A COMBINATORIAL PROOF FOR CAYLEY’S IDENTITY

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Abstract. In [3], Caracciolo, Sokal and Sportiello presented, \textit{inter alia}, an algebraic/combinatorial proof for Cayley's identity. The purpose of the present paper is to give a "purely combinatorial" proof for this identity; i.e., a proof involving only combinatorial arguments together with a generalization of Laplace's Theorem [6, section 148], for which a "purely combinatorial" proof is given in [4, proof of Theorem 6].

1. Introduction

For \( n \in \mathbb{N} \), denote by \([n]\) the set \( \{1, 2, \ldots, n\} \) and let \( X = X_n = (x_{i,j})_{(i,j) \in [n] \times [n]} \) be an \( n \times n \) matrix of indeterminates. For \( I \subseteq [n] \) and \( J \subseteq [n] \), we denote

\[ \begin{align*}
&\text{• the minor of } X \text{ corresponding to the rows } i \in I \text{ and the columns } j \in J \text{ by } X_{I,J}, \\
&\text{• the cominor of } X_{I,J} \text{ (which corresponds to the rows } i \notin I \text{ and the columns } j \notin J \text{) by } X_{I,J}.
\end{align*} \]

Let \( M = \{x_1 \leq x_2 \leq \cdots \leq x_m\} \) be a finite ordered set, and let \( S = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq M \) be a subset of \( M \). We define

\[ \text{sgn (} S \subseteq M \text{) := } (-1)^\sum_{j=1}^k i_j. \]

As pointed out in [3, Section 2.6], the following identity is conventionally but erroneously attributed to Cayley. (Muir [5, vol. 4, p. 479] attributes this identity to Vivanti [7].)

Theorem 1 (Cayley’s Identity). Consider \( X = (x_{i,j})_{(i,j) \in [n] \times [n]} \), and let \( \partial = \left( \frac{\partial}{\partial x_{i,j}} \right) \) be the corresponding \( n \times n \) matrix of partial derivatives. Let \( I, J \subseteq [n] \) with \( |I| = |J| = k \). Then we have for \( s \in \mathbb{N} \):

\[ \det (\partial_{I,J}) (\det (X))^s = s \cdot (s + 1) \cdots (s + k - 1) \cdot (\det (X))^{s-1} \cdot \text{sgn (} I \subseteq [n] \text{))} \cdot \text{sgn (} J \subseteq [n] \text{)} \cdot \det (X_{I,J}). \quad (1) \]

By the alternating property of the determinant, Cayley’s Identity is in fact equivalent to the following special case of (1).

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\(^1\partial \) is also known as \textit{Cayley’s }\( \Omega \)-\textit{process}. 

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Figure 1. View the permutation $\pi = (25143)$ as the corresponding perfect matching $m_\pi$ in the complete bipartite graph $K_{5,5}$. The intersections of edges are indicated by small circles; they correspond bijectively to $\pi$’s inversions:

$$\# \text{(inversions of } \pi) = |\{(1, 3), (2, 4), (2, 5), (4, 5)\}| = 4.$$ 

Assigning weight $x_{i,j}$ to the edge pointing from $i$ to $j$ gives the contribution of the permutation $\pi$ to the determinant of $X_5$:

$$\omega(\pi) = (-1)^4 \cdot x_{1,2} \cdot x_{2,5} \cdot x_{3,1} \cdot x_{4,4} \cdot x_{5,3}.$$ 

Corollary 1 (Vivanti’s Theorem). Specialize $I = J = [k]$ for some $k \leq n$ in Theorem 1. Then we have for $s \in \mathbb{N}$:

$$\det(\partial_{[k],[k]}) (\det(X))^s = s \cdot (s+1) \cdots (s+k-1) \cdot (\det(X))^{s-1} \cdot \det(X_{[k],[k]}) \cdot \omega(m_{\pi_{[k],[k]}}).$$ (2)

2. Combinatorial proof of Vivanti’s Theorem

We may view the determinant of $X$ as the generating function of all permutations $\pi$ in $\mathfrak{S}_n$, where the (signed) weight of a permutation $\pi$ is given as $\omega(\pi) := \text{sgn}(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$:

$$\det(X) = \sum_{\pi \in \mathfrak{S}_n} \omega(\pi).$$

2.1. View permutations as perfect matchings. For our considerations, it is convenient to view a permutation $\pi \in \mathfrak{S}_n$ as a perfect matching $m_\pi$ of the complete bipartite graph $K_{n,n}$, where the vertices consist of two copies of $[n]$ which are arranged in their natural order; see Figure 1 for an illustration of this simple idea. It is easy to see that the edges of such perfect matching can be drawn in a way such that all intersections are of precisely two (and not more) edges, and that the number of these intersections equals the number of inversions of $\pi$, whence the sign of $\pi$ is

$$\text{sgn}(\pi) = (-1)^{\# \text{(intersections in } m_\pi)}.$$ 

This simple visualization of permutations and their inversions is already used in [1 §15, p.32]: We call it the permutation diagram. So assigning weight $x_{i,j}$ to the edge pointing from $i$ to $j$ and defining the weight $\omega(m_\pi)$ of the permutation diagram $m_\pi$ to be the product of the edges belonging to $m_\pi$, we may write

$$\omega(\pi) = (-1)^{\# \text{(intersections in } m_\pi)} \cdot \omega(m_\pi).$$
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Figure 2. For \( n = 5 \), the picture shows a typical object of weight
\[-x_1 x_2^2 x_1 x_4 x_2 x_3 x_2 x_1 x_3 x_3 x_1 x_4 x_1 x_4 x_4 x_5 x_1 x_5 x_5^2 x_5^3,\]
which is counted by the generating function \( \det (X)^4 \). (The edge connecting lower vertex 3 to upper vertex 3 in the 4-th (right-most) matching is drawn as zigzag-line, just to avoid intersections of more than two edges in a single point.)

Given this view, the combinatorial interpretation of the \( s \)-th power of the determinant \( \det (X) \) is obvious: It is the generating function of all \( s \)-tuples \( m = (m_{\pi_1}, \ldots, m_{\pi_s}) \) of permutation diagrams, where the (signed) weight of such \( s \)-tuple \( m \) is given as
\[ \omega(m) = \prod_{i=1}^{s} (-1)^{\# \text{(intersections in } m_{\pi_i} \text{)}} \cdot \omega(m_{\pi_i}). \]
(See Figure 2 for an illustration.)

2.2. Action of the determinant of partial derivatives. Next we need to describe combinatorially the action of the determinant \( \det (\partial_{[k],[k]}) \) of partial derivatives. Let \( m = (m_{\pi_1}, \ldots, m_{\pi_s}) \) be an \( s \)-tuple of permutation diagrams counted in the generating function \( (\det (X))^s \), and let \( \tau \in S_k \). Then the summand
\[ \partial_\tau := \text{sgn} (\tau) \cdot \prod_{i=1}^{k} \frac{\partial}{\partial x_{i, \tau(i)}} \]
applied to \( \omega(m) \) yields
\[ \text{sgn} (\tau) \cdot \left( \prod_{i=1}^{k} \frac{\partial}{\partial x_{i, \tau(i)}} \right) \omega(m) = \text{sgn} (\tau) \cdot c_{\tau,m} \cdot \frac{\omega(m)}{\prod_{i=1}^{k} x_{i, \tau(i)}} , \]
where \( c_{\tau,m} \) is the number of ways to choose the set of \( k \) edges \( \{ (i \rightarrow \tau(i)) : i \in [k] \} \) from all the edges in \( m \) (this number, of course, might be zero). We may visualize the action of \( \delta_\tau \) as “erasing the edges constituting \( \tau \) in \( m \)”; see Figure 3 for an illustration.

Hence we have:
\[ \det (\partial_{[k],[k]}) (\det (X))^s = \sum_{m \in S_n} \omega(m) \sum_{\tau \in S_k} c_{\tau,m} \cdot \frac{\text{sgn} (\tau)}{\prod_{i=1}^{k} x_{i, \tau(i)}} . \]

2.3. Double counting. For our purposes, it is convenient to interchange the summation in (3). This application of double counting amounts here to a simple change of view: Instead of counting the ways to choose the set of edges corresponding to \( \tau \) from all the edges corresponding to some fixed \( s \)-tuple \( m \), we fix \( \tau \) and consider the set of \( m \)'s from which \( \tau \)'s edges might be chosen. This will involve two considerations:
Figure 3. Let \( n = 5, s = 4 \) and \( k = 3 \) in Corollary 1. The picture shows the four possible ways of “erasing” the edges constituting \( \tau \in S_3 \) from the 4-tuple \((m_{\pi_1}, m_{\pi_2}, m_{\pi_3}, m_{\pi_4})\) of matchings, where \((\pi_1, \pi_2, \pi_3, \pi_4) \in S_5^4\) is \(((31254), (51324), (14253), (23415))\). The erased edges are shown as grey dashed lines.

\[
\tau = (312) \in S_3:
\]

\[
\delta_t = \frac{\partial}{\partial x_{1,3}} \cdot \frac{\partial}{\partial x_{2,1}} \cdot \frac{\partial}{\partial x_{3,2}}
\]

- In how many ways can the edges corresponding to \( \tau \) be distributed on \( s \) copies of the bipartite graph \( K_{n,n} \)?
- For each such distribution, what is the set of compatible \( s \)-tuples of permutation diagrams?

For example, if \( k = 3 \) and \( s = 4 \) (as in Figure 3), there clearly

- is 1 way to distribute the three edges on a single copy of the 4 bipartite graphs (see the fourth row of pictures in Figure 3), and there are 4 ways to choose such single copy,
- are 3 ways to distribute the three edges on precisely two copies of the 4 bipartite graphs (see the second and third row of pictures in Figure 3), and there are \( 4 \cdot 3 \) ways to choose such pair of copies (whose order is relevant),
- is 1 way to distribute the three edges on precisely three copies of the 4 bipartite graphs (see the first row of pictures in Figure 3), and there are \( 4 \cdot 3 \cdot 2 \) ways to choose such triple of copies (whose order is relevant).

2.4. Partitioned permutations. A distribution of the edges corresponding to \( \tau \in S_k \) on \( s \) copies of the bipartite graph \( K_{n,n} \) may be viewed (see Figure 3).
as an $s$-tuple of \textit{partial matchings} (some of which may be empty) of $K_{k,k}$

such that the union of these $s$ partial matchings gives the \textit{perfect matching} $m_\tau$ of $K_{k,k}$.

Clearly, to each of such partial matching corresponds a \textit{partial permutation} $\tau_i$, which we may write in two-line notation as follows:

- the lower line shows the \textit{domain} of $\tau_i$ in its natural order,
- the upper line shows the \textit{image} of $\tau_i$,
- the \textit{ordering} of the upper line represents the permutation $\tau_i$.

We say that each of these $\tau_i$ is a \textit{partial permutation} of $\tau$, and that $\tau$ is a \textit{partitioned permutation}. We write in short:

$$\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s.$$ 

For example, the rows of pictures in Figure 3 correspond to the partitioned permutations (written in the aforementioned two-line notation)

- $\left(\frac{3}{1} \frac{1}{2} \frac{2}{3} \right) \star \left(\frac{3}{1} \frac{1}{2} \frac{2}{3} \right)$ for the first row,
- $\left(\frac{3}{1} \frac{1}{2} \frac{2}{3} \right) \star \left(\frac{1}{2} \frac{2}{3} \right)$ for the second row,
- $\left(\frac{3}{1} \frac{1}{2} \frac{2}{3} \right) \star \left(\frac{1}{2} \frac{2}{3} \right)$ for the third row,
- $\left(\frac{1}{2} \frac{2}{3} \right) \star \left(\frac{1}{2} \frac{2}{3} \right)$ for the fourth row.

### 2.5. Equivalence relation for partitioned permutations

For any partitioned permutation $\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s$, consider the $s$-tuple of the upper rows (in the aforementioned two-line notation) only: We call this $s$-tuple of \textit{permutation words} the \textit{partition scheme} of $\tau$ and denote it by $[\tau]$. We say that $\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s$ \textit{complies} to its partition scheme $[\tau] = [\tau_1 \star \tau_2 \star \ldots \star \tau_s]$ and denote this by $\tau \subseteq [\tau_1 \star \tau_2 \star \ldots \star \tau_s]$.

Now consider the following equivalence relation on the set of partitioned permutations:

$$\mu = \mu_1 \star \ldots \star \mu_s \sim \nu = \nu_1 \star \ldots \star \nu_s : \iff [\mu] = [\nu].$$

By definition, the corresponding equivalence classes are \textit{indexed} by a partition scheme, and $\mu = \mu_1 \star \mu_2 \star \ldots \star \mu_s$ belongs to the equivalence class of $\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s$ iff $\mu \subseteq [\tau]$. (For $s > 1$, a partitioned permutation $\tau$ is not uniquely determined by $[\tau]$.)

It is straightforward to compute the \textit{number} of these equivalence classes: In the language of \textit{combinatorial species} (see, for instance, [2]) the $s$-tuples of permutation words indexing these classes correspond bijectively to the (labelled) species $(\text{Permutations})^s$, and since the exponential generating function of \text{Permutations} is

$$\sum_{n=0}^\infty \frac{n! \cdot z^n}{n!} = \frac{1}{1 - z},$$

the exponential generating function of $(\text{Permutations})^s$ is simply

$$\left(\frac{1}{1 - z}\right)^s = (1 - z)^{-s} = \sum_{k=0}^\infty \binom{-s}{k} (-1)^k z^k = \sum_{k=0}^\infty s \cdot (s + 1) \cdots (s + k - 1) \frac{z^k}{k!}. $$
how the sign of a permutation $\pi$ is changed by removing a given partial permutation $\pi'$: We view this as erasing all the edges belonging to $\pi'$’s permutation diagram $m_{\pi'}$ from $\pi$’s permutation diagram $m_\pi$; see again Figure 3.

**Lemma 1.** Let $\pi \in \mathfrak{S}_n$ be a permutation, and let $\pi^*$ be the permutation corresponding to the permutation diagram $m_\pi$ with edge $(i, \pi(i))$ removed. Then we have

$$\sgn(\pi) = (-1)^{\pi(i) - i} \cdot \sgn(\pi^*).$$

**Proof.** We count the number of intersections with edge $(i, \pi(i))$ in $m_\pi$: Let $A = [i - 1], B = [n] \setminus [i], C = [n] \setminus [\pi(i)]$ and $D = [\pi(i) - 1]$ (see Figure 4 for an illustration).

Assume $|\pi^{-1}(C) \cap A| = k$: Then edge $(i, \pi(i))$ clearly intersects the $k$ edges joining vertices from $A$ to vertices from $C$ (see again Figure 4).

The only other intersections with $(i, \pi(i))$ come from edges joining vertices from $B$ to vertices from $D$: Since $\pi$ is a bijection, we have $|\pi^{-1}(C) \cap B| = n - \pi(i) - k$, whence $|\pi^{-1}(D) \cap B| = |B \setminus \pi^{-1}(C)| = n - i - (n - \pi(i) - k) = k + \pi(i) - i$.

Altogether, the removal of edge $(i, \pi(i))$ removes $2k + \pi(i) - i$ intersections of edges. \Halmos

**Corollary 2.** Let $\pi \in \mathfrak{S}_n$ be a partitioned permutation $\pi = \pi_1 \star \pi_2$, where $\pi_1$ is the partial permutation

$$\pi_1 = \begin{pmatrix} \pi(i_1) & \pi(i_2) & \cdots & \pi(i_k) \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$$

(with $\{i_1 \leq i_2 \leq \cdots \leq i_k\} \subseteq [n]$). Clearly, $\pi_2$ is the permutation corresponding to the matching $m_\pi$ with edges $(i_1, \pi(i_1)), (i_2, \pi(i_2)), \ldots, (i_k, \pi(i_k))$ erased, which we also denote by $\pi \setminus \pi_1$. Then we have

$$\sgn(\pi) = (-1)^{\sum_{j=1}^k \pi(i_k) - i_k} \cdot \sgn(\pi_1) \cdot \sgn(\pi_2).$$
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Figure 5. Erase several edges, corresponding to some partial permutation \( \pi_1 \) of \( \pi \): Start with the edge incident with the rightmost vertex from image \((\pi)_1\) in \(m_\pi\).

If we denote \( I = \{i_1, \ldots, i_k\} \) and \( J = \{\pi(i_1), \ldots, \pi(i_k)\} \), we may rewrite this as

\[
\text{sgn}(\pi) = \text{sgn}(I \leq [n]) \cdot \text{sgn}(J \leq [n]) \cdot \text{sgn}(\pi_1) \cdot \text{sgn}(\pi \setminus \pi_1).
\]

**Proof.** We proceed by induction: \( k = 1 \) simply amounts to the statement of Lemma 1.

For \( k > 1 \), let \( i_{\text{max}} := \pi^{-1}(\max(\text{image } \pi_1)) \) be the pre-image of the maximum of the image of \( \pi_1 \). Let \( l \) be the number of elements in the domain of \( \pi_1 \) which are greater than \( i_{\text{max}} \):

\[
l = |\{i : i \in \text{domain } \pi_1 \land i > i_{\text{max}}\}|.
\]

See Figure 5 for an illustration. Removing the edge \((i_{\text{max}}, \pi(i_{\text{max}}))\) leaves (the diagram of) a permutation \( \pi' \in \mathfrak{S}_{n-1} \) and a partial permutation

\[
\pi'_1 = \begin{pmatrix} \pi(i_1) & \cdots & \pi(i_{\text{max}} - 1) & \pi(i_{\text{max}} + 1) & \cdots & \pi(i_k - 1) \\ i_1 & \cdots & i_{\text{max}} - 1 & (i_{\text{max}} + 1) - 1 & \cdots & i_k - 1 \end{pmatrix}
\]

therein of length \( k - 1 \). By induction, we have

\[
\text{sgn}(\pi') = (-1)^{\left(\sum_{j=1}^k \pi(i_k) - i_k\right) - \left(\pi(i_{\text{max}}) - i_{\text{max}}\right) - l} \cdot \text{sgn}(\pi' \setminus \pi'_1) \cdot \text{sgn}(\pi'_1).
\]

Since we have

\[
\begin{align*}
& \bullet \ \pi \setminus \pi_1 = \pi' \setminus \pi'_1 \implies \text{sgn}(\pi_2) = \text{sgn}(\pi \setminus \pi_1) = \text{sgn}(\pi' \setminus \pi'_1), \\
& \bullet \ \text{sgn}(\pi_1) = (-1)^l \cdot \text{sgn}(\pi'_1) \ (\text{see again Figure 5}), \\
& \bullet \ \text{and } \text{sgn}(\pi) = (-1)^{\pi(i_{\text{max}}) - i_{\text{max}}} \text{sgn}(\pi') \ (\text{by Lemma 1}),
\end{align*}
\]

the assertion follows.

2.7. Sums of (signed) products of minors. Now consider a fixed equivalence class in the sense of Section 2.5 which is indexed by a partition-scheme

\[
[\tau_1 \ast \tau_2 \ast \cdots \ast \tau_s].
\]

We want to compute the generating function \( G_{[\tau]} \) of this equivalence class: Clearly, we may concentrate on the nonempty partial permutations; so w.l.o.g. we have to consider the partition-scheme

\[
[\tau_1 \ast \tau_2 \ast \cdots \ast \tau_m]
\]
which consists only of nonempty partial permutations \( \tau_j \) for \( 1 \leq j \leq m \leq s \). For any \( \sigma \in \mathfrak{S}_k \) with \( \sigma \subseteq [\tau_1 \ast \tau_2 \ast \cdots \ast \tau_m] \), such partition scheme corresponds to a unique ordered partition of the image of \( \sigma \):

\[
\text{image } \sigma = [k] = (\text{image } \tau_1) \cup (\text{image } \tau_2) \cup \cdots \cup (\text{image } \tau_m) = J_1 \cup J_2 \cup \cdots \cup J_m,
\]

and any specification of a compatible ordered partition \( I_{[\sigma]} = (I_1, I_2, \ldots, I_m) \), i.e.,

\[
[k] = I_1 \cup I_2 \cup \cdots \cup I_m \text{ where } |I_i| = |J_i|, i = 1, \ldots, m,
\]

uniquely determines such \( \sigma \), which we denote by \( \sigma (I_{[\sigma]}, [\tau]) \).

Equation (4) gives the sign-change caused by erasing the edges corresponding to \( \tau_j \) (with respect to any permutation in \( \mathfrak{S}_n \) which contains \( \tau_i \) as a partial permutation), whence we can write the generating function as

\[
G_{[\tau]} = \det (X)^{s-m} \times \sum_{I_{[\sigma]}} \text{sgn } (\sigma (I_{[\sigma]}, [\tau])) \cdot \prod_{l=1}^{m} (\text{sgn } (\tau_l) \cdot \text{sgn } (I_l \subseteq [n]) \cdot \text{sgn } (J_l \subseteq [n]) \cdot \det (X_{I_l \cup J_l})),
\]

where the sum is over all compatible partitions \( I_{[\sigma]} \). (The factor \( \text{sgn } (\sigma (I_{[\sigma]}, [\tau])) \) comes from the determinant of partial derivatives.) Clearly,

\[
\prod_{l=1}^{m} \text{sgn } (I_l \subseteq [n]) = \prod_{l=1}^{m} \text{sgn } (J_l \subseteq [n]) = 1,
\]

so it remains to show

\[
\sum_{I_{[\sigma]}} \text{sgn } \sigma (I_{[\sigma]}, [\tau]) \cdot \prod_{l=1}^{m} (\text{sgn } (\tau_l) \cdot \det (X_{I_l \cup J_l})) = \det (X)^{m-1} \det (X_{I_{[\sigma]} \cup J_{[\tau]}}).
\]

This, of course, is true for \( m = 1 \). We proceed by induction on \( m \).

For any ordered partition \( S_1 \cup S_2 \cup \ldots \cup S_m = [k] \), we introduce the shorthand notation

\[
S_i := [k] \setminus (S_1 \cup S_2 \cup \ldots \cup S_i).
\]

Moreover, write \( d_{I_j} := \det (X_{I_j \cup J_j}) \) for short. Then the lefthand-side of (5) may be written as the \((m-1)\)-fold sum

\[
\sum_{I_1 \subseteq I_0} \text{sgn } (\tau_1) d_{I_1} \sum_{I_2 \subseteq I_1} \text{sgn } (\tau_2) d_{I_2} \cdots \sum_{I_{m-1} \subseteq I_{m-2}} \text{sgn } (\tau_{m-1}) d_{I_{m-1}} \text{sgn } (\tau_m) d_{I_m} \cdot \text{sgn } (\sigma),
\]

where \( I_m = I_{m-2} \setminus I_{m-1} \) and \( \sigma = \sigma (I_{[\sigma]}, [\tau]) \).

Assume \( J_{m-2} = \{j_1, \ldots, j_a\} \), \( I_{m-2} = \{i_1, \ldots, i_a\} \) and \( J_m = \{j_1, \ldots, j_s\} \). Then the special choice \( \tau_{m-1} = \{i_{s_1}, \ldots, i_{s_a}\} \) (i.e., with respect to the relative ordering, “\( \tau_{m-1} \) is the same subset as \( J_m \)”) and \( \tau_{m-1} = I_{m-2} \setminus \tau_m \) determines uniquely a partial permutation \( \tau_{m-1} \)

\[
\tau_{m-1} : I_{m-2} \rightarrow J_{m-2}.
\]

According to (4), by construction we have

\[
\text{sgn } (\tau_{m-1}) = \text{sgn } (\tau_m) \cdot \text{sgn } (\tau_m).
\]

(7)
Now consider \( \sigma = \sigma (I_{[J]}, [\tau]) \) in the innermost sum of (6): Erasing the edges corresponding to \( \tau_{m-1} \) and \( \tau_{m-2} \) and replacing them by the edges corresponding to \( \overline{\tau_{m-1}} \) yields a permutation \( \overline{\sigma} = \tau_1 \cdots \tau_{m-2} \cdot \tau_{m-1} \) (which, of course, complies to the partition scheme \( [\tau] = [\tau_1 \cdots \tau_{m-2} \cdot \tau_{m-1}] \)). Since by (4) together with (7) we have

\[
\sgn (\overline{\tau_{m-1}}) = \sgn (\tau_{m-1} \cdot \tau_m) \cdot \sgn (I_m \subseteq I_{m-2}) \cdot \sgn (J_m \subseteq J_{m-2})
\]

and (clearly)

\[
\sigma \backslash (\tau_{m-1} \cdot \tau_m) = \overline{\sigma} \backslash \overline{\tau_{m-1}},
\]

we also have (again by (4))

\[
\sgn (\sigma) = \sgn (I_m \subseteq I_{m-2}) \cdot \sgn (J_m \subseteq J_{m-2}) \cdot \sgn (\overline{\sigma}).
\]

Hence the innermost sum of (6) can be written as

\[
\sgn ([\tau]) \cdot \left( \sum_{I_{m-1} \subseteq I_{m-2}} \sgn (I_m \subseteq I_{m-2}) \cdot \sgn (J_m \subseteq J_{m-2}) \cdot d_{I_{m-1}} \cdot d_{I_m} \right) \cdot \sgn (\overline{\sigma}).
\]

If we can show that this last sum equals \( \det (X) \cdot \det \left( X_{\overline{I_{m-2}} \cdot \overline{J_{m-2}}} \right) \), then (5) follows by induction, since the \((m-1)\)-fold sum in (6) thus reduces to an \((m-2)\)-fold sum, which corresponds to the partition-scheme \([\tau] = [\tau_1 \ast \tau_2 \ast \ldots \ast \tau_{m-2} \cdot \tau_{m-1}] \).

### 2.8. (A generalization of) Laplace’s theorem

Luckily, a generalization (see [6, section 148]) of Laplace’s Theorem serves as the closer for our argumentation:

**Theorem 2.** Let \( a \) be an \((m+k) \times (m+k)\)-matrix, and let \( 1 \leq i_1 < i_2 < \cdots < i_m \leq m+k \) and \( 1 \leq j_1 < j_2 < \cdots < j_m \leq m+k \) be (the indices of) \( k \) fixed rows and \( k \) fixed columns of \( a \). Denote the set of these (indices of) rows and columns by \( R \) and \( C \), respectively. Consider some fixed set \( I \subseteq R \). Then we have:

\[
det (a) \cdot \det (a_{\overline{R \cup C \cup J}}) = \sum_{J \subseteq C, |J|=|I|} \sgn (I \subseteq R) \cdot \sgn (J \subseteq C) \cdot \det \left( a_{\overline{R \cup C \cup J}} \right) \cdot \det (a_{\overline{I \cup J}}).
\]

A combinatorial proof for this identity is given in [4, proof of Theorem 6]: So altogether, we achieved a “purely combinatorial” proof for (2). \( \square \)

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