SADDLE POINT LEAST SQUARES PRECONDITIONING
OF MIXED METHODS

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ABSTRACT. We present a simple way to discretize and precondition mixed variational formulations. Our theory connects with, and takes advantage of, the classical theory of symmetric saddle point problems and the theory of preconditioning symmetric positive definite operators. Efficient iterative processes for solving the discrete mixed formulations are proposed and choices for discrete spaces that are always compatible are provided. For the proposed discrete spaces and solvers, a basis is needed only for the test spaces and assembly of a global saddle point system is avoided. We prove sharp approximation properties for the discretization and iteration errors and also provide a sharp estimate for the convergence rate of the proposed algorithm in terms of the condition number of the elliptic preconditioner and the discrete inf sup and sup sup constants of the pair of discrete spaces.

1. Introduction

We provide a general approach in preconditioning mixed problems of the form: Given \( f \in V^* \), find \( p \in Q \) such that
\[
(1.1) \quad b(v, p) = \langle f, v \rangle \quad \text{for all } v \in V,
\]
where \( V \) and \( Q \) are Hilbert spaces and \( b(\cdot, \cdot) \) is a continuous bilinear form on \( V \times Q \) satisfying an inf sup condition. In \([5, 12]\), a connection was made between problems of the form \((1.1)\) and a natural saddle point formulation. More specifically, if \( a(\cdot, \cdot) \) is the inner product on \( V \), then \( p \) is the unique solution of \((1.1)\) if and only if \((u = 0, p)\) is the unique solution to: Find \((u, p) \in V \times Q\) such that
\[
(1.2) \quad \begin{align*}
a(u, v) + b(v, p) &= \langle f, v \rangle \quad \text{for all } v \in V, \\
b(u, q) &= 0 \quad \text{for all } q \in Q,
\end{align*}
\]
where appropriate assumptions on \( f \) and the form \( b(\cdot, \cdot) \) hold, see Section 2.1. Thus, \((1.2)\) is a saddle point reformulation of \((1.1)\). It is clear that the solution component \( p \) of this reformulation is independent of the inner product (hence the norm) considered on \( V \). This observation is essential for the discretization and preconditioning of \((1.1)\).

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For finite dimensional approximation spaces \( V_h \subset V \) and \( M_h \subset Q \) satisfying a discrete inf-sup condition, we consider the discrete problem of finding \((u_h, p_h) \in V_h \times M_h\) such that
\[
\begin{align*}
  a(u_h, v_h) + b(v_h, p_h) &= \langle f, v_h \rangle & \text{for all } v_h \in V_h, \\
  b(u_h, q_h) &= 0 & \text{for all } q_h \in M_h,
\end{align*}
\]
which approximates the solution \((u = 0, p)\) of (1.2). The discrete variational formulation (1.3) is in fact a saddle point least squares discretization of (1.1), see Section 2.2. This saddle point discretization of (1.1) is also adapted by Demkowicz and Gopalakrishnan in [13, 14]. When solving the above problem, finding bases for the discrete trial space \( M_h \) and assembling a block stiffness matrix for (1.3) can be avoided by applying an Uzawa type algorithm. However, any attempt to solve (1.3) by an Uzawa iterative process requires the exact inversion of the operator \( A_h \) associated with the inner product \( a(\cdot, \cdot) \) on \( V_h \). To avoid exact inversion and to speed up the iterative solvers, we consider another form \( \tilde{a}(\cdot, \cdot) \) on \( V_h \), which leads to an equivalent norm on \( V_h \), and introduce a preconditioned discrete saddle point problem: Find \((\tilde{u}_h, \tilde{p}_h) \in V_h \times M_h\) such that
\[
\begin{align*}
  \tilde{a}(\tilde{u}_h, v_h) + b(v_h, \tilde{p}_h) &= \langle f, v_h \rangle & \text{for all } v_h \in V_h, \\
  b(\tilde{u}_h, q_h) &= 0 & \text{for all } q_h \in M_h,
\end{align*}
\]
where the action of the operator \( \tilde{A}_h^{-1} \) associated with the inner product \( \tilde{a}(\cdot, \cdot) \) on \( V_h \) is assumed to be fast and easy to implement.

The goals of this paper are: describe how well the component solution \( \tilde{p}_h \) of (1.4) approximates the solution \( p \) of (1.1), describe possible choices for the discrete pairs \((V_h, M_h)\), and propose an efficient iterative solver for (1.4) and estimate its convergence rate.

The paper is organized as follows. In Section 2, an abstract theory for saddle point least squares formulations is presented. Section 3 describes the general preconditioning theory and approximation results. In addition, convergence rates for the proposed iterative solver are estimated. Possible choices for discrete pairs of spaces are discussed in Section 4.

2. Abstract Saddle Point Least Squares Formulation for Mixed Methods

2.1. Notation and the continuous problem. Let \( V \) and \( Q \) be infinite dimensional Hilbert spaces and assume the inner products \( a(\cdot, \cdot) \) and \( \langle \cdot, \cdot \rangle \) induce the norms \( \| \cdot \|_V = | \cdot | = a(\cdot, \cdot)^{1/2} \) and \( \| \cdot \|_Q = | \cdot | = (\cdot, \cdot)^{1/2} \). The duals of \( V \) and \( Q \) will be denoted by \( V^* \) and \( Q^* \), respectively. The dual pairings on \( V^* \times V \) and \( Q^* \times Q \) will both be denoted by \( \langle \cdot, \cdot \rangle \). With the inner products \( a(\cdot, \cdot) \) and \( \langle \cdot, \cdot \rangle \), we associate the operators \( A : V \to V^* \) and \( C : Q \to Q^* \) defined by
\[
\langle Au, v \rangle = a(u, v) \quad \text{for all } u, v \in V,
\]
and
\[ (Cp, q) = (p, q) \quad \text{for all } p, q \in Q. \]

Assume that \( b(\cdot, \cdot) \) is a continuous bilinear form on \( V \times Q \) satisfying the inf-sup condition
\[ \inf_{p \in Q} \sup_{v \in V} \frac{b(v, p)}{|v| \|p\|} = m > 0, \tag{2.1} \]
and is bounded, i.e.,
\[ \sup_{p \in Q} \sup_{v \in V} \frac{b(v, p)}{|v| \|p\|} = M < \infty. \tag{2.2} \]

With the form \( b(\cdot, \cdot) \), we associate the linear operators \( B : V \to Q^* \) and \( B^* : Q \to V^* \) defined through the duality pairings
\[ \langle Bv, q \rangle = b(v, q) = \langle B^* q, v \rangle \quad \text{for all } v \in V, q \in Q. \]

It is well known that if, in addition to the assumptions on \( b(\cdot, \cdot) \), \( f \) satisfies the compatibility condition
\[ \langle f, v \rangle = 0 \quad \text{for all } v \in V_0 := \{ v \in V \mid b(v, q) = 0, \text{ for all } q \in Q \}, \tag{2.3} \]
then \((1.1)\) has a unique solution \( p \), see e.g. [1, 2]. Furthermore, \((u = 0, p)\) is the unique solution of \((1.2)\).

**Remark 2.1.** The saddle point problem \((1.2)\) has a unique solution \((u, p)\) regardless of the compatibility condition \((2.3)\). The operator form of problem \((1.1)\) is equivalent to finding \( p \in Q \) such that
\[ A^{-1}B^*p = A^{-1}f, \]
and solving for \( p \) from \((1.2)\) gives
\[ (C^{-1}B)(A^{-1}B^*)p = (C^{-1}B)A^{-1}f. \]

Since \( C^{-1}B \) is the Hilbert transpose of \( A^{-1}B^* \), we have that the \( p \) component of the solution of \((1.2)\) is the least squares solution of \((1.1)\).

For the rest of this paper we assume that the compatibility condition \((2.3)\) holds, and consequently, problem \((1.1)\) has a unique solution.

### 2.2. Saddle point least squares discretization.

Let \( V_h \subset V \) and \( \mathcal{M}_h \subset Q \) be finite dimensional approximation spaces and \( A_h \) be the discrete version of the operator \( A \), i.e., \( A_h \) satisfies
\[ \langle A_h u_h, v_h \rangle = a(u_h, v_h) \quad \text{for all } u_h, v_h \in V_h. \]

We define the discrete operators \( B_h : V_h \to \mathcal{M}_h \) and \( B_h^* : \mathcal{M}_h \to V_h^* \) by
\[ (B_h v_h, q_h) = b(v_h, q_h) = \langle B_h^* q_h, v_h \rangle \quad \text{for all } v_h \in V_h, q_h \in \mathcal{M}_h. \]

Note that the operator \( B_h \) is defined using the inner product on \( \mathcal{M}_h \) and not with the duality on \( \mathcal{M}_h^* \times \mathcal{M}_h \). Thus, we can define the discrete Schur
Hence, under assumption (2.4) restriction of \( f \). It is well known that the spectrum of \( S \) and that (2.5)
\[
\sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \|p_h\|} < M < \infty,
\]
and that \( m_h^2, M_h^2 \) are (the extreme) eigenvalues of \( S_h \). Define
\[
V_{h,0} := \{ v_h \in V_h \mid b(v_h, q_h) = 0 \quad \text{for all} \quad q_h \in M_h \},
\]
to be the kernel of the discrete operator \( B_h \). We define \( f_h \in V^*_h \) to be the restriction of \( f \) to \( V_h \), i.e., \( \langle f_h, v_h \rangle := \langle f, v_h \rangle \) for all \( v_h \in V_h \).

**Remark 2.2.** In the case \( V_{h,0} \subset V \), the compatibility condition (2.3) implies the discrete compatibility condition
\[
\langle f, v_h \rangle = 0 \quad \text{for all} \quad v_h \in V_{h,0}.
\]

Hence, under assumption (2.4), the problem of finding \( p_h \in M_h \) such that (2.6)
\[
b(v_h, p_h) = \langle f, v_h \rangle, \quad \text{for all} \quad v_h \in V_h, \quad \text{or} \quad B^*_h p_h = f_h, \quad \text{or} \quad A_h^{-1} B^*_h p_h = A_h^{-1} f_h,
\]
has a unique solution. In general, (2.3) may not hold on \( V_{h,0} \) and problem (2.6) may not be well-posed. However, if the form \( b(\cdot, \cdot) \) satisfies (2.4), then the problem of finding \( (u_h, p_h) \in V_h \times M_h \) satisfying (1.3) does have a unique solution. Solving for \( p_h \) from (1.3), we obtain
\[
S_h p_h = B_h(A_h^{-1} B^*_h) p_h = B_h A_h^{-1} f_h.
\]
Since the Hilbert transpose of \( B_h \) is \( B_h^T = A_h^{-1} B^*_h \), we call the component \( p_h \) of the solution \( (u_h, p_h) \) of (1.3) the saddle point least squares approximation of the solution \( p \) of the original mixed problem (1.1).

The following error estimate for \( \|p - p_h\| \) was proved in [6].

**Theorem 2.3.** Let \( b : V \times Q \to \mathbb{R} \) satisfy (2.1) and (2.2) and assume that \( f \in V^* \) is given and satisfies (2.3). Assume that \( p \) is the solution of (1.1) and \( V_h \subset V, M_h \subset Q \) are chosen such that the discrete inf – sup condition (2.4) holds. If \( (u_h, p_h) \) is the solution of (1.3), then the following error estimate holds:
\[
\frac{1}{M} \|u_h\| \leq \|p - p_h\| \leq \frac{M}{m_h} \inf_{q_h \in M_h} \|p - q_h\|.
\]
2.3. An Uzawa CG iterative solver. Note that a global linear system may be difficult to assemble when solving \((1.3)\) as bases for the trial spaces \(\mathcal{M}_h\), which are chosen to satisfy \((2.4)\), may be difficult to find. Nevertheless, we can solve \((1.3)\) and avoid building a basis for \(\mathcal{M}_h\) by using an Uzawa type algorithm, e.g., the Uzawa Conjugate Gradient (UCG) algorithm.

**Algorithm 2.4.** (UCG) Algorithm

**Step 1:** Choose any \(p_0 \in \mathcal{M}_h\). Compute \(u_1 \in V_h, q_1, d_1 \in \mathcal{M}_h\) by

\[
a(u_1, v_h) = \langle f, v_h \rangle - b(v_h, p_0) \quad \text{for all } v_h \in V_h,
\]

\[
(q_1, q_h) = b(u_1, q_h) \quad \text{for all } q_h \in \mathcal{M}_h, \quad d_1 := q_1.
\]

**Step 2:** For \(j = 1, 2, \ldots\), compute \(h_j, \alpha_j, p_j, u_{j+1}, q_{j+1}, \beta_j, d_{j+1}\) by

\[(UCG1) \quad a(h_j, v_h) = -b(v_h, d_j) \quad \text{for all } v_h \in V_h
\]

\[(UCG\alpha) \quad \alpha_j = -\frac{(q_j, q_j)}{b(h_j, q_j)}
\]

\[(UCG2) \quad p_j = p_{j-1} + \alpha_j d_j
\]

\[(UCG3) \quad u_{j+1} = u_j + \alpha_j h_j
\]

\[(UCG4) \quad (q_{j+1}, q_h) = b(u_{j+1}, q_h) \quad \text{for all } q_h \in \mathcal{M}_h
\]

\[(UCG\beta) \quad \beta_j = \frac{(q_{j+1}, q_{j+1})}{(q_j, q_j)}
\]

\[(UCG6) \quad d_{j+1} = d_{j+1} + \beta_j d_j.
\]

Note that the only inversions needed in the algorithm involve the form \(a(\cdot, \cdot)\) in **Step 1** and \((UCG1)\). In operator form, these steps become

\[(2.9) \quad u_1 = A_h^{-1}(f_h - B_h^* p_0), \quad \text{and} \quad h_j = -A_h^{-1}(B_h^* d_j),
\]

respectively. In practical implementations of Algorithm \((2.4)\) we would like to replace the action of \(A_h^{-1}\) with the action of a suitable preconditioner. The properties of the new preconditioned algorithm are discussed in the next section. The following sharp error estimation result was proved in \([3]\).

**Theorem 2.5.** If \((u_h, p_h)\) is the discrete solution of \((1.3)\) and \((u_{j}, p_{j-1})\) is the \(j^{th}\) iteration for Algorithm \((2.4)\), then \((u_{j}, p_{j-1}) \rightarrow (u_h, p_h)\) and

\[
\frac{1}{M^2} \|q_j\| \leq \|p_{j-1} - p_h\| \leq \frac{1}{m^2} \|q_j\|,
\]

\[(2.10) \quad \frac{m^2}{M^2} \|q_j\| \leq \|u_j - u_h\| \leq \frac{M^2}{m^2} \|q_j\|.
\]

**Remark 2.6.** In particular, Algorithm \((2.4)\) recovers the steps of the conjugate gradient algorithm for solving the Schur complement problem \((2.7)\). Hence, the rate of convergence for the iteration error \(\|p_j - p_h\|_{S_h}\) or \(\|p_j - p_h\|\) depends on the condition number of \(S_h\), which is \(\kappa(S_h) = \frac{M^2}{m^2}\).
3. Preconditioning techniques

In this section, we develop a general preconditioning framework to approximate the solution of (1.1) based on (1.3) and elliptic preconditioning of the operator associated with the inner product on $V_h$. More precisely, we replace the original form $a(\cdot, \cdot)$ in (1.3) with a uniformly equivalent form $\tilde{a}(\cdot, \cdot)$ on $V_h$ that leads to an implementably fast operator $\tilde{A}_h^{-1}$. We assume that $V_h \subset V$ and $M_h \subset Q$ are finite dimensional approximation spaces satisfying (2.4) and (2.5).

3.1. The preconditioned saddle point problem. First, we introduce a general preconditioner operator $P_h : V_h^* \rightarrow V_h$ that is equivalent to $A_h^{-1}$ in the following sense

\begin{align}
\langle g, P_h f \rangle = \langle f, P_h g \rangle \quad \text{for all } f, g \in V_h^*,
\end{align}

and

\begin{align}
m_1^2 |v_h|^2 \leq a(P_h A_h v_h, v_h) \leq m_2^2 |v_h|^2,
\end{align}

where the positive constants $m_1^2, m_2^2$ are the smallest and largest eigenvalues of $P_h A_h$, respectively.

**Remark 3.1.** Assumption (3.2) gives us that the condition number of $P_h A_h$ satisfies

\begin{align}
\kappa(P_h A_h) = \frac{m_2^2}{m_1^2}.
\end{align}

With the preconditioner $P_h : V_h^* \rightarrow V_h$, we define the form $\tilde{a} : V_h \times V_h \rightarrow \mathbb{R}$ by

\begin{align}
\tilde{a}(u_h, v_h) := a((P_h A_h)^{-1} u_h, v_h) \quad \text{for all } u_h, v_h \in V_h.
\end{align}

**Proposition 3.2.** Under assumptions (3.1) and (3.2), we have that $\tilde{a}(\cdot, \cdot)$ is symmetric and equivalent with $a(\cdot, \cdot)$ on $V_h$.

**Proof.** For symmetry, it suffices to prove that $P_h A_h$ is symmetric w.r.t. the $a(\cdot, \cdot)$ inner product. From the definition of the operator $A_h$ and (3.1), we have

\begin{align*}
a(P_h A_h u_h, v_h) &= \langle A_h v_h, P_h A_h u_h \rangle = \langle A_h u_h, P_h A_h v_h \rangle \\
&= a(u_h, P_h A_h v_h).
\end{align*}

For the equivalence, note that (3.2) and (3.4) imply

\begin{align}
\frac{1}{m_2^2} |v_h|^2 \leq \tilde{a}(v_h, v_h) \leq \frac{1}{m_1^2} |v_h|^2.
\end{align}

\qed
By Proposition 3.2, \( \tilde{a}(\cdot, \cdot) \) defines an equivalent inner product on \( V_h \). Let \( |v_h|_P := \tilde{a}(v_h, v_h)^{1/2} \) be the norm induced by the inner product \( \tilde{a}(\cdot, \cdot) \) and define the operator \( \tilde{A}_h : V_h \to V_h^* \) by
\[
\langle \tilde{A}_h u_h, v_h \rangle := \tilde{a}(u_h, v_h) \quad \text{for all } u_h, v_h \in V_h.
\]
Note that for any \( u_h, v_h \in V_h \)
\[
\langle \tilde{A}_h u_h, v_h \rangle = \tilde{a}(u_h, v_h) = a((P_h A_h)^{-1} u_h, v_h)
\]
which implies \( \tilde{A}_h = A_h (P_h A_h)^{-1} = P_h^{-1} \). Hence, we can view \( \tilde{a}(\cdot, \cdot) \) as a preconditioned version of the form \( a(\cdot, \cdot) \). The preconditioned discrete saddle point problem consists of finding \( (\tilde{u}_h, \tilde{p}_h) \in V_h \times M_h \) such that (1.4) holds. To simplify the notation, we will drop the \( \tilde{\cdot} \) notation from \((\tilde{u}_h, \tilde{p}_h)\).

Thus, for the remainder of this paper, the preconditioned saddle point least squares formulation is: Find \( (u_h, p_h) \in V_h \times M_h \) such that
\[
\begin{align*}
\tilde{a}(u_h, v_h) + b(v_h, p_h) &= (f, v_h) \quad \text{for all } v_h \in V_h, \\
b(u_h, q_h) &= 0 \quad \text{for all } q_h \in M_h.
\end{align*}
\]
(3.6)

Using that \( V_h \subset V \) and \( M_h \subset Q \) satisfy (2.4) and (2.5), we obtain
\[
\tilde{m}_h := \inf_{p_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h|_P \|p_h\|} \geq m_1 m_h > 0,
\]
and
\[
\tilde{M}_h := \sup_{p_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h|_P \|p_h\|} \leq m_2 M_h \leq m_2 M.
\]
(3.7) (3.8)
Hence, the preconditioned saddle point least squares formulation (3.6) has a unique solution.

The Schur complement associated with problem (3.6) is
\[
\tilde{S}_h = B_h \tilde{A}_h^{-1} B_h^* = B_h P_h B_h^*.
\]
Solving for \( p_h \) from (3.6), we obtain
\[
\tilde{S}_h p_h = B_h (P_h B_h^*) p_h = B_h P_h f_h.
\]
(3.9)
We call the component \( p_h \) of the solution \( (u_h, p_h) \) of (3.6) the (preconditioned) saddle point least squares approximation of the solution \( p \) of the original mixed problem (1.1). To estimate \( \|p - p_h\| \) in this case, we will prove the analog to Theorem 2.3 based on the Xu-Zikatanov argument, see [20].

**Theorem 3.3.** Let \( b : V \times Q \to \mathbb{R} \) satisfy (2.1) and (2.2) and assume that \( f \in V^* \) is given and satisfies (2.3). Assume that \( V_h \subset V, M_h \subset Q \) are chosen such that the discrete inf–sup condition (2.4) holds. If \( p \) is the solution of (1.1) and \( (u_h, p_h) \) is the solution of (3.6), then the following error estimate holds:
\[
\frac{1}{M m_2^2} |u_h| \leq \|p - p_h\| \leq \frac{M m_2}{m_1 m_1 q_h \in M_h} \inf \|p - q_h\|.
\]
(3.10)
Proof. Define the operator $T_h : Q \to Q$ by $T_h p = p_h$. Note that $T_h$ is linear and idempotent. To show the latter, consider the problem: Find $(u_h^*, p_h^*) \in V_h \times M_h$ such that

$$
\begin{align*}
\tilde{a}(u_h^*, v_h) + b(v_h, p_h^*) &= b(v_h, p_h) \quad \text{for all } v_h \in V_h, \\
b(u_h^*, q_h) &= 0 \quad \text{for all } q_h \in M_h.
\end{align*}
$$

(3.11)

Since $b$ satisfies (2.2), we have that (3.7) is satisfied as described above. Thus, problem (3.11) has a unique solution. Since $(u_h^*, p_h^*) = (0, p_h)$ solves the problem, we conclude $T_h p_h = p_h$ which gives us $T_h^2 = T_h$. From Kato [17] and Xu and Zikatanov [20], this implies

$$
\|I - T_h\|_{L(Q, Q)} = \|T_h\|_{L(Q, Q)}.
$$

Using the above equality, for an arbitrary $q_h \in M_h$ we have

$$
\|p - q_h\| \leq \|(I - T_h)p\| = \|(I - T_h)(p - q_h)\| \leq \|T_h\| \|p - q_h\|.
$$

(3.12)

We now estimate $\|T_h\|$. First, define $\tilde{V}_h, 0$ to be the orthogonal complement of $V_h, 0$ w.r.t. the $\tilde{a}(\cdot, \cdot)$ inner product. Note that from the first equation of (3.6) and the fact $p$ solves (1.1) we have that

$$
b(v_h, p_h) = b(v_h, p) - \tilde{a}(u_h, v_h).
$$

(3.13)

Also, since $b$ satisfies (2.2) we have that (3.8) holds. Hence, from (3.7), (3.8), and (3.13) we obtain

$$
\|T_h p\| \leq \frac{1}{m_h m_1} \sup_{v_h \in V_h} b(v_h, T_h p) = \frac{1}{m_h m_1} \sup_{v_h \in \tilde{V}_h, 0} b(v_h, p) \leq \frac{1}{m_h m_1} \sup_{v_h \in \tilde{V}_h, 0} \frac{b(v_h, p) - \tilde{a}(u_h, v_h)}{|v_h|_P} \leq \frac{M m_2}{m_h m_1} \|p\|.
$$

(3.14)

The right inequality now follows from (3.12) and (3.14). For the left inequality, note that

$$
|u_h|_P = \sup_{v_h \in V_h} \frac{\tilde{a}(u_h, v_h)}{|v_h|_P} = \sup_{v_h \in V_h} \frac{b(v_h, p) - p_h}{|v_h|_P} \leq M m_2 \|p - p_h\|,
$$

and

$$
|u_h| \leq m_2 |u_h|_P.
$$

□

3.2. An iterative solver for the preconditioned variational formulation. We use a modified version of Algorithm 2.4 to solve (3.6) by replacing the form $a(\cdot, \cdot)$ by $\tilde{a}(\cdot, \cdot)$ in Step 1 and (UCG1). With this modification, we obtain the following (Uzawa) Preconditioned Conjugate Gradient (PCG) algorithm for mixed methods.
Algorithm 3.4. (PCG) Algorithm for Mixed Methods

Step 1: Choose any $p_0 \in \mathcal{M}_h$. Compute $u_1 \in V_h$, $q_1, d_1 \in \mathcal{M}_h$ by

$$
\begin{align*}
    u_1 &= P_h(f_h - B_h^* p_0) \\
    q_1 &= B_h u_1, \quad d_1 := q_1.
\end{align*}
$$

Step 2: For $j = 1, 2, \ldots,$ compute $h_j, \alpha_j, p_j, u_{j+1}, q_{j+1}, \beta_j, d_{j+1}$ by

$$
\begin{align*}
    h_j &= - P_h(B_h^* d_j) \\
    \alpha_j &= - \frac{(q_j, q_j)}{b(h_j, q_j)} \\
    p_j &= p_{j-1} + \alpha_j d_j \\
    u_{j+1} &= u_j + \alpha_j h_j \\
    q_{j+1} &= B_h u_{j+1}, \\
    \beta_j &= \frac{(q_{j+1}, q_{j+1})}{(q_j, q_j)} \\
    d_{j+1} &= q_{j+1} + \beta_j d_j.
\end{align*}
$$

Note that only the actions of $P_h$, $B_h$, and $B_h^*$ are needed in the above algorithm. For any preconditioner $P_h$ and trial space $\mathcal{M}_h$ that is not defined via a global projection, these actions do not involve inversion processes, see Section 3.3 for the case $P_h$—an additive multilevel Schwarz preconditioner.

Similar to the remark in Section 2.3, we have the following:

Remark 3.5. Algorithm 3.4 recovers in particular the steps of the conjugate gradient algorithm for solving the problem (3.9). Hence, the rate of convergence for $\|p_j - p_h\|_{\tilde{S}_h}$ or $\|p_j - p_h\|$ depends on the condition number of $\tilde{S}_h$, which is $\kappa(\tilde{S}_h) = \frac{\tilde{m}_2^2}{m_h^2}$.

The following result is analogous to Theorem 2.5.

Theorem 3.6. If $(u_h, p_h)$ is the discrete solution of (3.6) and $(u_j, p_{j-1})$ is the $j$th iteration for Algorithm 3.4, then $(u_j, p_{j-1}) \to (u_h, p_h)$ and

$$
\begin{align*}
    \frac{1}{M^2} \frac{1}{m^2} \|q_j\| &\leq \|p_{j-1} - p_h\| \leq \frac{1}{m_h^2} \frac{1}{m_1^2} \|q_j\|, \\
    \frac{m_h m_1^2}{M^2 m_1^2} \|q_j\| &\leq \|u_j - u_h\| \leq \frac{M m_1^2}{m_h^2 m_1^2} \|q_j\|.
\end{align*}
$$

Proof. By induction over $j$, we have that

$$
\tilde{a}(u_j, v_h) + b(v_h, p_{j-1}) = \langle f, v_h \rangle \quad \text{for all } v_h \in V_h.
$$

Combining this with the first equation of (3.6) gives us

$$
\tilde{a}(u_j - u_h, v_h) = b(v_h, p_h - p_{j-1}) \quad \text{for all } v_h \in V_h.
$$
Note that \( \sigma(\tilde{S}_h) \subset [\tilde{m}_h^2, \tilde{M}_h^2] \). Hence,
\[
\tilde{m}_h \| q_h \| = (\tilde{S}_h q_h, q_h)^{1/2} \leq \tilde{M}_h \| q_h \| \quad \text{for all } q_h \in \mathcal{M}_h.
\]
By substituting \( v_h = \tilde{A}_h^{-1}B_h^*(p_h - p_{j-1}) \) into (3.16),
\[
|u_j - u_h|^2_p = (\tilde{S}_h(p_h - p_{j-1}), p_h - p_{j-1}) = \| p_h - p_{j-1} \|^2_{\tilde{S}_h}.
\]
The above equality, (3.15), and (3.17) gives us that
\[
m_1 \tilde{m}_h \| p_h - p_{j-1} \| \leq |u_j - u_h| \leq m_2 \tilde{M}_h \| p_h - p_{j-1} \|.
\]
From (PCG4), the second equation of (3.6), and (3.16) we have that
\[
q_j = B_h u_j = B_h (u_j - u_h) = \tilde{S}_h (p_h - p_{j-1}).
\]
Thus,
\[
\tilde{m}_h^2 \| p_h - p_{j-1} \| \leq \| \tilde{S}_h (p_h - p_{j-1}) \| = \| q_j \| \leq \tilde{M}_h^2 \| p_h - p_{j-1} \|.
\]
The inequalities (3.15) follow from (3.18), (3.19), and the fact that \( \tilde{m}_h \geq m_k m_1 \) and \( \tilde{M}_h \leq M m_2 \). From Remark 3.5 and the standard estimate for the convergence rate of the conjugate gradient algorithm, \([8, 15]\), we have that
\[
\| p_h - p_j \|_{\tilde{S}_h} \leq 2 \left( \frac{\tilde{M}_h - \tilde{m}_h}{\tilde{M}_h + \tilde{m}_h} \right)^j \| p_h - p_0 \|_{\tilde{S}_h}.
\]
Hence, \( p_j \rightarrow p_h \). From (3.15), we conclude that \( u_j \rightarrow u_h \) as well. \( \square \)

The following estimates are a direct consequence of (3.7), (3.8), (3.20), and the formula \( \kappa(\tilde{S}_h) = \frac{\tilde{M}_h^2}{\tilde{m}_h^2} \).

**Proposition 3.7.** The condition number of the Schur complement \( \tilde{S}_h = B_h P_h B_h^* \) satisfies
\[
\kappa(\tilde{S}_h) \leq \frac{M_h^2 m_2^2}{\tilde{m}_h^2 m_1^2} = \kappa(S_h) \cdot \kappa(P_h A_h).
\]
Consequently, the convergence rate \( \rho_h \) for \( \| p_j - p_h \|_{\tilde{S}_h} \) satisfies
\[
\rho_h \leq \frac{M_h m_2^2}{\tilde{m}_h m_1^2} \leq \frac{1}{\tilde{m}_h m_1^2 + 1}.
\]

**Remark 3.8.** We can relate our preconditioned SPLS discretization method for solving the general mixed problem (1.1) with the Bramble-Pasciak least squares approach presented in [9]. In our notation, the Bramble-Pasciak least squares discretization can be formulated as: Find \( p_h \in \mathcal{M}_h \) such that
\[
b(A_h^{-1}B_h^* q_h, p_h) = (f_h, A_h^{-1}B_h^* q_h) = b(A_h^{-1}f_h, q_h) \quad \text{for all } q_h \in \mathcal{M}_h.
\]
With a suitable preconditioner \( P_h \) replacing \( A_h^{-1} \), the problem becomes: Find \( p_h \in \mathcal{M}_h \) such that
\[
b(P_h B_h^* q_h, p_h) = b(P_h f_h, q_h) \quad \text{for all } q_h \in \mathcal{M}_h.
\]

We shall note that (3.22) is equivalent to our Schur complement problem (3.9). While we arrive at essentially the same normal equation for solving (2.6), our saddle point approach is more direct and allows sharper error estimates for the error \( \| p - p_h \| \). The two approaches are also essentially different in the way the trial spaces are chosen, see Section 4 for our choices of trial spaces. In [9], to iteratively solve (3.22), bases for both the test and trial spaces are needed. In contrast, we solve the coupled preconditioned saddle point problem (3.6) using Algorithm 3.4 which avoids the need of a basis for the trial space.

3.3. An example of a preconditioner. In order to illustrate the applicability of the theory presented thus far, we consider the case when \( P_h \) is given by the additive multilevel Schwarz or BPX preconditioner, see [10] [11] [21]. Assume that we have a nested sequence of approximation spaces \( V_1 \subset V_2 \subset \cdots \subset V_J = V_h \) and let \( \{ \phi_1^k, \phi_2^k, \ldots, \phi_n^k \} \) be a basis for \( V_k \). For \( f_h \in V_h^* \), the action of \( P_h \) is given by

\[
P_h f_h = \sum_{k=1}^J \sum_{i=1}^{n_k} \frac{\langle f_h, \phi_i^k \rangle a(\phi_i^k, \phi_i^k)}{b(\phi_i^k, \phi_i^k)} \phi_i^k.
\]

It is known that for \( V = H_0^1(\Omega) \) and a nested sequence \( \{ V_k \} \) of piecewise linear functions that, under standard mesh uniformity conditions, \( P_h \) is a preconditioner for \( A_h \) satisfying (3.1) and (3.2), see [10] [16] [18] [19] [21].

In this case, the first equation in (Step 1) of Algorithm 3.4 becomes

\[
u_1 = P_h (f_h - B_h^* p_0) = \sum_{k=1}^J \sum_{i=1}^{n_k} \frac{\langle f_h, \phi_i^k \rangle - b(\phi_i^k, p_0)}{a(\phi_i^k, \phi_i^k)} \phi_i^k.
\]

Furthermore, the iterates for \( h_j \) in (PCG1) are given by

\[
h_j = - \sum_{k=1}^J \sum_{i=1}^{n_k} \frac{b(\phi_i^k, d_j)}{a(\phi_i^k, \phi_i^k)} \phi_i^k,
\]

which implies that

\[
b(h_j, q_j) = - \sum_{k=1}^J \sum_{i=1}^{n_k} \frac{b(\phi_i^k, d_j) b(\phi_i^k, q_j)}{a(\phi_i^k, \phi_i^k)},
\]

in (PCG\(\alpha\)). Thus, the implementation of Algorithm 3.4 does not involve matrix inversion. Certainly, any elliptic preconditioner, including the standard multigrid ones, can be used for \( P_h \). We decided to show details of a general additive multilevel Schwarz (or BPX) preconditioner to emphasize the simplicity of implementation when dealing with mixed methods preconditioning. More details on implementing the matrix action of multilevel preconditioners (including BPX) can be found in [19].
4. Discrete spaces that satisfy an inf − sup condition

In this section, we describe two pairs of discrete spaces, introduced in [6], which satisfy the discrete inf − sup condition (2.4) in the general abstract framework of Section 2. In light of (3.21), we would like to provide families of spaces \{V_h, M_h\} such that \(\kappa(S_h)\) is small. Let \(V_h \subset V\) be a finite element test space and assume the action of \(C^{-1}\), where \(C\) was defined in Section 2, is easy to obtain at the continuous level.

4.1. No projection trial space. The first choice defines \(M_h \subset Q\) by

\[M_h := C^{-1}BV_h.\]

In this case, \(V_{h,0} \subset V_0\) and a discrete inf − sup condition holds. Indeed, for a generic \(p_h = C^{-1}Bw_h \in M_h\) where \(w_h \in V_{h,0}\), we have

\[m_{h,0} := \inf_{p_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{\|p_h\|} = \inf_{w_h \in V_{h,0}} \sup_{v_h \in V_h} \frac{(C^{-1}Bv_h, C^{-1}Bw_h)}{\|v_h\| \|C^{-1}Bw_h\|} \geq \inf_{w_h \in V_{h,0}} \frac{\|C^{-1}Bw_h\|^2}{\|w_h\| \|C^{-1}Bw_h\|} = \inf_{w_h \in V_{h,0}} \frac{\|C^{-1}Bw_h\|}{\|w_h\|} > 0.\]  

(4.1)

Hence, by Remark 2.2 we have that (2.6) has a unique solution \(p_h \in M_h\) and \((u_h = 0, p_h)\) solves (1.3). In this case, \(p_h\) is an optimal approximation to the solution \(p\) of (1.1). Indeed, for any \(v_h \in V_h\)

\[0 = b(v_h, p - p_h) = (Bv_h, p - p_h) = (C^{-1}Bv_h, p - p_h).\]

Thus, \(p_h\) is the orthogonal projection of \(p\) onto \(M_h\) which implies

\[\|p - p_h\| = \inf_{q_h \in M_h} \|p - q_h\|.\]  

(4.2)

While (4.2) gives optimal approximation error, to efficiently approximate \(p_h\) using Algorithm 2.4 or 3.4 it requires spaces \{\(V_h, M_h\)\} for which \(\kappa(S_h) = M^2/h^2\) is small or independent of \(h\).

4.2. Projection type trial space. The second choice defines \(M_h \subset Q\) by

\[M_h := R_h C^{-1}BV_h,\]

where \(R_h : Q \to \tilde{M}_h\) is defined by

\[(R_h p, q_h)_h := (p, q_h)\]  

for all \(q_h \in \tilde{M}_h,\)

and \(\tilde{M}_h\) is a finite dimensional subspace of \(Q\) equipped with the inner product \((\cdot, \cdot)_h\).

Remark 4.1. If the \((\cdot, \cdot)_h\) inner product coincides with the inner product on \(Q\), then by definition \(R_h\) is the orthogonal projection onto \(\tilde{M}_h\).
In general, the inner product on $\tilde{M}_h$ could be different from the inner product on $Q$, but we assume that $(\cdot, \cdot)_h$ induces an equivalent norm independent of $h$. The following proposition provides a sufficient condition on $R_h$ which implies well-posedness of problems (1.3) and (2.6) and relates the stability of the family of spaces $\{(V_h, R_hC^{-1}BV_h)\}$ with the stability of the family of spaces $\{(V_h, C^{-1}BV_h)\}$ defined in Section 4.1.

Proposition 4.2. Assume that

$$
\|R_h q_h\|_h \geq \tilde{c}\|q_h\| \quad \text{for all } q_h \in C^{-1}BV_h,
$$

with a constant $\tilde{c}$ independent of $h$. Then $V_{h,0} \subset V_0$. Furthermore, the stability of the family $(V_h, C^{-1}BV_h)$ implies the stability of the family $(V_h, R_hC^{-1}BV_h)$.

Proof. Let $v_h \in V_{h,0}$. Then, for any $p_h \in M_h$,

$$
0 = b(v_h, p_h) = (C^{-1}Bv_h, p_h) = (R_hC^{-1}Bv_h, p_h)_h.
$$

Taking $p_h = R_hC^{-1}Bv_h$ gives us $\|R_hC^{-1}Bv_h\|_h = 0$ and the inclusion $V_{h,0} \subset V_0$ follows from (4.1). For the stability, note that for a generic function $p_h = R_hC^{-1}Bw_h \in M_h$, where $w_h \in V_{h,0}$, we have

$$
m_h = \inf_{p_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{\|v_h\|_h} = \inf_{w_h \in V_{h,0}} \sup_{v_h \in V_h} \frac{(C^{-1}Bv_h, R_hC^{-1}Bw_h)_h}{\|v_h\| \|R_hC^{-1}Bw_h\|_h}
$$

$$
\geq \inf_{w_h \in V_{h,0}} \frac{\|R_hC^{-1}Bw_h\|_h^2}{\|w_h\| \|R_hC^{-1}Bw_h\|_h}
$$

$$
\geq \tilde{c} \inf_{w_h \in V_{h,0}} \frac{\|C^{-1}Bw_h\|_h}{\|w_h\|} = \tilde{c} m_{h,0},
$$

where $m_{h,0}$ is defined in (4.1). \qed

In this case, we have that $p_h$ is a quasi-optimal approximation of the solution $p$ of (1.1) by Theorem 3.3.

The benefit of using the projection type trial space is that it could lead to a better approximation of the continuous solution $p$. Indeed, for the case when preconditioning is not used, super-convergence of $\|p - p_h\|$ is observed, see [4] [6] [7]. Using uniform preconditioners and Theorems 3.3 and 3.6, we expect the same order of super-convergence for $\|p - p_h\|$.

Remark 4.3. With this choice of trial space, Algorithms 2.4 and 3.4 need to be modified to account for the $(\cdot, \cdot)_h$ inner product on $M_h \subset \tilde{M}_h$. This modification is nothing more than replacing the $(\cdot, \cdot)$ inner product with the $(\cdot, \cdot)_h$ inner product where it appears in the algorithms.
5. Conclusion

We presented a general preconditioning approach to mixed variational formulations of the form (1.1) that relies on the classical theory of symmetric saddle point problems and on the theory of preconditioning symmetric positive definite operators. First, a discrete saddle point variational formulation (1.3), that approximates the solution of the original mixed problem in a least squares sense, is considered. In this formulation, the inner product $a(\cdot, \cdot)$ is replaced by an equivalent bilinear form that give rise to efficient elliptic inversion or preconditioning. An Uzawa preconditioned conjugate gradient algorithm for solving the new saddle point system was proposed that requires bases only for the space $V_h$ and avoids costly inversion processes. Due to the saddle point interpretation of the preconditioned system, we were able to prove sharp approximability properties for the discretization and iteration errors and were able to provide practical estimates for the rate of convergence of the final preconditioned conjugate algorithm. Using a common test space, two choices of compatible discrete spaces were given.

We plan to apply preconditioned saddle point least squares discretization to first order systems of PDEs as well as second order elliptic problems. In addition, we plan to combine this approach with multilevel and adaptive techniques.

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