On the $\bar{\partial}$-dressing method applicable to heavenly equation

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Abstract

The $\bar{\partial}$-dressing scheme based on local nonlinear vector $\bar{\partial}$-problem is developed. It is applicable to multidimensional nonlinear equations for vector fields, and, after Hamiltonian reduction, to heavenly equation. Hamiltonian reduction is described explicitly in terms of the $\bar{\partial}$-data. An analogue of Hirota bilinear identity for heavenly equation hierarchy is introduced, $\tau$-function for the hierarchy is defined. Addition formulae (generating equations) for the $\tau$-function are found. It is demonstrated that $\tau$-function for heavenly equation hierarchy is given by the action for $\bar{\partial}$-problem evaluated on the solution of this problem.

1 Introduction

Dispersionless integrable equations represent themselves an important class of nonlinear PDEs with various applications in physics. Recently, considerable interest has been paid to hydrodynamic type equations which arise in the general relativity, Yang-Mills theory and theory of Einstein-Weyl spaces (see e.g. [1]-[9]). Heavenly equations [1] have been studied particularly intensively. Different techniques have been used to analyze these equations and their properties.

In the present paper we apply the $\bar{\partial}$-dressing method to such equations. The $\bar{\partial}$-dressing method has been proposed in [10] and it is applicable to various classes of PDEs (see e.g. [11][12]). During the last five years it was demonstrated that the quasiclassical $\bar{\partial}$-dressing method is an effective tool to study dispersionless integrable equations (see e.g. [13]-[19]). Here we will show that the $\bar{\partial}$-dressing method based on local nonlinear vector $\bar{\partial}$-problem effectively works for class of equations for vector fields including heavenly...
2 Basic dressing scheme. Integrable equations for vector fields.

We start with nonlinear local vector \( \bar{\partial} \)-problem

\[
\bar{\partial} S^i = W^i(z, \bar{z}; S^1, \ldots, S^N), \quad i = 1, \ldots, N,
\]

where \( z \in \mathbb{C} \), bar means complex conjugation, \( \partial = \frac{\partial}{\partial z} \), and functions \( W^i(\bar{\partial}-data) \) are smooth functions of \( S^i \). Similar to standard \( \bar{\partial} \)-dressing method [10], we suggest that \( \bar{\partial} \)-data \( W^i(z, \bar{z}; S^1, \ldots, S^N) \) are equal to zero in some domain (or set of domains) \( G \) of the complex plane, thus \( \bar{\partial} \)-data are localized on a closed subset \( \mathbb{C} \setminus G \). This subset may consist of a number of curves, thus giving rise to vector nonlocal Riemann problem. In the following we will usually suggest that \( \mathbb{C} \setminus G \) is a unit disc \( D \).

In the simplest case \( N = 1 \) equation (1) defines the so-called generalized analytic functions (see e.g. [20]). Various classes of elliptic systems of the type (1) are also studied quite well (see e.g. [21],[22]).

The \( \bar{\partial} \)-dressing method starts with specification of analytic properties of solutions of the system (1). Namely, the problem is to find the vector function \( S = (S^1, \ldots, S^N) \) of the form \( S = S_0 + \tilde{S} \), satisfying (1) in the unit disc, with \( \tilde{S} \) analytic outside the unit disc and decreasing at infinity, and \( S_0 \) analytic in the unit disc, for arbitrary \( S_0 \) (\( S_0 \) plays the role of the boundary condition). The function \( S \) in this case may be considered as a functional of \( S_0 \). Introducing a parameterization of \( S_0 \),

\[
S_0^i = \sum_{n=1}^{\infty} t_n^i z^{n-1},
\]

we may also consider \( S \) as a function of \( N \) infinite sets of times \( (t^1, \ldots, t^N) \).

First derivatives of \( S \) over times satisfy a linear system

\[
\bar{\partial} f^i = \sum_{j} W^i_j(z, \bar{z}; S^1, \ldots, S^N) f^j,
\]

where \( W^i_j = \frac{\partial W^i}{\partial S^j} \). The linear space of functions, defined by this equation, admits multiplication by arbitrary scalar function of times and by arbitrary scalar functions of \( z \), analytic inside the unit disc.
Solutions of equation (3) which are bounded on the whole complex plane and decrease at infinity play a special role. In the case \( N = 1 \) there is an analogue of the Liouville theorem (see e.g. Theorem 3.11 from [20]) which states that a solution of (3) continuous and bounded on the whole complex plane and vanishing at some fixed point (in particular, \( z = \infty \)) is identically zero. For systems (3) with \( N > 1 \) this is in general not true (see e.g. [21], [22]). In our approach we assume the validity of an analogue of Liouville theorem. It is obviously valid for the systems (3) with small \( \mathcal{W}_i, j \).

However, below we will demonstrate that for some special reductions of the problem \( 1 \) it is possible to prove an analogue of Liouville theorem for systems (3) for arbitrary \( \bar{\partial} \)-data satisfying the reduction (given the existence of the basis of solutions).

Using the properties of the linear space of solutions of the system (3) and the standard ideology of the \( \bar{\partial} \)-dressing method [10, 11, 12], it is possible to demonstrate that the function \( S \) satisfies an (infinite) set of linear equations. Compatibility condition for a pair of such equations defines a closed system of multidimensional nonlinear PDEs for the coefficients of linear operators. The first linear equations are

\[
\partial_i^2 S - z \partial_1^i S = \sum_{p=1}^{N} (\partial_1^i u^p) \partial_1^p S, \tag{4}
\]

where \( \partial_n^i = \frac{\partial}{\partial t_n^i} \), functions \( u^p(t^1, \ldots, t^N) \) represent first coefficients of expansion of functions \( S^p(z; (t^1, \ldots, t^N)) \) at infinity; \( S^p = \sum_{n=1}^{\infty} S_n^p z^{-n}, u^p = -\bar{S}_1^p \). Taking compatibility condition of a pair of linear equations (4) labelled by superscripts \( i \) and \( j \), we get a closed equation for the vector field \( u = \sum_{p=1}^{N} u^p \partial_1^p \),

\[
\partial_i^2 \partial_1^j u - \partial_j^2 \partial_1^i u - [\partial_1^i u, \partial_1^j u] = 0, \tag{5}
\]

where \([u, v]\) is a standard commutator of vector fields. In a similar manner one can construct higher linear problems (4) and nonlinear systems (5).

Lax pairs of the type (4) and corresponding systems (5) are known for a long time (see e.g. [23]). The method to solve Cauchy problem and the dressing method for (5) based on the spectral theory of operators of the type (4) has been recently developed by S.V. Manakov and P.M. Santini [24].

In the simplest case \( N = 1 \) our scheme provides us with the \( \bar{\partial} \)-dressing, linear problems and nonlinear equations for the universal hierarchy of hydrodynamic type studied in [25, 26]. \( \bar{\partial} \)-dressing for this hierarchy will be studied elsewhere. In this paper we will concentrate on the Hamiltonian
reductions of the equations for vector fields and associated heavenly type equations.

3 Heavenly equation

Let us consider two-component case \((N = 2)\) and denote \(x = t_1^1, y = t_2^1, t = t_1^2, \tilde{t} = t_2^2\). It is easy to check that the system \([5]\) admits the reduction to Hamiltonian vector field \(u = \Theta_y \partial_x - \Theta_x \partial_y\), for which it becomes a second heavenly equation \([1]\)

\[
\Theta_{ty} - \Theta_{tx} - \Theta_{xy}^2 + \Theta_{xx} \Theta_{yy} = 0.
\]

(6)

Hamiltonian reduction of the system \([5]\) surprisingly corresponds to the Hamiltonian reduction of the \(\bar{\partial}\)-system \([1]\). Characterization of this reduction in terms of problem \([1]\) is given by the following statements (for simplicity we formulate them for the two-component case, \(N\)-component case is completely analogous).

**Proposition 1** Let the \(\bar{\partial}\)-data for the problem \([7]\) be of the form

\[
W^1 = W, \quad W^2 = -W,
\]

(7)

where \(W(z, \bar{z}; S^1, S^2)\) is some function (potential for the \(\bar{\partial}\)-data). Then the two-forms

\[
w = \delta S^1 \wedge \delta S^2,
\]

(8)

or, equivalently,

\[
\tilde{w}(\delta S, \tilde{\delta} S) := \delta S^1 \tilde{\delta} S^2 - \tilde{\delta} S^1 \delta S^2,
\]

(9)

where \(\delta\) and \(\tilde{\delta}\) denote arbitrary variations, are analytic inside the unit disc (in general case in \(\mathbb{C} \setminus G\), which is the support of \(\bar{\partial}\)-data).

**Proof.** The problem \([1]\) in this case looks like Hamilton equations with complex time \(\bar{z}\) and Hamiltonian \(W\),

\[
\bar{\partial} S^1 = \frac{\partial W}{\partial S^2}, \quad \bar{\partial} S^2 = -\frac{\partial W}{\partial S^1},
\]

(10)

and in complete analogy with standard Hamiltonian mechanics, we get (analogue of Liouville theorem, see e.g. \([27]\))

\[
\bar{\partial}(\delta S^1 \wedge \delta S^2) = 0.
\]

(11)
It is also a simple check that
\[ \bar{\partial}(\delta S^1 \delta S^2 - \delta S^1 \delta S^2) = 0. \] (12)
\[ \square \]

Analyticity of the two-form \( \tilde{w} \) inside the unit disc readily implies that vector field \( \tilde{S}^1 \partial_x + \tilde{S}^2 \partial_y \) is Hamiltonian.

**Proposition 2** Let the \( \bar{\partial} \)-problem (1) be of the Hamiltonian form (10). Then the vector field \( \tilde{S}^1 \partial_x + \tilde{S}^2 \partial_y \) is Hamiltonian.

**Proof** Identity (12) means that for any pair of solutions \( f, \tilde{f} \) of linear equations (3) inside the unit disc one has
\[ \bar{\partial}(f \tilde{f}^2 - \tilde{f} f^2) = 0. \] (13)
Taking a pair of solutions \( f = (S^1_x, S^2_x), \tilde{f} = (S^1_y, S^2_y) \) and considering analytic properties of the form (9), we come to the conclusion that
\[ \{S^1, S^2\} := S^1_x S^2_y - S^1_y S^2_x = 1. \] (14)
The first term of expansion of this relation at \( z \to \infty \) gives the identity \( \partial_x \tilde{S}^1 + \partial_y \tilde{S}^2 = 0 \). Thus \( \tilde{S}^1 = -\Theta_y, \tilde{S}^2 = \Theta_x \), meaning that the vector field \( \tilde{S}^1 \partial_x + \tilde{S}^2 \partial_y \) is indeed Hamiltonian. \[ \square \]

Identity (13) can be used in many different ways. We will show now that it directly leads to the linear problems of the type (4). Indeed, let us chose \( f = S_x \) and \( \tilde{f} = \mathcal{L}S \), where \( \mathcal{L} \) is a first order differential operator in independent variables depending on \( z \) such that \( (\mathcal{L}S)(z) \) is bounded outside the unit disc and \( (\mathcal{L}S)(z) \to 0 \) as \( z \to 0 \). Then due to (13) \( \tilde{w}(S_x, \mathcal{L}S) = 0 \) and consequently \( \mathcal{L}S - \alpha S_x = 0 \), where \( \alpha \) is some function of independent variables and \( z \). On the other hand, choosing \( f = S_y \) and \( \tilde{f} = \mathcal{L}S \), we come to the conclusion that \( \mathcal{L}S - \beta S_x = 0 \). Then it is easy to see that \( \alpha = \beta = 0 \) and \( \mathcal{L}S = 0 \). So we have obtained linear problem for \( S \).

For the heavenly equation (6) one gets the known linear problems (see e.g. [4])
\[ S_t - z S_x - \{S, \Theta_x\} = 0, \]
\[ \tilde{S}_t - z S_y - \{S, \Theta_y\} = 0. \] (15)

Let us emphasize that our derivation of linear problems in the Hamiltonian case (10) was done for arbitrary \( \bar{\partial} \)-data (the only thing we need is
existence of solution), and we didn’t suggest validity of an analogue of Liouville theorem for the system [3] corresponding to Hamiltonian case (in fact in the process of derivation we have proved it).

4 τ-function and addition formulae for the heavenly equation hierarchy

Analyticity of the two-form $w$ [5] inside the unit disc or equation (13) play a role of Hirota identity for heavenly equation hierarchy. This property can be formulated in a standard way for the boundary value of $w$ on the unit circle, using a projection operator. It implies the existence of the τ-function for the heavenly equation hierarchy (which coincides with $Θ$ introduced above for the heavenly equation) and gives ‘addition formulae’ (generating equations) of the hierarchy and heavenly equation itself.

**Proposition 3** Identity (11) implies that one-form

$$θ = \frac{1}{2\pi i} \oint (\tilde{S}^2 \delta S_1^1 - \tilde{S}^1 \delta S_2^2)dz$$  (16)

is closed.

The proof is by simple direct calculation.

We define a τ-function $Θ(t_1, t_2)$ for heavenly equation hierarchy through closed one-form (11) by the relation $δΘ = θ$. Introducing vertex operators $D^1(z) = \sum_{n=1}^{\infty} z^{-n} \partial_n$, $D^2(z) = \sum_{n=1}^{\infty} z^{-n} \partial_n^2$, it is easy to demonstrate that

$$\tilde{S}^1(z) = -D^2(z)Θ, \quad \tilde{S}^2(z) = D^1(z)Θ.$$  (17)

Substituting this representation into (14), we get the equation

$$D^2(z)Θ_x - D^1(z)Θ_y - \{D^1(z)Θ, D^2(z)Θ\} = 0$$  (18)

The first nontrivial order of expansion of this equation at $z \to \infty$ gives exactly the heavenly equation [6].

To derive addition formulae (generating equations in terms of vertex operators) for $Θ$, we will consider the two-forms $\tilde{w}(D^1(z')S(z), D^1(z'')S(z))$, $\tilde{w}(D^2(z')S(z), D^2(z'')S(z))$, $\tilde{w}(D^1(z')S(z), D^2(z'')S(z))$. Taking into account the identity (13) and analytic properties of these forms, one gets

$$D^1(z')S^1(z) \cdot D^1(z'')S^2(z) - D^1(z'')S^1(z) \cdot D^1(z')S^2(z)$$

$$= \frac{1}{z' - z} D^1(z'')\tilde{S}^2(z') - \frac{1}{z'' - z} D^1(z')\tilde{S}^2(z''),$$
\[
D^2(z')S^1(z) \cdot D^2(z'')S^2(z) - D^2(z'')S^1(z) \cdot D^2(z')S^2(z) = \frac{1}{z''-z} D^2(z') \tilde{S}^1(z'') - \frac{1}{z'-z} D^2(z'') \tilde{S}^1(z'), \tag{19}
\]

\[
D^1(z')S^1(z) \cdot D^2(z'')S^2(z) - D^2(z'')S^1(z) \cdot D^1(z')S^2(z) = \frac{1}{z'-z} \frac{1}{z''-z} + \frac{1}{z'-z} D^2(z'') \tilde{S}^2(z') + \frac{1}{z''-z} D^1(z') \tilde{S}^2(z'').
\]

These relations directly imply the existence of the \(\tau\)-function (that is equivalent to the definition in terms of closed variational one-form \(\tilde{w}\)) and provide us with addition formulae for it. Indeed, taking into account the asymptotic behaviour of both sides of relations \(19\) (or using the analytic properties of the forms \(\tilde{w}\) in the l.h.s. of these relations and Cauchy formula for the domain \(G\)), one gets

\[
D^1(z'')S^2(z') - D^1(z')S^2(z'') = 0,
\]

\[
D^2(z'')S^1(z') - D^2(z')S^1(z'') = 0,
\]

\[
D^2(z'')S^2(z') + D^1(z')S^1(z'') = 0,
\]

that implies \(17\).

Substituting representation \(17\) to relations \(19\), we obtain a set of addition formulae

\[
\frac{1}{z'-z} D^1(z'')(D^1(z') - D^1(z))\Theta - \frac{1}{z''-z} D^1(z'')(D^1(z'') - D^1(z))\Theta = D^1(z'')D^2(z)\Theta \cdot D^1(z')D^1(z)\Theta - D^1(z'')D^1(z)\Theta \cdot D^1(z')D^2(z)\Theta,
\]

\[
\frac{1}{z''-z} D^2(z')(D^2(z') - D^2(z))\Theta - \frac{1}{z'-z} D^2(z')(D^2(z'') - D^2(z))\Theta = D^2(z')D^2(z)\Theta \cdot D^2(z'')D^1(z)\Theta - D^2(z')D^1(z)\Theta \cdot D^2(z'')D^2(z)\Theta, \tag{20}
\]

\[
\frac{1}{z''-z} D^2(z'')(D^1(z') - D^1(z))\Theta - \frac{1}{z'-z} D^1(z'')(D^2(z'') - D^2(z))\Theta = D^1(z')D^1(z)\Theta \cdot D^2(z'')D^2(z)\Theta - D^1(z')D^2(z)\Theta \cdot D^2(z'')D^1(z)\Theta.
\]

Expansion of these equations into powers of parameters \(z, z'', z''\) generates heavenly equation hierarchy.
5 $\bar{\partial}$-dressing method and $\tau$-function for the heavenly equation hierarchy

Similar to the case of dispersionless integrable hierarchies [18, 19], it is possible to obtain explicit formula for the $\tau$-function of heavenly equation hierarchy, which is given by the action for the system (10) evaluated on the solution of this system.

The problem (10) can be obtained by variation of the action

$$f = \frac{1}{2\pi i} \int \int_{C \setminus G} \left( \bar{S}^2 \bar{\partial}\bar{S}^1 - W(z, \bar{z}, S^1, S^2) \right) dz \wedge d\bar{z},$$

where one should consider independent variations of $\bar{S}$, possessing required analytic properties (analytic in $G$, decreasing at infinity), keeping $S_0$ fixed.

**Proposition 4** The function

$$\Theta(t) = \frac{1}{2\pi i} \int \int_{D} \left( \bar{S}^2(t) \bar{\partial}\bar{S}^1(t) - W(z, \bar{z}, S^1(t), S^2(t)) \right) dz \wedge d\bar{z},$$

where $t = (t_1, t_2)$, $D$ is a unit disc, i.e., the action (27) evaluated on the solution of the problem (11), is a $\tau$-function of the heavenly equation hierarchy.

**Proof.** Considering $\Theta(t)$ as a functional of $S_0$ and calculating its variation, we get

$$\delta \Theta = \frac{1}{2\pi i} \oint (\bar{S}^2 \delta S_0^1 - \bar{S}^1 \delta S_0^2) dz,$$

that coincides exactly with one-form (16) used to define the $\tau$-function. □

Let us consider also the action

$$\tilde{f} = \frac{1}{2\pi i} \int \int_{D} \left( S_0^2 \partial S^1 - W(z, \bar{z}, S^1, S^2) \right) dz \wedge d\bar{z},$$

which is an exact analogue of the classical action associated with the Hamiltonian equations (10). Then

$$\tilde{f} = \frac{1}{2\pi i} \oint S_0^2 S^1 dz + f.$$

So on the solutions of the $\bar{\partial}$-problem (10) one has ($\tilde{\Theta} = \tilde{f}$)

$$\tilde{\Theta} = \Theta - \sum_{n=1}^{\infty} t_n \frac{\partial \Theta}{\partial t_n}.$$
Similar to dispersionless case \[18, 19\], the formula (22) provides us with infinitesimal symmetries for the addition formulae (20) and in particular for the heavenly equation (6). They are given by the expression
\[
\delta \Theta = -\frac{1}{2\pi i} \int_D \delta W(z, \bar{z}, S)dz \wedge d\bar{z},
\]
where \(\delta W(z, \bar{z}, S)\) is an arbitrary function in the unit disc.

In the simplest case \(\delta W = \delta(z - z_0)F(S(z_0, t))\), where \(F\) is an arbitrary function, one has
\[
\delta \Theta = F(S(z_0, t)).
\]
It is a simple direct check that an arbitrary function \(F(S_1, S_2)\) of the solutions \(S_1\) and \(S_2\) of linear problems (15) solves the linear equation
\[
(\delta \Theta)_{ty} - (\delta \Theta)_{ix} + \{\Theta_{x}, (\delta \Theta)_{y}\} - \{\Theta_{y}, (\delta \Theta)_{x}\} = 0,
\]
which defines infinitesimal symmetries of the heavenly equation (6).

Finally, we note that, similar to the classical mechanics, canonical transformations \(S^i \rightarrow S'^i = f^i(S^1, S^2)\) preserve the two-form \(w\) and leave \(\bar{\delta}\)-problem invariant. In addition, associated linear problems \(\mathcal{L}S = 0\) are invariant under these transformations.

Under canonical transformation with the generating function \(F\) one has \(W \rightarrow W' = W + F\). So for canonical transformation independent of \(\bar{z}\) the \(\tilde{\Theta}\)-function corresponding to (23) and \(\tau\)-function (22) remain invariant while in general case \(\tilde{\Theta}' = \tilde{\Theta} + \frac{1}{2\pi i} \oint Fdz\).

6 Conclusion

The results presented above are generalizable naturally to multicomponent case. Corresponding systems admit various reductions. In particular, for even number of components \((2N)\), the Hamiltonian reduction,
\[
u = \sum_{i=1}^{N} (\Theta_{x_i} - \Theta_{x_{2i-1}}),
\]
where \(x_i = t^1_i, i = 1, \ldots, 2N\), gives rise to the equations
\[
\Theta_{x_i t_k} - \Theta_{x_k t_i} - \{\Theta_{x_i}, \Theta_{x_k}\} = 0,
\]
where \(x_i = t^2_i, i = 1, \ldots, 2N\), and
\[
\{F, H\} = \sum_{p=1}^{N} \left( \frac{\partial F}{\partial x_{2p}} \frac{\partial H}{\partial x_{2p-1}} - \frac{\partial F}{\partial x_{2p-1}} \frac{\partial H}{\partial x_{2p}} \right).
\]
These equations are associated with linear problems
\[ S_{ti} - zS_{xi} = \{S, \Theta_{xi}\}. \]

The reduction \( u = \sum_{i=1}^{2M} \partial_i + \nu \), where
\[ \nu = \sum_{p=M+1}^N (\Theta_{x2p} \partial_{x2p-1} - \Theta_{x2p-1} \partial_{x2p}), \]
leads to equations
\[ \Theta_{x,it} - \Theta_{xkt} - \{\Theta_{xi}, \Theta_{xk}\} = 0, \tag{26} \]
where the Poisson bracket is of the form
\[ \{F, H\} = \sum_{p=M+1}^N \left( \frac{\partial F}{\partial x_{2p}} \frac{\partial H}{\partial x_{2p-1}} - \frac{\partial F}{\partial x_{2p-1}} \frac{\partial H}{\partial x_{2p}} \right). \]

In the case \( N = 2, M = 1 \), taking \( i = 1, k = 2 \) one obtains six-dimensional generalization of the second heavenly equation [5]. The \( \partial \)-dressing method for these multidimensional equations will be studied in elsewhere.

Acknowledgments
The authors are grateful to S.V. Manakov for fruitful discussions. LVB was supported in part by RFBR grant 04-01-00508 and President of Russia grant 1716-2003 (scientific schools); BGK was supported in part by the grant COFIN 2004 ‘Sintesi’.

References
[1] J.F. Plebański, J. Math. Phys. 16 2395–2402 (1975)
[2] R. Penrose, Gen. Rel. Grav 7, 31–52 (1976)
[3] C.P. Boyer, J.D. Finley and J.F. Plebański, General Relativity and Gravitation vol. 2, 241–81, New York: Plenum (1980)
[4] V. Husain, Phys. Rev. Lett. 72(6), 800–803 (1994)
[5] J.F. Plebański and M. Przanowski, Phys. lett. A 212, 22–28 (1996)
[6] M. Dunajski, L.J. Mason and N.M.J. Woodhouse, J. Phys. A: Math. Gen. 31, 6019–6028 (1998)
[7] M. Dunajski, L.J. Mason and K.P. Tod, J. Geom. Phys. 37, 63–92 (2001)
[8] E.V. Ferapontov and M.V. Pavlov, Classical Quantum Gravity 20(11), 2429–2441 (2003)
[9] M. Dunajski, *J. Geom. Phys.* 51, 126–137 (2004)
[10] V.E. Zakharov and S.V. Manakov *Funct. Anal. Appl.* 19, 89–101 (1985).
[11] L.V. Bogdanov and S.V. Manakov, *J. Phys. A: Math. Gen.* 21, L537–L544 (1988)
[12] B.G. Konopelchenko, *Solitons in multidimensions*, World Scientific, Singapore (1993)
[13] B.G. Konopelchenko, L. Martínez Alonso and O. Ragnisco, *J. Phys. A: Math. Gen.* 34, 10209–10217 (2001).
[14] B. Konopelchenko and L. Martínez Alonso, *Phys. Lett. A* 286, 161–166 (2001).
[15] B. Konopelchenko and L. Martínez Alonso, *J. Math. Phys.* 43(7), 3807–3823 (2002)
[16] B. Konopelchenko and L. Martínez Alonso, *Stud. Appl. Math.* 109 (2002), no. 4, 313–336
[17] L.V. Bogdanov, B.G. Konopelchenko and L. Martínez Alonso, *Teor. Mat. Fiz.* 134(1), 46–54 (2003)
[18] L.V. Bogdanov and B.G. Konopelchenko, *Phys. Lett. A* 322(5-6), 330–337 (2004)
[19] L.V. Bogdanov and B.G. Konopelchenko, *Phys. Lett. A* 330(6), 448–459 (2004)
[20] I.N. Vekua, *Generalized analytic functions*, Pergamon Press, Oxford (1962)
[21] W.L. Wendland, *Elliptic systems in the plane*, Pitman, London (1979)
[22] R.P. Gilbert and J.L. Buchanan, *First order elliptic systems*, Academic Press, New York (1983)
[23] V.E. Zakharov, A.B. Shabat, *Funktsional. Anal. i Prilozhen.* 13(3), 13–22 (1979)
[24] S.V. Manakov and P.M. Santini, private communication
[25] L. Martínez Alonso and A.B. Shabat, *Phys. Lett. A* 299(4), 359–365 (2002)
[26] L. Martínez Alonso and A.B. Shabat, *J. Nonlinear Math. Phys* 10, 229–242 (2003)
[27] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York (1978)