Coherent-Squeezed State Representation of Travelling General Gaussian Wave Packets

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Abstract

Using the time-dependent annihilation and creation operators, the invariant operators, for a free mass and an oscillator, we find the coherent-squeezed state representation of a travelling general Gaussian wave packet with initial expectation values, $x_0$ and $p_0$, of the position and momentum and variances, $\Delta x_0$ and $\Delta p_0$. The initial general Gaussian wave packet takes, up to a normalization factor, the form $e^{ip_0x/\hbar}e^{-\left(1\mp i\delta\right)(x-x_0)^2/4(\Delta x_0)^2}$, where $\delta = \sqrt{(2\Delta x_0\Delta p_0/\hbar)^2 - 1}$ denotes a measure of deviation from the minimum uncertainty or the initial position-momentum correlation $\delta = 2\Delta xp_0/\hbar$. The travelling Gaussian wave packet takes, up to a time-dependent phase and normalization factor, the form $e^{ip_c x/\hbar}e^{-\left(1-2i\Delta x_c t/\hbar\right)(x-x_c)^2/4(\Delta x_c)^2}$ and the centroid follows the classical trajectory with $x_c(t)$ and $p_c(t)$. The position variance is found to have additionally a linearly time-dependent term proportional to $\delta$ with both positive and negative signs.

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I. INTRODUCTION

Localized wave packets have been used as an important method to describe quantum motions of matters since the advent of quantum theory (for review and references, see [1, 2]). Recently wave packets have attracted much attention, in particular, in quantum wave packet revivals predicted for long time evolution of bounded state systems [3] and their experimental confirmation [4]. The collapse and revival of matter field has also been observed in a Bose-Einstein condenstate [5]. The simplest and most well-known wave packet is the Gaussian wave packet which is useful not only to understand concepts of wave packets but also to study exactly a certain type of quantum motions. In the gravitational wave detection, for instance, a measuring apparatus is a free mass or an oscillator and the system should be monitored up to quantum limit due to extreme weak gravitational waves [6, 7, 8]. There the quantum motion of the free mass or oscillator that will provide important information about the external forces can be described by wave packets.

The Gaussian wave packet of minimum uncertainty is particularly of interest from the measurement point of view. It has been known that the Gaussian wave packet can have arbitrary position and momentum expectation values and the position variance or momentum variance [9, 10, 11, 12, 13]. Thus there seem to be at most three parameters to fix the profile of the Gaussian wave packet. However, the variances of the position and momentum can take arbitrary values as far as the uncertainty principle is satisfied. As Bohm has already explained [9], the independent variances of the position and momentum necessarily mean a correlation between these two noncommuting operators. This conditional freedom in choosing the variances of the position and momentum may raise the question of which wave packet will have all four parameters, \( x_0, p_0, \Delta x_0 \) and \( \Delta p_0 \) under the provisions of \( \Delta x_0 \Delta p_0 \geq \hbar / 2 \). Howard and Roy made an attempt to extend the Gaussian wave packet by introducing a position squared phase [14]. This additional position-dependent phase factor results in the position-momentum correlation and is recognized as a squeezing effect [15, 16].

On the other hand, various methods to construct quantum states and techniques to measure them have been developed in quantum optics and atomic physics (for review and references, see [17]). Glauber’s coherent state is a useful tool to find quantum states, particularly, of an oscillator that displace the initial wave packets and thus exhibit a classical feature [18]. The squeezed states deform the shape of wave packets by changing one variance
at the price of the other variance [19]. There is one complex parameter, equivalent to two real parameters, for a coherent state, and one squeeze parameter and one squeeze angle for a squeeze state. Then a relevant question is whether there is any connection between four parameters for initial wave packets and four parameters for the coherent and squeezed state. Years ago Yuen introduced such a coherent-squeezed state at the initial time to account for a negative contribution to the position variance from the initial correlation between the position and momentum [20]. However, there still remains unfound the exact form of the travelling general Gaussian wave packet evolved from an initial wave packet.

To exploit the connection between the parameters for Gaussian wave packets and the parameters for coherent and squeezed states, we shall employ the invariant operator method introduced by Lewis and Riesenfeld [21]. Though this method was introduced for time-dependent quantum systems, in particular, oscillators with time-dependent mass and/or frequency, it can be applied even to time-independent systems. The idea is that any operator satisfying the quantum Liouville-von Neumann equation provides exact solutions of the time-dependent Schrödinger equation up to time-dependent phase factors. For time-dependent oscillators, both quadratic invariant operators [21, 22, 23, 24] and linear invariant operators [25, 26, 27, 28, 29, 30, 31] were used to find the wave functions.

The purpose of this paper is to apply the invariant operator method to a free mass and an oscillator and find the travelling general Gaussian wave packet with four parameters. For that purpose we find a pair of linear invariant operators, the time-dependent annihilation and creation operators, for the free mass and the oscillator and then use them to find a coherent-squeezed state that is an exact solution of the time-dependent Schrödinger equation. And then we relate one complex parameter for the coherent state and two real parameters for the squeezed state with the four parameters for the initial Gaussian wave packet. This identification provides the explicit form of travelling general Gaussian wave packet that has the required initial parameters.

The organization of this paper is as follows. In Sec. II, it is shown that there are at most four parameters for initial Gaussian wave packets. In Sec. III, we apply the invariant operator method to find the time-dependent annihilation and creation operators for a free mass and an oscillator. It is shown that a general annihilation operator is a unitary transform of some preferred one via a squeeze operator. In Sec. IV, the general wave packet is obtained as a coherent-squeezed state of some preferred Gaussian wave packet. In. Sec.
V, we determine one complex parameter, one squeeze parameter and angle in terms of four parameters for an initial Gaussian wave packet. And then we find the evolution of variances of the position and momentum.

II. GAUSSIAN WAVE PACKETS

In the Schrödinger picture, many textbooks (for instance, see [11, 12, 13]) use the Gaussian wave packet in the position space

$$\psi(x, t = 0) = \frac{1}{(2(\Delta x_0)^2)^{1/4}} e^{ip_0 x/\hbar} e^{-(x-x_0)^2/4(\Delta x_0)^2},$$

and in the momentum space

$$\phi(p, t = 0) = \frac{1}{(2(\Delta p_0)^2)^{1/4}} e^{-i(p-p_0)x_0/\hbar} e^{-(p-p_0)^2/4(\Delta p_0)^2},$$

where the variances of the position and momentum are related by the minimum uncertainty

$$\Delta x_0 \Delta p_0 = \frac{\hbar}{2}.$$  (3)

The Gaussian wave packet (1) has the expectation values and the variances of the position and momentum

$$\langle \psi(x, 0)|\hat{x} |\psi(x, 0)\rangle = x_0, \quad \langle \psi(x, 0)|\hat{p} |\psi(x, 0)\rangle = p_0,$$

$$\langle \psi(x, 0)|(\hat{x} - \langle \hat{x} \rangle)^2|\psi(x, 0)\rangle = (\Delta x_0)^2, \quad \langle \psi(x, 0)|(\hat{p} - \langle \hat{p} \rangle)^2|\psi(x, 0)\rangle = (\Delta p_0)^2.$$  (4)

However, as far as the uncertainty principle is satisfied, one can have four parameters to fix the profile of a general Gaussian wave packet at the initial time: two expectation values and two variances

$$x_0 = \langle \hat{x} \rangle_0, \quad p_0 = \langle \hat{p} \rangle_0,$$

$$(\Delta x_0)^2 = \langle (\hat{x} - x_0)^2 \rangle_0, \quad (\Delta p_0)^2 = \langle (\hat{p} - p_0)^2 \rangle_0.$$  (5)

A simple way to make the momentum variance independent of the position variance without changing the probability distribution in the position space would introduce a phase factor [14]

$$\tilde{\psi}(x, t = 0) = e^{ix^2/\hbar} \psi(x, t = 0).$$  (6)
There is no new parameter in the wave function of the Schrödinger equation except for the four parameters in the initial wave packet, since the evolution of the wave packet is entirely determined by

\[ \Psi(x, t) = e^{-i\hat{H}t/\hbar}\psi(x, 0) = \int \frac{dp}{(2\pi\hbar)^{1/2}} e^{ipx/\hbar}\phi(p, t = 0). \] (7)

On the other hand, in the Heisenberg picture, an operator evolves according to the Heisenberg equation

\[ i\hbar \frac{d\hat{A}}{dt} = i\hbar \frac{\partial \hat{A}}{\partial t} + [\hat{A}, \hat{H}]. \] (8)

For a free mass, the Heisenberg equations become

\[ \frac{d\langle \hat{x} \rangle}{dt} = \frac{\langle \hat{p} \rangle}{m}, \quad \frac{d\langle \hat{p} \rangle}{dt} = 0, \]
\[ \frac{d^2\langle \hat{x}^2 \rangle}{dt^2} = \frac{\langle \hat{p}^2 \rangle}{m^2}, \quad \frac{d\langle \hat{p}^2 \rangle}{dt} = 0. \] (9)

Then the expectation values of the position and momentum at any time can be found as [20, 32]

\[ \langle \hat{x} \rangle_t = \langle \hat{x} \rangle_0 + \langle \hat{p} \rangle_0 \frac{t}{m}, \]
\[ \langle \hat{p} \rangle_t = \langle \hat{p} \rangle_0, \] (10)

and the variances as

\[ (\Delta x_t)^2 = (\Delta x_0)^2 + \left\{ (\langle \hat{x} - \langle \hat{x} \rangle_0 \rangle (\hat{p} - \langle \hat{p} \rangle_0) + (\hat{p} - \langle \hat{p} \rangle_0) (\hat{x} - \langle \hat{x} \rangle_0) \right\} \frac{t}{m} + (\Delta p_0)^2 \frac{t^2}{m^2}, \]
\[ (\Delta p_t)^2 = (\Delta p_0)^2. \] (11)

One can see from (10) and (11) that there are four independent initial data: \( \langle \hat{x} \rangle_0, \langle \hat{p} \rangle_0, \Delta x_0 \) and \( \Delta p_0 \). A similar argument holds for the oscillator [32]. The linearly time-dependent term in (11) was introduced earlier in [10] and was discussed later in the context of coherent states [20, 33]. It should be noted that the linearly time-dependent term is related with the correlation between the noncommuting operators \( \hat{x} \) and \( \hat{p} \) [9].

### III. INVARIANT OPERATOR METHOD

An immediate question may be asked which wave packet has the four parameters (5) to fit the initial profile. Instead of directly looking for such a wave packet, we go around by
solving the time-dependent Schrödinger equation

\[
  i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H}(t)\Psi,
\]

(12)

for the free mass or the oscillator

\[
  \hat{H}_f = \frac{\hat{p}^2}{2m}, \quad \hat{H}_o = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} \hat{x}^2.
\]

(13)

Though it would be straightforward to solve (12) for the free mass and the oscillator, we shall use the invariant operator method by Lewis and Riesenfeld [21]. In the invariant operator method one first solves the quantum Liouville-von Neumann equation

\[
  i\hbar \frac{\partial \hat{I}}{\partial t} + [\hat{I}, \hat{H}] = 0,
\]

(14)

and then finds the eigenfunction

\[
  \hat{I}(t)\psi_\lambda(x, t) = \lambda \psi_\lambda(x, t).
\]

(15)

Then the wave function of the Schrödinger equation is given by

\[
  \Psi_\lambda(x, t) = e^{i \int (\psi_\lambda(t)|i\partial/\partial t - \hat{H}/\hbar|\psi_\lambda(t))dt} \psi_\lambda(x, t),
\]

(16)

Though the invariant operator method was originally introduced to study time-dependent quantum systems, it can be used even for time-independent systems such as (13).

It is known that there are a pair of invariant operators of the form [25, 26, 27, 28, 29, 30, 31]

\[
  \hat{a}(t) = \frac{i}{\sqrt{\hbar}}(u^*(t)\hat{p} - m\dot{u}^*(t)\hat{x}), \quad \hat{a}^\dagger(t) = -\frac{i}{\sqrt{\hbar}}(u(t)\hat{p} - m\dot{u}(t)\hat{x}),
\]

(17)

where \( u \) is a complex solution of the motion either for the free mass

\[
  \ddot{u} = 0,
\]

(18)

or for the oscillator

\[
  \ddot{u} + \omega^2 u = 0.
\]

(19)

When the wronskian condition is imposed

\[
  m(u\dot{u}^* - \dot{u}u^*) = i,
\]

(20)
the invariant operators (17) become the time-dependent annihilation (lowering) and creation (raising) operators that satisfy the equal time commutation relation

\[[\hat{a}(t), \hat{a}^\dagger(t)] = 1.\]  

(21)

For instance, the complex solution to the oscillator

\[u_0(t) = \frac{e^{-i\omega t}}{\sqrt{2m\omega}},\]  

(22)

satisfies (20) and leads to the invariant operators

\[\hat{a}(t) = e^{i\omega t}\hat{a}, \quad \hat{a}^\dagger(t) = e^{-i\omega t}\hat{a}^\dagger,\]  

(23)

where \(\hat{a}\) and \(\hat{a}^\dagger\) are the standard annihilation and creation operators

\[\hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x} + \frac{i}{\sqrt{m\omega}}\hat{p}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x} - \frac{i}{\sqrt{m\omega}}\hat{p}\right).\]  

(24)

Note that the invariant operators have the opposite sign in the frequency from that of the Heisenberg operators

\[\hat{a}_H(t) = e^{-i\omega t}\hat{a}, \quad \hat{a}_H^\dagger(t) = e^{i\omega t}\hat{a}^\dagger.\]  

(25)

From now on we shall find a more general solution to (18) and (19). The general solution can be found in the form

\[u_r(t) = (\cosh r)u_0(t) + (e^{-i\vartheta}\sinh r)u_0^*(t), \quad (r \geq 0, \ 2\pi > \vartheta \geq 0)\]  

(26)

where \(u_0\) is a preferred solution satisfying (20) for the free mass

\[u_0(t) = \frac{1}{\sqrt{2}}\left(1 - i\frac{t}{m}\right),\]  

(27)

and (22) for the oscillator. Note that there are two arbitrary real parameters \(r\) and \(\vartheta\) in the general solution (26) which are required for a second order differential equation. The general solution (26) leads to a Bogoliubov transformation between the operators \(\hat{a}_r\) and \(\hat{a}_r^\dagger\) obtained by putting \(u_r\) and those operators \(\hat{a}_0\) and \(\hat{a}_0^\dagger\) obtained by putting \(u_0\) in (17):

\[\hat{a}_r(t) = (\cosh r)\hat{a}_0(t) - (e^{i\vartheta}\sinh r)\hat{a}_0^\dagger(t),\]

\[\hat{a}_r^\dagger(t) = (\cosh r)\hat{a}_0^\dagger(t) - (e^{-i\vartheta}\sinh r)\hat{a}_0(t).\]  

(28)

Here \(r\) is a squeeze parameter and \(\vartheta\) is a squeeze angle. In fact, in terms of the squeeze operator

\[\hat{S}(z,t) = e^{(z\hat{a}_0^\dagger(t) - z^*\hat{a}_0(t))/2}, \quad z = e^{i(\pi - \vartheta)}r,\]  

(29)
\( \hat{a}_r \) can be written as a unitary transformation [19]
\[
\hat{a}_r(t) = \hat{S}(z, t) \hat{a}_0(t) \hat{S}^\dagger(z, t).
\] (30)

Hence any wave function obtained by using \( \hat{a}_r \) and \( \hat{a}^\dagger_r \) is the squeezed state of the corresponding wave function by \( \hat{a} \) and \( \hat{a}^\dagger \). For instance, the number state \( \psi_{nr} \) of \( \hat{N}_r = \hat{a}^\dagger_r \hat{a}_r \) is the squeezed number state of \( \psi_{n0} \) of \( \hat{N}_0 = \hat{a}^\dagger_0 \hat{a}_0 \), that is, \( \psi_{nr} = \hat{S}\psi_{n0} \).

IV. COHERENT-SQUEEZED STATES

To find the general Gaussian wave packet, we introduce a coherent state
\[
\hat{a}_r(t) \psi_{\alpha r}(x, t) = \alpha \psi_{\alpha r}(x, t),
\] (31)
for a complex number \( \alpha \). It follows from (30) that \( \psi_{\alpha r} \) is the squeezed state of the coherent state \( \psi_{\alpha 0} \) defined by \( \hat{a}_0 \). In fact, the state (31) is the coherent-squeezed state (ideal squeezed state) [34]
\[
\psi_{\alpha r}(x, t) = \hat{D}(\alpha, t) \hat{S}(z, t) \psi_{00}(x, t),
\] (32)
where \( \hat{D} = e^{\alpha \hat{a}^\dagger_r - \alpha^* \hat{a}_r} \) is the displacement operator and \( \psi_{00} \) is the ground state of \( \hat{a}^\dagger_0 \hat{a}_0 \). Note that the coherent-squeezed state (31) has one complex constant \( \alpha \), two real constants \( r \) and \( \vartheta \), which will be related with four parameters in (5). From the position and momentum operators
\[
\hat{x} = \sqrt{\hbar}(u_r(t) \hat{a}_r(t) + u_r^*(t) \hat{a}_r^\dagger(t)),
\]
\[
\hat{p} = \sqrt{\hbar m}(\dot{u}_r(t) \hat{a}_r(t) + \dot{u}_r^*(t) \hat{a}_r^\dagger(t)),
\] (33)
their expectation values with respect to the coherent state (31) yield
\[
\langle \psi_{\alpha r}(t) | \hat{x} | \psi_{\alpha r}(t) \rangle = \sqrt{\hbar}(u_r(t) \alpha + u_r^*(t) \alpha^*),
\]
\[
\langle \psi_{\alpha r}(t) | \hat{p} | \psi_{\alpha r}(t) \rangle = \sqrt{\hbar m}(\dot{u}_r(t) \alpha + \dot{u}_r^*(t) \alpha^*). \] (34)

We shall choose \( \alpha \) for the free mass
\[
\alpha = \frac{1}{\sqrt{2\hbar}} \left\{ (\cosh r - e^{i\vartheta} \sinh r)x_0 + i(\cosh r + e^{i\vartheta} \sinh r)p_0 \right\}, \] (35)
and for the oscillator
\[
\alpha = \frac{1}{\sqrt{2\hbar}} \left\{ \sqrt{m\omega}(\cosh r - e^{i\vartheta} \sinh r)x_0 + \frac{i}{\sqrt{m\omega}}(\cosh r + e^{i\vartheta} \sinh r)p_0 \right\}. \] (36)
Then the expectation values follow the classical trajectory
\[
\langle \psi_{\alpha r}(t) | \hat{x} | \psi_{\alpha r}(t) \rangle = x_c(t), \quad \langle \psi_{\alpha r}(t) | \hat{p} | \psi_{\alpha r}(t) \rangle = p_c(t),
\]  
where the classical position \( x_c \) for the free mass or the oscillator is, respectively,
\[
x_c(t) = x_0 + \frac{p_0 t}{m},
\]
\[
x_c(t) = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t).
\]
(38)

The classical momentum is simply given by \( p_c = m \dot{x}_c \). Note that the classical position and momentum have the initial values \( x_c(0) = x_0 \) and \( p_c(0) = p_0 \) as expected.

Finally, we obtain the wave packet for the coherent state (31)
\[
\Psi_{\alpha r}(x, t) = \left( \frac{1}{\sqrt{2\pi\hbar}} \right)^{1/2} e^{-iS_c/\hbar} e^{i\hat{p}_0x/\hbar} e^{im\hat{u}_r^* x - x_c^2/2\hbar u_r^*},
\]
(39)
where \( S_c \) is the classical action
\[
S_c(t) = \int^t L_c(t')dt'.
\]
(40)

The classical Lagrangian \( L_c \) for the free mass is
\[
L_c(t) = \frac{p_c^2(t)}{2m} = \frac{p_0^2}{2m},
\]
(41)
and for the oscillator
\[
L_c(t) = \frac{p_c^2(t)}{2m} - \frac{m\omega^2}{2} x_c^2(t) = \left( \frac{p_0^2}{2m} - \frac{m\omega^2 x_0^2}{2} \right) \cos(2\omega t) - \omega x_0 p_0 \sin(2\omega t).
\]
(42)

As \( e^{-iS_c/\hbar} \) is a pure phase factor, the centroid of the Gaussian wave packet (39) follows the same classical trajectory as expected of a coherent state:
\[
\langle \Psi_{\alpha r}(t) | \hat{x} | \Psi_{\alpha r}(t) \rangle = x_c(t), \quad \langle \Psi_{\alpha r}(t) | \hat{p} | \Psi_{\alpha r}(t) \rangle = p_c(t).
\]
(43)

Thus we have obtained the travelling general wave packet (39) whose centroid is the classical trajectory (38).

V. SQUEEZE PARAMETERS

The free mass has the time-dependent variance of the position with respect to the Gaussian wave packet (39)
\[
(\Delta x_t)^2 = \hbar u_r(t) u_r(t) = \frac{\hbar}{2} \left\{ e^{2r} \left( \cos \frac{\vartheta}{2} + \frac{t}{m} \sin \frac{\vartheta}{2} \right)^2 + e^{-2r} \left( \sin \frac{\vartheta}{2} - \frac{t}{m} \cos \frac{\vartheta}{2} \right)^2 \right\},
\]
(44)
and the constant variance of the momentum

\[
(\Delta p_t)^2 = \hbar m^2 \dot{u}_r(t) \dot{u}_r(t) = \frac{\hbar}{2} \left( e^{2r} \sin^2 \frac{\vartheta}{2} + e^{-2r} \cos^2 \frac{\vartheta}{2} \right). \tag{45}
\]

Similarly, the oscillator has the time-dependent variances of the position and momentum

\[
(\Delta x_t)^2 = \frac{\hbar}{2 m \omega} \{ \cosh(2r) + \sinh(2r) \cos(2\omega t - \vartheta) \}, \quad (\Delta p_t)^2 = \frac{\hbar m \omega}{2} \{ \cosh(2r) - \sinh(2r) \cos(2\omega t - \vartheta) \}. \tag{46}
\]

However, the correlation between the position and momentum is not an independent quantity but is determined either by \( r \) and \( \vartheta \) or \( \Delta x_t \) or \( \Delta p_t \) since the parameters \( r \) and \( \vartheta \) are determined by the two variances or vice versa. The position-momentum correlation

\[
\Delta(xp)_t = \frac{1}{2} \left\{ (\hat{p} - p_c) \hat{x} - x_c \right\} + (\hat{x} - x_c)(\hat{p} - p_c) = \frac{\hbar m}{2} (\dot{u}_r^*(t) u_r(t) + u_r^*(t) \dot{u}_r(t)) \tag{47}
\]

is for the free mass

\[
\Delta(xp)_t = \frac{\hbar}{2} \left\{ \sinh(2r) \sin \vartheta + (\cosh(2r) - \sinh(2r) \cos \vartheta) \frac{t}{m} \right\}, \tag{48}
\]

and for the oscillator

\[
\Delta(xp)_t = -\frac{\hbar}{2} \sinh(2r) \sin(2\omega t - \vartheta). \tag{49}
\]

Now we determine the squeeze parameter \( r \) and squeeze angle \( \vartheta \) in terms of the initial variances at \( t = 0 \). The parameters \( r \) and \( \vartheta \) for the free mass are given by

\[
cosh(2r) = \frac{1}{\hbar} \{ (\Delta x_0)^2 + (\Delta p_0)^2 \}, \\
\cos \vartheta \sinh(2r) = \frac{1}{\hbar} \{ (\Delta x_0)^2 - (\Delta p_0)^2 \}, \tag{50}
\]

and for the oscillator are given by

\[
cosh(2r) = \frac{1}{\hbar} \left\{ m \omega (\Delta x_0)^2 + \frac{1}{m \omega} (\Delta p_0)^2 \right\}, \\
\cos \vartheta \sinh(2r) = \frac{1}{\hbar} \left\{ m \omega (\Delta x_0)^2 - \frac{1}{m \omega} (\Delta p_0)^2 \right\}. \tag{51}
\]

In both cases of the free mass and the oscillator, an important dimensionless parameter that denotes the amount of deviation from the minimum uncertainty or the position-momentum correlation at the initial time can be introduced

\[
\sin \vartheta \sinh(2r) = \frac{2}{\hbar} \Delta(xp)_0 = \pm \sqrt{(2\Delta x_0 \Delta p_0/\hbar)^2 - 1} \equiv \pm \delta. \tag{52}
\]
It should be noted that the position-momentum correlation is determined by the variances of the position and momentum as expected. Here and in this paper it is assumed that the variances satisfy the uncertainty principle

\[ \Delta x_0 \Delta p_0 \geq \frac{\hbar}{2}. \]  

(53)

Using (50) or (51), it is easy to show that the initial general Gaussian wave packet for both the free mass and the oscillator takes the form

\[ \Psi_{ar}(x, 0) = \left( \frac{1}{\sqrt{2\pi \hbar u_r^*(0)}} \right)^{1/2} e^{ip_0 x/\hbar} e^{-(1+i\delta)(x-x_0)^2/4(\Delta x_0)^2} e^{-iS_c(t)/\hbar}. \]  

(54)

For the minimum uncertainty, \( \Delta x_0 \Delta p_0 = \hbar/2 \) and so \( \delta = 0 \), we recover the wave packet (7) with the initial distribution (1) and (2). However, the minimum uncertainty does not always imply \( r = 0 \), since one may choose \( \vartheta = 0 \) and \( e^r = \sqrt{2/m \Delta x_0} \) for the free mass and \( e^r = \sqrt{2/\hbar m \Delta p_0} \) for the oscillator, or \( \vartheta = \pi \) and \( e^r = \sqrt{2/\hbar m \Delta p_0} \) for the free mass and \( e^r = \sqrt{2/\hbar m \Delta p_0} \) for the oscillator. The minimum energy will select \( r = 0 \) and \( \vartheta = 0 \) for both the free mass and oscillator. Using (20), we write the exponent of (39) as

\[ \frac{\hbar u_r^*}{2\hbar u_r^* u_r} = \frac{i + m(\dot{u}_r^* u_r + u_r^* \dot{u}_r)}{2\hbar u_r^* u_r} \]

and, finally, obtain the travelling wave packet in the form

\[ \Psi_{ar}(x,t) = \left( \frac{1}{\sqrt{2\pi \hbar u_r^*(t)}} \right)^{1/2} e^{-iS_c(t)/\hbar} e^{ip_0 x/\hbar} e^{-(1-2i\Delta(xp)/\hbar)(x-x_0(t))^2/4(\Delta x_0)^2}. \]  

(56)

From (50), (51) and (52), the variances at any time can be found for the free mass

\[ (\Delta x_t)^2 = (\Delta x_0)^2 + \hbar \delta t/m + (\Delta p_0)^2 t^2/m^2, \]  

(57)

and for the oscillator

\[ (\Delta x_t)^2 = (\Delta x_0)^2 \cos^2(\omega t) + \frac{\hbar \delta}{2m \omega} \sin(2\omega t) + \frac{(\Delta p_0)^2}{(\hbar m \omega)^2} \sin^2(\omega t). \]  

(58)

The linearly time-dependent term in (57) or (58) was interpreted due to the probability current of the initial wave packet [10, 33]. Later Yuen solved the Heisenberg equation for the position variance using an initial coherent-squeezed state to get such a linearly time-dependent term, in particular, with the negative sign [20]. Here we have found the travelling general wave packet from the initial one and thereby have obtained the variances. It is not hard to show that (52) comes from the correlation between \( \dot{x} \) and \( \dot{p} \) at the initial time.
There are two interesting cases of minimal- and maximal-uncertainty states. First, the variance of the position in the minimal-uncertainty \((\delta = 0)\) states monotonically increases for the free mass

\[
(\Delta x_t)^2 = (\Delta x_0)^2 + (\Delta p_0)^2 \frac{t^2}{m^2},
\]

but oscillates for the oscillator

\[
(\Delta x_t)^2 = (\Delta x_0)^2 \cos^2(\omega t) + \frac{(\Delta p_0)^2}{(m\omega)^2} \sin^2(\omega t).
\]

The spreading of the wave packet for the free particle is inevitable for minimal-uncertainty states. For nonminimal-uncertainty \((\delta > 0)\) state, however, there is one interesting case for the negative sign of (57). Though the late time evolution of variance is dominated by the last term, there is an interim when the variance first decreases until

\[
\tau = \frac{\hbar m \delta}{2 \Delta p_0^2};
\]

and then increases to the initial value \(\Delta x_0\) at \(t = 2\tau\). The minimum value of the variance is

\[
\Delta x^2 \approx \frac{3 \Delta x_0^2}{4} + \frac{\hbar^2}{4 \Delta p_0^2}.
\]

Second, the maximal-uncertainty \((\delta \gg 1)\) states, have approximately the position variance for the free mass

\[
\Delta x_t \approx |\Delta x_0 \pm \frac{\hbar \delta}{2 \Delta x_0 m} t|,
\]

and for the oscillator

\[
\Delta x_t \approx |\Delta x_0 \cos(\omega t) \pm \frac{\hbar \delta}{2 m \omega \Delta x_0} \sin(\omega t)|.
\]

For maximal-uncertainty states one has \(\Delta x_\tau \approx \sqrt{3} \Delta x_0/2\). It is thus possible, in principle, to measure the position the second time without much disturbance from the first measurement [20].

**VI. CONCLUSION**

Gaussian wave packets have many applications, for instance, in quantum optics or atomic physics or quantum measurement theory. In particular, the quantum motion of the measuring apparatus of a free mass or an oscillator will be monitored to detect the gravitational
waves. The general Gaussian wave packet can have four parameters to be fixed: the expectation values, $x_0$ and $p_0$, and the variances, $\Delta x_0$ and $\Delta p_0$ of the position and momentum. The only constraint on the variances is the uncertainty principle, $\Delta x_0 \Delta p_0 \geq \hbar/2$.

In this paper we have found a coherent-squeezed state representation of the travelling general Gaussian wave packet. The squeeze parameter and angle for a squeezed state are determined by the initial variances of the position and momentum, which are given by (50) for the free mass and by (51) for the oscillator. Further, the complex parameter for the coherent state is related with the centroid of the Gaussian wave packet and is determined by (35) for the free mass and by (36) for the oscillator. Thus we have been able to obtain the travelling general Gaussian wave packet (56) whose centroid moves along the classical trajectory with $x_c(t)$ and $p_c(t)$. The minimal-uncertainty states keep the well-known form whereas all the other nonminimal-uncertainty states have an additional position-squared phase proportional to a dimensionless measure of deviation from the minimum uncertainty or the initial position-momentum correlation. The effect of this phase is to correlate the initial position and momentum and modify the time evolution of the variances as given by (57) for the free mass and (58) for the oscillator.

An important consequence of the travelling general Gaussian wave packet (56) is that all non-minimal uncertainty states will have an additional contribution to the variances of the position, the linearly time-dependent term in (57) for the free mass and in (58) for the oscillator. There exists a specific initial Gaussian profile (56) with the negative sign for the free mass that will contribute negatively to the position variance and result in a minimum variance at a certain moment (61). Yuen proposed such a state, the so-called contractive state, and used it to test the standard limit for monitoring the free mass system [20]. However, the state was defined only at the initial time and the evolution of variances was found in the Heisenberg picture. In our coherent-squeezed state representation, the wave packet is the travelling general Gaussian wave packet from an initial general Gaussian one. It would be interesting to investigate how to prepare such a state with minimal variance of the position and how the system evolves during measurements. The travelling general wave packets are also expected to have useful applications in studying quantum revivals and collapses.
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