APPLICATIONS OF THE LIEB–THIRRING AND OTHER BOUNDS FOR ORTHONORMAL SYSTEMS IN MATHEMATICAL HYDRODYNAMICS

ALEXEI ILYIN\textsuperscript{1,2}, ANNA KOSTIANKO\textsuperscript{4,5}, AND SERGEY ZELIK\textsuperscript{1,3,4}

Abstract. We discuss the estimates for the $L^p$-norms of systems of functions that are orthonormal in $L^2$ and $H^1$, respectively, and their essential role in deriving good or even optimal bounds for the dimension of global attractors for the classical Navier–Stokes equations and for a class of $\alpha$-models approximating them. New applications to interpolation inequalities on the 2D torus are also given.

1. Introduction

The 2D Navier–Stokes system is probably one of the widest known and popular example of an evolution dissipative PDE possessing a global attractor in an appropriate phase space. Furthermore, many concepts and ideas of the theory of infinite dimensional dissipative dynamical systems have originated and have been developed from this example (see, for instance, \cite{3, 35} and the references therein).

The global attractor is a compact, strictly invariant and globally attracting set in the phase space, and one of main achievements of the theory was the proof that its Hausdorff and fractal dimension are finite. Then followed exponential and afterwards polynomial estimates of its dimension, which have saturated (see \cite{2, 9}) at the level of

$$\dim \mathcal{A} \leq \text{const} G^2, \quad G := \frac{\|f\|\|\Omega\|}{\nu^2},$$

(1.1)
where $\|f\|$ is the $L^2$-norm of the forcing term, $|\Omega|$ is the area of the spatial domain, $\nu$ is the viscosity coefficient (see (3.1)), and the dimensionless number $G$ built of the physical parameters of the system is called the Grashof number.

The idea to use Lieb–Thirring inequalities [26] for $L^2$-orthonormal families in the study of attractors of the Navier–Stokes equations was first suggested by D. Ruelle [31] and some conjectures of [31] have been proved by E. Lieb [24]. For the two dimensional Navier–Stokes system in a bounded domain with non slip boundary conditions R. Temam [34] using this technique obtained upper bounds for the Hausdorff and fractal dimension of the attractor which are linear with respect to the Grashof number and are probably optimal:

$$\dim \mathcal{A} \leq \text{const} \, G. \quad (1.2)$$

At least in terms of the physical parameters this upper bound stays unchanged for almost four decades, no lower bounds for the dimension in the case of the Dirichlet boundary conditions are available either.

On the other hand again for more than four decades the theory of the Lieb–Thirring inequalities is still a very active and dynamically developing area of functional analysis and mathematical physics. A current state of the art of many aspects of the theory is presented in [12].

In §2 we formulate the required Lieb–Thirring inequality for divergence free $L^2$-orthonormal vector functions in two dimensions and also the relevant Li–Yau-type lower bound for the eigenvalues of the Stokes operator. Then in §3 we describe in reasonable detail the proof of upper bounds (1.1) and (1.2) and single out the point, where the Lieb–Thirring inequality plays the vital role in going from one to the other.

While the first part of this work is essentially a brief review of the role of the Lieb–Thirring inequalities in the Navier–Stokes theory, the second part contains new results. We turn here from the classical models in hydrodynamics to a class of their approximations in terms of $\alpha$-models. Alpha models became popular over the last decades both in theory and in practice as subgrid scale models of turbulence. One of the characteristic features of these models is the smoothing of the velocity vector $u$ in certain parts of the bilinear convective term by replacing it with $\bar{u} := (1 - \alpha \Delta)^{-1} u$, where $\alpha = \alpha' L^2$, $L$ is the characteristic length, and $\alpha'$ is a small dimensionless parameter.

We also observe that in certain cases the energy space is not necessarily $L^2$. For instance, in the Euler–Bardina model that we have been interested
in the natural phase space is $H^1$ with scalar product
\[(u, v)_\alpha = (u, v) + \alpha(\nabla u, \nabla v).\] (1.3)
The corresponding global Lyapunov exponents are also estimated in $H^1$ and in this way we are led to find bounds for $\|\rho\|_{L^2}$, where
\[\rho(x) := \sum_{j=1}^{n} |v_j(x)|^2,
\]
where $\{v_j\}_{j=1}^{n}$ is an orthonormal family in $H^1$ with respect scalar product (1.3). This type of inequalities were discovered by E. Lieb in [23] and remarkably nicely fit in the estimates we required in [19, 17] (namely, the $L^2$-bound for $\rho$), where we have also given explicit expressions for the constants on $\mathbb{T}^2$ and $\mathbb{T}^3$ to be able to write down explicitly the estimate for fractal dimension of the global attractor.

The main result here (Theorem 4.2) gives explicit bounds for the $L^p$-norm on the torus $\mathbb{T}^2$ of the function $\rho$ for all $1 \leq p < \infty$.

The one-function corollary of this theorem is equivalent to the interpolation inequality for $\varphi \in H^1(\mathbb{T}^2)$:
\[\|\varphi\|_{L^q(\mathbb{T}^2)} \leq \left(\frac{1}{4\pi}\right)^{2\pi^2} \left(\frac{q}{2}\right)^{1/2} \|\varphi\|^{2/q} \|\nabla\varphi\|^{1-2/q}, \quad q \geq 2,\]
and the constant here should be compared with that in the corresponding inequality in $\mathbb{R}^2$, see (4.12).

In §5 we prove the key inequality for the 2D lattice sum
\[I_p(m) := \frac{(p-1)m^{2(p-1)}}{\pi} \sum_{n \in \mathbb{Z}^2} \frac{1}{(m^2 + |n|^2)^p} < 1,\]
which was previously proved for $p = 2$ in [19]. It is easy to see that $I_p(\infty) = 1$ so it suffices to establish monotonicity of $I_p(m)$. By using a special representation of $I_p(m)$ in terms of the Jacobi theta function $\theta_3$ the required monotonicity is proved by showing that $\frac{d}{dm} I_p(m) > 0$. We point out that this approach works simultaneously for all $p > 1$ (and with minor changes on $\mathbb{T}^3$ as well).

We finally observe that the monotonicity so obtained is a subtle property of $I_p(m)$. There are quite a few examples of similar lattice sums and series with respect to the eigenvalues of the Laplacian on the sphere $\mathbb{S}^3$ where the corresponding functions exhibit oscillations for $m$ not too large.
2. Lieb–Thirring and Li–Yau-type inequalities for divergence free orthonormal families

In this section formulate the Lieb–Thirring inequality for $L^2$-orthonormal families of divergence free vector functions in 2D and the Li–Yau-type lower for the eigenvalues of the Stokes operator.

**Theorem 2.1.** (See [16].) Let $\Omega$ be an arbitrary domain in $\mathbb{R}^d$ with finite volume $|\Omega| < \infty$. Let a family of vector functions $\{u_k\}_{k=1}^m \in H^1_0(\Omega)$ be orthonormal, and, further, let $\text{div} u_k = 0$, $k = 1, \ldots, m$. Then

$$
\sum_{k=1}^m \|\nabla u_k\|^2 \geq \frac{d}{2 + d} \left( \frac{(2\pi)^d}{\omega_d(d-1)|\Omega|} \right)^{2/d} m^{1+2/d}.
$$

(2.1)

If we take for $u_k$ the first $m$ eigenfunctions of the Stokes operator

$$
-\Delta u_k + \nabla p_k = \lambda_k u_k,
\text{div} u_k = 0, \quad u_k|_{\partial \Omega} = 0,
$$

(2.2)

then the left-hand side in (2.1) becomes $\sum_{k=1}^m \lambda_k$ and in view of the asymptotic formula (see [28] at least when $\partial \Omega$ is Lipschitz)

$$
\lambda_k \sim \left( \frac{(2\pi)^d}{\omega_d(d-1)|\Omega|} \right)^{2/d} k^{2/d}
$$

we see that the constant on the right-hand side in (2.1) is sharp in the sense that it cannot be taken greater rendering (2.1) valid for all $m$.

In the 2D case of our main concern we obtain

$$
\sum_{k=1}^m \|\nabla u_k\|^2 \geq \frac{2\pi}{|\Omega|} m^2 \quad \text{and} \quad \lambda_1 \geq \frac{2\pi}{|\Omega|}.
$$

(2.3)

We also point out that as is shown in [20] in the 2D case for all $k \geq 1$

$$
\lambda_k > \lambda_k^D,
$$

where $\lambda_k^D$ are the eigenvalues of the Dirichlet Laplacian.

The next result is crucial in finding good estimates for the dimension of attractors of the 2D Navier–Stokes system.

**Theorem 2.2.** Let $\Omega \subseteq \mathbb{R}^2$ be an arbitrary domain. Let a family of scalar functions $\{\varphi_k\}_{k=1}^m \in H^1_0(\Omega)$ be orthonormal in $L^2(\Omega)$. Then

$$
\rho(x) := \sum_{k=1}^m |\varphi_k(x)|^2
$$

satisfies the inequality
\[ \int_{\Omega} \rho(x)^2 \, dx \leq c_{\text{LT}} \sum_{k=1}^{m} \| \nabla \varphi_k \|^2. \] (2.4)

Let now a family of divergence free vector functions \( \{u_k\}_{k=1}^{m} \in H^1_0(\Omega), \) \( \text{div} \, u_k = 0, \) be orthonormal in \( L^2(\Omega) \). Then \( \rho(x) := \sum_{j=1}^{m} |u_k(x)|^2 \) satisfies
\[
\int_{\Omega} \rho(x)^2 \, dx \leq \bar{c}_{\text{LT}} \sum_{k=1}^{m} \| \nabla u_k \|^2,
\] (2.5)
where
\[
\bar{c}_{\text{LT}} \leq c_{\text{LT}}.
\] (2.6)

The constant \( c_{\text{LT}} \) is bounded from below by its ‘classical’ value, which is \( 1/(2\pi) \), and it is now customary [12] to write estimates for it in the form
\[
c_{\text{LT}} \leq R \cdot \frac{1}{2\pi}.
\]

Inequality (2.4) was originally proved in [26] with \( R = 3\pi \), followed by significant improvements in [15], \( R = 2 \), and in [10], \( R = \pi/\sqrt{3} = 1.8138\ldots \). The best to date estimate obtained in [11] is
\[
R = 1.456\ldots
\]

Finally, it was shown in [8] that in two dimensions the constant in the Lieb–Thirring inequality does not increase in going over from the scalar case to the divergence free vector case.

3. LIEB–THIRRING INEQUALITIES AND ATTRACTORS FOR NAVIER–STOKES EQUATIONS

We now consider the two-dimensional Navier–Stokes system
\[
\partial_t u + \sum_{i=1}^{2} u^i \partial_i u = \nu \Delta u - \nabla p + f, \quad (3.1)
\]
\[
\text{div} \, u = 0, \quad u|_{\partial \Omega} = 0, \quad u(0) = u_0, \quad \Omega \in \mathbb{R}^2
\]
in a domain \( \Omega \) with finite area \( |\Omega| < \infty \) and with Dirichlet boundary conditions for the velocity vector \( u \).

We denote by \( P \) the Helmholtz–Leray orthogonal projection in \( L^2(\Omega) \) onto the Hilbert space \( H \) which is the closure in \( L^2(\Omega) \) of the set of smooth
solenoidal vector functions with compact supports in $\Omega$. Applying $P$ and thereby excluding the pressure $p$ we obtain the evolution equation in $H$

$$\partial_t u + \nu A u + B(u, u) = f, \quad u(0) = u_0,$$

where $A = -P \Delta$ is the Stokes operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ and $B(u, v) = P \left( \sum_{i=1}^{2^2} u^i \partial_i v \right)$ is the bilinear operator satisfying the fundamental orthogonality relation

$$(B(u, v), v) = 0.$$  

The equation (3.2) has a unique solution in $H$, so that the the solution semigroup $S(t) : H \to H$, $S(t)u_0 = u(t)$ of continuous operators is well-defined (see, for instance [33]). The following two a priori estimates are essential in the proof. Taking the scalar product of (3.2) with $u$ and using (3.3) we obtain

$$\partial_t \|u\|^2 + 2\nu \|\nabla u\|^2 \leq 2\|f\|_{-1}\|\nabla u\| \leq \nu \|\nabla u\|^2 + \nu^{-1}\|f\|^2 \leq \nu \|\nabla u\|^2 + (\lambda_1 \nu)^{-1}\|f\|^2,$$

which gives

$$\partial_t \|u\|^2 + \nu \|\nabla u\|^2 \leq (\lambda_1 \nu)^{-1}\|f\|^2,$$

$$\partial_t \|u\|^2 + \nu \lambda_1 \|u\|^2 \leq (\lambda_1 \nu)^{-1}\|f\|^2.$$  

Integrating the first inequality (3.4) in time we obtain

$$\limsup_{t \to \infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t \|\nabla u(\tau)\|^2 d\tau \leq \frac{\|f\|^2}{\lambda_1 \nu^2}.$$  

It follows from the second inequality in (3.4) that the ball $B_0$ in $H$ of radius $2(\nu \lambda_1)^{-1}\|f\|$ is an absorbing set for the semigroup $S(t)$. Furthermore, the set $B_1 := S(1)B_0$ is bounded in $H_0^1(\Omega) \cap H$ (see [21]) and therefore compact in $H$. Hence, the $\omega$-limit set in $H$ of the set $B_1$ is well defined. This set is the global attractor of the Navier–Stokes system in the phase space $H$.

**Definition 3.1.** A set $\mathcal{A} \subset H$ is a global attractor of the semigroup $S(t)$ of continuous operators acting in a Banach space $H$ if

1) $\mathcal{A}$ is a compact set in $H$;
2) $\mathcal{A}$ is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$;
3) $\mathcal{A}$ attracts the images of all bounded sets in $H$, i.e. for every bounded set $B \subset H$ and every neighbourhood $O(\mathcal{A})$ of the attractor there exists $T = T(O, B)$ such that $S(t)B \subset O(\mathcal{A})$ for all $t \geq T$.

Next we consider the Navier–Stokes system linearized on the solution $u(t)$ lying on the attractor and parameterized by the initial point $u_0$:

$$\partial_t U = -\nu AU - B(U, u(t)) - B(u(t), U) =: \mathcal{L}(t, u_0)U, \quad U(0) = \xi.$$  


We define and estimate the numbers \( q(n) \), that is, the sums of the first \( n \) global Lyapunov exponents:

\[
q(n) := \lim_{t \to \infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t \sup_{\{v_j\}_{j=1}^n} \sum_{j=1}^n (\mathcal{L}(\tau, u_0)v_j, v_j) d\tau,
\]

where \( \{v_j\}_{j=1}^n \in H_0^1(\Omega) \cap \{\text{div} \, v = 0\} \) is and arbitrary divergence free \( L^2 \)-orthonormal system of dimension \( n \) [3, 9, 35].

The numbers \( q(n) \) control the expansion or contraction of the \( n \)-dimensional volumes transported by the variational equation along the solution lying on the attractor and their role in the dimension estimates is crucial (see [3, 9, 35] and [7, 8] for the Hausdorff and fractal dimension, respectively).

**Theorem 3.1.** Let for an integer \( n > 0 \) \( q(n) \geq 0 \) and \( q(n+1) < 0 \). Then both the Hausdorff and the fractal dimensions of \( \mathcal{A} \) satisfy

\[
\dim \mathcal{A} \leq n_L := n + \frac{q(n)}{q(n) - q(n+1)}.
\]

**Remark 3.1.** If the function \( q \) viewed as a function of a continuous variable is concave (at least near \( n \)), then it is geometrically clear that \( n_L \leq n^* \), where \( q(n^*) = 0 \).

Turning to estimating the numbers \( q(n) \) we integrate by parts and using the key orthogonality relation (3.3) we obtain

\[
\sum_{j=1}^n (\mathcal{L}(t, u_0)v_j, v_j) = -\nu \sum_{j=1}^n \|\nabla v_j\|^2 - \int \sum_{j=1}^n \sum_{k,i=1}^2 v_j^k \partial_k u^i v_i^j dx \leq
\]

\[
-\nu \sum_{j=1}^n \|\nabla v_j\|^2 + 2^{-1/2} \int \rho(x) |\nabla u(t, x)| dx \leq
\]

\[
-\nu \sum_{j=1}^n \|\nabla v_j\|^2 + 2^{-1/2} \|\rho\| \|\nabla u\|,
\]

where we used the pointwise inequality (see, [17, 24])

\[
\left| \sum_{k,i=1}^2 v^k \partial_k u^i v^i \right| = |\nabla u \cdot v| \leq 2^{-1/2} |\nabla u| |v|^2,
\]

and where

\[
\rho(x) := \sum_{j=1}^n |v_j(x)|^2.
\]
Prior to the use of the Lieb–Thirring inequalities in the context of the attractors for the Navier–Stokes equations the function $\rho$ was estimated (in a non-optimal way) by the Ladyzhenskaya inequality
\[
\|u\|_4^4 \leq c_{\text{Lad}} \|u\|^2 \|\nabla u\|^2,
\] (3.9)
where
\[
c_{\text{Lad}} \leq \frac{16}{27\pi}, \quad c_{\text{Lad}} = \frac{1}{\pi \cdot 1.8622 \ldots}
\]
see, respectively, (4.12), and [37], where the sharp value of the constant was found numerically, by calculating the norm of the ground state solution of the corresponding Euler–Lagrange equation.

Using (3.9) and the fact that the $v_j$’s are normalized (but not using orthogonality) we find that
\[
\rho(x)^2 = \sum_{i,j=1}^{n} |v_i(x)^2| |v_j(x)^2|^2 \leq \frac{1}{2} \sum_{i,j=1}^{n} (|v_i(x)|^4 + |v_j(x)|^4) = n \sum_{j=1}^{n} |v_j(x)|^4,
\]
so that
\[
\|\rho\|^2 = \int_{\Omega} \rho(x)^2 dx \leq n c_{\text{Lad}} \sum_{j=1}^{n} \|\nabla v_j\|^2.
\] (3.10)
Substituting this into (3.8) and splitting the second term there accordingly, we obtain
\[
\sum_{j=1}^{n} (\mathcal{L}(t,u_0)v_j, v_j) \leq -\nu \sum_{j=1}^{n} \|\nabla v_j\|^2 + \frac{nc_{\text{Lad}}}{4\nu} \|\nabla u(t)\|^2.
\]
It remains to use the lower bound for the sums of eigenvalues of the Stokes operator (2.3) including the lower bound $\lambda_1 \geq 2\pi/|\Omega|$ and estimate (3.5) for the solutions lying on the attractor. This finally gives
\[
q(n) \leq -\frac{\nu \pi}{|\Omega|} n^2 + \frac{c_{\text{Lad}} \|f\|^2 |\Omega|}{8\pi \nu^3} n,
\] (3.11)
so that $q(n^*) = 0$ for
\[
n^* = \frac{c_{\text{Lad}}}{8\pi^2} G^2, \quad G = \frac{\|f\| |\Omega|}{\nu^2},
\]
and the number $n^*$ is an upper bound both for the Hausdorff and fractal dimension of the global attractor $\mathcal{A}$:
\[
\dim \mathcal{A} \leq \frac{c_{\text{Lad}}}{8\pi^2} G^2.
\]
Up to explicit constants this is what the situation in this area looked like before the use the Lieb–Thirring inequalities, see, for instance, [2], [3].
The Lieb–Thirring bound for orthonormal families gives an optimal bound for the function \( \rho \) replacing (3.10) by the inequality with constant independent of the size \( n \) of the orthonormal family:

\[
\| \rho \| = \int_{\Omega} \rho(x)^2 \, dx \leq c_{LT} \sum_{j=1}^{n} \| \nabla v_j \|^2.
\] (3.12)

Replacing (3.10) with (3.12) in (3.8) and arguing as before we find that

\[
\sum_{j=1}^{n} (L(t, u_0)v_j, v_j) \leq -\nu \sum_{j=1}^{n} \| \nabla v_j \|^2 + 2^{-1/2} \| \rho \| \| \nabla u \| \leq \nu c_{LT} \sum_{j=1}^{n} \| \nabla v_j \|^2 \text{ with } c_{LT} \sum_{j=1}^{n} \| \nabla v_j \|^2 = 1/2 \| \nabla u(t) \| \leq \nu c_{LT} \frac{\| f \|^2 |\Omega|}{4\nu} \sum_{j=1}^{n} \| \nabla v_j \|^2 \leq \nu c_{LT} \frac{\| f \|^2 |\Omega|}{4\nu} \| \nabla u(t) \| \text{,}
\] (3.13)

so that (3.11) goes over to

\[
q(n) \leq -\frac{\nu \pi}{|\Omega|} n^2 + \frac{c_{LT} \| f \|^2 |\Omega|}{8\pi \nu^3},
\] (3.14)

which gives the estimate of the dimension that is linear with respect to the dimensionless number \( G \).

**Theorem 3.2.** The Hausdorff and the fractal dimension of the global attractor \( \mathcal{A} \) of the Navier–Stokes system in a domain \( \Omega \subset \mathbb{R}^2 \) with finite area satisfy the estimate

\[
dim \mathcal{A} \leq \frac{c_{LT}^{1/2} G}{2\sqrt{2\pi}}, \quad G = \frac{\| f \| |\Omega|}{\nu^2}.
\] (3.15)

**Remark 3.2.** One can avoid the Li–Yau-type lower bounds for the Stokes operator by using instead that

\[
n^2 = \left( \int_{\Omega} \rho(x) \, dx \right)^2 \leq |\Omega| \| \rho \|^2.
\]

Then we obtain

\[
\sum_{j=1}^{n} (L(t, u_0)v_j, v_j) \leq -\frac{\nu}{c_{LT}} \| \rho \|^2 + 2^{-1/2} \| \rho \| \| \nabla u \| \leq -\frac{\nu}{2c_{LT}} \| \rho \|^2 + \frac{c_{LT} \| \nabla u(t) \|^2}{4\nu} \leq -\frac{\nu}{2c_{LT}} n^2 + \frac{c_{LT} \| \nabla u(t) \|^2}{4\nu}.
\]
Using again (3.5) we obtain
\[
\dim \mathcal{A} \leq \frac{c_{LT}}{2\sqrt{n}} G, \quad G = \frac{\|f\|_{\Omega}}{\nu^2}.
\]
However, the factor of \( G \) in (3.15) is smaller, since \( c_{LT} \geq 1/(2\pi) \).

4. \( L^p \)-inequalities for families of functions with orthonormal derivatives and applications

We have seen in \S 2 and \S 3 that the Lieb–Thirring inequality for \( L^2 \)-orthonormal families is essential for finding good estimates for the attractor dimension of the 2D Navier–Stokes system in a bounded domain with Dirichlet boundary conditions.

In the 3D case the situation is drastically different, since the global well-posedness remains a mystery and therefore inspires a comprehensive study of various modifications/regularizations of the initial Navier-Stokes/Euler equations (such as various \( \alpha \) model, hyperviscous Navier-Stokes equations, regularizations via \( p \)-Laplacian, etc.), many of which have a strong physical background and are of independent interest, both in practice and theory, see e.g. \cite{4, 6, 14, 22, 27, 30} and the references therein.

In \cite{17, 19, 39} the authors have recently studied the following regularized damped Euler system:

\[
\begin{cases}
\partial_t u + (\vec{u}, \nabla)\vec{u} + \gamma u + \nabla p = g, \\
\text{div } \vec{u} = 0, \quad u(0) = u_0.
\end{cases}
\] (4.1)

with forcing \( g \) and Ekman damping term \( \gamma u, \gamma > 0 \). The damping term \( \gamma u \) makes the system dissipative and is important in various geophysical models. System (4.1) (at least in the conservative case \( \gamma = 0 \)) is often referred to as the simplified Bardina subgrid scale model of turbulence, see \cite{4} for the derivation of the model and further discussion.

The system is studied for \( d = 2, 3 \)
1) on the torus \( \mathbb{T}^d = [0, L]^d \) with standard zero mean condition;
2) in \( \Omega = \mathbb{R}^d \);
3) on the sphere \( S^2 \) or in a domain on it \( \Omega \subseteq S^2 \);
4) if \( \Omega \subseteq \mathbb{R}^d \) or \( \Omega \subseteq S^2 \), then \( \vec{u}|_{\partial\Omega} = 0 \) and \( \vec{u} \) is recovered from \( u \) by solving the Stokes problem

\[
\begin{cases}
(1 - \alpha\Delta)\vec{u} + \nabla q = u, \\
\text{div } \vec{u} = 0, \quad \vec{u}|_{\partial\Omega} = 0.
\end{cases}
\]

In the case when there is no boundary

\[
\vec{u} = (1 - \alpha\Delta)^{-1} u.
\]
Here $\alpha = \alpha' L^2$ and $\alpha' > 0$ is a small dimensionless parameter, so that $\bar{u}$ is a smoothed vector field with higher spatial modes filtered out.

The phase space with respect to $\bar{u}$ is the Sobolev space $H^1$ with divergence free condition

$$\bar{u} \in H^1 := \begin{cases} \dot{H}^1(T^d), & x \in T^d, \int_{T^d} \bar{u}(x) dx = 0, \\ H^1(\mathbb{R}^d), & x \in \mathbb{R}^d, \\ H^1_0(\Omega), & x \in \Omega \subset \mathbb{R}^d, S^2 \end{cases} \text{ div } \bar{u} = 0, \quad (4.2)$$

with scalar product (1.3).

The results obtained in [17, 19, 39] can be combined into the following theorem.

**Theorem 4.1.** Let $d = 2$. In each case of BC the system possesses a global attractor $\mathcal{A} \subset H^1$ with finite fractal dimension satisfying

$$\dim_F \mathcal{A} \leq \frac{1}{8\pi} \begin{cases} \frac{1}{\alpha^4} \min \left( \| \text{curl } g \|_{L^2}^2, \frac{\|g\|_{L^2}^2}{2\alpha} \right), & x \in T^2, \mathbb{R}^2, S^2 \\ \|g\|_{L^2}^2, & x \in \Omega \subset \mathbb{R}^2, S^2. \end{cases}$$

In the 3D case the estimates in all three cases look formally the same

$$\dim_F \mathcal{A} \leq \frac{1}{12\pi} \frac{\|g\|_{L^2}^2}{\alpha^5 \gamma^4}, \quad x \in T^3, x \in \mathbb{R}^3, x \in \Omega \subset \mathbb{R}^3.$$  

Furthermore, in the periodic case both on $T^2$ and $T^3$ the upper bounds are optimal in the limit as $\alpha \to 0^+$.

To the best of our knowledge this is the first example of a meaningful 3D hydrodynamic model with sharp two-sided estimates for the dimension of the global attractor.

Optimal lower bounds for the torus $T^2$ are based on the instability analysis of the specific stationary solutions — generalized Kolmogorov flows [19] and are carried over to $T^3$ by means of the Squire’s transformation [17].

Explicit upper bounds for all types of domains and boundary conditions as for the Navier–Stokes system are obtained by finding good estimates for the global Lyapunov exponents. However, the phase space is now $H^1$ and the $n$-traces of the linearized operator are now calculated with respect to the scalar product (1.3). The Lieb–Thirring inequalities for $L^2$-orthonormal families are replaced by inequalities for the $L^p$-forms of families of functions with orthonormal derivatives also obtained by E. Lieb in [23].
Proposition 4.1. Let \( \{v_j\}_{j=1}^n \in H^1 \), see (4.2). Suppose that this family is orthonormal with respect to the scalar product (1.3):
\[
(v_i, v_j) + \alpha(\nabla u_i, \nabla v_j) = \delta_{ij}.
\] (4.3)

Then the function
\[
\rho(x) = \sum_{j=1}^n |v_j(x)|^2
\]
satisfies
\[
\|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\pi}} \frac{n^{1/2}}{\alpha^{1/2}}, \quad d = 2,
\]
\[
\|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\pi}} \frac{n^{1/2}}{\alpha^{3/4}}, \quad d = 3.
\] (4.4)

Inequalities of this type in \( \mathbb{R}^d \) with \( L^p \)-norm on the left-hand side were proved in [23] for \( p = \infty \) (\( d = 1 \)), \( 1 \leq p < \infty \) (\( d = 2 \)), and for the critical \( p = d/(d-2) \) (\( d \geq 3 \)). No expressions for the constants were given. Our interest in what follows is in the case of the 2D torus \( \mathbb{T}^2 \). Furthermore, since the scalar product (1.3) is defined by system (1.1), and we are now interested just in inequalities themselves, we turn to the more convenient (and clearly equivalent) scalar product (4.5) which is also used in [23]. Finally, we consider the scalar case only, since in \( x \)-coordinates the vector case involves no problems at all, while the case of the sphere \( S^2 \) will be treated in a forthcoming work.

Theorem 4.2. Let \( \{\varphi_j\}_{j=1}^n \) be a family of zero mean functions on the torus \( \{\varphi_j\}_{j=1}^n \in \dot{H}^1(\mathbb{T}^2) \) or let \( \{\varphi_j\}_{j=1}^n \in H^1_0(\Omega) \), where \( \Omega \subseteq \mathbb{R}^2 \) is an arbitrary domain. Suppose further that in either case the family is orthonormal with respect to the scalar product:
\[
m^2(\varphi_i, \varphi_j) + (\nabla \varphi_i, \nabla \varphi_j) = \delta_{ij}.
\] (4.5)

Then for \( 1 \leq p < \infty \) the function
\[
\rho(x) := \sum_{j=1}^n |\varphi_j(x)|^2
\]
satisfies the inequality
\[
\|\rho\|_{L^p} \leq B_p m^{-2/p} n^{1/p},
\] (4.6)
where
\[
B_p \leq \left( \frac{p-1}{4\pi} \right)^{(p-1)/p}.
\] (4.7)
Proof. Since inequality (4.6) with constant (4.7) clearly holds for $p = 1$, we assume below that $1 < p < \infty$. We also first consider the periodic case.

Let us define two operators

$$
\mathbb{H} = V^{1/2}(m^2 - \Delta)^{-1/2}\Pi, \quad \mathbb{H}^* = \Pi(m^2 - \Delta)^{-1/2}V^{1/2},
$$

where $V \in L^p$ is a non-negative scalar function and $\Pi$ is the projection onto the space of functions with mean value zero:

$$
\Pi \varphi = \varphi - \frac{1}{4\pi^2} \int_{T^2} \varphi(x) \, dx.
$$

Then $K = \mathbb{H}^*\mathbb{H}$ is a compact self-adjoint operator in $L^2(T^2)$ and for $r = p' = p/(p-1) \in (1, \infty)$

$$
\text{Tr} K^r = \text{Tr} (\Pi(m^2 - \Delta)^{-1/2}V(m^2 - \Delta)^{-1/2}\Pi)^r \leq \text{Tr} (\Pi(m^2 - \Delta)^{-r/2}V^r(m^2 - \Delta)^{-r/2}\Pi) = \text{Tr} (V^r(m^2 - \Delta)^{-r}\Pi),
$$

where we used the Araki–Lieb–Thirring inequality for traces \cite{1,26,32}:

$$
\text{Tr}(BA^2B)^p \leq \text{Tr}(B^pA^{2p}B^p), \quad p \geq 1,
$$

and the cyclicity property of the trace together with the facts that $\Pi$ commutes with the Laplacian and that $\Pi$ is a projection: $\Pi^2 = \Pi$. Using the basis of orthonormal eigenfunctions of the Laplacian

$$
\frac{1}{2\pi} e^{in \cdot x}, \quad n \in \mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{0, 0\}
$$

in view of the key estimate (5.1) proved below we find that

$$
\text{Tr} K^r \leq \text{Tr} \left( V^r(m^2 - \Delta)^{-r}\Pi \right) = -\frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^2_0} \frac{1}{(m^2 + |n|^2)} \int_{T^2} V^r(x) \, dx \leq \frac{1}{4\pi} \frac{m^{-2(r-1)}}{r - 1} \|V\|_{L^r}^r.
$$

We can now argue as in \cite{23}. We observe that

$$
\int_{T^2} \rho(x)V(x) \, dx = \sum_{i=1}^n \|\mathbb{H}\psi_i\|_{L^2}^2,
$$

where

$$
\psi_j = (m^2 - \Delta)^{1/2} \varphi_j, \quad j = 1, \ldots, n.
$$
Next, in view of (4.5) the $\psi_j$'s are orthonormal in $L^2$ and in view of the variational principle
\[ \sum_{i=1}^{n} \| H\psi_i \|_{L^2}^2 = \sum_{i=1}^{n} (K\psi_i, \psi_i) \leq \sum_{i=1}^{n} \lambda_i, \]
where $\lambda_i$ are the eigenvalues of the operator $K$. Therefore
\[ \int_{T^2} \rho(x)V(x) \, dx \leq \sum_{j=1}^{n} \lambda_j \leq n^{1/p} \left( \text{Tr} \, K^{1/r} \right)^{1/r} \leq \]
\[ \leq n^{1/p} \left( \frac{p-1}{4\pi m^2/(p-1)} \right)^{(p-1)/p} \| V \|_{L^p/(p-1)} = \]
\[ = n^{1/p} m^{-2/p} \left( \frac{p-1}{4\pi} \right)^{(p-1)/p} \| V \|_{L^p/(p-1)}. \]
Finally, setting $V(x) = \rho(x)^{p-1}$ we obtain (4.6), (4.7). This completes the proof for the torus.

The proof of the theorem for $\Omega = \mathbb{R}^2$ is a word for word repetition of the above proof (without the projection $\Pi$, of course) and with use of the Fourier transform instead of the Fourier series. Furthermore, instead of a non-trivial inequality (5.1) for the Green's function on the diagonal we have the equality
\[ \frac{(p-1)m^{2/(p-1)}}{\pi} \int_{\mathbb{R}^2} \frac{dx}{(m^2 + |x|^2)^p} = 1. \]
Finally, if $\Omega \subsetneq \mathbb{R}^2$ is a proper domain, we extend by zero the vector functions $\varphi_j$ outside $\Omega$ and denote the results by $\tilde{\varphi}_j$, so that $\tilde{\varphi}_j \in H^1(\mathbb{R}^2)$. We further set $\tilde{\rho}(x) := \sum_{j=1}^{n} |\tilde{\varphi}_j(x)|^2$. Then setting in $\mathbb{R}^2$ $\tilde{\psi}_i := (m^2 - \Delta)^{1/2} \tilde{\varphi}_i$, we see that the system $\{ \tilde{\psi}_j \}_{j=1}^{n}$ is orthonormal in $L^2(\mathbb{R}^2)$. Since clearly $\| \tilde{\rho} \|_{L^p(\mathbb{R}^d)} = \| \rho \|_{L^p(\Omega)}$, the proof now reduces to the case $\Omega = \mathbb{R}^2$ and therefore is complete. \hfill $\Box$

**Corollary 4.1.** The following interpolation inequality holds:
\[ \| \varphi \|_{L^q} \leq \left( \frac{1}{4\pi} \right)^{\frac{q^2}{2q}} \left( \frac{q}{2} \right)^{1/2} \| \varphi \|^{2/q} \| \nabla \varphi \|^{1-2/q}, \quad q \geq 2. \quad (4.8) \]

**Proof.** For $n = 1$ inequality (4.4) goes over to
\[ \| \varphi \|_{L^{2p}} \leq B_p^{1/2} m^{-1/p} (m^{2}\| \varphi \|^2 + \| \nabla \varphi \|^2)^{1/2}, \quad p \geq 1. \]
Furthermore, writing this in the form
\[ \|\varphi\|_{L^2}^2 \leq B_p \left( m^{2-2/p} \|\varphi\|_2^2 + m^{-2/p} \|\nabla \varphi\|_2^2 \right) \tag{4.9} \]
and minimizing with respect to \( m \) we obtain
\[ \|\varphi\|_{L^2}^2 \leq B_p \left( \frac{p}{2} \right)^{(p-1)/p} p \|\varphi\|_2^{2/p} \|\nabla \varphi\|_2^{2-2/p}, \tag{4.10} \]
which is (4.8).

The one function inequality (4.9) for the torus \( \mathbb{T}^2 \) and the equivalent multiplicative inequality (4.10) can be proved in a more direct way in which, however, estimate (5.1) as before plays the essential role. For the case of \( \mathbb{R}^2 \), see Remark 4.2.

**Direct proof of Corollary 4.1 for the torus.** In fact, writing a zero mean function \( \varphi \) on the torus \( \mathbb{T}^2 \) in terms of the Fourier series
\[ \varphi(x) = \sum_{n \in \mathbb{Z}^2} a_n e^{i x \cdot n} \]
we have by the Parseval identity
\[ \|\varphi\|_{L^2} = 2\pi \|a\|_{l^2}, \]
and, furthermore, since the exponentials \( e^{i x \cdot n} \) have norm 1 in \( L^\infty \), we have
\[ \|\varphi\|_{L^\infty} \leq \|a\|_{l^1}. \]

This gives by the Riesz–Thorin interpolation theorem the well-known Hausdorff–Young inequality
\[ \|\varphi\|_{L^p} \leq (2\pi)^{2/p} \|a\|_{l^q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \geq 2. \]
Thus, by Hölder’s inequality for an arbitrary \( m > 0 \)
\[ \|\varphi\|_{L^p} \leq (2\pi)^{2/p} \|a\|_{l^q} = (2\pi)^{2/p} \left( m^2 + |n|^2 \right)^{-1/2} \cdot (m^2 + |n|^2)^{1/2} \|a_n\|_{l^q} \leq \leq (2\pi)^{2/p} \left( \sum_{n \in \mathbb{Z}^2} \frac{1}{(m^2 + |n|^2)^r/2} \right)^{1/r} \|\varphi\|_{L^p} \]
where \( \frac{1}{r} + \frac{1}{2} = \frac{1}{q} \), so that
\[ \frac{1}{r} = \frac{1}{2} - \frac{1}{p} = \frac{p - 2}{2p}, \quad \frac{r}{2} - 1 = \frac{2}{p - 2}. \]
We now use the key inequality (5.1) in (4.11)
\[
\left( \sum_{n \in \mathbb{Z}^2} \frac{1}{(m^2 + |n|^2)^{r/2}} \right)^{1/r} < \left( \frac{\pi}{(r/2 - 1)m^{2(r/2 - 1)}} \right)^{1/r} = \left( \frac{\pi(p - 2)}{2} \right) \frac{p-2}{2} m^{-2/r}
\]
and take into account that
\[
\| \langle m^2 + |n|^2 \rangle^{1/2} a_n \|_{L^2}^2 = \frac{1}{4\pi^2} \| (m^2 - \Delta)^{1/2} \varphi \|^2 = \frac{1}{4\pi^2} (m^2 \| \varphi \|^2_{L^2} + \| \nabla \varphi \|^2_{L^2}).
\]
We obtain
\[
\| \varphi \|_{L^p} \leq \left( \frac{p - 2}{8\pi} \right)^{\frac{p-2}{2p}} m^{-2/p} \left( m^2 \| \varphi \|^2_{L^2} + \| \nabla \varphi \|^2_{L^2} \right)^{1/2}, \quad p \geq 2.
\]
Taking the square and changing \( p \) to \( 2p \) gives the inequality
\[
\| \varphi \|^2_{L^{2p}} \leq \left( \frac{p - 1}{4\pi} \right)^{\frac{p-1}{2p}} m^{-2/p} \left( m^{2-2/p} \| \varphi \|^2 + m^{-2/r} \| \nabla \varphi \|^2 \right), \quad p \geq 1,
\]
which coincides with (4.9) and is equivalent to (4.10).

**Remark 4.1.** For \( q = 4 \) in (4.8), that is, in the Ladyzhenskaya inequality on the 2D torus \( \mathbb{T}^2 \), the constant is \( 1/\sqrt{\pi} \) and should be compared with (and is greater than) the recent estimate of it as a one particle Lieb–Thirring inequality [18]

\[
\frac{1}{\pi} > \frac{3\pi}{32}.
\]
On the other hand, (4.8) works for all \( q \geq 2 \) and provides a simple expression for the constant.

**Remark 4.2.** It is worth mentioning that the similar approach plus the knowledge of the sharp Babenko–Beckner inequality for the Fourier transform

\[
\| f \|_{L^p(\mathbb{R}^d)} \leq \left( \frac{2\pi}{p} \right)^{1/p} \frac{q^{1/q}}{p^p} \| \hat{f} \|_{L^q(\mathbb{R}^d)}, \quad p \geq 2, \quad \frac{1}{p} + \frac{1}{q} = 1
\]
in the analog of (4.11) for \( \mathbb{R}^2 \) gives the following improvement of inequality (4.8) for \( \mathbb{R}^2 \) with the best to date closed form estimate for the constant [29]:

\[
\| \varphi \|_{L^q(\mathbb{R}^2)} \leq \left( \frac{1}{4\pi} \right)^{\frac{q-2}{2q}} \frac{q^{q-2/q}}{(q - 1)(q-1)/q} \left( \frac{2}{q} \right)^{1/2} \| \varphi \|^{2/q} \| \nabla \varphi \|^{1-2/q}, \quad q \geq 2,
\]
where in comparison with (4.8) the middle factor in the constant here is due to the Babenko–Beckner inequality and is less than 1 for \( q \in (2, \infty) \);
see also [25, Theorem 8.5] where the equivalent result is obtained for the
inequality in the additive form.

Remark 4.3. The rate of growth as \( q \to \infty \) of the constant both in (4.8)
and (4.12), namely \( q^{1/2} \), is optimal in the power scale, since otherwise the
Sobolev space \( H^1 \) in two dimensions would have been embedded in the
Orlicz space with Orlicz function \( e^{t^2 + \varepsilon} - 1, \varepsilon > 0 \), which is impossible [13].

5. Appendix. Monotonicity of lattice sums

In this section we prove two key estimates for the lattice sums in dimen-
sion 2 and 3.

Proposition 5.1. The following inequality holds for \( p > 1 \) and all \( m \geq 0 \)
\[
I_p(m) := \frac{(p-1)m^{2(p-1)}}{\pi} \sum_{n \in \mathbb{Z}_0^2} \frac{1}{(m^2 + |n|^2)^p} < 1. \tag{5.1}
\]

Proof. Inequality (5.1) will obviously follow if we show that
\[
\lim_{m \to \infty} I_p(m) = 1 \tag{5.2}
\]
and \( I_p(m) \) is monotone increasing:
\[
\frac{d}{dm} I_p(m) > 0 \quad \text{for } m > 0. \tag{5.3}
\]

The proof of (5.2) is easy. Setting
\[
f(x) := \frac{1}{(1 + |x|^2)^p}, \quad x \in \mathbb{R}^2
\]
we use the Poisson summation formula and write:
\[
\sum_{n \in \mathbb{Z}_0^2} \frac{1}{(m^2 + |n|^2)^p} = m^{-2p} \left( \sum_{n \in \mathbb{Z}^2} f(|n|/m) - 1 \right) = \]
\[
m^{-2p} \left( m^2 \int_{\mathbb{R}^2} f(x) \, dx - 1 + 2\pi m^2 \sum_{n \in \mathbb{Z}_0^2} \hat{f}(2\pi|n|m) \right) = \]
\[
\frac{1}{m^{2(p-1)} \pi^{p-1}} \left( \frac{1}{p-1} - \frac{1}{m^{2p}} + \frac{2\pi}{m^{2(p-1)}} \sum_{n \in \mathbb{Z}_0^2} \hat{f}(2\pi|n|m) \right),
\]
where \( \hat{f}(\xi) \) is the Fourier transform of \( f \). Since the function \( f(z) \) is analytic
in the strip \( |\text{Im}z| < 1 \), it follows that its Fourier transform is exponentially
decaying and therefore the sum of the last two terms is negative for all sufficiently large $m$ and is of the order $O(1/m^{2p})$ as $m \to \infty$. This proves (5.2).

More precisely, since $f$ is radial

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x)e^{i\xi \cdot x} \, dx = g(t), \quad t = |\xi|,$$

and where

$$g(t) = \int_{0}^{\infty} J_0(tr) \frac{dr}{(1+r^2)^{p}} = \frac{1}{2^{p-1}\Gamma(p)} \, t^{p-1} K_{p-1}(t).$$

Here $K_\nu$ is the modified Bessel function of the second kind, and the second equality is formula 13.51 (4) in [36]. This finally gives

$$I_p(m) = 1 - \frac{p-1}{\pi} \frac{1}{m^2} + \frac{4(p-1)}{2^p\Gamma(p)} \sum_{n \in \mathbb{Z}_0^2} F_{p-1}(2\pi|n|m),$$

where

$$F_p(t) := t^p K_p(t).$$

It remains to recall that

$$K_p(t) = \sqrt{\frac{2}{\pi}} \frac{e^{-t}}{\sqrt{t}} \left( 1 + O\left(\frac{1}{t}\right) \right) \quad \text{as} \quad t \to \infty.$$

We now turn to the proof of (5.3). Using the formula

$$M^{-p} = \frac{1}{\Gamma(p)} \int_{0}^{\infty} x^{p-1} e^{-Mx} \, dx$$

with $M = m^2 + n_1^2 + n_2^2$ and summing over the lattice $n \in \mathbb{Z}_0^2$ we obtain

$$I_p(m) = \frac{(p-1)m^{2(p-1)}}{\pi\Gamma(p)} \int_{0}^{\infty} x^{p-1} e^{-m^2x} (\theta_3^2(e^{-x}) - 1) \, dx,$$

where $\theta_3(q)$ is the Jacobi theta function

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$  

Crucial for us is the following functional relation that is a corollary of the Poisson summation formula and the Fourier transform of the Gaussian

$$\varphi(x) = \frac{\varphi(x^{-1})}{\sqrt{x}}, \quad \varphi(x) := \theta_3(e^{-x}) = \sum_{n=-\infty}^{\infty} e^{-\pi x n^2}. \quad (5.5)$$
Rewriting (5.4) in terms of the function $\varphi$, changing the variable and then using (5.5) we arrive at

$$I_p(m) = (p-1) \frac{\pi^{p-1}}{\Gamma(p)} \int_0^\infty x^{p-1} e^{-x} \left( m^2 \varphi^2 \left( \frac{x}{\pi m^2} \right) - m^2 \right) dx =$$

$$= (p-1) \frac{\pi^{p-1}}{\Gamma(p)} \int_0^\infty x^{p-1} e^{-x} \left( \frac{\pi}{x} \varphi^2 \left( \frac{x}{\pi m^2} \right) \right) - m^2 dx.$$

Therefore

$$\frac{d}{dm} I_p(m) = 2(p-1) \frac{\pi^{p-1}}{\pi \Gamma(p) m^2} \int_0^\infty x^{p-1} e^{-x} \left( 2y^2 \varphi(y) \varphi'(y) + 1 \right) dx,$$

where $y = y(x) := \pi m^2 / x$, so that the monotonicity will be verified if we prove the following inequality

$$2y^2 \varphi(y) \varphi'(y) + 1 \geq 0, \quad y \in \mathbb{R}_+.$$  (5.6)

It is worthwhile to say that this sufficient condition for monotonicity is independent both of $m$ and $p$!

Since $\varphi(y)$ decays extremely fast as $y$ grows, it is most important to verify (5.6) near $y = 0$. For this purpose we again use (5.5) and taking into account that

$$2\varphi(t)\varphi'(t) = (\varphi^2(t))' = -\sum_{n \in \mathbb{Z}} \pi |n|^2 e^{-\pi |n|^2 t} = -\sum_{n \in \mathbb{Z}} \pi |n|^2 e^{-\pi |n|^2 t}$$

we obtain

$$2y^2 \varphi(y) \varphi'(y) = y^2 (\varphi^2(y))' = y^2 \left( y^{-1} \varphi^2(y^{-1}) \right)' = -\varphi^2(y^{-1}) - 2y^{-1} \varphi(y^{-1}) \varphi'(y^{-1}) = -1 + \sum_{n \in \mathbb{Z}} \left( y^{-1} \pi (n_1^2 + n_2^2) - 1 \right) e^{-\pi (n_1^2 + n_2^2)} y^{-1} > -1$$

for $y \leq \pi$, so that (5.6) holds in this case.

Thus, we only need to check (5.6) for $y \geq \pi$. We have a lot of free space here and this can be done in many ways, for instance, we may replace

$$\varphi(y) = \sum_{n=-\infty}^\infty e^{-\pi n^2 y} \leq \sum_{n=-\infty}^\infty e^{-\pi n^2 y} = 1 + \frac{2 e^{-\pi y}}{1 - e^{-\pi y}} = \coth(\pi y/2) := \psi(y)$$

with the similar estimate for the derivative:

$$0 > \varphi'(y) \geq \psi'(y), \quad y \geq \pi,$$

which holds in view of the elementary inequality

$$ne^{-an^2} \leq e^{-an}, \quad n \geq 1, \quad a (= \pi y) \geq 1.$$  (5.7)
Therefore, we may replace \( \varphi(y) \) by \( \psi(y) \) and verify instead that
\[
2y^2 \varphi(y) \varphi'(y) + 1 \geq 2y^2 \psi(y) \psi'(y) + 1 = -\pi y^2 \cosh(\pi y/2) \sinh^3(\pi y/2) + 1 =: -g(y) + 1 > 0.
\]
The function \( g(y) > 0 \) with derivative
\[
g'(y) = -\frac{t(4 \cosh t(t \cosh t - \sinh t) + 2t)}{(\cosh^2 t - 1)^2} \bigg|_{t=\pi y/2} < 0 \quad \text{for} \quad t \geq 1
\]
is monotone decreasing to 0, which finally gives for \( y \geq \pi \)
\[
2y^2 \varphi(y) \varphi'(y) + 1 > -g(\pi) + 1 = -0.0064 \cdots + 1 > 0
\]
and completes the proof. \( \square \)

The idea of the proof for the 3D lattice sum is similar and reduces to the 2D case even technically.

**Proposition 5.2.** The following inequality holds for \( p > 3/2 \) and all \( m \geq 0 \)
\[
I_p(m) := m^{2p-3} \sum_{n \in \mathbb{Z}^3} \frac{1}{(m^2 + |n|^2)^p} < \frac{\Gamma(p - 3/2)\pi^{3/2}}{\Gamma(p)}. \tag{5.8}
\]

**Proof.** It can easily be shown by the Poisson summation formula that
\[
\lim_{m \to \infty} I_p(m) = \int_{\mathbb{R}^3} \frac{dx}{(|x|^2 + 1)^p} = \frac{\Gamma(p - 3/2)\pi^{3/2}}{\Gamma(p)},
\]
so that inequality (5.8) will be proved once we have shown that \( I_p(m) \) is monotone increasing with respect to \( m \in [0, \infty) \). The proof of monotonicity, in turn, essentially reduces to that for the 2D torus.

Using (5.5), we write
\[
I_p(m) = m^{2p-3} \int_0^\infty x^{p-1} e^{-m^2 x} \left( \theta_3^3(e^{-x}) - 1 \right) dx =
\]
\[
= \frac{1}{m^3} \int_0^\infty x^{p-1} e^{-x} \left( \theta_3^3(e^{-x}) \frac{x}{\pi m^2} - \frac{1}{m^3} \right) dx =
\]
\[
= \int_0^\infty x^{p-1} e^{-x} \left( \frac{\pi^{3/2}}{x^{3/2}} \varphi^3 \left( \frac{\pi m}{x} \right) - \frac{1}{m^3} \right) dx.
\]
Therefore
\[ \frac{d}{dm} I_p(m) = \frac{1}{m^4} \int_0^\infty x^{p-1} e^{-x} \left( 6y^{5/2} \varphi^2(y) \varphi'(y) + 4 \right) dx, \quad y = y(x) := \frac{\pi m^2}{x}, \]
and it suffices to show that
\[ 3y^{5/2} \varphi^2(y) \varphi'(y) + 2 > 0, \quad y \in \mathbb{R}_+. \tag{5.9} \]

We first consider the case when \( y \) is small. Using (5.5), (5.6) and (5.7) we have for \( y \leq \pi \)
\[ 3y^{5/2} \varphi^2(y) \varphi'(y) + 2 = 3\varphi(1/y) y^2 \varphi(y) \varphi'(y) + 2 > \frac{3}{2} \varphi(1/y) + 2 > \frac{3}{2} \psi(1/y) + 2 > 0 \]
if
\[ \psi(1/y) \leq \frac{4}{3}, \]
that is, if, see (5.7)
\[ y \leq y_* := \frac{\pi}{2 \text{arcoth}(4/3)} = 1.6144 \ldots (\pi), \quad \psi(1/y_*) = \frac{4}{3}. \]

On the interval \( y \in (y_*, \infty) \) we have
\[ y^{5/2} \varphi^2(y) \varphi'(y) > y^{5/2} \psi^2(y) \psi'(y) = -\pi y^{5/2} \frac{\cosh^2(\pi y/2)}{\sinh^4(\pi y/2)} := -h(y). \]

The function \( h(y) \) with derivative
\[ h'(y) = -\frac{\pi^{1/2} \cosh t \left( \cosh t (4t \cosh t - 5 \sinh t) + 4t \right)}{2^{1/2}(\cosh^2 t - 1)^2 \sinh t}, \quad t = \frac{\pi y}{2} \]
is monotone decreasing for \( t > 5/4 \) (\( y > 5/(2\pi) < y_* \)), and for \( y \in [y_*, \infty) \)
\[ -h(y) > -h(y_*) = -0.270 \ldots > -\frac{2}{3}, \]
which completes the proof of (5.9) and the proposition. \( \square \)

**Remark 5.1.** For \( p = 2 \) direct proofs of inequalities (5.1) and (5.8) were given in [19] [17], respectively, where they were used in deriving explicit upper bounds for the dimension of the attractors for regularized damped Euler equations in dimension 2 and 3, respectively.
References

[1] H. Araki, On an inequality of Lieb and Thirring. Lett. Math. Phys. 19 (1990), no. 2, 167–170.

[2] A. V. Babin and M. I. Vishik, Attractors of partial differential equations and estimates of their dimension. Uspekhi Mat. Nauk. 38 (1983), 133–187; English transl. in Russian Math. Surveys 38 (1983).

[3] A. V. Babin and M. I. Vishik, Attractors of evolution equations. Nauka, Moscow, 1988; English transl. North-Holland, Amsterdam, 1992.

[4] J. Bardina, J. Feziger, and W. Reynolds, Improved subgrid scale models for large eddy simulation. Proceedings of the 13th AIAA Conference on Fluid and Plasma Dynamics, (1980).

[5] M. Bartuccelli, J. Deane and S. Zelik, Asymptotic expansions and extremals for the critical Sobolev and Gagliardo–Nirenberg inequalities on a torus. Proc. Royal Soc. Edinburgh 143A (2013), 445–482.

[6] C. Foias, D. D. Holm, and E. S. Titi, The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. J. Dynam. Diff. Equ. 14 (2002), 1–35.

[7] V. V. Chepyzhov and A. A. Ilyin, A note on the fractal dimension of attractors of dissipative dynamical systems. Nonlinear Anal. 44 (2001), 811–819.

[8] V. V. Chepyzhov and A. A. Ilyin, On the fractal dimension of invariant sets; applications to Navier–Stokes equations. Discrete and Continuous Dynamical Systems 10 (2004), nos. 1&2, 117–135.

[9] P. Constantin and C. Foias, Global Lyapunov exponents, Kaplan–Yorke formulas and the dimension of the attractors for the 2D Navier–Stokes equations. Comm. Pure Appl. Math. 38 (1985), 1–27.

[10] J. Dolbeault, A. Laptev and M. Loss, Lieb–Thirring inequalities with improved constants. J. European Math. Soc. 10 (2008), 1121–1126.

[11] R. L. Frank, D. Hundertmark, M. Jex, Phan Thành Nam. The Lieb–Thirring inequality revisited. J. European Math. Soc. 23 (2021), 2583–2600.

[12] R. L. Frank, A. Laptev, and T. Weidl, Lieb–Thirring inequalities. Cambridge University Press, Cambridge, (2022) in press.

[13] J. A. Hempel, G. R. Morris, and N. S. Trudinger, On the sharpness of a limiting case of the Sobolev imbedding theorem. Bull. Austral. Math. Soc. 3 (1970), 369–373.

[14] M. Holst, E. Lunasin, and G. Tsogtgerel, Analysis of a general family of regularized Navier-Stokes and MHD models. J. Nonlinear Sci. 20 (2010), no. 5, 523–567.

[15] D. Hundertmark, A. Laptev, and T. Weidl, New bounds on the Lieb-Thirring constants. Invent. Math. 140(3) (2000), 693–704.

[16] A. A. Ilyin, On the spectrum of the Stokes operator. Funktsional. Anal. i Prilozhen. 43 (2009), no. 4, 14–25; English transl. in Funct. Anal. Appl. 43 (2009), no. 4.

[17] A. A. Ilyin, A. G. Kostianko, S. V. Zelik, Sharp upper and lower bounds of the attractor dimension for 3D damped Euler–Bardina equations. p. 31. http://arxiv.org/abs/math/2106.09077, Physica D: Nonlinear Phenomena to appear.

[18] A. Ilyin, A. Laptev and S. Zelik, Lieb–Thirring constant on the sphere and on the torus. J. Func. Anal. 279 (2020) 108784.
[19] A. A. Ilyin and S. V. Zelik, Sharp dimension estimates of the attractor of the damped 2D Euler-Bardina equations. In *Partial Differential Equations, Spectral Theory, and Mathematical Physics*, pp. 209–229, European Math. Soc. Press, Berlin, 2021.

[20] J. P. Kelliher, Eigenvalues of the Stokes operator versus the Dirichlet Laplacian in the plane. *Pacific J. Math.* 244 (2010), no. 1, 99–132.

[21] O. A. Ladyzhenskaya, First boundary value problem for Navier–Stokes equations in domain with non smooth boundaries. *C. R. Acad. Sc. Paris* 314 serie 1 (1992), 253–258.

[22] A. Larios, B. Wingate, M. Petersen, E. S. Titi, The Euler-Voigt equations and a computational investigation of the finite-time blow-up of solutions to the 3D Euler Equations *Theor. Comp. Fluid Dyn.* 3 (2018), no. 1, 23–34.

[23] E. H. Lieb, An $L^p$ bound for the Riesz and Bessel potentials of orthonormal functions. *J. Func. Anal.* 51 (1983), 159–165.

[24] E. Lieb, On characteristic exponents in turbulence. *Comm. Math. Phys.* 92 (1984) 473–480.

[25] E. Lieb, M. Loss, *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[26] E. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, In *Studies in Mathematical Physics. Essays in honor of Valentine Bargmann*, pp. 269–303, Princeton University Press, Princeton NJ, 1976.

[27] M. Lopes Filho, H. Nussenzveig Lopes, E. Titi, A. Zang, Convergence of the 2D Euler-α to Euler equations in the Dirichlet case: indifference to boundary layers. *Phys. D* 292-293 (2015) 51–61.

[28] G. Metivier, Valeurs propres des opérateurs définis sur la restriction de systèmes variationnels à des sous-espaces. *J. Math. Pures Appl.* 57 (1978), 133–156.

[29] Sh. M. Nasibov, On optimal constants in some Sobolev inequalities and their application to a nonlinear Schrödinger equation. *Dokl. Akad. Nauk SSR.* 307 (1989), 538–542; English transl. in Soviet Math. *Dokl.* 40 (1990).

[30] E. Olson and E. Titi, Viscosity versus vorticity stretching: global well-posedness for a family of Navier-Stokes-α-like models. *Nonlinear Anal.* 66 (2007), no. 11, 2427–2458.

[31] D. Ruelle, Large volume limit of the distribution of characteristic exponents in turbulence. *Comm. Math. Phys.* 87 (1982), 287–302.

[32] B. Simon, *Trace ideals and their applications*, 2nd ed. Amer. Math. Soc., Providence RI, 2005.

[33] R. Temam, *Navier-Stokes equations, Theory and numerical analysis*. Amsterdam, North-Holland, 1977.

[34] R. Temam, Attractors for Navier–Stokes equations. *Research Notes in Mathematics* 122 (1985), 272–292.

[35] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, 2nd Edition. Springer-Verlag, New York, 1997.

[36] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press, Cambridge, 1995.

[37] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.* 87 (1983), 567–576.
[38] S. V. Zelik, A. A. Ilyin, Green’s function asymptotics and sharp interpolation inequalities. *Uspekhi Mat. Nauk* 69 (2014), no. 2, 23–76; English transl. in *Russian Math. Surveys* 69 (2014), no. 2.

[39] S. V. Zelik, A. A. Ilyin, and A. G. Kostianko, Dimension estimates for the attractor of the regularized damped Euler equations on the sphere. *Mat. zametki* 111 (2022), no. 1, 55–67; English transl. *Math. Notes* 111 (2022), no. 1, 47–57.

Email address: ilyin@keldysh.ru
Email address: a.kostianko@imperial.ac.uk
Email address: s.zelik@surrey.ac.uk

1 Keldysh Institute of Applied Mathematics, Moscow, Russia

2 Sirius Mathematics Center, Sirius University of Science and Technology, Russia, 354349 Sochi, Olimpiyskiy Ave. b.1

3 University of Surrey, Department of Mathematics, Guildford, GU2 7XH, United Kingdom.

4 School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, P.R. China

5 Imperial College, London SW7 2AZ, United Kingdom