The Noncommutative Anandan’s Quantum Phase

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In this work we study the noncommutative nonrelativistic quantum dynamics of a neutral particle, that possesses permanent magnetic and electric dipole moments, in the presence of an external electric and magnetic fields. We use the Foldy-Wouthuysen transformation of the Dirac spinor with a non-minimal coupling to obtain the nonrelativistic limit. In this limit, we study the noncommutative quantum dynamics and obtain the noncommutative Anandan’s geometric phase. We analyze the situation where magnetic dipole moment of the particle is zero and we obtain the noncommutative version of the He-McKellar-Wilkens effect. We demonstrate that this phase in the noncommutative case is a geometric dispersive phase. We also investigate this geometric phase considering the noncommutativity in the phase space and the Anandan’s phase is obtained.

I. INTRODUCTION

In 1959 Aharonov and Bohm [1] demonstrated that a quantum charge circulating a magnetic flux tube acquires a quantum topological phase. This effect was observed experimentally by Chambers [2, 3]. Aharonov and Casher showed that a particle with a magnetic moment moving in an electric field accumulates a quantum phase [4], which has been observed in a neutron interferometer [5] and in a neutral atomic Ramsey interferometer [6].

He and McKellar [7], and Wilkens [8] independently, have predicted the existence of a quantum phase that an electric dipole acquires while is circulating around and parallel to a line of magnetic monopoles. A simple practical experimental configuration to test this phase was proposed by Wei,
Han and Wei [9], where the electric field of a charged wire polarizes a neutral atom and an uniform magnetic field is applied parallel to the wire.

In a recent article, a topological phase effect was proposed by Anandan [10] which describes a unified and fully relativistic covariant treatment of the interaction between a particle with permanent electric and magnetic dipole moments and an electromagnetic field. This problem has been investigated in the non-relativistic quantum mechanics by Anandan [11] and one of the authors [12].

Recently, the interest in the study of physics in noncommutative space has attracted much interest in several areas of physics [13]. Noncommutative field theories are related to M-theory [14], string theory [15] and quantum Hall effect [16, 17, 18]. In quantum mechanics a great number of problems have been investigated in the case of the noncommutative space-time. Some important results obtained are related with geometric phases such as: Aharonov-Bohm effect [19, 20, 21, 22, 23], Aharonov–Casher effect [24, 25] and Berry’s quantum phase [26, 27] and others involving dynamics of dipoles [28]. In this paper we analyze the noncommutative quantum topological phase effect proposed by Anandan for a quantum particle with permanent magnetic and electric dipole moments in the presence of an external electric and magnetic fields, and we study the appearance of a geometrical quantum phase in their dynamics. We investigate the non-relativistic geometric phase, proposed by Anandan, for a quantum particle with permanent magnetic and electric dipole moments in the presence of external electric and magnetic fields in the noncommutative quantum mechanics. We also investigate the He-McKellar-Wilkens phase in noncommutative space.

This article is organized in the following way: in the next section, we discuss the quantum dynamics of a neutral particle in the presence of external electromagnetic field. In the section III the noncommutative non-relativistic quantum dynamics of quantum dipoles in the presence of external field are investigated. In section VI we study the noncommutative Aharonov-Casher effect, in section VII we extend the study for the He-McKellar-Wilkens phase in a noncommutative space-time. In section VIII we discuss this quantum phase considering the momentum-momentum non-commutativity or phase space non-commutativity. Finally, in section VIII the conclusions are presented.
II. THE NONRELATIVISTIC LIMIT

Now, we consider the relativistic quantum dynamics of a single neutral spin half particle with non-zero magnetic and electric dipole moving in external electromagnetic field, which is described by the following equation (we used $\hbar = c = 1$).

\[ [i\gamma_\mu \partial^\mu + \frac{1}{2} \mu \sigma_{\alpha\beta} F^{\alpha\beta} - \frac{i}{2} d \sigma_{\alpha\beta} \gamma_5 F^{\alpha\beta} - m] \psi = 0 , \]  

(1)

where $\mu$ is the magnetic dipole moment and $d$ is the electric dipole moment. We use the following convention for field strength [29]:

\[ F^{\mu\nu} = \{ \vec{E}, \vec{B} \}, \quad F^{\mu\nu} = -F^{\nu\mu} , \]

\[ F^{i0} = E^i , \quad F^{ij} = -\epsilon_{ijkl} B^k , \]

\[ F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^z \\ E^z & -B^y & B^x & 0 \end{bmatrix} , \]

(2)

\[ \sigma_{0i} = \frac{i}{2} [\gamma_0 \gamma_i - \gamma_i \gamma_0] = i\gamma_0 \gamma_i = -i\alpha_i , \]

\[ \sigma_{ij} = \frac{i}{2} [\gamma_i \gamma_j - \gamma_j \gamma_i] = i\gamma_i \gamma_j = \epsilon_{ijl} \Sigma_l , \]

(3)

and our $\gamma_5$ choice for convenience is

\[ \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \]

(4)

where 0 and $-1$ are corresponding $2 \times 2$ matrices [30]. Hence, we may write the equation (1) as

\[ [i\gamma^\mu \partial_\mu + \mu (i\bar{\alpha} \cdot \vec{E} - \vec{\Sigma} \cdot \vec{B}) - id (i\bar{\alpha} \cdot \vec{E} - \vec{\Sigma} \cdot \vec{B}) \gamma_5 - m] \psi = 0 , \]

(5)

with the Dirac matrices given by:

\[ \beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} , \quad \bar{\alpha} = \beta \gamma^5 = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} , \]
$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \vec{\Pi} = \beta \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix},$$

where $\sigma^j$ are the Pauli matrices obeying the relation $\{\sigma^i, \sigma^j + \sigma^j, \sigma^i\} = -2\delta^{ij}$. Then, the Hamiltonian given by equation (5) is reduced to the form

$$i\frac{\partial}{\partial t}\psi = H\psi = [\vec{\pi} \cdot \vec{\alpha} + \mu \vec{\Pi} \cdot \vec{B} + d \vec{\Pi} \cdot \vec{E} + \beta m] \psi; \quad (6)$$

where $\vec{\pi} = -i(\vec{\nabla} + \mu \beta \vec{E} - d \beta \vec{B})$.

From now on, we will use the Foldy-Wouthuysen method. This a very convenient method of description of the relativistic particle interaction with an external field and of transition to the semiclassical description is the Foldy-Wouthuysen transformation [31]. The Foldy-Wouthuysen representation provides the best opportunity for the transition to the classical limit of relativistic quantum mechanics. The Hamiltonian in equation (5) takes the form

$$\hat{\chi} = \beta (m + \hat{\epsilon}) + \hat{\chi}, \quad (7)$$

where $\hat{\epsilon} = \mu \vec{\Pi} \cdot \vec{B} + d \vec{\Pi} \cdot \vec{E}$ and $\hat{\chi} = \vec{\pi} \cdot \vec{\alpha}$ are the even and odd terms in the Hamiltonian. Hence we introduce the transformation

$$\hat{\chi}' = e^{i\hat{S}}(\hat{\chi} - i\partial_0)e^{-i\hat{S}}, \quad (8)$$

where $\hat{S}$ is a hermitian matrix. The purpose is to minimize the odd part of the Hamiltonian, or even to make it vanishing. Thus, we have

$$\hat{\chi}' = \hat{\chi} + i[\hat{S}, \hat{\chi}] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{\chi}]] - \frac{1}{6}i[\hat{S}, [\hat{S}, [\hat{S}, \hat{\chi}] + \ldots, \quad (9)$$

where $\hat{\epsilon}$ and $\hat{\chi}$ above obey the relations $\hat{\epsilon} \beta = \beta \hat{\epsilon}$ and $\hat{\chi} \beta = -\beta \hat{\chi}$.

For nonrelativistic particles in an electromagnetic field, the FW transformation can be performed with the operator $\hat{S} = -\frac{i}{2m} \beta \hat{\chi}$, such that we have

$$\hat{\chi}' = \hat{\chi} + \hat{\epsilon}' + \hat{\chi}', \quad (10)$$

where $\hat{\chi}'$ is at the order of $\frac{1}{2m}$, and we calculate the second-order FW transformation with $\hat{S}' = -\frac{i}{2m} \beta \hat{\chi}'$. This yields

$$\hat{\chi}'' = \hat{\chi} + \hat{\epsilon}' + \hat{\chi}'', \quad (11)$$
where $\hat{O}'' \approx \frac{1}{m^2}$. After that, the third FW approximation with $\hat{S}'' = -\frac{1}{2m} \hat{\beta} \hat{O}''$ makes the odd part of the nonrelativistic expansion vanishing; so finally we find the usual result

$$\hat{H}'' \approx \hat{\beta} m + \hat{\epsilon}'$$

$$= \hat{\beta}(m + \frac{1}{2m} \hat{O}^2 - \frac{1}{8m^3} \hat{O}^4) + \hat{\epsilon} - \frac{1}{8m^2} [\hat{O}, [\hat{O}, \hat{\epsilon}]] .$$ (12)

After replacing $\hat{\epsilon}$ and $\hat{O}$ in (12), we will consider only the terms up to order $1/m$. We obtain the following Hamiltonian

$$\hat{H}'' \approx \hat{\beta} m - \frac{\mu}{2m} \left( \vec{\nabla} - i\mu \vec{\beta}(\vec{\Sigma} \times \vec{E}) + i\beta \vec{d} \vec{\Sigma} \times \vec{B} \right)^2 - \frac{\mu^2 \vec{E}^2}{2m} - \frac{d^2 \vec{B}^2}{2m}$$

$$- \frac{\mu}{2m} \vec{\nabla} \cdot \vec{E} + \frac{d}{2m} \vec{\nabla} \cdot \vec{B} + \mu \vec{\Pi} \cdot \vec{B} + d \vec{\Pi} \cdot \vec{E} .$$ (13)

The equation (13) is the nonrelativistic quantum Hamiltonian for four-components fermions. However, for several applications in low energies in nonrelativistic quantum mechanics the two-components spinor field is considered, we may write (13) for two-components fermions in the form

$$\hat{H}'' \approx m - \frac{1}{2m} \left( \vec{\nabla} - i\tilde{\mu} \vec{\Sigma} \times \vec{E} + i\tilde{d} \vec{\Sigma} \times \vec{B} \right)^2 - \frac{\tilde{\mu}^2 \vec{E}^2}{2m} - \frac{\tilde{d}^2 \vec{B}^2}{2m}$$

$$- \frac{\tilde{\mu}}{2m} \vec{\nabla} \cdot \vec{E} + \frac{\tilde{d}}{2m} \vec{\nabla} \cdot \vec{B} + \tilde{\mu} \vec{\Pi} \cdot \vec{B} + \tilde{d} \vec{\Pi} \cdot \vec{E} ,$$ (14)

where $\tilde{\mu} = \mu \vec{\sigma}$ and $\tilde{d} = d \vec{\sigma}$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$; where $\sigma_i$ ($i = 1, 2, 3$) are the $2 \times 2$ Pauli matrices. The Hamiltonian (14) describes a system formed by a neutral particle, that possesses permanent electric and magnetic dipole moments, in the presence of electric and magnetic fields. Several topological and geometrical effects may be investigated by changing the fields-dipole configuration [7, 8, 9].

### III. NONRELATIVISTIC QUANTUM DYNAMICS OF DIPOLES

Considering the nonrelativistic quantum dynamics of a particle corresponding to the Hamiltonian (14) which describes several physical situations such as: the Aharonov–Casher effect $\mu \neq 0$ and $d = 0$, the He-McKellar-Wilkens effects $\mu = 0$ and $d \neq 0$, and the Anandan phase in general case $\mu \neq 0$ and $d \neq 0$. It is obvious that all this effects occur in specific field-dipole configuration. We analyze the quantum dynamics of a particle governed by the Hamiltonian (14). We
consider that the electric and magnetic fields, whose particle is immersed, are cylindrically radial. The electric and magnetic dipoles are aligned in z-direction. The Hamiltonian that describe the electric and magnetic dipoles in an external electric and magnetic fields can be written in following way:

\[ H = -\frac{1}{2m} \left( \mathbf{\nabla} - b_\mu \right)^2 + b_0, \]  

(15)

here the interaction with the electric and magnetic fields is similar as if it is minimally coupled to a non-Abelian gauge field with potential \( b_\mu \), where

\[ b_0 = -\frac{\mu^2 E^2}{2m} - \frac{d^2 B^2}{2m} - \frac{\mu}{2m} \mathbf{\nabla} \cdot \mathbf{E} + \frac{d}{2m} \mathbf{\nabla} \cdot \mathbf{B} + \mu \cdot \mathbf{B} + d \cdot \mathbf{E}, \]  

(16)

and \( b_i = (\mu \times \mathbf{E}) + (d \times \mathbf{B}) \). The first two terms in \( b_0 \) can be considered as an external potential and not contribute in the study of the geometric phase. Observe that the potential \( b_0 \), which depends on \( E^2 \) and \( B^2 \), represents a local influence to the wave function. We are interested in to study the asymptotic states for the dynamics. Then we will not consider this term, because it represent a local effect. Also Anadan has demonstrated that terms of the order \( O(E^2) \) and \( O(B^2) \) can be neglected in the study of geometric phase. The last four terms in potential \( b_0 \) in the Hamiltonian give null contribution in the dynamic of this dipole in the fields configuration since then the dipoles are aligned with z-direction. Assuming that the particle moving in plane \( x - y \), in presence of the external electric and magnetic fields. We also suppose that the fields generated by the source are radially distributed in the space. Now, we consider the commutative space version of this geometric phases. So that the only terms that contribute to geometric phase in are

\[ \hat{H} = -\frac{1}{2m} \left( \mathbf{\nabla} - i(\mu \times \mathbf{E}) + i(d \times \mathbf{B}) \right)^2 - \frac{\mu^2 E^2}{2m} - \frac{d^2 B^2}{2m}, \]  

(17)

the other terms of do not contribute to quantum phase due to choice of specific dipole-field configurations. The terms, in dynamical part of Hamiltonian, yields no force on the particle, while in quantum mechanics it affects the wave function of the particles by attaching to it a nondispersive geometric phase. The momentum operator can be written as

\[ k_i = mv_i = (p_i - (\mu \times \mathbf{E})_i + (d \times \mathbf{B})_i). \]  

(18)
The Schrödinger equation for this problem takes the form

\[
\left( -\frac{1}{2m}(\nabla - i(\vec{\mu} \times \vec{E}) + i(\vec{d} \times \vec{B}))^2 - \frac{\mu^2 E^2}{2m} - \frac{d^2 B^2}{2m} \right) \Psi = E \Psi ,
\]

(19)

To obtain the quantum phase we use the following ansatz

\[
\Psi = \Psi_0 e^{i\phi} ,
\]

(20)

where \( \Psi_0 \) is solution of following equation

\[
\left( -\frac{1}{2m}\nabla^2 - \frac{\mu^2 E^2}{2m} - \frac{d^2 B^2}{2m} \right) \Psi_0 = E \Psi_0 ,
\]

(21)

and the phase \( \phi \) is given by

\[
\phi = i \oint [\vec{\mu} \times \vec{E}] - (\vec{d} \times \vec{B})] \cdot d\vec{r} ,
\]

(22)

this phase was studied by Anandan [10]. It is a nondispersive effect due the independence on particle velocity [33]. Considering \( d = 0 \) in (22), we have the Aharonov-Casher geometric phase.

On the other hand, in the case \( d \neq 0 \) and \( \mu = 0 \) in (22) we have the He-McKellar-Wilkens phase,

\[
\phi_{HMW} = i \oint [-(\vec{d} \times \vec{B})] \cdot d\vec{r} ,
\]

(23)

that is usually known as topological phase but really it is a geometric phase [39].

IV. NONCOMMUTATIVE QUANTUM DYNAMICS OF DIPOLES

The usual noncommutative space canonical variables satisfy the following commutations relations

\[
[x_i, x_j] = i\Theta_{ij} , \quad [\hat{p}_i, \hat{p}_j] = 0 , \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij} ,
\]

(24)

where \( \hat{x}_i \) and \( \hat{p}_i \) are momentum and coordinate operators in a noncommutative space. The time-independent Schrödinger equation in the noncommutative (NC) space can be written in the form

\[
H(x, p) \star \psi = E \psi ,
\]

(25)

where \( H(x, p) \) is usual Hamiltonian and the Moyal-Weyl product (or star-product) is given by

\[
(f \star g)(x) = \exp\left(\frac{i}{2} \Theta_{ij} \partial_{x_i} \partial_{x_j}\right)f(x_i)g(x_j) ,
\]

(26)
here \( f(x) \) and \( g(x) \) are arbitrary functions. On the NC space the Moyal-Weyl product may be replaced by a Bopp’s shift \(^{[32]}\), i.e. the Moyal-Weyl product can be changed in the ordinary product by replacing \( H(x,p) \) with \( H(\hat{x},\hat{p}) \). This approach has been used by Li et al \(^{[23]}\). Hence, the Schrödinger equation can be written in the form

\[
H(\hat{x}_i, \hat{p}_i) = H(x_i - \frac{1}{2} \Theta_{ij} p_i, p_i) \psi = E \psi ,
\]

(27)

where \( x_i \) and \( p_i \) are the generalized position and momentum coordinates in the usual quantum mechanics. Therefore, the equation (27) is then actually defined on the commutative space, and the NC effect may be calculated by the terms that contain \( \Theta \). Note that \( \Theta \) in quantum mechanics may be taken as a perturbation considering \( \Theta_{ij} << 1 \).

When we have the presence of electric and magnetic fields as in \(^{[19]}\), the equation (25) becomes

\[
\left( -\frac{1}{2m} (\vec{\nabla} - i (\vec{\mu} \times \vec{E}) + i (\vec{d} \times \vec{B}))^2 - \frac{\mu^2 E^2}{2m} - \frac{\partial^2 B^2}{2m} \right) \ast \Psi = E \Psi ,
\]

(28)

To map the equation (28) from NC space to commutative space, we replace \( x_i \) and \( p_i \) by a Bopp’s shift \(^{[32]}\), as well as the fields \( E_i \) and \( B_i \) that will be replaced with a shift in the form

\[
(\vec{\mu} \times \vec{E}) \rightarrow (\vec{\mu} \times \vec{E}) + \frac{i}{2} \Theta_{lm} (\vec{\kappa} - (\vec{\mu} \times \vec{E}))_l \partial_m (\vec{\mu} \times \vec{E}) ,
\]

(29)

and

\[
(\vec{d} \times \vec{B}) \rightarrow (\vec{d} \times \vec{B}) + \frac{i}{2} \Theta_{lm} (\vec{\kappa} - (\vec{d} \times \vec{B}))_l \partial_m (\vec{d} \times \vec{B}) ,
\]

(30)

where the \( \kappa_l \) is the eigenvalue of momentum operator in the presence of the electric or magnetic field on NC space, and defined as

\[
(p_i - (\vec{\mu} \times \vec{E}) + (\vec{d} \times \vec{B})) \ast \psi = \kappa_i \psi ,
\]

(31)

where, \( \kappa_i = mv_i \) and \( v_i \) is the ordinary gradient. The relations (29) and (30) may be obtained in the same form by Taylor expansion up to first order of (26), for example, let us take the magnetic dipole case:

\[
\left( (\vec{\mu} \times \vec{E}) \ast \psi \right)(x) = \exp \left[ \frac{i}{2} \Theta_{ij} \partial_x \partial_x \right] (\vec{\mu} \times \vec{E})(x_i) \psi(x_j)
\]

\[
= (\vec{\mu} \times \vec{E}) \psi + \frac{i}{2} \Theta_{ij} \partial_i (\vec{\mu} \times \vec{E}) \partial_j \psi
\]

(32)
from (31) we have
\[ \partial_i \psi = (\kappa - (\vec{\mu} \times \vec{E}))_i \psi. \] (33)

Therefore, using (33) into (31) we obtain (29). In the same way we may obtain (30). Thus, the NC equation (28) mapped on commutative space is
\[ - \frac{1}{2m} \left( \vec{\nabla} - i(\vec{\mu} \times \vec{E}) \right) \partial_m (\vec{\mu} \times \vec{E}) + \left( \kappa - (\vec{\mu} \times \vec{E}) \right)_i \partial_m (\vec{\mu} \times \vec{E}) + i(\vec{d} \times \vec{B}) + \frac{i}{2} \Theta_{lm} (\kappa_l - (\vec{d} \times \vec{B})_l) \partial_m (\vec{d} \times \vec{B}) \right)^2 \psi = E \psi. \] (34)

In the same way of the usual quantum mechanics, the solution for (34) may be written as
\[ \psi = \psi_0 \exp(\phi), \] (35)
where \( \psi_0 \) is a solution of the Schrödinger equation in the absence of electric and magnetic fields and \( \phi \) is the Anandan’s geometric phase given in the form
\[ \phi = i \oint [(\vec{\mu} \times \vec{E}) - (\vec{d} \times \vec{B})] \cdot d\vec{r} + \frac{i}{2} \Theta_{lm} \oint \{ (\kappa - (\vec{\mu} \times \vec{E}))_l \partial_m (\vec{\mu} \times \vec{E}) - (\kappa - (\vec{d} \times \vec{B}))_l \partial_m (\vec{d} \times \vec{B}) \} \cdot d\vec{r}. \] (36)

The first term of the integral in the equation (36) is the usual Anandan’s phase in the commutative quantum mechanics. The other terms are the corrections due to NC effects. In the three-dimensional commutative space, we define the vector \( \vec{\theta} = (\theta_1, \theta_2, \theta_3) \) with \( \Theta_{ij} = \epsilon_{ijk} \theta_k \). Thus we rewrite the total phase (36) in the form
\[ \phi = i \oint [(\vec{\mu} \times \vec{E}) - (\vec{d} \times \vec{B})] \cdot d\vec{r} + \frac{i}{2m} \oint \vec{\theta} \cdot [\vec{v} \times \vec{\nabla} (\vec{\mu} \times \vec{E})]_i dr_i - \frac{i}{2m} \oint \vec{\theta} \cdot [(\vec{\mu} \times \vec{E}) \times \vec{\nabla} (\vec{\mu} \times \vec{E})]_i dr_i - \frac{i}{2m} \oint \vec{\theta} \cdot [\vec{v} \times \vec{\nabla} (\vec{d} \times \vec{B})]_i dr_i + \frac{i}{2m} \oint \vec{\theta} \cdot [(\vec{d} \times \vec{B}) \times \vec{\nabla} (\vec{d} \times \vec{B})]_i dr_i. \] (37)

The phase (37) is noncommutative version of a nonrelativistic quantum Anandan’s phase. Notice that the dependence of phase in the electric and magnetic field. Other properties of Eq. (37) is that geometric phase depend of the velocity of the particle. The noncommutativity of space introduces this dependence in the phase. In the next section we will discuss some special limits of this geometric phase.
V. NONCOMMUTATIVE AHARONOV–CASHER EFFECT

First we consider the case in the expression (36) where $d = 0$. In this case, we obtain the Aharonov–Casher (AC) phase given by

$$\phi_{AC} = i \oint [\vec{\mu} \times \vec{E} + \frac{1}{2} \Theta_{lm}(\kappa_l - (\vec{\mu} \times \vec{E}))_l] \partial_m (\vec{\mu} \times \vec{E}) \cdot d\vec{r}. \quad (38)$$

The first term in the integral in the equation (38) is the usual AC phase in the commutative quantum mechanics. The second term is the NC correction to the AC phase. In the three-dimensional commutative space, we define the vector $\theta = (\theta_1, \theta_2, \theta_3)$ with $\Theta_{ij} = \epsilon_{ijk}\theta_k$. Thus we rewrite the total phase (38) in the form

$$f = i \oint (\vec{\mu} \times \vec{E}) \cdot d\vec{r} + \frac{i}{2} m \oint \vec{\theta} \cdot (\vec{\nu} \times \vec{\Theta} \cdot \vec{E}) \cdot d\vec{r} - \frac{i}{2} m \oint \vec{\theta} \cdot [(\vec{\mu} \times \vec{E}) \times \vec{\nabla} \cdot (\vec{\mu} \times \vec{E})] \cdot d\vec{r}. \quad (39)$$

This geometric phase is the same obtained in [23, 24] in the relativistic case. Note that this is a dispersive geometric phase that depends on the velocity of the particle [33].

VI. THE NONCOMMUTATIVE HE–MCKELLAR–WILKENS EFFECTS

Now let us analyze the particular case of (22) in the noncommutative situation. This case is the noncommutative He-McKellar-Wilkens quantum phase in nonrelativistic limit. The He and McKellar and Wilkens independently [7, 8] have demonstrated that a quantum dynamics of an electric dipole in the presence of a radial magnetic field exhibits a geometric phase. The way to obtain the NC He–McKellar–Wilkens (HMW) effect is similar to the AC case. Taking $\mu = 0$ in the Pauli’s term in the equation (13). Hence, we find the following NC Schrödinger equation for the electric dipole in the presence a magnetic field. Applying this limit in the phase (36) we obtain the following expression

$$\phi_{HMW} = -i \oint \left(\vec{d} \times \vec{B} + \frac{1}{2} \Theta_{lm}(\kappa_l - (\vec{d} \times \vec{B}))_l\right) \partial_m (\vec{d} \times \vec{B}) \cdot d\vec{r}. \quad (40)$$

The first term in (40) is the commutative usual HMW quantum phase. The second term is the NC correction to HMW phase. In the same way as in the AC case, in the three-dimensional commutative space we define the vector $\theta = (\theta_1, \theta_2, \theta_3)$ with $\Theta_{ij} = \epsilon_{ijk}\theta_k$. Thus we rewrite the
total phase $\Phi_{HMW}$ in the form

$$\phi_{HMW} = -i \oint (\vec{d} \times \vec{B}) \cdot d\vec{r} - i \frac{1}{2m} \oint \bar{\theta} \cdot [\vec{v} \times \vec{\nabla} (\vec{d} \times \vec{B})] \cdot d\vec{r} + \frac{i}{2} m \oint \bar{\theta} \cdot [(\vec{d} \times \vec{B}) \times \vec{\nabla} \cdot (\vec{d} \times \vec{B})] \cdot d\vec{r}. \quad (41)$$

This equation gives the expression of the noncommutative version of He-MacKellar-Wilkens effect.

**VII. NONCOMMUTATIVE DYNAMICS OF DIPOLES IN PHASE SPACE**

In previous section we discuss the noncommutative version of geometric phase in the quantum dynamics of a neutral particle that possesses permanent electric and magnetic dipole moments. Now we will discuss the case where we are taking into account momentum-momentum noncommutativity. The Bose-Einstein statistics in noncommutative quantum mechanics requires both space-space and momentum-momentum noncommutativity [23, 25, 34, 35, 36]. This formulation has been denominated of phase space noncommutativity. In this case, the momentum commutation relation in (24) is replaced by

$$[\hat{p}_i, \hat{p}_j] = i \bar{\Theta}_{ij}, \quad (42)$$

where $\bar{\Theta}$ is the antisymmetric matrix, its elements represent the non-commutativity of the momenta. Thus the Schrödinger equation (25) is written in the form

$$- \frac{1}{2m} \left( \nabla - i(\vec{\mu} \times \vec{E}) + i(\vec{d} \times \vec{B}) \right)^2 \star \Psi = E \Psi. \quad (43)$$

On noncommutative phase space the star product can be replaced by a generalized Bopp’s shift [32], in this way the star product can be changed into ordinary product by shifting coordinates $x_\mu$ and momenta $p_\mu$ by

$$\tilde{x}_i = \lambda x_i - \frac{1}{2\lambda} \Theta_{ij} p_j \quad (44)$$

and

$$\tilde{p}_i = \lambda p_i - \frac{1}{2\lambda} \bar{\Theta}_{ij} x_j \quad (45)$$
where the scale factor $\lambda$ is an arbitrary constant parameter. The fields in equation change according to the formula (27) and assume the following form

\[
(\vec{\mu} \times \vec{E}) \to \lambda(\vec{\mu} \times \vec{E}) + \frac{i}{2\lambda} \Theta_{lm}(\kappa_l - (\vec{\mu} \times \vec{E})_l)\partial_m(\vec{\mu} \times \vec{E}) ,
\]

and the magnetic field term changes for the form

\[
(\vec{d} \times \vec{B}) \to \lambda(\vec{d} \times \vec{B}) + \frac{i}{2\lambda} \Theta_{lm}(\kappa_l - (\vec{d} \times \vec{B})_l)\partial_m(\vec{d} \times \vec{B}) .
\]

Now the Schrödinger equation for the neutral particle becomes

\[
-\frac{1}{2m'}\left(\nabla^2 + \frac{i}{\lambda^2} \Theta_{ij} x_i - i\lambda(\vec{\mu} \times \vec{E}) + \frac{1}{2\lambda^2} \Theta_{lm}(\kappa_l - (\vec{\mu} \times \vec{E})_l)\partial_m(\vec{\mu} \times \vec{E})
+ i\lambda(\vec{d} \times \vec{B}) - \frac{1}{2\lambda^2} \Theta_{lm}(\kappa_l - (\vec{d} \times \vec{B})_l)\partial_m(\vec{d} \times \vec{B})\right)^2 \psi = E\psi.
\]

We can rewrite the Schrödinger equation in the following form

\[
-\frac{1}{2m'}\left(\nabla^2 + \frac{i}{\lambda^2} \Theta_{ij} x_i - i(\vec{\mu} \times \vec{E}) + \frac{1}{2\lambda^2} \Theta_{lm}(\kappa_l - (\vec{\mu} \times \vec{E})_l)\partial_m(\vec{\mu} \times \vec{E})
+ i(\vec{d} \times \vec{B}) - \frac{1}{2\lambda^2} \Theta_{lm}(\kappa_l - (\vec{d} \times \vec{B})_l)\partial_m(\vec{d} \times \vec{B})\right)^2 \psi = E\psi ,
\]

where $m' = m/\lambda$. In the same way of the usual quantum mechanics, the solution for (49) may be written as

\[
\psi = \psi_0 \exp(\phi_{PS}) ,
\]

where $\psi_0$ is a solution of the Schrödinger equation for a particle of mass $m'$ in the absence of electromagnetic field and $\phi_{PS}$ is the Anandan’s geometric phase in non-commutative phase space given in the form

\[
\phi_{PS} = i \oint \{\vec{\mu} \times \vec{E} - \frac{1}{2\lambda^2} \Theta_{lm}(\kappa_l - (\vec{\mu} \times \vec{E})_l)\partial_m(\vec{\mu} \times \vec{E})\} \cdot d\vec{r} -
\]

\[-i \oint \{\vec{d} \times \vec{B} - \frac{1}{2\lambda^2} \Theta_{lm}(\kappa_l - (\vec{d} \times \vec{B})_l)\partial_m(\vec{d} \times \vec{B})\} \cdot d\vec{r} -
\]

\[-\frac{i}{2\lambda^2} \oint \Theta_{ij} x_j dx_i .
\]

The previous expression (51) has a contribution arisen due to a quantum phase in commutative space, other contribution due to a noncommutative space and one more contribution due to a noncommutative phase space. We can write the quantum phase (51) in the following form

\[
\phi_{PS} = \phi_{AP} + \phi_{NCS} + \phi_{NCP} ,
\]
where the $\phi_{AP}$ and the $\phi_{NCS}$ are the Anandan’s phase contribution and the contribution due to the space-space noncommutativity to the general dipole phase in the expression (51). The term $\phi_{NCPS}$ is the contribution due to noncommutativity of the momenta is given by

$$
\phi_{NCPS} = -\frac{i}{2\lambda^2} \oint \Theta_{ij} x_j dx_i + \frac{1 - \lambda^2}{2\lambda^2} \oint \left[ \Theta_{lm}(\kappa_l - (\vec{\mu} \times \vec{E})_l) \partial_m (\vec{\mu} \times \vec{E}) \right] \cdot d\vec{r} \quad -\frac{1 - \lambda^2}{2\lambda^2} \oint \left[ \Theta_{lm}(\kappa_l - (\vec{d} \times \vec{B})_l) \partial_m (\vec{d} \times \vec{B}) \right] \cdot d\vec{r}.
$$

This is the contribution to noncommutative geometric phase due to a noncommutativity in phase space. In this way we can write the He–McKellar–Wilkens phase in noncommutative phase space with a particular case where $\vec{\mu} = 0$, and this expression is given by

$$
\phi_{HMWPS} = -\frac{i}{2\lambda^2} \oint \Theta_{ij} x_j dx_i - \frac{1 - \lambda^2}{2\lambda^2} \oint \left[ \Theta_{lm}(\kappa_l - (\vec{d} \times \vec{B})_l) \partial_m (\vec{d} \times \vec{B}) \right] \cdot d\vec{r}.
$$

Therefore, we obtain the contribution due to NC phase space to the He–McKellar–Wilkens phase.

We can see that this phase depends on the magnetic field and also on the velocity of the particle. The first term in (54) is similar in appearance to the spin factor that occur in the partition function of spinning particle. The connection of this spin factor with the geometric phase was investigated by Kalhede et al [37] and Levay [38]. This similarity with spin factor and the physical implications of this terms is topic of a future contribution.

VIII. CONCLUDING REMARKS

In this paper, we study the nonrelativistic quantum dynamics of a neutral particle, that possesses permanent electric and magnetic dipole moments, in the presence of electric and magnetic external fields. We use the Foldy–Wouthuysen expansion to make transition to the classical limit of relativistic to nonrelativistic quantum mechanics. In this limit, we investigate the Aharonov-Casher and the He-McKellar-Wilkens effects in the noncommutative coordinates space. Here, we replace the $\star$-product by the Bopp’s shift [32] in the field terms, and then we obtain to AC and HMW quantum phases with the NC corrections. We obtain the noncommutative Anandan’s phase and demonstrate that this is a geometric dispersive phase. Usually, a geometric phase is a local effect, while a topological phase is nonlocal. Peshkin and Lipkin [39] have shown that the Aharonov-Bohm effect is nonlocal, because its value depends upon a physical quantity in a region
outside the closed path. It is a topological effect. The Aharonov-Bohm phase is proportional to the winding number of the path around the flux. It is a topological invariant and this phase depends on topology not on distance; hence it must be nonlocal. Therefore there are no electromagnetic fields along the paths of the charged particle and there are no change of physical quantities. They remarked that in the case of the Aharonov-Casher effect there are fields along the paths of the beams; then they concluded that the Aharonov-Casher effect is local due to the local interactions and is nontopological one because the phase shift depends on the local fields along the paths. In contrast with the topological phase, the geometric phase, in general, is a local effect because it depends on the geometry and topology in the space of parameters but not on the topology in spacetime. Here the quantum phase depends of the fields and the velocity of the particle. This fact characterize the noncommutative Anandan’s quantum phase with a geometric phase due to dependence in the fields and dispersive because of the dependence on the velocity. The NC Aharonov-Casher is obtained with a limit case of and agree with the results in the literature. The NC version He-McKellar-Wilkens phase is calculated for the NC quantum dynamics of electric dipoles and is a geometric dispersive phase. The noncommutative phase space version of Anandan’s phase and He-McKellar-Wilkens phase is obtained in this paper and we conclude they are geometric dispersive phases.

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