Existence of isometric immersions into nilpotent Lie groups

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Abstract

We establish necessary and sufficient conditions for existence of isometric immersions of a simply connected Riemannian manifold into a two-step nilpotent Lie group. This comprises the case of immersions into $H$-type groups.

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1 Introduction

The fundamental theorem of submanifold theory, usually referred to as Bonnet’s theorem, states that the Gauss, Codazzi and Ricci equations constitute a set of integrability conditions for isometric immersions of a simply connected Riemannian manifold in Euclidean space with prescribed second fundamental form. From the viewpoint of exterior differential systems, this result is a classical application of Frobenius’s theorem. At this respect, we refer the reader to [5], [10] and [18] for instance.

Versions of Bonnet’s Theorem for immersions in Riemannian spaces were recently achieved by Benoit Daniel in [6] and [7] and by P. Piccione and V. Tausk in [16] and [17]. In [7], Daniel consider immersions in three dimensional homogeneous spaces with four dimensional isometry group as Heisenberg spaces and Berger spheres. These ambients are regarded there as total spaces of Riemannian submersions over constant curvature surfaces, fibered by flow lines of a vertical Killing vector field $\xi$. It is proved that the ambient curvature tensor expressed in terms of a frame adapted to the immersion may be completely determined by the first and second fundamental forms and by the normal component $\nu$ and tangencial projection $T$ of $\xi$. Since Gauss and Codazzi equations involve these projections, it is necessary to consider two additional first order differential equations in $\nu$ and $T$. The augmented set of equations is then a complete integrability condition.

In [17], Piccione and Tausk prove a general existence result for affine immersions into affine manifolds endowed with a $G$-structure. The immersions should preserve the $G$-structure and the ambient spaces are required to be infinitesimally homogeneous. Roughly

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speaking, this last condition assures that the ambient curvature is constant when computed in terms of frames belonging to a $G$-reduction of the frame bundle. This method encompasses all classical results as well as Daniel’s results. Another applications of this technique in the context of Lie groups and Lorentzian Geometry may be found in [11], [12], [13] and [15].

The Heisenberg spaces studied in [7] are nilpotent Lie groups. Indeed, two-step nilpotent Lie groups form a distinguished class of geometric objects which include real, complex and quaternionic Heisenberg spaces and more generally $H$-type groups (see, e.g., [9], [8], [1] and [2]). These groups have some remarkable analytical properties and appear in distinct areas as Harmonic Analysis (v. [4]) and General Relativity (v. [14]).

These remarks motivate us to raise the question of extending Bonnet’s theorem from the classical case, which corresponds to Abelian groups, to two-step nilpotent Lie groups. Theorem 1 below yields such an extension in the spirit of the results in [7].

A brief outline of this paper may be given as follows. Let $N$ be a $(n+n')$-dimensional two-step nilpotent Lie group, where $n'$ is the dimension of the center $\mathfrak{z}$ in its Lie algebra. As occurs in [7], $\mathfrak{z}$ is spanned by left-invariant $n'$ Killing vector fields $E_{n+k}$, $k = 1, \ldots, n'$, whose covariant derivatives determine certain skew-symmetric tensors $J_k$, $k = 1, \ldots, n'$. The Section 2 is devoted to show that the curvature tensor in $N$ may be computed in an arbitrary frame $\{e_a\}^{n+n'}_{a=1}$ solely in terms of the tensors $J_k$ and the projections $U_k^a = \langle E_{n+k}, e_a \rangle$. The curvature form relative to the frame $\{e_a\}^{n+n'}_{a=1}$ is given by the tensor $Q$ defined in Section 2.1.2.

In the particular case of a frame adapted to an isometric immersion, this implies that Gauss, Codazzi and Ricci equations are completely written only in terms of the first and second fundamental forms and the normal and tangential projections of the Killing vector fields $E_{n+1}, \ldots, E_{n+n'}$ and their covariant derivatives. This is the content of Section 3.

In Sections 4 and 5, we establish sufficient conditions for immersing isometrically a simply connected Riemannian manifold $M$ into $N$, with prescribed second fundamental form. For this, we consider a real Riemannian vector bundle $\mathcal{E}$ over $M$ with rank $m' = n + n' - m$ so that the Whitney sum $\mathcal{S} = TM \oplus \mathcal{E}$ is a trivial bundle. We define an orthonormal global frame $\{\hat{E}_a\}^{n+n'}_{a=1}$ in $\mathcal{S}$ and then transplante the definitions of the tensors $J_k$ and $Q$ to this setting. This may be done in last analysis because these tensors depend on the structural constants of $N$. We then prove

**Theorem 1** a) Let $M^m$ be a Riemannian simply connected manifold and let $\mathcal{E}$ be a real Riemannian vector bundle with rank $m'$ so that $\mathcal{S} = TM \oplus \mathcal{E}$ is a trivial vector bundle. Let $\hat{\nabla}$ and $\hat{R}$ be respectively the compatible connection and curvature tensor in $\mathcal{S}$ and $\nabla$ and $\nabla^\mathcal{E}$ the compatible connections induced in $TM$ and $\mathcal{E}$, respectively. We fix an orthonormal frame $\{\hat{E}_k\}^{n+n'}_{k=1}$ in $\mathcal{S}$. Define $\hat{J}_k$ and $\hat{Q}$ as in (4.1) and (4.2), respectively. Assume that these fields satisfy the Gauss, Codazzi and Ricci equations

$$\hat{R} = \hat{Q}$$

(1.1)
and the additional equations
\[ \hat{\nabla} E_{n+k} = -1/2 \hat{J}_k, \quad k = 1, \ldots, n'. \] (1.2)

Thus, there exists an isometric immersion \( f : M \to N \) covered by a bundle isomorphism \( f^\perp_* : E \to TM^\perp_f \), where \( TM^\perp_f \) is the normal bundle along \( f \) so that \( f^\perp_* \) is an isometry when restricted to the fibers and satisfies
\[ f^\perp_* \nabla^E V = \nabla^N X f^\perp_* V, \quad X \in \Gamma(TM), \quad V \in \Gamma(E), \] (1.3)
\[ f^\perp_* II(X,Y) = \hat{\nabla}_{f^\perp_* X} f^\perp_* Y - f^\perp_* (\nabla_X Y), \quad X,Y \in \Gamma(TM), \] (1.4)
where \( \nabla \) and \( \nabla^\perp \) denote, respectively, the connections in \( N \) and \( TM^\perp_f \) and the tensor \( II \in \Gamma(T^* M \otimes T^* M \otimes E) \) is defined by
\[ \hat{\nabla}_X Y = \nabla_X Y + II(X,Y), \quad X,Y \in \Gamma(TM). \] (1.5)

b) Let \( f, \tilde{f} \) be two isometric immersions from \( M \) to \( N \) with second fundamental forms \( II_f \) and \( II_{\tilde{f}} \) satisfying
\[ II_f(X,Y) = \Phi II_{\tilde{f}}(X,Y), \quad X,Y \in \Gamma(TM), \] (1.6)
and normal connections \( \nabla^\perp \) and \( \hat{\nabla}^\perp \) on the respective normal bundles \( TM^\perp_f \) and \( TM^\perp_{\tilde{f}} \) related by
\[ \Phi \nabla^\perp X = \hat{\nabla}^\perp \Phi(V), \quad V \in \Gamma(TM^\perp_f), \] (1.7)
where \( \Phi : TM^\perp_f \to TM^\perp_{\tilde{f}} \) is a vector bundle isomorphism satisfying
\[ \langle \Phi(V), \Phi(W) \rangle = \langle V,W \rangle, \quad V,W \in \Gamma(TM^\perp_f). \] (1.8)

Fixed a left-invariant frame \( \{ E_k \}_{k=1}^{n+n'} \) in \( N \) we assume that
\[ \langle f_* X, E_{n+k} \rangle = \langle \tilde{f}_* X, E_{n+k} \rangle, \quad X \in \Gamma(TM) \] (1.9)
and that
\[ \langle V, E_{n+k} \rangle = \langle \Phi(V), E_{n+k} \rangle, \quad V \in \Gamma(TM^\perp_f) \] (1.10)
for \( k = 1, \ldots, n' \).

Then, there exists an isometry \( L : N \to N \) such that \( \tilde{f} = L \circ f \).

The ultimate reason for refer to (1.1) as Gauss, Codazzi and Ricci equations is that the tensor \( \hat{Q} \) imitates the curvature tensor in \( N \) when written in terms of a frame adapted to an isometric immersion. We point out that imposing that \( S \) is trivial allows us to give an intrinsic meaning to the tensors \( \hat{J}_k \). Hypothesis (1.1) and (1.2) play here the same role as the construction of a flat bundle endowed with parallel sections in the proof of the classical case.

Our method keeps some resemblance with the proof of Bonnet’s theorem given by P. Ciarlet and F. Larsonneur in [3]. Indeed, Theorem 1 may be regarded as establishing sufficient conditions for immersing an open set of the Euclidean space into a two-step nilpotent Lie group.
Two-step nilpotent Lie groups

Let \( N \) be a Lie group with Lie algebra \( \mathfrak{n} \) and Maurer-Cartan form \( \omega_{\mathfrak{n}} \). The Levi-Cività connection of a given left-invariant metric \( \langle \cdot, \cdot \rangle \) on \( N \) is

\[
2 \nabla_E F = [E, F] - \text{ad}_E^* \cdot F - \text{ad}_F^* \cdot E,
\]

where \( E, F \) are left-invariant vector fields in \( \mathfrak{n} \) and

\[
\langle \text{ad}_E^* \cdot F, G \rangle = \langle F, [E, G] \rangle, \quad E, F, G \in \mathfrak{n}.
\]

We suppose that \( \mathfrak{n} \) may be decomposed as \( \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \) with

\[
[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}, \quad [\mathfrak{z}, \mathfrak{n}] = \{0\},
\]

what implies that \( N \) is a two-step nilpotent Lie group. Let us denote by \( n \) and \( n' \) the dimensions of \( \mathfrak{v} \) and \( \mathfrak{z} \), respectively. We suppose that the direct sum \( \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \) is orthogonal. The relations (2.2) then yield

\[
\nabla_E F = \frac{1}{2} [E, F], \quad E, F \in \mathfrak{v},
\]

\[
\nabla_E Z = \nabla_Z E = -\frac{1}{2} J_Z E, \quad E \in \mathfrak{v}, \quad Z \in \mathfrak{z},
\]

\[
\nabla_Z Z' = 0, \quad Z, Z' \in \mathfrak{z},
\]

where the operator \( J_Z : \mathfrak{v} \to \mathfrak{v} \) associated to a vector field \( Z \in \mathfrak{z} \) is defined by \( J_Z = \text{ad}^* Z \).

This operator may be extended to the whole algebra \( \mathfrak{n} \) as

\[
J_Z := -2 \nabla Z.
\]

It is useful to consider also the \((0, 2)\) tensor field equally denoted by \( J_Z \) and defined by

\[
J_Z(E, F) = \langle J_Z E, F \rangle.
\]

2.1 Some auxiliary tensors

According to the decomposition \( \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z} \), we choose an orthonormal left-invariant frame field

\[
E_1, \ldots, E_n, E_{n+1}, \ldots, E_{n+n'},
\]

so that the first \( n \) vectors are in \( \mathfrak{v} \) and the next \( n' \) ones are in \( \mathfrak{z} \). Fixed this choice of frame, we define the structural constants of \( N \) by

\[
[E_k, E_l] = \sum_{r=1}^{n+n'} \sigma_{kl}^r E_r, \quad 1 \leq k, l \leq n + n'.
\]
If \( \{ \theta^k \}_{k=1}^{n+n'} \) denotes the co-frame dual to \( \{ E^k \}_{k=1}^{n+n'} \), then the corresponding connection forms in \( N \) are given by
\[
\theta^k_l = \frac{1}{2} \sum_{r=1}^{n+n'} \tau^k_{lr} \theta^r,
\] (2.9)
where
\[
\tau^k_{lr} = \sigma^k_{rl} + \sigma^l_{kr} + \sigma^r_{kl}.
\] (2.10)

We also define the curvature 2-form \( \Theta = \{ \Theta^k_l \}_{k,l=1}^{n+n'} \) of \( N \) associated to (2.7) by
\[
\Theta^k_l = \frac{1}{4} \sum_{r,s,t=1}^{n+n'} \left( \tau^k_{lr} \tau^r_{st} + \tau^k_{ls} \tau^s_{tr} \right) \theta^s \wedge \theta^t.
\] (2.11)

These forms satisfy the structural equations
\[
d\theta^k + \sum_{l=1}^{n+n'} \theta^k_l \wedge \theta^l = 0, \quad \theta^k_l = -\theta^l_k
\] (2.12)
and
\[
d\theta^k_l + \sum_{r=1}^{n+n'} \theta^k_r \wedge \theta^r_l = \Theta^k_l,
\] (2.13)
where \( 1 \leq k, l \leq n + n' \).

### 2.1.1 Christoffel tensor

Given the left-invariant frame above, we denote \( J_k = J_{E^k} \), \( 1 \leq k \leq n' \). Fixed this notation, we define in \( N \) the tensor field
\[
L(X, Y, V) = -\frac{1}{2} \sum_{k=1}^{n'} \langle J_k V, X \rangle \langle Y, E^k \rangle + \frac{1}{2} \sum_{k=1}^{n'} \langle J_k Y, X \rangle \langle V, E^k \rangle
\]
\[
-\frac{1}{2} \sum_{k=1}^{n'} \langle J_k Y, V \rangle \langle X, E^k \rangle, \quad X, Y, V \in \Gamma(TN).
\] (2.14)

In order to derive a \textit{local} expression for \( L \), we consider a frame \( \{ e_a \}_{a=1}^{n+n'} \) defined in an open set \( N' \) of \( N \) by
\[
e_a = \sum_{b=1}^{n+n'} E_b A_b^a,
\] (2.15)
for some map \( A : N' \to SO_{n+n'} \). For \( 1 \leq a \leq n + n' \) and \( 1 \leq k \leq n' \), we define the functions
\[
U^k_a = \theta^{n+k}(e_a) = A_a^{n+k}.
\] (2.16)
Thus, if \((\omega^a)_{a=1}^{n+n'}\) and \((\omega^b_\alpha)_{a,b=1}^{n+n'}\) are respectively the dual forms and the connection forms associated to the frame \(\{e_a\}_{a=1}^{n+n'}\), one has

\[
\omega^a(\bar{\nabla}E_{n+k}) = dU^k_a - \sum_c U^k_c \omega^c_a =: \frac{1}{2} \sum_b u^k_{ab} \omega^b.
\]

(2.17)

Hence, one gets

\[
J_k = \sum_{a,b=1}^{n+n'} u^k_{ab} \omega^a \otimes \omega^b.
\]

(2.18)

Notice that

\[
\langle J_k V, W \rangle = -2 \langle \bar{\nabla}V E_{n+k}, W \rangle = -2 \sum_{l,r} \langle V, E_l \rangle \langle W, E_r \rangle \langle \bar{\nabla}E_{n+k}, E_r \rangle
\]

\[
= \sum_{l,r} \langle V, E_l \rangle \langle W, E_r \rangle \sigma^{n+k}_{lr}.
\]

(2.19)

In local terms, that is, setting \(V = e_a, W = e_b\), one has

\[
u^k_{ab} = \sum_{l,r=1}^n A^l_a A^r_b \sigma^{n+k}_{lr}.
\]

(2.20)

Using the local frame, one computes

\[
L(X, e_a, e_b) = -\frac{1}{2} \sum_{k=1}^{n'} \langle J_k e_b, X \rangle \langle e_a, E_{n+k} \rangle + \frac{1}{2} \sum_{k=1}^{n'} \langle J_k e_a, X \rangle \langle e_b, E_{n+k} \rangle
\]

\[
- \frac{1}{2} \sum_{k=1}^{n'} \langle J_k e_a, e_b \rangle \langle X, E_{n+k} \rangle
\]

\[
= -\frac{1}{2} \sum_{c=1}^{n+n'} \sum_{k=1}^{n'} (U^k_c u^k_{bc} - U^k_b u^k_{ac} + U^k_c u^k_{ab}) \omega^c(X).
\]

One then defines the matrix of 1-forms \(\lambda^a = (\lambda^a_{\alpha})_{a,b=1}^{n+n'}\) by

\[
\lambda^a_{\alpha} = L(\cdot, e_a, e_b),
\]

(2.21)

that is,

\[
\lambda^a_{\alpha} = -\frac{1}{2} \sum_{c=1}^{n+n'} \sum_{k=1}^{n'} (U^k_c u^k_{bc} - U^k_b u^k_{ac} + U^k_c u^k_{ab}) \omega^c.
\]

(2.22)

We now use the equation (2.20) for obtaining an alternative expression for \(\lambda\), which will be useful later.
Proposition 1 The 1-form $\lambda = (\lambda^a_b)_{a,b=1}^{n+n'}$ defined in (2.21) satisfy

$$\lambda = A^{-1}\theta A,$$  

(2.23)

where $\theta = (\theta^k_l)_{k,l=1}^{n+n'}$. Thus, the connection form $\omega = (\omega^a_b)_{a,b=1}^{n+n'}$ is given by

$$\omega = A^{-1}dA + \lambda.$$  

(2.24)

Proof. Using (2.1) and (2.10), one obtains

$$\tau^k_{lr} = \begin{cases} 
\sigma^k_{rl}, & 1 \leq l, r \leq n \text{ and } k \geq n + 1, \\
\sigma^r_{kl}, & 1 \leq k, l \leq n \text{ and } r \geq n + 1, \\
\sigma^l_{kr}, & 1 \leq k, r \leq n \text{ and } l \geq n + 1.
\end{cases}$$

Thus, (2.22) and (2.20) yield

$$\lambda^a_b = -\frac{1}{2} \sum_{c=1}^{n+n'} \sum_{k=1}^{n+n'} (U^k_a U^k_b - U^k_b U^k_a - U^k_c U^k_a) \omega^c$$

$$= -\frac{1}{2} \sum_{c=1}^{n+n'} \sum_{k=1}^{n+n'} U^k_a A^k_c \sigma^{n+k}_c \omega^c + \frac{1}{2} \sum_{c=1}^{n+n'} \sum_{k=1}^{n+n'} A^k_c U^l_c \omega^c + \frac{1}{2} \sum_{c=1}^{n+n'} \sum_{k=1}^{n+n'} A^k_c U^l_c \omega^c$$

what implies that

$$\lambda^a_b = \frac{1}{2} \sum_{c,k,l,r=1}^{n+n'} A^k_a A^l_c \tau^k_{lr} \omega^c = \frac{1}{2} \sum_{k,l,r=1}^{n+n'} A^k_a \tau^k_{lr} \theta^r = \sum_{k,l=1}^{n+n'} A^k_a \theta^k_l$$

$$= \sum_{k,l=1}^{n+n'} (A^{-1})^a_b \theta^k_l A^k_b$$

$$= (A^{-1}\theta A)^a_b$$

This finishes the proof of the proposition. \hfill \Box

2.1.2 A curvature-type tensor

We then define a $(0, 4)$ covariant tensor $Q$ in $N$ by

$$Q(X,Y,V,W) = Q_1(X,Y,V,W) + Q_2(X,Y,V,W),$$

(2.25)
where $X, Y, V$ and $W$ are vector fields in $N$ and $Q_1$ and $Q_2$ are given by

$$Q_1(X, Y, V, W)$$

$$= \frac{1}{4} \langle J_k X, W \rangle \langle J_k Y, V \rangle + \frac{1}{2} \langle J_k Y, X \rangle \langle J_k W, V \rangle - \frac{1}{4} \langle J_k Y, W \rangle \langle J_k V, X \rangle$$

$$- \frac{1}{2} \sum_k \langle W, E_{n+k} \rangle \langle (\nabla_X J_k) V, Y \rangle + \frac{1}{2} \sum_k \langle V, E_{n+k} \rangle \langle (\nabla_X J_k) W, Y \rangle$$

$$+ \frac{1}{2} \sum_k \langle Y, E_{n+k} \rangle \langle (\nabla_Y J_k) W, V \rangle + \frac{1}{2} \sum_k \langle W, E_{n+k} \rangle \langle (\nabla_Y J_k) V, X \rangle$$

$$- \frac{1}{2} \sum_k \langle V, E_{n+k} \rangle \langle (\nabla_Y J_k) W, V \rangle - \frac{1}{2} \sum_k \langle X, E_{n+k} \rangle \langle (\nabla_Y J_k) W, V \rangle$$

and

$$Q_2(X, Y, V, W)$$

$$= -\frac{1}{4} \sum_{k,l} \langle E_{n+k}, W \rangle \langle E_{n+l}, V \rangle \langle J_k Y, J_l X \rangle + \frac{1}{4} \sum_{k,l} \langle E_{n+k}, W \rangle \langle E_{n+l}, X \rangle \langle J_k Y, J_l V \rangle$$

$$- \frac{1}{4} \sum_{k,l} \langle E_{n+k}, Y \rangle \langle E_{n+l}, V \rangle \langle J_k W, J_l X \rangle + \frac{1}{4} \sum_{k,l} \langle E_{n+k}, Y \rangle \langle E_{n+l}, X \rangle \langle J_k W, J_l V \rangle$$

$$+ \frac{1}{4} \sum_{k,l} \langle E_{n+k}, W \rangle \langle E_{n+l}, V \rangle \langle J_k X, J_l Y \rangle - \frac{1}{4} \sum_{k,l} \langle E_{n+k}, W \rangle \langle E_{n+l}, Y \rangle \langle J_k X, J_l V \rangle$$

$$+ \frac{1}{4} \sum_{k,l} \langle E_{n+k}, X \rangle \langle E_{n+l}, V \rangle \langle J_k W, J_l Y \rangle - \frac{1}{4} \sum_{k,l} \langle E_{n+k}, X \rangle \langle E_{n+l}, Y \rangle \langle J_k W, J_l V \rangle.$$

An important relation between $\lambda$ and $Q$ is given by the following lemma

**Lemma 1** The components $Q^b_a$ of $Q$ are given by the 2-forms

$$Q^b_a := Q(\cdot, \cdot, e_b, e_a) = (d\lambda + \lambda \wedge \omega + \omega \wedge \lambda - \lambda \wedge \lambda)_b^a. \tag{2.26}$$

**Proof.** Denoting the right hand side in (2.26) by $\Lambda^b_a$ and expanding it, it results that

$$-2\Lambda^a_c = \sum_c \sum_k \left( (dU^k_a - \sum_b U^k_b \omega_a^b) u_{dc} - (dU^k_d - \sum_b U^k_b \omega_d^b) u_{ac} - (dU^k_c - \sum_b U^k_b \omega_c^b) u_{ad} \right)$$

$$+ U^k_a (d u_{dc} - \sum_b u_{db} \omega_c^b - \sum_b u_{bc} \omega_d^b)$$

$$- U^k_d (d u_{ac} - \sum_b u_{ab} \omega_c^b - \sum_b u_{bc} \omega_d^b)$$

$$- U^k_c (d u_{ad} - \sum_b u_{ab} \omega_d^b - \sum_b u_{bd} \omega_a^b)$$

$$+ U^k_d \sum_b u_{bc} \lambda_a^b - \sum_b U^k_b \lambda_a^b u_{ac} - U^k_c \sum_b u_{ab} \lambda_d^b) \wedge \omega^c. \tag{2.27}$$
The covariant derivative of the $(0,2)$ tensor $J_k$ has components given in terms of the frame \( \{e_a\}_{a=1}^{n+n'} \) by

\[
\nabla J_k(e_a, e_b) = du^k_{ab} - u^k_{db} \omega^d_a - u^k_{ad} \omega^d_b =: \nabla u^k_{ab}. \tag{2.28}
\]

Using (2.17) and (2.28), one gets

\[
-2\Lambda^a_d = \sum_{k,c,c'} \left( -\frac{1}{2} u^k_{ca} u^k_{de} + \frac{1}{2} u^k_{d} u^k_{ac} + \frac{1}{2} u^k_{c} u^k_{ad} \right) \omega^e \wedge \omega^c
\]

\[
+ \sum_{k,c} \left( U^k_{a} \nabla u^k_{de} - U^k_{d} \nabla u^k_{ac} - U^k_{c} \nabla u^k_{ad} \right) \wedge \omega^e
\]

\[
+ \sum_{k,c} \left( U^k_{a} \sum_{b} u^k_{bc} \lambda^b_d - \sum_{b} U^k_{b} \lambda^b_d u^k_{ac} - U^k_{c} \sum_{b} u^k_{ab} \lambda^b_d \right) \wedge \omega^c.
\]

The last three terms may be calculated using that for $1 \leq k \leq n'$, $1 \leq a \leq n + n'$, one has

\[
dU^k_{a} - \sum_{c} U^k_{c} \omega^c_a + \sum_{c} U^k_{c} \lambda^c_a = 0. \tag{2.29}
\]

For proving (2.29), using (2.14), one computes

\[
\sum_{c} U^k_{c} \lambda^c_a = \sum_{c} U^k_{c} L(\cdot, e_c, e_a) = L(\cdot, E_{n+k}, e_a)
\]

\[
= -\frac{1}{2} \sum_{l} \left( \langle J_l e_a, \cdot \rangle \langle E_{n+k}, E_{n+l} \rangle - \langle J_l E_{n+k}, \cdot \rangle \langle e_a, E_{n+l} \rangle \right)
\]

\[
+ \langle J_l E_{n+k}, e_a \rangle \langle \cdot, E_{n+l} \rangle \right)
\]

\[
= -\frac{1}{2} \left( \langle J_k e_a, \cdot \rangle - \sum_{l} \langle J_l E_{n+k}, \cdot \rangle \langle e_a, E_{n+l} \rangle - \sum_{l} \langle J_l E_{n+k}, e_a \rangle \langle \cdot, E_{n+l} \rangle \right).
\]

However, given any vector field $V$ in $N$, one has

\[
\langle J_l E_{n+k}, V \rangle = \sum_{r,s} \langle E_{n+k}, E_r \rangle \langle V, E_s \rangle \sigma^{n+l}_{rs} = \sum_{s} \langle V, E_s \rangle \sigma^{n+l}_{n+k,s} = 0.
\]

Therefore, one concludes that

\[
\sum_{c} U^k_{c} \lambda^c_a = -\frac{1}{2} \langle J_k e_a, \cdot \rangle = -\frac{1}{2} \sum_{b} u^k_{ab} \omega^b_a = dU^k_{a} - \sum_{c} U^k_{c} \omega^c_a,
\]
as desired. This proves \((2.29)\). Thus, we may write

\[
-2\Lambda^a_d = \sum_{k,c,c'} \left( \frac{1}{2} u^k_{ca} u^k_{dc} + \frac{1}{2} u^k_{c,c} u^k_{ad} \right) \omega^c \land \omega^c \\
+ \sum_{k,c} \left( U^k_d \nabla u^k_{ac} - U^k_c \nabla u^k_{ad} - U^k_d \nabla u^k_{ac} \right) \land \omega^c \\
+ \sum_{k,c} \left( U^k_a \sum_{b} u^k_{bc} \lambda^b_d - U^k_c \sum_{b} u^k_{ab} \lambda^b_d \right) \land \omega^c.
\]

Nevertheless, in view of \((2.14)\), it follows that

\[
U^k_a \sum_{b} u^k_{bc} \lambda^b_d + U^k_c \sum_{b} u^k_{ab} \lambda^b_d = \sum_{b} \left( \langle E_{n+k}, e_a \rangle \langle J_k e_b, e_c \rangle + \langle E_{n+k}, e_c \rangle \langle J_k e_b, e_a \rangle \right) L(\cdot, e_b, e_d) \\
= \frac{1}{2} \sum_{l} \langle E_{n+k}, e_a \rangle \langle E_{n+l}, e_d \rangle \langle J_k e_c, J_l \cdot \rangle - \frac{1}{2} \sum_{l} \langle E_{n+k}, e_a \rangle \langle E_{n+l}, \cdot \rangle \langle J_k e_c, J_l e_d \rangle \\
+ \frac{1}{2} \sum_{l} \langle E_{n+k}, e_c \rangle \langle E_{n+l}, e_d \rangle \langle J_k e_a, J_l \cdot \rangle - \frac{1}{2} \sum_{l} \langle E_{n+k}, e_c \rangle \langle E_{n+l}, \cdot \rangle \langle J_k e_a, J_l e_d \rangle.
\]

Therefore, one concludes that

\[
\Lambda^a_d = \sum_{k,c,c'} \left( \frac{1}{4} u^k_{ca} u^k_{dc} - \frac{1}{4} u^k_{c,c} u^k_{ad} \right) \omega^c \land \omega^c \\
- \frac{1}{2} \sum_{k,c,c'} \left( U^k_d \nabla u^k_{ac} - U^k_c \nabla u^k_{ad} - U^k_d \nabla u^k_{ac} \right) \omega^c \land \omega^c \\
- \sum_{k,l,c,c'} \left( \frac{1}{4} \langle E_{n+k}, e_a \rangle \langle E_{n+l}, e_d \rangle \langle J_k e_c, J_l e_c \rangle - \frac{1}{4} \langle E_{n+k}, e_a \rangle \langle E_{n+l}, e_c \rangle \langle J_k e_c, J_l e_d \rangle \\
+ \frac{1}{4} \langle E_{n+k}, e_c \rangle \langle E_{n+l}, e_d \rangle \langle J_k e_a, J_l e_c \rangle - \frac{1}{4} \langle E_{n+k}, e_c \rangle \langle E_{n+l}, e_c \rangle \langle J_k e_a, J_l e_d \rangle \right) \omega^c \land \omega^c,
\]

what finishes the proof of the lemma. \(\square\)

This lemma has the following consequence, which characterizes geometrically the tensor \(Q\).

**Proposition 2** The tensor \(Q\) satisfies the equation

\[
Q = A^{-1} \Theta A
\]

\((2.30)\)

where \(\Theta = (\Theta^k_l)_{k,l=1}^{n+n'}\) are the curvature forms defined in \((2.11)\).
Proof. One has
\[ \begin{align*}
  d\omega + \omega \wedge \omega & = d(A^{-1}dA) + A^{-1}dA \wedge A^{-1}dA + d\lambda + \lambda \wedge \lambda + A^{-1}dA \wedge A^{-1}dA \\
  & = -A^{-1}dA \wedge A^{-1}dA + A^{-1}dA \wedge A^{-1}dA + d(A^{-1}\theta A) + A^{-1}\theta \wedge \theta A \\
  & \quad + A^{-1}\theta \wedge dA + A^{-1}dAA^{-1} \wedge \theta A \\
  & = dA^{-1} \wedge \theta A - A^{-1}d\theta A - A^{-1}\theta \wedge dA - A^{-1}\theta \wedge \theta A \\
  & \quad + A^{-1}\theta \wedge dA - dA^{-1} \wedge \theta A \\
  & = A^{-1}(d\theta + \theta \wedge \theta)A = A^{-1}\Theta A.
\end{align*} \] (2.31)

On the other hand we have
\[ d(\omega - \lambda) + (\omega - \lambda) \wedge (\omega - \lambda) = -A^{-1}dA \wedge A^{-1}dA + A^{-1}dA \wedge A^{-1}dA = 0, \] (2.32)
what implies that
\[ d\omega + \omega \wedge \omega = d\lambda - \lambda \wedge \lambda + \omega \wedge \lambda + \lambda \wedge \omega. \] (2.33)
Hence (2.31) and (2.33) give the desired result. \( \square \)

3 Isometric immersions into two-step nilpotent Lie groups

From now on, we consider a simply connected Riemannian manifold \( M^m \). We denote \( m' = n + n' - m \).

From the calculations above, we infer the following necessary conditions for the existence of isometric immersions in \( N \) with prescribed second fundamental form. In the statement, \( \bar{R} \) denotes the curvature tensor in \( N \).

Proposition 3 Let \( f : M \to N \) be an isometric immersion. Then, the Gauss, Ricci and Codazzi equations are given by
\[ \bar{R}(f_*X, f_*Y, V, W) = Q(f_*X, f_*Y, V, W), \quad X, Y \in \Gamma(TM), \quad V, W \in \Gamma(f^*TN). \] (3.1)
Moreover, the following additional equations are satisfied
\[ \bar{\nabla}_X E_{n+k} = -\frac{1}{2}J_k X, \quad X \in \Gamma(TM) \] (3.2)
for \( k = 1, \ldots, n' \).

Proof. After identifying \( M \) and the immersed submanifold \( f(M) \subset N \), we consider an orthonormal frame \( \{e_a\}_{a=1}^{m+m'} \) defined in an ambient open neighborhood of an arbitrary point in \( M \). This frame may be chosen adapted to the immersion, that is, in such a way
that, along points in \( M \), the first \( m \) fields in this frame are tangent to \( M \) and the other \( m' \) ones are local sections of the normal bundle \( TM^\perp \).

Let \( A \) be given as above by (2.15). Then, the connection forms \( \{ \omega^a \}_{a=1}^{m+m'} \) satisfy

\[
d\omega^a + \sum_{c} \omega_c^a \wedge \omega_c^b = (A^{-1} \Theta A)_b^a,
\]  

(3.3)

where \( \Theta \) is given by (2.11). Since the right-hand side in (3.3) corresponds to the ambient curvature tensor expressed in terms of the adapted frame, this equation corresponds to Gauss, Codazzi and Ricci equations, respectively, as we may easily verify considering suitable ranges of indices \( a, b \). Hence, (2.30) in Proposition 2 implies (3.1).

The equation (3.2) follows immediately from the preceding discussion. \( \square \)

4 Existence of an adapted frame

We now consider a real Riemannian vector bundle \( \mathcal{E} \) over \( M \) with rank \( m' \) and the Whitney sum bundle \( S = TM \oplus \mathcal{E} \). The metric in \( S \) is also represented by \( \langle \cdot, \cdot \rangle \). Let \( \hat{\nabla} \) and \( \hat{R} \) be respectively the compatible connection and curvature tensor in \( S \).

We suppose that \( S \) is a trivial vector bundle and then we fix a globally defined orthonormal frame \( \hat{E}_1, \ldots, \hat{E}_{n+n'} \) in \( S \). Hence, for any \( k = 1, \ldots, n' \), one defines

\[
\langle \hat{J}_k V, W \rangle = \sum_{l,r=1}^{n} \langle V, \hat{E}_l \rangle \langle W, \hat{E}_r \rangle \sigma_{l,r}^{n+k}, \quad V, W \in \Gamma(S),
\]  

(4.1)

where the constants \( \sigma_{l,r}^{n+k} \) are given by (2.8). It is obvious from the definition that

\[
\langle \hat{J}_k V, \hat{E}_{n+i} \rangle = 0
\]  

(4.2)

since \( \sigma_{r,n+i}^{n+k} = 0 \).

Now, we define in terms of \( \hat{J}_k \) tensors \( \hat{L} \) and \( \hat{Q} \) in \( S \) by

\[
\hat{L}(X,Y,V) = -\frac{1}{2} \sum_{k=1}^{n'} \langle \hat{J}_k V, X \rangle \langle Y, \hat{E}_{n+k} \rangle + \frac{1}{2} \sum_{k=1}^{n'} \langle \hat{J}_k Y, X \rangle \langle V, \hat{E}_{n+k} \rangle
\]

\[
-\frac{1}{2} \sum_{k=1}^{n'} \langle \hat{J}_k Y, V \rangle \langle X, \hat{E}_{n+k} \rangle, \quad X,Y,V \in \Gamma(S)
\]  

(4.3)

and for \( X,Y \in \Gamma(TM) \) and \( V,W \in \Gamma(S) \),

\[
\hat{Q}(X,Y,V,W) = \hat{Q}_1(X,Y,V,W) + \hat{Q}_2(X,Y,V,W),
\]  

(4.4)
where

\[ \hat{Q}_1(X, Y, V, W) \]
\[ = \frac{1}{4} \langle \hat{J}_k X, W \rangle \langle \hat{J}_k V, Y \rangle + \frac{1}{2} \langle \hat{J}_k X, \hat{J}_k W \rangle \langle \hat{J}_k V, X \rangle - \frac{1}{4} \langle \hat{J}_k Y, W \rangle \langle \hat{J}_k V, X \rangle - \frac{1}{2} \sum_k \langle W, \hat{E}_{n+k} \rangle \langle (\hat{\nabla}_X \hat{J}_k) V, Y \rangle + \frac{1}{2} \sum_k \langle V, \hat{E}_{n+k} \rangle \langle (\hat{\nabla}_X \hat{J}_k) W, Y \rangle + \frac{1}{2} \sum_k \langle Y, \hat{E}_{n+k} \rangle \langle (\hat{\nabla}_X \hat{J}_k) W, V \rangle + \frac{1}{2} \sum_k \langle W, \hat{E}_{n+k} \rangle \langle (\hat{\nabla}_X \hat{J}_k) V, X \rangle - \frac{1}{2} \sum_k \langle V, \hat{E}_{n+k} \rangle \langle (\hat{\nabla}_X \hat{J}_k) W, V \rangle \]

and

\[ \hat{Q}_2(X, Y, V, W) \]
\[ = -\frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, W \rangle \langle \hat{E}_{n+l}, V \rangle \langle \hat{J}_k Y, \hat{J}_l X \rangle + \frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, W \rangle \langle \hat{E}_{n+l}, X \rangle \langle \hat{J}_k Y, \hat{J}_l V \rangle - \frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, Y \rangle \langle \hat{E}_{n+l}, V \rangle \langle \hat{J}_k W, \hat{J}_l X \rangle + \frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, Y \rangle \langle \hat{E}_{n+l}, X \rangle \langle \hat{J}_k W, \hat{J}_l V \rangle + \frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, W \rangle \langle \hat{E}_{n+l}, V \rangle \langle \hat{J}_k X, \hat{J}_l Y \rangle - \frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, W \rangle \langle \hat{E}_{n+l}, Y \rangle \langle \hat{J}_k X, \hat{J}_l V \rangle + \frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, X \rangle \langle \hat{E}_{n+l}, V \rangle \langle \hat{J}_k W, \hat{J}_l Y \rangle - \frac{1}{4} \sum_{k,l} \langle \hat{E}_{n+k}, X \rangle \langle \hat{E}_{n+l}, Y \rangle \langle \hat{J}_k W, \hat{J}_l V \rangle. \]

We then suppose that

\[ \langle \hat{R}(X, Y)V, W \rangle = \hat{Q}(X, Y, V, W), \quad X, Y \in \Gamma(TM), \quad V, W \in \Gamma(S). \quad (4.5) \]

We also assume the following condition

\[ \hat{\nabla}_X \hat{E}_{n+k} = -\frac{1}{2} \hat{J}_k X, \quad X \in \Gamma(TM), \quad k = 1, \ldots, n'. \quad (4.6) \]

The connection in $S$ induces connections $\nabla$ in $M$ and $\nabla^E$ in $E$. More precisely, defining $II \in \Gamma(T^*M \otimes T^*M \otimes E)$ by

\[ \hat{\nabla}_X Y = \nabla_X Y + II(X, Y), \quad X, Y \in \Gamma(TM) \quad (4.7) \]

and defining, for $V \in \Gamma(E)$,

\[ \langle S_V(X), Y \rangle = \langle II(X, Y), V \rangle, \quad (4.8) \]

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one obtains
\[ \hat{\nabla}_X V = -SVX + \nabla^E_X V. \] (4.9)

In terms of the decomposition \( \tilde{E}_{n+k} = T_k + N_k, T_k \in \Gamma(TM), N_k \in \Gamma(E) \), the condition (4.6) becomes
\[ \nabla_X T_k - S_k(X) + \nabla^E_X N_k + II(T_k, X) = -\frac{1}{2}\hat{j}_k(X), \quad X \in \Gamma(TM), \] (4.10)

where \( S_k = S_{N_k} \).

**Definition 1**
Given a connected simply connected open subset \( M' \subset M \), we fix a map \( \hat{U} \in C^\infty(M', \mathbb{R}^{n'(n+n')}) \). A frame \( e : M' \to \mathcal{F}(S) \) with components
\[ e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+m'} \]
is admissible if the first \( m \) sections are vector fields in \( M' \) and the last \( m' \) ones are sections in \( E \) and, moreover, if it holds that
\[ \langle \tilde{E}_{n+k}, e_a \rangle = \hat{U}^k_a, \quad 1 \leq k \leq n'. \] (4.11)

In particular, this implies that
\[ \langle T_k, e_i \rangle = \langle \tilde{E}_{n+k}, e_i \rangle = \hat{U}^k_i \] (4.12)
for \( i = 1, \ldots, m \) and
\[ \langle N_k, e_\alpha \rangle = \langle \tilde{E}_{n+k}, e_\alpha \rangle = \hat{U}^k_\alpha \] (4.13)
for \( \alpha = m + 1, \ldots, m + m' \). The transition map from the frame \( \{ \tilde{E}_k \}_{k=1}^{n+n'} \) to an admissible frame \( \{ e_a \}_{a=1}^{m+m'} \) is given by an admissible map, that is, if
\[ e_a = \sum_{k=1}^{n+n'} \tilde{E}_k A^k_a, \] (4.14)
then \( A \) is if the form
\[ A(x) = \left( \begin{array}{c} * \\ \hat{U}(x) \end{array} \right), \] (4.15)
where the block \( \hat{U}(x) \) corresponds to the last \( n' \) lines.

We denote by
\[ \omega^1, \ldots, \omega^m, \omega^{m+1}, \ldots, \omega^{m+m'} \] (4.16)
the real-valued 1-forms dual to the frame \( \{e_a\}_{a=1}^{m+m'} \). The Riemannian connection \( \hat{\nabla} \) is given in terms of this frame by the matrix \( \omega = (\omega_a^b)_{a,b=1}^{n+n'} \). Hence, the first structural equation is written as

\[
d\omega^a + \sum_b \omega_b^a \wedge \omega^b = 0, \quad \omega_b^a = -\omega_a^b. \tag{4.17}
\]

Regarding each \( \hat{J}_k \) as \((0,2)\) tensor, we write them locally as

\[
\hat{J}_k = \sum_{a,b} \hat{u}_k^{ab} \omega^a \otimes \omega^b, \quad k = 1, \ldots, n'. \tag{4.18}
\]

Thus in local terms the equation (4.16) is rewritten as

\[
\sum_k \left( d\hat{U}_k^a - \sum_b \hat{U}_k^b \omega_a^b \right) = \frac{1}{2} \sum_k \hat{u}_k^{ab} \omega^b. \tag{4.19}
\]

The local expression for \( \hat{L} \) is given by the 1-forms

\[
\hat{\lambda}_a^b = \hat{L}(\cdot, e_a, e_b). \tag{4.20}
\]

Following the calculations in Section 2.1.1 we conclude that

\[
\hat{\lambda}_b^a = -\frac{1}{2} \sum_{c=1}^{n+n'} \sum_{k=1}^{n'} \left( \hat{U}_k^a \hat{u}_k^{bc} - \hat{U}_k^b \hat{u}_k^{ac} + \hat{U}_k^c \hat{u}_k^{ba} \right) \omega^c. \tag{4.21}
\]

The local expression for \( \hat{Q} \) is given by the 2-forms

\[
\hat{Q}_a^d = \hat{Q}(\cdot, \cdot, e_d, e_a). \tag{4.22}
\]

One notices that the hypothesis (4.5) is rephrased in terms of these forms as

\[
d\omega^a_b + \sum_c \omega^a_c \wedge \omega^c_b = \hat{Q}_b^a. \tag{4.23}
\]

We may verify proceeding as in the proof of the Lemma 1 and using (4.2) that

\[
\hat{Q}_d^a := \hat{Q}(\cdot, \cdot, e_d, e_a) = (d\hat{\lambda} + \hat{\lambda} \wedge \omega + \omega \wedge \hat{\lambda} - \hat{\lambda} \wedge \hat{\lambda})_d^a. \tag{4.24}
\]

Combining equations (4.23) and (4.24), one deduces that \( \hat{\omega} := \omega - \hat{\lambda} \) satisfies the zero curvature equation

\[
d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = 0. \tag{4.25}
\]

A suitable version of (2.29) allows us to claim that

\[
d\hat{U}_a^k - \sum_b \hat{U}_b^k \omega_a^b + \sum_b \hat{U}_b^k \hat{\lambda}_a^b = 0. \tag{4.26}
\]

We then prove the following result.
Proposition 4 Assume that (4.5) and (4.6) hold. Let $M' \subset M$ be a connected simply connected open subset. Then there exists an admissible map $A \in C^\infty(M', SO_{n+n'})$ so that

$$A^{-1}dA = \omega - \hat{\lambda}$$

(4.27)

with initial condition $A(x_0) = Id$, for a given $x_0 \in M'$.

Proof. We want to assure the existence of an admissible map so that

$$A^{-1}dA = \hat{\omega},$$

(4.28)

where $\hat{\omega} = \omega - \hat{\lambda}$.

If we denote by $\mu : M_{n+n'} \to \mathbb{R}^{n'(n+n')}$ the projection on the last $n'$ lines, the condition (4.14) means that $\mu(A(x)) = \hat{U}(x)$. The set of admissible maps define a submanifold of $M \times SO_{n+n'}$, namely

$$\mathcal{U} = \{(x, A) : A = \left( \begin{array}{c} \hat{U}(x) \\ \end{array} \right) \},$$

(4.29)

whose tangent space at a point $(x, A)$ is

$$T_{(x,A)}\mathcal{U} = \{(v, B) : B = \left( \begin{array}{c} d\hat{U}(x) \cdot v \\ \end{array} \right) \}.$$

(4.30)

Let $\tilde{\omega} \in \Lambda^1(SO_{n+n'}, \mathfrak{so}_{n+n'})$ be the Maurer-Cartan form in $SO_{n+n'}$. Thus, the equation (4.28) is written as

$$\hat{\omega} = A^*\tilde{\omega}.$$  

(4.31)

For solving this equation, we define a 1-form $\Upsilon$ in $M' \times SO_{n+n'}$ with values on $\mathfrak{so}_{n+n'}$ by

$$\Upsilon = \pi_1^*\tilde{\omega} - \pi_2^*\tilde{\omega},$$

(4.32)

where $\pi_1 : M \times SO_{n+n'} \to M$ and $\pi_2 : M \times SO_{n+n'} \to SO_{n+n'}$ are the natural projections. We then define the distribution $\mathcal{D} = \ker \Upsilon$ on $\mathcal{U}$. More precisely

$$(v, B) \in \mathcal{D}_{(x,A)} \text{ if and only if } \hat{\omega}_x(v) = \tilde{\omega}_A(B)$$

(4.33)

In order to prove that (4.33) defines a distribution we must verify that $\ker \Upsilon$ has constant rank. We begin by proving that the differential of $\pi_1$ restricted to $\mathcal{D}_{(x,A)}$ is a monomorphism. In fact, if $\pi_1(v, B) = 0$ for some $(v, B) \in \mathcal{D}_{(x,A)}$ then $v = 0$. Since $0 = \hat{\omega}_x(v) = \tilde{\omega}_A(B)$, it follows that $B = 0$. Therefore,

$$\dim \ker \Upsilon_{(x,A)} \leq m.$$  

In order to prove that (4.33) defines a distribution we must verify that $\ker \Upsilon$ has constant rank. We begin by proving that the differential of $\pi_1$ restricted to $\mathcal{D}_{(x,A)}$ is a monomorphism. In fact, if $\pi_1(v, B) = 0$ for some $(v, B) \in \mathcal{D}_{(x,A)}$ then $v = 0$. Since $0 = \hat{\omega}_x(v) = \tilde{\omega}_A(B)$, it follows that $B = 0$. Therefore,

$$\dim \ker \Upsilon_{(x,A)} \leq m.$$  

Now, given $(v, B) \in T_{(x,A)}\mathcal{U}$ we have

$$\mu(A\Upsilon_{(x,A)}(v, B)) = \mu(A\hat{\omega}_x(v) - A\tilde{\omega}_A(B)) = \mu(A\hat{\omega}_x(v) - AA^{-1} \cdot B)$$

$$= \mu(A)\hat{\omega}_x(v) - \mu(B) = \hat{U}\hat{\omega}_x(v) - d\hat{U}_x(v) = 0$$

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where in the last equality we used equation (4.26). We then had verified that
\[ \text{Im} \mathcal{Y}_{(x,A)} \subset \{ B \in \mathfrak{so}_{n+n'} : \mu(AB) = 0 \} \]

Thus, if \( B \in \text{Im} \mathcal{Y}_{(x,A)} \) then \( B = \bar{\omega}_A(B) \) for some \( B \) tangent to \( A \) such that \( \mu(B) = 0 \). This means that
\[ \text{Im} \mathcal{Y}_{(x,A)} \subset \bar{\omega}_A \left( \ker \mu_A \right) \]
where \( \ker \mu_A = \{ B \in T_ASO_{n+n'} : \mu(B) = 0 \} \). Since \( \bar{\omega}_A \) is an isomorphism, it follows that \( \bar{\omega}_A \left( \ker \mu_A \right) \) and \( \ker \mu_A \) have same dimension. Thus,
\[ \dim \ker \mathcal{Y}_{(x,A)} \geq m. \]

Hence, \( \mathcal{D}_{(x,A)} \) is \( m \) dimensional, for all \( x \in M' \), \( A \in \mathcal{U} \).

Now we verify the integrability of \( \mathcal{D} \). The zero curvature equation (4.25) implies that
\[ d\mathcal{Y} = d\bar{\omega} - d\hat{\omega} = \bar{\omega} \wedge \hat{\omega} - \hat{\omega} \wedge \bar{\omega} = (\bar{\omega} + \mathcal{Y}) \wedge (\hat{\omega} + \mathcal{Y}) - \bar{\omega} \wedge \hat{\omega}. \]

Thus if one calculates \( d\mathcal{Y} \) at some vector \((v, B) \in \mathcal{D}_{(x,A)} \) one obtains \( \mathcal{Y}(v, B) = 0 \) and then \( d\mathcal{Y}(v, B) = 0 \) too. So the ideal \( \ker \mathcal{Y} \) is differential and then the distribution \( \mathcal{D} \) is integrable.

Since \( \pi \) is a local diffeomorphism between the simply connected domain \( M' \) and the integral leaf of \( \mathcal{D} \) passing through \((x_0, \text{Id})\), a standard monodromy reasoning implies that this leaf as the graph \( x \mapsto A(x) \) of a certain map \( A \in C^\infty(M', SO_{n+n'}) \) which by definition satisfies (4.14) and (4.31).

Given an admissible map \( A : M' \to SO_{n+n'} \) solving (4.27), one defines a frame \( \{ e_a \}_{a=1}^{m+m'} \) in \( S \) along \( M' \) by (4.14). The corresponding sets of dual 1-forms are related by
\[ \hat{\theta}^k = \sum_{a=1}^{m+m'} A^k_a \omega^a. \quad (4.34) \]

It stems from (4.11) that the local expression for \( \hat{J}_k \) in the frame \( \{ e_a \}_{a=1}^{n+n'} \) is
\[ -\frac{1}{2} \hat{u}_{ab} = -\frac{1}{2} \langle \hat{J}_k e_a, e_b \rangle = \sum_{l,r} \langle e_a, \hat{E}_l \rangle \langle e_b, \hat{E}_r \rangle \sigma_{lr}^{n+k} = \sum_{l,r} A^l_a A^r_b \sigma_{lr}^{n+k}. \]

We then define
\[ \hat{\theta}^k = \frac{1}{2} \sum_r \tau^k_r \hat{\theta}^r = \frac{1}{2} \sum_a \sum_r \tau^k_r A^r_a \omega^a. \quad (4.35) \]

In view of these facts, we are able to restate Proposition 1 in the current context.
Proposition 5 The admissible frame obtained above as solution of the equation (4.27) satisfies
\[ \hat{\lambda} = A^{-1} \tilde{\theta} A, \] (4.36)
where \( \tilde{\theta} = (\tilde{\theta}_k^n)_{k,l=1}^{n+n'} \) is defined in (4.35).

Proof. It suffices to mimic the proof of Proposition 1 in Section 2.1.1.

We finally define the following 2-forms
\[ \hat{\Theta}^k_l = \frac{1}{4} \sum_{a,b} \sum_{r,s} \left( \tau^k_r \tau^r_s + \tau^k_l \tau^l_t \right) \hat{\omega}^a \wedge \hat{\omega}^b = \frac{1}{4} \sum_{a,b} \sum_{r,s} \left( \tau^k_r \tau^r_s + \tau^k_l \tau^l_t \right) A^a_r A^b_l \omega^a \wedge \omega^b. \] (4.37)

Then we are able to prove the following result.

Proposition 6 The admissible frame defined above as solution of the equation (4.27) satisfies
\[ \hat{Q} = A^{-1} \hat{\Theta} A, \] (4.38)
where \( \hat{\Theta} = (\hat{\Theta}^k_l)_{k,l=1}^{n+n'} \) is defined in (4.37).

Proof. From (4.27) and (4.36) it follows that
\[ d\hat{\lambda} = dA^{-1} \wedge \tilde{\theta} A + A^{-1} d\tilde{\theta} A - A^{-1} \tilde{\theta} A \wedge dA = -A^{-1} dA \wedge A^{-1} \hat{\Theta} A + A^{-1} d\hat{\Theta} A - A^{-1} \hat{\Theta} A \wedge A^{-1} dA = -A^{-1} \hat{\Theta} A \wedge \hat{\Theta} A = 2\hat{\lambda} \wedge \hat{\Theta} - \omega \wedge \hat{\Theta} = (\omega - \hat{\lambda}) \wedge \hat{\Theta} - \hat{\Theta} \wedge (\omega - \hat{\lambda}) = 2\hat{\lambda} \wedge \hat{\Theta} - \omega \wedge \hat{\Theta} = 2\hat{\lambda} \wedge \hat{\Theta} - \omega \wedge \hat{\Theta} = 2\hat{\lambda} \wedge \hat{\Theta} - \omega \wedge \hat{\Theta} = 2\hat{\lambda} \wedge \hat{\Theta} - \omega \wedge \hat{\Theta} = 2\hat{\lambda} \wedge \hat{\Theta} - \omega \wedge \hat{\Theta}.
\]

Therefore, in view of (4.24), we conclude that
\[ \hat{Q} = d\hat{\lambda} - \omega \wedge \hat{\Theta} = \hat{\Theta} A \wedge \hat{\Theta} A = A^{-1} \hat{\Theta} A \wedge A^{-1} \hat{\Theta} A = A^{-1} (d\hat{\Theta} \wedge \hat{\Theta} A). \] (4.39)

However, it follows from (4.35), (4.17) and (4.27) that
\[ d\hat{\Theta}^k_l = \frac{1}{2} \sum_a \sum_r \tau^k_r (dA^a_r \wedge \omega^a + A^r_a d\omega^a) = \frac{1}{2} \sum_a \sum_r \tau^k_r (dA^a_r \wedge \omega^b - A^r_a \omega^b \wedge \omega^b) = \frac{1}{2} \sum_a \sum_r \tau^k_r (dA^a_r - A^r_a \omega^b) \wedge \omega^b = -\frac{1}{2} \sum_b \sum_r \tau^k_r (A\hat{\lambda})^b_r \wedge \omega^b. \]
However $A \dot{\lambda} = AA^{-1} \dot{\theta} A = \dot{\theta} A$. Hence, one gets

$$d \hat{\theta}_k^l = -\frac{1}{2} \sum_b \sum_r \tau^k_{lr} (\dot{\theta} A)_b^r \wedge \omega^b = -\frac{1}{2} \sum_{r,s} \tau^r_s \hat{\theta}^s \wedge A^*_b \omega^b = -\frac{1}{2} \sum_{r,s} \tau^r_s \hat{\theta}^s \wedge \hat{\theta}^s.$$ 

On the other hand, one has

$$\sum_r \hat{\theta}^k_r \wedge \hat{\theta}^l_r = \frac{1}{4} \sum_{r,s,t} \tau^k_{rs} \hat{\theta}^s \wedge \hat{\theta}^t.$$ 

Therefore, one concludes that

$$d \hat{\theta} + \hat{\theta} \wedge \hat{\theta} = \hat{\Theta}.$$ 

(4.40)

Gathering (4.39) and (4.40) we finish the proof. □

5 Proof of the Theorem

Part a. In view of the hypothesis in Theorem 1, Proposition 4 implies that there exists an admissible map $A : M \to SO_{n+n'}$ which solves (4.27) and satisfies (4.36) and (4.38) for $\{\hat{\theta}_k^l\}_{k=1}^{n+n'}$ and $\{\hat{\omega}_k^l\}_{k=1}^{n+n'}$ defined in (4.35) and (4.37), respectively.

We fix in $n = \mathbb{R}^{n+n'}$ the orthonormal frame $\{\bar{e}_k = \omega_n(E_k)\}_{k=1}^{n+n'}$. We then define the following 1-form on $M \times N$ with values on $\Pi$

$$\Pi = \pi^*_N \omega_n - \sum_{k=1}^{n+n'} \sum_{a=1}^{m+m'} \bar{e}_k (A^*_a \circ \pi_M) \pi^*_M \omega^a,$$

where $\pi_N : M \times N \to N$ and $\pi_M : M \times N \to M$ are the canonical projections. We then consider the distribution $\mathcal{P} = \ker \Pi$ on $M \times N$. Thus, using (4.17) and (4.27), we calculate (omitting projections)

$$d\Pi = d\omega_n - \sum_{a,k} \bar{e}_k A^*_a \wedge \omega^a - \sum_{a,k} \bar{\omega}_k A^*_a d\omega^a$$

$$= -\frac{1}{2} \omega_n \wedge \omega_n - \sum_{a,k} \bar{e}_k (A \hat{\omega})^k_a \wedge \omega^a + \sum_{a,c,k} \bar{e}_k A^*_a \omega^a_c \wedge \omega_c$$

$$= -\frac{1}{2} \Pi + \sum_k \bar{e}_k \hat{\omega}^k \Pi + \sum_l \bar{e}_l \hat{\theta}^l \Pi - \sum_{a,k} \bar{e}_k \omega_n^a \wedge \omega^a + \sum_{a,c,k} \bar{e}_k A^*_a \omega^a_c \wedge \omega_c.$$
Hence, one has
\[ d\Pi = -\frac{1}{2}[\Pi,\Pi] - \frac{1}{2}[\Pi, \sum_k \tilde{e}_k \hat{\theta}^k] - \frac{1}{2}[\sum_l \tilde{e}_l \hat{\theta}^l, \Pi] - \frac{1}{2} \sum_{k,l} [\tilde{e}_k \hat{\theta}^k, \tilde{e}_l \hat{\theta}^l] \]
\[ - \sum_{a,k} \tilde{e}_k (A\omega)^k_a \wedge \omega^a + \sum_{a,k} \tilde{e}_k (A\hat{\lambda})^k_a \wedge \omega^a + \sum_{a,c,k} \tilde{e}_k A^k_c \omega^a_c \wedge \omega^c. \]

Thus considering equality modulo $\Pi$ it follows that
\[ d\Pi = -\frac{1}{2} \sum_{k,l} \hat{\theta}^k \wedge \hat{\theta}^l [\tilde{e}_k, \tilde{e}_l] - \sum_{a,c,k} \tilde{e}_k A^k_c \omega^a_c \wedge \omega^a + \sum_{a,c,k} \tilde{e}_k A^k_c \hat{\lambda}^c_a \wedge \omega^a + \sum_{a,c,k} \tilde{e}_k A^k_c \omega^a_c \wedge \omega^a \]
\[ = -\frac{1}{2} \sum_{k,l} \hat{\theta}^k \wedge \hat{\theta}^l [\tilde{e}_k, \tilde{e}_l] + \sum_{a,c,k} \tilde{e}_k A^k_c \hat{\lambda}^c_a \wedge \omega^a. \]

However using (4.36) one obtains
\[ d\Pi = -\frac{1}{2} \sum_{k,l,r} \hat{e}_r \sigma_{kl} \hat{\theta}^k \wedge \hat{\theta}^l + \sum_{a,b,c} \sum_{k,l} \tilde{e}_k A^k_c \hat{\lambda}^c_b (A^{-1})_b^a A^l_a \wedge \omega^a \]
\[ = -\frac{1}{2} \sum_{k,l} \sum_r \hat{e}_r \sigma_{rl} \hat{\theta}^r \wedge \hat{\theta}^l + \sum_{k,l} \tilde{e}_k \hat{\theta}^k \wedge \hat{\theta}^l = \sum_{k,l} \tilde{e}_k (\hat{\theta}^l - \frac{1}{2} \sum_r \sigma^r_{kl} \hat{\theta}^r) \wedge \hat{\theta}^l. \]

Therefore $\mathcal{P}$ is involutive since by (4.35) one has
\[ \hat{\theta}^l = \frac{1}{2} \sum_r \sigma^r_{kl} \hat{\theta}^r + \frac{1}{2} \sum_r \mu^k_{lr} \hat{\theta}^r, \quad (5.1) \]
where $\mu^k_{lr} = \sigma^l_{kr} + \sigma^r_{kl}$ satisfies $\mu^k_{lr} = \mu^k_{rl}$. This symmetry implies that
\[ \sum_l (\hat{\theta}^l - \frac{1}{2} \sum_r \sigma^r_{kl} \hat{\theta}^r) \wedge \hat{\theta}^l = \frac{1}{2} \sum_{l,r} \mu^k_{lr} \hat{\theta}^r \wedge \hat{\theta}^l = 0, \]
what gives the integrability condition
\[ d\Pi = 0 \mod \Pi. \]

We may verify that an integral leaf through the identity $y_0$ in $N$ is a graph over $M$. The function that graphics this leaf is an isometric immersion $f : M \to N$ with initial condition, say, $f(x_0) = y_0$, for a given point $x_0 \in M$.

Indeed, given a tangent vector $(v, w) \in \mathcal{P}(x,y)$ with $y = f(x)$, we have $f_*(x) \cdot v = w$ and
\[ \omega_m(w) - \sum_{k=1}^{n+n'} \sum_{a=1}^{m+m'} \tilde{e}_k A^k_a(x) \omega^a(v) = 0 \]

what yields after left translating both sides by $y$

$$f_*(x) \cdot v = w = \sum_{k=1}^{n+n'} \sum_{a=1}^{m+m'} E_k(f(x)) A^k_a(x) \omega^a(v).$$

Since $A(x)$ is an orthogonal matrix, we conclude that $f$ is an isometric immersion and that

$$e_a|_{f(x)} = \sum_{k=1}^{n+n'} E_k|_{f(x)} A^k_a(x), \quad 1 \leq a \leq m + m',$$

defines an adapted frame along $f$ with corresponding dual co-frame $\{\omega^a\}_{a=1}^{m+m'}$. Thus, it follows from (4.17) that $\{\omega^a\}_{a=1}^{m+m'}$ are the connection forms. Thus, (4.27) and (4.36) imply that $\{\hat{\theta}^k\}_{k,l=1}^{n+n'}$ are the connection forms in $N$ along $f$ with respect to the left-invariant frame $\{E_k\}_{k=1}^{n+n'}$. The equation (4.40) assures that $\{\hat{\Theta}^k\}_{k,l=1}^{n+n'}$ are the corresponding curvature forms along $f$. Finally, (4.37) guarantees that $\hat{Q}$ is the curvature form in $N$ at points of $f(M)$ associated to the adapted frame $\{e_a\}_{a=1}^{m+m'}$.

The choice of the initial condition $f(x_0) = y_0$ is not a serious restriction, since an isometric immersion with initial condition $y \in N$ is obtained merely composing $f$ and the left translation by $y y_0^{-1}$.

**Part b.** From (1.9) and (1.10) it follows that that there exist local orthonormal frames $\{e_a\}_{a=1}^{m+m'}$ and $\{\tilde{e}_a\}_{a=1}^{m+m'}$ respectively adapted to $f$ and $\tilde{f}$ such that the orthogonal matrices

$$A^k_a = \langle e_a, E_k \rangle, \quad \tilde{A}^k_a = \langle \tilde{e}_a, E_k \rangle$$

satisfy

$$\mu(A) = \mu(\tilde{A}).$$

Moreover, (1.6) and (1.7) imply that the connection forms $\omega$ and $\tilde{\omega}$ for adapted frames along $f$ and $\tilde{f}$ satisfy at corresponding points $\omega = \tilde{\omega}$.

Finally, (5.3), (2.22) and (2.17) imply that the Christoffel tensors $\lambda$ and $\tilde{\lambda}$ associated to these adapted frames are equal at corresponding points. We then conclude that $A$ and $\tilde{A}$ both satisfy the equation

$$A^{-1}dA = \omega - \lambda$$

Now, left translation by $f(x_0)\tilde{f}(x_0)^{-1}$ followed by a suitable rotation in $T_{f(x_0)}N$, if necessary, assure that we may suppose that $A(x_0) = \tilde{A}(x_0)$. Hence, the uniqueness of Darboux primitives in a simply connected domain implies that $A = \tilde{A}$.

Thus, we have

$$\omega_m|_{f(x)}(f_*e_a) = \sum_k \tilde{e}_k A^k_a$$

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and
\[ \omega_m |_{\tilde{f}(x)} (\tilde{f}_s e_a) = \sum_k \tilde{e}_k A_k^a. \] (5.6)

Therefore, \( f \) and \( \tilde{f} \) describe integral leaves of the distribution \( \mathcal{P} \) we defined above passing through the point \( f(x_0) \in N \). The uniqueness part of Frobenius’s theorem implies that \( f = \tilde{f} \).

This finishes the proof of Theorem 1.

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