A Fundamental Theorem of Calculus for Second-order Directional Derivative
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Abstract
Given a two-variable function $f$ without critical points and a compact region $R$ bounded by two level curves of $f$, this note proves that the integral over $R$ of $f$’s second-order directional derivative in the tangential directions of the interceding level curves is proportional to the rise in $f$-value over $R$. Also discussed are variations on this result when critical points are present or $R$ becomes unbounded. Several concrete examples exemplify the theory.

Let $f$ be a real-valued $C^2$ function on an open connected domain $\Omega \subset \mathbb{R}^2$. Suppose that $f$ has no critical points and that $a < b$ are values in $f[\Omega]$ such that $f^{-1}([a, b])$ is connected and compact (in which case $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}[a] \times [a, b]$ and $f^{-1}[t]$ is a simple closed $C^2$ curve for each $t \in [a, b]$). For $p \in \Omega$, let $T(p)$ be a unit tangent to the level curve $f^{-1}[f(p)]$ at $p$. With $D^2_{T(p)} f(p)$ denoting the second-order directional derivative of $f$ at $p$ in the direction $T(p)$, our main result is the following identity, unmistakably resembling the fundamental theorem of calculus:

$$\iint_{f^{-1}([a, b])} D^2_{T} f \, dA = \pm 2\pi (b - a),$$

with the positive sign in effect iff $f^{-1}[b]$ encircles $f^{-1}[a]$. As we shall see through several examples, the assumptions that $f^{-1}([a, b])$ be connected and compact and that $f$ have no critical points can be relaxed, allowing flexibility in application.

We establish this identity in §2 after treating some preparatory results in §1. In §3, we show how this result can be adapted for a variety of situations.

1 Second-order directional derivative and curvature of level curves

We review a few key notions, aiming to conceptualize curvature of level curves of a $C^2$ function $f$ in terms of its second-order directional derivative.

1.1 Second-order directional derivative

For notation, we often add a displacement vector $v$ to an initial point $p \in \mathbb{R}^2$ to express the terminal point $p + v$.

For each $p \in \Omega$ and any unit vector $v$, let $D(s) = f(p + sv)$ for $s$ sufficiently small so that $p + sv \in \Omega$. The first and second directional derivatives of $f$ at $p$ in the direction $v$, denoted by $D_v f(p)$ and $D^2_v f(p)$, are defined to be the two numbers $D'(0)$ and $D''(0)$. Using chain rule, we obtain the standard facts that

$$D_v f(p) = \nabla f(p) \cdot v \quad \text{and} \quad D^2_v f(p) = Q_p(v, v),$$
where $Q_p$ is the quadratic form associated with the Hessian matrix

$$H_p = \begin{bmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{yx}(p) & f_{yy}(p) \end{bmatrix}. $$

Simply put, $Q_p(v, v) = v \cdot (H_p v)$. We note another formula for $D^2_2 f(p)$.

**Lemma 1** Let $t \mapsto r(t)$ be a curve in $\Omega$ such that $r(0) = p$ and $r'(0) = v$. Then,

$$D^2_2 f(p) = \left. \frac{d}{dt} \right|_{t=0} (\nabla f(r(t)) \cdot v) = \left( \frac{d}{dt} \left|_{t=0} \nabla f(r(t)) \right. \right) \cdot v.$$

**Proof.** The second equality is clear. Checking the first amounts to verifying that $\frac{d}{dt} |_{t=0} (\nabla f(r(t)) \cdot v) = Q_p(v, v)$ by chain rule. □

### 1.2 Curvature of level curves

Let $C$ denote a level curve of $f$, which is a regular $C^2$ curve (as, by assumption, $f$ has no critical point). Install on $C$ the unit normal field $N := -\nabla f/|\nabla f|$ and the unit tangent field $T := (-f_y e_1 + f_x e_2)/|\nabla f|$. (The frame $(T, N)$ is positively-oriented.) The **signed curvature** $\kappa$ of $C$ at each point thereon is defined by the equation $dT/ds = \kappa N$, where $s$ is arc length along $C$ with its increasing direction induced by $T$. (The sign of $\kappa$ depends on the choice we make of $N$, but not of $T$.) For $p \in \Omega$, let $\kappa(p)$ be the signed curvature of $f^{-1}[f(p)]$ at $p$.

**Lemma 2** $\kappa(p) = D^2_2 f(p)/|\nabla f(p)|$.

**Proof.** Let $\gamma$ be the unit-speed parametrization of an arc on $C := f^{-1}[f(p)]$, with $\gamma(0) = p$ and $\gamma'(0) = T(p)$. By definition, $\gamma''(0) = \kappa(p)N(p)$. Because $C$ is a level curve, the two vectors $\nabla f(\gamma(t))$ and $\gamma'(t)$ are always orthogonal; hence,

$$0 = \frac{d}{dt} (\nabla f(\gamma(t)) \cdot \gamma'(t)) = \left( \frac{d}{dt} \nabla f(\gamma(t)) \right) \cdot \gamma'(t) + \nabla f(\gamma(t)) \cdot \gamma''(t).$$

For the last two terms, note that, by Lemma I

$$\left( \frac{d}{dt} |_{t=0} \nabla f(\gamma(t)) \right) \cdot \gamma'(0) = D^2_2 f(p),$$

whereas

$$\nabla f(\gamma(0)) \cdot \gamma''(0) = \nabla f(\gamma(0)) \cdot \left( \kappa(p) \frac{-\nabla f(p)}{|\nabla f(p)|} \right) = -\kappa(p) |\nabla f(p)|.$$ 

The claimed formula now follows, since $|\nabla f(p)| \neq 0$. □

In the literature, this formula for $\kappa$ is always written explicitly in terms of the partial derivatives of $f$ and is typically derived (using the implicit function theorem) from the curvature formula for graphs of one-variable functions; see, e.g., [1]. Not only do we formulate the result in a conceptually simpler form, we have given a conceptually simpler derivation not relying on any formula for curvature other than its definition.
2 Integrating Second-order Directional Derivative

We reiterate our assumptions, which will remain in effect in this section.

**Hypothesis.** \( f \) is a \( C^2 \) function with no critical points on an open connected set \( \Omega \subset \mathbb{R}^2 \); \( a < b \) are two numbers in \( f[\Omega] \) such that \( f^{-1}[[a, b]] \) is connected and compact.

We first recall two facts and also introduce a notation.

2.1 Two facts from calculus and geometry

First, the gradient flow originating from \( f^{-1}[a] \) induces a diffeomorphism between \( f^{-1}[[a, b]] \) and \( f^{-1}[a] \times [a, b] \), allowing this change of variables of integration:

\[
\int\int_{f^{-1}[[a, b]]} g \, dA = \int_a^b \left( \int_{f^{-1}[t]} \frac{g}{|\nabla f|} \, ds \right) \, dt \quad (1)
\]

where \( ds \) is the arc length element along \( f^{-1}[t] \); see [1, pp. 298–300].

Second, for a simple closed plane curve \( C \) with signed curvature \( \kappa \), \( \int_C \kappa \, ds = \pm 2\pi \); see [2, pp. 36-37]. The sign ambiguity is due to the dependence of \( \kappa \) on orientation, with the positive sign in force iff the chosen unit normal field on \( C \) points inward (relative to the Jordan domain enclosed by \( C \)). To encode the sign more effectively, we introduce \( \sigma := n \cdot N \), where \( n \) is the inward unit normal field along \( C \) and \( N \) is the chosen normal field. With \( \sigma \) tracking orientation, \( \int_C \kappa \, ds = 2\pi \sigma \).

Turning to level curves of \( f \), for \( p \in f^{-1}[[a, b]] \), let \( n(p) \) be the inward unit normal at \( p \) of the (simple closed) curve \( f^{-1}[f(p)] \) and let \( \sigma(p) := n(p) \cdot N(p) \). Being continuous and integer-valued, \( \sigma \) is constant on (the connected) \( f^{-1}[[a, b]] \).

(Note that \( \sigma \equiv 1 \) iff \( f^{-1}[b] \) encloses \( f^{-1}[a] \).) We then have, for every \( t \in [a, b] \),

\[
\int_{f^{-1}[t]} \kappa \, ds = 2\pi \sigma \quad (2)
\]

2.2 A fundamental theorem of calculus

We are ready for the main result.

**Theorem 3 (Fundamental Theorem)** Under the preceding Hypothesis,

\[
\int\int_{f^{-1}[[a, b]]} D^2_T f \, dA = \sigma \cdot 2\pi(b - a) .
\]

**Proof.** Using [1], Lemma [2] and [3], we calculate as follows:

\[
\int\int_{f^{-1}[[a, b]]} D^2_T f \, dA = \int_a^b \int_{f^{-1}[t]} D^2_T f |\nabla f| \, ds \, dt = \int_a^b \int_{f^{-1}[t]} \kappa \, ds \, dt = (2\pi \sigma)(b - a) ,
\]

establishing the claimed formula. \( \blacksquare \)

We note an immediate consequence concerning the integral of \( D^2_N f \).
Corollary 4 Under the preceding assumptions,
\[ \iint_{f^{-1}[a,b]} D^2_N f \, dA = \int_{f^{-1}[0]} \nabla f \, ds - \int_{f^{-1}[a]} |\nabla f| \, ds - 2\pi(b-a)\sigma. \]
If, in addition, \( f \) is harmonic, then
\[ \iint_{f^{-1}[a,b]} D^2_N f \, dA = -2\pi(b-a)\sigma. \]

We provide some hints for the proof and leave the details to the reader.

Note that the number \( Q_p(\mathbf{i}, \mathbf{j}) + Q_p(\mathbf{j}, \mathbf{i}) \), i.e., the trace of the Hessian form \( Q_p \), equals the Laplacian \( \Delta f(p) \), and that the trace of a symmetric bilinear form is invariant under an orthogonal change of coordinates. Hence,
\[ D^2_N f(p) + D^2_N f(p) = Q_p(\mathbf{T}, \mathbf{T}) + Q_p(\mathbf{N}, \mathbf{N}) = \text{Tr} Q_p = \Delta f(p). \]
As \( \Delta f = \text{div}(\nabla f) \), we may apply Green’s theorem to the integral of \( \Delta f \).

3 Variations on the fundamental theorem

We give three examples to illustrate some variations on the theme of Theorem 3.

3.1 Examples

In Examples 1 and 2, we will benefit from using complex-valued variables to express real functions. In both examples, we adopt the following notations.

Notation. We name the Cartesian and polar coordinates of the complex variables \( z \) and \( w \) as follows: \( z = x + iy = re^{i\theta} \) and \( w = u + iv = re^{i\theta} \). We let \( \mathcal{T}(0; t) \) denote the closed disc \( \{ w : |w| \leq t \} \).

Example 1. Let \( f(x, y) = |z + 1|/|z - 1| \), which is \( C^2 \) on \( \mathbb{C} \setminus \{-1, 1\} \). Except for \( f^{-1}[0] \) (the singleton \( \{-1\} \)) and \( f^{-1}[1] \) (the \( y \)-axis), the level curves of \( f \) are the so-called Apollonius’ circles with foci \( \pm 1 \). On the left half-plane, the level circles are oriented counterclockwise according to our earlier stipulation and \( \sigma = 1 \) as a result. Exploiting the simple relation between curvature of a circle and its radius, we find that \( \kappa(z) = -2\pi/|z^2 - 1| \).

We are to integrate \( D^2_N f \) over \( f^{-1}[0, 1] \), the entire left half-plane. According to Theorem 3, \[ \iint_{f^{-1}[0, 1]} D^2_N f \, dA = 2\pi(1 - \epsilon_2 - \epsilon_1) \]
for small positive \( \epsilon_1 \) and \( \epsilon_2 \). It follows that
\[ \iint_{f^{-1}[0, 1]} D^2_N f \, dA = 2\pi. \]

We verify this claim by calculation. Introduce the auxiliary complex function \( w(z) = (z + 1)/(z - 1) \), which maps \( f^{-1}[[0, t]] \) one-to-one onto \( \mathcal{T}(0; t) \). Taking advantage of Lemma 2 and the fact that \( |\nabla f| = |w'| \), we have
\[ D^2_N f(z) = \kappa(z) |\nabla f(z)| = \kappa(z) |w'(z)|. \]
Change the variables of integration from \((x, y)\) to \((u, v)\) at the cost of dividing the integrand by \( |\partial(x, y)/\partial(x, y)| = |w'|^2 \), we obtain
\[ \iint_{\mathbb{R} \times \mathbb{C}} D^2_N f \, dAz \, dAw = \iint_{|u| \leq 1} |w'|^2 \, dAw = \iint_{|u| \leq 1} \kappa(z) \, dAw. \]
(where the subscripts distinguish the area elements in the $z$-plane and $w$-plane).

Now,

$$\frac{\kappa}{|w'|} = -\frac{2x}{|z^2 - 1|} \frac{|z - 1|^2}{2} = -x \frac{|z - 1|}{z + 1} = -x \frac{|w|}{r} = \frac{-x}{r} .$$

Inverting the function $z \mapsto w(z)$ allows $x$ to be expressed in terms of $w$:

$$x = \text{Re} z = \text{Re} \frac{w + 1}{w - 1} = \frac{|w|^2 - 1}{|w - 1|^2} = \frac{r^2 - 1}{r^2 - 2r \cos \theta + 1} .$$

Finally, we have

$$\int_0^{2\pi} \left( \int_0^2 \frac{1 - r^2}{r^2 - 2r \cos \theta + 1} dr \right) d\theta = 2\pi ,$$

as modern computing technology capable of symbolic integration can verify.

**Example 2.** Let $f(x, y) = |z^2 - 1|$, which is $C^2$ on $\mathbb{C} \setminus \{-1, 1\}$ and has a saddle point at the origin. Let $C_t$ denote $f^{-1}[t]$. For each $t > 0$, the curve $C_t$ is a Cassini’s oval with foci $\pm 1$, which is the locus of points whose distances to $\pm 1$ have a fixed product $t$. For $t < 1$, $C_t$ has two components, both oriented counterclockwise by our stipulation; for $t = 1$, $C_t$ has a single self-intersection at the origin and is well known as a Bernoulli’s lemniscate.

Let $\Omega = f^{-1}[(0, 1)]$. In the spirit of Theorem\(\text{\textsuperscript{2}}\) we expect that

$$\int_\Omega \int_0^{2\pi} D_1^2 f dA = 2 \times 2\pi \times (1 - 0) = 4\pi ,$$

because, for $t \in (0, 1)$, $\int_{C_t} kds = 2 \times 2\pi$, $C_t$ being the union of two positively-oriented simple closed curves. Let’s verify this conclusion by purely computational means independent of any results established herein.

Define the auxiliary function $w(z) = z^2 - 1$, which is one-to-one on $R := \{re^{i\varphi} : \rho > 0; \varphi \in (-\pi/2, \pi/2)\}$ and maps $\overline{\Omega} \cap \Omega$ onto $\overline{D}(0; 1)$.

Using the Hessian form $Q$, we find

$$D_1^2 f(z) = \frac{2|z|^4 + (x^2 - y^2)}{|z|^2|z^2 - 1|} = \frac{2\rho^4 + 2\rho^2 \cos 2\alpha}{\rho^2 r} .$$

Note that $r^2 = |z^2 - 1|^2 = \rho^4 - 2\rho^2 \cos 2\alpha + 1$, enabling us to rewrite $D_1^2 f(z)$:

$$D_1^2 f(z) = \frac{3\rho^4 + 1 - r^2}{\rho^2 r} .$$

Finally, we change variables from $(x, y)$ to $(u, v)$ and apply the relations $\rho^2 = |w + 1|$ and $|w'| = 2\rho$ to obtain

$$\int_\Omega \int_0^{2\pi} D_1^2 f dA_z = \int_{\Omega \cap D(0; 1)} \int_0^2 \frac{D_1^2 f}{|w'|^2} dA_w = \int_{\Omega \cap D(0; 1)} \frac{3|w + 1|^2 + 1 - r^2}{4|w + 1|^2} dA_w$$

$$= \frac{1}{2} \int_0^{2\pi} \left( \int_0^1 \left( \int_0^1 \frac{r \cos \theta + 1}{r^2 + 2r \cos \theta + 1} dr \right) d\theta \right) .$$

As $\int_{\Omega \cap D(0; 1)} D_1^2 f dA = 2 \int_{\Omega \cap D(0; 1)} D_1^2 f dA_z$, it suffices to verify that

$$\int_0^{2\pi} \left( \int_0^1 \frac{r \cos \theta + 1}{r^2 + 2r \cos \theta + 1} dr \right) d\theta = 2\pi ,$$

5
which our computing technology can attest to.

**Example 3.** Let $f(x, y) = [x^2 + (y - 1)^2 - 4][x^2 + (y + 1)^2 - 4]$. The exercise of deducing the following claims from Theorem 3 is left to the reader.

$$\iint_{f^{-1}([-8, 0])} D^2_T f \, dA = 32\pi; \quad \iint_{f^{-1}([0, 8])} D^2_T f \, dA = 0; \quad \iint_{f^{-1}([1, 20])} D^2_T f \, dA = 22\pi.$$

### 3.2 Lessons from the examples

With the preceding examples in mind, we discuss ways in which Theorem 3 can be adapted. For convenience, we introduce two terms. If $p$ is a critical point of $f$, then $f(p)$ is a **critical value** of $f$ and $f^{-1}[f(p)]$ is a **critical level**.

Assume that $f$ has at most finitely many critical points in $\Omega$. We further impose the condition (sufficient for our purpose) that any critical level in $f^{-1}[[a, b]]$ has plane measure 0; a “naturally occurring” function easily meets this condition, as a critical level is often a finite union of rectifiable arcs.

Under these assumptions, we outline three cases.

**Case 1.** $f^{-1}[[a, b]]$ is compact, free of critical points, but disconnected. (Cf. Examples 2–3.) Theorem 3 can be applied to each component $K_j$ of $f^{-1}[[a, b]]$, resulting in

$$\iint_{f^{-1}[[a, b]]} D^2_T f \, dA = \sum_i 2\pi \sigma(K_i) \cdot (b - a). \quad (3)$$

**Case 2.** $f^{-1}[[a, b]]$ is compact and $[a, b]$ contains a critical value. (Cf. Examples 2–3.) For simplicity but without loss of generality, assume that $[a, b]$ contains exactly one critical value $c$. Either $c$ is an endpoint or $c \in (a, b)$. The latter case can be reduced to the former, as $[a, b] = [a, c] \cup [c, b]$. Suppose that $c = b$; the case $c = a$ is similar. Then

$$\iint_{f^{-1}[[a, b]]} D^2_T f \, dA = \lim_{\epsilon \to 0} \iint_{f^{-1}[[a, b-\epsilon]]} D^2_T f \, dA.$$

As the integral on the right is governed by (3), the limit exists. (Our assumption that a critical level has plane measure 0 is needed for the above equality. Also note that, although $D^2_T f$ is undefined at the critical points on the critical level $f^{-1}[b]$, its integral over $f^{-1}[[a, b]]$ after all exists.)

**Case 3.** $[a, b]$ contains no critical values but, for finitely many $c \in [a, b]$, $f^{-1}[c]$ is not compact. (Cf. Example 1.) As in Case 2, assume that there is exactly one such $c$ and $c = b$. The same limit argument in Case 2 shows that $D^2_T f$ (defined everywhere on $f^{-1}[[a, b]]$) is integrable on $f^{-1}[[a, b]]$ and given by (3).

### References

[1] Courant, R. (1936). *Differential and Integral Calculus, Vol. II*. Interscience Publishers.

[2] do Carmo, M. P. (1976). *Differential Geometry of Curves and Surfaces*. Prentice-Hall.