On the index instability for some nonlocal elliptic problems

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Abstract

The Fredholm index of unbounded operators defined on generalized solutions of nonlocal elliptic problems in plane bounded domains is investigated. It is known that nonlocal terms with smooth coefficients having zero of a certain order at the conjugation points do not affect the index of the unbounded operator. In this paper, we construct examples showing that the index may change under nonlocal perturbations with coefficients not vanishing at the points of conjugation of boundary-value conditions.

1 Introduction

The first one who began to study nonlocal problems in multidimensional case was Carleman. In [2], the problem of finding a function harmonic on a two-dimensional bounded domain and subjected to a nonlocal condition connecting the values of this function at different points of the boundary is considered. Bitsadze and Smarskii [1] suggested another setting of a nonlocal problem arising in plasma theory: to find a function harmonic on a bounded domain and satisfying nonlocal conditions on shifts of the boundary that can take points of the boundary inside the domain. Different generalizations of the above nonlocal problems were investigated by many authors (see [12] and references therein).

It turns out that the most difficult situation occurs if the support of nonlocal terms intersects the boundary. In this case, solutions of nonlocal problems can have power-law singularities near some points even if the boundary and the right-hand sides are infinitely smooth [9]. For this reason, such problems are naturally studied in weighted spaces introduced by Kondrat’ev [6] for boundary-value problems in nonsmooth domains. In particular, the Fredholm solvability of general nonlocal elliptic problems in weighted spaces is investigated by Skubachevskii [9, 10, 11] and his pupils.

In [4], an unbounded operator \( P : L_2(G) \to L_2(G) \) corresponding to an elliptic equation of order \( 2m \) in a plane bounded domain \( G \subset \mathbb{R}^2 \) with nonlocal boundary-value conditions is studied; the operator \( P \) is defined on generalized solutions of the nonlocal problem, i.e., the domain \( D(P) \) consists of the functions \( u \in W_2^m(G) \) that satisfy nonlocal conditions in the sense of traces and for which \( Pu \in L_2(G) \). In particular, it is proved that the operator \( P \) has the Fredholm property.

In [5], we investigate how lower-order terms in elliptic equations and nonlocal perturbations in boundary-value conditions affect the index of the unbounded operator \( P \).

∗Supported by the Russian Foundation for Basic Research (project No. 04-01-00256) and by the Russian President’s grant (project No. MK-980.2005.1).
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It is proved that the index of $P$ does not change if we add nonlocal terms with smooth coefficients having zero of a certain order at the points of conjugation of boundary-value conditions. It is also proved that lower-order terms in elliptic equation have no influence on the index.

In this paper, we construct examples showing that the index may change under nonlocal perturbations with coefficients not vanishing at the points of conjugation of boundary-value conditions (even if the coefficients in the perturbations are arbitrarily small). The case where the support of nonlocal terms contains the conjugation points and the case where it does not are both considered.

The reason of the index instability is that nonlocal terms change the domain of the corresponding unbounded operator; it turns out that if the coefficients at nonlocal terms do not vanish at the conjugation points, then these terms cannot be reduced to (relatively) small or compact perturbations.

2 Preliminaries

Let $G$ be a bounded domain in $\mathbb{R}^2$, and let

$$\partial G \setminus \{g_1, g_2\} = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_i$ are open (in the topology of $\partial G$) infinitely smooth curves, $\Gamma_1 \cap \Gamma_2 = \{g_1, g_2\}$, and $g_1$ and $g_2$ are the endpoints of the curves $\Gamma_1$ and $\Gamma_2$. We assume that the domain $G$ is a plane angle of nonzero opening in a neighborhood of the points $g_j$.

For integer $k \geq 0$, we denote by $W^k_2(G)$ the Sobolev space with the norm

$$\|u\|_{W^k_2(G)} = \left( \sum_{|\alpha| \leq k} \int_G |D^\alpha u|^2 dy \right)^{1/2},$$

(set $W^0_2(G) = L_2(G)$ for $k = 0$). For integer $k \geq 1$, we introduce the space $W^{k-1/2}_2(\Gamma)$ of traces on a smooth curve $\Gamma \subset \overline{G}$ with the norm

$$\|\psi\|_{W^{k-1/2}_2(\Gamma)} = \inf \|u\|_{W^k_2(G)},$$

where the infimum is taken over all $u \in W^k_2(G)$ such that $u|_\Gamma = \psi$.

Let $C^\infty_0(\overline{G} \setminus \{g_1, g_2\})$ denote the set of functions infinitely differentiable on $\overline{G}$ and vanishing near the points $g_1$ and $g_2$.

We introduce the Kondrat’ev space $H^k_\alpha(G)$ as the completion of the set $C^\infty_0(\overline{G} \setminus \{g_1, g_2\})$ with respect to the norm

$$\|u\|_{H^k_\alpha(G)} = \left( \sum_{|\alpha| \leq k} \int_G \rho^{2(a+|\alpha|-k)} \rho^2 \left| D^\alpha u \right|^2 dx \right)^{1/2},$$

where $k \geq 0$, $a \in \mathbb{R}$, and $\rho(y) = \text{dist}(y, \{g_1, g_2\})$.

Denote by $H^{k-1/2}_\alpha(G)$ ($k \geq 1$ is an integer) the space of traces on a smooth curve $\Gamma \subset \overline{G}$ with the norm

$$\|\psi\|_{H^{k-1/2}_\alpha(\Gamma)} = \inf \|u\|_{H^k_\alpha(G)},$$

where the infimum is taken over all $u \in H^k_\alpha(G)$ such that $u|_\Gamma = \psi$. 

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It is clear that $H^k_0(G) \subset W^k_2(G)$ for integer nonnegative $k$ and $H^{k-1/2}_0(\Gamma) \subset W^{k-1/2}_2(\Gamma)$ for integer $k \geq 1$. In the sequel, we also need the following result about the embedding of Sobolev spaces into weighted spaces.

**Lemma 2.1** (see Lemma 4.11 in [6]). The operators of embedding $W^{3/2}_2(\Gamma_j) \subset H^{3/2}_{a+1}(\Gamma_j)$, $j = 1, 2$, are bounded for all $a > 0$.

To conclude this section, we formulate a theorem about small perturbations of Fredholm operators in Hilbert spaces.

Let $H_1$ and $H_2$ be Hilbert spaces, and let $P : D(P) \subset H_1 \rightarrow H_2$ be a linear (generally speaking, unbounded) operator.

**Definition 2.1.** The operator $P$ is said to have the Fredholm property if it is closed, its image is closed, and the dimension of its kernel $\ker P$ and the codimension of its image $\mathcal{R}(P)$ are finite. The number $\text{ind } P = \dim \ker P - \text{codim } \mathcal{R}(P)$ is called the index of the Fredholm operator $P$.

Let $A : H_1 \rightarrow H_2$ be a linear operator.

**Theorem 2.1** (see Sec. 16 in [7]). Let the operator $P$ have the Fredholm property, $A$ be bounded, and $D(A) = H_1$. Then the operator $P + A$ has the Fredholm property, $\text{ind } (P + A) = \text{ind } P$, $\dim \ker (P + A) \leq \dim \ker P$, and $\text{codim } \mathcal{R}(P + A) \leq \text{codim } \mathcal{R}(P)$, provided that $\|A\|$ is sufficiently small.

## 3 The index instability in the case where the support of nonlocal terms contains the conjugation points

1. Let $G, \Gamma_i$, and $g_j$ be the same as above. Let a number $\varepsilon > 0$ be so small that the $\varepsilon$-neighborhoods $O_\varepsilon(g_j)$ of the points $g_j$ do not intersect with each other. We assume that the set $G \cap O_\varepsilon(g_j)$ coincides with the plane angle of opening $2\omega_0$, where $0 < \omega_0 < \pi$. Consider the following nonlocal problem in the domain $G$:

$$
\Delta u = f(y), \quad y \in G, \tag{3.1}
$$

$$
u|_{\Gamma_1} - (1 + t)u(\Omega_1(y))|_{\Gamma_1} = 0, \quad u|_{\Gamma_2} - (1 - t)u(\Omega_2(y))|_{\Gamma_2} = 0. \tag{3.2}
$$

Here $t \in \mathbb{C}$ is a parameter and $\Omega_i$ is a $C^\infty$-diffeomorphism defined on a neighborhood of $\Gamma_i$. We assume that $\Omega_i(\Gamma_i) \subset G$, $\Omega_i(g_1) = g_1$, $\Omega_i(g_2) = g_2$, and the transformation $\Omega_i$ is the rotation by the angle $\omega_0$ inside the domain $G$ near the points $g_1$ and $g_2$ (see Fig. 3.1).

We say that a function $u \in W^1_2(G)$ is a generalized solution of problem (3.1), (3.2) with right-hand side $f \in L^1_2(G)$ if $u$ satisfies Eq. (3.1) in the sense of distributions and nonlocal conditions (3.2) in the sense of traces.

Consider the unbounded operator $P_t : D(P_t) \subset L^1_2(G) \rightarrow L^1_2(G)$ given by

$$
P_t u = \Delta u, \quad u \in D(P_t),
$$

where

$$
D(P_t) = \{u \in W^1_2(G) : \Delta u \in L^1_2(G) \text{ and } u \text{ satisfies (3.2)} \}.
$$

The operator $P_t$ has the Fredholm property for all $t \in \mathbb{C}$ by Theorem 2.1 in [4].

Let us prove the following result.
Theorem 3.1. There is a number \( t_0 > 0 \) such that \( \text{ind} P_0 > \text{ind} P_t = \text{const} \) for \( 0 < |t| \leq t_0 \).

2. It is known (see, e.g., [9]) that the behavior of solutions for problem (3.1), (3.2) near the points \( g_1 \) and \( g_2 \) depends on the location of eigenvalues of model problems (with a parameter) corresponding to these points. To write the model problem corresponding to the point \( g_1 \), we assume that \( g_1 \) is the origin and the axis \( Ox_1 \) coincides with the bisectrix of the angle formed by the boundary of the domain \( G \) near the point \( g_1 \). Consider problem (3.1), (3.2) for \( y \in O_\varepsilon(g_1) \), formally setting \( f = 0 \):

\[
\Delta u = 0, \quad y \in O_\varepsilon(g_1),
\]

where \( (\omega, r) \) are the polar coordinates of the point \( y \). Further, writing the Laplace operator in the polar coordinates, setting \( \tau = \ln r \) and formally applying the Fourier transform \( u(\omega, \tau) \mapsto \tilde{u}(\omega, \lambda) \), we obtain the following model nonlocal problem with the parameter \( \lambda \in \mathbb{C} \):

\[
\varphi'' - \lambda^2 \varphi = 0, \quad |\varphi| < \omega_0,
\]

where \( \varphi(\omega) = \tilde{u}(\omega, \lambda) \) for a fixed \( \lambda \). Clearly, the same problem corresponds to the point \( g_2 \). The eigenvectors of problem (3.3), (3.4) are nonzero functions in finitely differentiable on the segment \([-\omega_0, \omega_0]\) such that they satisfy Eq. (3.3) and nonlocal conditions (3.4).

Straightforward calculation shows that the eigenvalues of problem (3.3), (3.4) do not depend on \( t \) and have the form

\[
\lambda_k = \frac{\pi k}{\omega_0} i, \quad k = 0, \pm 1, \pm 2, \ldots.
\]

In the sequel, we consider the eigenvalues in the strip \(-1 \leq \text{Im} \lambda \leq 0\). Since \( 0 < \omega_0 < \pi \), it follows that this strip contains the unique eigenvalue \( \lambda_0 = 0 \). The corresponding eigenvector has the form

\[
\varphi_0(\omega) = -\frac{t}{\omega_0} \omega + 1
\]

(1) If \( \lambda_0 \in \mathbb{C} \) is an eigenvalue of problem (3.3), (3.4) and \( \varphi_0(\omega) \) is the corresponding eigenvector, then the
Lemma 3.1. There is a number $t_0 > 0$ such that $\text{codim} \mathcal{R}(P_0) \leq \text{codim} \mathcal{R}(P_t) = \text{const}$ for $0 < |t| \leq t_0$.

Proof. 1. Consider the operator

$$\mathbf{N}_t : H_0^2(G) \to H_0^0(G) \times H_0^{3/2}(\Gamma_1) \times H_0^{3/2}(\Gamma_2)$$

given by

$$\mathbf{N}_t = (\Delta u, u|_{r_1} - (1 + t)u(\Omega_1(y))|_{r_1}, u|_{r_2} - (1 - t)u(\Omega_2(y))|_{r_2}).$$

Since the line $\text{Im} \lambda = -1$ contains no eigenvalues of problem (3.3), (3.4), it follows from Theorem 3.4 in [9] that the operator $\mathbf{N}_t$ has the Fredholm property for all $t$. Further, the operator $u \mapsto u(\Omega_j(y))|_{r_j}$ is a bounded operator acting from $H_0^2(G)$ to $H_0^{3/2}(\Gamma_j)$. On the other hand, small perturbations do not change the index of a Fredholm operator (see Theorem 2.1); therefore, $\text{ind} \mathbf{N}_t = \text{const}$ for all $t$ from a sufficiently small neighborhood of an arbitrary point $t' \in \mathbb{C}$, which yields

$$\text{ind} \mathbf{N}_t = \text{const}, \quad t \in \mathbb{C}. \quad (3.7)$$

2. Let us prove that

$$\text{codim} \mathcal{R}(\mathbf{N}_t) = \text{const}, \quad |t| \leq t_0, \quad (3.8)$$

where $t_0 > 0$ is sufficiently small. Due to (3.7), it suffices to show that

$$\dim \ker \mathbf{N}_t = 0, \quad |t| \leq t_0. \quad (3.9)$$

Let $t = 0$, and let $u \in \ker \mathbf{N}_0$. Lemma 2.1 in [4] implies that the function $u$ is infinitely differentiable outside an arbitrarily small neighborhood of the set $\{ g_1, g_2 \}$. On the other hand, $u \in H_0^2(G) \subset W_2^2(G)$; therefore, by the Sobolev embedding theorem, $u \in C^\infty(G) \cap C(\overline{G})$ and

$$u(g_1) = u(g_2) = 0. \quad (3.10)$$

Since the coefficients of the problem are real for $t = 0$, we can assume without loss of generality that the function $u(y)$ is real-valued. If the function $|u(y)|$ achieves its maximum inside the domain $G$, then the maximum principle implies that $u = \text{const}$ in $\overline{G}$; hence, $u = 0$ by (3.10). If $|u(y)|$ achieves its maximum on the part $\Gamma_i$ of the boundary, then nonlocal conditions (3.2), which take the form

$$u|_{r_1} = u(\Omega_1(y))|_{r_1}, \quad u|_{r_2} = u(\Omega_2(y))|_{r_2}$$

for $t = 0$, imply that $|u(y)|$ also achieves its maximum inside the domain $G$; hence, $u = 0$ by what has been proved. Finally, if $|u(y)|$ achieves its maximum at the point $g_1$ or $g_2$, then $u = 0$ by (3.10).

Thus, we have proved that $\dim \ker \mathbf{N}_0 = 0$. It follows from Theorem 2.1 that $\dim \ker \mathbf{N}_t \leq \dim \ker \mathbf{N}_0 = 0$ for sufficiently small $|t|$; this yields (3.9) and, hence, (3.8).

3. Now we prove that

$$\mathcal{R}(P_t) = \{ f \in L_2(G) : (f, 0, 0) \in \mathcal{R}(\mathbf{N}_t) \}, \quad 0 \neq t \in \mathbb{C}. \quad (3.11)$$

associate vector $\varphi_1(\omega)$ is a solution (possibly, zero) of the equation $\varphi''_1 - \lambda_0^2 \varphi_1 + \frac{d}{d\lambda}(\varphi''_0 - \lambda^2 \varphi_0)|_{\lambda = \lambda_0} = 0$ with nonlocal conditions (3.4). Therefore, if $\lambda_0 = 0$, then the associate vector $\varphi_1(\omega)$ is a solution of the equation $\varphi''_1 = 0$ with nonlocal conditions (3.4).
Since any solution \( u \in H_0^2(G) \) of problem (3.1), (3.2) with right-hand side \( f \in L_2(G) \) belongs to \( W_2^1(G) \), it follows that
\[
R(P_t) \supset \{ f \in L_2(G) : (f,0,0) \in R(N_t) \}, \quad t \in \mathbb{C}. \tag{3.12}
\]

To prove the inverse embedding for \( t \neq 0 \), we consider an arbitrary function \( f \in R(P_t) \). Let \( u \in W_2^1(G) \) be a solution of problem (3.1), (3.2) with the right-hand side \( f \). It follows from [5] that \( u \in H_0^2(G) \) for all \( a > 0 \). Due to (3.5), there is a number \( a > 0 \) such that the strip \(-1 \leq \text{Im}\lambda < a\) contains the unique eigenvalue \( \lambda_0 = 0 \) of problem (3.1), (3.2). Theorem 3.3 in [9] about the asymptotic behavior of solutions of nonlocal problems implies that
\[
u(y) = c_j \varphi_0(\omega) + d_j \varphi_0(\omega) \ln r + v_j(y), \quad y \in G \cap \mathcal{O}_\varepsilon(g_j), \tag{3.13}
\]
where \((\omega, r)\) are the polar coordinates with the pole at the point \( g_j \), \( \varphi_0(\omega) \) is given by (3.6), and \( v_j \in H_0^2(G \cap \mathcal{O}_\varepsilon(g_j)) \). Note that
\[
u \in W_2^1(G \cap \mathcal{O}_\varepsilon(g_j)), \quad v_j \in W_2^1(G \cap \mathcal{O}_\varepsilon(g_j)),
\]
but
\[
\varphi_0(\omega) \notin W_2^1(G \cap \mathcal{O}_\varepsilon(g_j)) \quad \varphi_0(\omega) \ln r \notin W_2^1(G \cap \mathcal{O}_\varepsilon(g_j))
\]
for \( t \neq 0 \). Therefore, \( c_j = d_j = 0 \) in (3.13), which yields \( u \in H_0^2(G) \). Thus, we have proved that \((f,0,0) \in R(N_t)\), i.e.,
\[
R(P_t) \subset \{ f \in L_2(G) : (f,0,0) \in R(N_t) \}, \quad 0 \neq t \in \mathbb{C}. \tag{3.14}
\]

Relations (3.12) and (3.14) imply (3.11).

4. Let us prove that \(^2\)
\[
\text{codim} \{ f \in L_2(G) : (f,0,0) \in R(N_t) \} = \text{codim} R(N_t), \quad t \in \mathbb{C}. \tag{3.15}
\]
Fix a number \( t \in \mathbb{C} \) and set
\[
J_1 = \text{codim} \{ f \in L_2(G) : (f,0,0) \in R(N_t) \}, \quad J_2 = \text{codim} R(N_t).
\]
Denote \( H_0^0(G, \Gamma) = H_0^0(G) \times H_0^{3/2}(\Gamma_1) \times H_0^{3/2}(\Gamma_2) \).

Let \( f \in L_2(G) \) and \((f,0,0) \in R(N_t)\). This is equivalent to the relations
\[
((f,0,0), F_j)_{H_0^0(G, \Gamma)} = 0, \quad j = 1, \ldots, J_2,
\]
where \( F_j \) are functions that form the basis for the orthogonal complement to the subspace \( R(N_t) \) in the space \( H_0^0(G, \Gamma) \). By the Riesz theorem on the general form of a linear bounded functional in a Hilbert space, these relations are equivalent to the following ones:
\[
(f, \hat{f}_j)_{L_2(G)} = 0, \quad j = 1, \ldots, J_2,
\]
where \( \hat{f}_j \) are some functions from the space \( L_2(G) \). Thus,
\[
J_1 \leq J_2 \tag{3.16}
\]
\(^2\)In (3.15), the codimension of the subspace \( \{ f \in L_2(G) : (f,0,0) \in R(N_t) \} \) is calculated in the space \( H_0^0(G) \), whereas the codimension of the subspace \( R(N_t) \) is calculated in the space \( H_0^0(G) \times H_0^{3/2}(\Gamma_1) \times H_0^{3/2}(\Gamma_2) \).
Using Lemma 3.1 in [8], we can construct a function $v \in H^2_0(G)$ such that
\begin{equation}
\|v\|_{H^2_0(G)} \leq k_1(\|f_1\|_{H^{3/2}_0(\Gamma_1)} + \|f_2\|_{H^{3/2}_0(\Gamma_2)}),
\end{equation}
where $k_1 > 0$ does not depend on $f_1$ and $f_2$.

Clearly, the function $w = u - v \in H^2_0(G)$ is a solution of the problem
\begin{align}
\Delta w &= f(y) - \Delta v, \quad y \in G, \\
w|_{\Gamma_1} - (1 + t)w(\Omega_1(y))|_{\Gamma_1} &= 0, \quad w|_{\Gamma_2} - (1 - t)w(\Omega_2(y))|_{\Gamma_2} = 0.
\end{align}

Therefore, $f - \Delta v \in L_2(G), \quad (f - \Delta v, 0, 0) \in \mathcal{R}(N_t)$,
which is equivalent to the relations
\begin{equation}
(f - \Delta v, f'_j)_{L_2(G)} = 0, \quad j = 1, \ldots, J_1,
\end{equation}
where $f'_j \in L_2(G)$ are functions that form the basis for the orthogonal supplement to the subspace $\{f \in L_2(G) : (f, 0, 0) \in \mathcal{R}(N_t)\}$ in the space $L_2(G)$. By the Riesz theorem on the general form of a linear bounded functional in a Hilbert space and by estimate (3.17), these relations are equivalent to the following ones:
\begin{equation}
(F, F'_j)_{H^0_0(G, \Gamma)} = 0, \quad j = 1, \ldots, J_1,
\end{equation}
where $F'_j$ are some functions from the space $H^0_0(G, \Gamma)$. Thus,
\begin{equation}
J_2 \leq J_1
\end{equation}
(the equality takes place if and only if the functions $F'_j$ are linearly independent).

Inequalities (3.16) and (3.18) imply (3.15).

It follows from relations (3.15) and (3.8) that
\begin{equation}
\text{codim} \{f \in L_2(G) : (f, 0, 0) \in \mathcal{R}(N_t)\} = \text{const}, \quad |t| \leq t_0.
\end{equation}
Combining (3.11), (3.12) for $t = 0$, and (3.19), we complete the proof. \hfill \Box

**Lemma 3.2.** Let the number $t_0 > 0$ be the same as in Lemma 3.1. Then $\dim \ker P_0 > \dim \ker P_t = 0$ for $0 < |t| \leq t_0$.

**Proof.** 1. Let $0 < |t| \leq t_0$, and let $u \in \ker P_t$. Similarly to item 3 of the proof of Lemma 3.1, one can show that $u \in H^2_0(G)$; therefore, $u \in \ker N_t$. It follows from (3.9) that $u = 0$; thus, $\dim \ker P_t = 0$.

2. Let $t = 0$. In this case, $u = \text{const}$ belongs to $\ker P_0$. \hfill \Box
**Proof of Theorem 3.1.** Applying Lemmas 3.2 and 3.1, we obtain

\[
\text{ind}\, \mathbf{P}_0 = \dim \ker \mathbf{P}_0 - \text{codim} \mathcal{R}(\mathbf{P}_0) > -\text{codim} \mathcal{R}(\mathbf{P}_0) \geq -\text{codim} \mathcal{R}(\mathbf{P}_t) = \text{ind} \, \mathbf{P}_t, \quad 0 < |t| \leq t_0.
\]

\[ \square \]

3. Now we show that the index of the unbounded operator may change even if nonlocal terms are supported in an arbitrarily small neighborhood of the conjugation points \( g_1 \) and \( g_2 \).

Let \( G, \Gamma_i \), and \( g_j \) be the same as above. Consider the following nonlocal problem in the domain \( G \):

\[
\Delta u = f(y), \quad y \in G, \tag{3.20}
\]

\[
u|_{\Gamma_1} - (1 + t)\xi(y)u(\Omega_1(y))|_{\Gamma_1} = 0, \quad u|_{\Gamma_2} - (1 - t)\xi(y)u(\Omega_2(y))|_{\Gamma_2} = 0, \tag{3.21}
\]

where \( \xi \in C^\infty(\mathbb{R}^2) \), the function \( \xi \) is supported in an arbitrarily small neighborhood of the points \( g_1 \) and \( g_2 \), and \( \xi(y) = 1 \) near these points (see Fig. 3.2).

![Figure 3.2: Problem (3.20), (3.21)](image)

Consider the unbounded operator \( \mathbf{P}_t' : D(\mathbf{P}_t') \subset L_2(G) \to L_2(G) \) given by

\[
\mathbf{P}_t'u = \Delta u, \quad u \in D(\mathbf{P}_t),
\]

where

\[
D(\mathbf{P}_t) = \{ u \in W_2^1(G) : \Delta u \in L_2(G) \text{ and } u \text{ satisfies (3.21)} \}.
\]

By Theorem 2.1 in [4], the operator \( \mathbf{P}_t \) has the Fredholm property for all \( t \in \mathbb{C} \).

The main result of this section is as follows.

**Theorem 3.2.** There is a number \( t_0 > 0 \) such that \( \text{ind} \, \mathbf{P}_0' > \text{ind} \, \mathbf{P}_t' = \text{const} \) for \( 0 < |t| \leq t_0 \).

**Proof.** Nonlocal conditions (3.2) differ from nonlocal conditions (3.21) by the operators

\[
u \mapsto (1 + t)(1 - \xi(y))u(\Omega_1(y))|_{\Gamma_1}, \quad u \mapsto (1 - t)(1 - \xi(y))u(\Omega_2(y))|_{\Gamma_2}.
\]

Since the coefficients \( (1 \pm t)(1 - \xi(y)) \) at the nonlocal terms vanish near the points \( g_1 \) and \( g_2 \), it follows that \( \text{ind} \, \mathbf{P}_t' = \text{ind} \, \mathbf{P}_t \) for all \( t \in \mathbb{C} \) due to [5]. Hence, the assertion of the theorem follows from Theorem 3.1. \( \square \)
4 The index instability in the case where the nonlocal terms are supported outside the conjugation points

In this section, we show that the index of the unbounded operator may also change in the case where the support of nonlocal terms does not contain the conjugation points (and even lies strictly inside the domain).

Let $G$, $\Gamma_i$, and $g_j$ be the same as above. We additionally assume that

$$0 < \omega_0 < \pi/2$$

and consider the following nonlocal problem in the domain $G$:

\[
\Delta u = f(y), \quad y \in G, \quad (4.1)
\]

\[
u|_{\Gamma_1} + tu(\Omega(y))|_{\Gamma_1} = 0, \quad u|_{\Gamma_2} = 0, \quad (4.2)
\]

where $t \in \mathbb{R}$ and $\Omega$ is a $C^\infty$-diffeomorphism defined on some neighborhood of the curve $\Gamma_1$. Moreover, let $\Omega(\Gamma_1) \subset G$ (see Fig. 4.1).

Consider the unbounded operator $P_t : D(P_t) \subset L^2(G) \to L^2(G)$ given by

$P_t u = \Delta u, \quad u \in D(P_t),$

where

$$D(P_t) = \{ u \in W^1_2(G) : \Delta u \in L^2(G) \text{ and } u \text{ satisfies (4.2)} \}.$$

By Theorem 2.1 in [4], the operator $P_t$ has the Fredholm property for all $t \in \mathbb{C}$.

The main result of this section is as follows.

**Theorem 4.1.** There is a number $t_0 > 0$ such that $0 = \text{ind } P_0 > \text{ind } P_t$ for $0 < |t| \leq t_0$.

It is well known that the operator $P_0$ is an isomorphism; in particular,

$$\text{ind } P_0 = 0. \quad (4.3)$$

Consider the operators $P_t$. One and the same problem

\[
\varphi'' - \lambda^2 \varphi = 0, \quad |\varphi| < \omega_0, \quad (4.4)
\]

\[
\varphi(-\omega_0) = \varphi(\omega_0) = 0 \quad (4.5)
\]
with the parameter $\lambda \in \mathbb{C}$ corresponds to the points $g_1$ and $g_2$ (this problem is local because the nonlocal term is supported outside the set $\{g_1, g_2\}$).

Straightforward calculation shows that the eigenvalues of problem (4.4), (4.5) have the form

$$\lambda_k = \frac{\pi k}{2\omega_0} i, \quad k = \pm 1, \pm 2, \ldots.$$  \hfill (4.6)

**Lemma 4.1.** We have dim ker $P_t = 0$ for $0 < |t| \leq 1$.

*Proof.* Let $u \in \ker P_t$. Since $0 < \omega_0 < \pi/2$, it follows from (4.6) that the strip $-1 \leq \Im \lambda < 0$ contains no eigenvalues of problem (4.4), (4.5). Therefore, $u \in W^2_2(G)$ by Theorem 1 in [3]. It follows from Lemma 2.1 in [4] that the function $\lambda < \Im$ is infinitely differentiable outside an arbitrarily small neighborhood of the set $\{g_1, g_2\}$. Thus, the Sobolev embedding theorem implies $u \in C^\infty(G) \cap C(\overline{G})$.

Since $t \in \mathbb{R}$, it follows that the coefficients in problem (4.1), (4.2) are real; hence, we can assume without loss of generality that the function $u(y)$ is real-valued. If the function $|u(y)|$ achieves its maximum inside the domain $G$, then $u = \text{const}$ in $\overline{G}$ by the maximum principle; in this case, $u = 0$ by the second condition in (4.2). If $|u(y)|$ achieves its maximum on $\Gamma_1$, then the first condition in (4.2) and the relation $|t| \leq 1$ imply that $|u(y)|$ achieves its maximum inside the domain $G$; in this case, $u = 0$ by what has been proved. Finally, if $|u(y)|$ achieves its maximum on $\overline{\Gamma_2}$, then $u = 0$ by its continuity and by the second condition in (4.2). \hfill \Box

**Lemma 4.2.** There is a number $t_0 > 0$ such that codim $\mathcal{R}(P_t) > 0$ for $0 < |t| \leq t_0$.

*Proof.* 1. Consider the bounded operator

$$M_t : H^2_{a+1}(G) \to H^0_{a+1}(G) \times H^{3/2}_{a+1}(\Gamma_1) \times H^{3/2}_{a+1}(\Gamma_2), \quad a > 0,$

given by

$$M_t = (\Delta u, \ u|_{\Gamma_1} + tu(\Omega(y))|_{\Gamma_1}, \ u|_{\Gamma_2}).$$

Since the operator of embedding $W^{3/2}_{a+1}(\Gamma_1) \subset H^{3/2}_{a+1}(\Gamma_1)$ is bounded by Lemma 2.1 and $\Omega(\Gamma_1) \subset G$, it follows that

$$\|u(\Omega(y))\|_{H^{3/2}_{a+1}(\Gamma_1)} \leq k_1\|u(y')\|_{W^{3/2}_{a+1}(\Gamma_1)} \leq k_2\|u\|_{H^2_{a+1}(G)}. \hfill (4.7)$$

Therefore,

$$M_t u \in H^0_{a+1}(G) \times H^{3/2}_{a+1}(\Gamma_1) \times H^{3/2}_{a+1}(\Gamma_2)$$

whenever $u \in H^2_{a+1}(G)$ and $a > 0$. Thus, the operator $M_t$ is well defined.

By Theorem 10.5 in [8], the local operator $M_0$ is an isomorphism for

$$0 < a < \pi/(2\omega_0). \hfill (4.8)$$

Fix a number $a$ satisfying (4.8). Since $M_0$ is an isomorphism and estimate (4.7) is true, it follows that the operator $M_t$ is also an isomorphism for $0 \leq |t| \leq t_0$, provided $t_0 = t_0(a)$ is sufficiently small.

2. Let us construct a function $u \in H^2_{a+1}(G)$ satisfying nonlocal conditions (4.2) and such that

$$u(\Omega(g_1)) = 1.$$ 

To this end, we consider a function $v \in C^\infty(G)$ such that $v(y) = 1$ for $y \in \overline{\Omega(\Gamma_1)}$ and supp $v \subset G$. In this case, we have $v(\Omega(y)) = 1$ for $y \in \Gamma_1$; therefore, $v(\Omega(y))|_{\Gamma_1} \in H^{3/2}_{a+1}(\Gamma_1)$. 

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Further, we consider a function \( w \in H^2_{a+1}(G) \) such that

\[
\begin{aligned}
w|_{\Gamma_1} &= -tv(\Omega(y))|_{\Gamma_1}, & w|_{\Gamma_2} &= 0, & \text{supp } w \cap \Omega(\Gamma_1) &= \emptyset
\end{aligned}
\]

(the existence of such a function \( w \) follows from Lemma 3.1 in [8]). One can easily see that \( u = v + w \) is the desired function (see Fig. 1).

3. We approximate the function \( f = \Delta u \in H^0_{a+1}(G) \) by the functions \( f_n \in L^2(G), n = 1, 2, \ldots \), in the space \( H^0_{a+1}(G) \):

\[
||f_n - f||_{H^0_{a+1}(G)} \to 0, \quad n \to \infty.
\] (4.9)

If \( \text{codim} \mathcal{R}(P_t) = 0 \) for \( 0 < |t| \leq t_0 \), then, for each function \( f_n \in L^2(G) \), there exists a generalized solution \( u_n \in W^2_2(G) \) of problem (1.1), (1.2) with the right-hand side \( f_n \) (this solution is unique by Lemma 4.1). Moreover, \( u_n \in H^2_{a+1}(G) \), see [5].

It follows from the fact that \( M_t \) is an isomorphism and from (4.9) that

\[
||u_n - u||_{H^2_{a+1}(G)} \leq k_3||f_n - f||_{H^0_{a+1}(G)} \to 0, \quad n \to \infty.
\]

Therefore, by the Sobolev embedding theorem, we have

\[
u_n(\Omega(g_1)) \to u(\Omega(g_1)) = 1, \quad n \to \infty.
\] (4.10)

On the other hand, the strip \(-1 \leq \text{Im} \lambda < 0\) contains no eigenvalues of problem (4.4), (4.5), and hence \( u_n \in W^2_2(G) \) by Theorem 1 in [3]. By the Sobolev embedding theorem, \( u_n \in C(G) \), and it follows from the second nonlocal condition in (1.2) that \( u_n(g_1) = 0 \). The first nonlocal condition in (1.2) now implies that \( u_n(\Omega(g_1)) = 0 \) (for \( t \neq 0 \)), which contradicts (4.10). Thus, we have proved that \( \text{codim} \mathcal{R}(P_t) > 0 \).

Theorem 4.1 follows from (4.3) and from Lemmas 4.1 and 4.2.

Remark 4.1. Let \( I \) denote the identity operator on \( L^2(G) \), and let \( \lambda \in \mathbb{C} \). It is proved in [5] that low-order terms in elliptic equation have no influence on the index of the nonlocal operator \( P_t \). Therefore,

\[
\text{ind } (P_t - \lambda I) = \text{ind } P_t < 0
\]

for \( 0 < |t| \leq t_0 \), where \( t_0 > 0 \) is sufficiently small. Thus, the spectrum of \( P_t \) for \( 0 < |t| \leq t_0 \) coincides with the whole complex plane.

The author is grateful to A. L. Skubachevskii for attention.

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