Quantum Theory of Cavityless Feedback Cooling of An optically Trapped Nanoparticle

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We present a quantum theory of cavityless feedback cooling of an optically trapped harmonically oscillating subwavelength dielectric particle, a configuration recently realized in several experiments. Specifically, we derive a Markovian master equation that treats the mechanical as well as optical degrees of freedom quantum mechanically. Employing this equation, we solve for the nanoparticle phonon number dynamics exactly, and extract analytic expressions for the cooling timescale and the steady state phonon number. We present experimental data verifying the predictions of our model in the classical regime, and also demonstrate that quantum ground state preparation is within reach of ongoing experiments. Our work provides a quantitative framework for future theoretical modeling of the cavityless quantum optomechanics of optically trapped dielectric particles.

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Spectacular advances in experimental science have recently made it possible to observe quantum effects at more macroscopic scales than ever before [1]. These studies investigate, on the one hand, fundamental quantum phenomena such as superposition [2, 3], entanglement [4, 5], backaction [6], and decoherence [7]. On the other hand, they probe the limits placed by quantum mechanics on practical devices such as displacement [8, 9], force [10, 11] and phonon noise [12, 13] sensors.

A crucial technique underlying these advances involves the laser cooling of mechanical oscillators [14–19]. In these optomechanical systems, the radiation modes that interact with the mechanical oscillator are typically confined in an optical resonator. The operation of such systems in the quantum regime has been achieved experimentally [20], and is well understood theoretically [21–23]. However, resonator-based optomechanical systems face some major limitations. For example, they can be addressed only by specific (i.e. resonant) optical wavelengths, offer limited physical access to the mechanical component(s), cannot accommodate large or numerous mechanical elements which lower the cavity finesse by absorption or scattering, and are difficult to scale up.

In response to these challenges, substantial efforts have recently been made to develop cavityless optomechanical systems, using freely propagating radiation modes. A typical configuration consists of a harmonically oscillating dielectric particle, trapped optically and cooled via feedback. Ongoing experiments are aiming to broach the quantum regime [24–30]. However, to the best of our knowledge, no theoretical model exists which specifies the physical conditions for approaching - or operating in - the quantum regime. In this Letter, we supply such a model for the case where the dielectric particle is smaller than an optical wavelength, corresponding to several laboratory realizations [24, 28, 30]. Using our model, we demonstrate agreement with experimental data in the classical regime, and answer a question of fundamental as well as pressing experimental interest for such systems: what are
the conditions for mechanical ground state occupation?

The physical system under consideration is shown in Fig. 1. A subwavelength polarizable dielectric sphere is confined at the focus of a Gaussian trapping beam, and its motion is detected using a probe beam, polarized orthogonal to the trap. The detected signal is fed back to the trap beam to cool the particle. We analyze this configuration by dividing it into a ‘system’ and a ‘bath’. The system consists of the nanomechanical oscillator and the optical probe and trap. The bath consists of the optical modes relevant to the problem, and the background thermal gas present in the experiment [24]. In the remainder of this Letter, we identify the electromagnetic modes into which light is scattered by the nanosphere, and the background thermal gas present in the experiment. The bath consists of the optical modes chosen orthogonal to that of the trap field, a Gaussian probe beam of frequency $\omega_p$ and beam waist $w_0$, and the Rayleigh range $z_R = \omega_0 w_0^2/2c$. Further, in Eq. (1), the quantized electric field of the paraxial Gaussian probe beam of frequency $\omega_p$, linewidth $\Delta \omega$ and beam waist $w_0$ is [34]

$$ E_p = i \left( \frac{\hbar \omega_p \Delta \omega}{4 \pi \epsilon_0 c} \right)^{1/2} e^{i \omega_p (t-z/c)} e_p G(r, \Delta r, \omega_p) a + h.c., $$

where $\Delta r$ is the shift from the trap focus, $e_p$ is the probe polarization chosen orthogonal to that of the trap field, and the annihilation and creation operators for the probe field obey $[a, a^\dagger] = 1$. Finally, in Eq. (1), the background field into which the photons are scattered by the nanoparticle is given by a superposition of plane waves in all directions

$$ E_b = i \int d^3k \left( \frac{\hbar \omega}{16 \pi^2 \epsilon_0} \right)^{1/2} \sum_{\mu} e_{\mu,k} a_{\mu}(k) e^{i(k \cdot r - \omega t)} + h.c., $$

$\mu$ indexes the polarization, $e_{\mu,k}$ is a polarization vector, $\omega = c |k|$ and $[a_{\mu}(k), a_{\mu}^\dagger(k')] = \delta(k - k')$. We note that since the plane waves form a complete basis, Eq. (1) involves an overcounting of modes due to the addition of the trap and probe fields. This is justified in optomechanics as the trap and probe modes are of measure zero [33].

The configuration Hamiltonian may now be written, $H = H_m + H_f + H_{int}$. In Eq. (4), the first term on the right hand side represents the mechanical kinetic energy $H_m = |p|^2/2m$, where $p$ is the three dimensional momentum of the nanoparticle, and $m$ its mass. The second term in Eq. (4) is the field energy $H_f = \epsilon_0 \int |E(r)|^2 d^3r$. Using the electric fields supplied above and ignoring the constant trap energy we find

$$ H_f = \hbar \omega_p a^\dagger a + \sum_{\mu} \int d^3k \hbar \omega_k a_{\mu}(k) a_{\mu}(k), $$

which is simply the sum of the probe and background field energies. Finally, the interaction Hamiltonian is given by $H_{int} = -\int_V P(r) \cdot E(r) d^3r/2 = -\alpha_p \int_V |E(r)|^2 d^3r/2$ [35], where we have assumed that the dielectric has a volume $V$, that its center-of-mass is located at $r$, and that it has a linear polarizability $\alpha_p$, i.e. the polarization $P(r) = \alpha_p E(r)$. Using Eq. (4) and the expressions for the trap, probe and background fields we can evaluate $H_{int}$ for small particle displacements $r$, and rewrite Eq. (4) as

$$ H = H_S + H_B + H_{SB}, $$

where the system Hamiltonian is that of standard optomechanics [10]

$$ H_S = \hbar \omega_p a^\dagger a + \sum_j \hbar \omega_j b_j^\dagger b_j - \sum_{j} h g_j a^\dagger a (b_j + b_j^\dagger), $$

with $j = \{x, y, z\}$, mechanical trapping frequencies $\omega_j = [2\epsilon_j I V/(m c^2 \epsilon_0^2 z_R^2)]^{1/2}$, and $\omega_x = \omega_j = [4\epsilon_j I V/(m c^2 \epsilon_0^2 z_R^2)]^{1/2}$, respectively, the trap beam intensity $I_t$ and $\epsilon_e = 3(\epsilon_r - 1)/(\epsilon_r + 2)$, where $\epsilon_e$ is the relative dielectric constant. The optomechanical couplings in $H_S$ are $g_x = 2C_m \Delta x \ell_y/\ell_x^2$, $g_y = 2C_m \Delta y \ell_x/\ell_y^2$, and $g_z = C_m \Delta z \ell_x^2/\ell_z$, where $C_m = (V \epsilon_e \omega_0 \omega_m)/(\pi^2 w_0^2 c^3)$, and $\Delta x$, $\Delta y$, and $\Delta z$ are the spatial shifts between the foci of the trap and probe beams along the respective axes. The quantities $\ell_j = \sqrt{\hbar/(2m \omega_j)}$ are the oscillator lengths along each Cartesian axis and the mechanical operators obey $[b_j, b_j^\dagger] = 1$. In Eq. (6), the bath Hamiltonian is $H_B = \sum_{\mu} \int d^3k \hbar \omega_k a_{\mu}(k) a_{\mu}(k)$ and the system-bath interaction Hamiltonian is $H_{SB} = -\epsilon_e \epsilon_0 \int_V d^3r |E_b(r) + \frac{\hbar \omega_p \Delta \omega}{4 \pi \epsilon_0 c} \right)^{1/2} e^{i \omega_p (t-z/c)} e_p G(r, \Delta r, \omega_p) a + h.c.,$
\( \mathbf{E}_p(\mathbf{r}) \cdot \mathbf{E}_q(\mathbf{r}) \), which represents the scattering of the trap and probe fields into the background, as shown more explicitly in the Supplementary Material.

We now trace over the bath modes, applying the standard Born and Markov approximations, since the system-bath coupling is weak and the bath correlations decay quickly \([3, 31, 32]\). We also trace over the \( x \) and \( y \) degrees of particle motion, since the dynamics along the three axes are independent of each other, and it suffices to analyze a single direction \([24]\). The net result of our calculation is a master equation for the density matrix \( \rho(t) \) describing the optical probe and the \( z \)-motion of the nanoparticle

\[
\dot{\rho}(t) = \frac{1}{\hbar}[H_S, \rho(t)] + D_m[\rho(t)] + \mathcal{L}_\text{sc}[\rho(t)], \tag{8}
\]

where the first term on the right hand side represents unitary evolution of the system with \( H_S = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b - \hbar g_a a^\dagger b(b + b^\dagger) \). The second term corresponds to the positional decoherence of the nanoparticle due to scattering of trap photons, \( D_m[\rho(t)] = -A_t[Q_z, [Q_z, \rho(t)]], \) where \( Q_z = b^\dagger + b \), and \( A_t = 7\hbar \omega t^3(\epsilon V)^2 / (60\pi \hbar c)^6 \). The third superoperator describes the loss of photons from the probe, also due to scattering by the nanoparticle, \( \mathcal{L}_\text{sc}[\rho(t)] = B(a^\dagger a \rho(t) - 2a \rho(t) a^\dagger + \rho(t) a^\dagger a) \), where \( B = \omega_0^2(\epsilon V)^2 \Delta \omega^2 / (c^3 12\pi^3 \hbar^5) \).

The nanoparticle also experiences collisions with background gas particles at the ambient temperature \( T \). This effect may be accounted for by adding to the right hand of Eq. (8) the superoperator \([35]\)

\[
\mathcal{B}[\rho(t)] = -D_p[Q_z, [Q_z, \rho(t)]] - D_q[P_z, [P_z, \rho(t)]] - \frac{m}{4m} [Q_z, [P_z, \rho(t)]] , \tag{9}
\]

where \( P_z = i(b^\dagger - b) \), and curly braces denote an anti-commutator. The first term on the right hand side corresponds to momentum diffusion and \( D_p = \eta_f \hbar k_B T m^2 \ell_z^4 / \hbar^2 \). The second term describes position diffusion with \( D_q = \eta_j \hbar^2 / (48k_B T m^2 \ell_z^4) \). The third term accounts for friction, and by Stokes law we have \( \eta_f = 6\pi \mu r_d \), where \( r_d \) is the radius of the nanoparticle and \( \mu \) is the dynamic viscosity of the background gas. Absorption, heating due to black-body radiation, and trap beam shot noise are negligible for our system \([33]\).

We now characterize the measurement of the oscillator motion using input-output theory from quantum optics \([35, 37]\) applied to the nanoparticle, as in ion-cooling theory \([38]\). Specifically, the incoming probe field \( a_{in} \) interacts with the nanoparticle, and the outgoing probe field \( a_{out} \) carries a signature of this interaction \([39]\)

\[
a_{out} = a_{in} + \frac{\chi}{4} Q_z(t), \tag{10}
\]

where \( \chi = 4g_z \Delta t \) is the scaled optomechanical coupling, with integration time \( \Delta t \) (ultimately determined by the detection bandwidth), and we have written the probe beam as a displaced vacuum (i.e. coherent state) \( a = -i\alpha + v \), with \( \alpha \) a classical number and \( v \) a bosonic annihilation operator. A homodyne measurement on the output field yields a current \([37]\)

\[
I_h = \chi \Phi \langle Q_z(t) \rangle(t) + \sqrt{\langle \Phi \rangle} \xi(t), \tag{11}
\]

where \( \Phi = \alpha^2 \Delta \omega \) is the average detected flux of probe photons, and \( \xi(t) \) is a stochastic variable with mean \( \langle \xi(t) \rangle = 0 \) and correlation \( \langle \xi(t) \xi(t') \rangle = \delta(t - t') \).

In the experiment, the detected current \( I_h \) is frequency doubled, phase shifted, and fed back to modulate the power of the trapping beam \([24]\). This results in a feedback Hamiltonian \( H_{fb} = -\partial H_\text{tot} / \partial Q_z \) which is equivalent to that used in experiments in the classical regime \([11]\). Taking the Markovian limit where the feedback occurs faster than any system timescale, and applying standard quantum feedback theory for homodyne detection \([42]\), we find that the following superoperator must be added to Eq. (8)

\[
\mathcal{F}[\rho(t)] = -i \Phi \chi G[Q_z^3, \{P_z, \rho\} - \frac{\Phi}{2} G^2 \left[ Q_z^3, \{Q_z^3, \rho\} \right], \tag{12}
\]

where the first term on the right hand side represents the desired effect of the feedback, and the second term the accompanying backaction.

The full master equation, assembled from Eqs. (8), (10), and (12) is then \( \dot{\rho}(t) = [H_S, \rho(t)] / i\hbar + D_m[\rho(t)] + \mathcal{L}_\text{sc}[\rho(t)] + \mathcal{B}[\rho(t)] + \mathcal{F}[\rho(t)] \), where the new system Hamiltonian

\[
H_S = \hbar \omega_p v^2 c + \hbar \omega_b b^\dagger b - \hbar g_z(v^2 + v)(b + b^\dagger), \tag{13}
\]

accounts for the linearization of the probe implemented above. Likewise \( D_m[\rho(t)] = (A_p + A_p \Phi) [Q_z, \{Q_z, \rho\}] \) now includes a mechanical decoherence term due to scattering from the probe, with \( A_p = 7\ell_z^4(\epsilon V)^2 \omega_p^6 / (60\hbar c^5 \eta_j^2) \).

Employing this master equation to consider the question of ground state occupation, tracing out the optical probe field, and using the resulting reduced master equation for the nanoparticle only, we find the equation for the dynamics of the phonon number \( \langle N \rangle \equiv b^\dagger b \), \( \langle N \rangle = -J(N^2) - K \langle N \rangle + L \), where \( J = 12\chi \gamma \omega - 6\gamma^2 \Phi, K = \eta_f / 2m + 9\chi G - 8\gamma^2 \Phi, \) \( L = D - [6\gamma G + 4\gamma^2 \Phi] \)).

The dot denotes a time derivative, and \( D = D'_p + D_q \) with \( D'_p = D_p + A_p + A_p \Phi \) accounting for positional decoherence. Since the nanoparticle is expected to be in a
thermal state $[21, 23, 31, 33]$, $\langle N^2 \rangle = 2\langle N \rangle^2 + \langle N \rangle$ [43], a relation which simplifies the phonon dynamics to

$$\langle \hat{N} \rangle = -2J\langle \hat{N} \rangle^2 - (J + K)\langle \hat{N} \rangle + L. \quad (14)$$

Below we will consider the case where $J \neq 0$. Before presenting the analytical solution to Eq. (14), we find the optimum value of the feedback, $G_{\text{opt}}$. Optimizing the rate of phonon number decrease in Eq. (14) with respect to $G$ we find

$$G_{\text{opt}} = \frac{1}{4J} \left( \frac{24\langle N \rangle^2 + 21\langle N \rangle + 6}{24\langle N \rangle^2 + 28\langle N \rangle + 8} \right) \chi, \quad (15)$$

Since $0 \leq \langle \hat{N} \rangle < \infty$, the optimal gain is also bounded, i.e. $3\chi/4 \leq G_{\text{opt}} \leq \chi$. From Fig. 2(a), it is seen that $G_{\text{opt}}$ saturates rather quickly with $\langle \hat{N} \rangle$, and keeping in mind that feedback cooling is typically initiated at large phonon numbers [24], we pick $G_{\text{opt}} = \chi$, assuming the feedback is not changed dynamically.

Starting with the initial condition $\langle \hat{N}(0) \rangle \equiv N_0 = D_P/2m/\eta \equiv k_B T_{\text{eff}}/\hbar \omega_z$, where $T_{\text{eff}}$ is the effective temperature of a thermal bath due to gas and optical scattering combined, the analytical solution to Eq. (14) is

$$\langle \hat{N}(t) \rangle = \frac{(J + K)}{4J} + \frac{1}{2J \tau} \tanh \left( \frac{t}{\tau} + \theta \right), \quad (16)$$

where $\theta = \tanh^{-1} \left[ (2JN_0 + J + K)/\tau \right]$ and the cooling timescale $\tau$ is given by

$$\frac{1}{\tau} = \frac{\sqrt{(J + K)^2 + 8JL}}{2}. \quad (17)$$

From Eq. (16) the steady state phonon number is

$$N_{ss} \equiv \lim_{t \to \infty} \langle \hat{N}(t) \rangle = \frac{1}{2J \tau} - \frac{(J + K)}{4J} \quad \approx \sqrt{\frac{\eta f N_0/2m - 6\chi G \Phi + 4\Phi G^2}{24\chi \Phi G^2 - 12\Phi G^2}}, \quad (18)$$

where the approximation is valid for $N_0 \gg 1$. In the numerator of the radical, the three terms represent heating from the environment, feedback, and feedback backaction, respectively. If we choose $G = G_{\text{opt}}$, then Eq. (18) becomes $N_{ss} \approx \sqrt{\eta f N_0/24m \chi \Phi}$.

Key plots using experimental parameters [24, 40] are shown in Fig. 2(b) and (c). Figure 2(b) shows the cooling timescale $\tau$ [Eq. (17)] versus the scaled feedback strength $G/\chi$ for different pressures. Figure 2(c) displays the steady state phonon number $N_{ss}$ as the vacuum pressure is tuned, see figure caption for details. As can be seen from this figure, our theoretical model matches well with experimental data in the classical regime, and also predicts that cooling to the ground state is possible, for optimized but realistic parameters.

To conclude, we have presented a model that describes the quantum optomechanics of an optically trapped subwavelength dielectric particle. We have shown that the predictions of this model are in very good agreement with experimentally measured steady state occupation values in the classical regime. Further, we have demonstrated that quantum ground state preparation lies within existing experimental capabilities. The master equation derived by us provides a general framework by which other aspects can be explored in future research. We are grateful to C. Stroud, A. Aiello, B. Zwickl, and S. Agarwal for useful discussions. This material is based upon work supported by the Office of Naval Research under Award Nos. N00014-14-1-0803 and N00014-14-1-0442. ANV thanks the Institute of Optics for generous support. LPN acknowledges support from a University of...
Supplementary Material

ELECTRIC FIELDS

The total electric field as defined in the main article is written as

\[ \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_t(\mathbf{r}, t) + \mathbf{E}_p(\mathbf{r}, t) + \mathbf{E}_b(\mathbf{r}, t). \]  

(S1)

Trapping field

The electric field of the trapping beam of frequency \( \omega_t \) is assumed to be a classical Gaussian beam

\[ \mathbf{E}_t(\mathbf{r}, t) = \frac{E_0 w_0}{2} \sqrt{\frac{\pi}{2}} G(\mathbf{r}, \omega_t) e^{i\omega_t(z/c-t)} + \text{c.c.}, \]  

(S2)

where \( w_0 \) is the beam waist and the mode profile

\[ G(\mathbf{r}, \omega) = \sqrt{\frac{2}{\pi w(z)}} \exp \left( \frac{-r_\perp^2}{w(z)^2} + i\omega \frac{r_\perp^2}{2cR(z)} - i \arctan(z/z_R) \right), \]  

(S3)

with \( r_\perp = \sqrt{x^2 + y^2} \), \( z \) is the axial propagation direction, \( E_0 \equiv E(0,0) \), \( w(z) = w_0 \sqrt{1 + (z/z_R)^2} \) is the beam waist, \( R(z) = z \left[ 1 + (z_R/z)^2 \right] \) is the radius of field curvature and \( z_R \equiv \pi w_0^2/\lambda = \omega w_0^2/2c \) is the Rayleigh range. Near focus \( z \ll z_R \) the radius of curvature becomes \( R(z) \approx z^2/cz^2 \), the Guoy phase becomes \( \arctan(z/z_R) \approx 0 \), and therefore our mode profile is approximately

\[ G(\mathbf{r}, \omega) \approx \sqrt{\frac{2}{\pi w(z)}} \exp \left( \frac{-r_\perp^2}{w(z)^2} + i\omega \frac{r_\perp^2 z}{2cz_R} \right). \]  

(S4)

Probe field

The electric field of the paraxial Gaussian probe beam is a quantized beam with frequency \( \omega_p \), linewidth \( \Delta \omega \), and beam waist \( w_0 \) and can be written as \[ S1 \]

\[ \mathbf{E}_p(\mathbf{r}, t) = i \left( \frac{\hbar \omega_p \Delta \omega}{4\pi \epsilon_0 c} \right)^{1/2} e^{i\omega_p(z/c-t)} \mathbf{e}_p G(\mathbf{r}', \omega_p) a + \text{h.c.}, \]  

(S5)

where \( \mathbf{r}' = \mathbf{r} + \Delta \mathbf{r} \), \( \mathbf{e}_p \) is the probe polarization assumed to be orthogonal to that of the trap field, and the annihilation and creation operators for the probe field obey the canonical commutation relation \( [a, a^\dagger] = 1 \).

Background field

The background field into which optical scattering occurs is also quantized and can be written in the standard plane wave expansion

\[ \mathbf{E}_b = i \int d^3k \left( \frac{\hbar \omega}{16\pi^2 \epsilon_0} \right)^{1/2} \sum_\mu \mathbf{e}_{\mu,k} a_{\mu}(\mathbf{k}) e^{i(k \cdot \mathbf{r} - \omega t)} + \text{h.c.}, \]  

(S6)

where \( \mu \) indexes the polarization, \( \mathbf{e}_{\mu,k} \) is a polarization vector, \( \omega = c|\mathbf{k}| \) and \( [a_\mu(k), a_\mu^\dagger(k')] = \delta(k - k') \).
FREE FIELD HAMILTONIAN

Our Hamiltonian as given in the main text is

\[ H = H_m + H_f + H_{\text{int}}, \]  

(S7)

where \( H_m = \frac{p^2}{2m} \) is the particle’s kinetic energy for the momentum \( p = \sqrt{p_x^2 + p_y^2 + p_z^2} \), and \( H_f \) and \( H_{\text{int}} \) are the free field interaction Hamiltonians respectively. The energy of the free field Hamiltonian is

\[ H_f = c_0 \int |E(r, t)|^2 \, d^3r. \]  

(S8)

Using Eq. (S1) we explicitly write out the contributions as

\[ H_f = \epsilon_0 \int \{|E_t(r)|^2 d^3r \} \] Trap beam energy  
{(S9a)}

\[ + \epsilon_0 \int \{|E_p(r)|^2 d^3r \} \] Probe beam energy  
{(S9b)}

\[ + \epsilon_0 \int \{|E_b(r)|^2 d^3r \} \] Background field energy  
{(S9c)}

\[ + 2\epsilon_0 \int \langle E_t + E_p \rangle \cdot E_b d^3r \] Field-background cross-term  
{(S9d)}

\[ + 2\epsilon_0 \int \langle E_t - E_p \rangle \cdot d^3r \] Probe-Trap cross-term  
{(S9e)}

The various terms on the right hand side of Eq. (S9) are dealt with as follows.

**Trap beam energy**

Equation (S9a) can be neglected as it is purely an offset in the energy due to the classical trap field.

**Probe beam energy**

Equation (S9b) corresponds to the energy of the free probe field, and can be evaluated, with some care, by using Eq. (S5),

\[
\epsilon_0 \int |E_p(r)|^2 \, d^3r = \epsilon_0 \lim_{\omega_p \rightarrow \omega_p} \int d^3r' \left( \frac{\hbar \sqrt{\omega_p \omega_p' \Delta \omega}}{4\pi \epsilon_0 c} \right) e^{i(\omega_p - \omega_p')(t-z/c)} \\
\times \left[ a(\omega_p) a^\dagger(\omega_p') G(r', \omega_p') G^*(r', \omega_p) + a^\dagger(\omega_p') a(\omega_p) G^*(r', \omega_p') G(r', \omega_p) \right] \\
\approx \lim_{\omega_p \rightarrow \omega_p} \left( \frac{\hbar \sqrt{\omega_p \omega_p' \Delta \omega}}{4\pi c} \right) \left[ a(\omega_p) a^\dagger(\omega_p') + a^\dagger(\omega_p') a(\omega_p) \right] e^{i(\omega_p - \omega_p')t} \\
\times \int e^{i(\omega_p - \omega_p')z/c} \, dz \int d^2r'_{\perp} |G(r', \omega_p')|^2 
\]  

(S10)

where we have assumed the paraxial limit in which \( G(r, \omega_p) \approx G(r, \omega_p') \), and we may then use \( \int d^2r'_{\perp} |G(r, \omega_p)|^2 = 1 \). Discretizing the \( z \) integral

\[
\int dz e^{i(\omega_p - \omega_p')z} = 2\pi c \delta(\omega_p - \omega_p') \rightarrow \frac{2\pi c \delta_{\omega_p, \omega_p'}}{\Delta \omega}, 
\]  

(S11)
Eq. (S10) becomes
\[
\epsilon_0 \int |E_p(r)|^2 \, d^3 r = \lim_{\omega_p \rightarrow \omega'_p} \int \left( \frac{\hbar \sqrt{\omega_p \omega'_p}}{4\pi c} \right) \left( \frac{2\pi c \delta \omega_p \omega'_p}{\Delta \omega} \right) e^{i(\omega_p - \omega'_p) t} \times \left[ a(\omega_p) a^\dagger(\omega'_p) + a^\dagger(\omega'_p) a(\omega_p) \right] \, d^3 r
\]
which is as expected for the probe energy.

**Background field energy**

Equation (S9c) corresponds to the energy of the background field and the evaluation can be found in several textbooks [S2],
\[
H_B = \sum_{\mu} \int d^3 k \hbar \omega_k a^\dagger_{\mu}(k) a_{\mu}(k). \tag{S13}
\]

**Field-background cross-term**

The cross term between the trap and background fields [Eq. (S9d)] vanishes as their mutual overlap is very small. This cancellation also represents the avoidance of self-interference and mode overcounting in our model. The cross term between the probe and the background fields vanishes due to the same reason.

**Probe-Trap cross-term**

The cross term between the trap and probe fields given in equation (S9e) vanishes due to polarization orthogonality.

**Free field Hamiltonian**

Finally, combining Eqs. (S12) and (S13), the free field Hamiltonian is
\[
H_f = \hbar \omega_p a^\dagger a + \sum_{\mu} \int d^3 k \hbar \omega_k a^\dagger_{\mu}(k) a_{\mu}(k), \tag{S14}
\]
which is simply the sum of the probe and background field energies.

**INTERACTION HAMILTONIAN**

The interaction Hamiltonian is given by
\[
H_{int} = -\frac{1}{2} \int P(r) \cdot E(r) \, d^3 r. \tag{S15}
\]
If we assume the particle has a linear polarizability described by \( P(r) = \alpha_p E(r) \), then the interaction Hamiltonian becomes
\[
H_{int} = -\frac{\epsilon_0 \epsilon_r}{2} \int_V |E(r)|^2 \, d^3 r, \tag{S16}
\]

3
where \( \epsilon_c = 3(\epsilon_r - 1)/(\epsilon_r + 2) \) is the Clausius-Mossotti relation for the effective relative permittivity of a dielectric due to local field effects, and \( V \) denotes integration over the volume of the dielectric particle.

Now if when we use the total electric field from Eq. (S1), we get

\[
H_{\text{int}} = -\frac{\epsilon_c \epsilon_0}{2} \int_V |\mathbf{E}_b(r)|^2 \, d^3r \quad \text{Vacuum fluctuations}
\]

\[
- \epsilon_c \epsilon_0 \int_V \mathbf{E}_b \cdot \mathbf{E}_p \, d^3r \quad \text{Orthogonally polarized}
\]

\[
- \frac{\epsilon_c \epsilon_0}{2} \int_V |\mathbf{E}_t(r)|^2 \, d^3r \quad \text{Trap potential}
\]

\[
- \frac{\epsilon_c \epsilon_0}{2} \int_V |\mathbf{E}_p(r)|^2 \, d^3r \quad \text{Optomechanical coupling}
\]

\[
- \epsilon_c \epsilon_0 \int_V (\mathbf{E}_t + \mathbf{E}_p) \cdot \mathbf{E}_b \, d^3r \quad \text{Optical scattering}
\]

(S17a)

(S17b)

(S17c)

**Trap potential**

First we consider the effect of the trap beam given by Eq. (S17a). Since the nanoparticle is smaller than the wavelength of any relevant optical field, the integration can be written as \( \int_V d^3r = V \int \delta(q) d^3r \), where \( q \) is the center of mass position of the particle. Therefore \( H_{\text{int}} \) for the trap-particle interaction is given by

\[
-\frac{\epsilon_c \epsilon_0}{2} V |\mathbf{E}_t(q)|^2 = -\frac{\epsilon_c \epsilon_0}{2} V E_0^2 \frac{2\pi}{\lambda} (G(q))^2
\]

\[
= \frac{\epsilon_c \epsilon_0}{2} V E_0^2 \left( -1 + \frac{q_z}{z_R} \right)^2 + 2 \left( \frac{q_x}{w_0} \right)^2 + 2 \left( \frac{q_y}{w_0} \right)^2 + O(q^4),
\]

(S18)

where \( q_z \) is the longitudinal coordinate, and \( q_x \) and \( q_y \) are the transverse coordinates. We can ignore the overall constant in the above equation and we finally have for our trap Hamiltonian

\[
H_{\text{trap}} = \frac{p^2}{2m} + \frac{\epsilon_c \epsilon_0}{2} V E_0^2 \left( \frac{q_z}{z_R} \right)^2 + 2 \left( \frac{q_x}{w_0} \right)^2 + 2 \left( \frac{q_y}{w_0} \right)^2 ,
\]

(S19)

which is a quadratic trap in along all three axes. Using the standard form for a harmonic oscillator, we can write \( H_{\text{trap}} \) as

\[
H_{\text{trap}} = \frac{p^2}{2m} + \frac{1}{2} m \left( \omega_z^2 q_z^2 + \omega_x^2 q_x^2 + \omega_y^2 q_y^2 \right),
\]

(S20)

where \( \omega_z = \sqrt{\epsilon_c \epsilon_0 E_0^2 V/(m z_R^2)} \) and \( \omega_{x,y} = \sqrt{2 \epsilon_c \epsilon_0 E_0^2 V/(m w_0^2)} \). Writing our canonical nanoparticle variables as \( q_j = \sqrt{\hbar/(2m \omega_j)} (b_j^+ + b_j^\dagger) = \ell_j (b_j^+ + b_j) = \ell_j Q_j \) and \( p_j = i\sqrt{\hbar m \omega_j}/2 (b_j^\dagger - b_j) \) allows us to rewrite Eq. (S20) as

\[
H_{\text{trap}} = \sum_j \hbar \omega_j b_j^\dagger b_j .
\]
Optomechanical coupling

Now we look at the coupling between the probe beam and the trapped nanoparticle given by line (S17b) in Eq. (S17). Our Hamiltonian for this term is

\[ H_{OM} = -\frac{\epsilon_{e_0} V}{2} \int |\hat{E}_p(r)|^2 \, d^3r \]

where we have assumed that the probe beam is shifted from the trap beam by a small amount \( \Delta r \equiv (\Delta x, \Delta y, \Delta z) \). Now the terms proportional to a constant times either \( a^\dagger a \) or \( q_j \) (for \( j \in \{x, y, z\} \)) simply act as small shifts to the probe frequency or harmonic oscillator position and are incorporated into the values of the frequency or position. The optomechanical coupling term between the probe and particle is then given by

\[ H_{OM} = -V \frac{\epsilon_e \hbar \omega_p \Delta \omega}{2 \pi c} a^\dagger a \left( \frac{\Delta z}{z_R} \ell_z Q_z + 2 \frac{\Delta r}{z_R} \ell_x Q_x + 2 \frac{\Delta y}{z_R} \ell_y Q_y \right) = -\sum_j h g_j a^\dagger a Q_j. \]  

Our system Hamiltonian is given by the combination of the energy of the probe field, optomechanical trap, and optomechanical coupling and is given by

\[ H_S = \hbar \omega_p a^\dagger a + \sum_j h \omega_j b^\dagger_j b_j - \sum_j h g_j a^\dagger a (b_j + b^\dagger_j). \]  

Optical scattering

Computing the cross coupling Hamiltonian between the field and the background due to scattering by the nanoparticle gives

\[ -\epsilon_e \epsilon_0 \int_{V} \mathbf{E}_t \cdot \mathbf{E}_b \, d^3r \approx -\frac{\epsilon_e V}{2} \left( \frac{\hbar \epsilon_0}{16 \pi^3} \right)^{1/2} \int d^3k \sqrt{\omega_k} \sum_{\mu} \left( k_\perp \cdot q_\perp + (k_z - k_0) q_z \right) \times \left( \mathbf{e}_\mu(k) \cdot \mathbf{E}_b^* \right) e^{-i(\omega_k - \omega_l)t} + \text{h.c.}, \]

for the trap beam and

\[ -\epsilon_e \epsilon_0 \int_{V} \mathbf{E}_p \cdot \mathbf{E}_b \, d^3r \approx -\frac{\hbar \epsilon_e V}{8 \pi^2 \epsilon_0} \sqrt{\frac{2 \omega_j \Delta \omega}{\pi c}} \times \int d^3k \sqrt{\omega_k} \sum_{\mu} \left( \mathbf{e}_\mu(k) \cdot \mathbf{E}_p \right) a^\dagger_\mu(k) a_\mu(k) e^{-i(\omega_k - \omega_l)t} + \text{h.c.} \]

for the probe beam, assuming elastic scattering and amplitude of the particle motion small relative to an optical wavelength \( \delta \). Applying standard Born-Markov theory and tracing over the background modes and particle motion transverse to the \( z \)-axis particle motion, allows us to derive a master equation for the evolution of the state \( \rho(t) \) which is given as

\[ \dot{\rho}(t) = -\frac{1}{\hbar c^3} \left( \frac{E_0^2 \epsilon_0 c^2 V}{120 \pi} \right) \left[ q_z, [q_z, \rho(t)] \right] + \frac{1}{\hbar^2 c^6} \left( \frac{E_0^2 \epsilon_0 c^2 V}{12 \pi^2 \epsilon_0^2 c^2 / (c / \Delta \omega)} \right) \left( a^\dagger_\mu a_\mu(t) \rho(t) - 2a_\mu(t) a^\dagger_\mu \rho(t) + \rho(t) a^\dagger_\mu a_\mu(t) \right) \]

\[ = D_m[\rho(t)] + \mathcal{L}_{sc}[\rho(t)]. \]

5
DERIVATION OF THE DETECTED HOMODYNE CURRENT AND FEEDBACK

In order to find the input-output relations for the probe field, we write the system Hamiltonian (for the single degree of freedom $q_z$) as $H_s = \hbar \omega_z a^\dagger a + \hbar \omega_z b^\dagger b_z - \hbar g a^\dagger a Q_z$. If we move into the interaction picture for the probe field where $a \rightarrow ae^{-i\omega_z t}$, then this becomes $\dot{H}_s = \hbar \omega_z b^\dagger b_z - \hbar g a^\dagger a Q_z$.

We assume the probe field is initially a coherent state which can be written as $a = -i\alpha + v$, where $\alpha$ is a constant and $v$ is a field annihilation operator. In this case the optomechanical coupling in our system Hamiltonian becomes

$$H_{OM} = -\hbar g z Q_z a^\dagger a = \hbar g_z Q_z \left(\alpha^2 + i\alpha v^\dagger - i\alpha^* v\right).$$  \hspace{1cm} (S27)

The term proportional to $|\alpha|^2 Q_z$ is responsible for simply shifting the mean position of the oscillator and can safely be ignored, leaving us with $H_{OM} = i\hbar g_z Q_z (\alpha v^\dagger - \alpha^* v)$.

The Heisenberg equation of motion for $v$ is given by

$$\dot{v} = \frac{1}{\hbar}[a, H] = \alpha g_z Q_z,$$  \hspace{1cm} (S28)

which can be integrated formally to give

$$v(t) = v(t_0) + \int_{t_0}^t \alpha g_z Q_z \, dt' \approx v(t_0) + \alpha g_z Q_z \Delta t,$$  \hspace{1cm} (S29)

where the integration is taken over a time $\Delta t$ short compared to $1/\omega_z$. If we had picked an initial time $t_f > t$ then we would have computed $v(t) = v(t_f) - \alpha g_z Q_z \Delta t$. By identifying the input state as $a_{in} = -i\alpha + v(t_0)$, and the output state as $a_{out} = -i\alpha + v(t_f)$, then we can relate the output and input fields by

$$a_{out} = a_{in} + 2\alpha g_z \Delta t \equiv a_{in} + \frac{\alpha \chi}{2} Q_z,$$  \hspace{1cm} (S30)

where we have defined the variable $\chi \equiv 4g_z \Delta t$.

Now the detected photon number operator is given by

$$a_{out}^\dagger a_{out} = \alpha^2 + \alpha^2 \chi Q_z + i\alpha(v_{in} - v_{in}^\dagger) + \alpha^2 \chi Q_z^2 + v^\dagger v,$$  \hspace{1cm} (S31)

where the last two terms are negligibly small. The detected homodyne current is found by subtracting off the constant offset, $\alpha^2$, and is therefore given as

$$I_h = \Phi \chi \langle Q_z \rangle (t) + \sqrt{\Phi} \xi(t),$$  \hspace{1cm} (S32)

where we have converted from photon numbers to rates by using the identity for the photon flux $\Phi \equiv \langle a^\dagger a \rangle \Delta \omega = \alpha^2 \Delta \omega$, and have introduced the stochastic variable $\xi(t)$ which accounts for the shot noise introduced by the term $i\alpha(v_{in} - v_{in}^\dagger)$.

As described in the main text, phase shifting the measured signal (or equivalently adding a short time delay) is equivalent to measuring a different quadrature of motion, i.e. the current that is fed back is $I_{fb} = \chi \Phi \langle P_z \rangle + \sqrt{\Phi} \xi(t)$, where $\xi(t)$ has the same properties as $\xi(t)$. Now writing $\sigma \equiv \chi P_z$, allows us to write the master equation for the feedback in standard notation \cite{S4}

$$\dot{\rho} = \mathcal{K} [\sigma \rho + \rho \sigma^\dagger] + \frac{1}{2\Phi} \mathcal{K}^2 [\rho] \equiv \chi \mathcal{K} [\{P_z, \rho\}] + \frac{1}{2\Phi} \mathcal{K}^2 [\rho],$$  \hspace{1cm} (S33)

where the Liouvillian superoperator $\mathcal{K}$ is defined as $\mathcal{K} [\rho] = [F, \rho] / i\hbar$ and where $F$ is the feedback term that comes from the feedback Hamiltonian $H_{fb} = I_{fb} F$, and is chosen to be $F = \hbar \Phi Q_z$ to match the classical physics of the problem as described in the main text. The gain coefficient $G$ may be related to the average relative trap power modulation $\Delta P/P_1$ in the experiment by using the relation $\Delta P/P_1 = G \Phi \langle Q \rangle / \omega_m \approx G \Phi \langle N \rangle / \omega_m$ \cite{S5}. 

6
The experimental data in Fig. 2(c) was obtained by using a scheme more involved, and of more general applicability, than indicated in the simplified detection model presented in the main article. In the laboratory, the position of the nanoparticle was determined from the optical interference between the unscattered probe and a spherical wave radiated by the induced dipole of the polarizable sphere. The interference signal varies linearly with the particle position for oscillation amplitudes small compared to the optical wavelength. In order to detect the particle position and also to eliminate the large constant background term (equal to the unscattered probe flux \( \Phi \)) in an experimental setting, we leveraged a balanced homodyne detection scheme, as described in [S3] and [S5].

In this configuration, one of the detectors sampled the entire spatial profile of the probe beam, while the other was apodized so that only the center of the beam was sampled. As the trapped particle moved in the axial direction, the relative amount of scattered light collected by each detector, and thus the AC term of the resulting homodyne current, was modulated. The optical channels were adjusted so that each detector recorded the same average power, and the resulting homodyne signals were subtracted to eliminate the common DC bias.

In our theoretical model, rather than include an additional quantized mode corresponding to the full induced dipole emission pattern, we considered for simplicity only the field scattered back into the probe mode by the nanoparticle. This simplification is admissible as long as the value of the linear optomechanical coupling constant \( \chi \) in the model is taken from experiment [S3]. However, we emphasize that our experimental method of feedback cooling works even if the trap and probe foci are very close to each other, in which case the optomechanical coupling is essentially quadratic in the particle position, while the position detection (using the dipole wave mode) is still linear. In the theoretical calculation of the phonon number, the linear (or quadratic) optomechanical coupling simply adds a position offset (or frequency shift) to the trap. Both the offset as well as the shift are negligible for weak probe light used in the experiment, rendering the theoretical predictions of ground state cooling identical for linear as well as quadratic optomechanical coupling.

**COMPUTING THE GAS DAMPING RATE AS A FUNCTION OF GAS PRESSURE**

The damping rate of the particle motion due to the background gas is given by \( \Gamma_0 \equiv \eta_f/m \), where \( m \) is the particle mass and \( \eta_f = 6\pi\mu r_d \) is the coefficient of friction for a spherical particle of radius \( r_d \) in a fluid of dynamic viscosity \( \mu \). This expression is for a particle much larger than the mean free path of the gas bath \( \lambda_{mfp} \). Instead, for a nanoparticle in a rarefied gas the damping constant becomes [S6]

\[
\Gamma_0 = \frac{6\pi\mu r_d}{m} \times \text{correction} = \frac{6\pi\mu r_d}{m} \frac{0.619}{0.619 + Kn} \left( 1 + \frac{0.31Kn}{0.785 + 1.152Kn + Kn^2} \right),
\]

(S34)

where the correction term is in terms of the Knudsen number \( Kn \equiv \lambda_{mfp}/r_d \propto 1/Pr_d \). Using the fact that \( \lambda_{mfp} \approx 70 \text{ nm} \) at atmospheric pressure, and matching the experimentally measured rates [S3, S5], we can write the damping for a general pressure and particle size as

\[
\Gamma_0 \approx \frac{r_d}{70 \text{ nm}} \frac{2\pi \times 10^6 \text{ Hz}}{0.619 + Kn} \left( 1 + \frac{0.31Kn}{0.785 + 1.152Kn + Kn^2} \right).
\]

(S35)

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