On metric chromatic number of comb product of ladder graph

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Abstract. All graphs in this paper are connected and nontrivial graph. Let $f : V(G) \rightarrow \{1, 2, ..., k\}$ be a vertex coloring of a graph $G$ where two adjacent vertices may be colored the same color. Consider the color classes $\Pi = \{C_1, C_2, ..., C_k\}$. For a vertex $v$ of $G$, the representation color of $v$ is the $k$-vektor $r(v | \Pi) = \{d(v, C_1), d(v, C_2), ..., d(v, C_k)\}$, where $d(v, C_i) = min\{d(v, c); c \in C_i\}$. If $r(u | \Pi) \neq r(v | \Pi)$ for every two adjacent vertices $u$ and $v$ of $G$, then $f$ is a metric coloring of $G$. The minimum $k$ for which $G$ has a metric $k$-coloring is called the metric chromatic number of $G$ and is denoted by $\mu(G)$. The metric chromatic number on comb product of ladder graphs namely path graph, star graph, fan graph, cycle graph, and complete graph.

1. Introduction

Graph in this paper are nontrivial graph and connected graph, for more detail definition of graph see [4]. A graph coloration is a way of giving color to each vertex of the graph so that no two adjacent vertices have the same color. The graph can be colored by assigning different colors to each of its vertices. However, in most graph coloring it can be found that the use of colors is less than the number of vertices in the graph [8]. A coloring of the vertices is called (not surprisingly) a vertex coloring. If the vertex coloring has the property that adjacent vertices are colored differently, then the coloring is called proper. Every graph has a proper vertex coloring [6].

There are a lot of type of colouring, namely metric coloring, graceful coloring, r-dynamic coloring, packing coloring, and others. Alfarisi et.al [3] founded graceful chromatic Number of Unicyclic Graphs $\chi_g(T_{n,m})$ and $\chi_g(M_n)$. Agustin et.al [1] describe exact value r-dynamic coloring $\chi_r(G)$ of some graph operations.

Let $f : V(G) \rightarrow \{1, 2, ..., k\}$ be a vertex coloring of a graph $G$ where two adjacent vertices may be colored the same color. Consider the color classes $\Pi = \{C_1, C_2, ..., C_k\}$. For a vertex $v$ of $G$, the representation color of $v$ is the $k$-vektor $r(v | \Pi) = \{d(v, C_1), d(v, C_2), ..., d(v, C_k)\}$, where $d(v, C_i) = min\{d(v, c); c \in C_i\}$. If $r(u | \Pi) \neq r(v | \Pi)$ for every two adjacent vertices $u$ and $v$ of $G$, then $f$ is a metric coloring of $G$. The minimum $k$ for which $G$ has a metric $k$-coloring is called the metric chromatic number of $G$ and is denoted by $\mu(G)$ [2]. In recent years, there are some results of the metric chromatic number of some well-known graph in Ping Zhang [10] as follows.
Proposition 1 A nontrivial connected graph $G$ has metric chromatic number 2 if and only if $G$ is bipartite.

Corollary 1.1 Let $G$ be a connected graph. If $\chi(G) = 3$, then $\mu(G) = 3$.

Corollary 1.2 Let $C_n$ be a cycle graph, then

$$\mu(C_n) = \begin{cases} 2, & \text{for } n \text{ is even} \\ 3, & \text{for } n \text{ is odd} \end{cases}$$

Proposition 2 For every complete $k$-partite graph $G$ where $k \geq 2$, then $\mu(G) = k$.

One way to get new graphs, it can be obtained by using a graph operation [5]. Comb operation is one of several types of graph operation. Let $G$ and $H$ be two connected graph and $o$ be a vertex of $H$ . The comb product between $G$ and $H$, denoted by $G \triangleright_o H$, is a graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and grafting the $i$-th copy of $H$ at the vertex $o$ to the $i$-th vertex of $G$ [9]. By definition, a comb graph is irregular graph which has no commutative properties [5]. The ladder graph $L_n$ is defined by $L_n = P_n \times K_2$ where $P_n$ is a path with $n$ vertices and $\times$ denotes the Cartesian product and $K_2$ is a complete graph with two-vertices [7].

For an example of comb product between $L_m$ and $P_n$ provided in Figure 1. Based on Definition comb product, $L_m \triangleright_{v_0} P_n$ is a graph obtained by taking one copy of $L_m$ and $|V(L_m)|$ copies of $P_n$. After that, grafting the $i-th$ copy of $P_n$ at the vertex $v_0$ to the $i-th$ vertex of $L_m$.

In this paper, we investigated about metric chromatic number of comb product of ladder graph and denoted by $\mu(L \triangleright G)$ where $G$ are path graph, star graph, fan graph, cycle graph, and complete graph.

2. Some Observation of Metric Chromatic Number

Prior to describe the research findings, we propose the following observations.

Observation 2.1 For any $P_m$ be a path graph, then

$$\mu(P_n) = 2, \text{ for } n \geq 2$$

By definition metric chromatic number, then metric chromatic number ($P_n$) for $n \geq 2$ that is 2.

Observation 2.2 For any $S_n$ be a star graph, then

$$\mu(S_n) = 2, \text{ for } n \geq 3$$

By definition metric chromatic number, then metric chromatic number ($S_n$) for $n \geq 3$ that is 2.

Observation 2.3 For any $F_n$ be a fan graph, then

$$\mu(F_n) = 3, \text{ for } n \geq 3$$

By definition metric chromatic number, then metric chromatic number ($F_n$) for $n \geq 3$ that is 3. There are some results of the metric chromatic number of some well-known graph in Alfarisi et.al [1] as follows.

Observation 2.4 For any two adjacent vertices $u, v \in V(T)$, then we have $d(u, x) \neq d(v, x)$ for $x \in V(T)$.
3. Results

In this paper, we investigate the metric chromatic number on comb product of ladder graphs namely path graph, star graph, cycle graph, fan graph, and complete graph, and fan graph. We construct the new lemma for any comb product of ladder and any graphs as follows.

**Lemma 1** Let ladder graph $L_n, n \geq 2$. If $G \cong P_m, m \geq 2$, $G \cong C_m, m \geq 3$, $G \cong S_m, m \geq 3$, $G \cong F_m, m \geq 3$, and $G \cong K_m, m \geq 2$, where $P_m, C_m, S_m, F_m,$, and $K_m$ are path, cycle, star, fan, and complete graph, then

$$\mu(L_n \triangleright G) \geq \mu(G).$$

**Proof.** $\mu(L_n \triangleright G) \geq \mu(G)$ for $n \geq 1$ has several subgraphs with $n$ vertices. So that, the coloring in the $\mu(L_n \triangleright G) \geq \mu(G)$ follows the coloring in the $\mu(G)$. Based on Corollary 2.3. that $\mu(C_m) = 2$ for $n$ is even and $\mu(C_m) = 3$ for $n$ is odd, and Proposition 2.4. that $\mu(G) = k$ for $G$ is a complete graph where $k \geq 2$. So, there are several condition in this proof as follows:

(i) Assume that $\mu(G) = m$, $m$ is $k$-minimum color in $G_i, G_k \neq G_l, k \neq l$.

(ii) Based on definition of comb that backbone $L_n$ and leaf subgraph $G_i$.

(iii) Subgraph $L_n(\text{Backbone})$ colored according to the color on $G_i$, such that $\mu(L_n \triangleright G) \geq m$.

(iv) If $m - 1$ color on subgraph $G_i$, hence contradicts the chromatic of $G$.

(v) Then, $\mu(L_n \triangleright G) \geq m = \mu(G)$.

**Theorem 3.1** For every natural number $n \geq 2$ dan $m \geq 2$, $\mu(L_n \triangleright P_m) = 2$.

**Proof.** Let $L_n \triangleright P_m$ be a comb operation graph between ladder graph $L_n$ for $n \geq 2$ with path graph $P_m$ for $m \geq 2$ has a set of vertices $V(L_n \triangleright P_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq m\}$ and subsets $E(L_n \triangleright P_m) = \{x_ix_{i+1}; 1 \leq i \leq n - 1\} \cup \{y_jy_{j+1}; 1 \leq j \leq m - 1\} \cup \{x_ix_{ik}; 1 \leq i \leq n; 1 \leq k \leq m\}$ and subsets $E(L_n \triangleright P_m) = \{x_ix_{i+1}; 1 \leq i \leq n - 1\} \cup \{y_jy_{j+1}; 1 \leq j \leq m - 1\} \cup \{x_ix_{ik}; 1 \leq i \leq n; 1 \leq k \leq m\}$.

Based on Observasi 2.1 and Lemma 2.1 that $\mu(L_n \triangleright P_m) \geq 2$. Furthermore, we proof that $\mu(L_n \triangleright P_m) \leq 2$. Let $f : V(G) \rightarrow \{1, 2\}$ be a vertex coloring(two adjacent vertices may be colored same color). For more detail the label color and the representation of vertices in graph $L_n \triangleright P_m$ respect to with class color $\Pi = \{C_1, C_2\}$ where $C_1 = \{x_i, y_j; i$ was odd, and $j$ was even$\} \cup \{x_{ik}, y_{jk}; i$ was odd $: j$ was even $; k$ was odd$\} \cup \{x_{ik}, y_{jk}; i$ was even $: j$ was odd $; k$ was even$\}$, $C_2 = \{x_i, y_j; i$ was even, and $j$ was odd$\} \cup \{x_{ik}, y_{jk}; i$ was odd $: j$ was even $; k$ was even$\} \cup \{x_{ik}, y_{jk}; i$ was even $: j$ was odd $; k$ was odd$\}$ as follows.

$$\mu(f(v)) = \begin{cases} 1, \{x_i, y_j; \text{for } i \text{ was odd and } j \text{ was even}\} \\
\cup \{x_{ik}, y_{jk}; \text{for } i \text{ was odd; } j \text{ was even; and } k \text{ was odd}\} \\
\cup \{x_{ik}, y_{jk}; \text{for } i \text{ was even; } j \text{ was odd; and } k \text{ was even}\} \\
2, \{x_i, y_j; \text{for } i \text{ was even and } j \text{ was odd}\} \\
\cup \{x_{ik}, y_{jk}; \text{for } i \text{ was odd; } j \text{ was even; and } k \text{ was even}\} \\
\cup \{x_{ik}, y_{jk}; \text{for } i \text{ was even; } j \text{ was odd; and } k \text{ was odd}\} \\
\end{cases}$$

Based on the color label $f$ in graph $L_n \triangleright P_m$. Thus, we have the representation as follows.
\[
\begin{align*}
    r(y_j|\Pi) &= r(x_i|\Pi) = (0, 1) \text{ for } i \text{ was odd and } j \text{ was even} \\
    r(y_j|\Pi) &= r(x_i|\Pi) = (1, 0) \text{ for } i \text{ was even and } j \text{ was odd} \\
    r(y_{jk}|\Pi) &= r(x_{ik}|\Pi) = (0, 1) \text{ for } i \text{ was even; } j \text{ was odd and} \\
    & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad k \text{ was even} \\
    r(y_{jk}|\Pi) &= r(x_{ik}|\Pi) = (1, 0) \text{ for } i \text{ was odd; } j \text{ was even and} \\
    & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad k \text{ was even}
\end{align*}
\]

It is clear that for every two adjacent vertices has distinct representation, we can see in
\[r(x_{2k-1}|\Pi) \neq r(x_{2k}|\Pi) \text{ or } r(y_{2k-1}|\Pi) \neq r(y_{2k}|\Pi)\] for \(k \in N\). Without the loss generality, we have \(\mu(L_n \triangleright P_m) \leq 2\). Thus \(\mu(L_n \triangleright P_m) = 2\).

Here is an example figure of \(\mu(L_n \triangleright P_m)\)

![Figure 1. \(\mu(L_n \triangleright P_m)\)](image)

**Theorem 3.2** For every natural number \(n \geq 2\) dan \(m \geq 3\), \(\mu(L_n \triangleright S_m) = 2\).

**Proof.** Let \(L_n \triangleright S_m\) be a comb operation graph between ladder graph \(L_n\) for \(n \geq 2\) with star graph \(S_m\) for \(m \geq 2\) has a set of vertices \(V(L_n \triangleright S_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq n\}\) and subsets \(E(L_n \triangleright S_m) = \{x_i x_{i+1}; 1 \leq i \leq n - 1\} \cup \{y_j y_{j+1}; 1 \leq j \leq n - 1\} \cup \{x_i x_{ik}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_j y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\}\). Based on Observation 2.1 and Lemma 2.1 that \(\mu(L_n \triangleright S_m) \geq 2\). Furthermore, we proof that \(\mu(L_n \triangleright S_m) \leq 2\). Let \(f : V(G) \to \{1, 2\}\) be a vertex coloring(two adjacent vertices may be colored same color). For more detail the label color and the representation of vertices in graph \(L_n \triangleright S_m\) respect to with class color \(\Pi = \{C_1, C_2\}\) where \(C_1 = \{x_i, y_j; i \text{ was odd, and } j \text{ was even}\} \cup \{x_{ik}, y_{jk}; i \text{ was even and } j \text{ was odd}\}\) and \(C_2 = \{x_i, y_j; i \text{ was even, and } j \text{ was odd}\} \cup \{x_{ik}, y_{jk}; i \text{ was odd and } j \text{ was even}\}\) as follows.

\[
f(v) = \begin{cases} 
    1, \{x_i, y_j; \text{for } i \text{ was odd and } j \text{ was even}\} \\
    \cup \{x_{ik}, y_{jk}; \text{for } i \text{ was even; } j \text{ was odd; and } 1 \leq k \leq m\}, \\
    2, \{x_i, y_j; \text{for } i \text{ was even and } j \text{ was odd}\} \\
    \cup \{x_{ik}, y_{jk}; \text{for } i \text{ was odd; } j \text{ was even; and } 1 \leq k \leq m\}.
\end{cases}
\]
Based on the color label \( f \) in graph \( L_n \triangleright S_m \). Thus, we have the representation as follows.

\[
\begin{align*}
    r(y_j \Pi) &= r(x_i \Pi) = (0, 1) \text{ for } i \text{ was odd and } j \text{ was even} \\
    r(y_j \Pi) &= r(x_i \Pi) = (1, 0) \text{ for } i \text{ was even and } j \text{ was odd} \\
    r(y_{jk} \Pi) &= r(x_{ik} \Pi) = (0, 1) \text{ for } i \text{ was even; } j \text{ was odd and } 1 \leq k \leq m \\
    r(y_{jk} \Pi) &= r(x_{ik} \Pi) = (1, 0) \text{ for } i \text{ was odd; } j \text{ was even and } 1 \leq k \leq m 
\end{align*}
\]

It is clear that for every two adjacent vertices has distinct representation, we can see in \( r(x_{2k-1} \Pi) \neq r(x_{2k} \Pi) \) or \( r(y_{2k-1} \Pi) \neq r(y_{2k} \Pi) \) for \( k \in N \). Without the loss generality, we have \( \mu(L_n \triangleright S_m) \leq 2 \). Thus \( \mu(L_n \triangleright S_m) = 2 \).

Here is an example figure of \( \mu(L_n \triangleright S_m) \)

![Figure 2. \( \mu(L_n \triangleright S_m) \)]

**Theorem 3.3** For every natural number \( n \geq 2 \) dan \( m \geq 3 \), \( \mu(L_n \triangleright F_m) = 3 \).

**Proof.** Let \( L_n \triangleright F_m \) be a comb operation graph between ladder graph \( L_n \) for \( n \geq 2 \) with fan graph \( F_m \) for \( m \geq 2 \) has a set of vertices \( V(L_n \triangleright F_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq \ell\} \cup \{x_{ik}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{jk}; 1 \leq j \leq \ell; 1 \leq k \leq m\} \) and subsets \( E(L_n \triangleright F_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{y_j y_{j+1}; 1 \leq j \leq \ell-1\} \cup \{x_i x_{ik}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_j y_{jk}; 1 \leq j \leq \ell; 1 \leq k \leq m\} \cup \{x_{ik} x_{ik+1}; 1 \leq i \leq n; 1 \leq k \leq m-1\} \cup \{y_{jk} y_{jk+1}; 1 \leq i \leq n; 1 \leq k \leq m-1\} \cup \{x_i y_j; 1 \leq i \leq n \leq 1 \leq j \leq m\} \).

Based on Observation 2.3 and Lemma 2.1 that \( \mu(L_n \triangleright F_m) \geq 3 \). Furthermore, we proof that \( \mu(L_n \triangleright F_m) \leq 3 \). Let \( f : V(G) \rightarrow \{1, 2, 3\} \) be a vertex coloring(two adjacent vertices may be colored same color). For more detail the label color and the representation of vertices in graph \( L_n \triangleright F_m \) respect to with class color \( \Pi = \{C_1, C_2, C_3\} \) where \( C_1 = \{x_i, y_j; i \text{ was odd, and } j \text{ was even}\} \cup \{x_{ik}, y_{jk}; i \text{ was even; } j \text{ was odd; and } k \text{ was even}\} \), \( C_2 = \{x_i, y_j; i \text{ was even, and } j \text{ was odd}\} \cup \{x_{ik}, y_{jk}; i \text{ was odd; } j \text{ was even; and } k \text{ was even}\} \), \( C_3 = \{x_{ik}, y_{jk}; 1 \leq i \leq n; 1 \leq j \leq n; \text{ and } k \text{ was odd}\} \) as follows.
Based on the color label $f$ in graph $L_n \bowtie F_m$. Thus, we have the representation as follows.

$$f(v) = \begin{cases} 
1, \{x_i, y_j\}; & \text{for } i \text{ was odd and } j \text{ was even} \\
\cup \{x_{ik}, y_{jk}\}; & \text{for } i \text{ was even; } j \text{ was odd; and } k \text{ was even} \\
2, \{x_i, y_j\}; & \text{for } i \text{ was even and } j \text{ was odd} \\
\cup \{x_{ik}, y_{jk}\}; & \text{for } i \text{ was odd; } j \text{ was even; and } k \text{ was even} \\
3, \{x_{ik}, y_{jk}\}; & \text{for } 1 \leq i \leq n; 1 \leq j \leq n; \text{ and } k \text{ was odd}
\end{cases}$$

It is clear that for every two adjacent vertices has distinct representation, we can see in $r(y_j|\Pi) = r(x_i|\Pi) = (0, 1, 1)$ for $i$ was odd and $j$ was even.

$$r(y_j|\Pi) = r(x_i|\Pi) = (1, 0, 1)$$ for $i$ was even and $j$ was odd.

$$r(y_{jk}|\Pi) = r(x_{ik}|\Pi) = (1, 0, 1)$$ for $i$ was odd; $j$ was even and $k$ was even.

$$r(y_{jk}|\Pi) = r(x_{ik}|\Pi) = (0, 1, 1)$$ for $i$ was even; $j$ was odd and $k$ was even.

Thus, this proof divided into two cases as follows.

\[ \mu(L_n \bowtie F_m) = \begin{cases} 
2, & \text{for } m \text{ was even} \\
3, & \text{for } m \text{ was odd}
\end{cases} \]

\[ \text{Figure 3. } \mu(L_n \bowtie F_m) \]

**Theorem 3.4** For every natural number $n \geq 2$ dan $m \geq 3$,

$$\mu(L_n \bowtie C_m) = \begin{cases} 
2, & \text{for } m \text{ was even} \\
3, & \text{for } m \text{ was odd}
\end{cases}$$

**Proof.** Let $L_n \bowtie C_m$ be a comb operation graph between ladder graph $L_n$ for $n \geq 2$ with cycle graph $C_m$ for $m \geq 3$ has a set of vertices $V(L_n \bowtie C_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq n\}$ and subsets $E(L_n \bowtie C_m) = \{x_{ik}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{jk}; 1 \leq j \leq n; 1 \leq k \leq m\}$ and subsets $E(L_n \bowtie C_m) = \{x_{i}x_{i+1}; 1 \leq i \leq n - 1\} \cup \{y_{j}y_{j+1}; 1 \leq j \leq n - 1\} \cup \{x_{i}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{j}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\}$ and subsets $E(L_n \bowtie C_m) = \{x_{i}x_{i+1}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{j}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{x_{i}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{j}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{x_{i}x_{i+1}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{j}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{x_{i}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{j}y_{jk}; 1 \leq i \leq n; 1 \leq k \leq m\}$. Thus, this proof divided into two cases as follows.
Case 1. For \( m \) is even

Based on Corollary 2.3 and Lemma 2.1 that \( \mu(L_n \supset C_m) \geq 2 \). Furthermore, we prove that \( \mu(L_n \supset C_m) \leq 2 \). Let \( f : V(G) \to \{1, 2\} \) be a vertex coloring (two adjacent vertices may be colored same color). For more detail the label color and the representation of vertices in graph \( L_n \supset C_m \) respect to with class color \( \Pi = \{C_1, C_2\} \) where \( C_1 = \{x_i, y_j; i \) was odd, and \( j \) was even \} \cup \{x_{ik}, y_{jk}; i \) was even; \( j \) was odd; and \( k \) was even \} \cup \{x_{ik}, y_{jk}; i \) was odd; \( j \) was even; and \( k \) was odd \}, \( C_2 = \{x_i, y_j; i \) was even, and \( j \) was odd \} \cup \{x_{ik}, y_{jk}; i \) was odd; \( j \) was even; and \( k \) was odd \} \cup \{x_{ik}, y_{jk}; i \) was even; \( j \) was odd; and \( k \) was even \} as follows.

\[
f(v) = \begin{cases} 
1, \{x_i, y_j; i \) was odd and \( j \) was even \} \\
- \{x_{ik}, y_{jk}; i \) was even; \( j \) was odd; and \( k \) was even \}
\end{cases}
\]

Based on the color label \( f \) in graph \( L_n \supset C_m \). Thus, we have the representation as follows.

\[
\begin{align*}
    r(y_j) & = r(x_j) = (0, 1) \text{ for } i \text{ was odd and } j \text{ was even} \\
    r(y_j) & = r(x_j) = (1, 0) \text{ for } i \text{ was even and } j \text{ was odd} \\
    r(y_{jk}) & = r(x_{jk}) = (0, 1) \text{ for } i \text{ was even; } j \text{ was odd and } \\
    r(y_{jk}) & = r(x_{jk}) = (1, 0) \text{ for } i \text{ was odd; } j \text{ was even and } \\
    & k \text{ was even}
\end{align*}
\]

It is clear that for every two adjacent vertices has distinct representation, we can see in

\[
\begin{align*}
    r(x_{2k-1}) & \neq r(x_{2k}) \text{ or } r(y_{2k-1}) \neq r(y_{2k}) \text{ for } k \in N. \\
    \text{Without the loss generality, we have } & \mu(L_n \supset C_m) = 2.
\end{align*}
\]

Case 2. for \( m \) is odd

Based on Observation 2.3 and Lemma 2.1 that \( \mu(L_n \supset C_m) \geq 3 \). Furthermore, we prove that \( \mu(L_n \supset C_m) \leq 3 \). Let \( f : V(G) \to \{1, 2, 3\} \) be a vertex coloring (two adjacent vertices may be colored same color). For more detail the label color and the representation of vertices in graph \( L_n \supset C_m \) respect to with class color \( \Pi = \{C_1, C_2, C_3\} \) where \( C_1 = \{x_i, y_j; i \) was odd, and \( j \) was even \} \cup \{x_{ik}, y_{jk}; i \) was even; \( j \) was odd; and \( k \) was even \} \cup \{x_{ik}, y_{jk}; i \) was odd; \( j \) was even; and \( k \) was odd \}, \( C_2 = \{x_i, y_j; i \) was even, and \( j \) was odd \} \cup \{x_{ik}, y_{jk}; i \) was odd; \( j \) was even; and \( k \) was odd \} \cup \{x_{ik}, y_{jk}; i \) was even; \( j \) was odd; and \( k \) was even \}, \( C_3 = \{x_{ik}, y_{jk}; 1 \leq i \leq n; 1 \leq j \leq n; \text{ and } k = m - 1 \} \) as follows.

\[
f(v) = \begin{cases} 
1, \{x_i, y_j; i \) was odd and \( j \) was even \} \\
- \{x_{ik}, y_{jk}; i \) was even; \( j \) was odd; and \( k \) was even \}
\end{cases}
\]

Based on the color label \( f \) in graph \( L_n \supset F_m \). Thus, we have the representation as follows.

\[
\begin{align*}
    r(y_j) & = r(x_j) = (0, 1, 1) \text{ for } i \text{ was odd and } j \text{ was even} \\
    r(y_j) & = r(x_j) = (1, 0, 1) \text{ for } i \text{ was even and } j \text{ was odd}
\end{align*}
\]
\[ r(y_{jk}|\Pi) = r(x_{ik}|\Pi) = (1, 1, 0) \text{ for } i \text{ was even; } j \text{ was odd and } k = m - 1 \]

\[ r(y_{jk}|\Pi) = r(x_{ik}|\Pi) = (0, 1, 1) \text{ for } i \text{ was odd; } j \text{ was even and } k = m - 1 \]

\[ r(y_{jk}|\Pi) = r(x_{ik}|\Pi) = (0, 1, 1) \text{ for } i \text{ was even; } j \text{ was odd and } k = m - 1 \]

It is clear that for every two adjacent vertices has distinct representation, we can see in 
\[ r(x_{2k-1}|\Pi) \neq r(x_{2k}|\Pi) \text{ or } r(y_{2k-1}|\Pi) \neq r(y_{2k}|\Pi) \text{ for } k \in N. \]

Without the loss generality, we have \( \mu(L_n \bowtie C_m) \leq 3. \) Thus \( \mu(L_n \bowtie C_m) = 3. \)

Here is an example figure of \( \mu(L_n \bowtie C_m) \)

![Figure 4. \( \mu(L_n \bowtie C_m) \)](image)

**Theorem 3.5** For every natural number \( n \geq 2 \) dan \( m \geq 2, \mu(L_n \bowtie K_m) = m. \)

**Proof.** Let \( L_n \bowtie K_m \) be a comb operation graph between ladder graph \( L_n \) for \( n \geq 2 \) with complete graph \( K_m \) for \( m \geq 2 \) has a set of vertices \( V(L_n \bowtie K_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq j \leq n\} \) and subsets \( E(L_n \bowtie K_m) = \{x_iy_{i+1}; 1 \leq i \leq n - 1\} \cup \{y_iy_{i+1}; 1 \leq j \leq n - 1\} \cup \{x_iy_k; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_iy_k; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{x_{ik}x_{ik+1}; 1 \leq i \leq n; 1 \leq k \leq m\} \cup \{y_{ik}y_{ik+1}; 1 \leq i \leq n; 1 \leq k \leq m\}. \)

Based on Observation Proposition 2.4 and Lemma 2.1 that \( \mu(L_n \bowtie K_m) \geq m. \) Furthermore, we proof that \( \mu(L_n \bowtie K_m) \leq m. \) Let \( f: V(G) \to \{1, 2, ..., m\} \) be a vertex coloring(two adjacent vertices may be colored same color). For more detail the label color and the representation of vertices in graph \( L_n \bowtie K_m \) respect to with class color \( \Pi = \{C_1, C_2, ..., C_m\} \) where \( C_1 = \{x_i, y_j; i \text{ was odd and } j \text{ was even}\} \cup \{x_{ik}, y_{jk}; i \text{ was even and } j \text{ was odd}\} \cup \{x_{ik}, y_{jk}; i \text{ was odd and } j \text{ was even; } k = 1\}, C_2 = \{x_i, y_{j}; i \text{ was even and } j \text{ was odd; } k = 2\}, ..., C_m = \{x_{ik}, y_{jk}; 1 \leq i \leq n; 1 \leq j \leq n; \text{and } k = m\} \) as follows.
Based on the color label $f$ in graph $L_n \bowtie K_m$. Thus, we have the representation as follows.

\[ f(v) = \begin{cases} 
1, \{x_i, y_j; \text{for } i \text{ was odd and } j \text{ was even}\} \\
\cup \{x_{ik}, y_{jk}; \text{for } i \text{ was even; } j \text{ was odd; and } k = 1\} \\
2, \{x_i, y_j; \text{for } i \text{ was even and } j \text{ was odd}\} \\
\cup \{x_{ik}, y_{jk}; \text{for } i \text{ was odd; } j \text{ was even; and } k = 2\} \\
m, \{x_{ik}, y_{jk}; \text{for } 1 \leq i \leq n; 1 \leq j \leq n; \text{and } k = m\} 
\end{cases} \]

It is clear that for every two adjacent vertices has distinct representation, we can see in $r(x_{2k-1} | \Pi) \neq r(x_{2k} | \Pi)$ or $r(y_{2k-1} | \Pi) \neq r(y_{2k} | \Pi)$ for $k \in \mathbb{N}$. Without the loss generality, we have $\mu(L_n \bowtie K_m) \leq m$. Thus $\mu(L_n \bowtie K_m) = m$.

Here is an example figure of $\mu(L_n \bowtie K_m)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_figure.png}
\caption{$\mu(L_n \bowtie K_m)$}
\end{figure}
4. Conclusion
We have obtained the exact value of metric chromatic number on comb product of ladder graph with any graph, namely path graph, star graph, cycle graph, fan graph, and complete graph, and fan graph. However, to obtain the exact values of some special graphs is hard problem. Hence we propose to determine metric chromatic number of other families graphs apart from the families that we have studied in this paper.

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