Some of semileptonic and nonleptonic decays of $B_c$ meson in a Bethe-Salpeter relativistic quark model

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The semileptonic decays $B_c^+ \rightarrow P(V) + \ell^+ + \bar{\nu}_\ell$ and the nonleptonic decays $B_c^+ \rightarrow P(V) + L$, where $P$ or $V$ denotes a $S$-wave charmonium or a $S$-wave ($b\bar{s}$) bound state and $L$ denotes a light meson, are studied under the framework of improved instantaneous Bethe-Salpeter (BS) equation and the Mandelstam formula. We present the numerical results about the width and branching ratio of each decay mode in tables. In order to compare with the others conveniently, the results obtained by other approaches are also presented in the relevant tables. Based on the fact that the ratio $BR(B_c^+ \rightarrow J/\psi \pi^+) / BR(B^+ \rightarrow J/\psi \pi^+)$ is 0.24 estimated in terms of the present framework is in agreement with the LHCb observation $BR(B_c^+ \rightarrow J/\psi (2S)\pi^+) / BR(B^+ \rightarrow J/\psi (2S)\pi^+) = 0.250 \pm 0.068 (stat) \pm 0.014 (syst) \pm 0.006 (B)$ quite well, one may see that the approach adopted here to the decays is really improved in comparison with the early one.

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$B_c$ meson carries two heavy flavor quantum numbers explicitly, and it decays only via weak interactions, although the strong and electromagnetic interactions can affect the decays. As consequences, $B_c$ meson has a comparatively long lifetime and very rich weak decay channels with sizable branching ratios. Moreover, as an explicit double heavy flavor meson, its production cross section can be calculated by perturbative QCD quite reliably and the conclusion can be drawn that only via strong interaction and at hadronic high energy collisions the meson can be produced so numerous that one can observe it experimentally. Therefore, the meson is specially interesting in studying its production and decays.

The first successful observation of $B_c$ was achieved through the semileptonic decay channel $B_c \rightarrow J/\psi + \ell^+ + \bar{\nu}_\ell$ by CDF collaboration in 1998 from Run-I at Tevatron. They obtained the mass of $B_c$: $m_{B_c} = 6.40 \pm 0.39 \pm 0.13$ GeV and the lifetime: $\tau_{B_c} = 0.46^{+0.18}_{-0.16} \pm 0.03$ ps. Later on CDF collaboration gave a more precise mass $m_{B_c} = 6275.6 \pm 2.9 (stat) \pm 5 (syst)$ MeV/$c^2$ obtained through the exclusive non-leptonic decay $B_c \rightarrow J/\psi \pi^+$ and upgraded their results. D0 collaboration at Tevatron has also carried out the observations and confirmed CDF results. Now LHCb has reported several observations on $B_c$ decays, and it is expected that in the near future the $B_c$ data will be largely enhanced.

In literatures, there has been many works studying various $B_c$ decays under different approaches. Among the approaches in the market, the one used in Ref. is based on the instantaneous Bethe-Salpeter (BS) equation (also called Salpeter equation) with a QCD-inspired kernel (interaction) for heavy quarks and the Mandelstam formula for the relevant hadron matrix elements. This approach has a firm foundation and may well take into account of the relativistic “recoil effects” in the decays because the BS equation and the Mandelstam formula both are well established on relativistic quantum field theory. Furthermore the components in $B_c$ meson and charmonium etc are heavy quarks i.e. the nature of the components is non-relativistic, so the BS equation is deduced into an instantaneous one in the ap-

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1 With the equation, the spectrum and relevant wave function as an eigenvalue problem derived from the BS equation can be computed.
proach. When solving an instantaneous BS equation the wave functions for heavy pseudoscalar and vector mesons should be formed in the spin-orbit basis of angular momentum with the spin of the components described by the Dirac spinors, whereas in Ref. 29 the authors followed Ref. 30 an additional approximation is taken. Since a way to solve an instantaneous BS equation without the approximation, i.e. full Salpeter equation, that a full wave function as a solution of the instantaneous BS equation is obtained by decomposing the wave function in the spinor space with all the possible Dirac matrices according to its quantum numbers 31, and a way to treat the relevant hadron matrix elements accordingly 32 are explored, so we think that, for a relativistic treatment of $B_c$ decays, using the full wave function in solving the instantaneous BS equation and the way for treating the hadron matrix elements would be a good improvement. Actually, the instantaneous BS equation with the full wave functions has been solved in Ref. 31 for heavy quarkonia. Thus we suspect the improvements may play roles in calculating $B_c$ decays, and we think that it is worth to recalculate some important weak decay channels of $B_c$ with this improved method. To see the consequences of the improvements in 31,32 and considering the progresses in experiments, in this paper we would like to restrict ourselves to focus lights on the decays, semileptonic and nonleptonic ones: $B_c^+ \to P(V) + \ell^+ + \bar{\nu}_\ell$ and nonleptonic ones: $B_c^+ \to P(V) + \pi(p, K, K^*)$ precisely, where $P(V)$ represents pseudoscalar (vector) meson and is limited to be a charmonium or a $b\bar{s}$ bound state.

The paper is organized as follows. In Sec. I we outline the useful formulas. In Sec. II we present numerical results for the semileptonic and nonleptonic decays and compare the results with those obtained by other approaches. Sec. III is contributed to discussions. We put the relativistic BS equation with covariant instantaneous approximation, the forms of relativistic wave functions for pseudoscalar and vector mesons, the formulations of the form factors, and the parameters used to solve the BS equation into Appendices.

I. THE FORMALISM FOR THE EXCLUSIVE SEMILEPTONIC AND NONLEPTONIC $B_c$ DECAYS

For the semileptonic decays $B_c^+ \to X + \ell^+ + \bar{\nu}_\ell$ shown in Fig. 1, the $T$-matrix element can be written as hadronic component and leptonic component:

$$T = \frac{G_F}{\sqrt{2}} V_{ij} \bar{u}_{\ell} \gamma^\mu (1 - \gamma_5) v_\ell \langle X(p', \epsilon) | J_\mu | B_c^+ (p) \rangle,$$

where $V_{ij}$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element, $J_\mu$ is the charged weak current responsible for the decays, $p, p'$ are the momenta of the initial state $B_c^+$ and the final state $X$ respectively, while $\epsilon$ is the polarization vector when $X$ is a vector particle. The square of the matrix element, summed and averaged over the spin (unpolarized), is:

$$\sum |T|^2 = \frac{G_F^2}{2} |V_{ij}|^2 l^{\mu\nu} h_{\mu\nu},$$

where the leptonic tensor:

$$l^{\mu\nu} = \bar{u}_{\ell} \gamma^\mu (1 - \gamma_5) v_\ell (1 + \gamma_5) \gamma^\nu u_{\ell},$$

is easy to compute, and the hadronic tensor is defined by:

$$h_{\mu\nu} = \sum_\epsilon \langle B_c^+ (p) | J_\nu | X(p', \epsilon) \rangle \langle X(p', \epsilon) | J_\mu | B_c^+ (p) \rangle.$$

By a straightforward calculation, the differential decay rate is obtained:

$$\frac{d^2 \Gamma}{dx dy} = |V_{ij}|^2 \frac{G_F^2 M_5^5}{32 \pi^3} \left\{ \frac{\alpha (y - m_\ell^2)}{M^2} + 2 \beta_{++} \right.$$

$$\times \left[ 2x(1 - \frac{M^2}{M_T^2} + y) - 4x^2 - y + \frac{m_\ell^2}{4M^2} \right]$$

$$\times \left( 8x + \frac{4M^2 - m_\ell^2}{M^2} - 3y \right)$$

$$+ 4(\beta_{++} - \beta_{--}) \frac{m_\ell^2}{M^2} (2 - 4x + y - \frac{2M^2 - m_\ell^2}{M^2})$$

$$+ 4\beta_{--} \frac{m_\ell^2}{M^2} (y - m_\ell^2) - \gamma \left[ y(1 - \frac{M^2}{M_T^2} - 4x + y) \right.$$

$$\left. + \frac{m_\ell^2}{M^2}(1 - \frac{M^2}{M_T^2} + y) \right],$$

where $x = p_c/M$ and $y = (p - p')^2/M^2$, $M$ is the mass of $B_c^+$ meson, $M'$ is the mass of the final state $X$. The coefficient functions $\alpha$, $\beta$ and $\gamma$ can be formulated in terms of form factors.

To evaluate the exclusive semileptonic differential decay rates of $B_c^+$ meson, one needs to calculate the hadron
matrix element of the weak current $J_\mu$ sandwiched by the $B_{c}^+$ meson state as the initial state and a single-hadron state of the concerned final state, i.e., $\langle X(p', \epsilon)|J_\mu|B_{c}^+(p)\rangle$ with $X$ being a given suitable meson. As that in Ref. [13], with the help of the Mandelstam formalism [30], which is one of proper approaches to compute the hadron matrix elements sandwiched by the BS wave functions of the two bound-state, no matter how great the recoil momentum carried by the elements is. With this method and the instantaneous approximation, the hadron matrix elements can be expressed in terms of momenta $p$ and $p'$ of the initial state and final state mesons, respectively. The coefficients, being the functions of the momentum transfer $(p-p')$, are Lorentz-invariant and usually called as form factors. To calculate these form factors, we adopt the method used in Refs. [9, 13] but with improvements, i.e., the form factors are still written as overlap integrals of the relevant wave functions for the bound states (mesons), but the wave functions are obtained by solving the relevant instantaneous BS equation with the improved method [31]. To have the general feature of the improved method, here we put an outline of the method in Appendix. Furthermore, here we temporarily constrain ourselves to consider the cases that $X$ is an $S$-wave meson in the decays only.

According to the Mandelstam formalism and with wave functions of instantaneous BS equation(s), in the leading order, the matrix element $\langle X(p')|J_\mu|B_{c}^+(p)\rangle$ can be written as [32]:

$$\langle X(p')|J_\mu|B_{c}^+(p)\rangle = \int \frac{d^4q'd^4q}{(2\pi)^4} \text{Tr} \left\{ \bar{\chi}_{p'} J_\mu \chi_p S_2(2)^{-1}(p_2-p'_2)\right\}$$

$$= \int \frac{d^3q}{(2\pi)^3} \text{Tr} \left[ \bar{\varphi}_{p'+}(q + \alpha'_r \vec{r})\gamma_\mu (1 - \gamma_5) \varphi_{p}^{++}(q) \right] \frac{p'_\mu}{M}(q_\mu) \tag{7}\right.$$  

here for the last equal sign we have chosen the center of mass system of initial meson $B_{c}^+$; $S_2(2)-p_2$ is the propagator of the second component ("spectator"); $\vec{r}$ is the three dimensional momentum of final hadron state $X$ and $\alpha'_2 = m_2^2/(m_1^2 + m_2^2)$; $\varphi_{p}^{++}$ is the component of BS wave function projected onto the "positive energy" for the relevant mesons, and may be obtained by solving the BS equation. Its definition can be found in Appendix. Since the initial and final states in the transition are both heavy mesons, as adopted in Eq. (7), it is a good approximation that only positive energy projected BS wave functions are included (the contributions from the component of the wave functions projected onto the "negative energy" are much smaller than that from the positive energy one).

The form factors can be generally related to the weak current matrix element as follows:

1. If $X$ is a $^3S_1$ state, the axial vector matrix element:

$$\langle X(p')|V_\mu|B_{c}^+(p)\rangle = f_+(p+p')_\mu + f_-(p-p')_\mu \tag{8}$$

2. If $X$ is a $^3S_1$ state, the axial vector matrix element:

$$\langle X(p', \epsilon)|A_\mu|B_{c}^+(p)\rangle = f_+^{*}_+(\epsilon \cdot p)(p+p')_\mu + a_-(\epsilon \cdot p)(p-p')_\mu \tag{9}$$

and the vector current matrix element:

$$\langle X(p', \epsilon)|V_\mu|B_{c}^+(p)\rangle = ig_\epsilon \epsilon_\nu p_\nu \epsilon^{*}_\mu (p+p')_\sigma (p-p')_\sigma \tag{10}$$

where $\epsilon$ is the polarization vector of the final hadron $X$.

Therefore using Eq. (7) and with the relation between the matrix element and the form factors above, we can calculate the form factors. Explicit expressions for the form factors as overlap integrals of meson wave functions are given in Appendix. Correspondingly, the coefficient functions $\alpha$, $\beta$ and $\gamma$ in Eq. (6) can be expressed in terms of the form factors. For example, for the decay $B_{c}^+ \rightarrow P \ell^+ \bar{\nu}_\ell$ ($P$ is a pseudoscalar meson) we have:

$$\begin{align*}
\alpha &= \gamma = 0,
\beta_{++} &= f_{++}^2, \\
\beta_{+-} &= f_{+}^2 + f_{-}^2, \\
\beta_{-+} &= f_{-}^2, \\
\beta_{--} &= f_{-}^2 + f_{+}^2 + 4 f_{a}^2 + \frac{1}{2 M^2} (M^2 - 1) f_{a_+} + f_{a_+}^2.
\end{align*} \tag{11}$$

For the decay $B_{c}^+ \rightarrow V \ell^+ \bar{\nu}_\ell$ ($V$ is a vector meson) we have:

$$\begin{align*}
\alpha &= f_{a}^2 + 4 M^2 g^2, \\
\beta_{++} &= \frac{f_{a}^2}{4 M^2} - M^2 y^2 g^2 + \left[ \frac{M^2}{M^2} - 1 \right] f_{a_+} + \frac{1}{2 M^2} f_{a_+}^2, \\
\beta_{+-} &= f_{a}^2 - f_{a_+}^2, \\
\beta_{-+} &= f_{a_+}^2 + 2 g^2 M^2 y^2 - 1 + \left[ \frac{M^2}{M^2} - 1 \right] f_{a_+} + \frac{1}{2 M^2} f_{a_+}^2, \\
\gamma &= 2 f_{a} g.
\end{align*} \tag{12}$$

Putting the above form factors into the formula for differential decay rates Eq. (6), the concerned semileptonic decay rates can be calculated.

For the nonleptonic decays $B_{c}^+ \rightarrow X + \pi(K, \rho, K*)$ concerned here, we follow Ref. [3] to take the CKM-favored effective Hamiltonian with QCD leading logarithm correction to be responsible for them:

$$H_{eff}^b = \frac{G_F}{\sqrt{2}} V_{cb} \left[ e_c^{cb} (\mu_b) Q_1^{cb} + e_c^{cb} (\mu_b) Q_2^{cb} \right] + h.c.,$$

$$H_{eff}^c = \frac{G_F}{\sqrt{2}} V_{cs} \left[ e_c^{cs} (\mu_c) Q_1^{cs} + e_c^{cs} (\mu_c) Q_2^{cs} \right] + h.c. \tag{13}$$
where $c_i^c(\mu_c) = c_i^s(m_c)$ and $c_i^b(\mu_b) = c_i^s(m_b)$ are the Wilson coefficients, and the four-fermion operators $Q_i^{ij}$ and $Q_{ij}^{q}$ are defined:

\[
Q_1^{bc} \equiv ([d\bar{u}]_{V-A} + (\bar{s}c)_{V-A})([\bar{c}b]_{V-A},
Q_2^{bc} \equiv (\bar{c}c)_{V-A}([s\bar{b}]_{V-A} + (\bar{c}u)_{V-A}([d\bar{b}]_{V-A},
Q_1^{cs} \equiv (\bar{c}s)_{V-A}([d\bar{u}]_{V-A},
Q_2^{cs} \equiv (\bar{d} s)_{V-A}([\bar{c}u]_{V-A},
\]

$d'$ and $s'$ denote 'down' and 'strange' weak eigenstates. Based on the QCD Renormalization Group calculation, and in terms of the combination operators $Q_{1,2} = (Q_1 \pm Q_2)$ which have diagonal anomalous dimensions, the corresponding Wilson coefficients read as follows:

\[
c_+^c(\mu) = \left[ \frac{\alpha_s(M_W)}{\alpha_s(M_b)} \right]^{6/23} \left[ \frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right]^{6/25},
\]

\[
c_-^c(\mu) = \left[ e^c_+ - (\mu) \right]^{-2},
\]

\[
c_+^b(\mu) = \left[ \frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right]^{6/23} \left[ \frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right]^{-3/25},
\]

\[
c_-^b(\mu) = \left[ \frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right]^{-12/23} \left[ \frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right]^{-12/25}.
\]

Then to use 'naive factorization' as done in Ref. [9], the $T$-matrix element can be written as:

\[
T = G_F \frac{V_{ij}V_{ik}}{\sqrt{2}} a_1 \langle L(k, \epsilon') | J^\mu | 0 \rangle \langle X(p', \epsilon') | J_\mu | B_c^+(p) \rangle,
\]

where $V_{ij}, V_{ik}$ are the relevant CKM matrix elements to $L$ and $X$ accordingly, $L = \pi(K, \rho, K^*)$, $p, p', k$ are the momenta of $B_c$, $X$ and $\pi(K, \rho, K^*)$ respectively, and $\epsilon', \epsilon$ are the polarization vectors for $\rho$ or $K^*$ and $X$ when $X$ is a vector meson. The parameter

\[
a_1 = c_1(\mu) + \xi c_2(\mu), \quad \xi = \frac{1}{N_c}
\]

in Eq. (15) is attributed to the contribution from the operators $Q_1$ and that from the Fierz-reordered $Q_2$ with a suppressed factor $\xi$ to the concerned decays.

For the two-body decays $B_c^+ \to X + L^+$ concerned here, having the $T$ matrix element Eq. (15), it is straightforward to calculate the decay widths.

\[\text{II. NUMERICAL RESULTS}\]

The components of the meson $B_c$ are $\bar{b}$ and $c$ quarks, and it happens that the contributions from each of them to the total decay rate are comparable in magnitude. Thus the semileptonic decay modes of $B_c$ meson can be classified into two: $b$-quark decays with the $c$ quark inside the meson as a spectator, and $c$-quark decays with the $\bar{b}$ quark as a spectator. The former causes $B_c$ decays into charmonium or $D$-meson pair, while the latter causes $B_c$ decays into $B_s$ or $B$ mesons. In this paper, we restrict ourselves to compute $B_c$ decays to charmonium or $B_s$ meson only because the approach adopted here is good for double heavy mesons.

Under the adopted approach, there are several parameters needed to be fixed when calculating the decays. The parameters are fixed by fitting well-measured experimental data and the established potential model. The parameters appearing in the potential (the kernel of Salpeter equation) used in this work are fixed by the spectra of heavy quarkonia as done in Ref. [32] and outlined in Appendix B4. The masses of the ground states are used as inputs, while the masses of excited states are considered as predictions. In this way we obtain $M_{q_s(2S)} = 3.576$ GeV and $M_{q_0(2S)} = 3.686$ GeV, and to compare with experimental data $M_{q_s}^{exp} = 3.637$ GeV and $M_{q_0}^{exp} = 3.686$ GeV, the predictions are quite good.

The values of the CKM matrix elements adopted in this paper are $V_{cb} = 0.0406, V_{cs} = 0.9735, V_{ud} = 0.974$ and $V_{us} = 0.2252$. The properties of relevant light mesons appearing in the concerned nonleptonic decays are served as phenomenological inputs, namely we take

\[
M_\pi = 0.140 \text{ GeV}, \quad f_\pi = 0.130 \text{ GeV},
M_\rho = 0.775 \text{ GeV}, \quad f_\rho = 0.205 \text{ GeV},
M_K = 0.494 \text{ GeV}, \quad f_K = 0.156 \text{ GeV},
M_{K^*} = 0.892 \text{ GeV}, \quad f_{K^*} = 0.217 \text{ GeV},
\]

where the masses and the decay constants are taken from PDG [34], except $f_\rho$ and $f_{K^*}$, which are quoted from Ref. [35].

The numerical results of semileptonic decays are presented in Table I. Results from other approaches are also listed in the table for comparison. To see the feature of the decays, we plot the lepton spectrum for the decays $B_c^+ \to P(V) + \ell^+ + \nu_\ell$ in Fig. 2 and Fig. 3 respectively.

The concerned nonleptonic decay modes for $c$-spectator decays and $b$-spectator decays computed precisely and the obtained results as well as those from other approaches are presented in Table II and Table III respectively.

\[\text{III. DISCUSSION AND CONCLUSION}\]

If comparing the semileptonic and nonleptonic decays estimated by various approaches in terms of Tables I-III, one may find that the deviations among the theoretical predictions by the various approaches are quite wide. Specifically, the results with new solutions of the Salpeter equation and new formulation are quite different from those in Ref. [3] too.

\[\text{2 Since we restrict ourselves to consider the decays $B_c^+ \to X + \pi(K, \rho, K^*)$ here, so we list the main operators the $Q_1^{ij}$ and $Q_2^{ij}$ only which relate and greatly contribute to the decays.}\]
TABLE I: The decay widths of the exclusive semileptonic decay modes (in $10^{-15}$ GeV).

| Mode                  | Ours  | [9]   | [10]  | [11]  | [15]  | [16]  | [17]  | [18]  | [19]  | [21]  | [22]  | [23]  |
|-----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $B_s^+ \to \eta_c e^+ \bar{\nu}_e$ | 8.02  | 14.2  | 11    | 11.1  | 13.05 | 8.9   | 14    | 10    | 4.3   | 10.6  | 8.3   | 6.3   |
| $B_c^+ \to B_c e^+ \bar{\nu}_e$    | 19.7  | 26.6  | 59    | 14.3  | 22.0  | 12    | 29    | 18    | 11.75 | 16.4  | 26.8  | 11.1  |
| $B_c^+ \to J/\psi e^+ \bar{\nu}_e$ | 25.2  | 34.4  | 28    | 30.2  | 26.6  | 17.7  | 33    | 42    | 16.8  | 38.5  | 20.3  | 21.8  |
| $B_c^+ \to B_s^+ e^+ \bar{\nu}_e$  | 39.9  | 44.0  | 65    | 50.4  | 51.2  | 25    | 37    | 43    | 32.56 | 40.9  | 34.6  | 43.7  |
| $B_c^+ \to \eta_c(2S)e^+ \bar{\nu}_e$ | 0.969 | 0.727 | 0.28  |       |       |       |       | 0.46  |       |       | 0.605 |       |
| $B_c^+ \to \psi(2S)e^+ \bar{\nu}_e$ | 1.49  | 1.45  | 1.36  |       |       |       |       | 0.44  |       |       | 0.186 |       |

TABLE II: The decay widths of the exclusive nonleptonic decay modes with $c$-quark spectator (in $10^{-15}$ GeV).

| Mode                  | Ours  | [9]   | [10]  | [11]  | [15]  | [16]  | [17]  | [18]  | [19]  | [20]  | [21]  | [22]  | [23]  |
|-----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $J/\psi + \pi$        | 1.24a | 1.97a | 1.43a | 1.22a | 0.82a | 0.67a | 1.79a | 1.01a |       |       |       |       |       |
| $J/\psi + K$          | 0.0949a | 0.152a | 0.12a | 0.090a | 0.079a | 0.052a | 0.130a | 0.0764a |       |       |       |       |       |
| $J/\psi + \rho$       | 3.59a | 5.95a | 4.37a | 3.48a | 2.32a | 1.8a | 5.07a | 3.25a |       |       |       |       |       |
| $J/\psi + K^*$        | 0.226a | 0.324a | 0.25a | 0.197a | 0.18a | 0.11a | 0.263a | 0.174a |       |       |       |       |       |
| $\psi(2S) + \pi$      | 0.298a | 0.251a |       |       |       |       |       |       |       |       |       |       |       |
| $\psi(2S) + K$        | 0.0218a | 0.018a |       |       |       |       |       |       |       |       |       |       |       |
| $\psi(2S) + \rho$     | 0.765a | 0.710a |       |       |       |       |       |       |       |       |       |       |       |
| $\psi(2S) + K^*$      | 0.0450a | 0.038a |       |       |       |       |       |       |       |       |       |       |       |
| $\eta_c + \pi$        | 1.18a | 2.07a | 1.8a | 1.59a | 1.47a | 0.93a | 1.71a | 1.49a |       |       |       |       |       |
| $\eta_c + K$          | 0.0919a | 0.161a | 0.15a | 0.119a | 0.15a | 0.073a | 0.127a | 0.115a |       |       |       |       |       |
| $\eta_c + \rho$       | 2.89a | 5.48a | 4.5a | 3.74a | 3.35a | 2.3a | 4.04a | 3.93a |       |       |       |       |       |
| $\eta_c + K^*$        | 0.172a | 0.286a | 0.22a | 0.200a | 0.24a | 0.12a | 0.203a | 0.198a |       |       |       |       |       |
| $\eta_c(2S) + \pi$    | 0.322a | 0.268a |       |       |       |       |       |       |       |       |       |       |       |
| $\eta_c(2S) + K$      | 0.0242a | 0.020a |       |       |       |       |       |       |       |       |       |       |       |
| $\eta_c(2S) + \rho$   | 0.711a | 0.622a |       |       |       |       |       |       |       |       |       |       |       |
| $\eta_c(2S) + K^*$    | 0.0408a | 0.031a |       |       |       |       |       |       |       |       |       |       |       |

In order to calculate the decay branching ratio of semileptonic and nonleptonic decays, the lifetime of $B_c$ meson is needed, whereas for the nonleptonic decays considered in the paper, the parameter $a_1$ for nonleptonic decays appearing in Eq. (13), additionally, is needed to be evaluated too. For this purpose, we take the experimental lifetime from PDG [34]. Note that $a_1$ for $b$ quark (denoted as $a_1^b$) decays should be different from $a_1$ for $c$ quark (denoted as $a_1^c$) decays, and we take $a_1^b = 1.14$ and $a_1^c = 1.2$ as in Refs. [11, 17, 18, 25, 27]. Having the lifetime and the parameter $a_1$ fixed, the branching ratio of the concerned decay modes are straightforwardly calculated and we put the results in Table IV and Table V respectively.

Recently, LHCb has reported an observation of decay $B_c^+ \to \psi(2S)\pi^+$, and obtained the ratio \[ \frac{BR(B_c^+ \to \psi(2S)\pi^+)}{BR(B_c^+ \to J/\psi\pi^+)} \] (17) \[ = 0.250 \pm 0.068(\text{stat}) \pm 0.014(\text{syst}) \pm 0.006(B). \]
TABLE III: The decay widths of the exclusive nonleptonic decay modes with \( b \)-quark spectator (in \( 10^{-13} \) GeV).

| Mode | Ours | \([9]\) | \([11]\) | \([15]\) | \([16]\) | \([17]\) | \([22]\) | \([23]\) |
|------|------|------|------|------|------|------|------|------|
| \( B_c^+ + \pi \) | 46.5a_1^2 | 58.4a_1^2 | 167a_1^2 | 15.8a_1^2 | 34.8a_1^2 | 25a_1^2 | 44.0a_1^2 | 66.1a_1^2 |
| \( B_c^+ + K \) | 3.55a_1^2 | 4.20a_1^2 | 10.7a_1^2 | 1.70a_1^2 | 2.1a_1^2 | 3.28a_1^2 | 4.60a_1^2 |
| \( B_c^+ + \rho \) | 26.5a_1^2 | 44.8a_1^2 | 72.5a_1^2 | 39.2a_1^2 | 23.6a_1^2 | 14a_1^2 | 20.2a_1^2 | 42.7a_1^2 |
| \( B_c^+ + K^* \) | 0.0862a_1^2 | 1.06a_1^2 | 66.3a_1^2 | 12.5a_1^2 | 19.8a_1^2 | 16a_1^2 | 34.7a_1^2 | 25.3a_1^2 |
| \( B_c^+ + \pi \) | 31.4a_1^2 | 51.6a_1^2 | 66.3a_1^2 | 12.5a_1^2 | 19.8a_1^2 | 16a_1^2 | 34.7a_1^2 | 25.3a_1^2 |
| \( B_c^+ + K \) | 1.66a_1^2 | 2.96a_1^2 | 3.8a_1^2 | 1.34a_1^2 | 1.1a_1^2 | 2.52a_1^2 | 1.34a_1^2 |
| \( B_c^+ + \rho \) | 139a_1^2 | 150a_1^2 | 204a_1^2 | 171a_1^2 | 123a_1^2 | 110a_1^2 | 152a_1^2 | 139a_1^2 |

TABLE IV: The branching ratio (in \%) of the exclusive nonleptonic decay modes with the lifetime of the \( B_c: \tau_{B_c} = 0.452ps \).

| Mode | BR (%) |
|------|--------|
| \( B_c^+ \to \eta \pi^+ \bar{\nu}_e \) | 0.55 |
| \( B_c^+ \to \eta_c \pi^+ \bar{\nu}_e \) | 1.35 |
| \( B_c^+ \to J/\psi \pi^+ \bar{\nu}_e \) | 1.73 |
| \( B_c^+ \to \rho \eta_c \pi^+ \bar{\nu}_e \) | 2.74 |
| \( B_c^+ \to \psi(2S) \eta_c \pi^+ \bar{\nu}_e \) | 0.0665 |
| \( B_c^+ \to \psi(2S) \pi^+ \bar{\nu}_e \) | 0.103 |

We think that the measured ratio, in contrary to the others measurements, without relating to the production of \( B_c \) meson at all, is an essential test of the decays and would like to point out that the corresponding ratio given by our results is

\[
\frac{\text{BR}(B_c^+ \to \psi(2S)\pi^+)}{\text{BR}(B_c^+ \to J/\psi \pi^+)} = 0.24,
\]

which is in good agreement with the observation. Moreover, the parameter \( a_1 \) which appears in Eq. \((15)\) and the theoretical uncertainties caused by naive factorization for the nonleptonic decays would be canceled in calculating the ratio, thus we suspected that the ratio is mostly determined by hadron transition, and this agreement between the experimental value and the result on the ratio indicates a quite strong support of the present approach.

In summary, we have calculated the decay width and branching ratio of the exclusive semileptonic decays of \( B_c \) meson to a charmonium or a \( B_s \) meson plus leptons and nonleptonic decays to a charmonium or a \( B_s \) meson plus a light meson under the improved instantaneous BS equation and Mandelstam approach. Under this approach, the full Salpeter equations for \((bc), (bs)\) and \((\bar{c}c)\) etc systems are solved with the respective full relativistic wave functions for \( J^P = 0^- \) and \( J^P = 1^- \) states. To calculate the hadron transition matrix elements, the Mandelstam formula has been used and it is suitably approximated to fit the instantaneous approximation. We find that the results with this approach seems to have been improved in comparison with those obtained by the early one in Ref. \([6]\) with the more approximated formulation. Finally, we would like to point out that so far only the experimental ratio \( \text{BR}(B_c^+ \to \psi(2S)\pi^+) \) is available, and the two involved decay modes in the ratio are two-body decays, so the test of the approaches are limited, therefore more experimental data on details of the semileptonic decays, e.g., the decay spectrum of the positron, and various decay modes for the nonleptonic decays, which are independent on the production of \( B_c \) meson, are requested, so as to conclude all the approaches in literature.

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Appendix A: INSTANTANEOUS BS EQUATION

BS equation for a quark-antiquark bound state generally is written as:

\[
\chi_p(q) = \frac{1}{p^2 - m^2} \int \frac{d^4k}{(2\pi)^4} V(P, k, q)\chi_p(k) \frac{1}{p^2 + m_2^2}(A1)
\]

where \( p_1, p_2; m_1, m_2 \) are the momenta and masses of the quark and anti-quark, respectively. \( \chi_p(q) \) is the BS
wave function with the total momentum $P$ and relative momentum $q$, $V(P, k, q)$ is the kernel between the quark-antiquark in the bound state. $P$ and $q$ are defined as:

$$p_1 = \alpha_1 P + q, \quad \alpha_1 = \frac{m_1}{m_1 + m_2},$$

$$p_2 = \alpha_2 P - q, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}.$$

Moreover, the BS wave function $\chi_{\mu}(q)$ satisfies the normalization condition:

$$\int \frac{d^4k d^4l}{(2\pi)^8} Tr \left\{ \chi_{\mu}(k) \frac{\partial}{\partial P_0} \left[ S_1^{-1}(p_1) S_2^{-1}(p_2) \delta^4(k - q) \right. \right.$$

$$\left. + V(P, k, q) \chi_{\mu}(q) \right\} = 2i F_0, \quad (A2)$$

where $S_1(p_1)$ and $S_2(p_2)$ are the propagators of the quark and anti-quark, respectively.

In general, the BS equation in four dimensional ‘relative’ space-time is hard to solve comprehensively. Whereas if the bound states are formed by heavy components (quarks) then the kernel of the equation may approximately become an instantaneous one, and one may overcome the difficulty to solve the equation in four dimensional ‘relative’ space-time instead by adopting a so-called instantaneous approximate approach to turn the equation into a one in three dimension. The proposal by Salpeter [29] is the approach, that the time-like component of the relative momentum is integrated out in terms of a contour integration so the BS equation in four dimension is reduced to a one in three dimension finally when the kernel is an instantaneous one. For double heavy bound states, here we follow the Salpeter approach but less approximations than he did. Let us outline our approach (Salpeter’s with less approximations) here. The approximately instantaneous kernel has the following form:

$$V(P, k, q) \sim V(|k - q|), \quad (A3)$$

especially, it is the case, when the two constituents of meson is very heavy.

Since the recoil in momentum may be great for the concerned $B_s$ semileptonic decays, so for convenience even under instantaneous approximation we reduce and solve the BS equation in a Lorentz covariant form, i.e., to divide the relative momentum $q$ into two parts, $q_{||}$ and $q_{\perp}$, a parallel part and an orthogonal one to $P$, respectively:

$$q^\mu = q_{||}^\mu + q_{\perp}^\mu, \quad (A4)$$

where $q_{||}^\mu \equiv (P \cdot q/M^2)P^\mu$, $q_{\perp}^\mu \equiv q^\mu - q_{||}^\mu$, and $M$ is the mass of the relevant meson. Correspondingly, we have two Lorentz invariant variables:

$$q_\perp = \frac{P \cdot q}{M}, \quad q_{\perp} = \sqrt{q_{\perp}^2 - q^2} = \sqrt{-q_{\perp}^2}. \quad (A5)$$

It is easy to see that they turn to the usual component $q_0$ and $|q|$ if in the frame of $\vec{P} = 0$. In the same sense, the volume element of a relative momentum $k$ can be written in an invariant form:

$$d^4k = dk_r k_r^2 dk_{\perp} d\theta d\phi, \quad (A6)$$

where $\phi$ is the azimuthal angle, $s = (k_r q_r - k_q)/k_{\perp} q_{\perp}$, and $k_{\perp}, q_{\perp}$ are the components of $k$ and $q$ transverse to $P$. So now the instantaneous interaction kernel Eq. (A3) can be rewritten as:

$$V(\tilde{k} - q) = V(k_{\perp}, s, q_{\perp}). \quad (A7)$$

If we introduce two notations as below:

$$\eta(q_{\perp}^\mu) \equiv \int \frac{k_2^2 dk_{\perp}}{(2\pi)^2} V(k_{\perp}, s, q_{\perp}) \varphi_{\perp}(k_{\perp}^\mu),$$

$$\varphi_{\perp}(q_{\perp}^\mu) \equiv i \int \frac{dq_{\perp}}{2\pi} \chi_{\perp}(q_{\perp}^\mu). \quad (A8)$$

Then the BS equation can be take the form as follows:

$$\chi_{\perp}(q_{\perp}^\mu) = S_1(p_{||}^\mu) \eta(q_{\perp}^\mu) S_2(p_{\perp}^\mu). \quad (A9)$$

The propagator of the relevant particles with masses $m_1$ and $m_2$ can be decomposed as:

$$S_i(p_{i}^\mu) = \frac{\Lambda_{i\mu}^+(q_{i\perp}^\mu)}{J(i) q_{\perp}^\mu + \alpha_i M - \omega_{i\perp} + i\varepsilon}$$

$$+ \frac{\Lambda_{i\mu}^-(q_{i\perp}^\mu)}{J(i) q_{\perp}^\mu + \alpha_i M + \omega_{i\perp} - i\varepsilon}, \quad (A10)$$

with

$$\omega_{i\perp} = \sqrt{m_i^2 + q_{i\perp}^2},$$

$$\Lambda_{i\mu}^\pm(q_{i\perp}^\mu) = \frac{1}{2\omega_{i\perp}} \left[ \frac{P}{M} \omega_{i\perp} \pm J(i) (m_i + q_{i\perp}) \right]. \quad (A11)$$

where $i=1, 2$ for the quark and anti-quark, respectively, and $J(i) = (-1)^{i+1}$. $\Lambda_{i\mu}^\pm(q_{i\perp}^\mu)$ satisfies the relations as follows:

$$\Lambda_{i\mu}^+(q_{i\perp}^\mu) + \Lambda_{i\mu}^-(q_{i\perp}^\mu) = \frac{P}{M},$$

$$\Lambda_{i\mu}^+(q_{i\perp}^\mu) \Lambda_{i\mu}^-(q_{i\perp}^\mu) = \frac{P}{M} \Lambda_{i\mu}^\pm(q_{i\perp}^\mu), \quad (A12)$$

In fact, $\Lambda^\pm$ may be considered as “covariant energy-projection” operators, i.e., in the rest frame $\vec{P} = 0$, they turn to the energy projection operator.

Introducing notations:

$$\varphi_{\perp}^\pm(q_{\perp}^\mu) \equiv \Lambda_{\perp\mu}^\pm(q_{\perp}^\mu) \frac{P}{M} \varphi_{\perp}(q_{\perp}^\mu) \frac{P}{M} \Lambda_{\perp\mu}^\pm(q_{\perp}^\mu), \quad (A13)$$

and taking into account $\frac{P}{M} \frac{P}{M} = 1$, we have:

$$\varphi_{\perp}(q_{\perp}^\mu) = \varphi_{\perp}^+ (q_{\perp}^\mu) + \varphi_{\perp}^- (q_{\perp}^\mu)$$

$$+ \varphi_{\perp}^-(q_{\perp}^\mu) + \varphi_{\perp}^-(q_{\perp}^\mu).$$
Let us further integrate $q$, out on both sides of Eq. (A9), and obtain:

$$
\varphi_p(q_{p_\perp}) = \frac{\Lambda_{1_p}^+(q_{p_\perp})\eta_p(q_{p_\perp})\Lambda_{2_p}^+(q_{p_\perp})}{M - \omega_1 - \omega_2} - \frac{\Lambda_{1_p}^+(q_{p_\perp})\eta_p(q_{p_\perp})\Lambda_{2_p}^+(q_{p_\perp})}{M + \omega_1 + \omega_2}.
$$

We decompose it into the coupled equations:

$$(M - \omega_1 - \omega_2)\varphi_p^+(q_{p_\perp}) = \Lambda_{1_p}^+(q_{p_\perp})\eta_p(q_{p_\perp})\Lambda_{2_p}^+(q_{p_\perp}),$$

$$(M + \omega_1 + \omega_2)\varphi_p^-(q_{p_\perp}) = -\Lambda_{1_p}^-(q_{p_\perp})\eta_p(q_{p_\perp})\Lambda_{2_p}^-(q_{p_\perp}).$$

Correspondingly, the normalization condition of Eq. (A2) in covariant form reads:

$$
\int \frac{d^2q_{p_\perp}}{(2\pi)^2} tr \left[ \varphi^+ P + \frac{P}{M} \varphi^+ - \frac{P}{M} \varphi - \frac{P}{M} \right] = 2P_0.
$$

If binding is weak, the positive energy components of the wave functions $\varphi^+$ are large owing to having a very small factor $(M - \omega_1 - \omega_2)$, so one can keep the first equation of Eq. (A14) only, and safely dropped the rest equations at the lowest-order approximation. In Ref. [9] it is the case for the heavy quarkonium and $B_c$ meson.

Appendix B: To Solve the INSTANTANEOUS BS EQUATION and the relativistic wave functions

Now let us solve the equations for heavy mesons.

1. Heavy Pseudoscalar Mesons

The relativistic wave function for heavy pseudoscalar mesons with the quantum numbers $J^P = 0^-$ can be generally written as the four terms constructed by $P$, $q_{p_\perp}$ and gamma matrices [30]:

$$
\varphi_{0-}(q_{p_\perp}) = \left[ f_1(q_{p_\perp}) P + f_2(q_{p_\perp}) M + f_3(q_{p_\perp}) q_{p_\perp} + f_4(q_{p_\perp}) \frac{P}{M} q_{p_\perp} \right] \gamma_5,
$$

where $M$ is the mass of the pseudoscalar meson. Due to the last two equations of Eq. (A14): $\varphi_{0-}^+ = \varphi_{0-}^- = 0$.

Then there are only two independent wave functions $f_1(q_{p_\perp})$ and $f_2(q_{p_\perp})$ being left in the Eq. (B1):

$$
\varphi_{0-}(q_{p_\perp}) = \left[ f_1(q_{p_\perp}) P + f_2(q_{p_\perp}) M \right] - f_2(q_{p_\perp}) M \frac{M(\omega_1 - \omega_2)}{m_2 \omega_1 + m_1 \omega_2} + f_1(q_{p_\perp}) M \frac{M(\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2} \gamma_5 (B3)
$$

According to Eq. (A13) we can further obtain the wave function corresponding to the positive projection:

$$
\varphi_{0+}(q_{p_\perp}) = \left[ f_1(q_{p_\perp}) P + f_2(q_{p_\perp}) M \right] - \frac{f_2(q_{p_\perp}) M}{m_2 \omega_1 + m_1 \omega_2} f_1(q_{p_\perp}) - \frac{f_1(q_{p_\perp}) M}{m_2 \omega_1 + m_1 \omega_2} f_2(q_{p_\perp}) \gamma_5 (B4)
$$

where

$$
L = \frac{M}{2}(f_1 + f_2 m_1 + m_2),
$$

$$
N = \frac{\omega_1 + \omega_2}{m_1 + m_2},
$$

$$
Y = \frac{m_2 - m_1}{m_2 \omega_1 + m_1 \omega_2},
$$

$$
Z = \frac{\omega_1 + \omega_2}{m_2 \omega_1 + m_1 \omega_2}.
$$

The normalization condition reads:

$$
\int \frac{d^4q}{(2\pi)^4} |f_1 f_2 M| \left( \frac{m_1 + m_2}{\omega_1 + \omega_2} + \frac{\omega_1 + \omega_2}{m_1 + m_2} \right) + \frac{2q_0 (m_1 \omega_1 + m_2 \omega_2)}{(m_2 \omega_1 + m_1 \omega_2)^2} = 2M. (B5)
$$

Putting Eq. (B3) into the first two equations of Eq. (A14), we obtain two coupled integral equations about $f_1(q_{p_\perp})$ and $f_2(q_{p_\perp})$, i.e., finally the numerical relativistic wave functions Eq. (B3) with $f_1(q_{p_\perp})$ and $f_2(q_{p_\perp})$ being given for the corresponding pseudoscalar mesons are obtained. Since the $B_c$ and $\eta_c$, $B_s$ etc. are pseudoscalar mesons, so the relativistic wave functions of them, which are needed in calculating the weak current matrix elements for the concerned semileptonic decays of $B_s$, are obtained in this way. Note that s-quark has a mass $m_s \sim 0.5$ GeV, here we consider it is still “heavy” although people consider it is light one, thus for the same reason we are quite sure that the results about $B_s$ are not so good as those about $\eta_c$ and $B_c$ etc. The same note for $B_s^*$ is applicable in the next subsection.

2. Heavy Vector Mesons

The relativistic wave function of heavy vector state $J^P = 1^-$ generally has 8 terms based on $P$, $q_{p_\perp}$, $\epsilon$ (polarization vector) and gamma matrices, so the general
form for the relativistic Salpeter wave function for $1^-$ states can be written as [32, 37]:

$$
\varphi_{1^-}^\lambda(q_{p_\lambda}) = q_{p_\lambda} \cdot \epsilon_{\lambda} \left[ f_1(q_{p_\lambda}) + f_2(q_{p_\lambda}) \frac{P}{M} 
+ f_3(q_{p_\lambda}) \frac{q_{p_\lambda}^2}{M} f_4(q_{p_\lambda}) \frac{P q_{p_\lambda}}{M^2} \right] 
+ f_5(q_{p_\lambda}) M \phi_{\lambda} + f_6(q_{p_\lambda}) \varphi_{\lambda} P 
+ f_7(q_{p_\lambda}) (\varphi_{\lambda} - q_{p_\lambda} \cdot \epsilon_{\lambda}) 
+ f_8(q_{p_\lambda}) \left[ (P \varphi_{\lambda} - P q_{p_\lambda} \cdot \epsilon_{\lambda}) \right].
$$

Putting the constrains into Eq. (B6), one can rewrite the relativistic Salpeter wave function for the states $1^-$ as:

$$
\varphi_{1^-}^\lambda(q_{p_\lambda}) = q_{p_\lambda} \cdot \epsilon_{\lambda} \left[ f_3(q_{p_\lambda}) q_{p_\lambda}^2 + f_5(q_{p_\lambda}) M^2 \right] 
\times \frac{(m_1 m_2 - \omega_1 \omega_2 + q_{p_\lambda}^2)}{M (m_1 + m_2)},
$$

where $M$ is the mass of the vector meson. The equations $\varphi_{0^-}^\lambda = \varphi_{0^-}^\lambda = 0$ give the following constrains on the components of the wave function:

$$
\begin{align*}
&f_1(q_{p_\lambda}) = \left[ f_3(q_{p_\lambda}) q_{p_\lambda}^2 + f_5(q_{p_\lambda}) M^2 \right] 
\times \frac{(m_1 m_2 - \omega_1 \omega_2 + q_{p_\lambda}^2)}{M (m_1 + m_2)}, \\
&f_7(q_{p_\lambda}) = \frac{f_5(q_{p_\lambda}) M (-\omega_1 + \omega_2)}{m_2 \omega_2 + m_1 \omega_2}, \\
&f_2(q_{p_\lambda}) = \left[ -f_4(q_{p_\lambda}) q_{p_\lambda}^2 + f_6(q_{p_\lambda}) M^2 \right] 
\times \frac{(m_1 \omega_2 - m_2 \omega_1)}{M (\omega_1 + \omega_2) q_{p_\lambda}^2}, \\
&f_8(q_{p_\lambda}) = \frac{f_6(q_{p_\lambda}) M (\omega_1 \omega_2 - m_1 m_2 - q_{p_\lambda}^2)}{(m_1 + m_2) q_{p_\lambda}^2}.
\end{align*}
$$

Furthermore, we can obtain the wave function corresponding to the positive projection by Eq. (A13):

$$
\varphi_{1^+}^\lambda(q_{p_\lambda}) = A \varphi_{1^+}^\lambda + B \varphi_{1^+}^\lambda P + C(q_{p_\lambda}^2 - q_{p_\lambda} \cdot \epsilon_{\lambda}) 
+ D(P \varphi_{\lambda} - P q_{p_\lambda} \cdot \epsilon_{\lambda}) + q_{p_\lambda} \cdot \epsilon_{\lambda} 
\times (E + F P + G q_{p_\lambda} + H P q_{p_\lambda}).
$$

where

$$
\begin{align*}
A &= \frac{1}{2} \left( f_5 - \frac{\omega_1 + \omega_2}{m_1 + m_2} \right), \\
B &= \frac{1}{2} \left( f_5 - \frac{\omega_1 + \omega_2}{m_1 + m_2} \right), \\
C &= \frac{1}{2} \left( f_5 - \frac{\omega_1 + \omega_2}{m_1 + m_2} \right), \\
D &= \frac{1}{2} \frac{\omega_1 + \omega_2}{m_1 m_2} \left( f_5 - \frac{\omega_1 + \omega_2}{m_1 + m_2} \right), \\
E &= \frac{1}{2} \frac{\omega_1 - \omega_2}{m_1 m_2} \left( f_5 - \frac{\omega_1 + \omega_2}{m_1 + m_2} \right), \\
F &= \frac{1}{2} \frac{\omega_1 - \omega_2}{m_1 m_2} \left( f_5 - \frac{\omega_1 + \omega_2}{m_1 + m_2} \right), \\
G &= \frac{1}{2} \frac{1}{M} \left( f_5 - \frac{f_5}{\omega_1 + \omega_2} \right) - \frac{2 f_5 M^2}{m_1 m_2 + \omega_1 + \omega_2}, \\
H &= \frac{1}{2} \frac{1}{M} \left( f_5 - \frac{f_5}{\omega_1 + \omega_2} \right) - \frac{2 f_5 M^2}{m_1 m_2 + \omega_1 + \omega_2}.
\end{align*}
$$

The normalization condition now is read as below:

$$
\int \frac{d\vec{q}}{(2\pi)^3} \frac{16 \omega_1 \omega_2}{3} \left[ 3 f_5 f_6 m_2 \omega_1 + m_1 \omega_2 
+ \frac{\omega_1 - \omega_2}{m_1 m_2 + \omega_1 + \omega_2} 
\times \left( f_5 f_6 - f_5 \frac{\omega_1 + \omega_2}{M^2} + f_6 \right) \right] = 2 M.\quad (B10)
$$

From the first two equations of Eq. (A14) and in terms of straightforward calculation, one may obtain four coupled integral equations about $f_3(q_{p_\lambda})$, $f_4(q_{p_\lambda})$, $f_5(q_{p_\lambda})$, and $f_6(q_{p_\lambda})$. By solving them one may obtain the numerical results for the mass $M$ and the relativistic wave function Eq. (B8) with $f_3(q_{p_\lambda}), f_4(q_{p_\lambda}), f_5(q_{p_\lambda})$, and $f_6(q_{p_\lambda})$ being given. Since the $J/\psi$ and $B^*_c$ etc are vector mesons, so for the concerned semileptonic decays of $B_c$, all the relativistic wave functions, which are needed in calculating the weak current matrix elements, are obtained in the present way.
3. Hadron Matrix Elements And Form Factors

For $B_c^+ \to P \ell^+ \bar{\nu}_\ell$ (here we take $P = B_s$ for example), the hadron matrix element Eq. (7) based on the positive energy wave function of pseudoscalar meson Eq. (10) and vector meson Eq. (19) can be written as:

$$
\langle B_s(P')|J_\mu|B_c^+(P) \rangle = \int \frac{d^3q}{(2\pi)^3} 4L'(P'_\mu + P'_\mu M^2 + q_{\nu \mu} s_3)
$$

where $E_f$ is the energy of the final meson, and

$$
L' = \frac{M'}{2} (f'_+ m'_1 + m'_2),
$$

$$
N' = \frac{\omega_1 + \omega_2}{m'_1 + m'_2},
$$

$$
Y' = \frac{m'_2 - m'_1}{m'_1 \omega'_1 + m'_2 \omega'_2},
$$

$$
Z' = \frac{\omega'_1 + \omega'_2}{m'_1 \omega'_1 + m'_2 \omega'_2}.
$$

$$
s_1 = N' N + \frac{Y}{M'} \epsilon_\lambda \cdot \hat{q} - Y' \alpha'_2 E_f + \frac{Y' Y'(q^2 + \alpha'_2 \hat{r} \cdot \hat{q}) + Z' Z' \alpha'_2 E_f \hat{r} \cdot \hat{q}}{M'} + \frac{Z' N}{M'} (\hat{r} \cdot \hat{q} + \alpha'_2 \hat{r}^2 + \alpha'_2 E_f^2)
$$

$$
s_2 = 2 + Y' \alpha'_2 M' + Z' N \alpha'_2 E_f + Z' Z' q^2,
$$

$$
s_3 = N' Z + Y M' \epsilon_\lambda \cdot \hat{q} + Y' + \frac{Z' Z}{M'} E_f + \frac{Z' Z}{2M'} (2 \hat{r} \cdot \hat{q} + \alpha'_2 \hat{r}^2),
$$

$$
S_1 = \int \frac{d^3q}{(2\pi)^3} A1'L_s,
$$

$$
S_2 = \int \frac{d^3q}{(2\pi)^3} A1'L_s,
$$

$$
S_3 = \frac{1}{|\hat{r}|} \int \frac{d^3q}{(2\pi)^3} |\hat{q}| \cos \theta A1'L_s.
$$

Then the form factors $f_+$ and $f_-$ in Eq. (8) are defined as:

$$
f_+ = \frac{1}{2} \left( \frac{S_1}{M} + \frac{S_2}{M'} + \frac{M - E_f}{M} S_3 \right),
$$

$$
f_- = \frac{1}{2} \left( \frac{S_1}{M} - \frac{S_2}{M'} - \frac{M + E_f}{M} S_3 \right).
$$

For $B_s^+ \to V \ell^+ \bar{\nu}_\ell$ (here we take $V = B_s^*$ for example), the hadron matrix element Eq. (7) based on the positive energy wave function of pseudoscalar meson Eq. (10) and vector meson Eq. (19) can be written as:

$$
\langle B_s^+(P', \epsilon)|J_\mu|B_s^+(P) \rangle = \int \frac{d^3q}{(2\pi)^3} 4L' \left\{ E_\parallel t_1 + \frac{P_\mu}{P'} \left[ (q_{\nu \mu} \cdot \epsilon_\lambda) t_2 + (P \cdot \epsilon_\lambda) t_3 \right] \right.
$$

$$
+ \frac{P_\mu}{P'} \left[ (q_{\nu \mu} \cdot \epsilon_\lambda) t_3 + (P \cdot \epsilon_\lambda) t_4 \right] - i\epsilon_{\mu \nu \rho \sigma} \left[ \frac{A Y}{M} \epsilon_{\lambda \nu} q_{\mu \rho} \right] P^\sigma - \left[ B' N + M \right] P_{\nu \rho} \epsilon_{\lambda \nu} P^\rho P^\sigma
$$

$$
- B' Z P^\nu \epsilon_{\lambda \nu} q_{\mu \rho} - C' N \epsilon_{\lambda \nu} q_{\mu \rho} P^\sigma - C' Z \alpha_2 \epsilon_{\lambda \nu} P^\nu P^\rho P_{\nu \rho}
$$

$$
+ \frac{C' Z}{M} \alpha_2 E_f \epsilon_{\lambda \nu} P^\nu P^\rho P_{\nu \rho} + \frac{D' Y}{M} \epsilon_{\lambda \nu} q_{\mu \rho} + \frac{D' Y}{M} \epsilon_{\lambda \nu} q_{\mu \rho}
$$

$$
- \frac{D' Y}{M} \alpha_2 M^2 \epsilon_{\lambda \nu} q_{\nu \rho} P^\rho - D' Y \alpha_2 \epsilon_{\lambda \nu} P^\nu q_{\nu \rho} P^\rho
$$

$$
- \frac{(q_{\nu \mu} \cdot \epsilon_\lambda)}{(M')^2} < P^\rho q_{\nu \rho} + F' Y \epsilon_{\lambda \nu} q_{\nu \rho} P^\rho
$$

$$
+ \frac{G' Y}{M} \alpha_2^2 P^\rho q_{\nu \rho} P^\rho + \frac{H' N}{M} \epsilon_{\lambda \nu} P^\rho q_{\nu \rho} P^\rho
$$

$$
- \frac{H' Z}{M} \alpha_2 E_f P^\rho q_{\nu \rho} P^\rho
$$

$$
\right\} \left\{ t_1 = A' - B' N E_f + B' Z \hat{r} \cdot \hat{q} - C' Z (q^2 + \alpha'_2 \hat{r}^2) + D' (\alpha'_2 \hat{r}^2 + \hat{r} \cdot \hat{q}) - D' Y E_f (q^2 + \alpha'_2 \hat{r}^2)
$$

$$
t_2 = \frac{A' Y}{M} + \frac{D' Y}{M} (\alpha'_2 M^2 - \hat{r} \cdot \hat{q}) - \frac{E' N}{M}
$$

$$
+ \frac{F' Y}{M} \hat{r} \cdot \hat{q} + \frac{G' Y}{M} (q^2 + \alpha'_2 \hat{r}^2)
$$

$$
- \frac{H' N}{M} (\alpha'_2 M^2 - \hat{r} \cdot \hat{q}) + \frac{C' Z}{M} \alpha_2 E_f
$$

$$
- \frac{G' Y}{M} \alpha_2 M^2 - \hat{r} \cdot \hat{q}) + \frac{C' Z}{M} \alpha_2 E_f
$$

$$
+ \frac{H' Z}{M} \alpha_2 E_f + \frac{H' Z}{M} \alpha_2 E_f \hat{r} \cdot \hat{q},
$$

where the definition of $A'$, $B'$, $C'$, $D'$, $E'$, $F'$, $G'$ and $H'$ is the same as Eq. (19) but for final meson, and

$$
t_1 = A' - B' N E_f + B' Z \hat{r} \cdot \hat{q} - C' Z (q^2 + \alpha'_2 \hat{r}^2) + D' (\alpha'_2 \hat{r}^2 + \hat{r} \cdot \hat{q}) - D' Y E_f (q^2 + \alpha'_2 \hat{r}^2)
$$

$$
t_2 = \frac{A' Y}{M} + \frac{D' Y}{M} (\alpha'_2 M^2 - \hat{r} \cdot \hat{q}) - \frac{E' N}{M}
$$

$$
+ \frac{F' Y}{M} \hat{r} \cdot \hat{q} + \frac{G' Y}{M} (q^2 + \alpha'_2 \hat{r}^2)
$$

$$
- \frac{H' N}{M} (\alpha'_2 M^2 - \hat{r} \cdot \hat{q}) + \frac{C' Z}{M} \alpha_2 E_f
$$

$$
- \frac{G' Y}{M} \alpha_2 M^2 - \hat{r} \cdot \hat{q}) + \frac{C' Z}{M} \alpha_2 E_f
$$

$$
+ \frac{H' Z}{M} \alpha_2 E_f + \frac{H' Z}{M} \alpha_2 E_f \hat{r} \cdot \hat{q},
$$

$$
\right\}
$$
\[ t'_{2} = \frac{D'Y}{M} \alpha_{2}E_{f} \vec{q} \cdot \vec{q} - \frac{F'Y}{M} \alpha_{2}E_{f} \vec{q} \cdot \vec{q} - \frac{G'Y}{M} \alpha_{2}E_{f} \left( q^{2} + \alpha_{f}^{2} \vec{q} \cdot \vec{q} \right) + \frac{H'N}{M} \alpha_{2}E_{f} \left( \alpha_{2}^{2} M^{2} - \vec{r} \cdot \vec{q} \right) - \frac{H'Z}{M} \alpha_{2}E_{f} \left( \vec{r} \cdot \vec{q} + E'N \alpha_{2}E_{f} \right) - \frac{C'N}{M} \alpha_{2}E_{f} + G' \alpha_{2}^{2} E_{f}^{2} + \frac{E'N}{M} \alpha_{2}^{2} E_{f}^{2} + \frac{H'N}{M} \alpha_{2}^{2} E_{f}^{2}, \]

\[ t_{3} = B'Z - D'Y \alpha_{2}E_{f} + F' + H'Z \vec{q}^{2} - C'Z \alpha_{2}^{2} + G' \alpha_{2}^{2} + H'N \alpha_{2}^{2} E_{f}, \]

\[ t'_{3} = \frac{B'N}{M} + \frac{D'Y}{M} \vec{q}^{2} - \frac{F'}{M} \alpha_{2}E_{f} - \frac{C'N}{M} \alpha_{2}^{2} - \frac{H'Z}{M} \alpha_{2}E_{f} \vec{q}^{2} - \frac{G'}{M} \alpha_{2}^{2} E_{f} - \frac{H'N}{M} \alpha_{2}^{2} E_{f}^{2}, \]

\[ t_{4} = -D'YE_{f} - E'Z + F'Y E_{f} - C'Z + G' + H'Z \alpha_{2}^{2} + H'N \alpha_{2}^{2} E_{f}, \]

\[ t'_{4} = \frac{A'Y}{M} + \frac{D'Y}{M} \alpha_{2}^{2} \vec{q}^{2} + \frac{E'Z}{M} \alpha_{2}^{2} E_{f} - \frac{C'N}{M} \alpha_{2}^{2} \vec{q}^{2} - \frac{F'Y}{M} \alpha_{2}^{2} E_{f} \vec{q}^{2} - \frac{G'}{M} \alpha_{2}^{2} E_{f} - \frac{H'N}{M} \alpha_{2}^{2} E_{f}^{2}. \]

\[ T_{1} = \int \frac{d^{3}q}{(2\pi)^{3}} A_{4t}, \]

\[ T_{2} = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{2}, \]

\[ T_{2}' = \int \frac{d^{3}q}{(2\pi)^{3}} A_{4t}', \]

\[ T_{3} = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{3}, \]

\[ T_{3}' = \int \frac{d^{3}q}{(2\pi)^{3}} A_{4t}' , \]

\[ T_{41} = \frac{1}{2M^{2}|r|^{2}} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} \times \left[ (M^{2} + 2E_{f}^{2}) \cos^{2} \theta - M^{2} \right] 4AT_{4}, \]

\[ T_{41}' = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{4}', \]

\[ T_{42} = -\frac{E_{f}}{2M|r|^{2}} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} (3\cos^{2} \theta - 1) 4AT_{4}, \]

\[ T_{42}' = \frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}| \cos 4AT_{4}', \]

\[ T_{43} = \frac{1}{2} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} (\cos^{2} \theta - 1) 4AT_{4}; \]

\[ M_{1} = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}| \cos 4AT_{4} \frac{A'Y}{M}, \]

\[ M_{2} = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{4} \frac{B'Z}{M}, \]

\[ M_{3} = \frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}| \cos 4AT_{4} \frac{C'N}{M}, \]

\[ M_{4} = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{4} \frac{C'Z}{M}, \]

\[ M_{5} = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{4} \frac{C'Z}{M}, \]

\[ M_{6} = \frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{4} \frac{D'Y}{M}, \]

\[ M_{7} = \frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{4} \frac{D'Y}{M}, \]

\[ M_{8} = -\frac{1}{|r|} \int \frac{d^{3}q}{(2\pi)^{3}} \frac{E_{f}}{M} |\vec{q}| \cos 4AT_{4} \frac{D'Y}{M}, \]

\[ M_{9} = \frac{1}{2} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} (\cos^{2} \theta - 1) 4AT_{4} \frac{E'Y}{M}, \]

\[ M_{10} = \frac{1}{2} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} (\cos^{2} \theta - 1) 4AT_{4} \frac{E'Y}{M}, \]

\[ M_{11} = \frac{1}{2} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} (\cos^{2} \theta - 1) 4AT_{4} \frac{E'Y}{M}, \]

\[ M_{12} = \frac{1}{2} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} (\cos^{2} \theta - 1) 4AT_{4} \frac{E'Y}{M}, \]

\[ M_{13} = \frac{1}{2} \int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} (\cos^{2} \theta - 1) 4AT_{4} \frac{E'Y}{M}, \]

\[ V_{1} = \int \frac{d^{3}q}{(2\pi)^{3}} A_{4} \frac{B'N}{M}, \]

\[ V_{2} = \int \frac{d^{3}q}{(2\pi)^{3}} \alpha_{2} \frac{C'N}{M}, \]

\[ V_{3} = \int \frac{d^{3}q}{(2\pi)^{3}} \alpha_{2} E_{f} \frac{D'Y}{M}, \]

\[ V_{4} = -\int \frac{d^{3}q}{(2\pi)^{3}} |\vec{q}|^{2} A_{4} \frac{D'Y}{M}, \]

Then the form factors \( f, a_{+}, a_{-} \) and \( g \) in Eq. (9) and (10) are defined as:

\[ f = T_{1} + T_{43}, \]

\[ a_{+} = \frac{1}{2} (T_{2} + T_{2}') + T_{41} + T_{41}' \]

\[ + T_{3} + T_{3}' + T_{42} + T_{42}' \]

\[ + T_{3} + T_{3}' - T_{42} - T_{42}' \]

\[ + M_{6} + M_{7} + M_{8} - M_{9} - M_{10} \]

\[ + M_{11} + M_{12} - M_{13} - V_{1} \]

\[ + V_{2} + V_{3} + V_{4}. \]

(B14)
4. The Parameters in BS Equation

When solving the equations, one has to fix the parameters appearing in the BS kernel. In our calculation, we refer the kernel to the Cornell potential [33]: a linear scalar interaction (confinement one) \( V_s(r) = \lambda r \) and a vector interaction (single gluon exchange) \( V_\alpha(r) = -\frac{4 \alpha_s(r)}{3r} \), i.e.:

\[
I(r) = V_\alpha(r) + V_0 + \gamma_0 \otimes \gamma^0 V_\alpha(r) = \lambda r + V_0 - \gamma_0 \otimes \gamma^0 \frac{4 \alpha_s(r)}{3r}, \quad (B15)
\]

where \( \lambda \) is the string constant, \( \alpha_s(r) \) is the running coupling constant, and \( V_0 \) is a constant which is often added to the scalar confining potential.

The kernel which corresponds to the potential Eq. [B15] in momentum space reads:

\[
I(\vec{q}) = V_\alpha(\vec{q}) + \gamma_0 \otimes \gamma^0 V_\alpha(\vec{q}), \quad (B16)
\]

\[
V_\alpha(\vec{q}) = -\frac{2}{3\pi^2} \frac{\alpha_s(\vec{q})}{(q^2 + \alpha^2)^2}, \quad (B17)
\]

\[
\alpha_s(\vec{q}) = \frac{12\pi}{27 \log(a + \frac{E}{\Lambda_{QCD}})} \quad \text{for } \vec{q} < \Lambda_{QCD}.
\]

In order to avoid infrared divergence in the Coulomb-like potential, people usually introduce a factor \( e^{-\alpha r} \) as below:

\[
V_\alpha(r) = \frac{\lambda}{\alpha}(1 - e^{-\alpha r}), \quad V_\alpha(r) = \frac{4 \alpha_s(r)}{3r} e^{-\alpha r}. \quad (B18)
\]

The parameters \( \lambda, \alpha, a \) and \( \Lambda_{QCD} \) characterizing the potential are fixed by fitting the mass spectrum of heavy quarkonium [32]. The fitted values are \( a = c = 2.7183, \alpha = 0.06 \text{ GeV}, \lambda = 0.21 \text{ GeV}^2 \) and \( \Lambda_{QCD} = 0.27 \text{ GeV} \). The parameter \( V_0 \) varies as the constituents and the quantum numbers vary. In this work, the relevant values are \( V_0 = -0.314 \text{ GeV for } \bar{c}c(0^+), V_0 = -0.176 \text{ GeV for } \bar{c}c(1^+), V_0 = -0.205 \text{ GeV for } bs(0^-), V_0 = -0.13 \text{ GeV for } bs(1^-) \) and \( V_0 = -0.185 \text{ GeV for } bc(0^-) \). In addition, the constituent quark masses are also parameters and are fixed by fitting the meson spectrum: \( m_u = 4.96 \text{ GeV}, m_c = 1.62 \text{ GeV}, m_s = 0.5 \text{ GeV} \). Solving the Salpeter equation with these parameters, we obtain the mass spectrum and the relevant wave functions.

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