Decomposition and Variation-of-Constants Formula
in the Phase Space for Integral Equations

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Abstract. The present paper is concerned with linear integral equations of the form
\[ x(t) = \int_{-\infty}^{t} K(t - s)x(s)ds, \]
where \( K(s) \) is a matrix measurable in \( s \), and satisfies some boundedness condition. We propose a dynamical-systems approach to studying the behavior of the equations by treating them as functional equations with infinite delay. As a result we obtain a decomposition of the phase space corresponding to a set of normal eigenvalues of the generator of solution semigroups associated with the equations. Moreover, we establish a representation formula (which is called “a variation-of-constants formula” in the phase space) for solutions of nonhomogeneous integral equations, together with a decomposed formula based on the decomposition of the phase space. Finally, we apply the formula to investigate the admissibility of function spaces with respect to these equations.

Key Words and Phrases. Integral equations, Phase space, Solution semigroups, Spectrum, A variation-of-constants formula, Admissibility.

2010 Mathematics Subject Classification Numbers. Primary: 45D05; Secondary: 45J05, 45M10.

1. Introduction

In this paper we are mainly concerned with linear (homogeneous) integral equation with infinite delay of the form

\[ x(t) = \int_{-\infty}^{t} K(t - s)x(s)ds, \]

that is also called “linear renewal equations” in the literature, together with nonhomogeneous equations

\[ x(t) = \int_{-\infty}^{t} K(t - s)x(s)ds + p(t), \]

* Partly supported by Grant-in-Aid for Young Scientists (B), No.21740103, Japanese Ministry of Education, Culture, Sports, Science and Technology.
† Partly supported by the Grant-in-Aid for Scientific Research (C), No.22540211, Japan Society for the Promotion of Science.
where $K$ is a (measurable) $m \times m$ matrix valued function with complex components satisfying the conditions

\begin{align}
\|K\|_{1,\rho,+} &:= \int_{-\infty}^{0} \|K(\tau)\| e^{\rho \tau} \, d\tau < \infty, \\
\|K\|_{\infty,\rho,+} &:= \text{ess sup}\{\|K(\tau)\| e^{\rho \tau} \mid \tau \geq 0\} < \infty;
\end{align}

here, $\rho$ is a (fixed) positive constant.

The purpose of this paper is to develop a new dynamical-systems approach to study the behavior of Eq. (1) that is based on the Theory of Semigroups of Operators and Spectral Theory of Functions. To this end, we consider the Banach space $X := L^1_\rho(\mathbb{R}^-; \mathbb{C}^m)$, $\mathbb{R}^- := (-\infty, 0]$, defined by the equivalent classes of functions

$$\varphi : \mathbb{R}^- \to \mathbb{C}^m$$

and

$$\varphi(\theta) e^{\rho \theta} \text{ is integrable on } \mathbb{R}^-;$$

equipped with norm

$$(\|\varphi\| :=) \|\varphi\|_{1,\rho} = \int_{-\infty}^{0} |\varphi(\theta)| e^{\rho \theta} \, d\theta \quad (\forall \varphi \in X),$$

that will be denoted for short by $L^1_\rho$, if this does not cause any confusion. Throughout this paper, $X$ is called the “phase space” for Eq. (1) and Eq. (2). To Eq. (1) we associate a mapping $L : X \to \mathbb{C}^m$ defined by

$$L(\varphi) = \int_{-\infty}^{0} K(-\theta)\varphi(\theta) d\theta, \quad \forall \varphi \in X.$$

Then Eq. (1) and Eq. (2) can be rewritten as

$$x(t) = L(x_t)$$

and

$$x(t) = L(x_t) + p(t),$$

respectively, where $x_t$ is an element in $X$ defined as $x_t(\theta) = x(t + \theta)$, $\forall \theta \in \mathbb{R}^-$. Therefore, the integral equations (1) and (2) induce functional equations (6) and (7) on $X$. In this way we can study several properties of solutions for the integral equations via the induced functional equations (6) and (7).

As is well known, to study the behavior of functional differential equations with finite or infinite delay one is usually based on analyzing the dynamics of the induced semigroups in the phase space associated with the equations. This approach uses extensively functional analytic methods. For more information in this direction we refer the reader to the following references [3, 7, 8, 10, 12].
13, 14, 15, 18, 19, 20, 21] and the references therein. To the best of our knowledge, it is not the case for integral equations, especially, for integral equations with infinite delay of the form (1). Diekmann and Gyllenberg’s paper [2], however, may be seen as a pioneering paper using extensively functional analytic methods in the study of integral equations with infinite delay. Indeed, using Adjoint Semigroup Theory developed in [3], the authors in [2] treated abstract (nonlinear) integral equations, which may be viewed as the variation-of-constants formula in the “sun-star”-like space of $X$ for equations with the trivial principal linear part (that is, $L = 0$), instead of some concrete nonlinear integral equations, and established the principle of linearized stability for nonlinear abstract integral equations. Also, we refer the reader to [6] for a result on stability of nonlinear equations for functions with values in a Banach space.

One of the main results in our paper is Theorem 3 that gives a representation formula of solutions of Eq. (2) in the phase space $X$, which we call a “variation-of-constants formula” (VCF, for short) in the phase space. Our VCF is different from the one in [2]. Indeed, to establish our VCF we take an approach similar to that in [5, 13, 14, 15, 20], without using the notion of “sun-star”-like space. Our VCF can be applied to equations with any principal linear part, so it seems to be an effective and simple tool in the study of several subjects in integral equations. For example, we can study Massera type results on the existence of almost periodic solutions for equations with almost periodic forcing term, several invariant manifolds for nonlinear equations, and so on, in the way that is discussed in [5, 14, 15, 18, 19, 20, 21]. In this paper as an application of the VCF we will restrict ourself to studying the admissibility of a translation-invariant closed space of functions with respect to homogeneous integral equations (Theorem 9).

Another crucial result in our paper is Theorem 2 that gives a decomposition of the phase space corresponding to a set of normal eigenvalues of generator of solution semigroup associated with Eq. (1). In Theorem 1, we establish an estimate on the essential spectrum radius of the solution semigroup generated by Eq. (1). By choosing several normal eigenvalues belonging to the spectrum of generator we obtain a decomposition of the phase space that plays an important role in this paper. In fact, via the decomposition of the phase space, we obtain a decomposition of VCF. And then, as an application of this we establish Theorem 9.

In Sections 2 and 3 we present preliminary results necessary for our later arguments that also complement several results in [2, 6]. Indeed, in Section 2 we give a definition for solutions of (nonlinear) functional equations on $X$ or (nonlinear) integral equations, which is slightly different from that in [7], and we establish the existence result on the (unique) locally or globally defined solutions for initial value problems under some mild condition. We emphasize that our
new definition for solutions of integral equations plays an essential role in our main results of this paper, especially in the decomposition of variation-of-constants formula in the phase space which is effectively employed to study the admissibility theory of function spaces in the final section. Moreover, in Section 3 we introduce the solution semigroup associated with Eq. (1) and its generator, and give a characterization of the generator of solution semigroup, for completeness.

2. Initial value problems for the induced functional equations

We first explain several notations employed throughout this paper. Let $N$, $R^-$, $R^+$, $R$ and $C$ be the sets of natural numbers, nonpositive real numbers, nonnegative real numbers, real numbers and complex numbers, respectively. For an $m \in N$, we denote by $C^m$ (resp. $R^m$) the space of all $m$-column vectors, whose components are complex (resp. real) numbers, with the Euclidean norm $|\cdot|$. For any $m \times m$ matrix $M$, the norm $\|M\|$ is the operator norm of $M$ which is defined as $\|M\| = \sup \{|Mx|/|x| : x \in C^m, x \neq 0\}$.

Let $\rho$ be a fixed positive constant, and consider the space $X := L^1_\rho(R^-; C^m)$ defined by

$$L^1_\rho := L^1_\rho(R^-; C^m) = \{\varphi : R^- \to C^m | \varphi(\theta)e^{\rho \theta} \text{ is integrable on } R^-\}$$

equipped with norm

$$\|\varphi\|_{1,\rho} = \int_{-\infty}^0 |\varphi(\theta)|e^{\rho \theta} \, d\theta \quad (\forall \varphi \in X).$$

Clearly, $(X, \| \cdot \|)$ is a (complex) Banach space.

For any function $x : (-\infty, a) \to C^m$ and $t < a$, we define a function $x_t : R^- \to C^m$ by $x_t(\theta) = x(t + \theta)$ for $\theta \in R^-$. Let us consider an abstract equation

$$x(t) = F(t, x_t),$$

where $F : [b, \infty) \times X \to C^m$ is a continuous function. For any given $\varphi \in X$ and $\sigma \in [b, \infty)$, we treat the initial value problem for Eq. (8) with the initial condition

$$x_\sigma \equiv \varphi \text{ on } R^-, \text{ that is, } x(\sigma + \theta) = \varphi(\theta) \text{ for all } \theta \in R^-.$$

Throughout this paper, we say that a function $x : (-\infty, a) \to C^m$ is a solution of the initial value problem (8)–(9) on $(\sigma, a)$ if $x$ satisfies the following three conditions (cf. [7, Sections 2.3, 12.2]);

(i) $x_\sigma \equiv \varphi \text{ on } R^-$;

(ii) $x \in L^1_{loc}[\sigma, a]$; that is, $x$ is locally integrable on $[\sigma, a]$;

(iii) $x(t) = F(t, x_t)$ for $t \in (\sigma, a)$.
It should be noticed that the above definition for solutions of integral equations is slightly different from the usual one, e.g., the one given in [7, Section 2.3] where the requirements (i) and (iii) are weakened. The following lemma plays a key role in the establishment of existence of solutions of the initial value problem (8)–(9), as well as for the developments in subsequent sections.

**Lemma 1.** A function \( x : (\mathbb{C}, a) \rightarrow \mathbb{C}^m \) is a solution of the initial value problem (8)–(9) on \((\sigma, a)\) if and only if \( x \) satisfies the conditions (i) and (iii) together with the condition

(ii) \( x \) is continuous on \((\sigma, a)\), the limit \( x(\sigma^+) := \lim_{t \rightarrow 0^+} x(\sigma + t) \) exists, and the relation \( x(\sigma^+) = F(\sigma, \varphi) \) holds true.

**Proof.** It is easy to establish the “if” part. In the following we will prove the “only if” part. Let \( x : (\mathbb{C}, a) \rightarrow \mathbb{C}^m \) be a solution of the initial value problem (8)–(9) on \((\sigma, a)\). For \( t \in [\sigma, a] \) it follows that

\[
\int_{-\infty}^{0} |x_t(\theta)| e^{\rho \theta} \, d\theta = \int_{-\infty}^{0} |\varphi(\theta)| e^{\rho (\sigma - \theta)} \, d\theta + \int_{\sigma}^{t} |x(\tau)| e^{\rho (\tau - \theta)} \, d\tau,
\]

and hence \( x_t \in X \) with

\[
\|x_t\|_{1, \rho} \leq \|\varphi\|_{1, \rho} e^{\rho \rho} + \int_{\sigma}^{t} |x(\tau)| d\tau.
\]

We assert that \( x_t \) is continuous on \([\sigma, a]\) as an \( X \)-valued function of \( t \). Indeed, let any \( t, \tau \in [\sigma, a] \) such that \( |t - \tau| \to 0 \) be given. Then there are some \( c \in (\sigma, a) \) and \( \delta > 0 \) satisfying \( \max\{|t, \tau|\} < c - \delta \). As seen in the preceding paragraph, it follows that \( x_t \in X \), and hence the function \( f \) defined by \( f(\theta) = x(c + \theta) e^{\rho \theta} \), \( \theta \leq 0 \) belongs to \( L^1(\mathbb{R}; \mathbb{C}^m) \), the space of all integrable functions on \( \mathbb{R}^- \) with values in \( \mathbb{C}^m \). By a well known result on the integrable functions, it follows that \( f \) satisfies the relation \( \int_{-\infty}^{0} |f(\theta) - f(\tau + \theta)| d\theta \to 0 \) as \( \tau \to 0 \), where \( f(\tau + \theta) = 0 \) if \( \tau + \theta > 0 \). In particular, we get \( \int_{-\infty}^{\delta} |f(\theta) - f(\tau + \theta)| d\theta \to 0 \) as \( \tau \to 0 \); hence

\[
\lim_{\tau \to 0} \int_{-\infty}^{\delta} |x(c + \theta) e^{\rho \theta} - x(c + \tau + \theta) e^{\rho (\tau + \theta)}| d\theta = 0.
\]

Observe that

\[
\lim_{\tau \to 0} \int_{-\infty}^{\delta} |x(c + \tau + \theta) e^{\rho \theta} - x(c + \tau + \theta) e^{\rho (\tau + \theta)}| d\theta = 0,
\]

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because of
\[
\int_{-\infty}^{-\delta} |x(c + \tau + \theta) - x(c + \tau + \theta)e^{\rho(\tau+\theta)}| d\theta \leq \int_{-\infty}^{0} |x(c + s)| e^{\rho s} ds \times |1 - e^{-\rho t}|
\]
for any \( \tau \) with \(|\tau| < \delta \). Combining this with (10), we have
\[
\lim_{t \to 0} \int_{-\infty}^{-\delta} |x(c + \theta) - x(c + \tau + \theta)| e^{\rho \theta} d\theta = 0;
\]
which yields that \( \lim_{t \to 0} \|x_t - x_t\|_{1, \rho} = 0 \) as required, because of
\[
\|x_t - x_t\|_{1, \rho} = \int_{-\infty}^{0} |x(t + \theta) - x(\bar{t} + \theta)| e^{\rho \theta} d\theta
= \int_{-\infty}^{t-c} |x(c + v) - x(c + \bar{t} - t + v)| e^{\rho (c+v-t)} \, dv
\leq \int_{-\infty}^{-\delta} |x(c + v) - x(c + \mu + v)| e^{\rho v} \, dv \times e^{\rho (c-s)}
\]
by \( t - c < -\delta \), where \( \mu := \bar{t} - t \).

Now, from the assertion and the continuity of \( F \) it follows that \( F(t, x_t) \) is a continuous function of \( t \in [\sigma, a) \). Note that \( x(t) \equiv F(t, x_t) \) on \((\sigma, a)\); hence \( x \) is continuous on \((\sigma, a)\), and moreover \( x(\sigma + ) = \lim_{t \to 0+} x(\sigma + \tau) = \lim_{t \to 0+} F(\sigma + \tau, x_{\sigma + \tau}) = F(\sigma, x_{\sigma}) = F(\sigma, \phi) \). Thus the condition \((ii)^*\) must hold true.

For any \( \varepsilon > 0 \) and \((\sigma, \phi) \in R \times X \), we set
\[
O_{\varepsilon}(\sigma, \phi) := \{(t, \psi) \in R \times X \mid |t - \sigma| < \varepsilon, \|\psi - \phi\|_{1, \rho} < \varepsilon\}.
\]
Now, let \( F : [b, \infty) \times X \to C^m \) be any continuous function satisfying the (local) Lipschitz condition (with respect to the second variable); that is, for any \((\sigma, \phi) \in [b, \infty) \times X \) there exist positive constants \( \varepsilon := \varepsilon(\sigma, \phi) \) and \( l := l(\sigma, \phi) \) such that
\[
|F(t, \psi_1) - F(t, \psi_2)| \leq l \|\psi_1 - \psi_2\|_{1, \rho}
\]
whenever \((t, \psi_i) \in O_{\varepsilon}(\sigma, \phi) \cap ([b, \infty) \times X) \) for \( i = 1, 2 \). In the following proposition, we will establish the existence and uniqueness result on the (local) solutions of the initial value problem (8)–(9).

**Proposition 1.** Assume that \( F : [b, \infty) \times X \to C^m \) is a continuous function which satisfies the condition (11). Then, given \((\sigma, \phi) \in [b, \infty) \times X \), there exists
where $k$ is a constant $c$

Clearly $\delta = \varepsilon(\sigma, \varphi)$ and $l := l(\sigma, \varphi)$ be the constants for the local Lipschitz continuity of the function $F$.

We will first establish the part for (local) existence of solutions. Since $\varphi \in X$, by the same reasoning as in the proof of Lemma 1 we see that there is a constant $c > 0$ such that $\int_{-\infty}^{s} |\varphi(\theta + \tau) - \varphi(\theta)|e^{\rho \theta} d\theta < e/2$ if $0 \leq \tau \leq c$. Let us choose a positive constant $\delta$ so that $\delta < \min\{e, \varepsilon, 1/(6l)\}$ as well as $\int_{\sigma}^{\sigma+\delta} |F(s, \varphi)| ds \leq e/6$ and $\int_{-\delta}^{0} |\varphi(s)| ds \leq e/6$, and define

$$\mathcal{P} = \left\{ \chi \in C((\sigma, \sigma+\delta); \mathbb{C}^m) \mid \chi(\sigma^+) \text{ exists with } \sup_{0 < t < \delta} |\chi(t + \sigma)|e^{-\kappa t} < \infty \right\},$$

where $\kappa$ is a positive constant such that $\kappa > 2l$. For any $\chi \in \mathcal{P}$, let us define $\tilde{\chi} : (-\infty, \sigma + \delta) \to \mathbb{C}^m$ by

$$\tilde{\chi}(s) = \begin{cases} \varphi(s - \sigma), & s \leq \sigma \\ \chi(s), & \sigma < s < \sigma + \delta. \end{cases}$$

Then $(\tilde{\chi})_t \in X$ for any $t \in [\sigma, \sigma + \delta)$, because of

$$\int_{-\infty}^{0} |\tilde{\chi}(t + \theta)|e^{\rho \theta} d\theta = \int_{-\infty}^{\sigma} |\tilde{\chi}(s)|e^{\rho(s-\tau)} ds + \int_{\sigma}^{t} |\tilde{\chi}(s)|e^{\rho(s-\tau)} ds$$

$$= \int_{-\infty}^{\sigma} |\varphi(s - \sigma)|e^{\rho(s-\tau)} ds + \int_{\sigma}^{t} |\chi(s)|e^{\rho(s-\tau)} ds$$

$$\leq \|\varphi\|_{1, \rho} + \int_{\sigma}^{t} \|\chi\|e^{\kappa(s-\sigma)} ds < \infty.$$ 

Let us consider a closed set $\Omega$ in $\mathcal{P}$ defined by

$$\Omega = \left\{ \chi \in \mathcal{P} \mid \chi(\sigma^+) = F(\sigma, \varphi), \sup_{\sigma < t < \sigma + \delta} \int_{-\infty}^{0} |\chi(t + \theta) - \varphi(t)| d\theta \leq e/2 \right\}.$$ 

As easily be checked, the function $F(\cdot, \varphi)$ restricted to the interval $(\sigma, \sigma + \delta)$ belongs to $\Omega$; hence $\Omega$ is a nonempty set. We claim that if $\chi \in \Omega$ and
By the same reasoning as in the proof of Lemma 1, we see that 
\( \| \hat{\mathcal{X}} - \varphi \|_{1, \rho} < \varepsilon \); consequently \((t, (\hat{\mathcal{X}})_t) \in O_t(\sigma, \varphi)\) for any \( t \in [\sigma, \sigma + \delta) \). Indeed, since \( 0 \leq t - \sigma < \delta < \varepsilon \), it follows that
\[
\| (\hat{\mathcal{X}}_t) - \varphi \|_{1, \rho} = \int_{-\infty}^{0} |\hat{\mathcal{X}}(t + \theta) - \varphi(\theta)| e^{\rho \theta} \, d\theta \\
\leq \int_{-\infty}^{\sigma - t} |\varphi(t + \theta - \sigma) - \varphi(\theta)| e^{\rho \theta} \, d\theta + \int_{0}^{\sigma - t} |\mathcal{X}(t + \theta) - \varphi(\theta)| \, d\theta \\
< \varepsilon / 2 + \varepsilon / 2 = \varepsilon,
\]
which proves the claim.

For any \( \chi \in \mathcal{P} \), put
\[
(\Phi \chi)(t) = F(t, (\hat{\mathcal{X}})_t), \quad t \in (\sigma, \sigma + \delta).
\]
By the same reasoning as in the proof of Lemma 1, we see that \((\hat{\mathcal{X}})_t\) is continuous on \([\sigma, \sigma + \delta)\) as an \( X \)-valued function of \( t \). Hence it follows that \((\Phi \chi)(\cdot) \in C([\sigma, \sigma + \delta]; C^m)\) and \((\Phi \chi)(\sigma^+) = F(\sigma, \varphi)\). We claim that \( \Phi \chi \in \Omega \) whenever \( \chi \in \Omega \). Indeed, if \( t \in (\sigma, \sigma + \delta) \), then \((s, (\hat{\mathcal{X}})_s) \in O_s(\sigma, \varphi)\) on \((\sigma, t)\) by the preceding claim and hence it follows that
\[
\int_{\sigma - t}^{0} \| (\Phi \chi)(t + \theta - \sigma) - \varphi(\theta) \| \, d\theta \\
= \int_{\sigma}^{t} \| (\Phi \chi)(s) - \varphi(s - t) \| \, ds \\
\leq \int_{\sigma}^{t} |F(s, (\hat{\mathcal{X}})_s)| \, ds + \int_{\sigma}^{t} |\varphi(s - t)| \, ds \\
\leq \int_{\sigma}^{t} |F(s, \varphi)| \, ds + \int_{\sigma}^{t} |F(s, (\hat{\mathcal{X}})_s) - F(s, \varphi)| \, ds + \int_{-\delta}^{0} |\varphi(\tau)| \, d\tau \\
\leq \varepsilon / 3 + l \int_{\sigma}^{t} \| (\hat{\mathcal{X}})_s - \varphi \|_{1, \rho} \, ds \\
\leq \varepsilon / 3 + l o \delta \leq \varepsilon / 2,
\]
which shows that \( \Phi \chi \in \Omega \), as required.

We will verify that
\[
\| | \Phi \chi - \Phi u || \leq \frac{1}{2} || \chi - u || \quad (\forall \chi, u \in \Omega).
\]
Indeed, if \( \chi \in \Omega \), \( u \in \Omega \) and \( t \in (0, \delta) \), then \((t + \sigma, (\hat{\mathcal{X}})_{t + \sigma})\) and \((t + \sigma, (\hat{u})_{t + \sigma})\) belong to \( O_{t+\sigma}(\sigma, \varphi) \) by the preceding claim; and hence
there exists some $t < F w w$ another solution of (8)–(9) on $t A \ldots$

which yields the inequality $||| \Phi \chi - \Phi u ||| \leq ||| \chi - u ||| / 2$.

By the Contraction Mapping Principle we see that the contraction mapping $\Phi : \mathcal{P} \to \mathcal{P}$ has a unique fixed point, say $\chi \in \mathcal{P}$. Then, $\tilde{\chi} : (-\infty, \sigma + \delta) \to C^m$ satisfies the conditions (i), (ii)* and (iii). Therefore, by Lemma 1, the function $\tilde{\chi}$ is a solution of (8)–(9) on $(\sigma, \sigma + \delta)$.

Next we will establish the uniqueness part of the proposition. Let $\mathcal{Y}$ be another solution of (8)–(9) on $(\sigma, \sigma + \delta)$. Since $||(\tilde{\mathcal{Y}})_t - \phi||_{1, \rho} \to 0$ as $t \to \sigma + 0$, there exists some $\delta_0, 0 < \delta_0 < \delta$, such that $||(\tilde{\mathcal{Y}})_t - \phi||_{1, \rho} < \varepsilon$ whenever $\sigma < t < \sigma + \delta_0$. We will first verify that $\chi(t) \equiv \mathcal{Y}(t)$ on $(\sigma, \sigma + \delta_0)$. Indeed, if $t \in (\sigma, \sigma + \delta_0)$, it follows that $(t, (\tilde{\mathcal{Y}})_t) \in O_\varepsilon(\sigma, \phi), (t, (\tilde{\chi})_t) \in O_\varepsilon(\sigma, \phi), \chi(t) = F(t, (\tilde{\chi})_t)$ and $\mathcal{Y}(t) = F(t, (\tilde{\mathcal{Y}})_t)$. Hence we get $|\chi(t) - \mathcal{Y}(t)|$

$$= |F(t, (\tilde{\chi})_t) - F(t, (\tilde{\mathcal{Y}})_t)|$$

$$\leq l ||| (\tilde{\chi})_t - (\tilde{\mathcal{Y}})_t |||_{1, \rho}$$

$$= l \left\{ \int_{-\infty}^{-\sigma} |\tilde{\chi}(t + \theta) - (\tilde{\mathcal{Y}})(t + \theta)| e^{\rho \theta} d\theta + \int_{-\sigma}^{0} |\chi(t + \theta) - \mathcal{Y}(t + \theta)| e^{\rho \theta} d\theta \right\}$$

$$= l \int_{-\sigma}^{0} |\chi(t) - \mathcal{Y}(t)| e^{\rho(t+\theta)} d\tau$$
for $\sigma < t < \sigma + \delta_0$. Thus,

$$0 \leq g(t) \leq \int _{\sigma} ^{t} g(\tau) d\tau, \quad t \in (\sigma, \sigma + \delta_0),$$

where $g$ is a function defined by

$$g(t) = \begin{cases} \exp(\int \varepsilon(t) \| \varphi(t) \| dt), & \sigma < t < \sigma + \delta_0 \\ 0, & t = \sigma. \end{cases}$$

Observe that $g$ is continuous on $[\sigma, \sigma + \delta_0)$ because of $\chi(\sigma^+) = \varphi(\sigma^+) = F(\sigma, \varphi)$ by (ii)*. Applying Gronwall's inequality we get $g(t) \equiv 0$ on $[\sigma, \sigma + \delta_0)$, which implies $\chi \equiv \varphi$ on $(\sigma, \sigma + \delta_0)$, as required.

Put $\sigma_1 := \sup \{ t \in (\sigma, \sigma + \delta] \mid \chi \equiv \varphi \text{ on } (\sigma, t) \}$. By the above argument, we see that $\sigma + \delta_0 \leq \sigma_1 \leq \sigma + \delta$. If $\sigma_1 < \sigma + \delta$, considering $\sigma_1$ and $\varphi_1 := \chi_{\sigma_1}$ (= $\varphi_0(\sigma_1)$) in place of $\sigma$ and $\varphi$ one can proceed the argument to get that $\chi \equiv \varphi$ on $(\sigma_1, \sigma_1 + \delta_1)$ for some $\delta_1 > 0$ that is a contradiction to the definition of $\sigma_1$. Thus, we must have that $\sigma_1 = \sigma + \delta$; hence $\chi \equiv \varphi$ on $(\sigma, \sigma + \delta)$, as required. The proof is now completed.

Let $x$ be a solution of the initial value problem (8)–(9) on $(\sigma, a)$. If there exists another solution $z$ of (8)–(9) on $(\sigma, c)$ with some $c > a$ which satisfies $x(t) \equiv z(t)$ on $(\sigma, a)$, the solution $x$ is said to be extendable, and the solution $z$ is called an extension of $x$.

The following result shows that as long as a solution of (8)–(9) is bounded, it can be extended to a solution of (8)–(9) on a larger interval.

**Proposition 2.** Assume that $F : [b, \infty) \times X \rightarrow C^m$ is a continuous function which satisfies the condition (11), and let $x : (-\infty, a) \rightarrow C^m$ be a solution of (8)–(9) on $(\sigma, a)$ with $\sigma < a < \infty$. If $\sup_{\sigma < t < a} |x(t)| < \infty$, then the limit $x(a^-) := \lim _{t \rightarrow a^-} x(a - t)$ exists, and $x$ is extended to a solution of (8)–(9) on $(\sigma, a + \delta)$ for some $\delta > 0$.

**Proof.** Define a function $\chi : R^+ \rightarrow C^m$ by $\chi(\theta) = x(a + \theta)$ if $\theta < 0$, and $\chi(\theta) = 0$ if $\theta = 0$. From the boundedness of $x$ on $(\sigma, a)$ and $x_\sigma \equiv \varphi \in L^1_\rho$ it follows that $\chi \in L^1_\rho$. Hence, repeating the argument employed in the proof of Lemma 1, we see that $\lim _{t \rightarrow a^-} \int _{\sigma} ^{t} |\chi(\theta - \tau) - \chi(\theta)| e^{\gamma \| \varphi \| d\tau} = 0$, and consequently,

$$\lim _{t \rightarrow a^-} \| x_{a - t} - \chi \| _{L^1_\rho} = \lim _{t \rightarrow a^-} \int _{\sigma} ^{a - t} |x(a - t + \theta) - \chi(\theta)| e^{\gamma \| \varphi \| d\theta} = \lim _{t \rightarrow a^-} \int _{\sigma} ^{a - t} |\chi(a - t + \theta) - \chi(\theta)| e^{\gamma \| \varphi \| d\theta} = 0,$$
which implies that \( x(a^-) = F(a, \chi) \), because of \( \lim_{t \to -0} x(a-t) = \lim_{t \to -0} F(a-t, x(a-t)) = F(a, \chi) \) by the continuity of \( F \).

We next consider the function \( \psi : \mathbb{R}^+ \to \mathbb{C}^m \) defined by \( \psi(\theta) = x(a + \theta) \) if \( \theta < 0 \), and \( \psi(\theta) = x(a^-) \) if \( \theta = 0 \). Notice that \( \psi \) is continuous at \( \theta = 0 \). Since \( \chi(\theta) \equiv \psi(\theta) \) on \( (-\infty, 0) \), we get \( \|\chi - \psi\|_{1, \rho} = 0 \), and hence \( \psi \in L^1_{\rho} \) and \( F(a, \psi) = F(a, \chi) = x(a^-) = \psi(0) \). By Proposition 1, there is a solution, say \( z \), of (8) on an interval \( (a, a + \delta) \) with the property that \( z(a + \theta) = \psi(\theta) \) for all \( \theta \in \mathbb{R}^+ \). In what follows, applying Lemma 1 we will prove that \( z \) is an extended solution of \( x \). It is easy to see that \( z(\sigma + \theta) = \phi(\theta) \) for all \( \theta \in \mathbb{R}^+ \), and \( z(\sigma^+) = x(\sigma^+) = F(\sigma, \phi) \). Also, \( z \) is continuous on the intervals \((\sigma, a)\) and \((a, a + \delta)\). In fact, \( z \) is continuous at \( t = a \), because \( z(a^-) = x(a^-) = \psi(0) = z(a) \) and \( z(a^+) = F(a, \psi) = F(a, \chi) = x(a^-) = z(a) \). Thus, the conditions (i) and (ii) in Lemma 1 are certified. It remains to verify the condition (iii), that is, the relation \( z(t) = F(t, z_t) \) for all \( t \in (\sigma, a + \delta) \). Clearly, the relation holds true when \( a < t < a + \delta \). Also, since \( z_a = \psi \), it follows \( F(a, z_a) = F(a, \psi) = z(a^+) = z(a) \). Moreover, if \( \sigma < t < a \), then \( x_t(\theta) = x(t + \theta) = \psi(t + \theta - a) = z(t + \theta) = z_t(\theta) \) for all \( \theta \in \mathbb{R}^+ \), and hence \( z(t) = \psi(t - a) = x(t) = F(t, x_t) = F(t, z_t) \), as required. The proof is now completed.

Let \( x \) be a solution of (8)–(9) on an interval. If \( x \) has no extensions of (8)–(9), then \( x \) is called a noncontinuable solution of (8)–(9). Applying the Zorn lemma we see that any solution of the initial value problem (8)–(9) can be extendable to a noncontinuable solution (with an open interval). The following result is an immediate consequence of this fact and Propositions 1 and 2.

**Corollary 1.** Assume that \( F : [b, \infty) \times X \to \mathbb{C}^m \) is a continuous function which satisfies the condition (11). Then there is a (unique) noncontinuable solution of (8)–(9) on the maximal interval \( (\sigma, a) \). If \( a < \infty \), then \( \limsup_{t \to a-0} |x(t)| = \infty \).

In what follows, we refer to the (unique) noncontinuable solution of (8)–(9) as the solution of (8)–(9), simply. If the solution of (8)–(9) is defined on \( (\sigma, a) \) with \( a = \infty \), then it is called a globally defined solution. The following result gives a sufficient condition for the existence of globally defined solutions.

**Proposition 3.** Let \( F : [b, \infty) \times X \to \mathbb{C}^m \) be a continuous function which satisfies the condition (11), and assume that there exist nonnegative continuous functions \( l(\cdot) \) and \( h(\cdot) \) such that
\[
|F(t, \phi)| \leq l(t)\|\phi\|_{1, \rho} + h(t), \quad \forall t \geq b, \, \phi \in X.
\]
Then the (noncontinuable) solution of (8)–(9) is globally defined.
Proof. Let \( x \) be the (noncontinuable) solution of (8)–(9) on \((\sigma, a)\). In what follows, we will certify that \( a = \infty \). In order to prove this by a contradiction, assume that \( a < \infty \) and consider a continuous function \( g: [\sigma, a) \rightarrow \mathbb{R}^+ \) defined by \( g(t) = |x(t)|e^{\rho t} \) if \( \sigma < t < a \), and \( g(t) = |F(\sigma, \varphi)|e^{\rho t} \) if \( t = \sigma \). Notice that \( x_\sigma \equiv \varphi \) on \( \mathbb{R}^+ \). Since \( x(t) = F(t, x_t) \) on \((\sigma, a)\), one can easily see by (12) that for \( \sigma < t < a \),

\[
|x(t)| \leq l(t) \left( \|\varphi\|_{1, \rho} e^{\rho(t-\sigma)} + \int_{\sigma}^{t} |x(\tau)|e^{\rho(t-\tau)} d\tau \right) + h(t);
\]
or

\[
g(t) \leq l(t) \left( \|\varphi\|_{1, \rho} e^{\rho t} + \int_{\sigma}^{t} g(\tau) d\tau \right) + h(t)e^{\rho t}
\]
for all \( \sigma \leq t < a \). Applying Gronwall’s inequality, we get \( g(t) \leq (He^{\rho a} + L\|\varphi\|_{1, \rho} e^{\rho(t-\sigma)}) \) on \([\sigma, a)\), where \( H := \sup_{\sigma < t < a} h(t) \) and \( L := \sup_{\sigma < t < a} l(t) \). Hence \( \sup_{\sigma < t < a} |x(t)| \leq (e^{\rho(a-\sigma)} + L\|\varphi\|_{1, \rho})e^{L(a-\sigma)} < \infty \). On the one hand, we have \( \limsup_{t \to a^-} |x(t)| = \infty \) by Corollary 1, a contradiction. \( \square \)

3. Solution semigroup for linear functional equations and its generator

Consider functional equations of the form

\[
x(t) = L(x_t) + p(t), \quad t > \sigma,
\]
where \( L : X := L^1_p(\mathbb{R}^+; C^m) \rightarrow C^m \) is a bounded linear operator and \( p \in C(\mathbb{R}^+; C^m) \), the space of all continuous functions mapping \( \mathbb{R}^+ \) into \( C^m \). Given \( \varphi \in X \) and \( \sigma \geq 0 \), there exists a unique globally defined solution, say \( x \), of Eq. (13) satisfying the initial condition \( x_\sigma \equiv \varphi \) on \( \mathbb{R}^+ \). Indeed, if one defines \( F : \mathbb{R}^+ \times X \rightarrow C^m \) by \( F(t, \psi) = L(\psi) + p(t) \) for \((t, \psi) \in \mathbb{R}^+ \times X \), then \( F \) satisfies the Lipschitz condition as well as the condition (12); and hence the existence result on the (unique) globally defined solutions for the initial value problems is a direct consequence of Proposition 3. In what follows, we call \( x \) the solution of Eq. (13) through \((\sigma, \varphi)\), and write it as \( x(\cdot; \sigma, \varphi, p) \). If \( \varphi = \psi \) in \( X \) from the uniqueness obtained in Proposition 1 it follows that \( x_t(\sigma, \psi, p) = x_t(\sigma, \psi, p) \) in \( X \) for any \( t \geq \sigma \) whenever \( \varphi = \psi \) in \( X \), and consequently \( x_t(\sigma, \varphi, p) \) can be considered as a function mapping \( X \) into \( X \).

Now, for any \( t \geq 0 \) and \( \varphi \in X \), we define \( T(t)\varphi \in X \) by

\[
(T(t)\varphi)(\theta) := x_t(\theta; 0, \varphi, 0) = \begin{cases} x(t+\theta; 0, \varphi, 0), & -t < \theta \leq 0 \\ \varphi(t+\theta), & \theta \leq -t. \end{cases}
\]
As noted in the preceding paragraph, \( T(t)\varphi = T(t)\psi \) in \( X \) whenever \( \varphi = \psi \) in \( X \); in other words, \( T(t) \) defines a mapping on \( X \). Indeed, \( T(t) \) is a bounded linear operator on \( X \). Recall that \( x(\cdot;0,\varphi,0) \) is the solution of the homogeneous linear equation
\[
(14) \quad x(t) = L(x_t)
\]
through \((0,\varphi)\). We call \( T(t) \) the “solution operator” for Eq. (14). In fact, \( \{T(t)\}_{t \geq 0} \) is a strongly continuous semigroup of bounded linear operators on \( X \), which is called the solution semigroup for Eq. (14).

Recall that a family of bounded linear operators \( \{T(t)\}_{t \geq 0} \) in \( X \) is said to be a strongly continuous semigroup of (bounded) linear operators in \( X \) if it satisfies
\[
\begin{align*}
(i) & \quad T(0) = Id; \\
(ii) & \quad T(t)T(s) = T(t+s) \text{ for all } t \geq 0, s \geq 0; \\
(iii) & \quad \lim_{t \to 0} T(t)x = x \text{ for each } x \in X.
\end{align*}
\]
The generator \( A \) of a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) is defined to be a closed linear operator with dense domain
\[
\mathcal{D}(A) := \left\{ \varphi \in X \mid \lim_{h \to +0} (1/h)(T(h)\varphi - \varphi) \text{ exists in } X \right\}
\]
in which
\[
A\varphi := \lim_{h \to +0} (1/h)(T(h)\varphi - \varphi), \quad \varphi \in \mathcal{D}(A).
\]

In what follows we will give a characterization for the generator of the solution semigroup \( \{T(t)\}_{t \geq 0} \). To do this, we consider a subset \( \tilde{X} \) of \( X \) given by
\[
\tilde{X} = \{ \tilde{\varphi} \in X \mid \tilde{\varphi} \text{ is locally absolutely continuous on } \mathbb{R}^-, \\
(\text{d}/\text{d}\theta)\tilde{\varphi} \in X \text{ and } \tilde{\varphi}(0) = L(\tilde{\varphi}) \}.
\]

**Proposition 4.** The generator \( A \) of the solution semigroup for Eq. (14) and its domain \( \mathcal{D}(A) \) are given by
\[
\mathcal{D}(A) = \{ \varphi \in X \mid \varphi(\theta) = \tilde{\varphi}(\theta) \text{ a.e. in } \theta \in \mathbb{R}^- \text{ for some } \tilde{\varphi} \in \tilde{X} \},
\]
\[
A\varphi = (\text{d}/\text{d}\theta)\tilde{\varphi}, \quad \varphi \in \mathcal{D}(A).
\]

**Proof.** The proposition is essentially the same as [7, Theorems 8.2.5 and 8.3.7]. In fact, our definition for solutions of Eq. (13) is slightly different from the one given in [7]. However, by Lemma 1 the argument in [7] can be used to establish the proposition. Therefore we omit the proof. \( \square \)
In the following, we will investigate the spectrum $\sigma(A)$ as well as the point spectrum $P_\sigma(A)$ of the generator $A$ of the solution semigroup $\{T(t)\}_{t \geq 0}$ on the space $X := L^1_\rho(\mathbb{R}^-; \mathbb{C}^m)$. To do this, for a given number $\rho$ we set

$$C_{-\rho} := \{ z \in \mathbb{C} \mid \Re z > -\rho \},$$

and consider a function $\omega_\lambda$ defined by

$$\omega_\lambda(\theta) := e^{i\theta}, \quad \forall \theta \leq 0$$

for each $\lambda \in C_{-\rho}$. One can easily check that if $\lambda \in C_{-\rho}$ and $z \in \mathbb{C}^m$, then the function $\omega_\lambda z$ defined by $(\omega_\lambda z)(\theta) = \omega_\lambda(\theta)z$, $\theta \leq 0$ belongs to the space $X$ with norm $\|\omega_\lambda z\| = |z|/(\Re \lambda + \rho)$. In particular, $\omega_\lambda e_i \in X$ and hence $L(\omega_\lambda e_i) \in \mathbb{C}^m$ for each $i = 1, \ldots, m$, where $e_i$ is the vector in $\mathbb{C}^m$ whose $j$-th component is 1 if $j = i$ and 0 otherwise. Notice that $E := (e_1, \ldots, e_m)$ is the $m \times m$ unit matrix. Set $L(\omega_\lambda E) = (L(\omega_\lambda e_1), \ldots, L(\omega_\lambda e_m))$. Then $L(\omega_\lambda E)$ is an $m \times m$ matrix, and it satisfies the relation

$$L(\omega_\lambda E)\alpha = L(\omega_\lambda \alpha), \quad \forall \alpha \in \mathbb{C}^m.$$ 

With the above notations, we have the following result on the spectrum $\sigma(A)$.

**Proposition 5.** Let $A$ be the generator of the solution semigroup for Eq. (14). Then, the following relation holds true:

$$\sigma(A) \cap C_{-\rho} = P_\sigma(A) \cap C_{-\rho} = \{ \lambda \in C_{-\rho} \mid \det (E - L(\omega_\lambda E)) = 0 \}.$$

For the equation with finite delay, the proposition has been established in [7, Theorem 8.2.7] by applying the characterization of the generator. We can follow the argument employed in [7] to prove the proposition for the equation with infinite delay by applying Proposition 4. Therefore, we omit details of the proof.

4. Decomposition of the phase space for integral equations

In the remainder of this paper, we always assume (without stating explicitly) that $K$ is a (measurable) $m \times m$ matrix valued function with complex components satisfying the conditions

$$\|K\|_{1,\rho,+} := \int_0^\infty \|K(\tau)\| e^{\rho \tau} \, d\tau < \infty;$$

$$\|K\|_{\infty,\rho,+} := \text{ess sup}\{\|K(\tau)\| e^{\rho \tau} \mid \tau \geq 0\} < \infty;$$

$$\|K\|_{1,\rho,-} := \int_0^\infty \|K(\tau)\| e^{-\rho \tau} \, d\tau < \infty;$$

$$\|K\|_{\infty,\rho,-} := \text{ess sup}\{\|K(\tau)\| e^{-\rho \tau} \mid \tau \geq 0\} < \infty;$$

$$\|K\|_{\infty,\rho} := \text{ess sup}\{\|K(\tau)\| \mid \tau \geq 0\} < \infty;$$

$$\|K\|_{\infty} := \text{ess sup}\{\|K(\tau)\| \mid \tau \geq 0\} < \infty.$$
here, \( p \) is a (fixed) positive constant. In what follows the notations \( \| \cdot \|_{1, \rho, +} \) and \( \| \cdot \|_{\infty, \rho, +} \) will often be shortened as \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \), respectively. To the function \( K \), let us associate a function \( L \) defined on the space \( X := L^1_p(\mathbb{R}^n; \mathbb{C}^m) \) by 
\[
L(\varphi) = \int_{-\infty}^{0} K(-\theta)\varphi(\theta) d\theta, \quad \forall \varphi \in X.
\]
Then, \( L : X \to C^m \) is a bounded linear operator with norm \( \| L \| \leq \| K \|_\infty \), because of the inequality
\[
|L(\varphi)| \leq \int_{-\infty}^{0} \| K(-\theta) \| e^{-\rho \theta} |\varphi(\theta)| e^{\rho \theta} d\theta
\]
\[
\leq \| K \|_{\infty, \rho, +} \int_{-\infty}^{0} |\varphi(\theta)| e^{\rho \theta} d\theta = \| K \|_\infty \| \varphi \|_{1, \rho}
\]
for any \( \varphi \in X \).

We now consider linear integral equations of the form
\[
(17) \quad x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + p(t), \quad t > \sigma,
\]
where \( p \) is an element in \( C(\mathbb{R}; \mathbb{C}^m) \) (the space of all continuous functions mapping \( \mathbb{R} \) into \( \mathbb{C}^m \)). Eq. (17) can be viewed as the functional equation (13) on the space \( X \). As in the previous sections, one can conclude that given \( \varphi \in X \) there exists a unique globally defined solution \( x \) of Eq. (17) satisfying \( x_\sigma \equiv \varphi \) on \( \mathbb{R}^+ \), that is, \( x \) satisfies Eq. (17) on \( (\sigma, \infty) \) together with the initial condition \( x(\sigma + \theta) = \varphi(\theta) \) for all \( \theta \leq 0 \). In the following, as a notation of the solution for Eq. (17) we will employ the same notation \( x(\cdot; \sigma, \varphi, p) \) as the one for Eq. (13). Similarly, we treat the solution semigroup and its generator with the notations \( \{ T(t) \}_{t \geq 0} \) and \( A \) for the homogeneous linear integral equation
\[
(18) \quad x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds, \quad t > 0.
\]
In particular, by virtue of Proposition 5 we get the following result on the spectrum \( \sigma(A) \) of the generator \( A \) of the solution semigroup for Eq. (18):
\[
(19) \quad \sigma(A) \cap C_{-\rho} = \mathcal{P}_\sigma(A) \cap C_{-\rho} = \{ \lambda \in C_{-\rho} \mid \det A(\lambda) = 0 \},
\]
where \( \det A(\lambda) := E - \int_{0}^{\infty} K(t)e^{-\lambda t} dt \) for \( \text{Re} \lambda > -\rho \).

Below we will establish a decomposition of the phase space \( X \) corresponding to a set of several eigenvalues of \( A \) that does not intersect its essential spectrum \( \text{ess}(A) \). Recall that the essential spectrum \( \text{ess}(T) \) of a closed linear operator \( T : X \to X \) with dense domain \( \mathcal{D}(T) \) is the set of all \( \lambda \) in \( \sigma(T) \), for which at least one of the following holds:

(i) the set \( \mathcal{M}(T - \lambda I) := \{ (T - \lambda I)\varphi \mid \varphi \in \mathcal{D}(T) \} \) is not closed;

(ii) the point \( \lambda \) is a limit point of \( \sigma(T) \);

(iii) the generalized eigenspace \( \mathcal{G}_\lambda(T) := \bigcup_{k \geq 1} \mathcal{N}((T - \lambda I)^k) \) for \( \lambda \) is infinite dimensional;
where \( \mathcal{N}((T - \lambda I)^k) \) is the null set of the operator \( (T - \lambda I)^k \); for details, see, e.g., [1, 12, 25]. A complex number \( \lambda \in \sigma(T) \setminus \text{ess}(T) \) is called a normal eigenvalue of \( T \). If \( \lambda \) is a normal eigenvalue, then it is in \( P_{\sigma}(T) \) with finite dimensional generalized eigenspace \( \mathcal{N}((T - \lambda I)^k) \) for some natural number \( k \), and \( X \) can be represented as the direct sum of \( \mathcal{N}((T - \lambda I)^k) \) and \( \mathcal{R}((T - \lambda I)^k) \); \( X = \mathcal{N}((T - \lambda I)^k) \oplus \mathcal{R}((T - \lambda I)^k) \). We define the essential spectral radius of \( T \) by

\[
\rho_e(T) = \sup\{ |\lambda| | \lambda \in \text{ess}(T) \}.
\]

If a bounded linear operator \( U : X \to X \) is compact, then the relation \( \rho_e(T + U) = \rho_e(T) \) holds true; see, e.g., [25].

In the following result, we give an estimate on the essential spectral radius of the solution operator \( T(t) \) for Eq. (18).

**Theorem 1.** Assume that the function \( K \) in Eq. (18) satisfies the condition (15), and let \( T(t) \) be the solution operator for Eq. (18). Then

\[
\rho_e(T(t)) \leq e^{-\rho t}, \quad \forall t \geq 0.
\]

**Proof.** At first, we recall that for the function \( K \) in Eq. (18), there exists a (unique) \( m \times m \) matrix valued function \( r \) on \( \mathbb{R}^+ \) (it is called the resolvent kernel of \( K \)) which is locally integrable on \( \mathbb{R}^+ \) and satisfies the relation \( r = K + K \ast r = K + r \ast K \), where the convolution \( f \ast g \) for two functions \( f \) and \( g \) is defined by \( (f \ast g)(t) := \int_0^t f(t-s)g(s)ds \). As is well known (e.g., [7]), the solution of the equation

\[
x(t) = \int_0^t K(t-s)x(s)ds + f(t), \quad t > 0
\]

is given by

\[
x(t) = f(t) + (r \ast f)(t) \quad \text{a.e. in } \mathbb{R}^+.
\]

Now, for any \( \varphi \in X := L^1_m(\mathbb{R}^+; C^m) \) we consider a function \( f^\varphi \) defined by

\[
f^\varphi(t) = \int_{-\infty}^0 K(t-s)\varphi(s)ds = \left[ \int_{t}^{\infty} K(t-\tau)\varphi(t-\tau)d\tau \right], \quad t \geq 0.
\]

It is easy to see that \( \|f^\varphi\|_{1,\rho,+} := \int_0^{\infty} |f^\varphi(t)|e^{\rho t} dt \leq \|K\|_1\|\varphi\|_{1,\rho} \). Notice that the solution \( x := x(\cdot; 0, \varphi, 0) \) of Eq. (18) is expressed as

\[
x(t; 0, \varphi, 0) = \begin{cases} 
\varphi(t), & t \leq 0 \\
f^\varphi(t) + (r \ast f^\varphi)(t), & \text{a.e. in } \mathbb{R}^+.
\end{cases}
\]
by using the resolvent kernel \( r \), because of
\[
\int_{-\infty}^{t} K(t-s)x(s)ds = \int_{0}^{t} K(t-s)x(s)ds + \int_{-\infty}^{0} K(t-s)\varphi(s)ds = \int_{0}^{t} K(t-s)x(s)ds + f^\theta(t).
\]

Given any positive constant \( \alpha \), let us consider an operator \( U^\alpha \) on \( X \) defined by
\[
(U^\alpha \varphi)(\theta) = \begin{cases} 
  f^\theta(a + \theta) + (r * f^\theta)(a + \theta), & -\alpha < \theta \leq 0 \\
  0, & \theta \leq -\alpha 
\end{cases}
\]
for any \( \theta \leq 0 \) and \( \varphi \in X \). Then it is straightforward to see that \( U^\alpha \) is a bounded linear operator on \( X \) with norm \(||U^\alpha||| \leq e^{-\alpha\rho}||K||_1(1 + \int_{0}^{\alpha} ||r(\tau)||e^{\rho\tau} d\tau)\).

We assert that \( U^\alpha : X \to X \) is a compact operator. If the assertion holds true, then \( T(a) - U^\alpha =: S_0(a) \) satisfies \( r^c(T(a)) = r^c(S_0(a)) \leq ||S_0(a)|| \); here \( S_0(a) \) is a bounded linear operator on \( X \) defined as \( [S_0(a)\varphi]\theta = \varphi(a + \theta) \) if \( \theta \leq -\alpha \), and 0 if \(-\alpha < \theta \leq 0\), for any \( \varphi \in X \). Hence the estimate (20) is a direct consequence of the relation \(||S_0(a)|| = e^{-\alpha\rho} \), which follows from the identity \(||S_0(a)\varphi||_{1,\rho} = \int_{-\infty}^{\alpha} |\varphi(a + \theta)|e^{\rho\theta} d\theta = e^{-\alpha\rho}||\varphi||_{1,\rho} \).

In order to prove the compactness of the operator \( U^\alpha \), we show that for any bounded subset \( \mathcal{M} \) of \( X \), the set \( U^\alpha \mathcal{M} \) is relatively compact in \( X \). Observe that the operator \( \mathcal{K} : X \to L^1 := L^1(R^+; C^n) \) defined by \( (\mathcal{K}\varphi)(\theta) = \varphi(\theta)e^{\rho\theta} \) for \( \theta \leq 0 \) and \( \varphi \in X \) is an isometrical isomorphism between \( X \) and \( L^1 \). As well known (e.g., [4, Chapter IV, Theorem 20, page 20]), any bounded set \( S \) in \( L^1 \) is relatively compact if and only if \( \lim_{\epsilon \to 0} \int_{-\infty}^{0} |\psi(\theta) - \psi(\theta + \epsilon)|d\theta = 0 \)
uniformly for \( \psi \in S \) and \( \lim_{R \to \infty} \int_{-\infty}^{R} |\psi(\theta)|d\theta = 0 \), uniformly for \( \psi \in S \); hence it is known that \( \psi(\theta + \epsilon) = 0 \) when \( \theta + \epsilon > 0 \). Since the set \( U^\alpha \mathcal{M} \) is clearly bounded in \( X \) and since \( (U^\alpha\varphi)(\theta) = 0 \) for any \( \varphi \in \mathcal{M} \) if \( \theta \in (-\infty, -\alpha - 1) \), the relative compactness of the set \( U^\alpha \mathcal{M} \) is equivalent to the condition \( \lim_{\epsilon \to 0} \int_{-\infty}^{0} |(U^\alpha\varphi)(\theta) - (U^\alpha\varphi)(\theta + \epsilon)|e^{\rho\theta} d\theta = 0 \), uniformly for \( \varphi \in \mathcal{M} \); hence we must verify the following two assertions to guarantee the compactness of the operator \( U^\alpha \):

\[
\begin{align*}
\lim_{\epsilon \to 0} \int_{-\infty}^{0} |(U^\alpha\varphi)(-\epsilon + \theta) - (U^\alpha\varphi)(\theta)|e^{\rho\theta} d\theta &= 0, & \text{uniformly for } \varphi \in \mathcal{M}; \\
\lim_{\epsilon \to 0} \int_{-\infty}^{0} |(U^\alpha\varphi)(\epsilon + \theta) - (U^\alpha\varphi)(\theta)|e^{\rho\theta} d\theta &= 0, & \text{uniformly for } \varphi \in \mathcal{M};
\end{align*}
\]

here \( (U^\alpha\varphi)(\epsilon + \theta) = 0 \) if \( \epsilon + \theta > 0 \).
Proof of (21). Let $0 < \varepsilon < a$ be any number. Then

$$\begin{align*}
\int_{-\infty}^{0} |(U^{a}\phi)(-\varepsilon + \theta) - (U^{a}\phi)(\theta)|e^{\rho \theta} \ d\theta \\
= \int_{-\infty}^{0} |f^{\phi}(a - \varepsilon + \theta) + (r \ast f^{\phi})(a - \varepsilon + \theta) \\
- f^{\phi}(a + \theta) - (r \ast f^{\phi})(a + \theta)|e^{\rho \theta} \ d\theta \\
+ \int_{-a}^{-a+\varepsilon} |f^{\phi}(a + \theta) + (r \ast f^{\phi})(a + \theta)|e^{\rho \theta} \ d\theta \\
=: (I)^{\phi,\varepsilon} + (II)^{\phi,\varepsilon}.
\end{align*}$$

Observe that

$$\begin{align*}
\int_{-a}^{-a+\varepsilon} |f^{\phi}(a + \theta)|e^{\rho \theta} \ d\theta = \int_{-a}^{-a+\varepsilon} \int_{-\infty}^{0} K(a + \theta - s)\phi(s)ds \ e^{\rho \theta} \ d\theta \\
\leq e^{-pa} \int_{-\infty}^{0} |\phi(s)|e^{p\theta} \left(\int_{-a}^{-a+\varepsilon} \|K(a + \theta - s)\|e^{\rho(a + \theta - s)} \ d\theta\right)ds \\
\leq e^{-pa} \|\phi\|_{1,\rho} \left(\sup_{s \leq 0} \int_{-s}^{-s+\varepsilon} \|K(\tau)\|e^{\rho \tau} \ d\tau\right)
\end{align*}$$

and that

$$\begin{align*}
\int_{-a}^{-a+\varepsilon} |(r \ast f^{\phi})(a + \theta)|e^{\rho \theta} \ d\theta &\leq e^{-pa} \int_{0}^{\varepsilon} \left(\int_{0}^{\tau} |(r(u) - u)| \|f^{\phi}(u)\|e^{\rho \tau} \ du\right) \ d\tau \\
&= e^{-pa} \int_{0}^{\varepsilon} \left(\int_{0}^{\tau} |(r(u) - u)| \|e^{\rho(\tau-u)} \|f^{\phi}(u)\|e^{\rho u} \ du\right) \\
&\leq e^{-pa} \left(\int_{0}^{\varepsilon} |r(v)|e^{\rho \tau} \ dv\right) \left(\int_{0}^{\varepsilon} \|f^{\phi}(u)\|e^{\rho u} \ du\right) \\
&\leq e^{-pa} \left(\int_{0}^{\varepsilon} |r(v)|e^{\rho \tau} \ dv\right) \|K\|_{1} \|\phi\|_{1,\rho},
\end{align*}$$

where we used the fact that $\|f^{\phi}\|_{1,\rho, +} \leq \|K\|_{1} \|\phi\|_{1,\rho}$. Therefore we get

$$\begin{align*}
(II)^{\phi,\varepsilon} := \int_{-a}^{-a+\varepsilon} |f^{\phi}(a + \theta) + (r \ast f^{\phi})(a + \theta)|e^{\rho \theta} \ d\theta \\
\leq e^{-pa} \|\phi\|_{1,\rho} \left(\|K\|_{1} \int_{0}^{\varepsilon} |r(v)|e^{\rho \tau} \ dv + \sup_{s \leq 0} \int_{-s}^{-s+\varepsilon} \|K(\tau)\|e^{\rho \tau} \ d\tau\right).
\end{align*}$$
Note that $\int_0^\varepsilon \|r(v)\|e^{\rho_0} dv \to 0$ as $\varepsilon \to +0$. Moreover, it follows from (15) that

$$\lim_{\varepsilon \to +0} \int_0^{b+\varepsilon} \|K(\tau)\|e^{\rho_\tau} d\tau = 0$$

uniformly for $b \in \mathbb{R}^+$. We thus obtain that $\lim_{\varepsilon \to +0} (II)^{\phi, \varepsilon} = 0$, uniformly for $\phi \in \mathcal{M}$.

Next, we will evaluate the term $(I)^{\phi, \varepsilon}$. First we have

$$\int_{-a+\varepsilon}^0 \left| f^\phi(a - \varepsilon + \theta) - f^\phi(a + \theta) \right| e^{\rho_\theta} d\theta$$

$$= \int_{-a+\varepsilon}^0 \left| \int_{-\infty}^0 K(a - \varepsilon + \theta - s) \varphi(s) ds - \int_{-\infty}^0 K(a + \theta - s) \varphi(s) ds \right| e^{\rho_\theta} d\theta$$

$$\leq \int_{-\infty}^0 \int_{-a+\varepsilon}^0 \|K(a - \varepsilon + \theta - s) - K(a + \theta - s)\| \|\varphi(s)\| e^{\rho_\theta} d\theta ds$$

$$\leq e^{-\rho_\theta} \int_{-\infty}^0 \left( \int_{-\varepsilon}^\infty \|K(\tau - \varepsilon) - K(\tau)\| e^{\rho_\tau} d\tau \right) \|\varphi(s)\| e^{\rho_\tau} ds$$

$$\leq e^{-\rho_\theta} \|\varphi\|_1 e^{\rho_\varepsilon} \int_{-\infty}^\infty \|K(\tau) - K(\tau + \varepsilon)\| e^{\rho_\tau} d\tau;$$

hence

(23) $\int_{-a+\varepsilon}^0 \left| f^\phi(a - \varepsilon + \theta) - f^\phi(a + \theta) \right| e^{\rho_\theta} d\theta$

$$\leq e^{-\rho_\theta} \|\varphi\|_1 e^{\rho_\varepsilon} \int_{-\varepsilon}^\infty \|K(\tau) - K(\tau + \varepsilon)\| e^{\rho_\tau} d\tau.$$

Moreover,

$$\int_{-a+\varepsilon}^0 \left| (r * f^\phi)(a - \varepsilon + \theta) - (r * f^\phi)(a + \theta) \right| e^{\rho_\theta} d\theta$$

$$= \int_{-a+\varepsilon}^0 \left| \int_{-\varepsilon+\theta}^{a+\theta} r(s) f^\phi(a - \varepsilon + \theta - s) ds - \int_{-\varepsilon+\theta}^{a+\theta} r(s) f^\phi(a + \theta - s) ds \right| e^{\rho_\theta} d\theta$$

$$\leq \int_{-a+\varepsilon}^0 \left( \int_{-\varepsilon+\theta}^{a+\theta} \|r(s)\| \left| f^\phi(a - \varepsilon + \theta - s) - f^\phi(a + \theta - s) \right| ds \right) e^{\rho_\theta} d\theta$$

$$+ \int_{-a+\varepsilon}^0 \left( \int_{-\varepsilon+\theta}^{a+\theta} \|r(s)\| \left| f^\phi(a + \theta - s) \right| ds \right) e^{\rho_\theta} d\theta$$

$$=: (III)^{\phi, \varepsilon} + (IV)^{\phi, \varepsilon}.$$

Observe that
\[
(IV)^{\varphi,\varepsilon} := \int_{a+\varepsilon}^{e} \left( \int_{a}^{0} \left\| r(a + \theta - \tau) \right\| e^{\rho \theta} d\theta \right) |f^{\varphi}(\tau)| d\tau
\]
\[
\leq e^{-\rho a} \left( \int_{0}^{a} \left\| r(v) \right\| e^{\rho v} dv \right) \left( \int_{0}^{e} |f^{\varphi}(\tau)| e^{\rho \tau} d\tau \right)
\]
\[
\leq e^{-\rho a} \left( \int_{0}^{a} \left\| r(v) \right\| e^{\rho v} dv \right) \|\varphi\|_{1,\rho} \left( \sup_{s \leq 0} \int_{-s}^{-s+\varepsilon} \|K(\tau)\| e^{\rho \tau} d\tau \right),
\]
and that
\[
(III)^{\varphi,\varepsilon} := \int_{a+\varepsilon}^{0} \left( \int_{0}^{a-\varepsilon+\theta} \left\| r(s) \right\| |f^{\varphi}(a - \varepsilon + \theta - s) - f^{\varphi}(a + \theta - s)| ds \right) e^{\rho \theta} d\theta
\]
\[
= \int_{0}^{a-\varepsilon} \left\| r(s) \right\| \left( \int_{0}^{a-\varepsilon+\theta} |f^{\varphi}(a - \varepsilon + \theta - s) - f^{\varphi}(a + \theta - s)| e^{\rho \theta} d\theta \right) ds
\]
\[
\leq \int_{0}^{a-\varepsilon} \left\| r(s) \right\| \left( e^{-\rho(a-s)} \|\varphi\|_{1,\rho} e^{\rho s} \int_{0}^{e} \|K(\tau) - K(\tau + \varepsilon)\| e^{\rho \tau} d\tau \right) ds,
\]
where we have used the inequality (23) with \(a - s\) in place of \(a\). Thus it follows that
\[
0 \leq (I)^{\varphi,\varepsilon} \leq e^{-\rho a} \|\varphi\|_{1,\rho} e^{\rho a} \int_{0}^{a} \left\| K(\tau) - K(\tau + \varepsilon) \right\| e^{\rho \tau} d\tau + (III)^{\varphi,\varepsilon} + (IV)^{\varphi,\varepsilon}
\]
\[
\leq \|\varphi\|_{1,\rho} \left( 1 + \int_{0}^{a} \left\| r(v) \right\| dv \right)
\]
\[
\times \left\{ \int_{0}^{e} \left\| K(\tau) - K(\tau + \varepsilon) \right\| e^{\rho \tau} d\tau + \left( \sup_{s \leq 0} \int_{-s}^{-s+\varepsilon} \|K(\tau)\| e^{\rho \tau} d\tau \right) \right\}.
\]
Since the terms \(\int_{0}^{e} \left\| K(\tau) - K(\tau + \varepsilon) \right\| e^{\rho \tau} d\tau\) and \(\sup_{s \leq 0} \int_{-s}^{-s+\varepsilon} \|K(\tau)\| e^{\rho \tau} d\tau\) tend to 0 as \(\varepsilon \to +0\) by (15), we get that \(\lim_{\varepsilon \to +0} (I)^{\varphi,\varepsilon} = 0\), uniformly for \(\varphi \in \mathcal{M}\), as required.

**Proof of (22).** Let \(0 < \varepsilon < a\) be any number. Then
\[
\int_{-\infty}^{0} \left| (U^a \varphi)(\varepsilon + \theta) - (U^a \varphi)(\theta) \right| e^{\rho \theta} d\theta
\]
\[
= \int_{-\varepsilon}^{0} \left| (U^a \varphi)(\theta) \right| e^{\rho \theta} d\theta + \int_{-\infty}^{-\varepsilon} \left| (U^a \varphi)(\varepsilon + \theta) - (U^a \varphi)(\theta) \right| e^{\rho \theta} d\theta
\]
\[
= \int_{-\varepsilon}^{0} \left| f^{\varphi}(a + \theta) + (r * f^{\varphi})(a + \theta) \right| e^{\rho \theta} d\theta
\]
\[
+ \int_{-\infty}^{0} \left| (U^a \varphi)(\tau) - (U^a \varphi)(\varepsilon + \tau) \right| e^{\rho(\tau-\varepsilon)} d\tau
\]
\[
=: (V)^{\varphi,\varepsilon} + (VI)^{\varphi,\varepsilon}.
\]
As shown in the proof of (21), $e^{\rho t(VI)^{\varphi,\varepsilon}} = \int_{-\infty}^{0} |(U^a \varphi)(\tau) - (U^a \varphi)(-\varepsilon + \tau)| e^{\rho t} \, dt \to 0$ as $\varepsilon \to +0$, uniformly for $\varphi \in \mathcal{M}$; hence $(VI)^{\varphi,\varepsilon} \to 0$ as $\varepsilon \to +0$, uniformly for $\varphi \in \mathcal{M}$. Also, observe that

$$\int_{-\varepsilon}^{0} |f^{\varphi}(a + \theta)| e^{\rho \theta} \, d\theta \leq \int_{-\infty}^{0} \left( \int_{-\infty}^{0} |K(a + \theta - s)| e^{\rho(a + \theta - s)} |\varphi(s)| e^{\rho(s-a)} \, ds \right) \, d\theta$$

$$\leq e^{-\rho a} \|\varphi\|_{1,R} \left( \sup_{s \leq 0} \int_{a-s-\varepsilon}^{a-s} \|K(u)\| e^{\rho u} \, du \right)$$

and that

$$\int_{-\varepsilon}^{0} |(r * f^{\varphi})(a + \theta)| e^{\rho \theta} \, d\theta$$

$$\leq e^{-\rho a} \int_{a-\varepsilon}^{a} \left( \int_{0}^{\tau} \|r(u)\| |f^{\varphi}(\tau-u)| e^{\rho \tau} \, du \right) \, d\tau$$

$$= e^{-\rho a} \int_{a-\varepsilon}^{a} \left( \int_{0}^{\tau} \|r(u)\| |f^{\varphi}(\tau-u)| e^{\rho \tau} \, du \right) \, d\tau$$

$$+ e^{-\rho a} \int_{a-\varepsilon}^{a} \|r(u)\| e^{\rho \tau} \, d\tau$$

$$\leq e^{-\rho a} \left( \int_{0}^{a-\varepsilon} \|r(u)\| e^{\rho u} \, du \right) \left( \sup_{0 \leq u \leq a-\varepsilon} \int_{a-\varepsilon}^{a} |f^{\varphi}(\tau-u)| e^{\rho(\tau-u)} \, d\tau \right)$$

$$+ e^{-\rho a} \left( \int_{a-\varepsilon}^{a} \|r(u)\| e^{\rho u} \, du \right) \left( \sup_{a-\varepsilon \leq u \leq a} \int_{a-\varepsilon}^{a} |f^{\varphi}(\tau-u)| e^{\rho(\tau-u)} \, d\tau \right)$$

$$\leq e^{-\rho a} \left( \int_{0}^{a-\varepsilon} \|r(u)\| e^{\rho u} \, du \right) \left( \sup_{0 \leq u \leq b+\varepsilon} \int_{b+\varepsilon}^{b+\varepsilon} |f^{\varphi}(v)| e^{\rho v} \, dv \right)$$

$$+ e^{-\rho a} \left( \int_{a-\varepsilon}^{a} \|r(u)\| e^{\rho u} \, du \right) \left( \int_{0}^{a} |f^{\varphi}(v)| e^{\rho v} \, dv \right)$$

$$\leq e^{-\rho a} \left( \int_{0}^{a} \|r(u)\| e^{\rho u} \, du \right) \left( \sup_{0 \leq u \leq a-\varepsilon} \int_{a-\varepsilon}^{a} \left\{ \int_{-\infty}^{0} \|K(v-s)\| |\varphi(s)| e^{\rho v} \, ds \right\} \, dv \right)$$

$$+ e^{-\rho a} \|K\|_{1} \|\varphi\|_{1,R} \left( \int_{a-\varepsilon}^{a} \|r(u)\| e^{\rho u} \, du \right)$$

$$\leq e^{-\rho a} \left( \int_{0}^{a} \|r(u)\| e^{\rho u} \, du \right) \left( \sup_{0 \leq u \leq c+\varepsilon} \int_{c+\varepsilon}^{c+\varepsilon} \|K(u)\| e^{\rho u} \, du \right) \|\varphi\|_{1,R}$$

$$+ e^{-\rho a} \|K\|_{1} \|\varphi\|_{1,R} \left( \int_{a-\varepsilon}^{a} \|r(u)\| e^{\rho u} \, du \right).$$
We thus obtain
\[
(V)^{\varphi,\varepsilon} := \int_{-\varepsilon}^{0} \left| f^{\varphi}(a + \theta) + (r \ast f^\varphi)(a + \theta) \right| e^{\varphi \theta} \, d\theta
\]
\[
\leq e^{-\rho \varphi} \| \varphi \|_{1, \rho} \left[ \sup_{s \leq 0} \int_{a-s-\varepsilon}^{a-s} \| K(u) \| e^{\mu u} \, du + \| K \|_{1} \int_{a-\varepsilon}^{a} \| r(u) \| e^{\mu u} \, du \right.
\]
\[
+ \left( \int_{\varepsilon}^{e+\varepsilon} \| r(u) \| e^{\mu u} \, du \right) \left( \sup_{c \geq 0} \int_{c-\varepsilon}^{c+\varepsilon} \| K(u) \| e^{\mu u} \, du \right),
\]
which shows that \((V)^{\varphi,\varepsilon} \to 0\) as \(\varepsilon \to +0\), uniformly for \(\varphi \in \mathcal{M}\). This completes the proof. \(\square\)

**Corollary 2.** Let the function \(K\) in Eq. (18) satisfy the condition (15), and let \(A\) be the generator of the solution semigroup \(\{T(t)\}_{t \geq 0}\) for Eq. (18). Then,
\[
\sup_{\lambda \in \text{ess}(A)} \Re \lambda \leq -\rho.
\]

**Proof.** By virtue of [25, Chapter 4, Proposition 4.13], one has that
\[
\{ e^{i\lambda} \mid \lambda \in \text{ess}(A) \} \subset \text{ess}(T(t)), \quad t > 0.
\]
Then, (24) is a direct consequence of Theorem 1. \(\square\)

Let \(c\) be a (fixed) constant such that \(c > -\rho\). Define \(\mathcal{C}_c := \{ z \in \mathbb{C} \mid \Re z \geq c \}\). We consider the set \(\sigma(A) \cap \mathcal{C}_c =: \Sigma_c^U\). By virtue of (19) and Corollary 2, we see that if \(\lambda_0 \in \Sigma_c^U\), then \(\lambda_0 \notin \text{ess}(A)\) and \(\det A(\lambda_0) = 0\). Therefore, since \(\det A(z)\) is an analytic function of \(z\) in the domain \(\mathbb{C}_{-\rho}, \Sigma_c^U\) is (at most) a finite set which consists of normal eigenvalues of \(A\). Then, from the well known result on the strongly continuous semigroup (see, e.g., [12, Section 5.3], [25, Chapter 4]) or periodic evolutionary process (see, e.g., [8, 11, 12]) one can get the following result on the decomposition of the phase space \(X\):

**Theorem 2.** For any real \(c > -\rho\), let \(\Sigma_c^U := \{ \lambda \in \sigma(A) \mid \Re \lambda \geq c \}\). Then, \(X\) is decomposed as a direct sum of closed subspaces \(U\) and \(S\)
\[
X = U \oplus S
\]
with the following properties:
(i) \(\dim U < \infty\);
(ii) \(T(t)U \subset U, T(t)S \subset S \quad (\forall t \geq 0)\);
(iii) \(\sigma(A|_U) = \Sigma^U, \sigma(A|_{\Sigma^U}) = \sigma(A) - \Sigma^U =: \Sigma^S\);
(iv) \(T^U(t) := T(t)|_U\) is extendable for \(t \in (-\infty, \infty)\), as a group of bounded linear operators on \(U\);
(v) \( T^S(t) := T(t)|_S \) is a strongly continuous semigroup of bounded linear operators on \( S \), and its generator is identical with the operator \( A|_{S \cap \sigma(A)} \);

(vi) for sufficiently small \( \varepsilon > 0 \) there exists a \( \gamma(\varepsilon) > 0 \) such that

\[
\|T^U(t)\| \leq \gamma(\varepsilon)e^{(c-\varepsilon)t} \quad (\forall t \leq 0),
\]

\[
\|T^S(t)\| \leq \gamma(\varepsilon)e^{(c+\varepsilon)t} \quad (\forall t \geq 0).
\]

In the above, the finite dimensional space \( U \) is indeed a direct sum of the generalized eigenspaces for normal eigenvalues of \( A \) belonging to the finite set \( \Sigma^U \). Throughout this paper, we denote by \( \Pi^U \) (or \( \Pi^S \)) the projection from \( X \) onto \( U \) (or \( S \) respectively) along the above decomposition. In fact, the projection \( \Pi^U \) is represented as

\[
\Pi^U = \frac{1}{2\pi i} \int \frac{(zI - A)^{-1}}{z} \, dz,
\]

where \( \mathcal{C} \) is a closed rectifiable Jordan curve which is disjointed with \( \sigma(A) \) and contains \( \Sigma^U \) in its interior but no element of \( \Sigma^S \).

5. Variation-of-constants formula in the phase space

In this section we will treat the following nonhomogeneous linear integral equation

\[
x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + p(t), \quad t > \sigma.
\]

(25)

Recall that \( x(\cdot; \sigma, \varphi, p) \) is the (unique) solution of Eq. (25) through \( (\sigma, \varphi) \); here \( \varphi \in X \). In what follows, we will establish a representation formula for \( x(t; \sigma, \varphi, p) \) in the space \( X \) by using \( T(t) \), \( \varphi \) and \( p \). To do this, we introduce a continuous function \( \Gamma^n : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) for each natural number \( n \) which is of compact support with \( \operatorname{supp} \Gamma^n \subset [-1/n, 0] \) and satisfies \( \int_{-\infty}^{0} \Gamma^n(\theta)d\theta = 1 \). Notice that \( \Gamma^n \beta \in X \) for any \( \beta \in C^m \).

Theorem 3. Let \( p \in C(\{\sigma, \infty\}; C^m) \). Then

(26)

\[
\lim_{n \to \infty} \left\| \int_{a}^{a+\sigma} \left( T(a+\sigma-s)(\Gamma^n p(s))ds - x_{a+\sigma}(\sigma, 0, p) \right) \right\|_{L^1, p} = 0
\]

for any \( a > 0 \). As a consequence, we get the following formula:

(27)

\[
x_t(\sigma, \varphi, p) = T(t-\sigma)\varphi + \lim_{n \to \infty} \int_{a}^{t} T(t-s)(\Gamma^n p(s))ds, \quad \forall t \geq \sigma
\]

in \( X \).
Proof. In order to establish the theorem, it suffices to prove the relation (26), because of \( x_t(\sigma, \varphi, p) = x_t(\sigma, \varphi, 0) + x_t(\sigma, 0, p) = x_{t-\sigma}(0, \varphi, 0) + x_t(\sigma, 0, p), \) \( t \geq \sigma, \) on \( \mathbb{R}^1 \) by the principle of superposition for Eq. (25). Also, putting \( h(t) = p(t + \sigma) \) one can see that \( x(t; \sigma, 0, p) \equiv x(t - \sigma; 0, 0, h); \) and hence (26) is equivalent to

\[
\lim_{n \to \infty} \left\| \int_0^a T(a - s)(\Gamma^n h(s))ds - x_a(0, 0, h) \right\|_{1, \rho} = 0,
\]

which is exactly the same one as (26) with 0 and \( h \in C([0, \infty); C^m) \) in place of \( \sigma \) and \( p \in C([\sigma, \infty); C^m) \). Thus, without loss of generality, it suffices to treat the case \( \sigma = 0 \) and \( p \in C([0, \infty); C^m) \) in (26). In what follows, under the assumption that \( \sigma = 0 \) and \( p \in C([0, \infty); C^m) \), we will prove (26) in three steps.

Step 1: Let \( n \) be any fixed natural number. Since \( p(s) \) is continuous in \( s \in [0, a] \), the \( X \)-valued function \( \Gamma^n(\cdot)p(s) \) is continuous in \( s \in [0, a] \). Therefore, the integral \( \int_0^a T(a - s)(\Gamma^n p(s))ds \) of a continuous \( X \)-valued function \( T(a - s) \cdot (\Gamma^n p(s)) \) is equal to the limit of a Riemann sum \( (a/N) \sum_{k=1}^N T(a - ka/N) \cdot (\Gamma^n p(ka/N)) \) in \( X \); that is,

\[
\lim_{N \to \infty} \left\| \left(\frac{a}{N}\right) \sum_{k=1}^N T(a - ka/N)(\Gamma^n p(ka/N)) \right\|_{1, \rho} = 0.
\]

Hence, applying a well known result in the theory of integration (e.g., [24, Theorem 3.12]) one can choose a sequence of natural numbers \( \{l_N\}, \) \( l_N \to \infty \) as \( N \to \infty \), such that, for almost all \( \theta \leq 0, \)

\[
(28) \quad \left[ \left(\frac{a}{l_N}\right) \sum_{k=1}^{l_N} T(a - ka/l_N)(\Gamma^n p(ka/l_N)) \right](\theta) \to \left[ \int_0^a T(a - s)(\Gamma^n p(s))ds \right](\theta)
\]

as \( N \to \infty \). It follows that

\[
\left[ \left(\frac{a}{l_N}\right) \sum_{k=1}^{l_N} T(a - ka/l_N)(\Gamma^n p(ka/l_N)) \right](\theta) = \left(\frac{a}{l_N}\right) \sum_{k=1}^{l_N} x(a - ka/l_N + \theta; 0, \Gamma^n p(ka/l_N), 0)
\]

\[
= \sum_{k=1}^{l_N} f(s_k, \theta) \Delta s_k,
\]
where \( f(s, \theta) := x(a-s+\theta;0, \Gamma^np(s), 0) \) for \( (s, \theta) \in D := [0, a] \times (-\infty, 0) \), \( s_k := ka/l_N \) and \( A_{s_k} = a/l_N \) for \( k = 0, 1, \ldots, l_N \). Notice that

\[
\|\Gamma^np(s)\|_{1,p} = \int_{-\infty}^{0} |\Gamma^np(\theta)p(s)| e^{\rho \theta} d\theta \leq |p(s)| \int_{-\infty}^{0} |\Gamma^n(\theta)| d\theta
\]

Hence, if \( \theta \in (s-a,0) \), then

\[
|x(a-s+\theta;0, \Gamma^np(s), 0)| = |L(x_{a-s+\theta}(0, \Gamma^np(s), 0))|
\]

\[
\leq \|L\| \|T(a-s+\theta)(\Gamma^np(s))\|
\]

\[
\leq \|L\| \|p(0,a)\|_{\sup_{0 \leq s \leq a} \|T(\tau)\|},
\]

where \( L \) is the bounded linear operator introduced in the beginning of Section 4. Also, for \( \theta \in (-\infty, s-a] \) it follows that \( |x(a-s+\theta;0, \Gamma^np(s), 0)| \leq |p|_{[0,a]} \max_{-1/n \leq v \leq 0} |\Gamma^n(v)| \). This observation leads to the fact that the function \( f(s, \theta) \) satisfies \( \sup_B |f(s, \theta)| =: C_n < \infty \) with some constant \( C_n \). Since \( f(s, \theta) \) is continuous on \( D \) except points on the line \( s = \theta + a \), for each \( \theta \) the bounded function \( f(s, \theta) \) is Riemann integrable on \( [0, a] \) as a function of \( s \); and hence it follows that

\[
\lim_{N \to \infty} \sum_{k=1}^{l_N} f(s_k, \theta)A_{s_k} = \int_{0}^{a} f(s, \theta)ds.
\]

Combining this with (28) we get

\[
\int_{0}^{a} f(s, \theta)ds = \int_{0}^{a} T(a-s)(\Gamma^np(s))ds(\theta) \quad \text{a.e. in } \theta \leq 0;
\]

that is,

\[
\int_{0}^{a} x(a-s+\theta;0, \Gamma^np(s), 0)ds = \int_{0}^{a} T(a-s)(\Gamma^np(s))ds(\theta) \quad \text{a.e. in } \theta \leq 0,
\]

for any \( n = 1, 2, \ldots \).

**Step 2:** By virtue of (29) it follows that

\[
\left\| \int_{0}^{a} T(a-s)(\Gamma^np(s))ds - x_{a}(0,0,p) \right\|_{1,p}
\]

\[
= \int_{-\infty}^{0} \left[ \left| \int_{0}^{a} T(a-s)(\Gamma^np(s))ds(\theta) - x(a+\theta;0,0,p) \right| e^{\rho \theta} d\theta \right.
\]

\[
- \int_{-\infty}^{0} \left| \int_{0}^{a} T(a-s)(\Gamma^np(s))ds(\theta) - x_{a}(0,0,p) \right| e^{\rho \theta} d\theta.
\]
\[
\leq \int_{-\infty}^{0} \left| \int_{0}^{a} x(a-s+\theta; 0, \Gamma^n p(s), 0) ds - x(a+\theta; 0, 0, p) \right| d\theta \\
= \int_{-\infty}^{-a} \left| \int_{0}^{a} x(a-s+\theta; 0, \Gamma^n p(s), 0) ds \right| d\theta \\
+ \int_{-a}^{0} \left| \int_{0}^{a} x(a-s+\theta; 0, \Gamma^n p(s), 0) ds - x(a+\theta; 0, 0, p) \right| d\theta \\
=: (I_n) + (II_n).
\]

Observe that if \(0 \leq s \leq a\) and \(\theta \leq -a - 1/n\), then \(a - s + \theta \leq a + \theta \leq -1/n\) and 
\[|x(a-s+\theta; 0, \Gamma^n p(s), 0)| = |\Gamma^n(a-s+\theta)p(s)| = 0.\]
Then
\[
(I_n) := \int_{-\infty}^{-a-1/n} \left( \int_{0}^{a} |x(a-s+\theta; 0, \Gamma^n p(s), 0)| ds \right) d\theta \\
\leq \int_{-\infty}^{-a-1/n} \left( \int_{0}^{a} |\Gamma^n(a-s+\theta)p(s)| ds \right) d\theta.
\]
Therefore, for any sufficiently large \(n\) satisfying \(1/n < a\), it follows that
\[
(I_n) \leq \int_{0}^{a} \left( \int_{-a-1/n}^{-a} \Gamma^n(a-s+\theta) d\theta \right) |p(s)| ds \\
= \int_{1/n}^{a} \left( \int_{-a-1/n}^{-a} \Gamma^n(a-s+\theta) d\theta \right) |p(s)| ds \\
+ \int_{0}^{1/n} \left( \int_{-a-1/n}^{-a} \Gamma^n(a-s+\theta) d\theta \right) |p(s)| ds \\
\leq \int_{0}^{1/n} |p(s)| ds,
\]
where we used the facts that \(\int_{-\infty}^{0} \Gamma^n(\theta) d\theta = 1\) and that \(a - s + \theta \leq -s \leq -1/n\) and \(\Gamma^n(a-s+\theta) = 0\) whenever \(1/n \leq s \leq a\) and \(-a - 1/n \leq \theta \leq -a\). We thus conclude that \(\lim_{n \to \infty} (I_n) = 0\) because of \(\lim_{n \to \infty} \int_{0}^{1/n} |p(s)| ds = 0\).

**Step 3:** It remains only to establish that \((II_n) \to 0\) as \(n \to \infty\). The term \((II_n)\) is written as
\[(I_n) := \left[ \int_{-a}^{a} \int_{0}^{\theta} x(a - s + \theta; 0, \Gamma^n p(s), 0) ds - x(a + \theta; 0, 0, p) \right] d\theta \]
\[= \int_{0}^{a} \left[ \int_{0}^{\theta} x(\tau - s; 0, \Gamma^n p(s), 0) ds - x(\tau; 0, 0, p) \right] d\tau.\]

Recall that \(x(\cdot) := x(\cdot; 0, 0, p)\) satisfies the relation
\[x(t) = \int_{-\infty}^{t} K(t - \tau)x(\tau)d\tau + p(t) = \int_{0}^{t} K(t - \tau)x(\tau)d\tau + p(t), \quad t > 0.\]

Hence, using the resolvent kernel \(r\) of \(K\) which was introduced in the proof of Theorem 1 \(x(\cdot; 0, 0, p)\) is expressed as
\[x(t; 0, 0, p) = p(t) + (r * p)(t) = p(t) + \int_{0}^{t} r(t - s)p(s)ds \quad \text{a.e. in } t \geq 0.\]

Similarly, \(x(\cdot; 0, \Gamma^n p(s), 0)\) is expressed as
\[x(t; 0, \Gamma^n p(s), 0) = \begin{cases} 
\Gamma^n(t)p(s), & t \leq 0 \\
\int_{t}^{0} f^{(n)}(t, s) + (r \ast f^{(n)})(\cdot, s))(t), & t > 0
\end{cases}
\quad \text{a.e. in } t \in \mathbb{R}, \]
where
\[f^{(n)}(t, s) := \int_{-\infty}^{t} K(t - v)(\Gamma^n p(s))(v)dv = \left( \int_{-\infty}^{0} K(t - v)(\Gamma^n(v))p(s) \right) p(s).\]

Thus it follows that
\[(II_n) \leq \int_{0}^{a} \int_{0}^{\tau} \left\{ f^{(n)}(\tau - s, s) + (r * f^{(n)}(\cdot, s))(\tau - s) \right\} ds \]
\[+ \int_{0}^{a} \Gamma^n(\tau - s)p(s)ds - \{ p(\tau) + (r * p)(\tau) \} d\tau \]
\[\leq \int_{0}^{a} \int_{0}^{\tau} \left\{ f^{(n)}(\tau - s, s) + (r * f^{(n)})(\cdot, s))(\tau - s) \right\} ds - (r * p)(\tau) d\tau \]
\[+ \int_{0}^{a} \int_{\tau}^{a} \Gamma^n(\tau - s)p(s)ds - p(\tau) d\tau \]
\[= (III_n) + (IV_n).\]

For any sufficiently large \(n\) such that \(1/n < a\), we get
\((IV_n) := \int_0^a \int_\tau^a \Gamma^n(\tau - s)p(s)ds - p(\tau) d\tau\)

\[= \int_0^{a-1/n} \int_\tau^a \Gamma^n(\tau - s)p(s)ds - p(\tau) d\tau + \int_{a-1/n}^a \int_\tau^a \Gamma^n(\tau - s)p(s)ds - p(\tau) d\tau\]

\[= \int_0^{a-1/n} \int_\tau^a \Gamma^n(\tau - s)p(s)ds - \left( \int_\tau^a \Gamma^n(\tau - s)ds \right) p(\tau) d\tau + (2/n)|p|_{[0,a]} ,\]

because \(\tau - a \leq -1/n\) and \(\int_\tau^a \Gamma^n(\tau - s)ds = \int_{\tau-a}^0 \Gamma^n(\theta)d\theta = 1\) whenever \(0 \leq \tau \leq a - 1/n\). Therefore

\[(IV_n) \leq (2/n)|p|_{[0,a]} + \int_0^{a-1/n} \left( \int_\tau^a |\Gamma^n(\tau - s)||p(s) - p(\tau)|ds \right) d\tau\]

\[\leq (2/n)|p|_{[0,a]} + \int_0^{a-1/n} \left( \int_\tau^{\tau+1/n} |\Gamma^n(\tau - s)||p(s) - p(\tau)|ds \right) d\tau\]

\[\leq (2/n)|p|_{[0,a]} + a \times \sup\{|p(s) - p(\tau)| : \tau \leq s \leq \tau + 1/n, 0 \leq \tau \leq a\},\]

which yields that \(\lim_{n\to\infty}(IV_n) = 0\) by the continuity of \(p\).

Also, it follows from the relation \(r(t) = K(t) + (r \ast K)(t)\) a.e. in \(R^+\) that

\((III_n) := \int_0^a \int_0^\tau \{f^{(n)}(\tau - s,s) + (r \ast f^{(n)})(\cdot,s)(\tau - s)\}ds - (r \ast p)(\tau)\) d\tau

\[= \int_0^a \int_0^\tau \left\{ [f^{(n)}(\tau - s,s) - K(\tau - s)p(s)]
\quad + [(r \ast f^{(n)})(\cdot,s)(\tau - s) - (r \ast K)(\tau - s)p(s)]
\quad + [K(\tau - s)p(s) + (r \ast K)(\tau - s)p(s)]ds - (r \ast p)(\tau) \right\} d\tau\]

\[\leq \int_0^a \int_0^\tau \{f^{(n)}(\tau - s,s) - K(\tau - s)p(s)\}ds d\tau\]

\[+ \int_0^a \int_0^\tau \{(r \ast f^{(n)})(\cdot,s)(\tau - s) - (r \ast K)(\tau - s)p(s)\}ds d\tau\]

\[+ \int_0^a \int_0^\tau r(\tau - s)p(s)ds - (r \ast p)(\tau) \right\} d\tau\]

\[= \int_0^a \int_0^\tau \{f^{(n)}(\tau - s,s) - K(\tau - s)p(s)\}ds d\tau\]

\[+ \int_0^a \int_0^\tau \{(r \ast f^{(n)})(\cdot,s)(\tau - s) - (r \ast K)(\tau - s)p(s)\}ds d\tau.\]
If \( 0 \leq \tau \leq a \), then

\[
\left| \int_{0}^{\tau} \left\{ f^{(n)}(\tau - s, s) - K(\tau - s)p(s) \right\} ds \right|
\]

\[
\leq \int_{0}^{\tau} \left( \left( \int_{-\infty}^{0} K(\tau - s - v)\Gamma^{n}(v)dv \right) p(s) - K(\tau - s) \left( \int_{-\infty}^{0} \Gamma^{n}(v)dv \right) p(s) \right) ds
\]

\[
\leq \int_{0}^{\tau} \left( \left| \int_{-\infty}^{0} K(\tau - s - v) - K(\tau - s) \right| |\Gamma^{n}(v)|dv \right) |p(s)|ds
\]

\[
= \int_{0}^{\tau} \left( \left| \int_{-1/n}^{0} K(\tau - s - v) - K(\tau - s) \right| |\Gamma^{n}(v)|dv \right) |p(s)|ds
\]

\[
\leq |p|_{[0,a]} \int_{-1/n}^{0} \left( \left( \sup_{-1/n \leq v \leq 0} \left| \int_{0}^{\tau} K(\tau - s - v) - K(\tau - s) \right| ds \right) \right) |\Gamma^{n}(v)|dv
\]

\[
\leq |p|_{[0,a]} \times \left( \sup_{-1/n \leq v \leq 0} \int_{0}^{\tau} \left| K(\mu - v) - K(\mu) \right| d\mu \right).
\]

Similarly, for any \( \tau \in [0,a] \) we get

\[
\left| \int_{0}^{\tau} \{ (r * f^{(n)}(\cdot, s))(\tau - s) - (r * K)(\tau - s)p(s) \} ds \right|
\]

\[
\leq \int_{0}^{\tau} \int_{0}^{\tau-s} |r(\tau - s - \mu)| |f^{(n)}(\mu, s) - K(\mu)p(s)| d\mu ds
\]

\[
= \int_{0}^{\tau} \int_{0}^{\tau-s} |r(\tau - s - \mu)|
\]

\[
\times \left( \left( \int_{-\infty}^{0} K(\mu - v)\Gamma^{n}(v)dv - K(\mu) \right) \int_{-\infty}^{0} \Gamma^{n}(v)dv \right) p(s) ds
\]

\[
\leq \int_{0}^{\tau} \int_{0}^{\tau-s} |r(\tau - s - \mu)| \left( \int_{-1/n}^{0} \left| K(\mu - v) - K(\mu) \right| |\Gamma^{n}(v)|dv \right) |p(s)|ds
\]

\[
\leq |p|_{[0,a]} \times \left( \sup_{-1/n \leq v \leq 0} \int_{0}^{\tau} |r(\tau - s - \mu)| \left| K(\mu - v) - K(\mu) \right| d\mu \right)
\]

\[
\leq |p|_{[0,a]} \times \left( \sup_{0 \leq l \leq \tau} \int_{0}^{\tau} |r(l)||dl| \right) \left( \sup_{-1/n \leq v \leq 0} \int_{0}^{\tau} |K(\mu - v) - K(\mu)||d\mu| \right).
\]
Therefore it follows that

\[(III)_n \leq a \left( 1 + \int_0^a \| r(l) \| \, dl \right) \sup_{-1/n \leq v \leq 0} \left( \int_0^a \| K(\mu - v) - K(\mu) \| \, d\mu \right).
\]

Then, using the integrability of $K$ on $[0,a]$ we see that $(III_n) \to 0$ as $n \to \infty$. This completes the proof of the theorem.

Let us consider a subset $\bar{X}$ consisting of all elements $\phi \in X$ which are continuous on $[-\epsilon_\phi,0]$ for some $\epsilon_\phi > 0$, and set

$$X_0 = \{ \phi \in X \mid \phi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \bar{X} \}.$$

For any $\phi \in X_0$, we define the value of $\phi$ at zero by

$$\phi[0] = \phi(0),$$

where $\phi$ is an element belonging to $\bar{X}$ satisfying $\phi = \phi$ a.e. on $\mathbb{R}^-$. We note that the value $\phi[0]$ is well-defined; that is, it does not depend on the particular choice of $\phi$ since $\phi(0) = \psi(0)$ for any other $\psi \in \bar{X}$ such that $\phi = \psi$ a.e. on $\mathbb{R}^-$. It is clear that $X_0$ is a normed space equipped with norm

$$\| \phi \|_{X_0} := \| \phi \|_X + |\phi[0]|, \quad \forall \phi \in X_0.$$

Also, by virtue of Lemma 1, the solution $x(\cdot;\sigma,\psi, p)$ of Eq. (25) through $(\sigma,\psi) \in \mathbb{R} \times X$ satisfies the relation $x_t(\sigma,\psi, p) \in X_0$ with $(x_t(\sigma,\psi, p))[0] = x(t;\sigma,\psi, p)$ whenever $t > \sigma$.

The following results yield intimate relations between solutions of Eq. (25) and $X$-valued functions satisfying an integral equation which arises from the variation-of-constants formula in the phase space.

**Theorem 4.** Let $p \in C((\sigma, \infty); \mathbb{C}^m)$. Then

(i) the segment $\xi(t) := x_t(\sigma, \phi, p)$ of the solution $x(\cdot; \sigma, \phi, p)$ of Eq. (25) through $(\sigma, \phi)$ satisfies the relations

(a) $\xi(t) = T(t - \sigma)\phi + \lim_{n \to \infty} \int_\sigma^t T(t - s)(\Gamma^n p(s))ds, \forall t \geq \sigma, \text{ in } X$;

(b) $\xi \in C((\sigma, \infty); X_0)$.

(ii) Conversely, if a function $\xi : [\sigma, \infty) \to X$ satisfies the relation

$$\xi(t) = T(t - \sigma)\phi(\sigma) + \lim_{n \to \infty} \int_\sigma^t T(t - s)(\Gamma^n p(s))ds, \forall t \geq \sigma,$$

then

(c) $\xi \in C((\sigma, \infty); X_0)$;
(d) if we set 
\[ u(t) = \begin{cases} (\xi(t))[0], & t > \sigma \\ (\xi(\sigma))(t - \sigma), & t \leq \sigma, \end{cases} \]
then \( u(t) \equiv x(t; \sigma, \xi(\sigma), p) \) on \((\sigma, \infty)\), and \( u_t = \xi(t) \) in \( X \) for any \( t \geq \sigma \).

Proof. The statement (i) immediately follows from Theorem 3 and the fact stated in the foregoing paragraph of Theorem 4. We will certify the statement (ii). By virtue of Theorem 3, we get 
\[ x(t) = x_t(\sigma, \xi(\sigma), p) \text{ in } X \text{ for all } t \geq \sigma. \]
Then (ii)-(c) is an immediate consequence of (i)-(b). Also, it follows from the definition of \( u \) that 
\[ u(t) = (\xi(t))[0] = (x_t(\sigma, \xi(\sigma), p))[0] = x(t; \sigma, \xi(\sigma), p), \quad \forall t > \sigma. \]
Therefore, for any \( t \geq \sigma \), we get 
\[ \|u_t - \xi(t)\|_{1, \rho} = \|u_t - x_t(\sigma, \xi(\sigma), p)\|_{1, \rho} \]
\[ = \int_{-\infty}^{0} |u(t + \theta) - x(t + \theta; \sigma, \xi(\sigma), p)| e^{\rho \theta} d\theta \\
= \int_{\sigma}^{t} |u(\tau) - x(\tau; \sigma, \xi(\sigma), p)| e^{\rho(\tau - \sigma)} d\tau \\
+ \int_{-\infty}^{\sigma} |u(\tau) - (\xi(\sigma))(\tau - \sigma)| e^{\rho(\tau - \sigma)} d\tau \]
\[ = 0, \]
and hence \( u_t = \xi(t) \) in \( X \) for \( t \geq \sigma \). This completes the proof of (ii)-(d). \( \square \)

If a function \( u : \mathbb{R} \to C^m \) satisfies the relations \( u_\sigma \in X \) and \( u(t) \equiv x(t; \sigma, u_\sigma, p) \) on \((\sigma, \infty)\) for any \( \sigma \in \mathbb{R} \), we call \( u \) a solution of Eq. (25) on \( \mathbb{R} \). Of course, if \( u \) is a solution of Eq. (25) on \( \mathbb{R} \), then it satisfies Eq. (25) for any \( t \in \mathbb{R} \); that is,
\[ u(t) = \int_{-\infty}^{t} K(t - s)u(s) ds + p(t), \quad \forall t \in \mathbb{R}. \]

**Theorem 5.** Let \( p \in C(\mathbb{R}; C^m) \).

(i) If \( x(t) \) is a solution of Eq. (25) on the entire \( \mathbb{R} \), then the \( X \)-valued function \( \xi(t) := x_t \) satisfies the relations

(a) \( \xi(t) = T(t - \sigma)\xi(\sigma) + \lim_{n \to \infty} \int_{0}^{t} T(t - s)(\Gamma^n p(s)) ds, \quad \forall (t, \sigma) \in \mathbb{R}^2 \) with \( t \geq \sigma \), in \( X \);

(b) \( \xi \in C(\mathbb{R}; X_0) \).
Conversely, if a function $x : \mathbb{R} \to X$ satisfies the relation

$$
\zeta(t) = T(t-\sigma)\xi(\sigma) + \lim_{n \to \infty} \int_{t-\sigma}^{t} T(t-s)(\Gamma^n p(s))ds, \quad \forall (t, \sigma) \in \mathbb{R}^2 \text{ with } t \geq \sigma,
$$

then

(c) $\xi \in C(\mathbb{R}; X_0)$;

(d) if we set $u(t) = (\zeta(t))[0], \quad \forall t \in \mathbb{R},$

then $u \in C(\mathbb{R}; C^m)$, $u_t = \zeta(t)$ (in $X$) for any $t \in \mathbb{R}$ and $u$ is a solution of Eq. (25) on $\mathbb{R}$.

**Proof.** The statements (i) and (ii)-(c) are direct consequences of Theorem 4. We will certify the statement (ii)-(d). Now, the fact that $u \in C(\mathbb{R}; C^m)$ follows from (ii)-(c). Notice that $u(t) = \frac{d}{dt}(\xi(t))[0], \quad \forall t \in \mathbb{R},$

by virtue of Theorem 4. Then, putting $\sigma = t - l$, $l = 1, 2, \ldots$, in the above, we get

$$
(x(t; t-l, \xi(t-l), p)) = u(t), \quad -l < 0
$$

for $l = 1, 2, \ldots$. Let $t$ be any fixed number, and set $\xi^l := x_{\theta}(t - l, \xi(t-l), p)$ for $l = 1, 2, \ldots$. By virtue of (30) and Theorem 4, it follows that

$$
\xi(t) = T(l)\bar{\xi}(t-l) + \lim_{n \to \infty} \int_{t-l}^{t} T(t-s)(\Gamma^n p(s))ds = x_{\theta}(t - l, \bar{\xi}(t-l), p) = \xi^l
$$

in $X$, and hence $(\xi(t))(\theta) = \xi^t(\theta)$ a.e. in $\theta \leq 0$ and $l = 1, 2, \ldots$. Combining this fact with (31), we have $(\xi(t))(\theta) = u(t+\theta)$ a.e. in $\theta \leq 0$; consequently $u_t = \zeta(t)$ in $X$. Then it follows from (30) that $u(t) = x(t; \sigma, \xi(\sigma), p) = x(t; \sigma, u_t,p)$ for any $t > \sigma$. This completes the proof of (ii)-(d).

6. Decomposition of variation-of-constants formula in the phase space, and
an application to admissibility theory

As remarked in the paragraph prior to Theorem 2, the set $\Sigma^U_{\epsilon} = \{ \lambda \in C \mid \text{Re } \lambda \geq c, \det A(\lambda) = 0 \}$ is (at most) a finite set when $c > -\rho$. Hence one can choose a negative constant $\delta$, $\delta > -\rho$ satisfying $\Sigma^U_{\epsilon} = \Sigma^U_{\delta}$. Applying Theorem 2 as $c = \delta$, we get the decomposition of the space $X$

$$
X = U \oplus S
$$

with the properties:
(i) \( \text{dim } U < \infty \);
(ii) \( T(t)U \subseteq U, \; T(t)S \subseteq S \; (\forall t \geq 0) \);
(iii) \( \sigma(A|_U) = \Sigma^U, \; \sigma(A|_{S \cup \rho(A)}) = \sigma(A) \setminus \Sigma^U =: \Sigma^S \);
(iv) \( T^U(t) := T(t)\vert_U \) is extendable for \( t \in (-\infty, \infty) \), as a group of bounded linear operators on \( U \);
(v) \( T^S(t) := T(t)\vert_S \) is a strongly continuous semigroup of bounded linear operators on \( S \), and its generator is identical with the operator \( A|_{S \cup \rho(A)} \);
(vi) there exist positive constants \( C \) and \( \alpha \) such that

\[
\|T^S(t)\| \leq Ce^{-\alpha t} \quad (\forall t \geq 0),
\]

\[
\|T^U(t)\| \leq Ce^{-\alpha t} \quad (\forall t \leq 0).
\]

Recall that \( P^U \) (or \( P^S \)) denotes the projection from \( X \) onto \( U \) (or \( S \)) respectively along the above decomposition of \( X \). Corresponding to the \( S \)-component and the \( U \)-component of the variation-of-constants formula (VCF) in Theorem 3, we consider the following two equations

\[
\xi(t) = T^S(t - \sigma)\xi(\sigma) + \lim_{n \to \infty} \int_{\sigma}^{t} T^S(t - s)P^S(\Gamma^n p(s))ds, \quad t \geq \sigma,
\]

(33) \( \eta(t) = T^U(t - \sigma)\eta(\sigma) + \lim_{n \to \infty} \int_{\sigma}^{t} T^U(t - s)P^U(\Gamma^n p(s))ds, \quad t \geq \sigma. \)

Henceforth, we refer to Eq. (32) and Eq. (33) as the stable part of VCF and the unstable part of VCF, respectively. Notice that \( P^U \) and \( P^S \) are bounded linear operators on \( X \). Then, as a direct consequence of Theorems 4 and 5, we get the following results on the decomposition of variation-of-constants formula in the phase space.

**Theorem 6.** Assume that \( X \) is decomposed as above, and let \( p \in C([\sigma, \infty); \mathbb{C}^{m}) \). Then, for the solution \( x(\cdot; \sigma, \varphi, p) \) of Eq. (25) through \( (\sigma, \varphi) \) the \( S \)-component \( \xi(t) := P^Sx_t(\sigma, \varphi, p) \) and the \( U \)-component \( \eta(t) := P^Ux_t(\sigma, \varphi, p) \) satisfy the stable part of VCF and the unstable part of VCF, respectively.

Conversely, if the functions \( \xi \) and \( \eta \) on \([\sigma, \infty)\) with \( \xi(t) \in S \) and \( \eta(t) \in U \) satisfy the stable part of VCF and the unstable part of VCF, respectively, for all \( t \geq \sigma \), then \( \xi + \eta \in C((\sigma, \infty); X_0) \), and the function \( u : \mathbb{R} \to \mathbb{C}^{m} \) defined by \( u(t) = (\xi(t) + \eta(t))[0] \) for \( t > \sigma \), and \( u(t) = (\xi(\sigma) + \eta(\sigma))(t - \sigma) \) for \( t \leq \sigma \) is a solution of Eq. (25) on \((\sigma, \infty)\), and satisfies \( u_t = \xi(t) + \eta(t), \forall t \geq \sigma, \) in \( X \).

**Theorem 7.** Assume that \( X \) is decomposed as above, and let \( p \in C(\mathbb{R}; \mathbb{C}^{m}) \). Then, for the solution \( x \) of Eq. (25) on \( \mathbb{R} \) the \( S \)-component \( \xi(t) := P^Sx_t \) and the \( U \)-component \( \eta(t) := P^Ux_t \) satisfy the stable part of VCF and the unstable part of VCF, respectively, for all \((t, \sigma)\) with \( t > \sigma > -\infty \).
Conversely, if the functions $\xi$ and $\eta$ on $[\sigma, \infty)$ with $\xi(t) \in S$ and $\eta(t) \in U$ satisfy the stable part of VCF and the unstable part of VCF, respectively, for all $(t, \sigma)$ with $t > \sigma > -\infty$, then $\xi + \eta \in C(R; X_0)$, and the function $u : R \to C^m$ defined by $u(t) = (\xi(t) + \eta(t))[0]$ for $t \in R$ is a solution of Eq. (25) on $R$, and satisfies $u_t = \xi(t) + \eta(t)$, $\forall t \geq \sigma$, in $X$.

As an application of the above decomposition results for VCF, we will study the existence of bounded solutions of Eq. (25) on $R$ under the condition that $p \in BC(R; C^m)$, the set of all bounded continuous functions mapping $R$ into $C^m$. As shown in the following result, the stable part of VCF possesses a unique bounded $X$-valued solution on $R$.

**Proposition 6.** Let $p \in BC(R; C^m)$. Then the limit

$$
\lim_{n \to \infty} \int_{-\infty}^{t} T^S(t-s)\Pi_S^S(\Gamma^n p(s))ds, \quad t \in R
$$

converges in $X$, and the $X$-valued function $\mathcal{Y}$ defined by

$$
\mathcal{Y}(t) = \lim_{n \to \infty} \int_{-\infty}^{t} T^S(t-s)\Pi_S^S(\Gamma^n p(s))ds, \quad t \in R
$$

is a bounded solution of the stable part of VCF on $R$; that is, $\mathcal{Y}(t)$ satisfies the stable part of VCF for any $(t, \sigma)$ with $t \geq \sigma > -\infty$, together with $\sup_{t \in R} \|\mathcal{Y}(t)\| < \infty$.

Moreover, if $\mathcal{Y} : R \to S$ is a bounded solution of the stable part of VCF on $R$, then $\mathcal{Y}(t) \equiv \mathcal{Y}(t)$.

**Proof.** We first observe that the limit

$$
\lim_{\sigma \to -\infty} \int_{\sigma}^{t} T^S(t-s)\Pi_S^S(\Gamma^n p(s))ds = \int_{-\infty}^{t} T^S(t-s)\Pi_S^S(\Gamma^n p(s))ds
$$

exists in $X$. Indeed, if $\sigma_1 < \sigma_2 < t$, then

$$
\left\| \int_{\sigma_1}^{t} T^S(t-s)\Pi_S^S(\Gamma^n p(s))ds - \int_{\sigma_2}^{t} T^S(t-s)\Pi_S^S(\Gamma^n p(s))ds \right\|
$$

$$
= \left\| \int_{\sigma_1}^{\sigma_2} T^S(t-s)\Pi_S^S(\Gamma^n p(s))ds \right\|
$$

$$
\leq \int_{\sigma_1}^{\sigma_2} Ce^{-z(t-s)}ds \|\Pi_S^S\| \|p\|_R
$$

$$
\leq (C/\alpha)e^{-z(t-\sigma_2)}\|\Pi_S^S\| \|p\|_R \to 0
$$

as $\sigma_2 \to -\infty$, where $\|p\|_R := \sup_{t \in R} |p(t)|$. Thus the limit exists in $X$. 

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Now, for any positive integers \( n \) and \( k \), we have

\[
\left\| \int_{-\infty}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds - \int_{-\infty}^{t} T^S(t - s) \Pi^S(\Gamma^k p(s) ) ds \right\|
\]

\[
\leq \left\| \int_{-\infty}^{\sigma} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds \right\| + \left\| \int_{\sigma}^{\gamma} T^S(t - s) \Pi^S(\Gamma^k p(s) ) ds \right\|
\]

\[
+ \left\| \int_{\sigma}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds - \int_{\sigma}^{t} T^S(t - s) \Pi^S(\Gamma^k p(s) ) ds \right\|
\]

\[
\leq (2C/2)e^{2(\sigma - t)} \| \Pi^S \| \| p \|_R
\]

\[
+ \left\| \Pi^S \left( \int_{\sigma}^{t} T(t - s)(\Gamma^n p(s) ) ds - \int_{\sigma}^{t} T(t - s)(\Gamma^k p(s) ) ds \right) \right\|.
\]

Therefore, since \( \lim_{n \to \infty} \int_{\sigma}^{t} T(t - s)(\Gamma^n p(s) ) ds \to \varphi(\sigma,0,p) \) in \( X \) by Theorem 3, it follows that

\[
\lim_{n,k \to \infty} \left\| \int_{-\infty}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds - \int_{-\infty}^{t} T^S(t - s) \Pi^S(\Gamma^k p(s) ) ds \right\|
\]

\[
\leq (2C/2)e^{2(\sigma - t)} \| \Pi^S \| \| p \|_R \to 0
\]

as \( \sigma \to -\infty \), and hence the limit \( \lim_{n \to \infty} \int_{-\infty}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds \) exists in \( X \).

We next prove that \( \vartheta(t) = \lim_{n \to \infty} \int_{-\infty}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds \), \( t \in R \) is a bounded solution of the stable part of VCF on \( R \). Indeed, \( \vartheta \) is bounded on \( R \) because of

\[
\| \vartheta(t) \| = \lim_{n \to \infty} \left\| \int_{-\infty}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds \right\|
\]

\[
\leq \int_{-\infty}^{t} C e^{-2(t - s)} \| \Pi^S \| \| p \|_R ds = (C/2) \| \Pi^S \| \| p \|_R.
\]

Also, if \( t \geq \sigma > -\infty \), then

\[
T^S(t - \sigma) \vartheta(\sigma) + \lim_{n \to \infty} \int_{\sigma}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds
\]

\[
= \lim_{n \to \infty} T^S(t - \sigma) \int_{-\infty}^{\sigma} T^S(\sigma - s) \Pi^S(\Gamma^n p(s) ) ds
\]

\[
+ \lim_{n \to \infty} \int_{\sigma}^{t} T^S(t - s) \Pi^S(\Gamma^n p(s) ) ds.
\]
unit matrix, and the projection operator $P$ on $\mathbb{R}_s$ as

$$\mathcal{V}_F(y) = \mathcal{V}_F(y)$$

thus, $\mathcal{V}$ satisfies the stable part of VCF on $\mathbb{R}$.

Let $\mathcal{V}: \mathbb{R} \to S$ be another bounded solution of the stable part of VCF on $\mathbb{R}$. Then $\mathcal{V}(t) - \mathcal{Y}(t) = T^S(t - \sigma)(\mathcal{V}(\sigma) - \mathcal{Y}(\sigma))$ for all $t \geq \sigma > -\infty$; hence

$$\|\mathcal{V}(t) - \mathcal{Y}(t)\| = Ce^{-\delta(t-\sigma)}\left\{\sup_{t}\|\mathcal{V}(\tau)\| + \sup_{t}\|\mathcal{Y}(\tau)\|\right\} \to 0$$

as $\sigma \to -\infty$. Thus $\mathcal{V}(t) \equiv \mathcal{Y}(t)$ in $X$. This completes the proof. \qed

We next discuss the existence of bounded solutions of the unstable part of VCF on $\mathbb{R}$. Let $d$ be the dimension of $U$, and take a basis $\{\varphi_1, \ldots, \varphi_d\}$ of $U$. We call $\Phi := (\varphi_1, \ldots, \varphi_d)$ a basis vector of $U$. Since $\{T^U(t)\}_{t \geq 0}$ is a strongly continuous semigroup on the finite dimensional space $U$, there exists a $d \times d$ matrix $G$ such that

$$T^U(t)\Phi = \Phi e^{Gt} \quad (\forall t \geq 0),$$

where the spectrum of the matrix $G$ is identical with the set $\Sigma^U_0$. As an application of Hahn-Banach’s theorem, we see that for the above basis vector $\Phi$, there exist uniquely $d$-elements $\psi_1, \ldots, \psi_d$ in $X^*$ (the dual space of $X$) such that $\langle \psi_i, \varphi_j \rangle = 1$ if $i = j$, 0 if $i \neq j$, and that $\psi_i = 0$ on $S$. Here and hereafter, $\langle , \rangle$ denotes the canonical pairing between $X^*$ and $X$. Denote by $\Psi$ the transpose of $(\psi_1, \ldots, \psi_d)$, and use the notation $\langle \Psi, \phi \rangle$ to express the transpose of $((\psi_1, \varphi_1), \ldots, (\psi_d, \varphi_d))$. Then, $\langle \psi_1, \varphi_1 \rangle, \ldots, \langle \psi_d, \varphi_d \rangle =: \langle \Psi, \Phi \rangle$ is the $d \times d$ unit matrix, and the projection operator $\Pi^U$ is given by

$$\Pi^U \varphi = \Phi \langle \Psi, \varphi \rangle, \quad \varphi \in X.$$

Recall that $e_i$ is the vector in $C^m$ whose $j$-th component is 1 if $j = i$ and 0 otherwise. With the above notations, we get the following preparatory lemma:

**Lemma 2.** There exists a $d \times m$ matrix $H$ such that

$$\lim_{n \to \infty} \langle \Psi, \Gamma^n e_i \rangle = He_i, \quad i = 1, \ldots, m.$$

**Proof.** For any $i = 1, \ldots, m$, let us consider a function $p_i \in BC(\mathbb{R}, C^m)$ defined by $p_i(t) \equiv e_i$, and set $z_i(t) = \langle \Psi, x_i(\sigma, 0, p_i) \rangle$ for $t \geq \sigma$. By virtue of Theorem 3, we get
\[
\Phi z_i(t) = \Pi^U x_i(\sigma, 0, p_i) = \Pi^U \left( \lim_{n \to \infty} \int_{\sigma}^t T(t-s)(\Gamma^n p_i(s))ds \right)
\]
\[
= \lim_{n \to \infty} \int_{\sigma}^t T^U(t-s)\Pi^U(\Gamma^n p_i(s))ds
\]
\[
= \lim_{n \to \infty} \int_{\sigma}^t T^U(t-s)\Phi(\Psi, \Gamma^n p_i(s))ds
\]
\[
= \lim_{n \to \infty} \int_{\sigma}^t \Phi e^{G(t-s)}\langle \Psi, \Gamma^n e_i \rangle ds;
\]

hence

\[
(34) \quad \Phi z_i(t) = \lim_{n \to \infty} \int_{\sigma}^t \Phi e^{G(t-s)}\langle \Psi, \Gamma^n e_i \rangle ds, \quad i = 1, \ldots, m.
\]

Observe that the sequence \( \{ \langle \Psi, \Gamma^n e_i \rangle \}_{n=1}^\infty \) in \( C^d \) is bounded with \( |\langle \Psi, \Gamma^n e_i \rangle| \leq ||\Psi|| \). Therefore there are a sequence \( \{n_k\}_{k=1}^\infty \) in \( N \) and \( \tilde{\xi}_i \in C^d \) such that \( \lim_{k \to \infty} \langle \Psi, \Gamma^{n_k} e_i \rangle = \tilde{\xi}_i \) for \( i = 1, \ldots, m \). Consequently, it follows from (34) that

\[
\Phi z_i(t) = \Phi \int_{\sigma}^t e^{G(t-s)}\tilde{\xi}_i ds, \quad i = 1, \ldots, m.
\]

We assert that

\[
\lim_{n \to \infty} \langle \Psi, \Gamma^n e_i \rangle = \tilde{\xi}_i, \quad i = 1, \ldots, m.
\]

Indeed, if the assertion is false, then, there are a sequence \( \{n_k\}_{k=1}^\infty \) in \( N \) and \( \tilde{\xi} := \langle \xi_1, \ldots, \xi_m \rangle \), \( \tilde{\xi} \neq \xi := \langle \xi_1, \ldots, \xi_m \rangle \), such that \( \lim_{k \to \infty} \langle \Psi, \Gamma^{n_k} e_i \rangle = \tilde{\xi}_i \) for \( i = 1, \ldots, m \); here \( \langle \xi_1, \ldots, \xi_m \rangle \) denotes the transpose of \( \langle \xi_1, \ldots, \xi_m \rangle \). Then, by virtue of (34) we get

\[
\Phi z_i(t) = \lim_{k \to \infty} \int_{\sigma}^t \Phi e^{G(t-s)}\langle \Psi, \Gamma^{n_k} e_i \rangle ds = \Phi \int_{\sigma}^t e^{G(t-s)}\tilde{\xi}_i ds;
\]

hence \( \Phi \int_{\sigma}^t e^{G(t-s)}\tilde{\xi}_i ds = \Phi \int_{\sigma}^t e^{G(t-s)}\xi_i ds \) for \( t \geq \sigma \), or \( \tilde{\xi}_i = \xi_i \) for \( i = 1, \ldots, m \); a contradiction. Thus the assertion must hold true, and \( H := \langle \xi_1, \ldots, \xi_m \rangle \) is the desired \( d \times m \) matrix.

Recall that \( z(t) := \langle \Psi, \Pi^U x_i(\sigma, \varphi, p) \rangle \) is the coefficient of the \( U \)-component \( \Pi^U x_i(\sigma, \varphi, p) \) for the basis vector \( \Phi \) in \( U \). Since \( \Pi^U x_i(\sigma, \varphi, p) \) satisfies the unstable part of VCF by Theorem 7, we see by using Lebesgue’s dominated convergence theorem and Lemma 2 that \( z \) satisfies the relation

\[
z(t) = e^{G(t-\sigma)}z(\sigma) + \int_{\sigma}^t e^{G(t-s)}H p(s)ds, \quad \forall t \geq \sigma > -\infty,
\]
in other words, \( z \) is a solution of the ordinary differential equation
\[
\dot{z}(t) = Gz(t) + Hp(t)
\]
on \( \mathbb{R} \). This observation leads to the following result on the existence of
(bounded) solutions of Eq. (25) by combining Theorem 7 and Proposition 6:

**Theorem 8.** Assume that \( X \) is decomposed as above, and let \( p \in BC(\mathbb{R}; C^m) \). With the above notations, the following statements hold true:

(I) If \( x : \mathbb{R} \to C^m \) is a (bounded) solution of Eq. (25) on \( \mathbb{R} \), then, the function \( z(t) := \langle \Psi, x_t \rangle \) is a (resp. bounded) solution of the ordinary differential equation (35) on \( \mathbb{R} \).

(II) If \( z \) is a (bounded) solution of the ordinary differential equation (35) on \( \mathbb{R} \), then, the function \( \eta : \mathbb{R} \to X \) defined by
\[
\eta(t) = \Phi z(t) + \lim_{n \to -\infty} \int_{-\infty}^t T^S(t-s)P^S(p(s))ds, \quad \forall t \in \mathbb{R}
\]
satisfies \( \eta \in C(\mathbb{R}; X_0) \), and the function \( x \) defined by \( x(t) = (\eta(t))[0] \) is a (resp. bounded) solution of Eq. (25) on \( \mathbb{R} \).

Finally we will give a result on the admissibility of some closed subspace of \( BC(\mathbb{R}; C^m) \) with respect to the homogeneous linear integral equation
\[
x(t) = \int_{-\infty}^t K(t-s)x(s)ds.
\]

A closed subspace \( \mathcal{M} \) of \( BC(\mathbb{R}; C^m) \) is said to be admissible with respect to Eq. (36), if for any \( p \in \mathcal{M} \), Eq. (25) on \( \mathbb{R} \) possesses a unique solution in \( \mathcal{M} \). We refer the reader to [7, 8, 9, 10, 12, 13, 15, 16, 17, 18, 19, 23] and the references therein for more information on the theory of admissibility of function spaces with respect to several linear equations including ordinary differential equations and functional differential equations. In what follows, as an application of the decomposition formula of VCF in the phase space, we will analyze the admissibility with respect to Eq. (36) for a translation-invariant subspace of bounded continuous functions whose spectra are contained in a closed subset of \( \mathbb{R} \).

We first recall the notion of a spectrum of \( f \in BUC(\mathbb{R}; C^m) \) which is defined as the set
\[
sp(f) := \left\{ \lambda \in \mathbb{R} \mid \forall \varepsilon > 0, \exists \chi \in L^1(\mathbb{R}), \supp \tilde{\chi} \subset (\lambda - \varepsilon, \lambda + \varepsilon), \int_{-\infty}^{\infty} \chi(t-s)f(s)ds \neq 0 \right\},
\]
where $L^1(R) := L^1(R; C)$ and $	ilde{\chi} := \int_{-\infty}^{\infty} e^{-ist} \chi(t) dt$. For any closed set $A$ in $R$, we set

$$A(C^m) = \{ f \in BUC(R; C^m) \mid sp(f) \subset A \}. $$

Then, $A(C^m)$ is a translation-invariant closed subspace of $BUC(R; C^m)$. For related results in the case of integro-differential equations we refer the reader to [16, 23].

**Theorem 9.** Assume that the function $K$ in Eq. (36) satisfies the conditions (15) and (16), and let $A$ be a closed set in $R$. Then, the space $A(C^m)$ is admissible with respect to Eq. (36), if and only if the following condition holds true:

$$\det \left( E - \int_0^{\infty} K(t)e^{-it\lambda} dt \right) \neq 0 \quad \text{for any } \lambda \in A. \tag{37}$$

**Proof of the "only if" part.** Assume that the space $A(C^m)$ is admissible with respect to Eq. (36) and that there is a $\lambda_0 \in A$ such that \( \det(E - \int_0^{\infty} K(t)e^{-it\lambda} dt) = 0 \). Then, there is a nonzero $\alpha \in C^m$ such that $\alpha = \int_0^{\infty} K(t)e^{-it\lambda_0} dt$. Then, $x(t) := e^{it\lambda_0}\alpha$ is a nontrivial bounded solution of Eq. (36) on $R$, and it belongs to the space $A(C^m)$ because of $sp(x) = \{ \lambda_0 \} \subset A$. This is a contradiction to the fact that the solution of Eq. (36) belonging to $A(C^m)$ is only the trivial one by the admissibility of $A(C^m)$ with respect to Eq. (36).

**Proof of the "if" part.** Assume the condition (37), and let $sp(p) \subset A$. We first assert that the ordinary differential equation (35) possesses a unique bounded solution $z$ belonging to $A(C^d)$. Indeed, since $\sigma(G) = \Sigma_{i=0}^{\infty} \{ \lambda \in C \mid \text{Re } \lambda \geq 0, \det A(\lambda) = 0 \}$, it follows that $\sigma(G) \cap iA = \emptyset$ by the condition (37). Therefore, in case $\sigma(G) \subset iR$, the assertion follows from the well known result; e.g., [19, Lemma 1]. Also, in case $\sigma(G) \not\subset iR$, the assertion can easily be certified by noting that the part in the decomposition of Eq. (35) which corresponds to the set $\{ \lambda \in C \mid \text{Re } \lambda > 0, \det A(\lambda) = 0 \}$ possesses a unique bounded solution on $R$ (cf. [9, Chapter IV]).

Now, by virtue of Theorem 8, we know that the function $x(t) := (\xi(t))_0$, $t \in R$ is a bounded solution of Eq. (25) on $R$; here $\xi(t) := \Phi z(t) + \lim_{n \to \infty} \int_{-\infty}^{t} T_S(t-\tau) \Pi^n \Pi_S(\tau) d\tau \in BC(R; X_0)$. In what follows, we will establish the relation $sp(x) \subset A$. Observe that for any continuous function $\chi : R \to C$ with compact support, the integration $\int_{-\infty}^{\infty} \chi(t-\tau) \xi(\tau) d\tau$ is expressed as the limit (in $X$) of a Riemann sum of the $X$-valued function $\chi(t-\tau) \xi(\tau)$ for the fixed $t$; consequently we see that

$$\int_{-\infty}^{\infty} \chi(t-\tau) L(\xi(\tau)) d\tau = L \left( \int_{-\infty}^{\infty} \chi(t-\tau) \xi(\tau) d\tau \right). \tag{38}$$
for any bounded linear operator $L : X \to C^m$. Note that the relation (38) holds true for any $\chi \in L^1(R)$. Indeed, to see this, let us take a continuous function $\chi'$ with compact support such that $\int_{-\infty}^{\infty} |\chi'(s) - \chi(s)| ds \to 0$. Then, since $\sup_{s \in R} \|\xi(s)\| < \infty$, it follows that $\int_{-\infty}^{\infty} \chi'(t-\tau)\xi(\tau) d\tau \to \int_{-\infty}^{\infty} \chi(t-\tau)\xi(\tau) d\tau$ in $X$; similarly, $\int_{-\infty}^{\infty} \chi'(t-\tau)L(\xi(\tau)) d\tau \to \int_{-\infty}^{\infty} \chi(t-\tau)L(\xi(\tau)) d\tau$ in $C^m$. Then, the relation (38) for a general $\chi \in L^1(R)$ follow from the one for $\chi'$. Let us consider $L$ defined by $L(\phi) = \int_{-\infty}^{0} K(-\theta)\phi(\theta)d\theta$, $\phi \in X$. Then, $x(t) = L(\xi(t)) + p(t)$ on $R$; and hence for any $\chi \in L^1(R)$, it follows that

$$\int_{-\infty}^{\infty} \chi(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} \chi(t-\tau)p(\tau)d\tau + L\left(\int_{-\infty}^{\infty} \chi(t-\tau)\xi(\tau)d\tau\right).$$

(39)

Here, observe that

$$\left\|\int_{-\infty}^{\infty} \chi(t-s)\xi(s) ds\right\|_X \leq \left\|\Phi_{-\infty}^{\infty} \chi(t-s)z(s) ds\right\|_X + \lim_{n \to \infty} \left\|\int_{-\infty}^{\infty} \chi(t-s) \int_{-\infty}^{s} T^S(s-\tau)\Pi^S(\Gamma^n p(\tau)) d\tau ds\right\|_X$$

$$\leq \left\|\Phi\right\|_X \int_{-\infty}^{\infty} \chi(t-s)z(s) ds + \lim_{n \to \infty} \left\|\int_{-\infty}^{\infty} T^S(s)\Pi^S(\Gamma^n \int_{-\infty}^{\infty} \chi(t-w-\mu)p(\mu) d\mu) dw\right\|_X$$

for any $\chi \in L^1(R)$. Since $sp(z) \subset A$ and $sp(p) \subset A$, it follows from the above relation that $sp(\xi) \subset A$. Then, the desired relation $sp(x) \subset A$ follows from (39).

Finally, we will prove that the solution $x$ is the unique one for Eq. (25) with the property that $sp(x) \subset A$. Let $y$ be any bounded continuous solution of Eq. (25) with the property that $sp(y) \subset A$. By almost the same reasoning as in Step 1 of the proof of Theorem 3, we see that

$$\left[\int_{-\infty}^{\infty} \chi(t-s)y_s ds\right](\theta) = \int_{-\infty}^{\infty} \chi(t-s)y(s+\theta) ds \quad \text{a.e. in} \ \theta \leq 0$$

for any continuous function $\chi : R \to C$ with compact support. In fact, repeating the same argument as for (39) one can deduce that the above relation holds true for any $\chi \in L^1(R)$. This observation leads to the relation $w \in A(C^d)$, here $w(t) := \langle \Psi, y(t) \rangle$. Since $w$ is a solution of Eq. (35) on $R$ by Theorem 8, from the uniqueness of the bounded solution of Eq. (35) belonging to $A(C^d)$ it follows that $w(t) \equiv z(t)$. Hence $y_t \equiv \xi(t)$, or $y(t) \equiv (\xi(t))[0] \equiv x(t)$, as required.
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(Ricevita la 30-an de novembro, 2011)
(Reviziita la 11-an de majo, 2012)