A TRILOGY, GIVEN BY COMPLETE TENSOR PRODUCT OF COMPLETE RINGS OVER THE COEFFICIENT RING

EHSAN TAVANF AR

Abstract. The purpose of this paper is triple. First off, for an equi-characteristic, or a p-torsion free (0, p)-mixed characteristic local ring, we settle, positively, the conjecture on the closedness of the category of modules of finite complete intersection dimension under taking finite direct sums. Secondly, a question proposed by Celikbas, Takahashi and Dao, is answered, again affirmatively, for all (local) rings. This second question asks whether a local ring possessing a (pd-)test module of finite complete intersection dimension has to be a complete intersection? Lastly, in the category of local rings with uncountable residue field, we show that a question raised by Celikbas and Wagstaff has positive answer. This third question asks whether the (pd-)test complex property is preserved under flat local homomorphisms with regular closed fiber?

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1. INTRODUCTION

Complete Intersection dimension, introduced by Avramov, Gasharov and Peeva in [ACP97], is a homological invariant lying between classical projective dimension and Auslander-Bridger Gorenstein dimension. Roughly speaking, a finitely generated module has finite complete intersection dimension if it, locally, has finite projective dimension up to taking deformation of a flat local extension. A typical, but quite naive, example is that every finitely generated module over a complete intersection local ring R of finite dimension, has finite projective dimension over a deformation of the completion of R. One

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of the significant outcomes of the present article is to answer positively the question below, for all equi-characteristic (not necessarily local) rings as well as mixed-characteristic ones whose non-zero residual characteristics are non-zero-divisor (see, first paragraph of the subsection 2.1 for the relevant definitions).

**Question 1.1.** Suppose that two homologically finite complexes $M$ and $N$, fitting into some exact sequence of homologically finite complexes, both have finite complete intersection dimension. Then does the third homologically finite complex in the exact sequence, have finite complete intersection dimension too?

To the best of my knowledge, this question is firstly proposed in 2004 by Wagstaff (see, [W04, Question 4.1]). Also among his lecture notes, entitled *Semidualizing Modules*, Wagstaff has recorded this question as [W1 Question 1.2.7], and stated it as a favorite question (the question appears also in Matsui-Takahashi [MT17, Remark 5.11]). In, [NW15 Theorem C], Nasseh and Wagstaff when the flat homomorphisms in the quasi-deformations have regular closed fiber and they induce separable field extension on residue fields had a progress on this question, but their Theorem is still insufficient to completely answer the question even in that case. In this paper, we answer the question positively with only restriction that, in mixed characteristic we require those residual characteristics which are not zero must be non-zero-divisor on $R$ ($R$ might be non-local). This condition on the residual characteristics is needed, only, to ensures us that a complete local ring and its deformation both have a common coefficient ring $C$ such that each of which is flat over $C$. Our proof is based on considering suitable complete tensor products which they are good enough to unify two quasi-deformations while preserving finite projective dimension property of both of modules, see Lemma 3.1.2. Along answering Question 1.1 by Proposition 2.3.2 we investigate and collect some useful properties of complete tensor products which will be used repeatedly throughout the paper. The only reference which might have been used in place of Proposition 2.3.2 seems to be the Grothendieck’s [GIV64], while our results in Proposition 2.3.2 are generalizing those of [GIV64] concerning complete tensor products. For instance (not the only instance), Proposition 2.3.2(v) has been stated in [GIV64 Lemma 19.7.1.2] but with a superfluous assumption that the residue field of one of the rings in the complete tensor product must be finitely generated over the base ring in the complete tensor product.

An immediate application to our result is that, over an equi-characteristic ring $R$ or a mixed characteristic ring $R$ admitting the aforementioned regular property for the residual characteristics, the complete intersection dimension can be measured in terms of resolutions consisting of modules of complete intersection dimension zero, and that a homologically finite complex of $R$-modules has finite complete intersection dimension if and only if it is quasi-isomorphic to a bounded complex consisting of modules of complete intersection dimension zero (Wagstaff, in [W04], extended the concept of complete intersection dimension, defined previously for finitely generated modules, to the realm of homologically finite complexes of modules). In particular, our result in conjunction with [DMTT19, Corollary 2.8], implies that the analogue of the New Intersection Theorem holds for any non-exact bounded complex with finite length homologies consisting of modules of complete intersection dimension zero (over local rings of equal-characteristic or $p$-regular $(0, p)$-mixed characteristic local rings).

As reminded above, over a complete intersection local ring every finitely generated module has finite complete intersection dimension. The reverse also holds, as, by virtue of [ACP97], a ring $(R, m)$ is complete intersection if and only if its residue field $R/m$ has finite complete intersection dimension. Thus, in view of the fact that the residue field tests finiteness of projective dimension of finitely generated modules via the eventual (equivalently single) vanishing of positive tor module(s) $\text{Tor}_i^R(R/m, -)$, so the following question is proposed by Celikbas, Dao and Takahashi in 2014 in [CDT14 Question 3.5].

1In other words, in case, the flat homomorphisms in the quasi-deformations are weakly Cohen homomorphisms.
Question 1.2. Is a local ring possessing a (pd-)test module of finite complete intersection dimension necessarily a complete intersection?

The G-dimension analogue of this question has been established in [CW16] (c.f. [CDT14] Corollary 3.4), i.e. $R$ is Gorenstein if and only if $R$ admits a test module of finite G-dimension. Particularly, the rings argued in the Question 1.2 are Gorenstein. In [M14], Majadas proves that if an $R$-finite module satisfies a stronger concept of test module, so called by him test module for flatness, and if it has finite upper complete intersection dimension$^1$ then $R$ is a complete intersection. We prove that the answer of this question is positive in general. Our proof, though, entirely, differs with the proof in [AGP97] establishing the case where the residue field has finite complete intersection dimension (although we use their result in our proof). In fact the above question is strongly connected to the (module version of the) question below, proposed by Celikbas and Wagstaff in [CW16] Question 3.7.

Question 1.3. Does being (pd-)test complex property remain stable under flat local homomorphisms with regular closed fiber? Namely, if $\varphi : R \to S$ is a flat local homomorphism of local rings with regular closed fiber and if $T$ is a (pd-)test complex for $R$, then is $T \otimes_R S$ a (pd-)test complex for $S$?

Towards answering the aforementioned related two questions, one major step was taken by Celikbas and Wagstaff, as they give affirmative answer to Question 1.3 when the extension $R \to S$ is a weakly unramified$^2$ flat local homomorphism inducing finite field extension on residue fields (see, [CW16, Theorem 3.5]). The proof of Celikbas and Wagstaff [CW16, Theorem 3.5] sounded more charming to me, once I noticed that a module (and homological) theoretic question as Question 1.2 (or module version of Question 1.3) found its solution only in Hyperhomological Algebra! Then, a further major step, has been taken, very recently, again by Wagstaff in [W19, Theorem 4.8], wherein the author settles Question 1.3 affirmatively for the case where $\varphi$ induces algebraic extension of fields. The residually purely transcendental case of Question 1.3, thus, were still open. We, by corollary 4.2.2 in the category of local rings with uncountable residue field, give positive answer to Question 1.3. The uncountable condition on residue fields, in our result, is firstly appeared in (and is due to of) Lemma 4.1.1 wherein we were going to establish the residually purely transcendental case of Question 1.3. Fortunately, to take one more step towards answering Question 1.2, we were able to relax the “uncountable condition” just mentioned in Lemma 4.1.1 for the case where the test module $T$ (in the statement of Question 1.3 and Question 1.2) has finite complete intersection dimension. To this end, an elegant discovery discussed by Eisenbud (see, [E80]), Avramov and Sun (see, e.g. [AS98]) and Guliksen (see, [G74]) has been applied, i.e. the graded module $\operatorname{Ext}^*_A(M, N) := \bigoplus_{i \in \mathbb{Z}_0} \operatorname{Ext}^i_A(M, N)$ is a finitely generated graded module over the ring of operators $\mathcal{S} := A[x_1, \ldots, x_c]$ provided $A$ deforms to a ring $B$ by a regular sequence of length $c$ in $B$ and $M$ has finite projective dimension over $B$. This finiteness property has been exploited to find a uniform annihilator for sufficiently large Ext-modules by which after applying the dualizing complex we find such a uniform annihilator for sufficiently large specific Tor-modules over a specific ring, (see, the proof of Lemma 4.1.1).

In the last section of the paper, for equi-characteristic rings with uncountable residue fields, in the case where $\varphi$ induces separable (possibly non-algebraic, but separable) field extension on residue fields, or

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$^1$That is, having finite projective dimension up to a deformation of a flat homomorphism with regular closed fiber, that might be stronger than having finite complete intersection dimension which the latter, by definition, imposes no condition on the closed fiber of the flat homomorphisms. However, it is noteworthy to stress here that by virtue of [W08, Theorem F], for complete intersection dimension, one can, without loss of generality, assume that the closed fiber of the flat homomorphism is Gorenstein.

$^2$That is a regular homomorphisms in the terminology of [SW15]. It means that the maximal ideal of the, source $R$, extends to that of the, target $S$.,
else if $R$ is Cohen-Macaulay, we give an alternative proof which settles Question 1.3 affirmatively (in the mentioned more special cases) and which avoids the use of [W19 Theorem 4.8]. While the proof of [W19 Theorem 4.8] is, elegantly, based on an translating the situation of Question 1.3 to a similar question on finitely generated DG-algebras over a field (by passing to the Koszul complex of a minimal generator of the maximal ideal) our alternative proof uses a different method which is based on reducing the question to the coefficient field base change extensions, $R \to R\hat{\otimes}_K C$. This reduction, for the residually separable case, is done by applying the Linquan Ma’s [M17, Lemma 5.1]. The Ma’s Lemma needs for the flat homomorphism $\phi : R \to S$ on complete rings $R$ and $S$, to map the coefficient field of $R$ to a coefficient field of $S$, which this coefficient field containment holds true whenever $R$ induces separable field extension on residue fields. For the case where $R$ is a Cohen-Macaulay ring, we present Lemma 4.3.1 by which we were able to relax the awkward condition on the containment of coefficient fields in the Ma’s Lemma. Thence, when $R$ is Cohen-Macaulay, as well, we were able to present our alternative proof.

2. The Foundation

2.1. Preliminaries. In this subsection we review notation and definitions used in our paper. All rings are Noetherian with identity and they are often, but not always, local. In our paper, by a residual characteristic, of a possibly non-local, ring $R$, we mean the characteristic of $R/m$ for some $m \in \text{max}(R)$ (which is either zero or a prime number). When the residual characteristics of $R$ all agree with the characteristic of $R$, we say that $R$ has equal-characteristic, or it is an equi-characteristic ring, otherwise it is said that $R$ has mixed characteristic. A local mixed characteristic $(0,p)$ ring $(R,m)$ is said to be unramified if $p \notin m^2$. The derived category of $R$ is denoted by $\mathcal{D}(R)$ and $\mathcal{D}_b(R)$ denotes the full subcategory of $\mathcal{D}(R)$ consisting of homologically bounded complexes, i.e. complexes whose homology modules are all zero but for at most finite number of homologies. Moreover, $\mathcal{D}^f_b(R)$ denotes the full subcategory of $\mathcal{D}_b(R)$ consisting of complexes all of whose homologies are finitely generated. Quasi-isomorphism of complexes are denoted by $\simeq$, and homotopy equivalence of morphism of complexes are denoted by $\approx$.

Conventions and Notation 2.1.1. Throughout this paper, $\varphi_{k,K} : k \to K$ denotes an extension of fields. The notation $C_k$ assigned to a fixed field $k$ either denotes the field $k$ itself (which would be the only possibility if Char $k = 0$) or denotes the, unique up to isomorphism, complete unramified mixed characteristic discrete valuation ring with the residue field $k$, that is called the p-ring for $k$ in the terminology of [MS9, page 223]. More strictly, if the characteristic of the ambient ring is mixed then $C_k$ is definitely a mixed characteristic complete unramified discrete valuation ring (and it plays the role of a coefficient ring of the ambient ring), otherwise since the characteristic of the ambient ring is not specified, so $C_k$ can be either a field, or a discrete valuation ring as above (playing the role of either a coefficient ring or a coefficient field of the ambient ring). When $k$ has prime characteristic $p > 0$, we shall, occasionally, consider additionally an extension $\varphi_{C_k,C_K} : (C_k,pC_k,k) \to (C_K,pC_K,K)$ of unramified complete discrete valuation domains of mixed characteristic $(0,p)$ lifting the extension $\varphi_{k,K}$ on the residue fields (see, [MS9 Theorem 29.1]).

Definition 2.1.2. Let $(R,m,K)$ be a complete local ring. In our paper, a coefficient ring of $R$ is a local homomorphism of complete rings, $\lambda_R : C_K \to R$, whose source is a domain and such that it contracts $m$ to the maximal ideal of $C_K$ which the latter is generated by the characteristic p of $C_K$ with the convention that p is zero if $R$ contains a field, and such that the induced map on residue fields is an isomorphism 4. In particular, $C_K$ is either a field or a p-ring. We say that a homomorphisms $\varphi_{C_k,C_K} : C_K \to C_K$ 5

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4That is, $R$ has characteristic zero and its residue field has prime characteristic $p > 0$.

5Our definition is similar to [SP, Definition 10.155.4.], but in contrast to it our coefficient ring is always a domain and they need not be a subring of the ambient ring.
is a homomorphism of coefficient rings provided \( \lambda_R : C_K \to R \) and \( \lambda_S : C_K \to S \) are coefficient rings, \( \varphi : R \to S \) is an \( R \)-algebra and \( \varphi_{C_K: C_K} \) makes the pertinent diagram commutative.

**Definition 2.1.3.** Let \( X \) be a homologically finite complex of \( R \)-modules, i.e. a complex, \( X \in \mathcal{D}_b^f(R) \).

(i) When \( R \) is local, a quasi-deformation of \( R \) is a diagram \( R \xrightarrow{g} S \xleftarrow{\pi} A \) of local homomorphisms of local rings wherein \( g \) is a flat local homomorphism and \( \pi \) is a surjection whose kernel is generated by a regular sequence on \( A \).

(ii) When \( R \) is a local ring and \( X \) is a homologically finite complex of \( R \)-modules, the complete intersection dimension of \( X \) is defined as,

\[
CI\text{-dim } X := \inf \{pd_A(X \otimes_R S) - pd_A S : R \hookrightarrow S \hookrightarrow A \text{ is a quasi-deformation of } R \}.
\]

(iii) Generally, the complete intersection dimension of \( X \) is defined as,

\[
CI\text{-dim } X = \sup \{CI\text{-dim}_{R_m} X_m : m \text{ is a maximal ideal of } R \}.
\]

**Definition 2.1.4.** Let, \( T \) be a finitely generated \( R \)-module. We say that \( T \) is a test module for \( R \), or equivalently a \( pd \)-test module, provided for any finitely generated \( R \)-module \( M \) it is the case that \( pd_R M < \infty \) if and only if \( Tor^R_{>0}(M, T) = 0 \), i.e. precisely when \( Tor^R_i(M, T) = 0 \) for sufficiently large \( i \in \mathbb{N} \).

**Definition 2.1.5.** Let, \( R \) be a local ring. A complex \( T \in \mathcal{D}_b^f(R) \) is said to be a test complex for \( R \), or equivalently a \( pd \)-test complex for \( R \), provided any other \( X \in \mathcal{D}_b^f(R) \) has finite projective dimension if and only if \( X \otimes^L_R T \in \mathcal{D}_b(R) \), i.e. if \( Tor^R_{>0}(X, T) = 0 \).

2.2. Well-known facts and Frequently used Results. In this subsection, for the convenience of the reader, we recall some elementary or well-known facts in Commutative Algebra without giving a proof, and we collect some frequently used theorems and results gathered from references of the paper.

**Fact 2.2.1.** Suppose that \( R \) is a Noetherian ring and \( N \) is an \( n \)-syzygy for some \( n \in \mathbb{N} \). Then any \( R \)-regular sequence \( x \) of length \( n \) is also a regular sequence on \( N \).

**Fact 2.2.2.** If \( T \) is a test module for a local ring \( R \), then, \( T \) considered as a homologically finite complex, is also a test-complex.

2.3. Complete Tensor Product. In the concluding subsection of the present section, the definition and certain properties of complete tensor product which are essential in our article will be given. The complete tensor product may have more nice properties in some more general settings, but we, more or less, have confined the scope of our investigation, to fulfill our requirements in this article. To the best of my knowledge, some results of this subsection are new, see the paragraph before Proposition 2.3.3.

**Definition 2.3.1.** Let \( \Lambda \) be a local ring. For a pair of \((\Lambda, m_A, K)\)-local algebra\(^6\) \((A, m_A)\) and \((B, m_B)\),

the complete tensor product of \( A \) and \( B \) over \( \Lambda \) is defined as the projective limit,

\[
A \widehat{\otimes}_\Lambda B := \lim_{\mathcal{N}} \left( (A/m_A^n) \otimes \Lambda (B/m_B^n) \right)
\]

It is noteworthy to stress that in the definition of complete tensor product of local rings \( A \) and \( B \), without loss of generality, we may assume that \( A \) and \( B \) are both complete local rings.

\(^6\)The local algebra means that the underlying algebra homomorphism is a local homomorphism of local rings.

\(^7\)In this paper, all of the complete tensor products are taken over complete regular rings, i.e. \( \Lambda \) always shall be a complete regular local ring. But in the given definition and Proposition 2.3.3, \( \Lambda \) is not necessarily regular.
We prove and collect some properties of complete tensor products which shall be used repeatedly in this paper. The part (v) of the next proposition is stated in [GIV64, Lemma 19.7.1.2] but with the superfluous condition that the residue field of one of the \( \Lambda \)-algebras is of finite type over \( \Lambda \). The parts (iv) and (vi) might be new result, as they are as far I know. The last part (vi) will not be applied in this paper, and is given for the sake completeness.

**Proposition 2.3.2.** The complete tensor product, \( A \hat{\otimes}_\Lambda B \), of a pair of \((\Lambda, m_\Lambda, K)\)-local algebras \((A, m_A, K)\) and \((B, m_B, L)\) is subject to the following properties.

(i) There is a natural ring homomorphism \( A \otimes \Lambda B \to A \hat{\otimes}_\Lambda B \) by which we may consider the extension of the ideal, \( \mathfrak{M} := m_A(A \otimes B) + m_B(A \otimes B) \), in \( A \hat{\otimes}_\Lambda B \). By abuse of notation, the extended ideal is also denoted by \( \mathfrak{M} \).

(ii) \( A \hat{\otimes}_\Lambda B \) is the completion of \( A \otimes \Lambda B \) with respect to its \( \mathfrak{M} \)-adic topology, and it is Noetherian and \( \mathfrak{M} \)-adically complete provided the residue ring \( A \otimes \Lambda B)/\mathfrak{M}(A \otimes \Lambda B) \) is Noetherian.

(iii) If \( \mathfrak{M} \) is a maximal ideal of \( A \otimes \Lambda B \) then \( A \hat{\otimes}_\Lambda B \) is a local ring, and if, moreover, \( a \) is an ideal of \( A \) then the natural map,

\[
(A \hat{\otimes}_\Lambda B)/\langle a(A \hat{\otimes}_\Lambda B) \rangle \to (A/a) \hat{\otimes}_\Lambda B,
\]

is an isomorphism.

(iv) If \( B \) is an Artinian ring and \( \mathfrak{M} \) is a maximal ideal of \( A \otimes \Lambda B \), then \( (A \otimes \Lambda B)_{\mathfrak{M}} \) is Noetherian. In particular, \( A \hat{\otimes}_\Lambda B \) is flat over \( (A \otimes \Lambda B)_{\mathfrak{M}} \).

(v) If \( B \) is a flat \( \Lambda \)-local algebra, and \( \mathfrak{M} \) is a maximal ideal of \( A \otimes \Lambda B \), then \( A \hat{\otimes}_\Lambda B \) is flat over \( A \).

(vi) If \( \Lambda = K \) is a field and \( \mathfrak{M} \) is a maximal ideal of \( A \otimes_K B \), then \( A \hat{\otimes}_K B \) coincides with the double completed ring \( (A \otimes_K B)^{\mathfrak{m}_A \mathfrak{m}_B} \), i.e. the \( (\mathfrak{m}_B) \)-adic completion of the Noetherian ring \( (A \otimes_K B)^{\mathfrak{m}_A}_{\mathfrak{m}_B} \).

**Proof.**

(i) Obvious.

(ii) Clearly, the pair of decreasing sequence of ideals \( \{ m^a_A(A \otimes B) + m^b_B(A \otimes B) \}_{n \in \mathbb{N}} \) and \( \{ m^a \}_{n \in \mathbb{N}} \), of \( A \otimes \Lambda B \), are cofinal. This, in conjunction with the definition of the complete tensor product, implies the leading statement of this part. Now, the latter statement is immediate, in the light of and beauty of [SP Lemma 10.96.5].

(iii) For the first assertion see, [C46, Theorem 2, page 59] (see also [C46, Theorem 3, page 61]) and for the second one see, [O91, Lemma 2].

(iv) In view of [M89, 7.9., page 53], [M89, Theorem 8.14.1 and [C46, Theorem 3] \((A \otimes \Lambda B)_{\mathfrak{M}} \) is Noetherian if and only if \( A \hat{\otimes}_\Lambda B \) is flat over \( A \otimes \Lambda B \). By virtue of [S77, Theorem 3.1], the maximality of \( \mathfrak{M} \), i.e. \( K \otimes_k \mathcal{L} \) being a field, yields that either \( \mathcal{L} \) or \( K \) is algebraic over \( K \). In particular, again by [S77, Theorem 3.1], for any subfield \( \mathcal{L}' \) of \( \mathcal{L} \), the subring \( K \otimes_K \mathcal{L}' \) of \( K \otimes_K \mathcal{L} \) is also a zero-dimensional domain, i.e. a field. Thence, in case \( B = \mathcal{L} \) is a field, the presentation of \( (A \otimes \Lambda \mathcal{L})_{\mathfrak{M}} = \left( \left( A/(\mathfrak{m}_A) \right) \otimes_K \mathcal{L} \right)_{\mathfrak{M}} \) as the direct limit of local rings and local homomorphisms,

\[
\left( \left( A/(\mathfrak{m}_A) \right) \otimes_K \mathcal{L} \right)_{\mathfrak{M}} = \lim_{\mathfrak{c} \in \text{finite sequence of} \; \mathcal{L} \setminus K} \left( \left( A/(\mathfrak{m}_A) \right) \otimes_K \left( K(\mathfrak{c}) \right) \right)_{\mathfrak{m}_A},
\]

\( ^* \)Here, it is perhaps worth to remark that completion is not necessarily flat if we drop the Noetherian condition. More strictly, by virtue of the nice example [N50, Appendix (2), page 69], due to Nagata, there exists a non-Noetherian (non-complete) generalized local ring \( R \) in the sense of Cohen’s [C46]. Therefore, \( \hat{R} \) is not flat over \( R \) otherwise in view of [M89, 7.9., page 53] and [C46, Theorem 3] together with [C46 Theorem 2] the non-Noetherian property of \( R \) would be violated.
satisfies all of the conditions of [O91] Theorem 1] and thus ensures us that \((A \otimes_A \mathcal{L})_{\mathfrak{n}}\) is Noetherian. More
generality, if \(B\) is an Artinian ring which possibly contains
a non-trivial zero-divisor, then \((A \otimes_A B)_{\mathfrak{n}}\) modulo
its nilpotent finitely generated ideal \(\mathfrak{m}_B(A \otimes_A B)_{\mathfrak{n}}\) is the
ring \((A \otimes_A \mathcal{L})_{\mathfrak{n}}\) whose Noetherianness has been
discussed already. Therefore each prime ideal of \((A \otimes_A B)_{\mathfrak{n}}\),
which certainly has to contain \(\mathfrak{m}_B(A \otimes_A B)_{\mathfrak{n}}\), is
finitely generated, i.e. \((A \otimes_A B)_{\mathfrak{n}}\) is Noetherian in view of
the Cohen’s criterion of Noetherianness in terms of the finite
generation of prime ideals.

(v) By virtue of [O91, Lemma 2], and in view of the previous
part, each quotient ring,

\[ A / (A \otimes_A B) = (A / \mathfrak{m}_A^\alpha) \otimes_A B, \]

is a flat extension of \((A / \mathfrak{m}_A^\alpha) \otimes_A B)_{\mathfrak{n}}\), while the latter
(localized tensor product) ring is flat over \(A / \mathfrak{m}_A^\alpha\)
by our hypothesis. Consequently, \(A \otimes_A B = \lim_{\mathfrak{n} \in \mathbb{N}} ((A \otimes_A B) /
((A / \mathfrak{m}_A^\alpha) \otimes_A B))\) is a flat \(A\)-algebra in the light
and art of [MS0 Theorem 22.3(5)].

(vi) According to the part (iii), the quotient ring \((A \otimes_K B)_{\mathfrak{n}}/
(m_A(A \otimes_K B)_{\mathfrak{n}}) = (K \otimes_K (B/pB))_{\mathfrak{n}}\) is
Noetherian, thus, in the light and art of [SP, Lemma 10.96.5.], the
\(m_A(A \otimes_K B)_{\mathfrak{n}}\)-adic completion
\((A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A\) of \((A \otimes_K B)_{\mathfrak{n}}\) is Noetherian as well. Notice that,

\[
\frac{(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}) \cap (m_A^\alpha(A \otimes_K B)_{\mathfrak{n}})}{(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}).m_A^\alpha(A \otimes_K B)_{\mathfrak{n}})} = \text{Tot}_1^{(A \otimes_K B)_{\mathfrak{n}}}(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}, m_A^\alpha(A \otimes_K B)_{\mathfrak{n}}),
\]

\[
= \text{Tot}_1(B/m_A^\ell(A \otimes_K B)_{\mathfrak{n}}/m_A^\alpha(A \otimes_K B)_{\mathfrak{n}}),
\]

\[
= \text{Tot}_1(B/m_A^\ell(A \otimes_K B)_{\mathfrak{n}}/(A / \mathfrak{m}_A^\alpha \otimes_K B)_{\mathfrak{n}}),
\]

\[
= 0,
\]

due to the flatness of \((A / \mathfrak{m}_A^\alpha) \otimes_K B)_{\mathfrak{n}}\) (respectively, \(A / \mathfrak{m}_A^\alpha\)) over \(B\) (respectively, \(K\)). This implies
that the \(m_A\)-adic topology (respectively, the \(m_A\)-adic completion) of the
ideal, \(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}\), is nothing but its linear topology
(respectively, completion) induced by the the filtration \(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}\) of
\((A \otimes_K B)_{\mathfrak{n}}\). Consequently, since \(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}\) is a finitely presented ideal, we have

\[
\frac{(A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A / m_A^\ell(A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A}{(A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A / \eta_m(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A)} = ((A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A / m_A^\ell(A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A),
\]

where \(\ell \in \mathbb{N}\) and \(\eta_m\) is the natural inclusion from
the completion of \(m_A^\ell(A \otimes_K B)_{\mathfrak{n}}\) to \((A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A\),
as stated in [A69, Corollary 10.3.]. The above identities for each \(m \in \mathbb{N}\),
altogether, constitute an inverse system whose projective limit,
on the one hand (with respect to the left hand side of the identities),
is \((A \otimes_K B)_{\mathfrak{n}}^\mathfrak{m}_A\) and on the other hand (for the right
hand side of the identities) is the complete tensor product \(A \otimes_K B\).

\[\square\]

3. Complete intersection dimension

In this section, Question [11] shall be answered positively for a vast
class of rings, namely, the equal-characteristic rings as well as mixed
characteristic rings whose non-zero residual characteristics are
non-zero-divisor. Then some applications to this result shall be
listed.

3.1. Complete intersection dimension under direct sum.

**Lemma 3.1.1.** Let \(M\) and \(N\) be two finitely generated 
\((R, m_R, K)\)-modules of finite complete intersection
dimension, witnessed by quasi-deformations \(R \leftrightarrow (S, m_S, K_1)\) and \(R \leftrightarrow (T, m_T, K_2)\). To prove that \(M \bigoplus N\) also has finite complete intersection we can deem that

\[\text{including}\] is more suitable than "namely", see Remark [61.30].
all rings are complete, \( \dim R = \dim S = \dim T \), \((L :=) K_1 = K_2 \) and, roughly speaking, all rings except possibly \( R \) have the same coefficient ring \( C_L \), namely there are ring homomorphisms \( \lambda_s : C_L \rightarrow * \), inducing isomorphism on residue fields, from a fixed \( p \)-ring (or a field) \( C_L \) where \(*\) varies in \( \{S, T, A, B\} \) and \( \lambda_S := (A \rightarrow S) \circ \lambda_A \) (respectively, \( \lambda_T := (B \rightarrow T) \circ \lambda_B \)).

\[ \text{Lemma 15.27.5} \] or \[ \text{M89, Theorem 22.3(3')} \] (c.f. \[ \text{S07, Theorem 7.6.} \]) says that (iii) which is a flat extension of \( S \), \( X \) is an acyclic complex again, i.e. \( \text{Tor}^\_i \) series ring. Thus, by analogy, it suffices only to inspect the flatness of \( \lambda_S \).

**Proof.** To attain equality on the dimension is just to proceed with the usual naive localization technique at a minimal prime ideal lying over \( m_R \). To arrive at identity on the residue fields, is just to apply \[ \text{[III64, 19.7.1.3]} \] (or our results in Proposition \[ \text{2.3.2} \]), to \( A \) (and \( B \)) with respect to a common field extension \( L \) of both of \( K_1 \) and \( K_2 \) (e.g. a residue field of \( K_1 \otimes \_ k K_2 \)). Thereafter, to address the completeness one may apply \[ \text{[M89, Theorem 22.4]} \]. The uniqueness, up to isomorphism (not equality), of the coefficient rings in equal-characteristic is obvious while in mixed characteristic it follows from the uniqueness of \( p \)-rings with the same residue field (see, \[ \text{[M89, Corollary, page 225]}. \])

**Lemma 3.1.2.** Following the hypothesis and statement of \[ \text{[5.1.1.4]} \] assume, additionally, that \( R \) is of equal-characteristic or is an \( (0,p) \)-mixed characteristic ring over which \( p \) is regular. Set, \( c := \text{depth} R = \text{depth} S = \text{depth} T \), \( h := \text{pd}_B T \) and \( h' := \text{pd}_A S \) with the convention \( h \leq h' \). Pick some, \( 0 \leq v \leq h \) and \( 0 \leq w \leq e \) and consider \( X := X_i, \ldots, X_w \), with \( i = 1 \) in the case of equal-characteristic while \( i = 2 \) in mixed characteristic, and \( Y := Y_1, \ldots, Y_v \) as a pair of sequences of indeterminates. Presume that, \( s_1, \ldots, s_e \) and \( t_1, \ldots, t_e \) are regular sequences in \( S \) and \( T \), respectively, such that \( s_1 = t_1 \) in the case of mixed characteristic. Assume, further, that \( a_1, \ldots, a_{e'} \) and \( b_1, \ldots, b_h \) are regular sequences in \( A \) and \( B \) respectively, with \( A/(a_1, \ldots, a_{e'}) = S \) and \( B/(b_1, \ldots, b_h) = T \). Consider \( S \) and \( T \) as \( C_L[[X]] \)-algebras via, \( C_L[[X]] \rightarrow S \) which extends \( \lambda_S \) by the rule \( X_i \mapsto s_i \), and \( C_L[[X]] \rightarrow T \) which extends \( \lambda_T \) with \( X_i \mapsto t_i \).

Similarly, consider \( A \) and \( B \) as \( C_L[[X,Y]] \)-algebras via \( \begin{align*}
X_i \mapsto s_i, \quad 1 \text{ (or } 2 \text{ in mixed char.) } \leq i \leq w \\
Y_i \mapsto a_i, \quad 1 \leq i \leq v \\
\lambda_A : C_L \rightarrow A
\end{align*} \)
and \( \begin{align*}
X_i \mapsto t_i, \quad 1 \text{ (or } 2 \text{ in mixed char.) } \leq i \leq w \\
Y_i \mapsto b_i, \quad 1 \leq i \leq v \\
\lambda_B : C_L \rightarrow B
\end{align*} \)

(i) each of \( S \) and \( T \) (respectively, \( A/(a_{e+1}, \ldots, a_{e+j}) \) and \( B/(b_{h+1}, \ldots, b_{h+j}) \) wherein \( j \) is possibly zero) is flat over \( C_L[[X]] \) (respectively, over \( C_L[[X,Y]] \)).

(ii) \( A \odot_{C_L[[X,Y]]} B \) is a complete local ring which deforms the complete local flat extension \( S \odot_{C_L[[X]]} T \) of \( R \), by a regular sequence of length \( h + h' - v \),

(iii) \( \text{pd}_A (S \otimes_R (S \odot_{C_L[[X]]} T)) = \text{pd}_A (S \otimes_R M) + h - v \), and, \( \text{pd}_A (T \otimes_R N) + h' - v \).

**Proof.** (i) The key point is the big-Cohen-Macaulayness of each of, \( S, T \) etc., over the ambient power series ring. Thus, by analogy, it suffices only to inspect the flatness of \( S \) over \( C_L[[X]] \). If \( p \) is a non-trivial non-invertible element of \( R \), then we add the element \( p \) to the sequence \( X \) of elements \( C_L[[X]] \) and denote the new sequence by \( X' := X \). In equal-characteristic \( X' := X \). Evidently, \( S \) is a big-Cohen-Macaulay \( C_L[[X]] \)-algebra, by definition, and thence tensoring the Koszul complex \( K_* (X'; C_L[[X]])) \) with \( S \) yields an acyclic complex again, i.e. \( \text{Tor}^\_1 \) (\( L, S \)) = 0 which in other words, in the light and art of \[ \text{[SP, Lemma 15.27.5]} \] or \[ \text{[M89, Theorem 22.3(3')] \} (c.f. \[ \text{[S07, Theorem 7.6.]} \]) says that \( S \) is flat over \( C_L[[X]] \) (even though \( S \) might not be a finite \( C_L[[X]] \)-module).

(ii) \( S \odot_{C_L[[X]]} T \) (respectively, \( A \odot_{C_L[[X,Y]]} B \) is a complete local ring (see, proposition \[ \text{2.3.2} \)) and (iii) which is a flat extension of \( S, T \) and \( R \) (respectively, of \( A \) and \( B \)), in sight of the previous part and
Proposition 2.3.2(v). Consider the sequence,
\[
\begin{array}{cccc}
\overline{a}_{v+1} \otimes 1, & \ldots , & a_{h'} \otimes 1, & b_1 \otimes 1, & \ldots , & b_h \otimes 1.
\end{array}
\]
of elements of \( A \otimes_{C_\ell}[x, y] B \) and \( \widehat{A} \otimes_{C_\ell}[x, y] B \) which is a regular sequence on both of them, in view of part (i) and Proposition 2.3.2(v) and (iii). Namely, since \( \widehat{A} \otimes_{C_\ell}[x, y] B \) is flat over \( A \) so \( \overline{a}_{v+1}, \ldots , \overline{a}_{h'} \) is regular on \( \widehat{A} \otimes_{C_\ell}[x, y] B \) and similarly since \( (A/(a_{v+1}, \ldots , a_{h'})) \otimes_{C_\ell}[x, y] B \) is flat over \( B \) we are done.

We shall inquire, \( A \otimes_{C_\ell}[x, y] B /(\overline{a}_{v+1}, \ldots , \overline{a}_{h'}, \overline{b}_1, \ldots , \overline{b}_h) = S \widehat{A} \otimes_{C_\ell}[x, y] T \), whose validity completes the proof of part (ii). In view of Proposition 2.3.2(iii), the left hand side of the desired identity is just the \( \mathfrak{m} := m_A (A \otimes_{C_\ell}[x, y] B) + m_B (A \otimes_{C_\ell}[x, y] B) \)-adic completion of
\[
(A \otimes_{C_\ell}[x, y] B)_{\mathfrak{m}} \big/ ((\overline{a}_{v+1}, \ldots , \overline{a}_{h'}, \overline{b}_1, \ldots , \overline{b}_h)(A \otimes_{C_\ell}[x, y] B)_{\mathfrak{m}})
\]
and similarly \( S \widehat{A} \otimes_{C_\ell}[x, y] T \) is completion of \( (S \otimes_{C_\ell}[x, y] T)_{m_S(S \otimes_{C_\ell}[x, y] T)} + m_T(S \otimes_{C_\ell}[x, y] T) \). So our desired identity reduces to,
\[
(A/(a_{v+1}, \ldots , a_{h'})) \otimes_{C_\ell}[x, y] B/(b_1, \ldots , b_h) = (A/(a_{v+1}, \ldots , a_{h'})) \otimes_{C_\ell}[x, y] T
\]
\[
= (A/(a_{v+1}, \ldots , a_{h'})) \otimes_{C_\ell}[x, y] ((C_\ell[x, y] / (y)) \otimes_{C_\ell}[x, y] T)
\]
\[
= S \otimes_{C_\ell}[x, y] T.
\]

(iii) We show, through several facts, that, \( \text{pd}_{A \otimes_{C_\ell}[x, y] B} (S \widehat{A} \otimes_{C_\ell}[x, y] T) \otimes_R N = \text{pd}_B(T \otimes_R N) + h' - v \). A similar argument settles the first assertion of this part as well.

**Fact 1:** By virtue of Proposition 2.3.2(iii) and the last display in the proof of part (ii) our desired module,
\[
(S \widehat{A} \otimes_{C_\ell}[x, y] T) \otimes_R N
\]
\[
= \left( \left( \frac{A}{(a_{v+1}, \ldots , a_{h'})} \otimes_{C_\ell}[x, y] B \bigg/ B \left( \frac{b_1, \ldots , b_h}{(b_1, \ldots , b_h)} \right) \right) \otimes_R N \right)
\]
has finite projective dimension, namely \( \text{pd}_B(T \otimes_R N) \), over \( (A/(a_{v+1}, \ldots , a_{h'})) \otimes_{C_\ell}[x, y] B \), because \( (A/(a_{v+1}, \ldots , a_{h'})) \otimes_{C_\ell}[x, y] B \) is a flat local extension of \( B \) by the first part in conjunction with Proposition 2.3.2(v).

**Fact 2:** According to the previous fact as well as, again, Proposition 2.3.2(iii), and following the notation of the proof of part (ii),
\[
(S \widehat{A} \otimes_{C_\ell}[x, y] T) \otimes_R N \text{ isomorphic also as } A \otimes_{C_\ell}[x, y] B \text{-modules to } \frac{(A \otimes_{C_\ell}[x, y] B) \otimes_B (T \otimes_R N)}{(a_{v+1}, \ldots , a_{h'})(A \otimes_{C_\ell}[x, y] B) \otimes_B (T \otimes_R N)}
\]
has projective dimension equal to, \( \text{pd}_B(T \otimes_R N) \), over the quotient ring,
\[
(A \otimes_{C_\ell}[x, y] B) / (a_{v+1}, \ldots , a_{h'}).
\]

**Fact 3:** Recall that, if a quotient, \( L/xL \), of an \( \Gamma \)-module \( L \), where \( \Gamma \) is a local ring and \( x \) is a regular sequence on both of \( \Gamma \) and \( L \), has finite projective dimension over \( \Gamma/x\Gamma \), then \( L/xL \) has also finite projective dimension as an \( \Gamma \)-module, namely \( \text{pd}_\Gamma(L/xL) = \text{pd}_{\Gamma/x\Gamma}(L/xL) + \ell \) where \( \ell = \text{pd}_\Gamma(\Gamma/x\Gamma) \) is the length of the sequence \( x \). As mentioned in the proof of part (ii), the sequence \( \overline{a}_{v+1}, \ldots , \overline{a}_{h'} \) is regular on \( A \otimes_{C_\ell}[x, y] B \). Hence, according to Fact 2, we are done once we show that the sequence, \( \overline{a}_{v+1}, \ldots , \overline{a}_{h'} \), is also regular on \( (A \otimes_{C_\ell}[x, y] B) \otimes_B (T \otimes_R N) \).
Fact 4: The desired regularity stated at the end of the previous paragraph holds true. Namely the augmented Koszul complex,

\[ K_*((\tilde{a}_{v+1}, \ldots, \tilde{a}_n) : A \otimes_{C_\ell[[X,Y]]} B) \to (A/(a_{v+1}, \ldots, a_n)) \otimes_{C_\ell[[X,Y]]} B \to 0, \]

can be considered as an exact complex of flat \( B \)-modules, by part (i) and Proposition 2.3.2(v). Therefore, it remains exact after tensoring with any \( B \)-module, specifically with the \( B \)-module \( T \otimes_R N \). Whence we are done, since our desired conclusion is equivalent to the acyclicity of the Koszul complex,

\[ K_*((\tilde{a}_{v+1}, \ldots, \tilde{a}_n) : (A \otimes_{C_\ell[[X,Y]]} B) \otimes_B (T \otimes_R N)). \]

\[ \square \]

Remark 3.1.3. Since a complete mixed characteristic \((0,p)\) ring \( R \) with \( p \) zero-divisor, is not flat over its coefficient ring, so our proof does not work in this case, unless when, e.g. \( R \) is such a mixed characteristic ring with extra conditions that \( p^n = 0 \) and \( R \) is flat over \( C/p^n C \); here \( C \) denotes a coefficient ring of \( R \).

Since complete intersection dimension is defined locally, so one can deduce the second part of the following corollary from its first part. In the first part of the next corollary, the fact that the complete intersection dimension of all modules are defined with respect to the unified quasi-deformation, can also be deduced by the Auslander-Buchsbaum Formula (or, Auslander-Bridger Formula), however our proof does not use them and is based on the precise computation of the projective dimensions.

Corollary 3.1.4. Suppose that \( R \) is an equi-characteristic ring, or a mixed characteristic ring such that its non-zero residual characteristics are non-zero-divisors. Let, \( M_1, \ldots, M_n \) be a couple of \( R \)-modules with \( CI\text{-dim}_R M_i < \infty \). Then,

(i) if \( R \) is local, then there is a (unique) quasi-deformation \( R \hookrightarrow S \leftarrow A \) of \( R \) such that,

\[ CI\text{-dim}_R M_i = pd_A(M \otimes_R S) - pd_A S, \forall 1 \leq i \leq n. \]

(ii) \( CI\text{-dim}_R(\bigoplus_{i=1}^n M_i) = \max\{CI\text{-dim} M_i : 1 \leq i \leq n\} \).

Proof. We prove only the first part. If \( n = 2 \), then we consider the quasi-deformation,

\[ R \hookrightarrow S \otimes_{C_\ell[[X]]} T \leftarrow A \otimes_{C_\ell[[X,Y]]} B, \]

of Lemma 3.1.2 associated to the quasi-deformation \( R \hookrightarrow S \leftarrow A \) (respectively, \( R \to T \leftarrow B \)) which defines the complete intersection dimension of \( M_1 \) (respectively \( M_2 \)), i.e. \( CI\text{-dim}_R M_1 = pd_A(M_1 \otimes_R S) - pd_A S \) (respectively, \( CI\text{-dim}_R M_2 = pd_B(M_2 \otimes_R T) - pd_B T \)) (for an arbitrary choice of \( v \) and \( w \) as in the statement of Lemma 3.1.2). Then, by Lemma 3.1.2 ii) and (iii), we have,

\[ pd_A \otimes_{C_\ell[[X,Y]]} B (M_1 \otimes_R (S \otimes_{C_\ell[[X]]} T)) - pd_A \otimes_{C_\ell[[X,Y]]} B (S \otimes_{C_\ell[[X]]} T) = pd_A(M_1 \otimes_R S) + pd_B T - v - pd_A \otimes_{C_\ell[[X,Y]]} B (S \otimes_{C_\ell[[X]]} T) = pd_A(M_1 \otimes_R S) + pd_B T - v - pd_A S - pd_B T + v = pd_A(M_1 \otimes_R S) - pd_A S = CI\text{-dim}_R M_i. \]

By analogy mentioned in Lemma 3.1.2 the desired identity for \( M_2 \) holds true also and we are done for \( n = 2 \). Now the case for \( n > 2 \) also follows by induction and a similar argument as in the case of \( n = 2 \). \[ \square \]
Remark 3.1.5. When $R$ is a Cohen-Macaulay local ring, the construction and proof of Lemma 3.1.2 shows that to unify two quasi-deformations assigned to modules $M$ and $N$ as asserted in the statement of lemma, we can consider the quasi-deformation, $R \hookrightarrow (S \otimes_{C_\ell[[X]]} T)_{\mathfrak{m}} \hookrightarrow (A \otimes_{C_\ell[[X,Y]]} B)_{\mathfrak{m}}$, of $R$, which is not given by complete tensor products. This is because when $R$ is Cohen-Macaulay (and thus so are $S$, $T$ and $B$, since the closed fibers are Artinian), $B$ (respectively, $S$ and $T$) is finite over $C_\ell[[X,Y]]$ (respectively, over $C_\ell[[X]]$) once we let $v$ and $w$, in the statement of Lemma 3.1.2, to attain their largest possible number, i.e. $h$ and $e$. Then, $(A \otimes_{C_\ell[[X,Y]]} B)_{\mathfrak{m}}$ and $(S \otimes_{C_\ell[[X]]} T)_{\mathfrak{m}}$ would be both Noetherian.

3.2. Applications. As the first application, Lemma 3.1.2 together with [W04, Theorem 4.2] immediately yields an affirmative answer to Question [1.1] for equal-characteristic rings or those whose non-zero residual characteristics are non-zero-divisor. Since in view of our lemma, when $R$ is local, there exists a unique quasi-deformation by which the flat base change of two modules of finite complete intersection both have finite projective dimension over the deformation of the flat extension, so the first part of the next corollary follows from its projective dimension analogue.

**Corollary 3.2.1.** Suppose that $R$ is a ring of equal-characteristic or, a mixed characteristic ring over which the non-zero residual characteristics are non-zero-divisor, and let $M$ be a finite $R$-module. Then,

(i) an exact sequence of homologically finite $R$-modules is consisting of complexes of finite complete intersection dimension if and only if two complexes in the exact sequence have finite complete intersection dimension.

(ii) (c.f. [W04, Corollary 3.9]) we have,

\[ CI \text{-dim}_R M = \inf \{ n : \exists \text{ an augmented resolution}, 0 \to C_n \to \cdots \to C_0 \to M \to 0, \text{ with } CI \text{-dim}_R C_i = 0, \forall i \}. \]

(iii) a bounded complex $X$ consisting of modules of finite complete intersection dimension, has $CI \text{-dim}_R X < \infty$.

**Proof.** (i) This is an immediate result of Lemma 3.1.2 in conjunction with [W04, Theorem 4.2].

(ii) If there is an exact sequence, $0 \to C_n \to \cdots \to C_0 \to M \to 0$, of $R$-modules such that each $C_i$ has complete intersection dimension zero, then we would have $CI \text{-dim}_R M \leq n$. Namely, since this is a local property, we assume that $R$ is local. Thus, by Corollary 3.1.3 there exists a unique quasi-deformation $R \hookrightarrow S \leftarrow A$ with $pd_A (C_i \otimes_R S) = pd_A S$, for each $0 \leq i \leq n$. Then, it is a well-known fact that, $pd_A M \leq n + pd_A S$, i.e. $CI \text{-dim}_R M \leq n$. The reverse implication also follows by applying some well-known facts on the behavior of the projective dimension under exact sequences, to the exact sequence, $0 \to \Omega_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ arising from some augmented free resolution of $M$.

(iii) This follows from [W04, Theorem 4.2].

In [DMTT19], it is shown that the analogue of the New Intersection Theorem for complexes of finite complete intersection holds. Since its analogue for complexes consisting of totally reflexive modules is unknown yet, so at that time it was unknown that whether non-exact bounded complexes with finite length homologies and consisting of modules of complete intersection dimension zero have New Intersection Theorem property? As one corollary to our results we give a positive answer to this question.

**Corollary 3.2.2.** Let, $(R, m, K)$, be a local ring and let, $C := 0 \to C_s \to \cdots \to C_0 \to 0$, be a non-exact complex consisting of modules of complete intersection dimension zero and such that $\ell (H_i (C)) < \infty$ for each $i$. Then, $s \geq \dim R$.

**Proof.** By Corollary 3.2.1(iii), we have $CI \text{-dim} C < \infty$. Thus we are done by [DMTT19, Corollary 2.8].
4. Two Theorems Concerning Test Modules

In this section, all rings are local unless otherwise is stated explicitly or is clear from the text. The main results of this section are Corollary 4.2.1 and Corollary 4.2.2. Corollary 4.2.1 answers positively Question 1.3 when the source of the flat homomorphism has uncountable residue field. Corollary 4.2.2 answers affirmatively Question 1.2 for all local rings.

4.1. Flat coefficient ring base change. To take the first step in the first subsection of the present section, we need the next lemma which, roughly speaking, ensures us that the test module property remains stable after enlarging the coefficient field of the ring by adding an indeterminate (i.e. a transcendental element), provided the coefficient field is uncountable, or the test module is of finite complete intersection dimension.

Lemma 4.1.1. Let \((R, m, K)\) be a complete ring with coefficient ring \(C_K\). Suppose that \(T\) is an \(R\)-test module. Assume that either of the following conditions holds,

(i) there exists a weakly unramified flat complete local \(R\)-algebra \((R', m')\) such that \(R'\) has uncountable residue field \(\mathcal{L}\) and coefficient ring \(C_{\mathcal{L}}\), that there is a map of coefficient rings \(C_K \to C_{\mathcal{L}}\) and that \(T \otimes_R R'\) is an \(R'\)-test module.\(^{10}\)

(ii) \(T\) has finite complete intersection dimension.

Then, \(T \otimes_R \left( (R \otimes_{C_K} (C_K[X] := X_1, \ldots, X_n)_{pC_K[X]})_\mathcal{L} \right)\), is a test module for \((R \otimes_{C_K} (C_K[X]_{pC_K[X]})_{\mathcal{L}}\).

Proof. Although in our paper the notation \(C_{\mathcal{L}}\), in mixed characteristic, stands for the \(p\)-ring of the field \(\mathcal{L}\) which has to be a complete discrete valuation ring, but let us in this proof, for the sake of simplicity and by abuse of notation, denote the incomplete discrete valuation ring \(C_K[X]_{pC_K[X]}\) also by the notation \(C_K[X]\). Note that, indeed, \(R \otimes_{C_K} (C_K[X]_{pC_K[X]})\) is a Noetherian ring.

Let, \(\text{Tor}_n^{(R \otimes_{C_K} C_K(X))_m} \left( N, T \otimes_R \left( R \otimes_{C_K} C_K(X)_{\mathcal{L}} \right)_\mathcal{L} \right) = 0\), for some finite \((R \otimes_{C_K} C_K(X))_{\mathcal{L}}\)-module \(N\). Let \(F_* := F_{n+t+3} \to F_{n+t+2} \to \cdots \to F_0 \to 0\) be the, length \(3 + n + t\) \((t = \text{depth } R\) and \(n\) is the number of indeterminates\), truncation of the \((R \otimes_{C_K} C_K(X))\)-minimal resolution of \(N\). Unifying all the denominators of the entries of the matrices of differentials of \(F_*\) and then eliminating the unique (unformed) invertible denominator, \(F_*\) comes, essentially, from \(R \otimes_{C_K} C_K(X)\) specifying, by 0-th homology, an \(R \otimes_{C_K} C_K(X)\)-module \(N'\) with \(N'_{\mathcal{L}} = N\). Due to the acyclicity over \((R \otimes_{C_K} C_K(X))_{\mathcal{L}}\), there is some, \(g \in \bigcap_{i=1}^{n+t+2} 0 : R \otimes_{C_K} C_K(X) H_i(F_*) \setminus \mathcal{L}\). Let, \(m\), be a minimal generating set of the maximal ideal \(m\) of \(R\). Although, possibly,

\[
\text{Tor}^{R \otimes_{C_K} C_K(X)}_n \left( N', T \otimes_R \left( R \otimes_{C_K} C_K(X) \right) \right) \neq 0,
\]

but considering the tensor product, \(N' \otimes_{R \otimes_{C_K} C_K(X)} \left( K_* (m; R \otimes_{C_K} C_K(X)) \right)\), of \(N'\) with the Koszul complex of \(m\) we have

\[
\text{Tor}^{R \otimes_{C_K} C_K(X)}_{\geq u} \left( N' \otimes_{R \otimes_{C_K} C_K(X)} \left( K_* (m; R \otimes_{C_K} C_K(X)) \right), T \otimes_R \left( R \otimes_{C_K} C_K(X) \right) \right) = 0,
\]

\(^{10}\)Of course we can set, \(R' = R\), provided \(R\), itself, has uncountable residue field. The condition on the existence of such extension \(R'\) is a technical condition which is necessary in page 24.
for some $u \in \mathbb{N}$.

Next, we can assume that $F_\bullet$ (the entries of the matrices of its differentials) and $g$ both come from the ring, $R \otimes_{C_K} (C_K[X]|_{c_0})$, for some invertible element $\zeta_0 \in C_K[X]$. The 0-homology of $F_\bullet$, defines an $R \otimes_{C_K} (C_K[X]|_{c_0})$-module $N''$ with

$$N'' \otimes_{R \otimes_{C_K} (C_K[X]|_{c_0})} (R \otimes_{C_K} C_K(X)) = N'.$$

Note that, $R \otimes_{C_K} C_K(X)$ coincides with the localization of $R \otimes_{C_K} (C_K[X]|_{\zeta_0})$ at its multiplicative closed subset $S := \{0 \neq \zeta \in f/\zeta_0^m ; f \notin pC_K[X] \text{ and } m \in \mathbb{N}_0 \} \subseteq R \otimes_{C_K} (C_K[X]|_{\zeta_0})$. Thus it is legitimate to presume that

$$\text{Tor}_a\left(R \otimes_{C_K} (C_K[X]|_{\zeta_0}) \left( N'' \otimes_{R \otimes_{C_K} (C_K[X]|_{c_0})} K_{\bullet} \left( m; (R \otimes_{C_K} (C_K[X]|_{\zeta_0})) \right), T \otimes_R (R \otimes_{C_K} (C_K[X]|_{\zeta_0})) \right) = 0$$

and that $F_{\bullet g}$ is acyclic in $\left(R \otimes_{C_K} (C_K[X]|_{\zeta_0})\right)_g$. By the same token, for each $s \geq 1$, inductively, we may find a multiple $\zeta_s$ of $\zeta_{s-1}$ such that

$$(4.2) \quad \text{Tor}_{\left[a, a+s+f \mod m\right]}\left(R \otimes_{C_K} (C_K[X]|_{\zeta_0}) \left( N'' \otimes_{R \otimes_{C_K} (C_K[X]|_{c_0})} K_{\bullet} \left( m; (R \otimes_{C_K} (C_K[X]|_{\zeta_0})) \right), T \otimes_R (R \otimes_{C_K} (C_K[X]|_{\zeta_0})) \right) = 0.$$

We consider a presentation of $g$ as, $g = \sum_{i=1}^{v} r_i \otimes (f_i(X)/\zeta_0^m)$.

In case (i) of the statement of the lemma, since the residue field $L$ of $R'$ is uncountable, so we are easily able to find some $c \subseteq C_L$ such that $\zeta_s(g) \notin pC_L$ for any $s \in \mathbb{N}_0$ and, moreover, such that the evaluation of (the image in $R'[X]$ of) the polynomial $P(g) := \sum_{i=1}^{v} r_i f_i(X)$, at $c$ in $R'$, does not belong to $m' \subseteq R'$.

\text{12} The sufficiently high homologies have empty support. If $p$ belongs to their support then $p \notin m$ by our tor-vanishing assumption over $\left(R \otimes_{C_K} (C_K[X])\right)_m$. On the other hand $p$ should contain the maximal ideal $m$, i.e. $p = m$, as $m$ kills all the homologies of,

$$N' \otimes_{R \otimes_{C_K} C_K(X)} \left( K_{\bullet} \left( m; R \otimes_{C_K} C_K(X) \right), T \otimes_R (R \otimes_{C_K} (C_K[X]|_{\zeta_0})) \right).$$

It is perhaps noteworthy to stress that, the last statement here can be deduced from the fact that the tensor product of two chain homotopic maps are again chain homotopic, more precisely if $f \approx f'$ and $g \approx g'$ then $f \otimes g \approx f' \otimes g'$ for chain maps of complexes $f, f', g, g'$. Recall that the multiplication map on the Koszul complex of some ideal $I$ with some element of $I$ is null-homotopic.

\text{13} To see this, just note that,

$$R \otimes_{C_K} C_K(X) = R \otimes_{C_K} \left( C_K[X]|_{\zeta_0} \otimes_{C_K[X]|_{c_0} C_K(X) \right) \cong \left( R \otimes_{C_K} C_K(X)|_{\zeta_0} \right) \otimes_{C_K[X]|_{c_0} C_K(X) \cong \left( R[X]|_{\zeta_0} \otimes_{C_K[X]|_{c_0} C_K[X]|_{(p)} \cong T^{-1}(R[X]|_{\zeta_0}) \right.$$

wherein $T$ is the image of $C_K[X]|_{(p)}$ in $R[X]|_{\zeta_0}$. Now, the isomorphism, $R[X]|_{\zeta_0} \cong R \otimes_{C_K} C_K(X)|_{\zeta_0}$, maps $T$ to the multiplicative closed subset $S$.

\text{14} Here, we tacitly used the natural isomorphism, $(R \otimes_{C_K} (C_K[X]|_{\zeta_0}))_{\zeta_{0}'} \cong R \otimes_{C_K} (C_K[X]|_{\zeta_0'})$, and we, if necessary, replace $\zeta_0$ with, $\zeta_{0}'$.

\text{15} Here, the image of $\zeta_s$ under $C_K[X] \to C_L[X]$ is considered.

\text{16} This is indeed possible, because, $g \notin M$, so $P(g) := \sum_{i=1}^{v} \tau_{g} f_i(X) \neq 0$, in $(R/m)[X] = R \otimes_{C_K} C_K[X]/2M$ (here, $\tau_{g} := r_i + m$ and $\tau_{g}$ is defined similarly by considering the residue classes under the isomorphism of fields $K \to R/(p) \to R/m$). So its image in $(R'/m')[X]$ is also non-zero. Then, indeed, as $L \cong R'/m'$ is uncountable,

$$P := \{ \zeta_{s} \}_{s \geq 0}, \quad \text{image of } \zeta_{s} \text{ in } C_L/pC_L[X] = C[X].$$
In case (ii) of the statement of lemma, where $T$ has finite complete intersection dimension, we will, at the end of the proof, show that the modules,
\[
(4.3) \text{Tor}^1_{\mathfrak{m}} \left( R \otimes_{\mathcal{L}} (\mathcal{L}[X], \zeta) \right) \left( N'' \otimes_{\mathfrak{m}} (\mathcal{L}[X], \zeta) \right) \to K \notag
\]
have a common annihilator $1 \otimes \zeta$. Thus, we may replace $\zeta$ in the preceding paragraphs with one fixed element $(\zeta = \zeta_0$ for any $s \geq 1$. In this case, if $K$ is an infinite field then we set $R'' = R$ (and $\mathcal{L} = K$, $\mathcal{L}_C = C_K$). Otherwise, we consider a finite extension $\mathcal{L}$ of $K$ whose cardinality is strictly greater than $n - 1 + \deg(\zeta_1) + \deg(\zeta_0) + \deg(P(g))$, so that $\zeta_0 \subseteq \mathcal{L}$ satisfying the desired non-membership property discussed in case (i) exists. Then choose a $p$-ring extension $C_K \to \mathcal{L}$ and set, $R'' := R \otimes_{C_K} \mathcal{L}$ which is Noetherian and a weakly unramified flat extension of $R$ by Proposition 2.3.2. Note that, by virtue of CW16 Proposition 3.5, $T \otimes_R R''$ is a test module for $R''$. The particular choice of $R''$ enables us to find $\zeta \subseteq \mathcal{L}$ as in the case (i).

Then, in both case (i) and case (ii), the natural map $\varphi_\ast : C_L[X] \to C_L$ which fixes $\mathcal{L}$ and maps $X_i$ to $c_i$, and whose kernel is generated by $(X_i - c_i)_{1 \leq i \leq n}$, extends to $C_L[X], C_L \to C_L$. So there are ring homomorphisms
\[
\psi_\ast : R \otimes_{C_K} (C_K[X], \zeta) \to R' \otimes_{C_C} (C_L[X], \zeta) \text{ id} \otimes \varphi_\ast, \quad R' \otimes_{C_C} C_L \cong R',
\]
which present $R''$ as the quotient
\[
(R' \otimes_{C_C} (C_L[X], \zeta))/(X_i - c_i) \leq i \leq n \cong R',
\]
by the regular sequence, \{(X_i - c_i)_{1 \leq i \leq n} \subseteq (R' \otimes_{C_C} (C_L[X], \zeta))\}. Then, again, we would still have
\[
R' \cong (R' \otimes_{C_C} (C_L[X], \zeta))/(X_i - c_i) \leq i \leq n, \quad s \in \mathbb{N}_0
\]
because inverting an invertible element is ineffective (g maps to invertible element of $R'$, by our assumption that $(P(g)) \not\subseteq 0$). The composition,
\[
R' \to (R' \otimes_{C_C} C_L[X], \zeta)_{g} \to (R' \otimes_{C_C} (C_L[X], \zeta))_{g}/(X_i - c_i) \leq i \leq n \cong R',
\]
is the identity map, therefore,
\[
\left( T \otimes_R R' \right) \otimes R' \left( R' \otimes_{C_C} (C_L[X], \zeta) \right)_{g}/(X_i - c_i) \leq i \leq n \left( T \otimes_R R' \right) \otimes R' \left( R' \otimes_{C_C} (C_L[X], \zeta) \right)_{g} \cong \left( T \otimes_R R' \right) \otimes R' \left( R' \otimes_{C_C} (C_L[X], \zeta) \right)_{g}/(X_i - c_i) \leq i \leq n \left( T \otimes_R R' \right) \otimes R' \left( R' \otimes_{C_C} (C_L[X], \zeta) \right)_{g}\]
and $R''$ in place of, $\left( R' \otimes_{C_C} (C_L[X], \zeta) \right)_{g}/(X_i - c_i) \leq i \leq n$.

If it is necessary, in view of Fact 2.2.3, we replace $N''$ with its high syzygy \[14\] to assume that $\{X_i - c_i\}_{1 \leq i \leq n}$ is a regular sequence on $L'' := N''_g \otimes_{R \otimes_{C_K} (C_K[X], \zeta)} (R' \otimes_{C_C} (C_L[X], \zeta))_{g}$ \[15\]. Then, have a common non-root, say $L$, in other words a sequence $\zeta \in C_L$ lifting $L$ such that $0 \not\subseteq (P(g)) \in R'/m'$ and $\zeta \in \overline{m'_r}$, for any $s \in \mathbb{N}_0$.

\[16\] At most n-th syzygy in $F_{\ast g}$ which was acyclic and resolves $N''_g$.

\[17\] $(R' \otimes_{C_C} (C_L[X], \zeta))_{g}$ is flat over $(R \otimes_{C_K} (C_K[X], \zeta))_{g}$, because so is $R' \otimes_{C_C} C_L[X] \cong R' \otimes_{C_C} C_L[X]$ over $R \otimes_{C_K} C_K[X] \cong R[X]$, and localization at a multiplicative closed subset of the domain of a flat homomorphism, remains flat (One way to see
from the double complex, of tors. Pick some \( \zeta \) already invertible element (spectral sequence collapses at the 0-th column, concluding that the total complex is quasi-isomorphic to \( X \) and first taking the homology of rows, as \( \ge \) because at any fixed number \( R \) that modules). Moreover, \( (R' \otimes_{C_L} (C_L[X]_{i,i})_g)/(X_i - c_{li})_{1 \leq i \leq n} \).

Before proceeding with the remainder of the proof we explain, by details, the just mentioned vanishing of tors. Pick some \( s' \) much greater than some fixed \( i \geq u + n \). Considering the spectral sequence arising from the double complex,

\[
\begin{aligned}
\left( \mathcal{L}N_1 \otimes_{\zeta, \mathcal{L}} \left( R' \otimes_{C_L} (C_L[X]_{i,i})_g \right) \right)_{K^\bullet(m; (R' \otimes_{C_L} (C_L[X]_{i,i})_g)_g)} \\
\text{sitting in columns}
\end{aligned}
\]

\[
\begin{aligned}
K^\bullet \left( (X_i - c_{li})_{1 \leq i \leq n}; (R' \otimes_{C_L} (C_L[X]_{i,i})_g) \right), \\
\text{sitting in rows}
\end{aligned}
\]

and first taking the homology of rows, as \( (X_i - c_{li})_{1 \leq i \leq n} \) acts regularly on \( \mathcal{L}N_1 \otimes_{\zeta, \mathcal{L}} \), the first page of the spectral sequence collapses at the 0-th column, concluding that the total complex is quasi-isomorphic to the complex,

\[
\begin{aligned}
\left( \left( \mathcal{L}N_1 \otimes_{\zeta, \mathcal{L}} \left( (X_i - c_{li})_{1 \leq i \leq n} \right) \right) \otimes_{(R' \otimes_{C_L} (C_L[X]_{i,i})_g)} K^\bullet(m; (R' \otimes_{C_L} (C_L[X]_{i,i})_g)_g) \right) \\
\text{sitting in columns}
\end{aligned}
\]

\[
\begin{aligned}
\left( \left( \mathcal{L}N_1 \otimes_{\zeta, \mathcal{L}} \left( (X_i - c_{li})_{1 \leq i \leq n} \right) \right) \otimes^L_{(R' \otimes_{C_L} (C_L[X]_{i,i})_g)} (T \otimes_R (R' \otimes_{C_L} (C_L[X]_{i,i})_g)_g) \right) \\
\text{sitting in rows}
\end{aligned}
\]

Therefore, all three complexes displayed above are quasi-isomorphic, and hence,

\[
\begin{aligned}
\left( \left( \mathcal{L}N_1 \otimes_{\zeta, \mathcal{L}} \left( (X_i - c_{li})_{1 \leq i \leq n} \right) \right) \otimes^L_{(R' \otimes_{C_L} (C_L[X]_{i,i})_g)} K^\bullet(m; (R' \otimes_{C_L} (C_L[X]_{i,i})_g)_g) \right) \\
\text{sitting in columns}
\end{aligned}
\]

\[
\begin{aligned}
\left( \left( \mathcal{L}N_1 \otimes_{\zeta, \mathcal{L}} \left( (X_i - c_{li})_{1 \leq i \leq n} \right) \right) \otimes^L_{(R' \otimes_{C_L} (C_L[X]_{i,i})_g)} (T \otimes_R (R' \otimes_{C_L} (C_L[X]_{i,i})_g)_g) \right) \\
\text{sitting in rows}
\end{aligned}
\]

has zero homologies at \( [i, i + n] \cap \mathbb{N} \), in view of the largeness of \( s' \) and \( [L, 2] \). In other words we have the vanishing,

\[
\begin{aligned}
\text{Tor}^R_{i, i+n} \left( (R' \otimes_{C_L} (C_L[X]_{i,i})_g) \left( \left( \mathcal{L}N_1 \otimes_{\zeta, \mathcal{L}} \left( (X_i - c_{li})_{1 \leq i \leq n} \right) \right) \otimes^L_{(R' \otimes_{C_L} (C_L[X]_{i,i})_g)} K^\bullet(m; (R' \otimes_{C_L} (C_L[X]_{i,i})_g)_g) \right) \right), \\
\text{where} \quad T \otimes_R (R' \otimes_{C_L} (C_L[X]_{i,i})_g)_g = 0,
\end{aligned}
\]

that \( R'[X] \) is flat over \( R[X] \) is that \( R' \) is a direct limit of free \( R \)-modules, thus \( R'[X] \) is a direct limit of free \( R[X] \)-modules. Moreover, \( (X_i - c_{li})_{1 \leq i \leq n} \mathcal{L}N \subseteq \mathcal{L}N \), because \( F_{s/g} \) is a minimal presentation for \( \mathcal{L}N/(X_i - c_{li})_{1 \leq i \leq n} \mathcal{L}N \) as \( F_{s} \) was a minimal complex.

\(^{18}\)Recall that \( \zeta \) is multiple of \( \zeta_0 \).
of tor modules. The sequence, $(X_i - c_i)_{1\leq i \leq n}$, is also a regular sequence on $T \otimes_R (R' \otimes_{cL} (C_L[X_{i^*}]))$. This, in conjunction with [BH98 Proposition 1.1.5], implies that the localization at $\zeta_{p'}(c_1)$ of $\text{Tor}_i^R \left( \left( N/(X_i - c_i)_{1 \leq i \leq n} \otimes_R K_{(m, R')} C_L[X_{i^*}] \right) \right)$ is zero, thus itself vanishes because $\zeta_{p'}(c_1)$ is invertible.

Setting, $\mathcal{L}F_* := F_* \otimes \left( B_{S} \otimes_{cK} (C_K[X_{i^*}]) \right)_g$, again by [BH98 Proposition 1.1.5], $\mathcal{L}F_*/(X_i - c_i)_{1 \leq i \leq n} \mathcal{L}F_*$ (or some part of it if some high syzygy of $N_g''$ is considered instead of $N_g''$) is a part of a minimal free resolution of $N/(X_i - c_i)_{1 \leq i \leq n} \mathcal{L}N$, which is forced to be less than or equal to $t$ by the Auslander-Buchsbaum formula. Thus, the $3n + t$-th free module in the complex $\mathcal{L}F_*/(X_i - c_i)_{1 \leq i \leq n} \mathcal{L}F_*$ is zero which is impossible unless either $F_{3n + t} = 0$ or $(X_i - c_i)_{1 \leq i \leq n}$ generates the improper ideal while the latter violates $R' \neq 0$.

This shows that $N$ is of finite projective dimension, as was to be proved.

To complete the proof, it remains to prove our claim that, under condition (ii) the tor modules in have a common annihilator $\zeta_1$. Consider a quasi-deformation $R \rightarrow (S, m_S, K) \leftarrow Q$ of $R$ with $\text{pd}_Q (T \otimes_R S) < \infty$, in which we are free to assume that $m_S$ is primary to $m_S$ and both of $S$ and $Q$ are complete rings (see, Lemma 3.1.1). By an argument as in the proof of [189 Theorem 29.2], as $C_K$ is $(p)$-smooth over $\mathbb{Z}Q$, and $Q$ is complete, we can see that the ring homomorphism $C_K \rightarrow S$, obtained from $C_K \rightarrow R \rightarrow S$, lifts to a ring homomorphism $C_K \rightarrow Q$. We can change the base ring to $S \otimes_{C_K} (C_K[X_{i^*}])$, because $R \otimes_{C_K} (C_K[X_{i^*}]) \rightarrow S \otimes_{C_K} (C_K[X_{i^*}])$ is a faithfully flat extension. Set, $T_S := T \otimes_R (S \otimes_{C_K} (C_K[X_{i^*}])$, $N_S := N'' \otimes_{B_S \otimes_{cK} (C_K[X_{i^*}])} (S \otimes_{C_K} (C_K[X_{i^*}]), R_S := R \otimes_{C_K} (C_K[X_{i^*}]), S_\zeta := S \otimes_{C_K} (C_K[X_{i^*}])$ and $Q_\zeta := Q \otimes_{C_K} (C_K[X_{i^*}]).$ Hence, our objective is to show that $1 \otimes_{\zeta} \left( \text{Tor}_{s_2}^S (NS \otimes_{S_\zeta} K_p(m; S_\zeta), T_S) \right) = 0$ for some $\zeta_3 \in C_K[X]$. 

Again, stating the reason of this tor vanishing might be helpful to the reader. We bear in mind the spectral sequence arising from the (first quadrant) double complex of the tensor product of two complexes, one determining the columns is the complex, 

$$
\left( \left( N_{i \otimes} \otimes_{(R' \otimes_{cL}(C_L[X_{i^*}]))_g} K_p(m; (R' \otimes_{cL}(C_L[X_{i^*}]))_g) \right) \right)_{0 \otimes (R' \otimes_{cL}(C_L[X_{i^*}]))_g},
$$

and the other one determining the rows is the complex $K_p \left( (X_i - c_i)_{1 \leq i \leq n}; \left( R' \otimes_{cL}(C_L[X_{i^*}]) \right) \right)$. The homology of the total complex are our desired tor homologies. We analyze the first filtration of this total complex which is obtained by restricting the number of columns that are considered at each step. Hence the first page of the spectral sequence is given by taking the homologies of the columns of the double complex. Thus in view of [123, the first page of our spectral sequence has lots of zero rows starting from row $u$ and (at least) up to the row $s' \leq s'$ which we can choose $s'$ to be much greater than $i + n$. At each $i \leq j \leq i + n$ then the homology of total complex is filtered by the filtration $\{ \phi_{j} H_i (\text{Tot}) \} \subseteq \ldots$, such that the quotient $\phi_{j} H_i (\text{Tot}) / \phi_{j} (H_i (\text{Tot}))$ is a subquotient of $E_{p, j - p}$. Since, $i \geq u + n$ so either $p \geq n + 1$ or $j - p \geq u$, which shows that $E_{p, j - p}$'s and (thus subquotients) are zero. Thus any module in the filtration is also zero, i.e our desired tor vanishings hold.

The complex, $K_p \left( (X_i - c_i)_{1 \leq i \leq n}; \left( R' \otimes_{cL}(C_L[X_{i^*}]) \right) \right) \rightarrow R' = 0$, is an exact complex consisting of flat $R$-modules, therefore it remains exact after tensoring with any $R$-module, including $T$. This is one way to observe the regularity of the sequence. 

One can easily verify that this minimal property of the complex is inherited from $(R \otimes_{cK} C_K[X_{i^*}])_{\otimes}$ as the entries of the matrices of differentials of the complex still belong to the extension of the maximal ideal $m$ of $R$.

This is the flat homomorphism $R[X_{i^*}] \rightarrow S[X_{i^*}]$. It is easily seen that if $A \rightarrow B$ is a faithfully flat homomorphism of rings and $S$ is a multiplicative closed subset of $A$ then $S^{-1}A \rightarrow S^{-1}B$ is also faithfully flat.
Let, $D^\bullet$ be a dualizing complex of $S_\zeta$, which exists e.g. by [CF06] (7.1.6) Remark, and let $G^S_\bullet$ be a free resolution of $T$ over $R$ yielding the free resolution, $G^S_\bullet := G^S_\bullet \otimes_R S_\zeta$, of $T_S$. We consider,

$$\text{Hom}_{S_\zeta}(G^S_\bullet \otimes_{S_\zeta} N_S \otimes_{S_\zeta} K_\bullet(m; S_\zeta), D^\bullet) \simeq \text{Hom}_{S_\zeta}(G^S_\bullet, \text{Hom}_{S_\zeta}(NS \otimes_{S_\zeta} K_\bullet(m; S_\zeta), D^\bullet)_{M^\bullet}) \simeq R\text{Hom}_{S_\zeta}(T_S, M^\bullet),$$

whose homologies are $\text{Ext}^i_{S_\zeta}(T_S, M^\bullet)$. Since $\text{pd}_Q(T \otimes_R S) < \infty$ and $M^\bullet$ is a bounded complex so $\text{Ext}^i_{S_\zeta}(T_S, M^\bullet) = 0$. Consequently, in the light of and beauty of [AS98] Theorem, page 708, the graded module, $\bigoplus_{c \in \mathbb{N}_0} \text{Ext}^i_{S_\zeta}(T_S, M^\bullet)$, is finitely generated over the ring of operators, $\mathcal{J} := S_\zeta[\xi_1, \ldots, \xi_n]; \text{deg}(\xi_i) = 2$ (here $c$ is the codimension of the ideal of $S_\zeta$ in $Q_\zeta$). Moreover,

$$\text{Ext}^\geq_w_{S \otimes_{C_K} C_K}(X)(T \otimes_R (S \otimes_{C_K} C_K(X)), M^\bullet \otimes_{S_\zeta} (S \otimes_{C_K} (C_K(X)))) = H^\geq_w(\text{Hom}_{S_\zeta}(G^S_\bullet, (S \otimes_{C_K} C_K(X))) \otimes_{S \otimes_{C_K} C_K(X)} \text{Hom}_{S_\zeta}(NS \otimes_{S_\zeta} K_\bullet(m; S_\zeta)) \otimes_{S_\zeta} (S \otimes_{C_K} C_K(X))),$$

for some natural number $w$, because, in view of (1.4),

$$\left(G^S_\bullet \otimes_{S_\zeta} (S \otimes_{C_K} C_K(X)) \otimes_{S \otimes_{C_K} C_K(X)} (NS \otimes_{S_\zeta} K_\bullet(m; S_\zeta)) \otimes_{S_\zeta} (S \otimes_{C_K} C_K(X))\right) \in D^b(S \otimes_{C_K} C_K(X)),$$

and furthermore $D^\bullet \otimes_{S_\zeta} (S \otimes_{C_K} C_K(X))$ is a bounded complex consisting of injective modules. Suppose that, $\text{Ext}^i_{S_\zeta}(T_S, M^\bullet), \ldots, \text{Ext}^j_{S_\zeta}(T_S, M^\bullet)$, generates $\text{Ext}^j_{S_\zeta}(T_S, M^\bullet)$ over $\mathcal{J}$. By our choice of $w$, there is some $0 \neq \zeta_w \in C_K[X]/pC_K[X]$ such that, $(1 \otimes \zeta_w)\left(\bigoplus_{j=w}^{w+v+1} \text{Ext}^j_{S_\zeta}(T_S, M^\bullet)\right) = 0$. Let, $i, v \geq w + 2$ be an arbitrary number and pick some, $e_j \in \text{Ext}^j_{S_\zeta}(T_S, M^\bullet)$. Then, $e_i = \sum_j h_j(\zeta_w) e_j$, for some homogeneous elements $h_j(\zeta_w) \in \mathcal{J}$ and $e_j \in \text{Ext}^j_{S_\zeta}(T_S, M^\bullet)$ with $j \leq v$. Then, $\text{deg}(h_j(\zeta_w)) \geq [w/2]$ and $h_j(\zeta_w) = \sum_{n_v + \ldots + n_j,=[w/2]} X^{i_1,1} \cdots X^{i_v,v} h_j(\zeta_w)$. Then, since each $X^{i_1,1} \cdots X^{i_v,v} e_j \in \left(\bigoplus_{j=w}^{w+v+1} \text{Ext}^j_{S_\zeta}(T_S, M^\bullet)\right)$ so $1 \otimes \zeta_w$ also kills $e_i$. It follows that $1 \otimes \zeta_w$ kills, $\text{Ext}^\geq_w(T_S, M^\bullet) = H^\geq_w(\text{Hom}_{S_\zeta}(G^S_\bullet, (S \otimes_{S_\zeta} K_\bullet(m; S_\zeta))), D^\bullet)$. Then, it is easily, by an spectral sequence argument, that some power, $1 \otimes \zeta_w$, of $1 \otimes \zeta_w$ kills sufficiently large homologies of, $\text{Hom}_{S_\zeta}(G^S_\bullet, (S \otimes_{S_\zeta} K_\bullet(m; S_\zeta))), D^\bullet) \simeq G^S_\bullet \otimes_{S_\zeta} (S \otimes_{S_\zeta} K_\bullet(m; S_\zeta)), as was claimed.

In the next lemma we generalize the result of the previous lemma concerning the finitely generated purely transcendental coefficient field extensions, to the context of arbitrary finitely generated extension of coefficient fields.

**Lemma 4.1.2.** Suppose that, $\varphi_{K^n}$ turns $K$ into a finitely generated field extension of a field $K$ of any characteristic, and that $R$ is a complete ring whose coefficient ring is $C_K$. Fix a lifting $\varphi_{K^n}$ of $\varphi_{K^n}$ by which, we can consider the complete tensor product $R \otimes_{C_K} C_K$. Let, $T$ be a test module for $R$. If either of two conditions in the statement of Lemma 4.1.1 holds, then, the extension, $T \otimes_R (R \otimes_{C_K} C_K)$, of $T$, is an $R \otimes_{C_K} C_K$-test module.

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22Here it is, perhaps, worth to stress that $\text{Tor}^S_{\bullet}(T_S, M^\bullet)$, as far we know, is not necessarily finitely generated over the ring of operators $\mathcal{J}$. If $\text{Tor}^S_{\bullet}(T_S, M^\bullet)$ is Artinian for all $i$, then the graded $\mathcal{J}$-module $\text{Tor}^S_{\bullet}(T_S, M^\bullet)$ is an Artinian module which might be non-finitely generated.

23We remind that $S \otimes_{C_K} C_K(X)$ is a localization of $S_\zeta$. 

A Trilogy

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Proof. We are allowed to consider a subfield \( \mathcal{L} = K(\tilde{\kappa}_1, \ldots, \tilde{\kappa}_n) \) of \( K \), assigned to a finite set of algebraically independent elements \( \tilde{\kappa}_1, \ldots, \tilde{\kappa}_n \) of \( K \) over \( K \), such that \( K \) is a finite algebraic extension of \( \mathcal{L} \). We are required to factor \( C_K \rightarrow C_K \) as a composition \( C_K \rightarrow C_\mathcal{L} \rightarrow C_K \). To observe that, the extension \( C_K[X_1, \ldots, X_n] \rightarrow C_K \), of \( \varphi_{C_K,C_K} \), given by the rule \( X_i \mapsto \kappa_i \), promotes to

\[
\varphi_{C_\mathcal{L},C_K} : C_\mathcal{L} := C_K[X_1, \ldots, X_n]^{(\widetilde{\kappa})} \rightarrow C_K
\]

is about to simply notice that any polynomial \( f \) having at least one coefficient not contained in \( pC_K \), has to satisfy \( f(\kappa_1, \ldots, \kappa_n) \notin pC_K \), because of the algebraic independence of the sequence \( \tilde{\kappa}_1, \ldots, \tilde{\kappa}_n \) over \( K \). Evidently, \( \varphi_{C_\mathcal{L},C_K} \) extends \( \varphi_{C_K,C_K} \).

We denote the extended ideal \( \mathfrak{m}(R \otimes_{C_K} C_\mathcal{L}) \) by \( \mathfrak{m} \), which is a maximal ideal of \( R \otimes_{C_K} C_\mathcal{L} \). Set, \( V := C_K[X_1, \ldots, X_n]^{(\widetilde{\kappa})} \), thus \( \widetilde{V} = C_\mathcal{L} \).

By Lemma 4.1.1 we already know that \( (T \otimes_{C_K} C_\mathcal{L})^{\mathfrak{m}} \) is a test module over \( (R \otimes_{C_K} C_\mathcal{L})^{\mathfrak{m}} \). Consequently, we, in view of \([\text{CW16, Theorem 3.5.}]\), already have \( T \otimes_R (R \widehat{\otimes}_{C_K} V) = T \otimes_R (R \widehat{\otimes}_{C_K} C_\mathcal{L}) \) is a test module for \( R \widehat{\otimes}_{C_K} V = R \widehat{\otimes}_{C_K} C_\mathcal{L} \).

Now a further use of [\text{CW16 Theorem 3.5}] in conjunction with Proposition 2.3.2(iii) (or alternatively, [\text{A69, Corollary 10.4}]), to the flat extension,

\[
\begin{align*}
R \widehat{\otimes}_{C_K} C_\mathcal{L} \rightarrow (R \widehat{\otimes}_{C_K} C_\mathcal{L}) \widehat{\otimes}_{C_K} C_K &= \lim_{\mathfrak{m} \in \mathfrak{m}} \left( (R \widehat{\otimes}_{C_K} C_\mathcal{L}) / (\mathfrak{m}^n) \right) \otimes_{C_K} C_K/p^n C_K \\
&= \lim_{\mathfrak{m} \in \mathfrak{m}} \left( (R / \mathfrak{m}^n) \otimes_{C_K/p^n C_K} C_\mathcal{L} / p^n C_\mathcal{L} \otimes_{C_K/p^n C_K} C_K/p^n C_K \right) \\
&= R \widehat{\otimes}_{C_K} C_K,
\end{align*}
\]

implies the statement, because the map induced on the residue fields, is the finite algebraic extension \( \varphi_{K,K}(\tilde{\kappa}_1, \ldots, \tilde{\kappa}_n) \rightarrow K \).

As the next step, we show that, in the statement of Lemma 4.1.2, the finitely generated condition of \([K : K]\) is superfluous. While for rings containing a field we only apply Lemma 4.1.2 (and thus tacitly Lemma 4.1.1), but for the mixed characteristic case of the next lemma we use the very recent Wagstaff’s [\text{W19 Theorem 4.8}]. Before, Wagstaff’s [\text{W19 Theorem 4.8}], in the mixed characteristic case, we had succeeded to find a long proof, using Hensel’s lemma, but our mixed characteristic proof needed the field extension \([K : K]\) to be separably generated, i.e. a separable extension of a purely transcendental extension of \( K \).

Lemma 4.1.3. Let \( T \) be a test module over a complete local ring \((R, \mathfrak{m}, K)\), with coefficient ring \( \lambda_K : C_K \rightarrow R \). If either of two conditions in the statement of Lemma 4.1.1 holds, then, for any extension \( \varphi_{C_K,C_K} \) of \( p \)-rings or fields, \( T_K := T \otimes_R (R \widehat{\otimes}_{C_K} C_K) \) a test module for \( R_K := R \widehat{\otimes}_{C_K} C_K \).

Proof. It suffices to reduce the statement to the case where \( K \) is a finitely generated field extension of \( K \), and then we are done by Lemma 4.1.2. This reduction is somewhat straightforward, as explained in the next few lines.

We first argue the equi-characteristic case, i.e. when \( R \) contains a field and so \([K = C_K : K = C_K]\) is a field extension. By virtue of [\text{CW16 Theorem 3.5} in conjunction with Proposition 2.3.2(iv)], our desired

25The completion of the source of \( \varphi_{C_\mathcal{L},C_K} \) is taken with respect to the \( p \)-adic topology. Moreover, \([K = C_K : K = C_K]\) is localized at \( pC_K[X_1, \ldots, X_n] \) and then, thereafter, the completion is taken over the localized ring.

26Here it is perhaps noteworthy to remark that the existence of such a factorization of \( \varphi_{C_K,C_K} \) (in mixed characteristic), in general, is obscure provided \([\mathcal{L} : K]\) is not a purely transcendental subextension of \([K : K]\), unless \([K : K]\) is a separably generated extension.

27To obtain the last identity in the above equation, we tacitly used the fact that \( \varphi_{C_K,C_K} \) factors through \( C_K \rightarrow C_\mathcal{L} \) followed by \( \varphi_{C_\mathcal{L},C_K} \).
test module property follows from test module property of \( T \otimes_R (R \otimes_K \mathbb{K})_{\mathfrak{m}} \) over \((R \otimes_K \mathbb{K})_{\mathfrak{m}}\), while \((R \otimes_K \mathbb{K})_{\mathfrak{m}}\) is a direct limit of the direct system, \( \{(R \otimes_K \mathbb{K})_{\mathfrak{m}}\}_{\mathfrak{m}} \) is a finite sequence in \( \mathbb{K} \), consisting of local rings and flat weakly unramified local homomorphisms. Moreover, an arbitrary finitely generated (thus finitely presented) \((R \otimes_K \mathbb{K})_{\mathfrak{m}}\)-module \( N \) essentially comes, via \(- \otimes (R \otimes_K \mathbb{K})_{\mathfrak{m}}\), from such a subring \((R \otimes_K \mathbb{K})_{\mathfrak{m}}\) of \((R \otimes_K \mathbb{K})_{\mathfrak{m}}\), because so does any information \((R \otimes_K \mathbb{K})_{\mathfrak{m}}^{n!} \rightarrow 0\). Finally, the (faithfully) flatness of \((R \otimes_K \mathbb{K})_{\mathfrak{m}}\) over \((R \otimes_K \mathbb{K})_{\mathfrak{m}}^{n!} \rightarrow 0\) implies that the eventually-tor-independence of the extension of \( T \) and \( N \), as well as the finiteness of the projective dimension of \( N \) are both independent of the base ring to be \((R \otimes_K \mathbb{K})_{\mathfrak{m}}\) or \((R \otimes_K \mathbb{K})_{\mathfrak{m}}^{n!} \rightarrow 0\).

Now, we deal with the mixed characteristic case. Consider a lift, \( \zeta \subseteq C_K \), of a transcendental basis, \( x \), of \( K \), over \( K \), such that \( x \) turns \( K \) to be an algebraic extension of \( K(x) \). Then, \( \zeta \) is algebraically independent over \( C_K \) as well and there is an induced natural map \( C_K[\zeta'](p) \rightarrow C_K[\zeta](p) \) for any finite subset \( \zeta' \) of \( \zeta \) as we argued in the proof of Lemma \[2.1.2\]. Thus, the direct limit, \[ \lim_{\zeta' \subseteq \zeta \text{ a finite subset of } \zeta} C_K[\zeta'](p) \] is unramified discrete valuation ring with residue field \( K(x) \) and there is a natural map, \[ \lim_{\zeta' \subseteq \zeta \text{ a finite subset of } \zeta} C_K[\zeta'](p) \rightarrow C_K, \] extending \( \varphi_{K,K} \). Thus, the \( p \)-adic completion of \[ \lim_{\zeta' \subseteq \zeta \text{ a finite subset of } \zeta} C_K[\zeta'](p), \] say \( C_K(p) \), is a \((\text{the}) p\)-ring for the field \( K(x) \). We have, \( (R \otimes_{C_K} C_K(x))_{\mathfrak{m}} = \lim_{\zeta' \subseteq \zeta \text{ a finite subset of } \zeta} (R \otimes_{C_K} C_K[\zeta'](p))_{\mathfrak{m}} \). Its Noetherian by the same reason as in the proof of Proposition \[2.3.2(iv)\]. Thus, arguing as in the equicharacteristic case, we can deduce that, \( T \otimes_R (R \otimes_{C_K} C_K(x))_{\mathfrak{m}} \), is a test module for \( R \otimes_{C_K} C_K(x) \). Now, the statement follows by applying \[19\] Theorem 4.8 to the flat local extension, \( R \otimes_{C_K} C_K(x) \rightarrow R \otimes_{C_K} C_K = (R \otimes_{C_K} C_K(x)) \otimes_{C_K(x)} C_K \), which is residually algebraic.

4.2. Celikbas-Wagstaff and Celikbas-Dao-Takahashi Questions. As a first corollary to the previous subsection, we show that the module version of the question \[CW16\] Question 3.7, raised by Wagstaff and Celikbas, has an affirmative answer in the category of local rings with uncountable residue field.

**Corollary 4.2.1.** Suppose that, \( \varphi : (R, \mathfrak{m}, K) \rightarrow (S, \mathfrak{n}, \mathbb{K}) \), is a flat local homomorphism with regular closed fiber and \( T \) is a test \( R \)-module. If the residue field of \( R \) is uncountable, or else if \( T \) has finite complete intersection dimension, then \( T \otimes_R S \) is a test module for \( S \).

**Proof.** In view of \[SP\] Lemma 10.98.15 (Critère de platitude par fibres; Noetherian case)] in conjunction with \[CW16\] Theorem 3.5, we may assume that both of \( R \) and \( S \) are complete local rings. When, \( R \) is equi-characteristic, by Zorn’s lemma, there exists maximal subfield, \( \mathcal{L} \), of \( S \) containing (the image under \( \varphi \)) of the coefficient field \( K \) of \( R \). Then, \( \varphi \) factors through \( R \rightarrow R \otimes_{K} \mathcal{L} \) and \( R \otimes_{K} \mathcal{L} ightarrow S \). One can easily verify that, \( R \otimes_{K} \mathcal{L} \rightarrow S \), is also a flat local homomorphism with regular closed fiber, as \( R \otimes_{K} \mathcal{L} \) is a flat weakly unramified extension of \( R \). Namely, we have, \( \text{Tor}_i^{R \otimes_{K} \mathcal{L}}((R \otimes_{K} \mathcal{L})/\mathfrak{m}, S) = \text{Tor}_i^{R \otimes_{K} \mathcal{L}}(R/\mathfrak{m}, S) = 0 \), and then we are done by \[SP\] Lemma 15.27.5 or \[M89\] Theorem 22.3(3’)]. By Lemma \[4.1.3\] \( T \otimes_R (R \otimes_{K} \mathcal{L}) \) is a test module for \( R \otimes_{K} \mathcal{L} \). By a straightforward inspection, or by \[C46\] page 73, forth paragraph, \( K \) is an algebraic extension of \( \mathcal{L} \), thence \( R \otimes_{K} \mathcal{L} \rightarrow S \) is residually algebraic. Hence, we are done in the light of and beauty of \[19\] Theorem 4.8.

---

28We again stress that, if the elements of \( x \) are not transcendental over the residue field \( K \), of \( C_K \), then we do not know how we can present a \( p \)-ring \( C_L \), assigned to an arbitrary sub-field extension \( |L : K| \) of \(|K : K|\), as a direct limit of \( p \)-rings over \( C_K \) such that each of \( p \)-rings in the direct system is residually finitely generated over \( K \), and such that \( C_L \rightarrow C_K \) extends \( C_K \rightarrow C_K \). However, we were able to do so whenever \( |L : K| \) is a separably generated field extension.
Now, for the case where $R$ has mixed characteristic, we deal as in the proof of Lemma 4.1.3. Namely, we consider a coefficient ring $\lambda : C_K \to R$, of $R$, and an lift, $\zeta$, of a transcendental basis, $x$, of $K$ over $K$. Then, similarly as we saw in the proof of Lemma 4.1.3 we can factor $C_K \xrightarrow{pR} S$ through,

$$C_K(\zeta) \left(= \lim_{\zeta \in \text{finite subset of } \zeta} C_K[\zeta](p) \right) \to S,$$

where $C_K(\zeta)$ is the $p$-ring for $K(x)$. Then, we factor $R \to S$ as,

$$R \text{ flat and weakly unramified } \xrightarrow{\otimes C_N C_K(\zeta)} \text{ residually algebraic and flat with regular fiber } S,$$

so again the result follows from [W19, Theorem 4.8] in conjunction with Lemma 4.1.3.

Now, to answer Question 1.3 for rings with uncountable residue field, we show that the complex case of Corollary 4.2.1 holds. The proof is certainly straightforward, and we give it only for the sake of completeness.

**Corollary 4.2.2.** Assume that all of the conditions and notation of the previous corollary holds, except that $T$ is a test complex for $R$. Then, $T \otimes_R S$ is also a test complex for $S$.

**Proof.** Suppose that, $X$, is a homologically finite complex of $S$-modules with $X \otimes_R (T \otimes_R S) \in D_b(S)$, and that,

$$P^X := \cdots \to P^X_{i+1} \to P^X_i \to P^X_{i-1} \to \cdots \to P^X_1 \to 0,$$

is a minimal projective resolution for $X$. As, $X$ is homologically finite, so there is a truncation, $P^X_{\geq m} := \cdots \to P^X_{m+1} \to P^X_m \to 0$, of $P$ which is an acyclic complex, with $m \geq \text{Sup } X+1$, wherein $\text{Sup } X$ denotes the largest integer $i$ with $H_i(X) \neq 0$. Set, $C^X := H_m(P^X_m)$, thus $P^X_m$ is a projective resolution of $C^X$. In the same vein, there is an $R$-module $CT$ by taking cokernel of the tail of an acyclic truncation of the minimal projective resolution of $T$, over $R$, such that $CT$ is a test module for $R$ and $\text{Tor}^R_S(C^T \otimes_R S, C^X) = 0$. By, Lemma 4.2.1, thus, $C^X$, and equivalently $P^X_{\geq m}$, has finite projective dimension. Then, the exact sequence, $0 \to P^X_{\geq m} \to P \to (0 \to P_{m-1} \to \cdots \to P_1 \to 0)$ shows that $\text{pd}_S X < \infty$.

**Remark 4.2.3.** If, in the statement of the preceding corollary, we drop the assumption on regularity of the closed fiber $S/mS$, then the extension $T \otimes_R S$ of a test $R$-module $T$ by $S$ may not be a test $S$-module, even when the closed fiber is a complete intersection (see, [CW16, Example 3.6]).

As the main result of this section, we shall present the promised positive answer to the question [CDT14, Question 3.5].

**Corollary 4.2.4.** Suppose that $(R, m, K)$ is local ring possessing a test module of finite complete intersection dimension. Then $R$ is complete intersection.

**Proof.** By our assumption, we are provided with a faithfully flat ring homomorphism $R \to R'$ and a ring epimorphism $A \to A'$ such that $\text{pd}_A(T \otimes_R R') < \infty$ and $R' = A\times A$ for a particular regular sequence $x$ of $A$. Set, $T' := T \otimes_R R'$. Furthermore, by virtue of [W08, Theorem F.], without loss of generality we can assume that the closed fiber $R'/mR'$ is a Gorenstein Artinian ring. Moreover, due to [ST, Lemma 10.98.15 (Criterie de platitude par fibres; Noetherian case)] in conjunction with [CW16, Theorem 3.5.], it is harmless to presume that $R$ and $A$ are complete local rings. In the light of and beauty of [A99, (3.5) Remark] and [A99, paragraph before definition, page 458]), there is a Cohen factorization, $R \to B \to R'$ such that $R \to B$ is a flat local homomorphism of local rings with regular closed fiber and $B \to R'$ is a surjective homomorphism making $R'$ an $B$-module of finite projective dimension. Using Corollary 4.2.1 we know that $T \otimes_R B$ is a test module for $B$. 

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By [W08 Corollary 3.9] (c.f. [CDT14 Corollary 3.4]), we already know that \( R \), and thence both of \( R' \) and \( A \) are Gorenstein rings. Let \( \Omega \) be a sufficiently high syzygy of the residue field, say \( K \), of \( A \) (over \( A \)), so that \( \Omega \) is a maximal Cohen-Macaulay \( A \)-module. Then since \( \Omega \) is a maximal Cohen-Macaulay module so \( x \subseteq A \) is also an \( \Omega \)-regular sequence and therefore by [RH98 Proposition 1.1.5] tensoring the minimal free resolution \( F_\bullet \) of \( \Omega \) over \( A \), with \( R' \), provides us with the minimal free resolution \( F_\bullet \otimes_A R' \) of \( \Omega/x\Omega \) over \( R' \). Consequently, by finiteness of the projective dimension of \( T' \), we have

\[
\text{Tor}_i^{R'}(T', \Omega/x\Omega) = H_i(F_\bullet \otimes_A R' \otimes_{R'} T') = H_i(F_\bullet \otimes_A T') = \text{Tor}_i^A(\Omega, T') = 0,
\]

for large enough \( i \). In other perspective, considering a free resolution \( G_\bullet \) of \( T \) (of course, over \( R \)), we get,

\[
\text{Tor}_i^{R}(T, \Omega/x\Omega) = H_i(G_\bullet \otimes_R (\Omega/x\Omega)) = H_i(G_\bullet \otimes_R (R' \otimes_{R'} (\Omega/x\Omega))) = 0.
\]

Thus, similarly, \( \text{Tor}_i^{B}(T \otimes_R B, \Omega/x\Omega) = \text{Tor}_i^{R}(T, \Omega/x\Omega) = 0 \), implying that \( \Omega/x\Omega \) has finite projective dimension over \( B \); by view of the test module property of the \( B \)-module \( T \otimes_R B \). Consequently, \( \text{Tor}_i^{R'}(K \otimes R', \Omega/x\Omega) = \text{Tor}_i^{R}(K, \Omega/x\Omega) = \text{Tor}_i^{B}(K \otimes B, \Omega/x\Omega) = 0 \). Then, again as we have seen above, \( \text{Tor}_i^{R'}(\Omega/mR', \Omega/x\Omega) = \text{Tor}_i^{R}(\Omega/mR', \Omega/x\Omega) = 0 \), so \( R'/mR' \) and \( \Omega \) are eventually-tor-independent over \( \mod A \). But \( \Omega \) is a syzygy of the residue field \( K \) of \( A \), therefore \( \text{Tor}_i^{A}(\Omega/mR', K) = 0 \), i.e. \( R'/mR' \) has finite projective dimension over \( A \). That is to say, the residue field of \( R \) has finite complete intersection dimension. Now the statement follows by virtue of [AGP97 (1.3) Theorem].

4.3. Alternative Proof for Special Cases. For the remainder of this section, we are going to show, for a flat local homomorphism \( \varphi : (R, \mathfrak{m}, K) \to (S, \mathfrak{n}, K) \) of equi-characteristic local rings, that when \( \varphi \) induces separable field extension (separable but not necessarily algebraic), or when \( R \) is Cohen-Macaulay, we can prove Corollary 4.2.1 without using the very recent (nice) result of Wagstaff, [W19 Theorem 4.8]. This shows that the Wagstaff’s result (that is about residually algebraic flat local morphisms with regular closed fiber) also can be deduced by an alternative technique in these cases. The results of this section also show that the equi-characteristic case of Corollary 4.2.4 can be deduced without applying [W19 Theorem 4.8], because Question 1.2 is a question on Cohen-Macaulay rings.

The main idea here, for the residually separable extension case, is to apply the Linquan Ma’s [ML7 Lemma 5.1]: If, \( \varphi : R \to S \), is a flat local homomorphism of complete equi-characteristic rings such that the coefficient field, say \( K \), of \( R \) is mapped by \( \varphi \) into a coefficient field, say \( K \), of \( S \), then \( S \) is a finite free \( R \otimes_K K \)-algebra. The condition on the containment of coefficient fields in Ma’s Lemma holds for any equi-characteristic zero ring, but it holds also in prime characteristic when the field extension of residue fields is separable 29.

For the Cohen-Macaulay case (\( R \) being Cohen-Macaulay), we were able to relax the annoying condition on the containment of coefficient fields in the Ma’s Lemma, by applying the following auxiliary lemma.

29We have a proof showing that, its analogue statement holds also in mixed characteristic when the residue field of the source \( R \) is perfect or when the extension of residue fields is separably generated. However, our proof is not given in this paper, to make the paper shorter.
**Proposition 4.3.1.** Suppose that \( \varphi : (R, m, K) \to (S, n, \hat{S}) \) is a flat local homomorphism of Artinian rings of prime characteristic \( p > 0 \). Then we can shrink \( \varphi \) to a flat local homomorphism \( \hat{\varphi} : \hat{R} \to \hat{S} \) such that \( \hat{R} \) (respectively, \( \hat{S} \)) is a local subring of \( R \) (respectively, of \( S \)) over which \( R \) (respectively, \( S \)) is a weakly unramified flat local extension, and such that \( \hat{\varphi} \) has a factorization,

\[
(\hat{R}, \hat{m}) \twoheadrightarrow \hat{R}_\mathcal{L} := \hat{R} \otimes_{F, \text{(coefficient field of } R)} \mathcal{L} \to (\hat{S}, \hat{\alpha}),
\]

to a coefficient field base change \( \hat{R} \to \hat{R}_\mathcal{L} \) (given by a field extension \( F \to \mathcal{L} \)) followed by a local homomorphism \( \hat{R}_\mathcal{L} \to \hat{S} \) which makes \( \hat{S} \) a finite free \( \hat{R}_\mathcal{L} \)-module.

Additionally, let \( T \) be a finitely generated \( R \)-module and \( N \) be a finitely generated \( S \)-module. Then, the factorization can be chosen, so that, there exist finitely generated \( R \)-module \( \hat{T} \) and finitely generated \( \hat{S} \)-module \( \hat{N} \) such that \( \hat{T} \otimes_R R = T, \hat{N} \otimes_S S = N \) and \( \hat{T} \) is a test module for \( \hat{R} \) provided \( T \) is so in mod \( R \).

**Proof.** Being Artinian, particularly, we can present \( R \) and \( S \) as quotients of polynomial rings, say \( R = K[X]/a \) and \( S = K[Y]/b \). Let, \( F_p \), be the prime subfield of \( K \) and of \( \mathcal{K} \), i.e. \( F_p = \mathbb{Z}/p\mathbb{Z} \). It is obvious that there exist natural maps from \( F_p \) to \( K \) and \( \mathcal{K} \), and thus to each of \( K[X], K[Y], R \) and \( S \). Let, \( \zeta \) be the set of coefficients, in \( K \), appearing in the matrix of a presentation (by finite free modules) of \( T \) over \( R \), as well as the coefficients appearing in the minimal generating set of the ideal \( a \) of \( R \). We have the natural field extension \( F_p(\zeta) \to K \) whose image contains all data needed to simulate \( R \), such that the simulation has coefficient ring \( F_p(\zeta) \). That is, \( K[X] \) is a flat extension of \( F_p(\zeta)[X] \) and there is an ideal \( \hat{a} \) of \( F_p(\zeta)[X] \) and an \( F_p(\zeta)[X]/\hat{a} \)-module \( \hat{T} \) such that \( R \) is a flat local extension of \( \hat{R} := \left( F_p(\zeta)[X]/\hat{a} \right) \).

Similarly, let \( \xi \) be the set of coefficients appearing in the polynomials \( \varphi(\zeta) \) and in \( \varphi(X), \zeta \in \xi \) and \( X \in X \), (note that \( \varphi(\zeta) \) is an element of \( S \), but not necessarily an element of \( \mathcal{K} \)), as well as coefficients appearing in the minimal generators of \( b \) along with those appearing in the matrix of a presentation of \( N \) over \( S \). Again, \( F_p(\xi)[Y] \) has an ideal \( \hat{b} \) whose residue ring \( \hat{S} \) admits a module \( \hat{N} \) with \( \hat{N} \otimes_{F_p(\xi)} \mathcal{K} = S, \hat{N} \otimes_S S = N \), \( \hat{S} \) is flat over \( \hat{S} \) and finally, and most notably, \( \varphi \) restricts to \( \hat{\varphi} : \hat{R} \to \hat{S} \) which has to be flat necessarily. Although the existence of \( \hat{\varphi} \) follows by a straightforward verification, but for the sake of completeness, we present a proof for its existence, later.\(^{32}\) For now, we assume the existence of \( \hat{\varphi} \) and we shall see what happens next.

Then as the residue fields of \( \hat{R} \) and \( \hat{S} \) are both finitely generated extensions of \( F_p \), so \( \hat{\varphi} \) also induces finitely generated extension of fields. It is well-known that, if we pick a maximal subfield, \( \mathcal{L} \), of \( \hat{S} \), containing \( \varphi(F_p(\zeta)) \subseteq \hat{S} \), then being maximal subfield implies that the extension of fields,

\[
\mathcal{L} \to \hat{S} \to \hat{S}/(Y),
\]

is algebraic (see, [4] page 73, forth paragraph). Consequently, \( \hat{R}_\mathcal{L} = \hat{R} \otimes_{F_p(\zeta)} \mathcal{L} \to \hat{S} \) induces finitely generated and algebraic, i.e. finite, extension on residue fields. It follows that \( \hat{S} \) is module-finite over \( \hat{R}_\mathcal{L} \), by complete version of Nakayama’s lemma (see, [95] Exercise 7.2, page 203).

It remains to deal with the existence of \( \hat{\varphi} \), as promised before. Let us present, \( F_p(\zeta) \) (respectively, \( F_p(\xi) \)) as \( \text{Frac}(F_p(\Theta)/\mathfrak{t}_1) \) (respectively, \( \text{Frac}(F_p(\Xi)/\mathfrak{t}_2) \)) wherein \( \mathfrak{t}_1 \) (respectively, \( \mathfrak{t}_2 \)) is the kernel of the

\[\begin{array}{c}
 R \\
 \downarrow \\
 S
\end{array}\]

\[\begin{array}{c}
 \hat{R} \\
 \downarrow \\
 \hat{S}
\end{array}\]

\(30\) The terminology shrinking, here, means that, as well, the homomorphism \( \varphi \) on those subrings forms a commutative diagram.

\(31\) The easiest way to see the flatness is just to simply notice that a basis \( \{e_i : i \in I\} \) of \( K \) over \( F_p(\zeta) \) forms also a basis for \( K[X] \) over \( F_p(\zeta[X] \), implying that \( K[X] \) is even free over \( F_p(\zeta)[X] \).

\(32\) Roughly speaking, the restriction, \( \hat{\varphi} \), of \( \varphi \), exists because all required data for its construction is finite.
map $\mathbb{F}_p[\Theta] \to \mathbb{F}_p(\zeta)$ (respectively, $\mathbb{F}_p[\Xi] \to \mathbb{F}_p(\xi)$) defined by $\theta_i \mapsto \zeta_i$ (respectively, $\Xi_i \mapsto \xi_i$) (here, each $\Theta$ and $\Xi$ denotes a set of finite number of indeterminates). First we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{F}_p[\Theta] & \xrightarrow{\eta} & \hat{R} = \mathbb{F}_p(\zeta)[X]/\hat{a} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\text{Frac}(\mathbb{F}_p[\Xi]/t_2)[Y]/\hat{b}_1 & \xrightarrow{\eta'} & \hat{S} = \mathbb{F}_p(\xi)[Y]/\hat{b} \\
\end{array}
$$

wherein $\hat{b}_1$ is the analogue of $\hat{b}$ in $\text{Frac}(\mathbb{F}_p[\Xi]/t_2)[Y]$ and $\alpha$ is defined so that the diagram is commutative, i.e. it maps $\theta_i$ to $\sum \Xi_{i,j}Y^j_{i,j}$ with the rule, $\varphi(\zeta_i) = (\sum \xi_{i,j}Y^j_{i,j}) + b$. The injectivity of $\beta'$ and $\eta'$ and the commutativity of the diagram shows that $\alpha(t_1) = 0$, hence we may replace $\mathbb{F}[\Theta]$ with $\mathbb{F}[\Theta]/t_1$ in the diagram. Similarly, since $\eta'$ and $\beta'$ are local homomorphisms and the diagram is commutative so any non-zero element, $f$, of $\mathbb{F}_p[\Theta]/t_1$ is mapped to an invertible element by $\alpha$, as $\varphi(\zeta_i) = \beta' \circ \eta' \circ \alpha(f) \in S^\times$. Therefore, $\alpha$, extends to $\text{Frac}(\mathbb{F}_p[\Theta]/t_1) \to \text{Frac}(\mathbb{F}_p[\Xi]/t_2)[Y]/\hat{b}_1$, which itself is extended to a ring homomorphism,

$$
\text{Frac}(\mathbb{F}_p[\Theta]/t_1)[X] \to \text{Frac}(\mathbb{F}_p[\Xi]/t_2)[Y]/\hat{b}_1,
$$

via $X_i \mapsto \sum \Xi_{i,j}Y^j_{i,j}$ with the rule, $\varphi(X_i) = \sum \xi_{i,j}Y^j_{i,j}$. Consequently, we can replace $\mathbb{F}_p[\Theta]$, in the commutative diagram above, by $\text{Frac}(\mathbb{F}_p[\Theta]/t_1)[X]$. The rest of the proof is straightforward. \qed

Now we prove Corollary 4.2.1 under the aforementioned situations, without applying [W19] Theorem 4.8.

**Alternative proof for Corollary 4.2.1 when either $R$ is Cohen-Macaulay or $\varphi$ is residually separable:** We can assume that $(R, m, K)$ and $(S, n, K)$ are complete local rings. We consider a finite $S$-module $N$ which is eventually-tor-independent with $T_S := T \otimes_R S$ and we prove that it is of finite projective dimension over $S$. Choosing a sequence $x$ in $S$, of length $\dim S - \dim R$, whose image forms a regular sequence on $S/mS$, by [A89] Corollary, page 177, $x$ is a regular sequence on $S$ and $S' := S/xS$ is flat and weakly unramified over $R$. Replacing $N$ with its sufficiently large syzygy, we may assume that $x$ is also a regular sequence on $N$. Set, $N' := N/xN$. By a straightforward induction on the length of $x$ we can easily conclude that $N'$ and $T_S$ are also eventually-tor-independent over $S$. In view of the flatness of $S$ over $R$, we easily can observe that $T$ and $N'$ are also eventually-tor-independent over $R$ and thus so are, $N'$ and $T \otimes_R S'$, over $S'$. If we have proven that $T \otimes_R S'$ is a test module for $S'$, then we would have $\text{pd}_{S'} N'$ is regular over $N' \cong \text{pd}_S N < \infty$. Thus, without loss of generality, we may and we do assume that $R \to S$ is weakly unramified (thus of the same dimension as well).

Suppose that $K \to K$ is a separable extension. Then, by [E95] Theorem 7.8], the coefficient field, say $K$, of $R$, maps by $\varphi$, to a coefficient, say $K$, of $S$. Thus, by [M17] Lemma 5.1), $S$ is a finite free weakly unramified extension of $R \otimes_K K$. Note that, in our situation, we have even much more that is, $S \cong R \otimes_K K$; to see this, simply apply the complete version of Nakayama’s Lemma [E95] Exercise 7.2, page 203). Thus, in this case, we are done by Lemma 4.1.3 whose proof in equal-characteristic is not using [W19] Theorem 4.8.

Now, suppose that, $R$ (and thus so $S$) is a Cohen-Macaulay ring. Recall that we have reduced ourselves to the case where $\varphi$ is weakly unramified. Considering, again, sufficiently high syzygies of $N$ and $T$, we may assume that $T$ and $N$ are both maximal Cohen-Macaulay modules. In particular, since we always can pick a regular sequence $x$ of $R$ such that $x_{i+1} \notin (x_1, \ldots, x_i) + m^2$, thus by applying [CDT14] Proposition
equipped with a factorization, we reduce ourselves to the case where $R$ and $S$ are Artinian rings. Now, by Proposition 2.3.1 and notation therein, the statement is reduced for the flat local homomorphism $\varphi : \hat{R} \to \hat{S}$ which is equipped with a factorization,

$$\hat{R} \to \hat{R} \otimes_F \mathcal{E} \text{ module-finite free extension } \hat{S},$$

$\hat{S}$-module $\hat{N}$ and $\hat{R}$-test module $T$. Then, again, Lemma 1.1.3 implies that $\hat{T} \otimes_{\hat{R}} \hat{R}_\mathcal{L}$ is a test module for $\hat{R}_\mathcal{L}$ while $\hat{N}$ is a finite $\hat{R}_\mathcal{L}$-module. This shows that, $\hat{N}$ has finite projective dimension over $\hat{R} \otimes_F \mathcal{L}$, because $\hat{N}$ and $\hat{T} \otimes_{\hat{R}} \hat{R}_\mathcal{L}$ are also eventually-tor independent over $\hat{R}_\mathcal{L}$. In particular, $\hat{R}_\mathcal{L}/m\hat{R}_\mathcal{L}$ and $\hat{N}$ are eventually-tor-independent over $\hat{R}_\mathcal{L}$, whence so are $\hat{R}$-modules $\hat{N}$ and $\hat{R}/m\hat{R}$ and also then $\hat{S}$-modules $\hat{S}/m\hat{S}$ and $\hat{N}$. In view of the weakly unramified property of $\varphi$, then one can see that $\hat{S}$ is also weakly unramified over $\hat{R}$, and so $\text{Tor}_n^S(\hat{N}, \hat{S}/\hat{n})$ is eventually zero, i.e. $\hat{N}$ has finite projective dimension over $\hat{S}$, as was to be proved.

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\[34\] If $x$ is regular on $N$, $S$ and $T_S$, then $N/xN$ and $T_S/xT_S$ are eventually tor-independent over $S/xS$ as well. First applying $\text{Tor}_n^S(T_S, -)$ to $0 \to N \to N \to N/xN \to 0$ we see that $\text{Tor}_n^S(T_S, N/xN) = 0$. Similarly, using $0 \to T_S \to T_S \to T_S/xT_S \to 0$ this time, we get $\text{Tor}_n^S(T_S/xT_S, N/xN) = 0$. Now, since $x$ is both of $S$ and $T_S$ regular sequence tensoring a free resolution of $T_S$ to $S/xS$ yields a free resolution of $T_S/xT_S$ over $S/xS$, and thus $\text{Tor}_n^S(T_S/xT_S, N/xN)$ is nothing but $\text{Tor}_{n+1}^S(T_S/xT_S, N/xN)$.

\[35\] Let $\mathcal{F}_n$ be a free resolution of $T$. Then, we have, $(\mathcal{F}_n \otimes_{\hat{R}} \hat{N}) \otimes_{\hat{S}} S \cong \mathcal{F}_n \otimes_{\hat{R}} N \cong \mathcal{F}_n \otimes_{\hat{R}} (R \otimes_{\hat{R}} N) \cong T \otimes_{\hat{R}} N$, showing that $N$ and $\hat{T} \otimes_{\hat{R}} \hat{S}$ are eventually-tor-independent.

\[36\] Here, the flat local map $R \to \hat{R}$ fulfills the required condition of existence of some flat local map $R \to R'$, as in the statement of Lemma 1.1.3 and Lemma 3.1.3.
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E-mail address: tavanfar@ipm.ir and tavanfar@gmail.com