FOLIATIONS OF CONTINUOUS Q-PSEUDOCONCAVE GRAPHS

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Abstract. We show that the graph of a continuous map $f : G \to \mathbb{R}^k \times \mathbb{C}^p$, defined on an open set $G$ in $\mathbb{C}^n_z \times \mathbb{R}^k_u$, is locally foliated by complex $n$-dimensional submanifolds if and only if its complement is $n$-pseudoconvex in $(G + i\mathbb{R}^k) \times \mathbb{C}^p \subset \mathbb{C}^n_z \times \mathbb{C}^k_u + iv \times \mathbb{C}^p$ (in the sense of Rothstein).

1. Introduction

One of the classical and fascinating theorems by Hartogs from 1909 states that a continuous function $f : \Delta^n \to \mathbb{C}$ is holomorphic on a polydisk $\Delta^n \subset \mathbb{C}^n_z$ if and only if the complement of its graph $\Gamma(f) = \{ (z, \zeta) : z \in G, \zeta = f(z) \}$ is a domain of holomorphy in $\Delta^n \times \mathbb{C}$. In this spirit, we prove the following result (see Theorem 5.7 of this paper).

Main Theorem. Let $n, k, p$ be integers with $n \geq 1$, $p \geq 0$ and let $k \in \{0, 1\}$ such that $N = n + k + p \geq 2$. Let $G$ be an open set in $\mathbb{C}^n_z \times \mathbb{R}^k_u$ and let $f : G \to \mathbb{R}^k_v \times \mathbb{C}^p$ be a continuous map such that the complement of the graph $\Gamma(f)$ in $\mathbb{C}^N_{z,u+iv,\zeta}$ is Hartogs $n$-pseudoconvex in the sense of Rothstein [Rot55]. Then $\Gamma(f)$ is locally foliated by $n$-dimensional complex submanifolds.

This statement generalizes not only Hartogs’ theorem (case $n \geq 1$, $k = 0$ and $p = 1$), but also results in [She93] (case $n = 1$, $k = 1$ and $p = 0$) and in [Chi01] (case $n \geq 2$, $k = 1$ and $p = 0$). The most important case in this paper is $n = k = 1$. From these results we easily derive all the other cases. Results in the case $n \geq 1$, $k \geq 2$ and $p \geq 0$ are not known to us yet.

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Since the graphs of functions $f : \mathbb{C}^n \to \mathbb{C}_\zeta$ are hypersurfaces in $\mathbb{C}^{n+1}_{z,\zeta}$, pseudoconvexity of their complement is meaningful. In contrast, the graphs of continuous maps, as in our setting above, are surfaces of higher codimension, so that pseudoconvexity is not a proper condition anymore and needs to be replaced by $q$-pseudoconvexity. The reason is that the continuity principle with discs fails if the dimension of the graph is too low. A continuity principle with analytic surfaces of higher dimension is therefore the key. It is equivalent to Hartogs $q$-pseudoconvexity in the sense of Rothstein [Rot55] and the $(n - q - 1)$-pseudoconvexity in the sense of Hunt and Murray [HM78]. Notice that in the smooth setting $q$-pseudoconvexity is $(q - 1)$-convexity in the sense of Grauert and Andreotti.

The paper is organized as follows. In Sections 2 and 3 we collect some (mainly known) facts on $q$-plurisubharmonic functions and $q$-pseudoconvex sets. In Section 4 we present results on the relation of Levi $q$-pseudoconvex sets and $q$-pseudoconvexity, which will form the essential tools to prove the main theorem. In the last Section 5 we prove the main theorem.

These results were obtained jointly by the authors of this paper and were included in the doctoral thesis [Paw15] of the first author. They have not been published yet, since we intended to proceed with further investigations in hope to prove more general statements. Meanwhile, Takeo Ohsawa proved in [Ohs20] a statement similar to our result in the case $k = 0$ using $L^2$-methods. As it appears that our result is more general ($k = 0, 1$) and its verification uses different techniques, we felt motivated to finalize our paper and present our results in its original form given in the above mentioned thesis.

2. On $q$-plurisubharmonic functions

In this section, we introduce the most important functions of this paper: the $q$-plurisubharmonic functions in the sense of Hunt and Murray [HM78]. A collection of results on $q$-plurisubharmonic functions (and $q$-pseudoconvex sets) can be found in the doctoral thesis [Paw15] of the first author.

**Definition 2.1.** Let $q \in \{0, \ldots, n-1\}$ and let $\psi$ be an upper semi-continuous function on an open set $\Omega$ in $\mathbb{C}^n$.

1. The function $\psi$ is called *subpluriharmonic on* $\Omega$ if for every ball $B \subseteq U$ and every function $h$ which is pluriharmonic on a neighborhood of $\overline{B}$ with $\psi \leq h$ on $bB$ we already have that $\psi \leq h$ on $\overline{B}$.

2. The function $\psi$ is *$q$-plurisubharmonic on* $\Omega$ if $\psi$ is subpluriharmonic on $\pi \cap \Omega$ for every complex affine plane $\pi$ of dimension $q + 1$. 
(3) If \( q \geq n \), every upper semi-continuous function on \( \Omega \) is by convention \( q \)-pluri-
subharmonic.

(4) The set of all \( q \)-plurisubharmonic functions on \( \Omega \) is denoted by \( \mathcal{PSH}_q(\Omega) \).

(5) An upper semi-continuous function \( \psi \) on \( \Omega \) is called \emph{strictly \( q \)-plurisubharmonic}
on \( \Omega \) if for every \( C^\infty \)-smooth non-negative function \( \theta \) with compact support in \( \Omega \) there is a positive number \( \varepsilon_0 \) such that \( \psi + \varepsilon \theta \) remains \( q \)-plurisubharmonic on \( \Omega \) for every real number \( \varepsilon \) with \( |\varepsilon| \leq \varepsilon_0 \).

We give a list of properties of \( q \)-plurisubharmonic functions.

\textbf{Proposition 2.2.} Every below mentioned function is defined on an open set \( \Omega \) in \( \mathbb{C}^n \) unless otherwise stated.

(1) The 0-plurisubharmonic functions are exactly the plurisubharmonic functions, and the \((n-1)\)-plurisubharmonic functions are the subpluriharmonic functions.

(2) Every \( q \)-plurisubharmonic function is \((q+1)\)-plurisubharmonic.

(3) \cite{Ste84} If \( \psi \) is \( q \)-plurisubharmonic and \( \varphi \) is \( r \)-plurisubharmonic, then \( \psi + \varphi \) is\((q+r)\)-plurisubharmonic.

(4) \cite{HM78} An upper semi-continuous function \( \psi \) is \( q \)-plurisubharmonic on \( \Omega \) if and only if it is locally \( q \)-plurisubharmonic on \( \Omega \), i.e., for each point \( p \) in \( \Omega \) there is a neighborhood \( U \) of \( p \) in \( \Omega \) such that \( \psi \) is \( q \)-plurisubharmonic on \( U \).

(5) \cite{Fu92, Die06} A function \( \psi \) is \( q \)-plurisubharmonic on an open set \( \Omega \) in \( \mathbb{C}^n \) if and only if \( \psi \circ f \) is \( q \)-plurisubharmonic for every holomorphic mapping \( f : D \to \Omega \), where \( D \) is a domain in \( \mathbb{C}^{q+1} \) (or even \( \mathbb{C}^k \) with \( k \geq q+1 \)).

(6) Let \( \Omega_1 \) be an open set in \( \Omega \), \( \psi \) be a \( q \)-plurisubharmonic function on \( \Omega \) and \( \psi_1 \) be a \( q \)-plurisubharmonic function on \( \Omega_1 \) such that

\[
\limsup_{w \to z \atop w \in \Omega_1} \psi(w) \leq \psi(z) \quad \text{for every} \quad z \in \partial \Omega_1 \cap \Omega.
\]

Then the subsequent function is \( q \)-plurisubharmonic on \( \Omega \),

\[
\varphi(z) := \begin{cases} 
\max\{\psi(z), \psi_1(z)\}, & z \in \Omega_1 \\
\psi(z), & z \in \Omega \setminus \Omega_1
\end{cases}
\]

A smooth (strictly) \( q \)-plurisubharmonic function can be characterized by counting the eigenvalues of its complex Hessian matrix.

\textbf{Definition 2.3.} Let \( \psi \) be twice differentiable at a point \( p \). For \( X, Y \in \mathbb{C}^n \) we define the \emph{Levi form} of \( \psi \) at \( p \) by

\[
\mathcal{L}_\psi(p)(X, Y) := \sum_{k,l=1}^n \frac{\partial^2 \psi}{\partial z_k \partial \overline{z}_l}(p) X_k Y_l.
\]
We have the following characterization of smooth \(q\)-plurisubharmonic functions (see Lemma 2.6 in [HM78]):

**Theorem 2.4.** Let \(q \in \{0, \ldots, n-1\}\) and let \(\psi\) be a \(C^2\)-smooth function on an open subset \(\Omega\) in \(\mathbb{C}^n\). Then \(\psi\) is (strictly) \(q\)-plurisubharmonic if and only if the Levi matrix \(L_\psi(p)\) has at most \(q\) negative (\(q\) non-positive) eigenvalues at every point \(p\) in \(\Omega\).

In the same paper [HM78] the local maximum property was shown.

**Theorem 2.5 (Local maximum property).** Let \(q \in \{0, \ldots, n-1\}\) and \(\Omega\) be a relatively compact open set in \(\mathbb{C}^n\). Then any function \(u\) which is upper semi-continuous on \(\overline{\Omega}\) and \(q\)-plurisubharmonic on \(\Omega\) fulfills

\[
\max_{\overline{\Omega}} \psi = \max_{\partial A} \psi.
\]

Słodkowski generalized in [Sło86] the previous results to analytic sets in his Proposition 5.2 and Corollary 5.3.

**Theorem 2.6 (Local maximum principle for analytic sets).** Fix an integer number \(q \in \{0, \ldots, n-1\}\). Let \(A\) be an analytic subset of an open set \(\Omega\) in \(\mathbb{C}^n\) with \(\dim_z A \geq q+1\) for all \(z \in A\) and let \(\psi\) be a \(q\)-plurisubharmonic function on \(A\), i.e. for every point \(z \in A\) the function \(\psi\) extends to a \(q\)-plurisubharmonic function on some open neighborhood of \(z\) in \(\Omega\). Then for every compact set \(K\) in \(A\) we have that

\[
\max_K \psi = \max_{b_A K} \psi.
\]

Here, by \(b_A K\) we mean the relative boundary of \(K\) in \(A\).

### 3. On \(q\)-pseudoconvex sets

Several characterizations of \(q\)-pseudoconvexity in \(\mathbb{C}^n\) can be found in the literature. We may refer, for example, to [Fuj64], [Sło86] and [Mat96]. Out of these notions we need the \(q\)-pseudoconvexity in the sense of Rothstein [Rot55] which is based on generalized Hartogs figures.

**Definition 3.1.** (1) We write \(\Delta^n_r := \Delta^n(0) = \{z \in \mathbb{C}^n : \max_j |z_j| < r\}\) for the polydisc with radius \(r > 0\) and \(A^n_{R,R} := \Delta^n_R \setminus \Delta^n_r\) for the open annulus with radii \(r > 0\) and \(R > 0\) centered at the origin in \(\mathbb{C}^n\).

(2) Let \(1 \leq k < n\) be fixed integers, and \(r\) and \(R\) be real numbers in the interval \((0, 1)\). An Euclidean \((n-k,k)\) Hartogs figure \(H_e\) is the set

\[
H_e := (\Delta_1^{n-k} \times \Delta_r^k) \cup (A_{R,1}^{n-k} \times \Delta_1^k) \subset \Delta_1^{n-k} \times \Delta_1^k = \Delta_1^n.
\]
(3) A pair \((H, P)\) of domains \(H\) and \(P\) in \(\mathbb{C}^n\) with \(H \subset P\) is called a (general) \((n-k, k)\) Hartogs figure if there is an Euclidean \((n-k, k)\) Hartogs figure \(H_e\) and a biholomorphic mapping \(F\) from \(\Delta_1^n\) onto \(P\) such that \(F(H_e) = H\).

(4) An open set \(\Omega\) in \(\mathbb{C}^n\) is called Hartogs \(k\)-pseudoconvex if it admits the Kontinuitätsatz with respect to the \((n-k, k)\)-dimensional polydiscs, i.e., given any \((n-k, k)\) Hartogs figure \((H, P)\) such that \(H \subset \Omega\), we already have that \(P \subset D\).

Another notion of \(q\)-pseudoconvexity is as follows.

**Definition 3.2.** We say that an open set \(\Omega\) in \(\mathbb{C}^n\) is \(q\)-pseudoconvex (in \(\mathbb{C}^n\)) if there exists a continuous \(q\)-plurisubharmonic exhaustion function \(\Phi\) for \(\Omega\), i.e., for every \(c \in \mathbb{R}\) the set \(\{z \in \Omega : \Phi(z) < c\}\) is relatively compact in \(\Omega\).

In regards to the definition of 0-plurisubharmonic functions and the classical Kontinuitätsatz, 0-pseudoconvexity and Hartogs \((n-1)\)-pseudoconvexity of sets are equivalent to their pseudoconvexity. Notice that every domain in \(\mathbb{C}^n\) is \((n-1)\)-pseudoconvex (see [5]). In fact, \(q\)-pseudoconvexity of sets is the same as their Hartogs \((n-q-1)\)-pseudoconvexity. We will use both properties later on.

A proof of the following statement and an extended list of notions which are equivalent to \(q\)-pseudoconvexity can be found in [10] or in [15].

**Theorem 3.3.** Let \(q \in \{0, \ldots, n-2\}\) and \(\Omega\) be an open set in \(\mathbb{C}^n\). Then the following statements are all equivalent.

1. The set \(\Omega\) is Hartogs \((n-q-1)\)-pseudoconvex.
2. For some/all complex norm(s) \(\| \cdot \|\) the boundary distance function
   \[
   z \mapsto -\log d_{\| \cdot \|}(z, b\Omega) = -\log \inf \{\|z - w\| : w \in b\Omega\}
   \]
   is \(q\)-plurisubharmonic on \(\Omega\).
3. \(\Omega\) is \(q\)-pseudoconvex.
4. Let \(\{A_t\}_{t \in [0,1]}\) be a family of \((q+1)\)-dimensional analytic subsets in some open set \(U\) in \(\mathbb{C}^n\) that continuously depend on \(t\) in the Hausdorff topology. Assume that the closure of \(\bigcup_{t \in [0,1]} A_t\) is compact. If \(\Omega\) contains the boundary \(bA_1\) and the closure \(\overline{A_t}\) for each \(t \in [0,1]\), then the closure \(\overline{A_1}\) also lies in \(\Omega\).

We recall the definition, the basic properties and some examples of relative \(q\)-pseudoconvex sets, which were originally introduced by Z. Słodkowski in chapter 4 of [5]. They will mainly serve to simplify our notations.

**Definition 3.4.** Given two open sets \(U \subset V\) in \(\mathbb{C}^n\), the set \(U\) is said to be \(q\)-pseudoconvex in \(V\) if there is a neighborhood \(W\) of \(bU \cap V\) in \(\mathbb{C}^n\) such that the function
$z \mapsto -\log d(z, bU)$ is $q$-plurisubharmonic on $U \cap W$. Here, $d(z, bU)$ is induced by the Euclidean distance.

The following proposition is a part of Theorem 4.3 and Corollary 4.7 in Słodkowski’s article [Sło86].

**Proposition 3.5.** Let $U \subset V$ be open sets in $\mathbb{C}^n$. Then the following statements are equivalent.

1. $U$ is $q$-pseudoconvex in $V$.
2. There exist a neighborhood $W$ of $bU \cap V$ in $V$ and a $q$-plurisubharmonic function $\psi$ on $W \cap U$ such that $\psi(z)$ tends to $+\infty$ whenever $z$ approaches the relative boundary $bU \cap V$.
3. For every point $p$ in $V \cap bU$ there exists an open ball $B_r(p)$ centered in $p$ such that the intersection $U \cap B_r(p)$ is $q$-pseudoconvex in $\mathbb{C}^n$.

We continue by presenting some examples of relative $q$-pseudoconvex sets.

**Example 3.6.** (1) Let $\varphi$ be $q$-plurisubharmonic on an open set $V$ in $\mathbb{C}^n$ and let $c$ be a real number. Then the set $U = \{z \in V : \varphi(z) < c\}$ is $q$-pseudoconvex in $V$. If, moreover, the set $V$ is $q$-pseudoconvex itself, then $U$ is $q$-pseudoconvex (in $\mathbb{C}^n$).

(2) Let $\Omega$ be an open set in $\mathbb{C}^n$ and let $h$ be a smooth $q$-holomorphic function on $\Omega$ in the sense of Basener [Bas76], i.e. $\overline{\partial}h \wedge (\partial \overline{\partial}h)^q = 0$. Let $\Gamma(h) := \{(z, h(z)) \in \mathbb{C}^{n+1} : z \in \Omega\}$ be the graph of $f$ over $\Omega$. Then the function $(z, w) \mapsto 1/(h(z) - w)$ is $q$-holomorphic on $U := (\Omega \times \mathbb{C}) \setminus \Gamma(h)$ by [Bas76], so the function $\psi(z, w) := -\log |h(z) - w|$ is $q$-plurisubharmonic on $U$ by [HM78]. It has the property that $\psi(z, w)$ tends to $+\infty$ whenever $(z, w)$ approaches the graph $\Gamma(h)$. Hence, the open set $U$ is $q$-pseudoconvex in $V := \Omega \times \mathbb{C}$ by Proposition 3.5(2). A converse statement is not known except for Hartog’s theorem which appears in the holomorphic case $q = 0$.

The smoothly bounded $q$-pseudoconvex sets can be characterized in terms of $q$-plurisubharmonic defining functions.

**Definition 3.7.** Let $U$ be an open set in $\mathbb{C}^n$.

1. The set $U$ is called Levi $q$-pseudoconvex (resp. strictly Levi $q$-pseudoconvex) at the point $p \in bU$ if there exist a neighborhood $W$ of $p$ and a $C^2$-function $\varphi$ on $W$ such that $\nabla \varphi(p) \neq 0$, $U \cap W = \{z \in W : \varphi(z) < 0\}$ and, moreover, such that its Levi form $L_\varphi$ at $p$ has at most $q$ negative (resp. $q$ non-positive) eigenvalues.
on the holomorphic tangent space

\[ H_p bU = \{ X \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \varrho}{\partial z_j}(p) X_j = 0 \}. \]

(Clearly, the definition of Levi pseudoconvexity does not depend on the defining function.)

(2) Let \( V \) be an open neighborhood of \( U \) with \( U \subseteq V \). Then \( U \) is called (strictly) Levi \( q \)-pseudoconvex in \( V \) if it is (strictly) Levi \( q \)-pseudoconvex at every point \( p \in bU \cap V \). A strictly Levi \( q \)-pseudoconvex set in \( \mathbb{C}^n \) is also simply called strictly \( q \)-pseudoconvex.

We present some facts about Levi \( q \)-pseudoconvex sets.

**Remark 3.8.** If \( U \subset \mathbb{C}^n \) is strictly \( q \)-pseudoconvex at a boundary point \( p \) and \( \psi \) is a defining function for \( U \) at \( p \), then for a large enough constant \( c > 0 \) the function \( \exp(c\psi) - 1 \) is strictly \( q \)-plurisubharmonic on some ball \( B \) centered at \( p \) and still defines \( U \) at \( p \). Therefore, in view of Example 3.6 (1), \( U \cap B \) is \( q \)-pseudoconvex.

The next theorem is the main result in [Sur84]. We will establish its converse statement later in Corollary 4.2.

**Theorem 3.9.** Every Levi \( q \)-pseudoconvex set is \( q \)-pseudoconvex. More precisely, it admits a \( C^2 \)-smooth \( q \)-plurisubharmonic exhaustion function.

### 4. Duality Principle of \( q \)-Pseudoconvex Sets

In this section, we study the link between strictly \( q \)-pseudoconvex sets and their complements. Their relation leads to two duality theorems. The first one is due to Basener (see Proposition 6 in [Bas76]).

**Theorem 4.1.** If an open set \( \Omega \) in \( \mathbb{C}^n \) is strictly \( q \)-pseudoconvex at some point \( p \in b\Omega \), then for every small enough neighborhood \( V \) of \( p \) the set \( V \cap (\mathbb{C}^n \setminus \overline{\Omega}) \) is not \( (n-q-2) \)-pseudoconvex at every point of \( b\Omega \cap V \). More precisely (and in view of Theorem 3.3 above), for each \( w \) in \( b\Omega \cap V \) and every neighborhood \( U \subset V \) of \( w \) there is a family \( \{A_t\}_{t \in [0,1]} \) of \( (n-q-1) \)-dimensional complex submanifolds of \( U \) which is continuously parameterized by \( t \) and fulfills

1. \( A_t \subset (\mathbb{C}^n \setminus \overline{\Omega}) \) for every \( t \in [0,1] \),
2. \( w \in A_1 \), but \( A_1 \setminus \{w\} \subset (\mathbb{C}^n \setminus \overline{\Omega}) \).

**Proof.** Fix the point \( p \in b\Omega \). By Proposition 6 in [Bas76] there exists a neighborhood \( V \) of \( p \) in \( \mathbb{C}^n \) such that for every point \( w \) in \( b\Omega \cap V \) the following properties hold after an appropriate holomorphic change coordinates on \( V \),

\[ w = 0 \quad \text{and} \quad \Re(z_1) < 0 \quad \text{for every} \quad z \in V \cap (\overline{\Omega} \setminus \{w\}) \cap (\mathbb{C}^{n-q} \times \{0\}^q). \]
Let \( U \Subset V \) be any neighborhood of \( w \). Then there are real numbers \( \varepsilon > 0 \) and \( r > 0 \) such that for each \( t \in [0, 1] \) the submanifold

\[
A_t = \{(1 - t)\varepsilon \times B_r^{n-q-1}(0) \times \{0\}\}
\]

is contained in \( U \). Finally, the properties \((4.1)\) imply that the family \( \{A_t\}_{t \in [0, 1]} \) has the desired properties.

As an application we can improve Suria’s observation (see Theorem 3.9) which fully clarifies the relation between Levi \( q \)-pseudoconvexity and \( q \)-pseudoconvexity.

**Corollary 4.2.** Let \( \Omega \) be an open set in \( \mathbb{C}^n \) which is \( C^2 \)-smoothly bounded. Then \( \Omega \) is Levi \( q \)-pseudoconvex if and only if it is \( q \)-pseudoconvex.

**Proof.** Due to Suria’s Theorem 3.9 it only remains to prove that, if \( \Omega \) is \( q \)-pseudoconvex, then it is Levi \( q \)-pseudoconvex. Suppose that \( \Omega \) is not Levi \( q \)-pseudoconvex at some boundary point \( p \) of \( \Omega \). Then there are a neighborhood \( W \Subset U \) of \( p \) and a \( C^2 \)-smooth defining function \( q \) for \( \Omega \) at \( p \) defined on \( W \) such that its Levi form has at most \( n-q-2 \) non-negative eigenvalues on the holomorphic tangent space to \( b\Omega \) at \( p \). Hence, \( -q \) is a defining function for \( D := \mathbb{C}^n \setminus \overline{\Omega} \) at \( p \) whose Levi form has at most \( n-q-2 \) non-positive eigenvalues on the holomorphic tangent space to \( b\Omega \) at \( p \). This means that \( D \) is strictly \((n-q-2)\)-pseudoconvex at \( p \). But then Theorem 4.1 implies that \( \Omega \) is not \( q \)-pseudoconvex near \( p \), which is absurd. Therefore, \( \Omega \) has to be Levi \( q \)-pseudoconvex at \( p \). \( \square \)

In order to establish a converse statement of Theorem 4.1 we need the following lemma.

**Lemma 4.3.** Let \( q \in \{0, \ldots, n-1\} \) and let \( \psi \) be a \( C^2 \)-smooth strictly \( q \)-plurisubharmonic function on an open set \( V \) in \( \mathbb{C}^n \). Assume that \( V \) contains two compact sets \( K \) and \( L \) which fulfill the following properties:

1. \( K, L \subset \{ z \in V : \psi(z) \leq 0 \} \)
2. \( L \cap \{ z \in V : \psi(z) = 0 \} = \emptyset \)
3. \( K \cap \{ z \in V : \psi(z) = 0 \} \neq \emptyset \)

Under these conditions, there exist a point \( z_0 \in bK \), a neighborhood \( U \Subset V \) of \( z_0 \) and a \( C^2 \)-smooth strictly \( q \)-plurisubharmonic function \( \varphi \) on \( U \) satisfying:

1. \( K, L \subset \{ z \in U : \varphi(z) \leq 0 \} \)
2. \( L \cap \{ z \in U : \varphi(z) = 0 \} = \emptyset \)
3. \( K \cap \{ z \in U : \varphi(z) = 0 \} = \{ z_0 \} \)
4. \( \nabla \varphi \neq 0 \) on \( \{ z \in U : \varphi(z) = 0 \} \)
In other words, the set \( G := \{ z \in U : \varphi(z) < 0 \} \) is strictly \( q \)-pseudoconvex in \( U \), contains \( L \), and \( K \) touches \( bG \) from the inside of \( G \) only at the point \( z_0 \).

**Proof.** We proceed similarly to the proof of Proposition 3.2 in [HST17]. Let \( \delta > 0 \) and \( V_\delta := B_{1/\delta}(0) \cap \{ z \in V : d(z,bV) > \delta \} \). We choose \( \delta > 0 \) so small that the conditions (1) to (3) of this lemma still hold if we replace \( V \) by \( U := V_\delta \).

Let \( B := B_\delta(0) \) and consider the function \( f: B \to \psi(U) \) defined by \( f(w) := \max_{z \in K} \psi(z + w) \). Pick a point \( p \in K \cap \{ z \in U : \psi(z) = 0 \} \). Since \( \psi \) is strictly \( q \)-plurisubharmonic, it follows from the local maximum property (see Theorem 2.6) that \( \{ \psi > 0 \} \cap W \) is not empty for any neighborhood \( W \) of \( p \). Hence, since \( p \) belongs to \( K \) and since \( f(0) = \psi(p) = 0 \), the image \( f(B) \) contains a non-empty open interval \( I = (0, \delta') \) for some \( \delta' > 0 \). Since \( f(B) \) lies in \( \psi(U) \), Sard’s theorem implies that there exists a regular value \( f(w_0) \) inside \( I \) which is so close to \( \psi(p) = 0 \) that the conditions (1) to (3) are still valid for the function \( \psi_0(z) := \psi(z + w_0) - f(w_0) \) instead of \( \psi \). Notice that 0 is now a regular value for \( \psi_0 \).

Let \( z_0 \) be a point in \( K \) with \( f(w_0) = \psi(z_0 + w_0) \), so that \( \psi_0(z_0) = 0 \). For \( \varepsilon > 0 \), we define \( \varphi(z) := \psi_0(z) - \varepsilon |z - z_0|^2 \). Then it is easy to see that \( K \cap \{ z \in U : \varphi(z) = 0 \} \) only contains the point \( z_0 \), so we also obtain property (c). Now if \( \varepsilon > 0 \) is small enough, then the function \( \varphi \) is still strictly \( q \)-plurisubharmonic. Besides of that, the function \( \varphi \) fulfills also the properties (a) and (b). Finally, by the choice of \( f(w_0) \), zero is a regular value for \( \varphi \), so we also gain the property (d).

The next result is the second duality theorem and a converse statement to Theorem 4.1.

**Theorem 4.4.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \) which is not \( q \)-pseudoconvex. Then there exist a point \( p \in b\Omega \), a neighborhood \( V \) of \( p \) and a strictly Levi \( (n-q-2) \)-pseudoconvex set \( G \) in \( V \) such that the set \( V \setminus \Omega \) is contained in \( G \cup \{ p \} \) and \( \Omega \) touches \( bG \) from the inside of \( G \) only at \( p \).

**Proof.** Since \( \Omega \) is not \( q \)-pseudoconvex, there exists a \((q+1,n-q-1)\)-Hartogs figure \((H,P)\) and a biholomorphic mapping \( F \) on \( \Delta := \Delta_q^+(0) \) onto its image in \( \mathbb{C}^n \) such that \( H = F(H_e) \) lies in \( \Omega \), but \( P = F(\Delta_q^+(0)) \) is not contained entirely in \( \Omega \) for the Euclidean Hartogs figure (recall Definition 5.1).

\[
H_e = (\Delta_{p+1}^+ \times \Delta_{n-q-1}^-) \cup (A_{R,1}^{q+1} \times \Delta_1^{n-q-1}) \subset \mathbb{C}_z^{q+1} \times \mathbb{C}_w^{n-q-1}.
\]

By shrinking \( \Delta \) if necessary, we can assume that \( F \) is defined on a neighborhood of the closure of \( \Delta \). We set \( M := F^{-1}(\mathbb{C}^n \setminus \Omega) \cap \Delta \). Now let \( \alpha, \beta \in (0,1) \) and set

\[
K_0 := (\Delta_1^{q+1} \times \Delta_0^{n-q-1}) \cap M \quad \text{and} \quad L_0 := (\Delta_1^{q+1} \times A_{\beta,1}^{n-q-1}) \cap M.
\]
Since $\Phi(H_{\alpha})$ lies in $\Omega$, we can find an appropriate $\alpha \in (0,1)$ such that $K_0$ is not empty. Fix some $\beta \in (0,1)$ with $\alpha < \beta$. Recall that $|w|_{\infty} = \max_{j=1,\ldots,n-q-1} |w_j|$ and consider the function $u(w) := -\log |w|_{\infty}$. By the assumptions made on $H$ and $P$, we can find a large enough number $c \in \mathbb{R}$ such that $M \subset D_c(u) := \{(z,w) \in \Delta : u(w) < c\}$. (4.2)

Let $k \in \mathbb{N}$ and define the function $u_k$ by

$$u_k(w) := -\frac{1}{k} \log |(w^1,\ldots,w^{n-q-1})| + \frac{1}{k} |w|^2.$$  

Then the function $u_k$ is $C^\infty$-smooth and strictly $(n-q-2)$-plurisubharmonic on $\mathbb{C}_w^{n-q-1} \setminus \{0\}$ \cite{PZ15}. Moreover, the sequence $(u_k)_{k \in \mathbb{N}}$ converges to $u$ uniformly on compact sets in $\mathbb{C}_w^{n-q-1} \setminus \{0\}$. Therefore, and in view of property (4.2), we can pick an integer $k_0$ so large that $M$ lies in $D_{c_k}(u_k) := \{(z,w) \in \Delta : u_k(w) < c_k\}$ for every $k \geq k_0$. Define

$$c_k := \inf \{a \in \mathbb{R} : M \subset D_a(u_k)\}.$$  

Now we fix an even larger $k \geq k_0$ so that $L_0 \cap D_{c_k}(u_k)$ is empty. Then it is easy to see that $K_0$ intersects $\{(z,w) \in \Delta : u_k(w) = c_k\}$ in a point $\zeta_0 \in \Delta$. Finally, we set $U := F^{-1}(\Delta)$, $K := F^{-1}(K_0)$, $L := F^{-1}(L_0)$ and $\psi := u_k \circ F^{-1}$ and verify that the conditions (1) to (3) in Lemma 4.3 all are satisfied. Thus, it follows from this lemma that there are a point $p$ in $b\Omega$, a neighborhood $V$ of $p$ and a strictly $(n-q-2)$-plurisubharmonic function on $V$ such that the set $G := \{z \in V : \varphi(z) < 0\}$ is the desired strictly Levi $(n-q-2)$-pseudoconvex set in $V$, whose boundary $bG$ shares only a single point with $b\Omega$ in $V$. \hfill $\square$

5. On $q$-PSEUDOCONCAVE GRAPHS

In this section, we will analyze whether submanifolds or graphs of a continuous function admit a local complex foliation under the condition that its complement is $q$-pseudoconvex. The goal is to generalize Hartogs’ theorem and the results in \cite{Shc93} and \cite{Chi01}.

**Theorem 5.1** (Hartogs, 1909). A continuous function $f : G \rightarrow \mathbb{C}_z$ is holomorphic on a domain $G \subset \mathbb{C}^2_z$ if and only if the complement of its graph $\Gamma(f) = \{(z,f(z)) : z \in G\}$ is pseudoconvex in $G \times \mathbb{C}_z$.

In order to simplify our notations, we introduce a generalized version of concavity.

**Definition & Remark 5.2.** Let $q \in \{0,\ldots,N\}$ and let $S$ be a closed subset of an open set $\Omega$ in $\mathbb{C}^N$. 


We say that $S$ is (Hartogs) $q$-pseudoconcave in $\Omega$ if $\Omega' := \Omega \setminus S$ is (Hartogs) $q$-pseudoconvex in $\Omega$, i.e., for every point $p$ in $bS$ there exists a ball $B$ in $\Omega$ such that $B \cap \Omega'$ is (Hartogs) $q$-pseudoconvex.

In view of Theorem 3.3 the set $S$ is $q$-pseudoconcave in $\Omega$ if and only if it is Hartogs $(N - q - 1)$-pseudoconcave in $\Omega$. For the sake of a better presentation, we shall prefer, only in this section, the notion of Hartogs $q$-pseudoconcavity rather than $q$-pseudoconcavity.

Using the duality theorems of the previous section, we obtain the first relation of foliated sets and $q$-pseudoconcavity.

**Proposition 5.3.** Let $q \in \{1, \ldots, N - 1\}$ and let $S$ be a closed subset of an open set $\Omega$ in $\mathbb{C}^N$. Assume that the boundary $b\Omega S$ of $S$ in $\Omega$ is locally filled by $q$-dimensional analytic sets, i.e., for every point $p$ in $b\Omega S$ there is a neighborhood $W$ of $p$ in $\Omega$ such that for each point $z$ in $b\Omega S \cap W$ there exists a $q$-dimensional analytic subset $A_z$ of $W$ with $z \in A_z \subset S$. Then $S$ is Hartogs $q$-pseudoconcave in $\Omega$.

**Proof.** Assume that the statement is false. Then, according to Theorem 4.4, there exist a boundary point $p$ of $S$ in $\Omega$, a neighborhood $V$ of $p$ and a strictly $(q - 1)$-pseudoconvex set $G$ in $V$ such that $S \cap V$ touches $bG$ from the inside of $G$ exactly in $p$. In view of Remark 3.8, we can construct a strictly $(q - 1)$-plurisubharmonic function $\psi$ on some neighborhood $U$ of $p$ in $V$ which defines $G$ near $p$. By the assumption made on $b\Omega S$, there are a neighborhood $W$ of $p$ and a $q$-dimensional analytic subset $A$ of $W$ with $p \in A \subset S$. But then $\psi(p) = 0$ and $\psi < 0$ on $A \cap U$ outside $p$, which contradicts the local maximum principle (see Theorem 2.6). Therefore, $S$ has to be Hartogs $q$-pseudoconcave in $\Omega$. \hfill \Box

We present a converse statement on the complex foliation of Hartogs $q$-pseudoconcave CR-submanifolds. For this we need to extend Definition 3.7 as follows.

**Definition 5.4.** Let $\Gamma = \{\varphi_1 = \ldots = \varphi_r = 0\}$ be a $C^2$-smooth submanifold in $\mathbb{C}^N$ such that $\nabla \varphi_j(p) \neq 0$ for each $j = 1, \ldots, r$.

1. The holomorphic tangent space $H_{p,\Gamma}$ to $\Gamma$ at some point $p$ in $\Gamma$ is given by

$$H_{p,\Gamma} := \bigcap_{j=1}^r \left\{ X \in \mathbb{C}^N : (\partial^\varphi_j(p), X) = \sum_{l=1}^N \frac{\partial^\varphi_j}{\partial z_l}(p)X_l = 0 \right\}.$$

2. If the complex dimension of $H_{p,\Gamma}$ has the same value $d$ at each point $p$ in $\Gamma$, then we say that $\Gamma$ is a CR-submanifold.

3. The Levi null space of $\Gamma$ at $p$ is the set

$$N_p := \bigcap_{j=1}^r \left\{ X \in H_{p,\Gamma} : L_{\varphi_j}(p)(X, Y) = 0 \text{ for every } Y \in H_{p,\Gamma} \right\}.$$
Proposition 5.5. Let $\Gamma = \{\varphi_1 = \ldots = \varphi_r = 0\}$ be a real $C^2$-smooth CR-submanifold of some open set $\Omega$ in $\mathbb{C}^N$ of codimension $r \in \{1, \ldots, 2N - 1\}$ and fix a number $q \in \{1, \ldots, N - 1\}$. Assume further that the complex dimension of the holomorphic tangent space to $\Gamma$ at every point in $\Gamma$ equals $q$ and that it is Hartogs $q$-pseudoconcave in $\Omega$. Then $\Gamma$ is locally foliated by complex $q$-dimensional submanifolds.

Proof. Since the Levi null space lies inside the holomorphic tangent space to $\Gamma$, it is clear that its complex dimension does not exceed $q$. We claim that the complex dimension of $N_p$ is equal to $q$ for each point $p \in \Gamma$, so that $N_p$ coincides with $H_p\Gamma$.

In order to get a contradiction, suppose that there is a point $p$ in $\Gamma$ such that $N_p$ is a proper subspace of $H_p\Gamma$. This implies that there is an index $j_0$ in $\{1, \ldots, r\}$ and a vector $X_0$ in $H_p\Gamma$ such that $L_{\varphi_{j_0}}(p, X_0, X_0) \neq 0$. Indeed, a priori, if $N_p \subseteq H_p\Gamma$, there exist two vectors $X'$ and $Y'$ in $H_p\Gamma$ such that

$$L_{\varphi_{j_0}}(p)(X', Y') \neq 0.$$ 

If $L_{\varphi_{j_0}}(p, X', X') \neq 0$ or $L_{\varphi_{j_0}}(p, Y', Y') \neq 0$, we are done and proceed by picking $X_0 = X'$ or, respectively, $X_0 = Y'$. Otherwise, if $L_{\varphi_{j_0}}(p, X', X')$ and $L_{\varphi_{j_0}}(p, Y', Y')$ both vanish, we can choose an appropriate complex number $\nu$ which satisfies

$$L_{\varphi_{j_0}}(p, X' + \nu Y', X' + \nu Y') = 2\text{Re}(\nu L_{\varphi_{j_0}}(p)(X', Y')) \neq 0.$$ 

Then we continue with $X_0 := X' + \nu Y'$. Now without loss of generality we can assume that $j_0 = 1$ and $L_{\varphi_1}(p, X_0, X_0) > 0$. For a positive constant $\mu$ we define another function

$$\varphi := \varphi_1 + \mu \sum_{j=1}^{r} \varphi_j^2.$$ 

Since, by the assumptions on $\Gamma$, the gradients $\nabla \varphi_1, \ldots, \nabla \varphi_r$ do not vanish at $p$, there is a neighborhood $U$ of $p$ such that the set $S := \{z \in U : \varphi(z) = 0\}$ is a real hypersurface containing $\Gamma \cap U$, so that $H_p\Gamma$ becomes a subspace of $H_pS$. Moreover, for $X \in H_p\Gamma$ we can easily compute the Levi form of $\varphi$ at $p$,

$$L_{\varphi}(p, X, X) = L_{\varphi_1}(p, X, X) + 2\mu \sum_{j=1}^{r} |(\partial \varphi_j(p), X)|^2, \quad (5.1)$$

We assert that $H_pS$ contains an $(N - q)$-dimensional subspace $E$ on which $L_{\varphi}(p, \cdot)$ is positive. To see this, consider the complex normal space $N_p\Gamma$ to $H_p\Gamma$ in $H_pS$,

$$N_p\Gamma := \{Y \in H_pS : \sum_{l=1}^{N} Y_lX_l = 0 \text{ for every } X \in H_p\Gamma\}.$$
Observe that $N_p \Gamma$ has dimension $d := N - q - 1$ and choose a basis $Y_1, \ldots, Y_d$ of $N_p \Gamma$. Let $E$ be the complex span of the vectors $X_0$ from above and $Y_1, \ldots, Y_d$. Since $X_0$ belongs to $H_p \Gamma$, but $Y_1, \ldots, Y_d$ do not, the dimension of $E$ equals $N - q$.

We set $E_0 := \{ Z \in E : |Z| = 1 \}$ and $M := \{ Z \in E_0 : \mathcal{L}_{\varphi_1}(p, Z, Z) \leq 0 \}$.

If $M$ is empty, then $\mathcal{L}_{\varphi}(p, \cdot, \cdot)$ is positive on $E$ and we can put $\mu = 0$.

If $M$ is not empty, notice first that, if $Z$ lies in $M$, then $R(p, Z) > 0$ (recall the equation (5.1) for the definition of $R(p, Z)$). Otherwise $Z$ belongs to $H_p \Gamma$ and, therefore, it is a multiple of $X_0$, i.e., $Z = \lambda X_0$ for some complex number $\lambda$. But then $\mathcal{L}_{\varphi_1}(p, Z, Z) = |\lambda|^2 \mathcal{L}_{\varphi_1}(p, X_0, X_0) > 0$ and $Z$ lies in $M$ at the same time, which is absurd. Hence, $R(p, Z) > 0$ for every vector $Z$ in $M$. Since $E_0$ is compact and $\Gamma$ is $C^2$-smooth, we can find constants $c_0 > 0$ and $c_1 > 0$ such that $R(p, Z) \geq c_0$ for every $Z$ in $M$ and $\mathcal{L}_{\varphi_1}(p, Z, Z) \geq -c_1$ for every $Z$ in $E_0$. Now we can choose $\mu$ so large that $-c_1 + \mu c_0 > 0$ in order to obtain that $\mathcal{L}_{\varphi}(p, Z, Z) > 0$ for each $Z$ in $E_0$. Since $\mathcal{L}_{\varphi}(p, \lambda X, \lambda X) = |\lambda|^2 \mathcal{L}_{\varphi}(p, X, X)$ for every $X$ in $E$ and $\lambda$ in $\mathbb{C}$, we have that $\mathcal{L}_{\varphi}(p, \cdot, \cdot)$ is positive on $E \setminus \{0\}$.

Therefore, in both cases, the Levi form $\mathcal{L}_{\varphi}$ at $p$ is positive definite on the $(N - q)$-dimensional space $E$. Hence, $\{ \varphi < 0 \}$ is strictly $(q - 1)$-pseudoconvex at $p$. But then, in view of Theorem 4.1, the submanifold $\Gamma$ cannot be Hartogs $q$-pseudoconcave near $p$, which is a contradiction. Finally, we can conclude that $N_p = H_p \Gamma$. By assumption, these two spaces have constant dimension $q$ on $\Gamma$, so Theorem 1.1 in \cite{Fre74} implies that $\Gamma$ admits a local foliation by complex $q$-dimensional submanifolds. \qed

In the previous statement we are in a comfortable situation of a smooth submanifold. Locally such a smooth submanifold can be described as the graph of a smooth mapping $f : G \subset \mathbb{C}^q \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{C}^p$. Now if we drop the smoothness assumption and replace it by continuity, the graph of $f$ can be considered as a “continuous CR-surface” whose complex structure we may study.

**Main setting.** Fix integers $n \geq 1$ and $k, p \geq 0$ such that $N = n + k + p \geq 2$. Then $\mathbb{C}^N$ splits into the product

$$
\mathbb{C}^N = \mathbb{C}^N_{z, w, \zeta} = \mathbb{C}^n_w \times \mathbb{C}^k_z \times \mathbb{C}^p_{\zeta} = \mathbb{C}^n_z \times (\mathbb{R}^k_u + i\mathbb{R}^k_{\zeta}) \times \mathbb{C}^p_{\zeta},
$$

where $w = u + iv$. Let $G$ be an open set in $\mathbb{C}^n_z \times \mathbb{R}^k_{u}$ and let $f = (f_v, f_{\zeta})$ be continuous on $G$ with image in $\mathbb{R}^k_u \times \mathbb{C}^p_{\zeta}$. Then the graph of $f$ is given by

$$
\Gamma(f) = \{(z, w, \zeta) \in \mathbb{C}^n_z \times \mathbb{C}^k_u \times \mathbb{C}^p_{\zeta} : (z, u, \zeta) \in G, (v, \zeta) = f(z, u)\}.
$$

Moreover, we denote by $\pi_{z, u}$ the natural projection

$$
\pi_{z, u} : \mathbb{C}^n_z \times \mathbb{C}^k_u \to \mathbb{C}^n_z \times \mathbb{R}^k_u, \quad \pi_{z, u}(z, w) \mapsto (z, u).
$$
We are interested in the question whether $\Gamma$ admits a local foliation by complex submanifolds. In this context, we first have to study the $q$-pseudoconcavity of the graph of $f$.

**Lemma 5.6.** Pick another integers $m \in \{1, \ldots, n\}$ and $r \in \{0, \ldots, p\}$ with $k + r \geq 1$. For $\mu_1, \ldots, \mu_r \in \{1, \ldots, p\}$ with $\mu_1 < \ldots < \mu_r$, we divide the coordinates of $\zeta$ into $\zeta' = (\zeta_{\mu_1}, \ldots, \zeta_{\mu_r})$ and the remaining coordinates $\zeta'' = (\zeta_j : j \in \{1, \ldots, p\} \setminus \{\mu_1, \ldots, \mu_r\})$ which we assume to be ordered by their index $\mu_j$, as well. Finally, let $\Pi$ be a complex $m$-dimensional plane in $\mathbb{C}_z^m$. We set $M = m + k + r$ and $\mathbb{C}_M := \Pi \times \mathbb{C}_w^k \times \mathbb{C}_\zeta'$, $G := \Pi \cap (\Pi \times \mathbb{R}_k^\mu)$ and $f_* := (f_v, f_{\zeta'})|_{G_*}$.

If the graph $\Gamma(f)$ is Hartogs $n$-pseudoconcave in $G \times \mathbb{R}_k^k \times \mathbb{C}_\zeta$, then the graph $\Gamma(f_*)$ is Hartogs $m$-pseudoconcave in $G_* \times \mathbb{R}_k^k \times \mathbb{C}_\zeta'$.

**Proof.** Since the Hartogs $n$-pseudoconcavity is a local property, after shrinking $G$ if necessary and after a biholomorphic change of coordinates we can assume without loss of generality that $\Pi = \{0\}^{n-m} \times \mathbb{C}_w^m \subset \mathbb{C}_z^n$, where $z' = (z_1, \ldots, z_{n-m})$ and $z'' = (z_{n-m+1}, \ldots, z_n)$, and that the $\zeta$-coordinates are ordered in such a way that $\zeta' = (\zeta_1, \ldots, \zeta_r)$ and $\zeta'' = (\zeta_{r+1}, \ldots, \zeta_p)$.

Assume that $\Gamma(f_*)$ is not Hartogs $m$-pseudoconvex in $G_* \times \mathbb{R}_k^k \times \mathbb{C}_\zeta'$ and recall that $M = m + k + r$. Then in view of Theorem 3.3 (3) and Theorem 3.3 there are a point $p$ in $\Gamma(f_*)$ and a ball $B = B_\rho(p)$ in $\mathbb{C}_M$ such that the set $(\mathbb{C}_M \setminus \Gamma(f_*)) \cap B$ is not $(M - m - 1) = (k + r - 1)$-pseudoconvex. Since $B$ is pseudoconvex, according to Theorem 3.3 (4) there is a family $\{A_t\}_{t \in [0, 1]}$ of $(k + r)$-dimensional analytic sets $A_t$ in $\mathbb{C}_M$ which depends continuously on $t$ and which fulfills the following properties:

- The closure of the union $\bigcup_{t \in [0, 1]} A_t$ is compact.
- For every $t \in [0, 1)$ the intersection $\overline{A_t} \cap \Gamma(f_*)$ is empty.
- $\partial A_t \cap \Gamma(f_*)$ is empty, as well.
- The set $A_1$ touches $\Gamma(f_*)$ at a point $p_0 = (z_0, w_0, \zeta'_0)$, where $z_0 = (z'_0, z''_0) = (0, z''_0)$ and $w_0 = u_0 + iv_0$.

Given some positive number $\rho$, consider the $(k + p)$-dimensional analytic sets

$$S_t := \{0\}^{n-m} \times A_t \times \Delta_{\rho}^{p-r}(f_{\zeta''}(z_0, u_0)) \subset \mathbb{C}_N.$$ 

It is easy to verify that the family $\{S_t\}_{t \in [0, 1]}$ of $(k + p)$-dimensional analytic sets violates the property (4) of Theorem 3.3. According to Theorem 3.3 (1) $\Gamma(f)$ cannot be Hartogs $n$-pseudoconvex, which is a contradiction to the assumption on $\Gamma(f)$. Hence, $\Gamma(f_*)$ has to be Hartogs $m$-pseudoconcave.

We are now able to prove the main theorem.
**Theorem 5.7.** Let $n, k, p$ be integers with $n \geq 1$, $p \geq 0$ and $k \in \{0, 1\}$ such that $N = n + k + p \geq 2$. Let $G$ be a domain in $\mathbb{C}^n \times \mathbb{R}^k$ and let $f : G \to \mathbb{R}^p \times \mathbb{C}_\zeta$ be a continuous function such that $\Gamma(f)$ is Hartogs $n$-pseudoconcave. Then $\Gamma(f)$ is locally the disjoint union of $n$-dimensional complex submanifolds.

**Proof.** The statement is of local nature, so we can assume that $G$ is an open ball $B$ in $\mathbb{C}^n \times \mathbb{R}^k$ and that $\Gamma(f)$ is bounded. We separate the problem into the subsequent cases.

**Case $n \geq 1$, $k = 0$, $p = 1$.** This is the classical Hartogs’ theorem (see [Sha92], Chapter III.42, Theorem 2).

**Case $n \geq 1$, $k = 0$, $p \geq 1$.** For each $j \in \{1, \ldots, p\}$ the set $\Gamma(f_{\zeta_j})$ is Hartogs $n$-pseudoconcave by Lemma 5.6. By Hartogs’ theorem, the functions $f_j$ and therefore, the mapping $f = (f_1, \ldots, f_p)$ are holomorphic which means that $\Gamma(f)$ is a complex hypersurface.

**Case $n = 1$, $k = 1$, $p = 0$.** This was proved by the second author in [She93].

**Case $n \geq 1$, $k = 1$, $p = 0$.** This case has been treated by Chirka in [Chi01].

**Case $n = 1$, $k = 1$, $p = 1$.** By Lemma 5.6 the graph $\Gamma(f_\zeta)$ is Hartogs 1-pseudoconcave. According to [She93], it is locally foliated by a family of holomorphic curves $\{\gamma_\alpha\}_{\alpha \in I}$ represented as graphs of holomorphic functions $g_\alpha$ which are all defined on a disc $D$ in $\mathbb{C}_z$ that does not depend on the indexes $\alpha \in I$. Denote by $\pi_z$ the standard projection of points in $\mathbb{C}^2_{z,u}$ into $\mathbb{C}_z$. We define another curves $f^\alpha_{\zeta}$ by the assignment

$$
\gamma_\alpha \ni t \mapsto f^\alpha_{\zeta}(t) := f_\zeta(\pi_z(t), \Re(g_\alpha)(\pi_z(t))).
$$

Since for each $\alpha \in I$ the curve $\gamma_\alpha$ is represented by the graph $\Gamma(g_\alpha)$, the function $f^\alpha_{\zeta} : \gamma_\alpha \to \mathbb{C}_z$ is well-defined, and its graph is given by $\Gamma(f^\alpha_{\zeta}) = \Gamma(f|_{\pi_z,\alpha}(\gamma_\alpha))$. Here, $\pi_{z,u}$ means the standard projection of $\mathbb{C}^2_{z,w}$ to $\mathbb{C}_z \times \mathbb{R}_u$. We claim that the curve $\Gamma(f^\alpha_{\zeta})$ is holomorphic.

Suppose that there is some graph $\Gamma(f^{\alpha_0}_{\zeta})$ which is not holomorphic in a neighborhood of a point $(z_0, w_0) \in \gamma_{\alpha_0}$. After a local holomorphic change of coordinates, we can assume that $\gamma_{\alpha_0} = \Delta_r(z_0) \times \{w = 0\}$, where $\Delta_r(z_0) \in D$ is a disc in $\mathbb{C}_z$ centered in $z_0$. After a reparametrization we can arrange that $\alpha_0 = 0$ and $(-1, 1) \subset I$. Since the curve $\gamma_0$ is of the form $\Delta_r(z_0) \times \{w = 0\}$ near $z_0$, we can treat $f^0_{\zeta}$ as a function $f^0_{\zeta} : \Delta_r(z_0) \to \mathbb{C}_z$. By our assumption that $f^0_{\zeta}$ is not holomorphic, in view of Hartogs’ theorem [5.1] it follows that the set $\Gamma(f^0_{\zeta})$ is not Hartogs 1-pseudoconcave in $\mathbb{C}^2_{z,\zeta}$. Then, by Theorem 4.3 there exist a point $p_1 = (z_1, \zeta_1) \in \Gamma(f^0_{\zeta})$, a small enough open neighborhood $V$ of $p_1$ in $\mathbb{C}^2_{z,\zeta}$, a $C^2$-smooth strictly plurisubharmonic function $\varrho_1 = \varrho_1(z, \zeta)$ on $V$ with $\nabla \varrho_1 \neq 0$, and radii $\sigma, r', r'' > 0$ with $r'' < r' < r$.
such that
\[
\Delta_{\sigma'} \times \Delta_{\sigma} \subset V, \quad \Gamma(f_{\xi}^0|_{\Delta_{\sigma'}}) \subset \Delta_{\sigma'} \times \Delta_{\sigma},
\]  
(5.2)

\[
\Gamma(f_{\xi}^0|_{\Delta_{\sigma'}}) \subset \{ \varrho_1 \leq 0 \}, \quad \Gamma(f_{\xi}^0|_{\Delta_{\sigma'}}) \cap \{ \varrho_1 = 0 \} = \{ (z_1, \zeta_1) \},
\]  
and \( \Gamma(f_{\xi}^0|_{\Delta_{\sigma'}}) \cap \{ \varrho_1 = 0 \} = \emptyset \),
(5.3)

where each disc \( \Delta_{\sigma} \) mentioned above is assumed to be centered in \( z_1 \), \( A_{\sigma',\sigma''} := \Delta_{\sigma'} \setminus \overline{\Delta_{\sigma}} \) and the disc \( \Delta_{\sigma} \) is assumed to be centered in \( \zeta_1 \). For \( \alpha \in (-1, 1) \) we set \( \gamma_{\alpha} := \Gamma(g_{\alpha}|_{\Delta_{\sigma'}}) \) and \( \Gamma_{\alpha} := \Gamma(f_{\xi}^0|_{\gamma_{\alpha}}) \). Since \( f \) is continuous and since the family \( \{ \gamma_{\alpha} \}_{\alpha \in I} \) depends continuously on \( \alpha \), it follows from (5.3) that there is a number \( \tau \in (0, 1) \) such that
\[
K := \bigcup_{\alpha \in [-\tau, \tau]} \Gamma_{\alpha} \subset \Delta_{\sigma'} \times \mathbb{C}_w \times \Delta_{\sigma},
\]  
(5.4)

and \( \varrho_1 < 0 \) on \( \Gamma_{\alpha} \cap \big( \overline{A_{\sigma',\sigma''}} \times \mathbb{C}^2_{w,\zeta} \big) \) for every \( \alpha \in [-\tau, \tau] \),

where \( \varrho_1 \) is now considered as a function defined on \( \{(z, w, \zeta) \in \mathbb{C}^3 : (z, \zeta) \in V \} \).

Since the curves in the family \( \{ \gamma_{\alpha} \}_{\alpha \in I} \) are holomorphic, the set \( A := (\gamma_{-\tau} \cup \gamma_{\alpha_0} \cup \gamma_{\tau}) \cap (\Delta_{\sigma} \times \mathbb{C}_w) \) is a closed analytic subset of the pseudoconvex domain \( \Delta_{\sigma} \times \mathbb{C}_w \). Let \( h \) be a holomorphic function on \( A \) defined by \( h \equiv 0 \) on \( \gamma_{\pm \tau} \) and \( h \equiv 1 \) on \( \gamma_{\alpha_0} \). Then there exists a holomorphic extension \( \hat{h} \) of \( h \) into the whole of \( \Delta_{\sigma} \times \mathbb{C}_w \) (see Theorem 4 in paragraph 4.2 of chapter V in [GR04]). Hence, the function \( \varrho_2(z, w) := \log |\hat{h}(z, w)| \) is plurisubharmonic on \( \Delta_{\sigma} \times \mathbb{C}_w \) and satisfies \( \varrho_2 \equiv -\infty \) on \( \gamma_{\pm \tau} \). Now for \( \varepsilon > 0 \) we define
\[
\psi_0(z, w, \zeta) := \varrho_1(z, \zeta) + \varepsilon \varrho_2(z, w),
\]
where \( \varrho_1 \) is the defining function from above. By the inequality (5.4) and the properties of \( \varrho_2 \), for a sufficiently small \( \varepsilon > 0 \) we obtain that
\[
\psi_0 < 0 \text{ on } L := \bigcup_{\alpha \in [-\tau, \tau]} \left( \Gamma_{\alpha} \cap \big( \overline{A_{\sigma',\sigma''}} \times \mathbb{C}^2_{w,\zeta} \big) \right) \cup \Gamma_{\tau} \cup \Gamma_{-\tau}.
\]  
(5.5)

By the choice of the point \( (z_1, \zeta_1) \) above and by the inclusion \( (z_1, 0) \in \gamma_{\alpha_0} \), we have that \( \varrho_1(z_1, \zeta_1) = 0 \), \( \varrho_2(z_1, 0) = 0 \) and, therefore, \( \psi_0(z_1, 0, \zeta_1) = 0 \). Since \( (z_1, 0, \zeta_1) \) belongs to \( K \), it follows from the inequality (5.5) that \( \psi_0 \) attains a non-negative maximal value on \( K \) outside \( L \). Since \( \psi_0 \) is plurisubharmonic on a neighborhood of \( K \), by using standard methods on the approximation of plurisubharmonic functions
we can assume without loss of generality that $\psi_0$ is $C^\infty$-smooth and strictly plurisubharmonic on a neighborhood of $K$, satisfies the property \((5.5)\) and still attains its maximum on $K$ outside $L$.

Now it is easy to verify that $K, L$ and $\psi := \psi_0 - \max_K \psi_0$ fulfill all the conditions (1) to (3) of Lemma \[4.3\]. Thus, there exist a point $p_2$ in $K \setminus L$, a neighborhood $U$ of $p_2$ containing $K$ and $L$ and a $C^2$-smooth strictly plurisubharmonic function $\varphi$ on $U$ so that $G := \{(z, w, \zeta) \in U : \varphi(z, w, \zeta) < 0\}$ is strictly pseudoconvex in $U$, $\varphi < 0$ on $L$, $\varphi \leq 0$ on $K$ and $\varphi(z, w, \zeta)$ vanishes on $K$ if and only if $(z, w, \zeta) = p_2$.

Since $G$ is strictly pseudoconvex at $p_2$, we derive from Theorem \[4.4\] that the graph $\Gamma(f)$ cannot be 1-pseudoconcave, which is a contradiction to the assumption made on $\Gamma(f)$. As a conclusion, the curves in $\{\Gamma(f_\zeta^\alpha)\}_{\alpha \in I}$ have to be holomorphic. This leads to the desired local complex foliation of $\Gamma(f)$.

**Case $n \geq 1$, $k = 1$, $p = 1$.** According to Lemma \[5.6\] with $m = n$ and $r = 0$, the graph $\Gamma(f_e)$ is Hartogs $n$-pseudoconcave in $B \times \mathbb{R}_+$, where $B$ is a ball in $\mathbb{C}_z^n \times \mathbb{R}_ u$.

Hence, by Chirka’s result (see the case $n \geq 1$, $k = 1$, $p = 0$), the graph $\Gamma(f_e)$ is foliated by a family $\{A_\alpha\}_{\alpha \in I}$ of holomorphic hypersurfaces $A_\alpha$. For $\alpha \in I$ define the function

$$
\left. f_\zeta^\alpha : A_\alpha \rightarrow C_\zeta \right| \text{ by } \left. f_\zeta^\alpha = f_\zeta|_{\pi_{z,u}(A_\alpha)} \right.
$$

and identify $\Gamma(f_\zeta^\alpha)$ with $\Gamma(f|_{\pi_{z,u}(A_\alpha)})$. Suppose that some function $f_\zeta^{\alpha_0}$ is not holomorphic. Then, by Hartogs’ theorem of separate holomorphicity, there is a complex one-dimensional curve $\sigma_{\alpha_0}$ in $A_{\alpha_0}$ on which $f_\zeta^{\alpha_0}$ is not holomorphic near a point $p_0 \in \sigma_{\alpha_0}$. After a change of coordinates we can assume that $p_0 = 0$, $f_\zeta^{\alpha_0}(0) = 0$ and $\sigma_{\alpha_0} = \Delta \times \{z_2 = \ldots = z_n = w = 0\}$ in a neighborhood of 0, where $\Delta$ is the unit disc in $\mathbb{C}_{z_1}$. We set $L := \mathbb{C}_{z_1} \times \{0\}^{n-1}$. By Lemma \[5.6\] the graph $\Gamma(f_\bullet)$ of $f_\bullet := f|_{(B \setminus (L \times \mathbb{R}_+))}$ is Hartogs 1-pseudoconcave in $\mathbb{C}_{z_1,w,\zeta}$. Thus, in view of the considered above case $n = k = p = 1$, the graph $\Gamma(f_\bullet)$ is foliated by complex curves of the form

$$
(f_\bullet|\zeta)^\beta : \gamma_\beta \rightarrow C_\zeta \quad \text{with} \quad (f_\bullet|\zeta)^\beta = (f_\bullet|_{\pi_{z,u}(\gamma_\beta)}),
$$

where $\{\gamma_\beta\}_{\beta \in I}$ is a family of holomorphic curves of a foliation of $\pi_{z_1,w}(\Gamma(f_\bullet))$. From the uniqueness of the foliation on $\pi_{z_1,w}(\Gamma(f_\bullet))$ we deduce that $\pi_{z_1,w}(\sigma_{\alpha_0})$ coincides (at least locally) with a curve $\gamma_{\beta_0}$ containing 0. Hence, in some neighborhood of 0 we have that $\gamma_{\beta_0} = \Delta \times \{0\}$ and therefore

$$
\left. f_\zeta^{\alpha_0} \right|_{\sigma_0} = \left. f_\zeta|_{\Delta \times \{z_2 = \ldots = z_n = 0\} \times \{u = 0\}} \right.
$$

$$
= \left. (f_\bullet|_{\Delta \times \{u = 0\}}) \right|_{\pi_{z,u}(\gamma_{\beta_0})} = (f_\bullet|_{\zeta})_{\beta_0}.
$$
This means that \( f^\alpha_0 \) has to be holomorphic on a neighborhood of 0 in \( \sigma_\alpha_0 \), which is a contradiction to the choice of \( f^\alpha_0 \) and \( \sigma_\alpha_0 \). Hence, \( \{ \Gamma(f^\alpha_0) \}_{\alpha \in I} \) is the desired foliation of \( \Gamma(f) \).

**Case \( n = 1, k = 1, p \geq 1 \).** We derive from Lemma 5.6 with \( m = 1 \) and \( r = 0 \) that the graph \( \Gamma(f_v) \) is Hartogs 1-pseudoconcave. It follows then from the theorem of the second author (see case \( n = k = 1, p = 0 \) above) that the graph \( \Gamma(f_v) \) is foliated by the family \( \{ \gamma_\alpha \}_{\alpha \in I} \) of holomorphic curves \( \gamma_\alpha \). Define similarly to the previous cases for \( \alpha \in I \) the mapping \( f^\alpha_\zeta = (f^{\alpha_1}_\zeta, \ldots, f^{\alpha_p}_\zeta) : \gamma_\alpha \to \mathbb{C}^p_\zeta \) by \( f^\alpha_\zeta := f^\alpha|_{\pi_{z,u}(\gamma_\alpha)} \).

**(5.6)**

Since \( \Gamma(f_v, f_j) \) are Hartogs 1-pseudoconcave due to Lemma 5.6 with \( m = 1 \) and \( r = 1 \), it follows by the same arguments as in the case \( n = k = p = 1 \) that for each \( j \in \{1, \ldots, p\} \) the component \( f^{\alpha_j}_\zeta : \gamma_\alpha \to \mathbb{C}_{\zeta_j} \) is holomorphic. Hence, the curve \( f^\alpha_\zeta \) is holomorphic, as well, so that \( \Gamma(f) \) is foliated by the family \( \{ \Gamma(f^\alpha_\zeta) \}_{\alpha \in I} \) of holomorphic curves.

**Case \( n \geq 1, k = 1, p \geq 1 \).** The proof is nearly the same as in the previous case \( n = k = 1, p \geq 1 \). We only need to replace the curves \( \{ \gamma_\alpha \}_{\alpha \in I} \) in (5.6) by complex hypersurfaces \( \{ A_\alpha \}_{\alpha \in I} \) obtained from Chirka’s result (case \( n \geq 1, k = 1, p = 0 \)) and to apply the case \( n \geq 1, k = 1, p = 1 \) to each \( j = 1, \ldots, p \) in order to show that \( f^{\alpha_j}_\zeta : A_\alpha \to \mathbb{C}_{\zeta_j} \) is holomorphic on \( A_\alpha \). Then \( \{ \Gamma(f^\alpha_\zeta) \}_{\alpha \in I} \) is a complex foliation of \( \Gamma(f) \).

The proof of the theorem is finally complete. \( \square \)

So far, we do not have techniques to treat the case \( n = 1, k = 2 \) and \( p \geq 0 \). The next example shows that it is not always possible to foliate a 1-pseudoconcave real 4-dimensional submanifold in \( \mathbb{C}^3 \) by complex submanifolds, but it is still possible to do this by analytic subsets.\(^1\)

**Example 5.8.** For a fixed integer \( k \geq 2 \) consider the function

\[
f(z_1, z_2) := \begin{cases} 
    z_1z_2^{2+k}/z_2, & \text{if } z_2 \neq 0 \\
    0, & \text{if } z_2 = 0
  \end{cases}
\]

It is \( \mathcal{C}^k \)-smooth on \( \mathbb{C}^2 \) and holomorphic on complex lines passing through the origin, since \( f(\lambda v) = \lambda^{2+k} f(v) \) for every \( \lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) and each vector \( v \in \mathbb{C} \). Therefore, in view of [Bas76], the function \( f \) is 1-holomorphic on \( \mathbb{C}^2 \), so \( \psi(z_1, z_2, w) := -\log |f(z_1, z_2) - w| \) is 1-plurisubharmonic outside \( \{ f = w \} \) by

---

\(^1\)Thanks to Prof. Kang-Tae Kim for this example.
Due to Theorem 3.3, this means that the graph $\Gamma(f)$ of $f$ is a 1-pseudo-concave real 4-dimensional submanifold of $\mathbb{C}^3$ which does not admit a regular foliation near the origin, but admits a singular one which is given by the family of holomorphic curves $\{\Gamma(f|_{C^*_v})\}_{v \in \mathbb{C}^2}$. Of course, the problem arises because the complex Jacobian of $f$ has non-constant rank near the origin.

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