Introducing Directionality to Anderson Localization:
the transport properties of Quantum Railroads

C. Barnes\textsuperscript{1,2} B. L Johnson\textsuperscript{2} G. Kirczenow\textsuperscript{2}

\textsuperscript{1} Cavendish Laboratory, Madingley Road, Cambridge, CB3 0HE, UK
\textsuperscript{2} Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

Abstract

We present a study of the transport properties of a general class of quantum mechanical waveguides: Quantum Railroads (QRR)\textsuperscript{3}. These waveguides are characterised by having a different number of adiabatic modes which carry current in one direction along the waveguide to the opposite direction; for example \(N\) forward modes and \(M\) reverse modes. Just as Anderson localization is a characteristic of the disordered \(N = M\) case and the quantum Hall effect a characteristic of the \(M = 0\) case we find that a mixed effect, directed localization, is a characteristic of QRR’s. We find that in any QRR there are always \(|N - M|\) perfectly transmitted effective adiabatic modes with the remainder being subject to multiple scattering and interference effects characteristic of the \(N = M\) case.

PACS numbers: 72.10.Bg, 72.20.Dp, 72.15.Rn
I. INTRODUCTION

It was Anderson who first proposed the idea that there would be ‘Absence of diffusion in certain random lattices’. This was the initial insight which gave rise to a broad class of problems now referred to collectively as ‘Anderson Localization’. The concept of a metal-insulator transition (MIT) in disordered systems was introduced by Mott and subsequently Kunz and Souillard, MacKinnon, Czycholl et al, and Thouless and Kirkpatrick showed in a variety of different ways that the conductance of a one-dimensional system should vanish with increasing length due to multiple scattering and interference effects. The conductance does not vanish in a simple manner but is characterised by strong fluctuations. In weakly disordered systems the fluctuations have the universal feature that root mean square conductance is approximately a constant $\sigma_g \sim e^2/h$. This has been seen experimentally by Blonder, Dynes and White, Umbach et al and Webb et al and shown theoretically by Al’tshuler and Lee, Stone and Fukuyama. Under more general conditions of disorder, away from the metallic limit, recent theoretical investigations by Pendry, MacKinnon and Pretre, and MacKinnon suggest that here the fluctuations are consistent with an occasional channel being open when compared to the mean exponential decay. This is the so-called ‘maximal fluctuation theorem’.

An implicit assumption in all of the theoretical work done on one-dimensional Anderson Localization is that scattering of waves in these systems occurs between equal numbers of modes which carry current in both the forward and reverse directions. Quasi-one-dimensional systems need not necessarily have this property however as has become increasingly clear following the discovery of the quantum Hall effect by von Klitzing, Dorda and Pepper. Laughlin used a gauge invariance argument to explain the observed quantisation of the Hall conductance in a 2D electron gas to integer multiples of $e^2/h$. Subsequently, however, following the introduction of magnetic edge states by Halperin, Streda, Kucera and MacDonald, Jain and Kivelson, and Büttiker proposed an alternate point of view, in which the quantum Hall effect is explained on the basis of the Landauer theory of
one-dimensional transport. In these theories edge states, which are the extensions of the quantised Landau levels at the edges of the system, form the one-dimensional adiabatic transport channels (or modes) between which disorder motivates scattering. This description of the integer quantum Hall effect in terms of edge states has gained wide acceptance. Although transport in the integer quantum Hall regime is effectively one-dimensional Anderson localization of the wave functions does not occur because all of the modes on a given edge of the sample propagate in the same direction so that in a macroscopic sample where the edges do not communicate backscattering of electrons is impossible. Thus there can be perfect transmission of electrons through a disordered macroscopic sample in the quantum Hall regime, in stark contrast to the zero average transmission found in more conventional 1D systems.

The aim of this paper is to report on a theoretical study of the general problem of quasi-one-dimensional transport in a disordered waveguide where scattering occurs between arbitrary numbers $N$ and $M$ of adiabatic modes which carry current in the forward and reverse directions. We have coined the term "quantum railroad" (QRR) as a general term for such waveguides since a very useful aid in understanding their transport properties is to think in terms of an imaginary railroad connecting two cities for example Los Angeles (LA) and New York (NY). The railroad consists of $N$ tracks which carry trains in the NY direction and $M$ tracks which carry trains in the LA direction. Along the tracks there are a number of different switching events, which represent disorder, and are set to switch a train from one track to another either changing its direction of travel or not depending on the tracks between which it switches. In studying the transport properties of QRR’s we are essentially asking questions about the likelihood that a train starting from one city will arrive at the other. There are two well understood special cases. The first is $N = M$ for which, when sufficiently strong disorder is introduced, Anderson localization of the wavefunctions will occur. In the railroad analogy this corresponds to all track forming closed loops or loops which return to the city they start from so that no trains pass from one city to the other in either direction. The other special case is $N \neq 0, M = 0$ for which irrespective of the
strength of the disorder we see the quantum Hall effect. This corresponds to all track passing in the same direction so that any switching between tracks does not alter the number of effective extended rails and hence the ability of trains to pass in the direction of the N tracks. For the general case $N \neq M$ we will show that disorder causes a QRR to exhibit a novel behaviour that we will refer to as “directed localization” in which the system appears to be in the Anderson localization regime as far as trains attempting to pass in one direction are concerned but in the quantum Hall regime for trains passing in the opposite direction. This corresponds to all the LA tracks and $M$ of the NY tracks forming closed loops similar to the Anderson case and $N - M$ tracks remaining extended as in the quantum Hall effect.

To our knowledge no example of a QRR has been reported experimentally however both the Azbel-Wannier-Hofstadter (AWH) system and two dimensional arrays of coupled quantum dots have been shown to exhibit rich spectra containing different combinations of rotating (forward) and counter rotating (reverse) edge states. The transmittance between two reservoirs of a Hall bar with an imprinted AWH system has been predicted within the edge state picture of Ramal et al and MacDonald to be the algebraic sum of edge states: $|N - M|$. We have also shown this to be the case for arrays of quantum dots. These systems are therefore potential realizations of the general QRR.

In section II we will introduce the underlying physics of QRR’s by expanding on the simple railroad concept. In section III we give a derivation of the limiting transmittance of a QRR and show how transmission through a QRR is characterised by directed localization. In section IV we give the results of a numerical investigation of the statistical properties of a number of different QRR’s. In section V we give a derivation for the Shubnikov de Haas and Hall measurements for a four terminal Hall bar in which each terminal is connected by a QRR. We do this both for the case where disorder is present and where it is not. Section VI is a summary of our results.
II. THE SIMPLE RAILROAD

The picture of a railroad linking NY and LA with trains being switched between tracks is a remarkably useful model for understanding the underlying physics of QRR’s because it gives a physical picture for the coexistence of extended and localized states and indicates what to expect for the distributions of quantum mechanical fluctuations.

First consider the simplest example of a railroad which when disorder is introduced has both extended and localized track. This is the case where there is one LA track (1) and two NY tracks (2,3) shown in figure (1) event ‘e’. In a QRR disorder promotes scattering between the adiabatic modes so here we represent it by connecting pairs of tracks. The six different ways of doing this are shown shown in figure (1). A typical disordered railroad will contain a random selection of these switching events. Figure (2) shows a typical section of railroad containing seven switching events $s_3, s_1, s_2, s_3, s_4, s_4, g$. As far as trains in this system are concerned it consists of a single extended NY rail and three closed loops reminiscent of the $N = M = 1$ Anderson case. The picture this figure gives is not an exception but in fact defines a rule. If we join any pair of switching events together as far as incident trains are concerned the resultant scattering is equivalent to one of the other six switching events. Table (I) shows the resultant scattering from joining all pairs. It is clear from this table that the events $s_1, ...s_4$ dominate once introduced into any railroad. Any railroad with even a single event from $s_1, ...s_4$ will be equivalent to one of these events and therefore have a single extended state and a set of closed loops and loops which return to the city they started from. For example as far as incident trains are concerned the railroad in figure (2) is equivalent to the event $s_1$.

This picture readily generalises to the case of $N$ NY tracks and $M$ LA tracks. Here the dominant set of events are those which have $N - M$ extended NY tracks and no extended LA tracks denoted by $H_M$. The subscript refers to the number of incident LA/NY tracks which are reflected into NY/LA tracks by the event. The symmetry comes about because for each NY track that is reflected it takes the space of an LA track thereby forcing and LA
track to also be reflected. When we join an event from $H_M$ to any other event the result must be in $H_M$ since we cannot increase the number of extended tracks which pass across any scattering event. Hence we expect that a typical railroad will consist of a set of $N - M$ extended NY tracks and that the remaining $M$ NY tracks and $M$ LA tracks will form closed loops similar in form to the $N = M$ case. The algebraic structure of the general case is discussed in the appendix.

For the quantum mechanical model we will show that we have a coexistence of extended and localised states similar to the simple railroad. In this model however when the localised states are closely degenerate or coupled to reservoirs we would expect to see random fluctuations due to tunnelling appearing above the simple railroad predictions for the transmittances.

As we have said for the general case disorder causes closed loops similar to the $N = M$ case the difference being that the $N - M$ extended rails fill space and therefore on average force the closed loops apart. Thus we might expect that the average transmittance would decrease with increasing $N - M$ but that the form of the distribution and the variance be dependent only upon $M$. This discussion will be the subject of our section on numerical work.

III. THE QUANTUM RAILROAD

A quantum mechanical picture of the railroad with $N$ NY tracks and $M$ LA tracks must include a number of features. We must talk in terms of electron wavefunction amplitudes; the tracks become the adiabatic modes of a quantum mechanical waveguide. Scattering probabilities must be continuous; when a mode is scattered it will be scattered with a different random amplitude into all available modes. If we wish to conserve the number of electrons entering and leaving the waveguide then each scattering event must be unitary. Experimentally for such waveguides we would be interested in making standard magneto-resistance measurements such as those which are used for the Hall effect and Shubnikov
de Haas effect\cite{17}. We will discuss this point in more detail in section V. The mathematical constituents to such measurements are the two point transmittances and reflectances. From them we can construct the experimentally measured quantities using the Büttiker formalism\cite{22}. At this stage we will switch to the notation ‘forward’ for the $N$ modes and ‘reverse’ for the $M$ modes we will also assume that the forward direction is the majority mode direction; $N \geq M$.

We will prove that as we introduce more scatterers into a QRR the transmittance in the forward direction reduces to $T = N - M$ and in the reverse direction to $T' = 0$. Also, in analogue with the simple railroads effective open rails and closed loops of track, we show that a QRR has a set of $N - M$ perfectly transmitting forward channels and a set $2M$ channels for which transmission coefficients are determined by multiple scattering and interference effects. Our proof comes in three parts. The first is to show that unitarity imposes lower limits on the values that $T$ and $T'$ may take. The second part is to make a justification for stating that these limits are achieved by increasing the number of scattering events in the QRR and the third part is to show that the transmission probabilities in the diagonal basis defined by the disorder in a QRR are equal to one for a set of $N - M$ channels and the remainder are equally matched between the forward and reverse directions.

First we will prove that there are lower limits imposed on the values of the forward transmittance $T$ and the reverse transmittance $T'$. The scattering matrix of a QRR will have the form

\[
S = \begin{pmatrix} T & R \\ R' & T' \end{pmatrix}
\] (1)

where $T$ is an $N \times N$ transmission matrix containing the complex amplitudes for scattering between the $N$ forward modes, $T'$ is the $M \times M$ transmission matrix for the $M$ reverse modes and $R$ and $R'$ are the corresponding reflection matrices.

The two-terminal transmittances $T, T'$ and reflectances $R, R'$ of the QRR, when connected at either end to perfectly emitting and absorbing reservoirs, are simply related to the elements of this $S$ matrix through the norms of the transmission and reflection matrices\cite{23, 28}. 

7
\[ T = ||T||^2T' = ||T'||^2R = ||R||^2R' = ||R'||^2 \]  
(2)

In this case the norm is defined by the inner product

\[(a, b) = \text{trace}(ab^\dagger)\]  
(3)

where \(a\) and \(b\) are matrices with suitable dimensions.

From these definitions it is easy to see that in a QRR containing no scattering the transmittances will have the form \(T = N\) in the forward direction and \(T' = M\) in the reverse direction since each mode carries the same current. For such a system where no scattering occurs it has been shown that a fractional quantum Hall effect will be observed\(^{28,29}\). We will discuss this point in section V.

If we now introduce a series of unitary scatterers into the QRR then the scattering matrix for the whole system must also be unitary. This condition implies the following set of relations between the reflection and transmission matrices

\[ TT^\dagger + RR^\dagger = 1_N \]  
(4)

\[ T'T'^\dagger + R'R'^\dagger = 1_M \]  
(5)

\[ T^\dagger T + R'^\dagger R' = 1_N \]  
(6)

\[ T'^\dagger T' + R^\dagger R = 1_M \]  
(7)

Taking the norms of these relations we find that they imply the following set of relations between the reflectances and transmittances

\[ T + R = N \quad R = R' \quad T' = T - (N - M) \]  
(8)

These three relations together with the fact that \(T, T', R, R'\) are real and positive imply that any such system will have transmittances in the ranges

\[ 0 \leq T' \leq M \]  
(9)

\[ N - M \leq T \leq N \]  
(10)
Hence, we see that the transmittance in the majority mode direction has a lower limit of $T = N - M$ and in the minority mode direction $T' = 0$.

The second part of our proof is to make a justification for asserting that as we add more scattering events to our QRR both $T$ and $T'$ will tend to decrease and therefore for a typical system will eventually reach their minimum values. Adding a unitary scatterer to a QRR will cause a change in its reflection matrix given by

$$\mathbf{R}_{+1} = \mathbf{R} + \mathbf{B} \tag{11}$$

where

$$\mathbf{B} = \text{Tr}(1 - \mathbf{R}^\prime \mathbf{r})^{-1} \mathbf{T}' \tag{12}$$

Capitals represent the transmission and reflection matrices of the initial QRR and lower case those of the added scatterer. Taking the norm of Eq. (11) and using Eqs. (8) we find

$$T_{+1} = T - \|\mathbf{B}\|^2 - (\mathbf{B}, \mathbf{R}) - (\mathbf{R}, \mathbf{B}) \tag{13}$$

Note that the same relation holds for $T'$. Since the last two terms in Eq. (13) may be positive or negative it is clear that adding the extra scatterer can cause the transmittance to increase or decrease. However the possible choices for making $T_{+1}$ larger are limited in comparison to those which make it smaller and also the range of those choices is highly dependent on the details of the $\mathbf{S}$ matrix of the initial QRR. Thus unless the the scatterers are added to the QRR with properties highly correlated to what has come before the transmittance will tend to decrease. For example, if we construct a QRR by adding uncorrelated random unitary scatterers, because there are no correlations between the reflection phases of such scatterers it is easy to show that $\overline{(\mathbf{R}, \mathbf{B})} = 0$ and $\overline{(\mathbf{B}, \mathbf{R})} = 0$. The bar indicates an average over all possible scatterers which may be added. The proof of this is simple. Both inner products may be expanded as multinomial series in the reflection amplitudes contained in $\mathbf{r}$. After averaging, these series become expansions in the multivariate moments of the elements of $\mathbf{r}$. Each of these moments is zero if there are no correlations between the reflection phases and
therefore on average the inner products are zero. This fact indicates that on average adding scatterers to the QRR will reduce its transmittance in either direction. This type of random phase model has been used many times in the investigation of Anderson localization and in particular in connection with its use in determining a scaling theory\[40\]–\[45\].

If adding an extra unitary scatterer to a QRR typically reduces its transmittance in both directions then for systems containing many scatterers experiments will measure the values

$$T' = 0$$

$$T = N - M$$

(14)

(15)

The possibility that they may reduce to different minima is excluded by the fact that $B = 0$ typically only if $T' = 0$. The QRR therefore is a system in which the transmittance is directed. In one direction the system is opaque and in the other it has a transmittance which tends to a quantised value with increasing length.

The last part of our proof is to show for any QRR that there are a set of $N - M$ perfectly transmitted effective channels and that scattering between the remaining effective channels has the signature of the $N = M$ case (for each forward channel with transmission probability $|\lambda_i|^2 < 1$ there is a reverse channel with the same transmission probability) for which it is known that multiple scattering and interference effects cause Anderson localization.

So far we have considered the transmission and reflection matrices in the basis of the adiabatic modes of the undisordered QRR however when disorder is introduced to the system it defines another unitary basis in which the transmission matrix is diagonal. In general the basis may be different on either side of the QRR so that (note that we only write the equations for the forward direction but identical expressions hold for the reverse direction)

$$PTQ^\dagger = \Lambda$$

(16)

Where $P$ and $Q$ are the matrices containing the bases for the left and right side respectively and $\Lambda$ is a diagonal matrix containing the transmission amplitudes $\lambda_i$ of the new basis. The transmission probabilities for these new channels $|\lambda_i|^2$ are simply the eigenvalues of $T^\dagger T$ since
\[ QT^\dagger P^\dagger PTQ^\dagger = QT^\dagger TQ^\dagger = \Lambda^\star \Lambda \] (17)

In order to show that \( N - M \) of these transmission probabilities are equal to unity and the others exactly match between the forward and reverse directions it is convenient to look at sums of their \( n \)th powers. From equations (17) we find

\[
\text{trace}((TT^\dagger)^n) = N + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (RR^\dagger)^i
\] (18)

\[
\text{trace}((T'^\dagger T')^n) = M + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (R'R)^i
\] (19)

Using the identities

\[
\text{trace}((AA^\dagger)^n) = \text{trace}((A^\dagger A)^n)
\] (20)

\[
= \sum_{i=1}^{m} |\nu_i|^{2n}
\] (21)

where \( A \) is any \( m \times m' \) matrix and \( \nu_i \) are the eigenvalues of \( A \), these expressions reduce to

\[
\sum_{i=1}^{N} |\lambda_i|^{2n} = \sum_{i=1}^{M} |\lambda'_i|^{2n} + N - M
\] (22)

When we take the limit \( n \to \infty \) of this expression all transmission probabilities for which \( |\lambda_i|^{2n} < 1 \) reduce to zero so that in order to satisfy the identity we must have at least \( N - M \) forward transmission probabilities equal to unity. If the system is sufficiently long then there will be exactly \( N - M \) with unit transmission probability in the forward direction and the rest will be less than unity. In this limit cancelling the unit transmission probabilities from each side of the expression we are left with

\[
\sum_{i=1}^{M} |\lambda_i|^{2n} = \sum_{i=1}^{M} |\lambda'_i|^{2n}
\] (23)

We arrange the transmission probabilities largest to smallest then take the limit \( n \to \infty \). The largest transmission probabilities must be equal since the other transmission probabilities contribute no weight to the sums in comparison. These then cancel from the sum and we can make the same argument for successive pairs.

Hence we have shown that transmission through a QRR is carried by \( N - M \) perfectly transmitted effective channels and that fluctuations are caused by the remaining \( 2M \) effective
channels which will have transmittances determined by multiple scattering and interference effects within the QRR. Fluctuations in the transmittance can of course play an important role in determining whether a particular QRR will have a forward transmittance $T = N - M$ and reverse transmittance $T' = 0$. Localized states or ‘necklaces’ of localized states can act like pinholes through an otherwise opaque system. We look more closely at the statistical properties of QRR’s in the next section using numerical techniques.

IV. NUMERICAL STUDIES

For the Anderson case $N = M$ it is well known that the transmittance from one disordered to sample to another fluctuates wildly over many orders of magnitude and that these fluctuations grow with system size. The distribution of possible transmittances for any ensemble of disordered systems does not obey the central limit theorem as its moments are always dominated by rare but strong fluctuations. However the distribution of the logarithm of the transmittance is a more tractable quantity which for the bulk of the probability is approximately normally distributed. The logarithm of the transmittance relates directly to the exponential decay of wave functions into a system from a reservoir or decay away from a localised centre of electron probability in the interior of a system. The usual notation is $T = e^{-\gamma L}$ where $\gamma$ is the inverse localization length and $L$ is the length of the system. Our calculations in the previous section have shown that the transport in a general QRR is carried by $N - M$ perfectly transmitted channels and that the remainder of the channels participate in multiple scattering and interference effects identical in character to the unitary scattering of the Anderson case. In this section we produce numerical simulations which show this relation explicitly and illuminate the differences which are easily explained with the insight gained from the simple railroad model.

Our numerical work is based on a matrix method which consists of generating a transfer matrix of the form
\[
T_i = \begin{pmatrix}
\begin{array}{cc}
-rt' - 1 & t' \\
-t'^{-1} & t'^{-1}
\end{array}
\end{pmatrix}
\] (24)

for each scattering event \(i = 1 \rightarrow L\) and multiplying them together to find the transfer matrix for the whole system

\[
T = \prod_{i=1}^{N} T_i
\] (25)

The algebraic expressions in the blocks of the resultant transfer matrix are the same as in equation (24) for the transmission and reflection matrices of the whole system and therefore are straightforwardly rearranged to find the transmittance and reflectance of the whole system. We consider a random unitary ensemble of scatterers because this ensemble contains both scattering with time reversal symmetry and without and therefore constitutes a rigorous test for the general case. Our method for generating the random unitary matrices for each scattering event is as follows: first we generate a set of \(N + M\) random complex vectors which lie in a complex unit \(N + M\) dimensional sphere; then we normalise each vector to have unit length and perform Gram-Schmidt orthogonalisation to find a set of \(N + M\) orthogonal vectors e.g \(u^\dagger u = 1\). These vectors then form the column vectors of a unitary matrix which may be partitioned to give the matrices \(t, t'r, r'\). We found that this method together with a good random number generator gave an even distribution for the phase of the determinants of the unitary matrices which was our criterion for randomness. For error checking in this method we always calculate both the reflectance and transmittance and check to see that unitarity is preserved to better than 0.1% of the minority direction transmittance.

This numerical method straightforwardly demonstrates how a typical QRR achieves the condition for directed localization. Figure (4a) shows the \(N = 2, M = 1\) case for seven different configurations of scattering events. Each trace shows the logarithm of the minority mode transmittance as a function of the number of scattering events. All traces show that the transmittance reduces to zero very quickly for these typical configurations. The relation (8) implies that if the minority mode transmittance reduces to zero then the majority
mode transmittance takes on the value $T = N - M$. Figure (4b) shows the $N = M = 2$ Anderson case for a number of different impurity configurations. It is clear here also that the transmittance reduces to zero.

By calculating the transmittances of large numbers of randomly generated QRR’s we can demonstrate their statistical properties. Figures (5a,b,c) show a series of histograms of the natural logarithm of the minority transmittance $\log(T')$ against its probability of occurring $P(\log(T'))$ after 30 random unitary scattering events for different choices of $N$ and $M$. Each histogram was generated by calculating the minority mode transmittance of approximately 32000 different samples. We list the mean, rms value and ratio of the mean to the rms value for each distribution in table (II). These histograms show in a quantitative manner that a typical experimental sample would be expected to have a transmittance well quantised to $T = N - M$. The worst case $N = M = 3$ has less than four percent of its transmittances greater than $T' = 0.03$. The approximate shape of these distributions is normal as is expected given our analytic comparison with the Anderson case. We can explain the other features easily on the basis of the simple railroad model.

First looking at figures (5a,b,c) it is clear that the shape distributions and rms values have only a weak dependence on $N$ but a strong dependence on $M$. This is accounted for by the simple railroad picture in which the fluctuations are caused by tunnelling through localised states because for the general case the form of localized states is dependent only upon $M$. Secondly we note that the distributions appear to increase in width with decreasing $M$. This may be accounted for by the fact that the larger the value of $M$ the easier it is to get around a particular impurity or be de-localized since there are more states available to scatter into and therefore the fluctuations will be weaker. This fact also accounts for the fact that the mean transmittance increases with increasing $M$. Finally we note that the mean of each distribution decreases with increasing $N - M$. This is consistent with our observation that for the simple railroad, although the scattering occurs between $M$ forward modes and $M$ reverse modes, the $N - M$ extended modes effectively force these states further apart making it harder to pass through the system.
V. HALL AND SHUBNIKOV DE HAAS MEASUREMENTS

Conditions of electrodynamic continuity prohibit connecting a single QRR between two electron reservoirs. With zero voltage difference between the reservoirs such a device would cause current to pass from one to the other in the majority mode direction. QRR’s must be connected between reservoirs such that the total number of quantum mechanical modes which carry current into and out of each reservoir is equal. For example a Hall bar configuration where the QRR’s are composed of edge states. Hall and Shubnikov de Haas measurements are made on these systems using four reservoirs passing current between two and measuring the resulting voltage between the other two. In this section we will calculate these quantities for the case where no scattering is present and then for the case where scattering is present and directed localization occurs.

The case where no scattering occurs leads to fractional Hall and Shubnikov de Haas conductances as has been discussed for the case of multiple edge state transport in arrays of quantum dot. Figure (6a) shows the picture we have in mind for this case. The expression for current passing out of a reservoir $\alpha$ in a multi-probe system is given by the Büttiker formalism:

$$I_{\alpha} = \frac{e}{h} \left( \mu_{\alpha}(N_{\alpha} - R_{\alpha}) - \sum_{\beta \neq \alpha} \mu_{\beta} T_{\beta\alpha} \right) \tag{26}$$

so that the four leads in figure (6a) we will have currents

$$\frac{h}{e} I_s = \mu_s(N + M) - \mu_1 M - \mu_d N \tag{27}$$
$$\frac{h}{e} I_1 = \mu_1(N + M) - \mu_2 M - \mu_s N \tag{28}$$
$$\frac{h}{e} I_2 = \mu_2(N + M) - \mu_d M - \mu_1 N \tag{29}$$
$$\frac{h}{e} I_d = \mu_d(N + M) - \mu_s M - \mu_2 N \tag{30}$$

For the Shubnikov de Haas measurement we pass a current $I$ between probes $\mu_s$ and $\mu_d$ and measure the voltage between probes $\mu_1$ and $\mu_2$. No current flows into or out of the voltage
probes by definition so that \( I_1 = I_2 = 0, I_s = -I_d = I \). The Shubnikov de Haas resistance is defined by

\[
R_{\text{deHaas}} = \frac{\mu_1 - \mu_2}{eI} \tag{31}
\]

which solving the above set of equations gives

\[
R_{\text{deHaas}} = \frac{\hbar}{e^2} \frac{NM}{N^2M + NM^2 + N^3 + M^3} \tag{32}
\]

For the Hall measurement we pass a current \( I \) between probes \( \mu_1 \) and \( \mu_d \) and measure a voltage between \( \mu_s \) and \( \mu_2 \). This gives us \( I_s = I_2 = 0 \) and \( I_1 = -I_d = I \). Using the definition,

\[
R_{\text{Hall}} = \frac{\mu_s - \mu_2}{eI} \tag{33}
\]

and solving the current relations we find

\[
R_{\text{Hall}} = \frac{\hbar}{e^2} \frac{N - M}{N^2 + M^2} \tag{34}
\]

As has been noted these equations for the Hall and Shubnikov de Haas resistances imply a fractional conductance for certain values of \( N \) and \( M \).

If we now consider the case where directed localization occurs then a different picture appears: figure (6b). The set new of current relations have the form

\[
\frac{\hbar}{e} I_s = \mu_s(N + M - 2M) - \mu_d(N - M) \tag{35}
\]

\[
\frac{\hbar}{e} I_s = \mu_1(N + M - 2M) - \mu_s(N - M) \tag{36}
\]

\[
\frac{\hbar}{e} I_s = \mu_2(N + M - 2M) - \mu_1(N - M) \tag{37}
\]

\[
\frac{\hbar}{e} I_s = \mu_d(N + M - 2M) - \mu_2(N - M) \tag{38}
\]

These equations when simplified are familiar: they are exactly those of the ordinary integer edge state picture but with the usual \( N \) replaced by \( N - M \). They solve to give

\[
R_{\text{deHaas}} = 0 \tag{39}
\]

\[
R_{\text{Hall}} = \frac{\hbar}{e^2} \frac{1}{N - M} \tag{40}
\]
The Hall bar with no scattering and therefore no directed localization yields fractional Shubnikov de Haas and Hall conductances for certain values of $N$ and $M$ and the Hall bar with scattering and therefore directed localization yields a simple Hall and Shubnikov-de Haas effect.

**VI. SUMMARY**

In summary, we have given a detailed theory for the transport properties of a general class of disordered waveguides the quantum railroads (QRR’s). We have shown analytically that the effect of disorder in a QRR with $N$ channels which carry current in the forward direction and $M$ channels which carry current in the reverse direction is to cause the transmittance to equilibrate to $T = |N - M|$. We have also shown analytically that current is carried through a QRR by $|N - M|$ perfectly transmitted effective channels and that the remaining $2M$ channels participate in multiple scattering and interference effects identical in nature to the Anderson case where $N = M$. Our numerical simulations confirm this by giving normal distributions for the logarithm of the minority mode transmittance and further show that the mean decay length in these systems decreases with increasing $|N - M|$ consistent with the notion that the $|N - M|$ fully transmitted states fill space and therefore force any localized states apart thereby making the transmittance on average smaller.

**VII. APPENDIX: ALGEBRAIC PROPERTIES**

The set of all scattering events for the general case of $N$ NY tracks and $M$ LA tracks contains $(N + M)!$ elements. Intuitively we might think that this set with the composition rule defined by joining two events would be related to the symmetric group of order $N + M$ it is not though. The under the composition rule the set in fact forms a monoid for which the identity element is the event where no scattering occurs. This monoid naturally splits into one sub-group and a set of $M$ semi-groups. The sub-group is the set $H_0$ which contains all the events with $N$ extended NY tracks and $M$ extended LA tracks. The $M$ semi-groups
are the sets $H_i$ with $i = 1, \ldots, M$ where $H_i$ is the set of events which have $N - i$ extended NY tracks and $M - i$ extended LA tracks. These semi-groups form a hierarchy such that when elements from $H_i$ are composed with elements from $H_j$ if $i \geq j$ the result is an element in $H_i$. This makes $H_M$ the ideal semi-group of the monoid and it is to this property that we may ascribe the dominant role of $H_M$ in any railroad.

Acknowledgments

We wish to thank A. H. MacDonald for a stimulating discussion. C. B. would like to thank the Royal Society of London for their financial support and SFU for their kind hospitality during his stay in Canada. B. L. J. and G. K. wish to acknowledge the financial support of the NSERC of Canada and the CSS at SFU. C.B. would like to thank D. Maslov and K. Yeo for useful discussion.
REFERENCES

1 C. Barnes, B. L. Johnson and G. Kirczenow, Phys. Rev. Lett. 70 8 1159 (1993).

2 P. W. Anderson, Phys. Rev. 109 1492 (1958).

3 B. Kramer and A. MacKinnon, submitted to Rep. Prog. Phys. and references therein.

4 N. F. Mott Metal Insulator Transitions, 2nd ed. (Taylor and Francis, London, 1991).

5 N. F. Mott and W. D. Twose, Adv. Phys. 10 107 (1961).

6 H. Kunz and B. Souillard, Commun. Math. Phys. 78 201 (1980).

7 A. MacKinnon, J. Phys. C: Solid State Physics 13 L1031 (1980).

8 G. Czycholl, B. Kramer and A. MacKinnon, Z. Physik B39 193 (1981).

9 D. J. Thouless and S. Kirkpatrick, J. Phys. C: Solid State Physics 14 235 (1981).

10 G. Blonder, R Dynes, and A. White, unpublished (1984).

11 C. P. Umbach, S. Washburn, R. B. Laibowitz, and R. A. Webb, Phys. Rev. B 30 4048 (1984)

12 R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, in ‘Localization, Interaction, and Transport Phenomena’, G. Bergmann, Y. Bruyseraede, and B. Kramer, eds., Springer Ser. in Sol. St. 61 121 (1985).

13 B. L. Al’tshuler, JETP Lett. 41 648 (1985).

14 P. A. Lee, A. D. Stone, and J. Fukuyama, Phys. Rev. B 35 1039 (1987).

15 J. B. Pendry, A MacKinnon and A. B. Pretre, Physica. A 168 400 (1990).

16 A. MacKinnon in ‘Quantum Coherence in Mesoscopic Systems’, editor B. Kramer, NATO ASI, Series B: Physics Vol. 254,page 415.

17 K. von Klitzing, G. Dorda and M. Pepper, Phys. Rev. Lett. 45 494 (1980).
18. R. B. Laughlin, Phys. Rev. B 23 5632 (1981).
19. B. I. Halperin, Phys. Rev. B 25 2185 (1982).
20. P. Streda, J. Kucera and A. H. MacDonald, Phys. Rev. Lett. 59 1973 (1987).
21. J. K. Jain and S. A. Kivelson, Phys. Rev. Lett. 60 1542 (1988).
22. M. Büttiker, Phys. Rev. B 38 9375 (1988).
23. R. Landauer, IBM J. Res. Dev. 1 223 (1957).
24. M. Y. Azbel, Zh. E. T. F. 46 929 (1964).
25. G. H. Wannier, Phys. Status Solidi (b) 88 757 (1978).
26. D. R. Hofstadter, Phys. Rev. B 14 2239 (1976).
27. G. Kirczenow, Phys. Rev. B 46 1439 (1992).
28. B. L. Johnson and G. Kirczenow, Phys. Rev. Lett. 69 672 (1992).
29. B. L. Johnson, C. Barnes and G. Kirczenow, Phys. Rev. B. 46 15 302 (1992).
30. M. A. Reed, J. N. Randall, R. J. Aggarwall, R. J. Matayi, T. M. Moore, and A. E. Westel, Phys. Rev. Lett. 60 535 (1988).
31. K. Ismail, T. P. Smith III, W. T. Masselink, H. I. Smith, Appl. Phys. Lett. 75 276 (1989).
32. R. Rammal, G. Toulouse, M. T. Jakel and B. I. Halperin, Phys. Rev. B 27 5124 (1983).
33. A. H. MacDonald, Phys. Rev. B 29 6563 (1984).
34. C. Barnes G. Kirczenow and T. Sugano, to be published.
35. R. Akis C. Barnes G. Kirczenow and T. Sugano, to be published
36. R. Landauer, Z. Phys. 68 217 (1987)
37. D. S. Fisher and P. A. Lee, Phys. Rev. B 23 6851 (1981).
A. D. Stone and A. Szafer, IBM J. Res. Dev. 32 384 (1988).

C. Barnes and J. B. Pendry, Proc. R. Soc. Lond. A 435 185 (1991).

P. W. Anderson, D. J. Thouless, E Abrahams and D.S. Fisher, Phys. Rev. B 22 3519 (1980).

P. W. Anderson Phys. Rev. B 23 4818 (1981).

E. Abrahams and M. Stephen J. Phys. C 13 L377 (1980).

B. Andereck and E. Abrahams J. Phys. C 13 L383 (1980).

J. Sak and B. Kramer Phys. Rev. B 24 1761 (1981).

A. D. Stone and J. D. Joannopoulous Phys. Rev. B 24 3592 (1981).

J. B Pendry J. Phys. C. 20 773 (1987).

A. D. Stone and J. D. Joannopulos, Phys. Rev. 25 2500 (1982).

P. D. Kirkman and J. B. Pendry, J. Phys. C: Solid State Physics 17 4327 (1984).

P. D. Kirkman and J. B. Pendry, J. Phys. C: Solid State Physics 17 5707 (1984).

A. A. Abrikosov, Solid State Commun. 37 997 (1981).

V. I. Mel’nikov Sov. Phys. JETP Lett. 32 225 (1981).

J. B. Pendry A. MacKinnon and P. J. Roberts, Proc. R. Soc. Lond. A 437 67 (1991).

A. J. O’Connor, Commun. Math. Phys. 45 63 (1975).

P. Markos and B. Kramer, Ann. Phys. to be published (1993a).

C. J. Isham, Lectures on Groups and Vector Spaces, (World Scientific Lecture Notes in Physics 1989 ) Vol 31, p.3. A monoid is a set of elements with an associative composition rule and in inverse. A semi-group is a set of elements with an associative composition rule.
FIGURES

FIG. 1. The six possible scattering events for the $N = 2, M = 1$ simple railroad.

FIG. 2. A typical stretch of railroad for the $N = 2, M = 1$ case showing a single extended track and a number of localized tracks.

FIG. 3. A typical stretch of railroad for the $N = 4, M = 2$ case showing two extended tracks (a) and a number of localized tracks (b).

FIG. 4. The log of the minority mode transmittance for a) $N = 2, M = 1$ and b) $N = M = 2$, as a function of the number of scatterers in the system each for seven different impurity configurations.

FIG. 5. Histograms of the probability distributions of the minority mode transmittances for a) $M = 1$ and $N = 1, 2, 3, 4$, b) $M = 2$ and $N = 2, 3, 4, 5$, c) $M = 3$ and $N = 3, 4, 5, 6$, calculated for railroads with 30 scattering events.

FIG. 6. a) Edge state transport with no scattering in a Hall bar. b) Edge state transport with directed localization in a Hall bar.
TABLES

TABLE I. Multiplication table for the $N = 2, M = 1$ railroad switching events.

| $\times$ | $e$ | $g$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---------|----|----|-------|-------|-------|-------|
| $e$     | $e$ | $g$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
| $g$     | $g$ | $e$ | $s_4$ | $s_3$ | $s_2$ | $s_1$ |
| $s_1$   | $s_1$ | $s_2$ | $s_1$ | $s_2$ | $s_2$ | $s_1$ |
| $s_2$   | $s_2$ | $s_1$ | $s_1$ | $s_2$ | $s_2$ | $s_1$ |
| $s_3$   | $s_3$ | $s_4$ | $s_4$ | $s_3$ | $s_3$ | $s_4$ |
| $s_4$   | $s_4$ | $s_3$ | $s_4$ | $s_3$ | $s_3$ | $s_4$ |

TABLE II. Table of the means and rms values of each histogram in figures (3a, b, c).

| N,M   | Mean | RMS  | $|\text{Mean}/\text{RMS}|$ |
|-------|------|------|----------------|
| 1,1   | -41.54 | 11.66 | 3.56           |
| 2,1   | -53.30 | 9.76  | 5.46           |
| 3,1   | -61.00 | 8.87  | 6.88           |
| 4,1   | -67.06 | 8.30  | 8.08           |
| 2,2   | -15.80 | 5.54  | 2.85           |
| 3,2   | -25.81 | 5.41  | 4.77           |
| 4,2   | -33.13 | 5.28  | 6.27           |
| 5,2   | -39.00 | 5.12  | 7.61           |
| 3,3   | -10.50 | 3.99  | 2.63           |
| 4,3   | -17.50 | 3.99  | 4.38           |
| 5,3   | -23.28 | 3.97  | 5.86           |
| 6,3   | -28.09 | 3.91  | 7.18           |