On an explicit representation of central \((2k+1)\)-nomial coefficients

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We propose an explicit representation of central \((2k+1)\)-nomial coefficients in terms of finite sums over trigonometric constructs. The approach utilizes the diagonalization of circulant boolean matrices and is generalizable to all \((2k+1)\)-nomial coefficients, thus yielding a new family of combinatorial identities.

I. INTRODUCTION

In the first volume of Monthly in 1894, De Volson Wood asked the question “An equal number of white and black balls of equal size are thrown into a rectangular box, what is the probability that there will be contiguous contact of white balls from one end of the box to the opposite end?” \[1\]. Though this was one of the first and most clearly visualizable examples in a set of combinatorial problems which later would give rise to the field of percolation theory \[2\], to this date it still eludes a definite solution.

Mathematically, this problem is linked to counting walks in random graphs, and recently an approach was proposed which translates this combinatorially hard problem into taking powers of a specific type of circulant boolean matrices \[3\]. Interestingly, here a specific type of sum over powers of fractions of trigonometric functions with fractional angle appears, specifically \(\sum_{m=1}^{m=1} \left(\frac{\sin[kl\pi/m]}{\sin[l\pi/m]}\right)^n\) for any given \(k, m \in \mathbb{N}: m > 1, 1 \leq k \leq \lfloor m/2 \rfloor\). The latter shows a striking similarity to the famous Kasteleyn product formula for the number of tilings of a \(2n \times 2n\) square with \(1 \times 2\) dominos \[4, 5\].

Numerically one can conjecture that these constructs yield integer numbers which are multiples of central \((2k+1)\)-nomial coefficients and, thus, provide an explicit representation of the integer sequences of multinomial coefficients in terms of finite sums over real-valued elementary functions. Although various recursive equations addressing these coefficients do exist, and the Almkvist-Zeilberger algorithm \[6\] allows for a systematic derivation of recursions for multinomial coefficients in the general case, no such explicit representation has yet been proposed.

Here, we provide a simple proof of the aforementioned conjecture and propose an explicit representation of central \((2k+1)\)-nomial coefficients. To that end, let

\[
P(x) = 1 + x + x^2 + \cdots + x^{2k}
\]

be a finite polynomial of even degree \(2k, k \in \mathbb{N}, k \geq 1\) in \(x \in \mathbb{Q}\). Using the multinomial theorem and collecting terms with the same power in \(x\), the \(n\)th power of \(P(x)\) is then given by

\[
P(x)^n = (1 + x + x^2 + \cdots + x^{2k})^n = \sum_{l=0}^{2kn} p_l^{(n)} x^l
\]

with

\[
p_l^{(n)} = \sum_{n_0+\cdots+n_{2k}=n} \left( \frac{n}{n_0, n_1, \cdots, n_{2k}} \right).
\]

The central \((2k+1)\)-nomial coefficients \(M^{(2k,n)}\) are then given by \(M^{(2k,n)} = p_{kn}^{(n)}\).

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II. A TRACE FORMULA FOR CENTRAL $(2k+1)$-NOMIAL COEFFICIENTS

Consider the $(2kn + 1) \times (2kn + 1)$ circulant matrix

$$A = \text{circ}\{\left(\frac{2k+1}{2}\right) \cdot 1, 0, \ldots, 0\} = \text{circ}\left\{\sum_{l=0}^{2k} \delta_{j,1+(m+l)\mod(2kn+1)}\right\}.$$  \hfill (4)

Multiplying $A$ by a vector $x = (1, x, x^2, \ldots, x^{2k}) \in \mathbb{Q}^{2k+1}$ will yield the original polynomial as the first element in the resulting vector $Ax$. Similarly, taking the $n$th power $(n' \leq n)$ of $A$ and multiplying the result with $x$ will yield $P(x)^{n'}$ as first element, thus $A^{n'}$ will contain the sequence of multinomial coefficients $p_l^{(n')}$ in its first row. Moreover, as the power of a circulant matrix is again circulant, this continuous sequence of non-zero entries in a given row will shift by one column to the right on each subsequent row, and wraps around once the row-dimension $(2kn+1)$ is reached. This behavior will not change even if one introduces a shift by $m$ columns of the sequence of $1$ in $A$, as this will correspond to simply multiplying the original polynomial by $x^m$. Such a shift, however, will allow, when correctly chosen, to bring the desired central multinomial coefficients on the diagonal of $A^{n'}$.

We can formalize this approach in the following

**Lemma 1.** Let

$$A^{(m)} = \text{circ}\left\{\sum_{l=0}^{2k} \delta_{j,1+(m+l)\mod(2kn+1)}\right\}$$  \hfill (5)

with $m \in \mathbb{N}_0$ be circulant boolean square matrices of dimension $2kn+1$ with $k, n \in \mathbb{N}$ and $k, n \geq 1$. The central $(2k+1)$-nomial coefficients are given by

$$M^{(2k,n)} = \frac{1}{2kn+1} \text{Tr}[A^{(2kn-k)}]^n].$$  \hfill (6)

**Proof.** Let $B = \text{circ}\{(0, 1, 0, \ldots, 0)\}$ be a $(2kn+1) \times (2kn+1)$ cyclic permutation matrix, such that

$$B^0 = I = B^{2kn+1}$$

$$B^m = B^r$$ with $r = m \mod(2kn+1)$

$$B^n B^m = B^{(nm) \mod(2kn+1)},$$  \hfill (7)

where $I$ denotes the $(2kn+1)$-dimensional identity matrix. The set of powers of the cyclic permutation matrix, $\{B^m\}, m \in [0, 2kn+1]$, then acts as a basis for the circulant matrices $A^{(m)}$.

Let us first consider the case $m = 0$. It can easily be shown that

$$A^{(0)} = I + \sum_{l=1}^{2k} B^l.$$

Applying the multinomial theorem and ordering with respect to powers of $B$, we have for the $n$th power of $A^{(0)}$

$$(A^{(0)})^n = \sum_{n_0+n_1+\cdots+n_{2k}=n} \binom{n}{n_0, n_1, \ldots, n_{2k}} I^{n_0} B^{n_0+n_1+\cdots+n_{2k}}$$

$$= b_0^{(n)} I + \sum_{l=1}^{2kn} b_l^{(n)} B^l$$

$$\equiv \text{circ}\{ b_0^{(n)}, b_1^{(n)}, \ldots, b_{2kn}^{(n)} \},$$  \hfill (8)

where $b_l^{(n)} = p_l^{(n)}$.

For $m > 0$, we have

$$A^{(m)} = B^m + \sum_{l=1}^{2k} B^{m+l} \equiv B^m \left( I + \sum_{l=1}^{2k} B^l \right),$$
and obtain, with (8), for the $n$th power
\[(A^{(m)})^n = B^{mn}(b_0^{(n)})I + \sum_{l=1}^{2kn} b_l^{(n)} B^l.\] (9)

Using (7), the factor $B^{mn}$ shifts and wraps all rows of the matrix to the right by $mn$ columns, so that
\[(A^{(m)})^n = b_0^{(n,m)} I + \sum_{l=1}^{2kn} b_l^{(n,m)} B^l \equiv \text{circ}\left\{ (b_0^{(n,m)}, b_1^{(n,m)}, \ldots, b_{2kn}^{(n,m)}) \right\},\]
where $b_l^{(n,m)} = b_{(l-nm)\mod(2kn)}$ with $b_l^{(n)}$ given by (3).

For $m = 0$, the desired central multinomial coefficients can be found in column $(kn)$, i.e. $M^{(2k,n)} = b_{kn}^{(n)}$. Observing that
\[(l - n(2kn - k))\mod(2kn) \equiv (l + kn)\mod(2kn),\]
a shift by $m = 2kn - k$ yields
\[M^{(2k,n)} = b_{(l+kn)\mod(2kn)}^{(n)} = b_{(n,2kn-k)}^{(n)\mod(2kn)}.\] (10)

That is, setting $l = 0$, the central $(2k + 1)$-nomial coefficients $M^{(2k,n)}$ reside on the diagonal of $(A^{(2kn-k)})^n$. Taking the trace of $(A^{(2kn-k)})^n$ thus proves (6).

With Lemma 11 the sequences of central $(2k + 1)$-nomial coefficients $M^{(2k,n)}$ are given in terms of the trace of powers of the $(2kn + 1) \times (2kn + 1)$-dimensional circulant boolean matrix
\[A^{(2kn-k)} = \text{circ}\left\{ (1,\underbrace{0, \ldots, 0}_{k},\ldots,\underbrace{0, \ldots, 0}_{k}) \right\} \]
\[= \text{circ}\left\{ \left( \sum_{l=0}^{k} \delta_{j,1+l} + \sum_{l=0}^{k-1} \delta_{j,1+2kn-l} \right)_j \right\}.\] (11)

This translates the original problem not just into one of matrix algebra, but, in effect, significantly reduces its combinatorical complexity to finding powers of circulant matrices.

III. A SUM FORMULA FOR CENTRAL $(2k + 1)$-NOMIAL COEFFICIENTS

Not only can circulant matrices be represented in terms of a simple base decomposition using powers of cyclic permutation matrices (see above), but circulant matrices also allow for an explicit diagonalization \[11.\] The latter will be utilized to prove the main result of this contribution, namely

**Proposition 1.** The sequence of central $(2k + 1)$-nomial coefficients $M^{(2k,n)}$ with $k, n \in \mathbb{N}, k, n > 0$ is given by
\[M^{(2k,n)} = \frac{1}{2kn + 1} \left\{ (2k + 1)^n + \sum_{l=1}^{2kn} \left( \frac{\sin \left( \frac{(2k+1)l}{2kn+1} \pi \right)}{\sin \left( \frac{l}{2kn+1} \pi \right)} \right)^n \right\}.\] (12)

**Proof.** As $A^{(2kn-k)}$ is a circulant matrix, we can utilize the circulant diagonalization theorem to calculate its $n$th power. The latter states that all circulants $c_{ij} = \text{circ}(c_j)$ constructed from an arbitrary $N$-dimensional vector $c_j$ are diagonalized by the same unitary matrix $U$ with components
\[u_{rs} = \frac{1}{\sqrt{N}} \exp \left[ -\frac{2\pi i}{N} (r-1)(s-1) \right],\] (13)
$r, s \in [1, N]$. Moreover, the $N$ eigenvalues are explicitly given by
\[\lambda_r(C) = \sum_{j=1}^{N} c_j \exp \left[ -\frac{2\pi i}{N} (r-1)(j-1) \right],\] (14)
such that

\[ c_{ij} = \sum_{r,s=1}^{N} u_{ir} e_{rs} u_{sj}^* \]  \hfill (15)

with \( e_{rs} = \text{diag}[E_r(C)] = \delta_{rs} E_r(C) \) and \( u_{rs}^* \) denoting the complex conjugate of \( u_{rs} \).

Using (11), the eigenvalues of \( A^{(2kn-k)} \) are

\[
E_r(A^{(2kn-k)}) = \left\{ \begin{array}{ll}
2kn+1 & k+1 \sum_{l=0}^{k} \left( \sum_{i=0}^{k-1} \delta_{j,1+i} + \sum_{l=0}^{k-1} \delta_{j,1+2kn-l} \right) e^{-2\pi i (r-1)(j-1)/(2kn+1)} \\
2kn+1 & \sum_{l=1}^{k+1} e^{-2\pi i (l-1)(r-1)/(2kn+1)} + \sum_{l=2kn+2-k}^{2kn+1} e^{-2\pi i (l-1)(r-1)/(2kn+1)} \\
= 1 + 2 \sum_{l=1}^{k} \cos \left[ \frac{r - 1}{2kn+1} \pi \right] s_j & \frac{k}{2k+1} \sum_{l=2}^{k} \cos \left[ \frac{(1+b)(r-1)}{2kn+1} \pi \right] \sin \left[ \frac{r-1}{2kn+1} \pi \right] & r > 1 \\
= 1 & \sin \left[ \frac{2k+1}{2kn+1} (r-1) \pi \right] \sin \left[ \frac{1}{2kn+1} (r-1) \pi \right] & r = 1.
\]

Using the product-to-sum identity for trigonometric functions, the last equation can be simplified, yielding

\[
E_r(A^{(2kn-k)}) = \left\{ \begin{array}{ll}
\sin \left[ \frac{2k+1}{2kn+1} (r-1) \pi \right] / \sin \left[ \frac{1}{2kn+1} (r-1) \pi \right] & r > 1 \\
1 & r = 1.
\end{array} \right. \hfill (16)
\]

With this, Eqs. (13) and (15), one obtains for the elements of the \( n \)th power of \( A^{(2kn-k)} \)

\[
(A^{(2kn-k)})^n_{pq} = \sum_{r,s=1}^{2kn+1} u_{pr} E_r^n u_{qs}^* = \frac{1}{2kn+1} \left\{ E_0^n + \sum_{r=2}^{2kn+1} E_r^n e^{-2\pi i (r-1)(p-q)/(2kn+1)} \right\} = \frac{1}{2kn+1} (2k+1)^n + \sum_{r=1}^{2kn} \left( \frac{\sin \left[ \frac{2k+1}{2kn+1} r \pi \right]}{\sin \left[ \frac{1}{2kn+1} r \pi \right]} \right)^n e^{-2\pi i r(p-q)/(2kn+1)} \right\}.
\]

Taking the trace, finally, proves (12).

Equation (12) is remarkable in several respects. First, it provides a general, explicit representation of the sequences of central \((2k+1)\)-nomial coefficients in terms of a linearly growing, but finite, sum, thus effectively translating a combinatorial problem into an analytical one. Note specifically \( k = 1 \), \( n \in \mathbb{N} \) yields the sequence of central trinomial coefficients (OEIS A002426), \( k = 2 \) the sequence of central pentanomial coefficients (OEIS A005191), and \( k = 3 \) the sequence of central heptanomial coefficients (OEIS A025012). Secondly, utilizing trigonometric identities, this explicit representation may help to formulate general recurrences not just for coefficients of a given sequence, but between different central multinomial sequences. Moreover, by using different shift parameters \( m \) (see proof of Lemma 11), each \((2k+1)\)-nomial coefficient could potentially be represented in a similarly explicit analytical form, thus allowing for a fast numerical calculation of arbitrary \((2k+1)\)-nomial coefficients.

Finally, Proposition 11 establishes a direct link between central \((2k+1)\)-nomial coefficients and the \( n \)th-degree Fourier series approximation of a function via the Dirichlet kernel \( D_k[\theta] \). Using the trigonometric representation of Chebyshev polynomials of the second kind,

\[
U_{2k}[\cos(\alpha)] = \frac{\sin[(2k+1)\alpha]}{\sin[\alpha]},
\]
equation (12) takes the form

$$M^{(2k,n)} = \frac{1}{2kn + 1} \left\{ (2k + 1)^n + \sum_{l=1}^{2kn} \left[ U_{2k} \left( \cos \left( \frac{l\pi}{2kn + 1} \right) \right) \right]^n \right\}.$$  \hspace{1cm} (17)

Observing $U_{2k} \cos(l\pi/(2kn + 1)) \equiv D_k[2l\pi/(2kn + 1)]$ makes explicit the link between central $(2k + 1)$-nomial coefficients and the Dirichlet kernels of fractional angles.

Returning to the original problem by De Volson Wood, however, a definite solution is still at large. Here, relation (12) allows so far only for a representation of the combinatorial complexity in terms of an interesting finite analytical construct, thus, in principle, expressing a hard combinatorial problem in a trigonometric framework.

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