Anomalous diffusion in stochastic systems with nonhomogeneously distributed traps

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The stochastic motion in a nonhomogeneous medium with traps is studied and diffusion properties of that system are discussed. The particle is subjected to a stochastic stimulation obeying a general Lévy stable statistics and experiences long rests due to traps the density of which depends on the position. The memory is taken into account by subordination of that process to a random time; then the subordination equation is position-dependent. The problem is approximated by means of a decoupling of the trap geometry and memory and exactly solved for a power-law trap density, corresponding to a fractal medium structure, in the case of the Gaussian statistics: the density distribution and moments are derived. Depending on geometry and memory parameters, the system may reveal both the subdiffusion and enhanced diffusion. A similar analysis is performed for the Lévy flights where the finiteness of the variance follows from a multiplicative noise, as a result of impurities and defects at the boundary. Two diffusion regimes are found: in the bulk and near the surface. The anomalous diffusion exponent as a function of the system parameters is derived.

I. INTRODUCTION

The diffusion is anomalous when the variance rises with time slower or faster than linearly and the density distribution differs from the normal distribution. This means that the central limit theorem is violated. One can expect that the theorem is not valid if the medium contains traps which hamper the transport introducing memory effects. As a consequence, a subdiffusion emerges – as well as a stretched-Gaussian asymptotics of the distribution. Subdiffusion is predicted by the continuous-time random walk (CTRW) if the waiting-time distribution has long tails [1]. Another reason that the central limit theorem may not apply is a nonhomogeneous structure of the environment which makes subsequent random stimulations mutually dependent and introduces a dependence of the random force on the process value. Then the stochastic system cannot be described by a simple Langevin equation with the additive white noise. We encounter such a situation for disordered media with impurities and defects where a particle exercises a space structure which slowly evolves with time and a quenched disorder emerges [2]. Taking into account the self-similar medium structure is crucial when one considers the diffusion on fractals; it is described by the Fokker-Planck equation with a variable diffusion coefficient [3] which may be fractional [4, 5]. CTRW involves, in general, a coupled jump density distribution and the waiting-time distribution may be position-dependent. The Fokker-Planck equation, corresponding to the master equation, contains the variable diffusion coefficient [6] and, in the non-Markovian case, a variable order of the fractional derivative [7]. For such inhomogeneous problems, CTRW implies stretched-Gaussian shape of the density distribution and predicts the anomalous diffusion.

If, on the other hand, the jump length does not obey the normal distribution but is governed by a general Lévy stable distribution (Lévy flights) the variance does not exist; such long jumps are frequently observed in many areas of science [8]. They may be directly related to a specific topology of the medium and then the transport description requires a variable diffusion coefficient: the folded polymers are a well-known example [9]. If a composite medium consists of many layers, the fractional equation is complicated and contains position-dependent both the diffusion coefficient and the order of the fractional derivative [10]. However, presence of the Lévy flights does not need to imply infinite fluctuations because any physical system is finite and, if one introduces a truncation of the distribution tail, the diffusion properties are well-determined. It has been recently demonstrated that cracking of heterogeneous materials reveals a slowly falling power-law tail of the local velocity distribution of the crack front [11] but, despite that, the authors were able to determine the variance due to the finiteness of the system.

Systems with traps can be conveniently handled in terms of a Langevin equation by introducing an auxiliary operational time. Process given by this equation is subsequently subordinated to the random physical time by means of a one-sided probability distribution with a long tail [12]. A subordination formalism, which directly takes into account the influence of the nonhomogeneous trap geometry on the local time lag via a variable intensity of the random time distribution, has been recently proposed [13]. The system is then described by a set of two Langevin equations,

\[
\begin{align*}
\frac{dx}{d\tau} &= \eta(d\tau) \\
\frac{dt}{d\tau} &= g(x)\xi(d\tau),
\end{align*}
\]

where a positive function \(g(x)\) models a trap density. Increments of the white noise \(\eta\) are determined, in general, by the \(\alpha\)-stable Lévy distribution, \(L_\alpha(x)\), and \(d\xi(\tau)\) stands for a stochastic process given by a one-sided distribution...
L_β(x). Then, the random subordinator is weighted by a relative probability of finding the particle at a given position, according to the function g(x). The case β ≥ 1 corresponds to a Markovian process and then L_β(x) has a finite first moment. Approximating ξ by this moment, ξ → ⟨ξ⟩, allows us to evaluate the operational time increment, \( \Delta t = (\langle \xi \rangle g(x))^{-1} \Delta t \), and reduce the system \((1)\) to a single equation with a multiplicative noise in the Itô interpretation, \(dx(t) = \nu(x)^{1/\alpha} \eta(dt)\), where \(\nu(x) = (\langle \xi \rangle g(x))^{-1}\). The above equation incorporates the medium structure but neglects the memory effects which are important if \(\beta < 1\). Those effects, in turn, can be approximately taken into account by the subordination of the process \(x(\tau)\) to a random time \(t\) and, after that decoupling of the medium structure and memory, Eq.\((1)\) takes the form

\[
\begin{align*}
\frac{dx(\tau)}{d\tau} &= \nu(x)^{1/\alpha} \eta(d\tau) \\
\frac{dt(\tau)}{d\tau} &= \xi(d\tau).
\end{align*}
\]

The anomalous transport predicted by Eq.\((2)\) is described by the variance for the Gaussian case and by the fractional moments for \(\alpha < 2\) \[13\]. In the present paper, we discuss a possibility of existence of the finite variance for the Lévy flights; we take into account that the medium – if the system is finite – may change its properties in the vicinity of the edge. In this region, one can expect an enhanced level of the disorder with additional impurities and external factors which make a simple description in terms of the additive noise, like Eq.\((1)\), problematic. We model those complicated dynamics on the edge by a multiplicative noise and require its intensity decreasing with the distance to ensure the finite variance. That procedure differs from the usual noise truncation because the distribution of \(\eta\) remains unaffected.

The paper is organised as follows. In Sec.II we discuss the case \(\alpha = 2\) exactly solving Eq.\((2)\) for the power-law trap density which corresponds to a fractal geometry. Sec.III is devoted to the Lévy flights. We discuss a system with the multiplicative noise in various interpretations and infer its general properties (Sec.IIIA). In Sec.IIIB, the multiplicative noise serves to model a boundary layer; we derive the variance as a function of time and all system parameters.

**II. THE GAUSSIAN CASE**

We consider a stochastic motion inside a medium containing traps that are distributed according to the function \(g(x)\). The particle is subjected to a white noise the increments of which obey the Gaussian statistics. Then the substrate nonhomogeneity is restricted to the time-characteristics of the system, determined by \(g(x)\), and the dynamics is described by Eq.\((1)\). We assume the function \(g(x)\) in a power-law form,

\[
g(x) \sim |x|^\theta \quad (\theta > -1).
\]

This form of the diffusion coefficient was assumed to describe, beside a problem of the diffusion on fractals, e.g. a turbulent two-particle diffusion \[14\] and transport of fast electrons in a hot plasma \[15\]. We look for the density distribution of the particle position, \(p(x,t)\), by solving Eq.\((2)\). The first equation \((2)\) in the Itô interpretation leads to the Fokker-Planck equation

\[
\frac{\partial p_0(x, \tau)}{\partial \tau} = \frac{\partial^2 [\nu(x) p_0(x, \tau)]}{\partial x^2},
\]

which has the solution \[16\]

\[
p_0(x, \tau) = N \tau^{-\frac{\beta}{2+\theta}} |x|^\theta \exp(-|x|^{2+\theta}/(2+\theta)^2 \tau),
\]

and corresponds to the Markovian process \(x(\tau)\). The random time, given by the second equation \((2)\), is defined by a one-sided, maximally asymmetric stable Lévy distribution \(L_\beta(\tau)\), where \(0 < \beta \leq 1\), and the inverse distribution we denote by \(h(\tau, t)\). The density \(p(x,t)\) results from the integration of the densities \(p_0(x, \tau)\) and \(h(\tau, t)\) over the operational time,

\[
p(x,t) = \int_0^\infty p_0(x, \tau) h(\tau, t) d\tau.
\]

To evaluate the integral, it is convenient to use the Laplace transform from Eq.\((3)\) taking into account that \(\tilde{h}(\tau, u) = u^{\beta-1} \exp(-\tau u^\beta)\) \[17\]. Then a direct integration yields a Laplace transform from the normalised density,

\[
\tilde{p}(x, u) = -2(2+\theta)^{\nu} \frac{|x|^{\theta+1/2} u^\nu}{\Gamma(-\nu)} \left( \frac{|x|^{1+\theta/2}}{1+\theta/2} \right) u^{\beta/2},
\]

\[(7)\]
where \( K_\nu(z) \) is a modified Bessel function, \( \nu = 1/(2 + \theta) \) and \( c = \beta - \beta/(4 + 2\theta) - 1 \). The above transform can be inverted when we express the Bessel function in terms of the Fox H-function [18] which formula, in our case, reads

\[
K_\nu(z) = \frac{1}{2} H^{2.0}_{0.2} \left[ \frac{z^2}{4} \left( -\nu/2, 1, (\nu/2, 1) \right) \right].
\]

After applying standard properties of the H-function and straightforward calculations we obtain the density in the form

\[
\bar{p}(x, u) = \frac{1}{\beta} \left( \frac{1 + \theta/2}{\Gamma(-\nu)} \right) |x|^{2(\theta)/\beta-1} H^{2.0}_{1.2} \left[ \frac{\xi}{(c/\beta - \nu/2, 1/\beta), (c/\beta + \nu/2, 1/\beta)} \right],
\]

where \( \xi = (2 + \theta)^{-2/\beta} |x|^{(2+\theta)/\beta} u = \kappa u \). Inversion of the transform enhances the order of the H-function [19]; in the case of the function in Eq.(9), the inversion formula yields

\[
H(t) = \frac{1}{t} H^{2.0}_{1.2} \left[ \frac{k}{t} (c/\beta - \nu/2, 1/\beta), (c/\beta + \nu/2, 1/\beta) \right]
\]

and the final expression for the density is of the form

\[
p(x, t) = -\frac{2}{\beta t} \left( \frac{1 + \theta/2}{\Gamma(-\nu)} \right) |x|^{2(\theta)/\beta-1} H^{2.0}_{1.2} \left[ \frac{|x|^{(2+\theta)/\beta}}{(2 + \theta)^{2/\beta} t} (c/\beta - \nu/2, 1/\beta), (c/\beta + \nu/2, 1/\beta) \right].
\]

Eq.(11) seems complicated but an expression for large \( |x| \) is simple: it has a stretched-Gaussian form which follows from an asymptotic formula for the H-function [20],

\[
p(x, t) \sim |x|^{\theta/\beta} t^{-\theta/(2+\theta)-1} \exp \left[ -A|x|^{(2+\theta)/(2-\beta)}/t^\theta \right],
\]

where \( A = (2/\beta - 1)/[\beta^3/(2-\beta)(2 + \theta)^2/(2-\beta)] \). The stretched-Gaussian shape of the tail is typical for diffusion on fractals [2] and emerges in the trap models [21].

We are interested in the moments of \( p(x, t) \); all of them are finite and given by a characteristic function which can be derived as a series expansion. Eq.(6) yields

\[
\bar{p}(k, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} k^{2n} \int_0^\infty \langle x^{2n} \rangle p_0(h(t, \tau)) d\tau,
\]

where the moments of \( p_0(x, \tau) \) directly follow from Eq.(6),

\[
\langle x^n \rangle_p = -(2 + \theta)^{2n/(2+\theta)+1} \frac{\Gamma[(1 + n + \theta)/(2 + \theta)]}{\Gamma[-1/(2 + \theta)]} \tau^{n/(2+\theta)}.
\]

Then the integral resolves itself to the moments of \( h(t, \tau) \) that are given by [22]

\[
\langle \tau^{2n/(2+\theta)} \rangle_h = \frac{\Gamma[2n/(2 + \theta) + 1]}{\Gamma[2n\beta/(2 + \theta) + 1]} t^{2n\beta/(2 + \theta)}
\]

and the final expression for the characteristic function reads

\[
\bar{p}(k, t) = -\frac{2 + \theta}{\Gamma[-1/(2 + \theta)]} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2 + \theta)^{4n/(2+\theta)} \Gamma[(1 + 2n + \theta)/(2 + \theta)] \frac{\Gamma[2n/(2 + \theta) + 1]}{\Gamma[2n\beta/(2 + \theta) + 1]} t^{2n\beta/(2 + \theta)} k^{2n}.
\]

Diffusion properties are determined by the variance the time-dependence of which follows from scaling arguments or from Eq.(10): \( \langle x^2 \rangle(t) = -\theta^2 \bar{p}(k = 0, t)/\partial k^2 \sim t^{\beta/(2+\theta)} \); this formula indicates both the sub- and superdiffusion. Comparison of the above approximate result with a numerical solution of Eq.(1) reveals a reasonable agreement; some discrepancies emerge for a small \( \beta \) and large \( \theta \) [13].
Finally, we will demonstrate that $p(x,t)$, Eq. (11), satisfies the fractional equation

$$\frac{\partial p(x,t)}{\partial t} = D_t^{1-\beta} \frac{\partial^2}{\partial x^2} \left(|x|^{-\theta} p(x,t)\right),$$

where $D_t^{1-\beta}$ is a Riemann-Liouville operator

$$D_t^{1-\beta} f(t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t dt' \frac{f(t')}{(t-t')^{1-\beta}},$$

and which equation constitutes a non-Markovian generalisation of a Fokker-Planck equation for CTRW with a position-dependent waiting-time distribution [23]. First, we integrate Eq. (17) over time,

$$p(x,t) - p_0(x) = D_t^{-\beta} \frac{\partial^2}{\partial x^2} \left(|x|^{-\theta} p(x,t)\right),$$

where $p_0(x)$ is the initial condition, and take the Laplace transform,

$$u^\beta \tilde{p}(x,u) - u^{\beta-1} p_0(x) = \frac{\partial^2}{\partial x^2} \left(|x|^{-\theta} \tilde{p}(x,u)\right).$$

The differentiation produces a differential equation

$$x^2 \tilde{p}'' - \theta x \tilde{p}' + [\theta(1+\theta) - u^\beta x^{\theta+2}]\tilde{p} + u^{\beta-1} x^{\theta+2} p_0 = 0$$

and the last term vanishes if $p_0(x) = \delta(x)$. Its particular solution has the form [24]

$$\tilde{p}(x,u) = |x|^{\theta+1/2} K_{\nu} \left(\frac{|x|^{1+\theta/2}}{1+\theta/2} u^{\beta/2}\right) f(u),$$

where $f(u)$ is an arbitrary function. Putting $f(u) \sim u^c$, we obtain Eq. (7).

Eq. (11) can be interpreted as a fractal problem where the trap positions are distributed according to the self-similar pattern and parametrised by $\theta$; the jump-size distribution, in turn, is homogeneous in space. Then $g(x)$ has a sense of the trap density where $\theta \in (-1,0)$ is related to the fractal (Hausdorff) dimension $d_f$; since the density of a fractal embedded in the one-dimensional space equals $|x|^{d_f-1}$ [23], we obtain $\theta = d_f - 1$. Then the expression for the variance takes the form,

$$\langle x^2 \rangle(t) \sim t^{2\beta/(1+d_f)}.$$  

We conclude that subdiffusion, observed in the homogeneous case ($d_f = 1$), may turn to the enhanced diffusion if the memory is sufficiently weak (large $\beta$) and the dimension of the trap structure small. In the limiting case $\beta \to 1$ and $d_f \to 0$, a ballistic diffusion emerges.

The present problem differs from the random walk on fractals when jump length – not the trap density – is nonhomogeneously distributed. That process is described by a fractional, non-Markovian equation [4] which has a form different from Eq. (17) and the motion is always subdiffusive. The density has a finite value at the origin in contrast to the our approach: $p(x,t)$, Eq. (11), diverges at $x = 0 (\theta < 0)$ where the trapping is strong and the evolution, as a function of the physical time, proceeds very slowly near the origin.

III. LÉVY FLIGHTS

Motion of a massive particle subjected to an additive noise which obeys the Lévy statistics with long jumps is characterised by the infinite variance; this situation is unacceptable for physical reasons. However, the additive noise is an idealisation and in realistic systems the stochastic stimulation may depend on a state of the system. For this – multiplicative – noise, the variance may be finite and the anomalous diffusion exponent well determined. In the next subsection, we discuss general properties of the non-Markovian Langevin dynamics with that noise; the memory effects are taken into account by a subordination of the dynamical process to the random time. One can expect that the multiplicative noise emerges near a boundary where the environment structure is more complicated than in the bulk. The boundary effects modelled in terms of the multiplicative noise will be discussed in Subsection IIIB.
A. Multiplicative noise

The generalisation of Eq. (1) including the multiplicative noise is the following

\[
\begin{align*}
\mathrm{d}x(\tau) &= f(x)\eta(\mathrm{d}\tau) \\
\mathrm{d}t(\tau) &= g(x)\xi(\mathrm{d}\tau)
\end{align*}
\]  

(24)

and, in the case of the first equation (24), we must decide how this equation is to be interpreted. In general, the stochastic integral is defined as a Riemann integral, and, in the case of the first equation (24), we must decide how this equation is to be interpreted. In general, the equation

\[
\int_0^t f[x(\tau)]\mathrm{d}\eta(\tau) = \sum_{i=1}^n f[(1 - \lambda_I)x(t_{i-1}) + \lambda_I x(t_i)][\eta(t_i) - \eta(t_{i-1})],
\]

(25)

where the parameter \(0 \leq \lambda_I \leq 1\) determines the interpretation and corresponds, in particular, to the Itô (II) \((\lambda_I = 0)\), Stratonovich (SI) \((\lambda_I = 1/2)\) and anti-Itô interpretation (AII) \((\lambda_I = 1)\). It is well-known that for the Gaussian processes the ordinary rules of the calculus are valid in the case of SI, in contrast to the other interpretations, which allows us to transform the equation with the multiplicative noise into an equation with the additive noise by a simple variable change. As regards the general stable processes, we observe the same property: the numerical solutions of the Langevin equation by using Eq. (25) with \(\lambda_I = 1/2\) agree with those where a standard technique of the variable change is applied [26]. Obviously, the ordinary rules of the calculus are always valid if one defines the white noise as a limit of the coloured noise [27]; equivalence of that limit and SI is an important property of the multiplicative Gaussian processes. In the case of Eq. (24), we obtain the equation with the additive noise by changing the variable, \(y(x) = \int_0^x \frac{\mathrm{d}x'}{f(x')}\), in the first of Eq. (24). Next, we decouple the medium structure and memory (cf. Eq. (2)) and get the equation \(\mathrm{d}y(\tau) = \nu(x(y))^{1/\alpha}\eta(\mathrm{d}\tau)\). It contains a multiplicative noise which should be interpreted in the Itô sense and corresponds to the Fokker-Planck equation

\[
\frac{\partial p_0(y, \tau)}{\partial \tau} = \frac{\partial^\alpha [\nu(x(y)) p_0(y, \tau)]}{\partial |y|^{\alpha}}.
\]

(26)

Solution in the original variable \(x\) directly follows from the solution of Eq. (26),

\[
p_0(x, \tau) = \frac{1}{f(x)} p_0(y(x), \tau),
\]

(27)

and the density as a function of the physical time – determined by the equation \(\mathrm{d}t(\tau) = \xi(\mathrm{d}\tau)\) – results from Eq. (6).

In the following, we restrict our considerations to a power-law form of \(g(x)\):

\[
\nu(x) = |x|^{-\theta\alpha} \quad (\theta > -1),
\]

(28)

similar to Eq. (3). Eq. (26) can be solved if one neglects terms higher than \(|k|^\alpha\) in the characteristic function [23, 29] which, after inverting the Fourier transform, yields tail of the density (27): \(p_0(y, \tau) \sim |y|^{-1-\alpha}\); it corresponds to the Lévy-stable asymptotics [30]. The backward transformation of the variable, \(y \rightarrow x\), produces the asymptotics \(p_0(x, \tau) \sim f(x)^{-1} y(x)^{-1-\alpha}\), which indicates that finite moments can exist. In particular, the variance is finite if \(f(x)\) satisfies the condition

\[
\lim_{x \to \infty} \frac{x^3}{f(x) y(x)^{1+\alpha}} = 0.
\]

(29)

We assume from now on that \(f(x)\) has the power-law form,

\[
f(x) = |x|^{-\gamma},
\]

(30)

for which Eq. (29) implies the condition for the finite variance: \(\gamma > 2/\alpha - 1\). Inserting the functions \(g(x)\) and \(f(x)\), given by Eq. (28) and Eq. (30), to Eq. (24), we obtain after elimination of the position-dependent factor in the subordination equation and straightforward calculations the following set of the Langevin equations,

\[
\begin{align*}
\mathrm{d}y(\tau) &= D |y|^{-\theta/(1+\gamma)} \eta(\mathrm{d}\tau) \\
\mathrm{d}t(\tau) &= \xi(\mathrm{d}\tau),
\end{align*}
\]

(31)
where \( D = (1 + \gamma)^{-\theta/(1+\gamma)} \). The first of the above equations corresponds to the Fokker-Planck equation,

\[
\frac{\partial p_0(y, \tau)}{\partial \tau} = D^\alpha \frac{\partial^\alpha}{\partial |y|^{\alpha}} \left[ |y|^{-\alpha/(1+\gamma)} p_0(y, \tau) \right]
\]

(32)

which in the limit of small wave numbers is satisfied by a density determined by the following characteristic function in respect to the variable \( y = \frac{x}{1+\gamma}|x|^{1+\gamma} \sign x \),

\[
\tilde{p}_0(k, \tau) \approx 1 - (A_0 \tau)\alpha |k|^{\alpha};
\]

(33)
in the above equation, \( c_\theta = 1/(1 + \theta/(1 + \gamma)) \) and

\[
A_0 = \frac{2D (1 + \theta + \gamma)}{\pi \alpha (1 + \gamma)} \Gamma(\theta/(1 + \gamma)) \Gamma(1 - \alpha \theta/(1 + \gamma)) \sin \left( \frac{\pi \alpha \theta}{2(1 + \gamma)} \right).
\]

(34)

Eq.(33) corresponds to the stable density \( L_\alpha \) if \( \theta \in (-1 + \gamma), (1 + \gamma)/\alpha \). The final density \( p(x, t) \) results from the transformation to the physical time by means of Eq.(6) which we rewrite as a Fourier transform in respect to \( y \).

Using Eq.(33) yields

\[
\tilde{p}(k, t) = 1 - A_0^\alpha |k|^{\alpha} \int_0^\infty \tau^\alpha h(\tau, t) d\tau
\]

(35)

and the integral can be easily evaluated if we express the function \( h(\tau, t) \) by the H-function \[31\],

\[
h(\tau, t) = \frac{1}{\beta \tau} H_{1,1}^{1,0} \left[ \frac{\tau^{1/\beta}}{t} \right] \left( \frac{1}{1,1/\beta} \right).
\]

(36)

It resolves itself to the Mellin transform from the H-function and the expression for that transform yields

\[
p(x, t) = A^{-1} t^{-\beta \alpha} |x|^\gamma L_\alpha \left( t^{-\beta \alpha} |y(x)/A| \right),
\]

(37)

where we applied Eq.\[24\] and denoted

\[
A = A_0^{\alpha/\alpha} \Gamma[1 + c_\theta]^{1/\alpha} / \Gamma[1 + \beta c_\theta]^{1/\alpha}.
\]

(38)
The solution \(37\) has the asymptotics

\[
p(x, t) \sim t^{\beta \alpha} |x|^{-1-\alpha-\alpha \gamma}.
\]

(39)

On the other hand, if one understands the noise in Eq.\[24\] in a sense of II, the Langevin equation for \( x(\tau) \) contains a multiplicative term \( |x|^{-\theta-\gamma} \) and the counterpart of Eq.\[32\] reads

\[
\frac{\partial p_0(x, \tau)}{\partial \tau} = \frac{\partial^\alpha}{\partial |x|^{\alpha}} \left[ |x|^{-\alpha(\theta+\gamma)} p_0(x, \tau) \right]
\]

(40)

Solving the above equation in the diffusion limit and evaluation of the integral \[3\] produces the distribution \( p(x, t) \) with the asymptotics \( \sim |x|^{-1-\alpha} \). Comparison with Eq.\[33\] indicates a qualitative difference between II and SI for the Lévy flights: whereas for II presence of the multiplicative factor in the noise influences only the time-dependence, for SI it modifies the tail shape and makes finiteness of the variance possible. In the face of that difference, it is interesting to check how the density distributions for the other interpretations look like. To find those distributions, we have to resort to the numerical analysis. According to Eq.\[25\], the discretized form of the Langevin equation for the arbitrary interpretation is given by the following expression,

\[
x_{n+1} = x_n + [(1 - \lambda_I) x_n + \lambda_I x_{n+1}]^{-\gamma} \eta h,
\]

(41)

where \( \eta \) is sampled from a symmetric stable distribution according to a well-known algorithm \[32\]. The expression for \( x_{n+1} \) is not explicit and Eq.\[41\] can be exactly solved only for a few values of \( \gamma \); in general, it must solved numerically at every integration step. For that purpose, we applied the parabolic interpolation scheme (the Muller method) \[33\]. A simple modification of the standard method \[17\] allows us to evaluate the variable \( x \) as a function of the physical time – determined by the second equation \[24\] – without an explicit derivation of the subordinator \( t(\tau) \). Examples of distributions for a few values of \( \lambda_I \) are presented in Fig.1, separately for the central part and for the tails. We observe a similar slope of the tail for all \( \lambda_I > 0.2 \); in particular, it is almost identical for SI and AII. Near the origin, in turn, some differences emerge and the height of the peak rises with \( \lambda_I \).
FIG. 1: (Colour online) Distributions obtained from a numerical integration of Eq. (24) with \( g(x) = |x|^{\theta \alpha} \) and \( f(x) = |x|^{-\gamma} \) for \( \alpha = 3/2, \theta = 2/3, \beta = 1/2, \gamma = 1/2 \) and \( t = 0.1 \). Curves correspond to the following values of \( \lambda_I \): 0 (magenta), 0.1 (blue), 0.2 (cyan), 0.5 (red) and 1 (green) (from bottom to top in the upper part and from top to bottom in the lower part). The straight lines correspond to a power-law function with the index 2.5, 2.9 and 3.25 (from top to bottom).

B. Boundary effects and anomalous diffusion

Let us assume that the particle is subjected to a noise the intervals of which are governed by the symmetric Lévy stable distribution. The transport proceeds inside a medium with traps and their density is given by the function \( g(x) \); the dynamics is described by the Langevin equation with the additive noise, Eq. (1). Next we take into account that the system is finite and the additional complication of the environment structure in the vicinity of the boundary requires a more general approach. One can expect that impurities and defects near the surface – which slowly evolve with time – modify the random driving introducing a dependence of the noise on the process value. We assume, in addition, that those effects do not affect the trap structure and model them by the Langevin equation with a multiplicative noise. Emergence of that noise near a boundary has been experimentally demonstrated for some physical systems. For example, a description of the colloidal particles diffusion in terms of a constant diffusion coefficient appears possible only if a particle remains far from any boundary [34]. Moreover, presence of the multiplicative noise in a description of particles near a wall is necessary to reach a proper thermal equilibrium [35]. The physical requirement that for massive particles the variance must be finite imposes a condition on the form of the position-dependent noise intensity.

The infinite variance of the Lévy flights has always been a problem due to physical reasons and attempts were undertaken to suppress long tails. A simple remedy is to introduce a modification of the stable distribution to make
the tail steeper. Such a truncation may be assumed as a simple cut-off or involve some rapidly falling function: e.g. an exponential or a power-law $|x|^{-\beta}$, where $\beta \geq 2 - \alpha$. Processes involving the truncated distributions actually converge to the normal distribution, according to the central limit theorem, but the power-law tails may be visible for a long time due to a slow convergence. On the other hand, variance becomes finite if one takes into account a finiteness of the particle velocity (Lévy walk). In the present approach, the finite variance results from a variable intensity of the noise in the region close to the boundary whereas intervals of the noise are always distributed according to the stable distribution.

We assume that the boundary effects become important at a distance $|x| = L$ and then the Langevin equation acquires the multiplicative noise. Its intensity is parametrised by $f(x)$ as a falling power-law function,

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq L \\ L^\gamma |x|^{-\gamma} & \text{for } |x| > L, \end{cases}$$

where $\gamma > 0$; the dynamics is governed by Eq. (24) and the noise in the first equation will be interpreted according to SL. A new variable,

$$y(x) = \begin{cases} \frac{x}{L^{1+\gamma}} [\gamma + (|x|/L)^{1+\gamma}] \text{sign } x & \text{for } |x| \leq L \\ \text{sign } x & \text{for } |x| > L, \end{cases}$$

allows us to get rid of the multiplicative factor $f(x)$ in Eq. (24) and the equation corresponding to the first equation (31) takes the form

$$dx(\tau) = |x|^{-\beta} \eta(d\tau)$$

$$dy(\tau) = [(\eta - L)(1 + \gamma)L^\gamma + L^{1+\gamma}]^{-\theta/(1+\gamma)} \eta(d\tau)$$

for $|x| \leq L$ and $|x| > L$. We consider, at the beginning, the case $\theta = 0$ corresponding to a uniform trap distribution and follow a similar method as in the precedent subsection: solve the Fokker-Planck equation and integrate over the operational time. The final result for $|x| > L$ reads

$$p(x,t) = A^{-1} t^{-\beta/\alpha} L^{-\gamma} |x|^\gamma L_\alpha(t^{-\beta/\alpha} |y(x)|/A),$$

where $A = \Gamma(1 + \beta)^{-1/\alpha}$. The diffusion problem is well-defined since the finite variance exists if $\gamma > 2/\alpha - 1$ and, in the following, we will calculate the variance on this assumption. The system defined by Eq. (41) changes its properties at $|x| = L$ and one can expect a different diffusion behaviour in the surface region, compared to the bulk. If $L$ is sufficiently large, the dynamics resolves itself to the truncated Lévy flights and the diffusion properties are equivalent to the Gaussian case, providing the time is relatively small; then one gets the standard anomalous diffusion law,

$$\langle x^2 \rangle(t) \sim t^\beta.$$  

Variance in the limit $t \to \infty$ follows from a direct evaluation of the integral. One can easily demonstrate that the contribution from $|x| < L$ is small for a large time. Then

$$\langle x^2 \rangle(t) = \frac{2t^{-\beta/\alpha}}{A L^\gamma} \int_L^\infty x^{\gamma+2} L_\alpha \left( \frac{|y(x)|}{A t^{\beta/\alpha}} \right) dx$$

and, introducing a new variable $x' = t^{-\beta/\alpha}(1+\gamma)x$, we obtain

$$y = L + \frac{L^\gamma}{1+\gamma} (x'^{1+\gamma} t^{\beta/\alpha} - L^{1+\gamma}) \to \frac{L^\gamma}{1+\gamma} x'^{1+\gamma} t^{\beta/\alpha} \quad (t \to \infty)$$

for any $x'$ which allows us to reduce Eq. (47) to the form

$$\langle x^2 \rangle(t) = \frac{2t^{2\beta/\alpha}(1+\gamma)}{A L^\gamma} \int_{x_0}^\infty x'^{\gamma+2} L_\alpha \left( \frac{L^\gamma}{A(1+\gamma)} x'^{1+\gamma} \right) dx,$$

where $x_0 = L/t^{\beta/\alpha}(1+\gamma) \to 0$. The integral can be evaluated by applying the standard properties of the H-function and, after lengthy but straightforward calculations, we obtain the final formula,

$$\langle x^2 \rangle(t) = -\frac{2}{\pi} L^{\alpha c_1}(1+\gamma)^{\alpha c_1} \Gamma(1+\beta)^{-c_1} \Gamma(-c_1) \Gamma(\alpha c_1) \sin \left( \frac{\pi}{1+\gamma} \right) t^{\beta c_1}.$$
FIG. 2: (Colour online) Variance as a function of time for $\alpha = 1$ and $\beta = 1/2$. The curves (from bottom to top) correspond to: 1. $\gamma = 4$ and $L = 10$; 2. $\gamma = 2$ and $L = 10$; 3. $\gamma = 2$ and $L = 100$. Straight-line segments (marked by the red lines) on the left-hand side have the slope 1/2 and those on the right-hand side: 1/5, 1/3 and 1/3.

FIG. 3: (Colour online) Variance as a function of time for a few sets of the parameters $\alpha, \theta$ and $\beta$ and for $\gamma = 2$. The left-hand part corresponds to $L = 10$. The right-hand part presents the numerical results for $L = 0.1$ with parameters $\alpha = 1.5$, $\theta = 0.33$, $\beta = 0.5$ and $\gamma = 2$ (dots) and slope of the straight line follows from Eq. (52).

where $c_{\gamma} = 2/\alpha(1 + \gamma)$. Since $c_{\gamma} < 1$, the motion is always subdiffusive: the variance in the limit $t \to \infty$ rises with time slower than linearly and also slower than in the case of a small time, Eq. (46). The slope explicitly depends on $\alpha$, in contrast to Eq. (46), and drops to zero for a sharp edge ($\gamma \to \infty$). Therefore, we observe two diffusion regimes; the standard form of the variance Eq. (46), which is typical for the Gaussian case and the truncated Lévy flights, emerges for relatively small times and corresponds to trajectories abiding not far from the origin. On the other hand, when at large time the surface region becomes important, diffusion is weaker.
FIG. 4: Slope of the time-dependence of the variance, \( t^{\mu} \), as a function of \( \theta \) for \( \beta = 0.5 \) and two values of \( \alpha \): 0.5 (lower points) and 1.5 (upper points). The other parameters: \( \gamma = 2 \) and \( L = 10 \). Solid lines mark the function \( \mu = 0.5/(1 + c(\alpha)\theta) \), where \( c(0.5) = 1.51 \) and \( c(1.5) = 0.51 \).

Those two diffusion regimes are illustrated in Fig.2 for the Cauchy distribution (\( \alpha = 1 \)) where the variance was obtained from \( p(x, t) \) by a direct integral evaluation: slopes agree with Eq.(46) and Eq.(50).

The general case \( \theta \neq 0 \) is manageable in a similar way but only if \( |x| \) is very large and its relation to \( |y| \), Eq.(13), becomes linear, \( |x| = (1 + \gamma)^{1/(1+\gamma)}L^{1/(1+\gamma)}|y|^{1/(1+\gamma)} \). Then the multiplicative factor in Eq.(44) is linear, the generalisation of Eq.(45) reads

\[
p(x, t) = A_1^{-1}t^{-\beta_{ca}/\alpha}|x|^\gamma L_\alpha(t^{-\beta_{ca}/\alpha}|y(x)|/A_1)),
\]

where \( A_1 = \text{const} \), and evaluation of the variance, analogous to Eq.(47), yields

\[
\langle x^2 \rangle(t) \sim t^{\frac{2\gamma L_{\alpha}}{L}},
\]

The nonhomogeneity of the medium makes the diffusion slower (\( \theta > 0 \)) or faster (\( \theta < 0 \)), compared to the homogeneous case. Eq.(52) constitutes a limiting form of the variance, valid for a very long time evolution, but it is rather formal since corresponds to large distances which can hardly be interpreted as the surface region.

For small and moderate times, variance is sensitive on the density not solely corresponding to very large \( |x| \) and one has to take into account Eq.(44) in its complete form which cannot be accomplished analytically. Then the variance was determined by a numerical solving of Eq.(24) where the multiplicative noise was treated according to Eq.(41) with \( \lambda_I = 1/2 \). The numerical calculations show that the time-dependence of the variance has the same form as Eq.(46) but with a modified index, \( \sim t^{\mu} \). This form was found for all the parameters – \( \theta \), \( \alpha \) and \( \beta \) – if \( L \) was large. The above observation is illustrated in Fig.3 where some examples of \( \langle x^2 \rangle(t) \) are presented. On the other hand, the calculation for a very large time and small \( L \) confirms the limiting form (52). Dependence of the slope on the parameters is presented in the subsequent figures. Fig.4 shows \( \mu \) as a function of \( \theta \) for two values of \( \alpha \); it diminishes with \( \theta \) and the numerical results reveal a dependence \( \mu = 0.5/(1 + c(\alpha)\theta) \) (the coefficient \( c(\alpha) \) is indicated in the figure). The dependence of \( \mu \) on \( \alpha \) is presented in Fig.5 for two values of \( \theta \), both negative and positive. Whereas in the former case \( \mu \) is almost constant – and larger than the value predicted by Eq.(46) – for the positive \( \theta \) we observe a linear growth in a wide range of \( \alpha \); \( \mu(\alpha) \) becomes flat only at large \( \alpha \). Finally, Fig.6 shows that the slope for a positive (negative)
θ rises with β weaker (stronger) than for θ = 0 and both dependences are linear. The superdiffusion emerges when θ is negative and β large.

In the above analysis, we interpreted the multiplicative noise \( f(x) \) in Eq. (24) according to SI. This interpretation is distinguish because it constitutes a white-noise limit of coloured noises for any \( \alpha \) \[27\]. However, the other interpretations may also be important. For example, the experimental analysis of the colloidal particles diffusion near the boundary favours AII (\( \lambda_I = 1 \)) \[34\], as we have already mentioned. We performed the numerical calculations for AII, similar to those for SI, and found the same diffusion properties in respect both to the exponent and the proportionality coefficient. This conclusion could be anticipated from Fig.1: tails of the distribution for both interpretations are very similar.
The stochastic motion in a medium with traps was studied in terms of the Langevin equation and the anomalous diffusion exponent was determined. The memory effects were taken into account by a subordination of a Markovian process to a physical, random time. The trap structure was assumed as nonhomogeneous which implied a dependence of the time lag on the trap density. If one decouples effects related to the trap structure and memory, the problem resolves itself to a multiplicative process subordinated to the random time. The density distribution can be exactly derived if the trap density has a power-law form, i.e. corresponds to a fractal structure. Then the trap density diminishes with the distance which qualitatively influences the diffusion properties: beside the subdiffusion, observed if the trap density has a power-law form, i.e. corresponds to a fractal structure. Then the trap density resolves itself to a multiplicative process subordinated to the random time. The density distribution can be exactly resolved if the trap density has a power-law form, i.e. corresponds to a fractal structure. Then the trap density diminishes with the distance which qualitatively influences the diffusion properties: beside the subdiffusion, observed for the homogeneous case, the enhanced diffusion emerges. The smaller the fractal dimension of the trap structure, the faster the variance grows with time.

Lévy flights are characterised by the infinite variance but a finite size of the system imposes a restriction on the jump size. Presence of impurities and defects near the boundary may require a generalisation of a simple modelling in terms of the additive noise. We took into account those effects by introducing a multiplicative noise to the Langevin equation and obtained the density distributions with fast falling tails. Therefore, diffusion inside the substrate is well-determined and may be quantified in terms of the variance the time-dependence of which does not depend of the system size. Moreover, the diffusion properties appear robust in respect to a particular interpretation of the multiplicative noise in the Langevin equation. On the other hand, diffusion inside a layer near the boundary is weaker than in the bulk and we observe two regimes with a different anomalous diffusion exponent $\mu$. This exponent reflects both the memory in the system, described by the parameter $\beta$, and the geometry of the medium, given by the trap density parameter $\theta$: it diminishes with $\theta$ and rises with $\beta$. In contrast to the homogeneous trap distribution, we observe a dependence of $\mu$ on $\alpha$ but only for small $\alpha$ and positive $\theta$. This conclusion points at a subtle relation between the nonhomogeneous trap structure and the noise statistics in respect to the diffusion properties of systems with the Lévy flights.

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The asymptotics $|y|^{-1-\alpha}$ is valid, in particular, for any positive $\theta$ but the approximation of the solution of Eq. (26) near the origin by the Lévy stable distribution imposes an upper limit on $\theta$ [20]. The solution for large $\theta$ requires a modification of the stable form [23].

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