Since Dynkin diagram is connected, the action of monodromy $\lambda_{c_a+2i}$ (resp. $\lambda_{c_1+2i}$) on $\delta_{c_1}^j$ can generate $\delta_{c_a+2i}$ (resp. $\delta_{c_1+2i+1}$). By repeating this procedure we can generate all $\delta_{\tilde{c}}^j$ because the degree of each vertex of the Dynkin diagram is at most 2. Since the number of tangency vanishing cycles is $n = \text{deg}(R)$ we can generate independent cycles $\delta_{\tilde{c}}^{j+1} - \delta_{\tilde{c}}^j$, where $i = 1, \ldots, n-1$. Therefore, these cycles can generate all $\delta_{\tilde{c}}^j - \delta_{\tilde{c}}^j$. \hfill $\square$

**Corollary 4.1.** The morphism $R_* : H_0((g \circ R)^{-1}(b), \mathbb{Z}) \to H_0(g^{-1}(b), \mathbb{Z})$ is surjective and

$\text{Ker}(R_*) = \langle \pi_1((C \setminus (C \cup \tilde{C})), b), \delta_{\tilde{c}} \rangle$, 

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where \(<\pi_1(C \setminus (C \cup \hat{C}), b), \hat{\delta}_t>\) is the group generated by the action of the monodromy group on the tangency vanishing cycle \(\hat{\delta}_t\).

**Proposition 4.1.** The group generated by the action of the monodromy group \(\pi_1(C \setminus (C \cup \hat{C}), b)\) on a vanishing cycle \(\hat{\delta}_c \in R^*_t(\gamma_c)\), where \(\gamma_c \in H_0(g^{-1}(b), \mathbb{Z})\), is equal to \(H_0((g \circ R)^{-1}(b), \mathbb{Z})\).

### 4.2 Direct Sum of Polynomials

Let \(F\) and \(f\) be the functions as in 2 and 4. We are going to study the topology of a regular fiber of \(f \circ F\).

**Notation 4.2.** We denote by \(C_1\) (resp. \(C_2\)) the set of critical values of \(g\) (resp. \(h\)), and also denote by \(\hat{C}_1\) (resp. \(\hat{C}_2\)) the set of the image of the critical points of \(R\) (resp. \(S\)) under \(g \circ R\) (resp. \(h \circ S\)). Thus \(C_1 \cup \hat{C}_1\) and \(C_2 \cup \hat{C}_2\) are the set of critical values of \(g \circ R\) and \(h \circ S\) respectively. Without loss of generality, we can suppose that \((C_1 \cup \hat{C}_1) \cap (C_2 \cup \hat{C}_2) = \emptyset\).

We take two systems of distinguished paths \(\lambda_c\) relative to the functions \(g \circ R\) and \(h \circ S\), where \(c \in (C_1 \cup \hat{C}_1) \cup (C_2 \cup \hat{C}_2)\) and \(\lambda_c\) starts from \(b = 0\) and ends at \(c\); see Figure (3). Note that for the function \(h \circ S\) we choose a distinguished system of paths such that all of the paths are in the lower half plane, and they preserve the order of \(C_2 \cup \hat{C}_2\) as in Notation 4.1. Let \(\delta \in H_0((g \circ R)^{-1}(0), \mathbb{Z})\) and \(\gamma \in H_0((h \circ S)^{-1}(0), \mathbb{Z})\) be two vanishing cycles along the paths \(\lambda_c\) and \(\lambda_a\) respectively. Let \(t_s : [0, 1] \rightarrow \mathbb{C}\) be a path defined by

\[
t_s := \begin{cases} 
\lambda_c(1 - 2s) & 0 \leq s \leq \frac{1}{2}, \ c \in (C_1 \cup \hat{C}_1) \\
\lambda_a(2s - 1) & \frac{1}{2} \leq s \leq 1, \ a \in (C_2 \cup \hat{C}_2) 
\end{cases}
\]

The cycle \(\delta\) vanishes along \(t_s^{-1}\) when \(s\) tends to zero and \(\gamma\) vanishes along \(t\) when \(s\) tends to 1.

![Figure 2: A distinguished system of paths where \(c_i \in C_1, a_i \in C_2\) and \(\hat{c}_i \in \hat{C}_1, \hat{a}_i \in \hat{C}_2\)](image)

**Definition 4.6.** The cycle

\[
\delta \ast \gamma := \delta \ast t_s \ast \gamma := \bigcup_{s \in [0,1]} \delta_{t_s} \times \gamma_{t_s} \in H_1((f \circ F)^{-1}(0), \mathbb{Z})
\]

is an oriented cycle. Note that its orientation changes when the direction of path \(t_s\) is changed. The triple \((t_s, \delta, \gamma) = (t_s, \delta_{t_s}, \gamma_{t_s})\) is called an admissible triple.

Let

\[
\delta^i_{c_i}, \delta^i_{\hat{c}_{a+k}} \in H_0((g \circ R)^{-1}(b), \mathbb{Z}) \quad \text{where} \quad i = 1, \ldots, a, \ j = 1, \ldots, n, \ k = 1, \ldots, n - 1,
\]

and

\[
\gamma^j_{c_i}, \gamma^j_{\hat{c}_{a+k}} \in H_0((h \circ S)^{-1}(b), \mathbb{Z}) \quad \text{where} \quad i = 1, \ldots, a, \ j = 1, \ldots, n, \ k = 1, \ldots, n - 1
\]

be the corresponding distinguished basis of vanishing cycles.
Theorem 4.5. The $\mathbb{Z}$-module $H_1((f \circ F)^{-1}(0), \mathbb{Z})$ is free and is generated by
\[ \alpha := \delta \ast \gamma \quad \text{where} \quad \delta \in H_0((g \circ R)^{-1}(b), \mathbb{Z}), \quad \gamma \in H_0((h \circ S)^{-1}(b), \mathbb{Z}), \]
and where we have taken the admissible triples
\[ (\lambda_c, \lambda_a^{-1}, \delta, \gamma) \text{ where } c \in C_1 \cup \tilde{C}_1 \text{ and } a \in C_2 \cup \tilde{C}_2. \]

See e.g. ([13] Chapter 7).

Take $h \circ S = b' - (h' \circ S')$, where $b'$ is a fixed complex number and $h' \circ S'$ is a perturbation of $h \circ S$. The set of critical values of $h' \circ S'$ is denoted by $(C'_2 \cup \tilde{C}'_2)$ and therefore the set of critical values of $h \circ S$ is $C_2 \cup \tilde{C}_2 = b' - (C'_2 \cup \tilde{C}_2)$. We define $(f \circ F)(x, y) := g \circ R(x) + h' \circ S'(y)$.

Assume that $(C_1 \cup \tilde{C}_1) \cap (C'_2 \cup \tilde{C}'_2) = \emptyset$, and since the set of critical values of $f \circ F$ is $(C_1 \cup \tilde{C}_1) + (C'_2 \cup \tilde{C}'_2)$ then $b'$ is a regular value of $f \circ F$. Let $(t_s, \delta, \gamma)$ be an admissible triple where $t_s$ starts from $c$ and ends at $b' - a'$ (here $c \in C_1 \cup \tilde{C}_1$ and $a' \in C'_2 \cup \tilde{C}'_2$). Therefore, the path $t_s + a'$ starts from $c + a'$ ands end at $b'$. For instance, see Figure 4.

Figure 4: A distinguished system of paths

Proposition 4.2. The topological cycle $\delta \ast \gamma$ is a vanishing cycle along the path $t_s + a'$ with respect to fibration $f \circ F = t$.

See e.g. ([13] Chapter 7).

Definition 4.7.

1. A vanishing cycle around the critical point $p$ where $p \in F^{-1}(\text{Sing}(f))$ is called a pull-back vanishing cycle.

2. A vanishing cycle around a tangency critical point is called a tangency vanishing cycle.
3. A vanishing cycle around a critical point \( p \) where \( p \in V(R_x) \cap V(S_x) \) is called an exceptional vanishing cycle.

For simplicity we denote by \( \delta_i^j \) the cycle \( \delta_i^j \) where \( c_i \in C_1, i = 1, \ldots, a \) and \( j = 1, \ldots, n \) (resp. by \( \gamma_i^j \) the cycle \( \gamma_i^j \) where \( a_i \in C_1, i = 1, \ldots, a \) and \( j = 1, \ldots, n \)). Also, we denote by \( \delta_k \) the cycle \( \delta_k \) where \( k = a + 1, \ldots, a + (n - 1) \) (resp. by \( \gamma_k \) the cycle \( \gamma_k \) where \( k = a + 1, \ldots, a + (n - 1) \)).

**Theorem 4.6.** Let \( b = 0 \) be the regular value of the function \( f \). Let \( \delta_i \) where \( i = 1, \ldots, a \) be the distinguished set of vanishing cycles in \( H_0(g^{-1}(b), \mathbb{Z}) \) and also, let \( \gamma_j \) where \( j = 1, \ldots, a \) be the distinguished set of vanishing cycles in \( H_0(h^{-1}(b), \mathbb{Z}) \). Therefore, the intersection matrix of \( H_1(f^{-1}(b), \mathbb{Z}) \) in the basis

\[
\delta_i \ast \gamma_j \quad \text{where} \quad i, j = 1, 2, \ldots, a,
\]

is of the form

\[
<\delta_i \ast \gamma_j, \delta_l \ast \gamma_k> = \begin{cases} 
(-1)^a & \text{if } \begin{cases} 
(i = l, k = j + 1, j = odd) \vee \\
(j = k, l = i + 1, i = even) \vee \\
(i = odd, j = even, l = i + 1 \text{ or } l = i - 1, k = j + 1) \vee \\
(i = l, k = j + 1, j = even) \vee \\
(j = k, l = i + 1, i = odd) \vee \\
(i = even, j = odd, l = i - 1 \text{ or } i + 1, k = l + 1)
\end{cases} \\
0 & \text{otherwise}.
\end{cases}
\]

See e.g. [13] Chapter 7, and [1].

The Dynkin diagram of \( f \), when \( a \) is even, is shown in Figure 5.

![Figure 5: Dynkin diagram](image)

In Figure 5, according to the distinguished set of paths, the paths such as \( t_{i,j} := \lambda_j \lambda_i^{-1} \) have transversal intersection with \( t_{i,j+1} = \lambda_{j+1} \lambda_i^{-1} \) (resp. \( t_{i+1,j} := \lambda_j \lambda_{i+1}^{-1} \)) at the point.
b = 0, and \(d(t_{ij}) ∧ d(t_{i,j+1}) = -d(t_{i,j+1}) ∧ d(t_{i,j+2})\) (resp. \(d(t_{ij}) ∧ d(t_{i+1,j}) = -d(t_{i+1,j}) ∧ d(t_{i+2,j})\)).

**Theorem 4.7.** For the regular value \(b = 0\) of \(f ∘ F = g(R) + h(S)\), we choose a distinguished set of vanishing cycles \(δ_i^j, δ_k\) where \(i = 1,2,\ldots,a,j = 1,\ldots,n\) and \(k = a+1,\ldots,a+(n-1)\) (resp. \(γ_i^j, γ_k\) where \(i = 1,2,\ldots,a,j = 1,\ldots,n\) and \(k = a+1,\ldots,a+(n-1)\)) in \(H_0((g ∘ R)^{-1}(b), \mathbb{Z})\), (resp. \(H_0((h ∘ S)^{-1}(b), \mathbb{Z})\)). The intersection matrix in this basis

\[
δ_i^j * γ_{i'}^j, δ_k * γ_i^j, δ_i^j * γ_k, δ_k * γ_k'
\]

where

\[ \begin{align*}
i, i' &= 1, \ldots, a, j, j' &= 1, \ldots, n, k, k' &= a+1, \ldots, a+(n-1),
\end{align*}\]

of \(H_1((f ∘ F)^{-1}(0), \mathbb{Z})\) is given by

\[
<δ_i^m * γ_j^s, δ_l^m * γ_k^s'> = <δ_i^1 * γ_j^1, δ_l^1 * γ_k^1>, \quad \text{for} \quad m = m', s = s', 1 \leq i, j, k, l \leq a,
\]

where

\[
<δ_i^1 * γ_j^1, δ_l^1 * γ_k^1> = (-1)^{a+1} <R(δ_i^1) * S(γ_j^1), R(δ_l^1) * S(γ_k^1)>.
\]

Here, the intersection can be explained by using Proposition 4.6 and

\[
\left\{ \begin{align*}
<δ_a^{0} * γ_j^1, δ_{a+i}^{0} * γ_k^1> &= <δ_a^{s} * γ_j^s, δ_{a+i}^{s} * γ_k^s> \\
<δ_i^m * γ_{a+j}^1, δ_i^m * γ_{a+j}^0> &= <δ_i^m * γ_a^m, δ_i^m * γ_a^m>,
\end{align*}\right.
\]

for the others we can use

\[
<δ_i^m * γ_a^s, δ_i^m * γ_{a+s}^0> =<δ_i^m * γ_{a+s}^0, δ_i^m * γ_a^{s+1}>
\]

\[
\left\{ \begin{align*}
-1 & \quad \text{if} \quad \begin{cases} (n = 2n' + 1 , s = 2t + 1) \lor \\ (n = 2n', a = 2a' + 1, s = 2t + 1) \end{cases} \\
1 & \quad \text{if} \quad \begin{cases} (n = 2n', a = 2a' + 1, s = 2t + 1), 
\end{cases}
\end{align*}\right.
\]

\[
<δ_i^m * γ_j^s, δ_{a+1}^m * γ_j^s> = <δ_a^{0} * γ_j^s, δ_{i+1}^m * γ_j^s>
\]

\[
\left\{ \begin{align*}
1 & \quad \text{if} \quad \begin{cases} (n = 2n' + 1 , m = 2t + 1) \lor \\ (n = 2n', a = 2a' + 1, m = 2t + 1) \end{cases} \\
-1 & \quad \text{if} \quad \begin{cases} (n = 2n', a = 2a' + 1, m = 2t + 1), 
\end{cases}
\end{align*}\right.
\]

See e.g. (13) Chapter 7.

Since the dimension of \((f ∘ F)^{-1}(b)\) is one, for the two vanishing cycles \(α, β \in H_1((f ∘ F)^{-1}(b), \mathbb{Z})\), we have \(<α, β> = -<β, α>\) and \(<α, α> = 0\), i.e. the intersection matrix is skew-symmetric.

The Dynkin diagram of \(f ∘ F\) when \(n\) is odd and \(a=3\) is shown in Figure 6.

In Figure 6, black vertices correspond to the tangency vanishing cycles, squires vertices correspond to the exceptional vanishing cycles and all the other vertices correspond to pull-back vanishing cycles. The white cycle vertices correspond to some of the pull-back vanishing cycles with the same image under \(F_n\). The direction of the intersections are to be considered from left to right and top to bottom in this figure.
Figure 6: Dynkin diagram of $f \circ F$, when $n$ is odd and $a=3$

**Definition 4.8.** An isomorphism of graphs $G$ and $H$ is a bijection between the vertex sets of $G$ and $H$

$$f : V(G) \rightarrow V(H),$$

such that any two vertices $u$ and $v$ of $G$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$.

Let us denote by $H$ (resp. $G$), the Dynkin diagram of $f \circ F$ (resp. $f$) with respect to the distinguished set of vanishing cycles related to the critical points of $f \circ F$ (resp. $f$). We consider the group generated by the action of the monodromy group $\pi := \langle \lambda_c | c \in C_1 + C_2 > \pi_1(C \setminus ((C_1 \cup C_2) + (\tilde{C}_1 \cup \tilde{C}_2)), b)$ on a pull-back vanishing cycle $\delta_c^i * \gamma_a^j$. We know that this group is generated by some pull-back vanishing cycles, so it introduces a sub-graph of $H$ which is denoted by $G_{ij}$.

For each $i, j = 1, \ldots, n$ the graph $G_{ij}$ is isomorphic to the graph $G$. Therefore, if we remove the vertices corresponding to the tangency and exceptional vanishing cycles, then $H$ is divided into $n^2$ graph $G_{ij}$.

**Definition 4.9.** The cycle $\delta$ in a regular fiber $f^{-1}(b)$ is called simple if the homology group $H_1(f^{-1}(b), \mathbb{Z})$ is generated by the action of monodromy group $\pi_1(C \setminus C, b)$ on $\delta$ (where $C$ is the set of critical values of $f$).

**Theorem 4.8.** Each vanishing cycle (respective to the distinguished set of paths related to the critical values) in a regular fiber of $f$ is simple.

See e.g. [14].

**Theorem 4.9.** For the regular value $b = 0$ of the Morse function $f \circ F = g \circ R + h \circ S$ (a composition of the functions which were defined in [2] and [4]), the action of the monodromy group on a tangency vanishing cycle generates

$$\delta_c^i * \gamma_a^j , \delta_c^i * \gamma_{\tilde{a}} , \delta_{\tilde{c}}^i * \gamma_a^j , \delta_{\tilde{c}}^i * \gamma_{\tilde{a}} , \delta_{\tilde{c}}^i * \gamma_{\tilde{a}}^j - \delta_{\tilde{c}}^{i'} * \gamma_{\tilde{a}}^{j'},$$

where $\tilde{c} \in \tilde{C}_1$, $c \in C_1$, $a \in C_2$, $\tilde{a} \in \tilde{C}_2$ and $i, j, i', j' = 1, 2, \ldots, n$. 

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Proof. 1. Let $L$ be a line of $D$. Each vanishing cycle around a critical point in $D$ is tangency or exceptional. Consider $f \circ F|_L - l$, when restriction of the functions $g \circ R$ or $h \circ S$ under $L$ are the constant value $l$. If $L = \{y = \text{cons}\}$ (resp. $L = \{x = \text{cons}\}$), then the cycles around the critical point in $L$ correspond to zero vanishing cycle $g \circ R$ (resp. $h \circ S$). Furthermore, a tangency vanishing cycle corresponds to a pull-back (zero) vanishing cycle, so by Proposition 4.1 the action of monodromy group on that cycle generates all the other vanishing cycles. This implies that, this action on our tangency vanishing cycle will generate the tangency and exceptional vanishing cycles around the critical points in $L$. The algebraic set $D$ consists of $n$ lines parallel with $x, y$ axes, so $V(R_x) \cap V(S_y) \neq \emptyset$. Since the critical point in two vertical lines of $D$ have different values except the point in the intersection, then the action of monodromy on a tangency vanishing cycle around a point of these lines can generate all the other vanishing cycles of these lines. In other words, this action on a tangency vanishing cycle can generate all the tangency and exceptional vanishing cycles.

2. If we remove the vertices of $H$ that correspond to the tangency and exceptional vanishing cycles, then the new graph contains $n^2$ sub-graphs $G_{i,j}$ where $1 \leq i, j \leq n$. The tangency vertex $\delta_i \ast \gamma_{a+j}^0$ (resp. $\delta_{a+i} \ast \gamma_i^0$) connects the sub-graphs $G_{i,j}$ to $G_{i,j+1}$ (resp. $G_{i,j}$ to $G_{i+1,j}$). In general $\delta_i \ast \gamma_{a+j}^0$ connects $\delta_i \ast \gamma_i^0$ to $\delta_i \ast \gamma_i^{j+1}$ when $j$ is odd and also, it connects $\delta_i \ast \gamma_i^0$ to $\delta_i \ast \gamma_i^{j+1}$ when $j$ is even. According to Picard-Lefschetz formula the action of monodromy $\lambda_{t,0}$ (resp. $\lambda_{t,1}$) around the critical point $t + 1$ (resp. $t + 2$) on the tangency vanishing cycle $\delta_i \ast \gamma_{a+j}^0$ is as follows:

$$
\lambda_{t,0}(\delta_i \ast \gamma_{a+j}^0) = \delta_i \ast \gamma_{a+j}^0 - \sum_{\alpha \in F_\ast^{-1}(\delta_i \ast \gamma_i^0)} < \alpha, \delta_i \ast \gamma_{a+j}^0 > \alpha
$$

$$
= \delta_i \ast \gamma_{a+j}^0 - (\delta_i \ast \gamma_i^0 - \delta_i \ast \gamma_{i+1}^0),
$$

when $j$ is even, the action of monodromy $\lambda_{t,1}$ on $\delta_i \ast \gamma_{a+j}^0$ generates $\delta_i \ast \gamma_{i+1}$.

According to Theorem 4.8 the action of the monodromy group $\pi$ on $\delta_i \ast \gamma_i^0$ generates corresponding vanishing cycles to the vertices in $G_{i,j}$. Therefore, the action of the monodromy group $\pi$ on the tangency vanishing cycle $\delta_i \ast \gamma_{a+j}^0$ generates the whole vanishing cycle $\delta_i \ast \gamma_k^0 - \delta_i \ast \gamma_{k+1}^0$ where $1 \leq l, k \leq a$. By the same process the action of the monodromy group $\pi$ on the tangency vanishing cycle $\delta_{a+i} \ast \gamma_i^0$ generates the whole vanishing cycle $\delta_i \ast \gamma_k^0 - \delta_i \ast \gamma_{i+1}^0 \ast \gamma_k^0$ where $1 \leq l, k \leq a$.

In general we can generate

$$
\nabla := \{\delta_i^{j+1} \ast \gamma_j^0 - \delta_i^j \ast \gamma_j^0, \delta_i^j \ast \gamma_{j+1}^0 - \delta_i^j \ast \gamma_j^0 | i, j = 1, \ldots, n - 1, c \in C_1, a \in C_2\}
$$

which is a basis for the group generated by

$$
\delta_i^c \ast \gamma_a^0 \ast \gamma_a^0 \ast \gamma_a^0 \forall c \in C_1, \forall a \in C_2, \ i, j, i', j' = 1, \ldots, n.
$$

\[ \square \]

Corollary 4.2. The group which is generated by the action of the monodromy group on a tangency vanishing cycle can also be generated by

$$
\delta_{\tilde{c}} \ast \gamma_a^0, \delta_{\tilde{c}} \ast \gamma_a^0, \delta_{\tilde{c}} \ast \gamma_a^0 \text{ and } \delta_{\tilde{c}}^i \ast \gamma_a^0 - \delta_{\tilde{c}}^i \ast \gamma_a^0 \cdot \gamma_a^0
$$

where $\tilde{c} \in \tilde{C}_1$, $c \in C_1$, $a \in C_2$, $\tilde{a} \in \tilde{C}_2$ and $i, j, i', j' = 1, 2, \ldots, n$. 

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**Theorem 4.10.** Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $(x, y) \to (R, S)$ as in [2] and let $f$ be a polynomial as in [4]. The linear map $F_* : H_1((f \circ F)^{-1}(b), \mathbb{Z}) \to H_1((f)^{-1}(b), \mathbb{Z})$ is surjective, and $\ker(F_*)$ is generated by the action of monodromy group $\pi_1(\mathbb{C} \setminus (C \cup \tilde{C}), b)$ on a tangency vanishing $\delta$.

**Proof.** It is clear that for each $c \in C$ the cycles $\delta^i_c - \delta^j_c$ belong to $\ker(F_*)$ where $\delta^i_c, \delta^j_c$ are the pull-back vanishing cycles around the singularities with value $c$. Each tangency vanishing cycle $\delta_t$ is divided into two paths with homotopic images under $F$ by $D$. Thus, $\delta_t \in \ker(F_*)$. Each exceptional vanishing cycle $\delta_e$ is divided into 4 paths by $D$. The images of those 4 paths under $F$ are homotopic, hence $F_*(\delta_e) = 0$. By using the Corollary 1.2 we conclude that $<\pi_1(\mathbb{C} \setminus (C \cup \tilde{C}), b)\delta_t >= \ker(F_*)$. It’s obvious that the morphism $F_*$ is surjective and so

$$null(F_*) = \#(V((g \circ R)_x) \cap V((h \circ S)_y)) - \#(V(g_{x}) \cap V(h_{y}))$$

(13)

If $F_*^{-1}(\gamma_c) = \{\delta^i_c | i = 1, \ldots, n^2\}$, where $\gamma_c \in H_1(f^{-1}(b), \mathbb{Z})$, and since all the elements in the set $\Delta_c =: \{\delta^i_{c+1} - \delta^i_{c} | i = 1, \ldots, n^2 - 1\}$ are independent, then its elements can generate all $\delta^i_t - \delta^j_t$ for all $i, j = 1, \ldots, n^2$. The number of all the elements of the set $\cup_{c \in C} \Delta_c$ is $a^2(n^2 - 1)$. The tangency and exceptional vanishing cycles, which are in $\ker(F_*)$, are independent elements and their number is $2na(n - 1) + (n - 1)^2$. Therefore, by considering the equality (13) we have $<\pi_1(\mathbb{C} \setminus (C \cup \tilde{C}), b)\delta_t >= \ker(F_*)$. □

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Yadollah Zare
Instituto Nacional de Matemática Pura e Aplicada-IMPA
Rio de Janeiro
Email: yadollah2806@gmail.com, yadollah@impa.br