Collision-induced amplitude dynamics of fast 2D solitons in saturable nonlinear media with weak nonlinear loss

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Abstract We study the amplitude dynamics of two-dimensional (2D) solitons in a fast collision described by the coupled nonlinear Schrödinger equations with a saturable nonlinearity and weak nonlinear loss. We extend the perturbative technique for calculating the collision-induced dynamics of two one-dimensional (1D) solitons to derive the theoretical expression for the collision-induced amplitude dynamics in a fast collision of two 2D solitons. Our perturbative approach is based on two major steps. The first step is the standard adiabatic perturbation for the calculations on the energy balance of perturbed solitons, and the second step, which is the crucial one, is for the analysis of the collision-induced change in the envelope of the perturbed 2D soliton. Furthermore, we also present the dependence of the collision-induced amplitude shift on the angle of the two 2D colliding solitons. In addition, we show that the current perturbative technique can be simply applied to study the collision-induced amplitude shift in a fast collision of two perturbed 1D solitons. Our analytic calculations are confirmed by numerical simulations with the corresponding coupled nonlinear Schrödinger equations in the presence of the cubic loss and in the presence of the quintic loss.

Keywords Soliton dynamics · 2D soliton interaction · Nonlinear Schrödinger equation · Saturable nonlinearity · Nonlinear dissipation

1 Introduction

Solitons are stable shape-preserving solitary waves propagating in nonlinear dispersive media. Solitons have attracted considerable attentions in recent years due to the broadened applications of solitons in modern science [1–6]. In fact, solitons appear in a variety of fields, including optics, nanophotonics, condensed matter physics [1,4], and plasma physics [7]. In optics, the one-dimensional soliton propagation is stable and can be described by the nonlinear Schrödinger (NLS) model [4]. Due to the stability of the temporal NLS solitons, they can be used as bits of information in the optical fiber transmission technology [4]. However, 2D solitons are generally unstable in nonlinear optical media [5]. In particular, 2D optical solitons do not propagate in uniform cubic (Kerr) nonlinear media because of the catastrophic beam collapse at high powers [8,9]. There have been several investigations to achieve the stabilization of 2D solitons [10–13]. It was shown that 2D solitons can be stabilized in a layered structure with sign-alternating Kerr nonlinearity [10]. They also...
can exist and be stabilized in Kerr nonlinear optical media with an external potential \([11–13]\). Recently, the existence and stability of 2D optical solitons in saturable nonlinear media are subjects of continuously renewed interest to achieve the stable transmission of light beams at high velocity. The theoretical analyses of the condition for the existence of 2D and 3D solitons in saturable media were developed in Ref. \([14]\). Saturable nonlinearities have been observed in many nonlinear materials including photorefractive materials such as \(\text{LiNbO}_3\) \([3,15]\). The 2D solitons can exist in photorefractive crystals due to the relatively slow nonlinear response of these materials. When the light goes through these media, the refractive index changes and the material might force the light to remain confined in its self-generated waveguide. As a result, the light can propagate without changing the shape. Additionally, it was shown that 2D solitons can be stabilized in the nonlinear medium where the cubic domains are embedded into materials with saturable nonlinearities \([16]\). In such optical media, the soliton propagation can also be described by \((2+1)\)-dimensional \((2+1)\)D NLS equation with a saturable nonlinearity \([3,15,17]\).

One of the most fundamental properties of ideal solitons is their shape-preserving property in a soliton collision, that is, a soliton collision is elastic \([18]\). In optics, the collisions of sequences of solitons are very frequent \([1,4]\). Therefore, the collisions of two and many 1D solitons have been intensively investigated in several studies, for example, see Refs. \([19–25]\) and references therein. More specifically, in Refs. \([23,24]\), the authors studied the 1D soliton collision-induced amplitude dynamics in the presence of the cubic loss and the generic nonlinear loss. In optics, the nonlinear loss arises due to multiphoton absorption (MPA) or gain/loss saturation in silicon media \([24,26]\). MPA has been received considerable attention in recent years due to the importance of MPA in silicon nanowaveguides, which are expected to play a crucial role in optical processing applications in optoelectronic devices, including pulse switching and compression, wavelength conversion, regeneration, etc. \([23–29]\). It has been uncovered that the presence of weak nonlinear loss leads to an additional downshift of the soliton amplitude in a fast collision of two 1D solitons \([23,24]\). The analytic expressions for the amplitude shift in two-soliton collisions, which is described by the \((1+1)\)D NLS model, in the presence of weak cubic loss, which can be a result of TPA or gain and loss saturation, were already found in Refs. \([23,30]\) and in the presence of the weak \((2m+1)\)-order loss, for any \(m \geq 1\), were found in Ref. \([24]\). In the previous studies for 1D soliton collision-induced change in the four parameters of solitons \([20,21,23,24]\), the perturbative techniques were based on the projections of the total collision-induced change in the soliton envelope on the four localized eigenmodes of the linear operator \(\hat{L}\) describing small perturbations about the fundamental NLS soliton, which was derived by Kaup in the 1990s \([31,32]\). However, in this original perturbation theory, the soliton solution of the unperturbed model, which is the sech-form soliton, was used for the calculations on the dynamics of perturbed 1D solitons. Consequently, it is very hard to apply a similar technique for studying the effects of small perturbations on the interactions of solitons in higher dimensions, in which the unperturbed equations are nonintegrable. One needs to develop a new approach for studying the soliton collision-induced dynamics in the presence of nonlinear dissipation in higher dimensions instead of using the Kaup’s perturbation theory. It is worthy to note that the collision-induced corrections to solitons amplitudes were investigated in Ref. \([33]\) in the framework of unperturbed nonintegrable wave models. However, the study for the soliton amplitude dynamics in the nonintegrable wave models with nonlinear dissipation has not been explored. So far, to the best of our knowledge, the study for the collision-induced amplitude dynamics of 2D solitons in saturable nonlinear media in the presence of nonlinear dissipation is a long-standing open problem.

In this work, this important and challenging problem will be addressed. We study fast collisions between two 2D solitons in weakly perturbed nonlinear optical media. The dynamics of the collision is described by the systems of coupled \((2+1)\)D NLS equations with the saturable nonlinearity, which are nonintegrable models, in the presence of the generic weak \((2m+1)\)-order of the nonlinear loss, for any \(m \geq 1\). We derive the analytic expression for the amplitude dynamics of a 2D single soliton and, particularly, for the collision-induced amplitude dynamics in a collision of two fast 2D solitons in the presence of weak nonlinear loss. For the aforementioned purposes, we develop a perturbative method for perturbed 2D solitons. Our perturbative method significantly extends the perturbative technique in Refs. \([20,21,23,24]\) for calculating the effects...
of weak perturbations on fast collisions between two 1D solitons of the NLS equation and the recent perturbative method presented in Ref. [34] for calculating the collision-induced amplitude dynamics of two 1D pulses in perturbed linear waveguides. The crucial points in the current perturbative approach are the uses of the solution of the perturbed NLS model instead of the unperturbed NLS model and the single soliton dynamics in calculating the total collision-induced change in the soliton envelope. These are the key improvements compared to the perturbation techniques for studying the perturbed 1D solitons presented in Refs. [20,21,23,24]. More specifically, our perturbative approach is based on a procedure of two steps. The first step is for calculations on the energy balance of perturbed solitons based on the perturbed solution and a standard adiabatic perturbation theory for solitons. The second step, which plays a crucial role for our perturbative approach, is for calculations on the collision-induced change in the soliton envelope and a technical approximation of integrals based on the assumption of a fast and complete collision. We verify the analytic expressions by the numerical simulations with the corresponding (2+1)D NLS models with the cubic loss \((m = 1)\) and with the quintic loss \((m = 2)\). Additionally, we also demonstrate that the current perturbative approach can be simply applied to calculate the collision-induced amplitude shift in a fast collision of two 1D solitons for a large class of perturbed (1+1)D NLS equations in a straightforward manner. As a concrete example, we use the current perturbative technique to derive the expression for the collision-induced amplitude shift in a fast collision of two 1D solitons of (1+1)D cubic NLS model in the presence of the delayed Raman response.

The rest of the paper is organized as follows. In Sects. 2.1, 2.2, and 2.3, we first study the dynamics of a single-soliton propagation in saturable nonlinear optical media in the presence of the generic weak nonlinearity. Then, we use the perturbative technique to calculate the collision-induced amplitude dynamics in a fast collision of two 2D solitons. The analytic predictions will be validated by simulations in Sect. 3. Section 4 is reserved for conclusions. In Appendix A, we demonstrate the robustness and the simplicity of the current perturbative method for other perturbed soliton equations.

2 Collision-induced amplitude dynamics of two 2D solitons

2.1 The perturbed coupled (2+1)D NLS equations and the ideal 2D solitons

We consider fast collisions between two 2D solitons propagating in saturable nonlinear optical media in the presence of the weak \((2m + 1)\)-order of the nonlinear loss, for any \(m \geq 1\). The dynamics of the collision is described by the system of coupled NLS equations as follows [3,15,17,24]:

\[
i\partial_z \psi_j + \Delta \psi_j + \frac{\alpha(|\psi_j|^2 + |\psi_l|^2)}{1 + (|\psi_j|^2 + |\psi_l|^2)/I_0} \psi_j
= -i\epsilon_{2m+1} |\psi_j|^{2m} \psi_j
- i\epsilon_{2m+1} \sum_{k=1}^{m} b_{k,m} |\psi_l|^{2k} |\psi_j|^{2(m-k)} \psi_j, \tag{1}
\]

where \(b_{k,m} = \frac{m!(m+1)!}{(k!)^2 (m+1-k)!(m-k)!}\) [24], \(1 \leq j, l \leq 2\) and \(j \neq l\), \(\Delta = \Delta_x^2 + \Delta_y^2\) is the transverse Laplace operator, \(\psi_j\) is the envelope of soliton \(j\), \(x\) and \(y\) are the spatial coordinates, \(z\) is the propagation distance, \(\alpha\) is the strength of the nonlinearity, \(I_0\) is the saturation parameter, and \(\epsilon_{2m+1}\), which satisfies \(0 < \epsilon_{2m+1} \ll 1\), is the \((2m + 1)\)-order of the nonlinear loss coefficient [24]. On the left-hand side (LHS) of Eq. (1), the second term corresponds to the second-order dispersion and the third term represents the effects of the saturable nonlinearity. On the right-hand side (RHS) of Eq. (1), the first and second terms describe the effects of intra-beam and inter-beam interaction due to the \((2m + 1)\)-order of the nonlinear loss, respectively. Note that the effect of the generic nonlinear loss on interaction of 1D cubic NLS solitons was uncovered in Ref. [24]. In addition, its specific cases, the weak cubic loss \((m = 1)\) and the weak quintic loss \((m = 2)\), were also investigated in Refs. [23,25,27,28], respectively.

We first discuss the form of the single ideal 2D soliton \(j\) which is the fundamental solution of the following
The soliton solution of Eq. (2) with the velocity vector 

\[ d_j = (d_{j1}, d_{j2}) \]

can be found in the form:

\[
\tilde{\psi}_{j0}(x, y, z) = U_j(x, y_j) \exp(i\mu_j z)
\]

\[ \exp\left[ i\alpha_j + i\chi_j(\tilde{X}_j, \tilde{Y}_j) \right], \]

(3)

where \( x = x - x_{j0} - d_{j1} z, \)
\( y = y - y_{j0} - d_{j2} z, \)
\( \tilde{X}_j = x - x_{j0} - d_{j1} z, \)
\( \tilde{Y}_j = y - y_{j0} - d_{j2} z, \)
\( \chi_j = d_{j1} \tilde{X}_j + d_{j2} \tilde{Y}_j, \)
\( d_{j1} = d_{j1}/2, d_{j2} = d_{j2}/2, (x_{j0}, y_{j0}) \)

is the initial position of soliton \( j, \alpha_j \) is related to the

phase, \( d_{j1} \) and \( d_{j2} \) correspond to the velocity

components in the \( x \) and \( y \) directions, respectively, \( \mu_j \)

is the propagation constant, and \( U_j \) is the localized

real-valued amplitude function. From Eqs. (2) and (3), it can

be shown that the function \( U_j \) satisfies the following

elliptic equation [3, 35]:

\[
\Delta_\bot U_j + \frac{\alpha U_j^3}{1 + U_j^2/I_0} = \mu_j U_j.
\]

(4)

2.2 The 2D soliton dynamics of the single-soliton

propagation

Next, we investigate the effects of the \((2m + 1)\)-order

of the nonlinear loss on the single-soliton propagation

described by the following perturbed equation:

\[
i\partial_z \psi_j + \Delta_\bot \psi_j + \frac{\alpha|\psi_j|^2}{1 + |\psi_j|^2/I_0} \psi_j = -i\epsilon_{2m+1}|\psi_j|^{2m} \psi_j.
\]

(5)

By using an energy balance calculation for Eq. (5), it implies:

\[
\partial_z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_j|^2 \, dx \, dy
\]

\[ = -2\epsilon_{2m+1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_j|^{2m+2} \, dx \, dy. \]

(6)

We assume that the initial envelopes of the 2D solitons

can be expressed in the general form

\[
\psi_{j0}(x, y, 0) = A_j(0)\tilde{\psi}_{j0}(x, y, 0), \]

(7)

where \( A_j(0) \) is the initial amplitude parameter, \( \tilde{\psi}_{j0} \)

(\( x, y, 0 \)) is the fundamental soliton solution of Eq. (2),

that is, \( \tilde{\psi}_{j0}(x, y, 0) \) is given by Eq. (3), and \( j = 1, 2 \).

We note that for an initial envelope of the unperturbed

soliton solution, one can define \( A_j(0) = 1 \), that is

\( \psi_{j0}(x, y, 0) = \tilde{\psi}_{j0}(x, y, 0) \). In the presence of the

nonlinear loss, we look for the solution of Eq. (5) in the

form of

\[
\psi_{j0}(x, y, z) = A_j(z)\tilde{\psi}_{j0}(x, y, z), \]

(8)

where \( A_j(z), 0 < A_j(z) < A_j(0), \) is the amplitude parameter taking into account of the effects of nonlinear

loss for \( z > 0 \), and \( \tilde{\psi}_{j0}(x, y, z) \) is given by Eq. (3). We

substitute the relation for \( \psi_{j0}(x, y, z) \) into Eq. (6) and

apply the standard adiabatic perturbation theory for the

NLS soliton [36]. It then yields:

\[
\frac{d}{dz} \left[ I_{2,j}(z)A_j^2(z) \right] = -2\epsilon_{2m+1}I_{2m+2,j}(z)A_j^{2m+2}(z),
\]

(9)

where

\[
I_{2,j}(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{\psi}_{j0}(x, y, z)|^2 \, dx \, dy
\]

\[ = \int_{-\infty}^{\infty} U_j^2 \, dx \, dy, \]

and

\[
I_{2m+2,j}(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{\psi}_{j0}(x, y, z)|^{2m+2} \, dx \, dy
\]

\[ = \int_{-\infty}^{\infty} U_j^{2m+2} \, dx \, dy. \]

By the definition of \( U_j \), one can obtain that \( I_{2,j}(z) \)

and \( I_{2m+2,j}(z) \) are constants. Solving Eq. (9) on the

interval \([0, z]\), it implies the equation for the amplitude

dynamics of a single soliton as follows:

\[
A_j(z) = \frac{A_j(0)}{\left[ 1 + 2m\epsilon_{2m+1}I_{2m+2,j0}/I_{2,j0}A_j^{2m}(0)z \right]^{1/(2m)}}.
\]

(10)

where \( I_{2,j0} = I_{2,j}(0) \) and \( I_{2m+2,j0} = I_{2m+2,j}(0) \).

Equation (10) describes the effects of the nonlinear

loss on the amplitude parameter of a single 2D soliton.

It also shows that in the leading order of perturbation

effects, the amplitude \( A_1(z) \) decays at order proportional to \( O(z^{-\frac{1}{2m}}) \).
2.3 The collision-induced amplitude dynamics of two 2D solitons

We now study the collision-induced amplitude dynamics in a fast two-soliton collision described by Eq. (1). For this purpose, we assume two solitons are well separated at the initial propagation distance \( z = 0 \) and at the final propagation distance \( z = z_f \) for a complete collision. By deriving the energy balance of Eq. (1), one then obtains:

\[
\partial_z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_j|^2 \, dx \, dy = -2\epsilon_{2m+1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_j|^{2m+2} \, dx \, dy - 2\epsilon_{2m+1} \sum_{k=1}^{m} b_{k,m} J_{k,m}^{(j,l)}, \tag{11}
\]

where \( J_{k,m}^{(j,l)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_l|^{2k} |\psi_j|^{2(m-k)} \, dx \, dy \).

Based on the perturbative calculation approach in Refs. [21, 23, 30], it is useful to look for the solution of Eq. (1) in the form:

\[
\psi_j(x, y, z) = \psi_{j0}(x, y, z) + \phi_j(x, y, z), \tag{12}
\]

where \( \psi_{j0} \) is the single-soliton propagation solution of Eq. (5) and \( \phi_j \) describes a small correction to \( \psi_{j0} \), i.e., the correction is solely due to collision effects. We substitute the relation (12) into Eq. (11) and take into account only leading-order effects, that is, the effects of order of \( \epsilon_{2m+1} \). Therefore, based on the standard adiabatic perturbation theory for the NLS soliton [36], the terms containing \( \phi_j \) on the RHS of the resulting equation can be neglected. It then leads to the following differential equation for soliton 1:

\[
\partial_z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_1|^2 \, dx \, dy = -2\epsilon_{2m+1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_1|^{2m+2} \, dx \, dy - 2\epsilon_{2m+1} \sum_{k=1}^{m} b_{k,m} K_{k,m}, \tag{13}
\]

where \( K_{k,m} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_{20}|^{2k} |\psi_{10}|^{2(m-k)} \, dx \, dy \).

Equation (13) represents the energy balance for soliton 1. The last term on the RHS of Eq. (13) is responsible for the contribution of the interaction term during the collision. We note that when \( \epsilon_{2m+1} = 0 \) then Eq. (13) becomes a conservation law for the energy and the calculations to obtain the equation for soliton 2 are the same. From Eqs. (6) and (13) and noting that \( \psi_{j0} \) satisfies Eq. (6), one then obtains the energy balance equation for soliton 1 via the use of the perturbed single-soliton propagation solution as follows:

\[
\partial_z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_1|^2 \, dx \, dy = -2\epsilon_{2m+1} \sum_{k=1}^{m} b_{k,m} L_{k,m}, \tag{14}
\]

In a fast collision, the collision takes place in a small interval \([z_c - \Delta z_c, z_c + \Delta z_c]\) around \(z_c\), where \(z_c\) is the collision distance, which is the distance at which the maxima of \(|\psi_j(x, y, z)|\) coincide at the same point \((x_0, y_0)\), and \(\Delta z_c\) is the distance along which the envelopes of the colliding solitons overlap \((\Delta z_c \ll 1)\). Integrating over \(z\) of Eq. (14), it implies:

\[
\int_{z_c - \Delta z_c}^{z_c + \Delta z_c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_1|^2 \, dx \, dy \, dz = -2\epsilon_{2m+1} \sum_{k=1}^{m} b_{k,m} L_{k,m}, \tag{15}
\]

where \( L_{k,m} = \int_{z_c - \Delta z_c}^{z_c + \Delta z_c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_{20}|^{2k} |\psi_{10}|^{2(m-k)+2} \, dx \, dy \, dz \).

Let us derive the expression for the collision-induced amplitude shift in a fast two-soliton collision from Eq. (15). These calculations are based on the approximations on the total collision-induced change in the soliton envelope via the use of the perturbed single-soliton solution and the conserved quantity of the unperturbed propagation equation. Let \( \Delta_1 \) and \( \Delta_{10} \) be the integral on the LHS and the first integral on the right-hand side.
of Eq. (15), respectively. That is,

\[
\Delta_1 = \int_{z_c - \Delta z_c}^{z_c + \Delta z_c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_1|^2 \, dx \, dy \, dz,
\]

and

\[
\Delta_{10} = \int_{z_c - \Delta z_c}^{z_c + \Delta z_c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_{10}|^2 \, dx \, dy \, dz.
\]

The expression for \(\Delta_1\) can be expressed in the term of the change in the soliton envelope:

\[
\Delta_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_1(x, y, z_c^+)|^2 \, dx \, dy \\
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_1(x, y, z_c^-)|^2 \, dx \, dy,
\]

where \(z_c^- = z_c - \Delta z_c\) and \(z_c^+ = z_c + \Delta z_c\). Let

\[
\psi_{10}(x, y, z) = U_j(x - x_{j0} - d_jz, y - y_{j0} - d_jz).
\]

We introduce the following approximation:

\[
|\psi_1(x, y, z_c^+)| = \left( A_1(z_c^-) + \Delta A_1^{(s)}(z_c) + \Delta A_1^{(c)} \right) \psi_{10}(x, y, z_c^+), \quad (17)
\]

where \(\Delta A_1^{(c)}\) is the total collision-induced amplitude shift of soliton 1, \(A_j(z_c^-)\) is the limit from the left of \(A_j(z)\) at \(z_c\), and \(\Delta A_1^{(s)}(z_c)\) is the amplitude shift of soliton 1 which is due to the single-soliton propagation from \(z_c^-\) to \(z_c^+\). By the definition of \(\psi_1, \psi_{10},\) and \(U_j\):

\[
|\psi_1(x, y, z_c^+)| = A_1(z_c^-) \psi_{10}(x, y, z_c^+). \quad (18)
\]

Substituting the relations (17) and (18) into Eq. (16), it yields the following key approximation for the total collision-induced change in the soliton envelope:

\[
\Delta_1 = \left( A_1(z_c^-) + \Delta A_1^{(s)}(z_c) + \Delta A_1^{(c)} \right)^2 \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{10}^2(x, y, z_c^+) \, dx \, dy \\
- A_1^2(z_c^-) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{10}^2(x, y, z_c^-) \, dx \, dy. \quad (19)
\]

We note that \(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{10}^2(x, y, z) \, dx \, dy\) is a conserved quantity of the propagation Eq. (5) when \(\epsilon_{2m+1} = 0\).

Therefore, the following relation holds

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{10}^2(x, y, z) \, dx \, dy = I_{2,10}
\]

for all \(z\). Substituting the relation (20) into Eq. (19) and taking into account only leading-order terms, it implies

\[
\Delta_1 = 2A_1(z_c^-) \left( \Delta A_1^{(c)} + \Delta A_1^{(s)}(z_c) \right) I_{2,10}. \quad (21)
\]

On the other hand, \(\Delta_{10}\) can be expressed as

\[
\Delta_{10} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_{10}(x, y, z_c^+)|^2 \, dx \, dy \\
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_{10}(x, y, z_c^-)|^2 \, dx \, dy. \quad (22)
\]

By the definition of \(\psi_{10}\), one can use the approximations \(|\psi_{10}(x, y, z_c^-)| = A_1(z_c^-) \psi_{10}(x, y, z_c^-)\) and

\[
|\psi_{10}(x, y, z_c^+)| = \left( A_1(z_c^-) + \Delta A_1^{(s)}(z_c) \right) \psi_{10}(x, y, z_c^+).
\]

Substituting these relations into Eq. (22) and then expanding the first integrand on the right-hand side while keeping only leading terms, it implies

\[
\Delta_{10} = 2A_1(z_c^-) \Delta A_1^{(s)}(z_c) I_{2,10}. \quad (23)
\]

We substitute Eqs. (21) and (23) into Eq. (15). It arrives at the equation for the collision-induced amplitude dynamics of soliton 1:

\[
A_1(z_c^-) \Delta A_1^{(c)} I_{2,10} = -\epsilon_{2m+1} \sum_{k=1}^{\infty} b_{k,m} M_{k,m}, \quad (24)
\]

where

\[
M_{k,m} = \int_{z_c - \Delta z_c}^{z_c + \Delta z_c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_2^{2k}(z) A_1^{2(m-k)+2}(z) \, dx \, dy \, dz.
\]

Finally, we simplify Eq. (24) by integrating \(M_{k,m}\) using the decompose approximation of the integrand based on the assumption of a fast soliton collision. We note that only functions on the right-hand side of Eq. (24) that contain fast variations in \(z\), which are the factors \(X_j\) and \(Y_j\), are \(U_1\) and \(U_2\). The slow varying amplitudes \(A_1(z)\) and \(A_2(z)\) can be approximated by \(A_1(z_c^-)\) and \(A_2(z_c^-)\), respectively. Therefore, Eq. (24) can be rewritten:

\[
\Delta A_1^{(c)} = -\epsilon_{2m+1} I_{2,10} \sum_{k=1}^{m} b_{k,m} A_2^{2k}(z_c^-) A_1^{2(m-k)+1}(z_c^-) N_{k,m}, \quad (25)
\]
where \( N_{k,m} = \int_{z_{c}^{-}}^{z_{c}^{-} + \Delta z_{c}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_{2}^{2k} U_{1}^{2(m-k)+2} \, dx \, dy \, dz \). Since the integrand on the right-hand side of Eq. (25) is sharply peaked at a small interval about \( z_{c} \), we can extend the limits of this integral to 0 and \( z_{f} \). Therefore, it yields

\[
\Delta A_{1}^{(e)} = -\epsilon_{2m+1}/I_{2,10} \sum_{k=1}^{m} b_{k,m} A_{2}^{2k}(z_{c}^{-}) A_{1}^{2(m-k)+1}(z_{c}^{-}) P_{k,m},
\]

where \( P_{k,m} = \int_{0}^{z_{f}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_{2}^{2k} U_{1}^{2(m-k)+2} \, dx \, dy \, dz \).

Equation (26) represents the collision-induced amplitude shift of two fast 2D solitons propagating in optical media with a saturable nonlinearity and the generic weak nonlinear loss. It shows that the collision-induced amplitude dynamics of two fast 2D solitons with nonlinear loss is independent of the phase of the initial solitons. This behavior is the same as one for fast 1D solitons. Therefore, the current perturbative method can be applied for other types of weak dissipative perturbations, which contribute on the RHS of Eq. (24), in a similar manner.

### 3 Numerical simulations

#### 3.1 Set up the measurements

First, let us describe a collision between two solitons as follows. For simplicity and without loss of generality, we assume the collision occurs at the origin \( O(0,0) \) in the \( xy \)-plane. Solitons 1 and 2, which are located at \( M_{1}(x_{10}, y_{10}) \) and \( M_{2}(x_{20}, y_{20}) \), respectively, are well separated at \( z = 0 \). These two solitons propagate toward \( O(0,0) \) with the velocity vectors \( \mathbf{d}_{1} \) and \( \mathbf{d}_{2} \). As a result, the group velocity difference of the colliding solitons is \( \mathbf{d} = \mathbf{d}_{1} - \mathbf{d}_{2} \). In simulations, \( \mathbf{d}_{1} \) and \( \mathbf{d}_{2} \) are chosen as:

\[
x_{10}/d_{11} = y_{10}/d_{12} = x_{20}/d_{21} = y_{20}/d_{22}.
\]

Therefore, there will be a collision at the origin at the propagation distance \( z_{c} = -x_{10}/d_{11} \). After the full collision, two solitons continue propagating away from \( O(0,0) \) and they are thus well separated at the final propagation distance \( z = z_{f} \). Denoting by \( \theta \) the collision angle between two colliding solitons, one can determine \( \cos \theta = (\mathbf{u}_{1} \cdot \mathbf{u}_{2})/(|\mathbf{u}_{1}| |\mathbf{u}_{2}|) \), where \( \mathbf{u}_{1} = M_{1} \mathbf{O} \) and \( \mathbf{u}_{2} = M_{2} \mathbf{O} \). It then yields \( \cos \theta = \mathbf{d}_{1} \cdot \mathbf{d}_{2}/(|\mathbf{d}_{1}| |\mathbf{d}_{2}|) \).

Second, we define the relative error in the approximation of \( \Delta A_{1}^{(e)} \) by \( |\Delta A_{1}^{(e)(num)} - \Delta A_{1}^{(e)(th)}|/\Delta A_{1}^{(e)(th)} \), where \( \Delta A_{1}^{(e)(th)} \) is calculated from the theoretical prediction of Eq. (26) and \( \Delta A_{1}^{(e)(num)} \) is defined by:

\[
\Delta A_{1}^{(e)(num)} = A_{1}(z_{c}^{+}) - A_{1}(z_{c}^{-}).
\]

In Eq. (28), \( A_{1}(z_{c}^{+}) \) is measured from Eq. (10) and \( A_{1}(z_{c}^{-}) \) is calculated by solving Eq. (9) with \( j = 1 \) on the interval \([z_{c}, z_{f}]\):

\[
A_{1}(z_{c}^{+}) = A_{1}(z_{f}) \left[ 1 - 2m \epsilon_{2m+1}/I_{2m+2,10}/I_{2,10} A_{1}^{2m}(z_{f})(z_{f} - z_{c}) \right]^{1/(2m)},
\]

where \( A_{1}(z_{f}) \) is measured by simulations of Eq. (1).

From Eq. (26), the expression for \( \Delta A_{1}^{(e)(th)} \) can be written as:

\[
\Delta A_{1}^{(e)(th)} = -2\epsilon_{3}/I_{2,10} A_{1}(z_{c}^{-}) A_{2}^{2}(z_{c}^{-}) P_{1,1}
\]
with cubic loss \((m = 1)\) and it is

\[
\Delta A_{1}^{(c)(th)} = -\varepsilon_5 I_{2,10} \left[ 6A_2^2(z_f)A_1(z_f)P_{1,2} + 3A_1^3(z_f)A_1(z_f)P_{2,2} \right]
\]  \( (31) \)

with quintic loss \((m = 2)\). Additionally, we define the relative error in measuring the soliton patterns at the propagation distance \(z\) by \(\left\| \psi^{(th)} - |\psi^{(num)}| \right\| / \|\psi^{(th)}\|\), where \(\|\psi\| = \left[ \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{max}} |\psi|^2 dxdy \right]^{1/2} \), \(\psi^{(num)}(x, y, z)\) is measured by simulations, \(\psi^{(th)}(x, y, z)\) is the theoretical prediction of the soliton pattern at the propagation distance \(z\), and \([x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]\) is the computational spatial domain. We then define the threshold levels \(E_a\) and \(E_s\). The small error domain in measuring the analytic prediction of Eq. \(26\) is determined when the relative errors in measuring \(\Delta A_{1}^{(c)}\) and in measuring the soliton patterns at \(z = z_f\) are less than or equal to \(E_a\) and \(E_s\), respectively. In contrast, the large error domain in measuring the analytic prediction of Eq. \(26\) is determined when the relative error in measuring \(\Delta A_{1}^{(c)}\) is greater than \(E_a\) or the relative error in measuring the soliton patterns at \(z = z_f\) is greater than \(E_s\). In simulations, \(E_a = 0.1\) and \(E_s = 0.04\) are used.

Next, we study the dependence of \(\Delta A_{1}^{(c)}\) on \(\theta\) when the values of \(\theta\) are changed over \([0, \pi]\) while the magnitudes of \(d_1\) and \(d_2\) are constant. One can define the relative change of \(\Delta A_{1}^{(c)}\) due to \(\theta\), for \(0 \leq \theta \leq \pi\), with respect to \(\Delta A_{1,\pi}\) as

\[
p = \frac{\Delta A_{1}^{(c)} - \Delta A_{1,\pi}^{(c)}}{\Delta A_{1,\pi}^{(c)}},
\]  \( (32) \)

where \(\Delta A_{1,\pi}^{(c)}\) is the value of \(\Delta A_{1}^{(c)}\) at \(\theta = \pi\). We note that \(\Delta A_{1,\pi}^{(c)}\) is the smallest value of \(\Delta A_{1}^{(c)}\) over \([0, \pi]\).

To validate Eq. \(26\), we carry out the simulations with Eqs. \(1\) and \(5\) using the split-step Fourier method with the second-order accuracy \([11]\). That is, the errors for numerically solving Eqs. \(1\) and \(5\) are of order \(O(h^3)\), where \(h = \Delta z\) is the propagation step-size \([11]\). As an example, we present the simulation results of Eq. \(1\) for \(\alpha = 1\) and \(I_0 = 1\) with \(m = 1\) and \(m = 2\). The initial conditions of Eq. \(1\) are defined from Eq. \(3\) at \(z = 0\). The ground state \(U_j\) in Eq. \(3\) at any fixed initial distance \(z_0 \geq 0\) is measured by simulations of Eq. \(4\) using the accelerated imaginary-time evolution method \([11,38]\). To implement simulations with Eq. \(4\), we use the input function \(U_j = \text{sech} \left[ (x - x_{j0} - d_{j1}z_0)^2 + (y - y_{j0} - d_{j2}z_0)^2 \right]\). Also we use the input power value of the beam \(j\) as \(P_{j0} = 22.5\). By the simulations of Eq. \(4\), one can obtain the power value of the soliton solution \(j\) as \(P_j = 22.5\) at \(\mu_j = 0.1629\), where \(P_j(\mu_j) = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{max}} U_j^2(X_j, Y_j; \mu_j) dxdy\). Additionally, the length of the computational spatial domain is \(L_x = L_y = 30\pi\) and the number of grid points in \(x\)-domain and in \(y\)-domain is \(N_x = N_y = 2048\).

As a result, the spacing of the grid points is \(\Delta x = \Delta y = L_x/N_x = 0.046\) and \([y_{\min}, y_{\max}] = [y_{\min}, y_{\max}] = [-Lx/2, Lx/2 - \Delta x]\). The propagation step-size is \(\Delta z = 5 \times 10^{-4}\). With this choice of \(\Delta z\), the errors for solving Eqs. \(1\) and \(5\) are thus in the order of \(10^{-10}\). Also, we use the initial phases \(\alpha_j = 0\) and emphasize that the simulation results are independent of the choices of the initial phases \(\alpha_j\).

### 3.2 Simulation results

Before validating Eq. \(26\), we first verify Eq. \(10\) by carrying out simulations for the single-soliton propagation described by Eq. \(5\) for \(j = 1\) with \(m = 1\) and \(m = 2\). The parameters are: \(\varepsilon_{2m+1} = 0.01\), \((x_{10}, y_{10}) = (-10, 9)\), \(d_1 = (d_{11}, d_{12}) = (2, -1.8)\). The final propagation distance is \(z_f = 10\). Figure 1 represents the initial soliton profile and the evolution of its profiles \(|\psi_1(x, y, z)|\) obtained by the simulation of Eq. \(5\) with \(\varepsilon_3 = 0.01\) at the propagation distances of \(z = 2, 4, 6, 8, 10\). The soliton profiles are presented using the color level colormap. In addition, the amplitude parameters \(A_{1}^{(num)}(z)\) and \(A_{1}^{(th)}(z)\) are calculated, where \(A_{1}^{(num)}(z)\) is measured by the simulation of Eq. \(5\) and \(A_{1}^{(th)}(z)\) is calculated from the theoretical prediction with Eq. \(10\). The agreement between the analytic calculations and the simulation results for \(m = 1\) is very good. In fact, the relative error in measuring \(A_{1}(z)\) for \(z \in [0, z_f]\), which is defined by \(|A_{1}^{(num)}(z) - A_{1}^{(th)}(z)|/A_{1}^{(th)}(z)\), is less than \(1.4 \times 10^{-3}\). The relative error in measuring the soliton patterns over \([0, z_f]\) is less than \(0.031\). In addition, by implementing the simulation of Eq. \(5\) with \(\varepsilon_5 = 0.01\) \((m = 2)\), we obtain the excellent agreement between the simulation results and the analytic calculations of the amplitudes and of the soliton patterns. The relative errors in measuring \(A_{1}(z)\) and in measuring the soliton patterns over \([0, z_f]\) are less than \(1.2 \times 10^{-3}\) and \(0.026\), respectively. Moreover, these numerical results also indicate that in average, the amplitude \(A_{1}(z)\) decays at order propor-
tional to \( O(z^{-0.492}) \) with \( m = 1 \) and proportional to \( O(z^{-0.246}) \) with \( m = 2 \), for \( z \in [0, z_f] \). These extensive numerical simulations above firmly validate Eq. (10) and the amplitude decay of order \( O(z^{-\frac{m}{2}}) \).

Second, let us illustrate the collision between two solitons by the simulation of Eq. (1) with cubic loss. We emphasize that the illustration with quintic loss is similar. The parameters are: \( \epsilon_3 = 0.01 \), \((x_{10}, y_{10}) = (-10, 9)\), \((x_{20}, y_{20}) = (9, 8)\), \(d_1 = (20, -18)\), and \(d_2 = (-18, -16)\). Two velocity vectors \(d_1\) and \(d_2\) satisfy the relation (27) with \( z_c = 0.5 \). One can measure \( \cos \theta = -0.1111 \) and \(|d| = 38.0526\). Figure 2a represents the initial soliton profiles \( |\psi_j(x, y, 0)| \) while Fig. 2c, e depicts the soliton profiles \( |\psi_j(x, y, z)| \), which are obtained by the simulation, at the intermediate distance \( z_i = 0.6 > z_c = 0.5 \), as an example, and at the final distance \( z_f = 1 \), respectively. Figure 2b, d, f represents the soliton profiles \( |\psi_j(x, y, z)| \) in form of the level colormap corresponding to the soliton profiles in Fig. 2a, c, e, respectively. The agreement between the analytic predictions and the simulation results is very good. In fact, the relative errors in measuring \( \Delta A_1^{(c)} \) and in measuring the soliton patterns at \( z = z_f \) are 0.02 and 0.009, respectively.

Next, we study the dependence of \( \Delta A_1^{(c)} \) on \(d_1\) and \(d_2\). In simulations, the magnitudes of \(d_1\) and \(d_2\) will be changed while the value of \( \theta \) is constant. The parameters are: \((x_{10}, y_{10}) = (-10, 9)\), \((x_{20}, y_{20}) = (9, 8)\), \(d_1 = (d_{11}, -0.9d_{11})\), \(d_2 = (-0.9d_{11}, -0.8d_{11})\), where \( 2 \leq d_{11} \leq 80 \), and \( z_f = 2z_c \). The velocities \(d_1\) and \(d_2\) satisfy the relation (27) with \( z_c = 10/d_{11} \). One can measure \( \cos \theta = -0.1111 \) and \(|d| = 1.9026d_{11}\). The loss coefficients are \( \epsilon_2m+1 = 0.01 \) and \( \epsilon_2m+1 = 0.02 \) for \( m = 1 \) and \( m = 2 \). The relative errors in the approximation of \( \Delta A_1^{(c)} \) are less than 0.31 for \( 2 \leq d_{11} < 10 \) and less than 0.07 for \( 10 \leq d_{11} \leq 80 \) for \( m = 1 \). They are less than 0.39 for \( 2 \leq d_{11} < 12 \) and less than 0.1 for \( 12 \leq d_{11} \leq 80 \) for \( m = 2 \). The dependence of \( \Delta A_1^{(c)} \) on \(d_1\) and \(d_2\) is depicted in Fig. 3 with \( \epsilon_3 = 0.01 \) (a) and \( \epsilon_5 = 0.01 \) (b). Figure 4 shows the simulation results for a wide range values of \( \epsilon_2m+1 \) and \( d_1 \) with \( m = 1 \) (a) and \( m = 2 \) (b). The blue domain corresponds to the small errors, i.e., the relative errors in measuring \( \Delta A_1^{(c)} \) and in measuring the soliton patterns at \( z = z_f \) are less than or equal to \( E_a \) and \( E_s \), respectively. Besides that the orange domain depicts the large errors, i.e., the relative error in measuring \( \Delta A_1^{(c)} \) is greater than \( E_a \) or the relative error in measuring the soliton patterns at \( z = z_f \) is greater than \( E_s \). Moreover, by implementing the simulations with \( 0.0001 \leq \epsilon_2m+1 \leq 0.05 \) and \( 2 \leq d_{11} \leq 80 \), one can observe the small errors for \( 12 \leq d_{11} \leq 80 \) with \( 0.0001 \leq \epsilon_2m+1 \leq 0.02 \), i.e., for fast collisions with weak nonlinear loss.

Finally, we describe the dependence of \( \Delta A_1^{(c)} \) on \( \theta \). In simulations, the values of \( \theta \) will be changed over the interval \([0, \pi]\) while \(|d_1|\) and \(|d_2|\) are constants. For example, the parameters can be chosen: \((x_{10}, y_{10}) = (18 \cos(9\pi/12), 18 \sin(9\pi/12))\), \((x_{20}, y_{20}) = (2 \cos(k\pi/12), 2 \sin(k\pi/12))\), \(d_{11} = \ldots \).
Fig. 2 (Color online) The initial soliton profiles (a) and the soliton profiles at $z = z_f = 0.6$ (c) and $z = z_f = 1$ (e) in a two-soliton collision obtained by the simulation of Eq. (1) with $\epsilon_3 = 0.01$ and $d_{11} = 20$. The soliton profiles $|\psi_j(x, y, z)|$ of a, c, e by using the level colormap, respectively.

Fig. 3 (Color online) The dependence of $\Delta A_1^{(c)}$ on $d_1$ and $d_2$ with $\epsilon_3 = 0.01$ (a) and $\epsilon_5 = 0.01$ (b). The red circles correspond to the $\Delta A_1^{(c)}$ obtained by simulations of Eq. (1). The solid blue curves represent the analytic prediction $\Delta A_1^{(c)}$ of Eq. (30) for $m = 1$ (a) and of Eq. (31) for $m = 2$ (b).

25, $d_{12} = y_{10}d_{11}/x_{10}$, $d_{21} = x_{20}d_{11}/x_{10}$, $d_{22} = y_{20}d_{11}/x_{10}$, where $k = -3, -2, -1, 0, 1, 2, \ldots, 9$, and $z_f = 2z_c$. Two velocity vectors $d_1$ and $d_2$ satisfy the relation (27) with $z_c = 0.5091$. One can measure $|d_1| = 35.3553$ and $|d_2| = 3.9284$. The agreement between the simulation results and the analytic predictions is very good. In fact, the maximal relative errors in the approximation of $\Delta A_1^{(c)}$ over $[0, \pi]$ are 0.02 and 0.038 for $\epsilon_3 = 0.01$ and $\epsilon_3 = 0.02$, respectively. They are 0.039 for $\epsilon_5 = 0.01$ and 0.071 for $\epsilon_5 = 0.02$. Figure 5 shows the dependence of $\Delta A_1^{(c)}$ on $\theta$ with $\epsilon_{2m+1} = 0.01$ and $\epsilon_{2m+1} = 0.02$ for $m = 1$ (a) and $m = 2$ (b). One can observe that the magnitude of $\Delta A_1^{(c)}$ is smaller for a larger value of $\theta$, i.e., for a faster collision. Figure 6 shows the dependence of the relative change $p$ on $\theta$ with $\epsilon_{2m+1} = 0.01$ and 0.02 for $m = 1$.
Fig. 4 (Color online) The simulation results of Eq. (1) for a wide range values of $\epsilon_{2m+1}$ and $d_j$ with $m = 1$ (a) and $m = 2$ (b). The initial position parameters are $(x_{10}, y_{10}) = (-10, 9)$ and $(x_{20}, y_{20}) = (9, 8)$.

Fig. 5 (Color online) The dependence of $\Delta A^{(c)}_1$ on $\theta$ for $m = 1$ (a) and $m = 2$ (b). The red circles and purple squares correspond to $\Delta A^{(c)}_1$ obtained by simulations of Eq. (1) with $\epsilon_{2m+1} = 0.01$ and $\epsilon_{2m+1} = 0.02$, respectively. The dashed blue and solid brown curves represent the analytic prediction $\Delta A^{(c)}_1$ with $\epsilon_{2m+1} = 0.01$ and $\epsilon_{2m+1} = 0.02$, respectively, of Eq. (30) for $m = 1$ and of Eq. (31) for $m = 2$.

Fig. 6 (Color online) The dependence of $p$, which is measured from Eq. (32), on $\theta$ for $m = 1$ (a) and $m = 2$ (b). The red squares and green circles represent the values of $p$ obtained by simulations of Eq. (1) with $\epsilon_{2m+1} = 0.01$ and $\epsilon_{2m+1} = 0.02$, respectively. The solid blue curves correspond to the theoretical prediction values of $p$.

(a) and for $m = 2$ (b). As can be seen, the relative difference $p$ is independent of the choices of $\epsilon_{2m+1}$ and $m$. The values of $p$ are decreasing from $p_{\text{max}} = 0.25$ at $\theta = 0$ to $p_{\text{min}} = 0$ at $\theta = \pi$. The maximal relative error in calculations of $p$ over $[0, \pi]$ is 0.016 for $m = 1$ and it is 0.026 for $m = 2$.

In summary, the very good agreement between the analytic calculations for $\Delta A^{(c)}_1$ and the simulation results of the perturbed coupled nonlinear Schrödinger model validated our theoretical calculations for $\Delta A^{(c)}_1$.
4 Conclusions

We derived the expressions for the amplitude dynamics of 2D NLS solitons in a fast collision in the saturable nonlinear media with the generic weak \((2m+1)\)-order loss, for any \(m \geq 1\). We first established the single soliton dynamics in the presence of the nonlinear loss. Then, we derived the expressions for the collision-induced amplitude dynamics in a fast collision of two 2D solitons in the presence of the nonlinear loss. Our perturbative method is quite different to the traditional perturbative method derived by Kaup [31, 32]. The previous method was based on the projections of the total collision-induced change in the soliton envelope on the four localized eigenmodes of the linear operator \(\hat{L}\) describing small perturbations about the fundamental 1D NLS soliton, where the unperturbed model is integrable and the ideal soliton solution was used [31, 32]. In fact, in the current paper, the unperturbed \((2+1)D\) NLS equation is nonintegrable. Our perturbative approach was based on the calculations on the energy balance of perturbed solitons, the analysis of the total collision-induced change in the soliton envelope, and the use of the perturbed single-soliton solution with the assumptions of weak nonlinear loss and fast collisions. Consequently, the current method allows us to study the fast soliton collision-induced amplitude dynamics of fast 2D solitons of the nonintegrable model such as the \((2+1)D\) NLS equations with a saturable nonlinearity. The theoretical calculations were confirmed by extensive simulations of the corresponding coupled nonlinear Schrödinger models in the presence of waves in Bose–Einstein condensates.

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

Appendix A: The efficiency of the current perturbation method

In this Appendix, we illustrate that the current perturbation method is robust and simple to study the collision-induced amplitude dynamics in fast collisions of solitons of perturbed NLS equations. More specifically, one can apply the current perturbative approach to simply derive the expression for the collision-induced amplitude shift in a fast collision of two 1D NLS solitons with delayed Raman response in a straightforward manner. This expression has been derived in Ref. [39] by the Taylor expansion and in Ref. [21] by the traditional perturbation technique developed by Kaup [31, 32] for 1D NLS solitons.

For this purpose, we consider a fast collision between two solitons under the framework of coupled \((1+1)D\) cubic NLS equations with the delayed Raman response [21, 40]:

\[
\begin{align*}
    i \partial_t \psi_j + \partial^2_z \psi_j + 2 |\psi_j|^2 \psi_j + 4 |\psi_l|^2 \psi_j \\
    &= -\epsilon_R |\psi_j|^2 - \epsilon_R |\psi_j| |\psi_l|^2 \\
    &\quad - \epsilon_R |\psi_j| |\psi_l|^2 
\end{align*}
\]

where \(\epsilon_R\) is the Raman coefficient, \(0 < \epsilon_R \ll 1\), \(1 \leq j, l \leq 2\), and \(j \neq l\). The first term on the RHS of equation (A1) describes the Raman-induced intra-pulse interaction while the second and third terms describe the Raman-induced inter-pulse interaction. We note that the unperturbed NLS equation \(i \partial_t \psi_j + \partial^2_z \psi_j + 2 |\psi_j|^2 \psi_j = 0\) has the fundamental soliton solution

\[\begin{align*}
    \psi &= \hat{\psi} \exp(-\frac{1}{2} |\hat{\psi}|^2) \\
    \hat{\psi} &= \sqrt{|l|^2 - |j|^2} \\
    l &= \sqrt{|l|^2} \\
    j &= \sqrt{|j|^2} \\
    \end{align*}\]
Collision-induced amplitude dynamics of fast 2D solitons

\[ \psi_{cs,j}(t, z) = \Psi_{cs,j}(z) \exp(i\chi_j), \]

where \[ \Psi_{cs,j}(t, z) = \frac{\eta_j}{\cosh(x_j)}, \]

\[ x_j = \eta_j(t - y_j - 2\beta_j z), \quad \chi_j = \alpha_j + \beta_j(t - y_j) + (\eta_j^2 - \beta_j^2)z, \]

and parameters \( \eta_j, \beta_j, \alpha_j, \) and \( y_j \) are related to the amplitude, frequency, phase, and position of the soliton \( j \), respectively. Similarly to Ref. [21], we assume that \( 1/|\beta| \ll 1 \) with \( \beta = \beta_2 - \beta_1 \) and that two solitons are well separated at the initial propagation distance \( z = 0 \) and at the final distance \( z = z_f \). One can look for the solution of Eq. (A1) in the following equation:

\[ i\partial_z \psi_j + \partial^2_t \psi_j + 2|\psi_j|^2\psi_j = -\epsilon_R \psi_j \partial_t |\psi_j|^2, \quad \text{if} \quad \phi_{c,j}(t, z) = \Phi_{c,j}(x_{\beta 0}) \exp(i\chi_{\beta 0}) \]

is the single-soliton propagation solution of Eq. (A1) in the absence of inter-pulse interaction terms, i.e., \( \psi_{c,0} \) satisfies the following equation:

\[ i\partial_z \psi_j + \partial^2_t \psi_j + 2|\psi_j|^2\psi_j = -\epsilon_R \psi_j \partial_t |\psi_j|^2, \quad \text{if} \quad \phi_{c,j}(t, z) = \Phi_{c,j}(x_{\beta 0}) \exp(i\chi_{\beta 0}) \]

We now apply the current perturbation technique to calculate the collision-induced amplitude shift in a fast collision of two solitons described by Eq. (A1). We first perform the calculations for the energy balance of Eq. (A1). It implies:

\[ i\partial_z \int_{-\infty}^{\infty} |\psi_j|^2 dt = -\epsilon_R \int_{-\infty}^{\infty} \psi_j \psi_j^* \partial_t (\psi_j \psi_j^*) dt + \epsilon_R \int_{-\infty}^{\infty} \psi_j^* \psi_j \partial_t (\psi_j \psi_j^*) dt. \quad \text{(A4)} \]

Equation (A4) can be written as:

\[ i\partial_z \int_{-\infty}^{\infty} |\psi_j|^2 dt = -\epsilon_R \int_{-\infty}^{\infty} |\psi_j|^2 (C_l - C_j^*) dt \]

where \( C_k = \psi_k \partial_t (\psi_k^*) \) with \( k = l, j \). By the definition of \( \psi_{c,k} \), it implies \( C_k = \Psi_{c,k} \partial_t \Psi_{c,k} - i\beta_k \Phi^2_{c,k} \), where \( \Psi_{c,k} = \Psi_{c,k0} + \Phi_{c,k} \). Substituting the relation for \( C_k \) into Eq. (A5) and using the adiabatic perturbation theory for the NLS soliton with concentrating only on the leading-order effects of the collision, it implies the energy balance equation for soliton 1:

\[ \partial_z \int_{-\infty}^{\infty} \psi^2_{c,1} dt = 2\epsilon_R \beta \int_{-\infty}^{\infty} \psi^2_{c,10} \psi^2_{c,20} dt. \quad \text{(A6)} \]

Integrating Eq. (A6) with respect to \( z \) over the collision-interval \([z_c - \Delta z_c, z_c + \Delta z_c]\), it yields

\[ \int_{z_c - \Delta z_c}^{z_c + \Delta z_c} \partial_z \int_{-\infty}^{\infty} \psi^2_{c,1} dz = 2\epsilon_R \beta M, \quad \text{(A7)} \]

where \( M = \int_{z_c - \Delta z_c}^{z_c + \Delta z_c} \int_{-\infty}^{\infty} \psi^2_{c,10}(t, z) \psi^2_{c,20}(t, z) dz \). By the definition of \( \psi_{c,1}(t, z) \) and \( \psi_{c,10}(t, z) \), one can use the approximation \( \psi_{c,1}(t, z_c^-) \simeq \psi_{c,10}(t, z_c^-) \). Equation (A7) then leads to

\[ \int_{-\infty}^{\infty} \psi^2_{c,1}(t, z_c^+) dt - \int_{-\infty}^{\infty} \psi^2_{c,10}(t, z_c^-) dt = 2\epsilon_R \beta M. \quad \text{(A8)} \]

We note that

\[ \int_{-\infty}^{\infty} \psi^2_{c,j0}(t, z) dt = 2\eta_{j0}(z), \quad \text{(A9)} \]

where \( \eta_{j0}(z) \) is the amplitude parameter of \( \psi_{c,j0} \) in the presence of the delayed Raman response. By using the standard adiabatic perturbation theory, one can obtain \( \eta_{j0}(z) = \eta_{j0}(0) \). On the other hand, \( \psi_{c,1}(t, z_c^-) \) can be expressed in the following manner:

\[ \int_{-\infty}^{\infty} \psi^2_{c,1}(t, z_c^+) dt = \int_{-\infty}^{\infty} \frac{\eta^2_{1}(z_c^+)}{\cosh^2[x_1(t, z_c^+)]} dt = 2\eta_1(z_c^+), \quad \text{(A10)} \]

where \( \eta_1(z_c^+) \simeq \eta_1(z_c^-) + \Delta \eta_1^{(c)} \simeq \eta_{10}(z_c^-) + \Delta \eta_1^{(c)} \)

and \( \Delta \eta_1^{(c)} \) is the total collision-induced amplitude shift of soliton 1. Next, we calculate the integral \( M \) by using the algebra approximations which were used to calculate the integral \( M_{k,m} \) in Eq. (24). That is, one can take into account only the fast dependence of \( \psi_{c,j0} \) on \( z \),
which is the factor \( v_j = t - y_j - 2\beta_j z \), and approximate other slow varying terms over \( [z_c - \Delta z_c, z_c + \Delta z_c] \) by their values at \( z_c \). This approximation of \( \Psi_{c,j0}(t, z) \) is denoted by \( \hat{\Psi}_{c,j0}(v_j, z_c) \). Moreover, since the integrand of \( M \) is sharply peaked at a small interval \( [z_c - \Delta z_c, z_c + \Delta z_c] \) about \( z_c \), the limits of the integral \( M \) can be extended to \(-\infty\) and \( \infty \). By changing the integration variable with \( v_j = t - y_j - 2\beta_j z \), it implies

\[
M = \frac{1}{2|\beta|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\Psi}_{c,20}(v_2, z_c) \hat{\Psi}_{c,10}(v_1, z_c)dv_1dv_2.
\]

Note that

\[
\int_{-\infty}^{\infty} \hat{\Psi}_{c,j0}(v_j, z_c)dv_j = \int_{-\infty}^{\infty} \Psi_{c,j0}(t, z_c)dt.
\]

It leads to:

\[
M = \frac{1}{2|\beta|} \int_{-\infty}^{\infty} \Psi_{c,20}(t, z_c)dt \int_{-\infty}^{\infty} \Psi_{c,10}(t, z_c)dt. \tag{A11}
\]

Substituting Eqs. (A9), (A10), and (A11) into Eq. (A9), it arrives at the equation for energy exchange:

\[
\Delta \eta^{(c)} = 2\epsilon_c \sqrt{\gamma} \eta_{10}\eta_{20}. \tag{A12}
\]

Equation (A12) is in the same form with Eq. (20) in Ref. [21] which was originally based on the perturbation technique developed by Kaup for 1D NLS solitons \([31,32]\). In addition, we note that one can apply the current perturbative approach to simply derive the expression for the collision-induced amplitude shift in a fast collision of two 1D NLS solitons with nonlinear loss in a straightforward manner.

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