ON THE HILBERT EVOLUTION ALGEBRAS OF A GRAPH
S. J. Vidal, P. Cadavid, and P. M. Rodriguez

Abstract: Evolution algebras are a special class of nonassociative algebras exhibiting connections with various fields of mathematics. Hilbert evolution algebras generalize the concept in the framework of Hilbert spaces. This allows us to deal with a wide class of infinite-dimensional spaces. We study Hilbert evolution algebras associated to a graph. Inspired by the definitions of evolution algebras we define the Hilbert evolution algebra that is associated to a given graph and the Hilbert evolution algebra that is associated to the symmetric random walk on a graph. For a given graph, we provide the conditions for these structures to be or not to be isomorphic. Our definitions and results extend to the graphs with infinitely many vertices. We also develop a similar theory for the evolution algebras associated to finite graphs.

DOI: 10.1134/S0037446622050184

Keywords: genetic algebra, evolution algebra, Hilbert space, infinite graph

1. Introduction

1.1. Evolution algebras. Evolution algebras are a special class of nonassociative algebras exhibiting connections with various fields of mathematics. The first reference to the theory of evolution algebras is due to Tian and Vojtechovsky (see [1]) who defined an evolution algebra as an algebra $\mathcal{A} := (\mathcal{A}, \cdot)$ over a field $\mathbb{K}$ with a basis $S := \{e_i\}_{i \in \Lambda}$ such that

$$e_i \cdot e_i = \sum_{k \in \Lambda} c_{ki} e_k \text{ for all } i \text{ and } e_i \cdot e_j = 0 \text{ if } i \neq j. \quad (1)$$

The concept was motivated by some specific evolution laws of genetics: If we think of alleles as generators of an algebra; then reproduction in genetics is represented by multiplication in the algebra. Indeed, the first properties stated by [1] are of interest from this biological point of view. The theory was developed further in the seminal work [2], where the author showed many interesting correspondences between these algebras and the subjects like discrete-time Markov chains, graph theory, group theory, and others. We refer the reader to [3–20] and the references therein for an overview of recent theoretical results, applications to other fields, and interesting open problems. We point out that the so-defined algebras are a special class of the genetic algebras introduced in [21].

The connections with other fields, suggested in part of the above literature, work mainly for finite-dimensional spaces. For the applications involving infinite-dimensional spaces, there still exist some gaps to be addressed as was been pointed out recently in [22]. The reason is that the previous definition considers a countable basis and assumes implicitly that the latter is a Hamel basis. Thus, the sum in (1) can have only finitely many nonzero terms. This constraint has implications, for instance, in the connection with Markov chains. Indeed, in [22, Example 1.2] the authors show that some Markov chains with a countably infinite state space have no associated evolution algebras in contrast with [2, Theorem 16, p. 54]. In order to avoid this circumstance in the series, in [22] there is introduced some generalization of the concept of evolution algebra which involves Hilbert spaces. This approach leads to considering other kind of bases; namely, Schauder bases. This gave rise to the concept of Hilbert evolution algebra which is an extension utilizing some evolution algebra structure in a given Hilbert space.

The article was submitted by the authors in English.

Original article submitted December 16, 2021; accepted June 15, 2022.
1.2. Hilbert evolution algebras. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $V$ be a finite or infinite-dimensional $\mathbb{K}$-vector space with inner product $\langle \cdot , \cdot \rangle$. A subset $\{ e_j \}_{j \in \Lambda} \subset V$, where $\Lambda$ is a countable set, is a Schauder basis for $V$ if each $v \in V$ has the unique representation 

$$v = \sum_{j \in \Lambda} v_j e_j, \text{ where } v_j \in \mathbb{K}.$$ 

We say that $V$ is a Hilbert space if $V$ is also a complete metric space with the distance function induced by the inner product. A subset $\{ e_j \}_{j \in \Lambda} \subset V$ is an orthonormal basis if every $v \in V$ can be expressed as 

$$v = \sum_{j \in \Lambda} \langle v, e_j \rangle e_j.$$ 

On the other hand, we say that $V$ is separable if $V$ has a countable dense subset. In this case, every orthonormal basis is countable. The Gram–Schmidt orthonormalization proves that every separable Hilbert space has an orthonormal basis. We highlight that if $V$ is finite-dimensional, then the notion of Schauder basis coincides with that of Hamel basis.

**Definition 1.1.** Let $A = (\mathcal{A}, \langle \cdot , \cdot \rangle)$ be a real or complex separable Hilbert space endowed with an algebra structure by the product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. We say that $A = (\mathcal{A}, \langle \cdot , \cdot \rangle, \cdot)$ is a separable Hilbert evolution algebra if $A$ satisfies the following properties:

(i) There exists an orthonormal basis $\{ e_i \}_{i \in \mathbb{N}}$ and scalars $\{ c_{ki} \}_{i,k \in \mathbb{N}}$ such that

$$e_i \cdot e_i = \sum_{k=1}^{\infty} c_{ki} e_k$$

and

$$e_i \cdot e_j = 0 \text{ if } i \neq j$$

for all $i, j \in \mathbb{N}$.

(ii) For every $v \in \mathcal{A}$, the left multiplications $L_v : \mathcal{A} \to \mathcal{A}$ defined by $L_v(w) := v \cdot w$ for each $w \in \mathcal{A}$ are continuous in the metric topology induced by the inner product; i.e., there exist constants $M_v > 0$ such that

$$\| L_v(w) \| \leq M_v \| w \| \text{ for all } w \in \mathcal{A}. \quad (4)$$

Note that (i) and (ii) are compatibility conditions between the structures. A basis satisfying (i) will be called a natural orthonormal basis. In the sequel we will work only with separable Hilbert spaces but, for the sake of simplicity, we omit the word separable and talk about Hilbert evolution algebras. It is useful to write the product explicitly in terms of the base $\{ e_i \}_{i \in \mathbb{N}}$. Let $v, w \in \mathcal{A}$ with $v = \sum_{k=1}^{\infty} v_k e_k$ and $w = \sum_{k=1}^{\infty} w_k e_k$; then, extending (2) by linearity, we have

$$v \cdot w = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i w_i c_{ki} \right) e_k.$$

**Proposition 1.2** [22, Corollary 2.4]. Let $\mathcal{A}$ be a separable Hilbert space. Consider an orthonormal basis $\{ e_i \}_{i \in \mathbb{N}}$ and suppose that the scalars $\{ c_{ki} \}_{i,k \in \mathbb{N}}$ satisfy

$$K := \sup \left\{ \sum_{k=1}^{\infty} |c_{ki}|^2 : i \in \mathbb{N} \right\} < +\infty. \quad (5)$$

Then $\mathcal{A}$ admits a Hilbert evolution algebra structure that satisfies (2) and (3).

**Example 1.3.** Let $\{ p_{ik} \}_{i,k \in \mathbb{N}}$ be the transition probabilities of a discrete-time Markov chain with state space $\mathbb{N}$; i.e., $p_{ik} \in [0,1]$ for all $i, k \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty} p_{ik} = 1.$$
for all $i \in \mathbb{N}$. It is not difficult to see that the sequence of scalars $\{c_{ki}\}_{i,k \in \mathbb{N}}$ such that $c_{ki} = p_{ik}$ satisfy (5). Indeed,

$$\sum_{k=1}^{\infty} |c_{ki}|^2 = \sum_{k=1}^{\infty} |p_{ik}|^2 \leq \sum_{k=1}^{\infty} p_{ik} = 1.$$  

In other words, contrary to what happens in evolution algebras, here we have a way of associating to each discrete-time Markov chain a Hilbert evolution algebra. We refer the reader to [22] for more details on this connection.

Let $\mathscr{A}$ be a Hilbert evolution algebra with a natural basis $\{e_i\}_{i \in \mathbb{N}}$ and structural constants $\{c_{ki}\}_{i,k \in \mathbb{N}}$. We define the evolution operator as the linear operator $C : D(C) \rightarrow \mathscr{A}$ given by its values on the natural orthonormal basis

$$C(e_i) := e_i^2 = \sum_{k=1}^{\infty} c_{ki} e_k$$

and with the domain $D(C) \subset \mathscr{A}$ defined as

$$D(C) := \left\{ v = \sum_{i=1}^{\infty} v_i e_i \in \mathscr{A} : \sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} v_i c_{ki} \right|^2 < \infty \right\}.$$  

Hence,

$$C(v) := \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i c_{ki} \right) e_k$$

for every $v = \sum_{i=1}^{\infty} v_i e_i \in D(C)$. In general, $C$ may be unbounded. Thus, it is important to find conditions on the structure constants for $C$ to be closable, closed, or bounded.

**Proposition 1.4** [22, Proposition 2.5]. Let $\mathscr{A}$ be a Hilbert evolution algebra with structure constants satisfying one of the following:

(i) $\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |c_{ki}|^2 < \infty$.

(ii) (Schur Test) There exist $\alpha_k, \beta_i > 0$, $i, k \in \mathbb{N}$, and $M_1, M_2 > 0$ such that

$$\sum_{k=1}^{\infty} |c_{ki}| \alpha_k \leq M_1 \beta_i \quad \text{for all } i \in \mathbb{N},$$

$$\sum_{i=1}^{\infty} |c_{ki}| \beta_i \leq M_2 \alpha_k \quad \text{for all } k \in \mathbb{N}.$$  

Then $D(C) = \mathscr{A}$ and the evolution operator $C : \mathscr{A} \rightarrow \mathscr{A}$ is bounded with $\|C\| \leq (M_1 M_2)^{1/2}$.

**Our contribution and outline of the paper.** The purpose of our work is to advance in the study of Hilbert evolution algebras by extending the results of the interplay between evolution algebras and graphs. This connection has been a subject of current research; see, for instance, [5, 7, 14, 15], and the references therein. One of the open problems on the subject is to understand the relationship between the evolution algebra determined by a graph and the evolution algebra induced by the symmetric random walk on the same graph. This question was stated in [2, 20] and motivated by the formulation of a new landscape in discrete geometry. Some partial results on this question are given in [5, 6], with focus on the search of the conditions for the graph to guarantee the existence of isomorphisms between these algebras.

Here we introduce the analogous definitions; namely, the Hilbert evolution algebra associated to a graph and the Hilbert evolution algebra associated to the symmetric random walk on a graph. Then we explore their connection through the analysis of the homomorphisms between them. We provide the conditions for these structures to be isomorphic, in a sense to be defined later but covering the finite-dimensional case, and we extend our discussion to the existence of unitary isomorphisms. As a sideline,
we point out that understanding the behavior of these structures related to an infinite graph can lead us to the development of new techniques to deal with general Hilbert evolution algebras by using the tools of graph theory.

The paper is organized as follows: Section 2 is devoted to a background of graph theory with some preliminary results that allow us to state our definitions of Hilbert evolution algebras associated to a graph. In Section 3 we formalize the concept of isomorphism between these new structures. We also state and prove our main theorem. Our discussion is then extended to the case of unitary isomorphisms, and the respective results are gathered in Section 4. Finally, we dedicate Section 5 to the discussion of open problems and suggestions about further research.

2. Hilbert Evolution Algebras Associated to a Graph

2.1. Brief background of graph theory. Although we assume that the reader is familiar with the basic notation of graph theory (see, for instance, [23]), we recall some part of it in the sequel for the sake of clarity. Let $G = (V, E)$ be a locally finite graph; i.e., a graph such that $\deg(i) < \infty$ for all $i \in V$, where $\deg(i)$ denotes the degree of a vertex $i$. We say that $G$ has uniformly bounded degree if there exists $M > 0$ such that $\deg(i) \leq M$ for all $i \in V$. Evidently if $G$ has uniformly bounded degree then $G$ is locally finite. The adjacency matrix of $G$ is denoted by $A := A(G) = (a_{ij})$, where $a_{ij}$ is the number of edges between vertices $i$ and $j$. Here we consider simple graphs, i.e., the graphs without multiple edges or loops, and so $a_{ii} = 0$ and $a_{ij} \in \{0,1\}$ for all $i, j \in V$.

Our first task is to define the Hilbert evolution algebra associated to a given graph. To this end, let us consider a locally finite graph $G = (V, E)$ such that $V = \mathbb{N}$ and let us add some additional structure to these objects. We will see later that our approach holds trivially for finite graphs. Following [24, 25], we will use the $\ell^2(\mathbb{N})$ Hilbert space of sequences $\{v_i\}_{i \in \mathbb{N}}$ of scalars such that $\sum_{i=1}^{\infty} |v_i|^2 < \infty$, and we denote by $\delta_j := \{\delta_{ij}\}_{i \in \mathbb{N}}$ the standard orthonormal basis for $\ell^2(\mathbb{N})$. Recall that the standard inner product in $\ell^2(\mathbb{N})$ is given by

$$\langle v, w \rangle := \sum_{i=1}^{\infty} v_i \overline{w_i},$$

where

$$v = \sum_{i=1}^{\infty} v_i \delta_i \quad \text{and} \quad w = \sum_{i=1}^{\infty} w_i \delta_i.$$

In this framework, the adjacency matrix of $G$ can be interpreted as an operator $A$ densely defined in $\ell^2(\mathbb{N})$. Hence, if $D_0$ is the dense subset of $\ell^2(\mathbb{N})$ formed by those sequences with only finitely many nonzero components, we can define the operator $A_0 : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ by

$$A_0(\delta_i) := \sum_{k=1}^{\infty} a_{ki} \delta_k$$

and we extend $A_0$ by linearity to all $D_0$ so that $D(A_0) = D_0$. Recall that $a_{ki}$ equals 1 or 0 according to whether or not there is an edge between vertices $k$ and $i$. This operator is well defined because $G$ is locally finite; i.e., $A_0(\delta_i) \in \ell^2(\mathbb{N})$ which follows from

$$\sum_{k=1}^{\infty} |a_{ki}|^2 = \sum_{k=1}^{\infty} a_{ki} = \deg(i) < \infty.$$

The domain $D_0$ is dense in $\ell^2(\mathbb{N})$, and so the adjoint $A_0^* : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ is well defined. Note that the operator $A_0$ is symmetric; i.e.,

$$\langle A_0(\delta_i), \delta_k \rangle = \langle \delta_i, A_0(\delta_k) \rangle \quad \text{for} \; i, j \in \mathbb{N},$$

998
because $a_{ik} = a_{ki}$. We recall now that a densely defined symmetric operator $T$ is always closable, because the adjoint $T^*$ is a closed linear operator and $T^*$ is an extension of $T$ (see [26, p. 38]). Thus $A_0$ is closable and its closure will be called the adjacency operator of $G$ and denoted by

$$A := A_0 : D(A) \to \ell^2(\mathbb{N})$$

where the domain of $A$ is

$$D(A) := \left\{ v = \sum_{i=1}^{\infty} v_i \delta_i \in \ell^2(\mathbb{N}) : \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} v_i a_{ki} \right|^2 < \infty \right\}.$$  \hfill (9)

Hence, $A$ is given by

$$A(v) = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i a_{ki} \right) \delta_k$$ \hfill (10)

for every $v = \sum_{i=1}^{\infty} v_i \delta_i \in D(A)$. Now we must analyze the properties of the domain of $A$ and the possibility of extending $A$ to the entire $\ell^2(\mathbb{N})$. This analysis is simplified because $A$ is symmetric. We use that the adjacency operator is real to ascertain that there exists a self-adjoint extension of $A$ which is not unique in general. Since this extension is unique if and only if $A$ is self-adjoint [24, 26], we will focus on the case of self-adjointness.

**Theorem 2.1.** Let $G$ be a locally finite graph. Then the adjacency operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is bounded if and only if $G$ has uniformly bounded degree. Specifically, if $M > 0$ is such that $\deg(i) \leq M$ for all $i \in \mathbb{N}$, \hfill (11)

then $\|A\| \leq M$. Moreover, $A$ is self-adjoint.

**Proof.** The first claim of Theorem 2.1 was proved by Mohar [24, Theorem 3.2].

Let us prove that $A$ is self-adjoint. We know that $A$ is continuous. Thus, the adjoint operator $A^* : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ exists which is defined by

$$\langle A^* v, w \rangle = \langle v, Aw \rangle$$

for all $v, w \in \ell^2(\mathbb{N})$. But $A$ is symmetric, and so $A = A^*$. \hfill $\Box$

**2.2. Hilbert evolution algebra associated to a graph.** The considerations of the previous subsection allow us to carry out the following analysis.

**Proposition 2.2.** Let $G$ be a graph with adjacency operator $A : D(A) \to \ell^2(\mathbb{N})$ given by (10) and let $\mathcal{A}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$. Then the algebra $\mathcal{A}(G)$ defined by

$$e_i \cdot e_i = \sum_{k=1}^{\infty} a_{ki} e_k \text{ and } e_i \cdot e_j = 0 \text{ if } i \neq j$$

is a Hilbert evolution algebra if and only if $G$ has uniformly bounded degree.

This $\mathcal{A}(G)$ will be called the Hilbert evolution algebra associated to $G$.

**Proof.** If $\mathcal{A}(G)$ is a Hilbert evolution algebra, then there exists a constant $K > 0$ such that $\|L_v w\| \leq K\|w\|$ for all $v, w \in \mathcal{A}(G)$. In particular $\|L_{e_i} e_i\| \leq K$ for all $i \in \mathbb{N}$; i.e.,

$$\deg(i) = \sum_{k=1}^{\infty} a_{ki} = \sum_{k=1}^{\infty} |a_{ki}|^2 = \|L_{e_i} e_i\|^2 \leq K^2$$
for all \( i \in \mathbb{N} \). Thus, \( G \) has uniformly bounded degree. Reciprocally, let \( M > 0 \) such that \( \deg(i) \leq M \) for all \( i \in \mathbb{N} \). Then
\[
\sum_{i=1}^{\infty} |a_{ki}|^2 = \sum_{i=1}^{\infty} a_{ki} = \deg(i) \leq M \quad \text{for all} \quad i \in \mathbb{N}.
\]
Thus, by Proposition 1.2 we can conclude that \( \mathcal{A}(G) \) is a Hilbert evolution algebra. \( \square \)

It is worth pointing out that our results show the strong relation between the concepts. On the one hand, we see that the adjacency operator is bounded if and only if it has uniformly bounded degree (see Theorem 2.1). On the other hand, Proposition 2.2 says that these are the only class of graphs which induces a Hilbert evolution algebra \( \mathcal{A}(G) \). Next, we see that in this framework the evolution operator is well behaved.

**Proposition 2.3.** If \( G \) has uniformly bounded degree, then \( C : \mathcal{A}(G) \to \mathcal{A}(G) \) is bounded.

**Proof.** Recall that the structure constants are given by the adjacency operator, i.e., \( c_{ij} = a_{ij} \). Again, the symmetry of \( A \) implies
\[
\deg(i) = \sum_{k=1}^{\infty} a_{ki} = \sum_{k=1}^{\infty} a_{ik} \leq M \quad \text{for all} \quad i \in \mathbb{N}.
\]
Thus we can apply the Schur test (8) of Proposition 1.4 to conclude that \( C \) is bounded. \( \square \)

For the next result we recall the following definition (see [27]): Let \( \mathcal{H} \) and \( \mathcal{K} \) be Hilbert spaces, and let \( T : \mathcal{H} \to \mathcal{K} \) and \( S : \mathcal{K} \to \mathcal{H} \) be two bounded operators. We say that \( T \) and \( S \) are unitarily equivalent if there exists a unitary operator \( U : \mathcal{H} \to \mathcal{K} \) such that \( S = UTU^* \). This concept describes when two bounded operators in Hilbert spaces are essentially the same.

**Proposition 2.4.** Let be a graph with uniformly bounded degree and let \( \mathcal{A} \) be a separable Hilbert space with an orthonormal basis \( \{e_i\}_{i \in \mathbb{N}} \). Then the adjacency operator \( A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) and the evolution operator \( C : \mathcal{A}(G) \to \mathcal{A}(G) \) are unitarily equivalent. In particular, \( C \) is self-adjoint.

**Proof.** Note first that, by the uniformly boundedness of \( G \), we can apply Theorem 2.1 to see that \( A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) is well defined, bounded, and self-adjoint.

On the other hand, by Proposition 2.3 \( C : \mathcal{A}(G) \to \mathcal{A}(G) \) is bounded. Note next that the spaces \( \ell^2(\mathbb{N}) \) and \( \mathcal{A} \) are both Hilbert and separable; hence, they are isometrically isomorphic. This implies that there is a unitary operator \( U : \ell^2(\mathbb{N}) \to \mathcal{A} \) taking the orthonormal basis \( \{\delta_i\}_{i \in \mathbb{N}} \) into the orthonormal basis \( \{e_i\}_{i \in \mathbb{N}} \), i.e., \( U(\delta_i) := e_i \). Thus, using the continuity of the operators, (7), and (10), we obtain
\[
C(U(v)) = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_ia_{ki} \right) e_k = U(A(v))
\]
for every \( v = \sum_{i=1}^{\infty} v_i\delta_i \in \ell^2(\mathbb{N}) \). Hence we can conclude that \( CU = UA \), i.e., \( C \) and \( A \) are unitarily equivalent, because \( U^* = U^{-1} \). \( \square \)

**Hilbert evolution algebra associated to the symmetric random walk on a graph.** There is a second way to define an evolution algebra associated to a graph \( G = (V, E) \); it is the one induced by a symmetric random walk (SRW) on \( G \). An SRW is a discrete-time Markov chain \( \{X_n\}_{n \geq 0} \) with the state space given by \( V \) and the transition probabilities given by
\[
P(X_{n+1} = k \mid X_n = i) = \frac{a_{ki}}{\deg(i)},
\]
where \( i, k, n \in \mathbb{N} \) and \( \deg(i) = \sum_{k=1}^{\infty} a_{ki} \). Roughly speaking, the sequence of random variables \( \{X_n\}_{n \geq 0} \) denotes the set of positions of a particle walking around the vertices of \( G \); at each discrete-time step the
next position is selected at random from the set of neighbors of the current state. In other words, we can introduce the transition operator of $G$ which is defined in the basis by

$$P(\delta_i) = \sum_{k=1}^{\infty} p_{ik} \delta_k$$

where $p_{ik} := a_{ki} / \deg(i)$ for $i, k \in \mathbb{N}$. This vector is well defined because

$$\|P(\delta_i)\|^2 = \sum_{k=1}^{\infty} \frac{a_{ki}}{\deg(i)} e_k = \frac{\sum_{k=1}^{\infty} a_{ki}}{\deg(i)^2} = \frac{1}{\deg(i)} \leq 1.$$

Note that we can do a similar analysis in the case of the adjacency operator; i.e., we can analyze the domain of the operator and conditions for boundedness. However, it is not necessary; because, as noticed in Example 1.3, the transition probabilities induce a Hilbert evolution algebra.

**Proposition 2.5.** Let $G$ be a graph, let $\mathcal{A}$ be a separable Hilbert space, and let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{A}$. Then we have a Hilbert evolution algebra associated to a symmetric random walk on $G$, denoted by $\mathcal{A}_{RW}(G)$ and defined by the product

$$e_i \cdot e_i = \sum_{k=1}^{\infty} \frac{a_{ki}}{\deg(i)} e_k \quad \text{and} \quad e_i \cdot e_j = 0 \text{ if } i \neq j.$$

**Proof.** As in Example 1.3, we can use Proposition 1.2, because

$$\sum_{k=1}^{\infty} \frac{|a_{ki}|}{\deg(i)} = 1. \quad \Box$$

Note that in the definition of $\mathcal{A}_{RW}(G)$ we do not need the condition of the uniform boundedness of the graph. The weights $1 / \deg(i)$ or, in other words, the probability condition over the structure constants is the reason for this.

### 3. Existence of Isomorphisms between $\mathcal{A}_{RW}(G)$ and $\mathcal{A}(G)$

**Definitions and the main result.** One of the open questions related to the study of evolution algebras and graphs is the connection between $\mathcal{A}_{RW}(G)$ and $\mathcal{A}(G)$. This issue was partially faced in [5,6], where the authors obtain some sufficient and necessary conditions under which both objects are isomorphic for a finite nonsingular graph. Our purpose is to extended that study to the context of Hilbert evolution algebras associated to a graph with infinite many vertices. We start with the preliminary definitions:

**Definition 3.1.** Let $\mathcal{A}$ and $\mathcal{A}'$ be Hilbert evolution algebras and let $f : \mathcal{A} \rightarrow \mathcal{A}'$ be a linear operator. We say that

(i) $f$ is a homomorphism of Hilbert evolution algebras if $f$ is an algebra homomorphism and $f$ is continuous.

(ii) $f$ is an isomorphism of Hilbert evolution algebras if $f$ is a homomorphism of Hilbert evolution algebras, there exists $f^{-1}$, and $f^{-1}$ is continuous. In this case we say that $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic as Hilbert evolution algebras, and we denote this fact by $\mathcal{A} \cong \mathcal{A}'$.

(iii) $f$ is a unitary homomorphism of Hilbert evolution algebras if $f$ is an isometric operator and an isomorphism of Hilbert evolution algebras. In this case we say that $\mathcal{A}$ and $\mathcal{A}'$ are unitarily isomorphic as Hilbert evolution algebras, and we denote this fact by $\mathcal{A} \cong_U \mathcal{A}'$.

**Definition 3.2.** We say that an infinite graph is nonsingular if the adjacency operator $A$ satisfies $\ker A = \{0\}$. 1001
Theorem 3.3. Let \( G \) be a graph with uniformly bounded degree. If \( G \) is a regular or a biregular graph; then \( \mathcal{A}_{RW}(G) \cong \mathcal{A}(G) \). Moreover, if \( G \) is nonsingular and \( \mathcal{A}_{RW}(G) \cong \mathcal{A}(G) \) then \( G \) is a regular or a biregular graph.

The proof of Theorem 3.3, which holds for graphs with infinitely many vertices, is carried out by adapting the arguments of [6, Theorem 2.3] to our framework. We recall now that the notion of isomorphism includes continuity. For the sake of clarity we left the details of the proof for Subsection 3.3.

Corollary 3.4 [6, Theorem 2.3]. Let \( G \) be a finite graph. If \( G \) is a regular or a biregular graph; then \( \mathcal{A}_{RW}(G) \cong \mathcal{A}(G) \). Reciprocally, if \( G \) is a nonsingular graph; then \( \mathcal{A}_{RW}(G) \cong \mathcal{A}(G) \) implies that \( G \) is a regular or a biregular graph.

Proof. Note that every finite graph is locally finite and has uniformly bounded degree. Moreover, if \( G \) is nonsingular then ker \( A = \{0\} \). □

We present Corollary 3.4 as stated by [6, Theorem 2.3]. However, as pointed out to us recently by [28], if \( G \) is a nonsingular graph then biregularity implies regularity.

Preliminary results for the proof of Theorem 3.3. In what follows we will use regular and biregular graphs. Recall that these are locally finite graphs with uniformly bounded degree, and so all definitions of this paper are effective.

Proposition 3.5. Let \( G \) be a regular or a biregular graph. Then \( \mathcal{A}(G) \cong \mathcal{A}_{RW}(G) \).

Proof. Let us prove the proposition for a biregular graph, as the regular case is analogous. Assume that \( G = (V_1, V_2, E) \) is a \((d_1, d_2)\)-biregular graph and consider, like in [6, Proposition 2.9], the linear map \( f : \mathcal{A}(G) \rightarrow \mathcal{A}_{RW}(G) \) defined by

\[
f(e_i) = \begin{cases} (d_1^2 d_2)^{1/3} e_i & \text{if } i \in V_1, \\ (d_1 d_2^2)^{1/3} e_i & \text{if } i \in V_2. \end{cases}
\]

Clearly, \( f \) is a linear isomorphism that behaves well with respect to the algebra product of \( \mathcal{A}(G) \). Thus \( f \) is an algebra isomorphism. To prove the continuity of \( f \), we write \( M_1 := (d_1^2 d_2)^{1/3} \) and \( M_2 := (d_1 d_2^2)^{1/3} \).

Hence,

\[
f(v) = \sum_{k=1}^{\infty} v_k f(e_k) = M_1 \sum_{k \in V_1} v_k e_k + M_2 \sum_{k \in V_2} v_k e_k
\]

for \( v = \sum_{k=1}^{\infty} v_k e_k \in \mathcal{A}(G) \). Thus

\[
\|f(v)\|^2 = M_1^2 \sum_{k \in V_1} |v_k|^2 + M_2^2 \sum_{k \in V_2} |v_k|^2 \leq M^2 \sum_{k=1}^{\infty} |v_k|^2 = M^2 \|v\|^2,
\]

where we define \( M := \max\{M_1, M_2\} \). Hence \( f \) is continuous and \( \|f\| \leq M \). Now, to prove the continuity of \( f^{-1} \) is equivalent to show that \( \|v\| \leq \|f(v)\| \) for all \( v \in \mathcal{A}(G) \). By the previous calculations

\[
\|v\|^2 = \sum_{k \in V_1} |v_k|^2 + \sum_{k \in V_2} |v_k|^2 \leq M_1^2 \sum_{k \in V_1} |v_k|^2 + M_2^2 \sum_{k \in V_2} |v_k|^2 = \|f(v)\|^2,
\]

where we use that \( M_1, M_2 \geq 1 \). Thus \( f \) is an isomorphism of Hilbert evolution algebras. □

Proposition 3.5 proves the first claim of Theorem 3.3. Note that this result has no constraint on ker \( A \) so it covers the graphs for which ker \( A \neq \{0\} \) as well. For proving the other claim of Theorem 3.3 we appeal to the following results that extend those in [6].

If \( \mathcal{A} \) is a Hilbert evolution algebra with orthonormal basis \( \{e_i\}_{i \in \mathbb{N}} \) and

\[
v = \sum_{i=1}^{\infty} v_i e_i \in \mathcal{A},
\]

then we denote by \( \Omega_v \) the set \( \Omega_v := \{i \in \mathbb{N} : v_i \neq 0\} \).

To simplify notation we write \( \Omega_i \) for \( \Omega_{f(e_i)} \).
Proposition 3.6. Let $G$ be a nonsingular graph with uniformly bounded degree. Let $\mathcal{A}$ be a separable Hilbert space, let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis, and let $\mathcal{A}(G)$ and $\mathcal{A}_{RW}(G)$ be the associated Hilbert evolution algebras. If $f : \mathcal{A}_{RW}(G) \to \mathcal{A}(G)$ is a homomorphism of Hilbert evolution algebras then either $f$ is the null map or satisfies the following:

(i) $f$ is injective.

(ii) $\{\Omega_i\}_{i \in \mathbb{N}}$ is a partition of $\mathbb{N}$.

Proof. Suppose that $f$ is not null. Let us write

$$f(e_i) = \sum_{k=1}^{\infty} t_{ki} e_k \quad \text{for all } i \in \mathbb{N}. \quad (13)$$

Note that the scalars $t_{ik}$ must satisfy

$$\sum_{k=1}^{\infty} |t_{ki}|^2 = \|f(e_i)\|^2 < \infty.$$ 

The condition $f(e_i) \cdot f(e_j) = 0$, for all $i \neq j$, implies

$$0 = \sum_{k=1}^{\infty} t_{ki} t_{kj} e_k^2 = \sum_{k=1}^{\infty} t_{ki} t_{kj} \left( \sum_{r=1}^{\infty} a_{rk} e_r \right) = \sum_{r=1}^{\infty} \left( \sum_{k=1}^{\infty} t_{ki} t_{kj} a_{rk} \right) e_r. \quad (14)$$

Given $i, j \in \mathbb{N}$, with $i \neq j$, we define

$$w_{ij} := \sum_{k=1}^{\infty} t_{ki} t_{kj} e_k.$$ 

We affirm that $w_{ij} \in \mathcal{A}$. In fact,

$$\sum_{k=1}^{\infty} |t_{ki} t_{kj}|^2 = \sum_{k=1}^{\infty} |t_{ki}|^2 |t_{kj}|^2 \leq \|f(e_j)\|^2 \sum_{k=1}^{\infty} |t_{ki}|^2 = \|f(e_j)\| \|f(e_i)\|^2 < \infty,$$

where we use that $|t_{kj}|^2 \leq \sum_{k=1}^{\infty} |t_{kj}|^2 = \|f(e_j)\|^2$ for all $k, j \in \mathbb{N}$. Then we can apply the evolution operator

$$C(w_{ij}) = \sum_{r=1}^{\infty} \left( \sum_{k=1}^{\infty} t_{ki} t_{kj} a_{rk} \right) e_r = 0,$$

where we use (14). Now, from (12) we obtain

$$C(w_{ij}) = C(U(v_{ij})) = U(A(v_{ij})) = 0,$$

where

$$v_{ij} := U^{-1}(w_{ij}) = \sum_{k=1}^{\infty} t_{ki} t_{kj} \delta_k \in \ell^2(\mathbb{N}).$$

But $U$ is an isomorphism. Thus, $A(v_{ij}) = 0$ for all $i \neq j$. This implies that $v_{ij} = 0$ for all $i \neq j$, because the graph is nonsingular; i.e., ker $A = \{0\}$. Hence

$$t_{ki} t_{kj} = 0 \text{ for all } i, j, k \in V \text{ with } i \neq j. \quad (15)$$

Thus, for $k \in \mathbb{N}$ we have $t_{ki} = 0$ for all $i \in \mathbb{N}$ or there exists at most one $i := i(k) \in \mathbb{N}$ such that $t_{ki} \neq 0$ and $t_{kj} = 0$ for all $j \neq i$. Since $\Omega_i = \{k \in \mathbb{N} : t_{ki} \neq 0\}$, we have that

$$\Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j. \quad (16)$$
To prove (i) we consider the two cases:

**Case 1.** Suppose that there exists \( i \in \mathbb{N} \) such that \( t_{ki} = 0 \) for all \( k \in V \). Then by (13) we have
\[
0 = f(e_i) = f\left( \sum_{\ell=1}^{\infty} \frac{a_{i\ell}}{\deg(i)} e_{\ell} \right) = \sum_{\ell=1}^{\infty} \frac{a_{i\ell}}{\deg(i)} f(e_{\ell}).
\]
By (16) we conclude that \( f(e_{\ell}) = 0 \) for any \( \ell \) such that \( a_{i\ell} = 1 \). Indeed, if \( f(e_{\ell}) \neq 0 \) then \( t_{ki} \neq 0 \) for some \( k \); i.e., \( k \in \Omega_{\ell} \) and (17) says that there must be some \( j \neq \ell \) with \( t_{kj} \neq 0 \); i.e., \( k \in \Omega_j \) which is not possible by (16). Then \( f(e_{\ell}) = 0 \) for all \( \ell \in \mathcal{N}(i) \). This procedure can be repeated for every \( \ell \in \mathcal{N}(i) \); i.e., \( f(e_{\ell}) = 0 \) for all \( v \in \mathcal{N}(\ell) \). As we deal with a connected graph, this procedure can be iterated finitely many times until we cover all vertices of \( G \). Therefore we can conclude that \( f(e_i) = 0 \) for every \( i \in \mathbb{N} \); a contradiction, because \( f \) is not null.

**Case 2.** For all \( i \in \mathbb{N} \) there exist \( k \in \mathbb{N} \) such that \( t_{ki} \neq 0 \). Then by (13) we have that \( f(e_i) \neq 0 \) for all \( i \in \mathbb{N} \) as desired. In this case we can prove that \( f \) is injective. Suppose that \( v \in \ker f \) and \( v \neq 0 \). Then \( v = \sum_{i=1}^{\infty} v_i e_i \) with some \( v_j \neq 0 \). For this \( j \in \mathbb{N} \) there exist at most one \( \ell \in \mathbb{N} \) such that \( t_{\ell j} \neq 0 \). Now, \( f(v) = 0 \) implies
\[
\sum_{i=1}^{\infty} v_i f(e_i) = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i t_{ki} \right) e_k = 0;
\]
i.e.,
\[
\sum_{i=1}^{\infty} v_i t_{ki} = 0 \quad \text{for all} \quad k \in \mathbb{N}.
\]
In particular \( \sum_{i=1}^{\infty} v_i t_{\ell i} = 0 \); and by (15) we can conclude that \( v_j t_{\ell j} = 0 \), which is false because both terms differ from 0.

For (ii), let us prove that
\[
\bigcup_{i \in \mathbb{N}} \Omega_i = \mathbb{N}.
\]
Let \( i \in \mathbb{N} \). We want to prove that \( i \in \Omega_{\ell} \); i.e., \( t_{i\ell} \neq 0 \) for some \( \ell \in \mathbb{N} \). To fix notation let us rewrite (13) as
\[
f(e_s) = \sum_{h \in \Omega_i} t_{hs} e_h \quad \text{for any} \quad s \in \mathbb{N}.
\]
In particular \( f(e_i) = \sum_{h \in \Omega_i} t_{hi} e_h \neq 0 \) and we can choose some \( j \in \Omega_i \). Now, as \( G \) is connected there is a finite path \( \{j = j_0, j_1, \ldots, j_n, j_{n+1} = i\} \) joining \( j \) to \( i \). We want to prove that every index in this set belongs to some \( \Omega_k \). Note that \( j_1 \in \mathcal{N}(j) \subset \mathcal{N}(\Omega_i) \) which implies
\[
f(e_i) \cdot f(e_i) = \sum_{h \in \Omega_i} t_{hi}^2 e_h = \sum_{s \in \mathcal{N}(\Omega_i)} \beta_s e_s = \beta_{j_1} e_{j_1} + \sum_{s \in \mathcal{N}(\Omega_i) \setminus \{j_1\}} \beta_s e_s.
\]
But \( f(e_i) \cdot f(e_i) = f(e_i^2) \). Hence,
\[
f(e_i^2) = \frac{1}{\deg(i)} \sum_{\ell \in \mathcal{N}(i)} f(e_{\ell}) = \beta_{j_1} e_{j_1} + \sum_{s \in \mathcal{N}(\Omega_i) \setminus \{j_1\}} \beta_s e_s
\]
which implies by (16) that there is some \( i_1 \in \mathcal{N}(i) \) satisfying \( j_1 \in \Omega_{i_1} \).

Repeating the process \( n \) times we obtain \( i_n \in \mathcal{N}(i_{n-1}) \) such that \( j_n \in \Omega_{i_n} \). But \( i \in \mathcal{N}(j_n) \subset \mathcal{N}(\Omega_{i_n}) \), and so
\[
f(e_{i_n}) \cdot f(e_{i_n}) = \sum_{s \in \mathcal{N}(\Omega_{i_n})} \beta_s e_s = \beta_i e_i + \sum_{s \in \mathcal{N}(\Omega_{i_n}) \setminus \{i\}} \beta_s e_s.
\]
Therefore,

\[ f(e_{i_n}^2) = \frac{1}{\deg(i_n)} \sum_{\ell \in \mathcal{N}(i_n)} f(e_{i_{\ell}}) = \beta_i e_i + \sum_{s \in \mathcal{N}(\Omega_n) \setminus \{i\}} \beta_s e_s \]

which implies that there exist some \( \ell \in \mathcal{N}(i_n) \) such that \( i \in \Omega_{\ell} \); i.e.,

\[ f(e_i) = t_{ii} e_i + \cdots , \quad \text{with } t_{ii} \neq 0. \]

Note that if \( i \in \Omega_l \) and \( i \in \Omega_k \) for \( k \neq l \), then we must have \( t_{ii} \neq 0 \) and \( t_{ik} \neq 0 \). But this is impossible by (15). Thus, \( \Omega_i \)'s are a partition of \( \mathbb{N} \). \( \square \)

In the previous result we find some properties of the homomorphisms of Hilbert evolution algebras between \( \mathcal{A}_{\text{RW}}(G) \) and \( \mathcal{A}(G) \). If we assume that \( f \) is an isomorphism then we have the stronger result:

**Corollary 3.7.** In the conditions of Proposition 3.6 suppose further that \( f : \mathcal{A}_{\text{RW}}(G) \rightarrow \mathcal{A}(G) \) is an isomorphism of Hilbert evolution algebras. Then

\[ f(e_i) = \alpha_i e_{\pi(i)} \quad (18) \]

where \( \alpha_i \neq 0 \), for all \( i \in \mathbb{N} \) and \( \pi : \mathbb{N} \rightarrow \mathbb{N} \) is a bijection. Moreover, there is \( M > 0 \) such that \( |\alpha_i| \leq M \) for all \( i \in \mathbb{N} \).

**Proof.** We know from the proof of Proposition 3.6 that if \( f \) is given by (13) then the scalars \( \{t_{ki}\}_{i,k \in \mathbb{N}} \) satisfy (15). Since \( f(e_i) \neq 0 \) for all \( i \in \mathbb{N} \); therefore, \( \Omega_i \neq \emptyset \). Moreover, we affirm that \( \Omega_i \) are unitary sets for all \( i \). In this case, the application \( \pi : \mathbb{N} \rightarrow \mathbb{N} \) defined by \( \pi(i) = k \), where \( k \in \Omega_i \) is well defined; and, by Proposition 3.6, \( \pi \) is a bijection. As a consequence, we have that

\[ f(e_i) = \sum_{k=1}^{\infty} t_{ki} e_k = t_{\pi(i)i} e_{\pi(i)} = \alpha_i e_{\pi(i)}, \]

where we put \( \alpha_i := t_{ki} \).

Let us see that \( \Omega_j \) is a unitary set for all \( j \in \mathbb{N} \). Suppose that for some \( i \in \mathbb{N} \) there are \( k, \ell \in \Omega_i \) and \( k \neq \ell \). That means that \( t_{ki} \neq 0 \) and \( t_{\ell i} \neq 0 \). Since \( f \) is surjective, there exist \( v^k \in \mathcal{A}_{\text{RW}}(G) \) such that \( f(v^k) = e_k \). If

\[ v^k = \sum_{j=1}^{\infty} v^k_j e_j \]

then

\[ e_k = f(v^k) = \sum_{j=1}^{\infty} v^k_j f(e_j) = \sum_{m=1}^{\infty} \left( \sum_{j=1}^{\infty} v^k_j t_{kj} \right) e_m. \]

Therefore,

\[ \sum_{j=1}^{\infty} v^k_j t_{mj} = \delta_{mk} \quad \text{for all } m \in \mathbb{N}. \]

If \( m = k \) then

\[ \sum_{j=1}^{\infty} v^k_j t_{kj} = 1. \]

Since \( t_{ki} \neq 0 \), by (15), we have that \( t_{kj} = 0 \) for \( j \neq i \). Then \( v^k_i t_{ki} = 1 \), i.e.,

\[ v^k_i = t_{ki}^{-1}. \quad (19) \]

If \( m = \ell \) then \( \sum_{j=1}^{\infty} v^k_j t_{\ell j} = 0 \). As before \( t_{\ell i} \neq 0 \) implies \( t_{\ell j} = 0 \) for \( j \neq i \), giving \( v^k_i t_{\ell i} = 0 \). Also \( v^k_i = 0 \), which is a contradiction of (19). Thus, \( k = \ell \), as desired.

The continuity of \( f \) implies that there exists \( M > 0 \) such that

\[ \|f(e_i)\| = |\alpha_i| \|e_{\pi(i)}\| = |\alpha_i| \leq M \quad \text{for all } i \in \mathbb{N}. \]
Proposition 3.8. Let $G$ be a graph. Assume that there exists a Hilbert evolution isomorphism $f : \mathcal{A}(G) \to \mathcal{A}_{RW}(G)$ defined by

$$f(e_i) = \alpha_i e_{\pi(i)} \quad \text{for all } i \in \mathbb{N}$$

where $\alpha_i \neq 0$ is a scalar and $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection. Then $G$ is a biregular graph or a regular graph.

Proof. The proof of this result is the same as that of [6, Proposition 2.9]. So we only sketch the proof referring the reader to [6] for details. If we assume that $f : \mathcal{A}(G) \to \mathcal{A}_{RW}(G)$ defined by (20) is a Hilbert evolution isomorphism; then, since $f(e_i^2) = f(e_i) \cdot f(e_i)$ for all $i \in \mathbb{N}$, we can conclude that $\deg(i) = \deg(\pi(i))$ for all $i \in \mathbb{N}$. From here it follows that $\alpha_\ell = \alpha_i^2 / \deg(i)$ for all $\ell \in \mathcal{N}(i)$. After some algebraic manipulations and appeal to the symmetry of the previous relations (note that if $\ell \in \mathcal{N}(i)$ then $i \in \mathcal{N}(\ell)$) we get

$$\frac{\alpha_{\ell_1}^2}{\deg(\ell_1)} = \alpha_i = \frac{\alpha_{\ell_2}^2}{\deg(\ell_2)} \quad \text{for } \ell_1, \ell_2 \in \mathcal{N}(i).$$

As a consequence, we obtain the following condition on the degrees in the graph: If $\ell_1, \ell_2 \in \mathcal{N}(i)$ then $\deg(\ell_1) = \deg(\ell_2)$ for each $i \in V$. This implies in turn that the graph must be a regular or a biregular graph, independently of whether the number of vertices is finite or infinite. \[\square\]

3.3. Proof of Theorem 3.3. The first claim is Proposition 3.5. Now, suppose that $G$ is nonsingular and $\mathcal{A}_{RW}(G) \cong \mathcal{A}(G)$. By Corollary 3.7 the isomorphism $f$ has the form of (18). Thus, we are in the conditions of Proposition 3.8; i.e., $G$ is regular or biregular, as desired.

4. Isometric Isomorphisms

Note that the isomorphism of Corollary 3.7 is not unitary in general. This is because $\{e_i\}_{i \in \mathbb{N}}$ is a orthonormal natural basis for both $\mathcal{A}(G)$ and $\mathcal{A}_{RW}(G)$. Thus,

$$\langle f(e_i), f(e_j) \rangle = \alpha_i \alpha_j \langle e_{\pi(i)}, e_{\pi(j)} \rangle = \alpha_i \alpha_j \delta_{\pi(i)\pi(j)}$$

which in general is not equal to $\langle e_i, e_j \rangle = \delta_{ij}$. If we want an isometric isomorphism, then the problem is that we can find an isomorphism of evolution algebras which is not unitary and there are unitary operators between the separable Hilbert spaces $\mathcal{A}(G)$ and $\mathcal{A}_{RW}(G)$ which are not homomorphisms of evolution algebras. In other words, respecting both structures at the same time is too restrictive. But (18) gives us a hint to define some new inner product on $\mathcal{A}_{RW}(G)$ to obtain an isometric isomorphism; i.e., a homomorphism which is a unitary operator.

Let $f : \mathcal{A}_{RW}(G) \to \mathcal{A}(G)$ be an isomorphism of Hilbert evolution algebras defined by (18) with $\alpha_i \neq 0$ and $i \in \mathbb{N}$. With this we can define a new basis on $\mathcal{A}_{RW}(G)$ given by

$$\tilde{e}_i := \alpha_{i}^{-1} e_i \quad \text{for all } i \in \mathbb{N}. $$

Thus we have the induced inner product on $\mathcal{A}_{RW}(G)$, turning the basis $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ into an orthonormal basis; i.e.,

$$\langle \tilde{e}_i, \tilde{e}_j \rangle_f := \delta_{ij}$$

and we define the new Hilbert space $\mathcal{A}^f = (\mathcal{A}, \langle \cdot, \cdot \rangle_f)$ with this structure. Note that in this new product the basis $\{e_i\}_{i \in \mathbb{N}}$ is no longer orthonormal; in fact

$$\langle e_i, e_j \rangle_f = \alpha_i \alpha_j \delta_{ij}.$$ 

This Hilbert space $\mathcal{A}^f$ induces the Hilbert evolution algebra $\mathcal{A}_{RW}^f(G)$, where the product is given by

$$\tilde{e}_i \cdot \tilde{e}_j = \sum_{k=1}^{\infty} \frac{a_{ki}}{\deg(i)} \tilde{e}_k \quad \text{and} \quad \tilde{e}_i \cdot \tilde{e}_j = 0 \text{ if } i \neq j.$$
Proposition 4.1. Let $f : \mathcal{A}_{\text{RW}}(G) \rightarrow \mathcal{A}(G)$ be an isomorphism of Hilbert evolution algebras defined by (18), where $\alpha_i \neq 0$, with $i \in \mathbb{N}$. Then the induced Hilbert evolution algebra $\mathcal{A}_{\text{RW}}^f(G)$ is unitarily isomorphic to $\mathcal{A}(G)$.

Proof. We want to prove that there exists some unitary Hilbert algebra isomorphism

$$\tilde{f} : \mathcal{A}_{\text{RW}}^f(G) \rightarrow \mathcal{A}(G).$$

Define the unitary isomorphism $\varphi : \mathcal{A}_{\text{RW}}^f(G) \rightarrow \mathcal{A}_{\text{RW}}(G)$ that takes an orthonormal basis into an orthonormal basis; i.e.,

$$\varphi(\tilde{e}_i) := e_i.$$

Then we can define the new isomorphism by

$$\tilde{f} := f \circ \varphi;$$

i.e., $\tilde{f}(\tilde{e}_i) = e_{\pi(i)}$ for all $i \in \mathbb{N}$. This $\tilde{f}$ is linear, bijective, continuous, and with the continuous inverse, because $f$ is a composition of some maps with these properties. It remains to prove that $\tilde{f}$ is unitary, but

$$\langle \tilde{f}(\tilde{e}_i), \tilde{f}(\tilde{e}_j) \rangle = \langle e_{\pi(i)}, e_{\pi(j)} \rangle = \delta_{\pi(i)\pi(j)} = \delta_{ij}.$$

Thus $\tilde{f}$ takes an orthonormal basis into an orthonormal basis which by linearity implies that

$$\langle \tilde{f}(v), \tilde{f}(w) \rangle = \langle v, w \rangle_f$$

for all $v, w \in \mathcal{A}_{\text{RW}}^f(G)$, as desired. $\square$

Now, we can formulate a similar result to Theorem 3.3.

Theorem 4.2. Let $G$ be a graph with uniformly bounded degree. If $G$ is a regular or a biregular graph, then $\mathcal{A}_{\text{RW}}(G) \cong_U \mathcal{A}(G)$ for some $f$ satisfying (18). Moreover, if $G$ is nonsingular and $\mathcal{A}_{\text{RW}}^f(G) \cong_U \mathcal{A}(G)$ for some $f$ satisfying (18), then $G$ is a regular or a biregular graph.

Proof. If $G$ is a regular or a biregular graph, then by Proposition 3.5 we know that there exists an isomorphism of the form

$$f(e_i) = \alpha_i e_i \quad \text{for all } i \in \mathbb{N},$$

where $\alpha_i \neq 0$. Thus, we are in the conditions of Proposition 4.1; i.e., $\mathcal{A}_{\text{RW}}^f(G)$ is unitarily isomorphic to $\mathcal{A}(G)$. On the one hand, suppose that $G$ is nonsingular and $\mathcal{A}_{\text{RW}}^f(G)$ is unitarily isomorphic to $\mathcal{A}(G)$ for some $f$ satisfying (18). In particular, we are in the conditions of Proposition 3.8. It follows that $G$ is a biregular or a regular graph. $\square$

5. Discussion

5.1. Homomorphisms that are not isomorphisms. In the finite case it was proved [6, Proposition 2.14] that, if $G$ is a nonsingular graph and $\mathcal{A}_{\text{RW}}(G)$ and $\mathcal{A}(G)$ are not isomorphic, then the only homomorphism of evolution algebras is the null map. In our case we prove that, for nonsingular graphs, the homomorphisms are always injective (see Proposition 3.6), but it was not possible to us, to prove that they are surjective (in the finite case this is a direct consequence of injectivity). Thus, the question remains open: Does there exist an injective but not surjective homomorphism between $\mathcal{A}_{\text{RW}}(G)$ and $\mathcal{A}(G)$ for a nonsingular graph?

On the other hand, it is worth pointing out that for some singular graphs we can find homomorphisms different from the isomorphisms identified in Corollary 3.7. Let us explain this by considering the twin partition of a given graph. Let us start with some additional definitions from graph theory. We say that vertices $i$ and $j$ of $G$ are twins if they have exactly the same set of neighbors, i.e., $\mathcal{M}(i) = \mathcal{M}(j)$. 

1007
We notice that by defining the relation $\sim_i$ on the set of vertices $V$ by $i \sim_i j$ whether $i$ and $j$ are twins, then $\sim_i$ is an equivalence. A coset of the twin relation is referred to as a twin class. In other words, the twin class of a vertex $i$ is $\{j \in V : i \sim_i j\}$. The set of all twin classes of $G$ is denoted by $\Pi(G)$ and is referred to as the twin partition of $G$. A graph is twin-free if it has no twins. We denote by $G/\Pi$ the quotient graph of $G$ with respect to $\Pi$; which is the graph obtained by merging each class of $\Pi$ into a single vertex. In this case, two vertices in $G/\Pi$ are neighbors if and only if their respective twin classes in $G$ are neighbors in the sense that each vertex of one of the classes is a neighbor of each vertex of the other one. Notice that this operation on the graph may be seen as the one called twin reduction in the literature; see, for instance, [29, Section 3.3]. It is not difficult to understand that this construction leads to a uniquely determined twin-free graph.

**Proposition 5.1.** Let $G$ be a graph with uniformly bounded degree. Let $\Pi(G)$ be the twin partition of $G$ and let $G/\Pi$ be the quotient graph of $G$ with respect to $\Pi$. If $G/\Pi$ is a regular or a biregular graph, then there exists a noninjective surjective homomorphism between $\mathcal{A}_{\text{RW}}(G)$ and $\mathcal{A}(G)$.

**Proof.** Let $G$ be a graph with uniformly bounded degree, and let $\mathcal{A}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$. Consider the Hilbert evolution algebras $\mathcal{A}(G)$ and $\mathcal{A}_{\text{RW}}(G)$ defined by
\[
e_i \cdot e_i = \sum_{k=1}^{\infty} a_{ki} e_k \quad \text{and} \quad e_i \cdot e_j = 0 \text{ if } i \neq j
\]
(see Proposition 2.2) and
\[
e_i \cdot e_i = \sum_{k=1}^{\infty} \frac{a_{ki}}{\deg(i)} e_k \quad \text{and} \quad e_i \cdot e_j = 0 \text{ if } i \neq j
\]
(see Proposition 2.4). Now, let $\Pi(G) = \{[i_k]_{k=1}^{\|\Pi(G)\|}\}$ be the twin partition of $G$, where $[i_k]$ denotes the twin class of a vertex $i_k$. This induces the closed linear span given by
\[
\mathcal{A}^{\Pi} := \overline{\text{span}}\{e_{i_k} : k = 1, \ldots, \|\Pi(G)\|\}
\]
which is a closed linear subspace of $\mathcal{A}$. Now consider the quotient graph $G/\Pi$ of $G$ with respect to $\Pi$ and note that $G/\Pi$ has uniformly bounded degree. Thus, we have the induced Hilbert evolution algebras $\mathcal{A}^{\Pi}(G/\Pi)$ and $\mathcal{A}_{\text{RW}}^{\Pi}(G/\Pi)$. If the quotient graph $G/\Pi$ is a regular or a biregular graph, then Theorem 3.3 guarantees the existence of an isomorphism $f : \mathcal{A}_{\text{RW}}^{\Pi}(G/\Pi) \rightarrow \mathcal{A}^{\Pi}(G/\Pi)$. This isomorphism allows us to construct a homomorphism denoted by the same letter, $f : \mathcal{A}_{\text{RW}}(G) \rightarrow \mathcal{A}(G)$, and defined by
\[
f(e_i) := f(e_{i_k}) \quad \text{for all } i \in \mathbb{N}, \ i \sim_i i_k.
\]
Note that $f$ is not injective because two different generators with twin indices have the same image. And $f : \mathcal{A}_{\text{RW}}(G) \rightarrow \mathcal{A}(G)$ is surjective because $f : \mathcal{A}_{\text{RW}}^{\Pi}(G/\Pi) \rightarrow \mathcal{A}^{\Pi}(G/\Pi)$ is surjective. $\square$

Thus, Proposition 5.1 gives examples of singular graphs where we have a noninjective surjective homomorphism between $\mathcal{A}_{\text{RW}}(G)$ and $\mathcal{A}(G)$.

**5.2. On the condition of the uniform boundedness of a graph.** Since we are dealing with graphs with infinitely many vertices we appeal to the self-adjointness of the adjacency operator, defined by (10), to guarantee that it is well defined into the entire space $l^2(\mathbb{N})$. Thus Theorem 2.1 lead us to confine our attention to the wide class of graphs with uniformly bounded degree. Moreover, Proposition 2.2 implies that only for these graphs we can define a Hilbert evolution algebra. In order to find examples of graphs with, and without, uniformly bounded degree we can appeal to the spherically symmetric trees. A rooted tree $T$, with root $0$, is called spherically symmetric if for each vertex $v$ its degree $\deg(v)$ depends only on $\text{dist}(0, v)$. See [30, Chapter 3, Section 2] for some properties. The 2 periodic tree with degrees 2 and 3, $\mathcal{T}_{2,3}$, is an example of an infinite graph with uniformly bounded degree; see Fig. 5.1(a). Then
Proposition 2.2 guarantees the existence of a well-defined Hilbert evolution algebra $\mathcal{A}(\mathbb{T}_{2,3})$. On the other hand, we cannot claim the same for the factorial tree $\mathbb{T}_1$, which is an example of an infinite graph which is locally finite but has not uniformly bounded degree. Indeed, $\mathbb{T}_1$ is the tree such that each vertex at a distance $n$ from the root has degree $n + 2$, and so the number of vertices at distance $n$ from the root is $n!$; see Fig. 5.1(b). The direction of further research could consist in extending our framework to deal with the graphs that are not of uniformly bounded degree. Then we are interested in extending the theory to the infinite graphs that are locally finite but not necessarily of uniformly bounded degree. Let $G$ be a locally finite graph, and let $\mathcal{A}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$. We consider the different induced Hilbert space

$$\mathcal{A}_d = \left\{ v = \sum_{i=1}^{\infty} v_i e_i : \sum_{i=1}^{\infty} |v_i|^2 \deg(i) < \infty \right\},$$

where we define the weighted inner product

$$\langle v, w \rangle_d := \sum_{i=1}^{\infty} v_i \overline{w_i} \deg(i)$$

for $v, w \in \mathcal{A}_d$, with corresponding norm $\|v\|_d := (\sum_{i=1}^{\infty} |v_i|^2 \deg(i))^{1/2}$. In this space the basis $\{e_i\}_{i \in \mathbb{N}}$ is no longer orthonormal. Instead, we have $\|e_i\|_d^2 = \deg(i)$. With this we have a result similar to Proposition 2.2.

Fig. 5.1. Example of infinite graphs with uniformly bounded degree (a), and without uniformly bounded degree (b).

**Proposition 5.2.** Let $G$ be a graph with adjacency operator $A : D(A) \to \ell^2(\mathbb{N})$ given by equation (10) and let $\mathcal{A}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$. If $G$ is a locally finite graph, then the Hilbert space $\mathcal{A}_d$ induces a Hilbert evolution algebra $\mathcal{A}_d(G)$ defined by

$$e_i \cdot e_i = \sum_{k=1}^{\infty} \frac{\alpha_{ki}}{\deg(k)^{1/2}} e_k \quad \text{and} \quad e_i \cdot e_j = 0 \text{ if } i \neq j.$$
PROOF. Let \( v \in \mathcal{A}_d(G) \). We must prove the continuity of the operator \( L_v : \mathcal{A}_d(G) \rightarrow \mathcal{A}_d(G) \). Write

\[
v = \sum_{i=1}^{\infty} v_i e_i \quad \text{and} \quad w = \sum_{i=1}^{\infty} w_i e_i.
\]

Then

\[
\|L_v w\|_d^2 = \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} v_i w_i a_{ki} / \deg(k)^{1/2} \right|^2 \deg(k).
\]

Hence

\[
\|L_v w\|_d^2 \leq \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} |v_i|^2 \right) \left( \sum_{i=1}^{\infty} |w_i|^2 a_{ki}^2 / \deg(k)^{1/2} \right) \deg(k)
= \left( \sum_{i=1}^{\infty} |v_i|^2 \right) \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |w_i|^2 a_{ki}^2
= \|v\|^2 \sum_{i=1}^{\infty} |w_i|^2 \deg(i)
= \|v\|^2 \|w\|_d^2
\]

for all \( w \in \mathcal{A}_d(G) \). It follows that \( \|L_v\|_d \leq \|v\| \) for all \( v \in \mathcal{A}_d(G) \). \( \square \)

In this case, we can consider the construction at the beginning of Section 2 to obtain the well-defined adjacency operator \( A : D(A) \rightarrow \ell^2(\mathbb{N}) \), where the domain is given by (9). Recall that, by construction, \( A \) is a closed operator. Now the question is when \( D(A) \) is dense in \( \ell^2(\mathbb{N}) \), which depends on the structure constants \( a_{ki} \). If \( G \) is locally finite but not of uniformly bounded degree, then \( D(A) \) is a proper subset of \( \ell^2(\mathbb{N}) \). If not, then \( D(A) = \ell^2(\mathbb{N}) \) and \( A \) is closed, which by the Closed Graph Theorem implies that \( A \) will be a bounded operator which is impossible because \( A \) has no uniformly bounded degree (see Theorem 2.1). Moreover, it is important to study the conditions on the structure constants in order to have some well-defined evolution operator \( C : \mathcal{A}_d(G) \rightarrow \mathcal{A}_d(G) \) which must be of the form \( C(e_i) = e_i^2 = \sum_{k=1}^{\infty} (a_{ki} / \deg(k)^{1/2}) e_k \) (recall that the condition of the uniform boundedness of the degree is not longer used in this case). In this framework the operators \( A \) and \( C \) are not unitarily conjugated, and so the analysis becomes more complicated.

Acknowledgments. Part of this work was carried out during a visit of S.J. Vidal to the Federal University of Pernambuco. P.M. Rodrigues thanks the FAPESP (Grant 2017/10555–0) for support.

References

1. Tian J.P. and Vojtechovsky P., “Mathematical concepts of evolution algebras in non-Mendelian genetics,” Quasigroups Relat. Syst., vol. 1, no. 4, 111–122 (2006).
2. Tian J.P., Evolution Algebras and Their Applications, Springer, Berlin and Heidelberg (2008).
3. Cabrera Y., Siles M., and Velasco M.V., “Evolution algebras of arbitrary dimension and their decompositions,” Linear Algebra Appl., vol. 495, 122–162 (2016).
4. Cabrera Y., Siles M., and Velasco M.V., “Classification of three-dimensional evolution algebras,” Linear Algebra Appl., vol. 524, no. 3, 68–108 (2017).
5. Cadavid P., Rodiño Montoya M.L., and Rodriguez P.M., “The connection between evolution algebras, random walks, and graphs,” J. Algebra Appl., vol. 19, no. 2, 2050023 (2020).
6. Cadavid P., Rodiño Montoya M.L., and Rodriguez P.M., “On the isomorphisms between evolution algebras of random walks and graphs,” Linear Multilinear Algebra, vol. 69, no. 10, 1858–1877 (2021).
7. Cadavid P., Rodiño Montoya M.L., and Rodriguez P.M., “Characterization theorems for the space of derivations of evolution algebras associated to graphs,” Linear Multilinear Algebra, vol. 68, no. 7, 1340–1354 (2020).
8. Cabrera Casado Y., Cadavid P., Rodiño Montoya M.L., and Rodriguez P.M., “On the characterization of the space of derivations in evolution algebras,” Ann. Mat. Pura Appl. (4), vol. 200, no. 2, 737–755 (2021).
9. Camacho L.M., Gómez J.R., Omirov B.A., and Turdibaev R.M., “Some properties of evolution algebras,” Bull. Korean Math. Soc., vol. 50, no. 5, 1481–1494 (2013).
10. Camacho L.M., Gómez J.R., Omirov B.A., and Turdibaev R.M., “The derivations of some evolution algebras,” Linear Multilinear Algebra, vol. 61, 309–322 (2013).
11. Cardoso M.I., Gonçalves D., Martín D., Martín C., and Siles M., “Squares and associative representations of two dimensional evolution algebras,” J. Algebra Appl., vol. 20, no. 6, 2150090 (2020).
12. Casas J.M., Ladra M., Omirov B.A., and Rozikov U.A., “On nilpotent index and dibaricity of evolution algebras,” Linear Algebra Appl., vol. 439, no. 1, 90–105 (2013).
13. Casas J.M., Ladra M., and Rozikov U.A., “A chain of evolution algebras,” Linear Algebra Appl., vol. 435, no. 4, 852–870 (2011).
14. Elduque A. and Labra A., “Evolution algebras and graphs,” J. Algebra Appl., vol. 14, no. 7, 1550103 (2015).
15. Elduque A. and Labra A., “Evolution algebras, automorphisms, and graphs,” Linear Multilinear Algebra, vol. 69, no. 2, 331–342 (2021).
16. Falcón O.J., Falcón R.M., and Núñez J., “Algebraic computation of genetic patterns related to three-dimensional evolution algebras,” Appl. Math. Comput., vol. 319, no. 3, 510–517 (2018).
17. Paniello I., “Evolution coalgebras,” Linear Multilinear Algebra, vol. 67, no. 8, 1539–1553 (2019).
18. Reis T. and Cadavid P., “Derivations of evolution algebras associated to graphs over a field of any characteristic,” Linear Multilinear Algebra (2020). doi 10.1080/03081087.2020.1818673
19. Rozikov U.A., Population Dynamics: Algebraic and Probabilistic Approach, World Sci., Singapore (2020).
20. Tian J.P., “Invitation to research of new mathematics from biology: evolution algebras,” in: Topics in Functional Analysis and Algebra, Amer. Math. Soc., Providence (2016), 257–272 (Contemp. Math.; vol. 672).
21. Schafer R.D., “Structure of genetic algebras,” Amer. J. Math., vol. 71, 121–135 (1949).
22. Vidal S.J., Cadavid P., and Rodriguez P.M., “Hilbert evolution algebras and its connection with discrete-time Markov chains,” Indian J. Pure Appl. Math. (in press).
23. Mohar B. and Woess W., “A survey on spectra of infinite graphs,” Bull. Lond. Math. Soc., vol. 21, no. 3, 209–234 (1989).
24. Mohar B. and Woess W., “A survey on spectra of infinite graphs,” Bull. Lond. Math. Soc., vol. 21, no. 3, 209–234 (1989).
25. Schmüdgen K., Unbounded Self-Adjoint Operators on Hilbert Space, Springer, Dordrecht (2012) (Graduate Texts in Mathematics; vol. 265).
26. Sunder V.S., Operators on Hilbert Space, Springer, Singapore (2016).
27. Henao C.A. and Rodiño Montoya M.L., Private communication.
28. Lovász L., Large Networks and Graph Limits, Amer. Math. Soc., Providence (2012) (Colloquium Publications 60).
29. Lyons R. and Peres Y., Probability on Trees and Networks, Cambridge University, Cambridge (2017).

S. J. VIDAL
DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE INGENIERÍA
UNIVERSIDAD NACIONAL DE LA PATAGONIA “SAN JUAN Bosco”
COMODORO RIVADAVIA, ARGENTINA
E-mail address: sebastianvidal79@gmail.com

P. CADAVID
UNIVERSIDADE FEDERAL DO ABC, SANTO ANDRÉ, BRASIL
E-mail address: pacadavid@gmail.com

P. M. RODRIGUEZ (corresponding author)
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
UNIVERSIDADE FEDERAL DE PERNAMBUCO, RECIFE, BRASIL
E-mail address: pablo@de.ufpe.br