We use the conjectured strong-weak coupling worldsheet duality between the $SL(2)/U(1)$ and Sine-Liouville conformal field theories to study some properties of degenerate operators and to compute correlation functions in CFT on $AdS_3$. The same quantities have been computed in the past by other means. The agreement between the different approaches provides new evidence for the duality. We also discuss the supersymmetric analog of this duality, the correspondence between SCFT on the cigar and $N = 2$ Liouville. We show that in the spacetime CFT dual to string theory on $AdS_3$ via the AdS/CFT correspondence, the central term in the Virasoro algebra takes different values in different sectors of the theory. In a companion paper we use the results described here to study D-branes in $AdS_3$. 

5/01
1. Introduction

In this paper we study string theory on $AdS_3$ (the infinite cover of the $SL(2)$ group manifold). This system received a lot of attention over the years (see e.g. [1,2,3,4,5] for some recent discussions and additional references), for a variety of reasons. Some of the reasons for our interest in this model are:

1. We regard it as a warmup exercise for the study of string theory in time dependent backgrounds. For instance, the coset $[SL(2) \times SU(2)]/R^2$ is a cosmological background corresponding to a closed universe which begins and ends with a singularity [6]. $AdS_3$ CFT is an important ingredient in analyzing the physics of this model.

2. It is relevant for the study of asymptotically linear dilaton theories, such as the Liouville model and the cigar CFT [7], which describes a Euclidean two dimensional black hole [8,9]. Liouville can be obtained from $SL(2)$ (after a certain twist) by gauging a Borel subgroup. The cigar corresponds to the coset $SL(2)/U(1)$ where the $U(1)$ is associated with the compact timelike direction in $SL(2)$. Gauging the non-compact spacelike $U(1)$ gives a Lorentzian two dimensional black hole.

3. Linear dilaton models have many applications. Liouville theory plays an important role in two dimensional string theory, which is holographically dual to a certain matrix quantum mechanics in the large $N$ limit (see e.g. [10] for a review). The cigar appears both in two dimensional string theory, where it describes the high energy thermodynamics (see [11] for a recent discussion) and in Little String Theory (LST), where it again plays a role in the thermodynamics (see e.g. [12]), and in a certain double scaling limit defined in [13]. The near-horizon geometry of NS fivebranes in the presence of fundamental strings interpolates between a linear dilaton region far from the strings and an $AdS_3$ geometry near the strings [14]. D-branes stretched between non-parallel fivebranes that give rise at low energies to four dimensional $N = 1$ SYM [15] are described in the near-horizon region of the intersection of the fivebranes as D-branes living near the tip of the cigar [16,17]. One of the main motivations for this work is to develop tools for studying such D-branes, with the hope of learning more about $N = 1$ SYM.

4. String theory on $AdS_3$ is an interesting special case of the AdS/CFT correspondence. It does not require turning on RR backgrounds, and thus can be studied (at weak string coupling) by standard worldsheet techniques. Also, the spacetime CFT is in this case two dimensional; hence the corresponding conformal symmetry is infinite.
dimensional. This symmetry can be realized directly in string theory \cite{1,2} and its presence might provide clues for the study of broken infinite dimensional symmetries in string theory in general.

5. String theory on $AdS_3$ is relevant for the study of the quantum mechanics of $d = 3, 4, 5$ black holes \cite{18,19}.

This is the first of two papers in which we discuss some aspects of the dynamics of strings on $AdS_3$. The main issues that we address here are the following:

Non-locality of string theory on $AdS_3$: the spacetime CFT corresponding to string theory on $AdS_3$ via the AdS/CFT correspondence has some non-local features. Some of the manifestations of this non-locality are:

(a) The spectrum of scaling dimensions in the spacetime CFT contains a continuum above a finite gap. This continuum corresponds to long strings living near the boundary of $AdS_3$ \cite{20,21,22}. These strings do not correspond to local operators in the spacetime CFT.

(b) Correlation functions in string theory on $AdS_3$ exhibit singularities at values of the scaling dimensions where short strings can scatter into long strings \cite{13,23}. Thus, the non-locality associated with long strings influences the physics of short strings as well.

(c) As mentioned above, after gauging a $U(1)$, one finds asymptotically linear dilaton theories such as Liouville and the cigar. The latter are related to LST and are non-local in spacetime.

In these notes we add another entry to this list. We show that the central term of the spacetime Virasoro algebra, which was constructed in \cite{1,2}, is given by a dimension zero operator which is not proportional to the identity. The central charge is different in different sectors of the theory; in a state with spacetime scaling dimension $h$, the central charge has a contribution that grows like $h$. This generalizes observations in \cite{1,2}, where it was shown that long strings carry non-zero central charge; we will see that short strings carry central charge as well.

The conjectured duality between $SL(2)/U(1)$ and Sine-Liouville: V. Fateev, A. B. Zamolodchikov and Al. B. Zamolodchikov conjectured \cite{24} that the CFT’s on the cigar and Sine-Liouville theory are equivalent under a strong-weak coupling duality. The detailed statement of the duality appears in \cite{11}. In the supersymmetric case, a similar duality between

\footnote{In the absence of RR backgrounds. The system is not understood in situations where RR backgrounds are turned on.}
the supersymmetric $SL(2)/U(1)$ coset and $N = 2$ Liouville was proposed in [13] based on considerations involving string dynamics near singularities of Calabi-Yau manifolds. Recently, there was some more work on these dualities [25,26]. We show here that, assuming the duality and using general properties of degenerate operators in CFT on $AdS_3$, leads to results for certain OPE coefficients and correlation functions on $AdS_3$ which agree with those obtained by solving the null state equations together with the Knizhnik-Zamolodchikov equations for the current algebra blocks. This provides new evidence for the duality.

**Properties of degenerate operators in CFT on $AdS_3$:** We compute the current algebra blocks that enter four point functions of two degenerate operators and two general ones. This was already done in [43], but we review the calculation here for two reasons. One is that this is needed for comparing to the results obtained using the duality mentioned in the previous paragraph. In addition, these blocks are needed for studying D-branes on $AdS_3$, which we do in [44].

These notes are organized as follows. In section 2 we begin with a very brief review of CFT on $AdS_3$. We establish the notations and quote some results that are needed later. In section 3 we discuss the central charge of the spacetime Virasoro algebra, which corresponds to the zero momentum dilaton. We show that this operator is not proportional to the identity operator in string theory on $AdS_3$ and compute its correlation functions with other operators. We explain the interpretation of the results in terms of the spacetime CFT and clarify the relation between the spacetime central charge operator and the Wakimoto screening operator in the free field realization of the model.

In section 4 we turn to a discussion of the properties of degenerate operators in CFT on $AdS_3$. There is an infinite set of such operators, labeled by two positive integers $(r, s)$. We consider in detail two such operators, corresponding to $(r, s) = (1, 2), (2, 1)$, which can be thought of as the generators of the set. We show that the OPE’s of these operators with other primaries contain a finite number of terms, and compute the structure constants by using the fact that they are dominated by the region near the boundary of $AdS_3$, where one can use perturbation theory in either the Wakimoto screening operator or the Sine-Liouville coupling. We also show that one can use the resulting structure constants to compute correlation functions in $SL(2)$ CFT and the results are in agreement with other methods of computing them. This provides a test of the duality of [24].

---

2 For other work on correlation functions in $SL(2)$ CFT see e.g. [28,43,20].
In section 5 we compute the current algebra blocks corresponding to the four point functions of two (identical) degenerate operators and two (identical) generic ones. This calculation originally appeared in [43], and is reviewed here for reasons that were mentioned above.

In section 6 we generalize the results to the supersymmetric case, where our results provide evidence for the duality of [13]. Some useful formulae are collected in an appendix.

2. Some properties of CFT on $AdS_3$

The WZNW model on $AdS_3$ is described by the following Lagrangian, written in Poincaré coordinates $(u, \gamma, \bar{\gamma})$

$$\mathcal{L} = 2k \left( \frac{1}{u^2} \partial_u \bar{\partial} u + u^2 \partial_\gamma \bar{\partial} \bar{\gamma} \right).$$  \hspace{1cm} (2.1)

The parameter $k$ is related to the radius of curvature of the space, $l$, via $k = l^2$. The boundary of $AdS_3$ is the two dimensional space labeled by $(\gamma, \bar{\gamma})$ at $u \to \infty$. In the Lorentzian case $\gamma$ and $\bar{\gamma}$ are independent real coordinates. In the Euclidean case $\bar{\gamma}$ is the complex conjugate of $\gamma$, and the boundary is the complex plane, or two-sphere.

The model described by the Lagrangian (2.1) is invariant under two copies of the $SL(2,\mathbb{R})$ current algebra. The left moving symmetry is generated by the currents $J^a(z)$, with $a = 3, \pm$, satisfying the OPE algebra

$$\begin{aligned}
J^3(z)J^\pm(w) &\sim \frac{\pm J^\pm(w)}{z-w} \\
J^3(z)J^3(w) &\sim -\frac{k}{(z-w)^2} \\
J^-(z)J^+(w) &\sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w}.
\end{aligned}$$  \hspace{1cm} (2.2)

A similar set of OPE’s holds for the right moving $SL(2)$ current algebra. The level $k$ of the current algebra (2.2) is related to the central charge of the CFT (2.1) via

$$c = \frac{3k}{k-2}.$$  \hspace{1cm} (2.3)

One is typically interested in $k > 2$.

---

3 We set the string length $l_s = 1$. 

---
A natural set of observables is given by the eigenfunctions of the Laplacian on \( AdS_3 \),

\[
\Phi_h = \frac{1 - 2h}{\pi} \left( \frac{1}{|\gamma - x|^2 e^{Q\phi} + e^{-Q\phi}} \right)^{2h} = -e^{Q(h-1)\phi} \delta^2(\gamma - x) + O(e^{Q(h-2)\phi}) + \frac{(1 - 2h)e^{-Qh\phi}}{\pi|\gamma - x|^{4h}} + O(e^{-Q(h+1)\phi}),
\]

where \( u = e^{\frac{Q\phi}{2}} \).

\( Q \) is related to \( k \) via

\[
Q^2 = \frac{2}{k - 2} \equiv -\frac{2}{t}.
\]

The last equality defines

\[
t = -(k - 2).
\]

\( x \) is an auxiliary complex variable whose role can be understood by expanding the operators \( \Phi_h \) near the boundary of \( AdS_3 \), \( \phi \to \infty \), as is done on the second line of (2.4). Note the difference between the behavior for \( h > 1/2 \) and \( h < 1/2 \) \footnote{Note that in (2.4) we have rescaled \( \phi \) and \( \Phi_h \) relative to equations such as (2.8) in \cite{2}.}. For \( h > 1/2 \), the operators \( \Phi_h \) are localized near the boundary at \((\gamma, \bar{\gamma}) = (x, \bar{x})\). For \( h < 1/2 \), the delta function is subleading, and the operators are smeared over the boundary. One can think of \( \Phi_h \) as the propagator of a particle with mass \( h(h - 1) \) from a point \((x, \bar{x})\) on the boundary to a point \((\phi, \gamma, \bar{\gamma})\) in the bulk of \( AdS_3 \). Thus, \( x \) labels the position on the boundary of \( AdS_3 \), which is the base space of the CFT dual to string theory on \( AdS_3 \) via the AdS/CFT correspondence.

The operators \( \Phi_h \) are primary under the \( \hat{SL}(2) \) current algebra (2.2); they satisfy

\[
J^3(z)\Phi_h(x, \bar{x}; w, \bar{w}) \sim -\frac{(x\partial_x + h)\Phi_h(x, \bar{x})}{z - w},
\]

\[
J^+(z)\Phi_h(x, \bar{x}; w, \bar{w}) \sim -\frac{(x^2\partial_x + 2hx)\Phi_h(x, \bar{x})}{z - w},
\]

\[
J^-(z)\Phi_h(x, \bar{x}; w, \bar{w}) \sim -\frac{\partial_x\Phi_h(x, \bar{x})}{z - w}.
\]

Their worldsheet scaling dimensions are

\[
\Delta_h = -\frac{h(h - 1)}{k - 2} = \frac{h(h - 1)}{t}.
\]
It is very convenient \cite{15} to “Fourier transform” the $SL(2)$ currents as well, and define

\[ J(x; z) \equiv -J^+(x; z) = 2xJ^3(z) - J^+(z) - x^2J^-(z). \]  

(2.10)

Since $J_0^- = -\partial_x$ is the generator of translations in $x$ (see (2.8)) we can think of (2.10) as a result of “evolving” the currents $J^a(z)$ in $x$:

\[ J^+(x; z) = e^{-xJ_0^-}J^+(z)e^{xJ_0^-} = J^+(z) - 2xJ^3(z) + x^2J^-(z) \]

\[ J^3(x; z) = e^{-xJ_0^-}J^3(z)e^{xJ_0^-} = J^3(z) - xJ^-(z) - \frac{1}{2}\partial_x J^+(x; z) \]  

(2.11)

\[ J^-(x; z) = e^{-xJ_0^-}J^-(z)e^{xJ_0^-} = J^-(z) - \frac{1}{2}\partial_x^2 J^+(x; z). \]

The OPE algebras (2.2) and (2.8) can be written in terms of $J(x; z)$ as follows:

\[ J(x; z)J(y; w) \sim \frac{k}{(z - w)^2} \frac{(y - x)^2}{z - w} \left[ (y - x)^2 \partial_y - 2(y - x) \right] J(y; w) \]  

(2.12)

\[ J(x; z)\Phi_h(y, \bar{y}; w, \bar{w}) \sim \frac{1}{z - w} \left[ (y - x)^2 \partial_y + 2h(y - x) \right] \Phi_h(y, \bar{y}). \]  

(2.13)

It is sometimes useful to expand the operators (2.4) in modes,

\[ \Phi_h(x, \bar{x}) = \sum_{m, \bar{m}} V_{h-1; m, \bar{m}} x^{-m-h} \bar{x}^{-\bar{m}-h} \]  

(2.14)

or

\[ V_{j; m, \bar{m}} = \int d^2x x^j x^{\bar{m}} \Phi_{j+1}(x, \bar{x}). \]  

(2.15)

Note that (2.8) implies that $V_{j; m, \bar{m}}$ transforms under $\widehat{SL(2)}$ as follows:

\[ J^3(z) V_{j; m, \bar{m}}(w) = \frac{m}{z - w} V_{j; m, \bar{m}} \]

\[ J^{\pm}(z) V_{j; m, \bar{m}}(w) = \frac{(m \mp j)}{z - w} V_{j; m \pm 1, \bar{m}}. \]  

(2.16)

The discussion above concerns the “short string” sector of the model. The theory has other sectors, which contain long strings located near the boundary of $AdS_3$. These are obtained by performing spectral flow on the short string sector \cite{4} (see also \cite{22}). We will not discuss the physics associated with long strings here, except for some comments in section 3. It would be interesting to generalize the results presented below to sectors with long strings. Since our analysis is algebraic, it should be possible to obtain such results by performing spectral flow \cite{1}, or twisting as in \cite{22}. Some results on correlation functions including long strings appeared recently in \cite{20}.
3. The Wakimoto representation and the spacetime central extension

Consider the Lagrangian:

\[ \mathcal{L} = \partial \phi \bar{\partial} \phi - Q \hat{R} \phi + \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} - \lambda \beta \bar{\beta} e^{-Q \phi} . \]  

(3.1)

Integrating out the fields \( \beta, \bar{\beta} \) one obtains the \( AdS_3 \) Lagrangian (2.1), (2.5). The description (3.1) is useful for studying the physics at large \( \phi \) (near the boundary of \( AdS_3 \)). The interaction term proportional to \( \lambda \) goes to zero there, and one gets a free linear dilaton theory for \( \phi \), as well as a free \((\beta, \gamma)\) system. Moreover, the effective string coupling \( g_s(\phi) \approx \exp(-Q \phi/2) \) also goes to zero there, so the system is weakly coupled in spacetime as well. All this should be contrasted with the original Lagrangian (2.1), which is singular at \( \phi \to \infty \).

Processes that are dominated by the large \( \phi \) region can be studied by viewing \((\phi, \beta, \gamma)\) as free fields with the propagators

\[ \langle \phi(z) \phi(0) \rangle = -\log |z|^2 , \quad \langle \beta(z) \gamma(0) \rangle = \frac{1}{z} \]  

(3.2)

and treating \( \lambda \) perturbatively (as a screening charge). This is a familiar technique in Liouville theory (see e.g. [46,47] for reviews); it has been applied to \( SL(2, R) \) CFT e.g. in [28,30,35,41,26,42]. Like in Liouville theory, generic correlation functions cannot be studied this way, since they are sensitive to the region \( \phi \to 0 \) (this is reasonable since the behavior of \( g_s(\phi) \) mentioned above implies that interactions turn off as \( \phi \to \infty \), so particles have to penetrate to finite \( \phi \) in order to interact). Formally, one cannot expand in \( \lambda \) in (3.1) since by shifting \( \phi, \lambda \) can be set to one.

Since we will be interested later in some situations were the physics is dominated by the large \( \phi \) region, we give next the form of some of the objects described in the previous section in the free field “Wakimoto variables” \((\phi, \beta, \gamma)\). The current algebra is represented by (normal ordering is implied):

\[ J^3 = \beta \gamma + \frac{1}{Q} \partial \phi , \]
\[ J^+ = \beta \gamma^2 + \frac{2}{Q} \gamma \partial \phi + k \partial \gamma , \]  

(3.3)
\[ J^- = \beta , \]

where \( Q \) is given in (2.6).
The $SL(2)$ primaries $\Phi_h$ behave (for $h > 1/2$) like (see (2.4))

$$\Phi_h \simeq -e^{Q(h-1)/2}(\gamma - x),$$

(3.4)
or, performing the transform (2.15),

$$V_{jm\bar{m}} = -\gamma^{j+m}\bar{\gamma}^{j+m}e^{Qj\phi}.$$ (3.5)

The powers of $\gamma$ and $\bar{\gamma}$ can be both positive and negative. The only constraint that follows from single valuedness on $AdS_3$ is that $m - \bar{m}$ must be an integer. One can check directly using free field theory that the scaling dimension of $V_{jm\bar{m}}$ is

$$\Delta(V_{jm\bar{m}}) = -\frac{j(j+1)}{(k-2)},$$

(3.6)
in agreement with (2.3) with

$$h = j + 1.$$ (3.7)

The coupling $\lambda$ in string theory on $AdS_3$ plays a role similar to that of the cosmological constant $\mu$ in Liouville theory. For example, the partition sum of the theory has a genus expansion of the form

$$Z(\lambda, g_s) = \sum_{n=0}^{\infty} Z_n \left( \frac{g_s^2}{\lambda} \right)^{n-1},$$

(3.8)

where $Z_n$ is the genus $n$ partition sum. Similarly, correlation functions of the operators $\Phi_h$ (2.4) scale as (for simplicity and future use we exhibit the form for the correlation functions on the sphere; similar formulae hold for higher genus correlation functions):

$$\langle \Phi_{h_1}(x_1; z_1) \cdots \Phi_{h_n}(x_n; z_n) \rangle = \lambda^s F_n(x_1, \cdots, x_n; z_1, \cdots, z_n),$$

(3.9)

where

$$s = 1 + \sum_{i=1}^{n} (h_i - 1),$$

(3.10)

and $F_n$ contains the non-trivial information about the correlation function. We see that, as mentioned above, the physics is essentially independent of $\lambda$ – it can be absorbed into $\phi$ (i.e. it can be set to one by rescaling $g_s$ and the operators). Nevertheless, as in Liouville theory, for some purposes it is convenient to keep the $\lambda$ dependence explicit.

---

5 For $n = 1$ (the torus) there is a logarithmic scaling violation.
It is natural to ask what is the invariant meaning of the Wakimoto coupling $\lambda$ in string theory on $AdS_3$? In other words, can one describe the Wakimoto interaction in (3.1) as an observable in the theory in a parametrization independent way? As we have just seen, changing $\lambda$ changes the effective string coupling of the model (3.8). Therefore, it is natural to expect that the operator that changes $\lambda$ should be the zero momentum mode of the dilaton. It is in fact well known that the dilaton is massive in string theory on $AdS_3$, but its zero mode is tunable. The corresponding vertex operator was constructed and discussed in [2]; its form is (see eq. (4.18) in [2])

$$I = -\frac{1}{k^2} \int d^2 z J(x; z) J(\bar{x}; \bar{z}) \Phi_1(x, \bar{x}; z, \bar{z}).$$

(3.11)

This operator has many special properties [2]. It commutes with the full $SL(2)_L \times SL(2)_R$ affine Lie algebra; this follows from the OPE

$$J(x; z) [J(y; w) \Phi_1(y, \bar{y}; w, \bar{w})] \sim k(x - y)^2 \partial_w \left[ \Phi_1(y, \bar{y}; w, \bar{w}) \right],$$

(3.12)

and its right moving analog. The operator $J(x; z) J(\bar{x}; \bar{z}) \Phi_1(x, \bar{x}; z, \bar{z})$ is marginal on the worldsheet. From the point of view of the spacetime CFT, $I$ is a dimension zero operator (but, as we will see soon it is not a multiple of the identity operator). It can be shown to satisfy $\partial_x I = \partial_{\bar{x}} I = 0$; hence it is constant in correlation functions (but not necessarily the same constant in different correlation functions).

Therefore, it is natural to conjecture that changing $\lambda$ in (3.1) corresponds to adding to the worldsheet action the operator $I$ (3.11). To see that this makes sense qualitatively, we next show that the large $\phi$ dependence of $I$ agrees with the Wakimoto interaction term in (3.1).

The leading behavior of $\Phi_1(x, \bar{x}; z, \bar{z})$ as $\phi \to \infty$ is given by (2.4):

$$\Phi_1 \simeq -\delta^2(\gamma - x) - \frac{e^{-Q\phi}}{\pi |\gamma - x|^4}. \tag{3.13}$$

As for $J(x; z)$, its behavior can be read off eq. (3.3) together with the definition (2.10):

$$J(x; z) = -\beta(x - \gamma)^2 + \frac{2}{Q} (x - \gamma) \partial \phi - k \partial \gamma. \tag{3.14}$$

We would now like to analyze the large $\phi$ behavior of $J(x; z) J(\bar{x}; \bar{z}) \Phi_1$. The first term in (3.13) (the $\delta^2(x - \gamma)$) contributes only

$$I = \int d^2 z \partial \gamma \partial \bar{\gamma} \delta^2(x - \gamma) + ... . \tag{3.15}$$
This is a vacuum contribution; the “...” stand for subleading contributions. As explained in [1,2], (3.15) measures the number of long strings in the vacuum.

The screening charge in (3.1) should thus come from the first subleading contribution, where we take the second term from $\Phi_1$ (3.13) and multiply by (3.14). Since, as proven in [2], the operator $I$ is independent of $x, \bar{x}$, the only term in the product $J(x; z) \bar{J}(\bar{x}; \bar{z}) \Phi_1$ that contributes to correlation functions is the first one in (3.14) (which can be seen e.g. by plugging (3.13), (3.14) in (3.11) and taking $x \to \infty$); thus, the first subleading contribution to $I$ is

$$I \simeq \int d^2z \partial \bar{\gamma} \bar{\partial} \gamma \delta^2(x - \gamma) + \frac{1}{\pi k^2} \int d^2z \bar{\beta} \beta e^{-Q\phi},$$

in agreement with the Wakimoto screening charge (3.1).

To establish the precise relation between $\lambda$ in (3.1) and a worldsheet deformation by $I$ of the form

$$S(\rho) = S_0 + \rho I,$$

one can proceed as follows. Consider the partition sum

$$Z(\rho) = \langle e^{-S(\rho)} \rangle.$$ (3.18)

As we will see shortly, on the sphere one has

$$\langle e^{-\rho I} \rangle = Z_0 e^{-\rho \langle I \rangle}.$$ (3.19)

Here and below $\langle I \rangle$ is the one point function of $I$ on the sphere normalized by dividing it by the partition sum on the sphere, unless stated otherwise. Comparing (3.19) to the tree level term in (3.8) we see that the relation between $\lambda$ in (3.1) and $\rho$ in (3.17) is

$$\lambda = e^{-\rho \langle I \rangle},$$

where we normalized $\lambda$ such that $\rho = 0$ corresponds to $\lambda = 1$. Note that (3.20) is a natural relation; $\rho$ changes the expectation value of the zero mode of the dilaton (see [2]), whereas $\lambda$ is proportional to $g_s^{-\frac{1}{2}}$ (see (3.8)). Thus, it is natural that $\rho$ goes like $\log \lambda$.

It remains to establish (3.13), but before getting to that note that combining (3.20) with (3.9), (3.17) and (3.18) leads to another interesting relation. Differentiating (3.9) once w.r.t. $\rho$ gives:

$$\langle I \Phi_{h_1} \cdots \Phi_{h_n} \rangle = \left(1 + \sum_{i=1}^{n} (h_i - 1) \right) \langle I \rangle \langle \Phi_{h_1} \cdots \Phi_{h_n} \rangle.$$ (3.21)
We will next prove (3.19) and (3.21) by a direct calculation, thereby establishing the correspondence (3.20).

Consider first the relation (3.21). To calculate the l.h.s. one notes [2] that the operator $I$ (3.11) is “almost” a total derivative. In fact, one has

$$
\bar{J} \Phi_1 = -\frac{k}{\pi} \partial \Lambda ,
$$

where $\Lambda$ is given explicitly in [2]; we will only need its large $\phi$ form,

$$
\lim_{\phi \to \infty} \Lambda = \frac{1}{x - \gamma} .
$$

Plugging (3.22) into (3.11) and using the fact that $J(x; z)$ is holomorphic (up to contact terms), we find that

$$
I = \frac{1}{\pi k} \int d^2 z \partial (J \Lambda) .
$$

Despite appearances, (3.24) does not imply that $I$ is trivial. The technical reason for that is that $\Lambda$ is not a good observable on AdS$_3$ (see [2] for a more detailed discussion). In particular, it transforms like a primary with $h = 0$ under $SL(2)_R$ (see (2.13)) but its transformation as an object with $h = 1$ under $SL(2)_L$ contains an anomalous term

$$
J(x; z) \Lambda(y; \bar{y}; w; \bar{w}) \sim \frac{[(y - x)^2 \partial_y + 2(y - x)] \Lambda(y; \bar{y}; w; \bar{w}) - 1}{z - w} .
$$

Therefore, the composite operator $J(x; z) \Lambda(x; z)$ requires normal ordering, and should be defined via a limiting procedure

$$
J(x; z) \Lambda(x; z) = \lim_{z \to z'} J(x; z) \Lambda(x; z') + \frac{1}{(z - z')} .
$$

In addition, the operator $J \Lambda$ in (3.24) has short distance singularities when it approaches other operators, which also give contributions to correlators involving $I$. From the space-time point of view, this happens because $\Lambda$ is associated with a gauge transformation that does not go to zero at infinity (the boundary of AdS$_3$).

At any rate, returning to (3.24), we conclude that the correlator $\langle I \Phi_{h_1} \cdots \Phi_{h_n} \rangle$ receives contributions from the boundaries of moduli space, which are small circles around the insertions $z_i$ (as described in detail in [2]), and an additional contribution $(1/k) \langle \Phi_{h_1} \cdots \Phi_{h_n} \rangle$ from the anomalous term in (3.26). To compute the contributions from the small circles around the insertions one uses the OPE algebra (2.13) and the relation (see [2,39])

$$
\lim_{z_1 \to z_2} \Lambda(x_1; \bar{x}_1; z_1, \bar{z}_1) \Phi_h(x_2, \bar{x}_2; z_2, \bar{z}_2) = \frac{1}{x_1 - x_2} \Phi_h(x_2, \bar{x}_2; z_2, \bar{z}_2) .
$$
This leads to:

\[ \langle I\Phi_{h_1}\cdots\Phi_{h_n} \rangle = \frac{1}{k} \left( 1 + \sum_{i=1}^{n} (h_i - 1) \right) \langle \Phi_{h_1}\cdots\Phi_{h_n} \rangle. \] (3.28)

Comparing to (3.21) we see that the structure is the same if

\[ \langle I \rangle = \frac{1}{k}. \] (3.29)

The result (3.29) can be proven directly as a part of the derivation of (3.19), to which we turn next. Consider the correlator

\[ \mathcal{I}_n = \langle I^n \rangle. \] (3.30)

\( \mathcal{I}_n \) can be computed using the same logic as before. Rewrite one of the \( n \) insertions of \( I \) as (3.24); there are again potential boundary contributions from small circles around the other insertions, and a contribution from the anomalous term in (3.26). In this particular case, the contributions from the vicinity of the other insertions vanish. This is not difficult to see using the OPE (3.12).

Hence, the analog of (3.28) for this case is

\[ \mathcal{I}_n = \frac{1}{k} \mathcal{I}_{n-1}. \] (3.31)

This is equivalent to (3.19) with the expectation value of \( I \) given by (3.29).

To summarize, we have proven the relation (3.20), by establishing its consequences (3.19) and (3.21).

The preceding discussion has interesting consequences for the structure of the spacetime Virasoro (and affine) algebras in string theory on \( AdS_3 \). As shown in [2], the spacetime stress tensor in string theory in \( AdS_3 \), \( T(x) \), is given by a certain integrated vertex operator. \( T(x) \) satisfies the spacetime Virasoro OPE

\[ T(x)T(y) \sim \frac{c_{st}/2}{(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial T(y)}{x-y}, \] (3.32)

where the central term is given by

\[ c_{st} = 6k I. \] (3.33)

Since in all known two dimensional CFT’s the central charge is proportional to the identity operator, it was assumed in [2] that this is the case here as well. We now see from (3.21) that the operator \( I \) appears to be a non-trivial dimension zero operator. Thus, string
theory on $AdS_3$ (with NS background) has the property that the central extension is not proportional to the identity operator.

This interesting behavior seems to be directly related to the non-locality of the theory. It is well known that in string theory on $AdS_3$ long strings carry central charge (see e.g. [1,2]). Each long string carries central charge $c = 6k$. This is in fact one of the consequences of (3.28). More generally, (3.28) predicts that “short strings” carry central charge as well. Indeed, consider a short string state $|h\rangle$, created by acting with the local operator $\Phi_h$ on the vacuum. The central charge (3.33) in this state is given by

$$c_{\text{st}}(h) = 6k \frac{\langle \Phi_h I \Phi_h \rangle}{\langle \Phi_h \Phi_h \rangle}.$$  \hfill (3.34)

One type of contribution comes from disconnected worldsheets [18], with the operator $I$ on one worldsheet and the two $\Phi_h$ operators on a second worldsheet. These give vacuum contributions to the spacetime central charge,

$$c_{\text{st}}^{(\text{vac})} = 2(x - y)^4 \langle T(x)T(y) \rangle = 6k \langle I \rangle.$$  \hfill (3.35)

Here $\langle I \rangle$ is the one point function of $I$, computed without dividing by the partition sum on the sphere (unlike eqs. like (3.29)). This vacuum central charge has an expansion in powers of $g_s$, but it is clearly independent of $h$ (by definition).

The leading correction to this vacuum contribution comes from the amplitude where all three operators in (3.34) are on the same worldsheet of spherical topology. Using (3.28) this gives

$$c_{\text{st}}(h) = c_{\text{st}}^{(\text{vac})} + 6(2h - 1).$$  \hfill (3.36)

Thus, we see that the central charge increases with $h$. When $h$ reaches $h \sim k/2$, the threshold for creating long strings, one finds that the central charge carried by the state is $c \simeq 6k$, in agreement with the long string picture. The result of [1,2], that long strings carry $c_{\text{st}} = 6k$ can be shown as follows. A long string vertex operator is obtained by taking a short string vertex operator from the continuous series, with $h = \frac{1}{2} + i\lambda$, and applying to it a certain twist operator described in [22] (which implements the spectral flow of [4]). The twist field contributes +1 to $I$ due to the presence of the first term in (3.16). The contribution of the vertex operator from the continuous series is given by (3.28):

$$\langle I \Phi_{\frac{1}{2} + i\lambda} \Phi_{\frac{1}{2} - i\lambda} \rangle = \frac{1}{k} \left( 1 + \left( -\frac{1}{2} + i\lambda \right) + \left( -\frac{1}{2} - i\lambda \right) \right) \langle \Phi_{\frac{1}{2} + i\lambda} \Phi_{\frac{1}{2} - i\lambda} \rangle = 0.$$  \hfill (3.37)

Therefore, we conclude that long string states with winding number one carry $c_{\text{st}} = 6k$, as expected.
4. Degenerate representations and correlation functions

In this section we compute the OPE coefficients of certain degenerate operators in CFT on AdS$_3$ with other primaries $\Phi_h$ (2.4). This can be done by using the conjectured duality between the $SL(2)/U(1)$ and Sine-Liouville CFT’s [24]; the results can be compared to a direct analysis in $SL(2)$ CFT, which we review in section 5. As we will see, the two approaches agree, providing a test of the duality.

The above OPE coefficients can be used to compute correlation functions in $SL(2)$ CFT, following a method used in Liouville theory [49]. As an example, we calculate the two point function

$$\langle \Phi_h(x, \bar{x}; z, \bar{z}) \Phi_{h'}(y, \bar{y}; w, \bar{w}) \rangle = \delta(h - h') \frac{D(h)}{|x - y|^{4h} |z - w|^{4\Delta_h}}, \quad (4.1)$$

which was obtained before in [43,24]. The duality provides an efficient way for calculating (4.1) and other correlation functions.

The degenerate operators in $SL(2)$ CFT are of the form (2.4) with (see [43] for more details):

$$h_{r,s} = \frac{1 - r}{2} - \frac{1 - s}{2} t, \quad r, s = 1, 2, 3, \ldots . \quad (4.2)$$

For irrational $k$, the Fock module corresponding to $h_{r,s}$ contains a single null state at level $r(s - 1)$. Consider, for example, the special case $s = 1$. The degenerate representations have $h_{r,1} = (1 - r)/2$, and the null state is at level zero. Looking back at (2.4) this is natural: $\Phi_{h_{r,1}}$ is in this case a polynomial of degree $r - 1$ in $x$ and $\bar{x}$, and the null state is

$$\partial^r_x \Phi_{(1-r)/2} = \partial^r_{\bar{x}} \Phi_{(1-r)/2} = 0, \quad r = 1, 2, 3, \ldots . \quad (4.3)$$

The operators $\Phi_{(1-r)/2}$ correspond to finite, $r$ dimensional, representations of $SL(2)$. They are direct generalizations of the finite dimensional spin $(r - 1)/2$ representations of $SU(2)$ which are described in the language used here in [45].

It should be emphasized that the fact that the quantum operators $\Phi_{(1-r)/2}$ satisfy the null state equations (1.3) is not completely obvious. It is certainly true that they satisfy these equations semiclassically, and that if the combination on the l.h.s. of (1.3) is not zero, it is a new $SL(2)$ primary in the theory, whose norm is zero. In a unitary CFT this would mean that this new primary vanishes, but $SL(2)$ CFT is not unitary, and so this argument does not apply. It seems that the fact that the null states associated with the degenerate representations (4.2) vanish is part of the definition of $SL(2)$ CFT.
As we will see, to calculate $D(h)$ in (4.1) it is enough to consider two degenerate operators:

(a) The first non-trivial operator in the series $\Phi_{h_{r,1}}$:

$$\Phi_{h_{2,1}} \equiv \Phi_{-\frac{1}{2}} .$$

(b) The first non-trivial operator in the series $\Phi_{h_{1,s}}$:

$$\Phi_{h_{1,2}} \equiv \Phi_{\frac{1}{2}} .$$

We next discuss these two cases.

4.1. $\Phi_{-\frac{1}{2}}$

Consider first the case $h = -1/2$. As mentioned above, a look at (2.4) makes it clear that in this case the operator $\Phi_h$ reduces to a finite polynomial,

$$\Phi_{-\frac{1}{2}} = \frac{2}{\pi} \left( |\gamma - x|^2 e^{\frac{Q\phi}{2}} + e^{-\frac{Q\phi}{2}} \right).$$

Thus, the mode expansion (2.14) is very simple:

$$\Phi_{-\frac{1}{2}} = \sum_{m, \bar{m} = -\frac{1}{2}, \frac{1}{2}} V_{-\frac{1}{2}; m, \bar{m}} x^{\frac{1}{2} - m} \bar{x}^{\frac{1}{2} - \bar{m}},$$

with

$$V_{-\frac{1}{2}; \frac{1}{2}, \bar{1}} = \frac{2}{\pi} \gamma \bar{\gamma} e^{\frac{Q\phi}{2}},$$

$$V_{-\frac{1}{2}; -\frac{1}{2}, \bar{-1}} = \frac{2}{\pi} e^{\frac{Q\phi}{2}},$$

etc. Note that, following standard practice, we have written in (4.8) only the large $\phi$ forms of the vertex operators.

There is actually an interesting subtlety here that will be important later. The formula (4.6) from which (4.8) follows is a semiclassical expression, which should be valid in the $t \to -\infty$ limit. It is possible that for finite $t$ it will receive finite renormalization, and this turns out indeed to be the case. To determine this finite renormalization we proceed as follows. We use the following two pieces of information:

---

6 The operator $\Phi_{h_{1,1}}$ is the identity.
(1) As \( \phi \to \infty \), the operators \( \Phi_h \) with \( h > 1/2 \) behave as (2.4):

\[
\Phi_h \simeq -e^{Q(h-1)\phi} \delta^2(\gamma - x) .
\] (4.9)

The fact that the coefficient of the exponential is \(-1\) is a choice of the normalization of \( \Phi_h \) with \( h > 1/2 \) in the quantum theory.

(2) Once the normalization of the operators with \( h > 1/2 \) is chosen, there is no further freedom. The normalization of the operators \( \Phi_h \) with \( h < 1/2 \) is fixed, since they are related to those with \( h > 1/2 \) by a reflection symmetry described in [43]:

\[
\Phi_h(x; z) = \mathcal{R}(h) \frac{2h-1}{\pi} \int d^2 x'|x - x'|^{-4h} \Phi_{1-h}(x'; z) ,
\] (4.10)

where the \( x' \) integral runs over the plane. We will see later that the reflection coefficient \( \mathcal{R}(h) \) depends on the coupling \( \lambda \) (3.1). For the value of \( \lambda \) that we will be using below, it is equal to

\[
\mathcal{R}(h) = \frac{\Gamma(1 + \frac{2h-1}{t})}{\Gamma(1 - \frac{2h-1}{t})} .
\] (4.11)

Using points (1), (2) above, we can determine the \( \phi \to \infty \) behavior of \( \Phi_{-\frac{1}{2}} \) by using the fact that

\[
\Phi_{\frac{3}{2}}(x) \simeq -e^{\frac{1}{2}Q\phi} \delta^2(\gamma - x) .
\] (4.12)

Plugging this into (4.10), we find that

\[
\Phi_{-\frac{1}{2}} \simeq \frac{2}{\pi} \mathcal{R}(\frac{1}{2}) |x - \gamma|^2 e^{\frac{1}{2}Q\phi} .
\] (4.13)

Comparing to the semiclassical expression (4.6), we see that the quantum correction is a multiplicative factor of

\[
\mathcal{R}(\frac{1}{2}) = \frac{\Gamma(1 - \frac{2}{t})}{\Gamma(1 + \frac{2}{t})} .
\] (4.14)

We see that in the classical limit \( t \to -\infty \) it goes to 1, as expected.

\( \Phi_{-\frac{1}{2}} \) satisfies the differential equations (4.3):

\[
\partial_x^2 \Phi_{-\frac{1}{2}} = \partial_{\bar{x}}^2 \Phi_{-\frac{1}{2}} = 0 .
\] (4.15)

Equation (4.15) places strong constraints on the OPE algebra of \( \Phi_{-\frac{1}{2}} \) with other operators. Indeed, consider the OPE

\[
\Phi_{-\frac{1}{2}}(x) \Phi_h(y) = \sum_{h'} C_{hh'} |x - y|^{2(\frac{1}{2} - h + h')} \Phi_{h'}(y) + \cdots .
\] (4.16)
The form of the r.h.s. is determined by spacetime conformal invariance of string theory on $AdS_3$, and the “…” stands for contributions of descendants (under spacetime and worldsheet Virasoro). Also, we have suppressed the dependence on the worldsheet locations of the operators. $C_{h,h'}$ are structure constants that need to be determined.

The differential equations (4.13) place the following constraint on the r.h.s. of (4.16):

$$C_{h,h'}\left(\frac{1}{2} - h + h'\right)\left(-\frac{1}{2} - h + h'\right) = 0 .$$

(4.17)

Thus, there are only two possible terms in the sum on the r.h.s. of (4.16), corresponding to $h' = h - \frac{1}{2}$ and $h' = h + \frac{1}{2}$:

$$\Phi_{-\frac{1}{2}}(x)\Phi_h(y) = C_{-}(h)\Phi_{h - \frac{1}{2}}(y) + |x - y|^2C_{+}(h)\Phi_{h + \frac{1}{2}}(y) + \cdots .$$

(4.18)

We would like next to determine the structure constants $C_{\pm}$.

Experience with Liouville theory and Feigin-Fuchs representations of various rational CFT’s leads one to expect that the physics of the degenerate operator $\Phi_{-\frac{1}{2}}$ should be perturbative in some description of the theory. In the present context this means that the physics of degenerate operators is dominated by the region $\phi \to \infty$, i.e. the vicinity of the boundary of $AdS_3$. As far as we know, the origin of this phenomenon is not well understood but, as we will see, one can use it to determine the structure constants in (4.18).

We will next show that OPE’s which involve $\Phi_{-\frac{1}{2}}$ are perturbative in the Wakimoto variables described in section 3. Consider first the free theory obtained by setting the Wakimoto coupling $\lambda$ in (3.1) to zero. This already gives rise to the second term in (4.18). Indeed, at large $\phi$ we have (see (2.4), (4.13); we are assuming that $h > \frac{1}{2}$)

$$\Phi_{-\frac{1}{2}}(x) \simeq \frac{2}{\pi} \mathcal{R}(x)\left|\gamma - x\right|^2\delta^2(\gamma - x) ,$$

$$\Phi_h(y) \simeq -e^{Q(h-1)}\delta^2(\gamma - y) .$$

(4.19)

Multiplying the two, using the fact that $\gamma(z)$ behaves as a c-number in these calculations, gives

$$\Phi_{-\frac{1}{2}}(x)\Phi_h(y) \simeq -\frac{2}{\pi} \mathcal{R}(x)\left|x - y\right|^2e^{Q(h-\frac{3}{2})}\delta^2(\gamma - y) + \cdots =$$

$$\frac{2}{\pi} \mathcal{R}(x)\left|x - y\right|^2\Phi_{h + \frac{1}{2}}(y) + \cdots .$$

(4.20)

Comparing to (4.18) we see that

$$C_{+} = \frac{2}{\pi} \mathcal{R}(x)\left|\frac{1}{2}\right) .$$

(4.21)
The fact that the OPE (4.20) does not involve $\lambda$ implies that the relevant interaction occurs very far from the “wall” provided by $\lambda$, i.e. it is a “bulk interaction” in the sense of $[16]$. It can occur anywhere in the infinite region near the boundary of $AdS_3$.

It remains to compute $C_-$. To do that, e.g. set $x = 0$, which gives (see (4.7))

$$\mathcal{R}(-\frac{1}{2})V_{-\frac{1}{2}+\frac{1}{2}} \Phi_h(y) = C_- \Phi_{h-\frac{1}{2}}(y) + \cdots . \quad (4.22)$$

For simplicity, set also $y = 0$ (this gets rid of contributions from descendants). We now have

$$\frac{2\lambda}{\pi} \mathcal{R}(-\frac{1}{2}) \gamma e^{\frac{1}{2}Q\phi(z_1)} e^{Q(h-1)\phi(z_2)} \delta^2(\gamma(z_2)) = C_- e^{Q(h-\frac{1}{2})\phi(z_2)} \delta^2(\gamma(z_2)) + \cdots . \quad (4.23)$$

We see that in this case the $\phi$ charges do not add up correctly, but if we bring down from the action (3.1) one power of the interaction, we seem to land on our feet:

$$\frac{2\lambda}{\pi} \mathcal{R}(-\frac{1}{2}) \gamma e^{\frac{1}{2}Q\phi(z_1)} e^{Q(h-1)\phi(z_2)} \delta^2(\gamma(z_2)) = C_- e^{Q(h-\frac{1}{2})\phi(z_2)} \delta^2(\gamma(z_2)) + \cdots . \quad (4.24)$$

The exponentials of $\phi$ simply give

$$|z - z_1|^Q|z - z_2|2Q^2(h-1)e^{Q(h-\frac{1}{2})\phi(z_2)} . \quad (4.25)$$

Using the fact that

$$\beta(z)f(\gamma(z_2)) \simeq \frac{1}{z - z_2}f'(\gamma(z_2)) , \quad (4.26)$$

for any function $f$, we have

$$\beta\bar{\beta}(z)\gamma\bar{\gamma}(z_1)\delta^2(\gamma(z_2)) = \left(\frac{1}{z - z_1} + \frac{1}{z - z_2} \gamma(z_1) \frac{\partial}{\partial \gamma(z_2)}\right) \left(\frac{1}{\bar{z} - \bar{z}_1} + \frac{1}{\bar{z} - \bar{z}_2} \bar{\gamma}(\bar{z}_1) \frac{\partial}{\partial \bar{\gamma}(\bar{z}_2)}\right) \delta^2(\gamma(z_2)) . \quad (4.27)$$

In the $\gamma \frac{\partial}{\partial \gamma}$ terms we can set $z_1 = z_2$ since we are only interested in the most singular terms as $z_1 \rightarrow z_2$. This allows one to integrate by parts so

$$\beta\bar{\beta}(z)\gamma\bar{\gamma}(z_1)\delta^2(\gamma(z_2)) = \left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right) \left(\frac{1}{\bar{z} - \bar{z}_1} - \frac{1}{\bar{z} - \bar{z}_2}\right) \delta^2(\gamma(z_2)) = \frac{|z_1 - z_2|^2}{|z - z_1|^2|z - z_2|^2} \delta^2(\gamma(z_2)) . \quad (4.28)$$
We see that the r.h.s. of (4.25) and (4.28) is exactly of the form expected in (4.23); hence

\[ C_- = \frac{2\lambda}{\pi} R\left(-\frac{1}{2}\right) \int d^2 z |z-1|^{Q^2-2} |z|^{2Q^2(h-1)-2}. \]  

(4.29)

Using standard results we get

\[ C_-(h) = 2\lambda R\left(-\frac{1}{2}\right) \frac{\Gamma\left(\frac{1}{2}Q^2\right)\Gamma\left(Q^2(h-1)\right)\Gamma\left(1-\frac{1}{2}Q^2(2h-1)\right)}{\Gamma\left(1-\frac{1}{2}Q^2\right)\Gamma\left(1-Q^2(h-1)\right)\Gamma\left(\frac{1}{2}Q^2(2h-1)\right)} \]

\[ = 2\lambda R\left(-\frac{1}{2}\right) \frac{\Gamma\left(-\frac{1}{t}\right)\Gamma\left(-\frac{2(h-1)}{t}\right)\Gamma\left(1+\frac{2h-1}{t}\right)}{\Gamma\left(1+\frac{1}{t}\right)\Gamma\left(1+\frac{2(h-1)}{t}\right)\Gamma\left(-\frac{2h-1}{t}\right)}. \]  

(4.30)

Physically, the analysis above means that the interaction implied by the first term on the r.h.s. of (4.18) is also a “bulk interaction” but it involves in addition to the fields \( \Phi_{-\frac{1}{2}} \), \( \Phi_h \) and \( \Phi_{h-\frac{1}{2}} \) that appear explicitly also a zero momentum dilaton, whose vertex operator is given by the Wakimoto screening charge, as discussed in section 3.

This concludes the calculation of \( C_\pm \) in (4.18). We can now use these structure constants to obtain a constraint on the correlators of \( \Phi_h \). Consider e.g. the three point function

\[ \langle \Phi_{-\frac{1}{2}}(x)\Phi_h(y_1)\Phi_{h+\frac{1}{2}}(y_2) \rangle. \]  

(4.31)

If we first send \( x \rightarrow y_1 \) and use the \( C_+ \) term in (4.18) we get

\[ C_+(h)D(h + \frac{1}{2}). \]  

(4.32)

Taking \( x \rightarrow y_2 \) first gives

\[ C_-(h + \frac{1}{2})D(h). \]  

(4.33)

Equating (4.32), (4.33) we conclude that

\[ C_-(h + \frac{1}{2})D(h) = C_+(h)D(h + \frac{1}{2}). \]  

(4.34)

Shifting \( h \) by 1/2 for convenience we conclude that

\[ \frac{D(h)}{D(h - \frac{1}{2})} = \frac{C_-(h)}{C_+(h - \frac{1}{2})} = \pi\lambda \frac{\Gamma\left(-\frac{1}{t}\right)\Gamma\left(-\frac{2(h-1)}{t}\right)\Gamma\left(1+\frac{2h-1}{t}\right)}{\Gamma\left(1+\frac{1}{t}\right)\Gamma\left(1+\frac{2(h-1)}{t}\right)\Gamma\left(-\frac{2h-1}{t}\right)}. \]  

(4.35)

One class of solutions (which turns out to be the relevant one, as we show below) is

\[ D(h) = \nu^{2h-1} \frac{\Gamma\left(1+\frac{2h-1}{t}\right)}{\Gamma\left(-\frac{2h-1}{t}\right)}, \]  

(4.36)
where $\nu$ is a constant (function of $k$ and $\lambda$):

$$
\nu = \pi \lambda \frac{\Gamma(-\frac{1}{t})}{\Gamma(1 + \frac{1}{t})} .
$$

(4.37)

The result of Teschner [43] is the same as (4.36) with

$$
\nu = \frac{\Gamma(1 - \frac{1}{t})}{\Gamma(1 + \frac{1}{t})} .
$$

(4.38)

By tuning $\lambda$ (to $1/\lambda = -t\pi$), the two results agree.

In any case, as is clear from the discussion above, $\lambda$ plays here a similar role to $\mu$ in Liouville theory, i.e. changing $\lambda$ rescales the operators $\Phi_h$ by an $h$-dependent factor. A nice choice of normalization that we will use below is

$$
\lambda = \frac{\Gamma(1 + \frac{1}{t})}{\Gamma(-\frac{1}{t})} ,
$$

(4.39)

which eliminates the factor $\nu$ from the two point function (4.36) (i.e. sets $\nu = 1$ in (4.37)).

4.2. $\Phi_\ell^2$

We now move on to a discussion of the second degenerate operator, (4.5). One way to introduce it is the following. Consider operators of the form (normal ordering implied)

$$
\theta(x) = a \partial_x^2 J \Phi_h + b \partial_x J \partial_x \Phi_h + c J \partial_x^2 \Phi_h .
$$

(4.40)

It is clearly a descendant of $\Phi_h$, but for some combination of the numbers $a, b, c, h$ it might be a primary again. To find $a, b, c, h$ it is convenient to use the language of states. Using (2.11) and the fact that

$$
\Phi_h(x) = e^{-xJ_0^-} \Phi_h(0)e^{xJ_0^-}
$$

(4.41)

(see e.g. [2] eq. (2.9)), and

$$
|h\rangle = \Phi_h(0)|0\rangle ,
$$

(4.42)

we have

$$
|h\rangle = \left[ -2a J_{-1}^- - 2b J_{-1} J_0^- - c J_{-1}^+(J_{-1}^-)^2 \right] |h\rangle .
$$

(4.43)

We would like to check that

$$
J_n^a |\theta\rangle = 0 , \quad \forall \ n \geq 1 , \quad a = 3, \pm .
$$

(4.44)
Only $J_i^a$ give non-trivial constraints. Applying $J^-_1$ and using

\[
[J^-_1, J^3_{-1}] = J^0_0, \\
[J^-_1, J^+_1] = 2J^3_0 + k,
\]

we have

\[
J^-_1 |\theta\rangle = [-2b - c(k + 2(-h - 2))] (J^-_0)^2 |h\rangle.
\]

Hence,

\[
2b + c[k - 2(h + 2)] = 0.
\]

Applying $J^3_1$ and using

\[
[J^3_1, J^\pm_{-1}] = \pm J^\pm_0, \\
[J^3_1, J^3_{-1}] = -\frac{1}{2} k,
\]

we have

\[
J^3_1 |\theta\rangle = \left[ 2a J^-_0 + 2b \frac{1}{2} k J^-_0 - c J^+_0 (J^-_0)^2 \right] |h\rangle.
\]

Since $J^+_0 |h\rangle = 0$, we can commute it through, using

\[
[J^+_0, J^-_0] = -2J^3_0.
\]

Collecting all the terms we get

\[
2a + kb - 2c(2h + 1) = 0.
\]

Similarly, applying $J^+_1$ and using

\[
[J^+_1, J^3_{-1}] = -J^+_0, \\
[J^+_1, J^-_{-1}] = -2J^3_0 + k,
\]

we have

\[
J^+_1 |\theta\rangle = \left[ -2a(k - 2J^3_0) - 2b(-J^+_0)J^-_0 \right] |h\rangle.
\]

Rearranging terms, we conclude that

\[
4bh - 2a(k + 2h) = 0.
\]
The equations (4.48), (4.50), (4.53) have a unique solution (up to rescaling $a, b, c$, which is of course a symmetry of (4.40), and up to $h \to 1 - h$ which is the reflection symmetry, (4.10))

\[
a = \frac{1}{2}t(t + 1) \\
b = t + 1 \\
c = 1 \\
h = \frac{t}{2}.
\]

(4.54)

Since (4.40), (4.54) is a primary of $\widetilde{SL}(2)$ which is also a current algebra descendant, and its norm is zero, it is natural to set it to zero. Again, as in the discussion of $\Phi_{-\frac{1}{2}}$ above, this is not obvious, since the theory is not unitary. In this case, one cannot even verify semiclassically that the combination (4.40) is zero, since the operator does not have a smooth $t \to -\infty$ limit. Nevertheless, we will assume that the null state should be set to zero as part of the definition of the theory.

Requiring that $\theta = 0$ imposes constraints on the OPE

\[
\Phi_{\frac{t}{2}}(x)\Phi_h(y) = \sum_{h'} |x - y|^{-2(\frac{1}{2}+h-h') + t} C_{h'h'} \Phi_{h'}(y) + \cdots .
\]

(4.55)

We must have

\[
0 = \theta(x)\Phi_h(y) = \frac{1}{2}t(t + 1)\partial^2_x J\Phi_{\frac{t}{2}}(x)\Phi_h(y) + (t + 1)\partial_x J\partial_x \Phi_{\frac{t}{2}}\Phi_h(y) + J(x)\partial^2_x \Phi_{\frac{t}{2}}(x)\Phi_h(y) .
\]

(4.56)

Using the OPE algebra (2.13), we have:

\[
t(t + 1)\Phi_{\frac{t}{2}}(x)\partial_y \Phi_h(y) +
(t + 1)\partial_x \Phi_{\frac{t}{2}}(x) [2(x - y)\partial_y - 2h] \Phi_h(y) + \partial^2_x \Phi_{\frac{t}{2}}(x) [(y - x)^2 \partial_y + 2h(y - x)] \Phi_h(y) = 0 .
\]

(4.57)

Substituting (4.55) into (4.57) we conclude that

\[
t(t + 1)\partial_y (x - y)^{-\frac{1}{2} - h + h'} +
(t + 1) [2(x - y)\partial_y - 2h] \partial_x (x - y)^{-\frac{1}{2} - h + h'} + [(y - x)^2 \partial_y + 2h(y - x)] \partial^2_x (x - y)^{-\frac{1}{2} - h + h'} = 0 .
\]

(4.58)

This gives a cubic equation for $h'$, whose solutions are:

\[
h' = h \pm \frac{t}{2}, \quad 1 - \frac{t}{2} - h .
\]

(4.59)
To summarize, we find that
\[
\Phi_{\frac{t}{2}}(x)\Phi_h(y) = C_1(h)\Phi_{h+\frac{t}{2}}(y) + C_2(h)|x-y|^{-2t}\Phi_{h-\frac{t}{2}}(y) + C_3(h)|x-y|^{2(t-2h+1)}\Phi_{-\frac{t}{2}-h+1}.
\] (4.60)

We will next show that the problem of determining \(C_1, C_2, C_3\) is perturbative in the Sine-Liouville description. First we need to find the large \(\phi\) behavior of \(\Phi_{\frac{t}{2}}\). Classically, since \(t/2\) is negative, the leading term is (see (2.4))
\[
\Phi_{\frac{t}{2}}(x) = \frac{1-t}{\pi} e^{-\frac{1}{2}Qt\phi} |\gamma - x|^{-2t}.
\] (4.61)

As for \(\Phi_{-\frac{t}{2}}\), we expect to find a multiplicative correction to this in the full quantum theory. We again use the reflection symmetry (4.11), and the fact that the dual operator \(\Phi_{1-\frac{t}{2}}\) behaves at large \(\phi\) like
\[
\Phi_{1-\frac{t}{2}} \simeq -e^{-\frac{1}{2}Qt\phi} \delta^2(\gamma - x).
\] (4.62)

Substituting this into (4.11) leads to
\[
\Phi_{\frac{t}{2}}(x) \simeq \frac{1-t}{\pi} R\left(\frac{t}{2}\right) e^{-\frac{1}{2}Qt\phi} |\gamma - x|^{-2t}.
\] (4.63)

For the choice of \(\lambda\) in (4.39), the reflection coefficient is given by (4.11), hence
\[
R\left(\frac{t}{2}\right) = \frac{\Gamma(2-\lambda)}{\Gamma\left(\frac{1}{2}\right)}.
\] (4.64)

Comparing (4.11) to (4.63) we see that the quantum correction is again a multiplicative factor given by the appropriate reflection coefficient.

Multiplying (4.63) by the leading behavior of \(\Phi_h(y)\) given by the second line of (4.13) gives the free field theory result
\[
\Phi_{\frac{t}{2}}(x)\Phi_h(y) = -\frac{1-t}{\pi} R\left(\frac{t}{2}\right)|x-y|^{-2t} e^{Q(h-\frac{t}{2}-1)\phi} \delta^2(\gamma - y)
\]
\[
= \frac{1-t}{\pi} R\left(\frac{t}{2}\right)|x-y|^{-2t}\Phi_{h-\frac{t}{2}}(y).
\] (4.65)

Comparing to (4.60) we see that
\[
C_2 = \frac{1-t}{\pi} R\left(\frac{t}{2}\right).
\] (4.66)

We next compute \(C_1\) in order to derive a recursion relation similar to (4.35).
As mentioned above, the calculation is perturbative in Sine-Liouville (or in the supersymmetric case in $N = 2$ Liouville). Therefore, we start by briefly reviewing the map (which is also reviewed in [11], but we use slightly different normalizations). The claim is that the CFT on the cigar $SL(2)_k/U(1)$, which has central charge

$$c = \frac{3k}{k - 2} - 1 = 2 + \frac{6}{k - 2}, \quad (4.67)$$

is equivalent to the CFT on $R_\phi \times S^1_x$, where $R_\phi$ is the theory of a real scalar $\phi$ with a linear dilaton, and $S^1_x$ is the theory of a scalar $x = x_l + x_r$ on a circle with radius

$$R = \sqrt{2k}. \quad (4.68)$$

The free part of the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ is

$$\mathcal{L}_0 = \partial x \bar{\partial} x + \partial \phi \bar{\partial} \phi - Q \hat{R} \phi. \quad (4.69)$$

The linear dilaton slope $Q$ and the corresponding central charge of the cylinder theory are

$$Q^2 = \frac{2}{k - 2} = -\frac{2}{l}; \quad c = 2 + 3Q^2 = 2 + \frac{6}{k - 2}, \quad (4.70)$$

as in (4.67). There is an interaction term in the Lagrangian given by

$$\mathcal{L}_{\text{int}} = \lambda_{sl} \cos \left( \frac{R}{2} (x_l - x_r) e^{\frac{1}{2} Q_\phi} \right). \quad (4.71)$$

One can check that it has dimension $(1, 1)$.

The operator map between the observables on the cigar (3.5) (for simplicity for $m = \bar{m}$, which is all we need here) and Sine-Liouville is

$$V_{j;m,m} \leftrightarrow e^{ip(x_l - x_r) + \beta \phi}, \quad (4.72)$$

with

$$p = m \sqrt{\frac{2}{k}}, \quad (4.73)$$

$$\beta = Qj. \quad (4.73)$$

One can check that

$$\frac{1}{2} p^2 - \frac{1}{2} \beta(\beta + Q) = \frac{m^2}{k} - \frac{j(j + 1)}{k - 2}. \quad (4.74)$$
Note that KPZ scaling implies that
\[
\langle V_j V_j \rangle \sim \lambda_{sl}^{-2} (2j+1). \tag{4.75}
\]
Comparing to (3.3) we see that
\[
\lambda_{sl}^{-2} \sim \lambda. \tag{4.76}
\]
Below we will determine the precise relation between the couplings:
\[
\pi \lambda \frac{\Gamma(-\frac{1}{t})}{\Gamma(1+\frac{1}{t})} = \left( \frac{\pi \lambda_{sl}}{t} \right)^{-\frac{2}{t}}. \tag{4.77}
\]
We would like to use the Sine-Liouville variables to calculate \( C_1 \) in (4.60). Start by sending \( x, y \to 0 \).
Looking back at (4.63) we see that
\[
\Phi_{\frac{t}{2}}(x = 0) = 1 - \frac{t}{\pi} \mathcal{R}(\frac{t}{2}) e^{-\frac{i}{2} t Q\phi(\gamma\bar{\gamma})}. \tag{4.78}
\]
Comparing to (3.7) we see that
\[
\Phi_{\frac{t}{2}}(x = 0) = \frac{1 - t}{\pi} \mathcal{R}(\frac{t}{2}) V_{-\frac{t}{2}; -\frac{t}{2}}. \tag{4.79}
\]
Similarly, from (2.14) we learn that
\[
\Phi_h(y = 0) = V_{h-1; -h; -h}. \tag{4.80}
\]
So, in the limit \( x, y \to 0 \), the \( C_1 \) term in (4.60) becomes
\[
\frac{1 - t}{\pi} \mathcal{R}(\frac{t}{2}) V_{-\frac{t}{2}; -\frac{t}{2}} V_{h-1; -h; -h} = C_1 V_{h+\frac{t}{2}; -h-\frac{t}{2}; -h-\frac{t}{2}}. \tag{4.81}
\]
This equation should be true also in \( SL(2)/U(1) \), since the OPE of the \( U(1) \) part is trivial. Thus we can use the map (4.72), (1.73), and write it as (replacing \( x_l - x_r \) by \( x \) for brevity)
\[
\frac{1 - t}{\pi} \mathcal{R}(\frac{t}{2}) e^{-i\frac{t}{2} \sqrt{\frac{e}{k}} x - \frac{i}{2} Q\phi} e^{-i h \sqrt{\frac{e}{k}} x + (h-1)Q\phi} = C_1 e^{-i(h+\frac{t}{2}) \sqrt{\frac{e}{k}} x + (h+\frac{t}{2}-1)Q\phi}. \tag{4.82}
\]
We see that the l.h.s. and r.h.s. have the same \( p \) but there is a mismatch of \( tQ \) in the value of \( \beta \) (in the notation of the r.h.s. of (4.72)). This is easy to fix; we simply expand \( \exp(-\int L_{\text{int}}) \) (1.71) to second order in \( \lambda_{sl} \):
\[
\frac{1 - t}{\pi} \mathcal{R}(\frac{t}{2}) \frac{\lambda_{sl}^2}{2} \int d^2 z_1 \int d^2 z_2 \cos \sqrt{\frac{k}{2}} x(z_1) e^{\frac{1}{2} t Q\phi(z_1)} \cos \sqrt{\frac{k}{2}} x(z_2) e^{\frac{1}{2} t Q\phi(z_2)}
\]
\[
e^{-i\frac{t}{2} \sqrt{\frac{e}{k}} x - \frac{i}{2} Q\phi} e^{-i h \sqrt{\frac{e}{k}} x + (h-1)Q\phi} = C_1 e^{-i(h+\frac{t}{2}) \sqrt{\frac{e}{k}} x + (h+\frac{t}{2}-1)Q\phi}. \tag{4.83}
\]
Performing the free field OPE’s as before we find that $C_1$ is given by (setting $z = 1$)

$$C_1 = \pi (1 - t) R \left( \frac{t}{2} \right) \lambda_{sl}^2 \int d^2 z_1 d^2 z_2 |z_1 - z_2|^{2(1-k)} |1 - z_1|^{2(k-2)} |z_1|^{-2} |z_2|^{2(2h-1)} .$$  \hfill (4.84)

Since the power of $1 - z_2$ vanishes, we can perform the integral over $z_2$, and then that over $z_1$, using standard formulae. One finds that all the $\Gamma$ functions cancel and

$$C_1(h) = -\frac{\pi (1 - t) \lambda_{sl}^2}{2h + t - 2} R \left( \frac{t}{2} \right) = -\frac{\pi (1 - t) \lambda_{sl}^2}{2h + t - 2} R \left( \frac{t}{2} \right) .$$  \hfill (4.85)

We can now derive a recursion relation for the two point function $D(h)$ \((4.1)\) by following the same logic as in eqs \((4.31)-(4.35)\) by considering the three point function

$$\langle \Phi_{\frac{t}{2}}(x) \Phi_h(x_1) \Phi_{h+\frac{t}{2}}(x_2) \rangle .$$

One finds:

$$\frac{D(h)}{D(h + \frac{t}{2})} = \frac{C_1(h)}{C_2(h + \frac{t}{2})} .$$  \hfill (4.86)

Plugging in our results \((4.66), (4.85)\) we find

$$\frac{D(h)}{D(h + \frac{t}{2})} = -\frac{\pi^2 \lambda_{sl}^2}{(2h + t - 2)} .$$  \hfill (4.87)

The correct answer \((1.36)\) satisfies

$$\frac{D(h)}{D(h + \frac{t}{2})} = -\frac{t^2 \nu^{-t}}{(2h + t - 2)} .$$  \hfill (4.88)

Clearly, we can set $\lambda_{sl}$ to a value such that the r.h.s. of \((4.88)\) coincides with that of \((4.87)\): 

$$\nu = \left( \frac{\pi \lambda_{sl}}{t} \right)^{-\frac{2}{t}} .$$  \hfill (4.89)

Comparing to \((4.37)\) we see that the relation between $\lambda_{sl}$ and $\lambda$ is indeed as advertised in eq. \((4.77)\). In particular, for the choice \((1.39)\) of the normalization of the operators, we have $\nu = 1$ thus

$$\lambda_{sl} = \frac{t}{\pi} .$$  \hfill (4.90)

To summarize, equations \((1.35)\) and \((4.87)\) determine the two point function $D(h)$ uniquely, at least in the case when $k$ (or $t$) is irrational. Of course, the calculations
described here can also be viewed as evidence for the strong-weak coupling duality of Sine-Liouville and the cigar CFT.

So far we have only determined $C_1$ and $C_2$ in (4.60). $C_3$ can be calculated by using the reflection symmetry (4.10). Substituting (4.10) into (4.60) we get

$$R(h)\frac{2h-1}{\pi}\Phi_{\frac{1}{2}}(x; z) \int d^2y' |y - y'|^{-4h}\Phi_{1-h}(y'; w) =$$

$$C_1(h)\Phi_{h+\frac{1}{4}}(y) + C_2(h)|x - y|^{-2t}\Phi_{h-\frac{1}{4}}(y) + C_3(h)|x - y|^{2(1-t-2h)}\Phi_{1-\frac{4}{t}-h}(y).$$

(4.91)

Using (4.60) directly on the l.h.s. gives

$$R(h)\frac{2h-1}{\pi}\int d^2y'|y - y'|^{-4h}[C_1(1-h)\Phi_{1-h+\frac{1}{4}}(y') +$$

$$C_2(1-h)|x - y'|^{-2t}\Phi_{1-h-\frac{1}{4}}(y') + C_3(1-h)|x - y'|^{2(2h-t-1)}\Phi_{h-\frac{1}{4}}(y')] .$$

(4.92)

The second term in (4.92) has the same behavior as the third term on the r.h.s. of (4.91). By comparing the two one can compute $C_3$. One finds:

$$C_3(h) = \frac{1}{\pi} \frac{(2h-1)(1-t)R\left(\frac{t}{2}\right)}{\Gamma(1+\frac{2h-1}{t})\Gamma(1-2h)\Gamma(1-t)\Gamma(2h+t-1)} \frac{\Gamma(1+\frac{2h-1}{t})\Gamma(1-2h)\Gamma(1-t)\Gamma(2h+t-1)}{\Gamma(1-2h-1)\Gamma(2h)\Gamma(t)\Gamma(2-2h-t)} .$$

(4.93)

As a check, note that $C_3$ satisfies the constraint coming from $\langle \Phi_{\frac{1}{2}} \Phi_h \Phi_{1-h-\frac{1}{4}} \rangle$,

$$C_3(h)D(1 - h - \frac{t}{2}) = C_3(1 - h - \frac{t}{2})D(h) .$$

(4.94)

5. Degenerate conformal blocks on the sphere

In this section we review the calculation of current algebra blocks relevant for the four point functions of the degenerate operators $\Phi_{-\frac{1}{2}}$ and $\Phi_{\frac{1}{2}}$ with operators with generic $h$. This provides a check on the fusion coefficients computed in the previous section; it is also needed for the study of D-branes in [44].

We start by deriving the generalization [43] of the Knizhnik-Zamolodchikov (KZ) equation for $SL(2)$ CFT. The four point function of $\Phi_{h_j}(x_j, \bar{x}_j; z_j, \bar{z}_j)$ ($j = 1, 2, 3, 4$) can

\footnote{Similarly, the third term in (4.92) has the same behavior as the second term on the r.h.s. of (4.91). On the contrary, it is not manifest how to identify the first term on the r.h.s of (4.91) with the first term in (4.92).}
be written using worldsheet and spacetime conformal invariance in the following form (see eqs. (44) – (48)):

\[ \langle \Phi_{h_1}(x_1, \bar{x}_1; z_1, \bar{z}_1) \cdots \Phi_{h_4}(x_4, \bar{x}_4; z_4, \bar{z}_4) \rangle = \]
\[ |z_1 - z_4|^{2(\Delta_3 + \Delta_2 - \Delta_1 - \Delta_4)} |z_3 - z_4|^{2(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)} \]
\[ |z_2 - z_4|^{-4\Delta_2} |z_1 - z_3|^{2(\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3)} \]
\[ |x_1 - x_4|^{2(h_3 + h_2 - h_1 - h_4)} |x_3 - x_4|^{2(h_1 + h_2 - h_3 - h_4)} \]
\[ |x_2 - x_4|^{-4h_2} |x_1 - x_3|^{2(h_4 - h_1 - h_2 - h_3)} \mathcal{F}(\eta_{ws}, \eta_{st}) . \]

Here \( \eta_{ws} \) and \( \eta_{st} \) are the worldsheet and spacetime crossratios,

\[ \eta_{ws} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} , \]
\[ \eta_{st} = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)} . \]

Note that if the dimensions are equal in pairs,

\[ \Delta_1 = \Delta_3 ; \quad \Delta_2 = \Delta_4 , \]
\[ h_1 = h_3 ; \quad h_2 = h_4 , \]

(5.1) simplifies:

\[ \langle \Phi_{h_1}(x_1, \bar{x}_1; z_1, \bar{z}_1) \Phi_{h_2}(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_{h_1}(x_3, \bar{x}_3; z_3, \bar{z}_3) \Phi_{h_2}(x_4, \bar{x}_4; z_4, \bar{z}_4) \rangle = \]
\[ |z_2 - z_4|^{-4\Delta_2} |z_1 - z_3|^{-4\Delta_1} |x_2 - x_4|^{-4h_2} |x_1 - x_3|^{-4h_1} \mathcal{F}(\eta_{ws}, \eta_{st}) . \]

We will next compute \( \mathcal{F} \) for the two cases of interest to us: \( h_1 = h_3 = -1/2 \) and \( h_1 = h_3 = t/2 \) with \( h_2 = h_4 = h \) in both cases.

5.1. The current algebra blocks for \( h_1 = h_3 = -1/2 \)

Substituting

\[ h_1 = h_3 = -\frac{1}{2} ; \quad h_2 = h_4 = h \]

in (5.4), we have:

\[ \langle \Phi_{-\frac{1}{2}}(x_1, \bar{x}_1; z_1, \bar{z}_1) \Phi_h(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_{-\frac{1}{2}}(x_3, \bar{x}_3; z_3, \bar{z}_3) \Phi_h(x_4, \bar{x}_4; z_4, \bar{z}_4) \rangle = \]
\[ |z_2 - z_4|^{-4\Delta_h} |z_1 - z_3|^{-4\Delta_h} |x_2 - x_4|^{-4h} |x_1 - x_3|^{2} \mathcal{F}(\eta_{ws}, \eta_{st}) . \]

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Due to (4.15), \( F \) must be a sum of terms like
\[
F(\eta_{ws}, \eta_{st}) = |F_0(\eta_{ws}) + \eta_{st}F_1(\eta_{ws})|^2.
\] (5.6)

The differential equation that we will derive will act on each factor separately. Therefore, from now on we can focus on the holomorphic part of \( F \) in (5.6):
\[
\langle \Phi_{-\frac{1}{2}} \Phi_h \Phi_{-\frac{1}{2}} \Phi_h \rangle = (z_2 - z_4)^{-2\Delta_h}(z_1 - z_3)^{-2\Delta_{\frac{1}{2}}}
\]
\[
[(x_2 - x_4)^{-2h}(x_1 - x_3)F_0(\eta_{ws}) + (x_2 - x_4)^{-2h-1}(x_1 - x_2)(x_3 - x_4)F_1(\eta_{ws})] .
\] (5.7)

We would next like to compute \( F_0, F_1 \) by solving the KZ equation for the four point function (5.7).

Let us review the derivation of the KZ equation for this case. The worldsheet stress tensor is (see [2] eq. (2.26))
\[
T_{ws} = -\frac{1}{2t} \left[ J\partial_x^2J - \frac{1}{2}(\partial_x J)^2 \right].
\] (5.8)

Consider
\[
\langle T_{ws}(z)\Phi_{h_1}(x_1; z_1) \cdots \Phi_{h_4}(x_4; z_4) \rangle ,
\] (5.9)
and focus on the coefficient of \( 1/(z - z_1) \). On the one hand we have
\[
\langle T_{ws}(z)\Phi_1 \cdots \Phi_4 \rangle \sim \frac{1}{z - z_1} \langle \partial_{z_1} \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle .
\] (5.10)

On the other hand, substituting (5.8) and using the OPE’s (2.13) we get\(^8\)
\[
\langle T_{ws}(z)\Phi_1 \cdots \Phi_4 \rangle \sim -\frac{1/t}{z - z_1} \sum_{i=2}^{4} \frac{1}{z_1 - z_i} \times
\]
\[
\left\{(x_i - x_1)^2 \frac{\partial}{\partial x_i} + 2h_i(x_i - x_1) \right\} \langle \partial_{x_1} \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle - 2h_1 \left[(x_i - x_1) \frac{\partial}{\partial x_i} + h_i \right] \langle \Phi_1 \cdots \Phi_4 \rangle
\]
\] (5.11)

Comparing the two gives the KZ equation \([15]\)
\[
-t \frac{\partial}{\partial z_1} \langle \Phi_1 \cdots \Phi_4 \rangle = \sum_{i=2}^{4} \frac{1}{z_1 - z_i} Q_i \langle \Phi_1 \cdots \Phi_4 \rangle ,
\] (5.12)

\(^8\) Note that \( T_{ws} \) is independent of \( x \) and hence we can choose \( x = x_1 \) when plugging (5.8) in (5.10).
where

\[ Q_i = (x_i - x_1)^2 \frac{\partial^2}{\partial x_i \partial x_1} + 2(x_i - x_1)(h_i \partial x_1 - h_1 \partial x_i) - 2h_1 h_i. \]  

(5.13)

Substituting (5.11) into (5.12) we get the following first order differential equations for \( F_0, F_1 \):

\[ t(\eta_{ws} - 1)\eta_{ws} \frac{\partial F_0}{\partial \eta_{ws}} = h F_0 + \eta_{ws} F_1, \]  

(5.14)

\[ t(1 - \eta_{ws})\eta_{ws} \frac{\partial F_1}{\partial \eta_{ws}} = 2h F_0 + (h - 1 + 2\eta_{ws}) F_1. \]

These equations have two independent sets of solutions which are:

\[ F_0^{(-)} = x^{-a}(1 - x)^{-a} F(-2a, 2b - 2a; b - 2a; x), \]

\[ F_1^{(-)} = \frac{2a}{b - 2a} x^{-a}(1 - x)^{-a} F(1 - 2a, 2b - 2a; b - 2a + 1; x), \]  

(5.15)

\[ F_0^{(+)} = x^{a-b+1}(1 - x)^{a-b} F(2a - 2b + 1, 2a + 1; 2a - b + 2; x), \]

\[ F_1^{(+)} = \frac{b - 2a - 1}{b} x^{a-b}(1 - x)^{-a} F(1 - b, b - 2a - b + 1; x), \]  

(5.16)

where

\[ a \equiv \frac{h}{t}; \quad b \equiv \frac{1}{t}; \quad x \equiv \eta_{ws}. \]  

(5.17)

The function \( F \) that appears in (5.3), (5.6) is a linear combination of the two solutions \( F_{\pm} \):

\[ F(\eta_{ws}, \eta_{st}) = A |F_0^{(-)}(\eta_{ws}) + \eta_{st} F_1^{(-)}(\eta_{ws})|^2 + B |F_0^{(+)}(\eta_{ws}) + \eta_{st} F_1^{(+)}(\eta_{ws})|^2. \]  

(5.18)

A and B can be determined by considering the limit \( \eta_{ws} \to 0 \). One finds:

\[ A = |C_-(h)|^2 D(h - \frac{1}{2}), \]

(5.19)

\[ B = \frac{|C_+(h)|^2}{(2h - 1 + t)^2} D(h + \frac{1}{2}). \]

\[ ^9 \text{We normalized } F_0 \text{ such that for small } \eta_{ws} \text{ it goes like } F_0 \approx \eta_{ws}^c \text{ where } c \text{ is different in the two cases, but the coefficient of the power is one.} \]
A non-trivial check on the structure constants that we derived above \((C_{\pm})\) comes from the requirement that if we exchange \(2 \leftrightarrow 4\) in \((5.5)\), we get a correct equation (crossing symmetry). Looking at \((5.2)\), \((5.5)\) we see that:

1. The prefactors in front of \(F\) in \((5.5)\) are invariant under \(2 \leftrightarrow 4\).
2. The transformation of \(\eta_{ws}, \eta_{st}\) under \(2 \leftrightarrow 4\) is: \(\eta \rightarrow 1 - \eta\).

Thus, we conclude that crossing symmetry implies that (denoting \(\eta_{ws}\) by \(x\) for brevity):

\[
A|F_0^-(x) + \eta_{st}F_1^-(x)|^2 + B|F_0^+(x) + \eta_{st}F_1^+(x)|^2 =
A|F_0^-(1-x) + (1-\eta_{st})F_1^-(1-x)|^2 + B|F_0^+(1-x) + (1-\eta_{st})F_1^+(1-x)|^2.
\]

We can write the r.h.s. of \((5.20)\) as:

\[
A|\mathcal{X}_1 - \eta_{st}\mathcal{X}_2|^2 + B|\mathcal{X}_3 - \eta_{st}\mathcal{X}_4|^2,
\]

which defines \(\mathcal{X}_i, i = 1, 2, 3, 4\). In order for \((5.20)\) to be valid it seems that we need a matrix relation of the sort:

\[
\mathcal{X}_1 - \eta_{st}\mathcal{X}_2 = \tilde{a} \left( F_0^-(x) + \eta_{st}F_1^-(x) \right) + \tilde{b} \left( F_0^+(x) + \eta_{st}F_1^+(x) \right),
\]

\[
\mathcal{X}_3 - \eta_{st}\mathcal{X}_4 = \tilde{c} \left( F_0^-(x) + \eta_{st}F_1^-(x) \right) + \tilde{d} \left( F_0^+(x) + \eta_{st}F_1^+(x) \right).
\]

After some algebra we get:

\[
\tilde{a} = \frac{\Gamma(b - 2a + 1)\Gamma(2a - b)}{\Gamma(b + 1)\Gamma(-b)},
\]

\[
\tilde{b} = \frac{b}{b - 2a - 1} \frac{\Gamma^2(b - 2a)}{\Gamma(-2a)\Gamma(2b - 2a)},
\]

\[
\tilde{c} = \frac{b - 2a}{b}(\frac{\Gamma(2a - b + 2)\Gamma(2a - b)}{\Gamma(2a + 1)\Gamma(2a - 2b + 1)}),
\]

\[
\tilde{d} = -\tilde{a}.
\]

A useful thing to note is that the determinant of the \(2 \times 2\) matrix we found is \(-1\):

\[
\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = -1.
\]

The fact that this determinant has absolute value one is a necessary consistency condition. Substituting \((5.23)\) into the previous equations we find the constraint

\[
\frac{|C_+(h)|^2D(h + \frac{1}{2})}{|C_-(h)|^2D(h - \frac{1}{2})} = \frac{\Gamma^2(b - 2a)\Gamma(2a + 1)\Gamma(2a - 2b + 1)}{\Gamma^2(2a - b + 1)\Gamma(-2a)\Gamma(2b - 2a)}.
\]

Plugging in the explicit formulae we found before for the l.h.s. we indeed find the r.h.s., which verifies the consistency of the procedure.
5.2. The current algebra blocks for $h_1 = h_3 = t/2$

In this case we would like to substitute

$$h_1 = h_3 = t/2; \quad h_2 = h_4 = h$$

in (5.4) and compute the conformal blocks $F$. The degeneracy equation following from the vanishing of $\theta$ (4.40) takes in this case the form

$$(D_2 + zD_3)F = 0 ,$$

with

$$D_2 = (e^y - 1)(t + \partial_y)(2h + \partial_y)(t + 1 + \partial_y) + 2(t + \partial_y)[h(t + 1) + \partial_y] ,$$

$$D_3 = (e^{-y} - 1)\partial_y^2(\partial_y - 1) + t(t - 1)\partial_y - 2\partial_y^2 ,$$

where $x = \exp(y)$. The KZ equation (5.13) takes the form

$$-tz\partial_z F = Q_2 F - \frac{z}{1-z} Q_4 F ,$$

where

$$Q_2 = e^y (t + \partial_y)(2h + \partial_y) - (\partial_y^2 - \partial_y + t\partial_y + 2h\partial_y + ht) ,$$

$$Q_4 = -e^y (2h + \partial_y)(t + \partial_y) + (h + \partial_y)(t + 2\partial_y) - e^{-y}\partial_y^2 .$$

These equations have three solutions with the boundary conditions that we are interested in:

$$F_A(x; z) = z^h(1 - z)^h F_1(2h, t, 2h + t - 1; 2h + t; x, z) ,$$

$$F_B(x; z) = x^{-t}z^{1-h}(1 - z)^h F_1(t, t, 1 - t; 2 - 2h; \frac{z}{x}; z) ,$$

$$F_C(x; z) = z^h(1 - z)^h e^{-i\pi(1-t)} \frac{\Gamma^2(2h)}{\Gamma(2h + 1 - t)\Gamma(2h + t - 1)} \times$$

$$\left[ Z_8 - \frac{\Gamma(2h + 1 - t)\Gamma(1 - 2h + t)}{\Gamma^2(1 - t)} e^{2i\pi h} Z_1 \right] ,$$

where

$$Z_8 = x^{-2h} F_1(2h, 1 - t, t + 2h - 1; 2h + 1 - t; \frac{1}{x}, \frac{z}{x}) ,$$

$$Z_1 = F_1(2h, t, t + 2h - 1; t + 2h; x, z) .$$

In the last equations we are using the hypergeometric function in two variables (whose definition and some properties are summarized in the appendix).
As $x, z \to 0$, the conformal blocks (5.30) have the following asymptotic behavior:

\begin{align*}
F_A &\simeq z^h , \\
F_B &\simeq x^{-t}z^{1-h} , \\
F_C &\simeq x^{1-t-2h}z^h .
\end{align*}

(5.32)

The solution for the four point function on the sphere (5.4) has the form

\[ \mathcal{F}(x; z) = A|F_A(x; z)|^2 + B|F_B(x; z)|^2 + C|F_C(x; z)|^2 , \]

(5.33)

where the coefficients $A, B, C$ are obtained by sending $x, z \to 0$ (in the order indicated in footnote 10) and comparing the behavior to the contributions of the terms in (4.60). This leads to:

\begin{align*}
A &= |C_1(h)|^2 D(h + \frac{t}{2}) , \\
B &= |C_2(h)|^2 D(h - \frac{t}{2}) , \\
C &= |C_3(h)|^2 D(1 - h - \frac{t}{2}) .
\end{align*}

(5.34)

Again, a non-trivial check on the structure constants $C_{1,2,3}$ can be done by requiring crossing symmetry:

\[ |\mathcal{F}(1-x; 1-z)| = |\mathcal{F}(x; z)| , \]

(5.35)

for the blocks in eq. (5.33). Defining a $3 \times 3$ matrix $\mathcal{M}$ by:

\[ F_I(1-x; 1-z) = \mathcal{M}_{IJ}F_J(x; z) , \quad I, J = A, B, C , \]

(5.36)

\footnote{More precisely, we are discussing here the limit $z \to 0$ first, followed by $x \to 0$. The order of limits is important in this case.}
one finds after some algebra that

\[
\begin{align*}
M_{AA} &= \frac{\sin \pi t}{\sin \pi (2h + t)}, \\
M_{AB} &= \frac{1 - 2h - t}{1 - 2h}, \\
M_{AC} &= \frac{\Gamma(2h + t)\Gamma(2h + t - 1)\Gamma(1 - 2h)}{\Gamma^2(t)\Gamma(2h)}, \\
M_{BA} &= \frac{1 - 2h}{1 - 2h - t}, \\
M_{BB} &= -e^{i\pi t} \frac{\sin \pi t}{\sin 2\pi h}, \\
M_{BC} &= -e^{i\pi (2h + t)} \frac{\Gamma(1 - 2h)\Gamma(2 - 2h)\Gamma(2h + t - 1)}{\Gamma^2(t)\Gamma(2 - 2h - t)}, \\
M_{CA} &= \frac{\Gamma(2 - 2h - t)\Gamma(1 - 2h - t)\Gamma(2h)}{\Gamma^2(1 - t)\Gamma(1 - 2h)}, \\
M_{CB} &= -e^{i\pi (2h + t)} \frac{\Gamma(2h)\Gamma(2h - 1)\Gamma(2 - 2h - t)}{\Gamma^2(1 - t)\Gamma(2h + t - 1)}, \\
M_{CC} &= e^{2\pi i(h + t)} + e^{i\pi (2h + t)} \frac{\sin^2 \pi t}{\sin \pi (2h + t) \sin 2\pi h}.
\end{align*}
\]

These \( M_{IJ} \) are the same as the ones found by J. Teschner; they can be read off from subsection 7.3.1 in [43]. Using this \( M \) in (5.33), (5.35), and comparing the coefficients \( A, B, C \) obtained this way to eq. (5.34), one finds the same absolute values of \( C_{1,2,3} \) as derived above.

6. \( N = 2 \) superconformal extension

In the supersymmetric case, the duality of [24] is replaced by the conjectured equivalence of \( N = 2 \) Liouville and the supersymmetric cigar CFT [13]. In this section we discuss calculations analogous to those performed in section 4, for the supersymmetric system. Since the calculations are similar to those of section 4, we will be rather schematic and focus mainly on the differences.

In the superconformal coset model \( SL(2)/U(1) \) there are observables \( V_{j;m,\bar{m}} \) with scaling dimensions

\[
\Delta(V_{j;m,\bar{m}}) = \frac{j(j + 1) - m^2}{t},
\]

where, as before,

\[
t \equiv -(k - 2).
\]
The right moving scaling dimension of $V_{j;m,\bar{m}}$ is given by a formula similar to (6.1), with $m \to \bar{m}$. $k$ is the bosonic level of the $SL(2)/U(1)$ sigma model while $k - 2 = -t$ is the total level.

Far from the tip of the cigar this SCFT looks like a sigma model on the cylinder $R_\phi \times S^1_Y$, where $R_\phi$ is the real line with a linear dilaton

$$\Phi(\phi) = -\frac{Q}{2}\phi , \quad (6.3)$$

with $Q$ given in (2.6), and the circle $S^1_Y$ is parametrized by a canonically normalized scalar $Y$ with radius

$$R_Y = \sqrt{-2t} . \quad (6.4)$$

The observables $V_{j;m,\bar{m}}$ (6.1) have the asymptotic form

$$V_{j;m,\bar{m}} \to e^{Q(j\phi + imY + i\bar{m}\bar{Y})} . \quad (6.5)$$

The two point functions of these observables can be computed as in the bosonic case (see [13,23]). One starts with correlators in the underlying $SL(2)$ SCFT; those are not affected by adding free fermions to the bosonic sigma model. Hence also the two point functions in the $SL(2)/U(1)$ quotient are not changed relative to the bosonic case, giving rise again to the relation (4.35).

On the other hand, the $N = 2$ Liouville interaction (the top component of the superpotential plus its complex conjugate) is

$$V_L = \mu \Psi \bar{\Psi} e^{-\frac{1}{Q}(\phi+iY)} + c.c. , \quad (6.6)$$

where

$$\Psi = \psi^\phi + i\psi^Y , \quad (6.7)$$

and the fermions $\psi^Y, \psi^\phi$ are the superpartners of the scalars $Y, \phi$, respectively. Following the same steps as in the previous sections, with the slight changes discussed above, leads to (the supersymmetric analog of (4.83)):

$$\mathcal{R}(\frac{t}{2}) \frac{(1-t)|\mu|^2}{2\pi} \prod_{i=1}^2 \int d^2 z_i \left( \Psi \bar{\Psi}(z_1) e^{-\frac{1}{Q}(\phi+iY)(z_1)} + c.c. \right) \left( \Psi \bar{\Psi}(z_2) e^{-\frac{1}{Q}(\phi+iY)(z_2)} + c.c. \right) e^{-\frac{1}{Q}(\phi+iY)\epsilon^{(h-1)}Q\phi - ihQY} = C_1 e^{(h+\frac{t}{2}-1)Q\phi - i(h+\frac{t}{2})QY} . \quad (6.8)$$

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Performing the free field OPE’s we find the same $C_1$ as in eq. (4.84), leading to (4.85) and (4.87).

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**Appendix A. Some useful formulae**

A useful integral is:

$$
\int d^2 x |x|^{2a} x^n |1-x|^{2b} (1-x)^m = \pi \frac{\Gamma(a+n+1)\Gamma(b+m+1)\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)\Gamma(a+b+m+n+2)}, \quad (A.1)
$$

$$
n, m \in \mathbb{Z},
$$

and the Gamma functions satisfy:

$$
\Gamma(a+1) = a\Gamma(a). \quad (A.2)
$$

The hypergeometric function is defined by the differential equation for a function $u(x)$

$$
x(1-x)u'' + [\gamma - (\alpha + \beta + 1)x] u' - \alpha \beta u = 0. \quad (A.3)
$$

This equation has two solutions:

$$
u_1 = F(\alpha, \beta; \gamma; x), \quad (A.4)
$$

$$
u_2 = x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2-\gamma; x).
$$

For small $x$, $F$ can be expanded as follows:

$$
F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\Gamma(n)(\beta)n}{n!(\gamma)n} x^n = 1 + \frac{\alpha \beta}{\gamma} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{2\gamma(\gamma + 1)} x^2 + \cdots, \quad (A.5)
$$
where
\[(a)_n \equiv a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{A.6}\]

Two other identities that are sometimes useful are:

\[
F(\alpha, \beta; \gamma; x) = (1 - x)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta; \gamma; x),
\]

\[
\frac{\partial F}{\partial x}(\alpha, \beta; \gamma; x) = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x). \tag{A.7}\]

Under \(x \to 1/x\):

\[
F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (-\frac{1}{x})^\alpha F(\alpha, \alpha + 1 - \gamma; \alpha + 1 - \beta; \frac{1}{x}) +
\]

\[
\frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} (-\frac{1}{x})^\beta F(\beta, \beta + 1 - \gamma; \beta + 1 - \alpha; \frac{1}{x}). \tag{A.8}\]

Under \(x \to 1 - x\):

\[
F(\alpha, \beta; \gamma; 1 - x) = \frac{\Gamma(\gamma)\Gamma(\gamma - \beta - \alpha)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta; \alpha + \beta + 1 - \gamma; x) +
\]

\[
x^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma - \alpha, \gamma - \beta; \gamma + 1 - \alpha - \beta; x). \tag{A.9}\]

The hypergeometric function in two variables \(F_1(x, y)\) can be defined as the analytic continuation of the small \(x, y (|x|, |y| < 1)\) expansion:

\[
F_1(\alpha, \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m}(\beta')_{n}}{m!n!(\gamma)_{m+n}} x^m y^n, \tag{A.10}\]

where \((a)_n\) is defined in (A.6).

More useful identities:

\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \tag{A.11}\]

\[
\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}. \tag{A.12}\]
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