Vortex tension as an order parameter in three-dimensional $U(1)+$Higgs theory

K. Kajantie$^{a,b,1}$, M. Laine$^{a,b,2}$, T. Neuhaus$^{c,d,3}$, J. Peisa$^{e,4}$, A. Rajantie$^{b,c,5}$ and K. Rummukainen$^{f,6}$

$^a$Theory Division, CERN, CH-1211 Geneva 23, Switzerland
$^b$Dept. of Physics, P.O. Box 9, FIN-00014 Univ. of Helsinki, Finland
$^c$Helsinki Institute of Physics, P.O. Box 9, FIN-00014 Univ. of Helsinki, Finland
$^d$Fakultät für Physik, Univ. Bielefeld, P.O. Box 100131, D-33501 Bielefeld, FRG
$^e$Dept. of Physics, Univ. of Wales Swansea, Singleton Park, Swansea SA2 8PP, UK
$^f$Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract

We use lattice Monte Carlo simulations to study non-perturbatively the tension, i.e. the free energy per unit length, of an infinitely long vortex in the three-dimensional $U(1)+$Higgs theory. This theory is the low-energy effective theory of high-temperature scalar electrodynamics, the standard framework for cosmic string studies. The vortex tension is measured as a function of the mass parameter at a large value of the Higgs self-coupling, where the transition between the phases is continuous. It is shown that the tension gives an order parameter that can distinguish between the two phases of the system. We argue that the vortex tension can describe the physics of long strings without lattice artifacts, unlike vortex network percolation.
1 Introduction

During the early stages of its evolution, the Universe went through a series of phase transitions. Topological defects, such as cosmic strings, domain walls and magnetic monopoles, may have been created as a result of these transitions [1]. It is possible that the topological defects have observable consequences even in the present Universe.

It is customary to view the topological defects as classical solutions obtained from the high-temperature effective potential of the system. However, the validity of this mean-field approach is uncertain, especially in gauge theories. Indeed, there is no symmetry breaking related to the Higgs mechanism [2], and in some cases the phases are analytically connected to each other [3]. Clearly the mean-field approximation does not work near the transition in such cases. It is therefore necessary to investigate how the inclusion of the full effect of fluctuations changes the picture of defect formation.

To study the properties of string-like defects in gauge theories, it is most convenient to concentrate on the Nielsen-Olesen vortices [4] of the U(1) gauge+Higgs theory. Although this is much simpler than realistic theories containing cosmic strings, it permits a first-principles field theoretic study. In Ref. [5] an effective three-dimensional (3d) theory was constructed to describe the thermodynamical behavior of the full four-dimensional (4d) theory at high temperatures near the phase transition. The effective theory, the 3d U(1)+Higgs theory, is equivalent to the Ginzburg-Landau theory of superconductivity [6]. It contains fewer degrees of freedom than the original theory and its ultraviolet behavior is also better understood [7], which makes it much more suitable for numerical simulations.

In discussing phase transitions, the first question is to find a proper variable to signal one. Although, in a gauge theory, there is no local gauge-invariant order parameter, the first order transition in the U(1)+Higgs theory can be localized by finding a discontinuity in, say, $|\phi|^2$ [8,9]. This does not help if the transition is continuous. Then a non-local order parameter may exist. For U(1)+Higgs one such is the photon mass [9] which vanishes identically in the symmetric (Coulomb) phase and is non-zero in the broken (Higgs) phase.

For the U(1)+Higgs theory one may suggest further order parameters related to the existence of vortices. In fact, several different effective models have been proposed (see, e.g., [6,10]). These cannot replace a first-principles numerical study. In Ref. [11] the total density of thermal vortex-loop excitations was studied as an order parameter. However, this suffers from the difficulty of
performing a continuum limit: small vortices also appear as lattice artifacts.

In scalar theories, it has been proposed that one should instead look at the density of infinitely long vortices, i.e., percolation [12] (see also [13]). However, as is well-known, this definition could be subject to lattice artifacts (for a review, see [14]). The basic problem is that “percolation” is a geometrical property, not directly related to the free energy of the system, and possibly sensitive to how the discretization needed for simulations is made.

In this paper, we suggest a better alternative in the context of gauge theories: the tension of an infinitely long vortex, i.e., its free energy per unit length. We show how one puts a vortex on a lattice and how one measures its free energy using Monte Carlo simulations. We show that it really is an order parameter, vanishing in the \( V \to \infty \) limit in the same phase as the photon mass. The result is given in continuum \( \overline{\text{MS}} \) units, which makes it possible to compare it with the mean-field value [15]. The vortex tension thus defined is a physical observable and has a well-defined continuum limit. We suggest that the vortex tension is a good first-principles tool for discussing the physics of long vortices, and might perhaps also be used as an input in some phenomenological non-equilibrium estimates in the regime where mean-field theory does not work.

The structure of the paper is the following. The U(1)+Higgs theory is defined and its properties are reviewed in Sec. 2. In Sec. 3 we describe how the vortex tension can be defined and measured, and in Sec. 4 we relate it to the phase structure of the theory. The details of the simulations, as well as their results, are given in Sec. 5. Sec. 6 contains the conclusions.

2 The U(1)+Higgs theory

In Ref. [5], the heavy degrees of freedom were integrated out from the 4d high-temperature U(1)+Higgs theory (see also [9,16]). The resulting 3d theory has the action

\[
S = \int d^3x \left[ \frac{1}{4} F_{ij} F_{ij} + |(\partial_i - iA_i) \phi|^2 + y |\phi|^2 + x |\phi|^4 \right],
\]

where \( F_{ij} = \partial_i A_j - \partial_j A_i \). Everything is expressed as dimensionless quantities, by scaling with appropriate powers of \( \epsilon_3^2 \), and the mass parameter \( y \) is renormalized in the \( \overline{\text{MS}} \) scheme with the scale \( \mu = 1 \).
Figure 1. The phase diagram of the 3d U(1)+Higgs theory. The vortex tension measurements in this paper are carried out in the region of continuous transitions \((x = 2)\) for \(-1.6 \leq y \leq 0.5\).

The phase diagram of the theory (see Fig. 1) has been studied with lattice simulations in Refs. [8,9]. On the lattice, there are two alternative formulations, the compact and the non-compact one, but they are expected to have the same continuum limit in 3d. It was shown in Ref. [17] that in the non-compact lattice formulation there are two phases, the Higgs and the Coulomb phase. There is no symmetry breaking and no local order parameter, but the mass of the photon acts as a non-local order parameter: it is zero in the Coulomb phase and non-zero in the Higgs phase. This differs from the compact lattice formulation at any finite lattice spacing, and from the SU(2)+Higgs model, in which cases the phases are analytically connected [3,18].

At the value \(x = 2\) that we will use in this paper, the transition between the two phases has been observed to be continuous, but quite different from standard symmetry breaking transitions [9]. At small \(x\), a first-order transition was found in accordance with perturbation theory [19].

The field equations derived from the action in Eq. (1) have Nielsen-Olesen vortices as solutions when \(y < 0\) [4]. The vortex tension can be calculated in the mean-field approximation by solving the field equations numerically. The
result can be written in the form
\[ T_{MF} = \frac{\Delta S}{L} = -\frac{y}{x} \pi \mathcal{E}(\sqrt{2x}), \]  
(2)

where the function \( \mathcal{E} \), with the value \( \mathcal{E}(1) = 1 \), has been calculated numerically in, e.g., Ref. [15]. From there we interpolate that \( \mathcal{E}(2) \approx 1.32 \), which value we use for comparison with our lattice results.

To study the tension \( T \) non-perturbatively in the quantum theory, we discretize the space and define the system on a lattice with lattice spacing \( a \). We use the non-compact formulation, i.e. the link field is a real number instead of a phase angle. The lattice action corresponding to the continuum theory in Eq. (1) is

\[
S = \beta_G \sum_{x,i<\hat{j}} \frac{1}{2} \alpha_{ij}^2(x) - \frac{2}{\beta_G} \sum_{x,i} \text{Re} \phi^*(x) U_i(x) \phi(x + i) \\
+ [...] \sum_x \phi^*(x) \phi(x) + \frac{x}{\beta_G^3} \sum_x \phi^*(x) \phi(x)^2, \tag{3}
\]

where \( \alpha_{ij}(x) = \alpha_i(x) + \alpha_j(x + \hat{i}) - \alpha_i(x + \hat{j}) - \alpha_j(x) \), \( U_i(x) = \exp[i \alpha_i(x)] \), \( x = (x_1, x_2, x_3) \) with \( 1 \leq x_i \leq N_i \),

\[
\beta_G = \frac{1}{a}, \tag{4}
\]

and the coefficient of the quadratic term is [7]

\[
[...] = \frac{1}{\beta_G} \left[ 6 + \frac{y}{\beta_G^2} - \frac{3.1759115(1 + 2x)}{2\pi \beta_G} \\
- \frac{(-4 + 8x - 8x^2)(\log 6\beta_G + 0.09) - 1.1 + 4.6x}{16\pi^2 \beta_G^2} \right]. \tag{5}
\]

Note how remarkably simple and analytic the lattice-continuum relation is.

To give a gauge-invariant definition for a vortex on a lattice, we define for each link [11,20] (We choose the opposite sign here!)

\[
Y_{(x,x+i)} = \alpha_i(x) - [\alpha_i(x) + \gamma(x + \hat{i}) - \gamma(x)]_\pi, \tag{6}
\]

where \( \gamma = \arg \phi \) and \( [X]_\pi = X + 2\pi n_X \) such that \( [X]_\pi \in (-\pi, \pi] \). For links in the opposite direction, \( Y_{(x,x-i)} = -Y_{(x-i,x)} \). For each closed curve \( C \) of links
For two vortices pass through one cell, the vortex tracing algorithm must
decide how to connect the vortices, and this leads to an ambiguity in the length
distribution of vortices.

on a lattice, we then define the winding number $n_C$ as

$$n_C = \frac{1}{2\pi} \sum_{l \in C} Y_l \in \mathbb{Z}. \quad (7)$$

In each configuration, the winding number gives the number of vortices passing
through the curve $C$. Note that we count vortices with multiple winding as
separate vortices on top of each other.

Recently, there have been many attempts to interpret the phase transition
in this and in related models as a percolation transition of vortices [12,13].
The percolation point $y_p$ is defined as the value of $y$ above which one can in
a typical configuration find a vortex that extends through the whole lattice.
However, there are ambiguities in this procedure if two or more vortices meet
in one cell (see Fig. 2), and it is not obvious that $y_p$ should coincide with $y_c$,
which is the transition point signalled by the non-analytic behavior in the free
energy. For example, in the Ising model, these two transitions agree at $d = 2$
[21], but $\beta_c \approx 0.95\beta_p$ at $d > 2$ [22].

The idea behind the percolation studies is that as the transition is approached
from below, the fluctuations can create larger and larger vortices, and at the
transition point, the typical size of a vortex loop diverges. In other words,
the free energy per unit length $T$ of a vortex decreases to zero. To avoid
the problems of the percolation approach, we have chosen here to measure $T$
directly. Since our definition of $T$ is based on the properties of the gauge field,
this approach cannot be used in globally symmetric theories.
3 Inserting a vortex and measuring its free energy

Free energies cannot be measured by Monte Carlo simulations, but changes of free energies can. To measure the free energy of a single vortex, we define a continuous vortex number (=winding number) \( m \), measure the change of the free energy when \( m \to m + dm \), \( 0 < m < 1 \), and integrate these changes of \( m \) from 0 to 1. The measurements can as well be done for \( m \) varying from 0 to any integer.

On the lattice, we call a field periodic, if it has the same value at \( x_i = N_i + 1 \) as at \( x_i = 1 \). If we require the fields \( \phi \) and \( \alpha \) to be periodic, i.e. use periodic boundary conditions, then for any planar cross section (say, in the \((x, y)\)-plane), the total flux \( \Phi = \sum_{\text{plane}} \alpha_{12} \) and the winding \( n_{\text{tot}} \) defined in Eq. (7), vanish in every configuration. The expectation value of an observable \( \mathcal{O}[\alpha, \phi] \) is then given by

\[
\langle \mathcal{O} \rangle = \int_{\text{periodic}} \mathcal{D}\alpha \mathcal{D}\phi \mathcal{O}[\alpha, \phi] e^{-S[\alpha, \phi]},
\]

with \( S[\alpha, \phi] \) given in Eq. (3).

We now define a non-periodic constant field \( \tilde{\alpha} \) such that \( \tilde{\alpha}_2(N_1 + 1, 1, x_3) = 2\pi \), and \( \tilde{\alpha}_i(x) = 0 \) otherwise. The magnetic flux through the system in the \( z \)-direction, calculated from the background \( \tilde{\alpha} \), is \( \tilde{\Phi} = \sum_{\text{boundary}} \tilde{\alpha}_i = 2\pi \). Note that there is no analogue to this in the compact formulation, where \( \alpha \) is only defined modulo \( 2\pi \).

Let us now replace \( \alpha \) by \( \alpha + m\tilde{\alpha} \) \((m \in \mathbb{Z})\) in the integrand of Eq. (8):

\[
\langle \mathcal{O} \rangle_m = \int_{\text{periodic}} \mathcal{D}\alpha \mathcal{D}\phi \mathcal{O}[\alpha + m\tilde{\alpha}, \phi] e^{-S[\alpha + m\tilde{\alpha}, \phi]}.
\]

Note that a change of variables \( \alpha + m\tilde{\alpha} \to \alpha \) transforms this to the form of Eq. (8) with non-periodic boundary conditions. Clearly, all the configurations in this integral have the total flux \( \Phi = 2\pi m \), and from Eqs. (6), (7) we notice that the total winding is \( n_{\text{tot}} = m \), i.e., the number of vortices going through the lattice is \( m \). It can be shown that any configuration in which \( \Phi = 2\pi m \) and \( n_{\text{tot}} = m \) and the physical quantities \( \phi^*\phi \), \( \phi^*(x)U_\ell(x)\phi(x + i) \) and \( \alpha_{ij} \) are periodic, can be gauge transformed to one that is included in the integral in Eq. (9). Therefore, Eq. (9) describes a periodic system in which the flux and
the winding number have been fixed\textsuperscript{7}.

Since $\tilde{U}_i(x) = 1$ always, and $\tilde{\alpha}_{ij}(x) = 0$ everywhere except $\tilde{\alpha}_{12}(N_1, 1, x_3) = 2\pi$, we see from Eq. (3) that

$$S[\alpha + m\tilde{\alpha}, \phi] = S[\alpha, \phi] + \beta_G \sum_{x_3} (2\pi m \alpha_{12}(N_1, 1, x_3) + 2\pi^2 m^2).$$

(10)

From this expression it appears as if we had effectively inserted $m$ vortices along the line $(N_1, 1, x_3), 1 \leq x_3 \leq N_3$. However, in Eq. (9) we integrate over all periodic field configurations, and thus all observables are translationally invariant. In fact, the precise form of $\tilde{\alpha}$ is irrelevant as long as its values are multiples of $2\pi$: only the total flux $\tilde{\Phi}$ affects the results.

Eq. (9) means that the free energy of a system with $m$ vortices can be written as

$$F_m = -\ln \int_{\text{periodic}} \mathcal{D}\alpha \mathcal{D}\phi e^{-S[\alpha + m\tilde{\alpha}, \phi]}. \quad (11)$$

Only when $m \in \mathbb{Z}$, are the action and the observables periodic, and boundary effects are avoided. This is the well-known flux quantization condition. Still, $F_m$ is mathematically well defined for any $m \in \mathbb{R}$, and the tension of the vortex can be written as (cf. Eq. (2))

$$T = \frac{F_1 - F_0}{aN_3} = \frac{1}{aN_3} \int_0^1 dm \frac{dF_m}{dm} = \frac{1}{aN_3} \int_0^1 dm \left( \frac{dS[\alpha + m\tilde{\alpha}, \phi]}{dm} \right)_m. \quad (12)$$

Substituting Eq. (10) to Eq. (12) yields

$$T = 2\pi^2 \beta_G^2 \int_0^1 dm W(m)$$

$$\equiv 2\pi^2 \beta_G^2 \int_0^1 dm \left[ \frac{1}{\pi N_3} \left( \sum_{x_3} \alpha_{12}(N_1, 1, x_3) \right)_m + 2m \right]. \quad (13)$$

It is the quantity $W(m)$ that we calculate with Monte Carlo simulations.

\textsuperscript{7} There are also periodic configurations in which $\Phi \neq 2\pi n_{\text{tot}}$, but their energy is assumed to diverge logarithmically with the area. Thus we neglect them here.
It is useful to inspect what kind of finite size effects one can expect for $T$. In the Coulomb phase, the massless photon gives rise to power-like finite size effects. Their magnitude can be estimated by considering the pure gauge theory, since the flux is expected to be distributed homogeneously in the whole system. In the absence of scalar fields, the flux quantization condition $m \in \mathbb{Z}$ does not apply, and the system is translationally invariant at all $m \in \mathbb{R}$. Thus

$$\langle \alpha_{12}(N_1, 1, x_3) \rangle_m = \frac{2\pi m}{N_1N_2} - m\tilde{\alpha}_{12}(N_1, 1, x_3) = 2\pi m \left( \frac{1}{N_1N_2} - 1 \right),$$

which yields

$$T_0 = 2\pi^2 \beta_G^2 \int_0^1 dm \frac{2m}{N_1N_2} = \frac{2\pi^2 \beta_G^2}{N_1N_2}.$$  

In the Higgs phase, all the fields are massive, and the finite size effects are exponentially suppressed.

4 The relation of vortex tension to thermodynamics

The behaviour of the vortex tension $T$ can be directly related to possible phase transitions in the system. This is simply because a vortex carries a magnetic flux, and a magnetic field contributes to the free energy. Let us discuss this in some more detail.

To begin with, note that a system with winding $m$ corresponds to the magnetic flux density $B_z = 2\pi m \beta_G^2/N_1N_2$. Thus, the free energy $F(B_z)$ of a system through which goes a non-vanishing magnetic flux, equals $F_m$ in Eq. (11), with $m = a^2 N_1N_2 B_z / 2\pi$. For symmetry reasons, $F(B_z) = F(|B_z|)$. In the thermodynamical limit, $B_z$ can have any real values. Hence the derivative of $F(B_z)$ becomes meaningful, and we can calculate it from the vortex tension, if we assume a repulsive interaction between the vortices, as is observed at large $x$. A system with a finite number $m$ of vortices becomes namely infinitely dilute as the volume increases, yielding $F_m = m F_1$, and thus (we denote $f(B_z) = V^{-1} F(B_z)$, the free energy density at a fixed flux density $B_z$),

$$\left. \frac{\partial f(B_z)}{\partial |B_z|} \right|_{B_z=0} = \frac{1}{V} \lim_{|B_z| \to 0} \frac{F(B_z) - F(0)}{|B_z|} = \frac{a^2 N_1N_2}{2\pi V} \left( F_1 - F_0 \right) = \frac{T}{2\pi},$$

$$\left. \frac{\partial f(B_z)}{\partial |B_z|} \right|_{B_z=0} = \frac{1}{V} \lim_{|B_z| \to 0} \frac{F(B_z) - F(0)}{|B_z|} = \frac{a^2 N_1N_2}{2\pi V} \left( F_1 - F_0 \right) = \frac{T}{2\pi},$$

$$\left. \frac{\partial f(B_z)}{\partial |B_z|} \right|_{B_z=0} = \frac{1}{V} \lim_{|B_z| \to 0} \frac{F(B_z) - F(0)}{|B_z|} = \frac{a^2 N_1N_2}{2\pi V} \left( F_1 - F_0 \right) = \frac{T}{2\pi},$$

$$\left. \frac{\partial f(B_z)}{\partial |B_z|} \right|_{B_z=0} = \frac{1}{V} \lim_{|B_z| \to 0} \frac{F(B_z) - F(0)}{|B_z|} = \frac{a^2 N_1N_2}{2\pi V} \left( F_1 - F_0 \right) = \frac{T}{2\pi},$$

$$\left. \frac{\partial f(B_z)}{\partial |B_z|} \right|_{B_z=0} = \frac{1}{V} \lim_{|B_z| \to 0} \frac{F(B_z) - F(0)}{|B_z|} = \frac{a^2 N_1N_2}{2\pi V} \left( F_1 - F_0 \right) = \frac{T}{2\pi},$$
Figure 3. If \( T > 0 \), there is a cusp in the free energy density \( f(B_z) \) at \( B_z = 0 \). Thus the function \( g(H_z) \), defined in Eq. (17), has a flat region at \(-H_c \leq H_z \leq H_c\).

where \( V = a^3 N_1 N_2 N_3 \) is the volume of the system.

Let us denote by \( g(H_z) \) the free energy density of a system in which the flux is allowed to fluctuate and an external field \( H_z \) is applied:

\[
g(H_z) = -\frac{1}{V} \ln \int_{-\infty}^{\infty} dB_z e^{-V(f(B_z) - H_z B_z)} \xrightarrow{V \to \infty} \min_{B_z} (f(B_z) - H_z B_z). \tag{17}
\]

In the thermodynamical limit, \( g(H_z) \) is thus given by the Legendre transform of \( f(B_z) \).

If \( T > 0 \), the derivative of \( f(B_z) \) is discontinuous at \( B_z = 0 \) (see Fig. 3). Its Legendre transform \( g(H_z) \) will therefore have a flat region, i.e. \( g(H_z) = g(0) \) when \(|H_z| \leq H_c \equiv T/2\pi\). This is precisely the Meissner effect, since \( \langle B_z \rangle = -\partial g/\partial H_z = 0 \), if \(|H_z| \leq H_c \). It also implies that

\[
\frac{\partial^2 g}{\partial H_z^2} \bigg|_{H_z=0} = 0, \quad \text{when } T > 0, \quad \text{and} \quad \frac{\partial^2 g}{\partial H_z^2} \bigg|_{H_z=0} > 0, \quad \text{when } T = 0. \tag{18}
\]

Thus, the values of \( T \) are related to the analytic structure (in particular, phase transitions) of the free energy.

At \( x = 2 \), we expect \( T \) to behave continuously. It is non-zero in the Higgs phase, but decreases with increasing \( y \), approaching zero at some value \( y = y_c \).
larger values of $y$, the tension stays zero. If this turns out to be true, Eq. (18) shows that the free energy density $g$ is non-analytic at $y = y_c$.

5 Simulations and results

The Monte Carlo simulations of the theory in Eq. (3) at zero and non-zero values of $m$ have been performed at the parameter value $x = 2$, at $\beta_G = 4$. For this value of $\beta_G$, the correlation lengths of massive modes like the scalar or the photon in the Higgs phase are large in units of the lattice spacing $a$. Indeed, for the values of $y$ considered here, $-1.6 \leq y \leq 0.5$, even the shortest correlation length, which is the scalar one, is always larger than about $3a$. Thus finite $a$ corrections to the measured quantities are expected to be small.

The major part of the numerical work has been performed with the help of Hybrid Monte Carlo (HMC) update algorithms, quite similar to those used in lattice QCD simulations. HMC updates were implemented separately for the gauge fields and for the scalar fields. Within both HMC algorithms a leapfrog discretization based upon 8 time steps into the molecular dynamics time direction was chosen. Acceptance rates were appropriately tuned. We measure our statistics in units of sweeps, where each sweep consists of two completed trajectories, one in the gauge and one in the scalar fields. The use of HMC updating schemes is favored by the fact that the usual checkerboard decomposition of degrees of freedom can be avoided and, therefore, the adaptation of the computer code to a parallel architecture is simplified. Most of our simulations have been executed on parallel computers. Some simulations have also been repeated on work stations with the use of different random number generation algorithms and therefore different random number sequences. These runs serve as a valuable cross-check, and no inconsistencies were detected.

We also experimented with over-relaxation updates of the gauge fields alone. As we failed to achieve a significant improvement in the statistical quality of the simulation, we will not report on the details of that approach here.

We simulate symmetric cubic lattices with volumes $N^3$. Typical lattice sizes range from $N = 16$ to $N = 32$, which currently represents our largest system. The expectation value $W(m)$ in Eq. (13) is evaluated by means of Monte Carlo simulations on a discrete set of 21 values $m_i$ partitioning the interval $m = [0, 1]$ into 20 equally spaced subintervals. The integral of Eq. (13) was then calculated by the trapezoidal rule and it was checked that systematic errors introduced by the discretization are much smaller than statistical errors. A
typical Monte Carlo result for the quantity $W(m)$ is displayed in Fig. 4 for the $32^3$ lattice at $y = -1$. Statistical errors on each individual $m_i$ measurement can be as large as 40%, as is expected for an energy difference calculation on such large systems. We employed between 10000 and 20000 Monte Carlo sweeps on each of the individual $m_i$ measurements. Statistical errors for the vortex tension $T$ were then calculated using jackknife error calculation and statistical error propagation.

In Fig. 5 we display our finite-volume data for the vortex tension $T$ as a function of $y$ in the whole interval of $y$ considered on a selected set of $16^3$, $20^3$, $24^3$ and $32^3$ lattices. One identifies three different regions. For $y \leq -1$ (Higgs phase), the vortex tension $T$ appears to be finite and finite size effects appear to be under control. Note that at $y = -1$, data from several lattice sizes cluster around a common mean value. For values of $y$ larger than about $y = 0$ (Coulomb phase), $T$ is non-vanishing in a finite system. However, it shows large finite size effects, which appear to be rather independent of $y$. There is also a region of intermediate values of $y$ in which finite size effects are present and presumably not of a simple form. In this paper we will concentrate on those values of $y$ that are clearly located in the Higgs or the Coulomb phase. A study of the intermediate $y$ region would require finite size scaling methods beyond the scope of the present paper.
Figure 5. Lattice results for $T$ as a function of $y$ on $16^3$, $20^3$, $24^3$ and $32^3$ lattices at $x = 2$ and $\beta_G = 4$. The solid straight line is the mean-field result. Actually, only its slope is unambiguously determined.

For two values of $y$, $y = 0$ and $y = 0.5$, we investigated in detail the possible existence of a power-like finite size behavior $T \propto N^{-2}$, as is characteristic of the pure gauge theory (see Eq. (15)). The observation of such a finite size behavior at the given values of $y$ within the Coulomb phase signals the homogeneity of the system under the response of an applied flux. It is a strong argument in favor of the existence of a massless mode or alternatively, a vanishing vortex tension $T$. In Fig. 6 we display our finite $N$ data for the vortex tension $T$. In Fig. 6 we display our finite $N$ data for the vortex tension $T$ as a function of $N^{-2}$ at $y = 0$ and $y = 0.5$. The lattice sizes considered are $N^3 = 16^3, 18^3, 20^3, 22^3, 24^3, 26^3, 28^3$ and $32^3$ at $y = 0$. The same lattices except $N^3 = 26^3$ have been considered at $y = 0.5$.

Both data sets have been independently fitted with the two-parameter form

$$T(N) = R_0 \frac{2\pi^2 \beta_G^2}{N^2} + T(N = \infty).$$

(19)

Deviations of the parameter $R_0$ from unity measure deviations from a behavior anticipated in pure gauge theory, and the parameter $T(N = \infty)$ corresponds to our infinite volume extrapolation of the vortex tension $T$. The fit to all the
Figure 6. Finite size scaling data for $T$ at $x = 2$, $\beta_G = 4$, $y = 0.0$ (left) and $y = 0.5$ (right), as a function of $N^{-2}$. The solid line corresponds to a straight line fit as described in the text. The triangle corresponds to the infinite volume extrapolation.

data, as indicated by the solid straight lines in Fig. 6, gives

$$R_0 = 1.06(7), \quad T(N = \infty) = -0.019(32) \quad \text{at} \quad y = 0,$$

$$R_0 = 0.97(6), \quad T(N = \infty) = +0.015(41) \quad \text{at} \quad y = 0.5. \quad (20)$$

We notice that $R_0$ agrees with unity within errorbars. It thus appears that the possible renormalization of the coefficient in front of the $N^{-2}$ behavior induced by the presence of the scalar field is negligible. Both fits exhibit reasonably small $\chi^2_{\text{dof}}$ values, namely $\chi^2_{\text{dof}} = 1.5$ at $y = 0$ and $\chi^2_{\text{dof}} = 1.4$ at $y = 0.5$. Nevertheless, it can be noted that there are some data points which scatter. Simulations in the Coulomb phase turn out to be statistically rather demanding. From Fig. 7 we observe that the Monte Carlo simulation is better behaved (all errorbars cross the central value) in the Higgs phase, which is natural since there is no massless mode there.

One may alternatively fix the parameter value $T(N = \infty)$ to zero in Eq. (19), and attempt to describe the data with

$$T(N) = \tilde{R}_0 \frac{2\pi^2 \beta_G^2}{N^\eta}, \quad (22)$$
where now any deviation of the parameter $\eta$ from the value 2 signals departure from homogeneity caused by the presence of a vortex. We find

\begin{align}
\eta &= 1.94(14) \quad \text{at} \quad y = 0, \quad (23) \\
\eta &= 1.93(12) \quad \text{at} \quad y = 0.5, \quad (24)
\end{align}

and comparable values of the parameter $\tilde{R}_0$ as in the previous case. These values clearly agree with the theoretical predictions for a Coulomb phase.

In the Higgs phase, at $y = -1.0$, the vortex tension $T$ was determined on $16^3$, $20^3$, $24^3$ and $32^3$ lattices, see Fig. 7. Within the statistical errors the data are constant. A fit of the data to a constant value has $\chi^2_{\text{dof}} = 0.3$ and gives

\[ T(N = \infty) = 2.51(9) \quad \text{at} \quad y = -1. \quad (25) \]

Comparing this value with the mean-field result $T_{\text{MF}} = 2.07$ from Eq. (2), we observe a 17% difference, i.e., the mean-field result is close to the measured value. One source for the discrepancy is that our result was measured at a finite lattice spacing $a$, while the mean-field calculation was performed in the continuum. One should also note that actually the mean-field calculation only predicts the slope $dT/dy$ (see Fig. 5). Namely, the choice of the renormalization scale $\mu$ is arbitrary at the mean-field level. If we chose some other scale, the value of $y$ would change by a constant term, $y(\mu) = y(1) + (4 - 8x + 8x^2)/(16\pi^2) \ln \mu$. In fact, changing from $\mu = 1$ to $\mu = 0.2$ would bring the results in perfect numerical agreement. To remove this ambiguity, corrections to the mean-field result would have to be computed up to 2-loop level.

Finally, we point out that since the infinite volume extrapolation of the vortex free energy at the values $y = 0.0$ and $y = 0.5$ is fully compatible with a vanishing vortex tension, and at $y = -1$ with a non-vanishing value, the vortex tension indeed behaves as an order parameter.

6 Conclusions

We have shown by numerical simulations that the vortex tension $T$ defined in Eq. (12) acts as an order parameter at large values of $x$, where the transition between the Higgs and the Coulomb phases is continuous. The point $y_c$, at which $T$ vanishes, coincides with the true transition point, since the free energy density $g$ was shown to be non-analytic at $y_c$. 
Figure 7. Finite size scaling data for $T$ at $y = -1$, $x = 2$ and $\beta_G = 4$, as a function of $N$. There are no visible finite size effects. The horizontal solid line corresponds to the infinite volume extrapolation $T = 2.51(9)$, and the dashed lines indicate the error interval.

At $y < -1$, the measured values of $T$ agree with the mean-field result in Eq. (2) within errorbars, when the ambiguity of the renormalization scale in the mean-field approximation is taken into account. At $y > 0$, the system was observed to behave very much like pure gauge theory, i.e. the change due to the presence of a scalar field was very small. Outside the vicinity of $y_c$, mean-field approximation thus seems to describe vortices well in both phases of the theory even at $x = 2$, where fluctuations are large. However, we expect that at $y \approx y_c$, there will be large deviations from the mean-field behavior. The study of this interesting region requires significantly more computational resources than used here.

We performed the simulations at a finite value of the lattice spacing $a$. Although we believe that the vortex tension $T$ has a well-defined continuum limit, we expect that there are $O(a)$ errors, in particular in the Higgs phase. However, as of now, we have not removed them by a continuum extrapolation $a \to 0$.

In cosmology, the time evolution of the vortex network after a phase transition is typically treated at the mean-field level [23]. Our results give confidence
to that approximation, at least deep in the Higgs phase. However, to see if modifications are needed to the picture of defect formation in gauge theories, a more detailed study is needed near the transition temperature to find out the role of the fluctuations. For a complete picture of cosmic string formation, one would also have to take the non-equilibrium effects into account.

Another interesting way to extend this first-principles numerical study would be to consider configurations with multiple vortices. In the context of superconductors, it has been suggested and also numerically verified in simplified models that the interactions between vortices give rise to new phases. It is then theoretically interesting and computationally demanding to explore, how the non-analytic structure in the multiple vortex case is related to the phase diagram at a vanishing magnetic flux density, Fig. 1.

Acknowledgements

Most of the simulations were carried out with a Cray T3E at the Center for Scientific Computing, Finland. This work was partly supported by the TMR network *Finite Temperature Phase Transitions in Particle Physics*, EU contract no. FMRX-CT97-0122.

References

[1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other Topological Defects* (Cambridge University Press, Cambridge, 1994); M.B. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. 58 (1995) 477.

[2] S. Elitzur, Phys. Rev. D 12 (1975) 3978; G.F. de Angelis, D. de Falco and F. Guerra, Phys. Rev. D 17 (1978) 1624.

[3] T. Banks and E. Rabinovici, Nucl. Phys. B 160 (1979) 349; E. Fradkin and S.H. Shenker, Phys. Rev. D 19 (1979) 3682.

[4] H.B. Nielsen and P. Olesen, Nucl. Phys. B 61 (1973) 45.

[5] K. Farakos, K. Kajantie, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B 425 (1994) 67 [hep-ph/9404201].

[6] H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989).

[7] M. Laine and A. Rajantie, Nucl. Phys. B 513 (1998) 471 [hep-lat/9705003].
[8] P. Dimopoulos, K. Farakos and G. Koutsoumbas, Eur. Phys. J. C 1 (1998) 711 [hep-lat/9703004].

[9] K. Kajantie, M. Karjalainen, M. Laine and J. Peisa, Phys. Rev. B 57 (1998) 3011 [cond-mat/9704056]; Nucl. Phys. B 520 (1998) 345 [hep-lat/9711048].

[10] A. Kovner, P. Kurzepa and B. Rosenstein, Mod. Phys. Lett. A 8 (1993) 1343; ibid. A 8 (1993) 2527 (Erratum).

[11] A. Rajantie, Physica B, in press [cond-mat/9803221]; K. Kajantie, M. Karjalainen, M. Laine, J. Peisa and A. Rajantie, Phys. Lett. B 428 (1998) 334 [hep-ph/9803367].

[12] A.V. Pochinsky, M.I. Polikarpov and B.N. Yurchenko, Phys. Lett. A 154 (1991) 194; A. Hulsebos, hep-lat/9406016; N.D. Antunes, L.M.A. Bettencourt and M. Hindmarsh, Phys. Rev. Lett. 80 (1998) 908 [hep-ph/9708215]; N.D. Antunes and L.M.A. Bettencourt, hep-ph/9807248.

[13] M.N. Chernodub, F.V. Gubarev, E.-M. Ilgenfritz and A. Schiller, hep-lat/9805016; hep-lat/9807016.

[14] M.B. Isichenko, Rev. Mod. Phys. 64 (1992) 961.

[15] L. Jacobs and C. Rebbi, Phys. Rev. B 19 (1979) 4486.

[16] J.O. Andersen, hep-ph/9709418.

[17] C. Borgs and F. Nill, J. Stat. Phys. 47 (1987) 877.

[18] K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B 466 (1996) 189 [hep-lat/9510020]; Phys. Rev. Lett. 77 (1996) 2887 [hep-ph/9605288].

[19] S. Coleman and E. Weinberg, Phys. Rev. Lett. 32 (1974) 292.

[20] J. Ranft, J. Kripfganz and G. Ranft, Phys. Rev. D 28 (1983) 360; M.N. Chernodub, M.I. Polikarpov and M.A. Zubkov, Nucl. Phys. B (Proc. Suppl.) 34 (1994) 256 [hep-lat/9401027]; M. Chavel, Phys. Lett. B 378 (1996) 227 [hep-lat/9603005].

[21] A. Coniglio, U. de Angelis, A. Forlani and G. Lauro, J. Phys. A 10 (1977) 219.

[22] H. Müller-Krumbhaar, Phys. Lett. A 50 (1974) 27.

[23] G. Vincent, N.D. Antunes and M. Hindmarsh, Phys. Rev. Lett. 80 (1998) 2277 [hep-ph/9708427].