Regge calculus from discontinuous metrics

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Abstract

Regge calculus is considered as a particular case of the more general system where the linklengths of any two neighbouring 4-tetrahedra do not necessarily coincide on their common face. This system is treated as that one described by metric discontinuous on the faces. In the superspace of all discontinuous metrics the Regge calculus metrics form some hypersurface. Quantum theory of the discontinuous metric system is assumed to be fixed somehow in the form of quantum measure on (the space of functionals on) the superspace. The problem of reducing this measure to the Regge hypersurface is addressed. The quantum Regge calculus measure is defined from a discontinuous metric measure by inserting the $\delta$-function-like phase factor. The requirement that this reduction would respect natural physical properties (positivity, well-defined continuum limit, absence of lattice artefacts) put rather severe restrictions and allows to define practically uniquely this phase factor.
Since the idea had been put forward that Regge calculus should be formulated in terms of the areas of the triangles rather than the edge lengths \(^1\), the so-called area Regge calculus model was of a certain interest \(^2, 3\). In this model the areas of the 2-faces (triangles) are treated as independent variables. Thereby configuration space of the theory becomes larger, and ordinary Regge calculus corresponds only to some subset (hypersurface) in this space. Despite of this, the equations of motion of the theory get simplified. In particular, a kind of the area Regge calculus, area tensor-connection one \(^4\) possesses the set of commuting constraints (in the continuous time limit) and can be quantised resulting in the finite expectation values for areas \(^5\). The problem is how to pass from areas to lengths. The constraints enforcing areas to be expressible in terms of the edge lengths have been discussed in \(^6, 7\). Now the problem is dynamical one, e. g. given the Feynman path integral on the configuration space of area Regge calculus, how could we define such an integral on the subspace corresponding to the usual length-based Regge calculus?

In the area Regge calculus the two 4-simplices \(\sigma^4\) sharing a given 3-face \(\sigma^3\) do not necessarily have coinciding linklengths on this face, only the 4 conditions equating the areas of the triangles \(\sigma^2\) on this 3-face are required. One can go further and discuss the 4-simplices with completely independent lengths, i. e. impose no conditions at all. Area Regge calculus will be a particular case of this configuration. Introducing metric tensor \(g_{\lambda\mu}\) one can say that it is discontinuous on the 3-faces. Regge calculus is then specified by setting continuity conditions on the induced 3-face metric \(g^\parallel_{ab}\) on the 3-faces \(\sigma^3\),

\[
\Delta_{\sigma^3} g^\parallel_{ab} \overset{\text{def}}{=} g^\parallel_{ab}(\sigma^4_1) - g^\parallel_{ab}(\sigma^4_2) = 0
\]

where \(\sigma^4_1, \sigma^4_2\) are the two 4-simplices sharing the 3-face \(\sigma^3 = \sigma^4_1 \cap \sigma^4_2\). (At the same time the normal-to-face metric \(g^\perp\) is discontinuous thus providing nonzero angle defects).

The conditions \(^1\) define some hypersurface \(\Gamma_{\text{cont}}\) in the superspace of simplicial (piecewise-flat) metrics including discontinuous ones. Suppose quantum theory on this superspace is fixed in the form of the Feynman path integral. That is, a quantum measure is given. The measure can be viewed as a linear functional \(\mu(\Psi)\) on the space of the functionals \(\Psi(\{g\})\) on the superspace. To pass to the quantum theory on the hypersurface \(\Gamma_{\text{cont}}\) we use some analogy with the quantum-mechanical notion of the ”state”, although full analogy is absent since we do not mean decomposing the spacetime into space and time. In our case the functionals \(\Psi(\{g\})\) are intended to describe the states. (Rather these could be considered as analogs of quantum-mechanical \(\Phi^\ast \Phi\).) We are interested in the particular case of these with support on \(\Gamma_{\text{cont}}\) of the form

\[
\Psi(\{g\}) = \psi(\{g\}) \delta_{\text{cont}}(\{g\})
\]

where \(\delta_{\text{cont}}(\{g\})\) is (many-dimensional) \(\delta\)-function with support on \(\Gamma_{\text{cont}}\). Generally derivatives of \(\delta\)-function also ensure given support, but these would violate positivity in what follows. If the measure can be defined on such the functionals then we define the measure on \(\Gamma_{\text{cont}}\),

\[
\mu_{\text{cont}}(\cdot) = \mu(\delta_{\text{cont}}(\{g\}) \cdot).
\]
To construct $\delta_{\text{cont}}$ first reveal irreducible set of the constraints on the metric of the type of (1) in terms of linklengths. For that consider all the 4-simplices containing the given link, i.e. the star of the link. The surface of the star is topologically the 2-sphere and is depicted in the Fig. 1.

In this picture the $k$-simplices $\sigma^k$ mean the 4D $k + 1$-simplices $O'\sigma^k$, in particular, the considered link $O'O$ is represented here by the vertex $O$, and it’s length squared defined in the 4-simplex, say, $O'OABC$ is denoted here by $s_O(ABC)$. The fact that these values are the same in all the 4-simplices $O'\sigma^k$ is taken into account by the product of $\delta$-functions over all the links on the 2-sphere Fig.1,

$$\prod_{\sigma^2 \supset O} \delta(\Delta_{\sigma^2} s_O) \quad (4)$$

where $\Delta_{OAB} s_O = s_{O(ABC)} - s_{O(ABG)}$, etc. This product contains redundant $\delta$-functions connected with occurence of cycles enclosing separate links $\sigma^1 = OA, OB, \ldots$ (corresponding to vertices on the 2-sphere Fig.1). To exclude cycles, draw some (2D in our picture) cut $H_O^{(2)}$ in the star of the point $O$ consisting of the triangles and passing through each radial link. By definition, $H_O^{(2)}$ is a collection of the triangles on which continuity conditions for $s_O$ should not be imposed. This cut should be chosen in such a way that it, first, would not consist of components intersecting only at the point $O$ (in order that cycles in the compliment of the cut in the star were absent indeed), and, second, would not divide interior of the star into the disconnected components (otherwise $s_O$ in these components would be allowed to be different). In other words, intersection of $H_O^{(2)}$ with the 2-sphere $S_O^{(2)}$ of Fig.1 should be (poligonal) continuous line without self-intersections.

To cancel redundant $\delta$’s in (1) one should divide (1) by many-dimensional $\delta$-function which would express continuity of $s_O$ on $H_O^{(2)}$. A sufficient condition for such the con-
tinuity is expressed by the product of \( \delta \)-functions over all the vertices on the 2-sphere Fig.1,

\[
\prod_{\sigma^1 \supset O} \delta(\Delta_{\sigma^1} s_O).
\] (5)

Here \( \Delta_{\sigma^1} s_O \stackrel{\text{def}}{=} \Delta_{\sigma^2(\sigma^1)} s_O \) where \( \sigma^2(\sigma^1) \) is a one of the two triangles in \( H_O^{(2)} \) containing \( \sigma^1, H_O^{(2)} \supset \sigma^2(\sigma^1) \supset \sigma^1 \).

In turn, (5) contains redundant \( \delta \)-functions (more accurately, one such function). Indeed, the number of edges in the continuous line without intersections \( H_O^{(2)} \cap S_O^{(2)} \) is by one smaller than the number of vertices through which the line passes. This means that \( \sigma^2(\sigma^1) \) involved in the definition of (5) above is not one-to-one correspondence, and there is a pair of the two neighbouring radial links, \( \sigma^1_1, \sigma^1_2 \), the edges of the same triangle \( \sigma^2(\sigma^1_1) = \sigma^2(\sigma^1_2) \) so that the \( \delta \)-function of \( \Delta_{\sigma^1_1} s_O = \Delta_{\sigma^1_2} s_O \) enters (5) twice. Therefore one should divide (5) by

\[
\delta(\Delta_O s_O)
\] (6)

where \( \Delta_O s_O \stackrel{\text{def}}{=} \Delta_{\sigma^1_1} s_O = \Delta_{\sigma^1_2} s_O \). Again situation can be described as occurrence of some reminder cycle, now in the cut \( H_O^{(2)} \) itself treated in some generalised way taking into account the number of times the cut passes the same triangle, and (6) just serves to cancel effect of this cycle. Such the cut OBABCDE... passing OAB twice is just shown in Fig.1.

The resulting \( \delta \)-function stating unambiguity of \( s_O \) looks like

\[
\prod_{\sigma^1 \supset O} \delta(\Delta_{\sigma^1} s_O) \left( \prod_{\sigma^1 \supset O} \delta(\Delta_{\sigma^1} s_O) \right)^{-1} \delta(\Delta_O s_O).
\] (7)

Here the number of \( \delta \)-functions is \( L - P + 1 \) where \( L, P \) are the numbers of the links and vertices on the 2-sphere Fig.1, respectively. In some functional integral these are to be integrated over

\[
d_{s_O(ABC)}d_{s_O(ACD)}d_{s_O(ADE)}\cdots,
\] (8)

the overall number of integrations being \( T \), the number of the triangles on the 2-sphere Fig.1. As a result, we are left with \( T - L + P - 1 = 1 \) integration as it follows from the Euler characteristic 2 for the 2-sphere.

Upon restoring the 4D notations the eq. (4) reads

\[
\prod_{\sigma^3 \supset O'} \delta(\Delta_{\sigma^3} s_{O'O}) \left( \prod_{\sigma^2 \supset O'} \delta(\Delta_{\sigma^2} s_{O'O}) \right)^{-1} \delta(\Delta_{O'O} s_{O'O}).
\] (9)

Take the product of these expressions over all the links O'O. Rearranging, we have for separate factors

\[
\prod_{\sigma^1} \prod_{\sigma^k \supset \sigma^1} \delta(\Delta_{\sigma^k} s_{\sigma^1}) = \prod_{\sigma^k} \prod_{\sigma^1 \subseteq \sigma^k} \delta(\Delta_{\sigma^k} s_{\sigma^1}) = \prod_{\sigma^k} \delta_{\frac{k+1}{2}}(\Delta_{\sigma^k} S_{\sigma^k})
\] (10)
thus expressing these in terms of the discontinuities of the edge component metric $S_{\sigma^k}$ induced on the $k$-simplex and being simply collection of $k^{k+1 \over 2}$ it’s edge lengths squared. The total product thus reads

$$\prod_{\sigma^3} \delta^6(\Delta_{\sigma^3} S_{\sigma^3}) \left( \prod_{\sigma^2} \delta^3(\Delta_{\sigma^2} S_{\sigma^2}) \right)^{-1} \prod_{\sigma^1} \delta(\Delta_{\sigma^1} S_{\sigma^1}).$$

(11)

Some remark should be made concerning the middle factor in this expression where the metric discontinuity $\Delta_{\sigma^2} S_{\sigma^2}$ for the different components of metric $S_{\sigma^2}$, in principle, may turn out to be defined between the different pairs of the neighbouring 4-simplices containing the given triangle $\sigma^2$, if one uses the above construction. Namely, for each $\sigma^2$ containing the given $\sigma^1$ we take $\sigma^3$, one of the two 3-faces containing $\sigma^2$ and belonging to the 3D cut $H^{(3)}_{\sigma^1}$ in the star of the link $\sigma^1$ (the above $H^{(2)}_O$ in 3D language). On this $\sigma^3$ the considered discontinuity of the $\sigma^2$-metric just should be taken. However, as we see, the choice of this $\sigma^3$ depends (by means of $H^{(3)}_{\sigma^1}$) on the edge $\sigma^1$ of this triangle, i.e. may be different for the different components. Fortunately, it is possible to change the choice of this 3-face not violating the result so that it would become the same for the edges of a given triangle. Indeed, singularity connected with the cycle enclosing the triangle is contained in the product of the type $\delta(s_1 - s_2)\delta(s_2 - s_3)\ldots \delta(s_N - s_1)$ over all $N$ 3-faces containing the triangle and can be cancelled by dividing it by any of the factors yielding the same result. Thereby definition of the discontinuity of the metric induced on the triangle can be adjusted to be the same for all the metric components.

The form looks suitable for setting requirements which would help to fix a factor multiplying the $\delta$ functions in the measure. A natural assumption is that the result of implementing continuity condition for the induced $k$-metric across a $k$-face should not depend on the real form and size of the $k$-face, only on the hyperplane spanned by this face. This assumption looks natural if we consider Regge links as imaginary objects only used for the needs of triangulation on some smooth background metric. Of course, something like ”interaction” of the Regge links may exist, but this is supposed to be introduced by ”physical” part of the theory by which we mean the theory of discontinuous metric distributions already fixed somehow in the form of the functional integral, while what do we consider here is a kind of kinematical phase factor in which we are trying to minimize any dynamical effect especially introduced ”by hand”. It is convenient to rewrite corresponding $\delta$ function in terms of metric tensor $g_{\lambda\mu}$ and of a set of $k$ 4-vectors $\tau^\lambda_a$ defining the $k$-face and the induced metric on it, $g_{ab}^\parallel = \tau^\lambda_a \tau^\mu_b g_{\lambda\mu}$. Then it is clear that the required factor is determinant of $g_{ab}^\parallel$ raised to the power $1 \over 2(k+1)$, i.e. the $(k+1)$-th power of the $k$-face volume $V_{\sigma^k}$. The resulting $\delta$ factor reads

$$[\det(\tau^\lambda_a \tau^\mu_b g_{\lambda\mu})]^{k+1 \over 2} \delta^{k+1 \over 2}(\tau^\lambda_a \tau^\mu_b \Delta_{\sigma^k} g_{\lambda\mu}) = V_{\sigma^k}^{k+1 \over 2} \delta^{k+1 \over 2}(\Delta_{\sigma^k} S_{\sigma^k}).$$

(12)

It is invariant w.r.t. the deformations of the $k$-face keeping it placed in a fixed $k$-plane (the metric $g_{\lambda\mu}$ being fixed) $\tau^\lambda_a \mapsto m^\lambda_a \tau^\lambda_b$ with arbitrary $k \times k$ matrix $m$. This property looks natural also from the viewpoint of existence of the continuum limit and, in particular, of absence of the lattice artefacts if we do not want to introduce these artefacts ”by hand”. troubleshooting
The final answer reads

$$\delta_{\text{cont}} = \prod_{\sigma^3} V_{\sigma^3}^4 \delta^6(\Delta_{\sigma^3} S_{\sigma^3}) \left( \prod_{\sigma^2} V_{\sigma^2}^3 \delta^3(\Delta_{\sigma^2} S_{\sigma^2}) \right)^{-1} \prod_{\sigma^1} V_{\sigma^1}^2 \delta(\Delta_{\sigma^1} S_{\sigma^1})$$

(13)

where the prime on the product means that the redundant $\delta$’s are omitted.

Consider consequences of taking a more simple expression for $\delta_{\text{cont}}$. For example, take in [13] only the primed product of $\delta$-functions and only slightly change it according to the recipe [12] at $k = 1$ considering each argument of each $\delta$-function there as discontinuity of the induced 1-metric (the length squared) across the 1-face (the link). Correspondingly, multiply each function by $s_{\sigma^1}$. Consider the star of a given link $O’O$, i.e. Fig. 1 in 3D language. The eq. (9) turns out to be multiplied by $(T - 1)$-th power of $s_{O’O}$, $T$ being the number of the 4-simplices containing this link. Upon integrating out the $\delta$-functions we are left with the factor

$$s_{O’O}^{-1} ds_{O’O}$$

(14)

in the measure corresponding to the $s_{O’O}$-dependence. Suppose the partial integration in the functional integral over $ds_{O’O}$ is made, and we are interested in the dependence on the distance between any two neighbouring vertices, say A and B, in the star of our link. Confluence of these two vertices together will lead to changing the index in (13) by $-2$. That is, there is discontinuity at the point $s_{AB} = 0$. This contradicts to our assumption on Regge links as purely triangulation objects on some smooth background metric since then tending A to B would be smooth process.

This “no-go” example advises to view our result [13] from a more general position. The issuing point are already discussed $\delta$-factors of the type of [12] invariant w.r.t. changing the form of the $k$-face. For a given $k$-face $\sigma^k$ these factors can differ by the choice $i = 1, \ldots, n_k$ of the 3-face $\sigma^3 = \sigma_i^3(\sigma^k) \supset \sigma^k$ across which discontinuity of the metric is defined. As we have discussed just now, the number of such factors per $k$-face $n_k$ should be fixed not depending on the neighbourhood of the $k$-face, and general form of $i$-th factor for all the $k$-faces looks like

$$\delta_{ik} = \prod_{\sigma^k} V_{\sigma^k}^{k+1} \delta^k \frac{k+1}{2} (\Delta_{\sigma^3(\sigma^k)} S_{\sigma^k}),$$

(15)

$i = 1, \ldots, n_k$, $n_3 = 1$ of which the overall factor is constructed,

$$\delta_{\text{cont}} = \delta_{13}^{\varepsilon_{13}} \prod_{i=1}^{n_2} \delta_{12}^{\varepsilon_{12}} \prod_{i=1}^{n_1} \delta_{11}^{\varepsilon_{11}}$$

(16)

where $\varepsilon_{ik}$ are some integers. These integers being larger than 1 or negative may have sense if, as above, upon formal cancellation of the $\delta$’s in the numerator and denominator we come to the well defined expression which in this case should take the form

$$\delta_{\text{cont}} = \left( \prod_{\sigma^3} V_{\sigma^3}^4 \right)^{\varepsilon_{13}} \left( \prod_{\sigma^2} V_{\sigma^2}^3 \right)^{\sum_{i=1}^{n_2} \varepsilon_{12}} \left( \prod_{\sigma^1} V_{\sigma^1}^2 \right)^{\sum_{i=1}^{n_1} \varepsilon_{11}} \prod_{\sigma^1, \sigma^3 \supset \sigma^1} \delta(\Delta_{\sigma^3} S_{\sigma^1}).$$

(17)
Original number of the length variables associated to the link is the number of the 4-simplices in the star of the link or the number of the triangles $T$ on the 2-sphere Fig. 1.

The final number of the variables with taking into account $\delta$-functions should be 1,

$$T - L\varepsilon_{13} - P \sum_{i=1}^{n_2} \varepsilon_{i2} - \sum_{i=1}^{n_1} \varepsilon_{i1} = 1.$$  \hspace{1cm} (18)

Comparing this to the Euler characteristic equation for the 2-sphere we return to the eq. (13).

This discussion admits extension to arbitrary spacetime dimensionality $d$, here without rigorous topological proof, however. The formula (13) is naturally generalised to

$$\delta^{(d)}_{\text{cont}} = \delta^{(d)}_{d-1} (\delta^{(d)}_{d-2})^{-1} \delta^{(d)}_{d-3} \ldots (\delta^{(d)}_{1})^{(-1)^d}$$

where

$$\delta^{(d)}_{k} = \prod_{\sigma^k} V_{\sigma^k}^{-k+1} \delta^{k+\frac{k-1}{2}} (\Delta_{d-1}(\sigma^k) S_{\sigma^k}).$$

(20)

This form ensures the required number 1 of the independent length variables per link upon taking into account the $\delta$-functions. Indeed, let $N^{(d)}_{k}(\sigma^1)$ denotes the number of $k$-simplices containing the link $\sigma^1$. Then the considered number of the independent length variables being read off from (19) is

$$N^{(d)}_{d}(\sigma^1) - N^{(d)}_{d-1}(\sigma^1) + \ldots + (-1)^d N^{(d)}_{2}(\sigma^1) + (-1)^{d+1}.$$  \hspace{1cm} (21)

At the same time $N^{(d)}_{k}(\sigma^1)$ can be viewed as the number of $(k-2)$-simplices on the surface of the star of the link $\sigma^1$ which possesses topology of the $(d-2)$-sphere, and (21) is just topology invariant, namely, Euler characteristic for the $(d-2)$-sphere minus 1 \cite{9}, i.e. just 1.

Thus, the assumption that in quantum theory Regge calculus is a kind of state of a more general quantised system where discontinuous metrics are allowed leads, in the natural physical assumptions of positivity and nonsingularity in the continuum limit, to the unique version of the Regge calculus quantum measure. Universality of the result is also seen from extendability to the arbitrary spacetime dimensionality and from the deep connection with the (local) topological properties of the spacetime.

The result obtained can be applied to the area Regge calculus. (The quantum measure in this theory can be identically rewritten by inserting some $\delta$-functions as the measure in the theory of discontinuous metrics.) A particular feature of the phase factor $\delta_{\text{cont}}$ is that it is invariant w.r.t. the overall rescaling linklengths. Therefore one can think that cutoff properties of the original area Regge calculus measure at small or large distances hold also for the usual (length-based) Regge calculus measure obtained with the help of the present construction, and finite expectation values of areas \cite{5} would mean also finite expectation values of the linklengths \cite{10}.

Another application is to the ordinary Regge calculus action,

$$\sum_{\sigma^2} V_{\sigma^2} \varphi_{\sigma^2}.$$  \hspace{1cm} (22)
Here angle defect $\varphi_{\sigma^2}$ is defined by the dihedral angles of separate 4-simplices $\sigma^4$ containing $\sigma^2$ and naturally extends to the case of discontinuous metric when separate 4-simplices become completely independent. Also the area $V_{\sigma^2}$ can be treated as area of any of the 4-simplices containing $\sigma^2$. Thus, (22) can be extended in many ways to the discontinuous metric so that it becomes a sum of independent terms referred to separate 4-simplices. It is natural to write down quantum measure for such the ultralocal system as the product of the measures for separate 4-simplices. Since there is one-to-one correspondence between the number of the linklengths of a 4-simplex $\sigma^4$ and the number of the components of the metric $g_{\lambda\mu} = const$ inside $\sigma^4$, there is also correspondence between possible 4-simplex measures and possible local measures at a point in the continuum theory. Thereby quantisation of the action (22) in the framework of our approach is formally defined, and the question is whether contours of integration in the path integral exist such that it would converge as in the quantised area Regge calculus model mentioned above.

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