Noncommutative $D_3$-brane, Black Holes and Attractor Mechanism

Supriya Kar $^{a,b,1}$ and Sumit Majumdar $^{b,2}$

$^a$The Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11, Trieste, Italy

$^b$Department of Physics & Astrophysics, University of Delhi
Delhi 110 007, India

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Abstract

We revisit the 4$D$ generalized black hole geometries, obtained by us [1], with a renewed interest, to unfold some aspects of effective gravity in a noncommutative $D_3$-brane formalism. In particular, we argue for the existence of extra dimensions in the gravity decoupling limit in the theory. We show that the theory is rather described by an ordinary geometry and is governed by an effective string theory in 5$D$. The extremal black hole geometry $AdS_5$ obtained in effective string theory is shown to be in precise agreement with the gravity dual proposed for $D_3$-brane in a constant magnetic field. Kaluza-Klein compactification is performed to obtain the corresponding charged black hole geometries in 4$D$. Interestingly, they are shown to be governed by the extremal black hole geometries known in string theory. The attractor mechanism is exploited in effective string theory underlying a noncommutative $D_3$-brane and the macroscopic entropy of a charged black hole is computed. We show that the generalized black hole geometries in a noncommutative $D_3$-brane theory are precisely identical to the extremal black holes known in 4$D$ effective string theory.
1 Introduction

Black holes are macroscopic objects with strong curvatures in space-time and they possess a non-zero temperature. Naturally, the thermodynamic properties of the black holes are characterized by its macroscopic entropy. In fact, the nature of thermodynamic entropy of a black hole is very similar to that of the Bekenstein-Hawking. It is known to be governed by one quarter of its area of the horizon in Planck units [2, 3]. Thus, on the one hand, the computation of thermodynamic entropy of a black hole involves the counting of microstates, which is based on statistical analysis. On the other hand, the Bekenstein-Hawking entropy of a black hole appears to be governed by some macroscopic interpretations.

In the recent past, the issue of microscopic analysis relevant for the computation of entropy of a black hole was revisited in string theory [4]-[13]. Interestingly, the microscopic entropy was computed for a certain magnetically charged black holes in its near horizon geometry [8] following an attractor mechanism in the theory. There, the variation of the moduli fields in string theory is governed by the damped geodesic equation on the moduli space. The damping is essentially caused by the presence of the electromagnetic field in the theory. The geodesic equation possesses an attractive fixed point at its event horizon. Since the area of the event horizon is determined precisely by the electromagnetic charges, the variation of the asymptotic moduli fields does not affect the near horizon geometry of the macroscopic black hole. In other words, the number of internal black hole states remain unchanged under the influence of an adiabatic change in its macroscopic environment. It leads to the statistical interpretation of the Bekenstein-Hawking entropy.

On the other hand, there are renewed interests to investigate some of the related issues in the quantum gravity with the developments of nonlinear electrodynamics on a D-brane [14]-[28],[1]. The stringy formulations have indeed motivated the construction of some of the realistic brane-world models, which are known to describe various effective theories of gravity. Interestingly, the construction of D-brane solutions leading to solitons, shock waves and black holes have been obtained in the folklore of string theory [15, 16, 17, 22, 23, 24, 25, 26, 1]. Thus, in a brane-world scenario, one needs a better understanding of the effective nature of gravity derived from the nonlinearity in the electromagnetic (EM-) field.

In the context, the computation of black hole microstates has been formalized by Wald [29]. Interestingly, the Bekenstein-Hawking entropy of an extremal black hole has been computed in presence of various different higher derivative terms in string theory [30]-[38]. For instance, in an arbitrary $d$-dimensions, the near horizon geometry of these black holes are governed by $AdS_2 \times S^{d-2}$. Then, the black hole entropy is shown to be defined as a function of electric and magnetic charges, respectively, associated with the one-form and the $(d-3)$-form gauge fields in theory [31].

In this paper, our primary motivation is to investigate some aspects of an effective theory
of gravity, formulated, in a noncommutative frame-work [39]-[45]. Very recently, one such attempt was made by us in a formulation based on a noncommutative $D_3$-brane [1]. Generalized Reissner-Nordstrom (RN-) and Schwarzschild black holes in $4D$ are obtained in the effective frame-work describing a curved $D_3$-brane. Most importantly, the generalized RN-black hole geometry obtained is new in the effective string theory. This is similar to the case for Einstein-Maxwell black hole with a generalized mass and a charge. A priori, the generalized RN-black holes are different than the charged black holes obtained in string theory [46]. This is due to the fact that the dilaton couples to the gauge field strength in string theory. As a result, every solution in presence of a gauge field in string theory must possess a nonconstant dilaton. On the contrary, the generalized RN-black holes are independent of the value of the dilaton in effective string theory. It provides hint towards an underlying attractor mechanism for the black holes in a noncommutative frame-work. In particular, we see that the generalized black hole geometries in a noncommutative $D_3$-brane theory are precisely identical to the ones obtained in effective string theory with an ordinary geometry [46].

In the context, we compute the entropy function of a generalized black hole. It is argued that the attractor mechanism established in string theory [8, 4, 10] may be extended to an effective string formulation based on a noncommutative $D_3$-brane. As a result, the 4-dimensional generalized black holes [1] are further investigated to explore the possibility of extra dimensions, if any, in the Planckian regime. Working out the constraints arising out of the noncommutativity, it is shown that the effective string theory governs an ordinary geometry in 5-dimensions. The additional small dimension in the theory is essentially due to the curved nature of $D_3$-brane and is transverse ($\perp$) to its flat world volume. Intuitively, the new $\perp$-dimension seemingly traverses into the bulk of the string from its boundary. Alternately, the bulk description is argued to govern an effective string in 5-dimensions. Various black hole solutions characterized by the effective mass $M_{\text{eff}}$ and charge $Q_{\text{eff}}$ are obtained in the frame-work, which are based on an underlying space-time noncommutativity on a $D_3$-brane at its gravity decoupling limit. Interestingly, in the limit, a black hole is governed by its near horizon geometry and is precisely described by the one obtained [47] using $AdS_5$ / noncommutative Yang-Mills correspondence. Furthermore, we perform Kaluza-Klein compactification of effective string theory and obtain $4D$ extremal black hole geometry in string and Einstein frames. Though, the extremal black holes are obtained in an effective theory of gravity, their geometrical fate is purely dependent on the nonlinear EM-field in the frame-work. The extremal black holes resemble to that obtained in string theory [46]. At the first sight, we would like to keep a note that our analysis is in agreement with the $AdS/CFT$ conjecture established in string theory [48, 47, 49, 50]. Most importantly, the $4D$ black holes obtained in a Kaluza-Klein compactified theory in the bulk are in precise correspondence with the ones obtained on a curved $D_3$-brane in its gravity decoupling limit. In otherwords, the black hole geometries
in an effective string theory are identical to that of the generalized geometries obtained in a noncommutative $D_3$-brane theory.

The plan of this paper is as follows. In section 2.1, we outline the relevant results from our recent work [1]. The possibility of extra dimensions in a formalism based on noncommutative $D_3$-brane in its gravity decoupling limit is argued in section 2.2. The required effective string description in 5$D$ is obtained in section 2.3. Subsequently, the black hole geometries are constructed in 5$D$ and is shown to be in precise agreement with the result obtained in a different context using $AdS_5$ / noncommutative gauge theory correspondence. The relevant 4$D$ extremal black hole geometries in Einstein and string frames are obtained, respectively, in sections 3.1 and 3.2 by using the Kaluza-Klein compactification in the frame-work. In section 3.3, the attractor mechanism is analyzed in the noncommutative $D_3$-brane formalism to compute the black hole entropy function. Finally, we conclude with some remarks in section 4.

2 Noncommutative $D_3$-brane and the notion of effective string

2.1 Preliminaries

Consider a $D_3$-brane, in presence of a constant two form $b$ induced on its world-volume. The uniform EM-field on the $D_3$-brane is governed by a nonlinear electrodynamics. Our starting point is in type IIB string theory. In principle, the gravity and the gauge dynamics in the frame-work may be approximated by coupling the $D_3$-brane dynamics to a generalization of Einstein’s action, i.e. in presence of higher derivatives terms. Since the present work is primarily confined to the gravity decoupling limit in the theory, the higher derivative terms in the gravity sector shall not contribute significantly in the frame-work.

In a static gauge for the space-time, the bulk metric may be viewed on the world-volume and the complete action becomes [1]

$$S = \int d^4 y \sqrt{g} \left( \frac{1}{16\pi G_N} R - \frac{1}{4} g^{\mu \nu} g^{\lambda \rho} \mathcal{F}_{\mu \lambda} \mathcal{F}_{\nu \rho} + O(F^4) + \ldots \right),$$

(1)

where the $U(1)$ gauge field $\tilde{\mathcal{F}}_{\mu \nu} = (b + 2\pi \alpha' F)_{\mu \nu}$. In absence of higher derivative terms in gauge field, the frame-work resembles to the Einstein’s theory coupled to the Maxwell’s. However in presence of higher derivatives, the action (1) can be given by

$$S = \int d^4 y \sqrt{g} \left( \frac{1}{16\pi G_N} R - \frac{1}{4} \left[ \mathcal{F}^2 - \frac{1}{2} \mathcal{F}\mathcal{F}^\perp F_+^2 K^2(\mathcal{F}) \right] \right),$$

(2)

where $\mathcal{F}_\pm = (\mathcal{F} \pm \star \mathcal{F})$ and $K(\mathcal{F})$ contains all the higher order terms in field strength. The Hodge dual of $\mathcal{F}$ is denoted as $\star \mathcal{F}$. The Minkowski’s inequality can be seen to yield the (anti-) self-duality condition $|E| = |B|$ in the theory. Since all the higher order terms in gauge fields
vanish, the framework leads to an exact stringy description. Then, the relevant action on a curved $D_3$-brane becomes

$$S = \int d^4y \sqrt{g} \left( \frac{1}{16\pi G_N} R - \frac{1}{4} g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} \right). \quad (3)$$

Interestingly, the Einstein’s equation is governed by the vacuum equations i.e. $T_{\mu\nu} = 0$, which is due to the self-dual nonlinear gauge field in the theory.

Now let us consider an equivalent noncommutative gauge dynamics on the $D_3$-brane [14]. The ordinary product is replaced by the Moyal $\star$-product on the world-volume, which introduces nonlocal terms in the gauge theory due to the infinite number of derivatives there. However, it does not affect the bulk dynamics i.e. the gravity sector. Then the action (3), for a curved $D_3$-brane, can alternately be given by

$$S = \int d^4y \sqrt{G} \left( \frac{1}{16\pi G_N} R - \frac{1}{4} G^{\mu\lambda} G^{\nu\rho} \hat{F}_{\mu\nu} \star \hat{F}_{\lambda\rho} \right). \quad (4)$$

where $G_{\mu\nu}$ denotes the effective metric. It can be checked that the $G_{\mu\nu}$ can be generalized to include higher order terms in the two-form potential $b$. On can re-express the effective metric as

$$G_{\mu\nu} = g_{\mu\nu} - \left( b g^{-1} b \right)_{\mu\nu} + \left( [b g^{-1} b] [b g^{-1} b] \right)_{\mu\nu} + \ldots. \quad (5)$$

With a gauge choice $G_{i\alpha} = 0$ for $(\alpha, \beta) = (y^0, y^1)$ and $(i, j) = (y^2, y^3)$, the action can be simplified using a noncommutative scaling [1]. The action takes a form

$$S = \int d^2y^{(a)} d^2y^{(i)} \sqrt{h} \sqrt{\hat{h}} \left[ \frac{1}{16\pi} R_h + \frac{1}{64\pi} h^{ij} \partial_i \hat{h}_{\alpha\beta} \partial_j \hat{h}_{\gamma\delta} e^{\alpha\gamma} e^{\beta\delta} - \frac{1}{2} \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} \right], \quad (6)$$

where $h_{ij}$ and $\hat{h}_{\alpha\beta}$ denote the components of $G_{\mu\nu}$, respectively, in the $\perp$- and longitudinal ($L$-) spaces. The action is derived by using the vacuum field configurations, i.e. $\partial_i h_{ij} = 0$, $R_h = 0$ and $\hat{F}_{\alpha\beta} = 0$.

Unlike to the nonlinear EM-theory (2), the energy-momentum tensor turns out to be significant in the noncommutative framework (6). The generalized black hole geometry in the theory can be derived. The computational details are beyond the scope of this paper and may be checked from our recent work [1]. Finally, the generalized black hole geometry is given by

$$ds^2 = - \left( 1 - \frac{2M_{\text{eff}}}{r} - \frac{Q_{\text{eff}}^2}{r^4} + \frac{2M_{\text{eff}} Q_{\text{eff}}^2}{r^5} \right) dt^2 + \left( 1 - \frac{2M_{\text{eff}}}{r} - \frac{Q_{\text{eff}}^2}{r^4} + \frac{2M_{\text{eff}} Q_{\text{eff}}^2}{r^5} \right)^{-1} dr^2 + \left( 1 - \frac{Q_{\text{eff}}^2}{r^4} \right) r^2 d\theta^2 + \left( 1 - \frac{Q_{\text{eff}}^2}{r^4} \right)^{-1} r^2 \sin^2 \theta d\phi^2, \quad (7)$$

$^3$The scaling may also be seen as that of Planckian energy limit as discussed in refs.[51, 52].
where mass and charge are generalized due to the noncommutative Θ-terms. Explicitly, they are given by

\[ M_{\text{eff}} = (G_N M) \left[ 1 - \frac{\Theta}{2r^2} + \mathcal{O}(\Theta^2) + \ldots \right] \]

and

\[ Q_{\text{eff}}^2 = (G_N Q^2) \left[ 1 - \frac{\Theta}{r^2} + \mathcal{O}(\Theta^2) + \ldots \right]. \tag{8} \]

2.2 Extra dimensions

The generalized black hole geometry \(^7\) in its gravity decoupling limit, \(i.e. M \to 0\), gives rise to a Schwarzschild-like geometry. It is given by

\[ ds^2 = \left( 1 - \frac{Q_{\text{eff}}^2}{r^4} \right) \left[ -dt^2 + r^2 d\theta^2 \right] + \left( 1 - \frac{Q_{\text{eff}}^2}{r^4} \right)^{-1} \left[ dr^2 + r^2 \sin^2 \theta \, d\phi^2 \right]. \tag{9} \]

In the limit, the black hole is characterized by its charge, which can also be interpreted as its light mass in the frame-work. In fact, the correction term \(Q_{\text{eff}}^2/r^4\) is essentially due to the non-zero energy-momentum tensor and hence is associated with the mass of the black hole. Since an \(n\)-dimensional Schwarzschild black hole mass is known to be associated with \(r^{3-n}\) term in its metric component \(G_{tt}\), a priori, the black hole (9) can be seen to be governed by a 7-dimensional space-time. However, these dimensions are not all independent due to the noncommutative constraints in the frame-work. The number of constraints on the space-time degrees of freedom can be computed from the EM-field configurations in the theory. For instance, two different length scales in the frame-work can be argued with two non-zero components of the \(E\)-field and one non-zero component of \(B\)-field or vice-versa. The self dual EM-field reduces the number of noncommutative constraints to two. As a result, the resulting 7\(D\) space-time effectively governs a 5\(D\) space-time with ordinary geometry. The orthogonality in the space-time enforces that the extra dimension is transverse to the \(D_3\)-brane world-volume. A priori, the world-volume coordinates together with three of the extra dimensions describe an effective string in 5\(D\) with an ordinary geometry. In otherwords, the noncommutativity is used to separate two large \(\perp\)-dimensions \((y^2, y^3)\) scaled apart from the longitudinal ones \((y^0, y^1)\). The presence of three small dimensions, in the regime, make the total dimension of space-time to five.

2.3 Effective string in 5\(D\) and charged black holes

We begin this section with an effective string dynamics in 5-dimensions. Since the notion of a curved \(D_3\)-brane is contained in a type IIB string theory, we consider the string compactified on \(K^3 \times S_1\). For instance, the generic form of the effective string action may be obtained from ref.[53] and is given by

\[ S = \int d^5 x \sqrt{-G} e^{-\Phi} \left[ R + (\partial \Phi)^2 - \frac{1}{4} F^{(\tilde{m})} C_{\tilde{m} \tilde{n}} F^{(\tilde{n})} - \frac{1}{12} \mathcal{H}^2 \right], \tag{10} \]
where $G = \det G$ and $\Phi$ denotes the dilaton. $F^{(\bar{m})}$ is a two form and $\mathcal{H}$ is a three form gauge field strength. $C_{\bar{m} \bar{n}}$ is a square matrix and it signifies the appropriate moduli field couplings in the theory. The irrelevant moduli term, i.e. $M_{pq} \partial_\alpha \phi^p \partial_\beta \phi^q$, is dropped from the action (10).

In the Einstein frame, i.e. $G_{\alpha\beta} = e^{2\Phi/3} G^E_{\alpha\beta}$, the action may be re-expressed as

$$S = \int d^5 x \sqrt{-G^E} \left[ R - \frac{1}{3} (\partial \Phi)^2 - \frac{1}{4} e^{-2\Phi/3} F^{(\bar{m})} C_{\bar{m} \bar{n}} F^{(\bar{n})} - \frac{1}{12} e^{-4\Phi/3} \mathcal{H}^2 \right]. \quad (11)$$

The action can be re-expressed using the duality

$$e^{-\Phi/3} H^{\mu\nu\lambda} = \frac{e^{\mu\nu\lambda\rho}}{2! \sqrt{-G^E}} \tilde{F}_{\sigma\rho}. \quad (12)$$

It is given by

$$S = \int d^5 x \sqrt{-G^E} \left[ R - \frac{1}{3} (\partial \Phi)^2 - \frac{1}{4} F^{(m)} \Lambda_{mn} [\Phi] F^{(n)} \right], \quad (13)$$

where

$$\Lambda_{mn} [\Phi] = e^{-2\Phi/3} \begin{pmatrix} C_{\bar{m} \bar{n}} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F^{(m)} = \begin{pmatrix} F^{(\bar{m})} \\ 0 \end{pmatrix}. \quad (14)$$

Though the EM-field, $E = (0, E_2, E_3)$ and $B = (0, B_2, B_3)$, are constants on a $D_3$-brane world-volume, their generalization to the 5-dimensional string bulk leads to nonconstant $E$- and $B$-fields. The E-field can be generalized appropriately, by using the effective metric (5), to yield a redefined electric field $\tilde{E}$ on the world-volume. It is given by

$$\tilde{E}^2 = \frac{E^2}{1 + E^2}. \quad (15)$$

Now the ansatz for a nontrivial 5-dimensionsal string metric may be constructed and is given by

$$ds^2 = dr^2 + G_{\mu\nu}(r) \, dx^\mu \, dx^\nu. \quad (16)$$

Using a Schwarzschild black hole geometry, i.e. a vacuum solution for the genuine metric $g_{\mu\nu}$ (7), the 5-dimensional solution to the effective string theory (13) is worked out to yield

$$ds^2 = dr^2 + \left( 1 - \frac{2M_{\text{eff}}}{r} \right) (1 - E^2) \, dt^2 + \left( 1 - \frac{2M_{\text{eff}}}{r} \right)^{-1} (1 - E^2)^{-1} \, d\rho^2$$

$$\quad + \left( 1 + B^2 \right) \rho^2 d\theta^2 + \left( 1 + B^2 \right)^{-1} \rho^2 \sin^2 \theta \, d\phi^2, \quad (17)$$

where $r$ denotes the $\perp$-coordinate to the $D_3$-brane world-volume $(t, \rho, \theta, \phi)$. Incorporating the self-duality of the EM-field, we re-express the metric in terms of the isotropic spherical coordinates for our purpose. Then, the 5D string solution in $(t, u, \rho, \theta, \phi)$-coordinates, for $u = 1/r$, becomes

$$ds^2 = \frac{du^2}{u^2} + u^2 \left[ - \left( 1 - \frac{M_{\text{eff}}}{2r} \right)^2 \left( 1 + \frac{M_{\text{eff}}}{2r} \right)^{-2} + \tilde{E}^2 \left( 1 + \frac{M_{\text{eff}}}{2r} \right)^{-4} \right] dt^2$$

6
where $\tilde{E}$ denotes the $E$-field in 5-dimensions. In the gravity decoupling limit, i.e. $M \to 0$, the geometry simplifies drastically to yield

$$ds^2 = \frac{du^2}{u^2} + \frac{d\rho^2}{\rho^2} + \rho^2 d\Omega^2.$$  \hspace{1cm} (19)

Using eq.(15), the $E$-field in the theory can be given by

$$\tilde{E} = \left[ \frac{Q_{\text{eff}}^2 u^4}{1 + Q_{\text{eff}}^2 u^4} \right]^{1/2}.$$  \hspace{1cm} (20)

It is straightforward to check that for large $u$, the constant value of the EM-field is recovered, which corresponds to that on the $D_3$-brane. The IR-limit, i.e. $u \to 0$, corresponds to the string bulk and in the UV-limit, i.e. $u \to \infty$, defines the string boundary. Interestingly, the gravity decoupled solution (19)-(20) obtained following a noncommutative $D_3$-brane formulation describes an extremal black hole there. It is precisely correspond to the gravity dual proposed for $D_3$-brane in presence of a constant magnetic field [47]. In other words, it provides evidence for the holographic correspondence between the bulk of the string and the noncommutative $U(1)$ theory on its boundary.

Re-writing the extremal black hole geometry (19) in $(t, r, \rho, \theta, \phi)$-coordinate, one gets

$$ds^2 = \frac{dr^2}{r^2} + \frac{r^2}{Q_{\text{eff}}^2 + r^4} \left[ -dt^2 + d\rho^2 + \rho^2 d\Omega^2 \right].$$  \hspace{1cm} (21)

Interestingly, the geometry remains unchanged under a change $r \to 1/r$. In $(t, u, \rho, \theta, \phi)$-coordinate system, the extremal black hole geometry governs a weakly coupled gravity in the string bulk and a strongly coupled noncommutative $U(1)$ gauge theory at its boundary. The situation reverses in the $(t, r, \rho, \theta, \phi)$-coordinate system, i.e. a weakly coupled gauge theory at the string boundary and a strongly coupled gravity in its bulk. It further reconfirms the strong-week coupling string duality between the (noncommutative) gauge and (ordinary) gravity theories.

Alternatively, the extremal limit can be incorporated by taking $M_{\text{eff}} \to Q_{\text{eff}}^2$. In the limit, the generalized Schwarzschild geometry (17) becomes

$$ds^2 = \frac{du^2}{u^2} + u^2 \left( -dt^2 + \left(1 + \frac{Q_{\text{eff}}^2}{2r} \right)^6 \left(1 - \frac{Q_{\text{eff}}^2}{2r} \right)^{-2} d\rho^2 \right) \right] \rho^2 d\Omega^2.$$  \hspace{1cm} (22)
In the limit $\tilde{E}^2 = -2Q_{\text{eff}}^2/r$. Then, the extremal black hole geometry can be approximated to yield
\[
\begin{align*}
    ds^2 &= \frac{du^2}{u^2} + u^2 \left[ -dt^2 + \left( 1 - \frac{Q_{\text{eff}}^2}{2r} \right)^{-2} \, d\rho^2 + \rho^2 \, d\Omega^2 \right]
    \end{align*}
\]
(23)

It is important to note that the extremal black hole geometry obtained following a noncommutative $D_3$-brane [1] is a precise generalization of that obtained in a 4D effective string theory[46]. The effective parameter $Q_{\text{eff}}^2$ in eq.(23) may be interpreted as a light mass of the extremal black hole. Thus in the limit, the extremal solution is well approximated by the $AdS_5$ geometry and is given by
\[
    ds^2 = -u^2 \, dt^2 + \frac{du^2}{u^2} + u^2 \, d\Omega^2_3,
\]
(24)
where $d\Omega^2_3$ governs the $S^3$ geometry. The analysis suggests that the generalized Schwarzschild black hole (17) in its gravity decoupling limit describes an $AdS_5$ geometry, which is in agreement with the $AdS/CFT$-correspondence established in string theory.

3 Extremal black hole geometries in 4D

3.1 Einstein metric

Let us consider the Kaluza-Klein compactification of the effective action (13) obtained in string bulk in Einstein frame. The 5D metric may be given explicitly in terms of the 4D metric $G_{\mu\nu}(x)$, $U(1)$ gauge field $A_\mu(x)$ and a scalar $\phi(x)$. It takes a form
\[
    G^E_{\alpha\beta} = e^{2\phi/\sqrt{3}} \begin{pmatrix}
    G^E_{\mu\nu} + e^{-2\sqrt{3}\phi} A_\mu A_\nu & e^{-2\sqrt{3}\phi} A_\mu \\
    e^{-2\sqrt{3}\phi} A_\nu & e^{-2\sqrt{3}\phi}
    \end{pmatrix}.
\]
(25)

Now, the Kaluza-Klein compactification of the 5D effective action (13) is performed. Ignoring the gauge Chern-Simon terms, the 4D action becomes
\[
    S = \int d^4x \sqrt{-G^E} \left[ R - 2(\partial\phi)^2 - \frac{1}{3}(\partial\Phi)^2 - \frac{1}{4} F^{(i)} D_{ij}[\Phi, \phi] F^{(j)} \right],
\]
(26)
The relevant vector field multiplet is
\[
    F^{(i)}_{\mu\nu} = \begin{pmatrix}
    F^1_{\mu\nu} \\
    F^{(m)}_{\mu\nu}
    \end{pmatrix}.
\]
(27)
The moduli matrix is given by
\[
    D_{ij}[\Phi, \phi] = \begin{pmatrix}
    e^{-2\sqrt{3}\phi} & 0 \\
    0 & e^{-2[2\phi + \sqrt{3}\phi]/3} \Lambda_{mn}
    \end{pmatrix}.
\]
(28)
The equations of motion for the gravity, the scalars and the gauge fields are, respectively, given by

\[ R_{\mu \nu} = 2 \partial_\mu \phi \partial_\nu \phi + \frac{1}{3} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} D_{ij}[\Phi, \phi] \left( F^{(i)}_{\mu \lambda} F^{(j)\lambda \nu} - \frac{1}{4} F^{(i)}_{\mu\nu} F^{(j)} \right), \]

\[ \partial_\mu \left( \sqrt{-G^\mathbb{E}} \partial^{\mu} \phi \right) = \frac{1}{16 \sqrt{-G^\mathbb{E}}} \partial D_{ij}[\Phi, \phi] F^{(i)} F^{(j)}, \]

\[ \partial_\mu \left( \sqrt{-G^\mathbb{E}} \partial^{\mu} \Phi \right) = \frac{3}{8 \sqrt{-G^\mathbb{E}}} \partial D_{ij}[\Phi, \phi] F^{(i)} F^{(j)}, \]

and

\[ \partial_\mu \left( \sqrt{G^\mathbb{E}} D_{ij}[\Phi, \phi] F^{(j)\mu \nu} \right) = 0. \] (29)

The most general static, spherically symmetric, solutions to these equations can be given by

\[ ds^2 = -a^2(r) dt^2 + a^{-2}(r) dr^2 + b^2(r) d\Omega^2. \] (30)

Let us first consider the case in presence of a magnetic field only. Then, the EM-fields are

\[ F^{(i)} = Q_{m(i)}^\text{eff} \sin \theta \, d\theta \wedge d\phi, \] (31)

where \( Q_{m(i)}^\text{eff} \) denote the effective magnetic charges for \( i = 1, 2, 3 \ldots \). The independent nonzero components of the Ricci tensor, in an orthonormal basis, are computed to yield

\[ R_{tt} = -\frac{a^2}{b^4} V_{\text{eff}}(\Phi, \phi), \]

\[ R_{rr} = 2(\partial_\phi \phi)^2 + \frac{1}{a^2 b^4} V_{\text{eff}}(\Phi, \phi) \]

and

\[ R_{\theta\theta} = -\frac{1}{b^2} V_{\text{eff}}(\Phi, \phi), \] (32)

where \( V_{\text{eff}}(\Phi, \phi) \) signifies the interaction between the moduli and gauge fields in the effective string theory. It can be expressed as

\[ V_{\text{eff}}(\Phi, \phi) = -\frac{1}{4} Q_{m(i)}^{n(i)} D_{ij}[\Phi, \phi] Q_{m(j)}^{n(j)}. \] (33)

For simplicity, we consider a constant \( \Phi \), i.e. \( V_{\text{eff}}(\Phi, \phi)|_{\Phi=\text{const.}} \to \tilde{V}_{\text{eff}}(\phi) \). Then, the eqs.(32) are worked out using the arbitrary metric (30). They simplify drastically to yield

\[ R_{rr} + \frac{1}{a^4} R_{tt} = 2(\partial_\phi \phi)^2 \]

and

\[ \partial_\phi^2 (a^2 b^2)^2 = 2. \] (34)

Further simplification gives rise to the following relations:

\[ (\partial_\phi \phi)^2 = -\frac{1}{b} \partial_\phi^2 b \]

and

\[ (a^2 b^3) \partial_\phi^2 b + ab^2 \left[ a + \partial_\phi a \right] \partial_\phi b^2 - b^2 = \tilde{V}_{\text{eff}}(\phi). \] (35)
The arbitrary functions $a(r)$ and $b(r)$ are worked out from the above relations to yield a charged black hole solution in the effective string theory. A priori, it can be written as

$$ds^2 = -\left(1 - \frac{2M_{\text{eff}}}{r}\right)dt^2 + \left(1 - \frac{2M_{\text{eff}}}{r}\right)^{-1}dr^2 + r\left(r - \frac{Q_{\text{eff}}^2 e^{-2\phi_h}}{2M_{\text{eff}}}\right)d\Omega^2,$$  \hspace{1cm} (36)

where $\phi_h$ is a constant value of the scalar field $\phi$ and shall be identified with its value on the event horizon. It satisfies

$$e^{2\phi} = e^{2\phi_h}\left(1 - \frac{Q_{\text{eff}}^2 e^{-2\phi_h}}{2rM_{\text{eff}}}\right).$$  \hspace{1cm} (37)

For instance when $Q = 0$, the charged black hole reduces to the Schwarzschild geometry. However for $Q \neq 0$, the radius of $S^2$ for a constant $r$ and $t$ depends on $Q_{\text{eff}}$. There, the area of $S^2$ is reduced in comparison to that of the Schwarzschild black hole. Importantly, the above black hole solution precisely resembles to that obtained in string theory [46]. At this point, we recall the fact that the charged black hole geometry (36) is obtained in presence of a noncommutative $D_3$-brane. Since the effective string theory is defined in the gravity decoupling limit, the charged black hole describes the near horizon geometry. Thus, the correct black hole geometry in the frame-work is governed by

$$ds^2 = -\left(1 - \frac{2Q_{\text{eff}}^2}{r}\right)dt^2 + \left(1 - \frac{2Q_{\text{eff}}^2}{r}\right)^{-1}dr^2 + r\left(r - \frac{1}{2} e^{-2\phi_h}\right)d\Omega^2.$$  \hspace{1cm} (38)

In the case, the effective potential (33) takes a simplified form

$$V_{\text{eff}} = \left(r - \frac{1}{2} e^{-2\phi_h}\right)^2.$$  \hspace{1cm} (39)

It implies that the area of the horizon is reduced due to the nonzero constant value of the moduli field there. In other words, the area of the event horizon is affected in presence of the $V_{\text{eff}}(\Phi, \phi)$ in the string frame-work.

### 3.2 String frame

In this section, we perform a similar analysis with the string metric, in presence of both non-zero electric and magnetic fields. We re-scale the string metric $G_{\mu\nu} = e^{-2\phi}G^E_{\mu\nu}$. The 4D effective string action in Einstein frame becomes

$$S = \int d^4x \sqrt{-G} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{3}(\partial\Phi)^2 - \frac{1}{4} F^{(i)} \tilde{D}_{ij}[\Phi, \phi] F^{(j)}\right].$$  \hspace{1cm} (40)

The moduli field in the case is

$$\tilde{D}_{ij}[\Phi, \phi] = \begin{pmatrix} e^{-2[\sqrt{3}+1]\phi} & 0 \\ 0 & e^{-2[(\sqrt{3}+3)\phi+2\Phi]/3}A_{mn} \end{pmatrix}.$$  \hspace{1cm} (41)
The equations of motion are

\[ R_{\mu\nu} = -4 \partial_\mu \phi \partial_\nu \phi + \frac{1}{3} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} \tilde{D}_{ij}[\Phi, \phi] \left( F^{(i)}_{\mu\lambda} F^{(j)\lambda} - \frac{1}{4} G_{\mu\nu} F^{(i)} F^{(j)} \right) , \]

\[ \partial_\mu \left( \sqrt{-G} \partial^\mu \phi \right) = \frac{1}{16} \sqrt{G} \left[ F^{(i)} \left( \tilde{D}_{ij}[\Phi, \phi] - \frac{1}{2} \frac{\partial \tilde{D}_{ij}[\Phi, \phi]}{\partial \phi} \right) F^{(j)} \right] , \]

\[ \partial_\mu \left( \sqrt{-G} \partial^\mu \Phi \right) = \frac{3}{8} \sqrt{-G} \frac{\partial \tilde{D}_{ij}[\Phi, \phi]}{\partial \Phi} F^{(i)} F^{(j)} , \]

and

\[ \partial_\mu \left( \sqrt{-G} e^{-2\phi} \tilde{D}_{ij}[\Phi, \phi] F^{(j)\mu\nu} \right) = 0 . \] (42)

Let us consider an arbitrary metric ansatz (30) in presence of both electric and magnetic field in the theory. They are given by

\[ F^{(i)} = \frac{Q^{(i)}_{\text{eff}}}{{b'}^{2}} e^{2\phi} dt \wedge dr + Q^{(i)}_{\text{eff}} \sin \theta \ d\theta \wedge d\phi . \] (43)

The non-zero components of Ricci tensor are worked out using eqs.(30) and (42). They satisfy the following relations:

\[ R_{tt} = -\frac{a^2}{b^4} V_{\text{eff}}(\Phi, \phi) , \]

\[ R_{rr} + 4 \left( \partial_r \phi \right)^2 = \frac{1}{a^2 b^4} V_{\text{eff}}(\Phi, \phi) \]

and

\[ R_{\theta\theta} = -\frac{1}{b^2} V_{\text{eff}}(\Phi, \phi) . \] (44)

where

\[ V_{\text{eff}}(\Phi, \phi) = -\frac{1}{4} \left( Q^{(m)}_{\text{eff}} \tilde{D}_{ij}[\Phi, \phi] Q^{(m)\text{eff}} + Q^{(e)}_{\text{eff}} \tilde{D}_{ij}[\Phi, \phi] e^{4\phi} Q^{(j)}_{\text{eff}} \right) . \] (45)

Further simplification yields

\[ \partial_r^2 (a^2 b^2) = 2 , \quad \frac{1}{b} \partial_r^2 b = 2 (\partial_r \phi)^2 \]

and

\[ 2 a b^2 (\partial_r a)(\partial_r b) + a^2 b^2 (\partial_r b)^2 - b^2 = \tilde{V}_{\text{eff}}(\phi) + 2 b^2 (\partial_r \phi)^2 . \] (46)

Then, the black hole solution in the string-frame can be given by

\[ ds^2 = -\frac{r^2}{\tilde{Q}^2_{\text{eff}}} dt^2 + \frac{\tilde{Q}^2_{\text{eff}}}{r^2} dr^2 + \tilde{Q}^2_{\text{eff}} d\Omega^2 , \] (47)

where \( \tilde{Q}_{\text{eff}} \) is the total effective charge due to all the electric and magnetic fields in the framework. As described in the Einstein-frame, the 4D solution (47) in the string frame describes a near horizon geometry \( AdS_2 \times S^2 \) of the Reissner-Nordstrom solution.

The stability analysis of the extremal black hole geometries (38) and (47) are performed by taking into account the attractor behaviour of the geometries at its event horizon. The total charge \( \tilde{Q}_{\text{eff}} \) is computed using the attractor mechanism, which can be expressed in terms of the charges associated with the gauge fields derived from the two-forms and a three-form in 5D.
3.3 Black hole entropy

Let us consider the 4D extremal black hole solution obtained in the string frame (47). Since the geometry $AdS_2 \times S^2$ is obtained in the gravity decoupling limit, it can be seen to be associated with two different length scales in a noncommutative framework [19, 25, 1]. If $l_\perp$ and $l_L$ are, respectively, the $\perp$- and $L$- length scales, then the extremal black hole geometry can be re-expressed as

$$ds^2 = l_\perp^2 \left( - \frac{r^2}{r^2} dt^2 + \frac{dr^2}{r^2} \right) + l_L^2 \, d\Omega^2 .$$

(48)

The appropriate EM-field components may be obtained from (43). They are

$$F_{rt}^{(i)} = E^{(i)}$$

and

$$F_{\theta\phi}^{(i)} = \frac{1}{2\pi} B^{(i)} \sin \theta ,$$

(49)

where $E^{(i)}$ and $B^{(i)}$ are the electric and magnetic fields respectively. Since the $\perp$- and $L$- spaces in the effective geometry are scaled apart, they give rise to two non-vanishing components of the Riemann tensor. They are

$$R_{\alpha\beta\gamma\delta} = \frac{R_{rtrt}}{\text{det}(G_{rt})} (G_{\alpha\gamma} G_{\beta\delta} - G_{\alpha\delta} G_{\beta\gamma})$$

and

$$R_{mnnpq} = \frac{R_{\theta\phi\theta\phi}}{\text{det}(G_{\theta\phi})} (G_{mp} G_{nq} - G_{mq} G_{np}) ,$$

(50)

where $(\alpha, \beta, \gamma, \delta)$ specify the $\perp$-space and $(m, n, p, q)$ there describe the $L$-space. The effective potential can be checked to yield

$$V_{\text{eff}} = - \frac{\text{det}(G_{rt})}{R_{rtrt}} = \frac{\text{det}(G_{\theta\phi})}{R_{\theta\phi\theta\phi}} .$$

(51)

Now the entropy function $f(l_\perp, l_L, E^{(i)}, B^{(i)}, \phi)$ of the 4D extremal black hole (48) is given by [31]

$$f(l_\perp, l_L, E^{(i)}, B^{(i)}, \phi) = \int d\theta d\phi \, \sqrt{-G} \, L ,$$

(52)

where $L$ denotes the lagrangian density in eq.(40). Using the on-shell condition, the action is simplified and can be expressed in terms of gauge fields only. Then, the attractor mechanism for the extremal black hole (48) at its event horizon radius $r_h$ is worked out to obtain the entropy function. Explicitly, it takes a form

$$f(l_\perp, l_L, E^{(i)}, B^{(i)}, \phi) = \frac{4\pi l_L}{l_1} \left[ E^{(i)} \tilde{D}_{ij}(\phi) E^{(j)} \right]_{\phi \rightarrow \phi_h} .$$

(53)

The Legendre transform of the entropy function becomes

$$S_{BH} = 2\pi \left( E^{(i)} \frac{\partial f}{\partial E^{(i)}} - f \right) .$$

(54)

Then the black hole entropy can be computed to yield

$$S_{BH} = \frac{1}{4} \left[ 4\pi \tilde{V}_{\text{eff}}(\phi) \right]_{\phi \rightarrow \phi_h} .$$

(55)

where $\tilde{V}_{\text{eff}}(\phi)$ on the event horizon of the extremal black hole defines $r_h^2$. It is in agreement with the Bekenstein-Hawking area law for a black hole.
4 Concluding remarks

To conclude, we have revisited the generalized RN- and Schwarzschild black hole geometries in 4D, recently obtained by us [1], following a noncommutative $D_3$-brane framework. Apart from the fact that the generalized RN-solution in string theory is new, it has provided a forum to investigate some aspects of quantum gravity. In the gravity decoupling limit, these black holes coincide to yield a Schwarzschild geometry and its mass term is shown to be associated with the $1/r^4$. This in turn prompts one to believe for the existence of an 7D effective theory in the decoupling regime. The noncommutative scaling in the framework is exploited to conclude that the effective space-time turns out to be 5D instead of 7D.

In the context, a relevant effective theory in 5D was obtained in type IIB string theory. The coupling of moduli fields to the gauge field strengths has been incorporated in terms of an effective potential in the theory. The black hole solutions in 5D were worked out with a static, spherically symmetric metric ansatz in presence of an arbitrary electric field $\tilde{E}$. These black holes were argued to describe the extremal geometries $AdS_2 \times S_3$ in the gravity decoupling limit. It was shown that a black hole solution is in precise agreement with the gravity dual of a $D_3$-brane in presence of a constant magnetic field [47]. As a result, our result provides evidence for a holographic correspondence between the boundary noncommutative gauge theory and the bulk of the string. The extremal limit was further analyzed to conclude an $AdS_5$ geometry in the bulk. Interestingly, the near horizon geometry of a generalized RN-black hole in 5D was shown to be a higher dimensional generalization of the charged black hole obtained in effective string theory [46]. In fact, our analysis provides evidences to the strong-week coupling duality between the noncommutative gauge and the ordinary gravity sector in the theory.

In order to compare the black hole geometries in a noncommutative $D_3$-brane to that in effective string theory, Kaluza-Klein compactification was performed in 5D. The relevant black hole geometries were obtained in Einstein and string frames in presence of an effective potential $V_{eff}$. The potential was shown to be characterized by the EM-charges and is independent of the value of moduli field there. Attractor mechanism was adopted to compute the entropy function of the black hole in a noncommutative frame-work. Then, the entropy of a macroscopic black hole was expressed in terms of its effective potential at the event horizon.

It is important to keep a note that, in the gravity decoupling limit the generalized black hole reduces to an appropriate $AdS$ geometry in the noncommutative frame-work. As the limit is intrinsic to the frame-work, it plays a vital role. For instance, the limit may be interpreted as the one leading to $AdS$ boundary in the theory. Now, let us recall the behaviour of Hawking temperature [1], i.e. it decreases to zero with an increase in Hawking radiation for a GRN-black hole and finally increases to attain the Hagedorn temperature for a Schwarzschild geometry. Taking into account the variation of Hawking temperature, the event horizon can be seen...
to be stretched between the GRN- and Schwarzschild geometries in the frame-work. Since a noncommutative $D_3$-brane is known to govern the event horizon of a black hole, the stretch at the eveny horizon can be interpreted as due to the noncommutative or new geometry there. Since the $D_3$-brane can be described by an appropriate $AdS$ geometry, it would be interesting to check the noncommutative formalism in presence of a cosmological constant.

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