The Complexity of the Spherical $p$-spin spin glass model, revisited

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Abstract. Some questions concerning the calculation of the number of “physical” (metastable) states or complexity of the spherical $p$-spin spin glass model are reviewed and examined further. Particular attention is focused on the general calculation procedure which is discussed step-by-step.

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1 introduction

The analysis of the equilibrium and non-equilibrium properties in terms of the energy landscape originally pushed forward for the structural glass transition [1], has risen in the recent years a new interest on the topological properties of the energy or free-energy landscapes of disordered and complex systems. In this approach an important theoretical tool is the logarithm of the number metastable states, called complexity or configurational entropy, identified as basins or valleys on the landscape.

Solvable models, such as mean-field models, have always played an important role in the theoretical study of physical problems. In this context classical calculations for the complexity of the Sherrington-Kirkpatrick (SK) model introduced by Crisanti and Sommers [8]. The aim of the paper is not the calculation of the complexity for the spherical $p$SP-SG model, which has been computed in Refs. [9,10,20], but the procedure of calculation itself to
clarify known results which could be useful for the understanding of the complexity of other systems. The spherical pSP-SG model is only used to enlighten subtle points of the procedure. We have tried to make the paper almost self-contained so that it can be also used by readers interested into the problem but not too familiar with all reported results.

The general approach to the calculation of complexity in mean-field spin glass model is discussed in Section 2. The procedure is illustrated in Section 4 using the spherical pSP-SG model introduced in Section 3. The correctness the procedure is discussed in Section 5. Finally Section 6 contains some conclusions and discussion.

2 How to compute the Complexity

The metastable states in mean-fields spin models are more easily studied using the Thouless-Anderson-Palmer (TAP) method [11], which introduces a mean-field free energy functional $F_{\text{TAP}}(\mathbf{m})$ of the local magnetizations $\mathbf{m} = (m_1, m_2, \ldots, m_N)$, where $N$ is the number of spins. At any temperature $T$ the metastable states of the system are identified with the local minima of $F_{\text{TAP}}(\mathbf{m})$, i.e., with the solutions of

$$\partial_{m_i} F_{\text{TAP}}(\mathbf{m}) = 0, \quad i = 1, \ldots, N$$

with the additional requirement that all eigenvalues of the matrix $\partial_{m_i} \partial_{m_j} F_{\text{TAP}}(\mathbf{m})$ evaluated on the solution are positive. At the mean-field level different local minima are separated by infinite barriers therefore the system cannot escape from a local minimum in a finite time and hence the minimum (and its basin of attraction) is a metastable state of infinite life-time. However, despite this simple intuitive picture, not all minima of $F_{\text{TAP}}(\mathbf{m})$ can be associated with physical metastable states but only those for which (Plefka’s criterion) [12,13]

$$x_p = 1 - \frac{c(q)}{N} \sum_{i=1}^{N} (1 - m_i^2)^2 \geq 0$$

where $\beta = 1/T$ and $c(q)$ is a function of $q = (1/N) \sum_i m_i^2$ whose form depends on the interactions. For example for the SK model $c(q) = \beta^2$ [12,13], while for p-spin interaction models $c(q) = (\beta^2 p/2)(p - 1)q^{p-2}$ [14]. Therefore within this approach the calculation of the number of metastable states is reduced to that of counting the number of solutions of (1) which are minima and satisfies the Plefka’s criterion (2) (“physical” minima). If it were found that physical minima must satisfy additional constraints, those must be also included.

Different physical minima may have different free-energy density, thus to have a better description of metastable states one can group together all minima with the same free-energy density and introduce the function $\rho(f)$ which gives the number of metastable states with $F_{\text{TAP}}(\mathbf{m}) = Nf$. The configurational entropy is then defined as:

$$\Sigma(f) = \frac{1}{N} \ln \rho(f).$$

We are eventually interested into the large $N$ limit, thus $\Sigma(f)$ is different from zero only if the number of physical minima with free energy density $f$ is exponentially large with $N$.

If we label the $N_{\text{sol}}$ solutions of the TAP equation (1) with the subscript $\alpha (\alpha = 1, \ldots, N_{\text{sol}})$ by definition $\rho(f)$
is given by

\[ \rho(f) \equiv \sum_{\alpha=1}^{N_{\text{sol}}} \prod_{i=1}^{N} \left[ \int dm_i \theta(\lambda_i^\alpha) \delta(m_i - m_i^\alpha) \right] \times \theta(x_p^\alpha) \delta [F_{\text{TAP}}(m) - Nf] \] (4)

where \( \lambda_i^\alpha \) (\( i = 1, \ldots, N \)) are the eigenvalues of the Hessian matrix for the \( \alpha \)-th solution:

\[ \chi_{ij}^\alpha = \partial_m m_i \partial_m m_j F_{\text{TAP}}(m) \big|_{m=m^\alpha} \] (5)

and \( \theta(x) \) is the Heaviside theta-function. As it stands, \( \rho(f) \) is difficult to handle, however using the properties of delta-function it can be transformed into the more manageable form:

\[ \rho(f) = \prod_{i=1}^{N} \left[ \int dm_i \delta[\partial_m F_{\text{TAP}}(m)] \right] \times \det(\chi(m)) \theta(x_p) \delta [F_{\text{TAP}}(m) - Nf] \] (6)

where \( \chi(m) \), the Hessian matrix, evaluated for a generic \( m \), is the Jacobian of the transformation and \( \lambda_i \) are its eigenvalues. The theta-functions ensure that the determinant of \( \chi \) is always positive and we have neglected the absolute value of the Jacobian.

In addition to \( \rho(f) \) we consider the definition without the theta-functions, which we denote by \( \rho_{\text{tot}}(f) \), which counts the total number of TAP solutions. The effect of the theta-functions is to eliminate all solutions with at least one negative eigenvalue, therefore the meaning of \( \rho_{\text{tot}}(f) \) is not exactly the same as that of \( \rho(f) \) since all solutions are now counted. There is just one case in which the two formulations, at least in the limit of our interest, \( N \gg 1 \), are indeed equivalent: if for large \( N \) the two integral – with and without theta-functions – are dominated by the same set of solutions, an assumption that must be verified in each case (and for each value of \( f \)) separately.

Keeping the sum over all solutions is not, however, completely free of difficulties: since all solutions are counted the determinant of the Jacobian can be negative and the absolute value must be retained making the subsequent calculation more problematic. To overcome this difficulties the absolute value is simply dropped leading to expression:

\[ \tilde{\rho}(f) = \prod_{i=1}^{N} \left[ \int dm_i \delta[\partial_m F_{\text{TAP}}(m)] \right] \times \det(\chi(m)) \delta [F_{\text{TAP}}(m) - Nf] , \] (7)

and arguments are given to justify under which circumstances this reproduces the correct result with the absolute value.

We have to compare \( \tilde{\rho}(f) \) given by (7) with \( \rho(f) \). The question is when \( \tilde{\rho}(f) \) yields the same result as \( \rho(f) \). The main difference between (6) and (7) is the support of the integrals, larger for the latter, hence the two expressions are equivalent if the integrals are dominated by the same support. Thus, to extract from \( \tilde{\rho}(f) \) the correct result for \( \rho(f) \) we should be able to isolate the contributions from the common support. For a generic value of \( N \) this could be quite a hard problem. However, in the limit of large \( N \) where the integrals are evaluated by saddle point methods, a simple rule can be applied.

In this case \( \rho(f) \) can be evaluated simply considering only the stationary points for which all eigenvalues of the Hessian are positive and the Plefka’s criterion is satisfied, disregarding all others. We stress that such constraint is
not contained into $\tilde{\rho}(f)$, so that the functional alone cannot give the desired result.

In the next Sections we shall illustrate this procedure (re)computing the complexity for the spherical $p$SP-SG model without external field \[9\] using both expressions \[7\] and \[6\].

### 3 TAP Equations for the spherical $p$SP-SG model

The spherical $p$SP-SG model consists of $N$ continuous spins $\sigma_i$ interacting via $p$-body interactions \[8\]:

$$H(\sigma) = \frac{r}{2} \sum_{i=1}^{N} \sigma_i^2 - \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$ (8)

The couplings are quenched independent Gaussian variables with zero mean and average $<(J_{i_1,\ldots,i_p})^2> = p!/(2N^{p-1})$.

The scaling with $N$ ensures a well defined thermodynamic limit. Here and in the following $\langle \cdots \rangle$ denotes disorder average. The parameter $r$ is a Lagrange multiplier to impose the global constraint $\sum_{i=1}^{N} \sigma_i^2 = N$ on the spins amplitude.

The study of both the static and dynamical properties shows that in the thermodynamic limit the model presents a (static) transition at a temperature $T_s$, between a high temperature replica symmetric phase and a low temperature phase with one step of replica symmetry breaking \[8\]. Despite its simplicity, the spherical $p$SP-SG model for $p > 2$ has an exponentially large number of locally stable states which dominate the dynamical behaviour above $T_s$.

As a consequence, two-time correlation functions acquire a time persistent part at a temperature $T_d > T_s$ which marks the dynamical transition \[10\]. The static transition can be seen as the point where the lowest accessible (metastable) states dominate. The dynamical transition, on the contrary, takes place at the point where the behaviour is ruled by higher, highly degenerate, metastable states.

The TAP functional has been derived in Refs. \[17,9\]:

$$\beta F_{\text{TAP}}(m) = -\frac{\beta}{p!} \sum_{i_1,\ldots,i_p} J_{i_1,\ldots,i_p} m_{i_1} \cdots m_{i_p}$$

$$-\frac{N}{2} \ln(1 - q)$$

$$-\frac{N \beta^2}{4} \left[1 + (p - 1)q^p - pq^{p-1}\right]$$ (9)

where $Nq = \sum m_i^2$, and taking the derivatives with respect to $m_i$ one obtains the TAP equations.

The structure of the solutions is better understood performing the change of variable $m_i = q^{1/2} \hat{m}_i (\sum \hat{m}_i^2 = N)$ which leads to TAP functional density:

$$f_{\text{TAP}}(q, E) = q^{p/2} E - \frac{T}{2} \ln(1 - q)$$

$$-\frac{\beta}{4} \left[1 + (p - 1)q^p - pq^{p-1}\right]$$ (10)

where $E = -(1/Np!) \sum J_{i_1,\ldots,i_p} \hat{m}_{i_1} \cdots \hat{m}_{i_p}$ is the $T = 0$ energy density. In general, $E$ is a random variable which depends on both the realization of couplings and on the orientation of the vector $m$. However all cases with the same value of $E$ will also have the same free energy, thus we can consider $E$ as given and study the solutions as a function of $E$. The TAP equations then reduce to $\partial_q f_{\text{TAP}}(q, E) = 0$ which can be written:

$$(1 - q) q^{p/2-1} = z T$$ (11)

where

$$z = \frac{1}{p - 1} \left[-E \pm \sqrt{E^2 - E_c^2}\right],$$ (12)
\[ E_c = -\sqrt{2(p-1)/p}. \]  

It is easy to understand that for any positive \( z \) and temperature \( T \) below

\[ T_a = (1 - q_a) q_a^{p/2 - 1} z^{-1} \]

where \( q_a = (p - 2)/p \), there are two solutions of the TAP equation (11), one larger and one smaller than \( q_a \). Anyway, possible candidates for physical solutions are only those which are local minima of \( f_{\text{TAP}}(q, E) \).

By using the TAP equation (11) the second derivative of \( f_{\text{TAP}}(q, E) \) with respect to \( q \) evaluated on the solutions can be expressed as

\[
\frac{\partial^2}{\partial q^2} f_{\text{TAP}}(q, E) = \frac{p}{4 \beta q} \left[ q - \frac{p - 2}{p} \right] \\
\times \left[ \frac{1}{(1-q^2)} - \mu(p-1)q^{p-2} \right] \\
= \frac{p}{4 \beta q} \left[ q - \frac{p - 2}{p} \right] \frac{z_c^2 - z^2}{z_c^2} \tag{15}
\]

where \( z_c = \sqrt{2/p(p-1)} \) and \( \mu = \beta^2 p/2 \). The requirement of positiveness of the second derivative thus selects the solutions

\[
q < \frac{p - 2}{p} \quad \text{for} \quad z > z_c \tag{16}
\]

\[
q > \frac{p - 2}{p} \quad \text{for} \quad z < z_c \tag{17}
\]

By comparing the two expressions in (15) we see that the condition \( z < z_c \) is equivalent to

\[
x_p = 1 - \mu(p-1)q^{p-2}(1-q)^2 > 0 \tag{18}
\]

which also follows from the stability requirement of the replica saddle point \([8]\) and of the dynamics \([10]\). This is the Plefka’s criterion \([2]\) for the physical relevance of TAP solutions \([12]\) for the spherical \( p \)-SP-SG model. Indeed it can be easily seen that the condition \([16]\), for which \( x_p < 0 \), leads to an unphysical \( q \) decreasing with temperature.

### 4 Complexity of the spherical \( p \)-SP-SG model: standard calculation

In this Section we report the main steps of the calculation of \( \hat{\rho}(f) \) for the spherical \( p \)-SP-SG model. Details can be found in the literature, see e.g. Refs. \([2,3]\) and for the specific case of the \( p \)-spin glass model Refs. \([18,9,29]\).

The starting point is \([\text{cfr. (7)}]\)

\[
\hat{\rho}(f) = N^2 \int_0^1 dq \prod_{i=1}^N \left[ \int_{-\infty}^{+\infty} dm_i \delta(G_i) \right] \det A \\
\times \delta \left( Nq - \sum_i m_i^2 \right) \delta \left[ F_{\text{TAP}}(m) - Nf \right] \tag{19}
\]

with

\[
G_i = \partial_{m_i} \beta f_{\text{TAP}}(m) \\
= a(q) m_i - \frac{\beta}{(p-1)!} \sum_j J_{i,j} m^{p-1} \tag{20}
\]

and

\[
A_{ij} = \partial_{m_j} G_i \\
= a(q) \delta_{ij} - \frac{\beta}{(p-2)!} \sum_k J_{i,k} m^{p-2} \\
+ \frac{2}{N} a'(q) m_i m_j \tag{21}
\]

where

\[
a(q) = \frac{1}{1-q} + \mu(p-1)(1-q)q^{p-2} \tag{22}
\]

\[
a'(q) = da(q)/dq \quad \text{and we have used the short-hand notation:}
\]

\[
\sum_j J_{i,j} m^{p-1} \overset{def}{=} \sum_{k_1,\ldots,k_{p-1}} J_{i,k_1\ldots,k_{p-1}} m_{k_1} \cdots m_{k_{p-1}} \tag{23}
\]
and similarly in [21]. The last term of A is of order $O(1/N)$, and can be neglected for $N \to \infty$ [see also below].

The structure of the minima is given by the couplings, therefore $\hat{\rho}(f)$ (and so $\rho(f)$) may change from sample to sample. Thus, in principle, to have a well defined complexity we should introduce replicas to compute $\langle \ln(\hat{\rho}(f)) \rangle$ [19]. However it can be shown [9,10] that for this model, in absence of a magnetic field, the annealed average $\ln(\hat{\rho}(f))$ is exact, so we can just average [19] over the disorder.

To perform the average over the couplings it is convenient to use the integral representation of the delta-function to exponentiate its argument. This introduces additional parameters which are usually denoted by $\hat{f}$, $\hat{q}$ and $\hat{m}_i$ [31] conjugated to $f$, $q$ and $G_i$ and the additional variable $\Delta$ coming from Hubbard-Stratonovich transformation. The calculation can be further simplified by substituting $\sum J_{i_1,\ldots,i_p}m_{i_1}\cdots m_{i_p}$ from equation [20] in $F_{TAP}(m)$ [eq. (23)] and noticing that the error involved in disorder-averaging the determinant of $A$ separately accounts for changing $A$ of terms of order $O(1/N)$ and hence negligible as $N \to \infty$ [21,24].

Performing the averages over the couplings results in
\[
\langle \hat{\rho}(f) \rangle = c \int^{+\infty}_{-\infty} df \int^{1}_{0} dq \int^{+\infty}_{-\infty} d\hat{q} \int^{+\infty}_{-\infty} d\Delta e^{N \Sigma} \tag{24}
\]
where $c$ is a constant and
\[
\Sigma = i\beta f [f - (f)] + i\beta q - \Delta(1 - q) - \frac{1}{\lambda} \Delta^2 + \ln I + G_{x_p}(q) \tag{25}
\]
with $f(q)$ the TAP density functional $f_{TAP}(m)$ evaluated on the solution of the TAP equation [20]:
\[
f(q) = -\frac{q}{p(1-q)} - \frac{q}{2}(p - 2)(1 - q)q^{p-1}
\]
and
\[
- \frac{qT}{p(1-q)} - \frac{T}{2} \ln(1 - q), \tag{26}
\]
\[
I = \int^{+\infty}_{-\infty} \frac{dm \, d\hat{m}}{2\pi} \exp\left\{ \frac{\mu q^{p-1}}{2}(m + i\hat{m})^2 \right\}
\]
\[
+ i\hat{m} \left( \frac{1}{1 - q} - \Delta \right) m - i\hat{m}^2 \right\}. \tag{27}
\]
and $\lambda = 2\mu(p - 1)q^{p-2}$.

The function $G_{x_p}(q)$ comes from the average of the determinant of $A$ which can be computed either using Grassmann variables [3] or introducing replicas [21,24]. The form depends on the sign of $x_p$ [eq. (22)]:
\[
G_{x_p}(q) = -\ln(1 - q), \quad \text{for } x_p > 0 \tag{28}
\]
this is $B = 0$ solution always adopted in standard calculations e.g. in [21,24] and
\[
G_{x_p} = \frac{1}{\lambda(1 - q)^2} \left[ 1 - \frac{\lambda^2}{4} (1 - q)^4 \right] + \ln \frac{\lambda}{2}(1 - q), \quad \text{for } x_p < 0. \tag{29}
\]
The two expressions coincide for $x_p = 0$, i.e. for $\lambda/2 = 1/(1 - q)^2$. Details of the calculation can be found in the Appendix A.

Integration over $\hat{m}, m, \hat{q}, \Delta$ can be done by the saddle point method [21], which turns out to be exact for the integrals over $\hat{m}, m, \Delta$ being Gaussian, while the integral over $\hat{f}$ can be easily performed giving a delta-function. The final results is then
\[
\langle \hat{\rho}(f) \rangle \sim c' \int^{1}_{0} dq \delta[f(q) - f] e^{N \Sigma(q)} \sim e^{N \Sigma(q^*)}, \quad N \to \infty \tag{30}
\]
where
\[
\Sigma(q) = G_{x_p}(q) + \frac{1}{2} + \frac{1}{2} \ln q - \frac{1}{2} \ln(\mu q^{p-1})
\]
\[ p - 1 \frac{2\mu pq^{p-2}}{2(1-q)^2q^{p-2}} \]

(31)

and \( q^* = q^*(f) \) is the solution of

\[ f(q) = f \]

(32)

which gives the largest value of \( \Sigma(q) \).  

The simplest way of studying the solution is using \( q \) as a free parameter to scan all values of \( f \). This is what is done, for example, in Ref. \([9]\) where the result \(31-32\) was first derived.

The solutions of eq. \(32\) can be found using the results of Section \(3\). The free energy \( f(q) \) as function of \( q \) for all stationary points of \( f_{\text{TAP}}(q, E) \) is shown in Figure \(1\) for \( p = 4 \) and temperature \( T = 0.51 \) between the static transition temperature \( T_s = 0.5030... \) and the dynamical transition temperature \( T_D = 0.5443... \) Thicker lines correspond to solutions for which the Plefka's criterion is satisfied, while full lines correspond to solution for which \(15\) is positive, i.e., to local minima of \( f_{\text{TAP}}(q, E) \).

Such configurations violate the Plefka criterion \( (x_p < 0) \), thus leading to a non-physical linear susceptibility.

If the \( x_p < 0 \) solutions are disregarded, we are left with the curves shown in Figure \(3\) corresponding to the solution of the TAP equations with \( q < (p-2)/p \) (dashed line) and \( q > (p-2)/p \) (full line). Again, if no other information is added, for each \( f \) the largest value must be selected to evaluate \( \Sigma(f) \). This means that there is a region of free energies where \( \Sigma(f) \) is dominated by solutions with \( q < (p-2)/p \). But these are not local minima of \( f_{\text{TAP}} \), see \(15\), and hence for these free energies \( \Sigma(f) \) does not give the desired result. This clearly shows that the condition \( x_p > 0 \) alone does not guarantees that only physical states are
Fig. 2. $\Sigma(q^*)$ as a function of $f$ for $p = 4$ and temperature $T = 0.51$ between the static transition temperature $T_s = 0.5030...$ and the dynamical transition temperature $T_D = 0.5443...$. Thicker lines correspond to solutions for which the Plefka’s criterion is satisfied, while full lines correspond to solution for which (15) is positive, i.e., to local minima of $f_{\text{TAP}}(q, E)$. Only values of $f$ for which $\Sigma > 0$ are reported.

For the spherical $p$SP-SG this information is easily obtained. Indeed, the analysis of the TAP solutions in Section 3 shows that only solutions with $x_p > 0$ and $q > (p - 2)/p$ do correspond to metastable states, so only the full line in Figure 3 must be considered. This leads to the result first derived in Ref. [9].

5 Complexity of the spherical $p$SP-SG model: Hessian Eigenvalues

In the previous Section we have revised step-by-step the standard calculation of the complexity for the spherical $p$SP-SG model showing which additional information, not included into the definition of $\tilde{\rho}(f)$, must be added to yield the correct answer. In this Section we show that that additional information is exactly the theta-functions needed to transform $\tilde{\rho}(f)$ into $\rho(f)$, see eqs. (6) and (7).

To prove the equivalence we must compute the eigenvalues $\lambda_i$ of the Hessian matrix (15) that for the spherical $p$SP-SG model is given by (33).

The eigenvalues are solutions of the equations

$$\sum_j A_{ij} \xi_j = a(q) \xi_i + 2 q a'(q) \bar{m}_i \frac{1}{N} \sum_j \bar{m}_j \xi_j$$

$$- \frac{\beta q(p-2)/2}{(p-2)!} \sum_{j,k} J_{ij,k} \bar{m}^{P-2} \xi_j = \lambda \xi_i$$

(33)

There are two classes of eigenvectors $\xi_i$: longitudinal and transversal.
5.1 Longitudinal Eigenvector

The longitudinal eigenvector is given by:

\[
\xi_i \propto \hat{m}_i, \quad \forall i
\]

and hence satisfies the equation:

\[
[a(q) + 2 q a'(q)] \hat{m}_i - \frac{\beta q^{(p-2)/2}}{(p-2)!} \sum_j J_{ij} \hat{m}_j^{p-1} = \lambda_L \hat{m}_i \quad (35)
\]

Since the Hessian must be evaluated on the solution of the TAP equation (1), we can use (20) to write

\[
\frac{\beta q^{(p-2)/2}}{(p-2)!} \sum_j J_{ij} \hat{m}_j^{p-1} = a(q) (p-1)! \hat{m}_i \quad (36)
\]

which inserted into (35) leads to

\[
\lambda_L = 2 q a'(q) - (p-2) a(q) = p \left[ q - \frac{p-2}{p} \right] - \frac{1}{(1-q)^2} - \mu(p-1)q^{p-2} \quad (37)
\]

The longitudinal eigenvalue is therefore, apart from positive multiplicative coefficients, equal to \( \frac{\partial^2}{\partial q^2} f_{\text{TAP}}(q, E) \) evaluated in Section 3 [eq. 13]. The different coefficients come from the derivative being taken with respect to \( q \) or to \( m_i \). The longitudinal eigenvalue has degeneracy 1.

Note that the term of \( O(1/N) \) in (21) yields a contribution of \( O(1) \) for longitudinal eigenvectors and cannot be neglected [12].

5.2 Transversal Eigenvectors

Transversal eigenvectors satisfy the orthogonality conditions:

\[
\sum_i \xi_i \hat{m}_i = 0 \quad (38)
\]

and hence span a space of dimension \( N - 1 \). The eigenvalues equation for transversal eigenvectors can be written as

\[
\tilde{J}_{ij} = \beta q^{(p-2)/p} \sum_{k_1 < \cdots < k_{p-2}} J_{ij, k_1, \ldots, k_{p-2}} \hat{m}_{k_1} \cdots \hat{m}_{k_{p-2}} \quad (40)
\]

For large values of \( N \) \( \tilde{J}_{ij} \) is a symmetric random matrix whose elements are independent Gaussian variables with zero average and variance:

\[
\langle (\tilde{J}_{ij})^2 \rangle = \frac{\mu(p-1)q^{p-2}}{N}. \quad (41)
\]

Therefore for \( N \to \infty \) the spectrum of \( \tilde{J}_{ij} \) is given by the Wigner’s semicircular law: [23, 29],

\[
\rho(\lambda_T) = \frac{1}{2\pi \mu(p-1)q^{p-2}} \times \sqrt{4\mu(p-1)q^{p-2} - (\lambda_T - a(q))^2} \quad (42)
\]

This gives a spectrum at the leading order in \( N \), displaying a non negative support. Since it can be shown that the tails of this distribution go to zero exponentially with \( N \) [23] we can safely exclude negative eigenvalues. The thermodynamic limit transversal fluctuations are, thus, always stable, regardless of the sign of \( x_P \) and the whole stability depends on the longitudinal eigenvalue. Note, however, that the \( N - 1 \) transversal eigenvalues dominate the calculation of

\[
\det \mathbf{A} = \exp (\text{Tr} \ln \mathbf{A}) \quad \text{for} \ N \to \infty \quad (43)
\]

and any information from the longitudinal eigenvalue is washed out when computing \( \tilde{\rho}(f) \).
In conclusion we see that the procedure described in Section 2 of selecting the saddle point solutions of $\hat{\rho}(f)$ according to their physical relevance obtained from the (independent) analysis of the TAP equations produces the correct result for the complexity.

6 Conclusion and Discussion

The study of the complex behaviour of glassy systems in terms of the topological properties of the energy or free-energy surfaces has recently put new interest into the calculation of the number of metastable states, also called complexity or configurational entropy, in mean-field spin glass models. In this context classical calculations done for the SK and other disordered spin models have been reconsidered, extended and also criticised [5,6,7,27].

Motivated by these controversies in this paper we have reviewed and examined further some questions concerning the calculation of the complexity of disordered systems. Particular care has been taken to distinguish between what we would like to compute, $\rho(f)$, and what we are able to compute, $\hat{\rho}(f)$. We have also discussed how information on $\rho(f)$ can be extracted, at least in the thermodynamic limit.

The general approach has been illustrated using the spherical $p$SP-SG model, showing the correctness of the reduction procedure. As by-product we have explicitly shown that the Plefka criterion separates all solutions of the TAP equations into two classes, both containing local minima as well as saddles. However, only local minima which satisfy the Plefka criterion do represent physical states. The Plefka’s criterion is indeed a necessary, but not sufficient, condition for physical states, and hence cannot be used alone for the reduction procedure but the requirement of local stability must be added.

In Ref. [5] the classical calculation of Bray and Moore [2] and DeDominicis and Young [3] for the SK model have been critically revised. The main criticism steams from the observation that $\hat{\rho}(f)$ (identified with $\rho(f)$ in those papers) can be written as saddle point calculation over a functional which possess a supersymmetry between commuting and anti-commuting variables used to express the Jacobian in (7) [26]. The classical solution breaks this symmetry and in Ref. [5] a different, supersymmetric solution was proposed.

In a separate paper [7] we have performed a carefully analysis of both the classical and the new supersymmetric solutions. The outcome is that both solutions have some limitations. For example, the neglected prefactor could be exponentially small in $N$ for the classical solution changing its prediction [15,7]. On the other hand the supersymmetric solution has a negative $x_p$, while it is positive for the classical solution. This point was recently used [6] to justify the correctness of the classical solution. However, as we have explicitly shown here for the spherical $p$SP-SG model, this condition is not a sufficient condition. In order to prove the correctness of the solution one should prove that it corresponds to a physical (stable) state. A rather difficult problem already at the annealed (replica symmetric) level used in these calculations. Moreover, since it is known that a (marginally) stable solution for the
SK model requires an infinite-number of replica symmetry breakings, the requirement of local stability may partially or totally wash-out the results from the annealed approximation. A complete calculation of the complexity of the SK model must include full replica-symmetry breaking, making not only the calculation but also the analysis of the saddle points more difficult [19,27,30]. Supersymmetry requirements introduce partial simplifications, since they lead to a connection between the complexity and the replica calculation. However, we stress that supersymmetry it is not an a-priori requirement for the complexity. Indeed, while \( \tilde{\rho}(f) \) is supersymmetric, the reduction procedure needed to go from \( \tilde{\rho}(f) \) to \( \rho(f) \) may destroy the supersymmetry, so that supersymmetry must be proved case by case. It turns out that in the case of the pSP-SG model discussed here the supersymmetry is conserved [10], a property which can be associated with the fact that all metastable states are uniquely identified by its energy at \( T = 0 \) and hence its number is conserved [17].

Indeed using the TAP equation [11] \( \Sigma(q) \) for the spherical pPS-SG can be rewritten as:

\[
\Sigma^+(z) = \frac{1}{2} \left[ \frac{2-p}{p} - \ln \frac{p z^2}{2} + \frac{p-1}{p} z^2 - \frac{2}{p^2 z^2} \right]
\]

(44)

for \( x_p > 0 \) [9], and

\[
\Sigma^-(z) = \Sigma^+(z) + \ln \left[ \frac{p(p-1)}{2} z^2 \right] + \frac{1}{p(p-1)z^2} \left[ 1 - \frac{p^2(p-1)^2 z^2}{4} \right]
\]

(45)

for \( x_p < 0 \). Such property is in agreement with the argument presented in Sec. I.D of Ref. [7] where the equivalence between supersymmetry and stability of the TAP equations under external field perturbation is shown.

### A Appendix

Here we calculate \( \langle \text{det} \ A \rangle \) for \( N \to \infty \) using the identity [see, e.g., Ref. 20]

\[
\text{det} \ A = \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \exp \left( \sum_{i<j} \eta_i^+ A_{ij} \eta_j \right)
\]

(46)

where \( \eta_i \) and \( \eta_i^+ \) are anti-commuting (Grassmann) variables. From eq. 21 we have

\[
\langle \text{det} \ A \rangle = \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \exp \left( a(q) \sum_i \eta_i^+ \eta_i \right)
\]

\[
\times \prod_{i_1 < \cdots < i_p} \exp \left[ \frac{\beta}{(p-2)!} \sum_{\pi} \eta_{\pi(i_1)}^+ \eta_{\pi(i_2)} m_{\pi(i_3)} \cdots m_{\pi(i_p)} \right]
\]

\[
= \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \exp \left( a(q) \sum_i \eta_i^+ \eta_i \right)
\]

\[
\times \prod_{i_1 < \cdots < i_p} \exp \left[ \frac{\mu}{2N^{p-1}} \int \frac{p!}{(p-2)!} \right]
\]

\[
\times \sum_{\pi} \eta_{\pi(i_1)}^+ \eta_{\pi(i_2)} m_{\pi(i_3)} \cdots m_{\pi(i_p)} \eta_{i_1}^+ \eta_{i_2} m_{i_3} \cdots m_{i_p}
\]

(47)

where \( \sum_{\pi} \) is a sum over all permutations of \( p \) different integers \( i_1, \ldots, i_p \). When the products are expanded only terms which contain pairs of \( \eta_i^+ \eta_i \) with the same index survive. Since since there are \( (p-2)! \) terms with the same pairs of Grassmann variables, we end up with

\[
\langle \text{det} \ A \rangle = \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \exp \left( a(q) \sum_i \eta_i^+ \eta_i \right)
\]

\[
\times \exp \left[ \frac{\mu}{2N^{p-1}} \int \frac{p!}{(p-2)!} \right]
\]

\[
\times \sum_{i_1 < \cdots < i_p} \eta_{i_1}^+ \eta_{i_2} \eta_{i_3}^+ \cdots m_{i_p} \eta_{i_1}^+ \eta_{i_2} m_{i_3} \cdots m_{i_p}
\]

\[
= \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \exp \left( a(q) \sum_i \eta_i^+ \eta_i \right)
\]

\[
+ \frac{\mu}{2N} q^{p-2} \left( \sum_i \eta_i^+ \eta_i \right)^2
\]

(48)
where we have used $Nq = \sum m_i^2$. The square in the exponential can be open using a Stratonovich-Hubbard transformation. The resulting expression is diagonal in $\eta_i^+ \eta_i$ and the integral over the Grassmann variable can be easily done. After a simple algebra we get

$$\langle \det A \rangle = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi\sigma^2(q)/N}} \times \exp N \left[ -\frac{z^2}{2\sigma^2(q)} + \ln(\alpha(q) + iz) \right]$$

where $\sigma^2(q) = \mu(p-1)q^{p-2}$. Finally performing the change of variable $iz + (1 - q)\sigma^2(q) = ix$ we end up with [28]:

$$\langle \det A \rangle = \exp \left[ \frac{\sigma^2(q)(1-q)^2}{2\sigma^2(q)/N} \int_{-\infty}^{+\infty} dx e^{G(x)} \right]$$

For $N \rightarrow \infty$ the integral can be done by saddle point method:

$$\frac{dG(x)}{dx} = x \left[ \frac{(1-q)^2}{1 + ix(1-q)} - \frac{1}{\sigma^2(q)} \right] = 0$$

which admits two solutions: $x = 0$ and $x \neq 0$. Stability requires that the saddle point be a maximum:

$$\frac{d^2G(x)}{dx^2} = \frac{(1-q)^2}{[1 + ix(1-q)]^2} - \frac{1}{\sigma^2(q)} < 0$$

For the $x = 0$ solution this implies that [cfr. eq. [13]]

$$x_P = 1 - \sigma^2(q)(1-q)^2 > 0$$

and $G(x)$ reduces to [28].

It is easy to see that the $x \neq 0$ solution is stable only if

$$x_P = 1 - \sigma^2(q)(1-q)^2 < 0$$

in which case $G(x)$ reduces to [24].

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20. Usually this function is written as $G_{x_P} = -B(1 - q) + B^2/\lambda + \ln[1/(1 - q) + B]$. The two expressions correspond to $B = 0$ and $B \neq 0$, respectively.

21. All integrals are well defined and no extra conditions must be added.

22. The coefficient in (30) contains the term $|\partial_q f(q)|_{q=q^*}|$, where $f(q)$ is the value of the TAP functional at its stationary point and not the TAP functional itself. A simple calculation shows that $\partial_q f(q) = -(q - q_a) x_P(q)/2\beta(1 - q)^2$ which is vanishes only for $q = q_a$ or $x_P = 0$.

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31. Not to be confused with the angular part of $m_i$ used in Sec. III. We leave it like this because it is the standard notation in literature.