Description of Complex Systems in terms of Self-Organization Processes of Prime Integer Relations

Victor Korotkikh† and Galina Korotkikh‡

Faculty of Informatics and Communication
Central Queensland University
Mackay, Queensland 4740, Australia

(Dated: October 26, 2021)

In the paper we present a description of complex systems in terms of self-organization processes of prime integer relations. A prime integer relation is an indivisible element made up of integers as the basic constituents following a single organizing principle. The prime integer relations control correlation structures of complex systems and may describe complex systems in a strong scale covariant form. It is possible to geometrize the prime integer relations as two-dimensional patterns and isomorphically express the self-organization processes through transformations of the geometric patterns. As a result, prime integer relations can be measured by corresponding geometric patterns specifying the dynamics of complex systems. Determined by arithmetic only, the self-organization processes of prime integer relations can describe complex systems by information not requiring further explanations. This gives the possibility to develop an irreducible theory of complex systems.

PACS numbers: 89.75.-k, 89.75.Fb

I. INTRODUCTION

Complex systems profoundly change human activities of the day. In order to understand and control them it becomes increasingly important to be confident in the theory of complex systems. Ultimately, this calls for clear explanations why the foundations of the theory are valid in the first place. The ideal situation would be to have an irreducible theory of complex systems not requiring a deeper explanatory base in principle. But the question arises: where such an irreducible theory may come from, when even the concept of space-time is questioned \[1\] as a fundamental entity?

As a possible answer to the question it is suggested that the concept of integers may take responsibility in the search for an irreducible theory of complex systems \[2\]. The aim of the paper is to present a description of complex systems in terms of self-organization processes of prime integer relations. In particular, it is considered that the prime integer relations control correlation structures of complex systems and may describe complex systems in a strong scale covariant form.

A prime integer relation is an indivisible element made up of integers as the basic constituents following a single organizing principle. Remarkably, the prime integer relations can be geometrized as two-dimensional patterns and the self-organization processes can be isomorphically expressed through transformations of the geometric patterns. As a result, the self-organization processes of prime integer relations are characterized geometrically and quantitatively. In fact, it becomes possible to measure prime integer relations by corresponding geometric patterns.

Due to the isomorphism the structure and the dynamics of a complex system are combined in our description: as a prime integer relation governs a correlation structure of a complex system, a corresponding geometric pattern specifies its dynamics. Applications of the geometrization of prime integer relations are considered to discuss potential advantages of the proposed description.

Determined by arithmetic only, the self-organization processes of prime integer relations can describe complex systems by information not requiring further explanations. This gives the possibility to develop an irreducible theory of complex systems.

II. INVARIANT QUANTITIES OF A COMPLEX SYSTEM AND UNDERLYING CORRELATIONS

We approach complex systems from a general perspective. In particular, we describe a complex system by its certain quantities and are concerned how many of them remain invariant as the system evolves from one state to another \[2\], \[3\].

Let \(I\) be an integer alphabet and

\[I_N = \{x = x_1 \ldots x_N, x_i \in I, i = 1, ..., N\}\]

be the set of sequences of length \(N \geq 2\). We consider a complex system consisting of \(N\) elementary parts with the state of an elementary part \(P_i\) specified by a space variable \(x_i \in I, i = 1, ..., N\) and the state of the complex system itself by a sequence \(x = x_1 \ldots x_N \in I_N\).

We use piecewise constant functions for a geometric representation of the sequences. Let \(\varepsilon > 0\) and \(\delta > 0\) be length scales of a two-dimensional lattice. Let \(\rho_{\varepsilon, \delta} : x \rightarrow f\) be a mapping that realizes the geometric representation of a sequence \(x = x_1 \ldots x_N \in I_N\) by associating it with a
The definite integrals give us certain quantities of the state $x$. The definite integrals are also equal $f^{[3]}(t_k) = g^{[3]}(t_k)$, but the third integrals are not $f^{[3]}(t_k) \neq g^{[3]}(t_k)$. Thus $C(x, x') = 2$.

We use the geometric representation to characterize a state $x = x_1 ... x_N \in I_N$ of a complex system in terms of the definite integrals

$$f^{[k]}(t_{m+N}) = \int_{t_m}^{t_{m+N}} f^{[k-1]}(t)dt, \quad k = 1, 2, \ldots$$

of a function $f^{[0]} = f = \rho_{\text{med}}(x) \in W_{\text{med}}[t_m, t_{m+N}]$. The definite integrals give us certain quantities of the complex system.

Remarkably, the integer code series expresses the quantities of a complex system in terms of the state $x = x_1 ... x_N \in I_N$ explicitly. In particular, it describes the definite integral

$$f^{[k]}(t_{m+N}) = \sum_{i=0}^{k-1} a_{kmi}((m+N)^i x_1 + \ldots + (m+1)^i x_N)v^i$$

of a function $f \in W_{\text{med}}[t_m, t_{m+N}]$ by using the code $c(f) = x_1 ... x_N$ of the function $f$, powers

$$(m+N)^i, \ldots, (m+1)^i, \quad i = 0, \ldots, k-1$$

of integers $(m+N), \ldots, (m+1)$ and combinatorial coefficients

$$a_{kmi} = \frac{(-1)^{k-i-1}(m+1)^{k-i} + (-1)^{k-i}m^{k-i}}{(k-i)!i!}$$

where $k \geq 1$ and $i = 0, \ldots, k-1$.

We are concerned how many quantities remain invariant as a complex system moves from one state $x = x_1 ... x_N \in I_N$ at a time $\tau$ to another state $x' = x'_1 ... x'_N \in I_N$ at a time $\tau'$.

$$f^{[k]}(t_{m+N}) = g^{[k]}(t_{m+N}), \quad k = 1, \ldots, C(x, x'), \quad (3)$$

$$f^{[C(x, x')+1]}(t_{m+N}) \neq g^{[C(x, x')+1]}(t_{m+N}), \quad (4)$$

where $f = \rho_{\text{med}}(x), g = \rho_{\text{med}}(x')$ (Figure 1).

We consider the conservation of the quantities in view of 1 as a consequence of the correlations between the parts of the complex system. Although the complex system moves from one state $x$ to another $x'$, yet the changes of the parts are correlated to preserve $C(x, x')$ of the quantities. The complex system can be characterized by the rate of change of the correlations with respect to time.

III. THE CONSERVING CORRELATIONS AS SPECIFIC LINEAR EQUATIONS

By using 2 it is proved that $C(x, x') \geq 1$ of the quantities of a complex system remain invariant, as it moves from one state $x = x_1 ... x_N \in I_N$ to another $x' = x'_1 ... x'_N \in I_N$. If the space-time changes $\Delta x_i = x'_i - x_i$, $i = 1, \ldots, N$ of the elementary parts satisfy a system of linear equations

$$(m+N)^0 \Delta x_1 + \ldots + (m+1)^0 \Delta x_N = 0$$

$$(m+N)^{C(x, x')-1} \Delta x_1 + \ldots + (m+1)^{C(x, x')-1} \Delta x_N = 0$$

and an inequality

$$(m+N)^{C(x, x')} \Delta x_1 + \ldots + (m+1)^{C(x, x')} \Delta x_N \neq 0.$$  

The coefficients of the system of linear equations become the entries of the matrix

$$
\begin{pmatrix}
(m+N)^0 & (m+N-1)^0 & \ldots & (m+1)^0 \\
(m+N)^1 & (m+N-1)^1 & \ldots & (m+1)^1 \\
& \ddots & \ddots & \ddots \\
(m+N)^{N-1} & (m+N-1)^{N-1} & \ldots & (m+1)^{N-1}
\end{pmatrix}
$$

with the Vandermonde determinant, when the number of the equations is $N$. This fact is important in order to prove that the number $C(x, x')$ of the conserved quantities of a complex system satisfies the condition $C(x, x') < N$.

The system of equations may bring interesting associations. For example:
1. It is suggested that the system of equations \( \text{(5)} \) may be connected with a smooth projective curve over a finite field and the number \( C(x,x') \) of the conserved quantities with the genus of the curve \( \text{(2)} \).

2. The system of equations \( \text{(5)} \) can be written for \( s = 0, -1, ..., -C(x,x') + 1 \) and \( m = 0 \) as

\[
\sum_{n=1}^{N} \frac{\Delta x_{N-n+1}}{n^s} = 0
\]

to have a naive resemblance with the Dirichlet zeta function

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi_n}{n^s},
\]

where \( \chi_n \) are some coefficients and \( s \) defined for proper complex numbers. This points to a possible link of the equations \( \text{(5)} \) with the zeroes of the Dirichlet zeta function

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi_n}{n^s} = 0
\]

and the zeroes of the Riemann zeta function

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 0
\]

in particular.

IV. SELF-ORGANIZATION PROCESSES OF PRIME INTEGER RELATIONS AND CORRELATION STRUCTURES OF COMPLEX SYSTEMS

The analysis of the equations \( \text{(5)} \) and inequality \( \text{(6)} \) reveals that a complex system is actually described in terms of hierarchical structures of prime integer relations. Such a hierarchical structure can be interpreted as a result of a self-organization process of prime integer relations \( \text{(2)} \). Namely, the self-organization process, starting with integers as the elementary building blocks and following a single principle, makes up the prime integer relations of one level of the hierarchical structure from the prime integer relations of the lower level.

We illustrate the results by considering two states of a complex system

\[
x = -1+1+1-1+1-1+1+1-1+1-1+1-1+1-1+1\]

\[
x' = +1-1+1-1+1+1-1+1-1+1-1+1-1+1+1+1-1+1
\]

specified by the Prouhet-Thue-Morse (PTM) sequences of length \( N = 16 \). In this case since \( C(x,x') = 4 \), the equations \( \text{(5)} \) can be written as four integer relations

\[
+16^0 - 15^0 - 14^0 + 13^0 - 12^0 + 11^0 + 10^0 - 9^0
\]

\[
+16^1 - 15^1 - 14^1 + 13^1 - 12^1 + 11^1 + 10^1 - 9^1
\]

\[
-8^0 + 7^0 + 6^0 - 5^0 + 4^0 - 3^0 - 2^0 + 1^0 = 0,
\]

\[
+16^1 - 15^1 - 14^1 + 13^1 - 12^1 + 11^1 + 10^1 - 9^1
\]

\[
-8^0 + 7^0 + 6^0 - 5^0 + 4^0 - 3^0 - 2^0 + 1^0 = 0,
\]

\[
+16^2 - 15^2 - 14^2 + 13^2 - 12^2 + 11^2 + 10^2 - 9^2
\]

\[
-8^2 + 7^2 + 6^2 - 5^2 + 4^2 - 3^2 - 2^2 + 1^2 = 0,
\]

\[
+16^3 - 15^3 - 14^3 + 13^3 - 12^3 + 11^3 + 10^3 - 9^3
\]

\[
-8^3 + 7^3 + 6^3 - 5^3 + 4^3 - 3^3 - 2^3 + 1^3 = 0
\]

and the inequality \( \text{(6)} \) takes the form

\[
+16^4 - 15^4 - 14^4 + 13^4 - 12^4 + 11^4 + 10^4 - 9^4
\]

\[
-8^4 + 7^4 + 6^4 - 5^4 + 4^4 - 3^4 - 2^4 + 1^4 \neq 0
\]

where \( m = 0 \) and a common factor 2 originated from the space-time variables \( \Delta x_i, i = 1, ..., 16 \) is not shown. This leaves the dynamics of the elementary parts represented by the signs only, but allows us to focus on the integer
relations and inequality, and identify prime integer relations.

It is worth to note that calculations in (4) and (5) and their results are completely determined by arithmetic.

There is a number of hierarchical structures of prime integer relations associated with the system of integer relations (7) and inequality (8). One of the hierarchical structures is shown in Figure 2. In the structure the relationships between the elements of neighboring levels can be interpreted as a consequence of a self-organization process of prime integer relations. The process starts with integers 1, ..., 16 in certain states, i.e., positive or negative, and proceeds level by level following the same organizing principle:

> *on each level the powers of the integers in the prime integer relations are increased by 1, so that through emerging arithmetic interdependencies the prime integer relations could self-organize as the components to form prime integer relations of the higher level.*

It is important to note that the formation of prime integer relations is more than their simple sum. The second level - the prime integer relations (5) to change our perspective from the self-organization processes of prime integer relations to the dynamics of the complex system in space-time. Starting with the prime integer relation +16⁰ − 15⁰ = 0 (Figure 2), which actually stands as

\[ +16^0 - 14^0 = 0, \quad -15^0 + 13^0 = 0, \quad -12^0 + 10^0 = 0, \]

\[ +11^0 - 9^0 = 0, \quad -8^0 + 6^0 = 0, \quad +7^0 - 5^0 = 0, \]

\[ +4^0 - 2^0 = 0, \quad -3^0 + 1^0 = 0. \]

The second level - the prime integer relations

\[ (+16^1 - 14^1) + (-15^1 + 13^1) = 0, \]

\[ (-12^1 + 10^1) + (+11^1 - 9^1) = 0, \]

\[ (-8^1 + 6^1) + (+7^1 - 5^1) = 0, \quad (+4^1 - 2^1) + (-3^1 + 1^1) = 0. \]

The third level - the prime integer relations

\[ ((+16^2 - 14^2) + (-15^2 + 13^2)) \]

\[ +((+12^2 + 10^2) + (+11^2 - 9^2)) = 0, \]

\[ ((-8^2 + 6^2) + (+7^2 - 5^2)) + ((+4^2 - 2^2) + (-3^2 + 1^2)) = 0. \]

The fourth level - the prime integer relation

\[ (((+16^3 - 14^3) + (-15^3 + 13^3)) + ((-12^3 + 10^3) + (+11^3 - 9^3))) \]

\[ +((+8^3 + 6^3) + (+7^3 - 5^3)) + ((+4^3 - 2^3) + (-3^3 + 1^3)) = 0. \]

Let us explain the notion of prime integer relation. A prime integer relation of the first level is made up of integers from the zero level. An integer comes to the first level in positive or negative state (Figure 2). Following the organizing principle prime integer relations of a level make up a prime integer relation of the higher level as an indivisible element. Namely, if even one of the prime integer relations is not involved, then according to the organizing principle the rest of the prime integer relations can not form an integer relation.

By our definition an integer relation

\[ +7^2 - 6^2 - 5^2 + 3^2 + 2^2 - 1^2 = 0 \]

is a prime integer relation. However, an integer relation

\[ +7^0 - 6^0 - 5^0 + 3^0 = 0 \]

is not prime, because it consists of two prime integer relations +7⁰ − 6⁰ = 0, −5⁰ + 3⁰ = 0. For simplicity prime integer relations such as

\[ +14^0 - 7^0 = 0, \quad +2 \cdot 14^0 - 2 \cdot 7^0 = 0 \]

are not distinguished. Multiple 2 means that we have two integers 14 in the positive state and two integers 7 in the negative state.

The space-time dynamics \( \Delta x_i \) of an elementary part \( P_i, i = 1, ..., N \) has the following interpretation. The absolute value of \( \Delta x_i \) is the number of integers \((m - N - i + 1)\) starting each of the self-organization processes associated with the system of equations (10) and inequality (11), while \( \text{sign}(\Delta x_i) \) determines the state, i.e., positive or negative, of the integers, provided that \( \Delta x_i \neq 0, i = 1, ..., N \).

The correlation structures underlying the conservation of the quantities (13) in view of (11) are defined by the hierarchical structures of prime integer relations associated with the system of equations (11) and inequality (12), (13), (14).

We illustrate the result by using the system of integer relations (17), where we return to the symbolic form of (15) to change our perspective from the self-organization processes of prime integer relations to the dynamics of the complex system in space-time. Starting with the prime integer relation +16⁰ − 15⁰ = 0 (Figure 2), which actually stands as

\[ +16^0 + 2^0 + 15^0(-2) = \]

\[ +16^0 \Delta x_1 + 15^0 \Delta x_2 = 0, \quad (9) \]

we can see that space-time changes \( \Delta x_1 \) and \( \Delta x_2 \) of the elementary parts \( P_1 \) and \( P_2 \) are correlated as

\[ +16^0 \Delta x_1 = -15^0 \Delta x_2 \quad (10) \]

and the elementary parts \( P_1 \) and \( P_2 \) thus make a composite part \( (P_1 \leftrightarrow P_2) \).
and form a larger composite part (Figure 2). A horizontal link denotes that the parts are correlated through a prime integer relation. As arithmetic behind the prime integer relations makes them sensitive to a minor change, so does the correlation structure. If the complex system deviates from the dynamic behavior even slightly, then some of the correlation links disappear and the complex system deviates.

The prime integer relation \( P \) does not contain information about a physical signal that may realize the correlation between the parts \( P_1 \) and \( P_2 \). But, if the dynamics \( \Delta x_2 \) of the elementary part \( P_2 \) is specified, then, according to \( (10) \), the dynamics \( \Delta x_3 \) of the elementary part \( P_1 \) is instantaneously determined and vice versa. We may also specify that the correlation is nonlocal, because the prime integer relation \( P \) does not have any reference to the distance between the parts \( P_1 \) and \( P_2 \).

Similarly, the prime integer relation \(-14^0 + 13^0 = 0\) leads to

\[
+14^0 \Delta x_3 + 13^0 \Delta x_4 = 0, 
\]

which specifies the correlation

\[
14^0 \Delta x_3 = -13^0 \Delta x_4 
\]

between the elementary parts \( P_3 \) and \( P_4 \) and describes a composite part \((P_3 \leftrightarrow P_4)\).

In its turn the prime integer relation

\[
+16^1 - 15^1 - 14^1 + 13^1 = 0, 
\]

made up of the prime integer relations \(+16^0 - 15^0 = 0\) and \(-14^0 + 13^0 = 0\), corresponds to

\[
(16^1 \Delta x_1 + 15^1 \Delta x_2) + (14^1 \Delta x_3 + 13^1 \Delta x_4) = 0, 
\]

which shows that the composite parts \((P_1 \leftrightarrow P_2)\) and \((P_3 \leftrightarrow P_4)\) are correlated as

\[
(16^1 \Delta x_1 + 15^1 \Delta x_2) = -(14^1 \Delta x_3 + 13^1 \Delta x_4) 
\]

and form a larger composite part

\[(P_1 \leftrightarrow P_2) \leftrightarrow (P_3 \leftrightarrow P_4).\]

Continuing the consideration we can associate the self-organization process of prime integer relations with the formation of a correlation structure of the complex system (Figures 2 and 3). The formation process starts with the elementary parts \( P_1, \ldots, P_{16} \) and combine them into composite parts to make up then larger composite parts and so on until the whole correlation structure is built.

A complex system can be described by self-organization processes of prime integer relations in a distinctive way. Information about a complex system can be given by prime integer relations, which are true statements not requiring further explanations. The prime integer relations are organized as hierarchical structures and there is no need for deeper principles to explain why the hierarchical structures exist the way they do and not otherwise.

For example, there is no need to explain a hierarchical structure, where prime integer relations

\[
+7^0 - 6^0 = 0, \quad -5^0 + 3^0 = 0, \quad +2^0 - 1^0 = 0 
\]

of level 1 form a prime integer relation

\[
+7^1 - 6^1 - 5^1 + 3^1 + 2^1 - 1^1 = 0 \quad (11) 
\]

of level 2. The prime integer relation \((11)\) alone makes up a prime integer relation

\[
+7^2 - 6^2 - 5^2 + 3^2 + 2^2 - 1^2 = 0 \quad (12) 
\]

of level 3. However, the prime integer relation \((12)\) on its own can not progress to level 4, because

\[
+7^3 - 6^3 - 5^3 + 3^3 + 2^3 - 1^3 \neq 0. 
\]

By using the self-organization processes of prime integer relations a concept of complexity is introduced \(2\) and its applications may be found in \(7\).

V. GEOMETRIZATION OF THE SELF-ORGANIZATION PROCESSES OF PRIME INTEGER RELATIONS AND ITS APPLICATIONS

Prime integer relations as abstract entities are not natives of mental pictures associated with formations of physical objects. At the same time in our description of complex systems prime integer relations behave like objects that under the control of arithmetic can transform into each other. This view would be more relevant with prime integer relations as geometric objects suitable for measurement to obtain information about complex systems.

Remarkably, by using the integer code series \(4\), the prime integer relations can be geometricized as two-dimensional patterns and the self-organization processes can be isomorphically expressed through transformations.
FIG. 4: A hierarchical structure of prime integer relations and an isomorphic hierarchical structure of geometric patterns. A prime integer relation can be positive (shown in black) and negative (shown in white). Under the integration of the function, the geometric patterns of one level form the geometric patterns of the higher level, so we can observe the geometric patterns length scale by length scale. A prime integer relation can be measured by the area of a corresponding geometric pattern or the length of its boundary curve. The boundary curve of a geometric pattern specifies the dynamics of an associated complex system.

The right side of Figure 4 presents an isomorphic hierarchical structure of geometric patterns as a result of the geometrization of the self-organization process of prime integer relations. The geometrization allows us to visualize the process and investigate it, as well as the complex system, geometrically and quantitatively. In particular, the self-organization process can be analyzed by measuring prime integer relations through their geometric patterns. For example, a prime integer relation can be measured by the area of a corresponding geometric pattern or the length of its boundary curve. These two characteristics of a prime integer relation are quantities of an associated complex system. We consider the quantities in applications to discuss potential advantages of the proposed description.

The first application shows that complex systems may be described by the prime integer relations in a strong scale covariant form. Namely, although a PTM geometric pattern at level \(N > 1\) of length scale \(2^N\epsilon\) is bounded by an intricate curve, nevertheless the PTM geometric pattern has a concise and universal description working for all levels. In particular, the area \(S\) of a PTM geometric pattern at level \(N \geq 1\) can be expressed as if it were a triangle

\[
S = \frac{LH}{2},
\]

where \(L\) and \(H\) are the length and the height of the PTM geometric pattern (Figure 4). Consequently, the law of PTM pattern area is the same for all length scales and in the simplest possible form. In other words, the description of \(S\) is strongly scale covariant, i.e.,

under the scale transformations the equation of \(S\) is preserved in the simplest possible form (13).

A PTM geometric pattern is a result of the formation, but for the description of its area \(S\) there is no need to know what happens at the lower levels. All information can be obtained by measuring the length \(L\) and the height \(H\) of the PTM geometric pattern at the level of consideration. Nevertheless, the history of the formation of a PTM geometric pattern is encoded by the boundary curve.

Because of (13) the PTM geometric patterns have a scale invariant property that divides the hierarchical levels into groups of three successive levels. Namely, the lengths and the heights of PTM geometric patterns at levels \(N = 1, 2, 3\) and \(N = 4, 5, 6\) are given in terms of \(\epsilon\) and \(\delta\) as

\[
(2\epsilon, \epsilon\delta), (4\epsilon, \epsilon^2\delta), (8\epsilon, 2\epsilon^3\delta)
\]

and

\[
(16\epsilon, 8\epsilon^4\delta), (32\epsilon, 64\epsilon^5\delta), (64\epsilon, 1024\epsilon^6\delta).
\]

By using the renomalization group transformation

\[
\epsilon' = 2^3\epsilon, \delta' = \epsilon^3\delta,
\]

of the geometric patterns 2. As a result, the self-organization processes of prime integer relations are characterized geometrically and quantitatively. In fact, it becomes possible to measure prime integer relations by corresponding geometric patterns.

Due to the isomorphism the structure and the dynamics of a complex system are combined in our description: as a prime integer relation governs a correlation structure of a complex system, a corresponding geometric pattern specifies its dynamics. The geometrization of the prime integer relations puts forward arithmetic to irreducibly explain complex systems through quantitative means.

To illustrate the results we consider a self-organization process of prime integer relations that can progress through the hierarchical levels 2. The process is connected with the PTM sequence and can be specified in terms of critical point 3, 4 features 2. This resonates with the fact that the PTM sequence gives a symbolic description of chaos resulting from the period-doubling 10 in a complex system 11.

The left side of Figure 4 shows one of the hierarchical structures of prime integer relations, when

\[
x = 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0\]

\[
x' = +1 \ -1 \ -1 \ +1 \ -1 \ +1 \ -1,
\]

\(m = -3\) and a complex system consists of \(N = 8\) elementary parts \(P_i, i = 1, \ldots, 8\). The sequence \(x'\) is the initial segment of length 8 of the PTM sequence starting with +1.
the lengths and the heights of PTM geometric patterns at levels \( N = 4, 5, 6 \) can be expressed in terms of \( \varepsilon' \) and \( \delta' \) in the same way
\[
(2\varepsilon', \varepsilon'\delta'), (4\varepsilon', \varepsilon'^2\delta'), (8\varepsilon', 2\varepsilon'^3\delta')
\]
as the lengths and the heights of PTM geometric patterns at levels \( N = 1, 2, 3 \) are given in terms of \( \varepsilon \) and \( \delta \). The situation repeats for levels \( N = 7, 8, 9 \) and so on.

The second application is connected with a long-standing problem to explain why constants of nature, such as the fine-structure constant \( \alpha \) measured to be equal to \( 1/137.03599976 \) and written as
\[
\alpha = 0.00729735... ,
\]
have the values they do and not even slightly different. Although the logic of digits in the fine-structure constant \( \alpha \) has not been established yet, it is known that \( \alpha \) is fragile. If the fine-structure constant \( \alpha \) varied even a bit, then complex physical systems would not be able to exist.

A prime integer relation is also an intricate entity, because arithmetic behind makes it sensitive to a minor change of the elements. However, when we read a prime integer relation, unlike a constant of nature, we can understand and accept it as a true statement not requiring further explanations. Moreover, a prime integer relation can describe a correlation structure of the complex system. In this capacity it encodes the parts of the correlation structure, the relationships between them, i.e., how the parts are connected, and the strengths of the relationships, i.e., how the dynamics of some parts of a relationship determine the dynamics of the other parts (Figures 2 and 3). In our description a minor change breaking a prime integer relation also leads to a collapse of a corresponding complex system, because some of the relationships between the parts of the correlation structure disappear.

The analogy with the sensitivity of the fine-structure constant \( \alpha \) could be made stronger, if a prime integer relation would be found sensitive to a single number. The geometrization of a prime integer relation provides a number with the required property. In particular, a prime integer relation can be isomorphically expressed as a two-dimensional geometric pattern, which is completely determined by the boundary curve (Figure 4).

The length of the curve, i.e., a number, encodes the geometric pattern and thus the prime integer relation. With even a minor change in the number the prime integer relation as well as a corresponding complex system cease to exist. Indeed, if the number i.e., the length, changes even slightly, then the boundary curve in its turn changes the geometric pattern, which leads the prime integer relation and the complex system to decay.

We consider the example above with \( m = 0, \varepsilon = 1 \) and \( \delta = 1 \) (Figure 4) to show how such numbers can be obtained. By using corresponding geometric patterns at levels 1, 2 and 3, we define the numbers for prime integer relations.

For the prime integer relation \(+8^0 - \tau^0 = 0\) we have
\[
\vartheta_1 = 2 \int_0^1 \sqrt{1 + \left( \frac{df[1]}{dt} \right)^2} dt =
\]
\[
2 \int_0^1 \sqrt{1 + \frac{t}{4} dt} = 2 \sqrt{2} = 2 \times 1.414223562... ;
\]
where \( f = \rho_{011} (x'), x' = +1 - 1 - 1 + 1 - 1 + 1 - 1 \).

For the prime integer relation
\[
8^1 - 7^1 - 6^1 + 5^1 = 0
\]
we obtain
\[
\vartheta_2 = 4 \int_0^1 \sqrt{1 + \left( \frac{df[2]}{dt} \right)^2} dt =
\]
\[
4 \int_0^1 \sqrt{1 + t^2} dt = 4 \times 1.14779... .
\]

For the prime integer relation
\[
8^2 - 7^2 - 6^2 + 5^2 - 4^2 + 3^2 + 2^2 - 1^2 = 0
\]
we get
\[
\vartheta_3 = 4 \int_0^2 \sqrt{1 + \left( \frac{df[3]}{dt} \right)^2} dt =
\]
\[
4 \left[ \int_0^1 \sqrt{1 + \frac{t^4}{4} dt} + \int_1^2 \sqrt{1 + \left( \frac{-t^2}{2} + 2t - 1 \right)^2} dt \right] =
\]
\[
4 \times (1.0242... + 1.30702... ) = 4 \times 2.33122... .
\]

The numerical results in the second and third cases are computed by using Mathematica.

We can write the prime integer relations and their corresponding numbers as
\[
+8^0 - \tau^0 = 0 \quad \Rightarrow \quad 2 \times 1.414223562...
\]
\[
+8^1 - 7^1 - 6^1 + 5^1 = 0 \quad \Rightarrow \quad 4 \times 1.14779...
\]
\[
+8^2 - 7^2 - 6^2 + 5^2 - 4^2 + 3^2 + 2^2 - 1^2 = 0 \quad \Rightarrow \quad 4 \times 2.33122... .
\]

On the side of prime integer relations we have confidence in the arithmetic statements, as we can check them. Moreover, we know how the prime integer relations are built, can observe symmetry in their corresponding geometric patterns (Figure 4) and associate them with correlation structures of complex systems.

But on the other side it is not clear what logic digits of the numbers may follow. The situation would become
intriguing, once the numbers turned to be found as constants by physical experiments.

When a constant of nature is measured, the information from physical devices comes for processing in the numerical form. Presenting numerical information through prime integer relations may give us a tool to understand experimental results and make our decisions relying on irreducible arguments. For instance, if some physical experiments identify a constant and as a number it corresponds in our description to a prime integer relation

\[ +7^2 - 6^2 - 5^2 + 3^2 + 2^2 - 1^2 = 0, \]

then it may be considered that the experiments through the constant actually reveal a true statement we can understand and agree with.

Therefore, we propose to explore the idea:

*constants of nature may be numerical expressions of prime integer relations or their metrics.*

If it were the case, then constants of nature would be understood without further explanations.

**VI. CONCLUSIONS**

We have shown that complex systems can be described in terms of self-organization processes of prime integer relations. The processes have the integers as the basic building blocks and controlled by arithmetic only make up the prime integer relations from one level to the higher level. In the description of complex systems prime integer relations demonstrate remarkable properties:

- following a single principle may self-organize at one level to form a prime integer relation at the higher level. A prime integer relation may participate in various self-organization processes and, as a result, be a component of different prime integer relations at the higher level;
- through the self-organization processes the prime integer relations are interconnected and inseparable in one hierarchical network. Not even a minor change can be made to any element of the network;
- control correlation structures of complex systems and may describe complex systems in a strong scale covariant form. Prime integer relations specify nonlocal and instantaneous correlations;
- provide a complexity order. The self-organization processes start with different integers and, as a result, progress to different levels thus producing the complexity order. It seems like a self-organization process of prime integer relations aims to progress as higher as possible in the direction of the order;
- can be geometrized and may be measured in renormalizable numbers by corresponding geometric patterns. The self-organization processes of prime integer relations can be isomorphically expressed through transformations of the geometric patterns. The geometrization of the prime integer relations puts forward arithmetic to irreducibly explain complex systems through quantitative means;
- determined by arithmetic only, the self-organization processes of prime integer relations can describe complex systems by information not requiring further reductions. This property could be particularly useful as irreducible arguments may be needed to explain the fundamental laws of complex physical systems.

Finally, we have presented self-organization processes of prime integer relations as a new way to describe complex systems. The processes can characterize complex systems by information not requiring further explanations. This gives the possibility to develop an irreducible theory of complex systems.

This work was supported by CQU Research Advance-ment Awards Scheme grants no. IN9022.

---

[1] L. Smolin, *Three Roads to Quantum Gravity* (Basic Books, 2001).
[2] V. Korotkikh, *A Mathematical Structure for Emergent Computation* (Kluwer, Dordrecht, 1999).
[3] V. Korotkiikh, *Integers: Irreducible Guides in the Search for a Unified Theory*, Braz. J. Phys., Special Issue on Decoherence, Information, Complexity and Entropy, 35(2B), 509 (2005).
[4] V. Korotkiikh, *Integer Code Series with Some Applications in Dynamical Systems and Complexity* (Computing Centre of the Russian Academy of Sciences, Moscow, 1993).
[5] G. Korotkiikh and V. Korotkiikh, in *Optimization and Industry: New Frontiers*, ed. by P. Pardalos and V. Korotkiikh (Kluwer, Dordrecht, 2003).
[6] V. Korotkiikh, in *Fuzzy Partial Differential Equations and Relational Equations*, ed. by M. Nikravesh, L. Zadeh and V. Korotkiikh (Springer, Berlin, 2004).
[7] V. Korotkiikh, G. Korotkiikh and D. Bond, *On Optimality Condition of Complex Systems: Computational Evidence*, arXiv:cs/0504092 April, 2005.
[8] L. Kadanoff, W. Gotze, D. Hamblen, R. Hecht, E.A.S. Lewis, V.V. Palciauskas, M. Rayl and J. Swift, Rev. Mod. Phys. 39(2), 395 (1967).
[9] K. G. Wilson and J. Kogut, Phys. Rep. C12, 75 (1974).
[10] M. Feigenbaum, Los Alamos Sci. 1, 4 (1980).
[11] J. Allouche and M. Cosnard, in *Dynamical Systems and Cellular Automata* (Academic Press, 1985).
[12] J.D. Barrow, *The Constants of Nature: From Alpha to Omega* (Jonathan Cape, London, 2002) and (Pantheon, New York, 2002); J.D. Barrow and J.K. Webb, Sci. Amer. 292, 6, 33 (2005).
[13] S. Weinberg, *Dreams of a Final Theory* (Vintage Books, New York, 1992).