Supersymmetry in the nonsupersymmetric Sachdev-Ye-Kitaev model

Jan Behrends\textsuperscript{1} and Benjamin Béri\textsuperscript{1,2}

\textsuperscript{1}T.C.M. Group, Cavendish Laboratory, University of Cambridge, J.J. Thomson Avenue, Cambridge, CB3 0HE, United Kingdom

\textsuperscript{2}DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

Supersymmetry is a powerful concept in quantum many-body physics. It helps to illuminate ground state properties of complex quantum systems and gives relations between correlation functions. In this work, we show that the Sachdev-Ye-Kitaev model, in its simplest form of Majorana fermions with random four-body interactions, is supersymmetric. In contrast to existing explicitly supersymmetric extensions of the model, the supersymmetry we find requires no relations between couplings. The type of supersymmetry and the structure of the supercharges are entirely set by the number of interacting Majorana modes, and are thus fundamentally linked to the model’s Altland-Zirnbauer classification. The supersymmetry we uncover has a natural interpretation in terms of a one-dimensional topological phase supporting Sachdev-Ye-Kitaev boundary physics, and has consequences away from the ground state, including in \( q \)-body dynamical correlation functions.

The Sachdev-Ye-Kitaev (SYK) model [1, 2] is a toy model that provides insight into diverse physical phenomena, ranging from the holographic principle [3–5] to quantum chaos [6, 7], and non-Fermi liquid behavior of black holes, it is believed to scramble quantum information with maximal efficiency [12, 13]. Similar to black holes, it is believed to scramble quantum information in a superconducting vortex [14], it represents a degree of freedom with support far away from the vortex where the SYK Majoranas \( \gamma \) reside. Since \( \{ T_\pm, \gamma \} \neq 0 \) \( \neq \infty \) at “infinity” must be included when \( k \) is odd [37]. The operator \( \gamma \) is not local to the SYK model; considering, e.g., a realization in a superconducting vortex [14], it represents a degree of freedom with support far away from the vortex where the SYK Majoranas \( \gamma_j \neq \infty \) reside. Since \( [H, P] = 0 \), all eigenstates of \( H \) can be labeled by their parity eigenvalue \( p = \pm 1 \), giving \( H | \psi_p \rangle = e^p | \psi_p \rangle \).

The number of interacting Majorana modes, specifically \( k \) mod 8, sets the model’s antiunitary symmetries [16, 37, 38]. These can be time-reversal symmetry \( T_+ \) or particle-hole symmetry \( T_- \), antiunitary operators satisfying \( T_\pm \gamma q \neq \infty T_\mp = -\gamma q \neq \infty \). Time-reversal commutes with fermion parity, \( [T_+, P] = 0 \), and particle-hole symmetry anticommutes with it, \( [T_-, P] = 0 \). For even \( k \), either time-reversal symmetry \( T_+ \) or particle-hole symmetry \( T_- \) is present. For odd \( k \), both \( T_+ \) and \( T_- \) are present; in this case \( T_+ \gamma \infty T_-^{-1} = (-1)^{(k+1)/2} \gamma \infty \). Their product, the unitary operator \( Z = T_+ T_- \), corresponds to a chiral symmetry [37, 38]. A key feature

The Hamiltonian we consider describes four-body interactions between \( k \) Majorana modes [2]

\[
H = \sum_{t=0}^{k-1} \sum_{s=0}^{t-1} \sum_{r=0}^{t-1} \sum_{q=0}^{t-1} J_{qrst} \gamma_q \gamma_r \gamma_s \gamma_t + E_0
\]

with real (as required by Hermiticity) but otherwise structureless couplings \( J_{qrst} \), and the constant \( E_0 \) that ensures positive energies. The Hermitian Majorana operators \( \gamma_q = \gamma_q^\dagger \) satisfy the anticommutation relation \( \{ \gamma_q, \gamma_r \} = 2 \delta_{qr} \) [35], and span an \( M \)-dimensional Hilbert space with \( M = 2^{[k/2]} \) [36]. Since each term in the Hamiltonian (2) contains an even number of Majoranas, it conserves fermion parity \( P \), given by

\[
P = \left\{ \begin{array}{ll}
\frac{1}{2} \gamma_1 \gamma_2 \cdots \gamma_k & \text{even } k \\
\frac{1}{2} \gamma_1 \gamma_2 \cdots \gamma_k \gamma_\infty & \text{odd } k.
\end{array} \right.
\]

To work in a Hilbert space with well-defined fermion parity, the additional Majorana \( \gamma_\infty \) “at infinity” must be included when \( k \) is odd [37]. The operator \( \gamma_\infty \) is not local to the SYK model; considering, e.g., a realization in a superconducting vortex [14], it represents a degree of freedom with support far away from the vortex where the SYK Majoranas \( \gamma_j \neq \infty \) reside. Since \( [H, P] = 0 \), all eigenstates of \( H \) can be labeled by their parity eigenvalue \( p = \pm 1 \), giving \( H | \psi_p \rangle = e^p | \psi_p \rangle \).

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implies SUSY: The operator $\tilde{\psi}$ anticommutes with fermion parity $[38]$. This in turn
follows from an operator that commutes with the Hamiltonian, but
$Q_{\pm}$ satisfies $\tilde{\psi}^P$ haves the same energy $\varepsilon_\mu = \varepsilon_\mu^P = \varepsilon_\mu^P$ (since $[T_-, H] = 0$), but opposite parity ($(T_-, P) = 0$) [37, 38].
Therefore, $|\psi_\mu^{(P)}\rangle$ is an odd-parity zero mode, i.e., an operator that commutes with the Hamiltonian, but anticommutes with fermion parity [38]. This in turn
implies SUSY: The operator $Q_\mu = \sqrt{\text{diag}(\varepsilon_\mu)}|\psi_\mu^+\rangle\langle\psi_\mu|$ satisfies
$\tilde{Q}_\mu Q_\mu = \varepsilon_\mu|\psi_\mu^+\rangle\langle\psi_\mu^+| \text{ and } Q_\mu |\psi_\mu^-\rangle\langle\psi_\mu^-|,$
and hence the linear combinations $Q_{1,\mu} = Q_\mu + Q_\mu^\dagger$ and $Q_{2,\mu} = i(Q_\mu - Q_\mu^\dagger)$ are Hermitian, anticommute, and square to $\varepsilon_\mu$ times the projector on the two parity-degenerate states. Consequently, the two supercharges

$$Q_1 = \sum_\mu (Q_\mu + Q_\mu^\dagger), \quad Q_2 = i \sum_\mu (Q_\mu - Q_\mu^\dagger)$$

satisfy Eq. (1). Particle-hole symmetry is present unless $k = 4n$. Thus, all but two of the symmetry classes are supersymmetric.

Given the presence of six supersymmetric classes, there are a number of questions regarding the interplay of SUSY and the symmetry classification. How does $N$ depend on the symmetry class? How do $Q_j$ transform under $T_\pm$ and how does this translate to the structure of the supercharges? We next turn to these questions.

We start with counting $N$. A direct approach is based on counting level degeneracies. This follows from the observation that the “spectrally flattened” Hermitian supercharges $\Gamma_j = Q_j/\sqrt{H}$ satisfy

$$\{\Gamma_j, \Gamma_k\} = 2\delta_{jk}, \quad [H, \Gamma_k] = 0, \quad \{P, \Gamma_k\} = 0.$$  (5)

They are thus many-body zero mode forms of Majorana fermions. An even $N$ of such zero modes give rise to a $2^{N/2}$-dimensional fermionic degeneracy space for each of the $\varepsilon_\mu$ with one of the $|\psi_\mu^{(P)}\rangle$ chosen as “vacuum”. (With a suitable choice, the $\Gamma_j$-fermion parity of an eigenstate matches the state’s physical fermion parity.) This procedure is similar in spirit to the standard construction of supermultiplets [40]. For the six supersymmetric SYK classes, a twofold degeneracy is guaranteed by $T_-$ and a further twofold (Kramers) degeneracy is present whenever $T_+^2 = -1$, resulting in an overall fourfold degeneracy. This suggests $N = 2$, except for DIII and CII where this count gives $N = 4$. What this counting does not address is how many $\Gamma_j$ (and hence $Q_j$) are local to the SYK model. Next we investigate this to obtain the decomposition $N = N_{\text{loc}} + N_{\text{loc}}$ with $N_{\text{loc}}$ counting the number of supercharges involving only $\gamma_{1,\text{loc}}$.

We begin with classes D and C. Here $k$ is even hence all $\gamma_0$ are local. Therefore, our argument above applies directly: we find $N = N_{\text{loc}} = 2$. All the other supersymmetric classes have odd $k$, thus potentially $N \neq N_{\text{loc}}$ due to $\gamma_{1,\text{loc}}$. As we shall see, in all of these classes $N = N_{\text{loc}} + 1$ with $N$ following its degeneracy-based value above. This is intuitive because $\gamma_{1,\text{loc}} = \Gamma_1$ automatically satisfies Eq. (5) (in particular, it anticommutes with any local parity-odd operator), thus $N_{\text{loc}}$ is at most $N - 1$. To formally establish $N_{\text{loc}}$, and the transformation of $\Gamma_j$ under $T_\pm$, we work in the energy eigenbasis, $H = \text{diag}(\varepsilon_\mu) \otimes \mathbb{1}_{2^{N/2}}$, with $P = \mathbb{1}_{2^{N/2}} \otimes \tau_3$. (Here and below, $\tau_j$ and $\sigma_j$ are Pauli matrices; $\tau_j$ act in parity grading and $\sigma_j$ in the space of Kramers doublets, where applicable. We will often omit trivial tensor factors.) In this basis, class D (C) has particle-hole symmetry (up to a phase diagonalisation omitted here and below) $T_+ = \tau_1(2)K$ (with $K$ for complex conjugation); this follows from $T_2^2 = \mp 1$ and parity being the only degeneracy, $\mathbb{1}_{2^{N/2}} = \tau_0$. We have $\Gamma_{1,2} = \tau_{1,2}$ [41].

To study classes BDI and CI, we focus on a degeneracy space with energy $\varepsilon_\mu$ and first establish the form of $T_\pm$ and thus $Z$ in this space. $T_\pm^2 = +1$ implies that parity is again the only degeneracy, so $T_- = \tau_1(2)K$ in class BDI (CI). $T_+|\psi_\mu^{(P)}\rangle \propto |\psi_\mu^{(P)}\rangle$ implies that the most general form is $T_+ = \exp(i\varphi_\mu\tau_3)K$. With a suitable choice of the relative phases between the two parity sectors we can thus use $Z = T_+T_\pm = \tau_1$; in this basis $T_{-} = K$ ($T_+ = \tau_3K$) for class BDI (CI). The two $\Gamma_j$ satisfying Eq. (5) can again be chosen as $\Gamma_{1,2} = \tau_{1,2}$. However, checking the (anti)commutation with $Z$ we find that only $\Gamma_1$ is local. Conversely, we can identify $\Gamma_2 \equiv \Gamma_2 \equiv \tau_0; \text{ this is consistent both with } \gamma_\infty \text{ itself satisfying Eq. (5) and its transformation under } T_+$. We thus find $N_{\text{loc}} = 1$.

In classes DIII and CII, we follow the same strategy. Now $T_2^2 = -1$, which implies Kramers degeneracy: The states $|\psi_\mu^{(P)}\rangle$ and $T_+|\psi_\mu^{(P)}\rangle$ are orthogonal. Together with parity degeneracy this gives $\mathbb{1}_{2^{N/2}} = \sigma_0\sigma_0$. $T_+$ preserves parity, $\{T_+, P\} = 0$. This, combined with $T_2^2 = -1$, gives its most general form as $T_+ = \exp(i\varphi_\mu\tau_3)\sigma_2K$. For $T_-,$ due to $\{T_+, P\} = 0$ and $T_2^2 = \pm 1$, the most general form is $T_- = \left(\pm w_\mu^\dagger U_\mu\right)K$ with $W_\mu \in \text{SU}(2)$ in $\sigma_j$ space. Thus, $Z = T_+T_- = \left(U_\mu^\dagger U_\mu\right)$ where $U_\mu = e^{i(\varphi+\chi)\tau_3}W_\mu\sigma_2$ with $\chi = 0$ ($\chi = \pi/2$) for class DIII (CII). Rotating by diag$(1, U_\mu)$, one can take, without loss of generality,
below, implies that, when expanding the supercharges, $n$ may change when applying $T_+$ on $\Upsilon$ of $\gamma$. That is, $T_+\Gamma_j = \gamma T_+\Gamma_j$. Furthermore, $Q_4 = \text{diag}(\{\sqrt{\sigma_m}\}) \otimes \Gamma_4$ squares to the Hamiltonian. It is, however, not a new supercharge because $\Gamma_4$ does not anticommute with $\Gamma_{1,2,3}$. (Nevertheless, $\Gamma_4$ contributes to correlation functions, as we shall see below.) We thus find $N_{\text{loc}} = 3$.

The values $N_{\text{loc}}$, together with the sign $s$ in $T_\pm\Gamma_j \leq N_{\text{loc}}, T_\pm^{-1} = s\Gamma_j \leq N_{\text{loc}}$ have a natural interpretation if one views the SYK model as arising at the end of a one-dimensional topological phase in class BDI [16, 37]. These systems admit a $Z_8$ classification: At one of their ends, they have $k_s$ Majoranas satisfying $T_\pm \gamma_q T_\pm^{-1} = s\gamma_q$; the topological index is $\nu = (k_+ - k_-)$ mod 8. Thus, the eight topological classes can be labeled by $\nu = 0, 1, 2, 3, 4, -3, -2, -1$ with the integers counting the number and sign of unpaired Majoranas. In the SUSY classes, we find the same pattern for $s N_{\text{loc}}$ against $k$ mod 8 ($T_\pm \gamma_q \neq \infty T_\pm^{-1} = \gamma_q$ implies $k_+ = k$, $k_- = 0$), see Table II. The $N_{\text{loc}}$ supercharges $\Gamma_j \leq N_{\text{loc}}$ can thus be viewed as the many-body emergence of the minimal number and type of unpaired Majoranas consistent with $k$.

Next we turn to the structure of the supercharges in terms of the Majorana fermions $\gamma_q$. For this, we employ another operator basis of the Hilbert space, the products of $n_a$ Majorana operators $\gamma_q$ [42]

$$T_a = \tau^{n_a(n_a-1)/2} \gamma_{i_1(a)} \gamma_{i_2(a)} \cdots \gamma_{i_{n_a}(a)} \tag{6}$$

with $i_j(a) \neq i_j'(a)$. $T_a$ are Hermitian, unitary, and orthonormal with respect to the trace, $\text{tr} [T_a T_b] / M = \delta_{ab}$. In total, there are $2^n$ local operators $T_a$ [42]. As we aim to expand $\Gamma_j \neq \infty$, i.e., Hermitian odd-parity operators in terms of $T_a$, we use only those $T_a$ with odd $n_a$, and use only real expansion coefficients.

Both time-reversal and particle-hole symmetry have the same (anti-) commutation properties when acting on $T_a$. Since $T_\pm \gamma_q T_\pm^{-1} = \gamma_q$, only the phase of $T_a$ [cf. Eq. (6)] may change when applying $T_\pm$, giving

$$T_\pm T_a T_\pm^{-1} = (-1)^{n_a(n_a-1)/2} T_a. \tag{7}$$

That is, $T_\pm$ and $T_a$ commute when $n_a = 4n + 1$, and anticommute when $n_a = 4n + 3$. This, together with $v_{j,a} \in \mathbb{R}$ below, implies that, when expanding the supercharges,

$$\Gamma_j = \sum_a v_{j,a} T_a, \quad \sum_a v_{j,a}^2 = 1, \tag{8}$$

only terms with $n_a = 4n + 1$ contribute to $\Gamma_j$ when $[T_\pm, \Gamma_j] = 0$, and only terms with $n_a = 4n + 3$ contribute when $\{T_\pm, \Gamma_j\} = 0$. In classes DIII and CII we also consider $\Gamma_4 = -i\Gamma_1 \Gamma_2 \Gamma_3$ whose transformation properties follow from those of $\Gamma_{1,2,3}$. The resulting expansion structure is summarized in Table II.

Having discussed the interplay of SUSY and the symmetry classification, we now ask what the signatures of SUSY, $N_{\text{loc}}$, and the supercharge structure are in various observables. A simple link between $N_{\text{loc}}$ and observables exists due to the fact that the number of different $\Gamma_j \leq N_{\text{loc}}$ and their linearly independent odd-parity products, i.e., including $\Gamma_4$ in classes CII and DIII, equals the degrees of freedom $\beta$ (i.e., the Dyson index linked to $T_+$ [43]) of the Hamiltonian’s off-diagonal matrix elements. In fact, the most general Hermitian linear combinations of these $\Gamma_j$ have the same type of oddf diagonals, up to an imaginary unit, as the Hamiltonian: real for $\beta = 1$ (classes BDI and CI), complex for $\beta = 2$ (classes D and C) and real quaternion for $\beta = 4$ (classes DIII and CII).

In the SUSY classes, the value of $\beta$ sets the energy level correlations, including the long-range spectral rigidity, across opposite parity sectors (these are uncorrelated without SUSY) which lead to “ramps” in time-dependent correlation functions of parity-odd observables. (For single-Majorana examples see Refs. 26 and 38.) These ramps occur at time scales below $2\pi$ times the inverse mean level spacing $1/\Delta_{\infty}$, and have $\beta$-dependent shape [27]. In particular, the ramp connects to a long-time plateau smoothly when $\beta = 1$, sharply when $\beta = 2$, and with a kink when $\beta = 4$. In Fig. 1, we show ensemble-averaged $q$-body correlation functions [Eq. (10) below] in classes D, C, DIII, and CI.

Besides this direct correspondence between the supercharges and ramp structure, we additionally find more subtle consequences of SUSY: The long-time ($t \gg 1/\Delta_{\infty}$) plateau in $q$-body correlation functions is also related to the number and structure of the supercharges, cf. Fig. 2. To quantify this relationship, we consider the retarded time-dependent $q$-body correlation function

$$C_q^+(t - t') = -i\Theta(t - t') \frac{1}{k_q!} \sum_{a_n = q} \langle \{T_a(t), \Upsilon_a(t')\} \rangle, \tag{9}$$

| $k \mod 8$ | 1 | 2 | 3 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|
| Label | BDI | D | DIII | CI | C | CI |
| $s N_{\text{loc}}$ | 1 | 2 | 3 | $-3$ | $-2$ | $-1$ |
| $\Gamma_j \leq N_{\text{loc}}$ | $4n + 1$ | $4n + 1$ | $4n + 1$ | $4n + 3$ | $4n + 3$ | $4n + 3$ |
| $\Gamma_4$ | $4n + 3$ | $4n + 1$ |
C

start by expanding \( \Upsilon \) composite way to \( \Gamma \) operators which holds as within an eigenspace, the (projected) and tr \( \Gamma \) energy \( \varepsilon \) where in converting the equal-energy sum to a trace, we reveal are present for any temperature, we find \( \langle \cdot \rangle \) where \( \Theta \) denotes thermal average. Although the signatures we reveal are present for any temperature, we find an especially transparent relationship at infinite temperatures we are studying in Fig. 2.

FIG. 1. \( q \)-body time-dependent correlation function at infinite temperature, averaged over an ensemble of up to \( 2^{16} \) Gaussian distributed \( J_{\mu \nu \tau} \), for classes (a) D, (b) C, (c) DIII, and (d) CI. The different colors denote different \( k \) and \( q \), cf. inset in (a), the dashed lines represent \( q = 3 \) and the solid lines \( q = 5 \). The ramp shape follows the Dyson index and hence links to the number of supercharges. The long-time plateau \( \mathcal{C}_{q,\infty} \) is studied in Fig. 2.

\[
C_q^+(t) = -i\Theta(t) \sum_{a,n_a=q} \frac{1}{M} \sum_{\mu\nu} \left\langle |\psi_\mu^0| \Upsilon_\alpha |\psi_\nu^p| \right\rangle^2 \times 2 \cos \left( t (\varepsilon_\mu^0 - \varepsilon_\nu^p) \right). \tag{10}
\]

When \( t \gg 1/\Delta_\infty \), terms with \( \varepsilon_\mu^0 \neq \varepsilon_\nu^p \) give a quickly oscillating contribution \( \delta C_q^+(t) \) that averages to zero. Only states with \( \varepsilon_\mu^0 = \varepsilon_\nu^p \) give a time-independent contribution \( C_{q,\infty} \). Thus, \( C_q^+(t) = -i\Theta(t) [C_{q,\infty} + \delta C_q^+(t)] \) with

\[
C_{q,\infty} = \frac{1}{(q)} \sum_{a,n_a=q} \frac{2}{M} \sum_{\mu} \Upsilon_{a\mu}^2, \quad \Upsilon_{a\mu} = P_\mu \Upsilon_a P_\mu, \tag{11}
\]

where in converting the equal-energy sum to a trace, we introduced the projection \( P_\mu \) to the eigenspace with energy \( \varepsilon_\mu \) and used that \( \Upsilon_a \) is Hermitian and parity odd.

Next we convert Eq. (11) into a sum over \( \Gamma_{j<\infty} \). We start by expanding \( \Upsilon_{a\mu} = \sum_{j<\infty} y_j P_\mu \Gamma_j \) with real \( y_j \) which holds as within an eigenspace, the (projected) operators \( P_\mu \Gamma_{j<\infty} \) form a basis for local, Hermitian, parity-odd operators. If \( \Upsilon_a \) transforms the same (opposite) way to \( \Gamma_j \) under \( T_\pm \) then generically \( y_j \neq 0 \) \((y_j = 0)\). Now using the trace-orthogonality of the \( \Gamma_{j<\infty} \) and tr \( \Gamma^2 = 2N/2 = N \) (for \( N = 2, 4 \)), we find

\[
C_{q,\infty} = \frac{1}{(q)} \sum_{a,n_a=q} \frac{2}{MN} \sum_{\mu} \sum_{j<\infty} \left[ \text{tr}(P_\mu \Upsilon_{a\mu} P_\mu \Gamma_j) \right]^2. \tag{12}
\]

A simple estimate for \( C_{q,\infty} \) can be given assuming that expanding \( P_\mu \Gamma_{j<\infty} = \sum_{a} v_{\mu j,a} \Upsilon_a \) results in random coefficients \( v_{\mu j,a} \) subject only to normalization and symmetry constraints. Denoting such a random vector average by \( \langle \cdot \rangle \), we find

\[
\mathcal{C}_{q,\infty} \frac{M}{4} = \begin{cases} \frac{N_{\text{loc}}}{4} & q : \Upsilon_a \triangleq \Gamma_j \leq N_{\text{loc}}, \\ \frac{N_{\text{loc}}}{4} \delta_{\beta,4} & \text{otherwise} \end{cases}. \tag{13}
\]

where \( \Upsilon_a \triangleq \Gamma_j \) here means that \( \Upsilon_a \) transforms the same way as \( \Gamma_{j<\infty} \) under \( T_\pm \). Thus, each \( \Gamma_{j<\infty} \) gives the same contribution to \( \mathcal{C}_{q,\infty} \) when they contain \( q \)-Majorana terms and zero otherwise. The nonzero value for \( \beta = 4 \) when \( \Upsilon_a \not\triangleq \Gamma_j \leq N_{\text{loc}} \) arises due to \( \Gamma_4 \) since \( \Gamma_4 \not\triangleq \Gamma_j \leq N_{\text{loc}} \). Considering the Majorana structure of \( \Gamma_{j} \) in Table II, Eq. (13) translates to an alternating pattern of \( \mathcal{C}_{q,\infty} \) as \( q \) is varied in a given symmetry class, with complementary \( \mathcal{C}_{q,\infty} \) values for classes with opposite \( sN_{\text{loc}} \).

FIG. 2. Normalized plateau \( \mathcal{C}_{q,\infty} M/4 \) of the \( q \)-body correlation function, averaged over an ensemble of up to \( 2^{14} \) Gaussian distributed \( J_{\mu \nu \tau} \). The color encodes the number \( k \) of Majoranas, cf. panels (d) and (e). In all classes, \( \mathcal{C}_{q,\infty} M/4 \) alternates with \( q \) approximately as predicted in Eq. (13); the agreement is excellent for sufficiently large \( (q) \). In panel (d), we show that \( \mathcal{C}_{q,\infty} M/(4c) \) [with \( c \) the random matrix expectation based on Eq. (13)] increases as a function of \( k \), but with a rate that decreases upon increasing \( q \) [panel (e)].
To summarize, we have shown that supersymmetry is (almost) always present in the SYK model with generic four-body interactions. It is only absent in those classes without particle-hole symmetry, i.e., in classes AI and AII. The type of SUSY, in particular the number $N_{\text{loc}}$ of local supercharges and their symmetry properties follow a pattern that finds a natural interpretation when the (spectrally flattened) supercharges $\Gamma_{j \leq N_{\text{loc}}}$ are viewed as emergent Majorana fermions in a one-dimensional topological phase with SYK model boundary physics. These SUSY features all link directly to features in time-dependent correlation functions of fermion-parity-odd observables. For $q$-body retarded correlation functions, this includes the shape of the ramp in the short-time regime, due to a link between $N_{\text{loc}}$ and the Dyson index $\beta$; and the value of the long-time plateau due to the imprint of how $\Gamma_j$ transforms under $T_\pm$ on its microscopic Majorana structure. These $q$-body correlation functions, even with large $q$, can be measured in digital quantum simulation of the SYK model [44]. The single-particle Green’s function ($q = 1$) is accessible through scanning tunneling microscopy [14, 15]. We stress that the features in the correlation functions are dynamical consequences of SUSY, which are less frequently considered than ground-state consequences [33, 45, 46].

The presence of SUSY opens up various further research directions. For example, while we focused on four-body interactions, one can consider generalizations to sums $\sum_n H_n$ of $n$-body terms [7, 9]. Our considerations apply for any $n \equiv 4 \pmod{4}$. Furthermore, for $n \equiv 2 \pmod{4}$, the $T_\pm$ is broken, chiral symmetry $Z$ remains for odd $k$ [38], giving rise to another variant of $N_{\text{loc}} = 1$ SUSY. It will be interesting to explore the consequences of SUSY in these generalized models. A spatial structure in the supercharges may arise in lattices of certain SYK model clusters [8–10, 47]. SUSY may also impose constraints on the SYK quantum field theory, potentially altering the description in the mesoscopic regime $1 \ll k < \infty$ [16, 26, 48–50] and perhaps even in the thermodynamic limit $k \to \infty$, in a manner dependent on the value of $k \pmod{8}$ setting the nature of SUSY.

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