On generalized cyclotomic derivations

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Abstract. In this article, we study the field of rational constants and Darboux polynomials of a generalized cyclotomic $K$-derivation $d$ of $K[X]$. It is shown that $d$ is without Darboux polynomials if and only if $K(X)^d = K$. The result is also studied in the tensor product of polynomial algebras.

Keywords. Darboux polynomial; Jouanolou derivation; cyclotomic derivation.

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1. Introduction

Throughout this article, $K$ denotes a field of characteristic zero, $K[X] = K[x_1, x_2, \ldots, x_n]$ is the polynomial algebra in $n$ variables over $K$ and $K(X)$ denotes the field of fractions of $K[X]$. Let $d$ be a $K$-derivation of $K[X]$ and $K[X]^d$ denote the algebra of constants of $d$. The $K$-derivation $d$ of $K[X]$ uniquely extends to a $K$-derivation of $K(X)$ and we continue to denote it by $d$. $K(X)^d$ represents the field of rational constants of $d$, that is, $K(X)^d = \{f \in K(X); d(f) = 0\}$ and $K[X]^d \subseteq K(X)^d$.

A non-constant polynomial $f \in K[X]$ is said to be a Darboux polynomial of $d$ if $d(f) = \lambda f$, for some $\lambda \in K[X]$ and in this case $\lambda$ is called the co-factor of $f$. We say $d$ is without Darboux polynomials if $d$ has no Darboux polynomials. It is easy to observe that if $d$ is without Darboux polynomials, then $K(X)^d = K$ but the converse of the above statement is not true, in general. One can refer to [1,2] for counter examples. In this paper, we study a class of monomial derivations for which $K(X)^d = K$ if and only if $d$ is without Darboux polynomials. Note that a derivation $d$ of $K[X]$ is said to be a monomial derivation if $d(x_i)$ is a monomial for every $1 \leq i \leq n$.

In [4], Nowicki and Ollagnier studied the Darboux polynomials and field of rational constants of a monomial derivation over the polynomial algebra $K[X]$. For $1 \leq i \leq n$, let $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in \mathbb{N}^n$. Consider the monomial derivation $d$ given by $d(x_i) = X^{\alpha_i}$, where $X^{\alpha_i}$ denotes the monomial $x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$. Then one can associate a matrix $A$ given by $A = [\alpha_{ij}] - I$ with the monomial derivation $d$. Let $w_d$ denote the determinant of the matrix $A$. A monomial derivation $d$ is said to be normal if $w_d \neq 0$ and $\alpha_{ii} = 0 \forall 1 \leq i \leq n$. In [4], Nowicki and Ollagnier proved that for a normal derivation $d$, $K(X)^d = K$ if and only
if \( d \) is without Darboux polynomials. In this article, they also raised a similar question in the case of \( w_d = 0 \). For \( n = 3 \), an independent proof was given to show that the result is true even if \( w_d = 0 \). It was also observed that the idea used to prove the result for \( n = 3 \) case could not be extended further. At the end of the article, they mentioned the following example of monomial derivation \( d \) on \( K[x, y, z, w] \):

\[
d(x) = w^2, \quad d(y) = zw, \quad d(z) = y^2, \quad d(w) = xy,
\]

and raised the same question about the field of constants and Darboux polynomials. Note that for this derivation, \( w_d = 0 \). Our study of the field of constants of monomial derivations is motivated by the aforesaid example.

In this article, we study a large class of monomial derivations for which \( K(X)^d = K \) if and only if \( d \) is without Darboux polynomial. The example mentioned above is a very particular case of our result. Further, our result is independent of the condition \( w_d \neq 0 \).

2. The main result

Let \( s \) be a non-negative integer. A derivation \( d \) on \( K[X] \) is said to be homogeneous of degree \( s \), if

\[
d(A^m) \subseteq d(A^{m+s}) \quad \forall \ m \geq 1,
\]

where \( A^m \) is the \( K \)-subspace of all the homogeneous polynomials of degree \( m \). In particular, a monomial derivation \( d \) on \( K[X] \) is homogeneous of degree \( s \) if for each \( 1 \leq i \leq n \), \( d(x_i) \)'s are monomials of total degree \( s + 1 \).

For a homogeneous \( K \)-derivation \( d \) of \( K[X] \), the following is a well known result ([5] Lemma 2.1).

**Lemma 2.1.** Let \( d \) be a homogeneous \( K \)-derivation of \( K[X] \) of degree \( s \). If \( f \in K[X] \) is a Darboux polynomial of \( d \) with the co-factor \( \lambda \), then \( \lambda \) is a homogeneous polynomial of degree \( s \) and all the homogeneous components of \( f \) are also Darboux polynomials with the same co-factor \( \lambda \).

Before proceeding further, we fix some notations and give some more definitions. Let \( n \geq 2 \) be a positive integer. A derivation \( d \) of \( K[X] \) is called cyclotomic if, for \( 1 \leq i \leq n-1 \), \( d(x_i) = x_{i+1} \) and \( d(x_n) = x_1 \). In the same line we have defined the generalized cyclotomic derivation. Let \( S = \{x_1, \ldots, x_n\} \) denote the set of \( n \)-variables, \( K[S] \) be the \( K \)-algebra generated by \( S \) and \( k > 1 \) be a positive integer. A monomial derivation \( d \) of \( K[S] \) is said to be a generalized cyclotomic derivation if we can split \( S \) into mutually disjoint \( k \) parts (say \( S_i, \ 1 \leq i \leq k \), \( k \in \mathbb{N} \) and \( 2 \leq k \leq n \)) such that \( d(S_i) \subseteq K[S_{i+1}] \) for \( 1 \leq i \leq k-1 \) and \( d(S_k) \subseteq K[S_1] \), where \( K[S_i] \) denotes the \( K \)-algebra generated by \( S_i \).

Let us redefine the variables as \( S_i = \{x_{i,1}, \ldots, x_{i,t_i}\} \) for \( 1 \leq i \leq k \) and take \( S = \cup S_i \). Then we have the following.

**DEFINITION 2.2**

A derivation \( d \) of \( K[S] \) is said to be generalized cyclotomic derivation if

\[
d(x_{i,j}) = \alpha_{i,j,1} x_{i+1,1}^{r_{i+1,1}} \cdots x_{i+1,t_i+1}^{r_{i+1,t_i+1}} \quad \forall \ 1 \leq j \leq t_i, \ 1 \leq i \leq k-1
\]
and
\[ d(x_{k,j}) = x_{1,1}^{\alpha_{k,j,1}} x_{1,2}^{\alpha_{k,j,2}} \cdots x_{1,t_i}^{\alpha_{k,j,t_i}} \quad \forall \ 1 \leq j \leq t_k, \]
where \( \alpha_{k,j,l} \) for every \( 1 \leq j \leq t_i, 1 \leq l \leq t_{i+1}, 1 \leq i \leq k - 1 \) and \( \alpha_{k,j,l} \) for every \( 1 \leq j \leq t_k, 1 \leq l \leq t_1 \) are non negative integers.

Let \( s \) be a positive integer. We say \( d \) is homogeneous generalized cyclotomic derivation of degree \( s - 1 \) if \( d(x_{ij}) \) are monomials of total degree \( s \). Now we state our main result.

**Theorem 2.3.** Let \( d \) be a homogeneous generalized cyclotomic derivation of \( K[S] \) of degree \( s - 1 \) as defined above. Then \( d \) is without Darboux polynomials if and only if \( K[S]^d = K \).

**Proof.**

\((\Rightarrow)\) Easy to prove.

\((\Leftarrow)\) Assume that \( K[S]^d = K \). Suppose \( d \) has a Darboux polynomial \( f \) such that \( d(f) = \lambda f \). Then by Lemma 2.1, we may assume that \( f \) is homogeneous of degree \( s - 1 \). Write \( \lambda \) as
\[
\lambda = \sum_{1 \leq i \leq k} a_\beta X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k},
\]
where \( X_i^{\beta_i} \) denotes the monomial \( x_{i,1}^{\beta_{i,1}} \cdots x_{i,t_i}^{\beta_{i,t_i}} \) and \( |\beta_i| = \sum_{j=1}^{t_i} \beta_{i,j} \) for \( 1 \leq i \leq k \).

Let \( N = (1 + s + s^2 + \cdots + s^{k-1}) \) and let \( \xi \) be the primitive \( N \)-th root of unity. For \( 0 \leq i \leq k - 1 \), define \( q_i = \sum_{j=0}^{i} s^j \). Observe that \( q_{k-1} = N \). Consider a \( K \)-automorphism \( \sigma \) of \( K[S] \) given by
\[
\sigma(x_{k-i,j}) = \xi^{q_i} x_{k-i,j} \quad \forall \ 1 \leq j \leq t_{k-i}.
\]
Moreover,
\[
\sigma^{-1} d \sigma(x_{k-i,j}) = \sigma^{-1} d(\xi^{q_i} x_{k-i,j})
\]
\[
= \xi^{q_i} \sigma^{-1} d(x_{k-i,j})
\]
\[
= \xi^{q_i} \sigma^{-1} \left( x_{k-i+1,1}^{\alpha_{k-i+1,1}} x_{k-i+1,2}^{\alpha_{k-i+1,2}} \cdots x_{k-i+1,t_{k-i+1}}^{\alpha_{k-i+1,t_{k-i+1}}} \right)
\]
\[
= \xi^{q_i} e^{-q_i-1} \left( \alpha_{k-i+1,1}^{\alpha_{k-i+1,1}} + \cdots + \alpha_{k-i+1,t_{k-i+1}}^{\alpha_{k-i+1,t_{k-i+1}}} \right) d(x_{k-i,j})
\]
\[
= \xi^{q_i} e^{-s(q_i-1)} d(x_{k-i,j})
\]
\[
= \xi d(x_{k-i,j}).
\]

Therefore, we have \( \sigma^{-1} d \sigma = \xi d \). Let \( F = \prod_{i=0}^{N-1} \sigma^i(f) \). Clearly, \( F \) is not a constant polynomial. Furthermore,
\[
d(F) = d \left( \prod_{i=0}^{N-1} \sigma^i(f) \right)
\]
\[
\begin{align*}
&= \sum_{i=0}^{N-1} \sigma^0(f) \cdots d(\sigma^i(f)) \cdots \sigma^{N-1}(f) \\
&= \sum_{i=0}^{N-1} \sigma^0(f) \cdots \xi^i \sigma^i d(f) \cdots \sigma^{N-1}(f) \\
&= \sum_{i=0}^{N-1} \sigma^0(f) \cdots \xi^i \sigma^i(\lambda f) \cdots \sigma^{N-1}(f) \\
&= \left( \sum_{i=0}^{N-1} \xi^i \sigma^i(\lambda) \right) \sigma^0(f) \cdots \sigma^{N-1}(f), \\
&= \Lambda F,
\end{align*}
\]

where \( \Lambda = \sum_{i=0}^{N-1} \xi^i \sigma^i(\lambda). \)

Now, let us do the precise calculation for \( \Lambda. \) If we look at the \( m \)-th term in the sum, we have

\[
\begin{align*}
\xi^m \sigma^m(\lambda) &= \xi^m \sigma^m \left[ \sum_{1 \leq l \leq k} \sum_{|\beta_l| = s-1} a_\beta \sigma^m(X_{\beta_1}^{l_1} \cdots X_{\beta_k}^{l_k}) \right] \\
&= \xi^m \left[ \sum_{1 \leq l \leq k} \sum_{|\beta_l| = s-1} a_\beta \sigma^m(X_{\beta_1}^{l_1} \cdots X_{\beta_k}^{l_k}) \right] \\
&= \xi^m \left[ \sum_{1 \leq l \leq k} \sum_{|\beta_l| = s-1} a_\beta \left( \prod_{l=1}^{k} \sigma^m(X_{l_{11}}^{\beta_{l1}}) \cdots \sigma^m(X_{l_{1l}}^{\beta_{ll}}) \right) X_{1}^{p_{l1}} X_{2}^{p_{l2}} \cdots X_{k}^{p_{lk}} \right],
\end{align*}
\]

where \( p_l = \beta_{l1} + \cdots + \beta_{ll} \) for all \( 1 \leq l \leq k. \) Therefore,

\[
\begin{align*}
\xi^m \sigma^m(\lambda) &= \xi^m \left[ \sum_{1 \leq l \leq k} \sum_{|\beta_l| = s-1} a_\beta \xi^{m \left( \sum_{i=1}^{l} \frac{1}{q_i} - 1 \right)} X_{1}^{\beta_{l1}} X_{2}^{\beta_{l2}} \cdots X_{k}^{\beta_{lk}} \right].
\end{align*}
\]
Observe that \( q_{k-1} = N \) and \( \xi^N = 1 \), therefore the above equation reduces to

\[
\xi^m \sigma^m(\lambda) = \xi^m \sum_{1 \leq i \leq k} a_{\beta} \xi^{m\left(\sum_{l=2}^{i} p_l q_{l-1}\right)} X_{1}^{\beta_{1}} X_{2}^{\beta_{2}} \cdots X_{k}^{\beta_{k}}
\]

where \( \delta = \sum_{l=2}^{k} p_l q_{l-1} \). More precisely,

\[
\delta = \sum_{l=2}^{k} q_{k-l} p_l
\]

\[
= \sum_{l=2}^{k} \left( \sum_{j=0}^{k-l} s^j \right) (\beta_{l,1} + \cdots + \beta_{l,t_l})
\]

\[
= \sum_{l=0}^{k-2} s^l (\beta_{2,1} + \cdots + \beta_{2,t_2} + \cdots + \beta_{k,1} + \cdots + \beta_{k-l,t_{k-l}})
\]

\[
\leq \sum_{l=0}^{k-2} s^l (s - 1) = s^{k-1} - 1.
\]

This implies that \( 0 < 1 + \delta \leq s^{k-1} < N \). Therefore, \( \xi^{1+\delta} \neq 1 \). Hence

\[
\Lambda = \sum_{m=0}^{N-1} \left[ \sum_{1 \leq i \leq k} \xi^{m(\delta+1)} a_{\beta} X_{1}^{\beta_{1}} X_{2}^{\beta_{2}} \cdots X_{k}^{\beta_{k}} \right]
\]

\[
= \sum_{1 \leq i \leq k} \left( \sum_{m=0}^{N-1} \xi^{m(\delta+1)} \right) a_{\beta} X_{1}^{\beta_{1}} X_{2}^{\beta_{2}} \cdots X_{k}^{\beta_{k}}
\]

\[
= \sum_{1 \leq i \leq k} \left( \frac{1 - \xi^{N(\delta+1)}}{1 - \xi^{\delta+1}} \right) a_{\beta} X_{1}^{\beta_{1}} X_{2}^{\beta_{2}} \cdots X_{k}^{\beta_{k}}
\]

\[
= 0.
\]

Therefore, \( d(F) = \Lambda F = 0 \). In other words, \( F \in K(S)^d \), a contradiction. \( \square \)

**Remark 2.4.** From Theorem 2.3, we can prove that the monomial derivation \( d \) of \( K[x, y, z, w] \) defined by

\[
d(x) = w^2, \ d(y) = zw, \ d(z) = y^2, \ d(w) = xy
\]
in [4] has no Darboux polynomials if and only if $K(x, y, z, w)^d = K$.

COROLLARY 2.5

The Jouanolou derivation $d$ of $K[x_1, x_2, \ldots, x_n]$ defined by

$$d(x_1) = x_2^3, d(x_2) = x_3^5, \ldots, d(x_{n-1}) = x_n^s, d(x_n) = x_1^s,$$

for $s \geq 1$ and $n \geq 2$ has no Darboux polynomials if and only if $K(x_1, x_2, \ldots, x_n)^d = K$.

3. Generalized cyclotomic derivation in tensor product

Let $m$ and $n$ be positive integers. Assume that $K[X] = K[x_1, \ldots, x_n]$ and $K[Y] = K[y_1, \ldots, y_m]$ are polynomial algebras. Then $K[X] \otimes_K K[Y] \cong K[X, Y] = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is a polynomial algebra. If $d_1$ and $d_2$ are $K$-derivations of $K[X]$ and $K[Y]$ respectively, then $d = d_1 \otimes 1 + 1 \otimes d_2$, denoted by $d_1 \oplus d_2$ is the $K$-derivation of $K[X, Y]$ such that $d|K[X] = d_1$ and $d|K[Y] = d_2$.

In [3], Nowicki and Ollagnier studied the Darboux polynomial of the tensor product of polynomial algebras and have proved the following result.

Lemma 3.1 ([3], Corollary 3.2). Let $d_1$ and $d_2$ be homogeneous $K$-derivations of $K[X]$ and $K[Y]$ of degree $s \geq 1$. If $d_1$ and $d_2$ are without Darboux polynomials, then $d_1 \oplus d_2$ is also without Darboux polynomials.

Using Lemma 3.1 and our result on generalized cyclotomic derivations, we have the following result.

Theorem 3.2. Let $d_1$ and $d_2$ be homogeneous generalized cyclotomic derivations of $K[X]$ and $K[Y]$ of degree $s \geq 1$. Then, $d_1 \oplus d_2$ is without Darboux polynomial if and only if $K(X, Y)^{d_1 \oplus d_2} = K$.

Proof.

$(\Rightarrow)$ Trivial to prove.

$(\Leftarrow)$ Let $K(X, Y)^{d_1 \oplus d_2} = K$. As $K(X)^{d_1} \subseteq K(X, Y)^{d_1 \oplus d_2} = K$, we have $K(X)^{d_1} = K$. Similarly, $K(Y)^{d_2} = K$. Therefore, from Theorem 2.3, $d_1$ and $d_2$ are without Darboux polynomials. Then by Lemma 3.1, $d_1 \oplus d_2$ is also without Darboux polynomials.

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