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A stochastic mass conserved reaction-diffusion equation

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Abstract

In this paper, we prove a well posedness result for an initial boundary value problem for a stochastic nonlocal reaction-diffusion equation with nonlinear diffusion together with a null-flux boundary condition in an open bounded domain of \( \mathbb{R}^n \) with a smooth boundary. We suppose that the additive noise is induced by a Q-Brownian motion.

1 Introduction

We study the problem

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = \text{div}(A(\nabla \varphi)) + f(\varphi) - \frac{1}{|D|} \int_D f(\varphi) dx + \frac{\partial W}{\partial t}, & x \in D, t \geq 0 \\
A(\nabla \varphi).\nu = 0, & \text{on } \partial D \times \mathbb{R}^+ \\
\varphi(x,0) = \varphi_0(x), & x \in D
\end{cases}
\]

where:

- \( D \) is an open bounded set of \( \mathbb{R}^n \) with a smooth boundary \( \partial D \);
- \( \nu \) is the outer normal vector to \( \partial D \);
- The initial function \( \varphi_0 \) is such that \( \varphi_0 \in L^2(D) \);
- We suppose that the nonlinear function \( f \) is a smooth function which satisfies the following properties:
  
  (F_1) There exist positive constants \( C_1 \) and \( C_2 \) such that
  
  \[ f(a + b)a \leq -C_1 a^{2p} + f_2(b), \quad |f_2(b)| \leq C_2 (b^{2p} + 1), \quad \text{for all } a, b \in \mathbb{R} \]
  
  (F_2) There exist positive constants \( C_3 \) and \( \tilde{C}_3(M) \) such that
  
  \[ |f(s)| \leq C_3 |s - M|^{2p-1} + \tilde{C}_3(M) \]
There exists a positive constant $C_4$ such that
\[ f'(s) \leq C_4. \]

We will check in the Appendix that the function $f(s) = \sum_{r=0}^{2p-1} b_r s^r$ with $b_{2p-1} < 0, p \geq 2$ satisfies the properties $(F_1) - (F_3)$.

- We assume that $A = \nabla_v \Psi(v) : \mathbb{R}^n \to \mathbb{R}^n$ for some strictly convex function $\Psi \in C^{1,1}$ (i.e. $\Psi(v) \in C^1(\mathbb{R}^n)$ and $\nabla \Psi(v)$ is Lipschitz-continuous) satisfying
\[
\begin{aligned}
    A(0) &= \nabla \Psi(0) = 0, \\
    \|D^2 \Psi\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} &\leq c_1,
\end{aligned}
\]
for some constant $c_1 > 0$. We remark that (1.1) implies that
\[ |A(a) - A(b)| \leq C|a - b| \]
for all $a, b \in \mathbb{R}^n$, where $C$ is a positive constant, and that the strict convexity of $\Psi$ implies that $A$ is strictly monotone, namely there exists a positive constant $C_0$ such that
\[ (A(a) - A(b))(a - b) \geq C_0|a - b|^2, \]
for all $a, b \in \mathbb{R}^n$.

We remark that if $A$ is the identity matrix, the nonlinear diffusion operator $-\text{div}(A(\nabla u))$ reduces to the linear operator $-\Delta u$.

- The function $W = W(x, t)$ is a Q-Brownian motion. More precisely, let $Q$ be a nonnegative definite symmetric operator on $L^2(D)$, $\{e_l\}_{l \geq 1}$ be an orthonormal basis in $L^2(D)$ diagonalizing $Q$, and $\{\lambda_l\}_{l \geq 1}$ be the corresponding eigenvalues, so that
\[ Qe_l = \lambda_l e_l \]
for all $l \geq 1$. Since $Q$ is of trace-class, it follows that
\[ \text{Tr } Q = \sum_{l=1}^{\infty} \langle Qe_l, e_l \rangle_{L^2(D)} = \sum_{l=1}^{\infty} \lambda_l \leq \Lambda_0, \]
for some positive constant $\Lambda_0$. We suppose furthermore that $e_l \in H^1(D) \cap L^\infty(D)$ for $l = 1, 2, \ldots$ and that there exist positive constants $\Lambda_1$ and $\Lambda_2$ such that
\[ \sum_{l=1}^{\infty} \lambda_l \|e_l\|_{L^\infty(D)}^2 \leq \Lambda_1, \]
and
\[ \sum_{l=1}^{\infty} \lambda_l \| \nabla e_l \|_{L^2(D)}^2 \leq \Lambda_2. \]

Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with a filtration \((\mathcal{F}_t)\) and \(\{\beta_l(t)\}_{l \geq 1}\) be a sequence of independent \((\mathcal{F}_t)\)-Brownian motions defined on \((\Omega, \mathcal{F}, P)\); the \(Q\)-Wiener process \(W\) is defined by
\[ W(x,t) = \sum_{l=1}^{\infty} \beta_l(t) Q^{1/2} e_l(x) = \sum_{l=1}^{\infty} \sqrt{\lambda_l} \beta_l(t) e_l(x) \]
in \(L^2(D)\). We recall that a Brownian motion \(\beta(t)\) is called an \((\mathcal{F}_t)\) Brownian motion if it is \((\mathcal{F}_t)\)-adapted and the increment \(\beta(t) - \beta(s)\) is independent of \(\mathcal{F}_s\) for every \(0 \leq s < t\).

We define:
\[ H = \left\{ v \in L^2(D), \int_D v = 0 \right\}, \quad V = H^1(D) \cap H \quad \text{and} \quad Z = V \cap L^{2p}(D) \]
where \(\| \cdot \|\) is the norm corresponding to the space \(H\). We also define \(\langle \cdot, \cdot \rangle_{Z^*, Z}\) as the duality product between \(Z\) and its dual space \(Z^* = V^* + L^{2p-1}(D)\) ([3], p.175).

The corresponding deterministic equation in the case of linear diffusion, when \(A\) is the identity matrix, has been introduced by Rubinstein and Sternberg [17] as a model for phase separation in a binary mixture. The well-posedness and the stabilization of the solution for large times for the corresponding Neumann problem were proved by Boussaïd, Hilhorst and Nguyen [4]. They assumed that the initial function was bounded in \(L^\infty(D)\) and proved the existence of the solution in an invariant set using a Galerkin approximation together with a compactness method.

The interfacial evolution process corresponding to a second order mass conserved Allen-Cahn equation shares many properties with the fourth order Cahn-Hilliard equation as discussed in [17]. Da Prato and Debussche proved the existence and the uniqueness of the solution of a stochastic Cahn-Hilliard equation in [6] with an additive space-time white noise.

In our work, inspired by this paper, we introduce a nonlinear stochastic heat equation, perform a change of functions in order to maintain a "deterministic style" mass conserved equation by hiding the noise term and prove the existence of the solution in suitable Sobolev spaces similar to those in [6].

Funaki and Yokoyama [8] derive a sharp interface limit for a stochastically perturbed mass conserved Allen-Cahn equation with a sufficiently mild additive noise. This is different from the stochastic term in this paper which is not smooth.

A singular limit of a rescaled version of Problem (P) with linear diffusion has been studied by Antonopoulou, Bates, Blömker and Karali [1] to model the motion of a droplet. However, they left open the problem of proving the existence and uniqueness of the solution, which we address here. The problem that we study is more general then the one in [1] since it has a nonlinear diffusion term. The proof is based on a Galerkin method together with
a monotonicity argument similar to that used in [14] for a deterministic reaction-diffusion equation, and that in [12] for a stochastic problem.

Our paper is organised as follows. In section 2 an auxiliary problem is introduced, more precisely the nonlinear stochastic heat equation and a change of function is defined to obtain an equation without the noise term; this simplifies the use of the Galerkin method in section 3, which yields uniform bounds for the approximate solution in \( L^\infty(0,T;L^2(\Omega \times D)) \), \( L^2(\Omega \times (0,T);H^1(D)) \) and in \( L^{2p}(\Omega \times (0,T) \times D) \). We deduce that the approximate weak solution weakly converges along a subsequence to a limits. The main problem is then to identify the limit of the elliptic term and the reaction term, which we do by means of the so-called monotonicity method.

We prove in section 4 the uniqueness of the weak solution which in turn implies the convergence of the whole sequence.

Finally, in section 5 we return to the study of the nonlinear stochastic heat equation and prove the existence and uniqueness of the solution.

## 2 A preliminary change of functions

We consider the Neumann boundary value problem for the stochastic nonlinear heat equation

\[
(P_1) \quad \begin{cases}
\frac{\partial W_A}{\partial t} = \text{div}(A(\nabla W_A)) + \frac{\partial W}{\partial t}, & x \in D, \ t \geq 0, \\
A(\nabla W_A).\nu = 0, & x \in \partial D, t \geq 0, \\
W_A(x,0) = 0, & x \in D.
\end{cases}
\]

Krylov and Rozovskii [12] proved the well-posedness result for a classes of problems similar to Problem \((P_1)\) using a definition of solution in the distribution sense, while Gess [10] defines a solution in the sense of \( L^2(D) \), namely almost everywhere in \( D \). More precisely, he defines a strong solution as follows (cf. [10], Definition 1.3).

**Definition 2.1. (Strong solution)** We say that \( W_A \) is a strong solution of Problem \((P_1)\) if:

(i) \( W_A \in L^\infty(0,T;L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T);H^1(D)) \);

(ii) \( W_A \in L^2(\Omega;C([0,T];L^2(D))) \);

(iii) \( \text{div}(A(\nabla W_A)) \in L^2(\Omega \times (0,T);L^2(D)) \);

(iv) \( W_A \) satisfies a.s. for all \( t \in (0,T) \) the problem

\[
\begin{cases}
W_A(t) = \int_0^t \text{div}(A(\nabla W_A(s)))ds + W(t), & \text{in } L^2(D), \\
A(\nabla W_A(t)).n = 0, & \text{in a suitable sense of trace on } \partial D.
\end{cases}
\]
We will show in Section 5 the existence and uniqueness of the strong solution $W_A$ of Problem $(P_1)$. Moreover we will prove that

$$W_A \in L^\infty(0,T;L^q(\Omega \times D)) \text{ for all } q \in [2,\infty).$$  \hfill (2.2)

We perform the change of functions

$$u(t) := \varphi(t) - W_A(t);$$

then $\varphi$ is a solution of (P) if and only if $u$ satisfies:

\[
\begin{align*}
(P_2) \quad \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} = \text{div}(A(\nabla(u + W_A)) - A(\nabla W_A)) + f(u + W_A) \\
- \frac{1}{|D|} \int_D f(u + W_A) \, dx, \quad x \in D, \; t \geq 0, \\
A(\nabla(u + W_A)) \cdot \nu = 0, \quad x \in \partial D, t \geq 0, \\
u(x,0) = \varphi_0(x), \quad x \in D.
\end{array} \right.
\]

We remark that $(P_2)$ has the form of a deterministic problem; however it is stochastic since the random function $W_A$ appears in the parabolic equation for $u$.

**Definition 2.2.** We say that $u$ is a solution of Problem $(P_2)$ if:

(i) $u \in L^\infty(0,T;L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T);H^1(D)) \cap L^{2p}(\Omega \times (0,T) \times D)$; $\text{div}[A\nabla(u + W_A)] \in L^2(\Omega \times (0,T);(H^1(D))^*)$;

(ii) $u$ satisfies almost surely the problem: for all $t \in [0,T]
\[
\begin{align*}
&u(t) = \varphi_0 + \int_0^t \text{div}[A(\nabla(u + W_A)) - A(\nabla W_A)] \, ds + \int_0^t f(u + W_A) \, ds \\
&- \frac{1}{|D|} \int_D f(u + W_A) \, dx \, ds, \quad \text{in the sense of distributions,} \\
&A(\nabla(u + W_A)) \cdot \nu = 0, \quad \text{in the sense of distributions on } \partial D \times \mathbb{R}^+.
\end{align*}
\]

In order to check the conservation of mass property, namely that

$$\int_D u(x,t) \, dx = \int_D \varphi_0(x) \, dx, \quad \text{a.s. for a.e. } t \in \mathbb{R}^+,$$

we recall that $Z^* = V^* + L^{\frac{2p}{2p-1}}(D)$ and take the duality product of (2.2) with 1 for a.e. $t$ and $\omega$.

### 3 Existence of a solution of Problem $(P_2)$

The main result is the following

**Theorem 3.1.** There exists a unique solution of Problem $(P_2)$.
Proof. In this subsection we apply the Galerkin method to prove the existence of a solution of Problem (P₂).

Denote by \( 0 < \gamma_1 < \gamma_2 \leq \ldots \leq \gamma_k \leq \ldots \) the eigenvalues of the operator \(-\Delta\) with homogeneous Neumann boundary conditions, and by \( w_k, k = 0, \ldots \) the corresponding unit eigenfunctions in \( L^2(D) \). Note that they are smooth functions.

**Lemma 3.1.** The functions \( \{w_j\} \) are an orthonormal basis of \( L^2(D) \) and satisfy:

\[
\int_D w_j w_0 dx = 0 \quad \text{for all } j \neq 0 \quad \text{and} \quad w_0 = \frac{1}{|D|}.
\]

**Proof.** We check below that \( \int_D w_j(x) dx = 0 \) for all \( j \neq 0 \). Indeed,

\[
\int_D w_j dx = -\frac{1}{\gamma_j} \int_D \Delta w_j dx = -\frac{1}{\gamma_j} \int_{\partial D} \frac{\partial w_j}{\partial n} dx = 0,
\]

which implies that \( \int_D w_j w_0 dx = 0 \) for all \( j \neq 0 \). Moreover, it is standard that the eigenfunctions corresponding to different eigenvalues are orthogonal. \( \square \)

We look for an approximate solution of the form

\[
u_m(x,t) - M = \sum_{i=1}^{m} u_{im}(t) w_i = \sum_{i=1}^{m} (u_m(t), w_i) w_i,
\]

where \( M = \frac{1}{|D|} \int_D \varphi_0(x) dx \) such that the function \( \nu_m - M \) satisfies the equations

\[
\int_D \frac{\partial}{\partial t} (\nu_m(x,t) - M) w_j dx
\]

\[
= -\int_D [A(\nabla(\nu_m - M + W_A)) - A(\nabla(W_A))] \nabla w_j dx
\]

\[
+ \int_D f(\nu_m + W_A) w_j - \frac{1}{|D|} \int_D (\int_D f(\nu_m + W_A) dx) w_j dx,
\]

\[(3.2)\]

for all \( w_j, j = 1, \ldots, m \). We remark that \( \nu_m(x,0) = M + \sum_{i=1}^{m} (\varphi_0, w_i) w_i \) converges strongly to \( \varphi_0 \) in \( L^2(D) \) as \( m \to \infty \).
Problem (3.2) is an initial value problem for a system of \( m \) ordinary differential equations with the unknown functions \( u_i(t) \), \( i = 1, \ldots, m \) so that it has a unique solution \( u_m \) on some interval \((0, T_m)\), \( T_m > 0\); in fact the following a priori estimates show that this solution is global in time.

First we remark that the contribution of the nonlocal term vanishes. Indeed for all \( j = 1, \ldots, m - 1 \)

\[
- \frac{1}{|D|} \int_D \left( \int_D f(u_m + W_A(t)) dx \right) w_j dx = - \frac{1}{|D|} \left( \int_D f(u_m + W_A(t)) dx \right) \times \int_D w_j dx
\]

Therefore (3.2) reduces to the equation:

\[
\int_D \frac{\partial}{\partial t}(u_m(x,t) - M) w_j dx = - \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))] \nabla w_j dx
\]

\[
+ \int_D f(u_m + W_A) w_j dx.
\] (3.3)

We multiply (3.3) by \( u_j = u_j(t) \) and sum on \( j = 1, \ldots, m \):

\[
\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)(u_m - M) dx
\]

\[
= - \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))] \nabla(u_m - M) dx
\]

\[
+ \int_D f(u_m + W_A)(u_m - M) dx.
\] (3.4)

Next we apply the monotonicity property of \( A(1.3) \) to bound the generalized Laplacian term, which yields

\[
\frac{1}{2} \frac{d}{dt} \int_D (u_m - M)^2 dx \leq -C_0 \int_D |\nabla(u_m - M)|^2 dx
\]

\[
+ \int_D f(u_m + W_A)(u_m - M) dx.
\] (3.5)

Using the property \((F_1)\) we deduce that

\[
\int_D f(u_m + W_A(t))(u_m - M) dx
\]

\[
= \int_D f(u_m - M + M + W_A(t))(u_m - M) dx
\]

\[
\leq \int_D [-C_1(u_m - M)^{2p} + C_2 ((M + W_A)^{2p}(t) + 1)] dx
\]

\[
\leq - \int_D C_1(u_m - M)^{2p} dx + C_2 \int_D [W_A(t)]^{2p} dx + \tilde{C}_2(M)|D|,
\]

which we substitute in (3.5) to obtain :

\[
\frac{1}{2} \frac{d}{dt} \int_D (u_m - M)^2 dx + C_0 \int_D |\nabla(u_m - M)|^2 dx + C_1 \int_D (u_m - M)^{2p} dx
\]

\[
\leq C_2 \int_D [W_A(t)]^{2p} dx + \tilde{C}_2(M)|D|.
\] (3.6)
3.1 A priori estimates

In what follows, we derive a priori estimates for the function $u_m$.

**Lemma 3.2.** There exists a positive constant $C$ such that

\[ \underset{t \in [0,T]}{\sup} \mathbb{E} \int_D (u_m - M)^2 \, dx \leq C, \]  
\[ \mathbb{E} \int_0^T \int_D |\nabla (u_m - M)|^2 \, dx \, dt \leq C, \]  
\[ \mathbb{E} \int_0^T \int_D (u_m - M)^{2p} \, dx \, dt \leq C, \]  
\[ \mathbb{E} \int_0^T \int_D (f(u_m + W_A))^{\frac{2p}{p-1}} \, dx \, dt \leq C, \]  
\[ \mathbb{E} \int_0^T \| \text{div} \, A(\nabla (u_m + W_A)) \|^2_{(H^1(D))'} \, dt \leq C. \]

**Proof.** Integrating (3.6) from 0 to $t$ and taking the expectation we deduce that for all $t \in [0,T]$

\[
\frac{1}{2} \mathbb{E} \int_D (u_m - M)^2(t) \, dx + C_0 \mathbb{E} \int_0^t \int_D |\nabla (u_m - M)|^2 \, dx \, ds + C_1 \mathbb{E} \int_0^t \int_D (u_m - M)^{2p} \, dx \, ds \\
\leq \frac{1}{2} \int_D (u_m(0) - M)^2 \, dx + C_2 \mathbb{E} \int_0^t \int_D |W_A(t)|^{2p} \, dx \, ds + \tilde{C}_2(M)|D|T \\
\leq \frac{1}{2} \sum_{i=1}^m |\langle u_0, w_i \rangle|^2_2 + \tilde{C}_2(M)|D|T + c_2T \\
\leq \frac{1}{2} \| u_0 - M \|_{L^2(D)}^2 + \tilde{C}_2(M)|D|T + c_2T \\
\leq K
\]

where we have used (2.2).

We deduce that :

\[ \mathbb{E} \int_D (u_m - M)^2(t) \, dx \leq 2K, \quad \text{for all} \quad t \in [0,T], \]
\[ \mathbb{E} \int_0^T \int_D |\nabla (u_m - M)|^2 \, dx \, dt \leq \frac{K}{C_0}, \]
\[ \mathbb{E} \int_0^T \int_D (u_m - M)^{2p} \, dx \, dt \leq \frac{K}{C_1}. \]

Therefore $\{u_m\}$ is bounded independently of $m$ in $L^\infty(0,T;L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T);H^1(D)) \cap L^{2p}(\Omega \times (0,T) \times D)$.
Using the property $(F_2)$ we deduce that

\[
E \| f(u_m + W_A) \|_{L^{2p-1}(0,T \times D)}^{2p-1} \\
= E \int_0^T \int_D |f(u_m + W_A)|^{2p-1} dx dt \\
\leq E \int_0^T \int_D \left[ C_3 |u_m + W_A - M|^{2p-1} + \tilde{C}_3(M) \right]^{2p-1} dx dt \\
\leq E \int_0^T \int_D \left[ C_3 |u_m - M| + |W_A|^{2p-1} dx + \tilde{C}_3(M) \right]^{2p-1} dx dt \\
\leq 2^{2p-1} E \int_0^T \int_D C_5 \left[ (|u_m - M| + |W_A|)^{2p-1} \right]^{2p-1} dx dt + \tilde{C}_3 |D| T \\
\leq c_3 E \int_0^T \int_D (|u_m - M|^{2p-1})^{2p-1} dx dt \\
+ c_3 E \int_0^T \int_D (|W_A|^{2p-1})^{2p-1} dx dt + \tilde{C}_3 |D| T \\
\leq c_3 E \int_0^T \int_D |u_m - M|^{2p} dx dt \\
+ c_3 E \int_0^T \int_D |W_A|^{2p} dx dt + \tilde{C}_3 |D| T \\
\leq K_1,
\]

by (3.9) and (2.2), where $c_3$ is a positive constant.

Finally we show that the elliptic term is bounded in $(H^1(D))^\prime$. We have that

\[
E \int_0^T \| \text{div} A(\nabla (u_m + W_A)) \|^2_{(H^1(D))^\prime} \\
= E \int_0^T \left( \sup_{v \in H^1, \|v\|_{H^1} \leq 1} |\langle \text{div} A(\nabla (u_m + W_A)), v \rangle| \right)^2 \\
= E \int_0^T \left( \sup_{v \in H^1, \|v\|_{H^1} \leq 1} \left| - \int_D A(\nabla (u_m + W_A)) \nabla v \right| \right)^2 \\
\leq E \int_0^T \left\{ \sup_{v \in H^1, \|v\|_{H^1} \leq 1} \left( \int_D |A(\nabla (u_m + W_A))|^2 \right)^{1/2} \left( \int_D \|\nabla v\|^2 \right)^{1/2} \right\}^2 \\
\leq E \int_0^T \sup_{v \in H^1, \|v\|_{H^1} \leq 1} \int_D |A(\nabla (u_m + W_A))|^2 \int_D \nabla v^2 \\
\leq E \int_0^T \int_D |A(\nabla (u_m + W_A))|^2. \quad (3.12)
\]
Next we use (1.2) and (1.1) to estimate the term on the right-hand-side of (3.12)

\begin{align*}
E \int_0^T \int_D |A(\nabla (u_m + W_A)|^2 & \leq C E \int_0^T \int_D |\nabla (u_m + W_A)|^2 \\
& \leq 2C (E \int_0^T \int_D |\nabla u_m|^2 + E \int_0^T \int_D |\nabla W_A|^2) \\
& \leq K_2.
\end{align*}

The last line follows from the a priori estimates and the regularity of the solution of Problem (P_1).

Hence there exist a subsequence which we denote again by \{u_m - M\} and a function

\[ u - M \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times (0, T) \times D) \cap L^\infty(0, T; L^2(\Omega \times D)) \]

such that

\begin{align*}
& u_m - M \rightharpoonup u - M \text{ weakly in } L^2(\Omega \times (0, T); V) \\
& \text{and } L^{2p}(\Omega \times (0, T) \times D) \\
& u_m - M \rightharpoonup u - M \text{ weakly star in } L^\infty(0, T; L^2(\Omega \times D)) \\
& f(u_m + W_A) \rightharpoonup \chi \text{ weakly in } L^{2p}(\Omega \times (0, T) \times D) \\
& \text{div}(A(\nabla (u_m + W_A))) \rightharpoonup \Phi \text{ weakly in } L^2(\Omega \times (0, T); (H^1)' )
\end{align*}

as \( m \to \infty \).

Next, we pass to the limit as \( m \to \infty \).

To that purpose we integrate in time the equation (3.3) to obtain

\begin{align*}
\int_D (u_m(x, t) - M) w_j &= \int_D (u_m(0) - M) w_j \\
&+ \int_0^t \langle \text{div}[A(\nabla (u_m - M + W_A)) - A(\nabla W_A)], w_j \rangle \\
&+ \int_0^t \int_D f(u_m + W_A) w_j, \text{ for all } j = 1, \ldots, m. \tag{3.17}
\end{align*}

Let \( y = y(\omega) \) be an arbitrary bounded random variable, and let \( \psi \) be an arbitrary bounded function on \((0, T)\). We multiply the equation (3.17) by the product \( y\psi \), integrate between
0 and $T$ and take the expectation to deduce

$$
\mathbb{E} \int_0^T \int_D y \psi(t)(u_m(t) - M) w_j dx dt
= \mathbb{E} \int_0^T \int_D y \psi(t)(u_m(0) - M) w_j dx dt
+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \langle \text{div}[A(\nabla(u_m - M + W_A))], w_j \rangle \right\}
- \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \langle \text{div}[A(\nabla(W_A))], w_j \rangle \right\}
+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \int_D f(u_m + W_A) w_j dx ds \right\} dt.
$$

(3.18)

for all $j = 1, \ldots, m$.

Next we pass to the limit in (3.18); we only give the proof of convergence for the last term using the a priori estimates and Hölder inequality. We have that

$$
\left| \psi(t) \mathbb{E} \int_0^t \int_D f(u_m + W_A) w_j dx ds \right|
\leq \|y\|_{L^\infty(\Omega)} |\psi(t)| \left( \mathbb{E} \int_0^t \int_D |f(u_m + W_A)|^{2p} dx ds \right)^{\frac{1}{2p}} \mathbb{E} \int_0^t \int_D |w_j|^{2p} dx ds
\leq \|y\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(0,T)} C.
$$

This shows that $|\psi(t)\mathbb{E} \int_0^t \int_D f(u_m + W_A) w_j dx ds|$ is uniformly bounded by a function belonging to $L^1(0,T)$. In addition using (3.15) we have that

$$
\psi(t)\mathbb{E} \int_0^t \int_D f(u_m + W_A) w_j dx ds \to \psi(t)\mathbb{E} \int_0^t \int_D \chi w_j dx ds
$$
for a.e. $t \in (0,T)$. Applying Lebesgue-dominated convergence theorem we deduce that:

$$
\lim_{m \to \infty} \int_0^T \psi(t) dt \mathbb{E} \int_0^t \int_D f(u_m + W_A) w_j dx ds
= \int_0^T \lim_{m \to \infty} \psi(t) dt \mathbb{E} \int_0^t \int_D f(u_m + W_A) w_j dx ds dt
= \int_0^T \psi(t) dt \mathbb{E} \int_0^t \int_D \chi w_j dx ds
= \mathbb{E} \int_0^T y \psi(t) dt \left\{ \int_0^t \int_D \chi w_j dx ds \right\}.
$$

Performing a similar proof for each term in (3.18), we pass to the limit by using Lebesgue-
dominated convergence theorem. This yields
\[
\mathbb{E} \int_0^T \int_D y\psi(t)(u(t) - M)w_j dx dt \\
= \mathbb{E} \int_0^T \int_D y\psi(t)(\varphi_0 - M)w_j dx dt \\
+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \Phi - \text{div} A(\nabla (W_A)), w_j \rangle \right\} dt \\
+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \chi w_j dx ds \right\} dt,
\]
for all \( j = 1, \ldots, m. \) \hspace{1cm} (3.19)

We remark that the linear combinations of \( w_j \) are dense in \( V \cap L^2p(D) \), so that
\[
\mathbb{E} \int_0^T \int_D y\psi(t)(u(t) - M)\tilde{w} dx dt = \mathbb{E} \int_0^T \int_D y\psi(t)(\varphi_0 - M)\tilde{w} dx dt \\
+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \Phi - \text{div} A(\nabla (W_A)), \tilde{w} \rangle ds \right\} dt \\
+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \chi \tilde{w} dx ds \right\} dt.
\]
for all \( \tilde{w} \in V \cap L^2p(D), \ y \in L^\infty(\Omega) \) and \( \psi \in L^\infty(0,T) \). This implies that for a.e. \( (t, \omega) \in (0,T) \times \Omega \)
\[
\langle u(t) - M, \tilde{w} \rangle = \langle \varphi_0 - M, \tilde{w} \rangle + \int_0^t \langle \Phi + \chi - \text{div}(A(\nabla W_A)), \tilde{w} \rangle ds
\]
for all \( \tilde{w} \in V \cap L^2p(D) \).

**Lemma 3.3.** The function \( u \) is such that \( u \in C([0,T]; L^2(D)) \) a.s.

**Proof.**
\[
Z \subseteq H \subseteq Z^*
\]
Since \( u - M \in L^2(0,T; Z) \) a.s. and \( \frac{du}{dt} \in L^2(0,T; V^*) + L^2(0,T; L^{\frac{2p}{p-1}}(D)) = L^2(0,T; Z^*) \) a.s., it follows by applying Lemma 1.2 p.260 in [18] that \( u - M \in C(0,T; H) \) a.s. \hfill \Box

It remains to prove that :
\[
\langle \Phi + \chi, \tilde{w} \rangle = \langle \text{div}(A(\nabla (u + W_A))) + f(u + W_A(t)), \tilde{w} \rangle \quad \text{for all} \quad \tilde{w} \in V \cap L^2p(D).
\]

We do so by means of the monotonicity method.
3.2 Monotonicity argument

Let $w$ be such that $w - M \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times D \times (0, T))$.

Let $c$ be a positive constant which will be fixed later. We define

$$
O_m = \mathbb{E} \left[ \int_0^T e^{-cs} \{ 2 \langle \text{div} (A(\nabla(u_m - M + W_A)) - A(\nabla W_A)) \rangle \\
- \text{div} (A(\nabla(w - M + W_A)) - A(\nabla W_A)), u_m - M - (w - M) \rangle_{Z^*, Z} \\
+ 2 \langle f(u_m + W_A) - f(w + W_A), u_m - M - (w - M) \rangle_{Z^*, Z} \\
- c \| u_m - M - (w - M) \|^2 \} \right] ds
$$

and prove below the following result

**Lemma 3.4.**

$$O_m \leq 0.$$  

**Proof.** First we estimate $J_1$ and apply (1.3)

$$
J_1 = \mathbb{E} \int_0^T e^{-cs} \{ 2 \langle \text{div} (A(\nabla(u_m - M + W_A)) \rangle \\
- \text{div} (A(\nabla(w - M + W_A))), u_m - M - (w - M) \rangle_{Z^*, Z} \} \\
= -2 \mathbb{E} \int_0^T e^{-cs} \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(w - M + W_A))] \\
[\nabla(u_m - M + W_A) - \nabla(w - M + W_A)] \\
\leq -2C_0 \mathbb{E} \int_0^T e^{-cs} \| \nabla(u_m - w) \|^2 \\
\leq 0.
$$

($F_3$) and the mean value theorem yield:

$$
J_2 = \mathbb{E} \int_0^T e^{-cs} 2 \langle f(u_m + W_A) - f(w + W_A), u_m - w \rangle_{Z^*, Z} ds \\
\leq \mathbb{E} \int_0^T e^{-cs} 2C_4 \| u_m - w \|^2 ds.
$$

Choosing $c \geq 2C_4$, we conclude the result.

\[ \square \]

We write $O_m$ in the form $O_m = O_m^1 + O_m^2$ where

$$
O_m^1 = \mathbb{E} \left[ \int_0^T e^{-cs} \{ 2 \langle \text{div} (A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), u_m - M \rangle_{Z^*, Z} \\
+ 2 \langle f(u_m + W_A), u_m - M \rangle_{Z^*, Z} - c \| u_m - M \|^2 \} \right] ds.
$$

(3.21)
We integrate the equation (3.3) between 0 and \( T \) to obtain
\[
\int_D (u_m(x,T) - M)w_j = \int_D (u_m(0) - M)w_j \\
+ \int_0^T (\text{div}[A(\nabla (u_m - M + W_A)) - A(\nabla W_A)], w_j)_{Z^*,Z} \\
+ \int_0^T \int_D f(u_m + W_A)w_j, \quad \text{for all } j = 1, \ldots, m. \quad (3.22)
\]

Next we recall a chain rule formula, which can be viewed as a simplified Itô’s formula.

**Proposition 3.1.** Let \( X \) be a real valued function such that
\[
X(t) = X(0) + \int_0^t h(s)ds, \quad 0 \leq s \leq t,
\]
and suppose that \( h \) is measurable in time such that \( h \in L^1(0,T) \). Suppose that the function \( F : [0,T] \times \mathbb{R} \rightarrow \mathbb{R} \) and its partial derivatives \( \frac{\partial F}{\partial t} \) and \( \frac{\partial F}{\partial X} \) are continuous on \([0,T] \times \mathbb{R}\). Then for all \( t \in [0,T] \)
\[
F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s))ds + \int_0^t \frac{\partial F}{\partial X}(s, X(s))h(s)ds.
\]

(3.23)

Applying (3.23) to the \( m \) equations in (3.22) with
\[
X_j = \int_D (u_m - M)w_j, \quad j = 1, \ldots, m, \quad F(s, q) = e^{-cs}q^2,
\]
and \( h(s) = (\text{div}[A(\nabla (u_m - M + W_A)) - A(\nabla W_A)] + f(u_m + W_A), w_j)_{Z^*,Z}, \)
we deduce that
\[
e^{-cT}(\int_D (u_m(x,T) - M)w_j)^2 \\
= (\int_D (u_m(0) - M)w_j)^2 - c \int_0^T e^{-cs}(\int_D (u_m - M)w_j)^2ds \\
+ 2 \int_0^T e^{-cs}\{\int_D (u_m - M)w_j\}(\text{div}[A(\nabla (u_m - M + W_A)) - A(\nabla W_A)], w_j) \\
+ 2 \int_0^T e^{-cs}\{\int_D (u_m - M)w_j\}f(u_m + W_A), w_j), \quad \text{for all } j = 1, \ldots, m.
\]

(3.24)

In what follows, we will use the identity

**Lemma 3.5.** Let \( F \in Z^* \) and \( B_m = \sum_{j=1}^m \langle B_m, w_j \rangle w_j. \)

Then
\[
\sum_{j=1}^m \langle F, w_j \rangle \langle B_m, w_j \rangle = \langle F, B_m \rangle.
\]

(3.25)
Proof.

\[
\sum_{j=1}^{m} \langle F, w_j \rangle \langle B_m, w_j \rangle = \sum_{j=1}^{m} \langle F, \langle B_m, w_j \rangle w_j \rangle = \langle F, \sum_{j=1}^{m} \langle B_m, w_j \rangle w_j \rangle = \langle F, B_m \rangle.
\]

\[\square\]

Summing (3.24) on \(j = 1, \ldots, m\) and applying the identity (3.25) yields

\[
e^{-cT} \|u_m(T) - M\|^2 = \|u_m(0) - M\|^2 - c \int_0^T e^{-cs} \|u_m - M\|^2 ds + 2 \int_0^T e^{-cs} \langle \text{div}[A(\nabla (u_m - M + W_A)) - A(\nabla W_A)], u_m - M \rangle \rangle_{Z^*,Z} + 2 \int_0^T e^{-cs} \langle f(u_m + W_A), u_m - M \rangle \rangle_{Z^*,Z}.
\]

(3.26)

Taking the expectation of the equation (3.26) yields

\[
\mathbb{E}[e^{-cT} \|u_m(T) - M\|^2] = \mathbb{E}[\|u_m(0) - M\|^2] - c \mathbb{E}[\int_0^T e^{-cs} \|u_m(s) - M\|^2 ds] + 2 \mathbb{E}[\int_0^T e^{-cs} \langle \text{div}[A(\nabla (u_m - M + W_A)) - A(\nabla W_A)], u_m - M \rangle \rangle_{Z^*,Z} + 2 \mathbb{E}[\int_0^T e^{-cs} \langle f(u_m + W_A), u_m - M \rangle \rangle_{Z^*,Z}].
\]

(3.27)

It follows from (3.21) and (3.27) that

\[
O_1^m = \mathbb{E}[e^{-cT} \|u_m(T) - M\|^2] - \mathbb{E}[\|u_m(0) - M\|^2].
\]

From this we obtain

\[
\lim_{m \to \infty} \sup O_1^m = \mathbb{E}[e^{-cT} \|u(T) - M\|^2] - \mathbb{E}[\|u(0) - M\|^2] + \delta e^{-cT},
\]

(3.28)

where

\[
\delta = \lim_{m \to \infty} \sup \mathbb{E}[\|u_m(T) - M\|^2] - \mathbb{E}[\|u(T) - M\|^2] \geq 0.
\]

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On the other hand, the equation (3.20) implies that
\[ u(t) - M = \varphi_0 - M + \int_0^t \Phi - \text{div}(A(\nabla W_A)) + \int_0^t \chi, \quad \forall t \in [0, T] \]
(3.29)
a.s. in \( Z^* = V^* + L_{2p}^{2p-1}(D) \).

Next we recall a second variant of the chain rule formula, which can be viewed as a simplified Itô’s formula as in [15] [p.75 Theorem 4.2.5], and involves different function spaces. Consider the Gelfand triple
\[ Z \subset H \subset Z^*, \]
where \( Z = V \cap L^{2p}(D) \) and \( Z^* \) are defined in the introduction.

**Proposition 3.2.** Let \( X \in L^2(0, T; V) \cap L^{2p}(0, T; L^{2p}(D)) \) and \( Y \in L^2(0, T; V^*) + L_{2p}^{2p-1}(0, T; L_{2p}^{2p-1}(D)) \) be such that
\[ X(t) := X_0 + \int_0^t Y(s) ds, \quad t \in [0, T]. \]

Suppose that the function \( F : [0, T] \times Z \rightarrow \mathbb{R} \) and its partial derivatives \( \frac{\partial F}{\partial t} \) and \( \frac{\partial F}{\partial X} \) are continuous on \([0, T] \times Z\). Then for all \( t \in [0, T] \)
\[ F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t} (s, X(s)) ds + \int_0^t \langle Y(s), \frac{\partial F}{\partial X}(s, X(s)) \rangle_{Z^*, Z} ds. \]
(3.30)

Applying Proposition 3.2 to the equation (3.29), we set \( X(t) = u(t) - M, F(s, q) = e^{-cs}||q||^2 \), and \( Y(s) = \Phi - \text{div}(A(\nabla W_A)) + \chi \), in (3.30) to deduce that
\[ \mathbb{E}[e^{-cT}||u(T) - M||^2] = \mathbb{E}[||u(0) - M||^2] - c\mathbb{E}[\int_0^T e^{-cs}||u(s) - M||^2 ds] \]
\[ + 2\mathbb{E}\int_0^T e^{-cs}(\Phi - \text{div}(A(\nabla W_A)), u - M)_{Z^*, Z} \]
\[ + 2\mathbb{E}[\int_0^T e^{-cs}(\chi, u - M)_{Z^*, Z}], \]
which we combine with (3.28) to deduce that
\[ \lim_{m \to \infty} \sup O_m^1 = 2\mathbb{E}\int_0^T e^{-cs}(\Phi - \text{div}(A(\nabla W_A)), u - M)_{Z^*, Z} \]
\[ + 2\mathbb{E}\int_0^T e^{-cs}(\chi, u - M)_{Z^*, Z} - c\mathbb{E}[\int_0^T e^{-cs}||u(s) - M||^2 ds] + \delta e^{-cT}. \]
(3.31)
It remains to compute the limit of $O^2_m$:

\[ O^2_m = O^1_m - O^1_m \]

\[ = \mathbb{E} \int_0^T e^{-cs} \{-2\langle \text{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], u_m - M \rangle_{Z^*, Z} \hspace{1cm} \]

\[ -2\langle \text{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w - M \rangle_{Z^*, Z} \hspace{1cm} \]

\[ -2\langle f(w + W_A), u_m - M \rangle_{Z^*, Z} - 2\langle f(u + W_A), w - M \rangle_{Z^*, Z} \hspace{1cm} \]

\[ -c\|w - M\|^2 + 2c\langle u_m - M, w - M \rangle \} ds. \]

In view of (3.13), (3.15) and (3.16), we deduce that

\[ \lim_{m \to \infty} O^2_m \]

\[ = \mathbb{E} \int_0^T e^{-cs} \{-2\langle \text{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], u - M \rangle_{Z^*, Z} \hspace{1cm} \]

\[ -2\langle \Phi - \text{div}(A(\nabla W_A)) - \text{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], w - M \rangle_{Z^*, Z} \hspace{1cm} \]

\[ -2\langle f(w + W_A), u - M \rangle_{Z^*, Z} - 2\langle \chi - f(w + W_A), w - M \rangle_{Z^*, Z} \hspace{1cm} \]

\[ -c\|w - M\|^2 + 2c\langle u - M, w - M \rangle \} ds. \]

Combining (3.31) and (3.32), and remembering that $O_m \leq 0$, yields

\[ \mathbb{E} \int_0^T e^{-cs} \{ 2\langle \Phi - \text{div}(A(\nabla(w - M + W_A)), u - M - (w - M) \rangle_{Z^*, Z} \hspace{1cm} \]

\[ + 2\langle \chi - f(w + W_A), u - M - (w - M) \rangle_{Z^*, Z} \hspace{1cm} \]

\[ -c\|u - M - (w - M)\|^2 \} + \delta e^{-cT} \leq 0. \]

Let $v \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times (0, T) \times D)$ be arbitrary and set

\[ w - M = u - M - \lambda v, \text{ with } \lambda \in \mathbb{R}_+. \]

We obtain the inequality:

\[ \mathbb{E} \int_0^T e^{-cs} \{ 2\langle \Phi - \text{div}(A(\nabla(u - \lambda v - M + W_A)), \lambda v \rangle_{Z^*, Z} \hspace{1cm} \]

\[ + 2\langle \chi - f(u - \lambda v + W_A), \lambda v \rangle_{Z^*, Z} - c\|\lambda v\|^2 \} dt \leq 0. \]

Dividing by $\lambda$ and letting $\lambda \to 0$, we find that:

\[ \mathbb{E} \int_0^T e^{-cs} \langle \Phi + \chi - \text{div}(A(\nabla(u - M + W_A)) - f(u + W_A), v \rangle_{Z^*, Z} dt \leq 0. \]

Since $v$ is arbitrary, it follows that

\[ \mathbb{E} \int_0^T \langle \Phi + \chi, v \rangle_{Z^*, Z} = \mathbb{E} \int_0^T \langle \text{div}[A(\nabla(u - M + W_A))] + f(u + W_A), v \rangle_{Z^*, Z}, \]

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for all $v \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times (0, T) \times D)$, 

\[ \Phi + \chi = \text{div}[A(\nabla (u - M + W_A))] + f(u + W_A) + \theta(t, \omega), \quad (3.33) \]

a.s. a.e. in $D \times (0, T)$. Taking the duality product of (3.33) with $\tilde{w} \in V \cap L^2_{2p}(D)$ we obtain that 

\[ \langle \Phi + \chi, \tilde{w} \rangle_{Z^*, Z} = \langle \text{div}[A(\nabla (u - M + W_A))] + f(u + W_A), \tilde{w} \rangle_{Z^*, Z}, \quad (3.34) \]

Substituting (3.34) in (3.20) we deduce that for a.e. $(t, \omega) \in (0, T) \times \Omega$ 

\[ \langle u(t) - M, \tilde{w} \rangle = \langle \phi_0 - M, \tilde{w} \rangle + \int_0^t \langle \text{div}[A(\nabla (u - M + W_A))] + f(u + W_A) - \text{div}(A(\nabla W_A)), \tilde{w} \rangle_{Z^*, Z}, \quad (3.35) \]

for all $\tilde{w} \in V \cap L^{2p}(D)$. 

This completes the identification of the limit terms by the monotonicity method. 

Next, we prove that $u$ satisfies the equation (2.3) in Definition 2.2. We define 

\[ V = H^1(D) \cap L^{2p}(D). \]

The equation (3.35) implies that a.s. in $V^* = (H^1(D))' + L^{\frac{2p}{p-1}}(D)$ 

\[ u(t) = \phi_0 + \int_0^t \text{div}[A(\nabla (u - M + W_A))] - \text{div}(A(\nabla W_A)) + \int_0^t f(u + W_A)ds + \int_0^t \lambda(s)ds, \quad (3.36) \]

for all $t \in [0, T]$. 

In order to identify the last term of (3.36), we take its duality product $\langle \ldots \rangle_{V^*, V}$ with 1. Remembering that the equation is mass conserved, we obtain 

\[ \int_D \int_0^t f(u + W_A)dsdx + \int_0^t \lambda(s)ds|D| = \int_D u(t)dx - \int_D \phi_0dx = 0. \quad (3.37) \]

Thus, 

\[ \int_0^t \lambda(s)ds = -\frac{1}{|D|} \int_D \int_0^t f(u + W_A)dsdx, \]

so that also 

\[ \lambda(t) = -\frac{1}{|D|} \int_D f(u(x, t) + W_A(x, t))dx. \]
4 Uniqueness of the solution of Problem \((P_2)\)

Let \(\omega\) be given such that two pathwise solutions of Problem \((P_2)\), \(u_1 = u_1(\omega, x, t)\) and \(u_2 = u_2(\omega, x, t)\) satisfy

\[
\begin{align*}
    u_i(\cdot, \cdot, \omega) & \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)) \cap L^{2p}((0, T) \times D), \\
    f(u_i + W_A) & \in L^{\frac{2p}{p-1}}((0, T) \times D), \\
    \text{div}(A(\nabla(u_i + W_A)) & \in L^2((0, T); (H^1(D))')
\end{align*}
\]

for \(i = 1, 2\), and \(u_1(\cdot, 0) = u_2(\cdot, 0) = \varphi_0\). Then

\[
\begin{align*}
    u_1(x, t) &= u_1(x, 0) + \int_0^t \text{div}(A(\nabla(u_1 + W_A))) - \text{div}(A(\nabla W_A)) + \int_0^t f(u_1 + W_A) \\
    & \quad - \frac{1}{|D|} \int_0^t \int_D f(u_1 + W_A) dx, \\
    u_2(x, t) &= u_2(x, 0) + \int_0^t \text{div}(A(\nabla(u_2 + W_A))) - \text{div}(A(\nabla W_A)) + \int_0^t f(u_2 + W_A) \\
    & \quad - \frac{1}{|D|} \int_0^t \int_D f(u_2 + W_A) dx,
\end{align*}
\]

so that the difference \(u_1 - u_2\) satisfies the equation

\[
\begin{align*}
    u_1(t) - u_2(t) &= \int_0^t \text{div}(A(\nabla(u_1 + W_A) - A(\nabla(u_2 + W_A)) \\
    & \quad + \int_0^t [f(u_1 + W_A) - f(u_2 + W_A)] \\
    & \quad - \frac{1}{|D|} \int_0^t \int_D [f(u_1 + W_A) - f(u_2 + W_A)] dx,
\end{align*}
\]

in \(L^2((0, T); V^*) + L^{\frac{2p}{p-1}}((0, T) \times D)\).

We take the duality product of this equation with \(u_1 - u_2 \in L^2((0, T); V^*) \cap L^{\frac{2p}{p-1}}((0, T) \times D)\), to deduce that
\[ \|u_1 - u_2\|_{L^2(D)}^2 = 2 \int_0^t \langle \text{div}(A(\nabla(u_1 + W_A)) - A(\nabla(u_2 + W_A)), u_1 - u_2 \rangle_{Z^*, Z} \\
+ 2 \int_0^t \langle f(u_1 + W_A) - f(u_2 + W_A), u_1 - u_2 \rangle_{Z^*, Z} \\
- 2 \int_0^t \frac{1}{|D|} \int_D (f(u_1 + W_A) - f(u_2 + W_A)) dx, u_1 - u_2 \rangle_{Z^*, Z} \\
= -2 \int_0^t \int_D (A(\nabla(u_1 + W_A)) - A(\nabla(u_2 + W_A))) \nabla(u_1 - u_2) \\
+ 2 \int_D \int_D (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) dx \\
- \frac{1}{|D|} \int_0^t \int_D (f(u_1 + W_A) - f(u_2 + W_A)) dx \int_D (u_1 - u_2) dx] \\
= -2 \int_0^t \int_D (A(\nabla(u_1 + W_A)) - A(\nabla(u_2 + W_A))) \nabla(u_1 - u_2) \\
+ 2 \int_0^t \int_D (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) dx, \tag{4.1} \]

where we remark that since \( \int_D u_1(x, t) dx = \int_D u_2(x, t) dx = \int_D \varphi_0(x) dx \), the nonlocal term vanishes.

In view of (1.3), (4.1) becomes

\[ \|u_1 - u_2\|_{L^2(D)}^2 \leq \int_0^t \int_D (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) dx dt - C_0 \int_0^t \int_D \nabla(u_1 - u_2)^2 dx dt, \tag{4.2} \]

for all \( t \in (0, T) \). In addition, the property \((F_3)\) implies that

\[ (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) \leq C_4 \int_D (u_1 - u_2)^2. \tag{4.3} \]

Substituting (4.3) in (4.2) yields

\[ \int_D (u_1 - u_2)^2(x, t) dx \leq C_4 \int_0^t \int_D (u_1 - u_2)^2(x, t) dx \quad \text{for all} \quad t \in (0, T), \]

which in turn implies by Gronwall’s Lemma that

\[ u_1 = u_2 \quad \text{a.e. in} \quad D \times (0, T). \]
5 Existence and uniqueness of the solution of Problem \((P_1)\)

In this section we return to the study of the solution \(W_A\) of Problem \((P_1)\), and derive a priori estimates for a Galerkin approximation in \(L^\infty(0,T;L^2(\Omega\times D))\) following an idea due to Gess [10]. We are then in a position to show that \(W_A\) is also bounded in \(L^\infty(0,T;L^q(\Omega\times D))\) for all \(q \geq 2\), which is necessary for the proof of Lemma 3.2.

We show below a priori estimates, which imply that the elliptic term \(\text{div}(A(\nabla W_A))\) is bounded in \(L^2(D)\) having in mind that Problem \((P_1)\) is a special case of Problem (4.33) in [10] (see also equation (2.8) in [10]). Whereas Gess concentrates on the special case of the p-Laplacian, we are interested in the uniformly parabolic case, which corresponds to \(m = 2\) in [10] p.280-281. We also remark that there are no reaction terms i.e. \(f_i = 0\) for all \(i\) from 1 to \(n\) and that the noise is additive. However, Gess assumes that the nonlinear function \(\Psi\) is twice continuously differentiable while we only suppose that \(\Psi \in C^{1,1}(\mathbb{R}^n)\).

We prove the following result.

**Theorem 5.1.** There exists a unique solution of Problem \((P_1)\).

**Proof.** To begin with, we approximate the function \(\Psi\) by a sufficiently smooth function \(\Psi^n\) such that

\[
\Psi^n \to \Psi \quad \text{in} \quad C^1(\mathbb{R}^n),
\]

and

\[
\|D^2\Psi^n\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} \leq c_1, \quad \nabla \Psi^n(0) = 0,
\]

and derive a priori estimates for a Galerkin approximation as in [10] (p. 2363 (2.13)). It turns out that the upper bounds which we find do not depend on \(n\).

We define \(W_{mn}^A\) by

\[
W_{mn}^A(t) = \int_0^t P_m[\text{div}(\nabla \Psi^n(\nabla W_{mn}^A(s)))]ds + \sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)\beta_l(t)
\]

a.s., where for \(v \in L^2(D)\) \(P_m v := \sum_{j=1}^m (\int_D v w_j) w_j\) and \(P_m : H^1(D) \to H_m = \text{span}\{w_1,...,w_m\}\), \(m \in \mathbb{N}\) is the continuous operator defined by

\[
\|a - P_m a\|_{H^1(D)}^2 = \inf_{v \in H_m} \|a - v\|_{H^1(D)}^2, \quad a \in H^1(D) \quad (\text{c.f [10] p. 2363}).
\]

Note that (cf. [5] p.193)

\[
\|P_m a\|_{H^1(D)} \leq \|a\|_{H^1(D)}.
\]

and that (cf. [10] Remark 2.3)

\[
P_m a \to a, \quad \text{in} \quad H^1(D) \quad \text{as} \quad m \to \infty.
\]

This implies in particular that

\[
P_m a \to a, \quad \text{in} \quad L^2(D) \quad \text{as} \quad m \to \infty.
\]
In addition, we have that (cf. [12] p. 49)
\[
\int_D u_m P_m [\text{div}(\nabla \Psi^n (\nabla W^{m,n}_A))] = - \int_D \nabla u_m \nabla \Psi^n (\nabla W^{m,n}_A). \tag{5.6}
\]
Indeed,
\[
\int_D u_m \sum_{j=1}^m (\int_D \text{div}(\nabla \Psi^n (\nabla W^{m,n}_A)) w_j) w_j = \sum_{j=1}^m \int_D u_m w_j \int_D \text{div}(\nabla \Psi^n (\nabla W^{m,n}_A)) w_j = \int_D \text{div}(\nabla \Psi^n (\nabla W^{m,n}_A)) (\sum_{j=1}^m \int_D u_m w_j) w_j = \int_D \text{div}(\nabla \Psi^n (\nabla W^{m,n}_A)) u_m = - \int_D \nabla u_m \nabla \Psi^n (\nabla W^{m,n}_A).
\]

**Lemma 5.1.** There exists a positive constant $\mathcal{K}$ such that
\[
\mathbb{E} \int_0^T \int_D (W^{m,n}_A)^2 dx dt \leq \mathcal{K}, \tag{5.7}
\]
\[
\mathbb{E} \int_0^T \int_D |\nabla (W^{m,n}_A)|^2 dx dt \leq \mathcal{K}, \tag{5.8}
\]
\[
\mathbb{E} \int_0^T \|P_m \text{div}(\nabla \Psi^n (\nabla W^{m,n}_A))\|_{L^2(D)}^2 \leq \mathcal{K}, \tag{5.9}
\]
\[
\sup_{t \in (0,T)} \mathbb{E} \int_D (W^{m,n}_A)^2 dx \leq \mathcal{K}. \tag{5.10}
\]

*Proof.* We first recall Itô’s formula as in [16] p.16-17 which is based on [11] [p.153, Theorem 3.6], and is applicable to systems of stochastic ordinary differential equations.

**Lemma 5.2.** For a smooth vector function $h$ and an adapted process $(g(t), t \geq 0)$ with $\int_0^T |g(t)| dt < \infty$ almost surely, for all $T > 0$ set
\[
X(t) := \int_0^t g(s) ds + \int_0^t h dW(s), \quad 0 \leq t \leq T,
\]
where $h$ is a vector of components $h_l$, $l = 1, \ldots, m$ and $dW$ is a vector of components $d\beta_l$, $l = 1, \ldots, m$ with $\beta_l$ a one-dimensional Brownian motion. Then, for $F$ twice continuously differentiable in $X$ and continuously differentiable in $t$, one has
\[
F(X(t), t) = F(X(0), 0) + \int_0^t F_t(X(s), s) + \int_0^t F_x(X(s)) g(s) ds + \int_0^t F_{xx}(X(s)) h dW(s) + \frac{1}{2} \sum_{l=1}^m \int_0^t F_{xx}(X(s)) h_l^2 ds. \tag{5.11}
\]
Next we apply Lemma 5.2 to (5.2) with \(hdW = \sum_{l=1}^{m} P_m \sqrt{\lambda_l} \epsilon_l \, \delta_l(t)\) and \(h_l = P_m \sqrt{\lambda_l} \epsilon_l\), supposing that \(F\) does not depend on time and setting

\[
X(t) = W_{A}^{m,n}(t), \\
F(X(t)) = (X(t))^2, \\
F'(X(t)) = 2X(t), \\
F''(X(t)) = 2, \\
g(s) = P_m \text{div}(\nabla \Psi^n(\nabla W_{A}^{m,n}(s))).
\]

We remark that in this case \(F\) does not depend on \(t\). After integrating on \(D\), we obtain almost surely, for all \(t \in [0,T]\),

\[
\int_{D} W_{A}^{m,n}(x,t)^2 \, dx = 2 \int_{0}^{t} \int_{D} W_{A}^{m,n} P_m \text{div}(\nabla \Psi^n(\nabla W_{A}^{m,n}(s))) \, dx \, ds \\
+ 2 \sum_{l=1}^{m} \int_{0}^{t} \int_{D} W_{A}^{m,n} P_m \sqrt{\lambda_l} \epsilon_l \, dx \, \delta_l(s) \\
+ \int_{0}^{t} \sum_{l=1}^{m} \|P_m \sqrt{\lambda_l} \epsilon_l\|^2_{L^2(D)} \, dx \, ds.
\]  

(5.12)

Substituting (5.6) into (5.12) we obtain,

\[
\|W_{A}^{m,n}(t)\|^2_{L^2(D)} + 2 \int_{0}^{t} \int_{D} \nabla W_{A}^{m,n} \nabla \Psi^n(\nabla W_{A}^{m,n}(s)) \, dx \, ds \\
= 2 \sum_{l=1}^{m} \int_{0}^{t} \int_{D} W_{A}^{m,n} P_m \sqrt{\lambda_l} \epsilon_l \, dx \, \delta_l(s) + \int_{0}^{t} \sum_{l=1}^{m} \|P_m \sqrt{\lambda_l} \epsilon_l\|^2_{L^2(D)} \, dx \, ds.
\]  

(5.13)

Taking the expectation, we obtain

\[
\mathbb{E}\|W_{A}^{m,n}(t)\|^2_{L^2(D)} + 2\mathbb{E} \int_{0}^{t} \int_{D} \nabla W_{A}^{m,n} \nabla \Psi^n(\nabla W_{A}^{m,n}(s)) \, dx \, ds \\
= \mathbb{E} \int_{0}^{t} \sum_{l=1}^{m} \|P_m \sqrt{\lambda_l} \epsilon_l\|^2_{L^2(D)} \, dx \, ds,
\]  

(5.14)

where we have used the fact that \(2\mathbb{E}[\sum_{l=1}^{m} \int_{0}^{t} \int_{D} W_{A}^{m,n} \sqrt{\lambda_l} \epsilon_l \, dx \, \delta_l(s)] = 0\) ([13] Theorem 2.3.4 - p.11).

We deduce from (5.3) that

\[
\sum_{l=1}^{m} \|P_m \sqrt{\lambda_l} \epsilon_l\|^2_{L^2(D)} \leq \sum_{l=1}^{m} (\|\sqrt{\lambda_l} \epsilon_l\|^2_{L^2(D)} + \|\nabla (\sqrt{\lambda_l} \epsilon_l)\|^2_{L^2(D)}) \\
\leq (\Lambda_0 + \Lambda_2).
\]  

(5.15)
Taking the supremum of equation (5.14) and substituting (5.15) into (5.14) we obtain
\[ \sup_{t \in (0,T)} \mathbb{E} \| W^{m,n}_A(t) \|_{L^2(D)}^2 \leq T(\Lambda_0 + \Lambda_2) \leq K. \]

This completes the proof of (5.10).

In order to obtain an $H^2$-type estimate for $W^{m}_A$, we take the gradient of the equation (5.2). For all $x \in D$, we have that
\[
\nabla W^{m,n}_A(t) = \int_0^t \nabla \{ P_m[\text{div}(\nabla \Psi^n(\nabla W^{m,n}_A))] \} ds + \sum_{l=1}^m \nabla \{ P_m[\sqrt{\lambda_l}e_l] \} \beta_l(t)
\]
\[
= \int_0^t \nabla \{ P_m[\text{div}(\nabla \Psi^n(\nabla W^{m,n}_A))] \} ds + \sum_{l=1}^m \int_0^t \nabla \{ P_m[\sqrt{\lambda_l}] \} d\beta_l(s).
\]
(5.16)

We fix $x \in D$ and apply below for a second time Itô’s formula Lemma 5.2 to the integral equation (5.16) where in this case $hdW = \sum_{l=1}^m \nabla \{ P_m[\sqrt{\lambda_l}e_l] \} d\beta_l(s)$ and $h_l = \nabla \{ P_m[\sqrt{\lambda_l}] \}$ with:

\[
X(t) = \nabla W^{m,n}_A(x,t),
F(X(t)) = \Psi^n(\nabla W^{m,n}_A(x,t)),
F'(X(t)) = \nabla \Psi^n(\nabla W^{m,n}_A(x,t)) = \nabla \Psi^n(\nabla W^{m,n}_A(x,t)),
F''(X(t)) = D^2 \Psi^n(\nabla W^{m,n}_A(x,t)), \text{ and}
g(s) = \nabla \{ P_m[\text{div}(\nabla \Psi^n(\nabla W^{m,n}_A(x,s)))] \}
\]

After integrating over $D$, we obtain almost surely, for all $t \in [0,T]$,
\[
\int_D \Psi^n(\nabla W^{m,n}_A(x,t)) dx = \int_0^t \int_D \nabla \Psi^n(\nabla W^{m,n}_A(x,s)) \nabla \{ P_m[\text{div}(\nabla \Psi^n(\nabla W^{m,n}_A(s)))] \} dx ds
\]
\[
+ \sum_{l=1}^m \int_0^t \int_D \nabla \Psi^n(\nabla W^{m,n}_A(x,s)) \nabla \{ P_m[\sqrt{\lambda_l}] \} dx d\beta_l(s)
\]
\[
+ \frac{1}{2} \sum_{l=1}^m \int_0^t \int_D D^2 \Psi^n(\nabla W^{m,n}_A(x,s)) |\nabla P_m[\sqrt{\lambda_l}]|^2 dx ds.
\]
In view of (1.1) and (5.6) we have that

\[
\int_D \Psi^n(\nabla W_{A}^{m,n}(t))dx \leq -\int_0^t \int_D [P_m \text{div}(\nabla \Psi^n(\nabla W_{A}^{m,n}(s)))^2]dxds + \sum_{l=1}^m \int_0^t \nabla \Psi^n(\nabla W_{A}^{m,n}(x,s))\nabla \{P_m(\sqrt{\Lambda_l e_l})\}dx\beta_l(s)
\]

\[
+ \frac{1}{2} \|D^2(\nabla W_{A}^{m,n}(s))\|_{L^\infty(D)} \sum_{l=1}^m \int_0^t |\nabla P_m(\sqrt{\Lambda_l e_l})|^2gs
\]

\[
\leq -\int_0^t \|P_m \text{div}(\nabla \Psi^n(\nabla W_{A}^{m,n}(s)))\|_{L^2(D)}^2ds + \sum_{l=1}^m \int_0^t \nabla \Psi^n(\nabla W_{A}^{m,n}(x,s))\nabla \{P_m(\sqrt{\Lambda_l e_l})\}dx\beta_l(s)
\]

\[
+ \frac{c_1}{2} \sum_{k=1}^m \int_0^t \|\nabla P_m\sqrt{\Lambda_l e_l}\|_{L^2(D)}^2ds.
\]  

(5.17)

Thus taking the expectation of (5.17) and using the fact that

\[
\mathbb{E}\left[\sum_{l=1}^m \int_0^t \nabla \Psi^n(\nabla W_{A}^{m,n}(x,s))\nabla \{P_m(\sqrt{\Lambda_l e_l})\}dx\beta_l(s)\right] = 0,
\]

we obtain

\[
\mathbb{E} \int_D \Psi^n(\nabla W_{A}^{m,n}(t))dx + \mathbb{E} \int_0^t \|P_m \text{div}(\nabla \Psi^n(\nabla W_{A}^{m,n}(s)))\|_{L^2(D)}^2ds
\]

\[
\leq \frac{c_1}{2} \sum_{l=1}^m \mathbb{E} \int_0^t \|\nabla P_m\sqrt{\Lambda_l e_l}\|_{L^2(D)}^2ds.
\]  

(5.18)

Adding (5.14) and (5.18), using (5.3), (1.5) and (1.6) we obtain

\[
\mathbb{E} \int_D \Psi^n(\nabla W_{A}^{m,n}(t))dx + \mathbb{E} \|W_{A}^{m,n}\|_{L^2(D)}^2 + \mathbb{E} \int_0^t \|P_m \text{div}(\nabla \Psi^n(\nabla W_{A}^{m,n}(s)))\|_{L^2(D)}^2ds
\]

\[
+ 2\mathbb{E} \int_0^t \int_D \nabla W_{A}^{m,n}\nabla \Psi^n(\nabla W_{A}^{m,n}(s))dxds
\]

\[
\leq c_0 \int_0^t \sum_{l=1}^m \left(\|P_m\sqrt{\Lambda_l e_l}\|_{L^2(D)}^2 + \|\nabla P_m\sqrt{\Lambda_l e_l}\|_{L^2(D)}^2\right)
\]

\[
\leq c_0 \int_0^t \sum_{l=1}^m \left(\|\sqrt{\Lambda_l e_l}\|_{L^2(D)}^2 + \|\nabla \sqrt{\Lambda_l e_l}\|_{L^2(D)}^2\right)
\]

\[
\leq c_0 \int_0^t \left(\sum_{l=1}^m \lambda_l ||e_l||_{L^2(D)}^2 + \sum_{l=1}^m \lambda_l ||\nabla e_l||_{L^2(D)}^2\right)ds
\]

\[
\leq c_0 T(\Lambda_0 + \Lambda_2),
\]

25
where $c_0 = \max(1, \frac{c_1}{2})$. In view of (1.3) we obtain,
\[
E \int_D \Psi^n(\nabla W_A^{m,n}(t)) \, dx + E\|W_A^{m,n}(t)\|_{L^2(D)}^2
+ E \int_0^t \|P_m[\text{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s)))]\|_{L^2(D)}^2 \, ds
\leq c_0 T(\Lambda_0 + \Lambda_2) \leq K,
\]
which completes the proof of (5.7), (5.8) and (5.9).

Hence there exist a subsequence which we denote again by $W_A^{m,n}$ and a function $W_A \in L^2(\Omega \times (0, T); H^1) \cap L^\infty(0, T; L^2(\Omega \times D))$ such that
\[
W_A^{m,n} \rightharpoonup W_A \text{ weakly in } L^2(\Omega \times (0, T); H^1(D)) \quad (5.19)
\]
\[
P_m \text{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s))) \rightharpoonup \tilde{\Phi} \text{ weakly in } L^2(\Omega \times (0, T); L^2(D)) \quad (5.21)
\]
as $m, n \to \infty$.

In addition, one can show the following result.

**Lemma 5.3.**
\[
\sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)\beta_l(t) \to \sum_{l=1}^\infty \sqrt{\lambda_l}e_l\beta_l(t) \quad \text{in } L^\infty((0, T); L^2(\Omega; L^2(D))). \quad (5.22)
\]

**Proof.** For all $t \in [0, T]$,
\[
E \int_D \left| \sum_{l=1}^\infty \sqrt{\lambda_l}e_l\beta_l(t) - \sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)\beta_l(t) \right|^2 \leq 2E \int_D \left| \sum_{l=1}^\infty \sqrt{\lambda_l}e_l\beta_l(t) - \sum_{l=1}^m \sqrt{\lambda_l}e_l\beta_l(t) \right|^2 \leq 2E \int_D \left| \sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)\beta_l(t) \right|^2
\]
\[
+ 2E \int_D \left| \sum_{l=m+1}^\infty \sqrt{\lambda_l}e_l\beta_l(t) \right|^2 \leq 2E \int_D \left| \sum_{l=m+1}^\infty \sqrt{\lambda_l}e_l\beta_l(t) \right|^2 \leq 2E \int_D \left| \sum_{l=1}^m [P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l]\beta_l(t) \right|^2 = W_1 + W_2.
\]
By [9] p. 20 we deduce that $W_1 \to 0$ in $C([0,T])$ as $m \to \infty$. For $W_2$, by the properties of the Brownian motion, we have that

$$2\int_D E \left| \sum_{l=1}^{m} [P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l] \beta_l(t) \right|^2$$

$$= 2\int_D E \left( \sum_{l=1}^{m} [P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l]^2 \beta_l^2(t) \right)$$

$$+ 2 \sum_{l \neq l_2} m [P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l] \beta_l [P_m(\sqrt{\lambda_{l_2}}e_{l_2}) - \sqrt{\lambda_{l_2}}e_{l_2}] \beta_{l_2} \right) dx$$

$$= 2\int_D \sum_{l=1}^{m} [P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l]^2 E[\beta_l^2(t)]$$

$$= 2 \sum_{l=1}^{m} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2 t$$

$$\leq 2T \sum_{l=1}^{\infty} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2, \quad (5.23)$$

In order to prove that the right-hand side of (5.23) tends to zero as $m \to \infty$, we use (5.3) and (5.5) to deduce that

$$\sum_{l=1}^{\infty} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2$$

$$= \sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2 + \sum_{l=K+1}^{\infty} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2$$

$$\leq \sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2 + 2 \sum_{l=K+1}^{\infty} \|\sqrt{\lambda_l}e_l\|_{H^1(D)}^2$$

$$\leq \sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2 + 4 \sum_{l=K+1}^{\infty} (\lambda_l + \lambda_l \|\nabla e_l\|_{L^2(D)})$$

$$\leq P_1 + P_2, \quad (5.24)$$

Let $\varepsilon > 0$ be arbitrary. We choose $K$ such that $P_2 \leq \frac{\varepsilon}{2}$. For a fixed $K$, we choose $m$ sufficiently large such that $P_1 \leq \frac{\varepsilon}{2}$. Therefore,

$$\sum_{l=1}^{\infty} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2 \leq \varepsilon, \quad (5.25)$$

so that $W_2 \to 0$ in $C([0,T])$ as $m \to \infty$. \hfill \Box

Let $y$ be an arbitrary bounded random variable, and let $\psi$ be an arbitrary bounded function on $(0,T)$. Next we multiply the equation (5.2) by the product $y \psi$, integrate on $D$.
between 0 and \(T\) and take the expectation to obtain
\[
\mathbb{E} \int_0^T \int_D y\psi(t)W_A^{m,n} w_j dx dt = \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle P_m(\text{div} \Psi^n (\nabla W_A^{m,n})), w_j \rangle ds \right\} dt \\
+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)\beta_l(t)w_j dx \right\} dt.
\]
Passing to the limit when \(m, n \to \infty\), using (5.19)-(5.21) and (5.22), and remembering that the linear combinations of \(w_j\) are dense in \(H^1(D)\), yields
\[
\mathbb{E} \int_0^T \int_D y\psi(t)W_A(t) \tilde{w} dx dt = \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \tilde{\Phi}, \tilde{w} \rangle ds \right\} dt \\
+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \sum_{l=1}^\infty (\sqrt{\lambda_l}e_l)\beta_l(t)\tilde{w} dx \right\} dt,
\]
for all \(\tilde{w} \in H^1(D)\). Therefore, we deduce that
\[
W_A(t) = \int_0^t \tilde{\Phi}(s) ds + \sum_{l=1}^\infty \sqrt{\lambda_l}e_l\beta_l(t) \text{ on } \Omega \times (0, T) \times D. \tag{5.26}
\]
We will prove below, using again the monotonicity method, that \(\tilde{\Phi} = \text{div}(\nabla \Psi(\nabla W_A))\).

### 5.1 Monotonicity argument

Let \(w\) be such that \(w \in L^2(\Omega \times (0, T); H^1(D))\) and let \(c\) be a positive constant. We define
\[
\mathcal{O}_{mn} = \mathbb{E} \left[ \int_0^T e^{-cs} \left\{ 2(P_m[\text{div} \Psi^n (\nabla W_A^{m,n})]) - P_m[\text{div} \Psi^n (\nabla w)] \right\}, W_A^{m,n} - w \right\} ds \\
- c||W_A^{m,n} - w||^2 \right\} ds
\]
\[
= J_1 + J_2.
\]
We will check as before the following result

**Lemma 5.4.**

\[ O_{mn} \leq 0. \]

**Proof.** Using (5.6) and (1.3) we have that
\[
J_1 = \left\langle P_m[\text{div} \Psi^n (\nabla W_A^{m,n})]) - P_m[\text{div} \Psi^n (\nabla w)] \right\rangle, W_A^{m,n} - w \right\} ds \\
= - \int_D [\nabla \Psi^n (\nabla W_A^{m,n}) - \nabla \Psi^n (\nabla w)] \nabla (W_A^{m,n} - w) ds \\
\leq -C_0 ||\nabla (W_A^{m,n} - w)||^2_{L^2(D)} \leq 0, \tag{5.27}
\]
which completes the proof. \(\blacksquare\)
We write $O_{mn}$ in the form $O_{mn} = O^1_{mn} + O^2_{mn}$ where

$$O^1_{mn} = \mathbb{E}\left[\int_0^T e^{-cs} \{2\langle P_m[\text{div}(\nabla \Psi^n(\nabla W_{A}^{mn}))], W_{A}^{mn}\rangle - c\|W_{A}^{mn}\|^2\}\}ds.\right]$$

(5.28)

We apply Ito formula Lemma 5.2 on (5.2) with $F(X,t) = e^{-ct}(X)^2$ and $F_t = -ce^{-ct}(X)^2$. After integrating on $D$ and taking the expectation, we obtain almost surely, for all $t \in [0,T]$,

$$\mathbb{E}[e^{-cT}\|W_{A}^{mn}(x,T)\|^2_{L^2(D)}] = -c\mathbb{E}\left[\int_0^T e^{-cs}\|W_{A}^{mn}(x,s)\|^2_{L^2(D)}ds\right]$$

$$+ 2\mathbb{E}\left[\int_0^T e^{-cs} \int_D W_{A}^{mn} P_m[\text{div}(\nabla \Psi^n(\nabla W_{A}^{mn}(s)))]dxds\right]$$

$$+ 2\mathbb{E}\left[\sum_{l=1}^m \int_0^T e^{-cs} \int_D W_{A}^{mn} P_m \sqrt{\lambda_l} \epsilon_l dxd\beta_l(s)\right]$$

$$+ \int_0^T e^{-cs} \sum_{l=1}^m \|P_m \sqrt{\lambda_l} \epsilon_l\|^2_{L^2(D)}ds.\right]$$

(5.29)

It follows from (5.28),(5.29) and the fact that

$$\mathbb{E}\left[\sum_{l=1}^m \int_0^T e^{-cs} \int_D W_{A}^{mn} P_m \sqrt{\lambda_l} \epsilon_l dxd\beta_l(s)\right] = 0$$

that

$$O^1_{mn} = \mathbb{E}[e^{-cT}\|W_{A}^{mn}(T)\|^2_{L^2(D)}] - \int_0^T e^{-cs} \sum_{l=1}^m \|P_m \sqrt{\lambda_l} \epsilon_l\|^2_{L^2(D)}ds.\right]$$

(5.30)
In view of [5] p.193 we have that
\[
\left| \int_0^T e^{-cs} \sum_{l=1}^m \| P_m \sqrt{\lambda_l} e_l \|^2_{L^2(D)} - \sum_{l=1}^\infty \| \sqrt{\lambda_l} e_l \|^2_{L^2(D)} \right| ds \\
\leq \int_0^T e^{-cs} \sum_{l=1}^m \| P_m \sqrt{\lambda_l} e_l \|^2_{L^2(D)} - \sum_{l=1}^\infty \| \sqrt{\lambda_l} e_l \|^2_{L^2(D)} \right| ds \\
+ \int_0^T e^{-cs} \sum_{l=m+1}^\infty \| \sqrt{\lambda_l} e_l \|^2_{L^2(D)} ds \\
\leq T \sum_{l=1}^\infty \| P_m \sqrt{\lambda_l} e_l \|^2_{L^2(D)} - \| \sqrt{\lambda_l} e_l \|^2_{L^2(D)} \right| + T \sum_{l=m+1}^\infty \lambda_l \\
\leq T \sum_{l=1}^\infty \| \sqrt{\lambda_l} e_l - P_m \sqrt{\lambda_l} e_l \|^2_{L^2(D)} + T \sum_{l=m+1}^\infty \lambda_l \\
\leq \varepsilon,
\]
which, in view of (5.25) and (1.4), tends to zero as \( m \to \infty \). Thus,
\[
\lim_{m \to \infty} \int_0^T e^{-cs} \sum_{l=1}^m \| P_m \sqrt{\lambda_l} e_l \|^2_{L^2(D)} ds = \int_0^T e^{-cs} \sum_{l=1}^\infty \| \sqrt{\lambda_l} e_l \|^2_{L^2(D)} ds.
\]

Letting \( m \) and \( n \) tend to infinity in (5.30), we deduce that
\[
\lim_{m,n \to \infty} \sup \mathcal{O}_{mn}^1 = \mathbb{E}[e^{-cT} \| W_A(T) \|_{L^2(D)}^2] - \int_0^T e^{-cs} \sum_{l=1}^\infty \lambda_l ds + \delta e^{-cT},
\] (5.31)
where
\[
\delta = \lim_{m,n \to \infty} \sup \mathbb{E}[\| W_A^{m,n}(T) \|^2] - \mathbb{E}[\| W_A(T) \|^2] \geq 0.
\]

On the other hand, the equation (5.26) implies that a.s. in \( L^2(D) \)
\[
W_A(t) = \int_0^t \Phi(s) ds + \int_0^t dW(s), \quad \forall t \in [0,T],
\] (5.32)

Next we recall a simplified form of the Itô’s formula given by [7] ( Theorem 4.32 p.106), which will suffice for our purpose. We do so since the Itô’s formula given in Lemma 5.2 only applies to finite dimensional problems.

**Lemma 5.5.** Let \( h \) be an \( L^2(D) \)-valued progressively measurable Bochner integrable process. Consider the following well defined process :
\[
X(t) = \int_0^t h(s) ds + W(t), \quad t \in [0,T].
\]


Assume that a function $F : [0, T] \times L^2(D) \to \mathbb{R}$ and its partial derivatives $F_t, F_x, F_{xx}$ are uniformly continuous on bounded subsets of $[0, T] \times L^2(D)$, and that $F(X(0), 0) = 0$. Then, a.s., for all $t \in [0, T]$,

$$F(X(t), t) = \int_0^t F_t(X(s), s) ds + \int_0^t \langle F_x(X(s), s), dW(s) \rangle_{L^2(D)} + \frac{1}{2} \int_0^t Tr[F_{xx}(X(s), s)Q] ds$$

where

$$Tr[F_{xx}(X(s))Q] = \sum_{l=1}^{\infty} \langle F_{xx}(X(s), s)Qe_l, e_l \rangle_{L^2(D)}$$

and

$$\langle u, v \rangle_{L^2(D)} = \int_D u(x)v(x) dx,$$

where we note that $TrA = \sum_{l=1}^{\infty} \langle Ae_l, e_l \rangle_{L^2(D)}$ is bounded linear operator on $L^2(D)$.

Applying Lemma 5.5 to (5.32) with $X = W_A, F(X(t), t) = e^{-ct}\|X\|^2_{L^2(D)}, F_t(X(t), t) = -ce^{-ct}\|X\|^2_{L^2(D)}, F_x(X(t), t) = 2e^{-ct}X, h = \tilde{\Phi}, F_{xx}(X(t), t) = 2e^{-ct}I$.

After taking the expectation, we deduce that

$$\mathbb{E}[e^{-cT}\|W_A\|^2] = -c\mathbb{E}[\int_0^T e^{-cs}\|W_A\|^2 ds] + 2\mathbb{E}[\int_0^T e^{-cs}(\tilde{\Phi}, W_A) ds] + 2\mathbb{E}[\sum_{l=1}^{\infty} \int_0^T e^{-cs} \int_D W_A \sqrt{\lambda_l} e_l(x) d\beta_l(s)] + \int_0^T e^{-cs} \sum_{l=1}^{\infty} \lambda_l ds,$$

which we combine with (5.31) to deduce that

$$\lim_{m,n \to \infty} \sup O_{1mn}^1 = 2\mathbb{E}[\int_0^T e^{-cs}(\tilde{\Phi}, W_A)] - c\mathbb{E}[\int_0^T e^{-cs}\|W_A\|^2 ds] + \delta e^{-cT}. \quad (5.33)$$

It remains to compute the limit of $O_{mn}^2$:

$$O_{mn}^2 = O_{mn} - O_{1mn}^1$$

$$= \mathbb{E}[\int_0^T e^{-cs}\{ -2\langle P_m(\text{div} \nabla \Psi^n(\nabla w)), W_A^{m,n} \rangle - 2\langle P_m(\text{div} \nabla \Psi^n(\nabla W_A^{m,n})), w \rangle$$

$$+ 2\langle P_m(\text{div} \nabla \Psi^n(\nabla w)), w \rangle - c\|w\|^2 + 2c(W_A^{m,n}, w) \} ds.$$
In view of (5.19), (5.21), using (5.1) and (5.6) we deduce that
\[
\lim_{m,n \to \infty} O_{mn}^2 = \mathbb{E} \int_0^T e^{-cs} \{-2 \langle \text{div} \nabla \Psi(w), W_A \rangle - 2 \langle \tilde{\Phi} - \text{div} \nabla \Psi(w), w \rangle \\
- c\|w\|^2 + 2c \langle W_A, w \rangle \} ds.
\] (5.34)

Combining (5.33) and (5.34), and remembering that \(O_{mn} \leq 0\), yields
\[
\mathbb{E} \int_0^T e^{-cs} \{ 2 \langle \tilde{\Phi} - \text{div} \nabla \Psi(w), W_A - w \rangle - c\|W_A - w\|^2 \} ds + \delta e^{-cT} \leq 0.
\]

Let \(\tilde{v} \in L^2(\Omega \times (0,T); H^1(D))\) be arbitrary and set
\[w = W_A - \lambda \tilde{v}, \text{ with } \lambda \in \mathbb{R}_+.\]
Dividing by \(\lambda\) and letting \(\lambda \to 0\), we find that :
\[
\mathbb{E} \int_0^T e^{-cs} \langle \tilde{\Phi} - \text{div} \nabla \Psi(W_A), \tilde{v} \rangle dt \leq 0.
\]
Since \(\tilde{v}\) is arbitrary, it follows that
\[
\mathbb{E} \int_0^T \langle \tilde{\Phi}, \tilde{v} \rangle = \mathbb{E} \int_0^T \langle \text{div} \nabla \Psi(W_A), \tilde{v} \rangle,
\]
for all \(\tilde{v} \in L^2(\Omega \times (0,T); H^1(D))\),
that is
\[
\tilde{\Phi} = \text{div} \nabla \Psi(W_A)
\] (5.35)
a.s. a.e. in \(D \times (0,T)\).
One finally concludes that \(W_A\) satisfies Definition 2.1.

Next, we prove below the boundedness of \(W_A\) in \(L^\infty(0,T; L^q(\Omega \times D))\), for all \(q \geq 2\).
The proof of this result is based on an article by Bauzet, Vallet, Wittbold [2] where a similar result was proved for a convection-diffusion equation with a multiplicative noise on \(\mathbb{R}^n\) involving a standard adapted one-dimensional Brownian motion. More precisely, we follow the proof of Proposition A.5 of [2].

**Theorem 5.2.** Let \(W_A\) be a solution of Problem \((P_1)\); then
\(W_A \in L^\infty(0,T; L^q(\Omega \times D))\), for all \(q \geq 2\).

**Proof.** For each positive constant \(k\), denote by \(\Phi_k : \mathbb{R} \to \mathbb{R}\) the function
\[
\Phi_k(\xi) = \begin{cases} 
|\xi|^q, & \text{if } |\xi| < k, \\
\frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + \left(q - 1\right)(q - 1)k^q, & \text{if } k \leq |\xi|.
\end{cases}
\]
\(\Phi_k\) is a convex \(C^2\) function and \(\Phi'_k\) is a Lipschitz-continuous function with \(\Phi'_k(0) = 0\). The
function $\Phi_k$ satisfies the inequalities $0 \leq \Phi_k'(\xi) \leq c(k)\xi$ and $0 \leq \Phi_k(\xi) = \int_0^\xi \Phi_k'(\zeta)d\zeta \leq \frac{c(k)}{2}\xi^2$ for all $\xi \in \mathbb{R}^+$. This yields in view of Definition 2.1 (i) that,

$$\mathbb{E}\int_D \Phi_k(W_A(x,t))dx \leq \frac{c(k)}{2}\mathbb{E}\int_D W_A^2(x,t)dx \leq \tau(k)$$

for a.e. $t \in [0,T]$.

**Lemma 5.6.** (i) One has $0 \leq \Phi_k''(\xi) \leq c_k$ for all $\xi \in \mathbb{R}$ where $c_k$ is a positive constant depending on $k$.

(ii) One has $0 \leq \Phi_k''(\xi) \leq q(q-1)(1+\Phi_k(\xi))$, for all $\xi \in \mathbb{R}$.

**Proof.** (i)

$$\Phi_k''(\xi) = \begin{cases} q(q-1)|\xi|^{q-2} & \text{if } 0 \leq |\xi| < k, \\ q(q-1)k^{q-2} & \text{if } k \leq |\xi|. \end{cases}$$

Thus,

$$\Phi_k''(\xi) \leq q(q-1)k^{q-2} =: c_k$$

(ii) If $|\xi| < k$, $\Phi_k''(\xi) = q(q-1)|\xi|^{q-2}$,

- if $1 \leq |\xi| < k$, $|\xi|^{q-2} \leq |\xi|^q$ which gives the result.
- if $0 \leq |\xi| < 1$, $|\xi|^{q-2} \leq |\xi|^q$ and $0 < 1 + |\xi|^q$.

If $|\xi| \geq k$, $\Phi_k''(\xi) = q(q-1)k^{q-2}$ the problem then reduces to prove that

$$H(\xi) = 1 + \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + \left(\frac{q}{2} - 1\right)(q-1)k^q - k^{q-2} \geq 0$$

Let us consider the function $H(\xi) = F(\xi) + G$ where

$$F(\xi) = \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi|$$

and

$$G = \left(\frac{q}{2} - 1\right)(q-1)k^q - k^{q-2} + 1.$$

- if $\xi \geq k$, $H'(\xi) = F'(\xi) \geq 0$ and $H(k) \geq 0$ for all $k > 0$, thus $H(\xi) \geq H(k) \geq 0$ for all $\xi \geq k$.
- if $\xi \leq -k$ then $H(-\xi) = F(-\xi) + G \geq 0$. Therefore

$$H(\xi) \geq H(-\xi) \geq 0.$$

Next we apply Lemma 5.5 to (2.1), supposing that $F$ does not depend on time and setting

$$X(t) = W_A(t),$$

$$F(X(t)) = \int_D \Phi_k(X(t))dx,$$

$$F'(X(t)) = \Phi_k'(X(t)),$$

$$h = \text{div}(A(\nabla W_A)),$$

$$F''(X(t)) = \Phi_k''(X(t)).$$

33
\[
\int_D \Phi_k(W_A(t)) \, dx = \int_0^t \langle \text{div}(A(\nabla W_A(s))), \Phi_k'(W_A(s)) \rangle \, ds \\
+ \int_0^t \int_D \Phi_k'(W_A(s)) \, dW(s) \\
+ \frac{1}{2} \sum_{l=1}^\infty \int_0^t \int_D \Phi_k''(W_A) \lambda_l |e_l|^2 \, dx \, ds \\
\leq - \int_0^t \int_D \Phi_k''(W_A) \nabla W_A(s) A(\nabla W_A(s)) \, ds \\
+ \int_0^t \int_D \Phi_k'(W_A(s)) \, dW(s) \\
+ \frac{1}{2} \sum_{l=1}^\infty \lambda_l \|e_l\|_{L^\infty}^2 \int_0^t \int_D \Phi_k''(W_A) \, dx \, ds.
\]
(5.36)

Taking the expectation of (5.36), and using the fact that \( \Phi_k'' \geq 0 \), we deduce from the fact that
\[
E \int_0^t \int_D \Phi_k'(W_A) \, dW(s) = 0 \quad (|13| \text{ Theorem 2.3.4 - p.11}),
\]
from the coercivity property (1.3) and from (1.5) that
\[
E \int_D \Phi_k(W_A(t)) \, dx \leq - C_0 E \int_0^t \int_D \Phi_k''(W_A) |\nabla W_A|^2 + \frac{1}{2} \Lambda_1 E \int_0^t \int_D \Phi_k''(W_A) \, dx \, ds.
\]

Then using Lemma 5.6 (ii) and Gronwall Lemma we obtain, defining
\[
C(q) = \frac{1}{2} q(q-1),
\]
\[
E \int_D \Phi_k(W_A(t)) \, dx \leq \frac{1}{2} q(q-1) \Lambda_1 E \int_0^t \int_D (1 + \Phi_k(W_A)) \, dx \, ds \\
\leq C(q) \Lambda_1 t |D| + C(q) \Lambda_1 E \int_0^t \int_D \Phi_k(W_A) \, dx \, ds \\
\leq C(q) \Lambda_1 t |D| e^{C(q) \Lambda_1 t}.
\]

Thus, \( E \int_D \Phi_k(W_A(t)) \, dx \) is bounded independently of \( k \).

Finally, since \( \Phi_k(W_A(x,t)) \) converges to \( |W_A(x,t)|^q \) for a.e. \( x \) and \( t \) when \( k \) goes to infinity, if follows from Fatou’s Lemma that
\[
E \int_D |W_A(x,t)|^q \, dx = E \int_D \lim_{k \to \infty} \Phi_k(W_A(x,t)) \, dx = E \int_D \lim inf_{k \to \infty} \Phi_k(W_A(x,t)) \, dx \\
\leq \lim inf_{k \to \infty} E \int_D \Phi_k(W_A(x,t)) \, dx \\
\leq C(q) \Lambda_1 t |D| e^{C(q) \Lambda_1 t}
\]
for all \( t > 0. \)

Therefore, \( W_A \in L^\infty(0,T; L^q(\Omega \times D)) \) for all \( q \geq 2. \)
### 5.2 Uniqueness of the solution $W_A$

Let $\omega$ be given such that two pathwise solutions of Problem $(P_2)$, $W^1_A = W^1_A(\omega, x, t)$ and $W^2_A = W^2_A(\omega, x, t)$ satisfy

\[
\begin{align*}
&u_i(\cdot \cdot, \omega) \in L^{\infty}(0, T; L^2(D)) \cap L^2(0, T; H^1(D)), \\
&\text{div}(A(\nabla(u_i + W_A))) \in L^2((0, T); L^2(D))
\end{align*}
\]

for $i = 1, 2$. The difference of the two solutions satisfies the equation

\[
W^1_A - W^2_A = \int_0^t \text{div}(A(\nabla W^1_A(s)) - A(\nabla W^2_A(s))) ds
\]

in $L^2((0, T) \times D)$.

We take the duality product of this equation with $W^1_A - W^2_A \in L^2((0, T); H^1(D))$. In view of (1.3) we obtain

\[
\begin{align*}
||W^1_A - W^2_A||^2_{L^2(D)} & = -\int_0^t [A(\nabla W^1_A) - A(\nabla W^2_A)]\nabla(W^1_A - W^2_A) \\
& \leq -C_0 \int_0^t ||\nabla(W^1_A - W^2_A)||^2_{L^2(D)},
\end{align*}
\]

which in turn implies that

\[
W^1_A = W^2_A \quad \text{a.e. in } D \times (0, T).
\]

\[\square\]

### A Appendix

In this appendix we prove the properties $(F_1), (F_2)$ and $(F_3)$ for the nonlinear function $f$.

\begin{enumerate}
  \item [(F_1)] There exist positive constants $C_1$ and $C_2$ such that
  \[f(a + b)a \leq -C_1 a^{2p} + f_2(b), \quad |f_2(b)| \leq C_2 (b^{2p} + 1), \quad \text{for all } a, b \in \mathbb{R}\]
\end{enumerate}

Proof. For simplicity we suppose that $b_j = 1$ for all $j = 0, \ldots, 2p - 2$ and that $b_{2p-1} = -1$.

\[
f(a + b) = \sum_{j=0}^{2p-1} b_j (a + b)^j
\]

\[
= -(a + b)^{2p-1} + (a + b)^{2p-2} + \cdots + (a + b)^2 + (a + b)
\]

\[
f(a + b)a = -(a + b)^{2p-1}a + (a + b)^{2p-2}a + \cdots + (a + b)^2a + (a + b)a
\]

\[
= L_{2p-1} + L_{2p-2} + \ldots L_1.
\]

(A.1)
We first estimate the term $L_{2p-1}$.

\[ L_{2p-1} = -(a + b)^{2p-1} a = -a^{2p} - C_{2p-1}^1 a^{2p-1} b - C_{2p-1}^2 a^{2p-2} b^2 - \cdots - C_{2p-1}^{2p-3} a^3 b^{2p-3} \]

We suppose that (A.4) is true for $n = 2p-1$. Thus,

\[ L_{2p-1} \leq -a^{2p} + C_{2p-1}^1 \frac{\varepsilon(2p-1)|a|^{2p}}{2p} + C_{2p-1}^2 \frac{|b|^{2p}}{2p\varepsilon} + \cdots + \frac{\varepsilon|a|^{2p}}{2p}. \]

Similarly, we find that.

**Lemma A.1.**

\[ L_q \leq \varepsilon C_1(p)|a|^{2p} + \frac{C_3(p)}{\varepsilon} |b|^{2p} + \frac{1}{\varepsilon} C_4(p), \quad \text{for all } q \in \{1, \ldots, 2p - 2\}. \]

**Proof.** By induction, we first prove that (A.4) is true for $q = 1$.

Using Hölder inequality, we deduce that

\[ L_1 = (a + b)a = a^2 + ab \leq \frac{\varepsilon}{p} |a|^{2p} + \frac{p - 1}{p\varepsilon} + \frac{\varepsilon}{2p} |a|^{2p} + \frac{2p - 1}{2p\varepsilon} |b|^{2p-1} \]

\[ \leq \frac{3\varepsilon}{2p} |a|^{2p} + \frac{p - 1}{p\varepsilon} + \frac{2p - 1}{2p\varepsilon} \left( \frac{1}{(2p - 1)} |b|^{2p} + \frac{(2p - 2)}{2p - 1} \right) \]

\[ \leq \frac{3\varepsilon}{2p} |a|^{2p} + \frac{1}{2p\varepsilon} |b|^{2p} + \frac{2p - 2}{p\varepsilon}. \]

We suppose that (A.4) is true for $q = 2p - 3$ and prove that it remains true for $q = 2p - 2$.

Using Hölder inequality, we obtain

\[ L_{2p-2} = (a + b)^{2p-2} a = a^{2p-1} + C_{2p-2}^1 a^{2p-2} b + \cdots + C_{2p-2}^{2p-3} a^3 b^{2p-3} + a b^{2p-2} \]

\[ \leq \frac{\varepsilon(2p - 1)}{2p} |a|^{2p} + \frac{1}{2p\varepsilon} + \cdots + \frac{\varepsilon}{2p} |a|^{2p} + \frac{2p - 1}{2p\varepsilon} |b|^{(2p-2)\varepsilon} \]

\[ \leq \varepsilon C_1(p)|a|^{2p} + \frac{1}{\varepsilon} C_3(p)|b|^{2p} + C_4(p). \]

\[ \text{Eq. A.5} \]
Combining (A.1),(A.3) and Lemma A.1 and choosing $\varepsilon < \frac{1}{2(C'(p) + C_1(p))}$ yields

$$f(a + b)a \leq \left(-1 + \varepsilon(C'(p) + C_1(p))\right)|a|^{2p} + \frac{1}{\varepsilon}C_3(p)|b|^{2p} + C_4(p) \leq -\frac{1}{2}a^{2p} + C_2(b^{2p} + 1),$$

with $C_2 = \max(\frac{1}{\varepsilon}C_3(p), C_4(p))$.

$(F_2)$ There exists a positive constant $C_3$ such that

$$|f(s)| \leq C_3|s - M|^{2p - 1} + \tilde{C}_3(M).$$

Proof. Again, we suppose that $b_j = 1$ for all $j = 0, \ldots, 2p - 2$ and that $b_{2p-1} = -1$.

$$f(s) = -s^{2p-1} + s^{2p-2} + \ldots + s^2 + s. \quad (A.6)$$

We estimate the leading term of (A.6)

$$|s|^{2p-1} = |s - M + M|^{2p-1} = |s - M|^{2p-1} + C_{2p-1}^{2p-1}|s - M|^{2p-2}M + \ldots + C_{2p-1}^{2p-2}|s - M|^{2p-2}M + M^{2p-1}.$$ 

By Hölder inequality, there holds

$$|s - M|^{2p-2}M \leq \frac{e(2p - 2)}{2p - 1} |s - M|^{2p - 1} + \frac{M^{2p-1}}{e(2p - 1)},$$

so that

$$|s|\leq (1 + \varepsilon C(p))|s - M|^{2p - 1} + (1 + \frac{1}{\varepsilon}C(p))M|^{2p - 1}. \quad (A.7)$$

Next, we estimate the last term on the right-hand-side of (A.6).

It follows from Hölder inequality that

$$|s| \leq |s - M| + |M| \leq \frac{e}{2p - 1} |s - M|^{2p - 1} + \frac{2p - 2}{e(2p - 1)} + |M|. \quad (A.8)$$

Computing all the other terms of (A.6) similarly and substituting them in (A.6) we obtain

$$|f(s)| \leq C_3|s - M|^{2p - 1} + \tilde{C}_3(M).$$
There exists a positive constant $C_4$ such that
\[ f'(s) \leq C_4. \]

Proof.
\[ f'(s) = -(2p - 1)s^{2p-2} + (2p - 2)s^{2p-3} + \ldots + 2s + 1. \tag{A.9} \]

By Hölder inequality
\[ |s|^{2p-3} \leq \frac{\varepsilon(2p - 3)}{(2p - 2)} |s|^{2p-2} + \frac{1}{(2p - 2)e}, \ldots \tag{A.10} \]
\[ |s| \leq \frac{\varepsilon}{2p - 2} |s|^{2p-2} + \frac{2p - 3}{(2p - 2)e}. \tag{A.11} \]

We compute all the other terms similarly, and substitute them in (A.9) to obtain
\[ f'(s) \leq -(2p - 1) + \varepsilon C(p)|s|^{2p-2} + \frac{\tilde{C}(p)}{\varepsilon} + 1. \]

Choosing $\varepsilon \leq \frac{2p - 1}{2C(p)}$ we conclude that
\[ f'(s) \leq C_4. \]

\[ \square \]

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