Double domination on bipolar fuzzy graphs with strong edge and its properties

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Abstract
In this article, the definition of double dominance is introduced in the bipolar fuzzy graph. It provides descriptions of the size, order, degree etc of a bipolar fuzzy graph. With sufficient examples, the double dominance number of a bipolar fuzzy graph has been clarified. It addresses the properties of double dominance on the bipolar fuzzy graph. Some simple theorems have also been proposed relating to the claimed supremacy.

Keywords
Bipolar fuzzy graph, dominating set, double dominating set, doubledomination number on bipolar fuzzy graph.

AMS Subject Classification
03E72, 03E55.

1 Introduction
From the definition of fuzzy relation introduced by L.A. Zadeh [13] in the year 1965, Kaufmann. A, first presented the idea of fuzzy graph. Another comprehensive definition was introduced by Rosenfeld [10] in 1975, including fuzzy vertex and fuzzy edges and other fuzzy analogues of graph theoretical concepts such as paths, cycles, connectedness, etc. In 1998, A. Somasundaram, S. Somasundaram [11] studied the definition of dominance in fuzzy graphs. In the year 2000, Harey and Haynes [2] introduced the idea of double dominance in graphs. In the year 2011, Muhammad Akram [5,6] first presented the idea of a bipolar fuzzy graph (BFG) and also presented the idea of regular BFG in 2012. In the year 2015, Nagoor Gani, Muhammed Akram and Anupriya [9] defined the concept of double dominance on intuitionistic fuzzy graph. In this article, the idea of double dominance is extended to BFG and discussed its properties.

2 Preliminaries

The basic definitions of a BFG are redefined and explained with suitable example. Throughout this paper,

(i) The edge between the vertices r and t as rt.

(ii) G = (A, B) be a BFG, mean that G be a BFG with underlying graph G* = (M, N).

Definition 2.1 ([6]). A fuzzy set α on a set U is a map α : U → [0, 1]. A map β : U × U → [0, 1] is called a fuzzy relation on X if β(r,t) = min{α(r), α(t)} for all r, t ∈ U. A fuzzy relation β is symmetric if β(r,t) = β(t,r) for r, t ∈ U.

Definition 2.2 ([5]). Let U be a non-void set. A bipolar fuzzy set H in U is an object having the form

\[
H = \{ (r, \alpha^P_H(r), \alpha^N_H(r)) / r \in U \},
\]

where, \( \alpha^P_H : U \to [0,1] \) and \( \alpha^N_H : U \to [-1,0] \) are mappings. The positive membership degree \( \alpha^P_H(r) \) to denote the degree of satisfaction of an element with the property corresponding to a bipolar fuzzy set H, and the negative membership degree...
α_H^P(r) to denote the degree of satisfaction of an element r with some implicit counter-property corresponding to a bipolar fuzzy set H. If α_H^P(r) ≠ 0 and α_H^N(r) ≠ 0 then H is known to have only positive satisfaction degree. If α_H^P(r) = 0 and α_H^N(r) ≠ 0, then the condition that x does not fulfill H’s property, but rather satisfies H’s counter property. It is possible for a factor r to be such that α_H^P(r) ≠ 0 and α_H^N(r) ≠ 0. When the property’s membership feature overlaps that of its counter property over some portion of U. We will use the symbol, H = (α_H^P, α_H^N) for the sake of simplicity, and for the bipolar fuzzy set H = \{(r, α_H^P(r), α_H^N(r)) / r ∈ U\}.

**Definition 2.3** ([6]). Let U be a non-void set. Then, the mapping H = (α_H^P, α_H^N) : X × X → [-1, 1] × [-1, 1] a bipolar fuzzy relation on X such that α_H^P(r,t) ∈ [0,1] and α_H^N(r,t) ∈ [-1,0].

**Definition 2.4** ([6]). A BFG, is denoted as a pair G = (A,B), where A = (α_A^P, α_A^N) and B = (α_B^P, α_B^N) are bipolar fuzzy sets and α_A^P:M → [0,1], α_A^N:M → [-1,0], and α_B^P:M × M → [0,1], α_B^N:M × M → [-1,0] are bipolar fuzzy mappings such that α_A^P(rt) ≤ min{α_A^P(r), α_A^P(t)} and α_A^N(α_A^P(rt)) ≥ max{α_A^N(r), α_A^N(t)} for all rt ∈ A. A is called the bipolar fuzzy vertex set of M and B the bipolar fuzzy edge set of M respectively. Note that B is a symmetric bipolar fuzzy relation on A. That is, G = (A,B) is a BFG of the underlying crisp graph G^* = (M,N), where M is a vertex set and the edge set N ⊆ M × M such that, α_A^P(1ρ) ≤ min{α_A^P(1r), α_A^P(1t)} and α_A^N(1ρ) = max{α_A^N(1r), α_A^N(1t)} for all 1ρ ∈ N.

**Definition 2.5** ([3]). A BFG G = (A,B) of a graph G^* = (M,N) is called strong if α_B^P(rt) = min{α_B^P(r), α_B^P(t)} and α_B^N(α_B^P(rt)) = max{α_B^N(r), α_B^N(t)} for all rt ∈ N.

**Definition 2.6.** For any BFG, G = (A, B), the cardinality of M or the order of G is defined by

\[ p = |M| = \sum_{r \in M} \frac{1 + α_A^P(r) + α_A^N(r)}{2} \]

**Definition 2.7.** For any BFG, G = (A, B), the cardinality of N or the size of G is defined as

\[ q = |N| = \sum_{r \in N} \frac{1 + α_B^P(r) + α_B^N(r)}{2} \]

**Definition 2.8.** For any BFG, G = (A, B), the degree of the vertex is denoted as deg(r) and it is defined as

\[ deg(r) = \sum_{r \in M} \frac{1 + α_A^P(r) + α_A^N(r)}{2} \]

**Definition 2.9.** For any BFG, G = (A, B), the maximum degree of a BFG is denoted by Δ(G) = max{deg(r) / r ∈ M}.

**Definition 2.10.** For any BFG, G = (A, B), the minimum degree of a BFG is denoted by δ(G) = min{deg(r) / r ∈ M}.

**Definition 2.11.** For any BFG, G = (A, B), the degree of an edge rt ∈ N is denoted as deg(rt) and it is defined as

\[ deg(rt) = \sum_{r \in N} \frac{1 + α_B^P(rt) + α_B^N(rt)}{2} \]

**Definition 2.12.** For any BFG, G = (A, B), the neighbors (neighborhood) of r or an open neighbor of r ∈ M of G is denoted by N(r) and is defined as N(r) = \{r ∈ M | α_B^P(rt) = min{α_B^P(r), α_B^P(t)} and α_B^N(α_B^P(rt)) = max{α_B^N(r), α_B^N(t)} for all rt ∈ N\}. The closed neighbors of r ∈ M of G is written by N[r] and is stated as N[r] = N(r) ∪ \{r\}.

**Definition 2.13.** For any BFG, G = (A, B), the neighborhood degree of r ∈ M is denoted as deg_N(r) and is defined as

\[ deg_N(r) = \sum_{r \in M(N) \cup \{r\}} \frac{1 + α_B^P(r) + α_B^N(r)}{2} \]

**Definition 2.14.** For any BFG, G = (A, B), the maximum neighborhood degree of a BFG is denoted by Δ_N(G) = max{deg_N(r) / r ∈ M}.

**Definition 2.15.** For any BFG, G = (A, B), the minimum neighborhood degree of a BFG is denoted by δ_N(G) = min{deg_N(r) / r ∈ M}.

**Definition 2.16.** For any BFG, G = (A, B), an edge of G is said to be an effective or strong edge if α_B^P(rt) = min{α_B^P(r), α_B^P(t)} and α_B^N(α_B^P(rt)) = max{α_B^N(r), α_B^N(t)} for all rt ∈ N.

**Definition 2.17.** For any BFG, G = (A, B), the effective degree of a vertex r ∈ M in G is defined as

\[ deg_E(r) = \sum_{r \in M} \frac{1 + α_B^P(r) + α_B^N(r)}{2} \]

where rt is an effective edge.

**Definition 2.18.** For any BFG, G = (A, B), the maximum effective degree of a BFG is denoted by Δ_E(G) = max{deg_E(r) / r ∈ M}.

**Definition 2.19.** For any BFG, G = (A, B), the minimum effective degree of a BFG is denoted by δ_E(G) = min{deg_E(r) / r ∈ M}.

**Example 2.20.**

![Figure 1. Bipolar Fuzzy graph G](image-url)
Definition 2.21 ([4]). Consider \( G = (A, B) \) be a BFG. Let \( r, t \in M \). The vertex \( r \) is said to dominate \( t \) in \( G \) if \( \alpha^B_r(t) = \min \{ \alpha^B_A(r), \alpha^B_C(t) \} \) and \( \alpha^N_r(t) = \max \{ \alpha^N_A(r), \alpha^N_C(t) \} \) for all \( rt \in N \). A subset \( D \) of \( M \) is said to be a dominating set if for every \( r \in M - D \) there exist \( t \in D \) such that \( r \) dominates \( t \).

A dominating set \( D \) of \( M \) is said to be a minimal dominating set if no proper subset of \( D \) is a dominating set of \( G \).

The minimum fuzzy cardinality of a minimal dominating set in \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \) or simply \( \gamma \) and the corresponding minimal dominating set is called the minimum dominating set of \( G \).

Example 2.22.

From the above example, Dominating set, \( D = \{a, c\} \), \( M - D = \{b, d\} \) The domination number of \( G \), \( \gamma(G) = 1.05 \).

3. Double Domination on BFG

The related principles of double dominance on the BFG and their properties are discussed in this section.

Definition 3.1. For any BFG, \( G = (A, B) \), a subset \( D_4 \) of \( M \) is a double dominating set of \( G \), if for each vertex in \( M - D_4 \), it is dominated by at least two vertices in \( D_4 \).

A double dominating set \( D_4 \) of \( M \) is said to be a minimal double dominating set if no proper subset of \( D_4 \) is a double dominating set of \( G \).

The minimum fuzzy cardinality of a minimal double dominating set in \( G \) is called the double domination number of \( G \) and is denoted by \( \gamma_4(G) \) and the corresponding minimal double dominating set is called the minimum double dominating set of \( G \).

Example 3.2. From the BFG \( G \) in Fig 3, we have

The dominating set \( D = \{a, b\} \).

The domination number of \( G \), \( \gamma(G) = 0.9 \).

The double dominating set of \( G \), \( D_4 = \{a, b, c\} \).

The double domination number of \( G \), \( \gamma_4(G) = 1.4 \).

Theorem 3.3. For any BFG, then \( \gamma(G) \leq \gamma_4(G) \).

Proof. Let \( G = (A, B) \) be any BFG. Let \( D \subseteq M \) be a dominating set and \( D_4 \subseteq M \) be a double dominating set of \( G \). If \( D = D_4 \), then \( \gamma(G) = \gamma_4(G) \). If \( D \neq D_4 \), then \( D_4 \) has at least one vertex more than \( D \) and hence, \( \gamma(G) < \gamma_4(G) \). Hence, \( \gamma(G) \leq \gamma_4(G) \).

Definition 3.4. Let \( G = (A, B) \) be a BFG. Then \( G \) is said to be a bipartite BFG if the vertex set \( M \) of a BFG \( G \) can be partitioned into two subsets \( M_1 \) and \( M_2 \) such that \( \alpha^B_r(t) = 0 \) and \( \alpha^N_r(t) = 0 \) for all \( r, t \in M_1 \) or \( r, t \in M_2 \). A bipartite BFG \( G = (A, B) \) is said to be a complete BFG if, \( \alpha^B_r(t) = \min \{ \alpha^B_A(r), \alpha^B_C(t) \} \) and \( \alpha^N_r(t) = \max \{ \alpha^N_A(r), \alpha^N_C(t) \} \) for all \( r \in M_1 \) and \( t \in M_2 \).

Theorem 3.5. Let \( G = (A, B) \) be any completely bipartite BFG with \( n > 2 \), \( n \) is the number of vertices of \( G \). Then double dominating set \( D_4 \) of \( G \) exists.

Proof. Let \( G = (A, B) \) be a completely bipartite BFG. Then, \( M = M_1 \cup M_2 \) and \( M_1 \cup M_2 = \emptyset \). Let \( t_1, t_2, \ldots, t_k \in M_1 \) or \( t_1, t_2, \ldots, t_k \in M_2 \) with either \( s > 1 \) or \( k > 1 \). Let \( n > 2 \), \( n \) be the number of vertices of \( G \).

If \( s = 1 \) and \( k > 1 \), then \( M_2 \) is the double dominating set of \( G \). If \( s > 1 \) and \( k = 1 \), then \( M_1 \) is the double dominating set of \( G \). If \( s > 1 \) and \( k > 1 \), then either \( M_1 \) or \( M_2 \) is the double dominating set of \( G \). Hence, \( G \). Then double dominating set \( D_4 \) exists for any completely bipartite BFG with \( n > 2 \), \( n \) is the number of vertices of \( G \).

Theorem 3.6. Let \( G = (A, B) \) be a BFG with double dominating set. Then, \( \gamma(G) + \gamma_4(G) \leq p \).

Proof. Let \( G = (A, B) \) be a BFG. Let \( D_4 \) be the double dominating set. Therefore, \( \gamma(G) \leq p - \gamma_4(G) \). Hence,

\[ \gamma(G) + \gamma_4(G) \leq p. \]

Theorem 3.7. For BFG, \( G = (A, B) \), then \( \gamma_4(G) < p \).

Proof. Let \( G = (A, B) \) be a BFG. Then by Theorem 3.6.

\[ \gamma(G) + \gamma_4(G) \leq p. \] Hence, \( \gamma_4(G) \leq p. \)
4. Conclusion

The idea of double domination on bipolar fuzzy graph was presented in this article and discussed some of its properties. We can extend our research work to double total domination and other various types of bipolar fuzzy graph.

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