Conditions of Gravitational Instability in Protoplanetary Disks

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Abstract

Gravitational instability is one of the considerable mechanisms to explain the formation of giant planets. We have studied the gravitational stability in protoplanetary disks around a protostar. The temperature and Toomre’s $Q$ value were calculated by assuming local equilibrium between viscous heating and radiative cooling (local thermal equilibrium). We assumed a constant $\alpha$ viscosity and used a cooling function with realistic opacity. Then, we derived the critical surface density, $\Sigma_c$, that is needed in order for a disk to become gravitationally unstable as a function of $r$. This critical surface density, $\Sigma_c$, is strongly affected by the temperature dependence of the opacity. At a radius of $r_c \sim 20$ AU, where ices form, the value of $\Sigma_c$ changes discontinuously by one order of magnitude. This $\Sigma_c$ is determined only by a local thermal process and the criterion of gravitational instability. By comparing a given surface density profile with $\Sigma_c$, one can discuss the gravitational instability of protoplanetary disks. As an example, we discuss the gravitational instability of two semianalytic models for protoplanetary disks. One is a steady state accretion disk, which is realized after viscous evolution. The other is a disk that has the same angular-momentum distribution as its parent cloud core, which corresponds to the disk that has just formed. As a result, it is found that the disk tends to become gravitationally unstable for $r \geq r_c$ because ices make the disk temperature low. In a region closer to the protostar than $r_c$, it is difficult for a typical protoplanetary disk to fragment because of the high temperature and the large Coriolis force. Based on this result, we conclude that fragmentation near the central star is possible, but difficult.

Key words: accretion, accretion disks — instabilities — stars: planetary systems: formation — planetary systems: protoplanetary disk

1. Introduction

Two major models are currently considered concerning the formation of gaseous giant planets. One is the core accretion model (CA model: Goldreich & Ward 1973; Hayashi 1981; Pollack et al. 1996), and the other is the gravitational instability model (GI model: Cameron 1978; Durisen et al. 2007). In the CA model, massive solid cores are made from dust first, and then gaseous giant planets are formed by the gas accretion onto these cores. In the GI model, a disk around a protostar fragments into pieces by gravitational instability, and these pieces become gaseous giant planets. Recently, extrasolar planets have been discovered by direct imaging (Marois et al. 2008; Kalas et al. 2008). These planets are more massive than Jupiter and farther away from the central star than Neptune. Without fresh ideas, it is difficult for the CA model to explain these planets, because it is thought that the gas disappears from the disk before the formation of a massive solid core (Dodson-Robinson et al. 2009). On the other hand, the GI model has the possibility of making gaseous giant planets far from the central star if the mass of a disk is greater than that of the protostar. Thus, the GI model has attracted our attention to explain their formation.

In order to understand the planet formation by the GI model, it is desirable to know the physical condition for disk fragmentation and the position where the fragmentation occurs. At present, two criteria are frequently used for discussing whether a protoplanetary disk is likely to fragment or not. The first is Toomre’s stability criterion (Toomre 1964),

$$ Q \equiv \frac{c_s \kappa_{ep}}{\pi G \Sigma} > 1, $$

(1)

with the gravitational constant, $G$, epicyclic frequency, $\kappa_{ep}$, sound speed, $c_s$, surface density, $\Sigma$, and stability parameter, $Q$. Toomre (1964) showed that the infinitesimally thin disk is stable if stability criterion (1) is satisfied. It is necessary to violate this criterion in order to cause the fragmentation. The other criterion is Gammie’s cooling criterion (Gammie 2001),

$$ \beta \equiv t_{cool} \Omega = u \left( \frac{du}{dt} \right)^{-1} \Omega \leq \beta_c, $$

(2)

where $\Omega$ is the angular velocity, $t_{cool} = u(du/dt)^{-1}$ the cooling time, $u$ the specific internal energy, $du/dt$ the total cooling rate, $\beta$ the cooling parameter, and $\beta_c$ the critical cooling parameter. Gammie (2001) suggested that rapid cooling is necessary for fragmentation, in addition to violating the Toomre criterion (1). Using shearing sheet simulations with a constant cooling rate and without heating, except for artificial viscosity, he showed that a self-gravitating disk can fragment only if $\beta \leq 3$. If not, the disk cannot fragment because it can be stabilized by internal heating due to turbulence arising from gravitational instability. This state is called “gravitoturbulence.” It is said that heating owing to gravitoturbulent
dissipation automatically controls the $Q$ value to be close to unity. Gravitoturbulence has a potential for stabilization of the system where the Toomre criterion is violated, while it does not arise in a disk that violates the Toomre criterion.

The $\beta_c$ obtained by Gammie (2001) is applicable in a specific case (local, 2D, etc.). Many three-dimensional simulations were carried out in order to obtain the general value of $\beta_c$ (e.g., Rice et al. 2005; Clarke et al. 2007; Cossins et al. 2010). However, each simulation suggested a different $\beta_c$, and a general $\beta_c$ has not yet been clarified. Furthermore, it is not obvious whether or not $\beta_c$ exists as some value. Meru and Bate (2011b) calculated disks with different values of initial parameters, such as the disk size, the surface-density profile, the mass of the disk, and the mass of the central star. From their calculations, it is found that $\beta_c$ is different among those disks with a different set of initial parameters. This result shows that it is not determined only by $\beta$ whether the disk can fragment or not. Moreover, Meru and Bate (2011b) calculated $\beta_c$ with various numerical resolutions. They showed that the disk was capable of fragmentation with a larger $\beta$ if the resolution is better, and that there was no indication of the convergence of $\beta$. From these results, it is not clear that the Gammie criterion is always applicable to discussing the fragmentation of a disk.

Some analytical studies concerning disk fragmentation have been conducted. Rafikov (2005) argued that a protoplanetary disk is unlikely to fragment near a protostar ($r \lesssim 100$ AU) by using the two criteria described above. However, as noted above, the Gammie criterion is uncertain, and thus this argument includes the same uncertainty. On the other hand, the Toomre criterion is probably assured in the sense that a disk violating the Toomre criterion is gravitationally unstable. Other previous analytical studies concerning disk fragmentation (e.g., Clarke 2009; Kratter & Murray-Clay 2011) have assumed gravitoturbulent disks with $Q = 1$. These studies do not have the ability to discuss the position where the Toomre criterion is satisfied. Since the Toomre criterion provides a robust necessary condition for fragmentation, those studies based solely on the Toomre criterion (1) are favorable for understanding the possibility of fragmentation.

In this study, we investigated the gravitational stability in protoplanetary disks based solely on the Toomre criterion, without the uncertainty associated with the Gammie criterion. We have aimed not to consider the time evolution of particular disks, but to derive an instability condition that is useful for various disks. Since recent observations show the diversity of protoplanetary disks and planetary systems, we consider that it is important to derive such a widely useful formulation. In order to calculate Toomre’s $Q$ value with given $\Sigma$ and $\kappa_{\text{op}}$, it is necessary to know the temperature. Thus, it is significant to consider realistic thermal processes. We construct an analytical model for a temperature calculation with a thermal process in section 2. In section 3, we derive the critical surface density, $\Sigma_c$, that is needed in order for a disk to become gravitationally unstable. This $\Sigma_c$ is useful for discussing the possibility and position of gravitational instability, regardless of the surface density profile or the formation process of the disks. In section 4, we introduce two semianalytic models for protoplanetary disks, and discuss the possibility and location of the gravitational instability by comparing the surface densities of these models with $\Sigma_c$. In section 5, our results are discussed and compared with those of previous studies. Finally, we summarize this study in section 6.

2. Temperature

In this section, an analytic model of temperature calculation is introduced. Local thermal equilibrium between viscous heating and radiative cooling is assumed. We use the Keplerian rotation law, an ideal gas, vertical hydrostatic equilibrium, and a geometrically thin disk. We assume that radiation escapes only in the vertical direction. Effective viscosity with a constant $\alpha$ introduced by Shakura and Sunyaev (1973) is adopted. Since these assumptions are similar to the standard accretion disk (Pringle 1981), we have

$$\Omega = \sqrt{\frac{GM_\star}{r^3}},$$

(3)

$$c_s^2 = \gamma k_B T / \mu,$$

(4)

$$H = c_s / \Omega,$$

(5)

$$\rho = \Sigma / 2H = \Sigma \Omega / 2c_s,$$

(6)

$$\Gamma = \frac{g}{8\alpha c_s H \Sigma \Omega^2},$$

(7)

$$\Lambda = \begin{cases} 
\frac{32\sigma T^4}{3\kappa \Sigma} & (\tau \geq 1) \\
\frac{8\sigma T^4 \kappa \Sigma}{3} & (\tau < 1),
\end{cases}$$

(8)

and

$$\Gamma = \Lambda,$$

(9)

where $\gamma$ is the specific heat, $\mu$ the mean molecular weight, $\rho$ the density, $H$ the scale height, $M_\star$ the mass of protostar, $\sigma$ the Stefan–Boltzmann constant, $k_B$ the Boltzmann constant, $\Gamma$ the viscous heating rate, $\kappa$ the opacity, $\Lambda$ the cooling rate per unit area, and $\tau = \Sigma \kappa / 2$ the optical depth in the vertical direction. We approximate the cooling rate by connecting the optically thin and thick limits continuously. Here, we use the Rosseland mean opacity, $\kappa(\rho, T)$, which is approximated by using power-law as

$$\kappa = \kappa_0 \rho^a T^b,$$

(10)

where the values of $\kappa_0, a, b$ and $b$ are summarized in table 1 (Bell & Lin 1994; Cossins et al. 2010). This table gives the characteristic temperature at which the main component of the opacity changes. For example, $T_1 = 166.8 \text{ K}$ is the temperature where the main component of the opacity changes between ices and the sublimation of ices, and $T_2 = 202.6 \text{ K}$ is between the sublimation of ices and dust grains. Other characteristic temperatures depend on the density.

In order to calculate the temperature as a function of $r$ and $\Sigma$, seven variables ($\Omega, c_s, H, \rho, \kappa, \Gamma$, and $\Lambda$) can be eliminated by using equations (3)–(10). Thus, we can obtain the equilibrium temperature analytically for given values of $r$ and $\Sigma$. Additionally, we introduce the minimum temperature, $T_{\text{min}}$, in order to imitate the effect of ambient heating sources in a molecular cloud. If the above equilibrium temperature becomes lower than $T_{\text{min}}$, we simply use $T_{\text{min}}$ instead of the equilibrium temperature. As a result, the temperature can be
From equation (11), we can calculate the temperature as a function of the opacity represented as

\[
T = \begin{cases} 
\left( \frac{27}{256} A B^a r^{-3(a+1)/2} \Sigma^{\alpha+2} \right)^{1/(3-b+0.5a)} & (\tau \geq 1) \\
\left( \frac{27 A}{64 \kappa_0} B^a r^{3(a-1)/2} \Sigma^{-\alpha} \right)^{1/(3+b-0.5a)} & (\tau < 1) 
\end{cases}
\]

(11)

where

\[
A = \frac{\alpha y k_B \sqrt{GM_*}}{m \sigma},
\]

(12)

and

\[
B = \frac{1}{2} \sqrt{\frac{m GM_*}{\gamma k_B}}.
\]

(13)

From equation (11), we can calculate the temperature as a function of \( \Sigma \) and \( r \) for given values of \( \alpha \), \( M_* \), and \( T_{\text{min}} \). In this study, we set \( \alpha = 0.01 \), which is a typical value of \( \alpha \) caused by the MRI (magnet rotational instability) turbulence, and assume \( T_{\text{min}} = 10 \) K, which is a typical temperature of the molecular cloud.

Figure 1 shows the temperature in the \( r-\Sigma \) plane for the case with \( M_* = 1 M_\odot \), \( \alpha = 0.01 \) and \( T_{\text{min}} = 10 \) K as the fiducial case in this study. The solid lines show the contours of temperature, and the dotted line shows the \( \tau = 1 \) line. The characteristic temperature, \( T_1 \), defined above is labeled as 166 K, and \( T_2 \) is labeled as 202 K. The 166 K line can be regarded as a snow line because ices become the main component of the opacity in the area below this line. Since other characteristic temperatures depend on the density, their lines do not coincide with the contour line of the iso-temperature. At these characteristic temperatures, the property of opacity drastically changes, as given in equation (10) and table 1, and has a great influence on the gravitational stability of protoplanetary disks.

As shown in figure 1, the temperature rises higher with a large \( \Sigma \) and a small \( r \). Above the dotted line, the temperature is high for a large surface density, because a large amount of matter hinders radiation escaping from the system. In a small radius, the dynamical time scale is short, and thus the heating rate is large. This is why the temperature is high in the region of large \( \Sigma \) and small \( r \). A drastic change in the cooling rate depends on whether or not the condition \( \tau \geq 1 \) is satisfied, as described in equation (8). Below the dotted line, which means that the disk is optically thin, it is seen that the temperature is independent of the surface density. This is because all radiation emitted from the disk can escape in the optically thin case. At the right-bottom area of the line with 10 K, the temperature becomes the minimum temperature, \( T_{\text{min}} \), due to the small viscous heating rate and large cooling rate. We henceforth name this area the "isothermal region." Note that the dotted line exists only in the low temperature area, such that the temperature ranges from 10 K to 20 K. This is due to the high efficiency of cooling in a disk with \( \tau \sim 1 \).

### 3. Critical Surface Density

We can now calculate Toomre’s \( Q(r, \Sigma) \) by using equations (1) and (11). The analytical expression for \( Q \) value is summarized in appendix 1. In figure 2, the solid line represents the critical surface density, \( \Sigma_c \), which is necessary to satisfy \( Q = 1 \) for a given value of \( r \). For reference, the dotted and dashed lines represent the surface densities that satisfy \( Q = 0.5 \) and \( Q = 2 \), respectively. A disk is expected to become gravitationally unstable if its surface density is larger than \( \Sigma_c \). This critical surface density, \( \Sigma_c \), is determined only by a local thermal process and on the criterion for gravitational instability. Thus, one can discuss the possibility and position of gravitational instability regardless of the disk-formation process by comparing a given surface-density profile with \( \Sigma_c \). We show examples of this discussion in the next section. However, one should keep in mind some cautious, for example, (1) gravitational instability is not the same as fragmentation and (2) the \( Q \) value is not universal for the case with a nonuniform medium, or with non-axisymmetric situations.

In figure 2, \( \Sigma_c \) changes discontinuously by one order of magnitude at \( r = 24 \) AU. We henceforth refer to this critical radius as \( r_c \). By comparing figures 1 and 2, it is confirmed that \( \Sigma_c \) is large enough to satisfy \( \tau \geq 1 \) in \( r < r_c \) because of the large Coriolis force and the high temperature. The analytic formula for \( \Sigma_c \) is derived from equation (1) and the \( \tau \geq 1 \) part of (11):

\[
\Sigma_c = \left( \frac{27}{256} \kappa_0 A B^a C^{6-2b+a} r^{-3(7+2a-2b)/2} \right)^{1/(4-2b)}
\]

(14)

where \( A, B, \) and \( C \) are defined by equations (12), (13), and

\[
C = \sqrt{\frac{\gamma k_B M_*}{\pi^2 G m}}.
\]

(15)
If these two effects are balanced, the value of Toomre’s $Q$ increases to stabilize a disk as the surface density increases. This balance is needed to realize this balance are satisfied in the shaded area in the optically thick regime (see appendix 1). The conditions following. If the surface density increases, the effect of self-regulated gravitoturbulence becomes independent of the surface density. This is because the viscous heating rate increases with a large $\alpha$ and a large $M_\ast$. However, the dependence on $\alpha$ and $M_\ast$ is weak. Thus, $r_c$ varies only severalfold, even if $\alpha$ or $M_\ast$ is different from the typical value by an order of magnitude.

In figure 2, it can be seen that $\Sigma_c$ becomes a multivalued function at $r ~ 1.5$ AU and $\Sigma ~ 2 \times 10^5$ g cm$^{-2}$. In this area, molecules are the main component of the opacity, and there is the innermost radius of the unstable region around this area. We explain this innermost radius in appendix 2.

Note that the critical surface density is derived using a constant viscous parameter, $\alpha$. It is considered that this assumption corresponds to the case that the viscosity originates not from self-gravity, but from MRI turbulence. This case assumes that the disk is MRI active everywhere. Thus, this result is specific to the case with a disk that has no region where the MRI is inactive (generally called “dead zones”). In other words, it is assumed that there is significant heating of nongravitational origin at all radii in the disk. On the other hand, previous studies, such as Clarke (2009), used the viscous parameter determined by the self-regulated gravitoturbulence with the condition $Q = 1$. In this case, the disk is assumed to have large dead zones without effective viscosity owing to the MRI turbulence. Realistic protoplanetary disks must be bracketed with the two extreme cases. Considering such disks is beyond the scope of this paper, and is left to us as future work.

### 4. Instability Condition of Protoplanetary Disks

In this section, the condition of gravitational instability is discussed by comparing the surface density of particular disks with $\Sigma_c$. We introduce two simple semianalytic models for protoplanetary disks. One is a steady state accretion disk (the steady model). This model is realized if angular momentum is sufficiently transported. The other is a disk that
has the same angular-momentum distribution as the cloud core before it collapses. In this model, we assume conservation of the angular-momentum distribution, and hence this model is hereafter called the AMC (angular-momentum conservation) model. The AMC model is realized if the angular momentum is not sufficiently transported. We consider the steady model in subsection 4.1. The AMC model is discussed in subsection 4.2. The interpretation of these two models is considered in subsection 4.3.

4.1. The Steady State Accretion Disk Model

First, we consider the model of a steady state accretion disk. The surface density profile is calculated within the framework of the standard disk model with $\alpha$ viscosity (Shakura & Sunyaev 1973; Pringle 1981). The disk structure is determined by three parameters: viscous parameter, $\alpha$, protostar mass, $M_*$, and mass accretion rate, $\dot{M}$. By the equation of continuity and angular-momentum conservation, we have the relation

$$\alpha c_s H \Sigma = \frac{\dot{M}}{3\pi}.$$  

(18)

This equation shows that the surface density, $\Sigma$, is determined by the mass-accretion rate, $\dot{M}$. We calculated the temperature in the same manner as described in section 2 by using the cooling function approximated at equation (8) and the opacity represented by equation (10) with table 1, and by introducing $T_{\text{min}}$. Using equations (3), (4), (5), (11), and (18), we obtain the analytic profile for the surface density in this model as

$$\Sigma_{\text{SD}} = \begin{cases} 
D_X \left( \frac{27 M_\alpha A B}{256} r^{-(3+\alpha)/2} \right)^{T_{\text{min}}^2} & (T \geq T_{\text{min}}) \\
D_X \left( \frac{127 A}{256 M_\alpha B} r^{-3(1-a)/2} \right)^{T_{\text{min}}^2} & (T < T_{\text{min}})
\end{cases},$$

where we use symbols $A$, $B$, $D$, and $X$, defined by equations (12) and (13),

$$D = \frac{m \dot{M}}{3\pi \alpha g H_k B} \sqrt{GM_*},$$

(20)

and

$$X = \begin{cases} 
3 + 0.5a - b & (r \geq r_{\text{crit}}) \\
3 + b - 0.5a & (r < r_{\text{crit}})
\end{cases},$$

(21)

respectively. This steady state accretion disk does not have an inner radius and outer one. In other words, the disk is infinitely extended.

From equation (19) and table 1, we can calculate $\Sigma_{\text{SD}}$. Figure 3 shows the surface densities, $\Sigma_{\text{SD}}$, in cases of $M = 10^{-6}$ (long-dashed line), $10^{-7}$ (short-dashed line), and $10^{-8} M_\odot \text{yr}^{-1}$ (dotted line) in the fiducial case defined in section 2. In figure 3, it is shown that the lines of $\Sigma_{\text{SD}}$ break at the radii where the main component of the opacity changes similarly in the case of $\Sigma_c$ line in figure 2. It can also be seen that $\Sigma_{\text{SD}}$ is large with a large $\dot{M}$. The critical surface density, $\Sigma_c$, derived in section 3 is superimposed in figure 3 by a solid line. The disk is unstable if the condition $\Sigma_{\text{SD}} > \Sigma_c$ is satisfied. In figure 3, it is shown that disks with $M = 10^{-6}$ and $10^{-7} M_\odot \text{yr}^{-1}$ are unstable for $r > r_c = 24 \text{AU}$, where $r_c$ is the critical radius defined by equation (17). On the other hand, the case of $M = 10^{-8} M_\odot \text{yr}^{-1}$ is expected to be stable for all radii. It can also be seen that all of the lines including $\Sigma_c$ and $\Sigma_{\text{SD}}$ have the same dependence on radius in the outer region ($r \gtrsim 100 \text{AU}$). From equations (16) and (19) for $T = T_{\text{min}}$, we can confirm that $\Sigma_{\text{SD}}$ and $\Sigma_c$ have the same dependence on the radius, owing to the isothermal state. By using this property, we can recast the instability condition, $\Sigma_{\text{SD}} \geq \Sigma_c$, as

$$\dot{M} \geq \dot{M}_{\text{crit}} \equiv \frac{3\alpha c_{\text{s,min}}^3}{G} = 7.7 \times 10^{-8} M_\odot \text{yr}^{-1} \left( \frac{\alpha}{0.01} \right) \left( \frac{T_{\text{min}}}{10^3 K} \right)^{3/2}.$$

(22)

Note that this instability condition (22) is independent of the protostar mass, $M_*$. It is also found that $\dot{M}_{\text{crit}}$ is an increasing function of $\alpha$ and $T_{\text{min}}$. As shown in figure 3, the minimum radius of the unstable region, defined as $r_{\text{SD,min}}$, equals the critical radius, $r_c$. The critical radius, $r_c$, depends on the protostar mass as $r_c \propto M_*^{1/3}$, and $r_c$ is independent of the mass-accretion rate. Thus, in the steady model, the mass-accretion rate determines whether or not the disk becomes unstable, and the protostar mass affects where the disk becomes unstable. This property described above is summarized in figure 4. Figure 4 shows the contour of the minimum radius of the unstable region, $r_{\text{SD,min}}$, in the $\dot{M} - M_*$ plane. It can be seen that the stable disk exists in the left area where the mass-accretion rate is low, and that the minimum radius is large for
a great protostar mass. It is also seen that the disk becomes unstable at a smaller radius than \( r_c \) for a large mass accretion rate, \( \dot{M} \geq 5.4 \times 10^{-6} M_\odot \text{yr}^{-1} \). This property appears where the surface density, \( \Sigma_{\text{SD}} \), is large enough to satisfy \( \Sigma_{\text{SD}} \geq \Sigma_c \) in \( r < r_c \). In figure 3, it is found that \( \Sigma_{\text{SD}} \) is required to be larger than \( 7.3 \times 10^2 \text{ g cm}^{-2} \) in order to become gravitationally unstable in \( r < r_c \).

### 4.2. The AMC Model

Next, we consider a disk that has the same angular-momentum distribution as the cloud core before it collapses. This model corresponds to a disk that has just formed.

Here, we introduce a simplified model for the formation of a disk and a protostar. Figure 5 schematically shows the formation process of the disk and protostar in this model. It is assumed that a rotating cloud core collapses to make a protostar and a disk. We divide the cloud core into two parts. The shaded part near the rotational axis is assumed to become the protostar. The disk is assumed to be formed from the other part. The boundary radius, \( r_{\text{cy,b}} \), between these two parts is determined so that the mass inside the shaded part equals a given protostar mass, \( M_\ast \), as a parameter. The cloud core is assumed to rotate rigidly. The angular velocity of the cloud core, \( \Omega_0 \), is determined by introducing the rotation parameter, \( \beta_0 \equiv E_{\text{rot}}/|E_{\text{grav}}| \), where \( E_{\text{rot}} \) is the rotation energy and \( E_{\text{grav}} \) is the gravitational energy. Note that the disk formed in this model has an inner radius of \( r_{\text{cy, in}} \) and an outer radius of \( r_{\text{cy, out}} \), different from the steady model, described in subsection 4.1. It is assumed that the protostar and disk form instantaneously, and that the disk has Keplerian rotation velocity when ignoring the self-gravity. We also assume the inviscid and axisymmetric formation of the disk. In other words, it is assumed that the relation between the mass and the angular momentum is maintained without redistribution. As the earlier cloud core, two types of density distributions are considered with the assumption of spherical symmetry. One is the Bonnor–Ebert sphere (Ebert 1955; Bonnor 1956), and the other is a sphere of uniform density. In this study, we consider both cases, and the result in the case of Bonnor–Ebert sphere is mainly described. We later discuss the difference between these two cases.

The Bonnor–Ebert sphere is an isothermal sphere in which the thermal pressure balances the self-gravity and the external pressure. This sphere is identified by three quantities: the cloud radius, \( R_\text{cy} \), the sound speed, \( c_{s,0} \), and the central density, \( \rho_c \). It is known that a Bonnor–Ebert sphere is unstable when the condition

\[
R_\text{cy} \geq R_{\text{crit}} \equiv 6.46 \frac{c_{s,0}}{\sqrt{4\pi G \rho_c}} \tag{23}
\]

is satisfied. The Bonnor–Ebert sphere, whose radius, \( R_\text{cy} \), equals \( R_{\text{crit}} \), is called the critical Bonnor–Ebert sphere, and we use this critical Bonnor–Ebert sphere in this model. We set \( c_{s,0} = 190 \text{ m s}^{-1} \), which corresponds to the typical sound speed in a molecular cloud. The central density is fixed at \( \rho_c = 7.45 \times 10^{-19} \text{ g cm}^{-3} \). With these parameters, the critical Bonnor–Ebert sphere has mass \( M_\text{cy} = 1 M_\odot \) and radius \( R_{\text{cy}} = 1.0 \times 10^4 \text{ AU} \). Here, we approximate the density profile of the Bonnor–Ebert sphere (K. Tomide 2011 private communication) at

\[
\rho(R) = \frac{\rho_c}{\left[1 + \left(R_c^2/R^2\right)\right]^{3/2}} \tag{24}
\]

where \( R \) is the radius in spherical coordinates and \( R_c = c_{s,0}/\sqrt{4\pi G \rho_c} \). Equation (24) approximates the density profile of a Bonnor–Ebert sphere with the accuracy to within a few percent. This approximation enables us to calculate the surface density of the Bonnor–Ebert sphere, \( \Sigma_{\text{BE}}(r_{\text{cy}}) \), analytically:

\[
\Sigma_{\text{BE}}(r_{\text{cy}}) = 2 \int_0^{(R_c^2 - r_{\text{cy}}^2)^{1/2}} \rho(R) d R z
\]

\[
= \frac{2 \rho_c}{1 + (R_c^2/R^2)^{2}} \sqrt{R_c^2 - r_{\text{cy}}^2} \left[1 + (R_c^2/R^2)^2\right] \tag{25}
\]

where \( r_{\text{cy}} \) is the radius in cylindrical coordinates.

In this model, the surface density of the disk \( \Sigma_{\text{AMC}}(r_d) \) is determined from the angular-momentum distribution of the earlier cloud core as follows. Once the rotation parameter, \( \beta_0 \), is given, the angular-momentum distribution of the cloud core is determined. Next, we give the ratio of the disk to protostar mass, \( M_d/M_\ast \), as a parameter, and the specific
angular-momentum distribution of the disk is determined. Suppose that a fluid element in the cloud core at cylindrical radius \( r_{cy} \) falls onto the disk at cylindrical radius \( r_d \). The disk is supported by centrifugal force and rotates with the Keplerian velocity at \( r_d \). By the conservation of angular momentum, the relation between \( r_{cy} \) and \( r_d \) is found to be

\[
r_{cy} = \left( \frac{GM_d r_d}{\Omega_0^2} \right)^{1/4}.
\]

By using relation (26) with the relation of mass conservation,

\[
r_d \Sigma_{AMC}(r_d) dr_d = r_{cy} \Sigma_{BE}(r_{cy}) dr_{cy},
\]

we derive the surface density of the disk \( \Sigma_{AMC} \):

\[
\Sigma_{AMC}(r_d) = \frac{1}{2} \left( \frac{\rho_c R_E}{1 + \left( \frac{r_{cy}^2}{R_E^2} \right)} \left[ 1 - \frac{\left( \frac{r_{cy}^2}{R_E^2} \right)}{1 + \left( \frac{r_{cy}^2}{R_E^2} \right)} \right] \right)^{1/2}.
\]

Note that the \( r_{cy} \) depends on \( r_d \), as given by equation (26). In equation (28), we can find that \( \Sigma_{AMC} \) depends on the radius as \( r_d^{-2} \) in the region where \( r_{cy} > R_c \). It is also found that \( \Sigma_{AMC} \) depends on the radius as \( r_d^{-1.5} \) for \( r_{cy} < R_c \). Thus, the enclosed mass in the disk does not diverge. In this study, we concentrate on the Keplerian rotating disk without self-gravity. In order to use this assumption with validity, the mass of the disk should be less than that of the protostar. If we request that the disk rotates with the Keplerian velocity, the boundary radius, \( r_{cy,b} \), shown in figure 5 is always larger than \( R_c \). Thus, \( \Sigma_{AMC} \) depends on radius as \( r_d^{-2} \) in this paper. By the way, realistically, we consider that a protostar has not such a large mass just after the formation of the protoplanetary disk. In this model, we consider only the outer region of the disk and regard the inner region of the disk as being a part of the protostar, at least in the sense of the gravitational field. The boundary radius, \( r_{d,\text{in}} \), between the inner region and the outer region of the disk is determined by equation (26) and \( r_{cy,b} \).

We can now calculate the surface density of the disk for given \( M_d/M_s \) and \( \beta_0 \). First, we look over the property of \( \Sigma_{AMC} \) in the case of a fixed \( M_d/M_s \). Figure 6 shows the surface densities, \( \Sigma_{AMC} \), in the cases of \( \beta_0 = 0.005 \) (long-dashed), 0.001 (short-dashed), and 0.0002 (dotted) with \( M_d/M_s = 0.25 \). In the case of a large \( \beta_0 \), it is seen that \( \Sigma_{AMC} \) is small and that the disk exists far from the protostar. The solid line in figure 6 represents the critical surface density, \( \Sigma_c \), derived in section 3. In the case of \( \beta_0 = 0.0002 \), the disk is stable because the stability condition \( \Sigma_c > \Sigma_{AMC} \) is satisfied owing to a large \( \Sigma_c \) in \( r < r_c = 22 \text{ AU} \), where \( r_c \) is the critical radius defined by equation (17) with \( M_s = 0.8 \text{ M}_{\odot} \). In the case of \( \beta_0 = 0.001 \), the stability condition, \( \Sigma_c > \Sigma_{AMC} \), is violated in \( r_c = 22 \text{ AU} \leq r \leq 54 \text{ AU} \), and thus the disk is unstable there. This is because \( \Sigma_c \) changes discontinuously at \( r = r_c \). In the case of \( \beta_0 = 0.005 \), the angular momentum is so large that the inner radius, \( r_{in} = 85 \text{ AU} \), is larger than the critical radius, \( r_c = 22 \text{ AU} \). The disk can be unstable around the inner radius in \( r_{in} = 85 \text{ AU} \leq r \leq 150 \text{ AU} \). Note that, in the cases of \( \beta_0 = 0.001 \) and 0.005, the outer regions of the disks are stable because \( \Sigma_{AMC} \) rapidly decreases as the radius increases.

Next, we consider \( \Sigma_{AMC} \) for various values of \( M_d/M_s \) as well as \( \beta_0 \). Figure 7 summarizes the results in the \( \beta_0-(M_d/M_s) \) plane. The symbols with “+” correspond to the cases in figure 6. The results are classified into five groups in the \( \beta_0-(M_d/M_s) \) plane. The area labeled as “stable” in figure 7 indicates that the disks with these parameters are stable for all radii. The case of \( \beta_0 = 0.0002 \) in figure 6 belongs to this area. These disks have a small outer radius of \( r_{out} < r_c \), owing to the small angular momentum. Thus, the condition \( \Sigma_{AMC} < \Sigma_c \) is satisfied due to the large \( \Sigma_c \) in \( r < r_c \). In figure 7, the areas labeled as “a few AU,” “+,” and “>30 AU” indicate the unstable disks. The area labeled as “a few AU” exists in the small \( \beta_0 \) and large \( M_d/M_s \) area. The disks with these parameters have a small radius and a large surface density sufficient to satisfy \( \Sigma_{AMC} > \Sigma_c \) in \( r < r_c \). Thus, they have a possibility of fragmentation at a few AU from the protostar. The area labeled as “+” exists around the middle \( \beta_0 \) area in figure 7. This area includes the case of \( \beta_0 = 0.001 \) in figure 6. Although the disks with this area are stable inside the critical radius, \( r_c \), due to a small \( \Sigma_{AMC} \), they are unstable in \( r \geq r_c \). This is because \( \Sigma_c \) drastically changes at \( r = r_c \). It is the formation of ices that is capable of making the disk unstable. The area
labeled as “> 30 AU” exists at the large \( \beta_0 \) area. This area includes the case of \( \beta_0 = 0.005 \) in figure 6. In this area, the inner radius, \( r_{\text{in}} \), of the disk is larger than the critical radius, \( r_c \), and the disk is unstable around the inner radius. The area labeled as “inner” in figure 7 is difficult to draw a conclusion. In the present AMC model, the disks with this area are regarded as being stable, because \( \Sigma_{\text{AMC}} \) is very small due to too large an inner radius, \( r_{\text{in}} \gtrsim 10^2 \) AU. However, this large \( r_{\text{in}} \) is artificial because the AMC model is too simplified to calculate analytically. Realistically, it is expected that the disk exists in \( r < r_{\text{in}} \). If the disk has a surface density represented by equation (28), the disks with the area labeled as “inner” in figure 7 have a possibility of being unstable.

Based on the above results, we found that the disk can be expected to be unstable with a large \( \beta_0 \) in the AMC model. Here, we estimate the critical rotation parameter, \( \beta_{0,c} \), which is necessary to become unstable for a disk with a small mass (\( \sim 0.01 M_\odot \)). In this model, the disk becomes unstable if it extends to a radius of \( r \gtrsim r_c \). The outer radius of the disk can be estimated by substituting \( R_c \) for \( r_c \) in equation (26). On the other hand, \( r_c \) is given by equation (17). By using the relation \( \beta_0 \propto \Omega_0^2 \), we derive the critical rotation parameter, \( \beta_{0,c} \):

\[
\beta_{0,c} = 4 \times 10^{-4} \left( \frac{r_c}{24 \text{AU}} \right) \left( \frac{R_E}{10^3 \text{AU}} \right)^4 \left( \frac{M_c}{1 M_\odot} \right). \tag{29}
\]

Note that \( r_c \) depends on the viscous parameter, \( \alpha \), and the protostar mass, \( M_c \): also, \( R_E \) and \( M_c \) depend on the central density, \( \rho_c \), and the sound speed, \( c_{s,0} \). In figure 7, it can be seen that the disk becomes unstable for \( \beta_0 \gtrsim \beta_{0,c} \), even with a small mass ratio, \( M_d/M_s \lesssim 0.1 \). The minimum ratio of the disk-to-protostar mass satisfying the unstable condition is \( M_d/M_{s,\text{min}} \approx 0.0092 \) with \( \beta_0 = 5.0 \times 10^{-4} \), which belongs to the “\( r_c \)” area.

We also performed a calculation in the case of the uniform-density sphere as the density distribution of the earlier core. In this case, the core was assumed to rotate rigidly in common with the case of the Bonnor–Ebert sphere. This uniform sphere is characterized by two parameters, \( \alpha_0 \equiv E_{\text{th}}/E_{\text{grav}} \) and \( \beta_0 \equiv E_{\text{rot}}/E_{\text{grav}} \), where \( \alpha_0 \) is the thermal parameter and \( E_{\text{th}} \) is the thermal energy. The thermal parameter is assumed to be \( \alpha_0 = 0.86 \), which is the same value as in the case with the critical Bonnor–Ebert sphere. The calculation was performed with various values of \( \beta_0 \) and \( M_d/M_s \). Figure 8 summarizes the results. The results are qualitatively the same as those with the Bonnor–Ebert sphere, but quantitatively different concerning some points. One is the critical rotation parameter, \( \beta_{0,c} \). In the case of a uniform sphere, the value \( \beta_{0,c} = 1.0 \times 10^{-3} \) is larger than that of the Bonnor–Ebert sphere by a factor of 2.5. This difference arises from the fact that the uniform-density sphere has a 1.2 times smaller radius, \( R_E \), and almost the same angular velocity, \( \Omega_0 \), as compared with those of the critical Bonnor–Ebert sphere with the same values of \( \alpha_0 \) and \( \beta_0 \). Since a sphere with a small radius makes a disk that has a small outer radius with a fixed angular velocity, the case of a uniform sphere needs a larger \( \beta_0 \) than that of the Bonnor–Ebert sphere in order to satisfy the condition \( r_{\text{out}} \gtrsim r_c \). Next, for the same radius, \( r \), in the disk, the mass ratio \( M_d/M_s \) that is necessary to the instability is smaller with the uniform sphere than with the Bonnor–Ebert sphere by nearly one order of magnitude. In the AMC model, the disk is formed from the outer part of the sphere, and the density of the outer part with the uniform sphere is larger than that with the Bonnor–Ebert sphere. Thus, if both spheres make disks of the same size, the disk formed from the uniform sphere would have a larger surface density than that from the Bonnor–Ebert sphere. Hence, for the same radius, \( r \), the disk formed from the uniform sphere needs a smaller \( M_d/M_s \) to become unstable than that from the Bonnor–Ebert sphere. Finally, the instability radius around the mid-\( \beta_0 \) is different. In the case of the Bonnor–Ebert sphere, the instability radius is clearly divided into two parts, labeled as “a few AU” and “\( r_c \)”.

On the other hand, in the case of a uniform sphere, the disk becomes unstable at an intermediate radius of 10–30 AU. This is due to the large surface density and its strong dependence on the radius of the disk formed from the uniform sphere. All of these differences between the disk from the Bonnor–Ebert sphere and that from the uniform sphere originate from the difference in the relation between the mass and the specific angular momentum (known as \( m \sim j \) relation). Moreover, these differences strongly depend on the assumption of rigid rotation.

4.3 Interpretation of the Steady Model and the AMC Model

In the two previous subsections, we considered the gravitational stability of protoplanetary disks by using two simple semianalytic models. Here, we interpret these models by considering the formation scenario of a protoplanetary disk.

It is believed that a cloud core with angular momentum collapses to form a protostar and a protoplanetary disk. First, the central part of the cloud core begins to contract, and makes a protostar and a small (typically a few AU) disk (e.g., Bate 1998). Next, the outer envelope of the core mainly falls onto the disk, and the mass accretion from the envelope makes the disk increase in mass and size, as typically larger than 100 AU (e.g., Adams et al. 1988). The time scale of the disk growth is estimated to be \( t_{\text{grow}} \sim r/c_{s,0} \), according to the self-similar solution for the collapse of a rotating isothermal cloud (Saigo & Hanawa 1998). This disk that has just formed corresponds to the AMC model introduced in subsection 4.2. Later, this disk experiences a period of the viscous evolution.
with the transport of angular momentum, and eventually there is no evidence to show the initial distribution of the angular momentum. The time scale of viscous evolution is estimated to be $t_{\text{vis}} \sim a^{-1}(H/r)^{-3/2}$ from the azimuthal component of the momentum equation. It is expected that a steady state disk, which is discussed in subsection 4.1, corresponds to the state after the viscous evolution. In the case of $c_{s,0} = 190 \text{ m s}^{-1}$ and $H/r = 0.1$, the time scales of viscous evolution, $t_{\text{vis}}$, and disk growth, $t_{\text{grow}}$, are estimated to be

$$t_{\text{vis}} \sim 1.6 \times 10^3 \text{yr} \left(\frac{a}{0.01} \right)^{-1} \left(\frac{r}{1 \text{AU}}\right)^{3/2}$$

and

$$t_{\text{grow}} \sim 25 \text{yr} \left(\frac{c_{s,0}}{190 \text{ m s}^{-1}}\right)^{-1} \left(\frac{r}{1 \text{AU}}\right).$$

From equations (30) and (31), it is seen that the disk-growth time is much shorter than the viscous evolution time for $r > 1 \text{ AU}$. Thus, it is expected that the viscous redistribution of angular momentum can be regarded as being negligible during a much longer time than the dynamical time scale, $\Omega^{-1}$, after formation of the disk. In equation (30), it is found that the viscous evolution time is short for a small radius. Hence, it is likely that the steady state accretion disk forms at the inner region first, which spreads outward on a viscous time scale.

In summary, we interpret the AMC model as the outer part of a young disk and the steady model as an old disk.

5. Discussion

5.1. Thermal Stability of Equilibrium State

We calculated the equilibrium temperature based on the approximated opacity given in equation (10) and table 1 (Bell & Lin 1994) in section 2. This opacity depends on the temperature; also, the property of the opacity varies when the main component of the opacity changes. Thus, the cooling rate also depends on the temperature similarly. If the temperature dependence of the cooling rate is weaker than that of the heating rate, this state is thermally unstable (Field 1965). In order to check the stability of the thermal equilibrium state that we used, we compare the exponent of the temperature in the heating rate with the exponent in the cooling rate. In our model, as described in section 2, the heating rate has the relation $\Gamma \propto T$, regardless of the optical depth.

In the optically thick case, the temperature dependence of the cooling rate is $\Lambda \propto T^{4+b-0.5a}$. Thus, the condition of thermal stability is given by $a - 2b > -6$. This condition is satisfied, except for the case where hydrogen scattering is the main component of the opacity. This case is realized where the temperature is larger than the critical temperature, $T \gtrsim T_{\text{TTI}} \sim 3300 \text{ K}$, for $\rho = 10^{-18} \text{ g cm}^{-3}$. Within the parameter range that we are interested in, the equilibrium temperature is safely smaller than $T_{\text{TTI}}$, and thus in the optically thick case the equilibrium state is stable.

In the optically thin case, the temperature dependence of the cooling rate is $\Lambda \propto T^{4+b-0.5a}$. The condition for thermal stability is obtained as $a - 2b < 6$. From the values given in table 1, this stability condition is violated when the main component of the opacity is the sublimation of ices ($T_1 = 166.8 \text{ K} < T < 202.6 \text{ K}$) or the sublimation of metal dust ($2286.7 \rho^{2/16} K < T < 2029.7 \rho^{1/8}$). According to results given in section 2, in the optically thin case, the equilibrium temperature is sufficiently smaller than $T_1$, above which the sublimation of ices becomes the main component of the opacity. Thus, in the optically thin case, the equilibrium state satisfies the condition of thermal stability. Therefore, the equilibrium state that we used in this study is thermally stable.

5.2. Comparison with Radiation Hydrodynamical Simulations

Here, we compare the results of our model with those of some radiation hydrodynamical (RHD) simulations. The first example is a simulation by Boley et al. (2006). They calculated the disk evolution at $\sim 4000 \text{ yr}$; the surface density of the disk at the final state was approximated at $\Sigma \simeq 150 \text{ g cm}^{-2} \times 10^{-1}(r/R_E)\theta^2$, where $R_E = 46.7 \text{ AU}$. This disk did not fragment in their simulation. Our model also predicts that this disk cannot fragment because this surface density satisfies the stable condition $\Sigma < \Sigma_c$. Thus, the result of our model is consistent with that of their simulation.

Next, we make a comparison between the result by Meru and Bate (2010) and ours. In their study, the evolution of disks was calculated in cases of different values of the opacity. Their results showed that an opacity that is smaller than the interstellar value promotes fragmentation. In our model, the values of $\Sigma_c$ and $r_c$ given by equations (14) and (17) are small with a small opacity. This means that the disk easily becomes unstable with small opacity. In this restricted sense, the prediction by our model is consistent with theirs.

Finally, we compare our result with that of a simulation by Stamatellos et al. (2011). Their calculation was performed on nine initial conditions; their result predicted that only three out of nine disks fragmented. Based on this result, they suggested that fragmentation is unlikely to occur when the disk satisfies the condition that $r_{\text{out}} < 100 \text{ AU}$ and $M_{\text{d}}/M_* < 0.36$, where $r_{\text{out}}$ is the outer radius of the disk and $M_{\text{d}}/M_*$ the mass ratio of the disk to the protostar. Our result is consistent with theirs in the sense that the fragmentation hardly occurs near the central star ($r \lesssim 20 \text{ AU}$). However, with their initial surface density profile, our model predicts that all of the nine disks used by Stamatellos et al. (2011) may become gravitationally unstable in the case of $a \lesssim 0.6$. The difference between their result and our prediction may be caused by heating due to gravito-turbulence. In this sense, we should notice that the instability condition investigated in this paper is not exactly the same as the fragmentation condition.

5.3. The Steady Disk Model

In subsection 4.1, we derive the critical mass accretion rate, $M_{\text{crit}}$, that is necessary in order that the steady disk becomes unstable. Hartmann et al. (1998) derived the mass accretion rate of T-Tauri stars in Taurus and Chamaeleon I, based on optical observations. The typical value of the accretion rate is as small as $M \sim 10^{-8} \text{ M}_\odot \text{ yr}^{-1}$. Since this value is less than the critical mass accretion rate, $M_{\text{crit}} \approx 7.7 \times 10^{-8} \text{ M}_\odot \text{ yr}^{-1}$, given by equation (22), the disks around T-Tauri stars are expected to be stable. However, in Hartmann et al. (1998), 7 out of 40 T-Tauri stars in Taurus and one out of 16 in
Chamaeleon I are estimated to have a large mass-accretion rate sufficient to become gravitationally unstable. Thus, these statistics indicate that ~18% of T-Tauri stars have a possibility of having an unstable disk in Taurus, whereas ~6% of T-Tauri stars in Chamaeleon I have the same possibility. By the way, around a protostar, the Keplerian disk has not yet been clearly observed. If such a disk exists, its mass-accretion rate is estimated to be

\[ M \sim \frac{M_1}{t_g} \sim \frac{c^3}{G} \sim 2 \times 10^{-6} M_\odot \text{yr}^{-1} \left( \frac{T}{10 \text{K}} \right)^{3/2}, \]  

(32)

where \( M_1 \) is the Jeans mass, and \( t_g \) is the free-fall time. Because this mass-accretion rate is much larger than \( \dot{M}_{\text{crit}} \), the disk around a protostar is expected to be gravitationally unstable. In summary, it is expected that most of disks around a T-Tauri star are stable, but that the disk around a protostar is likely to fragment. Anyway, the relation between the mass-accretion rate and gravitational instability may be a useful tool for probing fragmentation in disk systems observationally.

Clarke (2009) (hereafter C09) also discussed the fragmentation of a steady state accretion disk. Assumptions made in C09 are similar to those in our model, but there are critical differences between hers and ours. On the assumption of gravitoturbulence \( Q = 1 \), C09 uses a spatially variable \( \alpha \) parameter, and adopts the Gammie criterion for a fragmentation. On the other hand, without the gravitoturbulent state, we use a constant viscous parameter, \( \alpha \), and employ the Toomre criterion. These differences produce qualitatively different results. For example, in C09 the fragmentation is predicted even with a very small mass-accretion rate \( (M \sim 10^{-5} M_\odot \text{yr}^{-1}) \), whereas in our model the disk is expected to be stable with such a small mass-accretion rate. However, note that both studies imply that the formation of ices is an important process. In the regime of ice opacity, \( \kappa \propto T^2 \), the temperature dependence of the cooling rate becomes \( \Lambda \propto T^2 \), which is weaker than that of other opacities (sublimation of ices, metal dust, and sublimation of metal dust). This property makes the disk temperature low when the heating rate becomes small. The importance of this fact does not change regardless of the fragmentation criterion.

5.4. The AMC Model

In the AMC model introduced in subsection 4.2, the disk is unstable when the condition \( \beta_0 \geq \beta_{0,c} \sim 0.001 \) is satisfied. By the way, according to the result of radiation hydrodynamical calculations, the disk formed from the core with a large rotation parameter of \( \beta_0 \geq \beta_{0,b} \sim 0.01 \) is expected to fragment before the formation of a protostar (Bate 2011). From observations of the line emissions of NH\(_3\) and N\(_2\)H\(^+\), the rotation parameter, \( \beta_0 \), of the cloud core is estimated to range over \( 10^{-6} < \beta_0 < 0.07 \) with a typical value of \( \beta_0 \sim 0.02 \) (Caselli et al. 2002). Only 3 cores out of 20 have a rotation parameter of \( \beta_0 \leq \beta_{0,c} \sim 0.001 \). Thus, only 15% of the cores are expected to make a stable disk. The cores with a middle rotation parameter, \( \beta_{0,c} \sim 0.001 \leq \beta_0 \leq \beta_{0,b} \sim 0.01 \), have a possibility of fragmenting after formation of the protostar. Eight cores out of 20 exist within this parameter range. The number of cores with \( \beta_0 \geq \beta_{0,b} \sim 0.01 \) is 9. Those cases with such a large \( \beta_0 \) are outside the scope of our AMC model.

Matsumoto and Hanawa (2003), Machida, Inutsuka, and Matsumoto (2010), and Tsukamoto and Machida (2011) calculated the formation process of the disk from the cloud core by using the barotropic equation of state for various values of the rotation parameter, \( \beta_0 \). These studies showed that disks become gravitationally unstable in the case of a large \( \beta_0 \). This feature is consistent with the result of our AMC model. Matsumoto and Hanawa (2003) calculated the cloud collapse with a rotation parameter of \( 8 \times 10^{-4} \leq \beta_0 \leq 8 \times 10^{-2} \). They showed that a disk formed from the cloud with \( \beta_0 \simeq 8 \times 10^{-4} \) becomes gravitationally unstable, but not fragmenting until the final state of their calculation. Our AMC model predicts that a disk with \( \beta_0 \simeq 8 \times 10^{-4} \) is unstable. They also showed that the fragmentation occurs before formation of the protostar in the case of \( \beta_0 \gtrsim 2 \times 10^{-3} \). These cases are also outside the scope of our AMC model. The result in Machida, Inutsuka, and Matsumoto (2010) is quantitatively different from that of our model. For example, the value \( \beta_{0,c} \sim 3 \times 10^{-2} \) in their calculation is smaller than that in the AMC model by one order of magnitude. However, it is difficult to quantitatively compare the result of the AMC model with that of their calculation, because, in their calculation, the fragmentation tends to occur during the phase that the mass of the protostar is smaller than that of the disk. Tsukamoto and Machida (2011) is more approximately consistent with our AMC model. Despite that they used the barotropic equation of state which is different from ours, the value \( \beta_{0,c} = 3 \times 10^{-3} \) in their calculation is consistent with our AMC model; that is, their value is larger than ours by a factor of 3.

5.5. The Effects of Ignored Processes

In this study, we ignored some physical processes that are expected to affect the gravitational instability in the disk. Here, we comment on their effects.

First, the self-gravity affects the disk property through its effect on the transport of angular momentum and energy dissipation. In the form of gravitational torque due to the non-axisymmetric mode and the gravitoturbulence, the self-gravity affects the viscous parameter, \( \alpha \). It is discussed that \( \alpha \) becomes large if the \( Q \) value approaches unity (Kratter et al. 2008). However, this \( \alpha \) as a function of the disk properties has not yet been clarified. In this study, in order to dispel this uncertainty, we used a constant \( \alpha \) parameter, and derived the condition of gravitational instability as a function of \( \alpha \). Considering \( \alpha \) from the self-gravity remains as future work. The self-gravity also affects the scale height, \( H \). When considering the self-gravity in the vertical direction, the scale height with self-gravity, \( H_{\text{self}} \), is represented as a function of the \( Q \) value. This \( H_{\text{self}} \) in the case of \( Q = 1 \) is approximately half of the \( H \) without self-gravity given by equation (5). By this effect, the values of \( \Sigma_{\text{in}} \) and \( r_c \) are modified, but their modifications amount to some twenty or thirty percent at the most. Thus, we consider that our result does not qualitatively change, even if considering the compression by self-gravity in the vertical direction.

Secondly, the effect of irradiation from the protostar affects the disk property. At a region far from the protostar, the heating by irradiation from the protostar becomes greater than the viscous heating because it strongly decreases with radius. Thus, far from the protostar there is a possibility that the disk might be stabilized by irradiation heating. The heating rate by
irradiation is affected by disk flaring, which is determined by the temperature distribution of the disk. It is difficult to accurately treat the irradiation heating in our model, because we did not consider the radial distribution of the disk when deriving the instability condition in section 3. However, the effect of irradiation heating is weak in disks with large surface density. Thus, we consider that our results in the optically thick case do not critically change. This irradiation heating should be taken into account when investigating properties of a particular disk.

Thirdly, we comment on the effect of a magnetic field. The magnetic field has a great influence on angular-momentum transport during all phases between the protostar formation and the disk evolution. In the phase of protostar formation, it is conceivable that the magnetic field triggers outflow, and that it transports the mass and the angular momentum (Shu et al. 1994; Tomisaka 2002; Machida et al. 2008). Thus, it is expected that the assumption made in the AMC model is violated by the effect of outflow. As future work, it remains for us to construct a model including the effect of outflow. In the phase of disk evolution, the turbulence arising from the MRI is considered to become the main source of viscosity. In the region where the ionization degree is low (called “dead zone”), the MRI turbulence is inactive (Sano et al. 2000). Thus, in the dead zone, the viscous parameter, \( \alpha \), is expected to be much smaller than that in the MRI active region. Although we used a constant \( \alpha \) model in this study, as future work it will be important to consider the realistic viscous parameter based on the physical processes with relations among other physical quantities.

6. Conclusions

In the present work, the gravitational stability of protoplanetary disks has been studied. The temperature in a protoplanetary disk has been analytically calculated while considering thermal effects, such as radiative cooling, viscous heating, and an ambient heating source in a molecular cloud. It is found that the temperature becomes as low as \( T \lesssim 20 \text{ K} \) in the optically thin regime. We have derived a critical surface density, \( \Sigma_c \), which is needed in order for a disk to become unstable, as a function of the radius. A comparison between a given surface density and \( \Sigma_c \) can predict the possibility and position of the gravitational instability. The formation of ices is important for the gravitational instability, and we have found that \( \Sigma_c \) changes discontinuously at the critical radius, \( r_c \approx 20 \text{ AU} \), where ices form.

Two semianalytic models of protoplanetary disks with the above-mentioned critical surface density, \( \Sigma_c \), are used in order to discuss the possibility and position of the gravitational instability. In the model of a steady state accretion disk, which corresponds to the evolved disk, it is found that the disk becomes unstable when the mass-accretion rate is greater than the critical mass-accretion rate, \( M_{\text{crit}} \), given by equation (22). Based on this result, we expect that most of the disks around T-Tauri stars are gravitationally stable, and that the disk around the protostar is unstable in \( r \gtrsim r_c \). In the model of a disk with the same angular-momentum distribution as the earlier cloud core, which corresponds to the outer region of the young disk, the disk is expected to become unstable for \( r \gtrsim r_c \) in the case of a large rotation parameter (\( \beta_0 \sim 10^{-3} \)). Based on the results of these two models, we conclude that fragmentation near the central star (\( r < r_c \)) rarely occurs as long as we consider the parameter range indicated by observations.

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Appendix 1. Expressions of Temperature, \( Q \) Value, and Critical Surface Density

Here, we summarize the value and dependence of the temperature, the Toomre \( Q \) value, and the critical surface density, \( \Sigma_c \).

A.1.1. Equilibrium Temperature

The equilibrium temperature given by equation (11) is explicitly written below. The temperature range of each formula is given in table 1.

A.1.1.1. Optically thick case

(a) Ices

\[
T = 12K \left( \frac{r}{10 \text{AU}} \right)^{-3/2} \left( \frac{\Sigma}{10^2 \text{g cm}^{-2}} \right)^2 \times \left( \frac{M_s}{1 M_\odot} \right)^{1/2} \left( \frac{\alpha}{0.01} \right). \tag{A1}
\]

(b) Sublimation of ices

\[
T = 1.8 \times 10^7 K \left( \frac{r}{1 \text{AU}} \right)^{-3/20} \times \left( \frac{\Sigma}{10^2 \text{g cm}^{-2}} \right)^{1/5} \left( \frac{M_s}{1 M_\odot} \right)^{1/20} \left( \frac{\alpha}{0.01} \right)^{1/10}. \tag{A2}
\]

(c) Metal dust

\[
T = 8.0 \times 10^7 K \left( \frac{r}{1 \text{AU}} \right)^{-3/5} \times \left( \frac{\Sigma}{10^1 \text{g cm}^{-2}} \right)^{4/5} \left( \frac{M_s}{1 M_\odot} \right)^{1/5} \left( \frac{\alpha}{0.01} \right)^{2/5}. \tag{A3}
\]

(d) Sublimation of metal dust

\[
T = 1.2 \times 10^7 \left( \frac{r}{1 \text{AU}} \right)^{-6/55} \times \left( \frac{\Sigma}{10^4 \text{g cm}^{-2}} \right)^{6/55} \left( \frac{M_s}{1 M_\odot} \right)^{2/5} \left( \frac{\alpha}{0.01} \right)^{2/5}. \tag{A4}
\]

A.1.1.2. Optically thin case

As described in section 2, the temperature becomes very low in the optically thin case. Thus, we consider only the case of ices.

(a) Ices

\[
T = 21K \left( \frac{r}{1 \text{AU}} \right)^{-3/10} \left( \frac{M_s}{1 M_\odot} \right)^{1/10} \left( \frac{\alpha}{0.01} \right)^{1/5}. \tag{A5}
\]

A.1.2. \( Q \) Value

The \( Q \) value defined by equation (1) is represented in the following.
A.1.2.1. Isothermal region

\[ Q = \frac{\Omega_{\text{min}}}{\pi G \Sigma} = 21 \left( \frac{\Sigma}{1 \text{ g cm}^{-2}} \right)^{-1} \left( \frac{r}{100 \text{ AU}} \right)^{-3/2}. \]  
(A6)

A.1.2.2. Optically thick case

In the optically thick case, the \( Q \) value is represented as

\[ Q = C \left( \frac{27}{256} \kappa_0 A B^a \right) \frac{1}{r^2} \left( \frac{3}{8} \frac{2 b + 2 b - 1}{2 b + 2 b} \right) \left( \frac{2 b + 2 b - 1}{2 b + 2 b} \right), \]
(A7)

where \( A, B, \) and \( C \) are defined by equations (12), (13), and (15), respectively. Using the values given in table 1, we can see the dependence of the \( Q \) value. In equation (A7), it is found that the \( Q \) value is independent of the surface density in the case of \( b = 2 \), which is realized when ices dominate the opacity.

A.1.2.3. Optically thin case

In the optically thin case, the \( Q \) value is represented as

\[ Q = C \left( \frac{27 A}{64 \kappa_0 B^a} \right) \frac{1}{r^2} \left( \frac{3}{8} \frac{2 b + 2 b - 1}{2 b + 2 b} \right) \left( \frac{2 b + 2 b - 1}{2 b + 2 b} \right), \]
(A8)

where \( A, B, \) and \( C \) are defined by equations (12), (13), and (15), respectively.

A.1.3. Critical Surface Density, \( \Sigma_c \), and Critical Radius, \( r_c \)

(a) Isothermal region

\[ \Sigma_c = \frac{\Omega_{\text{min}}\Sigma}{\pi G} = 21 \text{ g cm}^{-2} \left( \frac{r}{100 \text{ AU}} \right)^{-3/2} \]
\[ \times \left( \frac{T_{\text{min}}}{10 \text{ K}} \right)^{1/2} \left( \frac{M}{M_\odot} \right)^{1/2}. \]
(A9)

This is available in \( r_c < r \), where \( r_c \) is the critical radius given by equation (17).

(b) Ices

In this regime, a unique \( \Sigma_c \) does not exist, but \( r_c \) does.

\[ r_c = \left( \frac{\gamma k u M_\odot^{3/4}}{\pi m G^{1/4}} \sqrt{\frac{27 \kappa_0 a}{256 \sigma}} \right)^{4/9} \]
\[ = 24 \text{ AU} \left( \frac{\alpha}{0.01} \right)^{2/9} \left( \frac{M}{M_\odot} \right)^{1/3}. \]
(A10)

(c) Sublimation of ices

\[ \Sigma_c = 3.4 \times 10^3 \text{ g cm}^{-2} \left( \frac{r}{10 \text{ AU}} \right)^{-7/4} \]
\[ \times \left( \frac{M}{M_\odot} \right)^{7/12} \left( \frac{\alpha}{0.01} \right)^{1/18}. \]
(A11)

This is available in \( r_1 < r < r_c \), where \( r_1 \) is the radius at which the main component of the opacity changes between the sublimation of ices and dust. This \( r_1 \) is represented as

\[ r_1 = 16 \text{ AU} \left( \frac{M}{M_\odot} \right)^{1/3} \left( \frac{\alpha}{0.01} \right)^{2/9}. \]
(A12)

(d) Metal dust

\[ \Sigma_c = 6.2 \times 10^3 \text{ g cm}^{-2} \left( \frac{r}{10 \text{ AU}} \right)^{-3} \left( \frac{M}{M_\odot} \right)^{1/18} \left( \frac{\alpha}{0.01} \right)^{1/3}. \]
(A13)

This is available in \( r_2 < r < r_1 \), where \( r_2 \) is the radius at which the main component of the opacity changes between metal dust and the sublimation of ices. This \( r_2 \) is represented as

\[ r_2 = 10 \text{ AU} \left( \frac{M}{M_\odot} \right)^{48/141} \left( \frac{\alpha}{0.01} \right)^{98/423}. \]
(A14)

(e) Sublimation of metal dust

\[ \Sigma_c = 6.3 \times 10^3 \text{ g cm}^{-2} \left( \frac{r}{10 \text{ AU}} \right)^{-171/104} \times \left( \frac{M}{M_\odot} \right)^{7/13} \left( \frac{\alpha}{0.01} \right)^{1/52}. \]
(A15)

This is available in \( r_{\text{edge}} < r < r_2 \), where \( r_{\text{edge}} \) is the innermost radius for the gravitational instability, represented as

\[ r_{\text{edge}} = 1.2 \text{ AU} \left( \frac{M}{M_\odot} \right)^{1/3} \left( \frac{\alpha}{0.01} \right)^{54/353}. \]
(A16)

We explain this \( r_{\text{edge}} \) in appendix 2.

Appendix 2. Innermost Radius for the Gravitational Instability

In the regime where molecules mainly contribute to the opacity, which is realized at \( T \gtrsim 1600 \text{ K} \), the temperature dependence of the cooling rate becomes weak as \( \Lambda \propto T^{4/3} \). Then, the temperature becomes too high with a large surface density, and the stabilizing effect of the thermal pressure becomes larger than the destabilizing effect of the surface-density increase. This means that increasing surface density stabilizes the disk. Thus, the line of \( \Sigma_c \) becomes a multivalued function of the radius. In figure 2, we see this feature for \( \Sigma \sim 2 \times 10^5 \text{ g cm}^{-2} \) and \( r \sim 1 \text{ AU} \). The upper line of \( \Sigma_c \) slopes upward when going from left to right. This is due to the lower heating rate at a larger \( r \). Because the \( \Sigma_c \) line slopes downward, where the main component of the opacity is the sublimation of dust grains, the unstable region has the inner edge at the point where the main component of the opacity changes between molecules and the sublimation of dust grains. In figure 2, we can see that this inner edge is located at \( r \simeq 1.2 \text{ AU} \) and \( \Sigma \simeq 2 \times 10^5 \text{ g cm}^{-2} \). We name this radius \( r_{\text{edge}} \). For \( r < r_{\text{edge}} \), the disk cannot become gravitationally unstable no matter how large the surface density becomes. By using equations (11) and (14) and the condition of the opacity change from the sublimation of dust grains to molecules, \( r_{\text{edge}} \) is represented as

\[ r_{\text{edge}} = 1.2 \text{ AU} \left( \frac{M}{M_\odot} \right)^{1/3} \left( \frac{\alpha}{0.01} \right)^{54/353}. \]
(A17)

This \( r_{\text{edge}} \) weakly depends on \( \alpha \) and \( M \), which is similar to the dependency of \( r_c \). Thus, even if \( \alpha \) or \( M \) is much different from the typical value, the value of \( r_{\text{edge}} \) varies only severalfold.
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