THERMODYNAMIC FORMALISM OF GL$_2(\mathbb{R})$-COCYCLES WITH CANONICAL HOLONOMIES

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Abstract. We study singular value potentials of Hölder continuous GL$_2(\mathbb{R})$-cocycles over hyperbolic systems whose canonical holonomies converge and are Hölder continuous. Such cocycles include locally constant GL$_2(\mathbb{R})$-cocycles as well as fiber-bunched GL$_2(\mathbb{R})$-cocycles. We show that singular value potentials of irreducible such cocycles have unique equilibrium states. Among the reducible cocycles, we provide a characterization for cocycles whose singular value potentials have more than one equilibrium states.

1. Introduction

In this paper, we study matrix cocycles over hyperbolic systems and their thermodynamic formalism. Let $(\Sigma_T, \sigma)$ be a subshift of finite type. For any continuous cocycle $A \in C(\Sigma_T, \text{GL}_2(\mathbb{R}))$, we define the associated singular value potential $\Phi_A := \{\log \varphi_{A,n}\}_{n \in \mathbb{N}}$ on $\Sigma_T$, where

$$\varphi_{A,n}(x) = \|A^n(x)\|$$

with $A^n(x) := A(\sigma^{n-1}x) \ldots A(x)$. The norm $\| \cdot \|$ is the standard operator norm on GL$_2(\mathbb{R})$. From the submultiplicativity of the norm, the singular value potential $\Phi_A$ is subadditive:

$$\log \varphi_{A,n+m} \leq \log \varphi_{A,n} + \log \varphi_{A,m} \circ \sigma^n \quad (1.1)$$

for all $m, n \in \mathbb{N}$.

Classical studies of thermodynamic formalism have been successfully extended to subadditive potentials such as $\Phi_A$; see [CFH08] and [Bar96]. Let $\mathcal{M}(\sigma)$ be the set of $\sigma$-invariant probability measures. Denoting the corresponding subadditive pressure of $\Phi_A$ by $P(\Phi_A)$, the subadditive variational principle [CFH08] states

$$P(\Phi_A) := \sup\{h_{\mu}(\sigma) + \mathcal{F}(\Phi_A, \mu) : \mu \in \mathcal{M}(\sigma), \mathcal{F}(\Phi_A, \mu) \neq -\infty\} \quad (1.2)$$

where

$$\mathcal{F}(\Phi_A, \mu) = \lambda_+(A, \mu) := \lim_{n \to \infty} \int \frac{1}{n} \log \|A^n(x)\| \, d\mu(x) \quad (1.3)$$

is the Lyapunov exponent of $A$ with respect to $\mu$. Any $\sigma$-invariant probability measure $\mu \in \mathcal{M}(\sigma)$ attaining the supremum in (1.2) is called an equilibrium state of $\Phi_A$.

We will focus on the singular value potentials of a class of $\alpha$-Hölder GL$_2(\mathbb{R})$-cocycles $A$ over $(\Sigma_T, \sigma)$ satisfying two extra conditions:

(a) Denoting the stable and unstable set of $\Sigma_T$ by $\mathcal{W}^{s/u}$, the following limits converge: for any $y \in \mathcal{W}^{s/u}(x)$,

$$H^s_{x,y} := \lim_{n \to \infty} A^n(y)^{-1}A^n(x) \quad \text{and} \quad H^u_{x,y} := \lim_{n \to -\infty} A^n(y)^{-1}A^n(x). \quad (1.4)$$

When they exists, such $H^{s/u}$ are called the canonical holonomies of $A$. 


The canonical holonomies are Hölder continuous with some exponent \( \beta \in (0, \alpha] \): there exists \( C > 0 \) such that for any \( y \in \mathcal{W}^{s/u}(x) \) we have
\[
\|H^{s/u}_{x,y} - I\| \leq Cd(x, y)^{\beta}.
\]
We denote by \( \mathcal{H} \) the set of \( \alpha \)-Hölder cocycles that meet such requirements:
\[
\mathcal{H} := \{ A \in C^\alpha(\Sigma_T, \text{GL}_2(\mathbb{R})) : A \text{ satisfies (a) and (b)} \}.
\]
There are many natural classes of cocycles that belong to \( \mathcal{H} \). Such cocycles include locally constant cocycles as well as cocycles that are close to being conformal; the later are called fiber-bunched cocycles. See Section 2 for details.

**Definition 1.1.** A cocycle \( A \in \mathcal{H} \) is reducible if there exists a proper \( A \)-invariant and biholonomy invariant line-bundle over \( \Sigma_T \). We say \( A \in \mathcal{H} \) is irreducible if \( A \) is not reducible.

From the upper semi-continuity of the entropy map \( \mu \mapsto h_\mu(\sigma) \), the singular value potential \( \Phi_A \) of any continuous cocycle \( A \in C(\Sigma_T, \text{GL}_d(\mathbb{R})) \) has at least one equilibrium state [Fen11]. The main theorem states that for cocycles in \( \mathcal{H} \), irreducibility implies the uniqueness of such equilibrium states.

**Theorem A.** Let \( A \in \mathcal{H} \). If \( A \) is irreducible, then the singular value potential \( \Phi_A \) has a unique equilibrium state.

Theorem A is similar to nowadays a folklore result that singular value potentials of locally constant cocycles generated by irreducible sets of matrices have unique equilibrium states. See Remark 4.7. In order to establish Theorem A, we introduce the notion of typical cocycles, which in some sense, contains most cocycles in \( \mathcal{H} \).

**Definition 1.2.** We say \( A \in \mathcal{H} \) is typical if

1. (pinching) There exists a periodic point \( p \in \Sigma_T \) such that \( A^{\text{per}}(p)(p) \) has simple eigenvalues of distinct norms with corresponding eigendirections \( v_+, v_- \in \mathbb{R}P^1 \);
2. (twisting) There exist \( z_+, z_- \in \mathcal{W}^s(p) \cap \mathcal{W}^u(p) \setminus \{p\} \) such that for each \( \tau \in \{+, -, \} \),
   
   the holonomy loop \( \psi^\tau_p := H^s_{z_-, p} \circ H^u_{p, z_+} \) twists \( v_\tau \):
   
   \[\psi^\tau_p(v_\tau) \neq v_\tau.\]

**Remark 1.3.** Two points \( z_+, z_- \in \mathcal{W}^s(p) \cap \mathcal{W}^u(p) \setminus \{p\} \) from the twisting condition above are homoclinic points of \( p \). More generally, given a periodic point \( p \in \Sigma_T \), we say \( z \in \Sigma_T \) is a homoclinic point of \( p \) if \( z \) belongs to the set \( \mathcal{H}(p) := \mathcal{W}^s(p) \cap \mathcal{W}^u(p) \setminus \{p\} \). Equivalently, the homoclinic points of \( p \) are characterized as the points other than \( p \) whose orbit synchronously approaches the orbit of \( p \), both in forward and backward time.

**Remark 1.4.** The notion of typical cocycles is first introduced by Bonatti and Viana in [BV04] for fiber-bunched \( \text{SL}_d(\mathbb{R}) \)-cocycles. Typicality as introduced in Definition 1.2 is weaker than that of [BV04]. We elaborate more on such differences in Remark 3.6.

Define
\[
\mathcal{U} := \{ A \in \mathcal{H} : A \text{ is typical} \}.
\]

Since the canonical holonomies \( H^{s/u} \) vary continuously in \( A \in \mathcal{H} \), \( \mathcal{U} \) is open in \( \mathcal{H} \). Bonatti and Viana [BV04] showed that \( \mathcal{U} \) is dense in the set of fiber-bunched \( \text{SL}_d(\mathbb{R}) \)-cocycles and that \( \mathcal{U}^c \) has infinite codimension. In fact, the proof there readily extends to establish the same properties for \( \mathcal{U} \) considered as a subset of \( \mathcal{H} \). Next theorem establishes the dichotomy among irreducible cocycles in \( \mathcal{H} \) with typicality being one of the alternatives.

**Theorem B.** Suppose \( A \in \mathcal{H} \) is irreducible. Then either
(1) \( A \) is typical (i.e., \( A \in \mathcal{U} \)), or
(2) there is a Hölder conjugacy of \( A \) into the group of linear conformal transformations of \( \mathbb{R}^2 \).

We also study thermodynamic formalism of reducible cocycles in \( \mathcal{H} \). For any \( A \in \mathcal{H} \), condition (b) implies that the map \( (x, y) \mapsto H_{x,y}^s \) is \( \beta \)-Hölder continuous for some \( \beta \in (0, \alpha] \). Hence, an \( A \)-invariant and bi-holonomy invariant line-bundle of a reducible cocycle from Definition 1.1 must also be \( \beta \)-Hölder continuous. By straightening out the line-bundle, the reducible cocycle \( A \) admits a \( \beta \)-Hölder conjugacy \( C: \Sigma_T \rightarrow \text{GL}_2(\mathbb{R}) \) such that \( B(x) := C(\sigma x)A(x)C(x)^{-1} \) is upper triangular for every \( x \in \Sigma_T \). Since \( \Phi_A \) and \( \Phi_B \) have the same set of equilibrium states (see Remark 2.6), the study of reducible cocycles in \( \mathcal{H} \) reduces to the study of Hölder cocycles taking values in upper triangular matrices.

**Theorem C.** Let \( B \in C^\beta(\Sigma_T, \text{GL}_2(\mathbb{R})) \) be an \( \beta \)-Hölder cocycle taking values in the group of upper triangular matrices:

\[
B(x) := \begin{pmatrix} a(x) & b(x) \\ 0 & c(x) \end{pmatrix}. \tag{1.5}
\]

The singular value potential \( \Phi_B \) has a unique equilibrium state, unless

1. \( \log |a| \) is not cohomologous to \( \log |c| \), and
2. \( P(\log |a|) = P(\log |c|) \).

If (1) and (2) hold, then \( \Phi_B \) has two ergodic equilibrium states.

**Remark 1.5.** Theorem C is a more general result than first two Theorems in that the assumptions are weaker; while we assume that the cocycle \( B \) is Hölder continuous with some exponent \( \beta > 0 \), we do not require that \( B \) belong to \( \mathcal{H} \). In particular, once a reducible cocycle \( A \in \mathcal{H} \) is conjugated to an \( \beta \)-Hölder cocycle \( B \) of the form (1.5) via the invariant line bundle, extra conditions (a) and (b) on \( A \) no longer play a role in studying thermodynamic formalism of \( \Phi_B \).

The result for reducible cocycles in \( \mathcal{H} \) is summarized in the following corollary whose proof appears in Section 5.

**Corollary 1.6.** Suppose \( A \in \mathcal{H} \) is reducible and admits an \( \beta \)-Hölder conjugacy \( C: \Sigma_T \rightarrow \text{GL}_2(\mathbb{R}) \) for some \( \beta > 0 \) such that \( B(x) := C(\sigma x)A(x)C(x)^{-1} \) takes values in upper triangular matrices. Then \( \Phi_A \) has a unique equilibrium state unless conditions (1) and (2) from Theorem C hold for \( B \), in which case there are two ergodic equilibrium states of \( \Phi_A \).

The paper is organized as follows. In Section 2, we introduce the setting of our results and survey relevant results in thermodynamic formalism. Then we prove Theorem B in Section 3, and using Theorem B, we prove Theorem A in Section 4. Theorem C and Corollary 1.6 are then established in Section 5. In Section 6, we explain how the results can be applied to the derivative cocycles, restricted to the unstable bundles, of certain Anosov diffeomorphisms.

2. Preliminaries

2.1. **Symbolic dynamics.** Let \( T \) be a \( q \times q \) square *adjacency matrix* with entries from \( \{0, 1\} \), and \( \Sigma_T \) be the set of bi-infinite \( T \)-admissible sequences in \( \{1, \ldots, q\}^\mathbb{Z} \) defined by

\[
\Sigma_T = \{(x_i)_{i \in \mathbb{Z}} \in \{1, \ldots, q\}^\mathbb{Z} : T_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.
\]

Throughout the paper, we will always assume that \( T \) is *primitive*, meaning that there exists \( N \in \mathbb{N} \) such that all entries of \( T^N \) are positive. The primitivity of \( T \) is equivalent to
topological mixing on \((\Sigma_T, \sigma)\). Denoting the left shift on \(\Sigma_T\) by \(\sigma\), the dynamical system \((\Sigma_T, \sigma)\) is called the subshift of finite type defined by \(T\). Fix \(\theta \in (0,1)\), and we put a metric \(d\) on \(\Sigma_T\) as follows: for \(x = (x_i)_{i \in \mathbb{Z}}\) and \(y = (y_i)_{i \in \mathbb{Z}} \in \Sigma_T\),
\[
d(x, y) := \theta^k,
\]
where \(k\) is the largest integer such that \(x_i = y_i\) for all \(|i| < k\). Equipped with such metric, \((\Sigma_T, \sigma)\) becomes a hyperbolic homeomorphism on a compact metric space. In particular, the local stable set of \(x\) is defined as
\[
W^s_{loc}(x) := \{y \in \Sigma_T : x_i = y_i \text{ for all } i \in \mathbb{N}_0\}.
\]
The stable manifold of \(x\) is defined as
\[
W^s(x) := \{y \in \Sigma_T : \sigma^n y \in W^s_{loc}(\sigma^n x) \text{ for some } n \in \mathbb{N}_0\},
\]
and is characterized by the set of \(y \in \Sigma_T\) such that \(d(\sigma^n x, \sigma^n y) \to 0\) as \(n\) tends to infinity. Similarly, we define the (local) unstable set \(W^u_{loc}(x)\) as the (local) stable set of \(x\) with respect to \(\sigma^{-1}\).

For any \(x, y \in \Sigma_T\) with \(x_0 = y_0\), we define the bracket of \(x\) and \(y\) by
\[
[x, y] := W^s_{loc}(x) \cap W^s_{loc}(y) \in \Sigma_T.
\]
An admissible word of length \(n\) is a word \(i_0 \ldots i_{n-1}\) with \(i_j \in \{1, \ldots, q\}\) such that \(T_{i_j, i_{j+1}} = 1\) for each \(0 \leq j \leq n - 2\). We denote the set of all admissible words of length \(n\) by \(\mathcal{L}(n)\), and let \(\mathcal{L} := \bigcup_{n=0}^{\infty} \mathcal{L}(n)\) be the set of all admissible words. For any \(I = i_0 \ldots i_{n-1} \in \mathcal{L}(n)\), we define the cylinder defined by \(I\) as
\[
[I] := \{(x_j)_{n \in \mathbb{Z}} \in \Sigma_T : x_j = i_j \text{ for all } 0 \leq j \leq n - 1\}.
\]

### 2.2. Holonomies and cocycles in \(\mathcal{H}\)

In this subsection, we describe natural classes of cocycles that belong to \(\mathcal{H}\). First, we introduce the formal definition of holonomies.

Let \(A \in C(\Sigma_T, \text{GL}_2(\mathbb{R}))\) be a continuous \(\text{GL}_2(\mathbb{R})\)-valued function. A **cocycle generated by** \(A\), denoted again by \(A\) by an abuse of notation, is the skew product map
\[
A : \Sigma_T \times \mathbb{R}^2 \to \Sigma_T \times \mathbb{R}^2,
\]
\[
(x, v) \mapsto (\sigma x, A(x)v).
\]
For any \(n \in \mathbb{N}\), we have \(A(x, v) := (\sigma^n x, A^n(x)v)\) where
\[
A^n(x) := A(\sigma^{n-1} x) \ldots A(x).
\]
Note that \(A^n\) satisfies the cocycle equation:
\[
A^{m+n}(x) = A^m(\sigma^n x)A^n(x) \text{ for all } m, n \in \mathbb{N}.
\]
As \(\sigma\) is invertible, we define \(A^0 \equiv 0\) and \(A^{-n}(x) := A^n(\sigma^{-n} x)^{-1}\). Then the cocycle equation above holds for all \(m, n \in \mathbb{Z}\).

**Definition 2.1.** A **local stable holonomy** for \(A\) is a family of matrices \(H^s_{x,y} \in \text{GL}_2(\mathbb{R})\) defined for any \(x, y \in \Sigma_T\) with \(y \in W^s_{loc}(x)\) such that
1. \(H^s_{x,x} = I\) and \(H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}\) for any \(y, z \in W^s_{loc}(x)\),
2. \(A(x) = H^s_{\sigma y, \sigma x} \circ A(y) \circ H^s_{x,y}\),
3. \(H^s : (x, y) \mapsto H^s_{x,y}\) is continuous.

A **local unstable holonomy** \(H^u_{x,y}\) is likewise defined for \(y \in W^u_{loc}(x)\) satisfying the analogous properties.
Even though a local stable holonomy \( H^{s}_{x,y} \) is defined only for \( y \in \mathcal{W}^{s}_{\text{loc}}(x) \) in Definition 2.1, it can be extended to a global stable holonomy \( H^{s}_{x,y} \) defined for any \( y \in \mathcal{W}^{s}(x) \) not necessarily in \( \mathcal{W}^{s}_{\text{loc}}(x) \):

\[
H^{s}_{x,y} = A^{n}(y)^{-1}H^{s}_{\sigma^{n}x,\sigma^{n}y}A^{n}(x),
\]

where \( n \in \mathbb{N} \) is any positive integer such that \( \sigma^{n}y \in \mathcal{W}^{s}_{\text{loc}}(\sigma^{n}x) \). Similarly, a local unstable holonomy can be extended to a global unstable holonomy. Such extension to global holonomies is coherent with the second property from Definition 2.1. It is easily verified that if the canonical holonomies \( H^{s/u} \) from (1.4) converge, then they satisfy the properties listed in Definition 2.1.

We now describe a natural class of cocycles, called fiber-bunched cocycles, that belongs to \( \mathcal{H} \). In the following definition, recall that \( \theta \in (0,1) \) is the constant defining the metric on \( \Sigma_{T} \).

**Definition 2.2.** Let \( A \in C^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) \). We say \( A \) is fiber-bunched if for every \( x \in \Sigma_{T} \),

\[
\|A(x)\| \cdot \|A(x)^{-1}\| \cdot \theta^{\alpha} < 1.
\]

Notice that conformal cocycles and their perturbations are fiber-bunched; indeed, fiber-bunched cocycles may be thought of as cocycles close to being conformal. Let

\[
C_{b}^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) := \{ A \in C^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) : A \text{ is fiber-bunched} \}
\]

be the set of fiber-bunched cocycles. It is clear from the definition that the set of fiber-bunched cocycles \( C_{b}^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) \) is open in \( C^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) \).

One important consequence of the Hölder continuity and the fiber-bunching assumption on \( A \in C_{b}^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) \) is the convergence of the canonical holonomies \( H^{s/u} \) from (1.4). Moreover, for \( A \in C_{b}^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) \) the canonical holonomies vary \( \alpha \)-Hölder continuously (i.e., condition (b) holds with the same exponent as the cocycle \( A \)) on the base points [KS13]: there exists \( C > 0 \) such that for any \( y \in \mathcal{W}^{s/u}_{\text{loc}}(x) \),

\[
\|H^{s/u}_{x,y} - I\| \leq Cd(x,y)^{\alpha}.
\]

This shows that the set of fiber-bunched cocycles \( C_{b}^{\alpha}(\Sigma_{T}, \text{GL}_{2}(\mathbb{R})) \) is a subset of \( \mathcal{H} \).

We note that Hölder continuity and the fiber-bunching assumption on the cocycle are sufficient but not necessary for the convergence of the canonical holonomies \( H^{s/u} \) from (1.4). For instance, the canonical holonomies \( H^{s/u} \) always converge for locally constant cocycles, another natural class of cocycles that belongs to \( \mathcal{H} \).

**Definition 2.3.** A cocycle \( A \) is locally constant if there exists \( k \in \mathbb{N}_{0} \) such that for every \( x \in \Sigma_{T} \), the value of \( A(x) \) depends only on \( x_{-k} \ldots x_{k} \in \mathcal{L}(2k+1) \).

**Remark 2.4.** For any locally constant cocycle \( A \) over \( \Sigma_{T} \), by re-coding the base dynamical system \( (\Sigma_{T}, \sigma) \) into a new subshift of finite type \( (\Sigma_{T}, \sigma, \zeta) \), we may and we will assume that \( A(x) \) only depends on the zero-th entry \( x_{0} \). Such cocycles are also known as one-step cocycles.

Locally constant cocycles are not necessarily fiber-bunched, but the canonical holonomies \( H^{s/u} \) from (1.4) trivially converge to the identity matrix. Hence, locally constant cocycles belong to \( \mathcal{H} \).

Another natural class of cocycles that belongs to \( \mathcal{H} \) is the derivative cocycles of certain Anosov diffeomorphisms restricted to 2-dimensional invariant subbundles. Via Markov partitions, such derivative cocycles can be realized as cocycles over subshifts of finite type,
and the results stated in the introduction applies. In Section 6, we will discuss such class of cocycles in further details.

We conclude the discussion on holonomies and cocycles in \( \mathcal{H} \) by describing a property called bounded distortion that is satisfied by singular value potentials of cocycles in \( \mathcal{H} \): for any \( \mathcal{A} \in \mathcal{H} \), there exists \( C \geq 1 \) such that for any \( n \in \mathbb{N} \), \( f \in \mathcal{L}(n) \), and \( x, y \in [I] \),

\[
C^{-1} \leq \frac{\varphi_{\mathcal{A}, n}(x)}{\varphi_{\mathcal{A}, n}(y)} = \frac{\|\mathcal{A}^n(x)\|}{\|\mathcal{A}^n(y)\|} \leq C. \tag{2.2}
\]

Indeed, \( \|\mathcal{A}^n(x)\|/\|\mathcal{A}^n(y)\| \) is equal to the product of two fractions \( \|\mathcal{A}^n(x)\|/\|\mathcal{A}^n(z)\| \) and \( \|\mathcal{A}^n(z)\|/\|\mathcal{A}^n(y)\| \) where \( z := [y, x] \) is the bracket (2.1) of \( y \) and \( x \). For the first fraction, notice that \( \mathcal{A}^n(x) \) is equal to \( H_{\sigma^n x, \sigma^n z}^s \mathcal{A}^n(z) H_{x, z}^s \). From Hölder continuity of the canonical holonomies (b), the norm of \( H_{\sigma^n x, \sigma^n z}^s \) and \( H_{x, z}^s \) are uniformly bounded above and below independent of \( x, z, \) and \( n \). It then follows that \( \|\mathcal{A}^n(x)\|/\|\mathcal{A}^n(z)\| \) is also bounded above and below by a uniform constant. Applying the same argument to \( \|\mathcal{A}^n(z)\|/\|\mathcal{A}^n(y)\| \) using instead the unstable holonomy establishes (2.2).

Note that locally constant cocycles satisfy the bounded distortion property (2.2) with the constant \( C = 1 \).

### 2.3. Thermodynamic formalism

In this subsection, we briefly survey the theory of both additive and subadditive thermodynamic formalism.

Let \( f \) be a homeomorphism on a compact metric space \( X \). A subset \( E \subset X \) is \((n, \varepsilon)\)-separated if any two distinct \( x, y \in E \) are at least \( \varepsilon \)-apart in the \( d_n \) metric:

\[
d_n(x, y) := \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)) \geq \varepsilon.
\]

From the compactness of \( X \), the cardinality of any \((n, \varepsilon)\)-separated set \( E \) is finite.

For any continuous function \( \varphi : X \to \mathbb{R} \) (often called a potential), the pressure \( P(\varphi) \) is defined as

\[
P(\varphi) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : E \text{ is a } (n, \varepsilon)\text{-separated subset of } X \right\} \tag{2.3}
\]

where \( S_n \varphi := \varphi + \varphi \circ f + \varphi \circ f^{n-1} \) is the \( n \)-th Birkhoff sum of \( \varphi \). Denoting the set of \( f \)-invariant probability measures by \( \mathcal{M}(f) \), the pressure \( P(\varphi) \) satisfies the following variational principle:

\[
P(\varphi) = \sup_{\mu \in \mathcal{M}(f)} \left\{ h_{\mu}(f) + \int \varphi \, d\mu \right\}, \tag{2.4}
\]

where \( h_{\mu}(f) \) is the measure-theoretic entropy of \( \mu \); see [Wal00]. Any \( f \)-invariant probability measure achieving the supremum in (2.4) is called an equilibrium state of \( \varphi \).

The existence and the number of equilibrium states depend on both the potential \( \varphi \) and the system \((X, f)\). For instance, any potential over a system whose entropy map \( \mu \mapsto h_{\mu}(f) \) is upper semi-continuous has at least one equilibrium state; such systems include hyperbolic systems and asymptotically entropy-expansive systems [Bow72], [Mis76].

The question on the finiteness or the uniqueness of the equilibrium state is more subtle. One result along this line is the following theorem of Bowen [Bow74] which establishes the uniqueness of the equilibrium states for any Hölder potentials over topologically mixing hyperbolic systems.

**Proposition 2.5.** [Bow74] Let \((\Sigma_T, \sigma)\) be a mixing subshift of finite type and \( \varphi : \Sigma_T \to \mathbb{R} \) be a Hölder continuous function. Then there exists a unique equilibrium state \( \mu_\varphi \) of \( \varphi \),
characterized as the unique $f$-invariant measure satisfying the following Gibbs property: there exists $C \geq 1$ such that for any $n \in \mathbb{N}$, $I \in \mathcal{L}(n)$, and $x \in [I]$, we have

$$C^{-1} \leq \frac{\mu_x(I)}{e^{-n\xi_\rho(x)}} \leq C.$$ 

Although we have stated Proposition 2.5 relevant to our setting of subshift of finite types, Bowen [Bow74] established sufficient conditions that guarantee the existence of a unique equilibrium in more general settings, and such conditions have been generalized in many different directions since then. In this paper, we focus on its generalization to subadditive potentials.

A sequence of non-negative and continuous function $\{\varphi_n\}_{n \in \mathbb{N}}$ on $X$ is submultiplicative if for any $m, n \in \mathbb{N}$,

$$0 \leq \varphi_{n+m} \leq \varphi_n \cdot \varphi_m \circ f^n.$$ 

From the submultiplicativity, $\Phi := \{\log \varphi_n\}_{n \in \mathbb{N}}$ becomes a subadditive potential on $X$, just as $\Phi_A$ in (1.1).

Given a potential $\varphi \in C(X)$, the Birkhoff sum $\{S_n \varphi\}_{n \in \mathbb{N}}$ is an additive sequence of functions on $X$. In a similar analogy, given a subadditive potential $\Phi = \{\log \varphi_n\}_{n \in \mathbb{N}}$, we may consider the $n$-th function $\log \varphi_n$ from $\Phi$ as a generalization of the $n$-th Birkhoff sum $S_n \varphi$ of some potential $\varphi$. Cao, Feng, and Huang [CFH08] define the subadditive pressure $P(\Phi)$ of $\Phi$ by generalizing (2.3) under such analogy:

$$P(\Phi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \varphi_n(x) : E \text{ is a } (n, \varepsilon) \text{-separated subset of } X \right\}. \quad (2.5)$$

As noted in the introduction, [CFH08] also established that $P(\Phi)$ satisfies the subadditive variational principle:

$$P(\Phi) := \sup \{ h_\mu(f) + \mathcal{F}(\Phi, \mu) : \mu \in \mathcal{M}(f), \mathcal{F}(\Phi, \mu) \neq -\infty \},$$

where

$$\mathcal{F}(\Phi, \mu) := \lim_{n \to \infty} \int \frac{1}{n} \log \varphi_n(x) \, d\mu(x).$$

We remark that Barreira [Bar96] introduced an alternative way to define a subadditive pressure using open covers. It is not known whether Barreira’s definition of the subadditive pressure coincides with Cao, Feng, and Huang’s definition (2.5) in the most general setting. However, it is shown in [CFH08] that two definitions coincide when the base system is entropy-expansive which includes our setting of subshifts of finite type $(\Sigma_T, \sigma)$.

In this paper, we will focus on subadditive potentials that arise as the singular value potential of $\text{GL}_2(\mathbb{R})$-cocycles over $\Sigma_T$. A continuous cocycle $A \in C(\Sigma_T, \text{GL}_2(\mathbb{R}))$ gives rise to the singular value potential

$$\Phi_A := \{\log \varphi_A, n\}_{n \in \mathbb{N}}, \text{ where } \varphi_A, n(x) = \|A^n(x)\|.$$ 

We note that Proposition 2.5 does not readily extend to subadditive potentials. Even restricted to singular value potentials $\Phi_A$ of locally constant cocycles $A$, there are examples where $\Phi_A$ admits multiple equilibrium states; see [FK10]. Moreover, while fiber-bunched cocycles are nearly conformal, the properties of their singular value potentials differ from those of conformal cocycles. Singular value potentials of conformal cocycles are additive due to conformality, and can be studied via tools from classical thermodynamic formalism such as Proposition 2.5. On the other hand, due to subadditivity, Proposition 2.5 does not necessarily hold for singular value potentials of fiber-bunched cocycles without extra assumptions; see Subsection 4.1.
However, restricted to singular value potentials of cocycles in $\mathcal{H}$, Proposition 2.5 remains valid for singular value potentials of large subset of $\mathcal{H}$. In particular, such subset includes the set of all typical cocycles $\mathcal{U}$. See Section 4 for more details on the statements in this paragraph.

The following remark shows that study of thermodynamic formalism for matrix cocycles is not affected under continuous conjugacies.

**Remark 2.6.** Two continuous functions $\varphi, \psi \in C(\Sigma_T)$ are *cohomologous* if there exists a continuous function $h$ such that $\varphi - \psi = h \circ \sigma - h$, and we denote it by $\varphi \sim \psi$.

It is clear from the variational principle (2.4) that if $\varphi \sim \psi$, then their pressures and the set of equilibrium states are the same.

Similarly, from the subadditive variational principle (1.2), if a cocycle $\mathcal{A} \in C(\Sigma_T, \text{GL}_d(\mathbb{R}))$ is continuously conjugated to another cocycle $\mathcal{B} \in C(\Sigma_T, \text{GL}_d(\mathbb{R}))$, then the pressures and the set of equilibrium states of the singular value potentials $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{B}}$ are the same.

This follows because $\mathcal{F}(\Phi_{\mathcal{A}}, \mu) = \mathcal{F}(\Phi_{\mathcal{B}}, \mu)$ for all $\mu \in \mathcal{M}(\sigma)$ as the norm $\|\mathcal{C}(x)\|$ of the continuous conjugacy $\mathcal{C}: \Sigma_T \to \text{GL}_2(\mathbb{R})$ is uniformly bounded from the compactness of $\Sigma_T$.

3. **Proof of Theorem B**

We now begin the proof of Theorem B. Recall that $\mathcal{H}(p)$ is the set of all homoclinic points of $p$, and for each $z \in \mathcal{H}(p)$, there is an associated holonomy loop $\psi^z_p := H_{z,p}^s \circ H_{p,z}^s$. As $(\Sigma_T, \sigma)$ is a mixing hyperbolic system, $\mathcal{H}(p)$ is dense in $\Sigma_T$ for any periodic point $p \in \Sigma_T$.

**Lemma 3.1.** Let $\mathcal{A} \in \mathcal{H}$ be an irreducible cocycle. For any fixed point $p \in \Sigma_T$ and any line $L \in \mathbb{R}^2$, either

1. $\mathcal{A}(p)(L) \neq L$, or
2. there exists a homoclinic point $z \in \mathcal{H}(p)$ such that $\psi^z_p(L) \neq L$.

**Proof.** Suppose the conclusion of the lemma does not hold. Then there exists an $\mathcal{A}(p)$-invariant line $L \in \mathbb{R}^2$ that is preserved under $\psi^z_p$ for all homoclinic points $z \in \mathcal{H}(p)$. For each homoclinic point $z \in \mathcal{H}(p)$, we define

$$L_z := H_{p,q}^s(L) = H_{p,q}^u(L).$$

The second equality holds because $L$ is invariant under $\psi^z_p$.

We will show that such extension of $L$ to $\mathcal{H}(p)$ is Hölder continuous. Suppose $x, y \in \mathcal{H}(p)$ with $d(x, y)$ small. Setting $z := [y, x]$, $z$ is also a homoclinic point of $p$. Then $H_{x,z}^s$ maps $L_x$ to $L_z$:

$$L_z = H_{p,z}^s(L) = H_{x,z}^s \circ H_{p,x}^s(L) = H_{x,z}^s(L_x).$$

Similarly, $L_z = H_{y,z}^u(L_y)$. Hence,

$$L_y = H_{z,y}^u \circ H_{x,z}^s(L_x).$$

Since $H_{x,y}^{s/u}$ varies $\beta$-Höder continuously in $x$ and $y$ from (b), there exists $C > 0$ depending only on $\mathcal{A}$ such that

$$\rho(L_x, L_y) \leq C d(x, y)^\beta,$$

where $\rho$ is the angular distance on $\mathbb{R}^2$.

Since $\mathcal{H}(p)$ is dense in $\Sigma_T$, it follows that $L$ can be uniquely extended to an $\mathcal{A}$-invariant and bi-holonomy invariant line bundle over $\Sigma_T$, contradicting the irreducibility of $\mathcal{A}$. \qed

The following corollary is an immediate consequence of Lemma 3.1.
Corollary 3.2. Let $A \in \mathcal{H}$ be an irreducible cocycle, $p \in \Sigma_T$ be a periodic point, and $L \in \mathbb{R}^1$ be an eigendirection of $A^{\text{per}}(p)$. Then there exists $z \in H(p)$ such that $f^z(L) \neq L$.

Let $\mathcal{E}(\sigma) \subset \mathcal{M}(\sigma)$ be the set of ergodic $\sigma$-invariant probability measures. For any $A \in C(\Sigma_T, \text{GL}_d(\mathbb{R}))$ and $\mu \in \mathcal{E}(\sigma)$, Kingman’s subadditive ergodic theorem [Kin73] ensures that the largest Lyapunov exponent $\lambda_+(A, \mu)$ of $A$ with respect to $\mu$ defined as in (1.3) satisfies

$$
\lambda_+(A, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)\| \text{ for } \mu \text{ a.e. } x \in \Sigma_T.
$$

Indeed, (3.1) may be taken as the definition of $\lambda_+(A, \mu)$. Similarly, the smallest Lyapunov exponent of $A$ with respect to $\mu$ may be defined as

$$
\lambda_-(A, \mu) := \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1} \text{ for } \mu \text{ a.e. } x \in \Sigma_T.
$$

We say $\lambda_+(A, \mu)$ are the extremal Lyapunov exponents of $A$ with respect to $\mu$. For any periodic point $p \in \Sigma_T$, we denote by $\lambda_+(A, p)$ the extremal Lyapunov exponents of the invariant measure $\mu_p$ supported on the orbit of $p$.

The following proposition from Kalinin and Sadovskaya [KS10] produces an $A$-invariant conformal (not necessarily non-trivial) sub-bundle when the extremal Lyapunov exponents of $A$ coincide for all periodic points.

Proposition 3.3. [KS10, Proposition 2.1, 2.7] Let $f$ be a transitive $C^2$ Anosov diffeomorphism on a compact manifold $M$, $\mathcal{E}$ a finite-dimensional vector bundle over $M$, and $A: \mathcal{E} \to M$ an $\alpha$-Hölder linear cocycle. Suppose for every periodic point $p \in M$, the invariant measure $\mu_p \in \mathcal{E}(\sigma)$ satisfies

$$
\lambda_+(A, p) = \lambda_-(A, p).
$$

Then either $A$ preserves an $\alpha$-Hölder continuous conformal structure on $\mathcal{E}$ or $A$ preserves an $\alpha$-Hölder continuous proper non-trivial sub-bundle $\mathcal{E}' \subset \mathcal{E}$ and an $\alpha$-Hölder continuous conformal structure on $\mathcal{E}'$.

Although it is not formulated in the statement of Proposition 3.3, the assumption (3.2) has other consequences as well. First of all, it implies that the canonical holonomies $H^{s/u}$ for $A$ converge and are as regular as the cocycle $A$ (see the proof of Corollary 3.6 in [KS13]). Moreover, the sub-bundle $\mathcal{E}'$ from Proposition 3.3 is $H^{s/u}$-invariant.

For fiber-bunched cocycles, the following proposition from Bochi and Garibaldi [BG18] shows that the converse also holds:

Proposition 3.4. [BG18, Corollary 3.5] Let $A$ be an $\alpha$-Hölder fiber-bunched cocyle of a vector bundle $\mathcal{E}$ over a hyperbolic homeomorphism. An $A$-invariant sub-bundle $\mathcal{F} \subset \mathcal{E}$ is $\alpha$-Hölder if and only if it is $H^{s/u}$-invariant.

Remark 3.5. While [BG18, Corollary 3.5] is stated for fiber-bunched cocycles, the same result holds for $\alpha$-Hölder cocycles whose canonical holonomies converge and are $\alpha$-Hölder continuous, including $\alpha$-Hölder cocycles satisfying (3.2). Moreover, Proposition 3.3 and 3.4 readily extend to our setting where the base dynamical system is a mixing subshift of finite type $(\Sigma_T, \sigma)$; see Section 6.

Hence, the conclusion of Proposition 3.3 for $A \in C^\alpha(\Sigma_T, \text{GL}_2(\mathbb{R}))$ satisfying (3.2) may be stated as follows: either $A$ preserves an $\alpha$-Hölder continuous conformal structure on $\Sigma_T \times \mathbb{R}^2$ or $A$ is reducible. The former alternative is equivalent to the existence of an $\alpha$-Hölder continuous conjugacy of $A$ into the group of linear conformal transformations of $\mathbb{R}^2$. The proof for Theorem B now easily follows.
Proof of Theorem B. Let $A \in \mathcal{H}$ be an irreducible cocycle.

If there exists a periodic point $p \in \Sigma_T$ such that $A^{\text{per}(p)}(p)$ has two eigenvalues of distinct absolute values, then $A$ is typical from Corollary 3.2.

If there is no such periodic point, then for every periodic point $p \in \Sigma_T$, the eigenvalues of $A^{\text{per}(p)}(p)$ are equal in modulus. In particular, the assumption (3.2) is satisfied and Proposition 3.3 applies. Proposition 3.3 and 3.4 imply that either there exists an $\alpha$-Hölder continuous conjugacy of $A$ into the group of conformal linear transformations of $\mathbb{R}^2$ or $A$ is reducible. Since $A$ is irreducible from the assumption, the first alternative must hold, completing the proof.

We end this section by commenting on the differences between the typicality assumptions from [BV04] and from Definition 1.2.

Remark 3.6. In Bonatti and Viana [BV04] where typicality is introduced among fiber-bunched $\text{SL}_n(\mathbb{R})$-cocycles, a cocycle is called typical if it satisfied the pinching and twisting assumption. While their pinching assumption is analogous to the pinching assumption from Definition 1.2, their twisting assumption is more restrictive. The twisting assumption of [BV04] requires that there exists a single homoclinic point $z \in \text{H}(p)$ whose holonomy loop $\psi^z_p$ twists all eigendirections. In our Definition 1.2, we allow each $v_+, v_- \in \mathbb{R}^1$ to be twisted under holonomy loops of different homoclinic points $z_+, z_- \in \text{H}(p)$, respectively. Our definition of typicality is flexible enough to establish the dichotomy in Theorem B, and yet has enough structures to guarantee the uniqueness of the equilibrium state; see Proposition 4.3 and 4.5.

Moreover, typicality in [BV04] implies that the cocycle is necessarily strongly irreducible, meaning that $A^n$ is irreducible for all $n \in \mathbb{N}$. However, our definition of typicality in Definition 1.2 does not necessarily imply that the cocycle is strongly irreducible. Indeed, we only require that $\psi^{z_\tau}_p(v_\tau) \neq v_\tau$ for each $\tau \in \{+, -\}$. In particular, it could happen that $\psi^{z_+}_p(v_-) = v_-$ and $\psi^{z_-}_p(v_+) = v_+$, and such possibility is the reason why the cocycle may fail to be strongly irreducible.

Lastly, our setting is slightly more general than [BV04] that we consider cocycles in $\mathcal{H}$ which contains the set of fiber-bunched cocycles $C^\alpha_b(\Sigma_T, \text{GL}_2(\mathbb{R}))$.

4. PROOF OF THEOREM A

We prove Theorem A in this section. In the following subsection, we first introduce the notion of quasi-multiplicativity, a property satisfied by all typical cocycles, and explain how Theorem A follows from Theorem B; see Definition 4.1, Proposition 4.3 and 4.5. Then in the subsequent subsection, we sketch the proof of Proposition 4.3.

4.1. Quasi-multiplicativity. From Theorem B, either $A$ is typical (i.e., $A \in \mathcal{U}$) or there exists a $\beta$-Hölder conjugacy $C : \Sigma_T \to \text{GL}_2(\mathbb{R})$ such that $B(x) = C(\sigma x)A(x)C(x)^{-1}$ is conformal.

In the later case, the conformality of $B$ implies that the norm of $B^n$ is multiplicative:

$$\|B^n(x)\| = \prod_{i=0}^{n-1} \|B(\sigma^i x)\|$$

for any $x \in \Sigma_T$ and $n \in \mathbb{N}$. Then $\Phi_B = \{\log \|B^n(\cdot)\|\}_{n \geq 0}$ becomes a Hölder continuous additive cocycle generated by $\varphi_B(x) := \log \|B(x)\|$ in the sense that $S_n \varphi(\cdot) = \log \|B^n(\cdot)\|$. Hence, $\Phi_B$ has a unique equilibrium state $\mu \in \mathcal{M}(\sigma)$ from Proposition 2.5. Remark 2.6 implies that $\mu$ is the unique equilibrium state of $\Phi_A$. 
Hence, in order to prove Theorem A, it suffices to establish the uniqueness of the equilibrium states for singular value potentials of typical cocycles. For typical cocycles \( \mathcal{A} \in \mathcal{U} \), the norm of \( \mathcal{A}^n \) is not necessarily multiplicative, but it is close to being multiplicative in the following sense. For every \( n \in \mathbb{N} \) and \( I \in \mathcal{L}(n) \), define
\[
\|\mathcal{A}(I)\| := \sup_{x \in [I]} \|\mathcal{A}^n(x)\|.
\]

**Definition 4.1.** We say \( \mathcal{A} \in \mathcal{H} \) is *quasi-multiplicative* if there exist \( c > 0 \), \( k \in \mathbb{N} \) such that for any \( I,J \in \mathcal{L} \), there exists \( K = K(I,J) \) with \( IJK \in \mathcal{L} \) such that \( |K| \leq k \) and
\[
\|\mathcal{A}(IKJ)\| \geq c \|\mathcal{A}(I)\| \|\mathcal{A}(J)\|.
\]

**Remark 4.2.** Quasi-multiplicativity resembles Bowen’s specification property [Bow74].

The following proposition from [Par19] states that typical cocycles are quasi-multiplicative.

**Proposition 4.3.** [Par19, Theorem A] Let \( \mathcal{A} \in \mathcal{U} \) be a typical cocycle. Then \( \mathcal{A} \) is quasi-multiplicative.

**Remark 4.4.** [Par19] proves Proposition 4.3 in the setting of fiber-bunched \( GL_d(\mathbb{R}) \)-cocycles. As noted in Remark 3.6, typicality in this paper defined as in Definition 1.2 are more general and differ slightly from typicality defined in [BV04] and [Par19]. However, Proposition 4.3 still holds for typical cocycles defined in our setting with little modifications, and we briefly sketch the proof in the following subsection.

When \( \mathcal{A} \in \mathcal{H} \) is quasi-multiplicative, then the following proposition from Feng and Käenmäki [FK10] and Feng [Fen11] establishes the uniqueness of the equilibrium state for the singular value potential \( \Phi_{\mathcal{A}} \).

**Proposition 4.5.** [FK10, Fen11] Suppose \( \mathcal{A} \in \mathcal{H} \) is quasi-multiplicative. Then \( \Phi_{\mathcal{A}} = \{ \log \varphi_{\mathcal{A},n} \}_{n \in \mathbb{N}} \) has a unique equilibrium state \( \mu_{\mathcal{A}} \), and \( \mu_{\mathcal{A}} \) has the following Gibbs property: there exists \( C \geq 1 \) such that for any \( n \in \mathbb{N} \) and \( I \in \mathcal{L}(n) \), we have
\[
C^{-1} \leq \frac{\mu_{\mathcal{A}}([I])}{e^{-n\Phi_{\mathcal{A}}} \|\mathcal{A}^n(x)\|} \leq C
\]
(4.1)
for any \( x \in [I] \).

Note that Theorem A now follows from Theorem B, Proposition 4.3 and 4.5. Before sketching the proof of Proposition 4.3, a few remarks on Proposition 4.5 are in order.

**Remark 4.6.** In [FK10], Proposition 4.5 is established for quasi-multiplicative locally constant cocycles over one-sided shifts. The result is then extended for quasi-multiplicative functions \( \varphi : \mathcal{L} \to \mathbb{R} \) in [Fen11], covering more general class of subadditive potentials. Moreover, the bounded distortion property (2.2) of the singular value potentials for cocycles in \( \mathcal{H} \) allows the result to be extended to singular value potentials of quasi-multiplicative cocycles. Such result for fiber-bunching cocycles \( C^0_b(\Sigma_T, GL_d(\mathbb{R})) \) are established in [Par19]. The only use of fiber-bunching assumption there is to establish the convergence as well as the Hölder continuity of the canonical holonomies \( H^{s/u} \), and hence, the same result holds for \( \mathcal{A} \in \mathcal{H} \) as well. This is similar in spirit to how Proposition 3.4 is stated and proved for fiber-bunched cocycles in [BG18], but the statement holds for more general cocycles; see Remark 3.5.

**Remark 4.7.** Given a finite set of matrices \( \mathcal{A} := \{A_1,\ldots,A_q\} \subset GL_d(\mathbb{R}) \), we say \( \mathcal{A} \) is *irreducible* if there does not exists a non-zero proper subspace \( V \subset \mathbb{R}^d \) such that \( A_i V = V \)
for all $1 \leq i \leq q$. Denoting by $\Sigma^+$ a one-sided full shift generated by $q$ alphabets, Feng and Käenmäki [FK10] established that if $A = \{A_1, \ldots, A_q\} \subset GL_d(\mathbb{R})$ is irreducible and $A: \Sigma^+ \to GL_d(\mathbb{R})$ is a locally constant cocycle given by $A(x) := A_{x_0}$ where $x_0$ is the 0-th entry of $x = (x_i)_{i \in \mathbb{N}_0}$, then the singular value potential $\Phi_A$ is quasi-multiplicative, and hence has a unique equilibrium state from Proposition 4.5. The same result extends via same methods to locally constant cocycles over two-sided full shifts $(\Sigma, \sigma)$ generated by irreducible sets of matrices.

We also remark that for such cocycles $A$, the irreducibility of its generating set $A$ as in the paragraph above is equivalent to the irreducibility of $A$ as in Definition 1.1; see [But19].

4.2. Proof of Proposition 4.3. We now sketch the proof of Proposition 4.3 by following the proof of [Par19, Theorem A]. The proof outlined below is simpler than the original proof appearing in [Par19] as we are working with $GL_2(\mathbb{R})$-cocycles. For statements not fully explained in the sketch of the proof, we refer the readers to [Par19, Section 4] for details.

Let $A \in \mathcal{U}$ be a typical cocycle with a periodic point $p \in \Sigma_T$ satisfying the pinching assumption and $z_\pm \in H(p)$ be homoclinic points satisfying the twisting assumption. We begin by making a few simplifying assumptions. By passing to a suitable power if necessary, we assume that $p$ is a fixed point of $\sigma$, and set $P := A(p)$. Note that if $z$ belongs to $H(p)$, then so does any point in its orbit. Moreover, the homoclinic loops of two homoclinic points in the same orbit are conjugated to each other by some power of $P$: for any $z \in H(p)$ and $r \in \mathbb{Z}$, we have

$$\psi_p^z = P^{-r} \circ \psi_p^{\sigma^r z} \circ P^r. \quad (4.2)$$

Hence, if $z_\pm \in H(p)$ satisfies the twisting assumption (i.e., $\psi_p^{\tau}(v_r) \neq v_r$ for $\tau \in \{+,-\}$), so does $\sigma^r z_\pm \in H(p)$ for any $r \in \mathbb{Z}$. So we may assume that $z_-$ belongs to $W_{\text{loc}}^u(p)$ by replacing it by its suitable pre-image under $\sigma$. Similarly, we may assume that $z_+$ belongs to $W_{\text{loc}}^s(p)$.

We now set up a few notations. As in the proof of Lemma 3.1, let $\rho$ be the angular distance on $\mathbb{R}P^1$. A $\delta$-ball in $\mathbb{R}P^1$ centered at $v$ with respect to $\rho$ will be denoted by $B_\rho(v, \delta)$. For any $A \in GL_2(\mathbb{R})$, we choose a singular value decomposition

$$A = UAV^\top$$

such that the singular values in $\Lambda$ are listed in a non-increasing order. We define $u(A)$ and $v(A)$ as the first columns of $U$ and $V$, respectively. They are related by the following equation:

$$\|A\|u(A) = A v(A).$$

When there is no confusion, we will not distinguish the notations between vectors in $\mathbb{R}^2$ and their projections to $\mathbb{R}P^1$.

Remark 4.8. $u(A)$ and $v(A)$ may be thought of as the most expanding direction of $A^*$ and $A$, respectively. Using linear algebra, it can be shown that given $\theta > 0$, there exists $c > 0$ such that for any $A, B \in GL_2(\mathbb{R})$,

$$\pi/2 - \rho(v(A), u(B)) > \theta \implies \|AB\| \geq c \|A\| \|B\|.$$ 

This will be used in establishing quasi-multiplicativity of typical cocycles.

We say $a, b, c, d \in \Sigma_T$ (in the prescribed order) form a holonomy rectangle, and denote it by $[a, b, c, d]$, if

$$b \in W^s_{\text{loc}}(a), \ c \in W^u_{\text{loc}}(b), \ d \in W^s_{\text{loc}}(c), \ \text{and} \ a \in W^u_{\text{loc}}(d).$$
For such a rectangle \([a, b, c, d]\), we define
\[
H[a, b, c, d] := H^u_{d,a} \circ H^s_{c,d} \circ H^u_{b,c} \circ H^s_{a,b}.
\]

**Lemma 4.9.** Given \(\varepsilon > 0\), there exists \(m := m(\varepsilon) \in \mathbb{N}\) such that for any holonomy rectangle \([a, b, c, d]\) and any \(v \in \mathbb{R}^2\),

1. If one of (hence, a pair of) the edges of the rectangle has length at most \(\theta^m\), then
   \[
   \rho(H[a, b, c, d](v), v) < \varepsilon/2.
   \]

2. If all edges of the rectangle have length at most \(\theta^m\), then
   \[
   \rho(H^u_{b,c} \circ H^s_{a,b}(v), v) < \varepsilon/2 \quad \text{and} \quad \rho(H^s_{d,c} \circ H^u_{a,d}(v), v) < \varepsilon/2.
   \]

**Proof.** Both claims follow from Hölder continuity (b) of the canonical holonomies \(H^{s/u}\). \(\square\)

For the proof of Proposition 4.3, we need to make use of the adjoint cocycle \(A_\ast\) over \((\Sigma_T, \sigma^{-1})\) defined as follows: for any \(x \in \Sigma_T\) and \(u, v \in \mathbb{R}^2\),
\[
\langle A_\ast(x)u, v \rangle = \langle u, A(\sigma^{-1}x)v \rangle.
\]
The adjoint cocycle \(A_\ast\) shares many properties with \(A\). For instance, if \(A\) admits holonomies \(H^{s/u}\), then \(A_\ast\) also admits holonomies \(H^{s/u, \ast}\) given by
\[
H^{s, \ast}_{x,y} = (H^u_{y,x})^\ast \quad \text{and} \quad H^{s, \ast}_{x,y} = (H^s_{y,x})^\ast.
\]
Hence, \(A\) is typical if and only if \(A_\ast\) is typical, with roles of \(z_+\) and \(z_-\) switched; see [Par19, Lemma 4.6].

Let \(v_\pm \in \mathbb{RP}^1\) be the eigendirections of \(P\) corresponding to eigenvalues \(\lambda_\pm\) with \(|\lambda_+| > |\lambda_-|\). Note that \(A_\ast(p) = P^\ast\) has the same eigenvalues \(\lambda_\pm\) as \(P\). Also, the eigendirections of \(w_+, w_- \in \mathbb{RP}^1\) of \(P^\ast\) are equal to \((v_-)^\perp, (v_+)^\perp \in \mathbb{RP}^1\), respectively.

Now we fix some constants. Let
\[
\beta := \rho(v_+, v_-) > 0.
\]
Since the top eigendirection \(w_+ \in \mathbb{RP}^1\) is equal to \((v_-)^\perp \in \mathbb{RP}^1\), \(\beta\) is also equal to \(\pi/2 - \rho(v_+, w_+) > 0\).

As \(\psi^-_p(v_-) \neq v_-\) from the twisting assumption, choose \(\varepsilon_0 \in (0, \beta/8)\) such that
\[
\rho(\psi^-_p(v_-), v_-) > \varepsilon_0 \quad \text{for any} \ v \in B_\rho(v_-, \varepsilon_0).
\]
Note that we will only make use of \(z_-\) when working with \(A\); another twisting homoclinic point \(z_+\) will be used when working with \(A_\ast\).

Let \(m := m(\varepsilon_0)\) from Lemma 4.9 and fix \(\ell \in \mathbb{N}\) such that
\[
(1) \quad d(\sigma^\ell z_-, p) \leq \theta^m, \quad \text{and} \quad \rho(v_+, v_-) > \varepsilon_0 / 2 \quad \text{for any} \ v \in B_\rho(v_-, \varepsilon_0 / 2).
\]

Such choice of \(\ell\) is possible because \(\sigma^n z_- \to p\) as \(n \to \infty\) and \(P\) has simple eigenvalues \(\lambda_\pm\) of distinct norms.

Lastly, by decreasing \(\varepsilon_0\) and increasing \(m, \ell\) in the sequential order they are defined if necessary, we assume that they also work for the adjoint cocycle \(A_\ast\) with the relevant modifications. By relevant modifications, we mean that (4.3) holds for \(\varepsilon_0 > 0\) with the roles of \(z_+\) and \(z_-\) switched and the role of \(v_- \in \mathbb{RP}^1\) replaced by \(v_+ \in \mathbb{RP}^1\). Moreover, we mean that \(m\) satisfies Lemma 4.9 for \(H^{s/u, \ast}\) and the defining properties of \(\ell\) hold for \(z_+, P^\ast, \) and \(w_\pm\). We are done with preliminary set up for the proof of Proposition 4.3.
Proof sketch of Proposition 4.3. Let $A \in U$ be a typical cocycle, $\varepsilon_0, m, \ell \in \mathbb{N}$ be as above, and $I, J \in \mathcal{L}$ be any admissible words.

Denoting the mixing rate of $\Sigma_T$ by $\bar{\tau} \in \mathbb{N}$ (i.e., $T^\tau > 0$), we can find $\omega \in [I]$ such that $\sigma^{+|I|}(\omega) \in \mathcal{W}^{u}_{\text{loc}}(p)$. By setting $\tau := \tau(I) = \bar{\tau} + m + |I|$, we ensure that $d(\sigma^{\tau}(\omega), p) \leq \theta^m$.

Lemma 4.10. There exists $x \in [I] \cap \mathcal{W}^u(p)$ such that $\sigma^\tau x \in \mathcal{W}^u_{\text{loc}}(\sigma^\tau \omega)$ and that

$$u_x := H^{s}_{\sigma^{\tau+\ell}x,p} \circ A^\ell(x) \circ H^{u}_{\sigma^{\tau}x,\sigma^\tau x} u(A^\tau(\omega))$$

(4.4)

belongs to $B_\rho(v_+,\varepsilon_0/2)$. Moreover, $d(\sigma^{\tau+\ell}x, p) \leq \theta^m$.

Proof. The construction of $x \in [I]$ depends on the following direction

$$u_\omega := H^{s}_{\sigma^{\tau}x,p} u(A^\tau(\omega)) \in \mathbb{R}^P.$$  

If $u_\omega \in B_\rho(v_-,\varepsilon_0/2)$, let $x := \sigma^{-\ell} (\sigma^\tau \omega, z_-)$, where the bracket operation is defined in (2.1). By using the defining properties of the holonomies and the definition of the holonomy rectangle, $u_x$ is equal to the following expression:

$$u_x = P^\ell \circ \psi^{-z} \circ H[p, \sigma^{\tau} \omega, \sigma^\tau x, z_-](u_\omega) \in \mathbb{R}^P.$$  

Since $d(\sigma^{\tau} w, p) \leq \theta^m$, the first statement of Lemma 4.9 applies to the holonomy rectangle $[p, \sigma^{\tau} \omega, \sigma^\tau x, z_-]$. It then follows from the assumption $u_\omega \in B_\rho(v_-,\varepsilon_0/2)$, and the choice of $\varepsilon_0 > 0$ and $\ell \in \mathbb{N}$ that $u_x$ belongs to $B_\rho(v_+,\varepsilon_0/2)$. The last claim of the lemma also follows because $d(\sigma^{\tau+\ell}x, p) = d(\sigma^\tau z_-, p) \leq \theta^m$ from the choice of $\ell$.

If instead $u_\omega \not\in B_\rho(v_-,\varepsilon_0/2)$, then set $x := \omega$ and consider the same expression $u_x \in \mathbb{R}^P$ in (4.4). Since $x$ is equal to $\omega$, the unstable holonomy $H^{s}_{\sigma^\tau \omega, \sigma^\tau x}$ is equal to the identity, and $u_x$ simplifies to $P^\ell u_\omega$ from the properties of the holonomies. Then from the assumption $u_\omega \not\in B_\rho(v_-,\varepsilon_0/2)$ and choice of $\ell \in \mathbb{N}$, it follows that $u_x$ belongs to $B_\rho(v_+,\varepsilon_0/2)$. Moreover, $d(\sigma^{\tau+\ell}x, p) = d(\sigma^\tau \omega, p) \leq \theta^m + \ell < \theta^m$.

So, in either cases, we have $x \in [I] \cap \mathcal{W}^u(p)$ with $d(\sigma^{\tau+\ell}x, p) \leq \theta^m$ such that $u_x \in \mathbb{R}^P$ is $\varepsilon_0/2$-close to $v_+ \in \mathbb{R}^P$. □

Similarly, given $J \in \mathcal{L}$, we apply the same argument to the adjoint cocycle $A_\ast$; this produces two points $\iota, \eta \in \sigma^{[J]}[J]$ where $y \in \sigma^{[J]}[J] \cap \mathcal{W}^u(p)$ is constructed depending on the direction $u_\iota := H^{s}_{\sigma^{-\tau}(J),p} u(A^{\tau}(J)(\iota)) \in \mathbb{R}^P$ such that $d(\sigma^{-(\tau(J)-\ell)}y, p) \leq \theta^m$ and that

$$u_\eta := H^{s}_{\sigma^{-\tau}(J),-\iota,y,p} \circ A^{\ell}(\sigma^{-\tau(J)}y) \circ H^{u}_{\sigma^{-(\tau(J))},\sigma^{-\tau(J)}y} u(A^{\tau}(J)(\iota)) \in \mathbb{R}^P$$

(4.5)

belongs to $B_\rho(w_+,\varepsilon_0/2)$.

Using $x \in [I]$ and $y \in \sigma^{[J]}[J]$, we create a new point $\chi := \sigma^{-(\tau(J)-\ell)}(\sigma^{\tau(J)} + \ell x, \sigma^{-(\tau(J))})$ in $[I]$. Since $d(\sigma^{\tau(J)+\ell}x, p)$ and $d(\sigma^{-\tau(J)-\ell}y, p)$ are at most $\theta^m$, the second statement of Lemma 4.9 applies to the rectangle $[p, \sigma^{\tau(J)+\ell}x, \sigma^{\tau(J)}x, \sigma^{\tau(J)-\ell}y]$. Then the fact that $u_\iota$ and $u_\eta$ are $\varepsilon_0/2$-close to $v_+$ and $w_+$, respectively, together the choice of $\varepsilon_0$ show that the $\rho$-distance between the directions

$$A^\ell(\sigma^{\tau(J)} \chi) \circ H^{u}_{\sigma^{\tau(J)\omega},\sigma^{\tau(J)\chi}} u(A^{\tau}(\omega)) \quad \text{and} \quad A^{\ell}(\sigma^{\tau(J)+2\ell} \chi) \circ H^{u}_{\sigma^{\tau(J)+2\ell} \chi,\sigma^{\tau(J)+2\ell} \chi} v(A^{\tau}(\sigma^{\tau(J)} \iota))$$

are bounded away from $\pi/2$ by at least $3\beta/4$. By applying the idea from Remark 4.8 (see [Par19, Lemma 4.5]), we obtain a connecting word $K \in \mathcal{L}$ of length $k := 2(\bar{\tau} + m + \ell)$ such that

$$\|A(\text{IKJ})\| \geq c\|A(\text{I})\|\|A(\text{J})\|$$

for some constant $c > 0$ independent of $I, J \in \mathcal{L}$. See [Par19, Section 4] for more details. □
Remark 4.11. In fact, [Par19, Theorem A] proves more than what is stated in Proposition 4.3. For typical cocycles $\mathcal{U}$ among the fiber-bunched cocycles $C^c_0(\Sigma_T, \text{GL}_d(\mathbb{R}))$, the constants $c, k > 0$ from quasi-multiplicativity can be chosen uniformly near $A$, and this implies the continuity of the pressure $P(\Phi_A)$ and the equilibrium state $\mu_A$ restricted to the set of typical cocycles $\mathcal{U}$; see [Par19, Theorem B].

In this direction of research, we remark that Cao, Pesin, and Zhao [CPZ19] recently established a more general result that the map $A \mapsto P(\Phi_A)$ is continuous on $C^\alpha(\Sigma_T; \text{GL}_d(\mathbb{R}))$ using a different approach.

5. Proof of Theorem C

We saw from Theorem A that the singular value potential $\Phi_A$ of an irreducible cocycle $A \in \mathcal{H}$ has a unique equilibrium state. In this subsection, we will show that the singular value potential $\Phi_A$ of a reducible cocycle $A \in \mathcal{H}$ has a unique equilibrium state unless the conjugated cocycle $B$ as in (1.5) satisfies condition (1) and (2) of Theorem C, in which case there are two ergodic equilibrium states for $\Phi_A$.

For reducible cocycles, we treat them by modifying the results from [FK10]. For locally constant cocycles matrix cocycles, Feng and Käenmäki [FK10] show that after simultaneously conjugating the cocycle into upper block triangle matrices of the same indices, the number of ergodic equilibrium states of the singular value potentials cannot exceed the number of the diagonal blocks. Since singular value potentials of cocycles in $\mathcal{H}$ has bounded distortion property (2.2), we may modify and apply the result of [FK10].

The following proposition states that the largest Lyapunov exponent and the singular value pressure of a GL$_2(\mathbb{R})$-cocycle taking values in upper-triangular matrices are coming from the diagonal entries.

**Proposition 5.1.** Suppose $B \in C(\Sigma_T, \text{GL}_2(\mathbb{R}))$ is of the form (1.5):

$$B(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & c(x) \end{pmatrix}.$$  

Then for any ergodic probability measure $\mu \in \mathcal{E}(\sigma)$,

(i) the Lyapunov exponent $\lambda_+(B, \mu)$ satisfies

$$\lambda_+(B, \mu) = \max \left\{ \int \log |a| \ d\mu, \int \log |c| \ d\mu \right\}.$$ 

(ii) $P(\Phi_B) = \max \left\{ P(\log |a|), P(\log |c|) \right\}.$

In order to prove Proposition 5.1, we need a lemma from ergodic theory.

**Lemma 5.2.** Let $(X, \mathcal{B}, \mu)$ be a probability space, $f : X \to X$ be a measure-preserving transformation, and $\varphi : X \to \mathbb{R}$ be a $\mu$-integrable function with $\sup_{x \in X} |\varphi(x)| < \infty$. Denoting $\alpha := \int \varphi \ d\mu$, for any $\varepsilon > 0$ and $\mu$-almost every $x \in X$, there exists $n_1 = n_1(x) \in \mathbb{N}$ such that

$$|S_n \varphi(f^m x) - n\alpha| < (n + m)\varepsilon$$

for any $n \geq n_1$ and any $m \in \mathbb{N}$.

**Proof.** Let $X_0 \subset X$ be a full measure subset from Birkhoff Ergodic Theorem such that the Birkhoff average $\frac{1}{n} \sum_{k=0}^{n-1} \varphi(x)$ converges to $\alpha$ for any $x \in X_0$. For each $x \in X_0$, choose $n_0 = n_0(x) \in \mathbb{N}$ such that

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \varphi(x) - \alpha \right| < \varepsilon/2$$
for each $n \geq n_0$. Denoting $a_n := S_n \varphi(x) - n\alpha$ for each $n$, define $n_1 = n_1(x) \geq n_0$ such that

$$n_1 \geq 2/\varepsilon \cdot \left( \max_{1 \leq i \leq n_0 - 1} |a_i| \right).$$

Consider any $n \geq n_1$ and $m \in \mathbb{N}$. If $m \geq n_0$, then

$$|S_n \varphi(f^m x) - n\alpha| = |(S_{n+m} \varphi(x) - (n+m)\alpha) - (S_m \varphi(x) - m\alpha)|,$$

$$\leq (n + 2m)\varepsilon/2,$$

$$\leq (n + m)\varepsilon.$$

If $m \leq n_0 - 1$, then

$$|S_n \varphi(f^m x) - n\alpha| = |(S_{n+m} \varphi(x) - (n+m)\alpha) - (S_m \varphi(x) - m\alpha)|,$$

$$\leq (n + m)\varepsilon/2 + |a_m|,$$

$$\leq (n + m)\varepsilon,$$

where the last inequality follows because $n \cdot \varepsilon/2 \geq n_1 \cdot \varepsilon/2 \geq |a_m|$. \qed

**Corollary 5.3.** Under the same assumptions of Lemma 5.2, suppose $C_0 := \sup_{x \in X} |\varphi(x)| < \infty$. Then for any $\varepsilon > 0$ and for $\mu$-almost every $x \in X$, there is $C(x) > 0$ such that

$$|S_n \varphi(f^m x) - n\alpha| < C(x) + (n + m)\varepsilon$$

for all $n, m \in \mathbb{N}$.

**Proof.** In view of Lemma 5.2, it suffices to set $C(x) = (C_0 + |\alpha|)(n_1(x) - 1)$ for each $x \in X_0$. \qed

We are now ready to prove Proposition 5.1.

**Proof of Proposition 5.1.** By considering $a(x)$ and $c(x)$ as multiplicative cocycles over $\Sigma_T$, let $\tau^n(x) := \prod_{i=0}^{n-1} \tau(f^i x)$ for $\tau = \{a, c\}$. Then for any $n \in \mathbb{N}$, we have

$$\mathcal{B}^n(x) = \begin{pmatrix} a^n(x) & \sum_{i=0}^{n-1} a^{n-i-1}(\sigma^{i+1} x)b(\sigma^i x)c^i(x) \\ 0 & c^n(x) \end{pmatrix}.$$

Denoting the $(i, j)$-entry of a matrix $A$ by $A_{i,j}$, we have

$$\max \left\{ |\mathcal{B}^n(x)_{1,1}|, |\mathcal{B}^n(x)_{2,2}| \right\} \leq \|\mathcal{B}^n(x)\| \leq 2^{2} \max_{1 \leq i,j \leq 2} |\mathcal{B}^n(x)_{i,j}|. \quad (5.1)$$

Here $\mathcal{B}^n(x)_{1,1} = a^n(x)$ and $\mathcal{B}^n(x)_{2,2} = c^n(x)$.

For any $\varepsilon > 0$, Corollary 5.3 applied to each $\varphi(x) = \log |a(x)|$ and $\varphi(x) = \log |c(x)|$ gives us $C(x) > 0$ for $\mu$-almost every $x \in \Sigma_T$ such that

$$|a^{n-i-1}(\sigma^{i+1} x)| \leq \exp \left( C(x) + (n - i - 1) \int \log |a| \, d\mu + n\varepsilon \right)$$

and

$$|c^i(x)| \leq \exp \left( C(x) + i \int \log |c| \, d\mu + i\varepsilon \right).$$
Denoting $L := \max_{x \in \Sigma_T} |b(x)|$, we have
\[ |\mathcal{B}^n(x)_{1,2}| = \left| \sum_{i=0}^{n-1} a^{n-i-1} (f^{i+1} x) b(f^i x) c'(x) \right|, \]
\[ \leq \sum_{i=0}^{n-1} L \exp \left( 2C(x) + (n - i - 1) \int \log |a| \, d\mu + i \int \log |c| \, d\mu + (n + i)\varepsilon \right), \]
\[ \leq nL \exp \left( 2C(x) + n \max \left\{ \int \log |a| \, d\mu, \int \log |c| \, d\mu \right\} + 2n\varepsilon \right). \]

Since $\varepsilon > 0$ was arbitrary, it follows from (5.1) that for $\mu$-a.e. $x \in \Sigma_T$, we have
\[ \lambda_+ (\mathcal{B}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \| \mathcal{B}^n (x) \| = \max \left\{ \int \log |a| \, d\mu, \int \log |c| \, d\mu \right\}, \]
establishing (i).

From (i) and the subadditive variational principle (1.2), (ii) also follows. Indeed, let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of measures in $\mathcal{E}(\sigma)$ such that $h_{\mu_n}(\sigma) + \lambda_+ (\mathcal{B}, \mu_n)$ limits to $\mathcal{P}(\Phi_B)$. By comparing $\int \log |a| \, d\mu_n$ to $\int \log |c| \, d\mu_n$ for each $n \in \mathbb{N}$, without loss of generality, we may assume that there exists $n_k \to \infty$ such that $\int \log |a| \, d\mu_{n_k} \geq \int \log |c| \, d\mu_{n_k}$ for each $k \in \mathbb{N}$. Then from (i) and (2.4), we have
\[ h_{\mu_{n_k}}(\sigma) + \lambda_+ (\mathcal{B}, \mu_{n_k}) = h_{\mu_{n_k}}(\sigma) + \int \log |a| \, d\mu_{n_k} \leq \mathcal{P}(\log |a|). \]

From the choice of $\mu_n$, the right hand side limits to $\mathcal{P}(\Phi_B)$ as $k \to \infty$ and this proves $\mathcal{P}(\Phi_B) \leq \max \left\{ \mathcal{P}(\log |a|), \mathcal{P}(\log |c|) \right\}$. Conversely, applying similar arguments to $\log |a|$ (i.e., by choosing a sequence $\mu_n \in \mathcal{E}(\sigma)$ such that $h_{\mu_n}(\sigma) + \int \log |a| \, d\mu_n$ limits to $\mathcal{P}(\log |a|)$ and making use of (i) and (1.2)) and $\log |c|$ establishes the reverse inequality. \(\square\)

If we further suppose that $\mathcal{B}$ from Proposition 5.1 is Hölder continuous, then each $\log |a|$ and $\log |c|$ is a Hölder potential over a mixing hyperbolic system $(\Sigma_T, \sigma)$ and has a unique equilibrium state from Proposition 2.5. Hence, the following corollary is a consequence of Proposition 5.1 and Remark 2.6. Also, it is clear that the corollary implies Theorem C.

**Corollary 5.4.** Suppose $\mathcal{B} \in C^3(\Sigma_T, \text{GL}_2(\mathbb{R}))$ is of the form (1.5). Then the following holds:

1. If $\mathcal{P}(\log |a|) \neq \mathcal{P}(\log |c|)$, then $\log |a| \nleq \log |c|$ and $\Phi_A$ has a unique equilibrium state.
2. If $\log |a| \sim \log |c|$, then $\mathcal{P}(\log |a|) = \mathcal{P}(\log |c|)$ and $\Phi_A$ has a unique equilibrium state $\mu_{\log |a|} = \mu_{\log |c|}$.
3. If $\log |a| \nleq \log |c|$ and $\mathcal{P}(\log |a|) = \mathcal{P}(\log |c|)$, then $\Phi_A$ has two ergodic equilibrium states $\mu_{\log |a|}$ and $\mu_{\log |c|}$.

**Remark 5.5.** The third alternative from Corollary 5.4 is not a vacuous option in that there are cocycles $\mathcal{B}$ satisfying such conditions. For instance, take any two positive Hölder continuous functions $\log |a|$, $\log |c| \in C^3(\Sigma_T, \mathbb{R}^+)$ such that there exist two periodic points $p, q \in \Sigma_T$ of some periods $n, m$ such that the Birkhoff sum $(S_n \log |a|)(p)$ equals $(S_m \log |c|)(p)$ while $(S_m \log |a|)(q)$ differs from $(S_m \log |c|)(q)$. The assumption on the Birkhoff sums along the orbit of $q$ ensures that $\log |a|$ is not cohomologous to $\log |c|$. If $\mathcal{P}(\log |a|) = \mathcal{P}(\log |c|)$, then by setting $b \equiv 0$, the cocycle $\mathcal{B}$ satisfies the conditions from the third alternative of Corollary 5.4. If not, then suppose $\mathcal{P}(\log |a|) > \mathcal{P}(\log |c|)$ without loss of generality. Since
log |c| is positive, from the variational principle (2.4), \(P(s \log |c|)\) limits to \(\infty\) as \(s \to \infty\). So there exists \(s_0 > 1\) such that \(P(\log |a|) = P(s_0 \log |c|)\), and the assumption on the Birkhoff sums along the orbit of \(p\) ensures that \(\log |a|\) is not cohomologous to \(s_0 \log |c|\). Then setting \(b \equiv 0\) again and replacing the function \(\log |c|\) by \(s_0 \log |c|\), the cocycle \(B\) satisfies the conditions from the third alternative of Corollary 5.4.

We may also choose such functions such that \(B\) is fiber-bunched as well. Indeed, start with any constant function \(\log |c| \equiv k\) with \(k \in \mathbb{R}^+\) sufficiently large compared to the entropy \(h_{\text{top}}(\sigma)\) of \((\Sigma_T, \sigma)\), and let \(\log |a|\) be a small perturbation of \(\log |c|\) obtained by slightly increasing the function in a neighborhood of some periodic orbit. If the perturbation is small enough, then \(s_0\) is sufficiently close to \(1\), and the resulting cocycle \(B\) will be fiber-bunched.

We conclude with the proof of Corollary 1.6.

**Proof of Corollary 1.6.** In view of Theorem C and Remark 2.6, we only need to show that the statement of Corollary 1.6 is well-posed, independent of the choice of conjugacy \(C\).

Indeed, if there are two Hölder conjugacies \(C_1, C_2: \Sigma_T \to \text{GL}_2(\mathbb{R})\) such that both cocycles \(B_i(x) := C_i(\sigma x)A(x)C_i(x)^{-1}, i \in \{1, 2\}\), take values in the group of upper-triangular matrices as in (1.5), then direct computation shows that \(\log |a_1| \sim \log |c_2|\) and \(\log |a_2| \sim \log |c_1|\). Hence, conditions (1) and (2) from Theorem C are intrinsic conditions on the reducible cocycles, independent of the choice of the conjugacy \(C\).

\(\square\)

6. Application to cocycles over hyperbolic systems other than \((\Sigma_T, \sigma)\)

We describe how the results in this paper can be applied to certain cocycles over hyperbolic bases other than the subshift \((\Sigma_T, \sigma)\), including Anosov diffeomorphisms of closed manifolds. Let \(f\) be a \(C^{1+\alpha}\) Anosov diffeomorphism of a closed manifold \(M\). This means that there exist a \(Df\)-invariant splitting \(TM = E^s \oplus E^u\) and constants \(C > 0, \nu \in (0, 1)\) such that for all \(n \in \mathbb{N}\),

\[
\|Df^n|_{E^s}\| \leq C\nu^n \quad \text{and} \quad \|Df^{-n}|_{E^u}\| \leq C\nu^n.
\]

For a \(C^{1+\alpha}\) Anosov diffeomorphism \(f\), the stable and unstable bundles \(E^s\) and \(E^u\) are \(\beta\)-Hölder for some \(\beta \in (0, \alpha]\).

Denoting the dimension of the unstable bundle \(E^u\) by \(d\), we then realize \(Df|_{E^u}\) as a \(\text{GL}_d(\mathbb{R})\)-cocycle over a suitable subshift of finite type \((\Sigma_T, \sigma)\) as follows: a Markov partition of \(f\) [Bow75] results in a Hölder continuous surjection \(\pi: \Sigma_T \to M\) such that \(f \circ \pi = \pi \circ \sigma\). By choosing a Markov partition of sufficiently small diameter, we may assume that the image of each cylinder \([j]\) of \(\Sigma_T\), \(1 \leq j \leq q\), is contained in an open set on which \(E^u\) is trivializable. For \(x \in [j]\), we let \(L_j(x): \mathbb{R}^d \to E^u_{\sigma x}\) be a fixed trivialization of \(E^u\) over \(\pi([j])\), and define

\[
A(x) := L_j(\sigma x)^{-1} \circ D\pi x f|_{E^u} \circ L_j(x).
\]  

This defines an \(\alpha\)-Hölder \(\text{GL}_d(\mathbb{R})\)-cocycle \(A\) over a subshift \((\Sigma_T, \sigma)\).

We say the derivative cocycle \(Df|_{E^u}\) is fiber-bunched if there exists \(N \in \mathbb{N}\) such that

\[
\|Df^n|_{E^u}\| : \|(Df^n|_{E^u})^{-1}\| \max\{\|Df^n|_{E^u}\|, \|(Df^n|_{E^u})^{-1}\|\} < 1.
\]

When \(Df|_{E^u}\) is fiber-bunched, the canonical holonomies \(H^{s/u}\) for \(Df|_{E^u}\) converge and are \(\beta\)-Hölder continuous [Via08]. Similar to Definition 1.1, we say \(Df|_{E^u}\) is reducible if there exists a proper \(Df|_{E^u}\)-invariant and \(H^{s/u}\)-invariant subbundle of \(E^u\), and irreducible otherwise. Via the same Markov partition, the \(\beta\)-Hölder canonical holonomies of \(H^{s/u}\) of
$Df|_{E^u}$ also lift to the $\beta$-Hölder canonical holonomies (denoted again by) $H^{\beta/\alpha}$ of $\mathcal{A}$ over $(\Sigma_T, \sigma)$.

The following corollary translates Corollary 5.4 for the subadditive potential $\Phi_f$ over $M$ defined by

$$\Phi_f := \{\log \|Df^n|_{E^u}\|\}_{n \in \mathbb{N}}.$$ 

**Corollary 6.1.** Let $f$ be a transitive $C^{1+\alpha}$ Anosov diffeomorphism of a closed manifold $M$. Suppose that $\dim(E^u) = 2$ and that $Df|_{E^u}$ is fiber-bunched. Then

1. If $Df|_{E^u}$ is reducible, let $\mathcal{L}$ be the $Df|_{E^u}$-invariant and bi-holonomy line bundle. Setting

   $$a(x) := Df|_{\mathcal{L}} \text{ and } c(x) := \text{Jac}(Df|_{E^u})/a(x),$$

   

2. If $Df|_{E^u}$ is reducible, let $L$ be the $Df|_{E^u}$-invariant and bi-holonomy line bundle. Setting

   $$a(x) := Df|_{L} \text{ and } c(x) := \text{Jac}(Df|_{E^u})/a(x),$$

   

(6.2) $\Phi_f$ has a unique equilibrium state unless $\log |a|$ and $\log |c|$ satisfy conditions (1) and (2) from Theorem C, in which case there are two ergodic equilibrium states.

**Proof.** For the reducible case, the proof follows the proof of Theorem B closely. If there exists a periodic point $p \in M$ such that $D_pf|_{E^u}$ has simple eigenvalues of distinct norms, then Corollary 3.2 gives homoclinic points $z_{\pm} \in M$ whose holonomy loops twist the eigendirections of $D_pf|_{E^u}$. The points in the subshift $\Sigma_T$ corresponding to $p$ and $z_{\pm}$ ensure that the cocycle $\mathcal{A}$ over $(\Sigma_T, \sigma)$ defined as (6.1) is typical. Then Proposition 4.3 and 4.5 give unique equilibrium state $\mu$ of $\Phi_f$. From the Gibbs property (4.1), $\mu$ gives zero measure to $\pi^{-1}(\partial R)$ where $\partial R$ is the union of the boundaries of the Markov partition (see [Bow75]), and hence, descends to the unique equilibrium state of $\Phi_f$.

If there exists no such periodic point $p \in M$, then Proposition 3.3 applies to show that $Df|_{E^u}$ is conformal with respect to some $\alpha$-Hölder continuous norm. By treating $\Phi_f$ as an additive potential over $M$ using conformality, Proposition 2.5 gives unique equilibrium state $\mu$ of $\Phi_f$. Note that this does not require passing to the subshift $(\Sigma_T, \sigma)$.

For the reducible case, we pass to the subshift $(\Sigma_T, \sigma)$ and the cocycle $\mathcal{A}$ as defined in (6.1). The $Df|_{E^u}$-invariant and bi-holonomy invariant line bundle $L$ over $M$ lifts to the $\mathcal{A}$-invariant and bi-holonomy invariant line bundle (also denoted by) $L$ over $\Sigma_T$. By straightening out $L$, we obtain a $\beta$-Hölder continuous conjugacy $C: \Sigma_T \to \text{GL}_2(\mathbb{R})$ of $\mathcal{A}$ into another cocycle $\mathcal{B}$ taking values in upper triangular matrices of the form (1.5) with $a, c$ defined as in (6.2). Then Theorem C provides criteria (1) and (2) for there to be two ergodic equilibrium states for $\Phi_f$. The same reasoning as in the first paragraph of this proof shows that two ergodic equilibrium states for $\Phi_f$ descend to equilibrium states for $\Phi_f$. \hfill \Box

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