Dual projection in a two-dimensional curved expanding universe

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I. INTRODUCTION

The study of chiral bosons in a $D = 2$ flat space has a wide interest. They occur as basic ingredients in the formulation of string theories and in a number of two-dimensional statistical systems, like the fractional quantum Hall effect phenomenology. More recently, the M-theory can be achieved by treating the chiral sectors in a more independent way. In superior dimensions, the six-dimensional chiral bosons and tensor multiplets in $N = 1$, $D = 6$ supergravity theories. They are necessary to complete the $N = 2$, $D = 6$ supermultiplet of the $M$-theory five-brane. Finally, a ten-dimensional chiral bosons appears in $IIB$, $D = 10$ supergravity.

The dual projection technique [1–5], strictly related to canonical transformations [6], has been presented initially in the study of electromagnetic duality groups [1]. The difference between both concepts is that the dual projection is performed at the level of the actions whereas the canonical transformations becomes manifest [6]. The most useful and interesting point in this dual projection procedure is that it is not based on evidently even-dimensional concepts and may be extended to the odd-dimensional situation [3] and also in non-Abelian systems [7]. In this work we will show that besides the use of dual projection in even and odd dimensions in flat space, we can obtain interesting results in curved space.

The noton is a nonmover field at the classical level [8], carrying the representation of the Siegel symmetry [9], that acquires dynamics upon quantization. As another feature, it is a well known fact that coupling chiral particles to external gravitational fields reveals the presence of notons [10]. At the quantum level in flat space, it was shown [2] that its dynamics is fully responsible for the Siegel anomaly. In other words, we can say that the importance of the inclusion of a normalized external noton - the Hull mechanism - is to cancel the Siegel anomaly [8] conveniently.

On the other hand, later than Siegel, Floreanini and Jackiw (FJ) introduced an action which describes a free chiral boson in two dimensions [11], but this action have some symmetry problems. In [2] it was shown that, in fact both models (Siegel and FJ) for the chiral bosons are related by a noton particle.

In this work we want to analyze the effect of the dual projection in a two-dimensional curved space (a Friedmann space) with a metric that describes an expanding universe model [12]. We shall consider here the universe as a curved space. Notice that the main idea here is to analyze the different spectra found.

Quantum field theory in curved spacetime is an effective theory which is able to make pretty reliable quantum gravity predictions in some specific regimes. In a nutshell we can say that it investigates the consequences of defining quantum fields on general background spacetime. Although it is unable to describe nature in extreme regimes, as in the Planck scale, some remarkable effects were already predicted in its context as, for instance, that quantum effects must induce black holes to evaporate, in contrast to the classical belief [13].

Firstly we will show that the dual projection of a scalar field in a generic gravitational background results in two notons. This result is quite different from the curved space one, which shows two dynamical chiral fields interacting with the external gravitational field. Notwithstanding, the final system is equivalent to a single dynamical non-chiral fields in the gravitational background, as it should be.

The mentioned chirality separation is confirmed when we perform the flat space limit, bearing out the results obtained by Fronsdal et al in a series of papers on curved space [14], i.e., “that a flat space result can be obtained from the curved space one through a determined approximation”.

The sequence of the paper is: in order to make this work self-consistent, in the next section we review the


11.10.Kk, 11.15.-q, 12.39.Fe, 98.80.-k
and substituting (2) in (1) the Lagrangian reads, \( \mathcal{L}_1 = \dot{\phi} p - \frac{1}{2} p^2 - \frac{1}{4} \phi'^2 \). Performing the canonical transformations convenient constructed in order to promote a complete field separation, we have,

\[ \phi = \rho + \sigma \quad \text{and} \quad p = \rho' - \sigma' . \]  

(3)

Substituting it in \( \mathcal{L}_1 \) above, we have a new action, \( \mathcal{L}_2 = \dot{\rho} \rho'^2 - \rho'^2 - \dot{\sigma} \sigma' - \sigma'^2 \), which shows clearly that the spectrum of a scalar field is formed by two FJ particles of opposite chirality as we said above. For an interested reader, the relation between two chiral particles was discussed at the Hamiltonian level in [20].

There are indications that a deeper understanding of such issues as string dynamics and fractional quantum Hall effect phenomenology can be achieved by treating the chiral sectors in a more independent way. However, coupling chiral fields to external gauge and gravitational fields is problematic. As we said above, it was discussed in [7], how the coupling of chiral (Abelian) fields to external gravitational backgrounds can be achieved by diagonalization (dual projection) of the first-order form of a covariant scalar action. The theory reduces then to a sum of a left and a right FJ’s actions [11], circumventing the problems caused by the lack of manifest Lorentz invariance.

B. The Siegel chiral boson. Now we will present a small version of the results obtained in [2]. We begin with the Siegel original classical Lagrangian density for a chiral scalar field [9] using the usual light-cone variables,

\[ \mathcal{L}_{\text{Siegel}} = \partial_+ \phi \partial_- \phi - \lambda_{++} (\partial_- \phi)^2 = \frac{1}{2} \sqrt{g^{++}} g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi \]

where the metric is \( g^{++} = 0, g^{+-} = 1, g^{-+} = -2 \lambda_{++} \), and we can say that the Lagrangian \( \mathcal{L}_{\text{Siegel}} \) describes a lefton [21]. A lefton (or a righton) is a particle which, besides to carry the dynamics of the theory, it is liable for the symmetry of the system too. This characterizes exactly a Siegel mode. Hence, it is different from a FJ’s particle [11] which carries only the dynamics of the system. Thereby, a FJ particle can not be classified either as a lefton or a righton.

The symmetry content of the theory is well described by the Siegel algebra, a truncate diffeomorphism that disappears at the quantum level. Hence \( \mathcal{L}_{\text{Siegel}} \) is invariant under Siegel gauge symmetry which is an invariance under the combined coordinate transformation and a Weyl rescaling of the form \( x^- \rightarrow \tilde{x}^- = x^- - \epsilon^- \) and \( \delta g_{\alpha \beta} = -g_{\alpha \beta} \partial_- \epsilon^- \). The fields \( \phi \) and \( \lambda_{++} \) transform under these relations as \( \delta \phi = \epsilon^- \partial_- \phi \), and \( \delta \lambda_{++} = -\partial_+ \epsilon^- + \epsilon^- \partial_+ \lambda_{++} - \partial_- \epsilon^- \lambda_{++} \), and \( \phi \) is invariant under the global axial transformation, \( \phi \rightarrow \tilde{\phi} = \phi + \phi \). It is easy to see that fixing the value of the multiplier as \( \lambda_{++} = 1 \) in \( \mathcal{L}_{\text{Siegel}} \) we can obtain the FJ model. As mentioned before, this was considered, for a long time in the literature, the unique constraint between these both representations of a chiral boson. The dual
projection permit us to see other intrinsic relations behind the model that can possibly (or not) be hidden in its spectrum.

We will now begin to apply the dual projection procedure. This is done introducing a dynamical redefinition in the phase space of the model. Using \( \mathcal{L}_{\text{Siegel}} \) in Lorentz coordinates we can obtain the canonical momentum as

\[
\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} - \lambda_{++} (\dot{\phi} - \phi')
\] (4)

or, in other words

\[
\dot{\phi} = \frac{\pi - \lambda_{++} \phi'}{1 - \lambda_{++}}.
\] (5)

After a little algebra, substituting (4) and (5) into \( \mathcal{L}_{\text{Siegel}} \), the Lagrangian in the first-order form reads

\[
\mathcal{L}_{\text{Siegel}} = \pi \dot{\phi} - \frac{\phi'^2}{2} - \frac{1}{2} \left( \frac{\pi - \lambda_{++} \phi'}{1 - \lambda_{++}} \right)^2 - \frac{\lambda_{++}}{2} \phi'^2.
\] (6)

As we said above, fixing the value of the multiplier as \( \lambda_{++} \rightarrow 1 \) in (6) we get the FJ form. This value of \( \lambda_{++} \) promotes a reduction of the phase space of the model to \( 22 \pi \rightarrow \phi' \), and consequently the third term in (6) reduces to zero as \( \lambda_{++} \rightarrow 1 \). Therefore the dynamics of the system will be described by a FJ action.

The above behavior in \( \pi \rightarrow \phi' \) suggests the following canonical transformations, analogous as in (3):

\[
\phi = \varphi + \sigma \quad \text{and} \quad \pi = \varphi' - \sigma',
\] (7)

and we stress that these fields are independent as they originate from completely different actions. After substituting (7) into (6) to perform the dual projection we find a diagonalized Lagrangian, \( \mathcal{L} = \varphi' \dot{\varphi} - \varphi'^2 - \sigma' \dot{\sigma} - \eta_+ \sigma'^2 \), where \( \eta_+ = \frac{1 + \lambda_{++}}{1 - \lambda_{++}} \). The effect of dual projection procedure into the first-order Siegel theory, equation (6), was the creation of two different internal spaces leading to the \( Z_2 \) group of dualities (a discrete group with two elements) [1,3] and the other is the diffeomorphism group of transformations. Clearly we can see that the chirality/duality group and the symmetry group are in different sectors. The first is obviously a FJ mode and the other is a noton mode. This result is complementary to the established knowledge, where the FJ action is interpreted as a gauge fixed Siegel action [9]. Under this point of view, we look at the gauge fixing process as the condition that sets the noton field to vanish. It can be proved [2] that this noton is totally responsible for the symmetries, both classically and quantitically.

**III. DUAL PROJECTION IN CURVED SPACE**

In this section we will use the dual projection to analyze the spectrum of a scalar field in an expanding universe. We will see that the dual projection procedure permit us to find a very different result from the flat space one (section II.A). Obeying the results obtained by Fronsdal et al [14], we will show that the flat result can be obtained by the application of a specific limit in the curved space outcome, as it should be.

**A. A scalar field in a gravitational background.**

Before we approach the principal issue of this article, let us make an application of the dual projection to the case of a scalar field embedded in a general gravitational background. In this way, we consider the following Lagrangian,

\[
\mathcal{L}_g = \frac{1}{2} \sqrt{-g} g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi
\] (8)

where

\[
h^{\mu \nu} = \sqrt{-g} g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix}
\] (9)

and \( h_{00} = h_{11} \).

Performing the decomposition in components of the action (8) we can write,

\[
\mathcal{L}_g = \frac{1}{2} h_{00} \dot{\varphi}^2 + \frac{1}{2} h_{11} \varphi'^2 + h_{01} \varphi \dot{\varphi}'
\] (10)

After the standard diagonalization process, as we explained in the last sections, we have as final result,

\[
\mathcal{L} = \varphi' \dot{\varphi} - \eta_+ \varphi'^2 - \sigma' \dot{\sigma} - \eta_- \sigma'^2,
\] (11)

where \( \eta_\pm = \frac{1}{h_{00}} (1 \pm h_{01}) \). And (11) shows the coupling of the chiral bosons (\( \varphi \) and \( \sigma \)) with the gravitational field. This behavior is characteristic of a system composed by notons, which couple to the gravitational field [2].

**B. A scalar field in a curved expanding universe.**

Now we will analyze, at the light of the dual projection procedure, the class of two-dimensional metrics of the form [12],

\[
ds^2 = -dt^2 + R(t)^2 \, dx^2
\] (12)

where \( R(t) \) is an unspecified positive function of \( t \). The two-dimensional universe with such a metric will be referred conveniently, as we said before, as an expanding universe. Consequently \( R(t) \) need be increasing with time. To work in two dimensions is, as it is well known, a laboratory where it can be possible to envision the results in four dimensions or even in higher dimensions.

The equations governing the fields are covariant generalizations of the special relativistic free-field equation. The gravitational metric is treated as an unquantized external field. No additional interactions are included.

As a first step, let us verify if our space (12) is really a curved one. It is well known that the Riemann-Christoffel tensor for two-dimensional spaces has only nonvanishing component equal to \( R_{0101} \) or to its negative. For the metric in (12), performing a standard calculation of the \( R_{0101} \) component, we have that, \( R_{0101} = -R \dddot{R} \). This gives us
the condition that we are working in a two-dimensional curved space only if \( R^2 \neq 0 \). On the other hand, for this space to be flat, we would have to have the following condition, \( R_{0101} = 0 \Rightarrow R \neq 0 \), so that \( R = 0 \) or \( \tilde{R} = 0 \). The first solution, \( R = 0 \) can be obviously ignored. The second one, \( \tilde{R}(t) = 0 \), has the straightforward solution \( R(t) = k_1 t + k_2 \), where \( k_1 > 0 \) (to pledge an expansion of the universe) and \( k_2 \) are arbitrary constants. So, as a condition for the space to be a curved one, \( R(t) \) can not be a linear function of time. Unless for this case, we can say that we are working in a curved space.

Back to the main issue, with the metric (12), the Lagrangian density for a two dimensional scalar particle in a curved space is [12]

\[
\mathcal{L} = \frac{1}{2} R^3 \left( \dot{\phi}^2 - \frac{1}{R^2} \phi'^2 \right) \tag{13}
\]

where we wrote for convenience that \( R \) represents \( R(t) \).

Now, let us perform the dual projection procedure following the steps of the last section. Reducing the order = 0. The first solution, \( R = 0 \) can be obviously ignored. The second one, \( \tilde{R} \), can be easily written as,

\[
\tilde{R}(t) = k_1 t + k_2 \quad \text{where} \quad k_1 > 0 \quad (to \quad pledge \quad an \quad expansion \quad of \quad the \quad universe) \quad and \quad k_2 \quad are \quad arbitrary \quad constants.
\]

So, as a condition for the space to be a curved one, \( \tilde{R}(t) \) can not be a linear function of time. Unless for this case, we can say that we are working in a curved space.

Back to the main issue, with the metric (12), the Lagrangian density for a two dimensional scalar particle in a curved space is [12]

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\mathcal{L} = \frac{1}{2} R^3 \left( \dot{\phi}^2 - \frac{1}{R^2} \phi'^2 \right) \tag{13}
\]

where we wrote for convenience that \( R \) represents \( R(t) \).

Now, let us perform the dual projection procedure following the steps of the last section. Reducing the order of the time derivative in the same way as we did in the last section, we can write

\[
\frac{1}{2} R^3 \phi'^2 \rightarrow p \dot{\phi} - \frac{1}{2} \frac{p^2}{R^3}. \tag{14}
\]

So, in (13) we have

\[
\mathcal{L} = p \dot{\phi} - \frac{1}{2} \frac{p^2}{R^3} - \frac{1}{2} R \phi'^2. \tag{15}
\]

Firstly, performing the generalized canonical transformations,

\[
\phi = a \varphi + b \rho \quad \text{and} \quad \rho = c \varphi' - d \rho'. \tag{16}
\]

where \( a, b, c \) and \( d \) are the coefficients that will be determined. Substituting (16) in (15) we have

\[
\mathcal{L} = a c \varphi' \dot{\varphi'} - b d \rho' \dot{\rho'} + (b c - a d) \dot{\varphi} \dot{\rho'} - \frac{1}{2} \left( \frac{c^2}{R^3} + R a^2 \right) \varphi'^2 - \frac{1}{2} \left( \frac{d^2}{R^3} + R b^2 \right) \rho'^2 + \left( \frac{c d}{R^3} - a b R \right) \varphi' \rho'. \tag{17}
\]

Our first attempt is to obtain a flat space-type result so that the cross-terms have to disappear. Hence, to compute the coefficients in (17) we have,

\[
a c = b d = 1 \quad \Rightarrow \quad a = \frac{1}{c} \quad \text{and} \quad b = \frac{1}{d}. \tag{18}
\]

\[
bc - ad = 0 \tag{19}
\]

and \( cd - ab R^4 = 0 \). \tag{20}

We will call the coefficients of the fourth and fifth terms in (17) as,

\[
COEF_1 = \frac{1}{2} \left( \frac{c^2}{R^3} + R a^2 \right) = \frac{1}{2} \left( \frac{1}{a^2 R^3} + R a^2 \right)
\]

\[
COEF_2 = \frac{1}{2} \left( \frac{d^2}{R^3} + R b^2 \right) = \frac{1}{2} \left( \frac{1}{b^2 R^3} + R b^2 \right). \tag{21}
\]

where we used equations (18). Afterwards we will be back to them again.

Substituting the equation (18) in (20) we have,

\[
a^2 b^2 = \frac{1}{R^4} \tag{22}
\]

and finally, using (18) in (19) we can write that,

\[
a = \pm b \quad \text{and} \quad c = \pm d \tag{23}
\]

and with these values it is easy to see that the coefficients \( COEF_1 \) and \( COEF_2 \) are identical, i.e.,

\[
\eta = COEF_1 = COEF_2 = \frac{1}{2} \left( \frac{1}{a^2 R^3} + R a^2 \right). \tag{24}
\]

Solving equations (22) and (23) together, we conclude that exist four possible values for \( a \) and \( b \),

\[
a = \pm \frac{1}{R}, \quad b = \pm \frac{1}{R} \tag{25}
\]

So with the values obtained in (23), and (25) our canonical transformation (3), are \( \phi = a (\varphi \pm \rho) \) and \( \rho = c (\varphi' \mp \rho') \), but we also know from (18) that \( a = 1/c \), therefore we can finally write, \( \phi = \frac{1}{c} (\varphi \pm \rho) \) and \( \rho = c (\varphi' \mp \rho') \). Making all the correspondent substitutions in (17), the final Lagrangian is

\[
\mathcal{L} = \varphi' \dot{\varphi} - \eta \varphi'^2 - \rho' \dot{\rho} - \eta \rho'^2, \tag{26}
\]

where

\[
\eta = \frac{1}{2} \left( \frac{1}{a^2 R^3} + R a^2 \right).
\]

Also, using (25), we can have two values for \( \eta, \eta_+ = \frac{1}{R} \) and \( \eta_- = -\frac{1}{R} \).

We can see from (26), as we said before, that \( \varphi, \rho \) are two dynamical chiral fields interacting with the external gravitational field. This result differs altogether from that found in the previous section, where we found two notons. There we can be naively convinced, at a first sight, that the final result is independent from the form of the gravitational background. We see now that it is not totally true. These two chiral fields coupling, in fact, describes a single dynamical non-chiral field in the given gravitational background, which was our starting point. This will be confirmed just below.

To corroborate the condition that the flat space result is a limit of the curved space one [14], let us fix the value
of \( \eta \) as \( \eta = 1 \) and hence from (24) \( \text{COEF}_1 = \text{COEF}_2 = 1 \), originating conditions on \( a \),

\[
a^4 - \frac{2}{R} a^2 + \frac{1}{R^4} = 0 \tag{27}
\]

which solutions are

\[
a_{1,2} = \pm \frac{1}{R^2} \sqrt{R + \sqrt{R^2 - 1}}
\]

\[
a_{3,4} = \pm \frac{1}{R^2} \sqrt{R - \sqrt{R^2 - 1}} . \tag{28}
\]

Applying simply \( \eta = 1 \) in (26) we have the same Lagrangian of the flat space,

\[
\mathcal{L} = \varphi' \dot{\varphi} - \varphi'^2 - \rho' \dot{\rho} - \rho^2 , \tag{29}
\]

i.e., it comprises two FJ’s particles with opposite chiralities, as stressed in [17,19] and confirms the existence of two chiral fields interacting with an external gravitational background as the result above in (26).

**IV. CONCLUSIONS**

The dual projection procedure is a technique that helps one to analyze the spectrum of first-order systems. It was used recently to study the notion of duality and self-duality creating an internal space of potentials [3].

Since the analysis was always effected for first-order systems, an equivalence between the Lagrangian and Hamiltonian approaches permitted us to use the concept of canonical transformations. In other words we can say that the dual projection demanded a change of variables which was, in the phase space, a canonical transformation.

In this work we consider the structure of a scalar field in a two-dimensional expanding universe embedded in a curved space. Firstly, we construct a scalar field embedded in a generic gravitational background and promote the dual projection of the system. We obtain, as expected, two notons. Differently from the generic case, now in a curved space, we have two dynamical chiral fields interacting with an external gravitational field. This is equivalent to the model describing a single dynamical non-chiral field in the gravitational background. In other words, only the final models are equivalent, the spectra \((\text{noton}_1 \oplus \text{noton}_2 \oplus \text{chiral particle}_1 \oplus \text{chiral particle}_2)\) are not. This final analysis is the main idea of this work.

However, as we can see, we could expect (naively based on the former result) that every other case involving this kind of coupling, i.e., whatever the form of the gravitational background, we always would find two notons comprising the spectrum. The only difference between the spectra would manifest itself in the explicit form of the metric.

We show here that any conclusion about the spectrum turns out to be premature and thereby must be preceded by a detailed (for example) dual projection analysis. In this way we can determine precisely if the particles that comprise the spectrum can be either notons or chiral fields coupled to the gravitational background although the final results are the same (a scalar field) as said above.

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