A Local Characterization of Lyapunov Functions and Robust Stability of Perturbed Systems on Riemannian Manifolds

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Abstract

This paper proposes several Converse Lyapunov Theorems for nonlinear dynamical systems defined on smooth connected Riemannian manifolds and characterizes properties of corresponding Lyapunov functions in a normal neighborhood of an equilibrium. We extend the methods of constructing of Lyapunov functions for ordinary differential equations on $\mathbb{R}^n$ to dynamical systems defined on Riemannian manifolds by employing the differential geometry. By employing the derived properties of Lyapunov functions, we obtained the stability of perturbed dynamical systems on Riemannian manifolds. The results are obtained by employing the notions of normal neighborhoods, the injectivity radius on Riemannian manifolds and existence of bump functions on manifolds.

Key words: Dynamical systems, Riemannian Manifolds, Geodesic Curves.

1 Introduction

The state spaces of many dynamical systems constitute Riemannian manifolds (see [1,3–5,32]) and consequently their analyses require differential geometric tools. Examples of such systems can be found in many mechanical settings, see [4,5]. Such systems arise naturally in classical mechanics (see [3–5]) where the state space of the dynamical system is restricted to such a manifold.

Stability theory is an important topic in the study of dynamical systems behavior. It addresses the stability of trajectories of dynamical systems as solutions of differential equations or differential inclusions, see [16,32,33]. It is well known that Lyapunov theory characterizes the stability of equilibrium points in the study of dynamical systems. The stability problem of dynamical systems in the sense of Lyapunov has been extensively analyzed in [16,17,22].

In this paper, we present several converse Lyapunov theorems for dynamical systems evolving on Riemannian manifolds and prove some local properties of such Lyapunov functions. To this end, we define the Lyapunov stability of dynamical systems on Riemannian manifolds based on the Riemannian distance function.

We employ the notion of geodesics on Riemannian manifolds and apply the stability results for dynamical systems on $\mathbb{R}^n$ to obtain the existence of Lyapunov functions for dynamical systems defined on Riemannian manifolds. The stability problem of dynamical systems on manifolds is a new topic which has been addressed in [23,25]. Some results for the existence of Lyapunov functions and their properties on general metric spaces are given in [7,10,13,15,21,28,30,33,35,37]. In particular, in [7,35], the existence of complete Lyapunov functions for dynamical systems on compact metric spaces is derived. In general, Riemannian manifolds can be considered as metric spaces by employing the notion of Riemannian distance function, see [18].

Using a version of stability theory for systems evolving on Riemannian manifolds (see [2,5,8]), the stability results for dynamical systems on $\mathbb{R}^n$ ([12,16,36]) are extended to those on Riemannian manifolds. By employing the notion of geodesics, we introduce a lift operator to convert the dynamical equations on a Riemannian manifold to a dynamical system on the tangent space (see [14,18,31]) of an equilibrium and invoke some of the standard results of the stability theory presented in [12,16]. It will be shown that in a normal neighborhood of an equilibrium of a dynamical system, the constructed Lyapunov functions satisfy certain properties which can be used to analyze the robustness and stability of numerous problems such as perturbed dynamical systems on Riemannian manifolds. We obtain the stability results for
perturbed dynamical systems on Riemannian manifolds in the last section of this paper. Geometric features of the normal neighborhoods such as existence of unique length minimizing geodesics and their local representations enable us to closely relate the results obtained for dynamical systems in \( \mathbb{R}^n \) to those in Riemannian manifolds.

In terms of exposition, Section 2 presents some mathematical preliminaries needed for the analyses of the paper. Section 3 presents the main results for the existence of Lyapunov functions for dynamical systems evolving on Riemannian manifolds. The results of Section 3 are employed to derive the stability of perturbed dynamical systems on Riemannian manifolds in Section 4.

## 2 Preliminaries

In this section we provide the differential geometric material which is necessary for the analyses presented in the rest of the paper. We define some of the frequently used symbols of this paper in Table 1:

### 2.1 Riemannian manifolds

**Definition 1 (see [20], Chapter 3)** A Riemannian manifold \((M, g)\) is a differentiable manifold \(M\) together with a Riemannian metric \(g\), where \(g\) is defined for each \(x \in M\) via an inner product \(g_x : T_xM \times T_xM \to \mathbb{R}\) on the tangent space \(T_xM\) (to \(M\) at \(x\)) such that the function defined by \(x \mapsto g_x(X(x), Y(x))\) is smooth for any vector fields \(X, Y \in \mathfrak{X}(M)\). In addition,

(i) \((M, g)\) is \(n\)-dimensional if \(M\) is \(n\)-dimensional;

(ii) \((M, g)\) is connected if for any \(x, y \in M\), there exists a piecewise smooth curve connecting them.

Note that in the special case where \(M \cong \mathbb{R}^n\), the Riemannian metric \(g\) is defined everywhere by \(g_x = \sum_{i=1}^n dx_i \otimes dx_i\), where \(\otimes\) is the tensor product on \(T_x^*M \times T_x^*M\), see [20].

As formalized in Definition 1, connected Riemannian manifolds possess the property that any pair of points \(x, y \in M\) can be connected via a path \(\gamma \in \mathcal{P}(x, y)\), where

\[
\mathcal{P}(x, y) = \left\{ \gamma : [a, b] \to M \left| \begin{array}{c} \text{\gamma piecewise smooth,} \\
\text{\gamma(a) = x, } \gamma(b) = y \end{array} \right. \right\} \tag{2.1}
\]

**Theorem 1** ([18], P. 94) Suppose \((M, g)\) is an \(n\)-dimensional connected Riemannian manifold. Then, for any \(x, y \in M\), there exists a piecewise smooth path \(\gamma \in \mathcal{P}(x, y)\) that connects \(x\) to \(y\).

Note that in the special case where \(M \cong \mathbb{R}^n\), the Riemannian distance \(d : M \times M \to \mathbb{R}\) is defined by the infimal path length between any two elements of \(M\), with

\[
d(x, y) = \inf_{\gamma \in \mathcal{P}(x, y)} \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \tag{2.2}
\]

The existence of connecting paths (via Theorem 1) between pairs of elements of an \(n\)-dimensional connected Riemannian manifold \((M, g)\) facilitates the definition of a corresponding Riemannian distance. In particular, the Riemannian distance \(d : M \times M \to \mathbb{R}\) is defined by the infimal path length between any two elements of \(M\), with

\[
d(x, y) = \inf_{\gamma \in \mathcal{P}(x, y)} \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \tag{2.2}
\]

Note that in the special case where \(M \cong \mathbb{R}^n\), the Riemannian distance (2.2) simplifies to \(d(x, y) = ||x - y||_e\).

Using the definition of Riemannian distance \(d\) of (2.2), it may be shown that \((M, d)\) defines a metric space.

**Theorem 2** ([18], P. 94) Any \(n\)-dimensional connected Riemannian manifold \((M, g)\) defines a metric space \((M, d)\) via the Riemannian distance \(d\) of (2.2). Furthermore, the induced topology of \((M, d)\) is the same as the manifold topology of \((M, g)\).
Definition 2 For a given smooth mapping \( F : M \rightarrow N \) from manifold \( M \) to manifold \( N \) the pushforward \( T F \) is defined as a generalization of the Jacobian of smooth maps in the Euclidean spaces as follows:

\[
TF : TM \rightarrow TN, \tag{2.3}
\]

where

\[
T_x F : T_x M \rightarrow T_{F(x)} N, \tag{2.4}
\]

and

\[
T_x F(X_x) \circ f = X_{f \circ F}, \quad X_x \in T_x M, f \in C^\infty(N). \tag{2.5}
\]

Definition 3 ([18]) A linear connection on a manifold \( M \) is a mapping \( \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \), written as \((X, Y) \mapsto \nabla_X Y \) for any smooth vector fields \( X, Y \in \mathfrak{X}(M) \), satisfying the following properties:

(i) \( \nabla_X Y \) is linear over \( C^\infty(M) \) in \( X \), i.e.,

\[
\nabla_f X_1 + h X_2 Y = f \nabla_X Y + h \nabla_{X_2} Y, \tag{2.6}
\]

for all \( f, h \in C^\infty(M) \), where \( X_1, X_2, Y \in \mathfrak{X}(M) \);

(ii) \( \nabla_X Y \) is linear over \( Y \), i.e.,

\[
\nabla_X (a Y_1 + b Y_2) = a \nabla_X Y_1 + b \nabla_X Y_2, \tag{2.7}
\]

for all \( a, b \in \mathbb{R} \), \( X, Y_1, Y_2 \in \mathfrak{X}(M) \);

(iii) \( \nabla \) satisfies the product rule, i.e.,

\[
\nabla_X (fY) = f \nabla_X Y + (X f) Y, \tag{2.8}
\]

for all \( f \in C^\infty(M) \), \( X, Y \in \mathfrak{X}(M) \).

\[ \blacksquare \]

Note that linear connections are also sometimes referred to as affine connections. Linear connections can be further specialized in the case where \((M, g)\) is a Riemannian manifold.

Definition 4 ([18]) A linear connection \( \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \) on a Riemannian manifold \((M, g)\) is

(1) compatible with Riemannian metric \( g \) if

\[
\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{2.9}
\]

for all \( X, Y, Z \in \mathfrak{X}(M) \);

(2) symmetric if it is torsion-free, i.e.,

\[
\nabla_X Y - \nabla_Y X = [X, Y], \tag{2.10}
\]

for all \( X, Y \in \mathfrak{X}(M) \), where

\[
[X, Y](f) \doteq X(Y(f)) - Y(X(f)), \quad f \in C^\infty(M). \]

\[ \blacksquare \]

On Riemannian manifolds, a unique linear connection which satisfies the properties above may be characterized by the following theorem.

Theorem 3 ([18], P. 68) Given a Riemannian manifold \((M, g)\), there exists a unique linear connection \( \nabla \) on \( M \) that is both

(i) compatible with the Riemannian metric \( g \); and

(ii) symmetric.

\[ \blacksquare \]

The unique, linear, compatible and symmetric connection specified by Theorem 3 is known as the Levi-Civita connection.

2.2 Dynamical systems on Riemannian manifolds

This paper focuses on dynamical systems governed by differential equations on a connected \( n \) dimensional Riemannian manifold \( M \). Locally these differential equations are defined by (see [20])

\[
\dot{x}(t) = f(x(t), t), \quad f(x(t), t) \in T_{x(t)} M, \quad x(0) = x_0 \in M, t \in [t_0, t_f]. \tag{2.11}
\]

The time dependent flow associated with a differentiable time dependent vector field \( f \) is a map \( \Phi_f \) satisfying:

\[
\Phi_f : [t_0, t_f] \times [t_0, t_f] \times M \rightarrow M, \tag{2.12}
\]

\[
(t_0, s, x) \rightarrow \Phi_f(s, t_0, x) \in M,
\]

and

\[
\frac{d\Phi_f(s, t_0, x)}{ds}|_{s=t} = f(x, t). \tag{2.13}
\]

One may show, for a smooth vector field \( f \), the integral flow \( \Phi_f(s, t_0, \cdot) : M \rightarrow M \) is a local diffeomorphism, see [20]. Here we assume that the vector field \( f \) is smooth and complete, i.e. \( \Phi_f \) exists for all \( t \in (t_0, \infty) \).

2.3 Geodesic Curves

As known (see [14]), geodesics are defined as length minimizing curves on Riemannian manifolds which satisfy

\[
\nabla_{\dot{s}}(t) = 0, \tag{2.14}
\]

where \( \gamma(\cdot) \) is a geodesic curve on \((M, g)\). The solution of the Euler-Lagrange variational problem associated with the
length minimizing problem shows that all the geodesics on an \( n \) dimensional Riemannian manifold \((M, g)\) must satisfy the following system of ordinary differential equations:

\[
\ddot{\gamma}_i(s) + \frac{n}{2} \Gamma^l_{jk,i} \dot{\gamma}_j(s) \dot{\gamma}_k(s) = 0, \quad i = 1, \ldots, n, \tag{2.15}
\]

where

\[
\Gamma^l_{jk,i} = \frac{1}{2} \sum_{l=1}^{n} g^{il}(g_{lj,k} + g_{ki,j} - g_{kj,l}), \quad g_{lj,k} = \frac{\partial^2 g_{lj}}{\partial x_k},
\tag{2.16}
\]

where all the indices \( i, j, k \) run from 1 up to \( n = \text{dim}(M) \) and \([g^{ij}] = [g_{ij}]^{-1}\). Note that \( g_{ij} \) is the \((i, j)\) entity of the matrix \( g \).

**Definition 5** ([18]) The restricted exponential map is defined by

\[
\text{exp}_x : T_x M \to M, \quad \text{exp}_x(v) = \gamma_v(1), \quad v \in T_x M, \tag{2.17}
\]

where \( \gamma_v(1) \) is the geodesic initiating from \( x \) with the velocity \( v \) up to one second. \[\blacksquare\]

For the economy of notation, in this paper we refer the restricted exponential maps as exponential maps. For \( x \in M \), consider a \( \delta \) ball in \( T_x M \) such that \( B_\delta(0) = \{ v \in T_x M \mid ||v||_g < \delta \} \). Then the geodesic ball is defined by the following definition.

**Definition 6** ([18]) In a neighborhood of \( x \in M \) where \( \text{exp}_x \) is a local diffeomorphism (this neighborhood always exits by Lemma 1), a geodesic ball of radius \( 0 < \delta \) is denoted by \( \text{exp}_x(B_\delta(0)) \subset M \). Also we call \( \text{exp}_x(TB_\delta(0)) \) a closed geodesic ball of radius \( \delta \). \[\blacksquare\]

**Lemma 1** ([18]) For any \( x \in M \) there exists a neighborhood \( B_\delta(0) \) in \( T_x M \) on which \( \text{exp}_x \) is a diffeomorphism onto \( \text{exp}_x(B_\delta(0)) \subset M \). \[\blacksquare\]

**Definition 7** For a vector space \( V \), a star-shaped neighborhood of \( 0 \in V \) is any open set \( U \) such that if \( u \in U \) then \( \alpha u \in U, \alpha \in [0, 1] \). \[\blacksquare\]

**Definition 8** ([18]) A normal neighborhood around \( x \in M \) is any open neighborhood of \( x \) which is a diffeomorphic image of a star shaped neighborhood of \( 0 \in T_x M \) under \( \text{exp}_x \) map. \[\blacksquare\]

**Definition 9** The injectivity radius of \( M \) is defined as follows:

\[
\forall x \in M, \quad i(x) = \sup(r),
\tag{2.18}
\]

where \( \text{exp}_x \) is diffeomorphic onto \( \text{exp}_x B_r(0) \). \[\blacksquare\]

**Definition 10** The metric ball with respect to \( d \) on \((M, g)\) is defined by

\[
B(x, r) = \{ y \in M \mid d(x, y) < r \}. \tag{2.19}
\]

The following lemma displays a relationship between the normal neighborhood and metric balls defined before on \( M \).

**Lemma 2** ([31]) If \( \text{exp}_x(\cdot) \), \( x \in M \) is a diffeomorphism on \( B_\epsilon(0) \subset T_x M, \quad \epsilon \in \mathbb{R}_{>0} \), and \( B(x, r) \subset \text{exp}_x B_\epsilon(0) \), then

\[
\text{exp}_x B_\epsilon(0) = B(x, r). \tag{2.20}
\]

\[\blacksquare\]

We note that \( B_\epsilon(0) \) is the metric ball of radius \( \epsilon \) with respect to the Riemannian metric \( g \) in \( T_x M \).

### 3 Lyapunov Analyses on Riemannian Manifolds

In this paper we extend the notion of stability for dynamical systems evolving on Riemannian manifolds. This problem has been addressed in [1, 5, 26] in a geometric framework.

**Definition 11** For the dynamical system \( \dot{x} = f(x), \quad f \in \mathcal{X}(M), \quad x \in M \) is an equilibrium if

\[
\Phi_f(t, t_0, \bar{x}) = \bar{x}, \quad t \in [t_0, \infty), \tag{3.21}
\]

where \( \Phi_f \) is the integral flow of \( f \) defined by (2.12). \[\blacksquare\]

**Definition 12** ([2, 5, 8, 16]) For the dynamical system \( \dot{x} = f(x), \quad f \in \mathcal{X}(M) \), an equilibrium \( \bar{x} \in M \) is

(i): Lyapunov stable if for any neighborhood \( U_{\bar{x}} \) of \( \bar{x} \), there exits a neighborhood \( W_{\bar{x}} \) of \( \bar{x} \), such that for all \( t_0 \in \mathbb{R} \), we have

\[
x(t_0) \in W_{\bar{x}} \Rightarrow \Phi_f(t, t_0, x(t_0)) \in U_{\bar{x}}, \quad t \in [t_0, \infty). \tag{3.22}
\]

(ii): locally asymptotically stable if it is Lyapunov stable and for all \( t_0 \in \mathbb{R} \), there exits \( U_{\bar{x}} \), such that

\[
\forall x(t_0) \in \bar{U}_{\bar{x}}, \quad \lim_{t \to \infty} \Phi_f(t, t_0, x(t_0)) = \bar{x}, \quad i.e.,
\lim_{t \to \infty} d(\Phi_f(t, t_0, x(t_0)), \bar{x}) = 0. \tag{3.23}
\]

(iii): globally asymptotically stable if it is Lyapunov stable and for all \( t_0 \in \mathbb{R} \),

\[
\forall x(t_0) \in M, \quad \lim_{t \to \infty} \Phi_f(t, t_0, x(t_0)) = \bar{x}. \tag{3.24}
\]
(iv): locally exponentially stable if it is locally asymptotically stable and for all $t_0 \in \mathbb{R}$, there exist $\mathcal{U}_\delta$ and $K, \lambda \in \mathbb{R}_{>0}$, such that

$$d(\Phi_f(t, t_0, x(t_0)), \bar{x}) \leq Kd(x, \bar{x}) \exp(-\lambda(t - t_0)), \quad K, \lambda \in \mathbb{R}_{>0}, x(t_0) \in \mathcal{U}_\delta. \quad (3.25)$$

(v): globally exponentially stable if it is globally asymptotically stable and for all $t_0 \in \mathbb{R}$, there exist $K, \lambda \in \mathbb{R}_{>0}$, such that

$$d(\Phi_f(t, t_0, x(t_0)), \bar{x}) \leq Kd(x, \bar{x}) \exp(-\lambda(t - t_0)), \quad K, \lambda \in \mathbb{R}_{>0}, x(t_0) \in \mathcal{M}. \quad (3.26)$$

We note that the convergence on $\mathcal{M}$ is defined in the topology induced by $d$ which is the same as the original topology of $\mathcal{M}$ by Theorem 2.

**Remark 1** The stability definition given by Definition 12 can be extended to time dependent dynamical systems $\dot{x} = f(x, t)$, where $\mathcal{U}_\delta$ and $\mathcal{W}_\delta$ above, depend upon the initial time $t_0$. In the case for which $\mathcal{U}_\delta$ and $\mathcal{W}_\delta$ are independent of $t_0$, the stability is uniform with respect to $t_0$.

**Definition 13** ([15, 16]) A function $v : \mathcal{M} \rightarrow \mathbb{R}$ is locally positive-definite (positive-semidefinite) around $\bar{x}$ if $v(\bar{x}) = 0$ and there exists a neighborhood $\mathcal{U}_v \subset \mathcal{M}$ such that for all $x \in \mathcal{U}_v - \{\bar{x}\}$, $0 < v(x)$ (respectively $0 \leq v(x)$).

Given a smooth function $v : \mathcal{M} \rightarrow \mathbb{R}$, the Lie derivative of $v$ along a time invariant vector field $f$ is defined by

$$\mathcal{L}_f v = dv(f), \quad (3.27)$$

where $dv : TM \rightarrow \mathbb{R}$ is the differential form of $v$, locally given by (see [20])

$$dv = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} dx_i \in T^*_\mathcal{M} \mathcal{M}, \quad (3.28)$$

where $n = \text{dim}(\mathcal{M})$.

**Remark 2** For time varying dynamical systems the Lie derivative of time varying Lyapunov function $v(x, t)$ is defined by

$$\mathcal{L}_{f(x, t)} v = dv\left(\frac{\partial}{\partial t}, f(x, t)\right), \quad (3.29)$$

where

$$dv = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} dx_i + d_x v \otimes d_t v \in T^*_\mathcal{M} \mathcal{M} \otimes T_t \mathbb{R}. \quad (3.30)$$

**Definition 14** ([1, 15, 16]) A smooth function $v : \mathcal{M} \rightarrow \mathbb{R}$ is a Lyapunov function for the time invariant vector field $f$, if $v$ is locally positive definite around the equilibrium $\bar{x}$ and $\mathcal{L}_f v$ is locally negative-definite on the same neighborhood on which $v$ is positive definite.

**Definition 15** The sublevel set $N_{\delta}(x)$ of a positive semidefinite function $v : \mathcal{M} \rightarrow \mathbb{R}$ is defined as $N_{\delta}(x) = \{x \in \mathcal{M}, \ v(x) \leq \delta\}$. By $N_{\delta}(\bar{x})$ we denote the connected sublevel set of $\mathcal{M}$ containing $\bar{x} \in \mathcal{M}$.

The following lemma shows that there exists a compact neighborhood of an equilibrium point $x$ of a dynamical system on a Riemannian manifold.

**Lemma 3** ([5]) Let $\bar{x} \in \mathcal{M}$ be an equilibrium of $\dot{x} = f(x)$, $x(t) \in \mathcal{M}$ and $v$ be a Lyapunov function on a neighborhood of $\bar{x}$. Then, for any neighborhood $\mathcal{U}_{\delta}$ of $\bar{x}$, there exists $\delta \in \mathbb{R}_{>0}$ such that $\mathcal{N}_{\delta}(\bar{x})$ is compact, $\bar{x} \in \text{intr}(\mathcal{N}_{\delta}(\bar{x}))$, and $\mathcal{N}_{\delta}(\bar{x}) \subset \mathcal{U}_{\delta}$, where $\text{intr}(A)$ is the interior set of $A$.

To analyze the behavior of dynamical systems on manifolds we employ the notion of comparison functions defined in [16].

**Definition 16** ([16]) A continuous function $\alpha : [0, r) \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_x$ if $\lim_{r \rightarrow \infty} \alpha(r) = \infty$.

**Definition 17** ([16]) A continuous function $\beta : [0, r) \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed $s$, $\beta(r, s)$ is of class $\mathcal{K}$ with respect to $r$ and for each fixed $r$, $\beta(r, s)$ is decreasing with respect to $s$ and $\lim_{s \rightarrow \infty} \beta(r, s) = 0$.

The following theorem gives the existence of $\mathcal{K}$ and $\mathcal{KL}$ functions for uniformly stable dynamical systems on Riemannian manifolds.

**Theorem 4** Consider a time varying dynamical system $\dot{x} = f(x, t)$ on an $n$ dimensional connected Riemannian manifold $(\mathcal{M}, g)$, then

- If an equilibrium $\bar{x} \in \mathcal{M}$ is uniformly stable with respect to $t_0$, then there exists a class $\mathcal{K}$ function $\alpha$ and a neighborhood $\mathcal{N}_{\delta}(\bar{x})$, such that
  $$d(\Phi_f(t, t_0, x(t_0)), \bar{x}) \leq \alpha(d(x_0, \bar{x})), \quad \forall x(t_0) \in \mathcal{N}_{\delta}(\bar{x}). \quad (3.31)$$

- If $\bar{x}$ is uniformly asymptotically stable then there exists a class $\mathcal{KL}$ function $\beta$ and a neighborhood $\mathcal{N}_{\delta}(\bar{x})$, such that
  $$d(\Phi_f(t, t_0, x(t_0)), \bar{x}) \leq \beta(d(x_0, \bar{x}), t - t_0), \quad \forall x(t_0) \in \mathcal{N}_{\delta}(\bar{x}). \quad (3.32)$$

**Proof.** Let us consider a neighborhood $\mathcal{U}_{\delta} \subset \exp_{\bar{x}} B_{i(\bar{x})}(0)$, where $i(\bar{x})$ is the injectivity radius at $\bar{x} \in \mathcal{M}$ and $B_{i(\bar{x})}(0) \subset \mathcal{U}_{\delta}$.
By Lemma 2 we have \( \exp_x B_r(0) = B(\bar{x}, r) \), provided \( 0 < r \leq i(\bar{x}) \). Hence, for any \( U_x \subset B(\bar{x}, r) \), \( 0 < r \leq i(\bar{x}) \), there exists \( \mathcal{W}^p_x \subseteq B(\bar{x}, r) \), such that \( x(t_0) \in \mathcal{W}^p_x \) results in \( \Phi_f(t, t_0, x_0) \) remains in a normal neighborhood of \( \bar{x} \).

To prove the first item we note that the uniform stability of \( \bar{x} \), implies that there exists \( W_{\epsilon} \subset M \), such that \( x(t_0) \in W_{\epsilon} \) results in \( \Phi_f(t, t_0, x_0) \in U_x \) for all \( t \in [t_0, \infty) \). Hence, \( W_{\epsilon} \subseteq U_x \subset \exp_x B_{\bar{x}(\epsilon)}(0) \) and \( \Phi_f(t, t_0, x_0) \) remains in a normal neighborhood of \( \bar{x} \).

We note that since \( \delta \) is a local diffeomorphism, implies \( 0 < \delta(r) \leq \bar{x} \). Define

\[
\delta(r) = \begin{cases} 
\arg\max_{x \in \mathbb{R}^n} \exp_x B_{l}(0) \subseteq \mathcal{W}^p_x, & r \leq i(\bar{x}) \\
\delta(i(\bar{x})) & i(\bar{x}) < r
\end{cases}
\]

We note that since \( \mathcal{W}^p_{\bar{x}} \) is an open set in \( M \) and the induced topology by the distance function \( d \) is the same as the manifold topology (Theorem 2), there always exists \( \epsilon \in \mathbb{R}^n \), such that \( \exp_x B_{\bar{x}}(0) \subseteq \mathcal{W}^p_{\bar{x}} \). Since our argument is local, without loss of generality, we assume \( i(\bar{x}) < \infty \).

Then for \( r \leq i(\bar{x}) \) we have \( \mathcal{W}^p_{\bar{x}} \subseteq \exp_x B_{l}(0) \), which together with the compactness of \( B_{\bar{x}}(0) \subset T_x M \), \( (T_x M) \) is a finite dimensional vector space, \( B_{\bar{x}}(0) \) is a closed and bounded set and \( \exp_x \) is a local diffeomorphism, implies \( 0 < \delta(r) < \infty \). Note that since \( \exp_x \) is a local diffeomorphism then \( \exp_x B_{\bar{x}}(0) = \exp_x B_{\bar{x}}(0) \). Now we show that \( \delta(r) \) is nondecreasing with respect to \( r \). Suppose \( \delta(r) \) is strictly decreasing then for \( r_1 < r_2 < i(\bar{x}) \) we have \( \delta(r_2) < \delta(r_1) \). Denote the associated neighborhoods of \( B(\bar{x}, r_1) \) and \( B(\bar{x}, r_2) \) by \( \mathcal{W}^p_{\bar{x}1} \) and \( \mathcal{W}^p_{\bar{x}2} \) respectively. Then \( \delta(r_2) < \delta(r_1) \) implies

\[
\exists x_0 \in \mathcal{W}^p_{\bar{x}1}, \text{ s.t. } x_0 \notin \mathcal{W}^p_{\bar{x}2},
\]

where \( B(\bar{x}, r_1) \subset B(\bar{x}, r_2) \). However, \( x_0 \in \mathcal{W}^p_{\bar{x}2} \) results in \( \Phi_f(t, t_0, x_0) \in B(\bar{x}, r_1) \subset B(\bar{x}, r_2) \), which contradicts \( x_0 \notin \mathcal{W}^p_{\bar{x}2} \). Hence, \( \delta(r_1) \leq \delta(r_2) \).

We choose a \( K \) class function \( \zeta(r) \) such that \( \zeta(r) \leq \delta(r) \) (this is always possible since \( \delta \) is nondecreasing), and \( \zeta^{-1} \) is a \( K \) class function. Now choose \( N_{\epsilon} = \exp_x B_{\delta(i(\bar{x}))}(0) \). Then select \( U_x = B(\bar{x}, \zeta^{-1}(d(x_0, \bar{x}))) \) where \( r = \zeta^{-1}(d(x_0, \bar{x})) \) implies

\[
d(x_0, \bar{x}) = \zeta(r) \leq \delta(r),
\]

and hence

\[
\forall x_0 \in B(\bar{x}, \delta(r)) \Rightarrow \Phi_f(t, t_0, x_0) \in U_x, \text{ } t \in [t_0, \infty).
\]
\[ \dot{z}(t) = T_x \exp_x^{-1} (f(x,t)) = T_{\exp_x z} \exp_x^{-1} (f(\exp_x z, t)) = \dot{f}(z,t), \]

\[ (3.40) \]

where \( \dot{z}(t) \in T_{x(t)} T_x M \approx T_x M. \) We note that the equilibrium \( \bar{x} \) of \( f(x,t) \) changes to \( z = 0 \in T_x M \) for the dynamical equations in \( z \) coordinates. In the case \( M = \mathbb{R}^n \), we have

\[ x = \exp_x z = \bar{x} + z \in \mathbb{R}^n. \]

\[ (3.41) \]

For any \( (x(t) \in \exp_x B_x(0), \) we have \( x(t_0) = \exp_x z(t_0) \) for some \( z(t_0) \in B_x(0). \) Now let us consider the geodesic curve \( \gamma(\cdot) : [0, 1] \to M, \gamma(\tau) = \exp_x \tau z(t_0). \) Employing the results of [18], Proposition 5.11, in normal coordinates of \( \bar{x}, \) we have

\[ \gamma(\tau) = (\tau z_1(t_0), ..., \tau z_n(t_0)), \]

\[ (3.42) \]

where \( d(x(t_0), \bar{x}) = \left( \sum_{j=1}^n z_j^2(t_0) \right)^{1/2} = ||z(t)||_e = ||z(t)||_g. \) The last equality is due to the fact that in normal coordinates of \( \bar{x}, \) the Riemannian metric is given by

\[ g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij} + O(r^2), \]

\[ (3.43) \]

where \( r \) is the distance function, see [31], Chapter 5. Hence, \( g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})|_x = \delta_{ij} \) and \( ||z(t)||_e = ||z(t)||_g. \) Therefore, we have

\[ ||z(t)||_g \leq \beta(||z(t)||_g, t - t_0), \]

\[ z(t_0) = z_0 \in B_x(0) \subset T_x M. \]

\[ (3.44) \]

The uniform boundedness of \( T_x f(x,t) \) together with smoothness of \( \exp^{-1} \) imply that \( \frac{\dot{\gamma}}{\gamma} \) is uniformly bounded on \( B_x(0) \subset T_x M. \) Hence, we can apply the results of [16], Theorem 4.16 for the existence of \( v(z,t) \) satisfying the following properties.

\begin{align*}
(i) & : \quad \alpha_1(||z(t)||_g) \leq v(z,t) \leq \alpha_2(||z(t)||_g), \\
(ii) & : \quad \mathcal{L}_{f(z,t)} v = -\alpha_3(||z(t)||_g), \\
(iii) & : \quad |T_v(T_x \exp_x^{-1} z)| \leq \alpha_4(||z(t)||_g) \\
& \quad T v : T T_x M \to TR \approx \mathbb{R} \times \mathbb{R}. \quad (3.45)
\end{align*}

Since \( \exp_x \) is a local diffeomorphism we have \( x = \exp_x \circ \exp_x^{-1} x. \) Hence,

\[ \text{Id} = T (\exp_x \circ \exp_x^{-1}) = T \exp_x (\exp_x^{-1} x) \circ T \exp_x^{-1} \]

\[ = T \exp_x^{-1} x \exp_x \circ T_x \exp_x^{-1}, \]

\[ (3.46) \]

where \( \text{Id} \) is the identity map. This shows

\[ T_x \exp_x^{-1} = \left( T_{\exp_x^{-1} x} \exp_x \right)^{-1}. \]

\[ (3.47) \]

The Lie derivative of \( v \) with respect to \( \dot{f} \) is locally given by

\[ \mathcal{L}_{f(z,t)} v = dv(\frac{\partial}{\partial t}, \dot{f}(z,t)) = d_t v(\frac{\partial}{\partial t}) + d_z v(\dot{f}(z,t)). \]

\[ (3.48) \]

Since \( v \) is a scalar function then \( dv(\frac{\partial}{\partial t}, \dot{f}(z,t)) = T_v(T_x(v(\frac{\partial}{\partial z}, \dot{f}(z,t))) + T_z(\dot{f}(z,t))). \) Employing \( \exp_x \), we define the following Lyapunov function on \( M. \)

\[ \tilde{v}(x,t) = v(\exp_x^{-1} x, t), \quad x \in \exp_x B_x(0). \]

\[ (3.49) \]

Then the Lie derivative of \( \tilde{v} \) along \( f \) is

\[ \mathcal{L}_{f(x,t)} \tilde{v} = d_t \tilde{v}(\frac{\partial}{\partial t}) + T_z \tilde{v}(f(x,t)) \]

\[ = d_t \tilde{v}(\frac{\partial}{\partial t}) + T_z \tilde{v} \left( T_x \exp_x^{-1} \circ T_z \exp_x(\dot{f}(z,t)) \right) \]

\[ = d_t \tilde{v}(\frac{\partial}{\partial t}) + T_z \tilde{v} \left( T_x \exp_x^{-1} \circ \exp_x(\dot{f}(z,t)) \right) \]

\[ = d_t \tilde{v}(\frac{\partial}{\partial t}) + T_z \tilde{v} \left( \dot{f}(z,t) \right) \]

by employing 3.47

\[ = d_t \tilde{v}(\frac{\partial}{\partial t}) + T_z \tilde{v} \left( \dot{f}(z,t) \right) \]

\[ = \mathcal{L}_{f(z,t)} \tilde{v}. \]

\[ (3.50) \]

Same argument applies to \( T_z \tilde{v}(\frac{\partial}{\partial z}) \) and shows that

\[ T_x \tilde{v}(\frac{\partial}{\partial x}) = T_z \tilde{v}(\frac{\partial}{\partial z}). \]

\[ (3.51) \]

Since the Lyapunov function constructed above is defined locally, it remains to extend the domain of the definition of \( \tilde{v} \) to \( M. \) For \( \delta \in (0, \epsilon), \) compactness of \( \overline{B_x(0)} \subset T_x M \) and smoothness of \( \exp_x \) together imply that \( \exp_x \overline{B_x(0)} \subset \exp_x B_x(0) \) is a compact set in \( M. \) Choose a bump function \( \psi \in C^\infty(M) \) such that (for the definition of bump functions see [20]) \( \psi \equiv 1 \) on \( \exp_x \overline{B_x(0)} \) and suppose \( \psi \in C^\infty B_x(0), \) where \( \supp \psi \subset \{ x \in M \ s.t. \ \psi(x) \neq 0 \}. \) As shown in [20], Proposition 2.26, such bump functions always exist. Hence, we consider \( U_x = \exp_x B_x(0) \) and \( w \equiv \psi \cdot \tilde{v} : M \times R \to R. \) The Lie derivative of \( w \) is given by

\[ \mathcal{L}_{f(x,t)} w = \mathcal{L}_{f(x,t)} \psi \cdot \dot{v} = \psi \mathcal{L}_{f(x,t)} \tilde{v} + \partial \mathcal{L}_{f(x,t)} \psi, \]

\[ (3.52) \]

where on \( U_x \) we have

\[ \mathcal{L}_{f(x,t)} w = \mathcal{L}_{f(x,t)} \tilde{v}. \]

\[ (3.53) \]
Same argument shows that on $U_{\bar{x}}, T_xw(\frac{\partial}{\partial x}) = T_xv(\frac{\partial}{\partial x})$, which completes the proof for the Lyapunov function $w$. ■

Theorem 6 Let $\bar{x}$ be uniformly exponentially stable on a neighborhood $\mathcal{N}_{\bar{x}} \subset U_{\bar{x}}$ ($U_{\bar{x}}$ is a normal neighborhood around $\bar{x}$) for the smooth dynamical system $\dot{x}(t) = f(x, t)$ on a $n$ dimensional Riemannian manifold $(M, g)$, and $||T_xf(x, t)||$ is uniformly bounded, where $||.||$ is the norm of the linear operator $Tf(x, t) : TM \to TM$. Then, for some $U_{\bar{x}} \subset U_{\bar{x}}^0, \forall x(t_0) \in U_{\bar{x}}$, there exists a Lyapunov function $v : M \times R \to R_{>0}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in R_{>0}$, such that for all $x \in U_{\bar{x}}$

\begin{align}
(i) : & \quad \lambda_1d^2(x, \bar{x}) \leq v(x, t) \leq \lambda_2d^2(x, \bar{x}), \\
(ii) : & \quad L_f(x, o)v \leq -\lambda_3d^2(x, \bar{x}), \\
(iii) : & \quad ||Tv|| \leq \lambda_4d(x, \bar{x}), \\
Tv : TM \to TR \simeq R \times R. \quad (3.54)
\end{align}

Proof. The proof parallels the proof of Theorem 5 and the results of [16], Theorem 4.14.

We note that employing the normal coordinate system used in the proof of Theorem 5, $d(\Phi(t, t_0, x(t_0)), \bar{x}) \leq Kd(x(t_0), \bar{x}) \exp(-\lambda(t-t_0))$ implies $||z(t)||_g \leq K\exp(-\lambda(t-t_0))||z(t_0)||_g$ which is required in the proof of Theorem 4.14 in [16]. ■

The Lyapunov functions in Theorems 5 and 6 are constructed in a normal neighborhood of an equilibrium where $exp$ is a local diffeomorphism. Hence, the properties derived in Theorems 5 and 6 hold locally and the corresponding neighborhoods are restricted by the injectivity radius of the equilibrium. Depending on the geometric features of $M$, the injectivity radius of a particular point might be very small. In this section we construct Lyapunov functions on a compact subset of a local chart of an equilibrium of a dynamical system on $M$ by scaling the Riemannian and Euclidean metrics. This is also a local method since we are restricted to work within a local coordinate system. However, in some cases, it may provide much larger neighborhood on which Theorems 5 and 6 hold.

Theorem 7 Let $\bar{x}$ be an equilibrium for the smooth dynamical system $\dot{x}(t) = f(x, t)$ on a coordinate chart $(U, \phi)$ of $\bar{x}$, such that there exists a $K\mathcal{L}$ function $\beta$, which satisfies

\begin{equation}
\tag{3.55}
d(\Phi(t, t_0, x(t_0)), \bar{x}) \leq \beta(d(x_0, \bar{x}), t - t_0), \quad x(t_0) = x_0 \in U.
\end{equation}

Assume $\langle |T_xf(x, t)| \rangle$ is uniformly bounded with respect to $t$ on $U$, where $\langle . \rangle$ is the norm of the linear operator $Tf(x, t) : TM \to TM$. Then, for some $U \subset U$, for all $x(t_0) = x_0 \in U$, there exist a Lyapunov function $w : M \times R \to R_{>0}$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K$, such that for all $x \in U_{\bar{x}}$

\begin{align}
(i) : & \quad \alpha_1(d(x, \bar{x})) \leq w(x, t) \leq \alpha_2(d(x, \bar{x})), \\
(ii) : & \quad L_f(x, o)w \leq -\alpha_3(d(x, \bar{x})), \\
(iii) : & \quad |T_xw| \leq \alpha_4(d(x, \bar{x})), \\
T_w : TM \to TR \simeq R \times R. \quad (3.56)
\end{align}

Proof. Consider a coordinate chart $(U, \phi)$ around $\bar{x}$, where $\phi : M \to R^n$. By definition, $\phi$ is a homeomorphism to an open set in $R^n$, see [19]. Without loss of generality we assume $\phi(\bar{x}) = (0, \ldots, 0)$, otherwise we can consider the map $\psi(x) = \phi(x) - \phi(\bar{x})$, where $\psi$ is a homeomorphism. Denote

$\Omega \doteq \text{argmax}_{r \in R_{>0}} B_r(r, 0), \text{ s.t. }$ \begin{equation}
B_r(0) \subset \phi(U), \phi^{-1}(B_r(0)) \subset U, \quad (3.57)
\end{equation}

where $B_r(0)$ is the Euclidean ball of radius $r$. In $R^n$, $B_r(0)$ is a compact set and since $\phi$ is a homeomorphism then, $\phi^{-1}(B_r(0)) \subset M$ is a compact set. By the stability assumption, there exists $W_{\bar{x}} \subset M$, such that for all $x_0 \in W_{\bar{x}}$, $\Phi(t, t_0, x_0) \in \phi^{-1}(B_{r}(0))$. Replace the Riemannian metric $||.||_g$ by the Euclidean metric $||.||_e$ on $\phi^{-1}(B_{r}(0))$. Since $\phi^{-1}(B_{r}(0))$ is compact, there exists $c_1, c_2 \in R_{>0}$, such that (see [18])

\begin{align}
&c_1||X||_g \leq ||X||_e \leq c_2||X||_g, \\
&X \in T_x M, x \in \phi^{-1}(B_{r}(0)) \subset M. \quad (3.58)
\end{align}

Since the state trajectory is bounded in $\phi^{-1}(B_{r}(0))$, by replacing the Riemannian metric with the Euclidean one, the state trajectory will be bounded in $B$. Employing (3.58), the Euclidean distance function is bounded by the Riemannian one as follows. Consider any piecewise smooth curve $\gamma : [a, b] \to M$ connecting $x \in \phi^{-1}(B_{r}(0))$ and $\bar{x}$, such that $\gamma(a) = x$ and $\gamma(b) = \bar{x}$. If $\gamma$ belongs to $\phi^{-1}(B_{r}(0)) \subset M$, then

\begin{align}
||x - \bar{x}||_e \leq \int_a^b ||\dot{\gamma}(s)||_e ds \leq c_2 \int_a^b ||\dot{\gamma}(s)||_g ds, \quad (3.59)
\end{align}

where $||x - \bar{x}||_e$ is the Euclidean distance between $x$ and $\bar{x}$. In case $\gamma$ does not entirely belong to $\phi^{-1}(B_{r}(0))$, then there exists a time $t \in [a, b]$, such that $\gamma(s) \in \phi^{-1}(S_t(0)), s \in [a, t]$ and $||x - \gamma(t)||_e = R$, where $S_t(0) = \{x \text{ s.t. } ||x||_e = R\}$. Hence, since $x \in \phi^{-1}(B_{r}(0))$, we have

\begin{align}
||x - \bar{x}||_e \leq R \leq \int_a^t ||\dot{\gamma}(s)||_e ds \leq c_2 \int_a^t ||\dot{\gamma}(s)||_g ds \\
\leq c_2 \int_a^b ||\dot{\gamma}(s)||_g ds. \quad (3.60)
\end{align}
Therefore, for any piecewise smooth $\gamma$, $||x - \bar{x}||_e \leq c_2 \Phi_2 ||\gamma(s)||_g ds$. Taking the infimum over all $\gamma$, we have

$$||x - \bar{x}||_e \leq c_2 d(x, \bar{x}).$$

(3.61)

Since $B_\varepsilon(\mathfrak{R}, 0)$ is a convex set then the straight line connecting $x$ and $\bar{x}$ is entirely in $\phi^{-1}(B_\varepsilon(\mathfrak{R}, 0))$. Hence, $c_1 d(x, \bar{x}) \leq c_1 \int_a^b ||\gamma(s)||_g ds \leq c_1 \int_a^b ||\gamma(s)||_e ds = ||x - \bar{x}||_e$.

(3.62)

Therefore

$$\frac{1}{c_2} ||x - \bar{x}||_e \leq d(\Phi_f(x, t, 0, x_0), \bar{x}) \leq \beta(d(x_0, \bar{x}), t - t_0) \leq \beta\left(\frac{1}{c_1} ||x - \bar{x}||_e, t - t_0\right).$$

(3.63)

Hence, $||x - \bar{x}||_e \leq c_2 \beta\left(\frac{1}{c_1} ||x - \bar{x}||_e, t - t_0\right) \leq \beta(||x - \bar{x}||_e, t - t_0)$.

The norm of $T_x f(x, t)$ is defined by

$$||T_x f(x, t)||_g \leq \sup_{X \in \mathcal{T}_x M} \frac{||T_x f(x, t)(X)||_e}{||X||_e} \leq \sup_{X \in \mathcal{T}_x M} \frac{c_2 ||T_x f(x, t)(X)||_g}{c_1 ||X||_g} \leq \frac{c_1}{c_2} ||T_x f(x, t)||_g.$$

(3.64)

Hence, boundedness of $||T_x f(x, t)||_g$ implies the bounded-ness of $||T_x f(x, t)||_e$. We can apply the results of [12, Theorem 4.16] to the dynamical system evolving on $M$, where $|| \cdot ||_g$ is replaced by $|| \cdot ||_e$. Therefore, there exists a Lyapunov function $\nu$ such that

(i) : $\alpha_1(||x||_e) \leq \nu(x, t) \leq \alpha_2(||x||_e),
(ii) : \nu_f(x, t) v \leq -\alpha_3(||x||_e) ,
(iii) : ||T_x v||_e \leq \alpha_4(||x||_e)$

$$Tv: T_{\bar{x}} M \rightarrow \mathbb{R} \times \mathbb{R}, x \in B_\varepsilon(\mathfrak{R}, 0),$$

(3.65)

where $||x||_e = ||x - \bar{x}||_e$. As a result of the scaling the Riemannian and Euclidean norms, we have

(i) : $\alpha_1(c_1 d(x, \bar{x})) \leq \nu(x, t) \leq \alpha_2(c_2 d(x, \bar{x})) ,
(ii) : \nu_f(x, t) v \leq -\alpha_3(c_1 d(x, \bar{x})) ,
(iii) : ||T_x v||_g \leq \frac{c_2}{c_1} \alpha_4(c_2 d(x, \bar{x}))$.

(3.66)

4 Stability of Perturbed Dynamical Systems

The properties of the constructed Lyapunov functions in Theorems 5 and 6 are employed to obtain the robust stability results for perturbed dynamical systems on Riemannian manifolds. Consider the following perturbed dynamical systems on $(M, g)$,

$$\dot{x}(t) = f(x, t) + h(x, t), f, h \in \mathcal{X}(M \times \mathbb{R}).$$

(4.67)

The term $g$ can be considered as a perturbation of the nominal system $f$. As stated in [9, 12, 24], stability results for (4.67) can be obtained based on technical assumptions on the stability of the nominal system $f$ and boundedness of $h$.

The following theorem gives the stability of (4.67), where the nominal system is locally uniformly asymptotically stable.

Theorem 8 Let $\bar{x}$ be an equilibrium of $\dot{x} = f(x, t), f \in \mathcal{X}(M \times \mathbb{R})$, which is uniformly asymptotically stable on a normal neighborhood $\mathcal{N}_\varepsilon$. Assume the perturbed dynamical system 4.67 is complete and the Riemannian norm of the perturbation $h \in \mathcal{X}(M \times \mathbb{R})$ is bounded on $\mathcal{N}_\varepsilon$, i.e., $||h(x, t)||_g \leq \delta, x \in \mathcal{N}_\varepsilon, t \in [t_0, \infty)$. Then, for sufficiently small $\delta$, there exists a neighborhood $U_{\bar{x}}$, such that

$$\lim_{t \rightarrow \infty} d(\Phi_{f+h}(t, 0, x_0), \bar{x}) \leq \rho(\delta), x_0 \in U_{\bar{x}},$$

(4.68)

for a class $K$ function $\rho$.

Proof. Following the proof of Theorem 5, there exists $U_{\bar{x}} \subset \mathcal{N}_\varepsilon$, such that (3.39) holds. First we show that the neighborhood $U_{\bar{x}}$ in Theorem 5 can be shrunk, such that $\Phi_{f+h} \subset \mathcal{N}_\varepsilon$ provided $x_0 \in U_{\bar{x}}$. By Lemma 3 there exists $\mathcal{N}_{\bar{x}}(\delta)$, such that $\mathcal{N}_{\bar{x}}(\delta) \subset U_{\bar{x}}$. The variation for $w$ along $f + h$ is given by

$$\mathcal{L}_{f+h} w = \mathcal{L}_f w + \mathcal{L}_h w \leq -\alpha_3(d(x, \bar{x})) + \mathcal{L}_h w, x \in \text{int}(\mathcal{N}_{\bar{x}}(\delta)).$$

(4.69)

By the Shrinking Lemma there exists a precompact set $W_{\bar{x}}$, such that $W_{\bar{x}} \subset \text{int}(\mathcal{N}_{\bar{x}}(\delta)) \subset \mathcal{N}_{\bar{x}}(\delta)$, see [12]. Hence, $M - W_{\bar{x}}$ is a closed set and $\mathcal{N}_{\bar{x}}(\delta) \cap M - W_{\bar{x}}$ is a compact set (compact sets of Hausdorff spaces are closed). The smoothness of $w$ and compactness of $\mathcal{N}_{\bar{x}}(\delta) \cap M - W_{\bar{x}}$ implies that there exists $\mathfrak{M}$,

$$\mathfrak{M} \triangleq \sup_{x \in \mathcal{N}_{\bar{x}}(\delta) \cap M - W_{\bar{x}}} -\alpha_3(d(x, \bar{x})) < 0.$$

(4.70)

Note that $\alpha_3 \in \mathcal{K}, x \in \mathcal{N}_{\bar{x}}(\delta) \cap M - W_{\bar{x}}$, and $d(x, \bar{x}) > 0, x \in \mathcal{N}_{\bar{x}}(\delta) \cap M - W_{\bar{x}}$, since $W_{\bar{x}}$ is a neighborhood of $\bar{x}$. Therefore, $\mathfrak{M} < 0$. Using (3.27) implies that $\mathcal{L}_w w = dw(h) \leq ||dw|| \cdot ||h||_g \leq \delta ||dw||$, where smoothness of $w$ and compactness of $\mathcal{N}_{\bar{x}}(\delta)$ together imply $||dw|| \leq \infty$. Note that $||dw||$ is the norm of the linear
operator \( dw : T_x \mathcal{M} \to \mathbb{R} \). Hence, for sufficiently small \( \delta \), we have \( \mathcal{L}_{f+h} w < 0, x \in \mathcal{N}_b(\bar{x}) \cap \mathcal{M} - \mathcal{W}_b \). Therefore, the state trajectory \( \Phi_{f+h}(\cdot, t_0, x_0) \) stays in \( \mathcal{U}_b \) for all \( x_0 \in \text{int}(\mathcal{N}_b(\bar{x})) \). The variation of \( w \) along \( f + h \) is then given by

\[
\mathcal{L}_{f+h} w = \mathcal{L}_f w + \mathcal{L}_h w \leq \alpha_3(d(\bar{x}, \bar{x})) + \mathcal{L}_h w
\]

\[
\leq -\alpha_3(d(\bar{x}, \bar{x})) + \mathcal{L}_h w = -\alpha_3(d(\bar{x}, \bar{x})) + dw(h)
\]

\[
\leq -\alpha_3(d(\bar{x}, \bar{x})) + ||T_x w|| \cdot ||h||_g
\]

\[
\leq -\alpha_3(d(\bar{x}, \bar{x})) + \delta \alpha_4(d(\bar{x}, \bar{x}))
\]

\[
\leq -(1 - \theta)\alpha_3(d(\bar{x}, \bar{x})) - \theta \alpha_3(d(\bar{x}, \bar{x}))
\]

\[
+ \delta \alpha_4(d(\bar{x}, \bar{x})) \leq -(1 - \theta)\alpha_3(d(\bar{x}, \bar{x})),
\]

if \( \frac{\alpha_3^{-1}(\delta \alpha_4(r_1))}{\theta} \leq d(x, \bar{x}) \leq r_2 \).

The variation of \( v \) along \( f + h \) is then given by

\[
\mathcal{L}_{f+h} v = \mathcal{L}_f v + \mathcal{L}_h v \leq -\lambda_3 d^2(\bar{x}, \bar{x}) + \mathcal{L}_h v
\]

\[
\leq -\lambda_3 d^2(\bar{x}, \bar{x}) + \mathcal{L}_h v = -\lambda_3 d^2(\bar{x}, \bar{x}) + dv(h)
\]

\[
= -\lambda_3 d^2(\bar{x}, \bar{x}) + ||T_x v|| \cdot ||h||_g
\]

\[
\leq -\lambda_3 d^2(\bar{x}, \bar{x}) + \delta \lambda_4 d(\bar{x}, \bar{x}).
\]

(4.73)

Hence,

\[
\mathcal{L}_{f+h} v = \dot{v} \leq -\frac{\lambda_3}{\lambda_2} v + \frac{\lambda_3}{\lambda_1} \sqrt{\frac{v}{\lambda_1}}.
\]

(4.74)

Following the Comparison Method presented in [16], Section 9.3, we have

\[
\sqrt{v(x, t)} \leq \sqrt{v(x_0, t_0)} \exp\left(-\frac{\lambda_3}{2\lambda_2}(t - t_0)\right)
\]

\[
+ \frac{\lambda_3\lambda_2}{\lambda_1^2} \left[1 - \exp\left(-\frac{\lambda_3}{2\lambda_2}(t - t_0)\right)\right].
\]

(4.75)

Therefore,

\[
d(\Phi_{f+h}(t, t_0, x_0), \bar{x}) \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \exp\left(-\frac{\lambda_3}{2\lambda_2}(t - t_0)\right) d(x_0, \bar{x})
\]

\[
+ \frac{\lambda_3\lambda_2}{\lambda_4\lambda_1} \left[1 - \exp\left(-\frac{\lambda_3}{2\lambda_2}(t - t_0)\right)\right]
\]

\[
\leq \sqrt{\frac{\lambda_2}{\lambda_1}} \exp\left(-\frac{\lambda_3}{2\lambda_2}(t - t_0)\right) d(x_0, \bar{x})
\]

\[
+ \frac{\lambda_3\lambda_2}{\lambda_4\lambda_1} \delta, \quad \beta \leq \frac{\lambda_3\lambda_2}{\lambda_4\lambda_1}
\]

which completes the proof for \( k = \sqrt{\lambda_1}, \gamma = \frac{\lambda_3}{\lambda_2} \) and \( \beta = \frac{\lambda_3\lambda_2}{\lambda_4\lambda_1} \).

(4.76)

\[
\text{Proof. By Theorem 6 there exists a Lyapunov function } v \text{ which satisfies (3.54). Hence, following the proof of Theorem 8, there exists a connected compact sublevel subset of } v \text{ such that } \mathcal{N}_b(\bar{x}) \subset \mathcal{U}_b, \text{ where } \Phi_{f+h}(t, t_0, \bar{x}_0) \in \mathcal{N}_b(\bar{x}), \; \bar{x}_0 \in \text{int}(\mathcal{N}_b(\bar{x})), \; t \in [t_0, \infty). \text{ Since } \text{int}(\mathcal{N}_b(\bar{x})) \text{ is an open set, for a given } x_0 \in \text{int}(\mathcal{N}_b(\bar{x})), \text{ we can choose } \bar{x}_0 \text{ sufficiently close to } x_0, \text{ such that } \bar{x}_0 \in \text{int}(\mathcal{N}_b(\bar{x})).
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