HOROCYCLE AVERAGES ON CLOSED MANIFOLDS AND TRANSFER OPERATORS

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Abstract. We study the ergodic integrals of the horocycle flows $h_\rho$ of $C^r$ codimension one mixing Anosov flows. In dimension three, for any suitably bunched $C^3$ contact Anosov flow with orientable strong-stable distribution $E_-$, we show $|\frac{1}{T} \int_0^T \varphi \circ h_\rho(x) d\rho - \mu(\varphi)| \leq \frac{C}{T^\epsilon} \|\varphi\|_{C^3}$ for some $\epsilon > 0$, with $\mu$ the invariant measure of $h_\rho$. We thereby implement the toy model program of Giulietti–Liverani [31] in the natural setting of geodesic flows in variable negative curvature, where nontrivial resonances exist.

1. Introduction

Anosov introduced a class of $C^2$ flows $g_\alpha : M \to M$, now bearing his name [3], on closed (i.e. compact and boundaryless) orientable manifolds $M$ of dimension $d \geq 3$. We focus on topologically mixing Anosov flows. A special class of mixing Anosov flows are those preserving a contact structure. Geodesic flows on the unit tangent bundle of a closed manifold with (possibly variable) negative sectional curvature are well-studied classes of contact Anosov flows.

Every Anosov flow $g_\alpha$ admits a strong stable foliation, tangent to a vector bundle denoted $E_-$. If this foliation is orientable and has dimension $d_-$ equal to one, and if $g_\alpha$ is mixing, one associates with $g_\alpha$ another flow, the horocycle flow $h_\rho : M \to M$, such that for every $x \in M$ the trajectory $h_\rho(x)$ is a strong stable leaf (defined up to speed reparametrisation). Horocycle flows were first introduced in the case of geodesic Anosov flows, [44, p.84], [38]. In a setting more general than ours (with $d_- \geq 1$), Bowen and Marcus [14] then proved that the horocycle flow is uniquely ergodic and minimal. Its invariant probability measure $\mu$ (related to, but distinct from, the measure of maximal entropy of $g_\alpha$, see Remark 4.16) plays an important role below.

Since the horocycle flow is induced by the Anosov flow, there exists $\tau(\rho, \alpha, x) > 0$ such that $g_\alpha \circ h_\rho(x) = h_{\tau(\rho, \alpha, x)} \circ g_\alpha(x)$, $\forall x \in M$, $\forall \alpha, \rho \in \mathbb{R}_+$. We call $\tau(\rho, \alpha, x)$ the renormalisation time. In the setting of unit speed geodesic flows on compact (or more generally, finite volume) surfaces of constant negative curvature, renormalisation has...
been used effectively in the work of Flaminio and Forni [28] to study the horocycle integrals

\[ \gamma_x(\varphi, T) := \int_0^T \varphi \circ h_\rho(x) d\rho, \quad x \in M, \quad T > 0, \]

for \( \varphi : M \to \mathbb{R} \) in Sobolev spaces of positive order. Flaminio and Forni found that the speed of convergence of \( \gamma_x(\varphi, T)/T \) to \( \mu(\varphi) \) as \( T \to \infty \) is controlled by invariant distributions under the push-forward of the horocyclic vector field. These distributions are also eigendistributions under the push-forward of the geodesic vector field, and the eigenvalues give the powers of \( T \) appearing in the expansion of \( \gamma_x(\varphi, T)/T - \mu(\varphi) \).

Their approach inspired Giulietti and Liverani [31] to study a toy model, replacing the Anosov flow with a hyperbolic diffeomorphism \( F \), using the renormalisation dynamics as a key to study \( \gamma_x(\varphi, T) \). Letting \( h_{\text{top}} \) be the topological entropy of \( F \), they show analogously (for the corresponding invariant measure \( \mu \)) that the speed of convergence to zero of \( \gamma_x(\varphi, T)/T - \mu(\varphi) \) is controlled by eigenvalues in the annulus \( 1 < |z| < e^{h_{\text{top}}} \) (and the corresponding eigendistributions) of a weighted transfer operator of \( F \). Unfortunately, in the setting of [31], there are in fact no eigenvalues in the annulus \( 1 < |z| < e^{h_{\text{top}}} \), see [6]. The approach of Giulietti and Liverani has been applied successfully in the meantime by Faure–Gouëzel–Lanneau [22] to the linear flow in the stable direction of a two-dimensional linear pseudo-Anosov map, and by Butterley–Simonelli [18] to parabolic flows on (3-dimensional) Heisenberg nilmanifolds which are renormalized by partially hyperbolic automorphisms (circle extensions of Anosov diffeomorphisms). In both these algebraic applications, nontrivial eigenvalues are present.

Giulietti and Liverani conjectured that a similar expansion exists for more general (non-algebraic) Anosov flows than in [28], e.g. for the geodesic flow of a surface with variable negative curvature [31, Conjecture 2.12]. More precisely, letting \( h_{\text{top}} \) be the topological entropy of the time-one map \( g_1 \), we expect that there exists \( \delta > 0 \) such that, for smooth enough observables \( \varphi \), the following expansion holds\(^2\) (analogously to [28], [31])

\[ \gamma_x(\varphi, T) = T \int \varphi d\mu + \sum_{\delta < \Re \lambda < h_{\text{top}}} T^{\frac{\delta}{\lambda}} \tilde{c}_\lambda(T, x) O_\lambda(\varphi) + \mathcal{E}_{T,x}(\varphi). \]

In the above formula, \( \mathcal{E}_{T,x} = O(T^{\delta/\lambda}) \), uniformly in \( x \), the \( O_\lambda \) are generalised eigendistributions associated to the eigenvalue \( \lambda \) for the adjoint (or dual) of the generator of a weighted transfer operator (see [3]) acting on an anisotropic Banach space, the real parameter \( \delta \) is an upper bound on the essential spectral bound of the generator, and \( \tilde{c}_\lambda(T, x) \in \mathbb{C} \) satisfies \( \sup_{x,T} \| \tilde{c}_\lambda(T, x) \| \log T |^{-J_\lambda} < \infty \), where \( J_\lambda \geq 0 \) is the size of the largest Jordan block of \( \lambda \).

The main result of this work, Theorem 4.8, provides an asymptotic expansion (1) for \( C^r \) time reparametrisations of the unit speed horocycle flow of codimension one topologically mixing \( C^r \) Anosov flows, if \( r > 2 \) and the distribution \( E_- \) is \( C^{r-1} \), under an essential spectral gap condition \( (\lambda_{\min}^{s.t. p} < h_{\text{top}}) \), and a weak Dolgopyat condition on the resolvent (Condition 3.12). As a consequence, we get power-law convergence of the ergodic averages (Corollary 4.9). In Proposition 4.10 we show that the conditions of Theorem 4.8 hold for \( C^3 \) contact Anosov flows in dimension three, with orientable strong stable bundle \( E_- \), under the following bunching assumption: Recalling that \( d_- = 1 \), define

\[ \lambda_+ = \lim_{\alpha \to \infty} \sup_x \log \| Dg_{-\alpha}(x)\|_{E_-}^{\alpha}, \quad \lambda_- = -\lim_{\alpha \to \infty} \sup_x \log \| Dg_\alpha(x)\|_{E_-}^{\alpha}, \quad \hat{\alpha} := 2\frac{\lambda_-}{\lambda_+} \in (0, 2]. \]

\(^2\)For Anosov flows, \( h_{\text{top}} > 0 \), see [3]. For geodesic flows on finite volume negatively curved surfaces, \( h_{\text{top}} = 1 \).
The bunching condition is

\[ \hat{\omega} > \frac{8}{5}. \]

For constant negative curvature geodesic flows, we have \( \hat{\omega} = 2 \). Assumption (2) thus holds for geodesic flows with variable strictly negative curvature close enough to a constant, but the reader is warned that it does not apply to generic three-dimensional contact Anosov flows.

For compact surfaces of constant negative curvature, Randol [45] proved that there exist eigenvalues of the Laplacian arbitrarily close to 1. This provides examples for which the expansion of Flaminio–Forni [28], and thus the expansion in Theorem 4.8, is not reduced to 

\[ \int T \phi d\mu + \mathcal{E}_{T,x}(\phi). \]

As in the work of Giulietti and Liverani [31], the key idea to study \( \gamma_x(\varphi, T) \) is to introduce the weighted semigroup of transfer operators, with generator \( X + V \), defined by

\[ \mathcal{L}_{\alpha,V} : W^{s,t,q}_p(M) \to W^{s,t,q}_p(M), \quad \mathcal{L}_{\alpha,V} \varphi = \phi_\alpha \cdot (\varphi \circ g_{-\alpha}), \quad \phi_\alpha(x) = e^{\int_0^\alpha V \circ g_\beta(x) d\beta}, \quad \alpha \geq 0, \]

where the potential is \( V = -\partial_\rho \tau(0,0,\cdot) \) (so that \( \phi_\alpha = \partial_\rho \tau(0,-\alpha,\cdot) \)), and where \( W^{s,t,q}_p(M) \) is an anisotropic Banach space with regularity parameters \( s < 0 < q \leq t < r-1+s \) and \( p \in (1,\infty) \). In the case of the unit speed parametrisation of \( h_\rho \), we shall see that \( \phi_\alpha = \det Dg_{-\alpha}|_{E^-} \) is just the Jacobian along the strong stable distribution at a negative time \( -\alpha \), and \( V = \text{div} (X|_{E^-}) \).

The paper is organised as follows: The transfer operator \( \mathcal{L}_{\alpha,V} \) is defined in Section 2.1 (for more general potentials). The new anisotropic Banach spaces \( W^{s,t,q}_p(M) \) are constructed in Section 2.3 after introducing admissible cones for \( C^r \) Anosov flows in Section 2.2 (if \( p = 2 \) we get Hilbert spaces). These spaces are a flow analogue to the spaces constructed in [9] to study hyperbolic diffeomorphisms. Anisotropic Banach spaces are now a standard tool for hyperbolic dynamics (see e.g. [12, 42, 8, 29, 7, 33, 52]). Although we do not study here the dynamical determinant or zeta function associated to the transfer operator \( \mathcal{L}_{\alpha,V} \), we believe that the spaces introduced in the present work are well suited for this purpose (see [5]). Guedes Bonthonneau and Lefeuvre very recently [36] applied a (microlocal) flow implementation of the spaces from [9] to study some dynamical and geometric problems.

In Section 3 we establish properties of the transfer operator semigroup, its generator \( X + V \) and the resolvent \( R_z \) (see (9)). Most of these results do not require the contact assumption. Among those are norm bounds yielding a Lasota–York inequality for the resolvent (Theorem 3.8). Then, in Corollary 3.9 we obtain a strip in the spectrum of the generator containing at most countable eigenvalues of finite multiplicity. Proposition 3.13 puts the weak Dolgopyat Condition 3.12 in more standard form. These tools are used in Section 4 to show the above-mentioned main results, Theorem 4.8 and Proposition 4.10 (Our proofs highlight sufficient conditions for intrinsicness of resonances and portability of Dolgopyat bounds on the resolvent when navigating between different Banach spaces.) Finally, Appendix A contains (elementary) integration by parts lemmas, adapted from [9], Appendix B recalls the fragmentation/reconstitution lemmas from [9], and Appendix C is devoted to interpolation and mollifiers.

We end this introduction with some remarks:

(a) The conjecture that the distributions \( \mathcal{O}_o \) in the expansion (11) are fixed by the (adjoint) of the horocycle flow, which was the starting point in [28], remains open for general codimension one mixing Anosov flows (see [31, Remark 2.10]). For smooth contact Anosov flows with \( d = 3 \), invariance was proved by Faure–Guillarmou [23].

(b) The anisotropic Banach spaces \( W^{s,t,q}_p(M) \) in this paper are based on those in [9]. We could also define spaces \( \mathcal{B}^{s,t}(M) \) based on those in [10] (or [5, Chapter 5]). We expect that the
following variational upper bound may be obtained\footnote{Here, \(h_{\mu}\) is the entropy of an ergodic \(g_1\) invariant probability measure, and \(\chi_{\lambda}(A)\) is the largest Lyapunov exponent of \(A\).} for the essential spectrum of the semigroup \(L_{\alpha,V}\) on \(B^{s,t}(M)\):

\[
\lambda_{\min}^{s,t}(X,V) := \sup_{\mu} \left\{ h_{\mu}(g_1) + \chi_{\mu} \left( \frac{\phi_1}{\det(Dg_1|_{E^s})} \right) + \max \left\{ t\chi_{\mu}(Dg_1|_{E^-}), |s|\chi_{\mu}(Dg_1|_{E^+}) \right\} \right\} .
\]

The above is in general better (even in the volume preserving case) than the bound \(\lambda_{\min}^{s,t,p}(X,V)\) we obtain in Corollary 3.9 (see \(54\)). Since \(\lambda_{\min}^{0,0}(X,V) = h_{\text{top}}\), the essential spectral gap condition \(\lambda_{\min}^{s,t} < h_{\text{top}}\) would thus hold for \(B^{s,t}(M)\) for arbitrarily small \(s < 0\) and \(t > 0\) (so that the assumptions of Proposition 4.10 could be weakened accordingly, and \(s'\) could be taken arbitrarily close to 0 in \(118\)). However, the scale \(B^{s,t}(M)\) is more messy\footnote{See also the caveat in \(5\) Remark 5.18 regarding the lack of validity of \(27\).} to define, it is not an interpolation scale, it does not include a Hilbert space, and showing (4) would require a thermodynamic analysis of the sums over subcovers in the proof of Lemma 3.6. To keep the paper short, we restrict to the scale \(W_p^{s,t,q}(M)\).

(c) The renormalisation time \(\tau(\rho,\alpha,\cdot)\) inherits the smoothness of the invariant bundle \(E_-\), which is only Hölder in general. We add the extra assumption that \(E_-\) is smooth enough and that an essential spectral gap holds in Theorem 4.8 (and Lemma 4.15), and we give settings where this is satisfied in Proposition 4.10. To work with anisotropic spaces with higher regularity (depending only on \(r\)), one could lift the dynamics to the Grassmannian \(33,31\). We have chosen to avoid the cumbersome corresponding technicalities for the sake of readability.

(d) Finally, we mention two directions of future research: First, the expansions of Flaminio and Forni \(28\) (or Faure–Tsuji\(i\) \(25,27\), see also \(19\)) are not limited to finite sets of eigenvalues. Our methods do not currently allow to go beyond the smallest \(\delta\) such that \(\Sigma_\delta = \sigma(X+V)|_{W_p^{s,t,q}(M)} \cap \{\Re z > \delta\}\) is finite (\(\delta = 1/2 \) for \(28\)). Second, even if the Dolgopyat Condition 3.12 holds for some \(\delta < 0\), we cannot improve the remainder due to the term with \(|\varphi||_0\) in Lemma 4.14. Although it is hoped that this term is spurious, an analogous error term is present in \(28\) Thm 1.5 or \(31\) Thm 2.8.

2. The Transfer Operators and the Banach Spaces

2.1. Transfer Operators Associated to a Flow \(g_\alpha\) and Weight \(\phi_\alpha\). The Generator \(X + V\). In the entire paper, \(M\) is a compact, boundaryless, connected, orientable, smooth manifold of dimension \(d \geq 3\), and \(r > 1\) is fixed, while \(g_\alpha: M \to M, \alpha \in \mathbb{R}\), is a \(C^r\) Anosov flow on \(M\). By definition, there is a \(Dg_\alpha\)-invariant splitting of the tangent space \(TM = E_- \oplus E_+ \oplus E_0\) of the tangent space such that for some \(C_\star \geq 1\) and \(0 < \theta < 1\), we have

\[
\|Dg_\alpha v\| \leq C_\star \theta^\alpha \|v\|, \forall v \in E_- , \|Dg_\alpha v\| \leq C_\star \theta^\alpha \|v\|, \forall v \in E_+ , \forall \alpha \geq 0 ,
\]

while \(E_0 = \langle X\rangle\), where the \(C^{r-1}\) vector field \(X\) is the generator of the flow defined by

\[
X := \partial_\alpha g_\alpha \big|_{\alpha = 0}.
\]

The (strong) stable and unstable distributions \(E_-\) and \(E_+\) are Hölder. For \(x \in M\), we split \(T_xM = E^-_{x,x} \oplus E^+_{x,x} \oplus E_{0,x}\). The cotangent space \(T^*M\) (the dual space of \(TM\)) is split analogously

\[
T^*M = E^*_- \oplus E^*_+ \oplus E^*_0, \quad T^*_xM = E^*_--x \oplus E^*_+ \oplus E^*_0, \quad x \in M.
\]
The splitting above is $(Dg_\alpha)^{tr}$-invariant and, up to taking larger $C_\gamma$, we have
\[
(7) \quad \begin{cases} 
C_\gamma^{-1} ||\xi|| \leq (Dg_{-\alpha})^{tr} ||\xi|| \leq C_\gamma ||\xi||, & \forall \xi \in E_0^\ast, \forall \alpha \geq 0, \\
|| (Dg_{-\sigma \alpha})^{tr} \xi || \leq C_\sigma \theta^\alpha ||\xi||, & \forall \xi \in E_\sigma^\ast, \sigma = \pm, \forall \alpha \geq 0. 
\end{cases}
\]

The dimensions of the spaces $E_{\sigma,x}$ do not depend on $x$, and we set $d_- := \dim E_- = \dim E_0^\ast$ and $d_+ := \dim E_+ = d - 1 - d_-$. Fixing a potential $V \in C^{r-1}(M, \mathbb{R})$, we introduce the $\phi_\alpha$-weighted transfer operators
\[
(8) \quad L_{\alpha,V}(\varphi) := \phi_\alpha \cdot (\varphi \circ g_{-\alpha}), \quad \alpha \geq 0, \quad \varphi \in C^{r-1}(M),
\]
where
\[
\phi_\alpha(x) := \exp(\int_{-\alpha}^0 V \circ g_{-\beta}(x) d\beta), \quad \text{i.e. } V = \partial_\alpha \phi_\alpha|_{\alpha=0^+}.
\]

For an “integrability” parameter $p \in (1, \infty)$, and suitable anisotropic “regularity” parameters $s$, $t$, and $q$ (see (21)), we will construct Banach spaces $W_p^{s,t,q}(M)$, containing $C^{r-1}(M)$ as a dense subspace, on which the operators $L_{\alpha,V}$ extend continuously to form a strongly continuous semigroup (Lemma 3.7). In particular, for all $\varphi \in C^{r-1}(M)$
\[
\partial_\alpha L_{\alpha,V}(\varphi)|_{\alpha=0^+} = X \varphi + V \varphi \in W_p^{s,t,q}(M).
\]

The generator of the semigroup is $X + V$, we denote by $R_z$ its resolvent
\[
(9) \quad R_z \varphi = (z - V - X)^{-1} \varphi, \quad z \notin \sigma(X + V)|_{W_p^{s,t,q}}, \quad \varphi \in W_p^{s,t,q}(M),
\]
where $\sigma(X + V)|_B$ denotes the spectrum of the operator $X + V$ on $B$. Theorem 3.8 will provide a Lasota–Yorke inequality for $R_z$ for large $R_z$. This gives a vertical strip in the complex plane in which $\sigma(X + V)|_{W_p^{s,t,q}(M)}$ contains only isolated eigenvalues of finite multiplicity (Corollary 3.9).

2.2. Cone Ensembles. The Atlas A. Cone Hyperbolicity. Admissible Cones for $g_\alpha$.

A cone is a nonempty convex set $C \subset \mathbb{R}^d$ such that $\lambda \xi \in C$ for all $\xi \in C$ and $\lambda \in \mathbb{R}$. We say that a cone $C$ is $d'$-dimensional if $d' \geq 1$ is the maximal dimension of a linear subset of $C$. A cone $C$ is compactly included in another cone $C'$, denoted by $C \subseteq C'$, if $\overline{C} \subseteq \text{int} C' \cup \{0\}$. Two cones $C$ and $C'$ are transversal if $C \cap C' = \{0\}$.

We identify $T^*M$ with $\mathbb{R}^d$, and for any $d' \geq 1$ we denote the norm of $\xi \in \mathbb{R}^{d'}$ by $||\xi|| = (\sum_j \xi_j^2)^{1/2}$. For $\xi \in T^*_xM$, write $\xi = \xi^- + \xi^+ + \xi^0$, where $\xi^\sigma \in E_{\sigma,x}^{\ast}$ for $\sigma \in \{\pm, 0\}$. For $\gamma > 0$, define two transversal closed cone fields on $T^*M$, of respective dimensions $d_-$ and $d_+$, by
\[
(10) \quad C^+_\gamma(x) := \{ \xi \mid \max\{||\xi^+||, ||\xi^0||\} \leq \gamma ||\xi^-|| \}, \quad C^+_\gamma(x) := \{ \xi \mid \max\{||\xi^-||, ||\xi^0||\} \leq \gamma ||\xi^+|| \},
\]
and define a one-dimensional closed cone field on $T^*M$ by
\[
C^0_\gamma(x) := \{ \xi \in T^*_xM \mid \max\{||\xi^-||, ||\xi^+||\} \leq \gamma ||\xi^0|| \}.
\]

For $\sigma \in \{\pm, 0\}$ we have $E_{\sigma,x}^\ast \subset C^0_\gamma(x)$ and, if $\gamma' > \gamma$, then $C^0_{\gamma'}(x) \subseteq C^0_{\gamma}(x)$. Moreover $T^*_xM \subset \cup_{\tau} C_{\gamma}(x)$, if $\gamma \geq 1$ (any line through the origin must cross one side of the unit cube in $\mathbb{R}^3$), while $C^0_{\gamma}(x)$ and $C^0_{\gamma}(x)$ are transversal if $\sigma \neq \tau$ and $\gamma < 1$. Last but not least, the lemma below is the key to construct admissible cones\footnote{These cones have non-empty interior while \cite{HOROCYCLE HOMOGENEOUS AND TRANSFER OPERATORS} Proposition 17.4.4] uses “flat” cones included in $E_\delta^\ast \oplus E_\delta^\ast$.}

\footnote{No such property holds for $C^0_{\gamma}(x)$. The cones in \cite{HOROCYCLE HOMOGENEOUS AND TRANSFER OPERATORS} are strictly expanding and contracting, respectively, and this is not true for $C^0_{\gamma}(x)$.}
Lemma 2.1. Let $C_* \in [1, \infty)$ and $\theta \in (0, 1)$ be the constants from (7). Then for any $\gamma, \gamma' \in (0, 1)$ and all $\alpha > 0$ such that $C_*^2 \theta^\alpha \gamma < \gamma'$, we have, recalling $C_\gamma^-(x)$ and $C_\gamma^+(x)$ from (10),

$$(Dg_{-\alpha})^u C_\gamma^-(x) \subset C_{\gamma'}(g_\alpha(x)), \quad (Dg_{\alpha})^u C_\gamma^+(x) \subset C_{\gamma'}(g_{-\alpha}(x)), \quad \forall x \in M.$$ 

Proof. We show the first claim. The proof of the second claim is analogous. Let $\xi = \xi^{-} + \xi^{+} + \xi^{0} \in C_\gamma^-(x)$. We estimate

$$\max\{||(Dg_{-\alpha})^u_\alpha \xi^{+}|, |(Dg_{-\alpha})^u_\alpha \xi^{0}|\} \leq C_* \max\{|\xi^{+}|, |\xi^{0}|\} \leq C_* \gamma |\xi^-| \leq C_*^2 \theta^\alpha \gamma |(Dg_{-\alpha})^u \xi^-|.$$ 

Since $C_{\gamma'-\epsilon}(g_\alpha(x)) \subset C_{\gamma'}(g_{-\alpha}(x))$ for all $\epsilon \in (0, \gamma')$, we conclude. \qed

Next, we adapt to flows the cone ensembles for hyperbolic diffeomorphisms from [9, 5]:

Definition 2.2 (Cone ensembles $\Theta$ for flows). Coverings $\tilde{\theta}$. A cone ensemble of $\mathbb{R}^d$, with $d = d_- + d_+ + 1$, is a pair $\Theta = (C, \Phi)$, where $C = (C_-, C_+, C_0)$ is a triplet of pairwise transversal closed cones with nonempty interiors, of respective dimensions $d_-, d_+$, and one, while $\Phi = (\Phi_-, \Phi_+, \Phi_0)$, where each $\Phi_\sigma$ is a $C^\infty$ map from the unit sphere $\mathbb{S}^d$ to $[0, 1]$, such that

$$\Phi_- + \Phi_+ + \Phi_0 \equiv 1, \quad \Phi_\sigma|_{C_\sigma \cap \mathbb{S}^d} \equiv 1, \quad \sigma \in \{\pm, 0\}.$$ 

In addition, we require that $C_0 = \{\xi \mid |\xi| \leq \gamma_0 |\xi_d|\}$ for some finite $\gamma_0$.

For two cone ensembles $\Theta$ and $\Theta'$ of $\mathbb{R}^d$, we say that $\tilde{\theta} \Theta' < \Theta$ if

$$\mathbb{R}^d \setminus (C_+ \cup C_0) \subset C'_- \quad \text{and} \quad \mathbb{R}^d \setminus C_0 \subset C'_0 \cup C'.$$

Finally, for a cone ensemble $\Theta$, we say that a triplet $\Phi = (\Phi_+, \Phi_-, \Phi_0)$ is a covering of $\Theta$ if each $\Phi_\sigma : \mathbb{S}^d \to [0, 1]$ is $C^\infty$, with $\Phi_\sigma|_{\text{supp} \Phi_\sigma} \equiv 1$.

Definition 2.3 (Cone hyperbolicity). Let $K \subset \mathbb{R}^d$ be compact with nonempty interior, and let $\Theta = (C, \Phi), \Theta' = (C', \Phi')$ be cone ensembles. A diffeomorphism $F : K \to F(K)$ is called cone-hyperbolic from $\Theta$ to $\Theta$ (on $K$) if we have\footnote{For our purposes, the second condition could be replaced by the weaker condition $\mathbb{R}^d \setminus (C_+ \cup C_0) \subset C'_0 \cup C'$.}

$$(D_x F)^u (\mathbb{R}^d \setminus (C'_+ \cup C'_0)) \subset C_-, \quad (D_x F)^u (\mathbb{R}^d \setminus C'_+) \subset C_0 \cup C_-, \quad \forall x \in K.$$ 

The conditions (11) ensure that no parts of higher regularity in the anisotropic Banach spaces of Section 2.3 are mapped to parts of lower regularity (see (37) in the proof of (39) below).

We next introduce a crucial ingredient to construct the anisotropic Banach spaces.

Lemma 2.4 (Admissible Cone Ensembles for $g_{-\alpha}$). There exists an atlas $A$, formed of a finite open cover $\{V_\omega \subset M \mid \omega \in \Omega\}$ of $M$ and $C^r$ local diffeomorphisms $\kappa_\omega : V_\omega \to \mathbb{R}^d$, such that\footnote{For our purposes, the second condition could be replaced by $(D_x F)^u (\mathbb{R}^d \setminus (C'_+ \cup C'_0)) \subset C_0 \cup C_.$}

$$(C_\omega := \kappa_\omega(V_\omega), \quad \text{we have } \min_{\omega \neq \omega'} d(K_\omega, K_{\omega'}) > 1, \quad \text{and } K_M := \cup_\omega \overline{K_\omega} \text{ is compact,}$$

and, fixing coordinates $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and recalling (9), the flow box condition

$$(D\kappa_\omega)(X|V_\omega) = \partial_{x_d}|_{\kappa_\omega(V_\omega)},$$

holds, and, further, setting $V_{-\alpha, \omega} := V_\omega \cap g_{-\alpha}(V_\omega)$ for each $\alpha \in \mathbb{R}$ and $\omega, \omega' \in \Omega$ such that $V_\omega \cap g_\alpha(V_\omega) \neq \emptyset$, and also defining $F_{-\alpha, \omega, \omega'} : \kappa_\omega(V_{-\alpha, \omega}) \to \kappa_{\omega'}(V_{\omega'})$ as

$$F_{-\alpha, \omega, \omega'} := \kappa_{\omega'} \circ g_{-\alpha} \circ \kappa^{-1}_\omega.$$
there exists $\alpha_0 > 0$ and, for each $\omega$, there exist cone ensembles $\Theta_\omega$ such that the cone $D\kappa_\omega^{-1}(C_{\sigma,\omega})$ in the cotangent space contains the normal subspace $E^*_\tau$ and is bounded away from $E^*_\tau$ for $\tau \neq \sigma$, and, for all $\alpha \geq \alpha_0$, the map $F := F_{-\alpha,\omega}$ is cone-hyperbolic from $\Theta_\omega$ to $\Theta_\omega$ on $K := \kappa_\omega(V_{\alpha,\omega})$.

**Remark 2.5** ([Remark 4.12]). For any $\Theta' < \Theta$ the identity is cone-hyperbolic from $\Theta$ to $\Theta'$. If $F$ is cone-hyperbolic from $\Theta'$ to $\Theta$, then there exists $\Theta' < \Theta$ such that $F$ is cone-hyperbolic from $\Theta'$ to $\Theta$, and there exists $\Theta > \Theta$ such that $F$ is cone-hyperbolic from $\Theta'$ to $\Theta$. Thus, Lemma 2.4 implies that there exist cone ensembles $\Theta'_\omega < \Theta_\omega$ such that for all $\alpha \geq \alpha_0$ the map $F_{-\alpha,\omega}$ is cone-hyperbolic from $\Theta'_\omega$ to $\Theta_\omega$, and there exists $\tilde{\Theta} \in \mathbb{R}$, cone ensembles $\Theta'_\omega < \Theta_\omega$, and $\alpha_0 > 0$, such that $\kappa_\omega \circ \kappa_\omega^{-1}$ is cone-hyperbolic from $\Theta'_\omega$ to $\Theta_\omega$ and $F_{-\alpha,\omega}$ is cone-hyperbolic from $\Theta'_\omega$ to $\Theta_\omega$ for all $\alpha \geq \alpha_0$. (Such pairs $\{\Theta_\omega\}$, $\{\Theta'_\omega\}$ are called adapted to $A$ and $g_\alpha$. They are used in Lemma C.2.)

**Proof of Lemma 2.4.** Let $cc(A)$ denote the convex closure of a set $A$. By uniform continuity of the stable and unstable distributions, setting

$$C_{\sigma,\omega}^\gamma := cc\left( \bigcup_{x \in V_\omega} (D\kappa_\omega^{-1})^\tau C_{\sigma}^\gamma(x) \right), \quad \sigma \in \{\pm\}, \quad \omega \in \Omega, \quad \gamma \in (0,1),$$

we may choose (small $V_\omega$, $\kappa_\omega$ satisfying (12)–(13) and $\gamma^* \in (0,1)$, $\tilde{\gamma}^* \in (0,\gamma^*)$ such that

$$D\kappa_\omega^{-1} C_{\sigma,\omega}^\gamma \subseteq C_{\sigma,\omega}^{\gamma^*}, \quad (D\kappa_\omega^{-1})^\tau C_{\sigma,\omega}^\gamma \subseteq C_{\sigma,\omega}^{\gamma^*}, \quad \forall \sigma \in \{\pm\}, \quad \omega \in \Omega, \quad \forall x \in V_\omega.$$

For $C_\sigma \geq 1$, $\theta < 1$ as in (7), and $\gamma$, $\tilde{\gamma}^*$, $\gamma^*$ as above, let $\alpha_0 > 0$ be such that $C_{\sigma,\omega}^\gamma < \tilde{\gamma}^*$ for all $\alpha \geq \alpha_0$. By (14) and Lemma 2.4, we have, using the transversal closed cones $C_{\sigma,\omega}^\gamma$,

$$(Dg_\alpha)^\tau (D\kappa_\omega^{-1})^\tau C_{\sigma,\omega}^{\gamma^*} \subseteq C_{\sigma,\omega}^{\tilde{\gamma}^*}, \quad (Dg_\alpha)^\tau (D\kappa_\omega^{-1})^\tau C_{\sigma,\omega}^{\gamma^*} \subseteq C_{\sigma,\omega}^{\tilde{\gamma}^*}, \quad (Dg_\alpha)^\tau C_{\sigma,\omega}^\gamma \subseteq C_{\sigma,\omega}^{\gamma^*}, \quad \forall \omega \in \Omega, \quad \forall x \in V_\omega.$$
Writing the inverse Fourier transform as $F^{-1}v(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi x} v(\xi) d\xi$, where $\xi x := \langle \xi, x \rangle$ denotes the scalar product, we have

$$
\|F^{-1}\Psi_n\|_{L^1} = \|F^{-1}\Psi_1\|_{L^1} < \infty, \quad \|F^{-1}\Psi_{\sigma,n}\|_{L^1} = \|F^{-1}\Psi_{\sigma,1}\|_{L^1} < \infty, \quad \sigma \in \{\pm, 0\}, \quad n \geq 1.
$$

Analogous estimates hold for $F^{-1}\Psi_{\sigma,0}$ and $F^{-1}\Psi_0$.

Using the convolution $v_1 * v_2(x) := \int_{\mathbb{R}^d} v_1(x-y)v_2(y)dy$ of two distributions $v_1, v_2$, we associate to any $\Psi$ with $F^{-1}\Psi \in L_1(\mathbb{R}^d)$ a pseudo-differential operator with symbol $\Psi$ via

$$
\Psi^{Op}v(x) := ((F^{-1}\Psi) * v)(x) = F^{-1}(\Psi \cdot Fv)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} \Psi(\xi) v(y) d\xi dy.
$$

Young’s inequality, $\|v_1 * v_2\|_{L^p} \leq \|v_1\|_{L^1} \|v_2\|_{L^p}$ for all $p \in (1, \infty)$, gives

$$
\|\Psi^{Op}v\|_{L^p} \leq \|F^{-1}\Psi\|_{L^1} \|v\|_{L^p}, \quad \forall p \in (1, \infty).
$$

The (Bochner) space $L_p(\mathbb{R}^d, \mathcal{H})$ associated to a Hilbert space $\mathcal{H}$ is defined by $\|v\|_{L_p(\mathbb{R}^d, \mathcal{H})} := \|\|v\||_{\mathcal{H}}\|_{L_p(\mathbb{R}^d)}$. The following is a variant of the Marcinkiewicz theorem, generalising (19):

**Theorem 2.6** (See e.g. [18] Thm 0.11.F). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $L(\mathcal{H}_1, \mathcal{H}_2)$ be the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ endowed with the operator norm. If $Q(\cdot) \in C^\infty(\mathbb{R}^d, L(\mathcal{H}_1, \mathcal{H}_2))$ satisfies

$$
\|\partial^\beta_x Q(\xi)\|_{L(\mathcal{H}_1, \mathcal{H}_2)} \leq C\beta(1 + \|\xi\|^2)^{-|\beta|/2}, \quad \text{for each multi-index } \beta,
$$

then the operator $Q^{Op}$ defined for compactly supported continuous $a : \mathbb{R}^d \to \mathcal{H}_1$ by

$$
(Q^{Op}a)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} Q(\xi) a(y) dy d\xi
$$

extends for each $1 < p < \infty$ to a bounded operator from $L_p(\mathbb{R}^d, \mathcal{H}_1)$ to $L_p(\mathbb{R}^d, \mathcal{H}_2)$, and

$$
\|Q^{Op}\|_{L_p(\mathbb{R}^d, \mathcal{H}_1) \to L_p(\mathbb{R}^d, \mathcal{H}_2)} \leq \|F^{-1}Q\|_{L_1(\mathbb{R}^d, L(\mathcal{H}_1, \mathcal{H}_2))}.
$$

We will mostly consider the three cases

$$
\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}; \quad \mathcal{H}_1 = \mathcal{H}_2 = L_2(\mathbb{C}); \quad \mathcal{H}_2 = L_2^c \text{ and } \mathcal{H}_1 = L_2^c \text{ or } \mathcal{H}_1 = L_2^c;
$$

where $L_2^c$, $L_2^c$ are the Hilbert spaces associated, for fixed

$$
-(r-1) < s < q < t < r - 1
$$

and $-(r-1) < s' < s, -(r-1) < q' \leq q, -(r-1) < t' < t$, to

$$
\|a\|_{L_2^c} := \left( \sum_{\sigma,n} 4^{c(\sigma)n} |a_{\sigma,n}|^2 \right)^{\frac{1}{2}}, \quad \|a\|_{L_2^c} := \left( \sum_{\sigma,n} 4^{c'(\sigma)n} |a_{\sigma,n}|^2 \right)^{\frac{1}{2}},
$$

where we set

$$
c(-) := s, \quad c(+) := t, \quad c(0) := q, \quad c'(-) := s', \quad c'(+) := t', \quad c'(0) := q'.
$$

Set $C_0^\infty(K) := \{ f : \mathbb{R}^d \to \mathbb{C} \mid f \text{ is } C^\infty, \text{ supp}(f) \subset K \}$ for $K \subset \mathbb{R}^d$ compact with nonempty interior and $r \in [0, \infty]$. We introduce the basic building block for our anisotropic spaces:

---

[10] With $F^{-1}Q(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi x} Q(\xi) d\xi$, for $x \in \mathbb{R}^d$, the notation (20) is compatible with (18).
Definition 2.7 (Local Anisotropic Norm and Banach Space). Fix a cone ensemble Θ and\(^\text{[1]}\)
\begin{equation}
    p \in (1, \infty), \quad -(r-1) < s \leq t < r - 1.
\end{equation}
For a compactly supported \(C^\infty\) function \(v : \mathbb{R}^d \rightarrow \mathbb{C}\), set
\begin{equation}
    \|v\|_{W^{s,t,q}_{p,Θ}} := \left\| \left( \sum_{n=0}^{\infty} 4^n s |\Psi_{-n}^p v|^2 + 4^n t |\Psi_{+n}^p v|^2 + 4^n q |\Psi_{0,n}^p v|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}^d)}.
\end{equation}
For \(K \subset \mathbb{R}^d\) compact with nonempty interior, the Banach space \(W^{s,t,q}_{p,Θ}(K)\) is defined to be the completion of \(C^\infty_0(K)\) under \(\| \cdot \|_{W^{s,t,q}_{p,Θ}}\).

We shall also use the auxiliary semi-norm \(\|v\|_{W_{p,Θ}^q} := \left\| \left( \sum_{n=0}^{\infty} 4^n q |\Psi_{0,n}^p v|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}^d)}\).

The definition of \(\|v\|_{W_{p,Θ}^q}\) is just like\(^\text{[2]}\)\(^\text{[9]}\) (2.4), except that we have three cones instead of two. We record the following result for convenience; the continuous inclusion claim is obvious, while the compact inclusion claim — which is in fact not used in the present work — is proved exactly like\(^\text{[9]}\) Prop. 5.1, using Arzelà–Ascoli:

Lemma 2.8 (Continuous and compact embeddings, local spaces). Let \(K \subset \mathbb{R}^d\) be compact with nonempty interior. For a compactly supported \(\Theta\), \(p,s,t,q\), the inclusion \(W^{s,t,q}_{p,Θ}(K) \subseteq W^{s',t',q'}_{p,Θ}(K)\) is continuous. If \(s' < s, t' < t, q' < q\), the inclusion \(W^{s,t,q}_{p,Θ}(K) \subseteq W^{s',t',q'}_{p,Θ}(K)\) is compact.

Next, let \(\|v\|_{W^2_{p,Θ}} = \|((1+\Delta)^{1/2}v)\|_{L_p}\) denote the classical isotropic Sobolev (Triebel–Lizorkin) norm, the argument\(^\text{[1]}\)\(^\text{[9]}\) App A] give, for \(p,s,t,q\) as in (23), a constant \(C \in (1, \infty)\) with
\begin{equation}
    C^{-2} \|v\|_{W^{2,p+1}_{p,Θ}} \leq C^{-1} \|v\|_{W^p_{p,Θ}} \leq \|v\|_{W^{s,t,q}_{p,Θ}} \leq C \|v\|_{W^p_{p,Θ}} \leq C^2 \|v\|_{W^{p-1}_{p,Θ}}, \quad \forall v \in C^\infty_0(K).
\end{equation}
It follows that \(C^{r-1}_0(K) \subset W^{s,t,q}_{p,Θ}(K)\) (as a dense subset).

We are finally ready to define our anisotropic space of distributions on \(M\):

Definition 2.9 (Anisotropic Banach space). Let \(Α = \{ V = \{ V_ω \}_{ω \in Ω}, \{ κ_ω : V_ω \rightarrow \mathbb{R}^d \}_{ω \in Ω} \},\) \(α_0 > 0,\) and cone ensembles \(\{ Θ_ω \}_{ω \in Ω}\) admissible for \(\{ g_α \}_{α \geq α_0}\) as be given by Lemma\(^\text{[2]}\)\(^\text{[4]}\). Fix a \(C^\infty\) partition of unity \(\{ θ_ω : \mathbb{R}^d_{\omega} \rightarrow [0, 1] \}_{ω \in Ω}\), subordinate to \(V\), that is, with \(\text{supp } θ_ω \subset V_ω\). For \(p, s, t, q\) as in (23), we put for \(ϕ \in C^\infty(Μ)\) (extending \(θ_ω \circ κ_ω^{-1}\) from \(κ_ω\) to \(R^d\) by zero),
\begin{equation}
    \left\| ϕ \right\|_{W^{s,t,q}_{p,Θ}} := \left( \sum_{ω \in Ω} \int_0^{α_0} \left\| (θ_ω \circ L_{α,V} ϕ) \circ κ_ω^{-1} \right\|_{W^{s,t,q}_{p,θ_ω}}^2 \, dα \right)^{\frac{1}{2}}.
\end{equation}

The Banach space \(W^{s,t,q}_{p,Θ}(Μ)\) is defined to be the completion of \(C^\infty(Μ)\) under \(\| \cdot \|_{W^{s,t,q}_{p,Θ}}\).
We show in Lemma C.1 that the scale $W_{p,t,q}^{s,t,q}(M)$ is an interpolation scale (the proof also shows this for the scale $W_{p,\Theta}^{s,t,q}(K)$). In Lemma C.2 we use mollifiers to approximate the identity.

Note that $W_{p,t,q}^{s,t,q}(M)$ depends on the atlas $\mathcal{A}$, the cone ensembles $\Theta_\omega$, and $\alpha_0$. It follows from [24] and the bound $\sup_{\alpha \in [0, \alpha_0]} \| L_\alpha \varphi \|_{C^{r-1}} \leq C(\alpha_0) \| \varphi \|_{C^{r-1}}$ that $C^{r-1}(M) \subset W_{p,t,q}^{s,t,q}(M)$, as a dense subset. It is not hard to show that $W_{p,t,q}^{s,t,q}(M)$ is contained in the set of distributions of order $r - 1 + d/p$ on $M$. For $p = 2$, the space $W_{p,t,q}^{s,t,q}$ is a Hilbert space since it satisfies the parallelogram law [11] Lemma 15.2

\[ \| \varphi_1 + \varphi_2 \|_{W_{p,t,q}^{s,t,q}}^2 + \| \varphi_1 - \varphi_2 \|_{W_{p,t,q}^{s,t,q}}^2 = 2 \| \varphi_1 \|_{W_{p,t,q}^{s,t,q}}^2 + 2 \| \varphi_2 \|_{W_{p,t,q}^{s,t,q}}^2. \]

Clearly, if $s' \leq s, q' \leq q$, and $t' \leq t$, we have the continuous injection $W_{p,t,q}^{s,t,q}(M) \subset W_{p,t,q}^{s',t',q'}(M)$. Due to the integration over $\| f \|_{C^{r-1}}$ uniformly in $\varphi$, Remark 2.22, 4.27]. Thus, the key fact is that the operator $F_{(\varphi_1, \varphi_2, \varphi')} \in W_{p,t,q}^{s,t,q}(M) \subset W_{p,t,q}^{s',t',q'}(M)$ is compact. Indeed, for every $\epsilon > 0$ there exists $C(\epsilon) < \infty$ such that, for any $v \in W_{p,t,q}^{s,t,q}$,

\[ \| \varphi \|_{W_{p,t,q}^{s',t',q'}} \leq \epsilon \| \varphi \|_{W_{p,t,q}^{s,t,q}} + C(\epsilon) \| \varphi \|_{W_{p,t,q}^{s',t',q'}} \]

This is easy to prove along the lines of [5] Remarks 2.22, 4.27]. Thus,

\[ \| \varphi \|_{W_{p,t,q}^{s',t',q'}} \leq \epsilon \| \varphi \|_{W_{p,t,q}^{s,t,q}} + C(\epsilon) \| \varphi \|_{W_{p,t,q}^{s',t',q'}} \quad \forall \varphi \in W_{p,t,q}^{s,t,q}(M). \]

We now show that $W_{p,t,q}^{s,t,q}(M) \subset W_{p,t,q}^{s',t',q'}(M)$ is compact. For a $C^r$ diffeomorphism $F : K \to F(K)$ and $f \in C^{r-1}(K)$, we introduce the local transfer operator

\[ \mathcal{M}_{\alpha, f} : C^{r-1}(F(K)) \to C^{r-1}(K), \quad \mathcal{M}_{\alpha, f}(v) = f \cdot (v \circ F). \]

The key fact is that the operator $\mathcal{M}_{\alpha, f}$ is bounded for the classical Sobolev norm $W_{p,t,q}^s$ on $\mathbb{R}^d$ if $s \in (-r - 1, -r - 1)$ (apply e.g. the results of [5] Chapter 2], with norm depending only on $\| f \|_{C^{r-1}}$ and $\| F \|_{C^{r}}$. In particular, since $F_{\alpha,\omega'} := \kappa_{\omega'} \circ g_{\omega} \circ \kappa^{-1}_{\omega}$ and $f_{\alpha,\omega'} := (\varphi \circ g_{\omega} \circ \kappa^{-1}_{\omega}) \circ \kappa_{\omega}$ are $C^r$, respectively $C^{r-1}$ (on $K_\omega$) uniformly in $\alpha \in [0, \alpha_0]$, with $\varphi \circ g_{\omega} \circ \kappa^{-1}_{\omega} \equiv 1$, decomposing

\[ (\varphi \circ g_{\omega} \circ \kappa^{-1}_{\omega}) \circ \kappa_{\omega} = (\varphi \circ g_{\omega} \circ \kappa^{-1}_{\omega}) \circ \kappa_{\omega} \circ \kappa^{-1}_{\omega} \circ \kappa_{\omega} \circ \kappa^{-1}_{\omega}, \]

we have, setting $M_{\alpha_0} = \sup_{\omega \omega', \alpha \in [0, \alpha_0]} \| \mathcal{M}_{\alpha,\omega, f_{\alpha,\omega'}} \|_{W_{p,t,q}^s} < \infty$,

\[ \| (\varphi \circ g_{\omega} \circ \kappa^{-1}_{\omega}) \circ \kappa_{\omega} \|_{W_{p,t,q}^s} \leq M_{\alpha_0} \sum_{\omega'} \| (\varphi \circ g_{\omega'} \circ \kappa^{-1}_{\omega'} \circ \kappa_{\omega'} \circ \kappa^{-1}_{\omega'} \circ \kappa_{\omega'} \circ \kappa^{-1}_{\omega'}) \|_{W_{p,t,q}^s}, \forall \varphi, \forall \omega, \forall \alpha \in [0, \alpha_0]. \]

Let now $\varphi_{\omega}$ be a sequence in the unit ball of $W_{p,t,q}^{s,t,q}(M)$. By definition [25] of the norm, for every $m$ there exists $\alpha(m) \in [0, \alpha_0]$ such that

\[ \| (\varphi \circ g_{\omega} \circ \kappa^{-1}_{\omega}) \circ \kappa_{\omega} \|_{W_{p,t,q}^s} \leq \| (\varphi \circ g_{\omega} \circ \kappa^{-1}_{\omega}) \circ \kappa_{\omega} \|_{W_{p,t,q}^s} \leq \frac{1}{\alpha_0}, \forall \omega'. \]

10See e.g. [11] and [24] for relevant results in this context.

17It is useful here that $|\phi_{\alpha}|$ is bounded away from zero.
Thus, using (29), and recalling C from (24),

\[(30) \|(\vartheta_\omega \varphi_m) \circ \kappa^{-1}_\omega \|_{W^s_{K}} \leq M_{a_0} \cdot \sum_{\omega'} \|(\vartheta_\omega' \cdot \mathcal{L}_{\alpha(m),\phi_{\alpha(m)}} \varphi_m) \circ \kappa^{-1}_\omega'\|_{W^s_{K}} \leq M_{a_0} \frac{C \cdot \#\Omega}{\alpha_0}, \forall \omega', \forall m.\]

Assume for a contradiction that there is \(\epsilon > 0\) and, for any \(k_0 \geq 1\), there are \(k, \ell \geq k_0\) with

\[(31) \|\varphi_k - \varphi_{\ell}\|_{W^s_{K}} > \epsilon.\]

By definition, this implies that there exists \(\omega \in \Omega\) with

\[\int_0^{\alpha_0} \|(\vartheta_\omega \cdot \mathcal{L}_{\alpha,V}(\varphi_k - \varphi_{\ell})) \circ \kappa^{-1}_\omega\|_{W^s_{K}}^2 \omega \cdot \#\Omega > \frac{\epsilon^2}{\#\Omega}.\]

Now, using again the key fact, and setting \(M'_{a_0} = \sup_{\omega,\omega',\alpha} \|M_{F_{\alpha_{\omega},f_\omega}}\|_{W^s_{K}}\), we get

\[(32) \leq C_{\alpha_0} \cdot (M'_{a_0})^2 \sum_{\omega'} \|(\vartheta_\omega'(\varphi_k - \varphi_{\ell})) \circ \kappa^{-1}_{\omega'}\|_{W^s_{K}}^2.\]

Since (30) implies that \((\vartheta_\omega'(\varphi_k - \varphi_{\ell})) \circ \kappa^{-1}_{\omega'}\) is a sequence in a bounded subset of \(W^s_{K}(K_M)\) for each \(\omega'\), and since the embedding \(W^s_{K}(K_M) \subset W^s_{K}(K_M)\) is compact, we find \(k_0\) such that (32) is smaller than \(\epsilon^2/\#\Omega\) for all \(k \geq \ell \geq k_0\) and thus the desired contradiction with (31). \(\square\)

3. Properties of the Transfer Operator, the Generator, the Resolvent

3.1. Basic Estimates on the Local Anisotropic Space. The natural ordering \(- < 0 < +\) on \(\{-, +, 0\}\) is compatible with our choice \(s = c(-) \leq q = c(0) \leq t = c(+)\) from (22). Inspired by [9], we introduce the following definition:

**Definition 3.1** (Arrow relation). For \(K \subset \mathbb{R}^d\) compact with nonempty interior, let \(F : K \to \mathbb{R}^d\) be a \(C^r\) cone hyperbolic diffeomorphism from \(\Theta'\) to \(\Theta\) on \(K\). For a covering \(\tilde{\Psi}'\) of \(\Theta'\), set

\[|F|_\tau := \sup_{\chi \in \overline{\tilde{\Psi}'}} |DF(x)|^\eta, \quad |F^{-1}|_\sigma := \sup_{\chi \in \overline{\tilde{\Psi}^{-1}}(K)} |DF^{-1}(x)|^\xi.\]

Fix \(s < 0 < q < t\). For \(n, \ell \geq 0, \) and \(\sigma, \tau \in \{\pm, 0\},\) we say that \((\tau, \ell) \to K (\sigma, n)\) if

\[(34) \quad (2^n \leq |F|_+^q \text{ or } 2^{-q} \leq \|F^{-1}|_\sigma^q\) \text{ and } \sigma \leq \tau \text{ and } |F^{-1}|^{-4} \leq 2^n \leq 2^{q}\|F|_\tau,\]

and we say that \((\tau, \ell) \not\to K (\sigma, n)\) otherwise.

Recalling (17), let \(\tilde{\Psi}_0, \tilde{\Psi}_1 \in C^\infty\) be such that \(\tilde{\Psi}_0|_{\supp \tilde{\Psi}_0} \equiv 1\) and \(\tilde{\Psi}_1|_{\supp \tilde{\Psi}_1} \equiv 1\). Set \(\tilde{\Psi}_n(\xi) := \tilde{\Psi}_1(2^{-n+1}\xi)\) for \(n \geq 2\). With [18] (39), (40), and (41), the following lemma shows the usefulness of the arrow relation:

**Lemma 3.2.** If \(F\) is a \(C^r\) cone hyperbolic diffeomorphism from \(\Theta'\) to \(\Theta\) on \(K\), there exist a covering \(\tilde{\Psi}'\) of \(\Theta'\) and a constant \(C_1 = C_1(F, K) > 0\) such that, setting,

\[\tilde{\Psi}_{\sigma,0}(\xi) := \frac{\chi^s + (\xi)}{3}, \quad \tilde{\Psi}_{\sigma,n}(\xi) := \tilde{\Psi}_n(\xi)\tilde{\Psi}'(\frac{\xi}{|\xi|})^n, \quad \xi \in \mathbb{R}^d, \sigma \in \{\pm, 0\}, n \geq 1.\]

\[\text{In our application below, } |F|_+ < 1 \text{ while } |F^{-1}|_\sigma > 1.\]
we have
\begin{equation}
\inf_{x \in K} d(\sup \Psi_{\sigma,n} - DF(x)^{\tau} \sup \Psi'_{\tau,\ell}) \geq C_1 2^{\max\{n,\ell\}}, \quad \forall (\tau, \ell) \not\rightarrow_K (\sigma, n).
\end{equation}

**Proof.** If $\sigma \leq \tau$, then (36) follows from (34) (without using cone-hyperbolicity): Indeed, if $n \geq \ell$,
\begin{equation}
d(\sup \Psi_{\sigma,n}, DF(x)^{\tau} \sup \Psi'_{\tau,\ell}) \geq 2^{n-1} - 2^{\ell+2}|F|_\tau = 2^{n-1} (1 - 2^{\ell-n+3}|F|_\tau) > 2^{n-2}, \quad \forall x \in K,
\end{equation}
while if $n < \ell$, we have
\begin{equation}
d(\sup \Psi_{\sigma,n} - DF(x)^{\tau} \sup \Psi'_{\tau,\ell}) \geq 2^{\ell-1} (2^{n-\ell} - 2^{3}|F|_\tau) > 2^{\ell+2}|F|_\tau, \quad \forall x \in K,
\end{equation}
If $\sigma > \tau$, then either $\tau = 0$ and $\sigma = +$, or $\tau = -$ and $\sigma \in \{0, +\}$. In both cases, cone-hyperbolicity of $F$ implies
\begin{equation}
\bigcup_{x \in K} (\sup \Psi_{\sigma}) \cap (DF(x)^{\tau} \sup \Psi'_{\tau}) = \{0\},
\end{equation}
which is a trivial intersection of closed cones. Hence there exists a covering $\Phi'$ such that $\bigcup_{x \in K} (\sup \Psi_{\sigma}) \cap (DF(x)^{\tau} \sup \Psi'_{\tau}) = \{0\}$, and (36) holds for suitable $C_1$. \qed

For a $C^r$ diffeomorphism $F : K \rightarrow F(K)$, cone-hyperbolic from $\Theta'$ to $\Theta$ and a covering $\Phi'$ of $\Theta'$ satisfying (34) and $f \in C_0^{r-1}(K)$, recalling the weighted composition operator $M_{F,f}(v) = f \cdot (v \circ F)$ from (29), set, for $a = (a_{\tau,\ell}) \in L_p(\mathbb{R}^d, \ell_2^2)$ (Lemmas 3.3 and 3.4 providing the necessary summability),
\begin{equation}
(Q_{\rightarrow_K}^{\Phi} a)_{\sigma,n} := \Psi_{\sigma,n} \bigoplus_{(\tau, \ell) \rightarrow_K (\sigma, n)} M_{\Phi,f}(a_{\tau,\ell}), \quad (Q_{\not\rightarrow_K}^{\Phi} a)_{\sigma,n} := \Psi_{\sigma,n} \bigoplus_{(\tau, \ell) \not\rightarrow_K (\sigma, n)} M_{\Phi,f}(\Psi'_{\tau,\ell} a_{\tau,\ell}).
\end{equation}
Then, taking $a_{\tau,\ell} := \Psi'_{\tau,\ell}^\Phi v$ for the ensemble $\Theta$, we have
\begin{equation}
\|M_{\Phi,f} v\|_{W^{s,t,q}} = \|Q_{\not\rightarrow_K}^{\Phi} a + Q_{\rightarrow_K}^{\Phi} a\|_{L_p(\mathbb{R}^d, \ell_2^2)}.
\end{equation}

Lemma 3.3 describes the $\rightarrow$ term in the decomposition above. It will give the “contracting” factor $C_\pm$ in Lemma 3.5 for $\sigma = \pm$, while the term with $C_0$ in Lemma 3.5 for $\sigma = 0$ will become compact for the resolvent, see Lemmas 3.11 and 3.6.

**Lemma 3.3 (The Bounded Term).** Fix $p \in (1, \infty)$ and $s, q, t$ as in (21). There exists $C < \infty$, such that for each compact $K \subset \mathbb{R}^d$ with nonempty interior, each $C^r$ cone-hyperbolic diffeomorphism $F : K \rightarrow F(K)$ from $\Theta'$ to $\Theta$, and each covering $\Phi'$ of $\Theta'$,
\begin{equation}
\|Q_{\rightarrow_K}^{\Phi} a\|_{L_p(\mathbb{R}^d, \ell_2^2)} \leq C \max\{|F|_+^{\ell}, |F^{-1}|_{-s}\} \sup_K |f| \cdot \det DF^{-1/p} \cdot \|a\|_{L_p(\mathbb{R}^d, \ell_2^2)}
\end{equation}
\begin{equation}
+ C |F|_+^{\ell} \sup_K |f| \cdot \det DF^{-1/p} \cdot \left\|\left(\sum_{\ell} 4^{\ell q} a_{0,\ell}^2\right)^{1/2}\right\|_{L_p} \cdot \forall f \in C_0^{r-1}(K).
\end{equation}

**Proof.** Recall (22). There exists $C < \infty$, independent of $F$, such that
\begin{equation}
\sum_{n: (\tau, \ell) \not\rightarrow_K (\sigma, n)} 2^{c(\sigma)n-c(\tau)\ell} = \sum_{n: (\tau, \ell) \not\rightarrow_K (\sigma, n)} 2^{c(\sigma)-c(\tau)n+c(\tau)(n-\ell)} \leq \sum_{n: (\tau, \ell) \not\rightarrow_K (\sigma, n)} 2^{c(\tau)(n-\ell)}
\end{equation}
\begin{equation}
\leq C \max\{|F|_+^{\ell}, |F^{-1}|_{-s}\}, \quad \forall (\tau, \ell), \quad \forall (\sigma, \tau) \neq (0, 0).
\end{equation}
Similarly, up to taking a larger constant \( C < \infty \), we have
\[
\sum_{\ell : (\tau, \ell) \rightarrow K(\sigma, n)} 2^{c(\sigma)n-c(\tau)\ell} \leq C \max\{|F|_{+}, |F^{-1}|_{\ell}\}^\ell, \forall (\sigma, n), \tau, \text{ with } (\sigma, \tau) \neq (0, 0).
\]

Theorem 2.6 applied to \( \mathcal{H}_1 = \mathcal{H}_2 = \ell_2^2 \) and (\( Qb \))\( \sigma,n,\xi \) = \( \Psi_{\sigma,n}(\xi) b_{\sigma,n}(\xi) \) gives \( D_1 \) such that
\[
\|Q_{\sigma,n}^{\alpha} a\|_{L^p(S^d, \ell_2^2)} \leq D_1 \left\| \left( \sum_{\sigma,n} 4^{c(\sigma)n} \sum_{(\tau, \ell) \rightarrow K(\sigma, n)} |\mathcal{M}_{F,\ell} a_{\tau,\ell}| \right)^\frac{1}{2} \right\|_{L_p}, \forall f, F, a.
\]

Set \( \lambda_{F,s,t} = \max\{|F|_{+}, |F^{-1}|_{\ell}\}^\ell \). By Cauchy–Schwarz, and (39)–(40), we find \( D_2, D_3 \) such that
\[
3D_1 \sum_{(\tau, \ell) \neq (0,0)} \left\| \left( \sum_{n: j(\tau, j) \rightarrow K(\sigma, n)} 2^{c(\sigma)n-c(\tau)j} \sum_{\ell : (\tau, \ell) \rightarrow K(\sigma, n)} 2^{c(\sigma)n+c(\tau)\ell} |\mathcal{M}_{F,\ell} a_{\tau,\ell}| \right)^\frac{1}{2} \right\|_{L_p}
\leq D_2 \sum_{(\tau, \ell) \neq (0,0)} \left\| \left( \lambda_{F,s,t} \sum_{n: j(\tau, j) \rightarrow K(\sigma, n)} 2^{c(\sigma)n+c(\tau)\ell} |\mathcal{M}_{F,\ell} a_{\tau,\ell}| \right)^\frac{1}{2} \right\|_{L_p}
= D_2 \sum_{(\tau, \ell) \neq (0,0)} \left\| \left( \lambda_{F,s,t} \sum_{n: j(\tau, j) \rightarrow K(\sigma, n)} 4^{c(\tau)\ell} |\mathcal{M}_{F,\ell} a_{\tau,\ell}| \right)^\frac{1}{2} \right\|_{L_p}
\leq D_3 \sup_{K} |f| \det DF^{-1/p} \sum_{\sigma, \tau} \left\| \left( \lambda_{F,s,t} \sum_{\ell} 4^{c(\tau)\ell} |a_{\tau,\ell}| \right)^\frac{1}{2} \right\|_{L_p}, \forall f, F, a.
\]

Finally, since there exists \( C_{00} < \infty \), independent of \( F \), such that
\[
\sum_{n: (\ell, n) \rightarrow K(0, n)} 2^{c(0)n-\ell} \leq C_{00} |F|_{0}^{2}, \forall \ell, \quad \text{and} \quad \sum_{\ell : (\ell, n) \rightarrow K(0, n)} 2^{c(0)n-\ell} \leq C_{00} |F|_{0}^{2}, \forall n,
\]
we find \( D_4 \) such that for all \( F, f, \) and \( a \)
\[
\left\| \left( \sum_{n} 4^{c(0)n} \sum_{(\ell, n) \rightarrow K(0, n)} |\mathcal{M}_{F,\ell} a_{\tau,\ell}| \right)^\frac{1}{2} \right\|_{L_p} \leq D_4 \sup_{K} |f| \det DF^{-1/p} |F|_{0}^{2} \left\| \left( \sum_{\ell} 4^{c(\ell)|a_{0,\ell}|} \right)^\frac{1}{2} \right\|_{L_p}.
\]

We next bound the other term in the decomposition (38) of \( \mathcal{M}_{F,\ell} \). For this, we need the following strengthening of condition (21):
\[
t - (r - 1) < s < 0 < q < t.
\]

**Lemma 3.4 (The Compact Term).** Fix \( p \in (1, \infty) \), and fix \( s, q, t \) as in (21). Let \( s' < s, q' < q \) and \( t' < t \) satisfy \( t - (r - 1) < s' < 0 < q' < t' \). Let \( F \) be a \( C' \) cone-hyperbolic diffeomorphism from \( \Theta' \) to \( \Theta \) on \( K \), let \( \tilde{\Phi}' \) be given by Lemma 3.2, and let \( f \in C_{0}^{-1}(K) \). Then there exists \( C(F, f) < \infty \) such that
\[
\|Q_{\sigma,n}^{\alpha} a\|_{L^p(S^d, \ell_2^2)} \leq C(F, f) \cdot \|a\|_{\ell_2^2 \rightarrow K(\sigma, n)}, \forall a.
\]

**Proof.** We shall use Lemma 3.2 and integration by parts, along the lines of [9] pp. 144–147].

Write, for \( x \in \mathbb{R}^d \) and \( (\tau, \ell) \rightarrow K(\sigma, n) \),
\[
(J^\tau, \ell_{\sigma,n})(x) := \frac{(2\pi)^d}{2^{(n+\ell)d}} \int_{\mathbb{R}^d} \mathcal{M}_{F,\tau} \tilde{\Phi}_{\tau,\ell}^{\alpha} b_{\tau,\ell}(x)
= \int_{y \in K} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} e^{i2^n(x-y) \cdot \tilde{\xi}} e^{2^n \tilde{\eta}(F(y)-w)} \Psi_{\sigma,1}(\tilde{\xi}) \tilde{\Phi}_{\tau,1}(\tilde{\eta}) f(y)a_{\tau,\ell}(w) dw d\tilde{\xi} d\tilde{\eta} dy,
\]
where we used the change of variables \( \tilde{\xi} = 2^{-n} \xi \) and \( \tilde{\eta} = 2^{-\ell} \eta \). Integrating by parts \( r - 1 \) times (see Lemmas \( \ref{lemma:1} \) \( \ref{lemma:2} \)) in \( y \), and using \( \ref{eq:36} \), we rewrite \((J_{\sigma,n}^r a(x))\) as

\[
\int_{y \in K} \int_{\mathbb{R}^d} e^{2\alpha_n \tilde{\xi}(x-y)} e^{2\beta_n \tilde{\eta} f(y) - \tilde{\eta} w} \Psi_{\sigma,1}(\tilde{\xi}) \tilde{\Psi}_{r,1}(\tilde{\eta}) \frac{f_{r-1,n,\ell}(\tilde{\eta}, \tilde{\xi}, y)}{2^{\max\{ n, \ell \} (r-1)}} a_{r,\ell}(w)dw \, d\tilde{\xi} \, d\tilde{\eta} \, dy,
\]

where all partial derivatives of \( f_{r-1,n,\ell}(\tilde{\eta}, \tilde{\xi}, y) \) with respect to \( \tilde{\eta} \) and \( \tilde{\xi} \) are bounded by a constant \( C_2(F, f) \) uniformly in \( n, \ell \), and \((\tilde{\xi}, \tilde{\eta}, y) \in \supp \Psi_{\sigma,1} \times \supp \tilde{\Psi}_{r,1} \times K \). Define \( b : \mathbb{R}^d \to [0,1] \) by

\[
b(y) := 1 \text{ if } |y| \leq 1, \quad b(y) := |y|^{-d-1} \text{ if } |y| > 1.
\]

If \(|x-y|^{2n} > 1\) we integrate \((d+1)\)-times by parts in \( \tilde{\xi} \), and if \(|w - F(y)|^{2\ell} > 1\) we integrate \((d+1)\)-times by parts in \( \tilde{\eta} \). Hence, we arrive at the following formula for \((J_{\sigma,n}^r a(x))\):

\[
\int_{y \in K} \int_{\supp \Psi_{\sigma,1}} \int_{\supp \tilde{\Psi}_{r,1}} \int_{\mathbb{R}^d} f_{r-1,n,\ell}(\tilde{\xi}, \tilde{\eta}, y) b_n(x-y) b_\ell(w - F(y)) a_{r,\ell}(w) \, dw \, d\tilde{\xi} \, d\tilde{\eta} \, dy,
\]

where \( b_n(w) = b(2^nw) \) \((m \geq 0)\), and \( f_{r-1,n,\ell}(\tilde{\xi}, \tilde{\eta}, y) \) is uniformly bounded by \( C_2'(F, f) \). Thus, there exists \( C_3 < \infty \) such that for all \( x \in \mathbb{R}^d \)

\[
|J_{\sigma,n}^r a(x)| \leq C_3 C_2(F, f) 2^{-\max\{ n, \ell \} (r-1)} \left( b_n \ast (b_\ell \circ F) \ast |a_{r,\ell}| \right)(x), \text{ if } (\tau, \ell) \not\succ K (\sigma, n).
\]

Since \( r - 1 > t - s' > 0 \) and \( c(\sigma) \leq t, c'(\tau) \geq s' \), there exists \( \epsilon > 0 \) such that for all \( \sigma \) and \( \tau \),

\[
2(c(\sigma)+\epsilon)n-c'(\tau)\ell - \max\{n,\ell\}(r-1) \leq 2^{(t+\epsilon)n-s'\ell - \max\{n,\ell\}(r-1)} \leq 2^{-\ell}, \forall n \geq 1, \forall \ell \geq 1.
\]

We can assume \( n \cdot \ell \neq 0 \) since if \( n = 0 \) or \( \ell = 0 \) then \( \xi \) or \( \eta \) is bounded. (By Footnote \[\ref{footnote:18}\] we have \( n \cdot \ell \neq 0 \) in our application.) Hence starting with the triangle inequality, then using \([\ref{eq:43}]\), we find \( C(\epsilon) < \infty \) such that for all \( a \)

\[
\|Q_{\gamma \prec \ell}^a a\|_{L_p(\mathbb{R}^d, \mathbb{E}_2')} \leq C(\epsilon) \sup_{\sigma, n} 2^{c(\sigma)+\epsilon} n A_{\Psi_{\sigma,0}} \sum_{(\tau, \ell) \succ K (\sigma, n)} \mathcal{M}_{F, f} \left( \tilde{\Psi}_{r, \ell}^0 a_{r, \ell} \right) \|_{L_p(\mathbb{R}^d)}
\]

\[
\leq C(\epsilon) \sup_{\sigma, n} \sum_{(\tau, \ell) \succ K (\sigma, n)} 2^{c(\sigma)+\epsilon} n - c'(\tau)\ell 2^{c'(\tau)\ell} \| \Psi_{\sigma,n} \mathcal{M}_{F, f} (\tilde{\Psi}_{r, \ell}^0 a_{r, \ell}) \|_{L_p}
\]

\[
= C(\epsilon) \frac{2^{d(n+\epsilon)}}{(2\pi)^{2d}} \sup_{\sigma, n} \sum_{(\tau, \ell) \succ K (\sigma, n)} 2^{c(\sigma)+\epsilon} n - c'(\tau)\ell 2^{c'(\tau)\ell} \| J_{\sigma,n}^r a \|_{L_p}.
\]

Applying Young’s inequality (with \( \|b_n\|_{L_1} = 2^{-dn} \|b\|_{L_1} \), for \( m = \ell \) and \( n \)), and \([\ref{eq:44}]\) then yields finite constants \( C_4, C_5, C_6 \) (depending on \( \epsilon, F \) and \( f \)) such that, again for all \( a \),

\[
\|Q_{\gamma \prec \ell}^a a\|_{L_p(\mathbb{R}^d, \mathbb{E}_2')}
\leq C_4 \sup_{\sigma, n} \sum_{\tau, \ell} 2^{c(\sigma)+\epsilon} n - c'(\tau)\ell - \max\{n,\ell\}(r-1) 2^{n+\ell} 2^{c'(\tau)\ell} \| b_n \ast (b_\ell \circ F) \ast a_{r,\ell} \|_{L_p}
\]

\[
\leq C_5 \sum_{\tau, \ell} 2^{-\ell} 2^{c'(\tau)\ell} \| a_{r,\ell} \|_{L_p}
\]

\[
\leq C_6 \sup_{\tau, \ell} 2^{c'(\tau)\ell} \| a_{r,\ell} \|_{L_p} \leq C(F, f) \| a \|_{L_p(\mathbb{R}^d, \mathbb{E}_2')},
\]

We end this subsection with a bound on the transfer operator \( \mathcal{M}_{F, f} \) from \([\ref{eq:28}]\).
Lemma 3.5 (Bounding the Local Transfer Operator). Let $K \subset \mathbb{R}^d$ be compact with nonempty interior. Fix $p \in (0, 1)$ and fix $s, q, t$ as in (22). Let $t - (r - 1) < s' < 0 < q' \leq t'$ satisfy $s' < s$, $q' < q$ and $t' < t$. Then there exists $C < \infty$ such that for any cone-hyperbolic $C^\tau$-diffeomorphism $F$ from $\Theta'$ to $\Theta$ on $K$, taking the covering $\tilde{\Psi}$ given by Lemma 3.2 we have for any $f : \mathbb{R}^d \to \mathbb{C}$ in $C_0^{-1}(K)$ and all $v \in W_p^{s,t,q}$

$$\|M_{F,v}\|_{W_p^{s,t,q}} \leq C_0(F,f)\|v\|_{W_p^{s',t',q'}_{p,\theta'},F(K)} + C_0(F,f)\|v\|_{W_p^{s',t',q'}_{p,\theta'},F(K)},$$

where $C_0(F,f)$ is from Lemma 3.4 and, recalling $|F|_+, |F|_0$, and $|F|_-$ from (33),

$$C_0(F,f) = \sup_K \frac{|f|}{|\det DF|^\frac{1}{p}} \cdot \max\{1, |F|_0^\alpha\}, \quad C_\pm(F,f) = \sup_K \frac{|f|}{|\det DF|^\frac{1}{p}} \cdot \max\{|F|_+, |F|_0^{-1}|s|\}.$$ 

Proof. Let $v \in W_p^{s,t,q}$ be as in Definition 3.1 be in decomposition (38) gives

$$\|M_{F,v}\|_{W_p^{s,t,q}} = \|Q_{\gamma +,F}^0 a + Q_{\gamma -,F}^0 a\|_{L_p(R^d, \ell^q_2)} \leq \|Q_{\gamma +,F}^0 a\|_{L_p(R^d, \ell^q_2)} + \|Q_{\gamma -,F}^0 a\|_{L_p(R^d, \ell^q_2)}.$$ 

Therefore, (45) is cone-hyperbolic from $\Theta'$ and $\Theta$ on $K$ and fix $v \in W_p^{s,t,q}$.

We conclude using Lemmas 3.3 and 3.4.

3.2. Lasota–Yorke Type Bounds on the Transfer Operator. For $s < 0 < t$ and $\alpha > 0$, set

$$\lambda^{(s,t,\alpha)}(x) := \max\{\|(Dg_{-\alpha})^\tau|_{E^+_+,x}|^t, \|(Dg_{\alpha})^\tau|_{E^+_{-g_{-\alpha}}}|^t\}, \quad x \in M.$$

There exists $C' < \infty$ such that $\sup_x \lambda^{(s,t,\alpha)}(x) \leq C'\theta_{\min(|t,s|)}$ by property (7).

Lemma 3.6 (Bounding the Transfer Operator). Fix $p \in (1, \infty)$ and fix $s, q, t$ as in (22). Let $t - (r - 1) < s' < s < 0 < q < t' < t$. There exist $A = A(X,V) < \infty$ and $C = C(X,V) < \infty$, such that for all $\varphi \in W_p^{s,t,q}(M)$

$$\|L_{\alpha,V}\varphi\|_{W_p^{s,t,q}} \leq C\alpha - \frac{1}{\beta} \cdot \lambda^{(s,t,\alpha)}(x) L_{\infty}\varphi\|_{W_p^{s,t,q}}, \forall \alpha \geq 0.$$

The bound in the above lemma shows that $L_{\alpha,V}$ is an operator semigroup on $W_p^{s,t,q}(M)$. As usual for flows, however, it is not a true Lasota–Yorke bound since $W_p^{s,t,q}(M)$ is not compactly embedded in $W_p^{s',t',q'}(M)$ if $q' = q$. However, using Lemma 3.11 we will prove in Theorem 3.8 that the resolvent $(z - V - X)^{-1}$ satisfies a Lasota–Yorke bound.

Proof. If $\alpha > \alpha_0$, using $f_0^{\alpha} = f_0^{\alpha - \alpha} + f_0^{\alpha_0 - \alpha}$, we find $\|L_{\alpha,V}\varphi\|_{W_p^{s,t,q}} \leq \|\varphi\|_{W_p^{s,t,q}} + \|L_{\alpha_0,\phi_0,0}\varphi\|_{W_p^{s,t,q}}$.

We may assume from now on that $\alpha \geq \alpha_0$. Recalling the charts $\kappa_{\omega} : V_{\omega} \to \mathbb{R}^d$, the partition of unity $g_{\omega}$, and the cone systems $\Theta_{\omega}$ (from Lemma 2.1), above Definition 2.9 write, as before, $V_{\alpha,\omega,\omega'} = V_{\omega} \cap g_{\omega}(V_{\omega'}), \text{ and } F_{-\alpha,\omega,\omega'}(x) = \kappa_{\omega'} \circ g_{-\alpha} \circ \kappa_{\omega}^{-1}(x), x \in \kappa_{\omega}(V_{\alpha,\omega,\omega'}), \alpha \geq \alpha_0, \omega, \omega' \in \Omega.$

Since $\alpha \geq \alpha_0$, each $F_{-\alpha,\omega,\omega'}$ is cone-hyperbolic from $\Theta_{\omega}$ to $\Theta_{\omega}$ on $\kappa_{\omega}(V_{\alpha,\omega,\omega'})$.

The intersection multiplicity of a family of sets is the maximal number of sets having nonempty intersection, while the intersection multiplicity of a family of functions is the intersection multiplicity of the family of the supports of the functions.
We claim that there exists an integer $\nu_d \geq 2$, depending only on $d$, such that the following holds: There exist $C = C_{\alpha_0} < \infty$ and, for each $\alpha \geq \alpha_0$, a finite refinement $\mathcal{W}_\alpha = \{W_{\alpha, \vec{\omega}}\}_{\vec{\omega} \in \Omega_\alpha}$ of $\mathcal{V}_\alpha = \{V_{\alpha, \vec{\omega}}\}_{(\omega, \vec{\omega}) \in \Omega^2}$, of intersection multiplicity at most $\nu_d$, such that

\begin{equation}
\sup_W |\phi_\alpha \cdot | \det D g_{-\alpha}|^{-\frac{1}{\nu_d}} | \leq C \inf_W |\phi_\alpha \cdot | \det D g_{-\alpha}|^{-\frac{1}{\nu_d}} |, \ \forall W \in \mathcal{W}_\alpha.
\end{equation}

Indeed, since there exists $K_\alpha < \infty$ such that $\sup_{t \in [0, \alpha]} d((g_{-\beta}(x), g_{-\beta}(y)) \leq K_\alpha d(x, y)$, while

$\log \phi_\alpha = \int_0^\alpha V(g_{-\beta}(x)) d\beta$, and (noting that $\alpha - \alpha_0(\alpha/\alpha_0) \in [0, \alpha_0])$

\[\log |\det D g_{-\alpha}| = \log |\det D g_{-(\alpha - \alpha_0(\alpha/\alpha_0))}| \circ g_{-((\alpha/\alpha_0) - 1)\alpha_0} + \sum_{\ell = 0}^{[\alpha/\alpha_0] - 1} \log |\det D g_{-\alpha}| \circ g_{-\ell \alpha_0},\]

where $V$ and $|\det D g_{-\alpha}|, \ |\det D g_{-(\alpha - \alpha_0(\alpha/\alpha_0))}|$ are uniformly continuous on $M$ (they are in fact $\gamma$-Hölder for $\gamma = \min\{r - 1, 1\}$), there exists a finite refinement $\tilde{\mathcal{V}}_\alpha$ of $\mathcal{V}_\alpha$ such that (46) holds for all $W \in \tilde{\mathcal{V}}_\alpha$. A finite refinement $\mathcal{W}_\alpha$ of $\tilde{\mathcal{V}}_\alpha$ satisfying the claimed intersection multiplicity bound can then be obtained e.g. by covering $M$ with $d$-dimensional balls of radius the Lebesgue number of $\tilde{\mathcal{V}}_\alpha$ centered on an appropriate lattice, see e.g. [5] Footnote 19 p. 46. (Note that the cardinality of $\tilde{\mathcal{V}}_\alpha$ or $\mathcal{W}_\alpha$ is immaterial in view of the use of the reconstitution Lemma 11.2 below.) Finally, fix a $C^r$ partition of unity $\{\vartheta_{\alpha, \vec{\omega}}\}_{\vec{\omega} \in \Omega_\alpha}$ of $M$, subordinate to the cover $\mathcal{W}_\alpha$, with intersection multiplicity at most $\nu_d$.

After these preliminaries, we perform the estimate: Let $\varphi \in W_{p, t, q}^r(M)$, by definition, we have

$$
\|\mathcal{L}_{\alpha, \ell, \varphi}\|_{W_{p, t, q}^r}^2 \leq \#\Omega \int_0^{\alpha_0} \|((\vartheta_{\omega} \cdot (\mathcal{L}_{\alpha', \ell, \varphi} \circ \mathcal{L}_{\alpha, \ell, \varphi})) \circ \kappa_\omega^{-1} \|_{W_{p, t, q}^r}^2 \, d\alpha'.
$$

Next, using $\mathcal{L}_{\alpha', \ell, \varphi} \circ \mathcal{L}_{\alpha, \ell, \varphi} = \mathcal{L}_{\alpha', \ell, \varphi}$, and setting $\varphi_{\alpha'} = \mathcal{L}_{\alpha', \ell, \varphi}$, we find

$$
\|((\vartheta_{\omega} \cdot (\mathcal{L}_{\alpha', \ell, \varphi}) \circ \mathcal{L}_{\alpha, \ell, \varphi})) \circ \kappa_\omega^{-1} \|_{W_{p, t, q}^r}^2 = \|\sum_{\omega' \in \Omega} \sum_{\vec{\omega} \in \Omega_\alpha} (\vartheta_{\omega} \vartheta_{\omega', \vec{\omega}} \cdot \phi_\alpha) \circ \kappa_\omega^{-1} \cdot (\vartheta_{\omega} \vartheta_{\omega', \vec{\omega}} \cdot \varphi_{\alpha'}) \circ \kappa_\omega^{-1} \circ F_{-\alpha, \omega', \vec{\omega}} \|_{W_{p, t, q}^r}^2
\leq C \nu_d \left(\frac{1}{d}ight)^{p-1} \max_{\omega' \in \Omega} \sum_{\vec{\omega} \in \Omega_\alpha} \|((\vartheta_{\omega} \vartheta_{\omega', \vec{\omega}} \cdot \phi_\alpha) \circ \kappa_\omega^{-1} \cdot (\vartheta_{\omega} \vartheta_{\omega', \vec{\omega}} \cdot \varphi_{\alpha'}) \circ \kappa_\omega^{-1} \circ F_{-\alpha, \omega', \vec{\omega}} \|_{W_{p, t, q}^r}^p \right)^{1/p}
$$

(47)

using the fragmentation Lemma 11.1. By Lemma 3.5 the term in the last line of (47) is bounded by

(48)

$$
\widetilde{C}_{0, \alpha}(X, \mathcal{W}_\varphi) \max_{\omega' \in \Omega} \|((\vartheta_{\omega'} \varphi_{\alpha'}) \circ \kappa_\omega^{-1} \|_{W_{p, t, q}^r}^2,
$$

for $\widetilde{C}_{0, \alpha}(X, \mathcal{W}_\varphi) < \infty$. Remark 2.5 gives systems $\Theta'_\omega < \Theta_\omega$ (independent of $\alpha$) such that $F_{-\alpha, \omega}$ is cone-hyperbolic from $\Theta'_\omega$ to $\Theta_\omega$ on $\kappa_\omega(V_{\alpha, \omega'})$. For $\alpha \geq \alpha_0$ and $\vec{\omega} \in \Omega_\alpha$, let $\vec{\vartheta}_{\alpha, \vec{\omega}} : M \to [0, 1]$
be $C^r$, supported in $W_{\alpha,\tilde{\omega}}$, and such that

$$f_{\alpha,\tilde{\omega}} = (\theta_\omega \tilde{\theta}_{\alpha,\tilde{\omega}} \phi_\alpha) \circ \kappa_\omega^{-1}, \quad \tilde{\theta}_{\alpha,\tilde{\omega}} = (\theta_\omega \theta_{\alpha,\tilde{\omega}}) \circ \kappa_\omega^{-1} \circ F_{-\alpha,\omega,\tilde{\omega}}^{-1}.$$ 

Then, Lemma \[5.5\] gives $C_p < \infty$ and $C_0,\alpha(X,V) < \infty$ such that each term in the sum on the second-to-last line of \[47\] is bounded by

$$C_p C_\pm (F_{-\alpha,\omega,\tilde{\omega}}, f_{\alpha,\tilde{\omega}}) \|\tilde{\theta}_{\alpha,\tilde{\omega}} \cdot ((\theta_\omega \cdot \varphi_\alpha') \circ \kappa_\omega^{-1})\|_{W_p,s,t,q}^p + \tilde{C}_0,\alpha(X,V) \|\tilde{\theta}_{\alpha,\tilde{\omega}} \cdot \varphi_\alpha' \circ \kappa_\omega^{-1}\|_{W_p,s',t',q}^p.$$

Due to the strict inequality between cone ensembles, the reconstitution Lemma \[B.2\] bounds the $p$th root of the sum of \[50\] over $\tilde{\omega}$, uniformly in $\alpha$. Recalling \[18\], taking the square, the maximum over $\omega$, and integrating over $\alpha$, we find $C < \infty$ and $C' = C'_\alpha(X,V) < \infty$ such that

$$\|\mathcal{L}_{\alpha,V}(\varphi)\|_{W_p,s,t,q}^p \leq C' \|\varphi\|_{W_p,s',t',q}^p + \tilde{C} \max (C_p (F_{-\alpha,\omega,\tilde{\omega}}, f_{\alpha,\tilde{\omega}}))^2 \|\varphi\|_{W_p,s,t,q}^2, \forall \alpha \geq \alpha_0.$$

We next estimate $C_p (F_{-\alpha,\omega,\tilde{\omega}}, f_{\alpha,\tilde{\omega}})$. By construction of $\Theta_\omega$ in the proof of Lemma \[2.4\] and since the covering $\Phi'$ from Lemma \[5.3\] used in Lemma \[5.5\] can be taken such that $\text{supp} \Phi_\sigma$ is bounded away from $E^*_\tau$ (in charts) if $\tau \neq \sigma$, there exists $C < \infty$ such that, recalling \[43\], we have for all $\alpha \geq \alpha_0$, all $\omega$, $\omega'$, and all $\tilde{\omega} \in \Omega_{\alpha}$, setting $K_{\alpha,\tilde{\omega}} = \kappa_\omega(\text{supp} \tilde{\theta}_{\alpha,\tilde{\omega}})$,

$$\|F_{-\alpha,\omega,\tilde{\omega}} + K_{\alpha,\tilde{\omega}} \|_{E^*_\alpha(x)} \leq C \sup_{x \in K_{\alpha,\tilde{\omega}}} \|\text{Dg}_\alpha\|_{E^*_\alpha(x)}^r \|E^*_\alpha + \text{D}g_\alpha\|_{E^*_\alpha} \leq C \sup_{x \in K_{\alpha,\tilde{\omega}}} \|\text{Dg}_\alpha\|_{E^*_\alpha} \|E^*_\alpha - \text{Dg}_\alpha\|_{E^*_\alpha}.$$

Thus, using \[46\] and $\inf |\psi_1| \sup |\psi_2| \leq \sup |\psi_1 \psi_2|$ for continuous $\psi_1$, $\psi_2$, we find $C < \infty$ such that

$$C_p (F_{-\alpha,\omega,\tilde{\omega}}, f_{\alpha,\tilde{\omega}}) \leq C \sup_{W \in \mathcal{W}_{\alpha}} \|\phi_\alpha \cdot \det \text{Dg}_\alpha^{-1/2} \cdot \sup_{W} |\lambda(s,t,\alpha)|\|
\leq \tilde{C} \max_{W \in \mathcal{W}_{\alpha}} \left(\inf_{W} \left|\frac{\phi_\alpha}{\det \text{Dg}_\alpha} \right| \cdot \sup_{W} |\lambda(s,t,\alpha)|\right) \leq \tilde{C} \sup_{W} \left|\frac{\phi_\alpha}{\det \text{Dg}_\alpha} \right| \lambda(s,t,\alpha), \forall \alpha \geq \alpha_0.$$

In view of \[51\], we have proved the lemma.

\[\square\]

Strong continuity suffices (it is not necessary \[15\]) to show that $X + V$ is the generator of $\mathcal{L}_{\alpha,V}$.

**Lemma 3.7** (Strong Continuity. Domain of the Generator $X + V$). Let $p \in (1,\infty)$ and fix $s$, $q$, $t$ as in \[12\]. The family $\{\mathcal{L}_{\alpha,V}\}_{\alpha \geq 0}$ of bounded operators on $W_{p}^{s,t,q}(M)$ forms a strongly continuous semigroup. The operator of this semigroup is $X + V : D(X + V) \to W_{p}^{s,t,q}(M)$, which is closed on its (dense) domain $D(X + V) \subseteq W_{p}^{s,t,q}(M)$. Moreover, if $q < r - 2$ or if $\phi_\alpha$ is $C^r$ in the flow direction, setting $D_{r-1} := C^{r-1}(M)$ if $q < r - 2$, and otherwise

$$D_{r-1} := C^{r-1}(M) = \{\varphi \in C^{r-1}(M) \mid \varphi \text{ is } C^r \text{ in the flow direction}\},$$

$D(X + V)$ contains $D_{r-1}$ as a dense subset for the graph norm $\|\cdot\|_{W_{p}^{s,t,q}(M)} + \|(X + V)(\cdot)\|_{W_{p}^{s,t,q}(M)}$. 

\[\square\]As observed in \[15\] strong continuity implies that the completion $\mathcal{D}$ of $\mathcal{D}_0 = \{ \int_0^1 \mathcal{L}_{\alpha,V} \varphi \, d\alpha \mid \varphi \in W_{p}^{s,t,q}(M), \beta > 0 \}$ under $\|\cdot\|_{W_{p}^{s,t,q}(M)}$ is a dense subset of $W_{p}^{s,t,q}(M)$, so that $\mathcal{D} = W_{p}^{s,t,q}(M)$. Clearly, $\mathcal{L}_{\alpha,V}(\mathcal{D}_0) \subset \mathcal{D}_0$ and $\mathcal{D}_0 \subset D(X + V)$. Thus, $\mathcal{D}_0$ is a dense subset of $D(X + V)$ for the graph norm, without any conditions on $q$ or $\phi_\alpha$. 

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\[\text{HOROCYCLE AVERAGES AND TRANSFER OPERATORS} 17\]
Proof. To establish strong continuity, it suffices to show \( \lim_{\alpha \to 0} \| \mathcal{L}_\alpha \varphi - \varphi \|_{W^{s,t,q}_p(M)} = 0 \) for all \( \varphi \in W^{s,t,q}_p(M) \) [21 Proposition I.1.3]. Lemmas 3.6 and 2.8 give \( C < \infty \) such that \( \| \mathcal{L}_{\alpha,V} \varphi \|_{W^{s,t,q}_p} \leq C \| \varphi \|_{W^{s,t,q}_p} \) for all \( \alpha \in [0,1] \). By density of \( C^\infty(M) \), for every \( \epsilon > 0 \) there is \( \tilde{\varphi} = \tilde{\varphi}_\epsilon \in C^{r-1}(M) \) such that \( \| \varphi - \tilde{\varphi} \|_{W^{s,t,q}_p} \leq \epsilon \). Therefore,

\[
\| \mathcal{L}_{\alpha,V} \varphi - \varphi \|_{W^{s,t,q}_p} \leq \| \mathcal{L}_{\alpha,V} (\varphi - \tilde{\varphi}) \|_{W^{s,t,q}_p} + \| \varphi - \tilde{\varphi} \|_{W^{s,t,q}_p} + \| \mathcal{L}_{\alpha,V} \tilde{\varphi} - \tilde{\varphi} \|_{W^{s,t,q}_p} \\
\leq (C + \epsilon) \| \mathcal{L}_{\alpha,V} \tilde{\varphi} - \tilde{\varphi} \|_{W^{s,t,q}_p}, \ \forall \epsilon > 0, \forall \alpha \in [0,1].
\]

Since \( \tilde{\varphi} \in C^{r-1}(M) \) (if \( 0 < q < r - 2 \) then the argument can be adapted to \( \tilde{\varphi} \in C^{r-1} \)) we have \( \| \mathcal{L}_{\alpha,V} \tilde{\varphi} - \tilde{\varphi} \|_{W^{s,t,q}_p} \) exists for \( \alpha \to 0 \) and, if \( \mathcal{L}_{\alpha,V} \tilde{\varphi} \in C^{r-1}(M) \), the final claim thus follows from [21, Proposition II.1.7], since \( \mathcal{L}_{\alpha,V}(D_{\epsilon-1}) \subset D_{\epsilon-1} \) and \( \lim_{\alpha \to 0} \alpha^{-1}(\mathcal{L}_{\alpha,V} \varphi - \varphi) \) exists for \( \varphi \in D_{\epsilon-1} \) in the two cases considered. \( \Box \)

3.3. Lasota–Yorke Bounds for the Resolvent. Discrete Spectrum of \( X + V \). Recall \( \lambda(s,t,\alpha) \) from [31]. We set (the limit below exists and is finite by superadditivity)

\[
\lambda_{\min}^{s,t,p}(X,V) := \lim_{\alpha \to \infty} \frac{1}{\alpha} \log \left| \phi_{\alpha} \right| \det Dg_{-\alpha}^{-1} \| \mathcal{L}_{\alpha,V} \varphi \|_{L^\infty(M)}.
\]

Recalling \( A(X,V) \) from Lemma 3.6 we may and shall replace \( A(X,V) \) by max\( \{A(X,V), \lambda_{\min}\} \) from now on. The following theorem will furnish an essential spectral bound for \( X + V \):

**Theorem 3.8** (Lasota–Yorke Inequality for the Resolvent). Let \( p \in (1,\infty) \) and let \( s, t, q \) be as in [22]. Let \( t - (r - 1) < s' < q < t' \) satisfy \( s' < s, t' < t, \) and \( q - 1 < q' < q \). For any \( \epsilon > 0 \) there exists \( C < \infty \) such that for all \( \delta > 0 \), all \( n \in \mathbb{N} \), all \( \varphi \in W^{s,t,q}_p(M) \), recalling our notation \( R_z = (z - X - V)^{-1} \),

\[
\delta \| R_z \| \leq \frac{C(z \in |z| + 1)(R_z - A(X,V))^n \| \varphi \|_{W^{s',t',q'}_p}}{(R_z - \epsilon - \lambda_{\min}^{s,t,p}(X,V))^n} + \frac{C}{(R_z - \epsilon - \lambda_{\min}^{s,t,p}(X,V))^n} \| \varphi \|_{W^{s,t,q}_p}.
\]

The above theorem implies that the spectral radius of the resolvent \( R_z \) on \( W^{s,t,q}_p(M) \) is bounded by \( |R_z - A(X,V)|^{-1} \) if \( R_z > A(X,V) \) (a very rough bound). In addition, we have:

**Corollary 3.9** (Essential Spectral Radius). For all \( z \in \mathbb{C} \) with \( R_z > A(X,V) \), the essential spectral radius of \( R_z \) on \( W^{s,t,q}_p(M) \) is bounded by \( |R_z - \lambda_{\min}^{s,t,p}(X,V)|^{-1} \). Moreover, the set \( \{ \lambda \in \sigma(X + V) \}_{|W^{s,t,q}_p(M)|} \) consists of isolated eigenvalues of finite multiplicity.

**Proof.** Since the inclusion \( W^{s',t',q'}_p(M) \subset W^{s,t,q}_p(M) \) is compact by Lemma 2.10 the first claim follows from a result of Hennion [37, Corollaire 1] and Theorem 3.8. The second claim then follows from the spectral mapping theorem [23] for the resolvent [21, Thm. V1.13]. \( \Box \)
If $\lambda_{\text{max}}^{s,t,q,p}(X,V) > \lambda_{\text{min}}^{s,t,q,p}(X,V)$, where

$$
\lambda_{\text{max}}^{s,t,q,p}(X,V) := \sup \Re \sigma(X + V)|_{W^{s,t,q,p}(M)},
$$

then the isolated eigenvalues furnished by Corollary 3.9 are called the Ruelle–Pollicott resonances of $X + V$ on $W^{s,t,q,p}(M)$. We will apply the following theorem to our scale $W^{s,t,q,p}(M)$ and, in Lemma 4.15 to the scale from [29]:

**Theorem 3.10** (Intrinsicness of Ruelle–Pollicott Resonances). Let $B_1$ and $B_2$ be two Banach spaces of distributions on $M$ on which $\{L_n,\}$ is a strongly continuous semigroup with generator $X + V$. Assume that both $B_1$ and $B_2$ contain $C^{r-1}(M)$ as a dense subset and are continuously embedded in the dual of $C^{r-1}(M)$. If there exists $\lambda_{\text{min}} > -\infty$ such that the sets $D_i = \{\lambda \in \sigma(X + V)|_{B_i} \mid \Re \lambda > \lambda_{\text{min}}\}, i = 1,2$, consist of isolated eigenvalues of finite multiplicity, then $D_1 = D_2$, including multiplicities. In particular, the corresponding generalised eigenvectors belong to $B_1 \cap B_2$ (in fact, to the intersection of the domains of $X + V$ on $B_1$ and on $B_2$).

Proof. If $r = \infty$, this is a special case of [32Thm 2.3], which refers to [24Thm 1.5]. If $r < \infty$ the proof of [24Thm 1.5] using meromorphic extensions of suitable resolvents applies, replacing $L_2(M)$ by the dual of $C^{r-1}(M)$ and using that $C^{r-1}(M)$ is a dense subset of both $B_1$ and $B_2$.

Lemma 4.15 says that $\lambda_{\text{max}}^{s,t,q,p}(X,V) = h_{\text{top}}$ (for suitable $s,t,q,p$) for $V$ as in Section 4.

The remainder of §3.3 is devoted to the proof of Theorem 3.8. Since the resolvent can be written as a Laplace transform (integrating along the flow), this proof will follow from the flow box condition [13], Lemma 3.6, and the lemma below:

**Lemma 3.11** (Integration Along the Flow). Fix $p \in (1,\infty)$. There exists $C < \infty$ such that

$$
\left\| \left( \sum_{n=0}^{\infty} 4^n |\Psi_{0,n}^{\text{Op}}(v)|^2 \right)^{1/2} \right\|_{L_p} \leq C \left( \sum_{n=0}^{\infty} 4^n |\Psi_{0,n}^{\text{Op}}(v)|^2 \right)^{1/2} \left\| \nabla \right\|_{L_p}, \forall \bar{v} > 0 .
$$

(Adapting the proof gives $\left\| \left( \sum_{n=0}^{\infty} 4^n |\Psi_{0,n}^{\text{Op}}(v)|^2 \right)^{1/2} \right\|_{L_p} \leq C \left( \sum_{n=0}^{\infty} 4^n |\Psi_{0,n}^{\text{Op}}(v)|^2 \right)^{1/2} \left\| \nabla \right\|_{L_p}$.)

Proof. It is enough to consider the terms with $n > 0$. The starting point is

$$
\Psi_{0,n}^{\text{Op}}(\partial_x v) = (\nabla^{-1} \Psi_{0,n}) \ast (\partial_x v) = (\partial_x \nabla^{-1} \Psi_{0,n}) \ast v = 2^n (D_d^{\text{Op}} v)_n, \forall v \in L_p(\mathbb{R}^d),
$$

where $(D_d(\xi)b)_n := i 2^\xi \Psi_{0,n}(\xi) b$, for $n \in \mathbb{N}, \xi \in \mathbb{R}^d$, and $b \in \mathbb{C}$.

For a sequence $a$ of complex numbers with $\|a\|_{L_2(\mathbb{R})} := \left( \sum_{n=1}^{\infty} 4^n |a_n|^2 \right)^{1/2} < \infty$, we put

$$
(Q_d(\xi)a)_n := -i \frac{2}{\xi} \tilde{\Psi}_{0,n}(\xi) a_n, \quad \xi \in \mathbb{R}^d, n \in \mathbb{N},
$$

where $\tilde{\Psi}_{0,n}$ is associated via [35] to a covering $\tilde{\Psi}'$ of $\Theta$ with $\text{supp} \tilde{\Psi}'_0$ contained in a cone around $\xi_d$. Then $(Q_d^{\text{Op}} D_d^{\text{Op}} v)_n = \Psi_{0,n}^{\text{Op}} v$. Since there exists $\tilde{\gamma}_0 < \infty$ such that $2^{n-1} \leq |\xi| \leq \tilde{\gamma}_0 |\xi_d|$ for any $\xi \in \text{supp} \tilde{\Psi}_{0,n}$, the map $Q_d$ satisfies the decay condition in Theorem 2.6. Hence, taking $H_1 = H_2 = \{a \mid \|a\|_{L_2(\mathbb{R})} < \infty\}$, the map $Q_d^{\text{Op}}$ is a bounded linear operator on $L_p(\mathbb{R}^d, L_2(\mathbb{R}))$. This concludes the proof, since it gives $C < \infty$ such that $\|\|Q_d^{\text{Op}} D_d^{\text{Op}} v\|_{L_2(\mathbb{R})}\|_{L_p} \leq C \|D_d^{\text{Op}} v\|_{L_2(\mathbb{R})}\|_{L_p}$.

---

24 We expect that intrinsicness can also be proved by using dynamical determinants.
Proof of Theorem 3.8. By Lemma 3.6, for $z \in \mathbb{C}$ such that $\Re z > A(X,V)$ (\cite[Cor. II.1.11]{AB}),

$$R^\alpha_z \varphi = \int_0^\infty \frac{\alpha^{n-1}e^{-z\alpha}}{(n-1)!} \mathcal{L}_{\alpha,V} \varphi \, d\alpha, \quad \forall n \in \mathbb{N},$$

for all $\varphi \in W^{s,d,q}_p(M)$. Introducing the truncated iterated resolvent

$$R_{tr,z}^\alpha \varphi := \int_0^\infty \frac{\alpha^{n-1}e^{-z\alpha}}{(n-1)!} \mathcal{L}_{\alpha,V} \varphi \, d\alpha,$$

we claim that

$$\|R_{tr,z}^\alpha \varphi\|_{W^{s,t,q}_p} \leq \frac{C}{(\Re z + \Delta)^n} \|\varphi\|_{W^{s,t,q}_p}, \quad \forall \Delta \geq 0, \forall \Re z > 0, \forall n > e \cdot \alpha_0 \cdot (\Re z + \Delta).$$

This bound holds because, using Lemma 3.6, $\sup_{\alpha \in [0,\alpha_0]} e^{-\Re z \alpha} \leq 1$, and $\int_0^{\alpha_0} \frac{\alpha^{n-1}}{(n-1)!} \, d\alpha = \frac{\alpha^n}{n^n} \leq 1$ if $n > e \cdot \alpha_0 \cdot (\Re z + \Delta)$ (recall that $n! \geq n^n e^{-n}$), we find

$$\int_0^{\alpha_0} \frac{\alpha^{n-1}e^{-z\alpha}}{(n-1)!} \|\mathcal{L}_{\alpha,V} \varphi\|_{W^{s,t,q}_p} \, d\alpha \leq C_0 \frac{n!}{\alpha^n} \|\varphi\|_{W^{s,t,q}_p} \leq C_1 \|\varphi\|_{W^{s,t,q}_p}, \quad \forall \Re z > 0, \forall n > e\alpha_0 (\Re z + \Delta).$$

We can therefore focus on times $\alpha \geq \alpha_0$ in (55) and invoke Remark 2.5. Lemma 3.6 gives $C_1 = C_1(\varepsilon) < \infty$ such that for all $n \in \mathbb{N}$

$$\|R_{z}^{\alpha+1} \varphi\|_{W^{s,t,q}_p} \leq \int_0^\infty \frac{\alpha^{n-1}e^{-z\alpha}}{(n-1)!} \|\mathcal{L}_{\alpha,V} R_z \varphi\|_{W^{s,t,q}_p} \, d\alpha$$

$$\leq \frac{C_1}{(\Re z - A(X,V))^n} \|R_z \varphi\|_{W^{s,t,q}_p},$$

where, for $\Theta' < \Theta_0$ as in Remark 2.5 we replaced $\Theta$ by $\Theta'$ in the first term of (58). Lemma 3.6 also gives $C_2 < \infty$ such that

$$\|R_z \varphi\|_{W^{s,t,q}_p} \leq \frac{C_2}{\Re z - A(X,V)} \|\varphi\|_{W^{s,t,q}_p},$$

so that the second term of (58) is bounded as claimed. The starting point to bound the first term is the fact that the flow box condition (13) gives (\cite{AB})

$$g(\hat{\mathcal{K}}^{-1}_\omega)(\hat{\omega}_x, \hat{\omega}_y) = X|\hat{\omega}_x|,$$

and hence

$$\partial_x((X \partial_x \cdot \hat{\varphi}) \circ \kappa^{-1}_\omega) = ((X \partial_x \cdot \hat{\varphi} + \theta_x \cdot (X \hat{\varphi})) \circ \kappa^{-1}_\omega).$$

Using the triangle inequality (and $-\infty < q'$) to separate the contribution of $\Theta'_0\omega_0$, and applying Lemma 3.11 with (60) (for $\hat{\varphi} = \mathcal{L}_{\alpha',V} R_z \varphi$) to bound this term, we find $C_3 < \infty$ such that

$$\|((X \partial_x \cdot \mathcal{L}_{\alpha',V} R_z \varphi) \circ \kappa^{-1}_\omega|_{W^{s,t,q}_p}(K_\omega) \leq \|((X \partial_x \cdot \mathcal{L}_{\alpha',V} R_z \varphi) \circ \kappa^{-1}_\omega|_{W^{s,t,q}_p}(K_\omega)$$

$$+ C_3 \|(X \partial_x \cdot \mathcal{L}_{\alpha',V} R_z \varphi) \circ \kappa^{-1}_\omega|_{W^{q-1}_{p,\theta_x}(K_\omega)} + C_3 \|(X \partial_x \cdot X R_z \mathcal{L}_{\alpha',V} \varphi) \circ \kappa^{-1}_\omega|_{W^{q-1}_{p,\theta_x}(K_\omega)},$$

where we used for the last term that (55) implies $\mathcal{L}_{\alpha',V} R_z \varphi = R_z \mathcal{L}_{\alpha',V} \varphi$. Since $(X \partial_x \omega) \circ \kappa^{-1}_\omega = \partial_x((\hat{\omega}_x \cdot \kappa^{-1}_\omega) \in C^{-1}_{\alpha_0}(\kappa_\omega(V_\omega))$ (using that $\hat{\omega}_x$ and $\kappa_\omega$ are $C^r$, with $\hat{\omega}_x$ is compactly supported in $V_\omega$) and $q - 1 \leq q'$, Lemma 3.5 for the identity map gives $C_4 < \infty$ such that

$$\|((X \partial_x \omega) \cdot \mathcal{L}_{\alpha',V} R_z \varphi) \circ \kappa^{-1}_\omega|_{W^{q-1}_{p,\theta_x}(K_\omega) \leq C_4 \sup_{\omega \in \Omega} \|((\hat{\omega}_x \cdot \mathcal{L}_{\alpha',V} \varphi) \circ \kappa^{-1}_\omega|_{W^{q-1}_{p,\theta_x}(K_\omega)}.\]
Using $X R_z \varphi = z R_z \varphi - V R_z \varphi - \varphi$, and, again, $R_z L_{\alpha', V} \varphi = L_{\alpha', V} R_z \varphi$, we find
\[
\| (\partial_{\omega} \cdot X R_z L_{\alpha', V} \varphi) \circ \kappa_{\omega}^{-1} \|_{W^{s, q-1}_{p, \Theta_{\alpha', q}}(K_{\omega})} \leq |z| \| (\partial_{\omega} \cdot L_{\alpha', V} R_z \varphi) \circ \kappa_{\omega}^{-1} \|_{W^{s', q-1}_{p, \Theta_{\alpha', q}}(K_{\omega})} \\
+ \| (\partial_{\omega} \cdot V L_{\alpha', V} R_z \varphi) \circ \kappa_{\omega}^{-1} \|_{W^{s', q-1}_{p, \Theta_{\alpha', q}}(K_{\omega})} + \| (\partial_{\omega} \cdot L_{\alpha', V} \varphi) \circ \kappa_{\omega}^{-1} \|_{W^{s', q-1}_{p, \Theta_{\alpha', q}}(K_{\omega})}.
\]
(61)

Since $V \in C^{r-1}(M)$, we may bound the first term in (61) with Lemma 3.5 for the identity map. Using the definition of $\| R_z \varphi \|_{W^{s', q-1}_{p, \Theta_{\alpha', q}}}$ as an integral of local norms over $\alpha' \in [0, a_0]$, this bounds the first term of (58) as claimed (we use (59) for the terms with $R_z \varphi$).

\[\square\]

3.4. **Dolgopyat Bounds for the Resolvent of Weighted Transfer Operators.** In the contact Anosov case and for the potential $V$ introduced in the next section, the spectrum of $X + V$ has already been studied [29], on a different Banach space. We will use in the proof of Lemma 4.15 that the discrete spectra of $X + V$ on our Banach spaces and the spaces of [29] coincide in a big enough half-plane (“intrinseness”), but it is not clear how to exploit this to obtain the bounds on the resolvent needed in Section 4. Indeed, in the self-adjoint case, there exist good bounds on the iterated resolvent $R^n_z$ in terms of the distance between $z$ and the spectrum. However, even when $W_{p, t, q}(M)$ is a Hilbert space, the operator $X + V$ is not self-adjoint a priori, so the existence of a spectral gap for $X + V$ does not imply good bounds on the resolvent in general (see [51] [27] [19] for special cases where such bounds are known). For this reason, we introduce the following condition:

**Condition 3.12 (Weak Dolgopyat Bounds on the Resolvent).** There exist $p \in (1, \infty)$, $s, q, t$ as in (12), constants
\[s'' \in \mathbb{R}, \quad \delta' \in (\lambda_{\min}^{s, t, p}, \lambda_{\max}^{s, t, q, p}),\]
and constants $a_0 > 0, b_0' > 1, c_1 < 1 < C_1$, such that, for all $a \geq a_0$ and $\gamma'$ in the range
\[a C_1 < \gamma' < \frac{C_1}{\log(1 + (\lambda_{\max}^{s, t, q, p} - \delta')/a)},\]
(62)

we have
\[\| R^n_{a + ib + \lambda_{\max}^{s, t, q, p}} \varphi \|_{W^{s''}_{p, t, q}} \leq C_1 |a + (\lambda_{\max}^{s, t, q, p} - \delta')^{-n} \| \varphi \|_{C^1} , \forall |b| \geq b_0' , \text{ where } n = \lceil \gamma' \log |b| \rceil.\]

Using the mollification ideas introduced by Liverani in [8] §5.7 and [7] §9, we will show that Condition 3.12 implies norm estimates on the resolvent (see [15] Remark 2.6):

**Proposition 3.13 (Strong Dolgopyat Bounds on the Resolvent).** If there exists $C_0 < \infty$ with
\[\| L_{\alpha, V} \|_{W^{s', q-1}_{p, \Theta_{\alpha', q}}} \leq C_0 e^{\lambda_{\max}^{s, t, q, p}} \alpha} , \forall \alpha \geq 0,\]
(63)

for $t - (r - 1) < s' < 0 < q' < t'$, with $t - t' = q - q' = s - s' > 0$, for some $s, q, t$ and $p > \max\{d/t, d/(r - 1 + s)\}$ such that Condition 3.12 holds for $c_1, C_1$, then there exist $\delta \in (\lambda_{\min}^{s, t, p}, \lambda_{\max}^{s, t, q, p}), a > 0, b_0 > 1, C \leq C_0$, and $\gamma \in (a C_1, C_1 / \log(1 + (\lambda_{\max}^{s, t, q, p} - \delta)/a))$, with
\[\| R^n_{a + ib + \lambda_{\max}^{s, t, q, p}} \|_{W^{p, q-1}_{p, t, q}} \leq C |a + (\lambda_{\max}^{s, t, q, p} - \delta)|^{-n} , \forall |b| \geq b_0' , \text{ where } n = \lceil \gamma \log |b| \rceil.\]

Before proving Proposition 3.13, we make further remarks. Bounds (64) on suitable anisotropic Banach spaces are used in many places in the literature, starting with Liverani’s breakthrough
paper [42] (see e.g. [8, 29, 7]). This has been axiomatized by Butterley in [15]: Together with a weak Lipschitz control on \( \alpha \mapsto \mathcal{L}_{\alpha,V} \), the bounds (63) and (64) imply the spectral gap property\(^2\)

\[
\sigma(X + V) |_{W^{s.t,q}(\mathcal{M})} \cap \{ \Re \lambda > \delta \}
\]

is a finite set.

The above spectral gap can be used to get exponential decay of correlations, but the implication in the other direction is not known in general. (Dolgopyat [20] obtained exponential decay of correlations for Gibbs measures with arbitrary Hölder potentials for geodesic flows on surfaces of strictly negative curvature or, more generally, \(C^5\) Anosov flows such that \(E_-\) and \(E_+\) are \(C^1\) and not jointly integrable, using symbolic dynamics. His ideas led to results of Liverani on the SRB measure of contact Anosov flows [42]. See [34] and [53], and references therein, for recent sufficient conditions ensuring exponential mixing for Gibbs measures and Anosov flows.)

The bounds (63) have been established [42, 31, 8, 29] for the generator \(X\) associated to contact Anosov flows and the potential \(V = 0\) replacing our spaces \(W^{s.t,q}_p\) by other anisotropic Banach spaces. For the potential \(V\) used in Section 4, Dolgopyat bounds are shown in [29] §7 (see also the argument sketched by Faure and Guillarmou before [23, Proposition 3.4]). We expect that (63) or Condition 3.12 can be proved directly in our setting. For our purposes it is sufficient instead to refer to [29] §7 in Section 4 to establish Condition 3.12 and then invoke Proposition 3.13. We thereby illustrate how to build bridges between results for different anisotropic spaces (once the essential radius is controlled, exact growth is obtained, and, for the Dolgopyat estimate, mollification bounds are known).

**Proof of Proposition 3.13** Let \(\{\Theta'_\omega\}\) and \(\{\Theta_\omega\}\) form an adapted pair for \(A\) and \(g_\alpha\) in the sense of Remark 2.5. Denote by \(\|\varphi\|_{W^{s',t',q'}(\Theta')}\) the norm constructed with \(\Theta'_\omega\) instead of \(\Theta_\omega\). We start with three trivial but useful observations: First, for any \(\beta > 0\), \(\delta_2 > 0\), and \(\delta_1 \geq 0\), we have for all \(|b| \geq 1\) and \(a > \delta_1\) that

\[
|a - \delta_1|^{-\gamma' \log |b|} |b|^{-\beta} \leq |a + \delta_2|^{-\gamma' \log |b|}, \quad \forall \gamma' \in (0, \frac{\beta}{\log(1 + \delta_2/a) - \log(1 - \delta_1/a)}) .
\]

(If \(\delta_1 = 0\) and \(\beta > 0\), taking \(\delta_2 > 0\) small enough, we can choose \(\gamma'\) arbitrarily large in (65).)

Second, for any \(\beta' > 0\), and \(\delta_3 > \delta_2 > 0\), we have for all \(|b| \geq 1\) and \(a > 0\)

\[
|a + \delta_3|^{-\gamma' \log |b|} |b|^{\beta'} \leq |a + \delta_2|^{-\gamma' \log |b|}, \quad \forall \gamma' > \frac{\beta'}{\log(1 + \delta_3/a) - \log(1 + \delta_2/a)} .
\]

Third, for any \(0 < \delta_0 < \delta_2\) and \(\delta_1 \geq 0\), we have for all \(a > \delta_1\) and \(m_1, m_2 \in \mathbb{N}\),

\[
|m_1 a + m_2 \delta | - m_1 |a - \delta_1|^{-m_2} \leq |a + \delta_0|^{-m_1 - m_2}, \quad \text{if } \frac{m_1}{m_2} \geq \frac{\log(1 + \delta_0/a) - \log(1 - \delta_1/a)}{\log(1 + \delta_2/a) - \log(1 + \delta_0/a)} .
\]

(If \(\delta_1 = 0\), for fixed \(\delta_2 > 0\), taking \(\delta_0 > 0\) small enough, we can choose \(m/n\) arbitrarily small.)

Set \(\lambda_{\max} = \lambda_{\max}^{s',t',q'}\). To deduce (64) from Condition 3.12 we use the Lasota–Yorke estimate: We may assume that \(s'' < \min\{-d - 1, 1\}\). Then, for any \(\delta_2 \in (0, \lambda_{\max} - \lambda_{\min}^{s,t,p})\), Theorem 3.8 with

\(^2\)Beware that this property does not imply a spectral gap (quasi-compactness) for the time-one transfer operator: we do not expect \(\mathcal{L}_{\alpha,V}\) to be eventually norm continuous [21, Thm II.5.3], so a priori we only have \(\sigma(\mathcal{L}_{\alpha,V}) \subset \exp(\alpha \sigma(X + V))\) for \(\alpha \geq 0\) (equality holds for eigenvalues and residual spectrum), see [21, §V.2.b]. A spectral gap for the time-one transfer operator is only known in special cases, [51, 27].
give $C, C(s'', m),$ and $A(X, V) \geq \lambda_{\text{max}},$ such that for all $\varphi \in W_{p}^{s, t, q}(M)$ and all $m, n \in \mathbb{N},$

$$
\|R_{a+b+\lambda_{\text{max}}}^{m+n} \varphi\|_{W_{p}^{s, t, q}} \leq \frac{2C(s'', m)|b|}{(a - A(X, V) - \lambda_{\text{max}})^{m}} \|R_{a+b+\lambda_{\text{max}}}^{n} \varphi\|_{W_{p}^{s', t', q}}, \forall |b| \geq 1, \forall a > A(X, V) - \lambda_{\text{max}} + 1.
$$

(68)

Then, we proceed as in [8 §5, §7] or [7 §9]: First, since (63) gives $\|R_{a+b+\lambda_{\text{max}}}^{n} \varphi\|_{W_{p}^{s, t, q}} \leq Ca^{-n}\|\varphi\|_{W_{p}^{s, t, q}},$ for any $\epsilon_{0} > 0$ (to be fixed in the next paragraph and by (65)) there exist $m$ and $n$ with $m \leq \epsilon_{0} n$, and such that the last term on the right-hand side of (63) satisfies the required condition: Indeed, apply (61) for $m_{1} = m, m_{2} = n, \delta_{1} = 0, \delta_{2} = \delta_{2},$ taking $\delta_{0} > 0$ small enough such that $m \leq \epsilon_{0} n$ is allowed, and choose $\delta \geq \min(\epsilon_{1}, \lambda_{\text{max}} - \delta_{0}).$

For the first term on the right-hand side of (68), it is enough to bound $\frac{2C(s'', m)|b|}{(a - A(X, V) - \lambda_{\text{max}})^{m}} \|R_{a+b+\lambda_{\text{max}}}^{n} \varphi - R_{a+b+\lambda_{\text{max}}}^{n} \varphi\|_{W_{p}^{s', t', q}}.$ Indeed, it is not hard to see that (s'') = 1 such that $C(s'') \leq 1,$ taking $\epsilon_{0}$ small enough so that

$$
(69)

n \geq m \frac{\log(1 + \delta_{0}/a) - \log(1 - \delta_{1}(s'')/a)}{\log(1 + \Delta/a) - \log(1 + \delta_{0}/a)},
$$

and taking $b_{0}$ large enough to ensure $n = \lceil \gamma' \log |b| \rceil > e\epsilon_{0}(a + \lambda_{\text{max}} + \Delta)$ if $|b| \geq b_{0},$ for $\gamma' \geq C_{1}a$ determined below. (Again, choose $\delta \geq \min(\epsilon_{1}, \lambda_{\text{max}} - \delta_{0})).$

Set $R_{a+b+\lambda_{\text{max}}}^{n} \varphi := R_{a+b+\lambda_{\text{max}}}^{n} \varphi - R_{a+b+\lambda_{\text{max}}}^{n} \varphi.$ Fixing $s', q', t'$ with $q' - q = t' - t = s' - s < 0,$ and $t - (r - 1) < s' < 0 < q' < t', we decompose, for any $s'' \leq s'$,

$$
(70)

|b||R_{a+b+\lambda_{\text{max}}}^{n} \varphi||_{W_{p}^{s', t', q'}} \leq |b||R_{a+b+\lambda_{\text{max}}}^{n} \varphi||_{W_{p}^{s', t', q'}} + C|b||R_{a+b+\lambda_{\text{max}}}^{n} \varphi||_{W_{p}^{s', t', q' - s'}},
$$

where $M_{\epsilon}$ is the mollification operator in charts defined by (115) for $\epsilon = |b|^{-\kappa},$ with $\kappa > 0,$ to be chosen later. Let $\{\Theta_{\omega}\}$ form an adapted pair with $\{\Theta_{\omega}\}$ by (63) we have

$$
(71)

\|R_{a+b+\lambda_{\text{max}}}^{n} \varphi - M_{\epsilon} \varphi\|_{W_{p}^{s', t', q'}} \leq \frac{C}{a^{n}} \|\varphi - M_{\epsilon} \varphi\|_{W_{p}^{s', t', q'}}.
$$

Then the mollification estimate Lemma C.2 gives

$$
(72)

\frac{C}{a^{n}} \|\varphi - M_{\epsilon} \varphi\|_{W_{p}^{s', t', q'}} \leq \frac{C}{a^{n}} e^{-s'} \|\varphi\|_{W_{p}^{s, t, q}} \leq \frac{C}{a^{n}} |b|^{-\kappa(s-s')} \|\varphi\|_{W_{p}^{s, t, q}},
$$

If $\kappa > 1/(s-s'),$ applying (63) with $\beta = \kappa(s-s') - 1 > 0$ and $a > \delta_{1} = 0, \delta_{2} = \lambda_{\text{max}} - \delta,$ the bounds (71) and (72) take care of the second term in the right-hand side of (70), assuming

$$
(73)

\gamma' < \frac{\kappa(s - s') - 1}{\log(1 + \lambda_{\text{max}} - \delta)}.
$$

Note that this inequality is compatible with $\gamma' > aC_{1}$ if $\kappa$ is large enough.
Fix \( \eta_0 \in (0, \min\{t, r - 1 + s\}) \), small. By the Sobolev embeddings for \( W^{1+\eta_0}_{p} = F^{1+\eta_0}_{p,2} \) and \( B^{1+\eta_0}_{\infty,\infty} \) [16, Thm 2.2.3(i)] in dimension \( d \), we have

\[
\|\tilde{\varphi}\|_{C^1} \leq \tilde{C}\|\varphi\|_{W^{1+\eta_0}_{p}} ; \text{ if } p > \frac{d}{\eta_0}.
\]

Thus, Condition 3.12 bounds the first term in the right-hand side of (70) by

\[
\frac{C_1|b|}{|a + \lambda_{\max} - \delta'|^n} \|M_{\epsilon}\varphi\|_{C^1} \leq \frac{\tilde{C}|b|}{|a + \lambda_{\max} - \delta'|^n} \|M_{\epsilon}\varphi\|_{W^{1+\eta_0}_{p}}.
\]

Since the charts in \( A \) are \( C^r \), the classical isotropic mollification estimate of [8, Lemma 5.3] (replacing \( X_0 \) by \( M \) and 2 by \( r \) there) becomes: For each \( p \in (1, \infty) \) and all \( -r + 1 < s \leq s' < r + s \leq r \), there exists \( C_{\#} \) so that for all small enough \( \epsilon > 0 \) and every \( \varphi \in W^s_p(M) \), we have

\[
\|M_{\epsilon}(\varphi)\|_{W^s_p(M)} \leq C_{\#} \epsilon^{s-s'}\|\varphi\|_{W^s_p(M)}.
\]

Therefore, since \( -r + 1 < s < 0 < 1 + \eta_0 < r + s \), taking \( s' = 1 + \eta \) and recalling (24), we have

\[
\frac{\tilde{C}|b|}{|a + \lambda_{\max} - \delta'|^n} \|M_{\epsilon}\varphi\|_{W^{1+\eta_0}_{p}} \leq \frac{\tilde{C}|b| \epsilon^{s-1-\eta_0}}{|a + \lambda_{\max} - \delta'|^n} \|\varphi\|_{W^{s,s,s}_{p}} = \frac{\tilde{C}|b|^{1+\kappa r}}{|a + \lambda_{\max} - \delta'|^n} \|\varphi\|_{W^{s,s,s}_{p}}.
\]

We need to multiply the above by \( C(a - \delta_1(s''))^{-m} \). For this, we use that

\[
\gamma' > \frac{1 + \kappa r}{\log(1 + (\lambda_{\max} - \delta')/a) - \log(1 + \delta_2/a)}
\]

is compatible with the upper bound (73) on \( \gamma' \), up to taking small enough \( \delta_2 \in (0, \lambda_{\max} - \delta') \) in the right-hand side of (66) for \( \delta' = 1 + \kappa r \) and \( \delta_2 = \lambda_{\max} - \delta' \).

We conclude the proof of the proposition by applying (67) for \( a > \delta_1 = \delta_1(s'') \), \( \delta_2 \) as in the previous paragraph, \( m_1 = n, m_2 = m \), for \( \delta_0 \in (0, \delta_2) \), and \( \epsilon_0 \in (0, 1) \) such that

\[
\log(1 + \delta_0/a) - \log(1 - \delta_1(s''))/a < \frac{1}{\epsilon_0}.
\]

Indeed, taking \( \delta > \epsilon_1 \) closer to \( \lambda_{\max} \) if necessary to ensure \( \delta \leq \lambda_{\max} - \delta_0 \), and for \( \gamma' > aC_1 \) satisfying (73)-(74), take \( \gamma > 0 \) such that (using \( m \leq \epsilon_0 n \))

\[
\gamma|\log|b|| = m + n \leq (\epsilon_0 + 1)\gamma|\log|b||.
\]

Then \( \gamma < 1/\log(1 + \frac{\lambda_{\max} - \delta}{a}) \) follows from (73), up to taking \( \delta < \lambda_{\max} \) closer to \( \lambda_{\max} \).

Remark C.3 explains why using mollifiers through isotropic spaces as in [8, Lemma 5.4, (7.5)–(7.6)] does not allow to carry out the bounds in the previous proof.

4. ASYMPTOTICS OF HOROCYCLE INTEGRALS

In this section, we assume throughout that \( r \geq 2 \), the \( C^r \) Anosov flow \( g_\alpha \) on \( M \) is topologically mixing with stable dimension \( d_- = 1 \), and that the strong-stable distribution \( E_- \) is orientable.
4.1. Horocycle Flow $h_\rho$. Horocycle Integral $\gamma_x(\varphi, T)$. Renormalisation Time $\tau(\rho, \alpha, x)$. We shall focus on stable horocycle flows. Analogous results exist for unstable horocycle flows.

**Definition 4.1** ((Stable) Horocycle Flow). A (stable) horocycle flow for a topologically mixing $C^r$ Anosov flow $g_\alpha$ on $M$ with $d_- = 1$ and $E_-$ orientable is a $C^0$ flow $h_\rho$ on $M$ such that $\partial_\rho h_\rho \in E_- \setminus \{0\}$ for all $\rho \in \mathbb{R}$.

**Remark 4.2** (Unit Speed Parametrisation). The stable manifolds of the flow $g_\alpha$ are the submanifolds tangent to the bundle $E_-$ (this bundle is in general only Hölder, existence is ensured by the stable manifold theorem, see e.g. [11] Thm 17.4.3). We can parametrise stable manifolds by the arc-length induced by the Riemannian metric on $M$. Since we assumed that $E_-$ is orientable, this defines uniquely a horocycle flow with $|\partial_\rho h_\rho| \equiv 1$, called the unit speed horocycle flow. All other horocycle flows are obtained by time reparametrisations. Topological mixing of $g_\alpha$ implies that each stable manifold is dense in $M$ [44, p. 84] so any horocycle flow is minimal.

Our main object of interest is the following (stable) horocycle integral:

**Definition 4.3** (Horocycle Integral). The horocycle integral of the horocycle flow $h_\rho$ for the observable $\varphi \in C^0(M)$ at $x \in M$ is defined by

$$
\gamma_x(\varphi, T) = \int_0^T \varphi \circ h_\rho(x) d\rho .
$$

Writing $\mu(\varphi) = \int \varphi d\mu$, where $\mu$ is the unique $h_\rho$-invariant probability measure, we have

$$
\gamma_x(\varphi, T) = T \cdot \mu(\varphi) + \mathcal{E}_{T,x}(\varphi) , \quad \lim_{T \to \infty} \frac{\mathcal{E}_{T,x}(\varphi)}{T} = 0, \forall x \in M, \ \forall \varphi \in C^0(M) .
$$

Our main result, Theorem 4.8 in [44, §27] gives a more precise asymptotic expansion, involving the spectrum and eigendistributions of a suitably weighted transfer operator $\mathcal{L}_{\alpha,V}$. A crucial ingredient in our analysis is the renormalisation time (first introduced by Marcus [44, p.83] to study ergodic properties of the horocycle flow):

**Definition 4.4** ((Pointwise) renormalisation time). A map $\tau : \mathbb{R}^2 \times M \to \mathbb{R}$ which satisfies

$$
g_\alpha \circ h_\rho(x) = h_{\tau(\rho, \alpha, x)} \circ g_\alpha(x) , \quad \forall \rho, \alpha \in \mathbb{R}, \forall x \in M ,
$$

is called a (pointwise) renormalisation time for the stable horocycle flow $h_\rho$.

For the unit speed horocycle flow of the geodesic flow on a compact surface of constant negative curvature, the renormalisation time is $\tau(\rho, \alpha, x) = \rho \cdot \exp(-\alpha \cdot h_{top})$. More generally:

**Lemma 4.5** (Properties of $\tau(\rho, \alpha, x)$). There exists a unique solution $\tau(\rho, \alpha, x)$ to (78). In addition $\tau(\rho, \alpha, x)$ is differentiable in $\rho$, and we have

$$
\tau(\rho, \alpha, x) = \gamma_x(\partial_\rho \tau(0, \alpha, \cdot), \cdot) , \quad \forall x \in M, \ \forall \alpha \in \mathbb{R}, \ \forall \rho \in \mathbb{R} ,
$$

$$
\partial_\rho \tau(0, \alpha, x) = \det Dg_\alpha|_{E_-}(x) \cdot \frac{(\partial_\rho h_0(x))^*(\partial_\rho h_0(x))}{(\partial_\rho h_0 \circ g_\alpha(x))^*(\partial_\rho h_0 \circ g_\alpha(x))} , \quad \forall x \in M, \ \forall \alpha \in \mathbb{R} .
$$

---

26See [44] for a proof of unique ergodicity. If $g_\alpha$ preserves a smooth measure see also [43]. See also Remark 4.16.

27In [39], we denote by $(\partial_\rho h_0)^* \in E^\perp$ the canonical dual of $\partial_\rho h_0 := \partial_\rho h_0|_{\rho=0}$. 


In particular, \( \partial_\rho \tau(0, \alpha, x) > 0 \), \( \tau(0, \alpha, x) = 0 \), \( \tau(\rho, 0, x) = \rho \). Moreover, there exists \( C < \infty \) with

\[
\frac{1}{C} \leq \frac{\tau(\rho, -\alpha, x)}{\rho} e^{-h_{top} \alpha} \leq C, \quad \forall x \in M, \quad \forall \rho \in \mathbb{R}, \quad \forall \alpha \geq 0 \text{ such that } |\rho| \geq 1,
\]

\[
\frac{1}{C} \leq \frac{\rho}{\tau(\rho, \alpha, x)} e^{-h_{top} \alpha} \leq C, \quad \forall x \in M, \quad \forall \rho \in \mathbb{R}, \quad \forall \alpha \geq 0 \text{ such that } |\tau(\rho, \alpha, x)| \geq 1.
\]

The bounds (81)–(82) will come from [29, App. C]. That \( \lim_{\rho \to -\infty} (\tau(\rho, \alpha, x)/\rho) = e^{-\alpha h_{top}} \) for all \( \alpha \geq 0 \) follows from [44], see the proof of Lemma 4.6.

The key fact behind our main result (Theorem 4.8) is the following consequence\(^{28}\) of (78)

\[
\int_0^T \varphi \circ h_\rho(x)d\rho = \int_0^\tau(T, \alpha, x) (L_{\alpha, V} \varphi) \circ h_\rho \circ g_\alpha(x)d\rho,
\]

(“renormalisation”) where the transfer operator \( L_{\alpha, V} \) is defined by (8), choosing

\[
V \equiv -\partial_\rho \partial_\alpha \tau(0, 0, \cdot), \quad \text{i.e. } \phi_\alpha = \partial_\rho \tau(0, -\alpha, \cdot),
\]

and assuming that \( \phi_\alpha \) is \( C^{r-1} \). The underlying idea will be to take \( \alpha = O(\log T) \) so that \( \tau(T, \alpha, x) = O(1) \), and then exploit the information on the spectrum of the semigroup \( L_{\alpha, V} \) obtained in the previous section.

Since the derivative of the Jacobian is the divergence and \( \partial_\alpha g_{-\alpha}|_{\alpha=0} = X \), we find for the unit speed horocycle flow that (83) implies

\[ V = \text{div} (X|_{E_-}) \text{ and } \phi_\alpha = \text{det } Dg_{-\alpha}|_{E_-}. \]

Hence, if \( E_- \) is \( C^{r-1} \) then \( \phi_\alpha \in C^{r-1}(M) \). More generally, if \( E_- \) is \( C^{r-1} \), for any \( C^r \) time reparametrisation of the unit speed horocycle flow, the weight \( \phi_\alpha \) is \( C^{r-1} \) by (80). (Cf. [31, Remark 2.4].) In order to fit in the Banach norm setting of Sections 2 and 3, we will need \( \phi_\alpha \) to be \( C^r \) for \( r \geq 2 \), and we will have to introduce the horocycle integrals (82) localised by smooth cutoff functions (following [31], see the proof of Lemma 4.14), replacing thus (83) by the more involved version of “renormalisation” in Sublemma 4.13.

Before proving Lemma 4.5, we state and prove a consequence of (83) and classical results:

**Lemma 4.6** (The Invariant Measure \( \mu \) as an Eigenvector). If \( \phi_\alpha \) from (84) is \( C^{r-1} \), we have

\[
\mu(L_{\alpha, V} \varphi) = e^{h_{top} \alpha} \mu(\varphi), \quad \forall \alpha \geq 0, \quad \forall \varphi \in C^0(M).
\]

**Remark 4.7** (Spectrum of \( X + V \) on \( L^1(\mu) \)). Lemma 4.6 gives \( \mu(|L_{\alpha, V} \varphi|) \leq \mu(L_{\alpha, V} |\varphi|) = \mu(|\varphi|) \) for all \( \alpha \geq 0 \) and any \( \varphi \in C^0(M) \). Therefore, since \( \mu \) is a Radon measure, for each \( \alpha \geq 0 \), the operator \( L_{\alpha, V} \) is bounded on the Banach space \( L^1(\mu) \), with norm equal to \( e^{\alpha h_{top}} \). Hence, using [21, Cor. II.1.11] and [55], the spectral radius of \( R_\varphi = (z - (X + V))^{-1} \) on \( L^1(\mu) \) is bounded by \( |R_\varphi - h_{top}| \) if \( R_\varphi > h_{top} \). The spectrum of \( X + V \) on \( L^1(\mu) \) thus lies in the half-plane \( \Re \varphi \leq h_{top} \).

**Proof of Lemma 4.6** Unique ergodicity (77) (twice), renormalisation (83), and a result of Marcus [44, Lemma 3.1, p 84] give that for all \( \alpha \geq 0 \) and \( \varphi \in C^0(M) \)

\[
\mu(\varphi) = \lim_{T \to \infty} \frac{1}{T} \gamma_\alpha(\varphi, T) = \frac{1}{\tau(T, \alpha, x)} \frac{1}{\tau(T, \alpha, x)} \gamma_{g_\alpha(x)}(L_{\alpha, V} \varphi, \tau(T, \alpha, x)) = e^{-\alpha h_{top} \mu(L_{\alpha, V} \varphi)}.
\]

\(^{28}\)The proof of (83) is a simplification of that of Sublemma 4.13 below.
Proof of Lemma 4.5 Since stable leaves are dense and the flow $h_\rho$ is non-singular, this flow does not admit any periodic orbits. For $x \in M$ and $\rho, \alpha \in \mathbb{R}$, set $h_{\alpha,\rho}(x) := g_\alpha \circ h_\rho \circ g_{-\alpha}(x)$. Then $\partial_\rho h_{\alpha,\rho} \in E_{-x} \setminus \{0\}$. Hence $h_{\alpha,\rho}(x)$ parametrizes the same stable manifold as $h_\rho(x)$. If there were two different pointwise times $\tau$, there would exist $\rho_1 < \rho_2 \in \mathbb{R}$ such that $h_{\alpha,\rho_1}(x) = h_{\rho_2}(x)$, and this would contradict the absence of periodic orbits. Thus, $\tau(\rho, \alpha, x)$ is uniquely defined and differentiable in $\rho$. We deduce from (78) that

$$h_{\tau(\rho_1 + \rho_2, \cdot)}(g_{\alpha_1 + \alpha_2} (\cdot)) = g_{\alpha_1 + \alpha_2} (h_\rho(\cdot)) = g_{\alpha_1} (h_{\tau(\rho, \alpha_2, \cdot)}(g_{\alpha_2} (\cdot))) = h_{\tau(\rho, \alpha_2, \cdot, \alpha_1, g_{\alpha_2})(x)}(g_{\alpha_1 + \alpha_2} (\cdot)),$$

and

$$h_{\tau(\rho_1 + \rho_2, \alpha, x)}(g_\alpha(x)) = g_\alpha(h_{\rho_1 + \rho_2}(x)) = g_\alpha (h_{\rho_1} \circ g_{-\alpha} \circ g_\alpha (h_{\rho_2} \circ g_{-\alpha} \circ g_\alpha (x))) = h_{\tau(\rho_1, \rho_2, \alpha, x)}(h_{\tau(\rho_2, \alpha, x)}(g_\alpha(x))).$$

This implies that for all $\alpha_1, \alpha_2, \rho_1, \rho_2 \in \mathbb{R}$,

\begin{equation}
\tau(\rho, \alpha_1 + \alpha_2, \cdot) = \tau(\tau(\rho, \alpha_2, \cdot), \alpha_1, g_{\alpha_2}(\cdot)), \quad \tau(\rho_1 + \rho_2, \cdot, x) = \tau(\rho_1, \cdot, h_{\rho_2}(x)) + \tau(\rho_2, \cdot, x).
\end{equation}

Then, using $\tau(0, \alpha, x) = 0$, and differentiating the identities in (86) at $\rho = 0$ and $\rho_1 = 0$, we find

\begin{equation}
\partial_\rho \tau(\rho, \alpha, x) = \partial_\rho \tau(0, \alpha, h_\rho(x)), \quad \partial_{\rho_1} \tau(0, \rho_1, g_{\alpha_2}(x)) \partial_\rho \tau(0, \alpha_2, x) = \partial_\rho \tau(0, \alpha_1 + \alpha_2, x), \quad \forall \alpha_i.
\end{equation}

Next (79) follows from the definition (76) of $\gamma_x$ and the first claim of (87).

To show (80), we take derivatives on both sides of (78) with respect to $\rho$

\begin{equation}
Dg_\alpha \partial_\rho h_\rho(x) = \partial_\rho \tau(\rho, \alpha, x) \cdot (\partial_\rho h_0) \circ h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x).
\end{equation}

We have

\begin{equation}
(\partial_\rho h_0 \circ g_\alpha)^*(Dg_\alpha \partial_\rho h_0) = (\partial_\rho h_0 \circ g_\alpha)^*(g_\alpha \circ \partial_\rho h_0) = (g_\alpha)^*(\partial_\rho h_0 \circ g_\alpha)^*(\partial_\rho h_0)
\end{equation}

\begin{equation}
= \det(Dg_\alpha|_{E_+})^*(\partial_\rho h_0)^*(\partial_\rho h_0) = \det Dg_\alpha|_{E_+} (\partial_\rho h_0)^*(\partial_\rho h_0).
\end{equation}

Setting $\rho = 0$ in (88), we obtain (80), using (89) and the non-singularity of the horocycle flow. That $\partial_\rho \tau(0, \alpha, x) > 0$ follows from (80), for instance.

Next, since $h_\rho$ is non-singular, the stable manifold $W_y := h_{[0,1]}(y)$ has length bounded from above and below uniformly in $y \in M$. Using (80) and [29, Lemma C.3, Remark C.4] (recalling that $g_\alpha$ is transitive), we find $C_3, C_4, C_5 < \infty$ such that

$$\tau(\rho, -\alpha, x) \leq C_3 \int_0^\rho \det Dg_{-\alpha|E_-} \circ h_\rho(x) d\rho \leq C_4 \rho \sup_{y \in M} \int \det Dg_{-\alpha|E_-} \circ dW_y$$

$$\leq C_5 \rho \sup_{y \in M} \text{vol}(g_{-\alpha}(W_y)) \leq C_6 \rho e^{h_{\text{top}} \alpha}, \quad \forall \rho \geq 1, \alpha \geq 0, x \in M.$$  

A lower bound for $\tau(\rho, -\alpha, x)$ is obtained analogously, using [29, Lemma C.1]. This shows (81) for all $\rho \leq -1$ since (79) implies $\tau(-\rho, \alpha, x) = -\tau(\rho, \alpha, h_{-\rho}(x))$.

Finally, (82) follows from (80) and the following consequence of the first claim of (86)

$$\rho = \tau(\rho, \alpha, x), -\alpha, g_\alpha(x)) = \tau(|\tau(\rho, \alpha, x)|, -\alpha, g_\alpha(x)).$$

\[\Box\]

29 There is a typo in [29, Section C] and the set $W$ there should actually be unstable.
4.2. Main Result: Asymptotic Expansion for the Horocycle Integral (Theorem 4.8).
We need some notation to state our main result: Denote by \((X + V)\)' the dual of \(X + V\) (acting on the dual of \(W_{p}^{s,t,q}(M)\)). Recall that \(\sigma((X + V)')|_{W_{p}^{s,t,q}(M)} = \sigma(X + V)|_{W_{p}^{s,t,q}(M)}\) (by [21, Section II.2.5], strong continuity of the dual semigroup is not needed for this). Therefore, by Corollary 3.9, each \(\lambda \in \sigma((X + V))\) with \(\Re \lambda > \lambda_{\min}^{s,t,p}\) is an eigenvalue of finite geometric multiplicity \(n_\lambda\) and finite algebraic multiplicities \(m_{\lambda,i}\), \(1 \leq i \leq n_\lambda\), of \((X + V)'\), with generalised eigenstates \(O_{\lambda,i,j}\) in the domain of \((X + V)'\), for \(1 \leq j \leq m_{\lambda,i}\), with

\[
((X + V)' - \lambda)^j O_{\lambda,i,j} = 0, \quad ((X + V)' - \lambda)^{-1} O_{\lambda,i,j} \neq 0, \quad 1 \leq j \leq m_{\lambda,i}.
\]

We may now state our main theorem:

**Theorem 4.8** (An Expansion for Horocycle Integrals). Let \(r > 2\) and let \(h_\rho\) be a \(C^r\) reparametrisation of the unit speed horocycle flow of a topologically mixing \(C^r\) Anosov flow \(g_\alpha\), such that \(d_\alpha = 1\), with \(E_\alpha\) orientable and \(C^{r-1}\). Assume that there exist \(t - (r - 1) < s < 0 < q < t < r - 2\) and \(p \in (d/\min\{t, r - 1 + s\}, \infty)\) with

\[
\lambda_{\min}^{s,t,p} < \delta_{\top}, \quad \text{and } \text{Condition 3.12 holds for some } \delta_{\top} > \delta > \max\{0, \delta\}.
\]

Then \(\Sigma_\delta := \sigma((X + V))|_{W_{p}^{s,t,q}(M)} \cap \{\lambda \in \mathbb{C} \mid \Re \lambda > \max\{0, \delta\}\}\) is finite, and there exists \(T_0 > 1\) such that for each \((\lambda, i, j)\) with \(\lambda \in \Sigma_\delta\), \(1 \leq i \leq n_\lambda\), and \(1 \leq j \leq m_{\lambda,i}\), there are functions

\[
c_{(\lambda,i,j)} : (T_0, \infty) \times M \to \mathbb{C}, \quad \text{with } \sup_{T > T_0, x \in M} |c_{(\lambda,i,j)}(T, x)| < \infty,
\]

and for any \(\tilde{\delta} > \max\{0, \delta\}\) there exists \(C_{\tilde{\delta}} < \infty\) such that for all \(\varphi \in C^r(M)\) and all \(T \geq T_0\)

\[
\int_0^T \varphi \circ h_\rho(x) d\rho = T \mu(\varphi) + \mathcal{E}_{T,x,\Sigma_\delta}(\varphi) + \sum_{\lambda \in \Sigma_\delta} \sum_{1 \leq i \leq n_\lambda} \sum_{1 \leq j \leq m_{\lambda,i}} T^\lambda_{\top} (\log T)^{j - 1} c_{(\lambda,i,j)}(T, x) \cdot O_{(\lambda,i,j)}(\varphi),
\]

where

\[
\sup_{x \in M} |\mathcal{E}_{T,x,\Sigma_\delta}(\varphi)| \leq C_{\tilde{\delta}}(T_{\delta/\top} \|\varphi\|_{C^r} + \|\varphi\|_{C^0}).
\]

The proof of Theorem 4.8 is given at the end of Section 4.4. We record an immediate corollary:

**Corollary 4.9** (Power Law Convergence). Under the assumptions of Theorem 4.8, there exist \(\epsilon \in (0, \min\{1 - 1/\delta_{\top}\})\) and \(C_{\epsilon} < \infty\) such that for all \(\varphi \in C^r(M)\)

\[
\left| \frac{1}{T} \int_0^T \varphi \circ h_\rho(x) d\rho - \mu(\varphi) \right| \leq C_{\epsilon} |\varphi|_{C^r} + C_{\epsilon} \|\varphi\|_{C^0}, \quad \forall T > 0.
\]

A contact form is a 1-form \(\nu \in T^*M\) such that \(\varphi \wedge (\Lambda_{\nu}^{d-1}(d\nu))\) vanishes nowhere, where \(d\nu\) denotes the exterior derivative. (A contact form can only exist if \(d\) is odd.) A flow \(g_\alpha\) on \(M\) is a contact flow if there exists a \(C^1\) contact form \(\nu\) which is preserved by the pullback of \(g_\alpha\). Geodesic flows on (the unit tangent bundle of) negatively curved compact manifolds are examples of an Anosov contact flow. Contact Anosov flows are topologically mixing [40, Thm 3.6].

---

30 Hence \(\partial_{\tau}(0, -\alpha, \cdot) \in C^{r-1}(M)\) for all \(\alpha \geq 0\).

31 Note that [109] gives a formula for \(c_{(\lambda,i,j)}\) using the generalised eigenvector of \((\lambda, i, j)\). We do not show \(\inf_{T > T_0, x} |c_{(\lambda,i,j)}(T, x)| > 0\).
Proposition 4.10. Let $g_\alpha$ be a $C^3$ contact Anosov flow on a compact manifold $M$ of dimension $d = 3$. Assume the strong-stable distribution $E_-$ is orientable. Then for any $\epsilon_1 > 0$, we may choose $r \in (2, 3)$ and $t - (r - 1) < s < 0 < t < r - 2$ such that $E_-$ is $C^{r-1}$ and such that, for any $C^r$ reparametrisation of the unit speed horocycle flow, we have $\lambda_{\text{min}}^{s,t,p} < \epsilon_1$ for all $p \in (1, \infty)$. In addition, if the flow satisfies the bunching condition \((\mathbf{2})\), there exists $p_0 > 1$ such that Condition \((\mathbf{3.12})\) holds for some $\delta' \in [\max(\lambda_{\text{min}}^{s,t,p}, 0), h_{\text{top}}]$ if $p > p_0$.

The proposition above is proved in §4.5. Its assumptions hold if $g_\alpha$ is the geodesic flow on a $C^3$ surface of strictly negative curvature ($E_-$ is $C^{2-\tilde{\eta}}$ for any $\tilde{\eta} \in (0,1)$ by [29] Thm 3.1); for the orientability of $E_-$, see [29] Lemma B.1]) with $\tilde{\omega}$ satisfying \((\mathbf{2})\). In particular, they hold if $g_\alpha$ is the geodesic flow on a $C^3$ compact surface of constant negative curvature, where $\tilde{\omega} = 2$.

We compare our main theorem and Proposition 4.10 with the results of Flaminio and Forni [28]. Let $M$ be the unit tangent bundle of a compact surface of constant negative curvature, and let $g_\alpha$ be its unit speed geodesic flow. Then the canonical volume form $\text{vol}$ on $M$ is the measure of maximal entropy for $g_\alpha$. The unit speed horocycle flow leaves $\text{vol}$ invariant as well, so that $\mu = \text{vol}$. Also, the vector fields $X$ and $V = h_{\text{top}}$ are constant, $r = \infty$, and $h_{\text{top}} = 1$ because $\tau(\rho, \alpha, x) = \rho \exp(-\alpha)$. In this setting [28] Thm 1.5] the noninteger obstructions to convergence, corresponding to our eigenvalues $\lambda$, are connected to the nonzero eigenvalues $\sigma$ of the Laplacian via $\lambda = \frac{1}{2} + \sqrt{\frac{3}{4} - \sigma}$. The eigenvalue $h_{\text{top}} = 1$ is simple, there are no other eigenvalues of real part equal to one, all eigenvalues with $\Re \lambda > 1/2$ are semi-simple, and there are only finitely many eigenvalues with $\Re \lambda > \frac{1}{2}$. Moreover, since $r = \infty$, for any $p \in (1, \infty)$ (including $p = 2$) the parameters $-s$, $t$ can be taken large enough to ensure $\lambda_{\text{min}}^{s,t,p} = \lambda_{\text{min}}^{s,t} < 0$. Since Condition 3.12 holds for some $\delta > \frac{1}{2}$ (see Proposition 4.10), we find $c_{(\lambda, 1)}(\varphi)$ and $\mathcal{E}_{T,x,\Sigma_\delta}$ as in Theorem 4.8 such that

\[
\int_0^T \varphi \circ h_{\rho}(x) \, d\rho = T \text{vol} (\varphi) + \sum_{\lambda \in \Sigma_\delta \setminus \{1\}} \sum_{i=1}^n T^\lambda c_{(\lambda,1)}(T, x) \mathcal{O}_{(\lambda,1)}(\varphi) + \mathcal{E}_{T,x,\Sigma_\delta}(\varphi). \]

4.3. Localised Horocycle Integrals, Properties of the Renormalisation Time $\tau$. In view of the smooth cutoff decomposition of $\gamma_{\omega}(\cdot, T)$ in Lemma 4.1.14 below, we introduce localised horocycle integrals as follows. For any bounded compactly supported $w : \mathbb{R} \to \mathbb{R}$ let

\[
\gamma_{w,x}(\varphi) := \int_{\mathbb{R}} w(\rho) \cdot (\varphi \circ h_{\rho}(x)) \, d\rho, \quad x \in M, \quad \varphi \in C^0(M). \tag{92}
\]

To show Theorem 4.18 it will be useful to view $\gamma_{w,x}$ as an element of the dual of $W_{p}^{s,t,q}(M)$:

Lemma 4.11. There exists $C < \infty$, depending on $\max_{\alpha \in [0, \alpha_0]} \|\phi_{-\alpha} \circ g_{-\alpha}\|_{C^{r-1}}$, the partition of unity $\vartheta_{\omega}$ (Definition 2.9), and the charts $\kappa_{\omega}$ (Definition 2.9), such that for any $w \in C_0^{r-1}(\mathbb{R})$ and $\varphi \in W_{p}^{s,t,q}(M)$

\[
sup_{x \in M} |\gamma_{w,x}(\varphi)| \leq C \sup_{x} |w| \cdot \|w\|_{C^1} \cdot \|\varphi\|_{W_{p}^{s,t,q'}}, \quad \forall t > (r - 1) < s < 0 < q' < t.
\]

Before proving it, we show an easy consequence of Lemma 4.11: The unique $h_{\rho}$ invariant measure $\mu$ belongs to the dual space of $W_{p}^{s,t,q}(M)$.  

Corollary 4.12. Let $p \in (1, \infty)$ and let $s, t$ be as in \((\mathbf{4.2})\). If $\phi_{\alpha}$ from \((\mathbf{8.3})\) is $C^{r-1}$ then $\mu \in (W_{p}^{r,t,q}(M))^\prime$. Also, by Lemma 4.6 we have $\lambda_{\text{max}}^{s,t,p} \geq h_{\text{top}}$ in $\sigma(X + V)|_{W_{p}^{s,t,q}(M)}$. 

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32 The corresponding result for the anisotropic norms of [31] is slightly more intuitive.
If $\lambda_{\text{min}}^{r,s,t,q} < \lambda_{\text{max}}^{r,s,t,q}$, the statement of this corollary could alternatively be obtained from [29, §4], see the proof of Lemma 4.15

Proof. Fix $\epsilon > 0$ small (much smaller than the diameter of $M$). For $x \in M$ denote by $C_{x,\epsilon}^{r-1}(M)$ the set of $\varphi \in C^{r-1}(M)$ which vanish in an $\epsilon$ neighbourhood of $x$. Then there exist $\delta(\epsilon) > 0$ and $C(\epsilon)$ such that for any $T > 1$ with $d(h_T(x), x) < \delta$ there exists $w_{T,\epsilon} \in \mathcal{C}_0^0(\mathbb{R}, [0,1])$ with $|\text{supp} (w_{T,\epsilon})| \leq T + 2$ and $\|w_{T,\epsilon}\|_{C^r} \leq C(\epsilon)$, such that

$$1_{[0,T]}(\rho)\varphi(h_\rho(x)) = w_{T,\epsilon}(\rho)\varphi(h_\rho(x)), \forall \varphi \in C_{x,\epsilon}^{r-1}(M), \forall \rho \in \mathbb{R}.$$ 

For any $x \in M$, since $h_\rho(x)$ is dense, there is a sequence $T_n = T_n(x, \epsilon)$ such that $d(h_{T_n}(x), x) < n^{-1}$ and $T_n \to \infty$. By unique ergodicity (77) and Lemma 4.11, we have

$$|\mu(\varphi)| \leq \lim_{n \to \infty} \frac{1}{T_n} \left| \frac{1}{T_n} \int_{T_n} \gamma_{w_{T_n,\epsilon,\alpha}}(\varphi) \right| \leq 2C(\epsilon)\bar{C}\|\varphi\|_{W_{p,s,t,q}^{r,s,t,q}}, \forall \varphi \in C_{x,\epsilon}^{r-1}(M).$$

Next, using a $C^\infty$ function $\psi = \psi_{x,y,\epsilon} : M \to [0,1]$, vanishing in an $\epsilon$ neighbourhood of $x$, and $\equiv 1$ in an $\epsilon$ neighbourhood of some $y \neq x$, we can write any $\varphi \in C^{r-1}(M)$ as $\varphi \psi + (1 - \psi)\varphi$, where $\varphi \in C_{x,\epsilon}^{r-1}(M)$, and $(1 - \psi)\varphi \in C_{y,\epsilon}^{r-1}(M)$. Applying (93) at $x$ and $y$ gives

$$|\mu(\varphi)| \leq 2C(\epsilon)\bar{C}(|\psi\varphi|_{W_{p,s,t,q}^{r,s,t,q}} + (1 - \psi)|\varphi|_W), \forall \varphi \in C^{r-1}(M),$$

where $W_{p,s,t,q}^{r,s,t,q}$ is defined like $W_{p,s,t,q}^{r,s,t,q}$, but using systems of cones $\tilde{\Theta}_\omega$ (see Remark 2.5) ensuring that $|\psi\varphi|_{W_{p,s,t,q}^{r,s,t,q}} \leq C\|\psi\|_{C^r}\|\varphi\|_{W_{p,s,t,q}^{r,s,t,q}}$ for some $C < \infty$ and all $\varphi$, $\psi$. We conclude by density of $C^{r-1}$ functions in $W_{p,s,t,q}^{r,s,t,q}(M)$. 

Proof of Lemma 4.11. Let $\delta_\star$ denote the Dirac distribution, fix $w \in \mathcal{C}_0^{r-1}(\mathbb{R})$, and set

$$w_{x,\phi,\omega_1,\omega_2,\alpha}(z) := (q\omega_2 \cdot \phi_{-\alpha} \circ g_{-\alpha}) \circ \kappa_{\omega_1}^{-1}(z) \cdot \int_{-\infty}^{\infty} w(\rho)\delta_\star(z - \kappa_{\omega_1} \circ g_{\alpha} \circ h_\rho(x))d\rho,$$

$$\varphi_{\omega,\alpha}(z) := (q\omega \cdot L_{\alpha}, \nu \varphi) \circ \kappa_{\omega_2}^{-1}(z), \ \ z \in \mathbb{R}^d, \ \ x \in M, \ \ \alpha \geq 0, \ \ \omega, \ \ \omega_1, \ \ \omega_2 \in \Omega, \ \ varphi \in C^{r-1}(M).$$

Since $h_\rho$ has no periodic orbits and $M$ is compact, $w_{x,\phi,\omega_1,\omega_2,\alpha}$ is a bounded function supported in the interior of a subset $J$ of the (one-dimensional) stable leaf at $g_\alpha(x)$ (using (78)) in charts. In addition, there exist $C_0$ such that

$$|J| \leq C_0|\text{supp} w| \quad \text{and} \quad \sup_{\alpha \in [0,\alpha_0], \ x \in J, \ \ \omega_1 \in \Omega} \|w_{x,\phi,\omega_1,\omega_2,\alpha}\|_{L^\infty} \leq C_0\|w\|_{L^\infty}.$$

Next, since $\phi_{-\alpha} \circ g_{-\alpha} = 1/\phi_{\alpha}$, we have, exchanging the integrals with respect to $z$ and $\rho$ (so that $z = \kappa_{\omega_1} \circ g_{\alpha} \circ h_\rho(x)$)

$$\gamma_{w,x}(\varphi) = \sum_{\omega_1, \omega_2 \in \Omega} \int_{\mathbb{R}^d} w_{x,\phi,\omega_1,\omega_2,\alpha}(z) \cdot \varphi_{\omega_1,\alpha}(z)dz, \ \ \forall \alpha \geq 0.$$ 

Recalling $\tilde{\Psi}_{\sigma,n}$ from (33), we find, using Plancherel’s theorem for the inner product of two functions, $\tilde{\Psi}_{\sigma,n} \Psi_{\sigma,n} = \Psi_{\sigma,n}$, Cauchy–Schwarz for the sum in $\sigma$ and $n$, Hölder’s inequality, a
constant $C < \infty$ such that, for all $p \in (1, \infty)$ and all $t - (r - 1) < s < q' \leq t$,
\[
\alpha_0 \cdot |\gamma_{w, x} (\varphi)| = \int_0^{\alpha_0} |\sum_{\omega_1, \omega_2} \int_{\mathbb{R}^d} w_{x, \varphi, \omega_1, \omega_2, \alpha}(z) \cdot \varphi_{\omega_1, \alpha}(z) dz| d\alpha \\
\leq \int_0^{\alpha_0} \sum_{\omega_1, \omega_2} |\int_{\mathbb{R}^d} 2^{-c(\sigma)n} \tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}(w_{x, \varphi, \omega_1, \omega_2, \alpha})(z) 2^{-c(\sigma)n} \tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}(\varphi_{\omega_1, \alpha})(z) dz| d\alpha \\
\leq C \sup_{\alpha \in [0, \alpha_0]} \sum_{\omega_1, \omega_2} \left\| \left( \sum_{\sigma, n} 2^{-c(\sigma)n} \tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}(w_{x, \varphi, \omega_1, \omega_2, \alpha})(z) \right)^2 \right\|_{L_{1-1/p}} \cdot \|\varphi\|_{W_p^{s,t,q'}} ,
\]
using the definition \((23)\) of the norm in \((94)\). To conclude, it suffices to find $C_0 < \infty$ such that
\[
\max_{\sigma \in \{+ , 0\}} \sup_{x \in M} \|\tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}(w_{x, \varphi, \omega_1, \omega_2, \alpha})\|_{L_{1-1/p}} \leq C_0 |\text{supp } w| \cdot \|w\|_{L_\infty} , \forall 0 \leq \alpha \leq \alpha_0 , \forall n \in \mathbb{N} ,
\]
and (since $c(+) > 0$ and $c(0) > 0$, it is enough to consider $\sigma = -$)
\[
\max_{\omega_1, \omega_2 \in \Omega} \sup_{x \in M} \|\tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}(w_{x, \varphi, \omega_1, \omega_2, \alpha})\|_{L_{1-1/p}} \leq \frac{C_0}{2^{(r-1)n}} |\text{supp } w| \cdot \|w\|_{C_\alpha} , \forall 0 \leq \alpha \leq \alpha_0 , \forall n \in \mathbb{N} .
\]
Young’s inequality for $\|\mathbb{P}^{-1}(\tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}) \ast w_{x, \varphi, \omega_1, \omega_2, \alpha}\|_{L_{1-1/p}}$ gives \((93)\) for $C_0 = C_0$ max diam $V_\omega$.

Finally, we show \((94)\). There are $\tilde{C}_0$ (depending on max $\alpha \in [0, \alpha_0]$, $\|\phi_{-\alpha} \circ g_{-\alpha}\|_{C_{r-1}, \tilde{\Psi}^{\text{Op}}}$, $\tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}$) a subset $\bar{J} \subset \mathbb{R}$, and a $C^{r-1}$ diffeomorphism $\tilde{y} : \bar{J} \to J \subset \mathbb{R}$ with $\tilde{y}$ is $\tilde{C}_0$-bounded supported in the interior of $\bar{J}$ such that $\sup_{\alpha \in [0, \alpha_0], x \in M, \omega \in \Omega} \|w_{x, \varphi, \omega_1, \omega_2, \alpha} \circ \tilde{y}\|_{C^r} \leq \tilde{C}_0 \|w\|_{C^r} , \forall \bar{r} \leq r - 1$.

Note that $J$ lies in a stable cone in charts. Thus, there exists $C_1 > 0$ such that $|\partial_y (\tilde{y}(\xi))| \geq C_1 2^n$ for any $\xi$ in the support of $\tilde{\Psi}^{\text{Op}}_{\sigma, \alpha}$ (which lies inside $E^\omega$ in charts). Finally, integrating $|r - 1|$ times by parts with respect to $y$, following by a regularised integration by parts if $r - 1$ is not an integer (Lemmas $A.1$ and $A.2$), and ending with Young’s inequality, we get \((90)\). \hfill \Box

The next two lemmas use the following version of the renormalisation equation \((83)\) for the localised horocycle integral \((92)\).

**Sublemma 4.13** (Renormalisation and Smooth Localisation). Fix $x \in M$ and $\varphi \in C^0$, then
\[
\gamma_{w, x} (\varphi) = \int_{\mathbb{R}} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot L_{\alpha, V} \varphi(h_\rho(g_\alpha(x))) d\rho , \forall \alpha \geq 0.
\]
\[\text{The bounds below can be viewed as yet another avatar of integration by parts.}\]
\[\text{As a warmup, the reader is invited to think of the case when $J$ is a subset of a coordinate axis in $\mathbb{R}^d$.}\]
Proof. By definition and our choice $\phi_\alpha = \partial_\rho \tau(0, -\alpha, \cdot)$,
\[
\int_{\mathbb{R}} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot \mathcal{L}_{\alpha, V} \varphi(h_\rho(g_\alpha(x))) d\rho = \gamma_{w(\tau(\cdot, -\alpha, g_\alpha(x)), g_\alpha(x))}(\mathcal{L}_{\alpha, \partial_\rho \tau(0, -\alpha, \cdot)} \varphi).
\]
Thus, the sublemma follows from Lemma 4.14 and the first claims of Lemma 4.15, since
\[
\gamma_{w, x}(\varphi) = \int_{-\infty}^{\infty} w(\rho) \cdot \varphi \circ g_{-\alpha} \circ h_\rho(\alpha, x) d\rho
\]
\[
= \int_{-\infty}^{\infty} w(\tau(\rho, -\alpha, g_\alpha(x)) \cdot \varphi \circ g_{-\alpha} \circ h_\rho \circ g_\alpha(x) d\rho
\]
\[
= \int_{-\infty}^{\infty} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot (\partial_\rho \tau(0, -\alpha, \cdot) \cdot \varphi \circ g_{-\alpha}) \circ h_\rho \circ g_\alpha(x) d\rho
\]
\[
= \gamma_{w, T, x}(\cdot, -\alpha, g_\alpha(x), g_\alpha(x)) (\partial_\rho \tau(0, -\alpha, \cdot) \cdot \varphi \circ g_{-\alpha}), \forall \alpha \geq 0.
\]
\[
\square
\]
Taking $w = w_T$ to be the characteristic function $w_T = 1_{[0, T]}$, we have $\gamma_{w_T, x}(\varphi) = \gamma_x(\varphi, T)$, and by choosing $\alpha = O(\log T)$ we can ensure (in view of Lemma 4.13) that the support of $w_T \circ \tau(\cdot, -\alpha, g_\alpha(x))$ has size $O(1)$, uniformly in $x$. In order to apply Lemma 4.11, some regularity of $w$ is required: we thus need a more clever choice of localisation function $w_T = w_{T, x}$. We state the corresponding result, similar to Lemma 5.16, Lemma 3.19.

Lemma 4.14 (Bounds for Localised Horocycle Integrals). Let $C > 1$ be as in Lemma 4.11 and fix $\tilde{C} > \max\{C, 4\}$. If $\phi_\alpha$ from Lemma 4.11 is $C^r - 1$, then for every $T > C\tilde{C}$ and $x \in M$ there exists a compactly supported $C^r$ function $w = w_{T, x} : \mathbb{R} \to [0, 1]$ with
\[
|\gamma_x(\varphi, T) - \gamma_{w_{T, x}}(\varphi)| \leq 2C\tilde{C} ||\varphi||_{C^0}, \forall \varphi \in C^0(M).
\]
Moreover, for $p \in (1, \infty)$ and $t - (r - 1) < s < 0 < q' \leq q \leq t$, there exists $\tilde{C} < \infty$ such that, if $\tilde{\varphi} \in W_p^{s, t, q'}(M)$ satisfies
\[
\|\mathcal{L}_{\alpha, V} \tilde{\varphi}\|_{W_p^{s, t, q'}} \leq \exp(\alpha \cdot a) \max\{1, |\alpha|^{j-1}\} C_{\tilde{\varphi}}, \forall \alpha \geq 0,
\]
for some $a > 0$, $j \geq 1$, and $C_{\tilde{\varphi}} < \infty$, then, setting $C(a) = 1/(1 - (C/\tilde{C})^{a/h_{\text{top}}})$, we have
\[
\sup_{x \in M} |\gamma_{w_{T, x}}(\tilde{\varphi})| \leq \tilde{C} \cdot C(a) T^{\tilde{a}_{\text{top}}}(\log T)^{j-1} C_{\tilde{\varphi}}, \forall T > C\tilde{C}.
\]
Proof. For $x \in M$ and $T > C\tilde{C}$, define inductively sequences $\alpha_k^\pm = \alpha_k^\pm(x, T) \in \mathbb{R}, k \geq 1$, by
\[
\tilde{C} = \tau(T, \alpha_1^+), \quad 1 = \tau(\tilde{C}, -\alpha_k^+, g_{\alpha_k^+}(x), \alpha_{k-1}^+, x) = \tau(\tilde{C}, \alpha_{k-1}^+, g_{\alpha_{k-1}^+}(x)),
\]
\[
\alpha_1^- = \alpha_1^+, \quad -1 = \tau(-\tilde{C}, \alpha_k^-, g_{\alpha_k^-}(h_T(x), -\alpha_{k-1}^-, \alpha_{k-1}^+), h_T(x)) = \tau(-\tilde{C}, -\alpha_{k-1}^-, g_{\alpha_{k-1}^-}(h_T(x)), -\alpha_{k-1}^+ - \alpha_k^+, g_{\alpha_k^+}(h_T(x))),
\]
where we used the first claim of Lemma 4.11. In the special case when $\tau(\rho, \alpha, x) = \rho e^{-ah_{\text{top}}}$ we find
\[
\alpha_1^- = \alpha_1^+ = \frac{\log(T/\tilde{C})}{h_{\text{top}}} > 0, \quad \alpha_k^+ - \alpha_{k-1}^+ = \alpha_k^- = \alpha_{k-1}^- = \frac{\log(1/\tilde{C})}{h_{\text{top}}} < 0, \quad k \geq 2.
\]
\[\]
More generally, since \( \tau(T,0,x) = T \) and \( \tau(T,\alpha,x) \) is continuous in \( \alpha \), the bounds (81)–(82) give \( 0 < \log(T/(CC)) \leq h_{\top} \cdot \alpha^+ \leq \log(TC/C) \). It is also easy to check that \( \alpha^+ \leq \alpha^- \) for all \( k \geq 2 \), and that (81) gives

\[
\{ e^{h_{\top}(\alpha^- - \alpha^-)} e^{h_{\top}(\alpha^+ - \alpha^-)} \} \subseteq \left[ \frac{1}{CC}, \frac{C}{C} \right], \quad \forall k \geq 2, \quad \forall x \in M, \forall T > \frac{1}{\tilde{C}}.
\]

Thus, we have

\[
T/(CC)^k \leq e^{h_{\top}\alpha^+} \leq TC^k/\tilde{C}^k, \quad \forall k \geq 1.
\]

Fixing a \( C^\infty \) function \( \chi : \mathbb{R} \rightarrow [0,1] \) such that \( \chi \equiv 1 \) and \( \chi \equiv 0 \), we put

\[
w_1(\rho) = \chi(\tau(\rho, \alpha^+), x) \cdot \chi(-\tau(\rho - T, \alpha^-), h_T(x)),
\]

and, for \( k \geq 2 \) (note that \( w_1 \) and the \( w_k^\pm \) depend on \( x \) and \( T \)),

\[
w_k^+(\rho) = \chi(\tau(\rho, \alpha^+_k), x) - \chi(\tau(\rho, \alpha^-_{k-1}), x),
\]

\[
w_k^-(\rho) = \chi(-\tau(\rho - T, \alpha^-_k), h_T(x)) - \chi(-\tau(\rho - T, \alpha^-_{k-1}), h_T(x)).
\]

Then, for any \( n_+ \geq 1 \), we have

\[
\begin{align*}
(102) \quad & w_1(\rho) + \sum_{k=2}^{n_+} w_k^+(\rho) + \sum_{k=2}^{n_-} w_k^-(\rho) = w_1(\rho) + \chi(\tau(\rho, \alpha^+), x) - \chi(\tau(\rho, \alpha^+), x) \\
& \quad + \chi(-\tau(\rho - T, \alpha^-), h_T(x)) - \chi(-\tau(\rho - T, \alpha^-), h_T(x)),
\end{align*}
\]

so that (82) and (101) imply \( w_1 + \sum_{k=1}^{\infty} (w_k^+ + w_k^-) = \chi \equiv 1 \). Define \( N_\pm \geq 1 \) by \( \min\{\alpha^+_N, \alpha^-_N\} \geq 0 \) and \( \max\{\alpha^+_N+1, \alpha^-_{N-1}\} \leq 0 \) (note that \( N_\pm \in [\log(T)/\log(C/C)] \) by (101)). Then put

\[
w_{T,x}(\rho) = w_1(\rho) + \sum_{k=2}^{N_+} w_k^+(\rho) + \sum_{k=2}^{N_-} w_k^-(\rho).
\]

Thus, setting \( n_\pm = N_\pm \) in (102), the definitions of \( \chi \) and \( \alpha^\pm_{N+1} \) give

\[
(103) \quad \text{supp} \left( (1_{[0,T]} - w_{T,x}) \subseteq [0, \tau(\tilde{C}, -\alpha^+_{N+1}, g_{\alpha^+_{N+1}}(x)) \cup [T + \tau(-\tilde{C}, -\alpha^-_{N-1}, g_{\alpha^-_{N-1}}(h_T(x)), T)].
\]

Using \( \max\{\alpha^+_{N+1}, \alpha^-_{N-1}\} \leq 0 \), the claim (97) then follows from (82).

Next, for \( \tilde{\varphi} \in W_p^{s,t,q}(M) \), using \( \gamma_{v,x}(\tilde{\varphi}) = \gamma_{v(\cdot + T), h_T(x)}(\tilde{\varphi}) \), Sublemma 4.13 gives

\[
(104) \quad \gamma_{w_{T,x}}(\tilde{\varphi}) = \gamma_{\tilde{w}_1, g_{\alpha^+}} (L_{\alpha^+_1}, \tilde{\varphi}) + \sum_{k=2}^{N_+} \gamma_{\tilde{w}_k, g_{\alpha^+_k}} (L_{\alpha^+_k}, \tilde{\varphi}) + \sum_{k=2}^{N_-} \gamma_{\tilde{w}_k, g_{\alpha^-_k}} (h_T(x), \tilde{\varphi}),
\]

where, recalling (86), we put \( \tilde{w}_1 = w_1(\tau(\cdot, -\alpha^+_{1}, g_{\alpha^+_{1}}(x))) = \chi \cdot \chi(\tilde{C} - \cdot) \), and

\[
\tilde{w}_k^+(\rho) = w_k^+(\tau(\rho, -\alpha^+_{k}, g_{\alpha^+_{k}}(x))), \quad \tilde{w}_k^-(\rho) = w_k^- (T + \tau(\cdot, -\alpha^-_{k}, g_{\alpha^-_{k}}(h_T(x))), k \geq 2.
\]

\[\text{This is analogous to the decomposition in [31] Lemma 3.1. We use more explicit smoothing functions.}\]
Since \( \tilde{w}_k^+(\rho) = \chi(\rho) - \chi(\tau(\rho, \alpha_{k-1}^+ - \alpha_k^+; g_{\alpha_k^+}(x))) \) and also \( \tilde{w}_k^-(\rho) = \chi(-\rho) - \chi(-\tau(\rho, \alpha_{k-1}^- - \alpha_k^-; g_{\alpha_k^-}(h_T(x)))) \), we find

\[
supp \tilde{w}_1 \subseteq [0, C\bar{C}], \quad supp \tilde{w}_k^+ \cup -supp \tilde{w}_k^- \subseteq [0, C\bar{C}] \quad \forall k \in \mathbb{N}.
\]

Since \( \phi_\alpha \in C^{r-1} \), (79)–(80) imply \( \sup \| \partial_\rho \tau(\cdot, \alpha, x) \|_{C^{r-1}} < \infty \), so (100) gives

\[
\sup_{x \in M} \max \{ \| \tilde{w}_1 \|_{C^r}, \sup_{k \geq 2} \| \tilde{w}_k^+ \|_{C^r} \} < \infty.
\]

Thus, if (98) holds for \( \tilde{\varphi} \), applying Lemma 4.11 and (101) to (104), we find \( \hat{C} < \infty \) such that

\[
\sup_x |\gamma_{w_{T\alpha,\bar{V}}}(\tilde{\varphi})| \leq \hat{C}C_{\tilde{\varphi}} T^{h_{top}} (\log T)^{-1} \sum_{k=1}^{\max\{N_-, N_+\}} (C/\hat{C})^{k-\frac{a}{h_{top}}},
\]

for all \( T > C\bar{C} \). Since \( 4 < C < \hat{C} \), summing \( \sum_{k=1}^\infty (C/\hat{C})^{k-\frac{a}{h_{top}}} \) gives (99). \( \square \)

### 4.4. Exact Bounds \( \sup_{\alpha \geq 0} \left\| e^{-a h_{top}} \mathcal{L}_{\alpha,V} \right\|_{W_{p,s,t,q}^{s,t,q}} < \infty \). Proof of Theorem 4.8.

We saw in Lemma 4.6 that \( \mu \), the unique \( h_\rho \) invariant probability, is a fixed point of \( e^{-h_{top}a} \mathcal{L}_{\alpha,V} \) acting on Radon measures, in Remark 4.7 that \( \sup_{\alpha \geq 0} \left\| e^{-h_{top}a} \mathcal{L}_{\alpha,V} \right\|_{L^1(\mu)} \leq 1 \), and in Corollary 4.12 that \( \mu \in (W_{p,s,t,q}(M))^\prime \) so that \( \lambda_{min}^{s,t,p} \geq h_{top} \). If \( \lambda_{max}^{s,t,p} < h_{top} \), we get more.\(^{37}\)

**Lemma 4.15** (Peripheral Spectrum and Exact Growth). If \( E_- \) is \( C^{r-1} \) and \( \lambda_{min}^{s,t,p} < h_{top} \) for some \( p \in (1, \infty) \) and \( s, q, t \) as in (12), then for all \( 0 < q < t \) we have \( \lambda_{max}^{s,t,p} = h_{top} \). Moreover, \( h_{top} \) is a simple eigenvalue and the only element of \( \{ \lambda \in \sigma(X + V)\}_{W_{p,s,t,q}(M)}, \Re(\lambda) = h_{top} \}. \) In particular, there are no maximal Jordan blocks, and\(^{38}\) \( \sup_{\alpha \geq 0} \left\| e^{-a h_{top}} \mathcal{L}_{\alpha,V} \right\|_{W_{p,s,t,q}^{s,t,q}} < \infty \).

For the potential \( V \) associated to the SRB measure, where \( \lambda_{max}^{s,t,q,p} = 0 \), the results above are well known, see [16] Lemma 5.1 and [17] (the claims there are for other Banach spaces, but intrinsicness can be applied as in our proof of Lemma 4.15).

**Remark 4.16** (MME and Bypassing Unique Ergodicity). Exploiting the results of [29] as in the proof of Lemma 4.15, it can be shown (without using Corollary 4.12), that the unique fixed point of \( e^{-h_{top}a} \mathcal{L}_{\alpha,V} \) in the dual of \( W_{p,s,t,q}^{s,t,q}(M) \) is a Radon measure \( \mu \), and letting \( \nu \in W_{p,s,t,q}^{s,t,q}(M) \) be the unique fixed point of \( e^{-h_{top}a} \mathcal{L}_{\alpha,V} \), that the distribution formally defined by \( \mu_{\nu}(\varphi) = \mu(\varphi \nu) \) is a Radon measure, and it is the unique measure of maximal entropy (MME) of \( g_{\alpha} \), which is (exponentially) mixing. (See e.g. [33] for the discrete-time analogue.) In fact, unique ergodicity of \( h_\rho \) (that is (77)) could be obtained from the information on the peripheral spectrum of \( \mathcal{L}_{\alpha,V} \), bypassing the results of Bowen and Marcus from [14]. To keep the paper short, we refer to [14].

Before showing Lemma 4.15 we state and prove consequences of the exact growth.

\(^{37}\)In fact, only the exact growth claim is needed from Lemma 4.15. The rest of the information about the peripheral spectrum could be obtained by an ad hoc argument based on [85], the identity in the proof of Lemma 4.16 (99), and Sublemma 4.13. See [1] Lemma 5.18 (v), and last claim of Lemma 5.14.

\(^{38}\)This exact growth estimate is a key ingredient, e.g., for [15] Assumption 1] used in the proof of Theorem 4.18. See e.g. the inverse Laplace transform in [15] Lemma 4.3.
Corollary 4.17 (Exact Growth for the Resolvent). Assume that $E_-$ is $C^{-1}$ and $\lambda^{s,t} \min < H_{\text{top}}$ for $p \in (1, \infty)$ and $s, t$ as in \cite{[12]}. Fix $0 < q < t$. There exists $C < \infty$ such that
\[ \| R_{z}^{n} \varphi \|_{W^{s,t,q}} \leq \frac{C}{(\mathbb{R}z - H_{\text{top}})^{n}} \| \varphi \|_{W^{s,t,q}}, \forall \mathbb{R}z > H_{\text{top}}, \forall n \geq 1. \]
Moreover, recalling \cite{[55]}, there exist $C < \infty$ and a system $\Theta = \{ \Theta'_\omega \}$ with $\Theta'_\omega < \Theta_\omega$ such that
\[ \| R_{z}^{n} \varphi - R_{tr,z}^{n} \varphi \|_{W^{s,t,q}} \leq \frac{C}{(\mathbb{R}z - H_{\text{top}})^{n}} \| \varphi \|_{W^{s,t,q}}, \forall \mathbb{R}z > H_{\text{top}}, \forall n \geq 1. \]

Proof. The first bound follows from exact growth ($\sup_{\alpha \geq 0} \| e^{-\alpha H_{\text{top}}} L_{\alpha,V} \|_{W^{s,t,q}} < \infty$), simplifying \cite{[38]}. The second claim follows from exact growth, using Remark 2.5 (for $\alpha \geq \alpha_0$) with \cite{[55]}. \[ \square \]

Proof of Lemma 4.15 By Corollary 3.9, since $\lambda^{s,t} \min < H_{\text{top}}$, we claim that we can exploit Theorem 3.10 about intrinsicness to transfer \cite{[47]} the results of Giulietti, Liverani, and Pollicott in \cite{[29]} to our spaces.

Indeed, first recall that a $C^{-1}$ one-form is a $C^{-1}$ section of the cotangent bundle $T^{*}M$, or equivalently, a $C^{-1}$ map from the tangent space $TM$ to $\mathbb{R}$ whose restriction to each fibre $T_{x}M$ is a linear functional on $T_{x}M$. Using that the Anosov flow $g_{\alpha}$ is topologically mixing (and $E_-$ is orientable), they showed \cite{[29]} (4.5), Lemma 4.7, Proposition 4.9, for $\ell = d_{=} = 1$ that $H_{\text{top}}$ (denoted $\sigma_{d_{=}}$ there, with $d_{=} - d_{=}$) is a simple eigenvalue and the only element $\lambda$ of the spectrum with $\Re \lambda \geq H_{\text{top}}$ for the generator $Y^{(d_{=})}$ of the pullback semigroup $\mathcal{L}_{d_{=}}^{(d_{=})}$ of $g_{\alpha}$ \cite{[29]} (2.9) acting on the closure $\overline{B^{1,|s|,d_{=}}_{\alpha}}$ of $C^{r_{=}}$ one-forms on $M$ vanishing in the flow direction, for an anisotropic Banach norm (see \cite{[29]} Def. 3.6 and (4.6)) for $\ell = d_{=} = 1, p = 1, \text{ and } q = t$, and note that this is equivalent to letting the pullback semigroup act on the Grassmannian of line bundles in $TM$ as in \cite{[33],[31]}). A key step for this is the fact that, setting $\lambda_{\min}^{1,|s|} := H_{\text{top}} + \min\{1, |s|\} \log \theta < H_{\text{top}}$, the intersection of the spectrum of $Y^{(d_{=})}$ on $\overline{B^{1,|s|,d_{=}}_{\alpha}}$ with the half-plane $\Re \lambda > \lambda_{\min}^{1,|s|}$ contains only isolated eigenvalues of finite multiplicity (this is shown by establishing the corresponding result for the resolvent $R_{z}^{(d_{=})}$ \cite{[29]} Def. 4.4, Lemma 4.8)).

Next, recall that, by our assumptions, $r \geq 2$ and $E_{-}$ is $C^{-1}$ (so that $E_{+}^{*}$ is $C^{-1}$ too). The (closed) subspace $\overline{B^{1,|s|,d_{=}}_{\alpha}}$ of $\overline{B^{1,|s|,d_{=}}_{\alpha}}$ obtained by taking the closure (for the norm of $\overline{B^{1,|s|,d_{=}}_{\alpha}}$) of the space $\Omega_{E_{-}^{*}}^{-1}$ of those $C^{r_{=}}$ one-forms taking values in $E_{+}^{*}$, is invariant under $\mathcal{L}_{d_{=}}^{(d_{=})}$. Using the natural bijection $\varphi \mapsto (\varphi(\cdot), E_{+}^{*}(\cdot))$ from $C^{r_{=}}(M)$ to $\Omega_{E_{-}^{*}}^{-1}$, we see that the restriction of $\mathcal{L}_{d_{=}}^{(d_{=})}$ to $\Omega_{E_{-}^{*}}^{-1}$ coincides with our operator $\mathcal{L}_{d_{=}}^{V_{d_{=}}}$ on $C^{r_{=}}(M)$. It is well-known \cite{[10]} that restricting a bounded operator $\mathcal{R}$ to a closed invariant subspace $\overline{B} \subset \overline{B}$ can fill up the holes (a hole in a compact set of $\mathbb{C}$ is a bounded connected component of its complement) in the original spectrum, but the spectrum of the restriction does not intersect the unbounded connected component of $\sigma(\mathcal{R}|_{\overline{B}})$ (see \cite{[17]} Corollary 4.1]). Hence, the intersection of the spectrum of $Y^{(d_{=})}$ on $\overline{B^{1,|s|,d_{=}}_{\alpha}}$ with the half-plane $\Re \lambda > \lambda_{\min}^{1,|s|}$ still contains only isolated eigenvalues of finite multiplicity. Some of the eigenvalues of $Y^{(d_{=})}$ on $\overline{B^{1,|s|,d_{=}}_{\alpha}}$ can disappear for the restricted operator, but we already established in Corollary 4.12 that $H_{\text{top}}$ is an eigenvalue in our space. Finally, since $\max\{\lambda^{s,t} \min, \lambda_{\min}^{1,|s|}\} < H_{\text{top}},$
we can apply Theorem 3.10 using also that $C^{r-1}$ functions are dense in $W^{s,t,q}_p$ and in $\mathcal{B}^{1,|s|}$, and that $W^{s,t,q}_p$ and $\mathcal{B}^{1,|s|}$ are both continuously embedded into the dual of $C^{[s]+\max\{1,t\}}$ [29] Lemma 3.10].

\[ \square \]

**Proof of Theorem 4.8** The starting point of the proof is

\[ \gamma_x(1,T)\mu(\varphi) = \int_0^T \mu(\varphi)d\rho = T\mu(\varphi) \]

(this is trivial for the unit speed horocycle flow, an easy computation otherwise). Thus, we may and shall assume $\mu(\varphi) = 0$, replacing $\varphi$ by $\varphi - \mu(\varphi)$ (constants belong to our Banach space).

Fix $0 < q < t$. By Lemma 4.15 $\lambda_{\text{max}} = \lambda_{\text{top}}$ is a simple eigenvalue and the only maximal eigenvalue of $X + V$ on $W^{s,t,q}_p(M)$. Hence, $O_{\text{top}} := O_{\text{top},1,1} = \mu$, so that $O_{\text{top}}(\varphi) = 0$.

For a general Ruelle–Pollicott resonance $\lambda \in \sigma(X + V)|_{W^{s,t,q}_p(M)}$ with $\Re \lambda > \lambda_{\text{min}}$, recalling the notation introduced above (90), we denote by $D(\lambda,i,j) \in D(X + V)$, for $1 \leq i \leq n_\lambda$ and $1 \leq j \leq m_{\lambda,i}$, its generalised eigenstates, i.e.,

\[ (X + V - \lambda)^2D(\lambda,i,j) = 0, \quad (X + V - \lambda)^{-1}D(\lambda,i,j) \neq 0. \]

We write $D_{\text{top}} := D_{\text{top},1,1}$. There is a curve $\Gamma_{\lambda}$ around $\lambda$ with $\frac{1}{2\pi i} \int_{\Gamma_{\lambda}} (z - (X + V))^{-1}\varphi dz = \sum_{j=1}^{m_{\lambda,i}} \Pi_{\lambda,j} \varphi$, where $\Pi_{\lambda,j}$ is a projector of rank $m_{\lambda,i}$, with

\[ \Pi_{\lambda,j} = \sum_{j=1}^{m_{\lambda,i}} D(\lambda,i,j) \otimes O(\lambda,i,j), \quad 1 \leq i \leq n_\lambda, \quad O(\lambda,i,j) \in D((X + V)'), \]

where $O(\lambda,i,j)(D(\lambda,2,j)) = 1$ if $(\lambda_1,i_1,j_1) = (\lambda_2,i_2,j_2)$ and $O(\lambda,i,j)(D(\lambda,2,j_2)) = 0$, otherwise. In addition, there are finite rank nilpotent operators $N_{\lambda,i}$, for $1 \leq i \leq n_\lambda$, such that

\[ \Pi_{\lambda_1,i_1} \Pi_{\lambda_2,i_2} = 0 \quad \text{and} \quad N_{\lambda_1,i_1} N_{\lambda_2,i_2} = 0 \quad \text{if} \quad \lambda_1 \neq \lambda_2 \text{ or } i_1 \neq i_2, \]

\[ N_{\lambda,i}^{m_{\lambda,i} - 1} = 0, \quad \Pi_{\lambda_1,i} N_{\lambda_2,i} = N_{\lambda_2,i} \Pi_{\lambda_1,i} = \begin{cases} N_{\lambda_2,i} & \text{if } \lambda_1 = \lambda_2 \text{ and } i_1 = i_2 \\ 0 & \text{if } \lambda_1 \neq \lambda_2 \text{ or } i_1 \neq i_2, \end{cases} \]

and, using the surjection from eigenvalues of $X + V$ to those of the semi-group [21] V (2.3)]

\[ L_{\alpha,V} \Pi_{\lambda,i} = \exp(\alpha \lambda) \exp(\alpha N_{\lambda,i}) \Pi_{\lambda,i}, \quad \forall \alpha \geq 0. \]

(See also e.g. [15].) Therefore, for each $(\lambda, i, j)$ there exists $C_{i,j} < \infty$ such that

\[ \|L_{\alpha,V}D(\lambda, i, j)\|_{W^{s,t,q}_p} \leq C_{i,j} \exp(\alpha \cdot \Re \lambda) \max\{1, |\alpha|^2 - 1\}\|D(\lambda, i, j)\|_{W^{s,t,q}_p}, \quad \forall \alpha \in \mathbb{R}. \]

In other words, $D(\lambda, i, j)$ satisfies (98) for all $\alpha \in \mathbb{R}$ if $a = \Re \lambda > \lambda_{\text{min}}$.

Assume that $\delta > 0$, let

\[ \lambda \in \Sigma_{\delta} = \sigma(X + V)|_{W^{s,t,q}_p(M)} \cap \{z \in \mathbb{C} \mid \Re z \geq \delta\} \]

and fix $x \in M$. Let $T \geq T_0 = CC > 1$ and $w_{T,x} \in C_0^{r-1}$ be given by Lemma 4.14 and define

\[ c(\lambda,i,j)(T,x) := T^{-\frac{1}{2}} \langle w_{T,x} \lambda \rangle (D(\lambda, i, j)) \in \mathbb{C}, \quad 1 \leq i \leq n_\lambda, \quad 1 \leq j \leq m_{\lambda,i}. \]
Then (99) from Lemma 4.14 implies that \( \sup_{x,T} |c_{(\lambda,i,j)}(T,x)| < \infty \). Decomposing
\[
\gamma_{w_{T,x}}(\Pi_{\lambda,i}\varphi) = \sum_{j=1}^{m_{\lambda,i}} O_{(\lambda,i,j)}(\varphi)\gamma_{w_{T,x}}(D(\lambda,i,j)) = \sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)} T^{\lambda_{\text{top}}}(\log T)^{j-1} O_{(\lambda,i,j)}(\varphi),
\]
and using Lemma 4.15 we find for any finite subset \( \Lambda_\delta \subset \Sigma_\delta \) that
\[
\gamma_x(\varphi, T) = \gamma_{w_{T,x}}(D_{h_{\text{top}}}) \mu(\varphi) + \sum_{\lambda \in \Lambda_\delta} \sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)} T^{\lambda_{\text{top}}}(\log T)^{j-1} O_{(\lambda,i,j)}(\varphi) + \mathcal{E}_{T,x,\Lambda_\delta}(\varphi),
\]
where \( \gamma_{w_{T,x}}(D_{h_{\text{top}}}) \mu(\varphi) = 0 \), and
\[
\mathcal{E}_{T,x,\Lambda_\delta}(\varphi) = \gamma_{w_{T,x}}(\varphi - \sum_{\lambda \in \Lambda_\delta} \sum_{j=1}^{n_{\lambda}} \Pi_{\lambda,i}\varphi) + \gamma_x(\varphi, T) - \gamma_{w_{T,x}}(\varphi).\]

To conclude, we show that finiteness of \( \Sigma_\delta \) and the claimed bound on \( \mathcal{E}_{T,x,\Sigma_\delta} \) follow from Condition 3.12. We first check that Assumptions 1, 2, and 3A from [15] hold for the semigroup \( e^{-h_{\text{top}} \alpha} L_{\alpha,V} \) on \( W_{p,s,t,q}(M) \) (with generator \( X + V - h_{\text{top}} \)), resolvent \( R_{z+h_{\text{top}}} \): Note that \( R_{h_{\text{top}}} = (h_{\text{top}} - (X + V))^{-1} \) is bounded on the codimension one subset \( W(h_{\text{top}}) \) of \( W_{p,s,t,q} \) formed of those \( \tilde{\varphi} \) such that \( \mu(\tilde{\varphi}) = 0 \). Therefore, the norm on \( W(h_{\text{top}}) \) defined by
\[
\|\tilde{\varphi}\|_{\text{weak}} = \frac{\|(h_{\text{top}} - (X + V))^{-1}(\tilde{\varphi})\|_{W_{p,s,t,q}^{\top}}}{\|R_{h_{\text{top}}}\|}
\]
satisfies \( \|\tilde{\varphi}\|_{\text{weak}} \leq \|\tilde{\varphi}\|_{W_{p,s,t,q}} \). The identity \( \tilde{\varphi} - e^{-h_{\text{top}} \alpha} L_{\alpha,V} \tilde{\varphi} = (h_{\text{top}} - (X + V)) \int_0^\alpha e^{-h_{\text{top}} \tilde{\alpha}} L_{\tilde{\alpha},V} \tilde{\varphi} \tilde{\alpha} \) thus implies Assumption 1 in [15], i.e.
\[
\sup_{\alpha > 0} \frac{1}{\alpha} \|\text{id} - e^{-h_{\text{top}} \alpha} L_{\alpha,V}\|_{W_{p,s,t,q}(M) \to \text{weak}} < \infty, \ \forall \tilde{\varphi} \in W(h_{\text{top}}).
\]
(Indeed, it is enough to consider \( \alpha \in (0,1] \) in (110) due to the exact growth.) Since \( h_{\text{top}} - \lambda_{\text{min}} > 0 \), the essential spectral radius of \( R_{z+h_{\text{top}}} \) is not larger than \( |R_{z+h_{\text{top}}} - \lambda_{\text{min}}|^{-1} \) by Corollary 3.9 giving Assumption 2 in [15]. Finally, since \( p > d/\min\{t, r - 1 + s\} \), Proposition 3.13 and Corollary 3.17 imply that Condition 3.12 gives (63), i.e., Assumption 3A from [15] for \( R_{z+h_{\text{top}}} \).

Thus [15 Thm 1] gives \( \#\Sigma_\delta < \infty \) and furnishes, for \( \delta > \delta_0 \), a constant \( C_B(\delta) < \infty \) with
\[
\|e^{-h_{\text{top}} \alpha} L_{\alpha,V}(\psi - \sum_{\lambda \in \Sigma_\delta} \sum_{i=1}^{n_{\lambda}} \Pi_{\lambda,i}\psi)\|_{\text{weak}} \leq C_B e^{\delta_0}(h_{\text{top}} - (X + V))\psi\|_{W_{p,s,t,q}^{\top}}, \ \forall \alpha \geq 0,
\]
for all \( \psi \in W(h_{\text{top}}) \).

Finally, Lemma 4.14 gives the bound (91) for \( \sup_x |\mathcal{E}_{T,x,\Sigma_\delta}(\varphi)| \): First, \( \text{97} \) implies that \( \sup_{x,T} |\gamma_x(\varphi, T) - \gamma_{w_{T,x}}(\varphi)| \leq C\|\varphi\|_{\alpha} \). Second, setting \( \psi_j = (h_{\text{top}} - (X + V))^{j}\varphi \), for \( j = 1, 2 \), we have \( \max_{j=1,2} \|\psi_j\|_{W_{p,s,t,q}} \leq C\|\varphi\|_{W_{p,s,t,q}} \), because \( V \in C^{r-1} \), and \( t < r - 2 \). Then (99) (with \( \tilde{\varphi} = \psi_1 \in W_{p,s,t,q}^{\top}(M) \) and \( C_{\tilde{\varphi}} = C_B((h_{\text{top}} - (X + V))\tilde{\varphi}\|_{W_{p,s,t,q}} \) gives \( T_0 < \infty \) such that
\[
\sup_x |\gamma_{w_{T,x}}(\varphi - \sum_{\lambda \in \Sigma_\delta} \sum_{i=1}^{n_{\lambda}} \Pi_{\lambda,i}\varphi)| \leq C\tilde{C}(\delta) C_B(\delta) \cdot T^{\delta/h_{\text{top}}}(h_{\text{top}} - (X + V))\psi_1\|_{W_{p,s,t,q}}, \ \forall T \geq T_0.
\]
4.5. Proof of Proposition 4.10. We assumed the flow fixes a $C^1$ contact 1-form $\nu \in T^* M$. In particular, $\nu$ is annihilated on $E_+ + E_-$ and the volume in $\wedge^3 T^* M$ is preserved by the flow. Then, since $d = 3$, we already mentioned that [29, Thm 3.1] gives that $E_-$ is $C^{2,\tilde{\eta}}$ for any $\tilde{\eta} \in (0,1)$. Taking $r = 3 - \tilde{\eta}$, we find $\partial_\nu \tau(0,-\alpha,\cdot) \in C^{r-1}$ for any $C^r$ reparametrisation of the unit speed horocycle flow.

It follows from (130) and Lemma 3.5 that the transfer operators associated to $C^r$ reparametrisations are conjugate to each other and thus have the same spectrum (using Remark 2.5). For the unit speed parametrisation we have $\phi_\alpha = \partial_\nu \tau(0,-\alpha,0) = \det Dg_\alpha|_{E_-}$. We claim that

$$\lambda^{s,t,p}(X,V) = \lim_{\alpha \to \infty} \frac{1}{\alpha} \log \|\phi_\alpha\| \det(Dg_\alpha)^{tr} |_{E_-^\prime}^{\min\{t,s\}} \|L_\infty(M).$$

Indeed, $d_- = d_+ = 1$, and since the flow $g_\alpha$ preserves volume, i.e. $\det Dg_\alpha \equiv 1$, we find

$$\|\det(Dg_\alpha)^{tr} |_{E_-^\prime} = \|\det(Dg_{-\alpha}g_\alpha)^{tr} |_{E_0^\prime} = \|\det(Dg_{-\alpha}g_\alpha)^{tr} |_{E_-^\prime}.$$ 

Using (145) and the upper and lower bounds on $\|\det(Dg_{-\alpha}g_\alpha)^{tr} |_{E_0^\prime}$, we get (112).

Then, taking $-s = t = \frac{r-1}{2} - \frac{\tilde{\eta}}{2} = 1 - \tilde{\eta}$, we have $t + r - 1 \leq s < 0 < t < r - 2$, and formula (112) together with (130) give that $\lambda^{s,t,p}_\alpha < \epsilon_1$, if $\tilde{\eta} > 0$ is small enough.

It remains to discuss Condition 3.12. (This condition is stable under reparametrizations of the horocycle flows, using (54) and the conjugacy mentioned in the previous paragraph.) Since we assumed (2), the second [14] claim of [29, Proposition 7.5] holds for $\ell = d_-$. Therefore, since our operator $\mathcal{R}_z$ coincides with the operator denoted $\mathcal{R}^{(d_z)}(z)$ in [29] restricted to those one-forms which take their images in $E_{-}^\prime$ (as in the proof of Lemma 4.15), the second claim of [29, Proposition 7.5] and [29, Lemma 7.4] combined with [29, (7.1)] (which holds due to the exact growth bounds) and (65) (for $\delta_1 = 0$, $\delta_2 > 0$ and $\beta = 3\gamma_0$) gives $\eta \in (0,1)$, $\rho_0 \geq 1$, $b_0' \geq 1$, $\delta_2 \in (0,\rho_{top})$, $C < \infty$, $\gamma_0 \in (0,1)$, $C_1 > 1$, and, for any $a \geq 0$ and $\gamma' \geq \alpha C_1$ such that $\gamma' < 3\gamma_0 / \log(1 + \delta_2/a)$, we have

$$\|\mathcal{R}_z^n_{a+b_0+h_{top}} \varphi \|_{\mathcal{B}^{0,1+\eta}} \leq C[a + \delta_2]^{-\eta} \|\varphi\|_{\mathcal{B}^{1,\eta}}, \forall |b| \geq b_0', \text{ where } n = \lceil \gamma / \log |b| \rceil,$$

for the anisotropic Banach spaces $\mathcal{B}^{j,\eta}$, $j = 0, 1$, in the scale of the proof of Lemma 4.15. (The statement of [29, Proposition 7.5] is for $a \in [a_0, 2a_0]$ and $\gamma' \geq C_1$, the proof gives (113).)

By [29, Lemma 3.10] the space $\mathcal{B}^{1,\eta}$ lies in the dual of $C^{1+\eta}(M)$. Thus, if $s'' < -1 - \eta - 3 - \frac{3}{p}$, we have $\|\varphi\|_{W_p^{s'',s''}} \leq C[\|\varphi\|_{\mathcal{B}^{1,\eta}}]$ (use Sobolev embeddings). (For the dual $\mathcal{B}^{1,-\eta}_0 \subset C^{1,1+\eta}(M)$ [46, Def. 2.1.3.1(ii), Remark 2.1.5.1] and $W_p^p = F_p^{p/2}$ in dimension $d = 3$.) The last bound of [29, Remark 3.8] gives $\|\varphi\|_{\mathcal{B}^{1,\eta}} \leq C[\|\varphi\|_{C^1}]$, ending the proof.

**Appendix A. Integration by Parts**

**Lemma A.1** (Integration by Parts (cf. text after [9, Remark 3.3])). Let $f: \mathbb{R}^d \to \mathbb{R}$ be $C^1$ and compactly supported. For any $C^2$ function $G: \mathbb{R}^d \to \mathbb{R}$ such that $\inf \sup f \sum_{j=1}^d (\partial_j G)^2 > 0$,

$$\int_{\mathbb{R}^d} e^{iG(z)} f(z) dz = i \int_{\mathbb{R}^d} e^{iG(z)} \sum_{k=1}^d \partial_k \sum_{j=1}^d (\partial_j G(z))^2 dz.$$

The proof of [29, Proposition 7.5] has a gap since a factor $e^{iW^{1,2}H_{\beta,1.1.1,1}}$ is missing from [29, (7.14)]. However, the statement is correct [30, Theorem 1 and its proof], replacing the condition $\min \{1, \frac{\eta}{2} \} > 2/3$ in [29] by: “$\lambda_+ - \lambda_- < \vartheta_0 \lambda_-$ with $\vartheta_0 \in (0, \frac{1}{2})$ and $\vartheta_0 \geq 1$,” which hold since we assumed (2).

We may take $\delta_2 > 0$ small enough in (54) to ensure $aC_1 < 3\gamma_0 / (1 + \delta_2/a)$. 

\footnote{The proof of [29, Proposition 7.5] has a gap since a factor $e^{iW^{1,2}H_{\beta,1.1.1,1}}$ is missing from [29, (7.14)]. However, the statement is correct [30, Theorem 1 and its proof], replacing the condition $\min \{1, \frac{\eta}{2} \} > 2/3$ in [29] by: “$\lambda_+ - \lambda_- < \vartheta_0 \lambda_-$ with $\vartheta_0 \in (0, \frac{1}{2})$ and $\vartheta_0 \geq 1$,” which hold since we assumed (2).}

\footnote{We may take $\delta_2 > 0$ small enough in (54) to ensure $aC_1 < 3\gamma_0 / (1 + \delta_2/a)$.}
For $z \in \mathbb{R}^d$ we write $\nabla_z G = (\partial_j G(z))_{j=1,\ldots,d}$ for the gradient and $\nabla^\mu_z G = \sum_{j=1}^d \partial_j G(z)$ for the divergence of $G : \mathbb{R}^d \to \mathbb{R}$. Let $\nu : \mathbb{R}^d \to \mathbb{R}_+$ be $C^\infty$, supported in the unit ball, and such that $\int_{\mathbb{R}^d} \nu(x)dx = 1$. Then we have:

**Lemma A.2 (Regularised Integration by Parts [9 (3.4)]).** Fix $1 < r < 2$. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a compactly supported $C^{r-1}$-map, let $G : \mathbb{R}^d \to \mathbb{C}$ be $C^r$ and such that $|\nabla_z G(z)|^2 = \sum_{j=1}^d (\partial_j G)^2 > 0$ on supp $\nu$. Set $^{\text{43}}$

$$h(z) := i \frac{\nabla_z G(z) f(z)}{|\nabla_z G(z)|^2}, \quad h_\delta := \delta^{-d} \cdot h * \nu\left(\frac{\cdot}{\delta}\right), \quad \delta > 0.$$  

Then, for every $L \geq 1$,

$$\int_{\mathbb{R}^d} e^{iLG}(z)f(z)dz = \frac{1}{L} \int_{\mathbb{R}^d} e^{iLG}(z)\nabla^\mu_z h_\delta(z)dz - i \int_{\mathbb{R}^d} e^{iLG}(z)\nabla^\mu_z G(z)(h(z) - h_\delta(z))dz.$$  

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**APPENDIX B. FRAGMENTATION AND RECONSTRUCTION**

A finite set of $C^r$ functions $\vartheta_j : \mathbb{R}^d \to [0,1]$ such that $\sum_j \vartheta_j(x) \leq 1$ for all $x \in \mathbb{R}^d$ is called a $C^r$ sub-partition of unity. The fragmentation and reconstitution lemmas of [9] and [5] extend straightforwardly to our anisotropic spaces. The first lemma is a variant of [9 Lemma 7.1]:

**Lemma B.1 (Fragmentation).** Let $1 < p < \infty$ and let $s, t, q$ as in (21) be real numbers, and let $K \subset \mathbb{R}^d$ be compact. For any $s', t', q' \in \mathbb{Z}$, there exists $C < \infty$ such that, for any $C^r$ sub-partition of unity $\{\vartheta_j\}_{j=1,\ldots,J}$ of $K$ with intersection multiplicity $\nu$, there exists $C_\vartheta < \infty$ such that (in the applications, we take $s' < s$, $t' < t$, and $q' \leq q$)

$$ \left(\sum_{j=1}^J \vartheta_j v\right)_{W^{s,t,q}_{p,\vartheta}} \leq C \nu^{(p-1)/p} \left(\sum_{j=1}^J \vartheta_j v\right)_{W^{s,t,q}_{p,\vartheta}}^{1/p} + C_\vartheta \|v\|_{W^{s',t',q'}_{p,\vartheta}}.$$

The last lemma, a variant of [9 Prop. 7.2] (see also [5 Lemma 4.29]), is useful to group partitions of unity associated with a fixed cone system:

**Lemma B.2 (Reconstitution).** Let $1 < p < \infty$, let $s, t, q$ as in (21) be real, and let $K \subset \mathbb{R}^d$ be compact. If $\Theta' < \Theta$ then for any $s', t', q' \in \mathbb{Z}$, there exists $C < \infty$ such that, for any $C^r$ sub-partition of unity $\{\vartheta_j\}_{j=1,\ldots,J}$ of $K$ with intersection multiplicity $\nu$, there exists $C_{\vartheta'} < \infty$ such that (in the applications, we take $s' < s$, $t' < t$, and $q' \leq q$)

$$ \left(\sum_{j=1}^J \vartheta_j v\right)_{W^{s,t,q}_{p,\vartheta}}^{1/p} \leq C \nu^{1/p} \|v\|_{W^{s,t,q}_{p,\vartheta}} + C_{\vartheta'} \|v\|_{W^{s',t',q'}_{p,\vartheta}}.$$  

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**APPENDIX C. INTERPOLATION, MOLLIFICATION, AND APPROXIMATIONS OF THE IDENTITY**

Let $[B_1, B_2]_u$, for $u \in (0,1)$, denote the complex (Calderón) interpolation of an interpolation pair of Banach spaces $B_1, B_2$. The Banach spaces in this paper are complex interpolation spaces:

**Lemma C.1 (Interpolation).** Setting $w(x_1, x_2) = (1 - w)x_1 + wx_2$ for $w \in (0,1)$, we have for any $p \in (1,\infty)$, all $t_j - (r - 1) < s_j < 0 < q_j \leq t_j$, $j = 1, 2$, and all $w \in (0,1)$ that

$$ W^{s_1,t_1,q_1}_{p,t_1,q_1}(M), W^{s_2,t_2,q_2}_{p,t_2,q_2}(M) \in W^{s,t,q}_{p,t,q}(M), \quad s = w(s_1, s_2), \quad t = w(t_1, t_2), \quad q = w(q_1, q_2).$$

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^{43}In particular, there exists $C \geq 1$ such that $\|\nabla_{\frac{\partial}{\partial x}} h_\delta\|_{C^{r-1}} \leq C\|h\|_{C^{r-1}}$ and $\|h - h_\delta\|_{L_\infty} \leq C\|h\|_{C^{r-1}} \delta^{r-1}$. 

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Proof. The norms on $\mathbb{R}$ given by $\|x\|_{n,u} = 2^n |x|$, $n \in \mathbb{N}$, form a complex interpolation scale with respect to $w \in \mathbb{R}$. The lemma thus follows from using [19] Thms 1.18.1, 1.18.4] to show that the local norms $\| \cdot \|_{W^{s,t,q}_\omega}$ have the desired interpolation property, and then applying [19] Thm 1.18.4] to the function $\alpha \mapsto \| (\partial_\omega \hat{L}(\varphi)) \| W^{s,t,q}_\omega$ on $[0, \alpha_0]$.

Recall the finite atlas $\mathcal{A}$, indexed by $\omega \in \Omega$, and the pair $\{\Theta_\omega\}, \{\Theta'_\omega\}$ of adapted cone systems from Lemma 2.2.4 and Remark 2.5. Let $\{U_\omega\}$ be an open cover of $M$ with $\bigcup U_\omega \subseteq V_\omega$. Let $\nu$ be as in Lemma 2.2.4, and set $\nu(x) = \epsilon^{-d} \nu(x/\epsilon)$. Fix $C^\infty$ functions $\bar{\vartheta}_\omega : M \to [0,1]$, with $\bar{\vartheta}_\omega$ supported in $U_\omega$, such that $\sum_\omega \bar{\vartheta}_\omega(x) = 1$ for all $x \in M$. Finally, let $\epsilon > 0$ be such that the $\epsilon$-neighbourhood of $\kappa_\omega(U_\omega)$ is contained in $\kappa_\omega(U_\omega)$ for each $\omega$. As in [8 (5.4)], define a mollifier operator $M_\epsilon$, by setting, for any distribution $\varphi$ of order at most $r$ on $M$,

$$M_\epsilon(\varphi)(u) = \int_{\mathbb{R}^d} \nu_\epsilon(u - v) \psi_\epsilon^{-1}(v) \, dv = [\nu_\epsilon * (\psi_\epsilon^{-1})](u), \quad \omega \in \Omega, \ u \in \kappa_\omega(U_\omega),$$

(116) $M_\epsilon(\varphi) = \sum_{\omega \in \Omega} \bar{\vartheta}_\omega \cdot (\{(M_\epsilon(\varphi))_\omega \circ \kappa_\omega\).$

Since $\{\Theta_\omega\}$ and $\{\Theta'_\omega\}$ are adapted to $\mathcal{A}$ and $g_\alpha$, the fact that $\Theta'_\omega < \Theta_\omega$ in the next lemma is not a problem:

Lemma C.2 (Approximation of the Identity). For any $p \in (1, \infty)$, all $s', t', q' \in \mathbb{R}$ and all $\eta > 0$ such that $-(r - 1) + t' + \eta < s' < -\eta < 0 < q' < t'$, there exists $C < \infty$ such that, letting $W^{s',t',q'}(\Theta')$ be the space constructed with $\Theta'$,

$$\| M_\epsilon \varphi - \varphi \|_{W^{s',t',q'}(\Theta')} \leq C \| \varphi \|_{W^{s,t,q}_\Theta} \quad \forall \varphi, \ \forall \epsilon > 0.$$

Proof. Minkowski-type integral bounds hold for the local norms $W^{s,t,q}_{\omega,\omega'}$: There exists $C_M < \infty$ such that for any $\psi : \mathbb{R}^d \to \mathbb{R}$ and any family $\varphi_y \in W^{s,t,q}_{\omega,\omega'}$, uniformly bounded in $y$,

$$\| \int_{\mathbb{R}^d} \psi(y) \varphi_y(\cdot) \, dy \|_{W^{s,t,q}_{\omega,\omega'}} \leq C_M \| \psi \|_{L^1(\mathbb{R}^d)} \sup_y \| \varphi_y \|_{W^{s,t,q}_{\omega,\omega'}}.$$

(See [8 Remark 5.1]. We already established interpolation for our spaces.) So we proceed as in the proof of [8 Lemma 5.4]. The changes of charts $\kappa_\omega \circ \kappa_\omega^{-1}$ (one chart is for the mollifier and the other for the norm) are cone-hyperbolic from $\Theta_\omega$ to $\Theta'_\omega$ by construction.

Remark C.3. If we attempted to show Dolgopyat bounds using mollifiers through isotropic spaces as in [8 Lemma 5.4, (7.5)–(7.6)], we would face a factor

$$\| \mathcal{R}_{a+b+h_{\text{top}}} \|_{W^{s',t',s'}_{\omega,\omega'}} \leq \frac{C}{(a + s' \log \Theta)^n}$$

instead of $Ca^{-n}$ in [72]. After applying [65] with $\beta = \kappa(s - s') - 1 > 0$, we would end up with an upper bound $\gamma' < \frac{\kappa(s - s') - 1}{\log(1 + \lambda_{\max} - \frac{s'}{a}) - \log(1 - \frac{\gamma}{\log \Theta})}$. In our main application, Proposition 4.10, we need to take $s'$ close to $-1$ to guarantee $\lambda_{\min} < \lambda_{\max}$. The upper bound would then conflict with [74]. This is why (proving [3] would give another solution to this problem) we used mollification through anisotropic spaces as in [29] Lemma D.2], taking advantage of the exact growth from Lemma [3.13. Our norms are different from those of [29]: Their drawback is that we need to go
through charts twice to prove Lemma \ref{lem:C.2} (we have Remark \ref{rem:2.5} to save us). Their strength is that we can use the interpolation\footnote{Interpolation was not available in \cite{8} due to the presence of a supremum in the norm there.} Lemma \ref{lem:C.1} and Minkowski inequalities to prove Lemma \ref{lem:C.2}.

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