Exceptional Lie Algebra $E_7(-25)$  
(Multiplets and Invariant Differential Operators)

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Abstract

In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact exceptional algebra $E_7(-25)$. Our choice of this particular algebra is motivated by the fact that it belongs to a narrow class of algebras, which we call ‘conformal Lie algebras’, which have very similar properties to the conformal algebras of $n$-dimensional Minkowskian space-time. This class of algebras is identified and summarized in a table. Another motivation is related to the AdS/CFT correspondence. We give the multiplets of indecomposable elementary representations, including the necessary data for all relevant invariant differential operators.

1. Introduction

1.1. Generalities

Recently, there was more interest in the study and applications of exceptional Lie groups, cf., e.g., [1-18].
Thus, in the development of our project [19] of systematic construction of invariant differential operators for non-compact Lie groups we decided to give priority to some exceptional Lie groups. We start with the more interesting ones - the only two exceptional Lie groups/algebras that have highest/lowest weight representations, namely, $E_6(-14)$, cf. [20], and $E_7(-25)$, which we consider in the present paper.

In fact, there are additional motivations for the choice of $E_7(-25)$, namely, it belongs to a narrow class of algebras, which we call 'conformal Lie algebras', which have very similar properties to the conformal algebras $so(n,2)$ of $n$-dimensional Minkowski space time. Another motivation is related to the AdS/CFT correspondence.

Thus, we expand our motivations in the next Subsection, where we also give the table of the conformal Lie algebras.

Further the paper is organized as follows. In section 2 we give the preliminaries, actually recalling and adapting facts from [19]. In Section 3 we specialize to the $E_7(-25)$ case. In Section 4 we present our results on the multiplet classification of the representations and intertwining differential operators between them. In Subsection 4.1 we make a brief interpretation of our results to relate to the usual conformal algebras.

1.2. Motivation: the class of conformal Lie algebras

The group-theoretical interpretation of the AdS/CFT correspondence [21], or more general holography, involves two standard decompositions valid for any non-compact semi-simple Lie group $G$ or Lie algebra $\mathcal{G}$ (also super-group/algebra): the Iwasawa decomposition:

\[ G = KAN, \quad \mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}, \quad (1.1) \]

where $K$ is the maximal compact subgroup of $G$, $A$ is abelian simply connected subgroup of $G$, $N$ is a nilpotent simply connected subgroup of $G$ preserved by the action of $A$, (and similarly for the algebra decomposition), and the Bruhat decomposition:

\[ G = MAN\tilde{N}, \quad \mathcal{G} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}, \quad (1.2) \]

where $M$ is a maximal subgroup of $K$ that commutes with $A$, $\tilde{N}$ is a subgroup conjugate to $N$ by the Cartan involution. The Iwasawa decomposition is used to define

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1 Actually, $A \cong SO(1,1) \times \cdots \times SO(1,1)$, $r = \dim A$ copies.

2 The group decomposition is global which means that each element $g$ of $G$ can be represented by the group multiplication of three elements from the respective subgroups $g = kan$, $k \in K$, $a \in A$, $n \in N$. Similarly, each element $W \in \mathcal{G}$, can be represented as the sum $W = X \oplus Y \oplus Z$, $X \in \mathcal{K}$, $Y \in \mathcal{A}$, $Z \in \mathcal{N}$.

3 This group decomposition is almost global, which means that the decomposition $g = man\tilde{n}$, $(m \in M, \tilde{n} \in \tilde{N})$ is valid except for a subset of $G$ of lower dimensionality. But the algebra decomposition $W = U \oplus Y \oplus Z \oplus \tilde{Z}$, $(U \in \mathcal{M}, \tilde{Z} \in \tilde{\mathcal{N}})$, is valid as above for each element $W \in \mathcal{G}$.
induced representations on the bulk, which in this approach is represented by the solvable subgroup \( AN \), while the Bruhat decomposition is used to define induced representations on the conformal boundary, i.e., on space-time, represented by the subgroup \( N \). [21].

The application of the group-theoretical approach in [21] for the Euclidean conformal group \( G = SO(n + 1, 1) \) was facilitated by the fact that in the group-subgroup chain \( G \supset K \supset M \) the subgroups were sufficiently large: \( K = SO(n + 1), \ M = SO(n) \). Thus, there was not much freedom when embedding representations, in particular, embedding the representations of \( SO(n) \) into those of \( SO(n + 1) \).

Since the non-compact exceptional Lie algebra \( E_7(+7) \) was prominently used recently, cf. [13], we would like to apply similar interpretation to its holography. However, there is the problem of subgroups being not large enough. In fact, while the maximal compact subalgebra is \( K = su(8) \), the corresponding subalgebra \( M \) is null, \( M = \{0\} \), and the Bruhat decomposition is just \( G = A \oplus N \oplus \tilde{N} \). The reason is that \( E_7(+7) \) is maximally split, in fact, it is just the restriction to the real numbers of the complex Lie algebra \( E_7 \).

In fact, that would be a general problem in the case when the dimension \( r \) of the subalgebra \( A \), called \textit{real rank} or \textit{split rank}, is bigger than 1. But that also contains possible solutions of the problem, since when \( r > 1 \) the algebra under consideration has more Bruhat decompositions, in fact, their number is \( 2^r - 1 \). They are written in a similar way (writing only the algebra version):

\[
G = \mathcal{G} = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}' \oplus \tilde{\mathcal{N}}', \quad (1.3)
\]

so that \( \mathcal{M}' \supset \mathcal{M} \), \( \mathcal{A}' \subset \mathcal{A} \), \( \mathcal{N}' \subset \mathcal{N} \), \( \tilde{\mathcal{N}}' \subset \tilde{\mathcal{N}} \). Especially useful are the so-called 'maximal' decompositions, when \( \dim \mathcal{A}' = 1 \), since they represent closer the case \( r = 1 \), and the idea that the dimensions of the bulk (with Lie algebra \( \mathcal{A}' \mathcal{N}' \)) and the boundary (with Lie algebra \( \mathcal{N}' \)) should differ by 1.

In the case of \( E_7(+7) \) there are several suitable Bruhat decompositions [19]^4: \(^4\)

\[
E_7(+7) = \mathcal{M}_1 \oplus \mathcal{A}_1 \oplus \mathcal{N}_1 \oplus \tilde{\mathcal{N}}_1, \quad \mathcal{M}_1 = so(6,6), \ \dim \mathcal{A}_1 = 1, \ \dim \mathcal{N}_1 = \dim \tilde{\mathcal{N}}_1 = 33
\]

\[
E_7(+7) = \mathcal{M}_2 \oplus \mathcal{A}_2 \oplus \mathcal{N}_2 \oplus \tilde{\mathcal{N}}_2, \quad \mathcal{M}_2 = E_6(+6), \ \dim \mathcal{A}_2 = 1, \ \dim \mathcal{N}_2 = \dim \tilde{\mathcal{N}}_2 = 27
\]  

Due to the presence of the subalgebra \( so(6,6) \) the first case deserves separate study. The decomposition (1.5) is mentioned, though not in our context, in [22], where it is called three-graded decomposition, and in [13], thus, it may be useful in applications to supergravity. However, instead of using the Bruhat decomposition (1.5), we shall use another non-compact real form of \( E_7 \), namely, the Lie algebra \( E_7(-25) \).

There are several motivations to use the non-compact exceptional Lie algebra \( E_7(-25) \). Unlike \( E_7(+7) \) it has discrete series representations. Even more important is that it is one of two exceptional non-compact groups that have highest/lowest weight representations.\(^5\)

\(^4\) The number of maximal Bruhat decompositions is equal to \( r \).

\(^5\) The other one is \( E_6(-14) \) which we have also started to study, [20].
The groups that have highest/lowest weight representations are called Hermitian symmetric spaces [23]. The corresponding non-compact Lie algebras are:

\[ su(m,n), \; so(n,2), \; sp(2n,R), \; so^*(2n), \; E_6(-14), \; E_7(-25), \]  

(1.6)

cf., e.g., [24]. The practical criterion is that in these cases, the maximal compact subalgebras are of the form:

\[ K = K' \oplus so(2) \]  

(1.7)

The most widely used of these algebras are the conformal algebras \( so(n,2) \) in \( n \)-dimensional Minkowski space-time. In that case, there is a maximal Bruhat decomposition that has direct physical meaning:

\[ so(n,2) = M_c \oplus A_c \oplus N_c \oplus \tilde{N}_c, \]

\[ M_c = so(n-1,1), \; \dim A_c = 1, \; \dim N_c = \dim \tilde{N}_c = n \]  

(1.8)

Indeed, \( M_c = so(n-1,1) \) is the Lorentz algebra of \( n \)-dimensional Minkowski space-time, the subalgebra \( A_c = so(1,1) \) represents the dilatations, the conjugated subalgebras \( N_c, \tilde{N}_c \) are the algebras of translations, and special conformal transformations, both being isomorphic to \( n \)-dimensional Minkowski space-time.\(^6\)

There are other special features which are important. In particular, the complexification of the maximal compact subgroup coincides with the complexification of the first two factors of the Bruhat decomposition (1.8):

\[ K^{\mathbb{C}} = so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) = so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = M_c^{\mathbb{C}} \oplus A_c^{\mathbb{C}} \]  

(1.9)

In particular, the coincidence of the complexification of the semi-simple subalgebras in (1.9), \( so(n,\mathbb{C}) = so(n-1,1)^{\mathbb{C}} \), means that the sets of finite-dimensional (nonunitary) representations of \( M_c \) are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of \( so(n) \). The latter leads to the fact that the induced representations that we consider in this paper (and which are of the type that is mostly used in physics), cf. next Section, are representations of finite \( K \)-type [23].

The role of the abelian factors in (1.9) for the construction of highest/lowest weight representations was singled out first in [28].

It turns out that some of the algebras in (1.6) share the above-mentioned special properties of \( so(n,2) \). That is why, in view of applications to physics, these algebras, together with the appropriate Bruhat decompositions should be called 'conformal Lie algebras', (resp. 'conformal Lie groups' in the group setting). We display all these algebras in the following table:

\(^6\) The Bruhat-decomposition interpretation of the conformal subgroups/subalgebras was done first in the Euclidean case, cf. [25], then in the Minkowski case, cf. [26], for the general picture see [27].
Table of conformal Lie algebras

| $\mathcal{G}$   | $\mathcal{K}'$            | $\mathcal{M}_c$                   | $\dim_{\mathbb{R}} \mathcal{N}_c$ |
|-----------------|---------------------------|---------------------------------|-----------------------------------|
| $\mathfrak{su}(n,n)$ | $\mathfrak{su}(n) \oplus \mathfrak{su}(n)$ | $\mathfrak{sl}(n,\mathcal{A})_{\mathbb{R}}$ | $n^2$                             |
| $\mathfrak{so}(n,2)$, $n > 4$ | $\mathfrak{so}(n)$ | $\mathfrak{so}(n-1,1)$ | $n$                               |
| $\mathfrak{sp}(n,\mathbb{R})$, $n \geq 2$ | $\mathfrak{su}(n)$ | $\mathfrak{sl}(n,\mathbb{R})$ | $\frac{1}{2}(n+1)n$               |
| $\mathfrak{so}^*(2n)$, $n$ - even, $n \geq 6$ | $\mathfrak{su}(n)$ | $\mathfrak{so}^*(n)$ | $\frac{1}{2}n(n-1)$              |
| $E_7(-25)$       | $e_6$                     | $E_6(-26)$                      | $27$                              |

where we display only the semisimple part $\mathcal{K}'$ of $\mathcal{K}$, $\mathfrak{sl}(n,\mathcal{A})_{\mathbb{R}}$ denotes $\mathfrak{sl}(n,\mathcal{A})$ as a real Lie algebra, (thus, $(\mathfrak{sl}(n,\mathcal{A})_{\mathbb{R}})^\mathcal{A} = \mathfrak{sl}(n,\mathcal{A}) \oplus \mathfrak{sl}(n,\mathcal{A})$), $e_6$ denotes the compact real form of $E_6$, and we have imposed restrictions to avoid coincidences or inconsistency due to well known isomorphisms: $\mathfrak{so}(1,2) \cong \mathfrak{sp}(1,\mathbb{R}) \cong \mathfrak{su}(1,1)$, $\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$, $\mathfrak{so}(3,2) \cong \mathfrak{sp}(2,\mathbb{R})$, $\mathfrak{so}(4,2) \cong \mathfrak{su}(2,2)$, $\mathfrak{so}^*(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2,1)$, $\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)$.

The same class was identified from different considerations in [29], where these groups/algebras were called ‘conformal groups of simple Jordan algebras’. It was identified from still different considerations also in [30], where the objects of the class were called simple space-time symmetries generalizing conformal symmetry.

Finally, we should mention that the algebra $E_7(-25)$ was applied to the classification of orbits of BPS black holes in $\mathbb{N}=2$ Maxwell-Einstein supergravity theories [31].

With these motivations in mind we continue with the algebra $E_7(-25)$ with the following maximal Bruhat decomposition:

$$E_7(-25) = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}' \oplus \tilde{\mathcal{N}}' , \quad \mathcal{M}' = E_{6(-26)} , \quad \dim \mathcal{A}' = 1 , \quad \dim \mathcal{N}' = \dim \tilde{\mathcal{N}}' = 27$$

(1.10)

The careful reader may notice that the above Bruhat decomposition is a Wick-rotation of the corresponding one for $E_7(+7)$, (1.5), yet there are crucial differences in their properties.
The next Section contains preliminaries which are general for our programme started in [19].

2. Preliminaries

This Section can be read independently from the Introduction. Let \( G \) be a semisimple non-compact Lie group, and \( K \) a maximal compact subgroup of \( G \). Then we have an Iwasawa decomposition \( G = KAN \), where \( A \) is abelian simply connected vector subgroup of \( G \), \( N \) is a nilpotent simply connected subgroup of \( G \) preserved by the action of \( A \). Further, let \( M \) be the centralizer of \( A \) in \( K \). Then the subgroup \( P_0 = MAN \) is a minimal parabolic subgroup of \( G \) (including \( G \) itself) which contains a minimal parabolic subgroup.\(^7\)

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of \( G \) [33]. For the classification of all irreducible representations it is enough to use only the so-called cuspidal parabolic subgroups \( P = M'A'N' \), singled out by the condition that \( \text{rank} M' = \text{rank} M' \cap K \) [34],[35], so that \( M' \) has discrete series representations [36]. However, often induction from non-cuspidal parabolics is also convenient, cf. [37],[19],[38].

Let \( \nu \) be a (non-unitary) character of \( A' \), \( \nu \in A'^* \), let \( \mu \) fix an irreducible representation \( D^\mu \) of \( M' \) on a vector space \( V_\mu \).

We call the induced representation \( \chi = \text{Ind}_G^G(\mu \otimes \nu \otimes 1) \) an elementary representation of \( G \) [25]. (These are called generalized principal series representations (or limits thereof) in [39].) Their spaces of functions are:

\[
C_\chi = \{ F \in C^\infty(G, V_\mu) \mid F(\text{gman}) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) F(g) \} \tag{2.1}
\]

where \( a = \exp(H) \in A' \), \( H \in A' \), \( m \in M' \), \( n \in N' \). The representation action is the left regular action:

\[
(T^\chi(g)F)(g') = F(g^{-1}g') , \quad g, g' \in G . \tag{2.2}
\]

For our purposes we need to restrict to maximal parabolic subgroups \( P \), (so that \( \text{rank} A' = 1 \)), that may not be cuspidal. For the representations that we consider the character \( \nu \) is parameterized by a real number \( d \), called the conformal weight or energy.

Further, let \( \mu \) fix a discrete series representation \( D^\mu \) of \( M' \) on the Hilbert space \( V_\mu \), or the so-called limit of a discrete series representation (cf. [39]). Actually, instead of the discrete series we can use the finite-dimensional (non-unitary) representation of \( M' \) with the same Casimirs.

An important ingredient in our considerations are the highest/lowest weight representations of \( G \). These can be realized as (factor-modules of) Verma modules \( V^\lambda \) over \( G^\mathfrak{g} \), where \( \Lambda \in (H^\mathfrak{g})^* \), \( H^\mathfrak{g} \) is a Cartan subalgebra of \( G^\mathfrak{g} \), weight \( \Lambda = \Lambda(\chi) \) is determined

\(^7\) The number of non-conjugate parabolic subgroups is \( 2^r \), where \( r = \text{rank} A \), cf., e.g., [32].
uniquely from $\chi$ \cite{27}. In this setting we can consider also unitarity, which here means positivity w.r.t. the Shapovalov form in which the conjugation is the one singling out $G$ from $G^\mathbb{C}$.

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}'$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use generalized Verma modules $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector $v_0$ is taken by the (finite-dimensional) space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight $d$. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called multiplets \cite{40,27}. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators.\footnote{For simplicity only the operators which are not compositions of other operators are depicted.} The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair $(\beta, m)$, where $\beta$ is a (non-compact) positive root of $G^\mathbb{C}$, $m \in \mathbb{N}$, such that the BGG \cite{41} Verma module reducibility condition (for highest weight modules) is fulfilled:

$$\quad (\Lambda + \rho, \beta^\vee) = m, \quad \beta^\vee \equiv 2\beta/(\beta, \beta).$$

When (2.3) holds then the Verma module with shifted weight $V^{\Lambda-m\beta}$ (or $\tilde{V}^{\Lambda-m\beta}$ for GVM and $\beta$ non-compact) is embedded in the Verma module $V^\Lambda$ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector $v_s$ determined by a polynomial $P_{m,\beta}(G^-)$ in the universal enveloping algebra $(U(G^-)) v_0$, $G^-$ is the subalgebra of $G^\mathbb{C}$ generated by the negative root generators \cite{42}. More explicitly, \cite{27}, $v^s_{m,\beta} = P_{m,\beta} v_0$ (or $v^s_{m,\beta} = P_{m,\beta} V_\mu v_0$ for GVMs).\footnote{For explicit expressions for singular vectors we refer to \cite{43}.} Then there exists \cite{27} an intertwining differential operator

$$\quad D_{m,\beta} : C_{\chi(\Lambda)} \longrightarrow C_{\chi(\Lambda-m\beta)}$$

given explicitly by:

$$\quad D_{m,\beta} = P_{m,\beta}(\hat{G}^-)$$

where $\hat{G}^-$ denotes the right action on the functions $\mathcal{F}$, cf. (2.1).

3. The non-compact Lie algebra $E_7(-25)$

Let $G = E_7(-25)$. The maximal compact subgroup is $K \cong e_6 \oplus so(2)$, dim$_{\mathbb{R}} \mathcal{P} = 54$, dim$_{\mathbb{R}} \mathcal{N} = 51$. This real form has discrete series representations and highest/lowest weight representations.

\begin{thebibliography}{99}

\bibitem{1} For simplicity only the operators which are not compositions of other operators are depicted.
\bibitem{2} For explicit expressions for singular vectors we refer to \cite{43}.
\end{thebibliography}
The split rank is equal to 3, while \( \mathcal{M} \cong so(8) \).

The Satake diagram is [44]:

\[
\begin{array}{c}
\circ \quad \bullet \quad \bullet \quad \bullet \quad \circ \quad \circ \\
\alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7
\end{array}
\]

\[\lambda_1 \Rightarrow \lambda_2 \Rightarrow \lambda_3\]  

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the \( C_3 \) Dynkin diagram:

\[\begin{array}{c}
\circ \Rightarrow \circ \Rightarrow \circ \\
\lambda_1 \quad \lambda_2 \quad \lambda_3
\end{array}\]  

but the short roots have multiplicity 8 (the long - multiplicity 1). Going to the \( C_3 \) diagram we drop the black nodes, (they give rise to \( M \) resp., of (3.2).

We choose a maximal parabolic \( P = M' A' N' \) such that \( A' \cong so(1,1) \), while the factor \( M' \) has the same finite-dimensional (nonunitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra of \( K \), i.e., \( M' = E_{6(-26)} \), cf. [19]. Thus, these induced representations are representations of finite \( K \)-type [23]. Relatedly, the number of ERs in the corresponding multiplets is equal to \( |W(G^d, H^d)|/|W(K^d, H^d)| = 56 \), cf. [45], where \( H \) is a Cartan subalgebra of both \( G \) and \( K \). Note also that \( K^d \cong M^d \oplus A^d \). Finally, note that \( \dim_{\mathbb{R}} N' = 27 \).

We label the signature of the ERs of \( G \) as follows:

\[\chi = \{ n_1, \ldots, n_6; c \}, \quad n_j \in \mathbb{N}, \quad c = d - 9 \]

(3.3)

where the last entry of \( \chi \) labels the characters of \( A' \), and the first 6 entries are labels of the finite-dimensional nonunitary irreps of \( M' \), (or of the finite-dimensional unitary irreps of the \( e_6 \)).

The reason to use the parameter \( c \) instead of \( d \) is that the parametrization of the ERs in the multiplets is given in a simpler way, as we shall see.

Further, we need the root system of the complex algebra \( E_7 \). With Dynkin diagram enumerating the simple roots \( \alpha_i \) as in (3.1), the positive roots are:

first there are 21 roots forming the positive roots of \( sl(7) \) with simple roots \( \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \), then 21 roots which are roots of the \( E_6 \) subalgebra and include the non-\( sl(7) \) root \( \alpha_2 \):

\[
\begin{align*}
\alpha_2, & \quad \alpha_2 + \alpha_4, \quad \alpha_2 + \alpha_4 + \alpha_3, \quad \alpha_2 + \alpha_4 + \alpha_5, \quad \alpha_2 + \alpha_4 + \alpha_3 + \alpha_5, \\
\alpha_2 + \alpha_4 + \alpha_3 + \alpha_1, & \quad \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \quad \alpha_2 + \alpha_4 + \alpha_3 + \alpha_5 + \alpha_1, \\
\alpha_2 + \alpha_4 + \alpha_3 + \alpha_5 + \alpha_6, & \quad \alpha_2 + \alpha_4 + \alpha_3 + \alpha_5 + \alpha_1 + \alpha_6, \quad \alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5, \\
\alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_1, & \quad \alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_6, \quad \alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_1 + \alpha_6, \\
\alpha_2 + 2\alpha_4 + \alpha_3 + 2\alpha_5 + \alpha_1, & \quad \alpha_2 + 2\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6, \quad \alpha_2 + 2\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6, \\
\alpha_2 + 3\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1, & \quad 2\alpha_2 + 3\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6.
\end{align*}
\]

(3.4)
finally there are the following 21 roots including the non-$E_6$ root $\alpha_7$:

$$
\begin{align*}
\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \quad \alpha_2 + \alpha_4 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7, \\
\alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7, & \quad \alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, \\
\alpha_2 + 2\alpha_4 + \alpha_3 + 2\alpha_5 + \alpha_6 + \alpha_7, & \quad \alpha_2 + 2\alpha_4 + 2\alpha_3 + \alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, \\
\alpha_2 + 2\alpha_4 + \alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, & \quad \alpha_2 + 2\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, \\
\alpha_2 + 3\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, & \quad 2\alpha_2 + 3\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, \\
\alpha_2 + 2\alpha_4 + \alpha_3 + 2\alpha_5 + 2\alpha_6 + \alpha_7, & \quad \alpha_2 + 3\alpha_4 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, \\
\alpha_2 + 2\alpha_4 + \alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, & \quad 2\alpha_2 + 3\alpha_4 + 2\alpha_3 + 3\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, \\
\alpha_2 + 4\alpha_4 + 2\alpha_3 + 3\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, & \quad 2\alpha_2 + 4\alpha_4 + 2\alpha_3 + 3\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7, \\
\alpha_2 + 4\alpha_4 + 3\alpha_3 + 3\alpha_5 + \alpha_1 + 2\alpha_6 + \alpha_7, & \quad 2\alpha_2 + 4\alpha_4 + 3\alpha_3 + 3\alpha_5 + \alpha_1 + 2\alpha_6 + \alpha_7, \\
\alpha_2 + 4\alpha_4 + 3\alpha_3 + 3\alpha_5 + 2\alpha_1 + 2\alpha_6 + \alpha_7 \equiv \tilde{\alpha},
\end{align*}
$$

where $\tilde{\alpha}$ is the highest root of the $E_7$ root system.

The differential intertwining operators that give the multiplets correspond to the non-compact roots, and since we shall use the latter extensively, we introduce more compact notation for them. Namely, the nonsimple roots will be denoted in a self-explanatory way as follows:

$$
\begin{align*}
\alpha_{ij} & = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, & \quad \alpha_{i,j} & = \alpha_i + \alpha_j, \quad i < j, \\
\alpha_{ij,k} & = \alpha_{k,ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k, \quad i < j, \\
\alpha_{ij,km} & = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k + \alpha_{k+1} + \cdots + \alpha_m, \quad i < j, \quad k < m, \\
\alpha_{ij,km,4} & = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k + \alpha_{k+1} + \cdots + \alpha_m + \alpha_4, \quad i < j, \quad k < m,
\end{align*}
$$

i.e., the non-compact roots will be written as:

$$
\begin{align*}
\alpha_7, \quad & \alpha_{67}, \quad \alpha_{57}, \quad \alpha_{47}, \quad \alpha_{37}, \quad \alpha_{1,37}, \\
& \alpha_{2,47}, \quad \alpha_{27}, \quad \alpha_{17}, \quad \alpha_{27,4}, \quad \alpha_{17,4}, \quad \alpha_{27,45}, \\
& \alpha_{17,34}, \quad \alpha_{17,45}, \quad \alpha_{27,46}, \quad \alpha_{17,35}, \quad \alpha_{17,46}, \quad \alpha_{17,36}, \\
& \alpha_{17,35,4}, \quad \alpha_{17,25,4}, \quad \alpha_{17,36,4}, \quad \alpha_{17,26,4}, \\
& \alpha_{17,36,45}, \quad \alpha_{17,26,45}, \quad \alpha_{17,26,45,4}, \quad \alpha_{17,26,35,4}, \quad \alpha_{17,16,35,4} = \tilde{\alpha},
\end{align*}
$$

where the first six roots in (3.7a) are from the $sl(7)$ subalgebra, and the 21 in (3.7b) are those from (3.5).

Further, we give the correspondence between the signatures $\chi$ and the highest weight $\Lambda$. The connection is through the Dynkin labels:

$$
m_i \equiv (\Lambda + \rho, \alpha_i^\vee) = (\Lambda + \rho, \alpha_i), \quad i = 1, \ldots, 7,
$$
where $\Lambda = \Lambda(\chi)$, $\rho$ is half the sum of the positive roots of $G^d$, $\alpha_i$ denotes the simple roots of $G^d$. The explicit connection is:

$$n_i = m_i, \quad c = -\frac{1}{2}(n_\alpha + n_\gamma) = -\frac{1}{2}(2n_1 + 2n_2 + 3n_3 + 4n_4 + 3n_5 + 2n_6 + 2n_7) \quad (3.9)$$

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta), \quad (3.10)$$

where $\beta$ is any positive root of $G^d$. These parameters are redundant, since obviously they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms.\(^{10}\)

There are several types of multiplets: the main type, which contains maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations, and some reduced types of multiplets.

In the next Section we give the main type of multiplets and the main reduced type.

4. Multiplets

4.1. Main type of multiplets

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $E_7$, i.e., they will be labelled by the seven positive Dynkin labels $m_i \in \mathbb{N}$. As we mentioned, it turns out that each such multiplet contains 56 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi_0^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6)\pm; \pm \frac{1}{2}(m_\alpha + m_7) \} \quad (4.1)$$
$$\chi_a^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_7) \}$$
$$\chi_b^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_6) \}$$
$$\chi_c^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_5) \}$$
$$\chi_d^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_4) \}$$
$$\chi_e^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_2, 4) \}$$
$$\chi_f^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_3, 7) \}$$
$$\chi_g^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_2, 7) \}$$
$$\chi_h^\pm = \{ (m_1, m_2, m_3, m_4, m_5, m_6, m_7)\pm; \pm \frac{1}{2}(m_\alpha - m_1, 7) \}$$

\(^{10}\) Clearly, both the Dynkin and Harish-Chandra labels have their origin in the BGG reducibility condition (2.3).
where we have used for the numbers \( m_\beta = (A(\chi) + \rho, \beta) \) the same compact notation as in (3.6) for the roots \( \beta \), and the notation \((\ldots)^\pm\) employs the natural conjugation of the subalgebra \( E_6 \), more precisely:

\[
(n_1, n_2, n_3, n_4, n_5, n_6)^- = (n_1, n_2, n_3, n_4, n_5, n_6) \\
(n_1, n_2, n_3, n_4, n_5, n_6)^+ = (n_1, n_2, n_3, n_4, n_5, n_6)E_6 \doteq (n_6, n_2, n_5, n_4, n_3, n_1)
\]

Note that in (4.1) the last entries with sign plus (resp. minus) are positive (resp. negative), except in the cases \( \chi_{m}^\pm, \chi_{n}^\pm, \chi_{n'}^\pm \).

The ERs in the multiplet are related by intertwining integral and differential operators. The integral operators were introduced by Knapp and Stein [46]. In fact, these operators are defined for any ER, not only for the reducible ones, the general action being:

\[
G_{KS} : \mathcal{C}_\chi \rightarrow \mathcal{C}_{\chi'}, \\
\chi = \{ n_1, n_2, n_3, n_4, n_5, n_6 ; c \} , \\
\chi' = \{ (n_1, n_2, n_3, n_4, n_5, n_6)^{E_6} ; -c \} = \{ n_6, n_2, n_5, n_4, n_3, n_1 ; -c \}
\]

Obviously, the pairs in (4.1) are related by Knapp-Stein integral operators, i.e.,

\[
G_{KS} : \mathcal{C}_{\chi^\pm} \rightarrow \mathcal{C}_{\chi^\pm}
\]
The action on the signatures is also called restricted Weyl reflection, since it represents the nontrivial element of the 2-element restricted Weyl group which arises canonically with every maximal parabolic subalgebra.\footnote{Generically, the Knapp-Stein operators can be normalized so that indeed $G_{KS} \circ G_{KS} = \text{Id}_{C^{\chi}}$. However, this usually fails exactly for the reducible ERs that form the multiplets, cf., e.g., [25].}

Matters are arranged so that in every multiplet only the ER with signature $\chi_{0}^-$ contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $E_7$ with signature $\{m_1, \ldots, m_7\}$. The subspace $\mathcal{E}$ is annihilated by the operator $G^+$, and is the image of the operator $G^-$. The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator acting from $\chi_{0}^-$ to $\chi_{b}^-$ (more about this operator below). When all $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $E_7(-25)$. Furthermore in that case the conformal weight is zero: $d = 9 + c = 9 - \frac{1}{2}(m_\tilde{\alpha} + m_7) |_{m_i=1} = 0$.

Analogously, in every multiplet only the ER with signature $\chi_{0}^+$ contains holomorphic discrete series representation. This is guaranteed by the criterion [36] that for such an ER all Harish-Chandra parameters for non-compact roots must be negative, i.e., in our situation, $n_\alpha < 0$, for $\alpha$ from (3.7). [That this holds for our $\chi_{0}^+$ can be easily checked using the signatures (4.1).]

In fact, the Harish-Chandra parameters are reflected in the division of the ERs into $\chi^-$ and $\chi^+$: for the $\chi^-$ modules less than half of the 27 non-compact Harish-Chandra parameters are negative (none for $\chi_{0}^-$, 13 for $\chi_{n}^-, \chi_{n'}^-, \chi_{n''}^-$), while for the $\chi^+$ modules more than half of the non-compact 27 Harish-Chandra parameters are negative (27 for $\chi_{0}^+$, 14 for $\chi_{n}^+, \chi_{n'}^+, \chi_{n''}^+$). In fact, as in the parenthesized examples, it is true that the sum of the number of negative Harish-Chandra parameters for any pair $\chi^{\pm}$ is equal to 27.

Note that the ER $\chi_{0}^+$ contains also the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the antiholomorphic representations are realized in an invariant subspace $\mathcal{D}$ of the ER $\chi_{0}^+$. That subspace is annihilated by the operator $G^-$, and is the image of the operator $G^+$. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series. The conformal weight of the ER $\chi_{0}^+$ has the restriction $d = 9 + c = 9 + \frac{1}{2}(m_\tilde{\alpha} + m_7) \geq 18$.

The intertwining differential operators correspond to non-compact positive roots of the root system of $E_7$, cf. [27], i.e., in the current context, the roots given in (3.7).

The multiplets are given explicitly in Fig. 1, where we use the notation: $\Lambda^{\pm} = \Lambda(\chi^{\pm})$. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j..k}$ encoding the root $\beta_{j..k}$ and the number $m_{\beta_{j..k}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data $\beta, m_{\beta}$, which is involved in the embedding $V^{\Lambda} \rightarrow V^{\Lambda - m_{\beta}}$, turns out to
involve only the \( m_i \) corresponding to simple roots, i.e., for each \( \beta, m_\beta \) there exists \( i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 7\} \), such that \( m_\beta = m_i \). Hence the data \( \beta_{j \ldots k}, m_{\beta_{j \ldots k}} \) is represented by \( i_{j \ldots k} \) on the arrows.

The pairs \( \Lambda^\pm \) are symmetric w.r.t. to the bullet in the middle of the figure, and the dashed line separates the \( \Lambda^- \) modules from the \( \Lambda^+ \) modules.

**Interpretation:** Since the relation to the usual conformal algebras in \( n \)-dimensional Minkowski space-time is one of our main motivations to study \( E_7(-25) \), we would like to mention briefly some analogies, using an exposition that is written in the same context, cf. [38], though the results are contained in much older work [25],[47],[26],[48], see also [27].

If we take the most basic example when the inducing \( E_6 \)-representation in the ERs \( \chi^\pm_0 \) is the trivial one: \( (m_1, m_2, m_3, m_4, m_5, m_6) = (1, 1, 1, 1, 1, 1) \), then the conformal fields represented by the ERs \( \chi^\pm_0 \) are scalar, while those represented by the ERs \( \chi^\pm_\alpha \) are 27-dimensional vectors. There are invariant differential operators depicted on Fig. 1:

\[
D_{m_7, \alpha_7} : C_{\chi^-_0} \longrightarrow C_{\chi^-_\alpha} \quad (4.5a)
\]
\[
D_{m_7, \alpha_{17,16,35,4}} : C_{\chi^+_0} \longrightarrow C_{\chi^+_\alpha} \quad (4.5b)
\]

Both are equations of order \( m_7 \). When the last free parameter \( m_7 = 1 \) then the ER \( \chi^-_0 \) is the analog of the vector potential \( A_\nu \), while the ER \( \chi^+_0 \) is the analog of the current \( J_\nu \). Then the equations in (4.5) are linear and can be written as:

\[
\begin{align*}
\partial_\nu \phi &= A_\nu, & \phi \in C_{\chi^-_0}, & A \in C_{\chi^-_\alpha} \\
\sum_{\nu=1}^{27} \partial^\nu J_\nu &= \Phi, & \Phi \in C_{\chi^+_0}, & J \in C_{\chi^+_\alpha}
\end{align*} \quad (4.6a)
\]

When the parameter \( m_7 > 1 \), then the analogs of (4.5) are also treated in the older references cited above (for instance (4.5b) would be an equation of partial conservation). In all cases, we stress that these are invariant differential equations, on- and off-shell. Naturally, this is only a glimpse in the analogies with the usual conformal case, much more will be said elsewhere, [49]. ♦

In the next Subsection we shall consider the main type of reduced multiplets.

### 4.2. Main type of reduced multiplet

The multiplets of reduced type \( R7 \) contain 42 ERs/GVMs and may be obtained formally from the main type by setting \( m_7 = 0 \). Their signatures are given explicitly by:

\[
\begin{align*}
\chi^\pm_0 &= \{ (m_1, m_2, m_3, m_4, m_5, m_6)\}^{\pm}; \pm \frac{1}{2} m_\alpha \\
\chi^\pm_\beta &= \{ (m_1, m_2, m_3, m_4, m_56, 0)\}^{\pm}; \pm \frac{1}{2} (m_\alpha - m_6) \\
\chi^\pm_\gamma &= \{ (m_1, m_2, m_3, m_4, m_56, 0)\}^{\pm}; \pm \frac{1}{2} (m_\alpha - m_56) \\
\chi^\pm_\delta &= \{ (m_1, m_2, m_3, m_4, m_5, 0)\}^{\pm}; \pm \frac{1}{2} (m_\alpha - m_56)
\end{align*} \quad (4.7)
\]
\[\chi^\pm = \{ (m_1, m_4, m_{24}, m_5, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{2,46}) \} \]
\[\chi^\pm_0 = \{ (m_1, m_{24}, m_4, m_5, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{36}) \} \]
\[\chi^\pm_f = \{ (m_{1,3}, m_{34}, m_{24}, m_5, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{26}) \} \]
\[\chi^\pm_j = \{ (m_3, m_{14}, m_4, m_5, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{1,36}) \} \]
\[\chi^\pm_g = \{ (m_{1,34}, m_3, m_2, m_{45}, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{26,4}) \} \]
\[\chi^\pm_g' = \{ (m_3, m_{1,34}, m_{24}, m_5, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{16}) \} \]
\[\chi^\pm_h = \{ (m_{1,35}, m_3, m_2, m_{45}, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{26,4}5) \} \]
\[\chi^\pm_h' = \{ (m_{34}, m_{1,3}, m_{24}, m_5, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{1,6,45}) \} \]
\[\chi^\pm_j = \{ (m_{1,36}, m_3, m_2, m_{45}, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{26,4}6) \} \]
\[\chi^\pm_j' = \{ (m_{35}, m_{1,3}, m_2, m_{45}, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{1,6,45}) \} \]
\[\chi^\pm_j'' = \{ (m_4, m_{1,1}, m_2, m_3, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{1,6,34}) \} \]
\[\chi^\pm_h'' = \{ (m_{45}, m_{1,1}, m_2, m_{34}, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{1,6,35}) \} \]
\[\chi^\pm_i = \{ (m_{36}, m_{1,1}, m_2, m_4, m_5, m_6)^\pm; \pm \frac{1}{2}(m_\alpha - m_{1,6,46}) \} \]
\[\chi^\pm_m = \{ (m_{46}, m_{1,1}, m_2, m_{34}, m_5, m_6)^\pm; \pm \frac{1}{2}(m_\alpha - m_{2,45,4}) \} \]
\[\chi^\pm_n = \{ (m_{45}, m_{1,1}, m_2, m_{34}, m_5, m_6)^\pm; \pm \frac{1}{2}(m_\alpha - m_{2,45,4}) \} \]
\[\chi^\pm_{n'} = \{ (m_5, m_{1,1}, m_2, m_{44}, m_6, 0)^\pm; \pm \frac{1}{2}(m_\alpha - m_{2,45,4}) \} \]
\[\chi^\pm_{m'} = \{ (m_5, m_{1,1}, m_2, m_{44}, m_6, 0)^\pm; \pm \frac{1}{2}(m_{56} - m_2) \} \]
\[\chi^\pm_{m''} = \{ (m_5, m_{1,1}, m_2, m_{44}, m_6, 0)^\pm; \pm \frac{1}{2}(m_{56} - m_2) \} \]

Here the ER \( \chi_0^+ \) contains limits of the (anti)holomorphic discrete series representations. This is guaranteed by the fact that for this ER all Harish-Chandra parameters for non-compact roots are non-positive, i.e., \( n_\alpha \leq 0 \), for \( \alpha \) from (3.7). The conformal weight has the restriction \( d = 9 + c = 9 + \frac{1}{2} m_\alpha \geq 17 \).

There are other limiting cases, where there are zero entries for the first six \( n_i \) values. In these cases the induction procedure would not use finite-dimensional irreps of the \( E_6 \) subgroup. The corresponding ERs would not have direct physical meaning, however, the fact that they are together with the physically meaningful ERs has important bearing on the structure of the latter.

Altogether, the analysis of the Harish-Chandra parameters reveals the following. For any ER there is exactly one Harish-Chandra parameter (counting all, not only the non-compact) that is zero. The compact ones are seen in the list above. The non-compact are as follows:

\[\chi^- : n_7 = 0, \quad \chi^+_0 : n_\alpha = 0, \]
\[\chi_0^+ : n_\alpha = 0, \quad \chi_j^\pm, \chi_j^\pm, \chi_m^\pm, \chi_n^\pm, n_{27,46} = 0. \quad (4.8)\]

As in the main type, for the \( \chi^- \) modules less than half of the 27 non-compact Harish-Chandra parameters are negative (none for \( \chi_0^- \), 13 for \( \chi_{n'}^- \)), while for the \( \chi^+ \) modules - except \( \chi_{n''}^+ \) - more than half of the non-compact 27 Harish-Chandra parameters are
negative (26 for $\chi_0^+$, 14 for $\chi_n^+$). In fact, it is true that for any pair $\chi^\pm$ the sum of the number of negative Harish-Chandra parameters is equal to 26.

These multiplets are depicted on Fig. 2. The Weyl-conjugated pairs $\Lambda^\pm$ are symmetric w.r.t. to the bullet in the middle of the figure, and the dashed line separates the $\Lambda^-$ modules from the $\Lambda^+$ modules. The fact that the pair $\chi_n^-, \chi_n^+$, sits on the dashed line signifies the fact that for these two ERs the number of negative non-compact Harish-Chandra parameters equals the number of positive non-compact Harish-Chandra parameters, and that equals 13. Note also that the ten ERs for which holds $n_{27,46} = 0$, cf. (4.8), are situated on two conjugated lines.

There are many other types of reduced multiplets, and their study may be done as in the case of $E_6(-14)$ in [20], but for $E_7(-25)$ it will need much more space, so we leave it for a future publication.

5. Outlook

In the present paper we continued the programme outlined in [19] on the example of the non-compact group $E_7(-25)$. Similar explicit descriptions are planned for the other non-compact groups, in particular those with highest/lowest weight representations. We plan also to extend these considerations to the supersymmetric cases and also to the quantum group setting. Such considerations are expected to be very useful for applications to string theory and integrable models, cf., e.g., [50].

In our further plans it shall be very useful that (as in [19]) we follow a procedure in representation theory in which intertwining differential operators appear canonically [27] and which procedure has been generalized to the supersymmetry setting [51],[52] and to quantum groups [53]. (For more references, cf. [19].)

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Fig. 1. Main Type
Fig. 2. Reduced Type R7