EXTINCTION IN STOCHASTIC PREDATOR-PREY POPULATION MODEL WITH ALLEE EFFECT ON PREY

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(Communicated by Xiaoying Han)

Abstract. This paper presents the analysis of the conditions which lead the stochastic predator-prey model with Allee effect on prey population to extinction. In order to find these conditions we first prove the existence and uniqueness of global positive solution of considered model using the comparison theorem for stochastic differential equations. Then, we establish the conditions under which extinction of predator and prey populations occur. We also find the conditions for parameters of the model under which the solution of the system is globally attractive in mean. Finally, the numerical illustration with real life example is carried out to confirm our theoretical results.

1. Introduction. Predator-prey interaction is one of the most extensively studied issue both in ecological and mathematical literature. Mathematical models represent a result of a desire to somehow predict the behavior of such systems. The classic predator-prey models are mostly variations of Lotka–Volterra model, which was proposed by Alfred Lotka [17] and Vito Volterra [21] in the 1920s. It is well known that a good model must be simple enough to be mathematically tractable, but complex enough to represent a system realistically. Realism is often sacrificed for simplicity, and one of the defects of the Lotka–Volterra model is its reliance on unrealistic assumptions. For example, the model ignores that prey population is also limited by food resources and not just by predation, and that no predator can consume infinite quantities of prey. In general, these features are better fit by models incorporating terms that represent carrying capacity (the maximum population size that a given environment can support) for the prey population, realistic functional responses (how a predator’s consumption rate changes as prey densities change) for the predator population, and complexity in the environment. Among the many processes that the Lotka–Volterra model ignores, the Allee effect may be the most important. The Allee effect was first proposed by Allee [2] as a result of his

2010 Mathematics Subject Classification. Primary: 60H10, 92D25; Secondary: 92D40.
Key words and phrases. Allee effect, extinction, global attractivity in mean, predator, prey.
The authors were supported by the Grant No 174007 of MNTRS.
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study of a relationship between capability of population to survive in the nature and its size. The Allee effect has been the subject of increased interest in the ecological literature, especially in the last two decades. It increases the likelihood of local and global extinction, and, hence, it is very important for conservation biology. Allee effect also plays a significant role in ecosystem invasions and, therefore, in biological control. In mathematical literature, many population models with the Allee effect have been considered (see \[1, 4, 5, 10, 14, 16, 25, 27, 28\], for instance).

Since population systems are often subjected to environmental noise, it is useful to see how changes in environment affect the relationship between predator and prey populations. There are many authors who investigate this phenomena (see \[1, 8, 9, 10, 14, 16, 18, 25\], for example). Among these papers, in \[1, 10, 14, 16, 25\] authors consider stochastic population models with Allee effect and investigate stability properties of their equilibrium states.

In this paper our goal is to investigate conditions under which stochastic Rosenzweig–MacArthur predator-prey model with Allee effect on prey population goes to extinction. For that purpose, let us briefly outline the deterministic analogue which is considered in \[28\]

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t) \left[ \frac{bx(t)}{A_1 + x(t)} - d_1 - \alpha x(t) - \frac{sy(t)}{1 + sh_1x(t)} \right], \\
\frac{dy(t)}{dt} &= y(t) \left[ \frac{c_1sx(t)}{1 + sh_1x(t)} - d_2 \right],
\end{align*}
\]

with initial value \(x(0) = x_0, y(0) = y_0\), where \(x(t)\) and \(y(t)\) represent size of prey and predator population at time \(t\), respectively. Moreover, the parameters of the model are:

- \(b\) - per capita maximum reproduction rate of prey population,
- \(d_i\) \((i = 1, 2)\) - per capita death rates of prey and predators respectively,
- \(\alpha\) - strength of intra-competition of prey population,
- \(s\) - effective search rate,
- \(h_1\) - handling time of predators,
- \(c_1\) - conversion efficiency of ingested prey into new predators.

Term \(\frac{sx(t)}{1 + sh_1x(t)}\) represents the predators functional response, i.e. the relationship between prey density \(x\), and the amount ingested by an average predator. All the parameters are supposed to be positive constants. Furthermore, since the difficulty in finding a mate may affect the population size and cause the Allee effect, there is a certain critical population size below which it is very difficult to find a mate, and, hence, the reproduction can not compensate mortality. Constant \(A_1\) represents Allee effect constant of the prey population. The bigger \(A_1\) is, the stronger Allee effect will be.

Extinction is one of the most important terms in population dynamics. A species is said to be extinct when the last existing member dies. Therefore, extinction becomes a certainty when there are no surviving individuals that can reproduce and create a new generation. In ecology, extinction is often used informally to refer to local extinction, in which a species ceases to exist in the chosen area of study, but may still exist elsewhere. This phenomenon is also known as extirpation. Local extinctions may be followed by a replacement of the species taken from other locations. Species which are not extinct are called extant. Those that are extant but threatened by extinction are referred to as endangered species. There are many environmental groups and governments in the world that try to preserve critically
endangered species. Currently, these organizations are concerned with the extinction of species caused by humanity, and they try to prevent further extinctions through a variety of conservation programs. Humans can cause extinction of a species through overharvesting, pollution, habitat destruction, introduction of invasive species (such as new predators and food competitors), overhunting, and other influences. Explosive, unsustainable human population growth is an essential cause of the extinction crisis. Also, there are variety of causes that can contribute directly or indirectly to the extinction of species or group of species, such as lack of food and space or toxic pollution of entire population habitat, competition for food to better adapted competitors, predation, etc.

Due to importance of this topic in population dynamics, our main goal in this paper is to investigate extinction in the considered model and, therefore, we have analyzed the effect of the noise perturbation impressed on the death rates. Thus, we perturb death rates \( d_1 \) of prey and \( d_2 \) of predator population in model (1) with Gaussian white noises and obtain the stochastic Rosenzweig–MacArthur predator–prey model with Allee effect on prey population

\[
\begin{align*}
    dx(t) &= x(t) \left[ \frac{bx(t)}{A_1 + x(t)} - d_1 - \alpha x(t) - \frac{sy(t)}{1 + sh_1 x(t)} \right] dt - \sigma_1 x(t) dw_1(t), \\
    dy(t) &= y(t) \left[ \frac{c_1 sx(t)}{1 + sh_1 x(t)} - d_2 \right] dt - \sigma_2 y(t) dw_2(t),
\end{align*}
\]

with initial value \( x(0) = x_0, y(0) = y_0 \), while \( w = \{ w_1(t), w_2(t), t \geq 0 \} \) represents the two-dimensional standard Brownian motion defined on a compete probability space \((\Omega, \mathcal{F}, P)\) with a filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) satisfying the usual conditions (it is right continuous and increasing, while \( \mathcal{F}_0 \) contains all \( P \)-null sets), and \( \sigma_1^2 \) and \( \sigma_2^2 \) represent the intensities of noises \( w_1(t) \) and \( w_2(t) \), respectively.

The paper is organized as follows: In the next section we show that system (2) has a unique positive global solution. In Section 3 we find the conditions for model parameters under which populations of prey and predator die out. In Section 4 we prove that the trivial solution of system (2) is globally attractive in mean. In Section 5 we consider interaction between wolf and moose populations on Isle Royale to confirm our theoretical results. Finally, in Section 6 we give summary of the results obtained in this paper and their biological interpretation.

2. Existence and uniqueness of the positive solution. Since \( x(t) \) and \( y(t) \) represent size of the prey and predator population at time \( t \), respectively, we are interested only in positive solutions of system (2). In order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of equation are generally required to satisfy the linear growth condition and local Lipschitz condition (see [19]). However, coefficients of system (2) satisfy local Lipschitz condition, but they do not satisfy linear growth condition, and hence, the solution of system (2) may explode in a finite time.

In this section, by using change of variables and comparison theorem of stochastic differential equations [7], we show that system (2) has unique positive global solution with positive initial value.

**Theorem 2.1.** For any initial value \((x_0, y_0) \in \mathbb{R}_+^2\), system (2) has unique global positive solution \((x(t), y(t))\) for \( t \geq 0 \).
\[ du(t) = \left[ \frac{b e^{u(t)}}{A_1 + e^{u(t)}} - d_1 - \alpha e^{u(t)} - \frac{se^{v(t)}}{1 + sh_1 e^{u(t)}} - \frac{\sigma_1^2}{2} \right] dt - \sigma_1 dw_1(t), \]
\[ dv(t) = \left[ \frac{c_1 s e^{u(t)}}{1 + sh_1 e^{u(t)}} - d_2 - \frac{\sigma_2^2}{2} \right] dt - \sigma_2 dw_2(t), \]  
with initial value \( u(0) = \ln x_0, v(0) = \ln y_0 \). Obviously, the coefficients of system (3) satisfy local Lipschitz condition, and, hence, there is a unique local solution \((u(t), v(t))\) of system (3) for \( t \in [0, \tau_e) \). Therefore, by using Itô formula, it follows that \((x(t), y(t)) = (e^{u(t)}, e^{v(t)})\) is the unique positive local solution of system (2) with the initial value \((x_0, y_0)\) for \( t \in [0, \tau_e) \).

Next step is to show that this solution is global, i.e. that \( \tau_e = \infty \).

Since the solution is positive on \([0, \tau_e)\), it follows that
\[ dx(t) \leq x(t) \left[ -\alpha x(t) + b - d_1 \right] dt - \sigma_1 x(t) dw_1(t). \]

Let \( \Phi(t) \) be the unique solution of the equation
\[ d\Phi(t) = \Phi(t) \left[ -\alpha \Phi(t) + b - d_1 \right] dt - \sigma_1 \Phi(t) dw_1(t), \]
\[ \Phi(0) = x_0, \]  
which has solution (see [13])
\[ \Phi(t) = \frac{e^{(b-d_1-\frac{\sigma_1^2}{2})t} - \sigma_1 w_1(t)}{\frac{1}{\alpha} + \int_0^t e^{(b-d_1-\frac{\sigma_1^2}{2})z - \sigma_1 w_1(z)} \, dz}. \]  
Then, the comparison theorem for stochastic differential equations yields \( x(t) \leq \Phi(t), \) \( t \in [0, \tau_e), \) a.s.

By the same reasoning, for the predator population \( y(t) \) we obtain
\[ dy(t) \leq y(t) \left( \frac{c_1}{h_1} - d_2 \right) dt - \sigma_2 y(t) dw_2(t), \]
and, hence \( \Psi(t) \) defined by
\[ \Psi(t) = y_0 e^{(\frac{c_1}{h_1} - d_2 - \frac{\sigma_2^2}{2})t - \sigma_2 w_2(t)} \]  
represents the solution of the linear stochastic differential equation
\[ d\Psi(t) = \Psi(t) \left( \frac{c_1}{h_1} - d_2 \right) dt - \sigma_2 \Psi(t) dw_2(t), \]
\[ \Psi(0) = y_0. \]

Also, for the predator population, it holds
\[ dy(t) \geq -d_2 y(t) dt - \sigma_2 y(t) dw_2(t). \]

Since \( \psi(t) \) defined by
\[ \psi(t) = y_0 e^{-\left( d_2 + \frac{\sigma_2^2}{2} \right)t - \sigma_2 w_2(t)} \]  
represents the solution of the linear stochastic differential equation
\[ d\psi(t) = \psi(t) \left( -d_2 \right) dt - \sigma_2 \psi(t) dw_2(t), \]  
\[ \psi(0) = y_0, \]  
we have
\[ \psi(t) \leq \frac{1}{\sigma_2} \ln \frac{y_0}{y(t)}. \]
represents the solution of the linear stochastic differential equation
\[ d\psi(t) = -d_2\psi(t)dt - \sigma_2\psi(t)dw_2(t), \]
\[ \psi(0) = y_0, \]
on the basis of comparison theorem for stochastic differential equations, we have
\[ \psi(t) \leq y(t) \leq \Psi(t) \text{ a.s for } t \in [0, \tau_e). \] (8)

On the other hand, for the prey population \( x(t) \) we have
\[ dx(t) \geq x(t)[-d_1 - \alpha x(t) - s \Psi(t)] dt - \sigma_1 x(t)dw_1(t). \]
The stochastic differential equation
\[ d\phi(t) = -\phi(t)[\alpha \phi(t) + d_1 + s \Psi(t)] dt - \sigma_1 \phi(t)dw_1(t) \]
with the initial value \( \phi(0) = x_0 \), has the solution
\[ \phi(t) = \frac{e^{-\left(\frac{d_1+\frac{\sigma_1^2}{2}}{\alpha x_0}\right)t} \int_0^t \Psi(z)dz - \sigma_1 w_1(t)}{1 + \alpha \int_0^t e^{-\left(\frac{d_1+\frac{\sigma_1^2}{2}}{z}\right)s - \sigma_1 w_1(z)} dz}. \] (9)

Finally, by the comparison theorem of stochastic differential equations we have
for prey population that
\[ \phi(t) \leq x(t) \leq \Phi(t) \text{ a.s for } t \in [0, \tau_e). \]

Since \( \phi(t), \Phi(t), \psi(t) \) and \( \Psi(t) \) exist for every \( t \geq 0 \), it follows that \( \tau_e = \infty \), which proves the theorem.

3. Extinction. Let us first give the definition of local extinction of a population.

**Definition 3.1.** The population \( x(t) \) is said to be extinct with probability 1 if
\[ \lim_{t \to \infty} x(t) = 0 \text{ a.s.} \]

In the sequel we investigate how Allee effect affects extermination of prey population and then the predator population.

**Theorem 3.2.** Let \((x(t), y(t))\) be the solution of system \( \text{[4]} \) for any initial value \((x_0, y_0) \in \mathbb{R}^2_+\). If parameters of the system satisfy any of following conditions:
1° \( d_1 > b \) and \( \alpha A_1 \leq \frac{d_1-b}{2}; \)
2° \( d_1 \leq b \) and
   a. \( \alpha A_1 \geq b - d_1 \), or
   b. \( \alpha A_1 < b - d_1 \) and \((b-d_1-\alpha A_1)^2(b-d_1+8\alpha A_1)-27d_1(\alpha A_1)^2 < 0; \)
3° \( \alpha A_1 \leq b \) and \( \sigma_1^2 > 2 \left( \sqrt{b - \sqrt{\alpha A_1}} - d_1 \right) \),
then
\[ \lim_{t \to \infty} x(t) = 0, \quad \text{and} \quad \lim_{t \to \infty} y(t) = 0 \text{ a.s.}, \]
i.e. prey and predator population become extinct exponentially with probability one.

**Proof.** Recall that from \( \text{[4]} \) we have
\[ dx(t) \leq x(t) \left[ \frac{bx(t)}{A_1+x(t)}d_1-\alpha x(t) \right] dt - \sigma_1 x(t)dw_1(t). \]
Let us consider function \( f(x) = x \left( \frac{b}{A_1 + x} - d_1 - \alpha x \right) \). We are interested in behavior of this function, so we compute it’s first derivative and get

\[
f'(x) = -2\alpha x^3 + (b - d_1 - 4\alpha A_1) x^2 + 2A_1 (b - d_1 - \alpha A_1) x - d_1 A_1^2 \quad \frac{1}{(A_1 + x)^2}.
\]

Since we want to investigate sign of first derivative, let us denote

\[
g(x) = -2\alpha x^3 + (b - d_1 - 4\alpha A_1) x^2 + 2A_1 (b - d_1 - \alpha A_1) x - d_1 A_1^2.
\]

Computing

\[
g'(x) = -6\alpha x^2 + 2(b - d_1 - 4\alpha A_1) x + 2A_1 (b - d_1 - \alpha A_1),
\]

we obtain that solutions of quadratic equation \( g'(x) = 0 \) are

\[
x_{1,2} = \frac{-2(b - d_1 - 4\alpha A_1) \pm 2|b - d_1 + 2\alpha A_1|}{-12\alpha}.
\]

We differ two cases:

1° If \( \alpha A_1 \leq \frac{b - d_1}{2\alpha} \) both solutions are negative, which means that function \( g \) is decreasing on \([0, \infty)\), so it’s maximum value is \( g(0) = -d_1 A_1^2 \). Thus, \( f'(x) \leq -\frac{d_1 A_1^2}{(A_1 + x)^2} < 0 \).

So, function \( f \) is also decreasing on \([0, \infty)\) and it’s maximum value is \( f(0) = 0 \). Now, we have

\[
dx(t) \leq -\sigma_1 x(t) dw_1(t),
\]

and by comparison theorem for stochastic differential equations we obtain

\[
x(t) \leq x_0 e^{-\frac{1}{2}\sigma_1^2 t - \sigma_1 w_1(t)},
\]

which implies that

\[
\ln x(t) \leq \ln x_0 - \frac{1}{2} \sigma_1^2 t - \sigma_1 w_1(t),
\]

and, finally after applying the strong law of large numbers for local martingales we get

\[
\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leq \lim_{t \to \infty} \left[ \frac{\ln x_0}{t} - \frac{1}{2} \sigma_1^2 - \sigma_1 \frac{w_1(t)}{t} \right] = -\frac{1}{2} \sigma_1^2 < 0,
\]

i.e.

\[
\lim_{t \to \infty} x(t) = 0 \quad \text{a.s.}
\]

2° If \( d_1 \leq b \) then \( \alpha A_1 > \frac{d_1 - b}{2\alpha} \), and we have one negative solution \( x_1 = -A_1 \) and second solution is \( x_2 = \frac{b - d_1 - \alpha A_1}{3\alpha} \) which can be either positive or negative, depending on the sign of \( b - d_1 - \alpha A_1 \).

2a. If \( \alpha A_1 \geq b - d_1 \), solution \( x_2 \) is also negative, and by the same reasoning as in the proof of case 1° we get desired assertion.

2°b. If \( \alpha A_1 < b - d_1 \), solution \( x_2 \) is positive and function \( g \) reaches it’s maximum value in point \( x_2 \), where

\[
g(x_2) = \left( \frac{b - d_1 - \alpha A_1}{2\alpha} \right)^2 \left( \frac{b - d_1 + 8\alpha A_1}{2\alpha} - 27d_1 (\alpha A_1)^2 \right).
\]

Bearing in mind condition of the theorem, \( g(x_2) < 0 \), which implies that \( f \) is decreasing function on \([0, \infty)\) and it’s maximum value is \( f(0) = 0 \). Repeating procedure from part 1° of this proof we get

\[
\lim_{t \to \infty} x(t) = 0 \quad \text{a.s.}
\]
3° On the other hand, by applying Itô formula to $\ln x(t)$, we have

$$d \ln x(t) \leq \left( \frac{bx(t)}{A_1 + x(t)} - \alpha x(t) - d_1 - \frac{\sigma_1^2}{2} \right) dt - \sigma_1 dw_1(t).$$

Function $f(x) = \frac{bx}{A_1 + x} - \alpha x - d_1 - \frac{\sigma_1^2}{2}$ has maximum value for $x = \sqrt{\frac{bA_1}{\alpha}} - A_1$, which is nonnegative, due to condition $\alpha A_1 \leq b$. Maximum value of function $f$ is $C = \left( \sqrt{b} - \sqrt{\alpha A_1} \right)^2 - d_1 - \frac{\sigma_1^2}{2}$. Therefore,

$$d \ln x(t) \leq C dt - \sigma_1 dw_1(t),$$

i.e.

$$\ln x(t) \leq \ln x_0 + Ct - \sigma_1 w_1(t),$$

and, finally, using the condition of the theorem and applying the strong law of large numbers for local martingales we get

$$\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leq \limsup_{t \to \infty} \frac{\ln x_0}{t} + \frac{Ct}{t} - \frac{\sigma_1 w_1(t)}{t} = C < 0,$$

i.e.

$$\lim_{t \to \infty} x(t) = 0 \quad \text{a.s.}$$

Therefore, for every $\varepsilon > 0$ there exist $t_0$ and set $\Omega_\varepsilon$ such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$ and $\frac{\ln x(t)}{t} \leq \frac{Ct}{t} - \frac{\sigma_1 w_1(t)}{t} = C < 0$, for every $t \geq t_0$ and $\omega \in \Omega_\varepsilon$. Using that fact, we get

$$d \ln y(t) = \left( \frac{c_1 sx(t)}{1 + sh_1 x(t)} - d_2 - \frac{\sigma_2^2}{2} \right) dt - \sigma_2 dw_2(t),$$

i.e.

$$\ln y(t) = \ln y_0 + \int_0^t \left( \frac{c_1 sx(s)}{1 + sh_1 x(s)} - d_2 - \frac{\sigma_2^2}{2} \right) ds - \sigma_2 \int_0^t dw_2(s) \leq \ln y_0 + \left( \varepsilon - d_2 - \frac{\sigma_2^2}{2} \right) t - \sigma_2 w_2(t).$$

This yields

$$\frac{\ln y(t)}{t} \leq \ln \frac{y_0}{t} + \varepsilon - d_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2 w_2(t)}{t}.$$ 

Bearing in mind that $\varepsilon$ is arbitrary, and using the strong law of large numbers for local martingales, we obtain

$$\limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq \varepsilon - d_2 - \frac{\sigma_2^2}{2} < 0,$$

which means that

$$\lim_{t \to \infty} y(t) = 0 \quad \text{a.s.}$$

From the previous theorem we can conclude that extinction in prey population occurs in several cases. The first one is when the death rate of prey population is greater than it’s maximum reproduction rate. Intuitively, this is expected condition because of the relation between reproduction rate and death rate. On the other hand, even if the reproduction rate is greater than death rate, this does not guarantee that population can survive in nature since strong Allee effect can drive population to extinction. Finally, the third reason that can cause extinction in prey population is great intensity of the noise, which means that unpredictable events
in nature can be fatal for the prey population. Extinction of prey population will drive predator population to extinction, too.

Let us now find out how does the intensity of noise affect population of predators. More precisely, by the next theorem we will prove that if intensity of the noise is large enough, then predator population becomes extinct regardless of the size of prey population.

**Theorem 3.3.** Let \((x(t), y(t))\) be the solution of system (2) for any initial value \((x_0, y_0) \in \mathbb{R}^2_+\). If \(\sigma_2^2 > 2 \left( \frac{c_1}{h_1} - d_2 \right)\), then

\[
\lim_{t \to \infty} y(t) = 0, \quad \text{a.s.}
\]

i.e. population \(y(t)\) becomes extinct exponentially with probability one.

**Proof.** From proof of Theorem 2.1, we have that

\[
y(t) \leq y_0 e^{\left( \frac{c_1}{h_1} - d_2 - \sigma_2^2 \sigma_2^2 \right) t - \sigma_2^2 \sigma_2^2},
\]

for every \(t \geq 0\). Hence,

\[
\ln y(t) \leq \ln y_0 + \left( \frac{c_1}{h_1} - d_2 - \frac{\sigma_2^2}{2} \right) t - \sigma_2^2 \sigma_2^2,
\]

i.e.

\[
\frac{\ln y(t)}{t} \leq \frac{\ln y_0}{t} + \frac{c_1}{h_1} - d_2 - \frac{\sigma_2^2}{2} \frac{\sigma_2^2}{t}.
\]

Application of the strong law of large numbers for local martingales and condition of the theorem yields

\[
\limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq \frac{c_1}{h_1} - d_2 - \frac{\sigma_2^2}{2} < 0, \quad \text{a.s.}
\]

which completes the proof. \(\blacksquare\)

4. **Global attractivity.** In the sequel we establish sufficient conditions under which we can claim that the solution of system (2) is globally attractive in mean. In order to do that let us first prove the following theorem.

**Theorem 4.1.** Let \((x(t), y(t))\) be the solution of system (2) for any initial value \((x_0, y_0) \in \mathbb{R}^2_+\). Then, for \(p > 1\),

\[
\mathbb{E} x^p(t) \leq K_1(p),
\]

where

\[
K_1(p) = \max \left\{ \left( \frac{\sigma_1^2}{\sigma_2^2} \right)^p \left( \frac{1 + b + \frac{\sigma_1^2 - d_1}{p+1}}{p+1} \right)^{p+1} x_0^p \right\},
\]

if

\[
\sigma_1^2 \geq \frac{2 \left( d_1 - b - \frac{1}{p} \right)}{p - 1},
\]

and

\[
\mathbb{E} x^p(t) \leq x_0^p
\]

if condition (10) is not satisfied. If conditions

\[
d_2 > \frac{c_1}{h_1} \quad \text{and} \quad \sigma_2^2 \leq \frac{2 \left( d_2 - \frac{\sigma_2^2}{h_1} \right)}{p - 1}
\]

(11)
Hence, that

\[ Ey^p(t) \leq y_0^p. \]

**Proof.** In order to show the theorem, let us define the function \( V(t, x) = e^t x^p \) for \( p > 1 \). By Itô formula we obtain

\[
d(V(t, x(t))) = p e^t x^p(t) \left[ \frac{1}{p} + \frac{bx(t)}{A_1 + x(t)} - \alpha x(t) - d_1 - \frac{sy(t)}{1 + s h_1 x(t)} + p - \frac{1}{2} \sigma_1^2 \right] dt
- p \sigma_1 e^t x^p(t) dw_1(t).
\]

Hence,

\[
E[e^t x^p(t)]
= x_0^p + p \int_0^t e^s E \left\{ x^p(s) \left[ \frac{1}{p} + \frac{bx(s)}{A_1 + x(s)} - \alpha x(s) - d_1 - \frac{sy(s)}{1 + s h_1 x(s)} + p - \frac{1}{2} \sigma_1^2 \right] \right\} ds
\leq x_0^p + p \int_0^t e^s E \left\{ x^p(s) \left[ \frac{1}{p} - \alpha x(s) + b - d_1 + p - \frac{1}{2} \sigma_1^2 \right] \right\} ds.
\]

Function \( f(x) = x^p \left[ -\alpha x + b + \frac{1}{p} - d_1 + \frac{p - 1}{2} \sigma_1^2 \right] \) reaches its maximum value for \( x = \frac{p(b - d_1 + \frac{1}{2} \sigma_1^2)}{\alpha p (p+1)} \) which is positive due to condition (10). Thus,

\[
E[e^t x^p(t)] \leq x_0^p + K (e^t - 1),
\]

where \( K = \left( \frac{p}{\alpha} \right)^p \left( \frac{\frac{1}{p} + b + \frac{p - 1}{2} \sigma_1^2 - d_1}{p+1} \right)^{p+1} \). Then,

\[
E x^p(t) \leq (x_0^p - K) e^{-t} + K.
\]

If condition (10) is not satisfied, then by applying Itô formula to \( x^p(t) \), one can see that

\[
E x^p(t) \leq x_0^p + p \int_0^t E \left\{ x^p(s) \left[ -\alpha x(s) + b - d_1 + p - \frac{1}{2} \sigma_1^2 \right] \right\} ds
\]

and we get \( E x^p(t) \leq x_0^p \). Finally, we obtain \( E[x^p(t)] \leq K_1(p) \).

On the other hand, from Theorem 2.1 we conclude that

\[
Ey^p(t) \leq E\Psi^p(t)
= y_0^p E e^{p \left( \frac{1}{p} - d_2 - \frac{\sigma_2^2}{2} \right) t - \sigma_2 w_2(t)}
= y_0^p e^{p \left( \frac{1}{p} - d_2 - \frac{\sigma_2^2}{2} \right) t + \frac{p^2 \sigma_2^2}{2} t}
= y_0^p e^{p \left( \frac{1}{p} - d_2 + \frac{p^2}{2} \sigma_2^2 \right) t}.
\]

Bearing in mind condition (11), we get \( Ey^p(t) \leq y_0^p \).

In the sequel we show that the positive solution of system 2 is uniformly Hölder-continuous. For that purpose, we give following lemma (see [11, 20]).

**Lemma 4.2.** Suppose that an \( n \)-dimensional stochastic process \( \{x_t, t \geq 0\} \) satisfies the condition

\[
E|x_t - x_s|^\alpha \leq C |t - s|^{1+\beta}, \quad 0 < s, t < \infty,
\]

then

\[
E x^p(t) \leq \tilde{K} e^{\tilde{r} t},
\]

where \( \tilde{K} \) and \( \tilde{r} \) are positive constants.
for some positive constants \(\alpha, \beta\) and \(C\). Then, there exists a continuous modification \(\tilde{x}_t\) of \(x_t\), which has the property that for every \(\gamma \in \left(0, \frac{2}{n}\right)\), there is a positive random variable \(h(\omega)\) such that

\[
P \left\{ \omega : \sup_{0 < |t-s| < h(\omega); 0 \leq s, t < \infty} \frac{|\tilde{x}_t - \tilde{x}_s|}{|t-s|^{\gamma}} \leq \frac{2}{1 - 2^{-\gamma}} \right\} = 1.
\]

In other words, almost every sample path of \(\tilde{x}_t\) is locally but uniformly Hölder-continuous with exponent \(\gamma\).

**Theorem 4.3.** Let \((x(t), y(t))\) be the solution of system (2) for any initial value \((x_0, y_0) \in \mathbb{R}^2_+\). Then, almost every sample path of \((x(t), y(t))\) is uniformly continuous on \(t \geq 0\) if conditions of Theorem 4.1 hold.

**Proof.** Let us first rewrite the first equation of system (2) in its integral form, that is,

\[
x(t) = x_0 + \int_0^t f_1(s, x(s), y(s)) \, ds + \int_0^t g_1(s, x(s), y(s)) \, dw_1(s),
\]

where

\[
f_1(s, x(s), y(s)) = x(s) \left[ \frac{b x(s)}{A_1 + x(s)} - \alpha x(s) - d_1 - \frac{s y(s)}{1 + s h_1 x(s)} \right],
\]

\[
g_1(s, x(s), y(s)) = -\sigma_1 x(s).
\]

From Theorem 4.1 we estimate

\[
E |f_1(s, x(s), y(s))|^p \leq \frac{1}{2} E |x(s)|^{2p} + \frac{1}{2} E \left[ \frac{b x(s)}{A_1 + x(s)} - \alpha x(s) - d_1 - \frac{s y(s)}{1 + s h_1 x(s)} \right]^{2p}
\]

\[
\leq \frac{1}{2} E x^{2p}(s) + \frac{1}{2} E \left[ -\alpha x(s) + b - d_1 - \frac{s y(s)}{1 + s h_1 x(s)} \right]^{2p}
\]

\[
\leq \frac{1}{2} E x^{2p}(s) + \frac{4^{p-1}-1}{2} \left[ \alpha^2 p E x^{2p}(s) + b^{2p} + d_1^{2p} + s^{2p} E y^{2p}(s) \right]
\]

\[
= \left[ \frac{1}{2} + \frac{4^{p-1}-1}{2} \alpha^2 p \right] E x^{2p}(s) + \frac{4^{p-1}-1}{2} b^{2p} + d_1^{2p} + \frac{4^{p-1}-1}{2} s^{2p} E y^{2p}(s)
\]

\[
\leq \left[ \frac{1}{2} + \frac{4^{p-1}-1}{2} \alpha^2 p \right] K_1(2p) + \frac{4^{p-1}-1}{2} b^{2p} + d_1^{2p} + \frac{4^{p-1}-1}{2} s^{2p} y_0^{2p} := F_1(p).
\]

On the other hand,

\[
E |g_1(s, x(s), y(s))|^p = E |-\sigma_1^{2p} x^{p}(s)| \leq \sigma_1^{2p} E x^{p}(s) \leq \sigma_1^{2p} K_1(p) := G_1(p).
\]

Hence, using the well-known moment inequality for the Itô integrals, for \(0 \leq t_1 < t_2 < \infty\) and \(p > 2\), we get

\[
E \left| \int_{t_1}^{t_2} g_1(s, x(s), y(s)) \, dw_1(s) \right|^p \leq \left[ \frac{p(p-1)}{2} \right] \frac{\gamma^p}{(t_2-t_1)^{\frac{p-2}{2}}} \int_{t_1}^{t_2} E |g_1(s, x(s), y(s))|^p \, ds.
\]
By Hölder inequality, for $t_2-t_1 \leq 1$ we compute

\[ E |x(t_2) - x(t_1)|^p \]

\[ \leq 2^{p-1} E \left[ \int_{t_1}^{t_2} |f_1(s, x(s), y(s))| \, ds \right]^p + 2^{p-1} E \left[ \int_{t_1}^{t_2} |g_1(s, x(s), y(s))| \, dw(s) \right]^p \]

\[ \leq 2^{p-1} (t_2-t_1)^{p-1} E \int_{t_1}^{t_2} F_1(p) \, ds + 2^{p-1} \left[ \frac{p(p-1)}{2} \right]^{\frac{p}{2}} (t_2-t_1)^{\frac{p}{2}} \int_{t_1}^{t_2} G_1(p) \, ds \]

\[ = 2^{p-1} (t_2-t_1)^{p} F_1(p) + 2^{p-1} \left[ \frac{p(p-1)}{2} \right]^{\frac{p}{2}} (t_2-t_1)^{\frac{p}{2}} \int_{t_1}^{t_2} G_1(p) \, ds \]

\[ \leq 2^{p-1} (t_2-t_1)^{\frac{p}{2}} \left[ 1 + \left[ \frac{p(p-1)}{2} \right]^{\frac{p}{2}} \right] \int_{t_1}^{t_2} [F_1(p) + G_1(p)] \, ds. \]

This means that almost every sample path of $x(t)$ is locally but uniformly Hölder-continuous with exponent $\gamma \in \left( 0, \frac{p-2}{2p} \right)$, and therefore almost every sample path of $x(t)$ is uniformly continuous on $t \geq 0$.

Similarly, we prove that almost every sample path of $y(t)$ is uniformly continuous on $t \geq 0$, and this completes the proof of the theorem.

In order to show global attractivity in mean of the solution of system (2), let us give the definition first.

**Definition 4.4.** Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two arbitrary solutions of system (2) with initial values $(x_1(0), y_1(0)) \in \mathbb{R}_+^2$ and $(x_2(0), y_2(0)) \in \mathbb{R}_+^2$, respectively. If

\[ \lim_{t \to \infty} E |x_1(t) - x_2(t)| + E |y_1(t) - y_2(t)| = 0 \quad \text{a.s.} \]

then system (2) is globally attractive in mean.

For that purpose we use

**Lemma 4.5.** [3] Let $f : [0, \infty) \to [0, \infty)$ be an integrable and uniformly continuous function. Then,

\[ \lim_{t \to \infty} f(t) = 0. \]

**Theorem 4.6.** If the parameters of system (3) satisfy conditions $d_1 < b < \alpha A_1$, $\frac{d_2}{h_1} < \frac{2\alpha}{h_1}$, and $\sigma_2^2 \leq \frac{2(d_2 - \frac{2\alpha}{h_1})}{p-1}$, for $p > 2$, then the solution of system (2) is globally attractive in mean.

**Proof.** Define $V(t) = |x_1(t) - x_2(t)| + k|y_1(t) - y_2(t)|$, where $k$ is constant which will be chosen in the sequel. Using Itô formula, we calculate

\[ LV(t) = \text{sgn}(x_1(t) - x_2(t)) \]

\[ \times \left[ \frac{bx_1^2(t)}{A_1 + x_1(t)} - \alpha x_2^2(t) - \frac{sx_1 y_1(t)}{1 + sh_1 x_1(t)} - \frac{bx_2^2(t)}{A_1 + x_2(t)} + \alpha x_2^2(t) + \frac{sx_2 y_2(t)}{1 + sh_1 x_2(t)} \right] \, dt \]

\[ + k \text{sgn}(y_1(t) - y_2(t)) \left[ -d_1 y_1(t) \frac{c_1 x_1(t)}{1 + sh_1 x_1(t)} - d_2 y_1(t) \frac{c_1 x_2(t) y_2(t)}{1 + sh_1 x_2(t)} + d_2 y_2(t) \right] \, dt. \]
Thus,
\[ dEV(t) = ELV(t)dt \]
\[ = E \left\{ \frac{b(A_1(x_1(t)+x_2(t))+x_1(t)x_2(t))}{A_1+x_1(t)(A_1+x_2(t))} \right\} E \left\{ \frac{x_1(t)-x_2(t)}{A_1+x_1(t)(A_1+x_2(t))} \right\} \]
\[ -d_1|x_1(t)-x_2(t)| - k d_2 |y_1(t)-y_2(t)| \]
\[ - s \sgn(x_1(t)-x_2(t)) \frac{x_2(t)(1+sh_1 x_1(t))(y_1(t)-y_2(t))+y_1(t)(x_1(t)-x_2(t))}{(1+sh_1 x_1(t))(1+sh_1 x_2(t))} \]
\[ + k c_1 s \sgn(y_1(t)-y_2(t)) \frac{x_2(t)(1+sh_1 x_1(t))(y_1(t)-y_2(t))+y_1(t)(x_1(t)-x_2(t))}{(1+sh_1 x_1(t))(1+sh_1 x_2(t))} \} dt \]
\[ \leq E \left\{ \left[ \left( \frac{b}{A_1} - \alpha \right)(x_1(t)+x_2(t)) + b - d_1 - \frac{s y_1 (1-k c_1)}{1+sh_1 x_1(t)(1+sh_1 x_2(t))} \right] |x_1(t)-x_2(t)| \right. \]
\[ \left. + \left[ - k d_2 + \frac{s x_2 (1+sh_1 x_1(t))(1+k c_1)}{1+sh_1 x_1(t)(1+sh_1 x_2(t))} \right] |y_1(t)-y_2(t)| \right\} dt. \]

For \( k = \frac{1}{c_1} \), we get
\[ EV(t) - V(0) \leq E \left\{ \int_0^t \left[ \left( \frac{b}{A_1} - \alpha \right)(x_1(r)+x_2(r)) + b - d_1 \right] |x_1(r)-x_2(r)| \right. \]
\[ \left. - \frac{1}{c_1} \left[ d_2 - \frac{2 c_1}{h_1} \right] |y_1(r)-y_2(r)| \right\} dr. \]

Since \( b < \alpha A_1 \), then
\[ EV(t) \leq V(0) - \int_0^t (d_1 - b) E |x_1(r)-x_2(r)| \right) dr - \frac{1}{c_1} \int_0^t \left( d_2 - \frac{2 c_1}{h_1} \right) E |y_1(r)-y_2(r)| \right) dr. \]

From condition of theorem \( d_1 < b \) and \( d_2 < \frac{2 c_1}{h_1} \), we conclude
\[ EV(t) + C_1 \int_0^t E |x_1(r)-x_2(r)| \right) dr + C_2 \int_0^t E |y_1(r)-y_2(r)| \right) dr \leq EV(0) < \infty, \]
where \( C_1 = d_1 - b \) and \( C_2 = d_2 - \frac{2 c_1}{h_1} \) are positive constants, which leads to
\[ E |x_1(t)-x_2(t)| \in L^1[0, \infty), \quad E |y_1(t)-y_2(t)| \in L^1[0, \infty). \]

Thus, on the basis of Lemma 4.5, Theorem 4.3 and previous discussion, the proof is complete.

\[ \square \]

**Remark 1.** One of the conditions of global attractivity in mean for system (4.2) is \( \alpha A_1 > b \). However, in that case \( \alpha A_1 \geq b - d_1 \), and this fulfills a condition 2° a of Theorem 4.2. Thus, the system can be globally stable in mean if the prey and predator populations become extinct.

5. **Numerical simulation.** In order to verify mathematical results obtained in previous sections, we try to put our theoretical research into the context of real-world settings.

We consider interaction between wolves (Canis lupus) and moose (Alces alces) on Isle Royale, the principal island of Isle Royale National Park in the United States. The relationship between wolves and moose on Isle Royale has been studied for very large amount of time. Moose settled there in the early 1900s, and in 1949 a pair of wolves wandered across an ice bridge made to the island during a harsh winter.
from Ontario. Since then a large oscillations in number of wolves and moose have been recorded due to changing conditions on the island.

The wolves on Isle Royale, which once numbered as many as 50, now are reduced to three. They have been mostly isolated for generations, and the population has been overrun with spinal defects likely as a result of the inbreeding. Also, the wolves may have simply left via the icy corridor to the mainland, seeking unrelated mates. Because of that, it is not a surprise that moose are booming. This year, they reach number of about 1250 despite the harsh winter. Wolves are so scarce that their effect on moose has been essentially nil for the past 4 years, during which time the moose population has grown about 22% annually (see [24]). This is not a sustainable state because such burgeoning populations of moose, deer, and elk have negative effects on vegetation of the island. More about proposed solutions of this problem can be found in [15, 22, 23, 24, 26] and references cited there in.

In order to confirm theoretical results obtained in this paper, we will use reliable data from the references cited above. Thus, let the parameters of system (2) be

\[ b = 0.22 \text{ per year (24)}, \quad d_1 = 0.008 \text{ per year (23)}, \]
\[ d_2 = 0.7 \text{ per year (24)}, \quad c_1 = 0.005 \text{ (0.5\%–1.5\%, 23)}, \]
\[ h_1 = 25 \text{ days (24.1 ± 1.94, 24)}, \quad s = 0.05 \text{ (15)}. \]

From Theorem 3.2 since \( b > d_1 \), if:
\[ \alpha A_1 \geq 0.212 \text{ (case 2°a)}, \]
\[ 0.152723 < \alpha A_1 < 0.212 \text{ (case 2°b)}, \]
\[ \alpha A_1 \leq 0.22 \text{ and } \sigma_1^2 > 2 \left( 0.469 - \sqrt{\alpha A_1} \right)^2 - 0.008 \text{ (case 3°)}, \]
moose and wolf population become extinct.

If \( \alpha = 0.0002 \) and \( A_1 = 0.5 \), for example, then for \( \sigma_1^2 = \sigma_2^2 = 0.005 \) moose population does not become extinct because of the fact that conditions of Theorem 3.2 are not satisfied, but since condition of Theorem 3.3 is fulfilled, extinction in wolf population is expected in about 14 years (Figure 1).

For the same \( \alpha \), and greater Allee effect constant, \( A_1 = 5 \), we can observe greater fluctuations in moose population (Figure 2).

**Figure 1.** Deterministic and stochastic trajectories of moose and wolf population described by (2) with parameters (12), \( \alpha = 0.0002, A_1 = 0.5, \sigma_1^2 = \sigma_2^2 = 0.005 \).
Due to the fact that on Isle Royale there are only three wolves remaining, the moose population was not limited by predation in the last five years, as Peterson et al. observed in [23]. In the past years the predation rate was smaller than 4%, and winters were not as severe as they used to be before. Hence, the growth of the moose population is very strong. Thus, the stronger competition among the individuals of the moose population is expected. If we choose $\alpha = 0.01$, $A_1 = 0.5$, we can observe that intensity of noise $\sigma_1^2 = \sigma_2^2 = 0.005$ does not lead moose population to extinction (Figure 3-left), but if intensity of the noise is $\sigma_1^2 = \sigma_2^2 = 0.31$ moose population becomes extinct (Figure 3-right). However, such high intensity of noise is not realistic.

Additionally, if $\alpha = 0.01$ and $A_1 = 5$, for example, then for $\sigma_1^2 = 0.14$ and $\sigma_2^2 = 0.005$, conditions of Theorem 3.2 are satisfied, so the moose become extinct in about 140 years, and wolf in 10 years (Figure 4). Hence, we can conclude that stronger Allee effect drives moose population to extinction, even if intensity of noise is not such high. In Figure 4 we can also notice that in appropriate deterministic model for moose population extinction does not occur.

**Figure 2.** Left: Deterministic and stochastic trajectories of moose population described by (2) with parameters (12), $\alpha = 0.0002$, $A_1 = 5$, $\sigma_1^2 = \sigma_2^2 = 0.005$; Right: Stochastic trajectories of moose and wolf populations in which we can observe behavior of these populations in 140 years.

**Figure 3.** Deterministic and stochastic trajectories of moose population described by (2) with parameters (12), $\alpha = 0.01$, $A_1 = 0.5$ and different intensities of noise.
Parameters (12) do not satisfy conditions of Theorem 4.6 and, hence system (2) is not globally attractive in mean. But, if we change parameters in the following way: $\alpha = 0.01, A_1 = 40, d_2 = 0.12, h_1 = 26$ days and $\sigma_2^2 = 0.002$, then the conditions of Theorem 4.6 hold and system (2) is globally attractive in mean, which can be observed in Figure 5.

6. Conclusions. In the paper we have extended the Rosenzweig–MacArthur predator-prey model with Allee effect on prey ([28]), by introducing a noise perturbation impressed on the death rates in it. It is natural extension because of the fact that the populations of predators and preys are subjected to environmental noise in the nature. We have considered how environmental fluctuations affect extinction of predator and prey population. Taking the environment noise into account, we first proved the existence and positivity of the solutions. Then, we conclude that extinction in prey population may occur when the death rate of prey population is greater than it’s maximum reproduction rate. Intuitively, this is expected condition, but, even if the reproduction rate is greater than death rate, Allee effect can drive population to extinction. The third reason that can cause extinction in prey population is large intensity of the noise, which means that unpredictable events in nature can be fatal for the prey population. Because of the interaction between predator and prey population, extinction of prey population will drive predator
population to extinction, too. However, extinction in predator population may occur, regardless of the size of prey population. More precisely, we prove that large intensity of the noise drives the predator population to extinction. We also examine the global attractivity in mean for considered stochastic predator-prey system and obtain that it is globally attractive if Allee constant is large enough. In that case we conclude that if system is globally attractive, predator and prey population go to extinction.

Our work shows that stochastic predator-prey model gives a different perspective of influence of Allee effect on extinction of predator and prey population.

Acknowledgments. The authors extend their appreciations to the anonymous referees for their very helpful suggestions which greatly improve the paper.

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