A supersymmetric exotic field theory in (1+1) dimensions.
One loop soliton quantum mass corrections

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We consider one loop quantum corrections to soliton mass for the \( N = 1 \) supersymmetric extension of the \( \phi^4 \) scalar field theory in (1+1) dimensions. First, we compute the one loop quantum soliton mass correction of the bosonic sector by using a mixture of the scattering phase shift and the Euclidean effective action technique. Afterwards the computation in the supersymmetric case is naturally extended by considering the fermionic phase shifts associated to the Majorana fields. As a result we derive a general formula for the one loop quantum corrections to the soliton mass of the SUSY kink, and obtain for this exotic model the same value as for the SUSY sine-Gordon and \( \phi^4 \) models.

I. INTRODUCTION

The calculations of quantum corrections to the kink mass in (1+1)-dimensional field theories have been an intensively studied subject since many years ago [1, 2]. Originally, authors calculated the quantum corrections to the kink mass in the bosonic \( \phi^4 \) and sine-Gordon field theories. Some years later, supersymmetric extensions of those models were also studied, and since then a large amount of different approaches to calculate quantum corrections to the supersymmetric kink mass and central charge have been exhaustively investigated [3]–[16]. Remarkable efforts were made on dealing with two interesting but tricky issues: whether or not the bosonic and fermionic contributions in the quantum corrections to the supersymmetric kink mass cancel each other, and if the BPS saturation condition survives at quantum level.

After many attempts of solving these two issues without having reached any consensus, Graham and Jaffe [16] unambiguously calculated the correction to the supersymmetric \( \phi^4 \) and sine-Gordon kink mass by using a simple renormalization scheme and obtained the known value \( \Delta M = -m/2\pi \), which is in complete agreement with results previously obtained by Schonfeld [6], and Nastase et al. [14]. Furthermore, authors also showed in [16] that the BPS bound remains saturated at one loop approximation. Soon after, Litvinsev and Nieuwenhuizen obtained the same result for the supersymmetric kink mass by using a generalized momentum cut-off regularization scheme [17].

In all above cited works, authors treated mainly sine-Gordon and \( \phi^4 \) models because their one loop solvability. This is possible since in those cases the soliton one loop fluctuations are described by exactly solvable one dimensional Schrödinger equation corresponding to the Poschl-Teller type potentials. Some time ago it was considered the problem of constructing one loop exactly solvable two-dimensional scalar models starting from exactly solvable one dimensional Schrödinger equations [18]. In this way, it was obtained the model \( U(\phi) = \phi^2 \cos^2(\ln(\phi^2)) \) from Scarf II hyperbolic potential. We have to stress that similar work was carried also in Ref. [19] where the same model was found. Also the exotic behaviour of above model has been subject of recent studies [20, 21]. However, to our knowledge the corresponding supersymmetric field theory and the first supersymmetric soliton quantum corrections have not been explored in the literature.

The purpose of this paper is to consider the \( N = 1 \) supersymmetric extension of the aforementioned bosonic scalar field theory and compute the one loop quantum correction to the mass of the supersymmetric soliton. The procedure will be based on a mixture of the scattering phase shift method, which requires the use of the bosonic and fermionic phase shifts, and the Euclidean effective action technique. In addition, we use the so-called sharp momentum cut-off regularization method, as well as the renormalization prescription that adding tadpole graphs cancel exactly with appropriate SUSY counterterms in the bare classical mass. In order to assure the correctness of our method we first consider the one loop computations for the soliton mass in the pure bosonic sector, where we will compare our results with well established ones. Afterwards we extend the method to the supersymmetric case where we have found results that agree with the expected ones.

This paper is organized as follows. In section 2, for the sake of clarity we present some basics on \( N = 1 \) supersymmetric field theory and then introduce the supersymmetric extension of the potential \( U(\phi) \). In section 3, we compute the one loop soliton quantum mass corrections of the bosonic sector by using a mixture of the scattering phase shift and the Euclidean effective action technique. In section 4, we consider the phase shift of the fermionic fluctuations and compute the one loop quantum corrections to the supersymmetric soliton mass. Some concluding remarks are made in section 5. Finally, we discuss in appendix A the derivation of the soliton quantum mass corrections by considering the continuous bosonic phase shift.

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II. THE $N=1$ SUPERSYMMETRIC FIELD THEORY

In this section we will introduce a supersymmetric field theory as an extension of the bosonic scalar model previously studied in [18]. In the standard superspace approach, the dynamics of the theory is derived from an action defined for some superfields. These superfields are functions in the superspace, which is constructed by adding a two-component Grassmann variable $\theta_\alpha = \{t, \theta\}$ to the two-dimensional space-time $x^\mu = \{t, x\}$. Starting from one real bosonic field $\phi(x)$, we can define a bosonic superfield as follows,

$$\Phi(x, \theta) = \phi(x) + \theta \bar{\phi}(x) + \frac{1}{2} \theta \theta F(x),$$  \hspace{1cm} (1)

where $\bar{\phi}(x)$ is a two-component Majorana spinor, and $F(x)$ a real auxiliary bosonic field. Here, we have used the usual convention $\theta = \theta^0$, where the representation of the $\gamma$-matrices in the two-dimensional space is chosen to be

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = i \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$  \hspace{1cm} (2)

with $\gamma^5$ given by,

$$\gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (3)

Under a translation in the superspace,

$$x^\mu \rightarrow x^\mu - i \theta \gamma^\mu \epsilon, \quad \theta_\alpha \rightarrow \theta_\alpha + \epsilon_\alpha,$$  \hspace{1cm} (4)

the superfield transforms as follows,

$$\delta \Phi(x, \theta) = \epsilon_\alpha Q_\alpha \Phi(x, \theta),$$  \hspace{1cm} (5)

where $Q_\alpha = \partial / \partial \theta_\alpha + i (\gamma^\mu \theta) \partial_\mu$, and $\epsilon_\alpha = \{\epsilon_1, \epsilon_2\}$ is a constant Grassmannian spinor. In components, they read as

$$\delta \phi = \bar{\epsilon} \psi, \quad \delta \psi = -i \bar{\partial}_\mu \phi \gamma^\mu \epsilon + F \epsilon, \quad \delta F = -i \bar{\epsilon} \gamma^\mu \partial_\mu \psi.$$  \hspace{1cm} (6)

The most general action invariant under the SUSY transformations (6) can be written in the following form,

$$S = i \int d^2x d^2\theta \left[ \frac{1}{4} (\bar{D}_\alpha \Phi)(D_\alpha \Phi) + W(\Phi) \right],$$  \hspace{1cm} (7)

where $W(\Phi)$ is the superpotential, and the covariant superderivatives are given by

$$D_\alpha = \frac{\partial}{\partial \theta_\alpha} - i (\gamma^\mu \theta) \partial_\mu, \quad \bar{D}_\beta = - \frac{\partial}{\partial \theta_\alpha} + i (\gamma^\mu \theta) \partial_\mu,$$  \hspace{1cm} (8)

which satisfy,

$$\{D_\alpha, \bar{D}_\beta\} = 2i (\gamma^\mu)_{\alpha \beta} \partial_\mu,$$  \hspace{1cm} (9)

whereas, the supercharges satisfy in general the following SUSY algebra [22],

$$\{Q_\alpha, \bar{Q}_\beta\} = 2 (\gamma^\mu)_{\alpha \beta} P_\mu + 2i (\gamma^5)_{\alpha \beta} Z,$$  \hspace{1cm} (10)

where $P_\mu$ are the total energy and momentum operators, and $Z$ is known as the central charge, which coincides with the topological charge

$$Z = \int_{-\infty}^{\infty} dx \partial_x W = W(\phi(+\infty)) - W(\phi(-\infty)).$$  \hspace{1cm} (11)

Then, the SUSY algebra is centrally extended whenever $Z$ is different from zero, which happens when the model supports kink excitations interpolating between several different vacua at $x = \pm \infty$.

After integrating on the Grassmann variables, the action (7) becomes

$$S = \int d^2x \left[ \frac{1}{4} (\partial_\mu \phi \partial^\mu \phi + i \bar{\psi} \gamma^\mu \partial_\mu \psi + F^2) + W'(\phi) F \right] - \frac{1}{2} W''(\phi) \bar{\psi} \psi,$$  \hspace{1cm} (12)

where the usual notation $W'(\phi) = dW(\phi) / d\phi$ has been used. The auxiliary field can be eliminated by using the equation of motion, that is $F = -W'(\phi)$, and finally write the action in the simple form,

$$S = \int d^2x \left[ \frac{1}{4} (\partial_\mu \phi \partial^\mu \phi + i \bar{\psi} \gamma^\mu \partial_\mu \psi) - \frac{1}{2} [W'(\phi)]^2 \right] - \frac{1}{2} W''(\phi) \bar{\psi} \psi.$$  \hspace{1cm} (13)

Expanding above expression around the classical field configuration, $\phi = \phi_c + \eta$, up to quadratic order in the fields $\eta$ and $\psi$, we get

$$S = \frac{1}{2} \int d^2x \left[ \partial_\mu \phi_c \partial^\mu \phi_c - (W'(\phi_c))^2 \right.$$

$$\left. + \bar{\psi} (i \gamma^\mu \partial_\mu - W''(\phi_c)) \psi + \partial_\mu \eta \partial^\mu \eta \right] - \frac{1}{2} [W'(\phi_c)]^2 + W'(\phi_c) W''(\phi_c) \eta^2,$$  \hspace{1cm} (14)

where $\phi_c$ follows the classical equation of motion

$$\square \phi_c + W'(\phi_c) W''(\phi_c) = 0.$$  \hspace{1cm} (15)

Considering finite energy static kink solutions of Eq. (15), after quantization of the quadratic action Lagrangian (14), we can find the energy ground state at one loop order. Subtracting from this the vacuum energy in the absence of the kink, we get that the mass of the soliton state at one loop order is given by

$$M_{bare} = E[\phi_c] + \frac{1}{2} \sum_n \omega_{nb} - \frac{1}{2} \sum_n \omega_{nf},$$  \hspace{1cm} (16)

where $E[\phi_c]$ is the energy of the static classical configuration,

$$E[\phi_c] = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \left( \frac{d\phi_c}{dx} \right)^2 + \frac{1}{2} [W'(\phi_c)]^2 \right],$$  \hspace{1cm} (17)
and $\omega_{nb}$ and $\omega_{nf}$ are solutions of the following eigenvalue equations,

$$\left[-\frac{d^2}{dx^2} + [W'(\phi_c)]^2 + W'(\phi_c)W''(\phi_c)\right] \eta_n = \omega_{nb}^2 \eta_n,$$

and

$$\left[-i\gamma^5 \frac{d}{dx} + W''(\phi_c)\gamma^0\right] \psi_n = \omega_{nf} \psi_n.$$  \hfill (19)

From now on, we will focus on the following superpotential,

$$W(\Phi) = \frac{8m^3B}{\lambda(1+4B^2)}(1+\beta\Phi)^2 \left[\cos \left(\frac{\ln(1+\beta\Phi)^2}{2B}\right) + \frac{1}{2B} \sin \left(\frac{\ln(1+\beta\Phi)^2}{2B}\right)\right],$$  \hfill (20)

where $m$ is a mass parameter, and the real parameters $B$, $\lambda$, and $\beta$ satisfy

$$\beta = \frac{\sqrt{2}}{2mB}.$$  

The superpotential (20) is the supersymmetric extension of the bosonic scalar potential introduced in [18, 19]. It is also worth noting that its particular form has been chosen in order to get a well-defined limit $B \to 0$, which corresponds to the $N = 1$ super sine-Gordon superpotential. After substituting in the action (13), we get the Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{2m^4}{\lambda}(1+\beta\Phi)^2 \cos \left(\frac{\ln(1+\beta\Phi)^2}{2B}\right) + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{m}{2} \left[B \cos \left(\frac{\ln(1+\beta\Phi)^2}{2B}\right) - \sin \left(\frac{\ln(1+\beta\Phi)^2}{2B}\right)\right] \bar{\psi} \psi.$$  \hfill (21)

As it was noted in [18], the bosonic potential in the Lagrangian (21),

$$U(\phi) = \frac{2m^4}{\lambda}(1+\beta\Phi)^2 \cos \left(\frac{\ln(1+\beta\Phi)^2}{2B}\right),$$  \hfill (22)

has infinitely degenerate trivial vacua at the points $\phi_n$ given by

$$\phi_n = \frac{2m}{B\sqrt{\lambda}} \left\{ \pm \exp \left[ \left( n + \frac{1}{2} \right) \pi B \right] - 1 \right\},$$  \hfill (23)

where $n = 0, \pm 1, \pm 2, \ldots$. It can be seen in the Figure 1 that this potential possesses a reflection symmetry around the point $\phi = -1/\beta$. In addition, we note that $W''(\phi_n)$ is equal to $-m$ for $n$ even, and $+m$ when $n$ is odd. This fact implies that the curvature of the potential $U''(\phi_n)$ is the same for all $n$, namely,

$$U''(\phi_n) = m^2, \quad n = 0, \pm 1, \pm 2, \ldots$$  \hfill (24)

Figure 1: Plot of the bosonic potential $U(\phi)$ for $\beta = 1$ and $B = 0.1$.

The classical kink and anti-kink solutions have the following explicit form,

$$\phi_c(x) = \frac{2m}{B\sqrt{\lambda}} \left\{ \pm \exp \left[ \frac{m}{\lambda} \pi B \right] - 1 \right\},$$  \hfill (25)

where $\epsilon = +1$ corresponds to a kink solution, while $\epsilon = -1$ corresponds to an anti-kink solution. Then, from a straightforward computation we found that the classical masses of the kink (or anti-kink) are given by,

$$E_n[\phi_c] = |V(\phi_{n+1}) - V(\phi_n)| = E_0 \exp (2m\pi B)$$  \hfill (26)

where

$$E_0 = \frac{8m^3}{\lambda(1+4B^2)} \cosh (\pi B).$$  \hfill (27)

Despite of the exponential dependence in eq. (26), the classical masses for a kink (or anti-kink) connecting any two neighbouring vacua is finite, and satisfy the relation $(E_{n+1}/E_n) = e^{2\pi B}$. It is also worth pointing out that in the limit $B \to 0$ we recover the kink configuration for the sine-Gordon model from Eq. (25), as well as its corresponding classical mass from Eq. (27).

In the next section we will focus on the computations of the bosonic one loop quantum mass corrections for the kink solution of model (20). Afterward we will extend the computation for the supersymmetric kink.

III. BOSONIC ONE LOOP QUANTUM MASS CORRECTIONS

Before considering the one loop quantum mass corrections for the supersymmetric model described in last section, let us consider first the purely bosonic case with the Fermi fields vanishing. Here, we will derive the main results for a general form of the potentials, and then we will apply them in our particular potential (22).
Let us consider the bosonic Lagrangian density,
\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi), \]
where the potential \( U(\phi) \) is given by Eq. (22). By quantizing around the static kink \( \phi_c \), we get for the soliton mass at one loop order,
\[ M^b_{\text{bare}} = E[\phi_c] + \frac{1}{2} \sum_{n} \omega_{nb} - \frac{1}{2} \sum_{k} \omega^0_{kb}, \]
where the index \( b \) stands for bosonic contributions, and \( \omega_{nb} \) is given by Eq. (18), which can be rewritten as
\[ \left[ -\frac{d^2}{dx^2} + V_+(x) \right] \eta_n = \omega^2_{nb} \eta_n, \]
with
\[ V_+(x) = U''[\phi_c(x)]. \]
Also, the free soliton eigenfrequencies \( \omega^0_b(k) \) are given by
\[ \left[ -\frac{d^2}{dx^2} + m^2 \right] \eta_k = [\omega^0_b(k)]^2 \eta_k, \]
where \( m \) is the mass of the quantum fluctuations around the trivial vacua. From Eq. (32) it is not difficult to see that the soliton-free eigenfunctions are given by \( \eta_k \propto e^{ikx} \) with \( \omega^0_b(k) = \sqrt{k^2 + m^2} \). In general the eigenvalues of Eq. (30) fall in two sets, a finite discrete set, that we denote by \( \omega^0_{nb} \) and a continuum set, that we denote by \( \omega^0_b(q) \) ranging from \( m^2 \) to \( \infty \). The continuous eigenfunctions behave asymptotically as \( e^{\pm i q x} \) with modes \( \omega^0_b(q) = \sqrt{q^2 + m^2} \), i.e. equal to the soliton-free modes \( \omega^0_b(k) \). Naively we can expect that the sum in Eq. (29) will be reduced to a sum over the discrete modes \( \omega^0_b(q) \). But this is not the case, since there is a difference between the distributions of \( \omega^0_b(q) \) and \( \omega^0_b(k) \) over \( q \) and \( k \) respectively, i.e. they have different densities of states. We can split the sum (29) into a sum over the discrete modes and a sum over the continuous modes. Doing that, we obtain for the bare one loop soliton quantum mass correction,
\[ \Delta M^b_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N^b} \omega_{ib} + \sum_{q=0}^{\infty} \omega_{ib}(q) - \sum_{k=0}^{\infty} \omega^0_{kb}(k). \]
Here, we will use the so-called sharp momentum cut-off regularization method. In order to get correct results, the basic idea is to use the same cut off \( \Lambda \) in the sums over \( q \) and \( k \) in Eq. (33), namely,
\[ \Delta M^b_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N^b} \omega_{ib} + \sum_{q=0}^{\Lambda} \omega_{ib}(q) - \sum_{k=0}^{\Lambda} \omega^0_{kb}(k). \]
In order to write the continuous sum in the above equation in an integral form, we enclose the system in a box of size \( L \), apply periodic boundary conditions and finally take the limit \( L \to \infty \). For simplicity we start with the simplest case in which \( V_+ \) is a reflectionless potential (this is the case for example, in the sine-Gordon and \( \phi^4 \) kink models), then we generalize the result for other potentials like the one we are considering here. For reflectionless potentials the asymptotic behaviour of \( \eta_q(x) \) is given by,
\[ \eta_q(x) = \begin{cases} e^{i q x} & x \to -\infty \\ e^{i q x + i \delta_+(q)} & x \to +\infty \end{cases}, \]
where \( \delta_+(q) \) is the bosonic phase shift. Imposing periodic boundary conditions for the free eigenfunctions \( \eta_k \propto e^{ikx} \), we have \( e^{-i k L/2} = e^{i k L/2} \), from which we obtain \( k_n = \frac{2 \pi n}{L} \), with \( n = 0, \pm 1, \pm 2 \). In this case the free density of states is \( 1/(k_{n+1} - k_n) = L/(2 \pi) \). On the other hand from Eq. (35), imposing periodic boundary conditions we get,
\[ q_n = \frac{2 \pi n}{L} - \frac{\delta_+(q_n)}{L}, \quad n = 0, \pm 1, \pm 2, \ldots \]
The phase shift \( \delta_+(q) \), can be chosen in two ways. The standard way in scattering theory is to choose it (for physical reasons) in such a way that \( \delta_+(\pm \infty) = 0 \). In that case the phase shift is in general discontinuous at \( q = 0 \). For the other choice we have that \( \delta_+(0) = 0 \), and then the phase shift is a continuous function. We will work with the first choice as in scattering theory, however it can be shown that the results obtained are independent of that particular choice (see discussion in appendix A).
In the discontinuous case, the phase shift for large momentum \( \Lambda \) is given by the first Born approximation
\[ \delta_+(\Lambda) = -\frac{\langle V_+ \rangle}{2\Lambda}, \]
where
\[ \langle V_+ \rangle = \int_{-\infty}^{\infty} V_+(x) dx, \quad V_+(x) = V_+(x) - m^2. \]
In terms of discretized eigenfrequencies, Eq. (34) can be written as
\[ \Delta M^b_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N^b} \omega_{ib} + \sum_{n=n_0}^{N'} \sqrt{\frac{q_n^2}{\pi} + m^2} - \sum_{n=0}^{N} \sqrt{k_n^2 + m^2}, \]
where $n_0$ in the second term is not necessarily equal to zero, and $N'$ is not necessarily equal to $N$, since we have required that both sums in Eq. (34) range from zero to the same cut-off $\Lambda$. For the free eigenfrequencies $N$ is related to the cut-off $\Lambda$ by,

$$\Lambda = 2\pi N/L \,.$$

(40)

To find $n_0$ and $N'$ we proceed as follows. Setting $q_{n_0} = 0^+$ in Eq. (36) and using one-dimensional Levinson theorem [31], $\delta_+(0^+) = N\pi$, we have $2\pi n_0/L - N\pi/L = 0$, from which we get $n_0 = N'/2$. On the other hand, by setting $\Lambda = q_{N'}$ in Eq. (36) and using $\delta_+(\Lambda) \to 0$, we obtain $\Lambda = 2\pi N'/L$. Now, by comparing with Eq. (40), we get $N' = N$. It is worth noting that the solution $n_0 = N'/2$, makes sense only when $N'$ is even, therefore we will assume for the moment that this is the case. Writing $N' = 2\mathcal{M}$, we have $n_0 = \mathcal{M}$ and Eq. (39) can be written as

$$\Delta M_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} + \sum_{n=\mathcal{M}+1}^{N} \sqrt{q_n^2 + m^2} - \sum_{n=0}^{N} \sqrt{k_n^2 + m^2}$$

$$= \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} - \mathcal{M} - \frac{\mathcal{M}-1}{2} \sum_{n=0}^{N} \sqrt{q_n^2 + m^2}$$

$$+ \sum_{n=0}^{N} \left( \sqrt{q_n^2 + m^2} - \sqrt{k_n^2 + m^2} \right).$$

(41)

From Eq. (36), we obtain

$$\sqrt{q_n^2 + m^2} = \sqrt{k_n^2 + m^2} - \frac{k_n \delta_+(k_n)}{L \sqrt{k_n^2 + m^2} + O(L^{-2})}.$$  

(42)

By taking the limit $L \to \infty$, and using $\delta_+(0^+) = 2\mathcal{M}\pi$, Eqs. (37) and (42), we get

$$\Delta M_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} - \mathcal{M}m - \left[ \frac{\omega_{b}(k)}{2\pi} \delta_+(k) \right]_{0^+}$$

$$+ \int_{0}^{\Lambda} \frac{dk}{2\pi} \omega_{b}(k) \frac{d}{dk} \delta_+(k)$$

$$= \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} + \frac{\langle V \rangle}{4\pi} + \int_{0}^{\Lambda} \frac{dk}{2\pi} \omega_{b}(k) \frac{d}{dk} \delta_+(k).$$

(43)

For odd $N$, $n_0$ will be a half integer, and then it will not be present in the spectrum as given by Eq. (36). Of course such a solution does not satisfy the periodic boundary conditions. Then, in this last case we have to drop out the term $n_0 = N'/2$ from the sum in the second term of Eq. (39), and the sum should start at the next integer larger than $N'/2$. In this case, writing $N' = 2\mathcal{M} + 1$, the sum must start at $n = \mathcal{M} + 1$. Also, in the last sum of Eq. (39) the contribution from the zero momentum, $n = 0$, must be multiplied by a factor $1/2$, since in the original sum given by Eq. (29) the corresponding term is counted only once. Therefore, Eq. (39) can be written as

$$\Delta M_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} + \sum_{n=\mathcal{M}+1}^{N} \sqrt{q_n^2 + m^2}$$

$$- \sum_{n=0}^{N} \sqrt{k_n^2 + m^2} - \frac{m}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} - \sum_{n=0}^{N} \sqrt{q_n^2 + m^2} - \frac{m}{2}$$

$$+ \sum_{n=0}^{N} \left( \sqrt{q_n^2 + m^2} - \sqrt{k_n^2 + m^2} \right).$$  

(44)

By taking limit $L \to \infty$, using $\delta_+(0^+) = (2\mathcal{M} + 1)\pi$, and Eqs. (42) and (37), we find

$$\Delta M_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} - \left( \mathcal{M} + \frac{1}{2} \right) m - \left[ \frac{\omega_{b}(k)}{2\pi} \delta_+(k) \right]_{0^+}$$

$$+ \int_{0}^{\Lambda} \frac{dk}{2\pi} \omega_{b}(k) \frac{d}{dk} \delta_+(k)$$

$$= \frac{1}{2} \sum_{i=1}^{N'} \omega_{ib} + \frac{\langle V \rangle}{4\pi} + \int_{0}^{\Lambda} \frac{dk}{2\pi} \omega_{b}(k) \frac{d}{dk} \delta_+(k).$$

(45)

an identical result we obtained in the even case, Eq. (43). Equation (43) or (45) can be generalized for potentials $V$ which are not necessarily reflectionless. We will found the same results as long as we generalize the expression for the phase shift in terms of the S-matrix associated to the one-dimensional scattering problem given by the continuous solutions of Eq. (30). In terms of the reflection and transmission coefficient amplitudes, $R$ and $T$, the S-matrix reads [31],

$$S(k) = \begin{pmatrix} T(k) & -R^*(k)T(k)/T^*(k) \\ R(k) & T(k) \end{pmatrix},$$

and in general the phase shift is given by

$$\delta(k) = \frac{1}{2\pi} \ln \det S(k)$$

$$= \frac{1}{2\pi} \ln \begin{pmatrix} T(k) \\ T^*(k) \end{pmatrix},$$

(47)

where we have used the well-known property $|R|^2 + |T|^2 = 1$ to get Eq. (47). The unitarity of the S-matrix ensures that $\delta(k)$ given by Eq. (47) is a real function of $k$. The equation (47) is trivially satisfied for reflectionless potentials since we have $T = e^{i\delta(k)}$.

As it was already mentioned, the bare quantum mass correction given by Eq. (43) or (45) is logarithmically divergent, since the phase shift behaves as $1/k$ for large $k$. Then, we have to renormalize such expression. In order to do that we write $\Delta M_{\text{bare}}$ in other equivalent form [24, 25],

$$\Delta M_{\text{bare}} = \frac{1}{2} \int \frac{d\omega}{2\pi} \ln \left[ 1 + \frac{\langle V \rangle}{\omega^2 - \frac{d^2}{d\omega^2} + m^2} \right].$$

(48)
The last expression is obtained using functional methods by defining the mass of the soliton as the Euclidean effective action per unit time evaluated at the static soliton configuration. The equation (48) can be expanded in terms of Feynman graphs,

\[ \Delta M^b_{\text{bare}} = \frac{1}{2} \sum_{i=1}^{N} \omega_{ib} + \frac{\langle \mathcal{V}_+ \rangle}{4\pi} + \int_{0}^{\Lambda} \frac{dk}{2\pi} \omega_{ib}(k) \frac{d}{dk} \delta_+(k) \]

where the background field is \( \mathcal{V}_+(x) \). From the expansion in Feynman graphs we note that the only divergent term is the tadpole graph, whose momentum cut-off regularized expression is given by

\[ \mathcal{V}_+ - \frac{\langle \mathcal{V}_+ \rangle}{2} = \int_{0}^{\Lambda} \frac{dk}{2\pi} \frac{1}{\sqrt{k^2 + m^2}}. \]  

(50)

Since Eqs. (45) and (49) are equivalent, we can subtract and add to it the tadpole graph given by Eq. (50) obtaining

\[ \Delta M^b_{\text{bare}} = \left\{ \sum_{i=1}^{N} \omega_{ib} + \frac{\langle \mathcal{V}_+ \rangle}{4\pi} + \int_{0}^{\Lambda} \frac{dk}{2\pi} \omega_{ib}(k) \frac{d}{dk} \delta_+(k) \right\} + \mathcal{V}_+ - \frac{\langle \mathcal{V}_+ \rangle}{2}. \]  

(51)

In the above equation, the first part is finite and the full one loop divergence is contained in the added tadpole graph only. Now we use the simplest renormalization prescription that the added tadpole graph cancels exactly with adequate counterterms (that are present in the classical bare mass), obtaining in this way unambiguously the renormalized soliton quantum mass correction at one loop order. Then, we can identify the finite renormalized one loop soliton quantum mass correction, that we denote by \( \Delta M^b \), with the first part of Eq. (51), namely

\[ \Delta M^b = \frac{1}{2} \sum_{i=1}^{N} \omega_{ib} + \frac{\langle \mathcal{V}_+ \rangle}{4\pi} + \int_{0}^{\Lambda} \frac{dk}{2\pi} \omega_{ib}(k) \frac{d}{dk} \delta_+(k) \]

\[ - \frac{\langle \mathcal{V}_+ \rangle}{2} \int_{0}^{\Lambda} \frac{dk}{2\pi} \frac{1}{\sqrt{k^2 + m^2}}, \]  

(52)

where we used Eq. (50), and it is understood that now the parameters that appear in Eq. (52) are the physical ones.

Let us now apply the general formula (51) for the bosonic scalar density potential (22). In this case, by substituting the static field configuration (25) in Eq. (31), we get the potential

\[ V_+(x) = m^2 \left[ 1 + \frac{(B^2 - 2)}{\cosh^2(mx)} - 3B \frac{\sinh(mx)}{\cosh^2(mx)} \right]. \]  

(53)

which belong to the Scarf II hyperbolic exactly solvable potentials [32]. This potential is shown in Figure 2. It has only one discrete eigenvalue, namely the zero mode \( \omega_{0b} = 0 \), and its corresponding eigenfunction has the following form,

\[ \eta_0(x) = \frac{c_0}{\cosh(mx)} \exp \left[ B \tan^{-1}(\sinh(mx)) \right], \]  

(54)

with \( c_0 \) a normalization constant. In addition, the transmission coefficient amplitude for this potential is given by [32],

\[ T_+(k) = \frac{\frac{1}{2} + i \frac{k}{m}}{\frac{1}{2} + i \frac{k}{m} + \frac{B}{2}}. \]  

(55)

Now, by substituting Eqs. (53) and (55) in Eqs. (38) and (47) respectively, we find that

\[ \langle \mathcal{V}_+ \rangle = 2m(B^2 - 2), \]  

(56)

and the bosonic phase shift is given by

\[ \delta_+(0) = 2 \arctan \left( \frac{m}{k} \right) + \frac{1}{2i} \ln \left[ \frac{\Gamma \left( \frac{1}{2} + iB + \frac{ik}{m} \right)}{\Gamma \left( \frac{1}{2} + iB + \frac{ik}{m} \right)} \right] \]

\[ + \frac{1}{2i} \ln \left[ \frac{\Gamma \left( \frac{1}{2} - iB + \frac{ik}{m} \right)}{\Gamma \left( \frac{1}{2} - iB + \frac{ik}{m} \right)} \right], \]  

(57)

which is plotted in Figure 3 for different values of the parameter \( B \). It is worth pointing out that \( \delta_+(0) = \pi \) and \( \delta_+ (+\infty) = 0 \), which is consistent with the Levinson theorem since we have only one bosonic bound state at \( \omega_{0b} = 0 \). This can be also verified graphically in Figure 3.

Now, by using the above results for the potential and the phase shift, and substituting in Eq. (52), we finally
we have plotted the results for $\Delta M_k$ in (58) analytically, however we can integrate numerically straightforwardly by using for example Maple software. In the Figure 4, we have plotted the results for $\Delta M^h/m$ as function of parameter $B$, ranging from $B = 0$ to $B = 3, 2$, where we have restricted to $B \geq 0$ because $\Delta M(-B) = \Delta M(B)$ as can be concluded from (58). For $B > 3, 2$, the behaviour of $\Delta M(B)$ is similar to the one depicted in figure (4), it varies smoothly as a function of parameter $B$ and $\Delta M(B + h) < \Delta M(B)$, for sufficiently large positive $h$.

IV. ONE LOOP SUSY KINK QUANTUM MASS CORRECTION

In this section, we compute the quantum correction to the SUSY kink mass by including both bosonic and fermionic fluctuations. First of all, let us now consider fluctuations of the fermion field $\psi(x)$. The Fermi field

\begin{equation}
\psi(x, t) = \xi(x)e^{-i\omega_j t} + \xi^*(x)e^{i\omega_j t},
\end{equation}

where $\omega_j$ is a real variable, and $\xi_{\pm}(x)$ are static normalizable solutions of the following system,

\begin{equation}
\left(\frac{d}{dx} + W''(\phi)\right)\xi_- = -i\omega_j \xi_+,
\end{equation}

\begin{equation}
\left(\frac{d}{dx} - W''(\phi)\right)\xi_+ = -i\omega_j \xi_-,
\end{equation}

where we have denoted $W''(\phi) = W'''(\phi)$. Then, by cross-differentiating eqs. (62) and (63) we obtain the decoupled equations,

\begin{equation}
\left\{\frac{d^2}{dx^2} + \left[\omega_j^2 - (W''(\phi))^2 + W'''W'_c\right]\right\}\xi_-(x) = 0,
\end{equation}

\begin{equation}
\left\{\frac{d^2}{dx^2} + \left[\omega_j^2 - (W''(\phi))^2 - W'''W'_c\right]\right\}\xi_+(x) = 0.
\end{equation}

We note that eq. (65) is the same bosonic fluctuation equation. The system of equations (64) and (65) can be rewritten in the following form,

\begin{equation}
\left\{-\frac{d^2}{dx^2} + \left[V_+(x) - \omega_j^2\right]\right\}\xi_\pm(x) = 0,
\end{equation}

where $V_+$ is given by (53), and

\begin{equation}
V_-(x) = (W''(\phi))^2 - W'''W'_c.
\end{equation}
Now, the bare one loop quantum mass corrections to the SUSY kink will be given by

$$\Delta M_{\text{bare}} = \frac{1}{2} \sum_q \omega_b(q) - \frac{1}{2} \sum_q \omega_f(q), \quad (68)$$

where $\omega_b(q)$ and $\omega_f(q)$ are respectively the bosonic and fermionic eigenfrequencies of Eqs. (18) and (19). Since the discrete eigenfrequencies are the same, we can rewrite above expression as

$$\Delta M_{\text{bare}} = \Delta M^b_{\text{bare}} - \Delta M^f_{\text{bare}}, \quad (69)$$

where $\Delta M^b_{\text{bare}}$ is given by Eq. (33) and $\Delta M^f_{\text{bare}}$ can be written as

$$\Delta M^f_{\text{bare}} = \frac{1}{2} \sum_i \omega_{if} + \frac{1}{2} \sum_q \omega_f(q) - \frac{1}{2} \sum_k \omega_f^0(k), \quad (70)$$

where $\omega_{if}$ denotes the discrete eigenfrequencies of (19).

In the continuum limit $\Delta M^b_{\text{bare}}$ is given by Eq. (43). In order to write the $\Delta M^f_{\text{bare}}$ in integral form we proceed as follows. First, since the eigenfrequencies of Eqs. (64) and (65) are the same, we rewrite (70) as

$$\Delta M^f_{\text{bare}} = \frac{1}{2} (\Delta M_{\text{bare}} + \Delta M^+_{\text{bare}}), \quad (71)$$

where $\Delta M^\pm_{\text{bare}}$ are given by expressions similar to (70), but now in relations to eigenfrequencies following respective Eqs. (64) and (65). As it was done for the bosonic case, enclosing the system in a box of size $L$, but now imposing anti-periodic boundary conditions $\xi(-L/2) = -\xi(L/2)$, we get

$$\Delta M^+_{\text{bare}} = \Delta M^b_{\text{bare}}, \quad (72)$$

and

$$\Delta M^-_{\text{bare}} = \frac{1}{2} \sum_i N^- \omega_{if} + \frac{\langle V_- \rangle}{4\pi} + \int_0^\Lambda \frac{dk}{2\pi} \omega(k) \frac{d}{dk} \delta_-(k), \quad (73)$$

where $N^-$ is the number of discrete eigenvalues of (65), $\omega(k) = \sqrt{k^2 + m^2}$, and

$$\langle V_- \rangle = \int_{-\infty}^{\infty} dx V_-(x), \quad V_-(x) = V_-(x) - m^2. \quad (74)$$

It is worth pointing out that $\delta_-(k)$ represents the fermionic phase shift associated to the upper component $\xi_-(x)$, and that the relation (72) implies that the fermionic phase shift associated to the lower component $\xi_+(x)$ is the same of the bosonic phase shift $\delta_+(k)$.

Now, by using Eq. (43), and Eqs. (72)–(73), we get

$$\Delta M_{\text{bare}} = \frac{1}{2} (\Delta M^+_{\text{bare}} - \Delta M^-_{\text{bare}})$$

$$= \frac{1}{2} \int_0^\Lambda \frac{dk}{2\pi} \omega(k) \frac{d}{dk} (\delta_+(k) - \delta_-(k))$$

$$+ \frac{1}{8\pi} \langle V_+ - V_- \rangle. \quad (75)$$

As it was done in the bosonic case, by adding and subtracting the tadpole graph from the above expression, properly obtained from the SUSY Euclidean effective action per unit time, and using the renormalization prescription that the added graph cancels exactly with the counterterms in the bare classical mass, we get that the renormalized one loop correction of the SUSY kink mass is given by

$$\Delta M = \frac{1}{8\pi} (V_+ - V_-) + \frac{1}{2} \int_0^\Lambda \frac{dk}{2\pi} \omega(k) \frac{d}{dk} (\delta_+(k) - \delta_-(k))$$

$$- \frac{1}{4} \langle V_+ - V_- \rangle \int_0^\Lambda \frac{dk}{2\pi} \frac{1}{\sqrt{k^2 + m^2}}. \quad (76)$$

Let us now apply the above formula for the susy exotic field theory. By substituting explicitly the kink configuration (25) in Eq. (67), we find that the potential for the upper component $V_-(x)$ takes the following form,

$$V_-(x) = m^2 \left[ 1 + \frac{B^2}{\cosh^2(mx)} - \frac{B \sinh(mx)}{\cosh^2(mx)} \right]. \quad (77)$$

which correspond to the associated superpartner potential of $V_+(x)$, and also belong to the Scarf II hyperbolic exactly solvable potentials [32]. This can be seen from Figure 5. In this case, the upper component $\xi_+$ does not posses any bound state, and only the zero mode solution ($\omega_{if} = 0$) is allowed for the lower component $\xi_-$, which leads to a single non-degenerate bound state. The corresponding transmission coefficient amplitude for the lower component potential $V_-$ is given by (55), and for the potential (77) has the following form [33],

$$T_-(k) = \frac{\Gamma \left( \frac{1}{2} + iB - \frac{i\kappa}{2m} \right) \Gamma \left( \frac{1}{2} - iB - \frac{i\kappa}{2m} \right)}{\Gamma^2 \left( \frac{1}{2} - \frac{i\kappa}{2m} \right)}. \quad (78)$$

From above results we find that the phase shift for the lower components $\xi_+(x)$ as the same form as the bosonic
phase shift $\delta_+(k)$ in (57), whereas the phase shift for the upper component can be written explicitly as follows,

$$\delta_+(k) = \frac{1}{2i} \ln \left[ \frac{\Gamma\left(\frac{1}{2} + iB - \frac{ik}{m}\right)\Gamma\left(\frac{1}{2} - iB - \frac{ik}{m}\right)}{\Gamma\left(\frac{1}{2} + iB + \frac{ik}{m}\right)\Gamma\left(\frac{1}{2} - iB + \frac{ik}{m}\right)} \times \frac{\Gamma^2\left(\frac{1}{2} + \frac{ik}{m}\right)}{\Gamma^2\left(\frac{1}{2} - \frac{ik}{m}\right)} \right].$$

(79)

By comparing Eqs. (57) and (79), we conclude that the fermionic phase shifts $\delta_\pm(k)$ satisfy an important relation, namely

$$\delta_+(k) - \delta_-(k) = 2 \arctan \left(\frac{m}{k}\right),$$

(80)

and,

$$\langle V_+ - V_- \rangle = -4m.$$  

(81)

By substituting the above results in Eq.(76), we finally obtain

$$\Delta M = -\frac{m}{2\pi},$$

(82)

which corresponds to the accepted value of the one loop supersymmetric quantum mass correction for any antisymmetric soliton [16]. It is important to note that a fermionic phase shift defined as the average of its values for the upper and lower components, namely

$$\delta_F(k) = \frac{1}{2} [\delta_+(k) + \delta_-(k)]$$

$$= \arctan \left(\frac{m}{k}\right) + \delta_-(k),$$

(83)

satisfies the Levinson theorem for $(1+1)$ dimensional Dirac equation [34], since we have only one bound state corresponding to the zero mode. This definition is consistent with the relation (71), and has been also checked previously for several models by using the Levinson theorem [35]–[37]. In Figure 6, we have plotted these fermionic phase shifts for different values of the parameter $B$.

V. CONCLUDING REMARKS

In this paper we have computed the one loop quantum correction to the kink mass in a supersymmetric theory described by the density Lagrangian (21) in $(1+1)$ dimensions. We have shown, by using the so-called sharp momentum cut-off regularization method, that up to one loop the quantum correction to the mass of the supersymmetric kink is $\Delta M = -m/2\pi$, which is in complete agreement with the results reported in the literature for antisymmetric solitons, $\phi_c(-x) \neq -\phi_c(x)$. First we established a formula for the purely bosonic sector and extended the method for the full supersymmetric case. One of the key points of our approach is the use of the one dimensional Levinson theorem in the form $\delta(0^+) = N\pi$

where $N$ is the number of bound states. Therefore before using our main formulas (52) or (76) we have checked that.

However, we note from Eq. (25) that the kinks of the model considered are not antisymmetric, $\phi_c(-x) \neq -\phi_c(x)$, and that our result (76) is model independent since it depends only on the relations (80) and (81) for
the super partner potentials \( V_+ \) and \( V_- \). Therefore, the result obtained for the one loop SUSY soliton mass is quite general.

In the purely bosonic sector, we have seen that the quantum correction of the kink mass is equivalent to the one of the sine-Gordon model in the limit \( B \to 0 \), namely \( \Delta M_{SG}^B = -m/\pi \). Then it behaves as an almost decreasing function for \( B > 0 \) as it was shown in the Figure 4. Since we have a smooth behaviour for \( \Delta M(B) \), interpolating the accepted value for the sine-Gordon Model \((B = 0)\) and other values of \( B \), we have confidence that our results are correct. After considering the corresponding contributions of the fermionic fluctuations for the quantum corrections we find that the dependence on the parameter \( B \) vanishes completely, as well as the divergent part of the SUSY kink mass by adding appropriate supersymmetric counterterms.

We would like to call attention to the following curious result. In the bosonic sector there is a special parameter value \( B = \sqrt{2} \), for which it is not necessary to regularized nothing, the tadpole graph given by Eq. (50) and (56) vanishes and there is no necessity for regularizing \( \Delta M_{bare}^B = \Delta M_\phi \). However for \( B = \sqrt{2} \) the one loop bare supersymmetric kink mass (75) is divergent and given by the last term of Eq. (76). This is a non intuitive unexpected result since it is widely believed that supersymmetry improves rather than spoiling ultraviolet divergences.

Finally, it should be interesting to examine the two-loop quantum corrections as it has been done in the sine-Gordon and \( \phi^4 \) models [14], [38]. This question is expected to be developed in future investigations.

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Appendix A: Continuous phase shift

Let us discuss the case in which the bosonic phase shift is chosen to be continuous. In this case \( \delta_+(0) = 0 \), and then we have for large momentum \( \Lambda \),

\[
\delta_+(\Lambda) = -\mathcal{N} \pi - \frac{\langle \mathcal{N}_+ \rangle}{2\lambda}, \tag{A1}
\]

where \( \mathcal{N} \) is the number of discrete eigenvalues of Eq. (30). In this case we find from (36) that \( n_0 = 0 \). On the other hand for the upper limit in Eq. (39) we have, from Eqs. (36) and (A1),

\[
\Lambda = \frac{2\mathcal{N} \pi}{L} + \frac{\mathcal{N} \pi}{L}, \tag{A2}
\]

and using (40) we get \( \mathcal{N} = N - N/2 \). Considering the case in which \( \mathcal{N} = 2M \), Eq. (39) can be written as

\[
\Delta M_{bare}^b = \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \omega_{ib} + \sum_{n=0}^{\mathcal{N}-1} \sqrt{q_n^2 + m^2} - \sum_{n=0}^{\mathcal{N}} \sqrt{k_n^2 + m^2}
\]

\[
= \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \omega_{ib} - \sum_{n=0}^{\mathcal{N}-1} \sqrt{q_n^2 - 2M + m^2}
\]

\[
+ \sum_{n=0}^{\mathcal{N}} \left( \sqrt{q_n^2 - 2M + m^2} - \sqrt{k_n^2 + m^2} \right). \tag{A3}
\]

Now, by using Eq. (36), we get

\[
\sqrt{q_n^2 + m^2} = \sqrt{k_n^2 + m^2} - \frac{k_n (\delta_+(k_n) + 2M \pi)}{L \sqrt{k_n^2 + m^2}} + O(L^{-2}). \tag{A4}
\]

By substituting the above expression in Eq. (A3), we get in the limit \( L \to \infty \),

\[
\Delta M_{bare}^b = \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \omega_{ib} - mM - \frac{\omega_b(k)}{2\pi} (\delta_+(k) + 2M \pi) \bigg|_0^\Lambda
\]

\[
+ \int_0^\Lambda \frac{dk}{2\pi} \omega_b(k) \frac{d}{dk} \delta_+(k) \bigg|_0^\Lambda
\]

\[
= \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \omega_{ib} + \frac{\langle \mathcal{N}_+ \rangle}{4\pi} + \int_0^\Lambda \frac{dk}{2\pi} \omega_b(k) \frac{d}{dk} \delta_+(k). \tag{A5}
\]

Subtracting from the above expression the tadpole graph contribution as given by expression (50) and taking \( \Lambda \to \infty \), we get again Eq. (52). In the case in which \( \mathcal{N} \) is odd, it can shown in a similar way that we get again the same formula given by Eq. (52).

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