Periodic Dynamics at Low Frequency of a Local Perturbation in the Isotropic XY Model

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Abstract

We continue our study on the isotropic XY chain with a time-periodic transverse magnetic field acting on a single site. Under a suitable technical assumption on the external forcing, we show the state to be asymptotically a periodic function synchronised with the forcing for any frequency. For small frequencies our result is valid if the amplitude of the transverse field is small.

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1 Introduction

We continue our investigations on the isotropic XY quantum spin chain with a periodically time-dependent transverse external field acting only on one site, namely the $\kappa$-th, and free boundary conditions. The Hamiltonian of the model reads

$$H_N(t) = -g \sum_{j=1}^{N-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) - hV(\omega t)\sigma_{\kappa}^z, \quad 1 < \kappa < N.$$ (1.1)

Here $\sigma^x, \sigma^y, \sigma^z$ denote the Pauli matrices, $g, h, \omega > 0$ are three parameters ruling respectively the strength of the spin-spin coupling, the magnitude of the external field and its frequency. We assume that $V(\omega t)$ is a real periodic analytic function with frequency $\omega$:

$$V(\omega t) = \sum_{n \in \mathbb{Z}} e^{i\omega t}V_n, \quad |V_n| \leq C_0 e^{-|\omega|n}.$$ (1.2)

For any $t \in \mathbb{R}$ and $N \in \mathbb{N}$ $H_N(t)$ is a self-adjoint operator on $\mathcal{H}_N := \mathbb{C}^{2^N}$, while the thermodynamic limit $N \to \infty$ is done as customary in the Fock space $\mathcal{F} := \bigoplus_N \mathcal{H}_N$.

It is well-known that this system is equivalent to a chain of quasi-free fermions and therefore the $N$-particle state is fully described by a one-particle wave function. For what concerns its equilibrium properties, the spectrum is given by the standard analysis of the rank-one perturbation of the Laplacian on $\mathbb{Z}$ (see [1]). As $N \to \infty$ we have a band $[-g, g]$ and an isolated eigenvalue given by

$$g \text{sign}(h) \sqrt{1 + \frac{h^2}{g^2}}.$$ (1.3)

The dynamics is not as simple because when $t$ varies the eigenvalue moves and can touch the band, creating resonances. More precisely, the dynamics in the time interval $[t_0, t]$ is governed by the following Floquet-Schrödinger equation on $\mathbb{Z}$ (for more details on the derivation we refer to [1, 2, 3, 4])

$$i\partial_t\psi(x, t) = -g\Delta\psi(x, t) + hH_F(t, t_0)\psi(x, t), \quad \psi(x, 0) = \delta(x), \quad x \in \mathbb{Z}.$$ (1.4)

Here $\Delta$ is the Laplacian on $\mathbb{Z}$ and

$$H_F(t, t_0)\psi(x, t) := V(\omega t)\psi(x, t) - g \int_{t_0}^{t} dt'J_1(g(t - t'))e^{i\Delta(t-t')}V(\omega t')\psi(x, t'),$$ (1.5)
with $J_1$ the Bessel function of first order. As we will soon need also the Bessel function of zero-th order $J_0$ we recall the general definition

$$J_k(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} dxe^{ixk + i\cos xt}, \quad k \in \mathbb{Z}.$$ 

The Floquet operator $H_F$ acts as a memory-term, accounting for the retarded effect of the rest of the chain on the site $\kappa$. This equation finds a more compact form in the Duhamel representation in the momentum space. We denote by $\xi \in [-1, 1]$ the points of the spectrum of $-\Delta$. Moreover with a slight abuse of notation throughout the paper we will systematically omit $\hat{}$ to indicate either Fourier transforms (when transforming in space) and Fourier coefficients (when transforming in time). Therefore we let $\psi(\xi, t), \xi \in [-1, 1]$, denote the Fourier transform of $\psi(x, t), x \in \mathbb{Z}$. The corresponding of equation (1.4) for $\psi(\xi, t)$ in its Duhamel form reads

$$\left( \mathbb{I} + i\hbar W_{t_0} \right) \psi(\xi, t) = 1, \quad (1.6)$$

where $\{W_{t_0}\}_{t_0 \in \mathbb{R}}$ is a family of Volterra operators for any $t > t_0$ and $\xi \in [-1, 1]$, defined via

$$W_{t_0} f(\xi, t) := \int_{t_0}^{t} dt' J_0(\hbar(t-t')) e^{ig(\xi - \omega t')} V(\omega t') f(\xi, t'). \quad (1.7)$$

Each $W_{t_0}$ is a linear map from $L^2(\mathbb{R} \times [-1, 1])$ into itself (here $C^\alpha(\mathbb{R})$ denotes the space of analytic functions on $\mathbb{R}$). For any $t_0$ finite $W_{t_0}$ is a compact integral operator, which ensures the existence of a unique solution for $t - t_0 < \infty$ (see for instance [3]). We denote this one-parameter family of functions with $\psi_{t_0}(\xi, t)$. As $t_0 \to -\infty$ the limit of the $W_{t_0}$ is an unbounded operator, denoted by $W_\infty$, defined through

$$W_\infty f(\xi, t) := \int_{-\infty}^{t} dt' J_0(\hbar(t-t')) e^{ig(\xi - \omega t')} V(\omega t') f(\xi, t'). \quad (1.8)$$

One can therefore use $W_\infty$ to define an asymptotic version of equation (1.6) as $t_0 \to -\infty$

$$\left( \mathbb{I} + i\hbar W_\infty \right) \psi(\xi, t) = 1, \quad (1.9)$$

whose solutions are denoted by $\psi_{\infty}(\xi, \omega t)$. It is easy to check that $W_\infty$ maps periodic functions of frequency $\omega$ into periodic functions of frequency $\omega$, thus it is somehow expected to find solutions of (1.9) in this class of functions. Our main result partially confirms this idea. We also need the following (technical) assumption on the potential.

**Hypothesis 1.1.** There is an integer $m$ such that

$$V(\omega t) = A \cos(m \omega t) + B \sin(m \omega t) + U(\omega t) \quad (1.10)$$

where $A, B$ are constants and $U(\theta)$ is any real analytic function on $\mathbb{T}$.

Then we have the following result.

**Theorem.** Let $\omega > 0$ and $\hbar/\omega$ sufficiently small and assume Hypothesis 1.1. Then there exists a periodic solution of (1.9) with frequency $\omega$, $\psi_\infty(x, \omega t) \in L^2(\mathbb{R} \times \mathbb{Z})$. Moreover

$$\psi_{t_0}(x, t) = \psi_{\infty}(x, \omega t) + O \left( \frac{1}{\sqrt{t - t_0}} \right).$$

The main relevance of this result is that it is valid for all frequencies. In particular, to the best of our knowledge this is the first result in the low frequency regime.

In [4, Proposition 2.1] we proved the existence of periodic solutions of (1.9) with frequency $\omega$ if $V_0 = O \left( \frac{1}{t} \right)$ and $h$ small or if $V_0 = 0, \omega > 2g$ (high frequencies) and $h/\omega$ small. The meaning of both these conditions is clear: if $V_0$ is large and $h$ is small, then the eigenvalue does not touch the band; if $\omega > 2g$ then the forcing cannot move energy levels within the band. In particular the high-frequency assumption appears in other related works in mathematical and theoretical physics [6, 7, 8, 9, 10, 11], as it is related to the so-called Magnus expansion.

Then in [4, Proposition 3.1] we also proved that if periodic solutions of (1.9) $\psi_{\infty}(\xi, t)$ with frequency $\omega$ exists, then $\psi_{t_0}$ must approach $\psi_{\infty}(\xi, t)$ as $t_0 \to -\infty$. We report the precise statement:
Proposition. Let $\psi_\infty(\xi, \omega t)$ a periodic solution of (1.9) with frequency $\omega$. For any $t \in \mathbb{R}$, $\xi \in [-1, 1]$ it holds

$$\psi_{t_0}(\xi, t) = \psi_\infty(\xi, \omega t) + O\left(\frac{1}{\sqrt{t-t_0}}\right). \quad (1.11)$$

Therefore in order to extend our previous result it suffices to establish the existence of a periodic solution of (1.9) for $\omega < 2g$, a condition defining the low frequency regime. This is a genuine PDE question, which is indeed the main focus of this paper.

More specifically, we are facing an unbounded time-dependent perturbation of the continuous spectrum of the Laplacian on $\mathbb{Z}$. Problems involving periodic forcing are typically dealt with via a KAM-approach, namely one tries to reduce the perturbation to a constant operator by means of a sequence of bounded maps (this is for instance the approach adopted in [9, 11]). Indeed some salient features of periodically driven systems, as for instance pre-thermalisation or slow heating, are from the mathematical angle essentially consequences of the KAM reduction. To the best of our knowledge however there is no example in literature in which this has been done for perturbations of operators with continuous spectrum. Another obstacle is that the operator in (1.9) is a perturbation of the identity, which makes trivial the homological equation at each KAM step. Thus we have to use a different approach. As in [4], we explicitly construct a solution of (1.9) by resumming the Neumann series. The main difficulty is represented by the occurrence of small denominators which actually vanish at some point within the spectrum of the Laplacian and also accumulate in the coefficients of the series. We cure these apparent divergences by a suitable renormalisation of the Neumann series.

The role of Hypothesis 1.1 is merely technical and very related to the method we use, and we cannot attach a physical meaning to it. The main point is that for those forcing excluded by Hypothesis 1.1 we see that the resummation we perform does not eliminate, but just shifts the divergences. So the small denominators remain such and it is not clear how to proceed. We leave the question to future investigation.

The interaction of a small system (the impurity) with an environment (the rest of the chain) while it is irradiated by monochromatic light is a question of primary interest in non-equilibrium statistical physics. Although more complicated systems have been considered [12, 13, 14, 15], quantum spin chains are particularly appealing as they present a rich phenomenology along with a limited amount of technical difficulties. Indeed the lack of ergodicity of such systems has already been object of study in the ’70s [16, 17]. The choice of considering the isotropic XY chain itself simplifies greatly the computations, as one gets exact formulas for Bessel functions. The dynamics of an impurity was first analysed in [2] with different forms of time-dependent external fields. In particular in the case $V(\omega t) = \cos \omega t$ the authors computed the magnetisation of the perturbed spin at the first order in $h$, observing a divergence at $\omega = 2g$. We stress that in our work we completely cover this case, showing absence of divergences in the state of the system.

The rest of the paper is organised as follows. In Section 2 the main objects needed for the proof are introduced while in Section 3 we prove the existence of a periodic solution of (1.9) with frequency $\omega$. Then the Theorem follows from the above proposition.

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2 Set-up

It is convenient to define

$$\varphi := \omega t, \quad \alpha := \frac{g}{\omega}, \quad \gamma := \frac{h}{\omega}. \quad (2.1)$$

Then a simple change of variables in (1.8) yields

$$W_\infty'\psi(\xi, \varphi) := \int_{-\infty}^{\varphi} d\varphi' J_0(\alpha(\varphi - \varphi')) e^{i\alpha\xi(\varphi - \varphi')} V(\varphi')\psi(\xi, \varphi').$$

Thus a periodic solution of (1.9) with frequency $\omega$ should satisfy

$$(1 + i\gamma W_\infty')\psi(\xi, \varphi) = 1.$$
Such a solution will be later explicitly constructed.

Next we introduce some quantities directly given by the problem.

Let $k \in \mathbb{N}_0$ and $I_k := \left[\frac{k}{2}, \frac{k+1}{2}\right)$. For any fixed $\alpha > 0$ there exists of course a unique

$$\bar{k} : \alpha \in I_k.$$ 

Moreover there is $\overline{\epsilon}$ such that $\alpha \in \left[\frac{\bar{k}}{2}, \frac{\bar{k}+1}{2} - \overline{\epsilon}\right]$.

Let

$$\mathcal{B} := \{ n \in \mathbb{Z} : V_n \neq 0 \}$$

and notice that by Hypothesis 1.1 we have

$$m := \text{GCD}\{\mathcal{B}\},$$

i.e. $m$ is the greatest common divisor of the elements of $\mathcal{B}$. Then $\psi_{\mathcal{N}, \mu} = 0$ if $m \not| \mu$ (m does not divide $\mu$). Note that there exists

$$q \in \mathbb{N}_0 : qm \leq \overline{\kappa} \leq (q + 1)m.$$ 

We stress that $\omega$ fixes the numbers $\bar{k}$ and $\overline{\epsilon}$, while for any $\omega$ the choice of the particular forcing $V(\omega t)$ (in particular its harmonics) fixes $q$.

The problem is simplified in terms of Fourier coefficients and the following functions will play the role of 

**small denominators**

$$j_k = j_k(\xi) := \int_0^\infty J_0(gt)e^{i(\xi + k)t}dt = \frac{\chi\{(\xi : |k + \alpha \xi| \leq \alpha\)}{\sqrt{|(\alpha \xi + k)^2 - \alpha^2|}}.$$ (2.4)

In this paper $\chi(\Omega)$ denotes the characteristic function of the set $\Omega$. Now we state few simple properties of the

$(2.2)$ $j_k$ which will be crucially exploited. $j_0(\xi)$ is a real function with singularities at $\pm \xi_0 = \pm 1$. For $1 \leq |k| \leq \overline{k}$ each $j_k$ is singular in

$$\xi_k := -\frac{k}{\alpha} + \text{sign} k.$$ (2.5)

**Remark 2.1.** Each $j_k(\xi)$, $k \neq 0$, is either real positive or imaginary according to the following prescription: for $0 < k < k' \leq \overline{k}$ one has

$$\xi_{k'} < \xi_k \quad \text{and} \quad j_k(\xi) = \begin{cases} \text{Re}(j_k(\xi)) & \xi < \xi_k, \\ \text{Im}(j_k(\xi)) & \xi > \xi_k. \end{cases}$$ (2.6)

On the other hand for $-\overline{k} \leq k' < k < 0$

$$\xi_{k'} > \xi_k \quad \text{and} \quad j_k(\xi) = \begin{cases} \text{Im}(j_k(\xi)) & \xi < \xi_k, \\ \text{Re}(j_k(\xi)) & \xi > \xi_k. \end{cases}$$ (2.7)

Finally for $|k| \geq \overline{k} + 1$ all the $j_k$ are purely imaginary. Moreover for $|k| \geq \overline{k} + 1$ it is easy to see that

$$|j_k(\xi)| \leq \frac{1}{\sqrt{2(k+1)\pi}},$$ (2.8)

while we have the lower bounds

$$j_0 \geq \frac{1}{\alpha}, \quad |j_k(\xi)| \geq \frac{1}{\sqrt{2k}}, \quad k \neq 0.$$ (2.9)

We conveniently localise the $j_k$ around their singularities. Let

$$r := \frac{1}{2} \min_{k \neq k'} |\xi_k - \xi_{k'}| > 0.$$ (2.10)

For every $|k| \leq \overline{k}$ we set

$$Lj_k(\xi) := j_k(\xi)\chi(|\xi - \xi_k| < r), \quad Rj_k(\xi) := j_k(\xi) - Lj_k(\xi).$$ (2.11)
and for $|k| \geq K + 1$
\[ Ljk(\xi) := 0, \quad Rjk(\xi) := jk(\xi). \] (2.12)

As $L$ localises $jk(\xi)$ near the divergence, there is $Z = Z_r$ such that
\[ |Ljk(\xi)| > Z \quad |Rjk(\xi)| \leq Z. \] (2.13)

Combining a (formal) expansion as a power series in $\gamma$ and Fourier series we can now obtain the Neumann series representation of the solution of (2.1) which is the starting point of our analysis. We write
\[ \psi = \sum_{k \geq 0} \gamma^k \psi_k, \] (2.14)
so that the coefficients satisfy
\[ \psi_0 = 1, \quad \psi_k = W_\infty[\psi_{k-1}]. \] (2.15)

We now represent the $\psi$ in Fourier series. Using
\[ (W_\infty \psi)_k = j_k \sum_{n \in Z} V_{k-n} \psi_n, \] (2.16)
by a direct computation we obtain
\[
\psi_1(\varphi) = \sum_{n_1 \in Z} j_{n_1} V_{n_1} e^{i n_1 \varphi},
\]
\[
\vdots
\]
\[
\psi_k(\varphi) = \sum_{n_1, \ldots, n_k \in Z} \left( \prod_{i=1}^k j_{\mu_i} V_{n_i} \right) e^{i \mu_k \varphi},
\] (2.17)

where we put
\[ \mu_i = \mu(n_1, \ldots, n_i) := \sum_{j=1}^i n_j. \] (2.18)

Therefore we arrive to write the Neumann series
\[
\widehat{\psi}(\xi; \varphi; \gamma) := \sum_{\mu \in Z} e^{i \mu \varphi} \psi_\mu(\xi; \gamma) = \sum_{\mu \in Z} e^{i \mu \varphi} \sum_{N \geq 0} (-i \gamma)^N \psi_{N,\mu}(\xi)
\]
\[
= \sum_{\mu \in Z} e^{i \mu \varphi} \sum_{N \geq 0} \sum_{n_1, \ldots, n_N \in Z} (-i \gamma)^N \left( \prod_{i=1}^N j_{\mu_i}(\xi)V_{n_i} \right),
\] (2.19)

which is a solution of (1.4) to all orders in $\gamma$. Note that for each $N \in \mathbb{N}$ the coefficient of $\gamma^N$ is a sum of singular terms. This makes difficult to show directly the convergence of (2.19) and we will instead prove the convergence of a rearranged series which solves the equation.

To explain the construction of such a resummed series it is useful to introduce a slightly modified version of the graphical formalism of [4], inspired by the one developed in the context of KAM theory (for a review see for instance [18]). We call reed a labelled rooted tree in which each vertex except the root has degree two, while the root has degree one. All vertex except the root are called nodes. In a reed the orientation induces a natural total ordering ($\preceq$) on the set of the nodes $N(\rho)$ and lines $L(\rho)$. If a vertex $v$ is attached to a line $\ell$ we say that $\ell$ exits $v$ is $v \preceq \ell$, otherwise we say that $\ell$ enters $v$.

Given a reed $\rho$ we associate labels with each node and line as follows. We associate with each node $v$ a mode label $n_v \in Z$ and with each line $\ell$ a momentum $\mu_\ell \in Z$ with the constraint
\[ \mu_\ell = \sum_{v \in \ell} n_v. \] (2.20)

Note that (2.20) above is a reformulation of the constraint (2.18). We call order of a reed $\rho$ the number of nodes in it and total momentum of a reed the momentum associated with the root line.

$\Theta_{N,\mu}$ denotes the set of reeds of order $N$ and total momentum $\mu$. With every line $\ell$ we attach a further operator label $O_\ell \in \{L, R\}$: We say that a line $\ell$ is regular if $|\mu_\ell| \geq K + 1$, otherwise it is singular; it is localised if $O_\ell = L$, otherwise it is regularised.
Remark 2.2. We remark that if in a reed $\rho$ there is a localised line $\ell$, all the lines with momentum different from $\mu_\ell$ are either regular or regularised.

We then associate with each node $v$ a node factor

$$F_v = V_{n_v},$$

and with each line $\ell$ a propagator

$$G_\ell = O_{j_\ell k_\ell} (\xi) ,$$

so that we can associate with each reed $\rho$ a value as

$$\text{Val}(\rho) = \left( \prod_{v \in N(\rho)} F_v \right) \left( \prod_{\ell \in L(\rho)} G_\ell \right).$$

Clearly

$$\psi_{N, \mu} = \sum_{\rho \in \Theta_{N, \mu}} \text{Val}(\rho).$$

Remark 2.3. Note that if a reed $\rho$ has a line $\ell$ with $m \neq \mu_\ell$ then $\text{Val}(\rho) = 0$.

Two lines are connected if they are attached to the same node (which has node factor $V_0$). We call consecutive two localised lines having exactly one line between them, which is either regular or regularised. By Remark 2.2 two consecutive localised lines have the same momentum.

Given a reed $\rho$ we say that a subset $s$ of nodes and lines in $\rho$ is a closed-subgraph if $\ell \in L(s)$ implies that $v, w \in N(s)$ where $v, w$ are the nodes $\ell$ exits and enters respectively. We say that a closed-subgraph $s$ has degree $d := |L(s)|$. A closed-subgraph has always an exiting line, as the root line cannot belong to any closed-subgraph. We say that a closed-subgraph $s$ is a resonance if it has an entering line and

$$\sum_{v \in N(s)} n_v = 0.$$

Therefore the exiting and entering lines of a resonance must carry the same momentum. We denote by $T_{d, \mu}$ the set of resonances with degree $d$ and entering and exiting lines with momentum $\mu$. Note that if $d = 0$ then $T_{0, \mu}$ is constituted by a single node $v$ with mode $n_v = 0$.

3 Proof

To facilitate readability we divide the proof in several steps.

Step 1: resummation. Let $\Theta_{N, \mu}^R$ be the set of reeds in which no pair of connected, consecutive lines appear. Let us set

$$M_{d, \mu}(\xi) := \sum_{s \in T_{d, \mu}} \text{Val}^R(s),$$

where

$$\text{Val}^R(s) := \left( \prod_{v \in N(s)} F_v \right) \left( \prod_{\ell \in L(s)} R_{j_\ell k_\ell}(\xi) \right).$$

Then we can define

$$M_{\mu}(\xi, \gamma) := (i\gamma) M_{0, \mu}(\xi) + (i\gamma)^2 M_{d, \mu}(\xi).$$

Note that

$$M_{\mu}(\xi, \gamma) = i\gamma V_0 - \gamma^2 \sum_{k \in \mathbb{Z}} V_k R_{jk+\mu}(\xi)V_{-k}.$$

Note that using (1.2), (2.8) and (2.13) we can estimate $M_{d, \mu} < C^d$ for some constant $C > 0$. We shall prove that for any $\xi \in [-1, 1]$ and $\gamma$ small enough one has indeed

$$|1 - M_{\mu}(\xi, \gamma)L_{j_\mu}(\xi)| > 0.$$
This allows us to set
\[
L_j^{R}(\xi) = \frac{L_j^{\mu}(\xi)}{1 - M^{\mu}(\xi, \gamma)L_j^{\mu}(\xi)}, \tag{3.4}
\]

For any \(\rho \in \Theta^{R}_{\rho, \mu}\) let us define the renormalised value of \(\rho\) as
\[
\text{Val}^R(\rho) := \left(\prod_{v \in N(\rho)} F_v\right) \left(\prod_{\ell \in L(\rho)} G^R_{\ell}\right), \tag{3.5}
\]
where
\[
G^R_{\ell} = \begin{cases} 
L^{R}_{\mu}(\xi), & |\mu_\ell| \leq \overline{\kappa}, \quad O_\ell = L, \\
R^{\mu}_j(\xi), & |\mu_\ell| \leq \overline{\kappa}, \quad O_\ell = R, \\
j^{\mu_\ell}(\xi), & |\mu_\ell| \geq \overline{\kappa}.
\end{cases} \tag{3.6}
\]

Note that if \(q = 0\) then we have to renormalise only \(j_0\). Then we can set
\[
\psi^{R}_\mu(\xi; \gamma) := \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \Theta^{R}_{\rho, \mu}} \text{Val}^R(\rho), \tag{3.7}
\]
and
\[
\psi^{R}(\varphi; \xi, \gamma) := \sum_{\mu \in \mathbb{Z}} e^{i\mu \varphi} \psi^{R}_\mu(\xi; \gamma), \tag{3.8}
\]
will be the renormalised series we want to prove to be a regular solution of (2.1).

Step 2: proof of (3.3) for \(\mu = 0\). Recall that \(j_0(\xi)\) is real for all \(\xi \in [-1, 1]\), while in general \(M(\xi, \gamma)\) is a complex number. Then it suffices to prove that
\[
\text{Im}(M_0(\xi, \gamma)) \neq 0. \tag{3.9}
\]

We distinguish two cases.

1) \(V_0 \neq 0\). In this case for \(\gamma\) small enough we can write
\[
M_0(\xi, \gamma) = i\gamma V_0 + O(\gamma^2)
\]
which clearly implies (3.9) for any \(\xi \in [-1, 1]\).

2) \(V_0 = 0\). In this case by Remark 2.1 we have for any \(p \geq \overline{\kappa} + 1\)
\[
|V_p|^2(R_j^{R}(\xi) + R_{j-p}^{R}(\xi)) = i|V_p|^2 \text{Im}(R_j^{R}(\xi) + R_{j-p}^{R}(\xi))
\]
and by (2.4) and (2.11)
\[
\text{Im}(R_j^{R}(\xi) + R_{j-p}^{R}(\xi)) < 0.
\]
for all \(p \geq 1\). Since
\[
M_{2,0}(\xi, \gamma) = \sum_{p=1}^{\overline{k}} |V_p|^2 (R_j^{R}(\xi) + R_{j-p}^{R}) + \gamma^2 \sum_{p \geq \overline{k} + 1} |V_p|^2 (R_j^{R}(\xi) + R_{j-p}^{R}),
\]
we have \(\text{Im}M_{2,0}(\xi, \gamma) < 0\) for any \(\xi \in [-1, 1]\). Writing for small \(\gamma\)
\[
M_0(\xi, \gamma) = -\gamma^2 M_{2,0}(\xi, \gamma) + O(\gamma^3),
\]
(3.9) follows.

Step 3: proof of (3.3) for \(\mu \neq 0\). Recall that \(j_\mu(\xi)\) is either real or purely imaginary for varying \(\xi \in [-1, 1]\), and thus we want to prove that
\[
\text{Im}(M_\mu(\xi, \gamma)L_j^{\mu}(\xi)) \neq 0. \tag{3.10}
\]

Remark 3.1. Note that, by Hypothesis 1.1 there is \(1 \leq p \leq |\mu|\) such that \(V_p \neq 0\).
1) there is $1 < p \leq |\mu|$ such that $V_\mu \neq 0$. Assume for concreteness $\mu > 0$: the other case is similar. By Remark 2.1
\[
j_{p+\mu}(\xi) = i \text{Im}(j_{p+\mu}(\xi)) \quad j_{\mu-p}(\xi) = \text{Re}(j_{\mu-p}(\xi)), \quad \text{for } |\xi - \xi_\mu| < r.
\]
Since $j_{\mu}$ is either real and positive or pure imaginary with negative imaginary part we have
\[
(Rj_{p+\mu}(\xi) + Rj_{-p+\mu})Lj_{\mu}(\xi) = \begin{cases}
(\text{Re}(Rj_{p+\mu}(\xi)) - i \text{Im}(Rj_{-p+\mu}(\xi)))\text{Re}(Lj_{\mu}(\xi)) & \xi : j_{\mu} \in \mathbb{R} \\
(\text{Im}(Rj_{p+\mu}(\xi)) - i \text{Re}(Rj_{-p+\mu}(\xi)))\text{Im}(Lj_{\mu}(\xi)) & \xi : j_{\mu} \in i\mathbb{R},
\end{cases}
\]
that is
\[
\text{Im}((Rj_{p+\mu}(\xi) + Rj_{-p+\mu})Lj_{\mu}(\xi)) < 0.
\]
Moreover again all the imaginary parts come with the same sign, therefore $\text{Im}(M_{p,2}(\xi, \gamma)) < 0$. Thus (3.10) follows.

Step 4: bounds. The propagators defined in (3.6) are bounded as follows. If $|\mu| \geq k + 1$ formula (2.8) yields
\[
|G_\ell^R| \leq \frac{1}{\sqrt{2(k+1)^3}}.
\]
For $|\mu| \leq k$, by (2.13) we have
\[
|Rj_{\mu}(\xi)| \leq Z.
\]
On the other hand, in Step 3 we saw that $Lj_{\mu}^R(\xi)$ has no zero denominator, so we can bound
\[
|Lj_{\mu}^R(\xi)| \leq \frac{C}{|M_{\mu}(\xi, \gamma)|} \leq \frac{C}{\gamma^3}.
\]
(3.11)
Note that if in a reed $\rho$ there are $l$ localised lines, we have
\[
|\text{Val}^R(\rho)| = \left( \prod_{v \in N(\rho)} |G_v^R| \right) \left( \prod_{\ell \in L(\rho)} |G_\ell| \right) \leq \left( C_0 e^{-\sigma \sum_{v \in N(\rho)} |v_v|} \right) \left( \prod_{\ell \in L(\rho)} |G_\ell| \right)
\leq e^{-\sigma \sum_{v \in N(\rho)} |v_v|} \left( \frac{C}{\gamma^3} \right)^l
\leq C_1 N_\gamma^{-2l} e^{-\sigma |\mu|},
\]
for some constant $C_1$. Moreover by construction, in a renormalised reed there must be at least two lines between two localised lines, which implies that a reed with $N$ nodes can have at most $N/3$ localised lines.

But then, setting $N_0 := (q(|a_1| + \ldots + |a_n|) + 1)(2q(|a_1| + \ldots + |a_n|) + 4)$, by (3.7) we obtain
\[
|\psi_\mu^R(\xi; \gamma)| \leq \gamma^{N/3} C_1^N e^{-\sigma |\mu|/2},
\]
(3.13)
so that the series above converge for $\gamma$ small enough. Therefore $\psi^R(\varphi; \xi, \gamma)$ (recall (3.8)) is analytic w.r.t. $\varphi \in \mathbb{T}$, uniformly in $\xi \in [-1, 1]$ and for $\gamma$ small enough

Step 5: $\psi^R(\varphi; \xi, \gamma)$ solves (2.1). Finally we want to prove that
\[
(\mathbb{1} + i\gamma W'_\infty)|\psi^R(\varphi; \xi, \gamma) = 1.
\]
This is essentially a standard computation. Using (3.7) and (3.8), the last equation can be rewritten as
\[
i\gamma W'_\infty \psi^R(\varphi; \xi, \gamma) = 1 - \psi^R(\varphi; \xi, \gamma) = -\sum_{\mu \in \mathbb{Z}} e^{i\mu \varphi} \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \Theta_N^R} \text{Val}^R(\rho).
\]
(14.14)
Moreover thanks to (2.16) we can compute
\[
    i\gamma W_\infty' \psi^R(\varphi; \xi, \gamma) = i\gamma \sum_{\mu \in \mathbb{Z}} \psi^R(\xi; \gamma)(W_\infty' e^{i\mu \varphi})
    = i\gamma \sum_{\mu \in \mathbb{Z}} e^{i\mu \varphi} j_\mu \sum_{k \in \mathbb{Z}} V_{\mu-k} \psi^R(\xi; \gamma)
    = i\gamma \sum_{\mu \in \mathbb{Z}} e^{i\mu \varphi} \sum_{N \geq 0} (-i\gamma)^N j_\mu \sum_{\mu_1+\mu_2 = \mu} V_{\mu_1} \sum_{\rho \in \Theta^R_{N,\mu_2}} \operatorname{Val}^R(\rho).
\]
 Thus we can write (3.14) in terms of Fourier coefficients as
\[
    \sum_{N \geq 1} (-i\gamma)^N j_\mu \sum_{\mu_1+\mu_2 = \mu} V_{\mu_1} \sum_{\rho \in \Theta^R_{N-1,\mu_2}} \operatorname{Val}^R(\rho) = \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \Theta^R_{N,\mu}} \operatorname{Val}^R(\rho). \tag{3.15}
\]

Note that the root line \( \ell \) of a reed has to be renormalised only if it carries momentum label \(|\mu_\ell| \leq k \) and operator \( O_\ell = L \), thus for \(|\mu_\ell| \geq k + 1 \), or \(|\mu_\ell| \leq k \) and \( O_\ell = R \) we see immediately that (3.15) holds.

Concerning the case \( \mu_\ell = \mu \) with \(|\mu| \leq k \) and \( O_\ell = L \), we first note that
\[
    j_\mu(\xi) \sum_{\mu_1+\mu_2 = \mu} V_{\mu_1} \sum_{\rho \in \Theta^R_{N-1,\mu_2}} \operatorname{Val}^R(\rho) = \sum_{\rho \in \Theta^R_{N,\mu}} \operatorname{Val}^R(\rho)
\]
where \( \Theta^R_{N,\mu} \) is the set of reeds such that the root line may exit a resonance of degree \( \leq 2q(|a_1| + \ldots + |a_n|) + 1 \), so that equation (3.15) reads
\[
    \psi^R_\mu(\xi; \gamma) = \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \Theta^R_{N,\mu}} \operatorname{Val}^R(\rho), \tag{3.16}
\]

Let us split
\[
    \Theta^R_{N,\mu} = \tilde{\Theta}^R_{N,\mu} \cup \tilde{\Delta}^R_{N,\mu}, \tag{3.17}
\]
where \( \tilde{\Theta}^R_{N,\mu} \) are the reeds such that the root line exits a resonance of degree zero or one, while \( \tilde{\Delta}^R_{N,\mu} \) is the set of all other renormalised reeds. Therefore we have
\[
    \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \tilde{\Theta}^R_{N,\mu}} \operatorname{Val}^R(\rho) = Lj_\mu^R(\xi) \sum_{\mu_1+\mu_2 = \mu} (i\gamma V_{\mu_1}) \psi^R_\mu(\xi; \gamma) \tag{3.18}
\]
and
\[
    \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \tilde{\Delta}^R_{N,\mu}} \operatorname{Val}^R(\rho) = Lj_\mu^R(\xi) M_\mu(\xi, \gamma) Lj_\mu^R(\xi) \sum_{\mu_1+\mu_2 = \mu} (i\gamma V_{\mu_1}) \psi^R_\mu(\xi; \gamma), \tag{3.19}
\]
so that summing together (3.18) and (3.19) we obtain \( \psi^R_\mu(\xi; \gamma) \).

This concludes the proof of the Theorem.

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