ON DEGENERATE SECANT AND TANGENTIAL VARIETIES AND LOCAL DIFFERENTIAL GEOMETRY

J.M. LANDSBERG

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Abstract. We study the local differential geometry of varieties $X^n \subset \mathbb{CP}^{n+a}$ with degenerate secant and tangential varieties. We show that the second fundamental form of a smooth variety with degenerate tangential variety is subject to certain rank restrictions. The rank restrictions imply a slightly refined version of Zak’s theorem on linear normality, a short proof of the Zak-Fantecchi theorem on the superadditivity of higher secant defects, and a short proof and extension of a result of Robert’s on the nondegeneracy of secant varieties of Veronese re-embeddings. We study the geometric structure of systems of quadrics generated by the second fundamental forms of varieties with degenerate tangential varieties. In particular, Clifford algebras make an appearance. We give a new proof of Zak’s theorem on Severi varieties that is substantially shorter than the original by utilizing the Clifford algebra structure and the rank restrictions. We also prove local and global results on the dimensions of Gauss images of degenerate tangential varieties.

§1. Introduction, Conventions

One way to study geometric properties of a variety $X^n \subset \mathbb{CP}^{n+a}$ is by studying coarse geometric properties of auxiliary varieties one constructs from $X$. The auxiliary varieties we will study in this paper are the secant variety $\sigma(X)$ and the tangential variety $\tau(X)$, and the coarse properties of $\sigma(X)$ and $\tau(X)$ we will study are their dimensions. For information on how this study fits into larger questions, see [LV]. It turns out that smooth varieties of small codimension with degenerate $\tau(X)$ (degenerate meaning that $\tau(X)$ is not the entire ambient space) carry a remarkable amount of infinitesimal geometric structure. Before going into details, we will need a few definitions.

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Given a variety $X^n \subset \mathbb{P}^{n+a}$, the secant variety $\sigma(X)$ of $X$ is defined to be the union of all points on all secant and tangent lines (i.e. $\mathbb{P}^1$'s) of $X$. More precisely, given $p, q \in \mathbb{P}^{n+a}$, let $\mathbb{P}^1_{pq} \subset \mathbb{P}^{n+a}$ denote the projective line containing $p$ and $q$. Then

$$\sigma(X) := \{ x \in \mathbb{P}^{n+a} | x \in \mathbb{P}^1_{pq} \text{ for some } p, q \in X \}.$$  

Secant varieties have been studied extensively. Two important results on them are:

Zak's Theorem on Linear Normality 1.1. ([FL],[Z1]) Let $X^n \subset \mathbb{C}^{n+a}$ be a smooth variety not contained in a hyperplane with $\sigma(X) \neq \mathbb{C}^{n+a}$. Then $a \geq \frac{n}{2} + 2$.

Zak's Theorem on Severi Varieties 1.2. ([LV],[Z1]) Let $X^n \subset \mathbb{C}^{n+a}$ be a smooth variety not contained in a hyperplane with $\sigma(X) \neq \mathbb{C}^{n+a}$. If $a = \frac{n}{2} + 2$, then $X$ is one of

i. Veronese $\mathbb{P}^2 \subset \mathbb{P}^5$

ii. Segre $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$

iii. Plücker embedded Grassmannian $G(\mathbb{C}^2, \mathbb{C}^6) \subset \mathbb{P}^{14}$

iv. $E_6/P \subset \mathbb{P}^{26}$.

These four varieties, now called Severi varieties, also have other special properties. For example, they classify the quadro-quadro Cremona transforms (see [ESB]).

Define the tangential variety $\tau(X)$ to be the union of all points on all tangent stars to $X$, where the tangent star at $p_0$ is defined to be the union of all points on all $\mathbb{P}^1$'s that are the limit of secant lines. More precisely, $\mathbb{P}^1_*$ is in the tangent star if there exist smooth curves $p(t), q(t)$ on $X$ such that $p(0) = q(0) = p_0$ and $\mathbb{P}^1_* = \lim_{t \to 0} \mathbb{P}^1_{pq}$. (See [F] or [Z1] for a more algebraic definition of the tangent star.) If we were to require $p(t) \equiv p_0$ and only allow one point to move, we would obtain the tangent cone. Of course at smooth points the two notions coincide and agree with the embedded tangent space.

For general $X^n \subset \mathbb{P}^{n+a}$ one expects:

$$\dim \sigma(X) = \min\{2n + 1, n + a\} \quad \dim \tau(X) = \min\{2n, n + a\}.$$  

and $\sigma(X)$ or $\tau(X)$ is said to be degenerate if these fail. In this paper we will study the local differential geometry of varieties with degenerate $\sigma(X)$ and $\tau(X)$ and the global geometric consequences of the local differential geometry. Of course $\dim \tau(X) \leq \dim \sigma(X)$, however a much stronger statement is true:
Connectnedness Applied to Secant Varieties 1.3. (Fulton-Hansen [FL])

Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Then either

\begin{enumerate}
  \item $\dim \tau(X) = 2n$ and $\dim \sigma(X) = 2n + 1$.
  \item $\sigma(X) = \tau(X)$.
\end{enumerate}

The paper is organized as follows: In §2 we explain some subtleties one encounters when studying $\tau(X)$ locally and give some examples of varieties $X$ with $\sigma(X)$ and $\tau(X)$ degenerate. In §3 we describe the Severi varieties as the complexification of the projective planes for the four division algebras. In §4 we define the second fundamental form of a mapping and state two ways to compute $\dim \tau(X)$ using $II_{X,x}$, the second fundamental form of $X$ at a general point $x \in X$, following [GH]. In §5 we describe the second fundamental forms of the Severi varieties. In §6 we describe geometric properties of systems of quadrics that could occur as second fundamental forms of varieties with degenerate tangential varieties, culminating in (6.26).

In §7 we state a version of the rank restriction theorem ([L1],(4.14)) for varieties with degenerate secant varieties, namely:

**Theorem 7.1 (Rank restrictions).** Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with degenerate secant variety that is a hypersurface. Let $x \in X$ be a general point. Let $r$ be the maximum rank of a quadric in $|\Pi|_x$ annihilating a $\Pi$-generic vector in $T_xX$. Then

$$r \geq n - a + 2.$$ 

Here $|\Pi|_x$ is the system of quadrics corresponding to $\Pi$ (see (4.12)) and a quadratic form $Q$ is said to annihilate $v \in T_xX$ if $Q(v, w) = 0$ for all $w \in T_xX$. See the notation section below for the definition of $\Pi$-generic. Corollary (7.3) provides a slight refinement of Zak’s theorem on tangencies.

In §8 we review some basic facts regarding moving frames and differential invariants of projective varieties. In §9 we prove the statements about how to compute $\dim \tau(X)$ stated in §4. In §10 we describe how to compute $\dim \sigma(X)$ infinitesimally. The computation immediately implies:

**Theorem 10.3.** Let $Y \subset \mathbb{P}V$ be a variety and let $X = v_d(Y) \subset \mathbb{P}S^dV$ be the Veronese re-embedding. If $d > 2$ or $d = 2$ and $Y$ is not a linear subspace of $\mathbb{P}V$, then $\sigma(X)$ is nondegenerate.

**Theorem 10.5.** Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety not contained in a hyperplane with degenerate secant variety of dimension $n + a_0$. Let $\sigma_k(X)$ denote the closure of the union of all secant $\mathbb{P}^{k-1}$’s of $X$. Then

$$\dim \sigma_k(X) \leq n + (k - 1)a_0.$$
(10.3) is a slight extension of a theorem of Roberts [R], where the \( d > 2 \) case was proved and (10.4) is the superadditivity result of Zak-Fantechi [F].

In §11 we prove the statements made in §6. In §12 we do a further local computation to prove a refinement of ([GH], 5.17):

**Theorem 12.1.** Let \( X^n \subset \mathbb{P}^{n+a} \) be a variety (respectively a patch of a complex manifold) with degenerate tangential variety (resp. manifold) of dimension \( n + a_0 \). Denote the the smooth points of \( X \) by \( X_{sm} \). Then the Gauss map of \( \tau(X) \) has fibers of dimension at least \( \delta_\tau(X) + 1 \), where \( \delta_\tau(X) \) is the tangential defect of \( X \).

In §13 we prove the key step in the rank restriction theorem (7.9) and we combine the rank restriction theorem and the results of §12 to prove

**Theorem 13.10.** Let \( X^n \subset \mathbb{P}^{n+a} \) be a smooth variety with degenerate tangential variety of dimension \( n + a_0 \). Then the Gauss map of \( \tau(X) \) has fibers of dimension at least \( \delta_\tau(X) + 2 \) dimensional fibers, where \( \delta_\tau(X) \) is the tangential defect of \( X \).

Finally, in §15 we do some further computations to complete our proof of Zak’s theorem on Severi varieties (1.2).

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**Notation:** By a *variety* we mean the zero locus in projective space of a collection of homogeneous polynomials that is reduced and irreducible. \( \mathbb{CP}^k \) will be denoted \( \mathbb{P}^k \). We will often write \( V = \mathbb{C}^{k+1} \) and \( \mathbb{PV} = \mathbb{P}^k \). We will generally denote varieties by \( X \), the smooth points of \( X \) will be denoted \( X_{sm} \), and the singular points \( X_{sing} \). If \( Y \subset \mathbb{P}^m \) then \( \hat{Y} \subset \mathbb{C}^{m+1} \) will denote its deprojectivization. Alternating products will be denoted with a wedge (\( \wedge \)), and symmetric products will not have anything (e.g. \( \omega \circ \beta \) will be denoted \( \omega \beta \)). In general we will supress reference to the base point of our manifold \( X \) when we abbreviate the names of fibers of bundles over \( X \subset \mathbb{P}V \), so \( T \) should be read as \( T_p X \) for some \( p \in X \), \( N \) as \( N p X \) etc... . \( \hat{T}_x X \subset V \) is the deprojectivization of the embedded tangent space \( \hat{T}_x X \). \( \{e_i\} \) means the span of the vectors \( e_i \) over the index range \( i \). If \( V \) is a vector space and \( W \) a subspace, and \( \{e_1, \ldots, e_n\} \) a basis of \( V \) such that \( \{e_1, \ldots, e_p\} = W \), we will occasionally write \( \{e_{p+1}, \ldots, e_n\} \mod W \) to denote the space \( V/W \). \( \cdot \) denotes contraction, i.e. given \( P \in S^k T^* \) and \( v \in T \), \( (v \cdot P) \in S^{k-1} T^* \). We will use the summation convention throughout (i.e. repeated indices are to be summed over). By a *II-generic vector* we mean a \( v \in T \) based at a general point such that the associated maps and spaces one constructs from \( \Pi \) (e.g. \( \ker \Pi_v \) as described in §6) are all locally of constant rank/dimension as \( v \) is varied in \( T \).
We will use the following conventions for indices

\[ 1 \leq A, B, C \leq n + a \]
\[ 1 \leq \alpha, \beta \leq n \]
\[ 1 \leq \lambda \leq a_0 \]
\[ a_0 + 1 \leq h \leq a \]
\[ 2 \leq \rho, \sigma \leq n \]
\[ 2 \leq i, j, k, l \leq r + 1 \]
\[ r + 2 \leq s, t \leq a_0 \]
\[ a_0 + 1 \leq \epsilon, \delta \leq n \]
\[ n + 1 \leq \mu, \nu \leq n + a. \]

§2. Some remarks on secant and tangential varieties.

In this paper we will describe how to compute the dimensions of the secant and tangential varieties of smooth varieties infinitesimally. The tests can compute the dimensions of the secant and tangential varieties of the smooth points of a singular variety. However it is not in general the case that \( \dim \tau(X_{\text{sm}}) = \dim \tau(X) \) (where \( X_{\text{sm}} \) denotes the smooth points of \( X \)). This is because the tangent star at singular points is larger than the tangent cone. A differential geometer might be inclined to define the tangential variety to be the union of all tangent cones. This would avoid having huge tangent spaces at singular points and allow one to compute dimensions infinitesimally even for singular varieties. Unfortunately, to have (1.3) hold, we need to use the tangent star as can be seen in the following:

Example (2.1): Let \( \mathcal{C} \) be a nondegenerate smooth curve in \( \mathbb{P}^4 \). Let \( p \in \mathbb{P}^4, p \notin \mathcal{C} \). Let \( X = C(\mathcal{C}, p) \) be the cone over \( \mathcal{C} \) with vertex \( p \). Then it is easy to see that \( \sigma(X) = C(\sigma(\mathcal{C}), p) \), so \( \dim \sigma(X) = \dim \sigma(X_{\text{sm}}) = 4 \). The infinitesimal tests we will develop in §9 and §10 yield \( \dim \tau(X_{\text{sm}}) = 3, \dim \sigma(X_{\text{sm}}) = 4 \). If one were to use the tangent cone in defining the tangential variety, call the resulting object \( \tau_{\text{cone}}(X) \), then one would get \( \dim \tau_{\text{cone}}(X) = 3 \) because except at the vertex, \( \tau(X) = C(\tau(C), p) \), and the tangent cone at the vertex is just \( X \) itself. But using the tangent star, there is a four dimensional space of tangent lines at the vertex \( p \), which allows \( \tau(X) = \sigma(X) \).

Schematic picture:
For this reason we will generally restrict ourselves to studying smooth varieties in this paper.

An example to keep in mind of a smooth variety with a degenerate secant variety is:

**Example 2.3. (The Segre)** $\mathbb{P}^{k-1} \times \mathbb{P}^{r-1} \to \mathbb{P}^{kr-1}$. Denote the $k \times r$ matrices by $M_{kr}$. Let $X = \mathbb{P}$(rank one matrices in $M_{kr}$) and consider $X$ as a subvariety of $\mathbb{P}^{kr-1}$. Here $\dim X = r + k - 1$. Then $\sigma(X) = \mathbb{P}$(rank two matrices in $M_{kr}$) and $\dim \sigma(X) = 2(r + k - 1) = 2\dim X - 1$. In some sense all known smooth varieties with degenerate secant varieties are like these or projections or hyperplane sections of such.

Following [LV], for $X$ with degenerate $\sigma(X)$, the secant defect is defined to be $\delta(X) := 2n + 1 - \dim \sigma(X)$. It is difficult to find smooth varieties with large secant defects. On the other hand there are singular varieties with arbitrarily large secant defects. One takes $X = \mathbb{P}$(rank $l$ matrices in $M_{kr}$) which has dimension $l(k + r - l)$. Then the secant variety is $\mathbb{P}$(rank $2l$ matrices) which is of dimension $2l(k + r - 2l)$ and thus $\delta(X) = 2l^2 + 1$. Such $X$ are singular along the rank $l - 1$ matrices.

§3. The Severi varieties.

For the purposes of this paper it will be better to look at the Severi varieties from a unified point of view. This point of view was first observed independently by J. Roberts and T. Banchoff.

Let $A_R$ denote $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (where $\mathbb{O}$ denotes the octonians, or Cayley numbers). Let $A = A_R \otimes \mathbb{C}$. (From now on $\mathbb{F} \otimes \mathbb{C}$ will be denoted by $\mathbb{F}^C$.) Let $H_R$ denote the $A_R$-Hermitian forms on $A^3_R$, i.e., the $3 \times 3$ $A_R$-Hermitian matrices. If $x \in H_R$, then we may write

$$x = \begin{pmatrix} r_1 & \bar{u}_1 & \bar{u}_2 \\ u_1 & r_2 & \bar{u}_3 \\ u_2 & u_3 & r_3 \end{pmatrix}, \quad r_i \in \mathbb{R}, \ u_i \in A_R.$$

Let $H = H_R \otimes \mathbb{C}$.

Define a cubic form $det$ on $H$ by

$$det(x) := \frac{1}{6}((\text{trace}(x))^3 + 2\text{trace}(x^3) - 3\text{trace}(x)\text{trace}(x^2)).$$

The reader may check that the notion of $x^2$ and $x^3$ make sense (one needs to use the Moufang identities (see [Hv]) in the case of the octonians). This is just the usual determinant of a $3 \times 3$ matrix when $A = \mathbb{C}$. When $A = \mathbb{O}^C$ one cannot define $det$ for $4 \times 4$ or larger matrices. The reader may find it amusing to check what happens in the $2 \times 2$ case.
Now, considering $\mathcal{H}$ as a vector space over $\mathbb{C}$, let $G$ be the subgroup of $GL(\mathcal{H}, \mathbb{C})$ preserving $\det$, i.e., define

\[(3.3) \quad G := \{g \in GL(\mathcal{H}, \mathbb{C}) | \det(gx) = \det(x) \}.
\]

The reader may check that the respective groups are:

\[(3.4) \quad A_\mathbb{R} = G =
\begin{align*}
\mathbb{R} & \quad Sl(3, \mathbb{R})^C = Sl(3, \mathbb{C}) \\
\mathbb{C} & \quad Sl(3, \mathbb{C})^C = Sl(3, \mathbb{C}) \times Sl(3, \mathbb{C}) \\
\mathbb{H} & \quad Sl(3, \mathbb{H})^C = Sl(6, \mathbb{C}) \\
\mathbb{O} & \quad 'Sl(3, \mathbb{O})^C' = E_6
\end{align*}
\]

where we have written ‘$Sl(3, \mathbb{O})^C$’ merely to be suggestive.

$\det$ tells us which elements of $\mathcal{H}$ are of less than full rank. One can also unambiguously define a notion of being rank one; either by taking $2 \times 2$ minors or by noting that under the $G$ action each $x \in \mathcal{H}$ is diagonalizable and one can take as the rank of $x$ the number of nonzero elements in the diagonalization of $x$.

Let

\[(3.5) \quad X := \mathbb{P}\{ \text{rank one elements of } \mathcal{H} \} = \mathbb{P}\{G \text{ orbit of any rank one matrix} \}.
\]

It is not difficult to check that $X = (A_\mathbb{R} \mathbb{P}^2)^C$. The four Severi varieties are exactly $X \subset \mathbb{P}(\mathcal{H})$ for $A_\mathbb{R} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (where again we treat $\mathcal{H}$ as a vector space over $\mathbb{C}$). Note that $\sigma(X) = \mathbb{P}\{\text{rank two elements}\} \neq \mathbb{P}(\mathcal{H})$.

§4 Second Fundamental Forms.

It is possible to define second fundamental forms in quite general settings. For the purposes of this paper, it will be useful to deal with the second fundamental form of a mapping to a projective space.

Let $Y$ be a (smooth) complex manifold and let $\phi : Y \to \mathbb{P}W$ be a smooth holomorphic mapping. (e.g. $\phi$ could be the restriction of a rational mapping to a neighborhood of a smooth regular point). Fixing $y \in Y$, one has the differential of $\phi$ at $y$;

\[(4.1) \quad \phi_\ast y : T_y Y \to T_{\phi(y)} \phi(Y) \subset T_{\phi(y)} \mathbb{P}W
\]

and the Gauss map of $\phi$, (where $\dim \phi(Y) = k$);

\[(4.2) \quad \gamma_\phi : Y \to G(k + 1, W) \\
y \mapsto T_{\phi(y)} \phi(Y) =: \overline{T}
\]
where $\hat{T}$ is the cone over the embedded tangent space and $G(k+1, W)$ denotes the Grassmanian of $(k+1)$-planes in $W$. Taking the derivative of $\gamma_\phi$, at $y$, one obtains a linear map

$$\gamma_{\phi \ast y}: T_y Y \to T_{\hat{T}} G(k+1, W) = \hat{T}^* \otimes W/\hat{T}$$

(4.3)

Let $\hat{\phi}(y) \subset \hat{T}$ denote the line corresponding to $\phi(y)$. Observe that

$$\forall v \in T_y Y, \quad \hat{\phi}(y) \subset \ker \gamma_{\phi \ast y}(v)$$

(4.4)

$$\ker \phi_{\ast y} \subseteq \ker \gamma_{\phi \ast y}.$$  

(4.5)

Thus $\gamma_{\phi \ast y}$ descends to an element of $(T_y Y/\ker \phi_{\ast y})^* \otimes (\hat{T}/\hat{\phi}(y))^* \otimes W/\hat{T}$. We may identify $T_y Y/\ker \phi_{\ast y} \simeq \phi_{\ast y}(T_y Y) \simeq T_{\phi(y)} \phi(Y) \subseteq T_{\phi(y)} \mathbb{P}W$. Moreover, since

$$T_{\phi(y)}^\ast \phi(Y) = (\hat{T}/\hat{\phi}(y))^* \otimes \hat{\phi}(y)$$

$$N_{\phi(y)} \phi(Y) = \hat{\phi}(y)^* \otimes W/\hat{T}$$

(4.6)

we obtain an element of $T_{\phi(y)}^\ast \phi(Y) \otimes T_{\phi(y)}^\ast \phi(Y) \otimes N_{\phi(y)} \phi(Y)$ and, in fact, because the Gauss map is already the derivative of a map and mixed partials commute, we obtain

$$II_{\phi, y} \in S^2(T_y Y/\ker \phi_{\ast y})^* \otimes N_{\phi(y)} \phi(Y)$$

(4.7)

the second fundamental form of the mapping $\phi$ at $y$. If $\phi$ is the inclusion map, we recover the second fundamental form of a subvariety.

Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be a general point. Let $II = II_{X,x}$ denote the second fundamental form at $x$. We will say a vector $v \in T = T_x X$ is $II$-generic if all the associated maps and spaces one constructs from $II$ and $v$ (see e.g. $II_v$ below) are locally of constant rank/dimension near $v$.

Fix a $II$-generic $v \in T$ and consider the map:

$$II_v: T \to N$$

$$w \mapsto II(v, w)$$

(4.8)

and the rational map:

$$ii: \mathbb{P}T \dashrightarrow \mathbb{P}N$$

$$[w] \mapsto [II(w, w)].$$

(4.9)
Proposition 4.10. Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety, let $x \in X$ be a general point and let $v \in T_x X$ be a $II$-generic vector. Then

\begin{equation}
\dim \tau(X) = n + \dim II_v(T) = n + \dim \text{ii}(\mathbb{P}T) .
\end{equation}

We give a proof of the first line of (4.11) in §9 (originally due to Terracini, [T] and rephrased in modern language in [GH]) and prove $\dim II_v(T) = \dim \text{ii}(\mathbb{P}T)$ in (6.6).

Considering the second fundamental form as a mapping $II^* : N^* \to S^2T^*$, let

\begin{equation}
|II| = \mathbb{P}(II^*(N^*)) \subset \mathbb{P}(S^2T^*) .
\end{equation}

§5. Second Fundamental Forms of the Severi Varieties

Let $Y \subset \mathbb{P}^{n-1}$ respectively denote $\emptyset, \mathbb{P}^1*\mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^3, S$, where respectively $n = 2, 4, 8, 16; *$ denotes disjoint union, $\mathbb{P}^1 \times \mathbb{P}^3$ occurs as the Segre, and $S$ is the 10 dimensional spinor variety obtained by taking the projectivized orbit of a highest weight vector in $\mathbb{C}^{16}$ under the Spin$(10, \mathbb{C})$ action.

The four Severi varieties have:

\begin{equation}
|II| = \text{the complete system of quadrics vanishing on } Y .
\end{equation}

If we write a Severi variety as $G/P$, the fastest way to see this is to observe that $Y$ must be an orbit of the $P$ action on $T_x(G/P)$. Now $\dim |II| = \text{codim}(G/P)$ since $II$ must surject onto $N$ (by (10.2)), so $Y$ must be small enough to have an $\frac{n}{2} + 2$ dimensional space of quadrics vanishing on it. But in each case here (except $n = 2$ which is easy) $Y$ is the projectivized orbit of a highest weight vector and therefore the smallest possible (nonempty) orbit. Alternatively, one may compute directly, see [GH] for the classical cases and the last chapter of [Hv] for a good model to study the $E_6/P$ case (actually [Hv] only deals with Spin$(9, \mathbb{R})$ on $\mathbb{R}^{16}$ but the argument for the case here is similar).

Another way to describe the second fundamental form is to use local affine coordinates. If we want to know the second fundamental form at $p$, we choose coordinates $[X_0, \ldots, X_N]$ of $\mathbb{P}^N$ such that $p = [1, 0, \ldots, 0]$, and affine coordinates $x^B = X_B/X_0$ where $X$ is locally describable as a graph over $x^1, \ldots, x^n$ and $T_p \simeq \{x^\alpha\}, 1 \leq \alpha, \beta \leq n$. In this case in some neighborhood of $(0, \ldots, 0)$,

$$x^\mu = x^\mu(x^\alpha) = q^\mu_{\alpha\beta} x^\alpha x^\beta + \text{terms of order three or greater}$$

where $n + 1 \leq \mu \leq n + a$, and $q^\mu_{\alpha\beta}$ are the coefficients of the second fundamental form (see [GH] for more details).
In the case of the Severi varieties, choose affine coordinates based at \( p \)
where
\[
p = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
and following the notation of (3.1), denote the affine coordinates \( u_1, u_2, u_3 \in A, r_2, r_3 \in C \) where the tangent space to \( p \) is \( \{u_1, u_2\} \) (the span is taken over \( C \)). Here we have (where \( \equiv \) means modulo terms of order three or greater)
\[
r_2(u_1, u_2) \equiv u_1 \bar{u}_1 \quad \text{as} \quad \det \begin{pmatrix}
1 & \bar{u}_1 \\
u_1 & r_2
\end{pmatrix} = 0
\]
\[
r_3(u_1, u_2) \equiv u_2 \bar{u}_2 \quad \text{as} \quad \det \begin{pmatrix}
1 & \bar{u}_2 \\
u_2 & r_3
\end{pmatrix} = 0
\]
\[
u_3(u_1, u_2) \equiv \bar{u}_2 u_1 \quad \text{as} \quad \det \begin{pmatrix}
1 & \bar{u}_1 \\
u_2 & u_3
\end{pmatrix} = 0
\]
where the last equation gives us one, two, four or eight quadratic forms. The determinants come from the vanishing of \( 2 \times 2 \) minors that must be zero to make the Severi variety consist only of rank one elements. (In fact, if one checks the other \( 2 \times 2 \) minors, one sees that the equations above are exact, i.e. all third and higher order terms vanish. The idea of the proof of Zak's theorem on Severi varieties given in this paper is to first show that any putative Severi variety must have one of the four fundamental forms in (5.1) and then to show all third and higher order invariants must be zero.)

In division algebra notation the second fundamental forms are
\[
|II| = \mathbb{P}\{u_1 \bar{u}_1, u_2 \bar{u}_2, \bar{u}_2 u_1\}.
\]
For example if \( A = \mathbb{O}^C \) and we write \( u = (u^0, \ldots, u^7) = u^0 + u^1 J_1 + \ldots + u^7 J_7 \) then
\[
|II| = \mathbb{P}\{(u_1^0)^2 + \ldots + (u_1^7)^2, (u_2^0)^2 + \ldots + (u_2^7)^2, u_1^0 u_2^0 + \ldots + u_1^7 u_2^7, u_1^0 u_2^1 - u_1^1 u_2^0 + u_1^2 u_3^2 - u_1^3 u_2^2 + \ldots + u_1^7 u_2^4, \ldots, u_1^0 u_2^7 + \ldots - u_1^4 u_2^1\}
\]
where we have used the octonionic multiplication table:
where if $J_1, \ldots, J_7$ are a basis of $\text{Imag}(\mathbb{O}^C)$ then the multiplication is to be read cyclicly and counter clockwise, e.g. $J_1J_2 = -J_2J_1 = J_3, J_2J_3 = J_1, J_1J_7 = J_4, J_4J_2 = -J_6$.

In fact, the original description of the Severi varieties used by Zak to prove his theorem was to consider $Y \subset \mathbb{P}^{n-1}$, and linearly embed $Y \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and to note that the image of the natural embedding by the linear system of quadrics on $\mathbb{P}^n$ vanishing along $Y$ of $\text{Bl}_Y \mathbb{P}^n \to \mathbb{P}^{\frac{4n}{2}+2}$ is the Severi variety.

§6. Properties of systems of quadrics with a (tangential) defect

Let $T$ be an $n$-dimensional vector space, $N$ an $a$ dimensional vector space with $a \leq n$, and let $II \in S^2T^* \otimes N$. We will write $II^*: N^* \to S^2T^*$ when we want to consider $II$ as a linear map. Assume $II^*$ is injective, which will be the case in our situation by (10.2). Let $|II| = \mathbb{P}(II^*(N^*)) \subseteq \mathbb{P}S^2T^*$, and let $\overline{II} = II^*(N^*) \subseteq S^2T^*$. Given $Q \in |II|$, let $[Q]$ denote the hypersurface in $\mathbb{P}T^*$ it determines. If $v \in T$ and $v \cdot Q = 0$, i.e. $[v] \in [Q]_{\text{sing}}$, we will say $Q$ annihilates $v$.

We will say $II$ has a (tangential) defect if for every $v \in T$ there is a $Q \in |II|$ annihilating $v$. If $X^n \subset \mathbb{P}^{n+a}$ is a variety with degenerate tangential variety, $x \in X$ is a general point and $a \leq n$, then $II_{X,x}$ has a defect (4.10). In what follows, we will fix a $II$-generic $v \in T$, which is natural, since in the geometric setting this amounts choosing a smooth point of $\tau(X)$ in $\tilde{T}_xX$ up to scalings.

Fixing a $II$-generic $v \in T$, consider the following linear subspace of $|II|$: 

(6.1) $\text{Ann}(v) = \{[Q] \in |II| \mid [v] \in [Q]_{\text{sing}}\}$ 

$= \mathbb{P}\{Q \in |\overline{II}| \mid v \cdot Q = 0\}$

the annihilator of $v$. Also consider 

(6.2) $\text{singloc}(\text{Ann}(v)) = \{v \in T \mid [v] \in [Q]_{\text{sing}} \forall [Q] \in \text{Ann}(v)\}$ 

$= \{v \in T \mid v \cdot Q = 0 \forall Q \in \text{Ann}(v)\}$,

the singular locus of the zero set of $\text{Ann}(v)$. Consider the linear map 

(6.3) $II_v: T \to N$ 

$w \mapsto II(v, w)$.

Note that 

(6.4) $\mathbb{P}(II^*(II_v(T)^\perp)) = \text{Ann}(v)$.
Lemma 6.5. \( \mathbb{P}\{v, \ker II_v\} \subseteq \text{singloc}(Ann(v)) \)

We will give two proofs of (6.5), one in frames in §11 and one that will follow from the following:

**Proposition 6.6.** Consider the rational map

\[ ii : \mathbb{P}T \longrightarrow \mathbb{P}N \]

\[ [v] \mapsto [II(v, v)]. \]

Let \( v \in T \) be \( II \)-generic.

1. Let \( w \in T \), and let \( \tilde{w} = v^* \otimes (w \mod v) \), so \( \tilde{w} \in T[v] \mathbb{P}T \), then

\[ (6.7) \quad ii_*[\tilde{w}] = II(v, v)^* \otimes (II(v, w) \mod \{II(v, v)\}). \]

In particular,

\[ (6.8) \quad \ker(ii_*[v]) = v^* \otimes (\ker II_v \mod \{v\}). \]

2. For all \( w_1, w_2 \in T \), let \( \tilde{w}_j = v^* \otimes (w_j \mod v, \ker II_v) \), so \( \tilde{w}_j \in T[v] \mathbb{P}T/\ker ii_*[v] \), then

\[ (6.9) \quad II_{ii_*[v]}(\tilde{w}_1, \tilde{w}_2) = II(v, v)^* \otimes (II(w_1, w_2) \mod II_v(T)). \]

where \( II_{ii_*[v]} \) is the second fundamental form of the mapping \( ii \) at \( [v] \) as in (4.7).

In particular, \( \text{singloc}[II_{ii_*[v]}] = v^* \otimes (\text{singloc}(Ann(v)) \mod \{v, \ker (II_v)\}) \)

Note that (6.8) proves the equivalence of the two lines in (4.11).

**Proof of (6.6).** Let \( v_t \) be a curve with \( v_0 = v \). We have

\[ (6.10) \quad ii(v_t) = II(v_t, v_t) \]

so

\[ (6.11) \quad \frac{d}{dt}ii(v_t)|_{t=0} \equiv 2II(v, v') \mod II(v, v) \]

where \( v'_0 = \frac{d}{dt}v_t|_{t=0} \), proving 1. To see 2, we compute \( II_{ii_*[v]} \) as in [GH];

\[ \frac{d^2}{(dt)^2}ii(v_t)|_{t=0} \equiv 2II(v, v'_0) + 2II(v'_0, v'_0) \mod II_v(T) \]

\[ \equiv 2II(v'_0, v'_0) \mod II_v(T) \]

polarizing (6.12) and putting in twists, one recovers \( II_{ii_*[v]} \). □
Proof of (6.5). Apply (4.5) with $\phi = ii$ and use (6.8) and (6.4). □

(6.5) is a statement about a special property of the subsystem of quadrics in $|II|$ annihilating a fixed vector. In general, one cannot expect to make any statements about all the quadrics in $|II|$, as it may be the case that a subsystem of $|II|$ already has a defect of the same size and therefore only this subsystem will have special properties. What we really want to study is the smallest subsystem of $|II|$ generated by quadrics annihilating the $II$-generic vectors.

Let $Z$ denote the (closure of the) image of the rational map $ii$. It may be the case that $Z$ is a cone. If so, its vertex is

\[(6.13) \quad V_Z = \cap_{z \in Z_{sm}} T_z Z \subset Z.\]

($Z_{sm}$ denotes the smooth points of $Z$.) Note that we may take the intersection over $\{z \in Z_{sm} | z = ii([v])$ for some $v \in T^gen\}$. Here and in what follows, $T^gen$ denotes the space of $II$-generic vectors.

**Proposition 6.14.** $II^*(\hat{V}_Z^\perp)$ is the smallest subsystem of $|\hat{II}|$ with the same size tangential defect.

**Proof.** By (6.7), $\hat{T}_{ii([v])}Z = II_v(T) \subset N$. So, $\hat{\text{Ann}}(v) = II^*(II_v(T)^\perp) = II^*((\hat{T}_{ii([v])}Z)^\perp)$ and therefore

$$\text{span}_{v \in T^gen} \hat{\text{Ann}}(v) = II^*((\cap_{v \in T^gen} \hat{T}_{ii([v])}Z)^\perp).$$ □

The remainder of the results in this section can be proved without using moving frames, but already our proof of (6.16) without frames was quite messy so we will only present proofs in frames and defer them to §11. Readers particularly allergic to frames may wish to write their own proofs of what follows. See [D] for a frame free argument of a slightly incorrect version of (6.16). (One mistake is that he implicitly assumes $\gamma_Z$ is nondegenerate.)

Let

\[(6.15) \quad F_v = \gamma_Z^{-1}(\gamma(ii([v])))\]

i.e., $F_v$ is the fiber of the Gauss map of $Z$ through $ii([v])$.

**Lemma 6.16.** $\text{singloc}(\text{Ann}(v)) \subseteq \text{baseloc}II^*(F_v^\perp)$.

By (6.5) we may consider the isomorphism

\[(6.17) \quad \hat{II}_v : T/\text{singloc}(\text{Ann}(v)) \to II_v(T)/\hat{F}_v.\]

By (6.16), for all $w \in \text{singloc}(\text{Ann}(v))$ we have

\[(6.18) \quad II_w(\text{singloc}(\text{Ann}(v))) \mod \hat{F}_v \subseteq II_v(\text{singloc}(\text{Ann}(v))) \mod \hat{F}_v\]
so the maps

\[ II'_w : T/\text{singloc}(\text{Ann}(v)) \to II_v(T)/\hat{F}_v \]

are well defined. In fact it will be useful to consider these as endomorphisms of \( T/\text{singloc}(\text{Ann}(v)) \). To this end, for \( w \in \text{singloc}(\text{Ann}(v)) \), let

\[ \phi_w = (II'_{\hat{w}})^{-1} \circ II'_w : T/\text{singloc}(\text{Ann}(v)) \to T/\text{singloc}(\text{Ann}(v)) \]

If \( Z \) is a hypersurface, then \( T/\text{singloc}(\text{Ann}(v)) \) comes equipped with a quadratic form defined up to scale, namely \( \text{Ann}(v) \). We will show that in this case, for \( w \in \ker II_v \) that \( \phi_w \in \text{so}(T/\text{singloc}(\text{Ann}(v)), \text{Ann}(v)) \). (Note that by construction, \( \phi_v = \text{Id}_{T/\text{singloc}(\text{Ann}(v))} \).) In fact it will be useful to choose a quadratic form \( P_v \in \hat{\text{Ann}}(v) \) so we do so.

**Lemma 6.21.** If \( Z \) is a hypersurface, then for all \( w \in \ker II_v \),

\[ \phi_w \in \text{so}(T/\text{singloc}(\text{Ann}(v)), P_v). \]

In general, if a variety \( Y \) is a cone with vertex \( V_Y \), then the fibers of \( \gamma_Y \), the Gauss map of \( Y \), all contain \( V_Y \). If \( Y \) is a cone over a variety having a nondegenerate Gauss map, then the dimension of the general fiber of \( \gamma_Y \) has dimension equal to \( \dim V_Y + 1 \). This will be the case if and only if \( Y \) is not “built out of a tangent developable”, in the language of ([GH], 2.27). Recall that a variety has a degenerate Gauss map if and only if it is ruled by linear spaces along which the embedded tangent space is constant. Two types of varieties having this property are cones and tangent developables. If one takes a cone over a tangent developable or visa versa, the resulting variety will still have a degenerate Gauss map, and is an example of a variety “built out of a tangent developable”.

From now on, assume \( Z \) is a hypersurface and that \( \dim V_Z + 1 = \dim F_v \), i.e., that \( Z \) is not built out of a tangent developable. By (6.16),

\[ II^*(V_Z^\perp) |_{\{v, \ker II_v\}} \]

is a well defined quadric up to scale. Let \( Q_v \in II^*(V_Z^\perp) |_{\{v, \ker II_v\}} \) be the unique quadric such that \( Q_v(v, v) = 1 \).

**Lemma 6.23.** Assume \( Z \) is a hypersurface such that \( \gamma_Z \) has fibers of dimension \( \dim V_Z - 1 \). (This condition is automatic if \( \gamma_Z \) is nondegenerate.) For all \( w_1, w_2 \in \{v, \ker II_v\} \),

\[ \phi_{w_1} \circ \phi_{w_2} + \phi_{w_2} \circ \phi_{w_1} + 2Q_v(w_1, w_2)\text{Id}_{T/\text{singloc}(\text{Ann}(v))} = 0 \]

where \( \phi_w \) is as in (6.20), \( Q_v \) is the quadric defined above, and \( \text{Id}_{T/\text{singloc}(\text{Ann}(v))} \) is the identity map on \( T/\text{singloc}(\text{Ann}(v)) \). In particular, \( T/\text{singloc}(\text{Ann}(v)) \)
is a \( Cl(\{ v, ker II_v \}, Q_v) \) module, where \( Cl(\{ v, ker II_v \}, Q_v) \) denotes the Clifford algebra.

Proof of “In particular”. The fundamental lemma of Clifford Algebras ([Hv] or [LM]) says: Given a linear map \( \phi : V \rightarrow A \) from a vector space \( V \) with inner product \( Q \) into an associative algebra \( A \) with unit, if

\[
\phi(x)\phi(y) + \phi(y)\phi(x) + 2IdQ(x,y) = 0
\]

then \( \phi \) has a unique extension \( \tilde{\phi} : Cl(V, Q) \rightarrow A \). Now let \( V = \{ v, ker II_v \}, Q = Q_v \), and \( A = \text{End}(T/singloc(Ann(v))) \). □

In the geometric setting, (6.23) implies

**Theorem 6.26 (The Clifford algebra structure).** Let \( X^n \subset \mathbb{CP}^{n+a} \) be a variety with degenerate tangential variety that is a hypersurface. Let \( x \in X \) be a general point and let \( v \in T_x X \) be a II-generic vector. Assume that the hypersurface \( Z \subset P_{N_x}X \) defined by \( II_x \) as above (6.13) is such that the Gauss map of \( Z, \gamma_Z \), has fibers of dimension \( \text{dim} V_Z - 1 \), where \( V_Z \) denotes the vertex of \( Z \) as in (6.13). (This condition is automatic if \( \gamma_Z \) is nondegenerate.) Then the space \( \{ v, ker II_v \} \) comes equipped with a quadratic form \( Q \) and \( T_x X/singloc(Ann(v)) \) is a \( Cl(\{ v, ker II_v \}, Q) \) module, where \( Cl(\{ v, ker II_v \}, Q) \) denotes the Clifford algebra.

In the critical codimension case of \( a = \frac{n}{2} + 2 \), the vertex dimension hypothesis in (6.26) is satisfied when \( X \) is smooth. One might hope to show the vertex dimension hypothesis in (6.26) always must hold when \( X \) is smooth using (7.1) below. Such a condition would have consequences for the admissible values of tangential defects.

### §7. Rank Restrictions

**Theorem 7.1 (Rank restrictions).** Let \( X^n \subset \mathbb{P}^{n+a} \) be a smooth variety with degenerate secant variety that is a hypersurface. Let \( x \in X \) be a general point. Let \( r \) be the maximum rank of a quadric in \( |II|_x \) annihilating a II-generic vector. Then

\[
r \geq n - a + 2.
\]

Since \( \text{dim} (singloc(Ann(v))) = n - r \), and \( \text{dim} (ker II_v) = n - a + 1 \), it follows that

\[
(7.2) \quad \text{dim} F_v = \text{dim} (singloc(Ann(v)))/\text{ker} II_v - 1 = a - r - 2
\]

where \( F_v \) is as in (6.15)
Corollary 7.3. \( X^n \subset \mathbb{P}^{n+a} \) be a smooth variety with degenerate secant variety that is a hypersurface. Let \( x \in X \) be a general point and \( v \in T_xX \) be a \( II \)-generic vector, and let \( F_v \) be as in (6.15). Then

\[
(7.4) \quad a \geq \frac{n}{2} + 2 + \frac{1}{2} \dim F_v.
\]

(7.3) gives a slightly refined version of (1.1) because one can always project to the case \( \sigma(X) \) is a hypersurface.

Proof of (7.1). Consider the incidence correspondences:

\[
(7.5) \quad \mathcal{I} := \{(x, H) | \tilde{T}_xX \subset H\} \subset X \times X^*
\]

\[
(7.6) \quad \Delta := \{(x, H) | \tilde{T}_xX \subset H \text{ and } Q_H = \text{Ann}(v) \text{ for some } v \in T^\text{gen}_xX\} \subseteq \mathcal{I}
\]

\[
X \quad X^* \quad X^* \Delta
\]

where \( X^* \) is the dual variety of \( X \), \( Q_H \in |II|_x \) corresponds to the image under \( II^* \) of the line in \( N^* \) determined by \( H \), \( T^\text{gen}_x \) is the space of \( II \)-generic vectors, and \( X^*_\Delta \subseteq X^* \) is defined to be \( \rho_\Delta(\Delta) \). Since \( \sigma(X) \) is degenerate, we can apply Zak’s theorem on tangencies [Z1] to the image of a projection of \( X \) to find that

\[
(7.7) \quad a - 2 \geq \dim \pi(\rho^{-1}(H)) \geq \dim \pi_\Delta(\rho_\Delta^{-1}(H)).
\]

Take \( H \) such that \((x, H) \in \Delta\) is a general point of \( \Delta \) in the sense that both \( \pi_\Delta \) and \( \rho_\Delta \) are smooth at \((x, H)\) and of maximal rank, and \( x \) is a general point of \( X \).

Note that

\[
(7.8) \quad \dim \Delta = n + r,
\]

where by this we mean the local dimension near \((x, H)\), because \( \dim \mathbb{P}(II^*(\hat{F}_v^\perp)) = r \) (\( II^* \) is injective by (10.2).)

We claim that

\[
(7.9) \quad \dim X^*_\Delta = 2r
\]
(where again, we mean the local dimension near \(H\)). To understand (7.9) heuristically, recall that \(\dim X^* = r' + (a - 1)\) where \(r'\) is the rank of a generic quadric in the second fundamental form of \(X\) at a general point. The way to think of \(\dim X^*\) is that there are \(a - 1\) “vertical” contributions to the dimension (all the hyperplanes tangent to a point) and \(r'\) “horizontal” contributions (where if \([Q_H] \in |II|\) is the quadric corresponding to a tangent hyperplane \(H\), then \(r' = \dim (T/singloc(Q_H))\)). The same is true for \(X^* \Delta\), there is a contribution of \(r\) “vertical” directions (corresponding to \(P_{II^*}(\hat{F}_v^\perp)\)) and \(r\) “horizontal” ones corresponding to \(T/singloc(Ann(v))\). (7.9) will be proved in §13 using moving frames.

Putting everything together, we have

(7.10) \(a - 2 \geq \dim \pi^\Delta(\rho_\Delta^{-1}(H)) = \dim \Delta - \dim X^*_\Delta = (n + r) - 2r = n - r.\)

§8. Moving frames

We will briefly review some notation here, the reader is referred to [L1], [L2] for more details. Let \(X^n\) be a (patch of a) smooth submanifold of \(P^n = \mathbb{P}^{n+a}\) and let \(x \in X\). Let \(\mathcal{F}^1\) be the bundle of all frames \(f = (A_0, \ldots, A_{n+a})\) (i.e. bases of \(V\)) adapted to the filtration \(\hat{x} \subset \hat{T} \subset V\), where \(\hat{x} \subset V\) is the line determined by \(x\) and \(\hat{T}\) is the deprojectivization of the embedded tangent space to \(X\) at \(x\). Write \(f = (A_0, A_0, A_\alpha, A_\mu)\) for an element of \(\mathcal{F}^1\), where \(1 \leq \alpha, \beta \leq n\), \(n + 1 \leq \mu, \nu \leq n + a\) where \([A_0] = x\), and \(\{A_0, \ldots, A_n\} = \hat{T}\). The motions in the fiber over \([A_0]\) are \(f \mapsto fg\) where

(8.1) \[
g = \begin{pmatrix}
g_0^0 & g_0^\beta & g_0^\nu \\
0 & g_\alpha^\beta & g_\alpha^\nu \\
0 & 0 & g_\mu^\nu
\end{pmatrix}, \quad \det g \neq 0.
\]

Let \(\Omega = f^{-1}df\) denote the Maurer-Cartan form. We write \(\Omega\) as

(8.2) \[
\Omega = \begin{pmatrix}
\omega_0^0 & \omega_0^\beta & \omega_0^\nu \\
\omega_\alpha^\beta & \omega_\alpha^\nu & \omega_\alpha^\alpha \\
0 & \omega_\beta^\mu & \omega_\mu^\nu
\end{pmatrix}
\]

where the entries of \(\Omega\) are one-forms on \(\mathcal{F}^1\) and we use the notation \(\omega^B\) for \(\omega_0^B\). The Maurer-Cartan equation

(8.3) \[
d\Omega = -\Omega \wedge \Omega
\]

will be used repeatedly in what follows. For example, using (8.3) and the Cartan lemma applied to \(0 = d\omega^\mu\) implies \(\omega_\beta^\mu = q_\beta^\mu \omega^\alpha\) with \(q_\alpha^\beta = q_\beta^\mu\). This gives an alternative definition of the second fundamental form because

(8.4) \[
\omega_\beta^\mu \omega^\beta \otimes A_\mu, \mod \hat{T} = q_\alpha^\beta \omega^\alpha \omega^\beta \otimes A_\mu, \mod \hat{T}
\]
descends (after twisting by $A_0^\beta$) to a well defined section of $S^2T^* \otimes N$ over $X$ which is $II$.

Similarly, the quantity

$$\omega^\nu_\mu \omega^\alpha_\beta \omega^\beta_\gamma A_\nu \mod \hat{T}^\alpha$$

(where $\hat{T}^\alpha = \{ \hat{T}, II(S^2 T) \}$) descends after twisting to a well defined section of $S^3T^* \otimes (N/II(S^2 T))$ over $X$, called the third fundamental form and denoted $III$.

Following [GH], for varieties with degenerate tangential varieties, or varieties where $a > n$, one can define a refinement of $III$. Namely, fixing a $II$-generic vector $v \in T$, define the refined third fundamental form

$$III^v \in S^3(\text{singloc}(\text{Ann}(v)))^* \otimes N/II_v(T)$$

by noting the quantity

$$\omega^\nu_\mu \omega^\alpha_\beta \omega^\gamma_\alpha \mid_{S^3(\text{singloc}(\text{Ann}(v)))} \otimes A_\nu \mod \{ \hat{T}, II_v(T) \}$$

descends (after twisting) to be well defined over $X$. $III^v$ is well defined as if $w \in \text{singloc}(\text{Ann}(v))$, then $II_w(T) \subseteq II_v(T)$. One can recover $III$ from knowing $III^v$ for all $v \in T$.

Differentiation (via (8.3)) of the equation

$$\omega^\beta_\mu - q^\mu_\alpha \omega^\alpha = 0$$

provides another third order invariant, which has been called the cubic form.

$$\partial II = (-dq^\mu_\alpha - q^\mu_\alpha \omega_0^0 + q^\mu_\beta \omega_\beta^0 + q^\mu_\beta \omega_\beta^\gamma - q^\nu_\gamma \omega_\nu^\gamma) \omega^\alpha_\beta \omega^\gamma_\alpha \otimes (A_\mu \mod \hat{T})$$

$$= r^\mu_\alpha \omega^\alpha_\beta \omega^\gamma_\gamma \otimes (A_\mu \mod \hat{T}).$$

The Cartan Lemma is what allows us to write the second line, i.e. that there are functions $r^\mu_\alpha \beta \gamma$, symmetric in their lower indices, such that

$$r^\mu_\alpha \beta \gamma \omega^\gamma = -dq^\mu_\alpha - q^\mu_\alpha \omega_0^0 - q^\nu_\beta \omega_\gamma^\nu - q^\nu_\alpha \omega_\gamma^\beta - q^\nu_\alpha \gamma \omega_\nu^\gamma - q^\nu_\alpha \gamma \omega_\nu^\beta.$$ 

Although $\partial II$ is well defined on $\mathcal{F}^1$ it varies on the fiber as follows: If we move along the fiber by motions

$$A_\beta \rightarrow A_\beta + g_\beta^0 A_0$$

then the expression (8.9) will change by

$$\partial II \rightarrow \partial II + \mathcal{E}_\alpha \beta \gamma g^0_\alpha \gamma q^\mu_\alpha \omega^\alpha_\beta \omega^\gamma_\gamma \otimes (A_\mu \mod \hat{T})$$
(where \( S \) denotes cyclic summation). If we move in the fiber by motions

\[
A_\mu \rightarrow A_\mu + g^\delta_\mu A_\delta
\]

then the expression (8.9) will change by

\[
\partial II \rightarrow \partial II - \mathcal{G}_{\alpha\beta\gamma} q^\mu_{\alpha\delta} q^\nu_{\beta\gamma} \omega^\alpha \omega^\beta \omega^\gamma \otimes (A_\mu \mod \hat{T}).
\]

If we differentiate (8.10), we obtain a fourth order invariant,

\[
\partial^2 II = (-dr^\mu_{\alpha\beta\gamma} - 2r^\mu_{\alpha\beta\gamma} \omega^0_\mu - r^\nu_{\alpha\beta\gamma} \omega^\mu_\nu + \mathcal{G}_{\alpha\beta\gamma} r^\mu_{\alpha\beta\gamma} \omega^\delta_\gamma \\
- \mathcal{G}_{\alpha\beta\gamma} q^\mu_{\alpha\delta} q^\nu_{\beta\gamma} \omega^\delta_\nu - \mathcal{G}_{\alpha\beta\gamma} q^\mu_{\alpha\beta} \omega^0_\gamma) \omega^\alpha \omega^\beta \omega^\gamma \otimes A_\mu \mod \hat{T} \\
= r^\mu_{\alpha\beta\gamma\delta} \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta \otimes A_\mu \mod \hat{T}
\]

where again the Cartan lemma says that there are functions \( r^\mu_{\alpha\beta\gamma\delta} \), symmetric in their lower indices, having the property that

\[
r^\mu_{\alpha\beta\gamma\delta} \omega^\delta = -dr^\mu_{\alpha\beta\gamma} - 2r^\mu_{\alpha\beta\gamma} \omega^0_\mu - r^\nu_{\alpha\beta\gamma} \omega^\mu_\nu + \mathcal{G}_{\alpha\beta\gamma} r^\mu_{\alpha\beta\gamma} \omega^\delta_\gamma \\
- \mathcal{G}_{\alpha\beta\gamma} q^\mu_{\alpha\delta} q^\nu_{\beta\gamma} \omega^\delta_\nu - \mathcal{G}_{\alpha\beta\gamma} q^\mu_{\alpha\beta} \omega^0_\gamma.
\]

\( \partial^2 II \) also varies as one moves along the fiber (see [L2] to see how it varies).

§9. Dimension of the tangential variety

The dimension of \( \tau(X) \) is the dimension of its tangent space at a point. Since we have adapted the frame bundle such that \( \{A_0, A_\alpha\} = \hat{T}_{[A_0]} \) we may take \([A_1]\) as a typical element of \( \tau(X) \). By the defining equation of the Maurer-Cartan form,

\[
dA_1 \equiv \omega^0_1 A_0 + \omega^\rho_1 A_\rho + \omega^\mu_1 A_\mu \mod \{A_1\}
\]

and \( \dim T_{[A_1]} \tau(X) = \{ \) the number of independent 1-forms in (9.1) \( \} \). Over \( X \), \( \{\omega^0_1, \omega^\mu_1\} \) are independent so we only need to know the number of independent one-forms among the \( \omega^\mu_1 \), but (up to twisting)

\[
\omega^\mu_1 \otimes (A_\mu \mod \hat{T}) \equiv II_{A_1}.
\]

where \( A_1 = A^*_0 \otimes (A_1 \mod A_0) \in T^* \).

(9.2) implies

\[
\dim \tau(X) = n + \dim II_v(T)
\]
where \( v \) is any \( II \)-generic tangent vector at a general point of \( X \), proving (4.10). Let \( a_0 := \max_{v \in T} (\dim(II_v(T))) \), so \( \dim(\tau(X)) = n + a_0 \).

Given \( X^n \subset \mathbb{P}^{n+a} \), \( \tau(X) \) is degenerate if \( \dim(\ker II_v) > \max\{0, n - a\} \) for generic (and therefore all) \( v \in T \). In other words, \( \tau(X) \) is degenerate in the case \( a \leq n \) if for all \( [v] \in \mathbb{P}^T \) there exists a \( [Q] \in |II| \) such that \( [v] \in [Q]_{\text{sing}} \), and \( \tau(X) \) is degenerate in the case \( a > n \) if there exists an \((a - n)\)-dimensional family of quadrics singular at \([v] \). By comparison, the Gauss map is degenerate if there exists an \( [v] \in \mathbb{P}^T \) such that every \( [Q] \in |II| \) is singular at \([v] \). Of course \( \tau(X) \) is degenerate if the Gauss map of \( X \) is degenerate, but the Gauss map of a smooth variety is nondegenerate unless \( X \) is a linear space (see [GH] or [L1]).

§10. Dimension of the Secant Variety

Although by the connectedness theorem it is sufficient to work with the tangential variety, we will gain additional information by studying the secant variety as well. We now infinitesimally compute the dimension of the secant variety \( \sigma(X) \). The computation is a straightforward generalization of the computation in [GH] of the case where the codimension is large. Let \([A_0]\) and \([B_0]\) denote points of \( X \). Points of \( \sigma(X) \) are of the form \([A_0 + sB_0]\) where \( s \in \mathbb{C} \). To find the dimension of \( \sigma(X) \) we compute the number of linearly independent entries in

\[
d(A_0 + sB_0) \equiv \omega_0^0 A_0 + \omega^\alpha A_\alpha + s(\phi_0^0 A_0 + \phi_0^\alpha A_\alpha) + dsB_0 \mod(A_0 + sB_0)
\]

where \( \phi^\beta_\alpha \) are the Maurer Cartan forms over \( B_0 \).

We see the dimension of \( \sigma(X) \) is one less than the number of linearly independent vectors among

\[
A_0, A_\alpha, B_0, B_\beta.
\]

(This gives an infinitesimal proof of Terracini’s lemma (see [FL]) that if \( X^n \subset \mathbb{P}^N \) is smooth \( x, y \in X \) and \( p \in \overline{xy} - X \) are general points then \( \tilde{T}_p \sigma(X) = \text{Span}(\tilde{T}_x X, \tilde{T}_y X) \).) Now we will use the assumption that \( X \) is smooth and connected. Let \((B_0(t), \ldots, B_N(t))\) be a coframing over an arc \( B_0(t) \) such that \((B_0(0), \ldots, B_N(0)) = (A_0, \ldots, A_N)\) and \( B_0(0)' = v \). Let \( \Lambda_0 = \{A_0, \ldots, A_n\} \). Expanding \( B_0(t), B_\alpha(t) \) in Taylor series we have:

\[
B_0(t) = A_0 + tv + \frac{t^2}{2} \frac{\partial^2 B_0}{\partial v \partial v} \big|_{t=0} + \frac{t^3}{3!} \frac{\partial^3 B_0}{\partial v^3} \big|_{t=0} + O(t^4)
\]

\[
B_\alpha(t) = A_\alpha + t \frac{\partial B_\alpha}{\partial v} \big|_{t=0} + \frac{t^2}{2} \frac{\partial^2 B_\alpha}{\partial v^2} \big|_{t=0} + \frac{t^3}{3!} \frac{\partial^3 B_\alpha}{\partial v^3} \big|_{t=0} + O(t^4).\]

Now \( \mod(\Lambda_0) \),

\[
\frac{\partial^2 B_0}{\partial v \partial v} \big|_{t=0} \equiv II(v, v)
\]

\[
\frac{\partial B_\alpha}{\partial v} \big|_{t=0} \equiv II(v, A_\alpha)
\]
and mod$\{\Lambda_0, II_v(T)\}$,

$$\frac{\partial^3 B_0}{\partial v \partial v \partial v} \bigg|_{t=0} \equiv III^v(v, v, v)$$

$$\frac{\partial^2 B_\alpha}{\partial v \partial v} \bigg|_{t=0} \equiv III^v(v, v, A_\alpha).$$

(where $A_\alpha = A_0^* \otimes (A_\alpha \text{ mod } A_0) \in T$). Higher order terms in the series cannot contribute to the dimension of $\sigma(X)$ because either the lower order terms (together with $\Lambda_0$) span a space of maximal dimension and $\sigma(X)$ is nondegenerate, or $III^v \equiv 0$ and since the higher fundamental forms lie in the prolongation of the lower fundamental forms, they are zero so the higher order terms cannot contribute any new directions. So letting $v \in T$ be $II$-generic we obtain

$$(10.1) \quad \dim \sigma(X) = \left\{ \begin{array}{ll}
n + \dim II_v(T) & \text{if } III^v(v, v, v) = 0 \\
n + \dim II_v(T) + 1 & \text{if } III^v(v, v, v) \neq 0 \end{array} \right..$$

(10.1) combined with (9.3) and (1.3) implies:

**Proposition 10.2.** Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with degenerate secant variety. Then for all $II$-generic vectors $v \in T$, $III^v = 0$ (and therefore $III = 0$). Conversely, if $X^n \subset \mathbb{P}^{n+a}$ is a variety and $III_X$ is not identically zero, then $\sigma(X)$ is nondegenerate.

**Theorem 10.3.** Let $Y \subset \mathbb{P}V$ be a variety and let $X = v_d(Y) \subset \mathbb{P}S^dV$ be the Veronese re-embedding. If $d > 2$ or $d = 2$ and $Y$ is not a linear subspace of $\mathbb{P}V$, then $\sigma(X)$ is nondegenerate.

**Proof.** By ([L2], 3.16), If $d > 2$ or $d = 2$ and $Y$ is not a linear space, then $III_{v_d(Y)} \neq 0$.

(10.3) extends a result of Roberts that follows from Proposition 2 of [R], where (10.3) is proved for $d > 2$. It was also shown by Kaji, [K], that (10.3) holds for homogeneous varieties.

One can describe the dimensions of higher secant varieties in a similar manner to (10.1) using fundamental forms of possibly higher order, and make similar apriori statements like (10.3). The general expression for $\sigma_k(X)$ is complicated, but if $III^v \equiv 0$ it becomes

$$(10.4) \quad \dim \sigma_k(X) = n + \dim \{II_{v_1}(T), \ldots, II_{v_{k-1}}(T)\}$$

where $v_1, \ldots, v_{k-1}$ are $II$-generic and in general position. (The notation is such that $\sigma(X) = \sigma_2(X)$.) In particular we recover the following result of Zak and Fantechi [F]:

DEGENERATE SECANT VARIETIES
Theorem 10.5. Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety not contained in a hyperplane with degenerate secant variety of dimension $n + a_0$. Let $\sigma_k(X)$ denote the closure of the union of all secant $\mathbb{P}^{k-1}$'s of $X$. Then

$$\dim \sigma_k(X) \leq n + (k-1)a_0.$$ 

This estimate enables one to recover the other valid results of $[Z2]$, e.g. Theorem 3. Notice that to have $\dim \sigma_k(X) = n + (k-1)a_0$ is rather difficult, so it might be interesting to study cases where equality occurs.

§11. Proofs of statements in §6

Although to keep notation consistent we will continue to work on frame bundles over $X$, all the results in this section are just about systems of quadrics with defects. This is reflected in that we always work modulo the semi-basic forms to prove the statements in §6, which is the same as saying that we are working with a fixed system of quadrics.

We now assume $v = A_1$ varies over $II$-generic vectors and adapt frames to reflect the structure described in §6. (Recall that we are using the notation $A_\alpha = A_0^* \otimes (A_\alpha \mod A_0)$.) Introduce the index ranges $a_0 + 1 = n - \dim (\ker II_v) + 1 \leq \epsilon, \delta \leq n$, $a_0 + 1 \leq h \leq a$ and restrict to frames where

(11.1) $\text{Ann}(v) = \{ \omega_{n+h}^\alpha \omega^\alpha \}$

(11.2) $\ker II_v = \{ A_\epsilon \}$.

(11.1) implies

(11.3) $\omega_{1+h} = 0$.

Introduce additional index ranges $2 \leq \rho, \sigma \leq n$. Using (8.10) (i.e. differentiating $\omega_1^{n+h}$), we have

(11.4) $r_{1\epsilon\beta}^{n+h} \omega^\beta = q_{\epsilon\rho}^{n+h} \omega_1^\rho$

Since the forms $\omega_1^\rho$ are all linearly independent, and independent of the semi-basic forms (since we are allowing $A_1$ to range over all $II$-generic vectors), (11.4) implies

(11.5) $q_{\epsilon\rho}^{n+h} = 0 \ \forall h, \epsilon, \rho$

which reproves (6.5). For later use we note that (11.4) also implies

(11.6) $r_{1\epsilon\beta}^{n+h} = 0 \ \forall h, \epsilon, \beta$. 

We reduce the frame bundle further by requiring

\begin{align}
\text{singloc}(\text{Ann}(v)) &= \{A_1, A_s, A_\epsilon\} \\
T/\text{singloc}(\text{Ann}(v)) &= \{A_j\} \mod \{A_1, A_s, A_\epsilon\}
\end{align}

where \(2 \leq i, j, k, l \leq r + 1, r + 2 \leq s, t \leq a_0\).

From now on, we will abuse notation, using \(A_B\) to denote \(A_{|B|}\).

Let \(1 \leq \lambda \leq a - 1\) so

\begin{align}
\text{singloc}(\text{Ann}(v)) &= \{A_\lambda\} \mod \{A_\epsilon\} = \{A_1, A_j, A_s\} \mod \{A_\epsilon\} = T/\ker (II_v).
\end{align}

Adapt frames to the map \(II_v = II_{A_1}\) such that

\begin{align}
II_{A_1}(A_{\lambda}) &= A_{n+\lambda}
\end{align}

On our bundle adapted to (11.1), (11.2), (11.7), (11.8), and (11.10), which we will call \(F^2\),

\begin{align}
\omega_1^{n+\lambda} &= \omega^\lambda.
\end{align}

Our adaptations have the effect that

\begin{align}
\{[Q] \in |II| \mid [v] \in [Q]\} &= \mathbb{P}\{\omega_\alpha^{n+j} \omega^\alpha, \omega_\alpha^{n+s} \omega^\alpha, \omega_\alpha^{n+h} \omega^\alpha\}
\end{align}

\begin{align}
F_v &= \mathbb{P}\{A_{n+1}, A_{n+s} \mod \hat{T}\}
\end{align}

\begin{align}
II^*(F_v^\perp) &= \{\omega_\alpha^{n+j} \omega^\alpha, \omega_\alpha^{n+h} \omega^\alpha\}
\end{align}

Continuing to use (8.10), i.e. to take third derivatives, we have

\begin{align}
r_{1\alpha}^{n+h} \omega^\alpha &= -\omega_1^{n+h} \\
r_{1\alpha}^{n+h} \omega^\alpha &= -\omega_{n+j}^{n+h} + q_{ij}^{n+h} \omega^i \\
r_{1s}^{n+h} \omega^\alpha &= -\omega_{n+s}^{n+h}
\end{align}

Introduce further index ranges \(\xi, \eta \in \{1, s, \epsilon\}\), i.e., let \(A_\xi\) be such that

\begin{align}
\{A_\xi\} &= \text{singloc}(\text{Ann}(v)).
\end{align}

Proof of (6.16). Using (8.10) again

\begin{align}
r_{\xi\eta}^{n+h} \omega^\beta &= -q_{\xi\eta}^{n+\lambda} \omega_1^{n+h}.
\end{align}
Substituting (11.15) into (11.17) implies
\begin{equation}
0 \equiv -q^{n+j}_{\xi\eta} q^{n+h}_{ji} \omega^i_1 \mod \{\omega^\beta\} \forall h.
\end{equation}
Since \(\text{singloc}(\{q^{n+h}_{ij} \omega^i_1 \omega^j\}|_{A_k}) = 0\), (11.18) implies
\begin{equation}
q^{n+j}_{\xi\eta} = 0 \forall j, \xi, \eta
\end{equation}
Comparing (11.14), (11.16) and (11.19) proves (6.16). \(\square\)

**Proof of (6.21).** Restrict to the case \(\tau(X)\) is a hypersurface (so \(a_0 = a - 1, h = a\) and \(1 \leq \lambda \leq a - 1\)). Continuing to use (8.10),
\begin{align}
r^{n+a}_{ke\beta} \omega^\beta &= -q^{n+a}_{ke} \omega^{n+a}_{n+k} - q^{n+t}_{ke} \omega^{n+a}_{n+t} + \omega^k_e \\
r^{n+j}_{k\epsilon\beta} \omega^\beta &= \omega^j_e + q^{n+j}_{ek} \omega^k_1
\end{align}
Combining (11.20), (11.21) with (11.15) yields
\begin{align}
\omega^j_e &\equiv q^{n+j}_{ke} \omega^k_1 \mod \{\omega^n\} \\
q^{n+a}_{ij} \omega^j_e &\equiv -q^{n+k}_{ie} \omega^{n+a}_{n+k} \mod \{\omega^n\} \\
&\equiv -q^{n+k}_{ie} (q^{n+a}_{kl} \omega^l_1) \mod \{\omega^n\}
\end{align}
Combining (11.22), (11.23) yields
\begin{equation}
q^{n+a}_{ij} q^{n+j}_{ie} + q^{n+k}_{ie} q^{n+a}_{kj} = 0.
\end{equation}
Here \(P_v = \omega^{n+a}_{i} \omega^a\). We restrict to frames where
\begin{equation}
q^{n+a}_{ij} = \delta_{ij}
\end{equation}
and (11.24) becomes
\begin{equation}
q^{n+j}_{ie} + q^{n+i}_{je} = 0
\end{equation}
proving (6.21). \(\square\)

**Proof of (6.23).** By the assumption that \(Z\) is not built out of a tangent developable, we can make a further adaptation that \(\mathbb{P}\{A_{n+s}\} = V_Z\), i.e. that \(\{A_{n+s}\} \mod \hat{T}\) is a fixed subspace as \(v\) varies in \(T_{\text{gen}}\). We need to work with the subsystem \(\Pi^*(V_Z^\perp)\). The notationally easiest way to do this is just set \(\{A_{n+s}\} = \{0\}\) in everything above. There is a potential for confusion here, because for the following argument only, \(r + 2 \leq \epsilon, \delta \leq n\), i.e. \(\{A_1, A_\epsilon\}\) will denote \(\text{singloc}(\text{Ann}(v))\) where \(\{A_1, A_\epsilon, A_\delta\}\) denoted that same space above. On the smaller system \(\Pi^*(V_Z^\perp)\), the spaces \(\text{singloc}(\text{Ann}(v))\) and \(\{v, \ker \Pi_v\}\) coincide.
Again, using (8.10),

$$r_{11 \beta}^{n+j} \omega^\beta = -\omega_{n+1}^{n+j} + 2\omega_1^j$$

i.e.

$$\omega_{n+1}^{n+j} \equiv 2\omega_1^j \mod \{\omega^\beta\}.$$ 

Finally

$$r_{\epsilon \delta \beta}^{n+j} \omega^\beta = -q_{\epsilon \delta}^{n+1} \omega_{n+1}^{n+j} + q_{\epsilon k}^{n+j} \omega_\delta^k + q_{\delta k}^{n+j} \omega_\epsilon^k \\ \equiv -(2q_{\epsilon \delta}^{n+1} \delta_{ij} + q_{\epsilon k}^{n+j} q_{i \delta}^{n+k} + q_{\delta k}^{n+j} q_{i \epsilon}^{n+k})\omega_1^i \mod \{\omega^\alpha\}$$

(12.29) implies

$$q_{\epsilon k}^{n+j} q_{\delta i}^{n+k} + q_{\delta k}^{n+j} q_{\epsilon i}^{n+k} + 2q_{\epsilon \delta}^{n+1} \delta_{ij} = 0 \quad \forall \epsilon, \delta, i, j$$

proving (6.23). □

§12. A refinement of a result in [GH]

Theorem 12.1. Let $X^n \subset \mathbb{P}^{n+a}$ be a variety (respectively a patch of a complex manifold) with degenerate tangential variety (resp. manifold) of dimension $n + a_0$. Then the Gauss map of $\tau(X)$ has fibers of dimension at least $\delta_\tau(X) + 1$, where $\delta_\tau(X)$ is the tangential defect of $X$.

(12.1) refines ([GH], 5.17) where it was observed that the Gauss map of a degenerate $\tau(X)$ has fibers of dimension at least 2.

Proof. The dimension of a fiber of $(\gamma(\tau(X_{sm})))$ is $\dim \text{singloc}[\Pi_{\tau(X)[A_1]}]$. From (9.1), we compute the second fundamental form of $\tau(X)$ at $[A_1]$: 

$$\Pi_{\tau(X)[A_1]} = (\omega_{j}^{n+h} \omega_1^j + \omega_{n+1}^{n+h} \omega_1^{n+\lambda}) \otimes A_{n+h} \mod \{A_0, \ldots, A_{n+a-1}\}.$$ 

However (11.15) combined with (11.6) implies $\omega_{n+\lambda}^{n+h} \equiv 0 \mod \{\omega^\lambda, \omega_1^j\}$.

Thus $\dim \text{singloc}[\Pi_{\tau(X)[A_1]}] \geq (n+a_0) - \dim \{\omega^\lambda, \omega_1^j\} = (n+a_0) - (a_0 - 1 + r) = n - r + 1$. But now $r \leq a_0 - 1$ and $\delta_\tau(X) = n - a_0$. □

Any tangential variety is ruled by parallel lines. (12.1) says that if a tangential variety is degenerate, it is ruled by parallel $(n - a_0 + 1)$-planes. (12.1) will be strengthened in the case $X$ is a smooth variety.
§13. Global Results

We will now restrict to the case $X^n \subset \mathbb{P}^{n+a}$ is a smooth variety with degenerate tangential variety that is a hypersurface, and use the global information that $X$ is smooth to assume $III^v = 0$ via (10.2). By (8.7),

\begin{equation}
III^v = \omega^{n+a}_{n+\lambda}\omega^{n+\lambda}_{n+\xi} |_{S^n(SA_1)} \otimes A_{n+a} \mod \{A_0, \ldots, A_{n+a-1}\}
\end{equation}

\begin{equation}
= -r^{n+a}_{n+\xi}\omega^{\xi}\omega^{n+\eta} \otimes A_{n+a} \mod \{A_0, \ldots, A_{n+a-1}\}
\end{equation}

where in the second line we have used (11.17). So $III^v = 0$ implies

\begin{equation}
r^{n+a}_{n+\xi}=0 \forall \xi, \eta, \zeta.
\end{equation}

Proof of (7.9). Let $A_B^r$ denote the dual basis element to $A_B$ at a point $f = (A_0, \ldots, A_{n+a})$ of $F^2$. Consider

\begin{equation}
\tilde{\rho} : F^2 \rightarrow X^*_\Delta
\end{equation}

\begin{equation}
(A_0, \ldots, A_{n+a}) \mapsto [A^{n+a}].
\end{equation}

The only restriction on $[A^{n+a}]$ comes from (11.1), so $\tilde{\rho}$ is surjective. Note that

\begin{equation}
\tilde{\rho}_*f = -\omega^{n+a}_j \otimes A^j - \omega^{n+a}_{n+\lambda} \otimes A^{n+\lambda} \mod \{A^{n+a}\}.
\end{equation}

By (13.4), the rank of $\tilde{\rho}_*f$ equals the number of independent forms among the $\{\omega^{n+a}_j, \omega^{n+a}_{n+\lambda}\}$.

Using (8.10) further we have

\begin{equation}
r^{n+a}_{n+\lambda} = -\omega^{n+a}_{n+1}
\end{equation}

\begin{equation}
r^{n+a}_{n+\lambda} = -\omega^{n+a}_{n+s}
\end{equation}

\begin{equation}
r^{n+a}_{n+\lambda} = -\omega^{n+a}_{n+k} + \omega^k
\end{equation}

where we have used (13.2) and (11.6).

To facilitate the computations, we will restrict to frames where some of the coefficients of $\partial II$ are zero. In fact we use the freedom in $g^{k}_{n+1}, g^k_{n+s}$ (see (8.14)) to restrict to frames where

\begin{equation}
r^{n+a}_{n+1}, r^{n+a}_{n+s} = 0.
\end{equation}

Call the new bundle $F^3$. On $F^3$,

\begin{equation}
\{\omega^{n+a}_{n+\lambda}, \omega^{n+a}_j\} = \{\omega^1, \omega^j\}
\end{equation}
Since the projections $F^3 \to X$, $F^3 \to X^*_\Delta$ factor through $I_\Delta$, we have
\begin{equation}
\text{rank } (\tilde{\rho}|_{F^3})_* f = \text{rank } (\rho_\Delta)_* ([A_0],[A^{n+a}]).
\end{equation}
But by (13.7), \( \text{rank } (\tilde{\rho}|_{F^3})_* f = 2r \). □

The computation above also allows us to improve (12.1) under the assumption that $X$ is a smooth variety.

Using the computations above, (12.2) reduces to
\begin{equation}
II_\tau(X)[A_1] = (\omega_{j+1}^n + \omega_{n+1}^a \omega_{1+1}^n) \otimes (A_{n+a} \mod \{A_0, \ldots, A_{n+a-1}\})
= (2\delta_{jk} - r_{1j}^{n+a}) \omega_{j+1}^n \omega_{k+1}^n \otimes (A_{n+a} \mod \{A_0, \ldots, A_{n+a-1}\}).
\end{equation}

**Theorem 13.10.** Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with degenerate tangential variety of dimension $n + a_0$. Then the Gauss map of $\tau(X)$ has fibers of dimension at least $\delta_\tau(X) + 2$, where $\delta_\tau(X)$ is the tangential defect of $X$.

**Proof.** If $\tau(X)$ is not a hypersurface, we can project $X$ smoothly to a smaller projective space where it is, and the dimension of the Gauss image of $\tau(X)$ will not change, and (13.9) proves the case $\tau(X)$ is a hypersurface.

(13.10) is sharp. Equality holds for Severi varieties and that strict inequality occurs for the Grassmannian $G(2,7) \subset \mathbb{P}^{20}$.

§14. Zak’s Theorem on Severi Varieties

Assume $a = \frac{n}{2} + 2$. (7.3) implies $\gamma_Z$ is nondegenerate at general points.

Assuming $A_2$ is $II$-generic (which is allowed since $A_2 \in \{A_j\}$), use $g_{n+k}^{n+1}$, $g_{n+a}^{n+1}$ to restrict to frames where
\begin{equation}
[q^{n+1}] = [g_{\alpha\beta}^{n+1} \omega^\alpha \omega^\beta] = \text{Ann}(A_2).
\end{equation}
This implies rank $(q^{n+1}) = r$. Using the Clifford algebra structure and the rank restriction theorem, one can check that $q^{n+1}|_{(A_1, A_4)}$ is non-degenerate. (The $n = 2, 4$ and 6 cases are clear, for the $n = 8$ case one would have a non-zero $4 \times 4$ skew symmetric matrix that squared to zero (which is possible) but that also skew commuted with a different non-zero $4 \times 4$ skew symmetric matrix that either squared to be the identity or zero (which is not possible). One argues similarly for larger $n$.) After using the remaining freedom in $g_j^i$, we may (and do) restrict to frames where
\begin{equation}
\text{singloc}(q^{n+1}) = \{A_j\}
\end{equation}
(Note that (14.2) makes sense since the $g_k^1$ are tied to the $g_{\alpha\beta}^{n+1}$, so $\{A_j\}$ is now a well defined subspace of $T$.)
Because of the restrictions on dimensions of Clifford modules, \( n = 2, 4, 8 \) or 16. (In the \( n = 2 \) case, \( \{ A_j \} = \{ A_2 \} \) and \( \{ A_\epsilon \} = \{ 0 \} \).)

Using some of the freedom in \( g^\epsilon_\delta \) we may restrict to frames where

\[
q^{n+1}_\epsilon\delta = \delta_{\epsilon\delta}.
\]

By using our remaining freedom in \( g^k_j \) and \( g^\epsilon_\delta \) we may arrange that \( \phi(A_\epsilon) = q^{n+i}_{j\epsilon} \omega^j \otimes A_i \) corresponds to \( \mathbb{A} \)-multiplication by \( J^{-(n+1)}\epsilon \) on the vector space \( \{ A_j \} \), where the \( J \)'s are as in \( \S 3 \). We are guaranteed that we are able to do this because the Clifford representation is nondegenerate, and in these dimensions there are unique irreducible representations.

(6.16) applied to \( A_2 \) implies that for all \( i, j, k \),

\[
q^{n+i}_{jk} = 0.
\]

Let \( J^{(j)} \) be the unique index \( k \) such that \( q^{n+j}_{k\epsilon} = \pm 1 \) and we use the sign convention that if \( q^{n+j}_{k\epsilon} = -1 \) then we take \( \omega^{j^{(j)}} = -\omega^k \). We have

\[
II = \mathbb{P}\{ (\omega^1)^2 + \Sigma (\omega^\epsilon)^2, \omega^1 \omega^j + \Sigma \omega^\epsilon \omega^j, \Sigma (\omega^j)^2 \}
\]

which is seen to be the second fundamental form of a Severi variety by comparing with (5.2). We also have

\[
0 = |III^A_j| = |(\omega^{n+1}_{n+k} \omega^i \omega^j + \omega^{n+1}_{n+a}\omega^j)_{\text{singloc(Ann(A_j))}}| = |r^{n+1}_{jki} \omega^i \omega^j \omega^k |
\]

giving

\[
r^{n+1}_{jki} = 0.
\]

Restrict frames further to normalize away more of \( \partial II \) by using \( g^1_{n+1}, g^1_{n+k}, g^\epsilon_{n+1}, g^\epsilon_{n+k} \) respectively to set

\[
r^{n+1}_{111} = 0, \ r^{n+1}_{1k1} = 0, \ r^{n+1}_{11\epsilon} = 0, \ r^{n+1}_{1k\epsilon} = 0.
\]

Fixing any \( k \), normalize away \( r^{n+k}_{1kk} \) (and therefore \( r^{n+j}_{1jj} \forall j \)) by using \( g^k_{n+a} \).
In summary, combining all the adaptations on our new frame bundle;

\[(14.9)\]

\[
0 = -\omega_0^0 - \omega_{n+1}^0 + 2\omega_1^1
\]

\[
r_{ij}^{n+1} \omega^j = -\omega_{n+j}^0 + \omega_j^0
\]

\[
r_{i\delta}^{n+1} \omega^\delta = \omega_1^1 + \omega_1^0
\]

\[
r_{j\epsilon}^{n+1} \omega^\epsilon = -\delta_{jk} \omega_{n+a}^{n+1}
\]

\[
r_{j\epsilon\rho}^{n+1} \omega^\rho = -\omega_{n+j, (j)}^0 + \omega_j^\epsilon
\]

\[
r_{e\delta\beta}^{n+k} \omega^\delta = -\delta_{e\delta} (\omega_0^0 + \omega_{n+1}^0) + \omega_\delta^0 + \omega_\delta^\epsilon
\]

\[
r_{11\beta}^{n+k} \omega^\beta = -\omega_{n+1}^0 + 2\omega_1^k
\]

\[
r_{ij\beta}^{n+k} \omega^\beta = \delta_j^i (\omega_1^0 - \omega_0^0) - \omega_{n+j}^0 + \omega_j^k + q_{ij}^{n+k} \omega_1^0 + \delta_j^i \omega_1^\epsilon
\]

\[
r_{i\beta}^{n+k} \omega^\beta = \omega_1^k + \omega_1^{J_2^k (i)}
\]

\[
r_{ij\beta}^{n+k} \omega^\beta = -\delta_{ij} \omega_{n+a}^{n+k} + q_{ie}^{n+k} \omega^j_\epsilon + q_{je}^{n+k} \omega^i_\epsilon
\]

\[
r_{j\epsilon\beta}^{n+k} \omega^\beta = -\delta_j^k (\omega_0^0 + \omega_{n+J_1^2 k (j)}^0) + \omega_j^j + \delta_j^k \omega_\epsilon^0
\]

\[
r_{e\delta\beta}^{n+k} \omega^\beta = -\delta_{e\delta} \omega_{n+1}^{n+k} + \omega_\delta^0 + \omega_\epsilon^{J_2^k (k)}
\]

\[
r_{j\epsilon\beta}^{n+a} \omega^\beta = -\delta_{jk} (\omega_0^0 + \omega_{n+a}^0) + \omega_0^0 + \omega_j^k
\]

\[
0 = \omega_{n+1}^{n+1}
\]

\[
0 = \omega_1^{J_2^k (k)} + \omega_1^k.
\]

Examining the terms in (14.9) combined with our new adaptations, we find that all the terms \(r_{\alpha\beta\gamma}^\mu = 0\) and the relations imposed on the \(\{\omega_\beta^\alpha, \omega_\nu^\mu\}\) are the ones obtained by setting the right hand side of (14.9) to zero. (Perhaps a better way to see all the coefficients of \(\partial II\) must be zero is, letting \(s = (s^1, \ldots, s^n)\), use the fact that \(\text{III}^\alpha \omega_\alpha = 0\) for generic and therefore all choices of \(s\). One gets a polynomial in the \(s^\alpha\) with coefficients the \(r_{\alpha\beta\gamma}^\mu\) that must be zero, which together with our normalizations, forces all coefficients of \(\partial II\) to be zero.) Examining the \(\partial^2 II\) equations, one gets that the \(r_{\alpha\beta\gamma\delta}^\mu\) are normalizable to zero by motions of the form \(A_\mu \to A_\mu + q_\mu^0 A_0\) and one obtains relations giving the forms \(\omega_\alpha^\mu\) in terms of the \(\omega_0^0\). Finally, the equation for the coefficients of \(\partial^3 II\) reduce to \(r_{\alpha\beta\gamma\delta\epsilon}^\mu \omega^\epsilon = \Sigma_{\alpha\beta\gamma\delta\epsilon} q_{\alpha\beta\gamma\delta\epsilon}^\mu q_{\gamma\delta\epsilon}^0 \omega_\nu^0\) which (using our explicit knowledge of \(II\)) tells us \(\omega_\nu^0 = 0\) and \(r_{\alpha\beta\gamma\delta\epsilon}^\mu = 0\). One is left with the structure equations for the corresponding Lie group proving the variety is indeed the expected homogeneous space. (One gets an aesthetically nicer basis of the Lie algebras if one switches the numberings of the \(A_\epsilon\) and the \(A_j\).) \(\square\)

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