Numerical analysis of a singularly perturbed convection diffusion problem with shift in space

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Abstract

We consider a singularly perturbed convection-diffusion problem that has in addition a shift term. We show a solution decomposition using asymptotic expansions and a stability result. Based upon this we provide a numerical analysis of high order finite element method on layer adapted meshes. We also apply a new idea of using a coarser mesh in places where weak layers appear. Numerical experiments confirm our theoretical results.

1 Introduction

In this paper we want to look at the static singularly perturbed problem given by

\[ -\varepsilon u''(x) - b(x)u'(x) + c(x)u(x) + d(x)u(x-1) = f(x), \quad x \in \Omega := (0, 2), \]
\[ u(2) = 0, \quad (1b) \]
\[ u(x) = \Phi(x), \quad x \in (-1, 0], \quad (1c) \]

where \( 0 < \varepsilon \ll 1, \quad b \geq \beta > 0, \quad d \geq 0, \quad c - \frac{b'}{2} \geq \frac{\|d\|_{L^\infty(1,2)}}{2} \geq \gamma > 0. \) For the function \( \Phi \) we assume \( \Phi(0) = 0, \) which is not a restriction as a simple transformation can always ensure this condition. Then, it holds \( u \in U := H^1_0(\Omega). \)

The literature on singularly perturbed problems is vast, see e.g. the book [13] and the references therein. But for problems that in addition also have a shift-operator, sometimes also called a delay-operator, with a large shift, there are not many. For the time-dependent

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case and a reaction-diffusion type problem there are e.g. [3, 5, 7]. Recently, we also investigated the time dependent version of a singularly perturbed reaction-diffusion problem in [2] using a finite element method in time and space.

For convection dominated singularly perturbed problems with an additional shift there are also publications in literature, see e.g. [9, 14, 15]. They all consider a negative coefficient, here called $d$, which supports a maximum principle. Then finite differences on layer adapted meshes of rather low order are used. In our paper we consider finite element methods of arbitrary order for positive coefficients $d$. Standard convection-diffusion problems with a fixed convection coefficient show one boundary layer near the outflow boundary in contrast to two boundary layers for the reaction diffusion problem. Therefore, we expect the behaviour of the problem with a shift also to have some different structure that those of reaction-diffusion type.

In e.g. [14] an asymptotic expansion of the solution is given, where the direction of shift and convection is opposite, but only to the lowest order. For the purpose of this paper we want a complete solution decomposition. Therefore, in Section 2 we provide a solution decomposition of $u$ into various layers and a smooth part using a different approach. We prove it rigorously for the constant coefficient case. In Section 3 a numerical analysis is provided for the discretisation using finite elements of arbitrary order on a classical S-type mesh and a new coarser type of mesh. Finally Section 4 provides some numerical results supporting our analysis. We finish this paper with a technical abstract on some terms involving Green’s function.

**Notation:** For a set $D$, we use the notation $\| \cdot \|_{L^p(D)}$ for the $L^p$-norm over $D$, where $p \geq 1$. The standard scalar product in $L_2(D)$ is marked with $\langle \cdot, \cdot \rangle_D$. If $D = \Omega$ we sometimes drop the $\Omega$ from the notation. Throughout the paper, we will write $A \lesssim B$ if there exists a generic positive constant $C$ independent of the perturbation parameter $\varepsilon$ and the mesh, such that $A \leq CB$. We will also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

## 2 Solution decomposition

The considered problem with a shift term has some different properties compared to a convection-diffusion problem without the shift. One of the major ones is, that it is unknown whether a maximum principle holds for $d \geq 0$. In the case $d \leq 0$ a maximum principle is proved in e.g. [14], but the proof cannot be applied here. For the following solution decomposition we will need a stability result that is provided in the next theorem for the case of constant coefficients and assumed to hold true in the general case of variable coefficient.

**Theorem 2.1.** Consider the problem: Find $u = u_1 \chi_{(0,1)} + u_2 \chi_{(1,2)}$, where $\chi_D$ is the characteristic function of $D$ such that it holds

\[-\varepsilon u_1'(x) - bu_1'(x) + cu_1(x) = f(x), \quad x \in (0,1),
\]

\[u_1(0) = 0, \quad u_1(1) = \alpha,
\]

\[-\varepsilon u_2''(x) - bu_2''(x) + cu_2(x) = f(x) - du_1(x - 1), \quad x \in (1,2),
\]

\[u_2(2) = \beta, \quad u_2(1) = \alpha + \delta\]
with constant $b > 0$ and $c > 0$, arbitrary boundary value $\beta \in \mathbb{R}$ and jump $\delta \in \mathbb{R}$, and $\alpha \in \mathbb{R}$ chosen, such that $u_1'(1^-) = u_2'(1^+)$. Then we have

$$\|u\|_{L^\infty} \lesssim \|f\|_{L^\infty(0,2)} + |\beta| + |\delta|.$$  

Proof. Each of these two sub-problems is a standard convection diffusion problem with a known Green’s function $G$ for the case of homogeneous boundary conditions on $(0,1)$ that can be constructed as shown e.g. in [8, Chapter 1.1]. Thus, we can, using $\hat{b}(t) := b - ct$ and $\hat{c}(t) := c + dt$, represent the solutions as

$$u_1(x) = \alpha x + \int_0^1 G(x,t) \left( f(t) + \alpha \hat{b}(t) \right) dt,$$

$$u_2(x) = (\alpha + \delta)(2 - x) + \beta(x - 1)
+ \int_0^1 G(x - 1,t) \left( f(t + 1) - (\alpha + \delta)(\hat{b}(t) + \hat{c}(t)) + \beta \hat{b}(t) \right)
- d \int_0^1 G(t,s) \left( f(s) + \alpha \hat{b}(s) \right) ds dt,$$

where in the second case we have used, that $G(x - 1, t - 1)$ is the Green’s function for the problem on $(1,2)$. The condition for $\alpha$ can now be written as

$$\alpha = \frac{N}{D}$$

where

$$N := \beta - \delta + \int_0^1 G_x(0, t) \left( f(t + 1) - \delta \hat{c}(t) + (\beta - \delta) \hat{b}(t) - d \int_0^1 G(t, s) f(s) ds \right) dt
- \int_0^1 G_x(1, t) f(t) dt$$

and

$$D := 2 + \int_0^1 G_x(1, t) \hat{b}(t) dt + \int_0^1 G_x(0, t) \left( \hat{b}(t) + \hat{c}(t) + d \int_0^1 G(t, s) \hat{b}(s) ds \right) dt.$$  

For the given problem we can compute all relevant information concerning the Green’s function, see Appendix A, and can estimate

$$|N| \lesssim |\beta| + |\delta| + \left( |\beta| + |\delta| + \|f\|_{L^\infty(0,2)} \right) \cdot \frac{1}{\varepsilon} \quad \text{and} \quad D \gtrsim \frac{1}{\varepsilon}.$$  

Therefore, we have

$$|\alpha| \lesssim |\beta| + |\delta| + \|f\|_{L^\infty(0,2)}.$$  

Now using the representations of $u_1$ and $u_2$ and

$$\|G(x, \cdot)\|_{L^1(0,1)} \lesssim 1, \quad \text{for all } x \in (0,1)$$
we have proved the assertion.
Contrary to the reaction-diffusion case, where in addition to the boundary layers a strong inner layer forms, see [2], the convection-diffusion case has only a strong boundary layer at the outflow boundary and a weak inner layer.

**Theorem 2.2.** Let \( k \geq 0 \) be a given integer and the data of (1) smooth enough. Then it holds
\[
u = S + E + W,
\]
where for any \( \ell \in \{0, 1, \ldots, k\} \) it holds
\[
\|S^{(\ell)}\|_{L_2(0,1)} + \|S^{(\ell)}\|_{L_2(1,2)} \lesssim 1, \quad |E^{(\ell)}(x)| \lesssim \epsilon^{-\ell} e^{-\beta x}, \quad x \in [0, 2],
\]
\[
|W^{(\ell)}(x)| \lesssim \begin{cases} 0, & x \in (0, 1), \\ \epsilon^{1-\ell} e^{-\beta(x-1)}, & x \in (1, 2). \end{cases}
\]

**Proof.** We prove this theorem using asymptotic expansions. For simplicity we assume \( b, c \) and \( d \) to be constant. Adjusting the proof for variable smooth coefficients is straightforward using Taylor expansions and assuming Theorem 2.1 to hold true for variable coefficients.

We start by writing the problem using \( u_1 \) and \( u_2 \) as the solution on \((0, 1)\) and \((1, 2)\) resp.
\[
-bu_1''(x) + cu_1(x) = f(x) - d\Phi(x - 1), \quad x \in (0, 1),
\]
\[
-bu_2''(x) - bu_2'(x) + cu_2(x) = f(x) - du_1(x - 1), \quad x \in (1, 2),
\]
\[
u_1(0) = 0, \quad u_1(1) = u_2(1), \quad u_1'(1) = u_2'(1), \quad u_2(2) = 0.
\]

Let \( \sum_{i=0}^k \epsilon^i (S_{i-}\chi_{[0,1]} + S_{i+}\chi_{[1,2]}) \) be the outer expansion and by substituting this into the differential system we obtain
\[
\sum_{i=0}^k \epsilon^i (-\epsilon S_{i-}''(x) - bS_{i-}'(x) + cS_{i-}(x)) = f(x) - d\Phi(x - 1), \quad x \in (0, 1),
\]
\[
\sum_{i=0}^k \epsilon^i (-\epsilon S_{i+}''(x) - bS_{i+}'(x) + cS_{i+}(x)) = f(x) - d \sum_{i=0}^k \epsilon^i S_{i-}(x - 1), \quad x \in (1, 2)
\]
plus boundary conditions and continuity conditions. For the coefficient of \( \epsilon^0 \) (including some of the additional conditions) we obtain
\[
-bS_{0-}'(x) + cS_{0-}(x) = f(x) - d\Phi(x - 1), \quad x \in (0, 1), \quad S_{0-}(1) = S_{0+}(1),
\]
\[
-bS_{0+}'(x) + cS_{0+}(x) = f(x) - dS_{0-}(x - 1), \quad x \in (1, 2), \quad S_{0+}(2) = 0.
\]

According to Lemma 2.3, after mapping the second line to \((0, 1)\), there exists a solution \( S_0 = S_{0-}\chi_{[0,1]} + S_{0+}\chi_{[1,2]} \), that is continuous and \( S_0(2) = 0 \), but \( S_0(0) \neq 0 \). Thus, we correct this with a boundary correction using the stretched variable \( \xi = \frac{x}{\epsilon} \) and \( \sum_{i=0}^k \epsilon^i \tilde{E}_i(\xi) \).

Substituting this into the differential equation yields
\[
\sum_{i=0}^k \epsilon^i \left( -\epsilon^{-1}(\tilde{E}_i''(\xi) + b\tilde{E}_i'(\xi) + c\tilde{E}_i(\xi)) + d\tilde{E}_i \left( \xi - \frac{1}{\epsilon} \right) \chi(\frac{\xi}{\epsilon^2}) \right) = 0.
\]
We deal with the shift term later and obtain for the coefficient of $\varepsilon^{-1}$ the boundary correction problem

$$E_0''(\xi) + bE_0'(\xi) = 0, \quad E_0(0) = -S_0(0), \quad \lim_{\xi \to \infty} E_0(\xi) = 0 \Rightarrow \tilde{E}_0(\xi) = -S_0(0)e^{-b\xi}.$$  

Furthermore, we correct the jump of the derivative of $S_0$ at $x = 1$ with an inner expansion and the variable $\eta = \frac{x - 1}{\varepsilon}$. Using $\sum_{i=1}^{k+1} \varepsilon^i \tilde{W}_i(\eta)$ we have

$$\sum_{i=1}^{k+1} \varepsilon^i \left( -\varepsilon^{-1}(\tilde{W}_i''(\eta) + b\tilde{W}_i'(\eta)) + c\tilde{W}_i(\eta) \right) = 0.$$  

For the coefficient of $\varepsilon^0$ and initial conditions at $\eta = 0$ it follows

$$\tilde{W}_1''(\eta) + b\tilde{W}_1'(\eta) = d\tilde{E}_0(\eta), \quad \tilde{W}_1'(0) = -[S_0'(0)], \quad \lim_{\eta \to \infty} \tilde{W}_1(\eta) = 0.$$  

Here we included the shift of $\tilde{E}_0$ into the differential equation. We obtain a solution

$$\tilde{W}_1(\eta) = \tilde{Q}_1(\eta)e^{-bn}$$

where $\tilde{Q}_1$ a polynomial of degree 1. Thus far we have

$$u_0 = S_0 + E_0 + \varepsilon W_1\chi_{[1,2]}, \quad u_0(0) = 0, \quad |u_0(2)| \lesssim e^{-\frac{b}{2}}$$

and in addition

$$[u_0'(1)] = 0, \quad [u_0(1)] = W_1(1) \lesssim \varepsilon.$$  

Thus we have corrected the jump in the derivative, but introduced a jump in the function value of order $\varepsilon$. In order to correct this jump we continue with the same steps, now for the coefficients of $\varepsilon^i$ for $i > 0$. We obtain the problems

$$-bS''_i(x) + cS_i(x) = S''_{i-1}(x), \quad x \in (0, 1), \quad S_i(1) = S_{i+1}(1) - W_i(1)$$

$$-bS''_i(x) + cS_i(x) = S''_{i-1}(x) - dS_i(x-1), \quad x \in (1, 2), \quad S_i(2) = 0$$

$$\begin{align*}
\text{Lemma} \rightarrow S_i &= S_{i-\chi_{[0,1]}} + S_{i+\chi_{[1,2]}}.
\end{align*}$$

and

$$E_i''(\xi) + bE_i'(\xi) = cE_{i-1}(\xi), \quad E_i(0) = -S_i(0), \quad \lim_{\xi \to \infty} E_i(\xi) = 0$$

$$\Rightarrow \quad E_i(\xi) = \tilde{P}_i(\xi)e^{-b\xi},$$

$$\tilde{W}_{i+1}''(\eta) + b\tilde{W}_{i+1}'(\eta) = c\tilde{W}_i(\eta) + dE_i(\eta), \quad \tilde{W}_{i+1}'(0) = -[S_i'(0)], \quad \lim_{\eta \to \infty} \tilde{W}_{i+1}(\eta) = 0$$

$$\Rightarrow \quad \tilde{W}_{i+1}(\eta) = \tilde{Q}_{i+1}(\eta)e^{-bn},$$

where $\tilde{P}_i$ and $\tilde{Q}_{i+1}$ are polynomials of degree $i$ and $i + 1$, resp. The following Figure 1 shows in a diagram the dependence of the problems. Dotted lines represent influence on
boundary values, while solid ones are via the differential equation.

Thus, for the expansion

\[ u_k := \sum_{i=0}^{k} \varepsilon^i S_i(x) + \sum_{i=0}^{k} \varepsilon^i P_i \left( \frac{x}{\varepsilon} \right) e^{-\frac{bk}{\varepsilon}} + \sum_{i=1}^{k+1} \varepsilon^i Q_i \left( \frac{x-1}{\varepsilon} \right) e^{-\frac{b(x-1)}{\varepsilon}} \chi_{[1,2]}(x) \]

we have

\[ [u_k(1)] =: \delta, \quad [u_k'(1)] = 0, \quad u_k(0) = 0, \quad u_k(2) =: \beta, \]

where

\[ |\delta| \lesssim \varepsilon^k \quad \text{and} \quad |\beta| \lesssim e^{-\frac{b}{\varepsilon}}, \]

and for the remainder \( R := u_k - u \) follows the same. Finally, it holds

\[ -\varepsilon R'' - bR' + cR = \varepsilon^k (S_k'' + c\varepsilon_k), \quad \text{in} \ (0,1) \]

\[ -\varepsilon R'' - bR' + cR = \varepsilon^k (S_k'' + c\varepsilon_k + cw_{k+1}) - dR(-1), \quad \text{in} \ (1,2) \]

Using the stability result of Theorem 2.1 we obtain

\[ \|R\|_{L^\infty} \lesssim \varepsilon^k \]

and we can set

\[ S := \tilde{S} + R. \]

**Lemma 2.3.** The ordinary differential system

\[-V'(x) + c_1(x)V(x) = g_1(x), \quad x \in (0,1), \quad V(1) = W(0) + \alpha, \]

\[-W'(x) + c_2(x)W(x) + d(x)V(x) = g_2(x), \quad x \in (0,1), \quad W(1) = 0 \]

has for positive \( d \) and any \( c_1, c_2, g_1, g_2, \alpha \) a unique solution.

**Proof.** For \( x \in (0,1) \) the system can be written as

\[
\begin{pmatrix} V' \\ W' \end{pmatrix} (x) = \begin{pmatrix} c_1(x) & 0 \\ d(x) & c_2(x) \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} (x) - \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \quad \begin{pmatrix} V \\ W \end{pmatrix} (1) = \begin{pmatrix} W(0) + \alpha \\ 0 \end{pmatrix}
\]
or short
\[
\begin{pmatrix} V \\ W \end{pmatrix}' = A \begin{pmatrix} V \\ W \end{pmatrix} - g, \quad \begin{pmatrix} V \\ W \end{pmatrix}(1) = \begin{pmatrix} W(0) + \alpha \\ 0 \end{pmatrix}.
\]

With \( B(x) = \int_0^x A(y)dy \) and the matrix exponential, the solution can be represented as
\[
\begin{pmatrix} V \\ W \end{pmatrix}(x) = e^{B(x)} \left( e^{-B(1)} \begin{pmatrix} W(0) + \alpha \\ 0 \end{pmatrix} + \int_0^1 e^{-B(y)} g(y)dy \right).
\]

Now, this solution is still recursively defined. In order to investigate this further, let
\[
B(1) = \begin{pmatrix} C_1 & 0 \\ D & C_2 \end{pmatrix}, \quad D := \int_0^1 d(x)dx, \quad C_i = \int_0^1 c_i(d)dx, \quad i \in \{1, 2\},
\]
\( \tilde{b}_{21} \) be the 2,1-component of \( e^{-B(1)} \) and \( T_2 \) the second component of \( T \). Then we have
\[
W(0) = \tilde{b}_{21} (W(0) + \alpha) + T_2(0) \quad \Rightarrow \quad W(0) = \frac{T_2(0) + \tilde{b}_{21} \alpha}{1 - \tilde{b}_{21}},
\]
if \( \tilde{b}_{21} \neq 1 \). Due to the assumption \( d > 0 \) we have \( D > 0 \) and therefore
\[
\tilde{b}_{21} := \begin{cases} -De^{-C_1}, & C_1 = C_2, \\ De^{-C_1}e^{-c_2} \frac{C_1}{C_2}, & C_1 \neq C_2, \end{cases}
\]
is always negative and thus not 1.

**Remark 2.4.** The condition \( d > 0 \) is sufficient, but not necessary. But some condition is needed, as can be seen by the example \( c_1 = c_2 = 1, d = -e, \alpha = 0 \) and for example \( g_1(x) = g_2(x) = 1 \) for which no solution \((V,W)\) exists:
\[
V(x) = 1 + (W(0) - 1)e^{-1}, \quad W(x) = e^x \cdot ((1 - x)W(0) + x - 2 - e^{-1}) + e + 1
\]
fulfils the system and the conditions \( W(1) = 0, V(1) = W(0) \), but
\[
W(0) = W(0) - 1 + e - e^{-1}
\]
is not defined.

**Remark 2.5.** The related problem
\[
-\varepsilon u''(x) - b(x)u'(x) + c(x)u(x) + d(x)u(x + 1) = f(x), \quad x \in \Omega := (0, 2),
\]
\[
u(0) = 0,
\]
\[
u(x) = \Phi(x), \quad x \in [2, 3),
\]
where the directions of shift and convection are opposing, can be analysed quite similarly, yielding the same solution decomposition as Theorem 2.2. Here the reduced problems are always solvable, independent of \( d \), but the problems for the boundary correction have to be split into the two subregions.
3 Numerical analysis

3.1 Preliminaries

Using standard $L^2$-products and integration by parts we define our bilinear and linear form by

$$B(u, v) := \varepsilon \langle u', v' \rangle_{\Omega} + \langle cu - bu', v \rangle_{\Omega} + \langle du(\cdot - 1), v \rangle_{(1, 2)}$$

$$= \langle f, v \rangle_{\Omega} - \langle d\phi(\cdot - 1), v \rangle_{(0, 1)} =: F(v)$$

for $u, v \in U$. With

$$- \langle bu', u \rangle_{\Omega} = \langle b'u, u \rangle_{\Omega} + \langle bu', u \rangle_{\Omega}$$

and

$$\langle du(\cdot - 1), u \rangle_{(1, 2)} \leq \frac{\|d\|_{L^\infty(1, 2)}^2}{2} \left( \|u\|_{L^2(0, 1)}^2 + \|u\|_{L^2(1, 2)}^2 \right) = \frac{\|d\|_{L^\infty(1, 2)}^2}{2} \|u\|_{L^2}^2$$

we have coercivity w.r.t. the energy norm $\|\cdot\|$.

$$B(u, u) = \varepsilon \|u'\|^2_{L^2} + \langle cu - bu', u \rangle_{\Omega} + \langle du(\cdot - 1), u \rangle_{(1, 2)}$$

$$\geq \varepsilon \|u'\|^2_{L^2} + \left( c - \frac{b'}{2} \right) \langle u, u \rangle_{\Omega} - \frac{\|d\|_{L^\infty(1, 2)}^2}{2} \|u\|_{L^2}^2$$

$$\geq \varepsilon \|u'\|^2_{L^2} + \gamma \|u\|_{L^2}^2 =: |||u|||^2$$

due to our assumptions on the data.

3.2 On standard S-type meshes

For the construction of an S-type mesh, see [12], let us assume the number of cells $N$ to be divisible by 4. Next we define a mesh transition value

$$\lambda = \frac{\sigma \varepsilon}{\beta} \ln(N),$$

with a constant $\sigma$ to be specified later. In order to have an actual layer we assume $\varepsilon$ to be small enough. To be more precise, we assume

$$\frac{\sigma \varepsilon}{\beta} \ln(N) \leq \frac{1}{2}$$

such that $\lambda \leq 1/2$ follows.

Then using a monotonically increasing mesh defining function $\phi$ with $\phi(0) = 0$ and $\phi(1/2) = \ln(N)$, see [12] for the precise conditions on $\phi$, we construct the mesh nodes

$$x_i = \begin{cases} \frac{\sigma \varepsilon}{\beta} \phi \left( \frac{2i}{N} \right), & 0 \leq i \leq \frac{N}{4}, \\ \frac{4i}{N} (1 - \lambda) + 2\lambda - 1, & \frac{N}{4} \leq i \leq \frac{N}{2}, \\ 1 + x_{i-N/2}, & \frac{N}{2} \leq i \leq N. \end{cases}$$
Let us denote the smallest mesh-width inside the layers by \( h \), for which holds \( h \leq \varepsilon \). Associated with \( \phi \) is the mesh characterising function \( \psi = e^\phi \), that classifies the convergence quality of the meshes by the quantity \( \max_{t \in [0,1/2]} |\psi'(t)| \). Two of the most common S-type meshes are the Shishkin mesh with 
\[
\phi(t) = 2t \ln N, \quad \psi(t) = N^{-2t}, \quad \max |\psi'| = 2 \ln N
\]
and the Bakhvalov-S-mesh 
\[
\phi(t) = -\ln(1 - 2t(1 - N^{-1})), \quad \psi(t) = 1 - 2t(1 - N^{-1}), \quad \max |\psi'| = 2.
\]
By definition it holds 
\[
|E(\lambda)| \lesssim N^{-\sigma} \quad \text{and} \quad |W(1 + \lambda)| \lesssim \varepsilon N^{-\sigma}.
\]
As discrete space we use 
\[
U_N := \{ v \in H^1_0(\Omega) : v|_\tau \in P_k(\tau) \},
\]
where \( P_k(\tau) \) is the space of polynomials of degree \( k \) at most on a cell \( \tau \) of the mesh. Let \( I \) be the standard Lagrange-interpolation operator into \( U_N \) using equidistant points or any other suitable distribution of points. The derivation of the interpolation error can be done like for a standard convection-diffusion problem, see e.g. [13]. We therefore skip the proof.

**Lemma 3.1** (Interpolation error estimates). For \( \sigma \geq k + 1 \), \( u = S + E + W \) assuming the solution decomposition and the Lagrange interpolation operator \( I \) it holds
\[
\|u - Iu\|_{L^2(\Omega)} \lesssim (h + N^{-1}\max |\psi'|)^{k+1},
\]
\[
\|(u - Iu)'\|_{L^2(\Omega)} \lesssim \varepsilon^{-1/2}(h + N^{-1}\max |\psi'|)^k
\]
and additionally
\[
\|E - IE\|_{L^2((0,\lambda) \cup (1+\lambda,2))} \lesssim \varepsilon^{1/2}(N^{-1}\max |\psi'|)^{k+1},
\]
\[
\|E - IE\|_{L^2((\lambda,1) \cup (1+\lambda,2))} \lesssim N^{-(k+1)},
\]
\[
\|(W - IW)\|_{L^2(\Omega)} \lesssim \varepsilon^{1/2}(N^{-1}\max |\psi'|)^k.
\]
The numerical method is now given by: Find \( u_N \in U_N \) such that for all \( v \in U_N \) it holds
\[
B(u_N, v) = F(v). \tag{4}
\]
Obviously, we have immediately Galerkin orthogonality
\[
B(u - u_N, v) = 0 \quad \text{for all} \ v \in U_N.
\]
Now the convergence of our method is easily shown.
Theorem 3.2. For the solution $u$ of (1) and the numerical solution $u_N$ of (4) holds on an $S$-type mesh with $\sigma \geq k + 1$

\[ \| u - u_N \| \lesssim (h + N^{-1} \max |\psi'|)^k. \]

Proof. We start with a triangle inequality

\[ \| u - u_N \| \leq \| u - Iu \| + \| Iu - u_N \| \]

where the first term can be estimated by Lemma 3.1. Let $\chi := Iu - u_N \in U_N$ and $\psi := u - Iu$. Then coercivity and Galerkin orthogonality yield

\[ \| \chi \|^2 \lesssim B(\eta, \chi) = \varepsilon \langle \eta', \chi' \rangle_{(0,\lambda) \cup (1,1+\lambda)} + \langle c\eta - b\eta', \chi \rangle_{(1,2)}, \]

\[ \lesssim (h + N^{-1} \max |\psi'|)^k \| \chi \| + \langle b(E - IE), \chi' \rangle_{\Omega}, \]

where Cauchy-Schwarz inequalities and the interpolation error estimates were used for all but the convection term including the strong layer, where integration by parts was applied. For the remaining term we decompose the resulting scalar product into fine and coarse regions.

\[ | \langle b(E - IE), \chi' \rangle_{\Omega} | \leq | \langle b(E - IE), \chi' \rangle_{(0,\lambda) \cup (1,1+\lambda)} | + | \langle b(E - IE), \chi' \rangle_{(\lambda,1) \cup (1+\lambda,2)} | \]

\[ \lesssim \varepsilon^{1/2} (N^{-1} \max |\psi'|)^k \| \chi' \|_{L^2(0,\lambda) \cup (1,1+\lambda))} + N^{-(k+1)} \| \chi' \|_{L^2(\lambda,1) \cup (1+\lambda,2))} \]

\[ \lesssim (N^{-1} \max |\psi'|)^k \| \chi \| + N^{-k} \| \chi \|_{L^2(\lambda,1) \cup (1+\lambda,2))}, \]

where an inverse inequality was used. Combining the results finishes the proof.

Remark 3.3. We could have also used a different layer adapted mesh, like a Durán mesh, introduced in [4], modified to our problem. The proof of interpolation errors and finally convergence follows again the standard ideas.

3.3 On a coarser mesh

Let us consider a mesh, see also [11] where a similar mesh is used for weak layers, that resolves the weak layer not by an S-type mesh, but just by an even simpler equidistant mesh and a specially chosen transition point, while the strong layer is still resolved by an S-type. Thus let

\[ \lambda := \frac{\sigma \varepsilon}{\beta} \ln N \leq \frac{1}{2} \]

and

\[ \mu := \frac{\varepsilon^{k+1}}{\beta} \leq \frac{1}{2} \]

that still implies the weak condition

\[ \varepsilon \lesssim (\ln N)^{-1}. \]
Note that in the case $k = 1$ we set $\mu = \frac{1}{2}$. The by the same ideas as in the previous subsection we construct the mesh nodes

$$x_i = \begin{cases} \frac{\sigma \varepsilon}{\beta} \left( \frac{2i}{N} \right), & 0 \leq i \leq \frac{N}{4}, \\
\frac{4i}{N}(1 - \lambda) + 2\lambda - 1, & \frac{N}{4} \leq i \leq \frac{N}{2}, \\
1 + \mu \left( \frac{4i}{N} - 2 \right), & \frac{N}{2} \leq i \leq \frac{3N}{4}, \\
\frac{4i}{N}(1 - \mu) + 4\mu - 2, & \frac{3N}{4} \leq i \leq N. \end{cases}$$

Note that for $i \geq N/4$ it is always piecewise equidistant, independent of the choice of $\phi$. For the (minimal) mesh width in the different regions it holds

$$h_1 \lesssim \varepsilon, \ H_1 \sim N^{-1}, \ h_2 \sim N^{-1} \varepsilon^{\frac{k+1}{k}} \quad \text{and} \quad H_2 \sim N^{-1}.$$

The proof of the interpolation errors uses local interpolation error estimates, given on any cell $\tau_i$ with width $h_i$ and $1 \leq s \leq k + 1$ and $1 \leq t \leq k$ by

$$\|v - Iv\|_{L^2(\tau_i)} \lesssim h_i^s \|v^{(s)}\|_{L^2(\tau_i)}, \quad (5a)$$
$$\|(v - Iv)'\|_{L^2(\tau_i)} \lesssim h_i^t \|v^{(t+1)}\|_{L^2(\tau_i)}, \quad (5b)$$

for $v$ smooth enough. In principle it is similar to proving interpolation error estimates on S-type meshes but the different layout of the mesh makes some changes in the proof necessary.

**Lemma 3.4.** Let us assume $\sigma \geq k + 1$ and

$$e^{-\varepsilon^{-1/k}} \leq N^{1-k}. \quad (6)$$

Then it holds

$$\|u - Iv\|_{L^2(\Omega)} \lesssim (h_1 + N^{-1} \max |\psi'|)^{k+1/2}, \quad (7a)$$
$$\|u - Iv\| \lesssim (h_1 + N^{-1} \max |\psi'|)^k \quad (7b)$$

and more detailed

$$\|W - IW\|_{L^2(\Omega)} \lesssim \varepsilon^{1/2} N^{-k}, \quad (7c)$$
$$\|E - IE\|_{L^2((\lambda,1) \cup (1+\mu,2))} \lesssim N^{-(k+1)}, \quad (7d)$$
$$\|E - IE\|_{L^2((0,\lambda) \cup (1+\mu))} \lesssim \varepsilon^{1/2} (N^{-1} \max |\psi'|)^k. \quad (7e)$$

**Proof.** Using $(5a)$ and $(5b)$ with $s = k + 1$ and $t = k$, resp. we obtain

$$\|S - IS\|_{L^2(\Omega)} \lesssim (h_1 + H_1 + h_2 + H_2)^{k+1} \lesssim (h_1 + N^{-1})^{k+1},$$
$$\|(S - IS)'\|_{L^2(\Omega)} \lesssim (h_1 + H_1 + h_2 + H_2)^k \lesssim (h_1 + N^{-1})^k.$$
For \( E \) we can proceed as on a classical \( S \)-type mesh and obtain with (5a) and \( s = k + 1 \)
\[
\| E - IE \|_{L^2(0,\lambda)} \lesssim \varepsilon^{1/2} (N^{-1} \max |\psi'|)^{k+1},
\]
while with a triangle inequality and the \( L^\infty \)-stability of \( I \) it follows
\[
\| E - IE \|_{L^2(\lambda,2)} \lesssim \| E \|_{L^2(\lambda,2)} + \| E \|_{L^\infty(\lambda,2)} \lesssim N^{-(k+1)}.\]
With (5b) and \( t = k \) we obtain
\[
\|(E - IE)'\|_{L^2(0,\lambda)} \lesssim \varepsilon^{-1/2} (N^{-1} \max |\psi'|)^{k},
\]
and with a triangle and an inverse inequality
\[
\|(E - IE)'\|_{L^2(\lambda,2)} \lesssim \| E' \|_{L^2(\lambda,2)} + N \| E \|_{L^\infty(\lambda,2)} \lesssim \varepsilon^{-1/2} N^{-k}.
\]
In the remaining part (5b) with \( t = k \) yields
\[
\|(E - IE)'\|_{L^2(1,1+\mu)} \lesssim h_{2}^{k} \| E^{(k+1)} \|_{L^2(1,1+\mu)} \lesssim N^{-k} \varepsilon^{k-1} \varepsilon^{-(k+1)} \varepsilon^{1/2} E(1) \lesssim N^{-k} \varepsilon^{-3/2} e^{-\beta/\varepsilon}
\]
due to
\[
\varepsilon^{-1} e^{-\beta/\varepsilon} \leq \frac{1}{\varepsilon \beta}.
\]
For the estimation of \( W \) we follow the idea given in (10) and apply (5a) with \( s = 1 \) and
\( s = 2 \) in order to obtain
\[
\| W - IW \|_{L^2(1+\mu,2)} \lesssim N^{-1} \| W' \|_{L^2(1+\mu,2)} \lesssim N^{-1} \varepsilon^{-1/2} W(1 + \mu) \lesssim N^{-1} \varepsilon^{-1/2} e^{-\varepsilon^{-1/2}},
\]
\[
\| W - IW \|_{L^2(1+\mu,2)} \lesssim N^{-2} \| W'' \|_{L^2(1+\mu,2)} \lesssim N^{-2} \varepsilon^{-1/2} e^{-\varepsilon^{-1/2}}.
\]
Combining these results we have
\[
\| W - IW \|_{L^2(1+\mu,2)} \lesssim N^{-3/2} e^{-\varepsilon^{-1/2}} \lesssim N^{-(k+1)/2},
\]
due to (6). Note that for \( k = 1 \) this approach can also be done on the interval \( (1,2) \), see (10). For \( k > 1 \) we also have with (5a) and \( s = k + 1 \)
\[
\| W - IW \|_{L^2(1+\mu,2)} \lesssim h_{2}^{k+1} \| W^{(k+1)} \|_{L^2(1,1+\mu)} \lesssim N^{-(k+1)} \varepsilon^{1/2}.
\]
For the derivative we obtain using (5b) with \( t = k \) and \( t = 1 \), resp.
\[
\|(W - IW)'\|_{L^2(1+\mu,2)} \lesssim h_{2}^{k} \| W^{(k+1)} \|_{L^2(1+\mu,2)} \lesssim \varepsilon^{-1/2} N^{-k},
\]
\[
\|(W - IW)'\|_{L^2(1+\mu,2)} \lesssim N^{-1} \| W'' \|_{L^2(1+\mu,2)} \lesssim \varepsilon^{-1/2} N^{-1} e^{-\varepsilon^{-1/2}} \lesssim \varepsilon^{-1/2} N^{-k},
\]
due to (6). Collecting the individual results gives (7a) and (7b). With (5a) and \( s = k \) we also obtain
\[
\| W - IW \|_{L^2(1,1+\mu)} \lesssim h_{2}^{k} \| W^{(k)} \|_{L^2(1,1+\mu)} \lesssim \varepsilon^{1/2} N^{-k}
\]
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and together with (11) and (6) we have (7c).

The result (7d) follows directly from (9). For the final results on $E$ we apply (5a) with $s = k$ and obtain

$$\|E - IE\|_{L^2(1,1+\mu)} \lesssim h^k_2\|E^{(k)}\|_{L^2(1,1+\mu)} \lesssim \varepsilon^{-1/2} N^{-k} e^{-\beta/\varepsilon} \lesssim \varepsilon^{1/2} N^{-k},$$
due to (10). Together with (8) we finish the proof. \qed

Remark 3.5. Assumption (6) restricts the application of the method for $k > 1$ slightly. We can rewrite it as

$$N \leq e^{\frac{1}{(k-1)\varepsilon^{1/k}}}$$

and Table 1 shows the bounds on $N$ obtained by this requirement. For small $k$ and reasonably small $\varepsilon$ the coarser mesh approach can be used. For higher polynomial degrees, the weak layer should be resolved by a classical layer-adapted mesh like the S-type mesh. Here we could still increase the value of the transition point, because

$$\mu = \frac{\sigma \varepsilon^{-k-1}}{\beta} \ln(N) > \frac{\sigma \varepsilon}{\beta} \ln(N)$$

would still be enough.

Theorem 3.6. For the solution $u$ of (1) and the numerical solution $u_N$ of (4) holds on the coarser S-type mesh with $\sigma \geq k + 1$ and $e^{-\varepsilon^{-1/k}} \leq N^{1-k}$

$$\|u - u_N\| \lesssim (h + N^{-1} \max |\psi'|)^k.$$

Proof. The proof follows that of Theorem 3.2 by considering $E$ and $W$ in the convective term separately and using the estimates of the previous lemma. \qed

4 Numerical example

Let us consider as example the following problem

$$-\varepsilon u''(x) - (2 + x)u'(x) + (3 + x)u(x) - d(x)u(x - 1) = 3, \quad x \in (0,2),$$

$$u(2) = 0,$$

$$u(x) = x^2, \quad x \in (-1,0],$$
where

\[ d(x) = \begin{cases} 
1 - x, & x < 1, \\
2 + \sin(4\pi x), & x \geq 1. 
\end{cases} \]

Here the exact solution is not known. On a Bakhvalov-S-mesh with \( \sigma = k + 1 \) and \( \varepsilon = 10^{-6} \) we obtain the results listed in Table 2. For other values of \( \varepsilon \) the results are similar. Obviously we see the expected rates of \( N^{-k} \) in \( |||u - u_N||| \). For the computation of these results instead of an exact solution, a reference solution on a finer mesh and higher polynomial degree was used.

On the coarsened mesh we obtain the results shown in Table 3. We observe for \( \varepsilon = 10^{-6} \) almost the same numbers as for the Bakhvalov-S-mesh. Here the conditions of Table 1 are fulfilled and we do not observe a reduction in the orders of convergence. But for the larger \( \varepsilon = 10^{-3} \) there is a visible reduction in the convergence orders for \( k = 4 \). This demonstrates clearly, that for higher polynomial degrees and rather large \( \varepsilon \) a classical layer adapted mesh should be chosen.

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Table 3: Errors $||u - u_N||$ on the coarsened mesh

| $N$  | $k = 1$       |       | $k = 2$       |       | $k = 3$       |       | $k = 4$       |       |
|------|---------------|-------|---------------|-------|---------------|-------|---------------|-------|
|      | $\varepsilon = 10^{-6}$ |       | $\varepsilon = 10^{-3}$ |       |
| 16   | 1.27e-01 0.96 | 2.16e-02 1.93 | 3.53e-03 2.89 | 5.74e-04 3.85 | 16   | 1.27e-01 0.96 | 2.18e-02 1.93 | 3.55e-03 2.86 | 5.94e-04 3.86 |
| 32   | 6.52e-02 0.98 | 5.67e-03 1.96 | 4.77e-04 2.94 | 3.98e-05 3.92 | 32   | 6.53e-02 0.98 | 5.72e-03 1.95 | 4.88e-04 2.96 | 4.08e-05 3.57 |
| 64   | 3.31e-02 0.99 | 1.46e-03 1.98 | 6.20e-05 2.97 | 2.63e-06 3.96 | 64   | 3.31e-02 0.99 | 1.48e-03 1.96 | 6.26e-05 2.97 | 3.43e-06 0.66 |
| 128  | 1.67e-02 0.99 | 3.69e-04 1.99 | 7.92e-06 2.98 | 1.69e-07 3.98 | 128  | 1.67e-02 0.99 | 3.80e-04 1.98 | 7.97e-06 2.99 | 2.17e-06 0.44 |
| 256  | 8.36e-03 1.00 | 9.29e-05 1.99 | 1.00e-06 2.99 | 1.08e-08 3.98 | 256  | 8.38e-03 1.00 | 9.63e-05 2.03 | 1.01e-06 2.98 | 1.59e-06 0.78 |
| 512  | 4.19e-03 1.00 | 2.33e-05 1.99 | 1.26e-07 3.00 | 6.81e-10 3.71 | 512  | 4.21e-03 1.00 | 2.36e-05 2.01 | 1.28e-07 2.77 | 9.29e-07 1.61 |
| 1024 | 2.10e-03 5.87e-06 1.58e-08 5.19e-11 | 1024 | 2.11e-03 5.86e-06 1.88e-08 3.05e-07 |

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When estimating $\alpha$ in (2) we have $\alpha = \frac{N}{D}$, where

\[ N := \beta - \delta + \int_0^1 G_x(0,t) \left( f(t + 1) - \delta \hat{c}(t) + (\beta - \delta)\hat{b}(t) - d \int_0^1 G(t,s)f(s)ds \right) dt \\
- \int_0^1 G_x(1,t)f(t)dt, \]

\[ D := 2 + \int_0^1 G_x(1,t)\hat{b}(t)dt + \int_0^1 G_x(0,t) \left( \hat{b}(t) + \hat{c}(t) + d \int_0^1 G(t,s)\hat{b}(s)ds \right) dt, \]

and $\hat{b}(t) := b - ct$ and $\hat{c}(t) := c + dt$. The Green’s function $G$ is defined as

\[ G(x,t) = \begin{cases} 
-\frac{v_1(x)v_2(t)}{\varepsilon w(t)}, & x \leq t, \\
-\frac{v_1(t)v_2(x)}{\varepsilon w(t)}, & x > t,
\end{cases} \]

where $v_1$ is the solution of

\[-\varepsilon v_1''(x) - bv_1'(x) + cv_1(x) = 0, \quad v_1(0) = 0, \quad v_1'(0) = 1,\]
\( v_2 \) is the solution of

\[-\varepsilon v''_2(x) - bv'_2(x) + cv_2(x) = 0, \quad v_2(1) = 0, \quad v'_2(1) = 1\]

and \( w \) their Wronskian

\[w(x) = v_1(x)v'_2(x) - v'_1(x)v_2(x).\]

We can expand the terms in \( D \) in powers of \( \varepsilon \), here done using the symbolic math program MAPLE, and obtain

\[
\int_0^1 G_x(1, t) dt = -\frac{1}{b} + O(\varepsilon),
\]

\[
\int_0^1 G_x(1, t) t dt = -\frac{1}{b} + O(\varepsilon),
\]

\[
\int_0^1 G_x(0, t) dt = \frac{b}{c}(1 - e^{-\frac{t}{\varepsilon}}) \frac{1}{\varepsilon} + \frac{b - (2b + c)e^{-\frac{t}{\varepsilon}}}{b^2} + O(\varepsilon),
\]

\[
\int_0^1 G_x(0, t) t dt = \frac{b}{c^2}(b - (b + c)e^{-\frac{t}{\varepsilon}}) \frac{1}{\varepsilon} + \frac{1}{b^2c}(2b^2 - (2b^2 + 3bc + c^2)e^{-\frac{t}{\varepsilon}}) + O(\varepsilon),
\]

\[
\int_0^1 G_x(0, t) \int_0^1 G(t, s) dsdt = \frac{b - (b + c)e^{-\frac{t}{\varepsilon}}}{c^2} \frac{1}{\varepsilon} - \frac{b + c}{b^3} e^{-\frac{t}{\varepsilon}} + O(\varepsilon),
\]

\[
\int_0^1 G_x(0, t) \int_0^1 G(t, s) sdsdt = \frac{2b^2 - (b^2 + (b + c)^2)e^{-\frac{t}{\varepsilon}}}{c^3} \frac{1}{\varepsilon} + \frac{2b^3 - (2b^3 + 2b^2c + 2bc^2 + c^3)e^{-\frac{t}{\varepsilon}}}{c^2b^3} + O(\varepsilon).
\]

Combining these expansions into the denominator \( D \) we have

\[D = (b + de^{-\frac{t}{\varepsilon}}) \frac{1}{\varepsilon} - \frac{d}{b^3}(2b^2 - c^2)e^{-\frac{t}{\varepsilon}} + O(\varepsilon)\]

and therefore, remember \( b, d > 0 \), it follows \( D \gtrsim \frac{1}{\varepsilon} \). Using above expansions again, we also have

\[
\int_0^1 |G_x(1, t)| dt \lesssim 1,
\]

\[
\int_0^1 |G_x(0, t)| \int_0^1 G(t, s) dsdt \lesssim \frac{1}{\varepsilon},
\]

\[
\int_0^1 |G_x(0, t)| dt \lesssim \frac{1}{\varepsilon},
\]

\[
\int_0^1 |G_x(0, t)| t dt \lesssim \frac{1}{\varepsilon},
\]

\[
\int_0^1 |G_x(0, t)| \int_0^1 G(t, s) sdsdt \lesssim \frac{1}{\varepsilon},
\]

and can estimate the numerator \( N \)

\[|N| \lesssim |\beta| + |\delta| + (|\beta| + |\delta| + \|f\|_{L^\infty(0, 2)}) \cdot \frac{1}{\varepsilon}.\]