CONTINUITY OF SOLUTIONS TO SPACE-VARYING
POINTWISE LINEAR ELLIPTIC EQUATIONS

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Abstract. We consider pointwise linear elliptic equations of the form $L_x u_x = \eta_x$
on a smooth compact manifold where the operators $L_x$ are in divergence form
with real, bounded, measurable coefficients that vary in the space variable $x$.
We establish $L^2$-continuity of the solutions at $x$ whenever the coefficients of $L_x$
are $L^\infty$-continuous at $x$ and the initial datum is $L^2$-continuous at $x$. This is
obtained by reducing the continuity of solutions to a homogeneous Kato square
root problem. As an application, we consider a time evolving family of metrics
$g_t$ that is tangential to the Ricci flow almost-everywhere along geodesics when
starting with a smooth initial metric. Under the assumption that our initial metric
is a rough metric on $\mathcal{M}$ with a $C^1$ heat kernel on a “non-singular” nonempty open
subset $\mathcal{N}$, we show that $x \mapsto g_t(x)$ is continuous whenever $x \in \mathcal{N}$.

Contents

1. Introduction 1
Acknowledgements 4
2. The structure and solutions of the equation 4
3. An application to a geometric flow 5
4. Proof of the theorem 9
References 14

1. INTRODUCTION

The object of this paper is to consider the continuity of solutions to certain linear
elliptic partial differential equations, where the differential operators themselves vary
from point to point. To fix our setting, let $\mathcal{M}$ be a smooth compact Riemannian
manifold, and $g$ a smooth metric. Near some point $x_0 \in \mathcal{M}$, we fix an open set $U_0$
containing $x_0$. We assume that $U_0 \ni x \mapsto L_x$, are space-varying elliptic, second-
order divergence form operators with real, bounded, measurable coefficients. The
equation at the centre of our study is the following pointwise linear problem

$$(PE) \quad L_x u_x = \eta_x$$

for suitable source data $\eta_x \in L^2(\mathcal{M})$. Our goal is to establish the continuity of
solutions $x \mapsto u_x$ (in $L^2(\mathcal{M})$) under sufficiently general hypotheses on $x \mapsto L_x$ and
$x \mapsto \eta_x$.

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problem.
There are abundant equations of the form (PE) that arise naturally. An important and large class of such equations arise as continuity equations. These equations are typically of the form

$$\text{(CE)} \quad - \text{div}_{g,y} f_x(y) \nabla u_{x,v}(y) = d_x(f_x(y))(v),$$

where $\gamma : I \to M$ is a smooth curve, $\gamma(0) = x$ and $\dot{\gamma}(0) = v$, and where this equation holds in a suitable weak sense in $y$. These equations play an important role in geometry, and more recently in mass transport and the geometry of measure metric spaces. See the book [19] by Villani, the paper [3] by Ambrosio and Trevisan, and references within.

The operators $L_x$ have the added complication that their domain may vary as the point $x$ varies. That being said, a redeeming quality is that they facilitate a certain disintegration. That is, considerations in $x$ (such as continuity and differentiability), can be obtained via weak solutions in $y$. This structural feature facilitates attack by techniques from operator theory and harmonic analysis as we demonstrate in this paper.

A very particular instance of the continuity equation that has been a core motivation is where, in the equation (CE), the term $f_x(y) = \rho^t(x,y)$, the heat kernel associated to the Laplacian $\Delta_g$. In this situation, Gigli and Mantegazza in [11] define a metric tensor $g_t(x)(v,u) = \langle L_x u_{x,v}, u_{x,u} \rangle$ for vectors $u,v \in T_x M$. The regularity of the metric is then regularity in $x$, and for an initial smooth metric, the aforementioned authors show that this evolving family of metrics are smooth. More interestingly, they demonstrate that

$$\partial_t g_t(\dot{\gamma}(s), \dot{\gamma}(s))|_{t=0} = -2\text{Ric}_g(\dot{\gamma}(s), \dot{\gamma}(s)),$$

for almost-every $s$ along geodesics $\gamma$. That is, this flow $g_t$ is tangential to the Ricci flow almost-everywhere along geodesics.

In [7], Bandara, Lakzian and Munn study a generalisation of this flow by considering divergence form elliptic equations with bounded measurable coefficients. They obtain regularity properties for $g_t$ when the heat kernel is Lipschitz and improves to a $C^k$ map ($k \geq 2$) on some non-empty open set in the manifold. Their study was motivated by attempting to describe the evolution of geometric conical singularities as well as other singular spaces. As an application we return to this work and consider the case when $k = 1$.

To describe the main theorem of this paper, let us give an account of some useful terminology. We assume that $L_x$ are defined through a space-varying symmetric form

$$J_x[u,v] = \langle A_x \nabla u, \nabla v \rangle,$$

where each $A_x$ is a bounded, measurable, symmetric $(1,1)$ tensor field which is elliptic at $x$: there exist $\kappa_x > 0$ such that $J_x[u,u] \geq \kappa_x \|\nabla u\|^2$.

Next, let us be precise about the notion of $L^p$-continuity. We say that $x \mapsto u_x$ is $L^p$-continuous if, given an $\varepsilon > 0$, there exists an open set $V_{x,\varepsilon}$ containing $x$ such that, whenever $y \in V_{x,\varepsilon}$, we have that $\|u_y - u_x\|_{L^p} < \varepsilon$. With this in mind, we showcase our main theorem.

**Theorem 1.1.** Let $M$ be a smooth manifold and $g$ a smooth metric. At $x \in M$ suppose that $x \mapsto A_x$ are real, symmetric, elliptic, bounded measurable coefficients
that are $L^\infty$-continuous at $x$, and that $x \mapsto \eta_x$ is $L^2$-continuous at $x$. If $x \mapsto u_x$ solves (PE) at $x$, then $x \mapsto u_x$ is $L^2$-continuous at $x$.

As aforementioned, a complication that arises in proving this theorem is that domains $D(L_x)$ may vary with $x$. However, since the solutions $x \mapsto u_x$ live at the level of the resolvent of $L_x$, there is hope to reduce this problem to the difference of its square root, which incidentally has the fixed domain $W^{1,2}(\mathcal{M})$. As a means to this end, we make connections between the study of the $L^2$-continuity of these solutions to solving a homogeneous Kato square root problem.

Let $B$ be complex and in general, non-symmetric coefficients and let $J_B[u,v] = \langle B\nabla u, \nabla v \rangle$ whenever $u, v \in W^{1,2}(\mathcal{M})$. Suppose that there exists $\kappa > 0$ such that $\Re J_B[u,u] \geq \kappa \|\nabla u\|$. Then, the Lax-Milgram theorem yields a closed, densely-defined operator $L_B u = -\text{div}_g B\nabla u$. The homogeneous Kato square root problem is to assert that $D(\sqrt{-\text{div}_g B\nabla}) = W^{1,2}(\mathcal{M})$ with the estimate $\|\sqrt{-\text{div}_g B\nabla} u\| \simeq \|\nabla u\|$.

The Kato square root problem on $\mathbb{R}^n$ is the case $\mathcal{M} = \mathbb{R}^n$ and this conjecture resisted resolution for almost forty years before it was finally settled in 2002 by Auscher, Hoffman, Lacey, McIntosh and Tchamitchian in [4]. Later, this problem was rephrased from a first-order point of view by Axelsson, Keith, and McIntosh in [5]. This seminal paper contained the first Kato square root result for compact manifolds, but the operator in consideration was inhomogeneous.

In the direction of non-compact manifolds, this approach was subsequently used by Morris in [14] to solve a similar inhomogeneous problem on Euclidean submanifolds. Later, in the intrinsic geometric setting, this problem was solved by McIntosh and the author in [8] on smooth manifolds (possibly non-compact) assuming a lower bound on injectivity radius and a bound on Ricci curvature. Again, these results were for inhomogeneous operators and are unsuitable for our setting where we deal with the homogeneous kind. In §4, we use the framework and other results in [8] to solve the homogeneous problem.

The solution to the homogeneous Kato square root problem is relevant to us for the following reason. Underpinning the Kato square root estimate is a functional calculus and due to the fact that we allow for complex coefficients, we obtain holomorphic dependency of this calculus. This, in turn, provides us with Lipschitz estimates for small perturbations of the (non-linear) operator $B \mapsto \sqrt{-\text{div}_g B\nabla}$. This is the crucial estimate that yields the continuity result in our main theorem.

To demonstrate the usefulness of our results, we give an application of Theorem 1.1 to the aforementioned geometric flow introduced by Gigli and Mantegazza. In §3, we demonstrate under a very weak hypothesis that this flow is continuous. We remark that this is the first instance known to us where the Kato square root problem has been used in the context of geometric flows. We hope that this paper provides an impetus to further investigate the relevance of Kato square root results to geometry,
particularly given the increasing prevalence of the continuity equation in geometric problems.

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2. The structure and solutions of the equation

Throughout this paper, let us fix the manifold $\mathcal{M}$ to be a smooth, compact manifold and, unless otherwise stated, let $g$ be a smooth Riemannian metric. We regard $\nabla : W^{1,2}(\mathcal{M}) \subset L^2(\mathcal{M}) \rightarrow L^2(\mathcal{T}^*\mathcal{M})$ to be the closed, densely-defined extension of the exterior derivative on functions with domain $W^{1,2}(\mathcal{M})$, the first $L^2$-Sobolev space on $\mathcal{M}$. Moreover, we let $\text{div}_g = -\nabla^*$, with domain $D(\text{div}_g) \subset L^2(\mathcal{T}^*\mathcal{M})$.

Indeed, operator theory yields that this is a densely-defined and closed operator (see, for instance, Theorem 5.29 in [13] by Kato). The $L^2$-Laplacian on $(\mathcal{M}, g)$ is then $\Delta_g = -\text{div}_g \nabla$ which can easily be checked to be a non-negative self-adjoint operator with energy $E[u] = \|\nabla u\|^2$.

In their paper [7], the authors prove existence and uniqueness to elliptic problems of the form

\[(E) \quad L_A u = - \text{div}_g A \nabla u = f,\]

for suitable source data $f \in L^2(\mathcal{M})$, where the coefficients $A$ are symmetric, bounded, measurable and for which there exists a $\kappa > 0$ satisfying $\langle Au, u \rangle \geq \kappa \|u\|^2$. The key to relating this equation to (PE) is that, the source data $f$ can be chosen independent of the coefficients $A$.

The operator $L_A$ is self-adjoint on the domain $D(L_A)$ supplied via the Lax-Milgram theorem by considering the symmetric form $J_A[u, v] = \langle A \nabla u, \nabla v \rangle$ whenever $u, v \in W^{1,2}(\mathcal{M})$. Since the coefficients are symmetric, we are able to write $J_A[u, v] = \langle \sqrt{L_A} u, \sqrt{L_A} v \rangle$. By the operator theory of self-adjoint operators, we obtain that $L^2(\mathcal{M}) = \mathcal{N}(L_A) \oplus \mathcal{R}(L_A)$, where by $\mathcal{N}(L_A)$ and $\mathcal{R}(L_A)$, we denote the null space and range of $L_A$ respectively. Similarly, $L^2(\mathcal{M}) = \mathcal{N}(\sqrt{L_A}) \oplus \mathcal{R}(\sqrt{L_A})$. See, for instance, the paper [9] by Cowling, Doust, McIntosh and Yagi.

First, we note that, due to the divergence structure of this equation, an easy operator theory argument yields $\mathcal{N}(L_A) = \mathcal{N}(\nabla) = \mathcal{N}(\sqrt{L_A})$. The characterisation of $\mathcal{R}(L_A)$ independent of $L_A$ rests on the fact that, by the compactness of $\mathcal{M}$ and smoothness of $g$, there exists a Poincaré inequality of the form

\[(P) \quad \|u - u_{\mathcal{M}, g}\|_{L^2} \leq C\|\nabla u\|_{L^2},\]
where \( u_{\mathcal{M},g} = \int_{\mathcal{M}} u \, d\mu_g \) (see, for instance Theorem 2.10 in [12] by Hebey). The constant \( C \) can be taken to be \( \lambda_1(\mathcal{M},g) \), the lowest non-zero eigenvalue of the Laplacian \( \Delta_g \) of \((\mathcal{M},g)\). The space \( \mathcal{R}(L_A) \) and \( \mathcal{R}(\sqrt{L_A}) \) can then be characterised as the set
\[
\mathcal{R} = \left\{ u \in L^2(\mathcal{M}) : \int_{\mathcal{M}} u \, d\mu_g = 0 \right\}.
\]
A proof of this can be found as Proposition 4.1 in [7].

Recall that, again as a consequence of the fact that \((\mathcal{M},g)\) is smooth and compact, the embedding \( E : W^{1,2}(\mathcal{M}) \to L^2(\mathcal{M}) \) is compact (see Theorem 2.9 in [12]). In Proposition 4.4 in [7], the authors use this fact to show that the spectrum of \( L_A \) is discrete, i.e., \( \sigma(L_A) = \{ 0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \} \). Coupled with the Poincaré inequality, we can obtain that the operator exhibits a spectral gap between the zero and the first-nonzero eigenvalues. That is, \( \lambda_0 < \lambda_1 \). Moreover, \( \kappa \lambda_1(\mathcal{M},g) \leq \lambda_1 \).

It is a fact from operator theory that the operator \( L_A \) preserves the subspaces \( \mathcal{N}(L_A) \) and \( \mathcal{R}(L_A) \). Consequently, the operator \( L_A^{-1} = L_A|_{\mathcal{R}(L_A)} \) has spectrum \( \sigma(L_A^{-1}) = \{ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \} \). Collating these facts together, we obtain the following.

**Theorem 2.1.** For every \( f \in L^2(\mathcal{M}) \) satisfying \( \int_{\mathcal{M}} f \, d\mu_g = 0 \), we obtain a unique solution \( u \in D(L_A) \subset W^{1,2}(\mathcal{M}) \) with \( \int_{\mathcal{M}} u \, d\mu_g = 0 \) to the equation \( L_A u = f \). This solution is given by \( u = (L_A^{-1} f) \).

For the purposes of legibility, we write \( L_A^{-1} \) in place of \( \left( L_A^{-1} \right)^{-1} \).

### 3. An Application to a Geometric Flow

In this section, we describe an application of Theorem 1.1 to a geometric flow first proposed by Gigli and Mantegazza in [11]. In their paper, they consider solving the continuity equation
\[(\text{GMC}) \quad - \text{div}_{g,y} \rho_t^g(x,y) \nabla \varphi_{t,x,v}(y) = d_x(\rho_t^g(x,y))(v),\]
for each fixed \( x \), where \( \rho_t^g \) is the heat kernel of \( \Delta_g \), \( \text{div}_{g,y} \) denotes the divergence operator acting on the variable \( y \), where \( v \in T_x\mathcal{M} \), and \( d_x(\rho_t^g(x,y))(v) \) is the directional derivative of \( \rho_t^g(x,y) \) in the variable \( x \) in the direction \( v \). They define a new family of metrics evolving in time by the expression
\[(\text{GM}) \quad g_t(x)(u,v) = \int_{\mathcal{M}} g(y)(\nabla \varphi_{t,x,u}(y), \nabla \varphi_{t,x,v}(y)) \, \rho_t^g(x,y) \, d\mu_g(y).\]

As aforementioned, this flow is of importance since it is tangential (a.e. along geodesics) to the Ricci flow when starting with a smooth initial metric. Moreover, in [11], the authors demonstrate that this flow is equal to a certain heat flow in the Wasserstein space, and define a flow of a distance metric for the recently developed RCD-spaces. These are metric spaces that have a notion of lower bound of a generalised Ricci curvature (formulated in the language of mass transport) and for which their Sobolev spaces are Hilbert. We refer the reader to the seminal work...
of Ambrosio, Savaré, and Gigli in [2] as well as the work of Gigli in [10] for a detailed
description of these spaces and their properties.

In [7], the authors were interested in the question of proving existence and regu-
larity of this flow when the metric g was no longer assumed to be smooth or even
continuous. The central geometric objects for them are rough metrics, which are
a sufficiently large class of symmetric tensor fields which are able to capture sin-
gularities, including, but not limited to, Lipschitz transforms and certain conical
singularities. The underlying differentiable structure of the manifold is always as-
sumed to be smooth, and hence, rough metrics capture geometric singularities.

More precisely, let ˜ g be a symmetric (2, 0) tensor field and suppose at each point
x ∈ M, there exists a chart (ψx, Ux) near x and a constant C = C(Ux) ≥ 1 satisfying

\[ C^{-1}|u|_{\psi x^* \delta(y)} \leq |u|_{\tilde{g}(y)} \leq C|u|_{\psi x^* \delta(y)}, \]

for y almost-everywhere (with respect to ψx∗L, the pullback of the Lebesgue mea-
sure) inside Ux, where u ∈ TyM, and where ψx∗δ is the pullback of the Euclidean
metric inside (ψx, Ux). A tensor field ˜ g satisfying this condition is called a rough
metric. Such a metric may not, in general, induce a length structure, but (on a
compact manifold) it will induce an n-dimensional Radon measure.

Two rough metrics ˜ g1 and ˜ g2 are said to be C-close (for C ≥ 1) if

\[ C^{-1}|u|_{\tilde{g}_1(x)} \leq |u|_{\tilde{g}_2(x)} \leq C|u|_{\tilde{g}_1(x)}, \]

for almost-every x and where u ∈ TxM. For any two rough metrics, there exists a
symmetric measurable (1, 1)-tensor field B such that ˜ g1(Bu, v) = ˜ g2(u, v). For C-
close rough metrics, C−2|u| ≤ |B(x)u| ≤ C2|u| in either induced norm. In particular,
this means that their Lp-spaces are equal with equivalent norms. Moreover, Sobolev
spaces exist as Hilbert spaces, and these spaces are also equal with comparable
norms. On writing θ = \sqrt{\det B}, which denotes the density for the change of measure
dμ_{\tilde{g}_2} = \sqrt{\det B} dμ_{\tilde{g}_1}, the divergence operators satisfy div_{\tilde{g}_2} = θ^{−1} div_{\tilde{g}_1} θB, and the
Laplacian Δ_{\tilde{g}_2} = θ^{−1} div_{\tilde{g}_1} θBV. Since we assume M is compact, for any rough
metric ˜ g, there exists a C ≥ 1 and a smooth metric g that is C-close.

As far as the author is aware, the notion of a rough metric was first introduced by
the author in his investigation of the geometric invariances of the Kato square root
problem in [6]. However, a notion close to this exists in the work of Norris in [15]
and the notion of C-closeness between two continuous metrics can be found in [17]
by Simon and in [16] by Saloff-Coste.

There is an important connection between divergence form operators and rough
metrics, and this is crucial to the analysis carried out in [7]. The authors noticed
that equation (GMC) and the flow (GM) still makes sense if the initial metric g was
replaced by a rough metric ˜ g. To fix ideas, let us denote a rough metric by ˜ g and
by g, a smooth metric that is C-close. In this situation, we can write the equation
(GMC) equivalently in the form

\[ \text{(GMC’)} \quad -\text{div}_{g,y} \rho_t^g(x,y) B\theta \nabla \varphi_{t,x,v} = \theta d_x(\rho_t^g(x,y))(v). \]
Indeed, it is essential to understand the heat kernel of $\Delta_{\tilde{g}}$ and its regularity to make sense of the right hand side of this equation. In [7], the authors assume $\rho_{t}^{g} \in C^{0,1}(M)$ and further assuming $\rho_{t}^{g} \in C^{k}(\mathcal{N}^{2})$, for $k \geq 2$ and where $\emptyset \neq \mathcal{N} \subset M$ represents a “non-singular” open set, they show the existence of solutions to (GMC’) and provide a time evolving family of metrics $g_{t}$ defined via the equation (GM) on $\mathcal{N}$ of regularity $C^{k-2,1}$. We remark that this set typically arises as $\mathcal{N} = M \setminus S$ where $S$ is some singular part of $g$. For instance, for a cone attached to a sphere at the north pole, we have that $S = \{p_{\text{north}}\}$, and on $\mathcal{N}$, both the metric and heat kernel are smooth.

The aforementioned assumptions are not a restriction to the applications that the authors of [7] consider as their primarily goal was to consider geometric conical singularities, and spaces like a box in Euclidean space. All these spaces are, in fact, RCD-spaces and such spaces have been shown to always have Lipschitz heat kernels. General rough metrics may fail to be RCD, and more seriously, even fail to induce a metric. However, for such metrics, the following still holds.

**Proposition 3.1.** For a rough metric $\tilde{g}$, the heat kernel $\rho_{t}^{g}$ for $\Delta_{\tilde{g}}$ exists and for every $t > 0$, there exists some $\alpha > 0$ such that $\rho_{t}^{g} \in C^{\alpha}(M)$.

This result is due to the fact that the notion of measure contraction property is preserved under $C$-closeness, and hence, by Theorem 7.4 in [18] by Sturm, one can obtain the existence and regularity of the heat kernel by viewing $\Delta_{\tilde{g}}$ as a divergence form operator on the nearby smooth metric $g$. A more detailed proof of this fact can be found in the proof of Theorem 5.1 in [7].

In order to proceed, we note the following existence and uniqueness result to solutions of the equation (GMC’).

**Proposition 3.2.** Suppose that $\rho_{t}^{g} \in C^{1}(\mathcal{N}^{2})$ where $\emptyset \neq \mathcal{N} \subset M$ is an open set. Then, for each $x \in \mathcal{N}$, the equation (GMC’) has a unique solution $\varphi_{t,x,v} \in W^{1,2}(M)$ satisfying $\int_{M} \varphi_{t,x,v} \ d\mu_{g} = 0$. This solution is given by

$$\varphi_{t,x,v} = L_{x}^{-1}(\theta \eta_{t,x,v}) - \int_{M} L_{x}^{-1}(\theta \eta_{t,x,v}) \ d\mu_{g},$$

where $L_{x}u = -\text{div}_{g,y} \rho_{t}^{g}(x,y) \nabla u$ and $\eta_{t,x,v} = d_{x}(\rho_{t}^{g}(x,Y))(v)$.

**Proof.** We note that the proof of this proposition runs in a very similar way to Proposition 4.6 and 4.7 in [7]. Note that the first proposition simply requires that $\rho_{t}^{g} \in C^{0}(M^{2})$, and that $\rho_{t}^{g} > 0$. This latter inequality is yielded by Lemma 5.4 in [7], which again, only requires that $\rho_{t}^{g} \in C^{0}(M^{2})$. □

**Remark 3.3.** When inverting this operator $L_{x}$ as a divergence form operator on the nearby smooth metric $g$, the solutions $\psi_{t,x,v} = L_{x}^{-1}(\theta \eta_{t,x,v})$ satisfy $\int_{M} \psi_{t,x,v} \ d\mu_{g} = 0$. The adjustment by subtracting $\int_{M} \psi_{t,x,v} \ d\mu_{g}$ to this solution is to ensure that $\int_{M} \varphi_{t,x,v} \ d\mu_{g} = 0$. That is, the integral with respect to $\mu_{g}$, rather than $\mu_{\tilde{g}}$, is zero.

Collating these results together, and invoking Theorem 1.1, we obtain the following.
Theorem 3.4. Let $\mathcal{M}$ be a smooth, compact manifold, and $\emptyset \neq \mathcal{N} \subset \mathcal{M}$, an open set. Suppose that $\tilde{g}$ is a rough metric and that $\rho_t^\tilde{g} \in C^1(N^2)$. Then, $g_t$ as defined by (GM) exists on $\mathcal{N}$ and it is continuous.

Proof. By Proposition 3.2, we obtain existence of $g_t(x)$ for each $x \in \mathcal{N}$ as a Riemannian metric. The fact that it is a non-degenerate inner product follows from similar argument to that of the proof of Theorem 3.1 in [7], which only requires the continuity of $\rho_t^\tilde{g}$.

Now, to prove that $x \mapsto g_t(x)$ is continuous, it suffices to prove that $x \mapsto |u|_{g_t(x)}^2$ as a consequence of polarisation. Here, we fix a coordinate chart $(\psi_x, U_x)$ near $x$ and consider $u = \psi^{-1}_x \tilde{u}$, where $\tilde{u} \in \mathbb{R}^n$ is a constant vector inside $(\psi_x, U_x)$. In this situation, we note that (GM) can be written in the following way:

$$|u|_{g_t(x)}^2 = \langle L_x \varphi_{t,x,u}, \varphi_{t,x,u} \rangle = \langle \eta_{t,x,u}, \varphi_{t,x,u} \rangle.$$ 

Now, to prove continuity, we need to prove that $\|u\|_{g_t(x)} - |u|_{g_t(y)}$ can be made small when $y$ is sufficiently close to $x$. This is obtained if, each of $|\langle \eta_{t,x,u} - \eta_{t,y,u}, \varphi_{t,x,u} \rangle|$ and $|\langle \eta_{t,y,u}, \varphi_{t,x,u} - \varphi_{t,y,u} \rangle|$ can be made small.

The first quantity is easy:

$$|\langle \eta_{t,x,u} - \eta_{t,y,u}, \varphi_{t,x,u} \rangle| \leq \|\eta_{t,x,u} - \eta_{t,y,u}\| \|\varphi_{t,x,u}\|,$$

and by our assumption on $\rho_t^\tilde{g}(x, z)$ that it is continuously differentiable for $x \in \mathcal{N}$ and $C^n$ in $z$, we have that $(x, y) \mapsto \eta_{t,x,u}(y)$ is uniformly continuous on $K \times \mathcal{M}$ for every $K \subset \mathcal{N}$ (open subset, compactly contained in $\mathcal{N}$) by the compactness of $\mathcal{M}$. Thus, on fixing $K \subset \mathcal{N}$, we have that for $x, y \in K$,

$$\|\eta_{t,x,u} - \eta_{t,y,u}\| \leq \mu_{\tilde{g}}(\mathcal{M}) \sup_{z \in \mathcal{M}} |\eta_{t,x,u}(z) - \eta_{t,y,u}(z)|$$

and the right hand side can be made small for $y$ sufficiently close to $x$.

Now, the remaining term can be estimated in a similar way:

$$|\langle \eta_{t,y,u}, \varphi_{t,x,u} - \varphi_{t,y,u} \rangle| \leq \|\eta_{t,y,u}\| \|\varphi_{t,x,u} - \varphi_{t,y,u}\|.$$

First, observe that $\|\eta_{t,y,u}\| = \|\eta_{t,y,u} - \eta_{t,x,u}\| + \|\eta_{t,x,u}\|$ and hence, by our previous argument, the first term can be made small and the second term only depends on $x$. Thus, it suffices to prove that $\|\varphi_{t,x,u} - \varphi_{t,y,u}\|$ can be made small. Note then that,

$$\|\varphi_{t,x,u} - \varphi_{t,y,u}\| \leq \|L_x^{-1} \vartheta_{t,x,u} - L_x^{-1} \vartheta_{t,y,u}\| + \mu_{\tilde{g}}(\mathcal{M}) \left( \int_{\mathcal{M}} L_x^{-1} \vartheta_{t,x,u} - L_x^{-1} \vartheta_{t,y,u} \, d\mu_{\tilde{g}} \right)$$

$$\leq (1 + \mu_{\tilde{g}}(\mathcal{M})) \|L_x^{-1} \vartheta_{t,x,u} - L_x^{-1} \vartheta_{t,y,u}\|,$$

where the last inequality follows from the Cauchy-Schwarz inequality applied to the average.

Again, by the assumptions on $\rho_t^\tilde{g}$,

$$\|B \vartheta_{t,x,u} - B \vartheta_{t,y,u}\| \leq \|B \vartheta\| \sup_{z \in \mathcal{M}} |\rho_t^\tilde{g}(x, z) - \rho_t^\tilde{g}(y, z)|,$$
Remark 3.5. If we assume that $\zeta$ bounded inverse on its range. It is easy to see that $(w, z) \mapsto \eta_{t,x,u}(z)$ is uniformly continuous on $K \times M$ for $K \subseteq \mathcal{N}$ and hence, since $\theta$ is essentially bounded from above and below, $x \mapsto \theta\eta_{t,x,u}$ is $L^2$-continuous. Thus, we apply Theorem 1.1 to obtain the conclusion. \hfill $\Box$

4. Proof of the theorem

In this section, we prove the main theorem by first proving a homogeneous Kato square root result. We begin with a description of functional calculus tools required phrase and resolve the problem.

4.1. Functional calculi for sectorial operators. Let $\mathcal{H}$ be a complex Hilbert space and $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ a linear operator. Recall that the resolvent set of $T$ denoted by $\rho(T)$ consists of $\zeta \in \mathbb{C}$ such that $(\zeta I - T)$ has dense range and a bounded inverse on its range. It is easy to see that $(\zeta I - T)^{-1}$ extends uniquely to bounded operator on the whole space. The spectrum is then $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Fix $\omega \in [0, \pi/2)$ and define the $\omega$-bisector and open $\omega$-bisector respectively as

$$S_\omega = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq \omega \text{ or } |\arg(-\zeta)| \leq \omega \text{ or } \zeta = 0\} \text{ and }$$

$$S_\omega^0 = \{\zeta \in \mathbb{C} : |\arg \zeta| < \omega \text{ or } |\arg(-\zeta)| < \omega \text{ and } \zeta \neq 0\}.$$

An operator $T$ is said to be $\omega$-bi-sectorial if it is closed, $\sigma(T) \subset S_\omega$, and whenever $\mu \in (\omega, \pi/2)$, there exist $C_\mu$ satisfying the resolvent bounds: $||\zeta||(|(I - T)^{-1}| \leq C_\mu$ for all $\zeta \in S_\mu \setminus S_\omega$. Bi-sectorial operators naturally generalise self-adjoint operators: a self-adjoint operator is $0$-bi-sectorial. Moreover, bi-sectorial operators admit a spectral decomposition of the space $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{R}(T)$. This sum is not, in general, orthogonal, but it is always topological. By $P_{\mathcal{N}(T)} : \mathcal{H} \to \mathcal{N}(T)$ we denote the continuous projection from $\mathcal{H}$ to $\mathcal{N}(T)$ that is zero on $\mathcal{R}(T)$.

Fix some $\mu \in (\omega, \pi/2)$ and by $\Psi(S_\mu^0)$ denote the class of holomorphic functions $\psi : S_\mu^0 \to \mathbb{C}$ for which there exists an $\alpha > 0$ satisfying

$$|\psi(\zeta)| \lesssim \frac{|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}}.$$

For an $\omega$-bi-sectorial operator $T$, we define a bounded operator $\psi(T)$ via

$$\psi(T)u = \frac{1}{2\pi i} \oint_{\gamma} \psi(\zeta)(\zeta I - T)^{-1}u \, d\zeta,$$

where $\gamma$ is an unbounded contour enveloping $S_\omega$ counter-clockwise inside $S_\mu^0$ and where the integral is defined via Riemann sums. The resolvent bounds for the
operator $T$ coupled with the decay of the function $\psi$ yields the absolute convergence of this integral.

Now, suppose there exists a $C > 0$ so that $\|\psi(T)\| \leq C\|\psi\|_\infty$. In this situation, we say that $T$ has a bounded functional calculus. Let $\text{Hol}^\infty(S_\mu^\circ)$ be the class of bounded functions $f : S_\mu^\circ \cup \{0\} \to \mathbb{C}$ for which $f|_{S_\mu^\circ} : S_\mu^\circ \to \mathbb{C}$ is holomorphic. For such a function, there is always a sequence of functions $\psi_n \in \Psi(S_\mu^\circ)$ which converges to $f|_{S_\mu^\circ}$ in the compact-open topology. Moreover, if $T$ has a bounded functional calculus, the limit $\lim_{n \to \infty} \psi_n(T)$ exists in the strong operator topology, and hence, we define

$$f(T)u = f(0) P_N(T)u + \lim_{n \to \infty} \psi_n(T)u.$$ 

The operator $f(T)$ is independent of the sequence $\psi_n$, it is bounded, and moreover, satisfies $\|f(T)\| \leq C\|f\|_\infty$. By considering the function $\chi^+$, which takes the value $1$ for $\text{Re} \zeta > 0$ and $0$ otherwise, and $\chi^-$ taking $1$ for $\text{Re} \zeta < 0$ and $0$ otherwise, we define $\text{sgn} = \chi^+ - \chi^-$. It is readily checked that $\text{sgn} \in \text{Hol}^\infty(S_\mu^\circ)$ for any $\mu$ and hence, for $T$ with a bounded functional calculus, the $\chi^\pm(T)$ define projections. In addition to the spectral decomposition, we obtain $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{R}(\chi^+(T)) \oplus \mathcal{R}(\chi^-(T))$.

Lastly, we remark that a quantitative criterion for demonstrating that $T$ has a bounded functional calculus is to find $\psi \in \Psi(S_\mu^\circ)$ satisfying the quadratic estimate

$$\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t} \simeq \|u\|^2, \quad u \in \overline{\mathcal{R}(T)}.$$ 

In particular, this criterion facilitates the use of harmonic analysis techniques to prove the boundedness of the functional calculus. We refer the reader to [1] by Albrecht, Duong and McIntosh for a more complete treatment of these ideas.

4.2. Homogeneous Kato square root problem. We have already given a brief historical overview of the Kato square root problem in the introduction. An important advancement, from the point of view of proving such results on manifolds, was the development of the first-order Dirac-type operator approach by Axelson, Keith and McIntosh in [5]. Their set of hypotheses (H1)-(H8) is easily accessed in the literature, and therefore, we shall omit repeating them here. For the benefit of the reader, we remark that the particular form that we use here is listed in [8].

Let $\mathcal{H} = L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M})$ and

$$\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \nabla \end{pmatrix}, \quad \text{and} \quad \Gamma^* = \begin{pmatrix} 0 & -\text{div} \\ 0 & 0 \end{pmatrix}.$$ 

Then, for elliptic (possibly complex and non-symmetric) coefficients $B \in L^\infty(T^{(1,1)}\mathcal{M})$, satisfying $\text{Re} \langle Bu, u \rangle \geq \kappa_1 \|u\|^2$, and $b \in L^\infty(\mathcal{M})$ with $\text{Re} b(x) \geq \kappa_2$, define

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$ 

Define the Dirac-type operators $\Pi_B = \Gamma + B_1\Gamma^*B_2$ and $\Pi = \Gamma + \Gamma^*$. The first operator is bi-sectorial and the second is self-adjoint (but with spectrum possibly on the whole real line).
First, we note that by bi-sectoriality,
\[ \mathcal{H} = \mathcal{D}(\Pi) \oplus R(\Pi) = \mathcal{D}(\Pi B) \oplus R(\Pi B), \]
where the second direct sum is topological but not necessarily orthogonal. In particular, the first direct sum yields that \( L^2(M) = \mathcal{N}(\nabla) \oplus R(\text{div}) \) and \( L^2(T^*M) = \mathcal{N}(\text{div}) \oplus R(\nabla) \).

We observe the following.

**Lemma 4.1.** The space \( R(\text{div}) = \{ u \in L^2(M) : \int_M u = 0 \} \).

**Proof.** Let \( u \in R(\text{div}) \). Then, there is a sequence \( u_n \in R(\text{div}) \) such that \( u_n \to u \).

Indeed, \( u_n = \text{div} v_n \), for some vector field \( v_n \in \mathcal{D}(\text{div}) \). Thus,
\[ \int_M u \, d\mu_{\tilde{g}} = \int_M \lim_{n \to \infty} \text{div} v_n \, d\mu_{\tilde{g}} = \lim_{n \to \infty} \langle \text{div} v_n, 1 \rangle = 0. \]

Now, suppose that \( \int_M u \, d\mu_{\tilde{g}} = 0 \). Then, since \((M, \tilde{g})\) admits a Poincaré inequality, we have that \( \langle u, v \rangle = 0 \) for all \( v \in \mathcal{N}(\nabla) \). But since we have that \( L^2(M) = \mathcal{N}(\text{div}) \oplus R(\nabla) \) via spectral theory, we obtain that \( u \in \mathcal{R}(\text{div}) \). \( \Box \)

With this lemma, we obtain the following coercivity estimate.

**Lemma 4.2.** Let \( u \in R(\Pi) \cap \mathcal{D}(\Pi) \). Then, there exists a constant \( C > 0 \) such that \( \| u \| \leq C \| \Pi u \| \).

**Proof.** Fix \( u = (u_1, u_2) = R(\Pi) = R(\text{div}) \oplus R(\nabla) \). Then, \( \| \Pi u \| = \| \nabla u_1 \| + \| \text{div} u_2 \| \). By the Poincaré inequality along with the previous lemma, we obtain that \( \| \nabla u_1 \| \geq C_1 \| u_1 \| \). For the other term, note that \( \text{div} u_2 = \text{div} \nabla v = \Delta v \) for some \( v \in \mathcal{D}(\nabla) \). Thus,
\[ \| \Delta v \| = \| \sqrt{\Delta} \sqrt{\Delta} v \| \geq C_1 \| \sqrt{\Delta} v \| = C_1 \| \nabla v \| = C_1 \| u_2 \|. \]

On setting \( C = C_1 \), we obtain the conclusion. \( \Box \)

Indeed, this is the key ingredient to obtain a bounded functional calculus for the operator \( \Pi_B \).

**Theorem 4.3** (Homogenous Kato square root problem for compact manifolds). On a compact manifold \( M \) with a smooth metric \( g \), the operator \( \Pi_B \) admits a bounded functional calculus. In particular, \( \mathcal{D}(\sqrt{b \text{div} B\nabla}) = W^{1,2}(M) \) and \( \| \sqrt{b \text{div} B\nabla} u \| \preceq \| \nabla u \| \). Moreover, whenever \( \| b \|_{\infty} < \eta_1 \) and \( \| B \|_{\infty} < \eta_2 \), where \( \eta_i < \kappa_i \), we have the following Lipschitz estimate
\[ \| \sqrt{b \text{div} B\nabla} u - \sqrt{(b + \tilde{b}) \text{div}(B + \tilde{B})\nabla} u \| \lesssim (\| \tilde{b} \|_{\infty} + \| \tilde{B} \|_{\infty}) \| \nabla u \| \]
whenever \( u \in W^{1,2}(M) \). The implicit constant depends on \( B_i \) and \( \eta_i \).

**Proof.** Our goal is to check the Axelsson-Keith-McIntosh hypotheses (H1)-(H8) as listed in [8] to invoke Theorem 4.2 and obtain a bounded functional calculus for \( \Pi_B \).
To avoid unnecessary repetition by listing this framework, we leave it to the reader to check [8]. However, for completeness of the proof, we will remark on why the bulk of these hypothesis are automatically true.

First, by virtue of the fact that we are on a smooth manifold with a smooth metric, we have that $|\text{Ric}| \leq 1$, and $\text{inj}(\mathcal{M}, g) > \kappa > 0$. Coupling this with the fact that $\Gamma$ is a first-order differential operator makes their hypotheses (H1)-(H7) and (H8)-1 valid immediately. The hypotheses (H1)-(H6) are valid as a consequence of their Theorem 6.4 and Corollary 6.5 in [8]. The proof of (H7) is contained in their Theorem 6.2, as is the proof of (H8)-1, which follows by bootstrapping the Poincaré inequality (P) and coupling this with their Proposition 5.3.

It only remains to prove their (H8)-2: that there exists a $C > 0$ such that $\|\nabla u\| + \|u\| \leq C\|\Pi u\|$, whenever $u \in R(\mathcal{B}) \cap D(\Pi)$. Fix such a $u = (u_1, u_2)$ and note that $u_1 = \text{div} v_2$ for some $v_2 \in D(\text{div})$ and $u_2 = \nabla v_1$ for some $v_1 \in D(\nabla)$. Then,

$$\|\nabla u\|^2 = \|\nabla u_1\|^2 + \|\nabla u_2\|^2 = \|\text{div} v_2\|^2 + \|\nabla^2 v_1\|^2.$$ 

Also,

$$\|\Pi u\|^2 = \|\text{div} \nabla v_1\|^2 + \|\nabla \text{div} v_2\|^2.$$ 

Thus, it suffices to estimate the term $\|\nabla^2 v_1\|$ above from $\|\Delta v_1\|$. By exploiting the fact that $C^\infty_c$ functions are dense in both $D(\Delta)$ and $W^{2,2}(\mathcal{M})$ on a compact manifold, the Bochner-Weitzenböck identity yields $\|\nabla^2 v_1\|^2 \lesssim \|\Delta v_1\|^2 + \|v_1\|^2$. Now, $u_2 = \nabla v_1 \in R(\nabla)$ and we can assume that $u_2 \neq 0$. Thus, $v_1 \notin \mathcal{N}(\nabla)$ and hence, $\int_M v_1 \, d\mu_{\tilde{g}} = 0$. Thus, by invoking the Poincaré inequality, we obtain that $\|v_1\| \leq C\|\nabla v_1\| = \|u_2\|$. On combining these estimates, we obtain that $\|\nabla u\| \lesssim \|\Pi u\|$. In Lemma 4.2, we have already proven that $\|u\| \lesssim \|\Pi u\|$.

This allows us to invoke Theorem 4.2 in [8], which says that the operator $\Pi_B$ has a bounded functional calculus. The first estimate in the conclusion is then immediate.

For the Lipschitz estimate, by the fact that that $\Pi_B$ has a bounded functional calculus, we can apply Corollary 4.6 in [8]. This result states that for multiplication operators $A_i$ satisfying satisfying

(i) $\|A_i\|_\infty \leq \eta_i,$

(ii) $A_1 A_2 \mathcal{R}(\Gamma), B_1 A_2 \mathcal{R}(\Gamma), A_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma),$ and

(iii) $A_2 A_1 \mathcal{R}(\Gamma^*), B_2 A_1 \mathcal{R}(\Gamma^*), A_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*),$

we obtain that for an appropriately chosen $\mu < \pi/2$, and for all $f \in \text{Hol}^\infty(S_\mu)$,

$$\| f(\Pi_B) - f(\Pi_{B+A}) \| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty) \|f\|_\infty.$$ 

Setting

$$A_1 = \begin{pmatrix} \hat{b} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{B} \end{pmatrix},$$ 

we have that $\|A_1\|_\infty \leq \eta_1 = \frac{1}{2}$, and $\|A_2\|_\infty \leq \eta_2 = \frac{1}{2}.$
it is easy to see that these conditions are satisfied, and by repeating the argument in Theorem 7.2 in [8] for our operator $\Pi_B$, we obtain the Lipschitz estimate in the conclusion. \hfill \Box

4.3. The main theorem. Let us now return to the proof of Theorem 1.1. Recall the operator $L_xu = -\text{div} A_x \nabla u$, and that $\langle A_x u, u \rangle \geq \kappa_x \|u\|^2$, for $u \in L^2(T^*M)$.

A direct consequence of the Kato square root result from our previous sub-section is then the following.

**Corollary 4.4.** Fix $x \in \mathcal{M}$ and $u \in W^{1,2}(\mathcal{M})$. If $\|A_x - A_y\| \leq \zeta < \kappa_x$, then for $u \in W^{1,2}(\mathcal{M})$,

$$\|\sqrt{L_x}u - \sqrt{L_y}u\| \lesssim \|A_x - A_y\| \|\nabla u\|.$$  

The implicit constant depends on $\zeta$ and $A_x$.

In turn, this implies the following.

**Corollary 4.5.** Fix $x \in \mathcal{M}$ and suppose that $\|A_x - A_y\| \leq \zeta < \kappa_x$. Then,

$$\|L_x^{-1}\eta_x - L_y^{-1}\eta_y\| \lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\|,$$

whenever $\eta_x, \eta_y \in L^2(\mathcal{M})$ satisfies $\int_\mathcal{M} \eta_x \, d\mu_x = \int_\mathcal{M} \eta_y \, d\mu_y = 0$. The implicit constant depends on $\zeta, \kappa_x$, and $A_x$.

**Proof.** First consider the operator $T_x = \sqrt{L_x}$, and fix $u \in L^2(\mathcal{M})$ such that $\int_\mathcal{M} u \, d\mu_x = 0$. We prove that $\|T_x^{-1}u - T_y^{-1}u\| \leq \|A_x - A_y\|_\infty \|u\|$.

Observe that $D(T_x) = W^{1,2}(\mathcal{M})$ and so $T_x^{-1}u = T_x^{-1}(T_y T_x^{-1})u = (T_x^{-1}T_y)T_x^{-1}u$ since $T_x^{-1}u \in W^{1,2}(\mathcal{M})$. Also, since $T_x^{-1}T_x = T_x T_x^{-1}$ on $W^{1,2}(\mathcal{M})$, we have that $T_y^{-1}u = T_x^{-1}T_x L_y^{-1}u$. Thus,

$$\|T_x^{-1}u - T_y^{-1}u\| = \|T_x^{-1}T_y T_x^{-1}u - T_x^{-1}T_x T_y^{-1}u\| = \|T_x^{-1}(T_y - T_x)T_y^{-1}u\| \lesssim \|T_y - T_x\| T_y^{-1}u\| \lesssim \|A_x - A_y\|_\infty \|\nabla T_y^{-1}u\|,$$

where the penultimate inequality follows from Corollary 4.4.

On letting $J_x[u] = \langle A_x \nabla u, \nabla u \rangle \geq \kappa_x \|\nabla u\|^2$, we note that, for $\|\nabla u\| \neq 0$,

$$\kappa_x - \kappa_y \leq \frac{J_x[u] - J_y[u]}{\|\nabla u\|^2} \leq \|A_x - A_y\|_\infty \leq \zeta < \kappa_x.$$  

This gives us that $\kappa_x - \zeta \geq \kappa_y$ and $\kappa_x - \zeta > 0$ by our hypothesis, and hence,

$$(\kappa_x - \zeta) \|\nabla u\|^2 \leq \kappa_y \|\nabla u\|^2 \leq J_y[u] = \|T_y u\|^2.$$  

Thus, $\|\nabla T_y^{-1}u\| \leq (\kappa_x - \zeta)^{-1} \|u\|$, and hence,

$$\|T_x^{-1}u - T_y^{-1}u\| \lesssim \|A_x - A_y\|_\infty \|u\|,$$

where the implicit constant depends on $\zeta, \kappa_x$ and $A_x$.  


Next, let \( v_x, v_y \in L^2(\mathcal{M}) \) satisfy \( \int_{\mathcal{M}} v_x \, d\mu_g = \int_{\mathcal{M}} v_y \, d\mu_g = 0 \) and note that
\[
\|T^{-1}_x v_x - T^{-1}_y v_y\| \leq \|T^{-1}_x v_x - T^{-1}_y v_y\| + \|T^{-1}_y(v_x - v_y)\| \\
\lesssim \|A_x - A_y\|_\infty \|v_x\| + \|(T^{-1}_x - T^{-1}_y)(v_x - v_y)\| + \|T^{-1}_y(v_x - v_y)\| \\
\lesssim \|A_x - A_y\|_\infty \|v_x\| + \|A_x - A_y\|_\infty \|v_x - v_y\| + \|v_x - v_y\| \\
\lesssim \|A_x - A_y\|_\infty \|v_x\| + \|v_x - v_y\|,
\]
where the constant depends on \( \zeta, \kappa_x, \) and \( A_x \). Now, putting \( v_x = L^{-\frac{1}{2}}_x \eta_x = T^{-1}_x \eta_x \), and similarly choosing \( v_y \), since we assume \( \int_{\mathcal{M}} \eta_x \, d\mu_g = \int_{\mathcal{M}} \eta_y \, d\mu_g = 0 \), the same is satisfied for \( v_x \) and \( v_y \). Hence, we apply what we have just proved to obtain
\[
\|L^{-1}_x \eta_x - L^{-1}_y \eta_y\| \lesssim \|A_x - A_y\|_\infty \|L^{-\frac{1}{2}}_x \eta_x\| + \|T^{-1}_x \eta_x - T^{-1}_y \eta_y\| \\
\lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\| \\
\lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\|.
\]
This proves the claim. \( \square \)

With the aid of this, the proof of Theorem 1.1 is immediate.

**Proof of Theorem 1.1.** Fix \( \varepsilon \in (0, \kappa_x) \) and by the assumption that \( x \mapsto \eta_x \) is \( L^2 \)-continuous at \( x \) and that \( x \mapsto A_x \) is \( L^\infty \)-continuous at \( x \), we have a \( \delta = \delta(x, \varepsilon) \) such that
\[
\|\eta_x - \eta_y\| < \varepsilon \quad \text{and} \quad \|A_x - A_y\|_\infty < \varepsilon.
\]
Thus, in invoking Corollary 4.5, we obtain \( \|u_x - u_y\| \lesssim \varepsilon \) where the implicit constant only depends on \( x \). \( \square \)

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