Effects of causality on the fluidity and viscous horizon of quark-gluon plasma

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The second order Israel-Stewart-Müller relativistic hydrodynamics has been applied to study the effects of causality on the acoustic oscillation in relativistic fluid. Causal dispersion relations have been derived with non-vanishing shear viscosity, bulk viscosity and thermal conductivity at non-zero temperature and baryonic chemical potential. These relations have been used to investigate the fluidity of Quark Gluon Plasma (QGP) at finite temperature ($T$). Results of the first order dissipative hydrodynamics have been obtained as limiting case of the second order theory. The effects of the causality on the fluidity near the transition point and on viscous horizon are found to be significant. We observe that the inclusion of causality increases the value of fluidity measure of QGP near $T_c$ and hence makes the flow strenuous. It has also been shown that the inclusion of large magnetic field in the causal hydrodynamics alters the fluidity of QGP.

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I. INTRODUCTION

The collision of heavy ions at relativistic energies create matter in a new state called quark-gluon plasma (QGP) [1, 2]. The QGP can be created with different temperatures ($T$) and net baryonic chemical potential ($\mu$) by altering the energy of the colliding beams [3]. For example, the system formed in the nuclear collisions at Large Hadron Collider (LHC) as well as at highest the Relativistic Heavy Ion Collider (RHIC) energy will have very small $\mu$ but large $T$. On the other hand the matter created at GSI-FAIR (Facility for Antiproton and Ion Research) energy, JINR-NICA (Nuclotron-based Ion Collider Facility) and at lower energy run of RHIC will have larger $\mu$ but smaller $T$. Nature of the transition from QGP to hadrons depends on the values of $T$ and $\mu$ [4]. It is expected that at high $\mu$ and low $T$ the phase transition is first order but at high $T$ low $\mu$ it is a continuous transition from QGP to hadrons [5–8]. When the QGP reverts to hot hadrons due to cooling caused by expansion, the system may encounter the critical point in the QCD phase diagram during the transition from QGP to hadrons. The characterization of the fluidity of the fluid at the critical point is one of the most crucial problem in heavy ion collision at relativistic energies.

Lattice QCD simulations at zero $\mu$ indicate that strongly interacting nuclear matter undergoes a rapid transition from a chirally broken confined hadronic phase to a chirally symmetric, deconfined QGP around $T_c \sim 155$ MeV [8]. The QGP expands very fast due to internal pressure and its evolution in space-time can be studied by using relativistic viscous hydrodynamics. In general, the presence of non-zero transport coefficients, like shear and bulk viscosities and thermal conductivity make the evolution and characterization of QGP very challenging and complex. The Navier-Stokes equation is not suitable to describe relativistic fluid as it suffers from severe flaws, e.g. it violates causality and leads to unstable solutions [9]. These unphysical behaviors were resolved by Müller [10] using Grad’s 14 moment method [11] and its relativistic covariant form is due to Israel and Stewart [12]. These theories are based on extended irreversible thermodynamics known as second order theories. The first order and second order hydrodynamical descriptions stem from the definition of entropy four-current. The conservations of energy-momentum and conserved charge (e.g. net baryon number) along with the second law of thermodynamics lead to the dynamical transport equations which are hyperbolic in nature and respect causality.

The transport coefficients such as shear viscosity, bulk viscosity, thermal conductivity etc. are taken as input in 1st order hydrodynamics. In addition to these standard transport coefficients, the causal or 2nd order theory contains a few more thermodynamic functions which are known as second-order coefficients. These coefficients along with the standard transport coefficients, correspond to different relaxation times and relaxation lengths for various dissipative fluxes which are absent in acasual theory. The results of acasual theory can be obtained by setting these extra coefficients to zero in causal theory. In this work we use the relativistic causal hydrodynamics to investigate propagation of acoustic wave through dissipative fluid with non-zero net (baryonic) charge, shear viscosity, bulk viscosity and thermal conductivity following the procedure outlined in Ref.[18]. In the present work we investigate the effects of causality on the fluidity of QGP in contrast to earlier work where the fluidity of QGP has been studied [13] within the scope of first order theory which is flawed due to causality violation in the relativistic domain. The aim of this work is to estimate the shift on the fluidity of relativistic fluid by using second order hydrodynamics which respects causality. Maartens et al. [14] has used causal hydrodynamics to explore the dissipation of acoustic waves in baryon-

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photon fluid in early universe.

The present article is organized as follows: In Section II, we will discuss the formalism used to derive the transverse and longitudinal dispersion relations for sound wave within the framework of causal hydrodynamics. Dispersion relations for sound wave in the dissipative system with the inclusion of magnetic field have been derived in section III. The impact of the causality and external magnetic field on the fluidity has been discussed in section IV. Section V has been devoted to present results and finally section VI has been dedicated to summary and discussions. We have used natural unit, i.e. $c = \hbar = k_B = 1$ here and the Minkowski metric is set as $g^\lambda_\mu = \text{diag}(-, +, +, +)$.

II. FORMALISM: DERIVATION OF CAUSAL DISPERSION RELATIONS

The relativistic energy-momentum tensor ($T^\lambda_\mu$) in the Israel-Stewart\cite{12} second order theory is given by:

$$T^\lambda_\mu = \epsilon u^\lambda u_\mu + P \Delta^\lambda_\mu + 2h^{(\lambda}_\mu u^{\mu)} + \tau^{\lambda}_\mu$$

(1)

where the dissipative viscous stress tensor $\tau^{\lambda}_\mu = \Pi \Delta^\lambda_\mu + \pi^{\lambda}_\mu$ with $\pi^\lambda_\mu = h^\lambda_\mu u_\lambda = \tau^{\lambda}_\mu u_\lambda = 0$ where the projection operator is defined by $\Delta^\lambda_\mu = g^{\lambda}_\mu + u^\alpha u_\alpha$ with $u^\mu u_\mu = -1$. The heat flux four vector is given by $q^\mu = h^\mu - n^\mu(\epsilon + P)/n$, the particle four flow $N^\mu = nu^\mu + n^\mu$ with $n^\mu u_\mu = 0$, where $n$ is the net number density, $\Pi$ is the bulk pressure, $P$ is the energy density, $\epsilon$ is the energy density, $h$ is the thermodynamic pressure and $h = (\epsilon + P)$ is the enthalpy density. The symmetric tensor $h^{(\lambda}_\mu u^{\mu)}$ is defined as $h^{(\lambda}_\mu u^{\mu)} = \frac{1}{2}(h^{\lambda}_\mu u^\mu + h^\mu_\mu u^\lambda)$.

The definition of fluid four velocity in Eq.1 can be fixed by choosing a proper reference frame attached to the fluid element either due to Landau-Lifshitz (LL) or Eckart. The Eckart frame\cite{15} represents a local rest frame for which the net charge dissipation is zero but the net energy dissipation is non-zero. The LL frame\cite{16} represents a local rest frame where the energy dissipation is zero but the net charge dissipation is non-zero. We consider LL frame here to study a system with net non-zero charge (baryon number).

In LL frame: $h^\mu = 0$, $n^\mu = -qq_\mu / (\epsilon + P)$ and the different viscous flows are given by\cite{12}

$$\Pi = -\frac{1}{3}q_\mu q_\mu + \beta_0 D\Pi - \alpha_0 q_\mu$$

$$q^\lambda = \chi T \Delta^\lambda_\mu [(\partial_\mu \alpha) n T / (\epsilon + P) - \beta_1 D q_\mu + \alpha_0 \partial_\mu \Pi + \alpha_1 \Pi']$$

(2)

$$\Pi^\lambda_\mu = -2\eta[q_{\lambda<\mu}> + \beta_2 D\Pi'_{\lambda\mu} - \alpha_1 q_{<\lambda\mu}>]$$

where $D$ = $u^\mu \partial_\mu$, is well known co-moving derivative or material derivative. In the local rest frame, $D\Pi = \partial_0 \Pi \equiv \Pi$. The different coefficients appearing in Eq.2 are $\alpha = \mu / T$, is known as thermal potential, $\chi$ is the coefficient of bulk viscosity, $\eta$ is the coefficient of shear viscosity, $\gamma$ is the coefficient of thermal conductivity, $\beta_0$, $\beta_1$, $\beta_2$ are relaxation coefficients, $\alpha_0$ and $\alpha_1$ are coupling coefficients. The relaxation times for the bulk pressure ($\tau_{11}$), the heat flux ($\tau_q$) and the shear tensor ($\tau_\pi$) are defined as\cite{17}

$$\tau_{11} = \zeta \beta_0, \tau_q = k_B T \beta_1, \tau_\pi = 2\eta \beta_2$$

(3)

The relaxation lengths which couple the heat flux and bulk pressure ($l_{1q}, l_{qq}$), the heat flux and shear tensor ($l_{q\pi}, l_{\pi q}$) are defined as

$$l_{1q} = \zeta \alpha_0, l_{qq} = k_B T \alpha_0, l_{q\pi} = k_B T \alpha_1, l_{\pi q} = 2\eta \alpha_1$$

(4)

The symmetric, trace free part of the spatial projection is defined by $A_{<\alpha\beta>} \equiv [\Delta_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta}] A_{\alpha\beta}$ and $q_\mu^\alpha = \partial_\mu u^\alpha$. Since in energy frame $h^\mu = 0$, then the energy-momentum tensor reduces to

$$T^\lambda_\mu = \epsilon u^\lambda u_\mu + P \Delta^\lambda_\mu + \Pi \Delta^\lambda_\mu + \pi^\lambda_\mu$$

(5)

We put the explicit forms of $\Pi$, $q^\lambda$ and $\pi^\lambda_\mu$ given by Eq.2 into Eq.1 to get

$$T^\lambda_\mu = \epsilon u^\lambda u_\mu + P \Delta^\lambda_\mu - \frac{1}{3} \zeta \epsilon_\mu u_\mu \Delta^\lambda_\mu + \frac{1}{9} \xi^2 \beta_0 \epsilon^\rho u_\rho \Delta^\lambda_\mu$$

$$+ \frac{n T^2 \zeta \alpha_0 \chi}{3 P + \epsilon} \partial_\mu [\Delta_{\alpha\beta} (\partial_\mu \alpha) \Delta^\lambda_\mu] - 2\eta q_{<\lambda\mu}>$$

$$+ 4\eta^2 \beta_2 u^\mu \Delta_{<\lambda\mu>} + \frac{2 n T^2 \alpha_1 \eta \chi}{P + \epsilon} \alpha_1 q_{<\lambda\mu>]}$$

(6)

$$- \frac{1}{3} \Delta_{\alpha\beta} \Delta^\lambda_\mu \partial_\rho \alpha$$

where we have kept terms upto second order in space-time derivatives and neglected all the higher order space-time derivatives. We impart small perturbations $P_1, \epsilon_1, n_1, T_1$ and $u^\alpha_1$ to $P, \epsilon, n, T$ and $u^\alpha$ respectively to study the acoustic oscillations set by these perturbations. In this work we consider a non-expanding fluid with $u^\alpha = (1, 0, 0, 0)$. Then the perturbation, $u^\alpha_1$ will be $u^\alpha_1 = (0, u^1_1)$ to satisfy the constraint, $u^\alpha u^\alpha_1 = u^\alpha_1 u^\alpha = -1$ where $u^\alpha = u^\alpha + u^\alpha_1$.

To analyze the fate of the perturbation in the dissipative medium we assume that the space time dependence of the perturbation is $\sim \exp(-i(kx - \omega t))$. The perturbations in different components of $T^\lambda_\mu$ appear as
follows VII A:

$$T_{0}^{0} = \epsilon_{1}$$

$$T_{i}^{0} = (\epsilon + P)u_{i} + \frac{\zeta nT^{2}}{3\epsilon + P}\chi\alpha_{0}u_{i}^{0}\nabla^{2}\alpha + 2\alpha_{1}\eta_{\omega}nT^{2}P + \epsilon$$

+ \{u_{i}^{0} \cdot \nabla\} \partial\alpha - \frac{1}{3} u_{i}^{0}\nabla^{2}\alpha \}$$

$$T_{ij} = P_{ij} - \frac{1}{3} \{i\vec{k} \cdot \vec{u} - \frac{1}{3} \zeta \beta_{0}(\vec{k} \cdot \vec{u})\} \delta^{ij}$$

+ \frac{1}{3} \zeta_{\alpha k}^{0} \{i\vec{k} \cdot \vec{u} - \frac{1}{3} \zeta \beta_{0}(\vec{k} \cdot \vec{u})\} \delta^{ij}$$

+ \{i\vec{k} \cdot \vec{u} - \frac{1}{3} \zeta \beta_{0}(\vec{k} \cdot \vec{u})\} \delta^{ij}$$

$$\n = \left\{ \frac{nT^{2} + 2TT_{1}n}{P + \epsilon} - \frac{nT^{2}(P + \epsilon)}{(P + \epsilon)^{2}} \right\}$$

The equations of motions (EoMs) of perturbations dictated by the conservations of energy-momentum and net-charge of the fluid are given by,

$$\partial_{\mu}T^{\mu\lambda} = 0, \quad \partial_{\mu}N^{\mu} = 0$$ (9)

The EoMs in the frequency-wave vector space take the following form:

$$0 = \omega T_{0}^{0} - k_{j}T_{j}^{i}$$

$$= \omega(\epsilon + P)u_{i} + \frac{\zeta nT^{2}}{3\epsilon + P}\chi\alpha_{0}u_{i}^{0}\nabla^{2}\alpha + 2\alpha_{1}\eta_{\omega}nT^{2}P + \epsilon$$

$$\times \{u_{i}^{0} \cdot \nabla\} \partial\alpha - \frac{1}{3} u_{i}^{0}\nabla^{2}\alpha \}$$

$$\times \{u_{i}^{0} \cdot \nabla\} \partial\alpha - \frac{1}{3} u_{i}^{0}\nabla^{2}\alpha \}$$

$$+ \frac{2}{3} \zeta \beta_{0}(\vec{k} \cdot \vec{u}) - 2\alpha_{1}\eta_{\omega}nT^{2}P + \epsilon$$

$$+ 2\eta_{\omega}2\alpha_{0}\chi^{(0)}(\alpha_{0}\zeta - 2\alpha_{1}\eta)\nabla^{2}\alpha$$

and the number conservation equation gives,

$$0 = \omega n_{1} - n(\vec{k} \cdot \vec{u})$$ (12)

$$P_{1}$$ and \(\epsilon_{1}\) can be expressed in terms of the independent variables, \(n_{1}\) and \(T_{1}\) as follows:

$$\epsilon_{1} = \left( \frac{\partial\epsilon}{\partial T} \right)_{n} T_{1} + \left( \frac{\partial\epsilon}{\partial n} \right)_{T} n_{1}$$ (13)

$$P_{1} = \left( \frac{\partial P}{\partial T} \right)_{n} T_{1} + \left( \frac{\partial P}{\partial n} \right)_{T} n_{1}$$ (14)

We decompose the fluid velocity into directions perpendicular and parallel to the direction of wave vector, \(\vec{k}\) as:

$$u_{i}^{0} = u_{i}^{\perp} + \vec{k}(\vec{k} \cdot \vec{u})/k^{2}$$ (15)

The modes propagating along the direction of \(\vec{k}\) are called longitudinal and those perpendicular to \(\vec{k}\) are called transverse modes. Inserting Eq.15 in the EoMs with the help of Eqs.13 and 14 and collecting the transverse components, we get the dispersion relation for the transverse mode as:

$$\omega^{\perp} = \frac{-ik^{2}(\eta - \alpha_{1}\alpha)}{P + \epsilon - 2\eta^{2}\beta_{2}k^{2} + \frac{nT^{2}}{3(P + \epsilon)}\chi(\alpha_{0}\zeta - 2\alpha_{1}n)\nabla^{2}\alpha}$$ (16)

In the acausal limit, \(\beta_{2} = \alpha_{1} = \alpha_{0} = 0\), Eq.16 reduces to:

$$\omega^{\perp} = \frac{-ik^{2}\eta}{P + \epsilon} = i\omega_{m}$$ (17)

which is the result obtained in acausal hydrodynamics[18]. We observe that \(\zeta\) does not appear in the imaginary part of \(\omega^{\perp}\) and it is purely imaginary if \(\chi = 0\).

The derivation of dispersion relation for the longitudinal component is lengthy and tedious to derive. The details are given in the appendix VII B. For \(\chi = 0\), the imaginary part of the longitudinal component of the dispersion relation is:

$$\omega_{m}^{\parallel} = \frac{-k^{2}\{\frac{1}{3}\zeta + \frac{4}{3}n\}}{2((P + \epsilon) - \frac{k^{2}\zeta^{2}\beta_{0}}{3} - \frac{2}{3}k^{2}\eta^{2}\beta_{2})}$$ (18)
In the acausal limit, taking $\beta_0 = \beta_2 = 0$ Eq.18 reduces to
\[
\omega_{1m} = \frac{-k^2\left(\frac{4}{3}\varsigma + \frac{4}{3}\eta\right)}{2(P + \epsilon)}
\]
(19)
which matches with results of [18] for $\chi = 0$. The coefficient of $\varsigma$ in Eq.18 differ from that of the one given in [18] due to the different numerical coefficient of $\varsigma$ in $T^{\mu\nu}$ used here.

The real part of the dispersion for the longitudinal modes turns out to be:
\[
\omega_{Re} = \left[-\frac{k^2}{3}\left(\frac{\partial e}{\partial T}n\right)^2 - 4\left(\frac{P + \epsilon}{\partial T}n\right)\right]^{1/2} \left[2\left(\frac{P + \epsilon}{\partial T}n\right)\right]
\]
(20)
In the acausal limit, considering vanishing net number density($n$), Eq.20 reduces to:
\[
\omega_{Re} = -\left[-\frac{k^2}{3}\left(\frac{\partial e}{\partial T}n\right)^2 - 4\left(\frac{P + \epsilon}{\partial T}n\right)\right]^{1/2} \left[2\left(\frac{P + \epsilon}{\partial T}n\right)\right]
\]
(21)
where $\varsigma$ is the speed of sound wave in the fluid. The acausal limit $\omega_{Re} = c_s|k|$ can be recovered by keeping only the linear term [18]. The causal dispersion relation derived here can reproduce all the known relations exist in the acausal limit.

III. EFFECTS OF MAGNETIC FIELD

It has been shown that a ultra-high but transient magnetic field is generated in the collision of heavy ions at RHIC and LHC energies [19]. Survivability of the magnetic field will depend on the value of the conductivity of the QGP medium formed in these collisions. The presence of magnetic field will affect the properties of the fluid through its contribution to the energy-momentum tensor. Considering constant magnetic field($B$), the magnetic contribution is given by[20]
\[
T_{m}^{\mu\nu} = \frac{B^2}{8\pi}\left(2u^\mu u^\nu + g^{\mu\nu} - 2n^\mu n^\nu\right)
\]
(23)
where $n^\mu$ is the unit vector in the direction of the magnetic field $n^\mu = B^\mu/B$ with $n^\mu n^\nu = -1$ and $u^\mu n^\nu = 0$
The conservation equation then reads:
\[
\partial_\mu T_{tot}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu T_{m}^{\mu\nu} = 0
\]
(24)
where
\[
T_{tot}^{\mu\nu} = T^{\mu\nu} + T_{m}^{\mu\nu}
\]
For small perturbation, $T_{tot}^{\mu\nu}$ to $T_{m}^{\mu\nu}$ the first and third term of Eq.23 will be changed. The changes in different components of $T_{tot}^{\mu\nu}$ are given in Appendix VII C. After taking Fourier transformation, the equations of motions in the presence of magnetic field becomes
\[
0 = \omega T_{1,tot}^{\mu\nu} - k_i T_{1,tot}^{\mu\nu} = F_1 - \frac{B^2}{8\pi}\left[2(k\cdot u_1) - \frac{(\vec{B} \cdot \vec{k})}{B^2}(\vec{B} \cdot u_1)\right]
\]
(25)
\[
0 = \omega T_{1,tot}^{i\mu} - k_i T_{1,tot}^{i\mu} = F_2 + \frac{\omega B^2}{8\pi}\left[2u_1^i - \frac{B^i}{B^2}(\vec{B} \cdot u_1)\right]
\]
(26)
where $F_1$ and $F_2$ are the expressions given in the right hand side of Eqs. 10 and 11 respectively. Decomposing the fluid velocity as Eq.15, Eq.25 becomes
\[
0 = F_1' - \frac{B^2}{8\pi}\left[2(k\cdot u_1) - \frac{(\vec{B} \cdot \vec{k})}{B^2}(\vec{B} \cdot u_{1\perp}) - \frac{(\vec{B} \cdot \vec{k})^2}{B^2 k^2}(\vec{k} \cdot u_1)\right]
\]
(27)
If we take constant magnetic field along the direction of wave vector $k$, then $(\vec{B} \cdot \vec{k}) = kB$ and $(\vec{B} \cdot u_{1\perp}) = 0$ as $u_1 \perp \vec{k}$. In that case Eq.27 becomes,
\[
0 = F_1' - \frac{B^2}{4\pi}(\vec{k} \cdot u_1)
\]
(28)
Similarly, after decomposition Eq.26 reads as:
\[
0 = F_2' + \frac{\omega B^2}{8\pi}\left[2u_1^i + \frac{2k^i}{k^2}(\vec{k} \cdot u_1) - \frac{B^i}{Bk}(\vec{k} \cdot u_1)\right]
\]
(29)
and the number conservation equation remains unchanged. Here $F_1'$ and $F_2'$ represents the right hand side (RHS) of Eq.10 and 11 respectively after decomposition of fluid velocity. The dispersion relation in the transverse direction in the presence of constant $B$ is given by,
\[
\omega^\perp = \frac{ik^2(\eta - \alpha_1\chi) + \frac{n^2}{(P + \epsilon)}\alpha}{\left[P + \epsilon - \frac{B^2}{4\pi} - 2\eta^2\beta_2k^2 + \frac{n^2}{(P + \epsilon)}\right]^{1/2}}
\]
(30)
For $B = 0$, Eq.30 reduces to Eq.16. The imaginary part of the dispersion relation in the longitudinal direction can be expressed as,
\[
\omega_{1m} = \frac{-k^2\left(\frac{4}{3}\varsigma + \frac{4}{3}\eta\right)}{2\left[P + \epsilon + \frac{B^2}{8\pi} - \frac{8}{9}k^2\varsigma^2\beta_0 - \frac{8}{3}k^2\eta^2\beta^2\right]}
\]
(31)
and the real part is:

\[ \omega_{\text{Re}}^{||} = \left[ -\left\{ k^2 \left( \frac{\partial \epsilon}{\partial T} \right)_n \left( \frac{1}{3} \zeta + \frac{4}{3} \eta \right) \right\}^2 - 4 \left( P + \epsilon + \frac{B^2}{8\pi} \right) \right]^{-1/2} \]

where we have considered \( \chi = 0 \) to keep the expression compact, however, the derivation of the dispersion relation for \( \chi \neq 0 \) is straight forward.

### IV. EFFECTS OF CAUSALITY ON FLUIDITY

What is the difference that it makes to characterize a relativistic fluid by using causal vis-a-vis acausal dispersion relations? In the following we will study this aspect in details. The fluidity of QGP can be studied [13] by introducing the ratio of two length scales - one of those is related to the wave length of the sound wave propagating through the fluid. The other one is the inter-particle distance in the fluid.

#### A. Viscous horizon

In the following we will provide the threshold value of wave vector, \( k_v \), above which no sound wave can propagate. The quantity, \( R_v \sim k_v^{-1} \) determines the length scale called viscous horizon [22]. The imaginary part of dispersion relation dictates the attenuation of sound wave in the fluid. A sound wave damps in time as \( \sim \exp(\omega_{1m}t) \) (for \( \omega_{1m} < 0 \)) in viscous medium. This can be expressed in terms of perturbation to \( T^{\mu\nu} \) as:

\[ T_1^{\mu\nu}(t) = T_1^{\mu\nu}(t_i) \exp(\omega_{1m}t) \]

where \( T_1^{\mu\nu}(t_i) \) represents the perturbation to \( T^{\mu\nu} \) at the initial time \( t_i \). The dispersion relation derived in the previous section may be used to determine upper limit of wave vector \( k_v \) of the sound wave that can propagate in the medium, which can be obtained by setting: \( |\omega_{1m}|t = 1 \)

\[ k_{v,\text{long}}^{\text{causal}} = \frac{1}{R_{v,\text{long}}^{\text{causal}}} = \sqrt{\frac{P + \epsilon}{\frac{1}{2}(\zeta \frac{1}{3} + \frac{2n}{3}) + \frac{1}{5}\beta_0 + \frac{2}{5}\eta^2\beta_2}} \]

We note that the viscous horizon scale, \( R_v \sim \sqrt{T} \) in contrast to sound horizon which varies linearly with \( t \). The above condition implies that a longitudinal mode with magnitude of \( k \) larger than \( k_{v,\text{long}}^{\text{causal}} = 1/R_{v,\text{long}} \) will be killed by dissipation and all other longitudinal modes with lower values of \( k \) will propagate. The known result [22] in the acausal limit (\( \beta_0 = \beta_2 = 0 \)) can be obtained as:

\[ k_{v,\text{long}}^{\text{acausal}} = \frac{1}{R_{v,\text{long}}^{\text{acausal}}} = \sqrt{\frac{P + \epsilon}{\frac{1}{2}(\zeta \frac{1}{3} + \frac{2n}{3})}} \]

Similarly for the causal transverse mode we have the upper limit,

\[ k_{v,\text{tran}}^{\text{causal}} = \frac{1}{R_{v,\text{tran}}^{\text{causal}}} = \sqrt{\frac{P + \epsilon}{\eta t}} \]

and in the acausal limit the above relation turns out to be

\[ k_{v,\text{tran}}^{\text{acausal}} = \frac{1}{R_{v,\text{tran}}^{\text{acausal}}} = \sqrt{\frac{P + \epsilon}{\eta t}} \]

We have already seen in the previous section that the application of magnetic field changes the dispersion relations. Therefore, the viscous horizon in presence of magnetic field should also change to:

\[ k_{v,\text{long},B}^{\text{causal}} = \frac{1}{R_{v,\text{long},B}^{\text{causal}}} = \sqrt{\frac{P + \epsilon + \frac{B^2}{4\pi}}{\frac{1}{2}(\zeta \frac{1}{3} + \frac{2n}{3}) + \frac{1}{5}\beta_0 + \frac{2}{5}\eta^2\beta_2}} \]

\[ k_{v,\text{tran},B}^{\text{causal}} = \frac{1}{R_{v,\text{tran},B}^{\text{causal}}} = \sqrt{\frac{P + \epsilon + \frac{B^2}{4\pi}}{\eta t}} \]

The viscous horizon has an impact on the flow harmonics. It is argued in [23] that the properties related to the ratio of higher order to second order harmonics, \( v_n/v_2 \) with \( n > 2 \) can be understood in terms of the propagation of sound wave through dissipative medium and hence such studies will help in estimating the size of the sound horizon and viscous horizon [22].

#### B. Measure of fluidity

Sound wave in a viscous fluid will stop propagating if its wave length is smaller than some threshold value, \( \lambda_{th} = 2\pi/k_v \). The value of \( \lambda_{th} \) will depend on the values of dissipative coefficients, \( \eta, \zeta, \chi \), etc. The fluidity of the system has been defined in Ref. [13, 21] with the introduction of a new quantity which depends on the intrinsic properties of the fluid and enables one to compare fluids of wide varieties such as non-relativistic fluid like water and relativistic, extremely dense and hot fluid like QGP. For example, the temperature of water and QGP differ by a factor \( \sim O(10^{16}) \). Now if we want to compare their fluidity we may find the dissipation per inter-particle separation. In Ref. [13] the linearized first order dispersion relation of the sound mode was used,

\[ \omega = c_s k - i \frac{k^2}{2} \frac{4\eta}{h/c^2} \]
The imaginary part of the dispersion relation represents the dissipation of sound wave in the medium. The sound mode with wave vector \( k \) will propagate if the imaginary part of frequency is small \( i.e.\):

\[
\left| \frac{\omega_{im}(k)}{\omega_{Re}(k)} \right| \ll 1
\]  

(41)

The limiting value can be found by setting \( | \omega_{im}/\omega_{Re}| = 1 \), which gives \( k = 3\hbar c_s/(2\eta) \) then the resulting threshold for wavelength of the sound mode becomes

\[
\lambda_{th} = \frac{2\pi}{k_s} = \frac{4\pi}{3} \frac{\eta}{\hbar c_s} = \frac{4\pi}{3} L_\eta
\]  

(42)

where \( L_\eta = \eta/(\hbar c_s) \). The \( L_\eta \) gives an estimation for lowest sound wavelength \( (\lambda_{th}) \) which can propagate through the viscous fluid. The \( L_\eta \) has the dimension of length and can be used to characterize fluids. However, introduction of a dimensionless scale will enable us to compare fluids with varying densities. Quantities like Reynolds or Knudsen numbers have been used in Refs. [24] and [25] respectively to study flow properties. However, both of these quantities involve parameters, like dimension of the system which is not connected with the intrinsic properties of the fluid. The particle number density \( (\rho) \) can be used to estimate the inter-particle distance, \( L_\rho \sim \rho^{-1/3} \), which is related to the intrinsic properties of the fluid. The ratio of \( L_\eta \) to \( L_\rho \) may be used to characterize the fluid. For relativistic QGP with vanishing net baryon number density, entropy density \( ( s) \) can be used to estimate \( \rho \) by using \( \rho \sim s/4 \). The ratio of these two length scales can be used as a measure of fluidity

\[
F \equiv \frac{L_\eta}{L_\rho}
\]  

(43)

What is corresponding expression of \( F \) for causal fluid dynamics involving other transport coefficients in addition to \( \eta \)? We use dispersion relations derived from causal relativistic hydrodynamics involving shear, bulk viscosities, thermal conductivity and different relaxation coefficients to estimate the fluidity. We would contrast our results to those obtained with acausal relation [13]. The length scale analogous to \( L_\eta \) for causal fluid dynamics is denoted by \( L_T \) depends on the transport coefficients like \( \zeta, \chi, \beta_0, \beta_2 \) in addition to \( \eta \). \( L_T \) for the longitudinal mode is given by,

\[
L_T = \left[ \frac{1}{4} \right] \left\{ \frac{\partial \kappa}{\partial T} \right\}_n \zeta^2 + 8\zeta \eta + 16\eta^2 \frac{\partial \kappa}{\partial T} T
+ 2(P + \varepsilon)\beta_0 \zeta^2 \frac{\partial P}{\partial n} T
+ 2n\beta_0^2 \frac{\partial^2 P}{\partial n^2} T
\nonumber
- 2n\zeta^2 \beta_0 \frac{\partial P}{\partial n} \frac{\partial \kappa}{\partial n} T
+ 2n\beta_0^2 \frac{\partial P}{\partial n} \frac{\partial \kappa}{\partial n} T
\nonumber
+ 48n^2 \beta_2 \eta \frac{\partial P}{\partial n} \frac{\partial \kappa}{\partial n} T
\nonumber
\right\}^{1/2}
\nonumber
\]  

(44)

We use \( (\partial P/\partial T) = (\partial \rho/\partial \varepsilon) (\partial \kappa/\partial T) \) to express \( F \) as:

\[
F = \left[ \frac{1}{4} \right] \left\{ \frac{\partial \kappa}{\partial T} \right\}_n \zeta^2 + 8\zeta \eta + 16\eta^2 + 2(P + \varepsilon)\beta_0^2 \frac{\partial P}{\partial \varepsilon} n
+ 2n\beta_0^2 \zeta^2 \frac{\partial P}{\partial \varepsilon} n
+ 2n\beta_0^2 \frac{\partial^2 P}{\partial \varepsilon \partial n} n
\nonumber
- 2n\zeta^2 \beta_0 \frac{\partial P}{\partial \varepsilon} n
+ 2n\beta_0^2 \frac{\partial P}{\partial \varepsilon} n
+ 48n^2 \beta_2 \eta \frac{\partial P}{\partial \varepsilon} n
\nonumber
\right\}^{1/2}
\nonumber
\]  

(45)

This is measure of fluidity of a relativistic fluid for \( \chi = 0 \).

For a fluid having vanishing net charge density \( (n = 0) \) the above equation becomes

\[
F = \left[ \frac{1}{4} \right] \left\{ \frac{\partial \kappa}{\partial T} \right\}_n \zeta^2 + 8\zeta \eta + 16\eta^2 + 2(P + \varepsilon)\beta_0 \zeta \frac{\partial P}{\partial \varepsilon} n
+ 48n^2 \beta_2 \eta \frac{\partial P}{\partial \varepsilon} n
\nonumber
\right\}^{1/2}
\nonumber
\]  

(46)

It is clear from this result that dispersion relations become more complex if relativistic causal hydrodynamics is used. Two more coefficients, \( \beta_0 \) and \( \beta_2 \) enter into the expression for fluidity. In the acausal limit i.e. for vanishing \( \beta_0 \) and \( \beta_2 \) as well as neglecting non linear terms in the real part of \( \omega \), the \( F \) reads,

\[
F = \frac{n^{1/3} \eta}{\hbar c_s}
\]  

(47)

which is exactly what is given in Ref.[13]. It may be noted from Eq. 45 that the fluidity measure, \( F \) of the causal fluid has a complicated functional dependence on various transport coefficients and thermodynamic variables of the fluid. In contrast to the causal case the \( F \) has simpler dependence on transport coefficients and thermodynamical variables in an acausal scenario (Eq. 47).
C. Fluidity in presence of magnetic field

We have already seen that non-zero $B$ affects the real and imaginary part of $\omega$ along the longitudinal direction and hence it modifies the fluidity measure also. For vanishing net charge and $\zeta = \chi = 0$ the $L_T$ becomes,

$$
L_T = \left[ \left( \frac{\partial \epsilon}{\partial T} \right) (\zeta + 4n) + 2\beta_2 \frac{B^2}{8\pi} (P + \epsilon) + 48\eta^2 \beta_2 \frac{B^2}{8\pi} (P + \epsilon) \right]^{1/2} / \left[ \left[ 4 \left( \frac{B^2}{8\pi} \right)^2 \right] (P + \epsilon) + (P + \epsilon)^2 \right]^{1/2}
$$

(48)

For simplicity we kept only $\eta$ as non-zero. However, it is straight forward to find $F$ with non-zero $n, \zeta$ and $\chi$.

V. RESULTS AND DISCUSSION

In this section we discuss the dispersion relation for the transverse and longitudinal modes for non-expanding fluid.

A. Transverse mode

In order to see how causality or causal hydrodynamics affects the damping of sound wave, first we consider the transverse component of the dispersion relation. For $\chi = 0$, Eq. 16 reads:

$$
\omega_T^{\perp} = \frac{-k^2 \eta}{P + \epsilon - 2n\beta_2 k^2}
$$

(49)

It is interesting to note that the bulk viscosity does not appear in the dispersion relation for the transverse mode. The coefficient $\beta_2$ appearing in the denominator is the signature of causal hydrodynamics. In the ultrarelativistic limit it has the limiting value $12$

$$
\beta_2 = \frac{3}{4P}
$$

(50)

We estimate the damping of the sound wave by using the thermodynamic relation for vanishing net charge density (such as baryon free QGP), $P + \epsilon = sT$. In Fig. 1 we display the damping of the transverse mode with $k$ for $\eta/s = 1/4\pi$ at $T = 200, 300$ and $400$ MeV. We find that damping is stronger for larger $\eta/s$, lower $T$ and larger wave numbers or smaller wave lengths. The imaginary part of the dispersion relation leads to the variation of amplitude with $k$ as $\sim \exp(-\Gamma_s k^2)$ where the $\Gamma_s$, square of the characteristic dissipation length that picks up different values at causal and acausal scenario resulting in different damping rate for different $k$. Although for small $k$ it is not significant but at large $k > 200$ MeV the difference is distinctly visible in the results displayed in Fig. 1. The decay of the perturbation with time is shown in Fig. 2 for $\eta/s = 1/4\pi$ for different $k$. We observe that at $T = 400$ MeV the perturbations decay faster in causal than acausal hydrodynamic as $k$ increases. Stronger damping is observed at $T = 200$ and $300$ MeV (not shown in the figure). At large $t$, the amplitude of the perturbations for causal and acausal scenarios are close because at large $t$ the amplitude decays to a very small value irrespective of the value of $\omega T$. Similarly at small $t$ the amplitude of the perturbation are also close. The enhanced magnitude of $\eta/s$ enforces faster decay. All these results represent a physically consistent picture because it is well-known that in the acausal (first order) hydrodynamics a non-equilibrium system evolves to the equilibrium instantly. However, in second order hydrodynamics the non-equilibrium system does not go to the equilibrium state instantaneously but takes some non-
zero time. This non-zero time lag is incorporated in the
second order hydrodynamics by means of relaxation co-
coefficients such as $\beta_0, \beta_1, \beta_2$. In other words the
second order hydrodynamics effectively enhances the dissipa-
tion of the system. As any disturbance will dissipate faster
in a higher order viscous hydrodynamics than the lower one,
the perturbations in causal disturbances fall faster than the acausal one. We have observed that the ampli-
tude of sound wave falls faster with increase in $\eta/s$ and
decrease in $T$.

**B. Longitudinal mode**

To study the perturbations in longitudinal direction, we encoun-
ter a new relaxation coefficient, $\beta_0$ that was absent in acausal theory. In the ultra-relativistic limit
$\beta_0$ is given by [12],

$$\beta_0 = \frac{216}{P_0}$$

(51)

where $\beta = m/T$. We have used thermal mass to estimate
$\beta$. To study the propagation of the longitudinal modes
in the fluid we consider the gluonic fluid. The thermal
mass of gluon is given by [26]

$$\frac{m_g}{T} = g \sqrt{\frac{C_A + \frac{N_f}{2}}{6}} \Rightarrow \beta = g \sqrt{\frac{C_A + \frac{N_f}{2}}{6}}$$

(52)

where $g = \sqrt{4\pi\alpha_s}$, $C_A = 3$ and $N_f = 2$ (for two flavours).
In the present work we have taken $\alpha_s = 0.2$. We use
Eq. 18 with the aid of $\beta_0$ to study the dissipation of
the longitudinal modes. One major difference with the
transverse mode is the appearance of bulk viscosity in the
longitudinal mode and it will be seen later that bulk vis-
cosity plays dominant role in the damping of the pertur-
bations. The nature of variation of the perturbations of
longitudinal mode is similar to that of transverse modes.
The perturbation decays faster with $k$ in causal than
acausal hydrodynamics (Fig. 3). At lower $T$ a faster de-

cay is observed. In Fig. 4, we depict the dissipation of
the perturbations with time for $\eta/s = 1/4\pi$ for differ-
tent $k$ values. A faster decay is observed at higher $\eta/s$ and
lower $T$. Similar to the transverse modes the differ-
ence in the decay of longitudinal amplitudes in causal
and acausal hydrodynamics is significant. We have dis-
cussed before that the longitudinal dispersion relation is
controlled not only by shear but by the bulk viscosity as
well. The damping of the longitudinal modes due to shear
and bulk viscous coefficients and the relative importance
of these coefficients are investigated. The variation of the
damping with $k$ has been depicted in Figs. 5. The result
indicates a bigger influence of the bulk viscosity on the
longitudinal modes than the shear viscosity.

As mentioned in section III the QGP fluid may be sub-
jected to the external magnetic field ($B$) created due to
the relativistic motion of the colliding nuclei. The mag-
nitude of the field during evolution of QGP will depend
on the rate of decay of the field which is controlled by
the value of electrical conductivity of QGP. We assume a
non-zero constant magnetic field in the QGP and study
its effects on the fluid properties. We find that the en-
ergy due to magnetic field appears with opposite sign in
the denominators of $\omega_\parallel$ and $\omega_\perp$ given by Eqs. 30 and
31 respectively. This is reflected in the results displayed
Figs. 6 and 7 for the variation of damping with $k$ and $t$ re-
spectively. The transverse modes decays faster in causal
hydrodynamics. An opposite trend is observed for the
longitudinal modes.
C. Quantitative changes in the viscous horizon

We would like to estimate the shift in the viscous horizon caused by causal hydrodynamics as compared to the acausal one. The viscous horizon size scales with time as: $R_v \sim 1/\sqrt{t}$. Through the relation, $R_v(fm) \approx 197/k_v(MeV)$, it determines the wave length that is unable to propagate in the dissipative medium, i.e. if the wave length is less than $2\pi/k_v$ then those waves will dissipate.

Using Eqs. 34, 35, and 36 we can estimate viscous horizon scales at different time. The variation of $k_v$ with $t$ for causal and acausal hydrodynamics has been depicted in Fig. 8 at $T = 400$ MeV. It is observed that $k_v$ for the causal scenario approaches the $k_v$ for the acausal scenario at large $t$. This trend can be understood from the mathematical expressions of Eqs. 34 and 35. However, if the time variation of pressure due to hydrodynamic evolution is considered then $\beta_1$ and $\beta_2$ will also increase with time as evident from Eqs. 50 and 51 and in such situation the difference between the causal and acausal scenario may survive at large $t$ also.

In Fig. 9, we display the ratio of viscous horizon lengths for causal and acausal hydrodynamics as a function of $t$ for $T = 200$ and 400 MeV. We find that the longitudinal scale in causal hydrodynamics is almost 3 times larger than acausal one at $t = 0.6$ fm for $T = 200$ MeV and $\eta/s = \zeta/s = 1/4\pi$. The same ratio becomes 2.07 for $T = 400$ MeV at $t = 0.6$ fm/c. We also note that the difference in the viscous horizon length for transverse modes is smaller than the longitudinal modes.

The viscous damping controls the highest order of flow harmonic ($n_v$) that will survive against the dissipative effects. The relation between $n_v$ and $R_v$ is given by [23]: $n_v = 2\pi R/R_v$ where $R$ is the size of the fluid system. Therefore, an increase in $R_v$ will reduce the value of $n_v$ resulting in a shift in its value between causal and acausal scenarios. Since the value of $n_v$ depends on $\eta/s$, measurement of amplitudes of various harmonics will help in determining the viscosity and consequently characterizing QGP [22].

D. Measure of Fluidity

First we consider a system devoid of bulk viscosity. Then the fluidity measure of such system can be obtained by putting $\zeta = 0$ in Eq. 46 which leads to:

$$F = \frac{\rho^2\{16\eta^2 + 48\eta^2\beta_2(P + \epsilon)\left(\frac{\partial P}{\partial \tau}\right)^2\}^{1/2}}{\left[4\{P + \epsilon\left(\frac{\partial P}{\partial \tau}\right)\}^2\right]^{1/2}}$$  (53)
where \( \beta_2 = 3/4P \) in the relativistic limit. We use Eq. 53 to display the variation of \( F \) with \( T \) for the following inputs. The particle number density \( (\rho) \) is estimated from entropy density \( (s) \) by using the relation, \( \rho \sim s/4 \). We have used the parametric form of specific viscosity given in Ref. [27] as:

\[
\frac{\eta(T)}{s(T)} \approx \frac{1}{4\pi} \left( \frac{s_Q}{s_H} \right) \left( \frac{T}{T_c} \right)^{\frac{1}{2}} \text{ for } T < T_c \\
\approx \frac{1}{4\pi} \left[ 1 + W \ln \frac{T}{T_c} \right]^2 \text{ for } T > T_c
\]

(54)

where \( s_Q \) and \( s_H \) are the entropy densities in the QGP and hadrons at the transition temperature \( (T_c = 150 \text{ MeV}) \). \( W \) is given by

\[
W^2 = \frac{9\beta_0^2}{80\pi^2 K_{SB} \ln \left\{ \frac{4\pi}{g^2(T)} \right\}}
\]

(55)

where

\[
[g^2(T)]^{-1} = \frac{9}{8} \pi^2 \ln(2\pi T/\Lambda) + \frac{4}{9} \pi^2 \ln 2[\ln(2\pi T/\Lambda)], \quad (56)
\]

and \( K_{SB} = 12, \ \Lambda = 190 \text{ MeV} \) and \( \beta_0 = 10 \). The value of entropy density \( (s) \) and \( c_s^2 \) for the hadronic and QGP phases have been estimated from hadronic resonance gas (HRG) model [28] and quasi-particle QGP model. The relevant thermodynamic quantities have been derived from the partition function using standard relations. The \( F \) is displayed as a function of \( T \) in Fig. 10 for \( \eta/s = 1/4\pi \). We observe that the value of \( F \) has increased in the causal scenario compared to the acausal dynamics. It is to be also noted that the enhancement is more with larger specific shear viscosity. The \( F \) has a non-linear dependence on the transport coefficients and thermodynamic variables in causal scenario. However, in the acausal case the dependence on the coefficient of viscosity is linear. This is reflected in the results already depicted in Fig. 10 as well as results displayed below. We observe a sharp decrease of \( F \) in the hadronic phase with the increase in temperature, \( i.e. \) hadrons flow easily with rise in temperature. However, the temperature variation of \( F \) in QGP phase is slower. As \( F \) is larger in causal limit the fluid flow becomes difficult compared to acausal case.

To study the sensitivity of the results on the velocity of sound we use the value of \( c_s^2 \) and other relevant thermodynamic variables, like entropy density, \( T \) from lattice QCD calculations [7]. The variation of \( F \) with \( T \) is displayed in Fig. 11. A larger discontinuity in \( F \) has been seen when \( T_c \) and \( c_s^2 \) are taken from lattice QCD calculations. The shift of fluidity in second order hydrodynamics from the first order is about 35% both in the hadronic as well as in QGP phase near \( T_c \). The same value of \( \eta/s \) has been used for second and first order hydrodynamics, therefore, the shift in \( F \) is due to stronger...
damping in causal hydrodynamics.

Fig. 12 shows the dependence of fluidity of QGP on bulk viscosity in a causal dynamical scenario determined by Eq. 46. \( \beta_0, \beta_2 \) and \( \beta \) are taken as \( 216/P\beta^4, 3/4P \) and 0.7 respectively. The bulk viscosity of the QGP phase has been taken in terms of shear viscosity as \(|\zeta|=18\times400\times200\times\text{in the presence of bulk viscosity}\)

\[ \zeta \approx 15\frac{\eta}{s}\left(1/3-c^2_s\right)^2 \quad (57) \]

where the parametric form of \( \eta/s \) is taken from Eq. 54. We find a peak in the value of \( \zeta/s \) around \( T \approx 150 \text{ MeV} \) (Fig. 13). This peak is reflected as a bump in the temperature variation of \( F \) just above \( T_c \), due to large conformal breaking \((1/3-c^2_s)^2\) near \( T_c \). It is also interesting to note that the bulk viscosity hardly play any role at higher \( T \) due to its small numerical value. As \( T \) increases, beyond \( T=250 \text{ MeV} \), conformal invariance restores and that results in almost vanishing \( \zeta/s \). However, a constant \( \zeta/s=1/4\pi \) represents a different picture as shown in Fig. 14. It is clear that non-zero value of \( \zeta/s \) will play a crucial role in determining the fluidity of the system.

We have shown before that the magnetic field alters both the transverse and longitudinal modes. Therefore, it will affect the fluidity of the QGP as shown in Fig. 15. The \( F \) for hadronic phase with magnetic field has not been shown, because the magnetic field will decay substantially and hence will have insignificant effects on fluidity of the hadronic phase which appears late in the evolution history. As we discussed earlier \( B \) makes the fluid less dissipative in the QGP phase. Near \( T_c \), \( F \) reduces significantly and hence the flow becomes easier near \( T_c \).

For AdS/CFT system, we have taken the well known KSS lower bound \((\eta/s=1/4\pi)\) of shear viscosity \([29]\) to show variation of \( F \) with \( T \) above \( T_c \) (Fig. 16). We have taken \( L_\rho=1/T \) \([13]\) which gives \( F \approx 0.2 \) in acausal hydrodynamics and \( F \approx 0.4 \) in its causal counterpart. The fluidity factor \( F \) gets enhanced as expected in Israel-Stewart hydrodynamics by a factor of 2 hence makes it harder for the fluid to flow.

In Fig. 17 the variation of the ratio of two length scales, \( L_T/L_\eta \) has been plotted as a function of \( T \). We find that the ratio remains above unity for the temperature range considered. It is discussed in Ref. \([13]\) that the applicability of hydrodynamics may be resolved from the ratio of \( L_\eta \) estimated in acausal hydrodynamics to some external length scale, say, the size of the system, \( R \). Since \( L_T/L_\eta > 1 \), therefore, the applicability of hydrodynamics become poorer when causality effects are included in the fluid dynamics, if all other relevant quantities kept same in causal and acausal scenarios.

VI. SUMMARY AND CONCLUSION

In summary, we have derived dispersion relations of relativistic fluid using Israel-Stewart second order causal viscous hydrodynamics. It is shown that the dispersion

\[ \zeta \]

\[ \frac{\eta}{s} \]

\[ \left(1/3-c^2_s\right)^2 \]

\[ \approx 15\frac{\eta}{s}\left(1/3-c^2_s\right)^2 \]

\[ \frac{\zeta}{s} \]

\[ \frac{\eta}{s} \]

\[ \left(1/3-c^2_s\right)^2 \]

\[ \zeta \approx 15\frac{\eta}{s}\left(1/3-c^2_s\right)^2 \]

\[ 150 \text{ MeV} \]

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relations in acausal hydrodynamics can be obtained from the causal results as a limiting case. The perturbations in viscous fluid damp faster within the scope of causal hydrodynamics than its acausal counterpart. The waves with large $k$ suffer more damping than the waves with short $k$. In both the longitudinal and transverse dispersion relations the difference between the causal and acausal hydrodynamics is significant. The difference increases with the magnitude of viscosities. It has also been noted that the bulk viscosity does not play any role in the dissipation of the transverse modes but it plays a crucial role in the dispersion for the longitudinal modes. The dispersion relations in the presence of magnetic field have also been derived and it is shown that the magnetic field affects the longitudinal and transverse modes oppositely. The magnetic field makes the fluid effectively less dissipative. The dispersion relations derived here have been used to find viscous measure of the fluid as well as the viscous horizon. We have seen that the use of causal relations enhances the size of the viscous horizon of the longitudinal mode by more than a factor two for the parameter values used here. Inclusion of the causal- ity enhances the $F$ of QGP near $T_c$. The bulk viscosity affects the fluidity strongly near $T_c$. However, its role becomes less important at higher temperature with the restoration of conformal symmetry resulting in lower $\zeta$. We also find that the effects of $\zeta$ on $F$ is more prominent than $\eta$ if $\eta$ and $\zeta$ have similar magnitudes. Magnetic field makes a fluid more perfect by compensating the effects of viscosity near $T_c$. The fluidity is enhanced by a constant factor for AdS/CFT fluid within the causal hydrodynamics.

In Ref. [13] the fluidity has been studied in the supercritical domain within the purview of acausal hydrodynamics. We observed a shift in fluidity due to causal hydrodynamics as compared to acausal hydrodynamics.
It is expected that similar shift will be seen in the supercritical region too.

In a nutshell the incorporation of causality in relativistic hydrodynamics makes the following changes with respect to acausal hydrodynamics: (i) The fluidity measure, $F$, increases and thus flow of the fluid becomes strenuous, (ii) the value of the highest order of flow harmonics ($n_\nu$) reduces as the viscous horizon, $R_\nu$ increase and (iii) applicability of the hydrodynamics becomes poorer because of $LT > L_\eta$ in the temperature range considered. In these conclusions it has been tacitly assumed that the relevant quantities, like $n/s$ etc are kept same in both the causal and acausal scenarios.

VII. APPENDIX

A. Perturbations in $T^{\lambda\mu}$

We evaluate the perturbation in the energy-momentum tensor ($T^{\lambda\mu}$), of the Israel-Stewart hydrodynamics. We denote perturbations in $P, \epsilon, n, T$ and $u^a$ by $P_1, \epsilon_1, n_1, T_1$ and $u_1^a$ respectively and decompose $T^{\lambda\mu}$ in Eq. 6 into sum of $A^{\lambda\mu}, B^{\lambda\mu}, C^{\lambda\mu}, D^{\lambda\mu}, E^{\lambda\mu}$, and $F^{\lambda\mu}$. We assume the perturbations as $P' = P + P_1, \epsilon' = \epsilon + \epsilon_1, n' = n + n_1, T' = T + T_1$ and $u' = u + u_1$. With perturbation $A^{\lambda\mu} = (\epsilon u^\lambda u^\mu + P \Delta^{\lambda\mu})$ changes to

\[
A_1^{\lambda\mu} = \epsilon u^{\lambda} u^{\mu} + \epsilon_1 u^{\lambda} u^{\mu} + \epsilon u^{\lambda} u^{\mu} + \epsilon u^{\lambda} u^{\mu} + P \Delta^{\lambda\mu} + P u_1^{\lambda} u^{\mu} + P u^{\lambda} u_1^{\mu} + P_1 \Delta^{\lambda\mu}
\]

where we keep only the linear terms in perturbations. Thus, the change in $A^{\lambda\mu}$ reads

\[
A_1^{\lambda\mu} = \epsilon_1 u^{\lambda} u^{\mu} + \epsilon_1 u^{\lambda} u^{\mu} + \epsilon u^{\lambda} u^{\mu} + \epsilon u^{\lambda} u^{\mu} + P \Delta^{\lambda\mu} + P u_1^{\lambda} u^{\mu} + P u^{\lambda} u_1^{\mu} + P_1 \Delta^{\lambda\mu}
\]

Similarly the change in the term $B^{\lambda\mu} = \frac{1}{3} \zeta u^\sigma \Delta^{\lambda\mu} + \frac{i}{3} \zeta^2 \beta_0 u_0^{\sigma} \Delta^{\lambda\mu}$ arising due to perturbation:

\[
B_1^{\lambda\mu} = \frac{1}{3} \zeta \partial_\sigma u_1^\sigma - \frac{1}{3} \zeta \beta_0 \partial_\sigma u_0^\sigma \Delta^{\lambda\mu}
\]

Perturbation in $C^{\lambda\mu} = \frac{nT^2}{P + \epsilon} \zeta a_0 \partial_\sigma [\Delta^{\sigma\rho} \partial_\rho \alpha (\Delta^{\lambda\mu})]$ is:

\[
C_1^{\lambda\mu} = \frac{1}{3} \zeta a_0 \partial_\sigma [(nT^2 + 2TT_1n) \Delta^{\lambda\mu}] + \frac{nT^2}{P + \epsilon} \partial_\rho a_0 \partial_\rho \partial_\sigma [\Delta^{\sigma\rho} \partial_\rho \alpha (\Delta^{\lambda\mu})] + \frac{nT^2}{P + \epsilon} \partial_\rho a_0 \partial_\rho \partial_\sigma [\Delta^{\sigma\rho} \partial_\rho \alpha (\Delta^{\lambda\mu})]
\]

Perturbation in $D^{\lambda\mu} = -2\eta u^{\lambda} u^{\mu}$ reads:

\[
D_1^{\lambda\mu} = -\eta (\Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta + \Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta - \frac{2}{3} \Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta) \partial_\sigma u_1^\sigma - \frac{2}{3} \Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta) \partial_\sigma u_1^\sigma
\]

Change in the term $E^{\lambda\mu} = (4\eta^2 \beta_2 u^k C^{\lambda\mu} >$ due to perturbation is:

\[
E_1^{\lambda\mu} = 2\eta^2 \beta_2 \partial_0 (\Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta - \frac{2}{3} \Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta) \partial_\sigma u_1^\sigma
\]

The $F^{\lambda\mu} = \frac{2\eta^2 \beta_2}{P + \epsilon} \partial_0 \eta \chi [\Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta - \frac{2}{3} \Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta] \partial_\sigma \partial_\rho \alpha$ is perturbed by the term:

\[
F_1^{\lambda\mu} = \alpha_1 \eta \chi [\Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta - \frac{2}{3} \Delta^{\lambda\mu} \Delta_\alpha \Delta_\beta] \partial_\sigma \partial_\rho \alpha
\]

The net change in $T^{\lambda\mu}$ due to perturbation is the sum of all terms discussed above:

\[
T_1^{\lambda\mu} = A_1^{\lambda\mu} + B_1^{\lambda\mu} + C_1^{\lambda\mu} + D_1^{\lambda\mu} + E_1^{\lambda\mu} + F_1^{\lambda\mu}
\]

B. Dispersion relation for the longitudinal mode.

The linearized equation of motion (EoM) of the Israel-Stewart hydrodynamics can be written in terms of the independent variables (perturbations) e.g. $(\vec{k} \cdot \hat{u}_i), T_i$ and $n_1$. Then the dispersion relation can be obtained by setting the determinant of the coefficients of the linear algebraic equations satisfied by $(\vec{k} \cdot \hat{u}_i), T_i$ and $n_1$ to zero. Expanding this determinant and solving for $\omega$ leads to the dispersion relation for longitudinal component. The determinant formed by three unknown coefficients in Eqs.10, 11 and 12 is:

\[
0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\]

where the values of the different matrix elements are given below:

\[
a_{11} = \omega (P + \epsilon) + \frac{nT^2}{3(\epsilon + P)} \zeta \chi a_0 \omega \nabla^2 \alpha + \frac{2nT^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|} \\
= \frac{nT^2}{3(\epsilon + P)} \zeta \chi a_0 \omega \nabla^2 \alpha + \frac{2nT^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|} \\
- \frac{nT^2}{3(\epsilon + P)} \zeta \chi a_0 \omega |\vec{k}|^2 \partial_\omega k^2
\]

\[
a_{12} = i \delta k^2 - \frac{8}{9} \eta^2 \beta_2 \omega k^2 - \frac{8nT^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|} \\
- \frac{4\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

\[
a_{13} = \frac{2\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

\[
a_{21} = i \delta k^2 - \frac{8}{9} \eta^2 \beta_2 \omega k^2 - \frac{8nT^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|} \\
- \frac{4\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

\[
a_{22} = \frac{2\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

\[
a_{23} = \frac{2\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

\[
a_{31} = \frac{2\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

\[
a_{32} = \frac{2\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

\[
a_{33} = \frac{2\eta^2 \beta_2 \omega k^2}{3(\epsilon + P) a_1 \eta \omega |\vec{k}|}
\]

where $\delta k^2 = k^2 - k_i^2$ is the dispersion relation for longitudinal component.
\[
\begin{align*}
  a_{12} &= -k^2 \left( \frac{\partial P}{\partial T} \right)_n - \Re \left[ 2nT \frac{\partial P}{P + \epsilon} - \frac{nT^2}{(P + \epsilon)^2} \left( \frac{\partial P}{\partial T} \right)_n \right] \\
  &\quad + \left( \frac{\partial \epsilon}{\partial T} \right)_n \\
  a_{13} &= -k^2 \left( \frac{\partial P}{\partial T} \right)_T - \Re \left[ \frac{T^2}{P + \epsilon} - \frac{nT^2}{(P + \epsilon)^2} \left( \frac{\partial P}{\partial T} \right)_T \right] \\
  &\quad + \left( \frac{\partial \epsilon}{\partial T} \right)_T \\
  a_{21} &= - (\epsilon + P) - \frac{\zeta nT^2}{3(\epsilon + P)} \chi_0 \nabla^2 \alpha - \frac{2nT^2}{(\epsilon + P)} \alpha_1 \eta \chi \\
  &\quad \times \left\{ \frac{\left( \vec{k} \cdot \nabla \right) \left( \vec{k} \cdot \nabla \right) \alpha}{k^2} - \frac{1}{3} \nabla^2 \alpha \right\} \\
  a_{22} &= \omega \left( \frac{\partial \epsilon}{\partial T} \right)_n \\
  a_{23} &= \omega \left( \frac{\partial \epsilon}{\partial T} \right)_T \\
  a_{31} &= -n \\
  a_{32} &= 0 \\
  a_{33} &= \omega
\end{align*}
\]

where
\[
\Re = \frac{i \zeta \chi_0 k^2 \left( \vec{k} \cdot \nabla \right) \alpha}{3} + \frac{\zeta \chi_0 k^2 \nabla^2 \alpha}{3} + i \frac{4 \alpha_1 \eta \chi}{3} k^2 \left( \vec{k} \cdot \nabla \right) \alpha 
\]

Expanding the above determinant and keeping terms upto 2nd order in \( \eta T / h, \zeta T / h \) and their products like \( \eta \chi, \eta \xi, \xi \chi \), we get an equation of the form
\[\omega (\omega^2 + k \omega + c) = 0\] (64)

which has a trivial solution \( \omega = 0 \) and the other two roots can be found by solving the quadratic equation \( \omega^2 + k \omega + c = 0 \). The coefficients of the quadratic equation is given by
\[
\begin{align*}
  a &= \left[ \left( \frac{P + \epsilon}{P + \epsilon} \right) + \frac{nT^2}{3(P + \epsilon)} \chi_0 \nabla^2 \alpha - \frac{1}{9} k^2 \zeta^2 \beta_0 \\
  &\quad + \frac{2nT^2 \eta \chi_0 (\vec{k} \cdot \nabla) \left( \vec{k} \cdot \nabla \right) \alpha}{k^2} - \frac{2nT^2}{3(P + \epsilon)} \eta \chi_0 \nabla^2 \alpha \\
  &\quad - \frac{8}{3} k^2 \eta \chi_0 \beta_2 + \frac{i nT^2}{3(P + \epsilon)} \chi_0 \left( \vec{k} \cdot \nabla \right) \alpha \left( \frac{\partial \epsilon}{\partial T} \right)_n \\
  b &= \left[ -\frac{2}{3} \frac{nT^2}{(P + \epsilon)} \chi_0 \left( \vec{k} \cdot \nabla \right) \alpha - \frac{8 nT^2 \eta \chi_0 (\vec{k} \cdot \nabla) \left( \vec{k} \cdot \nabla \right) \alpha}{3(P + \epsilon)} \\
  &\quad + \frac{i}{3} \left( \frac{1}{3} k^2 \zeta + \frac{4}{3} k^2 \eta - \frac{nT^2}{(P + \epsilon)} k^2 \zeta \chi_0 \alpha \right) \\
  &\quad - \frac{4}{3} \frac{nT^2}{(P + \epsilon)} k^2 \alpha \eta \chi \alpha \left( \frac{\partial \epsilon}{\partial T} \right)_n \\
  c &= -k^2 (P + \epsilon) \left( \frac{\partial P}{\partial T} \right)_n + k^2 n \left( \frac{\partial P}{\partial T} \right)_n \left( \frac{\partial \epsilon}{\partial n} \right)_T \\
  &\quad - k^2 n \left( \frac{\partial P}{\partial n} \right)_T \left( \frac{\partial \epsilon}{\partial n} \right)_T - \frac{2}{3} k^2 n T \zeta \alpha_0 \nabla^2 \alpha \\
  &\quad + \frac{2nT^2}{(P + \epsilon)} \alpha_0 \zeta \chi_0 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \nabla^2 \alpha \\
  &\quad - \frac{nT^2}{3(P + \epsilon)^2} \alpha_0 \zeta \chi_0 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \nabla^2 \alpha \\
  &\quad - \frac{nT^2}{3(P + \epsilon)^2} \alpha_0 \zeta \chi_0 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \nabla^2 \alpha \\
  &\quad + \frac{2nT^2}{3(P + \epsilon)} \alpha_1 \eta \chi k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \nabla^2 \alpha - \frac{2nT^2}{3(P + \epsilon)} \eta \chi \alpha_1 \left( \frac{\partial \epsilon}{\partial n} \right)_T n \\
  &\quad \times \left( \vec{k} \cdot \nabla \right) \left( \vec{k} \cdot \nabla \right) \alpha + \left[ -\frac{8}{3} k^2 n T \eta \chi_0 \right] + \frac{nT^2}{3(P + \epsilon)} \alpha_0 \zeta \chi_0 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \\
  &\quad + \frac{nT^2}{3(P + \epsilon)} \alpha_0 \zeta \chi_0 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \left( \frac{\partial \epsilon}{\partial n} \right)_T - \frac{8}{3} k^2 n T \eta \chi_0 \alpha_1 \\
  &\quad + \frac{4nT^2}{3(P + \epsilon)} \eta \chi \alpha_1 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T + \frac{8nT^2}{3(P + \epsilon)} \eta \chi \alpha_1 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \\
  &\quad - \frac{4nT^2}{3(P + \epsilon)} \eta \chi \alpha_1 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \left( \frac{\partial \epsilon}{\partial n} \right)_T - \frac{4nT^2}{3(P + \epsilon)} \eta \chi \alpha_1 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \nabla^2 \alpha \\
  &\quad + \frac{4nT^2}{3(P + \epsilon)} \eta \chi \alpha_1 k^2 \left( \frac{\partial \epsilon}{\partial n} \right)_T \left( \frac{\partial \epsilon}{\partial n} \right)_T \left( \vec{k} \cdot \nabla \right) \alpha \\
  &\quad \left( \frac{\partial \epsilon}{\partial T} \right)_n [\vec{k} \cdot \nabla] \alpha \left( \frac{\partial \epsilon}{\partial T} \right)_n \\
  &\quad \left[ \frac{\partial \epsilon}{\partial T} \right)_n
\end{align*}
\]

The physical solution of Eq. 64 gives the general dispersion relation for non-zero \( \eta, \zeta, \chi \) as well for non-zero (baryonic) conserved charge.

\section{C. Perturbations in \( T^{\mu \nu} \) in the presence of magnetic field}

The energy-momentum tensor in the presence of magnetic field \( B \) is given by Eq. 23 with \( \eta^\mu = B^\mu / B \) where \( B_\mu = (1/2) \epsilon_{\mu \nu \alpha \beta} F^{\nu \beta} u^\alpha \) and \( F^{\mu \nu} = (E^{\mu} u^{\nu} - E^{\nu} u^{\mu}) + (1/2) \epsilon^{\mu \nu \beta \gamma} (u_\beta B_\gamma - u_\gamma B_\beta) \). For vanishing electric field \( E \), the expression for energy-momentum tensor (Eq. 23) can be written as:
\[
T^\lambda_\mu = \frac{B^2}{8\pi} \left( 2u^\lambda u^\mu + g^\lambda \mu - \frac{1}{8} \lambda^\rho \eta^\sigma \epsilon_{\rho \sigma \alpha \beta} \epsilon^{\rho \sigma \alpha \beta} \right) (u^\alpha B^\beta - u^\beta B^\alpha) u_\eta u_\eta
\]

Using the following relation satisfied by the Levi-civita tensor
\[
\epsilon^{\alpha \beta \gamma} \epsilon_{\alpha \beta \gamma} = -2 (g^{\alpha}_0 g^{\beta}_0 - g^{\gamma}_0 g^{\gamma}_0)
\]
Eq.68 can be written as:
\[ T_m^{\lambda \mu} = \frac{B^2}{8\pi} \left( 2u^\lambda u^\mu + g^\lambda \mu - \frac{1}{2B^2} (g^\alpha_\beta \tilde{g}^\gamma_\delta - g^\alpha_\gamma g^\beta_\delta) (g^\mu_\alpha \tilde{g}^\nu_\beta - g^\mu_\beta \tilde{g}^\nu_\alpha) (u^\alpha B^\beta - u^\beta B^\alpha) \right) \]
\[ + \frac{1}{B^2} u^\nu B^\alpha (u^\alpha B^\beta - u^\beta B^\alpha) u_{\eta\eta'} \]
\[ \frac{1}{2B^2} (u^\alpha \tilde{B}^\beta u^\beta_\nu - u^\beta_\nu \tilde{B}^\alpha u^\alpha) (u^\mu B^\nu - u^\nu B^\mu) \]
\[ \frac{1}{B^2} (u^\alpha \tilde{B}^\beta u^\beta_\nu - u^\beta_\nu \tilde{B}^\alpha u^\alpha) (u^\mu B^\nu - u^\nu B^\mu) \]
\[ \frac{1}{B^2} (u^\alpha \tilde{B}^\beta u^\beta_\nu - u^\beta_\nu \tilde{B}^\alpha u^\alpha) (u^\mu B^\nu - u^\nu B^\mu) \]
\[ \frac{1}{B^2} (u^\alpha \tilde{B}^\beta u^\beta_\nu - u^\beta_\nu \tilde{B}^\alpha u^\alpha) (u^\mu B^\nu - u^\nu B^\mu) \]
\[ \frac{1}{B^2} (u^\alpha \tilde{B}^\beta u^\beta_\nu - u^\beta_\nu \tilde{B}^\alpha u^\alpha) (u^\mu B^\nu - u^\nu B^\mu) \]
\[ \frac{1}{B^2} (u^\alpha \tilde{B}^\beta u^\beta_\nu - u^\beta_\nu \tilde{B}^\alpha u^\alpha) (u^\mu B^\nu - u^\nu B^\mu) \]
\[ \frac{1}{B^2} (u^\alpha \tilde{B}^\beta u^\beta_\nu - u^\beta_\nu \tilde{B}^\alpha u^\alpha) (u^\mu B^\nu - u^\nu B^\mu) \]

The last term of the RHS of Eq.69 can be decomposed into 16 terms, each of which will contain product of four fluid velocity \((\mu,\nu,\sigma,\tau)\). If we write, \(u^\mu \rightarrow u^\mu + u_1^\mu\) and use \(u^0 = 1, u^i = 0\), then the only non-zero terms are:
\[ (1/2) u^\alpha u^\nu u_{1\mu} u_{1\nu} B^\beta (\lambda_\mu + \nu_\sigma + \mu_\nu + \mu_\nu) \]
\[ (1/2) u^\alpha u^\nu u_{1\mu} u_{1\nu} B^\beta (\lambda_\mu + \nu_\sigma + \mu_\nu + \mu_\nu) \]
\[ (1/2) u^\alpha u^\nu u_{1\mu} u_{1\nu} B^\beta (\lambda_\mu + \nu_\sigma + \mu_\nu + \mu_\nu) \]

These expressions for the energy momentum tensor due to presence of magnetic field in the fluid have been used to calculate the dispersion relation for the longitudinal and transverse wave in this work.

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