Measurements incompatible in Quantum Theory
cannot be measured jointly in any other local theory

Michael M. Wolf1, David Perez-Garcia2, Carlos Fernandez3
1 Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen, Denmark
2 Departamento de Análisis Matemático & IMI. Universidad Complutense de Madrid. 28040 Madrid, Spain
3 Departamento de Algebra & IMI. Universidad Complutense de Madrid. 28040 Madrid, Spain
(Dated: May 18, 2009)

It is well known that jointly measurable observables cannot lead to a violation of any Bell inequality—
independent of the state and the measurements chosen at the other site. In this letter we prove the converse:
every pair of incompatible quantum observables enables the violation of a Bell inequality and therefore must
remain incompatible within any other no-signaling theory. While in the case of von Neumann measurements
it is sufficient to use the same pair of observables at both sites, general measurements can require different
choices. The main result is obtained by showing that for arbitrary dimension the CHSH inequality provides
the Lagrangian dual of the characterization of joint measurability. This leads to a simple criterion for joint
measurability beyond the known qubit case.

"While [...] the wave function does not provide a complete
description of physical reality, we left open the question of
whether or not such a description exists. We believe, however,
that such a theory is possible." [1]

More than seventy years after Einstein, Podolsky and Rosen
(EPR) raised this puzzle we know, as a consequence of Bell’s
argument [2], that a complete theory in the sense of EPR
would force us to pay a high price—such as giving up Ein-
stein locality. Could there, however, be a theory which pro-
vides more information than quantum mechanics but still is
‘incomplete enough’ to circumvent such fundamental con-
flicts? In this work we address a particular instance of this
question, in the context of which the answer is clearly nega-
tive.

We consider observables which are not jointly measurable,
i.e., incompatible within quantum mechanics and show that
they all enable the violation of a Bell inequality. That is, there
exists a bipartite quantum state and a set of observables for an
added site together with which the given observables violate a
Bell inequality. As a consequence the observed probabilities
do not admit a joint distribution [3] unless this depends on the
observable chosen at the added site, which conflicts with Ein-
stein locality, i.e., the no-signaling condition (see appendix).
So, if a hypothetical no-signaling theory is a refinement of
quantum mechanics (but otherwise consistent with it [19]),
it can not render possible the joint measurability of observables
which are incompatible within quantum mechanics—even if
these observables are already almost jointly measurable in
quantum theory.

An enormous amount of work has been done in related di-
rections: Bell inequalities [4] and no-signaling theories [5] are
lively fields of research. It is well known (and used in con-
structing quantum states admitting a local hidden variable de-
scription [6, 7]) that jointly measurable quantum observables
can never lead to a violation of a Bell inequality [3]. The con-
verse, however, has hardly been addressed. For generalized
measurements (POVMs) this might partly be due to the fact
that no criterion for joint measurability is known beyond two-
level systems, for which it was derived only recently [8]. A
first indication of the present result can be found in [9] where
it has been observed that for particular two-level observables
the border of joint measurability [10] coincides with the one
for the violation of the CHSH Bell inequality [11] . In the
present work we show that the finding of [9] is not a mere
coincidence resulting from having only few parameters, but
that it holds in arbitrary dimension—even on a quantitative
level. The main tool in this analysis will be the identification
of the CHSH inequality with the Lagrangian dual of the joint
measurability problem. This connection allows us at the same
time to provide a simple criterion for joint measurability. We
can now, however, with a simpler case:

I. VON NEUMANN MEASUREMENTS

We begin with a warm-up on ‘sharp’ observables, i.e., those
described by Hermitian operators whose spectra represent the
possible measurement outcomes. If a set of Hermitian oper-
ators is not simultaneously diagonalizable, then it contains at
least one non-commuting pair. Similarly, such a pair of op-
"servers contains at least one non-commuting pair of spectral
projections. By relabeling outcomes we can therefore always
build a pair of non-commuting ±1-valued observables $A_1$, $A_2$
from a set of incompatible von Neumann measurements. For
each such pair we want to find now a bipartite quantum state
and ±1-valued observables $B_1$, $B_2$ which violate the CHSH
inequality $|\langle B \rangle| < 1$ where

$$B = \frac{1}{2} \left[ A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2) \right].$$

(1)

To this end note that for given observables the CHSH inequality
holds for all quantum states iff $B^2 \leq 1$ [20]. Using that the
observables have unit square one gets [13]

$$B^2 = 1 + \frac{1}{4} [A_1, A_2] \otimes [B_1, B_2].$$

(2)

Since the tensor product of the commutators is Hermitian and
traceless, $B^2$ has an eigenvalue larger than one iff the com-
mutators do not vanish. Hence $B_i = A_i$ enables a violation
whenever the observables $A_1$ and $A_2$ are incompatible.

For the optimal state (the respective eigenstate) this gives
the quantitative relation $|\langle B \rangle| = \sqrt{1 + ||[A_1, A_2]||^2/4}$
whereas an optimal choice of the $B'$s ($||[B_1, B_2]|| = 2$, e.g. by fulfilling Pauli commutation relations) yields

$$\max_{\rho, B_1, B_2} \left| \langle B \rangle \right| = \sqrt{1 + \frac{1}{2} ||[A_1, A_2]||}. \quad (3)$$

\section{General Measurements}

Let $A_1$ and $A_2$ now be described by $d$-dimensional POVMs, i.e., pairs of positive semidefinite 'effect' operators \{ $Q, 1 - Q$ \} and \{ $P, 1 - P$ \} whose expectation values give the probabilities of the assigned measurement outcomes. These observables are jointly measurable within quantum mechanics iff there is a measurement with four outcomes corresponding to four positive operators $R_{ij}$, ($i, j = \pm$) with correct 'marginals' $R_{++} + R_{+-} = Q$ and $R_{+0} + R_{-0} = P$.

Beyond the case of qubits \[\text{III}\] there is no explicit characterization of jointly measurable observables known, but we can easily get an implicit one:

**Proposition 1** Two observables characterized by the effects $P$ and $Q$ are jointly measurable iff there is a positive semidefinite operator $S$ satisfying $Q + P - 1 \leq S \leq P$, $Q$.

Necessity of this condition is proven by taking $S = R_{++}$ and sufficiency by simply constructing the other $R'$s from the given relations. A first look at Prop\[\text{III}\] suggests to just construct the 'largest' $S$ which is smaller than $P$ and $Q$ and then check the two inequalities on the lower side. However, such a largest element does exist in general not exist (unless $P$ an $Q$ fulfill trivial relations such as $P \geq Q$ \[\text{III}\]) since, mathematically speaking, the set of positive operators does not form a lattice.

Despite this fact, Prop\[\text{III}\] can be decided efficiently numerically as it can be phrased as a semidefinite program \[\text{III}\] of the form

$$\text{inf}\{ \lambda \in \mathbb{R} \mid Q + P \leq \lambda I + S \} \quad (4)$$

subject to the constraints $0 \leq S \leq Q, P$. The infimum of Eq\[\text{III}\], denote it $\lambda_0$, is larger than one iff two observables are not jointly measurable. Moreover, the magnitude of $\lambda_0$ provides a means of quantifying how incompatible the two measurements are: $\mu = \max_{0, \lambda_0 - 1}$ is the least amount of noise (or information loss) which has to be added to $Q$, $P$ in order to make the measurements $Q' = (1 - \mu)Q + \mu E$ and $P' = (1 - \mu)P + \mu E$ compatible for all $0 \leq E \leq I$.

Semidefinite programs always come with a dual (see Sec\[\text{IV}\]), and in case of \[\text{III}\] deciding whether $\lambda_0 > 1$ (meaning that $Q$ and $P$ are incompatible) is equivalent to checking strict positivity of

$$\lambda^* = \sup_{X, Y, Z \geq 0} \text{tr} [X (Q + P - 1)] - \text{tr} [QY] - \text{tr} [PZ] \quad (5)$$

under the additional constraints $X \leq \rho$ where $\rho = Y + Z$ is a density operator.

Our aim is now to show that $\lambda^*$ is exactly the maximal violation of the CHSH inequality to which $Q$ and $P$ can lead and, using this insight, to provide a simple way of computing it (without the need of setting up a semidefinite programming algorithm).

\section{CHSH as Lagrangian Dual}

Our main result is the following duality between the questions of whether two observables are jointly measurable and whether they enable a violation of the CHSH inequality:

**Proposition 2 (CHSH)** Two measurements characterized by effect operators $Q$ and $P$ are not jointly measurable iff they enable the violation of the CHSH inequality. Quantitatively,

$$\sup_{\psi, B_1, B_2} \left| \langle \psi | B | \psi \rangle \right| = 1 + 2\lambda^* \quad (6)$$

The supremum can be computed as $\lambda^* = \max_{\phi \in [0, \pi]} \mu(\phi)$ where $\mu(\phi)$ is the largest eigenvalue of

$$(Q + P - 1) \otimes \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} - Q \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - P \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7)$$

with $c = \cos(\phi)$ and $s = \sin(\phi)$.

**Proof.** We begin by rewriting the constraints in the dual problem \cite{5} by introducing $\rho := Z + Y, Q := \rho^{-1/2}\chi x\chi x^{-1/2}$ and $\tilde{P} := \rho^{-1/2}\chi y\chi y^{-1/2}$ (using the pseudo-inverse when necessary). The constraints in \cite{5} translate then to $0 \leq Q, \tilde{P} \leq 1$ (i.e., $Q$ and $\tilde{P}$ being effect operators) and $\rho$ being a density operator. We then exploit that the latter is the reduced density operator of a normalized pure state $|\psi\rangle := (\sqrt{\rho} \otimes 1) \sum_{i=1}^{d} |ii\rangle$ and that for instance $\text{tr} [QY] = \langle \psi | Q \otimes \tilde{P}^T | \psi \rangle$. In this way we obtain

$$\lambda^* = \sup \langle \psi | (Q + P - 1) \otimes \tilde{Q} - Q \otimes \tilde{P} - P \otimes (1 - \tilde{P}) | \psi \rangle = \sup \langle \psi | B - \lambda I | \psi \rangle / 2, \quad (8)$$

where the supremum is taken over all admissible effect operators $\tilde{Q}, \tilde{P}$ and state vectors $\psi$ and the last step is obtained by inserting $A_1 = 1 - 2P, A_2 = 2Q - 1, B_1 = 1 - 2\tilde{P}$ and $B_2 = 1 - 2Q$.

In order to arrive at the formula $\lambda^* = \max_{\phi \in [0, \pi]} \mu(\phi)$ we use that, due to convexity, the extremal value of $\langle B \rangle$ is attained for $\tilde{Q}, \tilde{P}$ being projections. Since two projections can be unitarily diagonalized simultaneously \cite{10} up to blocks of size at most $2 \times 2$, we obtain (again employing convexity) the same maximal violation when restricting to $\psi \in \mathbb{C}^d \otimes \mathbb{C}^2$. As the maximum over $\psi$ is nothing but computing the largest eigenvalue we can make further use of the unitary freedom we have to fix one of the observables, say $\tilde{P} = \text{diag}(1, 0)$ and make $\tilde{Q}$ a real projector with non-negative entries, which finally leads to the expression in \cite{7}.

\section{Discussion}

The two cases discussed above, von Neumann measurements and POVMs, differ in several respects: we saw in the von Neumann case that it is sufficient for a CHSH-violation to use the same observables at the added site. In the case of POVMs this is no longer true. To see this, note that a rescaling $A_i \mapsto \lambda A_i$ implies $\langle B \rangle \mapsto \lambda \langle B \rangle$. This means that starting
with von Neumann measurements on both sides and rescaling
the $A_i$’s until right before the violation vanishes will not allow
us anymore to depart from von Neumann observables for the
$B_i$’s.

A second difference is the reduction argument which al-
lowed us for von Neumann measurements to reduce the case
of many observables with several outcomes to two $\pm 1$-valued
observables. The importance of this step stems from the
fact that the ‘two $\pm 1$-valued observable case’ is the only one
where a single Bell inequality is sufficient to characterize the
existence of a joint probability distribution [12]. For more ob-
servables or outcomes the set of Bell inequalities becomes
fairly monstrous and is largely unexplored. Unfortunately, a
similar reduction to the CHSH-case is not always possible for
POVMs: incomparability of observables can in this case be
‘overlooked’ if one only considers pairwise incomparability of
several observables or if one groups measurement outcomes
together [17]. Hence, for the cases of POVMs (with more
than two outcomes or settings) where this happens the ques-
tion is still open. If one follows the same route as in the proof
of Prop. [one easily arrives at expressions which show that the
given observables can be used as one-side part of an entangle-
ment witness. However, not every witness corresponds to a
Bell inequality and whether or not this is the case highly de-
PENDS on the type of decomposition into local operators. We
have to leave this problem open for the moment.

Finally, it is worth mentioning that the followed approach
led to new insight into the joint measurability problem. On
the one hand Prop. provides a simple criterion for deciding
joint measurability for two two-valued observables beyond the
recently proven qubit case [8]. On the other hand, the fact that
the problem is a semidefinite program enables us to solve it in
practice (i.e., for any given instance). As this is an interesting
result in its own right we will provide more details about it in
the remaining part of this paper.

V. JOINT MEASURABILITY AS A SEMIDEFINITE
PROGRAM

The fact that the joint measurability problem is a semide-
finite program implies that for any instance of observables
(with not too many parameters) the problem can be solved in
an efficient and certifiable way. As we saw that the dual
problem is related to the violation of a Bell inequality we will state
the problems in a quantitative way (i.e., not as mere feasibility
problem).

The duality theorem for semidefinite programs [15] reads

$$\inf_{x \in \mathbb{R}^n} \left\{ c|x| \mid \sum_i x_i F_i \geq C \right\}$$

$$\geq \sup_{X \geq 0} \left\{ \operatorname{tr}[CX] \mid \operatorname{tr}[XF_i] = c_i \right\},$$

where $c \in \mathbb{R}^n$ and $C, F_i$ are Hermitian matrices. Moreover, if
one of the problems (say the primal problem) is bounded
and strictly feasible (i.e., $\exists x : \sum_i x_i F_i > C$), then the dual
attains its extremum and equality holds between (9) and (10).

Two dichotomic observables The joint measurability
problem can be cast as a semidefinite program by ex-
pressing $S = \sum x_i G_i$ in terms of a Hermitian operator base
set $G_i$ and setting

$$C = (Q + P) \oplus 0 \oplus (-Q) \oplus (-P),$$

$$F_0 = 1 \oplus 0 \oplus 0 \oplus 0,$$

$$F_i = G_i \oplus G_i \oplus (-G_i) \oplus (-G_i), \quad i \geq 1$$

and $c_0 = 1, x_0 = \lambda, c_i = 0$ for $i \geq 1$. From here we get the dual

$$\sup_{\rho, Y, Z} \operatorname{tr}[\rho(Q + P)] - \operatorname{tr}[QY] - \operatorname{tr}[PZ],$$

subject to the constraints $Y, Z \geq 0$ and $\rho \leq Y + Z$ being a
density operator. As this is strictly feasible, the supremum in
coincides with the minimum $\lambda_0$ of [4]. For our pur-
poses we slightly rewrite the problem and instead of checking
whether $\lambda_0 > 1$ (which means that $Q$ and $P$ are not jointly
measurable) we may as well study whether

$$\sup_{X, Y, Z \geq 0} \operatorname{tr}[X(Q + P - 1)] - \operatorname{tr}[QY] - \operatorname{tr}[PZ]$$

is positive under the constraint $X \leq Y + Z$ and $\operatorname{tr}[Y + Z] = 1$.
This is the form used in [9]. The corresponding primal
problem changes the constraints $S \leq P, Q$ and $P + Q \leq
\lambda + S$ in (10) to $S - \lambda \leq P, Q$ and $P + Q \leq S + 1$, and
minimizes $\lambda$ leading to the minimum $\lambda^\star$.

In a similar vein we can now treat more general scenarios.
All of them have a strictly feasible dual so that equality holds
between primal and dual problem.

Two arbitrary observables Consider two $N$-outcome
observables which are characterized by two sets of effect op-
erators $\{Q_i\}, \{P_j\}$ with $i, j = 1, \ldots, N$. These are jointly
measurable iff we can find $\{R_{ij} \geq 0\}$ such that $\sum_i R_{ij} = P_j$ and $\sum_j R_{ij} = Q_i$. One way, analogous to the previous one,
to express this as a semidefinite program is to minimize $\lambda \in \mathbb{R}$
w.r.t. $\{R_{ij} \geq 0\}$ such that

$$\sum_{i=1}^{N-1} Q_i + \sum_{j=1}^{N-1} P_j \leq \lambda \mathbb{1} + \sum_{i=1}^{N-1} R_{ij}$$

and $P_j \geq \sum_{i=1}^{N-1} R_{ij}$ and $Q_i \geq \sum_{j=1}^{N-1} R_{ij}$ for all $i, j$. The
corresponding dual is

$$\lambda_0 = \sup_{\rho, (Y_i, Z_j) \geq 0} \sum_{i=1}^{N-1} \operatorname{tr}[Q_i(\rho - Y_i) + P_i(\rho - Z_i)],$$

subject to the additional constraints $\rho \leq Y_i + Z_j$ for all $i, j$
and $\rho$ being a density operator.

Several dichotomic observables Let $M$ two-valued ob-
servables be characterized by effect operator $0 \leq T_\alpha \leq 1,$
$\alpha = 1, \ldots, M$. We will denote the effect operators of the
sought joint observable by $R_i$ using a multi-index $i \in \{0, 1\}^M$
This is again a semidefinite program which can be made quan-
titive by replacing $1 \rightarrow \lambda 1$ and minimizing $\lambda$. Again the minimum $\lambda_0$ can as well be obtained from the dual

$$\lambda_0 = \sup_{\rho, \{X_\alpha \geq 0\}} \sum_\alpha \tr [T_\alpha (\rho - X_\alpha)],$$

subject to $\forall i : (|i| - 1) \rho \leq \sum_\alpha \delta_{|i|,1} X_\alpha,$

where $\rho$ is constrained to be a density operator.

VI. APPENDIX: BELL INEQUALITIES AND NO-SIGNALING

For completeness we provide in this appendix the argument for the claim that observables which enable the violation of a Bell inequality cannot be measured jointly within any no-
signaling theory which is consistent with the predictions of quantum mechanics (but possibly a refinement thereof). Variants of this argument (or its main ingredients) can be found in [3, 5, 7, 18].

Suppose Alice can jointly measure two observables, which are labeled by $A_1$ and $A_2$, and yield outcomes $a_1, a_2$ with probability $p(a_1, a_2)$. If Bob, at a distance, measures an ob-
servable $B_1$ with outcome $b_1$, then they observe in a statistical experiment a joint probability distribution $p(a_1, a_2, b_1|B_1)$ so that [21]

$$p(a_1, a_2) = \sum_{b_1} p(a_1, a_2, b_1|B_1).$$

However, in a no-signaling theory this has to be indepen-
dent of Bob’s chosen observable, i.e., a possibly measured $p(a_1, a_2, b_2|B_2)$ has to have the same marginal $p(a_1, a_2)$. Assume that Bob chooses $B_1$ or $B_2$ at random so that they mea-
sure both triple distributions. From these we can write down a joint distribution

$$p(a_1, a_2, b_1, b_2) := \frac{p(a_1, a_2, b_1|B_1)p(a_1, a_2, b_2|B_2)}{p(a_1, a_2)},$$

which by construction correctly returns all measured distribu-
tions as marginals. As a result, the possibility of jointly mea-
suring $A_1$ and $A_2$ implies a joint probability distribution (22) if the no-signaling condition is invoked. A joint distribution, in turn, implies that no Bell inequality can be violated. So if a Bell inequality is violated, then either $A_1$ and $A_2$ are not jointly measurable, or the no-signaling condition is violated.

Note that this argument works independent of the numbers of measurement outcomes or observables.

This work has been funded by Spanish grants FPU, I-
MATH and MTM2008-01366, by QUANTOP and the Ole Roemer grant of the Danish Natural Science Research Coun-
cil (FNU). MMW thanks UCM/ISI for the hospitality.

[1] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. Lett. 47, 777 (1935).
[2] J.S. Bell, Physics 1, 195 (1964).
[3] A. Fine, J. Math. Phys. 23, 1306 (1982).
[4] R.F. Werner, M.M. Wolf, Quant. Inf. Comp. 1(3), 1 (2001); N. Gisin, quant-ph/0702021 (2007).
[5] L. Masanes, A. Acin, N. Gisin, Phys. Rev. A 73, 012112 (2006).
[6] R.F. Werner, Lett. Math. Phys. 17, 359 (1989); B. Terhal, A.C. Doherty, D. Schwab, Phys. Rev. Lett. 90, 157903 (2003).
[7] M.M. Wolf, F. Verstraete, J.I. Cirac, Int. J. Quant. Inf. 1, 465 (2003).
[8] P. Busch, H.-J. Schmidt, arXiv:0802.4167 (2008); S. Yu, N. Liu, L. Li, C.H. Oh, arXiv:0805.1538 (2008); P. Stano, D. Reitzner, T. Heinosaari, Phys. Rev. A 78, 012315 (2008).
[9] E. Andersson, S. Barnett, A. Aspect, Phys. Rev. A 72, 042104 (2005).
[10] P. Busch, Phys. Rev. D 33, 2253 (1986).
[11] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[12] A. Fine, Phys. Rev. Lett. 48, 291 (1982).
[13] L.J. Landau, Phys. Lett. A 120, 54 (1987).
[14] S. Gudder, J. Math. Phys. 37, 2637 (1996); A. Gheondea, S. Gudder, P. Jonas, J. Math. Phys. 46, 062102 (2005).
[15] L. Vandenberghe, S. Boyd, SIAM Review 38, 49 (1996).
[16] P.L. Halmos, Two subspaces, Trans. Amer. Math. Soc. 144, 381 (1969).
[17] T. Heinosaari, D. Reitzner, P. Stano, Found. Phys. 38, 1133 (2008).
[18] R.F. Werner in Quantum Information - an introduction to basic theoretical concepts and experiments (Springer tracts in modern physics, Berlin, 2001).
[19] Here consistency means that the theoretical theory provides the same distributions as quantum mechanics; possibly after ignoring (integrating over) a hidden variable which would provide extra information.
[20] An inequality of the form $H_1 \geq H_2$ between two Hermitian operators should be read as ‘$H_1 - H_2$ is positive semidefinite’.
[21] As common in classical probability theory we use the notation $p(\cdot | \cdot)$ where right of the dash is the condition which the prob-
ability is subject to. In our case this is the observable which is measured.