A solution to the Pompeiu problem

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Abstract

Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \cap S \), where \( S \) is the Schwartz class of distributions, and

\[
\int_{\sigma(D)} f(x) \, dx = 0 \quad \forall \sigma \in G, \tag{*}
\]

where \( D \subset \mathbb{R}^n \) is a bounded domain, the closure \( \bar{D} \) of which is diffeomorphic to a closed ball, and \( S \) is its boundary. Then the complement of \( \bar{D} \) is connected and path connected. By \( G \) the group of all rigid motions of \( \mathbb{R}^n \) is denoted. This group consists of all translations and rotations. A proof of the following theorem is given.

**Theorem 1.** Assume that \( n = 2 \), \( f \not\equiv 0 \), and (*) holds. Then \( D \) is a ball.

**Corollary.** If the problem \((\nabla^2 + k^2)u = 0 \) in \( D \), \( u_N|_S = 0 \), \( u|_S = \text{const} \not= 0 \) has a solution, then \( D \) is a ball.

Here \( N \) is the outer unit normal to \( S \).

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**Key words:** The Pompeiu problem; symmetry problems.

1 Introduction

Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \cap S \), where \( S \) is the Schwartz class of distributions, and

\[
\int_{\sigma(D)} f(x) \, dx = 0 \quad \forall \sigma \in G, \tag{1}
\]

where \( G \) is the group of all rigid motions of \( \mathbb{R}^n \), \( G \) consists of all translations and rotations, and \( D \subset \mathbb{R}^n \) is a bounded domain, the closure \( \bar{D} \) of which is diffeomorphic to a closed ball.
Under these assumptions the complement of $D$ in $\mathbb{R}^n$ is connected and path connected (\cite{5}). By $S$ the boundary of $D$ is denoted, and $N$ denotes the unit normal to $S$ pointing out of $D$. In \cite{6} the following question was raised by D. Pompeiu:

Does (1) imply that $f = 0$?

If yes, then we say that $D$ has $P$-property (Pompeiu’s property), and write $D \in P$. Otherwise, we say that $D$ fails to have $P$-property, and write $D \not\in P$. Pompeiu claimed that every plane bounded domain has $P$–property, but a counterexample was given 15 years later in \cite{2}. The counterexample is a domain $D$ which is a disc, a ball in $\mathbb{R}^n$ for $n > 2$. If $D$ is a ball, then there are $f \neq 0$ for which equation (1) holds. The set of all $f \neq 0$, for which equation (1) holds, was constructed in \cite{7}. A bibliography on the Pompeiu problem ($P$–problem) can be found in \cite{14}. The results on $P$–problem which are used in this paper are derived in \cite{12}. The $P$–problem is equivalent to a symmetry problem, see Corollaries 1,2 below. The author’s results on other symmetry problems are given in \cite{10} and \cite{11}. The modern formulation of the $P$–problem is the following:

Prove that if $D \in \overline{P}$ then $D$ is a ball.

We use the word ball also in the case $n = 2$, when this word means disc, and solve the $P$–problem. The proof of Theorem 1 we give assuming $n = 2$, but this proof is easily generalized to the case $n > 2$. Our standing assumptions are:

Assumptions A: a) $D$ is a bounded domain, the closure of which is diffeomorphic to a closed ball, the boundary $S$ of $D$ is a closed connected $C^1$–smooth surface, b) $D$ fails to have $P$–property, and c) $n = 2$.

Theorem 1. If Assumptions A hold, then $D$ is a ball.

Corollary 1. If problem (3) (see below) has a solution, then $D$ is a ball.

Corollary 2. If the problem $\left( \nabla^2 + k^2 \right)u = 0$ in $D$, $u_N|_S = 0$, $u|_S = \text{const} \neq 0$ has a solution, then $D$ is a ball.

In Section 2 these results are proved.

2 Proof of Theorem 1

If Assumptions A hold, then the boundary $S$ of $D$ is real-analytic (see \cite{13}) and

\begin{equation}
\int_D e^{ik\alpha \cdot x} dx = 0, \quad \forall \alpha \in S^1,
\end{equation}

where $S^1$ is the unit sphere in $\mathbb{R}^2$, and $k > 0$ is a fixed number, see \cite{12}.

The following Lemmas 1-3 are proved in \cite{12} (Lemma 1 is Lemma 3 in \cite{12}, Lemma 2 is Lemma 5 in \cite{12}, and Lemma 3 is formula (32) in \cite{12}):
Lemma 1. If and only if relation (2) holds then the overdetermined problem (3)
\[(\nabla^2 + k^2)u = 1 \quad \text{in} \quad D, \quad u|_S = 0, \quad u_N|_S = 0,\]
has a solution.

Lemma 2. If (2) holds for all \(\alpha \in S^1\) then it holds for all \(\alpha \in M\), where \(M := \{z : z \in \mathbb{C}^2, z_1^2 + z_2^2 = 1\}\).

The \(M\) is an algebraic variety intersecting \(\mathbb{R}^2\) over \(S^1\).

Let us assume that the boundary \(S\) is star-shaped. Let \(r = f(\phi)\) be the equation of \(S\), where \(0 < c_2 \leq f \leq c_2, c_j\) are constants, \(j = 1, 2\), and \(f\) is a smooth \(2\pi\)-periodic function.

Lemma 3. If (2) holds for all \(\alpha \in S^1\), then
\[
\int_{-\pi}^{\pi} f'(\phi) f(\phi) e^{ikf(\phi)\cos(\phi-\theta)} d\phi = 0, \quad \forall \theta \in \mathbb{C}.
\]

Let us choose \(\cos \theta = is\) and \(\sin \theta = (s^2 + 1)^{1/2}\). Then \(\{is, (s^2 + 1)^{1/2}\} \in M\), and (4) can be written as
\[
\int_{-\pi}^{\pi} f'(\phi) f(\phi) e^{-skf(\phi)\cos \phi + ik(s^2+1)^{1/2}f(\phi)\sin \phi} d\phi = 0, \quad \forall s > 0.
\]

Multiply (5) by \(e^{-As}\), where \(A > 0\) is a large constant, and integrate over \(s\) from 0 to \(\infty\). Then one gets
\[
\int_{-\pi}^{\pi} d\phi f'(\phi) f(\phi) \int_{0}^{\infty} ds e^{-s(a+A)+i(s^2+1)^{1/2}b} = 0, \quad \forall A > A_0,
\]
where \(A_0 > 0\) is a fixed large constant,
\[
a = a(\phi) = kf(\phi) \cos \phi, \quad b = b(\phi) = kf(\phi) \sin \phi, \quad A_0 > \max_{\phi \in [-\pi, \pi]} |a(\phi)|.
\]

One has
\[
\int_{0}^{\infty} e^{-s(a+A)+i(s^2+1)^{1/2}b} ds = (a + A)^{-1} e^{ib}[1 + O(A^{-1})] ds, \quad A \to \infty.
\]

Writing
\[
(a + A)^{-1} = \sum_{j=0}^{\infty} (-1)^j a^j A^{-1-j}, \quad A > A_0,
\]
one obtains from (6) and (7) the relation

\[ \int_{-\pi}^{\pi} f'(\phi)f(\phi)e^{ib}\sum_{j=0}^{\infty} (-1)^j a^j A^{-1-j}[1 + O(A^{-1})]d\phi = 0, \quad A \to \infty. \]

Multiply (8) by \( A \) and let \( A \to \infty \). This yields relation (9), see below, with \( j = 0 \). After getting relation (9) with \( j = 0 \), multiply (8) by \( A^2 \) and let \( A \to \infty \). This yields relation (9) with \( j = 1 \). Continue in this fashion to get

\[ \int_{-\pi}^{\pi} f'(\phi)f(\phi)a^j e^{ib}d\phi = 0, \quad \forall j = 0, 1, .... \]

Applying the Laplace method (see [3]) for calculating the asymptotic behavior of integral (9) as \( j \to \infty \), one concludes that (9) can hold if and only if \( f'' = 0 \), that is, if and only if \( f = \text{const} \).

Let us give details. Consider the function \( a^{2m} = e^{m\Psi} \), where \( \Psi := \ln[k^2 f^2(\phi) \cos^2 \phi] \), \( j = 2m \), so that the expression under the logarithm sign is non-negative. The stationary points of the function \( \Psi \) are found from the equation \( \frac{f'(\phi)}{f(\phi)} - \tan \phi = 0 \).

If \( D \) is not a ball, then the function \( f(\phi) \) attains its maximum \( F \) at a point, which one may denote \( \phi = 0 \). There can be finitely many points at which \( f \) attains local maximums, because \( f \) is analytic. There are finitely many points at which \( f \) attains the value \( F \). We assume for simplicity that these points are non-degenerate, so \( f'' < 0 \) at these points. Since \( f > 0 \), one has the inequality

\[ \frac{d}{d\phi} \left( \frac{f'(\phi)}{f(\phi)} - \tan \phi \right) = \frac{f''(\phi)}{f(\phi)} - \frac{(f'(\phi))^2}{f^2(\phi)} - \frac{1}{\cos^2 \phi} < 0, \]

if \( f'' < 0 \). Therefore, the critical points are non-degenerate and the main term of the asymptotic of the integral (9) with \( j = 2m \) as \( m \to \infty \), corresponding to the stationary point \( \phi = 0 \) can be calculated as follows. Let \( I \) denote the integral in (9). The stationary point \( \phi = 0 \) is a non-degenerate interior point of maximum of \( f \) and, therefore, of \( \Psi \). Since \( e^{ib(\phi)} = 1 + ikf(\phi)\sin \phi + ... \), \( f(\phi) = f(0) + f'(0)\phi + f''(0)\phi^2/2 + ... \) and \( f'(\phi) = f'(0) + f''(0)\phi + f'''(0)\phi^2/2 + ... \), \( \Psi(\phi) = \Psi(0) - \gamma \phi^2 + ... \), where \( \gamma := |\Psi''(0)| \), one multiplies the three terms \( ff'e^{ib} \), takes into account that \( f'(0) = 0 \) and \( \Psi'(0) = 0 \) at the critical point, and gets \( I \sim e^{m\Psi(0)} J \), where

\[ J = \int_{[-\delta, \delta]} \left( (ikf^2(0)f''(0) + f(0)f'''(0)/2 + ...) e^{-m\gamma \phi^2} d\phi. \]
As \( m \to \infty \), one extends the interval of integration to \((-\infty, \infty)\) and calculates the main term of the asymptotic of \( J \) as \( m \to \infty \) by using the formula
\[
\int_{-\infty}^{\infty} \phi^2 e^{-m\gamma\phi^2} d\phi = \frac{\Gamma(3/2)}{(m\gamma)^{3/2}},
\]
where \( \Gamma(z) \) is the Gamma-function, \( \Gamma(3/2) = \sqrt{\pi}/2 \). The result is
\[
I \sim e^{m\varphi(0)} \frac{\Gamma(3/2)}{(m\gamma)^{3/2}} \left( ikf^2(0)f''(0) + f(0)f'''(0)/2 \right), \quad m \to \infty.
\]
Since \( I = 0 \) and \( f(0) > 0 \), one concludes from (10), after taking the imaginary part, that \( f''(0) = 0 \), and after taking the real part, that \( f'''(0) = 0 \). This contradicts the non-degeneracy of the critical point \( \phi = 0 \). If one does not assume the non-degeneracy of this critical point, then one uses the analyticity of the function \( f \) and concludes that if for some \( j \) the derivative \( f^{(j)}(0) \neq 0 \), then this leads to a contradiction. Thus, all the derivatives \( f^{(j)}(0) = 0 \) for \( j > 0 \). Each critical point at which \( f = F \) can be taken to be the point \( \phi = 0 \), because the origin for \( \phi \) in formula (4) can be chosen arbitrarily on the interval of length of the period \([0, 2\pi]\).

If the critical point \( \phi = 0 \) is non-degenerate then the inputs of local maximums at which \( f = F \) cannot compensate each other since their imaginary parts are all of the same sign since \( f'' < 0 \) and \( f > 0 \) at these points. There can be at most finitely many critical points of \( f \) since \( f \) is analytic.

Thus, the only possibility to have equalities (9) for all large \( j \) is to have \( f = \text{const} \).

Theorem 1 is proved.

\[ \square \]

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