On Closed Geodesics on Ellipsoids

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Abstract
Closed geodesic lines on an ellipsoid in $d$-dimensional Euclidean space are considered. Explicit algebro-geometric condition for closedness of such a geodesic is given. The obtained condition is discussed in light of theta-functions theory and compared with some recent related results.

1 Introduction
Geodesic motion on the ellipsoid is one of the most celebrated and most important classical integrable systems. It has remained in the focus interest of many researchers for almost two centuries.

Introducing elliptical coordinates, Jacobi proved that the geodesic motion on the ellipsoid is integrable [8]. Weierstrass explicitly integrated Jacobi problem for two-dimensional ellipsoid in terms of theta-functions [14].

Twenty-century mathematicians also gave a great contribution to the knowledge on geodesic motion on ellipsoid. Let us mention results of Knörrer [9] [10] and Moser [12], where the deep connection between Jacobi and Neumann problems is discovered and interpreted in the modern language of isospectral theory. In [10], the explicit integration of Jacobi problem in arbitrary dimension is given.

Let us mention the connection between billiard motion inside an ellipsoid and Jacobi problem and the contribution of the authors in describing periodic billiard trajectories in [2] [3] [4] [5].

In this paper, we give analytical description of closed geodesic lines on an ellipsoid in $d$-dimensional Euclidean space. The article is organized as follows: in Section 2 we list the necessary prerequisites on geometry of quadrics and their geodesic lines, in Section 3 the analytical conditions for closed geodesics are derived, and in Section 4 the relations between our conditions and the theta-functions theory are discussed.

1 on leave at the Weizmann Institute of Science, Rehovot, Israel
2 Confocal Quadrics and the Caustics of a Geodesic Line on the Ellipsoid in $\mathbb{R}^d$

Consider an ellipsoid in $\mathbb{R}^d$:

$$\frac{x_1^2}{a_1} + \cdots + \frac{x_d^2}{a_d} = 1, \quad a_1 > \cdots > a_d > 0,$$

and the related system of Jacobian elliptic coordinates $(\lambda_1, \ldots, \lambda_d)$ ordered by the condition

$$\lambda_1 > \lambda_2 > \cdots > \lambda_d.$$

If we denote:

$$Q_\lambda(x) = \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_d^2}{a_d - \lambda},$$

then any quadric from the corresponding confocal family is given by the equation

$$Q_\lambda : Q_\lambda(x) = 1. \quad (1)$$

The famous Chasles theorem states that any line in the space $\mathbb{R}^d$ is tangent to exactly $d - 1$ quadrics from a given confocal family (see for example [1]). Next lemma gives an important condition on these quadrics.

**Lemma 1** Suppose a line $\ell$ is tangent to quadrics $Q_0, Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}$ from the family (1). Then Jacobian coordinates $(\lambda_1, \ldots, \lambda_d)$ of any point on $\ell$ satisfy the inequalities $P(\lambda_s) \geq 0$, $s = 1, \ldots, d$, where

$$P(x) = -x(a_1 - x) \cdots (a_d - x)(a_1 - x) \cdots (a_{d-2} - x).$$

**Proof.** Let $x$ be a point of $\ell$, $(\lambda_1, \ldots, \lambda_d)$ its Jacobian coordinates, and $y$ a vector parallel to $\ell$. The equation $Q_\lambda(x + ty) = 1$ is quadratic with respect to $t$. Its discriminant is:

$$\Phi_\lambda(x, y) = Q_\lambda(x, y)^2 - Q_\lambda(y)(Q_\lambda(x) - 1),$$

where

$$Q_\lambda(x, y) = \frac{x_1y_1}{a_1 - \lambda} + \cdots + \frac{x_dy}{a_d - \lambda}.$$ 

By (2),

$$\Phi_\lambda(x, y) = \frac{(a_1 - \lambda) \cdots (a_{d-1} - \lambda)}{(a_1 - \lambda) \cdots (a_d - \lambda)}.$$

For each of the coordinates $\lambda = \lambda_s, (1 \leq s \leq d)$, the quadratic equation has a solution $t = 0$; thus, the corresponding discriminants are non-negative. This is obviously equivalent to $P(\lambda_s) \geq 0$. \qed

It is well known that, for a given geodesic on $Q_0$, all its tangent lines touch, besides $Q_0$, the same $d - 2$ quadrics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}$ from the confocal family (1), see [1, 8]. We shall refer to these quadrics as caustics of the geodesic line.
The caustics cut out several domains on the ellipsoid $Q_0$. Due to the Lemma, the corresponding geodesic line can be placed only in some of the domains.

Denote by $\Omega$ a domain on $Q_0$, such that its boundary $\partial \Omega$ lies in the union of confocal quadrics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}$ from the family, and that there is a geodesic line in $\Omega$ with caustics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}$.

For any fixed geodesic line $g$ on $Q_0$ with the caustics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}$, and for any $s = 1, \ldots, d - 1$, denote by $\Lambda_s(g)$ the set of all values taken by the coordinate $\lambda_s$ on the geodesic line and

$$\Lambda'_s = \{ \lambda \in [a_{s+1}, a_s] : \mathcal{P}(\lambda) \geq 0 \}.$$

According to Lemma, we have the following:

**Corollary 1** $\Lambda_s(g) \subset \Lambda'_s$.

The converse is also true.

**Lemma 2** For a given geodesic line $g$,

$$\Lambda_s(g) \supset \Lambda'_s$$

for any $s = 1, \ldots, d - 1$.

**Proof.** By [10], each of the intervals $(a_{s+1}, a_s)$, $(2 \leq s \leq d - 1)$ contains at most two of the values $\alpha_1, \ldots, \alpha_{d-2}$, while none of them is included in $(-\infty, a_d) \cup (a_1, +\infty)$. Thus, for each $s$, the following three cases are possible:

**First case:** $\alpha_i, \alpha_j \in [a_{s+1}, a_s], \alpha_i < \alpha_j$. Since any line tangent to the geodesic line touches $Q_{\alpha_i}$ and $Q_{\alpha_j}$, the whole geodesic is placed between these two quadrics. The elliptic coordinate $\lambda_s$ has critical values at points where the geodesic touches one them, and remains monotonous elsewhere. Hence, meeting points with $Q_{\alpha_i}$ and $Q_{\alpha_j}$ are placed alternately along the geodesic and $\Lambda_s = \Lambda'_s = [\alpha_i, \alpha_j]$.

**Second case:** Among $\alpha_1, \ldots, \alpha_{d-2}$, only $\alpha_i$ is in $[a_{s+1}, a_s]$. $\mathcal{P}$ is non-negative in exactly one of the intervals: $[a_{s+1}, \alpha_i], [\alpha_i, a_s]$, let us take in the first one. Then the coordinate $\lambda_s$ has critical values at meeting points with the hyperplane $x_{s+1} = 0$ and the caustic $Q_{\alpha_i}$, and remains monotonous elsewhere. Hence, $\Lambda_s = \Lambda'_s = [a_{s+1}, \alpha_i]$. If $\mathcal{P}$ is non-negative in $[\alpha_i, a_s]$, then we obtain $\Lambda_s = \Lambda'_s = [\alpha_i, a_s]$.

**Third case:** The segment $[a_{s+1}, a_s]$ does not contain any of values $\alpha_1, \ldots, \alpha_{d-2}$. Then $\mathcal{P}$ is non-negative in $[a_{s+1}, a_s]$. The coordinate $\lambda_s$ has critical values only at meeting points with the hyperplanes $x_{s+1} = 0$, $x_s = 0$ and changes monotonously between them. This implies that the geodesic line meets them alternately. Obviously, $\Lambda_s = \Lambda'_s = [a_{s+1}, a_s]$. □

Denote $[\gamma'_s, \gamma''_s] := \Lambda_s = \Lambda'_s$. Notice that the geodesic line meets quadrics of any pair $Q_{\gamma'_s}, Q_{\gamma''_s}$ alternately. Thus, any closed geodesic has the same number of intersection points with each of them.
3 Analytical Conditions for Closed Geodesic Lines on Ellipsoid

Before formulating a criterion for sufficient and necessary condition for closedness of real geodesic lines on the ellipsoid, let us define the following projection of the Abel-Jacobi map. Consider a hyperelliptic curve

$$\Gamma : y^2 = P(x),$$

(2)

together with the standard basis of holomorphic differentials:

$$\omega^{st} = \left[\frac{dx}{y}, \frac{x dx}{y}, \ldots, \frac{x^{d-1} dx}{y}\right].$$

Denote

$$\bar{A}(P) = \begin{pmatrix}
0 \\
\int_0^P \frac{x dx}{y} \\
\int_0^P \frac{x^2 dx}{y} \\
\vdots \\
\int_0^P \frac{x^{d-1} dx}{y}
\end{pmatrix}.$$  

Theorem 1 A geodesic line on the ellipsoid $Q_0$, with caustics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}$, is closed with exactly $n_s$ intersection points with each of quadrics $Q_{\gamma'_s}, Q_{\gamma''_s}$, $(1 \leq s \leq d - 1)$ if and only if

$$\sum_{s=1}^{d-1} 2n_s (\bar{A}(P_{\gamma'_s}) - \bar{A}(P_{\gamma''_s})) = 0.$$  

(3)

Here,

$$[\gamma'_s, \gamma''_s] = \{ \lambda \in [a_{s+1}, a_s] : P(\lambda) \geq 0 \}$$

$P_{\gamma'_s}, P_{\gamma''_s}$ are the points on $\Gamma$ with coordinates $P_{\gamma'_s} = \left(\gamma'_s, (-1)^s \sqrt{P(\gamma'_s)}\right), P_{\gamma''_s} = \left(\gamma''_s, (-1)^s \sqrt{P(\gamma''_s)}\right).$

Proof. By $\mathcal{S}$, the system of differential equations of a geodesic line on $Q_0$ with the caustics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}$ is:

$$\sum_{s=1}^{d-1} \frac{\lambda_s d \lambda_s}{\sigma_s \sqrt{P(\lambda_s)}} = 0, \quad \sum_{s=1}^{d-1} \frac{\lambda_s^2 d \lambda_s}{\sigma_s \sqrt{P(\lambda_s)}} = 0, \ldots, \quad \sum_{s=1}^{d-1} \frac{\lambda_s^{d-1} d \lambda_s}{\sigma_s \sqrt{P(\lambda_s)}} = 0,$$  

(4)

with the same sign $\sigma_s \in \{-1, 1\}$ in all of the expressions, for any fixed $s$. Also,

$$\sum_{s=1}^{d-1} \frac{\lambda_s^d d \lambda_s}{\sqrt{P(\lambda_s)}} = 2d\ell,$$  

(5)
\[\text{where } dl \text{ is the length element.}\]

Attributing all possible combinations of signs \((\sigma_1, \ldots, \sigma_{d-1})\) to \(\sqrt{P(\lambda_1)}, \ldots, \sqrt{P(\lambda_{d-1})}\), we can obtain \(2^{d-2}\) non-equivalent systems \((4)\), which correspond to \(2^{d-2}\) different tangent lines to \(Q_0, Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}\) from a generic point of the ellipsoid \(Q_0\). Moreover, the systems corresponding to a line and its reflection to a given hyper-surface \(\lambda_s = \text{const}\) differ from each other only in signs of the roots \(\sqrt{P(\lambda_s)}\).

Solving \((4)\) and \((5)\) as a system of linear equations with respect to \(d\lambda_s \sqrt{P(\lambda_s)}\), we obtain:

\[
\frac{d\lambda_s}{\sqrt{P(\lambda_s)}} = \frac{2dl}{\prod_{i \neq s} (\lambda_s - \lambda_i)}.
\]

Thus, along the geodesic line, the differentials \((-1)^{s-1} \frac{d\lambda_s}{\sqrt{P(\lambda_s)}}\) stay always positive, if we assume that the signs of the square roots are chosen appropriately.

From these remarks and the discussion preceding this theorem, it follows that the value of the integral \(\int_{\gamma_s'} \frac{\lambda_i d\lambda_s}{\sqrt{P(\lambda_s)}}\) between two consecutive common points of the geodesic and the quadric \(Q_{\gamma_s'}\) (or \(Q_{\gamma_s''}\)) is equal to:

\[
2(-1)^{s-1} \int_{\gamma_s'}^{\gamma_s''} \frac{\lambda_i d\lambda_s}{\sqrt{P(\lambda_s)}} + \sqrt{P(\lambda_s)}.
\]

Now, if \(g\) is a closed geodesic having exactly \(n_s\) points at \(Q_{\gamma_s'}\) and \(n_s\) at \(Q_{\gamma_s''}\) \((1 \leq s \leq d - 1)\), then

\[
\sum \int_{K} \frac{\lambda_i d\lambda_s}{\sqrt{P(\lambda_s)}} = 2 \sum (-1)^{s-1} n_s \int_{\gamma_s'}^{\gamma_s''} \frac{\lambda_i d\lambda_s}{\sqrt{P(\lambda_s)} + \sqrt{P(\lambda_s)}}, \quad (1 \leq i \leq d - 2).
\]

Finally, the geodesic line is closed if and only if

\[
\sum (-1)^s 2n_s \int_{\gamma_s'}^{\gamma_s''} \frac{\lambda_i d\lambda_s}{\sqrt{P(\lambda_s)}} = 0, \quad (1 \leq i \leq d - 2),
\]

which was needed.

\[\square\]

**Example 1** An interesting class of closed geodesic lines on the ellipsoid in three-dimensional space, in obtained in [7].

**Corollary 2** If a geodesic line on the ellipsoid \(Q_0\), with caustics \(Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}}\), satisfies the condition

\[
\sum_{s=1}^{d-1} n_s (A(P_{\gamma_s'}) - A(P_{\gamma_s''})) = 0, \quad (6)
\]
then it is closed with exactly \( n_s \) intersection points with each of quadrics \( Q_{\gamma_s'}, Q_{\gamma'_{s'}} \), \( 1 \leq s \leq d-1 \).

**Proof.** After the reparametrization:

\[
ds = \lambda_1 \lambda_2 \cdots \lambda_d \, dl,
\]

the equation (6) transforms into:

\[
\sum_{s=1}^{d-1} \frac{d\lambda_s}{\sqrt{P(\lambda_s)}} = 2ds. \tag{7}
\]

Now, this equation, together with the system (4) is equivalent to the condition (3).

Let us note that the condition (6) will be satisfied if and only if (3) holds (i.e. the geodesic line is closed) and its length \( L \) with respect to the parameter \( s \):

\[
ds = \lambda_1 \lambda_2 \cdots \lambda_n \, dl
\]

is such that the vector

\[
\begin{pmatrix}
L/2 \\
0 \\
\vdots \\
0
\end{pmatrix} + \sum_{s=1}^{d-1} n_s (\bar{A}(P_{\gamma_s'}) - \bar{A}(P_{\gamma'_{s'}}))
\]

belongs to the period-lattice of the Jacobian of the corresponding hyper-elliptic curve.

### 4 Closed Geodesics and Theta Functions

In this section, we are going to present a different approach to finding an analytical condition for the closed geodesics on the ellipsoid, based on results of [6].

The equations of a geodesic line on the ellipsoid in the \( d \)-dimensional space are:

\[
x_1(t) = \alpha_1 \frac{\theta[\alpha_1, \beta_1](tU + z_0)\theta(z_0)}{\theta[\alpha_1, \beta_1](z_0)\theta(tU + z_0)},
\]

\[
x_2(t) = \alpha_2 \frac{\theta[\alpha_2, \beta_2](tU + z_0)\theta(z_0)}{\theta[\alpha_2, \beta_2](z_0)\theta(tU + z_0)},
\]

\[
\vdots
\]

\[
x_d(t) = \alpha_d \frac{\theta[\alpha_d, \beta_d](tU + z_0)\theta(z_0)}{\theta[\alpha_d, \beta_d](z_0)\theta(tU + z_0)}.
\tag{8}
\]
Here, 
\[ \alpha_i = \left[ \prod_{j \neq i} (a_i - a_j) \right]^{-1/2}, \]
the theta-functions are constructed over the Riemann surface \( \mathcal{R} \), \( z_0 \) is an arbitrary vector of the Jacobian \( \mathcal{J}(\Gamma) \), and \( U \) is a vector of \( b \)-periods of the differential \( \Omega \) of the second order with the pole at the point \( x = \infty \):
\[ \Omega = \frac{x^{d-1} + a_{d-1}x^{d-1} + \cdots + a_0}{2\sqrt{P(x)}} \]
normalized by the condition 
\[ \oint_{a_i} \Omega = 0, \quad U_i = \oint_{b_i} \Omega \quad (i = 1, \ldots, d - 1). \]
Pairs \( (\alpha_i, \beta_i) \) are the corresponding characteristics (for more detailed explanation, see [6] and [13]).

A sufficient condition for the periodicity of the curve given by (8) is:
\[ TU = n_1E_1 + n_2E_2 + \cdots + n_gE_g + m_1F_1 + m_2F_2 + \cdots + m_gF_g, \quad g = d - 1 \]
for some \( T > 0 \), where \( E_1, \ldots, E_g, F_1, \ldots, F_g \) are the basis vectors of the period lattice of a basis \( \omega \) of holomorphic differentials, and \( n_k, m_j \) are integers.

We can calculate explicitly the vector \( U \) in coordinate system associated with some basis \( \omega \) of holomorphic differentials. The order of the pole of \( \Omega \) at \( \infty \) is equal to 2, thus [13]:
\[ U_i = \oint_{b_i} \Omega = f_i(\infty). \]
Here, \( \omega_i = f_i(t)dt \), where \( t \) is a local coordinate around \( \infty \) and \( \omega = [\omega_1, \ldots, \omega_g] \).

For the standard basis
\[ \omega_1^{st} = \frac{dx}{y}, \ldots, \omega_g^{st} = \frac{x^{g-1}dx}{y}, \]
since \( x = 1/t^2 \), we have around \( \infty \):
\[ \omega_k^{st} = \frac{-2t^{2g-2k}dt}{\sqrt{P(t^2)}}. \]
Here
\[ \mathcal{P}(\xi) = -(a_1\xi - 1) \cdots (a_d\xi - 1)(a_1\xi - 1) \cdots (a_{d-2}\xi - 1). \]
Thus, \( \omega_k^{st} \) has a zero of order \( 2g - 2k \) at the infinity point, for \( 1 \leq k \leq g - 1 \), and \( \omega_g^{st} \) has no zero at this point.

Finally, we get the formula for the vector \( U \):
\[ U^{st} = [0, \ldots, 0, u_g]^T. \]
Denote now
\[
\bar{A}'(P) = \begin{pmatrix}
\int_0^p \frac{dx}{y} \\
\int_0^p \frac{x \, dx}{y} \\
\int_0^p \frac{x^2 \, dx}{y} \\
\vdots \\
\int_0^p \frac{x^{d-2} \, dx}{y} \\
0
\end{pmatrix}.
\]

**Theorem 2** Let a geodesic line \( g \) on the ellipsoid \( Q_0 \) associated with a curve \( \mathcal{C} \) where the ordered zeroes of the polynomial \( P \) are
\[
z_1 < \cdots < z_{2g}
\]
be given. A sufficient condition for \( g \) to be closed is
\[
\sum_{s=1}^{g} 2n_s (\bar{A}'(z_{2s}) - \bar{A}'(z_{2s-1})) + \sum_{s=1}^{g} m_s (\bar{A}'(b_s)) = 0,
\]
where \( b_s \) denotes a basis of \( b \)-cycles and basis of \( a \)-cycles of \( \Gamma \) is represented by double cuts \([z_{2s-1}, z_{2s}]\).

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