The direct problem for the perturbed
Kadomtsev-Petviashvili II one line solitons

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Abstract

We provide rigorous analysis for the direct scattering theory of perturbed Kadomtsev-Petviashvili II one line solitons. Namely, for generic small initial data, the existence of the eigenfunction is proved by establishing uniform estimates of the Green function and the Cauchy integral equation for the eigenfunction is justified by analysing the spectral transform.

1 Introduction

The well-posedness problem of the Kadomtsev-Petviashvili II (KPII) equation
\[(4u_t + uu_{xx} - 6uu_x)_x + 3u_{yy} = 0,\]
\[u(x, y, 0) = u_N(x) + v_0(x, y),\]  \hspace{1cm} (1.1)
where \(u_N\) is an \(N\) line soliton, has been initiated by Bourgain \[9\] and solved by Molinet-Saut-Tzvetkov \[16\]. Their results show that the deviation of the KPII solution from the initial (perturbed) line soliton could grow exponentially unbounded during the evolution which is not consistent with the isospectral property of integrable systems. Excellent \(L^2\)-orbital stability and \(L^2\)-instability theories were established by Mizumachi \[15\] for perturbed KPII one line solitons. But the approach has difficulties to be generalized to multi line solitons.

The inverse scattering theory (IST) is a powerful method to identify and solve classes of integrable nonlinear PDEs and integrable dynamical systems. Many important nonlinear PDEs, such as KdV, NLS, sine-Gordon, etc, have been studied by this method. Integrability of the KPII equation has been known since the beginning of the 1970s. It can be integrated via the Lax pair
\[
\begin{align*}
(\partial_y - \partial_x^2 + u) \Psi(x, y, t, \lambda) &= 0, \\
(\partial_t - (-\partial_x^3 + \frac{3}{2}u \partial_x + \frac{3}{4}u_x + \frac{3}{4} \partial^{-1}_x u_y + (-i\lambda)^3)) \Psi(x, y, t, \lambda) &= 0.
\end{align*}
\]  \hspace{1cm} (1.2)
The IST of the spectral operator $\partial_y - \partial_x^2 + u$ was solved, schematically, solvability of the spectral equation is transformed to that of a Cauchy integral equation or $\bar{\partial}$-equation, by [22], [12], [13] in case the potential $u$ rapidly decays at spatial infinity. In particular, an $L^2_T$-stability theorem can be implied by [22]. As $u$ is a perturbed multi line soliton, two research groups [3]-[8], [19] have published substantial and important works on algebraic characterization and formal IST. In particular, the most remarkable characteristic, discontinuities for the Green function and eigenfunction had been discovered by Boiti, Pempenelli, Pogrebkov, and Prinari (cf [5], [6], [8]). However, with incomplete rigorous analysis for the Green function and spectral data, both groups formulated Cauchy integral equations which hold only for non generic initial data [5], [6], [8], [19]. So the IST for perturbed KPII line solitons is still an important open problem in this field [17], [18].

This report is aimed to a rigorous analysis for the direct scattering theory of perturbed KPII one line solitons. Precisely, a uniform estimate of the Green function is established by decomposing the kernel into Gaussian parts, oscillatory parts, rational functions, and regular parts and analysing them separately via different techniques. The same proof also characterizes the discontinuities at singularities. Furthermore, due to discontinuities of the eigenfunction, both discrete and continuous scattering data are not meromorphic which make the kernel of the Cauchy integral equation complicated to analyze. Inspired by [8], we provide a regularized eigenfunction to simplify the formula. Still the scattering data blow up at singularities which, along with discontinuous, highly oscillating, and not fully symmetric kernels, cause difficulties for deriving uniform estimates for the spectral transform and solving the inverse problem. In this paper, we only present non-uniform estimates of the spectral transform which is sufficient to imply a Cauchy integral equation provided the initial data satisfying

$$(1 + |x| + |y|)\partial_y^j \partial_x^k v_0 \in L^1 \cap L^\infty, \ 0 \leq j, k \leq 4, \ |v_0|_{L^1 \cap L^\infty} \ll 1.$$  

The contents of the paper are as follows. In Section 2 for generic small initial data $v_0$, we derive a uniform estimate of the Green function and apply the result to prove the existence of the eigenfunction to KPII equation. In Section 3 we extract the scattering data of eigenfunction $m$, introduce the regularized eigenfunction $m$, define the scattering operator $T$, derive spectral analysis, and justify a singular Cauchy integral equation.

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2 The forward problem I: a class of eigenfunctions

We start the investigation of the inverse scattering theory of the KPII perturbed one-line solitons. Consider the spectral equation

\[
(\partial_y - \partial_x^2 + u(x,y) ) \Psi(x, y, \lambda) = 0, \\
u(x, y) = u_0(x) + v_0(x,y), \\
u_0(x) = -2\kappa^2 \text{sech}^2 \kappa x, \kappa > 0, \ v_0(x,y) \in \mathbb{R},
\]

with the boundary condition

\[
\lim_{(x,y)\to \infty} (\Psi(x, y, \lambda) - \vartheta_-(x, \lambda) e^{(-i\lambda)x+(i\lambda)^2 y}) = 0,
\]

where \(\vartheta_-(x, \lambda)\) is one of eigenfunctions of the Schrödinger operator corresponding to the KdV one soliton \(u_0(x)\), i.e., 

\[
\begin{align*}
(-\partial_x^2 + u_0(x) - \lambda^2) f &= 0, \quad f = \phi_{\pm}(x, \lambda), \psi_{\pm}(x, \lambda), \\
\phi_{\pm}(x, \lambda) &= \varphi_{\pm}(x, \lambda) e^{\mp i\lambda x}, \quad \psi_{\pm}(x, \lambda) = \vartheta_{\pm}(x, \lambda) e^{\mp i\lambda x}, \\
\varphi_{\pm} &= 1 + \frac{2i\kappa}{\lambda \mp 2\kappa}, \\
\vartheta_{\pm} &= 1 + \frac{2i\kappa}{\lambda \pm 2\kappa}.
\end{align*}
\]

Introducing the normalizations

\[
\Psi(x, y, \lambda) = \Phi(x, y, \lambda) e^{-\lambda^2 y} = m(x,y,\lambda) e^{(-i\lambda)x+(i\lambda)^2 y}
\]

the spectral equation \((2.1)\) turns into

\[
\begin{align*}
L_\lambda \Phi &= (\partial_y - \partial_x^2 + u_0(x) - \lambda^2) \Phi = -v_0(x,y)\Phi, \\
L_\lambda m &= (\partial_y - \partial_x^2 + 2i\lambda \partial_x + u_0(x)) \ m = -v_0(x,y) m.
\end{align*}
\]

Denote \(G\) and \(\bar{G}\) as the Green functions

\[
L_\lambda G(x, x', y - y', \lambda) = \delta(x - x') \delta(y - y'), \\
L_\lambda \bar{G}(x, x', y - y', \lambda) = \delta(x - x') \delta(y - y').
\]

Formula for the Green functions are available in \([6, 8\mbox{ Eq.(3.1)}], [19\mbox{ Eq.(17)}]\) for instance. For convenience, we sketch the approach in \([19]\) to derive \(G(x, x', y, \lambda)\) in the following lemma.
Lemma 2.1. For \( y \neq 0, \lambda \notin \mathbb{R} \cup i\mathbb{R} \cup \{ \lambda \in \mathbb{C} | \lambda \pm i\kappa \in \mathbb{R} \}, \)

\[
G(x, x', y, \lambda) = G_c(x, x', y, \lambda) + G_d(x, x', y, \lambda),
\]

\[
G_d(x, x', y, \lambda) = -2\theta(-y)\theta(\kappa - |\lambda_1|)e^{(\lambda^2 + \kappa^2)y\pm\kappa(x-x')}g(x, x', \pm i\kappa),
\]

\[
G_c = G_{C^+} = G_{C^-},
\]

\[
G_{C^+}(x, x', y, \lambda) = \int_{\mathbb{R}} (\theta(y)\chi_- - \theta(-y)\chi_+) e^{(\lambda^2 - |\lambda|^2)y} \times \frac{\phi_+(x, \lambda+\kappa')\psi_+(x', \lambda+\kappa')}{2\pi a(\lambda+\kappa')} d\lambda',
\]

\[
G_{C^-}(x, x', y, \lambda) = \int_{\mathbb{R}} (\theta(y)\chi_- - \theta(-y)\chi_+) e^{(\lambda^2 - |\lambda|^2)y} \times \frac{\phi_-(x, \lambda+\kappa')\psi_-(x, \lambda+\kappa')}{2\pi a(\lambda+\kappa')} d\lambda'.
\]

Here \( \phi_\pm, \psi_\pm \) are defined by (2.8),

\[
g(x, x', \lambda) = \begin{cases} 
\varphi_+(x, \lambda)\theta_+(x', \lambda), & \lambda \in \mathbb{C}^+, \\
\varphi_-(x', \lambda)\theta_-(x, \lambda), & \lambda \in \mathbb{C}^-, 
\end{cases}
\]

and

\[
\chi_+ \text{ the characteristic function for } \{ \lambda' | \text{Re}(\lambda^2 - [\lambda + \lambda']^2) > 0 \},
\]

\[
\chi_- \text{ the characteristic function for } \{ \lambda' | \text{Re}(\lambda^2 - [\lambda + \lambda']^2) < 0 \},
\]

\[
a(\lambda) = \begin{cases} 
\alpha_+(\lambda), & \lambda \in \mathbb{C}^+, \\
\alpha_-(\lambda), & \lambda \in \mathbb{C}^-,
\end{cases}
\]

\[
\alpha_+(\lambda) = \frac{\lambda - i\kappa}{\lambda + i\kappa}, \quad a_-(\lambda) = \frac{\lambda + i\kappa}{\lambda - i\kappa},
\]

\[
\theta(y) = 1 \text{ if } y > 0 \text{ and vanishes elsewhere},
\]

\[
\tilde{G}(x, x', y, \lambda) = G(x, x', y, \lambda)e^{i\lambda(x-x')}, \quad \tilde{G} = \tilde{G}_c + \tilde{G}_d.
\]

Proof. First of all, note that if the operator \( P = P(\frac{\partial^n}{\partial x^n}) \) admits a complete set of eigenfunctions \( \{ \phi(x, \lambda) \} \), i.e.,

\[
P\phi(x, \lambda) = \lambda \phi(x, \lambda), \quad \lambda \in \mathbb{R},
\]

\[
\int_{\mathbb{R}} \phi(x, \lambda)\phi(x', \lambda)d\lambda = \delta(x-x'),
\]

then

\[
\left( \frac{\partial}{\partial y} + P \right) K(x, x', y-y') = \delta(x-x')\delta(y-y'),
\]

\[
K(x, x', y) = \int_{\mathbb{R}} [\theta(y)\theta(\lambda) - \theta(-y)\theta(-\lambda)] e^{-y\lambda} \phi(x, \lambda)\phi(x', \lambda)d\lambda.
\]

Secondly, from (2.3),

\[
(-\partial^2_x + u_0 - \lambda^2)\phi_\pm(x, \lambda + \lambda') = [(\lambda + \lambda')^2 - \lambda^2]\phi_\pm(x, \lambda + \lambda'),
\]

\[
(-\partial^2_x + u_0 - \lambda^2)\psi_\pm(x, \lambda + \lambda') = [(\lambda + \lambda')^2 - \lambda^2]\psi_\pm(x, \lambda + \lambda'),
\]

\[
(-\partial^2_x + u_0 - \lambda^2)\phi_\pm(x, \pm i\kappa) = -(\lambda^2 + \kappa^2)\phi_\pm(x, \pm i\kappa),
\]

\[
(-\partial^2_x + u_0 - \lambda^2)\psi_\pm(x, \pm i\kappa) = -(\lambda^2 + \kappa^2)\psi_\pm(x, \pm i\kappa),
\]

(2.10)
and by the residue theorem,
\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \left[ \frac{\phi_+(x, \lambda, \lambda') \psi_+(x', \lambda + \lambda')}{a_+(\lambda + \lambda')} - e^{i(\lambda + \lambda')(x' - x)} \right] d\lambda' \\
= \Res_{\eta \in \mathbb{C}, \eta > \lambda_1 > 0} \left[ \frac{\phi_+(x, \eta) \psi_+(x', \eta)}{a_+(\eta)} \right],
\]
\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \left[ \frac{\phi_-(x', \lambda + \lambda') \psi_-(x, \lambda + \lambda')}{a_-(\lambda + \lambda')} - e^{i(\lambda + \lambda')(x' - x)} \right] d\lambda' \\
= -\Res_{\eta \in \mathbb{C}, \eta < \lambda_1 < 0} \left[ \frac{\phi_-(x', \eta) \psi_-(x, \eta)}{a_-(\eta)} \right].
\]
Plugging
\[
\frac{1}{2\pi} \int_{\lambda + \lambda' \in \mathbb{R}} e^{i(\lambda + \lambda')(x' - x)} d\lambda' = \delta(x - x')
\]
into (2.12), we prove the following orthogonality and completeness identities
\[
\delta(x - x') = \begin{cases} 
\frac{1}{2\pi} \int_{\mathbb{R}} \phi_+(x, \lambda, \lambda') \psi_+(x', \lambda + \lambda') a_+(\lambda + \lambda') d\lambda' + 2\theta(\kappa - \lambda_1) \phi_+(x, i\kappa) \psi_+(x', i\kappa), \\
\frac{1}{2\pi} \int_{\mathbb{R}} \phi_-(x', \lambda + \lambda') \psi_-(x, \lambda + \lambda') a_-(\lambda + \lambda') d\lambda' + 2\theta(\kappa + \lambda_1) \phi_-(x', -i\kappa) \psi_-(x, -i\kappa)
\end{cases}
\]  
(2.13)
for \( \lambda \in \mathbb{C}^\pm \). Therefore the formula of \( G(x, x', y, \lambda) \) follows from (2.10), (2.11), and (2.13).

The property \( G_{\mathbb{C}^+} = G_{\mathbb{C}^-} \) follows from
\[
\frac{\partial}{\partial \eta} \left[ \psi_+(x, \lambda) \right] = \psi_-(x, \lambda), \quad \frac{\partial}{\partial \eta} \left[ \psi_-(x, \lambda) \right] = \psi_+(x, \lambda),
\]
and
\[
\text{If } \lambda_R > 0, \text{ then } \chi_+(\lambda') = \begin{cases} 
1, & -2\lambda_R < \lambda' < 0, \\
0, & \lambda' \geq 0, \lambda' \leq -2\lambda_R;
\end{cases}
\]
\[
\text{If } \lambda_R < 0, \text{ then } \chi_+(\lambda') = \begin{cases} 
1, & 0 < \lambda' < -2\lambda_R, \\
0, & \lambda' \leq 0, \lambda' \geq -2\lambda_R;
\end{cases}
\]
\[
\text{If } \lambda_R > 0, \text{ then } \chi_-(\lambda') = \begin{cases} 
1, & \lambda' < 0, \lambda' \geq -2\lambda_R, \\
0, & \lambda' > 0, \lambda' \leq -2\lambda_R;
\end{cases}
\]
\[
\text{If } \lambda_R < 0, \text{ then } \chi_-(\lambda') = \begin{cases} 
1, & \lambda' > 0, \lambda' \leq -2\lambda_R, \\
0, & 0 \leq \lambda' \leq -2\lambda_R,
\end{cases}
\]
(2.15)
which are independent of \( \lambda \in \mathbb{C}^+ \) or \( \mathbb{C}^- \).

The basic existence theorem for this section is

**Theorem 1.** If \( \partial_y^j \partial_x^k v_0 \in L^1 \cap L^\infty, \) \( 0 \leq j, k \leq 2, \) \( \|v\|_{L^1 \cap L^\infty} \ll 1, \) then for fixed \( \lambda \neq \pm i\kappa, \) there is a unique solution \( \Psi(x, y, \lambda) = m(x, y, \lambda) e^{(-i\lambda)x + (-i\lambda)^2 y} \) to the problem (2.11), (2.22) and \( \partial_y^j \partial_x^k m \in L^\infty. \)
The proof of the theorem follows from the following uniform estimate

**Lemma 2.2.** There exists a uniform constant $C$ such that the Green function $G$, defined by (2.6) - (2.9), satisfies

\[
|\tilde{G}(x, x', y, \lambda)| \leq C \left(1 + \frac{1}{\sqrt{|y|}} \right) \tag{2.16}
\]

for $\forall x, x', y \in \mathbb{R}$, $y \neq 0$, $\lambda \notin \mathbb{R} \cup i\mathbb{R} \cup \{\lambda \in \mathbb{C}|\lambda \pm i\kappa \in \mathbb{R}\}$.

**Proof.** Step 1: From Lemma 2.1 and (2.15),

\[
|\tilde{G}_d| = \left| -2\theta(-y)|\tilde{\theta}(\kappa - |\lambda I|)e^{(\lambda^2 + \kappa^2)y \pm i\kappa(x-x') + i\lambda(x-x')}g(x, x', \pm i\kappa) \right|
= |2\theta(-y)|\tilde{\theta}(\kappa - |\lambda I|)g(x, x', \pm i\kappa) < C, \tag{2.17}
\]

and

\[
\tilde{G}_c(x, x', y, \lambda) = \theta(\lambda R)\tilde{\theta}(y) \left( \int_{-2|\lambda R|}^{-2|\lambda R|} e^{-\left(\lambda^2 + 2\lambda\lambda'\right)y \pm i\lambda'(x-x')}g(x, x', \lambda + \lambda')d\lambda + \int_{-|\lambda R|}^{0} e^{-\left(\lambda^2 + 2\lambda\lambda'\right)y \pm i\lambda'(x-x')}g(x, x', \lambda + \lambda')d\lambda \right)
\]

\[
+ \theta(-\lambda R)\tilde{\theta}(y) \left( \int_{0}^{|\lambda R|} e^{-\left(\lambda^2 + 2\lambda\lambda'\right)y \pm i\lambda'(x-x')}g(x, x', \lambda + \lambda')d\lambda \right)
\]

\[
- \theta(\lambda R)\tilde{\theta}(y) \left( \int_{-|\lambda R|}^{0} e^{-\left(\lambda^2 + 2\lambda\lambda'\right)y \pm i\lambda'(x-x')}g(x, x', \lambda + \lambda')d\lambda \right)
\]

\[
- \theta(-\lambda R)\tilde{\theta}(y) \left( \int_{-|\lambda R|}^{0} e^{-\left(\lambda^2 + 2\lambda\lambda'\right)y \pm i\lambda'(x-x')}g(x, x', \lambda + \lambda')d\lambda \right) \tag{2.18}
\]

By a change of variables $\eta = \lambda R + \lambda'$,

\[
\tilde{G}_c(x, x', y, \lambda)
= \theta(y)e^{-i\lambda R(x'-x-2\lambda y)}
\times \left( \int_{-|\lambda R|}^{0} e^{i\lambda R \eta^2(y \pm i\eta(x'-x-2\lambda y))}g(x, x', \eta + i\lambda I)\eta d\eta \right)
\times \left( \int_{-|\lambda R|}^{0} e^{-i\lambda R(x'-x-2\lambda y)}g(x, x', \eta + i\lambda I)\eta d\eta \right) \tag{2.19}
\]

where

\[
\tilde{\tilde{g}}(x, x', y, \lambda, \eta) = e^{i\lambda R \eta^2(y \pm i\eta(x'-x-2\lambda y))}g(x, x', \eta + i\lambda I),
\]

\[
\Xi(y, \lambda, \eta) = \theta(y)\chi_{\{\lambda R|\lambda R|\} \cup \{-\lambda R, \lambda R\}} - \theta(-y)\chi_{\{-\lambda R, \lambda R\}}, \tag{2.20}
\]

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and $\chi_A$ denotes the characteristic function of the set $A$. Since singularities of integrands are $\lambda_I = \pm \kappa$, $\lambda_R + \lambda' = 0$, and integrands consisting of both gaussian parts $e^{(\lambda_R^2 - \eta^2)y}$ and oscillatory parts $e^{\eta(x'-x-2\lambda_I y)}$, we decompose

\[
\tilde{G}_c(x, x', y, \lambda) = e^{-\lambda_R(x'-x-2\lambda_I y)}(I + I + III + IV + V), \quad \text{for } |\lambda_R| \leq 1,
\]

\[
\tilde{G}_c(x, x', y, \lambda) = e^{-\lambda_R(x'-x-2\lambda_I y)}(I + I + III + IV + V), \quad \text{for } |\lambda_R| \geq 1,
\]

\[(2.21)\]

with

\[
I = \pm \frac{i}{\pi} \int_{-1}^{1} \Xi(y, \lambda, \eta) \frac{e^{i\eta(x'-x-2\lambda_I y)} - 1}{\eta + i(\lambda_I + \kappa)} g(x, x', \eta + i\lambda_I) \, d\eta,
\]

\[
II = \pm \frac{i}{\pi} \int_{-1}^{1} \Xi(y, \lambda, \eta) \frac{e^{i\eta(\lambda_R^2 - \eta^2)y} - 1}{\eta + i(\lambda_I + \kappa)} g(x, x', \eta + i\lambda_I) \, d\eta,
\]

\[
III = \pm \frac{i}{\pi} \int_{-1}^{1} \Xi(y, \lambda, \eta) \frac{g(x, x', \eta + i\lambda_I) - g(x, x', \pm i\kappa)}{\eta + i(\lambda_I + \kappa)} \, d\eta,
\]

\[
IV = \pm \frac{i}{\pi} \int_{-1}^{1} \Xi(y, \lambda, \eta) \frac{g(x, x', \pm i\kappa)}{\eta + i(\lambda_I + \kappa)} \, d\eta,
\]

\[
V = (\int_{-\infty}^{-1} + \int_{1}^{\infty}) \Xi(y, \lambda, \eta) \frac{\tilde{g}(x, x', y, \lambda, \eta)}{2\pi(\eta + i\lambda_I)} \, d\eta + \int_{-1}^{1} \Xi(y, \lambda, \eta) \frac{\tilde{g}(x, x', y, \lambda, \eta)}{2\pi} \, d\eta.
\]

\[(2.22)\]

and

\[
I^\dagger = \theta(y) \left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \right) \frac{e^{i\eta(x'-x-2\lambda_I y)} - 1}{2\pi(\eta + i\lambda_I)} g(x, x', \eta + i\lambda_I) \, d\eta,
\]

\[
II^\dagger = \mp \theta(-y) \int_{-1}^{1} \frac{e^{i\eta(x'-x-2\lambda_I y)} - 1}{2\pi(\eta + i\lambda_I)} g(x, x', \eta + i\lambda_I) \, d\eta,
\]

\[
III^\dagger = -\theta(-y) \left\{ \left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \right) \frac{e^{i\eta(x'-x-2\lambda_I y)} - 1}{2\pi(\eta + i\lambda_I)} g(x, x', \eta + i\lambda_I) \, d\eta \right. \\
+ \int_{-1}^{1} \frac{e^{i\eta(x'-x-2\lambda_I y)} - 1}{2\pi(\eta + i\lambda_I)} g(x, x', \eta + i\lambda_I) \, d\eta \right\}.
\]

\[(2.23)\]

\[
|III| \leq C,
\]

\[(2.24)\]

since $\frac{g(x, x', \eta + i\lambda_I) - g(x, x', \pm i\kappa)}{\eta + i(\lambda_I + \kappa)}$ is a uniformly bounded function and $|\eta| \leq 1$. Moreover, note $\left| \frac{1}{2\pi(\eta + i\lambda_I)} \right| < C$ for $|\eta| \geq 1$. Therefore, via (2.13), for $\forall \lambda$,

\[
|V(x, x', y, \lambda)| \leq C |\theta(\lambda) \theta(y) (\int_{-\infty}^{-1} + \int_{1}^{\infty}) e^{-(\lambda'^2 + 2\lambda R \lambda') y} \, d\lambda' + \theta(-\lambda) \theta(y) (\int_{-\infty}^{\infty} + \int_{1}^{\infty}) e^{-(\lambda'^2 + 2\lambda R \lambda') y} \, d\lambda' + \theta(\lambda) \theta(-y) (\int_{-\infty}^{\infty} - \int_{-1}^{1}) e^{-(\lambda'^2 + 2\lambda R \lambda') y} \, d\lambda' + \theta(-\lambda) \theta(-y) (\int_{-\infty}^{\infty} + \int_{1}^{\infty}) e^{-(\lambda'^2 + 2\lambda R \lambda') y} \, d\lambda' | \\
\leq C |\theta(y) \int_{-\infty}^{\infty} e^{-y^2} \, ds + \theta(-y) e^{y|\lambda R|} \int_{-\infty}^{\infty} e^{-y^2} \, ds \| \\
\leq \frac{C}{\sqrt{|y|}}.
\]

\[(2.25)\]
Here we have used
\[
\theta(\lambda_R)\theta(y) \int_0^\infty e^{-(\lambda^2+2\lambda_R\lambda')y}d\lambda' \leq \theta(y) \int_0^\infty e^{-\lambda^2y}d\lambda' \leq \frac{C}{\sqrt{|y|}},
\]
\[
\theta(\lambda_R)\theta(y) \int_{-2|\lambda_R|}^0 e^{-(\lambda^2+2\lambda_R\lambda')y}d\lambda' = \theta(-y)e^{\lambda_R^2y} \int_{-|\lambda_R|}^{|\lambda_R|} e^{-\eta^2y}d\eta \leq \frac{C}{\sqrt{|y|}},
\]
\[
\theta(\lambda_R)\theta(y) \int_{-\infty}^{-2|\lambda_R|} e^{-(\lambda^2+2\lambda_R\lambda')y}d\lambda' = \theta(\lambda_R)\theta(y) \int_0^\infty e^{-(\xi^2+2\lambda_R\xi)y}d\xi \leq \frac{C}{\sqrt{|y|}}
\]
(via \(\eta = \lambda + \lambda_R, -\xi = \lambda' + 2\lambda_R\)), and similar arguments for terms with \(\theta(-\lambda_R)\) factors.

**Step 3 (Estimates for IV)**: Let us first claim: for \(\lambda = \lambda_R + i\lambda_I = \pm ik + se^{i\alpha}, 0 < s < 1, \pm \frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}\), we have
\[
\lim_{s \to 0} \int_{-|\lambda_R|}^{1} \frac{d\eta}{\eta + i(\lambda_I + \kappa)} = (\mp i) \left( 2\pi (1 - \theta(\kappa - |\lambda_I|)) - 2\cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|} \right),
\]
where
\[
\cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|} = \begin{cases}
\frac{\pi}{2} + \alpha, & -\frac{\pi}{2} < \alpha \leq \frac{\pi}{2}, 0 < |\lambda - ik| \leq 1, \\
\frac{\pi}{2} - \alpha, & 0 \leq \alpha < \frac{3\pi}{2}, 0 < |\lambda - ik| \leq 1, \\
\frac{\pi}{2} + \alpha, & -\frac{\pi}{2} \leq \alpha < \frac{\pi}{2}, 0 < |\lambda + ik| \leq 1, \\
\frac{\pi}{2} - \alpha, & -\frac{\pi}{2} \leq \alpha < \frac{3\pi}{2}, 0 < |\lambda + ik| \leq 1.
\end{cases}
\]

The claim is carried out by using the logarithmic functions. Precisely,
\[
\lim_{s \to 0} \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda_I + \kappa)} = \begin{cases}
+i\pi(2\theta(\kappa - |\lambda_I|) - 1), \lambda \in \mathbb{C}^+ \\
-i\pi(2\theta(\kappa - |\lambda_I|) - 1), \lambda \in \mathbb{C}^-,
\end{cases}
\]
\[
\int_{-|\lambda_R|}^{|\lambda_R|} \frac{d\eta}{\eta + i(\lambda_I + \kappa)} = \begin{cases}
\log \frac{|\lambda_R| + i(\lambda_I + \kappa)}{|\lambda_R| + i(\lambda_I - \kappa)}, \kappa < \lambda_I, \\
-\log \frac{-|\lambda_R| + i(\lambda_I - \kappa)}{-|\lambda_R| + i(\lambda_I + \kappa)}, -\lambda_I < \kappa < 0, \\
-2i \cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|}, \kappa < \lambda_I, \\
+2i \cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|}, 0 < \lambda_I < \kappa, \\
-2i \cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|}, -\kappa < \lambda_I < 0, \\
+2i \cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|}, \lambda_I < -\kappa
\end{cases}
\]
\[
= \begin{cases}
-i[2\pi(1 - \theta(\kappa - |\lambda_I|)) - 2\cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|}], \lambda \in \mathbb{C}^+, \\
+i[2\pi(1 - \theta(\kappa - |\lambda_I|)) - 2\cot^{-1} \frac{\kappa - |\lambda_I|}{|\lambda_R|}], \lambda \in \mathbb{C}^-.
\end{cases}
\]
Here we have used $0 < \cot^{-1} x < \pi$, $\cot^{-1}(-x) = \pi - \cot^{-1} x$, and the principal integration.

Plugging (2.26) into IV, one has

$$\lim_{s \to 0} IV = \pm \frac{i}{\pi} \Theta(y) \left( \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} + \frac{1}{\eta + i(\lambda I + \kappa)} \right)$$

$$= g(x, x', \pm i\kappa) \left[ \Theta(y) \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} - \Theta(y) \right] + \frac{2}{\pi} \cot^{-1} \frac{\kappa}{|\lambda I|},$$

and

$$|IV| \leq C.$$  \hspace{1cm} (2.29)

**Step 4 (Estimates for II):** By (2.24), the $L^\infty$-estimate of II reduces to that for

$$\tilde{II} = \tilde{I}_1 + \tilde{I}_2,$$

$$\tilde{I}_1 = \pm \frac{i}{\pi} \Theta(y) \left( \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} + \frac{1}{\eta + i(\lambda I + \kappa)} \right)$$

$$\tilde{I}_2 = \mp \frac{i}{\pi} \Theta(-y) \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)}.$$

In case of

$$(4a) \quad |\lambda I| \sqrt{y} \geq 1,$$

by the change of variables $\xi = \frac{\eta}{|\lambda I|^2},$ a Gaussian integration, and using logarithmic functions,

$$|\tilde{I}_1|$$

$$= \left| \pm \frac{i}{\pi} \Theta(y) \left( \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} + \frac{1}{\eta + i(\lambda I + \kappa)} \right) \right|$$

$$\leq C \Theta(y) \left( \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} + \frac{1}{\eta + i(\lambda I + \kappa)} \right) e^{-|y|^2} + C$$

$$\leq C \int_{-1}^{1} e^{-|y|^2} + C = C e^{1/2} + C \leq C.$$  \hspace{1cm} (2.30)

Besides, by the change of variables $\omega = \eta \sqrt{|y|}$ and (2.31),

$$|\tilde{I}_2|$$

$$= \left| \mp \frac{i}{\pi} \Theta(-y) \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} \right|$$

$$= \left| \mp \frac{i}{\pi} \Theta(-y) \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} \right|$$

$$\pm \frac{i}{\pi} \Theta(-y) \int_{|\lambda I| \sqrt{y} \sqrt{|y|}}^{1} \frac{d\omega}{\omega + i(\lambda I + \kappa)}$$

$$\pm \frac{i}{\pi} \Theta(-y) \int_{-1}^{1} \frac{d\eta}{\eta + i(\lambda I + \kappa)} \right|$$

$$\leq 2C.$$  \hspace{1cm} (2.32)
Therefore, applying the mean value theorem, (2.31), the logarithmic functions, a Gaussian integration, and (2.30),

\[
|I_2| \
\leq C\theta(-y) \int_0^1 \left| \frac{e^{y^2/\omega - \omega^2} - e^{y^2/\omega}}{(\omega + i(\lambda_I + \kappa))\sqrt{y}} \right| d\omega + C \int_1^1 \frac{1}{\omega + i(\lambda_I + \kappa)} \sqrt{y} d\omega 
+ C\theta(-y)(\int_{-\lambda_R/\sqrt{y}}^1 + \int_1^{\lambda_R/\sqrt{y}}) e^{\lambda_R y - \omega^2} d\omega + C 
\leq C.
\]  

(2.34)

Instead of (2.31), we now consider

\[
(4b) \ |\lambda_R| \sqrt{|y|} \leq 1.
\]  

(2.35)

Therefore, by the change of variables \( \xi = \frac{y}{\lambda_R} \) and the mean value theorem,

\[
|I_2| = \left| \int_{-1}^{1} \frac{e^{y^2(1-\xi^2)/\eta} - 1}{\xi + \frac{\lambda}{\lambda_R}} \right| d\xi 
\leq C\theta(-y) \int_{-1}^{1} \frac{1}{\xi - \lambda_R/\sqrt{y}} d\xi \leq C.
\]  

(2.36)

On the other hand, by the change of variables \( \omega = \eta \sqrt{|y|} \) and (2.35),

\[
|I_1| = \left| \int_{-1}^{\lambda_R/\sqrt{y}} + \int_{\lambda_R/\sqrt{y}}^{1} \frac{e^{y^2/\omega - \omega^2} - 1}{\omega + i(\lambda_I + \kappa)\sqrt{y}} d\omega 
\pm \frac{\lambda}{\pi} \theta(y)(\int_{-1}^{\lambda_R/\sqrt{y}} + \int_1^{\lambda_R/\sqrt{y}}) \frac{\sqrt{y}}{\theta} \right| d\eta.
\]  

(2.37)

By the mean value theorem, a Gaussian integral, (2.35), and the logarithmic functions,

\[
|I_1| \leq C\theta(y) \int_{\lambda_R/\sqrt{y}}^1 \frac{e^{y^2/\omega - \omega^2}}{\omega - 1} d\omega + C\theta(y) e^{y^2/\sqrt{y}} \sqrt{e^{-1} - e^{-y}} + C \leq C.
\]  

(2.38)

Combining cases (4a) and (4b), we obtain

\[
|I_2| \leq C.
\]  

(2.39)

Step 5 (Estimates for I) : By the estimate of III and II, the estimate of I reduces to considering \( \tilde{I} = I_1 + I_2 \), with

\[
\tilde{I}_1 = \int_{-1}^{\lambda_R/\sqrt{y}} \frac{e^{y^2(\sqrt{y} - s) - 2\lambda_I y}}{\eta + (\lambda_I + \kappa)} d\eta 
- \int_{1}^{\lambda_R/\sqrt{y}} \frac{e^{y^2(\sqrt{y} - s) - 2\lambda_I y}}{\eta + (\lambda_I + \kappa)} d\eta,
\]  

(2.40)
Observe, in either case

\( (5a) \ (\lambda_I \mp \kappa)(x' - x - 2\lambda_I y) > 0, \)

\( (5b) \ (\lambda_I \mp \kappa)(x' - x - 2\lambda_I y) \leq 0, \) and \( |\lambda_R| - |\lambda_I \mp \kappa| \geq \frac{1}{2}|\lambda_R|, \)

(2.41)

by the residue theorem [10 §6, Chapter III],

\[
|\int_{\gamma'}^{\lambda_R} \frac{e^{i\eta(x' - x - 2\lambda_I y)}}{\eta + (\lambda_I \mp \kappa)} d\eta| \\
\leq |\int_{\gamma'}^{\lambda_R} \theta(x' - x - 2\lambda_I y) \sin \beta) \frac{\lambda_R}{\eta + (\lambda_I \mp \kappa)} (x' - x - 2\lambda_I y) - 1| d\eta | \leq C \int_{\gamma'}^{\lambda_R} \theta(x' - x - 2\lambda_I y) \sin \beta) \\
\times |e^{i\lambda R}(x' - x - 2\lambda_I y) | = |\lambda_R| (x' - x - 2\lambda_I y) \sin \beta - 1| d\beta + C \\
\leq C. \quad (2.42)
\]

Instead of (2.41), if

\( (5c) \ (\lambda_I \mp \kappa)(x' - x - 2\lambda_I y) \leq 0, \) and \( |\lambda_R| - |\lambda_I \mp \kappa| \leq \frac{1}{2}|\lambda_R| \) (2.43)

holds, we will show the integral \( \tilde{I}_2 \) is basically a Hilbert transform. Precisely, by deforming the contours [10 §6, Chapter III],

\[
|\int_{\Gamma_\pm}^{\lambda_R} \frac{e^{i\eta(x' - x - 2\lambda_I y)}}{\eta + (\lambda_I \mp \kappa)} d\eta| \\
\leq |\int_{\Gamma_-}^{\lambda_R} \frac{e^{i\eta(x' - x - 2\lambda_I y)}}{\eta + (\lambda_I \mp \kappa)} d\eta| + |\int_{\Gamma_+}^{\lambda_R} \frac{e^{i\eta(x' - x - 2\lambda_I y)}}{\eta + (\lambda_I \mp \kappa)} d\eta| + 2\pi |e^{i\lambda R}(x' - x - 2\lambda_I y) | = |\lambda_R| (x' - x - 2\lambda_I y) \sin \beta - 1| d\beta + C \\
\leq C. \quad (2.44)
\]

where

\[
\Gamma_- = \{ \eta = -|\lambda_R| - i(\lambda_I \mp \kappa) t : 0 \leq t \leq 1 \}, \\
\Gamma_+ = \{ \eta = +|\lambda_R| - i(\lambda_I \mp \kappa) t : 0 \leq t \leq 1 \}, \\
\Gamma = \{ \eta = t - i(\lambda_I \mp \kappa) : -|\lambda_R| \leq t \leq |\lambda_R| \}. \\
\]

Due to (2.43) and (2.45), one has

\[
|\int_{\Gamma_\pm}^{\lambda_R} \frac{e^{i\eta(x' - x - 2\lambda_I y)}}{\eta + (\lambda_I \mp \kappa)} d\eta| \\
= |\int_0^1 \frac{e^{i\eta(x' - x - 2\lambda_I y)} + (\lambda_I \mp \kappa)(x' - x - 2\lambda_I y) t}{|\lambda_R| + (\lambda_I \mp \kappa)(1 - t)} - 1| (\lambda_I \mp \kappa) dt | \\
\leq \int_0^1 |e^{i\eta(x' - x - 2\lambda_I y)} + (\lambda_I \mp \kappa)(x' - x - 2\lambda_I y) t - 1| dt \leq C. \quad (2.46)
\]

On the other hand,

\[
|\int_{\Gamma_\pm}^{\lambda_R} \frac{e^{i\eta(x' - x - 2\lambda_I y)}}{\eta + (\lambda_I \mp \kappa)} d\eta| \\
= |\int_{-\lambda_R}^{\lambda_R} e^{i\eta(x' - x - 2\lambda_I y)} + t(x' - x - 2\lambda_I y) d\eta| \\
\leq |\int_{-\lambda_R}^{\lambda_R} e^{i\eta(x' - x - 2\lambda_I y)} d\eta| + C \quad (2.47)
\]

\[1\]
by using symmetries and the residue theorem [10, §6, Chapter III]. Therefore, combining cases (5a), (5b), and (5c), \(|\tilde{I}_2| \leq C\). The same method can be adapted to proving \(|\int_{-1}^{1} \frac{\cos(x' - x + \lambda y)}{y + i(\lambda_I + \kappa)} d\eta| < C\) which yields

\[
|\tilde{I}(x, x', y, \lambda)| \leq C
\]

is justified. Combining with results from previous steps, we obtain estimate (2.16) in case of \(|\lambda_R| \leq 1\).

Step 6 (Estimates for \(I^{\dagger}, II^{\dagger}, III^{\dagger}\)) : For \(II^{\dagger}\), we adapt the argument for (4a) in \(\tilde{II}\) in Step 4 and for \(\tilde{I}\) in Step 5; for \(I^{\dagger}, III^{\dagger}\), we apply Gaussian type estimates (2.25). □

Green functions \(G\) and \(\tilde{G}\), defined by (2.6) - (2.9), can be extended to \(y \neq 0\) and \(\lambda \neq \pm i\) by the following lemma.

**Lemma 2.3.** For \(y \neq 0\), the Green function \(G\), defined by (2.6) - (2.9),

\[
G(x, x', y, \lambda_R + i0^+) = G(x, x', y, \lambda_R + i0^-), \quad \forall \lambda_R;
\]

\[
G(x, x', y, 0^+ + i\lambda_I) = G(x, x', y, 0^- + i\lambda_I), \quad \forall \lambda_I \neq \pm \kappa;
\]

\[
G(x, x', y, \lambda_R + i(\pm \kappa + 0^+)) = G(x, x', y, \lambda_R + i(\pm \kappa + 0^-)), \quad \forall \lambda_R \neq 0.
\]

**Proof.** Follow from the proof of previous lemma. □

**Lemma 2.4.** Suppose \(f \in L^1 \cap L^\infty\). For \(\forall \lambda \neq \pm i\kappa\), the Green function \(\tilde{G}\), defined by (2.6) - (2.9), satisfies

\[
\tilde{G} * f \to 0 \text{ uniformly as } |x|, |y| \to \infty.
\]

Here the * operator is defined by

\[
\tilde{G} * f(x, y, \lambda) = \iint \tilde{G}(x, x', y - y', \lambda) f(x', y') dx' dy'.
\]

**Proof.** Since \(\lambda \neq \pm i\kappa\) fixed, both \(x\)-asymptotics and \(y\)-asymptotics can be obtained by Lemma 2.2 \(f \in L^1 \cap L^\infty\), and the dominated convergence theorem. □

**Proof of Theorem 7.** From Lemma 2.2 and \(v_0 \in L^1 \cap L^\infty\), for \(\lambda \neq \pm i\kappa\), if \(f(x, y) \in L^\infty(\mathbb{R}^2)\), then the map \(f \mapsto \tilde{G} * v_0 f\) is bounded from \(L^\infty(\mathbb{R}^2)\) to \(L^\infty(\mathbb{R}^2)\) and has a norm bounded by \(C|v_0|_{L^1 \cap L^\infty}\). Consequently, if \(|v_0|_{L^1 \cap L^\infty} \ll 1\), then for \(\lambda \neq \pm i\kappa\) the integral equation

\[
m = \partial_\lambda - \tilde{G} * v_0 m
\]

is justified. Combining with results from previous steps, we obtain estimate (2.16) in case of \(|\lambda_R| \leq 1\).
is uniquely solved for \( m \in L^\infty(\mathbb{R}^2) \). Moreover, \( \partial_y^j \partial_x^k m(x, y, \lambda) \in L^\infty(\mathbb{R}^2), \) \( 0 \leq j, k \leq 2 \), can be proved via the formula

\[
\begin{align*}
\partial_x m(x, y, \lambda) &= (1 - \tilde{G} * v_0)^{-1} \left[ \partial_x \tilde{G} \right] * v_0 (1 - \tilde{G} * v_0)^{-1} \partial_-
\end{align*}
\]

integration by parts, \( \partial_y^j \partial_x^k v_0 \in L^1 \cap L^\infty, 0 \leq j, k \leq 2 \), and an induction argument (Here we make remarks on \( \partial_x \tilde{G} \). To take \( x \)-derivatives of exponential terms, we need to use the antisymmetries in \( x, x' \) and apply integration by parts). From Lemma [2.4] \( \Psi(x, y, \lambda) = m(x, y, \lambda) e^{-(i\lambda)x + (-i\lambda)^2 y} \) is a solution to the problem (2.1), (2.2).

\[\square\]

3 The forward problem II: the forward scattering transform

The scattering data can be defined by the non-holomorphic part of the eigenfunctions, i.e., \( \tilde{\mathcal{D}} m = \partial_\lambda m \) [2], [11], [1]. Classically, \( \tilde{\mathcal{D}} m \) can often be computed in terms of \( m \) by noting that both \( m \) and \( \tilde{\mathcal{D}} m \) satisfy the same equation. Thus \( m \) can be reconstructed from a knowledge of this relationship by solving a \( \tilde{\mathcal{D}} \)-problem, namely, a Cauchy integral equation [2], [21], [22]. The main goals of this section are to compute \( \tilde{\mathcal{D}} m \), to define the scattering transform and to characterize its algebraic and analytical constraints, and to formulate a Cauchy integral equation.

3.1 Discrete scattering data of \( m(x, y, \lambda) \)

Major distinction of non-localized KPII equation from other integrable systems is the occurrence of non-meromorphic discrete scattering data. In the following lemma, we will prove the discontinuities at \( \lambda = \pm i\kappa \) of the Green function \( \tilde{G} \), defined by (2.6) - (2.9), which yield the non-meromorphic properties of the discrete scattering data.

**Definition 3.1.** For \( z \in \mathbb{Z} = \{ \pm i\kappa, \iota \}, \iota = 2i\kappa \), define

\[
\begin{align*}
D_z &= \{ \lambda \in \mathbb{C} : |\lambda - z| < 1 \}, \quad D_\times^z = \{ \lambda \in \mathbb{C} : 0 < |\lambda - z| < 1 \}; \\
D_{z,a} &= \{ \lambda \in \mathbb{C} : |\lambda - z| < a \}, \quad D^\times_{z,a} = \{ \lambda \in \mathbb{C} : 0 < |\lambda - z| < a \},
\end{align*}
\]
and characteristic functions

\[ E_z(\lambda) \equiv 1 \text{ on } D_z, \quad E_z(\lambda) \equiv 0 \text{ elsewhere}; \]
\[ E_{z,a}(\lambda) \equiv 1 \text{ on } D_{z,a}, \quad E_{z,a}(\lambda) \equiv 0 \text{ elsewhere}. \]  \hfill (3.2)

Moreover, define the polar coordinate for \( D_{z,a}^\infty \) to be \( \{(s,\alpha)|\lambda = z + se^{i\alpha}, \quad 0 < s < a, \quad -\frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}\}. \)

**Lemma 3.1.** For \( y \neq 0, \quad \lambda = \lambda_R + i\lambda_I \in D_{z,a}^\infty, \)

\[ \overline{G}(x, x', y, \lambda) = \mathcal{G}_\pm(x, x', y) + \frac{2}{i}g(x, x', \pm i\kappa) \cot^{-1} \frac{\pi - |\lambda I|}{|\lambda R|} + \omega_\pm(x, x', y, \lambda), \]  \hfill (3.3)

with

\[ \mathcal{G}_\pm(x, x', y) = \int_{|\eta| \geq 1} \frac{\theta(y) e^{-\eta y^2 + iy(x' - x + 2\pi y) g(x, x', \eta \pm i\kappa)}}{2\pi a(\eta \pm i\kappa)} d\eta \]
\[ + \int_{|\eta| \leq 1} \frac{\theta(y) e^{-\eta y^2 + iy(x' - x + 2\pi y) g(x, x', \eta \pm i\kappa) - g(x, x', \pm i\kappa)}}{2\pi a(\eta \pm i\kappa)} d\eta \]  \hfill (3.4)

and

\[ \omega_\pm(x, x', y, \lambda) = \theta(y) \int_{|\eta| \geq 1} \frac{\Xi(y, \lambda, \eta) e^{-i\lambda_R(x' - x + 2\pi y)} \delta(x, x', y, \lambda, \eta)}{2\pi a(\eta + i\lambda_I)} d\eta \]
\[ + \int_{|\eta| \leq 1} \Xi(y, \lambda, \eta) e^{-i\lambda_R(x' - x + 2\pi y)} \delta(x, x', y, \lambda, \eta) - g(x, x', \pm i\kappa) d\eta \]
\[ - \int_{|\eta| \leq 1} d\eta \theta(y) e^{-\eta^2 y + iy(x' - x + 2\pi y)} g(x, x', \eta \pm i\kappa) - g(x, x', \pm i\kappa) \]
\[ + \int_{|\eta| \leq 1} \Xi(y, \lambda, \eta) e^{-i\lambda_R(x' - x + 2\pi y)} - 1 g(x, x', \pm i\kappa) d\eta - \frac{8m(\lambda_R)\lambda_R}{\pi} g(x, x', \pm i\kappa) \]
\[ - 2\theta(-y) \theta(\kappa - 1) e^{(\lambda^2 + \kappa^2)(y + i\lambda(x - x'))} \pm (y - x') - 1 |g(x, x', \pm i\kappa) | = \pm \theta(y) g(x, x', \pm i\kappa) \log \frac{1 + i\sin \alpha}{1 + i\sin \alpha} - \log \frac{1 + i|\lambda_I|}{1 + i|\lambda_I|}, \]  \hfill (3.5)

where \( a(\lambda), \quad g(x, x', \lambda), \quad \cot^{-1} \frac{|\lambda I|}{|\lambda R|}, \quad \Xi(y, \lambda, \eta), \) and \( \delta(x, x', y, \lambda, \eta), \) are defined by \((2.9), (2.8), (2.27), (2.20), \) and \((s, \alpha)\) denotes the polar coordinates for \( D_{z,a}^\infty. \) Moreover,

\[ |\omega_\pm|_{L^\infty(D_{z,a}^\infty)} \leq C, \quad |\partial_\alpha \omega_\pm|_{L^\infty(D_{z,a}^\infty)} \leq C(1 + |x - x' + 2\kappa y|). \]  \hfill (3.6)

**Proof.** 

**Step 1 (Proof of (3.3) - (3.5):** Let \( \lambda_0 \) be a radial limit \( \lambda \) at \( \pm i\kappa. \) Write

\[ \overline{G} = e^{-i\lambda_R(x' - x + 2\pi y)} (I + II + III + IV + V + \overline{G}^0), \]
\[ \overline{G}^0 = -2\theta(-y) \theta(\kappa - |\lambda_I|) g(x, x', \pm i\kappa) e^{(\lambda_I \pm \kappa)(x' - x)} e^{(\lambda^2 + \kappa^2)(y + i\lambda(x - x'))} \]
where $I-V$ are defined as in the proof of Lemma 2.2. By the dominated convergence theorem \[10\] \S 6, Chapter III,

$$III, V, II \ (cf. \ (2.36), \ (2.37) \ in \ case \ (4b)), \ I$$

are continuous at $\lambda = \pm i\kappa$, for fixed $y \neq 0, x \in \mathbb{R}$.

Together with \[2.29\] and

$$\tilde{G}_d(x, x', y, \lambda_0) = \tilde{G}_d^0(x, x', y, \lambda_0) = -2\theta(-y)\theta(\kappa - |\lambda_0|)g(x, x', \pm i\kappa),$$

we then prove $\tilde{G}$ can be written as \[3.9\] where $\mathfrak{G}_\pm, \omega_\pm$ are defined by \[3.4\] and \[3.5\].

Step 2 (Proof of \[3.6\]) : The first inequality in \[3.6\] follows from Lemma 2.2 For simplicity, we only give the proof of the second inequality at $s = 0$ (i.e. $\lambda = \lambda_0$) because the computation is similar for $s \neq 0$. From Step 4 of the proof for Lemma 2.2

$$\lim_{s \to 0} \frac{\lambda^2}{\pi} \int_{\mathbb{R}} \{ -2\theta(-y)\theta(\kappa - |\lambda_0|) \} [e^{\lambda_0^2(x' - x)} + e^{\lambda_0^2(x' - x)}] g(x, x', \pm i\kappa)$$

we have

$$\lim_{s \to 0} \frac{\lambda^2}{\pi} \int_{\mathbb{R}} \{ -2\theta(-y)\theta(\kappa - |\lambda_0|) \} [e^{\lambda_0^2(x' - x)} + e^{\lambda_0^2(x' - x)}] g(x, x', \pm i\kappa) = \frac{2\theta(y) \sin \alpha g(x, x', \pm i\kappa)}{\pi}.$$ (3.7)

Furthermore, by

$$\frac{\partial f}{\partial \eta} = \cos \alpha \frac{\partial f}{\partial \lambda_R} + \sin \alpha \frac{\partial f}{\partial \eta_i \lambda_R}, \quad \frac{\partial f}{\partial \eta_i \lambda_R} = i \sin \alpha \frac{\partial f}{\partial \eta_i \lambda_R},$$

and integration by parts,

$$\lim_{s \to 0} \int_{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{s}^{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{\mathbb{R}} \{ -2\theta(-y)\theta(\kappa - |\lambda_0|) \} [e^{\lambda_0^2(x' - x)} + e^{\lambda_0^2(x' - x)}] g(x, x', \pm i\kappa)$$

we have

$$\lim_{s \to 0} \int_{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{s}^{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{\mathbb{R}} \{ -2\theta(-y)\theta(\kappa - |\lambda_0|) \} [e^{\lambda_0^2(x' - x)} + e^{\lambda_0^2(x' - x)}] g(x, x', \pm i\kappa) = \frac{2\theta(y) \sin \alpha g(x, x', \pm i\kappa)}{\pi}.$$ (3.7)

and integration by parts,

$$\lim_{s \to 0} \int_{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{s}^{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{\mathbb{R}} \{ -2\theta(-y)\theta(\kappa - |\lambda_0|) \} [e^{\lambda_0^2(x' - x)} + e^{\lambda_0^2(x' - x)}] g(x, x', \pm i\kappa)$$

we have

$$\lim_{s \to 0} \int_{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{s}^{\mathbb{R}} \frac{\lambda^2}{\pi} \int_{\mathbb{R}} \{ -2\theta(-y)\theta(\kappa - |\lambda_0|) \} [e^{\lambda_0^2(x' - x)} + e^{\lambda_0^2(x' - x)}] g(x, x', \pm i\kappa) = \frac{2\theta(y) \sin \alpha g(x, x', \pm i\kappa)}{\pi}.$$ (3.7)
we obtain
\[ (x' - x \mp 2\kappa y) \sin \alpha \tilde{G}(x, x', y, \lambda, \pm \kappa, \eta, ) \]

\[ \lim_{s \to 0} \frac{1}{s} \int_{-1}^{1} \tilde{G}(x, x' y, \pm \kappa, \eta, ) \delta_{\eta=\pm \kappa} \, d\eta = \delta_{\eta=\pm \kappa} \]

\[ \partial_{\lambda}^2 \tilde{g}(x, x', \pm \kappa, \eta, ) - \partial_{\lambda}^2 \tilde{g}(x, x', \pm \kappa, \eta, ) \delta_{\eta=\pm \kappa} \]

\[ \lim_{s \to 0} \frac{1}{s} \int_{-1}^{1} \tilde{G}(x, x', y, \pm \kappa, \eta, ) - \tilde{G}(x, x', \pm \kappa, \eta, ) \delta_{\eta=\pm \kappa} \, d\eta = \delta_{\eta=\pm \kappa} \]

\[ \lim_{s \to 0} \frac{V(x, x', y, \lambda) - V(x, x', y, \lambda_0)}{s} = (x' - x \mp 2\kappa y) \sin \alpha \bar{G}(x, x', y, \lambda, 0) \]

\[ \lim_{s \to 0} \frac{1}{s} \int_{-1}^{1} \tilde{G}(x, x', y, \pm \kappa, \eta, ) \delta_{\eta=\pm \kappa} \, d\eta \]

\[ \lim_{s \to 0} \frac{1}{s} \int_{-1}^{1} \tilde{G}(x, x', y, \pm \kappa, \eta, ) \delta_{\eta=\pm \kappa} \, d\eta = \delta_{\eta=\pm \kappa} \]

\[ \lim_{s \to 0} \frac{1}{s} \int_{-1}^{1} \tilde{G}(x, x', y, \pm \kappa, \eta, ) \delta_{\eta=\pm \kappa} \, d\eta \]

\[ \lim_{s \to 0} \frac{1}{s} \int_{-1}^{1} \tilde{G}(x, x', y, \pm \kappa, \eta, ) \delta_{\eta=\pm \kappa} \, d\eta \]

Therefore \ref{3.6} is justified for \( s = 0 \). The case for \( s \neq 0 \) can be proved by the same method.

\[ \square \]

**Theorem 2.** Let \( \partial_{\lambda}^j \partial_{\lambda}^k v_0 \in L^1 \cap L^\infty, 0 \leq j, k \leq 2, |v_0|_{L^1 \cap L^\infty} \ll 1, v_0(x, y) \in \mathbb{R}, \) and \( \lambda = \lambda_R + i\lambda_I \in D_{\pm i\kappa}^\times. \) Then

\[ m(x, y, \lambda) = \begin{cases} \\
\frac{m_{+, +}(x, y, \lambda)}{\lambda - i\kappa} + m_{+, -}(x, y, \lambda), & \lambda \in D_{+ i\kappa}^\times \\
m_{-, +}(x, y, \lambda) + m_{-, -}(x, y, \lambda), & \lambda \in D_{- i\kappa}^\times \end{cases} \]
To prove the lemma, we will compute the leading terms of (3.16) at \( \pm \nu \). Denote
\[
\varphi_{\pm} (x, y, \lambda) = 1 + [\mathcal{G}_\pm + \frac{2}{\pi} \mathcal{G}_0 (x, \pm \nu \lambda)] \cot^{-1} \frac{\kappa - |\lambda|}{|\lambda R|} \ast v_0
\]
(3.20)
as the leading terms in \((1 + \mathcal{G} \ast v_0)\) at \( \pm \nu \). Hence \( \varphi_{\pm} \) and \((1 + \mathcal{G}_0 \ast v_0)\) are invertible by Lemma 2.2. Together with (2.50), Lemma 3.1, and \( \varphi_{\pm} (x, \lambda) = \vartheta_{\pm} (x, \lambda) \),
\[
m(x, y, \lambda) = \frac{\mathcal{G}_0 (x, \nu \lambda)}{\alpha (\lambda)} \left( \varphi_{\pm} + \omega_+ \ast v_0 \right)^{-1} \varphi_+ \]
\[
= \frac{\mathcal{G}_0 (x, \nu \lambda)}{\alpha (\lambda)} \sum_{j=-\infty}^{\infty} (-1)^j \left( \varphi_{\pm}^{-1} \omega_+ \ast v_0 \right)^j \varphi_+^{-1} \varphi_+, \quad D_{+ \nu}^x;
\]
(3.21)
\[
m(x, y, \lambda) = (\varphi_- + \omega_- \ast v_0)^{-1} \vartheta_-
\]
\[
= \sum_{j=0}^{\infty} (-1)^j \left( \varphi_{\pm}^{-1} \omega_+ \ast v_0 \right)^j \varphi_+^{-1} \vartheta_, \quad D_{- \nu}^x.
\]
So \( \frac{2ik}{\lambda} \varphi_+^{-1}(x, y, +ik) \) and \( \varphi_-^{-1}(x, y, -ik) \) are leading terms at \( +ik, -ik \).

Defining \( \Theta_\pm(x, y) \) and \( \gamma_\pm \) by (3.14), (3.15), and using

\[
\left( 1 + \mathcal{G}_+ \ast v_0 \right)^{-1} \frac{2}{\pi} \delta(x, x', ik) \cot^{-1} \frac{\kappa - |\lambda|}{|\lambda R|} \ast v_0 \right) \Theta_+(x, y)
\]

we obtain

\[
\varphi_+^{-1} [\varphi_+(x, ik)] = \frac{\Theta_+(x, y)}{1 + \gamma_+ \cot^{-1} \frac{\kappa - |\lambda|}{|\lambda R|}}, \quad \varphi_-^{-1} [\varphi_-(x, -ik)] = \frac{\Theta_-(x, y)}{1 + \gamma_- \cot^{-1} \frac{\kappa - |\lambda|}{|\lambda R|}}.
\]

So the remainders are

\[ m_{+, \gamma}(x, y, \lambda) = \varphi_+^{-1} \varphi_+(x, ik) + \varphi_-^{-1} \varphi_+(x, ik) + \sum_{j=1}^{\infty} (-1)^j \left( \lambda_+^{-1} \omega_+ \ast v_0 \right)^j \varphi_+(x, \lambda), \]

\[ m_{-, \gamma}(x, y, \lambda) = \sum_{j=0}^{\infty} (-1)^j \left( \lambda_+^{-1} \omega_- \ast v_0 \right)^j \varphi_-(x, \lambda) - \varphi_-^{-1} [\varphi_-(x, -ik)]. \]

Along with Lemma 3.1, we then prove (3.13) - (3.16).

**Step 2 (Proof for (3.18))**: Condition (3.18) can be shown by the reality of variables \( \lambda' \mapsto -\lambda' \) in (2.15), and

\[
\phi_{\pm}(x, -\lambda) = \phi_{\pm}(x, \lambda), \quad \psi_{\pm}(x, -\lambda) = \psi_{\pm}(x, \lambda), \quad a(-\lambda) = a(\lambda)
\]

(3.24) to prove

\[
G_c(x, x', y, -\lambda) = G_c(x, x', y, \lambda),
G_d(x, x', y, -\lambda) = G_d(x, x', y, \lambda).
\]

**Step 3 (Proof for (3.17) and (3.19))**: We first claim

\[
G(x, x', y, ik + 0^+ e^{i\alpha}) = G(x, x', y, -ik + 0^+ e^{i(\pi + \alpha)}).
\]

Combining with \( \varphi_+(x, ik) = \varphi_-(x, -ik) e^{-2\kappa x}, \varphi_+(x, -ik) = \varphi_-(x, ik) e^{-2\kappa x}, \)

(2.49), and (3.33), we obtain the commutative condition

\[
\mathcal{G}_+(x, x', y) e^{-2\kappa x'} = e^{-2\kappa x} \mathcal{G}_-(x, x', y).
\]

Consequently,

\[
\Theta_\pm(x, y) = e^{-2\kappa x} \Theta_\mp(x, y), \quad \gamma_\pm = \gamma_\mp
\]

(3.27)
which, combining with (2.27), prove (3.17) and (3.19).

We now exploit the approach in [19, Proposition 9 (i)] to prove (3.25). In view of (2.7), (2.19), and, for fixed $x, y \neq 0$, $-\frac{\pi}{2} < \alpha \leq \frac{3\pi}{2}, \alpha \neq 0, \pi$, let

$$
\lambda_+ = \lambda_{+R} + i\lambda_{+I} = +i\kappa + 0^+ e^{i\alpha},
\lambda_- = \lambda_{-R} + i\lambda_{-I} = -i\kappa + 0^+ e^{(\pi + i\alpha)}.
$$

Then

$$
G_{C^-}(x, x', y, \lambda_-) = \theta(y) \int_{\mathbb{R}} e^{(\lambda_{-}^2 - [\eta + i\lambda_{-I}]^2)} y \phi_-(x, y + i\lambda_{-I}) \psi_-(x', y + i\lambda_{-I}) \, d\eta,
$$

$$
G_{C^+}(x, x', y, \lambda_+) = \theta(y) \int_{\mathbb{R}} e^{(\lambda_{+}^2 - [\eta + i\lambda_{+I}]^2)} y \phi_+(x, y + i\lambda_{+I}) \psi_+(x', y + i\lambda_{+I}) \, d\eta.
$$

Deforming the contour, applying the residue theorem and (2.21),

$$
\theta(y) \int_{\mathbb{R}} e^{(\lambda_-^2 - [\eta + i\lambda_{-I}]^2)} y \phi_-(x, y + i\lambda_{-I}) \psi_-(x', y + i\lambda_{-I}) \, d\eta
= \theta(y) \int_{\mathbb{R}} e^{-\eta^2} y \phi_-(x, y) \psi_-(x', y) \, d\eta + 2\kappa \theta(y) [1 - \theta(\kappa - |\lambda_{-I}|)] \phi_-(x, i\kappa) \psi_+(x', i\kappa)
= \theta(y) \int_{\mathbb{R}} e^{-\eta^2} y \phi_-(x', y) \psi_-(x, y) \, d\eta + 2\kappa \theta(y) [1 - \theta(\kappa - |\lambda_{-I}|)] \phi_-(x', -i\kappa) \psi_-(x, -i\kappa)
= \theta(y) \int_{\mathbb{R}} e^{(\lambda_-^2 - [\eta + i\lambda_{-I}]^2)} y \phi_-(x', y + i\lambda_{-I}) \psi_-(x, y + i\lambda_{-I}) \, d\eta.
$$

On the other hand, the residue theorem, (2.26), (2.27), (2.7), and the dominated convergence theorem imply

$$
\int_{-|\lambda_{+I}|}^{\lambda_{+I}} e^{(\lambda_{+}^2 - [\eta + i\lambda_{+I}]^2)} y \phi_+(x, y + i\lambda_{+I}) \psi_+(x', y + i\lambda_{+I}) \, d\eta
= \int_{-|\lambda_{-I}|}^{\lambda_{-I}} e^{(\lambda_{-}^2 - [\eta + i\lambda_{-I}]^2)} y \phi_-(x, y + i\lambda_{-I}) \psi_-(x', y + i\lambda_{-I}) \, d\eta.
$$

Consequently, $G_{C^+}(x, x', y, \lambda_0) = G_{C^-}(x, x', y, -\lambda_0)$ and (3.25) follow from (3.28), (3.30).
Example 3.1. If \( v_0(x, y) \equiv 0 \), then
\[
\begin{align*}
  s_d &\equiv 2i\kappa, \quad s_c(\lambda) \equiv 0, \quad \gamma_{\pm} = 0, \\
  m(x, y, \lambda) &= \frac{\varphi_{\pm}(x, \lambda)}{a_{\pm}(\lambda)} = \vartheta_{\pm}(x, \lambda), \\
  m_{+}(x, y, i\kappa + 0^+ e^{i\alpha}) &= 2i\kappa \Theta_{+}(x, y) \equiv \frac{2i\kappa}{1 + e^{-2\pi x}}, \\
  m_{-}(x, y, -i\kappa + 0^+ e^{i\alpha}) &= \Theta_{-}(x, y) \equiv \frac{2i\kappa}{1 + e^{2\pi x}}.
\end{align*}
\]
(3.31)

3.2 Continuous scattering data of \( m(x, y, \lambda) \)

Lemma 3.2. For \( \lambda_R \neq 0 \),
\[
\partial_{\lambda} \tilde{G}(x, x', y, \lambda) = -\frac{\text{sgn}(\lambda_R)}{2\pi a(-\lambda)} e^{(\lambda^2 - \lambda^2)y + i(\lambda + \lambda')(x - x')} g(x, x', -\lambda),
\]
(3.32)
where \( a(\lambda), g \) are defined by (2.8). So for \( \lambda_R \neq 0 \),
\[
\begin{align*}
  \partial_{\lambda} \tilde{G}(x, x', y, \lambda) &= \int_{\mathbb{R}} \frac{3(x, x', y, \lambda, \lambda')}{2\pi a(\lambda + \lambda')} \frac{\partial}{\partial \lambda'} \left( \theta(y) \chi_{-} - \theta(-y) \chi_{+} \right) d\lambda' \\
  &\quad + \int_{\mathbb{R}} \left[ 2\partial y \left( 1 - |\lambda_I| \right) \frac{e^{(\lambda^2 + 1)y + i(\lambda + \lambda')(x - x')}}{(e^{-x} + e^x)(e^{-x'} + e^{x'})} \right] d\lambda' \\
  &\quad - \frac{\partial}{\partial \lambda} \left[ 2\partial y \left( 1 - |\lambda_I| \right) \cdot \frac{e^{(\lambda^2 + 1)y + i(\lambda + \lambda')(x - x')}}{(e^{-x} + e^x)(e^{-x'} + e^{x'})} \right] \\
  &= A + B + C.
\end{align*}
\]
(3.33)

Using
\[
\frac{1}{\pi} \partial_{\lambda} \left( \frac{1}{\lambda - a} \right) = \delta_{\lambda_R = a_R} \delta_{\lambda_I = a_I}, \quad \partial_{\lambda} \theta(1 + \lambda I) = \frac{i}{2} \partial_{\lambda} \theta(1 + \lambda I) = \mp \frac{i}{2} \delta_{\lambda_I = \pm 1},
\]
one has
\[
\begin{align*}
  A &= \int_{\mathbb{R}} \frac{3(x, x', y, \lambda, \lambda')}{2\pi a(\lambda + \lambda')} \frac{\partial}{\partial \lambda'} \left( \theta(y) \chi_{-} - \theta(-y) \chi_{+} \right) d\lambda' \\
  &= \int_{\mathbb{R}} \frac{3(x, x', y, \lambda, \lambda')}{2\pi a(\lambda + \lambda')} \times \left( \mp \theta(y) \text{sgn}(\lambda_R) \delta_{\lambda_I = -2\lambda_R} \right) \\
  &= -\text{sgn}(\lambda_R) \frac{3(x, x', y, \lambda, \lambda')}{2\pi a(\lambda + \lambda')} \\
  B &= \int_{\mathbb{R}} \left( \theta(y) \chi_{-} - \theta(-y) \chi_{+} \right) \tilde{G}(x, x', y, \lambda, \lambda') \frac{\partial}{\partial \lambda} \left( \frac{1}{2\pi a(\lambda + \lambda')} \right) d\lambda' \\
  &= \int_{\mathbb{R}} \left( \theta(y) \chi_{-} - \theta(-y) \chi_{+} \right) \tilde{G}(x, x', y, \lambda, \lambda') \delta_{\lambda_I = \pm 1} d\lambda' \\
  &= \mp i \theta(-y) e^{(\lambda^2 + 1)y + i(\lambda + \lambda')(x - x')} (e^{-x} + e^x)(e^{-x'} + e^{x'}) \delta_{\lambda_I = \pm 1}.
\end{align*}
\]
(3.34)
Theorem 3. If $\partial_j^k \partial^k x v_0 \in L^1 \cap L^\infty$, $0 \leq j, k \leq 2$, $|v_0|_{L^1 \cap L^\infty} \ll 1$, then

$$\partial^\kappa m(x, y, \lambda) = s_c(\lambda) e^{i(\lambda R(x+y) + 2\lambda R^2)x} m(x, y, -\lambda), \quad \lambda_R \neq 0,$$

(3.35)

(cf. [19, § 6, Eq. (46), (51)]) where $s_c(\lambda)$ is defined by

$$s_c(\lambda) = \begin{cases} \frac{sgn(\lambda)}{2\pi a(\lambda)} \varphi_+ v_0 m(\lambda \pi, 2\lambda \pi; \lambda), \quad \lambda_R \neq 0, \lambda \in \mathbb{C}^+, \\ \frac{sgn(\lambda)}{2\pi a(\lambda)} \varphi_- v_0 m(\lambda \pi, 2\lambda \pi; \lambda), \quad \lambda_R \neq 0, \lambda \in \mathbb{C}^- \end{cases},$$

$$\varphi_+ v_0 m(\lambda \pi, 2\lambda \pi; \lambda) = \int e^{-i(4\lambda R(x+y) + 2\lambda R^2)} \varphi_+(x, -\lambda v_0 m(x, y, \lambda) dx dy,$$

(3.36)

and $\varphi_+, \varphi_-, a(\lambda)$ are defined by (2.3), (2.4), and (2.9).

Proof. Denote $\tilde{G}(x, x', y, \lambda)$ as $\tilde{G}_\lambda$ and $\rho(x, y, \lambda, -\lambda) = e^{(\lambda^2 - \lambda^2)^2 y + i(\lambda + \lambda) x}$. Note $\rho(x, y, \lambda, -\lambda)$ is annihilated by the heat operator $p_\lambda(D) \equiv \partial_y - \partial_x^2 + 2i\lambda \partial_x$. So

$$p_\lambda(D) f = p_\lambda(D) e^{(\lambda^2 - \lambda^2)^2 y + i(\lambda + \lambda) x} m(x, y, -\lambda) \tilde{G}_\lambda.$$

Therefore, for $\lambda_R \neq 0$, denoting $\mathfrak{c}(x, x', y, \lambda, -\lambda) = e^{(\lambda^2 - \lambda^2)^2 y + i(\lambda + \lambda) x + i(\lambda^2 - \lambda^2)^2 y + i(\lambda + \lambda) x}$, (2.50), (3.32), (3.36), and (3.37),

$$\partial^\kappa m(x, y, \lambda) = s_c(\lambda) (1 + \tilde{G}_\lambda \ast v_0)^{-1} e^{(\lambda^2 - \lambda^2)^2 y + i(\lambda + \lambda) x} m(x, -\lambda).$$
Lemma 3.3. Suppose \((1 + |x| + |y|)^2 \partial^2_x \partial^2_y v_0 \in L^1 \cap L^\infty, 0 \leq j, k \leq 2, \)|v_0|_{L^1 \cap L^\infty} \ll 1, \) and \(v_0(x,y) \in \mathbb{R}. \) Then

\[
|(1 - E_{+ik,1/2}(\lambda) - E_{-ik,1/2}(\lambda))s_c(\lambda)|_{L^2(\lambda R|d\lambda| \cap L^\infty)} \leq C|\partial^2_x \partial^2_y v_0|_{L^1},
\]

(3.38)

and

\[
s_c(\lambda) = \begin{cases} 
  +\frac{i \text{sgn}(\lambda_R)}{|\lambda|} r(\lambda) + \text{sgn}(\lambda_R) h^+(\lambda), & \lambda \in D^+_{+ik}, \\
  -\frac{i \text{sgn}(\lambda_R)}{|\lambda|} r(\lambda) + \text{sgn}(\lambda_R) h^-(\lambda), & \lambda \in D^-_{+ik},
\end{cases}
\]

(3.39)

where

\[
r(\lambda) = \frac{\gamma}{1 + \gamma \cot^{-1} \frac{\kappa}{|\lambda_R|}},
\]

(3.40)

and \(E_{z,a}, D^\infty_z, \text{cot}^{-1} \frac{\kappa}{|\lambda_R|} \) are defined by Definition \ref{def:3.1}, \(\gamma_+, \lambda_+ \) \ref{thm:3.11}.

Moreover,

\[
|\tau(\lambda)| \leq |v_0|_{L^1}, \quad \sum_{j=0,1} |\partial^j_h \pm|_{L^\infty} \leq C|\tau(\lambda)| |\partial^2_x \partial^2_y v_0|_{L^1}, \quad \tau(\lambda) = \tau(-\lambda) \in \mathbb{R}, \quad h^\pm(\lambda) = h^\pm(-\lambda).
\]

(3.41)

Proof. Step 1 (Proof for (3.38)): We restrict to \(\lambda \in \mathbb{C}^+\) since proofs for \(\lambda \in \mathbb{C}^-\) are identical. From (3.36), the Fourier theory, and Theorem \ref{thm:1},

\[
|\tau(\lambda)| \leq |v_0|_{L^1}, \quad \sum_{j=0,1} |\partial^j_h \pm|_{L^\infty} \leq C|\tau(\lambda)| |\partial^2_x \partial^2_y v_0|_{L^1}, \quad \tau(\lambda) = \tau(-\lambda) \in \mathbb{R}, \quad h^\pm(\lambda) = h^\pm(-\lambda).
\]

(3.42)

Therefore (3.38) follows from

\[
\begin{align*}
\left| \frac{\partial^2_x \partial^2_y v_0}{|\lambda_R|^{1+\gamma} \tau(\lambda)} \right| & \leq C \int \frac{\partial^2_x \partial^2_y v_0}{|\lambda_R|^{1+\gamma} \tau(\lambda)} |\lambda_R| \, d\lambda \\
& \leq C |\partial^2_x \partial^2_y v_0|_{L^1}.
\end{align*}
\]

(3.43)

Step 2 (Proof for (3.40) - (3.41)): For \(\lambda \in D^\infty_{\pm ik}, \lambda_R \neq 0, \) from (3.36), Theorem \ref{thm:2},

\[
\begin{align*}
s_c(\lambda) & = \begin{cases} 
  \frac{\text{sgn}(\lambda_R)}{2\pi(\lambda - ik)} \int e^{-i(4k\lambda + 2\lambda x + 2k\lambda y)} \varphi_+(x,\lambda) v_0(x,y) \varphi_+(x,y) \frac{d\lambda}{\lambda R} \, dx dy + \frac{\text{sgn}(\lambda_R)}{2\pi} \\
  \times \int e^{-i(4k\lambda + 2\lambda x + 2k\lambda y)} \varphi_+(x,\lambda) v_0(x,y) \varphi_+(x,y) \frac{d\lambda}{\lambda R} \, dx dy + \frac{\text{sgn}(\lambda_R)}{2\pi(\lambda - ik)} \\
  \times \int e^{-i(4k\lambda + 2\lambda x + 2k\lambda y)} \varphi_+(x,\lambda) v_0(x,y) \varphi_+(x,y) \frac{d\lambda}{\lambda R} \, dx dy + \frac{\text{sgn}(\lambda_R)}{2\pi(\lambda - ik)},
\end{cases}
\end{align*}
\]

(3.44)
Precisely, the data from the Cauchy integral equation [8, Eq.(5.10)] can not be achieved for generic initial data of the inverse problem. From (3.42), convergence of the singularities at \( \kappa_n \) for the perturbed KPII multi-line solitons equations.

Thus, another alternative is through a regularization to derive the Cauchy integral equation for the inverse problem. From [3.42], convergence of the Cauchy integral equation [8, Eq.(5.10)] can not be achieved for generic initial data \( v_0(x,y) \). Nevertheless, [8] made an important progress in dealing the singular integrals at \( \kappa_n \) for the perturbed KPII multi-line solitons equations.

3.3 The forward scattering transform and the eigenfunction space

Due to the discontinuities of \( m_{+,1} \), the kernel of the Cauchy equation (\( \partial_\lambda \)-equation) for \( m \) is not simply.

\[
\partial_\lambda m(x, y, \lambda) = s_\epsilon(\lambda)e^{i(4\lambda R(y+2\lambda R x))m(x, y, -\lambda)}
\]

simply. We need to correct it by subtracting the contribution from \( m_{+,1} \).

Another neater alternative is through a regularization to derive the Cauchy equation for the inverse problem. From [3.42], convergence of the Cauchy integral equation [8, Eq.(5.10)] can not be achieved for generic initial data \( v_0(x,y) \). Nevertheless, [8] made an important progress in dealing the singular integrals at \( \kappa_n \) for the perturbed KPII multi-line solitons equations.

Precisely,

- the singularities at \( \kappa_n \) of the Cauchy integral equation are regularized by renormalizing \( m(x, y, \lambda) \) by holomorphic functions vanishing at \( \kappa_n \) [8, Eq.(2.12)].
• Cauchy integrals of the leading singularities of the regularized eigenfunction at \( \kappa_n \) are integrated by applying Stokes’ theorem, and the reality property at \( \kappa_n \) is proved [8, Lemma 5.1];

• boundary terms of their Cauchy integral equation [8, Eq.(5.33)] are characterized by the asymptotic properties at \( \kappa_n \) (however, base on the convergence of [8, Eq.(5.10)]).

Inspired by these results, we introduce the regularized eigenfunction

\[
m(x, y, \lambda) = \frac{\lambda - \text{i} \kappa}{\lambda - 2\text{i} \kappa} m(x, y, \lambda),
\]

(3.45)
to tame the singularities at \(+ \text{i} \kappa, \infty\) and keep the symmetry.

**Theorem 4.** Suppose

\[
u_0(x) = -2 \kappa^2 \text{sech}^2 \kappa x, \; \kappa > 0,
\]

\[
(1 + |x| + |y|)^2 \partial_x^j \partial_y^k v_0 \in L^1 \cap L^\infty, \; 0 \leq j, k \leq 2,
\]

\[
|v_0|_{L^1 \cap L^\infty} \ll 1, \; v_0(x, y) \in \mathbb{R}.
\]

Then for fixed \( \lambda \in \mathbb{C} \setminus \{ \pm \text{i} \kappa, \text{i} \} \), there exists a unique eigenfunction \( m \) satisfying

\[
(\partial_y - \partial_x^2 + 2 \text{i} \lambda \partial_x + u_0(x)) m = -v_0(x, y)m,
\]

\[
\lim_{(x, y) \to \infty} (m - \frac{\lambda \text{i} \kappa}{\lambda - 2 \text{i} \kappa} \partial_-(x, \lambda)) = 0,
\]

(3.46)
and

\[
|(1 - E_\delta)m(x, y, \lambda)| \leq C;
\]

\[
m(x, y, \lambda) = m_{\text{res}}(x, y) + m_{\text{reg}}(x, y, \lambda), \; \lambda \in D^\lambda, \quad (3.47)
\]

\[
m_{\text{res}}(x, y) \in \mathbb{R}, \quad |m_{\text{res}}(x, y)|_{L^\infty} \leq C,
\]

\[
|\lambda - \kappa|^2 m_{\text{reg}}(x, y, \lambda)|_{L^\infty} \leq C, \quad |m_{\text{reg}}(x, y, \lambda)|_{L^\infty} \leq C(1 + |x| + |y|);
\]

\[
m(x, y, \lambda) = m_{\pm \text{i} \kappa, 0}(x, y, \lambda) + m_{\pm \text{i} \kappa, r}(x, y, \lambda), \; \lambda \in D^\pm_{\lambda},
\]

\[
m_{\pm \text{i} \kappa, 0}(x, y, \lambda) = \frac{-2 \Theta_+(x, y)}{1 + \gamma_+ \cot^{-1} \frac{\lambda - \text{i} \kappa}{|\lambda|}}, \quad m_{- \text{i} \kappa, 0}(x, y, \lambda) = \frac{2 \Theta_-(x, y)}{1 + \gamma_+ \cot^{-1} \frac{\lambda - \text{i} \kappa}{|\lambda|}},
\]

\[
m_{+ \text{i} \kappa, 0}(x, y, \pm \text{i} \kappa + 0^+ e^{\text{i} \alpha}) = s_\delta e^{-2 \kappa x} m_{- \text{i} \kappa, 0}(x, y, -\text{i} \kappa + 0^+ e^{\text{i} (\pi + \alpha)}),
\]

\[
m_{\pm \text{i} \kappa, r}(x, y, \pm \text{i} \kappa) = 0, \quad |m_{\pm \text{i} \kappa, r}(x, y, \lambda)|_{L^\infty} \leq C(1 + |x| + |y|),
\]

(3.48)

where \( \vartheta_-(x, \lambda) \) is defined by (2.3), \( \lambda = z + s \text{e}^{\text{i} \alpha}, \; z, s \in \mathbb{R} \), \( E_\delta, D^\pm_\lambda \), \( \vartheta \), are defined by Definition 3.14, \( \Theta_\pm, s_\delta = -3, \; \gamma_+, \) and \( \cot^{-1} \frac{\lambda - \text{i} \kappa}{|\lambda|} \) defined by (3.14), (3.15), (3.19), (2.27).
Moreover, for \( \lambda_R \neq 0, \lambda \neq \iota \),
\[
\partial \chi m(x, y, \lambda) = s_c(\lambda) e^{i(4\lambda_R \lambda y + 2\lambda_R x)} m(x, y, -\lambda),
\]
\[
s_c(\lambda) = \frac{\lambda - i\kappa}{\lambda - \iota} s_c(\lambda),
\]
(3.49)
with
\[
|(1 - E_{\pm i\kappa, 1/2}(\lambda) - E_{-i\kappa, 1/2}\delta)(\lambda)|_{L^2(\lambda_R, d\lambda)} \leq C|\partial^2_x \partial^2_y v_0|_{L^1};
\]
(3.50)
\[
s_c(\lambda) = \frac{1}{2} \text{sgn}(\lambda_R) e + \text{sgn}(\lambda_R) h_c(\lambda), \quad \lambda \in D^\times,
\]
\[
\sum_{0 \leq k \leq 1} |\partial^k h_c(\lambda)|_{L^\infty} \leq C(1 + |x| + |y|)^2 v_0|_{L^1},
\]
(3.51)
\[
s_c(\lambda) = \pm i\kappa \frac{\text{sgn}(\lambda_R)}{\lambda_{\pm i\kappa}} \tau(\lambda) + \text{sgn}(\lambda_R) h_{\pm i\kappa}(\lambda), \quad \lambda \in D_{\pm i\kappa},
\]
\[
\sum_{0 \leq k \leq 1} |\partial^k h_{\pm i\kappa}|_{L^\infty} \leq C(1 + |x| + |y|)^2 v_0|_{L^1},
\]
(3.52)
where \( s_c(\lambda), \tau(\lambda) \) are defined by (3.36), (3.40). Finally,
\[
m(x, y, \lambda) = m(x, y, -\lambda), \quad h_z(\lambda) = h_z(-\lambda), \quad z \in \{ \pm i\kappa, \iota \}.
\]
(3.53)

**Proof.** Properties (3.47) can be derived by the formula \( G_c, G_d, \) and the integral equation of \( m \), namely, (2.7), (2.9), (2.50), and (3.45). The others follow from Theorem 1, 2, 3, and Lemma 3.3 in particular, (3.13) - (3.16), (3.35), (3.36), and (3.37).

**Example 3.2.** If \( v_0(x, y) \equiv 0 \), then
\[
s_d \equiv -3, \quad s_c(\lambda) \equiv 0, \quad \gamma_\pm = 0,
\]
\[
m(x, y, \lambda) = \varphi_+(x, \lambda) \frac{\lambda_{+ i\kappa}}{\lambda_{-2i\kappa}} = 1 + \frac{3i\kappa}{\lambda - 2i\kappa}(1 - \frac{2}{1+e^{-2\kappa x}}),
\]
\[
m_{+i\kappa, 0}(x, y, \iota) \equiv \frac{2}{1+e^{2\kappa x}}, \quad m_{-i\kappa, 0}(x, y, -\iota + 0^+ e^{i\kappa}) \equiv \frac{3}{1+e^{-2\kappa x}},
\]
\[
m_{res}(x, y) \equiv 3i\kappa(1 - \frac{2}{1+e^{-2\kappa x}}).
\]

Based Theorem 4, we introduce the space of eigenfunctions \( W \) and the spectral operator \( T \) as follows.

**Definition 3.2.** The eigenfunction space \( W \equiv W_{x,y} \) is the set of functions

i. \( \phi(x, y, \lambda) = \phi(x, y, -\lambda); \)

ii. \( (1 - E_\iota)\phi(x, y, \lambda) \in L^\infty; \)

iii. \( \phi(x, y, \lambda) = \frac{\phi_{res}(x, y, \lambda)}{\lambda - \iota} + \phi_{res}(x, y, \lambda), \quad \lambda \in D^\times, \)
\[
\phi_{res}(x, y, (\lambda - \iota)\phi_{r,h}(x, y, \lambda), \quad \frac{\phi_{res}(x, y, \lambda)}{1 + |x| + |y|} \in L^\infty.
\]
iv. \( \phi(x, y, \lambda) = \phi_{\pm i\kappa, 0}(x, y, \lambda) + \phi_{\pm i\kappa, r}(x, y, \lambda), \quad \lambda \in D_{\pm i\kappa}^\times \),

\[
\phi_{\pm i\kappa, 0}(x, y, \lambda) = \begin{cases} 
1 + \gamma_+ \cot^{-1} \frac{\kappa - |\lambda I|}{\kappa - |\lambda R|}, \\
1 + \gamma_+ \cot^{-1} \frac{\kappa - |\lambda R|}{\kappa - |\lambda I|}, 
\end{cases} 
\]

\[
\phi_{\pm i\kappa, r}(x, y, \pm i\kappa) = 0, \quad \phi_{\pm i\kappa, r}(x, y, \lambda), \quad \frac{\partial \phi_{\pm i\kappa, r}}{\partial s}(x, y, \lambda) 
\in L^\infty.
\]

**Definition 3.3.** Define \( \{\iota, s_d, s_c(\lambda)\} \) as the set of scattering data, where the pole \( \iota = +2i\kappa \) and the norming constant \( s_d = -3 \) are the discrete scattering data; and \( s_c(\lambda) \), the continuous scattering data, is defined by (3.49). Denote \( T \) as the forward scattering transform by

\[
T(\phi)(x, y, \lambda) = s_c(\lambda)e^{i(4\lambda R\lambda I+y+2\lambda Rx)}\phi(x, y, -\lambda). \quad (3.54)
\]

**Definition 3.4.** Let \( C \) be the Cauchy integral operator defined by

\[
C(\phi)(x, y, \lambda) = C_\lambda(\phi) = -\frac{1}{2\pi i} \iint \frac{\phi(x, y, \zeta)}{(\zeta - \lambda)(\zeta - \zeta + i\kappa)} \frac{-\zeta - i\kappa}{1+|x|+|y|} d\zeta \wedge d\zeta. \quad (3.55)
\]

Decompose

\[
CT\phi = \sum_{\zeta \in \mathbb{Z}} C E_z T\phi + C \left[1 - \sum_{\zeta \in \mathbb{Z}} E_z\right] T\phi, \quad (3.56)
\]

where \( E_z \) and \( \mathbb{Z} = \{\pm i\kappa, \iota\} \) are defined by Definition 3.1.

### 3.4 The spectral analysis

Due to Theorem 4, outside the singularities \( \pm i\kappa, \iota \), the eigenfunction \( m \) and continuous scattering data \( (\tilde{\phi}-\text{data}) \) \( s_c \) possess the same analytical properties as those for the localized KPII solutions \([21],[22]\). As a result, spectral analysis there is the same as that for vacuum background (see Lemma 3.6 in below). On the other hand, from (3.51), (3.52), the Cauchy integrals at \( \pm i\kappa \) are two dimensional singular integrals with blowing ups of order two and highly oscillatory, not fully symmetric kernels which cause difficulties for deriving uniform estimates for the spectral transform. On the other hand, non-uniform estimates for these singular integrals can be achieved by applying Stokes’ or the Cauchy theorem to integrate the leading singularities (see Lemma 3.4 and 3.5).

**Lemma 3.4.** For \( \lambda \in D_{\pm i\kappa}^\times \),

\[
-\frac{1}{2\pi i} \iint \frac{\pm 4 \text{sgn}(\zeta)\tau(\zeta) E_{\pm i\kappa, m_{\pm i\kappa, 0}}(x, y, \zeta)}{(\zeta - \lambda)(\zeta - \zeta + i\kappa)} d\zeta \wedge d\zeta = m_{\pm i\kappa, 0}(x, y, \lambda) - \frac{1}{2\pi i} \int \frac{m_{\pm i\kappa, 0}(x, y, \zeta)}{\zeta - \lambda} d\zeta. \quad (3.57)
\]
Here and in the following the circular integration is taken counterclockwisely and \( E_z, D_x^z, \tau, m_{\pm ik,0} \) are defined by Definition 3.04, 3.08. Moreover,

\[
|CE_{\pm ik}Tm|_{L^\infty} \leq C(1 + |x| + |y|), \quad (3.58)
\]
\[
CE_{\pm ik}Tm \to 0 \text{ uniformly as } |\lambda| \to \infty. \quad (3.59)
\]

**Proof.** From 3.18, and for \( \zeta = \pm ik + se^{ib} \in D_x^{-ik}, \)

\[
-\frac{i}{2} \frac{1}{\zeta + ik} \partial_\zeta \beta, \partial_\zeta \cot^{-1} \frac{|\zeta|}{|R|} = \pm \frac{i}{2} \frac{\text{sng} (\zeta_R)}{-\zeta + ik}. \quad (3.60)
\]

Hence

\[
\partial_\zeta m_{\pm ik,0}(x, y, \zeta) = \frac{\pm i \text{sng} (\zeta_R) \tau (\zeta) m_{\pm ik,0}(x, y, \zeta)}{-\zeta + ik}. \quad (3.61)
\]

Applying Stokes’ theorem,

\[
-\frac{1}{2\pi i} \int_{D_{\pm ik}/(D_{\pm ik} \cup D_{\lambda,\epsilon})} \frac{\pm \text{sng} (\zeta_R) \tau (\zeta) E_{\pm ik} m_{\pm ik,0}(x, y, \zeta)}{(\zeta - \lambda)(-\zeta + ik)} d\zeta \wedge d\zeta = -\frac{1}{2\pi i} \int_{\partial(D_{\pm ik,\epsilon} \cup D_{\lambda,\epsilon})} \frac{m_{\pm ik,0}(x, y, \zeta)}{\zeta - \lambda} d\zeta \quad (3.62)
\]
\[
\begin{align*}
&\quad \quad \quad \quad \quad \quad + \frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} \frac{m_{\pm ik,0}(x, y, \zeta)}{\zeta - \lambda} d\zeta.
\end{align*}
\]

Note, by \( \lambda \neq \pm ik \) and Theorem \( 4 \)

\[
\begin{align*}
-\frac{1}{2\pi i} \int_{D_{\pm ik,\epsilon}} \frac{\pm \text{sng} (\zeta_R) \tau (\zeta) E_{\pm ik} m_{\pm ik,0}(x, y, \zeta)}{(\zeta - \lambda)(-\zeta + ik)} d\zeta \wedge d\zeta &\to 0, \\
-\frac{1}{2\pi i} \int_{D_{\lambda,\epsilon}} \frac{\pm \text{sng} (\zeta_R) \tau (\zeta) E_{\pm ik} m_{\pm ik,0}(x, y, \zeta)}{(\zeta - \lambda)(-\zeta + ik)} d\zeta \wedge d\zeta &\to 0, \\
+ \frac{1}{2\pi i} \int_{\partial D_{\pm ik,\epsilon}} \frac{m_{\pm ik,0}(x, y, \zeta)}{\zeta - \lambda} d\zeta &\to 0, \\
+ \frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} \frac{m_{\pm ik,0}(x, y, \zeta)}{\zeta - \lambda} d\zeta &\to m_{\pm ik,0}(x, y, \lambda), \quad \text{as } \epsilon \to 0.
\end{align*} \quad (3.63)
\]

Therefore 3.57 follows.

Moreover, writing

\[
C_\lambda E_{\pm ik} Tm = -\frac{i}{2\pi i} \int \frac{E_{\pm ik}(\zeta) e^{i4\zeta_R} e^{2\zeta_R \gamma}}{\zeta - \lambda} \left[ \pm \frac{i}{2} \frac{\text{sng} (\zeta_R)}{\zeta - \lambda} \tau (\zeta) + \text{sng} (\zeta_R) h_{\pm ik}(\zeta) \right] \times (m_{\pm ik,0}(x, y, -\zeta) + m_{\pm ik,0}(x, y, -\zeta)) d\zeta \wedge d\zeta
\]
\[
= I^\pm_1(x, y, \lambda) + I^\pm_2(x, y, \lambda),
\]

where

\[
I^\pm_1(x, y, \lambda) = C_\lambda \left( \pm \frac{i}{2} \frac{\text{sng} (\zeta_R)}{-\zeta + ik} \tau (\zeta) m_{\pm ik,0}(x, y, -\zeta) \right);
\]

\[
I^\pm_2(x, y, \lambda) = C_\lambda \left( \pm \frac{i}{2} \frac{\text{sng} (\zeta_R)}{-\zeta + ik} \tau (\zeta) m_{\pm ik,0}(x, y, -\zeta) \right);
\]
\[ I_2^\pm(x, y, \lambda) = C_\lambda \left( \frac{\pm \text{sgn}(\zeta) E_{\pm i\lambda}(\zeta)}{-\zeta + i} \right), \]

\[ F_{\pm}(x, y, \zeta) = \left[ e^{i(4\zeta x y + 2\zeta x^2)} - 1 \right] \mathbf{r}(\zeta) m_{\pm i\lambda, 0}(x, y, -\overline{\zeta}) \]

\[ + e^{i(4\zeta x y + 2\zeta x^2)} \mathbf{t}(\lambda) m_{\pm i\lambda, r}(x, y, -\overline{\zeta}) \]

\[ + e^{i(4\zeta x y + 2\zeta x^2)} \frac{2}{\lambda (-\zeta + i)} h_{\pm i\lambda}(\zeta) m(x, y, -\overline{\zeta}), \quad (3.64) \]

Using \[ \frac{m_{\pm i\lambda, r}(x, y, -\overline{\zeta}) - m_{\pm i\lambda, r}(x, y, +i\lambda)}{-\zeta + i} \]

and via (3.48), (3.57), we conclude

\[ |E_{\pm i\lambda} CE_{\pm i\lambda} T m|_{L^\infty} \leq C(1 + |x| + |y|), \]

\[ |1 - E_{\pm i\lambda} CE_{\pm i\lambda} T m|_{L^\infty} \leq C \]

which yield (3.58) and (3.59).

Lemma 3.5. Let \( E_t, D_t \) be defined by Definition 3.7. Then

\[ -\frac{1}{2\pi i} \int \frac{\text{sgn}(\zeta) E_t(\zeta)}{(\zeta - \lambda)(-\zeta - i)} d\zeta \wedge d\zeta \in L^\infty(D_t) \]

which vanishes at \( t = 2ik \). Consequently,

\[ |CE_t T m|_{L^\infty} \leq C(1 + |x| + |y|), \]

\[ CE_t T m \to 0 \quad \text{uniformly as } |\lambda| \to \infty. \]

Proof. By using the polar coordinates \( \zeta = \iota + se^{i\beta} \in D^\iota \),

\[ \frac{\lambda}{2\pi i} \int \frac{\text{sgn}(\zeta) E_t(\zeta)}{(\zeta - \lambda)(-\zeta - i)} d\zeta \wedge d\zeta \]

\[ = \frac{2}{\pi} \int_0^1 d\beta \int_0^\pi \text{sgn}(\zeta) \frac{1}{(\xi e^{i\beta} - \lambda)(s - (\lambda - i)e^{-i\beta})} s ds \]

\[ = \frac{2}{\pi} \int_0^1 d\beta \int_0^\pi \text{sgn}(\zeta) \frac{1}{(\xi - \lambda)(s - (\lambda - i)e^{-i\beta})} ds, \]

which is a composition of the Hilbert transform. Therefore the Hölder continuity and vanishing at \( t \) can be proved and (3.66) follows. Writing

\[ CE_t T m = -\frac{1}{2\pi i} \int E_t(\zeta) e^{i(\zeta x y + 2\zeta x^2)} \frac{\text{sgn}(\zeta) E_t(\zeta) + \text{sgn}(\zeta) h_t(\zeta)}{\zeta - \lambda} \]

\[ \times m_{\text{res}}(x, y) \theta_{\text{res}}(x, y, -\overline{\zeta}) d\zeta \wedge d\zeta \]

\[ + I_1^t(x, y, \lambda) + I_2^t(x, y, \lambda), \]

\[ I_1^t(x, y, \lambda) = C_\lambda \left( \frac{\text{sgn}(\zeta) E_t(\zeta) m_{\text{res}}(x, y)}{-\zeta + i} \right), \]

\[ I_2^t(x, y, \lambda) = C_\lambda \left( \frac{\text{sgn}(\zeta) E_t(\zeta) F(x, y, \zeta)}{-\zeta + i} \right), \]

\[ F(x, y, \zeta) = \left[ e^{i(4\zeta x y + 2\zeta x^2)} - 1 \right] \text{cm}_{\text{res}}(x, y) \]

\[ + e^{i(4\zeta x y + 2\zeta x^2)} \frac{2}{\lambda (-\zeta + i)} h_t(\zeta) m(x, y, -\overline{\zeta}), \]

\[ + e^{i(4\zeta x y + 2\zeta x^2)} \frac{2}{\lambda (-\zeta + i)} h_t(\zeta) m(x, y, -\overline{\zeta}). \]
Via (3.47) and (3.66), we conclude
\[
|E_i C E_i T m|_{L^\infty} \leq C(1 + |x| + |y|),
\]
\[
|(1 - E_i) C E_i T m|_{L^\infty} \leq C
\]
which yield (3.67) and (3.68).

\[\blacksquare\]

**Lemma 3.6.** Suppose \((1 + |x| + |y|)^2 \partial^j_x \partial^k_y v_0 \in L^1, 0 \leq j, k \leq 2, |v|_{L^1 \cap L^\infty} \ll 1\). Let \(E, E_{x,a}, t, \delta\) be defined by Definition 3.1.

\[
|C [1 - \sum_{z \in E} E_z] T m|_{L^\infty} \leq C,
\]
and
\[
C [1 - \sum_{z \in E} E_z] T m(x, y, \lambda) \to 0 \text{ uniformly as } |\lambda| \to \infty, \lambda_R \neq 0.
\]

**Proof.** Via a change of variables
\[
2\pi \xi = \zeta + \bar{\zeta}, \quad 2\pi i \eta = \zeta^2 - \bar{\zeta}^2,
\]
\[
\zeta = \pi \xi + i \frac{\eta}{2},
\]
\[
d\zeta \wedge d\zeta = \frac{i}{|\xi|} d\xi d\eta,
\]
and from (3.38), [21, Lemma 2.II], [22, Lemma 2.II]
\[
p_\lambda(\xi, \eta) = (2\pi \xi)^2 - 4\pi \xi \lambda + 2\pi i \eta,
\]
\[
\Omega_\lambda = \{ (\xi, \eta) \in \mathbb{R}^2 : |p_\lambda(\xi, \eta)| < 1 \},
\]
\[
\left| \frac{1}{p_\lambda} \right|_{L^1(\Omega_\lambda, d\xi d\eta)} \leq \frac{C}{(1 + |\lambda_R|^2)^{1/2}}, \quad \left| \frac{1}{p_\lambda} \right|_{L^2(\Omega_\lambda, d\xi d\eta)} \leq \frac{C}{(1 + |\lambda_R|^2)^{1/2}},
\]
we obtain
\[
|C[1 - \sum_{z \in E} E_z T m]| \leq C|1 - \sum_{z \in E} E_z|m|_{L^\infty} \int \frac{[1 - E_{+i\lambda, 1/2}(\zeta) - E_{-i\lambda, 1/2}(\zeta)] \xi_c(\zeta)}{|\zeta - \lambda|^2} d\zeta \wedge d\zeta \leq C|1 - \sum_{z \in E} E_z|m|_{L^\infty} \int \frac{[1 - E_{+i\lambda, 1/2}(\zeta) - E_{-i\lambda, 1/2}(\zeta)] \xi_c(\zeta)}{|(2\pi)^2 - 4\pi \lambda + 2\pi i \eta|^2} d\zeta d\eta \leq C|1 - \sum_{z \in E} E_z|m|_{L^\infty} \times \left\{ \left| 1 - E_{+i\lambda, 1/2}(\zeta) - E_{-i\lambda, 1/2}(\zeta) \right| \xi_c(\zeta) \right\}_{L^2(\Omega_\lambda, d\xi d\eta)} \left| \frac{1}{p_\lambda} \right|_{L^2(\Omega_\lambda, d\xi d\eta)} \right\}.
\]
Therefore we obtain (3.72) and (3.73).
3.5 The Cauchy integral equation

Theorem 5. If

\[ u_0(x) = -2\kappa^2 \text{sech}^2 \kappa x, \quad \kappa > 0, \]
\[ (1 + |x| + |y|)^2 \partial_y^j \partial_x^k v_0 \in L^1 \cap L^\infty, \quad 0 \leq j, k \leq 4, \]
\[ |v_0|_{L^1 \cap L^\infty} \ll 1, \quad v_0(x, y) \in \mathbb{R}, \]

then the eigenfunction \( m \) derived from Theorem 4 satisfies

\[ m(x, y, \lambda) \in W \quad \text{and the Cauchy integral equation} \]

\[ m(x, y, \lambda) = 1 + \frac{m_{\text{res}}(x, y)}{\lambda - \iota} + CTm, \quad \forall \lambda \neq \iota, \quad (3.77) \]

In particular, the residue \( m_{\text{res}}(x, y) \) at \( \lambda = \iota \) and leading singularities \( m_{\pm i\kappa,0} \) at \( \pm i\kappa \) satisfy the constraints

\[ m_{\text{res}}(x, y) - i\kappa = -1 + m_{+i\kappa,0}(x, y, +i\kappa + 0^+ e^{i\alpha}) - C_{+i\kappa+0^+ e^{i\alpha}} Tm, \]
\[ m_{-3i\kappa}(x, y) = -1 + m_{-i\kappa,0}(x, y, -i\kappa + 0^+ e^{i\alpha}) - C_{-i\kappa+0^+ e^{i\alpha}} Tm, \]
\[ m_{+i\kappa,0}(x, y, +i\kappa + 0^+ e^{i\alpha}) = \mathfrak{s}_d e^{-2\kappa x} m_{-i\kappa,0}(x, y, -i\kappa + 0^+ e^{i(\pi+\alpha)}) \]

for \( \forall -\frac{\pi}{2} < \alpha < \frac{3\pi}{2} \), with \( W, T, \mathfrak{s}_d \), and \( C \) defined by Definition 3.2 and 3.3.

Proof. Theorem 4 implies

\[ m(x, y, \lambda) - \frac{m_{\text{res}}(x, y)}{\lambda - \iota} \in L^\infty, \quad (3.79) \]
\[ E_{0,n} Tm(x, y, \lambda) \in L^1(d\lambda \wedge d\lambda), \quad (3.80) \]

for \( \forall n > 0 \). Here \( E_{z,a} \) is defined by Definition 3.1. Exploiting (3.80) and applying [20, §1, Theorem 1.13, Theorem 1.14], one derives

\[ \partial_\lambda C E_{0,n} Tm(x, y, \lambda) = E_{0,n} Tm(x, y, \lambda) \in L^1(d\lambda \wedge d\lambda). \quad (3.81) \]

Therefore, together with Theorem 4

\[ \partial_\lambda \left[ m(x, y, \lambda) - \frac{m_{\text{res}}(x, y)}{\lambda - \iota} - CTm(x, y, \lambda) \right] = 0. \quad (3.82) \]

On the other hand, Lemma 3.4, 3.5, and 3.6 imply

\[ |CTm| \leq C(1 + |x| + |y|), \quad (3.83) \]
\[ CTm(x, y, \lambda) \to 0 \quad \text{uniformly as } |\lambda| \to \infty, \quad \lambda_R \neq 0. \quad (3.84) \]

Applying (3.79), (3.82), (3.83), and Liouville’s theorem, one concludes

\[ m(x, y, \lambda) = g(x, y) + \frac{m_{\text{res}}(x, y)}{\lambda - \iota} + CTm(x, y, \lambda). \quad (3.85) \]
Equation (3.46) and a direct computation yield:

\[- u(x, y)m(x, y, \lambda) = (\partial_y - \partial_x^2 + 2i\lambda \partial_x)m(x, y, \lambda) + (\partial_y - \partial_x^2 + 2i\lambda \partial_x)C \varphi m. \tag{3.86}\]

Note that

\[
\begin{align*}
\partial_x C \varphi m &= C[(\lambda + \bar{\lambda})Tm + T(\partial_x m)], \\
\partial_x^2 C \varphi m &= C[-(\lambda + \bar{\lambda})^2Tm + 2i(\lambda + \bar{\lambda})T(\partial_x m) + T(\partial_x^2 m)], \\
\partial_y C \varphi m &= C[(\lambda^2 - \bar{\lambda}^2)Tm + T(\partial_y m)].
\end{align*}
\]

Applying the Fourier transform theory, namely, (3.42) and (3.43), if \(v(x, y)\) has 4 derivatives in \(L^1 \cap L^\infty\), then

\[
(1 - E_{+i\kappa,1/2}Tm + T(\partial_x m)), (1 - E_{-i\kappa,1/2}Tm + T(\partial_x m)), (1 - E_{+i\kappa,1/2}Tm + T(\partial_y m)), (1 - E_{-i\kappa,1/2}Tm + T(\partial_y m))\]

are all bounded in \(L^\infty \cap L^2(|\lambda| R \wedge d\lambda)\). Therefore by (3.75), if \(\partial_y \partial_x^2 v \in L^1 \cap L^\infty\), \(0 \leq j, k \leq 4\), one can adapt the proof for (3.84) and derive, as \(|\lambda| \to \infty, \lambda R \neq 0\),

\[
(\partial_y - \partial_x^2 + 2i\lambda \partial_x)C \varphi m \to 0.
\]

So comparing growth in (3.86), we conclude

\[
\partial_x g(x, y) = 0
\]

which turns (3.85) into

\[
m(x, y, \lambda) - 1 = g(y) - 1 + \frac{m_{\text{res}}(x, y)}{\lambda - i} + C \varphi m(x, y, \lambda). \tag{3.87}
\]

Fix \(y\), and let \(\epsilon > 0\) be given. Let \(\lambda \gg 1, \lambda R \neq 0\), be chosen such that

\[
|\frac{m_{\text{res}}(x, y)}{\lambda - i} + C \varphi m(x, y, \lambda)| < \frac{\epsilon}{2}
\]

by (3.84). For this \(\lambda\), by taking \(x \to \infty\), and using the boundary property (2.22), we justify \(g \equiv 1\) and establish (3.77).
One direct corollary from (3.77) is the uniform estimate

\[ |CTm|_{L^\infty} \leq C. \] (3.88)

**Example 3.3.** If \( v_0(x,y) \equiv 0 \), then \( \gamma_+ \equiv 0, \ s_c \equiv 0 \). So (3.77) and (3.78) reduce to

\[
m(x, y, \lambda) = 1 + \frac{m_{res}(x,y)}{\lambda - \iota},
\]

(3.89)

\[
\frac{m_{res}(x,y)}{-\iota \kappa} = -1 + m_{+i\kappa,0}(x,y),
\]

(3.90)

\[
\frac{m_{res}(x,y)}{-3i\kappa} = -1 + m_{-i\kappa,0}(x,y),
\]

(3.91)

\[
m_{+i\kappa,0}(x,y) = s_d e^{-2\kappa x} m_{-i\kappa,0}(x,y)
\]

(3.92)

which yield

\[
m_{-i\kappa,0}(x,y) = \frac{2}{1 + e^{-2\kappa x}}, \quad m_{+i\kappa,0}(x,y) = \frac{-2}{1 + e^{2\kappa x}},
\]

(3.93)

\[
m(x, y, \lambda) = 1 + \frac{3i\kappa}{\lambda - 2i\kappa}(1 - \frac{2}{1 + e^{-2\kappa x}}).
\]

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