Lie symmetry analysis and similarity solutions for the Camassa–Choi equations

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Abstract
The method of Lie symmetry analysis of differential equations is applied to determine exact solutions for the Camassa–Choi equation and its generalization. We prove that the Camassa–Choi equation is invariant under an infinity-dimensional Lie algebra, with an essential five-dimensional Lie algebra. The application of the Lie point symmetries leads to the construction of exact similarity solutions.

Keywords Lie symmetries · Similarity solutions · Camassa–Choi · Long waves

1 Introduction

The Camassa–Choi (CC) equation

\[(u_t + \alpha u_x - uu_x + u_{xx})_x + u_{yy} = 0.\]  

was derived by Choi and Camassa [1] to order to describe weakly nonlinear internal waves in a two-fluid system. Parameter \(\alpha = h^{-1}\) describes the depth in the two-fluid system. CC equation can be seen as the two-dimensional extension of the Benjamin-Ono equation, indeed when \(u_{yy} = 0\), from (1) the Benjamin-Ono equation is recovered. Because of the nonlinearity of (1) there are not any known-exact solutions in the literature. Only recently the existence of small data global solutions was proven by Harrop and Marzula in [2].

In this work, we propose that we apply the theory of Lie point symmetries in order to determine similarity solutions for the CC equation. The theory of Lie symmetries of differential equations is the standard technique for the computation of solutions and describes the algebra for nonlinear differential equations. The novelty of the Lie
symmetries is that invariant transformations can be found in order to simplify the given differential equation [3–9]. There is a plethora of applications in the Lie symmetries in the fluid dynamics with important results which have been used to understand the physical properties of the models.

The Lie symmetry analysis of the Camassa–Holm equation has been previously performed in [10]. Similarity solutions to the shallow water equations with a variable bottom were found in [11], while the Lie symmetry analysis of the rotating shallow-water equation was performed in [12]. The algebraic properties of the Benjamin-Ono equation were studied in [13]. For other applications in the point symmetries in fluid dynamics we refer the reader to [14–16,18] and references therein.

We extend our analysis to the generalized Camassa–Choi (GCC) equation

\[
(u_t + \alpha u_x - u'' u_x + \beta u_{xx})_x + u_{yy} = 0.
\]

which is the natural generalization of the generalized Benjamin-Ono equation [19]. For the two equations (1), (2) we determine the Lie point symmetries while we prove the existence of travel-wave similarity solutions for every value of parameter \( n \geq 1 \) and arbitrary depth \( \alpha \). The plan of the paper is as follows.

In Sect. 2, for the convenience of the reader we briefly discuss the basic properties and definitions of the theory of Lie point symmetries. Sections 3 and 4, include the main new material of our analysis, where we present the algebraic properties for Eqs. (1) and (2). Finally in Sect. 5 we draw our conclusions.

### 2 Preliminaries

In this section, we briefly discuss the theory of Lie point symmetries of differential equations which is the main mathematical tool that we apply in the following.

Consider function \( \Phi \) to describe the map of a one-parameter point transformation such as \( u' (t', x', y') = \Phi (u (t, x, y); \varepsilon) \) where the infinitesimal transformation is expressed as follows

\[
\begin{align*}
  t' &= t + \varepsilon \xi^t (t, x, y, u) \\
  x' &= x + \varepsilon \xi^x (t, x, y, u) \\
  y' &= y + \varepsilon \xi^y (t, x, y, u) \\
  u' &= u + \varepsilon \eta (t, x, y, u)
\end{align*}
\]

where \( \varepsilon \) is the infinitesimal parameter, that is, \( \varepsilon^2 \to 0 \).

From the latter one-parameter point transformation we can define the infinitesimal generator

\[
X = \frac{\partial t'}{\partial \varepsilon} \partial_t + \frac{\partial x'}{\partial \varepsilon} \partial_x + \frac{\partial y'}{\partial \varepsilon} \partial_y + \frac{\partial u'}{\partial \varepsilon} \partial_u,
\]
\[ \Phi (u(t,x,y) ; \varepsilon) = u(t,x,y) + \varepsilon X(u(t,x,y)) , \] (8)

that is,

\[ X(u(t,x,y)) = \lim_{\varepsilon \to 0} \frac{\Phi (u(t,x,y) ; \varepsilon) - u(t,x,y)}{\varepsilon} . \] (9)

The latter expression defines the Lie derivative of the function \( u(t,x,y) \) with respect to the vector field \( X \), also noted as \( L_X u \).

When

\[ L_X u = 0 \] (10)

then we shall say that \( u(t,x,y) \) is invariant under the action of the one-parameter point transformation with generator the vector field \( X \).

In terms of differential equations, i.e.

\[ \mathcal{H}(u, u_t, u_x, u_y, \ldots) = 0; \] (11)

then the symmetry condition reads

\[ L_X (\mathcal{H}) = 0 \text{ or } X^{[n]} (\mathcal{H}) = 0, \] (12)

where \( X^{[n]} \) describes the \( n \)th prolongation/extension of the symmetry vector in the jet-space of variables \( \{u, u_t, u_x, \ldots\} \) defined as

\[ X^{[n]} = X + \eta^{[1]}_i \frac{\partial}{\partial z^i} + \ldots + \eta^{[n]}_{ij} \frac{\partial}{\partial u_{ij \ldots n}} , \]

where \( u_i = \frac{\partial u}{\partial z^i}, \ z^i = (t, x, y) \) and

\[ \eta^{[n]}_i = D_i \eta^{[n-1]} - u_{i2 \ldots i_{n-1}} D_j \left( \xi^j \right), \ i \geq 1. \] (13)

The main application of the Lie point symmetries is based on the determination of the Lie invariants which are used to define similarity transformations and simplify the given differential equation. The exact solutions which follow by the application of the Lie point symmetries are called similarity solutions.

If \( X \) is an admitted Lie point symmetry, the solution of the associated Lagrange’s system,

\[ \frac{dt}{\xi^t} = \frac{dx}{\xi^x} = \frac{dy}{\xi^y} = \frac{du}{\eta}, \] (14)

provides the zeroth-order invariants, \( U^{A[0]}(t, x, u) \) which are applied to reduce the number of independent variables in partial differential equations, or the order in the case of ordinary differential equations.
For more details on the symmetry analysis of differential equations we refer the reader to the standard references [20–22].

3 Point symmetries of the Camassa–Choi equation

From the symmetry condition (12) for the CC equation (1) and for the one-parameter point transformation with generator \( X = \xi^1 (t, x, y, u) \partial_t + \xi^2 (t, x, y, u) \partial_x + \xi^3 (t, x, y, u) \partial_y + \eta (t, x, y, u) \partial_u \), we find the following system of differential equations

\[
\begin{align*}
\xi^1_{,t} &= 0, \quad \xi^2_{,t} = 0, \quad \xi^3_{,t} = 0, \quad \xi^4_{,x} = 0, \quad \xi^4_{,y} = 0, \quad \eta_{,uu} = 0, \\
\xi^3_{,x} + 2\xi^1_{,t} &= 0, \quad \xi^3_{,y} + 3\xi^2_{,x} = 0, \quad 2\xi^2_{,x} - \xi^1_{,t} = 0, \quad 2\xi^2_{,y} - \xi^3_{,t} = 0, \\
\eta_{,xxx} + (\alpha - u) \eta_{,xx} + \eta_{,yy} + \eta_{,tx} &= 0, \\
\eta_{,xx} - \xi^1_{,yy} &= 0, \quad 2\eta_{,yy} - \xi^3_{,yy} = 0, \\
3\eta_{,xu} - 3\xi^2_{,xx} + (\alpha - u) \xi^2_{,x} - \xi^2_{,t} - \eta &= 0, \\
3\eta_{,xuu} + (\alpha - u) \eta_{,uu} - \xi^2_{,x} - \eta_{,u} &= 0, \\
3\eta_{,xxx} - \xi^2_{,xxx} + (\alpha - u) (2\eta_{,xx} - \xi^2_{,x}) + \eta_{,tu} - \xi^2_{,yy} - 2\eta_{,x} - \xi^2_{,tx} &= 0.
\end{align*}
\]

The generic solution of the latter system is

\[
X = (c_1 + c_2 (2t)) \partial_t + (c_2 x + c_3 \phi (t) - \frac{1}{2}c_4 \psi_t (t) y) \partial_x + (\frac{1}{2}c_2 y + c_4 \psi (t)) \partial_y + (c_2 (\alpha - u) - c_3 \phi_t (t) + c_4 \frac{1}{2} \psi_{tt} (t) y) \partial_u,
\]

where \( c_1, c_2, c_3, c_4 \) are constants of integration and \( \phi (t), \psi (t) \) are arbitrary functions.

Therefore, the Lie point symmetries of the CC equation (1) are

\[
\begin{align*}
X_1 &= \partial_t, \quad X_2 = 2t \partial_t + x \partial_x + \frac{3}{2} y \partial_y - (u - \alpha) \partial_u, \\
X_3 (\phi) &= \phi (t) \partial_x - \phi_t (t) \partial_u, \quad X_4 (\psi) = \psi (t) \partial_y - \frac{1}{2} \psi_t (t) y \partial_x + \frac{1}{2} \psi_{tt} (t) y \partial_u.
\end{align*}
\]

Surprisingly, the CC equation admits infinity Lie point symmetries.

The commutators of the Lie point symmetries are

\[
\begin{align*}
[X_1, X_2] &= 2X_1, \quad [X_1, X_3 (\phi)] = (X_3 (\phi_t)), \\
[X_1, X_4 (\psi)] &= (X_4 (\psi_t)), \\
[X_2, X_3 (\phi)] &= X_3 (\phi - 2t \phi_t), \quad [X_2, X_4 (\psi)] = X_4 \left(\frac{3}{4} \psi - 2t \psi_t\right), \\
[X_3 (\phi), X_4 (\psi)] &= 0, \\
[X_3 (\phi), X_3 (\chi)] &= 0, \quad [X_4 (\psi), X_4 (\xi)] = \frac{1}{2} (X_3 (\xi \psi_t - \psi \xi_t)).
\end{align*}
\]

from where we observe that they form an infinity-dimensional Lie algebra. The existence of the infinity number of symmetries is not a real surprise. From \( X_3 \)
we determine the similarity transformation $u = - (\ln \phi (t))_{,x} x + U (t, y)$ where $U (t, y) = \frac{1}{2} \frac{\phi_{,u}}{\phi} y^2 + U_1 (t) y + U_0 (t)$ solves the reduced equation, functions $U_1 (t), U_0 (t)$ are arbitrary functions.

In the special case where $\phi (t)$ and $\psi (t)$ are constants, without loss of generality we assume that $\phi (t) = \psi (t) = 1$, the Lie point symmetries are simplified as

$$X_1' = \partial_t , \ X_2' = 2 \partial_t x + x \partial_x + \frac{3}{2} y \partial_y - (u - \alpha) \partial_u , \ X_3' = \partial_x , \ X_4' = \partial_y \quad (27)$$

with commutators

$$[X_1', X_2'] = 2X_1' , \ [X_1', X_3'] = 0 , \ [X_1', X_4'] = 0 \ , \quad (28)$$

$$[X_2', X_3'] = X_3' , \ [X_2', X_4'] = \frac{3}{2} X_3' , \ [X_3', X_4'] = 0 . \quad (29)$$

However, there are not any finite-dimensional closed Lie algebras for arbitrary functions of $\phi (t)$ and $\psi (t)$. The commutators of the latter finite-dimensional Lie algebra are presented in Table 1.

Let us demonstrate that by assuming $\phi (t) = \phi_1 + \phi_2 e^{\omega_1 t}$ and $\psi (t) = \psi_1 + \psi_2 e^{\omega_2 t}$. Then from (22), (23) it follows that the CC equation admits six Lie point symmetries which are the vector fields

$$X_1' , \ X_2' , \ X_3' , \ X_4' , \ X_5' = e^{\omega_1 t} (\partial_x - \omega_1 \partial_u) , \ X_6' = e^{\omega_1 t} \left( \partial_y - \frac{\omega_2}{2} y \partial_x + \frac{\omega_2^2}{2} y \partial_u \right) \quad (30)$$

with commutators (28), (29) and

$$[X_1', X_3'] = \omega_1 X_5' , \ [X_1', X_6'] = \omega_2 X_6' , \ [X_2', X_3'] = \omega_2 X_3' \ , \quad (31)$$

$$[X_2', X_6'] = e^{\omega_2 t}\left( \frac{3 - 4\omega_2 t}{2} \partial_y + \frac{1 + 4\omega_2 t}{4} \omega_2 y \partial_x - \frac{5 + 4\omega_2 t}{4} \omega_2^2 y \partial_u \right) \quad (32)$$

$$[X_3', X_5'] = 0 , \ [X_3', X_6'] = 0 , \ [X_4', X_6'] = \frac{\omega_2}{2} e^{\omega_2 t} (\partial_t - \omega_2 \partial_u) . \quad (33)$$

from where it is clear that the symmetry vectors (30) do not form a closed Lie algebra.
We want to constraint functions \( \phi (t) \), and \( \psi (t) \) such that the admitted Lie symmetries to form a closed Lie algebra of five-dimension with a different basis. In particular we focus on the case where the coefficients of the commutators (24)–(26) are constants. Thus we end up with the system of equations

\[
\{ \phi = c_1 \phi_t, \phi = c_2 \phi - 2t \phi_t \} \quad \text{or} \quad \{ \phi = c_1' \phi_t, \phi_t = c_2' (\phi - 2t \phi_t) \}, \quad (34)
\]

and

\[
\left\{ \begin{array}{l}
\psi = c_3 \psi_t, \psi = c_4 \left( \frac{3}{4} \psi - 2t \psi_t \right) \\
\psi' = c_3 \psi_t, \psi' = c_4 \left( \frac{3}{4} \psi - 2t \psi_t \right)
\end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l}
\psi = c_3 \psi_t, \psi = c_4 \left( \frac{3}{4} \psi - 2t \psi_t \right) \\
\psi' = c_3 \psi_t, \psi' = c_4 \left( \frac{3}{4} \psi - 2t \psi_t \right)
\end{array} \right\}, \quad (35)
\]

with constraint equations

\[
\xi = \xi \psi_t - \psi \xi_t, \text{ where } \xi = \phi, \text{ or } \xi = \phi_t \text{ or } \xi = (\phi - 2t \phi_t). \quad (36)
\]

Therefore, from (34), (35) and (36) it follows that the unique possible admitted five-dimensional Lie algebra is that of (27) for \( \phi (t) = \text{const.} \) and \( \psi (t) = \psi_0 + \psi_1 t \). Of course there are additional finite dimensional Lie algebras, for instance any set of generators constructed by \( X_3 \) form a Lie algebra; however this specific five-dimensional Lie algebra has the novelty that it can provide a plethora of different similarity transformations, while for instance the similarity transformations which follow by \( X_3 \) are all of the same family.

**Proposition 1** The CC equation (1) is invariant under infinity Lie point symmetries which form the Lie algebra \( \{ A_{2,1} \otimes_s A_\infty \otimes_s A_\infty \} \) in the Morozov–Mubarakzyanov classification scheme [25–28]. However, there exists a five-dimensional subalgebra consisted by the vector fields \( \{ X_1, X_2, X_3', X_4', X_5 = t \partial_y - \frac{1}{2} y \partial_x \} \) and form the Lie algebra \( A_{5,19}^{ab} \) in the Patera–Winternitz classification scheme [29,30]. This five-dimensional Lie algebra provides the maximum number of alternative families of similarity transformations.

As we shall see in the following, this five-dimensional Lie algebra plays a significant role in the study of the Lie point symmetries for the GCC equation (2). We proceed with the application of the Lie point symmetries for the derivation of similarity solutions.

### 3.1 Similarity solutions for the Camassa–Choi equation

Let us not apply the Lie point symmetries found in the previous section in order to find similarity solutions for the CC Eq. (1). The CC equation is a third equation of three independent variables. By applying the Lie point symmetries in partial differential equations we reduce the number of the independent variables. Hence, in order to reduce the CC equation to an ordinary differential equation we should apply two
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symmetry vectors. However, not all the symmetry vectors survive through the reduction process. In particular, if a given differential equation admits the two symmetry vectors $\Gamma_1, \Gamma_2$ with commutator $[\Gamma_1, \Gamma_2] = c \Gamma_2$, where $c$ may be zero, then reduction of the differential equation with respect to the symmetry vector $\Gamma_1$ provides that the reduced equation inherits the symmetry vector $\Gamma_1$, while reduction with $\Gamma_1$ provides a differential equation where $\Gamma_2$ is not a point symmetry when $c \neq 0$ [24]. It is clear, that if we want to perform a second reduction for the differential equation we start by considering the symmetry vector $\Gamma_1$.

Therefore, by using the results of Table 1 we find that the reduction with the symmetry vectors $\{X'_1, X'_3, X'_4, X'_3 + X'_4\}$ gives reduced equations which inherits some of the symmetries of the original equation. However, the application of the symmetry vectors $\{X'_1, X'_3, X'_4\}$ gives time-independent or static solutions which are not solutions of special interests solutions. Hence, we focus on the reduction which follows by the symmetry vector $X'_3 + X'_4$.

From the Lie point symmetry $X_{34} = X'_3 + X'_4$ we calculate the invariants

$$ t, \ w = y - x, \ u = U (t, w). \quad (37) $$

By using the latter invariant functions Eq. (1) is reduced to the following partial differential equation

$$ U_{www} + (U_w)^2 - (1 - U + h_0) U_{ww} + U_{wt} = 0. \quad (38) $$

In order to proceed with the reduction we should derive the Lie point symmetries of (38). Hence, by applying the Lie symmetry condition we find that Eq. (38) is invariant under the Lie point symmetries

$$ Z_1 = \partial_t, \ Z_2 = 2t \partial_t + w \partial_w + (h_0 + 1 - U) \partial_U, \quad (39) $$
$$ Z_3 = t^2 \partial_t + tw \partial_w + [(h_0 + 1 - U) t + w] \partial_U, \quad (40) $$
$$ Z_4 = \phi (t) \partial_w + \phi_t \partial_U. \quad (41) $$

Vector fields $Z_1$, $Z_2$ and $Z_4$ are reduced symmetries, while $Z_3$ is a new symmetry for the reduced Eq. (38). It is important to mention that $Z_4$ describes an infinity number of symmetries, hence the reduced Eq. (38) admits infinity number of Lie point symmetries as the “mother” Eq. (1). On the other hand, Lie point symmetries $\{Z_1, Z_2, Z_3\}$ form a closed Lie algebra, known as $SL(2, R)$.

The application of $Z_4$ in (38) provides the linear second-order ODE $\phi_{tt} = 0$, where $U (t, w) = U_0 (t) + \frac{\phi_t}{\phi_0} w$, where $U_0 (t)$ is an arbitrary function. Therefore, the similarity solution is derived to be

$$ U (t, w) = U_0 (t) + \frac{\phi_1}{\phi_1 t + \phi_0} w. \quad (42) $$
Reduction with respect the symmetry vector $Z_1$ of Eq. (38) provides the third-order ODE

$$\left. \begin{align*}
Y_{w w w} + (Y_w)^2 - YY_{w w} &= 0, \\
U (t, w) &= Y (w) + 1 + h_0, \quad w = x
\end{align*} \right\} \quad (43)$$

which admit two point symmetries the reduced symmetries $Z_2$, and $Z_3$. Equation (43) can be integrated as follows

$$Y_{w w} + YY_w + Y_0 = 0, \quad (44)$$

where the latter equation can be solved in terms of quadratics. Indeed for the integration constant $Y_0 = 0$, the general solution is

$$Y (w) = Y_0 \tanh \left( \frac{w - w_0}{2c} \right), \quad (45)$$

while in general Eq. (44) becomes

$$Y_w + \frac{1}{2} Y^2 + Y_0 w + Y_1 = 0. \quad (46)$$

The application of the Lie symmetry vector $Z_2$ provides the reduced third-order ODE

$$2\bar{Y}_{\sigma \sigma \sigma} + (\sigma - 2 (1 + \bar{Y})) \bar{Y}_{\sigma \sigma} - 2\bar{Y} = 0, \quad (47)$$

where now $U (t, w) = 1 + h_0 + \frac{y(\sigma)}{\sqrt{t}}, \quad \sigma = \frac{w}{\sqrt{t}}$. The latter equation can be easily integrated as follows

$$2\bar{Y}_{\sigma \sigma} - \bar{Y} - (2\bar{Y} - \sigma) \bar{Y} + \bar{Y}_0 = 0 \quad (48)$$

or

$$2\bar{Y}_{\sigma} + \bar{Y}^2 - \sigma \bar{Y} + \bar{Y}_0 \sigma + \bar{Y}_1 = 0. \quad (49)$$

In a similar way, the reduction with respect to the Lie symmetry vector $Z_3$ gives the solution

$$U (t, w) = \frac{w}{t} + h_0 + 1 + \frac{Y' (\lambda)}{t}, \quad \lambda = \frac{w}{t}, \quad (50)$$

where $Y' (\lambda)$ is given by the following first-order ODE

$$Y'_\lambda + \frac{1}{2} (Y')^2 + Y_0 \lambda + Y_1 = 0. \quad (51)$$
It comes as no surprise that the reduction with the three elements of the $SL(2, R)$ provides similar reduced equations. That is because the three symmetry vectors are related with similarity transformations as well as also the reduced equations are related, for more details we refer the reader to [23].

Finally, reduction with the vector field $Z_{1} + Z_{4}$, for $\phi(t) = 1$, provides travel-wave solution and the reduced equation is that of (43) where $w = t - x$.

Similarly, the reduction of CC Eq. (1) with respect the symmetry vector $X_{14} = X_{1} + X_{4}'$, provides a travel-wave solution, as before. Therefore, we conclude that travel-wave solutions exist for the CC equation.

We proceed our analysis by studying the invariant point transformations for the GCC Eq. (2).

4 Point symmetries of the generalized Camassa–Choi equation

The Lie point symmetries of the GCC Eq. (2) are

$$Y_{1} = \partial_{t}, \quad Y_{2} = 2t\partial_{t} + x\partial_{x} + \frac{3}{2}y\partial_{y} - \frac{1}{n}u\partial_{u}$$

(52)

$$Y_{3} = \partial_{x}, \quad Y_{4} = \partial_{y}, \quad Y_{5} = 2t\partial_{y} - y\partial_{x}$$

(53)

when $\alpha = 0$ and

$$\bar{Y}_{1} = \partial_{t}, \quad \bar{Y}_{2} = \partial_{x}, \quad \bar{Y}_{2} = 2t\partial_{t} + (x + \alpha t)\partial_{x} + \frac{3}{2}y\partial_{y} - \frac{1}{n}u\partial_{u}$$

(54)

$$\bar{Y}_{3} = \partial_{x}, \quad \bar{Y}_{4} = \partial_{y}, \quad \bar{Y}_{5} = 2t\partial_{y} - y\partial_{x}$$

(55)

for $\alpha \neq 0$.

The corresponding commutators for the admitted Lie symmetries are presented in Table 2. We observe that the two admitted Lie algebras are different. For $\alpha \neq 0$ the Lie symmetries form the Lie algebra $A_{5,23}^{b}$ and for $\alpha = 0$, the Lie symmetries form the Lie algebra $A_{5,19}^{ab}$ in the Patera–Winternitz classification scheme [29,30].

When the parameter $\alpha$ is zero, the Lie point symmetries $\{Y_{1}, Y_{2}, Y_{3}, Y_{4}\}$ are these which form a finite-dimensional Lie algebra for the CC Eq. (1), that is, vector fields (27). However, When $\alpha \neq 0$ things are different. The fifth symmetry $Y_{5}$ is a case of
\( X_4 (\psi) \) with \( \psi (t) = t \). Indeed, the admitted Lie point symmetries by the GCC are those which form the maximum finite-dimensional Lie algebra for the CC equation.

We continue our analysis by applying the Lie point symmetries to determine similarity solutions for the GCC equation.

### 4.1 Similarity Solutions for the generalized Camassa–Choi equation

As in the case of the CC we consider the similarity transformation provided by the vector field \( Y_{34} = Y_3 + Y_4 \), because it is the similarity transformation which provides a reduced equation which inherits symmetry vectors. Hence, we find that the GCC equation (2) is reduced to

\[
U_{ww} + U_{wt} + nU^{n-1} (U_w)^2 + (U^n + 1 - \alpha) U_{ww} = 0, \tag{56}
\]

where \( u (t, x, y) = U(t, w) \) and \( w = x - y \). We observe that Eq. (56) reduces into (38) when \( n = 1 \).

For \( n \neq 1 \), we calculate the Lie point symmetries of (56) which are found to be

\[
\tilde{Z}_1 = \partial_t, \quad \tilde{Z}_2 = \partial_w \quad \text{and} \quad \tilde{Z}_3 = 2t \partial_t + (t (1 + \alpha) - w) \partial_w - \frac{1}{n} U \partial_u.
\]

The application of the Lie symmetry vector \( \tilde{Z}_{12} = \partial_t + \partial_w \) in (56) provides the travel-wave solution

\[
Y_{\sigma \sigma} + nY^{n-1} (Y_\sigma)^2 + (Y^n - \alpha - 2) Y_{\sigma \sigma} = 0, \quad U(t, w) = Y(\sigma), \quad \sigma = w - t \tag{57}
\]

The latter equation can be easily integrated by quadratures as follows

\[
Y_\sigma + \frac{1}{n + 1} Y^{n+1} + (2 + A) Y + Y_1 \sigma + Y_0 = 0. \tag{58}
\]

On the other hand, the reduction of (56) with respect to the similarity transformation provided by the vector field \( \tilde{Z}_3 \) provides

\[
U(t, w) = H(\zeta) t^{-\frac{1}{2n}}, \quad \zeta = \frac{w + t (1 + \alpha)}{\sqrt{t}}, \tag{59}
\]

where \( H(\zeta) \) satisfies the third-order ordinary differential equation

\[
2nH H_{\zeta \zeta} + nH (2H^n - \zeta) H_{\zeta \zeta} - \left( (n + 1) H - 2n^2 H^n H_\zeta \right) H_\zeta = 0. \tag{60}
\]

Equation (60) can be integrated as follows

\[
H_{\zeta \zeta} - \frac{1}{2n} H + \left( H^n - \frac{\zeta}{2} H \right) H_\zeta + H_1 = 0. \tag{61}
\]
Fig. 1 Qualitative evolution of $H(\zeta)$ as it is given by the differential equation (61) for initial conditions $H(0) = 1$ and $H_\zeta(0) = -0.5$. The plots are for $H_1 = 0$ and $n = 2$ (red line), $n = 3$ (blue line) and $n = 5$ (yellow line) (color figure online)

The latter equation does not admit any point symmetry and we cannot perform further reduction. However in Fig. 1 we present some numerical solutions. What is also important to mention is that in Eq. (61) parameter $\alpha$ plays no role. Hence the same reduction holds and for the case $\alpha = 0$.

5 Conclusions

In this work, we applied the theory of symmetries of differential equations in order to determine exact similarity solutions for the Camassa–Choi equation (1) and its generalization (2). CC equation describes weakly nonlinear internal waves in a two-fluid system and it can be seen as the two-dimensional generalization of the Benjamin-Ono.

For the CC equation we found that it is invariant under an infinity-dimensional Lie algebra, with maximum finite Lie subalgebra of dimension five. That five-dimensional subalgebra is the one which form the complete group of invariant one-parameter point transformations for the GCC equation.

We apply the Lie point symmetries and we prove the existence of similarity solutions in the two-dimensional plane $\{x, y\}$. Specifically, we found that the similarity solutions can be expressed in terms of quadratures.

Surprisingly, the CC equation under the application of similarity transformations can be reduced into a three-dimensional ordinary differential equation which is invariant under the $SL(3, R)$, where all the possible reductions provide similarity solutions related under point transformations.
In a future work we plan to study the physical properties of those new similarity solutions.

Compliance with ethical standards

Conflict of interest The author does not have any conflict of interest.

References

1. Choi, W., Camassa, R.: Weakly nonlinear internal waves in a two-fluid system. J. Fluid Mech. 313, 83 (1996)
2. Harrop-Griffiths, B., Marzula, J.L.: Large data local well-posedness for a class of KdV-type equations II. Nonlinearity 31, 1868 (2018)
3. Paliathanasis, A., Krishnakumar, K., Tamizhmani, K.M., Leach, P.G.L.: The algebraic properties of the space- and time-dependent one-factor model of commodities. Mathematics 4, 28 (2016)
4. Xin, X.: Nonlocal symmetries, exact solutions and conservation laws of the coupled Hirota equations. Appl. Math. Lett. 55, 63 (2016)
5. Xin, X.: Nonlocal symmetries and interaction solutions of the (2+1)-dimensional higher order Broer-Kaup system. Acta Phys. Sin. 65, 240202 (2016)
6. Kallinikos, N., Meletlidou, E.: Symmetries of charged particle motion under time-independent electromagnetic fields. J. Phys. A: Math. Theor. 46, 305202 (2013)
7. Jamal, S., Paliathanasis, A.: Group invariant transformations for the Klein–Gordon equation in three dimensional flat spaces. J. Geom. Phys. 117, 50 (2017)
8. Webb, G.M.: Lie symmetries of a coupled nonlinear Burgerssheat equation system. J. Phys A: Math. Gen. 23, 3885 (1990)
9. Leach, P.G.L.: Symmetry and singularity properties of the generalised Kummer–Schwarz and related equations. J. Math. Anal. Appl. 348, 487 (2008)
10. Velan, M.S., Lakshmanan, M.: Lie Symmetries and Invariant Solutions of the Shallow–Water Equation. Int. J. Non-linear Mech. 31, 339 (1996)
11. Pandey, M.: Lie symmetries and exact solutions of shallow water equations with variable bottom. Int. J. Nonl. Sci. Num. Sim. 16, 337 (2015)
12. Paliathanasis, A.: Lie symmetries and similarity solutions for rotating shallow water. Zeitschrift für Naturforschung A, in press [https://doi.org/10.1515/zna-2019-0063]
13. Chetverikov, V.N.: Symmetry algebra of the Benjamin–Ono equation. Acta Appl. Math. 56, 121 (1999)
14. Szatmari, S., Bielho, A.: Symmetry analysis of a system of modified shallow-water equations. Commun. Nonl. Sci. Numer. Simul. 19, 530 (2014)
15. Chesnokov, A.A.: Symmetries and exact solutions of the shallow water equations for a two-dimensional shear flow. J. Appl. Mech. Tech. Phys. 49, 737 (2008)
16. Chesnokov, A.A.: Symmetries and exact solutions of the rotating shallow water equations. Eur. J. Appl. Math. 20, 461 (2009)
17. Xin, X., Zhang, L., Xia, Y., Liu, H.: Nonlocal symmetries and exact solutions of the (2+1)-dimensional generalized variable coefficient shallow water wave equation. Appl. Math. Lett. 94, 112 (2019)
18. Paliathanasis, A.: Benney-Lin and Kawahara equations: a detailed study through Lie symmetries and Painlevé analysis. Physica Scripta, in press [https://doi.org/10.1088/1402-4896/ab32ad]
19. Keing, C.E., Ponce, G., Vega, L.: On the generalized Benjamin-Ono equation. Trans. Am. Math. Soc. 342, 155 (1994)
20. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Springer, New York (1989)
21. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
22. Ibragimov, N.H.: CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws, CRS Press LLC, Florida (2000)
23. Jamal, S., Leach, P.G.L., Paliathanasis, A.: Nonlocal representation of the sl(2, R) algebra for the Chazy equations. Quaestiones Math. 42, 125 (2018)
24. Govinder, K.S.: Lie subalgebras, reduction of order, and group-invariant solutions. J. Math. Anal. Appl. 258, 720 (2001)
25. Morozov, V.V.: Classification of six-dimensional nilpotent Lie algebras. Izvestia Vysshikh Uchebn Zavedenii Matematika 5, 161 (1958)
26. Mubarakzyanov, G.M.: On solvable Lie algebras. Izvestia Vysshikh Uchebn Zavedenii Matematika 32, 114 (1963)
27. Mubarakzyanov, G.M.: Classification of real structures of five-dimensional Lie algebras. Izvestia Vysshikh Uchebn Zavedenii Matematika, 34, 99 (1963)
28. Mubarakzyanov, G.M.: Classification of solvable six-dimensional Lie algebras with one nilpotent base element. Izvestia Vysshikh Uchebn Zavedenii Matematika, 35, 104 (1963)
29. Patera, J., Sharp, R.T., Winternitz, P., Zassenhaus, H.: Invariants of real low dimension Lie algebras. J. Math. Phys. 17, 986 (1976)
30. Patera, J., Winternitz, P.: Subalgebras of real three- and four-dimensional Lie algebras. J. Math. Phys. 18, 1449 (1977)

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