SURFACE DENSITIES IN GENERAL RELATIVITY

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ABSTRACT

In this lecture we deal with the construction of surface densities for the angular momentum of the sources of asymptotically flat vacuum stationary axisymmetric spacetimes. These sources arise from the discontinuities of the twist potential. The result will be applied to the Kerr metric to obtain an integrable density which can be viewed as the regularized version of the density obtained using other formalisms.

1. Introduction

The construction of compact inner sources for stationary vacuum solutions of the Einstein equations is one of the challenges of general relativity. So far nobody has been able to match a physically reasonable matter spacetime to a nonstatic vacuum exterior. In this lecture we shall be less ambitious and restrict ourselves to shells that could act as minimal sources and calculate the surface densities for dipolar physical quantities such as the angular momentum. There is already a previous formalism due to Israel [1] which is grounded on the thin layer theory and Lanczos jump conditions [2]. Here we shall develop a different approach which is closer to the classical potential theory [3].

In order to accomplish this task it will be shown in section 2 a classical calculation of the magnetic moment density, which will be useful as a toy model for subsequent relativistic results. This will be generalized for the angular momentum in general relativity in section 3 and the result will be applied to the Kerr metric [4] in section 4 [5]. A discussion of the results is provided at the end of this essay.

2. Classical dipolar surface densities

In this section it is included a construction for obtaining classical densities of dipole moment from the discontinuities of the magnetic scalar potential. This result is consistent with the classical potential theory [3] and allows the calculation of sheets of magnetic dipoles as sources for the magnetostatic field.

For this purpose we need review both descriptions of the magnetostatic field in vacuum:
\[
\mathbf{B} = -\nabla V = \nabla \times \mathbf{A}
\] (1)

It is well known that the field can always be written as the rotor of the potential vector \(\mathbf{A}\), since the divergence of \(\mathbf{B}\) is zero. On the other hand the field can only be expressed as the gradient of a scalar potential \(V\) out of the sources and if the magnetic field is static. If the topology of the source happened not to be trivial, (think, for instance, of a ring of electric current) the scalar potential could not be defined as a continuous function in the vacuum surrounding the source.

The following formulae from integral calculus will also be useful:

\[
\int_{\Omega} d^3x \nabla V = \int_{\partial \Omega} dS \mathbf{n} \cdot V
\] (2)

\[
\int_{\Omega} d^3x \nabla \times \mathbf{A} = \int_{\partial \Omega} dS \mathbf{n} \times \mathbf{A}
\] (3)

These expressions relate integrals in a volume \(\Omega\) with integrals on its boundary. Although these formulae resemble much the Stokes theorem, they are not a direct consequence of it, since the Green identities (which are metric-dependent) are needed in their derivation. Therefore they rely much on the fact that the spacetime is flat and it is not straightforward to generalize them.

Let us assume that the magnetic field behaves at great distances from its source as the one produced by a magnetic moment \(M\) in the direction of \(z\). This means that the potentials can be written as:

\[
V = \frac{M \cos \theta}{r^2} + 0(r^{-3}) \quad A = \frac{M \sin \theta}{r^2} \mathbf{u}_z + 0(r^{-3})
\] (4)

The integral of the difference of both expressions for \(\mathbf{B}\) will be obviously zero:

\[
0 = \int_{\mathbb{R}^3} d^3x (\nabla \times \mathbf{A} + \nabla V) = \int_{\partial \mathbb{R}^3} dS (\mathbf{n} \times \mathbf{A} + \mathbf{n} V)
\] (5)

Let us assume that the scalar potential is discontinuous across a closed surface \(S\): The integral must be split into two pieces at \(S\) and this surface has to be taken as a part of the boundary, as well as the sphere at infinity.

The integral at infinity is straightforward to calculate since the asymptotic values of the potentials are known. We are left then with the following integral:

\[
M = \frac{1}{4\pi} \int_{\mathbb{R}^3} dS [V] \mathbf{n} \cdot \mathbf{u}_z
\] (6)

where \([V]\) denotes the jump of the scalar potential and \(\mathbf{n}\) is the outward unitary normal to the surface.

According to potential theory we can interpret the integrand as the magnetic moment surface density of the source:

\[
\sigma_M = \frac{1}{4\pi} [V] \mathbf{n} \cdot \mathbf{u}_z
\] (7)
3. Relativistic dipole densities

In order to generalize the previous expression to general relativity and apply it to angular momentum \([5]\), we shall make use of some results included in \([6]\) to obtain two different expressions for the rotation vector \(\omega\) of the congruence of worldlines of constant spatial coordinates (that is, the dragging of inertiels):

\[
\omega = -f^{-1} d\chi = -\rho^{-1} f \ast dA
\]

where \(\chi\) is the twist potential \([7]\), \(\ast\) stands for the Hodge dual in the space orthogonal to the orbits of the Killing vectors and the other functions can be read from the general expression for a stationary axisymmetric vacuum in Weyl coordinates:

\[
d s^2 = -f (d t - A d \phi)^2 + f^{-1}[e^{2k}(d \rho^2 + d z^2) + \rho^2 d \phi^2]
\]

Asymptotic flatness will be imposed on the metric functions and the twist potential in terms of the mass, \(m\), and angular momentum, \(J\), of the source in an adequate system of coordinates:

\[
f = 1 - \frac{2m}{r} + O(r^{-2}) \quad k = O(r^{-2})
\]

\[
A = -\frac{2J \sin^2 \theta}{r} + O(r^{-2}) \quad \chi = -\frac{2J \cos \theta}{r^2} + O(r^{-3})
\]

As we did in the previous section, we can calculate the following integral over the metric space \((V_3, g)\) orthogonal to the hipersurfaces of constant time:

\[
0 = \int_{V_3} \sqrt{3} g f^{-1/2} < f^{-1} d\chi - \rho^{-1} f \ast dA, dZ > dx^1 dx^2 dx^3
\]

The function \(Z\) involved in the scalar product is defined by the following elliptic differential equation:

\[
\partial_{\mu}(\sqrt{3}g f^{-3/2}g^{\mu \nu} \partial_{\nu} Z) = 0
\]

with boundary condition \(Z \sim r \cos \theta\) at infinity.

Under these conditions the previous integral can be reduced to a surface integral over the boundary of \(V_3\), that will be again the sphere at infinity and a surface \(S\) where \(\chi\) is assumed to be discontinuous. The final expression is rather similar to the one obtained previously for the magnetic moment:

\[
J = \int_S dS \sigma_J \quad \sigma_J = -\frac{1}{8 \pi} [\chi] f^{-3/2} g^{\mu \nu} n_\mu \partial_\nu Z
\]

where we denote by \([\chi]\) the jump of the twist potential across \(S\).
4. An example: The Kerr metric

Finally we shall apply the previous result to a solution of astrophysical interest, the Kerr metric:

\[ ds^2 = -(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta})(dt + \frac{2mar \sin^2 \theta}{r^2 - 2mr + a^2 \cos^2 \theta} d\phi)^2 + \]

\[ + (1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta})^{-1}\{(r^2 - 2mr + a^2) \sin^2 \theta d\phi^2 + (r^2 - 2mr + a^2 \cos^2 \theta)(\frac{dr^2}{r^2 - 2mr + a^2} + d\theta^2)\} \]  

(15)

whose twist potential is given by the following expression:

\[ \chi = -\frac{2 ma \cos \theta}{r^2 + a^2 \cos^2 \theta} \]  

(16)

This potential can be shown to be discontinuous [5] across the disk \( r = 0, \theta \in [0, \pi/2] \) if we identify events with collatitude \( \theta \) with those with \( \pi - \theta \) on the disk, as it is done in [8] to avoid the inclusion of regions with negative radius:

\[ [\chi]_{r=0} = -\frac{4 m}{a \cos \theta} \]  

(17)

The disk is flat and its surface element and unitary normal are given by the following expressions:

\[ ds^2 = a^2(\cos^2 \theta d\theta^2 + \sin^2 \theta d\phi^2) \quad n = \frac{1}{\cos \theta} \partial_r \]

(18)

Therefore we only need the required function \( Z \) to calculate the angular momentum surface density of the source hidden in \( S \). A solution for (13) satisfying the boundary condition at infinity is:

\[ Z = (r - 3m) \cos \theta + \frac{2 a^2 m \cos^3 \theta}{r^2 + a^2 \cos^2 \theta} \]  

(19)

The surface density constructed for the Kerr metric according to (14) is then:

\[ \sigma_J = \frac{m}{2\pi a \cos \theta} \]  

(20)

And if we integrate it over the disk \( r = 0 \), it yields the correct result for the angular momentum of the Kerr metric:
$J = ma$ (21)

5. Discussion

If we compare the angular momentum density for the disk $r = 0$ obtained with this new formalism with the one calculated by Israel in [8], we notice that they are somewhat different. In that reference, the surface density is non-integrable:

$$\sigma = -\frac{m \sin^2 \theta}{4 \pi a \cos^3 \theta}$$ (22)

and the singular ring has to be included [9] to correct the result. Therefore, what we have calculated is a regularized version of Israel’s result.

This is not unusual in potential theory: Think for instance of the magnetic field generated by a ring of constant electrical current. If the formula (7) is used, one obtains a uniform sheet of magnetic dipoles instead.

The extension of these results is discussed in references [10].

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