Effective field theory of modified gravity on the spherically symmetric background: leading order dynamics and the odd-type perturbations

Ryotaro Kase,1 László Á. Gergely,1,2 and Shinji Tsujikawa1
1Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
2Departments of Theoretical and Experimental Physics, University of Szeged, Dóm tér 9, 6720 Szeged, Hungary
(Dated: December 9, 2014)

We consider perturbations of a static and spherically symmetric background endowed with a metric tensor and a scalar field in the framework of the effective field theory of modified gravity. We employ the previously developed 2+1+1 canonical formalism of a double Arnowitt-Deser-Misner (ADM) decomposition of space-time, which singles out both time and radial directions. Our building block is a general gravitational action that depends on scalar quantities constructed from the 2+1+1 canonical variables and the lapse. Variation of the action up to first-order in perturbations gives rise to three independent background equations of motion, as expected from spherical symmetry. The dynamical equations of linear perturbations follow from the second-order Lagrangian after a suitable gauge fixing. We derive conditions for the avoidance of ghosts and Laplacian instabilities for the odd-type perturbations. We show that our results not only incorporate those derived in the most general scalar-tensor theories with second-order equations of motion (the Horndeski theories) but they can be applied to more generic theories beyond Horndeski.

I. INTRODUCTION

The unexpected discovery of the late-time cosmic acceleration from the supernovae type-Ia (SN Ia) observations [1,2] has pushed forward the idea that the gravitational law may be modified from General Relativity (GR) at large distances. The recent CMB measurement by Planck [3] combined with data of the WMAP polarization [4] and the recent SN Ia (from SNLS [5]) showed that the dark energy equation of state is constrained to be $w_{\text{DE}} = -1.13^{+0.13}_{-0.14}$ (95 % CL) for constant $w_{\text{DE}}$. In GR it is generally difficult to explain $w_{\text{DE}} < -1$ unless a ghost mode is introduced, but the modification of gravity allows a possibility of realizing such an equation of state while avoiding ghosts and instabilities [6].

Most of the dark energy models proposed in the literature—such as quintessence [7], k-essence [8], $f(R)$ gravity [9,10], Brans-Dicke theory [11,12], and Galileons [13–16]—belong to the category of the Horndeski theory [17–19], i.e., the most general scalar-tensor theory with second-order equations of motion. In the Horndeski theory, the conditions for avoiding ghosts and Laplacian instabilities of scalar and tensor perturbations have been derived in Refs. [20–22] on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background in the absence/presence of matter. Imposing these conditions and studying the background dynamics as well as the growth of density perturbations [22], we can test for theoretical consistent models of dark energy with numerous observational data.

For the unified description of modified gravitational theories, there is another approach based on the effective field theory (EFT) of cosmological perturbations [23–44]. The starting point of this formalism is a generic action in unitary gauge that depends on the lapse function and several geometric scalar quantities appearing in the 3+1 ADM decomposition on the flat FLRW background. Expanding the action up to second-order in perturbations, the resulting linear perturbation equations generally contain spatial derivatives higher than second order. Gleyzes et al. [35] showed that the Horndeski theory satisfies the conditions for absence of such higher-order derivatives by explicitly rewriting the Lagrangian in terms of the ADM variables. The EFT of cosmological perturbations can deal with a wide range of gravitational theories beyond the domain of the Horndeski theory.

Models of the large-distance modification of gravity are required to recover Newtonian gravity at short distances for the consistency with local gravity tests in the Solar System. There are several ways to suppress the propagation of the fifth force induced by a scalar degree of freedom $\phi$. One of them is the Vainshtein mechanism [45], under which non-linear scalar-field self interactions appearing e.g., in Galileon gravity, lead to the decoupling of the scalar field from baryons inside the radius much larger than the solar system [46]. Another is the chameleon mechanism [47] applicable to $f(R)$ gravity [48] and Brans-Dicke theory [12], under which the fifth force outside a spherically symmetric body is suppressed by the formation of a thin shell inside the body with a large effective mass of the scalar field.

For the purpose of understanding the screening mechanism of the fifth force in general, the equations of motion in the Horndeski theory were derived on the spherical symmetric background [49,50]. The stability of static and spherically symmetric vacuum solutions in the same theory was also studied in Ref. [51] by considering the odd-parity...

1Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
2Departments of Theoretical and Experimental Physics, University of Szeged, Dóm tér 9, 6720 Szeged, Hungary

(Dated: December 9, 2014)
mode of perturbations (associated with tensor perturbations). The analysis of the even-parity perturbations, which is much more involved due to a non-trivial coupling between the scalar field and gravity, was recently performed in Ref. [53]. The spherically symmetric background solutions of viable modified gravity models need to accommodate the screening mechanism of the fifth force, while satisfying the stability conditions against perturbations.

The EFT of modified gravity on the isotropic cosmological background allows a possibility of dealing with the theories beyond Horndeski in a systematic and unified way [28]-[43]. If we try to apply a similar formalism to the spherical symmetric background, there is another spatial direction singled out by the ADM decomposition besides the temporal direction. The EFT of modified gravity with the singled-out radial direction has not been worked out yet.

There are several ways to deal with the perturbations of spherically symmetric and static space-times. Some of the approaches monitor the metric perturbations and they heavily rely both on the decomposition of the perturbations into even and odd modes (under parity transformations on the sphere) and on a full gauge fixing. This line of research includes the pioneering work of Regge, Wheeler and Zerilli [54, 55], leading to the Regge-Wheeler equation for the odd modes and the Zerilli equation for the even modes of general relativistic black hole perturbations. The discussion of perturbations in the Horndeski class of theories presented in Refs. [52, 53] falls into this class.

A second approach, which also heavily relies on gauge fixing, was followed in Chandrasekhar’s monumental monograph [56], where general relativistic black hole perturbations are discussed in terms of Newman-Penrose spin coefficients and their perturbations. This approach is based on the introduction of a Newman-Penrose tetrad and requires to solve 76 coupled differential equations in 50 independent variables.

A third approach keeps the full covariance in the discussion of the perturbations and extensively uses gauge-invariant quantities. Such a covariant 1+1+2 formalism was worked out for general relativistic perturbations of the Schwarzschild space-time in Ref. [57], and later it was generalized for generic locally rotationally symmetric space-times in Ref. [58]. The advantages include a unified treatment of the even and odd sectors of the perturbations in the form of wave equations holding for both. The price to pay is a formalism involving a much larger number of kinematical (shear, expansion, vorticity and acceleration type rather than metric) variables, and the use of directional derivatives which do not commute (as they are not related to coordinate derivatives), hence the need for commutation relations, similarly as in the Newman-Penrose formalism.

We employ yet another formalism based on the s+1+1 decomposition, where s is an arbitrary positive integer [59, 60] (developed with the application to braneworld models in mind). This ADM inspired formalism relies on a canonical, rather than a covariant approach. Based on a double foliation of space-time, it is more restrictive than either of the Newman-Penrose or the covariant 1+1+2 formalisms. The clear advantage over them is a much lower number of variables. In comparison with the metric perturbation formalism (for s = 2) the number of variables is the same, nevertheless the variables in the 2+1+1 ADM formalism carry canonical interpretation, which is a clear virtue when it comes to the EFT approach.

In this paper, we study the EFT of modified gravity on a static and spherically symmetric background by employing the 2+1+1 ADM formalism. In the gravitational action, we take into account all the possible scalar combinations constructed from geometric quantities. We show that the Horndeski theory and its recent generalization by Gleyzes-Langlois-Piazza-Vernizzi (GLPV) [42] can be accommodated in our general framework, by explicitly rewriting the corresponding Lagrangians in terms of the 2+1+1 covariant variables. The three independent background equations of motion are derived in simple forms, which will be useful for the study of the screening mechanism in general modified gravitational theories.

We also obtain the second-order Lagrangian for odd-type perturbations in the EFT framework to discuss the stability of spherically symmetric and static vacuum solutions. We derive conditions for avoiding ghosts/Laplacian instabilities and apply our results to both Horndeski and GLPV theories (including a “covariantized” version of the original Galileon model [14] whose coordinate derivatives are replaced by covariant derivatives). We defer the study of the even-type perturbations to a follow-up work due to its non-triviality and complexity.

This paper is organized as follows.

In Sec. III the basic elements of the 2+1+1 ADM decomposition will be reviewed as a brief summary of the formalism developed in Refs. [59, 60].

In Sec. IV we present a variational principle for a general action in unitary gauge expressed in terms of scalars constructed from the geometric quantities arising in the 2+1+1 decomposition. Varying the action up to first order in perturbations allows us to derive three equations of motion for the background.

In Sec. V we express the Lagrangians of both Horndeski and GLPV theories in terms of the variables appearing in the 2+1+1 formalism and show that they belong to the sub-class of our general framework.

In Sec. VI we explore the diffeomorphism gauge freedom in dealing with perturbations on the static and spherically symmetric background. After choosing the unitary gauge δ0 = 0, there is still a remaining gauge degree of freedom associated with the time component of a coordinate transformation vector ξμ. We show that this residual gauge degree of freedom does not affect the odd-type perturbations studied in this paper.

In Sec. VII we derive the second-order perturbed Lagrangian density for the odd-mode perturbations expressed in
terms of a dynamical scalar variable and its derivatives.

In Sec. VII we discuss conditions for the absence of ghosts and Laplacian instabilities and apply the results to both Horndeski and GLPV theories. We also specialize our results for two covariant Galileon models.

Sec. VIII is devoted to conclusions.

We use the abstract index notation throughout the paper, hence tensors defined on the full space-time and on the 2-dimensional surface carry the same set of Latin indices, but the latter obey certain projection conditions. All quantities defined on the background will carry an overbar.

II. THE 2+1+1 FORMALISM

We assume that the 4-dimensional space-time allows for a double foliation in the 2+1+1 formalism, e.g., it can be foliated both by constant time hypersurfaces $\Sigma_t$ and by constant spatial coordinate hypersurfaces $\Sigma_r$. The time-like unit congruence $n^a$ (satisfying $n^a n_a = -1$) is orthogonal to $\Sigma_t$, while the unit vector $l^a$ of the singled-out spatial direction (satisfying $l^a l_a = 1$) is orthogonal to $\Sigma_r$. For convenience we choose them mutually orthogonal ($n^a l_a = 0$).

The 2-surface orthogonal to both congruences is labeled as $\Sigma$. The embedding of the co-dimension 2 surfaces is characterized by two types of extrinsic curvatures, related to each of the normal vector fields $n^a$ and $l^a$:

$$K_{ab} = h^c a h^d_b \nabla_c n_d, \quad L_{ab} = h^c a h^d_b \nabla_c l_d,$$  

(2.6)

where $\nabla$ denotes the $g$-metric compatible connection.
There are also two normal fundamental forms

\[ K_a = h^b_a l^c \nabla_c n_b , \quad L_a = -h^b_a n^c \nabla_c l_b , \]  

(2.7)

and two normal fundamental scalars

\[ K = l^a \nabla_a n_b , \quad L = n^a h^b_a \nabla_a l_b \]  

(2.8)

to consider\(^1\). To summarize, the covariant derivatives of the normal vectors can be expressed as

\[ \nabla_a n_b = K_{ab} + l_a K_b + l_b K_a + n_a n_b \alpha + \alpha_{ab} , \]  

(2.9)

\[ \nabla_a l_b = L_{ab} + n_a L_b + n_b L_a + n_a n_b \beta + \beta_{ab} , \]  

(2.10)

where \(\alpha^a\) and \(\beta^a\) are the curvatures of the congruences \(n^a\) and \(l^a\), defined by

\[ \alpha^a = n^c \nabla_c n^a = D^a (\ln N) - L^a , \]  

(2.11)

\[ \beta^a = l^c \nabla_c l^a = -D^a (\ln M) + K n^a . \]  

(2.12)

Occasionally, both \(\alpha^a\) and \(\beta^a\) will be referred as accelerations.

From the symmetric property of the extrinsic curvatures and the relation \(n^a l_a = 0\), it has been shown in Ref. 59 that \(K_a = L_a\) holds. The quantities \(L_{ab}\) and \(L\) are expressed in terms of \(r\)-derivatives and the covariant derivatives \(D_a\) associated with \(h_{ab}\) as 60:

\[ L_{ab} = \frac{1}{2M} \left( \frac{\partial h_{ab}}{\partial t} - 2D_a(M_b) \right) , \]  

(2.13)

\[ L = -\frac{1}{MN} \left( \frac{\partial N}{\partial r} - M^a D_a N \right) . \]  

(2.14)

Hence they are just convenient abbreviations for spatial derivatives.

By contrast, the quantities \(K_{ab}, K^a\) and \(K\) give the time evolution of \(h_{ab}, M^a\) and \(M\), respectively 59:

\[ K_{ab} = \frac{1}{2N} \left( \frac{\partial h_{ab}}{\partial t} - 2D_a(M_b) \right) , \]  

(2.15a)

\[ K^a = \frac{1}{2MN} \left( \frac{\partial M^a}{\partial t} - \frac{\partial N}{\partial r} + M^b D_b N^a - N^b D_b M^a \right) , \]  

(2.15b)

\[ K = \frac{1}{MN} \left( \frac{\partial M}{\partial t} - N^a D_a M \right) , \]  

(2.15c)

so that they are velocity-type variables.

Thus the coordinates in the velocity phase-space are

\[ \{ h_{ab}, M^a, M; K_{ab}, K^a, K \} , \]  

(2.16)

which is a feature any 2+1+1 covariant Lagrangian description of modified gravity should take into account.

Note that the time and spatial derivatives along the singled-out directions of any tensor \(T_{b_1...b_s}^{a_1...a_r}\) which has vanishing contraction with both \(n^a\) and \(l^a\) in all indices are defined as projected Lie-derivatives 59, 60:

\[ \frac{\partial}{\partial t} T_{b_1...b_s}^{a_1...a_r} = h^{c_1}_{c_1} ... h^{c_r}_{c_r} h_{b_1...b_s} d_{c_1} ... d_{c_r} (4) \mathcal{L}_{\frac{\partial}{\partial t}} T_{d_1...d_s}^{c_1...c_r} \equiv \mathcal{L}_{\frac{\partial}{\partial t}} T_{b_1...b_s}^{a_1...a_r} = N \mathcal{L}_T T_{b_1...b_s}^{a_1...a_r} + \mathcal{L} N T_{b_1...b_s}^{a_1...a_r} , \]  

(2.17)

\[ \frac{\partial}{\partial r} T_{b_1...b_s}^{a_1...a_r} = h^{c_1}_{c_1} ... h^{c_r}_{c_r} h_{b_1...b_s} d_{c_1} ... d_{c_r} (4) \mathcal{L}_{\frac{\partial}{\partial r}} T_{d_1...d_s}^{c_1...c_r} \equiv \mathcal{L}_{\frac{\partial}{\partial r}} T_{b_1...b_s}^{a_1...a_r} = M \mathcal{L}_T T_{b_1...b_s}^{a_1...a_r} + \mathcal{L} M T_{b_1...b_s}^{a_1...a_r} , \]  

(2.18)

\(^1\) The sets \((K_{ab}, K_a, K)\) and \((L_{ab}, L_a, L)\) can also be interpreted as the tensorial, vectorial and scalar contributions in the 2+1 split of the extrinsic curvatures of the hypersurfaces perpendicular to the congruences \(n^a\) and \(l^a\), respectively.
where \(^{(4)}L_{V}\) and \(L_{V}\) hold for the 4-dimensional and 2-dimensional Lie-derivatives along any vector congruence \(V\).

For a scalar quantity \(S\), one has

\[
\frac{\partial}{\partial t}S = N (\nabla^a \nabla_a) S + N^a D_a S, \quad (2.19)
\]

\[
\frac{\partial}{\partial r}S = M (\ell^a \nabla_a) S + M^a D_a S. \quad (2.20)
\]

From the above expressions, it is immediate to see that the time and spatial derivatives along the singled-out direction of scalars which are constant on \(\Sigma_{tr}\) (such that the last terms in Eqs. (2.19) and (2.20) vanish) are also expressible as projected covariant derivatives, a property we will employ in what follows. For the rest of our paper, we also denote time derivatives by a dot and the derivatives along the singled-out spatial direction by a prime.

### III. EQUATIONS OF MOTION ON THE SPHERICALLY SYMMETRIC BACKGROUND

We consider general gravitational theories with a single scalar degree of freedom \(\phi\). On the background the scalar field has only radial dependence. As will be discussed in detail in Sec. III, we choose a radial unitary gauge \(\phi = \phi(r)\).

Then the kinetic term of the scalar field can be expressed in terms of the radial lapse \(M\) and the radial derivative of the field. Hence we render the scalar field into the gravitational sector (the radial lapse) and into the explicit radial dependence of the action. We will therefore consider an action principle with the Lagrangian depending on variables constructed from the metric alone, however with explicit radial dependence allowed.

#### A. Action principle

We elaborate the variational principle developed for a cosmological setup \(\text{EFT}\) in a way that it applies to a spherically symmetric background. For this purpose we employ scalar quantities related to the velocity phase-space variables \(2.16\) emerging in the 2+1+1 decomposition. We introduce the gravitational action

\[
S_{\text{EFT}} = \int d^4x \sqrt{-g} L_{\text{EFT}} (N, \mathcal{L}; M, K; \mathcal{R}; \mathcal{L}, \kappa, L, \lambda; r), \quad (3.1)
\]

where we have denoted the gravitational Lagrangian by \(L_{\text{EFT}}\) and

\[
\mathcal{R} \equiv (2) R^a_a, \quad \mathcal{R} = M_a M^a, \quad \mathcal{R} \equiv K_a K^a = \mathcal{L}_a \mathcal{L}^a, \quad (3.2)
\]

Here \((2) R_{ab}\) is the 2-dimensional Ricci tensor.

The action \((3.1)\) depends on the lapse and the velocity phase-space variables \(2.16\) discussed in the previous section. Symmetry allows us to use fewer variables. While the scalar sector \(\{M; \mathcal{K}\}\) is fully included, the vectorial sector \(\{M^a; \mathcal{K}^a\}\) appears through the quantities \(\{\mathcal{R}, \mathcal{L}\}\). The tensorial sector \(\{h_{ab}; K_{ab}\}\) also appears through \(\mathcal{R}\) (which in two dimensions is the only independent component of the Riemann curvature tensor constructed from \(h_{ab}\)) and the quantities \(K, \kappa\). Besides, the scalars \(\{\mathcal{L}, L, \lambda\}\) formed from spatial derivatives of \(N\) and \(h_{ab}\) are also introduced.

In comparison to the corresponding action of the cosmological setup \(\text{EFT}\), the action \((3.1)\) does not depend on the variable \(Z \equiv (2) R_{ab}(2) R^{ab}\), as in two dimensions the intrinsic curvature has only one degree of freedom. In particular, the relation \((2) R_{ab} = (\mathcal{R}/2) h_{ab}\) holds, so that \(Z = \mathcal{R}^2/2\). By contrast, the extrinsic curvatures \(K_{ab}\) and \(L_{ab}\) of the 2-dimensional surface have two independent components each (related to the two sectional curvatures), hence we keep \(\kappa\) (denoted \(\kappa\) in Ref. \([5])\) and the new variable \(\lambda\).

In summary, we have taken into account scalars equivalent to the variables of the velocity phase-space. As the action contains a Lagrangian density \(\sqrt{-g} L_{\text{EFT}}\), scalars representing spatial derivatives have been also included. Instead of the induced 2-metric \(h_{ab}\), we have included the 2-dimensional scalar curvature (in two dimensions the curvature generated by the metric is equivalent to the scalar curvature). Finally, we included the scalars \(\kappa, \mathcal{R}\) and \(\lambda\) for later convenience, as they also appear in the 2+1+1 version of the twice contracted Gauss equation \(\text{EFT}\):

\[
\mathcal{R} = (4) R + K^2 - \kappa - 2 \mathcal{R} + 2 \mathcal{K} K - L^2 + \lambda + 2 \mathcal{L} L + 2 a^b \beta_b + 2 \nabla_a (\alpha^a - \beta^a - K n^a + L n^a). \quad (3.3)
\]
Note that Eq. (3.3) contains a 4-dimensional covariant derivative and is not completely written in 2-dimensional language, but it is adequate for our work. The expression for the Ricci scalar fully translated into 2-dimensional language can be found in Ref. 60.2.

B. Background equations of motion

In what follows, we will proceed in deriving the equations of motion by taking variations of the action on a spherically symmetric and static background. Under the assumption of spherical symmetry, the line element (2.4) contains only two free functions of \((t, r)\) and it simplifies to

\[
d s^2 = -\mathcal{N}^2 dt^2 + \mathcal{M}^2 dr^2 + r^2 d\Omega^2,
\]

where \(d\Omega^2 = d\theta^2 + (\sin^2 \theta)d\varphi^2\) is the surface element of the unit sphere. Since \(\mathcal{N}^a = \mathcal{M}^a = 0\), it follows that \(\mathcal{M} = \bar{\mathcal{M}} = \bar{R} = 0\). The one-forms \(n_a\) and \(l_a\) are given by \(n_a = (-N, 0, 0, 0)\) and \(l_a = (0, M, 0, 0)\), respectively. Note that these expressions of \(n_a\) and \(l_a\) stay valid to first order in the perturbations. The extrinsic curvatures obey \(\bar{K}_{ab} = \bar{K}_{\alpha\beta}/2\) and \(\bar{L}_{ab} = \bar{L}_{\alpha\beta}/2\), hence \(\mathcal{K} = \bar{K}^2/2\) and \(\mathcal{L} = \bar{L}^2/2\). If the background is further time-independent, then the relations \(\bar{K} = \mathcal{K} = \bar{K} = 0\) hold. Other non-vanishing geometric quantities are given by

\[
\bar{R} = \frac{2}{r^2}, \quad \bar{L} = -\frac{\mathcal{N}'}{\mathcal{M}}, \quad \bar{L} = \frac{2}{Mr}, \quad \bar{\lambda} = \frac{2}{M^2 r^2}.
\]

We expand the action (3.1) up to second order in perturbations of the geometric scalar quantities. In doing so, we define the variation of the velocity phase-space variables in the action as the difference between the background and perturbed variables. In particular, we have

\[
\delta \mathcal{R} \equiv \mathcal{R} - \frac{2}{r^2}, \quad \delta \mathcal{L} \equiv \mathcal{L} + \frac{\mathcal{N}'}{\mathcal{M}}, \quad \delta \mathcal{L} \equiv \mathcal{L} - \frac{2}{Mr}, \quad \delta \lambda \equiv \lambda - \frac{2}{M^2 r^2},
\]

and

\[
\delta \bar{R} \equiv \bar{R}, \quad \delta \mathcal{K} \equiv \mathcal{K}, \quad \delta \bar{M} \equiv \bar{M}, \quad \delta \mathcal{M} \equiv \mathcal{M}, \quad \delta \mathcal{K} \equiv \mathcal{K}.
\]

Alternatively, from the definitions of the variables, we obtain the following explicit expressions

\[
\delta \lambda = L^a_b L^b_a + \bar{L}^a_b \bar{L}^b_a = \frac{2}{Mr} \delta \mathcal{L} + \delta L^a_b \delta L^b_a,
\]

\[
\delta \bar{M} = M^a \delta M_a = \delta M_a \delta M^a,
\]

\[
\delta \bar{R} = K^a \delta K_a = \delta K_a \delta K^a,
\]

\[
\delta \mathcal{K} = \bar{K}^a \delta \bar{K}_a = \delta \bar{K}_a \delta \bar{K}^a.
\]

Hence the variables \(\mathcal{M}, \mathcal{R}\) and \(\mathcal{K}\) (which vanish on the background) are second order, while \(\lambda\) (non-vanishing on the background) is changed by the perturbations at both first and second order. We also see that the scalar variables \(\lambda\) and \(\mathcal{L}\) are not independent at first-order accuracy.

Next we expand the Lagrangian in the action (3.1) up to first order in perturbations. In doing so, we keep in mind that \(\mathcal{M}, \bar{R}\) and \(\mathcal{K}\) are second-order quantities, while at first order \(\delta \lambda\) is related to \(\delta \mathcal{L}\). This leaves us with the following Taylor expansion:

\[
L^{\text{EFT}}(N, \mathcal{L}; M, \mathcal{K}; \mathcal{M}, \bar{R}, \mathcal{K}, \mathcal{L}, \lambda; r) = L^{\text{EFT}} + L^{\text{EFT}}_N \delta N + L^{\text{EFT}}_L \delta \mathcal{L} + L^{\text{EFT}}_M \delta M + L^{\text{EFT}}_K \delta \mathcal{K} + L^{\text{EFT}}_{\bar{R}} \delta \bar{R} + L^{\text{EFT}}_{\mathcal{K}} \delta \mathcal{K} + F \delta \mathcal{L},
\]

where we introduced the notations \(L^{\text{EFT}}_G \equiv \frac{\partial L^{\text{EFT}}}{\partial G}\) for any \(G = N, \mathcal{L}; M, \mathcal{K}; \bar{R}; \mathcal{K}, \mathcal{L}, \lambda\) (evaluated on the background), and

\[
F \equiv L^{\text{EFT}}_L + \frac{2 L^{\text{EFT}}_\lambda}{Mr}.
\]

2 After the change in notation, \((\mathcal{R}, (4)^2 R, h_{ab}) \leftrightarrow (R, \bar{R}, g_{ab})\), one can show the equivalence between Eq. (3.9) in this paper and Eq. (A1) in Ref. 60.
In what follows, we explore further relations among the scalar variables. On using Eq. (2.1), we have that $L = h^{ab} \nabla_a b_b - \nabla_a t^a + \mathcal{L} + \delta \mathcal{L}$. Integrating by parts the term $\sqrt{-g} \mathcal{F} \delta L$ in the action and dropping the total covariant divergence term, finally employing Eq. (2.20) and the expression (3.5) of $\mathcal{L}$, we obtain

$$
\int d^4x \sqrt{-g} \mathcal{F} \delta L = - \int d^4x \sqrt{-g} \mathcal{F} \left( \frac{1}{M} \delta M \right) + \int d^4x \sqrt{-g} \mathcal{F} \left( -\frac{N^i}{NM} + \delta \mathcal{L} - \frac{2}{rM} \right),
$$

(3.11)

where we have also expanded $M^{-1}$ up to first order. In the same way, using $\delta K = K - \nabla_a a^a - \delta K$, integrating by parts, dropping the total covariant divergence term and taking into account Eq. (2.19), it follows that

$$
\int d^4x \sqrt{-g} L^{\text{EFT}} \delta K = - \int d^4x \sqrt{-g} L^{\text{EFT}} \delta K.
$$

(3.12)

Then the Lagrangian (3.9) is decomposed as

$$
L^{\text{EFT}} = L^0_{\text{EFT}} + \delta L^{\text{EFT}},
$$

(3.13)

where we have denoted

$$
L^0_{\text{EFT}} = \bar{L} - \frac{\mathcal{F}'}{M} \left( \frac{\bar{N}' r + 2 \bar{N}}{NM} \right),
$$

(3.14)

and

$$
\delta L^{\text{EFT}} = L^N \delta N + \left( L^\mathcal{E} + \mathcal{F} \right) \delta \mathcal{L} + \left( L^\mathcal{E} + \frac{\mathcal{F}'}{M} \right) \delta M + \left( L^N + L^{\text{EFT}} \right) \delta K + L^{\text{EFT}} \delta \mathcal{R}.
$$

(3.15)

It can be proven that the zeroth-order Lagrangians $L^0_{\text{EFT}}$ and $\bar{L}$ differ only by a total covariant divergence, which can be dropped.

The Lagrangian density is given by $\mathcal{L} = \sqrt{-g} L^{\text{EFT}}$, with $\sqrt{-g} = NM \sqrt{h}$ and $\sqrt{h} = r^2 \sin \theta$. It can be decomposed into a background contribution $\bar{L} = \sqrt{-g} L^0_{\text{EFT}}$ and a first-order contribution $\delta \mathcal{L} = \mathcal{L} - \bar{L}$ as follows:

$$
\delta \mathcal{L} = \sqrt{-g} \delta L^{\text{EFT}} + L^0_{\text{EFT}} \delta \sqrt{-g}.
$$

(3.16)

Up to first order in perturbations the metric is given by

$$
ds^2_{1} = - (\bar{N}^2 + 2 \bar{N} \delta N) dt^2 + 2 \bar{M} \delta N dtdr + 2 \delta N_a dtdx^a + (\bar{h}_{ab} + \delta h_{ab}) dx^a dx^b + 2 \delta M_a dx^a dr + (\bar{M}^2 + 2 \bar{M} \delta M) dr^2,
$$

(3.17)

and hence

$$
\delta \sqrt{-g} = \sqrt{-g} \left( \delta N \frac{1}{NM} + \frac{\delta M}{M} + \frac{1}{2} \bar{h}^{ab} \delta h_{ab} \right).
$$

(3.18)

We assume the form $h_{ab} = e^{2\zeta} \bar{h}_{ab}$, where $\zeta$ is the curvature perturbation. This is consistent with allowing only scalar perturbations and suitably fixing the gauge, like in the cosmological case [33, 40], see also the discussion of scalar perturbations in Sec. V. Hence the perturbed and unperturbed metrics are related by a conformal transformation and the respective curvature scalars can be expressed as

$$
\mathcal{R} = e^{-2\zeta} \left( \bar{\mathcal{R}} - 2 \bar{h}^{ab} \bar{D}_a \bar{D}_b \zeta \right),
$$

(3.19)

which to linear order gives

$$
\delta \mathcal{R} = -2 \zeta \delta \mathcal{R} - 2 \bar{h}^{ab} \bar{D}_a \bar{D}_b \zeta.
$$

(3.20)

In the generalized Stokes theorem, the integral of a differential form $\omega$ over the boundary of an oriented manifold $S$ is equivalent to the integral of the exterior derivative of $\omega$ over the manifold $S$, i.e., $\int_S d\omega = \int_{\partial S} \omega$. Since there is no boundary of a boundary, the rhs of the generalized Stokes theorem vanishes when $S$ is some closed surface, e.g., the 2-sphere as in our case. Using this and integrating the second term on the rhs of Eq. (3.20), we obtain

$$
\int d^4x \sqrt{-g} L^{\text{EFT}}_{\text{R}} (-2 \bar{h}^{ab} \bar{D}_a \bar{D}_b \zeta) = -2 \int dtdr \bar{N} \bar{M} r^2 L^{\text{EFT}}_{\text{R}} \int d\theta \ d\varphi D_a \left( \bar{\sqrt{h}} h^{ab} \bar{D}_b \zeta \right) = 0.
$$

(3.21)
Hence the variations in the scalar curvature and conformal factor are related by the simple expression
\[ \delta R = -\frac{4\zeta}{r^2}. \] (3.22)

Remarkably, the same expression emerges for restricting to spherically symmetric perturbations. Non-spherically symmetric modes in the perturbations do not contribute to the background equations of motion.

Similarly, to linear order in perturbations, we obtain
\[ \frac{1}{2} \bar{h}^{ab} \delta h_{ab} = 2\zeta, \] (3.23)
which, when employing Eq. (3.22) and the first equation (3.3), becomes
\[ \frac{1}{2} \bar{h}^{ab} \delta h_{ab} = -\frac{\delta R}{\bar{R}}. \] (3.24)

With this, we have completed the program of rewriting the linear-order variation exclusively into terms containing the variation of the scalar variables in the action.

In what follows we further reduce this set at linear order. Substitution of Eqs. (3.18) and (3.24) into the first-order Lagrangian density (3.16) leads to
\[ \delta L = \sqrt{-\bar{g}} \left[ L_N^{\text{EFT}} \delta N + \left( L_{\phi}^{\text{EFT}} + F \right) \delta \phi + \left( L_M^{\text{EFT}} + \frac{\dot{F}}{M^2} \right) \delta M + \left( L_K^{\text{EFT}} - L_K^{\text{EFT}} \right) \delta K + L_R^{\text{EFT}} \delta R \right] + \bar{L}_0^{\text{EFT}} \sqrt{-\bar{g}} \left( \frac{\delta \bar{N}}{N} + \frac{\delta \bar{M}}{M} - \frac{\delta \bar{R}}{\bar{R}} \right). \] (3.25)

By using Eqs. (2.14) and (2.15c), it follows that
\[ \delta L = \bar{N} \left( \frac{\delta \bar{N}'}{\bar{N}'} + \frac{\delta \bar{M}}{M} \right), \] (3.26)
\[ \delta K = \frac{\delta \bar{M}}{N}. \] (3.27)

Plugging these expressions into Eq. (3.25) and integrating by parts, we obtain
\[ \delta L = \sqrt{-\bar{g}} \left[ \left( L_N^{\text{EFT}} + \frac{\left( L_{\phi}^{\text{EFT}} + F \right)'}{MN} \right) \delta N + \left( L_M^{\text{EFT}} + \frac{\dot{F}}{M^2} + \frac{\bar{N}' \left( L_{\phi}^{\text{EFT}} + F \right)'}{NM^2} \right) \delta M \right] + \bar{L}_0^{\text{EFT}} \sqrt{-\bar{g}} \left( \frac{\delta \bar{N}}{N} + \frac{\delta \bar{M}}{M} - \frac{\delta \bar{R}}{\bar{R}} \right). \] (3.28)

Variation of the three scalars \( \delta N, \delta M, \) and \( \delta R \) leads, respectively, to
\[ \bar{L}^{\text{EFT}} + \bar{N} L_N^{\text{EFT}} + \frac{(\bar{N}' + 2\bar{N}) L_{\phi}^{\text{EFT}}}{NM} + \frac{\dot{L}^{\text{EFT}'} + \bar{N}' L_{\phi}^{\text{EFT}'}}{NM} = 0, \] (3.29)
\[ \bar{L}^{\text{EFT}} + \bar{M} L_M^{\text{EFT}} - \frac{2\dot{F}}{M} + \frac{\bar{N}' L_{\phi}^{\text{EFT}}}{MN} = 0, \] (3.30)
\[ \bar{L}^{\text{EFT}} - \frac{\dot{F}'}{M} - \frac{(\bar{N}' + 2\bar{N}) F}{NM} - \frac{2L_{R}^{\text{EFT}'} \bar{R}}{r^2} = 0, \] (3.31)

which are the equations of motion on the spherically symmetric and static background. For a given Lagrangian, they can be used for discussing the screening mechanism of the fifth force mediated by the scalar degree of freedom. In Appendix A we show that, in the Horndeski theory, the background equations of motion following from Eqs. (3.29)-(3.31) coincide with those derived in Refs. [50, 52] by the direct variation of the Horndeski action. In doing so, we need to express the Horndeski action in terms of the variables used in the 2+1+1 decomposition. In the next section we shall address this issue in both Horndeski and GLPV theories.
IV. 2+1+1 DECOMPOSITION OF HORNDESKI AND GLPV THEORIES

In what follows, we prove that, assuming a spherically symmetric and static background, both the Horndeski theory [17] and its recent GLPV [42] generalization are accommodated in the framework of the EFT of modified gravity.

In unitary gauge, the unit normal vector orthogonal to the constant $\phi$ hypersurfaces (which coincide with the constant $r$ hypersurfaces) can be expressed as

$$ l_a = \gamma \nabla_a \phi, \quad \gamma = \frac{1}{\sqrt{X}}. $$

By virtue of Eq. (2.10), the covariant derivative of $\nabla_a \phi = \gamma^{-1}l_a$ reads

$$ \nabla_a \nabla_b \phi = \gamma^{-1}(L_{ab} + n_a L_b + n_b L_a + n_a n_b \mathcal{L} + l_a \beta_b + l_b \beta_a) + \frac{\gamma^2}{2} \nabla^c \phi \nabla_c X l_a l_b. \quad (4.2) $$

Finally, the term $\Box \phi = g^{ab} \nabla_a \nabla_b \phi$ becomes

$$ \Box \phi = \gamma^{-1}(L - \mathcal{L}) + \frac{\nabla^c \phi \nabla_c X}{2X}. \quad (4.3) $$

With the help of these formulas, we will rewrite both the Horndeski and GLPV Lagrangians in terms of the 2+1+1 variables of the action [33, 31].

A. The Horndeski class of theories

The most general scalar-tensor theories with second-order equations of motion [17] can be given as a series of the Lagrangians [18]

$$ L^H = \sum_{i=2}^{5} L^H_i, \quad (4.4) $$

where

$$ L_2^H = G_2(\phi, X), $$

$$ L_3^H = G_3(\phi, X) \Box \phi, $$

$$ L_4^H = G_4(\phi, X) R - 2G_{4X}(\phi, X) \left[ (\Box \phi)^2 - \nabla^a \nabla^b \phi \nabla_a \nabla_b \phi \right], $$

$$ L_5^H = G_5(\phi, X) G_{ab} \nabla^a \nabla^b \phi + \frac{1}{3} G_{5X}(\phi, X) \left[ (\Box \phi)^3 - 3(\Box \phi) \nabla^a \nabla^b \phi \nabla_a \nabla_b \phi + 2\nabla_a \nabla_b \phi \nabla^c \nabla^d \phi \nabla_c \nabla_d \phi \right]. \quad (4.8) $$

Here $G_{2,3,4,5}$ are functions of a scalar field $\phi$ and of its kinetic term $X \equiv \nabla^a \phi \nabla_a \phi$.

The analysis of the background gravitational dynamics in the Horndeski theory have been presented in Refs. [49–51] on the spherically symmetric space-time and specialized for the weak gravity regime, allowing for confrontation with solar-system tests. In the presence of non-linear scalar-field self interactions, the Vainshtein mechanism can be efficient enough to suppress the propagation of the fifth force inside the solar system, provided that the non-minimal derivative coupling to the Einstein tensor is suppressed [49–51]. At a technical level, this translates into constraining the magnitude of the function $G_5$ in the $L_5^H$ contribution of the Horndeski Lagrangian to be subdominant as compared to the $L_4^H$ contribution. For the consistency with solar-system tests, we will consider the subclass of the Horndeski theory with $L_5^H = 0$ in the following.

The Lagrangian $L_2^H$ depends on the lapse $M$ according to

$$ L_2^H = G_2(\phi, X(M)), \quad X(M) = \frac{\phi'}{M^2}. \quad (4.9) $$

As for the Lagrangian $L_3^H = G_3 \Box \phi$, we introduce an auxiliary function $F_3(\phi, X)$ [33] such that

$$ G_3 \equiv F_3 + 2XF_{3X}. \quad (4.10) $$

Integrating the term $F_3 \Box \phi$ by parts and using Eq. (1.3) for the term $2XF_{3X} \Box \phi$, the Lagrangian $L_3^H$ reduces to

$$ L_3^H = 2X^{3/2}F_{3X}(L - \mathcal{L}) - F_{3\phi}X. \quad (4.11) $$
By using Eqs. (3.3), (4.2) and (4.3), the Lagrangian $L^H_4$ can be expressed as

$$L^H_4 = G_4 (R - K^2 + \omega) + (G_4 - 2XG_{4X}) \left[ L^2 - \lambda - 2L\mathcal{L} + 2\mathbb{K} - 2K\mathcal{K} + 2D^a \left( \ln N \right) D_a \left( \ln M \right) \right] + 2\sqrt{G_{4\phi}} (L - \mathcal{L}).$$

(4.12)

Thus we have shown that the Horndeski Lagrangians $L^H_{2,3,4}$ are fully expressed in terms of $2+1+1$ covariant quantities introduced in the action (3.1).

B. GLPV theories

We proceed to apply our formalism to the GLPV Lagrangian

$$L^{GLPV} = \sum_{i=2}^{5} L^{i, GLPV}_i,$$

(4.13)

where the series of Lagrangians $L^{GLPV}_{2-5}$ are given by [12

$$L^{2, GLPV}_2 = A_2(\phi, X),$$

(4.14)

$$L^{3, GLPV}_3 = \left[ C_3(\phi, X) + 2XC_{3X}(\phi, X) \right] \Box \phi + XC_{3\phi}(\phi, X),$$

(4.15)

$$L^{4, GLPV}_4 = B_4(\phi, X)R - \frac{B_4(\phi, X) + A_4(\phi, X)}{X} \left[ (\Box \phi)^2 - \nabla^a \nabla^b \phi \nabla_a \nabla_b \phi \right] + \frac{2[B_4(\phi, X) + A_4(\phi, X) - 2XB_{4X}(\phi, X)]}{X^2} \left( \nabla^a \nabla^b \phi \nabla_a \nabla_b \phi + \nabla^a \nabla_a \nabla_b \phi \right) - \nabla^a \nabla_b \phi \nabla^c \nabla_d \phi + 2\mathbb{K}_{\phi} \nabla^a \nabla_b \phi \nabla_c \nabla^d \phi$$

(4.16)

$$L^{5, GLPV}_5 = G_5(\phi, X)G_{a\phi} \nabla^a \nabla^b \phi - |X|^3/2 A_5(\phi, X) \left[ \left( \Box \phi \right)^3 - 3(\Box \phi) \nabla^a \nabla_b \phi \nabla^c \nabla_d \phi + 2\mathbb{K}_{\phi} \nabla^a \nabla_b \phi \nabla_c \nabla^d \phi \nabla_e \phi \right]$$

(4.17)

where

$$C_3 = \int dX \frac{A_3}{2X|X|^{3/2}}, \quad C_4 = - \int dX \frac{B_{4\phi}}{|X|}, \quad C_5 = \frac{XG_{5\phi} - |X|^{1/2}B_{5\phi}}{2}, \quad G_5 = - \int dX \frac{B_{5X}}{|X|^{1/2}},$$

(4.18)

with $A_{2,3,4,5}$ and $B_{1,5}$ arbitrary functions of a scalar field $\phi$ and its kinetic term $X$. The Lagrangians $L^{GLPV}_{2-5}$ arise as an extension of the Horndeski theory by generalizing the Horndeski Lagrangians written in terms of the ADM variables in the isotropic cosmological setup [42].

The Horndeski theory corresponds to

$$A_4 = -B_4 + 2XB_4X,$$

(4.19)

$$A_5 = -XB_{5X},$$

(4.20)

under which the terms on the second line of Eq. (4.16) and those in the second and third lines of Eq. (4.17) vanish. Then, the Horndeski Lagrangians (4.15)-(4.18) can be recovered by moving some of the terms (such as $XC_{3\phi}(\phi, X)$) in the Lagrangian $L^{GLPV}_i (i = 3, 4, 5)$ to the previous Lagrangian $L^{GLPV}_{i-1}$.

In comparison to the Horndeski Lagrangians characterized by the functions $G_{2,3,4,5}$, the theories (4.17) have two additional functions included in $A_{2,3,4,5}$ and $B_{1,5}$. Apparently, the equation for the scalar field allows for derivatives higher than second order. In the presence of higher-order derivatives$^3$, the theory can be plagued by

$^3$ Although such a higher-order dynamics is non-standard in physics, it has not been unaccounted either. An example for such a dynamics is provided by the (spin-orbit contribution to the) Lagrangian of spinning binary black holes. In this case the Lagrangian depends on the relative acceleration of the black holes, which leads to a third-order Euler-Lagrange equation [51].
Ostrogradski instabilities associated with the propagation of the extra degrees of freedom \[62\]. In the GLPV theory, however, a careful counting of the degrees of freedom in the Hamiltonian formulation on the isotropic cosmological background\[64\] indicates that no additional degrees of freedom would arise.

As in the discussion of the Horndeski theory, we will also drop the contribution of \(L_3^{\text{GLPV}}\). The Lagrangians \(L_2^{\text{GLPV}}, L_3^{\text{GLPV}}, L_4^{\text{GLPV}}\) can be expressed as

\[
\begin{align*}
L_2^{\text{GLPV}} &= A_2, \\
L_3^{\text{GLPV}} &= A_3 (L - L) , \\
L_4^{\text{GLPV}} &= B_4 (R - K^2 + \kappa) - 2 (B_4 - 2X B_{4X}) [K K - D^a (\ln N) D_a (\ln M)] - A_4 (L^2 - \lambda - L L + 2 \dot{R}) ,
\end{align*}
\]

fully rewritten in terms of the 2+1+1 covariant variables of the action \[37\]. Hence \(L_3^{\text{GLPV}}\) also belong to the class of the EFT of modified gravity. This illustrates that the latter accommodates theories beyond Horndeski.

In Appendix A we show the background equations of motion, as derived from Eqs. \[4.21\]-\[4.23\] for the GLPV Lagrangians \[4.21\]-\[4.23\]. Under the conditions \[4.19\] and \[4.20\], the equations of motion coincide\[5\] with those derived in Refs. \[50, 52\] in the Horndeski theory. In general, however, they differ from each other.

Thus we have shown that there are theories which at the level of the background are second order and more generic than the Horndeski theory. This seems to contradict the generic claim that the Horndeski theory represents the most generic second-order scalar-tensor dynamics. We have to keep in mind however that we are considering a spherically symmetric and static background. These additional symmetries may render some of the requirements imposed in order to achieve second-order dynamics unnecessarily restrictive.

Further, we comment that, under spherical symmetry and staticity imposed in the generic EFT of modified gravity, the tensorial sector is always governed by second-order dynamics. As we consider a static background, the equations of motion \[5.2\]-\[5.4\] represent constraints, containing no time derivatives. Due to the additional spherical symmetry, higher-order derivative terms could emerge only as radial derivatives. This could happen, if the Lagrangian \(L^{\text{EFT}}\) involves second radial derivatives. Nevertheless, this is forbidden by the very nature of the action. Indeed, the Lagrangian only depends on scalars constructed algebraically from the variables of the 2+1+1 formalism involving the induced metric, extrinsic curvatures, normal fundamental vectors and forms. The latter are related to first temporal and radial derivatives, as Eqs. \[2.18\]-\[2.19\] explicitly show. No second-order derivatives of the metric are included in these variables. Hence the background equations of motion (increasing the differential order of the Lagrangian at most by one) are free from third or higher order radial derivatives of the chosen variables of the action.

Nevertheless, at the level of perturbations, to be discussed in the rest of the paper, their second-order evolution cannot be guaranteed a priori.

V. GAUGE TRANSFORMATIONS AND FIXING

In this section we discuss the simplifications achieved by suitably employing the available gauge degrees of freedom (diffeomorphism invariance). In doing so, we will adapt the radial coordinate \(r\) to the hypersurfaces of constant scalar field even in the perturbed case by requiring

\[
\delta \phi = 0 . \tag{5.1}
\]

Next, we will simplify the perturbations of the induced 2-metric to a mere conformal rescaling. Finally we will adopt a gauge which maintains the geometrical interpretation of the variables as arising in the 2+1+1 canonical formalism (e.g., assure \(N = 0\) even in the presence of perturbations).

In a manner analogous to the Helmholtz theorem, any vector \(V_a = V_a (t, r, \theta, \varphi)\) on a sphere can be decomposed by using scalar potentials as follows:

\[
V_a = \bar{D}_a V_{\text{rot}} + E^a_{\, \, b} \bar{D}_b V_{\text{div}} , \tag{5.2}
\]

\[4\] In Ref. \[42\] this has been performed after the scalar degree of freedom is transferred into the lapse and coordinate associated with the constant \(\phi\) hypersurfaces. For the spatial hypersurfaces considered there, the usual lapse \(N\) and the time \(t\) were employed. On the spherically symmetric background the constant \(\phi\)-surfaces have spherical topology, so in this case the scalar degree of freedom is transferred into \(M\) and \(r\).

\[5\] In order to manifestly see this, one has to redefine the functions \(A_i\) and \(B_i\). These redefinitions will be discussed in Appendix A in the case when \(L_5^{\text{GLPV}}\) is dropped.
where $V_{\text{rot}} = V_{\text{rot}}(t, r, \theta, \varphi)$ and $V_{\text{div}} = V_{\text{div}}(t, r, \theta, \varphi)$ are arbitrary scalars generating a rotation-free part and a divergence-free part, respectively. Here $E_{ab} = \sqrt{h} \varepsilon_{ab}$ and $\varepsilon_{ab}$ stands for the antisymmetric tensor density, defined as $\varepsilon_{\theta \varphi} = 1$ [52]. Similarly, any rank-2 symmetric tensor $T_{ab} = T_{ab}(t, r, \theta, \varphi)$ on a sphere can be decomposed in terms of a scalar and a vector potential, e.g., $T_{\text{scalar}}$ and $T_{a}$, as $T_{ab} = \tilde{h}_{ab} T_{\text{scalar}} + \left( \tilde{D}_{a} T_{b} + \tilde{D}_{b} T_{a} \right) / 2$. Applying the decomposition (5.2) to $T_{a}$, the tensor $T_{ab}$ is uniquely expressed in terms of the scalar functions $T_{\text{scalar}}$, $T_{\text{rot}}$ and $T_{\text{div}}$, as

$$T_{ab} = \tilde{h}_{ab} T_{\text{scalar}} + \tilde{D}_{a} \tilde{D}_{b} T_{\text{rot}} + \frac{1}{2} \left( E^{c}_{a} \tilde{D}_{c} \tilde{D}_{b} + E^{c}_{b} \tilde{D}_{c} \tilde{D}_{a} \right) T_{\text{div}}. \quad (5.3)$$

We apply these decompositions to the metric perturbation (3.17), such that the perturbed quantities can be expressed as

$$\delta N_{a} = \tilde{D}_{a} P + E^{b}_{a} \tilde{D}_{b} Q, \quad (5.4a)$$

$$\delta M_{a} = \tilde{D}_{a} V + E^{b}_{a} \tilde{D}_{b} W, \quad (5.4b)$$

$$\delta h_{ab} = \tilde{h}_{ab} A + \tilde{D}_{a} \tilde{D}_{b} B + \frac{1}{2} \left( E^{c}_{a} \tilde{D}_{c} \tilde{D}_{b} + E^{c}_{b} \tilde{D}_{c} \tilde{D}_{a} \right) C. \quad (5.4c)$$

Here the perturbations $Q$, $W$ and $C$ correspond to either divergence-free terms or to derivatives of such terms (these terms have non-vanishing curls), whereas $P$, $V$, $A$, and $B$ represent either rotation-free terms or derivatives of such terms. As first shown in Ref. [54], after expanding in terms of spherical harmonics, the elements of the first set become infinitesimal displacement along the sphere is decomposed as

$$\xi^{a} = \tilde{D}^{a} \xi + E^{ba} \tilde{D}_{b} \eta, \quad (a = \theta, \varphi). \quad (5.5)$$

Then, the perturbed metric in the new coordinate system becomes $\tilde{g}_{ab} = \delta g_{ab} + \nabla_{a} \xi_{b} + \nabla_{b} \xi_{a}$.

The perturbations transform as

$$\tilde{\delta N} = \delta N - \tilde{N} \xi^{t} - \tilde{N} \xi^{r}, \quad (5.6a)$$

$$\tilde{\delta N} = \delta N - \frac{\tilde{N}^{2}}{2M} \xi^{t} + \frac{M}{2} \xi^{r}, \quad (5.6b)$$

$$\tilde{\delta M} = \delta M + M \xi^{t} + M \xi^{r}, \quad (5.6c)$$

$$\tilde{P} = P - \tilde{N}^{2} \xi^{t} + \tilde{\xi}, \quad (5.6d)$$

$$\tilde{Q} = Q + \dot{\eta}, \quad (5.6e)$$

$$\tilde{V} = V + M^{2} \xi^{r} + \xi^{t} - \frac{2}{r} \xi, \quad (5.6f)$$

$$\tilde{W} = W + \eta' - \frac{2}{r} \eta, \quad (5.6g)$$

$$\tilde{A} = A + \frac{2}{r} \xi^{r}, \quad (5.6h)$$

$$\tilde{B} = B + 2 \xi, \quad (5.6i)$$

$$\tilde{C} = C + 2 \eta. \quad (5.6j)$$

Additionally, the linear perturbation $\delta \phi$ of a scalar field $\phi(t, r, \theta, \varphi) = \tilde{\phi}(r) + \delta \phi(t, r, \theta, \varphi)$ transforms under an infinitesimal coordinate transformation as

$$\tilde{\delta \phi} = \delta \phi - \tilde{\phi}' \xi^{r}. \quad (5.7)$$

In the isotropic cosmological setting, the key ingredient in deriving the EFT of modified gravity is the $3 + 1$ decomposition with the time slicing determined by hypersurfaces of the uniform scalar field [83]. In an analogous way, we consider here the hypersurfaces of constant $\phi$ as defining the radial slicing with $r = \text{const}$, in a choice which
simplifies the EFT of modified gravity on the spherically symmetric background. Therefore, we first fix the gauge \( \xi^r \) to obtain \( \delta \tilde{\phi} = 0 \). Due to this gauge choice, the action (3.1) does not explicitly include the scalar field as a variable.

Next, we fix the two gauge degrees of freedom \( \xi \) and \( \eta \) such that the anisotropic contributions to \( \delta h_{ab} \) disappear, i.e., \( \tilde{B} = \tilde{C} = 0 \). By doing so, the perturbed and unperturbed induced metrics are simply related by a conformal transformation as \( h_{ab} = (1 + A)\bar{h}_{ab} \). After redefining \( A = e^{2\xi} - 1 \), the perturbed induced metric coincides with the one employed in Sec. III. Finally, we also need to fix the gauge \( \xi^t \) to achieve \( \delta \bar{N} = 0 \) [see Eq. (5.5)]\(^6\).

In summary, the gauge fixing is given by

\[
\xi^t = \int dt \frac{2\bar{M}}{\bar{N}^2} \left( \delta \bar{N} + \frac{\bar{M}}{2} \delta \bar{\phi} \right) + F(t, \theta, \varphi), \quad \xi^r = \frac{\delta \bar{\phi}}{\bar{\phi}'}, \quad \xi = -B, \quad \eta = -\frac{C}{2},
\]

where \( F(t, \theta, \varphi) \) is an integration function, yet to be fixed \(^7\).

With the new notation for the conformal factor in the transformation of the induced metric

\[
\delta h_{ab} = (e^{2\xi} - 1) \bar{h}_{ab},
\]

the line element up to first-order accuracy can be written as

\[
d\tau^2 = -\left(\tilde{N}^2 + 2\tilde{N}\delta N\right) dt^2 + 2\delta N_a dtdx^a + 2\delta M_a dtdx^a + (\bar{M}^2 + 2\bar{M}\delta M) dr^2 + e^{2\xi}\bar{h}_{ab}dx^adx^b,
\]

where \( \delta N_a \) and \( \delta M_a \) are given in terms of parity-related scalars through Eqs. (5.4a) and (5.4b). In the above expression we have omitted the tildes for notational simplicity, and we will do so hereafter.

We now discuss how the gauge fixing affects the even and odd modes. First, we stress that the residual gauge freedom in \( F \) does not affect the odd-parity perturbations as it does not appear in the transformation of the odd-sector variables \( (C, Q, W) \), as seen from Eqs. (5.10). In fact all these variables transform only in terms of \( \eta \), which has been fixed such that \( C \) could be eliminated. Then the other two odd-sector variables stay arbitrary, unaffected by the three other gauge choices.

Finally, we comment on the elimination of the even-sector variable \( \delta \bar{N} \). By doing so, the interpretation of the Lagrangian variables in terms of the geometric quantities defined in the 2+1+1 formalism continues to hold even in the presence of perturbations. Such a condition is equivalent to imposing hypersurface-orthogonality of the vector field \( l^t \). The last requirement could be relaxed such that the vector \( l^t \) acquires vorticity at a perturbative level. However, this would imply to develop a more involved formalism, allowing at least for a new scalar, a new vectorial and a new tensorial degree of freedom (and all the scalars formed from them). Then we can choose another gauge \( \tilde{P} = 0 \), as commonly used in past works. Such a generalization of the formalism for the even-parity perturbations is left for a subsequent work.

VI. ODD-MODE PERTURBATION DYNAMICS

We proceed with the analysis of the odd-parity perturbations by expanding the action up to second order to discuss the dynamical evolution of them.

A. Second-order perturbed Lagrangian

We expand the action (3.1) at second order for the odd-type perturbations in order to derive linear perturbation equations of motion. As the even and odd sectors decouple in the second-order perturbed Lagrangian, at a formal level, we could just switch off all even-type variables as

\[
P = V = \delta N = \delta M = \zeta = 0.
\]

\(^6\) Even if we would not choose \( \delta \bar{N} = 0 \), preserving at the level of perturbations the more general linear relation between \( \bar{N} \) and \( M \), Eq. (C2) of the Appendix C of Ref. [58] would consume this gauge degree of freedom.

\(^7\) In the particular case where \( P \) exhibits the radial dependence \( P(t, r, \theta, \varphi) = \bar{N}(r)^2 F(t, \theta, \varphi) \), the remaining gauge transformation \( t = t + F(t, \theta, \varphi) \) could be employed to eliminate \( \tilde{P} \). In general, however, this is not possible, so another fixing of the function \( F \) would be necessary in order to avoid the appearance of any non-physical gauge mode, similar to the one of the synchronous gauge in cosmology.
Then the second-order contribution to the Lagrangian density for the odd modes is given by
\[
\delta_2 \mathcal{L}^{\text{odd}} = L_0^{\text{EFT}} \delta_2 \sqrt{-g} + \delta_2 \sqrt{-g} \delta L^{\text{EFT}} + \sqrt{-g} \delta_2 L^{\text{EFT}},
\]
(6.2)
where \( \delta_2 \) represents second-order variations.

The second-order contribution to the line element reads
\[
\delta_2 \left( ds^2 \right) = (\delta N_a \delta N^a - \delta N^2) dt^2 + 2 \delta N_a \delta M^a dt dr + (\delta M_a \delta M^a + \delta M^2) dr^2 + 2 \zeta^2 \tilde{h}_{ab} dx^a dx^b.
\]
(6.3)
By employing Eqs. (5.10) and (6.3), it follows that
\[
\delta_2 \left( ds^2 \right) = \left( \delta M_a \delta M^a + \delta M^2 \right) dr^2 + 2 \zeta^2 \tilde{h}_{ab} dx^a dx^b.
\]
Thus the first term on the rhs of Eq. (6.2) vanishes identically. Similarly the second term on the rhs of Eq. (6.2) vanishes, since by virtue of Eq. (5.14) the first-order variation \( \delta \sqrt{-g} \) consists only of even-mode contributions.

Next we expand the Lagrangian up to second order. Before doing so, we note that the linear and quadratic perturbations of \( L, \mathcal{L}, K, \mathcal{K} \) and \( R \) arise from even modes only [see Eqs. (2.14), (2.15) and (3.19)], so they do not contribute to the odd-mode dynamics. As a result, the second-order Lagrangian for the odd-type perturbations becomes extremely simple (depending on 4 variables only out of 11):
\[
\delta_2 L^{\text{EFT}} = L_{2\mathcal{L}}^{\text{EFT}} \delta_2 \mathcal{L} + L_{2K}^{\text{EFT}} \delta_2 \mathcal{K} + L_{2\mathcal{K}}^{\text{EFT}} \delta_2 \mathcal{K} + L_{2R}^{\text{EFT}} \delta_2 R.
\]
(6.5)
Substituting Eqs. (6.4a) and (6.4b) into Eqs. (2.14) and (2.15), then integrating by parts (employing once again the generalized Stokes theorem for manifolds without boundaries), the second-order factors in \( \delta_2 \mathcal{L}^{\text{EFT}} \) can be explicitly expressed in terms of the odd-type variables:
\[
\delta_2 \mathcal{L} = (DW)^2, \quad \delta_2 \mathcal{L} = \frac{1}{2 M} \left[ (D^2 W)^2 - \frac{2}{r^2} (DW)^2 \right], \quad \delta_2 \mathcal{K} = \frac{1}{2 N_2} \left[ (D^2 Q)^2 - \frac{2}{r^2} (DQ)^2 \right],
\]
(6.6)
where the notations \( D^2 \equiv \tilde{D}^a \tilde{D}_a \) and \( (Df)^2 \equiv \tilde{D}^a f \tilde{D}_a f \) have been introduced for \( f \equiv (Q, W) \).

Substituting Eqs. (6.4c) (6.6) and \( \delta \sqrt{-g} = 0 \) into the second-order Lagrangian density (6.2) for the odd modes, we finally obtain
\[
\delta_2 \mathcal{L}^{\text{odd}} = \left\{ a_1 \left( D\hat{W} - DQ + \frac{2}{r} DQ \right) \right\}^2 + a_2 \left[ (D^2 Q)^2 - \frac{2}{r^2} (DQ)^2 \right] + a_3 (D^2 W)^2 + a_4 (DW)^2,
\]
(6.7)
where the coefficients \( a_i \) \( (i = 1, \cdots, 4) \) are
\[
a_1 = \frac{L_{2\mathcal{L}}^{\text{EFT}}}{4 N_2 M_2}, \quad a_2 = \frac{L_{2\mathcal{K}}^{\text{EFT}}}{2 N_2}, \quad a_3 = \frac{L_{2R}^{\text{EFT}}}{2 M_2}, \quad a_4 = \frac{L_{2\mathcal{L}}^{\text{EFT}}}{4 N_2^2} - \frac{2}{r^2} a_3.
\]
(6.8)
From the second-order Lagrangian density (6.7), we will derive the equations of motion for the odd-sector perturbations in the next subsection. We remark that the Lagrangian density (6.7) is quadratic in the odd-mode perturbations \( Q \) and \( W \), so in what follows we will refer to this Lagrangian contribution as quadratic.

### B. Perturbation equations in the harmonics expansion

We rewrite the quadratic action \( S_2 = \int d^4 x \delta_2 \mathcal{L}^{\text{odd}} \) in the following form
\[
\delta_2 \mathcal{L}^{\text{odd}} = \sqrt{-g} \left\{ a_1 \left( \tilde{D}^2 \left( W - Q + \frac{2}{r} Q \right) \right) + a_2 Q \tilde{D}^2 \left( \tilde{D}^2 + \frac{2}{r^2} \right) Q + W \tilde{D}^2 \left( a_3 \tilde{D}^2 - a_4 \right) W \right\},
\]
(6.9)
in which we have dropped covariant total divergence terms. The resulting equations of motion derived by varying \( W \) and \( Q \) are given, respectively, by
\[
\tilde{D}^2 \Psi^{(1)} = 0, \quad \Psi^{(1)} = a_1 \frac{\partial}{\partial t} \left( W - Q + \frac{2}{r} Q \right) + (a_3 \tilde{D}^2 - a_4) W,
\]
(6.10)
and
\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial r} \left[ \sqrt{-g} a_{l} r^{2} D^{2} \left( W - Q' + \frac{2}{r} Q \right) \right] - a_{2} D^{2} \left( D^{2} + \frac{2}{r^{2}} \right) Q = 0.
\] (6.11)

Since \(\sqrt{-g} = \tilde{N} \tilde{M} \sqrt{h} = \tilde{N} \tilde{M} r^{2} \sin \theta\) and \(D_{a}\) is the covariant derivative compatible with the metric \(h_{ab}\), it follows that \(D_{a} \sqrt{-g} = 0\). On using this identity and the fact that \(r^{2} D^{2}\) has no radial dependence (i.e., it commutes with \(\partial/\partial r\)), Eq. (6.11) reads
\[
\bar{D}^{2} \Psi^{(2)} = 0, \quad \Psi^{(2)} \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial r} \left[ \sqrt{-g} a_{l} \left( W - Q' + \frac{2}{r} Q \right) \right] - a_{2} \left( D^{2} + \frac{2}{r^{2}} \right) Q.
\] (6.12)

Hence Eqs. (6.10) and (6.12) are of the form \(\bar{D}^{2} \Psi^{(i)} = 0\) with \(i = 1, 2\). These are fourth-order coupled differential equations, but in the expressions of \(\Psi^{(i)}\) they contain time and radial derivatives up to second orders alone.

In the following, we expand the angular part of the odd-mode perturbations \(f \equiv (Q, W)\) in terms of spherical harmonics, i.e.,
\[
f(t, r, \theta, \varphi) = \sum_{l,m} f_{lm}(t, r) Y_{lm}^{m}.
\] (6.13)

A similar decomposition of the differential expressions \(\Psi^{(i)} (i = 1, 2)\) is given by
\[
\Psi^{(i)}(t, r, \theta, \varphi) = \sum_{l,m} \Psi^{(i)}_{lm}(t, r) Y_{lm}^{m}.
\] (6.14)

Each mode obeys the identity
\[
r^{2} D^{2} \left[ \Psi^{(i)}_{lm}(t, r) Y_{lm}^{m} \right] + l (l + 1) \left[ \Psi^{(i)}_{lm}(t, r) Y_{lm}^{m} \right] = 0.
\] (6.15)

The differential order of Eqs. (6.10) and (6.12) can be reduced by two, i.e.,
\[
\sum_{l,m} l (l + 1) \Psi^{(i)}_{lm}(t, r) Y_{lm}^{m} = 0, \quad (i = 1, 2),
\] (6.16)
or explicitly
\[
\sum_{l} l (l + 1) \Psi^{(1)}_{l}(t, r) Y_{lm}^{m} = 0, \quad \Psi^{(1)}_{l} \equiv a_{1} \frac{\partial}{\partial t} \left( \hat{W}_{l} - Q'_{l} + \frac{2}{r} Q_{l} \right) - a_{3} \frac{l (l + 1)}{r^{2}} + a_{4} \hat{W}_{l},
\] (6.17)
\[
\sum_{l} l (l + 1) \Psi^{(2)}_{l}(t, r) Y_{lm}^{m} = 0, \quad \Psi^{(2)}_{l} \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial r} \left[ \sqrt{-g} a_{1} \left( \hat{W}_{l} - Q'_{l} + \frac{2}{r} Q_{l} \right) \right] + a_{2} \frac{l (l + 1)}{r^{2}} - 2 Q_{l}.
\] (6.18)

Note that we have introduced the notations \(f_{l} \equiv \sum_{m} f_{lm} Y_{lm}^{m}\). The \(f_{l}\) modes are orthogonal to each other due to the orthogonality of spherical harmonics, so that \(\Psi^{(1)}_{l}\) and \(\Psi^{(2)}_{l}\) vanish for \(l \neq 0\). Hence we have derived a sequence of second-order differential equations \(\Psi^{(i)}_{l} = 0\) \((i = 1, 2)\) holding for each non-zero \(l\).

There exists a second time derivative of \(W_{l}\) in Eq. (6.17), so this corresponds to a dynamical equation of motion for \(W_{l}\). The variable \(Q_{l}\) appears only algebraically in the second-order Lagrangian density (6.7) and through a first temporal derivative in Eq. (6.17). Since Eq. (6.18) contains only a first time derivative of \(W_{l}\) with no time derivatives of \(Q_{l}\), this is a constraint equation in the Lagrangian sense. In Sec. VII we shall address the issue of a true dynamical degree of freedom for general \(l\) by using a method of the Lagrange multiplier. Before doing so, we shall discuss the specific cases of \(l = 0, 1\) in the next subsection.

### C. Monopolar and dipolar perturbations

#### 1. Monopolar mode \((l = 0)\)

The monopolar perturbations trivially obey Eqs. (6.17), (6.18), so they do not contribute to the dynamics. In fact, after integrations by parts, the quadratic odd-mode Lagrangian density (6.7) can be written in a form containing
exclusively Laplacian terms:

\[
\delta_2 \mathcal{L}^{\text{odd}} = \sqrt{-g} \left[ -a_1 \left( W - Q' + \frac{2}{r} Q \right) \left( \bar{D}^2 W - \bar{D}^2 Q' + \frac{2}{r} \bar{D}^2 Q \right) + a_2 \left( \bar{D}^2 + \frac{2}{r^2} \right) Q \\
+ (\bar{D}^2 W) \left( a_3 \bar{D}^2 - a_4 \right) W \right],
\]

(6.19)

all of which identically vanish for \( l = 0 \). In the following we consider only perturbations without a monopolar contribution.

2. Dipolar mode (\( l = 1 \))

For the dipolar perturbations, the last term of Eq. (5.4c), which contains the term \( C \), vanishes due to the identity (6.15). Hence there is no need to eliminate \( C \) by gauge fixing, so that the respective gauge degree of freedom can be used up as

\[
\eta = -r^2 \int dr \frac{W_1}{r^2} + r^2 C_0(t, \theta, \varphi),
\]

(6.20)

where \( C_0(t, \theta, \varphi) \) is an integration function. With this choice, \( \tilde{W}_1 = \tilde{Q}_1 = 0 \) and \( Q_1 + r^2 \dot{C}_0(t, \theta, \varphi) \). Omitting tildes as before and noting that the last term of Eq. (6.18) also vanishes due to the identity (6.15), Eqs. (6.17)-(6.18) is simplified as

\[
\frac{\partial}{\partial t} \left( Q_1' - 2 \frac{r}{r} Q_1 \right) = 0,
\]

(6.21)

\[
\frac{\partial}{\partial r} \left[ \sqrt{-g a_1} \left( Q_1' - 2 \frac{r}{r} Q_1 \right) \right] = 0.
\]

(6.22)

The dynamical degree of freedom \( W \) does not appear in Eqs. (6.21)-(6.22), suggesting that dipolar perturbations are non-dynamical. Indeed, direct integration of Eqs. (6.21)-(6.22) leads to

\[
Q_1 = r^2 C_1(\theta, \varphi) \int \frac{dr}{\sqrt{-\bar{g} a_1 r^2}} + r^2 C_2(t, \theta, \varphi),
\]

(6.23)

where \( C_{1,2} \) are integration functions. The remaining gauge degree of freedom can be exploited as \( \dot{C}_0 = -C_2 \), so the time dependence is completely eliminated from the dipolar odd-mode perturbations. As discussed in Ref. [55], the time-independent contribution to \( Q_1 \) appearing as the first term on the r.h.s. of Eq. (6.23) is related to the angular momentum induced by the dipolar perturbation.

D. Dynamical degree of freedom for \( l \geq 2 \)

The Lagrangian density (6.9) possesses first and second derivatives, which appear quadratically. Hence some of the terms would be of fourth order in spatial derivatives by partial integration (while the time derivatives remain of second order). This is why the perturbation Eqs. (6.10) and (6.12) involve fourth-order spatial differentiations. For \( l \geq 2 \) these equations of motion reduce to the form \( \Psi^{(1)}_l = 0 \) and \( \Psi^{(2)}_l = 0 \) under the expansion of spherical harmonics, where \( \Psi^{(i)}_l (i = 1, 2) \) are given by Eqs. (6.17) and (6.18).

As we already mentioned in Sec. VI B, the first equation (\( \Psi^{(1)}_l = 0 \)) describes the dynamical evolution of the variable \( W_l \), whereas the second one (\( \Psi^{(2)}_l = 0 \)) corresponds to a constraint involving a second spatial derivative of the field \( Q_l \). Since the latter constraint equation is not directly solved for \( Q_l \), it is difficult to derive a closed-form differential equation for \( W_l \) by eliminating the \( Q_l \)-dependent terms appearing in the equation \( \Psi^{(1)}_l = 0 \). This obstacle can be circumvented by using the method of a Lagrange multiplier. In fact, this method was employed to study the linear perturbations on a spherically symmetric background in modified Gauss-Bonnet gravity [63] and it was further applied to Horndeski theory [52].
Introducing the Lagrange multiplier vector $Y^a$, the Lagrangian density equivalent to Eq. (6.7) is given by

$$
\delta_2 \mathcal{L}^{\text{odd}} = \sqrt{-g} \left\{ a_1 \left[ 2Y^a \tilde{D}_a \left( \tilde{W} - Q' + \frac{2}{r} Q \right) - Y^2 \right] + a_2 \left[ (\tilde{D}^2 Q)^2 - \frac{2}{r^2} (\tilde{D}Q)^2 \right] + a_3 (\tilde{D}^2 W)^2 + a_4 (\tilde{D}W)^2 \right\} ,
$$

(6.24)

where $Y^2 = Y^a Y_a$. Variation of Eq. (6.24) with respect to $Y^a$ leads to $Y_a = \tilde{D}_a [\tilde{W} - Q' + (2/r) Q]$. Substituting this relation into Eq. (6.24), we recover the original Lagrangian density (6.7).

Defining the Lagrange multiplier potential $Z$ as $Y^a = \tilde{D} a Z$, the Lagrangian density (6.24) is characterized by two scalar fields $W$ and $Q$ plus the auxiliary scalar field $Z$. Varying Eq. (6.24) in terms of $W$ and $Q$, we obtain

$$
\tilde{D}^2 \left[ a_1 \dot{Z} + (a_3 \tilde{D}^2 a - a_4) W \right] = 0 ,
$$

(6.25)

$$
\tilde{D}^2 \left[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial r} (\sqrt{-g} a_1 Z) - a_2 \left( \tilde{D}^2 + \frac{2}{r^2} \right) Q \right] = 0 .
$$

(6.26)

For $l \geq 2$ the $\tilde{D}^2$ operators acting on the square brackets can be formally omitted, so the $l$-th multipolar components $W_l$ and $Q_l$ obey the following equations:

$$
W_l = \frac{a_1 r^2}{a_3 l (l + 1) + a_4 r^2} \dot{Z}_l ,
$$

(6.27)

$$
Q_l = - \frac{r^2}{a_2 (l + 2) (l - 1) \sqrt{-g}} \frac{\partial}{\partial r} (\sqrt{-g} a_1 Z_l) ,
$$

(6.28)

where $Z_l$ is the $l$-th component of $Z$.

Equations (6.27) and (6.28) show that both $W_l$ and $Q_l$ are directly known from $Z_l$. On using the last of Eq. (6.8), we can also write Eq. (6.24) of the form

$$
W_l = \frac{r^2}{a_3 (l + 2) (l - 1)} \left( a_1 \dot{Z}_l - L_{\text{EFT}} W_l \right) .
$$

(6.29)

Substituting Eqs. (6.28) and (6.29) into the $l$-th component of the multipolar decomposition of the Lagrangian density (6.24), using $Y^a = \tilde{D} a Z$, and integrating by parts, we finally obtain

$$
\delta_2 \mathcal{L}_l^{\text{odd}} = \frac{l(l + 1)}{(l + 2)(l - 1)} \sqrt{-g} \left[ - a_1^2 \tilde{Z}_l^2 / a_3 - \frac{a_1^2}{a_2} Z_l^2 - a_1 (\tilde{D} Z_l)^2 - U^H(r) Z_l^2 + \frac{a_1}{a_3} L_{\text{EFT}} W_l \dot{Z}_l \right] ,
$$

(6.30)

where the potential $U^H(r)$ is given by

$$
U^H(r) = - a_1 \frac{\partial}{\partial r} \left[ \frac{1}{\sqrt{-g} a_2} \frac{\partial}{\partial r} (\sqrt{-g} a_1) \right] - \frac{2 a_1}{r^2} ,
$$

(6.31)

or more explicitly,

$$
U^H(r) = - \frac{a_1}{a_2} \left[ \tilde{N}'' \frac{N''}{N} + \tilde{M}'' \frac{M''}{M} - \tilde{N}'' \frac{M''}{M} - \frac{2}{r^2} + \left( a_1 + a_2 \right) \left( \tilde{N}' \frac{N''}{N} + \tilde{M}' \frac{M''}{M} + \frac{2}{r} \right) + \frac{a_1}{a_1} - \frac{a_1 a_2'}{a_1 a_2} - \frac{2 a_1}{r^2} \right] .
$$

(6.32)

The superscript in $U^H(r)$ has been introduced to point out that in the Horndeski limit it reduces to the potential (24) of Ref. [52]. Using the relation $\tilde{D}^2 Z_l = - l(l + 1) Z_l / r^2$, the third term in the square bracket of Eq. (6.30) is equivalent to $- a_1 l(l + 1) Z_l^2 / r^2$ up to a boundary term.

The last term in the square bracket of Eq. (6.30) gives rise to a contribution $\dot{Z}_l^2$ with a coefficient including the multipolar index $l$ by virtue of Eq. (6.24). In this case the propagation speeds are different for each multipolar mode, so the global interpretation of the perturbation $Z_l$ and its propagation speeds become far from trivial. Hence, in the following, we impose the following condition

$$
L_{\text{EFT}} = 0 .
$$

(6.33)

In fact, this is satisfied both in the Horndeski theory and in the GLPV theory ($i = 2, 3, 4$ for our cases of interest).

Under the condition (6.33) the second-order Lagrangian density is expressed solely by the quantity $Z_l$ and its time and spatial derivatives, in a mode-independent way. As a result, $Z_l$ is a master variable governing the dynamics of
the odd-mode perturbations. Comparing Eqs. (6.25) and (6.26) with Eqs. (6.10) and (6.12), respectively, there is the correspondence \( Z \rightarrow W - Q' + 2Q/r \), which also arises by varying the Lagrangian density (6.24) for the Lagrange multiplier potential \( Z \). While \( Q \) and \( W \) were eliminated from the Lagrangian density by their respective equations of motion, Eqs. (6.27) and (6.28), we stress that this third equation of motion \( Z = W - Q' + 2Q/r \) was not exploited for deriving Eq. (6.30). In fact, after the substitution of Eqs. (6.27) and (6.28), the Lagrangian density (6.30) already contains the dynamics of the third field \( Z \). If we were to make the additional substitution \( Z_i \rightarrow W_i - Q'_i + 2Q_i/r \), the Lagrangian density (6.24) would reduce to a boundary term \( \delta Z_{\text{odd}} = -\left(\partial/\partial \tau\right)\left(\sqrt{-g}a_1Q(l+1)Z_i/r^3\right) \), which is irrelevant to the true dynamics of perturbations.

On using the equations of motion following from the variation of Eq. (6.30) with respect to \( Z_i \), we can discuss the stability of the odd-type perturbations. In the next section we shall address this issue.

VII. NO-OSTH CONDIONS AND AVOIDANCE OF LAPLACIAN INSTABILITIES

In the previous section we have seen that in an expansion with respect to spherical harmonics there is no monopolar contribution to the odd modes and the dipolar mode is non-dynamical. In the following we proceed with the stability analysis of quadrupolar and higher multipolar contributions to the odd-mode perturbations \( l \geq 2 \), governed by the quadratic Lagrangian density (6.30) under the condition (6.33).

A. Generalized Horndeski class

We categorize theories satisfying the condition (6.33) as the generalized Horndeski class (including the GLPV theory). In this case the quadratic Lagrangian could depend on the odd-mode variable \( W \), which generates the odd-mode \( \delta \)M through Eq. (5.4b). Nevertheless, the Lagrangian for the background dynamics does not depend on the particular combination \( \mathfrak{M} \equiv M_\lambda M^\lambda \). Whenever Eq. (6.33) holds, the quadratic Lagrangian density (6.30) leads to a second-order differential equation for the decoupled master variable \( Z_i \) and the usual stability conditions can be imposed on this equation.

The condition for avoidance of the scalar ghost (no negative kinetic term) is satisfied for \( a_3 < 0 \), i.e.,

\[
L^\text{EFT}_K < 0 .
\] (7.1)

For the modes with the large wave numbers along the radial or tangential directions, many terms of Eq. (6.30) are suppressed. In particular, \( U^\text{HE}(r) \) as well as the third (for radial modes) or second (for tangential modes) terms are sub-dominant in the high-frequency limit. In these two regimes the dispersion relations following from the Lagrangian density (6.30) are given, respectively, by

\[
\omega^2 + \frac{a_3}{a_2}k_r^2 = 0, \quad \omega^2 + \frac{a_3}{a_1}k_\Omega^2 = 0 ,
\] (7.2)

where \( \omega \) is the angular frequency, \( k_r \) and \( k_\Omega \) are the wave numbers along the radial and tangential directions respectively. Introducing proper time \( \tau = \int \mathfrak{N} \, dt \) and tortoise coordinate \( r_* = \int M \, dr \), the squared sound speeds of fluctuations along the radial and tangential directions read

\[
c^2_r \equiv \frac{\dot{M}^2k_r^2}{N^2\omega^2} = -\frac{\dot{M}^2a_3}{N^2a_2} = -\frac{L^\text{EFT}_\lambda}{L^\text{EFT}_K}, \quad c^2_\Omega \equiv \frac{k_\Omega^2}{N^2\omega^2} = -\frac{a_3}{N^2a_1} = -\frac{2L^\text{EFT}_\lambda}{L^\text{EFT}_R} ,
\] (7.3)

respectively. Under the no-ghost requirement (7.1), the conditions for the absence of Laplacian instabilities, i.e., \( c_r^2 > 0 \) and \( c_\Omega^2 > 0 \), take a remarkably simple form

\[
L^\text{EFT}_K > 0 , \quad L^\text{EFT}_R > 0 .
\] (7.4)

These simple stability conditions acquire geometrical significance, as \( \mathfrak{R} \) is the length squared of the normal fundamental vector, while \( \mathfrak{X} \) and \( \lambda \) are the traces of the squares of the two extrinsic curvature tensors of the spheres. These are quantities appearing in the 2+1+1 decomposition of the covariant derivatives of the two normal vectors to the spheres, Eqs. (2.23), (2.10). The additional quantities of these decompositions are the normal fundamental scalars and accelerations. They however do not contribute to the stability conditions for the odd modes as the normal fundamental scalars \( L \) and \( K \) are even-mode variables, while the accelerations \( \alpha^a \) and \( \beta^a \) appear in the action only through the curvature scalar \( \mathfrak{R} \) under divergences. Hence their rotation-free part alone survives under the Helmholtz decomposition, which again generates the even modes.
The stability conditions (7.1) and (7.4) can be further specified for the particular case of the Horndeski theory with \( L_{3,4}^H \) and the GLPV theory with \( L_{GLPV}^{\lambda,\zeta} \) discussed in Sec. [X]. For this we first remark that, according to Eqs. (4.12) and (4.23), only the contributions \( L_4^H \) and \( L_{GLPV}^\lambda \) depend on the variables \( \lambda, \zeta \) and \( \bar{R} \).

In the Horndeski theory, the stability conditions (7.1) and (7.4) read
\[
- L_\lambda^H = \frac{1}{2} L_{\bar{R}}^H = G_4 - 2X G_{4X} > 0, \quad L_\zeta^H = G_4 > 0. \tag{7.5}
\]

The first of these conditions exactly corresponds to Eq. (25) or Eq. (28) of Ref. [52] (these two conditions coincide when \( L_5^H = 0 \)). The second is the condition imposed in Ref. [52] for avoiding gradient instabilities, when \( L_5^H = 0 \). Since \( X > 0 \), the first condition (7.5) gives information beyond the second one only for \( G_{4X} > 0 \).

In the GLPV theory, the stability conditions reduce to
\[
L_\lambda^{GLPV} = - \frac{1}{2} L_{\bar{R}}^{GLPV} = A_4 < 0, \quad L_{\zeta}^{GLPV} = B_4 > 0. \tag{7.6}
\]

It is easy to see that, in the Horndeski limit characterized by Eq. (4.19), these reduce to Eq. (7.5).

### B. Stability conditions for covariant Galileon models

#### 1. Covariantized Galileons

The original Galileon model advocated in Ref. [14] is composed of five Lagrangians invariant under the Galilean symmetry \( \partial_\mu \phi \to \partial_\mu \phi + b_\mu \) in the Minkowski background. The equations of motion remain of second order by virtue of this symmetry. In the curved background, the original Galileon model can be covariantized by replacing coordinate derivatives with covariant derivatives. This “covariantized Galileon” belongs to a particular case of the GLPV theory given by the Lagrangians (4.14)-(4.16) with the functions
\[
A_2 = c_2 X, \quad A_3 = c_3 X^{3/2}, \quad A_4 = -\frac{M_{pl}^2}{2} - c_4 X^2, \quad B_4 = \frac{M_{pl}^2}{2}, \tag{7.7}
\]

where \( c_{2,3,4} \) are constants. Here we have taken into account the Einstein-Hilbert term \( M_{pl}^2 R/2 \) in the Lagrangian, where \( M_{pl} \) is the reduced Planck mass.

In general space-time, the theory described by (7.7) contains derivatives higher than second order. On the flat isotropic cosmological background, however, the equations of motion for the background and linear perturbations are second order without a new propagating degree of freedom [12]. This result was obtained by considering the constant-time hypersurfaces, such that the scalar field plays the role of time. A similar argument may also work for the spherically symmetric background due to the high degree of symmetry, in which case the scalar field takes the role of a radial coordinate \( r \). In fact, substituting Eq. (7.7) into the background equations of motion (A1)-(A3), we obtain
\[
\begin{align*}
\frac{M_{pl}^2}{r} & \left( \frac{1}{r} - \frac{1}{M^2 r} + \frac{2 \bar{M}'}{M^3} \right) + c_2 X + \frac{3 c_3 X}{M^2} \left( \frac{\phi' \bar{M}'}{M} - \phi'' \right) - \frac{2 c_4 X}{M^2 r} \left( \frac{X}{r} - \frac{10 X \bar{M}'}{M} + \frac{8 \phi' \phi''}{M^2} \right) = 0, \\
\frac{M_{pl}^2}{r} & \left( \frac{1}{r} - \frac{1}{M^2 r} - \frac{2 \bar{N}'}{M^2 N} \right) - c_2 X - \frac{3 c_3 X}{M^2} \left( \frac{2}{r} + \frac{\bar{N}'}{N} \right) - \frac{10 c_4 X^2}{2 r M^2} \left( \frac{1}{r} + \frac{2 \bar{N}'}{N} \right) = 0, \\
\frac{M_{pl}^2}{M^2} & \left[ \frac{\bar{M}'}{M} - \frac{\bar{N}''}{N} - \frac{\bar{N}'}{N} \left( \frac{1}{r} - \frac{\bar{M}'}{M} \right) \right] + c_2 X + \frac{3 c_3 X}{M^2} \left( \frac{\phi' \bar{M}'}{M} - \phi'' \right) \\
\frac{2 c_4 X}{M^2} & \left[ \frac{X \bar{N}'}{N} - \frac{5 X \bar{M}'}{M} + \frac{4 \phi' \phi''}{M^2} \right] + \frac{\bar{N}'}{N} \left( \frac{X}{r} - \frac{5 X \bar{M}'}{M} + \frac{4 \phi' \phi''}{M^2} \right) = 0, \tag{7.8}
\end{align*}
\]

which are of second order. The equations of motion for the odd-mode perturbations are also of second order. The stability conditions (7.6) translate to
\[
c_4 \left( \frac{X}{M_{pl}} \right)^2 > -\frac{1}{2}. \tag{7.9}
\]

The radial and tangential sound speeds read
\[
c_r^2 = 1 + 2 c_4 \left( \frac{X}{M_{pl}} \right)^2, \quad c_{\Omega}^2 = 1, \tag{7.10}
\]

respectively.
2. Covariant Galileons

Higher-order derivatives present for the covariantized Galileon in a general curved space-time can be eliminated by including a non-minimally coupled gravitational contribution to the Lagrangian\cite{[13, 16]}. The Galileon model with second-order equations of motion is dubbed “covariant Galileon”. This is a sub-class of the Horndeski Lagrangians\cite{[13, 17]} with the choice

\begin{equation}
G_2 = \hat{c}_2 X, \quad G_3 = \hat{c}_3 X, \quad G_4 = \frac{M_{\text{pl}}^2}{2} + \hat{c}_4 X^2,
\end{equation}

where \(\hat{c}_{2,3,4}\) are constants.

From Eqs.\cite{[A1], [A3]} the background equations of motion are given by

\begin{align}
\frac{M_{\text{pl}}^2}{r} \left( \frac{1}{r} - \frac{1}{M^2 r} + \frac{2\bar{M}'}{M^3} \right) + \hat{c}_2 X + 2\frac{\hat{c}_3 X}{M^2} \left( \frac{\phi' \bar{M}'}{M} - \phi'' \right) + 6\hat{c}_4 X \left( \frac{\bar{M}^2 X}{3r} + \frac{X}{r} - \frac{10X \bar{M}'}{M} + \frac{8\phi' \phi''}{M^2} \right) &= 0, \\
\frac{M_{\text{pl}}^2}{r} \left( \frac{1}{r} - \frac{2\bar{N}'}{M^2 N} \right) - \hat{c}_2 X - 2\frac{\hat{c}_3 X}{M^2} \left( \frac{2}{r} + \frac{\bar{N}'}{N} \right) + 30\hat{c}_4 X^2 \left( - \frac{\bar{M}^2}{5r} + \frac{1}{r} + \frac{2\bar{N}'}{N} \right) &= 0, \\
\frac{M_{\text{pl}}^2}{M^2} \left[ \frac{\bar{M}'}{M} - \frac{\bar{N}''}{N} - \frac{\bar{N}'}{N} \left( \frac{1}{r} - \frac{\bar{M}'}{M} \right) \right] + \hat{c}_2 X + 2\frac{\hat{c}_3 X}{M^2} \left( \frac{\phi' \bar{M}'}{M} - \phi'' \right) + 6\hat{c}_4 X \left[ \frac{XX' \bar{N}''}{N} - \frac{5X M'}{M r} + \frac{4\phi' \phi''}{M^2 r} + \frac{\bar{N}'}{N} \left( \frac{X}{r} - \frac{5X M'}{M} + \frac{4\phi' \phi''}{M^2} \right) \right] &= 0.
\end{align}

Compared to the covariantized Galileon, the difference arises from the \(B_4\)-dependent terms in Eqs.\cite{[A1], [A2]}. The stability conditions (7.3) translate to

\begin{equation}
-\frac{1}{2} < \hat{c}_4 \left( \frac{X}{M_{\text{pl}}} \right)^2 < \frac{1}{6},
\end{equation}

which is different from Eq. (7.10). The radial and tangential speeds of sound are given, respectively, by

\begin{equation}
c_r^2 = \frac{M_{\text{pl}}^2 - 6\hat{c}_4 X^2}{M_{\text{pl}}^2 + 2\hat{c}_4 X^2}, \quad c_\Omega^2 = 1,
\end{equation}

where \(\hat{c}_4^2\) differs from Eq. (7.10).

We have shown that the background and perturbation equations of motion for both the covariantized Galileon\cite{[17]} and the covariant Galileon\cite{[7,11]} are of second order on the spherically symmetric background. Their perturbations propagate identically along the spheres, but with different propagation speeds in the radial direction.

VIII. CONCLUDING REMARKS

We have studied the perturbations about a spherically symmetric and static background in the framework of the EFT of modified gravity. In this approach, the gravitational action is expressed in terms of scalar variables constructed from the canonical variables arising in the Arnowitt-Deser-Misner (ADM) decomposition of space-time. An additional scalar field can be included in the gravitational sector at the price of a partial gauge-fixing (unitary gauge), incorporating it into an explicit dependence of the dynamics of the coordinate and the lapse associated with the constant scalar-field hypersurfaces.

Since spherical symmetry selects a preferred radial direction besides the time direction, we employed a more intricate 2+1+1 decomposition, worked out earlier for arbitrary dimensions\cite{[52, 60]}. Due to the double foliation, there are two sets of extrinsic curvatures in the formalism. Some of them are related to temporal derivatives (\(K_{ab}, K_a, K\)), the others to radial derivatives (\(L_{ab}, L_a, L\)). We have started from a general action that depends on scalars formed from these quantities, the metric variables of the constant time hypersurfaces (\(h_{ab}, M_a, M\)) and the lapse \(N\).

We choose the gauge \(\mathcal{N} = 0\) to ensure the perpendicularity of the foliations on the spherical symmetric space-time. Then, the dynamics of the radial and temporal components proceeds in a hypersurface-orthogonal manner without vorticities. By this gauge choice, it is possible to avoid an unnecessary increase in the number of variables associated with vorticity-type quantities. A second gauge fixing is the radial unitary gauge \(\phi = \phi(r)\), which switches off the
perturbations of the scalar field \( \delta \phi = 0 \). In this case, the scalar field is absorbed in the gravitational sector (into the radial lapse \( M \)) and an explicit radial dependence of the action.

The gravitational action \( \delta \mathcal{A} \) incorporates a general system of a single scalar degree of freedom. Despite the relatively large number of scalar variables, variation of the action gives rise to three independent equations of motion at the background level. They are derived by the changes in the lapse \( \delta N \), in the radial lapse \( \delta M \), and in the scalar curvature on the sphere \( \delta R \), respectively. Equations (3.29)-(3.31) represent the most generic set of equations of motion in modified gravity theories on the spherically symmetric and static background.

The Horndeski theory and their recently suggested GLPV generalizations \(^{42}\) involve a single scalar degree of freedom beside the metric. They represent the most general class of theories with second-order equations of motion and the extended class that allows for higher-order derivatives in generic space-time, respectively. We have expressed both Lagrangians in terms of the 2+1+1 variables, proving that they belong to the class of the EFT of modified gravity studied in this paper. We also derived the background equations of motion explicitly for both under spherical symmetry and staticity. Under these symmetries the GLPV background is also second order, as in the case of the Horndeski theory.

Variation of the action up to second order leads to the linear perturbation equations of motion, with the even and odd modes decoupled. In this paper we focused on the analysis for the odd-parity mode of perturbations. The originally fourth-order differential equations were reduced to second order by employing a multipolar expansion into spherical harmonics. We derived the second-order Lagrangian density for odd-mode perturbations of the form \( \delta \mathcal{L} \). Under the condition \( \delta \mathcal{L} \), which is satisfied for both Horndeski and GLPV theories, the Lagrangian density is expressed solely by a dynamical scalar variable \( Z \) and its derivatives. We established extremely simple conditions for avoiding ghosts and Laplacian instabilities. The propagation speed of odd-mode perturbations depends on the direction of propagation. More specifically, the radial sound speed and the sound speed along the spheres are different, generalizing the corresponding result established for the Horndeski theory \(^{52}\).

As applications of our general stability analysis, we have i) confirmed the corresponding results for the Horndeski theory, ii) obtained the stability conditions for the recently proposed GLPV theory, iii) derived and compared both the tangential and the radial speeds of sound for two types of Galileon theories: “covariantized Galileon” (derived by replacing coordinate derivatives with covariant derivatives in the original Galileon model) and “covariant Galileon” with second-order dynamics in general space-time (obtained by adding a new term to eliminate higher-order derivatives). Although the background equations of motion are similar in the two Galileon theories, the stability conditions associated with the radial propagation speed \( c_r \) are different. This can be traced back to the terms \( B_4 \) and \( B_5 \) appearing in the Lagrangians \(^{42}\) and \(^{42}\) being different in these two theories. In the Horndeski theory \( B_4 \) and \( B_5 \) are related to the other terms \( A_4 \) and \( A_5 \) according to Eqs. \(^{42}\) and \(^{42}\), however in general no such restriction appears in the GLPV theory.

Recently, the cosmology based on the two Galileon theories was studied in Ref. \(^{64}\) on the flat Friedmann-Lemaître-Robertson-Walker background. It was shown that the propagation speeds of the field \( \phi \) for covariant and covariantized Galileons are different due to the different values of \( B_4 \) and \( B_5 \) in the two theories. On the isotropic cosmological background, the equations of motion for linear perturbations also remain of second order. In spite of the possible presence of derivatives higher than second order on general backgrounds, the GLPV theory remains healthy on both the static spherically symmetric and the isotropic cosmological backgrounds.

It is possible to extend our work to several interesting directions. First, the background equations of motion \( (3.29)-(3.31) \) can be generally applied to the discussion of the screening mechanism of the fifth force mediated by the scalar field \( \phi \). Second, the analysis of even-parity perturbations, which is much more involved than that of odd-parity modes, will be useful to discuss the full stability of the EFT of modified gravity on the spherically symmetric and static background. Third, the construction of theoretically consistent dark energy models in the framework of the GLPV theory will be also intriguing. We leave these issues for future works.

**Acknowledgements**

We are grateful to Riccardo Penco and Federico Piazza for interactions in the early stages of this project. R. K. and S. T. were supported by the Scientific Research Fund of the JSPS (Nos. 24-6770 and 24540286). L. G. was supported by the long-term Invitation Fellowship Program no. L13519 of the Japan Society for the Promotion of Science (JSPS).
Appendix A: Equations of motion in the Horndeski and GLPV theories on the spherically symmetric and static background

In this Appendix we present the background equations of motion for the spherically symmetric and static GLPV theory (including the Horndeski theory). Substituting the Lagrangians (4.21)-(4.23) into Eqs. (3.29)-(3.31), it follows that

\[
A_2 - \phi' A_{3, \phi} + \bar{X}' A_{3, X} + \frac{2A_4}{M^2 r^2} \left( \frac{1}{r} - \frac{M'}{M} \right) + \frac{4 \phi' A_{4, \phi} + \bar{X}' A_{4, X}}{M^2 r^2} + \frac{2B_4}{r^2} = 0, \tag{A1}
\]

\[
A_2 - 2 \bar{X} A_{2, X} - \frac{2 \bar{X} A_{3, X}}{M} \left( \frac{2}{r} + \frac{\bar{N}'}{N} \right) + \frac{2A_4 + 2 \bar{X} A_{4, X}}{M^2 r^2} \left( \frac{1}{r} + \frac{2 \bar{N}'}{N} \right) + \frac{2 (B_4 - 2 \bar{X} B_{4, X})}{r^2} = 0, \tag{A2}
\]

\[
A_2 - \phi' A_{3, \phi} + \bar{X}' A_{3, X} - \frac{2A_4}{M^2} \left[ \frac{\bar{N}''}{N} - \frac{\bar{N}'}{N} \left( \frac{1}{r} - \frac{\bar{M}'}{M} \right) \right] + \frac{2 (\phi' A_{4, \phi} + \bar{X}' A_{4, X})}{M^2} \left( \frac{1}{r} + \frac{\bar{N}'}{N} \right) = 0, \tag{A3}
\]

where \( \bar{X} \) represents the background value of the kinetic term \( X \), i.e., \( \bar{X} = \phi'^2/M^2 \). The last terms on the l.h.s. of Eqs. (A1), (A2), which include \( B_4 \) and its derivative with respect to \( X \), originate from the non-vanishing two-dimensional scalar curvature \( R \) on the spherically symmetric and static background. In the Horndeski theory \( B_4 \) is entirely determined by \( A_4 \) and \( X \), while in the GLPV theory it is not. Hence the equations of motion for the GLPV theory generally differ from those for the Horndeski theory.\(^8\)

Under the condition (1.19) and by redefining the functions \( A_2, A_3 \) and \( B_4 \) in terms of the new functions \( G_2, F_3 \) and \( F_4 \) as follows

\[
A_2 = G_2 - F_{3, \phi} X, \quad A_3 = 2X^{1/2} F_{3, X} + 2\sqrt{X} G_{4, \phi}, \quad B_4 = G_4, \tag{A4}
\]

the sum of the Lagrangians \( L_{2,3,4}^{\text{GLPV}} \) manifestly reduces to that of \( L_{2,3,4}^H \). Applying the same condition and redefinitions to the equations of motion (A1)-(A3), we obtain those for the Horndeski theory. In order to compare them with the equations of motion derived in Ref. [50] by a method entirely intrinsic to the Horndeski theory, we further need the conversion in the notations \( (\bar{N}, \bar{M}, X, G_3) \to (e^{\Psi(r)}, e^{\Phi(r)}, -2X, -G_3) \), after which a full agreement is reached.

\[\text{References:}[1]\ A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998) [astro-ph/9805201].
\[2]\ S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [astro-ph/9812133].
\[3]\ P. A. R. Ade et al. [Planck Collaboration], arXiv:1303.5076 [astro-ph.CO].
\[4]\ G. Hinshaw et al. [WMAP Collaboration], Astrophys. J. Suppl. 208, 19 (2013) [arXiv:1212.5226 [astro-ph.CO]].
\[5]\ K. Capozziello and S. Tsujikawa, Int. J. Mod. Phys. D 20, 539 (2011) [arXiv:1104.1443 [hep-th]].
\[6]\ E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006) [hep-th/0603057]; T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010) [arXiv:0805.1726 [gr-qc]]; A. De Felice and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010) [arXiv:1002.4928 [gr-qc]]; S. Tsujikawa, Lect. Notes Phys. 800, 99 (2010) [arXiv:1101.0191 [gr-qc]]; T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513, 1 (2012) [arXiv:1106.2476 [astro-ph.CO]].
\[7]\ Y. Fujii, Phys. Rev. D 26, 2580 (1982); L. H. Ford, Phys. Rev. D 35, 2339 (1987); C. Wetterich, Nucl. Phys. B 302, 668 (1988); T. Chiba, N. Sugiyama and T. Nakamura, Mon. Not. Roy. Astron. Soc. 289, L5 (1997) [astro-ph/9704199]; P. G. Ferreira and M. Joyce, Phys. Rev. Lett. 79, 4740 (1997) [astro-ph/9707286]; R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998) [astro-ph/9708069].
\[8]\ C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999) [hep-th/9904075]; T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62, 023511 (2000) [astro-ph/9912463]; C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000) [astro-ph/0004134].
\[9]\ S. Capozziello, Int. J. Mod. Phys. D 11, 483 (2002) [gr-qc/0201033]; S. Capozziello, S. Carloni and A. Troisi, Recent Res. Dev. Astron. Astrophys. 1, 625 (2003) [astro-ph/0303041]; S. M. Carroll, S. V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D 70, 043528 (2004) [astro-ph/0306438].
\[10]\ W. Hu and I. Sawicki, Phys. Rev. D 76, 064004 (2007) [arXiv:0705.1158 [astro-ph]]; S. A. Appleby and R. A. Battye, Phys. Lett. B 654, 7 (2007) [arXiv:0705.3199 [astro-ph]]; A. A. Starobinsky, JETP Lett. 86, 157 (2007) [arXiv:0706.2041 [astro-ph]]; S. Tsujikawa, Phys. Rev. D 77, 023507 (2008) [arXiv:0709.1391 [astro-ph]].

---

\(^8\) On the flat isotropic cosmological background the scalar curvature of the constant time hypersurfaces identically vanishes. We verified that no \( B_4 \) terms appear in the background equations of motion of the GLPV theory, which then coincide with those of the Horndeski theory at the background level.
[59] L. Á. Gergely and Z. Kovács, Phys. Rev. D 72, 064015 (2005) [gr-qc/0507020].
[60] Z. Kovács and L. Á. Gergely, Phys. Rev. D 77, 024003 (2008) arXiv:0709.2131 [gr-qc].
[61] L. E. Kidder, C. M. Will, A. G. Wiseman, Phys. Rev. D 47, 4183 (1993) arXiv:gr-qc/9211025.
[62] M. Ostrogradski, Mem. Ac. St. Petersbourg VI 4, 385 (1850).
[63] A. De Felice, T. Suyama and T. Tanaka, Phys. Rev. D 83, 104035 (2011) arXiv:1102.1521 [gr-qc].
[64] R. Kase and S. Tsujikawa, Phys. Rev. D 90, 044073 (2014) arXiv:1407.0794 [hep-th].