Hamilton-Ivey estimates for gradient Ricci solitons

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Abstract

We first show that any 4-dimensional non-Ricci-flat steady gradient Ricci soliton singularity model must satisfy $|Rm| \leq cR$ for some positive constant $c$. Then, we apply the Hamilton-Ivey estimate to prove a quantitative lower bound of the curvature operator for 4-dimensional steady gradient solitons with linear scalar curvature decay and proper potential function. The technique is also used to establish a sufficient condition for a 3-dimensional expanding gradient Ricci soliton to have positive curvature. This sufficient condition is satisfied by a large class of conical expanders. As an application, we remove the positive curvature condition in a classification result by [Cho14] in dimension three and show that any 3-dimensional gradient Ricci expander $C^2$ asymptotic to $(C(S^2), dt^2 + \alpha t^2 g_{S^2})$ is rotationally symmetric, where $\alpha \in (0, 1]$ is a constant and $g_{S^2}$ is the standard metric on $S^2$ with constant curvature 1.

1 Introduction

A triple $(M^n, g, f)$ consisting of a connected smooth Riemannian manifold $(M^n, g)$ and a function $f \in C^\infty(M)$ is called a gradient Ricci soliton, if the following equation is satisfied for some constant $\kappa$:

$$\text{Ric} + \nabla^2 f = \frac{\kappa}{2} g. \tag{1.1}$$

The soliton is called shrinking if $\kappa > 0$, steady if $\kappa = 0$, and expanding if $\kappa < 0$. By scaling the metric with a constant, we may always assume that $\kappa \in \{-1, 0, 1\}$. A complete gradient Ricci soliton induces a self-similar solution to the Ricci flow, called the canonical form. More precisely, if $\Phi_t$ is a family of self-diffeomorphisms on $M$ and $g_t$ is a family of metrics given by

$$\frac{\partial}{\partial t} \Phi_t = \frac{1}{1 - \kappa t} \nabla f \circ \Phi_t, \quad \Phi_0 = \text{id}, \quad g_t = (1 - \kappa t) \Phi_t^* g, \tag{1.2}$$

then $g_t$ evolves by the Ricci flow with $g_0 = g$. When $\kappa > 0$, $\kappa = 0$, or $\kappa < 0$, the canonical form is ancient, eternal, or immortal, respectively. The Ricci soliton is an important field of study, since they arise naturally as rescaled limits of Ricci flows near singularities. The blow-up limit at a Type I finite-time singularity, or the backward scaled limit of a Type I ancient solution, is the canonical form of a Ricci shrinker (see [Per02, Nab10, EMT11, MZ21] and the references therein), whereas Type II and Type III scaled limits of Ricci flows are closely related to steady and expanding solitons, respectively (see [Ham93, Ham95, Cao97, CZ00, Lot07, GZ08]). Moreover, a
non-Ricci-flat steady soliton may appear as a rescaled limit at the spatial infinity of a 4-dimensional shrinking soliton singularity model with unbounded curvature [CFSZ20]. Hence, the investigation of geometric properties of Ricci solitons will shed much light on the singularity analysis of the Ricci flow.

The classification of 3-dimensional shrinking gradient Ricci solitons has been completed. Any complete gradient shrinker in dimension three is isometric to either $\mathbb{R}^3$, or a quotient of $\mathbb{S}^3$, or a quotient of $\mathbb{R} \times \mathbb{S}^2$ (see [Ive93, Ham95, Per03a, CCZ08, NW08, Che09, Nab10]). However, the general picture of 3-dimensional steady solitons is far from clear. One significant result is due to Brendle [Br13]. He proved a conjecture due to Perelman [Per02], namely, that any nonflat and noncollapsed 3-dimensional steady gradient Ricci soliton is the Bryant soliton up to scaling (see also [Br14, Br20]). Deng-Zhu [DZ19, DZ20b] generalized Brendle’s result and classified steady gradient solitons with at least linear curvature decay in dimension three (see also [MSW19] and references therein). One may wonder whether the Bryant soliton and $\mathbb{R} \times \Sigma_0$ are the only non-trivial 3-dimensional steady gradient solitons, where $\Sigma_0$ denotes Hamilton’s cigar soliton. Indeed, 3-dimensional steady solitons are more complicated than their shrinking counterparts. Very recently, Lai [Lai20] has resolved a conjecture due to Hamilton and constructed a family of 3-dimensional steady solitons that are flying wings. These examples are collapsed, positively curved, and on them the curvature does not decay uniformly to 0 at infinity (new noncollapsed examples of steady solitons in higher dimensions are also constructed in [Lai20]).

One important feature which facilitates the study of shrinking and steady solitons in dimension three is that these solitons are nonnegatively curved, which is a consequence due to the celebrated Hamilton-Ivey pinching estimate [Ive93, Ham95, Che09] (see also [CXZ13]). As a consequence of this estimate, any 3-dimensional complete ancient Ricci flow must have nonnegative sectional curvature. Since the canonical form of a shrinking or steady gradient Ricci soliton is an ancient Ricci flow, these two types of solitons are nonnegatively curved.

**Theorem 1.1.** [Ham95, Ive93, Che09] Any nonflat 3-dimensional complete shrinking or steady gradient Ricci soliton must have nonnegative sectional curvature. Consequently,

$$|R_m| \leq CR,$$

for some numerical constant $C > 0$, where $R$ is the scalar curvature.

In view of Theorem 1.1, two natural questions arise immediately. The first one is whether or not higher dimensional shrinking or steady solitons have nonnegative sectional curvature. Another one is under what condition does a 3-dimensional expanding soliton have nonnegative curvature. The answer to the first question is negative. Some counterexamples are the shrinkers constructed by Feldman-Ilmanen-Knopf [FIK03] on $O(-k)$, where $1 \leq k < n$, the steady solitons constructed by Cao [Cao94], and the steady solitons constructed by Appleton [Ap17], where the latter two are constructed on line bundles over the complex projective space. All of these examples do not have nonnegative sectional curvature on the entire manifold. Nevertheless, one may still propose related questions, such as, whether a 4-dimensional Ricci soliton has bounded curvature, or whether an estimate like (1.3) is true in dimension four. One of the best results for shrinking solitons in these directions is obtained by Munteanu-Wang [MW15]:

**Theorem 1.2.** [MW15] Let $(M^4, g, f)$ be a complete 4-dimensional shrinking gradient Ricci soliton with bounded scalar curvature. Then the curvature operator $R_m$ has bounded norm, and

$$|R_m| \leq cR \quad \text{on} \quad M,$$
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for some positive constant $c$.

The curvature estimate (1.4) also has additional interest to Munteanu-Wang’s program of classifying 4-dimensional noncompact shrinkers according to the behavior of scalar curvature at infinity \cite{KW15, MW15, MW17, MW19, CDS19}. Moreover, the estimate (1.4) can be interpreted as a weak notion of curvature nonnegativity, namely, that the sectional curvatures of different 2-planes are pinched in a way that the negative ones are outweighed by the positive ones, and that a scalar multiple of the average (i.e., the scalar curvature) bounds the entire tensor norm $\|Rm\|$. The estimate of $\|Rm\|$ in the potentially unbounded curvature setting is lately studied by Chow, Freedman, Shin, and the third-named author \cite{CFSZ20}, and Cao, Ribeiro, and Zhou \cite{CRZ20}.

In the 4-dimensional steady case, Cao-Cui \cite{CC20} showed that $\|Rm\| \leq cR^\alpha$, where $\alpha$ is any number in $(0, 1/2)$, provided that the scalar curvature $R$ converges to 0 at infinity and that $R \geq Cr^{-k}$ near infinity for some $k > 0$. Later, the first-named author \cite{Cha19} removed the condition $R \geq Cr^{-k}$ and proved (1.4) for non-trivial steady gradient soliton with $\lim_{x \to \infty} R = 0$ (see also \cite{Cha20, Che20, CZh21}). However, there are steady solitons on which $R$ does not go to 0 at infinity, for instance, $\mathbb{R} \times B_3$, $\mathbb{R}^2 \times \Sigma_0$ or $\Sigma_0 \times \Sigma_0$, where $\Sigma_0$ and $B_3$ denote Hamilton’s cigar soliton and the 3-dimensional Bryant soliton. It is interesting to see if (1.4) is still true without curvature decay assumption. Very recently, (1.4) has been established by Cao-Liu \cite{CL21} on 4-dimensional expanding solitons under the assumptions that $\text{Ric} \geq 0$ and that $R$ has at most polynomial decay. See also \cite{Der17, Cha20, Che20, Zhl21} for other estimates on the expanding solitons. Ancient Ricci flows and steady solitons satisfying the estimate (1.4) have been recently studied in \cite{MZ21, CMZ21a}.

We first investigate the curvature of 4-dimensional steady Ricci soliton with bounded Riemann curvature tensor.

**Theorem 1.3.** Let $(M^4, g, f)$ be a 4-dimensional complete and non-Ricci-flat steady gradient soliton with bounded Riemann curvature. Then there exists a positive constant $c$ such that

$$\|Rm\| \leq cR \quad \text{on} \quad M,$$

where $\text{Rm}$ and $R$ denote the Riemann curvature tensor and the scalar curvature, respectively.

The main difference between Theorem 1.3 and the previous results in \cite{CC20, Cha19} is that, under the assumption $\sup_M \|Rm\| < \infty$, the scalar curvature decaying or the Ricci curvature positivity condition is no longer required. Bounded curvature is also a natural assumption. Indeed, among different Ricci solitons, the ones arising as a scaled limit of a compact Ricci flow with finite-time singularity are of particular interest, since they reveal the asymptotic behavior of the flow near the singular time. In \cite{CFSZ20}, it is proved that a 4-dimensional shrinking soliton singularity model has at most quadratic curvature growth, and that a 4-dimensional steady soliton singularity model has bounded Riemann curvature (see \cite{CFSZ20} for the precise definition of Ricci soliton singularity models). Applying the result \cite{CFSZ20} and Theorem 1.3, we see that

**Corollary 1.4.** Let $(M^4, g, f)$ be a 4-dimensional complete non-Ricci-flat gradient steady Ricci soliton singularity model. Then there exists a positive constant $c$ such that

$$\|Rm\| \leq cR \quad \text{on} \quad M.$$

On the other hand, Ricci soliton singularity models have also been extensively studied in the literature; see \cite{Per02, Zhz07, CDM20, Bam20a, Bam20b, Bam20c, Bam21, BCDMZ21, BCMZ21}, to list but a few.
Munteanu-Wang [MW15] also generalized the Hamilton-Ivey estimate to dimension four and gave a quantitative lower bound for 4-dimensional shrinkers with bounded scalar curvature. Previous works on Hamilton-Ivey type estimates under vanishing Weyl tensor condition in higher dimensions included [ELM08, Zh09].

**Theorem 1.5.** [MW15] Suppose that $(M^4, g, f)$ is a 4-dimensional complete and noncompact shrinking gradient Ricci soliton with bounded scalar curvature. Then there exists a positive constant $C$ such that

$$Rm \geq -\left(\frac{C}{\ln(r+1)}\right)^{1/4} \text{ on } M,$$

where $r$ is the distance function from a fixed base point.

Next, we shall prove an analogous result for steady solitons with proper potential function and linear scalar curvature decay.

**Theorem 1.6.** Let $(M^4, g, f)$ be a 4-dimensional complete and non-Ricci-flat steady gradient Ricci soliton with proper potential function and linear scalar curvature decay, i.e., $\lim_{x \to \infty} f(x) = -\infty$ and $R \leq C/(r + 1)$ for some constant $C$. Then

$$-Cr^{-3/2} \leq Rm \leq Cr^{-1},$$

(1.5)

where $r$ is the distance function from a fixed base point.

The $\nu$-entropy of a Riemannian manifold was introduced by Perelman in [Per02]:

$$\nu = \nu(M, g) := \inf \left\{ \overline{W}(g, u, \tau) : \tau > 0, u \geq 0, \sqrt{u} \in C^\infty_0(M), \int_M u \, dg = 1 \right\},$$

where $\overline{W}$ is Perelman’s $W$-functional ([Per02]) defined as

$$\overline{W}(g, u, \tau) := \int_M \left( \tau \left( |\nabla \log u|^2 + R \right) - \log u \right) u \, dg - \frac{n}{2} \log(4\pi \tau) - n.$$

Under the non-collapsing condition $\nu > -\infty$, the diameter estimate of the level set of $f$ in [DZ19] [DZ20b] and the argument in [BCDMZ21] can be used to obtain the following result. The details of the proof are left to the readers.

**Theorem 1.7.** Let $(M^4, g, f)$ be a 4-dimensional complete and non-Ricci-flat steady gradient Ricci soliton with proper potential function and linear scalar curvature decay, i.e., $R \leq C/(r + 1)$ for some constant $C$. Suppose, in addition, that $\nu$ is bounded from below, i.e.,

$$\nu(M, g) > -\infty.$$  \hspace{1cm} (1.6)

Then the canonical form of $(M, g, f)$ has $(S^3/\Gamma) \times \mathbb{R}$ as the tangent flow at infinity for some finite group $\Gamma$. Thus, by [BCDMZ21], $(M, g)$ has positive curvature operator outside a compact set and $R \sim r^{-1}$ near infinity.

**Remark 1.8.** By [CMZ21b, Theorem 1.1] or [CMZ21c, Theorem 1.14], it can be seen that the same conclusion in Theorem 1.7 holds if the condition (1.6) is replaced by bounded Nash entropy on the canonical form of the steady soliton.
Under the Riemann curvature linear decay assumption, the condition $\text{Ric} \geq 0$ near infinity, and the $\kappa$-noncollapsing condition, Deng-Zhu proved a dimension reduction principle for steady solitons $[\text{DZ}20a, \text{DZ}19, \text{DZ}20b]$, namely, that the rescaled limits of the canonical form of the soliton at spatial infinity split like $(\mathbb{R} \times P^{n-1}, dr^2 + g_P(t))$, where $g_P(t)$ is a Type I noncollapsed ancient Ricci flow on the compact manifold $P$. Moreover, when $n = 4$ and $\text{Ric} > 0$, they $[\text{DZ}20a]$ showed that $(P, g_P(t))$ is the shrinking sphere and the steady soliton have positive sectional curvature outside compact subset (see also $[\text{CDM}20, \text{BCDMZ}21]$). Under the same assumptions as Theorem 1.6, the first-named author and Zhu $[\text{CZh}21]$ applied Deng-Zhu’s method to prove a dichotomy for the asymptotic geometry at the spatial infinity of a steady soliton without volume noncollapsing condition. As an application of Theorem 1.6, the first-named author and Zhu $[\text{CZh}21]$ applied Deng-Zhu’s method to prove a dichotomy for the asymptotic limits in dimension four. We refer the reader to $[\text{CZh}21]$ for the definition of being smoothly asymptotic to a cylinder at the exponential rate.

**Corollary 1.9.** Under the conditions in Theorem 1.6, either one of the following holds:

1. For any sequence $p_i \to \infty$ in $M^4$, after passing to a subsequence, $(M^4, d_{R(p_i)}g, p_i)$ converges in the pointed Gromov-Hausdorff sense to a cylinder $(\mathbb{R} \times Y, \sqrt{d_e^2 + d_Y^2}, p_\infty)$, where $d_e$ is the flat metric on $\mathbb{R}$, $(Y, d_Y)$ denotes a compact Alexandrov space with nonnegative curvature and Hausdorff dimension $\leq 3$, and $\sqrt{d_e^2 + d_Y^2}$ is the product metric.

2. For any sequence $p_i \to \infty$ in $M^4$, $(M^4, d_{R(p_i)}g, p_i)$ converges in the pointed Gromov-Hausdorff sense (without passing to subsequence) to the ray $([0, \infty), d_e, 0)$, where $d_e$ is the flat metric restricted on $[0, \infty)$. In this case, $(M, g)$ is smoothly asymptotic to the flat cylinder $\mathbb{R} \times (\mathbb{T}^3/\sim)$ at exponential rate.

**Remark 1.10.** It is unclear at this point whether or not the Alexandrov space $(Y, d_Y)$ in Corollary 1.9 has nonempty boundary. The uniqueness of $(Y, d_Y)$ is also unknown. If a 4-dimensional non-Ricci-flat steady gradient Ricci soliton is $\kappa$-noncollapsed, satisfies $\text{Ric} \geq 0$ outside a compact set, and satisfies $|\text{Rm}| \leq C/(r + 1)$, then Deng-Zhu $[\text{DZ}20a]$ proved that the level sets near infinity are diffeomorphic to a quotient of $S^3$. Under the assumptions in Corollary 1.9, it follows from Theorem 1.6 $[\text{CZh}21]$ Corollary 1.6 and $[\text{CZh}21]$ Corollary 1.6 that the level sets $\Sigma$ of $f$ have almost nonnegative sectional curvature near infinity. Thus, by $[\text{FY}92, \text{Per}02, \text{Per}03a, \text{Per}03b]$, the level set $\Sigma$ of $f$ is either homeomorphic to a quotient of $\mathbb{T}^3$, $S^3$, $S^1 \times S^2$, or a nilmanifold (see $[\text{DZ}20a]$ for result in higher dimensions under the noncollapsing condition).

As mentioned above, motivated by Theorem 1.1 one may wonder if a 3-dimensional expanding gradient Ricci soliton must be nonnegatively curved. Bryant constructed a family of 3-dimensional negatively curved complete rotationally symmetric gradient expanders on $\mathbb{R}^3$ (see $[\text{CGGHKLLN}07]$). However, we will show that the sectional curvature of a 3-dimensional expanding gradient soliton is positive if the scalar curvature satisfies certain decay conditions outside a compact set. Let $h$ be any function on a noncompact manifold $M$ and $\alpha \in \mathbb{R}$ be a positive constant. We say that $h = o(r^{-\alpha})$ if $\lim_{x \to \infty} r^\alpha h = 0$, $h = O(r^{-\alpha})$ if $\limsup_{x \to \infty} r^\alpha |h| < \infty$, where $r$ is the distance function based at a fixed point.

**Theorem 1.11.** Let $(M^3, g, f)$ be a 3-dimensional complete and noncompact expanding gradient Ricci soliton. Assume that there exist nonnegative functions $h_1 = o(r^{-2})$ and $h_2 = o(r^{-1})$ near infinity such that

$$-h_1 \leq R \leq h_2$$

(1.7)

outside a compact set of $M$. Then $M$ has positive sectional curvature everywhere unless it is flat.
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Remark 1.12. In view of the Bryant expanding solitons with negative curvature \cite{CCGGIIKLN07}, we see that the lower bound $h_1 = o(r^{-2})$ in (1.7) is sharp and cannot be relaxed. Theorem 1.11 also fails in higher dimensions, since the expanding K"ahler Ricci soliton constructed by Feldman-Ilmanen-Knopf \cite{FIK03} with complex dimension $m \geq 2$ has mixed Ricci curvature signs and positive scalar curvature which decays exponentially in $r$.

It can also be seen from the proof of Theorem 1.11 that any 3-dimensional complete and noncompact expanding gradient Ricci soliton with nonnegative sectional curvature outside a compact set has nonnegative sectional curvature everywhere (see Remark 2.3). The same holds if one replaces the sectional curvature by the Ricci curvature (see Remark 2.3).

A special phenomenon arising from 3-dimensional Ricci expanders is that there exist asymptotically conical examples; the same is not true for 3-dimensional steadies and shrinkers, though higher dimensional asymptotically conical examples can be found in \cite{FIK03} \cite{AK19}. With the help of this extra structure at infinity, it is slightly more tractable to classify 3-dimensional asymptotically conical expanders according to their asymptotic cones. We first recall the definition of asymptotically conical expanding soliton. Let $X$ be a smooth $(n-1)$-dimensional closed manifold with Riemannian metric $g_X$. We shall denote by $C(X)$ the cone over $X$, i.e. $\{(t, \omega) : t > 0, \omega \in X\}$. Let $g_C$ and $\nabla_C$ be the metric $dt^2 + t^2 g_X$ on $C(X)$ and its Levi-Civita connection, respectively. For any positive constant $S_0$, $B(0, S_0) \subseteq C(X)$ is the subset given by $\{(t, \omega) : S_0 \geq t > 0, \omega \in X\}$. $X$ is also called the link of the cone $C(X)$.

Definition 1.13. (Cho14 Definition 1.1) Let $k \in \mathbb{N} \cup \{\infty\}$. A complete noncompact expanding gradient Ricci soliton $(M, g, f)$ is $C^k$-asymptotic to the cone $(C(X), g_C)$, if there exist constants $\varepsilon > 0$, $S_0 > 0$, and $c_0$, a compact set $K \subset M$, and a diffeomorphism $\phi : C(X) \setminus B(0, S_0) \longrightarrow M \setminus K$, such that for any $l = 0, 1, 2, \cdots, k$, there is a constant $C_l > 0$ such that for all $t > S_0$

$$\sup_{t \in \omega \in X} |\nabla_C^l (\phi^* g - g_C)|_{g_C} (t, \omega) \leq C_l t^{-\varepsilon l - 1}, \tag{1.8}$$

$$f \circ \phi (t, \omega) = -\frac{t^2}{4} + c_0 \quad \text{and} \quad \frac{2}{t} \frac{\partial \phi}{\partial t} = -\frac{\nabla f}{|\nabla f|^2}. \tag{1.9}$$

Remark 1.14. Note that the notion of conical expander in Definition 1.13 is slightly more restrictive than the one in Cho14 Definition 1.1, as here $|\nabla_C^l (\phi^* g - g_C)|_{g_C}$ is also assumed to be bounded for $t$ close to $S_0$ in (1.8). Hence the classification result in Cho14 also holds for conical expanding solitons satisfying Definition 1.13. Moreover, Definition 1.13 and the corresponding definition in Cho14 are equivalent if $f$ is proper, i.e., $\lim_{x \to \infty} f(x) = -\infty$, which holds in particular when $\text{Ric} \geq (\varepsilon_0 - 1/2) g$ outside a compact set for some positive constant $\varepsilon_0$ (see (2.6)). For general gradient expanding solitons, $f$ may not be proper and $M$ can have two ends (see \cite{R13} \cite{BM15}).

Cheeger-Colding (see \cite{CC96} and the references therein) showed that the tangent cones at infinity of a complete noncompact manifold with $\text{Ric} \geq 0$ and maximal volume growth are metric cones. In the case of expanding solitons, some sufficient conditions for the existence of an asymptotic metric cone were given in CD15 \cite{Der17} \cite{CL21}: if on an expander it holds that $r^2 |Rm|$ is bounded, then one can always find an asymptotic metric cone at infinity in the Gromov-Hausdorff sense (c.f. CD15 \cite{CL21}). Moreover, the asymptotic convergence can be made to be $C^\infty$ (as in Definition 1.13), if $r^{2k+4} |\nabla^k \text{Ric}|$ is also bounded for all integer $k \geq 0$ \cite{Der17}.

Given a cone $C(X)$, it has long been an interesting problem to find expanding solitons asymptotic to (smoothing out) $C(X)$ and to establish the uniqueness of these expanders under
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different conditions \([SS13, Si13, Der16, CDS19, CoD20]\). By constructing Killing vector fields as in \([Br13]\), Chodosh \([Cho14]\) proved the following uniqueness result for asymptotically conical expanding soliton (see also \([CCCMM14]\) for classification result under the Bach flat condition).

**Theorem 1.15.** \([Cho14]\) Let \(n \geq 3\) be an integer. If \((\mathcal{M}^n, g, f)\) is an \(n\)-dimensional complete expanding gradient Ricci soliton which has positive sectional curvature and is \(C^2\)-asymptotic to the cone \(C(S^{n-1}, dt^2 + \alpha t^2 g_{S^{n-1}})\), where \(\alpha \in (0, 1]\) is a constant, and \(g_{S^{n-1}}\) is the standard metric on \(S^{n-1}\) with constant curvature \(1\). Then \(\mathcal{M}\) is rotationally symmetric.

The uniqueness of \(C^2\)-asymptotically conical expanding Kähler soliton with positive bisectional curvature was also shown by Chodosh-Fong \([CF16]\). The proofs in \([Br14]\), as well as the ones in \([Cho14, CF16]\), require the information on the sectional (or bisectional) curvature for a maximum principle argument showing a Liouville-type theorem for a Lichnerowicz PDE. The existence and uniqueness of expander with prescribed asymptotic cone has been studied by Deruelle \([Der16]\) in a more general setting. Namely if \((X^{n-1}, g_X)\) is a simply connected closed manifold with

\[
\text{Rm}(g_X) > \text{id}_{\Lambda^2 TX},
\]

then there exists a positively curved gradient Ricci expander which is \(C^\infty\)-asymptotic to \(C(X)\) \([Der16]\). Moreover, two gradient expanders that satisfy \(Rm > 0\) and are \(C^\infty\)-asymptotic to \(C(X)\) must be isometric to each other \([Der16]\). It is natural to ask whether the uniqueness result holds within larger category of expanding solitons (see Remark 1.17 below). It follows from (1.8) that the decay condition (1.7) is fulfilled by those conical gradient expanders with link \(X\) satisfying (1.10) (or more generally \(Rm(g_X) \geq \text{id}_{\Lambda^2 TX}\)). Consequently, we have the following corollary.

**Corollary 1.16.** Suppose that \((\mathcal{M}^3, g, f)\) is a 3-dimensional complete non-compact expanding gradient Ricci soliton which is \(C^2\)-asymptotic to a cone \(C(X^2, dt^2 + t^2 g_X)\), where \((X^2, g_X)\) is a connected and closed 2-dimensional Riemannian manifold. Then the following are equivalent:

1. \(\mathcal{M}\) has nonnegative sectional curvature;
2. \(\mathcal{M}\) has nonnegative scalar curvature;
3. \(C(X)\) has nonnegative scalar curvature;
4. \(\text{Rm}(g_X) \geq \text{id}_{\Lambda^2 TX}\).

**Remark 1.17.** If instead we assume that \(\mathcal{M}^3\) in Corollary 1.16 is \(C^\infty\)-asymptotic to \(C(X)\) with \(\text{Rm}(g_X) > \text{id}_{\Lambda^2 TX}\) and that \(X\) is simply connected, then by Theorem 1.11 and the uniqueness result in \([Der16]\), \(\mathcal{M}^3\) is isometric to the positively curved conical expander (with asymptotic cone \(C(X)\)) constructed by Deruelle in \([Der16, Theorem 1.3]\). By the Morse theory, the level sets of \(f\) near infinity of a Ricci nonnegative expanding gradient soliton are diffeomorphic to \(S^{n-1}\), which is simply connected for \(n \geq 3\). Hence by Corollary 1.16 for any \(\alpha \in (0, 1]\), we have that \((C(RP^2), dt^2 + \alpha g_{RP^2})\) is not a \(C^2\)-asymptotic cone of any expander in the sense of Definition 1.13 where \(g_{RP^2}\) is the standard metric on \(RP^2\).

As another application of Theorem 1.11 we remove the positive curvature condition in Theorem 1.15 in dimension three.
Corollary 1.19. Suppose that \((M^3, g, f)\) is a 3-dimensional complete and noncompact expanding gradient Ricci soliton which is \(C^2\)-asymptotic to the cone \((C(S^2), dt^2 + \alpha t^2 g_{S^2})\), where \(\alpha \in (0, 1)\) is a constant, and \(g_{S^2}\) is the standard metric on \(S^2\). Then \(M\) is rotationally symmetric.

Remark 1.20. As pointed out by Chen-Deruelle [CD15, Remark 1.5] and Deruelle [Der16, Definition 1.1], the canonical form of a conical expander (with asymptotic cone \(C(X)\)) is a Ricci flow with singular initial data (in the Gromov-Hausdorff sense) given by \(C(X)\). Hence Corollaries 1.16 and 1.19 can be respectively interpreted as the preservation of nonnegative curvature and the uniqueness of 3-dimensional Ricci flow coming out of the cone \(C(X)\), when the flow is the canonical form of an expander. Singular Ricci flow on closed 3-manifold was introduced by Kleiner-Lott [KL17, KL18] and its uniqueness and stability in dimension three were established by Bamler-Kleiner [BK17].

Remark 1.21. The definition of conical expander in [Cho14, Definition 1.1] (see also Definition 1.13) is slightly different from the ones in [Der16, Der17]. More precisely, the condition in (1.9)
\[
\frac{2}{t} \frac{\partial \phi}{\partial t} = - \frac{\nabla f}{|\nabla f|^2}
\]
is required in [Cho14] but not in [Der16, Der17]. Indeed, Condition (1.11) is only used when we apply the result [Cho14] in Corollary 1.19 and hence Corollary 1.16 as well as Remark 1.17 also holds on those expanders satisfying all conditions in Definition 1.13 except for possibly (1.11).

We shall sketch the idea of proof very briefly. Let \(T_{ij} := R_{gij} - 2R_{ij}\). The tensor \(T\) was used extensively to study the curvature pinching in 3-dimensional Ricci flow (e.g., [Ham82, Ham95, CCGGIKLLN08, Cho09, CXZ13]). In general, sectional curvature being nonnegative implies \(T \geq 0\). When \(n = 3\), we have \(\text{Rm} \geq 0\) at \(p \in M\) if and only if \(T \geq 0\) at \(p \in M\) [Ham82, Corollary 8.2]. The key ingredient of the proof is to look at the differential equation satisfied by \(T\) and estimate the smallest eigenvalue of \(T\) via the maximum principle argument. When \(n = 3\), let \(\lambda_1 \leq \lambda_2 \leq \lambda_3\) be the eigenvalues of \(\text{Ric}\) and \(\nu_1 \leq \nu_2 \leq \nu_3\) be the eigenvalues of \(T\). In the case of expanding soliton, \(\nu_1\) satisfied the following nice inequality in the barrier sense (see (2.16))
\[
\Delta f \nu_1 \leq -\nu_1 - \nu_2^2 - \nu_3 \nu_3.
\]
The decay condition (1.7) then allows us to apply the asymptotic estimate in [Der17] and the maximum principle argument in [MW15] to conclude that \(\nu_1\) is nonnegative in the 3-dimensional expanding soliton case. For a 4-dimensional steady soliton, we invoke the dimension reduction trick via the level set of \(f\) by [MW15] and bound the smallest eigenvalue from below of a tensor \(U\) approximating \(T\) (see (3.24)). Theorem 1.6 follows from a similar argument as in the 3-dimensional case together with some delicate estimates of the error terms caused by the extra dimension.

This paper is organized as follows. We start with the 3-dimensional expanding soliton case to show Theorem 1.11 Corollary 1.16 and Corollary 1.19 in Section 2. Next, we move on to the 4-dimensional steady case in Section 3 and present the proofs of Theorem 1.6 and Corollary 1.9.

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2 Positive curvature in 3-dimensional expanding soliton case

In this section, we shall consider a 3-dimensional expanding gradient Ricci soliton \((M^3, g, f)\) satisfying (1.1) with \(\kappa = -1\). Recall that we have defined \(T_{ij} := R_{ij} - 2R g_{ij}\). As in the introduction, we denote the eigenvalues of Ric by \(\lambda_1 \leq \lambda_2 \leq \lambda_3\) and the eigenvalues of \(T\) by \(\nu_1 \leq \nu_2 \leq \nu_3\). Then it can be seen from the definition of \(T\) that

\[
\begin{align*}
\nu_1 &= \lambda_1 + \lambda_2 - \lambda_3 = R - 2\lambda_3 \\
\nu_2 &= \lambda_1 + \lambda_3 - \lambda_2 = R - 2\lambda_2 \\
\nu_3 &= \lambda_2 + \lambda_3 - \lambda_1 = R - 2\lambda_1.
\end{align*}
\]

(2.1)

To prove that \(M\) has nonnegative curvature, it suffices to show that \(\nu_1 \geq 0\) everywhere. We shall use the following well-known formula for \(R_{m}\) in dimension three (c.f. \([\text{CLN} 06, (1.62)]\)):

\[
R_{ijkl} = R_{il} g_{jk} + R_{jk} g_{il} - R_{ik} g_{jl} - R_{jl} g_{ik} - \frac{R}{2} (g_{il} g_{jk} - g_{ik} g_{jl}).
\]

(2.2)

It is a result by Hamilton \([\text{Ham}95]\) that \(\nabla (|\nabla f|^2 + R + f) = 0\), by adding a constant to \(f\), we have

\[
|\nabla f|^2 + R = -f
\]

(2.3)

On the other hand, it was proved by Pigola–Rimoldi–Setti \([\text{PRST}11]\) and S.-J. Zhang \([\text{Zhs}11]\) that the scalar curvature of any complete expanding gradient Ricci soliton satisfies

\[
R \geq -\frac{n}{2}.
\]

(2.4)

Throughout this article, for any smooth function \(\gamma\), we define the weighted laplacian \(\Delta_\gamma\) as \(\Delta_\gamma = \Delta - \nabla_\gamma \nabla_\gamma\), where \(\Delta\) is the standard laplacian operator. Let \(v = \frac{n}{2} - f\). By taking the trace of the soliton equation \((1.1), (2.4),\) and \((2.3)\), we see that \(\Delta v = R + \frac{n}{2}\), and

\[
\begin{align*}
v &\geq |\nabla v|^2; \\
\Delta f v &= v.
\end{align*}
\]

(2.5)

Integrating \((2.5)\), we have that on any complete gradient Ricci expander of any dimension, it holds that \(v \leq r^2/4 + Cr + C\) for some constant \(C > 0\) (see \([\text{CCGHIKLN07, Corollary 27.10}]\)). If, in addition, \(\liminf_{x \to \infty} \text{Ric} > -1/2\), then we have that \(\text{Ric} \geq (-1/2 + \varepsilon_0)g\) near infinity for some constant \(\varepsilon_0 > 0\), and hence \(\nabla^2 v \geq \varepsilon_0 g\) outside a compact set. By integrating the inequality along minimizing geodesics, we have

\[
v \geq \varepsilon_0 r^2/2 - C_1 r - C_1
\]

(2.6)

for some positive constant \(C_1\), and consequently \(v \sim r^2\).

The scalar curvature \(R\) of an expanding gradient Ricci soliton satisfies \([\text{CCGHIKLN07}]\)

\[
\Delta f R = -R - 2|\text{Ric}|^2 \leq -R - 2R^2/n = -(1 + 2R/n)R.
\]

(2.7)

By the strong maximum principle \([\text{GT01}]\), if \(M\) has nonnegative scalar curvature, then \(R > 0\) everywhere unless \(R \equiv 0\), and in the latter case, by \((2.7)\) and \((1.4)\), \(\text{Ric} \equiv 0\) and \(M\) is flat \([\text{PRS11, Theorem 1}]\). This also follows from the parabolic minimum principle applied to the scalar curvature of the canonical form \([1.2]\).
Lemma 2.1. If $(M^3, g, f)$ is a 3-dimensional complete noncompact expanding gradient Ricci soliton with the following scalar curvature decay,

$$|R| \leq o(r^{-1}).$$

Then the following estimate is satisfied:

$$|\nabla R_m| + |\nabla^2 R_m| \leq o(r^{-1}).$$

Proof. In view of (2.2), to prove the estimate on $R_m$, we only need to bound the Ricci tensor. By [Cha20, Theorem 7] and by applying Shi’s estimate [CLN06] to the canonical form of the soliton, it holds that $|R_m|$ is bounded and that there exists a finite positive constant $C_k$ for any $k \in \mathbb{N}$, such that

$$|\nabla^k R_m| \leq C_k \quad \text{on} \quad M.$$  \hfill (2.8)

It can be seen from (2.2) that in an orthonormal frame $\{e_1, e_2, n := \nabla f / |\nabla f|\}$, we have

$$R_{ij} = R_{nijn} + \frac{R}{2} (g_{ij} - g_{in}g_{jn}) - R_{nn}g_{ij} + R_{in}g_{jn} + R_{jn}g_{in},$$  \hfill (2.9)

where $i, j = 1, 2$, and we are not summing over $n$. Moreover, from the Ricci identity and the soliton equation we have

$$R_{ij,k} - R_{ik,j} = R_{kjil} f_l.$$  \hfill (2.10)

Thanks to [Cha20, Corollary 3] and (2.3), we have

$$- f \sim r^2$$  \hfill (2.11)

and there is a positive constant $C$ such that

$$C^{-1} r^2 \leq -f - R = |\nabla f|^2 \leq Cr^2$$  \hfill (2.12)

outside a compact set of $M$. By (2.8), (2.10), and $2 \text{Ric}(\nabla f) = \nabla R$, we see that

$$R_{nijn} = O \left( \frac{|\nabla \text{Ric}|}{|\nabla f|} \right) = O(r^{-1}) \quad \text{and} \quad R_{in} = O \left( \frac{|\nabla R|}{|\nabla f|} \right) = O(r^{-1}).$$

By virtue of (2.10), $|\text{Ric}|$ and hence $|R_m|$ are of $O(r^{-1})$. By the local Shi’s estimate [Der17, Lemma 2.6], there exists a positive constant $C$ such that for all $p \in M$ and $s \geq 1$

$$|\nabla R_m|(p) \leq C \sup_{B_s(p)} |R_m| \left[ 1 + \sup_{B_s(p)} |R_m| + \frac{\sup_{B_s(p) \setminus B_{s/2}(p)} |\nabla f|}{s} \right]^\frac{1}{2}.$$  \hfill (2.13)

Hence $|\nabla R_m| = O(r^{-1})$, and both $R_{nijn}$ and $R_{in}$ are of $O(r^{-2})$. $|R_m| = o(r^{-1})$ then follows from (2.2) and (2.9). The estimate on $\nabla R_m$ is now a consequence of the bound on $|R_m|$ and (2.13).

\hfill $\Box$

Lemma 2.2. Let $(M^3, g, f)$ be a 3-dimensional complete noncompact expanding gradient Ricci soliton with positive scalar curvature. Assume that the tensor $T$ satisfies either one of the following conditions:
1. $T/R$ is asymptotically nonnegative, i.e.,
\[
\liminf_{x \to \infty} \frac{T}{R} \geq 0;
\]
2. $\nu_1/R$ attains its minimum, where $\nu_1$ is the smallest eigenvalue of $T$ as in (2.1).

Then $M$ has nonnegative sectional curvature.

**Remark 2.3.** The above lemma implies that any 3-dimensional complete noncompact expanding gradient Ricci soliton with nonnegative sectional curvature outside a compact set has nonnegative sectional curvature everywhere.

**Proof.** By the differential equation of Ric [PW10, Lemma 2.1] and (2.2), we may compute in an orthonormal frame
\[
\Delta f R_{il} = -R_{il} - 2R_{ijkl} R_{jk} = -R_{il} - 2RR_{il} - 2|Ric|^2 g_{il} + 4R_{ik} R_{kl} + R^2 g_{il} - RR_{il}
\]
(2.14)
Hence, in the barrier sense, we have
\[
\Delta f 2 \lambda_3 \geq -2 \lambda_3 - 6R \lambda_3 + 8 \lambda_3^2 + 2R^2 - 4|Ric|^2.
\]
By virtue of the formula $\Delta f R = -R - 2|Ric|^2$, it holds that
\[
\Delta f \nu_1 = \Delta f (R - 2 \lambda_3) \leq -\nu_1 + 6R \lambda_3 - 2R^2 - 8 \lambda_3^2 + 2|Ric|^2.
\]
(2.15)
Using (2.1), we may express the R.H.S. of (2.15) in terms of the eigenvalues of $T$
\[
6R \lambda_3 - 2R^2 - 8 \lambda_3^2 + 2|Ric|^2 = 3(\nu_1 + \nu_2 + \nu_3)(\nu_2 + \nu_3) - 2(\nu_1 + \nu_2 + \nu_3)^2 - 2(\nu_2 + \nu_3)^2 + \frac{1}{2} [(\nu_2 + \nu_3)^2 + (\nu_1 + \nu_3)^2 + (\nu_1 + \nu_2)^2]
\]
\[
= (\nu_2 + \nu_3)^2 + 3(\nu_1 \nu_3 + \nu_1 \nu_2) - 4(\nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3)
\]
\[
-2(\nu_1^2 + \nu_2^2 + \nu_3^2) + [\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3]
\]
\[
= -\nu_1^2 - \nu_2 \nu_3.
\]
Hence (2.15) can be rewritten as
\[
\Delta f \nu_1 \leq -\nu_1 - \nu_1^2 - \nu_2 \nu_3.
\]
(2.16)
On the other hand, we have
\[
\Delta f R^{-1} = -R^{-2} \Delta f R + 2R^{-3} |\nabla R|^2
\]
\[
= -R^{-2}(-R - 2|Ric|^2) + 2R^{-3} |\nabla R|^2
\]
\[
= R^{-1} + 2R^{-2} |Ric|^2 + 2R^{-3} |\nabla R|^2.
\]
(2.17)
\[ \Delta f (R^{-1} \nu_1) = R^{-1} \Delta f \nu_1 + \nu_1 \Delta f R^{-1} + 2 \langle \nabla R^{-1}, \nabla (R^{-1} \nu_1) \rangle \]
\[ \leq -R^{-1} \nu_1 - R^{-1} (\nu_1^2 + \nu_2 \nu_3) + R^{-1} \nu_1 + 2R^{-2} |\text{Ric}|^2 \nu_1 + 2R^{-3} |\nabla R|^2 \nu_1 \]
\[ -2R^{-1} \langle \nabla R, \nabla (R^{-1} \nu_1) \rangle - 2R^{-3} |\nabla R|^2 \nu_1 \]
\[ = -R^{-1} (\nu_1^2 + \nu_2 \nu_3) + 2R^{-2} |\text{Ric}|^2 \nu_1 - 2R^{-1} \langle \nabla R, \nabla (R^{-1} \nu_1) \rangle. \]

Hence we can rewrite the above equation as
\[ \Delta f - 2 \ln R (R^{-1} \nu_1) \leq -R^{-1} (\nu_1^2 + \nu_2 \nu_3) + 2R^{-2} |\text{Ric}|^2 \nu_1 \]
\[ = -R^{-2} (R(\nu_1^2 + \nu_2 \nu_3) - 2 |\text{Ric}|^2 \nu_1). \]

Straightforward computation yields
\[ R(\nu_1^2 + \nu_2 \nu_3) - 2 |\text{Ric}|^2 \nu_1 = (\nu_1 + \nu_2 + \nu_3)(\nu_1^2 + \nu_2 \nu_3) - \frac{\nu_1}{2} \left[ (\nu_1 + \nu_2)^2 + (\nu_1 + \nu_3)^2 + (\nu_2 + \nu_3)^2 \right] \]
\[ = \nu_1^3 + \nu_2^2 \nu_2 + \nu_1^2 \nu_3 + \nu_2 \nu_3^2 + \nu_1 \nu_2 \nu_3 \]
\[ - [\nu_1^2 \nu_2 + \nu_1 \nu_2^2 + \nu_1^2 \nu_3 + \nu_2 \nu_3^2 + \nu_1 \nu_2 \nu_3] \]
\[ = \nu_2^2 (\nu_3 - \nu_1) + \nu_2^2 (\nu_2 - \nu_1). \]

Hence
\[ \Delta f - 2 \ln R (R^{-1} \nu_1) \leq -R^{-2} \left[ \nu_2^2 (\nu_3 - \nu_1) + \nu_2^2 (\nu_2 - \nu_1) \right] \leq 0. \tag{2.19} \]

Note that so far the above computation works well for any 3-dimensional gradient Ricci expander with positive scalar curvature. In particular, \[ (2.16) \] and \[ (2.19) \] also hold on these solitons without additional assumption.

If \( \nu_1 \) is negative somewhere, then either one of our assumptions will imply that \( R^{-1} \nu_1 \) attains its negative minimum, say, at a point \( q \in M \) and \( R^{-1} (q) \nu_1(q) < 0 \). Then we have
\[ 0 \leq \Delta f - 2 \ln R (R^{-1} \nu_1) (q) \leq -R^{-2} (q) \left[ \nu_2^2 (\nu_3 - \nu_1) + \nu_2^2 (\nu_2 - \nu_1) \right] (q) \]
in the barrier sense. As \( \nu_1 \leq \nu_2 \leq \nu_3 \), we have \( \nu_2^2 (\nu_3 - \nu_1) + \nu_2^2 (\nu_2 - \nu_1) = 0 \) at \( q \). Since \( 3 \nu_3(q) \geq R(q) > 0 \), we see that \( \nu_1(q) = \nu_2(q) < 0 \). However,
\[ 0 < \nu_2^2(q) (\nu_3(q) - \nu_1(q)) \leq 0, \]
which is impossible. This shows that \( \nu_1 \geq 0 \) on \( M \) and completes the proof of the lemma. \( \Box \)

The same line of argument of Lemma 2.2 also gives the following proposition which shall not be used in the rest of the article.

**Proposition 2.4.** If \((M^3, g, f)\) is a 3-dimensional complete noncompact expanding gradient Ricci soliton with positive scalar curvature \( R > 0 \) and \( \text{Ric} \) satisfies **either one** of the following conditions:

1. \( \text{Ric} / R \) is asymptotically nonnegative, i.e.
\[ \liminf_{x \to \infty} \frac{\text{Ric}}{R} \geq 0; \]
for some positive constant

\[ C > 0 \]

\[ x \]

\[ \text{noncompact gradient Ricci expander with } \liminf \]

\[ \text{The proof of Lemma 2.6 also shows that (2.21) is true on any 3-dimensional complete expanding soliton case.} \]

Remark 2.5. The above proposition implies that any 3-dimensional complete noncompact gradient Ricci soliton with nonnegative Ricci curvature outside a compact set has nonnegative Ricci curvature everywhere.

Proof. We sketch the proof and point out the essential differences from Lemma 2.2. It follows from (2.14) that in the sense of barrier

\[ \Delta f 2 \lambda_1 \leq -2 \lambda_1 - 6 R \lambda_1 + 8 \lambda_1^2 + 2 R^2 - 4 |\text{Ric}|^2. \]

Using \( \nu_3 = R - 2 \lambda_1 \) and the computation showing (2.16), we see that

\[ \Delta f \nu_3 \geq -\nu_3 + 6 R \lambda_1 - 2 R^2 - 8 \lambda_1^2 + 2 |\text{Ric}|^2 \]

\[ = -\nu_3 - \nu_3^2 - \nu_1 \nu_2. \]

By exchanging the roles of \( \nu_1 \) and \( \nu_3 \), we may compute as in (2.17) and (2.18) to get

\[ \Delta f 2 \ln R (R^{-1} \nu_3) \geq -R^{-2} (R (\nu_3^2 + \nu_1 \nu_2) - 2 |\text{Ric}|^2 \nu_3) \]

\[ = R^{-2} (\nu_3^2 (\nu_3 - \nu_2) + \nu_2^2 (\nu_3 - \nu_1)) \geq 0. \]

Since \( \nu_3 / R = 1 - 2 \lambda_1 / R \), it holds that

\[ \Delta f 2 \ln R (R^{-1} \lambda_1) \leq -R^{-2} (\nu_3^2 (\nu_3 - \nu_2) + \nu_2^2 (\nu_3 - \nu_1)) / 2 \leq 0. \]

(2.20)

If \( \lambda_1 \) is negative somewhere, then either one of our assumptions will imply that \( R^{-1} \lambda_1 \) attains its negative minimum, say, at a point \( q \in M \) and \( R^{-1}(q) \lambda_1(q) < 0 \). We may further suppose that \( \nu_1(q) < 0 \), otherwise \( \lambda_1(q) = (\nu_1(q) + \nu_2(q)) / 2 \geq \nu_1(q) / 2 \). By (2.20), we have at \( q \)

\[ \nu_3^2 (\nu_3 - \nu_2) + \nu_2^2 (\nu_3 - \nu_1) = 0. \]

Hence \( \nu_2 = \nu_3 > 0 \) and \( 0 < R / 3 \leq \nu_3 = \nu_1 < 0 \), which is impossible. The completes the proof of the proposition.

Lemma 2.6. Under the same assumptions of Theorem 1.11 it holds that

\[ |\text{Rm}| \leq C R \quad \text{on} \quad M, \]

(2.21)

for some positive constant \( C > 0 \).

Remark 2.7. The proof of Lemma 2.6 also shows that (2.21) is true on any 3-dimensional complete noncompact gradient Ricci expander with \( \liminf_{r \to \infty} r^2 |\text{Rm}| \geq 0 \) and \( |\text{Rm}| \leq C r^{-\delta} \) near infinity for some \( \delta > 0 \).

Proof. As \( \lim_{r \to \infty} R = 0 \) and that \( R \geq -o(r^{-2}) \), the scalar curvature \( R \) is nonnegative. \[ \text{Cha20} \] Theorem 6]. By the strong minimum principle, we may assume that \( R > 0 \), since otherwise the expander is flat. It follows from (1.7) that \( |R| \leq o(r^{-1}) \) near infinity. Hence, by Lemma 2.1, (2.10), and (2.12), we have

\[ R_{mijn} = O \left( \frac{|\nabla \text{Ric}|}{|\nabla f|} \right) = o(r^{-2}) \quad \text{and} \quad R_{in} = O \left( \frac{|\nabla R|}{|\nabla f|} \right) = o(r^{-2}). \]

(2.22)
where $n := \frac{\nabla f}{|\nabla f|}$. Using (2.20), (2.22) and the fact that $R > 0$ near infinity, we have

$$T_{ij} = R g_{ij} - 2 R_{ij} = -2 R_{nijn} + R_{g_{in} g_{jn}} + 2 R_{nn g_{ij}} - 2 R_{in g_{jn}} - 2 R_{jn g_{in}}$$

$$\geq -2 R_{nijn} + 2 R_{nn g_{ij}} - 2 R_{in g_{jn}} - 2 R_{jn g_{in}}$$

$$\geq -o(r^{-2}) g_{ij},$$

Hence $\nu_1 \geq -o(r^{-2})$ and

$$\limsup_{x \to \infty} \nu_1 r^2 \leq 0. \quad (2.24)$$

It is not difficult to see that (2.16) is still valid. As $\nu_1 + \nu_2 + \nu_3 = R \geq 0$ and $\nu_1 \leq \nu_2 \leq \nu_3$, we have that $3 \nu_3 \geq R \geq 0$ and that the following holds in the barrier sense.

$$\Delta f \nu_1 \leq -\nu_1 - \nu_1^2 - 2 \nu_2 \nu_3$$

$$= -\nu_1 - \nu_1^2 - \nu_1 \nu_3 - (\nu_2 - \nu_1) \nu_3$$

$$\leq -\nu_1 - \nu_1^2 - \nu_1 \nu_3. \quad (2.25)$$

Moreover by (2.24), $0 \leq \nu_3 \leq R - 2 \nu_1 \leq R + C r^{-2} \leq C r^{-1}$ near infinity. Let $u = -\nu_1$, from (2.11), we may rewrite (2.25) on the set $\{x : u(x) \geq 0\} = \{x : \nu_1(x) \leq 0\}$ as

$$\Delta f u \geq -u - c_0 v^{-1/2} u,$$

where $v = n/2 - f \sim r^2$ and $c_0$ is some positive constant. Together with the boundary condition at infinity $\limsup_{x \to \infty} v u \leq 0$, the maximum principle argument by Deruelle [Der17, Lemma 2.9(1)] shows that there are positive constants $C$ and $C_1$ such that on $M$

$$-\nu_1 = u \leq C v^{1-3/2} e^{-v}$$

$$\leq C_1 R, \quad (2.26)$$

where we also used the scalar curvature lower bound [Cha20, Theorem 3] and the fact that $M$ is not flat. Note that the argument in [Der17] requires $u \geq 0$. However, (2.26) becomes trivial if $u < 0$ at the point under consideration. Hence we can always assume that the maximum principle is applied on $\{u \geq 0\}$ and no extra sign assumption on $u$ is needed. From (2.26), we see that $-C_1 R \leq \nu_1 \leq \nu_2 \leq \nu_3 \leq R - 2 \nu_1 \leq (1 + 2 C_1) R$ and $|T| \leq c R$. Since $T = R g - 2 \text{Ric}$, we have

$$|\text{Ric}| \leq c(|T| + R) \leq CR.$$  

The estimate on $|\text{Ric}|$ follows from (2.22).

\[\square\]

**Proof of Theorem 1.11** By our assumption (1.7) and the argument in the proof of Lemma 2.6, we may assume that $R > 0$ on $M$. Using Lemma 2.1, (2.11), and (2.23), we see that $v \sim r^2$ and

$$R^{-1} T \geq -o(1) v^{-1} R^{-1} g. \quad (2.27)$$

We claim that there is a large positive constant $a$, such that outside a compact set $K_0 \subset M$, it holds that

$$\Delta_{f - 2 \ln R} \left( e^{a v^{-1/2} v^{-1} R^{-1}} \right) \leq -v^{-3/2} e^{a v^{-1/2} R^{-1} < 0. \quad (2.28)$$
We first assume (2.28) and prove the theorem. Fixing any positive integer \( k \), we define

\[
Q_k := R^{-1} \nu_1 + e^{av_{1/2}} v^{-1} R^{-1} k^{-1}.
\]

By (2.27), we have \( \lim_{x \to \infty} Q_k \geq 0 \). If \( Q_k \) is nonnegative everywhere for infinitely many \( k \), then \( \nu_1 \geq 0 \) follows by letting \( k \to \infty \). Arguing by contradiction, we may assume, without loss of generality, that for all \( k \), \( Q_k \) is negative somewhere and hence it attains its negative minimum at a point \( x_k \). We must have \( x_k \in K_0 \) for each \( k \), otherwise, by the minimum principle, (2.19), and (2.28), we have

\[
0 \leq \Delta_{f-2 \ln R} Q_k(x_k) \leq k^{-1} \Delta_{f-2 \ln R} \left( e^{av_{1/2}} v^{-1} R^{-1} \right)(x_k) \leq -k^{-1} v^{-3} e^{av_{1/2}} R^{-1} < 0,
\]

which is impossible. Hence \( \inf_M Q_k = Q_k(x_k) = \min_{K_0} Q_k \). By letting \( k \to \infty \), we conclude that \( \inf_M R^{-1} \nu_1 = \min_{K_0} R^{-1} \nu_1 \) which implies that \( R^{-1} \nu_1 \) attains its minimum. By Lemma 2.2, \( M \) has nonnegative sectional curvature.

Next we justify \( Rm \geq 0 \). Let \( \tilde{\pi} : \tilde{M} \to M \) be the universal covering map of \( M \). It is clear that \((\tilde{M}, \pi^*g, \pi^*f)\) is a complete nonflat expanding gradient Ricci soliton. If the sectional curvature is not strictly positive, then by the strong maximum principle of the Ricci flow [CLN06, Section 6.7] and the De Rham splitting theorem, \( \tilde{M} \) splits isometrically as \( \mathbb{R} \times N \), where \( N \) is a 2-dimensional expanding gradient Ricci soliton. As \( \lim_{x \to \infty} R_g = 0 \) and \( M \) is not flat, \( R_g \) attains its maximum, say at \( q_0 \in \tilde{M} \). Let \( (a, b) \in \mathbb{R} \times N \) such that \( \pi^*(a, b) = q_0 \). For any \((s, b) \in \mathbb{R} \times \{b\}\), it is evident that

\[
R_{\pi^*g}(s, b) = R_{\pi^*g}(a, b) = R_{\pi^*g}(q_0).
\]

As \( R_g \to 0 \) at \( \infty \), \( \pi(\mathbb{R} \times \{b\}) \subseteq K_1 \) for some compact set \( K_1 \). Furthermore, the soliton equation \( \frac{\partial^2 \pi^*f}{\partial s^2} = -1/2 \) implies that

\[
\pi^*f(s, b) = -s^2/4 + C_1 s + C_2
\]

for some constants \( C_1 \) and \( C_2 \) independent on \( s \). We have that for any \( s \in \mathbb{R} \)

\[
-s^2/4 + C_1 s + C_2 \geq \min_{K_1} f > -\infty,
\]

which is impossible. This completes the proof of Theorem 1.11 modulo formula (2.28). It remains to establish formula (2.28).

**Proof of formula (2.28):** By virtues of (2.5) and \( \Delta_f R = -R - 2|\text{Ric}|^2 \), it holds that

\[
\Delta_f v^{-1} = -v^{-2} \Delta_f v + 2|\nabla v|^2 v^{-3} = -v^{-1} + 2|\nabla v|^2 v^{-3},
\]

\[
\Delta_f R^{-1} = -R^{-2} \Delta_f R + 2R^{-3} |\nabla R|^2 = R^{-1}(1 + 2R^{-1}|\text{Ric}|^2) + 2R^{-3} |\nabla R|^2.
\]

By the product rule, we have

\[
\Delta_f \left( v^{-1} R^{-1} \right) = R^{-1} \Delta_f v^{-1} + v^{-1} \Delta_f R^{-1} - 2R^{-2} \langle \nabla R, \nabla \left( v^{-1} R^{-1} \right) \rangle
\]

\[
= -v^{-1} R^{-1} + 2|\nabla v|^2 v^{-3} R^{-1} + v^{-1} R^{-1}(1 + 2R^{-1}|\text{Ric}|^2) + 2v^{-1} R^{-3} |\nabla R|^2
\]

\[
- 2(\nabla \ln R, \nabla \left( v^{-1} R^{-1} \right)) - 2v^{-1} R^{-3} |\nabla R|^2
\]

\[
= 2|\nabla v|^2 v^{-3} R^{-1} + 2v^{-1} R^{-2} |\text{Ric}|^2 - 2(\nabla \ln R, \nabla \left( v^{-1} R^{-1} \right)).
\]
Hence from (1.7), (2.5), and Lemma 2.6, we have
\[
\Delta_{f-2 \ln R} \left( v^{-1} R^{-1} \right) \leq (2|\nabla v|^2 v^{-2} + 2R^{-1} |\mathrm{Ric}|^2) v^{-1} R^{-1} \\
\leq (2v^{-1} + CR) v^{-1} R^{-1} \\
\leq C_0 v^{-3/2} R^{-1}.
\] (2.29)

Let \( a \) be a large positive constant to be determined. By (2.25) again, we have
\[
\Delta f e^{av - 1/2} = \frac{a}{2} v^{-1/2} e^{av - 1/2} + \left( \frac{a^2}{4} v^{-3} + \frac{3a}{4} v^{-5/2} \right) |\nabla v|^2 e^{av - 1/2},
\]
\[
2(\nabla \ln R, \nabla e^{av - 1/2}) = -a R^{-1} (\nabla R, \nabla v) v^{-3/2} e^{av - 1/2}. \tag{2.30}
\]

Hence
\[
\Delta_{f-2 \ln R} e^{av - 1/2} \leq -\frac{a}{2} v^{-1/2} e^{av - 1/2} + C(a + a^2) v^{-3/2} e^{av - 1/2} - a R^{-1} (\nabla R, \nabla v) v^{-3/2} e^{av - 1/2},
\]
where \( C \) is some constant independent on \( a \).

On the other hand, we compute
\[
2(\nabla e^{av - 1/2}, \nabla (v^{-1} R^{-1})) = av^{-3/2} e^{av - 1/2} |\nabla v|^2 v^{-2} R^{-1} + av^{-3/2} e^{av - 1/2} (\nabla v, \nabla R) v^{-1} R^{-2} \\
= av^{-7/2} e^{av - 1/2} |\nabla v|^2 R^{-1} + av^{-5/2} e^{av - 1/2} (\nabla v, \nabla R) R^{-2}. \tag{2.31}
\]

Thanks to (2.29), (2.30) and (2.31), we have
\[
\Delta_{f-2 \ln R} \left( e^{av - 1/2} v^{-1} R^{-1} \right) = e^{av - 1/2} \Delta_{f-2 \ln R} \left( v^{-1} R^{-1} \right) + v^{-1} R^{-1} \Delta_{f-2 \ln R} \left( e^{av - 1/2} \right) \\
+ 2(\nabla e^{av - 1/2}, \nabla (v^{-1} R^{-1})) \\
\leq (C_0 - a/2) v^{-3/2} e^{av - 1/2} R^{-1} + C(a + a^2) v^{-5/2} e^{av - 1/2} R^{-1} \\
- a (\nabla R, \nabla v) v^{-5/2} e^{av - 1/2} R^{-2} + av^{-7/2} e^{av - 1/2} |\nabla v|^2 R^{-1} \\
+ av^{-5/2} e^{av - 1/2} (\nabla v, \nabla R) R^{-2} \\
\leq (C_0 - a/2) v^{-3/2} e^{av - 1/2} R^{-1} + C(a + a^2) v^{-5/2} e^{av - 1/2} R^{-1}.
\]

Hence, fixing \( a \gg C_0 \), outside a compact set \( K_0 \) (possibly depending on \( a \)), we have
\[
\Delta_{f-2 \ln R} \left( e^{av - 1/2} v^{-1} R^{-1} \right) \leq -v^{-3/2} e^{av - 1/2} R^{-1} < 0.
\]

This finishes the proofs of formula (2.28) and Theorem 1.11 \( \Box \)

Before proceeding to the proof of Corollary 1.16, we investigate some basic geometric properties of a \( C^2 \) conical expanding soliton.

**Proposition 2.8.** Let \((X^{n-1}, g_X)\) be a closed Riemannian manifold of dimension \( n - 1 \), where \( n \geq 3 \). Suppose that a complete noncompact expanding gradient Ricci soliton \((M^n, g, f)\) is \( C^2 \) asymptotic to \((C(X), g_C)\), where \( g_C = dt^2 + t^2 g_X \). Then the following conditions hold:
(a) \( f \) is proper, i.e. \( \lim_{x \to \infty} f = -\infty \);

(b) \( \text{Ric} \geq (\varepsilon_0 - 1/2)g \) near infinity for some positive constant \( \varepsilon_0 \);

(c) \( \text{Ric} \to 0 \) at infinity (or \( R \to 0 \) at infinity if \( n = 3 \)).

Proof. It suffices to show that \((c) \implies (b) \implies (a)\) and lastly \((c)\) holds.

\((c) \implies (b)\) : We only need to consider the case when \( n = 3 \) and \( \lim_{x \to \infty} R = 0 \). Under these assumptions, we may apply [Cha20, Theorems 11 and Theorem 12] to see that \( \text{Ric} \to 0 \) as \( x \to \infty \).

\((b) \implies (a)\) : \( \text{Ric} \geq (\varepsilon_0 - 1/2)g \) and the soliton equation imply that \( -\nabla^2 f \geq \varepsilon_0 g \) outside a compact subset. Integrating the inequality along minimizing geodesics as in (2.6) yields

\[ -f \geq \varepsilon_0 r^2/2 - cr - c \quad \text{on} \quad M \]

for some positive constant \( c \). It is then evident that \( \lim_{x \to \infty} f = -\infty \).

To see that \((c)\) holds, we recall that (1.8) implies

\[ |\phi^* g - g_C|_{g_C}(t, \omega) \leq C_0 S_0^{-3\varepsilon} \]

and hence for any \( S > S_0 \), the set \( \phi \left\{ (t, \omega) : t \in (S_0, S], \omega \in X \right\} \) is bounded in \( (M, g) \), where \( \varepsilon \) is the positive number in Definition 1.13. As a result, we see that \( \lim_{x \to \infty} t = \infty \). By the conical estimate (1.8) with \( l = 2 \), it holds that

\[ |\text{Ric}(g)|_g \leq c|\text{Ric}(g_C)|_{g_C} + O(t^{-2-3\varepsilon/2}) \]

\[ \leq O(t^{-2}) \to 0 \quad \text{as} \quad x \to \infty. \]

This complete the proofs of \((c)\) and Proposition 2.8.

With the above preparation, we are about to prove Corollary 1.16. For reader’s convenience, we recall the statement of the corollary:

**Corollary 2.9.** Suppose that \( (M^3, g, f) \) is a 3-dimensional complete noncompact expanding gradient Ricci soliton which is \( C^2 \) asymptotic to a cone \( (C(X^2), dt^2 + t^2 g_X) \), where \( (X^2, g_X) \) is a connected closed 2 dimensional Riemannian manifold. Then the following are equivalent:

(1) \( M \) has nonnegative sectional curvature;

(2) \( M \) has nonnegative scalar curvature;

(3) \( C(X) \) has nonnegative scalar curvature;

(4) \( \text{Rm}(g_X) \geq \text{id}_{\Lambda^2 TX} \)

**Proof.** (1) \( \implies \) (2) is immediate.

(2) \( \implies \) (3) : Let \( \phi \) be the diffeomorphism in Definition 1.13. By \( R_g \geq 0 \) and (1.8), for any \( \omega \in X \), we have

\[ 0 \leq \lim_{t \to \infty} 4v \circ \phi(t, \omega) R_g \circ \phi(t, \omega) = R_{g_C}(1, \omega). \]
Since \( g_C = dt^2 + t^2 g_X \) is a warped product metric with warping function \( t \), the scalar curvature satisfies [Li12 Appendix A]

\[
R_{g_C}(t, \omega) = R_{g_C}(1, \omega) = R_{g_X}(\omega) - 2 t^{-2} + O(t^{-3} - 2) 
\]

(2.32)

This proves (3).

(3) \( \implies \) (4) : Since \( X \) is of dimension 2, (4) is equivalent to \( R_{g_X} \geq 2 \). Thanks to (3) and the properties of warped product [Li12 Appendix A], we have for any \((t, \omega) \in C(X)\),

\[
R_{g_C}(t, \omega) = R_{g_X}(\omega) - 2 t^{-2} + O(t^{-3} - 2) 
\]

(2.32)

In particular, \( 0 \leq R_{g_C}(1, \omega) = R_{g_X}(\omega) - 2 \) for any \( \omega \in X \) and hence (4) holds.

(4) \( \implies \) (1) : By our assumptions and (4), \((M, g, f)\) is \( C^2 \) asymptotic to \((C(X), g_C)\), where \( g_C = dt^2 + t^2 g_X \) and \( \text{Rm}(g_X) \geq 1 \). Using Proposition 2.8 (b) and (2.6), we have \(-f \sim r^2 \) near infinity. Moreover, we can find a diffeomorphism \( \phi \) satisfying the conditions in Definition 1.13. By virtues of (1.8), (1.9) and (2.32), we have for all sufficiently large \( t \),

\[
R_g \circ \phi(t, \omega) = R_{g^*}(t, \omega) = R_{g_C}(t, \omega) + O(t^{-3} - 2) 
\]

\[
= (R_{g_X}(\omega) - 2) t^{-2} + O(t^{-3} - 2) 
\]

(2.33)

where \( c \) is some positive constant independent on large \( t \) and \( \varepsilon \) is the positive number in Definition 1.13. We then use (1.9), (2.33), and the fact \(-f \sim r^2 \) to conclude that outside some compact subset of \( M \), it holds that

\[
t \sim r \ ; \quad R_g \leq C r^{-2} = o(r^{-1}) 
\]

(2.34)

(2.35)

where \( C \) is a positive constant. To apply Theorem 1.11 It remains to justify the lower bound in (1.7), i.e. \( R_g \geq -o(r^{-2}) \). Indeed, by (1.8), it holds that

\[
R_g \geq (R_{g_X} - 2) t^{-2} - O(t^{-3} - 2) \geq -O(t^{-3} - 2), 
\]

where we used \( R_{g_X} \geq 2 \) from assumption (4) in the last inequality. Hence by (2.34) and (2.35), we have \( r^2 R_g \geq C r^{-3} \to 0 \) as \( x \to \infty \) and (1.7) is satisfied. We may then invoke Theorem 1.11 and conclude that \( M \) has nonnegative sectional curvature. This establishes (1) and completes the proof of Corollary 1.16.

We shall end this section by proving another application of Theorem 1.11

**Proof of Corollary 1.19** As \( \text{Rm}(\alpha g_{S^2}) \geq 1/\alpha \geq 1 \) for any \( \alpha \in (0, 1] \), it follows from Corollary 1.16 that \( M \) is nonnegatively curved. By the strong maximum principle argument as in the proof of Theorem 1.11 \( M \) is either flat or has positive sectional curvature. Corollary 1.19 becomes obvious in the former case. Hence we may assume that \( \text{Rm} > 0 \) on \( M^3 \). The rotational symmetry of \( M \) then follows from the result by Chodosh [Cho14 Theorem 1.2] (see also Theorem 1.15).
3 Curvature estimates in 4D steady soliton case

Let \((M^4, g, f)\) be a 4-dimensional complete steady gradient Ricci soliton, i.e.,
\[
\text{Ric} + \nabla^2 f = 0.
\]
(3.1)

By a result of Hamilton, it is well known that on a complete non-Ricci-flat (and hence noncompact) steady gradient Ricci soliton, upon scaling the metric if necessary, the following identity holds,
\[
|\nabla f|^2 + R = 1.
\]
(3.2)

Chen \cite{Che09} showed that the scalar curvature of any complete ancient Ricci flow must be nonnegative. Consequently, by the strong minimum principle, the scalar curvature of a steady gradient soliton is positive everywhere unless \(M\) is Ricci flat. Recall that as in Section 2, for any smooth function \(\gamma\), the weighted laplacian \(\Delta_\gamma\) is defined as
\[
\Delta_\gamma := \Delta - \nabla \nabla \gamma,
\]
where \(\Delta\) is the standard laplace operator. In dimension four, the curvature operator becomes more complicated than the 3-dimensional case. We shall apply the techniques by Munteanu-Wang \cite{MW15} and carry out the dimension reduction via the level sets of the potential function \(f\). The crucial observation is that the curvature tensor in \(\nabla f\) direction can be written as \(\nabla \text{Ric}\), namely,
\[
\nabla Z \text{Ric}(X, Y) - \nabla Y \text{Ric}(X, Z) = \mathcal{R}(Z, Y, X, \nabla f).
\]
(3.3)

The above equation is a consequence of the soliton equation (3.1) and the Ricci identity. It leads to the following lemma due to Munteanu-Wang \cite[Proposition 2.1]{MW15} which allows us to control the Riemann curvature tensor by the Ricci tensor and its derivatives.

**Lemma 3.1.** \cite{MW15} Suppose that \((M^4, g, f)\) is a 4-dimensional gradient Ricci soliton. Then there exists a universal positive constant \(A_0\), such that, if \(\nabla f \neq 0\) at \(q \in M\), then
\[
|R_m|(q) \leq A_0 \left( |\text{Ric}(q)| + \frac{|\nabla \text{Ric}|}{|\nabla f|}(q) \right).
\]
(3.4)

**Lemma 3.2.** Let \((M^4, g, f)\) be a 4-dimensional complete noncompact non-Ricci-flat steady gradient Ricci soliton with bounded Ricci curvature. Then there exists a positive constant \(c_1\) such that
\[
|\text{Ric}| \leq c_1 R \quad \text{on} \quad M.
\]
(3.5)

**Proof.** By the boundedness of \(|\text{Ric}|\), we can find a finite positive constant \(L\) such that
\[
|\text{Ric}| \leq L \quad \text{on} \quad M.
\]
(3.6)

Using the computation in \cite[p. 9001]{Cha19}, we see that wherever \(\nabla f \neq 0\),
\[
\Delta_f(|\text{Ric}| + |\text{Ric}|^2) \geq - \frac{A_0^2 |\text{Ric}|^2}{|\nabla f|^2} - 2A_0 |\text{Ric}|^2 - 4A_0 |\text{Ric}|^3 - \frac{4A_0^2 |\text{Ric}|^4}{|\nabla f|^2}.
\]
Hence, by (3.6), one can find a constant $Q_0 = Q_0(A_0, L) > 0$ such that
\[
\Delta_f(|\text{Ric}| + |\text{Ric}|^2) \geq -Q_0|\text{Ric}|^2 \quad \text{on} \quad \{ x \in M : |\nabla f|^2(x) > \frac{1}{2} \}.
\]
Let $Q_1$ be another large positive constant such that
\[
Q_1 \geq \frac{Q_0}{2} + 1 \quad \text{and} \quad \frac{Q_1}{2} \geq L^2 + L + 1.
\]
Letting $u := |\text{Ric}| + |\text{Ric}|^2 - Q_1 R$, from [CCGGIIKLLN07, (1.33)], we have
\[
\Delta_f R = -2|\text{Ric}|^2
\]
and
\[
\Delta_f u \geq 2|\text{Ric}|^2 \quad \text{on} \quad \{|\nabla f|^2 > \frac{1}{2}\}.
\]
If we can prove that $u \leq 0$ on $M$, then we are done with the lemma. Since the Ricci curvature is bounded, $u$ is also bounded. By a result of Pigola-Rimoldi-Setti [PRSI11 Corollary 10], the weak maximum principle of $\Delta_f$ holds on $M$ and there is a sequence $\{x_k\}_{k=1}^{\infty}$ on $M$ such that
\[
u(x_k) \geq \sup_M u - \frac{1}{k} \quad \text{and} \quad \Delta_f u(x_k) \leq \frac{1}{k}.
\]
Since the Ricci curvature is bounded from below and $|\nabla f|$ is uniformly bounded by (3.2), the classical Omori-Yau maximum principle can also be used to obtain a sequence $\{x_k\}_{k=1}^{\infty}$ satisfying (3.10) (see [AMP16 Theorem 2.3]).

**Case 1:** if $|\nabla f|^2(x_k) \leq \frac{1}{2}$ for some subsequence $\{x_{k_j}\}_{j=1}^{\infty}$, then by (3.2), (3.6), and (3.7), we see that $R(x_k) \geq \frac{1}{2}$ and
\[
\sup_M u \leq u(x_{k_j}) + \frac{1}{k_j} \\
\leq L + L^2 - \frac{Q_1}{2} + \frac{1}{k_j} \\
\leq -1 + \frac{1}{k_j}.
\]
Letting $j \to \infty$, we have $u \leq 0$ on $M$.

**Case 2:** If $|\nabla f|^2(x_k) > \frac{1}{2}$ for all large $k$, then by the differential inequality (3.9) and (3.10)
\[
2|\text{Ric}|^2(x_k) \leq \Delta_f u(x_k) \leq \frac{1}{k} \to 0
\]
and hence
\[
\sup_M u = \lim_{k \to \infty} u(x_k) = 0.
\]
This finishes the proof of the lemma.
With the above preparation, we are going to prove Theorem 1.3.

**Proof of Theorem 1.3:** By our assumption, there is a finite constant $L_1 > 0$ such that

$$|Rm| \leq L_1 \quad \text{on} \quad M. \quad (3.11)$$

It is well known that in any gradient Ricci soliton, the gradient of the scalar curvature $\nabla R$ can be expressed as (see [CCGHIKLLN07, (1.27)])

$$\nabla R = 2 \text{Ric}(\nabla f). \quad (3.12)$$

Then by the calculation in [Cha19, p.9002], (3.12), and Lemma 3.2, there exist constants $c_3$ and $c_3'$ depending on $A_0, L, \lambda$, and $c_1$ in (3.5) such that

$$\Delta f - 2 \ln R \geq \frac{|\nabla R|^2}{R^2} - 4A_0|\text{Ric}|^3 + \frac{4A_0^2|\text{Ric}|^4}{R^2} - 6|\nabla \ln R|^2|\text{Ric}|^2 \geq \frac{|\nabla f|^2}{R^2} - c_3 \quad (3.13)$$

for all large $\lambda$, where $c' = c'(A_0)$ is a positive constant. We also used (3.14) in the last inequality. By [Cha19 (48)], we have

$$\Delta f - 2 \ln R \geq -5 \frac{|Rm|^2}{R}. \quad (3.14)$$

Combining (3.13) and (3.14), we see that for all sufficiently large $\lambda$, we have

$$\Delta f - 2 \ln R \geq W^2 - c'' \quad \text{on} \quad \{|\nabla f|^2 > \frac{1}{2}\}, \quad (3.15)$$

where $W := R^{-1}|Rm| + \lambda R^{-2}|\text{Ric}|^2$ and $c'' = c''(A_0, L, \lambda, c_1)$ is some positive constant. We localize the function $W$ by considering

$$G := \phi^2 W,$$

where $\phi = \psi \left( \frac{d(x,p_0)}{\rho} \right)$, $\rho$ is a large positive number, $\psi : [0, \infty) \rightarrow [0, 1]$ is a cut off function satisfying

$$\psi(t) = \begin{cases} 1 & \text{if } t \leq 1; \\ 0 & \text{if } t > 2, \end{cases}$$

$\psi' \leq 0$, and $|\psi'| + |\psi''| \leq c$ for some constant $c$. Hence we have

$$|\nabla \phi| \leq \frac{c}{\rho}.$$

Thanks to the Laplacian comparison theorem for smooth metric measure space [WW09, Theorem 1.1], it holds that

$$\Delta f r \leq \frac{3}{r} + 1.$$
Then by (3.12) and Lemma 3.2, we have the estimate on the weighted Laplacian of $\phi$

\[
\Delta f^{-2\ln R} \phi = \Delta f \phi + 2(\nabla \ln R, \nabla \phi) \\
\geq \frac{\psi'}{\rho} \Delta f + \frac{\psi''}{\rho^2} |\nabla R|^2 - \frac{4c\lambda}{\rho} \\
\geq -c \left( \frac{3}{\rho} + 1 \right) - \frac{c}{\rho^2} - \frac{4c\lambda}{\rho} \\
\geq -c
\]
for all large $\rho$, see also [Per02, WW09, Cha19].

It follows from (3.15) that

\[
\phi^2 \Delta f^{-2\ln R} G = \phi^4 \Delta f^{-2\ln R} W + \left( 2\phi \Delta f^{-2\ln R} \phi + 2|\nabla \phi|^2 \right) G \\
+ 4\phi^3 \langle \nabla \phi, \nabla (\phi^{-2} G) \rangle \\
= \phi^4 \Delta f^{-2\ln R} W + \left( 2\phi \Delta f^{-2\ln R} \phi - 6|\nabla \phi|^2 \right) G \\
+ 4\phi \langle \nabla \phi, \nabla G \rangle \\
\geq \phi^4 W^2 + \left( 2\phi \Delta f^{-2\ln R} \phi - 6|\nabla \phi|^2 \right) G \\
+ 4\phi \langle \nabla \phi, \nabla G \rangle - c'' \\
\geq G^2 - c_4 G - c_4 + 4\phi \langle \nabla \phi, \nabla G \rangle \quad \text{on} \quad \{|f^2| > \frac{1}{2}\}
\]

for some positive constant $c_4$ independent on $\rho$. If the maximum of $G$ is attained on $\{|f^2| > \frac{1}{2}\}$, then by the maximum principle and (3.16), at the point where $G$ attains its maximum we have

\[
0 \geq G^2 - c_4 G - c_4
\]
and hence $G \leq c_5$ for some positive constant $c_5$ independent on $\rho$. If instead $G$ attains its maximum on $\{|f^2| \leq \frac{1}{2}\}$, then it follows from (3.2), (3.11), and the definition of $G$ that $R \geq \frac{1}{2}$ and

\[
G \leq 2L_1 + 4L^2 \lambda.
\]

By letting $\rho \to \infty$, we have $|Rm| \leq c_2 R$ for some constant $c_2 > 0$. This completes the proofs of Theorem 1.3.

Under the conditions that the scalar curvature decays at least linearly, i.e. $R \leq C/(r + 1)$ on $M$, we have, by [CC20, Theorem 4.1], that $Rm$ is bounded. It then follows from Theorem 1.3 that

\[
|Rm| \leq c_2 R \leq c_2 C/(r + 1).
\]

Therefore we may invoke Shi’s derivative estimates [CLN06] to obtain the derivative estimates for $Rm$, namely, for any nonnegative integer $k$, we have

\[
|\nabla^k Rm| \leq \frac{C_k}{(r + 1)^{(k+2)/2}}.
\]

In view of (3.3) and the derivative estimate (3.18), we see that the $Rm$ in $\nabla f$ direction is of the order $r^{-3/2}$ and hence we can restrict our consideration of $Rm$ on the orthogonal complement of...
\[ \nabla f, \text{ i.e. the tangent space of level set of } f \text{ which is of dimension three. This allows the use of } \]

\[ \text{Hamilton-Ivey estimate as in the previous section with error terms due to the curvature in } \nabla f \]

\[ \text{direction. By our assumption in Theorem 1.6, } R \to 0 \text{ as } x \to \infty \text{ and } f \text{ is proper. Hence outside a } \]

\[ \text{compact set, } |\nabla f|^2 = 1 - R \geq 1/2. \text{ Thanks to [Cha19] Lemma 2 and the properness of } f, \text{ there is constant } c > 0 \]

\[ \text{such that outside a compact subset, it holds that } \]

\[ e^{-1}r \leq v = -f \leq cr. \quad (3.19) \]

Moreover, for all large \( \tau \gg 1 \), the level set of the potential \( \Sigma := \{ f = -\tau \} \) is a compact hypersurface; this level set is also connected by a result of Munteanu-Wang [MW11]. Hence there exist a compact set \( K_0 \) and a positive constant \( \tau_0 \gg 1 \) such that \( M \setminus K_0 \) is foliated by the smooth and closed level sets, namely

\[ M \setminus K_0 = \bigcup_{\tau > \tau_0} \{ f = -\tau \} \quad (3.20) \]

and \( |\nabla f|^2 \geq 1/2 \) on \( M \setminus K_0 \). The second fundamental form \( A_{\Sigma} \) of \( \Sigma \) with respect to the normal \( \nabla f/|\nabla f| \) is given by

\[ A_{\Sigma} = \nabla^2 f/|\nabla f| = -\text{Ric}(g)/|\nabla f| = O(r^{-1}). \]

Let \( \tilde{g} \) be the metric on \( \Sigma \) induced by \( g \). Let \( \{ e_i \}_{i=1}^4 \) be an orthonormal frame such that \( e_4 = \nabla f/|\nabla f| \), which is well defined near infinity. As in [MW15], throughout this section, \( a, b, c, d \) and \( i, j, k, l \) shall denote the indices in \( \{1, 2, 3\} \) and \( \{1, 2, 3, 4\} \), respectively. We begin with a 4-dimensional analog of (2.2); its shrinker version was proved by Munteanu-Wang [MW15] Lemma 4.1.

**Lemma 3.3.** Under the assumptions of Theorem 1.6, outside a compact subset of \( M \), it holds that

\[ \text{Rm}(g)_{abcd} = \text{Ric}(g)_{ad9bc} + \text{Ric}(g)_{bc9ad} - \text{Ric}(g)_{ac9bd} - \text{Ric}(g)_{bd9ac} \]

\[ - R(g) \left( g_{ad9bc} - g_{ac9bd} \right) /2 + O(r^{-3/2}), \]

where \( \{ e_i \}_{i=1}^4 \) is any orthonormal frame such that \( e_4 = \nabla f/|\nabla f| \) and \( a, b, c, d \in \{1, 2, 3\} \).

**Proof.** Let \( K_0 \) be the compact set as in (3.20). For any \( x \in M \setminus K_0 \), \( x \) belongs to \( \Sigma := \{ f = -\tau \} \) for some large \( \tau \). Let \( \tilde{g} \) denote the induced metric on \( \Sigma \). By the Gauss equation, we have

\[ \text{Rm}(g)_{abcd} = \text{Rm}(\tilde{g})_{abcd} - f_{ad} f_{bc} /|\nabla f|^2 + f_{ac} f_{bd} /|\nabla f|^2 \]

\[ = \text{Rm}(\tilde{g})_{abcd} + O(r^{-2}). \quad (3.21) \]

Similarly, using (3.3) and (3.18), we also have

\[ \text{Ric}(g)_{ab} = \text{Ric}(\tilde{g})_{ab} + \text{Rm}(g)_{a44b} + O(r^{-2}) = \text{Ric}(\tilde{g})_{ab} + O(r^{-3/2}), \]

\[ R(g) = R(\tilde{g}) + O(r^{-3/2}). \quad (3.22) \]

Moreover, \( \Sigma \) is of dimension three. Hence, by (2.2), (3.21), and (3.22), we have

\[ \text{Rm}(g)_{abcd} = \text{Rm}(\tilde{g})_{abcd} + O(r^{-2}) \]

\[ = \text{Ric}(\tilde{g})_{ad9bc} + \text{Ric}(\tilde{g})_{bc9ad} - \text{Ric}(\tilde{g})_{ac9bd} - \text{Ric}(\tilde{g})_{bd9ac} \]

\[ - R(\tilde{g}) \left( g_{ad9bc} - g_{ac9bd} \right) /2 + O(r^{-3/2}) \]

\[ = \text{Ric}(g)_{ad9bc} + \text{Ric}(g)_{bc9ad} - \text{Ric}(g)_{ac9bd} - \text{Ric}(g)_{bd9ac} \]

\[ - R(g) \left( g_{ad9bc} - g_{ac9bd} \right) /2 + O(r^{-3/2}). \]

This finishes the proof of the lemma. \( \square \)
In view of (3.21), to bound \( R_m(g) \) from below, it suffices to provide a quantitative lower bound of \( R_m(\tilde{g}) \) by considering the tensor \( T(\tilde{g}) := R(\tilde{g})\tilde{g} - 2\text{Ric}(\tilde{g}) \) introduced in the previous section. Let

\[ \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \tilde{\lambda}_3 \] be the eigenvalues of \( \text{Ric}(\tilde{g}) \),
\[ \tilde{\nu}_1 \leq \tilde{\nu}_2 \leq \tilde{\nu}_3 \] be the eigenvalues of \( T(\tilde{g}) \).

Then it clearly follows from (2.1) that

\[ \tilde{\nu}_1 = R(\tilde{g}) - 2\tilde{\lambda}_3. \] (3.23)

We further let \( U \) be the tensor which approximates \( T(\tilde{g}) \)
\[ U := (R(g) - \text{Ric}(g))_{44}g - 2\text{Ric}(g); \] (3.24)
\[ \lambda_1 \leq \lambda_2 \leq \lambda_3 \] be the eigenvalues of \( \text{Ric}(g) \) when restricted on \( T\Sigma \),
\[ \nu_1 \leq \nu_2 \leq \nu_3 \] be the eigenvalues of \( U \) when restricted on \( T\Sigma \),

where \( T\Sigma \) denotes the tangent space of the level set \( \Sigma \) of \( f \). We also consider the trace of \( U \) with respect to \( \tilde{g} \)
\[ U := U_{aa} = \nu_1 + \nu_2 + \nu_3. \]

It follows from the definition of \( \lambda_i \) that \( U = R(g) - \text{Ric}(g)_{44} = \lambda_1 + \lambda_2 + \lambda_3 \) is smooth as long as \( \nabla f \neq 0 \). By an argument similar to (2.1), we have
\[ \nu_1 = \lambda_1 + \lambda_2 - \lambda_3 = U - 2\lambda_3 \]
\[ \nu_2 = \lambda_1 + \lambda_3 - \lambda_2 = U - 2\lambda_2 \]
\[ \nu_3 = \lambda_2 + \lambda_3 - \lambda_1 = U - 2\lambda_1. \] (3.26)

By virtues of (3.18), (3.22), and (3.23), it can be seen that \( \text{Ric}(g)_{44} = \mathcal{O}(r^{-3/2}) \), \( \tilde{\lambda}_3 \leq \lambda_3 + \mathcal{O}(r^{-3/2}) \) and

\[ \tilde{\nu}_1 \geq R(g) - 2\tilde{\lambda}_3 - \mathcal{O}(r^{-3/2}) \]
\[ \geq R(g) - 2\lambda_3 - \mathcal{O}(r^{-3/2}) \]
\[ \geq \nu_1 - \mathcal{O}(r^{-3/2}). \] (3.27)

Our goal is to bound \( \nu_1 \) from below near infinity by the maximum principle. We first compute the differential inequality satisfied by \( \nu_1 \). The computations involved in the proof are parallel to those in the proof of Lemma 2.6 except for the error terms due to the extra dimension.

**Lemma 3.4.** \( \nu_1 \) satisfies the following differential equation in the barrier sense,

\[ \Delta_f \nu_1 \leq -\nu_1^2 - \nu_2\nu_3 + O(r^{-5/2}). \]

**Proof.** For the sake of simplicity, in the rest of this section, the connections, norms, and various curvature quantities are taken with respect to the soliton metric \( g \) unless specified. As mentioned above, \( \{e_i\}_{i=1}^4 \) is an orthonormal frame with \( e_4 = \nabla f / |\nabla f| \). Let \( a, b, c, d \in \{1, 2, 3\} \) and \( i, j, k, l \in \{1, 2, 3, 4\} \). \( \{e_4\}_{a=1}^4 \) can be further chosen such that at the point under consideration, \( R_{ab} = \lambda_a \delta_{ab} \).
By the differential equation of Ric \cite[Lemma 2.1]{PW10}, (3.3), a computation similar to (2.14), and Lemma 3.3, it holds that

\[
\Delta f R_{ab} = -2R_{aijb}R_{ij} - 2R_{acdb}R_{cd} + O(r^{-5/2})
\]

where \(\nu_i\) and \(\lambda_i\) are given by (3.25). We first claim that in the barrier sense, it holds that

\[
\Delta f \lambda_3 \geq -3U\lambda_3 - 2R_{cd}R_{cd} + 4\lambda_3^2 + \nabla f + O(r^{-5/2})
\]  

(3.28)

outside a compact set. We include the details here, since \(\lambda_3\) is an eigenvalue of Ric restricted on \(T\Sigma\), and extra care needs to be taken.

Proof of Claim (3.28): At any point where \(\nabla f \neq 0\), let \(W\) be a tangent vector of \(M\). We denote the orthogonal projection of \(W\) onto \(\{\nabla f\}^\perp\) by \(W_T\), i.e.

\[
W_T := W - \langle W, \nabla f \rangle \nabla f / |\nabla f|^2.
\]

Let \(q\) be any point near infinity with \(\nabla f \neq 0\). We choose a smooth vector field \(Q\) of \(M\) near \(q\) as in \cite{CCGGIJKLLN08} such that \(Q(q) \in T_q\Sigma\), \(Q(q) = e_3(q)\) and thus \([\operatorname{Ric}(Q(q))]^T = \lambda_3 Q(q)\). Moreover, fixing any \(j \in \{1, 2, 3, 4\}\), we have

\[
(\nabla_j Q)(q) = 0 \quad \text{and} \quad (\nabla_j \nabla_j Q)(q) = 0.
\]  

(3.29)

Since \(Q\) is not necessarily a section of \(T\Sigma\) near \(q\), \(\operatorname{Ric}(Q, Q)\) may not be a barrier function required. Instead, we let \(\Psi\) be the smooth function \(\operatorname{Ric}(Q^T, Q^T)/|Q^T|^2\) near \(q\). By the choice of \(Q\), we have \(Q^T(q) = Q(q)\) and \(\Psi \leq \lambda_3\) near \(q\) with equality holding at \(q\). Furthermore, it follows from (3.12), (3.18), and (3.29) that at the point \(q\) we have

\[
\nabla_j (Q^T) = \operatorname{Ric}(Q, e_j)\nabla f / |\nabla f|^2 = \begin{cases} O(r^{-1}) & \text{if } j = 1, 2, 3; \\ O(r^{-3/2}) & \text{if } j = 4. \end{cases}
\]

and for \(j = 1, 2, 3, 4\),

\[
\nabla_j \nabla_j (Q^T) = O(r^{-3/2}).
\]

One may check using the above derivative estimates of \(Q^T\) to see that at the point \(q\) we have

\[
\Delta f \Psi \geq -3U\lambda_3 - 2R_{cd}R_{cd} + 4\lambda_3^2 + \nabla f + O(r^{-5/2}).
\]

This justifies Assertion (3.28). \(\Box\)

Next we claim that

\[
\Delta f U = -2R_{cd}R_{cd} + O(r^{-5/2})
\]  

(3.30)
Thus \( \Delta f = \Delta_f R - \Delta_f (R_{44}) \)
\[ = -2|\text{Ric}|^2 - \Delta_f (R_{44}) \]
\[ = -2R_{cd}R_{cd} - \Delta_f (R_{44}) + O(r^{-5/2}). \]

Using the facts that \( 2\text{Ric}(\nabla f) = \nabla R \) \((3.12)\) and \( 2\text{Ric}(\nabla f, \nabla f) = \langle \nabla R, \nabla f \rangle = \Delta R + 2|Ric|^2 \) \((3.8)\), we have
\[
\Delta_f (R_{44}) = \Delta_f \left( \frac{\text{Ric}(\nabla f, \nabla f)}{|\nabla f|^2} \right)
= |\nabla f|^{-2} \Delta_f (\text{Ric}(\nabla f, \nabla f)) + \text{Ric}(\nabla f, \nabla f) |\nabla f|^{-4} \Delta_f R
+ 2\text{Ric}(\nabla f, \nabla f) |\nabla f|^{-6} |\nabla R|^2 + 2\langle \text{Ric}(\nabla f, f), \nabla R |\nabla f|^{-4}
+ |\nabla f|^{-6} |\nabla R|^2 (\Delta R + 2|Ric|^2) + |\nabla f|^{-4} \langle \nabla \Delta R + 4(\nabla \text{Ric}, \nabla R) \rangle
= |\nabla f|^{-2} \Delta_f (\text{Ric}(\nabla f, \nabla f)) + O(r^{-4}).
\]

By a computation of Petersen-Wylie \cite[Lemma 2.4]{PW10}, \((3.8)\), and \((3.3)\), we have
\[
\Delta_f (\text{Ric}(\nabla f, \nabla f)) = -4 \langle \nabla f, \text{Ric}(\nabla f, \nabla f) \rangle + 2\text{Ric}(\nabla f, \nabla f) |\nabla f|^{-2} + 2R_{ij}R_{ij} |\nabla f|^2
= O(r^{-5/2}) + 2R_{jk}R_{ij}R_{ik} + 2(R_{ij} - R_{j4})R_{ij} |\nabla f|
= O(r^{-5/2}).
\]

Thus \( \Delta_f (R_{44}) = O(r^{-5/2}) \) and Claim \((3.30)\) follows.

Hence by Claims \((3.28)\) and \((3.30)\), in the barrier sense
\[
\Delta_f \nu_1 = \Delta_f (\overline{U} - 2\lambda_3) \leq 6\overline{U}\lambda_3 - 2\overline{U}^2 - 8\lambda_3^2 + 2R_{cd}R_{cd} + O(r^{-5/2}), \tag{3.31}
\]
Thanks to \((3.26)\) and a computation similar to the proof of \((2.19)\), we have
\[
6\overline{U}\lambda_3 - 2\overline{U}^2 - 8\lambda_3^2 + 2R_{cd}R_{cd} = 6(\nu_1 + \nu_2 + \nu_3)(\nu_2 + \nu_3)/2 - 2(\nu_1 + \nu_2 + \nu_3)^2
- 2(\nu_2 + \nu_3)^2 + [(\nu_1 + \nu_2)^2 + (\nu_2 + \nu_3)^2 + (\nu_1 + \nu_3)^2] /2
= -\nu_1^2 - \nu_2\nu_3.
\]
We then rewrite \((3.31)\) as the required inequality
\[
\Delta_f \nu_1 \leq -\nu_1^2 - \nu_2\nu_3 + O(r^{-5/2}).
\]

Hence by \((3.21)\) and \((3.27)\), Theorem \ref{thm:1.6} is a consequence of the following proposition.

**Proposition 3.5.** Under the above notations and the same assumptions in Theorem \ref{thm:1.6} the following inequality holds outside a compact set, we have
\[
\nu_1 \geq -Cr^{-3/2},
\]
where \( C \) is a positive constant.
Proof. Using (3.38), we see that \( \Delta f R^{-1} = 2R^{-2}|\text{Ric}|^2 + 2R^{-1}|\nabla \ln R|^2 \) and
\[
\Delta_f (R^{-1} \nu_1) = R^{-1} \Delta f \nu_1 + \nu_1 \Delta f R^{-1} + 2(\nabla R^{-1}, \nabla (RR^{-1} \nu_1)) \\
= R^{-1} \Delta f \nu_1 + \nu_1 \Delta f R^{-1} - 2(\nabla \ln R, \nabla (R^{-1} \nu_1)) - 2R^{-1}|\nabla \ln R|^2 \nu_1 \\
\leq -R^{-1}(\nu_1^2 + \nu_2 \nu_3) + 2R^{-2}|\text{Ric}|^2 \nu_1 - 2(\nabla \ln R, \nabla (R^{-1} \nu_1)) + O(R^{-1}r^{-5/2}).
\] (3.32)

Note that by (3.12) and (3.18), \(|\text{Ric}|^2 = R_{cd} R_{cd} + 2R_{44}R_{44} - R_{44}^2 = R_{cd} R_{cd} + O(r^{-3})\). Due to a computation similar to the proof of (2.19) and the facts that \( U = O(r^{-1}) \) and \( \mathcal{U} = R - R_{44} \), we have
\[
R(\nu_1^2 + \nu_2 \nu_3) - 2|Ric|^2 \nu_1 = \mathcal{U}(\nu_1^2 + \nu_2 \nu_3) - 2R_{cd} R_{cd} \nu_1 + O(r^{-7/2}) \\
= (\nu_1 + \nu_2 + \nu_3)(\nu_1^2 + \nu_2 \nu_3) - \frac{\nu_1}{2} [ (\nu_1 + \nu_2 + \nu_3)^2 + (\nu_2 + \nu_3)^2] \\
+ O(r^{-7/2}) \\
= \nu_2^2 (\nu_3 - \nu_1) + \nu_3^2 (\nu_2 - \nu_1) + O(r^{-7/2}).
\] (3.33)

As before, we define the weighted operator \( \Delta_{f-2\ln R} := \Delta_f + 2(\nabla \ln R, \nabla) \). Then, by (3.32) and (3.33), we obtain a 4-dimensional analog of (2.19)
\[
\Delta_{f-2\ln R} (R^{-1} \nu_1) \leq -R^{-2} [\nu_2^2 (\nu_3 - \nu_1) + \nu_3^2 (\nu_2 - \nu_1)] + O(R^{-2}r^{-7/2}) + O(R^{-1}r^{-5/2}).
\] (3.34)

To deal with the error terms in (3.34), we need the auxiliary function \( v = -f \). Since \( f \) is assumed to be proper (3.19), by adding a constant if necessary, we have \( v \geq 1 \) on \( M \) and \( v \sim r \) near infinity. Furthermore, by taking the trace of (3.1) and (3.2), we have \( \Delta_f v = 1 \). For any \( \alpha > 0 \), using \( \langle \nabla R, \nabla f \rangle = \Delta R + 2|Ric|^2 \) and (3.18), we have
\[
\Delta_{f-2\ln R} v^{-\alpha} = -\alpha v^{-\alpha-1} \Delta_{f-2\ln R} v + \alpha (\alpha + 1)v^{-\alpha-2} |\nabla v|^2 \\
= -\alpha v^{-\alpha-1} - 2\alpha v^{-\alpha-1}(\nabla \ln R, \nabla v) + \alpha (\alpha + 1)v^{-\alpha-2} |\nabla v|^2 \\
= -\alpha v^{-\alpha-1} + 2\alpha v^{-\alpha-1}(\Delta R + 2|Ric|^2) + \alpha (\alpha + 1)v^{-\alpha-2} |\nabla v|^2 \\
\leq -\alpha v^{-\alpha-1}/2 + O(R^{-2}v^{-\alpha-3}),
\] outside a compact set of \( M \). By (3.14) and [CZh21 Theorem 1.3], both \( R^{-1}U \) and \( R^{-1}v_1 \) are bounded, and there is a positive constant \( C \) such that either one of the following holds near infinity
\[
C^{-1}r^{-1} \leq R \leq Cr^{-1};
\] (3.36)
\[
C^{-1}e^{-r} \leq R \leq Ce^{-r}.
\] (3.37)

If \( R \) decays exponentially, namely, if (3.33) is true, then \( |v_1| \leq eR \leq Ce^{-r} \) and Proposition 3.5 is done. Therefore we may suppose that (3.36) holds. Using (3.19) and (3.34), one can find a constant \( C_1 > 0 \), such that
\[
\Delta_{f-2\ln R} (R^{-1} \nu_1) \leq -R^{-2} [\nu_2^2 (\nu_3 - \nu_1) + \nu_3^2 (\nu_2 - \nu_1)] + C_1 v^{-3/2}.
\]
Substitution \( \alpha = 1/2 \) in (3.33), we have \( \Delta_{f-2\ln R} 8v^{-1/2} \leq -v^{-3/2} \) near infinity and
\[
\Delta_{f-2\ln R} \left( R^{-1} \nu_1 + 8C_1 v^{-1/2} \right) \leq -R^{-2} [\nu_2^2 (\nu_3 - \nu_1) + \nu_3^2 (\nu_2 - \nu_1)].
\] (3.38)
Since $R^{-1}\nu_1$ is bounded near infinity, by the properness of $v = -f$ (3.19), there is a large positive constant $\tau_0$ such that (3.38) holds on $\{x : f(x) \leq -\tau_0\}$, and we can choose a large constant $C_1$ such that

$$R^{-1}\nu_1 + 8C_1v^{-1/2} > 0 \quad \text{on} \quad \{x : f(x) = -\tau_0\}.$$  

By (3.18), $\langle \nabla R, \nabla f \rangle = \Delta R + 2|\Ric|^2$, and the assumption that (3.36) is true, it is clear that

$$\Delta f_{-2\ln v} = v^{-1}\Delta f_v - v^{-2}|\nabla v|^2$$

$$= v^{-1}(1 + 2(\nabla \ln R, \nabla v)) - v^{-2}|\nabla v|^2$$

$$= v^{-1}(1 + O(\ln v^{-1})) - v^{-2}|\nabla v|^2 \leq 2v^{-1}.$$  

For all positive integer $k$, let $G_k := R^{-1}\nu_1 + 8C_1v^{-1/2} + \frac{1}{k}\ln v$. It can be seen that $G_k > 0$ on $\{x : f(x) = -\tau_0\}$. By the properness of $f$ (3.19), we see that $\lim_{k \to \infty} G_k = \infty$. If $G_k$ is negative somewhere in $\{x : f(x) \leq -\tau_0\}$, then it attains a negative minimum, say, at $q_k \in \{x : f(x) < -\tau_0\}$. It is evident that $\nu_1(q_k) < 0$ and $G_k$ satisfies the following inequality at $q_k$

$$0 \leq \Delta f_{-2\ln v} \left( R^{-1}\nu_1 + 8C_1v^{-1/2} + \frac{1}{k}\ln v \right) \leq -R^{-2}\left[ 2\nu_2(\nu_3 - \nu_1) + 3\nu^2(\nu_2 - \nu_1) \right] + 2v^{-1}/k. \quad (3.39)$$

Moreover, by (3.12), (3.18), and (3.36), we have $3\nu_3 \geq \|U\| = R - R_{44} \geq R/2$ for large $\tau_0$. From (3.39) and the linear scalar curvature decay condition we have, at $q_k$ it holds that

$$\nu_2^2 \leq 6R^{-1}\nu_2\nu_3 \leq 6R^{-1}\nu_2^2(\nu_3 - \nu_1) \leq CV^2/k$$

and thus $|\nu_2| \leq \sqrt{C/k}$. Hence again by (3.36), (3.19), (3.39), and the fact that $|R^{-1}U| \leq c_1$ near infinity, we have

$$|\nu_1|/36 \leq R^{-2}\nu_2^2|\nu_1| \leq 2v^{-1}/k + C_1v^{-2}\sqrt{C/k}$$

and $R^{-1}|\nu_1|(q_k) \leq C_2k^{-1/2}$ for some positive constant $C_2$ independent on $q_k$ and $k$. In conclusion

$$G_k \geq G_0(q_k) \geq -C_2k^{-1/2}.$$  

The above inequality is obvious when $G_k$ is nonnegative on $\{x : f(x) \leq -\tau_0\}$. By letting $k \to \infty$, we have $R^{-1}\nu_1 + 8C_1v^{-1/2} \geq 0$ on $\{x : f(x) \leq -\tau_0\}$ and Proposition 3.3 follows.

**Proof of Theorem 1.6:** The lower bound of $\text{Rm}$ in (1.5) is a consequence of (3.3), (3.18), (3.21), (3.27), and the estimate on $\nu_1$ in Proposition 3.5. The upper estimate of $\text{Rm}$ in (1.5) follows from (3.17). This completes the proof of Theorem 1.6.

Next, we move on to the proof of Corollary 1.9.

**Proof of Corollary 1.9:** By [CZh21, Theorem 1.3, Proposition 6.1], if the curvature of $M$ decays exponentially, then Corollary (1.9.2) holds. Therefore, we may assume that the scalar curvature satisfies (3.36) and show that Corollary (1.9.1) is true. For any sequence $p_i \to \infty$ in $M$, let $\Sigma_i := \{x : f(x) = f(p_i)\}$ and $\tau_i := f(p_i) \to \infty$ by the properness of $f$. From the discussion before Lemma 3.3 and (3.19), $\Sigma_i$ is a smooth compact connected hypersurface in $M$ and

$$c^{-1}r \leq \tau_i \leq cr \quad \text{on} \quad \Sigma_i.$$  

$\tilde{g}_i$ denotes the induced metric on $\Sigma_i$ by $M$ and $h_i := R(p_i)\tilde{g}_i$ is the scaled metric by the scalar curvature at $p_i$. In view of the proof of [CZh21, Proposition 6.1], the curvature and intrinsic diameter
of \((\Sigma_i, h_i)\) are uniformly bounded in \(i\). Thus by Gromov’s Compactness theorem [BBI01, Theorem 10.7.2], the sequence sub-converges to certain compact Alexandrov space \((Y, d_Y)\) of dimension \(\leq 3\). Moreover by Theorem 4.6, the Gauss equation, (3.36), and (3.40), we see that \((\Sigma_i, h_i)\) has almost nonnegative curvature in the following sense

\[
\begin{align*}
\text{Rm}(h) & = R(p_i) \text{Rm}(\tilde{g}_i) = R(p_i) \left( \text{Rm}(g) + \text{Ric} \ast \text{Ric} / |\nabla f|^2 \right) \\
& \geq -R(p_i) \left( C \tau_i^{-3/2} g \odot g + C \tau_i^{-2} g \odot g \right) \\
& \geq -2R(p_i)^{-1} \left( C \tau_i^{-3/2} h_i \odot h_i \right) \\
& \geq -C' \tau_i^{-1/2} h_i \odot h_i,
\end{align*}
\]

where \((g \odot g)_{abcd} := 2g_{ad}g_{bc} - 2g_{ac}g_{bd}\) is the Kulkarni-Nomizu product. Hence we have \((Y, d_Y)\), being a limit of the sequence \((\Sigma_i, h_i)\), is an Alexandrov space of nonnegative curvature. The pointed Gromov-Hausdorff convergence to \(\mathbb{R} \times Y\) then follows from the same argument as in the proof of [CZh21, Proposition 6.1].

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