Dynamics of Charged Plane Symmetric Gravitational Collapse

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Abstract

In this paper, we study dynamics of the charged plane symmetric gravitational collapse. For this purpose, we discuss non-adiabatic flow of a viscous fluid and deduce the results for adiabatic case. The Einstein and Maxwell field equations are formulated for general plane symmetric spacetime in the interior. Junction conditions between the interior and exterior regions are derived. For the non-adiabatic case, the exterior is taken as plane symmetric charged Vaidya spacetime while for the adiabatic case, it is described by plane Reissner-Nordström spacetime. Using Misner and Sharp formalism, we obtain dynamical equations to investigate the effects of different forces over the rate of collapse. In non-adiabatic case, a dynamical equation is joined with transport equation of heat flux. Finally, a relation between the Weyl tensor and energy density is found.

Keywords: Gravitational collapse; Junction conditions; Dynamical equations; Transport equations.

1 Introduction

Gravitational collapse is defined as the implosion of a celestial body under the influence of its own gravity. It is one of the basic processes driving evolution

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within galaxies, assembling giant molecular clouds and producing stars. The study of gravitational collapse is motivated by the fact that it represents one of the few observable phenomena in the universe. The gravitational collapse of a star proceeds to form compact objects as white dwarf, neutron star or black hole primarily depending upon the mass of the star. The end state of continual gravitational collapse of a massive star is an important issue in gravitation theory. According to Penrose, gravitational collapse of a massive star gives rise to a spacetime singularity [1].

Oppenheimer and Snyder [2] innovated the first mathematical model for the description of gravitational collapse of stars. They dealt with the dust case and concluded that gravitational collapse of massive stars might end to form a black hole. This work of Oppenheimer and Snyder attracted many people to explore it further. Markovic and Shapiro [3] extrapolated pioneer’s work involving positive cosmological constant and explained its effects on the rate of collapse. Misner and Sharp [4] did the same job by taking ideal fluid and also discussed some other aspects like thermodynamics and hydrodynamics of gravitational collapse.

The standard approach dealing with gravitational collapse problems requires junction conditions to join the two spacetimes. Three types of junction conditions are available in literature proposed by Darmois [5], Lichnerowicz [6] and O’Brien and Synge [7]. Bonnor and Vickers [8] studied these junction conditions and showed that the conditions suggested by Darmois and Lichnerowicz are equivalent. Also, the O’Brien and Synge conditions are unsatisfactory. It was concluded that Darmois junction conditions are the most convenient and appropriate.

Most of the work in this scenario has been done using spherical symmetric models [9]-[11]. Herrera et al. [12] used complicated fluid for this purpose and discussed the physical meaning of expansion free fluid evolution. In another paper [13], Herrera et al. explored the dynamics of dissipative gravitational collapse considering spherical symmetry. They also discussed applications of the results to astrophysical scenario.

The phenomenon of gravitational collapse has also been explored in the context of cylindrical, planar and quasi-spherical symmetries [14]-[19]. Kurita and Nakao [20] studied the null dust collapse for the cylindrically symmetric spacetimes. They concluded that singularity must form at symmetry axis and also discussed the behavior of geodesics arriving at the singularity. Wang et al. [21] worked on self-similar plane symmetric solutions. It was found that during collapse trapped surfaces are not formed and singularities are space-
like. Nath et al. [22] elaborated junction conditions for the quasi-spherical Szekeres spacetime in the interior and the Riessner Nordström Vaidya spacetime in the exterior region. The time difference between the formation of apparent horizon and central singularity was also discussed.

Using the concept of gravitational lensing (GL), Virbhadra et al. [23] introduced a new tool for examining naked singularities. Gravitational lensing is the process of bending of light around a massive object such as a black hole. Virbhadra and Ellis [24] discussed GL by the Schwarzschild black hole. It was found that the relativistic images guarantee the Schwarzschild geometry close to event horizon. The same authors [25] also analyzed GL by a naked singularity. Claudel et al. [26] proved that the necessary and sufficient condition for the black hole to be surrounded by a photon sphere is that a reasonable energy condition holds. Virbhadra and Keeton [27] showed that weak cosmic censorship hypothesis (CCH) can be examined observationally without any uncertainty. Virbhadra [28] found that Seifert's conjecture is supported by the naked singularities forming during Vaidya null dust collapse. The same author developed an improved form of CCH using GL phenomenon [29].

In a recent paper, Sharif and Abbas [30] found the effects of electromagnetic field on the gravitational collapse for perfect fluid in the presence of cosmological constant. It was concluded that charge increases the rate of collapse by decreasing the limit of cosmological constant. The same authors extended this work for 5D collapse [31] and found that the range of apparent horizon is greater than 4D case. Di Prisco et al. [32] figured out the consequences of charge and dissipation for spherical symmetric gravitational collapse of a real fluid. It was assumed that heat flow, free streaming radiation and shearing viscosity are the causes of dissipation. The dynamical and transport equations are coupled to observe the effects of dissipation over collapsing process.

This paper extends the work of Di Prisco et al. [32] to plane symmetry. The models exhibiting plane symmetry may swear out as test-bed for numerical relativity, quantum gravity and contribute for examining CCH and hoop conjecture among other important issues. The paper is organized in the following pattern. In the next section, the dynamics for non-adiabatic flow is discussed. The Einstein-Maxwell field equations and junction conditions are found. The dynamical and transport equations are obtained and then coupled. A relation between energy homogeneity and the Weyl tensor is also given. Section 3 describes dynamical equations for adiabatic case. The last section summarizes the results.
2 Dynamics of Viscous Non-adiabatic Case

Here the dynamics for non-adiabatic flow is discussed. We formulate dynamical and transport equations and then finally couple these equations.

2.1 Interior Spacetime and Some Physical Quantities

We consider a plane symmetric distribution of collapsing fluid bounded by a hypersurface $\Sigma$. The line element for the interior region has the following form

$$ds^2 = -A^2(t, z)dt^2 + B^2(t, z)(dx^2 + dy^2) + C^2(t, z)dz^2,$$

(2.1)

where we have assumed co-moving coordinates inside $\Sigma$. The interior coordinates are taken as $\chi^0 = t$, $\chi^1 = x$, $\chi^2 = y$, $\chi^3 = z$. It is assumed that fluid is locally anisotropic and suffering dissipation in the form of shearing viscosity, heat flow and free streaming radiation. The energy-momentum tensor has the following form

$$T_{ab} = (\mu + P_z)V_aV_b + P_\perp g_{ab} + (P_z - P_\perp)\chi_a\chi_b + q_aV_b + V_aq_b + \epsilon\ell_a\ell_b - 2\eta\sigma_{ab},$$

(2.2)

where $\mu$ is the energy density, $P_z$ is the pressure in $z$-direction, $P_\perp$ is the pressure perpendicular to $z$-direction (i.e., $x$ or $y$ direction), $V^a$ four velocity of fluid, $\eta$ is the coefficient of shear viscosity, $\chi^a$ is a unit vector in $z$-direction, $q_a$ is the heat flux, $\epsilon$ is the radiation density and $\ell_a$ is a null four vector in $z$-direction. Furthermore, these quantities satisfy the relations

$$V^aV_a = -1, \quad \chi^a\chi_a = 1, \quad \chi^aV_a = 0,$$

$$V^aq_a = 0, \quad \ell^aV_a = -1, \quad \ell^a\ell_a = 0.$$

(2.3)

Since the metric defined in Eq.(2.1) is co-moving, we can take

$$V^a = A^{-1}\delta^a_0, \quad \chi^a = C^{-1}\delta^a_3, \quad q^a = qC^{-1}\delta^a_3,$$

$$\ell^a = A^{-1}\delta^a_0 + C^{-1}\delta^a_3.$$

(2.4)

In the standard irreversible thermodynamics by Eckart, we have the following relation

$$\pi_{ab} = -2\eta\sigma_{ab}, \quad \Pi = -\zeta\Theta,$$

(2.5)

where $\eta$ and $\zeta$ stand for coefficients of shear and bulk viscosity, $\sigma_{ab}$ is the shear tensor, $\Theta$ is the expansion, $\Pi$ is the bulk viscosity and $\pi_{ab}$ is the shear
viscosity tensor. The algebraic nature of Eckart constitutive equations causes several problems but we are concerned with the causal approach of dissipative variables. Thus we would not assume (2.5) rather we shall resort to transport equations of Müller-Israel-Stewart theory.

The non-zero component of acceleration and expansion scalar are given by
\[ a_3 = \frac{A'}{A}, \quad \Theta = \frac{1}{A} \left( \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right), \] (2.6)

where dot and prime denote differentiation with respect to \( t \) and \( z \) respectively. The non-vanishing components of shear tensor are
\[ \sigma_{11} = -\frac{1}{3} B^2 F = \sigma_{22}, \quad \sigma_{33} = \frac{2}{3} C^2 F; \quad F = \frac{1}{A} \left( \frac{-\dot{B}}{B} + \frac{\dot{C}}{C} \right). \] (2.7)

The magnitude of the shear tensor, i.e., the shear scalar \( \sigma \) is defined as
\[ \sigma^2 = \frac{1}{2} \sigma^{ab} \sigma_{ab} = \frac{1}{9} F^2 \] (2.8)

which implies that \( F^2 = 9\sigma^2 \).

### 2.2 The Einstein and Maxwell Field Equations

The energy-momentum tensor of an electromagnetic field is defined as
\[ E_{ab} = \frac{1}{4\pi} (F^c_a F_{bc} - \frac{1}{4} F^{cd} F_{cd} g_{ab}), \] (2.9)

where \( F_{ab} \) is the electromagnetic field tensor given by
\[ F_{ab} = \phi_{b,a} - \phi_{a,b}, \] (2.10)

\( \phi_a \) is four potential. The Maxwell field equations are
\[ F^{ab} ;_b = \mu_0 J^a, \quad F_{[abc]} = 0, \] (2.11)

where \( \mu_0 = 4\pi \) is the magnetic permeability and \( J_a \) is the four current. In a co-moving frame \( \phi_a \) and \( J_a \) are given by
\[ \phi_a = \phi \delta_0^a, \quad J^a = \xi V^a, \] (2.12)
where $\xi$ and $\phi$ are the charge density and electric scalar potential respectively and both are functions of $t$ and $z$. The charge conservation $J^a_{;a} = 0$ gives the charge for interior region as

$$s(z) = \int_0^z \xi B^2 C dz. \quad (2.13)$$

For $a = 0, 3$, the first of Eq.(2.11) becomes

$$\phi'' - \left[ \frac{A'}{A} + \frac{C'}{C} - \frac{2B'}{B} \right] \phi' = \mu_0 \xi AC^2,$$  

$$\phi' - \left[ \frac{\dot{A}}{A} + \frac{\dot{C}}{C} - \frac{2\dot{B}}{B} \right] \phi' = 0,$$  

while for $a = 1, 2$, it is trivially satisfied. Also, the second of Eq.(2.11) is identically satisfied. Integration of Eq.(2.14) with respect to $z$, assuming $\phi'(t, 0) = 0$, gives

$$\phi' = \frac{\mu_0 s(z) AC}{B^2}. \quad (2.16)$$

The Taub’s mass function [34] for plane symmetric spacetime can be generalized to include the electromagnetic contributions as

$$m(t, z) = \frac{B}{2} \left( \frac{\dot{B}^2}{A^2} - \frac{B'^2}{C^2} \right) + \frac{s^2}{2B}. \quad (2.17)$$

For the interior spacetime, the Einstein field equations, $G_{ab} = 8\pi (T_{ab} + E_{ab})$, yield the following set of equations

$$8\pi (\mu + \epsilon) A^2 + \left( \frac{\mu_0 s A}{B^2} \right)^2 = \frac{\dot{B}}{B} \left( \frac{2\dot{C}}{C} + \frac{\dot{B}}{B} \right) + \frac{(A^2)^2}{C^2} \frac{-2B''}{B},$$

$$+\left( \frac{2C'}{C} - \frac{B'}{B} \right) \left( \frac{B'}{B} \right), \quad (2.18)$$

$$-8\pi AC(q + \epsilon) = -\frac{\dot{B}'}{B} + \frac{A' \dot{B}}{AB} + \frac{\dot{C} B'}{CB}, \quad (2.19)$$

$$8\pi (P_{\perp} + \frac{2}{3} \eta F) B^2 + \left( \frac{\mu_0 s}{B} \right)^2 = -\frac{(B A)^2}{B} \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{A}}{A} \frac{\dot{B}}{B} - \frac{A'}{A} \left( \frac{C'}{C} - \frac{\dot{B}}{B} \right) - \frac{B'C''}{BC},$$

$$+\frac{\dot{C}}{C} + \frac{\dot{B} \dot{C}}{BC} + \frac{B''}{B} \left( \frac{2}{A} \right) + \frac{A''}{A} \left( \frac{C'}{C} - \frac{\dot{B}}{B} \right) - \frac{B'C''}{BC}, \quad (2.20)$$

$$8\pi (P_z + \epsilon - \frac{4}{3} \eta F) C^2 - \left( \frac{\mu_0 s C}{B^2} \right)^2 = -\frac{(C A)^2}{B} \frac{\dot{B}}{B} + \frac{\dot{B}}{B} + \frac{2\dot{A} B'}{AB} \left( \frac{B'}{B} \right)^2 + \frac{2A'B'}{AB}. \quad (2.21)$$
In view of Eqs. (2.6) and (2.7), Eq. (2.19) can be written as

$$4\pi (q + \epsilon)C = \frac{1}{3}(\Theta - F)' - F'B' .$$  \hspace{1cm} (2.22)

### 2.3 Junction Conditions

Here we formulate junction conditions for the general plane symmetric space-time in the interior and charged Vaidya plane symmetric spacetime in the exterior. The line element for the exterior region is

$$ds^2_+ = \left( \frac{2M(\nu)}{Z} - \frac{e^2(\nu)}{Z^2} \right) d\nu^2 - 2dZd\nu + Z^2(dX^2 + dY^2),$$  \hspace{1cm} (2.23)

where $\chi^0 = \nu, \chi^1 = X, \chi^2 = Y, \chi^3 = Z$. The metric for hypersurface is defined as

$$(ds^2)_\Sigma = -(d\tau)^2 + f(\tau)^2(dx^2 + dy^2),$$  \hspace{1cm} (2.24)

where $w^i = (\tau, x, y)(i = 0, 1, 2)$ are the intrinsic coordinates of $\Sigma$. The equations of hypersurface in terms of interior and exterior coordinates are

$$k^-(t, z) = z - z_\Sigma = 0,$$  \hspace{1cm} (2.25)

$$k^+(\nu, Z) = Z - Z_\Sigma(\nu) = 0,$$  \hspace{1cm} (2.26)

where $z_\Sigma$ is constant. Using Eqs. (2.25) and (2.26), we get the interior and exterior metrics over the hypersurface as

$$(ds^2)_\Sigma = - A^2(t, z_\Sigma) dt^2 + B^2(t, z_\Sigma)(dx^2 + dy^2),$$  \hspace{1cm} (2.27)

$$(ds^2)_\Sigma = - [\left( -\frac{2M(\nu)}{Z_\Sigma} + \frac{e^2(\nu)}{Z_\Sigma^2} \right) + \frac{2dZ_\Sigma}{d\nu}] d\nu^2$$

$$+ Z_\Sigma^2(dX^2 + dY^2).$$  \hspace{1cm} (2.28)

Now we use the junction conditions proposed by Darmois [5], the first condition is

$$(ds^2)_\Sigma = (ds^2)_\Sigma = (ds^2)_\Sigma$$  \hspace{1cm} (2.29)

which yields the following equations

$$\frac{dt}{d\tau} = \frac{1}{A^2},$$  \hspace{1cm} (2.30)

$$Z_\Sigma = B,$$  \hspace{1cm} (2.31)

$$\frac{d\nu}{d\tau} = \left[ \left( -\frac{2M(\nu)}{Z_\Sigma} + \frac{e^2(\nu)}{Z_\Sigma^2} \right) + \frac{2dZ_\Sigma}{d\nu} \right]^{1/2}.$$  \hspace{1cm} (2.32)
The second junction condition is the continuity of extrinsic curvature (the second fundamental form)

\[ K_{ab} = K^a_0 = K_0^a. \] (2.33)

The unit normal in terms of interior and exterior coordinates are given respectively as

\[ n_a^- = C(0, 0, 0, 1), \quad n_a^+ = (-\hat{Z}_\Sigma, 0, 0, \hat{\nu}), \] (2.34)

here hat denotes differentiation with respect to \( \tau \). The surviving components of the extrinsic curvature for the interior spacetime are

\[ K_{00}^- = -\left[ \frac{A'}{A C} \right] \Sigma, \quad K_{11}^- = \left[ \frac{B B'}{C} \right] \Sigma = K_{22}^- . \] (2.35)

The non-null components of the extrinsic curvature for the exterior spacetime are given by

\[ K_{00}^+ = \left[ \frac{d^2 \nu}{d \tau^2} \right] - \left( \frac{M}{Z^2} - \frac{e^2}{Z^3} \right) \left( \frac{d \nu}{d \tau} \right) \Sigma . \] (2.36)

\[ K_{11}^+ = \left[ Z \frac{dZ}{d \tau} + \left( \frac{e^2}{Z} - 2M \right) \frac{d \nu}{d \tau} \right] \Sigma = K_{22}^+ . \] (2.37)

Thus the second junction condition yields the following equations

\[ - \left[ \frac{A'}{A C} \right] \Sigma = \left[ \frac{d^2 \nu}{d \tau^2} \right] - \left( \frac{M}{Z^2} - \frac{e^2}{Z^3} \right) \left( \frac{d \nu}{d \tau} \right) \Sigma , \] (2.38)

\[ \left[ \frac{B B'}{C} \right] \Sigma = \left[ Z \frac{dZ}{d \tau} + \left( \frac{e^2}{Z} - 2M \right) \frac{d \nu}{d \tau} \right] \Sigma . \] (2.39)

After some algebra, it follows that

\[ M(\nu) \overset{\Sigma}{=} m(t, z) \Leftrightarrow s \overset{\Sigma}{=} e , \] (2.40)

\[ q \overset{\Sigma}{=} P_z - \frac{4}{3} \eta F - \frac{s^2}{2B^4} (\mu_0^2 - 1) . \] (2.41)

These equations give necessary and sufficient conditions for the matching of interior and exterior spacetimes. Equation (2.41) describes a relationship between heat flux, effective pressure in \( z \)-direction and charge over the hypersurface. It shows that if the fluid has no charge then effective pressure and heat flux are equal over the hypersurface.
2.4 Dynamical Equations

Here we develop equations that govern the dynamics of non-adiabatic plane symmetric collapsing process by using Misner-Sharp formalism [4]. The proper time derivative and proper derivative in $z$-direction are defined respectively [16] as

$$D_{\tilde{T}} = \frac{1}{A} \frac{\partial}{\partial t}, \quad (2.42)$$

$$D_{\tilde{Z}} = \frac{1}{\tilde{Z}} \frac{\partial}{\partial \tilde{z}}, \quad (2.43)$$

where $\tilde{Z} = B$. The velocity of the collapsing fluid is the proper time derivative of $\tilde{Z}$ defined as

$$U = D_{\tilde{T}}(\tilde{Z}) = \frac{\dot{B}}{A} \quad (2.44)$$

which is always negative. Using this expression, Eq.(2.17) implies that

$$E = \frac{B'}{C} = \left[U^2 - \frac{2m}{B} + \frac{s^2}{B^2}\right]^{\frac{1}{2}}. \quad (2.45)$$

When we make use of Eq.(2.43) in (2.22), we have

$$4\pi C(q + \epsilon) = E\left[\frac{1}{3} D_{\tilde{Z}}(\Theta - F) - \frac{F}{Z}\right]. \quad (2.46)$$

The rate of change of mass (given in Eq.(2.17)) with respect to proper time is given by

$$D_{\tilde{T}}m = -4\pi[(P_z + \epsilon - \frac{4}{3}\eta F)U + E(q + \epsilon)]\tilde{Z}^2 + \frac{s^2U}{2\tilde{Z}^2}(\mu_0^2 - 1). \quad (2.47)$$

This equation shows how mass is varying within the plane hypersurface under the influence of matter variables. The first term represents effective pressure in $z$-direction and radiation density. When collapse takes place, this term is positive implying that energy increases by this factor. The second term in square brackets shows that energy is going out from the plane hypersurface while the last term is the charge contribution. During collapse, energy decreases due to these terms. Similarly, we calculate

$$D_{\tilde{Z}}m = 4\pi[\mu + \epsilon + (q + \epsilon)\frac{U}{E}]\tilde{Z}^2 + \frac{s}{Z}D_{\tilde{Z}}s + \frac{s^2}{2\tilde{Z}^2}(\mu_0^2 - 1). \quad (2.48)$$
This equation describes how different quantities influence the mass between neighboring hypersurfaces in the fluid distribution. The term \((\mu + \epsilon)\) indicates the effects of energy density and radiation density. Similarly, the second term shows the amount of heat and radiation which is getting out. The remaining two terms represent contribution of electric charge. Integration of Eq. (2.48) yields

\[
m = \int_{0}^{\tilde{Z}} 4\pi [\mu + \epsilon + (q + \epsilon) \frac{U}{E}] \tilde{Z}^2 d\tilde{Z} + \frac{s^2}{2\tilde{Z}} + \frac{1}{2} \int_{0}^{\tilde{Z}} (\frac{s^2}{\tilde{Z}^2}) d\tilde{Z} + \frac{(\mu_0^2 - 1)}{2} \int_{0}^{\tilde{Z}} (\frac{s^2}{\tilde{Z}^2}) d\tilde{Z}.
\]

(2.49)

The dynamical equations can be obtained from the contracted Bianchi identities \((T^{ab} + E^{ab})_{;b} = 0\). Consider the following two equations

\[
(T^{ab} + E^{ab})_{;b} V_a = (T_{0b}^{0b} + E_{0b}^{0b}) V_0 = 0,
\]

(2.50)

\[
(T^{ab} + E^{ab})_{;b} \chi_a = (T_{3b}^{3b} + E_{3b}^{3b}) \chi_3 = 0
\]

(2.51)

which yield

\[
\begin{align*}
\frac{(\mu + \epsilon)'}{A} + (\mu + 2\epsilon + P_z - \frac{4}{3} \eta F) \frac{\dot{C}}{AC} + 2(\mu + \epsilon + P_\perp + \frac{2}{3} \eta F) \\
\frac{\dot{B}}{AB} + \frac{(q + \epsilon)'}{AC} + \frac{2A'}{AC}(q + \epsilon) + \frac{2B'}{BC}(q + \epsilon) = 0,
\end{align*}
\]

(2.52)

\[
\begin{align*}
\frac{(q + \epsilon)'}{A} + \frac{1}{C}(P_z + \epsilon - \frac{4}{3} \eta F)' + \frac{2(q + \epsilon)(BC)'}{ABC} + (\mu + P_z + 2\epsilon \\
- \frac{4}{3} \eta F) \frac{A'}{AC} + 2(P_z - P_\perp + \epsilon - 2\eta F) \frac{B'}{BC} - \frac{\mu_0^2 s s'}{4\pi C B^3} = 0.
\end{align*}
\]

(2.53)

The acceleration of the collapsing fluid is defined as

\[
D_\parallel U = \frac{1}{A} \frac{\partial U}{\partial t} = \frac{\dot{B}}{A^2} - \frac{\dot{A} \dot{B}}{A^3}.
\]

(2.54)

Using Eqs. (2.21), (2.42) and (2.17), we have

\[
D_\parallel U = -4\pi (P_z + \epsilon - \frac{4}{3} \eta F) \tilde{Z} - \frac{m}{\tilde{Z}^2} + \frac{s^2}{2\tilde{Z}^3} (\mu_0^2 + 1) + \frac{EA'}{AC}
\]

(2.55)
which gives the value of $\frac{A'}{A}$

$$\frac{A'}{A} = \frac{C}{E} [D_T U + 4\pi(P_z + \epsilon - \frac{4}{3}\eta F)] + \frac{mC}{EZ^2} - \frac{Cs^2}{2EZ^3} (\mu_0^2 + 1). \tag{2.56}$$

Substituting this value in Eq. (2.53), it follows that

$$(\mu + P_z + 2\epsilon - \frac{4}{3}\eta F)D_T U = -(\mu + P_z + 2\epsilon - \frac{4}{3}\eta F)\frac{m}{Z^2} + 4\pi(P_z + \epsilon - \frac{4}{3}\eta F)$$

$$+ \epsilon - \frac{4}{3}\eta F)\tilde{Z} - \frac{s^2}{2Z^3} (\mu_0^2 + 1) - E^2[D_{\tilde{Z}}(P_z + \epsilon - \frac{4}{3}\eta F) + \frac{2}{Z}(P_z + \epsilon - \frac{4}{3}\eta F)]$$

$$- P_\perp + \epsilon - 2\eta F) - \frac{\mu_0^2 s D_{\tilde{Z}} s}{4\pi Z^4} - E[D_T(q + \epsilon) + 4(q + \epsilon)\frac{U}{Z}]$$

$$+ 2(q + \epsilon) F]. \tag{2.57}$$

This equation yields the effect of different forces on the collapsing process. It can be interpreted in the form of Newton’s second law of motion i.e., force = mass density × acceleration. The term within round brackets on LHS represents the inertial or passive gravitational mass density. This term shows that effective pressure, energy density and density of null fluid have effects on mass density while heat flux and charge have no contribution here. By equivalence principle, the round brackets factor on RHS is taken as active gravitational mass density. The quantities within square brackets in the first term show the influence of effective pressure, radiation density and electric charge on active gravitational mass. Using Eq. (2.49) in (2.57), it follows that charge increases the active gravitational mass if

$$\frac{s^2}{2Z^3} + \frac{\mu_0^2}{2Z^3} \int_0^{\tilde{Z}} (\frac{s^2}{2\tilde{Z}^2})d\tilde{Z} - \frac{s^2}{2Z^3} (\mu_0^2 + 1) > 0 \Rightarrow \frac{s}{Z} > D_{\tilde{Z}} s. \tag{2.58}$$

If this inequality holds then charge regeneration phenomenon analogous to pressure regeneration occurs [36]. The pressure regeneration means that the pressure which is trying to keep hydrostatic equilibrium by balancing gravitational attraction, at the same time contributes to the active gravitational mass. This implies that it promotes gravitational collapse. Otherwise, if the above inequality is not satisfied, charge will decrease active gravitational mass and consequently the Coulomb repulsion may prevent the gravitational collapse.

The first term in the second square brackets is the gradient of effective pressure in $z$-direction and radiation density. Since this gradient is negative,
it increases the rate of collapse. The second term is due to local anisotropy of pressure, radiation density and contribution of viscosity. If this term is positive then it contributes to increase collapse and vice versa. The last term depicts Coulomb repulsion that opposes gravitation implying that it decelerates the collapsing process.

Finally, the last square brackets is entirely due to dissipation. To see the role of $D_{\tilde{T}}q$, this equation is coupled with causal transport equation. The consequences of $D_{\tilde{T}}\epsilon$ have been discussed by Misner [37]. The outward flux of radiation accelerates collapsing process by increasing gravitational force. The third term is positive as $U < 0$, so it slows down rate of collapse. The last term shows the combine effect of viscosity and dissipation. From Eq. (2.57), the condition for hydrostatic equilibrium can be obtained by replacing $U = q = \epsilon = \eta = 0$ as

$$D_{\tilde{Z}}P_z = -\frac{(P_z + \mu)}{E^2}\left[\frac{m}{Z^2} - \frac{s^2}{2Z^3}(\mu_0^2 + 1)\right] + \frac{\mu_0^2 D_{\tilde{Z}}s}{4\pi Z^4} - \frac{2}{Z}(P_z - P_\perp). \quad (2.59)$$

### 2.5 Transport Equation

The transport equation for heat flux derived from the Müller-Israel-Stewart theory of dissipative fluids [33] is given by

$$\tau_0 h^{ab}V_c q_{bc} + q^a = -\kappa h^{ab}(T_b + T_a) - \frac{1}{2}\kappa T^2 h^{bc}q^b q^c, \quad (2.60)$$

where $h^{ab}$ is the projection tensor, $\kappa$ denotes thermal conductivity, $T$ is temperature and $\tau$ stands for relaxation time. This equation has only one independent component

$$D_{\tilde{Z}}q = -\frac{\kappa T^2 q}{2\tau_0}D_{\tilde{Z}}(\frac{\tau_0}{\kappa T^2}) - q\left[\frac{3U}{2Z} + \frac{F}{2} + \frac{1}{\tau_0}\right] - \frac{\kappa E}{\tau_0}D_{\tilde{Z}}T - \frac{\kappa T}{\tau_0 E}$$

$$\times \quad D_{\tilde{Z}}U - \frac{\kappa T}{\tau_0 E}[m + 4\pi(P_z + \epsilon - \frac{4}{3}\eta F)\tilde{Z}^3 - \frac{s^2}{Z} - \frac{1}{Z^2}]. \quad (2.61)$$

We now couple this equation with dynamical Eq. (2.57) to see the effects of heat flux or dissipation on collapsing process. Replacing Eq. (2.61) in (2.57),
we obtain
\[ (\mu + P_z + 2\epsilon - \frac{4}{3}\eta F)(1 - \alpha)\mathcal{D}_T U = (1 - \alpha)F_{\text{grav}} + F_{\text{hyd}} + \frac{\kappa E^2}{\tau_0} \]
\[ D_Z T + E[\frac{\kappa T^2}{2\tau_0}D_T(\frac{\tau_0}{\kappa T^2}) - D_T\epsilon] - Eq(\frac{5U}{2Z} + \frac{3}{2}F - \frac{1}{\tau_0}) - 2E\epsilon \]
\[ (\frac{2U}{Z} + F), \]  
(2.62)
where \( F_{\text{grav}}, F_{\text{hyd}} \) and \( \alpha \) are given by the following equations
\[ F_{\text{grav}} = -(\mu + P_z + 2\epsilon - \frac{4}{3}\eta F)[m + 4\pi(P_z + \epsilon + \frac{4}{3}\eta F)Z^3 - \frac{s^2}{2Z}] \]
\[ \times (\mu_0^2 + 1)]\frac{1}{Z^2}, \]  
(2.63)
\[ F_{\text{hyd}} = -E^2[D_Z(P_z + 2\epsilon - \frac{4}{3}\eta F) + \frac{2}{Z}(P_z - P_\perp + \epsilon - 2\eta F)] - \frac{sD_Zs}{4\pi Z^4}, \]  
(2.64)
\[ \alpha = \frac{\kappa T}{\tau_0}(\mu + P_z + 2\epsilon - \frac{4}{3}\eta F)^{-1}. \]  
(2.65)
The consequence of coupling transport and dynamical equations is that the inertial and active gravitational mass densities are affected by a factor \( \alpha \) given by Eq.(2.65). The gravitational force term defined in Eq.(2.63) is also affected by \( \alpha \) but the hydrodynamical forces Eq.(2.64) are not influenced by this term.

### 2.6 Relation Between the Weyl Tensor and Matter Variables

Here we find some relationship between the Weyl tensor and matter variables. The Weyl scalar \( \mathcal{C} \) in terms of Kretchman scalar \( \mathcal{R} \), the Ricci tensor \( R_{ab} \) and the Ricci scalar \( R \) is given by
\[ \mathcal{C}^2 = \mathcal{R} - 2R^{ab}R_{ab} + \frac{1}{3}R^2. \]  
(2.66)
The Kretchman scalar \( \mathcal{R}^2 = R^{abcd}R_{abcd} \) becomes
\[ \mathcal{R}^2 = 4\left[\frac{2}{A^2B^4}(R^{0101})^2 + \frac{1}{A^4C^4}(R^{0303})^2 + \frac{1}{B^8}(R^{1212})^2 \right. \]
\[ + \frac{2}{B^4C^4}(R^{2323})^2 - \frac{4}{A^2B^4C^2}(R^{0113})^2]. \]  
(2.67)
The non-zero components of the Riemann tensor can be written in terms of
the Einstein tensor and mass function as

\[ R_{0101} = (AB)^2 \left[ \frac{1}{2C^2}G_{33} + \frac{1}{B^3}(m - \frac{s^2}{2B}) \right] = R_{0202}, \]
\[ R_{0303} = (AC)^2 \left[ \frac{1}{2A^2}G_{00} - \frac{1}{2C^2}G_{33} + \frac{1}{B^3}G_{22} - \frac{2}{B^3}(m - \frac{s^2}{2B}) \right], \]
\[ R_{1212} = 2B(m - \frac{s^2}{2B}), \]
\[ R_{1313} = (BC)^2 \left[ \frac{1}{2A^2}G_{00} - \frac{1}{B^3}(m - \frac{s^2}{2B}) \right] = R_{2323}, \]
\[ R_{0113} = -\frac{B^2}{2}G_{03}. \]

Substituting these values in Eq. (2.67), after some algebra, we obtain

\[ \mathcal{R}^2 = \frac{48}{B^6}(m - \frac{s^2}{2B})^2 - \frac{16}{B^3}(m - \frac{s^2}{2B}) \left[ \frac{G_{00}}{A^2} - \frac{G_{33}}{C^2} + \frac{G_{22}}{B^2} \right] \]
\[- \frac{4}{A^2C^2}G_{03}^2 + 3 \left[ \left( \frac{G_{00}}{A^2} \right)^2 + \left( \frac{G_{33}}{C^2} \right)^2 \right] + \frac{4}{B^4}G_{22}^2 - 2 \frac{G_{00}G_{33}}{A^2C^2} \]
\[ + 4 \left( \frac{G_{00}}{A^2} - \frac{G_{33}}{C^2} \right) \frac{G_{22}}{B^2}. \] (2.68)

Now we calculate the remaining part of the Weyl scalars which need Ricci
tensor and Ricci scalar in terms of the Einstein tensor. These are

\[ R_{00} = A^2 \left[ \frac{G_{00}}{2A^2} + \frac{G_{33}}{2C^2} + \frac{G_{22}}{B^2} \right], \quad R_{03} = G_{03}, \]
\[ R_{11} = \frac{B^2}{2} \left[ \frac{G_{00}}{A^2} - \frac{G_{33}}{C^2} \right] = R_{22}, \quad R_{33} = C^2 \left[ \frac{G_{00}}{2A^2} + \frac{G_{33}}{2C^2} - \frac{G_{22}}{B^2} \right], \]
\[ R = \frac{G_{00}}{A^2} - \frac{G_{33}}{C^2} - \frac{2G_{22}}{B^2}, \]
\[ R^{ab}R_{ab} = \frac{G_{00}^2}{A^4} + \frac{G_{33}^2}{C^4} + \frac{2G_{22}^2}{B^4} - \frac{2G_{03}^2}{A^2C^2}. \]

Thus the remaining part of the Weyl scalar becomes

\[ \frac{1}{3}R^2 - 2R^{ab}R_{ab} = -\frac{5}{3} \frac{G_{00}^2}{A^4} - \frac{5}{3} \frac{G_{33}^2}{C^4} - \frac{8}{3} \frac{G_{22}^2}{B^4} + \frac{4G_{03}^2}{A^2C^2} - \frac{2}{3} \frac{G_{00}G_{33}}{A^2C^2} \]
\[ + \frac{4}{3} \frac{G_{22}G_{33}}{C^2B^2} - \frac{4}{3} \frac{G_{00}G_{22}}{A^2B^2}. \] (2.69)
Using Eqs. (2.68) and (2.69), the Weyl scalar takes the form

\[ C^2 = \frac{48}{B^6} \left( m - \frac{s^2}{2B} \right)^2 - \frac{16}{B^3} \left( m - \frac{s^2}{2B} \right) \left( \frac{G_{00}}{A^2} - \frac{G_{33}}{C^2} + \frac{G_{22}}{B^2} \right) \]

\[ - \frac{4}{A^2 C^2} G_{03}^2 + 3 \left[ \left( \frac{G_{00}}{A^2} \right)^2 + \left( \frac{G_{33}}{C^2} \right)^2 \right] + \frac{4}{B^4} G_{22}^2 - 2 \frac{G_{00} G_{33}}{A^2 C^2} \]

\[ + \frac{4(G_{00}}{A^2} - \frac{G_{33}}{C^2}) \frac{G_{22}}{B^2} - \frac{5 G_{00}}{3 A^4} - \frac{5 G_{33}}{3 C^4} - \frac{8 G_{22}}{3 B^4} + \frac{4 G_{03}^2}{A^2 C^2} \]

\[ - \frac{2 G_{00} G_{33}}{3 A^2} + \frac{4 G_{22} G_{33}}{3 C^2 B^2} - \frac{4 G_{00} G_{22}}{3 A^2 B^2}. \]  

(2.70)

After some algebra, it leads to the following equation

\[ \frac{C B^3}{(48)^{1/2}} = \left( m - \frac{s^2}{B} \right) - \frac{B^3}{6} \left[ \frac{G_{00}}{A^2} + \frac{G_{22}}{B^2} - \frac{G_{33}}{C^2} - \frac{3s^2}{B^4} \right]. \]  

(2.71)

Using the field equations, we have

\[ \frac{G_{00}}{A^2} + \frac{G_{22}}{B^2} - \frac{G_{33}}{C^2} - \frac{3\mu_0^2 s^2}{B^4} = 8\pi (\mu + P_{\perp} - P_z + 2\eta F). \]  

(2.72)

In view of the above equation and using \( \tilde{Z} = B \), Eq. (2.71) becomes

\[ \frac{C \tilde{Z}^3}{(48)^{1/2}} = \left[ m - \frac{4\pi}{3} (\mu + P_{\perp} - P_z + 2\eta F) \right] \tilde{Z}^3 - \frac{s^2}{2\tilde{Z}} (\mu_0^2 + 1)]. \]  

(2.73)

The derivatives of \( \frac{C \tilde{Z}^3}{(48)^{1/2}} \) with respect to \( \tilde{T} \) and \( \tilde{Z} \) are given by

\[ D_{\tilde{T}} \left( \frac{C \tilde{Z}^3}{(48)^{1/2}} \right) = -4\pi \left[ \frac{1}{3} \tilde{Z}^3 D_{\tilde{T}} (\mu + P_{\perp} - P_z + 2\eta F) + (\mu + P_{\perp} + \epsilon + \frac{2}{3}\eta F) \times \right] \tilde{Z}^2 U + (q + \epsilon) E \tilde{Z}^2] + \frac{s^2 U}{2\tilde{Z}^2} (\mu_0^2 + 1), \]  

(2.74)

\[ D_{\tilde{Z}} \left( \frac{C \tilde{Z}^3}{(48)^{1/2}} \right) = 4\pi [(q + \epsilon) \tilde{Z}^2 U \frac{E}{E} - \frac{1}{3} \tilde{Z}^2 D_{\tilde{Z}} (\mu + P_{\perp} - P_z + 2\eta F) + (\epsilon - P_{\perp}) \right] \tilde{Z}^2 - \frac{s D_{\tilde{Z}} \tilde{Z}}{\tilde{Z}} (\mu_0^2 + 1) + \frac{s^2}{2\tilde{Z}^2} (\mu_0^2 + 1). \]  

(2.75)

These equations give relationship between the Weyl scalar and the fluid properties like density, viscosity and pressure (anisotropy). For perfect and non-charged fluid, Eq. (2.75) reduces to the following form

\[ D_{\tilde{Z}} \left( \frac{C \tilde{Z}^3}{(48)^{1/2}} \right) = -\frac{4\pi}{3} \tilde{Z}^3 D_{\tilde{Z}} \mu. \]  

(2.76)
Using the regularity condition, it is concluded that $D_Z\mu = 0$ if and only if $C = 0$. This means that if energy density is homogeneous, the metric is conformally flat and vice versa.

We would like to mention here that a particularly simple relation between the Weyl tensor and density inhomogeneity such as (2.76), for perfect non-charged fluids, is at the origin of Penrose’s proposal to provide a gravitational arrow of time in terms of the Weyl tensor. The rationale behind this idea is that tidal forces tend to make the gravitating fluid more inhomogeneous as the evolution proceeds, thereby indicating the sense of time. However, the fact that such a relationship is no longer valid in the presence of local anisotropy of the pressure and/or dissipative processes and/or electric charge. This has already been discussed explaining its failure in scenarios where the above-mentioned factors are present. Here we see how the electric charge distribution affects the link between the Weyl tensor and density inhomogeneity, suggesting that electric charge (whenever present) should enter into any definition of a gravitational arrow of time.

## 3 Dynamics of Viscous Adiabatic Case

In this case, heat flux vanishes, also, we assume that radiation density is zero and hence dissipation is only due to shearing viscosity. The energy-momentum tensor is obtained by replacing $q_\alpha = \epsilon = 0$ in Eq.(2.2). Similarly, the Einstein-Maxwell field equations are found by using $q = \epsilon = 0$ in the corresponding equations derived for non-adiabatic case. For junction conditions, the line element for the exterior region is taken as plane symmetric Reissner-Nordström spacetime given by

$$ds^2_+ = -\left(\frac{-2M}{Z} + \frac{e^2}{Z^2}\right)dT^2 + \left(\frac{-2M}{Z} + \frac{e^2}{Z^2}\right)^{-1}dZ^2 + Z^2(dX^2 + dY^2), \quad (3.77)$$

where $(\chi^0, \chi^1, \chi^2, \chi^3) = (T, X, Y, Z)$. The equation of hypersurface in terms of exterior coordinates is

$$k^+(T, Z) = Z - Z_\Sigma(T) = 0. \quad (3.78)$$

Using Eq.(3.78), the exterior metric over the hypersurface becomes

$$ (ds^2_+)_\Sigma = -\left[\left(\frac{-2M}{Z_\Sigma} + \frac{e^2}{Z^2_\Sigma}\right) - \left(\frac{-2M}{Z_\Sigma} + \frac{e^2}{Z^2_\Sigma}\right)^{-1}\left(\frac{dZ_\Sigma}{dT}\right)^2\right]dT^2 \quad (3.79)$$

$$+ Z^2_\Sigma(dX^2 + dY^2).$$
The first junction condition yields the following equations
\[
\frac{dt}{d\tau} = \frac{1}{A}, \quad Z_\Sigma = B, \quad (3.80)
\]
\[
\frac{dT}{d\tau} = \left(-\frac{2M}{Z_\Sigma} + \frac{e^2}{Z_\Sigma^2}\right)^\frac{1}{2}\left(\frac{-2M}{Z_\Sigma} + \frac{e^2}{Z_\Sigma^2}\right)^2 - \left(\frac{dZ_\Sigma}{dT}\right)^2\frac{1}{2}. \quad (3.81)
\]
Equation (3.81) implies that
\[
\frac{d\tau^2}{dT^2} = \frac{1}{N} dZ_\Sigma^2, \quad N = \left(-\frac{2M}{Z_\Sigma} + \frac{e^2}{Z_\Sigma^2}\right). \quad (3.82)
\]
The unit normal in terms of exterior coordinates is given by
\[
n^\perp = (-\tilde{Z}_\Sigma, 0, 0, \tilde{T}), \quad (3.83)
\]
The non-null components of the extrinsic curvature for the exterior spacetime are
\[
K^0_0 = \left[\frac{dZ}{d\tau} \frac{dZ}{d\tau^2} - \frac{d^2Z}{d\tau^2} d\tau - \frac{N}{2} dN \left(\frac{dT}{d\tau}\right)^3 + \frac{3}{2N} dN d\tau \left(\frac{dZ}{d\tau}\right)^2\right] \Sigma. \quad (3.84)
\]
\[
K^1_{11} = [ZN \frac{dT}{d\tau}] \Sigma = K^2_{22}. \quad (3.85)
\]
The second junction condition yields
\[
M \Sigma = m(t, z) \iff s \Sigma = e. \quad (3.86)
\]
The rate of change of mass with respect to \(\tilde{T}\) and \(\tilde{Z}\) are given by the following equations
\[
D_{\tilde{T}}m(t, z) = -4\pi (P_z - \frac{4}{3} \eta F) U \tilde{Z}^2 + \frac{s^2 U}{2 \tilde{Z}^2} (\mu_0^2 - 1), \quad (3.87)
\]
\[
D_{\tilde{Z}}m(t, z) = 4\pi \mu \tilde{Z}^2 + \frac{s}{\tilde{Z}} D_{\tilde{Z}} s + \frac{s^2}{2 \tilde{Z}^2} (\mu_0^2 - 1). \quad (3.88)
\]
The description of these equations is the same as for the non-adiabatic case. Similarly, the dynamical equations can be obtained using \(q = \epsilon = 0\) in Eqs. (2.52)-(2.57). For this case, Eq. (2.57) becomes
\[
(\mu + P_z - \frac{4}{3} \eta F) D_{\tilde{T}} U = -\left(\mu + P_z - \frac{4}{3} \eta F\right)\left[\frac{m}{\tilde{Z}^2} + 4\pi (P_z - \frac{4}{3} \eta F) \tilde{Z} - \frac{s^2}{\tilde{Z}^2}\right]
\]
\[
-\frac{E^2}{2} [D_{\tilde{Z}}(P_z - \frac{4}{3} \eta F) + \frac{2}{\tilde{Z}} (P_z - P_{\perp} - 2\eta F) - \frac{\mu_0^2 s D_{\tilde{Z}} s}{4\pi \tilde{Z}^4}]. \quad (3.89)
\]
The inequality given in Eq. (2.58) remains the same. The interpretation of the terms in Eq. (3.89) is similar to that of the non-adiabatic case just excluding the effects of heat flux and radiation density. As there is no heat flux, so no transport equation is needed. The relationship between the Weyl tensor and energy homogeneity also remains the same.

4 Conclusions

Gravitational collapse is an outstanding phenomenon in gravitation theory. The aim of this work is to analyze the dynamics of gravitational collapse for plane symmetric configuration of real fluid. The conclusions are given in the following

1. The junction conditions for both cases yield that masses of the interior and exterior regions are equal if and only if their corresponding charges are equal. For the non-adiabatic case, junction conditions also give a relationship between heat flux, effective pressure in $z$-direction and charge over the hypersurface (2.41). This equation implies that if $\dot{s} = 0$, then the effective pressure in $z$-direction and heat flux are equal over the hypersurface.

2. For the non-dissipative case, Eq. (2.46) reduces to $D_\dot{z}(U_\dot{z}) = 0$ which shows that collapse is homologous, i.e., all the matter falls inward in a similar pattern.

3. Condition for hydrostatic equilibrium is given by Eq. (2.59). If this condition holds, then the collapsing process will stop and matter attains an equilibrium state.

4. It is observed that charge will increase the rate of collapse if $\dot{s} > D_\dot{z}s$. Thus the chance of becoming a black plane increases in this case.

5. In the non-adiabatic case, the radiation density increases inertial and active gravitational masses. Also, the outflow of radiation causes an increase in the rate of collapse and hence the collapsing process is expected to be faster than the adiabatic case.

6. The substitution of transport equation (2.61) in dynamical equation (2.57) yields an additional factor $\alpha$. This $\alpha$ term affects the inertial
mass density and gravitational force term. As $\alpha$ increases, the affected terms are decreased by the same amount and vice versa. The corresponding terms for $\alpha$ come from the term $a_0 T$ in Eq.(2.60) which is the Tolman’s inertial term. Hence the inertia of heat by increasing $\alpha$, causes a decrease in inertial mass and gravitational force [41]. Thus we can conclude that

- If $\alpha \to 0$, then inertial density and gravitational force are not affected by coupling.
- If $0 < \alpha < 1$, then inertia of heat causes a decrease in inertial and gravitational mass densities.
- If $\alpha \to 1$, then mass densities approach to zero.
- If $\alpha > 1$, then the gravitational force term becomes negative implying that reversal of collapse occurs due to the inertia of heat.

7. Under certain conditions homogeneity in energy density and conformal flatness of spacetime are necessary and sufficient condition for each other.

8. A relation (2.75) has been obtained exhibiting the way in which electric charge affects the link between the Weyl tensor and density inhomogeneity.

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