Coloring Grids *

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Abstract

A structure \( A = (A; E_i)_{i \in \mathbb{N}} \) where each \( E_i \) is an equivalence relation on \( A \) is called an \( n \)-grid if any two equivalence classes coming from distinct \( E_i \)'s intersect in a finite set. A function \( \chi : A \to \mathbb{N} \) is an acceptable coloring if for all \( i \in \mathbb{N} \), the set \( \chi^{-1}(i) \) intersects each \( E_i \)-equivalence class in a finite set. If \( B \) is a set, then the \( n \)-cube \( B^n \) may be seen as an \( n \)-grid, where the equivalence classes of \( E_i \) are the lines parallel to the \( i \)-th coordinate axis. We use elementary submodels of the universe to characterize those \( n \)-grids which admit an acceptable coloring. As an application we show that if an \( n \)-grid \( A \) does not admit an acceptable coloring, then every finite \( n \)-cube is embeddable in \( A \).

1 Introduction

Following [3], for a natural number \( n \geq 2 \) we shall call an \( n \)-grid a structure of the form \( A = (A; E_i)_{i \in \mathbb{N}} \) such that each \( E_i \) is an equivalence relation on the set \( A \) and \([a]_i \cap [a]_j \) is finite whenever \( a \in A \) and \( i < j < n \) (where \([a]_i \) denotes the equivalence class of \( a \) with respect to the relation \( E_i \)). An \( n \)-cube is a particular kind of \( n \)-grid where \( A \) is of the form \( A = A_0 \times \cdots \times A_{n-1} \) and each \( E_i \) is the equivalence relation on \( A \) whose equivalence classes are the lines parallel to the \( i \)-th coordinate axis (i.e. two \( n \)-tuples are \( E_i \)-related if and only if all of their coordinates coincide except perhaps for the \( i \)-th one).

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An *acceptable coloring* for an $n$-grid $\mathcal{A}$ is a function $\chi : A \to n$ such that $[a]_i \cap \chi^{-1}(i)$ is finite for all $a \in A$ and $i \in n$.

In [3], J.H. Schmerl gives a really nice characterization of those semialgebraic $n$-grids which admit an acceptable coloring:

**Theorem 1.1.** (Schmerl) Suppose that $2 \leq n < \omega$, $\mathcal{A}$ is a semialgebraic $n$-grid and $2^{\aleph_0} \geq \aleph_{n-1}$. Then the following are equivalent:

1. some finite $n$-cube is not embeddable in $\mathcal{A}$.
2. $\mathbb{R}^n$ is not embeddable in $\mathcal{A}$.
3. $\mathcal{A}$ has an acceptable $n$-coloring.

In this note, we present a characterization that works for any $n$-grid (see Definition 2.1 and Theorem 2.7). Then we use this characterization to show that $(1) \Rightarrow (3)$ in the previous theorem holds for arbitrary $n$-grids (see Theorem 3.1). In fact, the size of the continuum turns out to be irrelevant for this implication. The implication $(3) \Rightarrow (2)$ for arbitrary $n$-grids follows from a result of Kuratowski as it is mentioned in [3]. None of these implications can be reversed for arbitrary $n$-grids, regardless of the size of the continuum.

# 2 Twisted $n$-grids

In this section we use elementary submodels of the universe to obtain a characterization of those $n$-grids which admit an acceptable coloring. At first sight this characterization seems rather cumbersome, but it is the key to our results in the next section. The case $n = 3$ was already obtained in [1] with a bit different terminology and latter used in [2].

As it has become customary, whenever we say that $M$ is an *elementary submodel of the universe*, we really mean that $(M, \in)$ is an elementary submodel of $(H(\theta), \in)$ where $H(\theta)$ is the set of all sets of hereditary cardinality less than $\theta$ and $\theta$ is a large enough regular cardinal (e.g. when we are studying a fixed $n$-grid $\mathcal{A}$ on a transitive set $A$, $\theta = (2^{\aleph_0})^+$ is large enough).

Given an equivalence relation $E$ on a set $A$, we say that $B \subseteq A$ is $E$-small if the $E$-equivalence classes restricted to $B$ are all finite. Note that the $E$-small sets form an ideal in the power set of $A$. Using this terminology, an $n$-coloring $\chi : A \to n$ is acceptable for the $n$-grid $(A; E_i)_{i \in n}$ if and only if $\chi^{-1}(i)$ is $E_i$-small for each $i \in n$. 
A test set for an n-grid $\mathcal{A}$ is a set $\mathcal{M}$ of elementary submodels of the universe such that $\mathcal{A} \in \bigcap \mathcal{M}$, $|\mathcal{M}| = n - 1$ and $\mathcal{M}$ is linearly ordered by $\in$.

**Definition 2.1.** We say that an n-grid $\mathcal{A} = (A; E_i)_{i \in \mathbb{N}}$ is twisted if for every test set $\mathcal{M}$ for $\mathcal{A}$ and every $k \in \mathbb{N}$, the set

$$\{ x \in A \cup \mathcal{M} : [x]_i \in \mathcal{M} \text{ for all } i \neq k \}$$

is $E_k$-small.

The rest of this section is devoted to show that twisted n-grids are exactly the ones that admit acceptable colorings. For this, let us fix an arbitrary n-grid $\mathcal{A} = (A; E_i)_{i \in \mathbb{N}}$; our first task is to cover $\mathcal{A}$ with countable elementary submodels in a way that allows us to define a suitable rank function for elements of $\mathcal{A}$ and for $E_i$-equivalence classes of elements of $\mathcal{A}$.

We fix $M_\Lambda$ an elementary submodel such that $A \cup \{A\} \subseteq M_\Lambda$ and we let $\kappa = |M_\Lambda|$. Thinking of $\kappa$ as an initial ordinal, we let $T = \bigcup_{m \in \omega} \kappa^m$ be the set of finite sequences of ordinals in $\kappa$. We have two natural orders on $T$, the tree (partial) order $\subseteq$ and the lexicographic order $\leq$. In both orders we have the same minimum element $\Lambda$, the empty sequence. For $\sigma \in T$ and $\alpha \in \kappa$ we write $\sigma^\alpha = \sigma \cup \{(|\sigma|, \alpha)\}$. Given $\sigma \in T \setminus \{\Lambda\}$ we write $\sigma + 1$ for the successor of $\sigma$ in the lexicographic order of $\kappa^{\sigma}$; that is

$$\sigma + 1 = (\sigma \uparrow(|\sigma| - 1))^{-} (\sigma(|\sigma| - 1) + 1).$$

We shall write $\sigma \wedge \tau$ for the infimum of $\sigma$ and $\tau$ with respect to the tree order; thus for $\sigma \neq \tau$ we have:

$$\sigma \wedge \tau = \sigma \uparrow |\sigma \wedge \tau| = \tau \uparrow |\sigma \wedge \tau|$$

and

$$\sigma(|\sigma \wedge \tau|) \neq \tau(|\sigma \wedge \tau|).$$

Now we can find inductively (on the length of $\sigma \in T$) elementary submodels $M_\sigma$ such that:

i) The sequence $\langle M_{\sigma^\alpha} : \alpha \in cof(|M_\sigma|) \rangle$ is a continuous (increasing) elementary chain,

ii) $M_\sigma \subseteq \bigcup \{M_{\sigma^\alpha} : \alpha \in cof(|M_\sigma|)\}$,

iii) $\{A\} \cup \{M_\tau : \tau + 1 \subseteq \sigma\} \subseteq M_{\sigma^0}$, and
iv) If $\tau \subsetneq \sigma$ and $M_\tau$ is uncountable then $|M_\tau| > |M_\sigma|$.

We actually do not need to (and will not) define $M_{\sigma^\alpha}$ when $M_\sigma$ is countable or if $\alpha \geq \text{cof}(|M_\sigma|)$.

Although the lexicographic order on $T$ is not a well order, it is not hard to see that conditions ii and iv allow the following definition of rank to make sense:

**Definition 2.2.** For $x \in M_\Lambda$ we define $rk(x)$ as the minimum $\sigma \in T$ (in the lexicographic order) such that $M_\sigma$ is countable and $x \in M_\tau$ for all $\tau \subseteq \sigma$.

Note that by the continuity of the elementary chains in condition i, we have that $rk(x)$ is always a finite sequence of ordinals which are either successor ordinals or $0$. In particular, if $\sigma_x = rk(x)$, $\sigma_y = rk(y)$, $\sigma_x < \sigma_y$ and $m = |\sigma_x \land \sigma_y|$, then $\sigma_y(m)$ is a successor ordinal say $\alpha + 1$ and we can define

$$\Delta(x, y) = (\sigma_x \land \sigma_y)^\alpha.$$

This last definition will only be used in the proof of Lemma 2.5. The following remark summarizes the basic properties of $\Delta(x, y)$ that we will be using; all of them follow rather easily from the definitions.

**Remark 2.3.** If $rk(x) < rk(y)$ then

- $x \in M_{\Delta(x, y)}$ and $y \notin M_{\Delta(x, y)}$,
- $\Delta(x, y) + 1 \subseteq rk(y)$,
- if $\sigma \supseteq \Delta(x, y) + 1$ then $M_{\Delta(x, y)} \in M_\sigma$ (by conditions i and iii).

After assigning a rank to each member of $M_\Lambda$, we need a way to order in type $\omega$ all the elements of $M_\Lambda$ of the same rank. This is easily done by fixing an injective enumeration

$$M_\sigma = \{t^\sigma_m : m \in \omega\}$$

for each $\sigma$ for which $M_\sigma$ is countable, and defining the degree of an element of $M_\Lambda$ as follows:

**Definition 2.4.** For $x \in M_\Lambda$ we define $deg(x)$ as the unique natural number satisfying

$$x = t^{rk(x)}_{deg(x)}.$$
The following two lemmas will be used to construct an acceptable coloring for $A$ in the case that $A$ is twisted, although the second one does not make any assumptions on $A$.

**Lemma 2.5.** If $A$ is twisted then there is a set $B \subseteq A$ and a partition $B = \bigcup_{k \in \mathbb{N}} B_k$ such that:

1. Each $B_k$ is $E_k$-small and
2. $|\{i \in n : rk([x]_i) = rk(x)\}| \geq 2$ for any $x \in A \setminus B$.

**Proof.** For each $k \in n$ we let $B_k$ be the set of all $x \in A$ such that $rk([x]_i) > rk([x]_j)$ for all $i \neq k$. Let $B = \bigcup_{k \in \mathbb{N}} B_k$.

Note that for any $x \in A$ and $i \in n$ we have that $rk([x]_i) \leq rk(x)$. On the other hand if $\sigma = rk([x]_i) = rk([x]_j)$ for some $k \neq j$, then by elementarity and the fact that $[x]_k \cap [x]_j$ is finite, it follows that $rk(x) \leq \sigma$ and hence $rk(x) = \sigma$. This observation easily implies that condition b) is met. It also implies that if $x \in B_k$ then

$$rk([x]_{k_0}) < \cdots < rk([x]_{k_{n-2}}) < rk([x]_k) \leq rk(x)$$

for some numbers $k_0, \ldots, k_{n-2}$ such that $\{k_0, \ldots, k_{n-2}, k\} = n$.

Now we put $\mathcal{M} = \{M_{\Delta([x]_i), x} : i \in n - 1\}$, and use $\mathcal{M}$ as a test set for $A$ to conclude that, since $A$ is twisted, $B_k$ is $E_k$-small.

To see that $\mathcal{M}$ is indeed a test set, it is enough to show that $M_{\Delta([x]_{k_i}, x)} \in M_{\Delta([x]_{k_j}, x)}$ for $i < j$. So fix $i < j$ and note that since $[x]_{k_i} \cap [x]_{k_j}$ is finite we have $\Delta([x]_{k_i}, x) = \Delta([x]_{k_i}, [x]_{k_j})$ and therefore by Remark 2.3

$$\Delta([x]_{k_i}, x) + 1 \subseteq rk(x) \land rk([x]_{k_j}).$$

But then $\Delta([x]_{k_i}, x) + 1 \subseteq \Delta([x]_{k_j}, x)$ and again by Remark 2.3 we get $M_{\Delta([x]_{k_i}, x)} \in M_{\Delta([x]_{k_j}, x)}$.

\[\square\]

**Lemma 2.6.** For all $i, k \in n$ with $i \neq k$, the set

$$C_{i,k} = \{x \in A : rk([x]_i) = rk([x]_k) \text{ and } \deg([x]_i) < \deg([x]_k)\}$$

is $E_k$-small.
Proof. Fix $a \in A$ and let $\sigma = rk([a]_k)$ and $d = deg([a]_k)$. Note that if $x \in C_{i,k} \cap [a]_k$ then there is an $m < d$ (namely $m = deg([x]_i)$) such that $x \in t_m^\sigma \cap t_d^\sigma$ and $t_m^\sigma \cap t_d^\sigma$ is finite. Hence $C_{i,k} \cap [a]_k$ is contained in a finite union of finite sets.

We are finally ready to prove the main result of this section.

**Theorem 2.7.** The following are equivalent:

1) $\mathcal{A}$ is twisted.

2) $\mathcal{A}$ admits an acceptable coloring.

**Proof.** Suppose first that $\mathcal{A}$ is twisted. Let $B$ and $B_k$ for $k \in n$ be as in Lemma 2.5 and let $C_{i,k}$ for $i,k \in n$ be as in Lemma 2.6. For each $k \in n$ define $C_k$ as the set of all $x \in A \setminus B$ such that:

i) $rk(x) = rk([x]_k)$, and

ii) for all $i \in n \setminus \{k\}$, if $rk([x]_i) = rk([x]_k)$ then $deg([x]_i) < deg([x]_k)$.

By condition b) in Lemma 2.5 we have that $C_k \subseteq \bigcup_{i \in n} C_{i,k}$ and therefore each $C_k$ is $E_k$-small. It also follows that the $C_k$'s form a partition of $A \setminus B$ so that we can define an acceptable coloring for $\mathcal{A}$ by:

$$\chi(x) = k$$

if and only if $x \in B_k \cup C_k$.

Now suppose that $\mathcal{A}$ admits an acceptable coloring and fix a test set $\mathcal{M}$ and $k \in n$. We want to show that the set

$$X = \{x \in A \setminus \cup \mathcal{M} : [x]_i \in \cup \mathcal{M} \text{ for all } i \neq k\}$$

is $E_k$-small. For this let $\chi : A \to n$ be an acceptable coloring such that (using elementarity and the fact that $\mathcal{M}$ is linearly ordered by $\in$) $\chi$ belongs to each $M \in \mathcal{M}$. Now if $x \in X$ and $i \neq k$ then there is an $M \in \mathcal{M}$ such that $[x]_i \cap \chi^{-1}(i) \in M$ and hence $[x]_i \cap \chi^{-1}(i) \subseteq M$ (since $\chi$ is acceptable); this implies that $\chi(x) \neq i$. It follows that $X \subseteq \chi^{-1}(k)$ so that $X$ is $E_k$-small. 

\[\square\]
3 Embedding cubes into \( n \)-grids

Given an \( n \)-grid \( A = (A; E_i)_{i \in \mathbb{N}} \) it will be convenient in this section to have a name \( \rho_i : A \rightarrow A/E_i \) for the quotient maps \( \langle \rho_i(\cdot) \rangle = \left[ \cdot \right]_i \). Note that if \( i \neq k \), \( C \subseteq A \) is infinite and \( \rho_k \upharpoonright C \) is constant, then there is an infinite \( D \subseteq C \) such that \( \rho_i \upharpoonright D \) is injective. We will make repeated use of this fact without explicitly saying so, in the proof of the following:

**Theorem 3.1.** If \( A \) is a non-twisted \( n \)-grid then any finite \( n \)-cube \( l^n \) (with \( l \in \omega \)) can be embedded in \( A \).

**Proof.** By definition, since \( A \) is not twisted, there is a test set \( \mathcal{M} \) and a \( k \in n \) such that for some \( a \in A \), the set

\[
B = \{ x \in [a]_k \setminus \bigcup \mathcal{M} : [x]_i \in \bigcup \mathcal{M} \text{ for all } i \neq k \}
\]

is infinite. For each \( x \in B \) and each \( i \in n \setminus \{k\} \) there is an \( M_i^x \in \mathcal{M} \) such that \( [x]_i \in M_i^x \). Since \( \mathcal{M} \) is finite, there must be an infinite \( C \subseteq B \) on which the map \( x \mapsto \langle M_i^x : i \in n \setminus \{k\} \rangle \) is constant, say with value \( \langle M_i : i \in n \setminus \{k\} \rangle \). Note that since \( C \) is disjoint from \( \bigcup \mathcal{M} \), the map \( i \mapsto M_i \) must be injective and hence \( \mathcal{M} = \{ M_i : i \in n \setminus \{k\} \} \), because \( |\mathcal{M}| = n - 1 \). Finally, we can find an infinite set \( D \subseteq C \) such that \( \rho_i \upharpoonright C \) is injective for all \( i \neq k \).

Now taking \( k_1 = k \) and letting \( \varphi \) be any injection from \( l \) into \( D \), we easily see that the following statement is true for \( j = 1 \):

\[
P(j) : \text{There are distinct } k_1, \ldots, k_j \in n \text{ and an embedding } \varphi : l^j \rightarrow (A; E_{k_1}, \ldots, E_{k_j}) \text{ such that:}
\]

\[a) \text{ for } i \in n \setminus \{k_1, \ldots, k_j\}, \rho_i \circ \varphi \text{ is injective and belongs to } M_i,\]

\[b) \varphi \text{ takes values in } A \setminus \bigcup \{ M_i : i \in n \setminus \{k_1, \ldots, k_j\} \}.\]

Note that when \( j = n \), conditions \( a) \) and \( b) \) become trivially true, and \( P(n) \) just says that there is an embedding (modulo an irrelevant permutation of coordinates) of the finite cube \( l^n \) into \( A \), which is exactly what we want to show. We already know that \( P(1) \) is true, so we are done if we can show that \( P(j) \) implies \( P(j + 1) \) for \( 1 \leq j < n \).

Assuming \( P(j) \), let \( \varphi : l^j \rightarrow (A; E_{k_1}, \ldots, E_{k_j}) \) be such an embedding, and let \( k_{j+1} \in n \setminus \{k_1, \ldots, k_j\} \) be such that \( M_{k_{j+1}} \) is the \( \in \)-maximum element of \( \{ M_i : i \in n \setminus \{k_1, \ldots, k_j\} \} \). Let us call

\[
\delta := \rho_{k_{j+1}} \circ \varphi \in M_{k_{j+1}}.
\]
Now note that \( \varphi \notin M_{k+1} \) and at the same time \( \varphi \) satisfies the following properties (on the free variable \( \Phi \)), all of which can be expressed using parameters from \( M_{k+1} \):

- \( \Phi : \mathcal{L} \rightarrow (A; E_{k_1}, \ldots, E_{k_j}) \) is an embedding,
- \( \rho_{k_{j+1}} \circ \Phi = \delta \),
- for \( i \in n \setminus \{k_1, \ldots, k_j, k_{j+1}\} \), \( \rho_i \circ \Phi \) is injective and belongs to \( M_i \),
- \( \Phi \) takes values in \( A \setminus \bigcup \{ M_i : i \in n \setminus \{k_1, \ldots, k_j, k_{j+1}\} \} \).

This means that there must be an infinite set (in fact there must be an uncountable one, but we won’t be using this) \( \{ \varphi_m : m \in \omega \} \) of distinct functions satisfying those properties. Going to a subsequence \( \mathcal{L}' \)-many times, we may assume without loss of generality that for each \( t \in \mathcal{L}' \), the map \( m \mapsto \varphi_m(t) \) is either constant or injective. Now since they cannot all be constant, it is not hard to see that in fact all these maps have to be injective: just note that if \( t, t' \in \mathcal{L}' \) are in a line parallel to the \((r-1)\)-th coordinate axis then it cannot be the case that the map associated with \( t \) is constant while the one associated with \( t' \) is injective, since otherwise \( \{ \varphi_m(t') : m \in \omega \} \) would be an infinite set contained in \( [\varphi_0(t)]_{k_r} \cap [\varphi_0(t')]_{k_{j+1}} \). To see this, just note that in that situation we would have \( [\varphi_m(t')]_{k_r} = [\varphi_0(t)]_{k_r} = [\varphi_0(t')]_{k_r} \) and \( [\varphi_m(t')]_{k_{j+1}} = (\rho_{k_{j+1}} \circ \varphi_m)(t') = \delta(t') = (\rho_{k_{j+1}} \circ \varphi_0)(t') = [\varphi_0(t')]_{k_{j+1}} \).

Next we can find an infinite \( I \subseteq \omega \) such that for each \( t \in \mathcal{L}' \) and each \( i \in n \setminus \{k_{j+1}\} \) the map \( m \mapsto [\varphi_m(t)]_i \) is injective when restricted to \( I \). From here one can find (one at a time) \( l \) distinct elements \( m_0, \ldots, m_{l-1} \) of \( I \) such that for all \( t, t' \in \mathcal{L}' \), for all \( r, r' \in l \) with \( r \neq r' \) and for all \( i \in n \setminus \{k_{j+1}\} \), we have that \( [\varphi_{m_r}(t)]_i \neq [\varphi_{m_{r'}}(t')]_i \).

Finally we let \( \psi : \mathcal{L}^{l+1} \rightarrow (A; E_{k_1}, \ldots, E_{k_{j+1}}) \) be the function defined by \( \psi(t, r) = \varphi_{m_r}(t) \). By the way that we constructed the \( m_r \)'s and using the fact that all the \( \varphi_m \)'s are embeddings and also using that \( \delta \) is injective, one can see that \( \psi \) is in fact an embedding. From the fact that \( \psi \) is essentially a finite union of some \( \varphi_m \)'s and by the way we chose those \( \varphi_m \)'s, it follows that conditions \( a) \) and \( b) \) in \( P(j+1) \) are satisfied.

\[ \square \]

This last theorem only goes one way: for example, the \( n \)-cube \( \omega^n \) is twisted for \( n \geq 2 \), but of course any finite \( n \)-cube can be embedded in it. I suspect that only for very “nice” classes of \( n \)-grids one can reverse this.
theorem. Schmerl’s theorem does it for semialgebraic $n$-grids; perhaps some form of o-minimality is what is required.

The question of when can an infinite cube be embedded in an arbitrary $n$-grid seems more subtle. For instance, let us consider the case $n = 2$. Using the same idea as for the proof of 3.1 one can easily show:

**Theorem 3.2.** If $A$ is a non-twisted 2-grid then either $l \times \omega_1$ can be embedded in $A$ for all $l \in \omega$, or $\omega_1 \times l$ can be embedded in $A$ for all $l \in \omega$.

However, it is not true that $\omega \times \omega$ embeds in any non-twisted 2-grid. For example, fix an uncountable family $\{A_\alpha : \alpha \in \omega_1\}$ of almost disjoint subsets of $\omega$ and let $A = \{(n, \alpha) \in \omega \times \omega_1 : n \in A_\alpha\}$. Think of $A$ as a subgrid of the 2-cube $\omega \times \omega_1$. It is easy to see that this is a non-twisted grid, but not even $\omega \times 2$ can be embedded in it.

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