The Effect of Bulk Dimension in presence of String Clouds on Shear Viscosity Bound in Massive Gravity

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Abstract

In this paper, the Einstein AdS black brane solution in presence of string clouds in context of massive gravity in d dimensions is introduced. The ratio of shear viscosity to entropy density for this solution violates the KSS bound by applying the Dirichlet boundary and regularity on the horizon conditions. Our result shows that this value is independent of string clouds in arbitrary dimensions.

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1 Introduction

Black holes are the solution of Einstein equation which they have event horizon. In terms of the event horizon topology they are classified into three cases: Flat topology for black brane, hyperbolic topology for topological black hole and spherical topology of event horizon for black hole. Black holes are interesting subject in high energy physics. Thermodynamics and phase transition of black holes are studied extensively in a few decades. Another interesting aspect of black hole arises in the context of AdS/CFT duality \[1\, 2\] where Einstein equation in d-dimensions can be reduced to hydrodynamics equation in \((d - 1)\)-dimensions \[3\]. This map is called fluid/gravity duality \[3\, 4\] in literature. The important implication of this duality is the study of near-equilibrium property of strongly coupled systems. These fluids are strongly coupled and can not be described by perturbation methods but they can be described

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by holography, that is, by mapping them onto a weakly curved gravitational theory via fluid/gravity duality. In this duality, Fluid is located on the boundary of AdS space-time. For exact description of this fluid we should calculate the transport coefficients. The most important coefficient is the ratio of shear viscosity to entropy density \( \frac{\eta}{s} \). This value gets a lower bound \( \frac{1}{4\pi} \), that for Einstein gravity is known as KSS bound [5,6,7,8,9]. This value is proportional to inverse squared coupling of theory in boundary side. Membrane paradigm, pole method and Green-Kubo formula [5,6] are the ways for calculation of \( \frac{\eta}{s} \). Here, we calculate it by using Green-Kubo formula.

Hydrodynamics equations for perfect fluid [9] are the conservation of \( T_{\mu\nu} \) as follows,

\[
\nabla_\mu T^{\mu\nu} = 0, \\
T_{\mu\nu} = \epsilon u^\mu u^\nu + p P_{\mu\nu}, \\
P_{\mu\nu} = \eta_{\mu\nu} + u^\mu u^\nu, \\
\eta_{\mu\nu} = (-1, 1, 1, 1).
\]

and equation of state \( \epsilon = -p + Ts \) where \( T \) is temperature and \( s \) is entropy density. To illustrate the effects of dissipation we should add extra pieces to the \( T_{\mu\nu} \),

\[
T'_{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg_{\mu\nu} - \sigma_{\mu\nu}, \\
\sigma_{\mu\nu} = P^{\mu\rho} P^{\rho\beta} [\eta (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) + (\zeta - \frac{2}{3}\eta)g_{\alpha\beta} \nabla . u].
\]

where \( \eta, \zeta, \sigma^{\mu\nu} \) and \( P^{\mu\nu} \) are shear viscosity, bulk viscosity, shear tensor and projection operator, respectively [9]. Green-Kubo formula can be derived by linear response theory as follows,

\[
\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d\vec{x} e^{\omega t} \left< [T_{x_1} (x), T_{x_2} (0)] \right> = -\lim_{\omega \to 0} \frac{1}{\omega} \Im G_{x_1 x_2} (\omega, \vec{0}).
\]

The shear viscosity \( \eta \) is straightforwardly obtained from the Kubo formula and the entropy density \( s \) from the Bekenstein-Hawking formula.

The basic elements of the nature are considered one-dimensional objects in string theory. It’s worthwhile to study the gravitational effects by a cloud of strings. Cloud of strings as a gravitational source studied for the first time in Einstein gravity [10] and in Lovelock gravity [11,12] as a generalization of Einstein gravity. In [13] we studied string clouds as a gravitational source in massive gravity in 4-dimensions but in this paper we want to study it in d-dimensions.

Hierarchy problem and the brane-world gravity solutions [14,15] predict the existence of massive graviton. There are some problems such as the cosmological constant problem and the current acceleration that GR can not explain them. For these reasons GR must be modified. Massive gravity [16] is one of the generalization of Einstein gravity in which graviton is not massless and this theory enjoys of ghost-free. Massive gravity could explain the current observations related to dark matter [17] and also the accelerating expansion of universe without requiring any dark energy component [18,19]. C. Bachas and et al [20] show that massive gravity can be embedded in string theory. In this paper, we consider d-dimensional AdS solutions of black hole solution in massive gravity in presence of sting clouds.
Massive gravity can help to shed light on the quantum gravity effects\cite{21}. In this paper, our goal is further explore of a string cloud in the framework of massive theories of gravity.

In this paper we consider massive gravity with a cloud of strings \cite{10, 11, 12, 22, 23, 24, 25, 26, 27, 28, 29, 30} in $d$ dimensions and introduce the black brane solution. Finally, we study the effect of bulk dimension in presence of a string clouds on the value of $\frac{\eta}{s}$ and suggest some comments about the fluid dual to this gravity model.

2 The Einstein AdS Black Brane with a Cloud of String Background in Context of Massive Gravity

The action of this model is given by,

$$I = \frac{1}{2} \int d^d x \sqrt{-g} \left[ \mathcal{R} - 2\Lambda + m^2 \sum_{i=1}^{d-2} c_i \mathcal{U}_i(g,f) \right] + T_p \int_{\Sigma} \sqrt{-\gamma} d^0 d^1,$$

(5)

where $\mathcal{R}$ is the scalar curvature, $\Lambda = -\frac{d-2}{2l^2}$ is cosmological constant, $d_i = d - i$, $l$ is the radius of AdS spacetime, $f$ is a fixed rank-2 symmetric tensor known as reference metric and $m$ is the mass parameter. $c_i$’s are constants and $\mathcal{U}_i$ are symmetric polynomials of the eigenvalues of the $d \times d$ matrix $K_{\mu \nu} = \sqrt{g_{\mu \nu}} f_{\mu \nu}$. We choose a spatial reference metric (in the basis $(t, r, x_1, ..., x_{d-2}))$\cite{31, 32}

$$f_{\mu \nu} = (f_{sp})_{\mu \nu} = \text{diag}(0, 0, c_0^2 h_{12}).$$

(6)

where $c_0 > 0$ and $h_{ij} = \frac{1}{d} \delta_{ij}$

$$\mathcal{U}_1 = [\mathcal{K}], \quad \mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \quad \mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]

\mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}^2] + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]

\mathcal{U}_5 = [\mathcal{K}]^5 - 10[\mathcal{K}^2][\mathcal{K}]^3 + 20[\mathcal{K}^3][\mathcal{K}^2] - 20[\mathcal{K}^2][\mathcal{K}][\mathcal{K}^2] + 15[\mathcal{K}][\mathcal{K}^2]^2 - 30[\mathcal{K}][\mathcal{K}^3] + 24[\mathcal{K}^5]

...$$

(7)

The square root in $\mathcal{K}$ means $(\sqrt{\mathcal{K}})^{\mu}_\lambda = \mathcal{K}_\lambda^{\mu}$ and the rectangular brackets denote traces: $[A] \equiv A^{\mu \mu}$. According to reference metric Eq.(6), the values of $\mathcal{U}_i$ are as below,

$$\mathcal{U}_1 = \frac{d_2 c_0}{r}, \quad \mathcal{U}_2 = \frac{d_2 d_3 c_0^2}{r^2}, \quad \mathcal{U}_3 = \frac{d_2 d_3 d_4 c_0^3}{r^3},$$

$$\mathcal{U}_4 = \frac{d_2 d_3 d_4 d_5 c_0^4}{r^4}, ...$$

(8)

$\mathcal{U}_i$’s can be written as

$$\mathcal{U}_i = \left( \frac{c_0}{r} \right)^i \prod_{j=2}^{i+1} d_j,$$
in which \( \prod_{x}^{y} \) = 1 if \( x > y \). The last part in the action is called the Nambu-Goto action of a string and \((\lambda^0, \lambda^1)\) is a parametrization of the worldsheet, \( T_p \) is a positive quantity and is related to the tension of the string and \( \gamma \) is the determinant of the induced metric [27, 28, 29, 30].

\[
\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b},
\]

by this definition, we can write the Lagrangian part of the string as,

\[
S_{NG} = T_p \int_\Sigma \sqrt{-\frac{1}{2} \Sigma_{\mu\nu} \Sigma^{\mu\nu} d\lambda^0 d\lambda^1}
\]

where

\[
\Sigma^{\mu\nu} = \epsilon^{ab} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b}
\]

is the space-time bi-vector and satisfied in the following identities,

\[
\Sigma^{\mu[\alpha} \Sigma^{\beta\gamma]} = 0 \quad (12)
\]

\[
\nabla_{\mu} \Sigma^{\mu[\alpha} \Sigma^{\beta\gamma]} = 0 \quad (13)
\]

\[
\Sigma^{\mu\sigma} \Sigma_{\sigma\tau} \Sigma^{\tau\nu} = \gamma \Sigma^{\mu\nu}. \quad (14)
\]

Where the square brackets refer to antisymmetrization in the enclosed indices \( \epsilon^{ab} \) is two-dimensional Levi-Civita tensor, \( \epsilon^{01} = -\epsilon^{10} = 1 \) and \( \epsilon^{00} = \epsilon^{11} = 0 \).

The energy-momentum tensor for a single of string can be calculated by this formula,

\[
T^{\mu\nu}(\text{string}) = -2 \frac{\partial L}{\partial g^{\mu\nu}} = - T_p \frac{\Sigma^{\mu\sigma} \Sigma^{\nu}_{\sigma}}{\sqrt{-\gamma}}. \quad (15)
\]

We consider the energy-momentum tensor for a cloud of string as follows,

\[
T^{\mu\nu}(\text{cloud}) = \rho \frac{\Sigma^{\mu\sigma} \Sigma^{\nu}_{\sigma}}{\sqrt{-\gamma}}. \quad (16)
\]

where \( \rho \) is the density of the string cloud and conservation of the energy-momentum tensor \( \nabla_{\nu} T^{\mu\nu} = 0 \) results in [30],

\[
\nabla_{\mu}(\rho \Sigma^{\mu\nu}) \frac{\Sigma^{\nu}_{\sigma}}{(-\gamma)^{1/2}} + \rho \Sigma^{\mu\sigma} \nabla_{\mu} \left( \frac{\Sigma^{\nu}_{\sigma}}{(-\gamma)^{1/2}} \right) = 0. \quad (17)
\]

By applying the same procedure of [13] we have,

\[
\partial_{\mu} \left( \sqrt{-g} \rho \Sigma^{\mu\sigma} \right) = 0. \quad (18)
\]

In order to derive a d-dimensional AdS black hole solution with a cloud of strings in massive theory of gravity, we consider the following static metric as an ansatz,

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \sum_{i=1}^{d_2} h_{ij}dx^i dx^j, \quad (19)
\]
$h_{ij} dx^i dx^j$ is the line element for an Einstein space with constant curvature $d_2d_3k$ with the values of $k = 0$ for spherical, $k = 1$ for flat and $k = -1$ for hyperbolic topology of the black hole horizon. 

The field equations take the following forms by varying the action Eq. (5) with respect to the spacetime metric,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - m^2 \chi_{\mu\nu} = T_{\mu\nu} \text{(string-cloud)} \quad (20)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor of a string cloud and $\chi_{\mu\nu}$ is massive term,

$$\chi_{\mu\nu} = \frac{c_1}{2} (U_1 g_{\mu\nu} - K_{\mu\nu}) + \frac{c_2}{2} (U_2 g_{\mu\nu} - 2U_1 K_{\mu\nu} + 2K_{\mu\nu}^2) + \frac{c_3}{2} (U_3 g_{\mu\nu} - 3U_2 K_{\mu\nu} + 6U_1 K_{\mu\nu}^2 - 6K_{\mu\nu}^3) + \frac{c_4}{2} (U_4 g_{\mu\nu} - 4U_3 K_{\mu\nu} + 12U_2 K_{\mu\nu}^2 - 24U_1 K_{\mu\nu}^3 + 24K_{\mu\nu}^4) + \frac{c_5}{2} (U_5 g_{\mu\nu} - 5U_4 K_{\mu\nu} + 20U_3 K_{\mu\nu}^2 - 60U_2 K_{\mu\nu}^3 + 120U_1 K_{\mu\nu}^4 - 120K_{\mu\nu}^5) + ... \quad (21)$$

The $rr$ component of field equation Eq. (20) is,

$$d_r^2 = d_r^2 (k - f(r)) - d_r^2 \frac{df(r)}{dr} r - 2\Lambda r^2 + m_g^2 \left( \sum_{i=1}^{d_2} \frac{c_i^0 c_i}{r^{i-2}} \prod_{j=2}^{i+1} d_j \right) = 2r^2 T_{rr} \quad (22)$$

For the static and spherically symmetric string cloud the ansatz of the bivector $\Sigma^{\sigma\mu}$ is given by [11],

$$\Sigma^{\sigma\mu} = A(r) \left( \delta^{\sigma}_0 \delta^\mu_1 - \delta^{\mu}_0 \delta^\sigma_1 \right) \quad (23)$$

By plugging Eq. (23) in Eq. (16), the non-vanishing components of the energy-momentum tensor for the string cloud are as,

$$T^t_t = T^r_r = -\rho |A(r)| \quad (24)$$

By using Eq. (18) we get,

$$T^\mu_t = T^r_r = -\frac{a}{r d_2} \text{diag}[1, 1, 0, ..., 0] \quad (25)$$

where $a$ is a positive constant.

By inserting the value of $T^r_r$ in Eq. (22),

$$d_r^2 = d_r^2 (k - f(r)) - d_r^2 \frac{df(r)}{dr} r - 2\Lambda r^2 + m_g^2 \left( \sum_{i=1}^{d_2} \frac{c_i^0 c_i}{r^{i-2}} \prod_{j=2}^{i+1} d_j \right) = -\frac{2a}{r^{d_2}} \quad (26)$$

$f(r)$ is found as follows,

$$f(r) = k - \frac{b}{r^{d_3}} + \frac{2a}{d_2 d_4} \frac{2\Lambda r^2}{r^{d_2} d_2} + m_g^2 \sum_{i=1}^{d_2} \frac{c_i^0 c_i}{d_2 d_{i+1} r^{i-2}} \prod_{j=2}^{i+1} d_j \quad (27)$$
We know on event horizon \( g^{rr}(r_0) = 0 \). If we consider this constraint on this solution, we can find the constant of \( b \) as follows,

\[
b = r_0^2 \left[ k + \frac{2a}{d_2 r_0^2} - \frac{2\Lambda}{d_1 d_2} r_0^2 + m_g^2 \frac{d_2}{d_1^2} \left( \frac{c_0 c_i}{d_{i+1} d_2} \right)^{i+1} \prod_{j=2}^{d_2} d_j \right].
\] (28)

By plugging the value of \( b \) in \( f(r) \) Eq. (27) we have,

\[
f(r) = \left[ k \left( 1 - \frac{r_0}{r} \right)^{d_3} - \frac{2a}{d_2 r_0^2} \left( 1 - \frac{r_0}{r} \right) - \frac{2\Lambda}{d_1 d_2} \left( \frac{r}{r_0} \right)^2 - \frac{r_0}{r} \right]^{d_3} + m_g^2 \frac{d_2}{d_1^2} \left( \frac{c_0 c_i}{d_{i+1} d_2} \right)^{i+1} \prod_{j=2}^{d_2} d_j \right].
\] (29)

We want to find the black brane solution so we choose \( k = 0 \). The emblackening factor follows as,

\[
f(r) = \left[ - \frac{2a}{d_2 r_0^2} \left( 1 - \frac{r_0}{r} \right) - \frac{2\Lambda}{d_1 d_2} \left( \frac{r}{r_0} \right)^2 - \frac{r_0}{r} \right]^{d_3} + m_g^2 \frac{d_2}{d_1^2} \left( \frac{c_0 c_i}{d_{i+1} d_2} \right)^{i+1} \prod_{j=2}^{d_2} d_j \right].
\] (30)

We derive the entropy density for this solution by using of Hawking-Bekenstein relation,

\[
s = \frac{S}{V_{d_2}} = \frac{A}{4G V_{d_2}} = \frac{4\pi}{V_{d_2}} \int d^{d_2} x \sqrt{-g_{x_1 x_1} \cdots g_{x_2 x_2}} \bigg|_{r=r_0}
\]

\[
= \frac{4\pi}{V_{d_2}} \int d^{d_2} x \sqrt{\chi} = 4\pi \sqrt{\chi(r_0)} = 4\pi \left( \frac{r_0}{l} \right)^{d_2},
\] (31)

where \( V_{d_2} \) is the volume of the constant \( t \) and \( r \) hyper-surface with radius \( r_0 \), \( \chi(r_0) \) is the determinant of the spatial metric on the horizon and we used \( \frac{1}{l_{eq}} = 1 \) so \( \frac{1}{\chi} = 4\pi \).

### 3 Holographic Aspects of the Solution

The important claim of AdS/CFT is the equivalence of partition function of gravity and gauge theories. It can be understood by GKP-Witten relation [2] that stated:

\[
Z_{\text{gauge}} = \left< e^{i\int \phi^{(0)} O} \right> = Z_{\text{string}} = e^{iS[\phi^{(0)}]}
\] (32)

where \( \delta \) is the on-shell action, \( \phi \) is a field in the bulk theory and the \( \phi^{(0)} \) is the value of bulk field in the boundary and considered as a external source both the boundary operator and bulk field. By using GKP-Witten prescription we calculate two-point Greens function by differentiating the gravitational action with respect to the boundary values of fields then setting \( \phi^{(0)} = 0 \),

\[
< \mathcal{O} \mathcal{O} >_S = \frac{\delta^2 S[\phi^{(0)}]}{\delta \phi^{(0)} \delta \phi^{(0)}}
\] (33)
2-point functions of energy-momentum tensor is related to shear viscosity from Kubo formula Eq. (4). Therefore, we perturb the bulk metric by \((\delta g)^{x_1 x_2} = \phi(r)e^{-i\omega t}\). We consider the metric and energy-momentum tensor that they are homogeneous and isotropic in the field theory directions as the following,

\[
\begin{align*}
 ds^2 &= -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{x_1 x_1}(r)\sum_{i=1}^{d_2} dx_i^2 dx_i, \\
 T_{\mu\nu} &= \text{diag} \left( T_{tt}(r), T_{rr}(r), T_{x_1 x_1}(r), \ldots, T_{x_{d_2} x_{d_2}}(r) \right).
\end{align*}
\]

We apply stationary perturbation \(ds^2 = ds_0^2 + 2\phi(r)e^{-i\omega t}dx_1 dx_2\) on the bulk metric and the perturbation equation reduces to Eq. (33),

\[
\frac{1}{\sqrt{-g}} \partial_{\nu} \left( \sqrt{-g} g^{\nu\rho} \partial_{\rho} \phi \right) + [g^{tt}\omega^2 - m(r)^2] \phi = 0,
\]

\[
m(r)^2 = g^{x_1 x_2} T_{x_1 x_2} - \frac{\delta T_{x_1 x_2}}{\delta g^{x_1 x_2}}.
\]

Shear viscosity is calculated via Eq. (4),

\[
\eta = \lim_{\omega \to 0} \frac{1}{\omega} \Im G^R(\omega, k = 0) = \frac{\sqrt{\chi(r_0)}}{16\pi G_N} \phi_0(r_0)^2 = \frac{s}{4\pi} \phi_0(r_0)^2.
\]

Then, we will have,

\[
\frac{\eta}{s} = \frac{1}{4\pi} \phi_0(r_0)^2.
\]

where \(\phi_0\) is the solution of perturbation equation Eq. (36) at zero frequency \((\omega = 0)\). We demand two conditions for \(\phi\): it should be regular at horizon and the value of \(\phi\) near the boundary is 1.

By considering this perturbation \(\delta g^{x_1 x_2} = \frac{2}{r^3} \phi(r)e^{i\omega t}\) on the bulk metric Eq. (19), we have,

\[
\begin{align*}
 ds^2 &= -\frac{f_1(r)}{l^2} dt^2 + \frac{l^2}{f_1(r)} dr^2 + \frac{r^2}{l^2} \left( dx_1^2 + dx_2^2 + 2\phi(r)dx_1 dx_2 + dx_3^2 + \ldots + dx_{d_2}^2 \right),
\end{align*}
\]

\[
f_1(r) = \frac{l^2}{r^d_2} \left[ 2a_2(r - r_0) - \frac{2A}{d_1 d_2} (r^{d_1_2} - r_0^{d_1}) + m^2 \frac{c_0 c_1}{d_2} (r^{d_2} - r_0^{d_2}) + m^2 c_0^2 c_2 (r^{d_3} - r_0^{d_3}) \\
+ m^2 c_0^3 c_3 (r^{d_4} - r_0^{d_4}) d_3 + m^2 c_0^4 c_4 (r^{d_5} - r_0^{d_5}) d_3 d_4 \right] = l^2 f(r).
\]

By plugging this metric on the action Eq. (5) and keeping up to second order of \(\phi\), we have:

\[
S_2 = -\frac{1}{2} \int d^4x \left( K_1 \phi'^2 - K_2 \phi^2 \right).
\]
we set $\omega = 0$, where

$$K_1 = \frac{r_{d_2} f_1(r)}{l_{d_2}^2} = \frac{r_{d_2}}{l_{d_2}} f_1(r) = \frac{r_{d_2}}{l_{d_2}^2} \left[ \frac{2a}{d_{2}} (r - r_0) - \frac{2\Lambda}{d_{1}d_{2}} (r_{d_1} - r_{0_{d_1}}) \right] + m^2 c_0^2 c_2 (r_{d_3} - r_{0_{d_3}}) + m^2 c_0^2 c_3 (r_{d_4} - r_{0_{d_4}}) d_3 + m^2 c_0^4 c_1 (r_{d_5} - r_{0_{d_5}}) d_3 d_4 \right], \quad (42)$$

$$K_2 = \frac{m^2}{2l_{d_2}^2} \left( c_0 c_1 r_{d_5} + d_4 c_0^2 c_2 r_{d_4} + d_4 d_5 c_0^3 c_3 r_{d_5} \right) \quad (43)$$

Equation of motion for $\phi$ is found by variation of Eq. (41) with respect to $\phi$ then the EoM is,

$$(K_1 \phi')' + K_2 \phi = 0. \quad (44)$$

We try to solve the perturbation equation Eq. (44) perturbatively in terms of $m^2$ and $a$. For the leading order we choose $m = a = 0$ then the perturbation equation reduces to,

$$\left( r \left( r_{d_1} - r_{0_{d_1}} \right) \phi_0'(r) \right)' = 0, \quad d \neq 1 \quad (45)$$

The solution is found,

$$\phi_0(r) = C_1 + C_2 \left( d \log r - \log \left( r r_{d_0}^d - r_{0_{d_0}}^d \right) \right) \quad (46)$$

By imposing regularity on the horizon and a Dirichlet boundary condition at the AdS boundary, $\phi(r \rightarrow \infty) = 1$, as boundary conditions we can find the constants $C_1 = 1$ and $C_2 = 0$. So the value of $\frac{\eta}{s}$ from (58) is given,

$$\frac{\eta}{s} = \frac{1}{4\pi} \phi(r_0)^2 = \frac{1}{4\pi} \quad (47)$$

At the first order of perturbation we consider the solution as $\phi = 1 + m^2 \phi_1(r) + a \phi_2(r)$ and put this solution to Eq. (41) and expand EoM in terms of powers of $m^2$ and $a$, we have,

$$\frac{m^2}{2l_{d_2}^2} \left( c_0 c_1 r_{d_5} + d_4 c_0^2 c_2 r_{d_4} + d_4 d_5 c_0^3 c_3 r_{d_5} \right) - \left( \frac{2\Lambda m^2 r \left( r_{d_1} - r_{0_{d_1}} \right) \phi_1'(r)}{l_{d_2} d_{2} d_{1}} \right)' = 0, \quad (48)$$

$$a \left( r \left( r_{d_1} - r_{0_{d_1}} \right) \phi_2''(r) + \left( d \phi_2'(r) - r_{0_{d_1}}' \right) \phi_2'(r) \right) = 0. \quad (49)$$

The solutions are found,

$$\phi_1(r) \rightarrow C_2 + \int r^d r_{d_0}^d \frac{C_1}{r_{0_{d_0}}^d - r_{0_{d_0}}^d} dU - \int r^d c_0 d_{2} d_{1} r_{0_{d_0}}^d U_{d_1} \left( \frac{c_0 c_2 d_{d} U}{d_3} + \frac{c_1 U_{d_2}^2}{d_3} + c_0^2 c_3 d_{3} \right) dU \quad (50)$$
\[
\phi_2(r) \rightarrow C_3 + \frac{C_4 r_0^{-d}}{d_1} \left( d \log(r) - \log \left( r r_0^{d} - r_0 r^{d} \right) \right).
\]

\(\phi(r)\) is as follows,

\[
\phi(r) = \phi_0 + a C_3 + m^2 C_2 + \left( \frac{a C_4 + m^2 C_1}{r_0^d} \right) \frac{\left( \frac{a c_2 d r_0}{d_3} + \frac{a U^2}{d_2} + \frac{c_0^2 c_3 d_5}{d_2} \right)}{4 \Lambda \left( r_0^d U - r_0 U^d \right)}
\]

\[- m^2 \int \frac{c_0 d_2 d_1 r_0 U^d \left( \frac{a c_2 d r_0}{d_3} + \frac{a U^2}{d_2} + \frac{c_0^2 c_3 d_5}{d_2} \right)}{4 \Lambda \left( r_0^d U - r_0 U^d \right)} dU. \tag{52}\]

where \(\Phi_0 = \phi_0 + a C_3 + m^2 C_2\).

Near horizon of Eq. [52] gives us,

\[
\phi(r) = i \pi \left[ - \frac{a C_4 + m^2 C_1}{d_1 r_0^d} + \frac{c_0 d_2 d_1 r_0 U^d}{4 \Lambda} \left( \frac{a c_2 d r_0}{d_3} + \frac{a U^2}{d_2} + \frac{c_0^2 c_3 d_5}{d_2} \right) \right] \log(r - r_0) + ...
\]

... means finite terms. Now we apply the boundary conditions to find \(C_1\), \(C_4\) and \(\Phi_0\) constants. Regularity on horizon condition is deleted \(\log(r - r_0)\), so we must vanish the coefficient,

\[
C_1 = \frac{-a C_4 + m^2 C_1}{d_1 r_0^d} + \frac{c_0 d_2 d_1 r_0 U^d}{4 \Lambda} \left( \frac{a c_2 d r_0}{d_3} + \frac{a U^2}{d_2} + \frac{c_0^2 c_3 d_5}{d_2} \right) \tag{53}\]

By substituting the above result of \(C_1\) into Eq. [52] implies that,

\[
\phi(r) = \Phi_0 + m^2 \int \frac{c_0 r_0^d d_2 d_1 r_0 U^d \left( \frac{a c_2 d r_0}{d_3} + \frac{a U^2}{d_2} + \frac{c_0^2 c_3 d_5}{d_2} \right)}{4 \Lambda \left( r_0^d U - r_0 U^d \right)} dU \right) = \Phi_0 - m^2 B(r). \tag{55}\]

By applying \(\phi(r \to \infty) = 1\) as a second boundary condition on Eq. [55] The second boundary condition is at \(\phi(r = \infty) = 1\), which gives,

\[
\Phi_0 = \phi(r \to \infty) + m^2 B(r \to \infty) = 1 + m^2 B(r \to \infty) \tag{56}\]

So \(\phi(r)\) in \(r_0 < r < \infty\) is found,

\[
\phi(r) = 1 + m^2 B(r \to \infty) - m^2 B(r). \tag{57}\]

Thus for calculating \(\frac{\eta}{s}\) up to the first order in terms of \(a\) and \(m^2\) we evaluate the value of \(\phi(r)\) at \(r = r_0\) and by using Eq. [58] we will have,

\[
\frac{\eta}{s} = \frac{1}{4 \pi} \phi(r_0)^2 = \frac{1}{4 \pi} \left( 1 + 2 m^2 \left( B(r \to \infty) - B(r_0) \right) + O(m^4) \right). \tag{58}\]
It means the value of $\eta_s$ is independent of string clouds in arbitrary dimension of bulk and it behaves like massive gravity. So KSS bound is violated by applying regularity of the solution of perturbation equation at horizon and $\phi(r \to \infty) = 1$ on the boundary of AdS for this solution. KSS bound is violated for massive gravity theories much more general than dRGT theory [34]. Violation of KSS bound in massive gravity is because of mass term in perturbation equation whereas in higher derivative gravity is due to causality [35, 36, 37]. This bound is also preserved in Horava-Lifshitz gravity [38].

4 Conclusion

We consider string clouds in d-dimensions of massive gravity in AdS space and introduce the black brane solution of this model then study the aspects of holographic dual of this solution by calculating the $\eta_s$. The physical interpretation of this value is the inverse squared coupling of field dual. Our outcome shows the fluid dual of this model is the same as massive gravity and it’s independent of string clouds in arbitrary dimensions. It also shows that a cloud of strings acts like a charge comparison to [39, 40] results. If we consider string cloud as a matter, we understand that $\eta_s$ is independent of matter. There is a conjecture that states $\frac{\eta_s}{s} = \frac{1}{4\pi}$ for Einstein-Hilbert gravity, known as KSS bound [35] which it is violated for higher derivative gravity [32, 35, 36, 37, 41, 42] and massive gravity but the reason of them is different.

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