Operator product expansion in $SL(2)$ conformal field theory

Kazuo Hosomichi†, Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-8502, Japan

and

Yuji Satoh‡
Institute of Physics, University of Tsukuba
Tsukuba, Ibaraki 305-8571, Japan

Abstract

In the conformal field theories having affine $SL(2)$ symmetry, we study the operator product expansion (OPE) involving primary fields in highest weight representations. For this purpose, we analyze properties of primary fields with definite $SL(2)$ weights, and calculate their two- and three-point functions. Using these correlators, we show that the correct OPE is obtained when one of the primary fields belongs to the degenerate highest weight representation. We briefly comment on the OPE in the $SL(2, R)$ WZNW model.

May 2001

†hosomiti@yukawa.kyoto-u.ac.jp
‡ysatoh@het.ph.tsukuba.ac.jp
1. The conformal field theories having affine $SL(2)$ symmetry have been an interesting topic in recent string theory, since the $SL(2)$ symmetry expresses the isometry of the $AdS_3$ target space or its Euclidean counterpart known as simplest examples exhibiting the holography. The CFT on the Euclidean $AdS_3$, namely, the $H^+_3$ WZNW model is now well controlled \cite{1,2,3} and, starting from this, one may extract useful results for other models with the affine $SL(2)$ symmetry \cite{4,5}.

In this note, we continue this line of studies. Our point here is to focus on the primary fields with definite $SL(2)$ weights. These are important in dealing with highest weight representations, since highest weight conditions are expressed by certain relations between the $SL(2)$ spin and weight. In particular, we present pieces of properties of the primary fields mentioned above, and calculate their two- and three-point functions. Using these correlators, we discuss the operator product expansion including primary fields in highest weight representations. When one of the primary fields belongs to the degenerate highest weight representation, we show that the correct OPE is obtained. We briefly comment on the OPE in the $SL(2, R)$ WZNW model. Our analyses may serve also as preparatory steps for further studies.

2. The $H^+_3$ WZNW model \cite{1,2,3} has the action

$$S = \frac{k}{\pi} \int d^2 z \left[ \partial \phi \bar{\partial} \phi + e^{2\phi} \partial \bar{\gamma} \bar{\partial} \gamma \right].$$

The primary fields are organized by introducing “boundary coordinates” $x$ and $\bar{x}$:

$$\Phi_j(z, x) = \left(|\gamma - x|^2 e^\phi + e^{-\phi}\right)^{2j}.$$

They behave as if they were primary fields of conformal weight $-j$ on the $x$-plane. The global part of the affine $SL(2)$ symmetry of the model acts onto them as conformal transformations on the $x$-plane \cite{3,7}. They also have conformal weight $h \equiv -(j + 1)/(k - 2)$, and OPE’s with the $\hat{sl}_2$ currents,

$$J^a(z)\Phi_j(w, x) \sim -\frac{D^a\Phi_j(w, x)}{z - w},$$

$$D^- = \partial_x, \quad D^3 = x\partial_x - j, \quad D^+ = x^2\partial_x - 2jx.$$

The expressions with $J^a(\bar{z})$ are similar. The two- and the three-point functions of these fields are given by \cite{2,3,8,9}

$$\langle \Phi_{j_1}(z_1, x_1)\Phi_{j_2}(x_2, x_2) \rangle = |z_2|^{-4h_1} \left[ A(j_1)\delta^2(x_{12})\delta(j_1 + j_2 + 1) + B(j_1)|x_{12}|^{4j_1}\delta(j_1 - j_2) \right],$$

$$A(j) = -\frac{\pi^3}{(2j + 1)^2}, \quad B(j) = b^2\pi^2[k^{-1}\Delta(b^2)]^{2j+1}\Delta[-b^2(2j + 1)],$$
\[ \prod_{a=1}^{3} \Phi_{j_a}(z_a, x_a) = D(j_a) \prod_{a<b} |z_{ab}|^{-2h_{ab}} |x_{ab}|^{2j_{ab}}, \]

\[ D(j_a) = \frac{b^2 \pi}{2} \left[ k^{-1} b^{-2} \Delta(b^2) \Sigma_{j_a+1} \Upsilon[b] \Upsilon[-2j_1 b] \Upsilon[-2j_2 b] \Upsilon[-2j_3 b] \right], \]

where \( \Delta(x) = \Gamma(x)/\Gamma(1-x) \), \( b^{-2} = k - 2 \) and \( z_{ab} \equiv z_a - z_b \), \( j_{12} \equiv j_1 + j_2 - j_3 \), etc. An entire function \( \Upsilon \) was introduced in [10, 11] and is characterized by the spectrum of zeroes,

\[ \Upsilon(x) = 0 \text{ at } x = -mb - nb^{-1}, \quad x = (m + 1)b + (n + 1)b^{-1} \quad (m, n \in \mathbb{Z}_{\geq 0}). \]

Using the above correlators and the \( SL(2) \) symmetry, we can write down the OPE formula:

\[ \Phi_{j_1}(z_1, x_1) \Phi_{j_2}(z_2, x_2)^{z_1 \sim z_2} \int_{\mathcal{C}} d\bar{\tau} \int \frac{d^2 x_3}{x_{ab}} \prod_{a<b}^{3} |x_{ab}|^{2j_{ab}} \frac{D(j_a)}{A(j_3)} \Phi_{j_3^{-1}}(z_2, x_3). \quad (1) \]

where the \( j_3 \)-integration should be taken over all the normalizable representations on the Euclidean \( AdS_3 \), i.e., \( \mathcal{C} = \mathcal{P} = -\frac{1}{2} + i\mathbb{R} \), if the two operators both belong to the normalizable representations. For generic \( j_1 \) and \( j_2 \) we assume that certain deformations of contours should be made so as to go around the sequences of poles in the integrand and ensure the analyticity in \( j_{1,2} \). Those poles in the integrand are given by

\[ (1) \ j_{12} = S, \quad (1b) \ j_{12} = -1 - S; \quad (2a) \ N = S, \quad (2b) \ N = -1 - S; \]

\[ (3a) \ j_{13} = S, \quad (3b) \ j_{13} = -1 - S; \quad (4a) \ j_{23} = S, \quad (4b) \ j_{23} = -1 - S; \]

where \( N = \sum_{a=1}^{3} j_a + 1 \) and \( S = l + nb^{-2} \ (l, n \in \mathbb{Z}_{\geq 0}) \). These originate from the zeroes of \( \Upsilon \) functions as well as the exponents of \( \mid x_{ab} \mid \) via

\[ \text{Res } x^{\epsilon - l - 1} \bar{x}^{\epsilon - n - 1} |_{\epsilon = 0} = \frac{\pi}{l! n!} \delta^l \bar{\delta}^n \delta^2(x). \quad (3) \]

3. We would like to note that the above OPE formula has a semi-classical \( (k \rightarrow \infty) \) limit which agrees with the supergravity analysis on the Euclidean \( AdS_3 \) background. Using \( \Upsilon(x) \xrightarrow{b \to 0} \text{const} \cdot \Gamma(x/b)^{-1} \) we obtain the following semi-classical OPE formula,

\[ \Phi_{j_1}(x_1) \Phi_{j_2}(x_2) = \frac{1}{2} \int_{\mathcal{P}} d\tau \int d^2 x_3 \prod_{a<b} |x_{ab}|^{2j_{ab}} A(j_3)^{-1} D_0(j_a) \Phi_{-j_3^{-1}}(x_3), \]

\[ D_0(j_a) = \frac{\pi \Gamma(-\Sigma j_a - 1) \Gamma(-j_{12}) \Gamma(-j_{23}) \Gamma(-j_{31})}{\Gamma(-2j_1) \Gamma(-2j_2) \Gamma(-2j_3 - 1)}, \]

out of which the semi-classical four-point function can be expressed as [4]

\[ \left\langle \prod_{a=1}^{4} \Phi_{j_a}(x_a) \right\rangle \]

\[ = -\frac{1}{\pi^2} |x_{12}|^{2(j_1+j_2)+1} |x_{13}|^{2(j_1-j_2)-1} |x_{23}|^{2(-j_1+j_2+j_3-j_4)} |x_{24}|^{2(-j_3+j_4)-1} |x_{34}|^{2(j_3+j_4)+1} \]

\[ \times \int_{\mathcal{P}} dj \ (2j + 1) D_0(j_1, j_2, j) D_0(j_3, j_4, j) |x|^{-2j-1} |F(j_2 - j_1 - j, j_3 - j_4 - j; -2j; x)|^2, \]
where $x \equiv x_{12}x_{34}/x_{13}x_{24}$. This is obtained also by using the relations of completeness and orthogonality of $\Phi_j(x)$.

The behavior of the integrand for large $|j|$ is evaluated from the asymptotic behavior of the hypergeometric function $[12]$. Then, it turns out that the contour of $j$-integration can be closed in the left half-plane so that we may replace the $j$-integral with the sum over poles at

$$j = j_1 + j_2 - l, \ j_3 + j_4 - l \ (l \in Z_{\geq 0}). \tag{4}$$

The spectrum of intermediate states thus obtained agrees with the semi-classical result of Liu [13]. This further supports the prescription of the OPE given in (1). This also shows that there are two different expansions for the same quantity because of the existence of infinitely many primary fields: one is by $j \in P$ and the other is by (4). Similar phenomena are observed in Liouville theory [14] and in an $SL(2)$ model [15].

4. The primary fields with definite $SL(2)$ weights, i.e., eigenvalues of the zero-modes of $J^3(z)$ and $\bar{J}^3(\bar{z})$, are obtained by Fourier transforming the primaries $\Phi_j(z, x)$:

$$\Phi^j_{\bar{m}m} = \int d^2x \, x^{j+m} \bar{x}^{j+\bar{m}} \Phi_{-j-1}(z, x). \tag{5}$$

The above Fourier transformations are well defined only for $m - \bar{m} \in Z$. In fact, for the $H_3^+$ model, $m + \bar{m} \in iR$ and $m - \bar{m} \in Z$. These have the OPE’s, e.g.,

$$J^\pm(z) \Phi^j_{\bar{m}m}(w) \sim \frac{\mp j + m}{z - w} \Phi^j_{m \pm 1 \bar{m}}, \quad \bar{J}^3(z) \Phi^j_{\bar{m}m}(w) \sim \frac{m}{z - w} \Phi^j_{\bar{m}m}. \tag{6}$$

For evaluating (5), we use the Mellin transform of $(z + 1)^{-2(j+1)}$ with $z = |\gamma - x|^2 e^{2\phi}$ and Barnes’ representation of the hypergeometric function.

Here, we introduce a coordinate system $(\tau, \varphi, r)$ via

$$e^\phi = e^{-\tau} \cosh r, \quad \gamma = e^{\theta_L} \tanh r, \quad \bar{\gamma} = e^{\theta_R} \tanh r \quad (\theta_{L/R} \equiv \tau \pm i\varphi),$$

in which the metric reads

$$ds^2 = \cosh^2 r \, d\tau^2 + dr^2 + \sinh^2 r \, d\varphi^2.$$

Then, the explicit expression of $\Phi^j_{\bar{m}m}$ is given by

$$\Phi^j_{\bar{m}m} = \frac{\pi \Gamma(j + 1 + m) \Gamma(j + 1 - \bar{m})}{\Gamma(m - \bar{m} + 1) \Gamma(2j + 2)} \epsilon^{m\theta_L + m\theta_R} \cosh^{-m - \bar{m}} r \sinh^{m - \bar{m}} r \times F(-j - \bar{m}, j + 1 - \bar{m}; m - \bar{m} + 1; - \sinh^2 r) \quad (m - \bar{m} \geq 0), \tag{7}$$

and those for $m - \bar{m} \leq 0$ can be obtained by exchanging $m$ and $\bar{m}$ with $m\theta_L + \bar{m}\theta_R$ fixed. (See also [3].) $\Phi^j_{\bar{m}m}$ satisfy the reflection relation,

$$\Phi^{j-1}_{\bar{m}m} = \frac{2j + 1}{\pi} \epsilon^{j \bar{m}} \Phi^j_{\bar{m}m},$$

$$\epsilon^{j \bar{m}} = \frac{\pi \Gamma(m - j) \Gamma(-\bar{m} - j) \Delta(2j + 1)}{\Gamma(m + j + 1) \Gamma(-\bar{m} + j + 1)}.$$
For \( \phi \to \infty \), the asymptotic behavior of \( \Phi_j(z, x) \) (for generic \( j \)) is given by

\[
\Phi_j \sim e^{2j\phi}|z-x|^j + \cdots + \frac{-1}{2j+1} e^{-2(j+1)\phi} \delta^2(\gamma-x) + \cdots .
\]

Plugging this into (5), we obtain

\[
\Phi^j_{m\bar{m}} \sim c^{-j-1}_{m\bar{m}} \gamma^{m-j-1} \bar{\gamma}^{\bar{m}-j-1} e^{-2(j+1)\phi} + \cdots + \frac{1}{2j+1} \gamma^{m+j} \bar{\gamma}^{\bar{m}+j} e^{2j\phi} + \cdots .
\] (8)

For \( j \in \mathcal{P} \), the leading contribution comes from both series. However, for highest weight representations with

\[
m, \bar{m} \in j - Z_{\geq 0} , \quad \text{or} \quad m, \bar{m} \in -j + Z_{\geq 0} ,
\]

the coefficient \( c^{-j-1}_{m\bar{m}} \) and, hence, the first series vanish. This shows that the asymptotic behavior largely changes for highest weight representations. The precise form of the asymptotic behavior can be read off, e.g., from (7) by using the expression of the hypergeometric function around \( r \to \infty \).

For highest weight representations, the hypergeometric function in (7) reduces to a Jacobi polynomial. For example, for \( m = j - n, \bar{m} = j - \bar{n} \ (n, \bar{n} \in Z_{\geq 0}) \), we have

\[
\Phi^j_{m\bar{m}} = \frac{\pi \Gamma(m+j+1)}{\Gamma(2j+2)} e^{m\theta + \bar{m}\bar{\theta}} y^\frac{1}{2}(m-\bar{m}) (1+y)^{\frac{1}{2}(m+\bar{m})} n! P_n^{(m-\bar{m}, m+\bar{m})} (1+2y)
\]

(10)

where \( y = \sinh^2 r \). One can confirm that these expressions are symmetric with respect to \( m \) and \( \bar{m} \), and valid for both \( m - \bar{m} \in Z_{\geq 0} \) and \( m - \bar{m} \in Z_{\leq 0} \).

When the coordinate \( \tau \) is continued as \( \bar{t} = \tau \), so that \( (t, \varphi, r) \) parametrize the Lorentzian \( AdS_3 \) or \( SL(2, R) \), \( \Phi^j_{m\bar{m}} \) represent the wave functions on \( SL(2, R) \). Precisely, they are the matrix elements of the \( SL(2, R) \) representations of the principal continuous series and highest/(lowest) weight discrete series for \( j \in \mathcal{P}; m, \bar{m} \in R; \) and \( j \leq -\frac{1}{2}; m, \bar{m} \in j - Z_{\geq 0} \ (m, \bar{m} \in -j + Z_{\geq 0}) \); respectively. Note that the vanishing of the first series in (8) due to the highest weight conditions assures the correct normalizability of the wave functions.

For highest weight representations, \( \Phi^j_{m\bar{m}} \) can be associated with the power series expansions of (the analytic part of) \( \Phi_j \) around \( x = 0 \) or \( x = \infty \). To see this, let us define

\[
\Phi^{-j-1,-}_{m\bar{m}}(z) = \oint_0 \frac{dx}{2\pi i} \frac{d\bar{x}}{2\pi i} x^{m-j-1} \bar{x}^{\bar{m}-j-1} \Phi_j(z, x),
\]

\[
\Phi^{-j-1,+}_{m\bar{m}}(z) = \oint_0 \frac{dx}{2\pi i} \frac{d\bar{x}}{2\pi i} x^{j-m-1} \bar{x}^{\bar{m}-j-1} \Phi_j(z, x^{-1}),
\]

where \( m, \bar{m} \in j - Z_{\geq 0} \) for \( \Phi^{-j-1,-}_{m\bar{m}} \) and \( m, \bar{m} \in -j + Z_{\geq 0} \) for \( \Phi^{-j-1,+}_{m\bar{m}} \). Since \( \Phi_j \) is a function of a certain combination of the variables, derivatives of \( x \) can be converted to those of \( y \). We
thus obtain the explicit form of $\Phi_{m\bar{m}}^{j-1}$ similar to (10):

$$
\Phi_{m\bar{m}}^{j-1,-} = \frac{1}{(j-m)!(j-\bar{m})!} \frac{\partial^{j-m}_{x} \partial^{j-\bar{m}}_{\bar{x}} \Phi_{j} \big|_{x=x=0}}{2j+1} \Gamma(-j-m) \Gamma(-j-\bar{m}) \frac{\Gamma(\bar{m}) \Gamma(\bar{m}+j+1)}{\Gamma(2-j) \Gamma(j+1-m) \Gamma(j+1-\bar{m})} \Phi_{j}^{\bar{m}}.
$$

(11)

The expression for $\Phi_{m\bar{m}}^{j-1,1}$ is obtained in a parallel way by making use of the inversion relation $|x|^{j}\Phi_{j}(x^{-1}) = \Phi_{j}(x)|_{(\tau,\varphi)\rightarrow(-\tau,-\varphi)}$. The final result is the same as above up to the replacement $(m, \bar{m}) \rightarrow -(m, \bar{m})$ in the coefficient in front of $\Phi_{m\bar{m}}^{j}$. Because of the factors of the Gamma functions, $\Phi_{m\bar{m}}^{j,\pm}$ have OPE’s similar to (3), but with $j$ and $-j-1$ exchanged.

5. In order to discuss the OPE of $\Phi_{m\bar{m}}^{j}$, we need their correlation functions. The two- and three-point functions are obtained by Fourier transforming those of $\Phi_{j}(z, x)$.

First, the two-point functions are given by

$$
\langle \Phi_{m_{1},m_{1}}^{j_{1}}(z_{1}) \Phi_{m_{2},\bar{m}_{2}}^{j_{2}}(z_{2}) \rangle = |z_{1}|^{-4h_{1}} \delta^{2}(m_{1} + m_{2}) \left\{ A(j_{1}) \delta(j_{1} + j_{2} + 1) + c_{m_{1}m_{1}}^{-j_{1}^{-1}}B(-j_{1} - 1)\delta(j_{1} - j_{2}) \right\},
$$

where

$$\delta^{2}(m) = \int d^{2}x x^{m-1}x^{\bar{m}-1} = 4\pi^{2} \delta(m + \bar{m}) \delta_{m-\bar{m},0}.$$

If we concentrate, e.g., on $\Im j \geq 0$ and $\Re j \leq -1/2$, only the second term in (12) remains. For the highest weight representations in (3), the remaining expression can be reduced to that proportional to $\delta_{j_{1},j_{2}}$.

The three-point functions are given by

$$
\left\langle \prod_{a=1}^{3} \Phi_{m_{a},\bar{m}_{a}}^{j_{a}}(z_{a}) \right\rangle = \delta^{2}(\Sigma m_{a}) \prod_{a<b} |z_{ab}|^{-2h_{ab}} D(-j_{a} - 1) W(j_{a}; m_{a}),
$$

(13)

where $W(j_{a}; m_{a})$ is the following integral representing the group structure:

$$W(j_{a}; m_{a}) \equiv \int d^{2}x_{1}d^{2}x_{2} x_{1}^{j_{1}+m_{1}} x_{1}^{\bar{m}_{1}} x_{2}^{j_{2}+m_{2}} x_{2}^{\bar{m}_{2}} \times |1-x_{1}|^{-2j_{13}^{2}} |1-x_{2}|^{-2j_{23}^{2}} |x_{1}-x_{2}|^{-2j_{12}^{2}}.$$

Generically, the integral $W$ is expressed in terms of the generalized hypergeometric function $\,_{3}F_{2}$ and is therefore very complicated to evaluate. However, when

$$j_{1} + m_{1} = j_{1} + \bar{m}_{1} = 0,$$

the integral is simplified to

$$W(j_{a}; m_{a}) = (-)^{m_{3}^{-}\bar{m}_{3}} \pi^{2} \frac{\Delta(-N)\Delta(2j_{1}+1)}{\Delta(1+j_{12}) \Delta(1+j_{13})} \prod_{a=2,3} \frac{\Gamma(1+j_{a} + m_{a})}{\Gamma(-j_{a} - \bar{m}_{a})}.
$$

(14)
In a special case with \( m_a = \bar{m}_a \), the three-point function (13) with (14) reduces (up to a phase) to the result in [10].

These correlators are obtained also by appropriately adapting the approach in [3] to the present case. We do not go into details, though.

6. Given the two- and three-point functions of \( \Phi^j_{m \bar{m}} \), we would like to discuss the OPE. Here, we follow the argument in [2], [3]: we start from the OPE in the \( H^+_3 \) case as in (1), and deform the integration contour for generic cases so as to go around the poles in the integrand.

Thus, we begin with the following form of the OPE,

\[
\Phi^j_{m_1 \bar{m}_1}(z_1)\Phi^j_{m_2 \bar{m}_2}(z_2) \sim |z_{12}|^{-2h_{12}} \sum_{m_3, \bar{m}_3} \frac{1}{2} \int_{\mathcal{C}} dq_3 \, Q(q_a; m_a) \, \Phi^j_{m_3 \bar{m}_3}(z_2),
\]

with \( \mathcal{C} = \mathcal{P} \) for \( j_1, j_2 \in \mathcal{P} \), \( m + \bar{m} \in i\mathbb{R} \), \( m - \bar{m} \in \mathbb{Z} \). \( Q \) is obtained from the consistency with the two- and three-point functions through

\[
\langle \Phi^j_{m_1 \bar{m}_1}(z_1)\Phi^j_{m_2 \bar{m}_2}(z_2)\Phi^j_{m_4 \bar{m}_4}(z_4) \rangle \approx |z_{12}|^{-2h_{12}} \sum_{m_3, \bar{m}_3} \frac{1}{2} \int_{\mathcal{C}} dq_3 \, Q(q_a; m_a) \, \langle \Phi^j_{m_3 \bar{m}_3}(z_2)\Phi^j_{m_4 \bar{m}_4}(z_4) \rangle.
\]

When one of the primary fields is in the highest weight representation satisfying \( m = \bar{m} = -j \), we can use (12)-(14). Assuming an appropriate deformation of the contour, we find that

\[
Q(j_a; m_a) = \delta^2(m_1 + m_2 - m_3) \frac{\Gamma(j_2 + m_2 + 1) \Gamma(-j_3 - \bar{m}_3) \Gamma(-j_2 - \bar{m}_2) \Gamma(j_3 + m_3 + 1)}{\Gamma(j_2 + 1)R(j_2)A(j_3)D(j_a)} \Delta(-2j_2)\Delta(j_2 + 1),
\]

with \( R(j) = B(j)/A(j) \). In the above, we have repeatedly used a formula

\[
D(j_1, j_2, j_3) = \pi \Delta(-j_1)\Delta(-j_2)\Delta(2j_3 + 1)R(j_3)D(j_1, j_2, -j_3 - 1).
\]

Note that the two terms proportional to \( \delta(j_3 - j_4) \) and \( \delta(j_3 + j_4 + 1) \) in \( \langle \Phi^j_{m_3 \bar{m}_3}\Phi^j_{m_4 \bar{m}_4} \rangle \) give the same contributions to (10), because

\[
Q(j_1, j_2, j_3; m_a)c_{m_3 \bar{m}_3}^{-j_3-1}B(-j_3 - 1) = A(j_3)Q(j_1, j_2, -j_3 - 1; m_a).
\]

There are two types of poles in \( j_3 \) in \( Q(j_a; m_a) \): one is \( m \)-independent and the other is \( m \)-dependent. The \( m \)-independent poles develop at \( j_3 \in (-m_3 + \mathbb{Z}_{\geq 0}) \cap (-\bar{m}_3 + \mathbb{Z}_{\geq 0}) \). The \( m \)-independent poles come from \( D(j_a) \) and \( \Delta(j_2 + 1) \). Taking into account the zeros in the latter, we find the \( m \)-independent poles in \( Q \) at

\[
\begin{align*}
(1a) \ j_{12} &= S, \quad (1b) \ j_{12} = S' \quad (2a) \ N = S, \quad (2b) \ N = S'; \\
(3a) \ j_{13} &= S, \quad (3b) \ j_{13} = S' \quad (4a) \ -j_{23} - 1 = S, \quad (4b) \ -j_{23} - 1 = S'.
\end{align*}
\]

6
where \( S' = -1 - b^{-2} - S \). The structure of the poles here can be different from that in (2), since we are considering highest weight representations. Note that the \( \hat{sl}(2) \) representation is largely reduced by highest weight conditions.

7. Now, we would like to consider the OPE involving the degenerate highest weight representation whose spin is given by

\[
2j + 1 = (I) \ (l + 1) + nb^{-2} \quad \text{or} \quad (II) \ -(l + 1) - (n + 1)b^{-2},
\]

with \( l, n \in \mathbb{Z}_{\geq 0} \). From the representation theory of the current algebra, one can derive that the OPE can be non-vanishing among a degenerate primary with spin \( j_1 \) in (I), some generic primary with spin \( j_2 \), and primaries in highest weight representations with the following spin \( j_3 \) [17]:

\[
(Ia) \quad j_3 = j_2 - j_1 + u + wb^{-2} \quad (0 \leq u \leq l, \quad 0 \leq w \leq n),
\]
\[
(Ib) \quad j_3 = j_1 - j_2 - u - wb^{-2} \quad (1 \leq u \leq l + 1, \quad 1 \leq w \leq n),
\]
\[
(IIa) \quad j_3 = j_1 - j_2 + u + wb^{-2} \quad (0 \leq u \leq l, \quad 0 \leq w \leq n),
\]
\[
(IIb) \quad j_3 = j_2 - j_1 - u - wb^{-2} \quad (1 \leq u \leq l + 1, \quad 1 \leq w \leq n).
\]

Here, the first and second sequences correspond to case (I) in (19) and the third and fourth to (II). Note that these are not invariant under exchanging \( j_3 \) and \(-j_3-1\) and they represent non-equivalent cases, since we are considering the highest weight representations with \( m_{\text{max}} = j_3 \).

This OPE would be analyzed also by using the formula (1). When one of the two operators in the product has spin \( j \) as given in (19), \( D(j_3) \) vanishes for generic values of \( j_3 \) due to a factor \( \Upsilon(-2j_1b) \) in the numerator. However, by a careful analysis, we find that there are still some contributions to the right hand side of the OPE from pairs of poles pinching the contour and degenerating into double poles. In this process, we first need to separate such pairs of poles infinitesimally, and then take the pinching limit. There are other cases in which two poles are colliding and compensate the zero from \( \Upsilon(-2j_1b) \). However, in those cases, two contributions have opposite signs and cancel each other.

From the table of the poles in (2), we find the contributions from the pinching poles at

\[
(1a - 3a) \quad j_3 = j_2 - j_1 + u + wb^{-2} \quad (0 \leq u \leq l, \quad 0 \leq w \leq n),
\]
\[
(2a - 4b) \quad j_3 = j_1 - j_2 - u - wb^{-2} \quad (1 \leq u \leq l + 1, \quad 0 \leq w \leq n),
\]
\[
(2b - 4a) \quad j_3 = j_1 - j_2 + u + wb^{-2} \quad (0 \leq u \leq l, \quad 0 \leq w \leq n + 1),
\]
\[
(1b - 3b) \quad j_3 = j_2 - j_1 - u - wb^{-2} \quad (1 \leq u \leq l + 1, \quad 0 \leq w \leq n + 1).
\]

The first column stands for the sequences in (2) containing the pinching poles. The first two cases correspond to case (I) and the last two to case (II). This result is slightly different from
(20) (and correct the statement in [2]). Tracing the disagreements, we find that the additional pinchings all originate from the “contact terms” which appear in the right hand side of (1) multiplied by derivatives of $\delta^2(x_1 - x_2)$ via (3). If we work with $\Phi_{\ell m}$, not $\Phi_j$, this discrepancy is expected to be resolved, since we are considering highest weight representations. In fact, in the OPE for $\Phi_{\ell m}$ in the previous section, such contact terms did not appear, though some of the poles associated with them remain in (18) in disguise.

Thus, let us consider the above OPE following the formulation for $\Phi_{\ell m}$, namely, (15) and (17). The procedure is the same. The contributions to the OPE come only from the pinching poles. This assures that the $m$-dependent poles are irrelevant in our argument as long as $\Phi_{\ell m}$ is generic, although the prescription for the $m$-dependent poles is yet to be determined.

From (18), we then find the table of the allowed $j_3$:

\[
\begin{align*}
(1a - 3a) \quad j_3 &= \quad j_2 - j_1 + u + wb^{-2} \quad (0 \leq u \leq l), \quad 0 \leq w \leq n), \\
(2a - 4a) \quad j_3 &= \quad j_1 - j_2 - u - wb^{-2} \quad (1 \leq u \leq l + 1), \quad 0 \leq w \leq n), \\
(2b - 4b) \quad j_3 &= \quad j_1 - j_2 + u + wb^{-2} \quad (0 \leq u \leq l), \quad 1 \leq w \leq n), \\
(1b - 3b) \quad j_3 &= \quad j_2 - j_1 - u - wb^{-2} \quad (1 \leq u \leq l + 1), \quad 1 \leq w \leq n). 
\end{align*}
\]

(22)

In our formulation, spin $j$ and $-j - 1$ appear always in pairs, and they are regarded as giving equivalent representations with common $m$ and $\bar{m}$ as in (11). The first and second, and the third and fourth sequences represent such pairs. To compare this table with the results in [17] in which the primaries with $j_3$ belong to highest weight representations, we further need to impose highest weight conditions, taking into account the difference of conventions. Denoting a pair of $j_3$’s by $j_3^{(1)}$ and $j_3^{(2)} = -j_3^{(1)} - 1$, the condition for $m_3$ reads, e.g., $m_3 = j_3^{(1)} - Z_{\geq 0}$ (and similarly for $\bar{m}_3$). Since the primary with $j_2$ is in a generic representation and $m_3$ is determined by the conservation of $m$, i.e., $m_3 = m_1 + m_2$, we can always choose $m_2$ so that the condition is satisfied. $m_3$ is then fixed for given $\Phi_{m_1 \bar{m}_1}$ and $\Phi_{m_2 \bar{m}_2}$. In turn, this means that the contribution from $j_3^{(2)}$ vanishes, because $\Gamma^{-1}(j_3^{(2)} + m_3 + 1) = 0$ in $Q$. We note that either $j$ or $-j - 1$ was selected also in an observation in section 4 after imposing highest weight conditions.

From the two tables (20) and (22), we find that (1a-3a) ∼ (2a-4a) agrees with (Ia), and (2b-4b) ∼ (1b-3b) with (IIb). The results in [17] give the possible representations which are allowed in the OPE, and all of them do not necessarily appear. For example, correct gluing of the left and right sector may restrict them. In fact, in the case with both $j_1$ and $j_2$ in (19), there are examples in which only a part of the results in [17] appears in concrete models [18, 19]. Our results are analogous, and they are regarded as consistent with [17]. This was

\[1\] In the subsequent paper, the authors of [18] showed that the full results of [17] can be obtained by a careful analysis of integration contours and screenings [20]. The full results have also been obtained in [21, 19]. Regarding the OPE for the degenerate representations, see also [22]. We would like to thank J.
not the case for the results in (21) which was obtained by the OPE of $\Phi_j$. In this way, a puzzle about the OPE using $\Phi_j$ is resolved.

8. We have analyzed the properties of $\Phi^{j}_{m\bar{m}}$, and calculated their correlation functions. Using these correlators, we have considered the OPE in the case including the degenerate highest weight representation, and obtained the correct OPE.

Our arguments indicate an importance of $\Phi^{j}_{m\bar{m}}$ in considering highest weight representations, and further support the basic idea in [2, 3] that the OPE for the models with affine $SL(2)$ symmetry is obtained by continuations from the $H^+_3$ case. We also saw that $\Phi^{j}_{m\bar{m}}$ represent correct wave functions on $SL(2, R)$ after imposing highest weight conditions and continuing the parameters. Thus, we expect that the OPE in the $SL(2, R)$ WZNW model, which is of considerable interest, is obtained along this approach. Here, we encounter important problems yet to be clarified: One is how to deal with the $m$-dependent poles. The other is how to incorporate the spectral flowed sectors which would play a significant role in the $SL(2, R)$ case [5]. For the former problem, the representation theory of $SL(2, R)$ may be a good guide. In fact, one can give a prescription, by hand, so that the OPE becomes consistent with the representation theory. However, these problems need further substantial works, and they are beyond the scope of this short note.

*Note added*

While the first version of the manuscript was being written, a paper [23] appeared which has some overlap with our discussions. After the first version was written, the OPE of the $SL(2, R)$ WZNW model has been discussed in [24, 25].

*Acknowledgments*

We would like to thank H. Awata, T. Eguchi, T. Fukuda, K. Hamada, N. Ishibashi, S. Mizoguchi, N. Ohta, K. Okuyama, J. Rasmussen, Y. Yamada and S.K. Yang for useful discussions and correspondences. The work of K.H. was supported in part by JSPS Research Fellowships for Young Scientists, whereas the work of Y.S. was supported in part by Grant-in-Aid for Scientific Research on Priority Area 707 and Grant-in-Aid for Scientific Research (No.12740134) from the Japan Ministry of Education, Culture, Sports, Science and Technology.

Rasmussen, O. Andreev, P. Furlan and V. Petkova for bringing our attention to these references and useful comments.
References

[1] K. Gawędzki, Nucl. Phys. B 328 (1989) 733; NATO ASI: Cargese 1991: 0247-274.
[2] J. Teschner, Nucl. Phys. B 546 (1999) 390; Nucl. Phys. B 546 (1999) 369.
[3] J. Teschner, Nucl. Phys. B 571 (2000) 555.
[4] A. Giveon and D. Kutasov, JHEP 9910 (1999) 034; JHEP 0001 (2000) 023.
[5] J. Maldacena and H. Ooguri, J. Math. Phys. 42 (2001) 2929; J. Maldacena, H. Ooguri and J. Son, J. Math. Phys. 42 (2001) 2961.
[6] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, JHEP 9812 (1998) 026.
[7] D. Kutasov and N. Seiberg, JHEP 9904 (1999) 008.
[8] N. Ishibashi, K. Okuyama and Y. Satoh, Nucl. Phys. B 588 (2000) 149.
[9] K. Hosomichi, K. Okuyama and Y. Satoh, Nucl. Phys. B 598 (2001) 451.
[10] H. Dorn and H.J. Otto, Nucl. Phys. B 429, 375 (1994).
[11] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys.B 477 (1996) 577.
[12] Bateman Manuscript Project, “Higher transcendental functions”, Vol. 1, McGraw-Hill, New York (1953).
[13] H. Liu, Phys. Rev. D 60 (1999) 106005.
[14] G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. B 362 (1991) 665.
[15] A. Kato and Y. Satoh, Phys. Lett. B 486 (2000) 306.
[16] K. Becker and M. Becker, Nucl. Phys. B 418 (1994) 206.
[17] H. Awata and Y. Yamada, Mod. Phys. Lett. A 7 (1992) 1185.
[18] J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 457 (1995) 309.
[19] P. Furlan, A.Ch. Ganchev, R. Paunov and V.B. Petkova, Nucl. Phys. B 394 (1993) 665; P. Furlan, A.Ch. Ganchev and V.B. Petkova, Nucl. Phys. B 491 (1997) 635.
[20] J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 481 (1996) 577.
[21] O. Andreev, Phys. Lett. B 363 (1995) 166.
[22] B. Feigin and F. Malikov, Lett. Math. Phys. 31 (1994) 315; A.Ch. Ganchev, V.B. Petkova and G.M.T. Watts, Nucl. Phys. B 571 (2000) 457.
[23] G. Giribet and C. Núñez, JHEP 0106 (2001) 010.
[24] Y. Satoh, “Three-point functions and operator product expansion in the SL(2) conformal field theory”, to appear in Nucl. Phys. B, hep-th/0109059.
[25] J. Maldacena and H. Ooguri, “Strings in AdS$_3$ and the SL(2,R) WZW model. III: Correlation functions”, hep-th/0111180.