SMOOTH 3-DIMENSIONAL CANONICAL THRESHOLDS

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Dedicated to the memory of my advisor
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Abstract. If $X$ is an algebraic variety with at worst canonical singularities and $S$ is a $\mathbb{Q}$-Cartier hypersurface in $X$, the canonical threshold of the pair $(X, S)$ is the supremum of $c \in \mathbb{R}$ such that the pair $(X, cS)$ is canonical. We show that the set of all possible canonical thresholds of the pairs $(X, S)$, where $X$ is a germ of smooth 3-dimensional variety, satisfies the ascending chain condition. We also deduce a formula for the canonical threshold of $(\mathbb{C}^3, S)$, where $S$ is a Brieskorn singularity.

1. Introduction

Let $P \in X$ be a germ of a complex algebraic variety $X$ with at worst canonical singularities. Let $S$ be a hypersurface (not necessarily irreducible or reduced) in $X$ which is $\mathbb{Q}$-Cartier, i.e., for some integer $r$ the divisor $rS$ can locally be defined on $X$ by one equation.

Definition 1.1. The canonical threshold of the pair $(X, S)$ is
$$
ct_P(X, S) = \sup \{ c \in \mathbb{R} | \text{the pair } (X, cS) \text{ is canonical} \}.
$$

If we require in the above definition the variety $X$ and the pair $(X, cS)$ to be log canonical, we get the analogous notion of log canonical threshold $lct_P(X, S)$ which is perhaps better known (see, e.g., [7], Sections 8, 9, 10). In the same way as it is done for the log canonical threshold, considering an appropriate resolution of singularities of $(X, S)$ one shows that the number $ct_P(X, S)$ is rational and $ct_P(X, S) \in [0, 1] \cap \mathbb{Q}$.

Definition 1.2. The set of $n$-dimensional canonical thresholds is the set
$$
\mathcal{T}_n^{\text{can}} = \{ ct_P(X, S) | \dim X = n \},
$$
where $(X, S)$ varies over all pairs satisfying conditions of Definition 1.1.

We shall denote by $\mathcal{T}_n^{\text{lc}}$ the corresponding set of $n$-dimensional log canonical thresholds. The following conjecture is very important for the Minimal Model Program. Its first part was formulated by V. V. Shokurov and the second by J. Kollár.

Conjecture 1.3. (i): The set $\mathcal{T}_n^{\text{lc}}$ satisfies the ascending chain condition (ACC);
(ii): the set of accumulation points of $\mathcal{T}_n^{\text{lc}}$ is $\mathcal{T}_{n-1}^{\text{lc}} \setminus \{1\}$.

The research was supported by the Russian Grant for Scientific Schools 1987.2008.1 and by the Russian Program for Development of Scientific Potential of the High School 2.1.1/227.
An analog of Conjecture 1.3 (i) for canonical thresholds is

**Conjecture 1.4.** The set $T_n^{\text{can}}$ satisfies ACC.

Conjecture 1.3 is interesting for applications to birational geometry ([3]). It is also a particular case of conjecture of C. Birkar and V. V. Shokurov about K-lc thresholds ([1], Conjecture 1.7). We can not estimate whether the analog of Conjecture 1.3 (ii) for canonical thresholds would be plausible.

**Remark 1.5.** The log canonical threshold $\text{lct}_p(X, S)$ can be studied not only from algebraic geometry point of view. It has interpretations in terms of convergence of some integrals, Bernstein-Sato polynomials etc. (see [7], [8]). It would be interesting to find similar interpretations for the canonical threshold.

As far as we know, all the conjectures are still open in their general form. However, some important cases have been established. Let us denote by $T_{n, \text{smooth}}^{\text{lct}}$ the set $\{\text{lct}_p(X, S) \mid \dim X = n, X \text{ is smooth}\}$ of smooth $n$-dimensional log canonical thresholds. T. de Fernex and M. Mustaţă ([4]) and J. Kollár ([8]) proved the analog of Conjecture 1.3 (ii) for the set $T_{n, \text{smooth}}^{\text{lct}}$. A theorem of T. de Fernex, L. Ein and M. Mustaţă ([5]) states ACC for the set $T_{n, \text{smooth}}^{\text{lct}}$ for any $n$ (actually some restricted types of singularities of $X$ are allowed). Concerning canonical thresholds, there is a theorem of Yu. G. Prokhorov ([12], Theorem 1.4) describing the upper part of the set $T_3^{\text{can}}$ of 3-dimensional canonical thresholds.

**Theorem 1.6** (Prokhorov). If $c = \text{ct}_p(X, S)$ for some 3-dimensional variety $X$ and $c \neq 1$, then $c \leq 5/6$ and the bound is attained. Moreover, if $X$ is singular, then $c \leq 4/5$ and the bound is attained.

In this paper we also restrict ourselves to a particular case of Conjecture 1.3. Namely, we prove it for smooth 3-dimensional germs $P \in X$. Let $T_{3, \text{smooth}}^{\text{can}} = \{\text{ct}_p(X, S) \mid \dim X = 3, X \text{ is smooth}\}$ be the set of smooth 3-dimensional canonical thresholds. Our main result is the following.

**Theorem 1.7.** The set $T_{3, \text{smooth}}^{\text{can}}$ satisfies the ascending chain condition.

It is not difficult to check that $T_{n-1, \text{smooth}}^{\text{can}} \subseteq T_{n, \text{smooth}}^{\text{can}}$, so if we prove ACC in some dimension, we automatically have it in all smaller dimensions. The proof of Theorem 1.7 is contained in Section 2. Its main point is M. Kawakita’s classification of 3-dimensional contractions to smooth points ([6]). In that section we assume some familiarity of the reader with the Minimal Model Program ([9]). In Section 3 we deduce a formula for the canonical threshold of a 3-dimensional Brieskorn singularity. It is interesting that it turns out to be much more cumbersome than the corresponding formula for the log canonical threshold. In Section 4 we slightly strengthen Theorem 1.6 by showing that there are no 3-dimensional canonical thresholds between $4/5$ and $5/6$ (see Theorem 4.1).

The author is grateful to Yu. G. Prokhorov for attracting author’s attention to canonical thresholds and for useful suggestions during our work.
2. Ascending chain condition for smooth 3-dimensional canonical thresholds

2.1. Reduction to extremal contraction. Let \( P \in X \) be a germ of terminal \( \mathbb{Q} \)-factorial complex 3-dimensional algebraic variety and \( S \) an effective integer divisor on \( X \) such that the pair \((X, S)\) is not terminal. Let \( g: \tilde{X} \to X \) be an embedded resolution of the pair \((X, S)\). We denote by \( E_i, i \in I \), the prime exceptional divisors of \( g \) and by \( \tilde{S} \) the strict transform of \( S \). Then we can write the relations

\[
K_{\tilde{X}} = g^*K_X + \sum_{i \in I} a_i E_i, \quad g^*S = \tilde{S} + \sum_{i \in I} b_i E_i
\]

for some rational numbers \( a_i, b_i \). Here \( K_{\tilde{X}} \) and \( K_X \) stand for the canonical classes of \( \tilde{X} \) and \( X \) respectively. Now let \( c \in \mathbb{Q} \) be the canonical threshold of the pair \((X, S)\). This means that the pair \((X, cS)\) is canonical, i.e., if we write

\[
K_{\tilde{X}} + c\tilde{S} = g^*(K_X + cS) + \sum_{i \in I} (a_i - cb_i) E_i,
\]

then \( a_i - cb_i \geq 0 \) for all \( i \in I \). This implies the estimate \( c \leq \frac{a_i}{b_i} \) for all \( i \), and, since we assume that \( c \) is the threshold, the equality

\[
c = ct_P(X, S) = \min_{i \in I} \frac{a_i}{b_i}.
\]

(We can always assume that the resolution \( g \) has at least 1 exceptional divisor with discrepancy 0 over \((X, cS)\); this follows from \([7]\), Corollary 3.13.) This shows, in particular, that \( c \) is indeed rational. If for a divisor \( E_i \) we have \( a_i - cb_i = 0 \), we shall say that \( E_i \) (considered as a discrete valuation of the field \( \mathbb{C}(X) \) of rational functions on \( X \)) realizes the canonical threshold for the pair \((X, S)\).

Lemma 2.1 (cf. \([11]\), Section 3). Let \( c \) be the canonical threshold of a pair \((X, S)\) with terminal \( \mathbb{Q} \)-factorial 3-dimensional germ \( P \in X \). Then there exists an extremal divisorial contraction \( g': X' \supset E' \to X \ni P \) such that its exceptional divisor \( E' \) realizes the threshold \( c \). (We shall also say that the threshold \( c \) is achieved on the contraction \( g' \)).

Proof. We use the notation introduced before the lemma. Let us apply to \( \tilde{X} \) the \((K_{\tilde{X}} + c\tilde{S})\)-Minimal Model Program (MMP) relative over \( X \) (see \([9]\), Ch. 11). It stops with a \( \mathbb{Q} \)-factorial variety \( \hat{X} \) such that the pair \((\hat{X}, c\hat{S})\), where \( \hat{S} \) is the strict transform of \( \tilde{S} \), is terminal. Actually MMP contracts all the exceptional divisors of \( g \) which have positive discrepancies over \((X, cS)\). Next we apply \( K_{\hat{X}} \)-MMP over \( X \) to \( \hat{X} \). It contracts all the exceptional divisors which remain in \( \hat{X} \) and stops with the variety \( X \). Since \( X \) was supposed to be \( \mathbb{Q} \)-factorial, the last step of \( K_{\hat{X}} \)-MMP is an extremal divisorial contraction \( g': X' \to X \) from some \( \mathbb{Q} \)-factorial terminal variety \( X' \). Let \( E' \) be the exceptional divisor of \( g' \). All the divisors that we contract on this stage have discrepancy 0 over \((X, cS)\), thus \( E' \) realizes the canonical threshold of \((X, S)\). \( \square \)
For the rest of this section we assume $X$ to be smooth. The variety $X'$ obtained in Lemma 2.1 is $\mathbb{Q}$-factorial, so we again can write
\[ K_{X'} = g'^* K_X + a'E', \quad g'^* S = S' + b'E', \]
and since the canonical threshold $c$ is realized by $E'$, we have $c = a'/b'$. This reduces the calculation of the canonical threshold in the smooth 3-dimensional case to an extremal divisorial contraction, i.e., to a morphism with connected fibers $g': X' \rightarrow X$ subject to the following conditions:

(i): $X'$ is $\mathbb{Q}$-factorial with only terminal singularities;

(ii): the exceptional locus of $g'$ is a prime divisor;

(iii): $-K_{X'}$ is $g'$-ample;

(iv): the relative Picard number of $g'$ is 1.

In the 3-dimensional situation $g'$ can contract the divisor $E'$ either onto a curve $C \subset X$ or to the point $P \in X$. In the first case it follows from Mori’s classification of smooth extremal contractions \cite{10} that at a generic point of $C$ the morphism $g'$ is isomorphic to an ordinary blow up of $X$ at the curve $C$. Then it follows from \cite{1} that $c_P(X, S) = 1/\text{mult}_C(f)$, where $\text{mult}_C(f)$ is the multiplicity of the defining function $f$ of $S$ at a generic point of $C$. Thus the set $T_{3,\text{smooth}}$ contains the subset $\{1/n \mid n \in \mathbb{N}\}$ which satisfies ACC.

Now suppose that $g'$ contracts the divisor $E'$ to the smooth point $P \in X$. In this case we have an important result of M. Kawakita \cite{6}, Theorem 1.2 classifying extremal divisorial contractions to smooth points.

**Theorem 2.2** (Kawakita). Let $Y$ be a 3-dimensional $\mathbb{Q}$-factorial variety with only terminal singularities, and let $g: (Y \supset E) \rightarrow X \ni P$ be an algebraic germ of an extremal divisorial contraction which contracts its exceptional divisor $E$ to a smooth point $P$. Then we can take local parameters $x, y, z$ at $P$ and coprime positive integers $a$ and $b$ such that $g$ is the weighted blow up of $X$ with its weights $(1, a, b)$.

So further we may assume that the canonical threshold of the pair $(X, S)$ is realized by some weighted blow up. Also, since $X$ is smooth and the canonical threshold can be defined and calculated using analytic germs as well, we assume in the sequel that $P \in X$ is isomorphic to $0 \in \mathbb{C}^n$ and $S$ is determined by a convergent power series $f$.

### 2.2. Canonical threshold and weighted blow ups.

The affine space $\mathbb{A}^n_\mathbb{C}$ can be given a structure of a toric variety $X(\tau, N)$ where $N = \mathbb{Z}^n$ and the cone $\tau$ is the positive octant of the real vector space $\mathbb{R}^n \simeq N \otimes \mathbb{R}$. Let $w = (w_1, \ldots, w_n) \in N \cap \tau$ be a primitive vector. The weighted blow up $\sigma_w$ of the space $\mathbb{A}^n_\mathbb{C} \simeq \mathbb{C}^n$ with the weight vector $w$ is the toric morphism
\[ \sigma_w: X(\Sigma_w, N) \rightarrow \mathbb{C}^n \simeq X(\tau, N) \]
given by the natural subdivision $\Sigma_w$ of the cone $\tau$ with a help of the vector $w$. Certainly, a weighted blow up depends not only on its weights $w$ but also on the choice of a toric structure $\mathbb{A}^n_\mathbb{C} \simeq X(\tau, N)$. The variety $X(\Sigma_w, N)$ is $\mathbb{Q}$-factorial and can be covered by $n$ affine charts. The $i$th chart is isomorphic to $\mathbb{C}^n/\mathbb{Z}_{w_i}$ where the cyclic group acts with weights
(-w_1, \ldots, -w_{i-1}, 1, -w_{i+1}, \ldots, -w_n), \text{ and the morphism } \sigma_w \text{ is given in this chart by the formulae}

x_i = y_i^{w_i}, \quad x_j = y_j y_i^{w_i}, \quad j \neq i

where \(x_1, \ldots, x_n\) are the coordinates on the target and \(y_1, \ldots, y_n\) on the source space (see, e.g., [11], 3.7).

Given a hypersurface \(S = \{f = 0\}\) in \(\mathbb{C}^n\), we can estimate the canonical threshold \(c_t(\mathbb{C}^n, S)\) with a help of weighted blow ups. Namely, for any weight vector \(w\) (not equal to 0 or to a vector \(e_i\) of the standard basis) we have

\[c_t(\mathbb{C}^n, S) \leq w_1 + \cdots + w_n - 1 \frac{w(f)}{w_i},\]

where \(w(f)\) is the least weight of a monomial appearing in \(f\) with respect to the weights \(w_1, \ldots, w_n\) (see [12]). Moreover, if we know that the canonical threshold of \((\mathbb{C}^n, S)\) is achieved on some weighted blow up, we can calculate it as

\[c_t(\mathbb{C}^n, S) = \min_{w \neq 0, e_i, i = 1, \ldots, n} w_1 + \cdots + w_n - 1 \frac{w(f)}{w_i}.\]

where the minimum is taken over all integer vectors in \(\tau\). It is possible that the denominator in (2) is 0 for some weights \(w\); such fractions should be treated as \(+\infty\).

Recall that the extended Newton diagram \(\Gamma^+(f)\) of a polynomial (or a series) \(f = \sum a_m x^m\) is the convex hull in \(\mathbb{R}^n\) of the set \(\{m + \mathbb{R}_{\geq 0} | a_m \neq 0\}\). Note that the denominator \(w(f)\) in the above formulae depends only on \(\Gamma^+(f)\) but not on \(f\) itself. Let us introduce a new set \(\mathcal{T}_{\text{can}, w}^{n, \text{smooth}}\) = the set of smooth \(n\)-dimensional canonical thresholds which can be realized by some weighted blow up.

Remark 2.3. The set \(\mathcal{T}_{\text{can}, w}^{n, \text{smooth}}\) may look a bit artificial. However, it contains, for example, the set of smooth 3-dimensional canonical thresholds (for \(n = 3\); see subsection 2.1), or the set of canonical thresholds achieved on hypersurfaces \(S\) defined by series \(f\) nondegenerate with respect to their Newton diagrams.

Now Theorem 1.7 follows from the next result.

**Theorem 2.4.** The set \(\mathcal{T}_{\text{can}, w}^{n, \text{smooth}}\) satisfies ACC.

The proof of Theorem 2.4 will follow from the 2 lemmas below.

Let us denote by \(\mathcal{N}^+_n\) the set of all possible extended Newton diagrams in \(\mathbb{R}^n\). Clearly this set is ordered with respect to the inclusion relation \(\subseteq\). The next lemma is perhaps well known for specialists on singularity theory or on Gröbner bases. But since we do not know a good reference, we state it with a proof.

**Lemma 2.5.** Any infinite sequence of elements of the ordered set \(\mathcal{N}^+_n\) contains a monotonous non increasing subsequence. In particular, the set \(\mathcal{N}^+_n\) satisfies ACC.

**Proof.** It is more convinient to use the ideals of semigroup \((\mathbb{Z}_{\geq 0}, +)\) (see [2], § 1.1) in the proof instead of extended Newton diagrams. With any diagram \(\Gamma^+\) we can associate the ideal generated by its vertices. From any ideal
we can construct an extended Newton diagram by taking its convex hull in \( \mathbb{R}^n_{\geq 0} \). Despite this correspondence is not one-to-one, it should be clear that it suffices to prove the lemma for ideals.

The proof goes by induction on the dimension \( n \). For \( n = 1 \) the statement is obvious. Let \( \Gamma_1^+, \Gamma_2^+, \ldots \) be a sequence of ideals of the semigroup \( \mathbb{Z}^n_{\geq 0} \). We can project every ideal \( \Gamma_i^+ \) to the coordinate axes of the space \( \mathbb{R}^n \). Every projection gives an ideal in the semigroup \( \mathbb{Z}^n_{\geq 0} \). First let us suppose that for some axis, say, the last one, the so obtained sequence of projections is unbounded, i.e., for any \( M > 0 \) there is a number \( k \) such that the projection of the ideal \( \Gamma_k^+ \) to the last axis lies above \( M \). Choosing a subsequence we can assume that in fact the projection of any ideal \( \Gamma_i^+ \) lies above \( M \) for \( i \geq k \). The projection of every ideal \( \Gamma_i^+ \) to the coordinate hyperplane containing the first \( n-1 \) coordinate axes is an ideal in the semigroup \( \mathbb{Z}^{n-1}_{\geq 0} \). By the induction hypothesis we can choose a subsequence such that the corresponding sequence of projections to the given hyperplane is monotonous non-increasing. To simplify the notation we assume that already the sequence \( \Gamma_1^+, \Gamma_2^+, \ldots \) has this property. Consider the ideal \( \Gamma_1^+ \). Let us denote by \( M \) the maximal of the last coordinates of its vertices (we use here the fact that any ideal of \( \mathbb{Z}^n_{\geq 0} \) is a finite union of the shifted coordinate octants; this is called the \textit{first finiteness property} of the semigroup \( \mathbb{Z}^n_{\geq 0} \), see \cite{2}, § 1.1, Theorem 1).

Starting from some number \( k \) all the ideals \( \Gamma_i^+ \) lie above \( M \) with respect to the last coordinate, and thus they all are contained in \( \Gamma_1^+ \). Going on in the same way we construct the needed non-increasing subsequence of ideals.

Now suppose that the projections of the sequence \( \Gamma_1^+, \Gamma_2^+, \ldots \) to all the coordinate axes are bounded. This means that all the ideals \( \Gamma_i^+ \) have a vertex inside some fixed polytope. But every bounded polytope has only finite number of integer points, thus, again choosing a subsequence if necessary, we can assume that all \( \Gamma_i^+ \) have a common vertex \( m \). It follows that all the ideals \( \Gamma_i^+ \) contain also a common shifted coordinate octant \( m + \mathbb{R}^n_{\geq 0} \). Consider the complement to this octant in \( \mathbb{Z}^n_{\geq 0} \). It is clear that it is a union of finite number of shifted coordinate subsemigroups of the semigroup \( \mathbb{Z}^n_{\geq 0} \) (this is a particular case of the \textit{second finiteness property} of the semigroup \( \mathbb{Z}^n_{\geq 0} \), \cite{2}, § 1.1, Theorem 2). The dimension of such a semigroup is not greater than \( n-1 \), and the intersection of any ideal \( \Gamma_i^+ \) with such a semigroup is an ideal of it. Applying the induction hypothesis and repeatedly (but finite number of times!) choosing a subsequence, we conclude that outside the octant \( m + \mathbb{R}^n_{\geq 0} \) our sequence \( \Gamma_1^+, \Gamma_2^+, \ldots \) can be assumed to be non-increasing. But taking the union with the same octant obviously does not change this property. This finishes the proof. \( \square \)

The formula (2) formally defines the canonical threshold of an extended Newton diagram \( \Gamma^+ \) which we shall denote by \( \text{ct}(\Gamma^+) \). We can consider it as a map

\[
\text{ct}: \mathcal{N}^+_n \to \mathbb{R}.
\]

**Lemma 2.6.** The map \( \text{ct} \) is monotonous, i.e.,

\[
\Gamma_1^+ \subseteq \Gamma_2^+ \Rightarrow \text{ct}(\Gamma_1^+) \leq \text{ct}(\Gamma_2^+).
\]
Proof. Let $w$ be a vector in $\mathbb{Z}^n_{\geq 0}$, $w \neq 0, e_1, \ldots, e_n$. It defines a rational function $w: \mathbb{R}^n \rightarrow \mathbb{R}$

$$w(x) = \frac{w_1 + \cdots + w_n - 1}{w_1 x_1 + \cdots + w_n x_n}.$$  

The level set $w = c$ of this function (w is fixed, $x$ varies) is a hyperplane with a non negative normal vector $w$. The smaller $c > 0$ we take, the further from the origin the level set $w = c$ is.

Suppose that the canonical threshold $c = ct(\Gamma^+_2)$ of the diagram $\Gamma^+_2$ is realized by the weight vector $w$ and this threshold is attained on a vertex $m$ of the diagram $\Gamma^+_2$, $c = w(m)$. The minimum $c'$ of the function $w$ on the diagram $\Gamma^+_1$ is not greater than $c$ because $\Gamma^+_1$ is situated “above” the diagram $\Gamma^+_2$. The threshold $ct(\Gamma^+_1)$ can only be less or equal to $c'$. \qed

To finish the proof of Theorem 2.4, suppose that there exists a strictly increasing sequence $c_1 < c_2 < \ldots$ of canonical thresholds from $T_{\text{can, smooth, w}}^n$. Let $\Gamma^+_1, \Gamma^+_2, \ldots$ be a sequence of extended Newton diagrams such that $ct(\Gamma^+_k) = c_k$. We can not have an inclusion $\Gamma^+_i \supset \Gamma^+_j$ for any $i < j$ because of Lemma 2.6. But this contradicts Lemma 2.5.

3. Canonical threshold for Brieskorn singularities in $\mathbb{C}^3$

A Brieskorn singularity is a hypersurface singularity $S$ is $\mathbb{C}^n$ given by the equation

$$x^{a_1} + x^{a_2} + \cdots + x^{a_n} = 0.$$  

For $n = 3$ we shall assume that $S$ is given by

$$x^a + y^b + z^c = 0,$$

where $2 \leq a \leq b \leq c$. The log canonical threshold of the pair $(\mathbb{C}^n, S)$ can be determined by the formula ([7], 8.15)

$$\text{lct}_0(\mathbb{C}^n, S) = \min\left\{ \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, 1 \right\}.$$  

In this section we calculate the 3-dimensional canonical threshold $ct_0(\mathbb{C}^3, S)$. Brieskorn singularities are nondegenerate with respect to their Newton diagrams, thus they admit embedded toric resolutions (for the definition of nondegeneracy and construction of embedded toric resolution see [13]). It follows, in particular, that their canonical thresholds are realized by weighted blow ups and we can apply formula (2) from subsection 2.2. In the case of 3-dimensional Brieskorn singularities it takes the form

$$ct_0(\mathbb{C}^3, S) = \min_{w \neq e_1, e_2, e_3, 0} \frac{w_1 + w_2 + w_3 - 1}{\min\{aw_1, bw_2, cw_3\}},$$

where the minimum is taken over all vectors $w$ from $\mathbb{Z}^3_{\geq 0}$.

Remark 3.1. Formula (4) is not a direct consequence of Theorem 2.2 and subsection 2.1. Indeed, the theorem states that there exist some coordinates in which the extremal contraction realizing the canonical threshold is a weighted blow up. But in those coordinates the equation of our singularity must not be of Brieskorn type or even nondegenerate.
Lemma 3.2. Let \( S \subset \mathbb{C}^3 \) be a Brieskorn singularity given by equation (3). Suppose that a weight vector \( w = (p, q, r) \) realizes the minimum in (4). Then \( p \geq q \geq r \).

Proof. It is clear that the vector realizing the canonical threshold satisfies \( p, q, r \neq 0 \). Assume, for example, that \( p < q \). Then
\[
ct_0(\mathbb{C}^3, S) = \frac{p+q+r-1}{\min\{ap, cr\}}
\]
and \( q \geq 2 \). But in this case we could take \( w' = (p, q-1, r) \) instead of \( w \) and \( w' \) would give strictly smaller canonical threshold, a contradiction. Other inequalities can be considered similarly. \( \square \)

Lemma 3.3. A weight vector \( w \) giving the canonical threshold of a Brieskorn singularity (3) can always be chosen in the from \( w = (p, q, 1) \), where \( p \) and \( q \) are coprime positive integers.

Proof. Consider a piecewise rational function \( h \) determined on \( \mathbb{R}^3_{\geq 0} \) by the formula
\[
h(w) = \frac{w_1+w_2+w_3-1}{\min\{aw_1, bw_2, cw_3\}}.
\]
Its level set \( h(w) = s, s \geq 0 \), coincides with the lateral surface of a tetrahedron \( \Delta_s \) with vertices \((1,0,0), (0,1,0), (0,0,1)\) (forming the base face of the tetrahedron) and with the last vertex
\[
\frac{1}{1/a + 1/b + 1/c} - s \left( \frac{1}{a} \frac{1}{b} \frac{1}{c} \right)
\]
on the line \( aw_1 = bw_2 = cw_3 \). If \( s < s' \), then \( \Delta_s \subset \Delta_{s'} \). We see that the canonical threshold of a Brieskorn singularity (3) can be found with a help of the following process. For every \( s \geq 0 \) we construct the tetrahedron \( \Delta_s \) and find the minimal \( s_0 \) for which \( \Delta_{s_0} \) contains an integer point \((p, q, r)\) with \( p, q, r > 0 \) on its lateral border. Then \( ct_0(\mathbb{C}^3, S) = s_0 \) and the threshold is realized by the weight vector \((p, q, r)\).

Now suppose that \( ct_0(\mathbb{C}^3, S) = s_0 = h(p, q, r) \) and \( r \geq 2 \). From Lemma 3.2 we know that \( p \geq q \geq r \). Consider also a tetrahedron \( \Delta_w \) with vertices \((1,0,0), (0,1,0), (0,0,1), \) and \((p, q, r)\). Obviously \( \Delta_w \subset \Delta_{s_0} \) and we prove our lemma if we show that the intersection of \( \Delta_w \) with the plane \( w_3 = 1 \) contains a positive integer point. Indeed, if \((p', q', 1)\) is such a point and \( d \) is the greatest common divisor of \( p' \) and \( q' \), the point \((p'/d, q'/d, 1)\) also lies in \( \Delta_w \) and gives smaller value of \( h \). The intersection \( \Delta_w \cap \{w_3 = 1\} \) is a triangle with vertices \((0,0), 1/r(p+r-1, q), 1/r(p, q+r-1)\) (the third coordinate \( w_3 = 1 \) is omitted here). It is a bit more convenient to multiply everything by \( r \) and to show that the triangle \( OPQ, O = (0,0), \ P = (p, q+r-1), Q = (p+r-1, q), \) contains a positive integer point with coordinates \( 0 \mod r \) (see Figure 1). Other points in Figure 1 have the following meaning. \( E \) is the intersection point of the lines \( OP \) and \( w_2 = q \). Its coordinates are \((pq/(q+r-1), q)\). The segment \( BC \) is the middle line of the triangle \( EPQ \) and the segment \( AD \) lies on the line \( w_2 = q - r/2 \).

Note that the point \( Q \) lies under the diagonal \( w_1 = w_2 \). Thus if the point \( P \) lies above the diagonal, then the triangle \( OPQ \) contains already the point \((r, r)\). So let us assume \( p > q + r - 1 \). Choose an integer \( k \geq 2 \) such that
Figure 1. Integer points in the triangle

\[(k - 1)(q + r - 1) < p \leq k(q + r - 1)\]. Next we consider 2 cases: \(q \leq 2r - 1\) and \(q \geq 2r\). Suppose first that \(q \leq 2r - 1\). Consider a transformed triangle \(OP'Q'\) obtained from \(OPQ\) with a help of the unimodular transformation \(\begin{pmatrix} 1 & -(k-1) \\ 0 & 1 \end{pmatrix}\).

The point \(P'(p - (k - 1)(q + r - 1), q + r - 1)\) lies above the diagonal. Indeed, \(p - (k - 1)(q + r - 1) \leq q + r - 1\). On the other hand, the point \(Q'(p - (k - 1)q + r - 1, q)\) lies under the diagonal:

\[p - (k - 1)q + r - 1 > (k - 1)(q + r - 1) - (k - 1)q + r - 1 = 2r - 1 + (k - 2)r - (k - 1) \geq q + (k - 2)r - (k - 1) \geq q - 1\]

because \(r, k \geq 2\). It follows that the triangle \(OP'Q'\) contains the point \((r, r)\). But then the triangle \(OPQ\) also has a positive integer point \(0 \mod r\).

Now suppose that \(q \geq 2r\). Note that the length of \(E Q\) is

\[p + r - 1 - \frac{pq}{q + r - 1} = (r - 1)\frac{p + q + r - 1}{q + r - 1} > 2(r - 1)\]

and hence \(BC > r - 1\). \(AD \geq (3/4)EQ > (3/2)(r - 1)\). The last number is greater than \(r\) for \(r \geq 3\). If we have a segment \([x, y]\) with integer \(x\) or \(y\) on the real line \(\mathbb{R}\), and if this segment has length \(\geq r - 1\), then it necessarily contains a point \(0 \mod r\) (perhaps as one of its border points). If the length of the segment is \(\geq r\), it always contains a point \(0 \mod r\) no matter whether \(x\) or \(y\) are integers. From these observations it easily follows that for \(r \geq 3\) already the pentagon \(ABCQD\) contains a point \(0 \mod r\). We leave the details to the reader.

So it remains to consider the case when \(r = 2\) and \(q \geq 4\). Moreover, we can assume that \(p\) and \(q\) are odd because otherwise already one of the points \(P\), \(Q\), or \((p, q)\) will have coordinates \(0 \mod r\). Let us check if the point \((p - 1, q - 1)\) lies in the triangle \(OPQ\). This holds if

\[\frac{q - 1}{p - 1} \geq \frac{q}{p + 1},\]
or \( p \leq 2q - 1 \). On the other hand,

\[ AD = \frac{q - 1}{q} \cdot \frac{p + q + 1}{q + 1} \geq 2 \]

holds if

\[ p \geq \frac{(q + 1)^2}{q - 1} > q + 1. \]

Two inequalities \( p \leq 2q - 1 \) and \( p > q + 1 \) cover all possibilities for \( p \) and \( q \). Thus the triangle \( OPQ \) always contains the desired point. \( \square \)

**Remark 3.4.** Again Lemma 3.3 is not a direct consequence of Kawakita’s Theorem 2.2, see Remark 3.1

Now we are ready to deduce a formula for the canonical threshold of a Brieskorn singularity. To do this, let us introduce some new notation. Denote by \( L \) the point \((c/a, c/b, 1)\) of the intersection of the line \( aw_1 = bw_2 = cw_3 \) with the plane \( w_3 = 1 \). Fix a real number \( s \) and consider the intersection of the tetrahedron \( \Delta_s \) with the plane \( w_3 = 1 \) (see the proof of Lemma 3.3).

For \( s < 1/a + 1/b \) this intersection is empty; for \( s = 1/a + 1/b \) it is the segment \( OL \); for \( s > 1/a + 1/b \) it is a triangle \( OMN \) where \( M = (c/a, sc - c/a, 1) \), \( N = (sc - c/b, c/b, 1) \) (see Figure 2). This almost immediately implies

\[ w_2 \]

\[ k \]

\[ O \]

\[ w_1 \]

\[ M \]

\[ L \]

\[ N \]

**Figure 2. Minimizing the function \( h \)**

**Lemma 3.5.** Let \( S \) be a Brieskorn singularity of the form (3). Then \( ct_0(C^3, S) \geq 1/a + 1/b \). Moreover, if \( c \geq \text{l.c.m.}(a, b) \), where \( \text{l.c.m.} \) is the least common multiple, then \( ct_0(C^3, S) = 1/a + 1/b \).

**Proof.** Only the “moreover” part of the lemma needs a proof. If \( c \geq m = \text{l.c.m.}(a, b) \), then the segment \( OL \) contains the point \((m/a, m/b, 1)\). \( h(m/a, m/b, 1) = 1/a + 1/b \), but we know from the first part of the lemma that \( 1/a + 1/b \) is the least possible value of the canonical threshold. The proof is completed. \( \square \)

When \( c < \text{l.c.m.}(a, b) \), we have a kind of integer programming problem: minimize the piecewise rational function \( h \) on positive integer points of the plane \( w_3 = 1 \). Equivalently, we have to determine the minimal \( s \) such that the triangle \( OMN \) contains a positive integer point. The point \( w \) on which \( h \) attains its minimum is on the following list:
(i): $w$ is the nearest to $OL$ integer point lying on the horizontal line $w_2 = k$, $1 \leq k \leq \lfloor c/b \rfloor$ to the right from the segment $OL$ (see Figure 2); in this case $w = ([kb/a], k, 1)$; here $\lfloor \cdot \rfloor$ denotes the lower and $\lceil \cdot \rceil$ the upper integer;

(ii): $w$ is the nearest to $OL$ integer point lying on the vertical line $w_1 = k$, $1 \leq k \leq \lfloor c/a \rfloor$ above the segment $OL$; in this case $w = (k, \lceil ka/b \rceil, 1)$;

(iii): $w$ is “the first” integer point in the triangle $LMN$; then $w = (\lceil c/a \rceil, \lceil c/b \rceil, 1)$.

Let

$$s_1 = \min_{1 \leq k \leq \lfloor c/b \rfloor} \{ h([kb/a], k, 1) \} = \min_{1 \leq k \leq \lfloor c/b \rfloor} \left\{ \frac{1}{b} + \frac{1}{kb} \left\lceil \frac{kb}{a} \right\rceil \right\},$$

$$s_2 = \min_{1 \leq k \leq \lfloor c/a \rfloor} \{ h(k, [ka/b], k, 1) \} = \min_{1 \leq k \leq \lfloor c/a \rfloor} \left\{ \frac{1}{a} + \frac{1}{ka} \left\lceil \frac{ka}{b} \right\rceil \right\},$$

$$s_3 = h([c/a], [c/b], 1) = \frac{\lceil c/a \rceil + \lceil c/b \rceil}{c}.$$

We summarize what we did in this section in the following result.

**Theorem 3.6.** Let $S$ be a Brieskorn singularity [3]. If $\text{l.c.m.}(a, b) \leq c$, then $\text{ct}_0(C^3, S) = 1/a + 1/b$; otherwise $\text{ct}_0(C^3, S) = \min\{s_1, s_2, s_3, 1\}$ (notation as above).

**Example 3.7.** Consider a Brieskorn singularity $x^3 + y^7 + z^{11} = 0$.

Using our formulae we get

$$s_1 = \min_{1 \leq k \leq 4} \{ 1/7 + (1/7) \cdot 3 \} = \frac{4}{7},$$

$$s_2 = \min_{1 \leq k \leq 3} \{ 1/3 + (1/3k)\lceil (3k)/7 \rceil \} = \min\{1/3 + 1/3, 1/3 + 1/6, 1/3 + 2/9\} = \frac{1}{2},$$

$$s_3 = (4 + 2)/11 = \frac{6}{11}.$$

It follows that $\text{ct}_0(C^3, S) = 1/2 = s_2$ and it is achieved on the weighted blow up with weights $(2, 1, 1)$.

**Example 3.8.** Let $S$ be a Brieskorn singularity $x^5 + y^6 + z^{29} = 0$.

We have

$$s_1 = \min_{1 \leq k \leq 4} \{ 1/6 + 1/(6k)\lceil (6k)/5 \rceil \} = \min\{1/6 + 1/3, 1/6 + 1/4, 1/6 + 2/9, 1/6 + 5/24\} = \frac{3}{8},$$

$$s_2 = \min_{1 \leq k \leq 3} \{ 1/5 + 1/(5k)\lceil (5k)/6 \rceil \} = \frac{2}{5},$$

$$s_3 = (6 + 5)/29 = \frac{11}{29}.$$
It follows that $ct_0(\mathbb{C}^3, S) = 3/8 = s_1$ and it is achieved on the weighted blow up $(5, 4, 1)$.

**Example 3.9.** Now let $S$ be a Brieskorn singularity

$$x^{12} + y^{18} + z^{35} = 0.$$ 

We get

$$s_1 = \min_{1 \leq k \leq 1} \{1/18 + 1/9\} = \frac{1}{6},$$

$$s_2 = \min_{1 \leq k \leq 2} \{1/12 + 1/(12k)\} = \frac{1}{6},$$

$$s_3 = (3 + 2)/35 = \frac{1}{7}.$$ 

It follows that $ct_0(\mathbb{C}^3, S) = 1/7 = s_3$ and it is achieved on the weighted blow up $(3, 2, 1)$.

4. **The upper part of the canonical set**

In this section we strengthen Theorem 1.6 of Yu. G. Prokhorov describing the upper part of the set $\mathcal{T}_3^{\text{can}}$ of 3-dimensional canonical thresholds.

**Theorem 4.1.** The intersection $\mathcal{T}_3^{\text{can}} \cap [4/5, 1]$ is precisely $\{4/5, 5/6, 1\}$.

**Proof.** Recall that if $X$ is singular, then $ct_P(X, S) \leq 4/5$ (Theorem 1.6), and if $S$ has non isolated singularities in a neighborhood of $P$, then $ct_P(X, S) \leq 1/2$ (subsection 2.1). Thus we may assume that $S$ is a hypersurface in $\mathbb{C}^3$ with isolated singularity at the origin. Then by Kawakita’s Theorem 2.2 and subsection 2.1 the canonical threshold $ct_0(\mathbb{C}^3, S)$ is achieved on some weighted blow up.

Let $S$ be given in $\mathbb{C}^3$ by an equation $f = 0$. We shall analyze the Newton diagram $\Gamma(f)$ of $f$ and show that the canonical threshold of $S$ can be $1, 5/6, 3/4$ or smaller. First note that if the Newton diagram of $f$ lies above the plane $\alpha + \beta + \gamma = 3$, then $ct_0(\mathbb{C}^3, S) \leq 2/3$. Indeed, in this case $\Gamma^+(f)$ is contained in the extended Newton diagram of the singularity

$$x^3 + y^3 + z^3 = 0$$

which has canonical threshold $2/3$. Thus by Lemma 2.6 $ct_0(\mathbb{C}^3, S) \leq 2/3$.

We see that the function $f$ necessarily has monomials of degree 2. If the second differential of $f$ has rank 2 or 3, $f$ is isomorphic to a Du Val singularity of type $A_n$. In this case its canonical threshold is 1. But the same holds even if the second differential of $f$ has rank 1 and $f$ has at least 2 monomials of degree 2. Indeed, recall that the canonical threshold of $S$ depends only on the Newton diagram $\Gamma(f)$. But then we can perturb the coefficients of $f$ in such a way that the second differential becomes of rank $\geq 2$. It follows that we can assume that $f$ has the form

$$f = x^2 + \text{terms of degree } \geq 3.$$ 

Moreover, making a substitution $x' = x\sqrt{1 + \ldots}$ (it does not violate the property that the canonical threshold of $f$ is achieved on a weighted blow up) we can assume that $x^2$ is the only monomial of $f$ containing $x$ with degree $\geq 2$. 

Further, let us compare $f$ with the Brieskorn singularity
\[ x^2 + y^4 + z^4 = 0. \]
Its canonical threshold is $3/4$ (see Section 3). It follows that if $\Gamma(f)$ lies above the plane $2\alpha + \beta + \gamma = 4$, then $\text{ct}_0(\mathbb{C}^3, S) \leq 3/4$. Therefore we can suppose that $f$ has a monomial of degree 3. If this monomial is $y^2z$, we again conclude that $f$ is a Du Val singularity (this time of type $D_n$) and $\text{ct}_0(\mathbb{C}^3, S) = 1$. Thus we assume
\[ f = x^2 + y^3 + \text{other terms of degree } \geq 3. \]

Next we compare $f$ with the Brieskorn singularity
\[ x^2 + y^3 + z^6 = 0. \]
Its canonical threshold is $5/6$. Suppose that $f$ contains monomials $x^\alpha y^\beta z^\gamma$ lying below the plane $3\alpha + 2\beta + \gamma = 6$. Possible monomials are $z^3$, $z^4$, $z^5$, $yz^2$, $yz^3$, $xz^2$. In this case $f$ is a Du Val singularity of type $D_n$ or $E_n$ and its canonical threshold is 1. Therefore it remains to consider the case when $\Gamma(f)$ lies above the plane $3\alpha + 2\beta + \gamma = 6$ and $\text{ct}_0(\mathbb{C}^3, S) \leq 5/6$. Let us show that in fact $\text{ct}_0(\mathbb{C}^3, S) = 5/6$.

Since we suppose that $f$ defines an isolated singularity, it has monomials of the form $z^n$, $xz^n$, or $yz^n$. Hence we can estimate the canonical threshold of $S$ from below comparing it with the nondegenerate singularities
\[ x^2 + y^3 + z^n = 0, \quad n \geq 6, \]
\[ x^2 + y^3 + xz^n = 0, \quad n \geq 3, \]
or
\[ x^2 + y^3 + yz^n = 0, \quad n \geq 4. \]
The first and the second singularity are easily seen to be isomorphic to Brieskorn singularities with canonical threshold $5/6$. Let us show by a direct computation that the canonical threshold of the third singularity $S' \subset \mathbb{C}^3$ is also $5/6$.

Consider the weighted blow up $\sigma$ of $\mathbb{C}^3$ with weights $(3, 2, 1)$. The blown up variety $\widetilde{\mathbb{C}^3}$ is covered by 3 affine charts (see subsection 2.2). In the first isomorphic to $\mathbb{C}^3/\mathbb{Z}_3(1,1,2)$ the strict transform $S'_{\widetilde{\mathbb{C}^3}}$ of $S'$ is isomorphic to
\[ 1 + y^3 + x^{n-4}yz^n = 0 \]
and is nonsingular. In the second chart $\mathbb{C}^3/\mathbb{Z}_2(1,1,1)$ the strict transform
\[ x^2 + 1 + y^{n-4}z^n = 0 \]
is again nonsingular. Note that the quotient singularities of the first 2 charts are terminal and $S'_{\widetilde{\mathbb{C}^3}}$ does not pass through them. The third chart is isomorphic to $\mathbb{C}^3$ and the strict transform to
\[ x^2 + y^3 + yz^{n-4} = 0, \]
i. e., to a singularity of the same form but with smaller $n$. Computing discrepancy we get
\[ K_{\widetilde{\mathbb{C}^3}} + (5/6)S'_{\widetilde{\mathbb{C}^3}} = \sigma^*(K_{\mathbb{C}^3} + (5/6)S'). \]
Therefore by induction we show that $c_0(C^3, S') = 5/6$ and finish the proof of our theorem. □

References

[1] Birkar, C., Shokurov, V. V. Mld’s vs thresholds and flips, to appear in J. Reine Anrew. Math. arXiv:math.AG/0609539 2006.

[2] Chulkov, S. P., Khovanskiı̆, A. G. Geometry of the semigroup $\mathbb{Z}_{\geq 0}$. Applications to combinatorics, algebra, and differential equations (Russian), MCCME, Moscow, 2009.

[3] Corti, A. Factoring birational maps of threefolds after Sarkisov, J. Algebr. Geom. 4(2) (1995), 223–254.

[4] de Fernex, T., Mustaţă, M. Limits of log canonical thresholds, Ann. Sci. École Norm. Sup. 42 (2009), 493–517. arXiv:0710.4978 2007.

[5] de Fernex, T., Ein, L., Mustaţă, M. Shokurov’s ACC conjecture for log canonical thresholds on smooth varieties, to appear in Duke Math. J. arXiv:0905.3775 2009.

[6] Kawakita, M. Divisorial contractions in dimension 3 which contract divisors to smooth points, Invent. Math. 145 (2001), 105–119. arXiv:math.AG/0005207.

[7] Kollár, J. Singularities of pairs, in Algebraic geometry — Santa Cruz 1995, v. 62 of Proc. Sympos. Pure Math., 221–287, Amer. Math. Soc., Providence, RI, 1997.

[8] Kollár, J. Which powers of holomorphic functions are integrable, arXiv:0805.0750 2008.

[9] Matsuki, K. Introduction to Mori’s Program, Springer-Verlag, 2002.

[10] Mori, S. Threefolds whose canonical bundles are not numerically effective, Ann. Math. 116 (1982), 133–176.

[11] Prochorov, Yu. G. Lectures on complements on log surfaces. V. 10 of MSJ Memoirs, Mathematical Society of Japan, 2001.

[12] Prokhorov, Yu. G. Gap conjecture for 3-dimensional canonical thresholds, J. Math. Sci. Univ. Tokyo, 15 (2008), 449–459.

[13] Varchenko, A. N. Zeta-function of monodromy and Newton’s diagram, Invent. Math. 37(3) (1976), 253–262.