Robust evaluation of flow front data for in-plane permeability characterization by radial flow experiments

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ABSTRACT
A novel approach is presented for modeling the temporally advancing fluid flow front in radial flow experiments for in-plane permeability characterization of reinforcing fabrics. The method is based on fitting an elliptic paraboloid to the flow front data collected throughout such an experiment. This "paraboloid" approach is compared to the conventional "ellipse" method and validated by means of data sets of optically tracked experiments from two different research institutions. A detailed discussion of the results reveals the benefits of the "paraboloid" method in terms of numerical efficiency as well robustness against temporal or local data variations. The "paraboloid" method is tested on temporally and spatially limited data sets from a test rig involving linear capacitive sensors. There, the method shows advantages over the conventional approach as it incorporates the entirety of available measurement data, particularly in the last stages of the experiments which are most characteristic for the material under test.

Introduction
In liquid composite molding, dry preforms of reinforcing fabrics are placed in a mold and then impregnated with the liquid polymer matrix material. The impregnation process plays a key role as insufficiently saturated regions directly affect the mechanical properties of the final component. In order to avoid elaborate and expensive impregnation trials, filling simulations can be conducted. These simulations strongly rely on accurate and reliable information on the permeability of the fibrous reinforcement. In the late 1990s, Weitzenböck et al. introduced a framework for experimental permeability characterization techniques. For in-plane permeability characterization, the "rectilinear flow" (or "channel flow") method and the "radial flow" method were distinguished. The channel flow method is based on one-dimensional fluid flow through the reinforcing structure placed in a specifically designed characterization cell. In general, this method has two major disadvantages:

1. Race tracking is very likely to occur in gaps along the side edges of the material, deteriorating the rectilinear flow front advancement. Thus, specific care is required, typically by sealing the preform material at the side walls of the characterization cell.
2. At least three experiments are required to fully characterize the in-plane permeability tensor of the material under test. This, by nature, causes...
increased experimental effort compared to the radial flow method. Application of "multi-cavity parallel flow cells", which enable the simultaneous characterization of samples cut along different material directions, was proposed by various authors in order to reduce experimental efforts.

In contrast to the linear flow technique, the radial flow method allows for a full characterization of the in-plane permeability tensor from a single experiment and avoids race tracking effects on principle. The technique is based on the observation of radial flow experiments as introduced by Adams et al., which enable strictly planar fluid flow as a result of a circular injection opening punched into the preform stack under test. The experiments then comprise three major aspects:

(1) Flow front tracking: The radially advancing flow front is typically tracked by means of a camera system as originally proposed by Adams et al. For this, the optically transparent mold half has to be stiffened, e.g. by bars or cross-beam structures, in order to avoid extensive mold deflection during the experiments. Alternatively, two metal mold halves can be used, asking for alternative flow front tracking techniques. Liu et al. presented a cell with a star-like arrangement of line arrays of electrically sensitive point sensors. These are used to derive trigger signals as soon as the flow front reaches a sensor position. Morren et al. and Hoes et al. showed works with a similar approach based on arrays of dielectric sensors. Kissinger et al. presented a system involving linear capacitive sensors in a star-like setup around the central injection opening. These sensors are calibrated in such a manner that a linear relationship of sensor signal and saturated sensor area is obtained.

(2) Flow front modeling: Typically, the overall shape of the flow front at a specific point in time during the experiment is reconstructed with an elliptic geometric model fitted to the flow front data. Evaluating the major and minor ellipse axes lengths throughout the experiment results in the temporal flow front characteristics required for the subsequent calculation of in-plane permeability values. Surprisingly few information is available in existing literature on the specific ellipse fitting approaches used. This might be due to the common availability of routines for fitting elliptic geometry models to noisy measurement data in standard data evaluation software. However, the choice of elliptic geometry model fitted to the data can easily have an impact on the resulting in-plane permeability values as reported by Fauster et al.

(3) Computation of in-plane permeability data: this step involves evaluation strategies in combination with specifically developed mathematical algorithms. Adams and Rebenfeld presented an algorithm, which is based on transforming the problem to an elliptical coordinate system and uses an iterative numerical solution for the degree of anisotropy, i.e. the ratio of the two principal planar permeability values. Chan and Hwang presented an algorithm which introduces a transformation of the anisotropic problem into an equivalent isotropic system (EIS) following principles originally introduced by Bear et al. A modification of this algorithm was presented by Fauster et al. which avoids a violation of the isotropic data properties in the EIS inherent to the original method. Finally, a number of evaluation strategies can be distinguished as discussed by Ferland et al. for processing the data acquired during the experiments.

The paper at hand focusses on the aspect of flow front modeling by introducing a novel approach for processing flow front data acquired during radial flow experiments. In particular, the approach:

(1) ensures increased numerical efficiency as the entirety of flow front data is modelled by means of an elliptic paraboloid in a single step approach,

(2) provides a direct method for deriving a global value for the orientation angle of the principal permeability directions with respect to a testrig-specific coordinate frame and

(3) is particularly beneficial for modelling data, which lack completeness in terms of temporal and/or spatial resolution:

(a) in optically tracked radial flow experiments, the quasi-continuously acquired flow front data is partially occluded as a result of the need for mechanical elements increasing the structural stiffness of the optically transparent mold half, and

(b) the data provided by alternative techniques (e.g. electric or dielectric point sensors; linear capacitive sensors) is incomplete due to the limited number of available sensors.

The validity of this approach is demonstrated by means of examples from optical and capacitive permeameter systems, which were selected from datasets made available by three European research institutions collaborating for this piece of work: the Processing of Composites Group (LVV) at Montanuniversität Leoben (Austria), the Department of Polymer Materials and Plastics Engineering (PuK) at Clausthal University of Technology (Germany) and the Institut für Verbundwerkstoffe (IVW) in Kaiserslautern (Germany).
Basics of flow in porous media and geometric considerations

Fluid flow in fibrous, i.e. porous, structures is commonly modeled according to the fundamental work of Henry Darcy, whose empirical findings are commonly represented as a relation between the flow velocity \( v \) and the driving pressure gradient \( \nabla p \). It can be reproduced as:

\[
v = -\frac{1}{\eta} K \nabla p, \tag{1}
\]

with the dynamic fluid viscosity \( \eta \) as well as the permeability tensor \( K \). While Neuman analytically verified the correspondence of Darcy’s law with the Navier–Stokes equations, Liakopoulos proved the permeability tensor \( K \) to be symmetric and of second order. For planar fluid flow in anisotropic porous media, such as in an in-plane permeability characterization cell, the tensor is:

\[
K_{\text{anisotr.}} = \begin{bmatrix}
k_x & k_{xy} \\
k_{yx} & k_y
\end{bmatrix}, \quad \text{with: } k_{yx} = k_{xy}. \tag{2}
\]

However, by transformation to a coordinate frame aligned with the principal axes of fluid flow, the tensor simplifies to orthotropic shape:

\[
K_{\text{orthotr.}} = \begin{bmatrix}
k_1 & 0 & 0 \\
0 & k_2 & 0 \\
0 & 0 & k_3
\end{bmatrix} = k_1 \begin{bmatrix}1 & 0 & 0 \\
0 & 0 & \alpha
\end{bmatrix}, \tag{3}
\]

with \( \alpha = \frac{k_2}{k_1} \) denoting the degree of in-plane anisotropy, \( 0 < \alpha \leq 1 \). Substituting Darcy’s law in the continuity equation of incompressible fluid flow, \( \nabla \cdot v = 0 \), gives:

\[
\nabla \cdot (K_{\text{orthotr.}} \nabla p) = \begin{bmatrix} \frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix} \cdot \begin{bmatrix} k_1 & 0 & 0 \\
0 & k_2 & 0 \\
0 & 0 & k_3
\end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial x} \\
\frac{\partial p}{\partial y}
\end{bmatrix} = \frac{\partial^2 p}{\partial x^2} + \alpha \frac{\partial^2 p}{\partial y^2} = 0. \tag{4}
\]

This second order partial differential equation does not show a closed form solution, however for flow in isotropic porous media, i.e. \( \alpha = 1 \), Equation (4) simplifies to the well-known Laplace differential equation, which can be solved analytically.

Flow in isotropic porous media

For fluid flow in isotropic porous media, Adams et al. reported an analytical solution for the resulting radially symmetric pressure distribution. This in turn enables the derivation of an analytic solution for the temporally advancing radial flow front extent:

\[
r_f^2(2(\ln r_f - \ln r_0) - 1) + r_0^2 = \frac{4k\Delta p}{\eta e} t, \tag{5}
\]

with the radius \( r_0 \) of the injection opening and the pressure difference \( \Delta p \) between the injection point and the flow front. Furthermore, the following three material properties are included: fluid viscosity \( \eta \), fabric porosity \( \epsilon \) and isotropic fabric permeability \( k \). Although widely covered in literature, the mathematical derivations are briefly reproduced in Appendix I for the sake of completeness and consistency. Equation (5) can be reorganized to:

\[
f(r_f)r_f^2 + c_3 r_f^2 + 2c_4 t + c_6 = 0, \tag{6}
\]

with the nonlinear function \( f(r_f) = 2\ln r_f \) and the constant coefficients: \( c_3 = -2 \ln r_0 - 1 \), \( c_4 = -\frac{4k\Delta p}{\eta e} \) and \( c_6 = r_0^2 \) (the reason for the unordinary choice of coefficient indices \( c_i \) will be clarified at the end of this section). Studying the two terms involving \( r_f^2 \) in Equation (6) in a range relevant for typical in-plane permeability characterization cells (\( 0 < r_f < 0.25 \) m) reveals that the constant coefficient term, \( c_3 r_f^2 \), dominates over the nonlinear function term, \( f(r_f)r_f^2 \), as shown in Figure 1. This finding is of general validity, as (a) the coefficient \( c_3 \) as well as the function \( f(r_f) \) are independent of the material properties and (b) the only experimental parameter contained, the radius \( r_0 \) of the injection opening, shows insignificant variations in laboratory practice, typically being in the range: \( 5 < r_0 < 10 \) mm.

Figure 1 additionally illustrates that in the given range of \( r_f \), the sum of the two terms can be reasonably well approximated by a single, constant coefficient term:

\[
f(r_f)r_f^2 + c_3 r_f^2 \approx s c_3 r_f^2 = \tilde{c}_3 r_f^2, \tag{7}
\]

with the empirical scaling constant, \( 0 < s < 1 \). This indicates that the temporal advancement of the radially symmetric flow front can be well approximated by a simple equation of second order:

\[
\tilde{c}_3 r_f^2 + 2c_4 t + c_6 = 0. \tag{8}
\]

Comparison with the well-known general equation of quadratic forms,

\[
c_1 x^2 + 2c_2 xy + c_3 y^2 + 2c_4 x + 2c_5 y + c_6 = 0, \tag{9}
\]

now reveals the reason for the choice of coefficients indices \( c_i \) and more importantly, the fundamental nature of Equation (5): The temporal advancement of the fluid flow front in isotropic porous media exhibits the characteristics of a parabola, symmetric to the time axis. Obviously, the approximation of the flow front advancement described by Equation (7) could be improved by fitting even-degree polynomials of higher order to the sum of the two functional terms. However, this would add computational costs to the fitting task and raise questions for the subsequent steps of data processing, which require further investigations.
Validity for flow in anisotropic porous media

The derivation given in the previous section refers to fluid flow in isotropic porous media, however, the major finding of the parabolic nature inherent to the temporally advancing flow front extent is equally valid for fluid flow in anisotropic porous media. Although an analytic proof cannot be given here (the differential equation for the pressure distribution in anisotropic media does not show a closed form solution), empirical evidence is possible. As visualized in Figure 2 by means of exemplarily chosen experimental data, parabolic geometry models (Appendix 2 briefly reflects the mathematical basics) can be fitted to major and minor radial extent data \( r_1(t) \) and \( r_2(t) \), respectively, with a high level of accuracy.

However, some deviations can be observed especially in the early stages of the experiments. These are due to the parabolic fit being dominated by the large number of data points available in the later stages of the experiments (due to increasing flow ellipse circumference), which show rather low curvature characteristics. From a strictly mathematical point of view, this effect could easily be avoided, e.g. by weighting techniques. From a flow dynamics point of view however, the early stages of radial flow experiments are dominated by starting effects, where the flow front migrates from the shape of the circular injection opening to an elliptic shape specific to the material under test. Hence, this stage does actually not reflect the true material behavior. Thus, the deviations of the fitted parabolas from the measurement data

![Figure 1. Graphical analysis of functional terms contained in the solution for the temporal advancement of the radially symmetric flow front.](image1)

![Figure 2. Temporal characteristics of major and minor radial extent data acquired from radial flow experiments at LVV (left) and PuK (right) together with parabolic models fitted to the data.](image2)
In order to derive affine paraboloid parameters from the set of homogeneous quadric coefficients, principal component analysis (PCA) is applied. The smallest eigenvalue $\lambda_1$ of the upper-left $3 \times 3$ submatrix $Q_{33}$ turns out to be zero as a result of the infinite length of the primary paraboloid axis. The corresponding eigenvector $e_1 = [0 \ 0 \ 1]^T$ indicates its direction, i.e. the $z$-axis. Eigenvector $e_2 = [e_{21} \ e_{22} \ 0]^T$ corresponding to the second-smallest eigenvalue $\lambda_2$ of $Q_{33}$ is pointing towards the major vertex of the elliptic paraboloid cross-section. Thus, the orientation angle $\beta$ of the paraboloid with respect to the global $x$-axis can be directly derived from its components:

$$\beta = \tan^{-1}\left(\frac{e_{22}}{e_{21}}\right).$$

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(11)
This result is of major importance, as it reveals that the degree of anisotropy:

(1) is independent of time and
(2) can be analytically computed from the ratio of parabola coefficients.

Hence, a direct and analytical method for computing the degree of anisotropy $\alpha$ is available which avoids the need for an iterative optimization procedure as proposed in the original work of Adams and Rebenfeld. This results in a significant simplification of the algorithm and a reduction of computational costs.

**Experimental work**

Following the approach presented in the previous section, experimental data from three different test rigs for in-plane permeability characterization following the radial flow method was evaluated. Due to space limitations, the particular test rigs are not described here in detail. However, the interested reader is referred to relevant literature previously published by the authors:

(1) optical permeameter at LVV,30–35
(2) optical permeameter available at PuK36,37 and
(3) capacitive permeameter systems at IVW and LVV.18,38

**Validation of the evaluation method on optically tracked radial flow experiments**

In-plane permeability characterization through optical observation of radial flow experiments involves at least one optically transparent mold half. In order to avoid extensive mold deflection, specific arrangements for increased structural stiffness are required. However, these elements cause partial occlusions of the fluid flow.
front as it advances during the radial flow experiments. These occlusions have to be handled with digital image processing techniques such as application of mask images.\textsuperscript{39,40}

The sequence images acquired at the test rig of LVV (Figure 5, left) show a geometrically simple occluded region in terms of a single stiffening bar, vertically oriented in the central portion of the images. As a result, the amount of data points available to each side of the stiffening frame is well balanced, but obviously increasing over experimental time as depicted in Figure 6.

The images acquired with the test rig of PuK (Figure 5, right) show variations in the data point balance as the location and degree of flow front occlusion is varying during the experiment as visualized in the sequence images shown in Figure 7.

In order to validate the data evaluation approach proposed in this paper, two exemplarily chosen data sets were evaluated from optically tracked radial flow experiments conducted independently at LVV and PuK, respectively. In particular (a) radial extent data, (b) orientation angle and (c) in-plane permeability values are compared as determined according to the following two methods:

1. the conventional “ellipse” method, where elliptic geometry models are individually fitted to the flow front data extracted from the particular sequence images, and
2. the newly proposed “paraboloid” method, where a single elliptic paraboloid model is fitted to the entirety of available flow front data points as visualized in Figure 8.

Comparison of radial extent data
In Figure 9, the temporal characteristics of radial extent data $r_1(t)$ and $r_2(t)$ are compared. The data obtained with the “ellipse” method indicate the major and minor semi-axis length of the ellipse fitted to flow front data at a particular point of time. By contrast, the data of the “paraboloid” method are to be understood as major and minor elliptic extent of the paraboloid fitted to the data points available up to a specific point of time. The data characteristics are more or less perfectly coinciding.

Figure 5. Example images from radial flow experiments conducted independently at LVV (left) and PuK (right).

Figure 6. Flow front evolution in radial flow experiments at LVV: Raw images with overlay of injection opening center point and data points along the flow front.

Figure 7. Flow front evolution in radial flow experiments at PuK: Raw images with overlay of injection opening center point and data points found along the flow front.
experiments as shown in Figure 8, its orientation angle (which corresponds to the very last value of the temporal characteristics of “paraboloid” fits depicted in Figure 10) is very well in line with the final trend of the temporal characteristics derived with the “ellipse” method. This is in distinct contrast to the average of orientation angle characteristics derived with the “ellipse” method, which is clearly impacted by the strong variations in the early stages of the experiment. Although both approaches involve some kind of averaging effects, these are clearly different: The elliptic paraboloid fit involves the entirety of data points found along the temporally advancing fluid flow front and thus, inherently adds a stronger weight to the data in the later stages of the experiment as a result of the increasing amount of available data points. By contrast, averaging the temporal characteristics of orientation angles from the “ellipse” method provides equal weight to all of the values obtained throughout the experiment. Consequently, the results not only support the validity of the newly proposed data evaluation method, they also indicate a clear advantage:

**Comparison of orientation angles**

Figure 10 shows a comparison of orientation angle data $\beta(t)$ as determined with the two methods. The temporal characteristics obtained with the “ellipse” method exhibit a higher degree of scatter compared to those of the “paraboloid” method. This is a result of the amount of data points involved in the particular fitting tasks: while the number of data points is slightly increasing over time in the “ellipse” method, it is rising in a cumulative manner in the “paraboloid” method.

Furthermore, it can be clearly seen that the characteristics of the orientation angles are poorly defined in the early stages of the radial flow experiments and thus, show particularly strong variations. However, the characteristics trend towards a constant level in the later stages of the experiments. This transition reflects the flow front migration from an initially circular boundary to the elliptic shape characteristic for the material under test.

Considering the elliptic paraboloid fitted to the entirety of flow front data points collected during the experiments as shown in Figure 8, its orientation angle (which corresponds to the very last value of the temporal characteristics of “paraboloid” fits depicted in Figure 10) is very well in line with the final trend of the temporal characteristics derived with the “ellipse” method. This is in distinct contrast to the average of orientation angle characteristics derived with the “ellipse” method, which is clearly impacted by the strong variations in the early stages of the experiment. Although both approaches involve some kind of averaging effects, these are clearly different: The elliptic paraboloid fit involves the entirety of data points found along the temporally advancing fluid flow front and thus, inherently adds a stronger weight to the data in the later stages of the experiment as a result of the increasing amount of available data points. By contrast, averaging the temporal characteristics of orientation angles from the “ellipse” method provides equal weight to all of the values obtained throughout the experiment. Consequently, the results not only support the validity of the newly proposed data evaluation method, they also indicate a clear advantage:
While specific care is required when averaging the temporal characteristics of orientation angles found with the "ellipse" method, the "paraboloid" method robustly provides a representative value as a direct output.

**Comparison of in-plane permeability values**

The validity of the newly proposed "paraboloid" method is further demonstrated by computing major and minor in-plane permeability values $k_1$ and $k_2$ from the temporal characteristics of radial extent data $r_1(t)$ and $r_2(t)$, respectively, and by comparison with results obtained from the "ellipse" method. Following the systematics introduced by Ferland et al.\textsuperscript{23} for the computation of in-plane permeability data from channel flow experiments, three different strategies can be distinguished:

1. The "single point" strategy: The entire set of measurement data is evaluated by relating data from each point of time of the experiment to its start, i.e. $t_{\text{single},k} = \{t_1, t_k\}$ with $k = \{2 \ldots n\}$ and the total number of $n$ time steps.

2. The "elementary" method: Again, a multi-step approach is followed, whereas each evaluation step involves measurement data from a pair of consecutive time steps: $t_{\text{elem},k} = \{t_{k-1}, t_k\}$ with $k = \{2 \ldots n\}$.

3. The "interpolation" method: For each evaluation step, the entire set of measurement data acquired up to this point of time is considered, i.e. $t_{\text{interp},k} = \{t_1, t_2, \ldots, t_k\}$ with $k = \{2 \ldots n\}$. Figure 11 shows a comparison of temporal characteristics of major and minor in-plane permeability data computed according to the "interpolation" strategy based on radial extent data with the "ellipse" and the "paraboloid" method, respectively. The characteristics are more or less perfectly coinciding, which is not surprising as a result of the matching characteristics of radial extend data (see Figure 9).

Figure 10. Orientation angles from radial flow experiments at LVV (left) and PuK (right), respectively.

Figure 11. Major and minor in-plane permeability characteristics as derived from radial flow experiments at LVV (left) and PuK (right), respectively.
Following the “ellipse” evaluation method, the shape of the flow front can be reconstructed at each point of time during the experiment by fitting an elliptic geometry model to the sensor data. However, this requires to restrict the measurement data to a portion showing valid data on at least three or five sensors, depending on the choice of ellipse model being fitted. In other words, the measurement data needs to be cut when the required minimum of valid data points is reached and thus, a significant portion of measurement data from the last – and thus most accurate – stage of the radial flow experiment is excluded from being processed towards in-plane permeability data. By contrast, the “paraboloid” method proposed in this work allows for making full use of the available measurement data as the elliptic paraboloid can be fitted to the entirety of flow front data collected throughout the experiment without any restrictions as shown in Figure 13, right.

**Comparison of radial extent data and orientation angle**

Similar to the comparative analysis presented for the optically tracked experiments, the radial extent data and orientation angle obtained with the “ellipse” and “paraboloid” methods are compared as shown in Figure 14.

The characteristics of major and minor radial extent data clearly reveal the impact of the available number of data points on the radial extent values found with the “ellipse” method. As the number of valid sensor data reduces from eight to six at \( t \approx 60 \) s, the data characteristics (particularly the major radial extent) show significant changes in the overall trend and the scatter associated with the radial extent data increases significantly as well. This is because the two data points located...
closest to the major apex of the fitted ellipse fall away at this point of time, which leads to a poor definition of the major radial extent in the subsequent stage. By contrast, the radial extent data obtained with the “paraboloid” method shows steady characteristics throughout the entire experiment. Similar observations can be made for the temporal characteristics of the orientation angle.

**Comparison of in-plane permeability values**

Unsurprisingly, these variations directly translate into the temporal characteristics of major and minor in-plane permeability values as shown in Figure 15: The significant variation in the characteristics of the major radial extent data results in a strong change of the major principal in-plane permeability. Moreover, the characteristics reveal that in the section of concurrent availability of measurement data from all of the eight linear capacitive sensors, the experiment has not reached the final trend, i.e. the most characteristic behavior of the material under test is not adequately captured. The problem can generally be avoided when following the “paraboloid” method.

**Figure 13.** Temporal characteristics of flow front data tracked by linear capacitive sensors in a radial flow experiment (left) and elliptical paraboloid model fitted to the data (right).

**Figure 14.** Comparison of the major and minor radial extent data (left) as well as the orientation angle (right) as evaluated according to the “step-by-step” and “global” approach, respectively.

**Figure 15.** Major and minor in-plane permeability characteristics as derived from an exemplarily chosen radial flow experiment at LVV, tracked with linear capacitive sensors.
data evaluation method as the entirety of available flow front data is directly used for the fitting task as well as the calculation of in-plane permeability values.

Summary and conclusion

The paper at hand introduces a novel approach for evaluating flow front data acquired during radial flow experiments for in-plane permeability characterization of reinforcing fabrics. Based on the fitting of an elliptic paraboloid model, a data evaluation method is proposed, which directly and robustly delivers an estimate of the orientation angle as well as principal in-plane permeability values characteristic for the reinforcing material under test. In particular, the work comprises the following findings:

1. A mathematical study is given for the temporal advancement of the radial flow front in isotropic porous media, revealing that its fundamental nature can be well approximated by means of a parabolic geometry model. The validity of this finding for flow in anisotropic media is given by empirical evidence.

2. The approach is further extended to fitting an elliptic paraboloid model to the entirety of flow front data acquired during radial flow experiments. The fitting routine is based on singular value decomposition of a design matrix, specifically set up for the geometric model and shows increased numerical efficiency compared to standard nonlinear, iterative approaches. The paraboloid model allows for robustly determining the orientation angle of the principal directions of fluid flow with respect to a testrig-based coordinate frame. Furthermore, the major and minor intersecting parabolae can be extracted and directly processed towards principal in-plane permeability values.

3. An analytical method for computing the degree of anisotropy $a$ from the equations of major and minor intersecting parabolae is presented, which avoids the need for a non-linear, iterative search inherent to the original work of Adams and Rebenfeld for in-plane permeability calculation. This not only provides a simplification of the algorithm, it also ensures increased accuracy and reduced computational costs.

4. The applicability of the newly proposed "paraboloid" method is demonstrated by means of data from exemplarily chosen, optically tracked radial flow experiments conducted at two different research institutions. Comparison with the conventionally applied "ellipse" data evaluation technique reveals negligibly small deviations in the resulting in-plane permeability values for all of the three involved evaluation strategies, i.e. the "single step", the "elementary" as well as the "interpolation" method.

5. Further comparison on radial flow experiments tracked by linear capacitive sensors reveal the major advantages of the newly proposed evaluation method:

(a) direct incorporation of the entirety of available measurement data from the temporal tracking of the flow front during the radial flow experiments
(b) robust evaluation of the orientation angle of principal flow directions, and
(c) conformity with existing algorithms and strategies for the computation of in-plane permeability values of reinforcing fabrics.

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with the integration constant $C_1$. Back-substitution, once again separation of variables and subsequent integration lead to the general solution of the pressure distribution:

$$p(r) = C_1 \ln r + C_2,$$  \hspace{1cm} (21)

with a second integration constant $C_2$. Invoking the boundary conditions, i.e.: $p(r = r_f) = p_f$ and $p(r = r_i) = p_i$, i.e. the fluid pressure at the boundary of the injection opening $r_e$ and the fluid flow front $r_f$ are given by $p_f$ and $p_i$, respectively, finally results in:

$$p(r) = p_f - \frac{\ln r - \ln r_0}{\ln r_f - \ln r_0}(p_f - p_i) - p_i = -\frac{\ln r - \ln r_0}{\ln r_f - \ln r_0} \Delta p.$$  \hspace{1cm} (22)

**Temporal advancement of radial flow front**

Due to the change to the polar coordinate system, Darcy's law simplifies to:

$$v(r) = \frac{k}{\eta} \frac{\Delta p}{r \ln r_f - \ln r_0}.$$  \hspace{1cm} (23)

Thus, we need to compute the first derivative of Equation (21) at first:

$$\frac{dp}{dr} = -\frac{1}{r} \frac{\Delta p}{\ln r_f - \ln r_0},$$  \hspace{1cm} (24)

and then substitute in Equation (22):

$$v(r) = \frac{k}{\eta} \frac{\Delta p}{r(\ln r_f - \ln r_0)}.$$  \hspace{1cm} (25)

to finally obtain the solution for the volume average flow velocity in the porous medium:

$$v_f = \frac{dr_f}{dt} = \frac{1}{K_v} v(r_f) = -\frac{k}{\eta e \ln r_f - \ln r_0} \frac{\Delta p}{r(\ln r_f - \ln r_0)}.$$  \hspace{1cm} (26)

Separation of variables yields:

$$r_f \ln r_f dt_f - r_i \ln r_i dt_i = \frac{k \Delta p}{\eta e} dt,$$  \hspace{1cm} (27)

and subsequent integration by parts results in:

$$r_f^2 (2\ln r_f - \ln r_i) - 1 = \frac{4k \Delta p}{\eta e} t + C.$$  \hspace{1cm} (28)

The integration constant $C$ is found by means of the initial condition: $r_f(t = 0) = r_i$, i.e. the radial extent at the experimental start equals the radius of the injection opening. Thus, $C = -r_i^2$, and:

$$r_f^2 (2\ln r_f - \ln r_i) - 1 + r_i^2 = \frac{4k \Delta p}{\eta e} t.$$  \hspace{1cm} (29)

**Appendix 1. Derivations for fluid flow in isotropic porous media**

**Solution of the pressure distribution**

For isotropic porous media, Equation (4) simplifies to the well-known Laplace differential equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0.$$  \hspace{1cm} (17)

As the moving flow front is radially symmetric in this situation, the use of a polar coordinate system is beneficial as the second-order partial differential equation further simplifies to the following ordinary differential equation:\(^4\)

$$\frac{d^2 p}{dr^2} + \frac{1}{r} \frac{dp}{dr} = 0.$$  \hspace{1cm} (18)

Substitution of: $q(r) = \frac{4p(r)}{r}$, leads to the following first order differential equation:

$$\frac{dq}{dr} = -\frac{1}{r} q,$$  \hspace{1cm} (19)

which is easily solved by separation of variables and subsequent integration to:
Appendix 2. Basics for fitting of parabolae

Mathematical description of quadratic forms

The implicit equation of quadratic forms, i.e., conics, in general form is given as:\(^{(45)}\)
\[ c_1 x^2 + 2c_3 x y + c_4 y^2 + 2c_5 x + 2c_6 y + c_6 = 0, \]  
(30)
which can be reformulated as a matrix equation:
\[ x^T C x = 0, \]  
(31)
with the vector of homogeneous two-dimensional point coordinates, \( x = [x \ y 1]^T \), and the symmetric matrix of homogeneous conic coefficients \(^{(27)}\):
\[ C = \begin{bmatrix} c_1 & c_2 & c_4 \\ c_2 & c_3 & c_5 \\ c_4 & c_5 & c_6 \end{bmatrix}. \]  
(32)

Parabola of Normal Form

A parabola with parameter \( p \) in normal form, i.e., symmetric to the \( x \)-axis and the vertex at the point of origin is given by:
\[ y^2 = 2px, \]  
\(\text{i.e.:}\)
\[ C_{Par} = \begin{bmatrix} 0 & 0 & c_4 \\ 0 & c_2 & 0 \\ c_4 & 0 & 0 \end{bmatrix}, \]  
(33)
with the conic coefficients: \( c_1 = c_2 = c_5 = c_6 = 0, c_3 = 1 \) and \( c_4 = -p \).

Parabola with off-origin vertex

In order to investigate a parabola symmetric with the \( x \)-axis and the vertex shifted from the origin (see Figure 16), we simply apply a translating transformation of the parabola in normal form along the \( x \)-axis. Translation by \( \Delta x \) is realized by a matrix multiplication, \( C_{Par,\Delta x} = T_{\Delta x}^T C_{Par} T_{\Delta x}^{-1} \), involving the inverse and inverse transpose of the following translatory transformation matrix:
\[ T_{\Delta x} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow T_{\Delta x}^{-1} = \begin{bmatrix} 1 & 0 & -\Delta x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T_{\Delta x}^{-T} \]
(34)

Thus, multiplication gives:
\[ C_{Par,\Delta x} = \begin{bmatrix} 0 & 0 & c_4 \\ 0 & c_2 & 0 \\ c_4 & 0 & -2c_4 \Delta x \end{bmatrix}, \]  
(35)
and results in the following general equation for a parabola symmetric to the \( x \)-axis and the vertex at a distance \( \Delta x \) from the origin:
\[ c_4 y^2 + 2c_4 x - 2c_4 \Delta x = 0. \]  
(36)

Figure 16. Parabola in normal form (center plot) and translated along the \( x \)-axis (outward plots).

Fitting of a parabola with off-origin vertex to measurement data

Type-specific fitting of such a parabola model to a given set of \( n \) measurement data, \( \{(x_k, y_k), k = 1 \ldots n\} \), results in a system of equations which can be formulated by matrix multiplication as:
\[ D \tilde{c} = \begin{bmatrix} y_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ y_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} \tilde{c}_3 \\ \tilde{c}_4 \\ \tilde{c}_6 \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = r, \]  
(37)
with the set of modified conic coefficients: \( \tilde{c}_3 = c_3, \tilde{c}_4 = 2c_4 \) and \( \tilde{c}_6 = -2c_4 \Delta x \) as well as the vector \( r \) of algebraic residuals. The actual fitting task is mathematically set up as finding \( \tilde{c} \) such that the sum of squared residuals, i.e.:
\[ \sum_{k=1}^{n} r_k^2 = r^T r = (D \tilde{c})^T D \tilde{c} = \tilde{c}^T D^T D \tilde{c} = \tilde{c}^T S \tilde{c}, \]  
(38)
is minimized. This in turn is accomplished by singular value decomposition (SVD),\(^{(46)}\) of the scatter matrix \( S \), i.e. finding the matrix decomposition, \( S = U \Sigma V^T \), with the \( n \times n \) diagonal matrix \( \Sigma \) holding the singular values \( \sigma_j, j = 1 \ldots s \), of \( S \) along its diagonal as well as the matrices of left and right singular vectors, \( U \) and \( V \), respectively. Selecting the right singular vector corresponding to the smallest singular value yields the solution of the vector \( \tilde{c} \) of modified conic coefficients. The parabola parameters are finally found through back substitution.

Appendix 3. Basics and fitting of elliptic paraboloids

Mathematical description of quadrics

The implicit equation of quadric surfaces, i.e., quadrics, in general form is given as:\(^{(45)}\)
Elliptic paraboloid in normal form

An elliptic paraboloid in normal form, i.e. showing the orthonormal triad of coordinate axes and vertex at the point of origin and the major axis aligned with the z-axis, is realized by a matrix multiplication, involving the inverse and inverse transpose of the following rotatory transformation matrix:

\[
Q_{\text{Ell.Par}} = \begin{bmatrix}
q_1 & q_2 & q_3 & q_4 \\
q_5 & q_6 & q_7 & q_8 \\
q_9 & q_{10} & q_1 & q_7 \\
q_{10} & q_9 & q_8 & q_6 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

with the set of homogeneous quadric coefficients:
\[
q_2 = q_4 = q_5 = q_6 = q_8 = q_9 = q_{10} = 0,
\]

as well as:
\[
q_1 = \frac{x}{\rho}, q_3 = \frac{y}{\rho}, q_7 = \frac{z}{\rho}, q_0 = -1.
\]

Elliptic paraboloid with off-origin vertex

In order to investigate an elliptic paraboloid symmetric with the z-axis and the paraboloid vertex shifted from the origin (see Figure 17, left), we simply apply a translating transformation of the paraboloid in normal form along the z-axis.

\[
Q_{\text{Ell.Par,Äx}} = T_{\Delta z}Q_{\text{Ell.Par}}T_{\Delta z}^{-1},
\]

and results in the following general equation for an elliptic paraboloid symmetric to the z-axis and the vertex at a distance \( \Delta z \) from the origin:

\[
q_1 x^2 + q_3 y^2 + 2q_6 yz + q_9 z^2 + 2q_9 x y + 2q_9 y + q_{10} = 0.
\]

Fitting of an elliptic paraboloid to measurement data

Combining the two transformations described just above, the following implicit equation is obtained for an elliptic paraboloid with: (a) the z-axis as the primary axis, (b) the vertex at a distance \( \Delta z \) from the origin and (c) the major axis rotated by an angle \( \varphi \), with respect to the x-axis (see Figure 17, right):

\[
\tilde{q}_1 x^2 + \tilde{q}_2 x y + \tilde{q}_3 y^2 + \tilde{q}_6 y z + \tilde{q}_9 z^2 + \tilde{q}_{10} = 0,
\]
involving the modified quadric coefficients:
\[ \tilde{q}_1 = q_1 c + q_3 s, \quad \tilde{q}_2 = 2(q_1 - q_3) r, \quad \tilde{q}_3 = q_1 c + q_3 s, \quad \tilde{q}_9 = 2q_9, \quad \tilde{q}_{10} = -2q_9 \Delta. \]

Type-specific fitting of such an elliptic paraboloid model to a given set of \( n \) measurement data, \( \{x_k, y_k, z_k\}, k = 1 \ldots n \), results in a system of equations which can be formulated by matrix multiplication as:

\[
D\tilde{q} = \begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & z_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & x_n y_n & y_n^2 & z_n & 1 \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \tilde{q}_9 \\ \tilde{q}_{10} \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = r.
\]

Analogous to the procedure described in Appendix 2, the actual fitting task is mathematically set up as finding \( \tilde{q} \) such that the sum of squared residuals, i.e.:

\[
\sum_{k=1}^{n} r_k^2 = \tilde{q}^T D \tilde{q} = \tilde{q}^T D \tilde{q} = \tilde{q}^T S \tilde{q}.
\]

is minimized, which is solved by application of SVD on the scatter matrix \( S \). The parameters of the elliptic paraboloid are finally found through back substitution.