SINGULARITY OF MACROSCOPIC VARIABLES NEAR BOUNDARY FOR GASES WITH CUT-OFF INVERSE-POWER POTENTIAL

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Abstract. In this article, the boundary singularity for stationary solutions of the linearized Boltzmann equation with cut-off inverse power potential is analyzed. In particular, for cut-off hard-potential cases, we establish the asymptotic approximation for the gradient of the moments. Our analysis indicates the logarithmic singularity of the gradient of the moments.

1. Introduction

The Boltzmann equation is

\[ \frac{\partial F}{\partial t} + \xi \cdot \frac{\partial F}{\partial t} = Q(F, F), \]

where \( F = F(t, x, \xi) \). \( Q \) above is the collision operator only involves velocity as follows:

\[ Q(F, F) = \int \int \int_0^{2\pi} \int_0^{\pi/2} (F'F'_* - FF_*) B(|V|, \theta) d\theta d\epsilon d\xi, \]

where \( V = \xi_* - \xi \) and \( \alpha \) is a unit vector on a hemisphere parametrized by \( \theta \) and \( \epsilon \) such that \( \alpha \cdot V = |V| \cos \theta \) and

\[ F = F(\xi), \quad F_* = F(\xi_*), \quad F' = F(\xi'), \quad F'_* = F(\xi'_*), \]

\[ \xi' = \xi + (\alpha \cdot V)\alpha, \]

\[ \xi'_* = \xi_* - (\alpha \cdot V)\alpha. \]

The \( B(|V|, \theta) \geq 0 \) is called the cross-section. If we consider inverse power force between particles, i.e., \( \text{Force} = \frac{1}{r^s} \), then the cross-section is in the form

\[ B(|V|, \theta) = |V|^\gamma \beta(\theta), \]

where \( \gamma = \frac{s - 1}{s - 5} \). The fact \( \beta \sim \left( \frac{\pi}{2} - \theta \right)^{-\frac{s+1}{s-1}} \) as \( \theta \to \frac{\pi}{2} \), which is not integrable in \( \theta \), makes us unable to separate into gain and lost parts. To avoid this mathematical difficulty, it was Grad’s idea, \[13\], to consider the cross-section such that

\[ B(|V|, \theta) \leq C|V|^\gamma \cos \theta \sin \theta.\]
We will refer these cases as Grad’s angular cut-off potential. In particular, in our research we will consider the cases that $B$ as a product of a function of $|V|$ and one of $\theta$, i.e.,

\begin{equation}
B(|V|, \theta) = |V|^\gamma \beta(\theta), \quad \beta(\theta) \leq C \cos \theta \sin \theta.
\end{equation}

To make distinctions, we will refer the cases above, (1.8), as power-law potential with angular cut-off in this paper. We first non-dimensionsalize the equation so that the Maxwellian we are interested becomes the standard one:

\begin{equation}
w = \frac{1}{\sqrt{\pi}} e^{-|\zeta|^2}.
\end{equation}

We linearize the equation around standard Maxwellian so that

\begin{equation}
F = w + w^{1/2} f.
\end{equation}

We have,

\begin{equation}
sh \frac{\partial f}{\partial t} + \zeta \cdot \frac{\partial f}{\partial t} = \frac{1}{\kappa} L(f),
\end{equation}

where $L(f) = 2w^{-1/2} Q(w, w^{1/2} f)$. Under the assumption of Grad’s angular cut-off, the linearized collision operator can be decomposed into a damping multiplicative operator $-\nu$ and a smoothing integral operator $K$:

\begin{equation}
L(\phi)(\zeta) = -\nu(\zeta) \phi(\zeta) + K(\phi)(\zeta).
\end{equation}

The following properties of the linearized collision operator were studied by Grad [13] and Caflisch [6]. The collision frequency satisfies the following estimate

\begin{equation}
\nu_0(1 + |\zeta|)^\gamma \leq \nu(\zeta) \leq \nu_1(1 + |\zeta|)^\gamma,
\end{equation}

where $0 < \nu_0 < \nu_1$ and $-2 \leq \gamma \leq 1$ is a parameter from interaction between particles. $\gamma = 1$ is called the hard sphere model; $\gamma = 0$ is called the Maxwellian case. Positive $\gamma$’s corresponded to hard potential; negative $\gamma$’s correspond to soft-potential.

If we consider power-law potential with angular cut-off, we have further properties:

\begin{equation}
\nu(\zeta) \text{ is a function of } |\zeta|.
\end{equation}

\begin{equation}
\| \partial_{\zeta} K(f) \|_{L^p} \leq C \| f \|_{L^p}, \quad p \geq 1.
\end{equation}

In this paper, we restrict our study to the cases $0 < \gamma \leq 1$. We define

\begin{equation}
\| f(\zeta) \|_{L^\infty} = \sup_{\zeta} (1 + |\zeta|)^a |f(\zeta)|.
\end{equation}

The integral operator improves the decay:

\begin{equation}
\| K(f) \|_{L^\infty} \leq C \| f \|_{L^2},
\end{equation}

\begin{equation}
\| K(f) \|_{L^\infty} \leq C \| f \|_{L^\infty}.
\end{equation}
We consider the stationary equation:

\[(1.19) \quad \zeta_1 \partial_x f(x, \zeta) = L(f)(\zeta), \text{ for } 0 < x \leq l.\]

The functional space we are considering is as follows:

**Definition 1.1.**

\[(1.20) \quad L_\zeta^*(\mathbb{R}^3) = \{ f, \| f(\zeta) \|_* < \infty \}, \]

where

\[(1.21) \quad \| f(\zeta) \|_* = \left( \int f^2(\zeta) \nu(\zeta) d\zeta \right)^{\frac{1}{2}}. \]

Also,

\[(1.22) \quad ||| f ||| := \sup_{0 \leq x \leq l} ||| f |||_* . \]

We say \( f \in L_\infty^x([0, l], L_\zeta^*(\mathbb{R}^3)) \) is a solution to (1.19) if it satisfies the following integral equation:

\[(1.23) \quad f(x, \zeta) = \begin{cases} 
  e^{-\nu(\zeta)} f(0, \zeta) + \int_0^x \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} K(f)(s, \zeta) ds, & \text{for } \zeta_1 > 0, \\
  e^{-\nu(\zeta)} f(l, \zeta) + \int_l^x \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(s-x)} K(f)(s, \zeta) ds, & \text{for } \zeta_1 < 0.
\end{cases} \]

**Remark 1.2.** The solution spaces for both Milne and Kramar’s problems given in [11] are in \( L_\infty^x([0, l], L_\zeta^*(\mathbb{R}^3)) \) if \( x \) is restricted to \([0, l]\). 

The moments are defined as follows:

**Definition 1.3.** The \( \alpha \) moment is defined as

\[(1.24) \quad \sigma_\alpha(x) = \int f(x, \zeta) \phi_\alpha(\zeta) d\zeta, \]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), \( \alpha_i \)'s are nonnegative integers,

and

\[(1.25) \quad \phi_\alpha(\zeta) = \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} e^{-\frac{|\zeta|^2}{2}}. \]

We introduce a constant depending on \( \alpha \):

**Definition 1.4.** Set

\[(1.26) \quad A_\alpha = (2\alpha_1)^{\frac{1}{2}} (2\alpha_2)^{\frac{1}{2}} (2\alpha_3)^{\frac{1}{2}} e^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{2}}, \]

where we follow the convention \( 0^0 = 1 \).
The macroscopic variables are obtained through the moments. For example, \( \sigma_{0(0,0)} \) is the density, \( \sigma_{1(0,0)} \) is the flow velocity in \( x_1 \) direction, and 
\[
\frac{2}{3}(\sigma_{2(0,0)} + \sigma_{0(2,0)} + \sigma_{0(0,2)}) - \sigma_{0(0,0)}
\]
is the temperature. The following inequality will be frequently used later:

(1.27) \[ |\phi_0| \leq C_A e^{-\frac{|y|^2}{2}}. \]

The Main Theorem is as follows

**Theorem 1.5.** Suppose \( f \in L^\infty_x([0,l], L^\infty_\zeta(\mathbb{R}^3)) \) is a solution to (1.19) with power law potential with angular cut-off with \( 0 < p \leq 1 \) and \( \nabla f(0,\zeta) \in L^p_\zeta(\mathbb{R}^3+) \) for \( p > 1 \), \( f(0,\zeta) \in L^\infty_\zeta(\mathbb{R}^3+) \), and \( f(l,\zeta) \in L^\infty_\zeta(\mathbb{R}^3-) \). Then, for \( x \) small enough,

(1.28) \[
|\partial_x \sigma_0(x)| = -\ln x \int \phi_0(0,\zeta_2,\zeta_3)L(f)(0,0^+,\zeta_2,\zeta_3)d\zeta_2d\zeta_3 + O(A_\alpha \langle f \rangle'),
\]

where

(1.29) \[
L(f)(0,0^+,\zeta_2,\zeta_3) := \lim_{\zeta_1 \to 0^+} L(f)(0,\zeta_1,\zeta_2,\zeta_3)
\]

and

(1.30) \[
\langle f \rangle' := 1 + ||f|| + ||f(0,\zeta)|| L^\infty_\zeta(\mathbb{R}^3+) + ||f(\zeta)|| L^\infty_\zeta(\mathbb{R}^3-) + ||\nabla f(0,\zeta)|| L^p_\zeta(\mathbb{R}^3+).
\]

We first investigate the problem for Grad’s angular cut-off potential. We consider the the distribution function for \( \zeta_1 > 0 \). The case for \( \zeta_1 < 0 \) can be treated similarly. Differentiating (1.23) for \( \zeta_1 > 0 \), we have

(1.31) \[
\frac{\partial}{\partial x} f(x,\zeta) = -\frac{\nu(\zeta)}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|^2} x} f(0,\zeta) + \frac{1}{|\zeta_1|} K(f)(x,\zeta) \]

We observe that the first term is nice in \( \zeta \) when \( x \) is away from zero and has a singularity at \( x = 0 \). On the other hand, the second and third terms have factors \( |\zeta_1|^{-1} \) and \( |\zeta_1|^{-2} \) in \( \zeta \), which cause a difficulty in our analysis.

In order to overcome this difficulty, we reorganize the equation (1.31):

(1.32) \[
\frac{\partial}{\partial x} f(x,\zeta) = -\frac{\nu(\zeta)}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|^2} x} f(0,\zeta) + \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|^2} x} K(f)(x,\zeta) \]

\[ + \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|^2} (x-s)} (K(f)(x,\zeta) - K(f)(s,\zeta)) ds \]

\[ = \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|^2} x} L(f)(0,\zeta) + \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|^2} x} (K(f)(x,\zeta) - K(f)(0,\zeta)) \]

\[ + \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|^2} (x-s)} (K(f)(x,\zeta) - K(f)(s,\zeta)) ds. \]
The contribution from the second and third terms is uniformly bounded. Roughly speaking, we organize the equation in such a way in order to use the "Hölder continuity" of \( K(f) \) in \( x \) to obtain "differentiability" of \( f \) in a very week sense, which will be explained in detail in Section 2. Also in Section 2 we will deal with the contribution from \( \zeta_1 < 0 \). In section 3, with further assumption of invers-power potential with angular cut-off and regularity on boundary data, we can extract the singularity from the contribution of the first term on the right hand side of (1.32), which concludes the Theorem 1.5.

2. Upper Bound Estimates

As outlined in the introduction, the goal of this section is to prove

**Lemma 2.1.** Suppose \( f \in L^\infty_\zeta ([0, l], L^\infty_\zeta (\mathbb{R}^3)) \) is a solution to (1.19) with Grad’s angular cut-off potential with \( 0 < \gamma \leq 1 \) and \( f(0, \zeta) \in L^\infty_\zeta (\mathbb{R}^{3+}) \) and \( f(l, \zeta) \in L^\infty_\zeta (\mathbb{R}^{3-}) \).

Then,

(2.1)

\[
| \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\xi_1|^2}} L(f)(0, \zeta) d\zeta | \leq C (|\ln x| + 1) A_\alpha \langle f \rangle,
\]

(2.2)

\[
| \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\xi_1|^2}} (K(f)(x, \zeta) - K(f)(0, \zeta)) d\zeta | \leq C A_\alpha \langle f \rangle,
\]

(2.3)

\[
| \int_{\zeta_1 > 0} \phi_\alpha \int_0^x \frac{\nu(\zeta)}{|\xi_1|} e^{-\frac{\nu(\zeta)}{|\xi_1|^2}} (K(f)(x, \zeta) - K(f)(s, \zeta)) ds d\zeta | \leq C A_\alpha \langle f \rangle,
\]

where

(2.4)

\[ \langle f \rangle := |||f||| + \|f(0, \zeta)\|_{L^\infty_\zeta (\mathbb{R}^{3+})} + \|f(l, \zeta)\|_{L^\infty_\zeta (\mathbb{R}^{3-})}. \]

**Proof.** We observe that \( f \) is in fact bounded for all \( x \) and \( \zeta \) if \( f(0, \zeta) \in L^\infty_\zeta (\mathbb{R}^{3+}) \) and \( f(l, \zeta) \in L^\infty_\zeta (\mathbb{R}^{3-}) \). For \( \zeta_1 > 0 \),

\[
\left| f(x, \zeta) \right| \leq \left| f(0, \zeta) \right| + C \|f\|_s \int_0^x \frac{1}{|\xi_1|} e^{-\frac{\nu(\zeta)}{|\xi_1|^2}(x-s)} ds
\]

(2.5)

\[
\leq \|f(0, \zeta)\|_{L^\infty_\zeta (\mathbb{R}^{3+})} + C \|f\|_s \nu_0^{-1} \int_0^x \frac{1}{|\xi_1|} e^{-\frac{\nu(\zeta)}{|\xi_1|^2} z} dz
\]

\[
\leq C (\|f(0, \zeta)\|_{L^\infty_\zeta (\mathbb{R}^{3+})} + \|f\|_s).
\]

A similar inequality holds for \( \zeta_1 < 0 \). Therefore,

(2.6)

\[ \|f\|_{L^\infty_{\zeta, \xi}} \leq C \langle f \rangle. \]
We observe

\[ |\int_{\zeta_1 > 0} \phi_\alpha \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|}} L(f)(0, \zeta) d\zeta| \]

(2.7)

\[ \leq C A_\alpha \langle f \rangle \int_{\zeta_1 > 0} e^{-\frac{\nu|z|}{|\zeta_1|}} \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|}} d\zeta | \leq C A_\alpha \langle f \rangle (|\ln x| + 1), \]

which concludes (2.1).

We will present the proof for (2.3).

Replacing $0$, let us observe

\[ K(f)(x, \zeta) - K(f)(s, \zeta) = \int_{\zeta_1 > 0} k(\zeta, \zeta_*) (e^{-\frac{\nu(\zeta)}{|\zeta_1|}}|x-s| - 1) f(s, \zeta_*) d\zeta_* \]

(2.8)

\[ + \int_{\zeta_1 > 0} k(\zeta, \zeta_*) \int_s^x \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta_1)}{|\zeta_1|}} K(f)(t, \zeta_*) dt d\zeta_* \]

\[ + \int_{\zeta_1 < 0} k(\zeta, \zeta_*) \int_s^x \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta_1)}{|\zeta_1|}} |x-s| K(f)(x, \zeta_*) d\zeta_* \]

\[ =: H_1 + H_2 + H_3 + H_4. \]

The term $H_2$ and $H_4$ have the property to be proved later

(2.9)

\[ |H_2| \leq C \| f \|_* |x-s|^{\beta}, \ |H_4| \leq C \| f \|_* |x-s|^{\beta}, \]

where $0 < \beta < \frac{\gamma}{2 + \gamma}$. We let

\[ \int_{\zeta_1 > 0} \int_0^x \phi_\alpha(\zeta) \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}} K(f)(x, \zeta) - K(f)(s, \zeta) ds d\zeta \]

(2.10)

\[ = \int_{\zeta_1 > 0} \int_0^x \phi_\alpha(\zeta) \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}} (H_1 + H_2 + H_3 + H_4) ds d\zeta \]

\[ =: B_1 + B_2 + B_3 + B_4. \]

Therefore, for $i = 2$ and $4$,

\[ |B_i| \leq C A_\alpha \| f \|_* | \int_{\zeta_1 > 0} e^{-\frac{|z|^2}{|\zeta_1|^2}} \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}} |x-s|^{\beta} ds d\zeta | \]

(2.11)

\[ \leq C A_\alpha \| f \|_* \int_{\zeta_1 > 0} e^{-\frac{|z|^2}{2}} \nu(\zeta)^{-\beta} |\zeta_1|^{\beta-1} \int_0^x e^{-z} z^\beta e^{-z} dz d\zeta | \]

\[ \leq C A_\alpha \| f \|_* , \]

where $z = \frac{\nu(\zeta)}{|\zeta_1|} (x-s)$. Estimates for $B_1$ and $B_3$ are not so obvious and the analysis is more demanding. We shall present the case for $B_1$ only and the
case for \( B_3 \) can be done similarly. We claim

\[
(2.12) \quad |B_1| = \int_{\xi_1} f(s, \zeta_*) \left( e^{-\frac{\nu(\zeta)}{\|\zeta\|^2} (x-s)} - 1 \right) f(s, \zeta_*) d\zeta_1 ds d| \leq C A_\alpha(f).
\]

Change the order of integration, we have

\[
|B_1| = \int_{0}^{x} k(\xi_1) \phi_\alpha \int_{\xi_1} f(s, \zeta_*) \left( e^{-\frac{\nu(\zeta)}{\|\zeta\|^2} (x-s)} - 1 \right) f(s, \zeta_*) d\zeta_1 ds |.
\]

We observe, for \( a \geq 0 \)

\[
(2.14) \quad \| \phi_\alpha \frac{\nu(\zeta)}{\|\zeta\|^2} e^{-\frac{\nu(\zeta)}{\|\zeta\|^2} (x-s)} \| _{L^1_0} \leq C A_\alpha |x-s|^{-1} (1 + |\ln |x||),
\]

\[
(2.15) \quad \| \phi_\alpha \frac{\nu(\zeta)}{\|\zeta\|^2} e^{-\frac{\nu(\zeta)}{\|\zeta\|^2} (x-s)} \| _{L^\infty_0} \leq C A_\alpha |x-s|^{-2}.
\]

Interpolating the inequalities above, we obtain

\[
(2.16) \quad \| \phi_\alpha \frac{\nu(\zeta)}{\|\zeta\|^2} e^{-\frac{\nu(\zeta)}{\|\zeta\|^2} (x-s)} \| _{L^p} \leq C A_\alpha |x-s|^{-2+\frac{1}{p} (1 + |\ln |x||)^{\frac{1}{p}}},
\]

where \( 1 \leq p \leq \infty \). In particular,

\[
(2.17) \quad \| \phi_\alpha \frac{\nu(\zeta)}{\|\zeta\|^2} e^{-\frac{\nu(\zeta)}{\|\zeta\|^2} (x-s)} \| _{L^2} \leq C A_\alpha |x-s|^{-\frac{3}{2} (1 + |\ln |x||)^{\frac{1}{2}}}.
\]

Let \( h(\theta, \gamma, \alpha) = (\frac{3}{2} - \gamma) \theta + (2 + a - \gamma) (1 - \theta) = 2 + a(1 - \theta) - \gamma - \frac{1}{2} \theta, \)

where \( 0 \leq \theta \leq 1 \). We have

\[
(2.18) \quad \| K(f) \| _{L^\infty_{\theta, \gamma}} = \sup \mathbb{C} \left( |f(\zeta)|(1 + |\zeta|)^{\frac{3}{2}-\gamma} \right) \| K(f) \| _{L^\infty_{\theta, \gamma}} \leq \| K(f) \| _{L^2_{\theta, \gamma}} \leq \| f \| _{L^2_{\theta, \gamma}} \| f \| _{L^\infty_{\theta, \gamma}}.
\]

Combining \( (1.17), (1.18), (2.15), (2.17), \) and \( (2.18) \), we have

\[
(2.19) \quad \| \int_{\xi_1} f(s, \zeta_*) \left( e^{-\frac{\nu(\zeta)}{\|\zeta\|^2} (x-s)} - 1 \right) f(s, \zeta_*) d\zeta_1 ds | \leq CC A_\alpha |x-s|^{-2+\frac{1}{2} \theta (1 + |\ln |x-s||)^{\frac{1}{2} \theta}}.
\]
Applying (2.19) above with fixed $0 < \theta < 1$ and $a$ large enough, we have (2.20)

\[
|B_1| \leq \int_0^x \int_{\zeta_1 > 0} (1 + |\zeta_1|)^{-h(\theta, \gamma, \alpha)} |x - s|^{-(2/\theta - 1)} (1 + |\ln |x - s||)^{1/2} \left( e^{-\frac{\nu(\zeta_1)}{\nu(1)} |x - s|} - 1 \right) |f(s, \zeta_1)| d\zeta_1 ds
\]

\[
\leq CA_\alpha \int_0^x |x - s|^{-(2/\theta - 1)} (1 + |\ln |x - s||)^{1/2} \left( e^{-\frac{\nu(\zeta_1)}{\nu(1)} (1 + |\zeta_1|)^{-h(\theta, \gamma, \alpha)} |f(s, \zeta_1)| d\zeta_1 dt ds
\]

\[
\leq CA_\alpha (f) \int_0^x |x - s|^{-(1 - 1/\theta)} (1 + |\ln |x - s||)^{1+1/2} ds \leq CA_\alpha (f).
\]

The proof for (2.22) is similar and simpler. Replacing $s$ in (2.28) by 0 and denoting these terms as $H'_i$s, we write

\[
\int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta_1)}{\nu(1)} x} (K(f)(x, \zeta) - K(f)(0, \zeta)) d\zeta
\]

\[
= \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta_1)}{\nu(1)} x} (H'_1 + H'_2 + H'_3 + H'_4) d\zeta
\]

\[
= B'_1 + B'_2 + B'_3 + B'_4.
\]

Using the fact (2.22)

\[
|H'_2| \leq C|x|^\beta \|f\|_*, \quad |H'_4| \leq C|x|^\beta \|f\|_*,
\]

we have

\[
|B'_2 + B'_4| \leq \|f\|_* \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta_1)}{\nu(1)} x} |x|^\beta d\zeta
\]

\[
\leq C \|f\|_* \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\nu(\zeta_1) |\zeta_1|^{1-\beta}} d\zeta \leq C \|f\|_*.
\]

The treatment for $B'_1$ and $B'_2$ is similar, and therefore we present the case for $B'_1$ only. Changing the order of integration, we have

\[
B'_1 = \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta_1)}{\nu(1)} x} \int_{\zeta_1 > 0} k(\zeta, \zeta_*) (e^{-\frac{\nu(\zeta)}{\nu(1)} x} - 1) f(0, \zeta_*) d\zeta_1 d\zeta
\]

\[
= \int_{\zeta_1 > 0} (e^{-\frac{\nu(\zeta_1)}{\nu(1)} x} - 1) f(0, \zeta_*) \left( \int_{\zeta_1 > 0} k(\zeta, \zeta_*) \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{\nu(1)} x} d\zeta_1 \right) d\zeta_1.
\]

Note that

\[
\|\phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{\nu(1)} x}\|_{L^1} \leq CA_\alpha (1 + |\ln x|),
\]

\[
\|\phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{\nu(1)} x}\|_{L^\infty} \leq C A_\alpha \left( \frac{1}{x} \right).
\]
Suppose Lemma 2.3. Similar to (2.20), for fixed $0 < \theta < 1$ and $a$ large enough, we have

\begin{equation}
|B'_1| \leq CA_\alpha \int_{\zeta_1 > 0} \left( e^{-\frac{\nu(\zeta_1)}{K|\zeta_1|^2}} - 1 \right) f(0, \zeta_\ast)(1 + \zeta_\ast) \left( \frac{1}{x} \right)^{1 - \frac{\theta}{2}} (1 + |\ln x|)^{\frac{1}{2} \theta} d\zeta_\ast
\end{equation}

\begin{align}
&\leq CA_\alpha \left( \frac{1}{x} \right)^{1 - \frac{\theta}{2}} (1 + |\ln x|)^{\frac{1}{2} \theta} \int_0^x \int_{\zeta_1 > 0} \frac{\nu(\zeta_\ast)}{|\zeta_1|} \left( 1 + |\zeta_\ast| \right)^{-h(\theta, \gamma, a)} f(0, \zeta_\ast) d\zeta_\ast dt \\
&\leq CA_\alpha \langle f \rangle x^\theta (1 + |\ln x|) \frac{1}{1 + \frac{1}{2} \theta} \leq CA_\alpha \langle f \rangle.
\end{align}

We still have to prove (2.29). We will present the proof for $H_2$ only and the one for $H_4$ is similar. We will first present the following lemma

**Lemma 2.2.** If $f \in L^2_\sigma$ and $\theta \in \left( \frac{\theta}{2 + \gamma}, 1 \right)$, then

\begin{equation}
\int \frac{1}{|\zeta_\ast|^{2 - 2\theta \nu(\zeta_\ast)2\theta}} |K(f)|^2 d\zeta_\ast \leq C \|f\|^2.
\end{equation}

The proof Lemma 2.2 follows the idea of the one of Lemma 4.2 in [11]. We will present the proof at the end of this section to make this paper self-contained.

With the Lemma 2.2 above, we have

\begin{align}
|H_2| &= \left| \int_s^x \int_{\zeta_1 > 0} k(\zeta_\ast, \zeta_\ast) \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta_1)}{|\zeta_1|^2} |x-t|} K(f)(t, \zeta_\ast) d\zeta_\ast dt \right| \\
&\leq C \left( \int_s^x \left( \int_{\zeta_1 > 0} \left| \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta_1)}{|\zeta_1|^2} |x-t|} K(f)(t, \zeta_\ast) \right|^2 d\zeta_\ast \right)^{\frac{1}{2}} dt \right) \\
&\leq C \left( \int_s^x \frac{1}{|x-t|^{\theta}} \left( \int_{\zeta_1 > 0} \left| \frac{1}{|\zeta_1|^{2 - 2\theta \nu(\zeta_\ast)2\theta}} K(f)(t, \zeta_\ast) \right|^2 d\zeta_\ast \right)^{\frac{1}{2}} dt \right) \\
&\leq C|x - s|^{1 - \theta} = C|x - s|^{\beta}.
\end{align}

For $\zeta_1 < 0$, we can yield a similar lemma. Together, we have

**Lemma 2.3.** Suppose $f \in L^\infty_\sigma([0, 1], L^\sigma_\zeta(\mathbb{R}^3))$ is a solution to (1.19) with Grad’s angular cut-off potential with $0 < \gamma \leq 1$ and $f(0, \zeta) \in L^\infty_\zeta(\mathbb{R}^{3+})$ and $f(l, \zeta) \in L^\infty_\zeta(\mathbb{R}^{3-})$. Then,

\begin{equation}
|\partial_x \sigma^+_\alpha(x)| \leq C(|\ln |x|| + 1),
\end{equation}
\[ |\partial_x \sigma^-_\alpha(x)| \leq C (\ln |l - x|) + 1, \]

where
\[ \sigma^+_\alpha(x) = \int_{\zeta_1 > 0} \phi_\alpha f d\zeta, \quad \sigma^-_\alpha(x) = \int_{\zeta_1 < 0} \phi_\alpha f d\zeta. \]

**Proof for Lemma 2.2.** We observe
\[
\int |Kf|^2 d\zeta = \int \left( \int k(\zeta, \zeta_*) f(\zeta_*) d\zeta \right) d\zeta
\]
(2.35)
\[
\leq C \|f\|_{L^2} \int |f(\zeta_*)| \int k(\zeta, \zeta_*)(1 + |\zeta|)^{-(\frac{3}{2} - \gamma)} d\zeta d\zeta_*
\]
\[
\leq C \|f\|_{L^2} \int |f(\zeta_*)|(1 + |\zeta_*|)^{-(\frac{3}{2} - 2\gamma)} d\zeta_* \leq C \|f\|_{L^2} \|f\|_* \leq C \|f\|^2_*.
\]
Together with (1.17), we know \(|Kf|^2 \in L^\infty \cap L^1\). Interpolating between these two inequalities, we have
\[
\||Kf|^2\|_{L^p} \leq C \|f\|^2_* \text{ for } 1 \leq p \leq \infty.
\]
Therefore, we now only need to prove for some proper \(0 < \theta < 1\) and Hölder conjugate of \(p, p' \in [1, \infty]\),
\[
\int \frac{1}{|\zeta_1|^{(2 - 2\theta)p'} |1 + |\zeta_*||^{2\gamma} \theta p'} d\zeta_* < \infty,
\]
(2.37)
which yields the following condition:
\[
(2 - 2\theta)p' < 1; \quad (2 - 2\theta + 2\gamma) p' > 3.
\]
Such \(p'\) exists if and only if
\[
\theta < 1, \quad 3(2 - 2\theta) < 2 - 2\theta + 2\gamma,
\]
(2.39)
which concludes Lemma 2.2. \(\square\)

### 3. Asymptotic Formula

In the precious section, we obtain an upper bound for \(|\partial_x \sigma_\alpha|\), which diverges to infinity at boundary like a logarithmic function. Through the analysis, we also localize the source of singularity, which is the contribution from the first term on the right hand side in (1.32). In this section, restricted to the inverse-power potential with angular cut-off, the goal is to further single out and factorize the singularity and form an asymptotic formula, i.e.,

**Lemma 3.1.** Suppose \(f(0, \zeta) \in L_*\) and \(\nabla f(0, \zeta) \in L^P_\zeta(\mathbb{R}^3+)\). Then,
\[
\frac{\partial}{\partial x} \sigma^+_\alpha := \int_{\zeta_1 > 0} \phi_\alpha \frac{e^{-\alpha(\zeta)}}{|\zeta_1|^2} L(f)(0, \zeta) d\zeta
\]
(3.1)
\[
= - \ln x \int \phi_\alpha(0, \zeta_2, \zeta_3) L(f)(0, 0^+, \zeta_2, \zeta_3) d\zeta_2 d\zeta_3 + O(A_\alpha(g)'),
\]
where

\[(3.2)\quad L(f)(0, 0^+, \zeta_2, \zeta_3) := \lim_{\zeta_1 \to 0^+} L(f)(0, \zeta_1, \zeta_2, \zeta_3).\]

**Proof.** If we change to spherical coordinates so that

\[\zeta = (\rho \cos \theta, \rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi),\]

we have

\[(3.3)\quad \frac{\partial}{\partial x} \sigma_{a1}^+ = \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{\rho \cos \theta} e^{-\frac{\nu(\rho)}{\rho \cos \theta} x} F(\rho, \theta, \phi) \rho^2 \sin \theta d\theta d\phi d\rho,\]

where

\[(3.4)\quad F(\rho, \theta, \phi) = \pi^{-\frac{3}{2}} \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} L(f)(0, \zeta) =: g(\zeta).\]

Further letting \(z = \cos \theta\), we obtain

\[(3.5)\quad \frac{\partial}{\partial x} \sigma_{a1}^+ = \int_0^\infty \int_0^{2\pi} \left[ \int_0^1 \frac{1}{z} e^{-\frac{\nu(\rho)}{\rho x} x} F(\rho, z, \phi) dz \right] e^{-\frac{\nu^2}{\rho} z^2} \rho d\phi d\rho,\]

where

\[(3.6)\quad F(\rho, z, \phi) = \bar{F}(\rho, \cos^{-1} z, \phi).\]

Here, we introduce a well-known special function, exponential integral,

\[(3.7)\quad E_1(x) = \int_0^1 \frac{1}{z} e^{-\frac{1}{z}} dz.\]

The \(E_1(x)\) has the following properties, \[1]:

\[(3.8)\quad E_1(x) = -\gamma - \ln x + \sum_{k=1}^\infty \frac{(-1)^{k+1} x^k}{k \cdot k!},\]

\[(3.9)\quad \frac{1}{2} e^{-x} \ln(1 + \frac{2}{x}) \leq E_1(x) \leq e^{-x} \ln(1 + \frac{1}{x}) \text{ for } x > 0.\]

From the properties above, we have

\[(3.10)\quad E_1(x) = -\ln x + O(1) \text{, for } 0 < x \leq 1,\]

Let

\[(3.11)\quad H(z, x) = -\int_0^1 \frac{1}{u} e^{-\frac{\nu(\rho)}{\rho u} x} du.\]

Notice that \(\frac{\partial}{\partial z} H(z, x) = \frac{1}{x} e^{-\frac{\nu(\rho)}{\rho x} x}\) and \(H(0, x) = E_1(\frac{\nu(\rho)}{\rho} x)\). Performing integration by parts for the inner most integral in \[3.5\], we obtain

\[(3.12)\quad \int_0^1 \left( \frac{\partial}{\partial z} H(z) \right) F(\rho, z, \phi) dz = E_1(\frac{\nu(\rho)}{\rho} x) F(\rho, 0, \phi) - \int_0^1 H(z) \left( \frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz.\]
The first term on the right hand side above is the source of singularity and will be explained in detail later. We will prove the contribution from the second term above is finite. Let

\begin{equation}
I := \int_0^\infty \int_0^{2\pi} E_1(\frac{\nu(\rho)}{\rho}) F(\rho, 0, \phi) e^{-\frac{\rho^2}{2}} \rho d\phi d\rho.
\end{equation}

\begin{equation}
II := \int_0^\infty \int_0^{2\pi} \int_0^1 H(z) \left( \frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz e^{-\frac{\rho^2}{2}} \rho d\phi d\rho.
\end{equation}

Notice that

\begin{equation}
|H(z, x)| \leq |\ln z|,
\end{equation}

\begin{equation}
\frac{\partial}{\partial z} F(\rho, z, \phi) = \rho \frac{\partial}{\partial \zeta_1} g + \frac{\rho z}{\sqrt{1 - z^2}} \cos \phi \frac{\partial}{\partial \zeta_2} g + \frac{\rho z}{\sqrt{1 - z^2}} \sin \phi \frac{\partial}{\partial \zeta_3} g.
\end{equation}

We have

\begin{equation}
|II| \leq \left| \int_0^\infty \int_0^{2\pi} \int_0^1 H(z) \left( \frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz e^{-\frac{\rho^2}{2}} \rho d\phi d\rho \right|
\end{equation}

\begin{align*}
&\leq \left| \int_0^\infty \int_0^{2\pi} \left( \int_0^1 C(\left| \frac{\partial}{\partial \zeta_1} g \right| + \left| \frac{\partial}{\partial \zeta_2} g \right|) dz + \int_0^1 \left| \ln z \right| \left| \frac{\partial}{\partial \zeta_1} g \right| dz \right) e^{-\frac{\rho^2}{2}} \rho^2 d\phi d\rho \right|
\end{align*}

\begin{align*}
&\leq CA_\alpha \int_{\zeta_1 > 0} e^{-\frac{|\zeta|^2}{2}} \left( \left| L(0, \zeta) \right| + \left| \frac{\partial}{\partial \zeta_2} L(0, \zeta) \right| + \left| \frac{\partial}{\partial \zeta_3} L(0, \zeta) \right| \right) d\zeta
\end{align*}

\begin{align*}
+ A_\alpha \int_0^1 \int_0^{2\pi} \left( \int_0^1 \left| \ln z \right|^q dz \right)^{\frac{1}{q}} \left( \int_0^1 \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p dz \right)^{\frac{1}{p}} e^{-\frac{\rho^2}{2}} \rho^2 d\phi d\rho \right|
\end{align*}

where \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1. \)

Let the second term on the right hand side above be \( I'' \).

\begin{equation}
|I''| \leq \left( \int_0^\infty \int_0^{2\pi} \left( \int_0^1 \left| \ln z \right|^q dz \right)^{\frac{1}{q}} \left( \int_0^\infty \int_0^{2\pi} \left( \int_0^1 \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p dz \right)^{\frac{1}{p}} e^{-\frac{\rho^2}{2}} \rho^2 d\phi d\rho \right)
\end{equation}

\begin{align*}
&\leq C_p A_\alpha \left( \int_{\zeta_1 > 0} \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p e^{-\frac{|\zeta|^2}{2}} d\zeta \right)^{\frac{1}{p}}
\end{align*}

Therefore,

\begin{equation}
|II| \leq CA_\alpha \int_{\zeta_1 > 0} e^{-\frac{|\zeta|^2}{2}} \left( \left| L(0, \zeta) \right| + \left| \frac{\partial}{\partial \zeta_2} L(0, \zeta) \right| + \left| \frac{\partial}{\partial \zeta_3} L(0, \zeta) \right| \right) d\zeta
\end{equation}

\begin{align*}
+ C_p A_\alpha \left( \int_{\zeta_1 > 0} \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p e^{-\frac{|\zeta|^2}{2}} d\zeta \right)^{\frac{1}{p}},
\end{align*}

where \( p > 1. \)
Using the assumption and \( (1.15) \), we have
\[
|II| \leq C A_\alpha \langle \|\nabla f(0, \zeta)\|_{L^p(R^3)} + \|f(0, \zeta)\|_* \rangle
\]

Finally, we are going to extract the singularity from \( I \). We let
\[
|I_s| \leq C A_\alpha \langle f \rangle.
\]

Using the asymptotic formula \( (3.10) \), we obtain
\[
|I_l| = \frac{\nu(\rho)}{\rho} x > \frac{c_0}{c_1}.
\]

Applying \( (3.9) \), we have
\[
|I_s| \leq C A_\alpha \langle f \rangle.
\]

Using the asymptotic formula \( (3.10) \), we obtain
\[
I_l = -\ln(x) \int_0^\infty \int_0^{2\pi} F(\rho, 0, \phi) e^{\frac{-\rho^2}{2}} \rho d\phi d\rho + O(A_\alpha \langle f \rangle)
\]
\[
= -\ln x \int \int \phi(0, \zeta_2, \zeta_3) L(f)(0, 0^+, \zeta_2, \zeta_3) d\zeta_2 d\zeta_3 + O(A_\alpha \langle f \rangle (1 + \rho_0^2 |\ln x|)).
\]

The remaining task is to estimate \( \rho_0 \). If we assume \( x \leq \frac{1}{2c_1} \), then
\[
\frac{\nu(\rho_0)}{\rho_0} x \leq c_1 \frac{(1 + \rho_0)^\gamma}{\rho_0} x \leq \frac{(1 + \rho_0)^\gamma}{2\rho_0}.
\]

If \( \gamma = 1 \), we see \( \rho_0 \leq 1 \). Observe that, for \( 0 \leq \gamma < 1 \), \( \rho \) grows faster then the \( \frac{1}{2}(1 + \rho)^\gamma \) as \( \rho \to \infty \). Therefore, \( (3.26) \) implies \( \rho_0 \leq m \) for some \( m < \infty \). Therefore, we have
\[
1 = \frac{\nu(\rho_0)}{\rho_0} x \leq c_1 \frac{(1 + m)^\gamma}{\rho_0} x.
\]

We have
\[
|\rho_0^2 \ln x| \leq C x^2 |\ln x| \leq C
\]
and conclude the lemma.

Finally, combining \( (2.2) \) and \( (2.3) \) in Lemma \( 2.1 \), \( (2.33) \) in Lemma \( 2.3 \) and Lemma \( 3.1 \), we concludes Theorem \( 1.5 \).
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