M-theory on $S^1/Z_2$: new facts from a careful analysis

ADEL BILAL, JEAN-PIERRE DERENDINGER AND ROGER SAUSER

Institute of Physics, University of Neuchâtel, CH-2000 Neuchâtel, Switzerland
adel.bilal, jean-pierre.derendinger, roger.sauser @iph.unine.ch

ABSTRACT

We carefully re-examine the issues of solving the modified Bianchi identity, anomaly cancellations and flux quantization in the $S^1/Z_2$ orbifold of M-theory using the boundary-free “upstairs” formalism, avoiding several misconceptions present in earlier literature. While the solution for the four-form $G$ to the modified Bianchi identity appears to depend on an arbitrary parameter $b$, we show that requiring $G$ to be globally well-defined, i.e. invariant under small and large gauge and local Lorentz transformations, fixes $b = 1$. This value also is necessary for a consistent reduction to the heterotic string in the small-radius limit. Insisting on properly defining all fields on the circle, we find that there is a previously unnoticed additional contribution to the anomaly inflow from the eleven-dimensional topological term. Anomaly cancellation then requires a quadratic relation between $b$ and the combination $\lambda^6/\kappa^4$ of the gauge and gravitational coupling constants $\lambda$ and $\kappa$. This contrasts with previous beliefs that anomaly cancellation would give a cubic equation for $b$. We observe that our solution for $G$ automatically satisfies integer or half-integer flux quantization for the appropriate cycles. We explicitly write out the anomaly cancelling terms of the heterotic string as inherited from the M-theory approach. They differ from the usual ones by the addition of a well-defined local counterterm. We also show how five-branes enter our analysis.

* Work partially supported by the Swiss National Science Foundation.
1. Introduction and summary

Ever since the emergence of M-theory and its relation to the various perturbative string theories, considerations of anomaly cancellation have played a crucial role. In particular, in reconstructing the strongly-coupled $E_8 \times E_8$ heterotic string from M-theory compactified on $S^1/Z_2$ [1, 2], an important ingredient was the observation that the gauge and gravitational anomaly polynomial of the $E_8 \times E_8$ heterotic string can be written as a sum of two terms, each being associated with one $E_8$ factor, and that each of these terms separately factorises as required by anomaly cancellation through a Green-Schwarz mechanism. This enables local anomaly cancellation on each of the two ten-dimensional $S^1/Z_2$ fixed planes in the M-theory picture [1, 2, 3]. The anomaly-cancelling terms in M-theory are of two types: one is a Green-Schwarz term $\int G \wedge X_7$ where $G$ is the (modified) field strength of the three-form $C$ and $X_7$ is a purely gravitational Chern-Simons type seven-form. The presence of this Green-Schwarz term was well-known from string duality [4, 5] and as a cancelling term for the M-theory five-brane anomaly [5, 6]. The second contribution is anomaly inflow from the topological interaction term of eleven-dimensional supergravity $\int C \wedge G \wedge G$. In uncompactified eleven-dimensional M-theory both terms are gauge and local Lorentz invariant and no anomaly needs to be cancelled in this odd dimension. However, orbifold compactifications like $S^1/Z_2$ involve chiral projections on the even-dimensional fixed planes and then there are chiral anomalies which need to be cancelled. On the other hand, it has been shown in [2] that closure of the supersymmetry algebra for the $S^1/Z_2$ orbifold implies a modification of the Bianchi identity $dG = 0$, and this is why the $\int G \wedge X_7$ and $\int C \wedge G \wedge G$ terms have non-vanishing anomalous transformations under gauge and local $SO(9,1)$ Lorentz transformations on the fixed planes. This is the basic mechanism at the origin of local anomaly cancellation.

In ref. [2], the modified Bianchi identity $dG \neq 0$ was solved in a particular way and it was concluded that anomaly cancellation requires a certain fixed ratio $\lambda^3/\kappa^2$ of the gauge coupling $\lambda$ and the gravitational coupling $\kappa$. Subsequent analyses [8–13] have in particular emphasized that there actually is a one-parameter family of solutions $G(b)$, $b \in \mathbb{R}$, to the modified Bianchi identity.† It was concluded that anomaly cancellation alone does not fix the ratio $\lambda^3/\kappa^2$ but

† Of course, to any solution $G^{(0)} = dC + \ldots$ of $dG = \ldots$ one can add a $dA^{(3)}$ with any three-form $A^{(3)}$. The point is that most of these $A^{(3)}$ can be reabsorbed into $C$. So the only relevant $A^{(3)}$ must be made from the gauge and Lorentz Chern-Simons three-form $\omega^{(3)}$ on the fixed planes so that essentially the only freedom is $A^{(3)} \sim b\omega^{(3)}$ with one real parameter $b$. 
relates it to a cubic polynomial in this parameter $b$. It was argued that this parameter and hence $\lambda^3/\kappa^2$ can be fixed if one also takes into account the quantization of the flux of $G$ [14].

In this paper, we carefully reanalyse the issues of solving the modified Bianchi identity, anomaly cancellation and flux quantization for the $S^1/Z_2$ orbifold. We use the “upstairs” formalism where one works on the boundary-free circle $S^1$ and imposes a $Z_2$ projection on the fields. However, contrary to most of the previous papers, we insist on defining properly all fields on $S^1$, i.e. fields should be periodic. This immediately rules out the use of the step function $\theta(x^{11})$ as a primitive of a delta function $\delta(x^{11})$ on the circle. Using instead a correctly periodically defined function $\epsilon(x^{11})$ such that $\epsilon' \sim \delta - \frac{1}{2\pi}$ turns out to be crucial to obtain a consistent solution $G$ to the modified Bianchi identity. We carefully investigate gauge and local Lorentz invariance of $G$. It turns out that insisting on invariance under large transformations, i.e. insisting on having a globally well-defined $G$, is very powerful. It not only implies the well-known cohomology condition [15], but it also fixes the parameter $b$ of the solution $G$ to be 1, as long as there are topologically non-trivial gauge or gravity configurations with $\int (\mathrm{tr} F_i^2 - \frac{1}{2} \mathrm{tr} R^2) \neq 0$. The same conclusion $b = 1$ is obtained if five-branes are present. The value $b = 1$ also appears to be the only one which allows a safe truncation to the perturbative heterotic string: although the zero-modes on the circle correctly give the desired relation $H = dB - 4\omega_{YM} + 2\omega_L$ for all values of $b$, the neglected higher modes are gauge and local Lorentz invariant only if $b = 1$. Note that these arguments leading to $b = 1$ are not a consequence of $G$-flux quantization. We note that the downstairs approach automatically incorporates a correspondingly fixed value of $b$.

Looking carefully at anomaly cancellations also yields some surprises. We do the analysis for arbitrary parameter $b$. Again the use of the periodic $\epsilon(x^{11})$ is crucial, in particular the constant term in $\epsilon' \sim \delta - \frac{1}{2\pi}$ plays an important role. Due to this term, the relation between $b$ and the ratio $\lambda^3/\kappa^2$ required by anomaly cancellation is drastically modified with respect to previous analyses [2, 8–13]: the terms cubic in $b$ cancel, and one is left with a quadratic equation for $b$ namely $b^2 = \frac{12}{(4\pi)^2} \lambda^6/\kappa^4$. Also, to obtain this equation, one has to evaluate products of the form $\delta(x^{11})\epsilon(x^{11})\epsilon(x^{11})$ where $\epsilon(x^{11})$ is our periodic generalisation of the step function. But of course, this product is ill-defined and should be regularised. Any reasonable regularisation yields $\delta(x^{11})\epsilon(x^{11})\epsilon(x^{11}) \to \frac{1}{3} \delta(x^{11})$ providing an extra crucial factor of $\frac{1}{3}$ in the above equation determining the parameter $b$ from anomaly cancellations in terms of $\lambda^3/\kappa^2$. This completely
modified, now quadratic equation will have important consequences when considering further orbifold compactification [13], e.g. on $T^4/Z_2$, where the analysis has encountered certain difficulties.‡ This will be examined elsewhere [16]. We also show how the discussion of anomalies is affected by the presence of five-branes.

We then conclude that $b$ must be one, and hence $\lambda^6 = \frac{(4\pi)^2}{12}\kappa^4$. Indeed, in any topologically non-trivial sector of the theory where $\int (\text{tr} F_i^2 - \frac{1}{2}\text{tr} R^2) \neq 0$ we have $b = 1$. Anomaly cancellation then fixes the ratio $\lambda^6/\kappa^4 = \frac{(4\pi)^2}{12}b^2 = \frac{(4\pi)^2}{12}$. But this ratio does not depend on whether the integral is zero or not, so that by this same equation $b = 1$ always (strictly speaking: $b^2 = 1$). It is worthwhile noticing that $G$ differs from $dC$ also away from the fixed planes since $b \neq 0$. We further note that the Green-Schwarz term $\int G \wedge X_7$ also automatically ensures cancellation of the five-brane anomalies without any need of further modification. As a consistency check of our solution we show how the combination of the topological term and the Green-Schwarz term in eleven dimensions leads to the Green-Schwarz term of the heterotic string.

Another subtle point is flux quantization of $G$. In the “downstairs” approach where one considers M-theory compactified on the interval $I = S^1/Z_2$, i.e. in the presence of ten-dimensional boundary planes of the eleven-dimensional space-time, there is no modification of the Bianchi identity $dG = 0$, but there is a non-trivial boundary condition on $G$ [2] enabling the necessary anomaly inflow to cancel the one-loop anomaly on the boundary planes. This boundary condition does not admit a free parameter like $b$ (i.e. it is equivalent to a fixed value of $b$) and reads $G|_{\text{boundary}} = \frac{\kappa^2}{4\pi}(\text{tr} F_i^2 - \frac{1}{2}\text{tr} R^2)|_{\text{boundary}}$. Witten concluded [14] that for any four-cycle $C_4$ the flux $\frac{1}{(4\pi)^2\kappa^2} \int_{C_4} G$ has to be integer or half-integer. In particular, this ensures that the membrane functional integral is well-defined. Naively the latter would seem to be well-defined only for integer flux, but it was shown in ref. [14] that in the case of half-integer flux the three-dimensional membrane functional integral has a parity anomaly [17] which precisely cancels the sign ambiguity due to the half-integrality of the $G$-flux. In the boundary-free upstairs approach where the Bianchi identity for $G$ is modified this same flux quantization should appear as a consequence of our solution. We will show that this is indeed the case. Any four-cycle $C_4$ can be written as a sum of four-cycles not involving the $x^{11}$-direction and of four-cycles of the form $S^1 \times C_3$ with $C_3$ not in the $x^{11}$-direction. In the first case our solution

‡ We are grateful to D. Lüst for sharing his insights on this point.
$G$ to the modified Bianchi identity straightforwardly yields Witten’s result of integer or half-
integer flux (provided one correctly relates $\kappa^2_{\text{upstairs}}$ and $\kappa^2_{\text{downstairs}}$ with the necessary factor of $1/2$ [9]). Four-cycles of the second type wrap around the circle $S^1$. There is no analogue of these cycles in the downstairs approach, and we cannot conclude on any quantization from the results of [14]. Indeed, for such cycles, we find that the $b$ dependence trivially cancels (as always when integrating over the circle, thanks to the modification of the step function in order to be periodic!) and one is left with an integral $A \equiv \int_{S^1 \times \mathcal{C}_3} dC - \frac{1}{(4\pi)^2 \kappa^2} \sum_i \int_{\mathcal{C}_3} \tilde{\omega}_i$ where the sum is over the two fixed planes and $\tilde{\omega}_i$ is a combination of gauge and Lorentz Chern-Simons three-
forms on the $i$th plane. This integral is closely related to the flux $\int H = \int (dB - \omega_{YM} + \omega_L)$ in the heterotic string. The latter was discussed some time ago [18] and it was concluded (when appropriately normalised) to be of the form $n + \delta$ with $n \in \mathbb{Z}$, $\delta \in \mathbb{R}$. The same argument holds here and we conclude that the gauge and local Lorentz invariant combination $A$ cannot be determined further. The reason of course is that since $G \neq dC$ one cannot use the standard argument to obtain flux quantization. We conclude that for four-cycles not wrapping the circle $S^1$ we have standard integer or half-integer flux quantization, while if the four-cycle wraps the $S^1$ we cannot say anything interesting about the flux of $G$.

This paper is organised as follows. In Section 2, we discuss preliminaries: conventions and normalisations, and the correctly modified, periodically-defined step function $\epsilon(x^{11})$ on the circle, as well as the above-mentioned occurrence of an extra factor of $1/3$ when evaluating integrals involving $\delta \epsilon \epsilon$. In Section 3, we carefully solve the modified Bianchi identity for $G$ and discuss gauge and local Lorentz invariance and how the global definition of $G$ fixes $b$. We also comment on the relation to the perturbative heterotic string. Section 4 discusses anomaly cancellations and the quadratic relation between $b$ and $\lambda^6 / \kappa^4$ is obtained. Section 5 is devoted to flux (non) quantization of $G$ and in Section 6 we spell out the heterotic anomaly-cancelling terms as inherited from the present M-theory approach. We conclude in Section 7 and an appendix summarises the relevant anomaly polynomials [19, 20] needed in this paper.
2. Preliminaries

2.1. Conventions and normalisations

Since in the next section we want to pin down various numerical coefficients, normalisations are important. Our conventions are as follows.

Normalisations of fields and couplings

The anomaly polynomials $I_{12}(1-\text{loop})_i$, $X_8$ and $I_{4,i}$ are defined in the appendix. In particular we have

$$I_{4,i} = \frac{1}{(4\pi)^2} \left( \text{tr} F_i^2 - \frac{1}{2} \text{tr} R^2 \right) \Rightarrow \int_{C_4} I_{4,i} = m_i - \frac{1}{2} p_i, \quad m_i, p_i \in \mathbb{Z}, \quad (2.1)$$

for any four-cycle $C_4$. For differential forms, we use the standard normalisation: a $p$-form $A^{(p)}$ is given in terms of its completely antisymmetric components as

$$A^{(p)} = \frac{1}{p!} A_{[\mu_1 \ldots \mu_p]} \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \frac{1}{p!} A_{[\mu_1 \ldots \mu_p]} \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \quad (2.2)$$

and $d = dx^\mu \partial_\mu$ so that in particular $F = dA + A^2 = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$ and for the three-form field of eleven-dimensional supergravity $C = \frac{1}{3!} C_{\mu\nu\rho} \, dx^\mu \wedge dx^\nu \wedge dx^\rho$.

The normalisation of the eleven-dimensional supergravity action [7] is

$$S_{\text{sugra}} = \int d^{11} x \, e \left[ -\frac{1}{2\kappa^2} \mathcal{R} - \frac{1}{96\kappa^2} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} \right] + S_{\text{top}} + \ldots \quad (2.3)$$

where $\mathcal{R}$ is the curvature scalar and $e$ the square root of the determinant of the eleven-dimensional metric. $\kappa \equiv \kappa_{11}$ is the eleven-dimensional gravitational coupling. The so-called topological term is of central importance for us:

$$S_{\text{top}} = -\frac{1}{12\kappa^2} \int C \wedge G \wedge G, \quad (2.4)$$

where $G = dC + \ldots$ Omitted terms indicated by $\ldots$ involve the gravitino field.

\footnote{Our normalisation is related to the one used by Ho\v{r}ava and Witten [2] by $C_{\mu\nu\rho}^{\text{HW}} = \frac{1}{\sqrt{2}} C_{\mu\nu\rho}$ and $G_{\mu\nu\rho\sigma}^{\text{HW}} = \frac{1}{\sqrt{2}} G_{\mu\nu\rho\sigma}$.}
Similarly, the action for the ten-dimensional gauge fields is

\[ S_{\text{gauge}} = -\frac{1}{4\lambda^2} \int d^{10}x \sqrt{g_{10}} F_{AB} F^{AB} + \ldots , \]  

(2.5)

with gauge coupling constant \( \lambda \) and \( F = dA + A^2 \). On the \( S^1/\mathbb{Z}_2 \) fixed planes, \( g_{10} \) is the restriction of the eleven-dimensional metric to the planes.

As usual, we use \( \mu, \nu, \ldots \) for eleven-dimensional indices and \( A, B, \ldots \) for ten-dimensional ones.

**Engineering dimensions**

It is often useful to have in mind the dimensions of the various quantities and couplings. Taking \([\text{mass}] = 1\) as usual, one has \([\kappa] = -\frac{9}{2}\) and \([\lambda] = -3\). With these two parameters, a dimensionless combination is \( \frac{\lambda^3}{\kappa^2} \) while \([\frac{\kappa^2}{\lambda^2}] = -3\) and this latter combination may be used whenever a dimensionful constant is needed. Gauge and Lorentz connections have dimension one and then \([R] = [F] = 2\) so that \([J_{4,i}] = 4\) and \([X_8] = 8\). We also take \([C] = 0\) and \([G] = 1\) (for forms, by \([C]\) we really mean \([C_{\mu\nu\rho}]\), etc..).

**2.2. The orbifold \( S^1/\mathbb{Z}_2 \)**

We consider the eleven-dimensional manifold as the product of a ten-dimensional manifold \( M_{10} \) and a circle \( S^1 \). For notational convenience, and to emphasize that the coordinate on the circle is an angular variable (we are working in the upstairs approach), we use

\[ \phi \in [-\pi, \pi] \simeq S^1 , \]  

(2.6)
rather than \( x^{11} \). This definition also means that we work in general on a circle with unit radius. We will however reintroduce an arbitrary radius \( r \) when discussing the small-radius limit, i.e. the perturbative heterotic limit. This will be done by assuming that \( \phi \in [-\pi r, \pi r] \). The \( \mathbb{Z}_2 \) symmetry acts by \( \phi \to -\phi \) so that the two fixed planes are copies of \( M_{10} \) at \( \phi = 0 \equiv \phi_1 \) and \( \phi = \pi \equiv \phi_2 \). We then define the one-forms

\[ \delta_1 = \delta(\phi) \, d\phi , \quad \delta_2 = \delta(\phi - \pi) \, d\phi . \]  

(2.7)

These one-forms are well-defined on the circle \( S^1 \) by definition, i.e. \( \delta_2 = \delta(\phi - \pi) \, d\phi \equiv \delta(\phi + \pi) \, d\phi \). We further need zero-forms \( \epsilon_i \) such that \( d\epsilon_i \) includes \( \delta_i \). Determining a primitive
of a δ-function on a circle requires however a minimum of care: a simple step function \( \theta(\phi) \) will not do the job since it is not periodic. This simple fact apparently has been overlooked in previous discussions of anomaly inflow through the modified Bianchi identity. In fact there cannot be a function \( \theta(\phi) \) on the circle, i.e. a periodic function \( \theta(\phi) = \theta(\phi + 2\pi) \), such that \( \theta'(\phi) = \delta(\phi) \). Would it exist, one would also have \( \int_{-\pi}^{\pi} \theta' \delta \) = \( \int_{-\pi}^{\pi} \theta \delta \) = 0. The best one can do is to define for \( \phi \in [-\pi, \pi] \)

\[
\epsilon_1(\phi) = \text{sign}(\phi) - \frac{\phi}{\pi}. \tag{2.8}
\]

It is periodic, odd under \( \mathbb{Z}_2 \), and satisfies \( \epsilon_1(\pi) = \epsilon_1(-\pi) = 0 \). Similarly one defines \( \epsilon_2(\phi) = \epsilon_1(\phi \pm \pi) \). These functions satisfy \( \epsilon'_1(\phi) = 2\delta(\phi) - \frac{1}{\pi} \) and \( \epsilon'_2(\phi) = 2\delta(\phi - \pi) - \frac{1}{\pi} \). In form notation, these definitions can be concisely written as

\[
d\epsilon_i = 2\delta_i - \frac{1}{\pi}d\phi, \quad i = 1, 2. \tag{2.9}
\]

Later on we will need integrals of products of these \( \epsilon_i \) over the circle. They are elementary and can be concisely written as

\[
\int_{S^1} d\phi \epsilon_i \epsilon_j = \pi \left( \delta_{ij} - \frac{1}{3} \right), \quad i, j = 1, 2, \tag{2.10}
\]

where of course here and in the following \( \delta_{ij} \) stands for the Kronecker symbol. Since the \( \epsilon_i(\phi) \) are odd functions with respect to the \( \mathbb{Z}_2 \), one also has

\[
\int_{S^1} d\phi \epsilon_i = 0. \tag{2.11}
\]

Another delicate point concerns integrals of products of one δ function and two \( \epsilon \) functions, which are \( \mathbb{Z}_2 \)-even distributions. It is often argued that one can replace e.g. \( \delta_1 \epsilon_1 \epsilon_1 \) by \( \delta_1 \) since

---

† The issue of periodicity is correctly addressed in ref [21]. This paper studies background configurations for the four-form \( G \) and the same terms linear in \( \phi \) as in our \( \epsilon(\phi) \) below do appear. This paper does not discuss anomaly inflow.

‡ For the integral of a periodic function one can of course integrate over any interval of length \( 2\pi \). However, since the orbifold singularities are at 0 and \( \pi \equiv -\pi \), in order to avoid misinterpretations and have each singularity exactly once, it is preferable not to integrate from \( -\pi \) to \( \pi \) but to slightly shift the interval of integration. Thus, for example, \( \int_{-\pi}^{\pi} d\phi \) is meant to be \( \lim_{\eta \to 0} \int_{-\pi-\eta}^{\pi-\eta} d\phi \).
due to the $\delta$ only the immediate vicinity of $\phi = 0$ is important, and then $(\epsilon_1(0))^2 \simeq +1$. This argument is certainly incorrect. An expression like $\delta \epsilon_1 \epsilon_1$ is ill-defined and has to be regularised. One could as well say that $\epsilon_1(0) = 0$, implying $\delta \epsilon_1 \epsilon_1 = 0$ and this would be equally wrong. Any sensible regularisation should preserve the one relation we want to hold, namely (2.9), as well as $\mathbb{Z}_2$-oddness of the $\epsilon_i$. Then indeed in the vicinity of $\phi = 0$ the linear piece $-\phi/\pi$ in $\epsilon_1$ is unimportant and $\epsilon_1' = 2\delta(\phi)$ so that

$$\delta(\phi)\epsilon_1(\phi) = \frac{1}{2} \epsilon_1'(\phi) \epsilon_1(\phi) = \frac{1}{6} \frac{d}{d\phi} (\epsilon_1(\phi))^3 \simeq \frac{1}{3} \frac{d}{d\phi} \epsilon_1(\phi) \simeq \frac{1}{3} \delta(\phi)$$

(2.12)

and an important factor $\frac{1}{3}$ has appeared*. This can be rigorously verified using a specific regularisation†. One finally obtains:

$$\delta_i \epsilon_j \epsilon_k \rightarrow \frac{1}{3} (\delta_{ji} \delta_{ki}) \delta_i,$$  

(2.13)

where again $\delta_{ji}$ and $\delta_{ki}$ denote Kronecker symbols while $\delta_i$ is the Dirac $\delta$ one-form defined in (2.7). We need also the following relation

$$\delta_i \epsilon_j \rightarrow 0.$$  

(2.14)

### 2.3. The modified Bianchi identity for $G$

Hořava and Witten [2] consider the eleven-dimensional supergravity action (2.3) on the product of $M_{10}$ with the orbifold $S^1/\mathbb{Z}_2$, coupled to $E_8$ super-Yang-Mills actions located on the two $M_{10}$ fixed planes, at $\phi = 0$ and $\phi = \pi$. The orbifold projection eliminates fields odd under $\mathbb{Z}_2$. In particular, invariance of $S_{\text{top}}$ implies that $C_{ABC}$ is odd and projected out, while $C_{AB,11}$ is even and kept. The $\mathbb{Z}_2$ projection also breaks one half of the thirty-two supersymmetries. Preserving

---

* This factor of $\frac{1}{3}$ was noticed in ref. [9].
† One may e.g. take the regularized $\epsilon_1''(\phi)$ to be the continuous function which coincides with $\epsilon_1(\phi)$ for $\phi \notin [-\eta, \eta]$ and equals $\left(\frac{1}{2} - \frac{1}{\pi}\right) \phi$ for $\phi \in [-\eta, \eta]$. The corresponding $\delta_1''(\phi)$ is defined according to Eq. (2.9) and vanishes everywhere except in $[-\eta, \eta]$ where it equals $\frac{1}{\pi \eta}$. Then for any test function $f(\phi)$ one has $\int \delta''_1 \epsilon''_1 \epsilon''_1 f \sim \frac{1}{\eta} (1 - \frac{1}{\pi} )^2 f(0) \rightarrow \frac{1}{3} f(0), \int \delta''_1 \epsilon''_1 \epsilon''_1 f \sim \frac{1}{\eta} (1 - \frac{1}{\pi} ) f(0) \rightarrow 0, \int \delta''_1 \epsilon''_2 \epsilon''_2 f \sim \frac{1}{3} (\frac{2}{\pi})^2 f(0) \rightarrow 0,$
and similarly with 1 and 2 exchanged. Also, $\delta''_1 \epsilon''_1$ is an odd function of $\phi \in [-\pi, \pi]$ and $\int \delta''_1 \epsilon''_1 f = 0$ as well as $\int \delta''_1 \epsilon''_2 f = 0$, and similarly with 1 and 2 exchanged.
the sixteen remaining supersymmetries requires however a modification of the Bianchi identity of the four-form field $G$, involving the Yang-Mills curvatures. As usual in string effective actions, anomaly cancellation in turn requires the appearance in this modification of Lorentz curvatures: this contribution cannot be derived from supersymmetry of effective actions with up to two derivatives.

Instead of $dG = 0$, the modified Bianchi identity is postulated to be [2]

$$dG = -\frac{\kappa^2}{\lambda^2} \sum_i \delta_i \left( \text{tr} F_i^2 - \frac{1}{2} \text{tr} R^2 \right).$$

(2.15)

As already mentioned, the classical supersymmetry calculation only yields the $\text{tr} F^2$ term. The $\text{tr} R^2$ term is a higher-order effect. The factor $\kappa^2/\lambda^2$ can be inferred by simple dimensional analysis, since $[\delta_i] = 1$.

3. Solution of the modified Bianchi identity

3.1. A one-parameter family of solutions

We now proceed to solve the modified Bianchi identity (2.15). The combination of $\text{tr} F_i^2$ and $\text{tr} R^2$ that appears is usually called $I_{4,i}$. Our normalisations for all relevant anomaly-related polynomials in $F$ and $R$ are summarised in Appendix A. In particular, we have $I_{4,i} = \frac{1}{(4\pi)^2} \left( \text{tr} F_i^2 - \frac{1}{2} \text{tr} R^2 \right)$ and thus the Bianchi identity reads

$$dG = -(4\pi)^2 \frac{\kappa^2}{\lambda^2} \sum_i \delta_i \wedge I_{4,i}. \quad (3.1)$$

Since $\delta_i$ has support only on the $i^{th}$ fixed plane and is a one-form $\sim d\phi$, only the values of the smooth four-form $I_{4,i}$ on this fixed plane matter and only the components not including $d\phi$ are relevant. For the gauge piece $\text{tr} F_i^2$ this is automatic, but for the $\text{tr} R^2$ piece this is a non-trivial statement. It will prove convenient to use two-dimensional descent equations $I_4 = d\omega_3$, $\delta\omega_3 = d\omega_1$ with all forms having no $d\phi$ component. We will use the following convention: a tilde on a $p$-form $A_i^{(p)}$ means that $\widetilde{A_i}^{(p)}$ is obtained from $A_i^{(p)}$ by dropping all components $\sim d\phi$ and by setting the argument $\phi = \phi_i$, i.e. equal to 0 or $\pi$ depending on whether $i = 1$ or 2.
Clearly, \( d \) takes a tilde \( p \)-form to a tilde \((p + 1)\)-form: \( d\tilde{A}_i^{(p)} = \tilde{d}A_i^{(p)} \). In the Bianchi identity we can then replace \( I_{4,i} \) by \( \tilde{I}_{4,i} \) which has the effect of replacing \( R \rightarrow \tilde{R}_{i} \equiv \tilde{R}_{i|\phi=\phi} \). One has \( \tilde{R}_{i} = d\tilde{\Omega}_{i} + \bar{\Omega}_{i} \wedge \tilde{\Omega}_{i} \) with \( \tilde{\Omega}_{i} \) the correspondingly mutilated spin-connection. Using this \( \tilde{\Omega}_{i} \), one defines the Chern-Simons three-forms (except for \( I_{4} \), for convenience of notation, we will no longer write the degree of the forms explicitly: the subscript \( i = 1, 2 \) refers to the fixed plane, not the degree of the form)

\[
\tilde{\omega}_{i} = \frac{1}{(4\pi)^2} \left( \text{tr} \left( A_i dA_i + \frac{2}{3} A_i^3 \right) - \frac{1}{2} \text{tr} \left( \tilde{\Omega}_i d\tilde{\Omega}_i + \frac{2}{3} \tilde{\Omega}_i^3 \right) \right) \tag{3.2}
\]

which satisfies \( d\tilde{\omega}_{i} = \tilde{I}_{4,i} \). Also under a gauge and local Lorentz transformation with parameters \( \Lambda^k \) and \( \Lambda^L \) independent of \( \phi \) one has \( \delta \tilde{\omega}_{i} = d\tilde{\omega}_{i}^1 \) where the two-forms \( \tilde{\omega}_{i}^1 \) are

\[
\tilde{\omega}_{i}^1 = \frac{1}{(4\pi)^2} \left( \text{tr} \left( \Lambda^k dA_i - \frac{1}{2} \text{tr} \left( \Lambda^L d\tilde{\Omega}_i \right) \right) \right) . \tag{3.3}
\]

Then (3.1) is rewritten as

\[
dG = \gamma \sum_i \delta_i \wedge \tilde{I}_{4,i} , \quad \tilde{I}_{4,i} = d\tilde{\omega}_i , \quad \delta \tilde{\omega}_i = d\tilde{\omega}_i^1 , \tag{3.4}
\]

where to simplify notations we have introduced the dimensionful quantity

\[
\gamma = -(4\pi)^2 \frac{\kappa^2}{\lambda^2} . \tag{3.5}
\]

This is solved by

\[
G = d \left( C + \frac{b}{2} \gamma \sum_i \epsilon_i \tilde{\omega}_i \right) - \gamma \sum_i \delta_i \wedge \tilde{\omega}_i \\
= dC + (b - 1)\gamma \sum_i \delta_i \wedge \tilde{\omega}_i + \frac{b}{2} \gamma \sum_i \epsilon_i \tilde{I}_{4,i} - \frac{b}{2\pi} \gamma d\phi \wedge \sum_i \tilde{\omega}_i , \tag{3.6}
\]

where we used (2.9). In order to maintain full generality we should allow for a different parameter \( b \) for each fixed plane, i.e. \( G = d \left( C + \frac{1}{2} \gamma \sum_i b_i \epsilon_i \tilde{\omega}_i \right) - \gamma \sum_i \delta_i \wedge \tilde{\omega}_i \). But it is clear

* Note that \( \gamma \) can be related to the membrane tension \( T_2 \) by \( \gamma = -\frac{2\pi}{T_2} \). We will not use this fact here. We could of course write \( T_2 \) instead of \( \gamma \) in our formulae, but we prefer not to in order to emphasize that the discussions and anomaly cancellations of this paper have nothing to do with membranes (except for a short comment in Sect. 5).
that the conditions of anomaly cancellation are the same for both planes and in the end \( b_1 \) and \( b_2 \) are determined by the same equation so that \( b_1 = b_2 \equiv b \). Note the presence of the last term in the second expression for \( G \). This term is a direct consequence of enforcing periodicity of the \( \epsilon_i(\phi) \) and will turn out to play a most crucial role.

**Adding five-branes**

We will also consider situations where there are five-branes. Each five-brane has a world-volume \( W_{6,a} \) where \( a \) labels the different five-branes. Since five-branes wrapping the circle \( S^1 \) can easily be seen to play no particular role in the discussions of section 3.3 and 4.4 below, we will concentrate on five-branes that are perpendicular to \( S^1 \), and we denote by \( \phi_a \) the coordinate at which they intersect the circle. Let there be \( N_5 \) such five-branes. To each of them one associates a brane current \( \delta^{(5)}(W_{6,a}) \), analogous to the \( \delta_i \) of the ten-planes. This brane current then contains a piece \( \delta(\phi - \phi_a)d\phi \) with the remaining piece called \( \delta^{(4)}(W_{6,a}) \).

The Bianchi identity for \( G \) then gets an extra contribution [6]

\[
d G|_{5\text{-brane contribution}} = \gamma \sum_{a=1}^{N_5} \delta^{(5)}(W_{6,a}) = \gamma \sum_{a=1}^{N_5} \delta(\phi - \phi_a) d\phi \wedge \delta^{(4)}(W_{6,a}) .
\]  

\((3.7)\)

Accordingly, \( G \) gets an extra contribution. Just as before when integrating \( \delta_i T_{4,i} \), one now has a choice when integrating \( \delta(\phi - \phi_a)d\phi \wedge \delta^{(4)}(W_{6,a}) \), introducing more free parameters:

\[
G|_{5\text{-brane contribution}} = \gamma \sum_{a=1}^{N_5} \left\{ \frac{\beta}{2} \left[ \epsilon_1(\phi - \phi_a) \delta^{(4)}(W_{6,a}) - \frac{1}{\pi} d\phi \wedge (\theta \delta)^{(3)}(W_{6,a}) \right] 
- (1 - \beta) \delta(\phi - \phi_a) d\phi \wedge (\theta \delta)^{(3)}(W_{6,a}) \right\} .
\]  

\((3.8)\)

It is straightforward to verify, using eqs (2.9) and (2.7) that the exterior derivative of the r.h.s. of (3.8) indeed yields the r.h.s. of (3.7) for any choice of parameter \( \beta \) provided

\[
d(\theta \delta)^{(3)}(W_{6,a}) = \delta^{(4)}(W_{6,a}) .
\]  

\((3.9)\)

Again, there are many different choices for \( (\theta \delta)^{(3)}(W_{6,a}) \) but since \( \delta^{(4)}(W_{6,a}) \) does not involve the circle coordinate \( \phi \) which primitive \( (\theta \delta)^{(3)}(W_{6,a}) \) is chosen is irrelevant here.

\[\uparrow\] Actually, this equation will turn out to be quadratic in \( b_i \) and hence has another solution: \( b_1 = -b_2 \equiv \pm b \). However, we will see soon that \( b_1 = +b_2 = 1 \) is needed as soon as topologically non-trivial configurations exist.
3.2. Gauge and local Lorentz invariance

Now we want $G$ to be gauge and local Lorentz invariant. While $\tilde{I}_{4,i}$ is invariant, the Chern-Simons three-form $\tilde{\omega}_i$ is not and hence $C$ cannot be invariant either. Since the variation of $C$ will be crucial for anomaly cancellation, care must be taken. The most general $\delta C$ giving an invariant $G$ is†

$$\delta C = dB^1_2 - \gamma \sum_i \left( \frac{b}{2} \epsilon_i d\tilde{\omega}^1_i + \delta_i \wedge \tilde{\omega}^1_i \right)$$  \hspace{1cm} (3.10)

with some two-form $B^1_2$ linear in $\Lambda^g$ or $\Lambda^L$. To determine $B^1_2$ we must consider the $\mathbb{Z}_2$-orbifold projection on the field $C$. Recall that $C_{ABC}$ is odd and hence projected out, while $C_{AB,11}$ is even and kept. But $C_{ABC} = 0$ only makes sense if it is a gauge invariant statement, i.e. we must have $\delta C_{ABC} = 0$. This yields

$$(dB^1_2)_{ABC} = \gamma \frac{b}{2} \sum_i \epsilon_i (d\tilde{\omega}^1_i)_{ABC} = \gamma \frac{b}{2} \left( d \sum_i \epsilon_i \tilde{\omega}^1_i \right)_{ABC}$$

which is solved by $(B^1_2)_{AB} = \gamma \frac{b}{2} \sum_i \epsilon_i (\tilde{\omega}^1_i)_{AB}$. Hence we choose $B^1_2 = \gamma \frac{b}{2} \sum_i \epsilon_i \tilde{\omega}^1_i$ and

$$\delta C = \gamma \sum_i \left( \frac{b}{2} d\epsilon_i - \delta_i \right) \wedge \tilde{\omega}^1_i$$

$$= \gamma (b - 1) \sum_i \delta_i \wedge \tilde{\omega}^1_i - \gamma \frac{b}{2\pi} d\phi \wedge \sum_i \tilde{\omega}^1_i ,$$ \hspace{1cm} (3.11)

together with

$$C_{ABC} = 0 .$$  \hspace{1cm} (3.12)

Note that $C$ would be gauge invariant in the bulk (i.e. away from the fixed planes) were it not for the last term in (3.11), again due to enforcing periodicity of the “step functions” $\epsilon_i(\phi)$. Let us repeat that $\delta C$ is such that $\delta G = 0$.

† Note that the five-brane contribution to $G$ is field-independent and thus does not change $\delta C$. 

12
3.3. **Global definition of $G$: invariance under large transformations**

So far we have considered invariance under small transformations that can be continuously deformed to the identity. However, $G$ must also be invariant under large gauge and large local Lorentz transformations. This will be precisely the case if $G$ is globally well-defined. There is a simple criterion when this is true [15]. If $G$ is globally well-defined, i.e. $dG$ is exact, then for any (closed) five-cycle $C_5$ one has $\int_{C_5} dG = 0$. Take $C_5 = C_4 \times S^1$ where $C_4$ is an arbitrary four-cycle at fixed value of $\phi$ or homologous to such a cycle and rewrite $dG$ using the Bianchi identity (3.4) together with (3.7). One then gets

$$\sum_i \int_{C_4} \tilde{I}_{4,i} \equiv \frac{1}{(4\pi)^2} \int_{C_4} \left( \text{tr} F_1^2 - \frac{1}{2} \text{tr} \tilde{R}_1^2 \right) + \frac{1}{(4\pi)^2} \int_{C_4} \left( \text{tr} F_2^2 - \frac{1}{2} \text{tr} \tilde{R}_2^2 \right) = -N_5$$

where $N_5$ is the number of five-branes intersecting $S^1$ and $C_4$ at a point. The analogous cohomology condition is well-known from the heterotic string. So $G$ will be invariant also under large transformations precisely if this condition holds. Of course, since we do want $G$ to be invariant we assume this condition henceforth.

However, there is one more important piece of information we can obtain from requiring global definition of $G$ and using Stoke’s theorem. Consider a five-dimensional manifold $V$ which intersects exactly one of the fixed planes on a four-cycle. For example, we may take it of the form $V = I \times C$ where $C$ is a four-cycle and $I$ is the interval $[\phi_1, \phi_2]$ with $-\pi < \phi_1 < 0$ and $0 < \phi_2 < \pi$. Then $\partial V = C(\phi_2) - C(\phi_1)$ and Stoke’s theorem gives

$$\int_V dG = \int_{C(\phi_2)} G - \int_{C(\phi_1)} G . \quad (3.14)$$

Now, the integral on the l.h.s. is evaluated using the modified Bianchi identity. The contribution (3.4), due to the $\delta$-function, collapses to an integral of $\tilde{I}_{4,1}$ over the four-cycle $C$ on the fixed plane. The five-brane contribution (3.7) to $dG$ yields an extra term $\gamma N_5(I)$, where $N_5(I)$ is the number of five-branes that intersect the interval $I$ and the four-cycle $C$. On the other hand, the integrals on the r.h.s are evaluated using the solution (3.6) and (3.8) for $G$. Only the components $G_{ABCD}$ contribute, i.e. from (3.6) only the piece $\frac{b}{2} \gamma \sum_i \epsilon_i \tilde{I}_{4,i}$ and from (3.8) only the piece $\frac{b}{2} \gamma \sum_a \epsilon_1(\phi - \phi_a)\delta^{(4)}(W_{6,a})$ contribute. For the integral at $\phi = \phi_2$ one has
\( \epsilon_1(\phi_2) = 1 - \frac{\phi_2}{\pi} \) and \( \epsilon_2(\phi_2) = -\frac{\phi_2}{\pi} \), while for the integral at \( \phi = \phi_1 \) one has \( \epsilon_1(\phi_1) = -1 - \frac{\phi_1}{\pi} \) and \( \epsilon_2(\phi_1) = -\frac{\phi_1}{\pi} \). Of course \( \int_{C(\phi_2)} \tilde{I}_{4,i} = \int_{C(\phi_1)} \tilde{I}_{4,i} = \int_C \tilde{I}_{4,i} \) since \( \tilde{I}_{4,i} \) is independent of \( \phi \). Furthermore \( \int_{C(\phi_2)} \epsilon_1(\phi - \phi_a) \delta^{(4)}(W_{6,a}) - \int_{C(\phi_1)} \epsilon_1(\phi - \phi_a) \delta^{(4)}(W_{6,a}) \) equals \( \frac{\phi_1 - \phi_2}{\pi} \) if the five-brane does not intersect the interval \( I \) and equals \( 2 + \frac{\phi_1 - \phi_2}{\pi} \) if it does intersect \( I \). Using (3.13) and collecting all the pieces, eq. (3.14) becomes

\[
\gamma \int_C \tilde{I}_{4,1} + \gamma N_5(I) = \frac{b}{2\gamma} \left( 2 \int_C \tilde{I}_{4,1} - \frac{\phi_1 - \phi_2}{\pi} N_5 \right) + \frac{\beta}{2\gamma} \left( 2N_5(I) + \frac{\phi_1 - \phi_2}{\pi} N_5 \right).
\]  

(3.15)

But the exact positions of \( \phi_1 \) and \( \phi_2 \) are arbitrary, and upon slightly varying them such that \( N_5(I) \) remains unchanged one concludes that the terms linear in \( \phi_1 - \phi_2 \) and the terms independent of \( \phi_1 - \phi_2 \) must vanish separately. This yields two equations:

\[
(\beta - b)N_5 = 0,
\]

\[
(1 - \beta)N_5(I) + (1 - b) \int_C \tilde{I}_{4,1} = 0.
\]

(3.16)

Had we chosen the five-manifold \( \mathcal{V} \) to intersect the other fixed plane we would have obtained an analogous equation with \( \tilde{I}_{4,2} \).

Let first \( N_5 = 0 \) so that one simply gets \((1 - b) \int_C \tilde{I}_{4,1} = 0 \). There are two possibilities. If firstly \( \int_C \tilde{I}_{4,1} = 0 \), each of the two source terms in the modified Bianchi identity is cohomologically trivial, so that the \( \tilde{\omega}_i \) can be globally well-defined. In this case we see that \( b \) is unconstrained. Secondly, if \( \int_C \tilde{I}_{4,1} \neq 0 \) each of the two source terms individually is cohomologically non-trivial (although their sum is trivial by (3.13)), and one must take \( b = 1 \). Note that \( b = 1 \) eliminates the terms in \( G \) containing \( \delta \)-functions so that \( G \) becomes finite everywhere on \( S^1 \).

Now let \( N_5 \neq 0 \). Then the first eq. (3.16) gives \( \beta = b \) and the second eq. gives \((\int_C \tilde{I}_{4,1} + N_5(I))(1 - b) = 0 \). Upon varying the interval \( I \) we may change \( N_5(I) \) (unless all five-branes are stuck on the fixed planes) and conclude that \( b = 1 \). Thus global definition of \( G \) fixes \( b = \beta = 1 \).
The resulting four-form $G$ then reads:

$$
G|_{b=1, \beta=1} = dC + \frac{\gamma}{2} \left[ \sum_{i=1,2} \epsilon_i \tilde{I}_{4,i} + \sum_{a=1}^{N_5} \epsilon_1 (\phi - \phi_a) \delta^{(4)}(W_{6,a}) \right] - \frac{\gamma}{2\pi} \phi \wedge \left[ \sum_{i=1,2} \tilde{\omega}_i + \sum_{a=1}^{N_5} (\theta \delta)^{3}(W_{6,a}) \right].
$$

(3.17)

To summarize, with the $Z_2$ projection (3.12) on the three-index tensor, global definition of $G$ corresponds to the transformation (3.11) and the cohomology condition (3.13), as well as (3.16).

### 3.4. The case $b = 1$

As just noted, $b = 1$ is required whenever $\int \tilde{I}_{4,i} \neq 0$ or five-branes not wrapping the circle are present. In this case delta-function singularities are absent from $G$. The case $b = 1$ also presents some other interesting and important features when considering Eqs. (3.6) and (3.11) from the point of view of reduction to the perturbative heterotic string. For this we take $N_5 = 0$.

Expanding $G_{ABC,11}$ and $C_{AB,11}$ in Fourier modes along $S^1$ straightforwardly leads to

$$
G_{ABC,11}^{(0)} = d[A C_{BC,11}^{(0)}] + \frac{\gamma}{2\pi} [\tilde{\omega}_1 + \tilde{\omega}_2]_{ABC},
$$

$$
G_{ABC,11}^{(n)} = d[A C_{BC,11}^{(n)}] - \frac{\gamma}{2\pi} (b-1) [\tilde{\omega}_1 + (-1)^n \tilde{\omega}_2]_{ABC}, \quad n > 0.
$$

(3.18)

The zero mode is $b$-independent. Since $G$ is gauge and local Lorentz invariant we see that $C_{ABC,11}^{(0)}$ can be neither gauge nor local Lorentz invariant. On the other hand, the higher modes of $C_{AB,11}$ are gauge and local Lorentz invariant if and only if $b = 1$. To make contact with ten-dimensional heterotic strings in the field-theory (perturbative) limit, we want to truncate $C_{AB,11}$ to its zero-mode only, and this truncation is safe only if the higher modes are gauge invariant. This again points towards $b = 1$.

In the next section we will show that anomaly cancellations relate $b^2$ and $\lambda^6/\kappa^4$, but do not fix one or the other. However, the gauge and local Lorentz variation of the topological term will have a $b$-dependent contribution which is a variation of a (local) counterterm and
hence does not contribute to the formal twelve-form which characterizes the anomaly. The appearance of this term is related to the $b$-dependent contributions in the variation of $C$, and then to the higher modes in its expansion. On the other hand, a direct calculation of the anomaly-cancelling terms, by first truncating $C$ to $C^{(0)}$ and then calculating the resulting ten-dimensional action leads directly to a Green-Schwarz term which corresponds to $b = 1$, the case in which all truncated modes are gauge invariant. This will be done in Sect. 6.

The conclusion then is that anomaly cancellation alone does not fix $b$. The perturbative heterotic limit however (the small $S^1$ radius limit) selects $b = 1$ because it ensures gauge invariance of the massive modes. This condition is essentially due to compactification on a small space. As we have seen in the previous subsection, global considerations also impose $b = 1$, provided topologically non-trivial configurations occur.

### 3.5. A Consistency Check: Reduction to the Heterotic String

When the circle $S^1$ is very small the perturbative heterotic string theory must emerge. The fields of the latter are defined (modulo possible rescalings) as the zero-modes of the Fourier expansion on the circle of the corresponding M-theory fields. In particular, as mentioned already

\[
B_{AB} \equiv C_{AB,11}^{(0)} = \frac{1}{2\pi r} \int_{-\pi r}^{\pi r} C_{AB,11}(\phi) \, d\phi ,
\]

\[
H_{ABC} \equiv G_{ABC,11}^{(0)} = \frac{1}{2\pi r} \int_{-\pi r}^{\pi r} G_{ABC,11}(\phi) \, d\phi ,
\]

(3.19)

where we have introduced an arbitrary radius $r$ of $S^1$. Integrating the $ABC,11$ components of the solution (3.6) for $G$ over the circle $S^1$ yields

\[
H = dB + \frac{\gamma}{2\pi r} \sum_i \tilde{\omega}_i .
\]

(3.20)

As already observed in (3.18), the parameter $b$ disappears thanks to the last term in (3.6), a term due to the condition of $S^1$ periodicity of the $\mathbb{Z}_2$-odd “step function” $\epsilon_i(\phi)$. In the small radius limit, the Lorentz Chern-Simons forms contained in $\tilde{\omega}_1$ and $\tilde{\omega}_2$ become the same and
we can replace $\tilde{\omega}_1 + \tilde{\omega}_2$ by $\frac{1}{4\pi^2} (\Omega_{3YM} - \Omega_{3L})$ and (3.20) becomes

$$H = dB - \frac{\kappa_1^2}{2\pi r \lambda^2} (\Omega_{3YM} - \Omega_{3L}). \quad (3.21)$$

Upon reexpressing $\kappa \equiv \kappa_{11}$ in terms of the ten-dimensional gravitational coupling $\kappa_{10}$ and rescaling the fields to one’s favorite normalisations, one obtains the standard relation for the heterotic string. In our case, $\frac{\kappa_1^2}{2\pi r} = \kappa_{10}^2$, and we redefine $H$ and $B$ according to

$$H = \frac{\kappa_{10}^2}{\lambda^2} \hat{H}, \quad B = \frac{\kappa_{10}^2}{\lambda^2} \hat{B}, \quad (3.22)$$

so that one obtains the standard relation of the heterotic string:

$$\hat{H} = dB - \Omega_{3YM} + \Omega_{3L}. \quad (3.23)$$

The gauge variation of $\hat{B}$ can be directly obtained from its definition and the expression of $\delta C$, Eq. (3.11). The result is as expected

$$\delta \hat{B} = \Omega_{2YM}^1 - \Omega_{2L}^1,$$

with $\delta \Omega_{3YM} = d\Omega_{2YM}^1$ and $\delta \Omega_{3L} = d\Omega_{2L}^1$. Of course, $\hat{H}$ is invariant.

It is easy to see how these relations are modified by the presence of five-branes: an M-theory five-brane not wrapping $S^1$ (with $\beta = 1$) only contributes a zero-mode $\frac{2}{2\pi} (\theta \delta)^{(3)} (W_{6,a})$ to $G_{ABC,11}$. Such a term then appears on the r.h.s. of (3.23) and $d\hat{H}$ gets an extra term $\sim \delta^{(4)} (W_{6,a})$ which is the appropriate source term from a heterotic NS five-brane.

To complete the comparison with the low-energy effective supergravity of the heterotic string, we compute the Einstein and Yang-Mills terms and the kinetic terms for $\hat{H}$ using the

---

* $\Omega_{3YM}$ and $\Omega_{3L}$ are conventionally normalised Chern-Simons three-forms with $d\Omega_{3YM} = \text{tr} F_1^2 + \text{tr} F_2^2$ and $d\Omega_{3L} = \text{tr} R^2$. Of course, they should not be confused with the spin connection one-forms $\bar{\Omega}_1, \bar{\Omega}_2$ and $\bar{\Omega}$ used some time ago.
basic Lagrangians (2.3) and (2.5). With the metric ansatz
\[ g_{\mu\nu} = \left( \begin{array}{cc} \varphi^{-1/4} g_{AB} & 0 \\ 0 & -\varphi^2 \end{array} \right), \]
we obtain, in terms of ten-dimensional quantities only,
\[ \mathcal{L}_{\text{het}} = -\frac{\sqrt{g_{10}}}{2\kappa_{10}^2} \left[ \mathcal{R} + \frac{1}{2} \frac{\kappa_{10}^2}{\lambda^2} \varphi^{-3/4} \text{tr} (F_{AB} F^{AB}) - \frac{1}{12} \left( \frac{\kappa_{10}^2}{\lambda^2} \varphi^{-3/4} \right)^2 \widehat{H}_{ABC} \widehat{H}^{ABC} \right] + \ldots \] (3.24)

Notice that \( G_{ABCD} \) would contribute to four-derivative terms only. Finally, since \( \lambda^2 \kappa_{10}^{-3/2} \) is dimensionless, it can be absorbed in a redefinition of the field \( \varphi \) to obtain
\[ \mathcal{L}_{\text{het}} = -\frac{\sqrt{g_{10}}}{2\kappa_{10}^2} \left[ \mathcal{R} + \frac{1}{2} \kappa_{10}^{1/2} \varphi^{-3/4} \text{tr} (F_{AB} F^{AB}) - \frac{1}{12} \kappa_{10} \varphi^{-3/2} \widehat{H}_{ABC} \widehat{H}^{ABC} \right] + \ldots, \] (3.25)
and the field-dependent gauge coupling constant is as usual
\[ g^2 = \kappa_{10}^{3/2} \varphi^{3/4}. \]

This shows that \( \mathcal{L}_{\text{het}} \) has two parameters, \( \kappa_{10} \) and the expectation value of \( \varphi \) (related to the radius \( r \) of the circle), and that \( \lambda \) cannot be observed in the heterotic limit.

4. Anomaly cancellations

The \( \mathbb{Z}_2 \) orbifold projection generates a chiral spectrum and, as a consequence, a chiral gauge, mixed and gravitational quantum anomaly in \( M_{10} \) which can be characterized by a formal twelve-form. We have seen that the modification of the Bianchi identity is at the origin of a well-defined gauge and Lorentz variation of the three-form field \( C \). This variation will in turn produce an anomalous variation of the action. We now study the sources for this anomaly inflow and prove anomaly cancellation. To begin with, we will not consider five-branes. They will be taken into account separately in sect. 4.4. In this section we still keep \( b \) as a parameter, since from the M-theoretic point of view, the requirement \( b = 1 \) occurs in the first place only if \( \int \widetilde{I}_{4,i} \neq 0 \). Even in the topological sector where \( \int \widetilde{I}_{4,i} = 0 \) there are anomalies which need to be cancelled. In any case we will find that anomaly cancellation relates \( b^2 \) to \( \lambda^6/\kappa^4 \). In the sector where \( \int \widetilde{I}_{4,i} \neq 0 \) and \( b = 1 \) this then fixes the ratio \( \lambda^6/\kappa^4 \). But this ratio should not depend on the topological sector of the theory and this in turn will imply that \( b = 1 \) always.
4.1. Anomaly inflow from $S_{\text{top}}$

It is now straightforward to determine the anomaly inflow due to the topological term $S_{\text{top}}$:

$$\delta S_{\text{top}} = -\frac{1}{12\kappa^2} \int \delta C \wedge G \wedge G .$$  \hspace{1cm} (4.1)

Since $\delta C$ only contains components with a $d\phi$, only the components of $G$ not containing $d\phi$ can contribute, i.e. $G_{ABCD}$. Remembering that $C_{ABC} = 0$, we get from (3.6) $G_{ABCD} = \frac{b}{2\gamma} \sum_j \epsilon_j (\tilde{I}_{A,j})_{ABCD}$ and hence

$$\delta S_{\text{top}} = -\frac{1}{48\kappa^2} \int \sum_i \left[ (b-1) \delta_i - \frac{b}{2\pi} d\phi \right] \tilde{\omega}_i^1 \sum_j \epsilon_j \tilde{I}_{4,j} \sum_k \epsilon_k \tilde{I}_{4,k} .$$  \hspace{1cm} (4.2)

Now we use (2.10) and (2.13) to perform the $d\phi$ integrals over $S^1$. In the integrals not involving $\delta_i$ it is important that, apart from the $\epsilon_j \epsilon_k$, the rest of the integrand, namely $\tilde{\omega}_i^1 \tilde{I}_{4,j} \tilde{I}_{4,k}$ is independent of $\phi$. The purpose of introducing the tilde quantities was precisely to make manifest this $\phi$-independence. We get

$$\delta S_{\text{top}} = -\frac{\gamma^3 b^2}{48\kappa^2} \int \sum_i \left[ (b-1) \delta_i - \frac{b}{2\pi} d\phi \right] \tilde{\omega}_i^1 \sum_j \epsilon_j \tilde{I}_{4,j} \sum_k \epsilon_k \tilde{I}_{4,k} .$$

$$\equiv \delta S_{\text{top}}^{(1)} + \delta S_{\text{top}}^{(2)} .$$  \hspace{1cm} (4.3)

The second term $\delta S_{\text{top}}^{(2)}$ looks quite different from the first one $\delta S_{\text{top}}^{(1)}$, especially due to the multiple sum over $i, j, k$. However, we will see that both terms correspond to the same anomaly polynomial $I_{12}$. Carrying out the sums in $\delta S_{\text{top}}^{(2)}$, one gets

$$\delta S_{\text{top}}^{(2)} = -\frac{\gamma^3 b^3}{144\kappa^2} \int \sum_{M_{10}} \left( \tilde{\omega}_1^1 + \tilde{\omega}_2^1 \right) \left( (\tilde{I}_{4,1})^2 + (\tilde{I}_{4,2})^2 - \tilde{I}_{4,1} \tilde{I}_{4,2} \right) .$$  \hspace{1cm} (4.4)

Upon applying the descent equations, the corresponding invariant formal twelve-form is

$$I_{12}^{(\text{top},2)} = \frac{\gamma^3 b^3}{144\kappa^2} \left( \tilde{I}_{4,1} + \tilde{I}_{4,2} \right) \left( (\tilde{I}_{4,1})^2 + (\tilde{I}_{4,2})^2 - \tilde{I}_{4,1} \tilde{I}_{4,2} \right)$$

$$= \frac{\gamma^3 b^3}{144\kappa^2} \left( (\tilde{I}_{4,1})^3 + (\tilde{I}_{4,2})^3 \right).$$  \hspace{1cm} (4.5)
But this is of the same form as the invariant twelve-form corresponding to $\delta S_{\text{top}}^{(1)}$:

$$I_{12}^{(\text{top},1)} = -\frac{\gamma^3 b^2 (b - 1)}{144 \kappa^2} \left( (\tilde{I}_{4,1})^3 + (\tilde{I}_{4,2})^3 \right),$$

(4.6)

and adding them up, we see that the terms cubic in $b$ exactly cancel so that the twelve-form characterizing anomaly inflow from the topological term reads

$$I_{12}^{(\text{top})} = +\frac{\gamma^3 b^2}{144 \kappa^2} \sum_i (\tilde{I}_{4,i})^3 = -\frac{\pi}{3} \left( \frac{(4\pi)^5 \kappa^4}{12} \lambda^6 b^2 \right) \sum_i (\tilde{I}_{4,i})^3.$$  

(4.7)

Since the field $G$ is real, the parameter $b$ is real as well and $b^2$ is necessarily positive, as is $\frac{\kappa^4}{\lambda^6}$. So the sign of the coefficient in $I_{12}^{(\text{top})}$ is fixed. This reflects the fact that $N = 1$ ten-dimensional supersymmetry is chiral and specific signs appear in the Bianchi identity once the gravitino chirality is chosen. In any case, the above sign is as required to cancel the quantum anomaly generated by chiral fermions.

It is quite amazing that the two so different looking anomalies $\delta S_{\text{top}}^{(1)}$ and $\delta S_{\text{top}}^{(2)}$ correspond to the same anomaly twelve-form (apart from the different $b$-dependence). However, using the descent equations, as explained in Appendix A, it is not difficult to explicitly find a local ten-dimensional counterterm that relates the two forms:

$$\int_{M_{10}} \sum_{i=1,2} \tilde{\omega}_i^1 (\tilde{I}_{4,i})^2 = \int_{M_{10}} (\tilde{\omega}_1^1 + \tilde{\omega}_2^1) \left( (\tilde{I}_{4,1})^2 + (\tilde{I}_{4,2})^2 - \tilde{I}_{4,1} \tilde{I}_{4,2} \right) + \delta \int_{M_{10}} \Delta_{10}$$

(4.8)

with

$$\Delta_{10} = \frac{2}{3} \left( \tilde{\omega}_1 + \tilde{\omega}_2 \right) \left( \tilde{\omega}_1 \tilde{I}_{4,1} + \tilde{\omega}_2 \tilde{I}_{4,2} - \frac{1}{2} \tilde{\omega}_1 \tilde{I}_{4,2} - \frac{1}{2} \tilde{\omega}_2 \tilde{I}_{4,1} \right)$$

(4.9)

One may then rewrite $\delta S_{\text{top}}$ as

$$\delta S_{\text{top}} = \frac{\gamma^3 b^2}{144 \kappa^2} \left[ \int_{M_{10}} \left( \tilde{\omega}_1^1 (\tilde{I}_{4,1})^2 + \tilde{\omega}_2^1 (\tilde{I}_{4,2})^2 \right) - b \delta \int_{M_{10}} \Delta_{10} \right].$$

(4.10)

This form will be useful in Sect. 6.
4.2. Anomaly inflow from the Green-Schwarz term

We still have to determine the variation of the Green-Schwarz term $\int G \wedge X_7$. Here arises the question whether one should take $\int G \wedge X_7$ or $\int C \wedge X_8$ with $X_8 = dX_7$ and $X_8 = \frac{1}{(4\pi)^{12}} \left( \frac{1}{2} \text{tr} R^4 - \frac{1}{8} \text{tr} R^2 \right)^2$ (cf. Eq. (A.5) in the Appendix). Note that $X_8$ obeys $dX_8 = 0$ and $\delta X_8 = 0$. Both choices are equivalent if $G = dC$, but differ at present. Going through the same steps with $\int C \wedge X_8$ as above with $\int C \wedge G \wedge G$ we will show below that this form does not allow to cancel the one-loop anomalies. The correct form is $\int G \wedge X_7$. The appropriate normalisation of this term is known independently [4, 5], but we will rederive it from the present anomaly cancellation. Let

$$S_{GS} = \frac{c}{\gamma} \int G \wedge X_7$$

(4.11)

where $\gamma = -(4\pi)^2 \frac{a^2}{\lambda^2}$ as before and $c$ is a dimensionless constant to be determined. One has (using descent equations $X_8 = dX_7$, $\delta X_7 = dX_6^1$)

$$\delta S_{GS} = \frac{c}{\gamma} \int G \wedge \delta X_7 = \frac{c}{\gamma} \int G \wedge dX_6^1 = -\frac{c}{\gamma} \int dG \wedge X_6^1$$

$$= -c \int \sum_i \delta_i \wedge \tilde{I}_{4,i} \wedge X_6^1 = -c \int \sum_{M_{10}} \tilde{I}_{4,i} \wedge \tilde{X}_{6,i}^1 .$$

(4.12)

Note the replacement of the bulk $X_6^1$ by the $\tilde{X}_{6,i}^1$ defined only on the $i^{th}$ plane. Since $\tilde{I}_{4,i}$ is closed and gauge/local Lorentz invariant, this corresponds to the twelve-form

$$I_{12}^{(GS)} = -c \sum_i \tilde{I}_{4,i} \wedge \tilde{X}_{8,i} .$$

(4.13)

Had we started with $\tilde{S}_{GS} = \xi \int C \wedge X_8$, we would have obtained, using Eq. (3.11),

$$\delta \tilde{S}_{GS} = \frac{c}{\gamma} \int \delta C \wedge X_8 = c (b - 1) \int \sum_i \tilde{\omega}_i^1 \wedge \tilde{X}_{8,i} - c \frac{b}{2\pi} \int \sum_i d\phi \wedge \tilde{\omega}_i^1 \wedge X_8 .$$

(4.14)

While in the first term the $\delta$-function has the effect of replacing $X_8$ by $\tilde{X}_{8,i}$, in the second term $X_8$ truly depends on $\phi$. We see that the first term alone corresponds to a polynomial
\( \hat{I}^{(GS,1)}_{12} = c(b - 1) \sum_i \tilde{I}_{4,i} \wedge \tilde{X}_{8,i} \) and has the right form. Its coefficient however is wrong: if \( b = 1 \) it vanishes and if one is allowed to take \( b \neq 1 \) one would be forced into a non-standard choice of \( c \). In any case, in the second term however, \( X_8 \) genuinely depends on the circle coordinate \( \phi \) and there is no way to make it equal to \( \tilde{X}_{8,i} \) on the \( i \)th plane as needed for anomaly cancellation. We conclude that \( \hat{S}_{GS} \) could be suitable at best only for \( b = 0 \). But this case is certainly ruled out as it would lead to \( \delta S_{\text{top}} = 0 \). Note that the possibility to discriminate between \( S_{GS} \) and \( \hat{S}_{GS} \) relies on the presence of the second term in (4.14) and this term again is a consequence of enforcing periodicity of the “step functions” \( \epsilon_i(\phi) \).

### 4.3. Cancellation of the one-loop anomaly

The sum \( I_{12}^{(\text{top})} + I_{12}^{(GS)} \) must cancel the one-loop anomaly given in the Appendix, Eq. (A.5). (These expressions refer to ten-dimensional anomalies on a given fixed plane and should be understood as involving only tilde quantities.) It is

\[
I_{12}(1\text{-loop}) = \sum_i \left( \frac{\pi}{3} (\tilde{I}_{4,i})^3 + \tilde{I}_{4,i} \wedge \tilde{X}_{8,i} \right). \tag{4.15}
\]

We get anomaly cancellation, \( I_{12}^{(\text{top})} + I_{12}^{(GS)} + I_{12}(1\text{-loop}) = 0 \), if and only if

\[
b^2 = \frac{12}{(4\pi)^3 \kappa^4} \lambda^6 \tag{4.16}
\]

and

\[
c = 1. \tag{4.17}
\]

While \( c = 1 \) was known previously, from cancellation of the anomaly due to a five-brane (see below), earlier literature claims a cubic equation for the parameter \( b \). Instead, we have shown here that a careful treatment of periodicity along \( S^1 \) cancels the cubic terms and leads to a quadratic relation.

Using eqs. (4.9), (4.12) and (4.16) we may now rewrite the total anomalous variation as

\[
\delta S_{\text{top}} + \delta S_{GS} = -\frac{\pi}{3} \int_{M_{10}} \left[ \tilde{\omega}_1 \tilde{I}_{4,1}^2 + \tilde{\omega}_2 \tilde{I}_{4,2}^2 - b \delta \Delta_{10} \right] - \int_{M_{10}} \left[ \tilde{I}_{4,1} \tilde{X}_{6,1}^1 + \tilde{I}_{4,2} \tilde{X}_{6,2}^1 \right]. \tag{4.18}
\]

Note that the term \( \sim b \) is the variation of a local ten-dimensional counterterm, and this is why it does not contribute to the twelve-form of the descent equations. We will see in Section 6 that
in the small $S^1$ radius limit corresponding to the perturbative heterotic string, this particular form of the anomalous variation is at the origin of a local counterterm usually unexpected from standard ten-dimensional arguments.

4.4. The five-brane anomaly

The same Green-Schwarz term (4.11) is also able to cancel any additional five-brane anomaly$^\star$. The latter is a purely gravitational anomaly of the six-dimensional chiral theory living on the worldvolume of the five-brane (a chiral tensor multiplet). The invariant eight-form corresponding to this one-loop anomaly simply is

$$I_{S^8}^{5-\text{brane}}(1-\text{loop}) = X_8, \quad (4.19)$$

where the sign on the r.h.s actually depends on the choice of chirality. The Bianchi identity for $G$ is further modified by the presence of the five-brane as given by (3.7). Recall that $W_6$ denotes the five-brane worldvolume and $\delta^{(5)}(W_6)$ is a five-form such that for any six-form $I_6$,

$$\int_{M_{11}} \delta^{(5)}(W_6) \wedge I_6 = \int_{W_6} I_6. \quad (A \text{ possible switch of chirality would produce a minus sign on the r.h.s. of (4.19) and (3.7).})$$

Then one sees immediately from (4.12) that there is an additional five-brane contribution to $\delta S_{GS}$:

$$\delta S_{GS} |_{5-\text{brane}} = -\frac{1}{\gamma} \int dG|_{5-\text{brane}} \wedge X_6^1 = - \int \delta^{(5)}(W_6) \wedge X_6^1 = - \int_{W_6} X_6^1, \quad (4.20)$$

which is the descendent of $I_{S^8}^{(GS)} = -X_8$ thus cancelling the $I_{S^8}^{5-\text{brane}}(1-\text{loop})$.

We should still check that the new five-brane contribution to $dG$ does not spoil our previous results. This is a crucial point which, to the best of our knowledge, has not been addressed at the level of M-theory anomaly cancelling terms.$^\dagger$ We will see that the result is quite non-trivial.

Indeed, when solving the Bianchi identity, $G$ has an extra piece:

$$G|_{5-\text{brane}} = G|_{\text{without five-brane}} + \gamma(\theta \delta)^{(4)}(W_6) \quad (4.21)$$

where $d(\theta \delta)^{(4)}(W_6) = \delta^{(5)}(W_6)$. In section 3.1, we have explicitly given the form of $(\theta \delta)^{(4)}(W_6)$ for five-branes not wrapping the circle, and later on in sect 3.3 we have shown that in this

---

$^\star$ For five-branes, there is also the normal bundle anomaly, but it can be taken care of independently$^{[22]}$ and we will not consider it here.

$^\dagger$ A discussion of anomaly cancellation in a Calabi-Yau background with five-branes can be found in$^{[23]}$. 

23
case the parameter $\beta$ must be taken to be 1. For five-branes that wrap the circle, $\delta^{(5)}(W_6)$ does not contain $\phi$ and any primitive $(\theta \delta)^{(4)}(W_6)$ is allowed so far. For the time being we do not specify which type of five-branes we consider and thus we keep the more generic notation $(\theta \delta)^{(4)}(W_6)$. What is important is that the extra piece in $G$ does not depend on any fields and is thus trivially gauge and local Lorentz invariant. Thus the variation $\delta C$ as given in (3.11) is unchanged. The variation $\delta S_{\text{GS}}$ is modified by the additional piece (4.20) as required. What about $\delta S_{\text{top}} = -\frac{1}{12\kappa^2} \delta \int C \wedge G \wedge G$? Since $\delta C$ is the same as before, the only additional contribution is due to the additional piece in $G$. We then get an additional piece in $\delta S_{\text{top}}$:

$$\delta S_{\text{top}}|_{5\text{-brane}} = -\frac{1}{6\kappa^2} \int \delta C \wedge G|_{\text{without 5-brane}} \wedge G|_{\text{extra}}$$

$$= -\frac{\gamma^3 b}{12\kappa^2} \int \sum_i \left[(b - 1)\delta_i - \frac{b}{2\pi} d\phi\right] \wedge \tilde{\omega}_i^1 \wedge \sum_j \epsilon_j \tilde{I}_{4,j} \wedge (\theta \delta)^{(4)}(W_6),$$

(4.22)

where we used (3.11) for $\delta C$.

If the five-brane wraps the circle, $\delta^{(5)}(W_6)$ and $(\theta \delta)^{(4)}(W_6)$ are independent of $\phi$ and then the only $\phi$-dependence of the above integrand is $[(b - 1)\delta_i - \frac{b}{2\pi} d\phi] \epsilon_j$ which integrates to zero by virtue of Eqs. (2.11) and (2.14). Hence the anomalous variation of $S_{\text{top}}$ is not affected by these five-branes.

If the five-brane worldvolume does not extend in the $\phi$-direction, things are more subtle. We have seen that in this case necessarily $b = 1$ and $\beta = 1$. Then using the explicit form of (3.8) with $\beta = 1$ or eq. (3.17), eq. (4.22) becomes

$$\delta S_{\text{top}}|_{5\text{-brane}} = \frac{\gamma^3}{48\pi \kappa^2} \int \sum_i d\phi \wedge \tilde{\omega}_i^1 \wedge \sum_j \epsilon_j \tilde{I}_{4,j} \wedge \sum_{a=1}^{N_5} \epsilon_1(\phi - \phi_a) \delta^{(4)}(W_{6,a})$$

(4.23)

where we have now explicitly summed over all such five-branes. The integral over the circle can be performed explicitly, as well as the four integrals involving the $\delta^{(4)}(W_{6,a})$. The result is a non-vanishing contribution localised on the world volumes of the five-branes:

$$\delta S_{\text{top}}|_{5\text{-brane}} = \frac{\gamma^3}{48\pi \kappa^2} \sum_{a=1}^{N_5} \int_{W_{6,a}} (\tilde{\omega}_1^1 + \tilde{\omega}_2^1) \left(f_1(\phi_a)\tilde{I}_{4,1} + f_2(\phi_a)\tilde{I}_{4,2}\right),$$

(4.24)

where the $f_i(\phi_a) = \int d\phi \epsilon_i(\phi) \epsilon_1(\phi - \phi_a)$ are given by $f_1(\phi) = \frac{2\pi}{3} - 2|\phi| + \frac{\phi^2}{\pi}$ and $f_2(\phi) = -\frac{\pi}{3} + \frac{\phi^2}{\pi}$ and obey $f_1'(\phi) = -2\epsilon_i(\phi)$. Note that they are $\mathbb{Z}_2$ even functions. Thus we see that there is
an extra anomaly inflow from $S_{\text{top}}$ into the six-dimensional theory on the five-brane world volumes. The associated invariant eight-form is

$$I_8^{(\text{top}, 5\text{-brane})} = \frac{\gamma^3}{48\pi \kappa^2} \sum_{a=1}^{N_5} \left[ f_1(\phi_a) \left( (\tilde{I}_{4,1})^2 + \tilde{I}_{4,1} \tilde{I}_{4,2} \right) + f_2(\phi_a) \left( (\tilde{I}_{4,2})^2 + \tilde{I}_{4,1} \tilde{I}_{4,2} \right) \right]$$

which clearly is non-vanishing. This is an interesting new effect. A somewhat related issue appears in [23] where it is found, within the context of Calabi-Yau compactifications with less supersymmetry, that there are gauge fields that originate on the five-branes. The consequences of the non-vanishing term (4.24) go beyond the scope of the present paper and will be discussed elsewhere [16].

5. The issue of $G$-flux quantization

5.1. Does flux quantization hold?

Cancellation of all anomalies only required the validity of relation (4.16) between $b^2$ and $\lambda^6/\kappa^4$. To put it differently, whatever the ratio $\lambda^6/\kappa^4$ is, there is a choice of the parameter $b$ that cancels all anomalies. On the other hand, we have seen that if there are topologically non-trivial gauge/gravity configurations such that $\int_{C_{i}} \tilde{I}_{4,i} \neq 0$, one is forced to take $b = 1$ in order to have a globally well-defined four-form field $G$. We will now explore the consequences of this fact for the flux of $G$ and compare with Witten’s result on flux quantization [14] obtained in the downstairs approach which in a certain way also corresponds to a fixed value of $b$.

We will evaluate the integral of $G$ on the two different categories of four-cycles. To simplify the discussion, we suppose that there are no five-branes. Modifications due to the latter can be trivially incorporated.

Four-cycles not wrapping $S^1$

Consider a four-cycle $C(\phi)$ not wrapping the $S^1$ and at a fixed value of $\phi \in (0, \pi)$. Then

$$\frac{2}{\gamma} \int_{C(\phi)} G = b \sum_{i} \epsilon_i(\phi) \int_{C(\phi)} \tilde{I}_{4,i} = b \left[ \left( 1 - \frac{\phi}{\pi} \right) \int_{C(\phi)} \tilde{I}_{4,1} - \frac{\phi}{\pi} \int_{C(\phi)} \tilde{I}_{4,2} \right] = b \int_{C(\phi)} \tilde{I}_{4,1} \quad (5.1)$$

where the $\phi$-dependent terms have cancelled thanks to the cohomology condition (3.13). As shown in section 3.3, either $\int_{C(\phi)} \tilde{I}_{4,1} = 0$ and $b$ is arbitrary, or $\int_{C(\phi)} \tilde{I}_{4,1} = n_1 - \frac{1}{2} p_1 \neq 0$ with
\( n_1, p_1 \in \mathbb{Z} \) in which case \( b = 1 \). In either case we get

\[
\frac{2}{\gamma} \int_{\mathcal{C}} G = \left( n_1 - \frac{1}{2} p_1 \right) = - \left( n_2 - \frac{1}{2} p_2 \right)
\]

for any 4-cycle \( \mathcal{C} \) not wrapping \( S^1 \). \hspace{1cm} (5.2)

Usually one mentions another contribution to this integral which would come from the \( dC \) piece in \( G \). But with \( C_{ABC} = 0 \) there is no such contribution here.

This looks much like Witten’s flux quantization [14] which was obtained in the downstairs approach and which reads \( \frac{1}{\gamma_{\text{downstairs}}} \int_{\mathcal{C}} G = (n_1 - \frac{1}{2} p_1) \). Although the two conditions seem to differ by a factor of two, they are actually the same. The point is that the gravitational couplings \( \kappa^2 \) in the upstairs and downstairs approach differ precisely by this factor of two [9]: \( \kappa_{\text{downstairs}}^2 = \kappa_{\text{upstairs}}^2 / 2 \) and since \( \gamma = -(4\pi)^2 \kappa^2 / \lambda^3 \) one also has \( \gamma_{\text{downstairs}} = \gamma_{\text{upstairs}} / 2 \). Of course, in all our formulas \( \gamma \equiv \gamma_{\text{upstairs}} \) and one sees that eq (5.2) is exactly Witten’s flux quantization. Using Eq. (4.16) with \( b = 1 \) one can reexpress this flux quantization with only \( \lambda \) or only \( \kappa \) appearing as coefficient, but this is not too illuminating.

**Four-cycles wrapping \( S^1 \)**

We must also consider four-cycles of the form \( \mathcal{C} = \mathcal{C}_3 \times S^1 \) where \( \mathcal{C}_3 \) is some three-cycle at fixed value of \( \phi \). Obviously, such cycles do not exist in the downstairs approach and we cannot expect the corresponding flux to be related to Witten’s quantization condition. An arbitrary four-cycle then is homologous to \( m_1 \) times this four-cycle \( \mathcal{C} \) plus \( m_2 \) times any of the four-cycles of the previous subsection. One has from Eq. (3.6)

\[
\int_{\mathcal{C}_3 \times S^1} G = \int_{\mathcal{C}_3 \times S^1} \left( dC + \gamma \sum_i \left( (b - 1) \delta_i - \frac{b}{2\pi} \, d\phi \right) \right) \wedge \tilde{\omega}_i = \int_{\mathcal{C}_3 \times S^1} dC - \gamma \int_{\mathcal{C}_3} \sum_i \tilde{\omega}_i. \hspace{1cm} (5.3)
\]

As always when integrating over \( S^1 \), the \( b \)-dependent terms have cancelled. This integral is closely related to the flux \( \int H = \int (dB - \omega_{\text{YM}} + \omega_{\text{L}}) \) in the heterotic string. Indeed, as discussed in Section 3.5, it exactly reduces to the latter in the small radius limit. The flux of \( H \) was discussed some time ago [18] and it was concluded (when appropriately normalised) to be of the form \( n + \delta \) with \( n \in \mathbb{Z}, \delta \in \mathbb{R} \). A similar argument holds here and we conclude that the value of (5.3) cannot be determined further. The key difference with Witten’s analysis [14] is that since \( G \neq dC \) one cannot use the standard argument to obtain flux quantization.
We conclude that for four-cycles not wrapping the circle \( S^1 \) we have the standard flux quantization, while if the four-cycle wraps the \( S^1 \) we cannot say anything interesting about the flux of \( G \).

5.2. A REMARK ON THE MEMBRANE ACTION

Let us first review the standard argument [14]. In uncompactified M-theory or M-theory on a smooth manifold one simply has \( G = dC \). Since this needs not hold globally one typically argues that \( \int_C G = \int_{C^+} dC^+ + \int_{C^-} dC^- = \int_{C_3} (C^+ - C^-) \) where \( C_3 = \partial C^+ = -\partial C^- \). But since the membrane action \( T_2 \int_{C_3} C + \ldots \) \( (T_2 \) is the membrane tension) should be well-defined modulo \( 2\pi \) one concludes that \( T_2 \int_{C} G = 2\pi n \) with integer \( n \). Witten has argued [14] that the three-dimensional membrane theory does in certain cases have a so-called parity anomaly [17] which is a sign ambiguity \( e^{i\pi p} \), \( p \in \mathbb{Z} \) of the fermion determinant. This implies that actually the well-definedness of the membrane functional integral requires \( T_2 \int_{C} G = 2\pi (n - \frac{1}{2}p) \). This fits with the flux quantization since \( T_2 = -\frac{2\pi}{\gamma} \). In our present upstairs discussion however, this relation for the \( G \)-flux holds (with the appropriate redefinition of \( \kappa^2 \) by a factor of two as discussed above) if \( C \) does not wrap the \( S^1 \), but is replaced by (5.3) if it does. Does this mean that the membrane functional integral is no longer well-defined? Of course not. First, the above argument is spoiled since \( G \neq dC \) everywhere. Second, a coupling of \( C \) to the membrane worldvolume \( C_3 \) of the type \( \int_{C_3} C \) without modification certainly does not lead to a well-defined functional integral since we have observed that \( C \) is neither gauge nor local Lorentz invariant. Obviously then there must be corrective terms to restore the invariance. It would be interesting to explicitly construct these terms. A clue is probably provided by Eq. (5.3) which is gauge and local Lorentz invariant.
6. The heterotic anomaly-cancelling terms

As a last check of our scheme, we compute in the small-radius limit the anomaly-cancelling terms of the heterotic string. Taking the zero modes along the circle amounts to identifying $\tilde{R}_1$ and $\tilde{R}_2$ with $R$ as well as the seven-form $X_7$ in $M_{11}$ with its restriction $\tilde{X}_{7,1} \sim \tilde{X}_{7,2} \sim \tilde{X}_7$ to $M_{10}$ (no $d\phi$ components) and similarly for $X_8 = dX_7$ and $\tilde{X}_{8,1} \sim \tilde{X}_{8,2} \sim \tilde{X}_8$. We recall that $X_{8,i}$ is given in the appendix. As before, $B_{AB}$ is identified with the zero mode $\frac{1}{2\pi r} \int_{S^1} C_{AB,11}$, cf. Eq. (3.19), and $C_{ABC} = 0$ as usual. Using Eq. (3.6) (with an obvious insertion of a factor of $\frac{1}{r}$ in its last term), the Green-Schwarz term $\int G \wedge X_7$ then reduces to

$$\frac{1}{\gamma} \int G \wedge X_7 \to \frac{2\pi r}{\gamma} \int_{M_{10}} B \wedge \tilde{X}_8 + \int \int_{M_{10} S^1} \sum_i \left[ (b-1) \delta_i \wedge \tilde{\omega}_i - \frac{b}{2\pi r} d\phi \wedge \tilde{\omega}_i \right] \wedge \tilde{X}_7$$

$$= -\frac{2\pi r}{12(4\pi)^5 \kappa^2} \int_{M_{10}} B \wedge \left[ \frac{1}{2} \text{tr} R^4 - \frac{1}{8} (\text{tr} R^2)^2 \right] - \int \sum_i \tilde{\omega}_i \wedge \tilde{X}_7. \quad (6.1)$$

Similarly, the topological term yields, using Eq. (2.10) to perform the integral over $S^1$ and Eq. (4.16) to reexpress $b^2$ in terms of $\lambda^6/\kappa^4$,

$$-\frac{1}{12\kappa^2} \int C \wedge G \wedge G \to -\frac{b^2 \gamma^2}{48\kappa^2} \int_{M_{10} S^1} C \wedge \left( \sum_i \epsilon_i \tilde{I}_{4,i} \right) \wedge \left( \sum_j \epsilon_j \tilde{I}_{4,j} \right)$$

$$\to -\frac{2\pi r b^2 \gamma^2}{144\kappa^2} \int_{M_{10}} B \wedge \left( (\tilde{I}_{4,1})^2 + (\tilde{I}_{4,2})^2 - \tilde{I}_{4,1} \tilde{I}_{4,2} \right) \quad (6.2)$$

$$= -\frac{2\pi r}{12(4\pi)^5 \kappa^2} \int_{M_{10}} B \wedge \left[ (\text{tr} F_1^2)^2 + (\text{tr} F_2^2)^2 - (\text{tr} F_1^2)(\text{tr} F_2^2) \right]$$

$$+ \frac{1}{4} (\text{tr} R^2)^2 - \frac{1}{2} (\text{tr} R^2)(\text{tr} F_1^2 + \text{tr} F_2^2) \right].$$

At this point it is useful to switch to differently normalised heterotic variables: the ten-dimensional gravitational constant is $\kappa^2_{10} = \kappa^2/(2\pi r)$ and $B$ is rescaled to the heterotic $\hat{B}$ as in Eq. (3.22). It is also convenient to define* the heterotic eight-form $\tilde{X}_8$ and the seven-form

----

* To facilitate comparison with the literature, we note e.g. that the $X_8$ used in the textbook by Green, Schwarz and Witten [20] is $\frac{1}{2} \tilde{X}_8$. 

28
\( \hat{X}_7 \) such that \( d\hat{X}_7 = \hat{X}_8 \):

\[
\hat{X}_8 = (\text{tr} \, F_i^2)^2 + (\text{tr} \, F_2^2)^2 - \text{tr} \, F_i^2 \, \text{tr} \, F_2^2 - \frac{1}{2} \text{tr} \, R^2 \left( \text{tr} \, F_i^2 + \text{tr} \, F_2^2 \right) \\
+ \frac{1}{2} \text{tr} \, R^4 + \frac{1}{8} (\text{tr} \, R^2)^2 ,
\]

\[
\hat{X}_7 = \Omega_{3,1} \, \text{tr} \, F_i^2 + \Omega_{3,2} \, \text{tr} \, F_2^2 - \frac{1}{2} \Omega_{3,1} \, \text{tr} \, F_2^2 - \frac{1}{2} \Omega_{3,2} \, \text{tr} \, F_1^2 \\
- \frac{1}{4} \Omega_{3L}(\text{tr} \, F_i^2 + \text{tr} \, F_2^2) - \frac{1}{4}(\Omega_{3,1} + \Omega_{3,2}) \, \text{tr} \, R^2 + \frac{1}{2} \Omega_{7L} + \frac{1}{8} \Omega_{3L} \, \text{tr} \, R^2 .
\]

(6.3)

In the last expression, the gauge and gravitational Chern-Simons three-forms \( \Omega_{3,1} \), \( \Omega_{3,2} \) and \( \Omega_{3L} \) and seven-form \( \Omega_{7L} \) are such that \( d\Omega_{3,i} = \text{tr} \, F_i^2 \), \( i = 1, 2 \) and \( d\Omega_{3L} = \text{tr} \, R^2 \), \( d\Omega_{7L} = \text{tr} \, R^4 \) so that indeed \( \hat{X}_7 \) verifies \( d\hat{X}_7 = \hat{X}_8 \). Note that our previous definitions \( \tilde{X}_7 \) and \( \tilde{X}_8 \) only had gravitational contributions, while now, following the standard heterotic notation, \( \hat{X}_7 \) and \( \hat{X}_8 \) have both, gravitational and gauge, as well as mixed contributions. With these definitions, the sum of the contributions (6.1) and (6.2) becomes

\[
S_{\text{GS,het}} = -\frac{1}{12(4\pi)^5} \int_{M_{10}} \left( \hat{B} \wedge \hat{X}_8 + (\Omega_{3,1} + \Omega_{3,2} - \Omega_{3L}) \wedge \left( \frac{1}{2} \Omega_{7L} - \frac{1}{8} \Omega_{3L} \, \text{tr} \, R^2 \right) \right)
\]

(6.4)

which we may rewrite as

\[
S_{\text{GS,het}} = -\frac{1}{12(4\pi)^5} \int_{M_{10}} \left( \hat{B} \wedge \hat{X}_8 + (\Omega_{3YM} - \Omega_{3L}) \wedge \hat{X}_7 \right) \\
+ \frac{1}{8(4\pi)^5} \int_{M_{10}} (\Omega_{3,1} - \frac{1}{2} \Omega_{3L})(\Omega_{3,2} - \frac{1}{2} \Omega_{3L})(\text{tr} \, F_2^2 - \text{tr} \, F_1^2) .
\]

(6.5)

The first line is the standard Green-Schwarz counterterm of the heterotic string [20], while the second term is a local counterterm specific to the form of the anomaly-cancelling term generated by M-theory on \( S^1/\mathbb{Z}_2 \). Of course, it is our old friend from Eqs. (4.9) and (4.10), namely \( \frac{3}{4} \int_{M_{10}} \Delta_{10} = \frac{3}{2} \left( -\frac{3}{4} b^1 \int_{M_{10}} \Delta_{10} \right) \) with \( b = 1 \). As already explained in Sect. 3.4, the small-radius limit automatically produces this term with a coefficient corresponding to \( b = 1 \).

The extra factor of \( \frac{3}{2} \) with respect to Eq. (4.10) is due to the usual choice of including only part of this term, namely \(-\frac{1}{2}\) of it, into the standard heterotic counterterm of the first line of (6.5), leaving us with a factor of \(+\frac{1}{2} + 1\) = \( \frac{3}{2} \) for the second term.
7. Conclusions

We have carefully studied the solution for $G$ of the modified Bianchi identity. It appeared to depend on an arbitrary parameter $b$. When requiring $G$ to be globally well-defined, i.e. invariant under small and large gauge and local Lorentz transformations we have discovered that this parameter $b$ is fixed to $b = 1$ in all topologically non-trivial sectors of the theory (i.e. with non-vanishing gauge/gravitational instanton numbers and/or with five-branes) Anomaly cancellation equates $b^2$ and $\frac{12}{(4\pi)^4} \frac{\lambda^6}{\kappa^4}$, independently of the choice of topological sector. In those sectors where $b = 1$ this fixes $\frac{\lambda^6}{\kappa^4} = \frac{(4\pi)^5}{12}$. Since this ratio cannot depend on the topological sector of the theory, anomaly cancellation in turn implies that $b = 1$ always.

Thus anomaly cancellation and global well-definedness of $G$ have selected exactly one solution $G$ to the modified Bianchi identity. This is neither a consequence of flux quantization nor has it anything to do with membrane actions. Instead, we observe that for four-cycles not wrapping $S^1$, the flux of $G$ indeed automatically obeys Witten’s flux quantization while for cycles wrapping the $S^1$ the flux is more general. Also, in this $S^1/\mathbb{Z}_2$ orbifold for membranes wrapping the $S^1$ the naive membrane action is not gauge/local Lorentz invariant and needs to be modified.

We observed that $b = 1$ is also very natural when considering the reduction to the heterotic string in the small radius limit, since it is only for $b = 1$ that all higher modes of the Fourier expansion on the circle can be consistently decoupled. We explicitly wrote out the anomaly cancelling terms obtained from this reduction. They differ from what is usually taken in the heterotic string by the addition of a well-defined local counterterm.

Anomaly cancellations in the presence of five-branes are more subtle: For five-branes not wrapping the circle there is an extra contribution from $S_{\text{top}}$ signalling that non-trivial things are happening on those five-branes.

It is quite amazing that this simple $S^1/\mathbb{Z}_2$ orbifold of M-theory still had that many subtle points to be revealed. Now that we understand them, it is relatively straightforward to study other more complicated orbifolds of M-theory. This will be done elsewhere [16].
APPENDIX A

In this appendix, we give the relevant anomaly polynomials used in the paper. Different authors have conventions differing by factors of $2\pi$, etc. We use the conventions of [20], which were also used in the nice appendix of [13]. Recall that an anomaly $A_n$ in $n$ dimensions, i.e. $\delta S = \int d^n x A_n$, is described by a unique gauge-invariant* polynomial $I_{n+2}$ in $n+2$ dimensions (a twelve-form in our ten-dimensional case) which is related to $A_n$ by the descent equations

$$I_{n+2} = dI_{n+1}, \quad \delta I_{n+1} = dA_n.$$  \hspace{1cm} (A.1)

Recall that this is so because the anomaly $A_n$ itself is not uniquely defined. Indeed we always have the freedom to add an $n$-dimensional local counterterm to the action $S$: $S \rightarrow S' = S + \int d^n x \Delta_n$, so that $A_n \rightarrow A'_n = A_n + \delta \Delta_n + d\Delta_{n-1}$, with an arbitrary $\Delta_{n-1}$. On the other hand, the descent equations imply that $I_{n+1}$ is defined up to a $d\Delta_n$. But then $\delta I_{n+1}$ is defined up to a $\delta d\Delta_n = d\delta \Delta_n$ meaning that $A_n$ is only defined up to a $\delta \Delta_n$ and a $d\Delta_{n-1}$ as it should. When one applies the descent equations to different ways of writing the same invariant anomaly polynomial $I_{n+2}$ one is naturally led to different anomalies $A_n$ and $A'_n$, the two differing by the gauge variation $\delta \Delta_n$ of a local counterterm and possibly a total derivative $d\Delta_{n-1}$.

The relevant contributions to the gauge, gravitational and mixed anomalies due to the different chiral fields in ten dimensions are given by the following invariant twelve-forms of the gauge and Lorentz curvatures:

$$I_{\text{grav}}^{(3/2)}(R) = \frac{1}{(2\pi)^5 6!} \left( \frac{55}{56} \text{tr} R^6 - \frac{75}{128} \text{tr} R^4 \text{tr} R^2 + \frac{35}{512} (\text{tr} R^2)^3 \right),$$

$$I_{\text{grav}}^{(1/2)}(R) = \frac{1}{(2\pi)^5 6!} \left( -\frac{1}{504} \text{tr} R^6 + \frac{1}{384} \text{tr} R^4 \text{tr} R^2 - \frac{5}{4608} (\text{tr} R^2)^3 \right),$$

$$I_{\text{grav}}^{5-\text{form}}(R) = \frac{1}{(2\pi)^5 6!} \left( -\frac{496}{504} \text{tr} R^6 + \frac{7}{12} \text{tr} R^4 \text{tr} R^2 - \frac{5}{72} (\text{tr} R^2)^3 \right),$$

$$I_{\text{mixed}}^{(1/2)}(R, F) = \frac{1}{(2\pi)^5 6!} \left( \frac{1}{16} \text{tr} R^4 \text{Tr} F^2 + \frac{5}{64} (\text{tr} R^2)^2 \text{Tr} F^2 - \frac{5}{8} \text{tr} R^2 \text{Tr} F^4 \right),$$

$$I_{\text{gauge}}^{(1/2)}(F) = \frac{1}{(2\pi)^5 6!} \text{Tr} F^6,$$

where $\text{Tr}$ denotes as usual the trace over generators of the adjoint representation of the gauge

* By “gauge” invariant, we really mean invariant under gauge and local Lorentz transformations. Similarly, the variation $\delta$ designates both types of infinitesimal transformations.
group: the gauge curvature two-form $F$ has values in the adjoint representation. An additional factor $1/2$ must be included for Majorana-Weyl fermions, and opposite chiralities contribute with opposite signs.

The one-loop anomaly generated by each of the $S^1/Z_2$ fixed planes includes an $E_8$ gauge (and mixed) contribution, and a gravitational contribution that is the half of what would be expected in ten dimensions. This is because of the coupling of eleven-dimensional supergravity to ten-dimensional fields [1]. A pragmatic way to understand this extra factor of one half is to realise that, in the limit where the two fixed planes coincide, the sum of the anomalies of the two fixed planes should reproduce the standard ten-dimensional $E_8 \times E_8$ anomaly. Hence the one-loop anomaly on the $i^{th}$ plane ($i = 1, 2$) for a general gauge group is

$$I_{12}(1\text{-loop})_i = \frac{1}{2} \times \left( I_{\text{grav}}^{(3/2)}(R) - I_{\text{grav}}^{(1/2)}(R) \right) + \frac{1}{2} \left( n_i I_{\text{grav}}^{(1/2)}(R) + I_{\text{mixed}}^{(1/2)}(R, F_i) + I_{\text{gauge}}^{(1/2)}(F_i) \right),$$

where the overall factor of $\frac{1}{2}$ is due to the Majorana-Weyl condition and the extra factor of $\frac{1}{2}$ in the gravitational term has just been explained. The dimension of the gauge group on the $i^{th}$ plane is denoted by $n_i$. Note that $R$ is meant to be located on the $i^{th}$ plane. In the following, we write $R_i$ to emphasize this fact. Such an anomaly has a chance to be cancelled by an appropriate Green-Schwarz mechanism only if this twelve-form factorises into a four-form and an eight-form. For this to work the $\text{tr} F^6$ term must vanish and the $\text{Tr} F^6$ term must be expressible entirely as a combination of $\text{Tr} F^4 \text{Tr} F^2$ and $(\text{Tr} F^2)^3$. The first condition selects $n_i = 248$ while the only appropriate factorisation of $\text{Tr} F^6$ is obtained if

$$\text{Tr} F_i^6 = \frac{1}{7200} (\text{Tr} F_i^2)^3, \quad \text{Tr} F_i^4 = \frac{1}{100} (\text{Tr} F_i^2)^2.$$

These two conditions only hold for $E_8$, which also has the required dimension $n_i = 248$. The presence of large integers in the coefficients is due to the large value 30 of the quadratic Dynkin index for the adjoint representation of $E_8$. Expressions become nicer if one redefines the trace
as $\text{Tr} = 30 \text{tr}$. One then arrives at the factorised form

$$I_{12}(1-\text{loop})_i = \frac{\pi}{3} (I_{4,i})^3 + I_{4,i} X_{8,i} ,$$

$$I_{4,i} = \frac{1}{(4\pi)^2} \left( \text{tr} F_i^2 - \frac{1}{2} \text{tr} R_i^2 \right) ,$$

$$X_{8,i} = \frac{1}{(4\pi)^3 12} \left( \frac{1}{2} \text{tr} R_i^4 - \frac{1}{8} \left( \text{tr} R_i^2 \right)^2 \right) .$$

(A.5)

One sees that each of the $I_{12}(1-\text{loop})_i, i = 1, 2$ is factorised. For the perturbative $E_8 \times E_8$ heterotic string the relevant polynomial is the sum $I_{12}(1-\text{loop})_1 + I_{12}(1-\text{loop})_2$. How can this also be factorised? For the perturbative limit (weak coupling) the two fixed planes coincide and one should take the same value for $R$ in both expressions. Then the purely gravitational $X_8$ is the same and one gets

$$I_{12}(1-\text{loop})_1 + I_{12}(1-\text{loop})_2 \longrightarrow (I_{4,1} + I_{4,2}) X_8 + \frac{\pi}{3} \left( (I_{4,1})^3 + (I_{4,2})^3 \right)$$

$$= (I_{4,1} + I_{4,2}) \left[ X_8 + \frac{\pi}{3} \left( (I_{4,1})^2 + (I_{4,2})^2 - I_{4,1} I_{4,2} \right) \right] ,$$

(A.6)

which is factorised thanks to the trivial identity $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$. It is now easy to rewrite this expression in terms of traces over the representations of the product group $E_8 \times E_8$ to recover the well-known expression of the anomaly for $E_8 \times E_8$ super Yang-Mills coupled to $N = 1$ supergravity (heterotic string). But we do not need it here.

We have chosen the normalisation of the four-forms $I_{4,i}$ such that their integrals over a four-cycle equal the integral characteristic class of the $E_8$ bundle minus a quarter of the even first Pontryagin class. More precisely, for any four-cycle $C_4$ one has

$$\int_{C_4} I_{4,i} = m_i - \frac{1}{2} p_i , \quad m_i, p_i \in \mathbb{Z} .$$

(A.7)

Acknowledgements: J.-P. D. has benefitted from discussions with D. Lüst, H. P. Nilles and B. A. Ovrut. This work was partially supported by the Swiss National Science Foundation, by the European Union under TMR contract ERBFMRX-CT96-0045 and by the Swiss Office for Education and Science.
REFERENCES

1. P. Hořava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. B460 (1996) 506, hep-th/9510209.

2. P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, Nucl. Phys. B475 (1996) 94, hep-th/9603142.

3. S.P. de Alwis, *Anomaly cancellation in M-theory*, Phys. Lett. B392 (1997) 332, hep-th/9609211.

4. C. Vafa and E. Witten, *A one-loop test of string duality*, Nucl. Phys. B447 (1995) 261, hep-th/9505053.

5. M.J. Duff, J.T. Liu and R. Minasian, *Eleven dimensional origin of string/string duality: a one loop test*, Nucl. Phys. B452 (1995) 261, hep-th/9506126.

6. E. Witten, *Five-branes and M-theory on an orbifold*, Nucl. Phys. B463 (1996) 383, hep-th/9512219. *Five-brane effective action in M-theory*, J. Geom. Phys. 22 (1997) 103, hep-th/9610234.

7. E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. 76B (1978) 409.

8. E. Dudas and J. Mourad, *On the strongly coupled heterotic string*, Phys. Lett. B400 (1997) 71, hep-th/9701048.

9. J.O. Conrad, *Brane tensions and coupling constants from within M-theory*, Phys. Lett. B241 (1998) 119, hep-th/9708031.

10. J.X. Lu, *Remarks on M-theory coupling constants and M-brane tension quantizations*, hep-th/9711014.

11. M. Faux, *New consistent limits of M-theory*, hep-th/9801204. *Confluences of anomaly freedom requirements in M-theory*, hep-th/9803254.

12. T. Harmark, *Coupling constants and brane tensions from anomaly cancellation in M-theory*, Phys. Lett. B431 (1998) 295, hep-th/9802190.

13. M. Faux, D. Lüst and B.A. Ovrut, *Intersecting orbifold planes and local anomaly cancellation in M-theory*, Nucl. Phys. B554 (1999) 437, hep-th/9903028.
14. E. Witten, *On flux quantization in M-theory and the effective action*, J. Geom. Phys. **22** (1997) 1, [hep-th/9609122](https://arxiv.org/abs/hep-th/9609122).

15. E. Witten, *Strong coupling expansion of Calabi-Yau compactification*, Nucl. Phys. **B471** (1996) 135, [hep-th/9602070](https://arxiv.org/abs/hep-th/9602070).

16. A. Bilal, J.-P. Derendinger and R. Sauser, work in progress.

17. L. Alvarez-Gaumé, S. Della Pietra and G. Moore, *Anomalies and odd dimensions*, Ann. Phys. **163** (1985) 288.

18. R. Rohm and E. Witten, *The antisymmetric tensor field in superstring theory*, Ann. Phys. **170** (1986) 454.

19. L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, Nucl. Phys. **B234** (1984) 269.

   L. Alvarez-Gaumé and P. Ginsparg, *The structure of gauge and gravitational anomalies*, Ann. Phys. **161** (1985) 423, erratum-ibid **171** (1986) 233.

20. M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory*, Cambridge University Press, Cambridge, 1987, vol. 2.

21. A. Lukas, B. Ovrut and D. Waldram, *The ten-dimensional effective action of strongly coupled heterotic string theory*, Nucl. Phys. **B540** (1999) 230, [hep-th/9801087](https://arxiv.org/abs/hep-th/9801087).

22. D. Freed, J. A. Harvey, R. Minasian and G. Moore, *Gravitational anomaly cancellation for M-theory fivebranes*, Adv. Theor. Math. Phys. **2** (1998) 601, [hep-th/9803205](https://arxiv.org/abs/hep-th/9803205).

23. A. Lukas, B. Ovrut and D. Waldram, *Non-standard embedding and five-branes in heterotic M-Theory*, Phys. Rev. **D59** (1999) 106005, [hep-th/9808101](https://arxiv.org/abs/hep-th/9808101).