A MULTIVARIATE CENTRAL LIMIT THEOREM FOR LIPSCHITZ AND SMOOTH TEST FUNCTIONS

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Abstract

We provide an abstract multivariate central limit theorem with the Lindeberg type error bounded in terms of Lipschitz functions (Wasserstein 1-distance) or functions with bounded second or third derivatives. The result is proved by means of Stein’s method. For sums of i.i.d. random vectors with finite third absolute moment, the optimal rate of convergence is established (that is, we eliminate the logarithmic factor in the case of Lipschitz test functions). We indicate how the result could be applied to certain other dependence structures, but do not derive bounds explicitly.

Keywords: multivariate central limit theorem; Wasserstein distance; Stein’s method.
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1 Introduction

It is well-known that, roughly speaking, a sum of many random variables with sufficiently nice distributions, which do not differ too much in scale and are not too dependent, approximately follows a normal distribution. This fact is referred to as the central limit theorem and has been formulated in numerous variants. It can be readily extended to \( \mathbb{R}^d \)-valued random vectors.

One of the ways to make the sloppy statement above more precise is to provide a bound on the error in the normal approximation. One of the ways to measure the error is to consider expectations of test functions from a given class \( \mathcal{F} \): for a given \( \mathbb{R}^d \)-valued random vector \( W \) and a \( d \)-variate normal vector \( Z \), consider the supremum

\[
\sup_{f \in \mathcal{F}} \left| \mathbb{E}[f(W)] - \mathbb{E}[f(Z)] \right|.
\]

(1.1)

Several classes of test functions have been taken into consideration. In many cases, the error has been estimated optimally up to a constant. In particular, for properly scaled partial sums of a sequence of independent and identically distributed random vectors with finite third absolute moment, the optimal rate of convergence to the standard normal distribution is typically \( n^{-1/2} \), where \( n \) is the number of the summands. This rate has been established for many classes of test functions. For indicators of convex sets, see, e. g., Bentkus [6], Götze [17] or the author’s previous work [19].

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Another important example is the class of sufficiently smooth functions with properly bounded partial derivatives of a given order. Most of the results in the multivariate case are derived for classes based on the second or higher derivatives: see, for example, Goldstein and Rinott [16], Rinott and Reinert [15], Chatterjee and Meckes [8], and Reinert and Röllin [20]. For i. i. d. random vectors with finite third absolute moments, the optimal rate of convergence of \( n^{-1/2} \) has been established, too.

Surprisingly, classes based on the first-order derivatives seem to be more difficult. Typically, one simply considers the class of functions with the Lipschitz constant bounded from above by 1; for this class, the underlying supremum (1.1) is referred to as the 1-Wasserstein distance (or also Kantorovich–Rubinstein distance). In the context of the central limit theorem, this distance has been well-established only in the univariate case, where again the optimal rate of convergence of \( n^{-1/2} \) has been derived for i. i. d. random vectors with finite third absolute moments: it can be, for example, deduced from Theorem 1 of Barbour, Karoński and Ruciński [5]. To the best of the author’s knowledge, this is not the case in higher dimensions. In this case, a suboptimal rate of \( n^{-1/2} \log n \) has been derived by Galouet, Mijoule and Swan [14] as well as by Fang, Shao and Xu [13] (see also the references therein).

In the present paper, we succeed to remove the logarithmic factor – the latter only remains in the dependence on the dimension: see the bound (3.5). In addition, (3.5) does not require finiteness of the third absolute moments – it is a Lindeberg type bound.

The result is derived by Stein’s method, which has been introduced in [25]. The main idea of the method is to reduce the estimation of the error to the estimation of expectations related to a solution of a differential equation, which is now called Stein equation. For Lipschitz test functions, the third derivatives of the solution apparently play the key role. Unfortunately, they cannot be properly bounded – see a counterexample in the author’s previous paper [18], Remark 2 ibidem. However, we show that the third derivatives can be circumvented by the second and fourth ones, which behave properly. The key step is carried out in the estimate (4.27).

One of its major advantages of Stein’s method is that it is by no means limited to sums of independent random vectors. It works well under various dependence structures: for an overview, the reader is referred to Barbour and Chen [3, 4]. Moreover, the random variable to be approximated need not be a sum. We point out two approaches called size and zero biasing: see Baldi, Rinott and Stein [1], respectively Goldstein and Reinert [15].

In the present paper, we introduce a structure which generalizes the size and zero biasing, and state two abstract results, Theorems 2.9 and 2.15. In Section 3, the latter is applied to sums of independent random vectors, but we indicate how the abstract results can be used beyond independence. In particular, our approach may be applied in future for sums of random vectors where we can efficiently compare conditional distributions given particular summands with their unconditional counterparts – see Example 2.5. In addition, we indicate how it could be used for Palm processes – see Example 2.6. However, we do not derive explicit bounds for either of these two cases.

2 Notation, assumptions and general results

First, we introduce some basic notation:

- \( \mathbf{I}_d \) denotes the \( d \times d \) identity matrix.
- \( |\cdot| \) denotes the Euclidean norm.
• For $F: \mathbb{R}^d \rightarrow V$, where $V$ is a finite-dimensional vector space, define $\mathcal{N}F := \mathbb{E}[F(Z)]$, where $Z$ is a standard $d$-variate normal vector.

• For $f: \mathbb{R}^d \rightarrow \mathbb{R}$ and $r \in \mathbb{N}$, denote by $\nabla^r f(w)$ the $r$-th derivative of $f$ at $w$. This is a $r$-fold tensor; see Subsection 5.3 for more detail.

• Denote by $M_0(f)$ the supremum norm of a function $f$. Furthermore, for an $(r - 1)$-times differentiable function $f: \mathbb{R}^d \rightarrow \mathbb{R}$, define

$$M_r(f) := \sup_{x,y \in \mathbb{R}^d, x \neq y} \left| \frac{|\nabla^{r-1}f(x) - \nabla^{r-1}f(y)|}{|x - y|} \right|,$$

where $|\cdot|_\vee$ denotes the injective norm: see Subsections 5.2 and 5.3. If $f$ is not everywhere $(r - 1)$-times differentiable, we put $M_r(f) := \infty$.

For more details on notation and definitions, see Section 5.

**Remark 2.1.** This way, if $M_r(f) < \infty$, then $\nabla^{r-1}f$ exists everywhere and is Lipschitz. In this case, $|\nabla^r f(x)|_\vee \leq M_r(f)$ for all $x$ where $\nabla^{r-1} f$ is differentiable.

**Remark 2.2.** If $M_r(f) < \infty$, there exist constants $C$ and $D$, such that $|f(x)| \leq C + D |x|^r$ for all $x \in \mathbb{R}^d$.

Our main main results, Theorem 2.9 and 2.15, will be based on various assumptions which refer to various components. They are listed below and labelled. Components are labelled by the letter C and a number, other labels stand for properties.

(C1) Consider a $\mathbb{R}^d$-valued random vector $W$.

(St) Refers to (C1). Suppose that $\mathbb{E}|W|^2 < \infty$, $\mathbb{E}W = 0$ and $\text{Var}(W) = \mathcal{I}_d$.

(C2) Consider a measurable space $(\Xi, \mathcal{X})$.

(C3) For each $\xi \in \Xi$, consider a $\mathbb{R}^d$-valued random vector $V_\xi$. The random vectors $V_\xi$ may be defined on different probability spaces. Denote the underlying probability measures and expectations by $\mathbb{P}_\xi$ and $\mathbb{E}_\xi$. Assume that the maps $\xi \mapsto \mathbb{P}_\xi(V_\xi \in A)$ are measurable for all Borel sets $A \subseteq \mathbb{R}^d$.

(C4) Consider an $\mathbb{R}^d$-valued measure $\mu$ on $(\Xi, \mathcal{X})$.

(S) Refers to (C1)–(C4). Suppose that $\mathbb{E}|W| < \infty$ and that

$$\mathbb{E}[f(W)W] = \int_{\Xi} \mathbb{E}_\xi[f(V_\xi)] \, \mu(d\xi)$$

for all bounded measurable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$.

**Remark 2.3.** Assumption (S) can be regarded as a generalization of the size-biassed transformation (see Baldi, Rinott and Stein [1]): for $d = 1$ and $W \geq 0$, (S) is fulfilled with $\Xi = \{0\}$ and $\mu(\{0\}) = \mathbb{E}W$ if and only if the distribution of $V_0$ is the size-biassed distribution of $W$.

**Remark 2.4.** Under (S), we have $\mu(\Xi) = \mathbb{E}W$ (put $f(w) = 1$).

We list two more important examples where Assumption (S) is satisfied.

**Example 2.5.** Let $\mathcal{I}$ be a countable set and let $W = \sum_{i \in \mathcal{I}} X_i$—suppose that the latter sum exists almost surely. Define $\Xi := \mathcal{I} \times \mathbb{R}^d$ and let $\mathcal{X}$ be the product (in terms of $\sigma$-algebras) of the power set of $\mathcal{I}$ and the Borel $\sigma$-algebra on $\mathbb{R}$. Choose probability measures $\mathbb{P}_{i,x}$, $i \in \mathcal{I}$, $x \in \mathbb{R}^d$, so that for each $i \in \mathcal{I}$, they determine the conditional distribution of $W$ given $X_i$. 


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i. e., $P_{i,x}(A) = P(W_i \in A \mid X_i = x)$. For all $i$ and $x$, let $V_{i,x}$ be the identity on $\mathbb{R}^d$. Letting $\mu(\{i\} \times A) = \mathbb{E}[X_i \mathbb{1}(X_i \in A)]$, we find that

$$
\mathbb{E}[f(W)W] = \sum_{i \in J} \mathbb{E}[f(W)X_i] \\
= \sum_{i \in J} \int_{\mathbb{R}^d} f(w) P_{i,x}(dw) x \mathcal{L}(X_i)(dx) \\
= \int_{J \times \mathbb{R}^d} (\mathcal{E}_{i,x} f) \mu(di \otimes dx),
$$

implying $(S)$.

**Example 2.6.** Let $\mathcal{P}$ be a random point process on a space $\Xi$ admitting Palm processes $\mathcal{P}_\xi$, $\xi \in \Xi$. Intuitively, $\mathcal{P}_\xi$ is the conditional distribution of $\mathcal{P}$ given that there is a point at $\xi$. Strictly speaking, the Palm processes are characterized by the formula

$$
\mathbb{E} \left[ \int_{\Xi} \Phi(\xi, \mathcal{P})\mathcal{P}(d\xi) \right] = \int_{\Xi} \mathbb{E}_\xi \Phi(\xi, \mathcal{P}_\xi) m(d\xi),
$$

where $m$ is the mean measure of $\mathcal{P}$ (for details, see Proposition 13.1.IV of Daley and Vere-Jones [10]). Now take a function $F: \Xi \rightarrow \mathbb{R}^d$ and define $W := \int_{\Xi} F(\xi) \mathcal{P}(d\xi)$. Observe that

$$
\mathbb{E}[f(W)W] = \mathbb{E} \left[ \int_{\Xi} f \left( \int_{\Xi} F(\eta) \mathcal{P}(d\eta) \right) F(\xi) \mathcal{P}(d\xi) \right] \\
= \int_{\Xi} \mathbb{E}_\xi \left[ f \left( \int_{\Xi} F(\eta) \mathcal{P}(d\eta) \right) \right] F(\xi) m(d\xi).
$$

Thus, we can set $V_\xi = \int_{\Xi} F(\eta) \mathcal{P}_\xi(d\eta)$ and $\mu = F \cdot m$.

We continue listing components and properties required for Theorems 2.9 and 2.15.

(C5) For each $\xi \in \Xi$, consider a $\mathbb{R}^d$-valued measure $\nu_\xi$ and assume that the maps $\xi \mapsto \nu_\xi(A)$ are measurable for all $A \in \mathcal{X}$.

(Px) Refers to (C1)–(C3) and (C5). Suppose that $\mathbb{E}[W] < \infty$. For each $\xi \in \Xi$, suppose that $\mathbb{E}_\xi[V_\xi] < \infty$ and

$$
\mathbb{E}[f(W)] - \mathbb{E}_\xi[f(V_\xi)] = \int_{\Xi} \langle \mathbb{E}_\eta[\nabla f(V_\eta)] , \nu_\xi(d\eta) \rangle
$$

(2.3)

for all continuously differentiable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ with bounded derivative (observe that $\mathbb{E}[f(W)]$ and $\mathbb{E}_\xi[f(V_\xi)]$ are finite by Remark 2.2). The integral in (2.3) is defined according to Definition 5.14.

**Remark 2.7.** Under Assumption (Px), we can measure the proximity of the distribution of $V_\xi$ to the distribution of $W$ in terms of $\nu_\xi$. In particular, the 1-Wasserstein distance is straightforward to estimate, as $|\mathbb{E}[f(W)] - \mathbb{E}_\xi[f(V_\xi)]| \leq M_1(f) ||\nu_\xi||(\Xi)$. However, the latter distance will not be the only measure of proximity we shall need.

**Example 2.8.** If $W$ and $V_\xi$ are defined on the same probability space, observe that

$$
\mathbb{E}[f(W)] - \mathbb{E}[f(V_\xi)] = \int_0^1 \mathbb{E}[\langle \nabla f((1 - t)V_\xi + tW) , W - V_\xi \rangle] dt.
$$

(2.4)
Now suppose that there is a measurable map $\psi : \Xi \times [0, 1] \times \mathbb{R}^d \to \Xi$, such that the (unconditional) distribution of $V_{\psi(\xi, t, y)}$ agrees with the conditional distribution of $(1-t)V_\xi + tW$ given $W-V_\xi = y$. Then (2.4) can be rewritten as

$$
E[f(W)] - E[f(V_\xi)] = \int_{0}^{1} \int_{\mathbb{R}^d} \left\langle E_{\psi(\xi, t, y)}[\nabla f(V_{\psi(\xi, t, y)})], y \right\rangle \mathcal{L}(W-V_\xi)(dy) \, dt.
$$

(2.5)

Now put

$$
\nu_\xi(B) := \int_{0}^{1} \int_{\mathbb{R}^d} \mathbb{1}(\psi(\xi, t, y) \in B) \, y \, \mathcal{L}(W-V_\xi)(dy) \, dt \\
= \int_{0}^{1} E\left[(W-V_\xi) \mathbb{1}(\psi(\xi, t, W-V_\xi) \in B)\right] \, dt.
$$

A standard argument shows that

$$
\int h \, d\nu_\xi = \int_{0}^{1} \int_{\mathbb{R}^d} h(\psi(\xi, t, y)) \, y \, \mathcal{L}(W-V_\xi)(dy) \, dt = \int_{0}^{1} E\left[h(\psi(\xi, t, W-V_\xi))(W-V_\xi)\right] \, dt
$$

for all bounded measurable functions $h$. Combining with (2.5), (2.3) follows.

Before formulating the first main result, Theorem 2.9, we introduce some more quantities:

$$
\beta_1^{(\xi)} := |\nu_\xi|(\Xi), \quad \beta_2 := \int_{\Xi} \beta_1^{(\xi)} |\mu|(d\xi), \quad (2.6)
$$

$$
\beta_2^{(\xi)} := \int_{\Xi} \beta_1^{(n)} |\nu_\xi|(d\eta), \quad \beta_3 := \int_{\Xi} \beta_2^{(\xi)} |\mu|(d\xi), \quad (2.7)
$$

$$
\beta_{123}^{(\xi)}(a, b) := \int_{\Xi} \min\{a, b \beta_{1}^{(n)}\} |\nu_\xi|(d\eta), \quad \beta_{23}(a, b) := \int_{\Xi} \beta_{123}^{(\xi)}(a, b, c) |\mu|(d\xi), \quad (2.8)
$$

$$
\beta_{123}^{(\xi)}(a, b, c) := \int_{\Xi} \min\{a, b \beta_{1}^{(n)} + c\sqrt{\beta_{2}^{(n)}}\} |\nu_\xi|(d\eta), \quad \beta_{234}(a, b, c) := \int_{\Xi} \beta_{123}^{(\xi)}(a, b, c) |\mu|(d\xi). \quad (2.9)
$$

**Theorem 2.9.** Under Assumptions (St), (S) and (Px), and $\beta_2 < \infty$, we have

$$
\left| E[f(W)] - \mathcal{N}f \right| \leq \frac{\beta_2}{3} M_3(f),
$$

(2.10)

$$
\left| E[f(W)] - \mathcal{N}f \right| \leq \beta_{23} \left(1, \frac{\sqrt{2\pi}}{4}\right) M_2(f),
$$

(2.11)

$$
\left| E[f(W)] - \mathcal{N}f \right| \leq \beta_{234}(1.8, 3.58 + 0.55 \log d, 3.5) M_1(f). \quad (2.12)
$$

More precisely, for each of the inequalities, if the underlying $M_r(f)$ in the right hand side is finite, then $E[f(W)]$ and $\mathcal{N}f$ are also finite and the inequality holds true (under the convention $\infty \cdot 0 = 0$).

We defer the proof to Subsection 4.4.

Now we turn to a more direct construction with the underlying counterpart of the preceding result. Although the additional Component (C4) satisfying Assumption (Z) can be constructed from Component (C4) satisfying Assumption (S) (see Proposition 2.13), Theorem 2.15 might provide better bounds than Theorem 2.9. Moreover, as we shall see in Subsection 4.4, Theorem 2.9 is actually a direct consequence of Theorem 2.15 and Proposition 2.13.
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(C4) Consider an $\mathbb{R}^d \otimes \mathbb{R}^d$-valued measure $\tilde{\mu}$ on $(\Xi, \mathcal{H})$.

(Z) Refers to (C1)–(C3) and (C4). Suppose that $\mathbb{E}|W|^2 < \infty$ and that

$$\mathbb{E}[f(W)W] = \int_{\Xi} \tilde{\mu}(d\xi) \mathbb{E}_\xi[\nabla f(V_\xi)],$$

(2.13)

for all continuously differentiable functions $f: \mathbb{R}^d \to \mathbb{R}$ with bounded derivative, recalling the identification of a 2-tensor $\phi$ with a linear map $\tilde{L}_\phi$ in Subsection 5.2, as well as Definition 5.14 (observe that $\mathbb{E}|f(W)W|$ is finite by Remark 2.2).

Remark 2.10. Assumption (Z) can be regarded as a more flexible variant of the zero bias transformation introduced by Goldstein and Reinert [15]. Indeed, for $d = 1$, Assumption (Z) is fulfilled with $\Xi = \{0\}$, $\tilde{\mu}(\{0\}) = 1$, provided that the distribution of $V_0$ is the zero bias transform of the distribution of $W$.

Assumption (Z) can be formulated alternatively in the following way:

(Z') Refers to (C1)–(C3) and (C4). Suppose that $\mathbb{E}|W|^2 < \infty$ and that

$$\mathbb{E}[\langle F(W), W \rangle] = \int_{\Xi} \langle \mathbb{E}_\xi[\nabla F(V_\xi)], \tilde{\mu}(d\xi) \rangle$$

(14)

for all continuously differentiable maps $F: \mathbb{R}^d \to \mathbb{R}$ with bounded derivative (observe that $\mathbb{E}|F(W)||W|$ is finite by Remark 2.2).

Proposition 2.11. Assumptions (Z) and (Z') are equivalent.

Proof. Assumption (Z') remains the same if we require (2.14) only to hold for the vector functions of form $F(w) = f(w)u$, where $f$ is continuously differentiable with bounded derivative. Recalling (5.1), we find that in this case, (2.14) reduces to

$$\mathbb{E}[\langle f(W), W \rangle, u] = \int_{\Xi} \langle \tilde{\mu}(d\xi), u \otimes \mathbb{E}_\xi[\nabla f(V_\xi)] \rangle = \int_{\Xi} \langle \tilde{\mu}(d\xi), \mathbb{E}_\xi[\nabla f(V_\xi)], u \rangle.$$

However, this is equivalent to (2.13). \qed

Remark 2.12. Under (Z), we have $\mathbb{E}W = 0$ and $\text{Var}(W) = \tilde{\mu}(\Xi)$. The first equality follows by substituting $f \equiv 1$ into (2.13). Substituting $f(w) = \langle w, u \rangle$ and recalling (5.2), we find that $\tilde{\mu}(\Xi)u = \mathbb{E}[\langle W, u \rangle W] = \mathbb{E}(W \otimes W)u$ for all $u \in \mathbb{R}^d$, so that $\tilde{\mu}(\Xi) = \mathbb{E}(W \otimes W)$. Identifying 2-tensors with matrices, we can rewrite this as $\tilde{\mu}(\Xi) = \mathbb{E}(WW^T) = \text{Var}(W)$.

Proposition 2.13. Assume (S) and (P), recall (2.6) and suppose that if $\beta_2 < \infty$. Then Assumption (Z) is satisfied for Component $(\tilde{\beta}_i)$ defined by $\tilde{\beta}(B) := -\int_{\Xi} \int_{\Xi} 1_{(\eta \in B)} \tilde{\mu}(d\xi) \otimes \nu_\xi(d\eta)$ in view of Definition 5.17.

Remark 2.14. The finiteness of $\beta_2$ guarantees that the tensor-valued measure $\tilde{\mu}$ is well-defined.

The proof of Proposition 2.13 is deferred to Subsection 4.1.

Now we are about to formulate our second main result, Theorem 2.15. Before the statement, we need some more quantities, recalling that $|\cdot|_\Lambda$ denotes the projective norm – see Subsection 5.2:

$$\tilde{\beta}_3 := \int_{\Xi} \beta_1^{(\xi)} |\tilde{\mu}|_\Lambda(d\xi),$$

(2.15)

$$\tilde{\beta}_{23}(a, b) := \int_{\Xi} \min\{a, b \beta_1^{(\xi)}\} |\tilde{\mu}|_\Lambda(d\xi),$$

(2.16)

$$\tilde{\beta}_{234}(a, b, c) := \int_{\Xi} \min\{a, b \beta_1^{(\xi)} + c \beta_2^{(\xi)}\} |\tilde{\mu}|_\Lambda(d\xi).$$

(2.17)
Theorem 2.15. Under Assumptions (St), (Z) and (Px), the following inequalities hold true:

\[ |\mathbb{E}[f(W)] - Nf| \leq \tilde{\beta}_3 M_3(f), \quad (2.18) \]
\[ |\mathbb{E}[f(W)] - Nf| \leq \tilde{\beta}_{23} \left(1, \frac{\sqrt{2\pi}}{4}\right) M_2(f), \quad (2.19) \]
\[ |\mathbb{E}[f(W)] - Nf| \leq \tilde{\beta}_{234} \left(1.8, 3.58 + 0.55 \log d, 3.5\right) M_1(f). \quad (2.20) \]

More precisely, for each of the inequalities, if the underlying \( M_r(f) \) in the right hand side is finite, then \( \mathbb{E}|f(W)| \) and \( N|f| \) are also finite and the inequality holds true (under the convention \( \infty \cdot 0 = 0 \)).

We defer the proof to Subsection 4.4.

3 Application to sums of independent random vectors

Let \( \mathcal{I} \) be a countable set and let \( X_i, i \in \mathcal{I} \), be independent \( \mathbb{R}^d \)-valued random vectors with \( \mathbb{E} X_i = 0 \) for all \( i \in \mathcal{I} \). Suppose that \( \sum_{i \in \mathcal{I}} \mathbb{E} |X_i|^2 < \infty \). Then the sum \( W := \sum_{i \in \mathcal{I}} X_i \) exists almost surely. Suppose that \( \text{Var}(W) = \mathbf{1}_d \).

3.1 Construction of measure \( \tilde{\mu} \) satisfying (Z)

Let \( W_i := W - X_i \) and take a continuously differentiable map \( F: \mathbb{R}^d \to \mathbb{R}^d \) with bounded derivative. Using independence and applying Taylor’s expansion, write

\[ \mathbb{E}[\langle F(W), W \rangle] = \sum_{i \in \mathcal{I}} \mathbb{E}[\langle F(W_i + X_i) - F(X_i), X_i \rangle] = \sum_{i \in \mathcal{I}} \int_0^1 \mathbb{E}[\langle \nabla F(W_i + tX_i), X_i \otimes X_i \rangle] \, dt. \]

Now let \( \Xi := \mathcal{I} \times \mathbb{R}^d \) and let \( \mathcal{D} \) be the product (in terms of \( \sigma \)-algebras) of the power set of \( \mathcal{I} \) and the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \). Put \( V_{i,x} := W_i + x \). Then we may write

\[ \mathbb{E}[\langle F(W), W \rangle] = \sum_{i \in \mathcal{I}} \int_0^1 \int_{\mathbb{R}^d} \mathbb{E}[\langle \nabla F(V_{i,tx}), x \otimes x \rangle] \, \mathcal{L}(X_i)(dx) \, dt. \]

Letting

\[ \tilde{\mu}(B) := \sum_{i \in \mathcal{I}} \int_0^1 \int_{\mathbb{R}^d} 1\{(i, tx) \in B\} (x \otimes x) \, \mathcal{L}(X_i)(dx) \, dt, \]

a standard argument shows that

\[ \int_{\Xi} h \, d\tilde{\mu} = \sum_{i \in \mathcal{I}} \int_0^1 \int_{\mathbb{R}^d} h(i, tx) (x \otimes x) \, \mathcal{L}(X_i)(dx) \, dt = \sum_{i \in \mathcal{I}} \int_0^1 \mathbb{E}[h(i, tX_i) (X_i \otimes X_i)] \, dt \]

for all bounded measurable functions \( h: \Xi \to \mathbb{R} \). This proves (Z').
3.2 Construction of measures $\nu_x$ satisfying ($\text{Px}$)

We use the construction from Example 2.8. Observing that the conditional distribution of $(1-t)V_i + ty = W_i$ given $W - V_i = X_i - x = y$ agrees with the (unconditional) distribution of $W_i = x + ty$, we may set $\psi(i, x, t, y) := (i, x + ty)$. Therefore, there exist $\mathbb{R}^d$-valued vector measures $\nu_{i,x}$, such that

$$\int_{\mathbb{E}} h \, d\nu_{i,x} = \int_0^1 \mathbb{E} \left[ h(i, x + t(W - V_i, x)) (W - V_i, x) \right] dt = \int_0^1 \mathbb{E} \left[ h(i, (1-t)x + tX_i) (X_i - x) \right] dt$$

for all bounded measurable functions $h : \mathbb{E} \to \mathbb{R}$, and these measures satisfy ($\text{Px}$).

3.3 Estimation of $\beta_3$, $\beta_{23}$ and $\beta_{234}$

First, observe that

$$\int h \, d|\hat{\mu}|_\Lambda \leq \sum_{i \in \mathcal{I}} \int_0^1 \mathbb{E} \left[ |X_i|^2 h(i, tX_i) \right] dt,$$

$$\int h \, d|\nu_{i,x}| \leq \int_0^1 \mathbb{E} \left[ (|X_i| + |x|) h(i, (1-t)x + tX_i) \right] dt$$

for all measurable functions $h : \mathbb{E} \to [0, \infty]$. Recalling (2.6), (2.7) and (2.15)–(2.17), we estimate

$$\beta_1^{(i)} \leq \mathbb{E} |X_i| + |x|,$$

$$\tilde{\beta}_3 \leq \sum_{i \in \mathcal{I}} \int_0^1 \mathbb{E} \left[ |X_i|^2 \beta_1^{(i,tX_i)} \right] dt \leq \sum_{i \in \mathcal{I}} \int_0^1 \mathbb{E} \left[ |X_i|^2 (\mathbb{E} |X_i| + tX_i) \right] dt \leq \frac{3}{2} \sum_{i \in \mathcal{I}} \mathbb{E} |X_i|^3,$$

with the last inequality being due to Jensen’s inequality. Similarly,

$$\tilde{\beta}_{23}(a, b) \leq \sum_{i \in \mathcal{I}} \int_0^1 \mathbb{E} \left[ |X_i|^2 \min\left\{ a, b \beta_1^{(i,tX_i)} \right\} \right] dt \leq \sum_{i \in \mathcal{I}} \mathbb{E} \left[ |X_i|^2 \int_0^1 \min\left\{ a, b (\mathbb{E} |X_i| + tX_i) \right\} dt \right].$$

Applying the inequality $\int_0^1 \min\{ f(t), g(t) \} dt \leq \min\{ \int_0^1 f(t) dt, \int_0^1 g(t) dt \}$ and integrating, we find that

$$\tilde{\beta}_{23}(a, b) \leq \sum_{i \in \mathcal{I}} \mathbb{E} \left[ |X_i|^2 \min\left\{ a, b \mathbb{E} |X_i| + \frac{1}{2} b |X_i| \right\} \right]. \quad (3.1)$$

Finally, to bound $\tilde{\beta}_{234}(a, b, c)$, we first estimate

$$\beta_2^{(i,x)} \leq \int_0^1 \mathbb{E} \left[ (|X_i| + |x|) \beta_1^{(i,(1-t)x + tX_i)} \right] dt$$

$$\leq \int_0^1 \mathbb{E} \left[ (|X_i| + |x|) (\mathbb{E} |X_i| + (1-t)|x| + t|X_i|) \right] dt$$

$$= \frac{3}{2} \mathbb{E} |X_i|^2 + 2|x| \mathbb{E} |X_i| + \frac{1}{2} |x|^2$$

$$\leq \frac{3}{2} \mathbb{E} |X_i|^2 + 2|x| \sqrt{\mathbb{E} |X_i|^2} + \frac{1}{2} |x|^2$$

$$= \frac{1}{2} \left( 3 \sqrt{\mathbb{E} |X_i|^2} + |x| \right) \left( \sqrt{\mathbb{E} |X_i|^2} + |x| \right).$$
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An application of the inequality between the arithmetic and the geometric mean yields

\[ \sqrt{\beta_2^{(t,x)}} \leq \frac{5}{4} \sqrt{\mathbb{E}|X_i|^2 + \frac{3}{4}|x|}, \]

leading to the bound

\[ \beta_{234}(a, b, c) \leq \sum_{i \in \mathcal{I}} \int_0^1 \mathbb{E}\left[ |X_i|^2 \min\left\{ a, (b + \frac{5}{4} c) \sqrt{\mathbb{E}|X_i|^2 + (b + \frac{3}{4} c) t |X_i|} \right\} \right] \, dt \]
\[ \leq \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\left\{ a, (b + \frac{5}{4} c) \sqrt{\mathbb{E}|X_i|^2 + (\frac{1}{2} b + \frac{3}{8} c) |X_i|} \right\} \right], \tag{3.2} \]

For \(a, b \geq 0\), consider functions \(h_{a,b} : [0, \infty) \to [0, \infty)\) defined by \(h_{a,b}(u) := bu^{3/2}\) for \(u \leq \frac{a^2}{b^2}\) and \(h_{a,b}(u) := \frac{3}{2}au - \frac{a^3}{2b^2}\) for \(u \geq \frac{a^2}{b^2}\). Observe that \(h_{a,b}\) are convex and \(\min\{au, bu^{3/2}\} \leq h_{a,b}(u) \leq \min\{\frac{3}{2}au, bu^{3/2}\}\) for all \(u \geq 0\). Therefore, for any non-negative random variable \(X\), we have

\[ \min\{a \mathbb{E}X^2, b(\mathbb{E}X^2)^{3/2}\} \leq h_{a,b}(\mathbb{E}X^2) \leq \mathbb{E}[h_{a,b}(X^2)] \leq \mathbb{E}[\min\{\frac{3}{2}aX^2, bX^3\}]. \]

Further estimation of the right hand sides of (3.1) and (3.2) combined with the preceding observation leads to the following Lindeberg type bounds

\[ \beta_{233}(a, b) \leq \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\left\{ a, \frac{3}{2} a |X_i| \right\} \right] + \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\left\{ b, \frac{3}{2} b |X_i| \right\} \right] \]
\[ \leq \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\left\{ a, \frac{3}{2} b |X_i| \right\} \right], \]
\[ \beta_{234}(a, b, c) \leq \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\left\{ \frac{3}{2} a, (b + \frac{5}{4} c) |X_i| \right\} \right] + \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\left\{ a, (\frac{1}{2} b + \frac{3}{8} c) |X_i| \right\} \right] \]
\[ \leq \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\left\{ \frac{3}{2} a, (\frac{3}{2} b + \frac{13}{8} c) |X_i| \right\} \right] \]

and Theorem 2.15 yields

\[ |\mathbb{E}[f(W)] - \mathbb{N}f| \leq \frac{M_3(f)}{2} \sum_{i \in \mathcal{I}} \mathbb{E}|X_i|^3, \tag{3.3} \]
\[ |\mathbb{E}[f(W)] - \mathbb{N}f| \leq M_2(f) \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\{2.5, 0.94 |X_i|\} \right], \tag{3.4} \]
\[ |\mathbb{E}[f(W)] - \mathbb{N}f| \leq M_1(f) \sum_{i \in \mathcal{I}} \mathbb{E}\left[ |X_i|^2 \min\{4.5, (11.1 + 0.83 \log d) |X_i|\} \right]. \tag{3.5} \]

4 Proofs

4.1 Assumptions (S), (Px) and (Z)

Here, we prove Proposition 2.13 and derive stronger formulations of Assumptions (S) and (Z'), which will be necessary in the proofs of Theorems 2.9 and 2.15.
Proposition 4.1. Assume (S) and take a measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) with 
\[
\mathbb{E}_\xi |f(V_\xi)| \|\mu\|_\xi < \infty.
\] Then we have \( \mathbb{E}|f(W)W| < \infty \) and (2.2) remains true.

Proof. For each \( n \in \mathbb{N} \), define \( f_n(w) := f(w) \mathbb{I}(\{ |f(w)| \leq n \}) \). Next, for each \( n \in \mathbb{N} \) and each \( u \in \mathbb{R}^d \), define \( f_{n,u}(w) := f_n(w) \) if \( \langle w, u \rangle \geq 0 \) and \( f_{n,u}(w) := -f_n(w) \) if \( \langle w, u \rangle < 0 \). Observe that the functions \( f_n \) and \( f_{n,u} \) are measurable and bounded, so that (2.2) applies with \( f_n \) or \( f_{n,u} \) in place of \( f \). As a result, we have
\[
\mathbb{E}|f_n(W)(W, u)| = \mathbb{E}[f_{n,u}(W)(W, u)]
\]
\[
\quad = \left\langle \int_{\Xi} \mathbb{E}_\xi[f_{n,u}(V_\xi)] \mu(d\xi), u \right\rangle
\]
\[
\quad \leq \int_{\Xi} \mathbb{E}_\xi|f(V_\xi)| \|\mu\|_\xi
\]
\[
\quad < \infty.
\]
Noting that the functions \( f_n \) converge pointwise to \( f \), and applying Fatou’s lemma, we find that \( \mathbb{E}|f(W)(W, u)| < \infty \) for all \( u \in \mathbb{R}^d \). Therefore, \( \mathbb{E}|f(W)W| < \infty \). Now we can apply the dominated convergence theorem to the counterparts of (2.2) with \( f_n \) in place of \( f \), which implies that (2.2) remains true. \( \Box \)

Lemma 4.2. Under (P_x), we have \( \mathbb{E}|V_\xi| \leq \mathbb{E}|W| + |\nu_\xi|(\Xi) \) for all \( \xi \in \Xi \).

Proof. Take \( \varepsilon > 0 \), let \( f(w) := \sqrt{\varepsilon^2 + |w|^2} \) and compute \( \nabla f(w) = \frac{w}{\sqrt{\varepsilon^2 + |w|^2}} \). Clearly, \( |\nabla f(w)| \leq 1 \). By Assumption (P_x), we have
\[
\mathbb{E}_\xi|V_\xi| \leq \mathbb{E}_\xi[f(V_\xi)] = \mathbb{E}[f(W)] - \int_{\Xi} \langle \mathbb{E}_\xi[\nabla f(V_\xi)], \nu_\xi(d\eta) \rangle
\]
\[
\quad \leq \varepsilon + \mathbb{E}|W| + |\nu_\xi|(\Xi).
\]
Letting \( \varepsilon \) to zero, we obtain the desired inequality. \( \Box \)

Lemma 4.3. Assume (P_x), let \( D_\xi f := \mathbb{E}[f(W)] - \mathbb{E}_\xi[f(V_\xi)] \), and recall (2.6) and (2.7). Take \( \xi \in \Xi \) and a function \( f : \mathbb{R}^d \to \mathbb{R} \). If either \( f \) is measurable and bounded or \( M_1(f) < \infty \), then
\[
|D_\xi f| \leq 2 M_0(f).
\]
Next, if \( f \) is continuously differentiable with \( M_0(f) < \infty \) or \( M_1(f) < \infty \), then
\[
|D_\xi f| \leq M_1(f) \beta_1(\xi).
\]
Finally, if \( f \) is twice continuously differentiable with \( M_1(f) < \infty \), then
\[
|D_\xi f| \leq |\mathbb{E}[\nabla f(W)]| \beta_1(\xi) + M_2(f) \beta_2(\xi).
\]
All the bounds apply under the convention \( 0 \cdot \infty = \infty \cdot 0 = 0 \).

Remark 4.4. Under the condition specified for each particular bound, all underlying expectations exist. This follows from Remark 2.2 and Lemma 4.2.
Proof of Lemma 4.3. The bound (4.1) is immediate. To prove (4.2), assume first that $M_1(f) < \infty$. Combining (2.3) with Proposition 5.16, we find that

$$|D_\xi f| \leq \int_E \left| \mathbb{E}_\eta [\nabla f(V_\eta)] \right| |\nu_\xi|(d\eta) \tag{4.4}$$

and (4.2) follows. The latter is trivial if $M_1(f) = \infty$ and $\beta_1^{(\xi)} > 0$. If $\beta_1^{(\xi)} = 0$, then, by (2.3), we have $\mathbb{E}_\xi [f(V_\xi)] = \mathbb{E}[f(W)]$ for all continuously differentiable $f$ with bounded derivative. As a result, $V_\xi$ and $W$ have the same distribution and (4.2) again follows.

Applying (4.2) with the function $f_u(w) := \langle \nabla f, u \rangle$ in place of $f$, where $u \in \mathbb{R}^d$, and with $\eta$ in place of $\xi$, we obtain $|\mathbb{E}_\eta [\nabla f(V_\eta)] , u| \leq |\mathbb{E}[\nabla f(W)] , u| + M_2(f) |u| \beta^{(\eta)}$. Taking the supremum over $|u| \leq 1$, we derive $|\mathbb{E}_\eta [\nabla f(V_\eta)]| \leq |\mathbb{E}[\nabla f(W)]| + M_2(f) \beta^{(\eta)}$. Plugging into (4.4), (4.3) follows, completing the proof.

Proof of Proposition 2.13. First, we show that $\mathbb{E}|f(W)| < \infty$ and that (2.2) still applies for a continuously differentiable function $f$ with bounded derivative. By Proposition 4.1, it suffices to check that $\int_E \mathbb{E}_\xi [f(V_\xi)] |\mu|(d\xi) < \infty$. Since $|f(w)| \leq |f(0)| + |w| M_1(f)$, it suffices to check that $\int_E \mathbb{E}_\xi [V_\xi] |\mu|(d\xi) < \infty$. However, this follows from the finiteness of $\beta_2$ by Lemma 4.2.

As $f(w) := \sqrt{1 + |w|^2}$ is continuously differentiable with bounded derivative, $\mathbb{E}|f(W)|$ must be finite. Therefore, $\mathbb{E}|W|^2$ is finite, too.

Combining (2.2) with the fact that $\mu(\Xi) = 0$ (which follows from Remark 2.4 and the assumption $\mathbb{E} W = 0$), we obtain

$$\mathbb{E}[f(W)W] = \int_\Xi \left( \mathbb{E}_\xi [f(V_\xi)] - \mathbb{E}[f(W)] \right) \mu(d\xi).$$

Applying (Px) and (5.2), and recalling Definition 5.17, we rewrite this as

$$\mathbb{E}[f(W)W] = -\int_\Xi \int_\Xi \mathbb{E}_\eta [\nabla f(V_\eta)] , \nu_\xi(d\eta) \mu(d\xi)$$

$$= -\int_\Xi \int_\Xi (\mu(d\xi) \otimes \nu_\xi(d\eta)) \mathbb{E}_\eta [\nabla f(V_\eta)].$$

A standard argument shows that $\int_\Xi h d\mu = \int_\Xi \int_\Xi h(\eta) \nu_\xi(d\eta) \mu(d\xi)$ for all bounded measurable functions $\mu$. Property (Z) now follows.

Proposition 4.5. Assume (Z') and take a non-decreasing function $h$: $[0, \infty) \to [0, \infty)$, such that

$$\int_\Xi \mathbb{E}_\xi [h(|V_\xi|)] |\hat{\mu}|_\nu(d\xi) < \infty. \tag{4.5}$$

Let $F$: $\mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable vector function, such that $|\nabla F|_\nu(w) \leq h(|w|)$ for all $w \in \mathbb{R}^d$. Then $\mathbb{E}[\langle F(W) , W \rangle] < \infty$ and (2.14) remains true.

Proof. For each $n \in \mathbb{N}$, define function $\psi_n$: $[0, \infty) \to [0, 1]$ as $\psi_n(t) := 1$ for $t \leq n$, $\psi_n(t) := 1 - \frac{1}{2n^2} (t - n)^2$ for $n \leq t \leq 2n$, $\psi_n(t) := \frac{1}{n^2} (t - 3n)^2$ for $2n \leq t \leq 3n$ and $\psi_n(t) := 0$ for $t \geq 3n$. Observe that $\psi_n$ is well-defined and that for each fixed $n$, the expression $t \psi_n(t)$ is bounded in $t$. Differentiating, we obtain $\psi'_n(t) = 0$ for $t \leq n$, $\psi'_n(t) = \frac{1}{n^3} (t - n)$ for $n \leq t \leq 2n$, $\psi'_n(t) = \frac{1}{n^3} (t - 3n)$ for $2n \leq t \leq 3n$ and $\psi'_n(t) = 0$ for $t \geq 3n$. Thus, $\psi$ is continuously differentiable and observe that the expression $t |\psi'_n(t)|$ is uniformly bounded in $t$ and $n$. 

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Now let $F_n(w) := F(\psi_n(|w|) w)$. Identifying 2-tensors with linear transformations (see Section 5.2, in particular (5.1)) and applying the chain rule, compute

$$\nabla F_n(w) = \nabla F(\psi_n(|w|) w) \left( \frac{\psi'_n(|w|)}{|w|} w \otimes w + \psi_n(|w|) I_d \right)$$

and notice that $F_n$ is differentiable at the origin because the first term vanishes for $|w| \leq n$. Applying (5.1), (5.2) and again (5.1) in turn, we obtain

$$\langle \nabla F(\psi_n(|w|) w) (w \otimes w) , u \otimes v \rangle = \langle \nabla F(\psi_n(|w|) w) (w \otimes w) v , u \rangle = \langle \nabla F(\psi_n(|w|) w) , u \rangle \langle v , w \rangle$$

Taking the supremum over $u$ and $v$ with $|u|, |v| \leq 1$, we obtain

$$|\nabla F_n(w)|_\infty \leq (|w| |\psi'_n(|w|)| + \psi_n(|w|)) |\nabla F(\psi_n(|w|) w) , u \otimes v \rangle.$$

Since $t |\psi'(t)|$ is uniformly bounded in $t$ and $n$ and since $h$ is non-decreasing, there exists a constant $C$, such that $|\nabla F_n(w)|_\infty \leq C h(|w|)$ for all $n$ and $w$.

For each fixed $n$, the expression $\psi_n(|w|) w$ is bounded in $w \in \mathbb{R}^d$. Since $F$ is continuously differentiable, $|\nabla F_n(w)|_\infty$ is also bounded in $w \in \mathbb{R}^d$. Therefore, (2.14) holds true with $F_n$ in place of $F$.

Now observe that the functions $h_n$ converge pointwise to $F$ and that the functions $\nabla F_n$ converge pointwise to $\nabla F$ as well. Recalling (4.5) and applying the dominated convergence theorem, we obtain

$$\lim_{n \to \infty} \mathbb{E}[\langle F_n(W) , W \rangle] = \lim_{n \to \infty} \int_\Xi \langle \mathbb{E}_\xi[\nabla F_n(V_\xi)] , \tilde{\mu}(d\xi) \rangle = \int_\Xi \langle \mathbb{E}_\xi[\nabla F(V_\xi)] , \tilde{\mu}(d\xi) \rangle. \quad (4.6)$$

Now take another continuously differentiable vector function $\tilde{F} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, such that $|\nabla \tilde{F}|_\infty (w) \leq h(|w|)$ and, in addition, $\langle \tilde{F}(w) , w \rangle \geq 0$ for all $w \in \mathbb{R}^d$. Letting $\tilde{F}_n(w) := \tilde{F}(\psi_n(|w|) w)$, observe that we also have $\langle \tilde{F}_n(w) , w \rangle \geq 0$ for all $w \in \mathbb{R}^d$. Fatou’s lemma along with (4.6) with $\tilde{h}_n$ and $\tilde{F}$ in place of $F_n$ and $F$ implies

$$\mathbb{E}[\langle \tilde{F}(W) , W \rangle] \leq \int_\Xi \langle \mathbb{E}_\xi[\nabla \tilde{F}(V_\xi)] , \tilde{\mu}(d\xi) \rangle \leq \int_\Xi \mathbb{E}_\xi[h(|V_\xi|)] |\tilde{\mu}|(d\xi) < \infty. \quad (4.7)$$

Now put $\tilde{F}(w) := \frac{w}{|w|} \int_0^{|w|} \tilde{h}(t) \, dt$, where $\tilde{h}(t) := \frac{1}{3} (h(t) - h_0)$ and $h_0 := \lim_{s \uparrow 0} h(s)$. For $w \neq 0$, compute

$$\nabla \tilde{F}(w) = \left( \frac{I_d}{|w|} - \frac{w \otimes w}{|w|^3} \right) \int_0^{|w|} \tilde{h}(t) \, dt + \tilde{h}(|w|) \frac{w \otimes w}{|w|^2}$$

and estimate

$$|\nabla \tilde{F}(w)|_\infty \leq \frac{2}{|w|} \int_0^{|w|} \tilde{h}(t) \, dt + \tilde{h}(|w|) \leq 3 \hat{h}(|w|) \leq h(|w|);$$

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the second inequality is true because $h$ is nondecreasing. From the above, it also follows that $\tilde{F}$ is continuously differentiable at the origin if we put $\tilde{F}(0) := 0$. Now compute

$$\langle \tilde{F}(w), w \rangle = |w| \int_0^{|w|} h(t) \, dt = \frac{|w|}{3} \int_0^{|w|} h(t) \, dt - \frac{h_0 |w|^2}{3}$$

and estimate

$$|F_n(w)| = F(\psi_n(|w|) w) \leq |F(0)| + \int_0^{\psi_n(|w|)|w|} h(t) \, dt \leq |F(0)| + \int_0^{|w|} h(t) \, dt,$$

$$|\langle F_n(w), w \rangle| \leq |F_n(w)| |w| \leq |F(0)| |w| + |w| \int_0^{|w|} h(t) \, dt \leq |F(0)| |w| + h_0 |w|^2 + 3 \langle \tilde{F}(w), w \rangle.$$ Recalling (4.7), it follows that the sequence of random variables $\langle F_n(W), W \rangle$ is dominated by a non-negative random variable with finite expectation. Applying the dominated convergence theorem and combining with (4.6), the finiteness of $\mathbb{E}(\langle F(W), W \rangle)$ along with (2.14) follows. \(\square\)

4.2 Gaussian smoothing

Gaussian smoothing will be one of the key tools to prove Theorems 2.9 and 2.15. Let $\phi_d$ be the density of the standard $d$-variate normal density, i.e., $\phi_d(z) = (2\pi)^{-d/2} \exp(-|z|^2/2)$. For $\varepsilon \geq 0$ and a map $F: \mathbb{R}^d \to V$, where $V$ is a finite-dimensional vector space, define

$$\mathcal{N}_\varepsilon F(w) := \int_{\mathbb{R}^d} F(w + \varepsilon z) \phi_d(z) \, dz. \tag{4.8}$$

Notice that $\mathcal{N}_0 F = F$ and $\mathcal{N}_1 F(0) = \mathcal{N} h$. Next, define constants $c_0, c_1, c_2, \ldots$ as

$$c_s := \int_{-\infty}^{\infty} |\phi_1^{(s)}(z)| \, dz. \tag{4.9}$$

Observe that

$$\int_{-\infty}^{\infty} |\langle \phi_1^{(s)}(z), u \rangle| \, dz \leq c_s |u|^s$$

and compute

$$c_0 = 1, \quad c_1 = \frac{2}{\sqrt{2\pi}}, \quad c_2 = \frac{4}{\sqrt{2\pi e}}, \quad c_3 = \frac{2 + 8 e^{-3/2}}{\sqrt{2\pi}}. \tag{4.10}$$

Lemma 4.6. Let $\varepsilon > 0$. If $f: \mathbb{R}^d \to \mathbb{R}$ is either measurable and bounded or $M_r(f) < \infty$ for some $r \in \mathbb{N}$, then $\mathcal{N}_\varepsilon f(w) < \infty$ for all $w \in \mathbb{R}^d$ and $\mathcal{N}_\varepsilon f$ is infinitely differentiable. In addition, we have

$$M_{r+s}(\mathcal{N}_\varepsilon f) \leq \frac{c_s}{\varepsilon^s} M_r(f)$$

for all $r \in \mathbb{N}$ and all $s \in \mathbb{N} \cup \{0\}$.

Proof. First, $\mathcal{N}_\varepsilon f$ is finite by Remark 2.2 and the fact that $\int_{\mathbb{R}^d} |z|^r \phi_d(z) \, dz$ is finite. Substituting $z = y - w/\varepsilon$, we rewrite (4.8) as

$$\mathcal{N}_\varepsilon f(w) = \int_{\mathbb{R}^d} f(\varepsilon y) \phi_d \left( y - \frac{w}{\varepsilon} \right) \, dy. \tag{4.11}$$
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Differentiating (4.11) under the integral sign and substituting back, we obtain
\[ \nabla^s \mathcal{N}_f(w) = \frac{(-1)^s}{\varepsilon^s} \int_{\mathbb{R}^d} f(\varepsilon y) \nabla^s \phi_d \left(y - \frac{w}{\varepsilon}\right) dy \]
\[ = \frac{(-1)^s}{\varepsilon^s} \int_{\mathbb{R}^d} f(w + \varepsilon z) \nabla^s \phi_d(z) dz. \]

Further differentiation under the differential sign gives
\[ \nabla^{r+s-1} \mathcal{N}_f(w) = \frac{(-1)^s}{\varepsilon^s} \int_{\mathbb{R}^d} \nabla^{r-1} f(w + \varepsilon z) \otimes \nabla^s \phi_d(z) dz. \]

The verification of the validity of the differentiation under the integral sign is left to the reader as an exercise. Consequently,
\[ \left| \nabla^{r+s-1} \mathcal{N}_f(x) - \nabla^{r+s-1} \mathcal{N}_f(y), u^{\otimes (r+s-1)} \right| \leq \frac{|x - y|}{\varepsilon^s} |u|^{r-1} M_r(f) \int_{\mathbb{R}^d} \left| \nabla^s \phi_d(z) \right| dz \]
\[ \leq c_s \frac{|x - y|}{\varepsilon^s} |u|^{r+s-1} M_r(f). \]

By Proposition 5.8, this implies
\[ \left| \nabla^{r+s-1} \mathcal{N}_f(x) - \nabla^{r+s-1} \mathcal{N}_f(y) \right| \leq c_s \frac{|x - y|}{\varepsilon^s} M_r(f). \]

The result is now immediate. \(\square\)

4.3 Bounds on the Stein expectation

In this subsection, we turn to Stein’s method, which will be implemented in view of the proof of Lemma 1 of Slepian [24]. We recall the procedure briefly; for an exposition, see Röllin [21] and Appendix H of Chernozhukov, Chetverikov and Kato [9]. Recalling the definition of \(M_r(f)\) from (2.1), take a function \(f: \mathbb{R}^d \to \mathbb{R}\) with \(M_r(f) < \infty\) for some \(r \in \mathbb{N}\). For \(0 \leq \alpha \leq \pi/2\), define
\[ \mathcal{U}_\alpha f(w) := N \sin \alpha f(w \cos \alpha) = \int_{\mathbb{R}^d} f(w \cos \alpha + z \sin \alpha) \phi_d(z) dz \quad (4.12) \]
In particular, \(\mathcal{U}_0 f = f\) and \(\mathcal{U}_{\pi/2} f = N f\). By Lemma 4.6, \(\mathcal{U}_\alpha f\) is defined everywhere and is infinitely differentiable.

For a random variable \(W\), \(\mathbb{E}[\mathcal{U}_\alpha f(W)]\) can be regarded as an interpolant between \(\mathbb{E}[f(W)]\) and \(N f\). A straightforward calculation shows that
\[ \frac{d}{d\alpha} \mathcal{U}_\alpha f(w) = \mathcal{S} \mathcal{U}_\alpha f(w) \tan \alpha, \]
where \(\mathcal{S}\) denotes the Stein operator:
\[ \mathcal{S} f(w) := \Delta f(w) - \langle \nabla f(w), w \rangle \quad (4.13) \]
and where \(\Delta\) denotes the Laplacian. Integrating over \(\alpha\) and taking expectation, we find that
\[ \mathbb{E} \mathcal{U}_\varepsilon f(W) - N f = - \int_{\varepsilon}^{\pi/2} \mathbb{E} \left[ \mathcal{S} \mathcal{U}_\alpha f(W) \right] \tan \alpha \, d\alpha, \quad (4.14) \]
for all \(0 \leq \varepsilon \leq \pi/2\). More precisely, if \(f_{\pi/2}^{\varepsilon/2} \mathbb{E} \left| \mathcal{S} \mathcal{U}_\alpha f(W) \right| \tan \alpha \, d\alpha\) is finite, then, by Fubini’s theorem, \(\mathbb{E} \left| \mathcal{U}_\varepsilon f(W) \right|\) is also finite and (4.14) is true.
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**Lemma 4.7.** Let \( r \in \mathbb{N}, s \in \mathbb{N} \cup \{0\} \) and \( 0 < \alpha \leq \pi/2 \). Then any function \( f: \mathbb{R}^d \to \mathbb{R} \) with \( M_r(f) < \infty \) satisfies

\[
M_{r+s}(\mathbb{U}_\alpha f) \leq c_s \frac{\cos \alpha}{\sin \alpha} M_r(f),
\]

(4.15)

\[
|N \nabla^{r+s} \mathbb{U}_\alpha f| \leq c_s M_r(f) \cos \alpha^s.
\]

(4.16)

**Proof.** Letting \( f_\alpha(w) := f(w \cos \alpha) \), observe that \( \mathbb{U}_\alpha f = N \tan \alpha f_\alpha \). By Lemma 4.6, we have \( M_{r+s}(\mathbb{U}_\alpha f) \leq c_s M_r(f_\alpha) \cos \alpha \leq c_s M_r(f) \cos \alpha \cot \alpha \), proving (4.15). To derive (4.16), write \( N \nabla^{r+s} \mathbb{U}_\alpha f = N_1 \nabla^{r+s} \mathbb{U}_\alpha f(0) = \nabla^{r+s} N_1 \mathbb{U}_\alpha f(0) \) and observe that \( N_1 \mathbb{U}_\alpha f = N_1 N_\tan \alpha f_\alpha = N_1 / \cos \alpha f_\alpha \). Again by Lemma 4.6, we have \( M_{r+s}(N_1 \mathbb{U}_\alpha f) \leq c_s M_r(f_\alpha) \cos \alpha \leq c_s M_r(f) \cos \alpha^s \). Combining with Remark 2.1, we obtain (4.15).

Now we gradually turn to main result of this subsection, Lemma 4.10. We first need some results concerning finiteness of certain integrals.

**Lemma 4.8.** Let \( r \in \mathbb{N} \). Suppose that \( E|W|^r < \infty \) and take a function \( f: \mathbb{R}^d \to \mathbb{R} \) with \( M_r(f) < \infty \). Then:

1. If \( r \geq 2 \), then \( E|\mathcal{F} f(W)| < \infty \).
2. \( \int_0^{\pi/2} E|\mathcal{F} \mathbb{U}_\alpha f(W)| \tan \alpha \, d\alpha < \infty \) (this statement applies for either \( r \in \mathbb{N} \)).

**Proof.** Clearly, \( |\mathcal{F} f(w)| \leq d |\nabla^2 f(w)| \sqrt{1 + |\nabla f(w)||w|} \) for all \( w \) with \( M_2(f) < \infty \). Let \( \tilde{f}_\alpha := \mathbb{U}_\alpha f \). First, take \( r = 1 \). By Lemma 4.7, we have \( M_1(\tilde{f}_\alpha) \leq M_1(f) \cos \alpha \) and \( M_2(\tilde{f}_\alpha) \leq c_1 M_1(f) \cos^2 \alpha \). As a result, we have

\[
E|\nabla^2 \tilde{f}_\alpha(W)| \leq c_1 M_1(f) \frac{\cos^2 \alpha}{\sin \alpha} \quad \text{and} \quad E[|\nabla \tilde{f}_\alpha(W)| |W|] \leq M_1(f) E|W| \cos \alpha.
\]

Multiplying by \( \tan \alpha \) and integrating, we obtain the desired finiteness.

Now take \( r \geq 2 \). Observe that for each \( s = 0, 1, \ldots, r \), there exist constants \( C_{s,r} \) and \( D_{s,r} \), such that \( |\nabla^s f(x)| \leq C_{s,r} + D_{s,r} |x|^{-s} \) for all \( x \in \mathbb{R}^d \). Thus, \( E|\mathcal{F} f(W)| \leq d C_{2,r} + D_{2,r} E|W|^{-2} + C_{1,r} E|W| + D_{1,r} E|W|^r < \infty \).

Next, similarly as in the proof of Lemma 4.7, differentiation under the integration sign gives \( \nabla^s \tilde{f}_\alpha(w) = E[\nabla^s f(w \cos \alpha + Z \sin \alpha)] \cos^s \alpha \), where \( Z \) is a standard \( d \)-variate normal random vector. Therefore,

\[
|\nabla^s \tilde{f}_\alpha(w)| \leq C_{s,r} \cos^s \alpha + D_{s,r} \sum_{k=0}^{r-s} \binom{n-s}{k} |w|^k E|Z|^{r-s-k} \cos^{s+k} \alpha \sin^{r-s-k} \alpha.
\]

Replacing \( w \) with \( W \) and taking expectation, we obtain in particular

\[
E|\nabla^2 \tilde{f}_\alpha(W)| \leq C_{2,r} \cos^2 \alpha + D_{2,r} \sum_{k=0}^{r-2} \binom{n-2}{k} E|W|^k E|Z|^{r-2-k} \cos^{k+2} \alpha \sin^{r-k-2} \alpha,
\]

\[
E[|\nabla \tilde{f}_\alpha(W)| |W|] \leq C_{1,r} \cos \alpha + D_{1,r} \sum_{k=0}^{r-1} \binom{n-1}{k} E|W|^{k+1} E|Z|^{r-k-1} \cos^{k+1} \alpha \sin^{r-k-1} \alpha.
\]

Multiplying by \( \tan \alpha \) and integrating, we again obtain the desired finiteness.

**Lemma 4.9.** Assume \( (Z') \) and \( (Px) \). If the quantity \( \tilde{\beta}_3 \) defined in (2.15) is finite, then \( \int_\Xi E_{\xi} |V_{\xi}| |\tilde{\mu}_{\lambda}(d\xi) | E|W|^3 \) are finite, too.
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PROOF. The finiteness of $\int_\mathbb{E} \mathbb{E}_\xi |V_\xi| \|\bar{\mu}\|_\lambda (d\xi)$ follows from the finiteness of $\bar{\beta}_3$ by Lemma 4.2. To derive the finiteness of $\mathbb{E}|\mathcal{W}|^3$, apply Proposition 4.5 with $F(w) = |w| w$: clearly, $|\langle F(w), w \rangle| = |w|^3$. Noting that $\nabla F(w) = w \otimes w + |w| I_d$, we can set $h(t) := 2t$, so that the finiteness of $\int_\mathbb{E} \mathbb{E}_\xi |V_\xi| \|\bar{\mu}\|_\lambda (d\xi)$ implies (4.5) and the result follows. □

Lemma 4.10. Assume (St), (Z') and (Px), and recall (2.6), (2.7), (2.15) and (2.16). Take a three times differentiable function $f: \mathbb{R}^d \to \mathbb{R}$. If either $M_2(f) < \infty$, or $\bar{\beta}_3 < \infty$ and $M_3(f) < \infty$, then
\[
|\mathbb{E}[\mathcal{J} f(W)]| \leq \bar{\beta}_{23}(2 M_2(f), M_3(f)).
\] (4.17)
Moreover, if $f$ is four times continuously differentiable with $M_2(f) < \infty$ and $M_3(f) < \infty$, then
\[
|\mathbb{E}[\mathcal{J} f(W)]| \leq \int_\mathbb{E} \min\left\{ 2 M_2(f), \beta_1^{(\xi)}|\mathbb{E}[\nabla^3 f(W)]|_\nu + \beta_2^{(\xi)} M_4(f) \right\} |\bar{\mu}|_\lambda (d\xi).
\] (4.18)
Both bounds apply under the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

Remark 4.11. By Part (1) of Lemma 4.8 along with Lemma 4.9, $\mathbb{E}|\mathcal{J} f(W)|$ is finite in either case.

PROOF OF Lemma 4.10. First, observe that (2.14) applies with $F = \nabla f$ under the either of the specified conditions. If $M_2(f) < \infty$, this is true by Assumption (Z') itself. If $\bar{\beta}_3 < \infty$ and $M_3(f) < \infty$, we can apply Proposition 4.5. The latter applies if we can estimate $|\nabla^2 f(w)|_\nu \leq h(|w|)$, where $h$ is a non-decreasing function with $\int_\mathbb{E} \mathbb{E}_\xi [h(|V_\xi|)] |\bar{\mu}|_\lambda (d\xi) < \infty$. As $|\nabla^2 f(0)|_\nu \leq |\nabla^2 f(0)|_\nu + M_3(f) |w|$, it suffices to see that $\int_\mathbb{E} \mathbb{E}_\xi [V_\xi]|\bar{\mu}|_\lambda (d\xi) < \infty$. However, this is true by Lemma 4.9.

Recalling the identification $u \otimes v = w^T$ from Subsection 5.2 and Remark 2.12, and applying (St), consider
\[
\Delta f(w) = \langle \nabla^2 f(w), I_d \rangle = \langle \nabla^2 f(w), \bar{\mu}(\Xi) \rangle = \int_\mathbb{E} \langle \nabla^2 f(w), \bar{\mu}(d\xi) \rangle.
\] (4.19)
Taking expectation and applying (2.14), we obtain
\[
\mathbb{E}[\mathcal{J} f(W)] = \int_\mathbb{E} \langle D_\xi \nabla^2 f, \bar{\mu}(d\xi) \rangle,
\] (4.20)
where $D_\xi$ is as in Lemma 4.2, i.e., $D_\xi g := \mathbb{E}[g(W)] - \mathbb{E}_\xi [g(V_\xi)]$.

Let $u, v \in \mathbb{R}^d$ and suppose that $M_2(f) < \infty$. The bounds (4.1) and (4.2) applied to the function $f_{u,v}(w) := \langle \nabla^2 f, u \otimes v \rangle$ yield $|\langle D_\xi \nabla^2 f, u \otimes v \rangle| \leq \min\{ 2 M_2(f), M_3(f) \beta_1^{(\xi)} \}$. Taking the supremum over $|u|, |v| \leq 1$, we find that $|D_\xi \nabla^2 f|_\nu \leq \min\{ 2 M_2(f), M_3(f) \beta_1^{(\xi)} \}$. Plugging into (4.20) and applying Proposition 5.16, we obtain (4.17). By a similar argument, (4.18) can be derived from (4.1) combined with (4.3). □

4.4 Proofs of Theorems 2.9 and 2.15

PROOF OF Theorem 2.15. First, we verify that under the given assumptions, first, $\mathbb{E}|f(W)|$ and $\mathcal{N}[f]$ are finite, and, second, either (4.14) is valid or the result is trivially true. The finiteness of $\mathcal{N}[f]$ follows from Remark 2.2 and the fact that $\int_\mathbb{R}^d |z| \phi_\alpha(z) dz$ is finite. Now if $M_1(f)$ or $M_2(f)$ is finite, then, by Lemma 4.8, $\int_0^{\pi/2} \mathbb{E}|\mathcal{J} \mathcal{N}_\alpha f(W)| \tan \alpha d\alpha$ is also finite. Consequently, $\mathbb{E}|f(W)|$ is finite and (4.14) is valid (see the comment below (4.14)).
It remains to show that if $M_3(f) < \infty$, then, first, $\|f(W)\|_E$ is finite, and, second, either (4.14) is valid or (2.18) is trivially true. If $M_3(f) = 0$, then $f$ is a quadratic function, so that $\|f(W)\|$ must be finite and $\mathbb{E}[f(W)] = Nf$; as a result, (2.18) is trivially true. The latter is also trivially true if the right hand side is infinite. However, if $M_3(f) > 0$ and the right hand side is finite, then $\tilde{\beta}_3$ is also finite. By Lemma 4.9, $\mathbb{E}|W|^3$ must then be finite; by Lemma 4.8, $\int_0^{\pi/2} \mathbb{E}|\mathcal{S}_\alpha f(W)| \tan \alpha \, d\alpha$ must be finite. As a result, $\mathbb{E}[f(W)]$ is finite and (4.14) is valid.

First, observe that $\mathbb{E}[f(W)]$ and $Nf$ are finite and $\mathbb{E}[f(W)] = Nf$ if either of $M_1(f)$, $M_2(f)$ or $M_3(f)$ vanishes. Thus, (2.18)–(2.20) are all trivially true in this case. From now, assume that $M_r(f) > 0$ for all $r \in \{1, 2, 3\}$. In addition, to prove either of the inequalities (2.18)–(2.20), we can assume that the right hand side is finite.

Thus, in either case where we have not proved the result yet, we can estimate

$$\mathbb{E}[\tilde{f}_\varepsilon(W)] - Nf = \int_0^{\pi/2} \mathbb{E}[\mathcal{S}_\alpha \tilde{f}_\varepsilon(W)] \tan \alpha \, d\alpha \tag{4.21}$$

for all $0 \leq \varepsilon \leq \pi/2$; here, $\tilde{f}_\alpha := \mathcal{S}_\alpha f$.

In order to derive (2.18), an application of the second part of (4.17) along with (4.15) gives

$$\mathbb{E}[f(W)] - Nf \leq \tilde{\beta}_3 \int_0^{\pi/2} M_3(\tilde{f}_\alpha) \tan \alpha \, d\alpha \leq \tilde{\beta}_3 M_3(f) \int_0^{\pi/2} \cos^2 \alpha \sin \alpha \, d\alpha = \frac{\tilde{\beta}_3}{3} M_3(f).$$

To derive (2.19), observe that

$$\mathbb{E}[f(W)] - Nf \leq \int_0^{\pi/2} \tilde{\beta}_{23} (2 M_2(\tilde{f}_\alpha), M_3(\tilde{f}_\alpha)) \tan \alpha \, d\alpha \leq \int_\Xi \min \left\{ 2 \int_0^{\pi/2} M_2(\tilde{f}_\alpha) \tan \alpha \, d\alpha, \beta_1^{(\xi)} \int_0^{\pi/2} M_3(\tilde{f}_\alpha) \tan \alpha \, d\alpha \right\} |\tilde{\mu}|(d\xi).$$

Applying (4.15), recalling (2.16) and integrating, we obtain

$$\mathbb{E}[f(W)] - Nf \leq M_2(f) \int_\Xi \min \left\{ 2 \int_0^{\pi/2} \cos \alpha \sin \alpha \, d\alpha, c_1 \beta_1^{(\xi)} \int_0^{\pi/2} \cos^2 \alpha \, d\alpha \right\} |\tilde{\mu}|(d\xi) = M_2(f) \int_\Xi \min \left\{ 1, \frac{\pi}{4} c_1 \beta_1^{(\xi)} \right\} |\tilde{\mu}|(d\xi) = \tilde{\beta}_{23} \left( 1, \frac{\sqrt{2\pi}}{4} \right) M_2(f).$$

Finally, we turn to (2.20). The latter is trivially true if $M_1(f) = 0$, so that we can assume that $M_1(f) > 0$. Define

$$\delta := \sup \left\{ \frac{|\mathbb{E}[f(W)] - Nf|}{M_1(f)} : 0 < M_1(f) < \infty \right\}. \tag{4.22}$$

We first prove that $\delta$ is finite. To justify this, observe that $|\mathbb{E}[f(W)] - f(0)| \leq M_1(f) \mathbb{E}|W|$ and $|Nf - f(0)| \leq M_1(f) \mathbb{E}|Z|$. Since $\mathbb{E}|W|$ and $\mathbb{E}|Z|$ are both finite, $\delta$ must be finite, too.
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Now take $f: \mathbb{R}^d \to \mathbb{R}$ with $0 < M_1(f) < \infty$. We begin with smoothing: let $0 < \varepsilon < \pi/2$ and take a standard $d$-variate normal random vector $Z$, independent of $W$. Observe that
\[
\left| \mathbb{E}[\tilde{f}_\varepsilon(W)] - \mathbb{E}[f(W)] \right| \leq M_1(f) \mathbb{E}[(1 - \cos \varepsilon)W + Z \sin \varepsilon] \\
\leq M_1(f) \sqrt{\mathbb{E}[(1 - \cos \varepsilon)^2 W^2 + \mathbb{E}[Z]^2 \sin^2 \varepsilon]} \\
= 2\sqrt{d} M_1(f) \sin \frac{\varepsilon}{2}.
\]
Consequently,
\[
\left| \mathbb{E}[f(W)] - \mathcal{N}f \right| \leq \left| \mathbb{E}[\tilde{f}_\varepsilon(W)] - \mathcal{N}f \right| + 2\sqrt{d} M_1(f) \sin \frac{\varepsilon}{2}. \tag{4.23}
\]
Combining (4.21) with (4.18), we obtain
\[
\left| \mathbb{E}[\tilde{f}_\varepsilon(W)] - \mathcal{N}f \right| \leq \int_\mathbb{R} R_\varepsilon \left| \tilde{u} \right| \lambda(d\tilde{u}), \tag{4.24}
\]
where
\[
R_\varepsilon := \int_\varepsilon^{\pi/2} \min \left\{ 2 M_2(\tilde{f}_\alpha), \beta_1^{(\xi)} \mathbb{E}[|\nabla^3 \tilde{f}_\alpha(W)|] \right\} \tan \alpha \, d\alpha. \tag{4.25}
\]
Now take $u \in \mathbb{R}^d$ and let $\tilde{f}_{\alpha;u} := \langle \nabla^3 \tilde{f}_\alpha, u^{\otimes 3} \rangle$. Applying (4.15), we can estimate
\[
\left| \mathbb{E}[\nabla^3 \tilde{f}_\alpha(W), u^{\otimes 3}] \right| = \left| \mathbb{E}[\tilde{f}_{\alpha;u}(W)] \right| \leq M_3(\tilde{f}_{\alpha;u}) \leq M_3(\tilde{f}_\alpha) \left| u \right|^3.
\]
However, by (4.22), we can also estimate
\[
\left| \mathbb{E}[\nabla^3 \tilde{f}_\alpha(W), u^{\otimes 3}] \right| \leq \delta M_1(\tilde{f}_{\alpha;u}) \leq \delta M_4(\tilde{f}_\alpha) \left| u \right|^3
\]
and, consequently,
\[
\left| \mathbb{E}[\nabla^3 \tilde{f}_\alpha(W), u^{\otimes 3}] \right| \leq \left| \mathcal{N} \nabla^3 \tilde{f}_\alpha, u^{\otimes 3} \right| + \delta M_4(\tilde{f}_\alpha) \left| u \right|^3. \tag{4.26}
\]
Combining (4.4) and (4.26), taking the supremum over all $u$ with $|u| \leq 1$, and applying (4.15) and (4.16), we obtain
\[
\left| \mathbb{E}[\nabla^3 \tilde{f}_\alpha(W)] \right|_\nu \leq \min \left\{ M_3(\tilde{f}_\alpha), \left| \mathcal{N} \nabla^3 \tilde{f}_\alpha \right|_\nu + \delta M_4(\tilde{f}_\alpha) \right\} \\
\leq M_1(f) \left( c_2 \cos^3 \alpha + \min \left\{ c_2 \cos^2 \alpha \tan^2 \alpha, c_3 \delta \cos^4 \alpha \sin^2 \alpha \right\} \right).
\]
Plugging into (4.25), and estimating $M_2(\tilde{f}_\alpha)$ and $M_4(\tilde{f}_\alpha)$ by means of (4.15), we find that
\[
R_\varepsilon \leq M_1(f) \int_\varepsilon^{\pi/2} \min \left\{ 2 c_1 \cos \alpha, \beta_1^{(\xi)} \left( c_2 \cos^2 \alpha \sin \alpha + \min \left\{ c_2 \cos^2 \alpha \sin^2 \alpha, c_3 \delta \cos^3 \alpha \sin^2 \alpha \right\} \right) \right\} \tan \alpha \, d\alpha \\
\leq M_1(f) \min \left\{ 2 c_1, c_2 \beta_1^{(\xi)} + \beta_1^{(\xi)} \int_\varepsilon^{\pi/2} \min \left\{ c_2 \cos^2 \alpha \sin^2 \alpha, c_3 \delta \cos^3 \alpha \sin^2 \alpha \right\} \tan \alpha \, d\alpha \right\} \\
+ \int_0^{\pi/2} \min \left\{ 2 c_1, c_3 \beta_2^{(\xi)} \cos \alpha \right\} \cos \alpha \, d\alpha.
\]
The second integral can be estimated as
\[
\int_0^{\pi/2} \min\left\{ \frac{2}{s}, \frac{c_3 \beta_2(\xi)}{s^2} \right\} \cos \alpha \, d\alpha \leq \int_0^{\infty} \min\left\{ \frac{2}{s}, \frac{c_3 \beta_2(\xi)}{s^2} \right\} \, ds = 2\sqrt{2 c_1 c_3 \beta_2(\xi)}, \tag{4.27}
\]
and the first one can be estimated as
\[
\int_0^{\pi/2} \min\left\{ \frac{c_2 \cos^2 \alpha}{\sin \alpha}, c_3 \delta \frac{\cos^3 \alpha}{\sin^2 \alpha} \right\} \, d\alpha \leq \int_0^{\infty} \min\left\{ \frac{c_2}{s}, \frac{c_3 \delta}{2s^2} \right\} \, ds \leq \left( \int_0^{\sin(\epsilon/2)} \frac{c_2}{s} \, ds \right) + \int_{\sin(\epsilon/2)}^{\infty} \frac{c_3 \delta}{2s^2} \, ds \leq c_2 \left[ 1 + \left( \log \frac{c_3 \delta}{2c_2 \sin \frac{\epsilon}{2}} \right) \right].
\]

Collecting everything together, we obtain
\[
R_\xi \leq M_1(f) \min\left\{ 2 c_1, c_2 \beta_1(\xi) \right\} \left[ 1 + \left( \log \frac{c_3 \delta}{2c_2 \sin \frac{\epsilon}{2}} \right) \right] + 2\sqrt{2 c_1 c_3 \beta_2(\xi)}. \tag{5.12}
\]

Combining with (4.23) and (4.24), and recalling (2.17), we estimate the error in the normal approximation as
\[
\left| E[f(W)] - N \right| \leq M_1(f) \left[ 2\sqrt{d} \sin \frac{\epsilon}{2} + \bar{\beta}_{234} \left( 2 c_1, c_2 \left[ \frac{4}{3} + \left( \log \frac{c_3 \delta}{2c_2 \sin \frac{\epsilon}{2}} \right) \right] \right), \right. \tag{5.13}
\]
Taking the supremum over \( f \), dividing by \( M_1(f) \) and choosing \( \epsilon := 2 \arcsin \frac{\delta}{18\sqrt{d}} \) leads to the estimate
\[
\delta \leq \frac{\delta}{9} + \bar{\beta}_{234} \left( 2 c_1, c_2 \left[ \frac{4}{3} + \log \frac{9 c_3 \sqrt{d}}{2c_2} \right], \right. \tag{5.14}
\]
Resolving the latter and recalling that \( \delta \) is finite, the result follows after straightforward numerical computations.

**Proof of Theorem 2.9.** We derive the result from Theorem 2.15. Let \( \bar{\mu} \) be as in Proposition 2.13. By the latter, it satisfies Assumption (Z). Noting that \( \bar{\beta}_3 \leq \beta_3, \bar{\beta}_{23}(a,b) \leq \beta_{23}(a,b) \) and \( \bar{\beta}_{234}(a,b,c) \leq \beta_{234}(a,b,c) \), the inequalities (2.10)–(2.12) follow from the inequalities (2.18)–(2.20). \( \square \)

### 5 Appendix: theoretical preliminaries, notation and conventions

Throughout this appendix, \( U, U', V, V', W, W', Z \) and \( Z' \) will denote vector spaces. Unless specified otherwise, all vector spaces will be assumed to be real and finite-dimensional.
5.1 Dual pairs of vector spaces

Definition 5.1. A dual pair of vector spaces is a triplet \((V, V', \langle \cdot, \cdot \rangle)\), where \(\langle \cdot, \cdot \rangle\) is a non-singular pairing between \(V\) and \(V'\), i. e., a bilinear functional \(V \times V' \rightarrow \mathbb{R}\), such that for each \(v \in V \setminus \{0\}\), there exists \(v' \in V'\) with \(\langle v, v' \rangle \neq 0\), and that for each \(v' \in V' \setminus \{0\}\), there exists \(v \in V\) with \(\langle v, v' \rangle \neq 0\).

Observe that if \(V'\) is the space of all linear functionals on \(V\) and \(\langle v, v' \rangle = v'(v)\), then \((V, V', \langle \cdot, \cdot \rangle)\) is a dual pair of vector spaces. This is true because all linear functionals on subspaces can be extended to the whole space (a well known extension to the infinite-dimensional case is the Hahn–Banach theorem, see Theorem 3.3 of Rudin [23]). Conversely, if \((V, V', \langle \cdot, \cdot \rangle)\) is a dual pair of vector spaces, \(V'\) is naturally isomorphic to the space of all linear functionals on \(V\) and vice versa. Thus, \(V\) and \(V'\) are of the same dimension.

Definition 5.2. Let \((V, V', \langle \cdot, \cdot \rangle)\) be a dual pair of vector spaces. The bases \(e_1, \ldots, e_n\) of \(V\) and \(e'_1, \ldots, e'_n\) of \(V'\) are dual with respect to the pairing \(\langle \cdot, \cdot \rangle\) if \(\langle e_i, e'_j \rangle = \delta(i = j)\) for all \(i\) and \(j\).

Observe that for each basis of \(V\), there exists a unique dual basis if \(V'\).

Definition 5.3. A dual pair of normed spaces is a quintuplet \((V, V', \langle \cdot, \cdot \rangle, |\cdot|, |\cdot'|)\), where \((V, V', \langle \cdot, \cdot \rangle)\) is a dual pair of vector spaces, \(|\cdot|\) is a norm on \(V\), \(|\cdot'|\) a norm on \(V'\), and \(|\cdot|\) and \(|\cdot'|\) are dual norms (with respect to the pairing \(\langle \cdot, \cdot \rangle\)), i. e., \(|v| = \sup\{|\langle v, v' \rangle|; |v'| \leq 1\}\) for all \(v \in V\) and \(|v'| = \sup\{|\langle v, v' \rangle|; |v| \leq 1\}\) for all \(v' \in V'\).

Observe that one of the assumptions on norms is sufficient: if \(|v| = \sup\{|\langle v, v' \rangle|; |v'| \leq 1\}\) for all \(v \in V\), then also \(|v'| = \sup\{|\langle v, v' \rangle|; |v| \leq 1\}\) for all \(v' \in V'\). This is due to the Hahn–Banach theorem.

If \(V\) is an Euclidean space with a scalar product \(\langle \cdot, \cdot \rangle\) and the underlying norm \(|\cdot|\), then \((V, V, \langle \cdot, \cdot \rangle, |\cdot|)\) is a dual pair of normed spaces.

5.2 Tensors

Definition 5.4. Let \((U, U', \langle \cdot, \cdot \rangle_1)\) and \((V, V', \langle \cdot, \cdot \rangle_2)\) be dual pairs of vector spaces. The tensor product of \(U\) and \(V\) with respect to the preceding dual pairs is the space of all bilinear functionals \(U' \times V' \rightarrow \mathbb{R}\).

Observe that all tensor products of two fixed spaces \(U\) and \(V\) (with respect to different dual pairs) are naturally isomorphic and will be therefore all denoted by \(U \otimes V\).

The elements of \(U \otimes V\) are called tensors. For \(u \in U\) and \(v \in V\), we define the elementary tensor \(u \otimes v\) by \((u \otimes v)(u', v') := \langle u, u' \rangle_1(v, v')_2\).

Observe that each tensor is a sum of elementary tensors. Moreover, if \(e_1, \ldots, e_m\) is a basis of \(U\) and \(f_1, \ldots, f_n\) is a basis of \(V\), then the elementary tensors \(e_i \otimes f_j, i = 1, \ldots, m, j = 1, \ldots, n\), form a basis of \(U \otimes V\).

Furthermore, observe that for each bilinear map \(\Phi: U \times V \rightarrow W\), there exists a unique linear map \(L_\Phi: U \otimes V \rightarrow W\), such that \(L_\Phi(u \otimes v) = \Phi(u, v)\) for all \(u \in U\) and \(v \in V\). The latter fact often serves as a definition of the tensor product.

By our definition, however, each tensor \(\phi \in U \otimes V\) is a bilinear map \(U' \times V' \rightarrow \mathbb{R}\) and can be therefore assigned the linear map \(L_\phi: U' \otimes V' \rightarrow \mathbb{R}\). Observe that the map \((\phi, \phi') \mapsto \langle \phi, \phi' \rangle_\otimes := L_\phi \phi'\) is a non-singular pairing between \(U \otimes V\) and \(U' \otimes V'\) characterized by \((u \otimes v, u' \otimes v')_\otimes = \langle u, u' \rangle_1(v, v')_2\). Thus, \((U \otimes V, U' \otimes V', \langle \cdot, \cdot \rangle_\otimes)\) is a dual pair of vector spaces.
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Each tensor \( \phi \in U \otimes V \) can also be assigned a linear map \( \tilde{L}_\phi : V' \to U \) characterized by
\[
(\tilde{L}_\phi v', u'_1) = \phi(u'_1, v') = \langle \phi, u' \otimes v' \rangle.
\]
The latter will be identified with the tensor itself, so that we shall simply write \( \tilde{L}_\phi v' \) for \( \tilde{L}_\phi v' \).
Thus,
\[
(\tilde{L}_\phi v', u'_1) = \langle \phi, u' \otimes v' \rangle. \tag{5.1}
\]
Observe also that
\[
(\phi \circ v') = \langle \phi, u' \otimes v' \rangle. \tag{5.2}
\]
If \( e_1, \ldots, e_m \) is a basis of \( U \), \( f_1, \ldots, f_n \) a basis of \( V \) and \( f'_1, \ldots, f'_n \) the underlying dual basis of \( V' \), the tensor \( \sum_{i,j} a_{ij} e_i \otimes f_j \) is identified with the linear map with matrix \( [a_{ij}] \) with respect to the bases \( f'_1, \ldots, f'_n \) and \( e_1, \ldots, e_m \). If \( V \) is an Euclidean space, \( u \otimes v \) is identified with \( uv^T \).

The tensor product \( \mathbb{R} \otimes V \) is naturally isomorphic to \( V \); the latter two will therefore be identified. Next, the tensor products \( U \otimes V \otimes W \) and \( U \otimes (V \otimes W) \) are also naturally isomorphic and will be denoted by \( U \otimes V \otimes W \). Similarly, we write \( V_1 \otimes V_2 \otimes \cdots \otimes V_r \) and denote the elementary tensors by \( v_1 \otimes v_2 \otimes \cdots \otimes v_r \). We shall also denote \( V^\otimes r = v_1 \otimes v_2 \otimes \cdots \otimes v_r \) and \( v^\otimes r = v_1 \otimes v_2 \otimes \cdots \otimes v_r \).

**Definition 5.5.** Let \( (U, U', \langle \cdot, \cdot \rangle_1, \| \cdot \|_1, \| \cdot \|'_1) \) and \( (V, V', \langle \cdot, \cdot \rangle_2, \| \cdot \|_2, \| \cdot \|'_2) \) be dual pairs of normed spaces. The **injective norm** on \( U \otimes V \) is defined by \( \| \phi \|_\vee = \sup \{ |\langle \phi, u' \otimes v' \rangle| : \| u' \|'_1 \leq 1, \| v' \|'_2 \leq 1 \} \) on \( U \otimes V \). The **projective norm** on \( U \otimes V \) is the norm dual to the injective norm and will be denoted by \( \| \phi \|_\wedge = \sup \{ |\langle \phi, \phi' \rangle| : \| \phi \|_\vee \leq 1 \} \). For details on the projective and the injective norm, the reader is referred to Defant and Floret [11]. Notice that the authors use a different (in fact, more common) definition of the projective norm, but our definition is equivalent: see Section 3.2, Equation (1) ibidem.

The injective and the projective norm are both cross norms, i. e., \( \| u \otimes v \|_\vee = \| u \|_1 \| v \|_2 \) and \( \| u' \otimes v' \|'_\wedge = \| u' \|'_1 \| v' \|'_2 \). Next, observe that the natural isomorphism between \( (U \otimes V) \otimes W \) and \( U \otimes (V \otimes W) \) is an isometry if we take the injective norm in all tensor products. Similarly, the natural isomorphism between \( (U' \otimes V') \otimes W' \) and \( U' \otimes (V' \otimes W') \) is an isometry if we take the projective norm in all cases. Therefore, there is an unambiguous injective norm on \( U \otimes V \otimes W \) and an unambiguous projective norm on \( U' \otimes V' \otimes W' \).

Recall that, by the Hahn–Banach theorem, \( |\phi|_\vee = \sup \{ |\langle \phi, \phi' \rangle| : \| \phi \|_\wedge \leq 1 \} \). In other words, if \( U' \otimes V' \) is endowed with the projective norm, the injective norm of \( \phi \) matches the operator norm of the underlying linear functional \( L_\phi \). In addition, observe that it also matches the operator norm of the underlying linear map \( \tilde{L}_\phi : V' \to U \).

**Proposition 5.6.** Keeping the notation from **Definition 5.5**, take another dual pair of normed spaces \( (Z, Z', \langle \cdot, \cdot \rangle_3, \| \cdot \|_3, \| \cdot \|'_3) \). Recalling that each bilinear map \( \Phi : U' \times V' \to Z \) can be assigned a linear map \( L_\Phi : U' \otimes V' \to Z \), we have:
\[
\sup \{ |\Phi(u', v')|_3 : |u'|_1 \leq 1, |v'|_2 \leq 1 \} = \sup \{ |L_\Phi \phi'|_3 : |\phi'| \wedge \leq 1 \}.
\]

**Proof.** Take \( z' \in Z \) and consider the bilinear map \( \phi_z(u', v') := \langle \Phi(u', v'), z' \rangle_3 \), which is in fact a tensor in \( U \otimes V \). Observe that \( \langle \phi_z, \phi' \rangle_\otimes = \langle \Phi(u', v'), \phi' \rangle_\otimes = \langle 0_{U' \otimes V'}, \phi' \rangle_3 \) for all \( \phi' \in U' \otimes V' \). Now consider its injective norm \( |\phi_z|_\vee = \sup \{ |\langle \phi_z, u' \otimes v' \rangle| : |u'|_1 \leq 1, |v'|_2 \leq 1 \} = \sup \{ |\langle \phi_z, \phi' \rangle| : |\phi'| \wedge \leq 1 \} \). The latter equality can be rewritten as \( \sup \{ |\langle \phi_z, u' \otimes v' \rangle| : |u'|_1 \leq 1, |v'|_2 \leq 1 \} = \sup \{ |\langle \Phi(u', v'), z' \rangle_3| : |\phi'| \wedge \leq 1 \} \). Therefore, \( \sup \{ |\Phi(u', v')|_3 : |u'|_1 \leq 1, |v'|_2 \leq 1 \} = \sup \{ |\langle \Phi(u', v'), z' \rangle_3| : |u'|_1 \leq 1, |v'|_2 \leq 1, |z'|_3 \leq 1 \} = \sup \{ |L_\Phi \phi'|_3 : |\phi'| \wedge \leq 1 \} \). This completes the proof. \( \square \)
**Definition 5.7.** Let \((V, V', \langle \cdot , \cdot \rangle)\) be a dual pair of vector spaces. A tensor \(\phi \in V^\otimes r\) is symmetric if \(\langle \phi , v_1 \otimes \cdots \otimes v'_r \rangle = \langle \phi , v'_{\pi(1)} \otimes \cdots \otimes v'_{\pi(r)} \rangle\) for all permutations \(\pi\) of indices \(1, 2, \ldots, r\).

**Proposition 5.8 (Banach [2]; Bochnak and Siciak [7]).** Let \((V, V', \langle \cdot , \cdot \rangle, |\cdot|, |\cdot|^r)\) be a dual pair of normed spaces. Take \(r \in \mathbb{N}\). If \(\phi \in V^\otimes r\) is a symmetric tensor, then \(|\phi|_V = \sup_{|v'|_r \leq 1} |\phi , (v')^\otimes r|\).

In the sequel (and in the main part of the paper), we omit the indices at the pairings and the norms except for \(|\cdot|_V\) and \(|\cdot|_\lambda\), the injective and the projective norm.

### 5.3 Derivatives as tensors

Throughout this subsection, let \(D \subseteq V\) be an open set.

If \(H : D \to U\) is differentiable at \(x \in D\), denote its underlying derivative by \(\nabla H(x)\). This is a linear map \(V \to U\). If \((V, V', \langle \cdot , \cdot \rangle)\) is a dual pair of vector spaces, \(\nabla H(x)\) can be identified with a tensor in \(U \otimes V'\).

From now on, assume that \(U = \mathbb{R}^m\) and \(V = \mathbb{R}^n\) are Euclidean spaces with standard bases \(e_1, \ldots, e_m\) and \(f_1, \ldots, f_n\). Writing \(H(x) = \sum h_i(x) e_i\), we have \(\nabla H(x) = \sum_i \sum_j \frac{\partial h_i}{\partial x_j}(x) e_i \otimes f_j\).

Recall that \(\mathbb{R} \otimes V\) can be identified with \(V\). Thus, if \(g : D \to \mathbb{R}\) is differentiable at \(x\), we have \(\nabla g(x) = \sum_j \frac{\partial g}{\partial x_j}(x) f_j\). Thus, in this case, \(\nabla\) denotes the gradient, as usual. Observe also that for a fixed \(u \in U\), we have \(\nabla(u g) = u \otimes \nabla g\).

If \(g : D \to U\) is \(r\) times differentiable at \(x\), the \(r\)-fold derivative \(\nabla^r g(x) = \nabla \cdots \nabla g(x)\) at \(x\) can be regarded as an element of \(U \otimes V^\otimes r\) and we have

\[
\nabla^r g(x) = \sum_{1 \leq j_1, j_2, \ldots, j_r \leq n} \frac{\partial^r g(x)}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_r}} \otimes f_{j_1} \otimes f_{j_2} \otimes \cdots \otimes f_{j_r}.
\]

In addition, for \(U = \mathbb{R}\) and \(h(x) = \langle \nabla^r g(x) , v_1 \otimes \cdots \otimes v_r \rangle\), we have

\[
\langle \nabla h(x) , v_{r+1} \rangle = \langle \nabla^{r+1} g(x) , v_1 \otimes \cdots \otimes v_r \otimes v_{r+1} \rangle.
\]

### 5.4 Vector measures

Throughout this subsection, let \((\Xi, \mathcal{F})\) and \((\Upsilon, \mathcal{V})\) denote measurable spaces, i. e., \(\mathcal{F}\) is a \(\sigma\)-algebra on the set \(\Xi\) and \(\mathcal{V}\) is a \(\sigma\)-algebra on \(\Upsilon\).

Let \(\lambda\) be a positive measure on \((\Xi, \mathcal{F})\) and let \(V\) be a vector space. By \(L^1(\lambda, V)\), denote the space of all Borel measurable maps \(H : \Xi \to V\) with \(\int |H| \, d\lambda < \infty\), where \(|\cdot|\) is a norm on \(V\). Since vector spaces are assumed to be finite-dimensional, the definition is independent of the norm.

**Definition 5.9.** Let \((V, V', \langle \cdot , \cdot \rangle)\) be a dual pair of vector spaces. For each \(H \in L^1(\lambda, V)\) and each \(v' \in V'\), the function \(\xi \mapsto \langle H(\xi) , v' \rangle\) is \(\lambda\)-integrable. Moreover, there exists a unique vector \(v \in V\), such that \(\langle v , v' \rangle = \int_\Xi H(\xi) , v' \lambda(d\xi)\) for all \(v' \in V'\). The vector \(v\) is called the integral of \(H\) with respect to \(\nu\) and is denoted by \(\int_\Xi H \, d\nu\) or \(\int_\Xi H(\xi) \, \nu(d\xi)\). The integral is independent of the choice of \(V'\).

**Remark 5.10.** In infinite-dimensional spaces, the concept of integral is not so evident and there are several ones: see Chapter 2 of Diestel and Uhl [12].

**Remark 5.11.** For a linear map \(L\), we have \(\int_\Xi L H \, d\nu = L \int_\Xi H \, d\nu\).
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**Definition 5.12.** A $V$-valued measure on $(\Xi, \mathcal{X})$ is a map $\nu : \mathcal{X} \to V$, such that for each sequence $A_1, A_2, \ldots$ of disjoint measurable sets, we have $\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \nu(A_k)$. A sum of a $V$-valued series $\sum_{k=1}^{\infty} v_k$ is assumed to exist if $\sum_{k=1}^{\infty} |v_k| < \infty$; again, $|\cdot|$ can be an arbitrary norm on $V$.

If $\lambda$ is a positive measure on $(\Xi, \mathcal{X})$ and $H \in L^1(\lambda, V)$, the map $\nu := H \cdot \lambda : \mathcal{X} \to V$ defined by $\nu(A) := \int_A H \, d\lambda$ is a $V$-valued measure. Conversely, for each $\sigma$-finite positive measure $\lambda$ and each $\lambda$-continuous vector measure $\nu$ (i. e., $\nu(A) = 0$ for each $A$ with $\lambda(A) = 0$), there exists a map $H \in L^1(\lambda, V)$, such that $\nu = H \cdot \lambda$. Since $V$ is assumed to be finite-dimensional, this can be deduced from the classical Radon–Nikodým theorem. In general, this is not true (see Chapter 3 of Diestel and Uhl [12]).

Observe that for any finite collection $\nu_1, \ldots, \nu_n$ of vector measures on the same measurable space and with values in $V_1, V_2, \ldots, V_n$, respectively, there exists a finite positive measure $\lambda$, such that all measures $\mu_k$ are $\lambda$-continuous (one can take $\lambda = \sum_{k=1}^{n} |\mu_k|$). Thus, there also exist functions $H_k \in L^1(\lambda, V_k)$, such that $\nu_k = H_k \cdot \lambda$ for all $k = 1, \ldots, n$.

**Definition 5.13.** Let $(V, |\cdot|)$ be a normed space. The total variation of a $V$-valued vector measure $\nu$ is defined as $||\nu||(A) := \sup \sum_{k=1}^{n} |\nu(A_k)|$, where the supremum runs over all finite measurable partitions $A_1, A_2, \ldots, A_n$ of the set $A$.

The total variation of a vector measure is a positive measure. As $V$ is assumed to be finite-dimensional, it is finite. This can be deduced from the corresponding properties of real measures: see Theorems 6.2 and 6.4 of Rudin [22].

If $\nu = H \cdot \lambda$, where $H$ is a positive measure, we have $||\nu|| = |H| \cdot \lambda$: see Theorem 4 in Section 2 of Chapter 2 of Diestel and Uhl [12].

If $L : V \to \lambda$ is a linear map and $\nu$ is a $V$-valued vector measure, we define a new vector measure $L\nu$ by $(L\nu)(A) := L\nu(A)$. In particular, if $(V, V', \langle \cdot, \cdot \rangle)$ is a dual pair of vector spaces, we define a new real measure $\langle \nu, v' \rangle$.

**Definition 5.14.** Let $(U, U', \langle \cdot, \cdot \rangle)$ and $(V, V', \langle \cdot, \cdot \rangle)$ be dual pairs of vector spaces and let $\nu$ be a $V$-valued measure on $(\Xi, \mathcal{X})$. By $L^1(\nu, U)$, we denote the space of all Borel measurable functions $G : \Xi \to U$, such that $\int_{\Xi} |\langle G, u' \rangle| \, d|\nu(v', v)| < \infty$ for all $u' \in U'$ and all $v' \in V'$.

For $G \in L^1(\nu, U)$, define the integral $\int_{\Xi} G \, d\nu = \int_{\Xi} G(\xi) \, d\nu(\xi)$ as the unique tensor $\phi \in U \otimes V$ which satisfies $\langle \phi, u' \otimes v' \rangle = \int_{\Xi} \langle G, u' \rangle \, d\nu(v', v')$ for all $u' \in U'$ and all $v' \in V'$. Observe that the definitions of $L^1(\nu, U)$ and $\int_{\Xi} G \, d\nu$ are independent of the choice of $U'$ and $V'$.

This allows us to define a new $(U \otimes V)$-valued vector measure $G \otimes \nu$.

For a bilinear map $\Phi : U \otimes V \to W$, define $\int_{\Xi} \Phi(G, d\nu) := \int_{\Xi} \Phi(G(\xi), d\nu(\xi)) := L\Phi \int_{\Xi} G \, d\nu$.

**Proposition 5.15.** If $G$ and $\nu$ be as above. If $\nu = H \cdot \lambda$, where $\lambda$ is a positive measure, we have $\int_{\Xi} \Phi(G, d\nu) = \int_{\Xi} \Phi(G(\xi), H(\xi)) \, d\lambda$. In particular, $G \otimes \nu = (G \otimes H) \cdot \lambda$.

**Proof.** It suffices to prove that $\int_{\Xi} G \, d\nu = \int_{\Xi} (G \otimes H) \, d\lambda$ for all $G \in L^1(\nu, U)$, where $U$ is a vector space. Let $\nu$ be $V$-valued and let $(U, U', \langle \cdot, \cdot \rangle)$ and $(V, V', \langle \cdot, \cdot \rangle)$ be dual pairs of vector spaces. Then it suffices to check that $\int_{\Xi} \langle G, u' \rangle \, d\nu(v', v') = \int_{\Xi} \langle G, u' \rangle \, d\nu(H, v') \, d\lambda$ for all $G \in L^1(\nu, U)$, $u' \in U'$ and $v' \in V'$. However, the latter is equivalent to the claim that $\langle v', v' \rangle = \langle H, v' \rangle \cdot \lambda$, which follows from Remark 5.11.

**Proposition 5.16.** Let $U$, $V$ and $W$ be normed spaces. For each $V$-valued vector measure $\nu$ on $(\Xi, \mathcal{X})$ and each $G \in L^1(\nu, U)$, we have $|G \otimes \nu| = |G \otimes \nu| \leq |G| \cdot \nu$.

In addition, take a bilinear map $\Phi : U \times V \to W$. If $|\Phi(u, v)| \leq a |u| |v|$ for all $u \in U$ and all $v \in V$, we also have $|\int_{\Xi} \Phi(G, d\nu)| \leq a \int_{\Xi} |G| \, d\nu$. 23
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Proof. Write $\nu = H \cdot \lambda$, where $\lambda$ is a finite positive measure and where $H \in L^1(\lambda)$, $V$. Now observe that, by Proposition \ref{prop:multivariate-criteria}, $|G \otimes \nu|_V = |(G \otimes H)|_V = |G| |H| = |G| |\nu|$. An analogous observation holds for the the projective norm. To prove the second part, observe that $|f_{\Xi} \Phi(G, d\nu)| = |f_{\Xi} \Phi(G(\xi), H(\xi))\lambda(d\xi)| \leq a \int_{\Xi} |G| \lambda = a \int_{\Xi} |G| |d\nu|$. □

Now take measurable spaces $(\Xi_1, \mathcal{F}_1), \ldots, (\Xi_r, \mathcal{F}_r)$. Let $\nu_1$ be a $V_1$-valued vector measure on $(\Xi_1, \mathcal{F}_1)$. Next, for each $k = 1, 2, \ldots, r$ take a transition kernel $\nu_k$: $\Xi_1 \times \cdots \times \Xi_{k-1} \times \mathcal{F}_k \to V_k$, i.e., assume that the map $A \mapsto \nu_k(\xi_1, \ldots, \xi_{k-1}, A)$ is a $V_k$-valued vector measure for all $\xi_1 \in \Xi_1, \ldots, \xi_{k-1} \in \Xi_{k-1}$, and that the map $(\xi_1, \ldots, \xi_{k-1}) \mapsto \nu_k(\xi_1, \ldots, \xi_{k-1}, A)$ is measurable with respect to the product $\sigma$-algebra $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_{k-1}$ for all $A \in \mathcal{F}_k$. Finally, let $G: \Xi_1 \times \cdots \Xi_r = U$ be a product measurable function. Then one can consider the integral $J := \int_{\Xi_1} \cdots \int_{\Xi_{r-1}} \int_{\Xi_r} G(\xi_1, \ldots, \xi_r) \otimes \nu_r(\xi_1, \ldots, \xi_{r-1}, \mathcal{F}_r) \otimes \nu_{r-1}(\xi_1, \ldots, \xi_{r-2}, \mathcal{F}_{r-1}) \otimes \cdots \otimes \nu_1(\mathcal{F}_1)$, provided than all relevant functions are in the suitable $L^1$ spaces.

Definition 5.17. Let $G, \nu_1, \ldots, \nu_r$ be as before and let $\Phi: U \times V_1 \times \cdots \times V_r = \nu$ be a $(r+1)$-linear map. There exists a unique linear map $L_\Phi: U \otimes V_1 \otimes \cdots \otimes V_r = \nu$ such that $L_\Phi(u \otimes \nu_1 \otimes \cdots \otimes \nu_r) = \Phi(u, \nu_1, \ldots, \nu_r)$ for all $u \in U$, $\nu_1 \in V_1$, $\ldots$, $\nu_r \in V_r$. Define: $\int_{\Xi_1} \cdots \int_{\Xi_{r-1}} \int_{\Xi_r} \Phi(G(\xi_1, \ldots, \xi_r), \nu_r(\xi_1, \ldots, \xi_{r-1}, \mathcal{F}_r), \nu_{r-1}(\xi_1, \ldots, \xi_{r-2}, \mathcal{F}_{r-1}), \ldots, \nu_1(\mathcal{F}_1))$ to be $L_\Phi J$, provided that $J$ defined as above exists.

Remark 5.18. The preceding definition may be ambiguous as it may not be clear which variable is associated to which space. More strictly, one should write $\int_{\Xi_1} \cdots \int_{\Xi_{r-1}} \int_{\Xi_r} \Phi(G(\xi_1, \ldots, \xi_r), \nu_r(\xi_1, \ldots, \xi_{r-1}, \mathcal{F}_r), \nu_{r-1}(\xi_1, \ldots, \xi_{r-2}, \mathcal{F}_{r-1}), \ldots, \nu_1(\mathcal{F}_1))$ provided than all relevant functions are in the suitable $L^1$ spaces.

Definition 5.19. For a $U$-valued vector measure $\mu$ on $(\Xi, \mathcal{F})$ and a $V$-valued vector measure $\nu$ on $(\Psi, \mathcal{G})$, define the vector measure $\mu \otimes \nu$ on $(\Xi \times \Psi, \mathcal{F} \otimes \mathcal{G})$ as the unique $U \otimes V$-valued measure which satisfies $(\mu \otimes \nu)(A \times B) = \mu(A) \otimes \nu(B)$ for all $A \in \mathcal{F}$ and all $B \in \mathcal{G}$.

To see that $\mu \otimes \nu$ actually exists, we can define it componentwise in terms of product measures. Observe that in the case $U = V = \mathbb{R}$, $\mu \otimes \nu$ coincides with the usual product measure. Next, observe that if $\mu = G \cdot \kappa$ and $\nu = H \cdot \lambda$, where $\kappa$ and $\lambda$ are positive measures, we have $\int_{\Xi \times \Psi} \Phi(F, d\mu \otimes d\nu) = \int_{\Xi \times \Psi} \Phi(F(\xi, \eta), G(\xi) \otimes h(\eta) \kappa(d\xi) \otimes \lambda(d\eta))$ for all bilinear maps $\Phi: Z \times (U \otimes V) \to W$ and all maps $F \in L^1(\mu \otimes \nu, W)$. In particular, one can briefly write $\nu_1(\nu_2 \otimes \nu_3)(d\xi \otimes d\eta) = G(\xi) \otimes H(\eta) \kappa(d\xi) \otimes \lambda(d\eta)$. In other words, letting $\tilde{G}(\xi, \eta) := G(\xi)$ and $\tilde{H}(\xi, \eta) := H(\eta)$, we have $\mu \otimes \nu = (\tilde{G} \otimes \tilde{H}) \cdot (\kappa \otimes \lambda)$.

Proposition 5.20. For any two vector measures $\mu$ and $\nu$ with values in normed spaces, we have $|\mu \otimes \nu|_V = |\mu \otimes \nu|_\lambda = |\mu| \otimes |\nu|$. □

Proof. Write $\mu = G \cdot \kappa$ and $\nu = H \cdot \lambda$, where $\kappa$ and $\lambda$ are positive measures and $G$ and $H$ are in the suitable $L^1$ spaces. Letting $\tilde{G}$ and $\tilde{H}$ be as above, observe that $|\mu \otimes \nu|_V = |\tilde{G} \otimes \tilde{H}|_V \cdot (\kappa \otimes \lambda) = |\tilde{G}| |\tilde{H}| \cdot (\kappa \otimes \lambda) = (|G| \cdot \kappa) \otimes (|H| \cdot \lambda) = |\mu| \otimes |\nu|$ and similarly for the projective norm.

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