ON GENERALIZED LACUNARY SERIES

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Abstract. Given lacunary sequence of integers, \( n_k, n_{k+1}/n_k > \lambda > 1 \), we define a new sequence \( \{m_k\} \) formed by all possible \( l \)-wise sums \( \pm n_{k_1} \pm n_{k_2} \pm \ldots \pm n_{k_l} \). We prove if \( \lambda > \lambda_l \), then any series
\[
\sum_k c_k e^{i m_k x},
\]
with \( \sum_k |c_k|^2 < \infty \) converges almost everywhere after any rearrangement of the terms, where \( 1 < \lambda_l < 2 \) is a certain critical value. We establish this property, proving a new Khintchine type inequality
\[
\|S\|_p \leq C_{l,\lambda, p} \|S\|_2, \quad p > 2,
\]
where \( S \) is a finite sum of form (0.1). For \( \lambda \geq 3 \), we also establish a sharp rate \( p^{l/2} \) for the growth of the constant \( C_{l,\lambda, p} \) as \( p \to \infty \). Such an estimate for the Rademacher chaos sums was proved independently by Bonami [7] and Kiener [15]. In the case of \( \lambda \geq 3 \) we also establish some inverse convergence properties of series (0.1): 1) if series (0.1) converges a.e., then \( \sum_k |c_k|^2 < \infty \), 2) if it a.e. converges to zero, then \( c_k = 0 \).

1. Introduction

1.1. An historical overview. Let \( \{r_n\}_{n \geq 1} \) be the sequence of Rademacher functions \( r_n(x) = \text{sign} (\sin 2^n \pi x) \) on \( (0, 1) \). The classical Khintchine inequality states that for any \( 0 < p < \infty \) there are constants \( A_p, B_p \) such that
\[
A_p \|S\|_2 \leq \|S\|_p \leq B_p \|S\|_2,
\]
for every Rademacher polynomial \( S \). The Khintchine inequality is a well-known object in analysis and probability with various generalizations and applications. A special case of the inequality was first studied by Khintchine [14], proving the right bound of (1.1) with \( B_p = \sqrt{p/2 + 1} \) and \( p \geq 2 \). Further study of the inequality were given by Littlewood [17], Paley and Zygmund [21]. Let \( A_p \) and \( B_p \) denote the best constants, for which inequality (1.1) holds. It is trivial that \( A_p = 1 \) if \( 2 \leq p < \infty \) and \( B_p = 1 \) for all \( 0 < p \leq 2 \), while it took the work of many mathematicians to settle all the other cases. Stechkin [23] computed \( B_p \) for even integers \( p \geq 4 \), which then extended for all real numbers \( p \geq 3 \) by Young [28]. Then solving a long-standing problem of Littlewood (see. [17]) Szarek [24]...
proven that $A_1 = 1/\sqrt{2}$. Haagerup in [9] introduced a new method, computing the sharp constants in remaining cases and recovering the prior results newly.

For a given integer $l \geq 1$ denote by $D(l)$ the set of integers $m$, permitting a dyadic representation

$$m = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_l},$$

and let

$$w_m(x) = r_{k_1}(x) \ldots r_{k_l}(x)$$

be the corresponding Walsh function. In some literature the system $\{w_m : m \in D(l)\}$ is called Rademacher chaos of order $l$ ([1, 19]). This subsystem of the Walsh functions has been investigated from different point view: boolean functions, theory of orthogonal series, probability theory [1, 8, 19]. It has specific properties that the complete system of Walsh functions doesn’t. The following well-known inequality proved independently by Bonami [7] and Kiener [15] (see also [8] chap. 9, [6] chap. 7) provides a generalization of the Khintchine classical inequality for the Rademacher chaos polynomials

$$S(x) = \sum_{m \in D(l)} a_m w_m(x),$$

where $a_m$ are real numbers. It plays significant role in the theory of boolean functions (see [8], chap. 9).

**Theorem A (Bonami-Kiener).** For any integers $l \geq 2$ and any sum in (1.4) it holds the bound

$$\|S\|_p \leq (p - 1)^{l/2}\|S\|_2, \quad p > 2.$$  

(1.5)

It was also proved in [7] that the constant growth in (1.5) is optimal in the sense that for any $p > 2$ there exists a sum (1.4) such that $\|S\|_p \geq C_l p^{l/2}\|S\|_2$ with a constant $C_l > 0$, depending only on $l$. Using a standard argument one can deduce from (1.5) the bound

$$\|S\|_p \leq C_{p,q,l}\|S\|_q,$$

(1.6)

for any Rademacher chaos (1.5), where $1 \leq q < p < \infty$. In contrast to classical Rademacher case (when $l = 1$) to the best of our knowledge the optimal constant that can be in (1.6), is not known for any combination of the parameters $p > q$. Some estimates of the optimal constant of (1.6) one can find in papers [8, 10, 11, 16].

The following definition is well known in the theory of orthogonal series.

**Definition 1.1.** An orthogonal system $\{\phi_n, n = 1, 2, \ldots\} \subset L^2(0, 1)$ is said to be a convergence system, if the condition

$$\sum_{n=1}^{\infty} a_n^2 < \infty,$$

(1.7)
implies almost everywhere convergence of orthogonal series
\[
\sum_{n=1}^{\infty} a_n \phi_n(x).
\] (1.8)

If (1.8) converges a.e. after any rearrangement of the terms, then we say \( \{ \phi_n \} \) is an unconditional convergence system.

It is well-known that the trigonometric and Walsh systems are convergence systems (see [3, 5]), but none of them is an unconditional convergence system (see [20, 25, 26]). Moreover, Ul’yanov [25] and Olevskii [20] proved that unconditionality fails for any complete orthonormal system. According to a classical result of Stechkin [23] (see also [13], chap. 9.4), if an orthonormal system \( \{ \phi_n \} \) satisfies the Khintchine type inequality
\[
\left\| \sum_{k=1}^{n} a_k \phi_k \right\|_p \leq c \left\| \sum_{k=1}^{n} a_k \phi_k \right\|_2
\]
for some \( p > 2 \), then it is an unconditional convergence system. So from inequality (1.5) it follows that the Rademacher chaos of any order is an unconditional convergence system.

Inverse convergence properties of orthonormal systems give characterization of the coefficients \( \{ a_n \} \) based on certain convergence conditions on series (1.8). Classical examples of orthonormal systems having inverse properties are Rademacher and lacunary trigonometric systems. It is well-known if the Rademacher series
\[
\sum_{n=1}^{\infty} a_n r_n(x)
\] (1.9)
converges on a set of positive measure, then the coefficients satisfy (1.7) (Kolmogorov, [12], chap. 4.5), and if (1.9) converges to zero on a set of measure \( > 1/2 \), then \( a_n = 0 \) for all \( n \) (Stechkin-Ul’yanov [27]).

The Rademacher chaos in the context of inverse properties was first considered by Astashkin and Sukhanov [2] (see also [1], chap. 6).

**Theorem B** (Astashkin-Sukhanov, [2]). Let \( \{ m_k \} \) be the increasing numeration of \( D(l) \). If series
\[
\sum_{k\geq1} a_k w_{m_k}(x)
\] (1.10)
converges in measure, then
\[
\sum_{k\geq1} |a_k|^2 < \infty.
\]

**Theorem C** (Astashkin-Sukhanov, [2]). If series (1.10) converges to zero on a set \( E \subset (0,1) \) of measure \( |E| > 1 - 2^{-l} \), then \( a_k = 0 \) for all \( k \geq 1 \).

It was also shown in [2] the sharpness of the condition \( |E| > 1 - 2^{-l} \) in Theorem C. More precisely, it was given an example of non-trivial series (1.9), which terms vanish on a set of measure \( 1 - 2^{-l} \).
1.2. **Generalized lacunary trigonometric sums.** A sequence of positive integers \( \mathcal{N} = \{n_k: k = 1, 2, \ldots\} \) is said to be \( \lambda \)-lacunary if
\[
n_{k+1}/n_k > \lambda > 1.
\]
The trigonometric functions
\[
\{e^{\pm in_k x}: k = 1, 2, \ldots\}
\]
corresponding to a \( \lambda \)-lacunary \( \{n_k\} \) have many common properties with Rademacher functions. The analogue of the Khintchine inequality (1.1) was obtained by Zygmund ([29], chap. 5.8) and here again the optimal constant \( B_p \) satisfy the relation \( B_p \sim \sqrt{p} \) as \( p \to \infty \). Zygmund [30, 31] also proved inverse properties for lacunary trigonometric series. Namely, if the lacunary sums
\[
\sum_{k=-m}^m c_k e^{in_k x}, \quad m = 1, 2, \ldots, \quad (n_{-k} = n_k)
\]
converge on a set \( E \) of positive measure, then \( \sum_k |c_k|^2 < \infty \), if sums (1.11) converge to zero on \( E \), then \( c_k = 0 \).

To the best of our knowledge the analogue of Rademacher chaos for lacunary trigonometric systems was not previously considered.

**Definition 1.2.** Let \( \mathcal{N} = \{n_k: k = 1, 2, \ldots\} \) be an increasing sequence of integers, \( 1 \leq n_1 < n_2 < \ldots \). For \( l \geq 2 \) denote by \( \mathcal{N}(l) \) the set of integers \( m \in \mathbb{Z} \), having the representation
\[
m = \varepsilon_1 n_{k_1} + \varepsilon_2 n_{k_2} + \ldots + \varepsilon_l n_{k_l},
\]
where \( \varepsilon_j = \pm 1 \) and \( k_1 > k_2 > \ldots > k_l \), and let \( \mathcal{N}^*(l) = \cup_{1 \leq s \leq l} \mathcal{N}(s) \).

We will consider the \( \mathcal{N}(l) \) for a \( \lambda \)-lacunary sequences \( \mathcal{N} \). The corresponding generalized lacunary system
\[
\{e^{imx}: m \in \mathcal{N}(l)\}
\]
will serve as an analogue of Rademacher chaos of order \( l \). Let \( \lambda_l > 1 \) be the single solution of the equation
\[
x^{l-1} = x^{l-2} + \ldots + 1, \quad x > 1,
\]
where \( l \geq 2 \) is an integer.

**Theorem 1.1.** Let \( l \geq 2, \lambda > \lambda_l \) and \( \mathcal{N} \) be a \( \lambda \)-lacunary sequence of integers. Then for any finite sum
\[
S(x) = \sum_{m \in \mathcal{N}(l)} c_m e^{2\pi imx},
\]
it holds the bound
\[
\|S\|_p \leq C_{l, \lambda, p}\|S\|_2, \quad p > 2.
\]
Corollary 1.1. Let \( l \geq 2, \lambda > \lambda_l \) and \( \mathcal{N} \) be a \( \lambda \)-lacunary sequence. Then the trigonometric functions (1.13) form an unconditional convergence system. Namely, for any increasing sequence of integer sets \( G_n \subset \mathcal{N}(l) \) the sums
\[
\sum_{m \in G_n} c_m e^{imx}, \quad n = 1, 2, \ldots,
\]
converge a.e. as \( n \to \infty \), whenever
\[
\sum_{m \in \mathcal{N}(l)} |c_m|^2 < \infty.
\]
Conversely, the \( \lambda_l \) is the critical bound here. That is for any \( l \geq 2 \) there exists a \( \lambda_l \)-lacunary sequence \( \mathcal{N} \) and a choice of coefficients (1.18) such that for certain increasing sequence of integer sets \( G_n \subset \mathcal{N}(l) \) sums (1.17) diverge a.e. 

Remark 1.1. Note that Corollary 1.1 immediately follows from Theorem 1.1 combined with the above mentioned result of Stechkin [23]. So the condition \( \lambda > \lambda_l \) is sharp also in Theorem 1.1. Moreover, one can say that for a \( \lambda_l \)-lacunary sequence \( \mathcal{N} \) we have
\[
\sup_S \|S\|_p / \|S\|_2 = \infty
\]
for any \( p > 2 \). Here sup is taken over all finite nontrivial sums (1.15).

Remark 1.2. Observe that \( 1 < \lambda_l < 2 \) and \( \lambda_l \not\to 2 \), as \( l \to \infty \). Also, we have \( \lambda_2 = 1 \) and so in the case \( l = 2 \) results of Theorem 1.1 and Corollary 1.1 hold for any \( \lambda > 1 \).

Along with \( D(l) \) (see (1.2)) we consider also the enlarged family \( D^*(l) = \bigcup_{1 \leq s \leq l} D(s) \) of integers \( m \), permitting a dyadic representation
\[
m = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_s}, \quad 1 \leq s \leq l.
\]
(1.19)
Hence, as a particular case of Corollary 1.1 we can claim the following.

Corollary 1.2. Let \( l \geq 1 \) and \( \{m_k\} \) be a numeration of the integers in \( D^*(l) \). Then \( \{e^{im_kx}\} \) is an unconditional convergence system.

For 3-lacunary sequences we also prove the following full version of the Bonami-Kiener inequality (1.5) for the generalized lacunary sums.

Theorem 1.2. If \( l \geq 2 \) and \( \mathcal{N} \) is a 3-lacunary sequence, then for any finite sum (1.15) we have
\[
\|S\|_p \leq \left(8(p-1)\right)^{1/2} \|S\|_2.
\]
(1.20)
We will also consider inverse properties of generalized lacunary systems with the following definition of lacunarity.

Definition 1.3. Let \( l \geq 2 \) and \( \mathcal{N} = \{n_k : k = 1, 2, \ldots\} \) be an increasing sequence of integers. Denote by \( \mathcal{N}_+(l) \) the set of integers \( m \in \mathbb{Z} \), having a representation
\[
m = n_{k_1} + n_{k_2} + \ldots + n_{k_t},
\]
where \( k_1 > k_2 > \ldots > k_l \) and let \( \mathcal{M}_+^*(l) = \cup_{1 \leq s \leq l} \mathcal{M}_+^*(l) \).

The next results will be stated in the setting of general matrix summation methods \( T = \{ t_{n,m} : m \in \mathbb{Z}, n = 1, 2, \ldots \} \), satisfying the relations

\[
\begin{align*}
1) \quad & \{ t_{n,m} : m \in \mathbb{Z} \} \text{ is finite for any } n = 1, 2, \ldots, \\
2) \quad & |t_{n,m}| \leq M, \\
3) \quad & \lim_{n \to \infty} t_{n,m} = 1 \text{ for any } m = 1, 2, \ldots .
\end{align*}
\]

(1.21)

Here and everywhere below we say a numerical sequence \( \{ b_n \} \) is finite if it has finite number of non-zero terms. We say that a numerical series \( \sum_{m \in \mathbb{Z}} a_m \) is \( T \)-summable to \( S \) if

\[
\lim_{n \to \infty} \sum_m t_{n,m} a_m = S.
\]

An example of such a summation method can be given by an increasing sequence of finite integer sets \( G_n \subset \mathbb{Z} \). Then the convergence of the sums \( \sum_{m \in G_n} b_n \) is the summability, corresponding to the matrix

\[
t_{n,m} = \begin{cases} 
1 & \text{if } m \in G_n, \\
0 & \text{otherwise. }
\end{cases}
\]

(1.22)

Thus the convergence of series after some rearrangement of the terms is a case of such summability. In the next results we will always let \( T = \{ t_{m,n} \} \) be a summation method, satisfying 1)-3).

**Theorem 1.3.** Let \( l \geq 2 \), \( \mathcal{M} \) be a \( \lambda \)-lacunary sequence with \( \lambda > \lambda_{l+1} \), and suppose \( T = \{ t_{m,n} \} \) satisfies 1)-3). If

\[
\sup_n \left| \sum_{m \in \mathcal{M}_+^*(l)} t_{n,m} c_m e^{imx} \right| < \infty
\]

on a set \( E \) of measure

\[
|E| > \alpha(l, \lambda),
\]

(1.23)

then \( \sum_{m \in \mathcal{M}_+^*(l)} |c_m|^2 < \infty \). Here \( \alpha(l, \lambda) \in (0,1) \) is a constant, depending only on \( l \) and \( \lambda \).

**Theorem 1.4.** Under the conditions of Theorem 1.3, if the sums

\[
\sum_{m \in \mathcal{M}_+^*(l)} t_{n,m} c_m e^{imx}
\]

converge to zero on a set \( E \) of measure (1.23), then \( c_m = 0 \) for all \( m \in \mathcal{M}_+^*(l) \).
Remark 1.3. Theorem 1.4 provides a uniqueness property of rare trigonometric series, which is not true for the full trigonometric series. Moreover, Menshov [18] was the first who constructed a non-trivial trigonometric series, converging to zero almost everywhere. Such series in the theory of orthogonal series are called null-series.

A significant case of Theorems 1.3 and 1.4 is when the summation matrix coincides with (1.22). Thus we can state the following result, where it is important that the sequence \( \{m_k\} \) is not required to be increasing.

**Corollary 1.3.** Let \( \{m_k\} \) be a numeration of integers \( D^*(l) \) (not necessarily increasing). Then the series
\[
\sum_k c_k e^{im_k x}
\]
converges a.e. if and only if
\[
\sum_k |c_k|^2 < \infty.
\]

In fact, the proofs of both Theorems 1.3 and 1.4 immediately follows from Lemma 4.1 that is an inverse Parseval inequality. Proving a similar lemma, such results we establish for the Walsh functions too.

**Theorem 1.5.** Let \( \{m_k\} \) be a sequence of integers \( D^*(l) \) and a summation \( T = \{t_{m,n}\} \) satisfy 1)-3). If Walsh sums satisfy
\[
\sup_n \left| \sum_{k=1}^{\infty} t_{n,k} a_k w_{m_k}(x) \right| < \infty
\]
on a set \( E \subset (0, 1) \) of measure
\[
|E| > 1 - 2^{-l},
\]
then it follows that \( \sum_k a_k^2 < \infty \).

**Theorem 1.6.** Under the hypothesis of Theorem 1.5, if the Walsh sums
\[
\sum_k t_{n,k} a_k w_{m_k}(x)
\]
converge to zero on a set \( E \subset (0, 1) \) of measure (1.24) as \( n \to \infty \), then \( a_k = 0 \) for all \( k = 1, 2, \ldots \).

Note that both Theorems 1.5 and 1.6 use the sequence \( D^*(l) \). Taking \( D(l) \) in Theorem 1.6, the bound (1.24) may be improved.

**Theorem 1.7.** Let \( \{m_k\} \) be a sequence of integers \( D(l) \) and a summation \( T = \{t_{m,n}\} \) satisfy 1)-3). If the Walsh sums (1.25) converge to zero on a set \( E \subset (0, 1) \) of measure
\[
|E| > 1 - 2^{-l},
\]
then \( a_k = 0 \) for all \( k = 1, 2, \ldots \).
1.3. Final remarks.

Remark 1.4. Theorem 1.5 is a generalization of Theorem B. First, Theorem 1.5 is stated in the setting of general summation method and second, the convergence in measure is replaced by the boundedness of sums (1.25) on a set of measure (1.24). The method of the proof of Theorem 1.5 is quite different from the one used in Theorem B, where authors apply the decoupling technique that is applicable only for the series with the canonical order of the Walsh functions. In fact, our proof of Theorem 1.5 (as well as Theorem 1.6) is elementary and immediately follows from an inverse Parseval type inequality proved in Section 5 (Lemma 5.1).

Remark 1.5. In the proof of Theorem 1.7 we use the argument of Astashkin-Sukhanov [2]. The main ingredient here are Lemmas 6.1 and 6.2 proved in Section 6.

Remark 1.6. In fact, Theorem C of [2] was proved in more general setting, instead of the Walsh functions (1.3) considering the functions
\[ g_m(x) = \phi \left( 2^{k_1}x \right) \cdots \phi \left( 2^{k_j}x \right) \text{ (see (1.2))}, \]
where \( \phi \) is a 1-periodic finite-valued function such that
\[ \phi \left( x + \frac{1}{2} \right) - \phi(x) \neq 0 \text{ for a.e. } x \in (0, 1). \]

The same generalization permits also Theorem 1.7, it just needed to replace \( \pm 2^l \) in Lemma 6.1 by a non-zero number. One can apply this general results, taking \( \phi(x) \) to be either the Rademacher first function \( r_1(x) \) or \( \sin(2\pi x) \).

Remark 1.7. The Littlewood-Paley inequality is a basic tool in the proof of Theorem 1.1. Moreover, it is shown that the constants in (1.16) and (2.1) obey the estimate
\[ C_{l,\lambda,p} \leq c(l, \lambda) \cdot C_p. \]

The optimal rate of growth of the constant in (2.1) is an open problem yet. It is only known the bound \( \sqrt{p} \lesssim C_p \lesssim p \log p \) due to Pichorides [22], where it is also conjectured that \( C_p \lesssim \sqrt{p} \). Hence we conclude, in fact, (1.16) holds with a constant \( c(l, \lambda)p \log p \).

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2. Proof of Theorems 1.1 and 1.2

The proof of Theorem 1.1 uses the approach given in [19] in the case of Rademacher chaos. The main ingredient here is the Littlewood-Paley inequality
\[ \left\| \sum_{k \geq 1} c_k e^{2\pi ikx} \right\|_p \leq C_p \left\| \left( \sum_{j \geq 0} \left| \sum_{k \in [2^j, 2^{j+1})} c_k e^{2\pi ikx} \right|^2 \right]^{1/2} \right\|_p. \] (2.1)
The following standard properties of lacunary sequences are either well-known (see for example [29], chap. 5.7, or [4], chap. 11) or easy to verify. For an integer $m$ from (1.12) we denote $\text{head}(m) = \varepsilon_k n_k$, and let

(P1) If $l \geq 2$ and $\mathcal{N} = \{n_k\}$ is a $\lambda$-lacunary sequence with $\lambda > \lambda_l$, then

$$\mathcal{G}_j = \{m \in \mathbb{N}^*(l) : \text{head}(m) = \text{sign}(j) \cdot n_j \subset \pm(a_l \cdot n_j, b_l \cdot n_j),$$

where $b_l > a_l > 0$ are constants $j \in \mathbb{Z} \setminus \{0\}$.

(P2) If $\{n_k\}$ is $3$-lacunary, then every integer $m$ may have at most one representation

$$\varepsilon_1 n_{j_1} + \ldots + \varepsilon_s n_{j_s},$$

where $s = 1, 2, \ldots$, $\varepsilon_j = \pm 1$, $j_1 > \ldots > j_s$. Besides, 0 can not be written in the form (2.2).

(P3) Let $l \geq 2$ and $\{n_k\}$ be a $\lambda$-lacunary sequence with $\lambda > \lambda_{l+1}$. Then every integer $m$ has at most one representation

$$m = n_{j_1} + \ldots + n_{j_s},$$

where $j_1 > \ldots > j_s$, $1 \leq s \leq l$.

(P4) Let $l \geq 2$ and $\{n_k\}$ be a $\lambda$-lacunary sequence with $\lambda > \lambda_{l+1}$. Then every integer $m$ has at most $d(l, \lambda)$ number of representation

$$m = n_{j_1} + \ldots + n_{j_s} - n_{k_1} - \ldots - n_{k_t},$$

where

$$0 \leq s, t \leq l,$$

$$\{j_1, \ldots, j_s\} \cap \{k_1, \ldots, k_t\} = \emptyset,$$

$$j_1 > \ldots > j_s, \; k_1 > \ldots > k_t,$$

and $d(l, \lambda)$ is an integer depending only on $l$ and $\lambda$.

**Proof of Theorem 1.1.** We will use induction. Suppose we have already proved (1.16) for $l = s - 1$. Let us proceed the case of $l = s$. Using (P1) and the Littlewood-Paley inequality (2.1), for any polynomial (1.15) we can write

$$\|S\|_p \leq C \left\| \left( \sum_j \left| \sum_{m \in \mathcal{G}_j} c_m e^{2\pi i m x} \right|^2 \right)^{1/2} \right\|_p,$$

where $C = C_{l, \lambda, p}$. If $m \in \mathcal{G}_j$, $m = \varepsilon_j n_j + t_m$, where

$$\varepsilon_j = \pm 1, \quad t_m = \varepsilon_j n_{j_1} + \ldots + \varepsilon_{l-1} n_{j_{s-1}}, \; j > j_1 > \ldots > j_{s-1}.$$
Substituting it in (2.3) then applying Minkowski’s inequality in $L^{p/2}$, we obtain
\[
\|S\|_{p} \leq C \left( \sum_{j} \left( \sum_{m \in G_{j}} c_{m} e^{2\pi i t_{m}x} \right)^{2} \right)^{1/2} \leq C \left( \sum_{j} \left( \sum_{m \in G_{j}} c_{m} e^{2\pi i t_{m}x} \right)^{2} \right)^{1/2} = C \left( \sum_{j} \left( \sum_{m \in G_{j}} c_{m} e^{2\pi i t_{m}x} \right)^{2} \right)^{1/2}. \]

Since \( \{t_{m} : m \in G_{j}\} \subset \mathcal{N}(s-1) \) and \( \lambda > \lambda_{s} > \lambda_{s-1} \), applying the induction assumption, we obtain
\[
\left\| \sum_{m \in G_{j}} c_{m} e^{2\pi i t_{m}x} \right\|_{p} \leq C'_{L_{\lambda,p}} \left( \sum_{m \in G_{j}} |c_{m}|^{2} \right)^{1/2}
\]
and then \( \|S\|_{p} \leq C'_{L_{\lambda,p}} \|S\|_{2} \). This completes the proof of Theorem 1.1. \qed

**Lemma 2.1.** If \( \{n_{k}\} \) is a 3-lacunary sequence, then
\[
\int_{0}^{1} e^{2\pi i m(x+u)} \prod_{k=1}^{s} \cos(2\pi n_{k_{j}} u) du = \begin{cases} 2^{-s} e^{2\pi i m x}, & \text{if } m = \varepsilon_{1} n_{k_{1}} + \varepsilon_{2} n_{k_{2}} + \ldots + \varepsilon_{s} n_{k_{s}}, \\ 0, & \text{otherwise}, \end{cases}
\]
where \( \varepsilon_{j} = \pm 1 \), \( k_{1} > k_{2} > \ldots > k_{s} \).

**Proof.** Using the product of two cosines formula repeatedly, we get the formula
\[
\prod_{k=1}^{s} \cos(2\pi n_{k_{j}} x) = 2^{-s} \sum_{\varepsilon_{j}=\pm 1} \cos 2\pi (\varepsilon_{1} n_{k_{1}} + \varepsilon_{2} n_{k_{2}} + \ldots + \varepsilon_{s} n_{k_{s}}) x, \tag{2.4}
\]
where the summation is taken over all combinations of \( \varepsilon_{j} = \pm 1 \). Thus if \( m \neq \varepsilon_{1} n_{k_{1}} + \varepsilon_{2} n_{k_{2}} + \ldots + \varepsilon_{s} n_{k_{s}} \) for any choice of \( \varepsilon_{j} = \pm 1 \), then by orthogonality we obtain
\[
\int_{0}^{1} e^{2\pi i m(x+u)} \prod_{k=1}^{s} \cos(2\pi n_{k_{j}} u) du = 0.
\]
Now let \( m = \varepsilon'_1 n_{k_1} + \varepsilon'_2 n_{k_2} + \ldots + \varepsilon'_s n_{k_s} \), for some \( \varepsilon'_j = \pm 1 \). By (P2), such a representation of \( m \) is unique. Thus, using (2.4), we conclude

\[
\int_0^1 e^{2\pi i m(x+u)} \prod_{k=1}^s \cos(2\pi n_{k_j} u) du = 2^{-s} \int_0^1 e^{2\pi i m(x+u)} (\cos 2\pi mu + \cos(-2\pi mu)) du \\
= 2^{1-s} \int_0^1 e^{2\pi i m(x+u)} \cos(2\pi mu) du \\
= 2^{1-s} e^{2\pi imx} \int_0^1 \cos^2(2\pi mu) du \\
+ i \cdot 2^{1-s} e^{2\pi imx} \int_0^1 \sin(2\pi mu) \cos(2\pi mu) du \\
= 2^{-s} e^{2\pi imx},
\]

completing the proof. \( \square \)

Proof of Theorem 1.2. By (P2) each \( m \in \mathcal{N}(l) \) has a unique representation (1.12) and so \( m \) can be uniquely determined by a set \( A = \{j_1, \ldots, j_l\} \subset \mathbb{N} \), and a sequence \( \varepsilon = \{\varepsilon_j = \pm 1; j = 1, 2, \ldots, l\} \). Thus we can consider \( m = m(A, \varepsilon) \) as a function on \( A \) and \( \varepsilon \). Thus any finite sum may be written in the form

\[
S(x) = \sum_{m \in \mathcal{N}(l)} c_m e^{2\pi imx} = \sum_A \sum_\varepsilon c_m e^{2\pi imx}.
\]

For \( t \in (0, 1) \) we consider also the sum

\[
S_t(x) = \sum_A \left( \sum_\varepsilon c_m e^{2\pi imx} \right) w_A(t),
\]

where

\[
w_A(t) = \prod_{j \in A} r_j(t)
\]

denotes the Walsh function corresponding the set \( A = \{j_1, \ldots, j_l\} \). Since by Hölder’s inequality

\[
\left| \sum_{A} \sum_\varepsilon c_m e^{2\pi imx} \right|^2 \leq 2^l \sum_m |c_m|^2,
\]

from the Rademacher chaos bound (1.5) we obtain

\[
\int_0^1 \int_0^1 |S_t(x)|^p dt dx \leq (p - 1)^{p/2} \left( 2^l \sum_m |c_m|^2 \right)^{p/2}.
\]
Thus there exists a $t_0 \in (0, 1)$ such that
\[ \int_0^1 |S_{t_0}(x)|^p dx \leq (p - 1)^{p/2} \cdot \left(2^l \sum_m |c_m|^2\right)^{p/2}. \quad (2.5) \]

Consider the Riesz product
\[
S(x) = 2^l \int_0^1 S_{t_0}(x + u) g(u) du.
\]
Then, applying Jensen’s inequality, (2.5) and (2.6), we easily get
\[
\int_0^1 \left| \frac{S(x)}{2^l} \right|^p dx \leq \int_0^1 \int_0^1 |S_{t_0}(x + u)|^p g(u) du dx \leq (p - 1)^{p/2} \cdot \left(2^l \sum_m |c_m|^2\right)^{p/2}.
\]
This implies (1.20).

3. Proof of Corollary 1.1

As it was stated in the introduction that the first part of Corollary 1.1 immediately follows from Theorem 1.1. So we will only give here a counterexample showing sharpness of $\lambda_1$ both in Theorem 1.1 and Corollary 1.1. According to classical results stated in the introduction there is a trigonometric series
\[
\sum_{m=1}^{\infty} c_m e^{inx}, \quad \sum_{m=1}^{\infty} |c_m|^2 < \infty,
\]
which is divergent after some rearrangement of the terms. Thus it is enough to prove the following lemma.
Lemma 3.1. There exists a $\lambda_l$-lacunary sequence $\mathcal{N}$ such that $\mathcal{N}(l)$ contains all the integers $m \geq 3^l$.

Proof. For every $m \geq 3^l$ consider the group of integers $\{n_k(m) : k = 1, 2, \ldots, l\}$ defined by

$$
n_k(m) = \left[10^{md(l)} \lambda_l^k\right] + 3^k, \quad k = 1, 2, \ldots, l - 1,
$$

$$
n_l(m) = m + n_{l-1}(m) + \ldots + n_1(m).
$$

Recalling that $\lambda_l$ satisfies equation (1.14), a rough calculation show that all the groups together, i.e.

$$
\mathcal{N} = \{n_k(m) : 1 \leq k \leq l, m \geq 3^l\}
$$

form a $\lambda_l$-lacunary sequence. Besides, every $m \geq 3^l$ is written in the form

$$
m = n_l(m) - n_{l-1}(m) - \ldots - n_1(m)
$$

that means that $\mathcal{N}(l)$ contains all the integers $m \geq 3^l$. So the proof is complete. $\square$

4. Proofs of Theorems 1.3 and 1.4

Lemma 4.1. Let $l \geq 2$ and $\mathcal{N}$ be a $\lambda$-lacunary sequence with $\lambda > \lambda_{l+1}$, then for any finite sequence $b = \{b_k : k \in \mathcal{N}_+^l(l)\}$ and a set $E$ of measure (1.23) we have

$$
\int_E \left| \sum_{m \in \mathcal{N}_+^l(l)} b_m e^{2\pi i m x} \right|^2 > c \sum_{m \in \mathcal{N}_+^l(l)} |b_m|^2
$$

Proof. We have

$$
S = \int_E \left| \sum_{m \in \mathcal{N}_+^l(l)} b_m e^{2\pi i m x} \right|^2 = |E| \sum_{m \in \mathcal{N}_+^l(l)} |b_m|^2
$$

$$
- \sum_{t, s \in \mathcal{N}_+^l(l), t \neq s} b_t b_s \int_{E^c} e^{2\pi i (t-s) x} dx
$$

$$
= S_1 - S_2. \quad (4.1)
$$

We write $E^c$ in the last integral, since the integrals of $e^{2\pi i (t-s) x}$ over $(0, 1)$ are zero. Each $m \in \mathcal{N}_+^l(l)$ has a unique representation

$$
m = n_{j_1} + \ldots + n_{j_s}, \quad 1 \leq s \leq l, \quad (4.2)
$$

according to property (P3). This defines a one to one mapping $\tau$ assigning a subsets $A = \{j_1, \ldots, j_s\} \subset \mathbb{N}$ to $m$ by (4.2). If $t = \tau(A)$ and $s = \tau(B)$, then we define

$$
t \wedge s = \tau(A \cap B), \quad t \vee s = \tau(A \cup B), \quad (4.3)
$$
where in the last notation we additionally suppose that $A \cap B = \emptyset$ and $\#(A \cup B) \leq l$. For $m \in \mathfrak{N}_+^l(l)$, denote by $\#(m)$ the number of terms in representation (4.2). The second sum in (4.1) can be written in the form

$$S_2 = \sum_{u,v,z} b_{u,v,z} \int_{E^c} e^{2\pi i(u-v)x} \, dx,$$

where the summation is taken over all combinations of numbers $u, v, z \in \mathfrak{N}_+^l(l)$, satisfying

$$\#(z) \leq l - 1,$$

$$u \wedge v = u \wedge z = v \wedge z = 0,$$

$$0 \leq \#(u), \#(v) \leq l - \#(z), \quad u + v > 0.$$  

Using these notations and Hölder’s inequality, we obtain

$$|S_2| \leq \sum_{z: \#(z) \leq l - 1} \left( \sum_{u,v: \text{satisfying (4.5),(4.6)}} |b_{u,v,z}|^2 \right)^{1/2} \times \left( \sum_{u,v: \text{satisfying (4.5),(4.6)}} \left| \int_{E^c} e^{2\pi i(u-v)x} \, dx \right|^2 \right)^{1/2}. \tag{4.7}$$

From (P4) it follows that each integer $m$ has at most $d(l, \lambda)$ number of representation $u - v$, where $u, v$ satisfy (4.5), (4.6). So the set of functions $e^{2\pi i(u-v)x}$ are a union of $d(l, \lambda)$ orthonormal systems. Thus, by Bessel inequality we get

$$\sum_{u,v: \text{satisfying (4.5),(4.6)}} \left| \int_{E^c} e^{2\pi i(u-v)x} \, dx \right|^2 \leq d(l, \lambda) |E^c| < 2^{-l-1}, \tag{4.8}$$

where the last inequality is obtained if we choose $\alpha(l, \lambda) = 1 - (d(l, \lambda) \cdot 2^{l+1})^{-1}$ in (1.23). On the other hand for a fixed $z$, $\#(z) \leq l - 1$, we can write

$$\sum_{u,v: \text{satisfying (4.5),(4.6)}} |b_{u,v,z}|^2 \leq \sum_{u: u \wedge z = 0, \#(u) + \#(z) \leq l} |b_{u,z}|^2 \times \sum_{v: v \wedge z = 0, \#(v) + \#(z) \leq l} |b_{v,z}|^2$$

$$= \left( \sum_{u: u \wedge z = 0, \#(u) + \#(z) \leq l} |b_{u,z}|^2 \right)^2. \tag{4.9}$$

Then, any integer can have at most $2^l$ number of representation $u \vee z$, where $u, z$ satisfy (4.4)-(4.5). Thus from (4.7) and (4.8) we obtain

$$|S_2| < 2^{-l-1} \sum_{z: \#(z) \leq l - 1} \sum_{u: u \wedge z = 0, \#(u) + \#(z) \leq l} |b_{u,z}|^2 \leq \frac{1}{2} \sum_{m \in \mathfrak{N}_+^l(l)} |b_m|^2,$$
where we use the fact that the number of representation $s = u \lor z$ of an integer $1 \leq s \leq l$ doesn’t exceed $2^l$. Therefore,

$$S \geq S_1 - |S_2| > |E| \cdot \sum_{m \in \mathcal{N}_+^*(l)} |b_m|^2 - \frac{1}{2} \cdot \sum_{m \in \mathcal{N}_+^*(l)} |b_m|^2 > c \sum_{m \in \mathcal{N}_+^*(l)} |b_m|^2.$$

\[\Box\]

**Proof of Theorem 1.3.** Without loss of generality we can suppose that

$$\int_E \left| \sum_{m \in \mathcal{N}_+^*(l)} t_{n,m} b_m e^{2\pi i m x} \right|^2 < M, \quad n = 1, 2, \ldots,$$

as $x \in E$. Then, from Lemma 4.1 we obtain

$$\sum_{m \in \mathcal{N}_+^*(l)} |t_{n,m}|^2 |b_m|^2 \lesssim \left( \int_E \sum_{m \in \mathcal{N}_+^*(l)} t_{n,m} b_m e^{2\pi i m x} \right)^2 < M$$

and, applying (1.21), we conclude $\sum_{m \in \mathcal{N}_+^*(l)} |b_m|^2 \leq M$. \[\Box\]

**Proof of Theorem 1.4.** Using the Egorov theorem we can suppose that the sums converge to zero uniformly on $E$ of measure (1.24). Thus, applying Lemma 4.1, for bigger enough integers $n$ we can write

$$\varepsilon > \int_E \left| \sum_{m \in \mathcal{N}_+^*(l)} t_{n,m} b_m e^{2\pi i m x} \right|^2 \geq c \sum_{m \in \mathcal{N}_+^*(l)} |t_{n,m}|^2 |b_m|^2$$

and, once again using (1.21), we conclude $b_m = 0$, $m \in \mathcal{N}_+^*(l)$. \[\Box\]

## 5. PROOFS OF THEOREMS 1.5 AND 1.6

The approach used in Section 4 may be readily applied in the proofs of Theorems 1.3 and 1.4. So we will just briefly state the proof of lemma analogous to Lemma 4.1, leaving the proofs of the theorems to the reader.

**Lemma 5.1.** Let $l \geq 2$, then for any finite sequence $b = \{b_m : m \in \mathcal{D}^*(l)\}$ and a set $E$ of measure (1.23) we have

$$\int_E \left| \sum_{m \in \mathcal{D}^*(l)} b_m w_m(x) \right|^2 > c \sum_{m \in \mathcal{D}^*(l)} |b_m|^2.$$
Define a mapping $\tau,$ assigning a set $A = \{k_1, \ldots, k_s\} \subset \mathbb{N}$ to the number $m \in D^*(l)$ in (1.19). Using the notations (4.3), the second sum in (5.1) can be written in the form

$$ S_2 = \sum_{u,v,z} b_{u\vee z}b_{v\vee z} \int_E w_u(x)w_v(x)dx, $$

where the summation is taken over all the integers $u, v, z \in D^*(l)$, satisfying (4.4)-(4.6). Using Hölder’s inequality and writing $w_u \cdot w_v = w_{u+v}$, the second sum in (5.1) may be estimated by

$$ |S_2| \leq \sum_{z: \#(z) \leq l-1} \left( \sum_{u,v: \text{satisfying (4.5), (4.6)}} |b_{u\vee z}b_{v\vee z}|^2 \right)^{1/2} \times \left( \sum_{u,v: \text{satisfying (4.5), (4.6)}} \left| \int_{E^c} w_{u+v}(x)dx \right|^2 \right)^{1/2}. $$

Let $d(m)$ denote the number of all possible representation of an integer $m$ by $u + v$, where $u, v \in D^*(l), u \wedge v = 0, \#(u), \#(v) \leq l$. Observe that $d(m) < \frac{2^{2l}}{\sqrt{l}}$ for any $m$. Indeed, such a representation is possible if $\#(m) \leq 2l$. One can calculate that the $d(m)$ takes its biggest value when $\#(m) = 2l$ and in this case we will have

$$ d(m) = \binom{2l}{l} < \frac{2^{2l}}{\sqrt{l}}. $$

Then, by Bessel’s inequality we get

$$ \sum_{u,v: \text{satisfying (4.5), (4.6)}} \left| \int_{E^c} w_{u+v}(x)dx \right|^2 \leq \frac{2^{2l}|E^c|}{\sqrt{l}} < \frac{1}{2^{2l}\sqrt{l}}. $$

Besides, we can write (4.9), and therefore

$$ |S_2| < \frac{1}{2^{l}\sqrt{l}} \sum_{z: \#(z) \leq l-1} \sum_{u: u\wedge z = 0} |b_{u\vee z}|^2 \leq \frac{1}{\sqrt{l}} \sum_{m \in D^*(l)} |b_m|^2. $$

Thus we obtain

$$ S \geq S_1 - |S_2| > |E| \cdot \sum_{m \in D^*(l)} |b_m|^2 - \frac{1}{\sqrt{l}} \cdot \sum_{m \in D^*(l)} |b_m|^2 > c \sum_k |b_k|^2. $$
6. Proof of Theorem 1.7

We will use the notation $x \oplus y = x + y \mod 1$.

**Lemma 6.1.** If $\alpha \in [0, 1)$ and an integer $m \in \mathcal{D}(l)$ has a representation (1.2), then for any $n \in \mathcal{D}(l)$ we have

$$
\sum_{\varepsilon_j=0,1} (-1)^{\varepsilon_1+\ldots+\varepsilon_l} w_n \left( \alpha \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} \right) = \begin{cases} 0 & \text{if } n \neq m, \\ \pm 2^l & \text{if } n = m. \end{cases} \tag{6.1}
$$

**Proof.** If $n \in \mathcal{D}(l)$ and $n \neq m$, then there is a term in (1.2), say $2^{k_s}$, which is not included in the dyadic representation of $n$. One can check that

$$
r_k(x \oplus 2^{-s}) = \begin{cases} r_k(x) & \text{if } k \neq s, \\ -r_k(x) & \text{if } k = s, \end{cases} \tag{6.2}
$$

and so by the definition of a Walsh function we can say that the value of

$$
w_n \left( \alpha \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_s}{2^{k_s}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} \right)
$$

doesn’t depend on $\varepsilon_s$. Thus one can conclude that each term of the sum (6.1) has its opposite pair and so the whole sum is zero. If $n = m$, then the argument of (6.2) implies

$$
w_m \left( \alpha \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_s}{2^{k_s}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} \right) = \prod_{j=1}^l r_{k_j} \left( \alpha \oplus \frac{\varepsilon_j}{2^{k_j}} \right)
$$

and so the sum in (6.1) is equal

$$
\sum_{\varepsilon_j=0,1} \prod_{j=1}^l (-1)^{\varepsilon_j} r_{k_j} \left( \alpha \oplus \frac{\varepsilon_j}{2^{k_j}} \right) = \prod_{j=1}^l \left( r_{k_j}(\alpha) - r_{k_j}(\alpha \oplus 2^{-k_j}) \right) = \pm 2^l.
$$

Lemma is proved. □

**Lemma 6.2.** If $E \subset [0, 1)$ has a measure $|E| > 1 - 2^{-l}$, then there exists a $\alpha \in [0, 1)$ such that

$$
\left\{ \alpha \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} : \varepsilon_j = 0,1 \right\} \subset E.
$$

**Proof.** Define the sets $E_j \subset [0, 1)$ by the recursive formula

$$
E_0 = E, \quad E_{j+1} = E_j \cap (2^{-k_j} \oplus E_j), \quad j = 0,1,\ldots,l - 1.
$$

One can check inductively that $|E_{k+1}| > |E_k| - 2^{k-l}$. Indeed, we have

$$
|E_1| = 1 - |E^c \cup (2^{-k_1} \oplus E_1)^c| > 1 - 2^{1-l},
$$

and continuing similarly, we will get $|E_i| > 0$. Then on can check that $x \in E_i$ implies

$$
x \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} \in E$$
for any choice of $\varepsilon_j = 0, 1$. Thus as an $\alpha$ we can arbitrary point of the nonempty set $E_l$. □

**Proof of Theorem 1.7.** Let
\[
\sum_{k=1}^{\infty} t_{n,k} a_k w_{m_k}(x) \rightarrow 0 \tag{6.3}
\]
as $n \to \infty$ on a set $E$, $|E| > 1 - 2^{-l}$. Choose $m \in D(l)$ with a representation (1.2). According to Lemma 6.2, there exists $\alpha$ such that
\[
\alpha \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} \in E
\]
for any choice of $\varepsilon_j = 0, 1$. Substituting one of these points in (6.3) and applying Lemma 6.2, we obtain
\[
t_{n,m} a_m w_m \left( \alpha \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} \right) = 0, \quad n = 1, 2, \ldots.
\]
Since $t_{n,m} \to 1$ as $n \to \infty$, applying Lemma 6.1, we find
\[
a_m \sum_{\varepsilon_j=0,1} (-1)^{\varepsilon_1 + \ldots + \varepsilon_l} w_m \left( \alpha \oplus \frac{\varepsilon_1}{2^{k_1}} \oplus \ldots \oplus \frac{\varepsilon_l}{2^{k_l}} \right) = \pm a_m 2^l = 0
\]
and so $a_m = 0$. Theorem is proved. □

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