Boltzmann equation for non-equilibrium particles and its application to non-thermal dark matter production

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Abstract

We consider a scalar field (called \(\phi\)) which is very weakly coupled to thermal bath, and study the evolution of its number density. We use the Boltzmann equation derived from the Kadanoff-Baym equations, assuming that the degrees of freedom in the thermal bath are well described as “quasi-particles.” When the widths of quasi-particles are negligible, the evolution of the number density of \(\phi\) is well governed by a simple Boltzmann equation, which contains production rates and distribution functions both evaluated with dispersion relations of quasi-particles with thermal masses. We pay particular attention to the case that dark matter is non-thermally produced by the decay of particles in thermal bath, to which the above mentioned formalism is applicable. When the effects of thermal bath are properly included, the relic abundance of dark matter may change by \(O(10-100\%)\) compared to the result without taking account of thermal effects.
1 Introduction

In particle cosmology, it is inevitable to consider the behavior of quantum fields (or, in other words, particles) in thermal bath because the universe was filled with hot plasma in the early epoch. The detailed thermal effects depend on how the particle of our interest, which we call φ, interacts with degrees of freedom in thermal bath. Importantly, even if φ is so weakly interacting that it is not in thermal equilibrium, there can be non-negligible thermal effects on its dynamics. This is because the interaction rate of φ surrounded by thermal bath depends on the properties (in particular, the dispersion relation) of the degrees of freedom in thermal bath, which can of course be significantly affected by thermal effects.

One important example of such a very weakly interacting particle is non-thermal dark matter which is produced by the decay of particles in thermal bath. Although the existence of dark matter is strongly suggested by various cosmological observations [1], particle-physics properties of dark matter, as well as its production mechanism in the early universe, have not been understood yet. Various particle physics models including dark matter candidate have been proposed so far, like supersymmetric models, universal extra dimension models, and so on [2]. In the future, it is hoped that those models are tested by high energy experiments as well as by cosmological observations. In particular, the candidate of the dark matter particle may be discovered and studied by the LHC experiment as well as future linear colliders, based on which a large class of dark matter models are discriminated. For this program, precise theoretical calculation of the relic abundance of the dark matter candidate should be performed by using information about newly discovered particles [3]. The present dark matter density is very accurately determined by the WMAP collaboration as [4]:

\[
\Omega_c h^2 = 0.1126 \pm 0.0036, \tag{1.1}
\]

with \( h \) being the Hubble constant in units of 100 km/sec/Mpc, so the dark matter abundance is now known with \( O(1\%) \) accuracy. Thus, it is desirable to establish methods of calculating the dark matter density at the same level of accuracy. For this purpose, detailed understanding of thermal effects on the dark matter production process is required.

In the present study, we pay particular attention to the case that dark matter particle is non-thermally produced by the decay of heavier particles in thermal bath. There are many examples of such non-thermally produced dark matter, like gravitino [4], axino [6], a singlet field [7], the right handed sneutrino [8], and more generally, the recently proposed “freeze-in” particles [9]. In such a scenario, the relic density of dark matter is determined at the cosmic temperature comparable to the mass of decaying particle and is insensitive to the thermal history before that. In the previous studies, the production rate of dark matter has been calculated by using the decay rates of particles estimated in vacuum. However, in the actual situation, the particles decay in the thermal bath, so the production rate taking account of the thermal effects should be properly used in the calculation of the dark matter density. As we will see, the thermal effects may significantly change the resultant abundance of dark matter.
In this paper, we raise the question how important the thermal effects are in the production process of non-thermal dark matter. To answer this question, we first study the properties of Boltzmann equation derived from the Kadanoff-Baym equations [10] under the assumption that the production of dark matter does not affect the thermal bath. In particular, unless the dark matter production is almost kinematically blocked by thermal masses during the time when the production of dark matter is most effective, the full Boltzmann equation can be reduced to a simplified form which has the same structure as the conventional Boltzmann equation #1 but constructed with “thermal masses” of particles in thermal bath. (As we will see, such a simplified Boltzmann equation is obtained by taking the “zero-width approximation” of particles in thermal bath.) We evaluate the relic density of dark matter by solving (i) the full, (ii) zero-width approximated, and (iii) conventional Boltzmann equations. Comparing the three results, we discuss how important the thermal effects are and when the zero-width approximation breaks down. We will see that the dark matter abundance may be reduced by $O(10^{-100} \%)$ compared with the result of calculation where the thermal effects are neglected.

The organization of this paper is as follows. In Section 2, the relevant formulae to study the evolution of the number density of non-equilibrium particles are summarized. In particular, properties of the Boltzmann equation to be solved are discussed. Then, in Section 3, we apply the formalism to the non-thermal dark matter production process. We numerically solve the Boltzmann equation and discuss how important the thermal effects are. Section 4 is devoted to conclusions and discussion.

2 Formalism

First, let us introduce the formulae and equations relevant for our analysis. Although many of them can be found in literature (see, for instance, [10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]), we summarize the relevant equations to make this paper self-contained for the sake of readers. We assume that the thermal bath have a common temperature $T$, which have a large degrees of freedom, and the back reaction to the thermal bath from the production of $\phi$ is negligible.

In this paper, we study the evolution of the number density of a scalar field $\phi$ coupled to scalar fields in thermal bath, which are denoted as $\chi_i$. We introduce the interaction of the following form:

$$\mathcal{L}_{\text{int}} = g\phi \prod_{i=1}^{n} \chi_i = g\phi \mathcal{O}[\chi_0, \chi_1, \cdots],$$

(2.1)

where $g$ is a coupling constant and, for the convenience of the following discussion, we introduced the operator $\mathcal{O} \equiv \prod_{i} \chi_i$. The interaction of $\phi$ is assumed to be extremely small,

#1 In this paper, the “conventional” Boltzmann equation refers to the Boltzmann equation evaluated with zero-temperature dispersion relations [11].

2
i.e., \(gm_{\phi}^{-3} \ll 1\), where \(m_{\phi}\) is the mass of \(\phi\). In the following, we study the effects which are leading order in \(g\). For simplicity, we consider the case that \(\mathcal{O}\) is given by a product of scalar fields. However, the extension of the formalism to the case that \(\mathcal{O}\) includes derivatives of scalar fields is straightforward.

The evolution of the number density of \(\phi\) in the early universe is governed by two effects: one is the production of \(\phi\) due to the decay and scattering processes and the other is the cosmic expansion. We discuss these effects separately.

Because of the weakness of the interaction, \(\phi\) can be regarded as (almost) free particle, and the number density operator is given by

\[
\hat{N}_k(t) = \frac{1}{2\omega_k} \left[ \dot{\phi}(t; k)\dot{\phi}(t; -k) + \omega_k^2 \phi(t; k)\phi(t; -k) \right],
\]

where the “dot” denotes the derivative with respect to time, \(\dot{\cdots}\) is the normal ordering and

\[
\omega_k \equiv \sqrt{k^2 + m_{\phi}^2}.
\]

In addition,

\[
\hat{\phi}(t; k) \equiv L^{-3/2} \int d^3 x e^{-ik\cdot x} \hat{\phi}(t, x),
\]

where \(\hat{\phi}\) is the field operator for \(\phi\) and \(L^3\) is the volume of the system (where we have adopted box normalization). For the later convenience, the number density for each momentum eigenstate is defined. Then, the expectation value of the number density of \(\phi\) is obtained by using the density matrix \(\hat{\rho}\):\[
N_k(t) \equiv \langle \hat{N}_k(t) \rangle,
\]

where, for an operator \(\hat{A}\),

\[
\langle \hat{A} \rangle \equiv \text{tr}[\hat{\rho} \hat{A}].
\]

The \(\chi\) sector is in the thermal bath while \(\phi\) is always out of thermal equilibrium. In particular, we are interested in the case that \(\phi\) is initially absent in the system. Thus, we assume that the initial density matrix (at \(t = t_i\)) is given by the direct product of density matrices of two sectors:

\[
\hat{\rho} = \hat{\rho}_{\phi, i} \otimes \hat{\rho}_\chi.
\]
We set \( t_i = 0 \) without loss of generality. The \( \chi \) sector is in the thermal bath (with the temperature \( T \)), so \( \hat{\rho}_\chi \) is given by

\[
\hat{\rho}_\chi = e^{-\hat{H}_\chi/T},
\]

(2.8)

with \( \hat{H}_\chi \) being the Hamiltonian for the \( \chi \) sector. On the contrary, \( \hat{\rho}_{\phi,i} \) determines the initial distribution of \( \phi \). We assume that it has translational invariance, i.e., \([\hat{P}, \hat{\rho}_{\phi,i}] = 0\), where \( \hat{P} \) is the momentum operator. Furthermore, we assume \( \langle \hat{\phi}(t = 0) \rangle = \langle \hat{\phi}(t = 0) \rangle = 0 \).

In order to calculate the evolution of \( N_k(t) \), we define the Hadamard propagator and the Jordan propagator:

\[
G^\phi_H(t, t'; k) \equiv \langle \hat{\phi}(t; k)\hat{\phi}(t'; -k) \rangle + \langle \hat{\phi}(t'; -k)\hat{\phi}(t; k) \rangle, \tag{2.9}
\]

\[
G^\phi_J(t, t'; k) \equiv \langle \hat{\phi}(t; k)\hat{\phi}(t'; -k) \rangle - \langle \hat{\phi}(t'; -k)\hat{\phi}(t; k) \rangle. \tag{2.10}
\]

As can be seen from Eqs. (2.2) and (2.5), the expectation value of the number density is given by

\[
N_k(t) \equiv \frac{1}{4\omega_k} \left[ (\partial_t^2 + \omega_k^2)G^\phi_H(t, t'; k) \right]_{t \to t} - C_k, \tag{2.11}
\]

where \( C_k \) is normal-ordering constant. In the weak coupling limit, \( C_k = \frac{1}{2} \). The Hadamard propagator and the Jordan propagator satisfy the following equations, namely the Kadanoff-Baym equations [10] [12]

\[
(\partial_t^2 + \omega_k^2) G^\phi_J(t, t'; k) = -\int_0^t d\tau \Pi^\phi_{ret}(t - \tau; k) G^\phi_J(\tau, t'; k), \tag{2.12}
\]

\[
(\partial_t^2 + \omega_k^2) G^\phi_H(t, t'; k) = -\int_0^t d\tau \Pi^\phi_{ret}(t - \tau; k) G^\phi_H(\tau, t'; k)
- i\int_0^t d\tau \Pi^\phi_{ret}(t - \tau; k) G^\phi_J(\tau, t'; k), \tag{2.13}
\]

where

\[
\Pi^\phi_{ret}(t; k) = -i\theta(t) \left( \Pi^\phi_{\geq}(t; k) - \Pi^\phi_{\leq}(t; k) \right), \tag{2.14}
\]

\[
\Pi^\phi_{H}(t; k) = \Pi^\phi_{\geq}(t; k) + \Pi^\phi_{\leq}(t; k). \tag{2.15}
\]

Here, at the leading order (i.e, \( O(g^2) \)),

\[
\Pi^\phi_{\geq}(t; k) = \frac{g^2}{\text{tr}[e^{-\hat{H}_\chi/T}]} \text{tr} \left[ e^{-\hat{H}_\chi/T} \hat{O}(t; k) \hat{O}(0; -k) \right], \tag{2.16}
\]

\[
\Pi^\phi_{\leq}(t; k) = \frac{g^2}{\text{tr}[e^{-\hat{H}_\chi/T}]} \text{tr} \left[ e^{-\hat{H}_\chi/T} \hat{O}(0; -k) \hat{O}(t; k) \right]. \tag{2.17}
\]
with
\[ \hat{O}(t; k) = L^{-3/2} \int d^3xe^{-ik \cdot x} \hat{O}(t, x). \] (2.18)

Let us define the Fourier transformations
\[ \Pi^\phi_X(\omega, k) = \frac{1}{d} \int dt e^{i\omega t} \Pi^\phi_{X}(t; k), \] (2.19)
with \( \Pi_X = \Pi_{\text{ret}}, \Pi_{H}, \Pi_{<}, \) and \( \Pi_{>} \). Then, from Eq. (2.14),
\[ \Pi^\phi_{\text{ret}}(\omega, k) = \int d\omega' \frac{\Pi^\phi_{>} (\omega', k) - \Pi^\phi_{<} (\omega', k)}{\omega - \omega' + i0}. \] (2.20)

By using the relation \((\omega + i0)^{-1} = \mathcal{P}\omega^{-1} - i\pi \delta(\omega)\) (with \(\mathcal{P}\) denoting the principal value), we obtain
\[ \Im \Pi^\phi_{\text{ret}} (\omega, k) = -\frac{1}{2} \left[ \Pi^\phi_{>} (\omega, k) - \Pi^\phi_{<} (\omega, k) \right], \] (2.21)
and
\[ \Pi^\phi_{H} (\omega, k) = -2 \coth \left( \frac{\omega}{2T} \right) \Im \Pi^\phi_{\text{ret}} (\omega, k), \] (2.22)
where we have used the so-called Kubo-Martin-Schwinger (KMS) relation \[13\] \( \Pi_{>} (\omega, k) = \exp(\omega/T)\Pi_{<} (\omega, k) \) in deriving Eq. (2.22).

Using the fact that the Jordan propagator is time translational invariant within our setup\[4\] \( G^\phi_J (t, t'; k) = G^\phi_J (t - t', 0; k) \), it can be expressed in terms of the spectral density \( \rho_{\phi} \) as\[5\]
\[ G^\phi_J (t; k) \equiv G^\phi_J (t, 0; k) \equiv \int \frac{d\omega}{2\pi} e^{-i\omega t} \rho_{\phi} (\omega, k), \] (2.23)
and the solution to Eq. (2.12) is given by
\[ \rho_{\phi} (\omega, k) = \frac{-2 \Im \Pi^\phi_{\text{ret}} (\omega, k)}{[\omega^2 - \omega_k^2 - \Re \Pi^\phi_{\text{ret}} (\omega, k)]^2 + [\Im \Pi^\phi_{\text{ret}} (\omega, k)]^2}. \] (2.24)

We note that the initial condition is given by \( G^\phi_J (0; k) = 0 \) and \( \partial_t G^\phi_J (0; k) = -i \) because of equal time commutation relations.

\(^{#4}\)This is because, under no self interactions of \( \phi \) and truncating the perturbative expansion at \( O(g^2) \), the spectrum of \( \phi \) is determined only by the thermal bath regardless of the number density of \( \phi \). The time translational invariance of the Jordan propagator is not a general property. See also [12].

\(^{#5}\)The spectral density \( \rho_{\phi} \) should not be confused with the density matrix \( \hat{\rho}_{\phi,i} \).
With the initial condition of the Jordan propagator, the Hadamard propagator satisfying the Kadanoff-Baym equations (2.12) and (2.13) is obtained as follows [12]:

\[ G_H^{\phi}(t, t'; k) = G_{H}^{\text{hom}}(t, t'; k) + \int_0^t dt_1 \int_0^{t'} dt_2 G_{J}^{\phi}(t - t_1; k) \Pi^{\phi}_H(t_1 - t_2; k) G_{J}^{\phi}(t_2 - t'; k), \quad (2.25) \]

where the homogeneous solution is given by

\[ G_{H}^{\text{hom}}(t, t'; k) = -G_{H}^{\phi}(s, s'; k) \bigg|_{s, s' = 0} \partial_t \partial_{t'} G_{J}^{\phi}(t; k) G_{J}^{\phi}(t'; k) - \partial_s G_{H}^{\phi}(s, s'; k) \bigg|_{s, s' = 0} (\partial_t + \partial_{t'}) G_{J}^{\phi}(t; k) G_{J}^{\phi}(t'; k) - \partial_s \partial_{s'} G_{H}^{\phi}(s, s'; k) \bigg|_{s, s' = 0} G_{J}^{\phi}(t; k) G_{J}^{\phi}(t'; k). \quad (2.26) \]

In the calculation of the production rate of \( \phi \) in thermal bath, the most important effect is the shift of the pole of the spectral density because its imaginary part gives the production rate. Here, we are interested in the case that the interaction of \( \phi \) is so small that \( \omega^2_k \gg |\Pi^{\phi}_\text{ret}| \). Then, the spectral density can be well approximated by the Breit-Wigner form:

\[ \rho_{\phi}^{(BW)}(\omega, k) = \frac{2\omega \Gamma_{\phi}(k)}{(\omega^2 - \omega_k^2)^2 + (\omega \Gamma_{\phi}(k))^2}, \quad (2.27) \]

where

\[ \Gamma_{\phi}(k) \equiv -\frac{2\Pi^{\phi}_\text{ret}(\omega_k, k)}{\omega_k}. \quad (2.28) \]

Although \( \Gamma_{\phi} \) has the argument \( k \) in our expression, it depends only on \( |k| \) because of the rotational invariance of the thermal bath. Here, we neglected the correction to the real part of the pole, which is expected to be irrelevant. In addition, notice that \( \Gamma_{\phi}(k) \) is of \( O(g^2) \), and is much smaller than \( \omega_k \). Then, from Eq. (2.23), the Jordan propagator (for \( t \geq 0 \)) is well approximated as

\[ iG_{J}^{\phi}(t; k) \bigg|_{t \geq 0} = \frac{\sin \omega_k t}{\omega_k} e^{-\Gamma_{\phi}(k)t/2}, \quad (2.29) \]

resulting in the following expression for the expectation value of the number density defined in Eq. (2.5), at leading order in \( \Gamma_{\phi}/\omega_k \),

\[ N_{k}^{(BW)}(t) = f_B(\omega_k) \left( 1 - e^{-\Gamma_{\phi}(k)t} \right) + \frac{1}{4\omega_k} \left[ G_{H}^{\phi}(s, s'; k) \bigg|_{s, s' = 0} \omega_k^2 + \partial_s \partial_{s'} G_{H}^{\phi}(s, s'; k) \bigg|_{s, s' = 0} - 2\omega_k \right] e^{-\Gamma_{\phi}(k)t}, \quad (2.30) \]
with
\[ f_B(\omega) = \frac{1}{e^{\omega/T} - 1}. \tag{2.31} \]
Equivalently, irrespective of the initial condition, one finds
\[ \dot{N}_k^{(\text{Coll})} = \Gamma_\phi(k) [f_B(\omega_k) - N_k], \tag{2.32} \]
from which we can see that \( \Gamma_\phi(k) \) can be regarded as the production rate of \( \phi \) due to the decays and scatterings of particles in thermal bath. (Here, the superscript “(Coll)” implies that this is the collision term in the Boltzmann equation.)

As we have mentioned, there is another effect on the evolution of the \( \phi \)'s number density, which is the expansion of the universe. Effect of the cosmic expansion can be easily evaluated by taking into account the red-shift of the momentum and we obtain
\[ \dot{N}_k^{(\text{Exp})} = H_k \cdot \frac{\partial N_k}{\partial k}, \tag{2.33} \]
where the superscript “(Exp)” is for cosmic expansion, and \( H \) denotes Hubble parameter of the expanding universe. We assume that the energy density of the thermal bath is much larger than that of \( \phi \), and that the Hubble parameter depends only on the temperature \( T \).

Combining two effects, the Boltzmann equation to be solved is
\[ \dot{N}_k - H_k \cdot \frac{\partial N_k}{\partial k} = \Gamma_\phi(k; T) [f_B(\omega_k; T) - N_k], \tag{2.34} \]
or, for the total number density
\[ n_\phi \equiv \int \frac{d^3 k}{(2\pi)^3} N_k, \tag{2.35} \]
the Boltzmann equation is given by
\[ \frac{d n_\phi}{dt} + 3H n_\phi = \dot{n}_\phi^{(\text{Coll})} = \int \frac{d^3 k}{(2\pi)^3} \Gamma_\phi(k; T) [f_B(\omega_k; T) - N_k]. \tag{2.36} \]
In the above equations, we explicitly show that \( \Gamma_\phi \) and \( f_B \) depend on \( T \), which should be identified with the cosmic temperature. Thus, the most important quantity to study the evolution of the abundance of \( \phi \) is the production rate \( \Gamma_\phi(k; T) \) given in Eq. (2.28) or, equivalently, \( \Im \Pi^{\phi}_{\text{ret}} \) given in Eq. (2.21). With the operator \( \mathcal{O} \) given in Eq. (2.1), the leading contribution to \( \Im \Pi^{\phi}_{\text{ret}} \) is given by
\[ \Im \Pi^{\phi}_{\text{ret}}(\omega_k, k) = \frac{g^2}{2} \int \left[ \prod_i \frac{d^4 p_i}{(2\pi)^4} G^{\psi}_i(p_i^0, p_i) \right] (2\pi)^4 \delta(\omega_k - \sum_i p_i^0) \delta^{(3)}(k - \sum_i p_i) \]
\[ -\frac{g^2}{2} \int \left[ \prod_i \frac{d^4 p_i}{(2\pi)^4} G^{\phi^*}_i(p_i^0, p_i) \right] (2\pi)^4 \delta(\omega_k - \sum_i p_i^0) \delta^{(3)}(k - \sum_i p_i). \tag{2.37} \]
Thus, they are given by
\[ G_{\chi}(p^0, \mathbf{p}) = \frac{1}{\text{tr}[e^{-H_{\chi}/T}] \int d^4x e^{ipx} \text{tr}[e^{-H_{\chi}/T} \chi_i(x^0, \mathbf{x}) \chi_i(0, 0)]}, \] (2.38)
\[ G_{\chi}^{\chi}(p^0, k) = \frac{1}{\text{tr}[e^{-H_{\chi}/T}] \int d^4x e^{ipx} \text{tr}[e^{-H_{\chi}/T} \chi_i(0, 0) \chi_i(x^0, \mathbf{x})]}. \] (2.39)

These functions satisfy the KMS relation \( G_{\chi}^{\chi}(p^0, \mathbf{p}) = e^{ip^0/T} G_{\chi}^{\chi}(p^0, \mathbf{p}) \), whereas their difference, the Jordan propagator, is expressed in terms of the spectral density:
\[ G_{\chi}^{\chi}(p^0, \mathbf{p}) = G_{\chi}^{\chi}(p^0, \mathbf{p}) - G_{\chi}^{\chi}(p^0, \mathbf{p}) = \rho_{\chi_i}(p^0, \mathbf{p}). \] (2.40)

Thus, they are given by
\[ G_{\chi}^{\chi}(p^0, \mathbf{p}) = (f_B(p^0) + 1) \rho_{\chi_i}(p^0, \mathbf{p}), \] (2.41)
\[ G_{\chi}^{\chi}(p^0, \mathbf{p}) = f_B(p^0) \rho_{\chi_i}(p^0, \mathbf{p}). \] (2.42)

If the quasi-particle picture is applicable to \( \chi_i \), which we assume in the following analysis, the spectral density can be approximated by the Breit-Wigner form:
\[ \rho^{(BW)}_{\chi_i}(p^0, \mathbf{p}) = \frac{2p^0 \Gamma_{\chi_i}(\mathbf{p}; T)}{(p^0 - m_{\chi_i}^2(T))^2 + (p^0 \Gamma_{\chi_i}(\mathbf{p}; T))^2}, \] (2.43)
where \( \Omega_{\chi_i} \) and \( \Gamma_{\chi_i} \) are real and imaginary parts of the pole of the propagator, and are related to the real and imaginary parts of the self energy of \( \chi_i \). Contrary to the case of \( \phi \), we need to take account of the shift of the pole for particles which are thermalized. Thus, \( \Omega_{\chi_i} \) may significantly deviate from the frequency satisfying the on-shell condition in the vacuum. In a large class of models, including the case discussed in the following section, \( \Omega_{\chi_i}^2(\mathbf{p}; T) \) can be well approximated as
\[ \Omega_{\chi_i}^2(\mathbf{p}; T) = \mathbf{p}^2 + \vec{m}_{\chi_i}^2(T), \] (2.44)
where \( \vec{m}_{\chi_i}^2(T) \) is given by the sum of bare and thermal masses.

The effect of non-vanishing \( \Gamma_{\chi_i} \) will be numerically studied in the next section. Here, we comment that, if \( \Gamma_{\chi_i} \) is small enough, the evolution of the total number density of \( \phi \) is governed by a simple differential equation. If the interaction is perturbative, it is usually the case that \( \Gamma_{\chi_i}(\mathbf{p}; T) \ll \vec{m}_{\chi_i}(T) \). In such a case, the (Breit-Wigner) spectral density is well approximated as \( \rho_{\chi_i}(p^0, \mathbf{p}; T) \simeq 2\pi \text{sign}(p^0) \delta(p^0 - \mathbf{p}^2 - \vec{m}_{\chi_i}^2(T)) \), and hence \( G_{\chi}^{\chi}(p^0, \mathbf{p}) \) and \( G_{\chi}^{\chi}(\omega, \mathbf{k}) \) are also (approximately) proportional to the \( \delta \)-function. We call this limit as zero-width limit. Then, the collision term in Eq. (2.36) becomes
\[ \left[ \bar{n}_{\phi}^{(\text{Coll})} \right]_{\Gamma_{\chi_i} \to 0} = g^2 \int d\Pi^{(\omega > 0)}(k) \left[ \prod_i d\Pi_{\chi_i}(p_i) \right] (2\pi)^4 \delta(k^0 - \sum_i p_i^0) \delta(3)(\mathbf{k} - \sum_i \mathbf{p}_i) \left[ \prod_i (1 + f_B(p_i^0)) \text{sign}(p_i^0) - \prod_i f_B(p_i^0) \text{sign}(p_i^0) \right] [f_B(k^0) - N_k], \] (2.45)
where

\[ d\Pi_{\chi_i}(p_i) = \frac{d^4p_i}{(2\pi)^3}\delta(p_i^2 - \bar{m}_{\chi_i}^2(T)), \]

(2.46)

and \(d\Pi_\phi(k)\) is defined in the same way. (Here, we have introduced the four-component vector as \(p_i = (p_i^0, \mathbf{p}_i)\).) Notice that, in Eq. (2.45), the \(p_0^i\) integration is performed in the region \(-\infty < p_0^i < \infty\), while \(k_0\) integration is for \(k_0 > 0\).

It is notable that the collision term given in Eq. (2.45) includes all the relevant scattering and decay processes (and their inverse processes). Regarding \(\chi_i\) as scalar particles with masses \(\bar{m}_{\chi_i}(T)\), the integrand of the collision term becomes non-vanishing if (and only if) the momentum configurations are kinematically allowed. In addition, Eq. (2.45) contains the effect of induced emission. For example, if the scattering process \(\chi_1(p_1) \cdots \chi_I(p_I) \leftrightarrow \phi(k)\chi'_1(q_1) \cdots \chi'_F(q_F)\) is kinematically allowed, the collision term contains

\[
\left[ \hat{n}^{(\text{Coll})}_\phi \right]_{\Gamma_{\chi_i} \to 0} \supset g^2 \int \left[ \prod_i d\Pi_{\chi_i}^{(q^0_i>0)}(p_i) \right] \left[ \prod_f d\Pi_{\chi'_f}^{(q^0_f>0)}(q_f) \right] d\Pi_\phi^{(k^0>0)}(k) \nonumber \\
(2\pi)^4 \delta(k^0 + \sum_f q^0_f - \sum_i p_i) \delta^{(3)}(k + \sum_f q_f - \sum_i p_i) \\
\left\{ [\prod_i f_B(p_i^0)][\prod_f (1 + f_B(q_f^0))][1 + N_k] - [\prod_i (1 + f_B(p_i^0))][\prod_f f_B(q_f^0)]N_k \right\},
\]

(2.47)

where we have used the relation \(f_B(-\omega) = -(1 + f_B(\omega))\). The right-hand side of the above equation has the same structure as the collision term in conventional Boltzmann equation; however, notice that the thermally corrected dispersion relations should be used in evaluating both the phase-spaces and distribution functions.

### 3 Application to non-thermal dark matter production

Now, let us apply the formalism to the non-thermal dark matter production scenario, which is recently called freeze-in scenario, regarding \(\phi\) as dark matter. In such a scenario, the dark matter particle is always out of thermal equilibrium because of the weakness of its interaction, and is produced by the decay of particles in thermal bath, \(\chi_i\). Thus, this is the situation where we can safely use the formalism discussed in the previous section. We pay particular attention to the question how large the thermal effect can be. For this purpose, we numerically calculate the production rate and the relic abundance of \(\phi\) using the formalism presented in the previous section.

Here, we consider the simplest form of the interaction term, which is

\[ \mathcal{L}_{\text{int}} = g\phi \mathcal{O}[\chi_0, \chi_1] = g\phi \chi_0 \chi_1. \]

(3.1)

For simplicity, we concentrate on the case that \(\phi, \chi_0,\) and \(\chi_1\) are all real scalars, with masses satisfying \(m_{\chi_0} > m_\phi + m_{\chi_1}\). In our numerical calculations, we take \(m_{\chi_0} = 100\) GeV and
$m_{\chi_1} = 0$. Then, in the absence of thermal effects, $\phi$ can be produced by the two-body decay process $\chi_0 \to \phi \chi_1$. As we will see in the following, such a decay process may be blocked, or even the “decay” process $\chi_1 \to \phi \chi_0$ may occur, once thermal effects are taken into account.

At the leading-order in $g$, $\phi$ is produced only by $1 \leftrightarrow 2$ “decay” processes. Using the formulae given in the previous section, we obtain

$$\Gamma_{\phi}(k; T) = \frac{g^2}{2\omega_k} \int \frac{d^4q}{(2\pi)^4} \left[ 1 + f_B(q_0) + f_B(\omega_k - q_0) \right] \rho_{\chi_0}(q_0, q) \rho_{\chi_1}(\omega_k - q_0, k - q). \quad (3.2)$$

As we have discussed in the previous section, when the quasi-particle picture is applicable, the thermal effects on $\chi_i$ are imprinted in $\Omega_{\chi_i}$ and $\Gamma_{\chi_i}$. Importantly, these quantities depend on how $\chi_i$ interacts in thermal bath. To make our discussion concrete, we adopt the following form of the interaction

$$\mathcal{L}_{\text{int}} = -\frac{1}{4!} g_{\chi_1}^2 \chi_i^4. \quad (3.3)$$

Then, at the leading order in $g_{\chi_1}$, the thermal mass of $\chi_i$ is obtained as \[14\]

$$\tilde{m}_{\chi_1}^2 = m_{\chi_1}^2 + g_{\chi_1}^2 \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{f_B(\omega_p^N)}{\omega_p^N}, \quad (3.4)$$

where $\omega_p^N = \sqrt{m_{\chi_1}^2 + p^2}$. In the following analysis, we consider relatively large value of $g_{\chi_1}$. Thus, we include the NLO contribution to the thermal mass of $\chi_1$. In the present analysis, as we mentioned, we take $m_{\chi_1} = 0$. Then, the thermal mass of $\chi_1$ is given by, at NLO in $g_{\chi_1}$ \[14\]

$$\tilde{m}_{\chi_1}^2(T) = \frac{1}{24} g_{\chi_1}^2 T^2 \left( 1 - \frac{3}{\pi} \frac{g_{\chi_1}}{\sqrt{24}} \right). \quad (3.5)$$

Note that we drop the momentum dependence, since, at this order, the only contribution is a tad pole diagram. In addition, $\Gamma_{\chi_i}$ is given in the integral form as \[15\]

$$\Gamma_{\chi_i}(p; T) = \frac{\pi}{2 \Omega_{\chi_i}(p)} g_{\chi_i}^2 \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \delta(p + q_1 - q_2 - q_3) \delta(\Omega_{\chi_i}(p) + \Omega_{\chi_i,1} - \Omega_{\chi_i,2} - \Omega_{\chi_i,3}) \left[ f_{B,1}(1 + f_{B,2})(1 + f_{B,3}) - (1 + f_{B,1}) f_{B,2} f_{B,3} \right], \quad (3.6)$$

where, for the simplicity of the equation, we defined $\Omega_{\chi_i,J} \equiv \Omega_{\chi_i}(q_J)$ and $f_{B,J} \equiv f_B(\Omega_{\chi_i,J})$ (with $J = 1 - 3$).

With the above formulae, we can follow the evolution of the number density of $\phi$ in the early universe and calculate the relic abundance. For this purpose, it is convenient to define the “yield variable” as

$$Y_\phi \equiv \frac{n_\phi}{s}, \quad (3.7)$$

---

*#6* Although $\chi_i$ couples to other particles in the thermal bath, here we assume that the dominant effect on the dispersion relation of $\chi_i$ is from its self interaction.
where \( s = \frac{2\pi^2 g_* T^3}{45} \) is the entropy density. (Here, \( g_* \) is the effective number of relativistic degrees of freedom; in our numerical calculation, we use \( g_* = 100 \).) We are interested in the case that \( N_k \ll f_B(\omega_k) \) and, in such a case, we can neglect the term proportional to \( N_k \) in the right-hand side of Eq. (2.36). Then, the present value of \( Y_\phi \) is given by

\[
Y_\phi(T) \equiv \int_{-\infty}^{\log(m_{\chi_0}/T)} d\log z \frac{dY_\phi}{d\log z},
\]

where

\[
\frac{dY_\phi}{d\log z} = \frac{\dot{n}_\phi^{(\text{Coll})}}{sH} \bigg|_{N_k \to 0} = \int \frac{d^3 k}{(2\pi)^3} \frac{\Gamma_\phi(k; T)f_B(\omega_k)}{sH},
\]

with

\[
z \equiv \frac{m_{\chi_0}}{T}.
\]

Notice that the variable \( z \) increases as the universe expands. We will show how \( dY_\phi/d\log z \) and \( Y_\phi \) behave as a function of the cosmic temperature \( T \) in the following.

Before showing the numerical results, it is instructive to consider the zero-width limit; in the present case, Eq. (2.45) becomes

\[
\left[ \dot{n}_\phi^{(\text{Coll})} \right]_{\Gamma_{\chi_i} \to 0} = \theta(\tilde{m}_{\chi_0} - m_\phi - \tilde{m}_{\chi_1}) \int \frac{d^3 p}{(2\pi)^3} \frac{\tilde{m}_{\chi_0}}{\Omega_{\chi_0}(p)} \bar{\Gamma}_{\chi_0 \to \phi \chi_1} f_B(\Omega_{\chi_0}(p))[1 + \bar{f}_{\chi_1}(p)]
\]

\[
+ \theta(\tilde{m}_{\chi_1} - m_\phi - \tilde{m}_{\chi_0}) \int \frac{d^3 p}{(2\pi)^3} \frac{\tilde{m}_{\chi_1}}{\Omega_{\chi_1}(p)} \bar{\Gamma}_{\chi_1 \to \phi \chi_0} f_B(\Omega_{\chi_1}(p))[1 + \bar{f}_{\chi_0}(p)],
\]

where we have used the relation \( N_k \ll f_B(\omega_k) \). In the above expression, \( \bar{\Gamma}_{\chi_0 \to \phi \chi_1} \) and \( \bar{\Gamma}_{\chi_1 \to \phi \chi_0} \) are decay rates of \( \chi_0 \) and \( \chi_1 \) calculated with the conventional Feynman rules but with thermally corrected dispersion relation; for example, \( \bar{\Gamma}_{\chi_0 \to \phi \chi_1} \) is given by

\[
\bar{\Gamma}_{\chi_0 \to \phi \chi_1} = \frac{g^2}{16\pi \tilde{m}_{\chi_0}} \sqrt{1 - \frac{2(m_\phi^2 + \tilde{m}_{\chi_1}^2)}{\tilde{m}_{\chi_0}^2} + \frac{(m_\phi^2 - \tilde{m}_{\chi_1}^2)^2}{\tilde{m}_{\chi_0}^4}}.
\]

In addition, \( \bar{f}_{\chi_0} \) and \( \bar{f}_{\chi_1} \) are “averaged” distribution functions of \( \chi_0 \) and \( \chi_1 \) in thermal bath, respectively. For example, \( \bar{f}_{\chi_0} \) is given by

\[
\bar{f}_{\chi_0}(p) = \frac{1}{2} \int d\cos \theta_{\text{CM}} f_B(\tilde{E}_{\chi_1}(|p|, \theta_{\text{CM}})),
\]

where \( \tilde{E}_{\chi_1}(|p|, \theta_{\text{CM}}) \) denotes the energy of \( \chi_1 \) emitted from \( \chi_0 \) (carrying momentum \( |p| \)) to the direction \( \theta_{\text{CM}} \) relative to \( p \) (with \( \theta_{\text{CM}} \) being defined in the rest frame of \( \chi_0 \)). Notice
that, in the calculation of $\bar{E}_{\chi_1}(|p|, \theta_{\text{CM}})$, dispersion relations including the thermal masses should be used. Eq. (3.11) indicates that the production process of $\phi$ is active only when $\tilde{m}_{\chi_0}(T) > m_\phi + \tilde{m}_{\chi_1}(T)$ or $\tilde{m}_{\chi_1}(T) > m_\phi + \tilde{m}_{\chi_0}(T)$ otherwise, the production of $\phi$ is kinematically suppressed due to the thermal mass. As we will see below, this is indeed the case.

Now we are at the position to numerically calculate the production rate of $\phi$ in thermal bath. First, let us discuss how $dY_\phi/d\log z$ given in Eq. (3.9) behaves. In Fig. 1 we plot $dY_\phi/d\log z$ as a function of $z = m_{\chi_0}/T$. Here we take $m_\phi/m_{\chi_0} = 0.95$, $g_{\chi_0} = 0.1$, and $g_{\chi_1} = 0.3$ (left) and $g_{\chi_1} = 2$ (right). In each figure, the result of the full calculation, which is calculated by using Eq. (3.9) including the effects of the widths of $\chi_i$, $\Gamma_{\chi_i}$, is shown in the red solid line. In the same figure, in the black dashed line, we show the result taking the zero-width approximation (see Eq. (3.11)). The result of the calculation with conventional Boltzmann equation (which corresponds to the zero-width approximation with $\tilde{m}_{\chi_1} \rightarrow m_{\chi_1}$) is also shown in the blue dot-dashed line.

As one can see, $dY_\phi/d\log z$ is significantly suppressed when $0.027 \lesssim m_{\chi_0}/T \lesssim 1.2$ (for $g_{\chi_1} = 0.3$) or $0.16 \lesssim m_{\chi_0}/T \lesssim 6.3$ (for $g_{\chi_1} = 2$). This is due to the fact that, at leading order in $\Gamma_{\chi_i}/\tilde{m}_{\chi_i}$, the production process of $\phi$ is kinematically blocked because of the thermal mass; the upper and lower edges correspond to the temperature where $\tilde{m}_{\chi_1} \simeq m_\phi + \tilde{m}_{\chi_0}$ and $\tilde{m}_{\chi_0} \simeq m_\phi + \tilde{m}_{\chi_1}$ are realized, respectively. In the suppressed region between the edges, the $\phi$ production is from the off-shell effects of $\chi_i$ (especially $\chi_1$), namely due to non-vanishing $\Gamma_{\chi_i}$ [16] [17].

It is notable that, in the region where there is no kinematical suppression, the result of zero-width approximation agrees remarkably well with that of full calculation. Thus, the

\[ \text{In the present case, } \phi \text{ should play the role of dark matter, and hence is stable. Thus, we do not have to consider the case that } m_\phi > \tilde{m}_{\chi_0}(T) + \tilde{m}_{\chi_0}(T) \geq m_{\chi_0} + m_{\chi_0}. \]

Figure 1: $dY_\phi/d\log z$ as a function of $z = m_{\chi_0}/T$ for $m_\phi/m_{\chi_0} = 0.95$, $g_{\chi_0} = 0.1$, $g/m_{\chi_0} = 10^{-13}$, and $g_\ast = 100$. The left figure is computed with $g_{\chi_1} = 0.3$ and the right one is with $g_{\chi_1} = 2$. The red solid lines are the results of full calculation while the black dashed ones are zero-width results. Results of conventional Boltzmann equation are also shown in blue dot-dashed lines.
effect of $\Gamma_{\chi_i}$ can be safely neglected in the calculation of the relic abundance of $\phi$, unless the production channel is almost kinematically blocked at the epoch when the production of $\phi$ is most active. If the zero-width approximation can be adopted, the calculation of the production rate is extremely simplified. In the opposite case where $\phi$ is mostly produced at the epoch of kinematical suppression, production processes due to $\Gamma_{\chi_i} \neq 0$ may not be negligible in the calculation of the relic abundance of $\phi$. A typical example is given in the right hand side of Fig. 1.

From Fig. 1, we can also see that, if we perform a naive calculation without properly taking account of the thermal effect, the production rate may significantly deviate from the correct value. In particular, even at $\tilde{m}_{\chi_1} \gtrsim m_\phi + \tilde{m}_{\chi_0}$ or $\tilde{m}_{\chi_0} \gtrsim m_\phi + \tilde{m}_{\chi_1}$, the discrepancy between the results of full and conventional calculations is sizable in some parameter region. This is because, at $\tilde{m}_{\chi_1} \gtrsim m_\phi + \tilde{m}_{\chi_0}$, the dominant production process of $\phi$ is $\chi_1 \to \phi \chi_0$ contrary to the case of low enough temperature, and at $\tilde{m}_{\chi_0} \gtrsim m_\phi + \tilde{m}_{\chi_1}$, the phase space becomes smaller than that in the case of conventional calculation due to the thermal mass of $\chi_1$.

In the non-thermal dark matter production due to the decay, the dominant production occurs when $m_{\chi_0}/T \simeq 1 - 5$; for such a temperature, the actual value of $dY_\phi/d\log z$ may receive sizable thermal effects. Thus, for the accurate calculation of the relic density of non-thermally produced dark matter, it is dangerous to neglect the thermal effects. In Figs. 2 and 3 we show the evolution of the yield variable $Y_\phi$. In the same figure, we also show the result of the naive calculation neglecting the thermal effects. As one can see, the evolution of $Y_\phi$ changes once the thermal effects are taken into account, and the resultant value of $Y_\phi$ is suppressed compared to the result obtained by neglecting the thermal effects. In particular, for the case of $g_{\chi_1} = 2$, which gives relatively large value of $\Gamma_{\chi_1}$, significant amount of $\phi$ is produced even during the period of kinematical suppression and the resultant $Y_\phi$ differs from the zero-width result. This is because the production channel is suppressed during the time $m_{\chi_0}/T \simeq 1 - 5$, where the most effective production could occur if there were no thermal effects. Therefore, the production due to the off-shell effects with $\Gamma_{\chi_i} \neq 0$ becomes important. In such a case, the resultant value of $Y_\phi$ does not agree with the result of zero-width approximation nor that of the conventional Boltzmann equation.

In Fig. 4, we plot the ratio of the yield variable calculated with and without thermal effects,

$$R \equiv \frac{Y_\phi}{Y_\phi(\text{conventional Boltzmann equation})}|_{\text{now}},$$

as a function of $g_{\chi_1}$ and $m_\phi/m_{\chi_0}$, using the zero-width approximation. In the present set up, the present number density of $\phi$ decreases by taking account of the thermal effects. As one can see, $Y_\phi$ is more suppressed when $g_{\chi_1}$ becomes larger or when the mass of $\phi$ becomes closer to that of $\chi_0$.

Before closing this section, we note here that there is another possible mechanism of enhancing the production rate. With the emission of a massless particle in the thermal bath from the initial or final state particles (corresponding to $\chi_0$ or $\chi_1$ in the present setup), the
Figure 2: The evolution of the yield variable $Y_\phi$ for $m_\phi/m_{\chi_0} = 0.95$, $g_{\chi_0} = 0.1$, $g_{\chi_1} = 0.3$, $g/m_{\chi_0} = 10^{-13}$, and $g_* = 100$. The red solid line is computed with $\Gamma_{\chi_i} \neq 0$ and the black dashed one is done with $\Gamma_{\chi_i} = 0$. The blue dot-dashed one represents the conventional result (without thermal effects).

Figure 3: Same as Fig. 2 except for $g_{\chi_1} = 2$. Notice that, in the left figure, the line for the conventional result is almost parallel to the vertical axis.

production rate of the non-thermal dark matter at $z \lesssim 1$ may be affected, and may become larger by the factor of a few [23]. In realistic models of non-thermally produced dark matter, the particles corresponding to $\chi_0$ and $\chi_1$ may couple to massless gauge bosons, which may enhance the production rate at $z \lesssim 1$. In the present model, however, there is no massless particle responsible for such an enhancement. In addition, for the calculation of the relic density of the non-thermally produced dark matter, such an effect may not be important because, as we have shown, the relic abundance of the non-thermally produced dark matter is determined at $z \sim O(1)$.
4 Conclusions and Discussion

In this paper, we have discussed the evolution of the number density of a particle $\phi$ which is coupled to thermal bath very weakly and hence is in non-equilibrium. We first solved the Kadanoff-Baym equations for the case that (i) the effects of $\phi$ on the thermal bath is negligible and (ii) the self interaction of $\phi$ is sufficiently weak. Then, we derived Boltzmann equation describing the evolution of the $\phi$’s number density for the case that (iii) the real part of the $\phi$’s self energy is (almost) unchanged by thermal effects. We then studied the properties of Boltzmann equation, assuming that the degrees of freedom in the thermal bath are well described as the quasi-particles. In particular, in the situation that the widths of quasi-particles are negligible, the evolution of the number density is well described by the Boltzmann equation in the familiar form, which contains the matrix elements, phase-space integrals, and distribution functions evaluated with the “on-shell” condition modified by the thermal effects.

Then, we have applied the formalism to the scenario in which dark matter is non-thermally produced from the decay of “particles” in thermal bath, regarding $\phi$ as the dark matter. In such a case, the above conditions (i), (ii), and (iii) are satisfied, and the formalism we have studied can be safely applied. We calculated the number density of the non-thermally
produced dark matter, taking account of the effects of thermal bath. Because of the change of the dispersion relations of “particles” in thermal bath, the production rate of the dark matter may significantly change. In particular, in some cases, the decay process to produce $\phi$, which is kinematically allowed in the vacuum, may be blocked because the particles in thermal bath acquire thermal masses, which changes the mass relation among parent and daughter quasi-particles. Numerically, we found that, if the proper Boltzmann equation with the thermal effects is used, the dark matter density may change by $O(10-100\%)$ compared to the results of calculations neglecting the change of the dispersion relation of the “particles” in thermal bath. We have also studied the effect of the width in the spectral density of the quasi-particles in thermal bath, and found that the zero-width limit can be safely taken if the thermal blocking of the production of $\phi$ does not occur at the time when the production of $\phi$ is most effective.

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