Coupled Quintessence in a Power-Law Case and the Cosmic Coincidence Problem

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Abstract

The problem of the cosmic coincidence is a longstanding puzzle. This conundrum may be solved by introducing a coupling between the two dark sectors. In this Letter, we study a coupled quintessence scenario in which the scalar field evolves in a power law potential and the mass of dark matter particles depends on a power law function of $\phi$. It is shown that this scenario has a stable attractor solution and can thus provide a natural solution to the cosmic coincidence problem.
Before the accelerated expansion of the universe was revealed by high red-shift supernovae Ia (SNe Ia) observations [1], it could hardly be presumed that the main ingredients of the universe are dark sectors. The concept of dark energy was proposed for understanding this currently accelerating expansion of the universe, and then its existence was confirmed by several high precision observational experiments [2,3], esp. the Wilkinson Microwave Anisotropy Probe (WMAP) satellite experiment [2]. The WMAP shows that dark energy (DE) occupies about 73% of the energy of our universe, and dark matter (DM) about 23%. The usual baryon matter which can be described by our known particle theory occupies only about 4% of the total energy of the universe. Although we know that the ultimate fate of the universe is determined by the nature of DE, the information about its nature we can acquire is still very limited. So far the confirmed information about DE can be summarized as the following three items: it is a kind of exotic component with negative pressure such that it can accelerate the universe’s expansion; it is spatially homogeneous and non-clustering; and it dominates the universe today although it contributes little to the universe at the early times.

The investigation of the nature of DE is an important mission in the modern cosmology. Much work has been done on this issue, and there is still a long way to go. One candidate for DE is vacuum energy density or cosmological constant $\Lambda$ [4] for which the equation of state $w = -1$. The cosmological model that consists of a mixture of vacuum energy and cold dark matter (CDM) is called LCDM (or $\Lambda$CDM). Another possibility is the so-called QCDM cosmology which based upon a mixture of CDM and quintessence field [5]. The energy density and the negative pressure are provided by the quintessence scalar field $\phi$ slowly evolving down its potential $V(\phi)$. The equation of state of the quintessence $-1 < w < -1/3$ is guaranteed by the slow evolution. However, as is well known, there are two difficulties arise from all of these scenarios, namely, the “fine-tuning” problem, and the “cosmic coincidence” problem [6]. The cosmic coincidence problem states: Since the energy densities of DE and DM scale so differently during the expansion of the universe, why are they nearly equal today? To get this coincidence, it appears that their ratio must be set to a specific, infinitesimal value in the very early universe.

A possible solution to this cosmic coincidence problem may be provided by introducing a coupling between quintessence DE and CDM. This coupling is often described by the variable-mass particle (VAMP) scenario [7]. The VAMP scenario assumes that the CDM particles interact with the scalar DE field resulting in a time-dependent mass, i.e. the mass of the CDM particles evolves according to some function of the scalar field $\phi$. It has been shown that if we choose an exponential potential to the quintessence scalar field $\phi$ and at the same time assume the CDM particle mass also depends exponentially on $\phi$, the late-time ratio
between DM energy density $\rho_\chi$ and DE energy density $\rho_\phi$ will remain constant [8]. This behavior relies on the existence of an attractor solution, which makes the effective equation of state of DE mimic the effective equation of state of DM at the late times so that the late time cosmology insensitive to the initial conditions for DE and DM. In addition, the final effective equation of state of both components is negative and may lead, if this value is less than $-1/3$, to an accelerated expansion of the universe. Therefore, the scenario containing quintessence with exponential potential and VAMPs with exponential function of $\phi$ solves the cosmic coincidence problem in this sense. However, it should be pointed out that this case is not the unique case. We consider here another case — the power law case.

In this Letter, we explore the case in which the scalar field evolves in a power law potential and the mass of VAMPs depends on a power law function of $\phi$. We find that the power law case also has a stable attractor solution which is similar to the exponential case. We numerically solve the equation of motion of the scalar field $\phi$ and then illustrate the behavior of DE and DM. The numerical results show that the power law case of coupled DE with VAMPs also solves the cosmic coincidence problem.

Consider, now, the CDM particle $\chi$ with mass $M$ depending on a power law function of the DE field $\phi$,

$$M_\chi(\phi) = M_\star \phi^{-\alpha}, \quad (1)$$

where $\phi$ is expressed in units of the reduced Planck mass $M_\rho$ ($M_\rho \equiv 1/\sqrt{8\pi G} = 2.436 \times 10^{18}\text{GeV}$), and $\alpha$ is a positive constant. The scalar field has a power law potential

$$V(\phi) = V_\star \phi^\beta, \quad (2)$$

where $\beta$ is a positive constant. Since the CDM particle is stable, its number density $n_\chi$ must obey the equation

$$\dot{n}_\chi + 3Hn_\chi = 0, \quad (3)$$

where the dot denotes a derivative with respect to time, $H = \dot{a}/a$ represents the Hubble parameter, and $a(t)$ is the scale factor of the universe. The energy density of DM $\rho_\chi$ is also $\phi$-dependent, which is given by

$$\rho_\chi(\phi) = M_\chi(\phi)n_\chi, \quad (4)$$

and it follows that

$$\dot{\rho}_\chi + 3H\rho_\chi = -\alpha \frac{\dot{\phi}}{\phi}\rho_\chi. \quad (5)$$
Since the total energy of DE and DM is conserved, the equation of motion for DE can be obtained

\[ \dot{\rho}_\phi + 3H\rho_\phi(1 + w_\phi) = \frac{\dot{\phi}}{\phi}\rho_\chi, \quad (6) \]

where the usual parameter of equation of state for the homogeneous scalar field is given by

\[ w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \quad (7) \]

The equations of motion (5) and (6) can also be written in the form of effective equations of state for DM and DE

\[ \dot{\rho}_\chi + 3H\rho_\chi(1 + w_\chi^{(e)}) = 0, \quad (8) \]

\[ \dot{\rho}_\phi + 3H\rho_\phi(1 + w_\phi^{(e)}) = 0, \quad (9) \]

where

\[ w_\chi^{(e)} = \frac{\alpha}{3H}\frac{\dot{\phi}}{\phi} = \frac{\alpha}{3}\phi', \quad (10) \]

\[ w_\phi^{(e)} = w_\phi - \frac{\alpha}{3H}\frac{\dot{\phi}}{\phi}\rho_\chi = w_\phi - \frac{\alpha}{3}\phi\rho_\chi, \quad (11) \]

are the effective equation of state parameters for DM and DE, respectively. Primes denote derivatives with respect to \( u = \ln(a/a_0) = -\ln(1 + z) \), where \( z \) is the red-shift, and \( a_0 \) represents the current scale factor. From (6) or (9) one can get the equation of motion for the scalar field \( \phi \),

\[ \ddot{\phi} + 3H\dot{\phi} = \frac{\alpha}{\phi}\rho_\chi - \frac{\beta}{\phi}V. \quad (12) \]

The Friedmann equation for a spatially flat universe with DE, DM, baryons, and radiation reads

\[ 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_\chi(\phi) + \rho_b + \rho_{rad}, \quad (13) \]

the Planck normalization \( M_p = 1 \) has been used here. Using the Friedmann equation (13), the equation of motion of field \( \phi \) can be written in terms of the derivatives of \( \phi \) with respect to \( u \),

\[ H^2\phi'' + \frac{1}{2}(\rho_\chi + \rho_b + \frac{2}{3}\rho_{rad} + 2V)\phi' = \frac{\alpha}{\phi}\rho_\chi - \frac{\beta}{\phi}V, \quad (14) \]
where

$$H^2 = \frac{\frac{1}{3} \left( \rho_x + \rho_b + \rho_{\text{rad}} + V \right)}{1 - \frac{\phi'^2}{6}}.$$  \hspace{1cm} (15)

The attractor makes the ratio between energy densities of DE and DM be a constant, i.e. $\Omega'_{\phi} = 0$, which gives

$$2\phi^2 \phi'' - \alpha \phi \phi' + \alpha \phi'^2 = 0.$$  \hspace{1cm} (16)

We see explicitly that the situation in the case of power law is more complicated than in the case of exponential. In the exponential case, $\phi' = \text{const.}$, then using the Friedmann equation

FIG. 1. The phase plane for $\alpha = 11$ and $\beta = 4$. The units of $V_*$ and $M_*$ are in $\rho_{c0}$ and $\rho_{c0}/n_{\chi 0}$, respectively. The three lines correspond to cases $(V_*, M_*)$ taken to be $(0.1,230)$, $(0.2,23)$ and $(0.3,2.3)$, respectively.
this constant can be determined, furthermore it can be shown that $\Omega_\phi$ and $\Omega_\chi$ are both $\phi$-independent constant.\footnote{As an example, we take $M_\chi(\phi) = M_s e^{-\lambda \phi}$, and $V(\phi) = V_s e^{\eta \phi}$, then the attractor solution is given by $\phi = \phi_0 + \frac{-3}{\lambda + q} u$, $\Omega_\phi = 1 - \Omega_\chi = \frac{3 + \lambda(\lambda + q)}{(\lambda + q)^2}$. It can be seen that the attractor depends only on $\lambda$ and $q$.} However, in this case, since $\phi' \neq \text{const.}$, we can not identify the attractor easily. We are now required to solve the equation of motion of $\phi$ (14) numerically. Whether the attractor solution exists or not can be confirmed by the numerical results.

FIG. 2. Top panel: A typical solution for the differential equation (14), namely the evolution of the $\phi$ field. The corresponding parameter configuration is: $\alpha = 11$, $\beta = 4$, $V_s = 0.1 \rho_0$, and $M_s = 230 \rho_0 / n_\chi$. Bottom Panel: the evolution of $\phi'$ for the same parameters used in top panel.

The numerical results indeed show that there exists a stable attractor solution which depends only on the parameters $\alpha$ and $\beta$ while is very insensitive to the initial conditions and the chosen values of $V_s$ and $M_s$. As an example, we take the case $\alpha = 11$ and $\beta = 4$. The $(\phi, \phi')$ phase diagram is plotted in Fig.1. It is shown in the phase plane that lines corresponding to different chosen values of $(V_s, M_s)$ will converge together to the attractor solution with the cosmological evolution. Note that we express $V_s$ and $M_s$ in units of $\rho_0$ and $\rho_0 / n_\chi$, respectively, where $\rho_0 = 4.2 \times 10^{-47}$ GeV$^4$ is the present critical density of universe. It is of interest to notice that the lines in the phase plane will undergo a period of oscillation before they converge to the attractor. From Fig.1, we see that in the attractor region $\phi$ and...
\( \phi' \) both tend to 0 and at the same time \( \phi'/\phi \) becomes a constant. Using the fact that in the attractor region \( dV_{eff}/d\phi \sim 0 \), where \( V_{eff} = V + \rho_\chi \) is the effective potential, one gets \( \alpha \rho_\chi \approx \beta V \) in this limit. We re-express this condition as

\[ \alpha \Omega_\chi \approx \beta (\Omega_\phi - \frac{1}{6} \phi'^2). \]  

(18)

Considering \( \Omega_\chi = 1 - \Omega_\phi \) and \( \phi' \sim 0 \), one obtains

\[ \Omega_\phi \approx \frac{\alpha}{\alpha + \beta}. \]  

(19)

Combining with (16), we get

\[ \frac{\phi'}{\phi} \approx - \frac{3}{\alpha + \beta}, \]  

(20)

consequently the attractor solution in the field space can be expressed clearly as

\[ \phi = \phi_0 e^{-\frac{3}{\alpha + \beta} u}, \]  

(21)

such that

\[ w^{(e)}_\phi = w^{(e)}_\chi = - \frac{\alpha}{\alpha + \beta}. \]  

(22)

The existence of the attractor solution indicates that the role the coupled quintessence model in a power law case plays in solving the cosmic coincidence problem is similar to that of the model in an exponential case.
FIG. 3. Top panel: The evolution of the relative abundance of different species, expressed as fractions of the critical density. The corresponding model parameters are: $\alpha = 11$, $\beta = 4$, $V_* = 0.1 \rho_{c0}$, and $M_* = 230 \rho_{c0}/n_{\chi 0}$. Bottom panel: Effective equations of state for DE (solid line) and DM (dashed line) for the same parameters used in top panel.

We now illustrate the cosmological consequence of this scenario. The solution of the differential equation (14), namely the evolution of $\phi(u)$, is plotted in the top panel of Fig.2. Note that for concreteness the values of $(V_*, M_*)$ are taken as $V_* = 0.1 \rho_{c0}$ and $M_* = 230 \rho_{c0}/n_{\chi 0}$. We see explicitly in this case that $\phi$ increases in the beginning and begins to decrease around $u \sim -2.5$, which implies that the $\phi$ field climbs slowly up the potential initially and then rolls down with the evolution of the universe. Moreover, we notice that around $u \sim 0$ some wiggle appears. To make the picture of the kinematics of the field more clear, we plot the ‘velocity’ of the field $\phi'(u)$ (note that $\phi' = \dot{\phi}/H$) versus $u = -\ln(1+z)$ in the bottom panel of Fig.2. It is shown that the wiggle in the $\phi$-evolution diagram corresponds to the rapid oscillation of the rolling-down velocity. After the period of oscillation the field system will enter a stable attractor regime.

It is remarkable that the feature of the coupled quintessence model is quite different from the scalar DE model in the absence of the interaction with DM. In Fig.3, we show the evolution of density parameters of various components in universe and the effective equations of state for DE and DM. The model parameters are taken to be the same as above. It is exhibited that after a transient period of baryon domination, the universe enters into a DE-dominated epoch, and the interaction between DE and DM forces the ratio between energy densities of DE and DM to remain constant via DM particles varying mass. In the future, the universe will be governed by DE and DM which are coupled together, and $\rho_b$ and $\rho_{rad}$ are diluted away with the usual laws, $a^{-3}$ and $a^{-4}$, respectively. We see that in the attractor regime the effective equations of state for DE and DM converge together and both are negative. Notice that the attractor solution is going to be reached today in this example.

In summary, we investigate in this Letter the attractor solution of a coupled quintessence scenario in which the scalar field evolves in a power law potential and the mass of VAMPs depends on a power law function of $\phi$. It has been shown by numerical calculation that this interacting scalar field scenario has a stable attractor and can thus provide a natural solution to the cosmic coincidence problem.

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