RIGID AND STABLY BALANCED CURVES
ON CALABI-YAU AND GENERAL-TYPE HYPERSURFACES

ZIV RAN

ABSTRACT. A curve $C$ on a variety $X$ is stably balanced if the slopes of the Harder-Narasimhan filtration of its normal bundle $N$ are contained in an interval of length 1. For a general nonspecial curve of given genus $g \geq 1$ and large enough degree in $\mathbb{P}^n$, and a hypersurface $X$ of large enough degree containing $C$, we prove that $C$ is stably balanced and rigid on $X$ and that the family of hypersurfaces $X$ thus obtained is smooth of codimension $h^1(N)$ in the space of hypersurfaces of its degree.

CONFLICT OF INTEREST STATEMENT

The author has no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript

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There is no data set associated with this paper

INTRODUCTION

0.1. Known results: interpolation, rigidity, stability. A fundamental object of interest in algebraic geometry is a pair $(C, X)$ consisting of a curve $C$ on a hypersurface $X$ in $\mathbb{P}^n$. Generally, given a curve $C$ on a variety $X$, important attributes of $C \subset X$ such as the motion of $C$ on $X$ reside in in properties of the normal bundle $N = N_{C/X}$, i.e. the 1st-order infinitesimal neighborhood of $C$ in $X$ which controls motions both local and global of $C$ in $X$. Results on $N$ for curves on general hypersurfaces have thus far been largely restricted to the case where $X$ is in the Fano range, i.e of degree $\leq n$, so that $N$ is ‘positive’ (e.g. $\chi(N) \geq 0$). In this case, going beyond the regularity property, i.e. $H^1(N) = 0$, one natural property of $N$ is interpolation or balance, which states

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that for a general effective divisor $D$ of any degree, one has either $H^0(N(-D)) = 0$ or $H^1(N(-D)) = 0$, and there are now a number of papers that prove this property when $X$ is $\mathbb{P}^n$ itself (see Atanasov-Larson-Yang [11]) or a Fano hypersurface [13], [16].

Much less is known about curves with good normal bundles $N$ when the hypersurface $X$ is in the so-called CYGT range (degree $> n$ in $\mathbb{P}^n$) where $N$ need not be positive in any sense (e.g. $\chi(N) \leq 0$). One important good property of $N$ in case $\chi(N) \leq 0$ is rigidity, i.e. $H^0(N) = 0$. On Calabi-Yau complete intersection 3-folds (CYCI3fs) $X$, where $\chi(N) = 0$ for any $g$, there are examples in the literature of rigid curves going back to Clemens [3], as extended by Katz [10], who constructed infinitely many rigid rational curves on the general quintic in $\mathbb{P}^4$. This was extended to rational curves on all CICY3fs by Ekedahl, Johnsen and Sommervoll [7], and subsequently, based on Clemens’s method, by Kley [11] to any CICY3f $X$ and curves of sufficiently low genus $g$ (depending on $X$, e.g. $g < 35$ for the quintic) and sufficiently high degree (depending on the genus). For the quintic threefold, any $g \geq 0$ and large $d$, Zahariuc [19] constructs a degree-$d$ map from a smooth curve of genus $g$ to $X$ which is set-theoretically isolated (it is not proved the map is an embedding or infinitesimally rigid). A few examples of regular and rigid curves on Calabi-Yau and general type hypersurfaces, all with $\chi(N) \geq 0$, are given in [15].

Another property of $N$ which makes sense irrespective of positivity is (semi)stability, but results establishing this have so far been restricted to the case $X = \mathbb{P}^n$ or $X$ a (Fano) hypersurface in $\mathbb{P}^{n+1}$ [5], [14].

For curves on (very general) hypersurfaces of general type, what one has are a number of non-existence results due to Voisin, Clemens and others (see [4]), giving a lower bound on the genus in terms of the degree, but to my knowledge no existence results for good curves.

A naive way to construct pairs $(C, X)$ is to start with a good curve $C \subset \mathbb{P}^n$ and let $X$ be a hypersurface of sufficiently large degree containing $C$, which may be assumed smooth. Generally, the normal bundle $N_{C/X}$ has $\chi < 0$, hence $H^1 \neq 0$. The question then arises as to what good properties the pair $(C, X)$ has. This is addressed in some cases by our main result.

0.2. **New results: Stable balance, rigid regularity.** Our aim here is to study normal bundles from the point of view of stability, regularity and rigidity. Motivated by the case of rational curves whose normal bundle, though balanced, is not semistable unless it has integral slope, we introduce a property weaker than stability that we call stable balance. This stipulates that the slopes of graded pieces of the Harder-Narasimhan or stable filtration of $N$ should all lie in some interval of length 1 (which must then contain the slope of $N$ itself). Stable balance is obviously equivalent to balance for rational curves. For any genus curves stable balance is weaker than interpolation. Its advantage
is that it is meaningful for any bundle regardless of positivity or sections. Because interpolation results are known for \( \mathbb{P}^n \) and Fano hypersurfaces, we will focus here on the case of Calabi-Yau and general type hypersurfaces, i.e. those of degree \( \geq n + 1 \) in \( \mathbb{P}^n \). For curves on such hypersurfaces \( X \), one usually has \( \chi(N) < 0 \), so the best can hope for as regards \( N \) is that \( C \) is rigid in \( X \), i.e. \( H^0(N) = 0 \). And as for ‘regularity’, when \( H^1(N) \neq 0 \) the best kind of regularity one can hope for is what we call rigid-regularity for the pair \((C, X)\), which is expressed in terms of vanishing for a suitable ‘joint normal sheaf’ \( N_{C/X/P^n} \) in addition to rigidity of \( C \) in \( X \). This implies that the pair is regular as such, i.e. deforms in a smooth family of the expected dimension, and this family projects injectively to the family of hypersurfaces.

Our main result is essentially the following (see Theorem 16 §4):

**Theorem.** For each \( d \geq n + 1 \geq 5 \), and each \((e, g)\) in a suitable range depending on \( n, d \), a general pair \((C, X)\) where \( C \) is a (nonspecial) curve of degree \( e \) and genus \( g \) in \( \mathbb{P}^n \) and \( X \) is a hypersurface of degree \( d \) containing \( C \) has the properties
- the pair \((C, X)\) is rigid-regular;
- \( C \) has stably balanced normal bundle in \( X \);
- moreover the corresponding family of hypersurfaces \( X \) is smooth of codimension \(-\chi(N)\) in the space of all hypersurfaces of degree \( d \).

**Remarks:**

(i) Here
\[
\chi(N) = e(n + 1 - d) + (n - 4)(1 - g).
\]
In the cases considered, this is always \( \leq 0 \).

(ii) Because \( C \) is nonspecial and of maximal rank by [2], the family of pairs \((C, X)\) is irreducible.

(iii) As stated above, earlier examples of rigid curves on hypersurfaces included a few Calabi-Yau cases and general type examples satisfying \( \chi(N) \geq 0 \), which were not shown to be relatively regular or stably balanced. To my knowledge the above are the first examples with \( \chi(N) < 0 \) (a property which makes deformation/smoothing arguments trickier, and which leads us to consider deformations of the triple \( C \subset X \subset \mathbb{P}^n \)).

(iv) The range of hypersurface degrees for given \( C \) is not necessarily optimal - an optimal one might be \( (d + n)/n > ed + 1 - g \). Note that the degree, hence also the genus, of any rigid-regular curve of a hypersurface of degree \( d \) is a priori bounded by a function of \( d \).

(v) It remains an open question in genus \( g \geq 1 \) whether the normal bundles to the curves we construct may actually be stable or semistable rather than just stably balanced. However Example 6 suggests that this may be false for stability.
(n \geq 4) or semi-stability (n \geq 5), at least for the degenerate curves that we construct. By contrast, in the Fano case the normal bundle is often semistable (cf. [14]).

(vi) As for (in)completeness of our results, while there are numerous results from several viewpoints on the family of all curves on Fano hypersurfaces in \( \mathbb{P}^n \), the situation changes drastically as soon as one enters the CYGT range \( d > n \). In the very first such case, namely rational curves on the general quintic in \( \mathbb{P}^4 \), a well-known conjecture of Clemens stating that all such curves must be rigid (equivalently, regular, equivalently have semistable normal bundle), has remained open after some 30 years despite many attempts. Thus, anything close to completeness seems out of reach. Our purpose here is merely to construct some good examples beyond the Fano range including some in the general-type range, with no claim to sharpness or completeness.

0.3. Methods. We will use the fan method developed in the author’s earlier papers [13], [16]. After recalling some facts on fans, their hypersurfaces and degenerations in \( \S 1 \) some properties of, and operations on stably balanced bundles are developed in \( \S 2 \). In particular, we study elementary modifications and specialization/generalization. Next in \( \S 3 \) we develop some generalities on embedded deformations of a pair \((C, X)\) in an ambient space such as \( \mathbb{P}^n \), with focus on cases where \( \chi(N_{C/X}) < 0 \). We define a suitable normal sheaf \( N_{C/X/\mathbb{P}^n} \) controlling embedded deformations of the pair, such that the vanishing of \( H^1(N_{C/X/\mathbb{P}^n}) \) is equivalent to the pair \((C, X)\) having unobstructed deformations in \( \mathbb{P}^n \) of the expected dimension \( h^0(N_{C/X/\mathbb{P}^n}) = \chi(N_{C/X/\mathbb{P}^n}) \) (the cohomology of degree \( > 1 \) usually vanishes automatically). The curve \( C \subset X \) is said to be \textit{rigid-regular} if \( H^1(N_{C/X/\mathbb{P}^n}) = 0 \) and \( H^0(N_{C/X}) = 0 \). This implies that the space of deformations of the pair \((C, X)\) in \( \mathbb{P}^n \), the so-called flag Hilbert scheme, maps isomorphically to a smooth subvariety of codimension \( h^1(N_{C/X}) \) in the deformation space of \( X \).

The main result, Theorem [16] presented in \( \S 4 \) has been described above. Its proof, based on a suitable fan degeneration similar to [13], requires special care due to non-vanishing of \( H^1(N_{C/X}) \). It relies in an essential way on maximal rank theorems of Hartshorne–Hirschowitz [8] for general skew lines and Ballico-Ellia [2] for general non-special curves.

Conventions. We work over an arbitrary algebraically closed field. As general references for deformation theory, see the books by Sernesi [17] or Hartshorne [9]. The slope of a bundle \( E \) on a curve, i.e. \( \text{deg}(E)/\text{rk}(E) \) is denoted by \( \mu(E) \).
1. Fan and quasi-cone degenerations

See [13] for details. We recall that a fan (also called a 2-fan) is a reducible normal-crossing variety of the form
\[ P_0 = P_1 \cup E P_2 \]
where
\[ P_1 = B\mathbb{P}^n, P_2 = \mathbb{P}^n \]
and \( E \subset P_1 \) is the exceptional divisor and \( E \subset P_2 \) is a hyperplane. The family
\[ \mathcal{P} = B(\mathbb{P}^n \times \mathbb{A}^1) \]
is called a standard fan degeneration and realizes \( P_0 \) as the special fibre in a family with general fibre \( \mathbb{P}^n \).

A hypersurface of type \((d_1, d_2)\) in \( P_0 \) has the form
\[ X_0 = X_1 \cup Z X_2 \]
where
\[ X_1 \in |d_1 H_1 - d_2 E|_{P_1}, X_2 \in |d_2 H_2|_{P_2}, X_1 \cap Z = X_2 \cap Z \]
\((H_1, H_2)\) are the respective hyperplanes. If \( d_2 = d_1 - 1 \), \( X_0 \) is said to be of quasi-cone type. Given a family \( \mathcal{X} \subset \mathbb{P}^n \times \mathbb{A}^1 \) of hypersurfaces of degree \( d_1 \) whose special fibre has multiplicity \( d_2 \) at \( p \), its birational transform \( \mathcal{X} \subset \mathcal{P} \) is a family of hypersurfaces in \( \mathbb{P}^n \) specializing to one of type \((d_1, d_2)\).

2. Stably balanced bundles

2.1. Basics. Given a vector bundle \( E \) on a curve \( C \), we denote by
\[ (E_\bullet) = (0 \neq E_1 \subseteq E_2 \ldots) \]
its Harder-Narasimman (HN) filtration, characterized by the property that
\[ \mu_{\max}(E) = \mu_1(E) := \mu(E_1) \]
is the largest slope among slopes of subbundles \( E_1 \subset E \), and \( E_1 \) is maximal of slope \( \mu_1(E) \) (i.e. \( E_1 \) is the maximal-rank, maximal-slope subbundle and as such is unique and semi-stable); and likewise for \( \mu_{i+1}(E) := \mu(E_{i+1}/E_i), \forall i \geq 1 \). Thus we have a strictly decreasing slope sequence
\[ \mu_{\max}(E) > \ldots > \mu_{\min}(E) \]
(unless \( E \) is semi-stable, in which case \( \mu_{\max}(E) = \mu_{\min}(E) \)). The interval \([\mu_{\min}(E), \mu_{\max}(E)]\) is called the slope interval of \( E \) and its width \( \mu_{\max}(E) - \mu_{\min}(E) \) is called the slope width. \( E \) is said to be stably balanced if its slope width is \( \leq 1 \). Note that the slope \( \mu(E) \) is contained in the slope interval, hence if \( E \) is stably balanced then
\[ \mu_{\max}(E) - 1 \leq \mu(E) \leq \mu_{\min}(E) + 1. \]
By comparison, recall that $E$ is said to be \textit{balanced} if, denoting by $D_t$ a general effective divisor of degree $t$, we have for each $t \geq 0$ that either $H^1(E(-D_t)) = 0$ or $H^0(E(-D_t)) = 0$. Because this condition involves just negative ‘twists’ of $E$, it makes sense just for ‘positive’ bundles, unlike the stable balance condition. It is also stronger than stable balance:

\textbf{Lemma 1.} If $E$ is balanced then $E$ is stably balanced.

\textit{Proof.} Let $t$ be largest so that $H^1(E(-D_t)) = 0$ for $D_t$ general of degree $t$.

Then we have $H^1(E'(-D_t)) = 0$ for any quotient $E'$ of $E$. Also $H^0(E(-D_{t+1})) = 0$ and $H^0(E''(-D_{t+1})) = 0$ for any subsheaf $E'' \subset E$. Therefore $\chi(E(-D_t)) \geq 0$ and ditto for a minimal slope quotient, while $\chi((E(-D_{t+1}))) \leq 0$ and ditto for the max slope subsheaf. This easily implies $\mu_{\text{max}}(E) + 1 - g \leq t + 1$ and similarly $\mu_{\text{min}}(E) + 1 - g \geq t$, so $\mu_{\text{max}}(E) - \mu_{\text{min}}(E) \leq 1$. \qed

In fact the same argument yields the following stronger result, noted by Vogt [18]:

\textbf{Lemma 2.} If $E$ is balanced then we have for every subbundle $F \subset E$,

$$\mu(F) \leq \lceil \mu(E) \rceil$$

In particular, if $E$ is balanced and $\mu(E)$ is an integer then $E$ is semistable.

\textit{Proof.} Adding $1 - g$ to both sides, it suffices to prove $\chi(F)/\text{rk}(F) \leq \lceil \chi(E)/\text{rk}(E) \rceil$. If $t \geq \chi(E)/\text{rk}(E)$ then $\chi(E(-D_t)) \leq 0$, hence $H^0(E(-D_t)) = 0$, hence $H^0(F(-D_t)) = 0$, hence $\chi(F(-D_t)) \leq 0$, i.e. $t \geq \chi(F)/\text{rk}(F)$. \qed

Lemma 1 implies that the balanced curves constructed, e.g. in [1], [13] and [16], all with positive normal bundle in Fano hypersurfaces, are all stably balanced. So the interest in stably balanced curves on hypersurfaces is mainly for Calabi-Yau and general-type hypersurfaces.

A semistable bundle of slope $< 0$ has $H^0 = 0$. By Serre duality, a semistable bundle of slope $\mu > 2g - 2$ has $H^1 = 0$. Applying this to the gradeds of the HN filtration, we conclude

\textbf{Lemma 3.} Suppose $E$ is stably balanced of slope $\mu$. If $\mu < -1$ then $H^0(E) = 0$. If $\mu > 2g - 1$ then $H^1(E) = 0$.

If $C = \mathbb{P}^1$, stable balance is equivalent to balance. For general genus, balance implies some upper bound, not necessarily 1, on the length of the slope interval:

\textbf{Lemma 4.} Notation as above, assume $E$ is balanced. Let $r_{\text{max}}(E)$ (resp. $r_{\text{min}}(E)$) denote the smallest (resp. largest) rank of a maximal-slope subbundle (resp. minimal slope quotient bundle)
of $E$. Then we have

$$\mu_{\text{max}}(E) - \mu_{\text{min}}(E) \leq 1 + (2g - 2)(\frac{1}{r_{\text{min}}(E)} - \frac{1}{r_{\text{max}}(E)}) \leq 1 + (2g - 2)(1 - \frac{1}{rk(E) - 1}).$$

Moreover if $\chi(E) / rk(E) \in \mathbb{Z}$, then

$$\mu_{\text{max}}(E) - \mu_{\text{min}}(E) \leq (2g - 2)(\frac{1}{r_{\text{min}}(E)} - \frac{1}{r_{\text{max}}(E)}) \leq (2g - 2)(1 - \frac{1}{rk(E) - 1}).$$

In particular, if $g = 1$ or $rk(E) = 2$, $E$ is stably balanced.

Proof. For $t = \chi(E) / rk(E) = \mu(E) + (2g - 2) / rk(E)$, we have

$$H^1(E(-D_{\lceil t \rceil})) = 0 = H^0(E(-D_{\lfloor t \rfloor})).$$

Therefore if $E_{\text{min}}$ denotes the minimal-slope quotient of maximal rank (resp. maximal-slope subbundle of minimal rank), we have

$$H^1(E_{\text{min}}(-D_{\lfloor t \rfloor})) = 0 = H^0(E_{\text{max}}(-D_{\lceil t \rceil})).$$

Consequently

$$\chi(E_{\text{min}}(-D_{\lfloor t \rfloor})) \geq 0, \chi(E_{\text{max}}(-D_{\lceil t \rceil})) \leq 0.$$

Because $\lceil t \rceil - \lfloor t \rfloor \leq 1$, the inequality (1) follows from Riemann-Roch. Similarly for (2). □

It is noted in [16], Lemma 8 that if the graded of the HN filtration of $E$ are direct sums of line bundles with degrees in some interval $[a, a + 1], a \geq 2g - 1$ (so that $E$ is stably balanced), then $E$ is balanced. Otherwise, the relation between stable balance and balance in the sense used, e.g. in [16] is not understood for $g > 0$.

2.2. Modifications. Our first result is about the behavior of stable balance under general modifications. This was essentially done in [13] for $C = \mathbb{P}^1$ and the discussion there largely extends to the general case.

By a semi-general (resp. general) down modification of $E$ we mean an exact sequence

$$0 \to E' \to E \to \tau \to 0$$

such that $\tau$ is of the form $\bigoplus k(p_i)^{s_i}$ for some collection (resp. a general collection) of distinct points $p_1, ..., p_k \in C$, and such that the map

$$E|_{p_i} \to k(p_i)^{s_i}$$

is a general quotient for $i = 1, ..., k$.

Lemma 5. A semi-general down modification of a stably balanced bundle is stably balanced.
Proof. We may assume the modification $E \to \tau$ is at a single point $p$. Set $t = \ell(\tau), r = \text{rk}(E)$. Thus $t \leq r$. By generality we may assume that for every subbundle $E_i \subset E$ in the HN filtration of $E$, the map $E_i|_p \to \tau$ has maximal rank. Let $E_i \subset E_j$ be members of the HN filtration of $E$, of respective ranks $r_i < r_j$ and $E_i' \subset E_j'$ their (saturated) intersections with $E'$ (we only need the case $j = i + 1$). Note that $E_i, E_j$ are the respective saturations of $E_i', E_j'$ in $E$, hence every pair of saturated subbundles of $E'$ is obtained this way. There are the following cases.

Case 1: $r_j \leq t$ and $E_i|_p \to \tau$ injective.
In this case $E_i' = E_i(-p), E_j' = E_j(-p)$ and
$$\mu(E_j'/E_i') = \mu(E_j/E_i) - 1.$$

Case 2: $r_i \leq t < r_j, E_i|_p \to \tau$ injective and $E_2|_j \to \tau$ surjective. This case can happen for at most one pair $(i, i + 1)$.
In this case $E_i' = E_i(-p)$ and $E_j' \subset E_j$ has colength $t$. We compute that
$$\mu((E_j'/E_i')) = \mu(E_j/E_i) - \frac{t - r_i}{r_j - r_i}.$$

Case 3: $t < r_i$ and $E_i|_p \to \tau$ surjective.
In this case $E_i' \subset E_i, E_j' \subset E_j$ both have colength $t$ and we have
$$\mu(E_j'/E_i') = \mu(E_j/E_i).$$
The ‘case number’ increases (nonstrictly) as we go up the filtration.

Note that $\mu_{\text{max}}(E') \leq \mu_{\text{max}}(E)$. Now assume to begin with that $E$ is semistable. Then for any quotient $E'/G$, we have
$$\mu(E) \leq \mu(E/G) \leq \mu(E'/G) + 1.$$
Thus $\mu_{\text{min}}(E') \geq \mu(E) - 1$, so $E'$ is stably balanced.

In the general case consider the HN filtration
$$0 = E_0 \subsetneq E_1 \subsetneq ... \subsetneq E_a = E.$$
and the corresponding filtration $E^*_i$ on $E'$. There is a unique $k$ such that the map $E_i|_p \to \tau$ is injective for $i \leq k$ and surjective, non-injective for $i > k$. The we have
$$E_i' = E_i(-p), \mu(E_i'/E_{i-1}') = \mu(E_i/E_{i-1}) - 1, i < k,$$
$$\mu(E_k/E_{k-1}) - 1 \leq \mu_{\text{min}}(E_k'/E_{k-1}') \leq \mu_{\text{max}}(E_k'/E_{k-1}') \leq \mu(E_k/E_{k-1}),$$
$$\mu(E_i'/E_{i-1}') = \mu(E_i/E_{i-1}), i > k.$$
Now $E'$ admits a filtration, not necessarily the HN filtration, with semistable graded pieces consisting of

$$E_i(-p)/E_{i-1}(-p), i = 1, ..., k-1; E_i/E_{i-1}, i = k+1, ..., a;$$

plus the HN graded pieces of $E'_k/E'_{k-1}$. By the semistable case just discussed, the slopes of these all lie in the interval $[\mu(E_k/E_{k-1}) - 1, \mu(E_k/E_{k-1})]$, hence $E'$ is stably balanced.

□

Thus, semi-general elementary modifications stay within the class of stably balanced bundles. As the following results shows, this is false for the class of (semi)stable bundles.

**Example 6.** In any genus, if $E = 2O$ and $\tau = k(p)$, then $E$ is semistable but $E' = O \oplus O(-p)$ is stably balanced but not semistable. So in that sense the result is sharp.

It appears that a semi-general modification of a rank-2 stable bundle is stable, which may in part explain the result of [5] on the stability of normal bundles of curves in $\mathbb{P}^3$. However, in higher rank it is possible for $E$ to be stable while $E'$ is not, at least in genus $g \geq 2$. We start with an example of a non-stable but maybe semi-stable modification of a stable rank-3 bundle. Consider a general extension

$$0 \rightarrow O \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

where $G$ is a line bundle of degree 1 and the extension corresponds to a general element $e \in H^1(G^{-1}) = H^0(G \otimes K)^*$. I claim that $F$ is stable. In fact, consider any line bundle $L$ of degree $\geq \mu(F) = 1/2$ i.e. $\deg(L) \geq 1$. If $\deg(L) > 1$ or $\deg(L) = 1$ and $L \neq G$ then clearly $H^0(F \otimes L^{-1}) = 0$. If $L = G$, the same is true by nontriviality of the extension. Thus $F$ is stable.

Now consider an extension

$$0 \rightarrow O \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

(3)

where $G$ is a line bundle of degree 1 and the extension corresponds to a general element $e \in H^1(\tilde{G}) = H^0(\tilde{F} \otimes K)^*$ which comes from $H^1(\tilde{G})$, i.e. fits in an exact diagram

$$0 \rightarrow O \rightarrow E \rightarrow F \rightarrow 0$$

(4)

$$0 \rightarrow O \rightarrow E \rightarrow F \rightarrow 0.$$

Let $L$ be any line bundle with $\deg(L) \geq \mu(E) = 1/3$ (i.e. $\deg(L) > 0$). As we have seen $H^0(F \otimes L^{-1}) = 0$, hence $H^0(E \otimes L^{-1}) = 0$. So $E$ has no rank-1 destabilizing subsheaf.

Now we have an exact sequence

$$0 \rightarrow F \rightarrow \wedge^2 E \rightarrow G \rightarrow 0$$

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which is none other than the dual of (3), twisted by $G$, and we know $H^0(F \otimes L^{-1}) = 0$. Therefore an argument similar to the above shows that we have $H^0(\wedge^2 E \otimes L^{-1}) = 0$, for any $L$ of degree $> 0$, so $E$ has no rank-2 destabilizing subsheaf. Thus $E$ is stable (even cohomologically stable in the sense of [6]).

Now a general corank-1 modification $E'$ of $E$ at $p$ fits on an exact sequence

$$0 \to O(-p) \to E' \to F \to 0$$

which corresponds to an element of $H^1(\bar{F} \otimes O(-p))$. As with the original element $\eta$ this comes from an element of $H^1(\bar{G} \otimes O(-p))$ corresponding to a modification $E'_G$ of $E_G$. Therefore the section $s$ of $F$ lifts to $E'$. Then $E'$ has slope 0 but $H^0(E') \neq 0$ so $E'$ is not stable (but maybe semi-stable).

For an example of a non-semistable modification consider instead a similar extension

(5)  

$$0 \to 2O \to E_2 \to F \to 0$$

then a similar argument as above shows that for any line bundle $L$ with $\text{deg}(L) \geq \mu(E_2) = 1/4$, we have $H^0(E_2 \otimes L^{-1}) = H^0(\wedge^3 E_2 \otimes L^{-1}) = 0$. Moreover we have an exact sequence

$$0 \to O \to \wedge^2 E_2 \to E_1 \to 0$$

where $E_1$ is analogous to $E$ above. Then a similar argument again shows $H^0(\wedge^2 E_2 \otimes L^{-1}) = 0$. Therefore $E_2$ cannot admit a destabilizing subsheaf of rank 1,2 or 3, hence it is stable.

Now a general corank-2 modification at $p$, $E'_2$, fits in an exact sequence

$$0 \to 2O(-p) \to E'_2 \to F \to 0$$

and as above we have $H^0(E'_2) \neq 0$ while $\mu(E'_2) = -1/4$, so $E'_2$ is not semistable.  

\[ \Box \]

2.3. Degeneration. Next we come to an important technique for constructing stably balanced bundles via degeneration. We begin with the following situation:

$$\pi : C \to B$$

is a family of curves over a smooth curve with all fibres $C_t$ smooth except

$$C_0 = C_1 \cup_p C_2$$

with $C_1, C_2$ smooth and transverse; $E$ is a vector bundle on $C$, and we set $E_i = E|_{C_i}, i = 1,2, \text{ and } E_t = E|_{C_t}$.

\textbf{Lemma 7.} Notations as above, assume $E_1$ is semistable and $E_2$ is stably balanced. Then a general $E_t$ is stably balanced.
Proof. Let \([\alpha, \beta]\) be the slope interval of \(E_2\) and let \(F_t\) be a subbundle of the general fibre \(E_t\)
\((F_t\) may possibly be defined only after a base-change, though actually we may choose for \(F_t\) the maximal-rank, maximal-slope subbundle and then no base-change is required). We will show that
\[
\mu(F_t) \leq \mu(E_1) + \beta.
\]
Then applying this result to the dual will show that
\[
\mu(F_t) \geq \mu(E_1) + \alpha.
\]
Thus, the slope interval of \(E_t\) is contained in \([\mu(E_1) + \alpha, \mu(E_1) + \beta]\), proving our result.

Now to prove (6), after a suitable base-change \(B' \to B\) and replacing \(C\) by \(C' = C \times_B B'\), we may assume given a torsion-free subsheaf \(F \subset E\) extending \(F_t\). Then after further base-change and blowing up we may assume \(\det(F_t) = \wedge \mathcal{F}\) extends to an invertible sheaf \(L\) on \(C'\) and that the map \(L \to \wedge \mathcal{E}\) vanishes only on an effective divisor \(Z\) supported on the special fibre so the induced map \(L(Z) \to \wedge \mathcal{E}\) is nowhere vanishing. Then we have
\[
s\mu(F_t) = \deg(L(-Z)|_{C'_0}) \leq \deg(L_{C_1}) + \deg(L|_{C_2}) \leq s(\mu(E_1) + \beta).
\]

The same result with the same proof also holds for a comb-shaped curve:

Lemma 8. Let \(C_0\) be a connected nodal curve of the form \(D \cup C_{01} \cup \ldots \cup C_{0m}\) with \(D \cap C_{0i}\) a single point, \(i = 1, \ldots, m\), and no other singularities. Let \(C \to B\) be a family as above with special fibre \(C_0\) and general fibre \(C_t\), and let \(E\) a vector bundle on \(C\) such that the restriction of \(E\) on all but one of the components \(D, C_{01}, \ldots, C_{0m}\) is semistable, and the restriction is stably balanced on the remaining component. Then the restriction \(E|_{C_t}\) is stably balanced.

Remark 9. Note that a stably balanced bundle \(E\) of slope \(\mu(E) \leq -1\) must have \(\mu_{\text{max}}(E) < 0\), hence \(H^0(E) = 0\).

A lci subvariety \(C \subset X\) is said to be stably balanced if the normal bundle \(N = N_{C/X}\) is, and rigid if \(H^0(N) = 0\). When \(C\) is a curve of genus \(g\) and \(X\) is \(n\)-dimensional, we have
\[
\mu(N) = (-C.K_X + 2g - 2)/(n-1).
\]
Consequently, if \(C\) is stably balanced and \(2g - 2 \leq C.K_X - n + 1\) then \(C\) is rigid.

2.4. Relation to interpolation. Lemma 3 implies that a general divisor \(D_t = p_1 + \ldots + p_t\) on \(C\), we have \(H^1(N(-D_t)) = 0\) provided \(t < \mu(N) - (2g - 1)\). Therefore we conclude:

Lemma 10. If \(C \subset X\) is a stably balanced curve of genus \(g\) and \(t < (-C.K_X + 2g - 2)/(n-1) - 2g + 1\), the \(C\) is \(t\)-interpolating in the sense that \(t\) general points on a general deformation of \(C\) are general in \(X\).
The \(t\)-interpolating property in the Lemma is weaker than the full interpolation or balance property as studied e.g. in [16].

3. Generalities of deformations

Here we consider curves \(C\) on hypersurfaces \(X\) of degree \(d \geq n + 1\) in \(\mathbb{P}^n\). This case is of a different character because generally \(\chi(N_{C/X}) < 0\) so \(H^1(N_{C/X})\) cannot vanish, nor can \(X\) be general. Consequently at least the smoothing issue has to be handled differently. We shall do so by studying deformations and specializations of the triple \((C, X, \mathbb{P}^n)\).

3.1. Deformations of pairs. Compare [17]. Let \(C \subset X\) be an inclusion of smooth projective varieties. There are corresponding deformation functors and spaces

\[
\text{Def}(C/X) \to \text{Def}(C, X) \to \text{Def}(X)
\]

parametrizing deformations of, respectively, \(C\) fixing \(X\), the pair \((C, X)\), the variety \(X\). These correspond to an exact sequence of sheaves on \(X\)

\[
0 \to T_X(-\log C) \to T_X \to N_{C/X} \to 0.
\]

In fact, \(T_X\) is a sheaf of Lie algebras and \(T_X(-\log C)\) is a subalgebra sheaf, which endows \(N = N_{C/X}\) with the structure of Lie atom (see [12]), which yields the corresponding deformation theory, i.e. deformations of \(C\) in a fixed \(X\). Thus the complex \(N[-1]\) is endowed with a bracket in the derived category and deformations (resp. obstructions to deformations) of \(C\) fixing \(X\) are in \(H^0(N)\) (resp. \(H^1(N)\)); for the pair \((C, X)\) the corresponding groups are \(H^1(T_X(-\log C)), H^2(T_X(-\log C))\).

3.2. Embedded deformations of pairs. Let \(X \subset P\) be a locally complete intersection with \(P\) smooth, and let \(C \subset X\) be a locally complete intersection in \(X\). Then the normal bundles \(N_{C/P}\) and \(N_{X/P}\) admit \(N_{X/P}|_C\) as a common quotient. Define a coherent sheaf \(N_{C/X/P}\) on \(X\) by the exact sequence

\[
0 \to N_{C/X/P} \to N_{C/P} \oplus N_{X/P} \to N_{X/P}|_C \to 0.
\]

Then \(N_{C/X/P}\) controls deformations of the pair \((C, X)\) as subvarieties of \(P\), and we have an exact sequence

\[
0 \to N_{C/X} \to N_{C/X/P} \to N_{X/P} \to 0.
\]

We will say that the pair \((C \subset X)\) of subvarieties is rigid-regular in \(P\) provided the following vanishings hold.

\[
H^0(N_{C/X}) = 0, H^1(N_{C/P/X}) = 0, H^1(N_{X/P}) = 0.
\]

The following follows from general deformation obstruction theory (see e.g. [17]):
Lemma 11. Assume \((C, X)\) is rigid-regular in \(P\). Then near \((C, X)\), the Hilbert scheme of pairs in \(P\) embeds as a smooth subvariety of codimension \(-\chi(N_{C/X}) = h^1(N_{C/X})\) in the Hilbert scheme of \(X\) in \(P\).

There is another exact sequence

\begin{equation}
0 \to N^C_{X/P} \to N_{C/X/P} \to N_{C/P} \to 0
\end{equation}

where \(N^C_{X/P} \subset N_{X/P}\) is the subsheaf corresponding to deformations of \(X\) fixing \(C\) (or equivalently, deformations of the birational transform of \(X\) in the blowup of \(P\) in \(C\)).

Now suppose \(X\) is a hypersurface of degree \(d\) with equation \(f\) in \(P = \mathbb{P}^n\). We can identify

\begin{equation}
N_{X/P} = \mathcal{O}_X(d), N^C_{X/P} = \mathcal{I}_{C/X}(d) = \mathcal{I}_{C/P}(d)/f\mathcal{O}_P.
\end{equation}

Now assume \(C, X, P\) are smooth, and let \(T_{P/X/C} \subset T_P\) denote the subsheaf consisting of local vector fields tangent to both \(X\) and \(C\). Then we have an exact sequence

\[0 \to T_{P/X/C} \to T_P \to N_{C/X/P} \to 0\]

and also

\[0 \to T_P(-X) \to T_{P/X/C} \to T_X(-\log C) \to 0.\]

Then from the cohomology sequence we conclude:

Lemma 12. (i) Assume the vanishings

\[H^1(N_{C/X/P}) = 0, H^2(T_P) = 0, H^3(T_P(-X)) = 0.\]

Then \(C\) is rigid-regular in \(X\), i.e. the pair \((C, X)\) has unobstructed deformations of dimension \(h^1(T_X(-\log C))\).

(ii) If moreover \(H^1(N^C_{X/P}) = 0\), then \(C\) may be assumed general in the Hilbert scheme of subvarieties \(C \subset P\).

Remark 13. If \(C\) is a smooth curve and \(P = \mathbb{P}^n\), the vanishing \(H^1(N^C_{X/P}) = 0\) follows from maximal rank, specifically from \(H^1(\mathcal{I}_{C/P}(d)) = 0\).

Example 14. Let \(P = \mathbb{P}^n, n \geq 5\) and \(X \subset P\) a smooth hypersurface of degree \(d\). Then if \(C \subset X\) is a curve of degree \(e\) and genus \(g\) with \(H^1(N_{C/X/P}) = 0\), the pair \((C, X)\) has unobstructed deformations of dimension \(h^1(T_{X/C})\). If \((C, X)\) is relatively regular in \(P\), the family of hypersurfaces carrying deformations of \(C\) is smooth of codimension \(e(d-n-1) + (n-4)(g-1)\) near \(X\).
3.3. Singular case. Consider a lci diagram of connected normal-crossing varieties

\[ C_0 = C_1 \cup C_2 \subset X_0 = X_1 \cup_Z X_2 \subset P_0 = P_1 \cup_Q P_2 \subset P \]

where each \( C_i, X_i, P_i \) and \( P \) are smooth and there is a morphism \( \pi : P \to B \) to a smooth curve such that \( P_0 = \pi^{-1}(0) \). Then a general fibre \( P_t = \pi^{-1}(t) \) is smooth. We have an exact sequence

\[ 0 \to N_{C_0/X_0/P_0} \to N_{C_0/X_0/P} \to O_X \to 0. \]

We have an isomorphism \( H^0(O_{X_0}) \cong H^0(O_Z) \) and \( O_Z \) can be identified with the \( T^1_{X_0} \) which restricts on \( C_0 \) to \( T^1_{C_0} \), it follows that a deformation of \((C_0, X_0)\) in \( P \) is a smoothing of \( C_0 \) and \( X_0 \) iff its image in \( O_{X_0} \) is nonzero. Then from the long cohomology sequence we conclude

**Lemma 15.** Notations as above, assume

\[ H^1(N_{C_0/X_0/P_0}) = 0 = H^1(O_{X_0}). \]

Then \((C_0, X_0, P_0)\) is smoothable to \((C_t, X_t, P_t)\) in a general fibre \( P_t \) of \( \pi \), i.e. there is a smooth variety \( T \) of dimension \( h^0(N_{C_0/X_0/P}) \), a smooth morphism \( T \to B \) with a point \( 0' \) over \( 0 \), and a triple \((C, X, P \times_B T)\) flat over \( T \), with special fibre \((C_0, X_0, P_0)\) and general fibre \((C_t, X_t, P_t)\) with \( C_t, X_t, P_t \) smooth and moreover \( H^1(N_{C_t/X_t/P_t}) = 0 \).

As for the vanishing in question, note that

\[ N_{C_0/X_0/P_0}|_{X_i} = N_{C_i/X_i/P_i}, \quad i = 1, 2 \]

hence we have an exact sequence

\[ 0 \to N_{C_1/X_1/P_1}(-Z) \to N_{C_0/X_0/P_0} \to N_{C_2/X_2/P_2} \to 0. \]

4. Results

The main result of this paper is the following

**Theorem 16.** Assume

\[ d \geq n + 1 \geq 5, \]

\[ g \geq 1, \]

\[ e > 2g + 2n, \]

(10)

\[ \binom{d + n - 2}{n - 1} > \frac{ed}{2}. \]

Then for a general pair \((C, X)\), where \( C \) is a general smooth curve of genus \( g \) and degree \( e \) and \( X \) is a hypersurface of degree \( d \) in \( P^n \), \( C \) is rigid-regular stably balanced on \( X \).
Remark 17. Note that because \( d > n \), we have \( \binom{d+n-2}{n} \geq \binom{d+n-2}{n-1} \), hence our assumptions imply
\[
\binom{d+n}{n} \geq \binom{d+n-1}{n} = \binom{d+n-2}{n-1} + \binom{d+n-2}{n} \\
\geq ed/2 + ed/2 + 1 - g = ed + 1 - g.
\]

Ideally, one would like to replace the assumption (10) by a weaker one such as \( \binom{d+n}{n} > ed + 1 - g \), however our argument as is does not yield this. Likewise, our lower bound on \( e \) might be weakened to something like nonspeciality.

Remark 18. Assumptions as in Theorem 16, the family of pairs \((C, X)\) where \( C \) has maximal rank is clearly irreducible of the expected dimension, hence so is the corresponding family of hypersurfaces \( X \). As subfamily of the family of all hypersurfaces, the latter family has codimension
\[
h^1(N_{C/X}) = -\chi(N) = (d-n-1)e + (n-4)(g-1).
\]

Example 19. There are stably balanced, rigid-regular curves of every degree \( 5 < e < 72 \) and genus \( 0 < g < e/5 - 5 \) on smooth quintic fourfolds.

Proof of Theorem. We use induction on \( d \geq n + 1 \) and a suitable quasi-cone degeneration in a fan degeneration of \( \mathbb{P}^n \) as in §1. We construct a suitable curve on a quasi-cone degeneration as in §3.3
\[
C_0 = C_1 \cup C_2 \subset X_0 = X_1 \cup Z X_2 \subset P_0 = P_1 \cup Q P_2.
\]
Thus \( P_1 = B_p \mathbb{P}^n \) with exceptional divisor \( Q \), \( P_2 = \mathbb{P}^n \) with \( Q \subset P_2 \) a hyperplane, \( X_1 \subset P_1 \) is the blowup of a hypersurface of degree \( d \) with multiplicity \( d - 1 \) at \( p \), and \( X_2 \subset P_2 \) is a hypersurface of degree \( d - 1 \). Via projection from \( p \) we may realize \( X_1 \) as \( B_Y \mathbb{P}^{n-1} \) where \( Y \) is a \((d-1, d)\) complete intersection.

We now assume \( d \geq n + 1 \) and work by induction on \( d \). Assume to begin with that \( e \) is even. We first construct \( C_2 \subset X_2 \subset P_2 \). Begin with a general curve \( C_2 \in \mathbb{P}^n = P_2 \) of genus \( g \geq 0 \) and degree \( e_2 = e/2 > g + n \) satisfying the inequalities (10). Note that because \( d \geq n + 1 \) we have \( \binom{d+n-2}{n-1} \leq \binom{d+n-2}{n} \) hence
\[
\binom{d+n-2}{n} > e_2(d-2) + 1 - g.
\]
By the Maximal Rank Theorem of Ballico-Ellia [2], the restriction map
\[
H^0(\mathcal{O}_{\mathbb{P}^n}(d-2)) \rightarrow H^0(\mathcal{O}_{C_2}(d-2)h)
\]
is surjective and we may choose a general hypersurface \( X_2 \subset \mathbb{P}^n \) of degree \( d - 1 \) containing \( C_2 \) which will moreover be smooth. Moreover by induction or, in the initial case \( d = n + 1 \), by the results of [16] and Lemma 1, the normal bundle \( N_{C_2}/X_2 \) is stably balanced.

Furthermore, by the theorem of Atanasov-Larsen-Yang [1], we have \( H^1(N_{C_2}/P_2(-h)) = 0 \). From this, together with the basic exact sequence (7) it follows that

\[
H^1(N_{C_2}/X_2/P_2(-h)) = 0
\]

and also that by varying \( C_2, X_2 \) while fixing \( Z = X_2 \cap Q \), the points \( W \setminus C_2 \cap Z = \{p_1, \ldots, p_{e_2}\} \) will be \( e_2 \) general points on \( Z \).

Now to construct \( C_1 \subset X_1 \subset P_1 \), identify \( Z \) with \( F_{d-1} \subset \mathbb{P}^{n-1} \) and let \( \ell_i \subset \mathbb{P}^{n-1} \) be a general line through \( p_i \). By assumption, \( \binom{d+n-2}{n-1} > e_2(d-1) \). By the Maximal Rank Theorem of Hartshorne and Hirschowitz [8], the restriction map

\[
H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d-1)) \to \bigoplus H^0(\mathcal{O}_{\ell_i}(d-1))
\]

is surjective. Let \( F_d \subset \mathbb{P}^{n-1} \) be a general hypersurface containing \( \ell_i \cap F_{d-1} \setminus p_i \) but not \( p_i \) for \( i = 1, \ldots, e_2 \). Now I claim that for any \( q \in \ell_i \setminus p_i \), \( i = 1, \ldots, e_2 \), \( F_d \) has general tangent hyperplane at \( q \). Indeed this is clear because given \( q \) we may take an \( F_d \) of the form \( F_{d-1}' \cup F_1 \) with \( F_1 \) general through \( q \) and \( F_{d-1}' \) through all the other points. Therefore a general \( F_d \) has general tangent hyperplane at the given \( q \), hence also for all \( q \) simultaneously.

Now let \( Y = F_{d-1} \cap F_d \) and \( X_1 = B_Y \mathbb{P}^{n-1} \subset P_1 \) as above. Note that the lines \( \ell_i \) lift to conics \( D_i \) on \( X_1 \) and we set

\[
C_1 = D_1 \cup \ldots \cup D_{e_2}.
\]

Note that by the claim above, the normal bundle \( N_{D_i}/X_1 \) is a general modification of \( N_{\ell_i}/\mathbb{P}^{n-1} \), hence it is balanced. In particular, \( H^0(N_{C_1}/X_1) = 0 \).

Now on \( P_1 \) we have, where \( H \) is a hyperplane from \( \mathbb{P}^n \) and \( L_1 = H - Q \) is a hyperplane from \( \mathbb{P}^{n-1} \),

\[
X_1 \in |dH - (d-1)Q| = |dL_1 + Q|
\]

hence

\[
N_{X_1}/P_1 = \mathcal{O}(dL_1 + Z) = \mathcal{O}((2d-1)L_1 - E_Y).
\]

From this it follows easily that \( H^1(N_{X_1}/P_1) = 0 \). Also clearly \( H^1(N_{C_1}/P_1) = 0 \). Moreover by the aforementioned Maximal Rank Theorem of Hartshorne and Hirschowitz [8], the restriction map

\[
H^0(P_1, dL_1 + Q) \to H^0(C_1, (dL_1 + Q)|_{C_1})
\]

is surjective. Hence we conclude that

\[
H^1(N_{C_1}/X_1/P_1) = 0.
\]
Therefore by the obvious exact sequence
\[ 0 \to N_{C_2/X_2/P_2}(-1) \to N_{C_0/X_0/P_0} \to N_{C_1/X_1/P_1} \to 0 \]
(or just as well, the analogous one with $C_1, C_2$ etc. interchanged), we conclude

\[ H^1(N_{C_0/X_0/P_0}) = 0. \]

Next we claim that $C_0$ is rigid on $X_0$, hence likewise for the smoothing. To see this note that

\[ N_{C_0/X_0}|_{C_i} = N_{C_i/X_i}, i = 1, 2 \]

and as we have seen $N_{C_1/X_1}|_{D_i}$ is a general modification of $N_{i_i/\mathbb{P}^{n-1}}$ and in particular it is strictly negative and its upper subspace at $p_i$ is general. As for $N_{C_2/X_2}$, in the initial case $d = n + 2$, we know it is balanced and we have $\chi(C_2/X_2) = n - 4 < e_2$ so $h^0(N_{C_2/X_2}) < e_2$. In case $d > n + 2$ we have inductively that $h^0(N_{C_2/X_2}) = 0$. Thus in all cases it follows that

\[ H^0(N_{C_0/X_0}) = 0. \]

Moreover an argument as in the proof of [13], Theorem 6 or [15], Theorem 1 (essentially, blowing down the tails $D_i$) shows that for a smoothing $C' \subset X'$ of $C_0 \subset X_0$, the normal bundle $N_{C'/X'}$ is essentially a deformation of a semi-general down modification of $N_{C_2/X_2}$ corresponding to the upper subspaces of $T_{P_i}D_i$ and consequently by Lemma [5], $C' \subset X'$ is stably balanced.

To complete the proof it suffices by Lemma [12] to prove that $H^1(N_{C'/X'/\mathbb{P}^n}) = 0$ i.e. by Remark [13] that $H^1(\mathcal{I}_{C'/\mathbb{P}^n}(d)) = 0$. For this it would suffice to prove that $H^1(\mathcal{I}_{C_0/P_0}(L)) = 0$ where $L_0$ is the line bundle $\mathcal{O}_{P_1}(dH - (d - 1)Q) \cup \mathcal{O}_{P_2}(d - 1)H$ which restricts to $L_1$ on $P_1$. We have an exact diagram

\[
\begin{array}{cccc}
0 & \to & H^0(\mathcal{O}_{P_2}(d - 2)H) & \to & H^0(L_0) & \to & H^0(\mathcal{O}_{P_1}(dH - (d - 1)Q)) & \to & 0 \\
0 & \to & H^0(\mathcal{O}_{C_2}(d - 2)) & \to & H^0(\mathcal{O}_{C_0}(L_0)) & \to & H^0(\mathcal{O}_{C_1}(dH - (d - 1)Q)) & \to & 0
\end{array}
\]

As we have seen, the left vertical map is surjective. As for the right vertical map, we can identify $\mathcal{O}(dH - (d - 1)Q) = dL_1 + Q$ and use the exact sequence

\[ 0 \to \mathcal{I}_{C_1/P_1}(dL_1) \to \mathcal{I}_{C_1/P_1}(dL_1 + Q) \to I_{p_1,\ldots,p_2/Q}(dL_1 + Q) \to 0. \]

Here the left term has vanishing $H^1$ by Maximal Rank, while the right term can be identified with $I_{p_1,\ldots,p_2/Q}((d - 1)L_1)$ and has vanishing $H^1$ by the generality of the points. Therefore the middle term has vanishing $H^1$ so the right map in [13] is surjective, hence so is the middle map.

This completes the proof of Theorem [16] in case $e$ is even.
The proof for $e = 2e_2 - 1$ odd is similar except we make $D_{e_2}$ a 'line' (i.e. a ruling of $P_1/P^{n-1}$) instead of a conic by making $F_d$ go through the point $p_{e_2}$ as well. □
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