TIME EXPONENTIAL INTEGRATOR FOURIER PSEUDOSPECTRAL METHODS WITH HIGH ACCURACY AND MULTIPLE CONSERVATION LAWS FOR THREE-DIMENSIONAL MAXWELL’S EQUATIONS

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Abstract. Maxwell equations describe the propagation of electromagnetic waves and are therefore fundamental to understanding many problems encountered in the study of antennas and electromagnetics. The aim of this paper is to propose and analyse an efficient fully discrete scheme for solving three-dimensional Maxwell’s equations. This is accomplished by combining time exponential integrator and Fourier pseudospectral methods. Fast computation is implemented in the scheme by using the Fast Fourier Transform algorithm which is well known in scientific computations. An optimal error estimate which is not encumbered by the CFL condition is established and the resulting scheme is proved to be of spectral accuracy in space and infinite-order accuracy in time. Furthermore, the scheme is shown to have multiple conservation laws including discrete energy, helicity, momentum, symplecticity, and divergence-free field conservations. All the theoretical results of the accuracy and conservations are numerically illustrated by two numerical tests.

Keywords: Maxwell’s equations, exponential integrator, Fourier pseudospectral methods, structure-preserving algorithms, convergence

AMS Subject Classification: 65M12, 65M15

1. Introduction

The Maxwell’s equations are the fundamental laws in electromagnetism and they describe the propagation and scattering of electromagnetic waves. They play a crucial role in a wide variety of applications in science and engineering such as wireless engineering, antennas, microwave circuits, photonic crystals, radio-frequency, aircraft radar, integrated optical circuits, waveguides, and interferometers. This paper is devoted to the three-dimensional Maxwell’s equations in an isotropic, homogeneous, and lossless medium which are expressed in the following coupled form (see [27])

\[
\begin{cases}
\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \text{curl } \mathbf{E}, & \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \text{curl } \mathbf{H}, \quad \Omega \times [0, t_{\text{end}}], \\
\nabla \cdot (\varepsilon \mathbf{E}) = 0, & \nabla \cdot (\varepsilon \mathbf{H}) = 0, \quad \Omega \times [0, t_{\text{end}}],
\end{cases}
\]

where \( \mathbf{H} = (H_x, H_y, H_z)^T : \Omega \times [0, t_{\text{end}}] \to \mathbb{R}^3 \) stands for magnetic field intensity, \( \mathbf{E} = (E_x, E_y, E_z)^T : \Omega \times [0, t_{\text{end}}] \to \mathbb{R}^3 \) represents electric field intensity, and \( \mu \) and \( \varepsilon \) denote the magnetic permeability and electric permittivity, respectively. In this paper, the equations (1.1) and their initial values

\[
\mathbf{H}_0(x, y, z) = \mathbf{H}(x, y, z, 0), \quad \mathbf{E}_0(x, y, z) = \mathbf{E}(x, y, z, 0),
\]

are considered on the cuboid space domain \( \Omega = [x_L, x_R] \times [y_L, y_R] \times [z_L, z_R] \) and periodic boundary conditions are required on the domain \( \partial \Omega \times [0, t_{\text{end}}] \). It is well known that Maxwell’s equations (1.1) have unique smooth solutions for all time if the initial data (1.2) are suitably smooth [27]. The \textit{div-equations} (1.1b) can be derived from the \textit{curl-equations} (1.1a) by taking divergence, and they hold automatically if \( \mathbf{H}_0 \) and \( \mathbf{E}_0 \) are divergence-free.
Due to the great importance and diversity of applications, Maxwell’s equations have been researched for more than 150 years and there has been a great interest in their numerical analysis in the last couple of decades. Various numerical methods have been investigated for the Maxwell’s equations. The first kind of scheme was the finite-difference time domain (FDTD) method which was firstly proposed by Yee in [49], and further developed and analysed in [33, 37, 40, 42, 43, 50]. However, these Yee-based FDTD methods require very small temporal step-size to ensure their stability. In order to improve the efficiency, the alternating direction implicit (ADI) technique was proposed firstly proposed by Yee in [49], and further developed and analysed in [33, 37, 40, 42, 43, 50]. How-

ever, these Yee-based FDTD methods require very small temporal step-size to ensure their stability. In recent years, due to the superior properties in multiscale methods [23, 17], and exponential RK methods [36, 48]. Concerning the schemes of the time integration, some time methods are applied to common such as discontinuous Galerkin (dG) methods [8, 9, 31, 32] or the Fourier pseudo-spectral methods [29].

In order to improve the efficiency, the alternating direction implicit (ADI) technique was proposed and some unconditionally stable ADI-FDTD schemes were formulated [13, 25, 26, 35, 49, 51]. Besides finite-difference space discretization, other space discretization techniques have also become common such as discontinuous Galerkin (dG) methods [3, 9, 31, 32] or the Fourier pseudo-spectral method [29]. Concerning the schemes of the time integration, some time methods are applied to these physical invariants: energy conservation laws, symplectic conservation laws, helicity conservation laws, momentum conservation laws and divergence-free fields. These invariants have been proved to be very powerful in numerical simulations [16]. For the Maxwell’s equations (1.1), they admit many physical invariants: energy conservation laws, symplectic conservation laws, helicity conservation laws, momentum conservation laws and divergence-free fields. These invariants are very important in the long time propagation of the electromagnetic waves [13].

To present these physical invariants, we rewrite the two curl-equations (1.1a) as a bi-Hamiltonian system (30) which reads $\frac{d}{dt} \begin{pmatrix} H \\ E \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \times \\ \nabla \times & 0 \end{pmatrix} \begin{pmatrix} H \\ E \end{pmatrix}$, where the quadratic Hamiltonian functional is $H = \frac{1}{2} \int_\Omega \left( \varepsilon |E|^2 + \mu |H|^2 \right) dxdydz$ with the Euclidean norm $|\cdot|$. The solutions $E, H$ satisfy the following energy conservation laws (ECLs) for $w = x, y$ or $z$ (1.6)

$$
\begin{align*}
\frac{d}{dt} \mathcal{E}_1^{\text{exact}}(t) &= 0, \quad \mathcal{E}_1^{\text{exact}}(t) := \mathcal{H} = \frac{1}{2} \int_\Omega \left( \varepsilon |E|^2 + \mu |H|^2 \right) dxdydz, \\
\frac{d}{dt} \mathcal{E}_2^{\text{exact}}(t) &= 0, \quad \mathcal{E}_2^{\text{exact}}(t) := \frac{1}{2} \int_\Omega \left( \varepsilon |\partial_w E|^2 + \mu |\partial_w H|^2 \right) dxdydz, \\
\frac{d}{dt} \mathcal{E}_3^{\text{exact}}(t) &= 0, \quad \mathcal{E}_3^{\text{exact}}(t) := \frac{1}{2} \int_\Omega \left( \varepsilon |\partial_y E|^2 + \mu |\partial_y H|^2 \right) dxdydz, \\
\frac{d}{dt} \mathcal{E}_4^{\text{exact}}(t) &= 0, \quad \mathcal{E}_4^{\text{exact}}(t) := \frac{1}{2} \int_\Omega \left( \varepsilon |\partial_z E|^2 + \mu |\partial_z H|^2 \right) dxdydz.
\end{align*}
$$

With this Hamiltonian formulation, the Maxwell’s equations (1.1) also satisfy the symplectic conservation law (1.2)

$$
\frac{d}{dt} \int_\Omega \left( dE_x \wedge dH_x + dE_y \wedge dH_y + dE_z \wedge dH_z \right) dxdydz = 0,
$$

the helicity conservation laws (2)

$$
\begin{align*}
\frac{d}{dt} \mathcal{H}_1^{\text{exact}}(t) &= 0, \quad \mathcal{H}_1^{\text{exact}}(t) := \int_\Omega \left( \frac{E^T (\nabla \times E)}{2\mu} + \frac{H^T (\nabla \times H)}{2\varepsilon} \right) dxdydz, \\
\frac{d}{dt} \mathcal{H}_2^{\text{exact}}(t) &= 0, \quad \mathcal{H}_2^{\text{exact}}(t) := \int_\Omega \left( \frac{\partial_y E^T (\nabla \times E)}{2\mu} + \frac{\partial_z E^T (\nabla \times E)}{2\varepsilon} \right) dxdydz,
\end{align*}
$$

where $\varepsilon$ and $\mu$ are the permittivity and permeability of the medium.
and the momentum conservation laws \((\text{1.2})\)

\[
\frac{d}{dt} M_1^{\text{exact}}(t) = 0,
M_1^{\text{exact}}(t) := \int_{\Omega} (H^T \partial_w E) dx dy dz,
\text{ where } w = x, y, z,
\]

\[
\frac{d}{dt} M_2^{\text{exact}}(t) = 0,
M_2^{\text{exact}}(t) := \int_{\Omega} (E^T \partial_w H) dx dy dz,
\text{ where } w = x, y, z.
\]

These physical invariants are important for Maxwell’s equations. Naturally it is desirable to propose a numerical scheme preserving them in the discrete sense \((\text{1.6})\). Thus, structure-preserving algorithms which can inherit these original physical features as much as possible have gained remarkable success in the numerical analysis of Maxwell’s equations. Concerning the structure-preserving algorithms of Maxwell’s equations, three categories have been received much attention in recent years: symplectic methods, divergence-free methods and energy-preserving methods.

There has been a great interest in solving Maxwell’s equations by using symplectic methods (see, e.g. \([3, 18, 25, 39, 41, 42, 52]\)), which can preserve the symplectic conservation law \((\text{1.4})\) of the equations. In order to make the numerical solution satisfy the div-equations \((\text{1.1})\), divergence-free methods were analysed (see, e.g. \([7, 34]\)). Another important component of the structure-preserving methods is the energy-preserving method. A kind of energy-conserved splitting method was proposed in \([5]\) for two dimensional (2D) Maxwell’s equations and in \([6]\) for the three dimensional (3D) case. Some energy preserving and unconditionally stable splitting schemes \([1, 11, 25, 26, 28]\) were further proposed. However, most of these energy-conserved splitting schemes have only second order accuracy in both time and space at most. In order to improve the accuracy, another energy-conserved scheme with fourth order accuracy in time was given in \([2]\).

There is no doubt that the idea to make use of structure-preserving algorithms for Maxwell’s equations is by no means new, but there are still many key issues in such kind of algorithms which remain to be well researched. In this paper, a kind of scheme with high accuracy, low cost and multiple conservation laws is derived and analised for Maxwell’s equations. The main contributions of this paper are as follows:

a) Among all the existing structure-preserving methods for Maxwell’s equations, explicit methods usually suffer from step size restrictions due to stability requirements (CFL condition) and implicit methods can use larger time steps at the cost of solving linear or even nonlinear systems. Meanwhile, all the methods have limited accuracy (up to order four) in time. In this paper, we will formulate a kind of explicit fully discrete scheme for \((\text{1.1})\) which is not encumbered by the CFL condition (any large time step-size is acceptable) and has spectral accuracy in space and infinite-order accuracy in time.

b) Usually the structure-preserving algorithms of Maxwell’s equations can preserve some physical invariants. The new scheme proposed in this paper can simultaneously preserve the energy, symplecticity, helicity, momentum and divergence-free fields conservation laws.

c) For all the energy-preserving methods of Maxwell’s equations, they are implicit and the iterative procedure is needed. However, the fully discrete scheme proposed in this paper is explicit and can use any large time step-size, and thus its computational cost is very low. Meanwhile, the scheme can be implemented by using the matrix diagonalisation method and Fast Fourier Transform algorithm which are very efficient in scientific computing.

d) The proposed scheme needs the regularity \(C^1(0, t_{\text{end}}; [H^r_p(\Omega)]^3)\) with \(r > 3/2\) of \(H\) and \(E\) to get the spectral accuracy in space and infinite-order accuracy in time, which is lower than those methods (spectral accuracy in space and second or fourth-order accuracy in time) required in the publications (see, e.g. \([11, 2, 3]\)).
The rest of the paper is organised as follows. In Section 2 we present the formulation of the fully discrete scheme and discuss its cost. The convergent analysis is made in Section 3. Section 4 presents the conservative properties of the proposed scheme. Two numerical experiments are displayed in Section 5 and the results demonstrate the high accuracy and exact conservation laws of the proposed scheme.

2. Description of the fully discrete scheme

2.1. Analytic framework. We begin this subsection with presenting some notations. For a set $K \subset \Omega$ and the vectors fields $U, \tilde{U}, V, \tilde{V} : K \to \mathbb{R}^3$ the $L^2(K)$-inner product is denoted by $\langle U, \tilde{U} \rangle_K = \int_K U \cdot \tilde{U} \, dx \, dy \, dz$ and for $F \subset \partial K$ we denote $\langle U, \tilde{U} \rangle_F = \int_F U \cdot \tilde{U} \, d\sigma$. Denoting by $u = (U, V)$ and $\tilde{u} = (\tilde{U}, \tilde{V})$, the weighted inner products are given by

$$\langle \alpha U, \tilde{U} \rangle_{\alpha,K} = \langle \alpha \tilde{U}, U \rangle_K,$$

$$\langle u, \tilde{u} \rangle_{\alpha \times \beta,K} = \langle \tilde{u}, U \rangle_{\alpha,K} + \langle V, \tilde{V} \rangle_{\beta,K}$$

for the positive weight functions $\alpha, \beta : \Omega \to \mathbb{R}^+$. The corresponding norms are immediately obtained as $\|U\|^2_\alpha = \langle U, U \rangle_{\alpha,K}$ and $\|\tilde{u}\|^2_{\alpha \times \beta} = \|U\|^2_\alpha + \|V\|^2_{\beta}$.

The two curl-equations (1.1a) can be written as an abstract Cauchy problem

$$\frac{\partial}{\partial t} \left( \sqrt{\mu} H \right) = C \left( \sqrt{\epsilon} E \right),$$

(2.1)

where $C = \begin{pmatrix} 0 & -C_e \\ C_H & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\text{curl} \\ \sqrt{\mu} & 0 \end{pmatrix}$ is the Maxwell operator. For the curl operator, its graph space is given by $H(\text{curl}) = \{ U \in L^2(\Omega)^3 \mid \text{curl} U \in L^2(\Omega)^3 \}$, which is endowed with the inner product $\langle U, V \rangle_{H(\text{curl})} = \langle U, V \rangle_\Omega + \langle \text{curl} U, \text{curl} V \rangle_\Omega$ for all $U, V \in H(\text{curl}, \Omega)$ and the associated norm given by $\|U\|^2_{H(\text{curl}, \Omega)} = \langle U, U \rangle_{H(\text{curl}, \Omega)}$. Denote $H_0(\text{curl}, \Omega)$ the closure of $C_0^\infty(\Omega)^3 = \{ v \in C^\infty(\Omega) \mid \text{supp}(v) \subset \Omega \text{ is compact} \}$ with respect to the norm $\|\cdot\|_{H(\text{curl}, \Omega)}$. With these notations, it is well known that the Maxwell operator $C$ is skew-adjoint w.r.t. $(\cdot, \cdot)_{\alpha \times \beta, \Omega}$ (2.1), i.e. $\langle C_H H, E \rangle_{\epsilon, \Omega} = \langle H, C_E E \rangle_{\mu, \Omega}$ for $H \in D(C_H)$ and $E \in D(C_E)$. Meanwhile, the Maxwell operator $C$ with domain $D(C) = D(C_H) \times D(C_H) = H(\text{curl}, \Omega) \times H_0(\text{curl}, \Omega)$ generates a unitary $C_0$-group $e^{tC}$ (2.1) on a Hilbert space $X$.

Based on the above results, the well-posedness of Maxwell’s equations can be derived by Stone’s theorem. If the initial value $(H_0, E_0) \in D(C)$, the two curl-equations (1.1a) have a unique solution in $C^1(0, t_{\text{end}}; X) \cap C(0, t_{\text{end}}; D(C))$ (2.1) which can be given by $\left( \begin{pmatrix} \sqrt{\mu} H(t) \\ \sqrt{\epsilon} E(t) \end{pmatrix} \right) = e^{tC} \left( \begin{pmatrix} \sqrt{\mu} H_0 \\ \sqrt{\epsilon} E_0 \end{pmatrix} \right)$. According to this formula and the unitarity of $e^{tC}$, the solution is bounded by

$$\| (H, E) \|_{\mu \times \epsilon, \Omega} \leq \| (H_0, E_0) \|_{\mu \times \epsilon, \Omega}.$$

If these solutions $H, E$ are required to be smooth enough and the initial values (1.2) satisfy the two div-equations (1.1b), i.e. $\nabla \cdot (\epsilon E_0) = 0$, $\nabla \cdot (\mu H_0) = 0$, then the two div-equations (1.1b) hold true for $H, E$ at any $t \geq 0$. In the rest parts of this section, the numerical scheme will be derived for the abstract Cauchy problem (2.1). Then, in the next section we will show the fact that the proposed numerical solution satisfies the two div-equations (1.1b).
2.2. Spatial discretisation. To achieve high order accuracy in treating the space, the Fourier pseudo-spectral method is a very good discretisation. With this we are hopeful of obtaining high order accuracy and be implemented by the fast Fourier transform (FFT) algorithm \[29, 38, 44].

For the three-dimensional domain $\Omega = [x_L, x_R] \times [y_L, y_R] \times [z_L, z_R]$, define a series of collocation points $x_j = x_L + (j - 1)h_x$, $y_k = y_L + (k - 1)h_y$, $z_l = z_L + (l - 1)h_z$, $j = 1, 2, \ldots, N_x$, $k = 1, 2, \ldots, N_y$, $l = 1, 2, \ldots, N_z$, where $h_x = (x_R - x_L)/N_x$, $h_y = (y_R - y_L)/N_y$, $h_z = (z_R - z_L)/N_z$ with even integers $N_x, N_y, N_z$. Denote the time step-size by $\Delta t = t_{end}/N_t$ for some integer $N_t$ and let $t_n = n\Delta t$. The value of the function $E(x, y, z, t)$ at the node $(x_j, y_k, z_l, t_n)$ is denoted by $E_{j,k,l}^n$.

Denote a three-dimensional smooth function defined on $\Omega \times [0, t_{end}]$ by $U(x, y, z, t)$. The interpolation space of this function is considered as

$$S_{NS} = \text{span}\{g_j(x)g_k(y)g_l(z), \ j = 1, 2, \ldots, N_x, \ k = 1, 2, \ldots, N_y, \ l = 1, 2, \ldots, N_z\},$$

where $N_S = N_xN_yN_z$ and $g_j(x), g_k(y), g_l(z)$ are trigonometric polynomials of degree $N_x/2, N_y/2, N_z/2$, given respectively by

$$g_j(x) = \frac{1}{N_x} \sum_{|m| \leq N_x/2} e^{imv_x(x-x_j)}, \ \ g_k(y) = \frac{1}{N_y} \sum_{|m| \leq N_y/2} e^{imv_y(y-y_k)}, \ \ g_l(z) = \frac{1}{N_z} \sum_{|m| \leq N_z/2} e^{imv_z(z-z_l)}.$$

Here $i = \sqrt{-1}$ and the prime indicates that the first and last terms in the summation are taken with the factor 1/2, and $v_w = \frac{2\pi}{x_R-x_L}$ for $w = x, y, z$. Interpolating $U(x, y, z, t)$ at collocation points $(x_j, y_k, z_l)$ gives

$$U(x, y, z, t) \approx \mathcal{I}_{NS} U(x, y, z, t) = \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \sum_{l=1}^{N_z} U_{j,k,l}(t) g_j(x)g_k(y)g_l(z),$$

where $U_{j,k,l}(t) = U(x_j, y_k, z_l, t_{end})$ and its vector form is denoted by

$$U = (U_{1,1,1}, U_{1,1,2}, \ldots, U_{N_x,1,N_z}, U_{1,1,N_z}, U_{1,2,1}, U_{1,2,2}, \ldots, U_{N_x,2,N_z}, \ldots, U_{N_x,1,N_z})^\top.$$

Consider the following trigonometric polynomials as an approximation for the solution of (2.1)

$$\mathcal{I}_{NS} E^F = (\mathcal{I}_{NS} E_{x}^F, \mathcal{I}_{NS} E_{y}^F, \mathcal{I}_{NS} E_{z}^F)^\top, \ \ \mathcal{I}_{NS} H^F = (\mathcal{I}_{NS} H_{x}^F, \mathcal{I}_{NS} H_{y}^F, \mathcal{I}_{NS} H_{z}^F)^\top,$$

which is required to satisfy $\frac{\partial}{\partial t} \left( \sqrt{\mathcal{I}_{NS} H^F} \right) = C \left( \sqrt{\mathcal{I}_{NS} H^F} \right)$. In order to calculate $\text{curl}(\mathcal{I}_{NS} E^F)$, we make partial differential with respect to $x$ and evaluate the resulting expression at collocation points $(x_j, y_k, z_l)$:

$$\frac{\partial}{\partial x} \mathcal{I}_{NS} E_{x}^F(x_j, y_k, z_l) = \sum_{j'=1}^{N_x} \sum_{k'=1}^{N_y} \sum_{l'=1}^{N_z} E_{x,j',k',l'}(t) \frac{d}{dx} g_{j'}(x_j) g_{k'}(y_k) g_{l'}(z_l)$$

$$= \left( I_{NS} \otimes I_{Ny} \otimes D_x \right) E_{x}^F \big|_{N_xN_y(l-1)+N_x(k-1)+j},$$

where $\otimes$ is the Kronecker product, $I_{Nw}$ is the identity matrix of dimension $N_w \times N_w$,

$$E_{x}^F = (E_{x,1,1,1}, \ldots, E_{x,1,1,N_z}, E_{x,1,2,1}, \ldots, E_{x,1,2,N_z}, \ldots, E_{x,1,N_y,N_z}, \ldots, E_{x,N_x,N_y,N_z})^\top,$$

and the spectral differential matrix is explicitly given by $(D_w)_{j,l} = \frac{1}{2} \nu_w (-1)^{j+l} \cot(\nu_w(w_j - w_l)/2)$ for $j \neq l$ and others elements are zero with $j, l = 1, 2, \ldots, N_w$ and $w = x, y, z$. Similarly, one
has \( \frac{\partial}{\partial y} I_{N_x} E_x^F(x_j, y_k, z_l) = \left[ (I_{N_y} \otimes D_y \otimes I_{N_z}) E_x^F \right]_{N_x N_y (l-1) + N_y (k-1) + j} \) and \( \frac{\partial}{\partial z} I_{N_z} E_x^F(x_j, y_k, z_l) = \left[ (D_z \otimes I_{N_y} \otimes I_{N_z}) E_x^F \right]_{N_x N_y (l-1) + N_y (k-1) + j} \).

Based on the above results and with the notations \( \bar{E} = (E_x^F, E_y^F, E_z^F)^\top \), \( \bar{H} = (H_x^F, H_y^F, H_z^F)^\top \), the following ordinary differential equations are obtained
\[
\frac{d}{dt} \left( \sqrt{\bar{\mu}} \bar{H} \right) = \frac{1}{\sqrt{\bar{\mu} \bar{E}}} \left( \begin{array}{cc}
0 & -D \\
D & 0 \end{array} \right) \left( \sqrt{\bar{\mu}} \bar{H} \right),
\]
where
\[
D = \left( \begin{array}{ccc}
D_z \otimes I_{N_y} \otimes I_{N_z} & -D_z \otimes I_{N_y} \otimes I_{N_z} & I_{N_z} \otimes D_y \otimes I_{N_x} \\
-I_{N_z} \otimes D_y \otimes I_{N_x} & I_{N_z} \otimes I_{N_y} \otimes D_x \\
-I_{N_z} \otimes D_y \otimes I_{N_x} & I_{N_z} \otimes I_{N_y} \otimes D_x & 0 \end{array} \right).
\]

The initial values of (2.4) are obtained by considering the initial values (1.2) at the collocation points.

2.3. Time discretisation. In this part, we formulate the time discretisation of the system (2.4) by using exponential integrators. This kind of method has been well researched and has been made successful applications in many systems (see, e.g., [15, 22, 36]). However, exponential integrators suffer from the fact that direct discretisation of the three-dimensional Maxwell’s equations on a grid requires the storage and computation of a \( (6 N_x N_y N_z) \times (6 N_x N_y N_z) \) matrix exponential, which is prohibitive from a computational point of view in many cases. In order to make the computation of the matrix exponential be achievable and effective, matrix diagonalisation method and vector-valued trigonometric functions are employed in this section.

Denote \( F_{N_w} \) the matrix of DFT coefficients with entries given by \( (F_{N_w})_{j,k} = (e^{2\pi i/N_w})^{-jk} \) and \( (F_{N_w}^{-1})_{j,k} = \frac{1}{N_w} (e^{2\pi i/N_w})^{jk} \). Then it is known that \( D = F^{-1} \Lambda F \) with
\[
F = \text{diag} \left( F_{N_x} \otimes F_{N_y} \otimes F_{N_z}, F_{N_x} \otimes F_{N_y} \otimes F_{N_z}, F_{N_x} \otimes F_{N_y} \otimes F_{N_z} \right),
\]
\[
\Lambda = \left( \begin{array}{ccc}
\Lambda_z \otimes I_{N_y} \otimes I_{N_z} & -\Lambda_z \otimes I_{N_y} \otimes I_{N_z} & I_{N_z} \otimes \Lambda_y \otimes I_{N_x} \\
-I_{N_z} \otimes \Lambda_y \otimes I_{N_x} & I_{N_z} \otimes I_{N_y} \otimes \Lambda_x & 0 \\
-I_{N_z} \otimes \Lambda_y \otimes I_{N_x} & I_{N_z} \otimes I_{N_y} \otimes \Lambda_x & 0 \end{array} \right),
\]
\[
\Lambda_w = i\Omega_w := i \nu_w \text{diag} \left( 0, 1, \ldots, \frac{N_w}{2} - 1, 0, -\frac{N_w}{2} + 1, \ldots, -2, -1 \right) \quad \text{for} \quad w = x, y, z.
\]

Then letting \( \tilde{E} = F \bar{E}, \tilde{H} = F \bar{H} \), (2.4) can be reformulated as
\[
\frac{d}{dt} \left( \sqrt{\tilde{\mu}} \tilde{H} \right) = \tilde{\Lambda} \left( \sqrt{\tilde{\mu}} \tilde{E} \right),
\]
where \( \tilde{\Lambda} = \frac{1}{\sqrt{\bar{\mu}}} \left( \begin{array}{cc}
0 & -\Lambda \\
\Lambda & 0 \end{array} \right) \). The exact solution of (2.5) is
\[
\left( \begin{array}{c}
\sqrt{\tilde{\mu}} \tilde{H}(t) \\
\sqrt{\bar{\mu}} \bar{E}(t) \end{array} \right) = e^{t \tilde{\Lambda}} \left( \begin{array}{c}
\sqrt{\tilde{\mu}} \tilde{H}(0) \\
\sqrt{\bar{\mu}} \bar{E}(0) \end{array} \right).
\]
Here \( e^{t \tilde{\Lambda}} \) is the exponential of the matrix \( t \tilde{\Lambda} \). This exponential itself is again a \( 6 N_S \times 6 N_S \) matrix, or in other words a linear operator from \( \mathbb{C}^{6 N_S} \) to \( \mathbb{C}^{6 N_S} \). Even more, it is easy to see that \( e^{t \tilde{\Lambda}} \) with \( t \geq 0 \) is a strongly continuous semigroup. It is clear that \( \tilde{\Lambda} \) is a skew-hermitian matrix. Thus the matrix exponential \( e^{t \tilde{\Lambda}} \) is unitary and thus satisfies \( \| e^{t \tilde{\Lambda}} \| = 1 \).
Now the key step is to compute the matrix exponential $e^{t\hat{A}}$. The results are given by the following two propositions.

**Proposition 2.1.** For the matrix exponential, one has 

$$e^{t\hat{A}} = \left( \begin{array}{cc} \cos \left( \frac{t}{\sqrt{|\mu_e|}} \Lambda \right) & -\sin \left( \frac{t}{\sqrt{|\mu_e|}} \Lambda \right) \\ \sin \left( \frac{t}{\sqrt{|\mu_e|}} \Lambda \right) & \cos \left( \frac{t}{\sqrt{|\mu_e|}} \Lambda \right) \end{array} \right).$$

**Proof.** It is easy to see that 

$$\hat{A}^k = \frac{1}{(\sqrt{|\mu_e|})^k} \left( \begin{array}{cc} 0 & -\Lambda^k \\ \Lambda^k & 0 \end{array} \right)$$

for $k = 4m + 1$, 

$$\frac{1}{(\sqrt{|\mu_e|})^{k+1}} \left( \begin{array}{cc} 0 & -\Lambda^k \\ -\Lambda^k & 0 \end{array} \right)$$

for $k = 4m + 2$, 

$$\frac{1}{(\sqrt{|\mu_e|})^{k+2}} \left( \begin{array}{cc} \Lambda^k & 0 \\ 0 & \Lambda^k \end{array} \right)$$

for $k = 4m + 3$, and 

$$\frac{1}{(\sqrt{|\mu_e|})^{k+3}} \left( \begin{array}{cc} \Lambda^k & 0 \\ 0 & \Lambda^k \end{array} \right)$$

for $k = 4m$. Then one gets 

$$e^{t\hat{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} e^{\frac{t^k}{(\sqrt{|\mu_e|})^{k+3}}} = \sum_{m=0}^{\infty} \left( \frac{\Lambda^m}{(4m)(\sqrt{|\mu_e|})^{m+3}} - \frac{\Lambda^{m+2}}{(4m+2)(\sqrt{|\mu_e|})^{m+5}} - \frac{\Lambda^{m+1}}{(4m+1)(\sqrt{|\mu_e|})^{m+4}} + \frac{\Lambda^{4m+3}}{(4m+2)(\sqrt{|\mu_e|})^{m+4}} \right),$$

which confirms the result. $\square$

**Proposition 2.2.** It is deduced that 

$$\cos(t\Lambda/\sqrt{|\mu_e|}) = \left( \begin{array}{ccc} c_{11}(t\Lambda/\sqrt{|\mu_e|}) & c_{12}(t\Lambda/\sqrt{|\mu_e|}) & c_{13}(t\Lambda/\sqrt{|\mu_e|}) \\ c_{12}(t\Lambda/\sqrt{|\mu_e|}) & c_{22}(t\Lambda/\sqrt{|\mu_e|}) & c_{23}(t\Lambda/\sqrt{|\mu_e|}) \\ c_{13}(t\Lambda/\sqrt{|\mu_e|}) & c_{23}(t\Lambda/\sqrt{|\mu_e|}) & c_{33}(t\Lambda/\sqrt{|\mu_e|}) \end{array} \right),$$

$$\sin(t\Lambda/\sqrt{|\mu_e|}) = \left( \begin{array}{ccc} 0 & -s_{12}(t\Lambda/\sqrt{|\mu_e|}) & s_{13}(t\Lambda/\sqrt{|\mu_e|}) \\ -s_{12}(t\Lambda/\sqrt{|\mu_e|}) & 0 & -s_{23}(t\Lambda/\sqrt{|\mu_e|}) \\ s_{13}(t\Lambda/\sqrt{|\mu_e|}) & s_{23}(t\Lambda/\sqrt{|\mu_e|}) & 0 \end{array} \right),$$

where 

$$c_{11}(t\Lambda) = I + \frac{t^2(\Omega_2^2 + \Omega_2^2)}{\Psi(t\Omega)}(\cosh(-\sqrt{-\Psi(t\Omega)})) - I,$$ 

$$c_{12}(t\Lambda) = -\frac{t^2 \Omega_2 \Omega_1 (\cosh(-\sqrt{-\Psi(t\Omega)})) - I}{\Psi(t\Omega)},$$ 

$$c_{13}(t\Lambda) = -\frac{t^2 \Omega_3 \Omega_1 (\cosh(-\sqrt{-\Psi(t\Omega)})) - I}{\Psi(t\Omega)},$$ 

$$c_{22}(t\Lambda) = I + \frac{t^2(\Omega_2^2 + \Omega_2^2)}{\Psi(t\Omega)}(\cosh(-\sqrt{-\Psi(t\Omega)})) - I,$$ 

$$c_{23}(t\Lambda) = -\frac{t^2 \Omega_3 \Omega_2 (\cosh(-\sqrt{-\Psi(t\Omega)})) - I}{\Psi(t\Omega)},$$ 

$$c_{33}(t\Lambda) = I + \frac{t^2(\Omega_2^2 + \Omega_2^2)}{\Psi(t\Omega)}(\cosh(-\sqrt{-\Psi(t\Omega)})) - I,$$ 

$$s_{12}(t\Lambda) = \frac{it\Omega_1 \sinh(-\sqrt{-\Psi(t\Omega)})}{\sqrt{-\Psi(t\Omega)}},$$ 

$$s_{13}(t\Lambda) = \frac{it\Omega_2 \sinh(-\sqrt{-\Psi(t\Omega)})}{\sqrt{-\Psi(t\Omega)}},$$ 

$$s_{23}(t\Lambda) = \frac{it\Omega_3 \sinh(-\sqrt{-\Psi(t\Omega)})}{\sqrt{-\Psi(t\Omega)}},$$

with $\Omega_1 = I_N \otimes I_N \otimes \Omega_x$, $\Omega_2 = I_N \otimes \Omega_y \otimes I_N$, $\Omega_3 = \Omega_z \otimes I_N \otimes I_N$, and $\Psi(t\Omega) = t^2(\Omega_2^2 + \Omega_2^2 + \Omega_2^2)$. 

**Proof.** The proof is similar to that of Proposition 2.1 and we skip it for brevity. $\square$

### 2.4 Fully discrete scheme.

We now present the novel fully discrete scheme of the three-dimensional Maxwell’s equations (1.1).

**Definition 2.3.** (Fully discrete scheme.) For solving the three-dimensional Maxwell’s equations (1.1), the fully discrete scheme is defined as follows.

**Step 1.** (Space step-sizes and collocation points): Choose even integers $N_x, N_y, N_z$ to get the space stepsizes $h_x = (x_R - x_L)/N_x$, $h_y = (y_R - y_L)/N_y$, $h_z = (z_R - z_L)/N_z$ and collocation points $(x_j, y_k, z_l)$. 


Step 2. (Initial values): Considering the initial values (1.2) at these collocation points gives \((E_w^0, E_y^0, E_z^0)\) and \((H_x^0, H_y^0, H_z^0)\), for \(w = x, y\) or \(z\)

\[
E_w^0 = (E_w(x_1, y_1, z_1, 0), \ldots, E_w(x_N, y_1, z_1, 0), E_w(x_1, y_2, z_1, 0), \ldots, E_w(x_N, y_1, z_2, 0), \ldots, E_w(x_N, y_N, z_1, 0), \ldots, E_w(x_N, y_N, z_N, 0))^T,
\]
\[
H_w^0 = (H_w(x_1, y_1, z_1, 0), \ldots, H_w(x_N, y_1, z_1, 0), H_w(x_1, y_2, z_1, 0), \ldots, H_w(x_N, y_1, z_2, 0), \ldots, H_w(x_N, y_1, z_N, 0), \ldots, H_w(x_N, y_N, z_1, 0), \ldots, H_w(x_N, y_N, z_N, 0))^T. \tag{2.7}
\]

Then the initial values for the system (2.3) are given by

\[
E_w^0 = (F_{N_x} \otimes F_{N_y} \otimes F_{N_z}) E_w^0, \quad \hat{H}_w^0 = (F_{N_x} \otimes F_{N_y} \otimes F_{N_z}) H_w^0. \tag{2.8}
\]

Step 3. (Time integration): For solving the three-dimensional Maxwell’s equations (1.1) on the time interval \([0, t_{end}]\), the following scheme is considered

\[
\sqrt{\mu} \hat{H}_x^{t_{end}} = \sqrt{\mu} (c_{11}(t_{end})/\sqrt{\mu}) \hat{H}_0^x + c_{12}(t_{end})/\sqrt{\mu} \hat{H}_y^0 + c_{13}(t_{end})/\sqrt{\mu} \hat{H}_z^0 - \sqrt{\varepsilon} (s_{12}(t_{end})/\sqrt{\mu}) \hat{E}_y^0 + s_{13}(t_{end})/\sqrt{\mu} \hat{E}_z^0,
\]

\[
\sqrt{\mu} \hat{H}_y^{t_{end}} = \sqrt{\mu} (c_{12}(t_{end})/\sqrt{\mu}) \hat{H}_0^y + c_{22}(t_{end})/\sqrt{\mu} \hat{H}_y^0 + c_{23}(t_{end})/\sqrt{\mu} \hat{H}_z^0 - \sqrt{\varepsilon} (s_{12}(t_{end})/\sqrt{\mu}) \hat{E}_x^0 - s_{23}(t_{end})/\sqrt{\mu} \hat{E}_y^0,
\]

\[
\sqrt{\mu} \hat{H}_z^{t_{end}} = \sqrt{\mu} (c_{13}(t_{end})/\sqrt{\mu}) \hat{H}_0^z + c_{23}(t_{end})/\sqrt{\mu} \hat{H}_y^0 + c_{33}(t_{end})/\sqrt{\mu} \hat{H}_z^0 - \sqrt{\varepsilon} (s_{13}(t_{end})/\sqrt{\mu}) \hat{E}_x^0 + s_{23}(t_{end})/\sqrt{\mu} \hat{E}_y^0,
\]

\[
\sqrt{\varepsilon} \hat{E}_x^{t_{end}} = \sqrt{\varepsilon} (c_{11}(t_{end})/\sqrt{\mu}) \hat{E}_0^x + c_{12}(t_{end})/\sqrt{\mu} \hat{E}_y^0 + c_{13}(t_{end})/\sqrt{\mu} \hat{E}_z^0 + s_{11}(t_{end})/\sqrt{\mu} \hat{H}_y^0 + s_{12}(t_{end})/\sqrt{\mu} \hat{H}_z^0,
\]

\[
\sqrt{\varepsilon} \hat{E}_y^{t_{end}} = \sqrt{\varepsilon} (c_{12}(t_{end})/\sqrt{\mu}) \hat{E}_0^y + c_{22}(t_{end})/\sqrt{\mu} \hat{E}_y^0 + c_{23}(t_{end})/\sqrt{\mu} \hat{E}_z^0 + s_{12}(t_{end})/\sqrt{\mu} \hat{H}_x^0 + s_{22}(t_{end})/\sqrt{\mu} \hat{H}_z^0,
\]

\[
\sqrt{\varepsilon} \hat{E}_z^{t_{end}} = \sqrt{\varepsilon} (c_{13}(t_{end})/\sqrt{\mu}) \hat{E}_0^z + c_{23}(t_{end})/\sqrt{\mu} \hat{E}_y^0 + c_{33}(t_{end})/\sqrt{\mu} \hat{E}_z^0 + s_{13}(t_{end})/\sqrt{\mu} \hat{H}_x^0 + s_{23}(t_{end})/\sqrt{\mu} \hat{H}_y^0,
\]

where \(c.\) and \(s.\) are determined by (2.6).

Step 4. (Final result): The final results

\[
E_w^{t_{end}} \approx (E_w(x_j, y_k, z_l, t_{end}))_{j,k,l} \quad \text{and} \quad H_w^{t_{end}} \approx (H_w(x_j, y_k, z_l, t_{end}))_{j,k,l}
\]

approximating the solution of (1.1) at the collocation points \((x_j, y_k, z_l)\) and at the time \(t_{end}\) are given by

\[
E_w^{t_{end}} = (F_{N_x}^{-1} \otimes F_{N_y}^{-1} \otimes F_{N_z}^{-1}) \hat{E}_w^{t_{end}}, \quad H_w^{t_{end}} = (F_{N_x}^{-1} \otimes F_{N_y}^{-1} \otimes F_{N_z}^{-1}) \hat{H}_w^{t_{end}}. \tag{2.9}
\]

2.5. Fast computation and its cost. In this part, we discuss the complexity of the proposed scheme. A fast solver can be used to increase computational efficiency. The idea is based on the diagonal matrix and the Fast Fourier Transform (FFT) algorithm.

We first discuss the complexity of deriving initial values. Computing the collocation points requires \(O(N_x + N_y + N_z)\) arithmetic operations and storage. Then the storage cost and computational cost of the values \(E_w^0, \hat{H}_w^0\) [2.7] are both \(O(N_x N_y N_z)\). The Fast Fourier Transform (FFT) algorithm can be applied to obtain \(\hat{E}_w^0, \hat{H}_w^0\) [2.8] and the storage cost is \(O(N_x N_y N_z)\). The details of this procedure and its cost are presented in Algorithm 1.

We then discuss the computational characteristics of the proposed algorithm. We have to compute the coefficients [2.6] and this can be achieved by vector operations since \(\Lambda\) is diagonal. Thus the cost and storage of this step can be reduced to \(O(N_x N_y N_z)\) from \(O(N_x^2 N_y^2 N_z^2)\). Moreover, it is
Algorithm 1 (Initial values) The goal of the algorithm is to obtain the initial values $\mathbf{E}_w^0$ and $\mathbf{H}_w^0$ for the method proposed in this paper.

**Input:** $N_x, N_y, N_z$ (even integers)

**Output:** $\mathbf{E}_w^0, \mathbf{H}_w^0$ (initial values for our scheme)

1. Compute $h_x = (x_R - x_L)/N_x$, $h_y = (y_R - y_L)/N_y$, $h_z = (z_R - z_L)/N_z$.
   
   Cost: $O(1)$. Storage: $O(1)$.

2. Compute $x_j = x_L + (j - 1)h_x$, $y_k = y_L + (k - 1)h_y$, $z_l = z_L + (l - 1)h_z$, $j = 1, 2, \ldots, N_x$, $k = 1, 2, \ldots, N_y$, $l = 1, 2, \ldots, N_z$.
   
   Cost: $O(N_x + N_y + N_z)$. Storage: $O(N_x + N_y + N_z)$.

3. Compute the values $\mathbf{E}_w^0, \mathbf{H}_w^0$ from (12) such that
   
   $$(\mathbf{E}_w^0)_{N_xN_yN_z} = E_w(x_j, y_k, z_l, 0), \quad (\mathbf{H}_w^0)_{N_xN_yN_z} = H_w(x_j, y_k, z_l, 0)$$
   
   for $w = x, y, z$ and $j = 1, 2, \ldots, N_x$, $k = 1, 2, \ldots, N_y$, $l = 1, 2, \ldots, N_z$.

   Cost: $O(N_xN_yN_z)$. Storage: $O(N_xN_yN_z)$.

4. By Fast Fourier Transform (FFT), compute the initial values
   
   $$\mathbf{E}_w^0 = (\mathcal{F}_{N_x} \otimes \mathcal{F}_{N_y} \otimes \mathcal{F}_{N_z}) \mathbf{E}_w^0, \quad \mathbf{H}_w^0 = (\mathcal{F}_{N_x} \otimes \mathcal{F}_{N_y} \otimes \mathcal{F}_{N_z}) \mathbf{H}_w^0.$$

   Cost: $O(FFT)$. Storage: $O(N_xN_yN_z)$.

noted here that a great advantage of the scheme is that arbitrary large time step-size is accepted.

The final results at any $t_{end}$ are obtained from the initial values by only one step computation with a time step $\Delta t = t_{end}$. Therefore, the cost of the scheme is very low in comparison with the standard methods using a time step-size: $0 < \Delta t < 1$. The detailed complexity of the fully discrete scheme $\mathbb{E}_w^0$ is stated in Algorithm 2.

### 3. Convergence

In this proof, we focus on the error estimates of the proposed scheme. For simplicity we consider the cubic domain $\Omega = [0, 2\pi]^3$ with the spatial grid points $N_x = N_y = N_z = N$. A general cuboid domain can be linearly mapped into $\Omega = [0, 2\pi]^3$. Let $C^\infty(\Omega)$ be the set of infinitely differentiable periodic functions with period $2\pi$, and $H^r(\Omega)$ be the closure of $C^\infty(\Omega)$ in $H^r(\Omega)$. Define the inner product by $\langle u, v \rangle_{\Omega} = \frac{1}{8\pi^3} \int_{\Omega} u(x, y, z)v(x, y, z)dx dy dz$ and the discrete inner product and norm by, respectively, $\langle u, v \rangle_N = \frac{1}{N^3} \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \sum_{l=1}^{N_z} u(x_j, y_k, z_l)v(x_j, y_k, z_l)$, $\| u \|_N^2 = \langle u, u \rangle_N$. The norm and seminorm of $H^r(\Omega)$ are denoted by $\| \cdot \|_r$ and $\| \cdot \|_r$ respectively. In particular, $\| \cdot \|_0 = \| \cdot \|$. Let the interpolation space $S^l_N = \{ u | u = \sum_{|j|, |k|, |l| \leq N} \hat{u}_{j,k,l} e^{i(jx+kz)} : \hat{u}_{j,k,l} = \hat{u}/N \}$, where $c_1 = 1$ for $|l| < \frac{N}{2}$ and $c_1 = 2$ for $|l| = \frac{N}{2}$.

We denote $S^O_N = \{ u | u = \sum_{|j|, |k|, |l| \leq N} \hat{u}_{j,k,l} e^{i(jx+kz)} : \hat{u}_{j,k,l} = \hat{u}/N \}$.

We here remark that $S^l_N \subset S^O_N$. We denote $P^l_N : [L^2(\Omega)]^3 \to [S^l_N]^3$ as the orthogonal projection operator and recall the interpolation operator $P^l_N : [C(\Omega)]^3 \to [S^l_N]^3$.

**Lemma 3.1.** ([1]) For all $u \in [S^O_N]^3$, we have $\| u \|_0 \leq \| u \|_N \leq 2\sqrt{2} \| u \|_0$. If $u, v \in [S^O_N]^3$, then $\langle \partial_w u, v \rangle_N = -\langle u, \partial_w v \rangle_N$ for $w = x, y, z$. 

Algorithm 2 (Fully discrete scheme) The goal of the algorithm is to obtain numerical solution $E_{w}^{\text{end}} \in \mathbb{C}^{N \times N \times N}$ and $H_{w}^{\text{end}} \in \mathbb{C}^{N \times N \times N}$ such that $[E_{w}^{\text{end}}]_{N \times N \times N}^{(l-1)+N \times (k-1)+j} \approx E_{w}(x, y, z, t_{\text{end}})$ and $[H_{w}^{\text{end}}]_{N \times N \times N}^{(l-1)+N \times (k-1)+j} \approx H_{w}(x, y, z, t_{\text{end}})$ for $w = x, y, z$.

Input: $N_{x}, N_{y}, N_{z}, E_{w}^{0}, H_{w}^{0}$ (obtained by Algorithm 1) and $t_{\text{end}}$

Output: $E_{w}^{\text{end}}, H_{w}^{\text{end}}$ (such that $[E_{w}^{\text{end}}]_{N \times N \times N}^{(l-1)+N \times (k-1)+j} \approx E_{w}(x, y, z, t_{\text{end}})$ and $[H_{w}^{\text{end}}]_{N \times N \times N}^{(l-1)+N \times (k-1)+j} \approx H_{w}(x, y, z, t_{\text{end}})$)

1. Set $a_{w} := \frac{2 \pi}{w_{i} - w_{j}} \begin{pmatrix} 0, 1, \ldots, N_{x} - 1, 1, \ldots, N_{y} - 1, \ldots, -1 \end{pmatrix}^{T} \in \mathbb{R}^{N_{w}}$ for $w = x, y, z$.

   Cost: $O(N_{x} + N_{y} + N_{z})$. Storage: $O(N_{x}N_{y}N_{z})$.

2. Compute $b_{w} = 1_{N_{x}} \otimes 1_{N_{y}} \otimes a_{w}$, $b_{y} = 1_{N_{x}} \otimes a_{y} \otimes 1_{N_{z}}$, $b_{z} = a_{x} \otimes 1_{N_{y}} \otimes 1_{N_{z}}$, and $\Psi = \kappa^{2}(b_{x}^{2} + b_{y}^{2} + b_{z}^{2})$. Here $\kappa = t_{\text{end}}/\sqrt{N_{w}}$ and $1_{N_{w}} = (1, 1, \ldots, 1)^{T} \in \mathbb{R}^{N_{w}}$.

   Cost: $O(N_{x}N_{y}N_{z})$. Storage: $O(N_{x}N_{y}N_{z})$.

3. Compute (only one step)

   \begin{align*}
   r_{1} &= (\cosh \sqrt{-\Psi} - 1_{N_{x}N_{y}N_{z}}) / \Psi, \quad r_{2} = \sinh \sqrt{-\Psi} / \sqrt{-\Psi}, \\
   c_{11} &= 1_{N_{x}N_{y}N_{z}} + \kappa^{2}(b_{x}^{2} + b_{y}^{2} + b_{z}^{2}), \quad r_{1}, \quad c_{12} = -\kappa^{2}b_{y} \otimes b_{x} \otimes r_{1}, \quad s_{12} = i\kappa b_{y} \otimes b_{x}, \quad r_{2}, \\
   c_{22} &= 1_{N_{x}N_{y}N_{z}} + \kappa^{2}(b_{y}^{2} + b_{z}^{2}), \quad r_{1}, \quad c_{13} = -\kappa^{2}b_{x} \otimes b_{y} \otimes r_{1}, \quad s_{13} = i\kappa b_{x} \otimes b_{y}, \quad r_{2}, \\
   c_{33} &= 1_{N_{x}N_{y}N_{z}} + \kappa^{2}(b_{z}^{2} + b_{x}^{2} + b_{y}^{2}), \quad r_{1}, \quad c_{23} = -\kappa^{2}b_{z} \otimes b_{y} \otimes r_{1}, \quad s_{23} = i\kappa b_{z} \otimes b_{y}, \quad r_{2},
   \end{align*}

   where $\otimes$ and $\cdot$ denote the element-by-element division and multiplication of two vectors, respectively.

   Cost: $O(N_{x}N_{y}N_{z})$. Storage: $O(N_{x}N_{y}N_{z})$.

4. Compute (only one step)

   \begin{align*}
   \hat{H}_{x}^{0} &= \sqrt{-\Psi} \hat{H}_{y}^{0}, \quad \hat{H}_{y}^{0} := \sqrt{-\Psi} \hat{H}_{z}^{0}, \quad \hat{H}_{z}^{0} := \sqrt{-\Psi} \hat{H}_{x}^{0}, \\
   \hat{E}_{x}^{0} &= c_{11} \cdot \hat{H}_{y}^{0} + c_{12} \cdot \hat{H}_{z}^{0} + c_{13} \cdot \hat{H}_{x}^{0} + s_{12} \cdot \hat{E}_{y}^{0} + s_{13} \cdot \hat{E}_{z}^{0} - s_{12} \cdot \hat{E}_{y}^{0} - s_{13} \cdot \hat{E}_{z}^{0}, \\
   \hat{H}_{y}^{0} &= c_{12} \cdot \hat{H}_{x}^{0} + c_{22} \cdot \hat{H}_{z}^{0} + c_{23} \cdot \hat{H}_{x}^{0} + s_{22} \cdot \hat{H}_{x}^{0} + s_{23} \cdot \hat{H}_{z}^{0} - s_{22} \cdot \hat{H}_{x}^{0} - s_{23} \cdot \hat{H}_{z}^{0}, \\
   \hat{H}_{z}^{0} &= c_{13} \cdot \hat{H}_{x}^{0} + c_{23} \cdot \hat{H}_{y}^{0} + c_{33} \cdot \hat{H}_{y}^{0} + s_{23} \cdot \hat{H}_{z}^{0} + s_{33} \cdot \hat{H}_{z}^{0} - s_{23} \cdot \hat{H}_{z}^{0} - s_{33} \cdot \hat{H}_{z}^{0}, \\
   \hat{E}_{x}^{0} &= c_{11} \cdot \hat{E}_{y}^{0} + c_{12} \cdot \hat{E}_{y}^{0} + c_{13} \cdot \hat{E}_{y}^{0} + \hat{E}_{y}^{0} - s_{12} \cdot \hat{H}_{y}^{0} + s_{13} \cdot \hat{H}_{y}^{0}, \\
   \hat{E}_{y}^{0} &= c_{12} \cdot \hat{E}_{x}^{0} + c_{22} \cdot \hat{E}_{x}^{0} + \hat{E}_{x}^{0} - s_{22} \cdot \hat{H}_{x}^{0} + s_{23} \cdot \hat{H}_{x}^{0}, \\
   \hat{E}_{z}^{0} &= c_{13} \cdot \hat{E}_{x}^{0} + c_{23} \cdot \hat{E}_{y}^{0} + \hat{E}_{z}^{0} - s_{23} \cdot \hat{H}_{y}^{0} + s_{33} \cdot \hat{H}_{z}^{0},
   \end{align*}

   Cost: $O(N_{x}N_{y}N_{z})$. Storage: $O(N_{x}N_{y}N_{z})$.

5. Using Inverse Fast Fourier Transform (IFFT), compute for $w = x, y, z$.

   \begin{align*}
   E_{w}^{\text{end}} &= (\mathcal{F}_{N_{x}}^{-1} \otimes \mathcal{F}_{N_{y}}^{-1} \otimes \mathcal{F}_{N_{z}}^{-1}) \hat{E}_{w}^{\text{end}}, \\
   H_{w}^{\text{end}} &= (\mathcal{F}_{N_{x}}^{-1} \otimes \mathcal{F}_{N_{y}}^{-1} \otimes \mathcal{F}_{N_{z}}^{-1}) \hat{H}_{w}^{\text{end}}.
   \end{align*}

   Cost: $O(N_{x}N_{y}N_{z})$. Storage: $O(N_{x}N_{y}N_{z})$.

Lemma 3.2. \([I]\) $\langle P_{N}^{Q} u, v \rangle_{N} = \langle u, v \rangle_{N}$ for $v \in \mathcal{S}_{N}^{Q}$, By noting $P_{N}^{Q} \partial_{u} u = \partial_{u} P_{N}^{Q} u$ for $w = x, y, z$, we can see that $\partial u$ and $P_{N}^{Q}$ satisfy the commutative law.

Lemma 3.3. \([II]\) If $0 \leq \alpha \leq r$ and $u \in [H^{r}_{p}(\Omega)]^{3}$, then $\|P_{N}^{Q} u - u\|_{\alpha} \leq CN^{\alpha - r} \|u\|_{r}$, and in addition if $r > 3/2$ then $\|P_{N}^{Q} u - u\|_{\alpha} \leq CN^{\alpha - r} \|u\|_{r}$. 


Theorem 3.4. (Convergence.) Suppose that the exact solution $H, E \in C^1(0, t_{end}; [H^p_r(\Omega)]^3)$ and the initial values $H_0, E_0 \in [H^p_r(\Omega)]^3$, where $r > 3/2$ and the initial values are assumed to be bounded. Let $E^{\text{end}} = (E^{x, end}_y, E^{y, end}_x, E^{z, end}_z)^\top$, $H^{\text{end}} = (H^{x, end}_x, H^{y, end}_y, H^{z, end}_z)^\top$ be the solutions of the scheme \[2.3\]. Then, for any fixed $t_{end}$ there exists a positive constant $C$ independent of $\Delta t, h_x, h_y, h_z, t_{end}, N, \mu, \varepsilon$ such that

$$\left( \frac{\mu}{2} \left\| H^{\text{end}} - H(t_{end}) \right\|_{C^2}^2 + \varepsilon \left\| E^{\text{end}} - E(t_{end}) \right\|_N^2 \right)^{\frac{1}{2}} \leq C(\sqrt{\mu} + \sqrt{\varepsilon})N^{-r},$$

where $N = N_x = N_y = N_z$.

Proof. Let $E^* = P_{N-2}^Q E$, $H^* = P_{N-2}^Q H$. The projections of Eqs. \[2.1\] are written as

$$\partial \frac{\partial}{\partial x} \left( \frac{\sqrt{\mu} H^*}{\sqrt{\varepsilon} E^*} \right) = \frac{1}{\sqrt{\mu} \varepsilon} \left( \begin{array}{cc} 0 & -\text{curl} \\ \text{curl} & 0 \end{array} \right) \left( \frac{\sqrt{\mu} H^*}{\sqrt{\varepsilon} E^*} \right).$$

(3.1)

Noting that $E^* \in [S_{N-2}^l]^3 \subseteq [S_N^l]^3 \subseteq [S_N^l]^3$, we obtain that

$$\begin{align*}
\frac{\partial}{\partial x} E^*_w(x_j, y_k, z_l) &= \frac{\partial}{\partial x} P^*_w E^*_w(x_j, y_k, z_l) = \left[ D_1 E^*_w \right]_{N_x, N_y, 0 + 3}, \\
\frac{\partial}{\partial y} E^*_w(x_j, y_k, z_l) &= \frac{\partial}{\partial y} P^*_w E^*_w(x_j, y_k, z_l) = \left[ D_2 E^*_w \right]_{N_x, N_y, 0 + 3}, \\
\frac{\partial}{\partial z} E^*_w(x_j, y_k, z_l) &= \frac{\partial}{\partial z} P^*_w E^*_w(x_j, y_k, z_l) = \left[ D_3 E^*_w \right]_{N_x, N_y, 0 + 3},
\end{align*}$$

where $D_1 = I_{N_x} \otimes I_{N_y} \otimes D_x$, $D_2 = I_{N_x} \otimes D_y \otimes I_{N_z}$, $D_3 = D_z \otimes I_{N_y} \otimes I_{N_z}$, and

$$E^*_w = (E^*_w, 1, 1, \ldots, E^*_w, N_x, 1, 1, \ldots, E^*_w, 1, 2, 1, \ldots, E^*_w, N_x, 2, 1, \ldots, E^*_w, N_y, N_z, \ldots, E^*_w, N_y, N_z, \ldots, E^*_w, N_y, N_z)^\top \text{ for } w = x, y, z.$$ Similar results are obvious for $H^*$. Thus Eqs. \[3.1\] are transformed into

$$\frac{d}{dt} \left( \frac{\sqrt{\mu} H^*}{\sqrt{\varepsilon} E^*} \right) = D \left( \frac{\sqrt{\mu} H^*}{\sqrt{\varepsilon} E^*} \right),$$

(3.2)

with $D = \frac{1}{\sqrt{\mu} \varepsilon} \left( \begin{array}{cc} 0 & -D \\ D & 0 \end{array} \right)$ and $E^* = (E^*_x, E^*_y, E^*_z)^\top$, $H^* = (H^*_x, H^*_y, H^*_z)^\top$.

On the other hand, the Fully discrete scheme \[2.3\] is equivalent to finding the numerical solution $(\tilde{H}, \tilde{E})^\top \in [S_N^l]^6$ such that

$$\left\langle \frac{d}{dt} \left( \frac{\sqrt{\mu} H^*}{\sqrt{\varepsilon} E^*} \right), \left( \begin{array}{c} \tilde{\mu} \\ \tilde{\nu} \end{array} \right) \right\rangle_N = \left\langle D \left( \frac{\sqrt{\mu} H^*}{\sqrt{\varepsilon} E^*} \right), \left( \begin{array}{c} \frac{\tilde{\mu}}{\sqrt{\mu}} \\ \frac{\tilde{\nu}}{\sqrt{\nu}} \end{array} \right) \right\rangle_N,$$

(3.3)

for all $(\tilde{\mu}, \tilde{\nu})^\top \in [S_N^l]^6$.

Denote the errors $\mathcal{H}^t = \tilde{H}(t) - H^*(t)$ and $\mathcal{E}^t = \tilde{E}(t) - E^*(t)$. Based on the formulae \[3.2\]-\[3.3\], it is clear that

$$\left\langle \frac{d}{dt} \left( \frac{\sqrt{\mu} H^t}{\sqrt{\varepsilon} E^t} \right), \left( \begin{array}{c} \tilde{\mu} \\ \tilde{\nu} \end{array} \right) \right\rangle_N = \left\langle D \left( \frac{\sqrt{\mu} H^t}{\sqrt{\varepsilon} E^t} \right), \left( \begin{array}{c} \frac{\tilde{\mu}}{\sqrt{\mu}} \\ \frac{\tilde{\nu}}{\sqrt{\nu}} \end{array} \right) \right\rangle_N,$$

which is

$$\left\langle \left( \frac{\sqrt{\mu} H^t}{\sqrt{\varepsilon} E^t} \right), \left( \begin{array}{c} \tilde{\mu} \\ \tilde{\nu} \end{array} \right) \right\rangle_N = \left\langle e^D \left( \frac{\sqrt{\mu} H^0}{\sqrt{\varepsilon} E^0} \right), \left( \begin{array}{c} \tilde{\mu} \\ \tilde{\nu} \end{array} \right) \right\rangle_N.$$
Taking \( \left( \frac{\mu}{\nu} \right) = \left( \frac{\sqrt{\mu} H^t}{\sqrt{\nu} E^t} \right) + e^D \left( \frac{\sqrt{\mu} H^0}{\sqrt{\nu} E^0} \right) \) leads to

\[
0 = \left\| \left( \frac{\sqrt{\mu} H^t}{\sqrt{\nu} E^t} \right) \right\|^2_N - \left\| e^D \left( \frac{\sqrt{\mu} H^0}{\sqrt{\nu} E^0} \right) \right\|^2_N + \left( \left( \frac{\sqrt{\mu} H^t}{\sqrt{\nu} E^t} \right) , e^D \left( \frac{\sqrt{\mu} H^0}{\sqrt{\nu} E^0} \right) \right)_N
- \left( e^D \left( \frac{\sqrt{\mu} H^0}{\sqrt{\nu} E^0} \right) , \left( \frac{\sqrt{\mu} H^t}{\sqrt{\nu} E^t} \right) \right)_N = \left\| \left( \frac{\sqrt{\mu} H^t}{\sqrt{\nu} E^t} \right) \right\|^2_N - \left\| \left( \frac{\sqrt{\mu} H^0}{\sqrt{\nu} E^0} \right) \right\|^2_N.
\]

This shows that \( \mu \| H^t \|_N^2 + \varepsilon \| E^t \|_N^2 = \mu \| H^0 \|_N^2 + \varepsilon \| E^0 \|_N^2 \).

In what follows, we estimate \( \| H^0 \|_N^2 \) and \( \| E^0 \|_N^2 \). For \( H^0 = \tilde{H}(0) - P_{N-2}^O H(0) \in [S^t_N]^3 \), we transform the norm by using the result \( \| H^0 \|_N \leq \| \tilde{H}^0 \|_N \leq 2\sqrt{2} \| \tilde{H}^0 \|_0 \) and then study the bound of \( \| H^0 \|_0 \). To this end, we compute

\[
\| H^0 \|_0 = \| \tilde{H}(0) - P_{N-2}^O H(0) \|_0 = \| P_N^I(0) H(0) - P_{N-2}^O H(0) \|_0 \\
\leq \| P_N^I(0) H(0) - H(0) \|_0 + \| P_{N-2}^O H(0) - H(0) \|_0 \leq C N^{-r}.
\]

Similar result \( \| E^0 \|_N \leq 2\sqrt{2} \| E^0 \|_0 \leq C N^{-r} \) can be obtained. Therefore, \( \left( \mu \| H^t \|_N^2 + \varepsilon \| E^t \|_N^2 \right)^\frac{1}{2} \leq C(\sqrt{\mu} + \sqrt{\varepsilon}) N^{-r} \).

With these estimates, we are now in a position to present the error

\[
\left( \mu \| H^{t_{end}} - H(t_{end}) \|_N^2 + \varepsilon \| E^{t_{end}} - E(t_{end}) \|_N^2 \right)^\frac{1}{2} \\
\leq \left( \mu \| \tilde{H}(t_{end}) - H(t_{end}) \|_N^2 + \varepsilon \| E(t_{end}) - E(t_{end}) \|_N^2 \right)^\frac{1}{2} \\
\leq \left( \mu \| H^{t_{end}} + P_{N-2}^O H(t_{end}) - H(t_{end}) \|_N^2 + \varepsilon \| E^{t_{end}} + P_{N-2}^O E(t_{end}) - E(t_{end}) \|_N^2 \right)^\frac{1}{2} \\
\leq \sqrt{\mu} \| H^{t_{end}} + P_{N-2}^O H(t_{end}) - H(t_{end}) \|_N + \sqrt{\varepsilon} \| E^{t_{end}} + P_{N-2}^O E(t_{end}) - E(t_{end}) \|_N \\
\leq \sqrt{\mu} \| P_{N-2}^O H(t_{end}) - H(t_{end}) \|_N + \sqrt{\varepsilon} \| P_{N-2}^O E(t_{end}) - E(t_{end}) \|_N + C(\sqrt{\mu} + \sqrt{\varepsilon}) N^{-r} \\
\leq C(\sqrt{\mu} + \sqrt{\varepsilon}) N^{-r},
\]

where we have used the boundedness \([2,2]\) of the solution to get

\[
\sqrt{\mu} \| P_{N-2}^O H(t_{end}) - H(t_{end}) \|_N \leq 2\sqrt{2} \left( \sqrt{\mu} \| P_{N-2}^O H(t_{end}) - H(t_{end}) \|_0 + \sqrt{\varepsilon} \| P_{N-2}^O E(t_{end}) - E(t_{end}) \|_0 \right) \\
\leq C(\sqrt{\mu} + \sqrt{\varepsilon}) (N-2)^{-r}.
\]

**Remark 3.5.** It is noted that the scheme is of spectral accuracy in space and infinite-order accuracy in time. For the Maxwell’s equations with enough smoothness solutions, the scheme will converge with infinite-order accuracy both in space and in time.

4. **Structure preserving laws**

In this section, we rigorously prove the discrete structure preserving laws of the proposed scheme including the energy, helicity, momentum, symplecticity, and divergence-free field conservation laws.
Theorem 4.1. (Energy conservation laws.) The solutions $E^{t_{end}}$, $H^{t_{end}}$ produced by the fully discrete scheme \[ \{ E \}^{t_{end}} \] satisfy the discrete energy conservation laws

$$
\begin{align*}
\mathcal{E}_1^{t_{end}} &= \mathcal{E}_1^0, \\
\mathcal{E}_2^{t_{end}} &= \mathcal{E}_2^0, \\
\mathcal{E}_3^{t_{end}} &= \mathcal{E}_3^0, \\
\mathcal{E}_4^{t_{end}} &= \mathcal{E}_4^0, \\
\mathcal{E}_5^{t_{end}} &= \mathcal{E}_5^0, \\
\mathcal{E}_6^{t_{end}} &= \mathcal{E}_6^0,
\end{align*}
$$

where

$$
\begin{align*}
\mathcal{E}_1^t &= \frac{1}{2} \langle H, H \rangle + \epsilon \langle E, E \rangle, \\
\mathcal{E}_2^t &= \frac{1}{2} \sum_{w=x,y,z} \langle D_k H_w, D_k H_w \rangle + \epsilon \sum_{w=x,y,z} \langle D_k E_w, D_k E_w \rangle \\
\mathcal{E}_3^t &= \frac{1}{2} \sum_{w=x,y,z} \langle D_k H_w, D_k H_w \rangle - \frac{1}{2} \sum_{w=x,y,z} \langle D_k E_w, D_k E_w \rangle \\
\mathcal{E}_4^t &= \frac{1}{2} \sum_{w=x,y,z} \langle D_k E_w, D_k E_w \rangle \\
\mathcal{E}_5^t &= \frac{1}{2} \sum_{w=x,y,z} \langle D_k E_w, D_k E_w \rangle \\
\mathcal{E}_6^t &= \frac{1}{2} \sum_{w=x,y,z} \langle D_k E_w, D_k E_w \rangle
\end{align*}
$$

for $k = 1, 2, 3$.

Proof. Based on the formulation of fully discrete scheme, it is known that $E^{t_{end}}$ and $H^{t_{end}}$ satisfy

$$
\frac{d}{dt} \left( \sqrt{\mathcal{E}} H^{t_{end}} \right) = D \left( \sqrt{\mathcal{E}} \mathcal{H}^{t_{end}} \right). \tag{4.1}
$$

Therefore, we have that

$$
\frac{d}{dt} \epsilon_1^{t_{end}} = -\mu (H^{t_{end}})^T H^{t_{end}} + \epsilon (E^{t_{end}})^T E^{t_{end}} + (H^{t_{end}})^T (-DE^{t_{end}}) + (E^{t_{end}})^T (DH^{t_{end}}).
$$

Using the property that $D^T = D$, one gets

$$
\frac{d}{dt} \epsilon_2^{t_{end}} = (H^{t_{end}})^T (-DE^{t_{end}}) + (E^{t_{end}})^T (DH^{t_{end}}) = (H^{t_{end}})^T (-DE^{t_{end}}) + (H^{t_{end}})^T (DE^{t_{end}}) = 0.
$$

From (4.1), it follows that

$$
\frac{d}{dt} \mathcal{E}_2^{t_{end}} = \mu (H^{t_{end}})^T H^{t_{end}} + \epsilon (E^{t_{end}})^T E^{t_{end}} + (H^{t_{end}})^T (-DE^{t_{end}}) + (E^{t_{end}})^T (DH^{t_{end}})
$$

$$
= (H^{t_{end}})^T (-DE^{t_{end}}) + (H^{t_{end}})^T (DE^{t_{end}}) = 0.
$$

For the block diagonal matrices $B_k = \begin{pmatrix} D_k & D_k \\ D_k & D_k \end{pmatrix}$, it is easy to see that

$$
B_k D = \begin{pmatrix} D_k & D_k \\ D_k & D_k \end{pmatrix} \begin{pmatrix} 0 & -D_k \otimes I_{N_y} \otimes I_{N_z} \\ D_k \otimes I_{N_y} \otimes I_{N_z} & 0 \end{pmatrix} = DB_k,
$$

$$
= \begin{pmatrix} D_k \otimes I_{N_y} \otimes I_{N_z} & 0 \\ -I_{N_y} \otimes D_k \otimes I_{N_z} & -D_k \otimes I_{N_y} \otimes I_{N_z} \end{pmatrix} = DB_k.
$$
where we have used the commutative law of $D_k$ which can be shown for $D_2D_1$ as follows:

$$D_2D_1 = (I_N \otimes D_y \otimes I_N_x)(I_N \otimes I_N_y \otimes D_x) = I_N \otimes (D_y \otimes I_N_x)(I_N \otimes D_x \otimes D_x) = D_1D_2.$$  

Then, left-multiplying (4.1) with block diagonal matrix $\text{diag}(B_k, B_k)$, we have

$$\frac{d}{dt} \left( \sqrt{\mu} B_k \mathbf{H}^{t_{\text{end}}}_w \right) = \frac{1}{\sqrt{\mu}} \left( \begin{array}{ccc} 0 & -B_k D \\ B_k D & 0 \end{array} \right) \left( \sqrt{\mu} \mathbf{E}^{t_{\text{end}}}_w \right) \left( \sqrt{\mu} B_k \mathbf{H}^{t_{\text{end}}}_w \right).$$  

(4.2)

Based on this scheme and the same arguments as $\mathcal{E}_1^{t_{\text{end}}}$, it is obtained that $\frac{d}{dt} \mathcal{E}_3^{t_{\text{end}}} = 0$.

The statement of $\mathcal{E}_2^{t_{\text{end}}}$ can be proved by combining the proofs of $\mathcal{E}_2^{t_{\text{end}}}$ and $\mathcal{E}_3^{t_{\text{end}}}$.

For the energy $\mathcal{E}_5^{t_{\text{end}}}$, it is deduced that

$$\frac{d}{dt} \mathcal{E}_5^{t_{\text{end}}} = \mu \sum_{w=x,y,z} (\mathbf{H}^{t_{\text{end}}}_w)^T \mathbf{D}_k \mathbf{H}^{t_{\text{end}}}_w + \varepsilon \sum_{w=x,y,z} (\mathbf{E}^{t_{\text{end}}}_w)^T \mathbf{D}_k \mathbf{E}^{t_{\text{end}}}_w$$

$$= \mu (\mathbf{H}^{t_{\text{end}}})^T \mathbf{B}_k \mathbf{H}^{t_{\text{end}}} + \varepsilon (\mathbf{E}^{t_{\text{end}}})^T \mathbf{B}_k \mathbf{E}^{t_{\text{end}}} = -(\mathbf{H}^{t_{\text{end}}})^T \mathbf{D}_k \mathbf{E}^{t_{\text{end}}} + (\mathbf{E}^{t_{\text{end}}})^T \mathbf{D}_k \mathbf{H}^{t_{\text{end}}} = 0.$$  

The last result of $\mathcal{E}_6^{t_{\text{end}}}$ can be proved in a similar way to that stated above. \qed

**Remark 4.2.** It is noted that the first derivatives of $\mathbf{H}^{t_{\text{end}}}, \mathbf{E}^{t_{\text{end}}}$ are needed in the results and they are obtained by

$$\dot{\mathbf{H}}^{t_{\text{end}}}_w = (\mathcal{F}^{-1}_{-1} \otimes \mathcal{F}^{-1}_{-1} \otimes \mathcal{F}^{-1}_{-1}) \mathbf{H}^{t_{\text{end}}}_w, \quad \dot{\mathbf{E}}^{t_{\text{end}}}_w = (\mathcal{F}^{-1}_{-1} \otimes \mathcal{F}^{-1}_{-1} \otimes \mathcal{F}^{-1}_{-1}) \mathbf{E}^{t_{\text{end}}}_w,$$

with

$$\dot{\mathbf{H}}_x = -\frac{1}{\mu} (\Omega_3 \mathbf{H}^{t_{\text{end}}}_y + \Omega_2 \mathbf{H}^{t_{\text{end}}}_z), \quad \dot{\mathbf{H}}_y = -\frac{1}{\mu} (\Omega_3 \mathbf{H}^{t_{\text{end}}}_x + \Omega_2 \mathbf{H}^{t_{\text{end}}}_z),$$

$$\dot{\mathbf{H}}_z = -\frac{1}{\mu} (\Omega_3 \mathbf{H}^{t_{\text{end}}}_x + \Omega_2 \mathbf{H}^{t_{\text{end}}}_y), \quad \dot{\mathbf{E}}_x = \frac{1}{\varepsilon} (\Omega_3 \mathbf{E}^{t_{\text{end}}}_y + \Omega_2 \mathbf{E}^{t_{\text{end}}}_z),$$

$$\dot{\mathbf{E}}_y = \frac{1}{\varepsilon} (\Omega_3 \mathbf{E}^{t_{\text{end}}}_x + \Omega_2 \mathbf{E}^{t_{\text{end}}}_z), \quad \dot{\mathbf{E}}_z = \frac{1}{\varepsilon} (\Omega_3 \mathbf{E}^{t_{\text{end}}}_x + \Omega_2 \mathbf{E}^{t_{\text{end}}}_y).$$

On the other hand, these discrete energy conservation laws imply that the numerical solutions are bounded in the $L^2$ norm and do not blow up. Therefore, the scheme proposed in the paper is unconditionally stable.

**Theorem 4.3.** (Helicity conservation laws.) For the solutions given by the scheme 2.3, two discrete Helicity conservation laws $\mathcal{H}_1^{t_{\text{end}}} = \mathcal{H}_0^{t_{\text{end}}}$, $\mathcal{H}_2^{t_{\text{end}}} = \mathcal{H}_0^{t_{\text{end}}}$ hold, where

$$\mathcal{H}_1^{t_{\text{end}}} = \frac{1}{2\varepsilon} (\mathbf{H}^t, D \mathbf{H}^t)_N + \frac{1}{2\mu} (\mathbf{E}^t, D \mathbf{E}^t)_N, \quad \mathcal{H}_1 = \frac{1}{2\varepsilon} (\frac{d}{dt} \mathbf{H}^t, D \frac{d}{dt} \mathbf{H}^t)_N + \frac{1}{2\mu} (\frac{d}{dt} \mathbf{E}^t, D \frac{d}{dt} \mathbf{E}^t)_N.$$

**Proof.** Using (4.1), it is arrived at

$$\frac{d}{dt} \mathcal{H}_1^{t_{\text{end}}} = \frac{1}{\varepsilon} (\mathbf{H}^{t_{\text{end}}})^T D \mathbf{H}^{t_{\text{end}}} + \frac{1}{\mu} (\mathbf{E}^{t_{\text{end}}})^T D \mathbf{E}^{t_{\text{end}}} = \frac{1}{\varepsilon\mu} (\mathbf{H}^{t_{\text{end}}})^T D (\mathbf{H}^{t_{\text{end}}}) + \frac{1}{\mu\varepsilon} (\mathbf{E}^{t_{\text{end}}})^T D (\mathbf{H}^{t_{\text{end}}})$$

$$= -\frac{1}{\varepsilon\mu} (\mathbf{H}^{t_{\text{end}}})^T D \mathbf{E}^{t_{\text{end}}} + \frac{1}{\mu\varepsilon} (\mathbf{E}^{t_{\text{end}}})^T D \mathbf{H}^{t_{\text{end}}} = 0.$$  

The second statement can be proved by the similar arguments to $\mathcal{H}_1^{t_{\text{end}}}$. \qed

**Theorem 4.4.** (Momentum conservation laws.) The solutions of the proposed scheme 2.3 possess the discrete momentum conservation laws $\mathcal{M}_1^{t_{\text{end}}} = \mathcal{M}_0^{t_{\text{end}}}$, $\mathcal{M}_2^{t_{\text{end}}} = \mathcal{M}_2^{t_{\text{end}}}$, where for $k = 1, 2, 3$

$$\mathcal{M}_1^{t_{\text{end}}} = (\mathbf{H}^t, B_k \mathbf{E}^t)_N, \quad \mathcal{M}_2^{t_{\text{end}}} = (\mathbf{E}^t, B_k \mathbf{H}^t)_N$$  

with $B_k = \text{diag}(D_k, D_k, D_k)$. 

Proof. We first compute
\[
\frac{d}{dt} \mathcal{M}^{\text{end}} \cdot (H^{\text{end}})^{T} B_{k}^\text{end} E^{\text{end}} + (H^{\text{end}})^{T} B_{k} E^{\text{end}} = \frac{1}{\varepsilon} (H^{\text{end}})^{T} B_{k} D H^{\text{end}} - \frac{1}{\mu} (D E^{\text{end}})^{T} B_{k} E^{\text{end}}.
\]
With the properties $D^{T} = D$, $B_{k}^{T} = -B_{k}$ and $B_{k} D = D B_{k}$, one has $(B_{k} D)^{T} = -B_{k} D$. Thus it is clear that $\frac{d}{dt} \mathcal{M}^{\text{end}} = 0$. The other statement can be shown in the same way.

\[\Box\]

**Theorem 4.5. (Symplecticity conservation law.)** The solutions of the scheme \(2.3\) have the discrete symplecticity conservation law \(dE^{\text{end}} \wedge dH^{\text{end}} = dE^{0} \wedge dH^{0}\), where
\[
dE^{t} \wedge dH^{t} = dE_{x}^{t} \wedge dH_{x}^{t} + dE_{y}^{t} \wedge dH_{y}^{t} + dE_{z}^{t} \wedge dH_{z}^{t}.
\]

**Proof.** Considering \(4.1\), we deduce that
\[
(\sqrt{\varepsilon} H^{\text{end}} \wedge \sqrt{\varepsilon} E^{\text{end}}) = e^{t_{\text{end}} D}(\sqrt{\varepsilon} H^{0} \wedge \sqrt{\varepsilon} E^{0}) = \left( \begin{array}{cc} \cos \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) & -\sin \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) \\ \sin \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) & \cos \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) \end{array} \right) \left( \begin{array}{c} \sqrt{\varepsilon} H^{0} \\ -\sqrt{\varepsilon} E^{0} \end{array} \right).
\]
Therefore, one gets
\[
dE^{\text{end}} \wedge dH^{\text{end}} = \frac{1}{\sqrt{\varepsilon}} d \left( \sin \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) \sqrt{\varepsilon} H^{0} + \cos \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) \sqrt{\varepsilon} E^{0} \right) \wedge \frac{1}{\sqrt{\mu}} d \left( \cos \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) \sqrt{\varepsilon} H^{0} - \sin \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) \sqrt{\varepsilon} E^{0} \right)
\]
\[\cos^{2} \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) dE^{0} \wedge dH^{0} - \sin^{2} \left( \frac{t_{\text{end}}}{\sqrt{\mu \varepsilon}} D \right) dH^{0} \wedge dE^{0} = dE^{0} \wedge dH^{0},
\]
where we have used the fact that $dH^{0} \wedge dE^{0} = -dE^{0} \wedge dH^{0}$. 

\[\Box\]

**Theorem 4.6. (Divergence-free field conservation law.)** The following discrete divergence-free field conservation laws hold true for the solutions $E^{\text{end}}$, $H^{\text{end}}$ produced by the scheme \(2.3\)
\[
\tilde{\nabla} \cdot (\varepsilon E^{\text{end}}) = \tilde{\nabla} \cdot (\varepsilon E^{0}), \quad \tilde{\nabla} \cdot (\mu H^{\text{end}}) = \tilde{\nabla} \cdot (\mu H^{0}),
\]
where $\tilde{\nabla} \cdot (\varepsilon E^{t}) = D_{1}(\varepsilon E_{x}^{t}) + D_{2}(\varepsilon E_{y}^{t}) + D_{3}(\varepsilon E_{z}^{t})$, $\tilde{\nabla} \cdot (\mu H^{t}) = D_{1}(\mu H_{x}^{t}) + D_{2}(\mu H_{y}^{t}) + D_{3}(\mu H_{z}^{t})$.

**Proof.** Concerning the results of $D_{1}$, $D_{2}$, $D_{3}$ and $E^{\text{end}}$, it can be verified that
\[
\tilde{\nabla} \cdot E^{\text{end}} = D_{1} E_{x}^{\text{end}} + D_{2} E_{y}^{\text{end}} + D_{3} E_{z}^{\text{end}}
\]
\[= F^{-1}_{N_{y}}(\Lambda_{1} c_{11} + \Lambda_{2} c_{12} + \Lambda_{3} c_{13}) F_{N_{x}} E_{x}^{0} + F^{-1}_{N_{y}}(\Lambda_{1} c_{12} + \Lambda_{2} c_{22} + \Lambda_{3} c_{23}) F_{N_{x}} E_{y}^{0} + F_{N_{x}}(\Lambda_{1} c_{13} + \Lambda_{2} c_{23} + \Lambda_{3} c_{33}) F_{N_{y}} E_{z}^{0}
\]
\[+ \frac{\sqrt{\mu}}{\sqrt{\varepsilon}} F^{-1}_{N_{x}}(-\Lambda_{1} s_{12} + \Lambda_{3} s_{23}) F_{N_{y}} H_{x}^{0} + \frac{\sqrt{\mu}}{\sqrt{\varepsilon}} F^{-1}_{N_{x}}(\Lambda_{1} s_{13} - \Lambda_{2} s_{23}) F_{N_{y}} H_{y}^{0} + \frac{\sqrt{\mu}}{\sqrt{\varepsilon}} F^{-1}_{N_{x}}(\Lambda_{1} s_{13} - \Lambda_{2} s_{23}) F_{N_{y}} H_{z}^{0},
\]
where $F_{N_{x}} = F_{N} \otimes F_{N_{y}} \otimes F_{N_{z}}$ and we omit $(t_{\text{end}} A/\sqrt{\mu \varepsilon})$ for brevity. According to the results given in the formulation of the method, it can be checked that
\[
\Lambda_{1} c_{11} + \Lambda_{2} c_{12} + \Lambda_{3} c_{13} = I, \quad \Lambda_{1} c_{12} + \Lambda_{2} c_{22} + \Lambda_{3} c_{23} = I, \quad \Lambda_{1} c_{13} + \Lambda_{2} c_{23} + \Lambda_{3} c_{33} = I,
\]
\[- \Lambda_{1} s_{12} + \Lambda_{3} s_{23} = 0, \quad \Lambda_{2} s_{12} - \Lambda_{3} s_{13} = 0, \quad \Lambda_{1} s_{13} - \Lambda_{2} s_{23} = 0,
\]
which lead to the first result of this theorem. The second one can be proved in a similar way. 

\[\Box\]
Remark 4.7. These conservation laws stated above are established in a discrete form, i.e., those invariants are defined by $H^t$ and $E^t$. For the error between the discrete conservations and the exact conservations, it can be obtained by considering the convergence shown in Theorem 3.2 which leads to the estimate $O(N^{-r})$. For instance, we take the discrete divergence-free field conservation laws and it can be shown that \[ \| \nabla \cdot (\varepsilon E^0) - \nabla \cdot (\varepsilon E^0) \| \leq CN^{-r}. \]

5. Numerical experiments

In this section, we present numerical experiments to show the performance of our scheme. These two tests are conducted in a sequential program in MATLAB on a laptop ThinkPad X1 Nano (CPU: 11th Gen Intel(R) Core(TM) i7-1160G7 @ 1.20GHz 2.11 GHz, Memory: 16 GB, Os: Microsoft Windows 11 with 64bit).

5.1. Standing wave solutions. The first test is devoted to the standing wave solutions of Maxwell’s equations (1.1) \((1, 6)\)

\[
E_x = \frac{k_y - k_z}{\varepsilon \sqrt{\mu \omega}} \cos(\omega t) \cos(k_x \pi x) \sin(k_y \pi y) \sin(k_z \pi z), \quad H_x = \sin(\omega t) \sin(k_x \pi x) \cos(k_y \pi y) \cos(k_z \pi z),
\]

\[
E_y = \frac{k_z - k_x}{\varepsilon \sqrt{\mu \omega}} \cos(\omega t) \sin(k_x \pi x) \cos(k_y \pi y) \sin(k_z \pi z), \quad H_y = \sin(\omega t) \cos(k_x \pi x) \sin(k_y \pi y) \cos(k_z \pi z),
\]

\[
E_z = \frac{k_x - k_y}{\varepsilon \sqrt{\mu \omega}} \cos(\omega t) \sin(k_x \pi x) \sin(k_y \pi y) \cos(k_z \pi z), \quad H_z = \sin(\omega t) \cos(k_x \pi x) \cos(k_y \pi y) \sin(k_z \pi z),
\]

where $\varepsilon = \mu = 1$, $\omega = \sqrt{k_x^2 + k_y^2 + k_z^2}$, $k_x = 1$, $k_y = 2$, $k_z = -3$ and $\Omega = [0, 2]^3$.

Energy conservation behaviour. In this part, we test the performance of our scheme in the structure preserving laws, which begins with the energy invariants. Define the relative errors in discrete energy invariants $\text{Re}(E_k) = \frac{|E^{n+1}_k - E^n_k|}{|E^n_k|}$ and display the results of our scheme for $k = 1, 2, 3, 4$ in Table 5. We note here that the scheme has a similar behaviour for the other two energy invariants $E_5, E_6$ and the corresponding results are skipped for brevity. Then we present in Table 6 the relative changes in discrete helicity and momentum invariants $\text{Re}(H_k) = \frac{|H^{n+1}_k - H^n_k|}{|H^n_k|}$ and $\text{Re}(M_k) = \frac{|M^{n+1}_k - M^n_k|}{|M^n_k|}$ for $k = 1, 2$. Finally, the relative errors in divergence-free field discrete helicity $\text{Re}(D_1) = \| \nabla \cdot (\varepsilon H^{n+1}_k) \|$ and $\text{Re}(D_2) = \| \nabla \cdot (\varepsilon E^{n+1}_k) \|$ are displayed in Table 7. It can be observed from the results that our scheme preserves the invariants exactly since the relative errors are within the roundoff error of the machine, which supports the theoretical analysis proposed in this paper. To show the long time conservation, we plot the errors in discrete energy invariants on $[0, 10000]$ and the results are shown in Figure 1. It can be seen that our scheme has a persistent conservation over long times.

Accuracy analysis. For this problem, the regularities are infinite so that as shown in Theorem 4.2 the solutions of the schemes converge with infinite-order accuracy both in space and in time. In our simulations, we use two norms which are defined by $L_2 = \left( \| E^{n+1}_k - E(t_{end}) \|^2_N + \| H^{n+1}_k - H(t_{end}) \|^2_N \right)^\frac{1}{2}$ and $L_\infty = \max \{ \max |E^{n+1}_k - E(t_{end})|, \max |H^{n+1}_k - H(t_{end})| \}$ to scale the maximal and average errors in solution, respectively. The numerical errors as well as the CPU time...
Table 1. The relative errors in discrete energy invariants for Example 1.

| Spatial grid points | Time   | Re(\mathcal{E}_1) | Re(\mathcal{E}_2) | Re(\mathcal{E}_3) | Re(\mathcal{E}_4) |
|---------------------|--------|-------------------|-------------------|-------------------|-------------------|
| \( N_x = N_y = N_z = 8 \) | \( t_{\text{end}} = 1 \) | 2.9605e-16 | 2.7425e-16 | 2.1331e-16 | 0.0000e-16 |
|                     | \( t_{\text{end}} = 5 \) | 1.4802e-16 | 1.3712e-16 | 2.1331e-16 | 3.9521e-16 |
|                     | \( t_{\text{end}} = 10 \) | 5.9211e-16 | 4.1138e-16 | 4.2662e-16 | 5.9281e-16 |
|                     | \( t_{\text{end}} = 15 \) | 1.4802e-16 | 1.3712e-16 | 0.0000e-16 | 1.9760e-16 |
|                     | \( t_{\text{end}} = 20 \) | 2.9605e-16 | 4.1138e-16 | 4.2662e-16 | 1.9760e-16 |
| \( N_x = N_y = N_z = 16 \) | \( t_{\text{end}} = 1 \) | 2.9605e-16 | 2.7425e-16 | 2.1331e-16 | 0.0000e-16 |
|                     | \( t_{\text{end}} = 5 \) | 0.0000e-16 | 1.3712e-16 | 4.2662e-16 | 1.9760e-16 |
|                     | \( t_{\text{end}} = 10 \) | 2.9605e-16 | 4.1138e-16 | 4.2662e-16 | 3.9521e-16 |
|                     | \( t_{\text{end}} = 15 \) | 0.0000e-16 | 0.0000e-16 | 0.0000e-16 | 0.0000e-16 |
|                     | \( t_{\text{end}} = 20 \) | 1.4802e-16 | 1.3712e-16 | 4.2662e-16 | 1.9760e-16 |

Table 2. The relative errors in discrete helicity and momentum invariants for Example 1.

| Spatial grid points | Time   | Re(\mathcal{H}_1) | Re(\mathcal{H}_2) | Re(\mathcal{M}_1) | Re(\mathcal{M}_2) |
|---------------------|--------|-------------------|-------------------|-------------------|-------------------|
| \( N_x = N_y = N_z = 8 \) | \( t_{\text{end}} = 1 \) | 34575e-14 | 53756e-14 | 6.6189e-12 | 5.7559e-12 |
|                     | \( t_{\text{end}} = 5 \) | 10759e-14 | 44125e-14 | 8.2414e-12 | 7.5206e-12 |
|                     | \( t_{\text{end}} = 10 \) | 53193e-15 | 31357e-15 | 2.3899e-12 | 6.0112e-12 |
|                     | \( t_{\text{end}} = 15 \) | 14507e-14 | 29453e-14 | 6.4139e-12 | 6.6866e-12 |
|                     | \( t_{\text{end}} = 20 \) | 42917e-14 | 63387e-14 | 3.7772e-12 | 2.1877e-11 |
| \( N_x = N_y = N_z = 16 \) | \( t_{\text{end}} = 1 \) | 2.0793e-14 | 1.6924e-13 | 1.2774e-10 | 5.0600e-10 |
|                     | \( t_{\text{end}} = 5 \) | 1.3323e-14 | 1.2081e-13 | 1.2704e-10 | 5.6392e-10 |
|                     | \( t_{\text{end}} = 10 \) | 6.1976e-14 | 1.3128e-13 | 2.6356e-11 | 5.8609e-10 |
|                     | \( t_{\text{end}} = 15 \) | 6.4600e-14 | 1.0865e-13 | 1.7854e-10 | 5.4005e-10 |
|                     | \( t_{\text{end}} = 20 \) | 4.8450e-15 | 2.1487e-13 | 1.2504e-10 | 3.4773e-10 |

Table 3. The errors in discrete divergence-free field conservation for Example 1.

| Time | \( N_x = N_y = N_z = 8 \) | \( N_x = N_y = N_z = 16 \) |
|------|-----------------|-----------------|
| \( t_{\text{end}} = 1 \) | 2.5121e-15 | 4.3962e-14 |
| \( t_{\text{end}} = 5 \) | 2.5121e-15 | 4.3962e-14 |
| \( t_{\text{end}} = 10 \) | 5.0242e-15 | 4.3962e-14 |
| \( t_{\text{end}} = 15 \) | 2.5121e-15 | 2.5121e-14 |
| \( t_{\text{end}} = 20 \) | 2.5121e-15 | 3.7682e-14 |

used in the scheme are listed in Table 8. From the results, it can be observed that our scheme provides numerical results with a machine accuracy and the computational cost is very low, which demonstrates the effectiveness and efficiency of the new scheme.
Figure 1. Numerical energy conservations of the schemes over long times for Example 1.

Table 4. The errors in the solution for Example 1.

| Time  | $L_\infty$ ($10^{-16}$) | $L_2$ ($10^{-16}$) | CPU (s) | $L_\infty$ ($10^{-16}$) | $L_2$ ($10^{-16}$) | CPU (s) |
|-------|--------------------------|---------------------|--------|--------------------------|---------------------|--------|
| $t_{end}=1$ | 2.8588e-13 | 3.6606e-14 | 0.055 | 7.0558e-12 | 3.3915e-13 | 0.20 |
| $t_{end}=5$ | 7.8246e-13 | 1.0241e-13 | 0.023 | 1.1084e-11 | 4.4399e-13 | 0.16 |
| $t_{end}=10$ | 1.4865e-12 | 1.9932e-13 | 0.013 | 1.3474e-11 | 6.2014e-13 | 0.16 |
| $t_{end}=15$ | 2.6716e-12 | 3.9738e-13 | 0.015 | 2.1543e-11 | 1.1193e-12 | 0.17 |
| $t_{end}=20$ | 2.7995e-12 | 3.9228e-13 | 0.030 | 2.5181e-11 | 1.2169e-12 | 0.17 |

5.2. Traveling wave solutions. The second numerical example concerns the Maxwell’s equations \( \varepsilon = \mu = 1 \) (see [1])

\[ E_x = \cos(2\pi(x + y + z) - 2\sqrt{3}\pi t), \quad E_y = -2E_x, \quad E_z = E_x, \quad H_x = \sqrt{3}E_x, \quad H_y = 0, \quad H_z = -\sqrt{3}E_x, \]

where \( t \in [0, t_{end}] \) and \( \Omega = [0,1]^3 \).

Tables 5-7 display the relative errors in discrete energy invariants, discrete helicity and momentum invariants, and discrete divergence-free field conservation, respectively. A long term energy conservation is indicated in Figure 2. The errors in the solution and the corresponding CPU time are listed in Table 8. From the results, it can be observed that our scheme provides a similar numerical phenomena to the first example.

Finally, some numerical comparisons of the existing schemes are made in Table 9. We take some comparisons from [1], where the structure-preserving method [1], the ADI-FDTD method [51], and the energy conserved splitting FDTD (EC-S-FDTD) method [6] are simulated with \( N_x = N_y = N_z = 32 \). For comparison, our scheme (referred as TEIFP) is implemented with \( N_x = N_y = N_z = 16 \) and the results clearly show that our scheme has an infinite-order accuracy, which is much better than the existing schemes with finite-order accuracy.

The numerical results of these two tests highlight the favorable behavior of the scheme presented in this paper. It can be observed that in comparison with other existing structure-preserving methods, the proposed scheme has very high accuracy and exact conservation laws, and requires very low computing cost.
Table 5. The relative errors in discrete energy invariants for Example 2.

| Spatial grid points | Time   | Re(\(\mathcal{E}_1\)) | Re(\(\mathcal{E}_2\)) | Re(\(\mathcal{E}_3\)) | Re(\(\mathcal{E}_4\)) |
|--------------------|--------|------------------------|------------------------|------------------------|------------------------|
| \(N_x = N_y = N_z = 8\) | \(t_{end} = 1\) | 2.9605e-16             | 1.5998e-16             | 1.1998e-16             | 1.2967e-16             |
|                    | \(t_{end} = 5\) | 2.9605e-16             | 0.0000e-16             | 1.1998e-16             | 2.5935e-16             |
|                    | \(t_{end} = 10\) | 0.0000e-16             | 3.1996e-16             | 3.5996e-16             | 1.2967e-16             |
|                    | \(t_{end} = 15\) | 1.4802e-16             | 1.5998e-16             | 1.1998e-16             | 0.0000e-16             |
|                    | \(t_{end} = 20\) | 0.0000e-16             | 3.1996e-16             | 1.1998e-16             | 1.2967e-16             |
| \(N_x = N_y = N_z = 16\) | \(t_{end} = 1\) | 0.0000e-16             | 1.5998e-16             | 0.0000e-16             | 0.0000e-16             |
|                    | \(t_{end} = 5\) | 2.9605e-16             | 0.0000e-16             | 2.3997e-16             | 0.0000e-16             |
|                    | \(t_{end} = 10\) | 1.4802e-16             | 1.5998e-16             | 1.1998e-16             | 2.5935e-16             |
|                    | \(t_{end} = 15\) | 2.9605e-16             | 0.0000e-16             | 2.3997e-16             | 1.2967e-16             |
|                    | \(t_{end} = 20\) | 1.4802e-16             | 3.1996e-16             | 0.0000e-16             | 2.5935e-16             |

Table 6. The relative errors in discrete helicity and momentum invariants for Example 2.

| Spatial grid points | Time   | Re(\(\mathcal{H}_1\)) | Re(\(\mathcal{H}_2\)) | Re(\(\mathcal{M}_1\)) | Re(\(\mathcal{M}_2\)) |
|--------------------|--------|------------------------|------------------------|------------------------|------------------------|
| \(N_x = N_y = N_z = 8\) | \(t_{end} = 1\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
|                    | \(t_{end} = 5\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
|                    | \(t_{end} = 10\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
|                    | \(t_{end} = 15\) | 7.0187e-12             | 8.3126e-10             | 2.9103e-16             | 1.1641e-10             |
|                    | \(t_{end} = 20\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
| \(N_x = N_y = N_z = 16\) | \(t_{end} = 1\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
|                    | \(t_{end} = 5\) | 3.5454e-9              | 1.9930e-7              | 3.7252e-9              | 3.7252e-9              |
|                    | \(t_{end} = 10\) | 2.7610e-9              | 1.0640e-7              | 0.0000e-16             | 7.6798e-9              |
|                    | \(t_{end} = 15\) | 5.1680e-9              | 1.7087e-7              | 3.7252e-9              | 8.3300e-9              |
|                    | \(t_{end} = 20\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 1.4210e-14             |

Table 7. The errors in discrete divergence-free field conservation for Example 2.

| Time | \(N_x = N_y = N_z = 8\) | \(N_x = N_y = N_z = 16\) |
|------|------------------------|------------------------|
|      | Re(\(\mathcal{D}_1\)) | Re(\(\mathcal{D}_2\)) |
| \(t_{end} = 1\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
| \(t_{end} = 5\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
| \(t_{end} = 10\) | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             | 0.0000e-16             |
| \(t_{end} = 15\) | 0.0000e-16             | 1.0048e-14             | 0.0000e-16             | 1.4210e-14             |
| \(t_{end} = 20\) | 0.0000e-16             | 1.0048e-14             | 0.0000e-16             | 2.8421e-14             |

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Figure 2. Numerical energy conservations of the schemes over long times for Example 2.

Table 8. The errors in the solution for Example 2.

| Time   | $N_x = N_y = N_z = 8$ | $N_x = N_y = N_z = 16$ |
|--------|----------------------|------------------------|
|        | $L_{\infty}$  | $L_2$  | CPU (s) | $L_{\infty}$  | $L_2$  | CPU (s) |
| $t_{end} = 1$ | 1.6253e-12 | 1.5437e-13 | 0.034   | 3.5895e-10 | 7.4340e-12 | 0.16   |
| $t_{end} = 5$ | 6.6265e-12 | 4.2131e-13 | 0.013   | 3.3974e-10 | 6.7938e-12 | 0.15   |
| $t_{end} = 10$ | 1.4294e-11 | 8.5423e-13 | 0.0055  | 3.9946e-10 | 7.6605e-12 | 0.14   |
| $t_{end} = 15$ | 3.0699e-12 | 4.6944e-13 | 0.0033  | 2.8697e-10 | 6.2506e-12 | 0.14   |
| $t_{end} = 20$ | 2.7620e-11 | 1.6616e-12 | 0.0061  | 3.2438e-10 | 7.6824e-12 | 0.16   |

Table 9. The errors in the solution for different methods of Example 2.

| Method  | $\Delta t = 0.05$ | $\Delta t = 0.025$ |
|---------|-------------------|---------------------|
|         | $L_2$  | Order | $L_2$  | Order |
| SAVF(2) | 3.74e-2 | 2.06   | 9.53e-2 | 1.97   |
| ADI-FDTD | 1.76e-1 | 1.79   | 4.53e-2 | 1.95   |
| EC-S-FDTD | 1.41e-1 | 1.84   | 3.62e-2 | 1.96   |
| TEIFP   | 3.63e-13 | machine accuracy | 3.43e-13 | machine accuracy |

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