A NEGATIVE MODE ABOUT EUCLIDEAN WORMHOLE

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Abstract

Wormholes – solutions to the euclidean Einstein equations with non-trivial topology – are usually assumed to make real contributions to amplitudes in quantum gravity. However, we find a negative mode among fluctuations about the Giddings-Strominger wormhole solution. Hence, the wormhole contribution to the euclidean functional integral is argued to be purely imaginary rather than real, which suggests the interpretation of the wormhole as describing the instability of a large universe against the emission of baby universes.

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Euclidean wormhole configurations (fig. 1) may make non-trivial contributions into the functional integral in quantum gravity. Originally it was suggested 1, 2, 3 that they may lead to loss of quantum coherence for a macroscopic observer. It was then argued 4, 5 (see also refs. 3, 7) that the effects of wormholes on long distance physics can be absorbed into the redefinition of $c$-number coupling constants of the low energy theory, so that quantum coherence is not lost. All this discussion depends crucially on whether the wormhole contributions into the functional integral are real or imaginary. In particular, the arguments of refs. 4, 5 rely upon the assumption that these contributions are purely real. On the other hand, imaginary wormhole contributions would imply an instability of the large universe with respect to the emission of baby universes, in accord with the picture of refs. 1, 2, 3. The latter case is realized in a model which implements the ideas of refs. 8, 9: parent and baby universes are modelled in (1 + 1) dimensions by macroscopic and microscopic strings 10, 11; the wormhole contributions (string loops) into the forward amplitudes are complex in this model, and it has been argued 10 that the emission of baby universes leads to the loss of quantum coherence in the parent (1 + 1)-dimensional universe.

Clearly, it is hardly possible to decide whether generic wormhole configurations in (3+1) dimensions make real or complex contributions into the functional integral. One may try, however, to approach this problem semiclassically in models which admit Euclidean wormhole solutions. By analogy to the analysis of instantons/bounces in quantum mechanics and field theory 12, 13, 14, the wormhole contribution will be imaginary if there exists a negative mode among fluctuations about the classical solution 3. The purpose of this paper is to show that there exists a negative mode about the simplest solution, the Giddings–Strominger 3 wormhole.

3The analogy to field theory suggests that the decay interpretation would require the existence of exactly one negative mode 17. It is not straightforward to see, though, how this requirement would emerge in quantum gravity context.
The model with the Giddings–Strominger wormhole contains space-time metrics and anti-symmetric tensor $H_{\mu\nu\lambda}$ as field variables. As far as the wormhole dynamics is concerned, the equivalent formulations \cite{16, 17, 18} are provided by using, instead of $H_{\mu\nu\lambda}$, either conserved current $J^\mu(x)$ or axion field $a(x)$,

$$J_\mu = \partial_\mu a$$

(1)

The formulation convenient for our purposes is one in terms of $J^\mu$. The Euclidean action is the sum of the action of pure gravity (the Hilbert–Einstein action with boundary terms\cite{6}) and

$$S_{\text{matter}} = f^2 \int d^4x \sqrt{g} g_{\mu\nu} J^\mu J^\nu$$

(2)

with the instruction that the functional integration is performed over conserved current densities (see ref. \cite{18} for details),

$$\partial_\mu (\sqrt{g} J^\mu) = 0$$

(3)

$f$ in eq. (2) is the coupling constant.

The wormhole solution \cite{3} is $O(4)$-symmetric. In this paper we consider only $O(4)$-symmetric fluctuations about this solution. The general $O(4)$-symmetric Euclidean metrics is

$$ds^2 = N^2(\tilde{\rho}) d\tilde{\rho}^2 + \tilde{R}^2(\tilde{\rho}) d\Omega^2$$

where $d\Omega^2$ is the metrics on a unit 3-sphere. The $O(4)$-symmetric current density has one non-zero component, $J^0(\tilde{\rho})$, and its conservation, eq. (3), means that $J^0 \sqrt{g}$ is a constant independent of $\tilde{\rho}$. This constant is related to the global charge $Q$ flowing through the wormhole, $Q = 2\pi^2 \tilde{R}^3 N J^0$. The action for the $O(4)$-symmetric fields is then

$$S = \frac{3\pi}{4} (M_{Pl} L)^2 \int d\rho \left( -\frac{R}{N} R^2 - N R + \frac{N}{R^3} \right)$$

(4)

$^4$The boundary terms will play minor role in what follows.
where prime denotes derivative with respect to $\rho$,

$$L^4 = \frac{2f^2Q^2}{3\pi^3M_{Pl}^2}$$

is a fixed parameter for given wormhole type (i.e., for given $Q$), and we scaled this parameter out by introducing variables

$$R = \frac{\bar{R}}{L}, \quad \rho = \frac{\bar{\rho}}{L}$$

One may hope that the semiclassical analysis is relevant at $L \gg M_{Pl}^{-1}$.

The metrics of the wormhole solution [3] has $N_c = 1$, while $R_c(\rho)$ obeys the only non-trivial equation following from the action (4),

$$R_c'' = 1 - \frac{1}{R_c^4} \quad (5)$$

The metrics becomes flat at large $|\rho|$ (i.e., $R_c(\rho) \to |\rho|$ as $|\rho| \to \infty$). The origin of the coordinate $\rho$ can be chosen in such a way that $\rho = 0$ corresponds to the minimum size of the wormhole (turning point, $R'_c = 0$); this size is equal to $R_c(\rho = 0) = L$.

Let us now consider $O(4)$-symmetric fluctuations about the wormhole solution, i.e., set $R(\rho) = R_c(\rho) + r(\rho)$, $N(\rho) = 1 + n(\rho)$ and evaluate the quadratic in $(r, n)$ part of the action (4). This quadratic action is invariant under the gauge transformations ($O(4)$-symmetric general coordinate transformations) $n \to (n + \xi')$, $r \to (r + R_c'\xi)$, where $\xi(\rho)$ is the gauge function. Non-gauge modes can be chosen to satisfy $n(\rho) = 0$, while $r(\rho)$ is not subject to any constraint (for the discussion of admissible gauges in a similar context see ref. [19]). This gauge will be chosen in what follows. Before writing down the quadratic action for $r(\rho)$, we notice that this action is unbounded from below, because of the negative sign of the derivative term. This is the usual problem with fluctuations of the scale factor, and it is cured by performing the rotation $r \to i r$ (for further discussion of the rotation of the conformal factor see refs. [20, 22, 23]). After making this rotation one obtains

$$S^{(2)}[r] = \frac{3\pi}{4} (M_{Pl}L)^2 \int d\rho \left( R_c r'^2 - \frac{8}{R_c^5} r^2 \right) + \text{boundary terms} \quad (6)$$
Let us see that this action has one negative mode.

The argument is the usual one [11]. There exists a zero mode, \( r^{(0)} = R'_c \), which is the translational zero mode remaining after gauge fixing (it corresponds to the gauge parameter \( \xi \) independent of \( \rho \)). This function has a node, so there exists a negative energy ground state of the corresponding Schrödinger operator, which is the negative mode of the action (3). The fact that the determinant of the \( O(4) \)-symmetric fluctuation about the wormhole solution is imaginary, can be understood also on more general grounds within Maslov’s theory [24, 25]. However, one might worry that the zero mode does not vanish at large \( |\rho| \) because \( R_c(\rho) \to |\rho| \) (though this peculiarity can be dealt with by an appropriate choice of the integration measure for the corresponding quantum mechanical problem). One might also wonder whether the boundary terms in eq. (3) play any role. So, we present here the explicit check that there exists exactly one \( O(4) \)-symmetric negative mode, and find its form.

Let us consider the eigenvalue equation for fluctuations, which diagonalize the action (3),

\[
R_c \left[ - (R_c r')' - \frac{8}{R'_c^2} r \right] = \omega^2 r
\]  

(7)

The overall \( \rho \)-dependent factor multiplying the left hand side is arbitrary, and we have chosen it in such a way that eq. (7) can be solved explicitly (this is the same trick that works nicely in the calculation of determinants about the Yang–Mills instanton [24]). We are interested in negative \( \omega^2 \). Upon introducing a new variable \( y = R^{-4}_c(\rho) \) instead of \( \rho \), and writing \( r = R_c^{-\omega} \psi(y) \), one rewrites eq. (3) as follows,

\[
y(1 - y) \frac{d^2 \psi}{dy^2} + \left[ \left( 1 + \frac{\omega}{2} \right) - \left( \frac{3 + |\omega|}{2} \right) y \right] \frac{d\psi}{dy} + \left( \frac{1}{2} - \frac{|\omega|}{8} + \frac{\omega^2}{16} y \right) \psi = 0
\]

(8)

The variables \( y \) and \( \rho \) are not in one to one correspondence. This can be dealt with by requiring that \( r(\rho) \) is either symmetric or anti-symmetric in \( \rho \). As \( (R_c(\rho) - 1) \) is symmetric, the latter requirement means that \( \psi(y) \) is either a series in \((1 - y)\) (symmetric eigenfunctions of eq. (7)) or a series in odd powers of \( \sqrt{1 - y} \) (anti-symmetric
eigenfunctions) at small $(1 - y)$. We also impose the condition that the eigenfunctions $r(\rho)$ are square integrable with the weight $d\rho/R_c(\rho)$, which is appropriate for the choice of the pre-factor made in eq. (7). In terms of $\psi(y)$, this means square integrability with the weight $y^{\omega^2/2}dy/(y\sqrt{1-y})$. In fact, the precise conditions at $|\rho| \to \infty$ are not very important; it is sufficient to require that $r$ is finite.

Equation (8) is the hypergeometric equation. It is straightforward to see that there exists exactly one eigenfunction obeying above conditions, which is

$$\psi = \text{const} \quad \text{with} \quad \omega^2 = -4$$

Other eigenfunctions of eq. (7) with negative $\omega^2$ grow either as $\rho \to \infty$ or $\rho \to -\infty$. Thus, the only negative mode has the form

$$r^{(-)}(\rho) = \frac{1}{R_c^2(\rho)}$$

Making use of this expression, one can check that the boundary terms in eq. (8) (which are proportional to $R_c r r' r'$ or $R_c' r^2$) vanish.

It is worth pointing out that the irrelevance of the boundary terms in the gravitational action is the property of our gauge $N(\rho) = 1$. In other gauges the boundary terms may not vanish, and may even determine the sign of the quadratic action for fluctuations about the wormhole solution. The latter case is realized, for example, in conformal gauge, $N(\rho) = R(\rho)$. One can check that, when boundary terms are taken into account, the determinant of $O(4)$-symmetric fluctuations is imaginary in any gauge.

We conclude this paper by adding a few remarks.

i) The existence of the negative mode implies that the wormhole contribution into the functional integral is imaginary, which corresponds to the instability of the parent universe against the emission of baby universes. This fits nicely to the observation [27] that the analytical continuation in $\rho$ describes a baby universe evolving, after
its birth, in its intrinsic (real) time towards the singularity $R = 0$. The analogy to $(1 + 1)$-dimensional model of refs. [10, 11] is obvious.

ii) The wormhole contribution into the functional integral apparently has wrong dependence on the spatial volume $V$ of the parent universe and the normalization time $T$: the integration over the positions of the two ends of the wormhole ($x$ and $y$ in fig. 1) results in the factor $(VT)^2$. We think this is an infrared effect inherent in theories with Goldstone bosons. In the limit of small wormhole size, the contribution of the wormhole of global charge $Q$ into the vacuum–vacuum amplitude can be summarized as follows (cf. [13]),

$$\int dx \, dy \, A_Q < e^{iQa(x)}e^{-iQa(y)}>$$

where $A_Q$ is a purely imaginary factor (exponentially suppressed at large $Q$ by the wormhole action). This integral is indeed proportional to $(VT)^2$, which is the reflection of the existence of (almost) zero energy intermediate states with charge $Q$. In massive theories these states will be absent, and the wormhole contribution will be proportional to the usual factor $(VT)$.

iii) In our analysis of the $O(4)$-symmetric fluctuations about the wormhole solution, an important ingredient was the Gibbons–Hawking–Perry rotation, $r \rightarrow ir$. Although this prescription works well in other cases of tunneling in quantum gravity, an independent check that the wormhole contribution is indeed imaginary, is desirable. A promising formalism in this regard is the Wheeler–De Witt wave function approach. Also, the analysis of $O(4)$-asymmetric fluctuations about the wormhole is necessary. We hope to clarify these points in future publications.

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