Capacity-achieving Sparse Superposition Codes via Approximate Message Passing Decoding

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Abstract

Sparse superposition codes were recently introduced by Barron and Joseph for reliable communication over the AWGN channel at rates approaching the channel capacity. The codebook is defined in terms of a Gaussian design matrix, and codewords are sparse linear combinations of columns of the matrix. In this paper, we propose an approximate message passing decoder for sparse superposition codes, whose decoding complexity scales linearly with the size of the design matrix. The performance of the decoder is rigorously analyzed and it is shown to asymptotically achieve the AWGN capacity with an appropriate power allocation. We provide simulation results to demonstrate the performance of the decoder at finite block lengths, and investigate the effects of various power allocations on the decoding performance.

1 Introduction

This paper considers the problem of constructing low-complexity, capacity-achieving codes for the memoryless additive white Gaussian noise (AWGN) channel. The channel generates output $y$ from input $x$ according to

$$y = x + w,$$

where the noise $w$ is a Gaussian random variable with zero mean and variance $\sigma^2$. There is an average power constraint $P$ on the input $x$: if $x_1, \ldots, x_n$ are transmitted over $n$ uses of the channel, then we require that $\frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq P$. The signal-to-noise ratio $\frac{P}{\sigma^2}$ is denoted by $\text{snr}$. The goal is to construct codes with computationally efficient encoding and decoding, whose rates approach the channel capacity given by

$$C := \frac{1}{2} \log(1 + \text{snr}).$$

Sparse superposition codes, also called Sparse Regression Codes (SPARCs), were recently introduced by Barron and Joseph [1,2] for communication over the channel in (1). They proposed an efficient decoding algorithm called ‘adaptive successive decoding’, and showed that for any fixed rate $R < C$, the probability of decoding error decays to zero exponentially in $\frac{n}{\log n}$, where $n$ is the block length of the code. Despite the strong theoretical performance guarantees, the rates achieved by this decoder for practical block lengths are significantly less than $C$. Subsequently, a soft-decision iterative decoder was proposed by Cho and Barron [3,4], with theoretical guarantees similar to the earlier decoder in [2] but improved empirical performance for finite block lengths.

In this paper, we propose an approximate message passing (AMP) decoder for SPARCs. We analyze its performance and prove that the probability of decoding error goes to zero with growing block length for all fixed rates $R < C$. The decoding complexity is proportional to the size of the design matrix defining the code, which is a low order polynomial in $n$.

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1.1 Approximate Message Passing (AMP)

"Approximate message passing" refers to a class of algorithms [5–11] that are Gaussian or quadratic approximations of loopy belief propagation algorithms (e.g., min-sum, sum-product) on dense factor graphs. AMP has proved particularly effective for the problem of reconstructing sparse signals from a small number of noisy linear measurements. This problem, commonly referred to as compressed sensing [12], is described by the measurement model

\[ y = A\beta + w. \] (3)

Here \( A \) is an \( n \times N \) measurement matrix with \( n < N \), \( \beta \in \mathbb{R}^N \) is a sparse vector to be estimated from the observed vector \( y \in \mathbb{R}^n \), and \( w \in \mathbb{R}^n \) is the measurement noise. One popular class of algorithms to reconstruct \( \beta \) is \( \ell_1 \)-norm based convex optimization, e.g. [13–15]. Though these algorithms have strong theoretical guarantees and excellent empirical performance, the computational cost makes it challenging to implement the convex optimization procedures for problems where \( N \) is large. A fast AMP reconstruction algorithm for the model in (3) was proposed in [5]. Its empirical performance (for a large class of measurement matrices) was found to be similar to convex optimization based methods at significantly lower computational cost.

The factor graph corresponding to the model in (3) is dense, hence it is infeasible to implement message passing algorithms in which the messages are complicated real-valued functions. AMP circumvents this difficulty by passing only scalar parameters corresponding to these functions. For example, the scalars could be the mean and the variance if the functions are posterior distributions. The references [6,8–10] describe how various flavors of AMP for the model in (3) can be obtained by approximating the standard message passing equations. These approximations reduce the message passing equations to a set of simple rules for computing successive estimates of \( \beta \).

In [5], it was demonstrated via numerical experiments that the mean-squared reconstruction error of these estimates of \( \beta \) could be tracked by a simple scalar iteration called state evolution. In [7], it was rigorously proved that the state evolution is accurate in the large system limit\(^1\) for measurement matrices \( A \) with i.i.d. Gaussian entries.

In addition to compressed sensing, AMP has also been applied to a variety of related problems, e.g. [16–18]. We will not attempt a complete survey of the growing literature on AMP; the reader is referred to [10,11] for comprehensive lists of related work.

1.2 Main Contributions

- We propose an AMP decoder for sparse regression codes, which is derived via a first-order approximation of a min-sum-like message passing algorithm.
- The main result of the paper is Theorem 1, in which we rigorously show that the probability of decoding error goes to zero as the block length tends to infinity, for all rates \( R < C \).
- We demonstrate the performance of the decoder for finite block lengths via simulation results.

To prove our main result, we use the framework of Bayati and Montanari [7], who in turn built on techniques introduced by Bolthausen [19]. However, we remark that the analysis of the proposed algorithm does not follow directly from the results in [7,20]. The main reason for this is that the undersampling ratio \( n/N \) in our setting goes to zero in the large system limit, whereas previous

\(^1\)The large system limit considered in [7] lets \( n, N \to \infty \) with \( n/N \) held constant.
rigorous analyses of AMP consider the case where the undersampling ratio is a constant. This point is discussed further in Section 5.

1.3 Related work on communication with SPARC

The adaptive successive decoder of Joseph-Barron [2] and the iterative soft-decision decoder of Cho-Barron [3, 4] both have probability of error that decays as \( n/\log n \) for any fixed rate \( R < C \), but the latter has better empirical performance. Theorem 1 shows that the probability of error for the AMP decoder goes to zero for all \( R < C \), but does not give a rate of decay; hence we cannot theoretically compare its performance with the Cho-Barron decoder in [4]. We can, however, compare the two decoders qualitatively.

Both the AMP and the Cho-Barron decoder generate a succession of estimates \( \beta_1, \beta_2, \ldots \) for the message vector \( \beta \) based on test statistics \( s^0, s^1, \ldots \), respectively. At step \( t \), the Barron-Cho decoder generates statistic \( s^t \) based on an orthonormalization of the observed vector \( y \) and the previous ‘fits’ \( A\beta_1, \ldots, A\beta_t \). In contrast, the test statistic in the AMP decoder is based on a modified version of the residue \( (y - A\beta_t) \). Despite being generated in very different ways, the test statistics of the AMP and Cho-Barron decoders have a similar structure: they are asymptotically equivalent to an observation of \( \beta \) corrupted by additive Gaussian noise whose variance decreases with \( t \). However, the AMP statistic is faster to compute in each step, which makes it feasible to implement the decoder for larger block lengths, which in turn results in lower (empirical) probability of decoding error.

An approximate message passing decoder for sparse superposition codes was recently proposed by Barbier and Krzakala in [21]. This decoder has different update rules from the AMP proposed here. A replica-based analysis of the decoder in [21] suggested it could not achieve rates beyond a threshold which was strictly smaller than \( C \). Very recently, Barbier et al [22] reported empirical results which show that the performance of the decoder in [21] can be improved by using spatially coupled Hadamard matrices to define the code.

Finally, we mention bit-interleaved coded modulation [23] and the recently proposed low-density lattice codes [24] as two alternative approaches to designing high-rate codes for the AWGN channel. Though these schemes have very good empirical performance, in general there are no theoretical guarantees that they achieve the AWGN capacity with growing block length.

1.4 Paper outline and Notation

The paper is organized as follows. The sparse regression code construction is described in Section 2. We describe the AMP channel decoder in Section 3 and provide some intuition about its iterations. We also show how the decoder can be derived as a first-order approximation to a min-sum-like message passing algorithm. Section 4 contains the main result, which characterizes the performance of the AMP decoder for any rate \( R < C \) in the large system limit. In Section 4.1, we present simulation results to demonstrate the performance of the decoder at finite block lengths. Section 5 contains the proof of the main result.

Notation: The \( \ell_2 \)-norm of vector \( x \) is denoted by \( \|x\| \). The transpose of a matrix \( B \) is denoted by \( B^\ast \). \( \mathcal{N}(\mu, \sigma^2) \) denotes the Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \). For any positive integer \( m \), \( [m] \) is the set \( \{1, \ldots, m\} \). The indicator function of an event \( \mathcal{A} \) is denoted by \( 1(\mathcal{A}) \). \( f(x) = o(g(x)) \) means \( \lim_{x \to \infty} f(x)/g(x) = 0 \); \( f(x) = \Theta(g(x)) \) means \( f(x)/g(x) \) asymptotically lies in an interval \( [\kappa_1, \kappa_2] \) for some constants \( \kappa_1, \kappa_2 > 0 \). \( \log \) and \( \ln \) are used to denote logarithms with base 2 and base \( e \), respectively. Rate is measured in bits.
Section 1

Section 2

Section L

Figure 1: $A$ is an $n \times ML$ matrix and $\beta$ is a $ML \times 1$ vector. The positions of the non-zeros in $\beta$ correspond to the gray columns of $A$ which combine to form the codeword $A\beta$.

### 2 The Sparse Regression Codebook

A sparse regression code (SPARC) is defined in terms of a dictionary or design matrix $A$ of dimension $n \times ML$, whose entries are i.i.d. $\mathcal{N}(0, \frac{1}{n})$. Here $n$ is the block length, and $M, L$ are integers whose values will be specified shortly in terms of $n$ and the rate $R$. As shown in Fig. 1, one can think of the matrix $A$ being composed of $L$ sections with $M$ columns each. Each codeword is a linear combination of $L$ columns, with one column from each section. Formally, a codeword can be expressed as $A\beta$, where $\beta$ is an $ML \times 1$ vector ($\beta_1, ..., \beta_{ML}$) with the following property: there is exactly one non-zero $\beta_j$ for $1 \leq j \leq M$, one non-zero $\beta_j$ for $M + 1 \leq j \leq 2M$, and so forth. The non-zero value of $\beta$ in section $\ell$ is set to $\sqrt{nP_\ell}$, where $P_1, ..., P_L$ are positive constants that satisfy

$$\sum_{\ell=1}^L P_\ell = P$$

(4)

Denote the set of all $\beta$’s that satisfy this property by $B_{M,L}(P_1, ..., P_L)$. Since there are $M$ columns in each of the $L$ sections, the total number of codewords is $M^L$. To obtain a communication rate of $R$ bits/sample, we need

$$M^L = 2^{nR} \quad \text{or} \quad L \log M = nR.$$  

(5)

There are several choices for the pair $(M, L)$ which satisfy (5). For example, $L = 1$ and $M = 2^{nR}$ recovers the Shannon-style random codebook in which the number of columns in $A$ is $2^{nR}$. For our constructions, we will choose $M$ equal to $L^b$, for some constant $b > 0$. In this case, (5) becomes

$$bL \log L = nR.$$  

(6)

Thus $L = \Theta\left(\frac{n}{\log n}\right)$, and the size of the design matrix $A$ (given by $n \times ML = n \times L^{b+1}$) now grows polynomially in $n$.

**Encoding:** The encoder splits its stream of input bits into segments of $\log M$ bits each. A length $ML$ message vector $\beta_0$ is indexed by $L$ such segments—the decimal equivalent of segment $\ell$ determines the position of the non-zero coefficient in section $\ell$ of $\beta_0$. The input codeword is then computed as $X = A\beta_0$; note that computing $X$ simply involves the addition of $L$ columns of $A$, weighted by appropriate coefficients.

**Power Allocation:** The power allocation $\{P_\ell\}_{\ell=1}^L$, plays an important role in determining the performance of the decoder. We will consider allocations where $P_\ell = \Theta\left(\frac{1}{L}\right)$. Two examples are:
• Flat power allocation across sections: $P_{\ell} = \frac{P}{L}$, $\ell \in [L]$.
• Exponentially decaying power allocation: Fix parameter $\kappa > 0$. Then $P_{\ell} \propto 2^{-\kappa \ell / L}$, $\ell \in [L]$.

We use the exponentially decaying allocation with $\kappa = 2C$ for Theorem 1. In Section 4.1, we discuss other power allocations, and find that an appropriate combination of exponential and flat allocations yields good decoding performance at finite block lengths.

Both the design matrix $A$ and the power allocation $\{P_{\ell}\}$ are known to the encoder and the decoder before communication begins.

### 2.1 Some more notation

In the analysis, we will treat the message as a random vector $\beta$, which is uniformly distributed over $B_{M,L}(P_1, \ldots, P_L)$, the set of length $ML$ vectors that have a single non-zero entry $\sqrt{n}P_{\ell}$ in section $\ell$, for $\ell \in [L]$. We will denote the true message vector by $\beta_0$; $\beta_0$ should be understood as a realization of the random vector $\beta$.

We will use indices $i, j$ to denote specific entries of $\beta$, while the index $\ell$ will be used to denote the entire section $\ell$ of $\beta$. Thus $\beta_i, \beta_j$ are scalars, while $\beta_\ell$ is a length $M$ vector. We also set $N = ML$.

The performance of the SPARC decoder will be characterized in the limit as the dictionary size goes to $\infty$. We write $\lim x$ to denote the limit of the quantity $x$ as the SPARC parameters $n, L, M \to \infty$ simultaneously, according to $M = L^b$ and $bL \log L = nR$.

### 3 The AMP Channel Decoder

Given the received vector $y = A\beta_0 + w$, the AMP decoder generates successive estimates of the message vector, denoted by $\{\beta^t\}$, where $\beta^t \in \mathbb{R}^N$ for $t = 1, 2, \ldots$. Set $\beta^0 = 0$, the all-zeros vector. For $t = 0, 1, \ldots$, compute

$$z^t = y - A\beta^t + \frac{z^{t-1}}{\tau_{t-1}} \left( P - \frac{||\beta^t||^2}{n} \right), \quad (7)$$

$$\beta^{t+1}_i = \eta_i^t(\beta^t + A^*z^t), \quad \text{for } i = 1, \ldots, N = ML, \quad (8)$$

where quantities with negative indices are set equal to zero. The constants $\{\tau_t\}$, and the estimation functions $\eta_i^t(\cdot)$ are defined as follows for $t = 0, 1, \ldots$.

- Define

$$\tau_0^2 = \sigma^2 + P,$$
$$\tau_{t+1}^2 = \sigma^2 + P(1 - x_{t+1}), \quad t \geq 0, \quad (9)$$

where

$$x_{t+1} = \sum_{\ell=1}^L \frac{P_{\ell}}{P} \mathbb{E} \left[ \exp \left( \frac{\sqrt{n}P_{\ell}}{\tau_{t+1}} \left( U_{\ell}^t + \sqrt{nP_{\ell}} \right) \right) \right], \quad (10)$$

In (10), $\{U_{\ell}^t\}$ are i.i.d. $\mathcal{N}(0, 1)$ random variables for $j \in [M], \ell \in [L]$.

- For $i \in [N]$, define

$$\eta_i^t(s) = \sqrt{nP_{\ell}} \frac{\exp \left( \frac{s\sqrt{nP_{\ell}}}{\tau_{t+1}} \right)}{\sum_{j \in \text{sec}_\ell} \exp \left( \frac{s\sqrt{nP_{\ell}}}{\tau_{t+1}} \right)}, \quad \text{if } i \in \text{sec}_\ell, \ 1 \leq \ell \leq L. \quad (11)$$
The notation $j \in \text{sec}_\ell$ is used as shorthand for “index $j$ in section $\ell$”, i.e., $j \in \{(\ell-1)M+1, \ldots, \ell M\}$. Notice that $\eta^*_i(s)$ depends on all the components of $s$ in the section containing $i$. For brevity, the argument of $\eta^*_i$ in (8) is written as $A^* z^t + \beta^t$, with the understanding that only the components in the section containing $i$ play a role in computing $\eta^*_i$.

Before running the AMP decoder, the constants $\{\tau_t\}$ must be iteratively computed using (9) and (10). This is an offline computation: for given values of $M, L, n$, the expectations in (10) can be computed via Monte Carlo simulation. The relation (9), which describes how $\tau_{t+1}$ is obtained from $\tau_t$, is called state evolution, following the terminology in [5, Section I-C]. In Section 4 we derive closed form expressions for the trajectories of $x^{t+1}$ and $\tau_t^2$ as $n \to \infty$. For now, it suffices to note that for any fixed $R < C$, $\tau_t$ strictly decreases with $t$ for a finite number of steps $T_n$, at which point we have

$$\tau_{T_n+1} \geq \tau_n.$$ 

Having determined $\tau_0, \tau_1, \ldots, \tau_{T_n}$, the decoder iteratively computes codeword estimates $\beta^1, \ldots, \beta^{T_n}$ using (7) and (8). The algorithm is then terminated. Finally, in each section $\ell$ of $\beta^{T_n}$, set the maximum value to $\sqrt{n P_\ell}$ and remaining entries to 0 to obtain the decoded message $\hat{\beta}$.

**Computational Complexity:** The complexity is determined by the matrix-vector multiplications $A \beta^t$ and $A^* z^t$, whose running time is $O(nN)$ if performed in the straightforward way. The remaining operations are $O(N)$. As the number of iterations is finite, the complexity scales linearly with the size of the design matrix.

### 3.1 The Test Statistics $\beta^t + A^* z^t$

To understand the decoder let us first focus on (8), in which $\beta^{t+1}$ is generated from the test statistic

$$s^t := \beta^t + A^* z^t. \quad (12)$$

The AMP update step (8) is underpinned by the following key property of the test statistic: $s^t$ is asymptotically (as $n \to \infty$) distributed as $\beta + \tau_t Z$, where $\tau_t$ is the limit of $\tau_i$, and $Z$ is an i.i.d. $N(0,1)$ random vector independent of the message vector $\beta$. This property, which is proved in Section 5, is due to the presence of the “Onsager” term

$$\frac{z_{t-1}}{\tau_{t-1}^2} \left( P - \frac{\|\beta^t\|^2}{n} \right)$$

in the residue update step (7). The reader is referred to [7, Section I-C] for intuition about role of the Onsager term in the standard AMP algorithm.

In light of the above property, a natural way to generate $\beta^{t+1}$ from $s^t = s$ is

$$\beta^{t+1}(s) = \mathbb{E}[\beta | \beta + \tau_t Z = s], \quad (13)$$

i.e., $\beta^{t+1}$ is the Bayes optimal estimate of $\beta$ given the observation $s^t = \beta + \tau_t Z$. For $i \in \text{sec}_\ell$, $\ell \in [L]$, we have

$$\beta^{t+1}(s) = \mathbb{E}[\beta_i | \beta + \tau_t Z = s] = \mathbb{E}[\beta_i | \{\beta_j + \tau_t Z_j = s_j\}_{j \in \text{sec}_\ell}]$$

$$= \sqrt{n P_\ell} \ P(\beta_i = \sqrt{n P_\ell} | \{\beta_j + \tau_t Z_j = s_j\}_{j \in \text{sec}_\ell})$$

$$= \sqrt{n P_\ell} \ \frac{f(\{\beta_j + \tau_t Z_j = s_j\}_{j \in \text{sec}_\ell} | \beta_i = \sqrt{n P_\ell}) P(\beta_i = \sqrt{n P_\ell})}{\sum_{k \in \text{sec}_\ell} f(\{\beta_j + \tau_t Z_j = s_j\}_{j \in \text{sec}_\ell} | \beta_k = \sqrt{n P_\ell}) P(\beta_k = \sqrt{n P_\ell})} \quad (14)$$
where we have used Bayes Theorem with \( f \) denoting the joint density function of \( \{ \beta_j + \tau_t Z_j \}_{j \in \sec_t} \).

Since \( \beta \) and \( Z \) are independent, with \( Z \) having i.i.d. \( \mathcal{N}(0, 1) \) entries, for each \( k \in \sec_t \) we have

\[
f(\{ \beta_j + \tau_t Z_j = s_j \}_{j \in \sec_t} \mid \beta_k = \sqrt{nP_t}) \propto e^{-(s_k - \sqrt{nP_t})^2 / 2 \tau_t^2} \prod_{j \in \sec_t, j \neq k} e^{-s_j^2 / 2 \tau_t^2} = e^{s_k \sqrt{nP_t} / \tau_t^2} e^{-nP_t / 2 \tau_t^2} \prod_{j \in \sec_t} e^{-s_j^2 / 2 \tau_t^2}. \tag{15}
\]

Using (15) in (14), together with the fact that \( P(\beta_k = \sqrt{nP_t}) = \frac{1}{M} \) for each \( k \in \sec_t \), we obtain

\[
\beta_{i+1}^t(s) = \mathbb{E}[\beta_i \mid \beta + \tau_t Z = s] = \sqrt{nP_t} \frac{\exp \left( \frac{s_i \sqrt{nP_t}}{\tau_t^2} \right)}{\sum_{j \in \sec_t} \exp \left( \frac{s_j \sqrt{nP_t}}{\tau_t^2} \right)}, \tag{16}
\]

which is the expression in (11).

Thus, under the distributional assumption that \( s^t \) equals \( \beta + \tau_t Z \), \( \beta_{i+1}^t \) is the estimate of the message vector \( \beta \) (based on \( s^t \)) that minimizes the expected squared estimation error. Also, for \( i \in \sec_t \), \( \beta_{i+1}^t / \sqrt{nP_t} \) is the posterior probability of \( \beta_i \) being the non-zero entry in section \( \ell \), conditioned on the observation \( s^t = \beta + \tau_t Z \).

### 3.2 State Evolution and its Consequences

We now discuss the role of the quantity \( x_{t+1} \) in the state evolution equations \((9)\) and \((10)\).

**Proposition 3.1.** Under the assumption that \( s^t = \beta + \tau_t Z \), where \( Z \) is i.i.d. \( \mathcal{N}(0, 1) \) and independent of \( \beta \), the quantity \( x_{t+1} \) defined in \((10)\) satisfies

\[
x_{t+1} = \frac{1}{nP} \mathbb{E}[\beta^* x_{t+1}], \quad \text{and} \quad 1 - x_{t+1} = \frac{1}{nP} \mathbb{E}[\| \beta - \beta_{t+1} \|^2]. \tag{17}
\]

**Proof.** For convenience of notation, we relabel the \( N \) i.i.d. random variables \( Z_k, k \in [N] \) as \( \{ U^\ell_j \}, j \in [M], \ell \in [L] \). Then

\[
\frac{1}{nP} \mathbb{E}[\beta^* x_{t+1}] = \frac{1}{nP} \mathbb{E}[\beta^* U^\ell_t (\beta + \tau_t Z)] \overset{(a)}{=} \frac{1}{nP} \sum_{\ell = 1}^L \mathbb{E}[\sqrt{nP_t} U^\ell_t \beta^*(\beta + \tau_t Z)] \overset{(b)}{=} \frac{1}{nP} \sum_{\ell = 1}^L \mathbb{E}[\sqrt{nP_t} \cdot \sqrt{nP_t} \exp \left( \frac{(\sqrt{nP_t} + \tau_t U^\ell_t) \sqrt{nP_t}}{\tau_t^2} \right) \exp \left( \frac{(\sqrt{nP_t} + \tau_t U^\ell_t) \sqrt{nP_t}}{\tau_t^2} \right) + \sum_{j = 2}^M \exp \left( \tau_t U^\ell_j \sqrt{nP_t} / \tau_t^2 \right) \sum_{j = 2}^M \exp \left( \tau_t U^\ell_j \sqrt{nP_t} / \tau_t^2 \right)] = \frac{1}{nP} \mathbb{E}[\beta^* x_{t+1}] = x_{t+1}. \tag{18}
\]

In \((a)\) above, the index of the non-zero term in section \( \ell \) is denoted by \( \text{sent}(\ell) \). \((b)\) is obtained by assuming that \( \text{sent}(\ell) \) is the first entry in section \( \ell \) — this assumption is valid because the prior on \( \beta \) is uniform over \( \mathcal{B}_{M,L}(P_1, \ldots, P_L) \).
Next, consider
\[
\frac{1}{n^2}E[\|\beta - \beta^{t+1}\|^2] = 1 + \frac{E[\|\beta^{t+1}\|^2] - 2E[\beta^*\beta^{t+1}]}{n^2}.
\] (19)
Under the assumption that \(s^t = \beta + \tau_t Z\), recall from Section 3.1 that \(\beta^{t+1}\) can be expressed as \(\beta^{t+1} = E[\beta \mid s^t]\). We therefore have
\[
E[\|\beta^{t+1}\|^2] = E[\|E[\beta]s^t\|^2] = E[(E[\beta]s^t - \beta + \beta)^*E[\beta]s^t] = (a) E[\beta^*E[\beta^*]] = E[\beta^*\beta^{t+1}],
\] (20)
where step (a) follows because \(E[(E[\beta]s^t - \beta)^*E[\beta]s^t] = 0\) due to the orthogonality principle. Substituting (20) in (19) and using (18) yields
\[
\frac{1}{n^2}E[\|\beta - \beta^{t+1}\|^2] = 1 - \frac{E[\beta^*\beta^{t+1}]}{n^2} = 1 - x_{t+1}.
\]

Hence \(x_{t+1}\) can be interpreted as the expectation of the (power-weighted) fraction of correctly decoded sections in step \(t + 1\). We emphasize that this interpretation is accurate only in the limit as \(n, M, L \to \infty\), when \(s^t\) is distributed as \(\beta + \tau Z\), with \(\tau := \lim \tau_t\). In Section 5 (Lemmas 1 and 2), we derive a closed-form expression for \(\bar{x}_{t+1} := \lim x_{t+1}\) under an exponentially decaying power allocation of the form \(P_t \propto 2^{-2Ct/L}\). We show that for rates \(R < C\),
\[
\bar{x}_{t+1} = \frac{(1 + \text{snr}) - (1 + \text{snr})^{1-\xi_t}}{\text{snr}}, \quad \tau^2_t = \sigma^2 + P(1 - \bar{x}_{t+1}), \quad \text{for } t \geq 0
\] (21)
where
\[
\xi_0 = \frac{1}{2C} \log \left(\frac{C}{R}\right), \quad \xi_{t+1} = \min \left\{ \left(\frac{1}{2C} \log \left(\frac{C}{R}\right) + \xi_t\right), 1 \right\}.
\] (22)
A direct consequence of (21) and (22) is that \(\bar{x}_t\) strictly increases with \(t\) until it reaches one, and the number of steps \(T^*\) until \(\bar{x}_{T^*} = 1\) is \(T^* = \left\lceil \frac{2C}{\log(C/R)} \right\rceil\).

The constants \(\{\xi_t\}_{t \geq 0}\) have a nice interpretation in the large system limit: at the end of step \(t + 1\), the first \(\xi_t\) fraction of sections in \(\beta^{t+1}\) will be correctly decodable with high probability, i.e., the true non-zero entry in these sections will have almost all the posterior probability mass. The other \((1 - \xi_t)\) fraction of sections will not be correctly decodable from \(\beta^{t+1}\) as the power allocated to these sections is not large enough. An additional \(\frac{1}{2C} \log \left(\frac{C}{R}\right)\) fraction of sections become correctly decodable in each step until \(T^*\), when all the sections are correctly decodable with high probability.

As \(\bar{x}_t\) increases to 1, (21) implies that \(\tau^2_t\), the variance of the “noise” in the AMP test statistic, decreases monotonically from \(\tau^2_0 = \sigma^2 + P\) down to \(\tau^2_{T^*} = \sigma^2\). In other words, the initial observation \(y = A\beta + w\) is effectively transformed by the AMP decoder into a cleaner statistic \((sT^*) = \beta + w'\), where \(w'\) is Gaussian with the same variance as the measurement noise \(w\).

To summarize, in the large system limit:
- The AMP decoder terminates within a finite number of steps for any fixed \(R < C\).
- At the termination step \(T^*\), the limit \(\frac{1}{n^2}E[\beta - \beta^{T^*}]^2\) equals zero.

For finite-sized dictionaries, the test statistic \(s^t\) will not be precisely distributed as \(\beta + \tau Z\). Nevertheless, computing \(x_{t+1}\) numerically via the state evolution equations (9) and (10) yields an estimate for the expected weighted fraction of correctly decoded sections after each step. Figure 2 shows the trajectory of \(x_t\) vs \(t\) for a SPARC with the parameters specified in the figure. The empirical average of \((\beta^*_t \beta^t)/nP\) matches almost exactly with \(x_t\). The theoretical limit \(\bar{x}_t\) given in (21) is also shown in the figure.
Figure 2: Comparison of state evolution and AMP. The SPARC parameters are $M = 512, L = 1024, \text{snr} = 15, R = 0.7C, P_t \propto 2^{-2Ct/L}$. The average of the 200 trials (green curves) is the dashed red curve, which is almost indistinguishable from the state evolution prediction (black curve).

3.3 Derivation of the AMP

We describe a min-sum-like message passing algorithm for SPARC decoding from which the AMP decoder is obtained as a first-order approximation. The aim is to highlight the similarities and differences from the derivation of the AMP in [7]. The derivation here is not required for the analysis in the remainder of the paper.

Consider the factor graph for the model $y = A\beta + w$, where $\beta \in B_{M,L}(P_1, \ldots, P_L)$. Each row of $A$ corresponds to a constraint (factor) node, while each column corresponds to a variable node. We use the indices $a, b$ to denote factor nodes, and indices $i, j$ to denote variable nodes. The AMP updates in (7)–(8) are obtained via a first-order approximation to the following message passing algorithm that iteratively computes estimates of $\beta$ from $y$.

For $i \in [N], a \in [n]$, set $\beta^0_{i \rightarrow a} = 0$, and compute the following for $t \geq 0$:

$$z^t_{a \rightarrow i} = y_a - \sum_{j \in [N] \setminus i} A_{aj} \beta^t_{j \rightarrow a},$$

$$\beta^{t+1}_{i \rightarrow a} = \eta^t_i(s_{i \rightarrow a}),$$

where $\eta^t_i(.)$ is the estimation function defined in (11), and for $i \in \text{sec}_{\ell}$, the entries of the test statistic $s_{i \rightarrow a} \in \mathbb{R}^M$ are defined as

$$(s_{i \rightarrow a})_i = \sum_{b \in [n] \setminus a} A_{bi} z^t_{b \rightarrow i},$$

$$(s_{i \rightarrow a})_j = \sum_{b \in [n]} A_{bj} z^t_{b \rightarrow j}, \quad j \in \text{sec}_{\ell} \setminus i.$$  

It is useful to compare the $\beta$-update in (24) to the message passing algorithm from which the traditional AMP is derived (cf. equation (1.2) in [7]). In [7], the vector $x$ to be recovered is
assumed to be i.i.d. across entries; hence we have a single estimating function $\eta^t$ in this case, which for $i \in [N], a \in [n]$, generates the message

$$x_{i \rightarrow a}^{t+1} = \eta^t \left( \sum_{b \in [n] \setminus a} A_{bi} z_{b \rightarrow i}^t \right). \tag{26}$$

In (26), each outgoing message from the $i$th variable node depends only on its own incoming messages. In contrast, in (24), each outgoing message from a variable node depends on the incoming messages of all the other nodes in the same section. This is due to the constraint that $\beta^t$ has exactly one non-zero entry in each section, which ensures that entries of $\beta^t$ within each section are dependent, while entries in different sections are mutually independent.

The derivation of the AMP updates in (7)–(8) starting from the messaging passing algorithm (23)–(24) is given in Appendix A.1.

4 Performance of the AMP Decoder

Before giving the main result, we state two lemmas that specify the limiting behaviour of the state evolution parameters defined in (9), (10). Treating $x_t^{t+1}$ in (10) as a function of $\tau$, we can define

$$\bar{x}(\tau) := \lim x(\tau) = \lim_{L \rightarrow \infty} \sum_{\ell=1}^{L} \frac{P_\ell}{P} E \left[ \frac{\exp \left( \sqrt{\frac{nP}{\tau}} (U_\ell^{t} + \sqrt{\frac{nP}{\tau}}) \right)}{\exp \left( \sqrt{\frac{nP}{\tau}} (U_\ell^{t} + \sqrt{\frac{nP}{\tau}}) \right) + \sum_{j=2}^{M} \exp \left( \sqrt{\frac{nP}{\tau}} U_j^{t} \right)} \right], \tag{27}$$

where $\{U_j^{t}\}$ are i.i.d. $\sim \mathcal{N}(0, 1)$ for $j \in [M], \ell \in [L]$.\n
**Lemma 1.** For $t = 0, 1, \ldots$, we have

$$\bar{x}(\tau) := \lim x(\tau) = \lim_{L \rightarrow \infty} \sum_{\ell=1}^{L} \frac{P_\ell}{P} 1\{c_\ell > 2(\ln 2) R \tau^2\} \tag{28}$$

where $c_\ell := \lim_{L \rightarrow \infty} LP_\ell$.

**Proof.** In Appendix A.2

We remark that the $\ln 2$ term appears because $R$ and $C$ are measured in bits. The performance of the AMP decoder will be analyzed with the following exponentially decaying power allocation:

$$P_\ell = P \cdot \frac{2^{2\ell C/L} - 2^{-2\ell C/L}}{2^{-2C} - 2^{-2C}} \cdot 2^{-2\ell C/L}, \quad \ell \in [L]. \tag{29}$$

For the power allocation in (29),

$$c_\ell = \lim_{L \rightarrow \infty} LP_\ell = 2(\ln 2)C(P + \sigma^2) \lim_{L \rightarrow \infty} \left( \frac{\sigma^2}{\sigma^2 + P} \right)^{\ell/L}. \tag{30}$$

**Lemma 2.** For the power allocation $\{P_\ell\}$ given in (29), we have for $t = 0, 1, \ldots$:

$$\bar{x}_t := \lim x_t = \frac{(1 + snr) - (1 + snr)^{1-\xi_{t-1}}}{snr}, \tag{31}$$

$$\bar{x}_t^2 := \lim x_t^2 = \sigma^2 + P(1 - \bar{x}_t) = \sigma^2 (1 + snr)^{1-\xi_{t-1}}. \tag{32}$$
where $\xi_{-1} = 0$, and for $t \geq 0$,

$$
\xi_t = \min \left\{ \left( \frac{1}{2C} \log \left( \frac{C}{R} \right) + \xi_{t-1} \right), 1 \right\}.
$$

(33)

Proof. In Appendix A.3

We observe from Lemma 2 that $\xi_t$ increases in each step by $\frac{1}{2C} \log \left( \frac{C}{R} \right)$ until it equals 1. Also note that $\tau_t$ strictly decreases with $t$ until it reaches $\sigma^2$ (when $\xi_t$ reaches 1), after which it remains constant. Thus the number of steps until $\xi_t$ reaches one (i.e., $\tau_t^2$ stops decreasing) equals

$$
T^* = \left\lceil \frac{2C}{\log(C/R)} \right\rceil.
$$

(34)

Recall from Section 3 that the termination step $T_n$ is the smallest $t$ for which $\tau_t^2 \leq \tau_{t+1}^2$. Hence we have shown that in the large system limit, the number of steps until the AMP decoder terminates is $\lim T_n = T^*$. We remark that since $T_n$ and $T^*$ are both integers, $\lim T_n = T^*$ implies that for sufficiently large $n$ we will have $T_n = T^*$.

Our main result is proved for the following slightly modified AMP decoder, which runs for exactly $T^*$ steps. Set $\beta_0 = 0$ and compute

$$
z^t = y - A\beta^t + \frac{z^{t-1}}{\tau_t^2} \left( P - \frac{||\beta^t||^2}{n} \right),
$$

(35)

$$
\beta_{t+1}^i = \eta_t^i(\beta^t + A* z^t), \quad \text{for } i \in [N]
$$

(36)

where for $i \in \sec_\ell$, $\ell \in [L],

$$
\eta_t^i(s) = \sqrt{nP_\ell} \frac{\exp(s_i \sqrt{nP_\ell / \tau_t^2})}{\sum_{j \in \sec_\ell} \exp(s_j \sqrt{nP_\ell / \tau_t^2})}.
$$

(37)

The only difference from the earlier decoder described in (9)–(11) is that we now use the limiting value $\tau_t^2$ defined in Lemma 2 instead of $\tau_t^2$.

The algorithm terminates after generating $\beta^{T^*}$, where $T^*$ is defined in (34). The decoded codeword $\hat{\beta} \in B_{M,L}(P_1, \ldots, P_L)$ is obtained by setting the maximum of $\beta^{T^*}$ in each section $\ell$ to $\sqrt{nP_\ell}$ and the remaining entries to 0.

The section error rate of a decoder for a SPARC $S$ is defined as

$$
\mathcal{E}_{\sec}(S) := \frac{1}{L} \sum_{\ell=1}^L 1\{\hat{\beta}_\ell \neq \beta_0, \ell\}
$$

(38)

Theorem 1. Fix any rate $R < C$, and $b > 0$. Consider a sequence of rate $R$ SPARCs $\{S_n\}$ indexed by block length $n$, with design matrix parameters $L$ and $M = L^b$ determined according to (6), and an exponentially decaying power allocation given by (29). Then the section error rate of the AMP decoder (described in (35)–(37), and run for $T^*$ steps) converges to zero almost surely, i.e., for any $\epsilon > 0$,

$$
\lim_{n_0 \to \infty} P(\mathcal{E}_{\sec}(S_n) < \epsilon, \forall n \geq n_0) = 1.
$$

(39)
Remarks:

1. The probability measure in (39) is over the Gaussian design matrix $A$, the Gaussian channel noise $w$, and the message $\beta$ distributed uniformly in $B_{M,L}(P_1, \ldots, P_L)$.

2. As in [2], we can construct a concatenated code with an inner SPARC of rate $R$ and an outer Reed-Solomon (RS) code of rate $(1 - 2\epsilon)$. If $M$ is a prime power, a RS code defined over $GF(M)$ defines a one-to-one mapping between a symbol of the RS codeword and a section of the SPARC. The concatenated code has rate $R(1 - 2\epsilon)$, and decoding complexity that is polynomial in $n$. The decoded message $\hat{\beta}$ equals $\beta$ whenever the section error rate of the SPARC is less than $\epsilon$. Thus for any $\epsilon > 0$, the theorem guarantees that the probability of message decoding error for a sequence of rate $R(1 - 2\epsilon)$ SPARC-RS concatenated codes will tend to zero, i.e.,

$$\lim P(\hat{\beta} \neq \beta) = 0.$$ 

The proof of Theorem 1 is given in Section 5.

4.1 Experimental Results and the Effect of Power Allocation

Fig. 3 shows the performance of the AMP at different rates for a SPARC with the parameters specified in the figure. The block length $n$ is determined by the rate $R$ according to (5), e.g., $n = 7680$ for $R = 0.6C$, and $n = 5120$ for $R = 0.9C$.

The upper solid curve shows the average section error rate of the AMP (over 1000 runs) with an exponentially decaying power allocation with $P_\ell \propto 2^{-2\epsilon \ell / L}$. The upper dashed curve is the section error rate prediction obtained from state evolution as follows. Recall from Section 3.2 that $x_{t+1}$ in (10) can be interpreted as the expectation of the (power-weighted) fraction of correctly decoded sections in step $t + 1$. Using arguments similar to Proposition 3.1, we can show that under the assumption that the test statistic $s^t \sim \beta + \tau_t Z$, the (non-weighted) expectation of the correctly decoded sections after step $(t + 1)$ is given by

$$\frac{1}{nP} \sum_{\ell=1}^L \frac{P/L}{P_\ell} \mathbb{E}[\beta^*_t \beta^t_{t+1}] = \sum_{\ell=1}^L \frac{1}{L} \mathbb{E} \left[ \frac{\exp \left( \frac{\sqrt{n}P_\ell}{\tau_t} (U_1^\ell + \sqrt{n}P_\ell) \right)}{\exp \left( \frac{\sqrt{n}P_\ell}{\tau_t} (U_1^\ell + \sqrt{n}P_\ell) \right) + \sum_{j=2}^M \exp \left( \frac{\sqrt{n}P_\ell}{\tau_t} U_j^\ell \right)} \right] := v_{t+1}.$$

(40)

In Fig. 3 the dashed curve on top is the state evolution prediction $(1 - v_{T^n})$ computed using (40), where $T^n$ denotes the termination step. We see that the average section error rate agrees closely with this prediction, but it decays rather slowly with decreasing $R$.

The bottom solid curve shows the section error rate of the AMP with the following power allocation, characterized by two parameters $a, f$. For $f \in [0, 1]$, let

$$P_\ell = \begin{cases} 
\frac{P \cdot 2a2C/L - 1}{1 - 2^{-aC/L}} \cdot 2^{-a2C/L} & \text{for } 1 \leq \ell \leq fL, \\
\frac{P}{(1-f)L} \left( \frac{2a2C(1-f)}{2^{-2aC} - 1} \right) & \text{for } fL + 1 \leq \ell \leq L.
\end{cases}$$

(41)

For intuition, first assume that $f = 1$. Then (41) implies that $P_\ell \propto 2^{-a2C/L}$ for $\ell \in [L]$. Setting $a = 1$ recovers the original power allocation, and $a = 0$ allocates $P/L$ to each section. Increasing $a$ increases the power allotted to the initial sections which makes them more likely to decode correctly, which in turn helps by decreasing the effective noise variance $\tau_t^2$ in subsequent AMP iterations. However, if $a$ is too large, the final sections may have too little power to decode correctly.
Hence we want the parameter $a$ to be large enough to ensure that the AMP gets started on the right track, but not much larger. This intuition can be made precise in the large system limit using Lemma 1, which specifies the condition for a section $\ell$ to be correctly decoded in step $(t + 1)$: the limit of $L P_t$ must exceed a threshold proportional to $R T^2$. For rates close to $C$, we need $a$ to be close to 1 for the initial sections to cross this threshold and get decoding started correctly. On the other hand, for rates such as $R = 0.6C$, $a = 1$ allocates more power than necessary to the initial sections, leading to poor decoding performance in the final sections. Thus the section error rate in the top curve of Fig. 3 can be improved by setting $a$ to an appropriate value smaller than 1.

In addition, we found that further improvement can be obtained by flattening the power allocation in the final sections. For a given $a$, (41) has an exponential power allocation until section $f L$, and distributes the remaining power equally among the last $(1 - f) L$ sections. Flattening boosts the power given to the final sections compared to an exponentially decaying allocation. The two parameters $(a, f)$ let us trade-off between the conflicting objectives of assigning enough power to the initial sections and ensuring that the final sections have enough power to be decoded correctly.

The bottom solid curve in Fig. 3 shows the average section error rates with values of $(a, f)$ obtained via a rough optimization around an initial guess of $a = f = R/C$. Again, these values are close to the state evolution prediction $(1 - v T^n)$ computed using (40). Across trials, we observed good concentration around the average section error rates. For example, at $R = 0.75C$, 674 of the 1000 trials had zero errors, and all but three trials had four or fewer section errors. Further, all the section errors were in the flat part of the power allocation, as expected.

It is evident that judicious power allocation can yield significant improvements in section error rates. An interesting open question is to find good rules of thumb for the power allocation as a function of rate and $\text{snr}$. For any given allocation, one can determine whether the section error
rate goes to zero in the large system limit. Indeed, using Lemma 1 with \( \tau^2 = \hat{\tau}_0^2 = \sigma^2 + P \), we see that those sections \( \ell \) for which the indicator in (28) is positive are decoded in the first step; this also gives the value of \( \bar{x}_1 \). Then with \( \hat{\tau}_2^2 = \sigma^2 + P(1 - \bar{x}_1) \) we can determine which sections are decoded in step 2, and so on. The section error rate goes to zero if and only if \( \bar{x}_T = 1 \), where \( T \) is the termination step in the limit. The proof of this is essentially identical to that of Theorem 1.

Thus Lemma 1 gives a straightforward way to check whether a power allocation is good in the large system limit. This can provide some guidance for the finite length case, but the challenge is to choose between several power allocations for which \( \bar{x}_T = 1 \). One way to compare these allocations may be via the state evolution prediction \( v_T \) from (40), but this needs additional investigation.

The main implementation bottleneck for SPARCs with large dictionaries is memory rather than computation time. One way to reduce the memory footprint at the expense of increased computation time is to generate the \( A \) matrix procedurally during the operation, so that only one section (or column) needs to be stored at once. Such a set-up can be then parallelized in hardware quite effectively. Another way to address this issue and scale the decoder to large values of \( (n, M, L) \) is via structured dictionaries such as Hadamard matrices, as proposed recently in [22]. Investigating the performance-complexity trade-off of SPARCs with Hadamard design matrices is a promising direction for further research.

5 Proof of Theorem 1

The main ingredient in the proof is a technical lemma (Lemma 3) which shows that the performance of the AMP decoder in the large system limit is accurately predicted by the state evolution equations (31) and (32). In particular, it is shown that the squared error \( \frac{1}{n} \| \beta^t - \beta \|_2^2 \) converges almost surely to \( P(1 - \bar{x}_t) \), for \( 0 \leq t \leq T^* \).

Lemma 3 is similar to [7, Lemma 1], with several modifications to account for the differences between the two settings (e.g., the undersampling ratio \( n/N \) in our case goes to zero in the limit).

5.1 Definitions and Notation for Lemma 3

For consistency and ease of comparison, we use notation similar to [7]. Define the following column vectors recursively for \( t \geq 0 \), starting with \( \beta^0 = 0 \) and \( z^0 = y \).

\[
\begin{align*}
    h^{t+1} &= \beta_0 - (A^*z^t + \beta^t), \\
    q^t &= \beta^t - \beta_0, \\
    b^t &= w - z^t, \\
    m^t &= -z^t.
\end{align*}
\]  

(42)

Recall that \( \beta_0 \) is the message vector chosen by the transmitter. Due to the symmetry of the code construction, we can assume that the non-zeros of \( \beta_0 \) are in the first entry of each section.

Define \( \mathcal{F}_{t_1,t_2} \) to be the sigma-algebra generated by

\[
    b^0, ..., b^{t_1-1}, m^0, ..., m^{t_1-1}, h^1, ..., h^{t_2}, q^0, ..., q^{t_2}, \text{ and } \beta_0, w.
\]

Lemma 3 recursively computes the conditional distributions \( b^t|_{\mathcal{F}_{t,t}} \) and \( h^{t+1}|_{\mathcal{F}_{t+1,t}} \), as well as the limiting values of various inner products involving \( h^{t+1}, q^t, b^t, \) and \( m^t \). A key ingredient in proving the lemma is the conditional distribution of the design matrix \( A \) given \( \mathcal{F}_{t_1,t_2} \). For \( t \geq 1 \), let

\[
    \lambda_t = \frac{-1}{\tau_{t-1}^2} \left( P - \frac{\| \beta^2 \|^2}{n} \right).
\]  

(43)
We then have
\[ b^t + \lambda t m^{t-1} = A q^t, \quad (44) \]
which follows from \((7)\) and \((42)\). We also have
\[ h^{t+1} + q^t = A^* m^t. \quad (45) \]
From \((44)\) and \((45)\), we have the matrix equations
\[ X_t = A^* M_t, \quad Y_t = AQ_t, \quad (46) \]
where
\[ X_t = [h^1 + q^0 | h^2 + q^1 | \ldots | h^t + q^{t-1}], \quad Y_t = [b^0 | b^1 + \lambda t m^0 | \ldots | b^{t-1} + \lambda t-1 m^{t-2}], \]
\[ M_t = [m^0 | \ldots | m^{t-1}], \quad Q_t = [q^0 | \ldots | q^{t-1}]. \quad (47) \]

The notation \([c_1 | c_2 | \ldots | c_k]\) is used to denote a matrix with columns \(c_1, \ldots, c_k\). Note that \(M_0\) and \(Q_0\) are the all-zero vector. We use the notation \(m_t\) and \(q_t\) to denote the projection of \(m^t\) and \(q^t\) onto the column space of \(M_t\) and \(Q_t\), respectively. Let \(\alpha_t = (\alpha_0, \ldots, \alpha_{t-1})\) and \(\gamma_t = (\gamma_0, \ldots, \gamma_{t-1})\) be the coefficient vectors of these projections, i.e.,
\[ m^t = \sum_{i=0}^{t-1} \alpha_i m^i, \quad q^t = \sum_{i=0}^{t-1} \gamma_i q^i. \quad (48) \]

The projections of \(m^t\) and \(q^t\) onto the orthogonal complements of \(M^t\) and \(Q^t\), respectively, are denoted by
\[ m^\perp = m^t - m^||, \quad q^\perp = q^t - q^||. \quad (49) \]

Given two random vectors \(X, Y\) and a sigma-algebra \(\mathcal{F}\), \(X \big|_{\mathcal{F}} \overset{d}{=} Y\) implies that the conditional distribution of \(X\) given \(\mathcal{F}\) equals the distribution of \(Y\). For random variables \(X, Y\), the notation \(X \overset{a.s.}{=} Y\) means that \(X\) and \(Y\) are equal almost surely. We use the notation \(\bar{\alpha}(n^{-\delta})\) to denote a vector in \(\mathbb{R}^t\) such that each of its coordinates is \(o(n^{-\delta})\) (here \(t\) is fixed). The \(t \times t\) identity matrix is denoted by \(I_{t \times t}\), and the \(t \times s\) all-zero matrix is denoted by \(0_{t \times s}\).

The notation ‘lim’ is used to denote the large system limit as \(n, M, L \to \infty\); recall that the three quantities are related as \(L \log M = nR\), with \(M = L^b\). We keep in mind that (given \(R\) and \(b\)) the block length \(n\) uniquely determines the dimensions of all the quantities in the system including \(A, \beta_0, w, h^{t+1}, q^t, b^t, m^t\). Thus we have a sequence indexed by \(n\) of each of these random quantities, associated with the sequence of SPARCs \(\{S_n\}\).

Finally, we recall the definition of pseudo-Lipschitz functions from \([7]\).

**Definition 5.1.** A function \(\phi : \mathbb{R}^m \to \mathbb{R}\) is **pseudo-Lipschitz of order** \(k\) (denoted by \(\phi \in PL(k)\)) if there exists a constant \(C > 0\) such that for all \(x, y \in \mathbb{R}^m\),
\[ |\phi(x) - \phi(y)| \leq C(1 + \|x\|^{k-1} + \|y\|^{k-1})\|x - y\|. \quad (50) \]

We will use the fact that when \(\phi \in PL(k)\), there is a constant \(C'\) such that \(\forall x \in \mathbb{R}^m,\)
\[ |\phi(x)| \leq C'(1 + \|x\|^{k}). \quad (51) \]
5.2 Main Lemma

In the lemma below, $\delta \in (0, \frac{1}{2})$ is a generic positive number whose exact value is not required. The value of $\delta$ in each statement of the lemma may be different. We will say that a sequence $x_n$ converges to a constant $c$ at rate $n^{-\delta}$ if $\lim_{n \to \infty} n^\delta (x_n - c) = 0$.

Lemma 3. The following statements hold for $0 \leq t \leq T^*$, where $T^* = \left[ \frac{2C}{\log(C/R)} \right]$.

(a)

$$h^{t+1}_{t+1} = \sum_{i=1}^{t-1} \alpha_i h^{t+1}_{i+1} + \tilde{A}^* m_{i}^t + \tilde{Q}_{t+1} \tilde{o}_{t+1}(n^{-\delta}),$$  \hspace{1cm} (52)

$$b^{t}_{t+1} = \sum_{i=1}^{t-1} \gamma_i b^{t}_{i+1} + \tilde{A} q_{i}^t + \tilde{M}_t \tilde{o}_t(n^{-\delta})$$  \hspace{1cm} (53)

where $\tilde{A}$ is an independent copy of $A$ and the columns of the matrices $\tilde{Q}_t$ and $\tilde{M}_t$ form an orthogonal basis for the column space of $Q_t$ and $M_t$, respectively, such that

$$\tilde{Q}_t^* \tilde{Q}_t = \tilde{M}_t^* \tilde{M}_t = nI_{t \times t}.$$  \hspace{1cm} (54)

(b) i) Consider the following functions $\phi_h$ defined on $\mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$:

$$\phi_h(h_t, \tilde{h}_t, \beta) = \begin{cases} \frac{(h_t)^* \tilde{h}_t}{M}, & 0 \leq r \leq t, \\
\frac{||\tilde{\eta}^* (\beta_t - h_t)||^2}{\log M}, & 0 \leq r \leq s \leq t, \\
\frac{||\tilde{\eta}^* (\beta_t - h_t) - \beta_t||}{\log M}, & 0 \leq r \leq t, \\
\frac{h_t^* \tilde{\eta}^* (\beta_t - h_t) - \beta_t}{\log M}, & 0 \leq r \leq t. \\
\end{cases}$$  \hspace{1cm} (55)

For each function in (55) and arbitrary constants $(a_0, \ldots, a_t, b_0, \ldots, b_t)$, we have:

$$\lim n^\delta \left[ \frac{1}{L} \sum_{t=1}^{L} \phi_h \left( \sum_{r=0}^{t} a_r h_r^{t+1} \sum_{s=0}^{t} b_s h_s^{t+1}, \beta_{0r} \right) \right] = \lim \frac{1}{L} \sum_{t=1}^{L} \left\{ \phi_h \left( \sum_{r=0}^{t} a_r \tilde{r}_r Z_{t,r} \sum_{s=0}^{t} b_s \tilde{r}_s Z_{s,t}, \beta_t \right) \right\}.$$  \hspace{1cm} (56)

where $\tilde{r}_r$ is defined in Lemma 2 and $Z_0, \ldots, Z_t$ are length-$N$ Gaussian random vectors independent of $\beta$, with $Z_{t,r}$ denoting the $r$th section of $Z_t$. For $0 \leq s \leq t$, $\{Z_{s,j}\}_{j \in [N]}$ are i.i.d. $\sim \mathcal{N}(0, 1)$, and for each $i \in [N]$, $(Z_{0,i}, \ldots, Z_{t,i})$ are jointly Gaussian. The inner limit in (56) exists and is finite for each $\phi_h$ in (55).

ii) For all pseudo-Lipschitz functions $\phi_h : \mathbb{R}^{t+2} \to \mathbb{R}$ of order two, we have

$$\lim n^\delta \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_h(b_i^0, \ldots, b_i^t, w_i) - \mathbb{E}\{\phi_h(\tilde{\sigma}_0 \tilde{Z}_0, \ldots, \tilde{\sigma}_t \tilde{Z}_t, \sigma Z_w)\} \right] = 0. \hspace{1cm} (57)$$

where for $s \geq 0$,

$$\sigma_s^2 := \tilde{\sigma}_s^2 - \sigma^2 = P(1 - \tilde{x}_s),$$  \hspace{1cm} (58)

with $\tilde{x}_s$ defined in Lemma 2. The random variables $(\tilde{Z}_0, \ldots, \tilde{Z}_t)$ are jointly Gaussian with $\tilde{Z}_s \sim \mathcal{N}(0, 1)$ for $0 \leq s \leq t$. Further, $(\tilde{Z}_0, \ldots, \tilde{Z}_t)$ are independent of $Z_w \sim \mathcal{N}(0, 1)$. 

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(c) For all \(0 \leq r \leq s \leq t\),
\[
\lim \left( \frac{(h^{r+1}_t)^{s}}{N} \right)_{a.s} = \lim \left( \frac{(m^{r}_t)^{s}}{n} \right)_{a.s} = \mathbb{E}[(\sigma_r \hat{Z}_r - \sigma Z_w)(\sigma_s \hat{Z}_s - \sigma Z_w)],
\]
where the random variables \(\hat{Z}_r, \hat{Z}_s, Z_w\) are those in (54), and \(\sigma_s\) is defined in (58). The convergence rate in both (59) and (60) is \(n^{-\delta}\).

(d) For all \(0 \leq r \leq s \leq t\),
\[
\lim \left( \frac{(h^{r+1}_t)^{s+1}}{n} \right)_{a.s} = \lim \lambda_{s+1} \lim \left( \frac{(m^{r}_t)^{s}}{n} \right)_{a.s} = \frac{-\sigma^2_{s+1}}{\sigma^2 + \sigma^2_{s}} \mathbb{E}[(\sigma_r \hat{Z}_r - \sigma Z_w)(\sigma_s \hat{Z}_s - \sigma Z_w)],
\]
\[
\lim \left( \frac{(b^{r}_t)^{s}}{n} \right)_{a.s} = \lim \left( \frac{(b^{r}_t)^{s}}{n} \right)_{a.s} = \sigma^2_{s}.
\]
The convergence rate in both (61) and (62) is \(n^{-\delta}\).

(e) For all \(t \geq 0\),
\[
\lim \left( \frac{(h^{r+1}_t)^{s}}{n} \right)_{a.s} = 0.
\]

(f) The following hold almost surely.
\[
\lim \left( \frac{q_{i}^{0}}{n} \right)_{a.s}^2 = \bar{\sigma}_0^2 = P, \quad \lim \left( \frac{q_{i}^{1}}{n} \right)_{a.s}^2 = \bar{\sigma}_r^2 \left( 1 - \frac{\sigma^2_r}{\sigma^2_{r-1}} \right) \quad \text{for} \quad 1 \leq r \leq t,
\]
\[
\lim \left( \frac{m_{i}^{0}}{n} \right)_{a.s}^2 = \bar{\sigma}_s^2 = \sigma^2 + P, \quad \lim \left( \frac{m_{i}^{1}}{n} \right)_{a.s}^2 = \bar{\sigma}_s^2 - u^* C^{-1} u, \quad \text{for} \quad 1 \leq s \leq t - 1,
\]
where for \(1 \leq i, j \leq s\),
\[
u_i = \mathbb{E} \left[ (\sigma_s \hat{Z}_s - \sigma Z_w)(\sigma_i \hat{Z}_i - \sigma Z_w) \right], \quad n_{ij} = \mathbb{E} \left[ (\sigma_{i-1} \hat{Z}_{i-1} - \sigma Z_w)(\sigma_{j-1} \hat{Z}_{j-1} - \sigma Z_w) \right].
\]
The limits in (64) and (65) are strictly positive for \(r, s < T^*\).

The lemma is proved in Section 5.5. The main difference from [7, Lemma 1] is in part (b).i of the lemma, which is a key ingredient in proving Theorem 1. The functions involving \(\eta\) in (55) all act section-wise when applied to vectors in \(\mathbb{R}^N\), in contrast to the component-wise functions considered in [7] (and in part (b).ii above). To prove (56) for the section-wise functions as the section size \(M \to \infty\), we need that the limits in the other parts of the lemma (particularly in (52) and (63)) have convergence rates of \(n^{-\delta}\) for some \(\delta > 0\). Minimum rates of convergence were not needed for [7, Lemma 1].

5.3 Proof of Theorem 1

From the definition in (38), the event that the section error rate is larger than \(\epsilon\) can be written as
\[
\{\mathcal{E}_{sec}(S_n) > \epsilon\} = \left\{ \sum_{t=1}^{L} 1\{\hat{\beta}_t \neq \beta_0\} > L\epsilon \right\}.
\]
When a section $\ell$ is decoded in error, the correct non-zero entry has no more than half the total mass of section $\ell$ at the termination step $T^*$. That is,

$$\beta_{\text{sent}}^{T^*} \leq \frac{1}{2} \sqrt{n P_\ell} \quad (67)$$

where $\text{sent}(\ell)$ is the index of the non-zero entry in section $\ell$ of the true message $\beta_0$. Since $\beta_{\text{sent}}(\ell) = \sqrt{n P_\ell}$, we have

$$1\{\hat{\beta}_\ell \neq \beta_0\} \Rightarrow \|\beta_{T^*} - \beta_0\|_2 \geq \frac{n P_\ell}{4}, \quad \ell \in [L]. \quad (68)$$

Hence when (66) holds, we have

$$\|\beta_{T^*} - \beta_0\|_2 = \sum_{\ell=1}^L \|\beta_{T^*} - \beta_0\|_2 \geq \sum_{\ell=1}^L 1\{\hat{\beta}_\ell \neq \beta_0\} \frac{n P_\ell}{4} \geq L \epsilon \frac{n P_L}{4} \geq \frac{n \epsilon \sigma^2 \ln(1 + \text{snr})}{4}, \quad (69)$$

where (a) follows from (68); (b) is obtained using (66), and the fact that $P_\ell > P_L$ for $\ell \in [L - 1]$ for the exponentially decaying power allocation in (29); (c) is obtained using the first-order Taylor series lower bound $LP_L \geq \sigma^2 \ln(1 + \frac{P^2}{\sigma^2})$. We therefore conclude that

$$\{E_{\text{sec}}(S_n) > \epsilon\} \Rightarrow \left\{\frac{\|\beta_{T^*} - \beta_0\|_2}{n} \geq \frac{\epsilon \sigma^2 \ln(1 + \text{snr})}{4}\right\}. \quad (70)$$

Now, from (60) of Lemma 3(c), we know that

$$\lim_{n \to \infty} \frac{\|\beta_{T^*} - \beta_0\|_2}{n} = \lim_{n \to \infty} \frac{\|q_{T^*}\|_2}{n} \overset{a.s.}{\to} P(1 - x_{T^*}) = 0, \quad (71)$$

where (a) follows from Lemma 2, which implies that $\xi_{T^* - 1} = 1$ for $T^* = \left\lceil \frac{2C}{\log(C/R)} \right\rceil$, and hence $x_{T^*} = 1$. Thus we have shown in (71) that $\frac{\|\beta_{T^*} - \beta_0\|_2}{n}$ converges almost surely to zero, i.e.,

$$\lim_{n_0 \to \infty} P\left(\frac{\|\beta_{T^*} - \beta_0\|_2}{n} < \epsilon, \forall n \geq n_0\right) = 1 \quad (72)$$

for any $\epsilon > 0$. From (70), this implies that for $\epsilon' = \frac{4 \epsilon}{\sigma^2 \ln(1 + \text{snr})}$,

$$\lim_{n_0 \to \infty} P\left(E_{\text{sec}}(S_n) \leq \epsilon', \forall n \geq n_0\right) = 1. \quad (73)$$

### 5.4 Useful Probability and Linear Algebra Results

We now list some results that will be used in the proof of Lemma 3. Most of these can be found in [7, Section III.G], but we summarize them here for completeness.

**Fact 1.** Let $u \in \mathbb{R}^N$ and $v \in \mathbb{R}^n$ be deterministic vectors such that $\lim_{n \to \infty} \frac{\|u\|^2}{n}$ and $\lim_{n \to \infty} \frac{\|v\|^2}{n}$ both exist and are finite. Let $A \in \mathbb{R}^{n \times N}$ be a matrix with independent $\mathcal{N}(0, 1/n)$ entries. Then:

(a) \quad \tilde{A}u \overset{d}{=} \frac{\|u\|}{\sqrt{n}} Z_u \quad \text{and} \quad \tilde{A}^*v \overset{d}{=} \frac{\|v\|}{\sqrt{n}} Z_v, \quad (74)
where \( Z_u \in \mathbb{R}^n \) and \( Z_v \in \mathbb{R}^N \) are Gaussian random vectors distributed as \( \mathcal{N}(0, I_{n \times n}) \) and \( \mathcal{N}(0, I_{N \times N}) \), respectively. Consequently,

\[
\lim_{n \to \infty} \frac{\| \bar{A} u \|^2}{n} \xrightarrow{a.s.} \lim_{n \to \infty} \frac{\| u \|^2}{n} = \lim_{n \to \infty} \frac{\| Z_{u,i} \|^2}{n} \xrightarrow{a.s.} 0
\]

(75)

\[
\lim_{n \to \infty} \frac{\| \bar{A}^* v \|^2}{N} \xrightarrow{a.s.} \lim_{n \to \infty} \frac{\| v \|^2}{n} = \lim_{n \to \infty} \frac{\| Z_{v,j} \|^2}{N} \xrightarrow{a.s.} 0
\]

(76)

(b) Let \( W \) be a \( d \)-dimensional subspace of \( \mathbb{R}^n \) for \( d \leq n \). Let \( (w_1, \ldots, w_d) \) be an orthogonal basis of \( W \) with \( \| w_i \|^2 = n \) for \( i \in [d] \), and let \( P_W \) denote the orthogonal projection operator onto \( W \). Then for \( D = [w_1 \mid \ldots \mid w_d] \), we have \( P_W \bar{A} u \equiv \frac{\| u \|}{\sqrt{n}} D x \) where \( x \in \mathbb{R}^d \) is a random vector with i.i.d. \( \mathcal{N}(0, 1/n) \) entries. Therefore \( \lim_{n \to \infty} n^{-\delta} \| x \| \xrightarrow{a.s.} 0 \) for any constant \( \delta \in [0, 0.5) \). (The limit is taken with \( d \) fixed.)

**Fact 3.** (Strong Law for Triangular Arrays) Let \( \{X_{n,i} : i \in [n], n \geq 1\} \) be a triangular array of random variables such that for each \( n \) (\( X_{n,1}, \ldots, X_{n,n} \)) are mutually independent, have zero mean, and satisfy

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} |X_{n,i}|^{2+\kappa} \leq cn^{\kappa/2} \quad \text{for some } \kappa \in (0, 1) \text{ and } c < \infty.
\]

(77)

Then \( \frac{1}{n} \sum_{i=1}^{n} X_{n,i} \to 0 \) almost surely as \( n \to \infty \).

**Fact 4.** Let \( Z_1, \ldots, Z_t \) be jointly Gaussian random variables with zero mean and an invertible covariance matrix \( C \). Then

\[
\text{Var}(Z_t \mid Z_1, \ldots, Z_{t-1}) = \mathbb{E}[Z_t^2] - \mathbb{E}[Z_t|Z_1, \ldots, Z_{t-1}] - u^* C^{-1} u,
\]

where for \( i \in [t-1] \), \( u_i = \mathbb{E}[Z_i Z_t] \).

**Fact 5.** Let \( Z_1, \ldots, Z_t \) be jointly Gaussian random variables such that for all \( i \in [t] \),

\[
\mathbb{E}[Z_i^2] \leq K \quad \text{and} \quad \text{Var}(Z_i \mid Z_1, \ldots, Z_{i-1}) \geq c_i,
\]

for some strictly positive constants \( c_1, \ldots, c_t \). Let \( Y \) be a random variable defined on the same probability space, and let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a Lipschitz function with \( z \to g(z, Y) \) non-constant with positive probability. Then there exists a positive constant \( c'_t \) such that

\[
\mathbb{E}[|g(Z_t, Y)|^2] - u^* C^{-1} u > c'_t,
\]

where \( u \in \mathbb{R}^{t-1} \) and \( C \in \mathbb{R}^{(t-1) \times (t-1)} \) are given by

\[
u_i = \mathbb{E}[g(Z_t, Y)g(Z_i, Y)], \quad C_{ij} = \mathbb{E}[g(Z_i, Y)g(Z_j, Y)], \quad i, j \in [t-1].
\]

(The constant \( c'_t \) depends only on the \( K \), the random variable \( Y \) and the function \( g \).)
Fact 6 (Stein’s lemma). For zero-mean jointly Gaussian random variables $Z_1, Z_2$, and any function $f : \mathbb{R} \rightarrow \mathbb{R}$ for which $\mathbb{E}[Z_1 f(Z_2)]$ and $\mathbb{E}[f'(Z_2)]$ both exist, we have $\mathbb{E}[Z_1 f(Z_2)] = \mathbb{E}[Z_1 Z_2] \mathbb{E}[f'(Z_2)]$.

Fact 7. Let $v_1, \ldots, v_t$ be a sequence of vectors in $\mathbb{R}^n$ such that for $i \in [t]$

$$\frac{1}{n} \|v_i - P_{i-1}(v_i)\|^2 \geq c,$$

where $c$ is a positive constant and $P_{i-1}$ is the orthogonal projection onto the span of $v_1, \ldots, v_{i-1}$. Then the matrix $C \in \mathbb{R}^{t \times t}$ with $C_{ij} = v_i^* v_j / n$ has minimum eigenvalue $\lambda_{\min} \geq c'$, where $c'$ is a strictly positive constant (depending only on $c$ and $t$).

Fact 8. Let $\{S_n\}_{n \geq 1}$ be a sequence of $t \times t$ matrices such that $\lim_{n \to \infty} S_n = S_\infty$ where the limit is element-wise. Then if $\lim \inf_{n \to \infty} \lambda_{\min}(S_n) \geq c$ for a positive constant $c$, then $\lambda_{\min}(S_\infty) \geq c$.

5.5 Proof of Lemma 3

A key ingredient in the proof is the distribution of $A$ conditioned on the sigma algebra $\mathcal{A}_{t_1, t}$ where $t_1$ is either $t + 1$ or $t$. Observing that conditioning on $\mathcal{A}_{t_1, t}$ is equivalent to conditioning on the linear constraints

$$AQ_{t_1} = Y_{t_1}, \ A^* M_t = X_t,$$

the following lemma from [7] specifies the conditional distribution $A | \mathcal{A}_{t_1, t}$.

**Lemma 4.** [4] Lemma 10 | For $t_1 = t + 1$ or $t$, the conditional distribution of the random matrix $A$ given $\mathcal{A}_{t_1, t}$ satisfies

$$A | \mathcal{A}_{t_1, t} \overset{d}{=} E_{t_1, t} + P_{M_t} \hat{A} P_{Q_{t_1}}.$$

Here $\hat{A} \overset{d}{=} A$ is random matrix independent of $\mathcal{A}_{t_1, t}$, and $P_{M_t} = I - P_{M_t}$ where $P_{M_t} = M_t(M_t^* M_t)^{-1} M_t^*$ is the orthogonal projection matrix onto the column space of $M_t$; similarly, $P_{Q_{t_1}} = I - P_{Q_{t_1}}$, where $P_{Q_{t_1}} = Q_{t_1}(Q_{t_1}^* Q_{t_1})^{-1} Q_{t_1}^*$. The matrix $E_{t_1, t} = \mathbb{E}[A | \mathcal{A}_{t_1, t}]$ is given by

$$E_{t_1, t} = \mathbb{E}[AP_{Q_{t_1}} + P_{M_t} \hat{A} P_{Q_{t_1}} | AQ_{t_1} = Y_{t_1}, A^* M_t = X_t] = Y_{t_1}(Q_{t_1}^* Q_{t_1})^{-1} Q_{t_1}^* + M_t(M_t^* M_t)^{-1} X_t^* - M_t(M_t^* M_t)^{-1} M_t^* Y_{t_1}(Q_{t_1}^* Q_{t_1})^{-1} Q_{t_1}^*.$$

**Lemma 5.** [7] Lemma 12 | For the matrix $E_{t_1, t}$ defined in Lemma 4, the following hold:

$$E_{t_1, t}^* m_t^* = X_t(M_t^* M_t)^{-1} M_t^* m_t^*, \quad Q_{t+1}(Q_{t+1}^* Q_{t+1})^{-1} Y_{t+1} m_t^*, \quad E_{t, t} q_t^* = Y_t(Q_t^* Q_t)^{-1} Q_t q_t^* + M_t(M_t^* M_t)^{-1} X_t^* q_t^*,$$

where $m_t^*, m_t^*, q_t^*, q_t^*$ are defined in (48) and (49).

We mention that Lemmas 4 and 5 can be applied only when $M_t^* M_t$ and $Q_t^* Q_{t_1}$ are invertible. We are now ready to prove Lemma 3. The proof proceeds by induction on $t$. We label as $H_{t+1}$ the results (52), (50), (59), (61), (63), (64) and similarly as $B_t$ the results (53), (57), (60), (62), (65). The proof consists of four steps:

---

[2] While conditioning on the linear constraints, we emphasize that only $A$ is treated as random.
1. \( B_0 \) holds.
2. \( \mathcal{H}_1 \) holds.
3. If \( B_r, \mathcal{H}_s \) holds for all \( r < t \) and \( s \leq t \), then \( B_t \) holds.
4. if \( B_r, \mathcal{H}_s \) holds for all \( r \leq t \) and \( s \leq t \), then \( \mathcal{H}_{t+1} \) holds.

5.5.1 Step 1: Showing \( B_0 \) holds

We wish to show that (53), (57), (60), (62), and (65) hold when \( t = 0 \).

(a) The sigma-algebra \( \mathcal{F}_{0,0} \) is generated by \( q^0 = -\beta_0 \) and \( u \). Both \( M_0 \) and \( Q_0 \) are empty matrices, and therefore \( \tilde{M}_0 \) is an empty matrix and \( q^0_\perp = q^0 \). The result follows by noting that \( b^0 = -A\beta_0 = Aq_0 \), from the definitions in (42).

(b) We will first use Fact 2 to show that

\[
\lim n^\delta \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_b(b^0_i, w_i) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_A \{ \phi_b(b^0_i, w_i) \} \right] \overset{a.s.}{=} 0. \tag{83}
\]

To apply Fact 2 we need to verify that

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[n^\delta \phi(b^0_i, w_i)] - n^\delta \mathbb{E}_A \{ \phi_b(b^0_i, w_i) \} \leq cn^{\kappa/2}. \tag{84}
\]

for some \( 0 \leq \kappa \leq 1 \) and \( c \) some constant. Using \( b^0 = Aq^0 \),

\[
\mathbb{E}[\phi_b(b^0_i, w_i)] - \mathbb{E}_A \{ \phi_b([Aq^0]_i, w_i) \} = \mathbb{E}_{\tilde{A}_i} \{ \phi_b([\tilde{A}q^0]_i, w_i) \} - \mathbb{E}_A \{ \phi_b([Aq^0]_i, w_i) \} \leq \mathbb{E}_{\tilde{A}_i} \{ \phi_b([\tilde{A}q^0]_i, w_i) \} - \mathbb{E}_A \{ \phi_b([Aq^0]_i, w_i) \} \leq \mathbb{E}_{\tilde{A}_i} \{ \phi_b([\tilde{A}q^0]_i, w_i) \} - \mathbb{E}_A \{ \phi_b([Aq^0]_i, w_i) \} \leq c_0 \mathbb{E}_{\tilde{A}_i} \{ \phi_b([\tilde{A}q^0]_i, w_i) \} - \mathbb{E}_A \{ \phi_b([Aq^0]_i, w_i) \} \leq c_1 + c_2 |w_i|^{2+\kappa}, \tag{85}
\]

where \( c', c_0, c_1, c_2 \) are positive constants. In the chain above, \( (a) \) uses Jensen’s inequality, \( (b) \) holds because \( \phi_b \in \text{PL}(2) \), and \( (c) \) is obtained using the fact that \( [Aq^0]_i = -[A\beta_0]_i = \sqrt{P}Z \), and \( [\tilde{A}q^0]_i = \sqrt{PZ} \), where \( Z, \tilde{Z} \) are i.i.d. \( \mathcal{N}(0, 1) \). Using (85) in (84), we obtain

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[n^\delta \phi(b^0_i, w_i)] - n^\delta \mathbb{E}_A \{ \phi_b(b^0_i, w_i) \} \leq \frac{n^{\delta(\kappa+2)}}{n} \sum_{i=1}^{n} (c_1 + c_2 |w_i|^{2+\kappa}) \leq cn^{\kappa/2}, \tag{86}
\]

for \( \delta < \kappa/2 \kappa+2 \) since the \( w_i \)’s are i.i.d. \( \mathcal{N}(0, \sigma^2) \). Thus (83) holds.

Since \( b^0 = Aq^0 = \sqrt{P}Z \), where \( Z \in \mathbb{R}^n \) is i.i.d. \( \mathcal{N}(0, 1) \), we have

\[
\mathbb{E}_A \{ \phi_b(b^0_i, w_i) \} = \mathbb{E}_A \{ \phi_b([Aq^0]_i, w_i) \} = \mathbb{E}_{Z_0} \{ \phi_b(\sigma_0 Z_0, w_i) \}, \tag{87}
\]
where $\sigma_0^2 = P$ and $Z_0 \sim \mathcal{N}(0, 1)$. Thus

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[A[b_i, w_i]] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z_0[b_i, \sigma_0 Z_0, \sigma Z_w]] \rightarrow_{\text{a.s.}} \mathbb{E}[\phi_b(\sigma_0 Z_0, \sigma Z_w)]$$

(88)
due to Fact 3, which also guarantees that the convergence rate in (88) is $o(n^{-\delta})$. Combining (88) and (88) yields the result.

(c) Using the definition $b^0 = Aq^0$ and conditioning on $q^0 = -\beta_0$, we have using Fact 1(a):

$$\frac{\|b^0\|^2}{n} = \frac{\|Aq^0\|^2}{n} = \frac{\|q^0\|^2}{n} \sum_{i=1}^{n} Z_i^2$$

(89)

where $Z_1, \ldots, Z_n$ are i.i.d. $\mathcal{N}(0, 1)$. Taking the limit of (89) gives the desired result since $\|q^0\|^2 / n = P$ and by the central limit theorem, $\frac{1}{n} \sum_{i=1}^{n} Z_i^2 - 1 = o(n^{-\delta})$ almost surely for any $\delta \in (0, 1/2)$.

(d) Since $m_0^0 = b^0 - w_0$, $(b^0)^* m_0^0 = \|b^0\|^2 - (b^0)^* w$. By Step 1(c) above, $\frac{\|b^0\|^2}{n} \rightarrow P$ almost surely at rate $n^{-\delta}$. Using using Fact 1(a), we have

$$(b^0)^* w = (Aq^0)^* w = \frac{(q^0)^* A^* w}{n} = \frac{\|q^0\| \|w\|}{\sqrt{n}} \frac{Z}{\sqrt{n}} = \sqrt{\frac{P}{n}} \|w\| \frac{Z}{\sqrt{n}}$$

(90)

where the random variable $Z \sim \mathcal{N}(0, 1)$ is independent of $w$. The result follows by noting that $(\frac{\|w\|}{\sqrt{n}} - \sigma)$ is $o(n^{-\delta})$ almost surely.

(f) Since $M_0$ is the empty matrix, $m_0^0 = m_0 = (b^0 - w)$. Applying $B_0(b)$ to the function $\phi_b(b^0, w_i) = (b^0 - w_i)^2$, we obtain

$$\lim \frac{\|m_0^0\|^2}{n} = \lim \frac{1}{n} \sum_{i=1}^{n} (b^0_i - w_i)^2 = \lim \frac{1}{n} \sum_{i=1}^{n} \phi_0(b^0, w_i) \rightarrow_{\text{a.s.}} \mathbb{E}\{(\sigma_0 Z_0 - \sigma Z_w)^2\} = \sigma_2^2 + \sigma_0^2$$

(91)

5.5.2 Step 2: Showing $\mathcal{H}_1$ holds

(a) The conditioning sigma-algebra $\mathcal{J}_{1,0}$ is generated by $b^0, m_0, q^0 = -\beta_0$ and $w$. From Lemmas 4 and 5, we have

$$A|\mathcal{J}_{1,0} = Y_1(Q_1^{-1}Q_1)^{-1}Q_1^* + \tilde{A}P_{Q_1}^\perp = \frac{b^0 q^0}{\|q^0\|^2} + \tilde{A}P_{q^0}^\perp$$

(92)
as $M_0$ and $Q_0$ are empty matrices, and $Q_1 = q^0$. Since $h^1 = A^* m_0^0 - q^0$, (92) implies

$$h^1|\mathcal{J}_{1,0} = \frac{q^0 b^0 m_0}{\|q^0\|^2} + P_{q^0}^\perp \tilde{A}^* m_0^0 - q^0$$

(93)

First note that

$$\frac{q^0 b^0 m_0}{\|q^0\|^2} - q^0 = \frac{q^0}{P} \left(\frac{b^0 m_0}{n} - P\right) \rightarrow_{\text{a.s.}} \frac{q^0}{P} o(n^{-\delta}),$$

(94)

where the last equality follows from $B_0(d)$. Substituting (94) in (93), we see that the result follows if we prove that

$$P_{q^0}^\perp \tilde{A}^* m_0^0 \xrightarrow{d} \tilde{A}^* m_0^0 + \frac{q^0}{\sqrt{P}} o(n^{-\delta}).$$

(95)
To show (95), we observe that \( P^0 \hat{A} m^0 = \hat{A} m^0 - P^0 \hat{A} m^0 \). Further, since \( M^0 \) is an empty matrix \( \hat{A} m^0 = \hat{A} m^0 \). Thus, all that is left to show is that \( P^0 \hat{A} m^0 = \frac{q^0}{\sqrt{p}} o(n^{-\delta}) \) almost surely. Since \( q^0, m^0 \) are in the conditioning sigma-algebra and are independent of \( \hat{A} \), we obtain using Fact 1(a),

\[
P^0 \hat{A} m^0 = \frac{q^0}{\|q^0\|} \hat{A} m^0 = \frac{q^0}{\|q^0\|} \left( \frac{q^0}{\|q^0\|} \hat{A} m^0 \right) = \frac{q^0}{\sqrt{p}} \left( \frac{\|m^0\|}{\sqrt{n}} \right),
\]

where \( Z \) is a standard normal random variable. It was shown in \([91]\) that \( \frac{\|m^0\|^2}{n} \overset{a.s.}{\to} \sigma^2 + \sigma_0^2 = \frac{\tau_0^2}{n} \), which implies that that \( \frac{Z}{\sqrt{n}} \overset{a.s.}{\to} o(n^{-\delta}) \) almost surely.

(c) From \( \mathcal{H}_1(a) \) shown above, \( h^1|_{\mathcal{H}_1} \overset{d}{=} \hat{A} m^0 + \frac{q^0}{\sqrt{p}} o(n^{-\delta}) \), and so

\[
\frac{\|h^1\|^2}{N} \overset{d}{=} \frac{\|\hat{A} m^0\|^2}{N} + \frac{\|q^0\|^2}{N P} \sigma_1(n^{-2\delta}) - 2\frac{(q^0)^* \hat{A} m^0}{N \sqrt{p} \sqrt{N}} o(n^{-\delta}).
\]

The last two terms in (97) are \( o(n^{-\delta}) \). Indeed, \( \|q^0\|^2 = n P \), \( \frac{N}{n} = \Theta\left(\frac{\log M}{M}\right) \), and by Fact 1(a),

\[
\frac{(q^0)^* \hat{A} m^0}{N \sqrt{p} \sqrt{N}} \overset{d}{=} \frac{\|m^0\|}{\sqrt{n}} \frac{\|q^0\|}{\sqrt{N P}} \frac{Z}{\sqrt{N}} \text{ where } Z \sim \mathcal{N}(0, 1).
\]

It was shown in \([91]\) that \( \frac{\|m^0\|}{\sqrt{n}} \overset{a.s.}{\to} \tau_0 \), hence the term in (98) is \( o(n^{-\delta}) \).

Applying Fact 1(a) to the first term in (97), we obtain

\[
\lim \frac{\|\hat{A} m^0\|^2}{N} \overset{a.s.}{=} \lim \frac{\|m^0\|^2}{n} \|Z\|^2 \overset{a.s.}{=} \frac{\tau_0^2}{n} \cdot 1
\]

where \( Z \in \mathbb{R}^N \) is i.i.d. \( \mathcal{N}(0, 1) \). By \( \mathcal{B}_0(b) \) and the central limit theorem, the convergence rate in (99) is \( n^{-\delta} \).

(b) The proof of this part involves several claims which are fairly straightforward but tedious to verify, so we only give the main steps, referring the reader to [25] for the details. From \( \mathcal{H}_1(a) \),

\[
h^1|_{\mathcal{H}_1} \overset{d}{=} \hat{A} m^0 + \tilde{Q}_1 \tilde{\sigma}_1(n^{-\delta}),
\]

where \( \hat{A} \) is an independent copy of \( A \) and \( \tilde{Q}_1 = \frac{q^0}{\sqrt{p}} \). Then

\[
\phi_h(a_0 h^1, b_0 h^1, \beta_0,)|_{\mathcal{H}_1} \overset{d}{=} \phi_h \left( a_0 [\hat{A} m^0]|_\ell + a_0 [\tilde{Q}_1 \tilde{\sigma}_1(n^{-\delta})]|_\ell, b_0 [\hat{A} m^0]|_\ell + b_0 [\tilde{Q}_1 \tilde{\sigma}_1(n^{-\delta})]|_\ell, \beta_0 \right).
\]

First, we show that the error term \( \tilde{Q}_1 \tilde{\sigma}_1(n^{-\delta'}) \) can be dropped. For each section \( \ell \in [L] \), let \( h_\ell = a_0 [\hat{A} m^0]|_\ell \) and \( \Delta_\ell = a_0 [\tilde{Q}_1 \tilde{\sigma}_1(n^{-\delta'})]|_\ell \). Similarly define \( \hat{h}_\ell \) and \( \hat{\Delta}_\ell \) with \( a_0 \) replaced by \( b_0 \).

Then it is shown in [25] that for each of the functions in (55), we almost surely have

\[
\frac{1}{L} \sum_{\ell=1}^L \left| \phi_h(h_\ell + \Delta_\ell, \hat{h}_\ell + \hat{\Delta}_\ell, \beta_0,)- \phi_h(h_\ell, \hat{h}_\ell, \beta_0) \right| = \Theta(n^{-\delta' \log M}).
\]

for some \( \delta' > 0 \). Choosing \( \delta \in (0, \delta') \) ensures that we can drop the \( \tilde{Q}_1 \tilde{\sigma}_1(n^{-\delta}) \) terms.
In what follows, we use the notation $h_\ell[\bar{A}] = a_0[\bar{A}^*m^0]_\ell$ and $\bar{h}_\ell[\bar{A}] = b_0[\bar{A}^*m^0]_\ell$, making explicit the dependence on $\bar{A}$. We will appeal to Fact 2 to show that

$$
\lim n^\delta \left[ \frac{1}{L} \sum_{\ell=1}^L \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) - \frac{1}{L} \sum_{\ell=1}^L \mathbb{E}_{\bar{A}} \left\{ \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) \right\} \right] \overset{a.s.}{=} 0 \quad (102)
$$

To invoke Fact 2 (conditionally on $\mathcal{F}_{1,0}$), we need to verify that

$$
\frac{1}{L} \sum_{\ell=1}^L \mathbb{E}_{\bar{A}} \left| n^\delta \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) - n^\delta \mathbb{E}_{\bar{A}} \left\{ \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) \right\} \right|^{2+\kappa} \leq c L^{\kappa/2} \quad (103)
$$

for some $0 \leq \kappa \leq 1$ and some constant $c$. In (103), $\bar{A}, \bar{\bar{A}}$ are i.i.d. copies of $A$. From Jensen’s inequality, we have

$$
\mathbb{E}_{\bar{A}} \left| \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) - \mathbb{E}_{\bar{A}} \left\{ \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) \right\} \right|^{2+\kappa} \leq \mathbb{E}_{\bar{A}, \bar{\bar{A}}} \left| \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) - \phi_h \left( h_\ell[\bar{\bar{A}}], \bar{h}_\ell[\bar{\bar{A}}], \beta_{0\ell} \right) \right|^{2+\kappa},
$$

and in (25), it is shown that for each function in (55),

$$
\mathbb{E}_{\bar{A}, \bar{\bar{A}}} \left| \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0\ell} \right) - \phi_h \left( h_\ell[\bar{\bar{A}}], \bar{h}_\ell[\bar{\bar{A}}], \beta_{0\ell} \right) \right|^{2+\kappa} \overset{a.s.}{=} O((\log M)^{2+\kappa}), \quad \ell \in [L]. \quad (104)
$$

The bound in (104) implies (103) holds if $\delta(2 + \kappa)$ is chosen to be smaller than $\kappa/2$. (Recall that $L = \Theta(n/\log n)$). We have thus shown (102).

Recall that for each $\ell \in [L]$, we have $[\bar{A}^*m^0]_\ell \overset{d}{=} (\|m^0\|/\sqrt{n}) Z_{0\ell}$ where $Z_{0\ell} \sim \mathcal{N}(0, I_{M \times M})$. Therefore, in (102), $h_\ell[\bar{A}] \overset{d}{=} a_0 \frac{\|m^0\|}{\sqrt{n}} Z_{0\ell}$, and $\bar{h}_\ell[\bar{A}] \overset{d}{=} b_0 \frac{\|m^0\|}{\sqrt{n}} Z_{0\ell}$. We will next show that

$$
\lim n^\delta \left[ \frac{1}{L} \sum_{\ell=1}^L \mathbb{E}_{Z_0} \left| \phi_h \left( a_0 \frac{\|m^0\|}{\sqrt{n}} Z_{0\ell}, b_0 \frac{\|m^0\|}{\sqrt{n}} Z_{0\ell}, \beta_{0\ell} \right) - \phi_h \left( a_0 \tilde{\tau}_0 Z_{0\ell}, b_0 \tilde{\tau}_0 Z_{0\ell}, \beta_{0\ell} \right) \right| \right] \overset{a.s.}{=} 0. \quad (105)
$$

Let us redefine $h_\ell = a_0 \frac{\|m^0\|}{\sqrt{n}} Z_{0\ell}$ and $\Delta_\ell = a_0 \left( \tilde{\tau}_0 - \frac{\|m^0\|}{\sqrt{n}} \right) Z_{0\ell}$. Define $\bar{h}_\ell$ and $\bar{\Delta}_\ell$ similarly with $b_0$ replacing $a_0$. Then (105) can be written as

$$
\lim n^\delta \left[ \frac{1}{L} \sum_{\ell=1}^L \mathbb{E}_{Z_0} \left| \phi_h \left( h_\ell, \bar{h}_\ell, \beta_{0\ell} \right) - \phi_h \left( h_\ell + \Delta_\ell, \bar{h}_\ell + \bar{\Delta}_\ell, \beta_{0\ell} \right) \right| \right] \overset{a.s.}{=} 0. \quad (106)
$$

Note that from $\mathcal{H}_1(c)$ and the fact that $Z_{0\ell} \sim \mathcal{N}(0, I_{M \times M})$,

$$
\max_{j \in \sec(\ell)} |h_{0\ell}| = a_0 \frac{\|m^0\|}{\sqrt{n}} \max_{j \in \sec(\ell)} |Z_{0\ell}| \overset{a.s.}{=} \Theta(\sqrt{\log M}),
$$

$$
\max_{j \in \sec(\ell)} |\Delta_{0\ell}| = a_0 \left| \tilde{\tau}_0 - \frac{\|m^0\|}{\sqrt{n}} \right| \max_{j \in \sec(\ell)} |Z_{0\ell}| \overset{a.s.}{=} \Theta(n^{-\delta'} \sqrt{\log M}) \quad (107)
$$

for some $\delta' > 0$. The almost-sure equality in each line of (107) holds for sufficiently large $M$. (This can be shown using the standard normal distribution of $Z_0$ and the Borel-Cantelli lemma).
Similarly \( \max_{j \in \text{sec}(\ell)} |\tilde{h}_{\ell,j}| = \Theta(\sqrt{\log M}) \) and \( \max_{j \in \text{sec}(\ell)} |\tilde{\Delta}_{\ell,j}| = \Theta(n^{-\delta'} \sqrt{\log M}) \). Using (107), it is shown in [25] that

\[
|\phi_h \left( h_{\ell}, \tilde{h}_{\ell}, \beta_0 \right) - \phi_h \left( h_{\ell}, \Delta_{\ell}, \tilde{h}_{\ell} + \tilde{\Delta}_{\ell}, \beta_0 \right) | \overset{a.s.}{=} o(n^{-\delta'} \log M) \tag{108}
\]

for some \( \delta' > 0 \). Thus (105) holds for \( \delta < \delta' \). To complete the proof, we need to show that

\[
\lim n^\delta \left| \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{Z_0} \left[ \phi_h \left( a_0 \tilde{\tau}_0 Z_{\ell_0}, b_0 \tilde{\tau}_0 Z_{\ell_0}, \beta_0 \right) \right] - \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{(Z_0, \beta)} \left[ \phi_h \left( a_0 \tilde{\tau}_0 Z_{\ell_0}, b_0 \tilde{\tau}_0 Z_{\ell_0}, \beta_0 \right) \right] \right| \overset{a.s.}{=} 0 \tag{109}
\]

But (109) holds because the uniform distribution of the non-zero entry in \( \beta_\ell \) over the \( M \) possible locations and the i.d.d. distribution of \( Z_\ell \) together ensure that for all \( \beta_0 \in \mathcal{B}_{M,L} \), we have

\[
\mathbb{E}_{Z_0} \left[ \phi_h \left( a_0 \tilde{\tau}_0 Z_{\ell_0}, b_0 \tilde{\tau}_0 Z_{\ell_0}, \beta_0 \right) \right] = \mathbb{E}_{(Z_0, \beta)} \left[ \phi_h \left( a_0 \tilde{\tau}_0 Z_{\ell_0}, b_0 \tilde{\tau}_0 Z_{\ell_0}, \beta_0 \right) \right], \quad \forall \ell \in [L].
\]

(d) By definition \( q^1 = \eta^0_b(\beta_0 - h^1) - \beta_0 \), and hence \( \frac{(h_{\ell})^* q^1}{n} = \frac{1}{n} \sum_{\ell=1}^{L} \phi_h(h_{\ell}^1, \beta_0) \), where the function \( \phi_h : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R} \) is \( \phi_h(h_{\ell}^1, \beta_0) := (h_{\ell})^* \eta^0(\beta_0 - h^1) - \beta_0 \). Applying \( \mathcal{H}_1 \) to \( \phi_h \) yields

\[
\lim n^\delta \left[ \frac{1}{n} \sum_{\ell=1}^{L} \phi_h(h_{\ell}^1, \beta_0) - \lim \frac{1}{n} \sum_{\ell=1}^{L} \mathbb{E}\{ \tilde{\tau}_0 Z_{0,i}[\eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) - \beta_0] \} \right] \overset{a.s.}{=} 0. \tag{110}
\]

Consider a single term in the expectation in (110), say \( \ell = 1 \). We have

\[
\mathbb{E}\{ \tilde{\tau}_0 Z_{0,i}[\eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) - \beta_0] \} = \tilde{\tau}_0 \sum_{i=1}^{M} \mathbb{E}\{ Z_0, [\eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) - \beta_0] \} \tag{111}
\]

where \( \beta_0(i) = (\beta_0, \beta_0), \ldots, \beta_0, \ldots, \beta_0 \) and \( Z_0 = (Z_0, Z_0, \ldots, Z_0, Z_0) \). Note that for each \( i \), the function \( \eta^0_i(\cdot) \) depends on all the \( M \) indices in the section containing \( i \). For each \( i \in [M] \), we evaluate the expectation on the RHS of (111) using the law of iterated expectations:

\[
\mathbb{E}\{ Z_0, [\eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) - \beta_0] \} = \mathbb{E}\left[ \mathbb{E}\left\{ Z_0, [\eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) - \beta_0] | \beta_0(i), Z_0(i) \right\} \right] \tag{112}
\]

where the inner expectation is over \( Z_i \) conditioned on \( \{ \beta_0(i), Z_0(i) \} \). Since \( Z_0 \) is independent of \( \{ \beta_0(i), Z_0(i) \} \), the latter just act as constants in the inner expectation, which is over \( Z_0 \sim \mathcal{N}(0,1) \). Applying Stein’s lemma (Fact 6) to the inner expectation, we obtain

\[
\mathbb{E}\left[ \mathbb{E}\left\{ Z_0, [\eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) - \beta_0] | \beta_0(i), Z_0(i) \right\} \right] = \mathbb{E}\left[ \frac{\partial}{\partial Z_0} [\eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) - \beta_0] | \beta_0(i), Z_0(i) \right] \right]\]

\[
\overset{(a)}{=} -\frac{\tilde{\tau}_0}{\sigma_0^2} \mathbb{E}\left[ \eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) \left( \sqrt{n} P_{\ell i} - \eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) \right) | \beta_0(i), Z_0(i) \right] \]

\[
\overset{(b)}{=} -\frac{1}{\tilde{\tau}_0} \mathbb{E}\left[ \eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) \left( \sqrt{n} P_{\ell i} - \eta^0_i(\beta_0 - \tilde{\tau}_0 Z_0) \right) \right] \tag{113}
\]

where (a) holds because the definition of \( \eta^0_i \) in (11) implies that

\[
\frac{\partial \eta^0_i(s)}{\partial s_i} = \frac{\eta^0_i(s)}{\tilde{\tau}_0^2} \left( \sqrt{n} P_{\ell i} - \eta^0_i(s) \right) \text{ for } i \in \text{section } \ell,
\]
and (b) follows from the law of iterated expectation. Using (113) in (112) and (111), we have

\[
E \left\{ \bar{t}_0 Z_{01}^* \right\} = \sum_{i=1}^M E \left[ \eta_0^i (\beta_0 - \bar{t}_0 Z_0) \right].
\]  

(114)

The argument above can be repeated for each section \( \ell \in [L] \) to obtain a relation analogous to (114). Using this for the expectation in (110), we obtain

\[
\lim n^\delta \left[ \frac{1}{n} \sum_{\ell=1}^L \phi_{h_\ell} \right] \left( \frac{1}{n} \sum_{\ell=1}^L \eta_0^\ell (\beta_0 - h_\ell) - \beta_0 \right) + \bar{\sigma}_1^2 = 0.
\]  

(115)

It is shown in Appendix A.4 that \( \lim \left( P - \frac{E \{ \eta^0 (\beta_0 - \bar{t}_0 Z_0) \} }{n} \right) = \sigma^2_1 \). Therefore (115) becomes

\[
\lim n^\delta \left[ \frac{1}{n} \sum_{\ell=1}^L \eta_0^\ell (\beta_0 - h_\ell) \right] \left( \frac{1}{n} \sum_{\ell=1}^L \eta_0^\ell (\beta_0 - h_\ell) - \beta_0 \right) + \bar{\sigma}_1^2 = 0,
\]  

(116)

where we have used \( \phi_{h_\ell} (\beta_0 - h_\ell) = (h_\ell)^* \eta_0^\ell (\beta_0 - h_\ell) - \beta_0 \).

To complete the proof, recall from \( \mathcal{H}_1(c) \) that \( \frac{\|m_0\|^2}{n} \rightarrow \sigma^2 + \bar{\sigma}_0^2 \) at rate \( n^{-\delta} \). Further, from (43), we observe that

\[
\lambda_1 = \frac{1}{\bar{\tau}_0^2} \left( \frac{\|\beta_1\|^2}{n} - P \right) \xrightarrow{a.s.} \frac{1}{\bar{\tau}_0^2} \left( \frac{E \{ \|\eta^0 (\beta_0 - \bar{t}_0 Z_0)\|^2 \} }{n} - P \right) = -\frac{\bar{\sigma}_1^2}{\bar{\tau}_0^2} = -\frac{\bar{\sigma}_1^2}{\sigma^2 + \bar{\sigma}_0^2},
\]  

(117)

where the convergence at rate \( n^{-\delta} \) follows from \( \mathcal{H}_1(b) \) applied to the function \( \frac{\|\eta^0(\beta - h_1)\|^2}{n} = \frac{\|\beta_1\|^2}{n} \).

(e) We use \( \mathcal{H}_1(a) \) to represent

\[
\frac{1}{n} (\eta_0^0)_{\mathcal{J}_{1,0}} = \hat{A}^* m_+ + \hat{Q}_1 o(n^{-\delta}) = \hat{A}^* m_0 + \frac{q_0^0}{\sqrt{P}} o(n^{-\delta}).
\]  

(118)

Therefore

\[
\frac{(q_0^0)_{\mathcal{J}_{1,0}}}{n} \xrightarrow{d} \frac{(q_0^0)^* \hat{A}^* m_0}{n} + \frac{\|q_0^0\|^2}{n \sqrt{P}} o(n^{-\delta}) \xrightarrow{d} \sqrt{P} \frac{\|m_0\|}{\sqrt{n}} \frac{Z}{\sqrt{n}} + \sqrt{P} o(n^{-\delta}),
\]  

(119)

where we have used Fact (a) as \( q_0^0, m_0 \) are in the sigma-field and independent of \( \hat{A} \). By \( \mathcal{H}_1(c) \), \( \lim \frac{\|m_0\|^2}{n} \xrightarrow{a.s.} \bar{\tau}_0^2 \) and therefore (119) goes to zero almost surely in the limit at rate \( n^{-\delta} \).

(f) Since \( Q^0 \) is the empty matrix, \( q_+ = q_0^0 \) and so \( \lim \frac{\|q_0^0\|^2}{n} = \lim \frac{\|q_0^0\|^2}{n} = P \).

5.5.3 Step 3: Showing \( B_t \) holds

(f) By the induction hypothesis \( B_{t-1} \), (65) is true for \( 0 \leq s \leq t - 2 \), so we prove the \( s = t - 1 \) case. Let \( P_{M_{t-1}} = M_{t-1}(M_{t-1}^* M_{t-1})^{-1} M_{t-1}^* \) be the projection matrix onto the column space of \( M_{t-1} \). Then,

\[
\frac{\|m_{t-1}^{-1}\|^2}{n} = \|(1 - P_{M_{t-1}}) m_{t-1}^{-1}\|^2 = \frac{\|m_{t-1}^{-1}\|^2}{n} - \frac{(M_{t-1}^* M_{t-1})^{-1} M_{t-1}^* m_{t-1}^{-1}}{n}. \]

(120)
Consider the matrix inverse in (120). By the induction hypothesis \( B_{t-1}(f) \),
\[
\lim_{n} \frac{\|m^*\|}{n} = \lim_{n} \frac{\|m^* - P_{M_{t-1}}m^*\|}{n} > \xi_r \quad \text{for} \quad 0 \leq r \leq t - 2,
\]
(121)
for positive constants \( \xi_r \). Using (121), Facts 7 and 8 imply that the smallest eigenvalue of \( \lim_{M_{t-1}} M_{t-1} \)
is greater than some positive constant; hence its inverse exists.

Let \( \phi_0(b^*_r, b^*_s, w_i) = (b^*_r - w_i)(b^*_s - w_i) = m^*_r m^*_s \). It can be verified that \( \phi_0 \in PL(2) \). Using the induction hypothesis \( B_{t-1} \) (b), we have for \( 0 \leq r, s \leq t - 1 \):
\[
\lim_{n} \frac{1}{n} \sum_{i=1}^{n} \phi_0(b^*_r, b^*_s, w_i) = \lim \frac{(m^r)^* m^s}{n} = \mathbb{E} \left[ (\bar{\sigma}_r \hat{Z}_r - \sigma Z_w)(\bar{\sigma}_s \hat{Z}_s - \sigma Z_w) \right]
\]
(122)
where \((\hat{Z}_r, \hat{Z}_s)\) are jointly Gaussian with \( \mathcal{N}(0, 1) \) marginals, and independent of \( Z_w \). Using (122) in (120), we obtain
\[
\lim \frac{\|m^{t-1}\|}{n} = \mathbb{E} \left[ (\sigma_{t-1} \hat{Z}_{t-1} - \sigma Z_w)^2 \right] - u^* C^{-1} u
\]
(123)
where for \( 1 \leq i, j \leq (t - 1) \),
\[
u_i = \mathbb{E} \left[ (\sigma_{t-1} \hat{Z}_{t-1} - \sigma Z_w)(\sigma_{i-1} \hat{Z}_{i-1} - \sigma Z_w) \right], \quad C_{ij} = \mathbb{E} \left[ (\sigma_{i-1} \hat{Z}_{i-1} - \sigma Z_w)(\sigma_{j-1} \hat{Z}_{j-1} - \sigma Z_w) \right].
\]
(124)

Now the result follows from Fact 5 if we can show that there exists strictly positive constants \( c_1, \ldots, c_{t-1} \) such that \( \text{Var}(\bar{\sigma}_r Z_r|\sigma_0 Z_0, \ldots, \sigma_{r-1} Z_{r-1}) \geq c_r \), for \( 1 \leq r \leq (t - 1) \). Indeed, we will now prove that
\[
\text{Var}(\bar{\sigma}_r Z_r|\sigma_0 Z_0, \ldots, \sigma_{r-1} Z_{r-1}) = \sigma_r^2 \left( 1 - \frac{\sigma_r^2}{\sigma_{r-1}^2} \right).
\]
(125)
Since \( \sigma_r^2 = \sigma^2 \left( 1 + nr^1 - 1 \right) \), the definition of \( \xi_{r-1} \) in (33) implies that the RHS of (125) is strictly positive for \( r \leq T^* - 1 \), where \( T^* = \left[ \frac{2C}{\log(t/\beta)} \right] \).

For \( r \in [t - 1] \), we have
\[
\lim \frac{\|b^*_r\|^2}{n} = \lim \frac{\|b^*_r\|^2}{n} - \left( (b^*_r)^* B_r \frac{B_r^* B_r}{n} \right)^{-1} B_r^* b^*_r = \lim \frac{\|q^\prime\|^2}{n} \frac{Q_r^* Q_r}{n} = \frac{(Q_r^* Q_r)^{-1} Q_r^* q^\prime}{n}
\]
(126)
where the second equality follows from the induction hypothesis \( B_{t-1}(c) \) which says that
\[
\lim \frac{(b^\prime)^* b^\prime}{n} = \lim \frac{(q^\prime)^* q^\prime}{n} = \sigma_{r'}^2 \quad \text{for} \quad 0 \leq r' \leq r \leq (t - 1).
\]
(127)
Denoting \( \lim \frac{B_r^* B_r}{n} = \lim \frac{Q_r^* Q_r}{n} \) by \( \tilde{C} \), we have \( \tilde{C}_{ij} = \tilde{C}_{ji} = \lim \frac{(q^\prime)^* q^\prime}{n} = \sigma_j^2 \), for \( 0 \leq i \leq j \leq (r - 1) \). The induction hypothesis \( H_t(f) \) guarantees that \( \frac{\|q^\prime\|^2}{n} \) is strictly positive for \( 0 \leq r \leq t - 1 \). Consequently, Facts 7 and 8 imply that \( \tilde{C} \) is invertible. Hence
\[
\lim \frac{\|b^*_r\|^2}{n} = \lim \frac{\|q^\prime\|^2}{n} - \left( (q^\prime)^* Q_r \frac{Q_r^* Q_r}{n} \right)^{-1} Q_r^* q^\prime \equiv \sigma_r^2 - \sigma_r^2 (\tilde{e}^*_r \tilde{C}^{-1} e_r) \sigma_r^2 \equiv \sigma_r^2 \left( 1 - \frac{\sigma_r^2}{\sigma_{r-1}^2} \right).
\]
(128)
In \([128]\), (a) is obtained using \([127]\) with \(e_r \in \mathbb{R}^r\) denoting the all-ones column vector. The equality (b) is obtained using the fact that \(C^{-1}e_r\) is the solution to \(\tilde{C}x = e_r\); since all the entries in the last column of \(\tilde{C}\) are equal to \(\sigma^2_{r-1}\), by inspection the solution to \(\tilde{C}x = e_r\) is \(x = [0, \ldots, 0, (\sigma^2_{r-1})^{-1}]^T\), which yields equality (b) in \([128]\).

Using the induction hypothesis \(B_{t-1}(b)\) for the \(PL(2)\) function \(\phi_b(x, y) = xy\), we have

\[
\lim \frac{1}{n}(b^*)^Tb^s = \lim \sum_{i=1}^{n} \frac{1}{n}b_i^r b_i^s = \mathbb{E}[\tilde{\sigma}_r \tilde{Z}_r \tilde{Z}_s], \quad 0 \leq r, s \leq (t - 1). \tag{129}
\]

Using this, we obtain

\[
\lim \frac{\|b_i^r\|^2}{n} = \lim \frac{\|b_i^r\|^2}{n} - \frac{(b^*)^T B_r (B_r^T B_r)^{-1} B_r^T b_i^s}{n} = \sigma_r^2 - v^s D^{-1}v \quad (a) \equiv \text{Var}(\tilde{\sigma}_r \tilde{Z}_r | \tilde{\sigma}_0 \tilde{Z}_0, \ldots, \tilde{\sigma}_{r-1} \tilde{Z}_{r-1}) \tag{130}
\]

where for \(0 \leq i, j \leq (r - 1), v_i = \mathbb{E} \left[ \sigma_{ij} \tilde{Z}_r \tilde{Z}_j \right] \) and \(D_{ij} = \mathbb{E} \left[ \sigma_{ij} \tilde{Z}_r \tilde{Z}_j \right] \). Equality (a) in \([130]\) follows from Fact 4. We have proved \([125]\) via \([128]\) and \([130]\), which completes the proof of \(B_t(f)\).

We now state a couple of lemmas that will be useful for proving the remainder of \(B_t\) and \(H_{t+1}\).

**Lemma 6.** For \(t \leq T^*\), the vectors of coefficients in \([48]\), given by

\[
\tilde{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{t-1}) = \left( \frac{M_t^* M_t}{n} \right)^{-1} \frac{M_t^* m_t}{n}, \quad \tilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{t-1}) = \left( \frac{Q_t^* Q_t}{n} \right)^{-1} \frac{Q_t^* q_t}{n}
\]

converge to finite limits at rate \(n^{-\delta}\) as \(n \to \infty\).

**Proof.** From the induction hypothesis \(H_t(c)\), \((m^r)^s m^s\) converges almost surely to a constant at rate \(n^{-\delta}\) for \(r, s \leq (t - 1)\). Further, \(B_t(f)\) proved above and Fact 7 together imply that the smallest eigenvalue of the matrix \(\frac{M_t^* M_t}{n}\) is bounded from below by a positive constant for all \(n\); then Fact 8 implies that its inverse has a finite limit. Further, the inverse converges to its limit at rate \(n^{-\delta}\) as each entry in \(\frac{M_t^* M_t}{n}\) converges at this rate. The statement for \(\tilde{\gamma}\) is proved in an analogous manner using the induction hypotheses \(B_{t-1}(c)\) and \(H_t(f)\), together with Facts 7 and 8.

**Lemma 7.** The following statements hold for \(t \leq T^*\):

\[
h_t^{i+1} \big|_{X_t, t} \overset{d}{=} H_t(M_t^* M_t)^{-1} M_t^* m_t + P_{Q_t^* Q_t}^{\perp} X_t^{i+1} + Q_{t+1} \sigma_{t+1} (n^{-\delta}) \tag{131}
\]

\[
b_t^i \big|_{X_t, t} \overset{d}{=} B_t(Q_t^* Q_t)^{-1} Q_t^* q_t^{i+1} + P_{M_t}^{\perp} \hat{A} q_t^i + M_t \tilde{\sigma}_t (n^{-\delta}) \tag{132}
\]

where \(B_t = [b^0 \mid \ldots \mid b^{t-1}]\) and \(H_t = [h_1 \mid \ldots \mid h_t]\).

**Proof.** The proof is very similar to that of 7 Lemma 13. We use Lemmas 4 and 5 to write

\[
b_t^i \big|_{X_t, t} = (A q_t - \lambda_t m_t^{-1}) \big|_{X_t, t} \overset{d}{=} Y_t (Q_t^* Q_t)^{-1} Q_t^* q_t^i + M_t (M_t^* M_t)^{-1} X_t^i + P_{M_t}^{\perp} \hat{A} q_t^i - \lambda_t m_t^{-1} \]

\[
= B_t(Q_t^* Q_t)^{-1} Q_t^* q_t^{i+1} + \left[ 0 | M_t \right] \Lambda_t (Q_t^* Q_t)^{-1} Q_t^* q_t^{i+1} + M_t (M_t^* M_t)^{-1} H_t^i q_t^j + P_{M_t}^{\perp} \hat{A} q_t^i - \lambda_t m_t^{-1}, \tag{133}
\]

where \(\Lambda_t = \text{diag} (\lambda_0, \ldots, \lambda_{t-1})\). The last equality above is obtained using \(Y_t = B_t + [0 | M_{t-1}] \Lambda_t\), and \(X_t = H_t + Q_t\). Thus, to show \([132]\), we need to prove that

\[
[0 | M_{t-1}] \Lambda_t \tilde{\gamma} + M_t (M_t^* M_t)^{-1} H_t^i q_t^j - \lambda_t m_t^{-1} = M_t \tilde{\sigma}_t (n^{-\delta}). \tag{134}
\]
Observe that each side of (134) is a linear combination of \{m^k\}, \(0 \leq k \leq (t-1)\). The coefficient of \(m^k\) on the LHS equals
\[
\lambda_{k+1} \gamma_{k+1} + \left[ \left( \frac{M_t^* M_t}{n} \right)^{-1} H_t^* q^t_i \right]_{k+1} \quad \text{for } 0 \leq k \leq t-2,
\]
\[-\lambda_t + \left( \frac{M_t^* M_t}{n} \right)^{-1} H_t^* q^t_i \tag{135}, \quad \text{for } k = t-1.
\]

We prove (134) by showing that each of the coefficients above is \(o(n^{-\delta})\). Indeed, for \(1 \leq i \leq t\),
\[
\left[ \frac{H^*_t q^t_i}{n} \right] = \frac{(h^i)^* q^t_i}{n} = \frac{(h^i)^* (q^t_i - q^t_{i+1})}{n} = \frac{(h^i)^* q^t_i}{n} - \sum_{r=0}^{t-1} \lambda_r (h^i)^* q^r \tag{136},
\]
where the convergence (at rate \(n^{-\delta}\)) follows from \(H_t(d)\); Lemma 6 guarantees the convergence of the \(\lambda_r\) coefficients. Therefore
\[
\left[ \frac{H^*_t q^t_i}{n} \right] \xrightarrow{a.s.} \lim \left[ \lambda_t (M_t)^* m^{t-1} - \sum_{r=1}^{t-2} \gamma_{r+1} (M_t)^* m^r \right] \text{ at rate } n^{-\delta}. \tag{137}
\]
Substituting (137) in (135) yields (134), and completes the proof of (132). The other part of the lemma, (131), is proved in a similar manner.

\(\Box\)

(a) From Lemma 7, we have
\[
b^t |_{A_t} \overset{d}{=} B_t(Q^t_t Q_t)^{-1}Q^t_t q^t_{\perp} + P^\perp_{M_t} \tilde{A} q^t_{\perp} + M_t \tilde{\sigma}_t(n^{-\delta}). \tag{138}
\]
Using the fact that \(q^t_{\perp} = \sum_{i=0}^{t-1} \gamma_i q^t_i\), we have
\[
B_t(Q^t_t Q_t)^{-1}Q^t_t q^t_{\perp} = \sum_{i=0}^{t-1} \gamma_i B_t(Q^t_t Q_t)^{-1}Q^t_t q^t_i = \sum_{i=0}^{t-1} \gamma_i b^t \tag{139}
\]
as \((Q^t_t Q_t)^{-1}Q^t_t q^t_{\perp} \in \mathbb{R}^d\) is a vector of zeros with a 1 in position \((i+1)\). Next, observe that \(P^\perp_{M_t} \tilde{A} q^t_{\perp} = \tilde{A} q^t - P^\parallel_{M_t} \tilde{A} q^t\). Hence the result follows if we can show that \(P^\parallel_{M_t} \tilde{A} q^t = M_t \tilde{\sigma}_t(n^{-\delta})\). Indeed, using Fact 1(b), we see that
\[
P^\parallel_{M_t} \tilde{A} q^t = \frac{\|q^t_{\perp}\|^2}{\|q^t_{\perp}\|^2} M_t \tilde{\sigma}_t(n^{-\delta}) = M_t \tilde{\sigma}_t(n^{-\delta})
\]
where the last equality follows since \(\|q^t_{\perp}\|^2 \leq \|q^t_{\perp}\|^2 \leq 2P\).

(c) By the induction hypothesis, the result holds for all \(r, s < t\), so we only consider the \(r < t, s = t\) and \(r = s = t\) cases. From \(B_t(a)\) above, we have
\[
b^t |_{A_t} \overset{d}{=} \sum_{i=0}^{t-1} \gamma_i b^t + \tilde{A} q^t_{\perp} + M_t \tilde{\sigma}_t(n^{-\delta}) \tag{140}
\]
where we have used $M_t \bar{g}_t(n^{-\delta}) = \bar{M}_t \bar{g}_t(n^{-\delta})$. For $r < t$, $s = t$, we have from (140):

$$
\left(\frac{(b^*)^{s}b^i}{n}\right)_r \frac{d}{n} = \sum_{i=1}^{t-1} \gamma_i (b^*)^{s}b^i - \frac{(b^*)^{s}\bar{A}_q^t}{n} + \frac{(b^*)^{s}M_t\bar{g}_t(n^{-\delta})}{n} + \sum_{i=0}^{t-1} o(n^{-\delta}) \left(\frac{(b^*)^{s}m^i}{n}\right). 
$$

Applying Fact 1(a), the second term in (141) is for $\frac{(b^*)^{s}\bar{A}_q^t}{n} = \frac{\|b^*\|\|q^t\| Z}{n}$, where $Z \sim \mathcal{N}(0, 1)$. Therefore the last two terms in (141) are $o(n^{-\delta})$ since $\frac{\|q^t\|}{n} \leq 2P$ and $B_{t-1}(c)$, (d) imply that $\frac{\|b^*\|}{n}$ and $\frac{(b^*)^{s}m^i}{n}$ converge to finite limits. Using $B_{t-1}(c)$ again, the limit of first term in (141) can be written as

$$
\lim \sum_{i=0}^{t-1} \gamma_i \left(\frac{(b^*)^{s}b^i}{n}\right) = \lim \sum_{i=1}^{t-1} \gamma_i \left(\frac{(q^t)^*q^i}{n}\right) = \lim \left(\frac{(q^t)^*q^t}{n}\right) = \lim \left(\frac{(q^t)^*q^t}{n}\right) = \sigma_t^2 \text{ a.s.} 
$$

where the $\gamma_i$'s have finite limits due to Lemma 6. Equality (a) in (142) holds because $q^t_1 \perp q^t$, while (b) is obtained by applying $H_t(b)$ to the function

$$
\phi_b(h^t_\ell, h^{s}_\ell, \beta_0) := [\gamma^{r-1}_\ell (\beta_0 - h^r) - \beta_0][\gamma^{s-1}_\ell (\beta_0 - h^s) - \beta_0] = (q^t_\ell)^*q^t_\ell,
$$

which yields

$$
\lim \left(\frac{(q^t)^*q^t}{n}\right) \overset{a.s.}{=} \lim \frac{1}{n} E \left\{ [\gamma^{r-1}_\ell (\beta - \tau_{t-1} Z_{t-1}) - \beta][\gamma^{s-1}_\ell (\beta - \tau_{t-1} Z_{t-1}) - \beta] \right\} = \sigma_t^2,
$$

where the second equality above is proved in Appendix A.4. From $B_{t-1}(c)$ and $H_{t-1}(b)$, it follows that the rate of convergence in (142) is $n^{-\delta}$.

For $r = s = t$, using (140), we have

$$
\left(\frac{\|b^t\|^2}{n}\right)_r \frac{d}{n} = \sum_{i=0}^{t-1} \gamma_i \left(\frac{(b^t)^*b^i}{n}\right) + \frac{\|\bar{A}_q^t\|^2}{n} + 2 \sum_{i=0}^{t-1} \gamma_i \left(\frac{(b^t)^*\bar{M}_t\bar{g}_t(n^{-\delta})}{n}\right) + 2 \left(\frac{\bar{A}_q^t}{n}\right)^* \left(\frac{\bar{M}_t\bar{g}_t(n^{-\delta})}{n}\right) + \left(\frac{\|\bar{M}_t\bar{g}_t(n^{-\delta})\|^2}{n}\right).
$$

Using arguments similar to those for the $r < t$ case, the last four terms in (143) can be shown to be $o(n^{-\delta})$, and by Fact 1, $\frac{\|\bar{A}_q^t\|^2}{n} = \frac{\|q^t_1\|^2 \|Z\|^2}{n}$, where $Z \sim \mathbb{R}^n$ is standard normal. Therefore,

$$
\left(\frac{\|b^t\|^2}{n}\right)_r \frac{d}{n} = \sum_{i=0}^{t-1} \sum_{i'}^{t-1} \gamma_i \gamma_{i'} \left(\frac{(b^t)^*b^{i'}}{n}\right) + \frac{\|q^t_1\|^2}{n} + o(n^{-\delta})
$$

$$
\left(\frac{n^{-\frac{\delta}{2}}}{n}\right) \lim \sum_{i=0}^{t-1} \sum_{i'}^{t-1} \gamma_i \gamma_{i'} \left(\frac{(q^t)^*q^{i'}}{n}\right) + \frac{\|q^t_1\|^2}{n} = \lim \frac{\|q^t_1\|^2}{n} + \frac{\|q^t_1\|^2}{n} = \lim \frac{\|q^t_1\|^2}{n},
$$

where the convergence at rate $n^{-\delta}$ follows from $B_{t-1}(c)$.

(b) Using the characterization for $b^t$ obtained in $B_t(a)$ above, we have

$$
\phi_b(b^t_1, \ldots, b^t_i, w_i) \left(\frac{d}{n}\right) = \phi_b \left(\left[\sum_{r=0}^{t-1} \gamma_r b^r + \bar{A}_q^t + \bar{M}_t\bar{g}_t(n^{-\delta})\right]_i, w_i\right).
$$

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for some $\delta' > 0$. The term $\tilde{M}_t\tilde{a}_t(n^{-\delta'})$ in the RHS can be dropped. Indeed, defining
\[
a_i = \left( b_i^0, \ldots, b_i^{t-1}, \left[ \sum_{r=0}^{t-1} \gamma_r b^r + \tilde{M}_t\tilde{a}_t(n^{-\delta'}) \right]_i, w_i \right), \quad c_i = \left( b_i^0, \ldots, b_i^{t-1}, \left[ \sum_{r=0}^{t-1} \gamma_r b^r + \tilde{M}_t\tilde{a}_t(n^{-\delta'}) \right]_i, w_i \right),
\]
we can show that
\[
\begin{align*}
\frac{1}{n} \sum_{i=1}^{n} \phi_b(a_i) - \frac{1}{n} \sum_{i=1}^{n} \phi_b(c_i) & \leq \frac{n^{\delta}}{n} \sum_{i=1}^{n} \left[ \phi_b(a_i) - \phi_b(c_i) \right] \\
& \leq \frac{C}{n} \sum_{i=1}^{n} \left( 1 + \|a_i\| + \|c_i\| \right) \left( \tilde{M}_t\tilde{a}_t(n^{-\delta'}) \right)_i \\
& \leq C \sqrt{n} \sum_{i=1}^{n} \left( 1 + \|a_i\| + \|c_i\| \right) \sum_{r=0}^{t-1} \frac{\|\tilde{m}_r\|^2}{n} 2^r o(n^{-\delta'}) \equiv o(n^{-\delta'}). \tag{145}
\end{align*}
\]
In (145), (a) holds because $\phi_b \in PL(2)$. (b) is obtained using Holder’s inequality and the fact that $\sum_{i=1}^{n} \left[ \tilde{M}_t\tilde{a}_t(n^{-\delta'}) \right]^2_i \leq 2^t \tilde{a}_t(n^{-\delta'}) \sum_{r=0}^{t-1} \|\tilde{m}_r\|^2$. Equality (c) can be shown by verifying that $\sum_{i=1}^{n} \|a_i\|^2$ and $\sum_{i=1}^{n} \|c_i\|^2$ are bounded and finite. The details are similar to Fact 2(b) and are omitted. Thus by choosing $\delta < \delta'$, we can work with $c_i$ instead of $a_i$.

Next, we use Fact 2 to show that
\[
\lim_{n \to \infty} n^{\delta} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_b(c_i) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_A \{ \phi_b(c_i) \} \right] \overset{a.s.}{=} 0. \tag{146}
\]
To appeal to Fact 2 we need to verify that
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ n^{\delta} \phi_b(c_i) - \mathbb{E}_A \{ n^{\delta} \phi_b(c_i) \} \right]^{2+\kappa} \leq cn^{\kappa/2}. \tag{147}
\]
Using steps similar to (55), we can show that
\[
\mathbb{E} \left[ \phi_b(c_i) - \mathbb{E}_A \{ \phi_b(c_i) \} \right]^{2+\kappa} \leq \kappa' \mathbb{E}_A, A \left[ \left[ |\tilde{A}'q_{i1}'|_i - |\tilde{A}q_{i1}'|_i \right]^{2+\kappa} \left( 1 + |\tilde{A}'q_{i1}'|_i^{2+\kappa} + |\tilde{A}q_{i1}'|_i^{2+\kappa} \right) \right] \\
+ \kappa' \left( \sum_{r=0}^{t-1} \left( |1 + \gamma_r| b^{(r)}_i \right)^{2+\kappa} + |w_i|^{2+\kappa} \right) \mathbb{E}_A, A \left[ \left[ |\tilde{A}'q_{i1}'|_i - |\tilde{A}q_{i1}'|_i \right]^{2+\kappa} \right] \\
\overset{(a)}{\leq} \kappa_1 + \kappa_2 \left( |w_i|^{2+\kappa} + \sum_{r=0}^{t-1} \left( 1 + \gamma_r \right)^{2+\kappa} |b^{(r)}_i|^{2+\kappa} \right) \tag{148}
\]
for some constants $\kappa', \kappa_1, \kappa_2 > 0$, where $\tilde{A}, \tilde{A}'$ are independent copies of $A$. In (148), (a) holds because $\tilde{A}q_{i1}' \equiv \|\tilde{q}_i\| \tilde{Z}$ and $|\tilde{q}_i|^{2+\kappa} \leq P$; similarly, $\tilde{A}'q_{i1}' \equiv \|\tilde{q}_i\| \tilde{Z}'$, where $\tilde{Z}, \tilde{Z}'$ are $\mathcal{N}(0,1)$. Substituting (148) in the LHS of (147), and applying induction hypothesis $\mathcal{B}_t(b)$ shows that the condition (148) is satisfied if $\delta < \frac{\kappa_2 / \kappa + 2}{\kappa + 2}$.

Thus we now need to show that
\[
\lim_{n \to \infty} \frac{n^{\delta}}{n} \sum_{i=1}^{n} \mathbb{E}_A \left[ \left\{ \phi_b(b_i^0, \ldots, b_i^{t-1}, \sum_{r=0}^{t-1} \gamma_r b^{(r)}_i) + |\tilde{A}q_{i1}'|_i, w_i \right\} - \mathbb{E} \{ \phi_b(\sigma_0 Z_0, \ldots, \sigma_1 Z_t, W) \} \right] \overset{a.s.}{=} 0. \tag{149}
\]
Recalling that \([\tilde{A}q'_{\perp i}]_i \overset{d}{=} \frac{1}{\sqrt{n}} \tilde{Z}\) where \(\tilde{Z} \sim \mathcal{N}(0,1)\), we have

\[
\mathbb{E}_{\tilde{A}} \left\{ \phi_b(b_{i}^{0}, \ldots, b_{i}^{t-1}, \sum_{r=0}^{t-1} \gamma_{r}b_{i}^{r} + [\tilde{A}q'_{\perp i}]_i, w_{i}) \right\} = \mathbb{E}_{\tilde{Z}} \left\{ \phi_b(b_{i}^{0}, \ldots, b_{i}^{t-1}, \sum_{r=0}^{t-1} \gamma_{r}b_{i}^{r} + \frac{q'_i}{\sqrt{n}} \tilde{Z}, w_{i}) \right\}. \tag{150}
\]

Define the function

\[
\phi_b^{NEW}(b_{i}^{0}, \ldots, b_{i}^{t-1}, w_{i}) := \mathbb{E}_{\tilde{Z}} \left\{ \phi_b(b_{i}^{0}, \ldots, b_{i}^{t-1}, \sum_{r=0}^{t-1} \gamma_{r}b_{i}^{r} + \frac{q'_i}{\sqrt{n}} \tilde{Z}, w_{i}) \right\}. \tag{151}
\]

It can be verified that \(\phi_b^{NEW} \in PL(2)\), and hence the induction hypothesis \(\mathcal{B}_{t-1}(b)\) implies that

\[
\lim n^{\delta} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_b^{NEW}(b_{i}^{0}, \ldots, b_{i}^{t-1}, w_{i}) - \mathbb{E} \left\{ \phi_b^{NEW}(\sigma_{0}\tilde{Z}_0, \ldots, \sigma_{t-1}\tilde{Z}_{t-1}, \sigma Z_w) \right\} \right] \overset{a.s.}{=} 0. \tag{152}
\]

Thus from (150) – (152), we see that

\[
\lim n^{\delta} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\tilde{A}} \left\{ \phi_b(b_{i}^{0}, \ldots, b_{i}^{t-1}, \sum_{r=0}^{t-1} \gamma_{r}b_{i}^{r} + [\tilde{A}q'_{\perp i}]_i, w_{i}) \right\} \right. \nonumber
\]

\[
- \mathbb{E}_{\tilde{Z}} \left\{ \phi_b(\sigma_{0}\tilde{Z}_0, \ldots, \sigma_{t-1}\tilde{Z}_{t-1}, \sum_{r=0}^{t-1} \gamma_{r}\tilde{Z}_{r} + \frac{q'_i}{\sqrt{n}} \tilde{Z}, \sigma Z_w) \right\} \left] \overset{a.s.}{=} 0. \tag{153}
\]

In (153), Lemma 6 implies that the \(\gamma_{r}\)’s converge to a finite limit as \(n \to \infty\). Further,

\[
\frac{\|q'_i\|^2}{n} = \frac{\|q'^2\|^2}{n} - \frac{\|q'_i\|^2}{n} = \frac{\|q'^2\|^2}{n} - \frac{1}{n} \sum_{r=1}^{t-1} \frac{\gamma_{r}\|q'^2\|^2}{n} - \frac{\sum_{r,s=1}^{t-1} \gamma_{r}\gamma_{s}(q'^2)q'^2}{n}.
\]

Hence \(\frac{\|q'_i\|}{\sqrt{n}}\) also converges to a finite limit due to \(\mathcal{B}_{t}(c)\), proved above. The final step is to show that the variance of the Gaussian random variable \((\sum_{r=0}^{t-1} \gamma_{r}\tilde{Z}_{r} + \frac{q'_i}{\sqrt{n}} \tilde{Z})\) converges to \(\tilde{\sigma}_{t}^{2}\) at rate \(n^{-\delta'}\) for some \(\delta' > 0\). Applying (153) to the \(PL(2)\) function \(\phi_b(b_{i}^{0}, \ldots, b_{i}^{t-1}, w_{i}) := (b_{i}^{t})^{2}\), we obtain

\[
\lim n^{\delta} \left[ \frac{\|b'^2\|^2}{n} - \mathbb{E} \left\{ \frac{1}{n} \sum_{r=0}^{t-1} \gamma_{r}\tilde{Z}_{r} + \frac{q'_i}{\sqrt{n}} \tilde{Z} \right\}^{2} \right] \overset{a.s.}{=} 0. \tag{154}
\]

Using the induction hypothesis \(\mathcal{H}_{t}(b)\) for the function \(\phi_{\bar{t}}(h_{\bar{t}}, \beta_{\bar{t}}) = \|\eta_{\bar{t}}^{-1}(\beta - h_{\bar{t}}) - \beta_{\bar{t}}\|^{2} = \|q'_{\bar{t}}\|^2\), we have

\[
\lim n^{\delta} \left[ \frac{\|q'^2\|^2}{n} - \mathbb{E} \left\{ \|\eta_{\bar{t}}^{-1}(\beta - \tilde{Z}_{t-1}) - \beta\|^{2} \right\} \right] \overset{a.s.}{=} 0 \tag{155}
\]

since Appendix A.4 shows that \(\lim \mathbb{E}\{\|\eta_{\bar{t}}^{-1}(\beta - \tilde{Z}_{t-1}) - \beta\|^{2}\} = \tilde{\sigma}_{t}^{2}\). Further, induction hypothesis \(\mathcal{B}_{t}(c)\) implies that \(\lim n^{\delta} \left[ \frac{\|b'^2\|^2}{n} - \frac{\|q'^2\|^2}{n} \right] \overset{a.s.}{=} 0\). Combining this with (154) and (155) completes the proof.

(d) By definition \(m^{\bar{t}} = b^{\bar{t}} - w\) and so
\textbf{Step 4: Showing } \mathcal{H}_{t+1} \text{ holds}

\begin{enumerate}[(f)]
  \item By the induction hypothesis, \( \mathcal{H}_t(f) \) is true for \( 0 \leq r \leq (t-1) \). For \( r = t \), we have
    \[
    \lim \frac{\|q^t_\perp\|^2}{n} = \lim \frac{\|q^t\|^2}{n} - \frac{(q^t)^* Q_t}{n} \left( Q_t^* Q_t \right)^{-1} (Q_t)^* q^t
    \]
    We note the matrix inverse in (157) exists almost surely. Indeed, from the induction hypothesis \( \mathcal{H}_t(f) \) we have
    \[
    \lim \frac{\|q^t_\perp\|^2}{n} = \sigma^2 (1 - \frac{\sigma^2}{\sigma^2_{r-1}}) > 0 \text{ for } 0 \leq r \leq (t-1).
    \]
    Then Facts 7 and 8 imply that the matrix \( \lim Q_t^* Q_t/n \) is invertible.
    
    From \( \mathcal{B}_t(c) \), we know that \( \frac{\|q^t\|^2}{n} \overset{a.s.}{\to} \sigma^2_t \). Substituting this in (157), and using arguments identical to those used to prove (128) in \( \mathcal{B}_t(f) \), we obtain
    \[
    \lim \frac{\|q^t_\perp\|^2}{n} - \frac{(q^t)^* Q_t}{n} \left( Q_t^* Q_t \right)^{-1} Q_t^* q^t = \sigma^2_t \left( 1 - \frac{\sigma^2_t}{\sigma^2_{t-1}} \right).
    \]
    Since \( \sigma^2_t = \sigma^2 \left( 1 + \text{snr} \right)^{-\xi_{t-1}} - 1 \), the definition of \( \xi_{t-1} \) in (33) implies that the RHS of (158) is strictly positive for \( t \leq T^* - 1 \).

  \item We start with the characterization for \( h^{t+1} \) in (131) of Lemma 7. The proof from there on is along the same lines as \( \mathcal{B}_t(a) \), with \( (H_t, M_t, m_t, Q_{t+1}) \) replacing \( (B_t, q^t, M_t) \), respectively.

  \item From \( \mathcal{H}_{t+1}(a) \), we have
    \[
    h_t^{t+1}|_{\mathcal{H}_{t+1,t}} = \sum_{i=1}^{t-1} \alpha_i h_i^{t+1} + \tilde{A}^* m_{i\perp}^t + \tilde{Q}_{t+1} \tilde{o}_{t+1}(n^{-\delta})
    \]
    where we have used \( Q_{t+1} \tilde{o}_{t+1}(n^{-\delta}) = \tilde{Q}_{t+1} \tilde{o}_{t+1}(n^{-\delta}) \). For \( r < t, s = t \), we have
    \[
    \frac{(h^{t+1})^* h^{t+1}}{N} \bigg|_{\mathcal{H}_{t+1,t}} = \sum_{i=1}^{t-1} \alpha_i \frac{(h^{t+1})^* h_i^{t+1}}{N} + \frac{(h^{t+1})^* \tilde{A}^* m_{i\perp}^t}{N} + \frac{t}{N} o(n^{-\delta}) \frac{(h^{t+1})^* q^i}{N}.
    \]
    Applying Fact 1(a), the second term in (160) is \( \frac{(h^{t+1})^* \tilde{A} q_i^t}{n} \overset{d}{=} \frac{\|h^{t+1} \tilde{A} q_i^t\|}{Z/n} Z \), where \( Z \sim N(0, 1) \). Therefore, \( \mathcal{H}_t(c) \) and \( \mathcal{B}_t(f) \) imply that the second term is \( o(n^{-\delta}) \). The third term is also \( o(n^{-\delta}) \).
since $\mathcal{H}_t(c)$ implies that the inner products $\frac{(h^{r+1})^q}{N^q}$ go to zero. Using $\mathcal{H}_t(c)$ and Lemma 6 the first term in (160) converges at rate $n^{-\delta}$ to

$$
\lim_{t \to 1} \sum_{i=0}^{t-1} \alpha_i \frac{(h^{r+1})^q h^{r+1}}{N^n} = \lim_{t \to 1} \sum_{i=0}^{t-1} \alpha_i \frac{(m^r)^q m^r}{n} = \lim_{t \to 1} \frac{(m^r)^q m^r}{n} = \frac{(m^r)^q m^r}{n} \quad (161)
$$

where the last equality is obtained by applying $\mathcal{B}_t(b)$ to $\phi_b(b^t, b^t, w_i) = (b^t - w_i)(b^t - w_i) = m^t m^t$.

For $r = s = t$, using (159) we have

$$
\frac{\|h^{t+1}\|^2}{N} \bigg|_{\mathcal{T}_{t+1}, t} = \sum_{i=0}^{t-1} \sum_{j=0}^{t-1} \alpha_i \alpha_j \frac{(h^{j+1})^q h^{j+1}}{N^n} + \frac{\|\tilde{A}^* m^t\|^2}{N} + 2 \sum_{i=0}^{t-1} \frac{(h^{i+1})^q Q_{t+1} \tilde{o}_{t+1}(n^{-\delta})}{N} + 2 \sum_{i=0}^{t-1} \frac{(h^{i+1})^q Q_{t+1} \tilde{o}_{t+1}(n^{-\delta})}{N} + \frac{\|Q_{t+1} \tilde{o}_{t+1}(n^{-\delta})\|^2}{N} \quad (162)
$$

Using arguments similar to those for the $r < t$ case, the last four terms in (162) can be shown to be $o(n^{-\delta})$, and by Fact 1(a) $\frac{\tilde{A}^* m^t}{\sqrt{n}} = \frac{\|m^t\|}{N} Z$ where $Z \sim \mathbb{R}^N$ is standard normal. Therefore,

$$
\lim_{t \to 1} \frac{\|h^{t+1}\|^2}{N} \bigg|_{\mathcal{T}_{t}, t} = \lim_{t \to 1} \sum_{i=0}^{t-1} \sum_{j=0}^{t-1} \alpha_i \alpha_j \frac{(h^{j+1})^q h^{j+1}}{N^n} + \frac{\|m^t\|^2}{n} \quad (163)
$$

where the second equality is obtained using $\mathcal{H}_t(c)$, which together with the central limit theorem also gives the $n^{-\delta}$ rate of convergence.

(b) From $\mathcal{H}_{t+1}(a)$, we have

$$
\frac{h^{t+1}}{\mathcal{T}_{t+1}, t} = \sum_{u=0}^{t-1} \alpha_u h^{u+1} + \tilde{A}^* m^t + \tilde{Q}_{t+1} \tilde{o}_{t+1}(n^{-\delta}) \quad (164)
$$

where $\tilde{A}$ is an independent copy of $A$ and the columns of the matrix $\tilde{Q}_{t+1}$ form an orthogonal basis for the columns of $Q_{t+1}$ with $\tilde{Q}_{t+1} \tilde{Q}_{t+1}^* = n_{t+1} I$. We therefore have

$$
\phi_{h} \left( \sum_{u=0}^{t} a_u h^{u+1}, \sum_{v=0}^{t} b_v h^{v+1}, \beta_0 \right) \bigg|_{\mathcal{T}_{t+1}, t} = \phi_{h}(h_{t+1} + \Delta_{t}, \bar{h}_{t+1} + \bar{\Delta}_{t}, \beta_{0t}) \quad (165)
$$

where $h_{t+1} = \sum_{u=0}^{t-1} (a_u + \alpha_t \alpha_u) h^{u+1} + a_t [\tilde{A}^* m^t]_t$ and $\Delta_{t} = a_t [\tilde{Q}_{t+1} \tilde{o}_{t+1}(n^{-\delta})]_t$. Similarly define $\bar{h}_{t+1}$ and $\bar{\Delta}_{t}$, with the $b_v$’s replacing the $a_u$’s. Note that for each $r \geq 0$, we have $\|q^r\| \leq c \sqrt{n M^r} = \Theta(\sqrt{\log M})$. Therefore, $\max_{j \in [M]} |\Delta_{t}| = \Theta(n^{-\delta} \sqrt{\log M})$ for $\ell \in [L]$. Using this, it is shown in (25) that for each of the functions in (55), we have

$$
\frac{1}{L} \sum_{\ell=1}^{L} \left| \phi_{h}(h_{t+1} + \Delta_{t}, \bar{h}_{t+1} + \bar{\Delta}_{t}, \beta_{0t}) - \phi_{h}(h_{t+1}, \bar{h}_{t+1}, \beta_{0t}) \right| = o(n^{-\delta} \log M) \quad (165)
$$

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for some $\delta' > 0$. Consequently, by choosing $\delta \in (0, \delta')$ we can drop the $[\bar{Q}_{t+1} \bar{o}_{t+1} (n^{-\delta'})]_\ell$ terms. In what follows, we use the notation $h_\ell[A] = \sum_{t=0}^{t-1} (a_u + a_t \alpha_u) h_{t+1}^{u+1} + a_t [\bar{A}^* m_{t+1}]_\ell$ and $\bar{h}_\ell[\bar{A}] = \sum_{t=0}^{t-1} (b_u + b_t \alpha_u) h_{t+1}^{u+1} + b_t [\bar{A}^* m_{t+1}]_\ell$, making explicit the dependence on $\bar{A}$. We now appeal to Fact 2 to show that

$$\lim n^\delta \left[ \sum_{\ell=1}^{L} \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0t} \right) - \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E} \left\{ \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0t} \right) \right\} \right] \overset{a.s.}{=} 0 \quad (166)$$

To invoke Fact 2 (conditionally on $\mathcal{F}_{t, 0}$), we need to verify that

$$\frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E} \left[ \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0t} \right) - \mathbb{E} \left\{ \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0t} \right) \right\} \right]^{2+\kappa} \leq c L^{\kappa/2} \quad (167)$$

for constants $\kappa \in (0, 1)$. In (167), $\bar{A}, \bar{A}$ are i.i.d. copies of $A$. In [25], it is shown that for each function in $[55]$

$$\mathbb{E} \left\{ \phi_h \left( h_\ell[\bar{A}], \bar{h}_\ell[\bar{A}], \beta_{0t} \right) \right\} = \mathbb{E}_Z \left[ \phi_h \left( \sum_{t=0}^{t-1} a_u h_{t+1}^{u+1} + a_t \frac{m_{t+1}}{\sqrt{n}} Z_{\ell}, \sum_{t=0}^{t-1} b_u h_{t+1}^{u+1} + b_t \frac{m_{t+1}}{\sqrt{n}} Z_{\ell}, \beta_{0t} \right) \right]$$

where we have defined $a_u' = (a_u + a_t \alpha_u)$ and $b_u' = (b_u + b_t \alpha_u)$. Using Jensen's inequality, it can be shown that the induction hypothesis $\mathcal{H}_t(b)$ holds for the function $\phi_h^{new}$ holds whenever $\mathcal{H}_t(b)$ holds for the function $\phi_h$ inside the expectation defining $\phi_h^{new}$ in (169). We therefore have

$$\lim n^\delta \left[ \sum_{\ell=1}^{L} \phi_h^{new} \left( \sum_{t=0}^{t-1} a_u h_{t+1}^{u+1}, \sum_{t=0}^{t-1} b_u h_{t+1}^{u+1}, \beta_{0t} \right) - \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E} \left\{ \phi_h^{new} \left( \sum_{t=0}^{t-1} a_u \bar{r}_u Z_{u_{\ell}}, \sum_{t=0}^{t-1} b_u \bar{r}_u Z_{v_{\ell}}, \beta_{\ell} \right) \right\} \right] \overset{a.s.}{=} 0 \quad (170)$$

It is then shown in [25] that

$$\lim n^\delta \sum_{\ell=1}^{L} \mathbb{E} \left\{ \phi_h \left( \sum_{t=0}^{t-1} a_u \bar{r}_u Z_{u_{\ell}} + a_t \frac{m_{t+1}}{\sqrt{n}} Z_{\ell}, \sum_{t=0}^{t-1} b_u \bar{r}_u Z_{v_{\ell}} + b_t \frac{m_{t+1}}{\sqrt{n}} Z_{\ell}, \beta_{\ell} \right) \right\}
- \mathbb{E} \left\{ \phi_h \left( \sum_{t=0}^{t-1} a_u \bar{r}_u Z_{u_{\ell}} + a_t \bar{z}_t Z_{\ell}, \sum_{t=0}^{t-1} b_u \bar{r}_u Z_{v_{\ell}} + b_t \bar{z}_t Z_{\ell}, \beta_{\ell} \right) \right\} \overset{a.s.}{=} 0 \quad (171)$$
where $\zeta_t$ is the limit of $\frac{\|m_t\|}{\sqrt{n}}$. That $\zeta_t$ is well-defined and finite can be seen as follows.

$$\frac{\|m_t\|^2}{n} = \frac{\|m_t\|^2}{n} - \frac{\|m_t\|^2}{n} = \frac{\|m_t\|^2}{n} - \sum_{t=1}^{n} \sum_{t'=1}^{n} \alpha_t \alpha_{t'} \frac{(m_t)^* m_{t'}}{n}. \quad (172)$$

Each of the terms in (172) converges to a finite limit at rate $n^{-\delta}$ by $H_{t+1}(c)$ and Lemma 6. Using the definitions $a'_u = (a_u + a_t \alpha_u)$ and $b'_u = (b_v + b_t \alpha_v)$, we have for $\ell \in [L]$

$$\phi_h \left( \sum_{u=0}^{t-1} a'_u \tau_u Z_u + a_t \zeta_t Z_t, \sum_{v=0}^{t-1} b'_v \tau_v Z_v + b_t \zeta_t Z_t, \beta_t \right)$$

$$= \phi_h \left( \sum_{u=0}^{t-1} a_u \tau_u Z_u + a_t \left( \sum_{u=0}^{t-1} \alpha_u Z_u + \zeta_t Z_t, \sum_{v=0}^{t-1} b_v \tau_v Z_v + b_t \left( \sum_{v=0}^{t-1} \alpha_v Z_v + \zeta_t Z_t, \beta_t \right) \right. \right). \quad (173)$$

Thus the proof is complete if we show that the i.i.d. entries of the Gaussian random vector $\sum_{u=0}^{t-1} \alpha_u Z_u + \zeta_t Z$ have variance $\tau_t^2$. To see this, apply the proof thus far (from (164) - (173)) to the function $\phi_h(h_t, \hat{h}_t, \beta_t) = \frac{(h_0)^* h_t}{\hat{M}}$ with $a_t = b_t = 1$ and $a_u = b_u = 0$ for $0 \leq u \leq (t - 1)$. We thus obtain

$$\lim n^\delta \left[ \frac{\|h_t + \gamma_t Z_t\|^2}{M} - \frac{1}{M} \sum_{\ell=1}^{L} \mathbb{E} \|\sum_{u=0}^{t-1} \alpha_u \tau_u Z_u + \gamma_t Z_t\|^2 \right] \overset{a.s.}{=} 0. \quad (174)$$

Further, since $\sum_{u=0}^{t-1} \alpha_u Z_u + \zeta_t Z$ has i.i.d. entries, $\frac{1}{\hat{M}} \sum_{\ell=1}^{L} \mathbb{E} \|\sum_{u=0}^{t-1} \alpha_u \tau_u Z_u + \gamma_t Z_t\|^2$ equals $\mathbb{E} \left( \sum_{u=0}^{t-1} \alpha_u \tau_u Z_u + \gamma_t Z_t \right)^2$ for any $i \in [N]$. On the other hand, from $H_{t+1}(c)$ we know that

$$\lim n^\delta \left[ \frac{\|m_t\|^2}{n} - \mathbb{E} \left( \bar{\zeta}_t \zeta_t - \bar{\zeta}_t \sigma Z_w \right)^2 \right] \overset{a.s.}{=} 0,$$

The result follows since $\mathbb{E} \left( \bar{\zeta}_t \zeta_t - \bar{\zeta}_t \sigma Z_w \right)^2 = \sigma_t^2 + \sigma^2 = \tau_t^2$.

(d) By definition $q^{s+1} = q^s (\beta_0 - h^{s+1}) - \beta_0$, and hence

$$\frac{(h^{r+1})^* q^{s+1}}{n} = \frac{1}{n} \sum_{\ell=1}^{L} \phi_h(h^{r+1}, h^{s+1}, \beta_0) \quad (175)$$

for $\phi_h : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ defined as $\phi_h(h^{r+1}, h^{s+1}, \beta_0) = (h^{r+1})^* [q^s (\beta_0 - h^{s+1}) - \beta_0]$. Applying $H_{t+1}(b)$ to $\phi_h$ yields

$$\lim n^\delta \left[ \frac{1}{n} \sum_{\ell=1}^{L} \phi_h(h^{r+1}, h^{s+1}, \beta_0) - \lim \frac{1}{n} \sum_{\ell=1}^{L} \mathbb{E} \{ \tilde{\tau}_r Z_{r, s} \} \right] \overset{a.s.}{=} 0. \quad \text{Using arguments very similar to those in } H_1(d) \text{ (iterated expectations and Stein’s lemma), we obtain that}

$$\mathbb{E} \{ \tilde{\tau}_r Z_{r, s} \} \overset{\text{a.s.}}{=} \frac{\tau_s}{\hat{r}_s} \mathbb{E} \{ Z_{r, s} \} \left( \mathbb{E} \| q^s (\beta - \tilde{\tau}_s Z_s) \|^2 - n P_{\ell} \right), \quad \ell \in [L]. \quad (176)$$
Here $Z_{r_1}, Z_{s_1}$ refer to the first entries of the vectors $Z_r, Z_s$, respectively. Thus (175) becomes

$$
\lim n^\delta \left[ \frac{1}{n} \sum_{t=1}^{L} \phi_{t}^n(h_{t}^{\beta+1}, h_{t}^{s+1}, \beta_{0_t}) - \lim \frac{\tau_{t}}{\tau_{s}} \mathbb{E}[Z_{r_1}Z_{s_1}] \left( \frac{\mathbb{E}\|\eta^z(\beta - \tau s Z_s)^{2})}{n} - P \right) \right]^{\alpha s} = 0.
$$

From (43), we observe that

$$
\lambda_{s+1} = \frac{1}{\tau_{s}} \left( \frac{\|\beta^{s+1}\|^{2}}{n} - P \right) \xrightarrow{a.s.} \lim \frac{1}{\tau_{s}} \left( \frac{\mathbb{E}\|\eta^z(\beta - \tau s Z_s)^{2})}{n} - P \right) = \frac{-\sigma_{s+1}^{2}}{\sigma_{s}^{2} + \sigma^{2}}, \tag{178}
$$

where the convergence at rate $n^{-\delta}$ follows from $\mathcal{H}_{t+1}(b)$ applied to the function $\|\eta^z(\beta-h^{s+1})\|^{2} = \frac{\|\beta^{s+1}\|^{2}}{n}$. The last equality in (178) holds because $\left( P - \frac{\mathbb{E}\|\eta^z(\beta_{0} - \tau s Z_s)^{2})}{n} \right) \to \sigma_{s+1}^{2}$ (cf. Appendix A.4). Substituting (178) in (177), we see that what remains to be shown is

$$
\tilde{\tau}_{t} \mathbb{E}[Z_{r_1}Z_{s_1}] \xrightarrow{a.s.} \lim \left( \frac{m^{r}}{n} \right)^{m^{s}} \xrightarrow{a.s.} \mathbb{E}\left[ (\sigma_{r} \tilde{Z}_{r} - \sigma Z_{r}) (\sigma_{s} \tilde{Z}_{s} - \sigma Z_{s}) \right] \tag{179}
$$

The second equality above is due to $\mathcal{H}_{t+1}(c)$, which also says that $\lim \left( \frac{m^{r}}{n} \right)^{m^{s}} \xrightarrow{a.s.} \lim \left( h^{r+1}s^{1} \right) \xrightarrow{a.s.} (h^{r+1}s^{1})$. Then the first equality in (179) is obtained by applying $\mathcal{H}_{t+1}(b)$ to the function $(h^{r+1}s^{1})$ to see that

$$
\lim \left( \frac{h^{r+1}s^{1}}{N} \right) \xrightarrow{a.s.} \tilde{\tau}_{t} \mathbb{E}[Z_{r_1}Z_{s_1}] \tag{179}
$$

(e) By $\mathcal{H}_{t+1}$ part (a),

$$
\left( q^{0} \right)^{h^{r+1}} \left| q^{0}_{i}Z_{s_1} \right| = \frac{d}{\bar{\alpha}_{i}} \sum_{i=1}^{L} \left( q^{0} \right)^{h^{r+1}} \frac{m_{i}^{l}}{n} + \left( q^{0} \right)^{h^{r+1}} \frac{w_{i}^{l}}{n} + \left( q^{0} \right)^{h^{r+1}} \frac{w_{i}^{l}}{n} \tag{180}
$$

We argue that each term on the RHS approaches 0 almost surely with rate $n^{-\delta}$. This is true for the first term by the induction hypothesis $\mathcal{H}_{t}(e)$ and Lemma 6. Next, Fact 1(a) implies that $\left( q^{0} \right)^{h^{r+1}} \frac{m_{i}^{l}}{n} \xrightarrow{a.s.} \mathbb{E}[Z_{r_1}Z_{s_1}] \tag{180}$ where $Z \sim \mathcal{N}(0, 1)$. Thus the second term in (180) approaches 0 almost surely with rate $n^{-\delta}$ since $\|q^{0}\|/\sqrt{n} = \sqrt{P}$ and $\lim \|m_{i}^{l}\|/\sqrt{n}$ is a constant by $\mathcal{H}_{t}(f)$. For the third term, the result holds because $\frac{\left( q^{0} \right)^{h^{r+1}}}{n}$ converges to a constant for $r = 0, \ldots, t$, due to $\mathcal{B}_{t}(c)$.

### A Appendices

#### A.1 AMP Derivation

In [23], the dependence of $z^{t}_{a \rightarrow i}$ on $i$ is only due to the term $A_{ai}\beta^{t}_{i \rightarrow a}$ being excluded from the sum. Similarly, in [24] the dependence of $\beta^{t}_{i \rightarrow a}$ on $a$ is due to excluding the term $A_{ai}z^{t}_{a \rightarrow i}$ from the argument. We begin by estimating the order of these excluded terms.

Note that $A_{ai} = O(n^{-1/2})$, and $\beta^{t}_{i \rightarrow a} = O(\sqrt{\log n})$. The latter is true since for $i$ in section $\ell$, $\beta_{i} \leq \sqrt{n}P_{t}$, where $P_{t} = O(1/L)$, and $L = \Theta(n/\log n)$. Therefore $A_{ai}\beta^{t}_{i \rightarrow a} = O\left( \sqrt{\log n/n} \right)$. In [24], the excluded term $A_{bi}z^{t}_{b \rightarrow i}$ is $O(n^{-1/2})$ because $z^{t}_{b \rightarrow i} = O(1)$.

We set

$$
z^{t}_{a \rightarrow i} = z^{t}_{a} + \delta z^{t}_{a \rightarrow i}, \quad \text{and} \quad \beta^{t+1}_{i \rightarrow a} = \beta^{t+1}_{i} + \delta \beta^{t+1}_{i \rightarrow a}, \tag{181}
$$
Comparing (181) with (23), we can write

\[ z^t_a = y_a - \sum_{j \in [N]} A_{aj} \beta^{t}_{j \to a}, \quad \delta z^t_{a \to i} = A_{ai} \beta^{t}_{i \to a}. \]  

(182)

For \( i \in [N] \), let \( \text{sec}(i) \) denote the set of indices in the section containing \( i \). To determine \( \delta \beta^{t}_{i \to a} \), we expand \( \eta^t_i \) in (24) in a Taylor series around the argument \( \left\{ \sum_{b \in [n]} A_{bj} \hat{z}^t_{b \to j} \right\}_{j \in \text{sec}(i)} \), which does not depend on \( a \). We thus obtain

\[ \beta^{t+1}_{i \to a} \approx \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} \hat{z}^t_{b \to j} \right\}_{j \in \text{sec}(i)} \right) - A_{ai} \hat{z}^t_{a \to i} \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} \hat{z}^t_{b \to j} \right\}_{j \in \text{sec}(i)} \right), \]  

(183)

where \( \partial_i \eta^t_i(.) \) is the partial derivative of \( \eta^t_i \) with respect to the component of the argument corresponding to index \( i \). (Recall from (11) that the argument is a length \( M \) vector.) From (11), the partial derivative can be evaluated as

\[ \partial_i \eta^t_i(s) = \eta^t_i(s) \partial_i \ln \eta^t_i(s) = \eta^t_i(s) \left( \frac{\sqrt{n P_0}}{\tau^2} - \frac{\sqrt{n P_0}}{\tau^2} \sum_{j \in \text{sec}(i)} \exp \left( \frac{s_j \sqrt{n P_0}}{\tau^2} \right) \right) = \eta^t_i(s) \left( \frac{\sqrt{n P_0}}{\tau^2} - \eta^t_i(s) \right). \]  

(184)

Using (184) in (183) yields

\[ \beta^{t+1}_{i \to a} = \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} \hat{z}^t_{b \to j} \right\}_{j \in \text{sec}(i)} \right) - \frac{A_{ai} \hat{z}^t_{a \to i}}{\tau^2} \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} \hat{z}^t_{b \to j} \right\}_{j \in \text{sec}(i)} \right) \left[ \sqrt{n P_0} - \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} \hat{z}^t_{b \to j} \right\}_{j \in \text{sec}(i)} \right) \right]. \]  

(185)

Notice that we have replaced the stand-alone term \( A_{ai} \hat{z}^t_{a \to i} \) in (183) with \( A_{ai} \hat{z}^t_{a} \) because the difference \( A_{ai} \delta \hat{z}^t_{a \to i} \) is \( O(\sqrt{\log n}/n) \), which can be ignored — we only keep terms as small as \( O(n^{-1/2}) \).

Since only the second term on the right-hand side of (185) depends on \( a \), we can write

\[ \beta^{t+1}_{i \to a} = \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} (z^t_b + \delta z^t_{b \to j}) \right\}_{j \in \text{sec}(i)} \right), \]  

(186)

and

\[ \delta \beta^{t}_{i \to a} = -\frac{A_{ai} \hat{z}^t_{a}}{\tau^2} \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} (z^t_b + \delta z^t_{b \to j}) \right\}_{j \in \text{sec}(i)} \right) \left[ \sqrt{n P_0} - \eta^t_i \left( \left\{ \sum_{b \in [n]} A_{bj} (z^t_b + \delta z^t_{b \to j}) \right\}_{j \in \text{sec}(i)} \right) \right]. \]  

(187)

We observe that \( \delta \beta^{t}_{i \to a} = O(\log n/\sqrt{n}) \). Hence, in (182), we can write

\[ \delta z^t_{a \to i} = A_{ai} \beta^t_i. \]  

(188)
because the difference $A_{a_k} \delta \beta^{t}_{k \to a} = O(\log n/n)$. Substituting (188) in (186), we see that

$$\beta^{t+1} = \eta^{t}_1 \left( \left\{ \sum_{b \in [n]} A_{b_j} z^b_k + A_{b_j} \beta^t_j \right\}_{j \in \text{sec}(i)} \right) = \eta^{t}_1 \left( \left\{ (A^* z^t + \beta^t_j) \right\}_{j \in \text{sec}(i)} \right),$$

(189)

where (a) holds because $\sum_{b} A_{b_j}^2 \rightarrow 1$ as $n \to \infty$. Analogously, using (188) in (187) gives

$$\frac{\delta \beta^{t+1}_{k \to a}}{\tau^t} \eta^{t}_1 \left( \left\{ (A^* z^t + \beta^t_j) \right\}_{j \in \text{sec}(i)} \right) \left[ \sqrt{n P_{\text{sec}(k)}} - \eta^{t-1}_k (A^* z^{t-1} + \beta^{t-1}) \right].$$

(190)

Finally, we use (189) and (190) in (182) to obtain

$$z^t_a = y_a - \sum_{k \in [N]} A_{a_k} (\beta^t_k + \delta \beta^{t+1}_{k \to a}) = y_a - \sum_{k \in [N]} A_{a_k} \eta^{t-1}_k (A^* z^{t-1} + \beta^{t-1}) + \frac{A_{a_k}^2 z^t_a}{\tau^{t-1}} \eta^{t-1}_k (A^* z^{t-1} + \beta^{t-1}) \left[ \sqrt{n P_{\text{sec}(k)}} - \eta^{t-1}_k (A^* z^{t-1} + \beta^{t-1}) \right] \left( b \right) a + \frac{z^t_a}{n \tau^{t-1}} (n P - \| \beta^t \|^2),$$

(191)

where (b) is obtained as follows. Then, we use $A_{a_k}^2 \approx \frac{1}{n}$. Next, (11) implies that for all $s$, \n
$$\sum_{k \in [N]} \sqrt{n P_{\text{sec}(k)}} \eta^{t}_k (s) = \sum_{\ell = 1}^{L} n P_{\ell} = n P.$$

Finally, note from (189) that $\sum_k (\eta^{t-1}_k (A^* z^{t-1} + \beta^{t-1}))^2 = \sum_k (\beta^t_k)^2 = \| \beta^t \|^2$. The AMP update equations are thus given by (191) and (189).

A.2 Proof of Lemma 1

From (27), $x(\tau)$ can be written as

$$x(\tau) := \sum_{\ell = 1}^{L} \frac{n P_{\ell}}{\tau} E_{\ell}$$

(192)

where

$$E_{\ell} = \mathbb{E} \left[ \exp \left( \frac{\sqrt{n P_{\ell}}}{\tau} U_{1,\ell} \right) \left( \exp \left( \frac{\sqrt{n P_{1}}}{\tau} U_{1,\ell} \right) + \exp \left( -\frac{n P_{1}}{\tau^2} \right) \sum_{j = 2}^{M} \exp \left( \frac{\sqrt{n P_{j}}}{\tau} U_{j,\ell} \right) \right) \right].$$

(193)

We will prove the lemma by showing that for $\ell = 1, \ldots, L$:

$$\lim E_{\ell} = \left\{ \begin{array}{ll} 1, & \text{if } c_{\ell} > 2(\ln 2)R \tau^2, \\ 0, & \text{if } c_{\ell} < 2(\ln 2)R \tau^2, \end{array} \right.$$

(194)

where $c_{\ell} = \lim LP_{\ell}$ as defined in (30). Using the relation $n R = \frac{L \ln M}{2}$, we can write

$$\frac{n P_{\ell}}{\tau^2} = \nu_{\ell} \ln M,$$

(195)

\footnote{We can also prove that $\lim E_{\ell} = \frac{1}{2}$ if $c_{\ell} = 2(\ln 2)R \tau^2$, but we do not need this for the exponentially decaying power allocation since $c_{\ell}$ exactly equals $2(\ln 2)R \tau^2$ for only a vanishing fraction of sections. Since $E_{\ell} \in [0, 1]$, these sections do not affect the value of $\lim x(\tau)$ in (192).}
where \( \nu_\ell = \frac{LP_t}{N_{\ell}^2 \ln 2} \). Hence \( \mathcal{E}_\ell \) in (193) can be written as

\[
\mathcal{E}_\ell = E \left[ \frac{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right)}{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right) + M^{-\nu_\ell} \sum_{j=2}^M \exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_j^\ell \right)} \right],
\]

(196)

\[
= E \left[ \frac{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right)}{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right) + M^{-\nu_\ell} \sum_{j=2}^M \exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_j^\ell \right) \mid U_1^\ell} \right].
\]

(197)

The inner expectation in (197) is of the form

\[
E \left[ \frac{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right)}{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right) + M^{-\nu_\ell} \sum_{j=2}^M \exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_j^\ell \right) \mid U_1^\ell} \right] = \mathcal{E}_X \left[ \frac{c}{c + X} \right],
\]

(198)

where \( c = \exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right) \) is treated as a positive constant, and the expectation is with respect to the random variable

\[
X := M^{-\nu_\ell} \sum_{j=2}^M \exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_j^\ell \right).
\]

(199)

**Case 1:** \( \lim \nu_\ell > 2 \). Since \( \frac{c}{c + X} \) is a convex function of \( X \), applying Jensen’s inequality we get

\[
\mathcal{E}_X \left[ \frac{c}{c + X} \right] \geq \frac{c}{c + \mathcal{E}X}.
\]

(200)

The expectation of \( X \) is

\[
\mathcal{E}X = M^{-\nu_\ell} \sum_{j=2}^M \mathcal{E} \left[ \exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_j^\ell \right) \right] = M^{-\nu_\ell} (M - 1) M^{\nu_\ell/2} \leq M^{1-\nu_\ell/2}
\]

(201)

where \( (a) \) is obtained using the moment generating function of a Gaussian random variable. We therefore have

\[
1 \geq \mathcal{E}_X \left[ \frac{c}{c + X} \right] \geq \frac{c}{c + \mathcal{E}X} \geq \frac{c}{c + M^{1-\nu_\ell/2}}.
\]

(202)

Recalling that \( c = \exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right), \) (202) implies that

\[
\mathcal{E}_X \left[ \frac{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right)}{\exp \left( \sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right) + X \mid U_1^\ell} \right] \geq \frac{1}{1 + M^{1-\nu_\ell/2} \exp \left( -\sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell \right)}.
\]

(203)

When \( \{U_1^\ell > -(\ln M)^{1/4}\} \), the RHS of (203) is at least \( [1 + M^{1-\nu_\ell/2} \exp \left( (\ln M)^{3/4} \sqrt{\nu_\ell} \right)]^{-1} \). Using this in (197), we obtain that

\[
1 \geq \mathcal{E}_\ell \geq P(U_1^\ell > -(\ln M)^{1/4}) \cdot \frac{1}{1 + M^{1-\nu_\ell/2} \exp \left( (\ln M)^{3/4} \sqrt{\nu_\ell} \right)} M^{\ell \to \infty} 1 \text{ since } \nu_\ell > 2.
\]

Hence \( \mathcal{E}_\ell \to 1 \) when \( \nu_\ell > 2 \).
Case 2: \(\lim \nu_\ell < 2\). The random variable \(X\) in (199) can be bounded from below as follows.

\[
X \geq M^{-\nu_\ell} \max_{j \in \{2, \ldots, M\}} \exp \left(\sqrt{\ln M} \sqrt{\nu_\ell} U_j^\ell\right) = M^{-\nu_\ell} \exp \left(\max_{j \in \{2, \ldots, M\}} U_j^\ell\right) \sqrt{\ln M} \sqrt{\nu_\ell}.
\] (204)

Using standard bounds for the standard normal distribution, it can be shown that

\[
P \left( \max_{j \in \{2, \ldots, M\}} U_j^\ell < \sqrt{2 \ln M} (1 - \epsilon) \right) \leq \exp(-M^{\epsilon(1 - \epsilon)}),
\] (205)

for \(\epsilon = \omega \left(\frac{\ln \ln M}{\ln M}\right)\). \footnote{Recall that \(f(n) = \omega(g(n))\) if for each \(k > 0\), \(|f(n)|/|g(n)| \geq k\) for sufficiently large \(n\).} Combining (205) and (204), we obtain that

\[
\exp(-M^{\epsilon(1 - \epsilon)}) \geq P \left( \max_{j \in \{2, \ldots, M\}} U_j^\ell < \sqrt{2 \ln M} (1 - \epsilon) \right)
\]

\[
\geq P \left( X < M^{-\nu_\ell} \exp \left(\sqrt{2 \ln M} (1 - \epsilon) \sqrt{\ln M} \sqrt{\nu_\ell}\right) \right) = P \left( X < M^{2\nu_\ell(1 - \epsilon) - \nu_\ell} \right).
\] (206)

Since \(\lim \nu_\ell < 2\) and \(\epsilon > 0\) can be arbitrary small, there exists a strictly positive constant \(\delta\) such that \(\delta < \sqrt{2\nu_\ell(1 - \epsilon) - \nu_\ell}\) for all sufficiently large \(L\). Therefore, for sufficiently large \(M\), the expectation in (198) can be bounded as

\[
\mathbb{E}_X \left[ \frac{c}{c + X} \right] \leq P(X < M^\delta) \cdot 1 + P(X \geq M^\delta) \cdot \frac{c}{c + M^\delta}
\]

\[
\leq \exp(-M^{\epsilon(1 - \epsilon)}) + 1 \cdot \frac{c}{c + M^\delta} \leq \frac{2}{1 + c^{-1}M^\delta}.
\] (207)

Recalling that \(c = \exp \left(\sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell\right)\), and using the bound of (207) in (197), we obtain

\[
\mathcal{E}_\ell \leq \mathbb{E} \left[ \frac{1}{1 + M^\delta \exp \left(-\sqrt{\ln M} \sqrt{\nu_\ell} U_1^\ell\right)} \right]
\]

\[
\leq P(U_1^\ell > (\ln M)^{1/4}) \cdot 1 + P(U_1^\ell \leq (\ln M)^{1/4}) \cdot \frac{1}{1 + M^\delta \exp(-\sqrt{\nu_\ell}(\ln M)^{3/4})}
\]

\[
\overset{(a)}{\leq} \exp(-\frac{1}{2}(\ln M)^{1/2}) + 1 \cdot \frac{1}{1 + \exp \left(\delta \ln M - \sqrt{\nu_\ell}(\ln M)^{3/4}\right)} \overset{(b)}{\rightarrow} 0 \text{ as } M \rightarrow \infty.
\] (208)

In (208), (a) is obtained using the bound \(\Phi(x) < \exp(-x^2/2)\) for \(x \geq 0\), where \(\Phi(\cdot)\) is the Gaussian cdf; (b) holds since \(\delta\) and \(\lim \nu_\ell\) are both positive constants.

This proves that \(\mathcal{E}_\ell \rightarrow 0\) when \(\lim \nu_\ell < 2\). The proof of the lemma is complete since we have proved both statements in (194).

A.3 Proof of Lemma 2

For \(t = 0, \tau_0 = \sigma^2 + P\). From Lemma 1, we have

\[
\bar{x}_1 = \lim_{L \rightarrow \infty} \sum_{\ell = 1}^L \frac{P_\ell}{P} 1\{c_\ell > 2(\ln 2)R(\sigma^2 + P)\} = \lim_{L \rightarrow \infty} \sum_{\ell = 1}^L \frac{P_\ell}{P} 1 \left\{ \ell < \frac{\log(C/R)}{2C} \right\},
\] (209)
where the second equality is obtained using the expression for \( c_\ell \) in (30) and simplifying. Substituting \( \frac{\log(C/R)}{2c} = \xi_0 \), and using the geometric series formula

\[
\sum_{\ell=1}^{k} P_\ell = (P + \sigma^2)(1 - 2^{-2ck/L}),
\]

(210) becomes

\[
\bar{x}_1 = \lim_{L \to \infty} \sum_{\ell=1}^{[\xi_0L]} \frac{P_\ell}{P} = \frac{P + \sigma^2}{P} (1 - 2^{-2c\xi_0}) = \frac{(1 + \text{snr}) - (1 + \text{snr})^{1-\xi_0}}{\text{snr}}. \tag{211}
\]

The expression for \( \bar{x}_2^2 \) is a straightforward simplification of \( \sigma^2 + P(1 - \bar{x}_1) \).

Assume towards induction that (31) and (32) hold for \( x_t, \bar{x}_t^2 \). For step \( (t+1) \), from Lemma 1, we have

\[
\bar{x}_{t+1} = \lim_{L \to \infty} \sum_{\ell=1}^{L} \frac{P_\ell}{P} 1\{c_\ell > 2(\ln 2)R \bar{x}_t^2\} = \lim_{L \to \infty} \sum_{\ell=1}^{L} \frac{P_\ell}{P} 1\{ \ell < \frac{1}{2C} \log \frac{C(\sigma^2)}{R \bar{x}_t^2} \}, \tag{212}
\]

where the second equality is obtained using the expression for \( c_\ell \) in (30) and simplifying. Using the induction hypothesis for \( \bar{x}_t^2 \), we get

\[
\frac{(P + \sigma^2)}{\bar{x}_t^2} = \frac{(P + \sigma^2)}{\sigma^2 (1 + \text{snr})^{1-\xi_{t-1}}} = (1 + \text{snr})^{\xi_{t-1}} = 2^{2c\xi_{t-1}}. \tag{213}
\]

Hence

\[
\frac{1}{2C} \log \frac{C(\sigma^2)}{R \bar{x}_t^2} = \frac{1}{2C} \log \left( \frac{C}{R} \right) + \xi_{t-1}. \tag{214}
\]

Using (214) in (212), we obtain

\[
\bar{x}_{t+1} = \lim_{L \to \infty} \sum_{\ell=1}^{[\xi_tL]} \frac{P_\ell}{P} = \frac{P + \sigma^2}{P} (1 - 2^{-2c\xi_t}) = \frac{(1 + \text{snr}) - (1 + \text{snr})^{1-\xi_t}}{\text{snr}}. \tag{215}
\]

The proof is concluded by using (215) to compute \( \bar{x}_{t+1}^2 = P + \sigma^2(1 - \bar{x}_{t+1}) \).

A.4 The limit of \( \frac{1}{n} \mathbb{E}\{ \eta^r(\beta - \bar{\tau}_r Z_r) - \beta^* \eta^s(\beta - \bar{\tau}_s Z_s) - \beta \} \) equals \( \bar{\sigma}_{s+1}^2 \) for \( r \leq s \).

Since \( \|\beta\|_2^2 = P \), the required limit is

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\{ \eta^r(\beta - \bar{\tau}_r Z_r) \} = \frac{1}{n} \mathbb{E}\{ \beta^* \eta^r(\beta - \bar{\tau}_r Z_r) \} - \frac{1}{n} \mathbb{E}\{ \beta^* \eta^s(\beta - \bar{\tau}_s Z_s) \} + P. \tag{216}
\]

For \( r \leq s \), we prove that the limit in (216) equals \( \bar{\sigma}_{s+1}^2 = \sigma^2((1 + \text{snr})^{1-\xi_s} - 1) \) by showing the following:

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\{ \beta^* \eta^r(\beta - \bar{\tau}_r Z_r) \} = \sigma^2 \left( (1 + \text{snr}) - (1 + \text{snr})^{1-\xi_r} \right), \tag{217}
\]

\[
\frac{1}{n} \mathbb{E}\{ ||\eta^r(\beta - \bar{\tau}_r Z_r)||^2 \} = \frac{1}{n} \mathbb{E}\{ \beta^* \eta^r(\beta - \bar{\tau}_r Z_r) \}, \tag{218}
\]

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\{ \eta^r(\beta - \bar{\tau}_r Z_r) \} = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}\{ \beta^* \eta^r(\beta - \bar{\tau}_r Z_r) \}, \quad \text{for } r < s. \tag{219}
\]
Since $\beta$ is distributed uniformly over the set $B_{M,L}$, the expectation in (217) can be computed by assuming that $\beta$ has a non-zero in the first entry of each section. Thus

$$\lim \frac{1}{n} \mathbb{E}\{\eta^*(\beta - \bar{\eta}_sZ_r)\} = \lim \sum_{l=1}^L P_l \mathbb{E}\left[ \frac{\exp\left(\frac{nP_r}{\tau_r}\right) \exp\left(\frac{\sqrt{nP_r} U^l_r}{\tau_r}\right)}{\exp\left(\frac{nP_r}{\tau_r}\right) \exp\left(\frac{\sqrt{nP_r} U^l_r}{\tau_r}\right) + \sum_{j=2}^M \exp\left(\frac{\sqrt{nP_r} U^l_j}{\tau_r}\right)} \right].$$

$$= \sum_{l=1}^L P_l 1\{c_l > 2(\ln 2) R_2\} \quad \text{(a)} \quad \text{(220)}$$

In (220), $\{U^l_j\}$ with $\ell \in [L], j \in [M]$ is just a relabeled version of $-Z_r$, and is thus i.i.d. $\mathcal{N}(0,1)$.

The equality (a) is obtained from (193) and (194) in Appendix A.2 noting that $c_l = \lim L P_l$ while (b) follows from Lemmas 1 and 2 (cf. (28) and (31)).

Since $\beta^{r+1}(s) = \eta^r(s)$, (218) was proved in Proposition 3.1 (cf. (19) and (20)). Next, from the Cauchy-Schwarz inequality, we have

$$\frac{1}{n} \mathbb{E}\{\eta^r(\beta - \bar{\eta}_sZ_r)\}^{\eta^g(\beta - \bar{\eta}_sZ_s)} \leq \frac{1}{n} \sum_{l=1}^L (\mathbb{E}\{\|\eta^r(\beta - \bar{\eta}_sZ_r)\|^2\} \mathbb{E}\{\|\eta^g(\beta - \bar{\eta}_sZ_s)\|^2\})^{1/2}$$

$$= \sum_{\ell} P_l 1\{c_{\ell} > 2(\ln 2) R_{2\ell}^2\} \quad \text{(a)} \quad \text{(221)}$$

where (a) follows from (218) and (220), and (b) holds because $\bar{\eta}_{2r}^2 > \bar{\eta}_{2s}^2$ since $r < s$.

Since $\beta$ is distributed uniformly over the set $B_{M,L}$, the expectation $\mathbb{E}\{[\eta^r(\beta - \bar{\eta}_rZ_r)]^* [\eta^g(\beta - \bar{\eta}_sZ_s)]\}$ can be computed by assuming that $\beta$ has a non-zero in the first entry of each section:

$$\frac{1}{n} \mathbb{E}\{\eta^r(\beta - \bar{\eta}_rZ_r)\}^{\eta^g(\beta - \bar{\eta}_sZ_s)} = \frac{1}{n} \sum_{\ell} \mathbb{E}\{[\eta^r(\beta - \bar{\eta}_rZ_r)]^* [\eta^g(\beta - \bar{\eta}_sZ_s)]\} = \sum_{\ell} P_l \mathcal{E}_{rs,\ell} \quad \text{(222)}$$

where

$$\mathcal{E}_{rs,\ell} = \mathbb{E}\left[ \frac{\exp\left(\frac{nP_r}{\tau_r}\right) \exp\left(\frac{\sqrt{nP_r} U^l_r}{\tau_r}\right)}{\exp\left(\frac{nP_r}{\tau_r}\right) \exp\left(\frac{\sqrt{nP_r} U^l_r}{\tau_r}\right) + \sum_{j=2}^M \exp\left(\frac{\sqrt{nP_r} U^l_j}{\tau_r}\right)} \right].$$

In (223), the pairs of random variables $\{U^l_{rj}, U^l_{sj}\}, j \in [M]$ are i.i.d. across index $j$, and for each $j, U^l_{rj}$ and $U^l_{sj}$ are jointly Gaussian with $\mathcal{N}(0,1)$ marginals.
The expectation of the first term on the right-hand side of (223) can be written as
\[
\mathbb{E} \left[ \frac{\exp(\frac{\sqrt{nR}}{\tau_r} U_{r1}^{\ell})}{\exp(\frac{\sqrt{nR}}{\tau_r} U_{r1}) + \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \sum_{j=2}^{M} \exp(\frac{\sqrt{nR}}{\tau_r} U_{rj}^{\ell})} \right] \left[ \frac{\exp(\frac{\sqrt{nR}}{\tau_r} U_{s1}^{\ell})}{\exp(\frac{nP_{\ell}}{2\tau_r^2}) + \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \sum_{j=2}^{M} \exp(\frac{\sqrt{nR}}{\tau_r} U_{sj}^{\ell})} \right] | U_{r1}^{\ell}, U_{s1}^{\ell} \]
\[
\geq \mathbb{E} \left[ \frac{\exp(\frac{\sqrt{nR}}{\tau_r} U_{r1}^{\ell})}{1 + M \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \exp(-\frac{\sqrt{nR}}{\tau_r} U_{r1}^{\ell})} \right] \left[ \frac{\exp(\frac{\sqrt{nR}}{\tau_r} U_{s1}^{\ell})}{1 + M \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \exp(-\frac{\sqrt{nR}}{\tau_r} U_{s1}^{\ell})} \right]
\]
\[
= \mathbb{E} \left[ \frac{1}{1 + M \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \exp(-\frac{\sqrt{nR}}{\tau_r} U_{r1}^{\ell})} \right] \left[ \frac{1}{1 + M \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \exp(-\frac{\sqrt{nR}}{\tau_r} U_{s1}^{\ell})} \right]
\]
\[
\geq P \left( U_{r1}^{\ell} > -\left( \frac{\sqrt{nR}}{\tau_r} \right)^{1/2} \right) P \left( U_{s1}^{\ell} > -\left( \frac{\sqrt{nR}}{\tau_r} \right)^{1/2} \right)
\cdot \left( \frac{1}{1 + M \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \exp(-\frac{\sqrt{nR}}{\tau_r} U_{r1}^{\ell})} \right) \left( \frac{1}{1 + M \exp(-\frac{nP_{\ell}}{2\tau_r^2}) \exp(-\frac{\sqrt{nR}}{\tau_r} U_{s1}^{\ell})} \right)
\]
\[
\xrightarrow{(b)} 1 \text{ as } M \to \infty \text{ if } \lim_{M \to \infty} \frac{nP_{\ell}}{2\tau_r^2} \ln M > 1.
\]

In (224), (a) is obtained as follows. The inner expectation on the first line of the form \( \mathbb{E}_{X,Y} [f(X,Y)] \) with \( f(X,Y) = \frac{\kappa_1}{\kappa_1 + X} \cdot \frac{\kappa_2}{\kappa_2 + Y} \), where \( \kappa_1, \kappa_2 \) are positive constants. Since \( f \) is a convex function of \( (X,Y) \), Jensen’s inequality implies \( \mathbb{E}[f(X,Y)] \geq f(\mathbb{E}X, \mathbb{E}Y) \), with \( \mathbb{E}[\exp(\frac{\sqrt{nR}}{\tau_r} U_{r1}^{\ell})] = \exp(\frac{nP_{\ell}}{2\tau_r^2}) \).

Since \( \mathcal{E}_{rs,\ell} \) in (223) lies in \([0,1]\), (224) implies that
\[
\lim \mathcal{E}_{rs,\ell} = 1 \text{ if } \lim_{M \to \infty} \frac{nP_{\ell}}{2\tau_r^2} \ln M = \frac{c_\ell}{2R\tau_r^2 \ln 2} > 1.
\]
\[
(225)
\]
where we have used \( nR = L \log M \), noting that \( c_\ell := \lim L P_{\ell} \). Using this in (222), we conclude that
\[
\frac{1}{P_{\ell}} \mathbb{E} \left( \eta^\ell(\beta - \tau_r Z) \right) < \frac{n}{P_{\ell}} \left( \frac{1}{P_{\ell}} \mathbb{E} \left( \eta^\ell(\beta - \tau_r Z) \right) \right) = \frac{1}{P_{\ell}} \left( \frac{1}{P_{\ell}} \mathbb{E} \left( \eta^\ell(\beta - \tau_r Z) \right) \right) \geq \sum_{\ell} P_{\ell} \mathbb{1} \left( c_\ell > 2(2n)R\tau_r^2 \right). \]
Together with the upper bound in (221), this proves (219), and hence completes the proof.

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