Approximate indicators for closed subgroups of locally compact
groups with applications to weakly amenable groups

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Abstract

We generalize the notion of an approximate indicator for a closed subgroup $H$ of a locally compact
group $G$ introduced in [1] and extend their characterization of the existence of such nets in terms of the
approximability of $\chi_H$ in an appropriate weak* topology. We find that this equivalent condition
is satisfied whenever $H$ is weakly amenable and $\chi_H$, considered as acting on $l^1(G)$ by multiplication,
extends to a bounded map on $VN(G)$. This occurs in particular when a natural projection
$VN(G) \rightarrow I(A(G), H)^+$ exists. Applications are obtained to the existence (and non-existence) of
natural and invariant projections onto $I(A(G), H)^+$ and $I(A_{cb}(G), H)^+$ and to the existence of
($\Delta$-weak) bounded approximate identities in ideals of $A(G)$ and $A_{cb}(G)$. In particular, we exhibit a
locally compact group without the invariant complementation property.

1 Introduction

In [17], the idempotents of the Fourier-Stieltjes algebra $B(G)$ of a locally compact group $G$ are identified
as the characteristic functions of subsets lying in the open coset ring $R(G)$ of $G$, that is, the ring of subsets
of $G$ generated by open cosets. The closed coset ring $R_c(G)$ of $G$ is the collection of sets in $R(G)$ that
are closed in $G$. When the group $G$ is amenable, there is a one-to-one correspondence between elements of
$R_c(G)$ and those closed ideals $I$ of the Fourier algebra $A(G)$ possessing a bounded approximate identity
[12, Theorem 2.3]. Since $A(G)$ has a bounded approximate identity when $G$ is amenable, these are exactly
the ideal that are weakly invariantly complemented [9, Proposition 6.4].

For discrete groups, the correspondence is easier to verify and doing so elucidates the results that are
absent in the general setting. Let $G$ be an amenable discrete group. The coset ring and closed coset
ring of $G$ coincide and are exactly the subsets whose characteristic functions lie in $B(G)$. Now suppose
$A \subset G$ for which $\chi_A \in B(G)$. As $A(G)$ is an ideal in $B(G)$, the map sending $u$ in $A(G)$ to $\chi_A u$ is an
invariant projection onto $\chi_A A(G) = I(A(G), G \setminus A)$. Conversely, suppose $I$ is a closed ideal such that
$I^\perp$ is invariantly complemented in $A(G)^*$ by a projection $P$. Since every subset of the amenable discrete
group $G$ is a set of synthesis for $A(G)$ [21, Proposition 2.2], there is a subset $A$ of $G$ such that $I = I(A)$.Arguing as in the proof of [9, Proposition 8.3] yields that $P$ must be weak* continuous, hence the adjoint
of a multiplication map $u \mapsto vu$ on $A(G)$ for some multiplier $v$ of $A(G)$. That $P$ is a projection forces $v$
to be an idempotent and the amenability of $G$ then implies that $v$ must be in the Fourier-Stieltjes algebra
[21]. The correspondence is thus established.

The complete characterization of the closed ideals with bounded approximate identity in the Fourier
algebra of a general amenable group uses ingredients that are similar to the above, but the arguments are
more sophisticated and their development, culminating in [12], was the work of several mathematicians.
For nonamenable groups, no such complete characterization is known and pursuing analogues of the
concepts and relations present in the above argument has proven fruitful. In particular, we are interested
in the following two questions:

1. When is a closed ideal of $A(G)$ weakly invariantly complemented?

2. What is a useful analogue for the condition $\chi_A \in B(G)$ for nondiscrete, nonamenable groups $G$?

When a subset $A$ of $G$ satisfies an appropriate analogous condition, what can be deduced about the
ideal $I ( A ( G ), A )$? Specifically, can an equivalence between weak invariant complementation and the existence of (some form of) approximate identities in ideals be recovered?

Both questions have been studied. The invariant complementation question is classical, having been studied thoroughly in the context of abelian groups, where $A ( G ) = \mathcal{L}^1 ( \hat{G} )$ and one is concerned with weak* -closed invariantly complemented subspaces of $L^\infty ( \hat{G} )$. For nonabelian groups, the theory is an active area of research, see for example [1, 14, 25]. Recently, a weaker notion than invariant projection, that of a natural projection, was introduced and studied in [25]. We may ask Question 1 for such projections as well.

In [4], a potential answer to the first part of question 2 was introduced. They defined a bounded approximate indicator for a closed subgroup $H$ of $G$ and characterize the existence of such nets in terms of the condition $\chi_H \in B^d ( G )$, where $B^d ( G )$ is the weak* closure of $B ( G )$ in $B ( G_\delta )$. A bounded approximate indicator for $H$ yield an invariant projection onto $I ( A ( G ), H )^\perp$ (this follows from [1, Lemma 2.2] or see Proposition 6.1 below). These nets exist in several concrete contexts studied earlier, for example when $H$ is neutral in $G$ or when $G$ has the $H$-separation property. A bounded approximate diagonal for $A ( G )$ is an example of a bounded approximate indicator for the diagonal subgroup of $G \times G$, one which is required to lie in $A ( G \times G )$ and be bounded in the norm of that algebra. These examples are discussed in [4].

For a nonamenable group $G$, the multipliers of the Fourier algerba do not coincide with $B ( G )$ [27]. The completely bounded multipliers of $A ( G )$, denoted $M_{cb} ( G )$, satisfy the conditions requires in Aristov, Runde, and Spronk’s study of approximate indicators in $B ( G )$ and have a useful representation theorem (Theorem 2.1 below). These too differ from the $B ( G )$ when $G$ is not amenable. Moreover, the Cohen-Host theorem fails to characterize the idempotents in $M_{cb} ( G )$, even for discrete groups [20].

Studying questions 1 and 2 for the algebra $A_{cb} ( G )$, the closure of $A ( G )$ in $M_{cb} ( G )$, can provide insights for $A ( G )$ [13, 10]. Here, the notion of weak amenability $G$ is more relevant than amenability (the latter implying that $A_{cb} ( G )$ and $A ( G )$ coincide). Indeed, we may take the definition of weak amenability of $G$ to assert the existence of a bounded approximate identity in $A_{cb} ( G )$. The weakly amenable groups form a strictly larger class of groups than the amenable groups (see for example the main result of [2]).

2 Notations and definitions

For an algebra $A$ of functions on a locally compact group $G$ and a subset $S$ of $G$, we write $I ( A, S )$ for the functions in $A$ that are zero on $S$.

Given a commutative Banach algebra $A$ that is an ideal in another commutative Banach algebra $M$, we consider the canonical module action of $M$ on $A$ (by multiplication in $M$) and its dual action on $A^*$ given by $( m \cdot \varphi, a ) = \langle \varphi, ma \rangle$ for $\varphi \in A^*, m \in M, a \in A$. Let $M_m : A^* \to A^*$ denote the bounded operator implementing the dual action of an element $m$ of $M$. If $M$ and $A$ are completely contractive Banach algebras, the map $M_m$ is completely bounded.

For the definitions and basic properties of the Fourier algebra $A ( G )$ and the Fourier-Stieltjes algebra $B ( G )$ associated to a locally compact group $G$, we refer to [7]. Here, we highlight only the essential facts required in the sequel. The full group $C^*-$algebra $C^* ( G )$ is the completion of $L^1 ( G )$ with respect to the norm induced by the universal representation of $G$. The dual of $C^* ( G )$ may be identified with the space of coefficient functions of continuous unitary representations of $G$,

$$B ( G ) = \{ x \mapsto \langle \pi ( x ) \xi | \eta \rangle : \pi : G \to \mathcal{H} \text{ is a continuous unitary representation and } \xi, \eta \in \mathcal{H} \}. \quad (*)$$

This space, equipped with the norm

$$\| u \|_{B ( G )} = \inf \{ \| \xi \| \| \eta \| : u = \langle \pi ( \cdot ) \xi | \eta \rangle \text{ with } \pi, \xi, \eta \text{ as in } (*) \}$$

and pointwise multiplication forms a Banach algebra of continuous functions on $G$. Note that $B ( G )$ is a dual space and so has a weak* topology.

Let $\lambda$ denote the left regular representation of $G$. The group von Neumann algebra $VN ( G )$ is the von Neumann algebra generated in $B ( L^2 ( G ) )$ by the left translation operators $\{ \lambda ( x ) : x \in G \}$. The unique
predual of \(VN(G)\) may be identified with the space of coefficient functions of \(\lambda\) and forms the closed ideal \(A(G)\) in \(B(G)\). Equivalently, \(A(G)\) is the closed subalgebra of \(B(G)\) generated by the compactly supported functions. The algebra \(A(G)\) is regular and has spectrum \(G\).

A completely bounded multiplier of \(A(G)\) is a function \(m\) on \(G\) such that \(mA(G) \subset A(G)\) and \(M_m : VN(G) \to VN(G)\) is completely bounded (see \[6\] for a general reference on operator spaces and completely bounded maps). These functions are continuous and bounded. They form a Banach algebra under pointwise multiplication and the norm \(\|m\|_{\mathcal{M}_b(G)} := \|M_m\|_c\), the cb-multiplier algebra \(\mathcal{M}_b(G)\) of \(G\). Many equivalent characterizations of \(\mathcal{M}_b(G)\) exist \[2\] p. 508. We will require the following representation theorem due to Gilbert (\[18\] gives a short proof).

**Theorem 2.1.** Let \(G\) be a locally compact group. A function \(m\) on \(G\) is in \(\mathcal{M}_b(G)\) if and only if there exists a Hilbert space \(\mathcal{H}\) and bounded continuous maps \(P, Q : G \to \mathcal{H}\) such that \(m(x^{-1}y) = \langle P(y) | Q(x) \rangle\) for all \(x, y \in G\), in which case \(\|m\|_{\mathcal{M}_b(G)}\) is the infimum of the quantities \(\|P\|_\infty \|Q\|_\infty\) taken over all such maps \(P, Q\) and Hilbert spaces \(\mathcal{H}\).

It follows from Theorem 2.1 that \(B(G) \subset \mathcal{M}_b(G)\) and that \(\|\cdot\|_{\mathcal{M}_b(G)} \leq \|\cdot\|_{B(G)}\) on \(B(G)\) (see also \[3\] Corollary 1.8). The norm closure of \(A(G)\) in \(\mathcal{M}_b(G)\), denoted \(\mathcal{A}_b(G)\), forms a regular, Tauberian Banach algebra with spectrum \(G\) \[11\] Lemma 1.

As completely bounded multipliers of \(A(G)\) lie in \(L^\infty(G)\), we may consider \(L^1(G)\) as a subspace of the dual of \(\mathcal{M}_b(G)\). Taking the completion of \(L^1(G)\) with respect to the norm
\[
\|f\|_{\mathcal{Q}(G)} = \sup \left\{ \left| \int_G f \, m \right| : m \in \mathcal{M}_b(G) \text{ and } \|m\|_{\mathcal{M}_b(G)} \leq 1 \right\} \quad (f \in L^1(G))
\]
yields a predual \( \mathcal{Q}(G) \) for \( \mathcal{M}_b(G) \) \[3\] Proposition 1.10]. Thus \( \mathcal{M}_b(G) \) is a dual space and has a weak* topology. Since \(\|\cdot\|_{\mathcal{Q}(G)} \leq \|\cdot\|_{L^1(G)}\) on \(L^1(G)\), it follows that \(\|\cdot\|_\infty \leq \|\cdot\|_{\mathcal{M}_b(G)}\).

In summary,
\[
A(G) \subset \mathcal{A}_b(G) \quad \text{and} \quad A(G) \subset B(G) \subset \mathcal{M}_b(G) \subset \mathcal{C}_b(G)
\]
with the first and third containments strict unless \(G\) is amenable and the second strict unless \(G\) is compact. For \(u \in A(G)\) and \(v \in B(G)\),
\[
\|u\|_\infty \leq \|u\|_{\mathcal{A}_b(G)} = \|u\|_{\mathcal{M}_b(G)} \leq \|u\|_{A(G)} = \|u\|_{B(G)} \quad \text{and} \quad \|v\|_\infty \leq \|v\|_{\mathcal{M}_b(G)} \leq \|v\|_{B(G)},
\]
so the first, third, and fourth inclusions are contractive while the second is isometric.

The locally compact group \(G\) is amenable if \(A(G)\) has a bounded approximate identity and is weakly amenable if it has an approximate identity bounded in the cb-multiplier norm. As mentioned before, the latter is equivalent to the assertion that \(\mathcal{A}_b(G)\) has a bounded approximate identity \[10\] Proposition 1]. Let \(\Lambda_G\) denote the Cowling-Haagerup constant, i.e. the infimum of the norm bounds of bounded approximate identities for \(\mathcal{A}_b(G)\).

We will have need to consider the locally compact group \(G\) equipped with the discrete topology, denoted \(G_d\). The inclusions \(B(G) \subset B(G_d)\) and \(\mathcal{M}_b(G) \subset \mathcal{M}_b(G_d)\) are isometric \([7, 29, \text{Corollary } 6.3]\), respectively. For each \(x \in G\) the evaluation functional \(\delta_x\) lies in \(\ell^1(G_d)\), which is contained in \(C^*(G_d)\) and in \(Q(G_d)\), whence convergence in the weak* topology of \(B(G_d)\) or \(\mathcal{M}_b(G_d)\) implies pointwise convergence. For bounded nets, the converse holds (see \[7\] for the assertion regarding \(B(G_d)\) and \[13\] Lemma 2.6] or the useful Appendix A of \[23\] regarding \(\mathcal{M}_b(G_d)\)).

Let \(A\) be either \(A(G)\) or \(\mathcal{A}_b(G)\), or more generally a regular Banach algebra of functions on \(G\) with pointwise multiplication, norm dominating the supremum norm, and spectrum \(G\). For \(x \in G\), let \(\delta_x\) denote the corresponding character in \(A^*\). If \(I\) is a closed ideal in \(A\), an invariant (natural) projection onto \(I^1\) is a bounded projection \(P : A^* \to I^1\) that is also an \(A\)-module map (that satisfies \(P(\delta_x) = \chi_S(x) \delta_x\) for some subset \(S\) of \(G\), for all \(x \in G\)). When such a projection exists, the ideal \(I\) is called weakly invariantly complemented (naturally complemented). Natural projections were introduced in \[25\], where it was shown that an invariant projection is always natural if \(A\) has a bounded approximate identity \[25\] Lemma 3.2(b)]. The weaker notion of a \(\Delta\)-weak bounded approximate identity was introduced in \[19, 26\] and shown in \[26\].
to be closely related to the existence of natural projections. For our purposes, we may define a $\Delta$-weak bounded approximate identity to be a bounded net $(u_{\alpha})_{\alpha}$ in $A$ such that $u_{\alpha} \to 1$ pointwise. Any bounded approximate identity in $A$ satisfies this condition since $A$ is regular with norm dominating the supremum norm.

### 3 Characteristic functions of closed subgroups and the discretized cb-multiplier algebra

Let $G$ be a locally compact group. In [14], the discretized Fourier-Stieltjes algebra $B^d(G)$, defined to be the weak$^*$ closure of $B(G)$ in $B(G_d)$, is discussed, and it is noted that a weak$^*$ continuous quotient map $B(G)^{**} \to B^d(G)$ exists. We make the analogous definition for the cb-multipliers of $G$.

**Definition 3.1.** For a locally compact group $G$, let $A^d(G)$ and $M^d_{cb}(G)$ denote the weak$^*$ closures of $A(G)$ and $M_{cb}(G)$ in $M_{cb}(G_d)$, respectively.

**Remark 3.2.** For any locally compact group $G$ the algebra $M_{cb}(G)$ is a dual Banach algebra (see Example 4 following Definition 1.4 in [14]). The weak$^*$ closed subalgebra $M^d_{cb}(G)$ of $M_{cb}(G_d)$ is then itself a dual Banach algebra. In particular, multiplication in $M^d_{cb}(G)$ is separately weak$^*$ continuous.

**Remark 3.3.** The Banach space $M^d_{cb}(G)$ has predual $Q^d(G) := Q(G_d)/M^d_{cb}(G_d)_\perp$. An application of the bipolar theorem yields that $M^d_{cb}(G)_\perp = M_{cb}(G_d)_\perp$. Let $\iota_d : M_{cb}(G) \to M_{cb}(G_d)$ denote the inclusion. The map $\iota^*_d : Q(G_d)^{**} \to Q(G)^{**}$ satisfies $\ker(\iota^*_d) = \iota_{\perp} = M_{cb}(G)_\perp$ and, if $\kappa_Q : Q(G_d) \to Q(G)^{**}$ is the canonical inclusion, then $\kappa_Q(M_{cb}(G)_\perp) \subset M_{cb}(G)_\perp$ implies that $\iota^*_d\kappa_Q$ induces a map $Q^d(G) \to Q(G)^{**}$. We denote the adjoint of the induced map by

$$\tau : M_{cb}(G)^{**} \to M^d_{cb}(G).$$

**Remark 3.4.** Let $q : Q(G_d) \to Q^d(G)$ denote the quotient map. Since $\langle q^*(m), a \rangle = \langle m, q(a) \rangle = \langle m, a \rangle$ for all $m \in M^d_{cb}(G)$ and $a \in Q(G_d)$, the adjoint $q^*$ is the inclusion map $M^d_{cb}(G) \to M_{cb}(G_d)$. In particular, if $m \in M^d_{cb}(G)$ then

$$m(x) = \langle q^*(m), \delta_x \rangle = \langle m, q(\delta_x) \rangle$$

where $x \in G$ and $\delta_x \in \ell^1(G_d) \subset Q(G_d)$ is the point mass at $x$.

**Proposition 3.5.** Let $G$ be a locally compact group and $H$ a closed subgroup. If there is a bounded map $P : VN(G) \to VN(G)$ satisfying $P(\lambda(x)) = \lambda_H(x) \lambda(x)$ then $\lambda_H A(G) \subset M^d_{cb}(G)$. If moreover $1_G \in A^d(G)$ then $\chi_H \in M^d_{cb}(G)$.

**Proof.** Let $\kappa_A : A(G) \to A(G)^{**}$ and $\iota_A : A(G) \to M_{cb}(G)$ be the inclusions and denote the composition

$$A(G) \xrightarrow{\kappa} A(G)^{**} \xrightarrow{P^*} A(G)^{**} \xrightarrow{\iota^*_d\kappa_Q} M_{cb}(G)^{**} \xrightarrow{\tau} M^d_{cb}(G)$$

by $\sigma$. For $u \in A(G)$ and $x \in G$,

$$\sigma(u)(x) = \langle \sigma(u), q(\delta_x) \rangle = \langle \iota^*_d P^* \kappa_A(u), \delta_x \rangle = \langle P^* \kappa_A(u), \lambda(x) \rangle = \chi_H(x) u(x),$$

where we have used that $\iota^*_d P^* \kappa_A(u)(\delta_x) = \lambda(x)$. Thus $\chi_H u = \sigma(u) \in M^d_{cb}(G)$ for all $u \in A(G)$.

Suppose $(u_\alpha)_\alpha$ is a net in $A(G)$ converging $\sigma(M_{cb}(G_d), Q(G_d))$ to $1_G$. By the Cohen-Host idempotent theorem $\chi_H$ is in $B(G_d)$ [17], hence in $M_{cb}(G_d)$, so $u_\alpha \chi_H \to \chi_H$ by $w^*$-continuity of multiplication. Thus $\chi_H \in M^d_{cb}(G)$.
Remark 3.6. Let $G$ be a locally compact group and $H$ a closed subgroup. Any invariant projection $P$ of $VN(G)$ onto $I\left(A(G), H^\perp\right) = \overline{\text{span}}_{\text{approx}}\{\lambda_G(x) : x \in H\}$ satisfies the hypothesis of the Proposition 3.5. Indeed, a routine argument shows that any $A(G)$-module map $R$ on $VN(G)$ satisfies $\text{supp} R(T) \subset \text{supp}T$ for all $T \in VN(G)$, whence $P(\lambda(x)) = \chi_H\lambda(x)$ because $\text{supp} P(\lambda(x)) \subset H$ [30 Theorem 3].

Natural projections of $VN(G)$ onto $I\left(A(G), H^\perp\right)$ also supply maps with the desired property. Whether the existence of a natural projection always implies the existence of an invariant projection with the same range is Question 7 posed in [25]. In Section 6, we show that this is the case for projections onto $I(A_G, H^\perp)$ whenever $1_H \in A^d(H)$, so in particular when $H$ is weakly amenable, and for projections onto $I(A_{cb}(G), H^\perp)$ whenever $G$ is weakly amenable.

Remark 3.7. The condition $1_G \in A^d(G)$ is satisfied when $A_{cb}(G)$ has a $\Delta$-weak bounded approximate identity: we may assume such a net lies in $A(G)$ and on bounded subsets of $M_{cb}(G)$ pointwise and weak* convergence coincide. It is shown in Corollary 3.11 that when $\chi_H A(G) \subset M_{cb}(G)$ we need only require $1_H \in A^d(H)$ to deduce that $\chi_H \in M_{cb}(G)$.

The construction of the map $\tau$ and the proof of Proposition 3.5 can be carried out with $M_{cb}(G)$ replaced by $B(G)$. Then, to obtain the conclusion that $\chi_H$ is in the weak* closure of $B(G)$ in $B(G_d)$, we require $1_G \in A(G)_{\sigma(B(G_d), C^*(G_d))}$. It is often difficult to satisfy the condition $1_G \in A^d(G)$ without utilizing a $\Delta$-weak bounded approximate identity: any bounded net in $A(G)$ with the desired weak* limit is already such an approximate identity, and any attempt to find an unbounded net cannot reduce the task to verifying only pointwise convergence. But when $A(G)$ has a $\Delta$-weak bounded approximate identity the group $G$ is already amenable [22 Theorem 5.1]. It is the availability of $\Delta$-weak bounded approximate identities in $A_{cb}(G)$ for a broader class of groups (at least containing the weakly amenable groups) that is responsible for the utility of Proposition 1.5. Whether the existence of a $\Delta$-weak bounded approximate identity in $A_{cb}(G)$ implies weak amenability of $G$ appears to be an open question.

Let $G$ be a locally compact group and $H$ a closed subgroup. For a function $f$ on $H$ let $\dot{f}$ denote its extension by zero to $G$. Let

$$r_H : M_{cb}(G_d) \rightarrow M_{cb}(H_d), \quad e_H : M_{cb}(H_d) \rightarrow M_{cb}(G_d)$$

denote the restriction map and the map that extends by zero. The map $r_H$ is a complete quotient map, $e_H$ is a complete isometry ([29 Corollary 6.3] or [28 Proposition 4.1]), $\tau_H e_H = \text{id}_{M_{cb}(H_d)}$, and $e_H r_H = L_{\chi_H}$, the multiplication by $\chi_H$.

Lemma 3.8. Let $G$ be a locally compact group and $H$ a closed subgroup. The maps $r_H$ and $e_H$ are weak*–continuous.

Proof. Let $f = \sum_{j=1}^{n} \alpha_j \delta_{x_j} \in \ell^1(H_d) \cap C_c(H_d)$. Then $\dot{f} \in \ell^1(G_d)$ and for $m \in M_{cb}(G_d)$,

$$\langle r_H^* (f), m \rangle = \sum_{j=1}^{n} \alpha_j m(x_j) = \left\langle \dot{f}, m \right\rangle,$$

denoting $r_H^* \left(\ell^1(H_d) \cap C_c(H_d)\right) \subset Q(G_d)$. Since $\ell^1(H_d) \cap C_c(H_d)$ is dense in $Q(H_d)$, it follows that $r_H^*$ is weak*–continuous.

Now let $f = \sum_{j=1}^{n} \alpha_j \delta_{x_j} \in \ell^1(G_d) \cap C_c(G_d)$. For $m \in M_{cb}(H_d),$

$$\langle e_H^* (f), m \rangle = \left\langle \dot{f}, m \right\rangle = \sum_{j=1}^{n} \alpha_j \chi_H(x_j) m(x_j) = \left\langle \sum_{j=1}^{n} \alpha_j \chi_H(x_j) \delta_{x_j}, m \right\rangle,$$

and $\sum_{j=1}^{n} \alpha_j \chi_H(x_j) \delta_{x_j} \in \ell^1(H_d)$. Thus $e_H^* \left(\ell^1(G_d) \cap C_c(G_d)\right) \subset Q(H_d)$ and the claim follows by density as above.

Proposition 3.9. Let $G$ be a locally compact group and $H$ a closed subgroup. Then $r_H \left(M_{cb}^d(G)\right) \subset M_{cb}^d(H)$ and the following are equivalent:
1. \( \chi_H A(G) \subset A^d(G) \).

2. \( e_H \left( A^d(H) \right) \subset A^d(G) \).

3. \( A^d(G) = I \left( A^d(G), G \setminus H \right) \oplus I \left( A^d(G), H \right) \).

These equivalent conditions imply that \( r_H \left( A^d(G) \right) \neq A^d(H) \).

**Proof.** The first claim follows from weak*-continuity of \( r_H \).

\( 1 \Rightarrow 2 \) By Herzw’s restriction theorem \( A(H) = r_H \left( A(G) \right) \) so \( e_H \left( A(H) \right) = e_H \left( r_H \left( A(G) \right) \right) = \chi_H A(G) \subset A^d(G) \) and the claim follows by weak*-continuity of \( e_H \).

\( 2 \Rightarrow 3 \) If \( m \in A^d(G) \), weak*-continuity of \( r_H \) and (2) imply that \( \chi_H m = e_H r_H \left( m \right) \in A^d(G) \), whence \( \chi_H m = m - \chi_H m \in A^d(G) \) and \( m = \chi_H m + \chi_H m \in I \left( A^d(G), G \setminus H \right) + I \left( A^d(G), H \right) \). The ideals \( I \left( A^d(G), G \setminus H \right) \) and \( I \left( A^d(G), H \right) \) have empty intersection {0}.

\( 3 \Rightarrow 1 \) If \( m \in A(G) \) and \( m = m_1 + m_2 \) for \( m_1 \in I \left( A^d(G), G \setminus H \right) \) and \( m_2 \in I \left( A^d(G), H \right) \) then \( \chi_H m = m_1 \in A^d(G) \).

If (2) holds then, given \( m \in A^d(H) \), we have \( e_H \left( m \right) \in A^d(G) \) and thus \( m = r_H e_H \left( m \right) \in r_H \left( A^d(G) \right) \subset A^d(H) \).

**Remark 3.10.** When the equivalent conditions of Proposition 3.9 hold, condition (2) implies that \( e_H \left( A^d(H) \right) = I \left( A^d(G), G \setminus H \right) \) and, because \( e_H \) is a complete isometry, it follows from condition (3) that \( A^d(G) = A^d(H) \oplus I \left( A^d(G), H \right) \).

**Corollary 3.11.** Let \( G \) be a locally compact group and \( H \) a closed subgroup. If \( \chi_H A(G) \subset A^d(G) \) and \( 1_H \in A^d(H) \) then \( \chi_H \subset A^d(G) \).

**Proof.** \( \chi_H = e_H \left( 1_H \right) \in A^d(G) \) by condition 2 of the previous proposition.

## 4 Averaging over closed subgroups

Throughout this section, let \( G \) be a locally compact group, \( H \) be a closed subgroup, and fix \( f \in C_c(H) \) such that \( f \geq 0 \) and \( \int_H f = 1 \). Set \( K = \text{supp}f \).

For a bounded continuous function \( u \) on \( G \) define another function on \( G \) by

\[
\Omega_f \left( u \right) \left( x \right) := \int_H f \left( h \right) u \left( h^{-1}x \right) dh.
\]

We show that \( \Omega_f \left( u \right) \) is in \( M_{cb}(G) \) when \( u \) is, which requires a couple of lemmas.

**Lemma 4.1.** Let \( H \) be a Hilbert space. If \( u \in C_c(G, H) \) then for any \( \epsilon > 0 \) there is an open neighbourhood \( U \) of the identity \( e \) such that \( \sup_{z \in U} \left\| u \left( xz \right) - u \left( z \right) \right\| < \epsilon \) for all \( x \in U \).

**Proof.** The standard proof in the case that \( H = \mathbb{C} \), for example [8, Proposition 2.6], works for any Hilbert space.

If \( g \) is a function on a group \( G \) and \( x \in G \) let \( xg \left( y \right) = g \left( xy \right) \) and \( gx \left( y \right) = g \left( yx \right) \).

**Lemma 4.2.** Let \( H \) be a Hilbert space. If \( u \in C_c(G, H) \), \( x_0 \in G \), and \( \epsilon > 0 \), then there is an open neighbourhood \( U \) of \( x_0 \) in \( G \) such that \( \sup_{h \in H} \left\| f \left( h \right) u \left( xh \right) - f \left( h \right) u \left( x_0h \right) \right\| < \epsilon \) for all \( x \in U \).

**Proof.** Since \( H \) is closed in \( G \), the function \( f \) extends to a continuous compactly supported function \( f' \) on \( G \). Assume that \( x_0 = e \). Since \( f' \) is compactly supported, Lemma 4.1 yields an open neighbourhood \( U \) of \( e \) such that

\[
\sup_{z \in G} \left\| f' \left( uz \right) - f' \left( z \right) \right\| < \frac{\epsilon}{2} \quad \text{and} \quad \sup_{z \in G} \left( f' \left( uz \right) - f' \left( z \right) \right) < \frac{\epsilon}{2 \left\| u \right\|_\infty}.
\]
for all \( x \in U \). Then
\[
\sup_{h \in H} \| f(h) u(xh) - f(h) u(h) \| \leq \sup_{z \in G} \| f'(z) u(xz) - f'(z) u(z) \|
\leq \sup_{z \in G} (\| f'(z) u(xz) - f'u(xz) \| + \| f' u(xz) - f'(z) u(z) \|)
< \| u \|_\infty \sup_{z \in G} | f'(xz) - f'(z) | + \frac{\epsilon}{2} < \epsilon.
\]

If \( x_0 \neq e \) then the above argument with \( u \) replaced by \( x_0 u \) produces a neighbourhood \( U \) of \( e \) and \( x_0 U \) is the desired neighbourhood of \( x_0 \).

**Proposition 4.3.** If \( u \in M_{cb}(G) \) then \( \Omega_f(u) \in M_{cb}(G) \). The map \( \Omega_f : M_{cb}(G) \to M_{cb}(G) \) is contractive.

**Proof.** Let \( u \in M_{cb}(G) \) and apply Theorem 2.1 to obtain a Hilbert space \( \mathcal{H} \) and \( P, Q \in \mathcal{C}_b(G, \mathcal{H}) \) such that \( u(x^{-1}y) = \langle P(y) | Q(x) \rangle \) for all \( x, y \in G \). If \( x, y \in G \) then
\[
\Omega_f(u)(x^{-1}y) = \int_H f(h) u(h^{-1}x^{-1}y) \, dh = \left\langle P(y) \left| \int_H f(h) Q(xh) \, dh \right. \right\rangle.
\]

We show that \( q(x) = \int_H f(h) Q(xh) \, dh \) is a bounded continuous function on \( G \), from which it will follow that \( \Omega_f(u) \) is in \( M_{cb}(G) \), again by Theorem 2.1. Define \( Q' : G \to L^1(H, \mathcal{H}) \) by \( Q'(x) = f(xQ) \), which maps into \( L^1(H, \mathcal{H}) \) since \( f \) has compact support. Given \( x_0 \in G \) and \( \epsilon > 0 \), Lemma 4.2 yields an open neighbourhood \( U \) of \( x_0 \) in \( G \) such that
\[
\| Q'(x) - Q'(x_0) \|_{L^1(H, \mathcal{H})} = \sup_{h \in H} \| f(h) Q(xh) - f(h) Q(x_0h) \| < \frac{\epsilon}{|K|}
\]
for all \( x \in U \). Since \( Q'(x) \) is supported in \( K \) for every \( x \) in \( G \), it follows that
\[
\| Q'(x) - Q'(x_0) \|_{L^1(H, \mathcal{H})} = \| \chi_K (Q'(x) - Q'(x_0)) \|_{L^1(H, \mathcal{H})} \leq \| \chi_K \|_{L^1(H, \mathcal{H})} \| Q'(x) - Q'(x_0) \|_{L^\infty(H, \mathcal{H})} < \epsilon
\]
for all \( x \) in \( U \). Thus \( Q' \) is continuous and so too is \( q \), the latter being the composition of \( Q' \) with the bounded map \( L^1(H, \mathcal{H}) \to \mathcal{H} : g \mapsto \int_H g \).

Using that \( f \) is nonnegative with mass one, if \( x \in G \) then \( || q(x) || \leq \int_H f(h) || Q(xh) || \, dh \leq || Q ||_\infty \), so \( q \) is bounded with \( || q ||_\infty \leq || Q ||_\infty \). By the norm characterization of Theorem 2.1, \( || \Omega_f(u) ||_{M_{cb}(G)} \leq || P ||_\infty || Q ||_\infty \leq || P ||_\infty || Q ||_\infty \), and since \( P, Q \) and \( \mathcal{H} \) were an arbitrary representation of \( u \), we deduce that \( || \Omega_f(u) ||_{M_{cb}(G)} \leq || u ||_{M_{cb}(G)} \).

The following clarifies in what sense \( \Omega_f \) averages over the subgroup \( H \).

**Proposition 4.4.** \( r_H \Omega_f(u) = f * r_H(u) \) for all \( u \in M_{cb}(G) \).

**Proof.** If \( u \in M_{cb}(G) \) and \( x \in G \)
\[
r_H \Omega_f(u)(x) = \int_H f(h) u(h^{-1}x) \, dh = \int_H f(h) r_H(u)(h^{-1}x) \, dh = f * r_H(u)(x).
\]

We conclude the section with some additional observations regarding the map \( \Omega_f \).

**Remark 4.5.** An argument similar to the proof of Proposition 4.3 shows that \( \Omega_f(u) \) is bounded and continuous for any bounded continuous function \( u \) on \( G \). It is clear that Proposition 4.4 holds for functions in \( \mathcal{C}_b(G) \) as well.
Remark 4.6. If \( u = \langle \pi (\cdot) \xi | \eta \rangle \in B (G) \), where \( \pi \) is a continuous unitary representation of \( G \) on a Hilbert space \( H \) and \( \xi, \eta \in H \) then
\[
\Omega_f (u) (x) = \int_H f (h) \langle \pi (x) \xi | \pi (h) \eta \rangle \, dh = \left\langle \pi (x) \xi \left( \int_H f (h) \pi (h) \, dh \right) \eta \right\rangle ,
\]
so \( \Omega_f (u) \in B (G) \), and from \( \left\| \int_H f (h) \pi (h) \, dh \right\| \leq \int_H \| \pi (h) \| \, dh = 1 \) it follows that \( \| \Omega_f (u) \|_{B(G)} \leq \| u \|_{B(G)} \). Thus \( \Omega_f \) restricts to a contraction on \( B (G) \) and moreover restricts to a contraction on \( A (G) \), since \( \Omega_f (u) \) is a coefficient of the same representation as \( u \).

Remark 4.7. An argument similar to that establishing the weak* continuity of the map \( \Phi_f \) in the proof of [16, Lemma 1.16] shows that \( \Omega_f \) is weak* continuous on \( M_{cb} (G) \) with preadjoint mapping \( g \in L^1 (G) \) to the \( L^1 (G) \) function \( x \mapsto \int_H f (h) g (hx) \, dh \).

5 Approximate indicators for closed subgroups

As the inclusion of the Fourier-Stieltjes algebra of a locally compact group \( G \) into its cb-multiplier algebra is contractive, the following definition is a generalization of [1, Definition 2.1].

Definition 5.1. Let \( G \) be a locally compact group and \( H \) be a closed subgroup. A bounded approximate indicator for \( H \) is a bounded net \( \{ m_\alpha \}_\alpha \) in \( M_{cb} (G) \) satisfying
\[
\begin{align*}
1. & \quad \| u \cdot r_H (m_\alpha) - u \|_{A(H)} \to 0 \text{ for all } u \in A (H), \\
2. & \quad \| u \cdot m_\alpha \|_{A(G)} \to 0 \text{ for all } u \in I (A (G), H).
\end{align*}
\]

It is often difficult to show directly that a net has the property 1 above. In [1], a theorem of Granirer and Leinert [15, Theorem B2] is invoked to strengthen the uniform convergence on compact subsets of \( H \) of a net in \( B (G) \) to property 1. For nets in \( M_{cb} (G) \) no such theorem is available, however the device of the previous section can be applied to fill a similar role.

Let \( G \) be a locally compact group and \( A \subset G \) be any subset. We say a net \( \{ m_\alpha \}_\alpha \) in \( M_{cb} (G) \) converges locally eventually to zero on \( A \) if for any compact subset \( K \) of \( A \) there is \( \alpha_0 \) such that \( \alpha \geq \alpha_0 \) implies \( m_\alpha = 0 \) on \( K \).

Proposition 5.2. Let \( G \) be a locally compact group and \( H \) a closed subgroup. If \( \chi_H \in M_{cb}^d (G) \) then there exists a net \( \{ m_\alpha \}_\alpha \) in \( M_{cb} (G) \) of norm bound one such that:
\[
\begin{align*}
1. & \quad r_H (m_\alpha) \to 0 \text{ uniformly on compact subsets of } H, \\
2. & \quad m_\alpha \text{ converges locally eventually to zero on } G \setminus H.
\end{align*}
\]

Proof. We need only note that the arguments of [1, Section 3] establishing their Proposition 3.6 are valid when \( B (G) \) is replaced by \( M_{cb} (G) \) throughout.

Proposition 5.3. Let \( G \) be a locally compact group and \( H \) be a closed subgroup. Let \( f \) be as in Section 4. Let \( \{ m_\alpha \}_\alpha \) be a bounded net in \( M_{cb} (G) \) and set \( m'_\alpha = \Omega_f (m_\alpha) \). The net \( \{ m'_\alpha \}_\alpha \) has the same norm bound as \( \{ m_\alpha \}_\alpha \) and
\[
\begin{align*}
1. & \quad r_H (m_\alpha) \to 1_H \text{ uniformly on compact sets then } \| u \cdot r_H (m'_\alpha) - u \|_{A(H)} \to 0 \text{ for all } u \in A (H), \\
2. & \quad \text{if } \{ m_\alpha \}_\alpha \text{ converges locally eventually to zero on } G \setminus H \text{ then the net } \{ m'_\alpha \}_\alpha \text{ has this same property.}
\end{align*}
\]

Proof. Since \( \Omega_f \) is a contraction on \( M_{cb} (G) \), the claim regarding the norm bound is clear.

(1) Suppose that \( r_H (m_\alpha) \to 0 \) uniformly on compact sets. The restriction map \( r_H \) takes \( M_{cb} (G) \) contractively into \( M_{cb} (H) \) [2, Proposition 1.12] so \( \{ r_H (m_\alpha) \}_\alpha \) is a bounded net in \( M_{cb} (H) \). The argument of [2, Proposition 1.1], which remains valid for nets of cb-multipliers, yields that \( (f \ast r_H (m_\alpha))_\alpha \) is an approximate identity for \( A (H) \). The conclusion of 1 follows since \( r_H (m'_\alpha) = f \ast r_H (m_\alpha) \) by Proposition 4.4.
(2) Suppose that \((m_\alpha)_\alpha\) converges locally eventually to zero on \(G \setminus H\). Let \(K \subset G \setminus H\) be compact and choose \(a_0\) such that \(\alpha \geq a_0\) implies \(m_\alpha\)\((\supp f)^{-1} K\) = 0. If \(x \in K\) then \(f(h) m_\alpha(h^{-1} x) \neq 0\) implies \(h^{-1} x \in (\supp f)^{-1} K\) for any \(h \in H\), whence \(m'_\alpha(x) = \int_H f(h) m_\alpha(h^{-1} x) \, dh = 0\). Thus \(m'_\alpha|_K = 0\).

We can now deduce the analogue of [1, Proposition 3.7], although we should not expect their condition (iii) to be among our equivalent conditions: any approximate indicator for \(H\) consisting of positive definite functions must necessarily be a bounded net in \(B(G)\) since, for a positive definition function \(u\) on \(G\), \(u(e) = \|u\|_\infty \leq \|u\|_{\mathcal{M}_H(\mathcal{G})} \leq \|u\|_{B(G)} = u(e)\).

**Corollary 5.4.** Let \(G\) be a locally compact group and \(H\) be a closed subgroup. The following are equivalent:

1. There is a bounded net in \(\mathcal{M}_c^d(\mathcal{G})\) converging pointwise to \(\chi_H\).
2. \(\chi_H \in \mathcal{M}_c^d(\mathcal{G})\).
3. There is a bounded approximate indicator for \(H\) (equivalently, of norm bound one).

**Proof.** (1) \(\Rightarrow\) (2) As noted in Section 2, pointwise convergence of a bounded net implies \(\sigma(\mathcal{M}_c^d(\mathcal{G}) , \mathcal{Q}(\mathcal{G}))\) convergence.

(2) \(\Rightarrow\) (3) If \(\chi_H \in \mathcal{M}_c^d(\mathcal{G})\), it follows from Propositions 5.2 and 5.3 that there is a net \((m_\alpha)_\alpha\) in \(\mathcal{M}_c(\mathcal{G})\) of norm bound one satisfying condition 1 of Definition 5.1 and converging locally eventually to zero on \(G \setminus H\). Since closed subgroups are of synthesis for \(A(G)\), the functions in \(I(A(G), H)\) with compact support off \(H\) are norm dense. Condition 2 of Definition 5.1 thus follows immediately from the facts that \((m_\alpha)_\alpha\) converges locally eventually to zero on \(G \setminus H\) and \((m_\alpha)_\alpha\) is a bounded net.

(3) \(\Rightarrow\) (1) Let \((m_\alpha)_\alpha\) be a bounded approximate indicator for \(H\). Since \(A(G)\) is regular and the \(A(G)\) norm dominates the supremum norm, condition 1 of Definition 5.1 implies that \(m_\alpha \to 1\) uniformly on compact subsets of \(H\) and condition 2 implies that \(m_\alpha \to 0\) uniformly on compact subsets of \(G \setminus H\).

## 6 Applications to invariant projections and to bounded approximate identities in ideals of \(A(G)\) and \(A_{cb}(G)\)

**Proposition 6.1.** Let \(G\) be a locally compact group and \(H\) be a closed subgroup. If \(\chi_H \in \mathcal{M}_c^d(\mathcal{G})\) then there is a completely bounded invariant projection \(P : VN(G) \to I(A(G), H)'\) of norm one.

**Proof.** By Corollary 5.4 there is an approximate indicator \((m_\alpha)_\alpha\) for \(H\) of norm bound one. Write \(M_m\) for adjoint of the map on \(A(G)\) that multiplies by \(m\), so that \(M_m\) is a completely bounded \(A(G)\)-module map on \(VN(G)\). The net \((M_{m_\alpha})_\alpha\) in \(CB(VN(G))\) is bounded by one and thus has a weak* operator topology (w*ot) cluster point in the unit ball of \(CB(VN(G))\). Say without loss of generality that \(P = w*ot - \lim A_{m_\alpha}\). The map \(P\) is an \(A(G)\)-module map since the maps \(M_{m_\alpha}\) are.

If \(T \in I(A(G), H)'\) then \(T = r_H S\) for some \(S \in VN(H)\) and if \(u \in A(G)\),

\[
\langle P(T), u \rangle = \lim_\alpha \langle S, r_H(u_m) \rangle = \lim_\alpha \langle S, r_H(u_m) r_H(u_m) \rangle = \langle S, r_H(u_m) \rangle = \langle T, u \rangle,
\]

showing that \(P\) is the identity map on \(I(A(G), H)'\). For any \(T \in VN(G)\) and \(u \in I(A(G), H)\),

\[
\langle P(T), u \rangle = \lim_\alpha \langle S, uu_m \rangle = 0,
\]

so \(\text{im}P \subset I(A(G), H)'\) and \(P\) is a projection onto \(I(A(G), H)'\).

**Corollary 6.2.** Let \(G\) be a locally compact group and \(H\) be a closed subgroup such that \(1_H \in A^d(H)\), which holds in particular if \(H\) is weakly amenable. If there is a natural projection or an invariant projection \(VN(G) \to I(A(G), H)'\) then there is a completely bounded invariant projection \(VN(G) \to I(A(G), H)'\) of norm one.
Proof. It follows from Proposition 3.5 and Corollary 3.11 that $\chi_H \in M_{cb}^d(G)$. \hfill \Box

Example 6.3. Let $G = SL(2, \mathbb{R})$, which contains $H = \mathbb{F}_2$ as a closed subgroup. In Remark 2 following Proposition 4.1 of [1] it is shown that no norm one projection $VN(G) \to I(A(G), H)^\perp$ exists. Since $\mathbb{F}_2$ is weakly amenable, Corollary 6.2 implies that no natural projection or invariant projection $VN(G) \to I(A(G), H)^\perp$ exists. The latter assertion is the statement that $G$ is does not have the Lau’s invariant complementation property [20], and answers the question immediately following the Remark 2 of [1] in the negative. We emphasize:

Corollary 6.4. Let $G$ be a locally compact group. If $VN(G)$ is injective and $G$ contains a closed subgroup that is non-amenable but is weakly amenable and inner amenable, then $G$ does not have the invariant complementation property.

Assuming the stronger hypothesis that $G$ is amenable, we recover Theorem 2.2 and Corollary 2.3 of [13].

Corollary 6.5. Let $G$ be an amenable locally compact group. If $H$ is a closed subgroup then there is a completely bounded invariant projection $VN(G) \to I(A(G), H)^\perp$ of norm one and $I(A(G), H)$ has an approximate identity of norm bound 2.

Proof. An $A(G)$-module projection $VN(G) \to I(A(G), H)^\perp$ exists by [12] Theorem 1.3. Since $G$ is weakly amenable, so is $H$ [2] Proposition 1.3 and the desired projection exists by Corollary 6.2. The claim regarding the approximate identity then follows from [5] Proposition 11. \hfill \Box

Corollary 6.6. Let $G$ be a locally compact group and $H$ be a closed subgroup. If $I(A(G), H)$ has a bounded approximate identity, it has one of norm bound 2.

Proof. This follows from Corollary 6.5, given that $G$ is amenable when $I(A(G), H)$ has a bounded approximate identity [12] Corollary 1.6. \hfill \Box

We now obtain some results regarding bounded approximate identities in ideals and (natural) projections for $A_{cb}(G)$.

Theorem 6.7. Let $G$ be a weakly amenable locally compact group and $H$ be a closed subgroup. The following are equivalent:

1. $\chi_H \in M_{cb}^d(G)$.
2. There is a completely bounded invariant projection $A_{cb}(G)^* \to I(A_{cb}(G), H)^\perp$ (equivalently, of norm one).
3. There is a natural projection $A_{cb}(G)^* \to I(A_{cb}(G), H)^\perp$ (equivalently, of norm one).
4. $I(A_{cb}(G), H)$ has a bounded approximate identity.
5. $I(A_{cb}(G), H)$ has a $\Delta$-weak bounded approximate identity.

Proof. $(1) \Rightarrow (2)$ If $\chi_H \in M_{cb}^d(G)$ then there is a bounded approximate indicator $(m_\alpha)_\alpha$ for $H$ of norm bound one, by Corollary 5.4. The density of $A(G)$ in $A_{cb}(G)$ implies that $A_{cb}(G)$ is an ideal in $M_{cb}(G)$, so in particular multiplication by each $m_\alpha$ is a completely bounded $A_{cb}(G)$-module map on $A_{cb}(G)$, say with adjoint denoted by $M_{m_\alpha}$. As in the proof of Proposition 6.1 we may suppose that $P = w^* ot - \lim_{n} M_{m_\alpha}$ exists, in which case $P$ is a completely bounded norm one $A_{cb}(G)$-module map on $A_{cb}(G)^*$. Let $P_A \in CB(A(G))(VN(G))$ be the projection constructed in Proposition 6.1 and let $\iota : A(G) \to A_{cb}(G)$ be the inclusion. If $u \in A(G)$ and $T \in A_{cb}(G)^*$ then

$$\langle P_Au^*(T), u \rangle = \lim_{\alpha} \langle u^*(T), um_\alpha \rangle = \lim_{\alpha} \langle T, \iota(um_\alpha) \rangle = \lim_{\alpha} \langle T, \iota(u) m_\alpha \rangle = \langle P(T), \iota(u) \rangle = \langle \iota^* P(T), u \rangle$$
and we conclude that $P A \iota^* = \iota^* P$ by density of $A(G)$. It follows from

$$\iota^* P^2 = P A \iota^* P = P A \iota^* = \iota^* P$$

and injectivity of $\iota^*$ that $P^2 = P$. If $T \in I(A_{cb}(G), H)^\perp$, then $\iota^* (T) \in I(A(G), H)^\perp$ and so for $u \in A(G)$,

$$\langle P(T), \iota (u) \rangle = \langle P A \iota^* (T), u \rangle = \langle \iota^* (T), u \rangle = \langle T, \iota (u) \rangle,$$

whence $P(T) = T$ again by density of $A(G)$. Thus $I(A_{cb}(G), H)^\perp \subset \text{im} P$. Finally, for any $T \in A_{cb}(G)^*$ if $u \in I(A(G), H)$ then $\langle P(T), \iota (u) \rangle = \langle P A \iota^* (T), u \rangle = 0$ since $\text{im} P \subset I(A(G), H)^\perp$, so $P(T) \in I(A(G), H)^\perp = \left( \frac{I(A(G), H)}{A_{cb}(G)^*} \right)^\perp$. As $A_{cb}(G)$ has bounded approximate identity, $T(A(G), H)^{A_{cb}(G)} = I(A_{cb}(G), H)$. It follows that $\text{im} P \subset I(A_{cb}(G), H)^\perp$ and $P$ is a projection onto $I(A_{cb}(G), H)^\perp$.

(2) $\Rightarrow$ (3) As noted in Section 2, any invariant projections is natural since $A_{cb}(G)$ has bounded approximate identity.

(3) $\Rightarrow$ (1) Recall from Section 2 that $A_{cb}(G)$ has spectrum $G$. For $x \in G$ let $\delta_x$ denote the character corresponding in $A_{cb}(G)^*$. When a bounded map $P : A_{cb}(G)^* \rightarrow A_{cb}(G)^*$ satisfying $P(\delta_x) = \chi_G(x) \delta_x$ exists the argument of Proposition 3.5 establishes that $\chi_G A_{cb}(G) \subset M_{cb}^2(G)$. Any natural projection $A_{cb}(G)^* \rightarrow I(A_{cb}(G), H)^\perp$ is such a map and thus (1) follows from the second assertion of Proposition 3.5.

(2) $\Rightarrow$ (4) This follows from [9, Proposition 6.4].

(4) $\Rightarrow$ (5) This implication is trivial.

(5) $\Rightarrow$ (1) Since $\Delta(I(A_{cb}(G), H)) = G \setminus H$ a $\Delta$-weak bounded approximate identity for $I(A_{cb}(G), H)$ is a bounded net in $M_{cb}(G)$ that converges pointwise to $\chi_{G\setminus H}$. It follows that $\chi_{G\setminus H} \in M_{cb}^2(G)$ and (1) holds.

Remark 6.8. For the implication (1) implies (2), the assumption that $G$ is weakly amenable may be replaced by the assumption that $H$ is a set of synthesis for $A_{cb}(G)$. Indeed, since compactly supported elements of $A_{cb}(G)$ (or in fact $M_{cb}(G)$) are necessarily in $A(G)$, that $H$ is of synthesis for $A_{cb}(G)$ is exactly the assertion $T(A(G), H)^{A_{cb}(G)} = I(A_{cb}(G), H)$ required to complete the proof above. The weak amenability of $G$ implies that every set of synthesis for $A(G)$ is one for $A_{cb}(G)$ [13, Proposition 3.1].

Remark 6.9. Corollary 6.2 and the implication (3) implies (2) of Theorem 6.7 both provide conditions under which the existence of a natural projection implies the existence of an invariant projection. This yields a partial answer to Question 6 posed in [25].

Remark 6.10. Examining the proof of Proposition 6.4 in [9], if $G$ is weakly amenable and $H$ is a closed subgroup for which the equivalent conditions of Theorem 6.7 hold, we may conclude that $I(A_{cb}(G), H)$ has an approximate identity of norm bound $2\Lambda_G$.

For weakly amenable groups, Theorem 6.7 allows us to strengthen a convergence property of bounded approximate indicators for closed subgroups at the cost of increasing their norm bounds.

Corollary 6.11. Let $G$ be a weakly amenable locally compact group and $H$ be a closed subgroup. If an approximate indicator for $H$ exists then an approximate indicator for $H$ of norm bound $1 + 2\Lambda_G$ exists that is identically one on $H$.

Proof. Theorem 6.7 yields a bounded approximate identity $(e_\alpha)_\alpha$ for $I(A_{cb}(G), H)$ and is it clear that $(1 - e_\alpha)_\alpha$ is the desired approximate indicator for $H$. By Remark 6.10, $\sup_\alpha \|1 - e_\alpha\|_{A_{cb}(G)} \leq 1 + 2\Lambda_G$.  

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