An estimate of accuracy for interpolant numerical solutions of a PDE problem

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Abstract

In this paper we present an estimate of accuracy for a piecewise polynomial approximation of a classical numerical solution to a non linear differential problem. We suppose the numerical solution \( u \) is computed using a grid with a small linear step and interval time \( T_u \), while the polynomial approximation \( v \) is an interpolation of the values of a numerical solution on a less fine grid and interval time \( T_v \ll T_u \). The estimate shows that the interpolant solution \( v \) can be, under suitable hypotheses, a good approximation and in general its computational cost is much lower of the cost of the fine numerical solution. We present two possible applications to linear case and periodic case.

1 The problem

Let \([a, b], a < b\), an interval on the real line \(\mathbb{R}\), and let \(C^1([0, +\infty) \times [a, b])\) the space of continuously differentiable real functions defined on \([0, +\infty) \times [a, b]\). Let \(F : C^1([0, +\infty) \times [a, b]) \to \mathbb{R}\) a continuously differentiable real functional. Then, if \(u \in C^1([0, +\infty) \times [a, b])\), the partial differential equation, usually associated to hyperbolic conservation laws,
\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0
\]

is in general non linear on the unknown \( u \). Let \( u_0 \in C^1([a, b]) \) and \( u_a, u_b \) two real numbers. We consider the boundary problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(x) + \frac{\partial F(u)}{\partial x}(x) = 0 \quad \forall x \in (a, b) \\
u(t, a) = u_a, \ u(t, b) = u_b \quad \forall t \in [0, +\infty) \\
u(0, x) = u_0(x) \quad \forall x \in (a, b)
\end{cases}
\]

and its numerical resolution. In this paper we consider only the Euler forward in time - centered in space (EFC) schema (see [4]) for Finite Differences method, but for other numerical schemas one can apply the same considerations with analogous, but more complicated, demonstrations. Let \( h > 0 \) a linear step for a partition \((x_j)\), \( 0 \leq j \leq P \), \( P = \frac{b-a}{h} \), of the interval \([a, b]\) such that \( x_0 = a, x_P = b \), and let \( \Delta t \) the time step of the numerical schema. We denote by \( u^n = (u^n_j) \) the numerical solution of problem (2) computed at time step \( n \)-th, hence at instant \( t = n\Delta t \).

In EFC schema the values at time step \( n+1 \) are so computed (see [1],[4]):

\[
u_{j+1}^n = u_j^n - F'(u_j^n) \frac{\Delta t}{2h} (u_{j+1}^n - u_{j-1}^n), \quad 0 < j < P - 1
\]

where \( F'(u_j^n) \) is the value of \( \frac{dF}{du}(u) \) obtained by substitution of \( u \) with the discretized value \( u_j^n \). We suppose that the schema (3) satisfies the Courant-Friedrichs-Lewy condition (CFL) (see [4]):

\[
|F'(u_j^n)| \frac{\Delta t}{h} \leq 1 \quad \forall j \in \mathbb{Z}, \forall n \in \mathbb{N}, \forall h \in \mathbb{R}_+
\]

This condition is usually admitted in the literature about numerical resolution of PDEs and is related to the question of stability of solutions.

Let \( N \in \mathbb{N}, N > 1, r \in \mathbb{N}_+, k = \frac{h}{r}, \ dt = \frac{\Delta t}{r}, \ u^N \) the numerical solution of (2) at time step \( N \) with linear step \( h \), \( u^M \) the solution at time step \( M = Nr \) with linear step \( k \). Note that CFL condition is satisfied in the second case too, because \( \frac{\Delta t}{k} = \frac{\Delta t}{r} \). The solution \( u^M \) is computed on a grid finer than that of \( u^N \), being \( h > k \), and hence it can be considered a better approximation of the exact solution, if exists, of the problem (2). But \( u^M \) has the disadvantage of a much greater computational cost, which can be very high for \( r >> 1 \).
We construct a piecewise polynomial interpolation \( v \) of \( u^N \) on the nodes of the partition with grid step \( h \). Function \( v \) can be considered as a \textit{global continuous and differentiable} representation of the discrete values \( u^N \) on the interval \([a, b]\). Then one can compute, for every couple \((x_j, x_{j+1})\) of nodes, the interpolating polynomial \( v_j = v_j(t) \) on a set of \( r \) values of \( t \). In the following, we present an estimate of the error \( |v_j(t) - u^M_x| \), where \( t = \frac{x - x_j}{h} \) (see Section 3), with the aim of establishing if \( v \) can be considered a good approximation for the solution of the problem \([2]\).

2 Construction of interpolating polynomials

Let \((x_j, x_{j+1})\) a couple of nodes internal to \([a, b]\), hence \(1 \leq j \leq P - 2\). For simplicity we don’t consider the boundary nodes \( x_0 \) and \( x_P \), but with a suitable choice of a discretization formula for the spatial derivative \( \frac{\partial u}{\partial x} \) (see \([1]\), \([5]\)) we could apply analogous considerations. We write, for more simple notation, \( p_1 = x_j \), \( p_2 = x_{j+1} \), \( d_1 = u^N_j \), \( d_2 = u^N_{j+1} \). We construct a cubic polynomial \( q = q(t) = at^3 + bt^2 + ct + d \), \( 0 \leq t \leq 1 \), imposing the following four conditions:

1. \( q(0) = p_1 \);
2. \( q(1) = p_2 \);
3. \( q'(0) = d_1 \);
4. \( q'(1) = d_2 \);

where \( q'(t) = 3at^2 + 2bt + c \). Then, solving the following linear system for the unknowns \( a, b, c, d \)

\[
\begin{align*}
    d &= p_1 \\
    a + b + c + d &= p_2 \\
    c &= d_1 \\
    3a + 2b + c &= d_2
\end{align*}
\]

one obtains the expression for the derivative of each cubic:

\[
q'(t) = (6p_1 - 6p_2 + 3d_1 + 3d_2)t^2 + (-6p_1 + 6p_2 - 4d_1 - 2d_2)t + d_1
\]
3 Auxiliary technical results

Let \( w = u^N \) the numerical solution of problem (2) obtained with a EFC scheme with grid step \( h \) (h-grid) and initial condition \( u_0 \) sampled using step \( h \).

We consider a couple of internal nodes \((x_j, x_{j+1})\) on h-grid, and a subpartition \((x_{ij})_{0 \leq i \leq r}\), where \( r \in \mathbb{N}_+, \mod(r, 2) = 0, x_j = x_{j,0}, ..., x_{j+1} = x_{j,r} \). Let \( v = v_j(t) = q_j'(t) \), where \( q_j \) is the cubic computed solving the system (3). We call \( v \) an interpolant solution of the problem (2). We define the set \( t_m = \frac{m}{r} \), where \( m \in \mathbb{N}, 0 \leq m \leq r \). If \( s = \frac{x_{j+1} - x_j}{r} \), we have \( x_m = x_{j,m} = ms + x_j \), therefore

\[
t_m = \frac{m}{r} = \frac{m x_{j+1} - x_j}{h} = \frac{m x_{j+1} - x_j}{r} \cdot \frac{1}{h} = \frac{m s}{h} = \frac{x_m - x_j}{h} \tag{7}
\]

Let \( k = \frac{h}{r} \) and let \( u = u^M \) the solution obtained with grid step \( k \) and \( u_0 \) sampled using \( k \). For simplicity we denote \( u_m \) the value of \( u \) on the node \( x_m = ms + x_j \), where \( j \), as before, is index of the h-grid. In the following we demonstrate some preliminary results for estimate the error \(|v_m - u_m|\), where \( v_m = v(t_m) \).

**Proposition 1** If \( t \in [0, 1] \), exists a value \( s \in [0, 1] \) such that

\[
t^2 = t - \frac{s}{4} \tag{8}
\]

**Proof.** If \( y = t^2 - t \), then \( y \leq 0 \) and \( y' = 2t - 1 \), therefore \( t_0 = \frac{1}{2} \) is the solution of \( y'(t) = 0 \). At \( t_0 \) the function \( y \) has a minimum, because \( y'' = 2 > 0 \). The value is \( y(\frac{1}{2}) = -\frac{1}{4} \). Therefore we have \(-\frac{1}{4} \leq t^2 - t \leq 0 \), from which (8) follows with \( s \in [0, 1] \). \( \square \)

**Proposition 2** For every \( m \) such that \( 0 \leq m \leq r \):

\[
|v_m - v_r| \leq \frac{3}{2} h + 2|d_1 - d_2| + \frac{3}{4} |d_1 + d_2| \tag{9}
\]

**Proof.** From (6) we have \( v_r = v(1) = d_2 \), therefore

\[
|v_m - v_r| \leq \|(6p_1 - 6p_2 + 3d_1 + 3d_2)t^2 + (-6p_1 + 6p_2 - 4d_1 - 2d_2)t + d_1 - d_2| \\
\leq \|(6p_1 - 6p_2)(t^2 - t)| + |(3d_1 + 3d_2)t^2 - (4d_1 + 2d_2)t| + |d_1 - d_2|
\]

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Then, using (8), we obtain
\[ |v_m - v_r| \leq \frac{3}{2}|p_1 - p_2| + 2|d_1 - d_2| + \frac{3}{4}|d_1 + d_2| \tag{10} \]
and hence (9), because \(|p_1 - p_2| = h|\). □

In the same manner, using the fact that \(v_0 = v(0) = d_1\), one proofs the following

**Proposition 3** For every \(m\) such that \(0 \leq m \leq r\):
\[ |v_m - v_0| \leq \frac{3}{2}h + 2|d_1 - d_2| + \frac{3}{4}|d_1 + d_2| \tag{11} \]

In the following propositions \(\epsilon\) is a fixed positive real number, \(w^0\) is the initial condition \(u_0\) of (2) sampled on the nodes of \(h\)-grid and \(u^0\) the initial condition sampled on \(k\)-grid nodes.

**Proposition 4** If \(|u^0_i - u^0_{i+1}| \leq \frac{\epsilon}{3M} \forall i\), then \(|u^n_i - u^n_{i+1}| \leq \frac{\epsilon}{3(M-n)} \forall i, \forall n \leq M, n \in \mathbb{N}_+\).

**Proof.** By induction. Let \(n = 1\). We have
\[ |u^1_i - u^1_{i+1}| \leq |u^1_i - u^0_i| + |u^0_i - u^0_{i+1}| + |u^0_{i+1} - u^1_{i+1}| \tag{12} \]
Using the EFC schema (3), the CFL condition (4) and the hypothesis, we have
\[ |u^1_i - u^1_{i+1}| \leq \frac{1}{2}|u^0_{i+1} - u^0_i| + |u^0_i - u^0_{i+1}| + \frac{1}{2}|u^0_{i+2} - u^0_i| \leq \frac{1}{2}|u^0_{i+1} - u^0_i| + \frac{1}{2}|u^0_i - u^0_{i-1}| + \frac{1}{2}|u^0_{i+1} - u^0_{i+1}| + \frac{1}{2}|u^0_{i+2} - u^0_{i+1}| + \frac{1}{2}|u^0_{i+1} - u^0_i| \leq 3\frac{\epsilon}{3M} = \frac{\epsilon}{3(M-1)} \]
that is the first inductive step. The intermediate inductive step is \(|u^n_i - u^n_{i+1}| \leq \frac{\epsilon}{3(M-n)}\). Then
\[ |u^{n+1}_i - u^{n+1}_{i+1}| \leq |u^{n+1}_i - u^n_i| + |u^n_i - u^n_{i+1}| + |u^n_{i+1} - u^{n+1}_{i+1}| \]

5
\[
\leq \frac{1}{2}|u_{i+1}^n - u_{i-1}^n| + |u_i^n - u_{i+1}^n| + \frac{1}{2}|u_{i+2}^n - u_i^n|
\leq \frac{1}{2}|u_{i+1}^n - u_i^n| + \frac{1}{2}|u_i^n - u_{i-1}^n| + |u_i^n - u_{i+1}^n| + \frac{1}{2}|u_{i+2}^n - u_{i+1}^n| + \frac{1}{2}|u_i^n - u_{i-1}^n|
\leq 3\frac{\epsilon}{3(M-n)} = \frac{\epsilon}{3(M-(n+1))}
\]
and this is the final inductive step. □

**Proposition 5** If \(|u_i^0 - u_{i+1}^0| \leq \frac{\epsilon}{3r} \quad \forall i, \text{ then } |u_i^{n+1} - u_i^n| \leq \frac{\epsilon}{3(M-n)} \quad \forall i, \forall n \leq M.

*Proof.* Let \(n \geq 0\). Then

\[
|u_i^{n+1} - u_i^n| \leq \frac{1}{2}|u_{i+1}^n - u_{i-1}^n|
\leq \frac{1}{2}(|u_{i+1}^n - u_i^n| + |u_i^n - u_{i-1}^n|)
\]
and hence, from Proposition (4)

\[
|u_i^{n+1} - u_i^n| \leq \frac{\epsilon}{3(M-n)}
\] (13)
that is the thesis. □

In the next proposition, the index \(i\) is associated to the node \(x_i\) in the \(k\)-grid, while the index \(j\) is associated to the node \(x_j\) in the \(h\)-grid.

**Proposition 6** If \(|u_i^0 - u_{i+1}^0| \leq \frac{\epsilon}{3r} \quad \forall i, \text{ then } |u_j^0 - u_{j+1}^0| \leq \frac{\epsilon}{3r} \quad \forall j.

*Proof.* Using the fact that \(r < 3r'\) and \(M = Nr \geq (N + r)\) if \(r, N > 1\), we have

\[
|u_j^0 - u_{j+1}^0| \leq \sum_{m=0}^{r-1} |u_{j+m}^0 - u_{j+1+m}^0| \leq \sum_{m=0}^{r-1} \frac{\epsilon}{3N'} \leq \sum_{m=0}^{r-1} \frac{\epsilon}{r3N} = \frac{\epsilon}{3N}
\]
therefore the proposition is verified. □

Note that by definition \(w_j^0 = u_j^0\), hence from Proposition 6 follows that
\[ |w^0_j - w^0_{j+1}| \leq \frac{\epsilon}{3^N} \quad (14) \]

Then the Propositions (4) and (5) hold for the solutions \((w^k)\) too, using \(N\) instead of \(M\):

**Corollary 1** If \(|u^0_i - u^0_{i+1}| \leq \frac{\epsilon}{3^M} \ \forall i\), then \(|w^{k+1}_j - w^k_j| \leq \frac{\epsilon}{3^{(N-k)}}, \ |w^k_j - w^k_{j+1}| \leq \frac{\epsilon}{3^{(N-k)-1}} \ \forall j, \ \forall k \leq N\).

In particular, for \(k = N\), writing \(w^N_j = d_1\) and \(w^N_{j+1} = d_2\), follows that

**Corollary 2** If \(|u^0_i - u^0_{i+1}| \leq \frac{\epsilon}{3^M} \ \forall i\), then \(|d_1 - d_2| \leq \epsilon\).

In the next proposition the simplified symbol \(u_0 = u^M_j\) is the value of solution \(u\) at node \(x_{jr}\) of the \(k\)-grid, i.e. the node \(x_j\) of the \(h\)-grid.

**Proposition 7** If \(|u^0_i - u^0_{i+1}| \leq \frac{\epsilon}{3^M} \ \forall i\), then \(|v_0 - u_0| \leq \epsilon\).

**Proof.** By definition, \(v_0 = d_1 = w_j = w^N_j\), where \(w^N\) is the solution at time step \(N\) of (2) with initial condition \(w^0\). Then, using a telescopic sum like \((a_N - a_0) = \sum_{k=1}^{N} (a_k - a_{k-1})\):

\[ |v_0 - u_0| = |w_j - u_0| \leq \sum_{k=1}^{N} |w^k_j - w^{k-1}_j| + |w^0_j - u^0_j| + \sum_{h=1}^{M} |u^h_{j-1} - u^h_j| \quad (15) \]

By definition, \(w^0_j = w^0_j\), hence the second term of the second member is null. For the first sum, from hypothesis and from Corollary (1) follows

\[ \sum_{k=1}^{N} |w^k_j - w^{k-1}_j| \leq \sum_{k=1}^{N} \frac{\epsilon}{3^{(N-k+1)}} = \frac{\epsilon}{3^{(N+1)}} \sum_{k=1}^{N} 3^k = \frac{\epsilon}{3^{(N+1)}} \frac{3^{(N+1)} - 1}{2} \leq \frac{\epsilon}{2} \quad (15) \]

For the second sum the computation, using the Proposition (5), is the same. Therefore the proposition is verified. □
Using the simplified symbol \( u_r = u_{(j+1)r}^M \) for the value of \( u \) at the node \( x_{(j+1)r} \) at time step \( M \), in the same manner we can proof the following

**Proposition 8** If \( |u_i^0 - u_{i+1}^0| \leq \frac{\epsilon}{3M} \) \( \forall i \), then \( |v_r - u_r| \leq \epsilon \).

In the next Proposition we remember that \( u_m \) is the value of \( u = u^M \) on the node \( x_m = ms + x_j \).

**Proposition 9** Let \( 1 \leq m \leq \frac{r}{2} \). If \( |u_i^0 - u_{i+1}^0| \leq \frac{\epsilon}{3M} \) \( \forall i \), then \( |u_0 - u_m| \leq m\epsilon \).

*Proof.* Using Proposition (4) with \( n = M \), we have
\[
|u_0 - u_m| \leq \sum_{k=1}^{m} |u_k^M - u_{k-1}^M| \leq m\epsilon \quad \square
\]

**Proposition 10** Let \( \frac{r}{2} \leq m \leq r \). If \( |u_i^0 - u_{i+1}^0| \leq \frac{\epsilon}{3M} \) \( \forall i \), then \( |u_m - u_r| \leq (r - m)\epsilon \).

*Proof.* Using Proposition (4) with \( n = M \), we have
\[
|u_m - u_r| \leq \sum_{k=1}^{r-m} |u_{m+k-1}^M - u_{m+k}^M| \leq (r - m)\epsilon \quad \square
\]

### 4 The main results

Now we can stated the main results on the estimate about the accuracy of the interpolant solution \( v \).

**Theorem 1** Let \( \epsilon > 0 \), \( m \in \mathbb{N}_+ \), \( 1 \leq m \leq r \). If \( |u_i^0 - u_{i+1}^0| \leq \frac{\epsilon}{3M} \) \( \forall i \), then
\[
|v_m - u_m| \leq \frac{3}{2} h + \frac{3}{4} |d_1 + d_2| + (\min[m, r - m] + 3)\epsilon \quad (16)
\]

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Proof. If $m \leq \frac{r}{2}$, let $p = 0$, else $p = r$. Using Propositions (2) or (3), (7) or (8), (9) or (10) and Corollary (1), we have

\[
|v_m - u_m| \leq |v_m - v_p| + |v_p - u_p| + |u_p - u_m| \\
\leq \frac{3}{2} h + 2\epsilon + \frac{3}{4}|d_1 + d_2| + (\min[m, r - m] + 1)\epsilon
\]

because if $m \leq \frac{r}{2}$ then $\min[m, r - m] = m$, else $\min[m, r - m] = r - m$. □

Corollary 3 Let $\epsilon > 0$, $m \in \mathbb{N}_+$, $1 \leq m \leq r$. There exists $\delta > 0$ such that if $|u_i^0 - u_{i+1}^0| \leq \delta$ ∀$i$, then

\[
|v_m - u_m| \leq \frac{3}{2} h + \frac{3}{4}|d_1 + d_2| + \epsilon
\]  

(17)

Proof. Let $\mu = \frac{\epsilon}{\min[m, r - m] + 3}$. Then $\mu > 0$, and from Theorem (11), defined $\delta = \frac{\mu}{\mu^2}$, follows that if $|u_i^0 - u_{i+1}^0| \leq \delta$, then

\[
|v_m - u_m| \leq \frac{3}{2} h + \frac{3}{4}|d_1 + d_2| + (\min[m, r - m] + 3)\mu
\]  

(18)

But $(\min[m, r - m] + 3)\mu = \epsilon$, so the thesis is verified. □

The estimate stated in Corollary (3) is of the same kind of the estimates established, in the case of finite element method, by Kuznetsov and Tadmor-Tang for approximated numerical solutions to scalar conservation laws equations (see [6]).

In particular, inequality (17) shows that the accuracy, respect to the fine $u$ solution computed using a grid step $k$, of the interpolant solution $v$, computed using a grid step $h = rk$, is of the first order on $h$. In (17) the role of the time steps $N$ and $M$ is hidden in the third addendum $\epsilon$, as stated by Corollary (2).

The next theorem gives a sufficient condition on the initial function $u_0$ for the validity of the hypothesis in Theorem (1).

Theorem 2 Let $\epsilon > 0$, $h > 0$, $N \in \mathbb{N}$, $N > 1$. If $u_0 \in C^0([a, b])$, then exists $r \in \mathbb{R}_+$ such that, if $0 \leq m \leq r$,

\[
|v_m - u_m| \leq \frac{3}{2} h + \frac{3}{4}|d_1 + d_2| + \epsilon
\]  

(19)
Proof. The function \( u_0 \), continuous on a closed and limited subset of \( \mathbb{R} \), is uniformly continuous, therefore exists \( \delta > 0 \) such that if \( x_1, x_2 \in [a, b] \), \(|x_1 - x_2| \leq \delta\), then \(|u_0(x_1) - u_0(x_2)| \leq \epsilon\). We can choose the parameter \( r \in \mathbb{N}_+ \), and hence the time step \( M = rN \), such that \( \frac{\Delta t}{3^r} \leq \delta \). Therefore, with this choice of \( r \), it can be applied the theorem \( 1 \) and, with the right choice of \( \delta \), the Corollary \( 3 \). \( \square \)

In the next section we discuss the role of the term \(|d_1 + d_2|\) in some particular cases.

5 Applications to particular cases

As first application of the previous estimate, let \( F(u) = au \), with \( a \in \mathbb{R} \), \( a \neq 0 \). Hence we suppose that \( F \) is linear. In this case the numerical schema EFC is stable in the norm \(|u|_2 = \left( h \sum_j |u_j|^2 \right)^{\frac{1}{2}}\) until time step \( N \), i.e. \(|u|_2 \leq C_N|u_0|_2\) with \( C_N = C(N) \in \mathbb{R}_+ \), if \( \Delta t \leq \left( \frac{h}{|a|} \right)^2 \); the constant \( C_N \) is equal to \( e^{\frac{N\Delta t}{h}} \) (see \( 4 \)), while \( j \) is the index on \( h \)-grid. If we choose the grid step \( h \) such that \( h \leq |a| \), than the CFL condition is satisfied because

\[ |a| \frac{\Delta t}{h} \leq \frac{h}{|a|} \leq 1 \] (20)

**Lemma 1** Let \( m \in \mathbb{N} \), \( m \geq 1 \), and \((C_j)_{0 \leq j \leq m}\) a set of positive real numbers. If \( A, B \in (C_j)_j \), then

\[ A + B \leq 2 \left( \sum_{j=0}^{m} C_j^2 \right)^{\frac{1}{2}} \] (21)

**Proof.** We have \( A^2 \leq \sum_{j=0}^{m} C_j^2 \) and \( B^2 \leq \sum_{j=0}^{m} C_j^2 \), therefore

\[ A \leq \left( \sum_{j=0}^{m} C_j^2 \right)^{\frac{1}{2}}, \quad B \leq \left( \sum_{j=0}^{m} C_j^2 \right)^{\frac{1}{2}} \] (22)

hence the thesis is verified. \( \square \)

If \( u \in C^0([a, b]) \), let \(|u|_\infty = \sup_{x \in [a, b]} |u(x)|\).
Corollary 4 Let $\epsilon > 0$, $N \in \mathbb{N}$, $N > 1$, $h > 0$, $a \neq 0$, $h \leq |a|$, $\Delta t \leq \left(\frac{h}{a}\right)^2$ and $u_0 \in C^0([a, b])$. Then exists $r \in \mathbb{R}_+$ such that, if $0 \leq m \leq r$,

$$|v_m - u_m| \leq \frac{3}{2} \left[ h + \left( \frac{b - a}{h} + 1 \right)^{\frac{1}{2}} e^{\frac{N\Delta t}{2}} |u_0|_{\infty} \right] + \epsilon$$

(23)

Proof. From theorem (2) we have

$$|v_m - u_m| \leq \frac{3}{2} h + \frac{3}{4} |d_1 + d_2| + \epsilon$$

(24)

where $d_1 = w_j^N$ and $d_2 = w_j^{N+1}$, for a generic node $x_j$ of $h$-grid. Let $s = \frac{b - a}{h}$ and $|w_0|_{\infty} = sup_j |w_{0,j}|$. From Lemma (1) and from stability in the norm $|\cdot|_2$ follows that

$$\frac{3}{4} |d_1 + d_2| \leq \frac{3}{4} (|w_j^N| + |w_{j+1}^N|) \leq \frac{3}{2} \left( \sum_{j=0}^{s} (w_j^N)^2 \right)^{\frac{1}{2}}$$

$$= \frac{3}{2 h^{\frac{1}{2}}} |w_j^N|_2 \leq \frac{3}{2 h^{\frac{1}{2}}} e^{\frac{N\Delta t}{2}} |w_0|_2$$

$$\leq \frac{3}{2} (s + 1)^{\frac{1}{2}} e^{\frac{N\Delta t}{2}} |w_0|_{\infty}$$

and the thesis is verified because $|w_0|_{\infty} \leq |u_0|_{\infty}$. □

The Corollary (4) shows that the error $|v_m - u_m|$ is uniformly limited on interval $[a, b]$ for every time step $n \leq N$. The presence of some kind of norm on $u_0$, like $|u_0|_{\infty}$, in the upper limitation is typical in literature about accuracy of numerical solutions (see (4)).

As second application, let $w$ the numerical solution at time step $N$ of the problem (2) on the $h$-grid, and let $x_j$, $x_{j+1}$ two nodes of this grid. We call the interval $[x_j, x_{j+1}]$ an interval of rising turbulence at time step $N$ for the $h$-grid if $(w_j w_{j+1}) < 0$. The definition is based on the intuitive fact that a change of sign for the velocity field of a linear flow induces a superposition of particles in the motion of a fluid. In one dimensional case, in this situation the velocity $u$ in the problem (2) might be not a classic function.
Corollary 5 Let $\epsilon > 0$, $N \in \mathbb{N}$, $N > 1$, $h > 0$, $u_0 \in C^0([a, b])$ and $[x_j, x_{j+1}] \subseteq (a, b)$ an interval of rising turbulence at time step $N$ for the $h$-grid. Let $w^N$ the numerical solution at time step $N$. Then exists $r \in \mathbb{R}_+$ such that, if $0 \leq m \leq r$, $v_0 = w^N_j$ and $v_r = w^N_{j+1}$,

$$|v_m - u_m| \leq \frac{1}{2}(3h + 5\epsilon) \quad (25)$$

where $u$ is the solution on $k$-grid, $k = rh$.

Proof. From theorem (2) we have

$$|v_m - u_m| \leq \frac{3}{2}h + \frac{3}{4}|w_j^N + w_{j+1}^N| + \epsilon \quad (26)$$

From hypothesis we suppose $w_j = w^N_j < 0$ and $w_{j+1} = w^N_{j+1} > 0$. Then exists $w > 0$ such that $w_j = -ww_{j+1}$. Theorem (2) holds, and from Corollary (1) we have $|w_j - w_{j+1}| \leq \epsilon$. Therefore

$$|w_j - w_{j+1}| = |-ww_{j+1} - w_{j+1}| = w_{j+1}| - w - 1| = w_{j+1}(1 + w) \leq \epsilon \quad (27)$$

and hence $w_{j+1} \leq \frac{\epsilon}{1+w} \leq \epsilon$. Also

$$|w_j - w_{j+1}| = -w_j + w_{j+1} \leq \epsilon \quad (28)$$

hence $|w_j| = -w_j \leq \epsilon - w_{j+1} \leq \epsilon$. Therefore we have

$$|w_j + w_{j+1}| \leq |w_j| + |w_{j+1}| \leq 2\epsilon \quad (29)$$

and from (26) follows that

$$|v_m - u_m| \leq \frac{3}{2}h + \frac{3}{2}\epsilon + \epsilon \quad (30)$$

The thesis is verified. □

From the proof of the Corollary we may note that in an interval of rising turbulence the values of the solution at extremes are small. The particular limit case $w_j = w_{j+1} = 0$, for which, from theorem (2), $|v_m - u_m| \leq \frac{3}{2}h + \epsilon$, might be interesting for searching periodic solution $u$ for which $u_j = u_{j+1} = 0$ on the nodes of the $h$-grid:

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\[ u \]

\[ x_{j-1} \quad x_j \quad x_{j+1} \]

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