AdS$_2$ dilaton gravity from reductions of some nonrelativistic theories

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Abstract

We study dilaton-gravity theories in 2-dimensions obtained by dimensional reduction of higher dimensional nonrelativistic theories. Focussing on certain families of extremal charged hyperscaling violating Lifshitz black branes in Einstein-Maxwell-scalar theories with an extra gauge field in 4-dimensions, we obtain AdS$_2$ backgrounds in the near horizon throats. We argue that these backgrounds can be obtained in equivalent theories of 2-dim dilaton-gravity with an extra scalar, descending from the higher dimensional scalar, and an interaction potential with the dilaton. A simple subcase here is the relativistic black brane in Einstein-Maxwell theory. We then study linearized fluctuations of the metric, dilaton and the extra scalar about these AdS$_2$ backgrounds. The coefficient of the leading Schwarzian derivative term is proportional to the entropy of the (compactified) extremal black branes.
1 Introduction

Gravity in two dimensions, trivial as such, is rendered dynamical in the presence of a dilaton scalar and additional matter. Such dilaton-gravity theories arise generically under dimensional reduction from higher dimensional theories of gravity coupled to matter. There is interesting interplay with \( AdS_2 \) holography, which arises in the context of extremal black holes and branes: the near horizon regions typically acquire an \( AdS_2 \times X \) geometry, and a 2-dimensional description arises after compactifying the transverse space \( X \). Almheiri and Polchinski [1] considered toy models of 2-dim dilaton-gravity of this sort, with backgrounds involving \( AdS_2 \) with a varying dilaton. Analyzing the backreaction of a minimally coupled scalar perturbation on the \( AdS_2 \) background reveals nontrivial scaling of boundary 4-point correlation functions thereby indicating the breaking of \( AdS_2 \) isometries in the deep IR. This breaking amounts to breaking of local reparametrizations of the boundary time coordinate (modulo global \( SL(2) \) symmetries), which would have been preserved in the presence of exact conformal symmetry. In [2], as well as [3, 4, 5], it was argued that the leading effects describing such nearly \( AdS_2 \) theories are captured universally by a Schwarzian derivative action governing boundary time reparametrizations modulo \( SL(2) \), which arises from keeping the leading nonconstant dilaton behaviour. This picture dovetails with the absence of finite...
energy excitations in $AdS_2$ discussed previously in [6, 7]. Parallel exciting developments involve various recent investigations of the SYK model [8, 9, 10], a quantum mechanical model of interacting fermions. This exhibits approximate conformal symmetry at low energies: the leading departures from conformality are governed by a Schwarzian derivative action for time reparametrizations modulo $SL(2)$, as above. A recent review is [11].

$AdS_2$ throats arise quite generally in the near horizon regions of extremal black holes and black branes, where other fields acquire near constant “attractor” values. This attractor mechanism, first discussed in [12] for BPS black holes in $N = 2$ theories, arises from extremality rather than supersymmetry, as studied in [13, 14]. In the last several years, this has been ubiquitous rather than conformal symmetry, as studied in [13, 14]. In the last several years, this has been ubiquitous in the context of nonrelativistic generalizations of holography: a nice recent review is [15]. A large family of such theories is obtained by considering Einstein-Maxwell-scalar theories with a negative cosmological constant and potential: the $U(1)$ gauge field and scalar serve to support the nonrelativistic background, typically of the form of a Lifshitz, or hyperscaling violating (conformally Lifshitz) theory. The duals to the bulk uncharged black branes in these hvLif theories capture many features of finite density condensed matter-like systems. Towards studying extremal black branes, we note that charge can be added to these theories by adding an additional $U(1)$ gauge field, as discussed in e.g. [16, 17, 18]. Now at extremality, the infrared region approaches an $AdS_2 \times X$ throat, with $X$ typically of the form of an extended transverse plane $R^d$. The discussion above of $AdS_2$ holography now applies upon compactifying $X$ taken as e.g. a torus $T^d$. This was in fact the broad context for [1]: other recent discussions of reduction from higher dimensional theories appear in e.g. [19, 20, 21, 22, 23, 24, 25]; see also [26].

Towards studying such $AdS_2$ theories arising in this nonrelativistic context, we study effective gravity theories of the above form, with two $U(1)$ gauge fields and a scalar field $\Psi$ with a negative cosmological constant and potential. We focus for concreteness on the charged hyperscaling violating Lifshitz black branes in 4-dimensions described in [17]. In the extremal limit, the near horizon geometry of these charged hyperscaling violating Lifshitz black branes becomes $AdS_2 \times \mathbb{R}^2$. These charged hyperscaling violating Lifshitz attractors arise for certain regimes of the Lifshitz $z$ and hyperscaling violating $\theta$ exponents allowed by the energy conditions, with the additional requirement that the theory exhibits hvLif boundary conditions in the ultraviolet: these are perhaps best regarded as intermediate infrared phases themselves in some bigger phase diagram. Then compactifying the two spatial directions as a torus $T^2$, we dimensionally reduce this charged hvLif extremal black brane to obtain a 2-dimensional dilaton-gravity-matter theory. This theory is equivalent to gravity with a dilaton $\Phi$ and an additional scalar $\Psi$ that descends from the hvLif scalar in the higher
dimensional theory, along with an interaction potential \( U(\Phi, \Psi) \). The interaction potential raises the question of whether the extra scalar destabilizes the \( \text{AdS}_2 \) regime, possibly in some region of parameter space. Towards understanding this, we study small fluctuations about the extremal \( \text{AdS}_2 \) background in these theories and argue that these are in fact stable, the stability stemming from the restrictions imposed on \( z, \theta \) stated above from energy conditions and asymptotic boundary conditions. Studying the action for small fluctuations up to quadratic order, it can be seen that the leading corrections to \( \text{AdS}_2 \) arise at linear order in \( \delta \Phi \) leading again to a Schwarzian derivative action from the Gibbons-Hawking term, although there are subleading coupled quadratic corrections (sec. 3). The coefficient of the Schwarzian is proportional to the entropy of the compactified extremal black branes, which being the number of microstates of the background is akin to a central charge of the effective theory. In sec 2.1 we first describe in detail the simpler case of the relativistic black brane, which has \( z = 1, \theta = 0 \), arising in Einstein-Maxwell theory, the extra scalar being absent: at leading order this shows how the Jackiw-Teitelboim theory \cite{27, 28} arises, with subleading terms at quadratic order. We finally study in sec. 5 a null reduction of the charged relativistic black brane: this results in charged hvLif black brane backgrounds with specific exponents, but with an extra scalar background profile (for the uncharged case, these coincide with \cite{29}). Sec. 6 contains a brief Discussion and an Appendix contains some technical details.

2 Einstein-Maxwell theory in 4-dimensions

Einstein-Maxwell theory with a negative cosmological constant is a useful playground for various interesting physics: see e.g. \cite{15} for a review. We focus on 4-dimensions for simplicity: as a consistent truncation of M-theory on appropriate 7-manifolds, the bulk gauge field can be taken as the dual to the \( U(1)_R \) current. The action is

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G_4} \left( \mathcal{R} - 2\Lambda \right) - \frac{1}{4} F_{MN} F^{MN} \right],
\]

where \( \Lambda = -3 \) is the cosmological constant in 4-dimensions. The field equations are

\[
\mathcal{R}_{MN} - \mathcal{R} g_{MN} - 8\pi G_4 \left( F_{MP} F^P_N - \frac{g_{MN}}{4} F^2 \right) = 0, \quad \partial_M \left( \sqrt{-g} F^{MN} \right) = 0.
\]

These equations have both electrically and magnetically charged black branes as solutions.

**Magnetic branes:** These are slightly simpler and we discuss them first, mostly reviewing discussions already in the literature. The metric and field strength \cite{30} are

\[
ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 (dx^2 + dy^2), \quad f(r) = 1 - \left( \frac{r_0}{r} \right)^3 + \frac{Q_m^2}{r^4} \left( 1 - \frac{r}{r_0} \right),
\]

\[
F_{xy} = Q_m,
\]

\[
(2.3)
\]
where $Q_m$ is related to the magnetic charge of the black brane, $r_0$ is the location of the horizon and $r \to \infty$ is the boundary. In the extremal limit, the Hawking temperature vanishes, fixing the horizon location in relation to the charge,

$$T = \frac{3r_0}{4\pi} \left(1 - \frac{Q_m^2}{3r_0^4}\right) = 0 \implies Q_m^2 = 3r_0^4. \quad (2.4)$$

The near horizon geometry of the magnetic black brane becomes $AdS_2 \times \mathcal{R}^2$,

$$ds^2 = -r_0^2 f(r) dt^2 + \frac{dr^2}{r_0^2 f(r)} + r_0^2 (dx^2 + dy^2), \quad f(r)|_{r\to r_0} \approx \frac{6}{r_0^2}(r-r_0)^2. \quad (2.5)$$

We compactify the two spatial dimensions $x^i$ as $T^2$ and dimensionally reducing with an ansatz for the metric

$$ds^2 = g^{(2)}_{\mu\nu} dx^\mu dx^\nu + \Phi^2 (dx^2 + dy^2), \quad (2.6)$$

with $g^{(2)}_{\mu\nu}$ and $\Phi$ being independent of the compact coordinates $x,y \in T^2$. The action $(2.1)$ for the magnetic black brane solution then reduces to

$$S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g^{(2)}} \left[ \Phi^2 \mathcal{R}^{(2)} - 2\Lambda \Phi^2 - \frac{Q_m^2}{2\Phi^2} + 2\partial_\mu \Phi \partial^\mu \Phi \right], \quad (2.7)$$

where $G_2 = G_4/V_2$ is the dimensionless Newton constant in 2-dimensions. A Weyl transformation $g_{\mu\nu} = \Phi g^{(2)}_{\mu\nu}$ absorbs the kinetic term for the dilaton $\Phi$ in the Ricci scalar giving

$$S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left( \Phi^2 \mathcal{R} - 2\Lambda \Phi - \frac{Q_m^2}{2\Phi^4} \right) \equiv \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left( \Phi^2 \mathcal{R} - U(\Phi) \right). \quad (2.8)$$

The equations of motion from this action are

$$U(\Phi) = 2\Lambda \Phi + \frac{Q_m^2}{2\Phi^4} ; \quad \mathcal{R} - \frac{\partial U}{\partial \Phi^2} = 0 , \quad g_{\mu\nu} \nabla^\mu \Phi^2 - \nabla_\mu \nabla^\mu \Phi^2 + \frac{g_{\mu\nu}}{2} U(\Phi) = 0 . \quad (2.9)$$

This 2-dimensional dilaton-gravity theory admits $AdS_2$ as a solution with a constant dilaton. This constant dilaton, $AdS_2$ solution is just the near horizon $AdS_2$ geometry of the extremal magnetic black brane in 4-dimensions (which asymptotically, as $r \to \infty$, is $AdS_4$).

The purpose of this section was to simply illustrate that the original theory with the gauge field is equivalent to a dilaton-gravity theory with an appropriate dilaton potential: this will be a recurrent theme. A simple toy model capturing many features of 2-dim dilaton gravity is the Jackiw-Teitelboim theory \[27, 28\]. In the discussion above, we have not been careful with length-scales: in the next section for the relativistic electric brane, we will reinstate various scales.
2.1 Relativistic electric black brane, reduction to 2-dimensions

The electric black brane solution to (2.1), (2.2), is

\[ ds^2 = -\frac{r^2 f(r)}{R^2} dt^2 + \frac{R^2}{r^2 f(r)} dr^2 + \frac{r^2}{R^2} (dx^2 + dy^2), \quad f(r) = 1 - \left( \frac{r_0}{r} \right)^3 + \frac{Q_e^2}{r^4} \left( 1 - \frac{r}{r_0} \right), \]

\[ A_t = \frac{Q_e}{2\sqrt{\pi G_4} R r_0} \left( 1 - \frac{r_0}{r} \right), \quad F_{rt} = \frac{Q_e}{2\sqrt{\pi G_4} R} \frac{1}{r^2}. \]  \hspace{1cm} (2.10)

The gauge field \( A_t \) vanishes at the horizon. The charge parameter \( Q_e \) is related to the chemical potential \( \mu \) and the charge density \( \sigma \) of the black brane as

\[ \frac{Q_e}{2\sqrt{\pi G_4} R r_0} = \mu, \quad \sigma = \mu \frac{r_0}{R^2} = \frac{Q_e}{2\sqrt{\pi G_4} R^3}. \]  \hspace{1cm} (2.11)

Reinstating the dimensionless gauge coupling \( e^2 \) in \( \mu \) and \( \sigma \) as \( \mu \rightarrow \frac{\mu}{e^2} \) and \( \sigma \rightarrow e^2 \sigma \) and using (2.11), we recover the expressions for the gauge field, field strength and the thermal factor in terms of \( r_0, \mu, \sigma \) as given in sec. 4.2.1 in [15]. Note that in (2.10) the charge parameter \( Q_e \) has dimensions of charge times length-squared, and the gauge field \( A_t \) has mass dimension one. In the extremal limit, the temperature vanishes giving

\[ T = \frac{3r_0}{4\pi R^2} \left( 1 - \frac{Q_e^2}{3r_0^4} \right) = 0 \quad \Rightarrow \quad Q_e^2 = 3r_0^4. \]  \hspace{1cm} (2.12)

The near horizon geometry of the electric black brane becomes \( AdS_2 \times \mathbb{R}^2 \),

\[ ds^2 = -\frac{r_0^2}{R^2} f(r) dt^2 + \frac{R^2}{r_0 f(r)} dr^2 + \frac{r_0^2}{R^2} (dx^2 + dy^2), \quad f(r)|_{r \rightarrow r_0} \simeq \frac{6}{r_0^2} (r - r_0)^2, \]  \hspace{1cm} (2.13)

as in the magnetic case. The Bekenstein-Hawking entropy is the horizon area in Planck units

\[ S_{BH} = \frac{r_0^2}{4G_4} V_2 \frac{Q_e}{4\sqrt{3} R^2} \frac{V_2}{4G_4} = \frac{Q_e}{\sqrt{3}} \frac{V_2}{4G_4} . \]  \hspace{1cm} (2.14)

With \( V_2 = \int dx dy \) the area, this is finite entropy density for noncompact branes.

It is worth noting that asymptotically, these branes (2.10) give rise to an \( AdS_4 \) geometry, with scale \( R \). In the near horizon region, we obtain an \( AdS_2 \) throat with scale \( \frac{R}{\sqrt{6}} \); this is a well-defined \( AdS_2 \) throat in the regime \( \frac{r_0}{R} \gg 1 \) and \( \frac{r_0}{r_C} \ll 1 \). The \( AdS_2 \) region is well-separated from the boundary of the \( AdS_4 \) geometry at \( r \sim r_C \gg r_0 \) if \( \frac{r_0}{r_C} \ll 1 \).

Compactifying the two spatial dimensions \( x^i \) as \( T^2 \) and dimensionally reducing with the metric ansatz (2.6) reduces the action (2.1) for the electric black brane solution to

\[ S = \int d^2 x \sqrt{-g} \left[ \frac{1}{16\pi G_2} \left( \Phi^2 R^{(2)} - 2\Lambda \Phi^2 + 2\partial_\mu \Phi \partial^\mu \Phi \right) - V_2 \Phi^2 \frac{4}{F_{\mu\nu} F^{\mu\nu}} \right] , \]  \hspace{1cm} (2.15)
and we have suppressed a total derivative term which cancels with a corresponding term arising from the dimensional reduction of the Gibbons-Hawking boundary term (more on this later). Performing a Weyl transformation \( g_{\mu\nu} = \Phi^2 g_{\mu\nu}^{(2)} \) to absorb the kinetic term for the dilaton \( \Phi \) in the Ricci scalar, we get

\[
S = \int d^2x \sqrt{-g} \left[ \frac{1}{16\pi G_2} (\Phi^2 R - 2\Lambda \Phi) - \frac{V_2 \Phi^3}{4} F_{\mu\nu} F^{\mu\nu} \right].
\] (2.16)

The Maxwell equations for the gauge field are

\[
\partial_\mu (\sqrt{-g} \Phi^3 F^{\mu\nu}) = 0.
\] (2.17)

The two components \( b = t, r \) of (2.17), i.e. \( \partial_t (\sqrt{-g} \Phi^3 F^{tr}) = 0 = \partial_r (\sqrt{-g} \Phi^3 F^{tr}) \), imply

\[
\sqrt{-g} \Phi^3 F^{tr} = \text{const}.
\] (2.18)

Using the gauge field solution in (2.10) to fix this constant as \( \frac{Q e}{2\sqrt{\pi G_4 R^3}} \), we get

\[
F^{\mu\nu} = \frac{Q e}{2\sqrt{\pi G_4 R^3}} \frac{1}{\sqrt{-g} \Phi^3} \varepsilon^{\mu\nu},
\] (2.19)

where \( \varepsilon^{\mu\nu} \) is defined as \( \varepsilon^{tr} = 1 = -\varepsilon^{rt} \) and \( \varepsilon_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} \varepsilon^{\rho\sigma} \). Substituting \( F_{\mu\nu} F^{\mu\nu} = -\frac{Q_e^2}{2\pi G_4 R^6 \Phi^6} \) and \( F_{\mu\rho} F^{\rho}_{\nu} = -\frac{Q_e^2}{4\pi G_4 R^6 \Phi^6} g_{\mu\nu} \) in eqns. (A.1), we get

\[
g_{\mu\nu} \nabla^2 \Phi^2 - \nabla_\mu \nabla_\nu \Phi^2 + \frac{g_{\mu\nu}}{2} \left( 2\Lambda \Phi + \frac{2Q^2_e}{R^6 \Phi^3} \right) = 0,
\]

\[
R - \frac{\Lambda}{\Phi} + \frac{3Q^2_e}{R^6 \Phi^5} = 0.
\] (2.20)

These field equations can be obtained by varying the following equivalent action

\[
S = \frac{1}{16\pi G_2} \int d^2x \sqrt{\alpha} \left( \Phi^2 R - 2\Lambda \Phi - \frac{2Q^2_e}{R^6 \Phi^3} \right) \equiv \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left( \Phi^2 R - U(\Phi) \right),
\] (2.21)

This equivalent action is obtained by substituting the solution for \( F^{\mu\nu} \) (in terms of the dilaton \( \Phi^2 \)) in the action (2.16) and changing the sign of the \( F^2 \) term which contains a minus sign for electric branes alone, arising from \( g_{tt} \) (a similar treatment appears also in e.g. [3]). Note that this is also consistent with and expected from electric-magnetic duality \( Q_e \to Q_m, Q_m \to -Q_e \), which would suggest that the effective dilaton potential for magnetic branes (2.8) is unchanged in going to electric branes. Now for instance the second equation in (2.20) becomes \( R - \frac{\partial U}{\partial \Phi} = 0 \). The constant dilaton, \( AdS_2 \) solution to the equations (2.20), consistent with the \( T^2 \) compactification of the near horizon geometry in (2.13), is

\[
ds^2 = L^2 \left( -\frac{r_0^2}{L^4 R^2} (r - r_0)^2 dt^2 + \frac{dr^2}{(r - r_0)^2} \right), \quad \Phi = \frac{r_0}{R}, \quad L^2 = \frac{R r_0}{6}, \quad Q_e^2 = 3 r_0^4,
\] (2.22)
with $L$ the $AdS_2$ scale. Changing the radial coordinate to $\rho = \frac{R^2}{6(r - r_0)}$, we write the metric in conformal gauge

$$ds^2 = e^{2\omega}(-dt^2 + d\rho^2) = e^{2\omega}(-dx^+ dx^-), \quad e^{2\omega} = \frac{L^2}{\rho^2},$$

(2.23)

where the lightcone coordinates are $x^\pm = t \pm \rho$. To see that (2.21) admits the above $AdS_2$ solution, we compute $\frac{\partial U}{\partial \Phi^2}$ for the above solution, which gives

$$\frac{\partial U}{\partial \Phi^2} = -\frac{12}{Rr_0} = -\frac{2}{2L^2} \implies \mathcal{R} = \frac{\partial U}{\partial \Phi^2} = -\frac{2}{L^2},$$

(2.24)

using (2.20) for the Ricci scalar. This constant dilaton, $AdS_2$ solution (2.22) is just the compactification of the near horizon $AdS_2$ geometry of the 4-dim extremal electric black brane.

### 2.1.1 Perturbations about the constant dilaton, $AdS_2$ background

The 4-dimensional theory has a large spectrum of tensor, vector and scalar perturbations, which upon reduction to 2-dimensions give a corresponding spectrum: we will discuss this briefly later, in sec. 3.2.3. In this section, we focus on perturbations to only those fields that have nontrivial background profiles in the effective 2-dimensional dilaton-gravity theory: thus we turn on perturbations to the metric and the dilaton

$$\Phi = \Phi_b + \phi(x^+, x^-), \quad \omega = \omega_b + \Omega(x^+, x^-),$$

(2.25)

where $\Phi_b$ and $\omega_b$ denote the background (2.22). We expand the action (2.21) (in conformal gauge) about this background up to quadratic order to get

$$S = \frac{1}{16\pi G_2} \int d^2x \left( 4\Phi^2 \partial_+ \partial_- \omega - \frac{e^{2\omega}}{2} U(\Phi) \right) \equiv S_0 + S_1 + S_2,$$

(2.26)

where

$$S_0 = \frac{1}{16\pi G_2} \int d^2x \left( 4\Phi_b^2 \partial_+ \partial_- \omega_b - \frac{e^{2\omega_b}}{2} U(\Phi_b) \right)$$

(2.27)

is the background action and $S_1$ is linear in perturbations and vanishes by equations of motion. $S_2$ is quadratic in perturbations given by

$$S_2 = \frac{1}{16\pi G_2} \int d^2x \left( \frac{4r_0^2}{3L^2} \phi \partial_+ \partial_- \Omega + \frac{1}{(x^+ - x^-)^2} \left( \frac{8r_0^2}{3L^2} \Omega \phi - 16\phi^2 \right) \right).$$

(2.28)

Varying this action, we get the linearized equations of motion for the perturbations,

$$\partial_+ \partial_- \phi + \frac{2}{(x^+ - x^-)^2} \phi = 0,$$

$$\partial_+ \partial_- \Omega + \frac{1}{(x^+ - x^-)^2} \left( 2\Omega - \frac{24L^2}{r_0^2} \phi \right) = 0.$$  

(2.29)
These equations are consistent at linear order with the “constraint” equations for the $++$ and $−−$ components of the Einstein equation in (2.20). From these linearized equations, we see that the dilaton fluctuation $\phi$ is decoupled from the metric fluctuation $\Omega$. Solving the equation for $\phi$ in (2.29), we get

$$
\phi = \frac{a + bt + c(t^2 - \rho^2)}{\rho},
$$

(2.30)

where $a$, $b$, $c$ are independent constants. Substituting the solution (2.30) for $\phi$ in the equation for $\Omega$ in (2.29), we can solve for the metric perturbation $\Omega$, which implies that the AdS$_2$ metric gets corrected at the same order as the dilaton. The on-shell (boundary) action obtained then by using the linearized field equations in (2.28) gives terms at quadratic order in the perturbations,

$$
S_2 = \frac{1}{16\pi G_2} \int dt \sqrt{-\gamma} n^\mu \left(2r_0^2 \left(\Omega \partial_\mu \phi - \phi \partial_\mu \Omega\right)\right),
$$

(2.31)

where $n^\mu$ is the outward unit normal to the boundary.

### 2.1.2 The Schwarzian effective action

In this section, we switch to Euclidean time $\tau = it$. The Gibbons-Hawking boundary term in the 2-dimensional theory arises from the reduction of the corresponding term in the higher dimensional theory. The Gibbons-Hawking term on the 3-dimensional boundary of the 4-dimensional theories described by the Euclidean form of the action (2.1) is

$$
S_{GH}^{4d} = -\frac{1}{8\pi G_4} \int d^3x \sqrt{\gamma^{(3)}} K^{(4)},
$$

(2.32)

where the extrinsic curvature is defined as $K^{(4)}_{AB} = \frac{1}{2}(\nabla_A n_B + \nabla_B n_A)$, $n^A$ being the outward unit normal to the 3-dimensional boundary. Using the ansatz (2.6) for the $T^2$-compactification, dimensionally reducing and performing the Weyl transformation of the 2-dimensional metric $g_{\mu\nu} = \Phi g^{(2)}_{\mu\nu}$, the Gibbons-Hawking term reduces to

$$
S_{GH}^{4d} = -\frac{1}{16\pi G_2} \int d\tau \sqrt{-\gamma} \left(2 \Phi^2 K + \frac{3}{2} n_\mu \partial^\mu \Phi^2\right).
$$

(2.33)

The Ricci scalar term in the bulk 4-dim Euclidean action upon dimensional reduction and after the Weyl transformation becomes

$$
-\sqrt{g^{(4)}} R^{(4)} = -\sqrt{-\gamma} \left(\Phi^2 R - \frac{3}{2} \nabla^2 \Phi^2\right).
$$

(2.34)

$^1$We have $K^{(4)} = \gamma^{(3) AB} K^{(4)}_{AB} = \gamma^{(3) \tau \tau} K^{(4)}_{\tau \tau} + 2\gamma^{(3) xx} K^{(4)}_{xx}$, with $K^{(4)}_{xx} = -\Gamma^\tau_{xx} n_\tau = \frac{1}{2} n_\tau \partial^\tau \Phi^2 = \frac{1}{2} n_\mu \partial^\mu \Phi^2$ becomes $K^{(4)} = K^{(2)} + \Phi^{-2} n_\mu \partial^\mu \Phi^2$. Then (2.32) gives (2.33) after the Weyl transformation.
Note also that $\sqrt{g^{(4)}} = \sqrt{g^{(2)}} \Phi^2$ and $\Phi^2 = g_{xx}$. We write the total derivative term (the second term) in (2.34) as a boundary term

$$- \frac{1}{16\pi G_2} \int d^2 x \sqrt{g} \left( - \frac{3}{2} \nabla^2 \Phi^2 \right) = \frac{1}{16\pi G_2} \int d\tau \sqrt{\gamma} \left( \frac{3}{2} n_\mu \partial^\mu \Phi^2 \right). \tag{2.35}$$

We see that this boundary term which comes from the dimensional reduction of the bulk action in 4-dimensions cancels the second term in (2.33), thereby giving the Gibbons-Hawking term on the boundary of the 2-dimensional theory as

$$S_{GH} = - \frac{1}{8\pi G_2} \int d\tau \sqrt{\gamma} \Phi^2 \kappa. \tag{2.36}$$

Expanding the Gibbons-Hawking term in the perturbations (2.25) and adding it to the Euclidean form of $S_2$ (which is $S_2^E = -i S_2$, with $t = -i \tau$ in $S_2$), the leading term in the total boundary action $I_{bdy} = S_2^E + S_{GH}$ arises at linear order in the dilaton perturbation (with subleading terms at quadratic order). To illustrate this in greater detail, it is important that we define the dilaton perturbation in (2.25) in a physically appropriate manner. Since the background value $\Phi_b$ is constant, it is sensible to define the dilaton perturbation as

$$\Phi = \Phi_b (1 + \tilde{\phi}) \ , \quad \Phi_b = \frac{r_0}{R} \quad \Rightarrow \quad \tilde{\phi} = \frac{\Phi - \Phi_b}{\Phi_b} \ll 1. \tag{2.37}$$

Thus with this redefinition, the perturbation is reasonable since it automatically satisfies $\tilde{\phi} \ll 1$. In terms of the dilaton background value $\Phi_b$, the entropy (2.14) is simply

$$S_{BH} = \frac{\Phi_b^2 V_2}{4G_4} = \frac{\Phi_b^2}{4G_2}. \tag{2.38}$$

This gives

$$S_{GH}^{(1)} = - \frac{2\Phi_b^2}{8\pi G_2} \int d\tau \sqrt{\gamma} \tilde{\phi} \kappa \quad \rightarrow \quad - \frac{\Phi_b^2}{4\pi G_2} \int du \phi_r(u) \{ \tau(u), u \}. \tag{2.39}$$

In evaluating the last term, we take the boundary of $AdS_2$ as a slightly deformed curve $(\tau(u), \rho(u))$ parametrized by the boundary coordinate $u$, and define $\tilde{\phi} = \frac{\phi_r(u)}{e}$, as discussed in [2] (reviewed in [11]). Now using the outward unit normal $n^\mu$ to the boundary, we expand the extrinsic curvature. Expanding $S_{GH}^{(1)}$ then leads to a Schwarzian derivative action $Sch(\tau(u), u) = \{ \tau(u), u \} = \frac{\rho'''}{\rho} - \frac{3}{2} (\frac{\rho''}{\rho})^2$. The integral above pertains only to $AdS_2$ does not contain any further scales besides the $AdS_2$ scale $L$ which also appears in the extrinsic curvature giving the Schwarzian (also $\sqrt{\gamma} = \frac{L}{4}$). The various length scales in the original extremal brane have been absorbed into the $AdS_2$ scale $L$. Now we note that the coefficient of the Schwarzian is in fact proportional to the entropy (2.38) of the compactified extremal
black brane with $V_2$ finite (the dependence on $\Phi_b$ is expected since it controls the transverse area). Since the entropy captures the number of microstates of the unperturbed background, this is akin to a central charge of the effective theory. Similar comments appear in [10] (see also [19], the Schwarzian arising in some cases from the conformal anomaly).

It is worth noting that the coefficient in the Schwarzian term above is proportional to the extremal entropy after the reasonable definition of the perturbation as (2.37) by scaling out $\Phi_b$; apart from this, the Schwarzian term here is as in [2]. As discussed there, we note that the perturbation makes this nearly $AdS_2$ and contributes to the near-extremal entropy via the Schwarzian. This can be obtained as in the analysis there by a transformation $\tau(u) = \tan \frac{\bar{\tau}(u)}{2}$ which gives $S_{\text{GH}}^{(1)} = -\frac{\Phi_b^2}{4\pi G_2} \tilde{\phi}_r \int du \left( \{\bar{\tau}(u),u\} + \frac{1}{2} \bar{\tau}'^2 \right)$, treating $\tilde{\phi}_r$ as constant. Solutions with $\bar{\tau} = \frac{2\pi}{\rho} u$ have $\bar{\tau} \sim \bar{\tau} + 2\pi$, giving the action $S_{\text{GH}}^{(1)} = -2\pi^2 \frac{\Phi_b^2}{4\pi G_2} \tilde{\phi}_r T = -\log Z$, giving the near-extremal correction to the entropy $\Delta S = 4\pi \frac{\Phi_b^2}{4\pi G_2} \tilde{\phi}_r T$ (which, being linear in temperature, can also be seen to be the specific heat): this again is proportional to the background entropy with the perturbation defined as (2.37).

The remaining terms in the expansion of $S_{\text{GH}}$ and $S_{\xi}^2$ are all quadratic in perturbations and thus subleading compared to $S_{\text{GH}}^{(1)}$. See also e.g. [19, 22, 24, 25], for $AdS_2$ backgrounds obtained from reductions of higher dimensional theories (see also [26]). In particular there are parallels with some of the analysis on the reduction of near extremal black holes in [25].

Overall, expanding in the perturbations $\tilde{\phi}, \bar{\Omega}$, we have $I = S_E + S_{\text{GH}} = I_0 + I_1 + I_2 + \ldots$, with

$$I_0 = -\frac{\Phi_b^2}{16\pi G_2} \left( \int d^2x \sqrt{g} R + 2 \int_{\text{bndry}} \sqrt{\gamma} K \right)$$

(2.40)

is the background Euclidean action (see (2.27)): it can be checked that $U(\Phi_b) = 0$. The action $I_0$ is a topological term and gives the extremal entropy $S_{\text{BH}} = \frac{\Phi_b^2}{4G_2}$ after regulating this as a near-extremal background. The linear terms are contained in

$$I_1 = -\frac{2\Phi_b^2}{16\pi G_2} \int d^2x \sqrt{g} \tilde{\phi} \left( R - \frac{\partial U}{\partial \Phi^2} \right) - \frac{2\Phi_b^2}{8\pi G_2} \int_{\text{bndry}} \sqrt{\gamma} \tilde{\phi} K$$

(2.41)

\footnote{Here the Euclidean time periodicity, large for a small near-extremal temperature, precisely cancels the small regularized change in the extremal horizon. In more detail, expanding $f(r)$ in (2.10) about extremality, we have $f(r) \simeq \frac{6(r-r_0)}{r_0^2}(r-r_0 + \frac{\rho^2}{2}(3 - \frac{27}{4} \delta r^2)) \equiv \frac{4\delta}{r_0^2}(r-r_0 - \frac{\rho^2}{2})$ where $\delta = \frac{2\pi}{\rho}(3 - \frac{27}{4} \delta r^2)$ and $r_0 = r_0 - \frac{\rho^2}{2}$. Then the nearly $AdS_2$ throat acquires a small horizon with metric $ds^2 \sim \frac{4\delta}{r_0^2} \rho^2 dt^2 + d\rho^2$ near the origin: the Euclidean time periodicity then is $\Delta \tau = \beta = \frac{2\pi \rho^2}{3\delta}$ consistent with (2.12). The horizon contribution to the action gives $I_0 = -\frac{4\delta}{16\pi G_2} \Delta \tau \frac{2\pi}{3\delta} \left( \frac{12}{r_0^2} \right) \equiv -\beta F$ and thereby the background extremal entropy $S_{\text{BH}} = -I_0$. The boundary terms in the action above cancel: to elaborate, we have the $AdS_2$ metric $ds^2 = \frac{L^2}{\rho^2} (dt^2 + d\rho^2)$, the boundary at $\rho = \epsilon$ has outward unit normal $n_\rho = -\frac{1}{\rho}$. The extrinsic curvature defined as $K_{\mu\nu} = \frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ gives $K_{\tau\rho} = -\Gamma^\tau_\rho \tau n_\rho = \frac{1}{\rho^2}$ and $K = \gamma^{\tau\nu} K_{\tau\nu} = \frac{1}{L^2}$. Then the terms at the boundary cancel as $-\frac{4\delta}{16\pi G_2} \left( \int dt \frac{L^2 \rho^2}{\epsilon} dr \gamma^{\tau\nu} (-\frac{1}{\rho^2}) + 2 \int dt \frac{L^2}{\epsilon} (-\frac{1}{L^2}) \right)$.
}
with \( \frac{\partial U}{\partial \phi_b} \big|_{\Phi_b} = -\frac{2}{L^2} \), which is the Jackiw-Teitelboim theory \cite{27,28}, which serves as a simple model for \( AdS_2 \) physics (with parallels with the SYK model). The bulk term vanishes by the \( \tilde{\phi} \) equation giving the fixed background \( AdS_2 \) geometry, while the boundary term gives the Schwarzian as explained above. The analysis here of the higher dimensional realization serves to recover the background entropy as expected and reveal the various subleading terms beyond the Jackiw-Teitelboim theory emerging from reduction: \( I_2 \) is second order in perturbations, from \( S_{GH}^\varnothing \) (see (2.31)) and the second order terms in the expansion of \( S_{GH} \).

\[
I_2 = -\frac{1}{16\pi G_2} \int d\tau \sqrt{\gamma} \left[ \frac{2\gamma^2}{3L^2} \Phi \partial_\rho \tilde{\phi} - \frac{\tilde{\phi}}{\rho} \Phi \Omega \right] \frac{\partial_\rho \tilde{\phi}}{} + 2\Phi_b^2 (\tilde{\phi}^2 K - 2\tilde{\phi} e^{-\omega_b} \partial_\rho \Omega) ) , \tag{2.42}
\]
expanding in conformal gauge.

3 Charged hyperscaling violating Lifshitz black branes

Over the last several years, nonrelativistic generalizations of holography have been investigated extensively: see e.g. \cite{15} for a review of various aspects. A particular family of interesting theories comprises the so-called hyperscaling violating Lifshitz (hvLif) theories, which are conformal to Lifshitz theories. These arise as solutions to Einstein-Maxwell-scalar theories, the \( U(1) \) gauge field and dilaton scalar necessary to support the nonrelativistic background. For the most part, we regard these as effective gravity theories: in certain cases these can be shown to arise from gauge/string realizations (see e.g. \cite{29}).

These nonrelativistic black branes are uncharged. A minimal way to construct charged black branes is to add an additional \( U(1) \) gauge field, which serves to supply charge to the black brane: see e.g. \cite{16}, \cite{17}, \cite{18}. For these latter charged black branes, there exist extremal limits where the near horizon geometry takes the form \( AdS_2 \times X \), and contains an \( AdS_2 \) throat. Compactifying the transverse space now allows us to study the extremal limits of these theories in the context of a 2-dimensional dilaton gravity theory with additional matter, notably the scalar descending from higher dimensions as well as gauge fields.

3.1 4-dimensional charged hvLif black brane

Consider Einstein-Maxwell-scalar theory with a further \( U(1) \) gauge field, with action \cite{17}

\[
S = \int d^4 x \sqrt{-g^{(4)}} \left[ \frac{1}{16\pi G_4} \left( \mathcal{R}^{(4)} - \frac{1}{2} \partial_M \Psi \partial^M \Psi + V(\Psi) - \frac{Z_1}{4} F_{1 MN} F_1^{MN} \right) - \frac{Z_2}{4} F_{2 MN} F_2^{MN} \right] , \tag{3.1}
\]

Note that in the \( AdS/CMT \) literature, these theories are referred to Einstein-Maxwell-dilaton theories: we here use Einstein-Maxwell-scalar since the 2-dim dilaton \( \Phi \) here is distinct from the hvLif scalar \( \Psi \).
where the scalar field dependent couplings and the scalar potential are

\[ Z_1 = e^{\lambda_1 \Psi}, \quad Z_2 = e^{\lambda_2 \Psi}, \quad V(\Psi) = V_0 e^{\gamma \Psi}. \quad (3.2) \]

The field equations following from the above action are

\[
\mathcal{R}^{(4)}_{MN} - \frac{1}{2} \partial_M \Psi \partial_N \Psi + g_{MN} \frac{V}{2} - \frac{Z_1}{2} \left( F_{1MP} F_{1NP} - \frac{g_{MN}}{4} (F_1)^2 \right) - 8\pi G_4 Z_1 \left( F_{2MP} F_{2NP} - \frac{g_{MN}}{4} (F_2)^2 \right) = 0,
\]

\[
\frac{1}{\sqrt{-g^{(4)}}} \partial_M (\sqrt{-g^{(4)}} \partial^M \Psi) + \gamma V - \frac{\lambda_1 Z_1}{4} F_{1MN} F_{1}^{MN} - 4\pi G_4 \lambda_2 Z_2 F_{2MN} F_{2}^{MN} = 0,
\]

\[
\partial_M (\sqrt{-g^{(4)}} Z_1 F_{1}^{MN}) = 0, \quad \partial_M (\sqrt{-g^{(4)}} Z_2 F_{2}^{MN}) = 0. \quad (3.3)
\]

The charged hvLif black brane solution to these equations is

\[
ds^2 = \left( \frac{r}{r_{hv}} \right)^{-\theta} \left[ -\frac{r^2 f(r)}{R^{2z}} dt^2 + \frac{R^2}{r^2 f(r)} dr^2 + \frac{r^2}{R^2} (dx^2 + dy^2) \right],
\]

\[
f(r) = 1 - \left( \frac{r_0}{r} \right)^{2z - \theta} + \frac{Q^2}{r^{2(1+z-\theta)}} \left( 1 - \left( \frac{r}{r_0} \right)^{z - \theta} \right),
\]

\[
F_{1rt} = \sqrt{2(z - 1)(2 + z - \theta)} e^{-\frac{\lambda_1 \Psi_0}{2z}} r_{hv}^2 R^{\theta - z - \theta} r^{1 + z - \theta},
\]

\[
F_{2rt} = Q \sqrt{2(2 - \theta)(z - \theta)} e^{-\frac{\lambda_1 \Psi_0}{2z}} \frac{4\sqrt{\pi G_4}}{r_{hv}^{\theta - z - \theta + 1} r^{1 + z - \theta}},
\]

\[
e^\Psi = e^{\Psi_0} \left( \frac{r_{hv}^2 R^2}{R^2} \right)^{\sqrt{2(2 - \theta)(z - \theta)}}, \quad (3.4)
\]

being explicit with length scales, and

\[
V_0 = \frac{(2 + z - \theta)(1 + z - \theta) e^{-\gamma \Psi_0}}{R^{2z - 2\theta} r_{hv}^{2z}} , \quad \gamma = \frac{\theta}{\sqrt{(2 - \theta)(2z - 2 - \theta)}},
\]

\[
\lambda_1 = \frac{-4 + \theta}{\sqrt{(2 - \theta)(2z - 2 - \theta)}}, \quad \lambda_2 = \frac{\sqrt{2z - 2 - \theta}}{2 - \theta}. \quad (3.5)
\]

Here \( r_{hv} \) is the hyperscaling violating scale arising in the conformal factor in the metric, and the charge parameter \( Q \) has dimensions of \( r^{1 + z - \theta} \); this is equivalent to absorbing factors of \( r_{hv}, R \) into \( Q \). For \( z = 1, \theta = 0 \), this scaling coincides with that for the relativistic black brane in sec. 2.1.

In these charged hyperscaling violating Lifshitz black brane solutions to the action (3.1), the gauge field \( A_1 \) and the scalar field \( \Psi \) source the hyperscaling violating Lifshitz background while the gauge field \( A_2 \) giving charge to the black brane, as mentioned above. This
action (3.1) has also been defined by absorbing the Newton constant into the definition of the hyperscaling violating gauge field $A_1$ and scalar $\Psi$ (which thus makes $A_1$ and $\Psi$ dimensionless) while retaining the gauge field $A_2$ in $F_2$ as having mass dimension one. Thus the field strength $F_{2\,rt}$ in (3.4) has mass dimension 2, as for the relativistic brane.

The null energy conditions for the metric follow from the asymptotic hvLif geometry [15] and are given by

$$ (z - 1)(2 + z - \theta) \geq 0, \quad (2 - \theta)(2(z - 1) - \theta) \geq 0. \quad (3.6) $$

In addition, we require the gauge field $A_2t$ to vanish at the boundary ($r \to \infty$) so that the theory does not ruin the hvLif boundary conditions we have assumed: this is equivalent to assuming that these charged black branes represent finite temperature charged states in the boundary hvLif theory. The background profile $A_{2t} \sim 1 - (\frac{r_0}{r})^{z-\theta}$ then implies that

$$ z - \theta \geq 0. \quad (3.7) $$

These conditions together constrain the range of $z, \theta$ for these extremal nonrelativistic black brane backgrounds, which will be important in the discussion of perturbations later. Specifically:

(i) First, the last condition (3.7) is specific to the charged case: using this, the first of the null energy conditions (3.6) implies that $z \geq 1$.

(ii) From the second of the conditions (3.6), we have either $2 - \theta \geq 0$, $2z - 2 - \theta \geq 0$, or $2 - \theta < 0$, $2z - 2 - \theta < 0$. Considering the second possibility, we obtain $z \geq \theta \geq 2$, but this implies $2z - 2 - \theta = z - 2 + z - \theta > 0$, which is a contradiction. This forces $2 - \theta \geq 0$, $2z - 2 - \theta \geq 0$.

Overall, this gives the conditions

$$ z \geq 1, \quad 2z - 2 - \theta \geq 0, \quad 2 - \theta \geq 0, \quad (3.8) $$

for the regime of validity of the $z, \theta$ exponents of the charged hvLif background above. For the special case of $z = 1$, the NEC becomes $(2 - \theta)(-\theta) \geq 0$, which forces $\theta \leq 0$ by (3.8).

The relativistic limit of this charged hvLif black brane gives the relativistic electric black brane discussed previously in sec. 2.1. From the constraint (3.8), we see that the correct relativistic limit is to take first $\theta = 0$ and then $z = 1$. In this limit, we get

$$ \gamma = 0, \quad \lambda_1 \to -\infty, \quad \lambda_2 = 0, \quad V_0 = 6/R^2, \quad \Psi = \Psi_0. \quad (3.9) $$

With this the Einstein-Maxwell-scalar action (3.1) reduces to the Einstein-Maxwell action (2.1), where $F_2$ and $V_0$ in (3.1) are identified with $F$ and $-2\Lambda$ in (2.1).
3.1.1 Extremality and attractors

In the extremal limit,

\[
T = \frac{(2 + z - \theta)}{4\pi R^{z+1}} \left(1 - \frac{(z - \theta)Q^2}{(2 + z - \theta)} \right) = 0 \implies Q^2 = \frac{(2 + z - \theta)}{(z - \theta)} r_0^{2(1+z-\theta)}, \tag{3.10}
\]

and the near horizon geometry becomes \(AdS_2 \times \mathbb{R}^2\),

\[
ds^2 = \left(\frac{r_0}{r_{hv}}\right)^{-\theta} \left[ -\frac{2z^2 f(r)}{R^{2z}} dt^2 + \frac{R^2}{r_0^2 f(r)} dr^2 + \frac{r_0^2}{R^2} (dx^2 + dy^2) \right],
\]

\[
f(r) \big|_{r \to r_0} \approx \left(\frac{2 + z - \theta}{1 + z - \theta}\right) (r - r_0)^2,
\]

the \(AdS_2\) scale being \(R (\frac{r_0}{r_{hv}})^{-\theta/2}\). The Bekenstein-Hawking entropy is the horizon area in Planck units

\[
S_{BH} = \left(\frac{r_0^2}{R^2}\right) \left(\frac{r_0}{r_{hv}}\right)^{-\theta} \frac{V_2}{4G_4} = \left(\frac{z - \theta}{2 + z - \theta}\right) \frac{r_0^2}{2(1+z-\theta)} \left(\frac{r_0}{r_{hv}}\right)^{2(\theta/(1+z-\theta)}} \frac{Q^2}{4G_4 R^2},
\]

where \(V_2 = \int dx dy\) is the transverse area of the brane. For \(z = 1, \theta = 0\), this coincides with the relativistic brane.

It is worth noting that the full metric in (3.4) is asymptotically of hvLif form, for \(r \gg r_0\). The boundary of the theory could be taken as \(r \sim r_{hv}\), i.e. the theory flows to hvLif below this scale, in some bigger phase diagram. The \(AdS_2\) throat, well-defined if \(\frac{r_0}{r_{hv}} \ll 1\) and \(\frac{r - r_0}{R} \gg 1\), is well-separated from the asymptotic hvLif region if \(\frac{r - r_0}{r_{hv}} \ll 1\) and the \(AdS_2\) scale satisfies \(R (\frac{r_0}{r_{hv}})^{-\theta/2} \ll r_{hv}\) i.e. \(R \ll r_{hv} (\frac{r_0}{r_{hv}})^{\theta/2}\). Note that this is not vacuous since \(r_0 \ll r_{hv}\) so that \(\frac{r_0}{r_{hv}} \ll 1\) is a small factor.

Along the lines of the attractor mechanism discussion in [13], we would like to convert this theory to a dilatonic gravity theory in 4-dimensions with a potential (and no gauge fields). Towards this end, we integrate Maxwell’s equations in (3.3) and use the solutions for field strengths in (3.4) to get

\[
F_1^{tr} = \sqrt{2(z - 1)(2 + z - \theta)} e^{\frac{\lambda_1 \Psi}{2}} r_0^{\theta - 2} \frac{R^1 - \theta}{\sqrt{-g} e^{\lambda_1 \Psi}} , \quad F_2^{tr} = \frac{Q}{4\pi G_4} \sqrt{2(z - 1)} e^{(2 - \theta)(z - \theta)} e^{\lambda_2 \Psi} r_0^{z - 1} \frac{R^2 - \theta}{\sqrt{-g} e^{\lambda_2 \Psi}}.
\]

Substituting (3.13) in (3.3), we obtain equations of motion for the metric and the scalar field \(\Psi\), which can be derived from the following equivalent action

\[
S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left( R - \frac{1}{2} (\partial \Psi)^2 - V_{eff}(\Psi) \right),
\]

\[
V_{eff}(\Psi) = -\frac{(2 + z - \theta)(1 + z - \theta)}{R^{2-2\theta} \frac{r_0^{2\theta}}{r_{hv}^{2\theta}}} e^{\gamma(\psi - \psi_0)}
\]

\[
+ \frac{1}{g_{xx}^2} \frac{(z - 1)(2 + z - \theta) r_0^{2 - 2\theta} R_0^{2 - 2\theta}}{e^{\lambda_1(\psi - \psi_0)}} + \frac{(2 - \theta)(z - \theta) Q^2}{e^{\lambda_2(\psi - \psi_0)}}(2^{z_2 - 2 + 2\theta})
\]

\[
\tag{3.14}
\]
The explicit scales show that the potential term-by-term has mass dimension 2. This equivalent action is obtained by substituting the solutions for $F_{tr}^1$ and $F_{tr}^2$ in the action (3.1) and changing the signs of $F_{21}$, $F_{22}$ terms, as earlier. At the critical point (extremality),

$$g_{xx} = \left( \frac{r_0}{r_{hv}} \right)^{-\theta} \left( \frac{r_0}{R} \right)^2, \quad e^\Psi = e^{\Psi_0} \left( \frac{r_{hv} r_0}{R^2} \right)^{\sqrt{(2-\theta)(2z-2-\theta)}}, \quad Q^2 = \frac{(2+z-\theta)}{(z-\theta)} r_0^{-2(1+z-\theta)},$$

(3.15)

the first and second derivatives of $V_{eff}$ ((A.2), (A.3)) are

$$\frac{\partial V_{eff}}{\partial \Psi} \bigg|_{ext} = 0, \quad \frac{\partial^2 V_{eff}}{\partial \Psi^2} \bigg|_{ext} = \frac{4(z-1)(2+z-\theta)(1+z-\theta)}{2z-2-\theta} r_0^\theta r_{hv}^\theta R^2 > 0,$$

(3.16)

which imply that the extremal point is stable for all values of $z, \theta$ allowed by the conditions (3.8). It is worth mentioning that for $z = 1$ and $\theta$ nonzero, these and all higher derivatives of $V_{eff}$ in fact vanish (see (A.5)): thus we obtain no insight into the stability of these attractors in this case and we will not discuss this subcase in what follows.

### 3.2 Dimensional reduction to 2-dimensions

Compactifying the two spatial dimensions, $x^i$ as $T^2$, we dimensionally reduce with the metric ansatz (2.6), taking the lower dimensional fields $g^{(2)}_{\mu\nu}, \Phi, \Psi, A_1, A_2$, to be $T^2$-independent: then the action (3.1) reduces to (A.6), Performing a Weyl transformation, $g_{\mu\nu} = \Phi g^{(2)}_{\mu\nu}$ to absorb the kinetic term for the dilaton $\Phi$ in the Ricci scalar, the 2-dimensional action (A.6) becomes

$$S = \int d^2x \sqrt{-g} \left[ \frac{1}{16\pi G_2} \left( \Phi^2 \mathcal{R} - \frac{\Phi^2}{2} \partial_\mu \Psi \partial^\mu \Psi + V\Phi - \frac{\Phi^3}{4} Z_1 F_{1\mu
u} F_{1\mu
u} - \frac{V\Phi^3}{4} Z_2 F_{2\mu
u} F_{2\mu
u} \right) \right].$$

(3.17)

We only retain fields with nontrivial background profiles: more general comments appear later. The Maxwell equations for the gauge fields are

$$\partial_\mu (\sqrt{-g} \Phi^3 Z_1 F_{1\mu}^\nu) = 0, \quad \partial_\mu (\sqrt{-g} \Phi^3 Z_2 F_{2\mu}^\nu) = 0.$$

(3.18)

Integrating and using $F_{1rt}, F_{2rt}$ from (3.1) to fix the integration constants gives

$$F_{1\mu\nu} = \sqrt{2(z-1)(2+z-\theta)} \frac{\sqrt{\frac{\lambda \Phi_0}{G_4} r_{hv} \theta^{-2} R^{1-\theta}}}{\sqrt{-g} Z_1 \Phi^3} \varepsilon^{\mu\nu}, \quad F_{2\mu\nu} = \frac{Q \sqrt{2(2-\theta)(z-\theta)} \sqrt{\frac{\lambda \Phi_0}{G_4} r_{hv} \theta^{-1}}}{4 \sqrt{\pi G_4 R^{2z+1-\theta} \sqrt{-g} Z_2 \Phi^3}} \varepsilon^{\mu\nu},$$

(3.19)
where \( \varepsilon^{\mu\nu} \) satisfies \( \varepsilon^{tr} = 1 = -\varepsilon^{rt} \) and \( \varepsilon_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}\varepsilon^{\rho\sigma} \). We substitute the solutions (3.19) in the remaining field equations obtained by varying the action (3.17) (i.e. eq. (A.7)) to obtain

\[
\begin{aligned}
g_{\mu\nu} \nabla^2 \Phi^2 - \nabla_{\mu} \nabla_{\nu} \Phi^2 + \frac{g_{\mu\nu}}{2} \left( \frac{\Phi^2}{2} (\partial \Psi)^2 + U \right) - \frac{\Phi^2}{2} \partial_{\mu} \Psi \partial_{\nu} \Psi &= 0, \\
\mathcal{R} - \frac{1}{2} (\partial \Psi)^2 - \frac{\partial U}{\partial (\Phi^2)} &= 0, \\
\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} \Phi^2 \partial^\mu \Phi) - \frac{\partial U}{\partial \Psi} &= 0,
\end{aligned}
\]

where \( U(\Phi, \Psi) \) is an effective interaction potential. These equations can then be obtained from the following equivalent action

\[
S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left( \Phi^2 \mathcal{R} - \frac{\Phi^2}{2} (\partial \Psi)^2 - U(\Phi, \Psi) \right),
\]

\[
U(\Phi, \Psi) = -\frac{(2 + z - \theta)(1 + z - \theta)}{R^2 - 2\theta r_{hv}^2} e^{(\Psi - \Psi_0)} \Phi
\]

\[
+ \frac{1}{\Phi^3} \left( \frac{(z - 1)(2 + z - \theta)r_{hv}^2}{e^{\lambda_1(\Psi - \Psi_0)}} + \frac{(2 - \theta)(z - \theta)Q^2 e^{2z - 2} R^2 - 4z - 2 + 2\theta}{e^{\lambda_2(\Psi - \Psi_0)}} \right),
\]

where \( V_0, \gamma, \lambda_1, \lambda_2 \) are given in (3.5). This equivalent action is obtained by substituting the solutions for \( F_1^{\mu\nu}, F_2^{\mu\nu} \) in terms of the dilaton \( \Phi^2 \) and the scalar \( \Psi \) in the action (3.17) and changing the signs of \( F_1^{\mu\nu}, F_2^{\mu\nu} \) terms, as discussed in the case for relativistic electric black brane, sec. 2.1. Also note that the relativistic electric black brane is a special case of the dilaton-gravity-matter theory, considered here, for \( \theta = 0 \) and \( z = 1 \).

We note that the scalar \( \Psi \) that descends from the hyperscaling violating scalar in higher dimensions is not minimally coupled in the 2-dimensional theory. The potential \( U(\Phi, \Psi) \) contains nontrivial interactions between the dilaton \( \Phi \) and the hvLiF scalar \( \Psi \). Thus the small fluctuation spectrum of the dilaton and \( \Psi \) are coupled, and one might worry about the stability of the 2-dimensional attractor. This is reminiscent of multi-field inflation models, where one scalar field provides a slow-roll phase while another scalar provides a waterfall phase, ending inflation. In the current context, stability would require that no tachyonic modes arise from the interaction induced by \( U(\Phi, \Psi) \) between \( \Phi \) and \( \Psi \). We will address this soon.

The field equations (3.20) admit a constant dilaton, \( AdS_2 \) solution as

\[
\begin{aligned}
ds^2 &= L^2 \left[ - \frac{r_0^{2z - 2\theta}}{R^2 r_{hv}} \frac{dt^2}{(r - r_0)^2} + \frac{dr^2}{(r - r_0)^2} \right], \\
L^2 &= \frac{R r_0^{1 - \frac{3\theta}{2}} r_{hv}^{\frac{3\theta}{2}}}{(2 + z - \theta)(1 + z - \theta)}, \\
\Phi^2 &= \left( \frac{r_0}{r_{hv}} \right)^{\theta} \left( \frac{r_0}{R} \right)^2, \\
Q^2 &= \left( \frac{2 + z - \theta}{z - \theta} \right) r_0^{2(1 + z - \theta)}.
\end{aligned}
\]

(3.22)
Let us choose conformal gauge by doing a coordinate transformation,

$$\rho = \frac{R^{z+1}r^{1-z}r_0}{(2 + z - \theta)(1 + z - \theta)(r - r_0)}.$$  \hfill (3.23)

In conformal gauge, the \(AdS_2\) metric in (3.22) can be written as

$$ds^2 = e^{2\omega}(-dt^2 + d\rho^2) = e^{2\omega}(-dx^+ dx^-), \quad e^{2\omega} = \frac{L^2}{\rho^2},$$  \hfill (3.24)

where the lightcone coordinates are \(x^\pm = t \pm \rho\) and \(L\) is the radius of \(AdS_2\). To see that (3.21) admits the above \(AdS_2\) solution, we compute \(\partial U / \partial \Phi^2\) for the above solution, which gives

$$\frac{\partial U}{\partial \Phi^2} = -2\frac{(2 + z - \theta)(1 + z - \theta)}{Rr_0^{1-\frac{z}{2}}rs} = -\frac{2}{L^2}.$$  \hfill (3.25)

From (3.20) for \(\Psi = \text{constant}\) (from (3.22)), we get the Ricci scalar as

$$R = \frac{\partial U}{\partial \Phi^2} = -\frac{2}{L^2}.$$  \hfill (3.26)

### 3.2.1 Perturbations about \(AdS_2\)

As before, we turn on perturbations to fields with background profiles, \(i.e.\) to the metric, the dilaton \(\Phi\) and the scalar field \(\Psi\),

$$\Phi = \Phi_b + \phi(x^+, x^-), \quad \omega = \omega_b + \Omega(x^+, x^-), \quad \Psi = \Psi_b + \sqrt{2(2 - 2 - \theta)} \psi(x^+, x^-),$$  \hfill (3.27)

where \(\Phi_b, \omega_b\) and \(\Psi_b\) denote the (3.22) background solution. Expanding the action (3.21) (in conformal gauge) about this background gives

$$S = \frac{1}{16\pi G_2} \int d^2x \left(4\Phi^2 \partial_+ \partial_- \omega - 2\frac{e^{2\omega}}{2} U(\Phi, \Psi)\right) \equiv S_0 + S_1 + S_2,$$  \hfill (3.28)

where

$$S_0 = \frac{1}{16\pi G_2} \int d^2x \left(4\Phi_b^2 \partial_+ \partial_\omega_b + \Phi_b^2 \partial_+ \Psi_b \partial_- \Psi_b - 2\frac{e^{2\omega_b}}{2} U(\Phi_b, \Psi_b)\right)$$  \hfill (3.29)

is the background action and \(S_1\) vanishes by the equations of motion. \(S_2\) is quadratic in perturbations and is given by

$$S_2 = \frac{1}{16\pi G_2} \int d^2x \frac{r_0^2r_{hv}^2}{L^2(2 + z - \theta)(1 + z - \theta)} \left[8\phi \partial_+ \partial_- \Omega + \frac{16}{(x^+ - x^-)^2}\phi \Omega\right]$$

$$+ \frac{r_0^2r_{hv}^2}{L^2(2 + z - \theta)(1 + z - \theta)} \left[(2 - 2 - \theta)\partial_+ \psi \partial_- \psi - 4(z - 1)\phi \Omega\right]$$

$$+ \frac{1}{(x^+ - x^-)^2} \left[-16L^2(2 + z - \theta)(1 + z - \theta)\phi^2 + \frac{8\theta}{\sqrt{(2 - \theta)} \psi \phi}\right].$$  \hfill (3.30)
Varying this action, we get the linearized equations of motion for the perturbations,

\[ \partial_+ \partial_- \phi + \frac{2}{(x^+ - x^-)^2} \phi = 0 , \]

\[ (2z - 2 - \theta) \partial_+ \partial_- \psi + \frac{1}{(x^+ - x^-)^2} \left( 4(z - 1) \psi - \frac{L^2 (2 + z - \theta) (1 + z - \theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \frac{4\theta}{\sqrt{(2 - \theta)}} \phi \right) = 0 , \]

\( (3.31) \)

These equations are consistent at linear order with the “constraint” equations for the ±± components of the Einstein equation in (3.20): see Appendix, eq.(A.8)-(A.10).

We see that the equation for \( \psi \) is coupled to \( \phi \) as well: defining a new field \( \zeta \),

\[ \zeta = \psi - \frac{2}{\sqrt{2 - \theta}} \frac{L^2 (2 + z - \theta) (1 + z - \theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \phi , \]

(3.32)

decouples the equations for \( \zeta \) and \( \phi \), which now become

\[ \partial_+ \partial_- \phi + \frac{2}{(x^+ - x^-)^2} \phi = 0 , \]

\[ (2z - 2 - \theta) \partial_+ \partial_- \zeta + 2(z - 1) \frac{2}{(x^+ - x^-)^2} \zeta = 0 , \]

\[ \partial_+ \partial_- \Omega + \frac{1}{(x^+ - x^-)^2} \left( 2 \Omega + \frac{2(3\theta - 4)}{(2 - \theta)} \frac{L^2 (2 + z - \theta) (1 + z - \theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \phi + \frac{\theta}{\sqrt{(2 - \theta)}} \right) \zeta = 0 . \]

(3.33)

In this form, the perturbations \( \phi \) and \( \zeta \) are equivalent to scalars with positive mass propagating in a perturbed \( AdS_2 \) background, with equation of motion \( \frac{1}{\sqrt{-g}} g^{\mu\nu} \partial_\mu (\sqrt{-g} g^{\nu\rho} \partial_\rho \phi) - m^2 \phi = 0 \): in conformal gauge this is \( \partial_+ \partial_- \phi + \frac{m^2 L^2}{(x^+ - x^-)^2} \phi = 0 \). Let us look at a few special cases here:

- For \( z = 1, \ \theta = 0 \), we have seen that this system reduces to the relativistic brane case studied earlier (2.29), and the \( \Psi \) scalar (the nonrelativistic scalar in higher dimensions) can be then seen to decouple from the system: in particular, the terms containing \( \psi \)-perturbations vanish in the action (3.30) for quadratic perturbations. This is expected from the fact that the original action for the higher dimensional nonrelativistic theory reduces to the relativistic brane theory as \( z \to 1, \ \theta \to 0 \), as discussed after (3.1). In effect, we have defined the \( \psi \)-perturbation in (3.27) so that the relativistic brane limit arises smoothly, and the \( \Psi \)-scalar freezes out. This is also reflected in the linearized equations for perturbations.
• For $\theta = 0$ and $z > 1$, both $\phi$ and $\zeta$ have positive mass term coefficients, and further $\zeta$ decouples entirely from the $\Omega$ equation. This means that in fact any linear combination of the fields $A\phi + B\zeta$ also in fact has a positive mass term coefficient in its linearized fluctuation equation, as can be seen by taking that linear combination of the two equations $\partial_+ \partial_- (A\phi + B\zeta) + \frac{2}{(x^+ - x^-)^2} (A\phi + B\zeta) = 0$. The linear fluctuation analysis thus suggests that the attractor point is in fact perfectly stable for small fluctuations.

• For $\theta \neq 0$ and $z = 1$, we see that the $\zeta$ field is a massless mode and further it does not decouple from the $\Omega$ equation. This suggests that the linear stability analysis is insufficient to determine stability of the attractor point. However in this case, there is a more basic concern: looking back at the higher dimensional system (3.16), we see that in fact $\frac{\partial^2 \mathcal{V}_{\text{eff}}}{\partial \Psi^2} = 0$ in this case (in fact all derivatives vanish, (A.5)), so that the higher dimensional theory is also not manifestly a stable attractor. Thus the relevance of the 2-dimensional theory is less clear in this case.

• For generic $z, \theta$ values satisfying the energy conditions (3.6), (3.7), (3.8), we see that the mass term coefficients for both $\phi$ and $\zeta$ perturbations are positive. Now a generic linear combination of the fields $A\phi + B\zeta$ satisfies

$$\partial_+ \partial_- (A\phi + (2z - 2 - \theta)B\zeta) + \frac{2}{(x^+ - x^-)^2} (A\phi + (2z - 2 - \theta)B\zeta) = -\frac{2}{(x^+ - x^-)^2}\theta B\zeta.$$  

(3.34)

This is akin to a scalar field $A\phi + (2z - 2 - \theta)B\zeta$ with positive mass, driven by the source field $\zeta$. Since $\zeta$ is also a positive mass scalar, small fluctuations do not contain any unstable modes growing in time. Thus the general perturbation also is stable. To elaborate a bit further, imagine long-wavelength modes of $\phi, \zeta$ which are spatially uniform, i.e. $\phi = \phi(t), \zeta = \zeta(t)$. Now the linearized equations are of the form $\ddot{\phi} + m_\phi^2 \phi = 0, \ddot{\zeta} + m_\zeta^2 \zeta = 0$, so that these fields are effectively decoupled harmonic oscillators. Then the general field is a driven oscillator, with the driving force itself executing small oscillations: so there are no unstable modes growing in time. It is important to note that the positivity of the mass term coefficients and the stability they imply stems from the energy conditions and asymptotic boundary conditions, which force $z > 1$ and $2z - 2 - \theta > 0$ for generic $z, \theta$ values.

It is worth noting that for fixed $\zeta$, the relative sizes of the dilaton $\phi$ and hvLif scalar $\psi$ perturbations are $\frac{\dot{\phi}}{\phi} \sim \frac{L_2^2}{r_0^2} \frac{(\frac{\dot{\psi}}{r_0})^2}{r_{hv}} \ll \frac{L_2^2}{r_0^2} \quad \text{for} \quad \theta > 0 \quad \text{since} \quad \frac{\dot{\psi}}{r_0} \ll 1.$

It is worth comparing this analysis with that for the higher dimensional theory discussed earlier in (3.14), (3.16): the scalar $\Psi$ has a canonical kinetic term and the equation governing
small fluctuations of $\Psi$ about the attractor point acquires a mass term from $\frac{\partial^2 U}{\partial \Psi^2}$, whose positivity dictates the stability of the attractor point. For a theory with two scalars $\phi_1, \phi_2$ with canonical kinetic terms, the stability of the linearized fluctuations can again be studied by studying the second derivative matrix of the potential $U(\phi_1, \phi_2)$ or the Hessian $[\frac{\partial^2 U}{\partial \phi_i \partial \phi_j}]$. Positivity of the Hessian then translates to stability of the attractor extremum. In the present case however, the effective action is (3.21), and the kinetic terms for $\Phi, \Psi$ are not canonical: thus the naive Hessian analysis to study the stability of $U(\Phi, \Psi)$ about the attractor point is not valid. Instead we must analyze perturbations about the attractor point, which are governed by the above equations. From these equations, we see that the mass terms for the decoupled fields $\zeta$ and $\phi$ are positive.

In terms of $\phi$ and $\zeta$, the quadratic action becomes

\[
S_2 = \frac{1}{16\pi G_2} \int d^2 x \left[ 8 \phi \partial_+ \partial_- \Omega + \frac{16}{(x^+ - x^-)^2} \phi \Omega 
+ \frac{2^2 - 2\theta}{L^2(2 + z - \theta)(1 + z - \theta)} \left( (2z - 2 - \theta) \partial_+ \zeta \partial_- \zeta - \frac{4(z - 1)}{(x^+ - x^-)^2} \zeta^2 \right) 
+ \frac{L^2(2 + z - \theta)(1 + z - \theta)}{r_0^{2-2\theta}} \left( \frac{4(2z - 2 - \theta)}{(2 - \theta)} \partial_+ \phi \partial_- \phi - \frac{16(z + 1 - 2\theta)}{(2 - \theta)(x^+ - x^-)^2} \phi^2 \right) 
+ 2 \sqrt{\frac{2z - 2 - \theta}{2 - \theta}} \left( \partial_+ \zeta \partial_- \phi + \partial_- \zeta \partial_+ \phi \right) - \frac{8(2z - 2 - \theta)}{\sqrt{2 - \theta}} \frac{\zeta \phi}{(x^+ - x^-)^2} \right].
\]

It can be checked that varying this action leads to the linearized equations written in terms of $\phi, \zeta$ above.

### 3.2.2 The Schwarzian

In this section, we switch to Euclidean time $\tau = it$. From the linearized equations (3.31), we see that the dilaton fluctuation $\phi$ is decoupled from the metric and scalar fluctuations $\Omega$ and $\psi$, as in the case of the relativistic brane. So solving the equation for $\phi$ (i.e. the Euclidean form of (3.31)) gives, as before,

\[
\phi = \frac{a + b\tau + c(\tau^2 + \rho^2)}{\rho},
\]

where $a, b, c$ are independent constants. Substituting $\phi$ in the equation for $\psi$ in (3.31), we can solve for the scalar perturbation $\psi$. Using these solutions for $\phi$ and $\psi$ in the equation for $\Omega$ in (3.31), we can solve for the metric perturbation $\Omega$. We see that the $AdS_2$ metric gets corrected at the same order as the dilaton and the scalar field. The Euclidean on-shell (boundary) action obtained by using linearized field equations in (3.35) and changing to
Euclidean time $\tau = it$ is

$$S_2^E = -\frac{1}{16\pi G_2} \int d\tau \sqrt{\gamma} n^\mu \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2 + z - \theta)(1 + z - \theta)} \left[ 4(\Omega \partial_\mu \phi - \phi \partial_\mu \Omega) - \frac{(2z - 2 - \theta)}{\sqrt{2 - \theta}} (\phi \partial_\mu \zeta + \zeta \partial_\mu \phi) - \frac{L^2(2 + z - \theta)(1 + z - \theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \frac{2(2z - 2 - \theta)}{(2 - \theta)} \phi \partial_\mu \phi \right] (3.37)$$

The discussion of the Gibbons-Hawking term is very similar to that in sec. 2.1.2 so we will not be detailed. The Gibbons-Hawking boundary term for the Euclidean form of the bulk action (3.21) is

$$S_{GH} = -\frac{1}{8\pi G_2} \int d\tau \sqrt{\gamma} \Phi^2 K , \quad (3.38)$$

arising as discussed in the case of the relativistic electric brane earlier. As in sec. 2.1.2, we now redefine the dilaton perturbation after rescaling the background value $\Phi_b$ out, so that the perturbation satisfies $\frac{\Phi - \Phi_b}{\Phi_b} \equiv \tilde{\phi} \ll 1$. A similar redefinition is appropriate for the $hvL$ scalar $\Psi$ as well (we have however retained the perturbations in (3.27) without this rescaling simply with a view to not cluttering the resulting expressions). Then the perturbation, the background value (3.22) and the entropy (3.12) are

$$\Phi = \Phi_b (1 + \tilde{\phi}) , \quad \Phi_b^2 = (\frac{r_0}{r_{hv}})^{-\theta} \left( \frac{r_0}{R} \right)^2 , \quad S_{BH} = \Phi_b^2 \frac{V_2}{4G_4} = \frac{\Phi_b^2}{4G_2} . \quad (3.39)$$

This gives

$$S_{GH}^{(1)} = -\frac{2\Phi_b^2}{8\pi G_2} \int d\tau \sqrt{\gamma} \tilde{\phi} K \rightarrow -\frac{\Phi_b^2}{4\pi G_2} \int du \phi_r(u) \{\tau(u), u\} . \quad (3.40)$$

In evaluating the last term, we take the boundary of $AdS_2$ as a slightly deformed curve $(\tau(u), \rho(u))$ parametrized by the boundary coordinate $u$, as discussed in [2] (reviewed in [11]), and expand the extrinsic curvature using the outward unit normal $n^\mu$ to the boundary. Expanding $S_{GH}^{(1)}$ leads to the action above, which contains the Schwarzian derivative

$$Sch(\tau(u), u) = \{\tau(u), u\} = \frac{\tau'''}{\tau'} - \frac{3}{2}(\frac{\tau''}{\tau'})^2 .$$

The integral above pertains simply to the $AdS_2$ scale $L$, into which the various length scales in the nonrelativistic theory have been absorbed. We have also as before defined $\tilde{\phi} = \frac{\phi_r(u)}{\epsilon}$ and $\sqrt{\gamma} = \frac{L}{\epsilon}$.

As for the relativistic brane sec. 2.1.2 and (2.39), we note that the coefficient of the Schwarzian effective action is proportional to the entropy (3.12), (3.39) of the compactified black brane, with $V_2$ finite. As in sec. 2.1.2 this coefficient as the entropy arises after making the reasonable definition of the dilaton perturbation as in (3.39), scaling out the background $\Phi_b$. The entropy now contains only $\Phi_b$, which controls the transverse area. Since the entropy
captures the number of microstates of the unperturbed background, this is akin to a central charge.

This is the leading term in the total boundary action $I_{\text{bdy}} = S^E + S_{\text{GH}}$. The remaining terms in the expansion of $S_{\text{GH}}$ and $S^E_2$ are all quadratic in perturbations and hence are subleading compared to $S_{\text{GH}}^{(1)}$ which contains the dilaton perturbation alone at linear order, as for the relativistic brane discussed earlier. This universal behaviour is in accord with the general arguments in e.g. [2].

Thus overall, expanding in the perturbations $\tilde{\phi}, \Omega, \psi$, we have $I = S^E + S_{\text{GH}} = I_0 + I_1 + I_2 + \ldots$, where

$$I_0 = -\frac{\Phi^2_b}{16\pi G_2} \left( \int d^2 x \sqrt{g} R + 2 \int_{\text{bdry}} \sqrt{\gamma} K \right),$$

(3.41)
is the background action (see (3.29)): here $\Psi_b$ is constant and it can be checked that $U(\Phi_b, \Psi_b) = 0$. This is a topological term and gives the extremal entropy, very similar to the detailed discussion for the relativistic brane sec. 2.1.2. The linear terms are contained in

$$I_1 = -\frac{2\Phi^2_b}{16\pi G_2} \int d^2 x \sqrt{g} \tilde{\phi} \left( R - \frac{\partial U}{\partial \Phi} - \frac{1}{2} (\partial \Psi_b)^2 \right) - \frac{2\Phi^2_b}{8\pi G_2} \int_{\text{bdry}} \sqrt{\gamma} \tilde{\phi} \tilde{K}$$

$$- \frac{1}{16\pi G_2} \int d^2 x \sqrt{g} \left( -\frac{\Phi^2_b}{2} \partial_{\mu} \Psi_b \partial^\mu \psi - \psi \frac{\partial U}{\partial \Psi} \right).$$

(3.42)

On the $AdS_2$ background with a constant dilaton $\Phi_b$ and a constant hvLif scalar field $\Psi_b$, we get $\frac{\partial U}{\partial \Phi}(\Phi_b, \Psi_b) = -\frac{2}{L^2}$ and the second line in the expression for $I_1$ above vanishes by the $\Psi$ equation in (3.20). Thus, $I_1$ reduces to

$$I_1 = -\frac{2\Phi^2_b}{16\pi G_2} \int d^2 x \sqrt{g} \tilde{\phi} \left( R + \frac{2}{L^2} \right) - \frac{2\Phi^2_b}{8\pi G_2} \int_{\text{bdry}} \sqrt{\gamma} \tilde{\phi} \tilde{K},$$

(3.43)

which is the Jackiw-Teitelboim theory. The fluctuations of the scalar $\Psi$ now propagate on the fixed $AdS_2$ background at this order. However we see as in sec. 2.1.2 that there are various subleading terms at quadratic order ((3.37) and from the Gibbons-Hawking term, see (2.42), as well as possible counterterms), containing the perturbations to the dilaton $\Phi$, metric and scalar $\Psi$, which all mix (at the same order as the metric): the fluctuation spectrum is stable for physically sensible theories satisfying the energy conditions as we have seen. These encode information about the regularization of the $AdS_2$ theory by the particular higher dimensional hvLif theory.

3.2.3 More general perturbations

In the above analysis we have restricted ourselves to the dimensional reduction of perturbations to only those components of fields (metric, gauge fields, scalar) which have non-trivial
background values in the higher dimensional theory. More generally, considering the dimensional reduction of perturbations to all the components of all the fields (some of which have trivial background values) gives

\[ h_{MN} \rightarrow h_{\mu \nu}, \quad h_{\mu i}, \quad h_{ij}; \quad A^{(1,2)}_M \rightarrow A^{(1,2)}_\mu, \quad A^{(1,2)}_{1i}, \quad A^{(1,2)}_{2i}; \quad \phi \rightarrow \phi, \quad (3.44) \]

i.e. tensor, vector and scalar perturbations in the 2-dimensional theory (note that the 2-dim dilaton is \( g_{xx} \)). For instance this includes the shear perturbation \( h_{xy} \) in the higher dimensional theory as well the spatial components of the gauge fields \( A_{1i}, A_{2i} \) for \( i = x, y \) which reduce respectively to a non-minimally coupled scalar \( h = g^{(4)xx}h_{xy} \) and minimally coupled scalars \( A_{1i} = \chi_1^{(1)}, \quad A_{2i} = \chi_1^{(2)} \) in the 2-dimensional theory. The terms in the full 2-dimensional action which govern these perturbations are

\[ S = \frac{1}{16\pi G} \int d^2x \sqrt{-g}\left[ \cdots - \frac{\Phi^2}{2} (\partial h)^2 - \frac{e^{\lambda_1 \psi}}{2} (\partial \chi_1^{(1)})^2 - \frac{e^{\lambda_2 \psi}}{2} (\partial \chi_1^{(2)})^2 \right]. \quad (3.45) \]

The terms involving \( h_{xy} \) arise from the higher dimensional Ricci scalar and so contain the overall dilaton factor \( \Phi^2 \) under reduction to 2-dimensions. The linearized equations for \( h_{xy} \) in the higher dimensional theory in \( e.g. \ [31] \) can be dimensionally reduced to 2-dimensions: at zero momentum, this is consistent with the Kaluza-Klein ansatz for reduction and the action above. Expanding these terms around the background \( AdS_2 \), the leading contributions from these terms appear at quadratic order in perturbations

\[ S_2 = \frac{1}{16\pi G} \int d^2x \sqrt{-g}\left[ \cdots - \frac{\Phi^2_b}{2} (\partial h)^2 - \frac{e^{\lambda_1 \psi_b}}{2} (\partial \chi_1^{(1)})^2 - \frac{e^{\lambda_2 \psi_b}}{2} (\partial \chi_1^{(2)})^2 \right]. \quad (3.46) \]

These are subleading compared to \( S_{GH}^{(1)} \) and thus do not contribute to the Schwarzian.

### 4 On a null reduction of the charged \( \text{AdS}_5 \) black brane

In [29] (see also [32]), it was argued that the null reduction of \( \text{AdS} \) plane waves, highly boosted limits of uncharged black branes, gives rise to hvLif theories with certain specific \( z, \theta \) exponents. The lower dimensional hvLif gauge field and scalar arise as the KK gauge field and scalar under \( x^+ \)-reduction. One might imagine that considering such a null reduction of the charged relativistic black brane might be interesting along these lines. In this section, we describe an attempt to obtain the charged hvLif black branes here by a null \( x^+ \)-reduction of the charged relativistic black brane in one higher dimension. Unfortunately this turns out to be close, but not quite on the nose: while the charge electric gauge field upstairs does give rise to an electric field in the lower dimensional theory, it also leads to an additional background scalar profile. It would be interesting to understand if this can be refined further.
The action for a charged $AdS_5$ black brane \cite{15} is\footnote{In this section, $r \to 0$ is the boundary.}

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left[R - 2\Lambda - \frac{2\kappa^2 F^2}{e^2 F} \right].$$ \hspace{1cm} (4.1)

The charged $AdS_5$ black brane metric is

$$ds^2 = \frac{L^2}{r^2} \left( -f(r) dt^2 + \frac{dr^2}{f(r)} + dx_1^2 + dx_2^2 + dx_3^2 \right),$$ \hspace{1cm} (4.2)

$$f(r) = 1 - \left(1 + \frac{r_0^2 \mu^2}{\gamma^2} \right) \left(1 - \frac{r_0^2}{r^2} \right), \quad \gamma^2 = \frac{3e^2 L^2}{2\kappa^2},$$ \hspace{1cm} (4.3)

where the horizon is at $r = r_0$ and the boundary at $r \to 0$. The gauge field $A_t$, charge density $\rho$ and temperature are

$$A_t = \mu \left(1 - \left(\frac{r}{r_0}\right)^2\right), \quad \rho = \frac{2L}{e^2 r_0^2 \mu}, \quad T = \frac{1}{4\pi r_0} \left(4 - 2\frac{r_0^2 \mu^2}{\gamma^2}\right).$$ \hspace{1cm} (4.4)

Transforming to lightcone coordinates, $x^\pm = \frac{t \pm dx}{\sqrt{2}}$ and performing a boost $x^\pm \to \lambda^\pm x^\pm$, the metric becomes

$$ds^2 = \frac{L^2}{r^2} \left( -f(r) \left(\frac{\lambda dx^+ + \lambda^{-1} dx^-}{\sqrt{2}}\right)^2 + \left(\frac{\lambda dx^+ - \lambda^{-1} dx^-}{\sqrt{2}}\right)^2 + \frac{dr^2}{f(r)} + dx_1^2 + dx_2^2 \right).$$ \hspace{1cm} (4.5)

Completing squares in $dx^+$, we get

$$ds^2 = -\frac{2L^2 r_0^2 f(r)}{\lambda^2 r_0^6 \left(1 + \frac{r_0^2 \mu^2}{\gamma^2} \right)(1 - \frac{r_0^2}{r^2})} (dx^-)^2 + \frac{L^2}{r^2} \left( \frac{dr^2}{f(r)} + dx_1^2 + dx_2^2 \right)$$

$$+ \frac{L^2 \lambda^2 r_0^2}{2r_0^4 \gamma^2} \left(1 + \frac{r_0^2 \mu^2}{\gamma^2} \right)(1 - \frac{r_0^2}{r^2}) (dx^+ + A_- dx^-)^2,$$ \hspace{1cm} (4.6)

$$A_- = -1 + \frac{r_0^4}{2r_0^2 \gamma^2} \left(1 + \frac{r_0^2 \mu^2}{\gamma^2} \right)(1 - \frac{r_0^2}{r^2}) \frac{\lambda^2 r_0^4}{\gamma^2} \left(1 + \frac{r_0^2 \mu^2}{\gamma^2} \right)(1 - \frac{r_0^2}{r^2}) \left(1 - \frac{r_0^2}{r^2} \right).$$ \hspace{1cm} (4.7)

The first line in \eqref{4.6} after incorporating the conformal factor from $x^+$-reduction leads approximately to the 4-dim hvLif metric with $z = 3, \theta = 1$, in the vicinity of $r \to 0$ and $r \to r_0$. The KK-gauge field becomes the $F_1$ gauge field in the lower dimensional theory: its form becomes that of $F_{1rt}$ only in the vicinity of the horizon $r \to r_0$, giving $A_{1-} \equiv A_{1t} \sim -\frac{1}{(\lambda^2/r_0^2) r^4 + 1/\lambda^2}$, where we hold $\lambda^2/r_0^2$ fixed which preserves the first term (while the 2nd term dies). This reduction to hvLif is exact if $\mu = 0$, as in \cite{29} for zero temperature (and \cite{32, 31} for finite temperature).
Likewise the $A_t \equiv A_{2t}$ gauge field giving charge becomes in the lower dimensional theory

$$A_{2+} = \lambda A_{2t}, \quad A_{2-} = \frac{1}{\lambda} A_{2t} \to A^{4d}_{2t}.$$  \hspace{1cm} (4.8)

Scaling the chemical potential as $\mu \to \frac{\mu}{\lambda} = fixed$, we obtain precisely the gauge field profile for $A_{2t}$: however $A_{2+}$ survives as a scalar background in the lower dimensional theory.

It can also be seen that the relativistic brane action (2.1) gives rise upon $x^+-$reduction to the hvLif action (3.1), up to the extra scalar arising from $A_{2+}$. It would be interesting look for refinements of the discussion here, towards decoupling this extra scalar.

5 Discussion

We have studied dilaton-gravity theories in 2-dimensions obtained by dimensional reduction of certain families of extremal charged hyperscaling violating Lifshitz black branes in Einstein-Maxwell-scalar theories with an extra gauge field in 4-dimensions. We have argued that the near horizon $AdS_2$ backgrounds here can be obtained in equivalent theories of 2-dim dilaton-gravity with an extra scalar, descending from the higher dimensional scalar, and an interaction potential with the dilaton. A simple subcase is the relativistic black brane with $z = 1, \theta = 0$ (which has no extra scalar), which we have analysed in detail. Studying linearized fluctuations of the metric, dilaton and the extra scalar about these $AdS_2$ backgrounds suggests stability of the attractor background generically. This is correlated with the requirements imposed by the energy conditions on these backgrounds. From the study of small fluctuations, we have seen that the leading corrections to $AdS_2$ arise at linear order in the dilaton perturbation resulting in a Schwarzian derivative effective action from the Gibbons-Hawking term, and Jackiw-Teitelboim theory at leading order. We have also seen that the coefficient of the Schwarzian derivative term, (2.39), (3.40), is proportional to the entropy of the (compactified) extremal black branes after defining the perturbations by scaling out the background values (2.37), (3.39): this being the number of microstates of the unperturbed background is thus akin to a central charge. The background entropy arises automatically as a topological term from the compactification. There are of course various subleading terms in the action at quadratic order which mix at the same order as the metric: these encode information on the higher dimensional realization of these $AdS_2$ backgrounds.

We have explored certain classes of such extremal backgrounds: it would be interesting to understand the space of such $AdS_2$ theories in a more systematic manner. One might imagine that the parameters in these theories, for instance the dynamical exponents, are reflected in the spectrum of correlation functions, thus distinguishing the specific ultraviolet
regularization of the AdS$_2$ regimes. This requires better understanding of the subleading terms beyond the Schwarzian, which in turn requires a systematic treatment of counterterms and holographic renormalization. We hope to explore these further.

From the point of view of the dual theories, it would seem that the present 2-dim backgrounds are dual to 1-dimensional theories arising from $T^2$ compactifications of the dual field theories. It would be interesting to understand these better, in part towards possibly exploring parallels with the SYK models [8, 9], discussed more recently in e.g. [33, 2, 34, 10] and related SYK/tensor models (see e.g. [35]).

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A Some details

Relativistic electric black brane: The Einstein equation and the dilaton equation from the action (2.16) are

$$g_{\mu\nu} \nabla^2 \Phi^2 - \nabla_{\mu} \nabla_{\nu} \Phi^2 + \frac{g_{\mu\nu}}{2} \left( 2\Lambda \Phi + \frac{16\pi G_2 V_2 \Phi^3 F_{\mu\nu} F^{\mu\nu}}{4} \right) - \frac{R}{\Phi} - \frac{16\pi G_2 V_2 \Phi^3}{2} F_{\mu\nu} F^{\mu\nu} = 0 ,$$

$$R - \frac{\Lambda}{\Phi} - \frac{(6\pi G_2) V_2 \Phi F_{\mu\nu} F^{\mu\nu}}{2} = 0 .$$

Charged hvLif black brane

Effective scalar potential in 4-dimensional hvLif black brane and its derivatives:
The first and second derivatives of the effective scalar potential in 4-dimensional charged hvLif black brane are

$$\frac{\partial V_{eff}}{\partial \Psi} = -\frac{\gamma (2 + z - \theta) (1 + z - \theta) e^{\gamma (\Psi - \Psi_0)}}{R^{2-2\theta} r_{hv}^{2\theta}} - \frac{\lambda_1 (z - 1) (2 + z - \theta) r_{hv}^{2\theta - 4} R^{2-2\theta}}{e^{\lambda_1 (\Psi - \Psi_0)}} + \frac{\lambda_2 (2 - \theta) (z - \theta) Q^2 r_{hv}^{2\theta - 2} R^{-4z - 2 + 2\theta}}{e^{\lambda_2 (\Psi - \Psi_0)}} ,$$

$$\frac{\partial^2 V_{eff}}{\partial \Psi^2} = \frac{\gamma^2 (2 + z - \theta) (1 + z - \theta) e^{2\gamma (\Psi - \Psi_0)}}{R^{2-2\theta} r_{hv}^{4\theta}} - \frac{\lambda_1^2 (z - 1)^2 (2 + z - \theta) r_{hv}^{4\theta - 4} R^{2-2\theta}}{e^{2\lambda_1 (\Psi - \Psi_0)}} + \frac{\lambda_2^2 (2 - \theta)^2 (z - \theta)^2 Q^4 r_{hv}^{4\theta - 2} R^{-4z - 2 + 2\theta}}{e^{2\lambda_2 (\Psi - \Psi_0)}} ,$$
\[
\frac{\partial^2 V_{\text{eff}}}{\partial \Psi^2} = -\frac{\gamma^2(2 + z - \theta)(1 + z - \theta)e^{\gamma(\Psi - \Psi_0)}}{R^2 - 2\theta R^{2\theta} r_{hv}^{2\theta}} + \frac{1}{g_{xx}^2} \left( \frac{\lambda_1^2(z - 1)(2 + z - \theta)r_{hv}^{2\theta - 4} R^{2\theta}}{e^{\lambda_1(\Psi - \Psi_0)}} + \frac{\lambda_2^2(2 - \theta)(z - \theta)Q^2 r_{hv}^{2\theta - 2} R^{-4z + 2 + 2\theta}}{e^{\lambda_2(\Psi - \Psi_0)}} \right). 
\]

Differentiating \( V_{\text{eff}} \) \( n \) times, we get
\[
\frac{\partial^n V_{\text{eff}}}{\partial \Psi^n} = -\frac{\gamma^n(2 + z - \theta)(1 + z - \theta)e^{\gamma(\Psi - \Psi_0)}}{R^2 - 2\theta R^{2\theta} r_{hv}^{2\theta}} + \frac{(-)^n}{g_{xx}^2} \left( \frac{\lambda_1^n(z - 1)(2 + z - \theta)r_{hv}^{2\theta - 4} R^{2\theta}}{e^{\lambda_1(\Psi - \Psi_0)}} + \frac{\lambda_2^n(2 - \theta)(z - \theta)Q^2 r_{hv}^{2\theta - 2} R^{-4z + 2 + 2\theta}}{e^{\lambda_2(\Psi - \Psi_0)}} \right),
\]
which at the extremal point becomes
\[
\frac{\partial^n V_{\text{eff}}}{\partial \Psi^n} = \frac{\gamma^n(1 + z - \theta)}{r_{hv}^{2\theta} R^2} \left[ \frac{1}{(2 - \theta)(2z - 2\theta)} \frac{(-)^n (2z - 2 - \theta) \hat{\Phi}(2 - \theta)}{(2 - \theta) \hat{\Phi}} \right].
\]

At \( z = 1, \theta \neq 0 \), we see that \( \frac{\partial^n V_{\text{eff}}}{\partial \Psi^n} = 0 \ \forall \ n \) at the extremal point.

**Dimensional reduction to 2-dimensions:** The 2-dim action obtained by reducing (3.1) on \( T^2 \) is (retaining only fields with background profiles)
\[
S = \int d^2x \sqrt{-g}(\frac{1}{16\pi G_2} \left( \Phi^2 \mathcal{R}^{(2)} + 2\partial_{\mu} \Phi \partial^\mu \Phi - \frac{\Phi^2}{2} \partial_{\mu} \Psi \partial^\mu \Psi + V\Phi^2 \right) - \frac{\Phi^2}{4} Z_1 F_{1\mu\nu} F^{1\mu\nu} - \frac{V\Phi^2}{4} Z_2 F_{2\mu\nu} F^{2\mu\nu} ) . 
\]

**Equations of motion from 2-dimensional action (3.17):** The equations of motion obtained by varying the action (3.17) are
\[
g_{\mu\nu} \nabla^2 \Phi^2 - \nabla_\mu \nabla_\nu \Phi^2 + \frac{g_{\mu\nu}}{2} \left( \frac{\Phi^2}{2} \partial(\Phi)^2 - V\Phi + \frac{\Phi^3}{4} (Z_1(F_1)^2 + 16\pi G_2 V_2 Z_2(F_2)^2) \right) \\
- \frac{\Phi^2}{4} \partial_{\mu} \Psi \partial_{\nu} \Psi - \frac{\Phi^3}{2} (Z_1 F_{1\mu\rho} F_{1\nu}^{\rho} + 16\pi G_2 V_2 Z_2 F_{2\mu\rho} F_{2\nu}^{\rho} ) = 0 , \\
\mathcal{R} - \frac{1}{2} (\partial \Phi)^2 + \frac{V}{2\Phi} - \frac{3}{8} \Phi (Z_1(F_1)^2 + 16\pi G_2 V_2 Z_2(F_2)^2) = 0 , \\
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \Phi^2 \partial^\mu \Psi) + \gamma V \Phi = \frac{\Phi^3}{4} (\lambda_1 Z_1(F_1)^2 + \lambda_2 16\pi G_2 V_2 Z_2(F_2)^2) = 0 . 
\]

The equations of motion (3.20) in conformal gauge and in lightcone coordinates are
\[
- e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \Phi^2) - \frac{\Phi^2}{2} \partial_+ \Psi \partial_+ \Psi = 0 , \\
\partial_+ \partial_- \Phi^2 - \frac{\Phi^2}{4} U = 0 , \\
4\partial_+ \partial_- \omega + \partial_+ \Psi \partial_- \Psi - \frac{e^{2\omega}}{2} \frac{\partial U}{\partial (\Phi^2)} = 0 , \\
\partial_+ (\Phi^2 \partial_- \Psi) + \partial_- (\Phi^2 \partial_+ \Psi) + \frac{e^{2\omega}}{2} \frac{\partial U}{\partial \Psi} = 0 . 
\]
Expanding the constraint equations in the first line of (A.8) to linear order in perturbations (3.27) gives

$$\partial_+ \partial_\pm \phi \pm \frac{2}{(x^+ - x^-)} \partial_\pm \phi = 0 ,$$

(A.9)

the other terms vanishing at linear order. To see that these linearized constraint equations are consistent with the linearized equations (3.31), we differentiate the ++ constraint equation with respect to $x^-$ to get

$$\partial_+ (\partial_+ \partial_\pm \phi) + \frac{2}{(x^+ - x^-)} \partial_+ \partial_- \phi + \frac{2}{(x^+ - x^-)^2} \partial_+ \phi = 0 ,$$

(A.10)

which is satisfied after using the equation for $\phi$ in (3.31). Similarly differentiating the -- constraint equation with respect to $x^+$, we can show that the resulting equation is satisfied upon substituting the equation for $\phi$ in (3.31).

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