Plasmoid Instability in Forming Current Sheets

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Abstract

The plasmoid instability has revolutionized our understanding of magnetic reconnection in astrophysical environments. By preventing the formation of highly elongated reconnection layers, it is crucial in enabling the rapid energy conversion rates that are characteristic of many astrophysical phenomena. Most previous studies have focused on Sweet–Parker current sheets, which are unattainable in typical astrophysical systems. Here we derive a general set of scaling laws for the plasmoid instability in resistive and visco-resistive current sheets that evolve over time. Our method relies on a principle of least time that enables us to determine the properties of the reconnecting current sheet (aspect ratio and elapsed time) and the plasmoid instability (growth rate, wavenumber, inner layer width) at the end of the linear phase. After this phase the reconnecting current sheet is disrupted and fast reconnection can occur. The scaling laws of the plasmoid instability are not simple power laws, and they depend on the Lundquist number ($S$), the magnetic Prandtl number ($P_m$), the noise of the system ($\psi_0$), the characteristic rate of current sheet evolution ($1/\tau$), and the thinning process. We also demonstrate that previous scalings are inapplicable to the vast majority of astrophysical systems. We explore the implications of the new scaling relations in astrophysical systems such as the solar corona and the interstellar medium. In both of these systems, we show that our scaling laws yield values for the growth rate, wavenumber, and aspect ratio that are much smaller than the Sweet–Parker–based scalings.

Key words: ISM: magnetic fields – magnetic reconnection – magnetohydrodynamics – plasmas – stars: coronae – Sun: flares

1. Introduction

It is now generally acknowledged that magnetic reconnection powers some of the most important and spectacular astrophysical phenomena in the universe such as coronal mass ejections, stellar flares, nonthermal signatures of pulsar wind nebulae, and gamma-ray flares in blazar jets (Tajima & Shibata 1997; Kulsrud 2005; Zweibel & Yamada 2009; Benz & Güdel 2010; Ji & Daughton 2011; Kagan et al. 2015; Kumar & Zhang 2015). Although the importance of magnetic reconnection has been recognized since the 1950s, it has recently witnessed an upsurge in popularity due to the realization of the importance of the plasmoid instability in facilitating fast reconnection (and energy release).

The plasmoid instability can be understood in terms of a tearing instability occurring in a reconnecting current sheet (see Figure 1). Numerical simulations providing clear indications that thin reconnecting current sheets may be unstable to the formation of plasmoids date back at least to the 1980s (Biskamp 1982; Steinolfson & van Hoven 1984; Matthaeus & Lamkin 1985; Biskamp 1986; Lee & Fu 1986), but it is only in the last decade that their role in speeding up the reconnection process has been widely appreciated. Indeed, very narrow reconnection layers would form in the absence of the plasmoid instability, which, in turn, have the effect of throttling the reconnection rate. However, reconnecting current sheets exceeding a certain aspect ratio cannot form because they become unstable to the formation of plasmoids, which break the reconnection layer into shorter elements, consequently leading to a significant increase in the reconnection rate (Daughton et al. 2006, 2009). Hence the predictions of the classical Sweet–Parker reconnection model (Parker 1957; Sweet 1958) break down for sufficiently large systems such as those typically encountered in astrophysical environments— in these cases, it was shown that the reconnection rate becomes nearly independent of the magnetic diffusivity (Bhattacharjee et al. 2009; Huang & Bhattacharjee 2010; Uzdensky et al. 2010; Loueiro et al. 2012; Huang & Bhattacharjee 2013; Takamoto 2013; Comisso et al. 2015a; Ebrahimi & Raman 2015; Comisso & Grasso 2016).

The ability of plasmoid-mediated reconnection to enable fast energy release has been exploited in explaining multiple phenomena in a wide range of astrophysical settings with considerable success. They include solar flares (Shibata & Tanuma 2001; Báta et al. 2011a, 2011b; Li et al. 2015; Shibata et al. 2016; Janvier 2017), coronal mass ejections (Milligan et al. 2010; Karpen et al. 2012; Mei et al. 2012; Ni et al. 2012; Lin et al. 2015), chromospheric jets (Shibata et al. 2007; Ni et al. 2015, 2017), blazar emissions (Giannios 2013; Sironi et al. 2015; Petropoulou et al. 2016; Beloborodov 2017), gamma-ray bursts (Giannios 2010; McKinney & Uzdensky 2012; Kumar & Zhang 2015), and nonthermal signatures of pulsar wind nebulae (Sironi & Spitkovsky 2014; Guo et al. 2015, 2016; Sironi et al. 2016; Werner et al. 2016). Plasmoid formation can also produce self-generated turbulent reconnection (Daughton et al. 2011; Oishi et al. 2015; Huang & Bhattacharjee 2016; Kowal et al. 2016; Wang et al. 2016), implying that large-scale current sheets are likely to become turbulent during the advanced stages of the reconnection process (del Valle et al. 2016). Given the importance of nonlinear plasmoids in the reconnection process, several studies have also been devoted to the understanding of their statistical properties (Fermo et al. 2010; Uzdensky et al. 2010;
In order to circumvent this problem, Pucci & Velli (2014) constrained the growth rate such that it was not divergent with respect to $S$. This led them to the conclusion that the final inverse-aspect ratio in the resistive regime depends only on the Lundquist number $S$ and corresponds to $a/L \sim S^{-1/3}$. For this choice of $a/L = S^{-1/3}$, they obtained $\gamma S \approx 0.623$, which was independent of $S$. On the other hand, Uzdensky & Loureiro (2016) posited that the linear stage of the plasmoid instability essentially ends when $\gamma S = 1$, with $\gamma$ being the characteristic timescale of the current sheet evolution. However, each of these assumptions is open to question. If one “terminates” the linear dynamics at this stage, then the growth rate is only comparable to the timescale of the current sheet thinning, implying that one cannot use the static dispersion relations of the tearing instability to carry out the calculations. Moreover, at this stage, the effects of the reconnection layer outflow are also nonnegligible, since the modes are subject to stretching.

These difficulties are mostly rendered void when one observes that typically $\gamma S \gg 1$ at the end of the linear phase. Furthermore, it is also important to note that the most interesting dynamics occur when $\gamma > 1/\tau$. This has been shown in a recent Letter published by us (Comisso et al. 2016), where we demonstrated that the plasmoid instability exhibits a quiescence period followed by a rapid growth. Furthermore, we showed that the scaling relations of the plasmoid instability were no longer simple power laws, as they included nonnegligible logarithmic contributions and also depended upon the noise of the system, the characteristic rate of the current sheet evolution, and even the nature of the thinning process. This has direct implications for the onset of fast magnetic reconnection, because the correct identification of the scaling laws of the plasmoid instability is necessary for understanding when and how plasmoids become nonlinear and disrupt the reconnecting current sheet.

In this work, we extend the analysis presented in the aforementioned Letter by formulating a detailed treatment of both the inviscid and viscous regimes of the plasmoid instability. A proper treatment of the latter is very important since viscosity (or equivalently, the magnetic Prandtl number, defined later in this paper) plays a major role in several astrophysical systems such as accretion discs around neutron stars and black holes (Balbus & Henri 2008), the warm interstellar medium (Brandenburg & Subramanian 2005), protogalactic plasmas (Kulsrud et al. 1997), and the intergalactic medium (Subramanian et al. 2006). Our work accords four major advantages over prior studies: (1) the scaling laws for the plasmoid instability in general time-evolving current sheets are derived in both the resistive and the visco-resistive regimes, (2) a clear demarcation of the limited domain in which the previous scalings are applicable, (3) the presentation of accurate results in astrophysically relevant regimes with very high $S$ values, and (4) the exploration of the astrophysical implications of the plasmoid instability in the stellar and interstellar medium contexts.

The outline of the paper is as follows. In Section 2, the least-time principle, which is used to compute the properties of the dominant mode at the end of the linear phase, is introduced. This is followed by a derivation of the resistive and visco-resistive scaling laws for the plasmoid instability in Sections 3 and 4, respectively. We discuss the astrophysical relevance of the derived scaling relations by choosing two systems (the solar corona and the interstellar medium) in Section 5. Finally, we summarize our results in Section 6.

![Figure 1. Sketch of linear plasmoids forming in a reconnecting current sheet. The shaded orange region indicates the out-of-plane current sheet whose total width and length are $2a$ and $2L$, respectively. Plasmoids are represented by the thin magnetic islands in the current sheet.](image-url)
Figure 2. Sketch of a typical tearing mode dispersion relation for a time-evolving current sheet, assuming that the current sheet is thinning in time ($t_1 < t_2 < t_3$). $\gamma$ is the growth rate and $k$ is the absolute value of the wave vector that does not change in time.

2. Least-Time Principle for Plasmas

In this section, we provide a general framework to evaluate the properties of the plasmoid instability in general current sheets that can evolve over time. In such general current sheets, tearing modes (Biskamp 2000; Goedbloed et al. 2010; Fitzpatrick 2014) do not begin to grow at the same time, i.e., they are rendered unstable at different times. Moreover, their growth rate does not depend solely on the wavenumber $k$ but also depends on the time $t$ that has elapsed since the current sheet evolution commenced at some initial aspect ratio (see example in Figure 2).

The amplitude of the tearing modes evolves as per

$$\psi(k, t) = \psi_0 \exp \left( \int_{t_0}^{t} \gamma(k, t') dt' \right),$$  \hspace{1cm} (1)

where $\gamma(k, t)$ indicates the time-dependent growth rate, $t_0$ is the initial time, and $\psi_0 := \psi(k, t_0)$ represents the perturbation from which the modes can start to grow. If $\gamma$ were constant, then the amplitude evolution would be identical to that obtained from conventional linear eigenmode theory. Since $\gamma$ itself depends on time, it is more instructive to regard Equation (1) as the WKB solution (Bender & Orszag 1978) to the linearized equations governing the tearing mode process. Notice that in the linear phase the amplitudes of the modes are “small,” and therefore they do not affect the current sheet evolution and there is no mode–mode coupling. Note also that we neglect mode stretching due to the reconnection outflow, which proves to be a good approximation for the very large $S$ plasmas of interest in this work. We point out, however, that mode stretching is important to correctly evaluate the critical Lundquist number $S_c$ above which the plasmoid instability is manifested (see Huang et al. 2017).

It should be noted that in a real-world system, the initial noise is not constituted of only the unstable eigenmodes. In our study, we do not consider the effects of transients due to the fact that the initial conditions for the noise do not necessarily correspond to pure normal modes. It takes a transient period for the noise to attain the unstable eigenmode structures, and this is ignored in our analysis as implied in Equation (1). This approximation is guided by the rationale that the initial evolution of the plasmoid size is determined essentially by the current sheet formation rate, as explained later in the paper. We have also neglected possible effects associated with transient growth of damped eigenmodes. This assumption is consistent with the neglect of in-plane flows, which can be a source of nonmodality. As a matter of fact, behavior associated with nonmodal transient growth has not been observed in numerical simulations, e.g., in Figure 4 of Huang et al. (2017). However, future theoretical and numerical analyses should explore the role of transient phenomena in greater detail to confirm the validity of our assumption.

The linear phase of the plasmoid growth terminates when the plasmoid half-width $w(k, t)$ is on the same order as the inner layer width $\delta_{in}(k, t)$. The former is given by (see, for example, Fitzpatrick 2014)

$$w(k, t) = 2 \left( \frac{\psi \cdot a}{B} \right)^{1/2},$$  \hspace{1cm} (2)

where $\psi$, defined in Equation (1), must be understood as being evaluated at the resonant surface. Here, $B$ represents the reconnecting magnetic field evaluated upstream of the current sheet, while $a$ is the half-width of the current sheet. The latter, namely $\delta_{in}(k, t)$, is not the same for all physics models, and hence we leave it unspecified at this stage.

As described above, we identify the end of the linear phase with the condition $w = \delta_{in}$. Of course, this condition is not a sharp cutoff for the end of the linear phase. In reality, there could be a $O(1)$ factor that needs to be included prior to the equality sign. However, for the purposes of simplicity, we terminate the linear phase when these two quantities are exactly equal to one another. Finally, a caveat regarding $w$ must be introduced—it represents the plasmoid half-width only when its associated mode is much more dominant than the rest. This assumption can be slightly relaxed to cases where the perturbation amplitude is sufficiently localized in the spectrum. It turns out that this condition is typically met at the end of the linear stage of the plasmoid instability. A more precise evaluation of the plasmoid width can still be obtained by considering the contribution of a proper range of the fluctuation spectrum (Huang et al. 2017).

Although we have now specified the end of the linear phase, we have still not identified the tearing mode that emerges dominant at the end. At this stage, we introduce the primary physical principle behind the paper. We follow the approach espoused in Comisso et al. (2016): the mode that emerges “first” at the end of the linear phase is the one that has taken the least time to traverse it. This “principle of least time for the plasmoid instability” shares some apparent similarities with Fermat’s principle of least time, but there is also one essential difference—the latter relies upon a variational principle (Born & Wolf 1980), whereas in the former the extremum of a function (the time) is computed.

Some modes may become unstable from an early stage and continue growing at a steady (relatively slow) pace. Others may remain stable for a long time, therefore remaining quiescent, until they become unstable at a later stage and are subject to explosive growth (see example in Figure 3). Thus, among this wide range of possibilities, the above principle enables us to select the mode that exits the linear stage first. In mathematical terms, these conditions are expressible as follows. First, we have

$$w(k_s, t_s) = \delta_{in}(k_s, t_s),$$  \hspace{1cm} (3)

where the symbol “$\leftrightarrow$” denotes the end of the linear phase. The above expression implies that the time $t_s$ is solely a function of
Then the principle of least time amounts to stating that
\[ \frac{dt_m}{dk_{**}} = 0. \] (4)

Hence, the conditions given by Equations (3) and (4) permit us to compute the mode that takes the least value of \( t_m \). It can also be shown a posteriori that in the neighborhood of \( k_{**} \), \( w \) is localized and has a stronger dependence than \( \delta_m \) on \( k \). Therefore, the mode that completes the linear phase in the least time can be seen as the dominant one that enters the nonlinear phase (see example in Figure 4).

We can explicitly rewrite Equation (3) as
\[ \left\{ \ln \left( \frac{w_0 g^{1/2}}{\delta_m \sqrt{t}} \right) - \frac{1}{2} \int_{t_o}^{t} \gamma(t') dt' \right\}_{k_{**},t_0} = 0, \] (5)

where \( w_0 := w(k, t_0) = 2(\psi_0 a_0 / B_0)^{1/2} \) is the label for the initial perturbation amplitude and the functions \( f \) and \( g \) are defined such that \( a(t) = a_0 f(t) \) and \( B(t) = B_0 g(t) \). We can also rewrite Equation (4) as
\[ \frac{\partial F(k, t)}{\partial k} \bigg|_{k_{**},t_0} = 0 \] (6)

if \( \partial / \partial t [F(k, t)]_{k_{**},t_0} = 0 \). Here we have defined
\[ F(k, t) := \delta_m (k, t) - w(k, t) \] (7)
as in our previous Letter (Comisso et al. 2016).

Hitherto, our discussion has been completely general as the above Equations (5) and (6) describing the principle of least time are equally applicable for a wide range of plasma models that can include resistive, viscous, and collisionless contributions. In this paper, we focus primarily on the resistive and visco-resistive regimes.

### 3. Resistive Regime

In what follows, we move to dimensionless quantities. We adopt a normalization convention such that all the lengths are normalized to the current sheet half-length \( L \), the time to the Alfvén time \( \tau_A = L / v_A \), and the magnetic field to the upstream field \( B_0 \). Thus, the other quantities are normalized as
\[ (\hat{\kappa}, \hat{\gamma}, \hat{\psi}, \hat{\eta}, \hat{\nu}) = \left( kL, \gamma \tau_A, \frac{\psi}{LB_0}, \frac{\eta}{Lv_A}, \frac{\nu}{Lv_A} \right), \] (8)

where we use carets for denoting dimensionless quantities.

Note that the normalized magnetic diffusivity corresponds to the inverse of the Lundquist number, i.e., \( \hat{\eta}^{-1} = S \), while the normalized kinematic viscosity corresponds to the inverse of the kinetic Reynolds number when the Alfvén velocity is the typical velocity scale of the system. Therefore, the ratio \( \hat{\nu} / \hat{\eta} = P_m \) defines the magnetic Prandtl number.5

In this section we consider the plasmoid instability in the resistive regime, which is characterized by \( P_m \ll 1 \), while the next section is devoted to the visco-resistive regime, in which \( P_m \gg 1 \).

Although the framework provided in Section 2 is fully general, here we are interested in the case where the current sheet half-length \( L \) and the reconnecting magnetic field \( B \) remain approximately constant, while the current sheet width decreases in time. This is indeed a classic case of current sheet formation (see also Huang et al. 2017). The function \( f(\hat{\tilde{t}}) = \hat{a}(\hat{\tilde{t}}) / a_0 \) that takes into account the current sheet thinning must obey \( f(\hat{\tilde{t}}_0) = 1 \) and
\[ \lim_{\hat{\tilde{t}} \rightarrow \infty} f(\hat{\tilde{t}}) = \frac{(1 + P_m)^{1/4}}{a_0 S^{1/2}}. \] (9)

Indeed, \( \hat{a} = S^{-1/2}(1 + P_m)^{1/4} \) is the natural lower limit to the thickness of a reconnection layer (Parker 1957; Sweet 1958; Park et al. 1984). In the resistive regime, we have simply \( \lim_{\hat{\tilde{t}} \rightarrow \infty} f(\hat{\tilde{t}}) = \hat{a}_0^{-1} S^{-1/2} \).

When the modes grow slower than the evolution of the current sheet, i.e., \( \hat{\gamma} < \hat{a}^{-1} d\hat{a} / d\hat{\tilde{t}} \), the change in \( \hat{w} \) is dominated by the change in \( \hat{a} \) and the growth rate \( \hat{\gamma} \) is negligible in this respect. On the other hand, when the modes grow faster than the evolution of the current sheet, i.e., \( \hat{\gamma} > \hat{a}^{-1} d\hat{a} / d\hat{\tilde{t}} \), the change in \( \hat{w} \) is mainly due to the growth rate of the perturbed magnetic flux. In this case, the tearing mode growth rate can be

5 We specify that the magnetic Prandtl number that appears in the formulas of this paper is defined using the cross-field collisional velocity of \( \hat{\nu} \).
computed using the instantaneous value of $\dot{a}$ in the standard tearing mode dispersion relations. Depending on the value of the tearing stability parameter $\Delta'$ (Furth et al. 1963), two simple algebraic relations can be considered. For $\Delta'\delta_m \ll 1$ (the small-$\Delta'$ regime), the tearing modes grow as per the following relation (Furth et al. 1963):

$$\gamma_t \simeq c_1 k^2/5 \dot{a}^{-2/5} \Delta'^{4/5} S^{-3/5}, \quad (10)$$

where $c_1 = [(2\pi)^{-1/4}I(1/4)/I(3/4)]^{3/5} \approx 0.55$. On the other hand, for $\Delta'\delta_m \gtrsim 1$ (the large-$\Delta'$ regime), the growth rate becomes (Coppi et al. 1976; Ara et al. 1978):

$$\gamma_t \simeq k^2/3 \dot{a}^{-2/3} S^{-1/3}. \quad (11)$$

In our analysis, we are interested in the complete $\Delta'$-domain. Therefore, we seek an expression for $\gamma_t$ that (1) is a reasonable approximation of the exact dispersion relation (Coppi et al. 1976; Ara et al. 1978), (2) reduces to Equations (10) and (11) in the proper limits, and (3) is simple enough to be analytically tractable. For this purpose we adopt the half-harmonic mean of these two relations, namely

$$\gamma_t = \gamma_t^{\star}/(\gamma_t + \gamma_t^{\star}), \quad (12)$$

which fulfills all of the criteria described above. If a different approximation for $\gamma_t$ is adopted, such as the simpler one employed in several previous works (Tajima & Shibata 1997; Bhattacharjee et al. 2009; Huang & Bhattacharjee 2013; Loureiro et al. 2013; Pucci & Velli 2014) or the more complex one used by Huang et al. (2017), the same scaling relations as the ones derived below are obtained, albeit with slightly different numerical factors.

At this point we only need to specify the inner resistive layer width, which corresponds to (Fitzpatrick 2014)

$$\tilde{k}_m = \gamma_t^{1/4}(\dot{a}/\tilde{k})^{1/2} S^{-1/4}. \quad (13)$$

Using this expression, Equations (5) and (6) can be combined to obtain the least-time equation

$$\left\{ (i\tilde{\gamma} - 1/2) \frac{\partial \gamma_t}{\partial k} + \frac{\gamma_t}{k} + \frac{\gamma_t}{\psi_0} \frac{\partial \psi_0}{\partial k} \right\}_{k, s} = 0, \quad (14)$$

where $i = \partial / \partial \gamma_t \int_{\tilde{k}_m}^{\tilde{k}} (\tilde{r}) d\tilde{r}$. In our subsequent discussion, we assume that the natural noise of the system has a general power-law form, namely $\psi_0 = \tilde{c} k^{-\alpha}$, but other cases can be treated on an equal footing by considering different perturbation spectra. We also assume that the current sheet has the common Harris-type structure (Harris 1962), for which $\Delta' = 2[\tilde{k}(\tilde{a})^{-1} - \tilde{k}\tilde{a}]/\dot{\tilde{a}}$. This expression for $\Delta'$ can be simplified by considering only the regime $k\tilde{a} \ll 1$, since the very slow-growing part of the mode evolution does not affect the results of the linear phase. Then, from Equation (14) we get

$$\frac{1}{\bar{\tilde{k}}_m} \left[ 5 - \frac{15\alpha}{2} \right] \left( 1 + \hat{k}_s^{-16/15} \hat{a}_s^{-4/3} S^{-4/15} \right)$$

$$+ \left[ 9 - \frac{15\alpha}{2} \right] \left( 1 + \hat{k}_s^{-16/15} \hat{a}_s^{4/3} S^{4/15} \right)$$

$$\approx 3\hat{k}_s^{2/3} \hat{a}_s^{-2/3} S^{-1/3} - 5\hat{k}_s^{-2/3} \hat{a}_s^{-2} S^{-3/5}. \quad (15)$$

It can be shown a posteriori that the two terms on the right-hand side must approximately balance each other for $\psi(\tilde{k}_m, \tilde{a}_0) \ll \tilde{k}_s^{-3/2} S^{-2/3}$. Hence, the emergent mode satisfies the relation

$$\hat{k}_s \approx c_k \hat{a}_s^{-5/4} S^{-1/4}, \quad (16)$$

where $c_k$ is a $O(1)$ coefficient. This implies that

$$\gamma_t \approx c_s \hat{a}_s^{-3/2} S^{-1/2}, \quad (17)$$

$$\delta_m \approx c_s \hat{a}_s^{3/2} S^{-1/4}, \quad (18)$$

where $c_s$ and $c_k$ are also $O(1)$ coefficients. The above relations indicate that the dominant mode at the end of the linear phase has the same scaling properties of the fastest-growing mode (Furth et al. 1963), the latter of which satisfies the equation $\partial^2 \gamma_t / \partial k^2 |_{k, s} = 0$.

For moderately high values of the Lundquist number $S$, the reconnecting current sheet might be capable of attaining the Sweet–Parker inverse-aspect ratio $\hat{a}_s \approx S^{-1/2}$. In this case, it is straightforward to see that

$$\gamma_t \approx S^{1/4}, \quad \hat{k}_s \approx S^{3/8}, \quad \delta_m \approx S^{-5/8}. \quad (19)$$

It is therefore not surprising to discover that these relations match the ones obtained in previous studies of the plasmoid instability (Tajima & Shibata 1997; Loureiro et al. 2007; Bhattacharjee et al. 2009; Baalrud et al. 2012; Huang & Bhattacharjee 2013; Comisso & Grasso 2016) that were undertaken assuming a fixed Sweet–Parker current sheet.

On the other hand, for very high Lundquist numbers, which are widely prevalent in most astrophysical plasmas, the plasmoids complete their linear evolution well before the Sweet–Parker aspect ratio is reached. Thus we need to calculate $\hat{a}_s$ for a more general case. This can be done by substituting the relations for the dominant mode at the end of the linear phase into Equation (5), which yields the following equation for the inverse-aspect ratio:

$$\ln \left( \frac{c_s}{2 \hat{k}_s^{1/2} S^{1/4}} \right) = \frac{1}{2} \int_{\hat{a}_0}^{\hat{a}_s} \tilde{\gamma}(\tilde{a}) \frac{d\tilde{a}}{d\tilde{a}}. \quad (20)$$

This equation gives us the final inverse-aspect ratio $\hat{a}_s$ for a general current sheet evolution $\tilde{\gamma}(\tilde{a})$. It is evident that $\hat{a}_s$, and thus the scaling relations of $\gamma_t^{\star}$, $\hat{k}_s$, $\delta_m^{\star}$ and $\bar{\tilde{k}}_m$, cannot be universal, because they depend on the specific form of the function $\tilde{\gamma}(\tilde{a})$.

Since we must specify a specific form of $\tilde{\gamma}(\tilde{a})$, in what follows, we first consider what is probably the most typical case of current sheet thinning: the exponential thinning. This is indeed known to be the standard case for instability-driven current sheets. We then generalize the results of the exponential thinning to also include algebraic cases. Other less common possibilities could also be investigated, since the developed framework is general.

### 3.1. Exponentially Shrinking Current Sheet

It can be shown from first principles that the exponential thinning of a reconnecting current sheet evolves according to
the following expression (Kulsrud 2005):

\[ \hat{a}(\tilde{t})^2 = (\hat{a}_0^2 - \hat{a}_\infty^2) e^{-2\theta/\tau} + \hat{a}_\infty^2, \]  

(21)

where \( \hat{a}_\infty = S^{-1/2} \) in the resistive regime (Parker 1957; Sweet 1958). This expression slightly differs from the one we adopted in our previous work (Comisso et al. 2016), but it shares the same asymptotic behaviors for small and large \( \tilde{t} \), therefore leading to the same asymptotic relations for the plasmoid instability. Other cases for \( \hat{a}_\infty \), such as the ones imposed in the numerical simulations by Tenerani et al. (2015b), where \( \hat{a}_\infty = S^{-1/3} \), are not considered here because they are not supported by physical evidence (Kulsrud 2005; Huang et al. 2017). In this respect, it is worth noting that the plasmoid half-width given by Equation (2) starts to grow only when \( \tilde{\gamma} > 1/\tau \) (for \( \tilde{\gamma} < 1/\tau \) it is straightforward to check that \( \dot{\tilde{a}}(\tilde{t}) \) decreases because of the rapid decrease of \( \dot{a}(\tilde{t}) \)). We see later that this condition occurs when \( \tilde{a} < \hat{k}_\* S^{-1/2} \tau^{-3/2} \), which is smaller than \( S^{-1/3} \) for \( \tau \) of order unity, which is indeed the case for an ideal exponentially thinning current sheet (Kulsrud 2005; Huang et al. 2017).

Using Equation (21) we can compute the transitional time \( \tilde{t}_T \) that separates the two asymptotic behaviors for small and large \( \tilde{t} \). For \( \tilde{t} > \tilde{t}_T \), with

\[ \tilde{t}_T = \tau \ln \left( 1 + \hat{a}_0 / \hat{a}_\infty^2 \right), \]  

(22)

we have \( \hat{a}_* \simeq \hat{a}_\infty \). On the other hand, for \( \tilde{t} < \tilde{t}_T \) we have \( \hat{a} = \hat{a}_0 e^{-\tilde{t}/\tau} \). While for \( \tilde{t} > \tilde{t}_T \) one recovers the relations (19), the case \( \tilde{t} < \tilde{t}_T \), which is more relevant for astrophysical environments because it occurs for larger \( S \) values, necessitates further analysis. In this case we have to solve Equation (20). Using \( \hat{a}_* \ll \hat{a}_0 \), we obtain

\[ \hat{a}_* \simeq 6^{2/3} c_a \frac{\tau^{2/3}}{S^{1/3}} \left( \ln \left( \frac{c_k^{6/5} \hat{a}_0^{3/2} - 50^{1/4}}{25 \varepsilon^{3} \hat{a}_\infty^{3(2+\alpha)/4}} \right) \right)^{-2/3}, \]  

(23)

where

\[ c_a = \left( \frac{3}{4} c_k c_k^{2/3} \right)^{2/3}, \]  

(24)

and

\[ \tilde{c}_k = 1 + c_k^{4/15} (\cot^{-1}(1 + \sqrt{2} c_k^{4/15}) + \cot^{-1}(1 - \sqrt{2} c_k^{4/15})). \]  

(25)

The coefficient \( c_a \) turns out to be \( c_a \approx 0.3 \) for \( c_k \) of order unity. Therefore, we can neglect the factor \( 6^{2/3} c_a \approx 1 \) in Equation (23). This equation for the inverse-aspect ratio can be solved exactly in terms of the Lambert W function (Corless et al. 1996), but here we prefer to consider an asymptotic solution that yields more transparent results. As was done in Comisso et al. (2016) we solve Equation (23) by iteration, obtaining

\[ \hat{a}_* \simeq \frac{\tau^{2/3}}{S^{1/3}} \left( \ln \theta_R \right)^{-2/3}, \]  

(26)

where

\[ \theta_R := \frac{\sqrt{2} \varepsilon^{3/50} \hat{a}_0^{3/2} - 50^{1/4}}{25 \varepsilon^{3} \hat{a}_\infty^{3(2+\alpha)/4}}, \]  

(27)

and the subdominant term proportional to \( \ln (c_k^{6} c_k^{3(2+\alpha)}) \) has been neglected.

Given the final inverse-aspect ratio, we can easily determine the growth rate, wavenumber, and inner layer width at the end of the linear phase:

\[ \tilde{\gamma}_* \simeq c_\gamma \frac{\ln \theta_R}{\tau}, \]  

(28)

\[ \hat{k}_* \simeq c_k S^{1/6} \left( \ln \theta_R \right)^{5/6}, \]  

(29)

and

\[ \hat{\delta}_{\text{in}}* \simeq \frac{c_\delta}{S^{1/2}} \left( \frac{\tau}{\ln \theta_R} \right)^{1/2}. \]  

(30)

These relations exhibit a nontrivial dependence with respect to the Lundquist number \( S \), the noise level \( \tilde{\psi}_0 \) (through both \( \varepsilon \) and \( \alpha \)), and the timescale of the driving process \( \tau \). Note that these scaling relations of the plasmoid instability are not pure power laws, as they also include nonnegligible logarithmic factors. This has important implications for very large-\( S \) plasmas like those typically encountered in astrophysical environments (Ji \\& Daughton 2011), since the scaling properties of the plasmoid instability change considerably with respect to those obtained for not-so-large-\( S \) plasmas.

To better evaluate the implications of the new scaling relations, we focus on the case in which the natural noise amplitude is approximately the same for all wavelengths. In this case we can set \( \alpha = 0 \) and, recalling that \( \tilde{\psi}_0 = 2(\varepsilon \hat{a}_0^{1/2}) \), Equations (26)–(30) reduce to

\[ \hat{a}_* \simeq \frac{\tau^{2/3}}{S} \left( \ln \left( \frac{\tau \hat{a}_0^3}{S^{2} \tilde{\psi}_0^6} \right) \right)^{-2/3}, \]  

(31)

\[ \tilde{\gamma}_* \simeq c_\gamma \frac{\ln \left( \tau \hat{a}_0^3 \right)}{S^{2} \tilde{\psi}_0^6}, \]  

(32)

\[ \hat{k}_* \simeq c_k S^{1/6} \left( \ln \left( \frac{\tau \hat{a}_0^3}{S^{2} \tilde{\psi}_0^6} \right) \right)^{5/6}, \]  

(33)

and

\[ \hat{\delta}_{\text{in}}* \simeq c_\delta \left( \frac{\tau}{S} \right)^{1/2} \left( \ln \left( \frac{\tau \hat{a}_0^3}{S^2 \tilde{\psi}_0^6} \right) \right)^{-1/2}. \]  

(34)

Note that these expressions are identical to those obtained in Comisso et al. (2016) provided that the quantity \( 6^{2/3} c_a \) is not explicitly set to unity.

From Equation (31), together with Equation (21), we can see that the final inverse-aspect ratio turns out to be bounded between \( S^{-1/2} < \hat{a}_* < \tau^{2/3} S^{-1/3} \). Equation (31) also indicates that \( \hat{a}_* \) decreases for smaller perturbation amplitudes of \( \tilde{\psi}_0 \). The final inverse-aspect ratio as a function of \( S \) for two different values of \( \tilde{\psi}_0 \) is plotted in Figure 5. An inspection of this figure reveals that the Sweet–Parker aspect ratio can be attained only for moderately high \( S \) values. The domain of existence of the Sweet–Parker aspect ratio may be slightly extended in low-noise systems, but for most of the astrophysically relevant regimes the final width of the reconnecting current sheet remains thicker as predicted by Equation (31).
The dependence of the growth rate $\hat{\gamma}_* \text{ and the wavenumber } \hat{k}_* \text{ as a function of the Lundquist number } S \text{ significantly upon considering large } S \text{ systems. This is clearly shown in Figures 6 and 7, where the black dashed lines represent the earlier scalings, which are clearly not applicable to large- } S \text{ plasmas, while the solid curves represent the results that have been obtained by means of this new theoretical approach. The behavior of } \hat{\gamma}_* \text{ is nonmonotonic in } S, \text{ while } \hat{k}_* \text{ displays a monotonic behavior but with much lower values with respect to the Sweet–Parker–based solution for large values of } S. \text{ While it could seem counterintuitive at first glance, the decrease of the final growth rate for very large } S \text{ can be understood by noting that the inner layer width decreases for an increasing } S, \text{ therefore, a given noise amplitude leads to perturbation amplitudes closer to the condition for the end of the linear phase if } S \text{ is larger. This in turn reduces the time available for the acceleration of the perturbation growth.}

Our approach also enables a quantification of the effects of noise: lower values of the noise can increase the final instantaneous growth rate and the number of plasmoids (proportional to $\hat{k}_*$), as can be seen from the orange curves in Figures 6 and 7. Finally, note that for large-$S$ astrophysical environments, Equation (34) (not plotted here) indicates that the inner resistive layer width at the end of the linear phase is thicker than what would be predicted using the Sweet–Parker–based solution (19).

An important observable that can be duly obtained from Equation (26) is the time that has elapsed since the current sheet evolution commenced at the initial inverse-aspect ratio $\hat{a}_0$. This timescale corresponds to

$$\hat{t}_s \simeq \tau \ln \left[ \frac{\hat{a}_0 S^{1/3}}{\hat{\gamma}_*^{1/3}} (\ln \theta_R)^2/3 \right]. \quad (35)$$

Figure 8 shows that the elapsed time computed by means of the principle of least time has a nonmonotonic behavior, confirmed also by recent numerical simulations (Huang et al. 2017), and after reaching a minimum value at moderate $S$ values it increases as predicted by Equation (35). Note that the time $\hat{t}_s$ does not correspond to the time required for the plasmoids to
grow. This is because the final (dominant mode) wavenumber \( \hat{k}_s \) remains quiescent for a certain period of time before being subject to violent growth over a short timescale (as shown in the example in Figure 3).

The actual time that it takes for the final wavenumber to undergo finite growth is \( \tau_p = \tau \ln(\hat{a}_{an}/\hat{a}_s) \), where \( \hat{a}_{an} \) is the inverse-aspect ratio at the onset time, i.e., when \( \tilde{\gamma}(\hat{k}_s, \hat{a}_{an}) = 1/\tau \). Using Equation (12) and retaining the dominant terms, we find \( \hat{a}_{an} \approx \hat{k}_s \hat{S}^{-1/2} \tau^{-1/2} \). Therefore, the intrinsic timescale \( \tau_p \) of the plasmoid instability becomes

\[
\tau_p \simeq \tau \ln [C_k (\ln \theta_R)^{3/2}] .
\]

(36)

This timescale exhibits a very weak dependence on the Lundquist number and the natural noise of the system, thereby implying that the intrinsic timescale of the plasmoid instability is nearly universal for exponentially thinning current sheets.

Finally, we want to evaluate the value of the Lundquist number above which the scaling laws of the plasmoid instability change behavior as described by the previously obtained equations. We refer to this value as to the transitional Lundquist number. It can be obtained by equating the two asymptotic behaviors of \( \hat{a}_s \), which yields the equation

\[
S_T^{-1/4} \ln \left( \frac{\tau^{(2 - 5\alpha)}/2}{2\varepsilon \hat{S}^{(\alpha - 1)}/2} \right) = \tau .
\]

(37)

The exact explicit solution of this equation is

\[
S_T = \left[ \frac{\hat{a}_s}{\tau} \right]^{4} W \left( \frac{1}{\hat{a}_s} \left( \frac{\tau^{(2 - 5\alpha)}/2}{2\varepsilon \hat{S}^{(\alpha - 1)}/2} \right) \right) ,
\]

(38)

where \( \hat{a}_s = 2(4 - \alpha) \) and \( W(z) \) is the Lambert W function, which is defined such that \( W(z)e^{W(z)} = z \). This expression exhibits a complex dependence on the noise level (\( \varepsilon \) and \( \alpha \)), and the timescale of the driving process (\( \tau \)). A simpler asymptotic approximation can be constructed when considering large arguments of the Lambert W function. In this case (Corless et al. 1996),

\[
W(z) = \ln(z) - \ln(\ln(z)) + o(1).
\]

(39)

Keeping only the first term of this expansion, we obtain

\[
S_T = \left[ \frac{1}{\tau^4} \ln \left( \frac{\tau^{(2 - 5\alpha)}/2}{2\varepsilon \hat{S}^{(\alpha - 1)}/2} \right) \right] .
\]

(40)

From this expression we can see that \( S_T \) decreases if the timescale of the current sheet thinning becomes larger. Furthermore, an increase of \( S_T \) occurs for lower values of \( \varepsilon \) and/or increasing values of \( \alpha \). The accurate behavior of the transitional Lundquist number as a function of the system noise for a wide range of noise amplitudes is shown in Figure 9. The transitional Lundquist number turns out to be fairly modest even for very low noise amplitudes, implying that the plasmoid instability in most astrophysical systems should follow the newly obtained formulas.

3.2. Generalized Current Sheet Shrinking

It is possible to generalize the results obtained for exponentially thinning current sheets in order to also describe current sheets whose thinning depends algebraically on time.

Figure 9. Transitional Lundquist number \( S_T \) as a function of the noise amplitude \( \varepsilon \) for three different values of the spectral index \( \alpha \). Recall that \( \varepsilon \sim \varepsilon \sim \tau^{-1} \). The curves are given by Equation (38) with \( \tau = 1 \).

While the former is the natural consequence of an instability-driven current sheet, the latter has been shown to occur in several cases of forced magnetic reconnection (Hahm & Kulsrud 1985; Wang & Bhattacharjee 1992; Fitzpatrick 2003; Bhattacharjee 2004; Birn et al. 2005; Hosseinpour & Vekstein 2008; Comisso et al. 2015a, 2015b). To encompass both exponential and algebraic behaviors, we consider a generalized current shrinking function of the form

\[
\hat{a}(\hat{t})^2 = \left( \hat{a}_0^2 - \hat{a}_\infty^2 \right) \left( \frac{\tau}{\tau + 2\hat{t}/\chi} \right)^\chi + \hat{a}_\infty^2,
\]

(41)

where \( \hat{a}_\infty = S^{-1/2} \). This expression recovers the exponential thinning specified in Equation (21) when taking the limit \( \chi \to \infty \), while other cases can be obtained by considering different values of \( \chi \). For example, setting \( \chi = 2 \), we obtain a current sheet thinning that is inversely proportional in time. This is relevant for various forced reconnection models, most notably the Taylor model (Hahm & Kulsrud 1985; Wang & Bhattacharjee 1992; Fitzpatrick 2003; Comisso et al. 2015a, 2015b; Zhou et al. 2016; Beidler et al. 2017), which has applications in both laboratory and astrophysical plasmas. For the adopted generalized current shrinking function, the plasmoid half-width (2) starts to grow when \( \hat{t} > \hat{t}_f \), and the transitional time that separates the two asymptotic solutions for the plasmoid instability is

\[
\hat{t}_f = \frac{\chi\tau^2}{2} \left( 1 + \left( \frac{\hat{a}_0^2}{\hat{a}_\infty^2} \right)^{1/\chi} - 1 \right).
\]

(42)

As before, we are especially interested in the case \( \hat{t} < \hat{t}_f \) since it is astrophysically more relevant. However, to derive the analytical solution in this case we consider only the small \( \hat{\Delta}' \) branch of the dispersion relation in consideration of the fact that \( \hat{\gamma}_{\Delta} \approx \hat{\gamma}_f (\hat{t}_f) \), where \( \hat{\gamma}_f \) is the instantaneous growth rate of the fastest-growing mode. Thus we approximate \( \hat{\gamma}(\hat{a}) \) with \( \hat{\gamma}_f (\hat{a}) \) in Equation (20), again using \( \hat{\Delta}' \hat{a} \simeq 2(\hat{k}_d)^{-1} \) and
\[ \hat{a}_s \ll \hat{a}_0. \] Therefore we obtain
\[ \hat{a}_s^{\prime}/\zeta = \frac{c_e \chi}{2 \pi k} \frac{\hat{a}_s^{3(2-5\alpha)/4}}{S^{3(2+\alpha)/4}} \sim \frac{\hat{a}_0^{2/\zeta}}{S^{1/2}}, \quad (43) \]
where
\[ \zeta := \frac{2\chi}{4 + 3\chi}. \quad (44) \]
Solving this equation along the same lines as Equation (26), we obtain
\[ \hat{a}_s \sim \left( \frac{\hat{a}_0^{2/\zeta}}{S^{1/2} \ln \Theta_R} \right)^{\zeta}, \quad (45) \]
where
\[ \Theta_R := \frac{(\alpha_0^{2/\zeta} \tau (\alpha^{-1/2} + (\alpha-5\alpha)/4)}{2 \pi k \chi (\alpha-2+\alpha)/4}. \quad (46) \]
From Equation (45), we obtain the following generalized scaling relations for the plasmoid instability:
\[ \hat{\gamma}_s \sim S^{(3\chi-2)/4} \left( \frac{\ln \Theta_R}{\hat{a}_0^{2/\zeta} \tau} \right)^{3\chi/4}, \quad (47) \]
\[ \hat{k}_s \sim S^{(3\chi-2)/8} \left( \frac{\ln \Theta_R}{\hat{a}_0^{2/\zeta} \tau} \right)^{5\chi/4}, \quad (48) \]
and
\[ \hat{\kappa}_{im} \sim S^{-(3\chi+2)/12} \left( \frac{\hat{a}_0^{2/\zeta} \tau}{\ln \Theta_R} \right)^{3\chi/4}. \quad (49) \]
For \( \chi \to \infty \), we recover the scaling relations given by Equations (26)–(30), while different choices of \( \chi \) give us the scaling relations relevant for different algebraic thinning possibilities. These expressions indicate that faster current sheet shrinking rates (a larger \( \chi \)) or a smaller \( \tau \), lead to a larger aspect ratio (1/\( \hat{a}_s \)), growth rate, and wavenumber. On the other hand, the inner resistive layer width at the end of the linear phase decreases for faster current sheet formation. It is also interesting to observe that for \( \zeta > 2/5 \) (i.e., \( \chi > 2 \)), the number of plasmoids increases with \( S \) in the astrophysically relevant regimes, but the opposite trend is possibly manifested for \( \chi < 2 \). In other words, for the latter case, the number of plasmoids can actually decrease in this regime as \( S \) increases. For \( \chi = 2 \), where the thinning is inversely dependent on the time, the scaling of the number of plasmoids with \( S \) is weak (logarithmic).

As should be expected, the elapsed time from the initial aspect ratio can change significantly for different current sheet formation rates. Indeed, from Equations (45) and (41), with \( \hat{I} < \hat{I}_T \), the elapsed time turns out to be
\[ \hat{t}_e \sim \frac{\chi \tau}{2} \left[ \frac{\hat{a}_0^{2\chi - \zeta} / \chi^2}{\tau} \left( \frac{S^{1/2} \ln \Theta_R}{\alpha_0^{2/\zeta} \tau} \right)^{2\chi / \zeta} - 1 \right]. \quad (50) \]
Lower values of \( \chi \) lead to much higher values of the elapsed time \( \hat{t}_e \), implying that the final dominant wavenumber remains quiescent for a much longer period when the current sheet evolution is slower.

The transitional Lundquist number \( S_L \) for this class of generalized thinning current sheets can be computed by equating the two asymptotic branches for \( \hat{a}_s \) in a manner analogous to that of exponential thinning. Thus we are led to the equation
\[ S_L^{(\zeta-1)/2\zeta} \ln \Theta_R \sim \hat{a}_0^{2/\zeta} \tau, \quad (51) \]
which can be inverted in a straightforward manner, by means of the Lambert \( W \) function, to obtain \( S_L \). Similarly, it is also possible to compute the timescale \( \tau_p \) for the plasmoid instability in this generalized scenario by following the procedure outlined for exponential thinning sheets.

We explore the implications of our preceding results for astrophysical plasmas in Section 5. Next, we consider the visco-resistive regime and carry out a similar analysis.

### 4. Visco-resistive Regime

In this section, we derive the corresponding scaling laws of the plasmoid instability in the presence of strong plasma viscosity, namely when \( \rho_m \gg 1 \). Plasma viscosity is indeed important in several astrophysical environments such as (1) the warm interstellar medium, (2) protogalactic plasmas, (3) the intergalactic medium, and (4) accretion discs around neutron stars and black holes (Kulsrud & Anderson 1992; Brandenburg & Subramanian 2005; Balbus & Henri 2008). In this case, \( \lim_{t \to \infty} f (\hat{t}) = \hat{a}_0^{1/2} S^{-1/2} P_m^{1/4} \). Furthermore, two different relations for the growth rate in the small-\( \hat{\Lambda} \) and large-\( \hat{\Lambda} \) regimes must be considered. For \( \hat{\Lambda} \hat{\kappa}_i \ll 1 \), the growth rate of the tearing modes modified by strong plasma viscosity is (Bondeson & Sobel 1984)
\[ \hat{\gamma}_i \approx c_1 \hat{k}^{1/3} \hat{a}^{-1/3} \hat{\Lambda}^{1/3} (S^{-2/3}) P_m^{-1/6}, \quad (52) \]
where \( c_1 = (6^{1/2} \pi)^{-1} I(1/6)/I(5/6) \approx 0.48 \). On the other hand, for \( \hat{\Lambda} \hat{\kappa}_i \gtrsim 1 \) the growth rate satisfies the following relation (Porcelli 1987):
\[ \hat{\gamma}_i \approx c_2 \hat{k}^{2/3} \hat{a}^{-2/3} S^{-1/3} P_m^{-1/3}, \quad (53) \]
where \( c_2 \approx 1.53 \). An effective approximation for \( \hat{\gamma}_i \) across the entire domain of \( \hat{\Lambda} \) can be constructed as before, by using Equation (12).

In a manner analogous to the resistive regime, by combining Equations (5) and (6) and specifying the inner visco-resistive layer width (Porcelli 1987),
\[ \hat{\kappa}_{im} = (\hat{a} / \hat{k})^{1/3} S^{-1/3} P_m^{1/6}, \quad (54) \]
it is possible to obtain the least-time equation
\[ \left\{ \frac{\partial \hat{\gamma}_i}{\partial \hat{k}} + \frac{2}{3 \hat{k}} + \frac{1}{\hat{\psi}_0} \left( \frac{\partial \hat{\psi}_0}{\partial \hat{k}} \right) \right\} \hat{\kappa}_{im} \hat{\kappa}_{im} = 0, \quad (55) \]
which follows from a careful application of Equation (7). By repeating the procedure delineated in the previous section, we find the counterpart of Equation (15) that is valid in the
visco-resistive regime. This corresponds to

\[
1 - \frac{3}{\hat{t}_m} \left[ 1 + c_2^2 \hat{k}_*^{-4/3} \hat{a}_*^{-5/3} S^{-1/3} \hat{p}_m^{1/6} \right] + \left( 1 + c_2^2 \hat{k}_*^{-4/3} \hat{a}_*^{-5/3} S^{-1/3} \hat{p}_m^{-1/6} \right) = c_2^2 \hat{k}_*^{-2/3} \hat{a}_*^{-2/3} S^{-1/3} \hat{p}_m^{-1/3} - \hat{k}_*^{-2/3} \hat{a}_*^{-2/3} S^{-1/3} \hat{p}_m^{-1/3},
\]

(56)

It can be shown a posteriori that for \( \hat{w}(\hat{k}_*, \hat{t}_0) \ll \hat{\delta}_m(\hat{k}_*, \hat{t}_0) \), the two terms on the right-hand side must approximately balance each other. Thus we end up with

\[
\hat{k}_* \approx \lambda_\kappa \hat{a}_*^{-5/4} S^{-1/4} \hat{p}_m^{1/8},
\]

(57)

\[
\hat{\gamma}_* \approx \lambda_\gamma \hat{a}_*^{-3/2} S^{-1/2} \hat{p}_m^{-1/4},
\]

(58)

\[
\hat{\delta}_{\text{in}} \approx \lambda_\delta \hat{a}_*^{-3/4} S^{-1/4} \hat{p}_m^{1/8},
\]

(59)

where \( \lambda_\kappa \), \( \lambda_\gamma \), and \( \lambda_\delta \) are \( O(1) \) coefficients.

By assuming that the reconnecting current sheet has the time to reach the asymptotic value \( \hat{a}_* \approx S^{-1/2} \hat{p}_m^{1/4} \), one can find

\[
\hat{\gamma}_* \approx S^{1/4} \hat{p}_m^{-5/8}, \quad \hat{k}_* \approx S^{3/8} \hat{p}_m^{-3/16}, \quad \hat{\delta}_{\text{in}} \approx S^{-5/8} \hat{p}_m^{5/16},
\]

(60)
as in Loureiro et al. (2013) and Comisso & Grasso (2016). Note that plasma viscosity leads to a decrease of the asymptotic aspect ratio \( (1/\hat{a}_*) \) and of the growth rate associated to it. As a consequence of these two factors, the validity of relations (60) can be extended to a larger \( S \)-domain, as shown in the subsequent analysis. On the other hand, for large enough Lundquist numbers, it is necessary to evaluate \( \hat{a}_* \) from the inverse-aspect ratio equation that is valid in the \( \hat{p}_m \gg 1 \) regime, which corresponds to

\[
\ln \left( \frac{\lambda_\kappa \hat{a}_*^{1/4}}{2^{2/3} S^{1/4} \hat{p}_m^{1/8}} \right) = \frac{1}{2} \int_{\hat{a}_0}^{\hat{a}_*} \hat{\gamma}(\hat{a}) \frac{d\hat{a}}{\hat{a}} + \hat{\delta}_{\text{in}}(\hat{a}).
\]

(61)

In a manner similar to the \( \hat{p}_m \ll 1 \) case, we first evaluate \( \hat{a}_* \) and the properties of the plasmoid instability for an exponentially thinning reconnection layer and then generalize the obtained results to also encompass algebraic thinning layers.

### 4.1. Exponentially Shrinking Current Sheet

For strong plasma viscosity we have \( \hat{a}_\infty = S^{-1/2} \hat{p}_m^{1/4} \) (Park et al. 1984) in the exponential thinning function described by Equation (21). Considering the case \( \hat{t} < \hat{t}_1 \), and adopting the same approximations employed for the resistive regime, the solution of Equation (61) can be written as

\[
\hat{a}_* \approx \frac{\tau^{2/3}}{S^{1/3} \hat{p}_m^{1/6}} \left( \ln \theta_V \right)^{-2/3},
\]

(62)

where

\[
\theta_V = \theta_R \hat{p}_m^{1+2\alpha}/2.
\]

(63)

Therefore, the growth rate, wavenumber, and inner layer width at the end of the linear phase become

\[
\hat{\gamma}_* \approx \lambda_\gamma \frac{\ln \theta_V}{\tau},
\]

(64)

\[
\hat{k}_* \approx \frac{\lambda_\kappa}{\lambda_\gamma} \frac{\tau^{2/3}}{\hat{p}_m^{1/3}} \left( \ln \left( \frac{\tau \hat{p}_m^{1/2} \hat{a}_0^3}{S^{2/3} \hat{w}_0^{1/6}} \right) \right)^{1/2}
\]

(65)

\[
\hat{\gamma}_* \approx \frac{\lambda_\gamma}{\tau} \ln \left( \frac{\tau \hat{p}_m^{1/2} \hat{a}_0^3}{S^{2/3} \hat{w}_0^{1/6}} \right),
\]

(66)

\[
\hat{k}_* \approx \frac{\lambda_\kappa}{\lambda_\gamma} \frac{S^{1/6} \hat{p}_m^{1/3}}{\tau^{5/6}} \left( \ln \left( \frac{\tau \hat{p}_m^{1/2} \hat{a}_0^3}{S^{2/3} \hat{w}_0^{1/6}} \right) \right)^{5/6},
\]

(67)

Figure 10. Final inverse-aspect ratio \( \hat{a}_* \) as a function of the Lundquist number \( S \) for two values of the magnetic Prandtl number \( \hat{p}_m \). In both cases \( \hat{w}_0 \equiv 2(\epsilon \hat{a}_0^{1/2})/10^{-35} \tau = 1 \), and \( \hat{a}_1 = 1/\tau \). The solid lines refer to the numerical solution of the system given by Equations (5)–(6). The purple dashed line refers to Equation (67), while the black dashed line denotes the viscous Sweet–Parker scaling \( \hat{a}_* \approx S^{-1/2} \hat{p}_m^{1/4} \) with \( \hat{p}_m = 10 \).

Plasma viscosity enters in all scaling laws through the logarithmic contributions and also as power-law factors in the relations for the final aspect ratio of the current sheet and the wavenumber of the plasmoids.

For definiteness, we consider again the case in which the natural noise amplitude is approximately constant \( (\alpha = 0) \). In this case Equation (62) reduces to

\[
\hat{a}_* \approx \frac{\tau^{2/3}}{S^{1/3} \hat{p}_m^{1/6}} \left( \ln \left( \frac{\tau \hat{p}_m^{1/2} \hat{a}_0^3}{S^{2/3} \hat{w}_0^{1/6}} \right) \right)^{-2/3},
\]

(68)

This relation indicates that an increase of the magnetic Prandtl number leads to a decrease of the final inverse-aspect ratio. Note that this trend is the opposite to what occurs in the viscous Sweet–Parker limit that is valid for lower \( S \) values. It can be understood by noticing that plasma viscosity reduces the growth rate for a fixed value of the aspect ratio, which implies that smaller values of the inverse-aspect ratio are required for the onset and rapid growth of the final dominant mode. The described behavior is also clearly evident from Figure 10, which illustrates the final inverse-aspect ratio as a function of \( S \) for two different values of the magnetic Prandtl number.

For constant noise \( \hat{w}_0 \) we also have

\[
\hat{\gamma}_* \approx \frac{\lambda_\gamma}{\tau} \ln \left( \frac{\tau \hat{p}_m^{1/2} \hat{a}_0^3}{S^{2/3} \hat{w}_0^{1/6}} \right),
\]

(69)
and

$$\hat{\delta}_{\text{in}} \simeq \lambda (\frac{\tau}{S})^{1/2} \left[ \ln \left( \frac{\tau P_m^{1/2} \delta_0^3}{S^2 \tilde{v}_0^3} \right) \right]^{-1/2}. \quad (70)$$

We see that the final growth rate exhibits a weak dependence on the magnetic Prandtl number. This can also be appreciated from Figure 11, where the curves for $P_m = 0$ and $P_m = 10$ begin to overlap at very large values of $S$. A weak dependence on the magnetic Prandtl number also occurs for the final inner layer width, as seen from Equation (70). In contrast, a stronger dependence on $P_m$ is found in the final dominant mode wavenumber. In particular, Equation (69) predicts an increase in the number of plasmoids for larger values of $P_m$. This behavior is verified for large values of $S$ in Figure 12. Notice that the opposite trend occurs for lower $S$ values, where $k_0$ decreases as $P_m$ increases—this is evident from the second of relations (60) and Figure 12. Thus on interstellar scales with high values of $P_m$ (and $S$), one would expect the production of a higher number of plasmoids compared to the number of plasmoids in the resistive case.

In analogy with the resistive case, the elapsed time since the beginning of the current sheet evolution is given by

$$t \approx \frac{\tau}{\lambda} \ln \left[ \tilde{a}_0 \left( \frac{S^{1/3} P_m^{1/6}}{\tau^{2/3} \ln \theta_T} \right) \right]. \quad (71)$$

while the intrinsic timescale of the plasmoid instability is

$$\tau_p \approx \frac{\tau}{\lambda} \ln \left[ \lambda (\ln \theta_T)^{3/2} \right]. \quad (72)$$

The elapsed time $t$ exhibits a weak dependence on the magnetic Prandtl number. In particular, Equation (71) reveals that the total time $t$ slightly increases for increasing $P_m$ values, which is also apparent upon inspecting Figure 13. In addition, the minimum of $t$ occurs at a larger $S$ since $S_T$ increases with $P_m$ The timescale $\tau_p$ displays an even weaker $P_m$ dependence to the point that it is fair to say that $\tau_p$ remains almost unchanged with respect to the results obtained for the resistive regime.

An important point that must be recognized is that increasing the plasma viscosity eventually leads to the persistence of the viscous Sweet–Parker regime even for extremely large $S$ values. Hence for a system with a given Lundquist number there exists a value of $P_m$ for which the final inverse-aspect ratio $\hat{a}_0$ is minimum. This fact is confirmed by inspecting Figure 14, where the final inverse-aspect ratio is displayed as a function of $P_m$ for different Lundquist numbers. The value of $P_m$ corresponding to the minimum inverse-aspect ratio can be obtained by equating the two asymptotic behaviors of $\hat{a}_0$. This gives us the transitional magnetic Prandtl number

$$P_m = S^{2/5} \left[ \frac{\hat{c}_0}{\tau} \left( \frac{\tau/\theta_T}{S^{2\alpha-10}/10} \right)^{1/\alpha} \right]^{1/5}, \quad (73)$$

where $\hat{c}_0 = (4 + 8\alpha)/5$. This expression indicates that the transitional magnetic Prandtl number should increase monotonically with $S$, which is indeed confirmed from Figure 14. Finally, we can also easily obtain the transitional Lundquist number.
Sweet are the same as in Figure 10. Solid lines refer to the numerical solution of the system given by Equations (5–6). The black dashed lines denote the viscous Sweet–Parker scaling \( \hat{a}_* \sim S^{-1/2} P_m^{1/4} \), while the other dashed lines refer to Equation (67).

\[
S_T = P_m^{5/2} \left[ \frac{\hat{a}}{\tau} W \left( \frac{1}{\alpha} \left( \frac{r^2/P_m}{2^\epsilon 3} \right)^{1/6} \right) \right]^{\epsilon/4}.
\]

This relation tells us that \( S_T \) has a strong dependence on \( P_m \), implying that plasma viscosity can significantly extend the domain of existence of (viscous) Sweet–Parker current sheets.

4.2. Generalized Current Sheet Shrinking

We complete the visco-resistive scalings by extending the previously obtained relations to the various current thinning possibilities described by Equation (41) with \( \hat{a}_\infty = S^{-1/2} P_m^{1/4} \).

For \( i < i_T \), when using the same approximations employed for the resistive regime, the final inverse-aspect ratio of the current sheet becomes

\[
\hat{a}_* \sim \left( \frac{\hat{a}_0^{2/\chi}}{S^{1/2} P_m^{1/4} \ln \Theta_V} \right)^{\chi},
\]

where

\[
\Theta_V := \Theta_K P_m^{3(4+2\alpha-2\zeta+5\alpha\chi)/16}.
\]

Therefore, the final growth rate, wavenumber, and inner layer width in the visco-resistive regime are

\[
\begin{align*}
\hat{\gamma}_* & \sim S^{(3-2)/4} P_m^{(3-2)/8} \left( \frac{\ln \Theta_V}{\hat{a}_0^{2/\chi/\tau}} \right)^{3/2}, \\
\hat{k}_* & \sim S^{(5-2)/8} P_m^{(5-2)/16} \left( \frac{\ln \Theta_V}{\hat{a}_0^{2/\chi/\tau}} \right)^{5/4}, \\
\hat{c}_m & \sim S^{-(2+3\chi)/8} P_m^{(2-3\chi)/16} \left( \frac{\hat{a}_0^{2/\chi/\tau}}{\ln \Theta_V} \right)^{3/4},
\end{align*}
\]

and the final elapsed time is

\[
\hat{t}_* \sim \frac{\chi \tau}{2} \left[ \hat{a}_0^{2(\chi-\zeta)/\chi} \left( \frac{S^{1/2} P_m^{1/4}}{\ln \Theta_V} \right)^{2\chi/\chi} - 1 \right].
\]

For \( \chi \to \infty \) we recover the scaling relations obtained for exponentially thinning current sheets, while other choices of \( \chi \) enable us to obtain the scaling laws for different cases of algebraic thinning. Moreover, we note that a particularly important and robust effect of plasma viscosity, which is common to both exponential and algebraically thinning current sheets, is that it gives rise to a significant increase in the number of plasmoids that enter the nonlinear evolutionary phase for most of the astrophysically relevant (very large \( S \)) plasmas.

5. Discussion

In this section, we discuss some of the primary astrophysical consequences of the obtained scaling laws. We have shown that in astrophysical environments, reconnecting current sheets break up before they can reach the aspect ratio predicted by the widely employed Sweet–Parker model (Parker 1957; Sweet 1958). The degree of discrepancy as compared to this classic prediction depends on several factors: the Lundquist number \( S \), the magnetic Prandtl number \( P_m \), the noise (both the amplitude and the spectrum) of the system \( \nu_0 \), the characteristic rate of current sheet evolution \( 1/\tau \), and the thinning process (taken into account by \( \chi \) in our generalized current shrinking function). The importance of the noise and of the thinning process has often been underestimated in previous studies. Of the two, the thinning process \( \chi \) is the one that has stronger effects on the behavior of the plasmoid instability, as can be seen from the generalized scaling laws derived in Sections 3.2 and 4.2. This points out the importance of understanding the mechanism that drives the current sheet thinning in order to apply the correct scaling relations.

We emphasize that during the current sheet thinning the reconnection rate remains slow. It is only after the plasmoids become nonlinear and break up the reconnecting current sheet that a sudden increase in the reconnection rate is manifested. When fast reconnection is triggered, during the highly nonlinear evolutionary phase of the plasmoids, the reconnecting current sheet is replaced by a chain of plasmoids of different sizes separated by secondary current sheets (Shibata & Tanuma 2001; Bhattacharjee et al. 2009; Cassak et al. 2009; Huang & Bhattacharjee 2010; Uzdensky et al. 2010; Loureiro et al. 2012; Takamoto 2013; Comisso et al. 2015a; Shibayama...
et al. 2015). This fragmentation could reach kinetic scales, allowing an even faster Hall/collisionless reconnection regime (Daughton et al. 2006, 2009; Shepherd & Cassak 2010; Huang et al. 2011; Ji & Daughton 2011; Huang & Bhattacharjee 2013; Comisso & Bhattacharjee 2016). Therefore, the general analysis presented in this paper enables us to determine the onset of fast magnetic reconnection by means of the plasmoid instability, while the problem of the reconnection rate during the fast reconnection regime is not addressed in this work.

Finally, we observe that the growth of the plasmoids slows down when they enter into the early nonlinear phase, which occurs for \( \delta \gtrsim \hat{w} < \hat{a} \). But in spite of this “deceleration,” they can complete the early nonlinear evolution in a short timescale. This is because the nonlinear evolution of the plasmoid instability does not exhibit a slow Rutherford evolution as it occurs for nonlinear \( m \gtrsim 2 \) magnetic islands in fusion devices. Indeed, a Rutherford evolution (Rutherford 1989) requires \( \hat{\Delta} \lesssim 1 \), while the plasmoids enter into the nonlinear phase with \( \hat{\Delta} \gtrsim 1 \). This can be shown by using Equations (16) and (18) for the resistive case, or Equations (57) and (59) for the visco-resistive case. In both cases, at the beginning of the nonlinear phase, we obtain

\[
\hat{\Delta} \hat{w}_* \approx 2. \tag{81}
\]

Therefore, after the linear regime the plasmoid instability will evolve only through a fast Waelbroeck phase (Waelbroeck 1989), as detailed in Comisso & Grasso (2016) for the case of a time-independent current sheet. An important consequence is that the end of the linear phase practically corresponds to the disruption of the reconnecting current sheet when it is defined by the condition \( \hat{w} \sim \hat{a} \) (Comisso & Grasso 2016; Uzdensky & Loureiro 2016). One could also define the disruption of the current sheet to correspond exactly to the end of the linear phase. Indeed, at the end of the linear phase the current density fluctuations are of the same order as the background current density, implying that the reconnecting current sheet loses its integrity and can be regarded as having been disrupted (Huang et al. 2017).

In the following, we apply the obtained scaling relations to reconnecting current sheets in two different astrophysical systems: the solar corona (where \( P_m \ll 1 \)) and the warm interstellar medium (where \( P_m \gg 1 \)).

### 5.1. The Solar Corona

The solar corona is one of the most typical environments where reconnection is expected to play a fundamental role. In fact, magnetic reconnection is considered the leading mechanism for energy release in the form of solar flares (Shibata & Magara 2011; Bárá et al. 2011a, 2011b; Shen et al. 2011; Li et al. 2015; Shibata et al. 2016; Janvier 2017). It is also responsible for coronal mass ejections (Milligan et al. 2010; Chen 2011; Karpen et al. 2012; Mei et al. 2012; Ni et al. 2012; Lin et al. 2015). Furthermore, it is thought that reconnection may play a fundamental role in heating the solar corona (Klimchuk 2006; Parnell & De Moortel 2012; De Moortel & Browning 2015). Due to the relevance of magnetic reconnection in the solar corona, we explore some of the predictions of our scaling laws in this context and compare them against the scalings based on Sweet–Parker current sheets (Tajima & Shibata 1997; Loureiro et al. 2007; Bhattacharjee et al. 2009; Baalrud et al. 2012; Huang & Bhattacharjee 2013; Loureiro et al. 2013; Comisso & Grasso 2016).

We consider a typical case of an exponentially thinning current sheet in the solar corona. In this environment, we can adopt \( B_0 = 50 \) G, \( n = 10^{10} \) cm\(^{-3} \), \( T = 10^6 \) K, and \( L = 2 \times 10^4 \) km (Ji & Daughton 2011). We assume that the density \( n \) and the temperature \( T \) are the same for electrons and ions. From these parameters, we obtain \( \gamma_* \approx 10^3 \) km s\(^{-1} \) and \( S \approx 10^{13} \). Since the (perpendicular) magnetic Prandtl number is low in the solar corona, we can adopt the scaling laws obtained in Section 3. Exponentially shrinking current sheets form on the Alfvénic timescale (Kulsrud 2005), enabling us to set \( \tau = 1 \). We also set \( c_s = 1 \) and \( c_A = 1/2 \), which is consistent with what can be found from the full solution of the principle of least time, and choose \( \hat{a}_0 = 1/\pi \). The latter quantity is chosen such that the starting time corresponds to the moment at which the longest mode (with \( k = 1 \)) becomes marginally unstable. Finally, we assume a normalized perturbation \( \hat{w}_0 = 10^{-15} \) for all wavelengths; this corresponds to \( \hat{w}_0 \approx 4 \times 10^{-8} \), or, in dimensional units, \( w_0 \approx 70 \) cm, which is more than one order of magnitude larger than the electron skin depth (Ji & Daughton 2011), thereby indicating that the corresponding fluctuation is larger than a physically relevant (kinetic) length scale. The natural noise of the system is clearly subject to a great degree of uncertainty; however, since the scaling laws of the plasmoid instability are weakly (logarithmically) dependent on the noise level, the final results will not be very sensitive to the actual choice of \( \hat{w}_0 \). Furthermore, this choice is consistent with the final growth rates obtained in the numerical simulations by Huang et al. (2017).

Expressing our final results in dimensional units, we find

\[
\begin{align*}
& a_* \approx 8 \times 10^3 \text{ cm}, \\
& \gamma_* \approx 20 \tau_A^{-1}, \\
& k_* \approx 3 \times 10^3 \text{ L}^{-1}, \\
& \delta_{in*} \approx 84 \text{ cm},
\end{align*}
\tag{82}
\]

where \( \tau_A = L/v_A \approx 20 \) s. For our choice of parameters, we have \( S_T \approx 2 \times 10^7 \), which validates the use of Equations (31)–(34). On the other hand, using the Sweet–Parker–based scalings would have yielded the following results:

\[
\begin{align*}
& a_* \approx 6 \times 10^2 \text{ cm}, \\
& \gamma_* \approx 1 \times 10^3 \tau_A^{-1}, \\
& k_* \approx 7.5 \times 10^4 \text{ L}^{-1}, \\
& \delta_{in*} \approx 13 \text{ cm}.
\end{align*}
\tag{83}
\]

From the results (82) and (83), we see that the final inverse-aspect ratio predicted by Equation (31) is higher than the Sweet–Parker scaling by more than an order of magnitude, implying that the usual Sweet–Parker current sheets cannot form in the solar corona. Similarly, the inner resistive layer width at the end of the linear phase is also thicker than the one obtained from the Sweet–Parker–based scaling.

A significant discrepancy between the two theoretical approaches is also evident from inspecting the final growth rate and the wavenumber of the instability. The growth rate based on a Sweet–Parker current sheet overestimates the actual growth rate by two orders of magnitude for this particular example. Furthermore, the dominant wavenumber that emerges from the linear phase is more than an order of magnitude lower.
than the Sweet–Parker–based solution. This implies that a lower number of plasmoids will appear at the beginning of the nonlinear phase.

An exponentially shrinking current sheet leads to a fast disruption of the current sheet—for this example, we find \( t_\text{s} \approx 11 \tau_A \). An exponential thinning can occur for a current sheet that is being driven by an ideal MHD instability. However, much longer timescales are involved if the current sheet formation is due to other driving processes that lead to an algebraic thinning of the current sheet. For instance, this could be the case when current sheet formation is driven by the motion of the photospheric footpoints of the magnetic field lines.

In the case of an algebraic thinning current sheet, Equations (45)–(49) reveal that the discrepancies, when compared to the Sweet–Parker–based predictions, are even larger than those obtained for an exponentially thinning current sheet. This serves to highlight the importance of adopting an appropriate time-evolving current sheet ansatz when carrying out analyses of the plasmoid instability in astrophysical environments.

5.2. The Interstellar Medium

Another astrophysical environment where magnetic reconnection is thought to play a crucial role is the interstellar medium. The importance of magnetic reconnection stems from the fact that it has been advanced as a possible heating source for the interstellar medium (Parker 1992; Raymond 1992; Zweibel 1999; Tanuma et al. 2001; Elmegreen & Scalo 2004). Furthermore, the nature of magnetic reconnection under interstellar conditions has pivotal implications for the viability of both the galactic dynamo and theories of primordial magnetic fields (Zweibel & Brandenburg 1997; Zweibel & Heiles 1997; Vichnev & Lazarian 1999; Ebrahimi 2016). In dealing with large-scale systems such as the interstellar medium, it is important to recognize that magnetic reconnection operates on timescales that are too large to be observed directly. Therefore it is all the more essential to apply the correct theoretical model to gain information about how reconnection occurs in this system.

Here we consider the case of an exponentially thinning current sheet in an ionized interstellar medium. For this system we can assume the typical parameters \( B_0 = 5 \times 10^{-6} \text{ G} \), \( n = 0.1 \text{ cm}^{-3} \), \( T = 10^4 \text{ K} \), and \( L = 10^{16} \text{ cm} \) (Arévalo et al. 2011). The density \( n \) and the temperature \( T \) are considered to be the same for electrons and ions. From these parameters we have \( v_\text{A} \approx 3 \times 10^6 \text{ cm s}^{-1} \), \( S \approx 10^{20} \), and \( P_m \approx 10 \). We therefore must adopt the scaling laws obtained in Section 4. We set \( \lambda_3 = 1 \) and \( \lambda_1 = 1/2 \) as in the previous case and choose \( \tau = 1 \) and \( \tilde{a}_0 = 1/\pi \). Finally, we are left with a noise perturbation that needs to be specified. We assume the normalized noise perturbation to be a constant \( \tilde{\psi}_0 = 10^{-20} \), which corresponds to \( w_0 \approx 10^{-10} \). In dimensional units we have \( w_0 \approx 10^6 \text{ km} \), which happens to be several orders of magnitude larger than physically relevant (kinetic) scales such as the electron and ion skin depths (Arévalo et al. 2011). As explained earlier, this does not significantly affect the final results since they depend only weakly on the perturbation amplitude. Note that for these parameters \( S_T \approx 2 \times 10^{10} \). Therefore the scaling relations that are valid for very large \( S \) values are the correct ones to be used. As with the solar corona, we note that our choice leads to final growth rates that are consistent with those obtained in the numerical simulations by Huang et al. (2017).

From Equations (67)–(70), expressing the final results in dimensional units, we arrive at

\[
\begin{align*}
\alpha & \approx 1.2 \times 10^6 \text{ km}, \\
\gamma & \approx 1.2 \times 10^6 \tau_A^{-1}, \\
k & \approx 2 \times 10^7 \text{ L}^{-1}, \\
\delta_{\text{in}} & \approx 6.5 \times 10^3 \text{ km}.
\end{align*}
\]

Due to the large Lundquist number of the system, the discrepancies with the viscous Sweet–Parker–based predictions are very large. The final inverse-aspect ratio is two orders of magnitude higher than the corresponding viscous Sweet–Parker value. Therefore, it is not possible to even come close to attaining such current sheets in this system. The inner viscous-resistive layer width is also more than one order of magnitude thicker than the one obtained assuming a viscous Sweet–Parker current sheet.

The plasmoid instability at the end of the nonlinear phase is characterized by an instantaneous growth rate that is very similar to the value obtained for the solar corona. It differs by three orders of magnitude from the predictions obtained by assuming a Sweet–Parker sheet, which serves to highlight the fact that the latter scalings are highly inapplicable in the interstellar medium. Finally, a difference of more than two orders of magnitude is found for the number of plasmoids produced at the end of the nonlinear phase; once again, the Sweet–Parker–based theory overestimates the actual number.

The exponential thinning of the current sheet leads to the final aspect ratio in a time \( t_\text{s} \approx 17 \tau_A \) for this application. After this time period has elapsed, fast reconnection can occur at the reconnection rate of \( \sim 10^{-2}(1 + P_m)^{-1/2}v_\text{A}B_0 \) (Comisso et al. 2015a; Comisso & Grasso 2016). Since the Alfvénic timescale is extremely large in this system, we conclude that the final aspect ratio is reached over a period that is \( \sim 1/100 \) the age of the universe. On the other hand, a much longer time is expected to occur for algebraic thinning of the current sheet. In other words, there exists a very long time during which the energy buildup occurs, conceivably even on the order of Hubble time.

6. Conclusions

As described in Section 1, the plasmoid instability has a great impact on astrophysical systems, ranging from solar flares (Shibata & Tanuma 2001; Shibata & Magara 2011) and coronal mass ejections (Mei et al. 2012; Lin et al. 2015) to blazar emissions (Sironi et al. 2015; Petropoulou et al. 2016) and pulsar wind nebulae (Sironi & Spitkovsky 2014; Guo et al. 2015; Sironi et al. 2016). The importance of the plasmoid instability arises from its capacity to prevent highly elongated current sheets from forming. Hence, in breaking up the current
Unlike most previous studies (fast reconnection will occur), we enable us to understand and predict the conditions under which fast reconnection will occur—e.g., the timescales involved. Unlike most previous studies (Tajima & Shibata 1997; Loureiro et al. 2007; Bhattacharjee et al. 2009; Baalrud et al. 2012; Loureiro et al. 2013; Comisso & Grasso 2016) that focused on stationary Sweet–Parker current sheets, we consider dynamically evolving current sheets. One of our most important results is that the Sweet–Parker–based scalings are invalid for a large majority of space and astrophysical plasmas that are typically characterized by very high values of $S$. In this domain, the effects of the outflow in the reconnect layer become negligible and thus do not need to be taken into account (Ni et al. 2010; Huang et al. 2017), implying that the validity of the new scaling relations is preserved.

The dynamical picture of the plasmoid instability is very complex since different modes become unstable at different times and are then subject to rapid growth at different rates. Thus, in order to determine the scaling laws of the current sheet and the plasmoid instability at the end of the linear stage, we utilized a principle of least time that was delineated in a recent Letter (Comisso et al. 2016). We calculated the growth rate, wavenumber, and inner layer width at the end of the linear phase as well as the final aspect ratio of the reconnecting current sheet and the total time elapsed from a given time of its evolution. The analysis was carried out for both resistive and visco-resistive plasmas, since many well-known astrophysical plasmas fall within these regimes.

One of the most important results is that the scaling laws of the plasmoid instability are not simple power laws. These scaling relations depend on the Lundquist number ($S$), the magnetic Prandtl number ($P_m$), the noise (both the amplitude and the spectrum) of the system ($\psi_0$), the characteristic rate of current sheet evolution ($1/\tau$), and the thinning process ($\chi$). We validated the obtained analytical scaling relations by comparing them against the full numerical solutions of the principle of least time and demonstrated that the two are in excellent agreement. Furthermore, we showed that the plasmoid instability comprises a relatively long period of quiescence followed by rapid growth over a shorter timescale. This is important because it enables us to precisely estimate when the plasmoid instability becomes nonlinear and disrupts the reconnecting current sheet.

We contrasted the new scalings against those derived using a Sweet–Parker equilibrium by considering two different astrophysical systems. The first is the solar corona, where (perpendicular) $P_m \ll 1$, enabling us to use the resistive scalings. We showed that the final aspect ratio of the current sheet, the number of plasmoids produced, and the growth rate at the end of the linear stage are much lower (by 1–2 orders of magnitude) than those obtained by assuming a Sweet–Parker current sheet. As a matter of fact, the latter theory is not applicable to the solar corona, which has a very high value of $S \sim 10^{13}$. In addition, we computed the associated timescale and showed that the duration of the linear phase is about 10 times the Alfvén time for an exponentially thinning current sheet.

We also considered the warm interstellar medium, which has (perpendicular) $P_m \gg 1$ and $S \sim 10^2$. The former implies that the visco-resistive scalings must be used since the magnetic Prandtl number is not negligible. As in the resistive case, we found significant differences compared to scalings based on viscous Sweet–Parker current sheets, since the warm interstellar medium does not fall under their domain of applicability. We also found that the linear stage of the instability is $\gtrsim 1/100$ the Hubble time, which implies that fast magnetic reconnection cannot occur (in the considered system) before this time period.

To summarize, we have rendered a dynamical picture of the linear stage of the plasmoid instability in general, time-evolving current sheets for both the resistive and the visco-resistive cases. An important outcome of our analysis is that the earlier Sweet–Parker–based scalings have a limited domain of validity and are thus not applicable to the majority of astrophysical systems. We have derived new scaling relations that are no longer simple power laws and exhibit a complex dependence on the appropriate physical parameters. By applying these scaling relations to the solar corona and the warm interstellar medium, we have highlighted some of the main advantages of the adopted theoretical framework. We anticipate that future studies can gainfully employ these scaling relations to obtain a detailed characterization of the plasmoid instability and the onset of fast reconnect in different astrophysical systems.

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References

Ara, G., Basu, B., Coppi, B., et al. 1978, AnPhy, 112, 443
Baalrud, S. D., Bhattacharjee, A., & Huang, Y.-M. 2012, PhPl, 19, 022101
Baalrud, S. D., Bhattacharjee, A., Huang, Y.-M., & Germschewski, K. 2011, PhPl, 18, 092108
Balsb, S. A., & Henri, P. 2008, ApJ, 674, 408
Bärta, M., Büchner, J., Karlický, M., & Kotrč, P. 2011a, ApJ, 730, 47
Bärta, M., Büchner, J., Karlický, M., & Skála, J. 2011b, ApJ, 737, 24
Beidler, M., Callen, J., Hegna, C., & Sovinec, C. 2017, PhPl, 24, 052508
Beloborodov, A. M. 2017, arXiv:1701.02847
Bender, C. M., & Orszag, S. A. 1978, Advanced Mathematical Methods for Scientists and Engineers (New York: McGraw-Hill)
Benz, A. O., & Güdel, M. 2010, ARA&A, 48, 241
Bhattacharjee, A. 2004, ARA&A, 42, 365
Bhattacharjee, A., Huang, Y.-M., Yang, H., & Rogers, B. 2009, PhPl, 16, 112102
Birn, J., Galsgaard, K., Hesse, M., et al. 2005, GeoRL, 32, L06105
Biskamp, D. 1982, PhLA, 87, 357
Biskamp, D. 1986, PhPI, 29, 1520

6 However, it must be noted that an important effect of viscosity is to extend the validity of the (viscous) Sweet–Parker–based scalings, albeit not to extremely high values of $S$. 
