A WEAK APPROXIMATION FOR THE EXTREMA’S DISTRIBUTIONS OF LÉVY PROCESSES

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Abstract. Suppose \( X_t \) is a one-dimensional and real-valued Lévy process started from \( X_0 = 0 \), which (1) its nonnegative jumps measure \( \nu \) satisfying \( \int_{\mathbb{R}} \min\{1, x^2\} \nu(dx) < \infty \) and (2) its stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of \( X_t \) and \( \tau(0) = \infty \). This article employs the Wiener-Hopf Factorization (WHF) to find, an \( L^p \) (where \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( 1 < p \leq 2 \)), approximation for the extrema’s distributions of \( X_t \).

Approximating the finite (infinite)-time ruin probability as a direct application of our findings has been given. Estimation bounds, for such approximation method, along with two approximation procedures and several examples are explored.

Keywords: Lévy processes; Positive-definite function; Extrema’s distributions; the Fourier transform; the Hilbert transform.

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1. Introduction

Suppose that \( X_t \) is a one-dimensional and real-valued Lévy process started from \( X_0 = 0 \) and defined by a triple \((\mu, \sigma, \nu)\) : the drift \( \mu \in \mathbb{R} \), the volatility \( \sigma \geq 0 \), and the jumps measure \( \nu \) which is given by a nonnegative function defined on \( \mathbb{R} \setminus \{0\} \) satisfying \( \int_{\mathbb{R}} \min\{1, x^2\} \nu(dx) < \infty \). Moreover, suppose that the stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of the Lévy process \( X_t \) and \( \tau(0) = \infty \). The Lévy-Khintchine formula states that the characteristic exponent \( \psi \) (i.e., \( \psi(\omega) = \ln(E(\exp(i\omega X_1))) \), \( \omega \in \mathbb{R} \)) can be represented by

\[
\psi(\omega) = i\mu \omega - \frac{1}{2} \sigma^2 \omega^2 + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x I_{[-1,1]}(x)) \nu(dx), \quad \omega \in \mathbb{R}.
\]

The extrema of the Lévy process \( X_t \) are given by

\[
M_q = \sup\{X_s : s \leq \tau(q)\} \quad \text{and} \quad I_q = \inf\{X_s : s \leq \tau(q)\}.
\]

The Wiener-Hopf Factorization (WHF) is a well known technique to study the characteristic functions of the extrema random variables (see [1]). Namely, the WHF states that: (i) product of their characteristic functions equal to
the characteristic function of Lévy process $X_t$ at its stopping time $\tau(q)$, say $X_{\tau(q)}$ and (ii) random variable $M_q$ ($I_q$) is infinitely divisible, positive (negative), and has zero drift.

In the cases that, the characteristic function of Lévy process $X_t$ either a rational function or can be decomposed as a product of two sectionally analytic functions in the closed upper, i.e., $\mathbb{C}^+ := \{ \lambda : \lambda \in \mathbb{C} \text{ and } \Im(\lambda) \geq 0 \}$, and lower half complex planes, i.e., $\mathbb{C}^- := \{ \lambda : \lambda \in \mathbb{C} \text{ and } \Im(\lambda) \leq 0 \}$.

Then, the characteristic functions of random variables $M_q$ and $I_q$ can be determined explicitly (see [24]). [16] considered a Lévy process $X_t$ which its negative jumps is distributed according to a mixture-gamma family of distributions and its positive jumps measure has an arbitrary distribution. They established that the characteristic function of such a Lévy process can be decomposed as a product of a rational function in an arbitrary function, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. They also provided an analog result for a Lévy process whose its corresponding positive jumps measure follows from a mixture-gamma family of distributions while its negative jumps measure is an arbitrary one, more details can be found in [17].

Unfortunately, in the most situations, the characteristic function of the process neither is a rational function nor can be decomposed as a product of two sectionally analytic functions in $\mathbb{C}^+$ and $\mathbb{C}^-$. Therefore, the characteristic functions of $M_q$ and $I_q$ should be expressed in terms of a Sokhotsky-Plemelj integral (see Equation, 2.1). But, this form, also, presents some difficulties in numerical work due to slow evaluation and numerical problems caused by singularities near the integral contour (see [11]). To overcome these difficulties, an appropriate (in some sense) approximation method has to be considered. It is well known that a Lévy process $X_t$ which its jumps distribution follows from the phase-type distribution has a rational characteristic function (see [7]). [13] utilized this fact and approximated a jumps measure $\nu$ of a ten-parameter Lévy processes (named $\beta-$family of Lévy process) by a sequence of the phase-type measures. Then, he determined the characteristic functions of random variables $M_q$ and $I_q$, approximately. [14] extended [13]'s findings to class of Meromorophic Lévy processes. Moreover, [15] provided a uniform approximation for the cumulative distribution function of $M_{\tau(q)}$ whenever $X_t$ is a symmetric Lévy process. [12] employed the Shannon sampling method to find the distributions of the extrema for a wide class of Lévy processes.

This article begins with an extension of [11]'s results for the multiplicative WHF

\begin{equation}
\Phi^+(\omega)\Phi^-(\omega) = g(\omega) \quad \omega \in \mathbb{R},
\end{equation}

where $g(\cdot)$ is a given function with some certain conditions (see below) and $\Phi^\pm(\cdot)$ are to be determined. Then, it utilizes such results to approximate the extrema's distributions of a class of Lévy processes. Estimation bounds, for such approximate method, along with two approximation procedures are given.
Section 2 collects some useful elements for other sections. Moreover, it provides an $L^p(\mathbb{R}), 1 < p \leq 2$ approximation technique for solving a multiplicative WHF (1.2). Section 3 considers the problem of approximating the extrema’s density functions for a class of Lévy processes. Then, it develops two approximate techniques for situations where those density functions cannot be determined, explicitly. Error bounds for such techniques are given. Several examples are given in Sections 4. Section 5 provides concluding remarks along with some suggestions for other application of our techniques.

2. Preliminaries

The Sokhotskyi-Plemelj integral for $s(\cdot)$, which satisfies the Hölder condition, is defined by a principal value integral, as follows

$$\phi_s(\lambda) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s(x)}{x - \lambda} dx, \quad \text{for } \lambda \in \mathbb{C}. \tag{2.1}$$

It is worth mentioning that, the Sokhotskyi-Plemelj integral can be existed for non-integrable function such as $\sin(x)$. Therefore, the Sokhotskyi-Plemelj integral $\phi_s(\cdot)$ should be viewed different from the usual integral over $\mathbb{R}$.

The radial limits of the Sokhotskyi-Plemelj integral of $s(\cdot)$, are given by $\phi_s^+(\omega) = \lim_{\lambda \to \omega + i0^+} \phi_s(\lambda)$ and satisfy the following jump formulas: (1)

$$\phi_s^+(\omega) = \pm s(\omega)/2 + \phi_s(\omega), \quad \text{for } \omega \in \mathbb{R} \quad \text{and} \quad (2) \quad \phi_s^+(\omega) = \pm s(\omega)/2 + H_s(\omega)/(2i),$$

where $H_s(\omega)$ stands for the Hilbert transform of $s(\cdot)$ and $\omega \in \mathbb{R}$.

The multiplicative WHF is the problem of finding an analytic and bounded, except on the real line, function $\Phi(\cdot)$ where its upper and lower radial limits $\Phi^\pm(\cdot)$ satisfy Equation (1.2). Given function $g(\cdot)$ is a bounded above by 1, zero index\(^1\), continuous, and positive function which satisfies the Hölder condition on $\mathbb{R}$, $g(0) = 1$, and $g(\omega) \neq 0$ for all $\omega \in \mathbb{R}$.

The following extends [11]'s results to the multiplicative WHF (1.2). We begin with what we term the Resolvent Equation for Sokhotskyi-Plemelj integrals.

**Lemma 2.1.** The Sokhotskyi-Plemelj integral of a function $f(\cdot)$ satisfies $\phi_f(\lambda) - \phi_f(\mu) = (\lambda - \mu)\phi_{f(\mu)}(\lambda)$, where $\lambda$ and $\mu$ are real or complex values.

**Proof.** In general, $(x-\lambda)^{-1} - (x-\mu)^{-1} = (\lambda - \mu)(x-\mu)^{-1}(x-\lambda)^{-1}$. Then, see [8], we have an equation of Cauchy integrals, where $\Gamma = \mathbb{R}$:

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(x)}{x - \lambda} dx - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(x)}{x - \mu} dx = \frac{\lambda - \mu}{2\pi i} \int_{\Gamma} \frac{f(x)}{(x-\mu)(x-\lambda)} dx.$$

The above is valid only for $\lambda$ and $\mu$ not on the real line. However, by Equation (2.1) the values of $\phi_f(\cdot)$ on the real line are obtained by averaging

\(^1\)The index of a complex-valued function $f$ on a smooth oriented curve $\Gamma$, such that $f(\Gamma)$ is closed and compact, is defined to be the winding number of $f(\Gamma)$ about the origin (see [23], §1), for more technical details.
Lemma 2.2. Suppose $\Phi^\pm(\cdot)$ are sectionally analytic functions that satisfy the multiplicative WHF (1.2). Moreover, suppose that given function $g(\cdot)$ is a zero index function which satisfies the Hölder condition and $g(0) = 1$. Then $\Phi^\pm(\lambda) = \exp\{\pm(\phi_{in\cdot g}(\lambda) - \phi_{in\cdot g}(0))\}$, where $\phi_{in\cdot g}(\cdot)$ stands for the Sokhotskyi-Plemelj integral of $\ln g(\cdot)$.

Proof. Using the [10]'s suggestion for solving the homogeneous WHF (1.2) gives, see also [17]:

$$\Phi^\pm(\lambda) = \exp\{\pm\frac{\lambda}{2\pi i} \int_\mathbb{R} \frac{\ln g(x)/x}{x - \lambda} dx\}.$$ 

Lemma (2.1) with $f \equiv \ln g$ gives $\phi_{in\cdot g}(\lambda) - \phi_{in\cdot g}(\mu) = (\lambda - \mu)\phi_{in\cdot g}(\mu)$. Letting $\lambda$ goes to zero from the above, in the complex plane, and using the fact that $\ln g(0) = 0$, Equation (2.1) lets us to conclude that $\phi_{in\cdot g}(0) - \phi_{in\cdot g}(\mu) = -\mu\phi_{in\cdot g}(\mu)$. Substituting this into the above equation for $\Phi^\pm(\cdot)$ gives our claimed result. □

Using the jump formula one can conclude that

$$(2.2) \quad \Phi^\pm(\omega) = \sqrt{g(\omega)} \exp\{\pm\frac{i}{2}(H_{in\cdot g}(0) - H_{in\cdot g}(\omega))\},$$

where $H_{in\cdot g}(\cdot)$ stands for the Hilbert transform of $\ln g(\cdot)$.

The Carlemann’s method explores a situation which one may evaluate solutions of the multiplicative WHF (1.2) directly, rather than using the Sokhotskyi-Plemelj integrations. The Carlemann’s method states that: if $g(\cdot)$ can be decomposed as a product of two sectionally analytic functions $g^+(-\cdot)$ and $g^-(-\cdot)$, respectively in $\mathbb{C}^+$ and $\mathbb{C}^-$. Then, solutions of the multiplicative WHF (1.2) are given by $\Phi^+ \equiv g^+$ and $\Phi^- \equiv g^-$.

In a situation that $g(\cdot)$ is a rational function $\frac{P(x)}{Q(x)}$ that has no poles or zeros on $\mathbb{R}$. Using the Carlemann’s method, we may conclude that the multiplicative WHF problem can be solved by factoring the polynomial $P(Q)$, and then let $P^+(Q^+)$ be the product of those factors of $P(Q)$ that have zeros in $\mathbb{C}^-$, and $P^-(Q^-)$ be the product of those factors that have zeros in $\mathbb{C}^+$. Then, setting $g^+(x) = \frac{P^+(x)}{Q^+(x)}$ and $g^-(x) = \frac{P^-(x)}{Q^-(x)}$ gives us (up to a scalar multiple) our desired factorization.

The Hausdorff-Young theorem (see [22]) states that: if $s(\cdot)$ is an $L^p(\mathbb{R})$ function. Then, $s(\cdot)$ and its corresponding the Fourier transform, say $\hat{s}(\cdot)$, satisfy $||s||_{L^p} \leq (2\pi)^{-1/p}||\hat{s}||_{L^p}$, where $1 \leq p \leq 2$ and $1/p + 1/p^* = 1$. From the Hausdorff-Young Theorem, one can observe that if $\{s_n(\cdot)\}$ is a sequence of functions converging in $L^p(\mathbb{R})$, $1 \leq p \leq 2$, to $s(\cdot)$. Then, the Fourier

\footnote{The condition $g(0) = 1$ does not always hold in the multiplicative WHF, but happen to arise in our application, and can lead to complications. Lemma (2.2) is used to simplifying this case.}
transforms of the $s_n(\cdot)$ converges in $L^p(\mathbb{R})$, to the Fourier transform of $s(\cdot)$, where $1/p + 1/p^* = 1$. The converse is false.

A similar property for the Hilbert transform is well known as the Titmarsh-Riesz lemma (see [22]). The Titmarsh-Riesz lemma says that: if $s(\cdot)$ is an $L^p(\mathbb{R})$ function, where $1 < p \leq 2$. Then, $\|H_s\|_p \leq \tan(\pi/(2p))\|s\|_p$, where $H_s(\cdot)$ stands for the Hilbert transform of $s(\cdot)$. Using the Titmarsh-Riesz lemma, one may conclude that if \{f_n(\cdot)\}, is a sequence of functions which converge, in $L^p(\mathbb{R})$, $1 < p \leq 2$, to $f(\cdot)$. Then, the Hilbert transforms $H(f_n)$ converge, in $L^p(\mathbb{R})$, $1 < p \leq 2$, to the Hilbert transform of $f(\cdot)$.

The well known Paley-Wiener theorem states that: if $F(\cdot)$ is a function in $L^2(\mathbb{R})$. Then, the real-valued function $F(\cdot)$ vanishes on $\mathbb{R}^-$ if and only if the Fourier transform $F(\cdot)$, say, $\hat{F}(\cdot)$ is holomorphic on $\mathbb{C}^+$ and the $L^2(\mathbb{R})$-norm of the functions $x \mapsto \hat{F}(x + iy_0)$ are uniformly bounded for all $y_0 \geq 0$.

The following, from [11], recalls some further useful properties of functions in $L^p(\mathbb{R})$, for $1 < p \leq 2$, space.

**Lemma 2.3.** Suppose $s(\cdot)$ and $r(\cdot)$ are two $L^p(\mathbb{R})$, $1 < p \leq 2$, functions. Then,

i): $\|\sqrt{s} - \sqrt{r}\|_p \leq \frac{1}{2\sqrt{a}}\|s - r\|_p$, whenever both $s(\cdot)$ and $r(\cdot)$ are bounded, above by $a$, functions;

ii): $\|\ln s - \ln r\|_p \leq a^{-1}\|s - r\|_p$, whenever both $s(\cdot)$ and $r(\cdot)$ are positive-valued and bounded, above by $a$, functions;

iii): $\|e^{-i\sqrt{s}} - e^{-i\sqrt{r}}\|_p \leq \frac{1}{2}\|s - r\|_p$, whenever $s(\cdot)$ and $r(\cdot)$ are real-valued functions.

In many situations, WHF (1.2) cannot be solved explicitly and has to be solved approximately (see [11]). The following develops an approximation technique to solve a multiplicative WHF (1.2).

**Theorem 2.4.** Suppose $\Phi^\pm(\cdot)$ are two sectionally analytic functions satisfying the multiplicative WHF (1.2) where

A1): $g(\cdot)$ is real, positive, bounded above by $a$, index zero, satisfies the Hölder condition, and $g(0) = 1$;

A2): There exist a sequence of functions $g_n(\cdot)$ where converge, in $L^p(\mathbb{R})$, $1 < p \leq 2$, to $g(\cdot)$.

Then, $\Phi^\pm(\cdot)$ can be approximated by $\Phi_n^\pm(\cdot)$, where

$$\|\Phi^\pm_n - \Phi^\pm\|_p \leq \frac{1}{2}\tan\left(\frac{\pi}{2p}\right)\|g_n - g\|_p^2 + \left(\tan\left(\frac{\pi}{2p}\right) + \frac{1}{2}\right)\|g_n - g\|_p.$$  

**Proof.** Set $k(\omega) := -H_{\ln g}(\omega) + H_{\ln g}(0)$ and $k_n(\omega) := -H_{\ln g_n}(\omega) + H_{\ln g_n}(0)$. Now, from Equation (2.2) and Lemma (2.3) observe that $\|\Phi^\pm_n -
Definition 2.5. A positive-definite function is a complex-valued function $f: \mathbb{R} \rightarrow \mathbb{C}$ such that for any real numbers $x_1, \ldots, x_n$ the $n \times n$ square matrix $A = (f(x_i - x_j))_{i,j=1}^n$ is a positive semi-definite matrix.

In the theory of the Fourier transform, it is well known that “$f(\cdot)$ is a continuous positive-definite function on $\mathbb{R}$ if and only if its corresponding Fourier transform is a (positive) measure”, see [4] for more details.

Lemma 2.6. Suppose $\phi: \mathbb{R} \rightarrow \mathbb{C}$ is a positive-definite function which two equations $q_1 - \phi(\omega) = 0$ and $1 - q_2 \exp\{-\phi(\omega)\} = 0$ have not any solution on $\mathbb{R}$, where $q_1 > 0$ and $q_2 \in (0,1)$. Then, $h_1(\omega) = q_1/(q_1 - \phi(\omega))$ and $h_2(\omega) = (1 - q_2)/(1 - q_2 \exp\{-\phi(\omega)\})$ are positive-definite functions.

Proof. Using the Taylor expansion of $q_1/(q_1 - x)$ and $(1 - q_2)/(1 - q_2 \exp\{-x\})$, about zero, one may restated $h_1(\cdot)$ and $h_2(\cdot)$ as

\begin{align*}
h_1(\omega) &= \sum_{k=1}^{\infty} \frac{\phi^k(\omega)}{q_1^k} \\
h_2(\omega) &= 1 + \frac{q_2}{q_2 - 1}\phi(\omega) + \frac{q_2(q_2 + 1)}{2(q_2 - 1)^2}\phi^2(\omega) + \frac{q_2(q_2^2 + 4q_2 + 1)}{6(q_2 - 1)^3}\phi^3(\omega) + \cdots
\end{align*}

Now, the desired proof arrives from the fact that the product of two positive-definite functions is again a positive-definite function (see [30]). □

Now, we provide two classes of positive-definite rational functions which play a vital role in numerical section of this article.
Lemma 2.7. Consider the following two class of rational functions $D$ and $D^*$.

\[
D : = \{ r(\omega); \ r(\omega) = A_0 + \sum_{k=1}^{n} \sum_{j=1}^{m_k} \sum_{l=1}^{4} C_{kj} r_{lk}^j(\omega); \ A_0 \ \& \ C_{kj} \geq 0 \}; \\
D^* : = \{ r(\omega); \ r(\omega) = A_0 + \sum_{k=1}^{n} \sum_{l=1}^{2} C_{kj} r_{lk}(\omega); \ A_0 \ \& \ C_{kj} \geq 0 \},
\]

where

\[
\begin{align*}
    r_{1k}(\omega) &= \frac{1}{i \omega + \beta_k} \ (\text{where } \beta_k > 0); \\
    r_{2k}(\omega) &= \frac{1}{-i \omega + \beta_k} \ (\text{where } \beta_k > 0); \\
    r_{3k}(\omega) &= \frac{1}{(i \omega + \beta_k)(i \omega + \beta_k + \alpha_k i)(i \omega + \beta_k - \alpha_k i)} \ (\text{where } \alpha_k, \beta_k > 0); \\
    r_{4k}(\omega) &= \frac{1}{(-i \omega + \beta_k)(-i \omega + \beta_k + \alpha_k i)(-i \omega + \beta_k - \alpha_k i)} \ (\text{where } \alpha_k, \beta_k > 0).
\end{align*}
\]

Then, (i) the Fourier transform of functions in $D$ are nonnegative and real-valued functions. (ii) the Fourier transform of functions in $D^*$ are nonnegative, real-valued, and completely monotone functions.

**Proof.** Nonnegativity of the Fourier transform of functions in $D$ (or $D^*$) arrives from the fact that $r_{lk}$ (for $l = 1, \ldots, 4$) and their powers are positive-definite rational functions. Now, from the Bernstein’s theorem observe that a real-valued function defined on $\mathbb{R}^+$ is a completely monotone function, whenever it is a mixture of exponential functions, see [29] for more details. □

The following may be concluded from properties of the WHF given by [1].

Lemma 2.8. Suppose $g(\cdot)$ in the multiplicative WHF (1.2) is a positive-definite function. Then, solutions, of the multiplicative WHF (1.2), $\Phi^\pm(\cdot)$ are two positive-definite functions.

**Proof.** First observe that, using the multiplicative WHF (1.2) the characteristic function of Lévy process $X_t$ at its stopping time $\tau(q)$, say $X_{\tau(q)}$, can be decomposed as a product of the characteristic functions of two random variables $I_q$ and $M_q$, see [1] for more details. Moreover, [3]’s theorem states that “An arbitrary function $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$ is the characteristic function of some random variable if and only if $\phi(\cdot)$ is positive-definite, continuous at the origin, and if $\phi(0) = 1$”. The desired proof arrives using these observations. □

One may readily observe that the characteristic function of the mixed gamma family of distributions (given below) are belong to $D$. 
Definition 2.9. (Mixed gamma family of distributions) A nonnegative random variable \( X \) is said to be distributed according to a mixed gamma distribution if its density function is given by

\[
p(x) = \sum_{k=1}^{\nu} \alpha_k \sum_{j=1}^{n_k} c_{kj} x^{j-1} e^{-\alpha_k x} I_{[0,\infty)}(x) + \sum_{k=1}^{\nu^*} \beta_k \sum_{j=1}^{n_{k^*}} c^{*}_{kj} (-x)^{j-1} e^{-\beta_k x} I_{(-\infty,0]}(x)
\]

where \( c_{kj} \) and \( \alpha_k \) are positive values which satisfy \( \sum_{k=1}^{\nu} \sum_{j=1}^{n_k} c_{kj} = 1 \).

We now from [2] recall some useful properties of the characteristic function, which plays an important role for the next sections.

Lemma 2.10. Suppose \( \hat{p}(\cdot) \) stands for the characteristic function of a distribution. Then,

i): \( \hat{p}(\cdot) \) is a positive-definite function;

ii): \( \hat{p}(\cdot) \) is a positive-definite rational function whenever its characteristic function belongs to \( D \) given by Lemma (2.7);

iii): \( \hat{p}(0) = 1; \) and the norm of \( \hat{p}(\cdot) \) is bounded by 1.

The next section provides an application of Theorem (2.4) to the problem of finding the distributions of the extrema of Lévy process \( X_t \), approximately.

3. Main results

The following lemma restates the characteristic function of Lévy process \( X_t \) at its stopping time \( \tau(q) \), say \( X_{\tau(q)} \).

Lemma 3.1. Suppose \( X_{\tau(q)} \) represents Lévy process \( X_t \) at its stopping time \( \tau(q) \). Then, the characteristic function of \( X_{\tau(q)} \) can be restated as:

(i): \( q/(q - \psi(\omega)) \), for an exponential stopping time \( \tau(q) \) with parameter \( q > 0 \);

(ii): \( (1 - q)/(1 - q \exp\{-\psi(\omega)\}) \), for a geometric stopping time \( \tau(q) \) with parameter \( q \in (0, 1) \).

Proof. Conditioning on stopping time \( \tau(q) \), one may restates the characteristic function of \( X_{\tau(q)} \) as:

For part (i):

\[
E(e^{i\omega X_{\tau(q)}}) = \int_0^\infty E(e^{i\omega X_{\tau(q)}|\tau(q) = t}) f_{\tau(q)}(t) dt = \int_0^\infty E(e^{i\omega X_t}) q e^{-qt} dt
\]

\[
= \int_0^\infty e^{\psi(\omega)t} q e^{-qt} dt = \frac{q}{q - \psi(\omega)};
\]

For part (ii):

\[
E(e^{i\omega X_{\tau(q)}}) = \sum_{n=0}^\infty E(e^{i\omega X_{\tau(q)}|\tau(q) = n}) P(\tau(q) = n) = \sum_{n=0}^\infty E(e^{i\omega X_n})(1 - q) q^n
\]

\[
= \sum_{n=0}^\infty e^{\psi(\omega)n}(1 - q) q^n = \frac{1 - q}{1 - q \exp\{-\psi(\omega)\}}.
\]
where for both cases, the second equality arrives from the fact that \( X_t \) and \( \tau(q) \) are independent and the third equality obtains from definition of the characteristic exponent \( \psi \) and infinitely divisibility of Lévy process \( X_t \).

The following theorem represents an error bound for approximating the density functions of extrema of a Lévy process.

**Theorem 3.2.** Suppose \( X_t \) is a Lévy process defined by a triple \((\mu, \sigma, \nu)\). Moreover, suppose that:

- \( A_1 \): The stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of \( X_t \) and \( \tau(0) = \infty \);
- \( A_2 \): The \( r_n(dx) \) are a sequence of positive-definite rational functions which converge, in \( L^p(\mathbb{R}) \) (where \( 1/p^* + 1/p = 1 \) and \( 1 < p \leq 2 \)), to characteristic exponent \( q/(q-\psi(dx)) \) (or \((1-q)/(1-q\exp\{-\psi(dx)\})\)) for geometric stopping time.

Then, the density function of the suprema and infima of the Lévy process \( X_t \), denoted \( f^+_q \) and \( f^-_q \), respectively, can be approximated, in \( L^p(\mathbb{R}) \) sense, by a sequence of the density functions \( f^{\pm}_{q,n} \) and \( f^{\pm}_{q,n} \) where:

1. For exponentially distributed stopping time \( \tau(q) \), for \( q > 0 \),
\[
||f^+_q - f^+_{q,n}||_{p^*} \leq \frac{1}{2} \tan\left(\frac{\pi}{2p^*}\right)||r_n - \frac{q}{q - \psi}\|^2_{p^*} + (\tan\left(\frac{\pi}{2p^*}\right) + \frac{1}{2})||r_n - \frac{q}{q - \psi}\|_{p^*};
\]

2. For geometric stopping time \( \tau(q) \), for \( q \in (0,1) \),
\[
||f^+_q - f^+_{q,n}||_{p^*} \leq \frac{1}{2} \tan\left(\frac{\pi}{2p^*}\right)||r_n - \frac{1 - q}{1 - qe^{-\psi}}\|^2_{p^*} + (\tan\left(\frac{\pi}{2p^*}\right) + \frac{1}{2})||r_n - \frac{1 - q}{1 - qe^{-\psi}}\|_{p^*}.
\]

**Proof.** From [1] and Lemma (3.1), one can observe that the Fourier transform of the density functions of random variables \( M_q \) and \( I_q \), say respectively \( \Phi^+ \) and \( \Phi^- \), satisfy either the multiplicative WHF \( \Phi^+(\omega)\Phi^-(\omega) = q/(q - \psi(\omega)) \), where \( \omega \in \mathbb{R} \) (for exponentially distributed stopping time) or the multiplicative WHF \( \Phi^+(\omega)\Phi^-(\omega) = (1 - q)/(1 - q\exp\{-\psi(\omega)\}) \), where \( \omega \in \mathbb{R} \) (for geometric stopping time). Now, from the fact that the expressions
\[
q/(q - \psi(\cdot)) \quad \text{and} \quad (1 - q)/(1 - q\exp\{-\psi(\cdot)\})
\]
are the characteristic function of the Lévy process \( X_t \), at exponential and geometric stopping time, respectively, we observe that both expressions are bounded above by 1 because of the property of the characteristic function given by Lemma (2.10, part ii).

For part (i), from Theorem (2.4) observe that
\[
||\Phi^+ - \Phi^+||_p \leq \frac{1}{2} \tan\left(\frac{\pi}{2p}\right)||r_n - \frac{q}{q - \psi}\|^2 + (\tan\left(\frac{\pi}{2p}\right) + \frac{1}{2})||r_n - \frac{q}{q - \psi}\|_p.
\]

The rest of proof arrives from an application of the Hausdorff-Young Theorem. The proof of part (ii) is quite similar. \( \Box \)

**Remark 3.3.** In case that the distribution of \( I_q \) or \( M_q \) has an atom at \( x = 0 \). Then, it corresponding probability mass function at zero can be found, approximately, by
\[
P(I_q = 0) = \lim_{\omega \to \infty} \Phi^-(-i\omega) \quad \text{&} \quad P(M_q = 0) = \lim_{\omega \to \infty} \Phi^+(i\omega).
\]
Using the fact that the Compound Poisson has bounded characteristic exponent \( \psi(\cdot) \). The following formulates result of the above theorem in terms of the jumps measure \( \nu(dx) \).

**Theorem 3.4.** (Compound Poisson) Suppose \( X_t \) is a Compound Poisson process defined by a triple \((\mu, \sigma, \nu)\). Moreover, suppose that

\( A_1 \): the stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of \( X_t \) and \( \tau(0) = \infty \);
\( A_2 \): the \( \nu_n(dx) \) are a sequence of the density functions which converge in \( L^2(\mathbb{R}) \), to jumps measure \( \nu \) and \( \int_{-1}^{1} x\nu_n(dx) = \int_{-1}^{1} x\nu(dx) \).

Then, the density functions of the suprema and infima of the Compound Poisson process \( X_t \), denoted by \( f_{q,n}(\cdot) \) and \( f_{q,n}(\cdot) \), respectively, can be approximated by a sequence of the density functions \( f_{q,n}(\cdot) \) and \( f_{q,n}(\cdot) \) where:

\[
\begin{align*}
&\text{i): For exponentially distributed stopping time } \tau(q), \\
&||f_q^\pm - f_{q,n}^\pm||_2 \leq \frac{1}{q^2\sqrt{8\pi}}||\nu - \nu||_2^2 + \frac{3}{2q}||\nu - \nu||_2^2 + \frac{3}{2q}||\nu - \nu||_2^2;
\end{align*}
\]

\[
\begin{align*}
&\text{ii): For geometric stopping time } \tau(q), \\
&||f_q^\pm - f_{q,n}^\pm||_2 \leq \frac{(1 - q)^2}{q^2\sqrt{8\pi}}||\nu - \nu||_2^2 + \frac{3(1 - q)}{2q}||\nu - \nu||_2^2.
\end{align*}
\]

**Proof.** Suppose \( \psi_n(\cdot) \) are sequence of the characteristic exponent corresponding to \( \nu_n(dx) \). For part (i) using result of Theorem (3.2), one may conclude that

\[
\begin{align*}
||\Phi_n^\pm - \Phi^\pm||_2 &\leq \frac{1}{2}||q - \psi_n - q - \psi||_2^2 + \frac{3}{2}||q - \psi_n - q - \psi||_2^2 \\
&\leq \frac{1}{2q^2}||\psi_n - \psi||_2^2 + \frac{3}{2q}||\psi_n - \psi||_2^2 \\
&\leq \frac{1}{4q^2\sqrt{8\pi}||\nu - \nu||_2^2} + \frac{3}{2q}||\nu - \nu||_2^2.
\end{align*}
\]

The second inequality arrives from the fact that the characteristic function \( q/(q - \psi(\cdot)) \) is bounded above by 1, while the third inequality comes from the Levy-Khintchine representation (Equation, 1.1) along with conditions \( A_2 \) and the Hausdorff-Young Theorem. The rest of proof arrives from an application of the Hausdorff-Young Theorem. The proof of part (ii) is quite similar. □

**4. Application to the Finite (Infinite)-time Ruin Probability**

Suppose surplus process of an insurance company can be restated as

\[
U_t = u + X_t,
\]

where Lévy process \( X_t \) and \( u > 0 \) stands for initial wealth/reserve of the process.
The finite-time ruin probability for the such surplus process is denoted by \( R^{(q)}(u) \) and defined by
\[
R^{(q)}(u) = P(T \leq \tau_q|U_0 = u),
\]
where \( T \) is the hitting time, i.e., \( T := \inf\{t : U_t \leq 0|U_0 = u\} \) and \( \tau_u \) is a random stopping time. Such the stoping time has been distributed corroding to either an exponential distribution (with mean \( 1/q \)) or a geometric distribution (with mean \( (1 - q)/q \)).

The infinite-time ruin probability for the surplus process (4.1) is denoted by \( R(u) \) and defined by
\[
R(u) = P(T < \infty|U_0 = u).
\]
The infinite-time ruin probability \( R(u) \), also, can be evaluated by \( R(u) = \lim_{q \to 0} R^{(q)}(u) \).

Using Alili & Kyprianou (2005, Lemma 1 with setting \( \beta = 0 \) and replacing \( X \) by \( -X \))’s findings, one may conclude that: in a situation that the infima density function \( f_q \) of the Lévy process \( X_t \) is available, the finite-time ruin probability under the above surplus process can be restated as
\[
R^{(q)}(u) = P(I_q < -u) = \int_{-\infty}^{-u} f_q(y)dy.
\]

Now using an \( L_p(\mathbb{R}) \)-norm for an integral operator (see Theorem 3.36 in [6]), one may restate results of Theorem (3.2) and Theorem (3.4) for approximating the finite-time ruin probability under the surplus process (4.1) as the following two corollaries.

**Corollary 4.1.** Suppose \( X_t \) in the surplus process (4.1) is a Lévy process defined by a triple \((\mu, \sigma, \nu)\). Moreover, suppose that:

- **A1:** The stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of \( X_t \) and \( \tau(0) = \infty \);
- **A2:** The \( r_n(dx) \) are a sequence of positive-definite rational functions which converge, in \( L^p(\mathbb{R}) \) (where \( 1/p + 1/p = 1 \) and \( 1 < p \leq 2 \)), to characteristic exponent \( q/(q - \psi(dx)) \) (or \( (1 - q)/(1 - q\exp\{-\psi(dx)\}) \)) for geometric stopping time

Then, the finite-time ruin probability under the surplus process (4.1), say \( R^{(q)}(u) \), can be approximated, in \( L^p(\mathbb{R}) \) sense, by a sequence of the ruin probability, say \( R^{(q)}_n(u) \), where:

i): For exponentially distributed stopping time \( \tau(q) \), for \( q > 0 \),
\[
||R^{(q)} - R^{(q)}_n||_{L^p} \leq \frac{1}{2} \tan\left(\frac{\pi}{2p}\right)||r_n - \frac{q}{q - \psi}||_{L^p}^2 + \left(\tan\left(\frac{\pi}{2p}\right) + \frac{1}{2}\right)||r_n - \frac{q}{q - \psi}||_{L^p}^2;
\]

ii): For geometric stopping time \( \tau(q) \), for \( q \in (0,1) \),
\[
||R^{(q)} - R^{(q)}_n||_{L^p} \leq \frac{1}{2} \tan\left(\frac{\pi}{2p}\right)||r_n - \frac{1 - q}{1 - qe^{-\psi}}||_{L^p}^2 + \left(\tan\left(\frac{\pi}{2p}\right) + \frac{1}{2}\right)||r_n - \frac{1 - q}{1 - qe^{-\psi}}||_{L^p}^2.
\]
Corollary 4.2. (Compound Poisson) Suppose \( X_t \) in the surplus process (4.1) is a Compound Poisson process defined by a triple \( (\mu, \sigma, \nu) \). Moreover, suppose that

A1): the stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of \( X_t \) and \( \tau(0) = \infty \);

A2): the \( \nu_n(dx) \) are a sequence of the density functions which converge in \( L^2(\mathbb{R}) \), to jumps measure \( \nu \) and \( \int_{-\infty}^{1} x \nu_n(dx) = \int_{-\infty}^{1} x \nu(dx) \).

Then, the finite-time ruin probability under the surplus process (4.1), say \( R^{(q)}(u) \), can be approximated, in \( L^p(\mathbb{R}) \) sense, by a sequence of the finite-time ruin probability, say \( R^{(q)}_n(u) \), where:

i): For exponentially distributed stopping time \( \tau(q) \),

\[
\| R^{(q)} - R^{(q)}_n \|_2 \leq \frac{1}{q^2\sqrt{8\pi}} \| \nu_n - \nu \|_2^2 + \frac{3}{2q} \| \nu_n - \nu \|_2;
\]

ii): For geometric stopping time \( \tau(q) \),

\[
\| R^{(q)} - R^{(q)}_n \|_2 \leq \frac{(1-q)^2}{q^2\sqrt{8\pi}} \| \nu_n - \nu \|_2^2 + \frac{3(1-q)}{2q} \| \nu_n - \nu \|_2.
\]

It is worth mentioning that the above results may be obtained for the infinite-time ruin probability by letting \( q \to 0 \).

The next section provides some practical applications of the above findings.

5. Examples

In the first step, this section provides two particle procedures for the problem of finding the density functions of the suprema and infima of a Lévy process.

Using the fact that the characteristic exponent \( \psi(i\omega) \), \( \omega \in \mathbb{R} \), is a real-valued function, (see [1]) along with Lemma (2.8), we suggest the following two procedures to generate approximation density functions for \( M_q \) and \( I_q \).

Procedure 5.1. Suppose \( X_t \) is a Meromorphic Lévy process\(^3\) with the characteristic exponents \( \psi(\cdot) \). Moreover, suppose that stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of \( X_t \) and \( \tau(0) = \infty \). Then, by the following steps, one can approximate, in \( L^p(\mathbb{R}) \) (where \( 1/p^* + 1/p = 1 \) and \( 1 < p \leq 2 \)) sense, the density functions of the extrema random variables \( M_q \) and \( I_q \).

**Step 1-:** 1): Find out all zeros and poles of \( q/(q - \psi(\omega)) \) (or \( (1-q)/\{1-q\exp\{-\psi(\omega)\}\});

2): Define \( f^+(\omega) \) as product over all zeros/poles lying in \( \mathbb{C}^- \) and \( f^-(-\omega) \) as product over all zeros/poles lying in \( \mathbb{C}^+ \);

---

\(^3\)Lévy process \( X_t \) is said to belong to the Meromorphic class of Lévy process if and only if \( \tilde{\nu}^+(x) = \nu(x, \infty) \) and \( \tilde{\nu}^-(-x) = \nu(-\infty, -x) \) are two completely monotone functions and characteristic exponents \( \psi(\cdot) \) is a Meromorphic function, see [14] for more details.
Step 2): Determine error of approximating \( q/(q - \psi(\omega)) \) (or \((1 - q)/(1 - q \exp(-\psi(\omega))) \)) by \( f^+(\omega)f^-(\omega) \); 

Step 3): Obtain the density functions of \( M_q \) and \( I_q \) by the inverse Fourier transform of \( f^+(\cdot) \) and \( f^-(\cdot) \), respectively.

Proof. For an exponential stopping time, \([14]\) showed that zeros and poles of \( q/(q - \psi(\omega)) \), respectively, appear as \( \{-i\alpha_n, i\alpha_n\} \) and \( \{-i\beta_n, i\beta_n\} \), where \( \cdots < -\beta_1 < -\alpha_1 < 0 < \beta_1 < \alpha_1 < \cdots \) \([13]\) proved that \( f^+(\omega)f^-(\omega) \)

where
\[
f^+(\omega) = \prod_{n \geq 1} \frac{1 + i\omega/\alpha_n}{1 + i\omega/\beta_n} \quad \text{and} \quad f^-(\omega) = \prod_{n \geq 1} \frac{1 - i\omega/\alpha_n}{1 - i\omega/\beta_n}
\]
uniformly approximates \( q/(q - \psi(\omega)) \). Now observe that, all terms of \( f^+(\cdot) \) and \( f^-(\cdot) \) (e.g. \( \frac{1 + i\omega/\alpha_n}{1 + i\omega/\beta_n} \) or \( \frac{1 - i\omega/\alpha_n}{1 - i\omega/\beta_n} \)) are positive-definite rational functions. Therefore, \( f^+(\cdot) \) and \( f^-(\cdot) \) are two positive-definite rational functions and analytical in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \), respectively. An application of the Paly-Winer theorem warrants that the inverse Fourier transform of \( f^+(\cdot) \) and \( f^-(\cdot) \) are two positive density functions which vanish on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \), respectively.

For the geometric stopping time, using the fact that \( q < 1 \) again one may show that all poles of \((1 - q)/(1 - q \exp(-\psi(\cdot))) \) evaluated by equation \( 1 - q \exp(-\psi(\cdot)) = 0 \) or equivalently by \( \ln(q) + \psi(\cdot) = 0 \). Now, \([14]\)’s findings shows that all poles will be appear as \( \{-i\beta_n, i\beta_n\} \). On the other hands, zeros of \((1 - q)/(1 - q \exp(-\psi(\cdot))) \) are points where \( \psi(\omega) = \infty \). Therefore, zeros of appear as \( \{-i\alpha_n, i\alpha_n\} \). The rest of proof is similar. □

The following examples show application of the above procedure.

Example 5.2. Stable processes have been successfully fitted to stock returns, excess bond returns, foreign exchange rates, commodity price returns, real estate return data (see, e.g., \([18]\) and \([27]\)), financial data (see, e.g., \([5]\)), Market- and Credit-Value-at-Risk, Value-at-Risk, credit risk management (see, e.g., \([26]\)). With the exception of the normal distribution \((\alpha = 2)\), stable distribution are the heavy tailed distributions which paly an important role in heavy-tail modeling of economic data (see, e.g., \([19]\) and \([20]\)) and finance data (see, e.g., \([27]\)).

Now consider a symmetric stable process \( X_t \) with the homomorphic characteristic exponent function \( \psi(\omega) = 1/(i\mu\omega - \lambda^n|\omega|^\alpha) \), where \( \alpha \in (0, 2] \).

Using the fact that the real value \( \alpha \), in the above characteristic exponent, can be constructed from the rational numbers \( m/n \), where \( m \) and \( n \) respectively are even and odd numbers. Now, an expression \( q/(q - \psi(\omega)) \) can be restated as
\[
\frac{q i\mu\omega^n - q\lambda^m/n\omega^m}{1 - q i\mu\omega^n + q\lambda^m/n\omega^m} = \frac{(q i\mu\omega^n - q\lambda^m/n\omega^m)}{1} \prod_{i=1}^{n^+} \frac{1}{\omega - z^+_i} \prod_{i=1}^{n^-} \frac{1}{\omega - z^-_i},
\]
where \( n^+ + n^- \) is number of solutions for equation \( 1 - q i\mu\omega^n + q\lambda^m/n\omega^m = 0 \) in \( \omega \). Moreover, \( z^+_i \) and \( z^-_i \) are solutions of the recent equation where
belong to $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. Therefore, approximate solutions for the density function of extrema, $f_{q}^{\pm}$, are the inverse Fourier transform of $\phi_{q}^{\pm}(\omega) := \sqrt{q}i\mu\omega - q\lambda^{m/n}\omega^{m/n}/\prod_{i=1}^{n}(\omega^{1/n} - z_i^\mp)$.

To implement Procedure (5.1) for the Meromorphic Lévy process, one has to determine all zeros and poles of $q/(q - \psi(\cdot))$ (or $(1 - q)/(1 - q \exp\{-\psi(\cdot)\})$) which is a difficult task in may cases. Moreover, in the case where zeros or poles of $q/(q - \psi(\cdot))$ (or $(1 - q)/(1 - q \exp\{-\psi(\cdot)\})$) appear as $\{\alpha_n \pm \beta_n i\}$ (where at least one of $\alpha_n > 0$). Some terms of decomposition $f^+(\cdot)$ (or $f^-(\cdot)$) are not positive-definite rational function. Therefore, the inverse Fourier transform of $f^+(\cdot)$ and $f^-(\cdot)$ can be negative in some interval. The following procedure extents result of Procedure (5.1) for such cases and the non-homomorphic Lévy processes.

Before stating the second procedure, we need the following lemma.

**Lemma 5.3.** Suppose $\psi(\cdot)$ stands for the characteristic exponent of a Lévy process. Moreover, suppose that $\alpha_0 + \beta_0 i$ is a root of $q - \psi(\lambda) = 0$, $\lambda \in \mathbb{C}$. Then, $-\alpha_0 + \beta_0 i$ also is root of $q - \psi(\lambda) = 0$.

**Proof.** Using the Lévy Khintchine formula (Equation, (1.1)), equation of $q - \psi(\lambda) = 0$ at point $\alpha_0 + \beta_0 i$ can be restated as

\[
\begin{align*}
-\sigma^2\alpha_0\beta_0 i + \alpha_0\mu i + i \int_{\mathbb{R}} \left(e^{-\beta_0 \sin(\alpha_0 x)} - \alpha_0 x I_{[-1,1]}(x)\right) \nu(dx) = 0; \\
-\frac{1}{2}\sigma^2(\alpha_0^2 - \beta_0^2) - \mu\beta_0 + \int_{\mathbb{R}} \left(e^{-\beta_0 \cos(\alpha_0 x)} - 1 + \beta_0 x I_{[-1,1]}(x)\right) \nu(dx) = q.
\end{align*}
\]

Since $\sin(\cdot)$ and $\cos(\cdot)$, respectively, are odd and even functions. Therefore, one may conclude that point $-\alpha_0 + \beta_0 i$ satisfies the above system of equations, as well. $\square$

**Procedure 5.4.** Suppose $X_t$ is a Lévy process with characteristic exponents $\psi(\cdot)$. Moreover, suppose that the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter $q$ independent of $X_t$ and $\tau(0) = \infty$. Then, by the following steps, one can approximate, in $L^p(\mathbb{R})$ (where $1/p^* + 1/p = 1$ and $1 < p \leq 2$) sense, the density functions of the extrema random variables $M_q$ and $I_q$.

**Step 1:** Approximating $h(\omega) := q/(q - \psi(\omega))$, for the exponential stopping time, (or $h(\omega) := (1 - q)/(1 - q \exp\{-\psi(\omega)\}$), for the geometric stopping time) by a positive-definite rational function by the following steps:

1) Find out all poles of $h(\omega)$;
2) Based upon such poles pick up some positive-definite rational functions given in Lemma (2.7);
3) Approximate $h(\omega)$ by positive-definite rational function $r(\omega)$, given by Lemma (2.7);
4) Set $A_0 := \lim_{\omega \to \infty} h(\omega)$ and $m_k$ equal to order of $k^{th}$ pole.
5): Determine positive coefficients $C_{lk}$ by a visual investigation or

$$C_{lk} = \max \left\{ 0, \text{argmin} \int_{\mathbb{R}} (h(\omega) - r(\omega))^p \, d\omega \right\}$$

Step 2): Determine error of approximating $h(\omega)$ by $r(\omega)$;

Step 3): Decompose the positive-definite rational function $r(\omega)$ as a product of two functions, say $f^+(\omega)$, which are sectionally analytic and bounded in $\mathbb{C}^+$;

Step 4): Obtain the density functions of $M_q$ and $I_q$ by the inverse Fourier transform of $f^+(\cdot)$ and $f^-(\cdot)$, respectively.

Proof. Since $h(\omega)$ is a characteristic function, Lemma (2.10) warranties that, it is a positive-definite function and consequently its limit at infinity, say $A_0$, is a positive real number. Moreover Lemma (5.3) warranties that, one may use positive rational functions $r_{3k}(\cdot)$ and $r_{4k}(\cdot)$ whenever pole with form $\alpha \pm \beta i$ has been observed. The rest of proof is similar to Procedure (5.1). □

Example 5.5. Suppose $X_t$ is a Lévy process with independent and continuous $\tau(q)$ and a jumps measure $\nu(dx) = \exp\{\alpha x\} \text{cosech}(x/2) dx$. The characteristic exponent for such Lévy process is given by

$$\psi(\omega) = -\frac{\sigma^2 \omega^2}{2} - i\rho \omega - 4\pi(\omega - i\alpha) \coth(\pi(\omega - i\alpha)) + 4\gamma,$$

where $\gamma = \pi\alpha \cot(\pi\alpha)$, $\rho = 4\pi^2\alpha + \frac{4\gamma(\gamma - 1)}{\alpha}$, $\omega \in \mathbb{R}$, and $\alpha$, $\mu$, and $\sigma$ are given. Note that it is impossible to solve equation $q - \psi(\omega) = 0$ in the general case. Now consider special cases, whenever $\sigma = \mu = 2$ and $\alpha = 0$. Now, we compute the Wiener-Hopf factorization for $q = 5$. Finding all poles of $5/(5 - \psi(\omega))$ is difficult task. Using Maple 15, one may readily compute the three first poles as $\{0.4781i, 0.5658i, 1.4921i\}$. On the other hands $A_0 = \lim_{\omega \to \infty} 5/(5 - \psi(\omega)) = 0$. Now, we approximate $5/(5 - \psi(\omega))$ by

$$r(\omega) = \frac{C_1}{-i\omega + 0.4781} + \frac{C_2}{i\omega + 0.5658} + \frac{C_3}{i\omega + 1.4921}.$$  

A graphical illustration shows that, one may readily chose $C_1 = C_2 = C_3 = 1/4.5$ see Figure 1-a. Error of this approximation is about 0.08719956902. $r(\omega)$ can be restarted as

$$r(\omega) = \frac{(i\omega + 0.9560)(-i\omega + 1.9123)}{4.5(-i\omega + 0.4781)(i\omega + 0.5658)(i\omega + 1.4921)} = \frac{i\omega + 0.9560}{\sqrt{4.5(i\omega + 0.5658)(i\omega + 1.4921)}\sqrt{4.5(-i\omega + 0.4781)}} = f^-(\omega)f^+(\omega).$$

Therefore, the density function of $I_{r(5)}$ and $M_{r(5)}$ can be approximated by

$$f_{I_{r(5)}}(x) = 0.5110841035e^{1.4921x} + 0.3719983876e^{0.5658x}, \text{ for } x \leq 0;$$

$$f_{M_{r(5)}}(x) = 0.2857404120\text{Dirac}(x) + 0.4097937547e^{-0.4781x}, \text{ for } x \geq 0,$$
where $\text{Dirac}(x)$ stands for the dirac delta at point $x = 0$. Figures 1-b and 1-c illustrate behavior of $f_{\tau(\cdot)}(\cdot)$ and $f_{M\tau(\cdot)}(\cdot)$, respectively.

Example 5.6. Suppose $X_t$ in the surplus process (4.1) is the Lévy process in Example (5.5). Moreover, suppose that the random stopping time $\tau(q)$ has an exponential distribution with mean 0.2. Using result of Example (5.5), Figure 2 illustrates behavior of the finite-time ruin probability for different initial value $u$.

The following example explores situation that roots of $q - \psi(\omega) = 0$ appears in form of $\alpha + i\beta$.

Example 5.7. Consider a generalized hyperbolic process with the characteristic function

$$
\phi(\omega) = e^{i\psi(\omega)} = e^{i\mu \omega} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + i\omega)^2} \right)^{\lambda/2} K_\lambda(\delta \sqrt{\alpha^2 - (\beta + i\omega)^2}) K_\lambda(\delta \sqrt{\alpha^2 - \beta^2}),
$$
where $\lambda, \mu \in \mathbb{R}$, $\alpha, \delta > 0$, $\beta \in (-\alpha, \alpha)$, and $K_\lambda(\cdot)$ is the Modified Bessel functions of the third kind with index $\lambda$. Many well known processes are member of the class of generalized hyperbolic Lévy processes. For $\lambda > 0$ and $\delta \to 0$ one gets a Variance-Gamma process. The case $\lambda = -1/2$ corresponds to the normal inverse Gaussian process, see [9] for some analytic facts and applications about the generalized hyperbolic processes. The generalized hyperbolic process $X_t$ is a pure jump process which can be considered as a Brownian motion with drift that evolves according to an increasing Levy process (i.e., subordinator). Such properties make the generalized hyperbolic process an appealing process to model the financial returns, see [21] for more details.

Note that it is impossible to solve Equation $q - \psi(\omega) = 0$ in the general case. Now consider special cases, whenever $\alpha = \mu = 2$, $\beta = -\lambda = 1$, and $\delta = 3$. Now, we compute the Wiener-Hopf factorization for $q = 5$. Finding all poles of $5/(5 - \psi(\omega))$ is difficult task. Using Maple 15, one may readily compute the sixth first poles as $(\pm 0.4809389066 \pm 4.280110446; \pm 0.9030763690 + 2.340695867i; \pm 2.516794346 + 0.4442175550i; \pm 3.756731426 - 0.9399774855i; \pm 4.853043564 - 2.318278971i; \pm 5.894258220 - 3.713000684i; \pm 6.909960755 - 5.121014155i; \pm 7.912034845 - 6.538310520i; \pm 8.905992610 - 7.962083725i; \pm 9.894699100 - 9.390493630i)$. On the other hands $A_0 = \lim_{\omega \to \infty} 5/(5 - \psi(\omega)) = 0.

[28] established that the generalized hyperbolic process has completely monotone jump density. Therefore, One has to approximate $q/(q - \psi(\omega))$ by function class $D^*$, given by Lemma (2.7). Therefore, $5/(5 - \psi(\omega))$ can be approximated by

$$r(\omega) = \frac{C_1}{i\omega + 4.280110446} + \frac{C_2}{i\omega + 2.340695867} + \frac{C_3}{i\omega + 0.4442175550}$$
$$\quad + \frac{C_4}{-i\omega + 0.9399774855} + \frac{C_5}{-i\omega + 2.318278971} + \frac{C_6}{-i\omega + 3.713000684}$$
$$\quad + \frac{C_7}{-i\omega + 5.121014155} + \frac{C_8}{-i\omega + 6.538310520} + \frac{C_9}{-i\omega + 7.962083725}$$
$$\quad + \frac{C_{10}}{-i\omega + 9.390493630}.$$

A graphical illustration shows that, one may readily chose $C_1 = \cdots = C_{10} = 0.4$, see Figure 2-a. An $L_2(\mathbb{R})$ error of this approximation is about $0.000002527687170$, which can be improved by choosing more appropriate coefficients. $r(\omega)$ can be restarted as

$$r(\omega) = \frac{0.4}{i\omega + 4.280110446} + \frac{0.4}{i\omega + 2.340695867} + \frac{0.4}{i\omega + 0.4442175550}$$
$$\quad + \frac{0.4}{-i\omega + 0.9399774855} + \frac{0.4}{-i\omega + 2.318278971} + \frac{0.4}{-i\omega + 3.713000684}$$
$$\quad + \frac{0.4}{-i\omega + 5.121014155} + \frac{0.4}{-i\omega + 6.538310520} + \frac{0.4}{-i\omega + 7.962083725}$$
$$\quad + \frac{0.4}{-i\omega + 9.390493630} = f^-(\omega)f^+(\omega),$$
Therefore, the density function of $I_{r(5)}(\cdot)$ and $M_{r(5)}(\cdot)$ can be approximated by

$$f_{I_{r(5)}}(x) = 0.3268288347e^{0.9399774846x} + 0.6308685531e^{0.390493235x} + 0.6059253078e^{7.962085726x} + 0.4905298019e^{3.713000220x} + 0.5383620597e^{5.121016047x} + 0.5757366525e^{6.538307461x} + 0.4259214977e^{2.318279006x} \text{e}^{-0.444217554x}, \text{ for } x \leq 0;$$

$$f_{M_{r(5)}}(x) = 0.2367700968\text{Dirac}(x) + 0.4078345184e^{-2.340695867x} + 0.5390749086e^{-4.280110443x} + 0.2582813546e^{-0.444217554x}, \text{ for } x \geq 0.$$
Using result of Example (5.7), Figure 4 illustrates behavior of the finite-time ruin probability for different initial value $u$.

![Figure 4. Behavior of the finite-time ruin probability for different initial value $u$.](image)

6. Conclusion and suggestion

This article considers approximately the extrema’s density functions of a class of Lévy processes. It provides two approximation techniques for approximating such the density functions. Namely, it suggests to replace $q/(q - \psi(\cdot))$ (or $(1 - q)/(1 - q \exp\{-\psi(\cdot)\})$) by a sequence of positive-definite rational functions. Two practical approximation procedures along several examples are given. The methods presented in this article can be generalized to other situations where the multiplicative WHF is applicable, such as finding first/last passage time and the overshoot, the last time the extrema was archived, several kind of option pricing, etc. Using [25]’s findings, result of this article may be generalized to a class of multivariate Lévy processes.

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