Irreducible Multiparty Correlations in Quantum States without Maximal Rank

D. L. Zhou

Beijing National Laboratory for Condensed Matter Physics, and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
(Received 9 April 2008; published 30 October 2008)

The correlations of an n-partite quantum state are classified into a series of irreducible k-party ones (2 ≤ k ≤ n), with the irreducible k-party correlation being the correlation in the states of k parties but nonexistent in the states of (k − 1) parties. A measure of the degree of irreducible k-party correlation is defined based on the principle of maximal entropy. Adopting a continuity approach, we overcome the difficulties in calculating the degrees of irreducible multiparty correlations for the multipartite states without maximal rank. In particular, we obtain the degrees of irreducible multiparty correlations in the n-qubit pure states, the n-qubit stabilizer states and the n-qubit generalized Greenberger-Horne-Zeilinger states, which reveals the distribution of multiparty correlations.

Introduction.—How to classify and quantify correlations in a multipartite quantum state is a fundamental problem in many-particle physics and quantum information science. Traditionally, this is done by introducing correlation functions between or among experimental observables of different parties. A modern method of characterizing those correlations is based on entropy, in which different types of correlations in a multipartite state are regarded as different types of nonlocal information, and entropy is used as a measure of information [1].

The concept of irreducible n-party correlation in an n-party quantum state was first proposed in Ref. [2] by Linden et al. based on the principle of maximum entropy. This concept describes how much more information in the n-party state than what is already contained in its reduced states of (n − 1) parties. The degree of irreducible two-party correlation in a bipartite quantum state is equal to the two-party mutual entropy [2], which is also obtained in different contexts [3,4]. Among n-qubit pure states, the irreducible n-party correlation is nonzero only for the Greenberger-Horne-Zeilinger (GHZ) type pure states [5]. Most n-partite pure states are completely determined by their reduced states of just over half the parties [6,7].

It is worth pointing out that the idea of using the maximum entropy principle to quantify classical correlation has developed independently in the classical information community [8–10]. Remarkably, the connected information of order k (2 ≤ k ≤ n) for a probability distribution of n classical variables was defined in Ref. [8] by Schneidman et al.

In this Letter, we define a measure for the degree of irreducible k-party correlation in an n-partite state. This definition can be regarded as not only a direct generalization of the concept in Ref. [2], but also a quantum version of connected correlation of order k in Ref. [8].

The measure of the degree of irreducible k-party correlation we define relies on a constrained optimization problem over n-partite quantum states. Its explicit calculation for a general n-partite state (n > 2) is quite challenging, even for a 3-qubit state. To the best of our knowledge, no explicit calculations for irreducible multiparty correlations exist in the available literature.

The main purpose of this Letter is to calculate the degrees of irreducible multiparty correlations for the multipartite states without maximal rank based on a continuity approach. We obtain analytic results for the degrees of irreducible multiparty correlations of the stabilizer states [11–13] and the generalized GHZ states [5].

Notation and definitions.—Let [n] be the set {1, 2, . . . , n}. An m-element subset of [n] is denoted as a(m) = {a1, a2, . . . , am}, and the relative complement of a(m) in [n] is denoted as a(n − m) = {̄a1, ̄a2, . . . , ̄an−m}. The state of an n-partite quantum system is specified by an n-partite density matrix ρa(m). The irreducible k-party correlation (2 ≤ k ≤ n) in the state is defined as the information appearing in the k-partite reduced density matrices ρa(k), but nonexistent in the (k − 1)-partite reduced density matrices ρa(k−1). To define a measure for the degree of irreducible multiparty correlation in the state ρa(m), we introduce an n-partite density matrix ̂ρ̂l for l = 1, 2, . . . , n as

\[ ̂ρ̂l = \arg\max S(\sigma^{a(l)}) = \rho^{a(l)} \]  

(1)

for any subset a(l), which is similar to the method adopted in Ref. [14]. Function S is the von Neumann entropy defined as S(\sigma) = −Tr(\sigma \log_2 \sigma). Namely, the n-partite density matrix ̂ρ̂l has the same l-partite reduced density matrices as those of ρl, but it is maximally noncommittal to other missing information contained in the state ρl [15]. A measure for the degree of irreducible k-party correlation in the state ρl is then defined as

\[ C^{(k)}(\rho^{a(n)}) = S(\rho^{a(n)}) - S(\rho^{a(k)}) \]  

(2)
The total correlation in the state $\rho^{[n]}$ is then referred to as the nonlocal information appearing in $\rho^{[n]}$, but nonexistent in the 1-partite states $\rho^{[1]}$. A measure of the degree of the total correlation in the state $\rho^{[n]}$ is then defined as

$$C^T(\rho^{[n]}) = S(\rho_1^{[n]}) - S(\rho_n^{[n]}).$$  

Substituting Eqs. (2) into Eq. (3), we find that

$$C^T(\rho^{[n]}) = \sum_{k=2}^{n} C^{(k)}(\rho^{[n]}),$$  

Equation (4) not only justifies Eq. (3) as a legitimate measure of the total correlation, but also implies that all irreducible multiparty correlations construct a classification of the total correlation. In other words, the degrees of irreducible $k$-party correlation tell us how the total correlation is distributed in the $n$-partite quantum state.

As shown in Eqs. (2) and (3), the degrees of different types of correlations are intimately related to the von Neumann entropy. The underlying reason is as follows. The von Neumann entropy of a quantum state is a measure of the degree of uncertainties of the state. The existence of correlation in the multipartite quantum state decreases the uncertainties of the state. Therefore the decrease of uncertainties, i.e., the entropy difference, becomes a reasonable measure for the degree of correlation.

**Standard exponential form.**—Since the $n$-partite density matrices $\tilde{\rho}_l^{[n]}$ are essential elements in the definitions in Eqs. (2) and (3), we give the following important Theorem on the standard exponential form of the state $\tilde{\rho}_l^{[n]}$.

**Theorem 1.**—For an $n$-partite quantum state $\rho^{[n]}$ with maximal rank, a state $\tilde{\rho}_l^{[n]}$ ($1 \leq l \leq n$) satisfying Eq. (1) can be expressed in the exponential form

$$\tilde{\rho}_l^{[n]} = \exp\left(\sum_{a(l)} \Lambda^{a(l)} \otimes 1^{a(n-l)}\right),$$

where $1^{a(n-l)}$ is the identity operators on the Hilbert space of parties $a(n-l)$, and the operators $\Lambda^{a(l)}$ are determined by the constrained conditions in Eq. (1).

**Proof.**—We solve the constrained maximization problem defined by Eq. (1) by the method of Lagrange multipliers.

The equality is satisfied if and only if Eq. (5) is satisfied, and the Lagrange multipliers $\Lambda^{a(l)}$ are the operators in Eq. (5). To prove the above inequality, we have used the Klein inequality [16]: $\text{TrA}(\ln A - \ln B) \geq \text{Tr}(A - B)$ for positive operators $A$ and $B$, where the equality is satisfied if and only if $A = B$. Because the Klein inequality involves only positive operators, we need to limit ourselves to the states with maximal rank.

A direct result derived from Theorem 1 is

$$\tilde{\rho}_1^{[n]} = \exp\left(\sum_{a(1)} \Lambda^{a(1)} \otimes 1^{a(n-1)}\right) = \prod_{i=1}^{n} \otimes \rho^{(i)}.$$  

Therefore the degree of the total correlation (4) in the state $\rho^{[n]}$ is given by

$$C^T(\rho^{[n]}) = \sum_{i=1}^{n} S(\rho^{(i)}) - S(\rho^{[n]}),$$

where we used $\tilde{\rho}_n^{[n]} = \rho^{[n]}$. Although the degree of the total correlation has an analytical expression (6), we failed to find similar analytical results for the degrees of irreducible multiparty correlations $C^{(k)}(\rho^{[n]})$.

**Theorem 1** is a direct generalization of Eq. (4) in Ref. [2]. It shows that the feature of multiparty correlation in the state $\tilde{\rho}_1^{[n]}$ is directly embodied in the exponential form of the state. As noted in Ref. [2], Theorem 1 is inapplicable for the multipartite states without maximal rank. However, most multipartite states of interest in many-particle physics or quantum information have nonmaximal ranks, e.g., the $n$-qubit stabilizer states and the generalized GHZ states to be discussed below.

Our strategy to treat states without maximal rank is based on the fact that a multipartite state without maximal rank can always be regarded as the limit of a series of multipartite states with maximal rank. If the degrees of irreducible multiparty correlations for the series of states with maximal rank can be obtained by using Theorem 1, we can take the limit to get the degrees of irreducible multiparty correlations for the state without maximal rank. We call the above method the continuity approach. The proofs of Theorems 2 and 3 below are typical applications of this approach.

**Correlations in stabilizer states.**—An $n$-qubit stabilizer state $\rho_s^{[n]}$ is defined as

$$\rho_s^{[n]} = \frac{1}{2^m} \sum_{\alpha_1, \ldots, \alpha_n = 0, 1} \prod_{i=1}^{m} g_i^{\alpha_i},$$

where the operators $g_i$ are $m (m \leq n)$ independent commuting $n$-qubit Pauli group elements. The set $\mathcal{G}(\rho_s^{[n]}) = \{g_i\}$ is called the stabilizer generator for the state $\rho_s^{[n]}$, and the group generated by the generator $g_i$, denoted as $\mathcal{G}(\rho_s^{[n]}) = \{\prod_{i=1}^{m} g_i^{\alpha_i}, \alpha_i = 0, 1\}$, is called the stabilizer of the state. To make Eq. (7) a legitimate state, the minus identity operator is required not to be an element of the
stabilizer \(G(p_{\lambda}^{[n]})\). We remark that our definition of the n-qubit stabilizer state is a generalization of the usual definition [13], which corresponds to the case when \(m = n\). When \(m < n\), the stabilizer states defined by Eq. (7) are no longer pure states.

According to the definition of the n-qubit Pauli group, an element \(h \in G\{p_{\lambda}^{[n]}\}\) can be written as \(h = \pm \prod_{i=1}^{n} O^{(i)}\) for \(O \in \{I, X, Y, Z\}\), where \(I\) is the 2 \(\times\) 2 identity operator, and \(X, Y, Z\) are three Pauli matrices. The number of identity operators \(I\) in the element \(h\) is \(N_I(h) = \sum_i \text{Tr} O^{(i)}/2\). The stabilizer \(G(p_{\lambda}^{[n]})\) can be classified into a series of sets \(G_k(p_{\lambda}^{[n]})\) such that any element in \(G_k(p_{\lambda}^{[n]})\) can be written as a unique product of elements in the set. We remark that \(G_k(p_{\lambda}^{[n]})\) can be any generator set of the Abelian Pauli subgroup generated by \(G_k(p_{\lambda}^{[n]})\). Even though the generator \(g_k(p_{\lambda}^{[n]})\) is not uniquely defined, its cardinality \(|g_k(p_{\lambda}^{[n]})|\) is.

**Theorem 2.** —The irreducible \(k\)-party irreducible correlation in an n-qubit stabilizer state \(p_{\lambda}^{[n]}\) is

\[
C^{(k)}(p_{\lambda}^{[n]}) = |g_k(p_{\lambda}^{[n]})| - |g_{k-1}(p_{\lambda}^{[n]})|.
\]

**Proof.**—Because \(G_1(p_{\lambda}^{[n]}) \subseteq G_2(p_{\lambda}^{[n]}) \subseteq \cdots \subseteq G_n(p_{\lambda}^{[n]})\), we can always take \(g_1(p_{\lambda}^{[n]}) \subseteq g_2(p_{\lambda}^{[n]}) \subseteq \cdots \subseteq g_n(p_{\lambda}^{[n]})\). Then the elements contained in \(g_k(p_{\lambda}^{[n]})\) but not in \(g_{k-1}(p_{\lambda}^{[n]})\) are reexpressed as \(g_{ki}\) for \(i \in [|g_k| - |g_{k-1}|]\). Thus we can construct an n-qubit state with a real parameter \(\lambda\) as

\[
\rho_{\lambda}^{[n]}(\lambda) = \exp \left( \eta + \lambda \sum_{k=1}^{n} \sum_{i=1}^{|g_k|} g_{ki} \right).
\]

where \(\eta = -\ln(2\cosh|\lambda|)\), which is determined by the normalization condition \(\text{Tr}(\rho_{\lambda}^{[n]}) = 1\). Then the above state can be expanded as

\[
\rho_{\lambda}^{[n]}(\lambda) = \frac{1}{2^n} \left( 1 + \sum_{d=1}^{n} \tanh^d \lambda \sum_{\alpha_{k} = d}^{k=m} \prod_{a_{ki} = d}^{k=m} g_{ki}^{a_{ki}} \right).
\]

Note that if \(\exists \alpha_{ki} = 1\) for \(k > m\), then \(\forall \alpha(m)\), we have \(\text{Tr}_{\alpha(m)}(\prod_{\alpha_{k} = 1}^{k=m} g_{ki}^{a_{ki}}) = 0\). Thus the \(m\)-partite reduced density matrix \(\rho_{\lambda}^{[m]}(\lambda) = \rho_{\lambda}^{[m]}(\lambda)\). According to Theorem 1, the degree of irreducible \(k\)-party correlation in the n-qubit state \(p_{\lambda}^{[n]}(\lambda)\) is

\[
C^{(k)}(p_{\lambda}^{[n]}(\lambda)) = S(p_{\lambda}^{[n-1]}(\lambda)) - S(p_{\lambda}^{[n]}(\lambda)).
\]

From Eq. (10), we observe that when the parameter \(\lambda\) takes the limit of positive infinity, the states \(\rho_{\lambda}^{[n]}(\lambda)\) are stabilizer states. In particular,
matrices. Therefore, the state $\rho^{[n]}(\gamma, \lambda')$ has the same $(n-1)$-partite reduced density matrices as the state $\rho^{[n]}(\gamma, \lambda)$ if the following condition is satisfied:

$$\lambda_x' = \lambda_y' = 0, \quad \tanh \lambda_z' = \frac{\lambda_z}{\lambda} \tanh \lambda.$$  (17)

According to Theorem 1, we find

$$\hat{\rho}^{[n]}_m(\gamma, \lambda') = \rho^{[n]}(\gamma, \lambda')$$  (18)

for $m = 2, 3, \ldots, n-1$. Therefore, the degrees of irreducible multiparty correlations for the state $\rho^{[n]}(\gamma, \lambda)$ can be obtained via Eqs. (2).

Without loss of generality, we assume that in Eq. (13) $\alpha = \cos(\theta/2)$ and $\beta = \sin(\theta/2)e^{i\phi}$. Then we define the Bloch vector $\hat{\alpha} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Let us take $\lambda = \hat{\alpha}$, then $\Sigma = |G_{\alpha}\rangle = |G_{\alpha}\rangle$. The operators $\{Z^{(i)}X^{(j)}\}$ and $\Sigma_n$ can be regarded as the stabilizer generator of the state $\rho^{[n]}_G$. According to Theorem 2, the relation between the generalized GHZ state $\rho^{[n]}_G$ and $\rho^{[n]}(\gamma, \lambda)$ is

$$\rho^{[n]}_G = \lim_{\lambda \to +\infty} \rho^{[n]}(\lambda, \lambda \hat{\alpha}).$$  (19)

In this case, we find that, for $m = 2, 3, \ldots, n-1$,

$$\lim_{\lambda \to +\infty} \rho^{[n]}_m(\lambda, \lambda \hat{\alpha}) = |\alpha|^2[00\cdots0](00\cdots0)$$

$$+ |\beta|^2[11\cdots1](11\cdots1).$$  (20)

A direct calculation yields the results of Theorem 3.

Theorem 3 shows that in the generalized $n$-qubit GHZ state (13), only irreducible 2-party and $n$-party correlation exist, and the former is $(n-1)$ times of the latter.

Summary.—The definition of the degree of irreducible $k$-party correlation in an $n$-partite state is given as a natural generalization of those defined in [2,8]. The significance of the exponential form of a multipartite state in characterizing irreducible multiparty correlation is emphasized by Theorem 1. Adopting the continuity approach, we are capable of applying Theorem 1 to deal with the irreducible multiparty correlations in multipartite states without maximal rank. Particularly, we successfully obtained the degrees of irreducible $k$-party correlation in the $n$-qubit stabilizer states and the $n$-qubit generalized GHZ states. The multiparty correlation structures in these states are revealed by our results. We hope that the concept of irreducible multiparty correlation will shed light on the characterizations of multiparty correlations in condensed matter system, e.g., topological orders [18–20] in degenerate ground states.

The author thanks L. You, Z. D. Wang, and C. P. Sun for helpful discussions. This work is supported by NSF of China under Grant No. 10775176, and NKBRSF of China under Grants No. 2006CB921206 and No. 2006AA06Z104.

[1] C. E. Shannon, Bell Syst. Tech. J. 27, 379 (1948).
[2] N. Linden, S. Popescu, and W. K. Wootters, Phys. Rev. Lett. 89, 207901 (2002).
[3] B. Groisman, S. Popescu, and A. Winter, Phys. Rev. A 72, 032317 (2005).
[4] B. Schumacher and M. D. Westmoreland, Phys. Rev. A 74, 042305 (2006).
[5] S. N. Walck and D. W. Lyons, Phys. Rev. Lett. 100, 050501 (2008).
[6] N. Linden, D. L. Zhou, and C. P. Sun, Phys. Rev. A 71, 012324 (2005).
[7] N. Ay, E. Olbrich, N. Bertschinger, and J. Jost, in Proceedings of ECCS’06 (Santa Fe Institute Working Paper No. 06-08-028, 2006).
[8] D. Gottesman, Ph.D. thesis, Caltech, 1997, arXiv:quant-ph/9705052.
[9] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[10] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, and H. J. Briegel, in Quantum Computers, Algorithms and Chaos, Proceedings of the International School of Physics “Enrico Fermi,” Course 162, edited by G. Casati et al. (IOS Press, Amsterdam, 2006); arXiv: quant-ph/0602096.
[11] D. L. Zhou, B. Zeng, Z. Xu, and L. You, Phys. Rev. A 74, 052110 (2006).
[12] E. T. Jaynes, Phys. Rev. 106, 620 (1957).
[13] A. Wehrl, Rev. Mod. Phys. 50, 221 (1978).
[14] A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
[15] M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006).
[16] S. Yang, D. L. Zhou, and C. P. Sun, Phys. Rev. B 76, 180404(R) (2007).