SMALL DATA GLOBAL EXISTENCE FOR THE SEMILINEAR
WAVE EQUATION WITH SPACE-TIME DEPENDENT DAMPING

YUTA WAKASUGI

ABSTRACT. In this paper we consider the critical exponent problem for the
semilinear wave equation with space-time dependent damping. When the
damping is effective, it is expected that the critical exponent agrees with that
of only space dependent coefficient case. We shall prove that there exists a
unique global solution for small data if the power of nonlinearity is larger than
the expected exponent. Moreover, we do not assume that the data are comp-
pactly supported. However, it is still open whether there exists a blow-up
solution if the power of nonlinearity is smaller than the expected exponent.

1. INTRODUCTION

We consider the Cauchy problem for the semilinear damped wave equation

\[
\begin{aligned}
\begin{cases}
  u_{tt} - \Delta u + a(x)b(t)u_t = f(u), & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\
  u(0,x) = u_0(x), & u_t(0,x) = u_1(x), 
\end{cases}
\end{aligned}
\]

where the coefficients of damping are

\[ a(x) = a_0 \langle x \rangle^{-\alpha}, \quad b(t) = (1 + t)^{-\beta}, \]

where \( \langle x \rangle = (1 + |x|^2)^{1/2} \). Here \( u \) is a real-valued unknown function and \( (u_0, u_1) \) is
in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). We note that \( u_0 \) and \( u_1 \) need not be compactly supported. The nonlinear term \( f(u) \) is given by

\[ f(u) = \pm |u|^p \quad \text{or} \quad |u|^{p-1}u \]

and the power \( p \) satisfies

\[ 1 < p \leq \frac{n}{n-2} \quad (n \geq 3), \quad 1 < p < \infty \quad (n = 1, 2). \]

Our aim is to determine the critical exponent \( p_c \), which is a number defined by the following property:

If \( p < p_c \), all small data solutions of (1.1) are global; if \( 1 < p \leq p_c \), the time-local solution cannot be extended time-globally for some data.

It is expected that the critical exponent of (1.1) is given by

\[ p_c = 1 + \frac{2}{n - \alpha}. \]

In this paper we shall prove the existence of global solutions with small data when \( p > 1 + 2/(n - \alpha) \). However, it is still open whether there exists a blow-up solution when \( 1 < p \leq 1 + 2/(n - \alpha) \).

Key words and phrases. semilinear damped wave equations; critical exponent; small data global existence.
When the damping term is missing and \( f(u) = |u|^p \), that is
\[
\begin{align*}
  u_{tt} - \Delta u &= |u|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]
(1.2)
it is well known that the critical exponent \( p_w(n) \) is the positive root of \((n-1)p^2-(n+1)p-2 = 0\) for \( n \geq 2 \) \((p_w(1) = \infty)\). This is the famous Strauss conjecture and the proof is completed by the effort of many mathematicians (see \cite{[2],[3],[13],[21],[30],[34],[41]}).

For the linear wave equation with a damping term
\[
\begin{align*}
  u_{tt} - \Delta u + c(t, x)u_t &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]
(1.3)
there are many results about the asymptotic behavior of the solution. When \( c(t, x) = c_0 > 0 \) and \((u_0, u_1) \in (H^1 \cap L^1) \times (L^2 \cap L^1)\), Matsumura \cite{[22]} showed that the energy of solutions decays at the same rate as the corresponding heat equation. When the space dimension is 3, using the exact expression of the solution, Nishihara \cite{[24]} discovered that the solution of (1.3) with \( c(t, x) = 1 \) is expressed asymptotically by
\[
u(t, x) \sim v(t, x) + e^{-t/2}w(t, x),
\]
where \( v(t, x) \) is the solution of the corresponding heat equation
\[
\begin{align*}
  v_t - \Delta v &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \\
  v(0, x) &= u_0(x) + u_1(x), \quad x \in \mathbb{R}^3
\end{align*}
\]
and \( w(t, x) \) is the solution of the free wave equation
\[
\begin{align*}
  w_{tt} - \Delta w &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \\
  w(0, x) &= u_0(x), \quad w_t(0, x) = u_1(x), \quad x \in \mathbb{R}^3.
\end{align*}
\]
These results indicate a diffusive structure of damped wave equations. On the other hand, Mochizuki \cite{[23]} showed that if \( 0 \leq c(t, x) \leq C(1 + |x|)^{-1-\delta} \), where \( \delta > 0 \), then the energy of solutions of (1.3) does not decay to 0 for nonzero data and the solution is asymptotically free. We can interpret this result as (1.3) loses its “parabolicity” and recover its “hyperbolicity”. Wirth \cite{[35],[36]} treated time-dependent damping case, that is \( c(t, x) = b(t) \) in (1.3). By the Fourier transform method, he got several sharp \( L^p - L^q \) estimates of the solution and showed that there exists diffusive structure for general \( b(t) \) including \( b(t) = b_0(1 + t)^{-\beta}(-1 < \beta < 1) \). Todorova and Yordanov \cite{[37]} considered the case \( c(t, x) = a(x) = a_0(x)^{-\alpha} \) with \( \alpha \in [0, 1) \) and J. S. Kenigson and J. J. Kenigson \cite{[16]} considered space-time dependent coefficient case \( c(t, x) = a(x)b(t), a(x) = a_0(x)^{-\alpha}, b(t) = (1 + t)^{-\beta}, (0 \leq \alpha + \beta < 1) \). They established the energy decay estimate that also implies diffusive structure even in the decaying coefficient cases. From these results, the decay rate \(-1\) of the coefficient of the damping term is the threshold of parabolicity. This is the reason why we assume \( \alpha + \beta < 1 \) for (1.1). We mention that recently, Ikehata, Todorova and Yordanov \cite{[12]} treated the case \( c(t, x) = a_0(x)^{-1} \) and obtained almost optimal decay estimates.

There are also many results for the semilinear damped wave equation with absorbing semilinear term:
\[
\begin{align*}
  u_{tt} - \Delta u + a(x)b(t)u_t + |u|^{p-1}u &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]
(1.4)
It is well known that there exists a unique global solution even for large initial data. When \( a(x)b(t) = 1 \), that is constant coefficient case, Kawashima, Nakao and Ono \[15\], Karch \[14\], Hayashi, Kaikina and Naumkin \[7\], Ikehata, Nishihara and Zhao \[9\] and Nishihara \[25\] showed global existence of solutions and that their asymptotic profile is given by a constant multiple of the Gauss kernel for \( 1+2/n < p \) and \( n \leq 4 \). For \( 1 < p \leq 1+2/n \), Nishihara and Zhao \[28\], Ikehata, Nishihara and Zhao \[9\], Nishihara \[25\] proved that the decay rate of the solution agrees with that of a self-similar solution of the corresponding heat equation. Hayashi, Kaikina and Naumkin \[17\] proved the large time asymptotic formulas in terms of the weighted Sobolev spaces. These results indicate the critical exponent for (1.4) with \( a(x)b(t) = 1 \) is given by \( p_c = 1 + \frac{2}{n} \). In this case the critical exponent means the turning point of the asymptotic behavior of the solution. When \( b(t) = 1 \), namely space-dependent damping case, Nishihara \[26\] established decay estimates of solutions and conjectured the critical exponent is given by \( p_c = 1 + 2/(n - \alpha) \).

Li and Zhou \[17\] considered the semilinear damped wave equation

\[
u_{tt} - \Delta u + u_t = |u|^p.
\]

They proved that if \( n \leq 2, 1 < p \leq 1 + \frac{2}{n} \) and the data are positive on average, then the local solution of (1.5) must blow up in a finite time. Todorova and Yordanov \[35, 36\] developed a weighted energy method using the function which has the form \( e^{2\psi} \) and determined that the critical exponent of (1.5) is

\[ p_c = 1 + \frac{2}{n} \]

which is well known as Fujita’s critical exponent for the heat equation \( u_t - \Delta u = u^p \) (see \[1\]). More precisely, they proved small data global existence in the case \( p > 1 + 2/n \) and blow-up for all solutions of (1.5) with positive on average data in the case \( 1 < p < 1 + 2/n \). Later on Zhang \[40\] showed that the critical exponent \( p = 1+2/n \) belongs to the blow-up region. We mention that Todorova and Yordanov \[35, 36\] assumed data have compact support and essentially used this property. However, Ikehata and Tanizawa \[10\] removed this assumption. Ikehata, Todorova and Yordanov \[11\] investigated the space-dependent coefficient case:

\[
u_{tt} - \Delta u + a(x)u_t = |u|^p,
\]

where

\[ a(x) \sim a_0 (x)^{-\alpha}, |x| \to \infty, \] radially symmetric and \( 0 \leq \alpha < 1 \).

They proved that the critical exponent of (1.5) is given by

\[ p_c = 1 + \frac{2}{n - \alpha} \]

by using a refined multiplier method. Their method also depends on the finite propagation speed property. Recently, Nishihara \[27\] and Lin, Nishihara and Zhai...
considered the semilinear wave equation with time-dependent damping
\[ u_{tt} - \Delta u + b(t)u_t = |u|^p, \tag{1.7} \]
where \[ b(t) = b_0(1 + t)^{-\beta}, \quad \beta \in (-1, 1). \]
They proved that the critical exponent of (1.7) is
\[ p_c = 1 + \frac{2}{n}. \]
This shows that, roughly speaking, time-dependent coefficients of damping term do not influence the critical exponent. Therefore we expect that the critical exponent of the semilinear wave equation (1.1) is
\[ p_c = 1 + \frac{2}{n - \alpha}. \]
To state our results, we introduce an auxiliary function
\[ \psi(t, x) := A \langle x \rangle^{2-\alpha} \frac{|u(t,x)|^2}{(1 + t)^{1+\beta}} \tag{1.8} \]
with
\[ A = \frac{(1 + \beta)a_0}{(2 - \alpha)(2 + \delta)^2}, \quad \delta > 0 \tag{1.9} \]
This type of weight function is first introduced by Ikehata and Tanizawa [10]. We have the following result:

**Theorem 1.1.** If
\[ p > 1 + \frac{2}{n - \alpha}, \]
then there exists a small positive number \( \delta_0 > 0 \) such that for any \( 0 < \delta \leq \delta_0 \) the following holds: If
\[ I_0^2 := \int_{\mathbb{R}^n} e^{2\psi(0,x)} (u_t^2 + |\nabla u_0|^2 + |u_0|^2) dx \]
is sufficiently small, then there exists a unique solution \( u \in C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n)) \) to (1.1) satisfying
\[ \int_{\mathbb{R}^n} e^{2\psi(t,x)} |u(t,x)|^2 dx \leq C_\delta (1 + t)^{-\frac{2(1+\beta)}{2-\alpha} - \frac{1}{2} + \varepsilon}, \tag{1.10} \]
\[ \int_{\mathbb{R}^n} e^{2\psi(t,x)} (|u_t(t,x)|^2 + |\nabla u(t,x)|^2) dx \leq C_\delta (1 + t)^{-\frac{2(1+\beta)}{2-\alpha} + 1} + \varepsilon, \]
where
\[ \varepsilon = \varepsilon(\delta) := \frac{3(1 + \beta)(n - \alpha)}{2(2 - \alpha)(2 + \delta)} \tag{1.11} \]
and \( C_\delta \) is a constant depending on \( \delta \).

**Remark 1.2.** When \( 1 < p \leq 1 + 2/(n - \alpha) \), it is expected that no matter how small the data are, if the data have some shape, then the corresponding local solution blows up in finite time. However, we have no result.

**Remark 1.3.** We do not assume that the data are compactly supported. Hence our result is an extension of the results of Ikehata, Todorova and Yordanov [11] to noncompactly supported data cases. However, we prove only the case \( a(x) = a_0 \langle x \rangle^{-\alpha} \).
As a consequence of the main theorem, we have an exponential decay estimate outside a parabolic region.

**Corollary 1.4.** If

\[ p > 1 + \frac{2}{n-\alpha}, \]

then there exists a small positive number \( \delta_0 > 0 \) such that for any \( 0 < \delta \leq \delta_0 \) the following holds: Take \( \rho \) and \( \mu \) so small that

\[ 0 < \rho < 1 - \alpha - \beta, \quad \text{and} \quad 0 < \mu < 2A, \]

and put

\[ \Omega_{\rho}(t) := \{ x \in \mathbb{R}^n; \langle x \rangle^{2-\alpha} \geq (1+t)^{1+\beta+\rho} \}. \]

Then, for the global solution \( u \) in Theorem 1.1, we have the following estimate

\[
\int_{\Omega_{\rho}(t)} \left( u_t^2 + |\nabla u|^2 + u^2 \right) dx \leq C_{\delta, \rho, \mu} (1+t)^{(1+\beta)(n-2\alpha)/2\alpha + \varepsilon} e^{-2A(1-\mu)(1+t)^\rho},
\]

(1.12)

here \( \varepsilon \) is defined by (1.11) and \( C_{\delta, \rho, \mu} \) is a constant depending on \( \delta, \rho \) and \( \mu \).

Namely, the decay rate of solution in the region \( \Omega_{\rho}(t) \) is exponential. We note that the support of \( u(t) \) and the region \( \Omega_{\rho}(t) \) can intersect even if the data are compactly supported. This phenomenon was first discovered by Todorova and Yordanov [36]. We can interpret this result as follows: The support of the solution is strongly suppressed by damping, so that the solution is concentrated in the parabolic region much smaller than the light cone.

2. **Proof of Theorem 1.1**

In this section we prove our main result. At first we prepare some notation and terminology. We put

\[ \| f \|_{L^p(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}, \quad \| u \| := \| u \|_{L^2(\mathbb{R}^n)}. \]

By \( H^1(\mathbb{R}^n) \) we denote the usual Sobolev space. For an interval \( I \) and a Banach space \( X \), we define \( C^r(I;X) \) as the Banach space whose element is an \( r \)-times continuously differentiable mapping from \( I \) to \( X \) with respect to the topology in \( X \). The letter \( C \) indicates the generic constant, which may change from line to the next line.

To prove Theorem 1.1 we use a weighted energy method which was originally developed by Todorova and Yordanov [35][36]. We first describe the local existence:

**Proposition 2.1.** For any \( \delta > 0 \), there exists \( T_m \in (0, +\infty] \) depending on \( I_0^2 \) such that the Cauchy problem (1.1) has a unique solution \( u \in C([0, T_m); H^1(\mathbb{R}^n)) \cap C^1([0, T_m); L^2(\mathbb{R}^n)) \), and if \( T_m \) < +\( \infty \) then we have

\[
\liminf_{t \to T_m} \int_{\mathbb{R}^n} e^{\psi(t,x)} (u_t^2 + |\nabla u|^2 + u^2) dx = +\infty.
\]

We can prove this proposition by standard arguments (see [10]). We prove a priori estimate for the following functional:

\[
M(t) := \sup_{0 \leq \tau < t} \left\{ (1+\tau)^{B+1-\varepsilon} \int_{\mathbb{R}^n} e^{2\psi(u_t^2 + |\nabla u|^2)} dx + (1+\tau)^{B-\varepsilon} \int_{\mathbb{R}^n} e^{\psi a(x)b(t)u^2} dx \right\},
\]

(2.1)
where

\[ B := \frac{(1 + \beta)(n - \alpha)}{2 - \alpha} + \beta \]

and \( \varepsilon \) is given by (1.11). From (1.8), (1.9), it is easy to see that

\[ -\psi_t = 1 + \frac{\beta}{1 + t} \psi, \quad (2.2) \]

\[ \nabla \psi = A \left( \frac{2 - \alpha}{(1 + t)^{1+\beta}} \langle x \rangle - \alpha x \right) (1 + t) 1 + \frac{\beta}{1 + t}, \quad (2.3) \]

\[ \Delta \psi = A(2 - \alpha)(n - \alpha) \left( \frac{\langle x \rangle - \alpha}{(1 + t)^{1+\beta}} + A(2 - \alpha) \frac{\langle x \rangle - 2 - \alpha}{(1 + t)^{1+\beta}} \right) \]

\[ \geq \frac{(1 + \beta)(n - \alpha) a(x) b(t)}{(2 - \alpha)(2 + \delta)} \frac{1 + t}{1 + t} \]

\[ =: \left( \frac{(1 + \beta)(n - \alpha)}{2(2 - \alpha)} - \delta_1 \right) \frac{a(x) b(t)}{1 + t}. \quad (2.4) \]

Here and after, \( \delta_i (i = 1, 2, \ldots) \) is a positive constant depending only on \( \delta \) such that \( \delta_i \to 0^+ \) as \( \delta \to 0^+ \).

We also have

\[ (-\psi_t a(x) b(t) = Aa_0(1 + \beta) \frac{\langle x \rangle^{2-2\alpha}}{(1 + t)^{2+2\alpha}} \]

\[ \geq \frac{a_0(1 + \beta)}{(2 - \alpha)^2 A^2 (2 - \alpha)^2} \frac{\langle x \rangle^{-2\alpha} |x|^2}{(1 + t)^{2+2\beta}} \]

\[ = (2 + \delta) |\nabla \psi|^2. \quad (2.5) \]

By multiplying (1.1) by \( e^{2\psi} u_t \), it follows that

\[ \frac{\partial}{\partial t} \left[ \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \right] - \nabla \cdot (e^{2\psi} u_t \nabla u) \]

\[ + e^{2\psi} \left( a(x) b(t) - \frac{|\nabla \psi|}{-\psi_t} - \psi_t \right) u_t^2 + \frac{e^{2\psi}}{-\psi_t} |\psi_t \nabla u - u_t \nabla \psi|^2 \]

\[ = \frac{\partial}{\partial t} \left[ e^{2\psi} F(u) \right] + 2 e^{2\psi} (-\psi_t) F(u), \quad (2.6) \]

where \( F \) is the primitive of \( f \) satisfying \( F(0) = 0 \), namely \( F'(u) = f(u) \). Using the Schwarz inequality and (2.5), we can calculate

\[ T_1 = \frac{e^{2\psi}}{-\psi_t} (\psi_t^2 |\nabla u|^2 - 2 \psi_t u_t \nabla u \cdot \nabla \psi + u_t^2 |\nabla \psi|^2) \]

\[ \geq \frac{e^{2\psi}}{-\psi_t} \left( \frac{1}{5} \psi_t^2 |\nabla u|^2 - \frac{1}{4} u_t^2 |\nabla \psi|^2 \right) \]

\[ \geq e^{2\psi} \left( \frac{1}{5} (-\psi_t) |\nabla u|^2 - \frac{a(x) b(t)}{4(2 + \delta) u_t^2} \right). \]
From this and (2.5), we obtain

$$\frac{\partial}{\partial t} \left[ \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \right] - \nabla \cdot (e^{2\psi} u_t \nabla u)$$

$$+ e^{2\psi} \left\{ \left( \frac{1}{4} a(x)b(t) - \psi \right) u_t^2 + \frac{-\psi_t}{5} |\nabla u|^2 \right\}$$

$$\leq \frac{\partial}{\partial t} \left[ e^{2\psi} F(u) \right] + 2e^{2\psi} (-\psi_t) F(u).$$

(2.7)

By multiplying (2.7) by $(t_0 + t)^{B+1-\varepsilon}$, here $t_0 \geq 1$ is determined later, it follows that

$$\frac{\partial}{\partial t} \left[ (t_0 + t)^{B+1-\varepsilon} \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \right]$$

$$- (B + 1 - \varepsilon) (t_0 + t)^{B-\varepsilon} e^{2\psi} \frac{1}{2} (u_t^2 + |\nabla u|^2)$$

$$- \nabla \cdot ((t_0 + t)^{B+1-\varepsilon} e^{2\psi} u_t \nabla u)$$

$$+ e^{2\psi} (t_0 + t)^{B+1-\varepsilon} \left\{ \left( \frac{1}{4} a(x)b(t) - \psi \right) u_t^2 + \frac{-\psi_t}{5} |\nabla u|^2 \right\}$$

$$\leq \frac{\partial}{\partial t} \left[ (t_0 + t)^{B+1-\varepsilon} e^{2\psi} F(u) \right] - (B + 1 - \varepsilon) (t_0 + t)^{B-\varepsilon} e^{2\psi} F(u)$$

$$+ 2(t_0 + t)^{B+1-\varepsilon} e^{2\psi} (-\psi_t) F(u).$$

(2.8)

We put

$$E(t) := \int_{\mathbb{R}^n} e^{2\psi} (u_t^2 + |\nabla u|^2) dx, \quad E_\psi(t) := \int_{\mathbb{R}^n} e^{2\psi} (-\psi_t) (u_t^2 + |\nabla u|^2) dx,$$

$$J(t; g) := \int_{\mathbb{R}^n} e^{2\psi} g dx, \quad J_\psi(t; g) := \int_{\mathbb{R}^n} e^{2\psi} (-\psi_t) g dx.$$

Integrating (2.8) over the whole space, we have

$$\frac{1}{2} \frac{d}{dt} \left[ (t_0 + t)^{B+1-\varepsilon} E(t) \right] - \frac{1}{2} (B + 1 - \varepsilon) (t_0 + t)^{B-\varepsilon} E(t)$$

$$+ \frac{1}{4} (t_0 + t)^{B+1-\varepsilon} J(t; a(x)b(t)u_t^2) + \frac{1}{5} (t_0 + t)^{B+1-\varepsilon} E_\psi(t)$$

$$\leq \frac{d}{dt} \left[ (t_0 + t)^{B+1-\varepsilon} \int e^{2\psi} F(u) dx \right]$$

$$+ C(t_0 + t)^{B+1-\varepsilon} J_\psi(t; |u|^{p+1}) + C(t_0 + t)^{B-\varepsilon} J(t; |u|^{p+1})$$

(2.9)
Therefore, we integrate (2.9) on the interval \([0, t]\) and obtain the estimate for \((t_0 + t)^{B+1-\varepsilon} E(t)\), which is the first term of \(M(t)\):

\[
(t_0 + t)^{B+1-\varepsilon} E(t) - C \int_0^t (t_0 + \tau)^{B-\varepsilon} E(\tau) d\tau \\
+ \int_0^t (t_0 + \tau)^{B+1-\varepsilon} J(\tau; a(x)b(t)u^2_t) + (t_0 + \tau)^{B+1-\varepsilon} E_\psi(\tau) d\tau \\
\leq \left[ C B_0^2 + C (t_0 + t)^{B+1-\varepsilon} J(t; |u|^{p+1}) \\
+ C \int_0^t (t_0 + \tau)^{B+1-\varepsilon} J_\psi(\tau; |u|^{p+1}) d\tau \\
+ C \int_0^t (t_0 + \tau)^{B-\varepsilon} J(\tau; |u|^{p+1}) d\tau \right].
\] (2.10)

In order to complete a priori estimate, however, we have to manage the second term of the inequality above whose sign is negative, and we also have to estimate the second term of \(M(t)\). The following argument, which is little more complicated, can settle both these problems.

At first, we multiply (1.1) by \(e^{2\psi} u\) and have

\[
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( uu_t + \frac{a(x)b(t)}{2} u^2 \right) \right] - \nabla \cdot (e^{2\psi} u \nabla u) \\
+ e^{2\psi} \left\{ |\nabla u|^2 + \left( -\psi_t + \frac{\beta}{2(1+t)} \right) a(x)b(t)u^2 + 2u \nabla \psi \cdot \nabla u - 2\psi_t uu_t - u_t^2 \right\} \\
= e^{2\psi} u f(u).
\] (2.11)

We calculate

\[
e^{2\psi} T_2 = 4e^{2\psi} u \nabla \psi \cdot \nabla u - 2e^{2\psi} u \nabla \psi \cdot \nabla u \\
= 4e^{2\psi} u \nabla \psi \cdot \nabla u - \nabla \cdot (e^{2\psi} u^2 \nabla \psi) + 2e^{2\psi} u^2 |\nabla \psi|^2 + e^{2\psi} (\Delta \psi) u^2
\]

and by (2.8) we can rewrite (2.11) to

\[
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( uu_t + \frac{a(x)b(t)}{2} u^2 \right) \right] - \nabla \cdot (e^{2\psi} (u \nabla u + u^2 \nabla \psi)) \\
+ e^{2\psi} \left\{ |\nabla u|^2 + 4u \nabla u \cdot \nabla \psi + (-\psi_t) a(x)b(t) + 2|\nabla \psi|^2 u^2 \right\} \\
+ (B - 2\delta_1) \frac{a(x)b(t)}{2(1+t)} u^2 - 2\psi uu_t - u_t^2 \right\} \leq e^{2\psi} u f(u).
\] (2.12)
It follows from (2.5) that

\[
T_3 = |\nabla u|^2 + 4u\nabla u \cdot \nabla \psi \\
+ \left(1 - \frac{\delta}{3}\right)(-\psi_t) a(x)b(t) + 2|\nabla \psi|^2 \right) u^2 + \frac{\delta}{3}(-\psi_t) a(x)b(t)u^2 \\
\geq |\nabla u|^2 + 4u\nabla u \cdot \nabla \psi \\
+ \left(4 - \frac{\delta^2}{3}\right)|\nabla \psi|^2 u^2 + \frac{\delta}{3}(-\psi_t) a(x)b(t)u^2 \\
= \left(1 - \frac{4}{4 + \delta_2}\right)|\nabla u|^2 + \delta_2|\nabla \psi|^2 u^2 \\
+ \left|\frac{2}{4 + \delta_2}\nabla u + \sqrt{4 + \delta_2} \nabla \psi \right|^2 + \frac{\delta}{3}(-\psi_t) a(x)b(t)u^2 \\
\geq \delta_3(|\nabla u|^2 + |\nabla \psi|^2 u^2) + \frac{\delta}{3}(-\psi_t) a(x) b(t)u^2,
\]

where

\[
\delta_2 := \frac{\delta}{6} - \frac{\delta^2}{6}, \quad \delta_3 := \min(1 - \frac{4}{4 + \delta_2}, \delta_2).
\]

Thus, we obtain

\[
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( u u_t + \frac{a(x)b(t)}{2} u_t^2 \right) \right] - \nabla \cdot \left( e^{2\psi} \left( u\nabla u + u^2 \nabla \psi \right) \right) \\
+ e^{2\psi} \delta_3 |\nabla u|^2 \\
+ e^{2\psi} \left( \delta_3 |\nabla \psi|^2 + \frac{\delta}{3}(-\psi_t) a(x)b(t) + (B - 2\delta_1) \frac{a(x)b(t)}{2(1 + t)} \right) u^2 \\
+ e^{2\psi} \left( -2\psi_t uu_t - u_t^2 \right) \\
\leq e^{2\psi} uf(u). \tag{2.13}
\]

Following Nishihara [19], related to the size of \(1 + |x|^2\) and the size of \((1 + t)^2\), we divide the space \(R^n\) into two different zones \(\Omega(t; K, t_0)\) and \(\Omega^c(t; K, t_0)\), where

\[
\Omega = \Omega(t; K, t_0) := \{ x \in R^n; (t_0 + t)^2 \geq K + |x|^2 \},
\]

and \(\Omega^c = \Omega^c(t; K, t_0) := R^n \setminus \Omega(t; K, t_0)\) with \(K \geq 1\) determined later. Since \(a(x)b(t) \geq a_0(t + t_0)^{-(\alpha + \beta)}\) in the domain \(\Omega\), we multiply (2.7) by \((t_0 + t)^{\alpha + \beta}\) and obtain

\[
\frac{\partial}{\partial t} \left[ e^{2\psi} \frac{(t_0 + t)^{\alpha + \beta}}{2} (u_t^2 + |\nabla u|^2) \right] - \nabla \cdot \left( e^{2\psi} \frac{(t_0 + t)^{\alpha + \beta}}{2} u \nabla u \right) \\
+ e^{2\psi} \left[ \left( \frac{a_0}{4} - \frac{\alpha + \beta}{2(t_0 + t)^{1 - \alpha - \beta}} \right) + (t_0 + t)^{\alpha + \beta} (-\psi_t) \right] u_t^2 \\
+ e^{2\psi} \left[ -\psi_t (t_0 + t)^{\alpha + \beta} - \frac{\alpha + \beta}{2(t_0 + t)^{1 - \alpha - \beta}} \right] |\nabla u|^2 \\
\leq \frac{\partial}{\partial t} \left[ (t_0 + t)^{\alpha + \beta} e^{2\psi} F(u) \right] - \frac{\alpha + \beta}{(t_0 + t)^{1 - \alpha - \beta}} e^{2\psi} F(u) \\
+ 2(t_0 + t)^{\alpha + \beta} e^{2\psi} (-\psi_t) F(u). \tag{2.14}
\]
Let $\nu$ be a small positive number depends on $\delta$, which will be chosen later. By (2.14) + (2.13), we have
\[
\begin{align*}
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( \frac{(t_0 + t)^{\alpha+\beta}}{2} u_t^2 + \nu u u_t + \frac{\nu a(x)b(t)}{2} u^2 + \frac{(t_0 + t)^{\alpha+\beta}}{2} |\nabla u|^2 \right) \right] \\
- \nabla \cdot (e^{2\psi}(t_0 + t)^{\alpha+\beta} u_t \nabla u + \nu e^{2\psi}(u \nabla u + u^2 \nabla \psi)) \\
+ e^{2\psi} \left[ \left( \frac{a_0}{4} - \frac{\alpha + \beta}{2(t_0 + t)^{1-\alpha-\beta}} - \nu \right) + (t_0 + t)^{\alpha+\beta} (-\psi_t) \right] u_t^2 \\
+ e^{2\psi} \left[ \nu \delta_3 - \frac{\alpha + \beta}{2(t_0 + t)^{1-\alpha-\beta}} + \frac{-\psi_t}{5} (t_0 + t)^{\alpha+\beta} \right] |\nabla u|^2 \\
+ e^{2\psi} \left[ \nu (\delta_3 |\nabla \psi|^2 + \frac{\delta}{3} (-\psi_t) a(x)b(t) + (B - 2\delta_1) \frac{a(x)b(t)}{2(1+t)} \right] u^2 \\
+ 2\nu e^{2\psi} (-\psi_t) u u_t \\
\leq \frac{\partial}{\partial t} \left[ (t_0 + t)^{\alpha+\beta} e^{2\psi} F(u) \right] - \frac{\alpha + \beta}{(t_0 + t)^{1-\alpha-\beta} e^{2\psi} F(u)} \\
+ 2(t_0 + t)^{\alpha+\beta} e^{2\psi} (-\psi_t) F(u) + \nu e^{2\psi} u f(u).
\end{align*}
\]
By the Schwarz inequality, the last term of the left hand side in the above inequality can be estimated as
\[
|2\nu (-\psi_t) u u_t| \leq \frac{\nu \delta}{3} (-\psi_t) a(x)b(t) u^2 + \frac{3\nu}{a_0 \delta} (-\psi_t) (t_0 + t)^{\alpha+\beta} u_t^2.
\]
Thus, we have
\[
\begin{align*}
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( \frac{(t_0 + t)^{\alpha+\beta}}{2} u_t^2 + \nu u u_t + \frac{\nu a(x)b(t)}{2} u^2 + \frac{(t_0 + t)^{\alpha+\beta}}{2} |\nabla u|^2 \right) \right] \\
- \nabla \cdot (e^{2\psi}(t_0 + t)^{\alpha+\beta} u_t \nabla u + \nu e^{2\psi}(u \nabla u + u^2 \nabla \psi)) \\
+ e^{2\psi} \left[ \left( \frac{a_0}{4} - \frac{\alpha + \beta}{2(t_0 + t)^{1-\alpha-\beta}} - \nu \right) + (t_0 + t)^{\alpha+\beta} (-\psi_t) \right] u_t^2 \\
+ e^{2\psi} \left[ \nu \delta_3 - \frac{\alpha + \beta}{2(t_0 + t)^{1-\alpha-\beta}} + \frac{-\psi_t}{5} (t_0 + t)^{\alpha+\beta} \right] |\nabla u|^2 \\
+ e^{2\psi} \left[ \nu (\delta_3 |\nabla \psi|^2 + (B - 2\delta_1) \frac{a(x)b(t)}{2(1+t)} \right] u^2 \\
\leq \frac{\partial}{\partial t} \left[ (t_0 + t)^{\alpha+\beta} e^{2\psi} F(u) \right] - \frac{\alpha + \beta}{(t_0 + t)^{1-\alpha-\beta} e^{2\psi} F(u)} \\
+ 2(t_0 + t)^{\alpha+\beta} e^{2\psi} (-\psi_t) F(u) + \nu e^{2\psi} u f(u).
\end{align*}
\]
Now we choose the parameters $\nu$ and $t_0$ such that
\[
\frac{a_0}{4} - \frac{\alpha + \beta}{2(t_0 + t)^{1-\alpha-\beta}} - \nu \geq c_0, \quad 1 - \frac{3\nu}{a_0 \delta} \geq c_0,
\]
\[
\nu \delta_3 - \frac{\alpha + \beta}{2(t_0 + t)^{1-\alpha-\beta}} \geq c_0, \quad \frac{\delta_3}{5} \geq c_0,
\]
hold for some constant $c_0 > 0$. This is possible because we first determine $\nu$ sufficiently small depending on $\delta$ and then we choose $t_0$ sufficiently large depending on $\nu$. Therefore, integrating (2.16) on $\Omega$, we obtain the following energy inequality:
\[
\frac{d}{dt} E_{\psi}(t; \Omega(t; K, t_0)) - N_1(t) - M_1(t) + H_{\psi}(t; \Omega(t; K, t_0)) \leq P_1,
\]
(2.17)
where

\[
\overline{E}_\psi(t; \Omega) = \overline{E}_\psi(t; \Omega(t; K, t_0)) = \int_\Omega e^{2\psi} \left( \frac{(t_0 + t)^{\alpha+\beta}}{2} u_t^2 + \nu uu_t + \frac{\nu a(x)b(t)}{2} u^2 + \frac{(t_0 + t)^{\alpha+\beta}}{2} |\nabla u|^2 \right) dx,
\]

\[
N_1(t) := \int_{S^{n-1}} e^{2\psi} \left( \frac{(t_0 + t)^{\alpha+\beta}}{2} u_t^2 + \nu uu_t + \frac{\nu a(x)b(t)}{2} u^2 \right) \left[ \frac{(t_0 + t)^{\alpha+\beta}}{2} |\nabla u|^2 \right]_{|x|=\sqrt{(t_0+t)^2-K}} dx,
\]

\[
M_1(t) := \int_{\partial\Omega} e^{2\psi} (t_0 + t)^{\alpha+\beta} u_t \nabla u - \nu e^{2\psi} (u \nabla u + u^2 \nabla \psi)) \cdot \vec{n} dS,
\]

\[
H_\psi(t; \Omega) = H_\psi(t; \Omega(t; K, t_0)) = c_0 \int_\Omega e^{2\psi} (1 + (t_0 + t)^{\alpha+\beta} (-\psi_t))(u_t^2 + |\nabla u|^2) dx + \nu(B - 2\delta_1) \int_\Omega e^{2\psi} a(x)b(t) \frac{u^2}{2(1 + t)} dx,
\]

\[
P_1 := \frac{d}{dt} \left[ (t_0 + t)^{\alpha+\beta} \int_\Omega e^{2\psi} F(u) dx \right]
\]

\[
- \int_{S^{n-1}} (t_0 + t)^{\alpha+\beta} e^{2\psi} F(u) \left[ \frac{(t_0 + t)^{\alpha+\beta}}{2} |\nabla u|^2 \right]_{|x|=\sqrt{(t_0+t)^2-K}} dx,
\]

Here \( \vec{n} \) denotes the unit outer normal vector of \( \partial\Omega \). We note that by \( \nu \leq a_0/4 \) and \( |\nu uu_t| \leq \frac{\nu a(x)b(t)}{4} u_t^2 + \frac{\nu(t_0 + t)^{\alpha+\beta}}{a_0} u_t^2 \), it follows that

\[
c \int \Omega e^{2\psi} (t_0 + t)^{\alpha+\beta} (u_t^2 + |\nabla u|^2) dx + c \int \Omega e^{2\psi} a(x)b(t) u^2 dx
\]

\[
\leq \overline{E}_\psi(t; \Omega(t; K, t_0))
\]

\[
\leq C \int \Omega e^{2\psi} (t_0 + t)^{\alpha+\beta} (u_t^2 + |\nabla u|^2) dx + C \int \Omega e^{2\psi} a(x)b(t) u^2 dx
\]

for some constants \( c > 0 \) and \( C > 0 \).

Next, we derive an energy inequality in the domain \( \Omega^c \). We use the notation

\[
\langle x \rangle_K := (K + |x|^2)^{1/2}.
\]
Since \( a(x)b(t) \geq a_0(x)(\alpha + \beta) \) in \( \Omega(t, ; K, t_0) \), we multiply (2.17) by \( \langle x \rangle^\alpha K^\beta \) and obtain
\[
\frac{\partial}{\partial t} \left[ \frac{e^{2\psi}}{2} \langle x \rangle^\alpha K^\beta (u_t^2 + |\nabla u|^2) \right] - \nabla \cdot \left( e^{2\psi} \langle x \rangle^\alpha K^\beta u_t \nabla u \right) \\
+ e^{2\psi} \left( \frac{a_0}{4} + \langle x \rangle^\alpha K^\beta \right) u_t^2 + \frac{1}{5} e^{2\psi} (\langle x \rangle^\alpha K^\beta |\nabla u|^2) \\
+ (\alpha + \beta) e^{2\psi} \langle x \rangle^\alpha K^\beta - 2 x \cdot u_t \nabla u \\
\leq \frac{\partial}{\partial t} \left[ e^{2\psi} \langle x \rangle^\alpha K^\beta F(u) \right] + 2 e^{2\psi} \langle x \rangle^\alpha K^\beta (-\psi_t) F(u).
\]
(2.18)

By (2.18) + \( \hat{\nu} \times (2.13) \), here \( \hat{\nu} \) is a small positive parameter determined later, it follows that
\[
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( \frac{\langle x \rangle^\alpha K^\beta}{2} u_t^2 + \hat{\nu} u u_t + \frac{\hat{\nu} a(x)b(t)}{2} u^2 + \frac{\langle x \rangle^\alpha K^\beta}{2} |\nabla u|^2 \right) \right] \\
- \nabla \cdot \left( e^{2\psi} \langle x \rangle^\alpha K^\beta u_t \nabla u + \hat{\nu} e^{2\psi} (u \nabla u + u^2 \nabla \psi) \right) \\
+ e^{2\psi} \left[ \frac{a_0}{4} - \hat{\nu} + \langle x \rangle^\alpha K^\beta \right] u_t^2 + e^{2\psi} \left[ \hat{\nu} \delta_3 + \frac{-\psi_t}{5} \langle x \rangle^\alpha K^\beta \right] |\nabla u|^2 \\
+ e^{2\psi} \left[ \hat{\nu} \left( \delta_3 |\nabla \psi|^2 + \frac{\delta}{3} (\langle x \rangle^\alpha K^\beta) a(x)b(t) + (B - 2\delta_1) a(x)b(t) \right) \right] u^2 \\
+ e^{2\psi} \left[ (\alpha + \beta) \langle x \rangle^\alpha K^\beta - 2 x \cdot u_t \nabla u - 2\hat{\nu} \psi_t u u_t \right] \\
\leq \frac{\partial}{\partial t} \left[ e^{2\psi} \langle x \rangle^\alpha K^\beta F(u) \right] + 2 e^{2\psi} \langle x \rangle^\alpha K^\beta (-\psi_t) F(u) + \hat{\nu} e^{2\psi} u f(u).
\]
(2.19)
The terms \( T_4 \) can be estimated as
\[
|\langle x \rangle^\alpha K^\beta - 2 x \cdot u_t \nabla u| \leq \frac{\hat{\nu} \delta_3}{2} |\nabla u|^2 + \frac{\delta}{2} \frac{\delta_3}{K^2(1 - \alpha - \beta)} u_t^2,
\]
\[
|2\hat{\nu} (-\psi_t) u u_t| \leq \frac{\hat{\nu} \delta_3}{3} (-\psi_t) a(x)b(t) u^2 + \frac{3\hat{\nu}}{a_0 \delta} (-\psi_t) \langle x \rangle^\alpha K^\beta u_t^2.
\]
From this we can rewrite (2.19) as
\[
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( \frac{\langle x \rangle^\alpha K^\beta}{2} u_t^2 + \hat{\nu} u u_t + \frac{\hat{\nu} a(x)b(t)}{2} u^2 + \frac{\langle x \rangle^\alpha K^\beta}{2} |\nabla u|^2 \right) \right] \\
- \nabla \cdot \left( e^{2\psi} \langle x \rangle^\alpha K^\beta u_t \nabla u + \hat{\nu} e^{2\psi} (u \nabla u + u^2 \nabla \psi) \right) \\
+ e^{2\psi} \left( \frac{a_0}{4} - \hat{\nu} + \langle x \rangle^\alpha K^\beta \right) u_t^2 + e^{2\psi} \left[ \hat{\nu} \delta_3 + \frac{-\psi_t}{5} \langle x \rangle^\alpha K^\beta \right] |\nabla u|^2 \\
+ e^{2\psi} \left[ \hat{\nu} \left( \delta_3 |\nabla \psi|^2 + \frac{\delta}{3} (\langle x \rangle^\alpha K^\beta) a(x)b(t) + (B - 2\delta_1) a(x)b(t) \right) \right] u^2 \\
\leq \frac{\partial}{\partial t} \left[ e^{2\psi} \langle x \rangle^\alpha K^\beta F(u) \right] + 2 e^{2\psi} \langle x \rangle^\alpha K^\beta (-\psi_t) F(u) + \hat{\nu} e^{2\psi} u f(u).
\]
(2.20)
Now we choose the parameters \( \hat{\nu} \) and \( K \) in the same manner as before. Indeed taking \( \hat{\nu} \) sufficiently small depending on \( \delta \) and then choosing \( K \) sufficiently large
depending on \( \dot{\nu} \), we can obtain
\[
\frac{a_0}{4} - \dot{\nu} - \frac{(\alpha + \beta)^2}{2p\delta_3K^{2(1-\alpha-\beta)}} \geq c_1, \quad 1 - \frac{3\dot{\nu}}{a_0\delta} \geq c_1, \quad \nu\delta_3 \geq c_1, \quad \frac{1}{5} \geq c_1
\]
for some constant \( c_1 > 0 \). Consequently, By integrating (2.20) on \( \Omega^c \), the energy inequality on \( \Omega^c \) follows:
\[
\frac{d}{dt} \mathcal{E}_{\psi}(t; \Omega^c(t; K, t_0)) + N_2(t) + M_2(t) + H_{\psi}(t; \Omega^c(t; K, t_0)) \leq P_2, \quad (2.21)
\]
where
\[
\mathcal{E}_{\psi}(t; \Omega^c) = \mathcal{E}_{\psi}(t; \Omega^c(t; K, t_0))
\]
\[
= \int_{\Omega^c} e^{2\psi} \left( \frac{\langle x \rangle_{K}^{\alpha+\beta}}{2} u_t^2 + \dot{\nu}uu_t + \frac{\dot{\nu}a(x)b(t)}{2} u^2 + \frac{\langle x \rangle_{K}^{\alpha+\beta}}{2} |\nabla u|^2 \right) dx,
\]
\[
N_2(t) := \int_{S^{n-1}} e^{2\psi} \left( \frac{\langle x \rangle_{K}^{\alpha+\beta}}{2} u_t^2 + \dot{\nu}uu_t + \frac{\dot{\nu}a(x)b(t)}{2} u^2 + \frac{\langle x \rangle_{K}^{\alpha+\beta}}{2} |\nabla u|^2 \right) \bigg|_{|x|=\sqrt{(t_0+t)^2-\delta}} dx,
\]
\[
M_2(t) := \int_{\partial \Omega^c} e^{2\psi} \langle x \rangle_{K}^{\alpha+\beta} u_t \nabla u + \dot{\nu}e^{2\psi}(u \nabla u + u^2 \nabla \psi) \cdot \nu n dS,
\]
\[
H_{\psi}(t; \Omega^c) := H_{\psi}(t; \Omega^c(t; K, t_0))
\]
\[
= c_1 \int_{\Omega^c} e^{2\psi} (1 + \langle x \rangle_{K}^{\alpha+\beta} (-\psi)) (u_t^2 + |\nabla u|^2) dx \nonumber
\]
\[
+ \frac{\dot{\nu}B - 2\delta_1}{2(1+t)} \int_{\Omega^c} e^{2\psi} \frac{a(x)b(t)}{2} u^2 dx,
\]
\[
P_2 := \frac{d}{dt} \left[ \int_{\Omega^c} e^{2\psi} \langle x \rangle_{K}^{\alpha+\beta} F(u) dx \right] + \int_{S^{n-1}} \langle x \rangle_{K}^{\alpha+\beta} e^{2\psi} F(u) \bigg|_{|x|=\sqrt{(t_0+t)^2-\delta}} d\theta \cdot \frac{d}{dt} \sqrt{(t_0+t)^2 - K} + C \int_{\Omega^c} e^{2\psi} (1 + \langle x \rangle_{K}^{\alpha+\beta} (-\psi)) |u|^{p+1} dx.
\]
In a similar way as the case in \( \Omega \), we note that
\[
c \int_{\Omega^c} e^{2\psi} (t_0 + t)^{\alpha+\beta} (u_t^2 + |\nabla u|^2) dx + c \int_{\Omega^c} e^{2\psi} a(x)b(t)u^2 dx \leq \mathcal{E}_{\psi}(t; \Omega^c(t; K, t_0)) \leq C \int_{\Omega^c} e^{2\psi} (t_0 + t)^{\alpha+\beta} (u_t^2 + |\nabla u|^2) dx + C \int_{\Omega^c} e^{2\psi} a(x)b(t)u^2 dx
\]
for some constants \( c > 0 \) and \( C > 0 \).
We add the energy inequalities on $\Omega$ and $\Omega^c$. We note that replacing $\nu$ and $\hat{\nu}$ by $\nu_0 := \min\{\nu, \hat{\nu}\}$, we can still have the inequalities \[(2.17)\] and \[(2.21)\], provided that we retake $t_0$ and $K$ larger.

By \[(2.17) + (2.21)\] \times \((t_0 + t)^{B - \epsilon}\), we have
\[
\frac{d}{dt}[(t_0 + t)^{-\epsilon}(E_{\psi}(t; \Omega) + E_{\psi}(t; \Omega^c))]
\]
\[
- (B - \epsilon)(t_0 + t)^{-1 - \epsilon}(E_{\psi}(t; \Omega) + E_{\psi}(t; \Omega^c))
\]
\[
+ (t_0 + t)^{-\epsilon}(H_{\psi}(t; \Omega) + H_{\psi}(t; \Omega^c))
\]
\[
\leq (t_0 + t)^{-\epsilon}(P_1 + P_2),
\]
here we note that
\[N_1(t) = N_2(t), \quad M_1(t) = M_2(t)\]
on $\partial \Omega$. Since
\[
|\nu_0 u_t| \leq \frac{\nu_0 \delta_4}{2} a(x) b(t) u^2 + \frac{\nu_0}{2 \delta_4 a_0} (t_0 + t)^{\alpha + \beta} u_t^2
\]
on $\Omega$ and
\[
|\nu_0 u_t| \leq \frac{\nu_0 \delta_4}{2} a(x) b(t) u^2 + \frac{\nu_0}{2 \delta_4 a_0} (x)^{\alpha + \beta} u_t^2
\]
on $\Omega^c$, we have
\[- T_5 + T_6 \geq (t_0 + t)^{B - \epsilon} I_1 + (t_0 + t)^{B - \epsilon} I_2,
\]
where
\[
I_1 := \int_{\Omega} e^{2\psi} \left\{ \frac{c_0}{2} (1 + (t_0 + t)^{\alpha + \beta} (-\psi_t)) - \frac{B - \epsilon}{2(t_0 + t)} \left( (t_0 + t)^{\alpha + \beta} \right) \right\} u_t^2
\]
\[+ e^{2\psi} \left\{ \frac{c_0}{2} (1 + (t_0 + t)^{\alpha + \beta} (-\psi_t)) - \frac{B - \epsilon}{2(t_0 + t)} \left( (t_0 + t)^{\alpha + \beta} \right) \right\} |\nabla u|^2 \, dx
\]
\[+ \int_{\Omega^c} e^{2\psi} \left\{ \frac{c_1}{2} (1 + (x)^{\alpha + \beta} (-\psi_t)) - \frac{B - \epsilon}{2(t_0 + t)} \left( (x)^{\alpha + \beta} \right) \right\} u_t^2
\]
\[+ e^{2\psi} \left\{ \frac{c_1}{2} (1 + (x)^{\alpha + \beta} (-\psi_t)) - \frac{B - \epsilon}{2(t_0 + t)} \left( (x)^{\alpha + \beta} \right) \right\} |\nabla u|^2 \, dx
\]
\[=: I_{11} + I_{12},
\]
\[
I_2 := \nu_0 (B - 2 \delta_1 - (1 + \delta_4) (B - \epsilon)) \left( \int_{\Omega} + \int_{\Omega^c} \right) e^{2\psi} \frac{a(x) b(t)}{2(1 + t)} u^2 \, dx
\]
\[+ \frac{c_2}{2} \int_{\mathbb{R}^n} e^{2\psi}(u_t^2 + |\nabla u|^2) \, dx,
\]
where $c_2 := \min\{c_0, c_1\}$. Recall the definition of $\varepsilon$ and $\delta_1$ (i.e. \[(1.11)\] and \[(2.3)\]). A simple calculation shows $\varepsilon = 3 \delta_1$. Choosing $\delta_4$ sufficiently small depending on $\varepsilon$, we have
\[
(t_0 + t)^{B - \epsilon} I_2 \geq c_3 (t_0 + t)^{B - 1 - \epsilon} \int_{\mathbb{R}^n} e^{2\psi} a(x) b(t) u^2 \, dx + \frac{c_2}{2} (t_0 + t)^{B - \epsilon} E(t)
\]
for some constant $c_3 > 0$. Next, we prove that $I_1 \geq 0$. By noting that $\alpha + \beta < 1$, it is easy to see that $I_{11} \geq 0$ if we retake $t_0$ larger depending on $c_0, \nu_0$ and $\delta_4$. To estimate $I_{12}$, we further divide the region $\Omega^c$ into
\[
\Omega^c(t; K, t_0) = (\Omega^c(t; K, t_0) \cap \Sigma_L) \cup (\Omega^c(t; K, t_0) \cap \Sigma_L^c),
\]
where
\[ \Sigma_L := \{ x \in \mathbb{R}^n; \langle x \rangle^{2-\alpha} \leq L(1+t)^{1+\beta} \}, \quad \Sigma^c_L := \mathbb{R}^n \setminus \Sigma_L \]
with \( L \gg 1 \) determined later. First, since \( K + |x|^2 \leq K(1+|x|^2) \leq KL^{2/(2-\alpha)}(1+t)^{2(1+\beta)/(2-\alpha)} \) on \( \Omega^c \cap \Sigma_L \), we have
\[
\frac{c_1}{2}(1 + \langle x \rangle^{\alpha+\beta}(-\psi_t)) - \frac{B - \varepsilon}{2(t_0 + t)} \left( 1 + \frac{2\nu_0}{\delta_4 a_0} \right) \langle x \rangle^{\alpha+\beta}_K \\
\geq \frac{c_1}{2} - \frac{B - \varepsilon}{2(t_0 + t)} \left( 1 + \frac{2\nu_0}{\delta_4 a_0} \right) K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{(1+\beta)(\alpha+\beta)/2-\alpha}.\]

We note that \(-1 + \frac{(1+\beta)(\alpha+\beta)}{2-\alpha} < 0\) by \( \alpha + \beta < 1 \). Thus, we obtain
\[
\frac{c_1}{2} - \frac{B - \varepsilon}{2(t_0 + t)} \left( 1 + \frac{2\nu_0}{\delta_4 a_0} \right) K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{(1+\beta)(\alpha+\beta)/2-\alpha} \geq 0
\]
for large \( t_0 \) depending on \( L \) and \( K \). Secondly, on \( \Omega^c \cap \Sigma^c_L \), we have
\[
\frac{c_1}{2}(1 + \langle x \rangle^{\alpha+\beta}(-\psi_t)) - \frac{B - \varepsilon}{2(t_0 + t)} \left( 1 + \frac{2\nu_0}{\delta_4 a_0} \right) \langle x \rangle^{\alpha+\beta}_K \\
\geq \left\{ \frac{c_1}{2}(1+\beta) \frac{\langle x \rangle^{2-\alpha}}{(1+t)^{2+\beta}} \right\} \left( 1 + \frac{2\nu_0}{\delta_4 a_0} \right) \langle x \rangle^{\alpha+\beta}_K \\
\geq \left\{ \frac{c_1}{2}(1+\beta) \frac{L}{1+t} \right\} \left( 1 + \frac{2\nu_0}{\delta_4 a_0} \right) \langle x \rangle^{\alpha+\beta}_K.
\]

Therefore one can obtain \( I_{12} \geq 0 \), provided that \( L \geq \frac{B - \varepsilon}{c_1(1+\beta)}(1 + \frac{2\nu_0}{\delta_4 a_0}) \). Consequently, we have \( I_1 \geq 0 \). By \[2.23\] and that we mentioned above, it follows that
\[
-T_3 + T_6 \geq c_3(t_0 + t)^{B-1-\varepsilon} \int_{\mathbb{R}^n} e^{2\psi} a(x) b(t) u^2 dx + \frac{c_0}{2}(t_0 + t)^{B-\varepsilon} E(t).
\]

Therefore, we have
\[
\frac{d}{dt}(t_0 + t)^{B-\varepsilon}(E_\psi(t; \Omega) + E_\psi(t; \Omega^c)) + \frac{c_2}{2}(t_0 + t)^{B-\varepsilon} E(t) \\
+ c_3(t_0 + t)^{B-1-\varepsilon} J(t; a(x)b(t)u^2) \\
\leq (t_0 + t)^{B-\varepsilon}(P_1 + P_2). \tag{2.24}
\]

Integrating (2.24) on the interval \([0, t]\), one can obtain the energy inequality on the whole space:
\[
(t_0 + t)^{B-\varepsilon}(E_\psi(t; \Omega) + E_\psi(t; \Omega^c)) + \frac{c_2}{2} \int_0^t (t_0 + \tau)^{B-\varepsilon} E(\tau) d\tau \\
+ c_3 \int_0^t (t_0 + \tau)^{B-1-\varepsilon} J(\tau; a(x)b(\tau)u^2) d\tau \\
\leq C t_0^2 + \int_0^t (t_0 + \tau)^{B-\varepsilon}(P_1 + P_2) d\tau. \tag{2.25}
\]
By (2.25) + $\mu \times (2.10)$, here $\mu$ is a small positive parameter determined later, it follows that
\[
(t_0 + t)^{B-\varepsilon}E_\psi(t; \Omega) + (t_0 + t)^{B-\varepsilon}E_\psi(t; \Omega^c)
\]
\[
+ \int_0^t \frac{c_2}{2}(t_0 + \tau)^{B-\varepsilon}E(\tau) - \mu C(t_0 + \tau)^{B-\varepsilon}E(\tau) d\tau
\]
\[
+ c_3 \int_0^t (t_0 + \tau)^{B-1-\varepsilon}J(\tau; a(x)b(\tau)u^2) d\tau + \mu (t_0 + t)^{B+1-\varepsilon}E(t)
\]
\[
+ \mu \int_0^t (t_0 + \tau)^{B+1-\varepsilon}J(\tau; a(x)b(\tau)u_\tau^2) + (t_0 + \tau)^{B+1-\varepsilon}E_\psi(\tau) d\tau
\]
\[
\leq CI_\psi^2 + P
\]
\[
+ C(t_0 + t)^{B+1-\varepsilon}J(t; |u|^{p+1})
\]
\[
+ C \int_0^t (t_0 + \tau)^{B+1-\varepsilon}J_\psi(\tau; |u|^{p+1}) d\tau
\]
\[
+ C \int_0^t (t_0 + \tau)^{B-\varepsilon}J(\tau; |u|^{p+1}) d\tau,
\]
(2.26)

where
\[
P = \int_0^t (t_0 + \tau)^{B-\varepsilon}(P_1 + P_2) d\tau.
\]

Now we choose $\mu$ sufficiently small, then we can rewrite (2.26) as
\[
(t_0 + t)^{B+1-\varepsilon}E(t) + (t_0 + t)^{B-\varepsilon}J(t; a(x)b(t)u^2)
\]
\[
\leq CI_\psi^2 + P + C(t_0 + t)^{B+1-\varepsilon}J(t; |u|^{p+1})
\]
\[
+ C \int_0^t (t_0 + \tau)^{B+1-\varepsilon}J_\psi(\tau; |u|^{p+1}) d\tau
\]
\[
+ C \int_0^t (t_0 + \tau)^{B-\varepsilon}J(\tau; |u|^{p+1}) d\tau.
\]
(2.27)

We shall estimate the right hand side of (2.27). We need the following lemma.

**Lemma 2.2** (Gagliardo-Nirenberg). Let $p, q, r(1 \leq p, q, r \leq \infty)$ and $\sigma \in [0, 1]$ satisfy
\[
\frac{1}{p} = \sigma \left( \frac{1}{r} - \frac{1}{n} \right) + (1 - \sigma) \frac{1}{q}
\]
except for $p = \infty$ or $r = n$ when $n \geq 2$. Then for some constant $C = C(p, q, r, n) > 0$, the inequality
\[
\|h\|_{L^p} \leq C\|h\|_{L^r}^{1-\sigma}\|\nabla h\|_{L^r}^{\sigma}, \quad \text{for any } h \in C_0^1(\mathbb{R}^n)
\]
holds.

We first estimate $(t_0 + t)^{B+1-\varepsilon}J(t; |u|^{p+1})$. From the above lemma, we have
\[
J(t; |u|^{p+1}) \leq C \left( \int_{\mathbb{R}^n} e^{\frac{1}{n} |u|^2} dx \right)^{(1-\sigma)(p+1)/2}
\]
\[
\times \left( \int_{\mathbb{R}^n} e^{\frac{1}{n} |\nabla \psi|^2 u^2} dx \right)^{\sigma(p+1)/2}
\]
(2.28)
with $\sigma = \frac{n(p-1)}{2(p+1)}$. Since

$$e^{\frac{n}{p+1} \psi} u^2 = (e^{2\psi} a(x)b(t)u^2) a(x)^{-1} b(t)^{-1} e^{\left(\frac{n}{p+1} - 2\right) \psi}$$

$$\leq C (e^{2\psi} a(x)b(t)u^2) \left[ \left(\frac{(x)^{2-\alpha}}{(1+t)^{1+\beta}} \right)^{\frac{\alpha}{\beta}} e^{\left(\frac{n}{p+1} - 2\right) \psi} \right] \times (1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)}$$

$$\leq C(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)} e^{2\psi} a(x)b(t)u^2$$

and

$$e^{\frac{n}{p+1} \psi} |\nabla \psi|^2 u^2 \leq C (\frac{(x)^{2-2\alpha}}{(1+t)^{2+2\beta}}) e^{\frac{n}{p+1} \psi} \left[ \left(\frac{(x)^{2-\alpha}}{(1+t)^{1+\beta}} \right)^{\frac{2-2\alpha}{2-\alpha}} e^{\left(\frac{n}{p+1} - 2\right) \psi} \right] \times (1+t)^{-2(1+\beta)+(1+\beta)(2-2\alpha)/(2-\alpha)} u^2$$

$$\leq C(1+t)^{-2(1+\beta)/(2-\alpha)} (1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)} e^{2\psi} a(x)b(t)u^2$$

we can estimate (2.28) as

$$J(t; |u|^{p+1}) \leq C(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)} (1-\sigma)(p+1)/2 J(t; a(x)b(t)u^2)^{(1-\sigma)(p+1)/2}$$

$$\times [(1+t)^{-1} J(t; a(x)b(t)u^2) + E(t)]^{\sigma(p+1)/2}$$

and hence

$$(t_0 + t)^{B+1-\varepsilon} J(t; |u|^{p+1}) \leq C \left( (t_0 + t)^{\gamma_1} M(t)^{(p+1)/2} + (t_0 + t)^{\gamma_2} M(t)^{(p+1)/2} \right),$$

where

$$\gamma_1 = B + 1 - \varepsilon + \left[ \beta + (1+\beta) \frac{\alpha}{2-\alpha} \right] \frac{1 - \sigma}{2}(p+1) - \frac{\sigma}{2}(p+1) - (B-\varepsilon) \frac{\sigma}{2}(p+1),$$

$$\gamma_2 = B + 1 - \varepsilon + \left[ \beta + (1+\beta) \frac{\alpha}{2-\alpha} \right] \frac{1 - \sigma}{2}(p+1) - (B-\varepsilon) \frac{1 - \sigma}{2}(p+1) - (B+1 - \varepsilon) \frac{\sigma}{2}(p+1).$$

By a simple calculation it follows that if

$$p > 1 + \frac{2}{n - \alpha},$$

then by taking $\varepsilon$ sufficiently small (i.e. $\delta$ sufficiently small) both $\gamma_1$ and $\gamma_2$ are negative. We note that

$$J_\psi(t; |u|^{p+1}) = \int_{\mathbb{R}^n} e^{2\psi} (-\psi_t)|u|^{p+1} dx$$

$$\leq \frac{C}{1+t} \int_{\mathbb{R}^n} e^{(2+\rho)\psi} |u|^{p+1} dx,$$
where $\rho$ is a sufficiently small positive number. Therefore, we can estimate the terms
\[
\int_0^t (t_0 + \tau)^{B+1-\varepsilon} J_\psi(\tau; |u|^{p+1}) d\tau \quad \text{and} \quad \int_0^t (t_0 + \tau)^{B-\varepsilon} J(\tau; |u|^{p+1}) d\tau
\]
in the same manner as before. Noting that
\[
P_1 + P_2 = \frac{d}{dt} \left[ (t_0 + t)^{\alpha+\beta} \int_\Omega e^{2\psi} F(u) dx + \int_\Omega e^{2\psi} \langle x \rangle_{K}^{\alpha+\beta} F(u) dx \right]
\]
we have
\[
P = \int_0^t (t_0 + \tau)^{B-\varepsilon} (P_1 + P_2) d\tau
\]
\[
\leq CI_0^2 + C(t_0 + t)^{B-\varepsilon} \int_\Omega e^{2\psi} (t_0 + t)^{\alpha+\beta} F(u) dx 
+ C(t_0 + t)^{B-\varepsilon} \int_\Omega e^{2\psi} \langle x \rangle_{K}^{\alpha+\beta} F(u) dx 
+ C \int_0^t (t_0 + \tau)^{B-1-\varepsilon} \int_\Omega e^{2\psi} (t_0 + \tau)^{\alpha+\beta} F(u) dx d\tau 
+ C \int_0^t (t_0 + \tau)^{B-1-\varepsilon} \int_\Omega e^{2\psi} \langle x \rangle_{K}^{\alpha+\beta} F(u) dx d\tau 
+ C \int_0^t (t_0 + \tau)^{B-\varepsilon} \int_\Omega e^{2\psi} (1 + (t_0 + \tau)^{\alpha+\beta} (-\psi_t)) |u|^{p+1} dx d\tau 
+ C \int_0^t (t_0 + \tau)^{B-\varepsilon} \int_\Omega e^{2\psi} (1 + \langle x \rangle_{K}^{\alpha+\beta} (-\psi_t)) |u|^{p+1} dx d\tau.
\]
We calculate
\[
e^{2\psi} \langle x \rangle_{K}^{\alpha+\beta} = e^{2A \frac{x^{2-\alpha}}{(1+x)^{\alpha+\beta}}} \langle x \rangle_{K}^{\alpha+\beta}
\]
\[
\leq C e^{2A \frac{x^{2-\alpha}}{(1+x)^{\alpha+\beta}}} \left( \frac{\langle x \rangle^{2-\alpha}}{(1+t)^{1+\beta}} \right)^{\frac{\alpha+\beta}{2-\alpha}} (1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}}
\]
\[
\leq C e^{(2+\rho)\psi} (1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}}
\]
for small $\rho > 0$. Noting that $\frac{(\alpha+\beta)(1+\beta)}{2-\alpha} < 1$ and taking $\rho$ sufficiently small, we can estimate the terms $P$ in the same manner as estimating $(t_0 + t)^{B+1-\varepsilon} J(\tau; |u|^{p+1})$. Consequently, we have a priori estimate for $M(t)$:
\[
M(t) \leq CI_0^2 + CM(t)^{(p+1)/2}.
\]
This shows that the local solution of (1.1) can be extended globally. We note that
\[
e^{2\psi} a(x) b(t) \geq c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha} - \beta}
\]
with some constant $c > 0$. Then we have
\[
\int_{\mathbb{R}^n} e^{2\psi} a(x) b(t) u^2 dx \geq c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha} - \beta} \int_{\mathbb{R}^n} u^2 dx. \quad (2.29)
\]
This implies the decay estimate of global solution (1.10) and completes the proof of Theorem 1.1.

Proof of Corollary 1.4. In a similar way to derive (2.29), we have
\[
\int_{\mathbb{R}^n} e^{2\psi} a(x) b(t) u^2 dx \geq c(1 + t)^{-1+\beta} e^{-\beta} \int_{\mathbb{R}^n} e^{(2A-\mu)(x)^2-n} u^2 dx.
\]
By noting that
\[
\frac{(x)^2-n}{(1+t)^{\frac{1}{\alpha}}} \geq (1+t)^\rho
\]
on \Omega(t) and Theorem 1.1 it follows that
\[
(1+t)^{-1+\beta} e^{-\beta} \int_{\Omega(t)} e^{(2A-\mu)(1+t)\rho} (u^2 + |\nabla u|^2 + u^2) dx \\
\leq C(1+t)^{-1+\beta} e^{-\beta} \int_{\Omega(t)} e^{(2A-\mu)(x)^2-n} (u^2 + |\nabla u|^2 + u^2) dx \\
\leq C \int_{\mathbb{R}^n} e^{2\psi} (u^2 + |\nabla u|^2 + a(x)b(t)u^2) dx \\
\leq C(1+t)^{-1+\beta} e^{\rho}.
\]
Thus, we obtain
\[
\int_{\Omega(t)} (u^2 + |\nabla u|^2 + u^2) dx \leq C(1+t)^{-1+\beta} e^{\rho}.
\]
This proves Corollary 1.4. \qed

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Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka, 560-0043, Japan

E-mail address: y-wakasugi@cr.math.sci.osaka-u.ac.jp