Color-kinematics dual representations of one-loop matrix elements in the open-superstring effective action

Alex Edison\textsuperscript{a,b} and Micah Tegevi\textsuperscript{b}
\textsuperscript{a}Department of Physics and Astronomy, Northwestern University, Evanston, Illinois, 60208, U.S.A.
\textsuperscript{b}Department of Physics and Astronomy, Uppsala University, Box 516, 75120 Uppsala, Sweden
E-mail: alexander.edison@northwestern.edu, tegevi@protonmail.com

Abstract: The $\alpha'$-expansion of string theory provides a rich set of higher-dimension operators, indexed by $\zeta$ values, which can be used to study color-kinematics duality and the double copy. These two powerful properties, actually first noticed in tree-level string amplitudes, simplify the construction of both gauge and gravity amplitudes. However, their applicability and limitations are not fully understood. We attempt to construct a set of color-kinematics dual numerators at one loop and four points for insertions of operator combinations corresponding to the lowest four $\zeta_2$-free operator insertions from the open superstring: $\zeta_3$, $\zeta_5$, $\zeta_3^2$, and $\zeta_7$. We succeed in finding a representation for the first three in terms of box, triangle, and bubble numerators. In the case of $\zeta_7$ we find an obstruction to a fully color-dual representation related to the bubble-on-external-leg type diagrams. We discuss two paths around this obstruction, both of which signal an incompatibility between color-kinematics duality and manifesting certain desired properties. Using the constructed color-dual numerators, we find two different Bern-Carrasco-Johansson double copies that produce candidate closed-string-insertion numerators. Both approaches to the double copy match the kinematics of the cuts, with relative normalization set by either summing over both double copies including degeneracy or by including an explicit prefactor on the double-copy numerator definitions.

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1 Introduction

The study of scattering amplitudes has long built on the interplay between generating new theoretical data, manifesting different properties of the data, and finding new ways of encoding the results. One significant achievement of this progression is the discovery of double-copy relations between gauge theory and gravity amplitudes. At tree level, the connections between three different manifestations of the double copy: Kawai-Lewellen-Tye (KLT) [1], Bern-Carrasco-Johansson (BCJ) [2], and Cachazo-He-Yuan (CHY) [3], have opened up a wide range of methods for studying gauge and gravity amplitudes. All three representations can also be useful in studying loop-level amplitudes. The KLT and CHY formulations are effective at generating on-shell data for both gauge and gravity theories. The loop level incarnation of the BCJ double copy allows calculating gravitational diagram numerators directly from gauge theory numerators.

The tree-level BCJ double copy starts from the observation that when writing the four-point Yang-Mills (YM) amplitude in terms of cubic diagrams, the group Jacobi relations between the color factors of the $s$, $t$ and $u$ channels

$$c_s - c_t - c_u = 0 \quad (1.1)$$

are a necessary relation to ensure gauge invariance of the amplitude. If we then suppose that we can write a gravitational amplitude in terms of the square of some numerator factor $N_g$, linearized diffeomorphism invariance requires that

$$N_s - N_t - N_u = 0 \quad (1.2)$$
The algebraic duality between the color Jacobi relations, eq. (1.1), and eq. (1.2) is referred to as color-kinematics (CK) duality. The full BCJ double copy states that if one can write an amplitude in terms of numerators that obey eq. (1.2) for any s, t, u triplet of cubic graphs whose color obeys eq. (1.1), then a “gravitational” amplitude can be obtained by replacing all color factors with their corresponding numerators. This process can even work at loop level, with the caveat that in the color Jacobi relations one should only sum over the color contractions within the triplet. It has also been shown to work for a wide range of theories. Of interest in the current work is that when $N_g$ is a set of numerators for an n-point $\mathcal{N} = 4$ super-Yang-Mills (sYM) amplitude obeying eq. (1.2), then $N_g^2$ is an n-point numerator for $\mathcal{N} = 8$ supergravity (SG). See ref. [4] for a comprehensive review of the subject.

Recent decades have also seen a growing effort to explore ways that scattering amplitudes and string theoretic methods can inform each other. Even just limiting the scope to the $\alpha'$-expansion of the open superstring, we find a rich playground to: study higher-dimension operators, generalize the notion of double copies [5], find interesting representations of theories [6, 7], and even inform number theory [8–13]. Working in the other direction, amplitudes methods were recently applied to calculating one-loop matrix elements in both the open- and closed-string effective actions [14]. These methods were able to verify pieces of the closed-string results of D'Hoker and Green [15], produce new predictions for the open string, and target terms currently unreachable by standard string methods.

The goal of this paper is to continue building the groundwork for exploring loop-level BCJ with higher-dimension operators. We will attempt to construct CK dual numerators for open-superstring matrix elements up through $\alpha'^7$. The paper is structured as follows. We begin by reviewing the structure of amplitudes related to color-kinematics duality in section 2, including discussion on loop-level considerations and tree-level constructions in sYM and the open superstring. Next, in section 3 we demonstrate the simplicity of constructing one-loop unitarity cuts from well-structured tree amplitudes, even with the higher dimension operators of the open string low energy effective action. Finally, in section 4 we apply the method of maximal cuts [16–20] in conjunction with CK duality to construct color-dual representations for insertions of open string operators, and double copy the numerators to find representations for closed-string operator insertions.

## 2 Review

We begin with a review of the structure of tree amplitudes in (super-)Yang-Mills, with a particular eye towards the duality between color and kinematics. A tree-level gauge amplitude can be written

$$ A_{\text{tree}}^n = -i g^2 n^{-2} \sum_{g \in \Gamma_{3,n}} c_g N_g \prod_i p_{g_i}, \quad (2.1) $$

where the sum runs over the set of all cubic graphs with n external legs, $\Gamma_{3,n}$, including permutations of external legs. The index n is the number of external particles, and $g_s$ is the coupling constant and other overall normalizations for the theory. The kinematic denominator is the product of all propagators in the graph, $p_{g_i}$. The numerators are decomposed into color factors $c_g$, and kinematic factors $N_g$. The color factors are gauge
group structure constants. The kinematic contain Lorentz products of polarization vectors for the external particles, $\varepsilon_i$, as well as the momenta for the particles, $p_i$. In this paper, we will use all-outgoing external momenta.

We work in a normalization where the Mandelstam variables of massless momenta carry a factor of 2 and no explicit factors of $\alpha'$, i.e.

$$ s_{ij} \equiv 2p_i \cdot p_j = (p_i + p_j)^2 |_{p^2=0} \quad (2.2) $$

with a natural extension to multi-labeled $s$

$$ s_{ij...} \equiv (p_i + p_j + \ldots)^2 |_{p^2=0} = \sum_{a,b \in \{i,j,...\}} 2p_a \cdot p_b. \quad (2.3) $$

When off-shell momenta, like loop momenta, are included we use the sum-of-products definition

$$ s_{\ell i...} = s_{i...} + \sum_{a \in \{i,...\}} 2\ell \cdot p_a. \quad (2.4) $$

The color factors $c_g$ come from dressing every cubic-vertex in the graph with group theory structure constants $f^{abc} = \text{Tr}([T^a, T^b]T^c)$, where the $T^a$'s are generators of the gauge group. We assume that the structure constants belong to a Lie group so that they satisfy the Jacobi relations

$$ f^{12a} f^{a34} - f^{41a} f^{a23} - f^{31a} f^{a42} = 0. \quad (2.5) $$

Identifying the three products of structure constants in eq. (2.5) with the color factors for the $s$-, $t$-, and $u$-channel cubic graphs, figure 1, the Jacobi relation can be simplified to

$$ c_s - c_t - c_u = 0. \quad (2.6) $$

The full color-dressed amplitude $A_{\text{tree}}^n$ can be decomposed into an $(n-2)!$ basis of color-ordered amplitudes through judicious application of cyclicity, reflection (anti-)symmetry, the photon decoupling identity, and the Kleiss-Kuijf relations $[21–27]$. At four points, the Jacobi relation tells us that the color coefficients of the color-ordered amplitudes can be chosen as any two of $c_s, c_t$, and $c_u$.

Bern and collaborators $[2, 28, 29]$ discovered that one can arrange the kinematic numerators such that they obey the same algebraic relations as the color factors, e.g.

$$ N \left( \begin{array}{c} b \\ a \\ c \\ d \\ e \\ b \end{array} \right) - N \left( \begin{array}{c} b \\ a \\ c \\ d \end{array} \right) - N \left( \begin{array}{c} b \\ a \\ c \\ d \\ e \\ b \end{array} \right) = 0, \quad (2.7) $$

Figure 1. The $s$-, $t$-, and $u$-channel cubic 4-point tree level graphs.
for all diagrams that only differ on the four-point sub diagram. Further, once such numerators are found, gravity amplitudes can be constructed via a “BCJ double copy” [2] by simply replacing the $c_g$ in eq. (2.1) with a second copy of $N_g$

$$A^{\text{tree}}_n(1, 2, \ldots, n) = \sum_{g \in \Gamma_{3,n}} c_g N_g \Rightarrow M^{\text{tree}}_n(1, 2, \ldots, n) = \sum_{g \in \Gamma_{3,n}} \frac{N_g^2}{D_g}.$$  

Since the numerator relations, eq. (2.7), are algebraically the same as the group theory Jacobi relations, they are referred to as the kinematic Jacobi relations. A set of numerators which obey all possible kinematic Jacobi relations are referred to as color-kinematics (CK) dual numerators. Asymmetric double copies are also possible between two distinct theories, as long as they share a basis of cubic graphs and at least one of theories is written in terms of CK dual numerators. There are a wide variety of methods for calculating manifestly CK dual numerators at tree level [3, 30–39], one of which we will recap below in section 2.2. Significant work has also gone into finding double copy representations of a wide range of theories; we direct interested readers to ref. [4], a recent comprehensive review of the topic.

### 2.1 CK duality at one loop

The general structure of a loop level amplitude is a straightforward extension of the tree-level structure (dropping the coupling for simplicity)

$$A^{L-\text{loop}}_n = \sum_{g \in \Gamma_{(L),n}} \int \left( \frac{d^D\ell}{(2\pi)^D} \right)^L \frac{1}{S_g} \frac{c_g N_g}{\prod_i p_{g_i}^2},$$

where the diagram sum is now over diagrams with exactly $L$ loops, and the symmetry factors $S_g$ are the same ones that appear in Feynman diagrams and take care of any over-count due to external orderings or internal automorphisms. Since the amplitude is again written in terms of pairings between color and kinematic structures, color kinematics duality can still be explored and exploited to simplify both gauge theory and gravity calculations. The gauge theory simplifications come from the fact that the kinematic Jacobi relations can be used to express all cubic loop diagram numerators in terms of a small number of basis diagrams. This can be useful even in the one-loop four-point case. The relevant graph topologies that will enter our calculations are the box, the triangle, and the 2-2 bubble

$$\Gamma^{(1)}_{3,4} = \left\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array} \right\} + \text{ relabelings}.$$  

Through the kinematic Jacobi relations, we can relate the triangle and bubble numerators to different labelings of the box numerator. We start by applying the Jacobi relation on the edge connecting externals 1 and 2 of the box-diagram with color-order 1234, giving the
following relation between diagrams

\[
\begin{align*}
- \quad & \quad -
\end{align*}
\]

\[= 0. \quad (2.11)\]

Expressed as the numerators of the graphs, eq. (2.11) fixes the numerator for the triangle graph, \( N_\Delta(1234) \), in terms of the two box numerators

\[
N_\Box(1234) - N_\Box(1243) - N_\Delta(1234) = 0
\]

\[\Rightarrow N_\Delta(1234) = N_\Box(1234) - N_\Box(1243). \quad (2.12)\]

When referring to functions of loop diagrams (both numerators and cuts), we use vertical bars to separate vertices in the loop, commas to separate external legs which meet at a three-point vertex, and concatenated labels for when external legs meet at a higher-point vertex. We can get another relation involving the 2-2 bubble diagram by applying the Jacobi relation on the loop edge connecting externals 3 and 4 of the triangle

\[
\begin{align*}
- \quad & \quad -
\end{align*}
\]

\[= 0. \quad (2.13)\]

This relates the bubble numerator, \( N_\Box(1234) \), to the triangle and, using the relation we just found in eq. (2.12), the box numerators

\[
N_\Delta(1234) - N_\Delta(2134) - N_\Box(1234) = 0
\]

\[\Rightarrow N_\Box(1234) = N_\Box(1234) - N_\Box(2134)
\]

\[= N_\Box(1234) - N_\Box(1243) - N_\Box(2134) + N_\Box(2143). \quad (2.14)\]

Since eqs. (2.12) and (2.14) are used to define triangle and bubble numerators in terms of the box numerator, we refer to them as the defining kinematic Jacobi relations. In addition to these defining relations, there are also boundary kinematic Jacobi relations that relate a difference of two non-zero diagrams to bubble-on-external-leg (BEL) or tadpole diagrams, whose numerators we often set to 0. The main boundary relation stems from applying the Jacobi relation on one of the triangle edges only adjacent to one external leg, which yields

\[
\begin{align*}
- \quad & \quad -
\end{align*}
\]

\[= 0. \quad (2.15)\]

In the “snail” regularization of ref. [19], the third diagram in the relation is either exactly zero or carries a numerator factor of \( k_1^2 \sim 0 \) meaning that the kinematic Jacobi relation reduces to a consistency condition on the triangle numerators. Alternatively, one can use
“Minahaning” regularization [40, 41], in which the BEL numerators are not defined to be 0 (and can thus be defined via eq. (2.15)), but are regulated by relaxing momentum conservation from $n$-point to $n+1$-point and requiring that they conspire between themselves on the cuts and at the level of the integrand to cancel the unphysical $k_1^{-2}$ pole. We adopt the “snail” regularization in this work, and will generally assume that the BEL numerator is identically 0 unless explicitly stated otherwise.

While the four-point one-loop Jacobi relations are rather simple, the same basic method can become quite complicated yet powerful for higher loops and legs. For example, the four-loop four-point integrand of maximal sYM can be expressed in terms of 85 diagrams. The numerators for 83 of these diagrams can be expressed in terms of only two planar diagrams, or one non-planar diagram [19]. Beyond four loops there are problems manifesting CK duality [42], and effort has been put into finding more-accessible examples for testing the assumptions that go into finding CK numerators [43–47].

2.2 Tree amplitudes with manifest CK duality in sYM and the open superstring

A systematic and efficient method of constructing tree amplitudes that manifests their double copy structures is using the Cachazo, He, and Yuan (CHY) formalism [30]. In this form, tree amplitudes are given by an integral over $n$ punctures on a sphere against a product of theory-dependent half-integrands $I_L(\sigma), I_R(\sigma)$

$$A_n = \int \frac{d^n\sigma}{\text{Vol}(\text{SL}(2,\mathbb{C}))} \prod_a' \delta \left( \sum_{b \neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} \right) I_L(\sigma) I_R(\sigma).$$

The $\prod'\delta(\ldots)$ fully localizes the integrals, and the arguments of the $\delta(\ldots)$ are known as the scattering equations. For (super-)Yang-Mills, one of the half-integrands encodes the color information and the other half-integrand encodes the kinematics, while for (super)gravity both half-integrands carry kinematics. In both cases, the integrals can be resolved in terms of bi-adjoint scalar amplitudes $m(\alpha|\beta)$ [3, 48–50] as

$$A^\text{tree}_{\text{sYM}} = \sum_{\sigma,\rho \in S_{n-2}} c \left( \begin{array}{ccc|ccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho(1) & \rho(2) & \cdots & \cdots & \cdots & \cdots \\ \sigma(1) & \sigma(2) & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right) m(1, \rho, n|1, \sigma, n) N \left( \begin{array}{ccc|ccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array} \right)$$

$$A^\text{tree}_{\text{GR}} = \sum_{\sigma,\rho \in S_{n-2}} \tilde{N}(1, \rho, n) m(1, \rho, n|1, \sigma, n) N(1, \sigma, n)$$

where the sums run over permutations of $2, 3, \ldots, n-1$, the $c(1, \rho, n)$ are DDM half-ladder color structures [51], and the $N, \tilde{N}$ are corresponding kinematic half-ladder numerators. These two equations manifest the double-copy relations between gauge theory and gravity: a gravitational amplitude can be obtained by simply replacing the color structures of eq. (2.17) with a second copy of the kinematic numerator. The (s)YM representation also manifests the decomposition into an $(n-2)!$ basis of color-ordered amplitudes [27, 51], by simply...
restricting to a single element in the sum over $\rho$ and stripping the color factor

$$A_{sY M}^\text{tree}(1, \rho, n) = \sum_{\sigma \in S_{n-2}} m(1, \rho, n|1, \sigma, n)N(1, \sigma, n). \quad (2.19)$$

The $N(\beta)$ are functions of external polarizations and momenta, and are half-ladder basis numerators for the BCJ relations: all non-half-ladder graph numerators can be obtained from them via repeated application of the kinematic Jacobi relations, eq. (2.7). For example, at four points we have

$$A_{sY M}^\text{tree}(1, 2, 3, 4) = \frac{1}{s} N(1, 2, 3, 4) - \frac{1}{t} N(1, 3, 2, 4), \quad (2.20)$$

so that the $N(1, \sigma, 4)$ are equivalent to the numerators in the standard presentation of the amplitude of eq. (2.1)

$$N(1, 2, 3, 4) \equiv N_s \quad N(1, 3, 2, 4) \equiv N_u \quad N_t \equiv N(1, 2, 3, 4) - N(1, 3, 2, 4). \quad (2.21)$$

The composition of $N(\ldots)$ is dependent on the theory under consideration. Since the original presentation by CHY, significant work has gone into finding efficient calculations and compact representations of $N(\beta)$ for (super-)Yang-Mills \cite{31–36, 52, 53}. Most convenient for the current work is the construction developed by one of the current authors and Teng \cite{35}, which calculates the numerators via auxiliary combinatoric structures known as increasing trees. The resulting numerators for external gluons can be rearranged as

$$N(1, \sigma, n) = \sum_{\sigma = A \cup B} \left( \varepsilon_1 \cdot f_{b_1} \cdot f_{b_2} \cdots f_{b_{|B|}} \cdot \varepsilon_n \right) \mathfrak{N}_\sigma(1; A; n) \quad (2.22)$$

with linearized field strengths $f_{bi}^{mn} = p_i^m \varepsilon_i^n - p_i^n \varepsilon_i^m$, and where $\mathfrak{N}_\sigma(1; A; n)$ is independent of $n$ and dependent on the momenta of 1 and $A$, but only the polarizations of $A$. The exact content of $\mathfrak{N}_\sigma(\ldots)$ will not be relevant for the current discussion; we will only encounter $\mathfrak{N}_\sigma(a; \emptyset; b) = 1$ here. There is also a natural extension when particles 1 and $n$ are fermions

$$N(1_f, \sigma, n_f) = \sum_{\sigma = A \cup B} \left( \chi_1 f_{b_1} \cdots f_{b_{|B|}} \xi_n \right) \mathfrak{N}_\sigma(1; A; n) \quad (2.23)$$

with fermion polarizations

$$\not\!p_i \cdot \chi_i = 0, \quad \not\!p_n \cdot \xi_n = \chi_n \quad (2.24)$$

and $\gamma$-slash conventions

$$p \equiv p^\mu \gamma_\mu, \quad f_i \equiv \frac{1}{8} f_i^{\mu\nu} [\gamma_\mu, \gamma_\nu] = \frac{1}{2} \not\!p_i \not\!f_i \quad (2.25)$$

which is convenient for computing loops in sYM \cite{14, 54}. With an eye towards maximal sYM in $D = 10$ and its dimensional reductions, it is most straightforward to work in the $16 \times 16$ Weyl representation of the 10D Dirac matrices, in which case the fermion polarizations
will be 16-component SO(1, 9) spinors. We will shorthand the products of $f$s appearing in eqs. (2.22) and (2.23) as

\[ W_{2g}(1, B, n) = \varepsilon_1 \cdot f_{B_1} \cdots f_{B_{|B|}} \cdot \varepsilon_n, \]  
(2.26)

\[ W_{2f}(1_f, B, n_f) = \chi_1 f_{B_1} \cdots f_{B_{|B|}} \varepsilon_n. \]  
(2.27)

The particular usefulness of this representation for the task at hand stems from the fact that the polarizations for 1 and $n$ are structurally distinguished from the other particles, and thus behave well when used as the sewn legs in the construction of unitarity cuts.

The CHY integral representation for sYM, eq. (2.16), also has an interpretation as the \( \alpha' \to 0 \) limit of superstring tree amplitudes computed via disk integrals. We limit ourselves to a brief presentation here, and refer the reader to ref. [14] and references therein for a deeper discussion of the subject. From the disk integral representation of open-superstring amplitudes

\[ A_{\text{tree}}^{\text{super}}(1, 2, \ldots, n) = \int_{0<z_2<\ldots<z_{n-2}} \frac{dz_2 dz_3 \cdots dz_{n-2}}{\text{Vol}(\text{SL}(2, \mathbb{R}))} \langle V_1(z_1) \cdots V_n(z_n) \rangle \]  
(2.28)

the worldsheet correlator \( \langle V_1(z_1) \cdots V_n(z_n) \rangle \) can be rearranged with the help of total $z$ derivatives to collect all $\alpha'$ dependence into the Koba-Nielsen factor

\[ A_{\text{tree}}^{\text{super}}(1, 2, \ldots, n) = \int_{0<z_2<\ldots<z_{n-2}} \frac{dz_2 dz_3 \cdots dz_{n-2}}{\text{Vol}(\text{SL}(2, \mathbb{R}))} \prod_{1 \leq i < j}^{n-1} |z_i - z_j|^{2\alpha' k_i k_j} \times \sum_{\rho, \tau \in S_{n-3}} A_{\text{tree}}^{\text{sYM}}(1, \rho, n - 1, n) \frac{S_0[\rho|\tau]}{(z_1 - z_\tau) \cdots (z_n - z_{n-1})}, \]  
(2.29)

where \( S_0[\rho|\tau] \) is the field-theory KLT matrix, and is related to the matrix inverse of \( m(\alpha|\beta) \) with legs $1, n-1, n$ at fixed positions in $\alpha, \beta$ [55, 56]. The \( A_{\text{tree}}^{\text{sYM}} \) and \( S_0[\rho|\tau] \) are independent of the $z_i$. As such, all of the $z_i$ and $\alpha'$ dependent terms can be collected into a single object [57, 58]

\[ Z(a_1, a_2, \ldots, a_n|b_1, b_2, \ldots, b_n) = \int_{z_{a_1} < z_{a_2} < \ldots < z_{a_n}} \frac{dz_1 dz_2 \cdots dz_n}{\text{Vol}(\text{SL}(2, \mathbb{R}))} \prod_{1 \leq i < j}^{n-1} |z_i - z_j|^{2\alpha' k_i k_j} \]  
(2.30)

known as $Z$ functions or $Z$-theory amplitudes. Explicit expressions for the $Z$-theory amplitudes can be computed using the BGap package [59] or extracted from the $F$ matrix representation [8, 60, 61]. In the limit $\alpha' \to 0$, $Z$-theory amplitudes become bi-adjoint amplitudes

\[ \lim_{\alpha' \to 0} Z(\rho|\sigma) = m(\rho|\sigma). \]  
(2.31)

Using $Z$ functions, the open-superstring tree amplitude can be written as

\[ A_{\text{super}}(a_1, \ldots, a_n) = \sum_{\rho, \tau \in S_{n-3}} Z(a_1, \ldots |1, \rho, n, n-1) S_0[\rho|\tau] A_{\text{sYM}}(1, \tau, n-1, n). \]  
(2.32)
Thus we see that sYM tree amplitudes can be promoted to open-superstring tree amplitudes
the entries of which contain the kinematic information paired with $\zeta$ values. The matrices $P$ and $Q$ are
infinite series containing all of the $\zeta_k$ and the higher-depth multiple-$\zeta$ values, respectively.
The entries of $M_w$ are homogenous polynomials of degree $w$ in $s_{ij}$ with rational coefficients.
The ordering colons prescribe that all products of $M_i$ resulting from the expansion of the
exponential only appear in descending order of $i$, i.e. we only have $M_1M_2 \ldots$ when $i_1 > i_2 > \ldots$

Through the use of eq. (2.19) and the relation between $S_0$ and $m(\alpha|\beta)$ we can further rewrite eq. (2.32) as
$$A_{\text{tree}}^{\text{super}}(a_1, \ldots, a_n) = \sum_{\rho, \tau \in S_{n-3}} Z(a_1 \ldots |1, \rho, n, n-1) S[\rho|\tau]_1 \sum_{\sigma \in S_{n-2}} m(1, \tau, n-1, n|1, \sigma, n) N(1, \sigma, n)$$
$$= \sum_{\rho \in S_{n-2}} Z(a_1 \ldots |1, \rho, n) N(1, \rho, n).$$

Thus we see that sYM tree amplitudes can be promoted to open-superstring tree amplitudes
by simply upgrading bi-adjoint amplitudes to Z-theory amplitudes. Further, we can interpret eq. (2.32) as writing the open superstring as a double copy between the amplitudes of Z-theory and super-Yang-Mills. Expanding both sides of the double copy in $\alpha'$ leads to the schematic identification of operators between the respective effective actions [62]

\begin{align}
(\alpha')^0 : F^2 &= sYM \otimes \phi^3, \\
\zeta_2(\alpha')^2 : F^4 &= sYM \otimes (\partial^2 \phi^4 + \phi^5), \\
\zeta_3(\alpha')^3 : D^2 F^4 + F^5 &= sYM \otimes (\partial^4 \phi^4 + \partial^2 \phi^5 + \phi^6), \\
\zeta_4(\alpha')^4 : D^4 F^4 + D^2 F^5 + F^6 &= sYM \otimes (\partial^6 \phi^4 + \partial^4 \phi^5 + \partial^2 \phi^6 + \phi^7).
\end{align}

For the first four orders in $\alpha'$, there is only one combination of operators that appears. Starting at $\alpha'^5$ multiple combinations appear. However, these operators can be grouped
according to their $\zeta$-value coefficients [63–67]. For instance, $\alpha'^5$ dresses two different operator combinations, one with coefficient $\zeta_2 \zeta_3$ and another with $\zeta_5$ [8]. Throughout the rest of this work, we will often refer to terms in the expansion of Z-theory and their corresponding operator insertions via their $\zeta$-value coefficient, e.g. we will talk about $\zeta_3$ insertions as a shorthand for insertions of the $D^2 F^4 + F^5$ operator combination appearing in Z theory at $\alpha'^3$. Similarly, we will often index objects by the $\zeta$ value of the operator they correspond to, as in $Z_{\zeta_3}$ or $N_{\zeta_5}^3$.

Closed-string amplitudes can be expanded in a very similar manner using the KLT
relation [1] which equates closed-string amplitudes to products of open-string amplitudes
via the $\alpha'$ uplift of the kinematic kernel $S_0 \to S_{\alpha'}$

$$M_{\text{tree}}^\text{closed} = (A_{\text{tree}}^\text{open})^T \cdot S_{\alpha'} \cdot A_{\text{tree}}^\text{open}. \quad (2.36)$$

Inserting eq. (2.33), noting that the transposition operation reverses the color ordering prescription, and with the help of $P^T S_{\alpha'} P = S_0$ and $M_i^T S_0 = S_0 M_i$ we get [8]

$$M_{\text{tree}}^\text{closed} = (A_{\text{sYM}}^\text{tree})^T S_0 (1 + 2 \zeta_3 M_3 \cdots + 2 \zeta_3 \zeta_5 (M_5 M_3 + M_3 M_5) + \cdots) A_{\text{sYM}}^\text{tree}. \quad (2.37)$$

Making use of eq. (2.19), we find a multiplication of two sYM numerator factors via the matrix kernel

$$m \cdot S_0 \cdot (1 + 2 \zeta_3 M_3 + \cdots) \cdot m \text{ (suppressing the various permutation labels)},$$

which can be identified as the result of applying the single-valued map “sv” [68, 69]¹

$$\text{sv} 1 = 1 \quad \text{sv} \zeta_{2k} = 0 \quad \text{sv} \zeta_{2k+1} = 2 \zeta_{2k+1} \quad \text{for} \ k \in \mathbb{N} \quad (2.38)$$

order-by-order in $\alpha'$ on the $Z$-functions [8–13]

$$m(\gamma, \ldots) \cdot S_0 \cdot (1 + 2 \zeta_3 M_3 + \ldots) \cdot m(\ldots | \rho) = \text{sv} Z(\gamma | \rho), \quad (2.39)$$

leading to a compact representation of the closed-string tree amplitude [58]

$$M_{\text{tree}}^\text{closed} = \sum_{\sigma, \rho \in S_{n-2}} \tilde{N}(1, \rho, n) \cdot [\text{sv} Z(1, \rho, n | 1, \sigma, n)] \cdot N(1, \sigma, n). \quad (2.40)$$

### 3 Unitarity cuts in bi-adjoint and $Z$-theory decompositions

#### 3.1 Open string

As alluded to above, the packaging of tree amplitudes into a propagator-matrix decomposition via eqs. (2.19) and (2.34) and the further arrangement of the numerator terms is particularly conducive to calculating one-loop unitarity cuts, which we now demonstrate. We restrict our attention to four points and only external gluons in the current work.

Generically, the unitarity cut for a graph $G$ with vertices $V(G)$ and edge $E(G)$ is given by

$$C(G) = \sum_{\text{states over } E(G)} \prod_{v \in V(G)} A_{\text{tree}}^\text{open}(v). \quad (3.1)$$

Immediately restricting to one-loop four-point, one of the most useful cuts is the two-particle (2P) cut, which is given as

$$C(12|34) = \sum_{\text{states}} A_{\text{tree}}^\text{open}(\ell_1, 1, 2, -\ell_2) A_{\text{tree}}^\text{open}(\ell_2, 3, 4, -\ell_1) \quad (3.2)$$

¹Since we only deal with single $\zeta$s in this work, we will omit discussion of the subtleties related to the “sv” map and multiple $\zeta$ values.
in the planar color ordering. The gluonic state sums are resolved as \[54, 70, 71\]

\[
\sum_{\text{states of } \ell_i} \mathcal{C}_{\mu
u} \varepsilon^\mu_{\ell_i} \varepsilon^\nu_{-\ell_i} + \mathcal{C} (\varepsilon_{\ell_i} \cdot \varepsilon_{-\ell_i}) \rightarrow \mathcal{C}_{\mu
u} \eta^{\mu\nu} + (D - 2)\mathcal{C} \tag{3.3}
\]

where \(\mathcal{C}_{\mu
u}\) is the component of \(\mathcal{C}(12|34)\) in which \(\varepsilon^\mu_{\ell_i} \varepsilon^\nu_{-\ell_i}\) are not contracted with each other, and \(\mathcal{C}\) the one in which they are. Note that during the sewing of the first leg, \(\mathcal{C} = 0\) since \(\varepsilon_{\ell_i}\) and \(\varepsilon_{-\ell_i}\) come from separate tree amplitudes and thus cannot begin contracted. However, after sewing the first leg subsequent legs may have been contracted by previous sewings. The fermionic state sums are similarly \([54, 70]\)

\[
\sum_{\text{states of } \ell_i} \chi^{\alpha}_{\ell_i} (\xi_{\ell_i})^{\beta} = \delta^{\alpha\beta}. \tag{3.4}
\]

Amplitudes expressed via eqs. (2.22) and (2.23) are particularly well-suited to applying state sums on legs \(1\) and \(n\), with the state sums closing chains of \(W_s\) into appropriate traces.

In the case of 2P cuts, we find

\[
\sum_{\text{states}} W_{2g}(\ell_1, X, -\ell_2) W_{2g}(\ell_2, Y, -\ell_1) = \begin{cases} D - 2 & |X| = |Y| = 0 \\ \text{tr}_v(X, Y) & \text{otherwise} \end{cases} \tag{3.5}
\]

where

\[
\text{tr}_v(X, Y) = f^{\mu}_{x_1 \nu} f^{\nu}_{x_2 \rho} \cdots f^{\sigma}_{y_1 \rho}, \tag{3.6}
\]

for sewing gluons, while for fermions

\[
\sum_{\text{states}} W_{2f}(\ell_1, X, -\ell_2) W_{2f}(\ell_2, Y, -\ell_1) = \begin{cases} \text{tr}_s(I) = 2^{D-1} & |X| = |Y| = 0 \\ \text{tr}_s(X, Y) & \text{otherwise} \end{cases} \tag{3.7}
\]

with

\[
\text{tr}_s(X, Y) = \left(f_{x_1} \cdots f_{y_1} \cdots \right)^{\alpha}_{\beta} \delta^{\beta}_{\alpha}, \tag{3.8}
\]

following our Dirac matrix conventions eq. (2.25). The numerator factors \(\Re\) are unaffected other than renaming the legs.

These trace structures were studied in detail in ref. \([54]\), and are particularly well-behaved in the case of maximal SUSY\(^3\)

\[
\sum_{\text{states}} W_s(\ell_1, X, -\ell_2) W_s(\ell_2, Y, -\ell_1) = \begin{cases} 0 & |X| + |Y| < 4 \\ \text{tr}_v(X, Y) - \frac{1}{2} \text{tr}_s(X, Y) & \text{otherwise} \end{cases} \tag{3.9}
\]

The relative factor of \(-\frac{1}{2}\) between the two traces is discussed in more detail in refs. \([41, 54]\). Further, when \(|X| + |Y| = 4\) the combination of traces is the permutation invariant contraction of four field strengths via the \(t_8\)-tensor:

\[
\text{tr}(1234) - \frac{1}{2} \text{tr}_s(1234) = \frac{1}{2} \left( \text{tr}(1234) + \frac{1}{4} \text{tr}(12) \text{tr}(34) + \text{cyc}(234) \right) = \frac{1}{2} t_8(1, 2, 3, 4). \tag{3.10}
\]

\(^2\)This is not strictly true, but along with the other prescriptions below properly captures the parity even part of the state sewing. See \([41]\) for a more comprehensive treatment.

\(^3\)For ease of discussion we focus on \(\mathcal{N} = 1\) sYM in \(D = 10\), but as demonstrated in \([54]\) the various dimensional reductions with maximal SUSY behave identically, with a minor change in the way in which the cancellations occur for \(|X| + |Y| = 0\).
We will ignore the overall normalization factor of \( \frac{1}{2} \) from now on, as it will appear on all cuts calculated in this manner so can be absorbed into the definition of the coupling constants.

Thus, we have all of the components needed to evaluate the four-point 2P cut from eq. (3.2) in maximal sYM
\[
\mathcal{C}_{\text{SYM}}(12|34) = \sum_{\text{states}} A(\ell_1, 1, 2, -\ell_2) A(\ell_2, 3, 4, -\ell_1) = \sum_{\rho \in S_{(1,2)}} m(\ell_1, 1, 2, -\ell_2|\ell_1, \rho, -\ell_2)m(\ell_2, 3, 4, -\ell_1|\ell_2, \sigma, -\ell_1) \\
\times \mathcal{M}_\rho(\ell_1; \emptyset; -\ell_2) \mathcal{M}_\sigma(\ell_2; \emptyset; -\ell_1) \sum_{\text{states}} W_\rho(\ell_1, \rho, -\ell_2) W_\sigma(\ell_2, \sigma, -\ell_1) \quad (3.11)
\]

From ref. [35] we identify \( \mathcal{M}(\alpha; \emptyset; b) = 1 \). Additionally, the state sum is resolved to \( t_8(1, 2, 3, 4) \) as discussed above, giving the well-known result
\[
\mathcal{C}_{\text{SYM}}(12|34) = t_8(1, 2, 3, 4) \sum_{\rho \in S_{(1,2)}} m(\ell_1, 1, 2, -\ell_2|\ell_1, \rho, -\ell_2)m(\ell_2, 3, 4, -\ell_1|\ell_2, \sigma, -\ell_1) \\
= t_8(1, 2, 3, 4) \left[ \frac{1}{s_{\ell_11}} + \frac{1}{s_{\ell_12}} - \frac{1}{s_{\ell_21}} \right] \times \left[ \frac{1}{s_{\ell_23}} + \frac{1}{s_{\ell_34}} - \frac{1}{s_{\ell_41}} \right] = t_8(1, 2, 3, 4) \frac{t_8(1, 2, 3, 4)}{s_{\ell_11} s_{\ell_23}}. \quad (3.12)
\]

This construction is straightforward to extend from sYM trees to superstring higher-dimension-operator insertions by noting that the doubly-color-ordered amplitudes \( m(\cdot|\cdot) \) are given by the \( \alpha' \to 0 \) limit of \( Z(\cdot|\cdot) \). Thus, to calculate a cut for an open-superstring matrix element, all the steps in eq. (3.12) remain the same save the evaluation of the \( m(\cdot|\cdot) \) which is replaced by the relevant \( Z(\cdot|\cdot) \) expansions. Importantly, the operator identification via series expansion and \( \zeta \) value coefficient matching must be taken at the level of the entire cut, e.g. for \( \zeta_4 \propto \zeta_2^2 \leftrightarrow D^4 F^4 + \ldots \) we must sum over \( Z_1(L) Z_{\zeta_4}(R), Z_{\zeta_4}(L) Z_{\zeta_4}(R) \) and \( Z_{\zeta_4}(L) Z_1(R) \) to get the correct cut.

For a simple example, we compute the \( \zeta_2 \leftrightarrow t_8 F^4 \) 2P cut via the above method
\[
\mathcal{C}_{\zeta_2}^{\text{open}}(12|34) = t_8(1, 2, 3, 4) \sum_{\rho \in S_{(1,2)}} Z(\ell_1, 1, 2, -\ell_2|\ell_1, \rho, -\ell_2) Z(\ell_2, 3, 4, -\ell_1|\ell_2, \sigma, -\ell_1) \big|_{\zeta_2} \\
= t_8(1, 2, 3, 4) \sum_{\rho \in S_{(1,2)}} \left( m(\ell_1, 1, 2, -\ell_2|\ell_1, \rho, -\ell_2) Z_{\zeta_2}(\ell_2, 3, 4, -\ell_1|\ell_2, \sigma, -\ell_1) \right) \\
= -t_8(1, 2, 3, 4) s_{12} \left( \frac{1}{s_{\ell_11}} + \frac{1}{s_{\ell_32}} \right) \quad (3.13)
\]

As expected, we find that this two-particle cut supports additional residues corresponding to triangle topology cuts, but not the two simultaneous residues to have non-zero contribution to the box topology.

The triangle and box cuts can be extracted by taking additional residues on the 2P cut. The 1-3 bubble cut and one-particle cut can be calculated in a similar manner via
where we identify \( \varepsilon \) with vertex separately to cancel potentially-singular terms before applying the overall four-point polarization [72].

### 3.2 Closed string

Gravitational cuts at one loop are nearly as simple as gauge theory cuts. The abstract definition of unitarity cuts via products of trees is the same, except with gravity tree amplitudes as the constituent objects, e.g. in the 2P case

\[
\mathcal{C}_{\text{grav}}(12|34) = \sum_{\text{states}} \mathcal{M}_{\text{tree}}(\ell_1, 1, 2, -\ell_2) \mathcal{M}_{\text{tree}}(\ell_2, 3, 4, -\ell_1).
\]

(3.14)

Since gravitons are not colored, the tree amplitudes are fully permutation invariant, and the cut is invariant under \( 1 \leftrightarrow 2 \) and \( 3 \leftrightarrow 4 \). As spin-two particles, the state sum for gravitons is more complicated than the one for gauge bosons [71]

\[
\sum_{\text{states of } \ell_i} \mathcal{C}(\mu_\nu, i) \varepsilon^{i \mu, \nu} - \frac{2}{D-2} \varepsilon^{i \mu, \nu} + \mathcal{C}(\mu_\nu, \ell_i, \ell_i, \mu_\nu, 2) \varepsilon^{\ell_i, \mu, \ell_i, \nu}
\]

\[
\rightarrow \frac{1}{2} \left( \varepsilon^{i \mu, \nu} + \varepsilon^{i \nu, \mu} - \frac{2}{D-2} \varepsilon^{i \mu, \nu} \right) + \frac{D(D-3)}{2(D-2)} \mathcal{C}_{\mu} + \frac{D(D-3)}{2} \mathcal{C},
\]

with \( \varepsilon^{\mu \nu} \) the graviton polarizations and the various \( \mathcal{C} \) are pieces of the cut with the specified polarization contractions. Using eq. (2.18) for the tree amplitudes and evaluating the state sums in eq. (3.14) for maximal supergravity yields

\[
\mathcal{C}_{\text{max SG}}(12|34) = t_8(1, 2, 3, 4)^2 \sum_{s \in S_{1,2}} \sum_{t \in S_{1,2}} m(\ell_1, \alpha, -\ell_2|\ell_0, \rho, -\ell_2) m(\ell_2, \beta, -\ell_0|\ell_2, \sigma, -\ell_0)
\]

\[
= t_8(1, 2, 3, 4)^2 \left( \frac{1}{s_{\ell_1,1}s_{\ell_2,3}} + \frac{1}{s_{\ell_1,2}s_{\ell_2,3}} + \frac{1}{s_{\ell_1,1}s_{\ell_2,4}} + \frac{1}{s_{\ell_1,2}s_{\ell_2,4}} \right)
\]

(3.16)

where we identify \( t_8(1, 2, 3, 4)^2 = (stA_{\text{SYM}}(1, 2, 3, 4))^2 = stuM_{\text{GR}}(1, 2, 3, 4) \). Similar to lifting sYM→open superstring, gravity cuts also have a natural uplift to string-insertion cuts via replacing the gravity tree amplitudes with closed-string ones, eq. (2.40),

\[
\mathcal{C}_{\text{closed}}(12|34) = t_8(1, 2, 3, 4)^2 \sum_{s \in S_{1,2}} \sum_{t \in S_{1,2}} s_{\ell_1} s_{\ell_2} Z(\ell_1, \alpha, -\ell_2|\ell_1, \rho, -\ell_2) Z(\ell_2, \beta, -\ell_1|\ell_2, \sigma, -\ell_1)
\]

(3.17)

A direct calculation of closed-string-insertion cuts is invaluable for verifying the double-copy behavior of our CK-dual numerators. In particular, it is \textit{a priori} unclear exactly how

\textsuperscript{4}Showing this result via direct state sewing in maximal supergravity is somewhat tedious. However, for maximal supersymmetry no information is lost or corrupted by performing the state sums separately on each of the two sYM numerators \( N \) and \( N \) and then identifying \( \varepsilon_i \varepsilon_i = \varepsilon_i \mu \nu \) as the external graviton polarizations [72].
to apply the BCJ double copy to get the desired closed-string insertion. Some natural choices are

\[ N^{\text{closed}}_{\zeta_1 \zeta_2}(g) \propto N^{\text{YM}}_{\zeta_1 \zeta_2}(g) N^{\text{open}}_{\zeta_3 \zeta_4}(g) \]  

(3.18)

\[ N^{\text{open}}_{\zeta_1 \zeta_2}(g) \propto N^{\text{open}}_{\zeta_3 \zeta_4}(g) N^{\text{open}}_{\zeta_3 \zeta_4}(g) \]  

(3.19)

or a linear combination of the two. The first is calculationally desirable, as at four points we have

\[ N^{\text{SYM}}(g) = \begin{cases} 1 & g = \text{perms of } 1|2|3|4 \\ 0 & \text{otherwise} \end{cases} \Rightarrow N^{\text{closed}}_{\zeta_1 \zeta_2}(g) \propto \begin{cases} N^{\text{open}}_{\zeta_3 \zeta_4}(g) & g = \text{perms of } 1|2|3|4 \\ 0 & \text{otherwise} \end{cases} \]  

(3.20)

i.e. the closed-string insertion numerator would only be written in terms of box diagrams.

An important thing to note is that both eq. (3.18) and eq. (3.19) can be valid double copies, while eq. (3.20) implies that

\[ N^{\text{YM}}(g) N^{\text{open}}_{\zeta_1 \zeta_2}(g) \propto N^{\text{open}}_{\zeta_1 \zeta_2}(g) N^{\text{open}}_{\zeta_3 \zeta_4}(g) ; \]  

(3.21)

they would only need to agree on the cuts such that the actual amplitudes would only differ by a normalization.

We use “∝” in all of these relations because the CK construction does not care about the overall normalization of the numerators. A more-complete treatment may include at least a prefactor of $sv(\zeta_1 \zeta_2)$ as part of the definition of $N^{\text{closed}}_{\zeta_1 \zeta_2}$, but then we would still be interested if the double copy choices produce different normalizations. Comparing cuts constructed from eqs. (3.18) and (3.19) against those constructed directly via eq. (3.17) can identify which combinations actually produce the desired physical observables. We will explore this in more detail in section 4.3, with explicit CK numerators in hand.

4 Construction of color-dual representations

With the foundations of the theory and cut construction established, we turn our attention to the assembly of color-kinematics-dual representations for the odd-$\zeta$-indexed matrix elements. From eqs. (2.12) and (2.14) we know that four-point one-loop color-kinematics-dual numerators can be written in terms of a single basis numerator: that of the cubic box.

We will construct our color-dual representations following the method of maximal cuts [16–20], which can be briefly summarized in the current context as:

Method of maximal cuts:

1. Assemble an ansatz for the box diagram as a homogeneous polynomial of momentum products. A quick analysis of how the $Z$-functions enter the cuts (or the contact-diagram representation of [14]) shows that the degree of the box numerator must be equal to the desired $\alpha'$ order.

2. Impose powercounting constraints on the triangle and bubble numerators (as constructed from the box numerator via kinematic Jacobi relations).

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3. Impose boundary Jacobi relations on the ansatz. These are the kinematic Jacobi relations which would relate triangles and bubbles to bubble-on-external-leg and tadpole diagrams. They generally impose that an antisymmetrization around an internal edge is zero.

4. Impose diagram symmetries on the ansatz. For instance, the triangle diagram has a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ symmetry corresponding to

$$N_\Delta(1, 2|3|4) = -N_\Delta(2, 1|3|4) \quad \text{and} \quad N_\Delta(1, 2|3|4) = -N_\Delta(1, 2|4|3)|_{\ell \rightarrow \ell+k_1+k_2}$$

In the current construction, the symmetries are always a subset of the kinematic Jacobi relations.

5. Match the cuts, as calculated via section 3. We impose the matching in order of “distance from the box”: first matching the box itself, then the triangle cut, followed by the 2-2 (product of two four-point amplitudes) and 1-3 (product of three-point and five-point amplitude) bubbles, and finally the one-particle cut.

Let us now turn our attention to the details of ansatz construction. As we saw in section 3, the external state data factors out of all cuts as an overall $\ell_s(1, 2, 3, 4)$. We require that this factorization holds for the full integrand as well, so will leave the $\ell_s$ factor implicit from now on. We will assume that the numerators are not secretly rational functions themselves, and thus will be polynomials of the external Mandelstams and $k \cdot \ell$. We use

$$\mathcal{P} = a_1s_{12} + a_2s_{23} + a_3s_{1\ell} + a_4s_{2\ell} + a_5s_{3\ell} + a_6\ell^2$$

(4.1)

to denote a degree one arbitrary polynomial in the variables, and $\mathcal{P}_n$ to refer to a degree-$n$ arbitrary polynomial. For example the degree-two ansatz $\mathcal{P}_2$ is given by

$$\mathcal{P}_2 = a_1s_{12}^2 + a_2s_{12}s_{1\ell} + a_3s_{12}s_{23} + a_4s_{12}s_{2\ell} + a_5s_{12}s_{3\ell} + a_6s_{12}\ell^2 + a_7s_{1\ell}^2 + a_8s_{1\ell}s_{23} + a_9s_{1\ell}s_{2\ell} + a_{10}s_{1\ell}s_{3\ell} + a_{11}s_{1\ell}\ell^2 + a_{12}s_{23}^2 + a_{13}s_{23}s_{2\ell} + a_{14}s_{23}s_{3\ell} + a_{15}s_{23}\ell^2 + a_{16}s_{2\ell}^2 + a_{17}s_{2\ell}s_{3\ell} + a_{18}s_{2\ell}\ell^2 + a_{19}s_{3\ell}^2 + a_{20}s_{3\ell}\ell^2 + a_{21}\ell^4.$$  

(4.2)

From the explicit representations of the $Z$-functions [8, 59, 60], it is clear that only sYM supports a maximal (box) cut. Using this observation, we can refine the initial ansatz for the box with degree $w$ as

$$N^w_{\Delta} = \ell_1^2 \mathcal{P}_{w-1} + \text{cyc}(1, 2, 3, 4)$$

(4.3)

where $\text{cyc}(1, 2, 3, 4)$ sums the various relabelings of $\mathcal{P}_{w-1}$, as well as the four possible inverse propagators

$$\ell_1^2 = (\ell + k_1)^2 \quad \ell_2^2 = (\ell + k_{12})^2 \quad \ell_3^2 = (\ell + k_{123})^2 \quad \ell_4^2 = \ell^2.$$ 

(4.4)

Note that this type of ansatz is in general overcomplete, as terms like $\ell_1^2\ell_3^2$ can be sourced from the explicit $\ell_2^2$ prefactor multiplying against the linear combination of terms from $\mathcal{P}_2$.
that build $\ell_2^4$, or from the explicit $\ell_2^2$ multiplying a different linear combination from $P_2$ generating $\ell_2^2\ell_2^2$. However, this ansatz trivializes the vanishing of the box cut as well as the cyclic symmetry properties of the box diagram. We will additionally impose that the triangle and bubble ansatz have one and two powers of $s_{12}$ respectively

$$N_\triangle(1|2|3,4) \propto s_{12} \quad N_\Box(1,2|3,4) \propto s_{12}^2$$

so that the loop amplitude has no explicit poles in $s_{ij}$. These conditions are nontrivial to impose a priori on $N_\Box$, and enter as constraints on the $a_i$.

We proceed to construct color-dual representations for the four lowest odd-only $\zeta$ values: $\zeta_3$, $\zeta_5$, $\zeta_3^2$, and $\zeta_7$. For all insertions with a factor of $\zeta_2$ we conjecture that it should be impossible to construct a standard color-dual representation due to two main reasons. First, the non-trivial monodromy relations in the open superstring manifest in the $\zeta_2$ terms but not the odd-only terms [73–75]. Second, we expect the $\zeta_2$ dropout from the closed superstring to manifest itself in the framework of numerator-level double-copies via the inability to construct double-copiable numerators for those operators. We have explicitly verified the construction failure for $\zeta_2$, $\zeta_2^2 \sim \zeta_4$, $\zeta_2\zeta_3$, $\zeta_3^2$, $\zeta_2\zeta_5$, and $\zeta_2^2\zeta_3$. These constructions fail by being unable to match $C_w(1|2|34)$ while also satisfying the color-kinematics relations.

### 4.1 Detailed construction of $\zeta_3 \leftrightarrow D^2F^4 + F^5$

We begin with a detailed account of the construction of the $\zeta_3$ representation. Examining the color-ordered cuts we find

$$C_{\zeta_3}(1|2|34) = s_{12} (s_{12} + s_{14} + s_{23} + s_{34}) \, s_{12} \, s_{14} \, s_{23} \, s_{34} \, (1|2|34) = s_{12} (s_{12} + s_{14} + s_{23} + s_{34}),$$

$$C_{\zeta_3}(12|34) = s_{12}(s_{12} + s_{14} + s_{23} + s_{34}),$$

and that the one-particle cut is purely rational, thus containing no additional information.

We can diagrammatically expand these cuts onto cubic diagrams as

$$C(1|2|34) = \begin{array}{c|c|c} 2 & 3 & 1 \\ \hline 1 & 4 & \end{array} + \begin{array}{c|c|c} 2 & 3 & 1 \\ \hline 1 & 4 & \end{array} + \begin{array}{c|c|c} 2 & 3 & 1 \\ \hline 1 & 4 & \end{array},$$

$$C(12|34) = \begin{array}{c|c|c} 2 & 3 & 1 \\ \hline 1 & 4 & \end{array} + \begin{array}{c|c|c} 2 & 3 & 1 \\ \hline 1 & 4 & \end{array} + \begin{array}{c|c|c} 2 & 3 & 1 \\ \hline 1 & 4 & \end{array},$$

(4.7) (4.8)
Here the dotted lines indicate cut propagators.

Noting that the explicit expression for the triangle cut in eq. (4.6) is a degree-2 polynomial, we see that the ansatz for \( N^{\triangle}_{\square} \) is covered by

\[
N^{\triangle}_{\square}(1|234) = \ell_1^2 \ell_2^3 + \text{cyc}(1, 2, 3, 4)
\] (4.10)

exactly as in eq. (4.2) with 21 free parameters, \( a_i \). Through the Jacobi relations, eqs. (2.11) and (2.13), we get the numerators for the cubic triangle and 2-2 bubble, \( N^{\triangle}_{\Delta} \) and \( N^{\triangle}_{\triangle} \). Imposing the power counting of eq. (4.5) fixes 10 of the free parameters, with the remaining 11 parameters only appearing in three distinct combinations. For this case the overlap of the defining Jacobi relations, eqs. (2.12) and (2.14), and power counting, eq. (4.5), turn out to be a superset of the symmetry conditions and boundary Jacobi relations; both are trivially satisfied by the resulting numerators. With the numerators defined we can start imposing the cuts.

Inserting the numerator definitions into the triangle cut, eq. (4.7), we get

\[
C^{\triangle}_{\square}(1|234) = \text{Cut}_{1;1;2} \left[ \frac{N^{\triangle}_{\Delta}(1|234)}{\ell_2^3} + \frac{N^{\triangle}_{\triangle}(1|234)}{s_{34}} \right]
\] (4.11)

where \( \text{Cut}_{1;1;2} \) imposes \( \ell_1^2 = \ell_2^3 = \ell_4^1 = 0 \). Performing the explicit matching between the left and right hand sides of eq. (4.11) fixes one of the remaining \( a_i \). Continuing to the bubbles, we have

\[
C^{\triangle}_{\square}(12|34) = \text{Cut}_{2;2} \left[ \frac{N^{\triangle}_{\Delta}(1, 234)}{s_{12} \ell_1^2} + \frac{N^{\triangle}_{\triangle}(1|234)}{s_{12} \ell_2^3} + \frac{N^{\triangle}_{\triangle}(1|234)}{s_{12} \ell_3^2} + \frac{N^{\triangle}_{\triangle}(1, 2|34)}{s_{12}^2} \right]
\] (4.12)

and

\[
C^{\triangle}_{\square}(1|234) = \text{Cut}_{1;3} \left[ \frac{N^{\triangle}_{\Delta}(1|234)}{s_{12} \ell_2^2} + \frac{N^{\triangle}_{\triangle}(1, 23|4)}{s_{23} \ell_3^2} + \frac{N^{\triangle}_{\triangle}(1, 23|4)}{s_{23} \ell_3^2} \right]
\] (4.13)

which can of course be written exclusively in terms of labelings of the box numerator via eqs. (2.12) and (2.14). The matching of the numerators against the cuts from eq. (4.6) fixes
the last two $a_i$ and we have unique color-kinematics-dual numerators for the $\zeta_3$ insertion

$$N_{\zeta_3}^C(1|2|3|4) = -4s_{12}^2s_{1\ell} + 3s_{12}^3s_{23} - s_{12}s_{1\ell}s_{23} + 2s_{1\ell}^2s_{23} + 2s_{12}s_{23}^2 + 2s_{1\ell}s_{23}^2$$

$$- 2s_{12}^3s_{2\ell} - 4s_{12}s_{1\ell}s_{2\ell} + 8s_{12}s_{23}s_{2\ell} + 2s_{23}^2s_{2\ell} - 2s_{12}s_{2\ell} - 2s_{23}s_{2\ell}$$

$$- 2s_{12}^3s_{3\ell} + 4s_{12}s_{1\ell}s_{3\ell} - 3s_{12}s_{23}s_{3\ell} + 4s_{1\ell}s_{23}s_{3\ell} - 4s_{23}s_{23}s_{3\ell}$$

$$+ 2s_{12}s_{3\ell}^2 + 4s_{12}s_{23}s_{3\ell}^2 + 4s_{23}s_{23}^2, \quad (4.14)$$

$$N_{\zeta_3}^C(1|2|3,4) = s_{12}^3 - 2s_{12}s_{1\ell} + 2s_{12}^2s_{23} - 3s_{12}s_{1\ell}s_{23} + s_{12}^2s_{23} + 3s_{12}s_{23}s_{2\ell}$$

$$- s_{12}^3s_{2\ell}, \quad (4.15)$$

$$N_{\zeta_3}^C(1,2|3,4) = 2s_{12}^3(s_{23} - s_{13}). \quad (4.16)$$

These numerators are also provided in the supplementary material attached to this paper.

4.2 Higher orders

The construction procedure for the higher $\zeta$ values remains nearly identical to the $\zeta_3$ case, just applied to larger initial ansatze. However, there is one major difference. For $\zeta_3$, we only required bubble cuts. With the higher $\zeta$s, the Z-theory power counting is such that one-particle cuts contain polynomials of $s_{\ell\ell}$, in addition to rational functions, which signals a potential contribution that is undetectable on other cuts. Thus, we must also match the cubic numerators against this class of cuts. The color structure of the contact tadpole is such that the two external orderings, $(1,2,3,4)$ and $(4,3,2,1)$ contribute to the same color ordering so to get the complete color-ordered cut we need the expansion:

$$C(1234) = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (1,0) circle (0.5cm);
\draw (0,0) -- (1,0);
\draw (0,0) -- (0.5,0.866);
\draw (0,0) -- (0.5,-0.866);
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (1,0) circle (0.5cm);
\draw (0,0) -- (1,0);
\draw (0,0) -- (0.5,0.866);
\draw (0,0) -- (0.5,-0.866);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (1,0) circle (0.5cm);
\draw (0,0) -- (1,0);
\draw (0,0) -- (0.5,0.866);
\draw (0,0) -- (0.5,-0.866);
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (1,0) circle (0.5cm);
\draw (0,0) -- (1,0);
\draw (0,0) -- (0.5,0.866);
\draw (0,0) -- (0.5,-0.866);
\end{tikzpicture}
\end{array} (4.17)
\end{array} + (\ell \leftrightarrow -\ell).$$

At $(\alpha')^2$ the boundary Jacobi relations are no longer a subset of the defining Jacobi relations and thus we need to impose their constraints on the numerator. We find that the symmetries are automatically satisfied after application of all kinematic Jacobi relations including the boundary relations.

Beyond the need to actually impose boundary Jacobi relations, the $\zeta_5$ and $\zeta_3^2$ constructions follow exactly the same process as the $\zeta_3$ construction. The triangle and bubble cuts impose a small number of conditions on the ansatz, after which the one-particle cuts are

\footnote{This can be seen by considering the six-point half-ladder color structures \cite{51} $c(a, 1, 2, 3, 4, b)$ and $c(a, 4, 3, 2, 1, b)$. Under normal circumstances these two color structures are distinct; attempting to convert one into the other eventually results in a color Jacobi identity involving a commutator of $a$ and $b$. For the tadpole color structure, $a$ and $b$ are identified and thus commute, removing the obstruction to mapping them into each other.}
satisfied without imposing additional constraints. Unlike the $\zeta_3$ numerator, the $\zeta_5$ and $\zeta_3^2$ numerators are not unique. After all conditions are imposed, the $\zeta_5$ numerator has two free parameters remaining, while the $\zeta_3^2$ numerator has only one.

At $\zeta_7$ our construction fails. The first point of trouble occurs in the matching of $C_{\zeta_7}(12|34)$ and $C_{\zeta_7}(1|234)$. Each cut can be matched separately, but the two cannot be matched simultaneously. This difficulty can be surmounted by introducing a non-trivial numerator for the bubble-on-external-leg (BEL) cubic diagram (the third diagram in eq. (2.15)). In the “snail” regularization [19] the singular $\frac{1}{k_1^2}$ propagator that would be expected in such a diagram is allowed to be formally non-zero at the level of the integrand, and is then cancelled by introducing an overall factor of $k_1^2$ in the diagram’s numerator. This procedure effectively introduces a contact correction with the problematic propagator collapsed, carrying only the color structure of the cubic diagram. After double copying to gravitational theories, these new diagrams will drop out from the representation as the double-copied numerator will contain a $k_1^4$ while there will only be one factor of $\frac{1}{k_1^2}$ as a propagator, leaving a $k_1^2$ which can be safely set to zero using on-shell conditions.

While this process solves the BEL obstruction, problems continue to arise when attempting to match the one-particle cut. Unlike in the three previous cases, the local terms of the one-particle cut are not automatically matched by the constrained numerators. By itself this is not a problem. Unfortunately, we further find that the matching cannot be completed with the remaining freedom in the ansatz. As with the BEL numerators, one might hope to introduce only appropriately-regularized cubic tadpole contributions that will drop out of the resulting double-copied theory. Our attempts to absorb the necessary local contributions into symmetry-obeying cubic tadpoles have failed. However, there are significantly more subtleties with introducing cubic tadpole numerators to absorb this mismatch, so we cannot claim that finding appropriate cubic tadpoles is impossible. Even though the single-copy numerators do not match all cuts, since they match the triangle and bubble cuts and obey color-kinematics duality they will compute the triangle and bubble cuts correctly in the double-copy theory.

The above problems arise as a result of the tension between various assumptions in the construction: cubic and color dual versus unitarity cuts versus powercounting for manifest pole structures. Ref. [14] dodges the BEL and tadpoles entirely by working purely with contact diagrams, thus never needing to introduce spurious poles but at the cost of not being able to directly perform a numerator double copy. As mentioned above, the “snail” BEL regularization is effectively trying sneak contact contributions back into the representation. A full “Minahaning” treatment [40, 41] of the BEL numerators could overcome the obstacle at the cost of no longer manifesting the amplitude’s pole structures. However, the required computational overhead for that approach is prohibitive for the current work.

Rather than breaking the manifest pole structure via BEL, we can alternatively loosen the powercounting constraints on the bubble diagram, i.e. relax eq. (4.5). These constraints are imposed to manifest the property that the amplitude has no poles in external Mandelstams, but strictly speaking this only needs to hold after integration. Since the construction obstacle occurs at the level of the bubble cuts, we loosen the powercounting restrictions on the bubble to

$$ N_{\Box}(1, 2|3, 4) \propto s_{12}. \quad (4.18) $$
Table 1. Number of free parameters for ζ₅, ζ₂₃ and ζ₇ at each step in the construction. The exact number of parameters at each step depends on the representation of the initial ansatz. The final number of free parameters is independent on such choices. For ζ₇, we report on two methods. First we use an additional 30 parameter ansatz for the bubble-on-external-leg numerator which is unaffected by powercounting and Jacobi relations. Since this construction of ζ₇ is incomplete, we do not present the final free parameter count. Second, we relax the bubble powercounting constraint which allows a complete solution.

| ζ weight | Initial ansatz | After powercounting | After boundary Jacobies | Free params after cuts |
|-----------|----------------|---------------------|-------------------------|-----------------------|
| ζ₃        | 21             | 3                   | 3                       | 0                     |
| ζ₅        | 126            | 43                  | 25                      | 2                     |
| ζ₂₃       | 252            | 135                 | 45                      | 1                     |
| ζ₇        | 462 (+30)      | 223 (+30)           | 118 (+30)               | -                     |
|           | 462            | 352 (relaxed bubble)| 280                     | 4                     |

Repeating the construction with the weaker powercounting succeeds, with four free parameters left over after all conditions are imposed.

The number of free parameters at each step in the construction for each of ζ₅, ζ₂₃, ζ₇ are summarized in table 1. Since the degree of the box numerator is the same as the α' order, the resulting numerators are sufficiently large that they are not worth presenting in text. They may be found in the supplementary material attached to this paper, with the remaining free parameters left unfixed.

Color-kinematics representations of numerators are often also manifest loop powercounting representations, isolating the leading ultraviolet divergences to particular diagram classes. For instance, there are two different representations for the four-loop four-point intergrand in maximal super-Yang-Mills: one without CK [18], and one with manifest CK [19]. The one with manifest CK numerators also manifests the UV structure of the theory by pushing the bad powercounting terms from the box-like topologies into the bubble-like ones. The UV critical dimension and leading divergence can be extracted from only these bubble-like numerators.

We can ask the same questions of our new representations (only using the ζ₇ with the relaxed bubble power counting). Namely, how does the CK representation assign loop momentum power counting, and does the CK dual representation generate the leading UV divergence from only the bubble? In table 2 we present the relevant data. We find that the ζ₃, ζ₅, and ζ₇ representations do have manifest UV properties. The scalar bubble integral first develops a UV divergence in $D_C = 4$, and the bubble numerator for ζ₅ (ζ₇) carries two (four) additional powers of ℓ to produce its critical dimension $D_C = 2$ (0). Interestingly, while the ζ₂₃ representation does not have manifest UV behavior, the two powers of ℓ enter the bubble numerator in such a way that the leading UV divergence cancels at the level of the individual diagram. Note that while this is not the best imaginable power counting, it is significantly closer than the contact representation from ref. [14]. That representation
Table 2. The loop momentum power counting for each of the topologies, and the lowest dimension in which the representation develops a UV divergence. The critical dimensions are in agreement with those reported in ref. [14].

also carries $\ell^2$ on bubble diagrams, but in addition it contains explicit tadpole numerators, thus requiring a careful interplay between multiple diagrams to see the UV cancellations.

4.3 Double-copy cut comparison

With color-dual numerators in hand, we can easily construct a large number of closed-string-insertion matrix elements. In particular, the double copy gives easy access to all closed-string matrix elements in \{sYM, $\zeta_3$, $\zeta_5$, $\zeta_3^2$, $\zeta_7$\} $\otimes$ \{sYM, $\zeta_3$, $\zeta_5$, $\zeta_3^2$, $\zeta_7$\}. Using the direct cut construction method from section 3.2, we can verify the double-copy integrands and explore the different methods of arriving at representations for products of $\zeta$ values. In table 3, we present the various ratios between cuts constructed by taking double copies of our representations and cuts calculated directly from eq. (3.17), without adopting any shifts in normalization. We find that both double copy prescriptions, eqs. (3.18) and (3.19), reproduce the correct kinematic structure of the cuts but lead to the wrong overall normalization.

To account for the normalization mismatch, we need to dig a bit deeper into what the various double-copy constructions are doing. First, note that the BCJ double copy, eq. (2.8), is generally commutative: for a product between two theories it does not matter which is assigned to $N$ and which to $\tilde{N}$. As such the notation $X \otimes Y$ in the table and double-copy numerator definitions in eqs. (3.18) and (3.19) do not distinguish between $N_X \tilde{N}_Y$ or $N_Y \tilde{N}_X$, and so have no reason to sum over the contributions. On the other hand, the KLT double copy used to define the direct cut construction, eqs. (2.36), (2.37), (2.40) and (3.17), sum over contributions to each $\zeta$ value from both the “left” theory and the “right” theory. We can make this clear by artificially breaking the symmetry between the “left” and “right” copies of the open string by putting “L” and “R” tag labels on eq. (2.36),

$$M_{\text{closed}}^{\text{closed}} = (A_{\text{open}, L})^T S_{\alpha} A_{\text{open}, R}$$

and following the same expansion and commutation process, yielding

$$M_{\text{closed}}^{\text{closed}} = A_{\text{SYM}} S_0 (1 + \zeta_3 (M_{3, L} + M_{3, R}) + \ldots) A_{\text{SYM}}.$$  

(4.20)

Clearly the closed-string tree amplitude sees contributions from both the “left” and “right” theories, even though they are identical. Thus, in matching the cuts using BCJ double-copy numerators, we should really be summing over all relevant combinations of $N$ and $\tilde{N}$, even
Double-copy form | double-copy cuts | Analogous tree-level structure
--- | --- | ---
sYM $\otimes \zeta_3$ | $\frac{1}{2}$ | $M_{3,L} + M_{3,R}$
sYM $\otimes \zeta_5$ | $\frac{1}{2}$ | $M_{5,L} + M_{5,R}$
sYM $\otimes \zeta_7$ | $\frac{1}{2}$ | $M_{7,L} + M_{7,R}$
$\zeta_3 \otimes \zeta_3$ | $\frac{1}{2}$ | $\frac{1}{2}(2M_{3,L}M_{3,R})$
sYM $\otimes \zeta_3^2$ | $\frac{1}{2}$ | $\frac{1}{2}(M_{3,L}^2 + M_{3,R}^2)$
$\zeta_3 \otimes \zeta_5$ | conjecture: $\frac{1}{4}$ | $M_{5,L}M_{5,R} + M_{5,L}M_{3,R}$
sYM $\otimes \zeta_3\zeta_5$ | conjecture: $\frac{1}{4}$ | $M_{5,L}M_{3,L} + M_{3,R}M_{5,R}$
$\zeta_5 \otimes \zeta_5$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(2M_{5,L}M_{5,R})$
sYM $\otimes \zeta_5^2$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(M_{5,L}^2 + M_{5,R}^2)$
$\zeta_3 \otimes \zeta_7$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(M_{7,L}M_{3,L} + M_{3,R}M_{7,R})$
sYM $\otimes \zeta_3\zeta_7$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(M_{7,L}M_{5,L} + M_{5,R}M_{7,R})$
$\zeta_5 \otimes \zeta_7$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(M_{7,L}M_{5,L} + M_{5,R}M_{7,R})$
sYM $\otimes \zeta_5\zeta_7$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(M_{7,L}M_{5,L} + M_{5,R}M_{7,R})$
$\zeta_7 \otimes \zeta_7$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(2M_{7,L}M_{7,R})$
sYM $\otimes \zeta_7^2$ | conjecture: $\frac{1}{4}$ | $\frac{1}{2}(M_{7,L}^2 + M_{7,R}^2)$

Table 3. A comparison between the various BCJ double-copy constructions buildable from sYM, $\zeta_3$, $\zeta_5$, $\zeta_3^2$, and $\zeta_7$ representations. The first column lists the possible constructions, and $\otimes$ should be understood as commutative. The second column highlights the differences between the symmetric and asymmetric constructions. While we do not have access to both forms of double-copy numerators for $\alpha'^{2\leq 8}$, we conjecture that if such numerators exist then the normalization pattern we observe in the asymmetric double copy continues. The third column recalls analogous tree-level kinematic structures appearing in the KLT representation of closed-string amplitudes, eq. (2.37). The $\zeta_7$ entries used the relaxed-bubble-powercounting representation. The starred entries in the second column were only checked on the triangle cut, $\epsilon(1|2|34).

those that look degenerate. For example, consider the $\zeta_3^2$ contributions, which on all cuts behave like

$$\mathcal{C}_{\zeta_3^2 \otimes sYM} + \mathcal{C}_{sYM \otimes \zeta_3^2} + \mathcal{C}_{\zeta_3 \otimes \zeta_3} = \mathcal{C}_{\zeta_3^2 \left( \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \right)} = \mathcal{C}_{\zeta_3^2}. \quad (4.21)$$

We list the relevant terms in the expansion of eq. (4.20) in the right-most column of table 3.

Through this lens, the relative combinatorics is mundane while the amazing thing is that both of the double copy forms correctly reproduce the kinematics of the cuts individually. Cut-matching could have instead required cancellations between the different double-copy numerators. We conjecture that this normalization pattern continues for higher $\alpha'$ operators. Due to the difficulties with the $\zeta_7$ numerators mentioned above, we limit ourselves to the more-symmetric double copy numerators which do continue to follow the observed pattern. It is also interesting that adopting an additional normalization on $N_{\zeta_i\zeta_j}^{\text{lead}}$ of $\text{sv}(\zeta_i\zeta_j)$ leads to an exact match for all of the asymmetric double-copy numerators, and a degeneracy overcount for the symmetric numerators.

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5 Conclusions

In this work, we constructed new color-dual representations at one loop and four points that contain operator insertions from the open-string low-energy effective action. The three lowest mass dimension operators for which this was possible, \( D^2 F^4 + F^5 \) (with coefficient \( \zeta_3 \alpha'^3 \)), \( D^6 F^4 + \ldots \) (with coefficient \( \alpha'^5 \zeta_5 \)), and \( D^8 F^4 + \ldots \) (with coefficient \( \alpha'^6 \zeta_7^2 \)), are represented in terms of box, triangle, and 2-2 bubble numerators. Via the kinematic Jacobi relations, the latter two numerators are expressed purely in terms of linear combinations of box numerators with different labelings. For the next mass dimension operator, the \( D^{10} F^4 + \ldots \) combination linked with \( \alpha'^7 \zeta_7 \), we introduced a bubble-on-external-leg contribution via “snail” regularization to match both bubble cuts, and could not fully match the one-particle cuts. Interestingly, when double copied with sYM this operator becomes a \( D^8 R^4 \) type operator, the same type which was found to cause a deviation between the \( \mathcal{N} = 4 \) sYM and \( \mathcal{N} = 8 \) SUGRA UV behavior at five loops [20]. The known five-loop sYM representations also have unresolved tension between color-kinematics duality and unitarity constraints [20, 42]. We saw that it was possible to overcome the \( \zeta_7 \) construction obstacle using a relaxed powercounting condition on the bubble. It is also possible that the failure of the \( \zeta_7 \) construction may be overcome by adopting the less-restrictive “Minahan” [40, 41] regularization for BEL diagrams. Both approaches provide guidance for a re-examination of the five loop construction with considerations of relaxing either the “snail” scheme or diagrammatic powercounting restrictions related to forced powers of external mandelstams.

The \( \zeta_3 \), \( \zeta_5 \), and \( \zeta_7 \) representations have manifest UV behavior, with the critical dimension and actual divergence coming from only the bubble numerators. On the other hand, the \( \zeta_3^2 \) representation does not have manifest power counting. The UV critical dimension of \( D_C = 4 \) suggests that the numerators should at worst be a tensor triangle or scalar bubble. Instead, the CK dual numerator picks an \( \ell^2 \) bubble. However, the cancellation of the potential \( D_C = 2 \) UV divergence happens at the level of an individual bubble diagram, and does not require summing over separate diagrams or channels, in contrast to the contact representation of ref. [14].

We hope that the representations we constructed will be useful for studying loop-level color-kinematics duality and BCJ relations in a simpler context than five-loop sYM or two- and three-loop \( \mathcal{N} = 0, 1, 2 \) YM. In particular, it would be interesting to investigate if the tree-level kinematic composition construction of Carrasco, Rodina, Yin, and Zekioglu [38, 39] can be lifted to loop level, for which our representations could serve as seed data.

As an additional bonus to constructing color-dual numerators, the BCJ double copy allowed us to construct closed-string-insertion numerators from pairings of the open-string-insertion numerators. We found that regardless of the prescription used, the double-copy numerators produced the correct kinematic structures to match against directly calculated cuts, but not the correct normalization. In fact, the mismatch in normalization is explained by observing that the sum over all allowed prescriptions (with degeneracy) exactly matches the directly calculated cuts. This sum including degeneracy is in direct analogy with terms and normalizations in the KLT representation of the tree-level closed-string amplitudes, and may suggest a similar decomposition for the one-loop numerators.
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