Complexity of syntactical tree fragments of Independence-Friendly logic (DRAFT)

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Abstract

In 1986 (2), Blass and Gurevich proved that any non-linear Henkin quantifier can be applied to a quantifier-free first-order formula in such a way that the resulting sentence characterizes an NP-complete problem. In 2014 Sevenster (31) proved a more general result for regular quantifier prefixes of Independence-Friendly (IF) logic; he showed that these prefixes can express (in the sense described above) either 1) only FO problems or 2) also NP-complete problems. The latter class is constituted by 2a) prefixes that mimic non-linear Henkin quantifiers, and 2b) prefixes that encode game-theoretical phenomena of signalling. Furthermore, the dichotomy result yielded a new sufficient (and recursive) criterion for recognizing IF sentences that are equivalent to first-order sentences.

In the present paper we develop the machinery which is needed in order to extend the results of Sevenster to non-prenex (regular) IF sentences. This involves shifting attention from quantifier prefixes to a (rather general) class of syntactical trees. Instrumentally, we explicitate and prove a number of equivalence rules for incomplete syntactical trees.

We partially classify the fragments of IF logic that are thus determined by syntactical trees; in particular, we identify three syntactical structures that are neither signalling nor Henkin, and yet express concepts that are beyond the reach of first-order logic.

1 Introduction

Starting with Fagin’s theorem (10), the enterprise of descriptive complexity has systematically developed correspondences between classes of computational complexity, on one side, and logics and their fragments, on the other. One of the main connections between the two fields (and which our paper shall focus on) is given by the problem of model checking: given a fixed formula \( \psi \) expressed in some logical language, and a class \( K \) of finite structures, the problem asks whether an input structure \( M \in K \) satisfies \( \varphi \) (\( M \models \varphi \)). Furthermore, the choice of opportune encodings of input instances, and of the class \( K \), allows reducing decision problems, that may seem to be completely unrelated to logic,
into model checking problems. A decision problem is described by a sentence $\varphi$ if there is an encoding of the instances of the problem onto $K$, so that $M \models \varphi$ if and only if $M$ encodes a “yes” instance of the problem.

Fagin’s theorem amounts to the statement that the NP problems (those solvable in polynomial time by a nondeterministic Turing machine) are exactly those that can be described by sentences of existential second-order logic (ESO). Given the amount of unsolved issues about the internal structure of NP, it was a natural choice to investigate the descriptive complexity of ESO fragments. The approach most often taken was the study of syntactical fragments determined by specific quantifier prefixes: given a quantifier prefix $Q$, one can study the fragment of prenex ESO sentences of the form $Q\psi$, $\psi$ being a quantifier-free formula. To the best of our knowledge, most of the results obtained in the literature have taken, up to now, the form of classifications of prefix classes up to reduction closure; that is, for each fragment one aims to find an upper bound (all sentences in the fragment describe problems of a certain complexity class C) and an explicit description of at least one difficult problem of the class C (typically, a problem that is C-complete under some kind of significant reduction). In the literature, we find a systematical classification of the quantifier prefixes of relational ESO over graph structures, which was begun in [13] and recently completed in [32]: from it, it emerged that prefix fragments of relational ESO can capture (up to reduction closure) the complexity classes FO, L, NL, and NP. In [7], a similar systematical analysis of relational ESO on string structures has been carried out, showing the surprising dichotomy that prefix fragments either fall in REG or they express NP-complete problems (the result is remarkable because it is known that REG is a small class, REG $\subset$ NP). In general, see [8] for an overview of the results on relation ESO (up to 2010). For what regards functional ESO, prefix classes seem to be of lesser interest, since it is known (15) that the minimal interesting prefix, $\exists f\forall x$, already allows expressing NP-complete problems.

Now, it is interesting that many systems of logic of imperfect information (for example, positive Henkin quantification [2], Independence-Friendly logic [27], Dependence logic [34], Inclusion logic [11], Independence logic [14]) are expressively equivalent to ESO logic, and thus they capture NP. So, under the perspective of descriptive complexity, these logics can be seen as alternative cartographies of the NP class. It is thus natural to study the descriptive complexity of fragments of these logic; and it is perhaps to be hoped that these kinds of investigations lead to a better understanding of the fine structure of (functional and relational) ESO. In the present paper we will focus on the system of Independence-Friendly (IF) logic[1] (first developed in [17] and [29]; see [27] and [3] as references). IF logic is first-order logic (with logical constants $\exists, \forall, \land, \lor, \neg$) enriched with slashed quantifiers of the forms ($\exists v/V$) and ($\forall v/V$), where $V$ is a finite set of variables (slash set). The former can be read as “there exists a $v$ independent from (the variables in) $V$”; the latter has a less intu-

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1See [3] for recent results and a survey on the descriptive complexity study of other logics of imperfect information.
itive appeal. The slash set added to a universal quantifier has no impact at all on the evaluation of the truth of a sentence; but it does for what regards the evaluation of falsity (IF logic can be thought of as a three-valued logic). It should be thought of as a constraint limiting the search for a counterexample. In this paper we will mostly focus on truth only. The main results about the descriptive complexity of IF logic can be found in a recent paper of M. Sevenster (31), building on earlier works of Blass and Gurevich on positive Henkin quantification (2). Blass and Gurevich showed that (unless one applies special restrictions to the range of variables) all Henkin prefixes define NP-complete problems. Sevenster gave a dichotomy result for regular IF logic: IF prefixes are either equivalent to first-order ones (and thus capture the small complexity class FO) or they allow expressing NP-complete problems. The latter can happen in two different situations: first, in case the IF prefix mimicks a Henkin quantifier; and secondly, in case the IF prefix contains a signalling sequence.

Such a neat and minimal classification of regular IF prefixes gives us the courage to investigate more complex fragments of regular IF logic; those that are characterized by syntactical tree prefixes. We will focus on trees that do not contain atomic formulas nor negation symbols (positive initial trees). The study of these kinds of syntactical structures has some formal similarities with the study of restricted Henkin quantifiers (2), partially ordered connectives (see e.g. 30) and Boolean Dependence Logic (35), but yet it seems not to be easily reducible to any of these. Furthermore, we want to stress that the study of the syntactical structures of IF logic considered in the present paper is not easily reducible to the study of quantifier prefix classes of existential second-order logic. To make a concrete example, we will study the fragment of IF sentences of the form $\forall x(\forall y(\exists u/x)\epsilon_1(x, y, u) \lor \forall z(\exists v/x)\epsilon_2(x, z, v))$, with $\epsilon_1(x, y, u), \epsilon_2(x, z, v)$ quantifier-free; by a well-known Skolemization procedure, sentences of this form are equivalent to the functional ESO sentences $\exists f\exists g\forall x\forall y(\forall z(\epsilon_1(x, y, f(y)/u) \lor \epsilon_2(x, z, g(z)/v)))$. Yet, sentences of this form do not fully cover the fragment of ESO corresponding to the quantifier prefix $\exists f\exists g\forall x\forall y\forall z$, because not all quantifier-free formulas $\epsilon(x, y, z)$ are equivalent to quantifier-free formulas of the form $\epsilon_1(x, y, f(y)/u) \lor \epsilon_2(x, z, g(z)/v)$; each disjunct is a two-variable formula, and notice also that there are restrictions on the form of terms: all occurrences of $f$ in this last formula are applied to $y$, and all occurrences of $g$ are applied to $x$.

Sevenster’s results give an answer to the question: what new and interesting dependence patterns are expressible in regular, prenex IF logic, that are not explicitly expressible in the (positive) logic of Henkin quantifiers? And the

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2 A sentence is said to be regular if the same variable is never requantified; e.g., $\forall x P(x) \lor \forall x Q(x)$ is regular, but $\forall x \forall x P(x)$ is not.

3 See [35] and [21] for the state-of-the-art of the study of finite variable fragments of IF logic.

4 The positive logic of Henkin quantifiers is the set of sentences of the form

$$
\left( \begin{array}{c}
\forall x_1 \ldots x_{i-1} & \exists y_1 \\
\vdots & \vdots \\
\forall x_m \ldots x_{m+1} & \exists y_m
\end{array} \right) \psi,
$$

with $\psi$ first-order. Here each $y_i$ depends on $x_1 \ldots x_m$ and
answer was: intransitive, signalling patterns, and nothing else. In this paper we work with a similar question in mind. That is: in IF logic the slashed quantifiers are not required (as in the positive logic of Henkin quantifiers) to occur in an initial quantifier prefix, but they might occur within the scope of conjunctions and disjunctions; do then new interesting dependence patterns arise, that have not the form of Henkin quantification, nor of signalling?

In the present paper we give a partial classification of the (positive initial) tree fragments (up to reduction closure). Analyzing these kinds of fragments, at least two questions suggest themselves: 1) are there other ‘ingredients” in regular IF logic, aside from Henkin quantification and signalling, that allow expressing concepts beyond first-order? 2) Shifting attention from quantifier to tree prefixes, does the FO/NP-C dichotomy still hold (under the restriction that tree prefixes do not contain atomic formulas and negations)? The results in the present paper give a YES answer to the first question: we found three syntactical patterns (Sections 8.1, 9.1 and 9.2) that are not Henkin nor signalling, and yet express higher-order concepts. These patterns are strenghtenings of the signalling by disjunction pattern which was individuated in some examples of T.M.V. Janssen ([20]). For what regards question number 2), we do not yet have a definite answer; all the tree fragments considered up to now have turned out to be either in FO or to define NP-complete problems.

After Section 2 in which IF logic is reviewed, and Section 3, in which syntactical trees are introduced, two sections (4 and 5) are dedicated to the development of a calculus of tree prefixes, by means of which we can extend some of Sevenster’s results by purely syntactical means. Probably, the reader who is only interested in the complexity results can skip these sections and consult them as needed. The methods developed therein are used in section 6 in order to prove that a large fragment of IF logic (which contains all the FO quantifier prefix fragments, and other tree prefixes), the fragment of modest trees, has first-order complexity. In the same section, two classes of “signalling by disjunction” trees are introduced: the generalized Henkin and the coordinated ones. In section 7 we generalize Sevenster’s extension lemma, showing that taking extensions of syntactical trees preserves properties such as NL,P,NP-hardness; and we use it to show that all trees that contain Henkin or signalling patterns are NP-complete. Section 8 divides the generalized Henkin fragment into four subclasses; of these, one is shown to contain only NP-complete trees, which can express the SAT problem; for the other three classes, we give partial results, showing that many of the trees they contain are in FO. Section 9 considers a first kind of coordinated trees, which are classified into three subclasses, all shown to be NP-complete (the first two define SAT, and the third the SET SPLITTING problem). We also show that the trees in the third class can express 2-COLORABILITY (a logspace, non first-order problem). Section 10 takes briefly into account the remaining coordinated trees (of “second kind”).
2 The semantics of IF logic

Independence-Friendly sentences are usually given a meaning by means of certain semantic games; the slash sets are interpreted in the games as constraints of imperfect information. We will not need this viewpoint here, and so we will not review it; the reader may consult [27], Chapters 3-4 for details and motivation.

It was shown by Hodges ([18], [19]) that a certain compositional semantics, known as team semantics, properly extends the game-theoretical semantics giving a meaning also to open formulas. Team semantics can be thought as a technical instrument for the study of the game-theoretical semantics; but we must say that, although a number of alternative semantics have been shown to extend properly the game-theoretical semantics (for example, the lax and strict interpretations of logical operators considered in [9], or the 1-semantics of [28]), there are results (Theorems 4.13 and 4.28 in [27]) which strongly suggest that team semantics might be the correct way of extending game-theoretical semantics. In any case, we have no need to make any commitment about these matters: we are satisfied with the fact that replacement of formulas that are equivalent in an appropriate team-theoretical sense is an operation which preserves truth values of sentences (Theorem 5.18 of [27]).

In team semantics, formulas are interpreted over sets of assignments of a common variable domain (teams), and thus their “meanings” are sets of teams. Indeed, intuitively the notion of independence has no meaning over single assignments, and this intuition has been assessed by a combinatorial argument ([4]). We write $M, X \models \varphi$ to say that the formula $\varphi$ is satisfied by the team $X$ on model $M$; we say that a sentence $\varphi$ is true ($M \models \varphi$) if $M, \{\emptyset\} \models \varphi$. Dual notions of negative satisfaction and falsity are represented with the symbol $\models \neg$.

We present the compositional clauses in the style of [33].

**Def 2.1.** A team on a structure $M$ is a set of assignment such that, for all $s, s' \in X$, $\text{dom}(s)$ is a finite set of variables, and $\text{dom}(s) = \text{dom}(s') =: \text{dom}(X)$.

A team $X$ is suitable for a formula $\psi$ in case $\text{FV}(\psi) \subseteq \text{dom}(X)$.

**Def 2.2.** Given a team $X$ over a structure $M$ and a variable $v$, the duplicated team $X[M/v]$ is defined as the team $\{s(a/v) | s \in X, a \in M\}$.

Given a team $X$ over a structure $M$, a variable $v$ and a function $F : X \to M$, the supplement team $X[F/v]$ is defined as the team $\{s(F(s)/v) | s \in X\}$.

**Def 2.3.** Given two assignments $s, s'$ with the same domain, and a set of variables $V$, we say that $s$ and $s'$ are $V$-equivalent, and we write $s \sim_V s'$, if $s(x) = s'(x)$ for all variables $x \in \text{dom}(s) \setminus V$.

Given a team $X$, a structure $M$ and a set $V$ of variables, a function $F : X \to M$ is $V$-uniform if $s \sim_V s'$ implies $F(s) = F(s')$ for any $s, s' \in X$.

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5 An annotation for lovers of bibliographical precision: although actually the strict/lax distinction appears in an earlier paper [11], it is acknowledged by Galliani himself (page 4, footnote 4 of his paper) that the idea comes from an earlier draft of [9].
Def 2.4. We say that a suitable team $X$ satisfies (resp. negatively satisfies) an IF formula $\varphi$ over a structure $M$, and we write $M,X \models \varphi$ (resp. $M,X \models \neg \varphi$) in any of the following circumstances:

- $M,X \models R(t_1,\ldots,t_n)$ if $M,s \models R(t_1,\ldots,t_n)$ in the classical sense for every $s \in X$; $M,X \models \neg R(t_1,\ldots,t_n)$ if $M,s \not\models \neg R(t_1,\ldots,t_n)$ for every $s \in X$

- $M,X \models \neg \psi$ if $M,X \models \neg \psi$

- $M,X \models \psi \land \chi$ if $M,X \models \psi$ and $M,X \models \chi$

- $M,X \models \psi \lor \chi$ if there are $Y,Z \subseteq X$ such that $Y \cup Z = X$, $M,Y \models \psi$, and $M,Z \models \chi$

- $M,X \models (\forall v/V)\psi$ if $M,X[M/v] \models \psi$

- $M,X \models (\exists v/V)\psi$ if $M,X[F/v] \models \psi$ for some $V$-uniform function $F : X \rightarrow M$

- Dual clauses, obtained interchanging $\land$ with $\lor$ and $\forall$ with $\exists$, define inductively the relation $\models \neg$.

In the present work, we will mainly operate at the level of team semantics; but often, instead of the semantical clauses, we will make use of some syntactical equivalence rules which have been developed in [3]. Notice that IF logic is three-valued (sentences that are neither true nor false are called undetermined), so that there is more than one reasonable candidate for the notion of equivalence of IF sentences. Here we will consider mainly the notion of truth-equivalence.

Def 2.5. Two IF sentences are truth-equivalent ($\equiv$) if they are true in the same structures (i.e., $\psi \equiv \chi$ if for all structures $M$, $M \models \psi \iff M \models \chi$).

Most equivalence rules of IF logic, however, hold for a stricter notion of equivalence:

Def 2.6. Two IF sentences are strongly equivalent ($\equiv^*$) if they assume the same truth value (true, false or undeterminate) on each structure (i.e., $\varphi \equiv^* \chi$ if for all structures $M$ we have $M \models \varphi \iff M \models \chi$ and $M \models \neg \varphi \iff M \models \neg \chi$).

We will need a well known fact about the expressivity of IF sentences:

Proposition 2.7. ([27], Theorems 6.10, 6.16) On the sentence level, IF logic is equiexpressible with existential second order logic. Thus, by Fagin’s theorem ([10]), the set of IF sentences characterizes the complexity class NP.

For what regards equivalence of open formulas, many possibilities have been considered in the literature; probably the simplest option would be to consider two formulas $\psi, \theta$ equivalent if in all structures they are satisfied by the same
teams, provided that we only consider teams whose variable domain contains $FV(\psi) \cup FV(\theta)$. However, many important equivalence rules of IF logic are context-dependent; they hold only if some kinds of restrictions are imposed on the contexts in which the formulas may appear; that is, these rules only hold if the formulas do not occur in the scope of certain quantifiers. Thus, it is in many occasions more convenient to consider notions of equivalence relativized to contexts. This has been been done in two different ways in the literature: 1) in the style of Caicedo, Dechesne and Janssen ([3]), specifying which variables should not appear in the context, and 2) in the style of Mann, Sandu and Sevensen, expressing equivalence in a fixed context. The latter has the advantage of involving only a finite domain, but it leads to more cumbersome formulations of the equivalence rules. We stick here to the former option.

**Def 2.8.** Let $\psi, \theta$ be IF formulas. A team $X$ is said to be **suitable** for $\psi$ and $\theta$ if $\text{dom}(X) \supseteq FV(\psi) \cup FV(\theta)$.

**Def 2.9.** Let $\psi$ be an IF formula, $Z$ a finite set of variables. Then $\psi$ is **$Z$-closed** if $FV(\psi) \cap Z = \emptyset$.

**Def 2.10.** Let $\psi, \theta$ be IF formulas, let $Z$ be a finite set of variables. We say that $\psi$ and $\theta$ are **$Z$-equivalent**, and we write $\psi \equiv_Z \theta$, if they are $Z$-closed and, furthermore, $M, X \models \psi \iff M, X \models \theta$ and $M, X \models \neg \psi \iff M, X \models \neg \theta$ for all structures $M$ and for all teams $X$ that are suitable for $\psi$ and $\theta$ and such that $\text{dom}(X) \cap Z = \emptyset$.

If we have an explicit description $\{z_1, z_2, \ldots, z_n\}$ of $Z$, we can also write, for brevity, $\psi \equiv_{z_1, z_2, \ldots, z_n} \theta$.

So, the subscripts to the equivalence symbols mean that the equivalence only holds for those teams whose domain does not contain any of the subscripted variables; and also, in order to avoid triviality, the subscripted variables must not occur free in the formulas under consideration. This notion of equivalence of formulas works well because of the following two facts:

**Proposition 2.11.** 1) ([3], remarks on page 22) If $\phi$ and $\chi$ are IF sentences, then, for any finite set $Z$ of variables, $\phi \equiv_Z \chi$ if and only if $\phi \equiv \chi$.

2) ([3], Lemma 6.16) More generally, if $\phi$ and $\chi$ are $Z$-closed IF formulas, then $\phi \equiv_Z \chi$ if and only if $\phi \equiv_{Z \cap \text{Bound}(\phi) \cap \text{Bound}(\chi)} \chi$.

**Proposition 2.12.** ([3], Theorem 6.14) If $\phi, \psi, \psi'$ are IF formulas, $Z$ a finite set of variables, $\psi'$ is obtained from $\psi$ by replacing a subformula occurrence of $\psi$ with $\psi'$, and $\psi \equiv_Z \psi'$, then $\phi \equiv_Z \phi'$.

### 3 Synctactical trees: basic definitions

We define here the class of synctactical trees which is of our interest – we are seeking for the simplest possible generalization of what a quantifier prefix is if
we do not restrict attention to prenex sentences. This requires including in the prefixes also connectives, and taking into account the binary ramifications they induce in the structure of formulas. This class of trees (the positive initial trees) has already been introduced elsewhere ([1]), but here we will require some more precision in the formal details.

**Def 3.1.** A *synctactical tree* is a finite tree whose nodes are (occurrences of) atomic formulas, negations, conjunctions, disjunctions or quantifiers (with their slash sets), and which respects the following constraints: 1) atomic formulas are leaves (i.e., they have no successors) 2) each negation has at most one successor 3) each binary connective has at most two successors 4) each quantifier has at most one successor.

It should be clear in what sense to each IF sentence we can associate its syntactical tree, and in the following we will always indentify a formula with its tree. On the contrary, there are many syntactical trees (according to the above definition) that are not the syntactical tree of any formula, for example:

\[
\begin{array}{cccccc}
\forall x & \forall x & (\forall x/\{z\}) & \forall x & \forall \\
\exists y & \lor & \lor & \lor & \exists x \exists y \\
B(x) & A(x) & \neg & \land & A(x) \\
& & B(y) & C(z)
\end{array}
\]

**Def 3.2.** A *positive initial tree* is a syntactical tree which contains no occurrence of atomic formulas nor of negations.

Said otherwise, a positive initial tree can be obtained from the syntactical tree of some negation normal IF formula by removing from it all nodes that correspond to atomic formulas and negations (a formula is said to be negation normal if negation symbols occur only in front of atomic formulas). The word positive refers to the fact that we do not allow negation symbols to occur in the tree, while the word initial refers both to the fact that none of the paths end with an atomic formula and to fact that, if we think the tree as obtained removing nodes from the syntactical tree of a formula, whenever a node is deleted also all nodes below it are removed. This generalizes the fact that a quantifier prefix is an initial segment of the syntactical tree of a formula; the parallel is made more explicit in the following two definitions (in which we use standard set-theoretical terminology for the description of trees).

**Def 3.3.** A *dense open subset* \(Y\) of a tree \(T\) is \(Y \subseteq T\) such that

\[
\forall y \in Y \forall t \in T (t \preceq y \Rightarrow t \in Y)
\]

(where \(\preceq\) is the partial ordering of the tree in which the root is the minimal element).
**Def 3.4.** An IF formula $\varphi$ **begins with** $T$ if $T$ is a dense open subset of the syntactical tree of $\varphi$.

**Def 3.5.** Given an IF formula $\varphi$, we define the **tree prefix of** $\varphi$, and denote it as $PTr(\varphi)$, the maximal positive initial tree contained in the syntactical tree of $\varphi$.

(We could equally well define the tree prefix of $\varphi$ to be the maximal positive initial tree contained in the syntactical tree of $\varphi$ **that does not intersect any quantifier-free subformula of** $\varphi$, since it is reasonable to expect that the connectives in quantifier-free subformulas do not yield any expressive power beyond first-order; this is the kind of prefixes that we will consider in the rest of the paper.)

Notice that, for example, we can say that the formula $\varphi = \forall x (A(x) \lor \neg B(x))$ begins with the tree

```
          \forall x
         /    \    \\
        /     \   \\
     [ ]   \   \\
        \   B(x) \\
```

even though this tree is not positive initial (it contains a negation, and also an atomic formula). Instead, $PTr(\varphi)$ is

```
          \forall x
         /    \\
        /    \\
[ ] [ ] \\
```

In these graphical representations, we have added some “gap” nodes [ ] whenever there were less than two successors to a connective, or no successors to a quantifier; the gaps mark the points in which the tree may potentially be filled with (the syntactical tree of) an IF formula. This convention grants us with a more comfortable linear notation for syntactical trees (e.g., we might write the tree above as $\forall x (\ [ \lor \ [ \ ]))$, and it is made precise by the forthcoming definition of **path** of a (possibly incomplete) syntactical tree; this does not coincide with the usual notion of paths of a tree from set theory, as maximal suborders (it would, if we had included as possible elements of the trees occurrences of “the gap”). Notice indeed that a connective followed by a single gap

```
          ...  \\
        /    \\
        /    \\
[ ] [ ] \\
```

is not the end point of a maximal suborder, but we may want to attach a formula in that gap. All we need to do is to associate to each of our syntactical trees $T$ another tree, $\hat{T}$, which is identical to $T$ except for the fact that it has a
“gap node” in all position where $T$ has gaps; the “paths” of $T$ can be identified with the maximal suborders of $\hat{T}$.

**Notation 3.6.** Given a formula $\varphi$ (resp. a quantifier prefix $\overrightarrow{Q}$, a syntactical tree $T$), we denote the relation of superordination between pairs of logical operators as $\prec_\varphi$ (resp. $\prec_{\overrightarrow{Q}}$, $\prec_T$). So, for example, $\forall x \prec_\varphi \exists y$ means that (a certain occurrence of) $\exists y$ occurs within the scope of (a certain occurrence of) $\forall x$.

Two quantifier prefixes $R$, $S$ can obviously always be concatenated in order to obtain a longer prefix $RS$; this notation can sometimes be extended to trees:

**Notation 3.7.** Whenever $R$ is a finite linearly ordered set whose last element is a gap, and $S$ is a tree, we can unambiguously denote as $RS$ the concatenation of $R$ and $S$, that is, the tree obtained removing the last element of $R$ and replacing it with the tree $S$.

**Def 3.8.** Let $R = R' \{O\}$ be a linear suborder of a tree $T$ which is an initial segment of it (that is, $d <_T e$ and $e \in R$ imply $d \in R$).
We then define the **arity of** $R$ in $T$, $ar_T(R)$.
In case $O$ is a connective,

$$ar_T(R) = \begin{cases} 
2 & \text{if } O \text{ is a maximal element of } T \\
1 & \text{if } O \text{ has exactly one successor} \\
0 & \text{otherwise.}
\end{cases}$$

In case $O$ is a quantifier,

$$ar_T(R) = \begin{cases} 
1 & \text{if } O \text{ is a maximal element of } T \\
0 & \text{otherwise.}
\end{cases}$$

**Def 3.9.** Given a syntactical tree $T$, let $\hat{T}$ be the tree obtained attaching as new leaves, to each initial segment $R$ of $T$, a number of occurrences of the gap $[,]$ equal to the arity of $R$ in $T$.

For example, if $T$ is

\[
\forall x \\
\quad \exists y
\]

then $\hat{T}$ will be the much more transparent

\[
\forall x \\
\quad \exists y
\]

Notice that, since connectives in IF logic are commutative, it is not really important whether we place gap nodes on the left or on the right of an existing node.
Def 3.10. Given a tree $T$, we will call any maximal linear suborder of $\hat{T}$ a path of $T$. The set of these objects will be called $\text{Path}(T)$.

Notice that each path of $T$ corresponds to a gap node of $T'$, and vice versa; and that linear suborders ending with an occurrence of an atomic formula are not paths in the sense defined here.

The main results of [31] worked properly only for a restricted class of quantifier prefixes, the sentential class. We will often need to impose a somewhat stronger condition on our tree prefixes: regularity, in the sense of [3].

Def 3.11. A syntactical tree $T$ is regular if the following hold:
1) No variable occurs both free and bound in $T$
2) If a quantification $(Qv/V)$ occurs in $T$, then it is not subordinated to any quantification of the form $(Qv/W)$.

In particular, we denote as $\text{Reg}(\text{IF})$ the set of regular IF formulas. Notice that sentences automatically satisfy condition 1).

Furthermore, we say that a tree $T$ is strongly regular if it satisfies 1) plus the following:
2') No variable is quantified more than once in $T$.

4 Synctactical reductions of trees

In [31], the analysis of low complexity quantifier prefixes is based on a somewhat intuitive notion of equivalence of prefixes. We develop here something which is analogous; but, perhaps a bit surprisingly, the needed notion for trees is not an equivalence relation, but an order relation, that we shall simply call reduction.

Def 4.1. Let $\psi$ be an IF formula. The set $\text{Bound}(\psi)$ of bound variables of $\psi$ is defined as usual. The set $\text{FV}(\psi)$ of free variables is defined as for first-order logic, except for the clause $\text{FV}((Qv/V)\chi) := \text{FV}(\chi) \setminus \{v\} \cup V$. Similar definitions can be given for trees, paths, etc.

Notation 4.2. If $R$ is a tree/path/formula/etc., we denote as $\text{FV}(R)$ and $\text{Bound}(R)$ the set of free variables, respectively of bound variables, occurring in $R$. We write $\text{Var}(R)$ for $\text{FV}(R) \cup \text{Bound}(R)$. We denote as $\text{QFree}(\text{IF})$ the set of quantifier-free IF formulas.

In [31], two quantifier prefixes $R, S$ are defined to be equivalent if, whenever the same quantifier-free formula $\psi$ is postfixed to them, one obtains truth-equivalent formulas $R\psi \equiv S\psi$. We have not yet specified what we mean by equivalence of (open) formulas; in any case, we will not need, here, this degree of generality. But one can instead make the weaker requirement that two prefixes are equivalent in case $R\psi \equiv S\psi$ whenever $R\psi$ and $S\psi$ are regular sentences. This is the idea behind definitions [31]. However, if $R, S$ are trees, we will rather require that $R\psi \equiv S\psi$ whenever $R\psi$ is a regular sentence. We make a requirement only on $R$, not on $S$; the asymmetry of our notion of reduction stems from here. The reason for this choice is the following: there are seemingly
innocuous equivalence rules between formulas that actually *increase* (or, in the opposite direction, decrease) the expressive power of a syntactical tree (this even happens if we restrict attention to first-order logic). One such example is given by the extraction of quantifier rules. Let us think of the trees $R = \forall x(\exists y( [ ] \lor [ ] ))$ and $R' = \forall x \exists y( [ ] \lor [ ] )$. $R'$ is actually more expressive than $R$, since its rightmost gap can be filled (if we want to obtain a sentence) only with formulas having $x, y$ as free variables – while the rightmost gap of $R$ can be filled only with formulas having just $x$ as free variable.

The first of the following definitions generalizes the operation of postfixing a formula to a quantifier prefix. Here we may need to attach *many* formulas, one for each gap in the tree.

**Def 4.3.** Let $T$ be a syntactical tree. We will call any function $e : \text{Path}(T) \to \text{Reg}(\text{IF})$ a **completing function for** $T$. A completing function can be:

- **weak**: if $\text{ran}(e) \subseteq \text{QFree}(\text{IF})$
- **sentential**: if, for each $R \in \text{Path}(T)$, $\text{FV}(e(R)) \subseteq \text{Bound}(R)$.
- **regularity-preserving**: if, for each $R \in \text{Path}(T)$, we have $\text{Bound}(e(R)) \cap \text{Bound}(R) = \emptyset$.
- **nice**: if it is both sentential and regularity-preserving.

Notice that weak completing functions are automatically regularity-preserving.

**Def 4.4.** We call $\hat{e}(T)$ the formula obtained attaching $e(R)$ at the end of $R$, for each $R \in \text{Path}(T)$.

Similarly, if $S \subseteq T$ is a tree, we denote by $\hat{e}(S)$ the smallest subformula of $\hat{e}(T)$ which contains $S$.

It should be clear that, if $T$ is a regular tree, then asserting that $e$ is sentential amounts to saying that $\hat{e}(T)$ is a sentence; and that if $T$ is regular and $e$ is regularity-preserving, then $\hat{e}(T)$ is a regular formula.

**Example 4.5.** 1) Consider the simplest possible example of a positive initial tree: a quantifier prefix. So, let $T = \exists y(\forall x(\{y\}))$; then $\hat{T} = \exists y(\forall x(\{y\}))(x)$, which is still linear. There is only one gap, so only one path, which is $T$ itself; so, a completing function for $T$ is just a function from the singleton set $\{T\}$ to $\text{Reg}(\text{IF})$. Set for example $e(\hat{T}) = (\exists z(\{z\}))Q(x, z)$. Applying this completing function to $T$, you obtain the formula $\hat{e}(T) = \exists y(\forall x(\{y\}))(\exists z(\{z\}))Q(x, z)$. Notice that we have $\text{FV}(e(\hat{T})) = \{x\} \subseteq \{y, x\} = \text{Bound}(\hat{T})$: $e$ is sentential. And indeed $\hat{e}(T)$ is a sentence. We also have $\text{Bound}(\hat{T}) \cap \text{Bound}(\hat{e}(T)) = \{y, x\} \cap \{z\} = \emptyset$, so that $e$ is regularity preserving. And indeed $\hat{e}(T)$ is a regular formula. Putting things together, $e$ is a nice completing function, but it is not weak, because $e(\hat{T})$ contains the quantifier $(\exists z(\{x\}))$. Instead, $f(\hat{T}) := P(x, y) \land Q(x, y)$ is nice and also weak.

Let instead $g(\hat{T}) = (\exists u(\{u\}))(x)$, This function is regularity-preserving but not sentential: $\text{FV}((\exists u(\{u\}))Q(x, z)) = \{x, z\} \not\subseteq \{y, x\} = \text{Bound}(\hat{T})$. And indeed, $\hat{g}(T) = \exists y(\forall x(\{y\}))(\exists u(\{u\}))(x, z)$ is a regular formula, but not a sentence, because $z$ is free in it.
Let \( h(T) = (\exists y/\{x\})Q(x,y) \). This is sentential but not regularity preserving: \( \text{Bound}(T) \cap \text{Bound}(g(T)) = \{y,x\} \cap \{y\} = \{y\} \neq \emptyset \). Indeed, \( h(T) = \exists y(\forall x/\{y\})(\exists y/\{x\})Q(x,z) \) is a sentence, but it is not regular, because there are two quantifications of \( y \), one superordinated to the other.

Finally, let \( k(T) = \forall y(\exists y/\{x\})Q(x,z) \). This is not a completing function, because \( k(T) \) is not a regular formula.

2) Consider again a linear tree, but this time of the form \( T' = \exists y(\forall x/\{y\}) \lor \). The corresponding tree with gap nodes is \( \hat{T}' = \exists y(\forall x/\{y\})([ ] \lor [ ] ) \), which is not a linear tree anymore. And it has two paths (call \( A \) the path containing the leftmost gap, and \( B \) the other one). A completing function for \( T' \) will be a function \( \{A,B\} \rightarrow \text{Reg}(IF) \), for example

\[
\begin{cases}
  j(A) = \exists zP(y,z) \\
  j(B) = Q(y,z)
\end{cases}
\]

which is not weak because of \( j(A) \), not sentential because of \( z \) occurring free in \( j(B) \), and it is regularity preserving because neither \( j(A) \) nor \( j(B) \) contain quantifications over \( x \) or \( y \). The result of the completion is the sentence \( j(T') = \exists y(\forall x/\{y\})(\exists zP(y,z) \lor Q(y,z)) \).

If we want to define reductions between tree prefixes, we face one more difficulty: while with linear prefixes \( R,S \) it is obvious what it means to say “when \( R,S \) are prefixed to the same formula”, here we should say that two trees \( T,U \) are “completed in the same way by the same completing function”. But 1) no function is a completing function for both \( T \) and \( U \), unless \( T = U \), so we will need to establish some kind of correspondence between completing functions of different trees, and 2) the location “in the same way” makes any sense only after we have fixed a bijection between the gaps in \( T \) and the gaps in \( U \); this bijection will establish which pairs of gaps must be filled with the same formula. This also means that we will manage to define reductions only between pairs of trees which have the same number of gaps. Noticing that we have defined the paths of a tree so that paths and gaps are in a natural bijection (each gap is the last element of a path, and each path ends with a certain gap), we can proceed and define equivalence without any reference to gaps.

**Def 4.6.** Given two trees \( T,T' \), a bijection \( \iota : \text{Path}(T) \rightarrow \text{Path}(T') \) and a completing function \( e \) for \( T \), we define a completing function \( e_\iota \) for \( T' \) (a function \( \text{Path}(T') \rightarrow \text{Reg}(IF) \) by the clauses \( e_\iota(\iota(P)) = e(P) \), where \( P \) varies in \( \text{Path}(T) \)).

**Def 4.7.** A reduction between two syntactical trees \( T \) and \( T' \) is a bijection \( \iota : \text{Path}(T) \rightarrow \text{Path}(T') \) such that, for all fine completing functions \( e \), \( e(T) \equiv e_\iota(T') \).

We say \( T \) is reducible to \( T' \) if there is a reduction between them. Whenever the name of the equivalence is not specified, we will abuse notation and denote both \( e \) and \( e_\iota \) as \( e \), unless this can be a source of confusion.

Notice, again, the asymmetry of reductions, given by the fact that we impose niceness only on \( e \), and not on the corresponding \( e_\iota \).
As was mentioned in the introduction, the complexity notions studied in
[2] and [31] are expressed in terms of “postfixing all possible quantifier-free
formulas” to a quantifier prefixes. This leads us to a still weaker notion of
reduction.

**Def 4.8.** A weak reduction $\iota$ between two syntactical trees $T$ and $T'$ is a
bijection between the paths of $T$ and the paths of $T'$ such that, for all finite weak completing functions $e, \hat{e}(T) \equiv \hat{e}(T')$. We say $T$ is weakly reducible to $T'$ if there is a weak equivalence between them.

The notion of weak reduction will turn out to be flexible enough for our
purposes; its most important feature is that it does not decrease the complexity
of a tree, while at the same time it allows the formulation of a number of proof-
theoretic rules (to be develop in the following section, up to a prenex form theorem) that would fail for mere reductions.

The precise connection with complexity theory will be explicitated in section
for now, let us just see how reductions relate to a measure of expressive power:

**Def 4.9.** To any IF sentence $\varphi$, we associate the class $F_\varphi = \{M | M$ finite, $M \models \varphi\}$ of its finite models. Given a syntactical tree $T$, we define the complexity class of $T$:

$C(T) = \{F_\hat{e}(T) | e$ weak completing function for $T\}$.

If we have $C(T) = C(T')$, resp. $C(T) \subseteq C(T')$... then we say that $T$ is as complex as $T'$, resp. $T$ is less complex than $T'$...

**Theorem 4.10.** If there is a reduction or a weak reduction from tree $T$ to tree $T'$, then $C(T) \subseteq C(T')$.

*Proof.* If $\iota$ is a (weak) reduction, it means that, in particular, for each weak completing function $e$ for $T$ there is a corresponding weak completing function $e_\iota$ for $T'$, so that $\hat{e}(T) \equiv \hat{e}_\iota(T')$: that is, such that $F_\hat{e}(T) = F_{\hat{e}_\iota}(T')$. This means that, for each weak completing function $e$ of $T$, $F_{\hat{e}(T)} \subseteq C(T')$, that is, $C(T) \subseteq C(T')$.

This result holds for both kinds of reductions, but only the weak reductions are in abundant number enough for our purpose; this will emerge in the next section.

**5 Prenex transformations and other formal rules for regular trees**

A number of the results proved in [31] depend on manipulations of quantifier prefixes that preserve an opportune notion of equivalence of prefixes; similarly,

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7The requirement of $\iota$-fineness, here, is a bit redundant, since weak completing functions are always regularity-preserving.
here, we develop proof-theoretical rules for the manipulation of trees that either preserve the complexity of trees or are weak reductions. We also prove, along the way, a sort of prenex form theorem: each tree can be turned into a prenex one without losing complexity. This result will not be used in the following, but it is perhaps of some intrinsic interest.

Turning equivalence rules for sentences into reduction rules for initial trees is not a trivial matter, as it may be thought. Problems arise for example with rules for the extraction of quantifiers, and rules for the renaming of variables; we recall them below in a form for formulas; they are notational variants of (the first half of) Proposition 5.37 and 5.35 from [27].

**Proposition 5.1.** Suppose $u$ is not quantified in $\psi$. If $v$ does not occur in $(Qu/U)\psi$, then

$$ (Qu/U)\psi \equiv_{uv} (Qv/U)Subst(\psi, u, v). $$

where $Subst(\psi, u, v)$ is the formula obtained by replacing, in $\psi$, all free occurrences of $u$ with $v$.

**Proposition 5.2.** Let $\psi$ and $\chi$ be IF formulas. If $u$ occurs neither in $\chi$ nor in $U$, then

$$ (Qu/U)\psi \circ \chi \equiv_{u} (Qu/U)(\psi \circ \chi/\{u\}). $$

where $\chi/\{u\}$ is the formula obtained adding the variable $u$ to all the slash sets of $\chi$.

(The variable $u$ must be added to the slash sets of the right disjunct in order to prevent it to be used as a source of signals – which could be used to circumvent the restrictions imposed by slash sets. See [3].)

These rules are problematic, because they are not “local”, in the sense that they depend on global conditions: that some variable does not occur in a certain subformula, that a whole subformula is slashed. Since a syntactical tree may be extended in many ways to a well-formed IF sentence – perhaps also attaching to the conjuncts/disjuncts some formula containing the variable which is mentioned in the global conditions – the extraction and renaming rule do not produce “equivalent” syntactical trees. We show it with an example (involving irregular trees).

Consider the tree $QxQ'y([ ] \lor Q'y[ ]) \ (\text{where } Q, Q' \text{ are quantifiers})$. One may be tempted to state it is equivalent to $QxQ'yQ'z([ ] \lor [ ])$, to be obtained applying renaming followed by extraction of quantifiers. But, for example, in general $QxQ'y(P(x) \lor Q'yP(y)) \neq QxQ'yQ'z(P(x) \lor P(y))$; yet, setting $P(y)$ in that position is allowed in both trees, since it is inserted in the scope of $Q'y$ quantifiers. Notice, furthermore, that we cannot aim at a prenex transformation by renaming the outmost $Q'y$ quantifier, since the renaming rule requires $y$ not to occur bound in the scope of $Q'y$.

\textsuperscript{8}A note of warning. This proposition is nothing else than Theorem 6.12 of [3]. If the reader confronts our formulation with the rule stated in that paper, he might think that we have forgotten a clause; that we should have specified that $u$ must not be in $U$. Yet, this is already implied by $uv$-closedness: the formulation in [3] was redundant.
It seems clear that the pathological aspect of these kinds of examples lies in the irregularity of the trees involved, and that we may expect the extraction and renaming rules to hold for regular trees. However, regularity is not enough to account for the fact that the slash sets of a certain subformula should be extended, when performing extraction of quantifiers in IF sentences. This is the reason why the quantifier extraction rule for trees only holds as a weak reduction (if $\chi$ is a quantifier-free formula, then $\chi = \chi/\{u\}$, so that adding slashed variable to the quantifiers of the tree is sufficient).

**Def 5.3.** Denote with $\text{Subst}(R, u, v)$ the tree which is obtained from the tree $R$ replacing all free occurrences of $u$ in $R$ with $v$. Here, by a subtree of $R$ we mean a subborder which is constituted by all the elements $a$ of $R$ such that $r \prec a$, for a fixed $r \in R$. If $b \leq_R c$, the closed interval $A = [b, e]_R$ is the set $\{x \in R | b \leq_R x \leq_R c\}$. We omit indices if they are clear from the context. If there are two distincts trees $T, T'$ and $S$ occurs as a subtree of both $T, T'$, we will abuse notation and refer to both subtrees as $S$ (unless there is risk of ambiguity).

**Theorem 5.4.** Suppose two regular syntactical trees $T, T'$ differ only for one subtree, which is $S = (Qu/U)R$ in $T$ and $S' = (Qv/U)\text{Subst}(R, u, v)$ in $T'$. Suppose $v$ does not occur in $S$ and $u \notin U$. Then $C(T) = C(T')$.

**Proof.** Let $c, d, c', d'$ be the roots of $T, S, T', S'$, respectively. We define $\iota : \text{Path}(T) \rightarrow \text{Path}(T')$ as the identity on paths which do not intersect $S$; if instead $P$ is a path of the form $[c, d]B$, for some linear order $B$, define $\iota(P) = [c', d']\text{Subst}(B, u, v)$. This is clearly a well-defined bijection.

Let $e$ be a weak completing function for $T$. We want to find a completing function $f$ for $T'$ such that $\bar{e}(T) \equiv f(T')$.

For each $P \in \text{Path}(T)$ that intersects $S$, let $f(\iota(P)) = \text{Subst}(e(P), u, v)$; otherwise, let $f(\iota(P)) = e(P)$. Function $f$ is weak, since $e$ is (and thus it is regularity-preserving). The condition for sententiality obviously holds for paths that do not intersect $S'$; if instead $\iota(P)$ intersects $S'$, we have $\text{Bound}(\iota(P)) = (\text{Bound}(P) - \{u\}) \cup \{v\} \supset (\text{FV}(e(P)) - \{u\}) \cup \{v\}$, where, in the inequality, we used the fact that $e$ is sentential. We want to show that $\bar{e}(S)$ and $\bar{f}(S')$ satisfy the assumptions of Prop. 5.1, that is: 1) $u$ is not quantified in $\bar{e}(R), 2) v$ does not occur in $\bar{e}(S)$, and 3) $f(R') = \text{Subst}(\bar{e}(R), u, v)$.

The condition 1) follows from the fact that $u$ is not quantified in $R$ (by regularity of $T$) nor in $e(P)$, for any $P \in \text{Path}(T)$ (since $e$ is weak).

Condition 2) follows from the fact that, by the hypothesis, $v$ does not occur in $S$; and furthermore, by the fact that $v$ does not occur in $e(S)$ (it occurs not as quantified, because $e$ is weak; and not as free, since $e$ is sentential).

Condition 3) holds by the definitions of $S'$ and $f$, also taking into account the fact that the $e(P)s$ do not contain quantifications over $u$.

Applying Prop. 5.1 we obtain $\bar{e}(S) \equiv_{uv} \bar{f}(S')$. Substitution of $\{u, v\}$-equivalent formulas (Prop. 2.12) yields then $\bar{e}(T) \equiv_{uv} \bar{f}(T')$. Since $e$ and $e_s$ are sentential.

\[9\] We could obtain that $e(T) \equiv e_s(T')$ without using sententiality; thus, the present result
it follows that \( \hat{e}(T) \) and \( \hat{f}(T') \) are sentences. So, it is legitimate to apply part
1) of Prop. 2.11 and obtain \( \hat{e}(T) \equiv \hat{f}(T') \). Since \( e \) was arbitrary among weak, nice completing functions, we may conclude that \( C(T) \subseteq C(T') \).

To obtain the opposite direction, just observe that, if we define the tree \( R' = \text{Subst}(R, u, v) \), then \( S = (Qu/U)\text{Subst}(R', v, u) \). Furthermore, since \( u \in U \), and by regularity \( u \notin \text{Bound}(R) \), \( u \) does not occur in \( S' \). So, a completely symmetric argument applies.

Given a tree \( S \), we denote as \( S/\{u\} \) the corresponding tree in which \( u \) has been added to all slash sets of quantifiers. Coherently with earlier notations, \( (Qu/U)S_1 \circ S_2 \) will denote the tree whose root is a binary connective \( \circ \), below which are attached two subtrees \( (Qu/U)S_1 \) and \( S_2 \). Instead, in order to avoid confusion about precedence of operators, we will sometimes enclose descriptions of trees in angular parentheses; for example, we will denote as \( (Qu/U)(S_1 \circ S_2) \) the tree whose root is \( (Qu/U) \), below which is attached \( \circ \), below which are attached two subtrees \( S_1 \) and \( S_2 \).

**Theorem 5.5.** Suppose two regular syntactical trees \( T, T' \) differ only for one subtree, which is \( S = (Qu/U)S_1 \circ S_2 \) in \( T \) and \( S' = (Qu/U)(S_1 \circ S_2/\{u\}) \) in \( T' \). Suppose \( u \) does not occur in \( U \) nor \( S_2 \). Then \( T \) is weakly reducible to \( T' \). An analogous result holds for trees of the form \( S_1 \circ (Qu/U)S_2 \).

**Proof.** Let

\[
\begin{align*}
\text{c} & \text{ be the root of } T \\
n\text{c'} & \text{ be the root of } T' \\
\text{d} & \text{ be the root of } S_1 \text{ as a subtree of } T \\
n\text{d'} & \text{ be the root of } S_1 \text{ as a subtree of } T' \\
\text{f} & \text{ be the root of } S \\
n\text{f'} & \text{ be the root of } S_1 \circ S_2/\{u\}
\end{align*}
\]

Define the function \( \iota : \text{Path}(T) \rightarrow \text{Path}(T') \) to be the identity on paths which do not intersect \( S \); paths of the form \([c, d](Qu/U)B\), with \( B \subseteq S_1 \), must be sent to \([c', d']B\); paths of the form \([c, f]C\), with \( C \subseteq S_2 \), must be sent to \([c', f'](C/\{u\})\) (if it happens that \( S_1 = S_2/\{u\} \), we specify that the above mentioned copy of \( C/\{u\} \) is the one which is attached to \( \circ \) on the rightmost node).

Let \( e \) be any weak, nice\(^{10}\) completing function for \( T \). We must verify that \( \hat{e}(S) \) and \( \hat{e}_e(S') \) satisfy the hypotheses of Prop. 5.2. These amount to: 1) \( u \) does not occur in \( \hat{e}(S_2) \), 2) \( u \) does not occur in \( U \), and 3) \( \hat{e}_e(S_2/\{u\}) = \hat{e}_e(S_2/\{u\})/\{u\} \) (while it is obvious that all slash sets in \( S_2/\{u\} \) contain \( u \), we need a proof that this still holds for \( \hat{e}_e(S_2/\{u\}) \)).

\(^{10}\) should also hold for equivalence relations between trees which take into account open formulas (the same can not be said for the forthcoming theorem on extraction of quantifiers). The argument reads as follows: by regularity, \( u \) and \( v \) are not quantified superordinated to \( S, S' \), and from this it follows that \( u, v \notin \text{Bound}(\hat{e}(T)) \cap \text{Bound}(\hat{e}(T')) \); so, we can apply Prop. 2.11.2.

\(^{11}\) Notice that, since the completing functions considered are weak, we never need to make explicit mention, in this proof, of the regularity-preserving property.
1) Since \( T, T' \) are regular, \( u \) is not quantified superordinated to \( S, S' \) and by the assumptions, it is not quantified in \( S_2 \). So, since \( e \) is sentential, i.e., for any \( P \in \text{Path}(T) \) we have \( \text{FV}(e(P)) \subseteq \text{Bound}(P) \), we can conclude that \( u \notin \text{FV}(e(P)) \). Furthermore, since \( e \) is weak, \( u \notin \text{Bound}(e(P)) \). In conclusion, \( u \notin \text{Var}(\hat{e}(S_2)) \).

2) That \( u \) is not \( U \), is one of our hypotheses.

3) Since \( e \) is weak, also \( e_i \) is, which means that there are no quantifiers in \( e_i(\hat{\iota}(P)) \) for any \( P \in \text{Path}(T) \). Thus, \( \hat{e}_i(S_2/\{u\}) = \hat{e}_i(S_2/\{u\})/\{u\} \).

Then we can apply Prop. 5.2 to get \( \hat{e}(S) \equiv_u \hat{e}_i(S') \). Substitution of equivalents (Prop. 2.12) yields \( \hat{e}(T) \equiv_u \hat{e}_i(T') \). Now \( \hat{e}(T) \) is a sentence, because \( e \) is sentential. But observe also that, for any \( P \in \text{Path}(T) \), we have, in this particular correspondence, \( \text{Var}(e_i(P)) \subseteq \text{Bound}(P) \subseteq \text{Bound}(\hat{\iota}(P)) \). So, also \( \hat{e}_i(T') \) is a sentence. Thus we can use Prop. 2.11 to conclude \( \hat{e}(T) \equiv \hat{e}_i(T') \). Since we proved this for any nice, weak completing function of \( T \), we have that \( T \) weakly reduces to \( T' \). □

We can also prove a strong regularization theorem for regular trees.

**Theorem 5.6.** For every regular tree \( T \) there is a strongly regular tree \( T' \) such that \( C(T) \subseteq C(T') \); if \( T \) is positive initial, we can require also \( T' \) to be such.

**Proof.** Let \( T \) be a regular tree, let \( u \) be a variable which is quantified more than once, and let \( (Qu/U)S \) be a quantifier over \( u \) which occurs with maximum depth in \( T \). Call \( (Qu/U)S \) the subtree beginning with this occurrence of \( (Qu/U) \). Let \( v \) be a new variable. Call \( T' \) the tree obtained replacing \( (Qu/U)S \) with \( (Qu/U)\text{Subst}(S, u, v) \). By Theorem 5.4, \( C(T) \subseteq C(T') \). Notice also that, since \( v \) did not occur in \( T \), this replacement does not create new instances of irregularity. Repeat until no variable is quantified twice.

Since no atomic formulas nor negations are added, the process preserves the property of being positive initial. □

Also forming a prenex form does not decrease complexity:

**Theorem 5.7.** For any regular tree \( T \) there is another regular tree \( AB \) in prenex normal form (i.e., \( A \) is a quantifier prefix and \( B \) does not contain quantifiers) such that \( C(T) \subseteq C(AB) \).

**Proof.** By Theorem 5.6 we can assume that \( T \) is strongly regular. Thanks to strong regularity, any subtree of the form \( (Qu/U)S_1 \circ S_2 \) is such that \( u \) does not occur in \( S_2 \) nor in \( U \). Let \( T' \) be the tree that differs from \( T \) only in that the subtree \( (Qu/U)S_1 \circ S_2 \) is replaced with \( (Qu/U)(S_1 \circ S_2/\{u\}) \). Then, Theorem 5.5 guarantees that \( C(T) \subseteq C(T') \). The operation preserves strong regularity, since it preserves both the number of quantifiers and the number of distinct bound variables. Repeat until the quantifiers form a prefix. □

We will need some more rules for trees: a stronger extraction rule that has the advantage of not transforming any first-order quantifier into an IF one; a distribution rule, and a quantifier swapping rule. We state here the versions for IF formulas.
Def 5.8. Given an IF formula $\psi$, we define $\psi\mid_v$ to be the formula obtained adding the variable $v$ to all nonempty slash sets of $\psi$; and similarly for syntactical trees.

Proposition 5.9. (A special case of Theorem 8.3 of [3]) If $u$ does not occur in $\psi$ nor $U$, then:

$$(Qu/U)(\varphi \land \psi) \equiv_u (Qu/U)(\varphi \land \psi\mid_u)$$

and similarly for disjunctions.

We list two more useful equivalence rules, distribution of universal quantifiers (see [27], 5.23) and quantifier swapping ([27]):

Proposition 5.10.

1) $\forall u(\varphi \land \psi) \equiv_u \forall u\varphi \land \forall u\psi$.

2) $\exists u(\varphi \lor \psi) \equiv_u \exists u\varphi \lor \exists u\psi$.

Proposition 5.11. Let $Q, Q'$ be either existential or universal quantifiers. Then:

$$(Qu/U)(Q'v/V \cup \{u\})\psi \equiv_{uv} (Q'v/V)(Qu/U \cup \{v\})\psi.$$  

(Notice that, in this logic, adjacent quantifiers of the same kind are not always allowed to commute; for example, notice that in the left member of the above formula we require $u$ to occur in the slash set of $v$.)

And now we translate these rules into the language of trees.

Theorem 5.12. Suppose two regular syntactical trees $T, T'$ differ only for one subtree, which is $S = (Qu/U)S_1 \circ S_2$ in $T$ and $S' = (Qu/U)(S_1 \circ (S_2\mid_u))$ in $T'$. Suppose $u$ does not occur in $U$ nor $S_2$. Then $T$ is weakly reducible to $T'$. An analogous result holds for $S = S_1 \circ (Qu/U)S_2$.

Theorem 5.13. Suppose two regular syntactical trees $T, T'$ differ only for one subtree, which is $S = Qu(S_1 \circ S_2)$ in $T$ and $S' = QuS_1 \circ QuS_2$ in $T'$. Then $T$ is weakly reducible to $T'$, and:

1) if $Q = \forall$ and $\circ = \land$, then $C(T) = C(T')$

2) if $Q = \exists$ and $\circ = \lor$, then $C(T) = C(T')$.

Theorem 5.14. Suppose two regular syntactical trees $T, T'$ differ only for one subtree, which is $S = (Qu/U)(Q'v/V \cup \{u\})R$ in $T$ and $S' = (Q'v/V)(Qu/U \cup \{v\})R$ in $T'$. Then $T$ is weakly reducible to $T'$, and $C(T) = C(T')$.

Theorem 5.12 can be proved in the same way as 5.8; the proofs of 5.13 and 5.14 are quite trivial, since the corresponding rules for formulas hold without any global requirement on formulas.

Finally, we look at a tree rule which is specific of IF quantification.

Theorem 5.15. Suppose two regular syntactical trees $T, T'$ differ only for one quantifier, which is $(\exists v/V)$ in $T$ and $\exists v$ in $T'$; suppose furthermore that all variables in $V$ are existentially quantified. Then $T$ is weakly reducible to $T'$, and $C(T) = C(T')$. 

19
Proof. A proof of the analogous claim for truth equivalence of formulas was given in [31], Lemma 12, and \( C(T) = C(T') \) follows easily from it. The weak reduction is obtained identifying each path of \( T \) either with itself (if the path does not contain \((\exists v/V)\)) or with a corresponding path of \( T' \) which differs only in that \((\exists v/V)\) is replaced with \( \exists v \).

Summarizing: the variant rule preserves complexity, but is not a weak reduction; the extraction rules are weak reductions, but do not preserve complexity classes; distribution of universals over conjunctions, quantifier swapping and the last IF rule are weak reductions and preserve complexity classes.

6 IF trees of low complexity

We assume the reader has a minimum of familiarity with notions of complexity theory, in particular reductions, hardness, completeness and the complexity classes FO, L, NL, P and NP. In the following, when we speak of NP-completeness, we are thinking of completeness up to polynomial reductions. It is known that the following inclusions hold:

\[
\text{FO} \subseteq \text{AC}^0 \subset \text{TC}^0 \subseteq \text{L} \subseteq \text{NL} \subseteq \text{P} \subseteq \text{NP}
\]

where AC\(^0\) and TC\(^0\) are two classes of computation by circuits (AC\(^0\): problems decidable by boolean circuits of unbounded fan-in and constant depth, TC\(^0\): problems solvable by threshold circuits of constant depth. AC\(^0\) \(\subset\) TC\(^0\) is one of the few strict inclusions that are known of within NP; it has the interesting consequence that first-order formulas cannot even express all L problems.

We study the complexity of IF positive initial trees, in the sense given by the following definitions (given along the lines of [2]):

**Def 6.1.** The model-checking problem for an IF sentence \( \varphi \) is the problem of establishing whether \( M \models \varphi \) when a (representation of a) finite structure \( M \) is given as input.

**Def 6.2.** We shall say that a regular positive initial tree \( T \) is in complexity class \( K \) if for all weak, sentential completing functions \( e \) the model-checking problem for \( \hat{e}(T) \) is in \( K \) (equivalently: if \( C(T) \subseteq K \)).

We say that \( T \) is \textbf{K-hard}, or that it \textbf{encodes a K-hard problem}, if there is at least one weak, sentential completing function \( e \) such that the model-checking problem for \( \hat{e}(T) \) is in \( K \) (equivalently: if \( C(T) \cap \text{K-hard} \neq \emptyset \)).

If \( T \) is in \( K \) and it is \textbf{K-hard}, we say it is \textbf{K-complete} (equivalently: if \( C(T) \cap \text{K-complete} \neq \emptyset \) and \( C(T) \subseteq K \)).

Theorem 4.10 above states that weak reductions do not make the complexity class of a tree smaller; so, if \( T' \) is in \( K \) and \( T \) reduces to \( T' \), we can also conclude that \( T \) is in \( K \).

We are now in the condition to enunciate in our formalism the dichotomy result given by Sevenster ([31], Theorem 29), restricted to the case of IF regular prefixes:
Proposition 6.3. Every regular IF prefix either encodes an NP-complete problem, or it is in the class FO of first-order definable problems.

This result can be stated in a stronger form, saying 1) that the FO prefixes are equivalent, in a rather strong sense, to first-order prefixes, and 2) giving a complete (and effective) classification of the NP-complete vs. the FO prefixes. We define analogous classes for trees.

Def 6.4. We say that a quantifier $(Qy/Y)$, occurring in a regular formula or a regular tree, depends on $(Q’x/X)$ (or, shortly, depends on $x$) if $(Q’x/X) \prec (Qy/Y)$ and $x \notin Y$. If any of these two conditions does not hold, we say it does not depend on $(Q’x/X)$.

We define two “path properties” mimicking and generalizing the properties defined in 31. They identify paths which mimic the behaviour of Henkin quantifiers (16) and paths which contain signalling phenomena (18, 20).

Def 6.5. A path of a positive initial tree is Henkin if it contains quantifiers $(\forall x/X), (\exists y/Y), (\forall z/Z), (\exists w/W)$ such that:
1) $(\exists y/Y)$ depends on $(\forall x/X)$ but neither on $(\forall z/Z)$ nor $(\exists w/W)$
2) $(\exists w/W)$ depends on $(\forall z/Z)$ but neither on $(\forall x/X)$ nor $(\exists y/Y)$.

Def 6.6. A path of a positive initial tree is signalling if it contains quantifiers $(\forall x/X), (\exists y/Y), (\exists z/Z)$ such that:
1) $(\exists y/Y)$ depends on $(\forall x/X)$
2) $(\exists z/Z)$ depends on $(\exists y/Y)$ but not on $(\forall x/X)$.

Def 6.7. Given any path property $\mathbb{P}$, we say that a tree $T$ has property $\mathbb{P}$ if there is a path of $T$ which has property $\mathbb{P}$.

Of course, not every interesting property of a syntactical tree is a path property; for example

Def 6.8. A syntactical tree $T$ is first-order if all of its slash sets are empty.

Def 6.9. A syntactical tree $T$ is primary if it is neither Henkin nor signalling.

Then, the classification result of Sevenster can be summarized thus (restricting attention to regular prefixes):

Proposition 6.10. 1) Henkin and signalling (regular) IF prefixes are NP-complete 2) Primary regular IF prefixes are in FO.

The apparatus developed so far allows us to (partially) extend the result on primary prefixes to trees. We define three new classes of trees:

Def 6.11. A tree $T$ is generalized Henkin if it contains logical operators $\forall x, \forall y, (\exists u/U), (\exists v/V)$ such that:
1) $\forall x \prec_T (\exists u/U), (\exists v/V)$
2) $u$ depends on $x$ but not on $y$ nor $v$
3) $v$ depends on $y$ but not on $x$ nor $u$. 

21
Condition 1) allows excluding some first-order trees, such as $(\forall x \exists u[ ]) \lor (\forall y \exists v[ ])$, from this class of trees. Notice that this class includes all Henkin trees, but also others such as $\forall x (\exists u[ ]) \lor (\forall y (\exists u/\{x\})[ ])$, in which the two existentials occur in distinct branches of the tree. Notice also that if such a tree is not Henkin, then there is a connective $\circ$ which is superordinated to both $(\exists u/U), (\exists v/V)$ and which occurs below $\forall x$. We then say that $\forall x, \forall y, (\exists u/U), (\exists v/V)$ and $\circ$ form a generalized Henkin pattern.

Def 6.12. A syntactical IF tree is coordinated if it contains logical constants $(\forall x/X), \lor, (\forall y/Y), (\forall z/Z), (\exists u/U), (\exists w/W)$ such that:
1) $(\exists u/U)$ depends on $y$ and $\lor$, but not on $x, z, w$
2) $(\exists w/W)$ depends on $z$ and $\lor$, but not on $x, y, u$
3) $\lor$ depends on $\forall x$, and $(\exists u/U), (\exists w/W)$ depend on $\lor$.
In case $(\exists u/U), (\exists w/W)$ occur in different disjuncts below $\lor$, we say the coordinated tree is of first kind; otherwise, we say it is of second kind.

Def 6.13. A regular tree is modest if it is neither signalling nor generalized Henkin nor coordinated.

Lemma 6.14. Let $T$ be a regular, modest tree. Let $\circ$ be a binary connective occurring with maximal depth in $T$. Suppose below $\circ$ an existential quantifier $(\exists d/D)$ occurs. Then, by quantifier swapping, we can transform $T$ into a tree $T'$ of the same complexity, and such that, if a universal quantifier $\forall c$ occurs between $\circ$ and $(\exists d/D)$, then $d$ depends on $c$.

Proof. Suppose there is a $\forall c$ such that $\forall c \prec_T \circ \prec_T (\exists d/D)$ and $c \in D$. We show how to push $\forall c$ below $(\exists d/D)$ by quantifier swapping. First of all, notice that there are no connectives between $\circ$ and $(\exists d/D)$; so, we can rearrange this part of the tree in Hintikka normal form ([27], Theorem 5.45), as a sequence of universals followed by a sequence of existentials. This does not alter the dependence relations between quantifiers.

Secondly, one can push $\forall c$ below the other universal quantifiers, until it is immediately above the sequence of existential quantifiers.

If the sequence of existential quantifiers begins with quantifiers that are independent of $\forall c$, swap them above $\forall c$.

Then, below $\forall c$ and before $\exists d$, find the first pair of existential quantifiers $(\exists u/U), (\exists v/V)$ such that: 1) $(\exists u/U)$ is immediately above $(\exists v/V)$, 2) $u$ depends on $c$, and 3) $v$ does not depend on $c$. Now, if $v$ depended on $u$, then $\forall c, (\exists u/U), (\exists v/V)$ would form a signalling sequence, contradicting the hypothesis that $T$ is modest. So, $v$ is independent of $u$; thus, we can swap $(\exists v/V)$ above $(\exists u/U)$; then, for the same reason, we can push it above all the existential quantifiers that were between $\forall c$ and $(\exists u/U)$; and finally, above $\forall c$. Iterating this process, one can push above $\forall c$ all the existential quantifiers that are independent of $c$, including $(\exists d/D)$.

Theorem 6.15. Any regular, modest IF tree is in FO.
Proof. Suppose there is a \( \forall c \) such that \( \forall c \prec T \circ \prec T (\exists d/D) \) and \( c \in D \). We show how to push \( \forall c \) below \( (\exists d/D) \) by quantifier swapping. First of all, notice that there are no connectives between \( \circ \) and \( (\exists d/D) \); so, we can rearrange this part of the tree in Hintikka normal form (Theorem 5.45), as a sequence of universals followed by a sequence of existentials. This does not alter the dependence relations between quantifiers.

Secondly, one can push \( \forall c \) below the other universal quantifiers, until it is immediately above the sequence of existential quantifiers.

If the sequence of existential quantifiers begins with quantifiers that are independent of \( \forall c \), swap them above \( \forall c \).

Then, below \( \forall c \) and before \( \exists d \), find the first pair of existential quantifiers \( (\exists u/U), (\exists v/V) \) such that: 1) \( (\exists u/U) \) is immediately above \( (\exists v/V) \), 2) \( u \) depends on \( c \), and 3) \( v \) does not depend on \( c \). Now, if \( v \) depended on \( u \), then \( \forall c, (\exists u/U), (\exists v/V) \) would form a signalling sequence, contradicting the hypothesis that \( T \) is modest. So, \( v \) is independent of \( u \); thus, we can swap \( (\exists v/V) \) above \( (\exists u/U) \); then, for the same reason, we can push it above all the existential quantifiers that were between \( \forall c \) and \( (\exists u/U) \); and finally, above \( \forall c \).

Iterating this process, one can push above \( \forall c \) all the existential quantifiers that are independent of \( c \), including \( (\exists d/D) \).

*Theorem 6.16.* Any regular, modest IF tree is in FO.

*Proof.* We want to show that given any modest tree \( T \) we can construct a modest tree \( T' \) which is in prenex normal form and which is more complex than \( T \) (that is, \( C(T) \subseteq C(T') \)). If \( T' = QR \), where \( Q \) is a quantifier prefix and \( R \) is a quantifier-free tree, then we will clearly have \( C(T') \subseteq C(Q) \); and, \( Q \) being primary, by Sevenster’s result (Prop. 6.10) \( C(Q) \subseteq FO \) (and \( Q \) will even be equivalent to a first order prefix).

We know that the prenex transformations do not decrease the complexity of a tree, but we face a problem: extraction of quantifiers does not preserve in general the modesty of the tree. We will thus need to choose carefully the order in which the extractions are performed.

So, let \( \circ \) be a connective below which occur some quantifiers, and of maximum height among such connectives. We can assume, by Theorem 5.15 that all purely existential slash sets are empty. Furthermore, by Lemma 6.14 we can assume that no existential quantifier below \( \circ \) is independent of universals occurring below \( \circ \) and above the existential itself (we call this normalization).

Immediately below \( \circ \) occurs a quantifier \( (Qa/A) \), and we extract it above \( \circ \) (after renaming \( a \) to a new variable \( a' \), if necessary). We want to check that the resulting tree is still modest. Let \( R \) be the subtree immediately below \( \circ \) that does not contain \( Qa \). We use the strong extraction rule, transforming \( R \) into \( R' = R_{(a)} \).
1) Suppose \( Q = \forall \).

1a) If the new tree is signalling, then this must be witnessed by \( \forall a \) itself and by two existentials \((\exists u/U), (\exists v/V)\) occurring in \( R' \) and such that \( a \not\in U \), \( a \in V \), \( u \not\in V \). But, in that case it means that \( U \) is empty and that the slash set of \( v \) contains another variable \( b \) (otherwise, the strong extraction rule would have preserved the empty slash set); by our normalization assumptions, we can assume \( b \) is universally quantified above \( \circ \). But then, the quantifiers over \( b, u, v \) prove that the old tree was signalling: contradiction.

1b) Suppose the new tree \( T' \) is (linear) Henkin. Then there are quantifiers \((\exists u/U), (\exists v/V)\), occurring in \( R' \), such that \( a \not\in U \) and \( a \in V \), and a universal quantifier \( \forall b \), above \( (\exists v/V) \), such that \( b \not\in V \). Since strong extraction was used, \( a \not\in U \) implies that \( U = \emptyset \); and \( a \in V \), plus the normalization assumptions, implies that there is a universal quantifier \( \forall c \) occurring above \( (\exists v/V) \) and such that \( c \in V \).

FIRST SUBCASE: \( (\exists v/V) \prec T' \exists u \). Then by definition of Henkin pattern it must be \( v \in U \); but this contradicts the above observation that \( U = \emptyset \).

SECOND SUBCASE: \( \exists u \prec T' (\exists v/V) \). Then \( u \in V \). In this case we find no contradiction, but we show that the case could have been avoided with some extra normalization work. \( \forall b \) cannot occur above \( \exists u \): if it did, then \( \forall b, \exists u, \exists v/V \) would be a signalling pattern already occurring in \( T \): contradiction. The same reasoning applies to \( \forall c \). So, \( \exists u \prec \forall b, \forall c \prec (\exists v/V) \). But no connective occurs between \( \forall b \) and \( (\exists v/V) \); this means that (by an argument similar to Lemma 6.14) \( \forall c \) could have been pushed below \( \exists v \) before the extraction rule was applied; so that the present case would not have occurred.

1c) Suppose instead the new tree is generalized Henkin. Our normalization excludes the possibility that the left subtree below \( \circ \) contains an existential quantifier independent of \( a \); so, the new tree being generalized Henkin means that \( R' \) contains a quantifier \((\exists v/V)\) depending on some superordinated \( \forall z \) but not on \( a \); and the left branch contains \((\exists u/U)\) such that \( u \) depends on \( a \) (but not on \( z \)). However, since the strong extraction rule was used, \( a \in V \) implies that \( V \setminus \{a\} \neq \emptyset \); the normalization assumptions give us a quantifier \( \forall b \) occurring above \( \circ \), superordinated to \( v \), such that \( v \) does not depend on \( b \). Now there are three cases.

a) \( u \) depends on \( b \). In this case, the old tree was already generalized Henkin (as witnessed by \( \forall b, \exists u, \forall z, \circ, \exists v \): contradiction.

b) \( u \) does not depend on \( b \), and there is a disjunction \( \forall b \prec \forall \leq \circ \). Then the old tree was coordinated (as witnessed by \( \forall b, \forall z, \forall a, \forall, \exists u, \exists v \): a contradiction.

c) \( u \) does not depend on \( b \), and there is no disjunction \( \forall b \prec \forall \leq \circ \). In this case we find no contradiction, but we show that we could have performed some
It means that, in the new tree, $\forall a$ plays either the role of $\forall x$ or that of $\forall y$ in the definition of coordinated tree.

In the first case, it means that $\circ$ is a disjunction, and in the new tree there are quantifiers $\forall z, \forall x$ and, in $R'$, a quantifier ($\exists v/V$) such that $v$ depends on $z$ but not on $a$; and in the other subtree below $\circ$ there is a quantifier ($\exists u/U$) such that $u$ depends on $x$ but not on $a$. Since the strong extraction rule was used, $a \in V$ implies that $V \setminus \{a\} \neq \emptyset$; the normalization assumptions give us a quantifier $\forall b$ occurring above $\circ$, superordinated to $v$, such that $v$ does not depend on $b$.

We check again the three cases a), b), c) as above; the case a) works as before, case b) similarly (coordination of the old tree is certified by $\forall b, \forall x, \forall z$ instead of $\forall b, \forall a, \forall z$); finally, c) cannot happen.

In the second case ($\forall a$ playing the role of $\forall x$ in the definition of coordinated tree), there is an existential quantifier ($\exists v/V$) in $R'$ such that $v$ depends on $a$. Since the strong extraction rule was applied, it means that the quantifier over $v$, in $R'$, is first-order; but we also require, for the tree to be coordinated, that $v$ is independent of some universally quantifier variable superordinated to it, so that the slash set of $\exists v$ is nonempty: a contradiction.

1c) Suppose $T'$ is coordinated, second kind. Then, $T'$ contains a coordinated pattern $\forall a, \exists y, \forall z, \forall, (\exists u/U), (\exists v/V)$ of the second kind. Suppose $a$ plays the role of a quantifier which is seen by one of the existentials (say $\exists u$ depends on $a$ but not on $y$ nor $z$) but not by the other. But since $y, z \in V$, the strong extraction rule should have added $a$ to the slash set of $u$: contradiction.

So, we must suppose instead that $a \in U$ and $a \in V$ (and: $u$ sees $y$ and not $z$, $v$ sees $z$ and not $y$).

Furthermore, we can suppose that $(\exists u/U), (\exists v/V)$ occur in distinct paths of the tree (if they occurred in the same path, then $\forall y, \forall z, (\exists u/U), (\exists v/V)$ would have formed an Henkin pattern already in $T$: contradiction). So, there must be a conjunction $\land$ occurring below $\lor$, such that $(\exists u/U)$ occurs (say) in the left conjunct and $(\exists v/V)$ occurs in the right conjunct.

Now, since $a \in U$ and the strong extraction rule has been applied, there must be some other universally quantified variable $c_u$ in $U$: similarly, there must be some $c_v \in V$. We consider separately the cases that $c_u$ is $z$ (or, symmetrically, $c_v$ is $y$), or both $c_u, c_v$ are new variables that are not part of the coordinated pattern.

**FIRST SUBCASE:** $z \in U$. Then $\forall y, \forall z, \land, \exists u, \exists v$ was generalized Henkin in $T$: contradiction. (The case “$y \in V$” is analogous).

**SECOND SUBCASE:** $c_u \in U$ and $c_v \in V$, with $c_u, c_v \neq z, y$. We can exclude
that both \( \forall c_u \) and \( \forall c_v \) occur below \( \land \) (we could have pushed, for example, \( \forall c_u \) below \( (\exists u/U) \) by a procedure similar to that used in Lemma 6.14).

Suppose then that, for example, \( \forall c_u \) occurs above \( \land \). In case there is no disjunction symbol between \( \forall c_u \) and \( \land \), then \( \forall c_u \) could have been pushed below \( \land \) by means of quantifier swapping and distribution, falling again in the previous contradictory subcase. If instead there is a disjunction \( \lor \) between \( \forall c_u \) and \( \land \), we observe that either:

1) \( c_u \in V \), and \( \forall c_u, \forall y, \land, \exists u, \exists v \) form a generalized Henkin pattern; or,

2) \( c_u \notin V \), and \( \forall c_u, \forall y, \forall z, \lor, \exists u, \exists v \) form a coordinated pattern.

In both cases we have a contradiction.

2) Suppose \( Q = \exists \).

Then the new tree is not signalling (the existential quantifiers in \( R' \) are either first-order, in which case they cannot play the role of the rightmost quantifier of a signalling pattern, or they have nonempty slash set, in which case \( a \) has been added to their slash set in \( R' \), and they cannot receive signals from \( \exists a \)).

Suppose the new tree is (linear) Henkin. Then there are quantifiers \( \forall x, \forall y, \exists \exists b/B \) such that \( \forall x, (\exists a/A), \forall y, (\exists b/B) \) form a linear Henkin pattern in \( T' \). But then the logical operators \( \forall x, (\exists a/A), \forall y, (\exists b/B'), \circ \) (where \( B' = B \cup \{ a \} \)) formed a generalized Henkin pattern in \( T \): contradiction.

The tree cannot become generalized Henkin nor coordinated, because in the new tree there is no new universal-existential dependence pair.

\[ \square \]

7 Trees of high complexity

Def 7.1. Let \( T, U \) be positive initial trees. We say that \( U \) extends \( T \) if there is an injective function \( \mu : T \rightarrow U \) such that:

1) for every quantifier \( (Qv/V) \) in \( T \), \( \mu((Qv/V)) = (Qv/V') \), and for every connective \( c \), \( \mu(c) \) is an occurrence of the same connective

2) \( \mu \) preserves the subordination ordering \( \prec_T \)

3) if \( (Qv/V) \) and \( (Qw/W) \) occur in \( T \), then the latter depends on the former if and only if \( \mu((Qw/W)) \) depends on \( \mu((Qv/V)) \).

The clause 2) makes our definition stricter than the corresponding notion for quantifier prefixes given by Sevenster. The correct generalization should allow some form of swapping of independent quantifiers. However, we will not need such subtleties.

Lemma 7.2. If a regular IF positive initial tree extends an NL-hard (resp. P, NP-hard) positive initial tree, then it is NL-hard (resp. P, NP-hard) itself.

Proof. Suppose \( T \) is NL-hard (resp. P,NP-hard). If \( U \) extends \( T \), then there is a \( \mu \) such that \( U \) can be obtained by adding, one by one, quantifiers and

\[ \text{[11]} \]Could there be any existential quantifier \( (\exists w/W) \), between \( \forall c_u \) and \( \land \), such that \( c_u \in W \)? Such an occurrence would make it impossible to push down the quantifier \( \forall c_u \). However, in such case \( \forall c_u, \exists w, \forall y, \exists u \) would have formed already a signalling or a Henkin pattern (depending on whether \( \exists u \) sees \( w \) or not).
connectives to $\mu(T)$, in some arbitrary order (and possibly adding the newly quantified variables to subordinated quantifiers). We prove the theorem by induction on one of these possible constructions; what we want to prove is that, given any weak, sentential function $f$ for $T$, and any structure $M$ (suitable for $\hat{f}(T)$), i.e., containing at least interpretations for the symbols in $\hat{f}(T)$, there is another such completing function $f'$ for $U$, and a structure $R(M)$, so that $M \models \hat{f}(T)$ iff $R(M) \models \hat{f}'(U)$. Thus, $R$ is shown to be a reduction from the problem of checking $\hat{f}(T)$ on the class of structures suitable for $\hat{f}(T)$ to the problem of checking $\hat{f}'(U)$ on structures suitable for $\hat{f}'(U)$. We will then show that the reduction is logspace.

Suppose $T_n$ and $T_{n+1}$ differ only in that one maximal linear suborder $AB$ of $T_n$ is replaced by $A(Qv/V)B'$ in $T_{n+1}$ (either $A$ or $B$ may be empty); and that $B'$ differs from $B$ only for the addition of variable $v$ to some slash sets. Denote as $B$, resp. $B'$ the smallest subtree of $T_n$, resp. of $T_{n+1}$, which contains $B$, resp. $B'$. Define a structure $R(M)$ which has the same domain as $M$, and interprets a constant $c$. For any path $P$ of $T_n$, call $P'$ the corresponding path of $T_{n+1}$. For any weak, sentential completing function $e$ of $T_n$, define $g$ as the completing function which assigns, to each path $P'$ of $T_{n+1}$, the formula $(v = c \land e(P))$ in case $P'$ intersects $B'$ and $Q = \exists$; and, otherwise, just formula $e(P)$. Then, one can easily prove, by induction on the syntactical tree of $\hat{e}(B)$, that

$$M, X \models \hat{e}(B) \upharpoonright \hat{g}(B')$$

for any suitable team $X$ and for every completing function $e$ of $T_n$. See [31], Lemma 20, for a detailed proof of a similar claim. The idea is this: the fact that $U$ is regular implies that $T$ does not contain occurrences of $v$; and since $e$ is sentential, also $e(T)$ does not contain $v$. If we were discussing first-order logic, this would be enough to ensure that $Qv$ is a dummy, eliminable, quantifier; in IF logic this is not sufficient, because an existentially quantified $v$ could be used to signal to some subordinated quantifier ($\exists u/U$) the value of some variable in $U$. But the additional clause $v = c$ makes it impossible to store values in $v$.

Thus, by substitution of equivalents (2.12 plus 2.11), $g$ is the completing function for $T_{n+1}$ that we were looking for.

Suppose instead that $T_{n+1}$ differs from $T_n$ in that a certain path $AB$ is turned into a tree $A([ \ ] \land B)$ (the ordering of the conjuncts is unimportant). For any completing function $e$ of $T_n$, we can define the completing function $g$ for $T_{n+1}$ as that function which differs from $e$ only in that it assigns $\forall x(x = x)$ to the path we marked with a $[ \ ]$. Then it is clear that

$$M \models \hat{e}(T_n) \upharpoonright \hat{g}(T_{n+1}).$$
The case that $T_{n+1}$ differs from $T_n$ in that a certain path $AB$ is turned into a path $A( \blacklozenge \lor B)$ can be treated analogously, using $\forall x(x \neq x)$ instead of $\forall x(x = x)$.

It remains to show that the reduction $R$ is logspace. But since $R(M)$ differs from $M$ only in that it contains an interpretation for the constant $c$, this addition can be performed using logarithmic work space (in order to make space for the interpretation of $c$, and to write this interpretation in the freed space, one only needs a fixed number of counters over the elements of the structure; the values assumed by each counter can be stored in binary digits, occupying logarithmic space.

**Theorem 7.3.** Any regular positive initial tree which is Henkin encodes an NP-complete problem.

*Proof.* Any such tree $T$ extends a regular Henkin quantifier prefix. It was shown in [31], Theorem 22, that regular Henkin prefixes encode the NP-complete problem of 3-COLORABILITY. So, from Lemma 7.2 it follows that $T$ is NP-hard. Since $IF$ sentences are in NP (Prop. 2.7), $T$ is NP-complete.

**Theorem 7.4.** Any regular positive initial tree which is signalling encodes an NP-complete problem.

*Proof.* Use Lemma 7.2 and Prop. 2.7 again, on the basis that regular signalling quantifier prefixes codify the NP-complete problem EXACT COVER BY 3-SETS ([31], Theorem 23).

### 8 Complexity of Generalized Henkin trees

The minimal examples of (nonlinear) generalized Henkin trees are of the following forms:

\[
\forall x \\
\quad \blacklozenge \\
\exists u \quad \forall y \\
\quad \blacklozenge \\
\quad \exists v/x \\
\quad \exists u/y \\
\quad \exists v/x
\]

where $\blacklozenge$ is either $\lor$ or $\land$. We will call GH1($\blacklozenge$) the first type, and GH2($\blacklozenge$) the second.

---

12 We also have an unpublished and rather simple description of SAT by means of the minimal signalling prefix. From [35], Prop. 3.9.4., one can also obtain a description of DOMINATING SET by means of the minimal signalling prefix.
8.1 SAT by a minimal generalized Henkin sentence

Here we express the NP-complete problem SAT by means of an IF sentence whose (positive initial) tree is generalized Henkin (specifically, GH2(∨)), but not Henkin nor signaling nor coordinated.

SAT Problem: Given a proposition in conjunctive normal form, with exactly two literals per clause, decide whether there is an assignment which satisfies the proposition.

How we model the problem: each instance of it is a structure of signature $P^2, N^2, C^1, 0, 1$ ($0, 1$ are constants denoting two distinct elements; $C(y)$: “$y$ is a clause”; $¬C(y)$: “$y$ is a propositional letter”; $P(x, y)$: “$x$ occurs positively in $y$”; $N(x, y)$: “$x$ occurs negatively in $y$”); we allow only structures such that for each clause $y$ there are at least two prop.letters $x$ such that $P(x, y) ∨ N(x, y)$.

It is well known that, even with this restriction, the SAT problem stays NP-complete.

For brevity, we shall write $O(x, y)$, “$x$ occurs in $y$”, as a shortening for $¬C(x) ∧ C(y) ∧ (P(x, y) ∨ N(x, y))$.

The sentence is:

$$\varphi : \forall x \forall y((\exists u/y)\psi_1 ∨ (\exists v/x)\psi_2)$$

where

$$\psi_1 : O(x, y) ∧ (P(x, y) → u = 1) ∧ (N(x, y) → u = 0)$$

and

$$\psi_2 : O(v, y) ∧ (O(x, y) → x \neq v).$$

**Theorem 8.1.** If $M$ is a suitable structure, then $M \models \varphi$ iff $M$ is a “yes” instance of SAT.

The idea behind this description is similar to that of Jarmo Kontinen’s Theorem 4.3.3 from his PhD thesis (although, he deals with a very different kind of descriptive complexity; and although his method seems to capture just the 2-SAT problem). See [22] or [23] for a comparison. Think of $x$ as a propositional letter, $u$ as the truth value which is assigned to $x$, $y$ as a clause, $v$ as a propositional letter which corresponds to a literal of $y$ which is made true by the truth assignment. The left disjunct enforces $u$ to be a truth assignment; the $y$-uniformity of the function which picks $u$ guarantees that the assignment is correctly defined, i.e., a function of the propositional letters. The right disjunct ensures that, for every clause $\hat{y}$, there is at least one literal in it (corresponding to a prop.letter $\hat{v}$) which is made true by the assignment (because the pair $(\hat{v}, \hat{y})$ is necessarily sent to the left disjunct).

**Proof.** 1) Suppose $M$ is a “yes” instance. Then there is a truth assignment $T$ on propositional letters which makes the proposition $\bigwedge\{c ∈ M | c ∈ CM\}$ true. This means that to each clause $c$ we can associate a propositional letter $f(c)$
which either occurs positively in \(c\) and \(T(f(c)) = 1\), or it occurs negated in \(c\) and \(T(f(c)) = 0\). Let \(R = \{(f(c), c) | c \in M\}\), and \(S = M^2 \setminus R\). Let \(Y = Team_{xy}(R)\) and \(Z = Team_{xy}(S)\) be the corresponding teams of domain \(\{x, y\}\). They form a partition of \(\{\emptyset\}[M/x, M/y]\). Let \(Y' := Y[T/u]\); clearly \(M, Y' \models \psi_1\). Instead, define \(Z' = Z[f/v]\). Any triple \((x, y, v) \in Z'\) either is such that \(x\) occurs not in the clause \(y\), or, if it does, \(x\) is not \(f(y)\) (because the pair \((f(y), y)\) is not in \(Z\)). So, \(Z'\) satisfies \(\psi_2\).

2) Suppose \(M\) is a “no” instance. Let \(Y, Z\) be any partition of \(\{\emptyset\}[M/x, M/y]\); let \(T\) be a \(y\)-uniform function \(Y \rightarrow M\); let \(f\) be an \(x\)-uniform function \(Z \rightarrow M\). Define \(Y', Z'\) from \(X, Y, T, f\) as was done above. Since \(T\) cannot be a satisfying assignment, there must be a clause \(\hat{y}\) such that, for any propositional letter \(x\), the triple \((x, \hat{y}, u)\) falsifies either \(P(x, y) \rightarrow u = 1\) or \(N(x, y) \rightarrow u = 0\) or \(O(x, y)\). So, if \(M, Y' \models \psi_1\), then for every \(x \notin C^M\), \((x, \hat{y}, u) \notin Y'\); so, \((x, \hat{y}) \notin Y\); so, \((x, \hat{y}) \in Z\). But then, if \((x', \hat{y}, v) \in Z'\), by \(x\)-uniformity of \(f\), then also \((x', \hat{y}, v) \in Z'\) for any propositional letter \(x'\). So, \(v\) must be equal to some such \(x'\). Thus, \(M, Z' \models O(x, y) \rightarrow x \neq v\): contradiction.

\[\text{Corollary 8.2.} \quad \text{The minimal generalized Henkin tree}\]

\[
\forall x \\
\forall y \\
\forall z
\]

\[\begin{array}{c}
\forall z \\
\forall y \\
\forall x \\
(\exists u/y) \\
(\exists v/x)
\end{array}
\]

\[\text{and any positive initial tree extending it is NP-complete.}\]

It is perhaps of some interest that the SAT-describing sentence above can be rewritten as an Henkin prefix sentence

\[
\left( \forall x \exists u \forall y \exists v \right) (\psi_1 \lor \psi_2).
\]

This is an example of an \(H^1_2\) sentence which cannot be reduced in any obvious way to a \(F^1_2\) sentence, since the variables \(u\) and \(v\) describe here two very different functions (see e.g. [24] for the definition of the function quantifier \(F^1_2\) and related discussions).

\subsection*{8.2 Disjunction-free Generalized Henkin trees}

The minimal trees \(GH1(\infty)\) and \(GH2(\land)\) can be easily shown to be first-order (use quantifier distribution for \(GH1(\land)\) and \(GH2(\land)\); use the strong extraction rule in the longest path of \(GH1(\lor)\)). This tells us nothing about their extensions. We might conjecture that, if one tree falls in one of these classes but not in any
other class that we have isolated\textsuperscript{13}, then it is in FO. We fall short of proving such results in full generality; for example, in the case of extensions of \(GH1(\land)\) and \(GH2(\land)\) we prove this to hold only under the additional assumption that the tree in question does not contain disjunctions.

**Theorem 8.3.** 1) If a tree \(T\) is in \(GH1(\land)\) but it does not contain disjunctions, and is not Henkin, nor signalling, then it has first-order complexity.

2) If a tree \(T\) is in \(GH2(\land)\) but it does not contain disjunctions, and is not in \(GH1(\land)\), Henkin, nor signalling, then it has first-order complexity.

*Proof.* 1) Suppose \(T\) satisfies the hypotheses; then, it contains at least one pattern

\[
\vdots \\
\forall x \\
\vdots \\
\land \\
\vdots \\
(\exists u/U) \\
\forall y \\
\vdots \\
(\exists v/V) \\
\vdots
\]

where \(x \notin U, y \notin V, x \in V\), witnessing that \(T\) is \(GH1(\land)\).

*Notice:*

1. There are, by hypothesis, no disjunctions between \(\forall x\) and \((\exists v/V)\).

2. Every existential quantifier \((\exists w/W)\) between \(\forall x\) and \((\exists v/V)\) is independent of \(\forall x\) (otherwise either \(\forall x, \exists w, \forall y, \forall v\) form a Henkin pattern, or \(\forall x, \exists w, \forall v\) form a signalling pattern). Consequently, \(\forall x\) can be pushed below any such quantifier by the quantifier swapping rule.

3. \(\forall x\) can be pushed below any other universal quantifier by quantifier swapping.

4. \(\forall x\) can be pushed below any conjunction by means of quantifier distribution.

None of these transformations generates new dependence patterns, nor disjunction symbols; so, applying them preserves the hypotheses of the theorem. Using these transformations, one can push \(\forall x\) below \(\land\), so that there is one less witness of the \(GH1(\lor)\) pattern.

Iterating the process, one can remove all witnesses of the \(GH1(\lor)\) pattern, until the resulting tree is modest (and thus of first-order complexity, Theorem 6.16).

2) Analogous.

\textsuperscript{13}Such a tree will be called an extension* of \(GH1(\land)\), resp. of \(GH2(\land)\), \(GH1(\lor)\). We use this terminology in the summary table at the end of the paper.
8.3 Conjunction-free GH1 trees

Here we prove that extensions* of $\text{GH1}(\lor)$ are in $\text{FO}$, under the additional assumption that they do not contain conjunction symbols.

**Theorem 8.4.** Suppose an IF positive initial tree has no conjunctions, and it is not $\text{GH2}$, coordinated, Henkin nor signalling. Then it can be reduced to a first-order tree, and is thus in the $\text{FO}$ complexity class.

**Proof.** Suppose the tree is not modest; then it is $\text{GH1}(\lor)$, and so it has the following form:

$$
\vdots
\forall x
\vdots
\forall 1
\vdots
\forall n
\vdots
(\exists u/U) \forall y
\vdots
(\exists v/V)
\vdots
$$

where $\forall x, \forall n, \exists u, \forall y, \exists v$ form a $\text{GH1}(\lor)$ pattern (i.e., $x \notin U, y \notin V, x \in V$), (*) $\forall n$ has maximal depth among disjunctions that are part of a $\text{GH1}(\lor)$ pattern, and (***) $(\exists v/V)$ has minimal depth among existential quantifiers that form a $\text{GH1}(\lor)$ pattern together with $\forall x, \forall n, \exists u, \forall y$; and $\forall 1, \ldots, \forall n$ is an exhaustive list of all disjunctions occurring between $\forall x$ and $\forall n$.

We can also assume, without loss of generality, that:

1. All universal quantifiers have empty slash sets.

2. All nonempty slash sets contain at least one universally quantified variable.

Our final aim is to push the quantifiers $\forall y$ and $\exists v$ above $\forall x$, so that $x$ can be removed from the slash set of $\exists v$. The purpose is to obtain a new tree which still satisfies the hypotheses (it has no conjunctions, and it is not $\text{GH2}$, coordinated, Henkin nor signalling) of the theorem, but has one less witness of the $\text{GH1}(\lor)$ pattern (we must also check that new witnesses of $\text{GH1}(\lor)$ are not generated in the process). So, one can repeat the procedure until there are no more such witnesses: the resulting tree is modest, and thus we already know (Theorem 6.10) it is reducible to a first-order tree.

Notice that it might be necessary to push above also some other quantifiers $(Qr/R)$, occurring between $\forall y$ and $(\exists v/V)$, such that $r \notin V$ (they cannot be pushed below $\exists v$); and (***) if $(Qr/R)$ is existential, then $x \notin R$, otherwise...
∀x, ∃r, ∃v would form a signalling pattern. It is also safe to assume that (****) for every such quantifier (Qr/R), r /∈ V (otherwise (Qr/R) might be pushed below (∃v/V)).

We divide this whole process into four phases, which will constitute four parts of the proof.

Phase 1: Push ∀y and ∃v (and the quantifiers in between) upwards, until they are immediately below ∨n.
Phase 2: push the quantifiers above ∨n
Phase 3: push the quantifiers above ∨1
Phase 4: push the quantifiers above ∀x.

In each phase, we check that the transformation performed cannot generate any new higher-order or GH1(∨) patterns. We will proceed by contradiction: “suppose that the transformed tree T' has a certain pattern. Then, already the untransformed tree T had some forbidden pattern...”

PHASE 1: We must show how to push the quantifiers ∀y and (∃v/V) (together with those in between) above disjunctions. We always use the weak extraction rule: this prevents the formation of new signalling patterns. We also always assume w.l.o.g. that the quantifiers we extract are in the right conjunct. So, suppose that after pushing the quantifiers in question above some disjunction ∨, the new tree is:

- Henkin: then there is, in the left subformula, an ex. quantifier (∃w/W) which depends on some ∀z, and quantifiers ∀ŷ, (∃v/V), occurring between ∀y and ∃v, such that w does not depend on ∀ŷ nor ∃v, and ũ does not depend on z nor w.

Suppose x ∈ W. Then ∀x, ∨n, ∀ŷ, ∀z, ∃w, ∃ũ already formed a coordinated pattern in T: contradiction.

Suppose instead x /∈ W. Then ∀x, ∨, ∃w, ∀ŷ, ∃ũ already formed a GH1(∨) pattern in T. But ∨ has greater depth than ∨n, contradicting the assumption (**).

- GH2, coordinated: then there is some existential quantifier (∃w/W) (below (∃v/V), by (**)), and ∀ŷ, (∃ũ/V) as above, such that ŷ, ũ ∈ W, and a quantifier ∀z on which w depends. Suppose first that (∃w/W) is in the right disjunct: then either ∀ŷ, ∃ũ, ∃w already formed a signalling pattern, or they formed a Henkin pattern together with some other universal quantifier ∀z: contradiction. Suppose instead that (∃w/W) is in the left disjunct: then there are two possibilities: 1) x /∈ W. Then ∀x, ∃w, ∨, ∀ŷ, ∃ũ already formed a GH1(∨) with ∨ of greater depth than ∨n: contradiction. 2) x ∈ W. Then ∀x, ∨, ∀z, ∃w, ∀ŷ, ∃ũ already formed a coordinated tree: contradiction.

- GH1(∨): There are two possibilities.

---

14 Thus, ∀ŷ, ∃ũ might be ∀y, ∃v, or some other quantifiers occurring between ∀y and ∃v. We follow the same naming convention in the rest of the proof.
Case 1: There is a quantifier ($\exists w/W$) in the left subformula, and $\forall y$ as above, such that $\hat{y} \notin W$, and quantifiers $\forall z$ and ($\exists s/S$), occurring below ($\exists v/V$), such that $s$ sees $z$ but not $\hat{y}$. Now notice that, if $\hat{y} \notin V$, then $\forall y, \exists v, \exists s$ would have already formed a signalling pattern, or $\forall y, \exists v, \forall z, \exists s$ a Henkin pattern: contradiction. So we can suppose that $\hat{y} \in V$. But this contradicts (**).

Case 2: There are quantifiers $\forall z, (\exists w/W)$ in the left subformula, and $\forall y$ as above, such that $z \notin W$ but $\hat{y} \in W$, and a quantifier ($\exists s/S$) occurring below ($\exists v/V$) such that $\hat{y} \notin S$. Notice that it must also be $x \in W$, otherwise $\forall x, \forall, \exists w, \exists y$ would have formed a GH1($\forall$) pattern with $\forall$ of greater depth than $\forall_n$ (violating (**)). And it must also be the case that $x \in S$, otherwise $\forall x, \forall, \exists s, \forall z, \exists w$ would have formed a GH1($\forall$) pattern with $\forall$ of greater depth than $\forall_n$.

But then, $\forall x, \forall, \forall z, \exists w, \forall y, \exists s$ already formed a coordinated pattern: contradiction.

**PHASE 2:** First of all, we prove that $U = \emptyset$. Suppose it is nonempty: then (by 2.) there is some quantifier $\forall x$ such that $z \in U$. Suppose first that $z \notin V$: then $\forall x, \forall x, \forall z, \forall w, \exists y$ form a GH2 pattern, a contradiction. If instead $z \in V$, we have that $\forall x, \forall x, \forall z, \forall w, \exists y$ form a coordinated pattern: again, a contradiction.

Notice, then, that the same can be proved of any existential quantifier occurring in the left disjunct and dependent on $\forall x$. We call # this observation.

Consequently, if we raise quantifiers from the right disjunct using the **strong** quantifier extraction rule, no Henkin patterns can be generated. Suppose instead that some other pattern is generated:

- **Signalling:** then the left disjunction contains an existential quantifier ($\exists w/\emptyset$) and another quantifier ($\exists s/S$) occurring below it, with $S \neq \emptyset$. But then, by our assumption 2., $S$ must contain some universally quantified variable $t$. Thus $\forall t, \exists w, \exists s$ formed a signalling pattern already before the transformation: a contradiction.

- **Coordinated:** then there is an ($\exists w/W$) in the left disjunct which depends on some universal quantifier $\forall s$ and is independent of some other quantifier (so that, after the strong extraction, also $y$ is in the slash set of $\exists w$); and some $\forall y, (\exists v/V)$, occurring between $\forall y$ and $\exists v$, such that $s, x \in V$ and $\hat{y} \in W$. But the observation # above implies that $\exists w$ is independent of $\forall x$. Then, $\forall x, \forall n, \forall s, \forall y, \exists w, \exists v$ formed a coordinated pattern: contradiction.

- **GH2:** Case 1: there was some existential quantifier ($\exists w/W$), occurring below ($\exists v/V$), and a $\forall y$ between $\forall y$ and $\exists v$, such that $\hat{y} \in W$. But then $\forall x, \exists v, \exists w$ would have formed a signalling pattern: contradiction.

Case 2: there was some existential quantifier ($\exists w/W$) occurring in the left disjunct, which depended on some universal $\forall z$ occurring above $\forall_n$, and such that $W$ is nonempty (so that, after the application of strong
extraction, $y$ is inserted in the slash set of $w$). But $W \neq \emptyset$ implies, by observation #, that $x \in W$. But then $\forall x, \forall z, \forall u, \exists v, \forall y, \exists v$ already formed a coordinated pattern: contradiction.

- GH1(∨): The proof is identical to the GH2 case (except for the phrase “$\forall z$ occurring above $\forall u$”).

PHASE 3: Just observe that # can be proved for each of the $\forall_i$. Then, all cases can be treated as in phase 2.

PHASE 4: use quantifier swapping until $\exists v$ is above $\forall x$, and $x$ does not occur anymore in its slash set.

We attempted in vain to prove a similar result without the assumption that the tree does not contain conjunctions. Adapting the above proof seems to fail because there may occur patterns that are similar to coordinated ones, but with a $\land$ instead of an $\lor$: 

\[
\forall x \\
\land \\
\forall y \\
\forall y \\
(\exists u/x) \\
(\exists u/x)
\]

So, we might suspect that some GH1(∨) tree which also contains this pattern (and falls not in any other relevant category) could be able to express something beyond first-order logic.

9 Coordinated trees of the first kind

The minimal examples of coordinated trees of the first kind can have the following forms:

\[
\forall x \\
\forall y \\
\forall z \\
(\exists u/x) \\
(\exists v/x)
\]

\[
\forall x \\
\forall y \\
\forall z \\
(\exists u/x) \\
(\exists u/x, z)
\]

\[
\forall x \\
\forall y \\
\forall z \\
(\exists u/x) \\
(\exists u/x, y)
\]

We call these trees (and the corresponding fragments of IF logic) C1, C2, and C3, from left to right. It is apparent that $C(C1) \subseteq C(C2) \subseteq C(C3)$.
### 9.1 SAT by coordinated trees

First: observe that the coordinated tree $C_3$ is an extension of the generalized Henkin tree $GH_2(\lor)$. So, by the Extension Lemma, it permits defining the SAT problem. So, all trees extending $C_3$ are NP-complete (and we can exclude them from our classification, since they are a special case of $GH_2(\lor)$ trees).

Secondly: we give a different (but similar in spirit) description of SAT by means of the coordinated tree $C_2$. This will prove that all trees extending $C_2$ are NP-complete. We use the same notations and conventions as in the previous section, with the following exception: now $P(x, y)$ is interpreted as “$y$ is a prop. letter occurring positively in the clause $x$”, and not viceversa; similarly for the relations $N$ and $O$. $O(x, y)$ is an abbreviation for $C(x) \land \lnot C(y) \land (P(x, y) \lor N(x, y))$. The SAT-defining sentence is:

$$\theta : \forall x \forall y (\exists u/x) \chi_1 \lor \forall z (\exists v/xy) \chi_2$$

where

$$\chi_1 : O(x, y) \land [(P(x, y) \rightarrow u = 1) \land (N(x, y) \rightarrow u = 0)]$$

and

$$\chi_2 : (z = x \land O(x, y)) \rightarrow (v \neq y \land O(x, v)).$$

**Theorem 9.1.** If $M$ is a suitable structure, then $M \models \theta$ iff $M$ encodes a “yes” instance of SAT.

**Proof.** 1) Suppose $M$ encodes a “yes” instance of SAT. Then there is an assignment $T$ of truth values to the propositional variables which makes the conjunction of clauses true. This means that to each clause $a$ we can associate a proposition $b = g(a)$ (we functionally choose one) such that either $b$ occurs positively in $a$, and $T(b) = 1$, or $b$ occurs negatively in $a$, and $T(b) = 0$. We say that such pairs are in a relation $R(a, b)$. Now define $Y, Z \subseteq \{\emptyset\} \cup \{MM/xy\}$ as: $Y := \{(x, a), (y, b) | (a, b) \in R\}$, $Z := \{\emptyset\} \cup MM/xy \setminus Y$. Define the function $F : Y \rightarrow M$, by $F(s) := T(s(y))$ whenever $s(y)$ is a propositional letter, and arbitrarily otherwise. Then, by the comments on $T$ above, $M, Y[F/u] \models \chi_1$.

Suppose $s \in Z[MG/\chi v]$ satisfies $s(z) = s(x)$. Then $g(s(z)) = g(s(x))$. Define the function $G : Z[M/z] \rightarrow M$ as $G(s) := g(s(z))$. Then, if $s \in Z[MG/\chi v]$, $s(y) \neq s(v)$, because $s(v) = g(s(x))$, and the pair $((x, s(x)), (v, g(s(x))))$ is in $Y$, so not in $Z$. Furthermore, by the definition of $g$, $(s(x), s(y)) = (s(x), g(s(x))) \in R$, which implies $(s(x), s(v)) \in O^M$. So $M, Z[MG/\chi v] \models \chi_2$.

2) Suppose $M$ encodes a “no” instance of SAT. Fix an assignment $T$ to the propositional letters. Since the propositional formula encoded by $M$ is not satisfiable, it must contain a clause $\hat{a}$ such that, for each propositional letter $b$, one of the following holds: *) $b$ occurs not in $\hat{a}$, or **) $b$ occurs positively in $\hat{a}$ and $T(b) = 0$, or *** $b$ occurs negatively in $\hat{a}$ and $T(b) = 1$. Call $B$ the set of $bs$ that satisfy either ** or ***; it is nonempty, by the restriction we made on structures (that each clause contain at least one literal).
Let \( Y, Z \subseteq \{0\}[MM/x,y] \) s.t. \( Y \cup Z = \{0\}[MM/x,y] \). Define \( F \) from \( T \) as above. Suppose that for some \( b \), the assignment \( \{(x, \hat{a}), (y, b)\} \) is in \( Y \). Then \( M, Y \not\models \chi_1 \).

Suppose instead that for all \( b \), \( \{(x, \hat{a}), (y, b)\} \not\in Y \), which implies \( \{(x, \hat{a}), (y, b)\} \in Z \). Let \( G : Z[M/z] \rightarrow M \) be an \( \{x, y\}\)-uniform function. Then \( Z[MG/zv] \) contains, for each \( b \in B \), at least one assignment \( t_b \) such that \( t_b(z) = t_b(x) = \hat{a} \) and \( t_b(y) = b \).

If \( M, Z[MG/zv] = \chi_2 \), then, for any \( b \in B \), we have \( t_b(v) \neq t_b(y) \) and \( (t_b(x), t_b(y)) \in O^M \). Fix a \( \hat{b} \in B \). By \( y \)-uniformity of \( G \), \( b = t_b(y) \neq t_b(v) \) for any \( b \in B \); therefore, \( t_b(v) \notin B \), a contradiction. So, \( M, Z[MG/zv] \not\models \chi \). Thus, our choice of \( X, Y, F, G \) does not witness \( M \models \theta \). Our choice of \( X, Y, F, G \) was arbitrary, except for the fact that \( F \) was induced by some assignment \( T \). In case \( F \) is not obtained this way, it means that \( F(s) \neq 0,1 \) for some \( s \in Y \) such that \( s(y) \) is a propositional letter. But this is impossible: the assignment \( s(F/u) \) does not satisfy \( \chi_1 \). So, our choice holds for any choice of \( X, Y, F, G \); we may conclude that \( M \not\models \theta \).

2. alternative proof) Suppose \( M \models \theta \). Then there are \( Y, Z \subseteq \{0\}[MM/xy] \), a \( x \)-uniform function \( F : Y[M/y] \rightarrow M \) and a \( xy \)-uniform \( G : Z[M/z] \rightarrow M \) such that \( Y \cup Z = \{0\}[MM/xy] \), \( M, Y[MF/yu] \models \chi_1 \) and \( M, Z[MG/zv] \models \chi_2 \). Suppose for sake of contradiction that, for some clause \( \hat{a} \in C^M \), \( s \in Y \) implies \( s(x) \neq \hat{a} \). By our assumption on structures, that in each clause at least one literal occurs, there must be an \( s \in Z \) such that \( b := s(y) \) occurs in \( \hat{a} \). Pick \( s' \in Z[MG/zv] \) such that \( s'(x) = \hat{a}, s'(y) = b \) and \( s'(z) = s'(x) \). Since \( M, Z[MG/zv] = \chi_2 \), we have \( s'(v) \neq s(y) \) and \( (s'(v), s(x)) \in O^M \). By \( xy \)-uniformity of \( G \), \( s(v) = G(s'(x,y,z)) \) is different from any \( s''(v) \) such that \( s'' \in Z \). Then, the assignment \( \{(x, \hat{a}), (y, s(v))\} \) must be in \( Y \), contradicting our hypothesis.

So, for each clause \( a \), there is a propositional letter \( g(a) \) occurring in \( a \) such that \( s_a = \{(x, a), (y, g(a))\} \in Y \). Define \( T(g(a)) := F(s_a) \), and extend it arbitrarily to a propositional assignment over propositional letters that are not of the form \( g(a) \). Since \( M, Y[MF/yu] = \chi_1 \) and \( Y \) contains \( s_a \) for each clause \( a \), \( T \) is an assignment that satisfies the instance of SAT which is encoded by \( M \).

In this proof we used the restriction that each clause contain at least one literal; eliminating such restriction on the class of structures would require the usage of an extra existential quantifier (independent of \( x \)) in each disjunct; the resulting tree would be an extension of the one considered, and (because of the right disjunct) either signalling or a Henkin tree. The result above, in any case proves that the model checking problem for the tree \( C_2 \) is (modulo Cobham’s thesis) unfeasible on all structures: indeed, the sentence above defines the NP-complete problem “\( M \) is in SAT or \( M \) is not the encoding of an instance of SAT”. So:

**Corollary 9.2.** The coordinated tree \( C_2 \) (and any tree extending it) is NP-complete.
9.2 NP-completeness of C1

We show here that the NP-complete problem SET SPLITTING (see e.g. [22]) is definable by means of the coordinated tree C1. This result was obtained in collaboration with Lauri Hella.

Input: a set $A$, a family $\mathcal{B} \subseteq \wp(A)$ s.t., for every $B \in \mathcal{B}$, $\text{card}(b) \geq 2$.

Measure: $\text{card}(A \cup \mathcal{B})$.

Problem: Is there a partition $\{U,V\}$ of $A$ such that, for each $B \in \mathcal{B}$, $B \cap U \neq \emptyset$ and $B \cap V \neq \emptyset$?

We encode input instances as structures of domain $A \cup \mathcal{B}$ (with $\mathcal{B} \subseteq \wp(A)$ such that each of its element has at least cardinality 2) which interprets in the obvious way unary predicates $A$ and $\mathcal{B}$, and a binary set membership” relation $R_\in$ (with the restriction that, if $(a,B) \in R_\in$, then $a \in A$, $B \in \mathcal{B}$ and $a \in B$).

The requirement that the sets in $\mathcal{B}$ have at least cardinality 2 is our addition to the original problem; it obviously does not decrease its complexity, and it makes the problem easier to define in our fragment of IF logic.

The defining sentence is:
\[
\eta : \forall x(\forall y(\exists u/\{x\})\epsilon_1 \lor \forall z(\exists v/\{x\})\epsilon_2)
\]

where
\[
\epsilon_1 : (A(x) \land B(y)) \rightarrow (u \neq x \land R_\in(u,y))
\]
and
\[
\epsilon_2 : (A(x) \land B(z)) \rightarrow (v \neq x \land R_\in(v,z))
\]

**Theorem 9.3.** For every suitable structure $M$, $M \models \eta$ iff $M$ encodes a “yes” instance of SET SPLITTING.

**Proof.** $\Leftarrow$ Let $M$ be a “yes” instance of SET SPLITTING. Let $\{U,V\}$ be a partition of $A$ which satisfies the requirement of the problem: for every $B \in \mathcal{B}$, $B \cap U \neq \emptyset$ and $B \cap V \neq \emptyset$. For each $B \in \mathcal{B}$, choose a $u_B \in B \cap V$ and a $v_B \in B \cap U$.

Define teams $Y := \{s \in \{\emptyset\}[M/x] | s(x) \in U\}$ and $Z := \{\emptyset\}[M/x] \setminus Y$; they form a partition of $\{\emptyset\}[M/x]$. Let $F : Y[M/y] \rightarrow M$ be defined as $F(s) := u_B$ if $s(y) = B \in \mathcal{B}$, and as an arbitrary function of $y$ otherwise. Let $G : Z[M/z] \rightarrow M$ be defined as $G(s) := v_B$ if $s(z) = B \in \mathcal{B}$, and arbitrarily otherwise. Since the $u_B$s are in $V$, they are not in $U$, and so every $s \in Y[M/yz]$ is such that, if $s(y)$, then $s(u) = F(s\mid_{\{x,y\}}) = u_B \neq s(x) \in U$. So, $M,Y[M/yz] \models \epsilon_1$. A symmetrical argument shows that $M,Z[M/yz] \models \epsilon_2$.

$\Rightarrow$ Suppose $M \models \eta$. Then there are $Y,Z \subseteq \{\emptyset\}[M/x]$ such that $Y \cup Z = \{\emptyset\}[M/x]$, and $x$-uniform functions $F : Y[M/y] \rightarrow M$ and $G : Z[M/z] \rightarrow M$, such that $M,Y[M/yz] \models \epsilon_1$ and $M,Z[M/yz] \models \epsilon_2$. Define $U$ as $\{a \in A | \exists s \in Y(s(x) = a)\}$, and $V := A \setminus U$. Since $U \cup V = A$, at least one out of $U$ and $V$
is nonempty. We suppose w.l.o.g. that $U$ be nonempty, which implies that $Y$ is nonempty.

Let $B \in B$. Let $s_B \in M, Y[[MF/uy]]$ be an assignment such that $s_B(y) = B$ and $s_B(x) \in A$ (there is at least one such $s_B$, because of the nonemptyness of $Y$ and the fact that $y$ is universally quantified). The fact that $M, s_B \models \epsilon_1$ implies that $s_B(u) \in s_B(y) = B$ and $s_B(u) \neq s_B(x)$; since $s_B(u) = F(s_{(x,y)})$, the $x$-uniformity of $F$ implies that $s_B(u) \neq s(x)$ for any $s \in Y[[MF/uy]]$, that is, $s_B(u) \neq a$ for any $a \in U$. So $s_B(u) \in B \cap V$.

This furthermore implies that $V \neq \emptyset$. So, by a symmetric argument one can prove the existence of one element in $B \cap U$.

\textbf{Corollary 9.4.} \textit{The coordinated tree $C_1$, as all trees extending it, is NP-complete.}

This concludes the classification of coordinated trees up to reduction closure. However, since we do not known whether coordinated trees capture NP or any smaller complexity class, it might be of interest that we found a description of an L-complete problem, 2-COLORABILITY, by means of the minimal $C_1$ tree. This problem is known not to be in FO.

The sentence, in the language of graphs, is:

$$\xi : \forall x(\forall y(\exists u/\{x\})\xi_1 \lor \forall z(\exists v/\{x,y\})\xi_2)$$

where

$$\xi_1 : E(x,y) \to (u = y \land u \neq x)$$

and

$$\xi_2 : E(x,z) \to (v = z \land v \neq x).$$

\textbf{Theorem 9.5.} \textit{A graph structure $M$ satisfies $\xi$ if and only if it encodes a “yes” instance of 2-COLORABILITY.}

\textit{Proof.} 1) Suppose $M$ is a “yes” instance of 2-COLORABILITY. Then, it can be partitioned into two subsets $A, B$ such that $c \in A$ plus $(c, d) \in E^M$ implies $d \in B$, and viceversa.

Define $Y := \{s \in \emptyset[[M/x]]|s(x) \in A\}$ and $Z = \{\emptyset[[M/x]] \setminus Y\}$. Define $F : Y[[M/y]] \to M$, $F(s) := s(y)$, and $G : Z[[M/z]] \to M$, $G(s) := s(z)$.

Notice that, if $s \in Y$, then $s(x) \in A$; and if $(s(x), s(y)) \in E^M$, then $s(y) \in B$; so, since $A$ and $B$ are disjoint, $s(y) \neq s(x)$. Furthermore, $s(u) = s(y)$ by the definition of $F$. Thus, $M, Y[[MF/uy]] \models \xi_1$. The proof that $M, Z[[MG/zv]] \models \xi_2$ is completely analogous.

2) Suppose $M \models \xi$. Then there are $Y, Z \subseteq \emptyset[[M/x]]$, a $x$-uniform function $F : Y[[M/y]] \to M$ and a $x$-uniform $G : Z[[M/z]] \to M$ such that $Y \cup Z = \{\emptyset[[M/x]], M, Y[[MF/uy]] \models \xi_1$ and $M, Z[[MG/zv]] \models \xi_2$. By downward closure, we can assume that $Y \cap Z = \emptyset$. Let $A := \{a \in M|(x, a) \in Y\}$, and $B = M \setminus A$. Now suppose, for the sake of contradiction, that $a \in A$, $(a, c) \in E^M$ and $c \in A$. There is an $s \in Y[[MF/uy]]$ such that $s(x) = a$ and $s(y) = c$. Since $M, Y[[MF/uy]] \models \xi_1$ and $M, s \models E(x, y)$, we have $s(u) = s(y)$ and $s(u) \neq s(x)$.

39
Since \( s(u) = F(s_I(x,y)) \) and \( F \) is \( x \)-uniform, we have \( s(u) \neq s'(x) \) for any \( s' \in Y[MF/yu] \). But \( s(u) = s(y) \); so, \( s(y) \) is different from any \( s''(x) \) such that \( s'' \in Y \). Thus \( c = s(y) \in B \): a contradiction.

Similarly one proves that \( b \in B, (b,c) \in E^M \) implies \( c \in A \).

10 Coordinated trees, second kind

A coordinated tree is of the second kind if contains some logical operators \( \forall x, \forall y, \forall z, \lor, (\exists u/U), (\exists u/V) \) that form a coordinated pattern, and such that \( (\exists v/V), (\exists v/V) \) occur in the same disjunct below \( \lor \). If we want to avoid the trivial (linear Henkin) case that \( (\exists v/V), (\exists v/V) \) are in the same path of the tree, we must suppose that below \( \lor \) there is a conjunction \( \land \) such that \( (\exists u/U) \) occurs (say) in the left conjunct, and \( (\exists v/V) \) occurs in the right conjunct. Taking into account all the different positions in which \( \forall y, \forall z \), and ignoring permutations of quantifiers of the same kind, we can isolate six minimal coordinated trees of the second kind:

From left to right, we call these trees \( C_1' \), \( C_2' \), \( C_3' \), \( C_4' \), \( C_5' \), \( C_6' \). Before the reader starts worrying because of this explosion of cases, we point out that only the \( C_1' \) case is genuinely new, while \( C_2', C_3', C_4', C_5', C_6' \) are extensions of
either tree \(GH1(\wedge)\) or \(GH2(\wedge)\); in a certain sense, they fall in cases that we had already left open before. Notice, furthermore, that \(C(C1') \subseteq C(C2') \subseteq C(C3') \subseteq C(C5') \subseteq C(C6')\) and \(C(C1') \subseteq C(C2') \subseteq C(C4') \subseteq C(C5') \subseteq C(C6')\).

We show that the trees \(C1', C2', C3'\) are first-order (but we leave as an open problem the study of their extensions).

**Theorem 10.1.** The \(C1', C2'\) and \(C3'\) trees are in FO.

*Proof.* It is sufficient to prove it for \(C3'\). By quantifier distribution and quantifier swapping, the tree \(C3'\) can be transformed into

\[
\forall x \\
\bigvee \\
\left( \left( \forall y \left( \left( \exists u/x \right) \left( \left( \exists v/x \right) \left( \forall z \right) \left( \forall u \left( \forall y \left( \exists v \left( x, y, u, z \right) \land \psi_3(x, y, v, z) \right) \right) \right) \right) \right) \right) \\
\left( \left( \forall z \left( \forall u \left( \forall y \left( \exists v \left( x, y, u, z \right) \land \psi_3(x, y, v, z) \right) \right) \right) \right) \right) \\
\left( \left( \forall u \left( \forall y \left( \exists v \left( x, y, u, z \right) \land \psi_3(x, y, v, z) \right) \right) \right) \right)
\]

(which is what one obtains attaching two extra quantifiers \(\forall z, \forall y\) below the rightmost paths of the coordinated tree \(C1'\).) Let

\[\varphi : \forall x(\psi_1(x) \lor (\forall y(\exists u/x))\forall z(\psi_2(x, y, u, z) \land \forall z(\exists v/\{x\}\forall y\psi_3(x, z, v, y)))\]

be a generic \(IF\) sentence beginning with it; we want to prove it to be equivalent to

\[\varphi' : \forall x(\psi_1(x) \lor (\forall y(\exists u/\{x\})(\exists v/\{x, u\}))\forall z(\psi_2(x, y, u, z) \land \psi_3(x, y, v, z)))\]

Then by quantifier extraction one obtains

\[\forall x\forall y(\exists u/\{x\})(\exists v/\{x, u\})\forall z(\psi_1(x) \lor (\psi_2(x, y, u, z) \land \psi_3(x, y, v, z)))\]

and by quantifier swapping

\[\forall y\exists u(\exists v/\{u\})\forall x\forall z(\psi_1(x) \lor (\psi_2(x, y, u, z) \land \psi_3(x, y, v, z)))\]

Since the slash set of \(v\) contains only an existentially quantified variable, \(u\), it can be made empty, thus obtaining a first-order sentence.

But let us show the equivalence of \(\varphi\) and \(\varphi'\). 1) Suppose \(M, X \models \varphi\). Then there are \(Y, Z \subseteq \{0\}[M/x]\) such that \(Y, Z \subseteq \{0\}[M/x]\), and \(x\)-uniform functions \(F : Z[M/y] \to M\) and \(G : Z[M/z] \to M\), such that \(M, Y \models \psi_1(x), M, Z[MFM/yuz] \models \psi_2(x, y, u, z)\) and \(M, Z[MGM/zuy] \models \psi_3(x, z, v, y)\).

Then, define \(G' : Z[MFM/yu] \to M\) as \(G'(s) := G(s_{\text{dom}(s) \setminus \{y, u\}}(s(y)/z))\). It should be clear that \(M, Z[MFG'M/yuvz] \models \psi_2(x, y, u, z) \land \psi_3(x, y, v, z)\), and that \(G'\) is \(\{x, u\}\)-uniform.
2) Suppose $M, X \models \varphi'$, and thus we have $Y, Z, F, G'$ as above. For any $s \in Z[M/z]$, let $s' \in Z[M/y]$ be $s_-(s(z)/y)$; and let $s'' \in Z[MF/yu]$ be $s'(F(s')/u)$. Then one can define $G : Z[M/z] \rightarrow M$ as $G(s) := G'(s'')$. Then it is immediate to verify that $M, Z[MG/zy] \models \psi_3(x, y, u, z)$, and that $G$ is $x$-uniform.

With a different argument, we prove that the tree $C_6'$ (and thus $C_5', C_4'$) is in FO (the key idea of the proof is due to Lauri Hella). Again, the result tells us nothing about extensions of these trees.

**Theorem 10.2.** The trees $C_4', C_5', C_6'$ are in FO.

**Proof.** We prove that a sentence $\varphi$ which begins with tree $C_6'$

\[ \forall x \]
\[ \forall y \]
\[ \forall z \]
\[ \bigwedge \]
\[ \psi_1(x, y, z) \]
\[ \bigwedge \]
\[ (\exists u/x, z) \]
\[ (\exists v/x, y) \]
\[ \psi_2(x, y, z, u) \]
\[ \psi_3(x, y, z, v) \]

is equivalent to a sentence $\varphi'$ of the form

\[ \forall x \]
\[ \forall y \]
\[ \bigwedge \]
\[ (\exists u/x, z) \]
\[ (\exists v/x, y) \]
\[ \forall x \]
\[ \bigwedge \]
\[ \psi_1(x, y, z) \]
\[ \psi_2(x, y, z, u) \]
\[ \psi_3(x, y, z, v) \]

Notice that the resulting formula can also be obtained as completion of the positive initial tree.
which is a disjunction-free extension* of the GH2($\land$) tree, and thus in FO, by Theorem 8.3. So, $\varphi$ itself is equivalent to a first-order sentence.

So, now we have to prove the above equivalence.

$\implies$) $M \models \varphi$ iff there are teams $X_1, X_2 \subseteq \{\emptyset\}[MMM/xyz]$ such that $X_1 \cup X_2 = \{\emptyset\}[MMM/xyz]$, a $\{x, z\}$-uniform function $F : X_1 \to M$, and a $\{x, y\}$-uniform function $G : X_2 \to M$, such that $M, X_1 \models \psi_1(x, y, z)$, $M, X_2[F/u] \models \psi_2(x, y, z, u)$, and $M, X_2[G/v] \models \psi_3(x, y, z, v)$.

Now fix an $a \in M$, and define functions $F', G' : \{\emptyset\}[M/yz] \to M$ as $F'(s) := F(s(a/x))$ and $G'(s) := G(s(a/x))$. Obviously $F'$ is $z$-uniform and $G'$ is $y$-uniform.

Define teams $X'_1 := \{s \in \{\emptyset\}[MFM'/yzux] | M, s \models \psi_1(x, y, z)\}$ and $X'_2 := \{s \in \{\emptyset\}[MFM'/yzux] \setminus X'_1 \}$. We have to verify that, then, $M, X'_2 \models \psi_2(x, y, z, u)$. Suppose this is not the case, that is, there is an assignment $s \in X'_2$ such that $M, s \models \neg \psi_2(x, y, z, u)$. Then we also have that $M, \{s-u\} \models \exists u/\{x, z\} \psi_2(x, y, z, u)$ (where $s-u$ is the assignment $s$ restricted to $\text{dom}(s) \setminus \{u\}$). This implies that $s-u \in X'_1$; so, that $s \in X'_1$; this contradicts the initial assumption that $s \in X'_2$.

One can then analogously define $X'_3 := \{s \in \{\emptyset\}[MGM'/yzvx] | M, s \models \psi_3(x, y, z)\}$ and $X'_4 := \{\emptyset\}[MGM'/yzvx] \setminus X'_3$, and prove that $M, X'_4 \models \psi_3(x, y, z, v)$.

$\iff$) $M \models \neg \varphi$ iff there are functions $F', G' : \{\emptyset\}[MM/yz] \to M$ ($F'$ $z$-uniform, and $G'$ $y$-uniform) such that $M, \{\emptyset\}[MFM'/yzux] \models \psi_1(x, y, z) \lor \psi_2(x, y, z, u)$, and $M, \{\emptyset\}[MGM'/yzvx] \models \psi_1(x, y, z) \lor \psi_3(x, y, z, v)$. Calling $X'_1 := \{s \in \{\emptyset\}[MFM'/yzux] | M, s \models \psi_1(x, y, z)\}$ and $X'_3 := \{s \in \{\emptyset\}[MGM'/yzvx] | M, s \models \psi_3(x, y, z)\}$, the last two statements above are equivalent to the existence of a team $X'_1 \subseteq \{\emptyset\}[MFM'/yzux]$ such that $X_1 \cup X'_2 = \{\emptyset\}[MFM'/yzux]$ and $M, X'_2 \models \psi_2$, and, respectively, to the existence of a team $X'_3 \subseteq \{\emptyset\}[MGM'/yzvx]$ such that $X_2 \cup X'_3 = \{\emptyset\}[MGM'/yzvx]$ and $M, X'_3 \models \psi_3$.

Let $X_1 := \{s \in \{\emptyset\}[MMM/xyz] | M, s \models \psi_1(x, y, z)\}$. Let $X_2$ be its complement $\{\emptyset\}[MMM/xyz] \setminus X_1$. Define $F : X_2 \to M$ as $F(s) := F'(s-v)$ and $G : X_2 \to M$ as $G(s) := G'(s-v)$. Obviously $F'$ is $\{x, z\}$-uniform and $G'$ is $\{x, y\}$-uniform.

Does $M, X_2[F/u] \models \psi_2(x, y, z, u)$? Yes, because $s \in X_2[F/u]$ implies $s \in \{\emptyset\}[MFM'/yzux]$, and we already know that $M, \{\emptyset\}[MFM'/yzux] \models$
Similarly, one can see that $M, X_2[G/u] \models \psi_3(x, y, z, v)$.

11 Conclusions

In this paper we have classified, up to reduction closure, many of the syntactical fragments of $IF$ logic that are individuated by positive initial trees (see the Table at the end). All the trees that we have examined fall in the FO/NPC dichotomy. So, the question whether positive initial trees respect the dichotomy is still open.

Interestingly, we have found three patterns (GH2($\lor$), C2, C1) which express NP-complete problems even though they contain no Henkin nor signalling quantifier patterns; and for all we know, there might still be other unrecognized higher-order patterns (to be found among extensions of the GH1($\land$), GH1($\lor$), GH2($\land$) and C1’ trees). All the corresponding descriptions we have found are quite atypical; in particular, they can be easily translated into $H_2^1$ sentences ($H_2^1$ being the smallest, four-place Henkin quantifier) but not into $F_2^1$ sentences ($F_2^1$ being the smallest function quantifier, see [25]).

For what regards trees of low complexity, our theorem on modest trees (6.16) together with further results on generalized Henkin trees (8.3) and coordinated trees of the second kind (10.1, 10.2), provides a rather general sufficient (and effective) criterion for an $IF$ sentence to have first-order expressive power, extending the primality test of Sevenster, which in turn generalized the Knowledge Memory test ([1]) and the Perfect Recall test ([27]). A different criterion for first-orderness is given by checking the absence of broken signalling sequences, in the sense of [1]; however, a moment of thought shows that this is also a special case of the modest tree criterion. Sufficient, effective criteria for first-orderness have an interest because recognizing the $IF$ sentences (resp. ESO sentences, etc.) that are equivalent to first-order ones is an undecidable problem [13].

The present results can be seen as a step forward in the understanding of fragments of $IF$ logic. Future work should be addressed to a more systematical understanding of the classes GH1, GH2($\land$) and C1’-C6’, although it is not clear whether a complete and satisfying classification is possible. Further work might be directed at relaxing the requirement of regularity that we imposed on trees, or at finding exact characterizations of the expressive power of fragments (not just up to reduction closure). Thirdly, it might be interesting to investigate what happens abandoning the restriction that trees be positive initial; although,

\[\psi_2(x, y, z, u).\]
surely in this case a satisfying classification is impossible (a complete classification of syntactical trees would yield in particular a sufficient and necessary criterion of first-orderness – which, as we said, is an undecidable problem). One interesting example, in this sense, is the tree

$$\forall x \exists \alpha \exists \exists \exists \mu (\exists x, \alpha)(\exists \exists \exists \mu (x, \alpha))((\alpha = 0 \lor \alpha = 1) \land (\mu = 0 \lor \mu = 1) \land [ ]$$

which is equivalent to the smallest of the so-called partially ordered connectives. By results of [2], it follows that this tree is NL-complete, a possibility that, so far, we have not individuated among positive initial trees.

A fourth direction of work might be the analysis of logics similar to IF, such as the system IF*, which also allows slashed connectives, or Dependence-friendly logic, or their extensions via generalized quantifiers (see [9]). We also hope that the understanding of fragments of IF logic might be of help in the understanding of other logics that cover NP, and that are structurally very different. One example is given by the logics of imperfect information based on atoms; for these, an approach based on prefixes has perhaps little meaning, since their higher-order expressive power is in part generated at the level of atomic formulas. The other main example is, of course, existential second-order logic; in particular, its functional version, for which a prefix approach only yields a trivializing dichotomy.

In the following pages, a table summarizes all that we know about positive initial tree prefixes. As a convention, if T is the name of a specific tree, we refer to its extensions* to mean trees that extend T and do not fall in any other of the categories described in the table.

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| Tree          | Lower bound                        | Upper bound |
|--------------|------------------------------------|-------------|
| Henkin       | $\forall x \exists y \forall z (\exists u/{x,y})$, $\forall x \exists y (\exists z/{x,y}) (\exists u/{x,y})$ and their extensions | NP-complete (3-COLORING) | NP |
| Signaling    | $\forall x \exists z (\exists y/x)$ and its extensions | NP-complete (EXACT COVER BY 3-SETS, SAT, DOMINATING SET) | NP |
| Generalized Henkin | GH1($\land$): $\forall x$ $\land$ $\forall y$ ($\exists u/y$) $\land$ ($\exists v/x$) $\land$ ($\exists u/y$) and its disjunction-free extensions* | FO |
|              | GH2($\land$): $\forall x$ $\forall y$ $\land$ ($\exists v/x$) ($\exists u/y$) $\land$ ($\exists u/y$) and its disjunction-free extensions* | FO |
|              | GH1($\lor$): $\forall x$ $\lor$ $\forall y$ ($\exists u/y$) ($\exists v/x$) $\lor$ ($\exists u/y$) and its conjunction-free extensions* | FO |
| Extensions* of GH1($\land$), GH2($\land$) and GH1($\lor$) | ??? | ??? |
GH2(\lor):
\forall x
\forall y
\lor
(\exists y/x)
(\exists u/y)
\land
\land
and its extensions

NP-complete (SAT)  NP

| Coordinated | Coordinated | Coordinated |
|-------------|-------------|-------------|
| C1: \forall x \lor \forall y \land \forall z
(\exists u/x)
(\exists v/x)
\land
\land
and its extensions
| C1: \forall x \lor \forall y \land \forall z
(\exists u/x)
(\exists v/x, y)
\land
\land
and its extensions
| C1: \forall x \lor \forall y \land \forall z
\land
(\exists u/x, z)
(\exists v/x, y)
\land
\land
and its extensions
| NP-complete (SET SPLITTING)  NP
| NP-complete (SAT)  NP
| FO

Extensions* of C1'-C6'  ???  ???
Modest (everything else)  FO
References

[1] F. Barbero, On existential declarations of independence in IF logic, The Review of Symbolic Logic 6 (2013) 254–280.

[2] A. Blass, Y. Gurevich, Henkin quantifiers and complete problems, Annals of Pure and Applied Logic 32 (1986) 1–16.

[3] X. Caicedo, F. Dechesne, T. M. V. Janssen, Equivalence and quantifier rules for logic with imperfect information, Logic Journal of the IGPL 17 (2009) 91–129.

[4] P. Cameron, W. Hodges, Some combinatorics of imperfect information, Journal of Symbolic Logic 66 (2001) 673–684.

[5] L. A. Chagrova, An undecidable problem in correspondence theory, J. Symbolic Logic 56 (4) (1991) 1261–1272.

[6] A. Durand, J. Kontinen, N. de Rugy-Altherre, J. Väänänen, Tractability frontier of data complexity in team semantics, pre-print, arXiv:1509.06858.

[7] T. Eiter, G. Gottlob, Y. Gurevich, Existential second-order logic over strings, J. ACM 47 (1) (2000) 77–131.

[8] T. Eiter, G. Gottlob, T. Schwentick, The model checking problem for prefix classes of second-order logic: a survey, vol. 6300 of LNCS, Springer-Verlag Berlin Heidelberg, 2010, pp. 227–250.

[9] F. Engström, Generalized quantifiers in Dependence Logic, Journal of Logic, Language and Information 21 (2012) 299–324.

[10] R. Fagin, Generalized first-order spectra and polynomial-time recognizable sets, Complexity of Computation, ed. R. Karp, SIAM-AMS Proceedings 7 (1974) 27–41.

[11] P. Galliani, Inclusion and exclusion dependencies in team semantics - on some logics of imperfect information, Annals of Pure and Applied Logic 163 (1) (2012) 68–84.

[12] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman & Co., New York, NY, USA, 1979.

[13] G. Gottlob, P. G. Kolaitis, T. Schwentick, Existential second-order logic over graphs: Charting the tractability frontier, Journal of the Association for Computing Machinery 51 (2000) 664–674.

[14] E. Grädel, J. Väänänen, Dependence and independence, Studia Logica 101 (2013) 399–410.
[15] E. Grandjean, First-order spectra with one variable, Journal of Computer and System Sciences 40 (1990) 136153.

[16] L. Henkin, Some remarks on infinitely long formulas, in: Infinitistic methods, Pergamon Press, Oxford-London-New York-Paris, 1961.

[17] J. Hintikka, G. Sandu, Informational independence as a semantical phenomenon, in: J. E. Fenstad et al (ed.), Logic, Methodology and Philosophy of Science VIII, Elsevier Science Publishers B.V., 1989, pp. 571–589.

[18] W. Hodges, Compositional semantics for a language of imperfect information, Logic Journal of the IGPL 5 (1997) 539–563.

[19] W. Hodges, Some strange quantifiers, in: J. Mycielski, G. Rozenberg, A. Salomaa (eds.), Structures in Logic and Computer Science. Lecture Notes in Computer Sci. vol.1261, Springer-Verlag London, UK, 1997, pp. 51–65.

[20] T. M. V. Janssen, Independent choices and the interpretation of IF logic, Journal of Logic, Language and Information 11 (2002) 367–387.

[21] J. Kontinen, A. Kuusisto, J. Väänänen, Decidability of Predicate Logics with Team Semantics, in: P. Faliszewski, A. Muscholl, R. Niedermeier (eds.), 41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016), vol. 58 of Leibniz International Proceedings in Informatics (LIPIcs), Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2016.

[22] J. A. Kontinen, Coherence and complexity in fragments of dependence logic, Ph.D. thesis, Institute for Logic, Language and Computation (2010).

[23] J. A. Kontinen, Coherence and computational complexity of quantifier-free dependence logic formulas, Studia Logica 101 (2013) 267–291.

[24] M. Krynicki, M. Mostowski, Henkin quantifiers, in: M. Krynicki, M. Mostowski, L. Szczerba (eds.), Quantifiers: Logics, Models and Computation, Kluwer Academic Publishers, 1995, pp. 193–263.

[25] M. Krynicki, J. Väänänen, Henkin and function quantifiers, Annals of Pure and Applied Logic 43 (3) (1989) 273 – 292.

[26] L. Löwenheim, Über möglichkeiten im relativkalkül, Mathematische Annalen 76 (1915) 447470, translated as “On possibilities in the calculus of relatives” in Jean van Heijenoort, 1967. A Source Book in Mathematical Logic, 18791931, Harvard Univ. Press: 228251.

[27] A. L. Mann, G. Sandu, M. Sevenster, Independence-Friendly Logic - a Game-Theoretic Approach, vol. 386 of London Mathematical Society lecture note series, Cambridge University Press, 2011.

[28] V. Nurmi, Dependence Logic: Investigations into higher-order semantics defined on teams, Ph.D. thesis, University of Helsinki (2009).
[29] G. Sandu, On the logic of informational independence and its applications, Journal of Philosophical Logic 22 (1993) 29–60.

[30] G. Sandu, J. Väänänen, Partially ordered connectives, Mathematical Logic Quarterly 38 (1) (1992) 361–372.

[31] M. Sevenster, Dichotomy result for independence-friendly prefixes of generalized quantifiers, The Journal of Symbolic Logic 79(04) (2014) 1224–1246.

[32] T. Tantau, Existential second-order logic over graphs: A complete complexity-theoretic classification, in: 32nd International Symposium on Theoretical Aspects of Computer Science (STACS 2015), Volume 30 of Leibniz International Proceedings in Informatics (LIPIcs), Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2015.

[33] J. Väänänen, On the semantics of informational independence, Logic Journal of the IGPL 10 (2002) 337–350.

[34] J. Väänänen, Dependence Logic: A New Approach to Independence Friendly Logic, vol. 70 of London Mathematical Society Student Texts, Cambridge University Press, 2007.

[35] J. Virtema, Approaches to finite variable dependence, Ph.D. thesis, University of Tampere (2014).