On Küchle varieties with Picard number greater than 1

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Abstract. We describe the geometry of Küchle varieties (that is, Fano fourfolds of index 1 contained in Grassmannians as zero loci of sections of equivariant vector bundles) with Picard number greater than 1. We also describe the structure of their derived categories.

Keywords: Fano varieties, special varieties, semiorthogonal decompositions of derived categories.

§ 1. Introduction

In 1995 Küchle [1] classified all Fano varieties of dimension 4 and index 1 that can be represented as zero loci of sections of equivariant vector bundles on Grassmannians. His list, which consists of 21 examples, may be regarded as a first step towards the classification of Fano 4-folds and is still the main source of examples of them. However, the actual geometry of Küchle varieties has not yet been investigated. This paper is a first step in that direction.

We consider only those Küchle varieties whose Picard number is greater than 1. There are four such examples in the list. They are the zero loci of global sections of the following vector bundles on Grassmannians:

(b4) \( S^2U^\vee \oplus O(2) \) on \( \text{Gr}(2, 6) \);
(b9) \( S^2U^\vee \oplus S^2U^{\perp} \) on \( \text{Gr}(2, 7) \);
(c7) \( \Lambda^2U^{\perp}(1) \oplus O(1) \) on \( \text{Gr}(3, 8) \);
(d3) \( \Lambda^2U^\vee \oplus \Lambda^2U^{\perp} \oplus O(1) \) on \( \text{Gr}(5, 10) \).

Here \( U \) (resp. \( U^{\perp} \)) is the tautological subbundle of rank \( k \) (resp. \( n - k \)) on the Grassmannian \( \text{Gr}(k, n) \), and \( O(1) \) stands for the ample generator of the Picard group. Our main result is an alternative description of these varieties, which gives a much better understanding of their geometry. In particular, we use it to describe the structure of the derived categories of coherent sheaves on them.

Our alternative description is actually obvious for varieties of type (b4) and is given by a recent result of Casagrande [2] in case (b9). The case (d3) is also quite simple and should be known to the experts, although we could not find a reference. The case (c7) is more difficult but still quite manageable. The description of this variety is our main result.

Of course, the Küchle varieties under consideration admit a simplified description because their Picard group has large rank and hence they have additional structures...
that can be used. It is hard to expect something similar for other Küchle varieties. Their geometry should be more complicated, but at the same time more interesting.

We mention two questions of special interest about the geometry of other Küchle varieties. First, Küchle’s list contains two varieties with equal sets of discrete invariants:

- (b3) the zero locus of a global section of the bundle $\Lambda^3 U \perp (2)$ on $\text{Gr}(2, 6)$;
- (b7) the zero locus of a global section of the bundle $O(1) \oplus 6$ on $\text{Gr}(2, 7)$.

**Question 1.1.** Are Küchle varieties of types (b3) and (b7) deformation equivalent?

**Remark 1.2.** Soon after a preliminary version of this paper appeared, Manivel [3] gave an affirmative answer to this question.

The second question concerns yet another variety:

- (c5) the zero locus of a global section of $\Lambda^2 U \vee \oplus U \perp (1) \oplus O(1)$ on $\text{Gr}(3, 7)$.

Computations in Küchle’s paper show that the Hodge diamond of this variety contains the Hodge diamond of a K3 surface. Hence one can expect this variety to be similar to a cubic 4-fold. In particular, it is natural to expect that its derived category contains a non-commutative K3 surface as one of the components (compare with [4]) and that the Hilbert schemes of rational curves on it give rise to hyper-Kähler varieties as in the case of the Fano scheme of lines [5] and the Hilbert scheme of twisted cubics [6] on a cubic fourfold. The rationality questions for these varieties may also be interesting.

**Question 1.3.** Is there a semiorthogonal decomposition of the derived category of a Küchle variety of type (c5) with a non-commutative K3 surface as one of the components? Does it give a hyper-Kähler structure on appropriate Hilbert schemes of curves? What can be said about rationality of varieties of this type?

The varieties of types (d3) and (c7) also contain the Hodge diamond of a K3 surface, and we may pose Question 1.3 for these. However, our simplified description of these varieties in §§3, 4 answers this question (to a certain extent). For varieties of type (d3) the answer is trivial (all varieties are rational, their derived category contains a commutative K3 surface as a component, and the Hilbert schemes should be equal to moduli spaces of sheaves on the associated K3 surface). For varieties of type (c7) the question reduces to the same question for special cubic fourfolds.

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§ 2. Varieties of types (b4) and (b9)

**2.1. Type (b4).** We recall that a Küchle variety of type (b4) is the zero locus of a global section of the vector bundle $S^2 U \vee \oplus O(2)$ on $\text{Gr}(2, 6)$.

**Proposition 2.1.** A smooth Küchle variety $X_{b4}$ of type (b4) is an intersection of divisors of bidegree $(1, 1)$ and $(2, 2)$ in $\mathbb{P}^3 \times \mathbb{P}^3$.

**Proof.** A global section of $S^2 U \vee$ on $\text{Gr}(2, 6)$ is given by a quadratic form on the underlying 6-dimensional vector space $V = \mathbb{C}^6$, and its zero locus is nothing but the Fano scheme of lines on the associated quadric in $\mathbb{P}(V)$. If the quadric
were degenerate, the zero locus would have a singularity of dimension at least 3, and hence $X_{b4}$ would also be singular. Thus we can assume that the quadric is non-degenerate. But a non-degenerate quadric in $\mathbb{P}(V) = \mathbb{P}^3$ can be identified with the Grassmannian $\text{Gr}(2, W) \subset \mathbb{P}(\Lambda^2 W)$ of a 4-dimensional vector space $W = \mathbb{C}^4$, and the Fano scheme of lines on $\text{Gr}(2, W)$ is then identified with the flag variety $\text{Fl}(1, 3; W)$. Furthermore, the flag variety $\text{Fl}(1, 3; W)$ is a divisor of bidegree $(1, 1)$ in $\mathbb{P}(W) \times \mathbb{P}(W^\vee)$ (this divisor corresponds to the natural pairing between $W$ and $W^\vee$), and the restriction of the generator of $\text{Pic}(\text{Gr}(2, V))$ corresponds under this identification to the class of divisors of bidegree $(1, 1)$ on $\text{Fl}(1, 3; W) \subset \mathbb{P}(W) \times \mathbb{P}(W^\vee)$. Hence the additional zero locus of a section of the line bundle $\mathcal{O}(2)$ on $\text{Gr}(2, V)$ is equivalent to the zero locus of a section of the line bundle $\mathcal{O}(2, 2)$ on $\text{Fl}(1, 3; W)$. □

Denote by $h$ the pullback of the positive generator of $\text{Pic}(\mathbb{P}(W))$ under the projection $X_{b4} \to \mathbb{P}(W)$.

**Corollary 2.2.** A Küchle variety $X_{b4}$ of type (b4) has the structure of a conic bundle over $\mathbb{P}^3$. For generic $X_{b4}$, the discriminant of the conic bundle is an octic surface in $\mathbb{P}^3$ with 80 ordinary double points.

**Proof.** The projection $\text{Fl}(1, 3; W) \to \mathbb{P}(W)$ is the projectivization of the bundle $\Omega_{\mathbb{P}(W)}$, and a divisor of bidegree $(2, 2)$ gives a quadratic form on $\Omega_{\mathbb{P}(W)}$, that is, a symmetric map $\Omega_{\mathbb{P}(W)} \to T_{\mathbb{P}(W)}$. The cohomology classes of the discriminant and the corank 2 locus of this map can be computed by [7]. They are equal to $2c_1(T_{\mathbb{P}(W)})$ and $4(c_1(T_{\mathbb{P}(W)})c_2(T_{\mathbb{P}(W)}) - c_3(T_{\mathbb{P}(W)}))$ respectively. Substituting $c_1(T_{\mathbb{P}(W)}) = 4h$, $c_2(T_{\mathbb{P}(W)}) = 6h^2$ and $c_3(T_{\mathbb{P}(W)}) = 4h^3$, we conclude that the discriminant is an octic surface whose singular locus has class $4(4h \cdot 6h^2 - 4h^3) = 4(24h^3 - 4h^3) = 80h^3$. For generic $X_{b4}$ it follows that the corank 2 locus consists of 80 points, and these points are the only singularities of the discriminant. □

Using this description, one can decompose the derived category.

**Proposition 2.3.** Let $X = X_{b4}$. Then there is a semiorthogonal decomposition

$$D(X_{b4}) = \langle \mathcal{O}_X, \mathcal{O}_X(h), \mathcal{O}_X(2h), \mathcal{O}_X(3h), D(\mathbb{P}(W), \mathcal{C}_0) \rangle,$$

where

$$\mathcal{C}_0 = \mathcal{O}_{\mathbb{P}(W)} \oplus \Omega^2_{\mathbb{P}(W)}$$

is the sheaf of even parts of the Clifford algebras associated with the conic bundle $X \to \mathbb{P}(W)$. The category $D(\mathbb{P}(W), \mathcal{C}_0)$ is a twisted 3-Calabi–Yau category.

**Proof.** By Theorem 4.2 in [8], a conic bundle structure on $X$ gives a two-component semiorthogonal decomposition $D(X) = \langle D(\mathbb{P}(W)), D(\mathbb{P}(W), \mathcal{C}_0) \rangle$. Taking into account the standard exceptional collection in the derived category of the projective space $\mathbb{P}(W)$, we deduce the first claim. An explicit formula for the sheaf of even parts of the Clifford algebras in this case is obtained from formula (12) in [8]. Finally, to check the Calabi–Yau property of $D(\mathbb{P}(W), \mathcal{C}_0)$, we use Theorem 5.3 in [9]. Note that $\text{Fl}(1, 3; W)$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}(W)$. Hence it has a rectangular Lefschetz decomposition

$$D(\text{Fl}(1, 3; W)) = \langle D(\mathbb{P}(W)), D(\mathbb{P}(W)) \otimes \mathcal{O}(1, 1), D(\mathbb{P}(W)) \otimes \mathcal{O}(2, 2) \rangle.$$


Using Theorem 5.3 in [9], we conclude that the square of the Serre functor of the category \( D(\mathbb{P}(W), \mathcal{O}_0) \) is isomorphic to the shift by 6. Hence the composition of the Serre functor and the shift by \(-3\) is an involution \( \tau \), and the category may be regarded as a \( \tau \)-twisted 3-Calabi–Yau category. □

Remark 2.4. One can consider the category \( D(\mathbb{P}(W), \mathcal{O}_0) \) equivariant with respect to the action of \( \mathbb{Z}/2 \) determined by the involution \( \tau \) (in the sense of [10]). This category is an untwisted 3-Calabi–Yau category. It would be interesting to investigate it.

2.2. Type (b9). These varieties have recently been thoroughly studied by Casagrande [2]. In particular, she proved the following theorem.

Theorem 2.5 ([2], §3.2). Let \( X_{b9} \) be a Küchle fourfold of type (b9). Then there are seven points \( x_1, \ldots, x_7 \in \mathbb{P}^4 \) in general position such that \( X_{b9} \) is obtained from the blow-up \( \text{Bl}_{x_1,\ldots,x_7} \mathbb{P}^4 \) in these seven points by 22 antiflips in the proper transforms of the 21 lines \( L_{ij} \) passing through \( x_i \) and \( x_j \), and the proper transform of the rational normal quartic \( C \) passing through all seven points.

This result has a corollary for the structure of the derived category.

Corollary 2.6. The category \( D(X_{b9}) \) has a full exceptional collection of length 48.

Proof. The projective space \( \mathbb{P}^4 \) has an exceptional collection of length 5. When we blow up seven points, Orlov’s blow-up formula shows that three exceptional objects must be added for each of these points. This gives \( 7 \cdot 3 = 21 \) new exceptional objects. Finally, by [11], for each of 22 antiflips we must add one more object. In total we get a full exceptional collection of length \( 5 + 21 + 22 = 48 \). □

§ 3. Varieties of type (d3)

By definition, a Küchle fourfold of type (d3) is the zero locus of a generic section of the bundle \( \Lambda^2 U^\vee \oplus \Lambda^2 U^\vee \oplus \mathcal{O}(1) \) on \( \text{Gr}(5, 10) \). Thus it is a hyperplane section of a fivefold \( M \) defined as the zero locus of a section of \( \Lambda^2 U^\vee \oplus \Lambda^2 U^\vee \) on \( \text{Gr}(5, 10) \). We start with a description of \( M \). In fact we prove a more general result replacing \( \text{Gr}(5, 10) \) by \( \text{Gr}(n, 2n) \) with \( n \) arbitrary.

3.1. Lagrangian spaces for a pencil of skew-forms. Let \( V \) be a vector space of dimension \( 2n \), \( \lambda: \mathbb{C}^2 \to \Lambda^2 V^\vee \) a pencil of skew-forms, and \( M_\lambda \) the zero locus of the corresponding global section of the bundle \( \Lambda^2 U^\vee \oplus \Lambda^2 U^\vee \) on \( \text{Gr}(n, V) \). Clearly, \( M_\lambda \) parametrizes those half-dimensional subspaces in \( V \) that are Lagrangian for all skew-forms in the pencil.

Theorem 3.1. If \( M_\lambda \) is smooth, then \( M_\lambda \cong (\mathbb{P}^1)^n \).

Proof. Let \( L_\lambda \subset \mathbb{P}(\Lambda^2 V^\vee) \) be the line corresponding to \( \lambda \), and let \( D \subset \mathbb{P}(\Lambda^2 V^\vee) \) be the discriminant hypersurface that parametrizes all degenerate skew-forms. Since the degeneracy condition for a skew-form is equivalent to the vanishing of its determinant, which is in turn equal to the square of the Pfaffian, we see that \( D \) is a hypersurface of degree \( n \). Hence the intersection \( L_\lambda \cap D \) consists of \( n \) points (counted with appropriate multiplicity) unless the line \( L_\lambda \) is contained in \( D \). For
every point $\lambda_i \in L_\lambda \cap D$ we denote the kernel of the skew-form $\lambda_i$ by $K_i \subset V$. Let us show that the smoothness of $M_\lambda$ is equivalent to the following two conditions:

1) the intersection $L_\lambda \cap D = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \}$ consists of $n$ distinct points;

2) we have $\dim K_i = 2$ for every $i$, $1 \leq i \leq n$, and $V = K_1 \oplus \cdots \oplus K_n$ is a direct-sum decomposition.

To prove this, note that for any skew-form $\lambda_0$, the zero locus of the corresponding section of $\Lambda^2 U'$ on $\text{Gr}(n, V)$ is singular at the point corresponding to an $n$-dimensional subspace $U \subset V$ if and only if $\dim(U \cap \text{Ker} \lambda_0) \geq 2$. This can be checked by a simple local computation.

We first assume that $\dim K_i > 2$ for some $i$. Choose a skew-form $\lambda_0$ (distinct from $\lambda_i$) in $\lambda$. Let $K_0 \subset K_i$ be a two-dimensional subspace which is isotropic with respect to $\lambda_0$ (it exists since $\dim K_i > 2$). Let $K_0^\perp \subset V$ be the orthogonal complement to $K_0$ with respect to $\lambda_0$. Then $K_0 \subset K_0^\perp$ because $K_0$ is $\lambda_0$-isotropic. Putting $V' := K_0^\perp / K_0$, we have $\dim V' = 2n - 4$. Note that $K_0$ lies in the kernel of the restriction to $K_0^\perp$ of both $\lambda_i$ and $\lambda_0$, whence both forms induce skew-forms $\lambda_i'$ and $\lambda_0'$ on $V'$. Let $U' \subset V'$ be an $(n - 2)$-dimensional subspace isotropic for $\lambda_i'$ and $\lambda_0'$ (it exists by a dimension count). Let $U \subset K_0^\perp \subset V$ be the pre-image of $U'$ under the projection $K_0^\perp \to K_0^\perp / K_0 = V'$. By construction, $U$ is an $n$-dimensional subspace of $V$ isotropic with respect to $\lambda_i$ and $\lambda_0$. Moreover, $\dim(U \cap K_i) \geq 2$. Hence the zero locus of $\lambda_i$ on $\text{Gr}(n, V)$ is singular at the point $U$. On the other hand, $U$ is also contained in the zero locus of $\lambda_0$. Hence $U$ is a singular point of the intersection $M_\lambda$ of the zero loci. This proves that for a smooth $M_\lambda$ we have $\dim K_i = 2$ for all $i$.

This argument also proves that the kernel $K_i$ of $\lambda_i$ is not isotropic with respect to the other skew-forms in the pencil (otherwise we could take $K_0 = K_i$ and repeat the argument). It follows that the line $L_\lambda$ is transversal to $D$. Indeed, the tangent space to $D$ at a point $\lambda_i$ is the space of all skew-forms vanishing on $K_i$, and we have just seen that it does not contain the line $L_\lambda$. By transversality, there are precisely $n$ distinct points of intersection of $L_\lambda$ with $D$.

It remains to establish the direct-sum decomposition. Take a non-degenerate skew-form $\lambda_0$ in the pencil. We claim that the kernel spaces $K_i$ are mutually orthogonal with respect to $\lambda_0$, that is, $\lambda_0(K_i, K_j) = 0$ for all $i \neq j$. Indeed, since $\lambda_0$ is a linear combination of $\lambda_i$ and $\lambda_j$, it suffices to show that $\lambda_i(K_i, K_j) = 0$ and $\lambda_j(K_i, K_j) = 0$. But both equalities hold because $K_i$ is the kernel of $\lambda_i$ and $K_j$ is the kernel of $\lambda_j$. Since the restriction of $\lambda_0$ to each $K_i$ is non-degenerate, the direct-sum decomposition follows.

**Remark 3.2.** We have actually proved that $M_\lambda$ is smooth if and only if the pencil of skew-forms in an appropriate basis can be written in the block-diagonal form

$$
\lambda = u \text{ diag} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) + v \text{ diag} \left( \begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -a_2 \\ a_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -a_n \\ a_n & 0 \end{pmatrix} \right),
$$

where $u$ and $v$ are appropriate coordinates on the pencil and $a_1, \ldots, a_n \in \mathbb{C}$ are pairwise distinct. This is an analogue of the standard form for a pencil of quadrics.
Now let us show that $M_\lambda$ is isomorphic to the product
\[ \mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_n) \cong (\mathbb{P}^1)^n. \]
To do this, we define a map $M_\lambda \to \prod \mathbb{P}(K_i)$ in the following way. Let $U \subset V$ be an $n$-dimensional vector subspace isotropic for $\lambda$. The pencil $\lambda$ of skew-forms induces a pencil of maps $\lambda: U \to U^\perp$. The Pfaffian of a skew-form in $\lambda$ is equal to the determinant of the corresponding map $U \to U^\perp$. Hence the maps $\lambda_1, \ldots, \lambda_n$ are degenerate. This means that $U$ intersects each of the kernel spaces $K_i$ non-trivially. We claim that
\[ U = (U \cap K_1) \oplus (U \cap K_2) \oplus \cdots \oplus (U \cap K_n) \tag{3.1} \]
and $\dim(U \cap K_i) = 1$ for all $i$. Indeed, since each summand on the right-hand side of (3.1) is at least one-dimensional, the dimension of the right-hand side of (3.1) is not less than $n$. On the other hand, the right-hand side is obviously contained in the left-hand side, whose dimension is equal to $n$. Hence both sides coincide and the dimension of each summand is equal to 1. Therefore the map
\[ M \to \mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_n), \quad U \mapsto (U \cap K_1, U \cap K_2, \ldots, U \cap K_n) \]
is well defined. The inverse map is given by the formula
\[ \mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_n) \to M, \quad (u_1, u_2, \ldots, u_n) \mapsto C u_1 \oplus C u_2 \oplus \cdots \oplus C u_n. \]
To check that it is well defined, we must show that $\lambda(u_i, u_j) = 0$ for all skew-forms $\lambda$ in the pencil and all $u_i \in K_i$, $u_j \in K_j$. But this follows from the fact that $\lambda$ is a linear combination of $\lambda_i$ and $\lambda_j$ and $\lambda_i(u_i, u_j) = \lambda_j(u_i, u_j) = 0$ because $u_i \in K_i = \text{Ker} \lambda_i$ and $u_j \in K_j = \text{Ker} \lambda_j$ (we used this argument above).

Clearly, the two maps constructed are inverse to each other. This proves the theorem. □

3.2. A hyperplane section. It was proved in § 3.1 that a smooth zero locus of a section of the bundle $\Lambda^2 U^\vee \oplus \Lambda^2 U^\vee$ on $\text{Gr}(n, 2n)$ is isomorphic to $(\mathbb{P}^1)^n$. We can now describe the required hyperplane section.

Lemma 3.3. The isomorphism defined in Theorem 3.1 identifies the line bundle $\mathcal{O}_{\text{Gr}(n, 2n)}(1)|_{M_\lambda}$ with the line bundle $\mathcal{O}(1, 1, \ldots, 1)$ on $(\mathbb{P}^1)^n$. In particular, the zero locus of a general section of $\Lambda^2 U^\vee \oplus \Lambda^2 U^\vee \oplus \mathcal{O}(1)$ on $\text{Gr}(n, 2n)$ is isomorphic to a divisor of multidegree $(1, 1, \ldots, 1)$ in $(\mathbb{P}^1)^n$.

Proof. The explicit formula for the isomorphism in Theorem 3.1 shows that the restriction of the tautological bundle $\mathcal{U}$ from $\text{Gr}(n, 2n)$ to $(\mathbb{P}^1)^n$ splits into the following direct sum:
\[ \mathcal{U}_{(\mathbb{P}^1)^n} = \bigoplus_{i=1}^n p_i^* \mathcal{O}(-1), \]
where $p_i$ is the projection of $(\mathbb{P}^1)^n$ to the $i$-th factor. Since
\[ \mathcal{O}_{\text{Gr}(n, 2n)}(-1) = \text{det} \mathcal{U}, \]
it follows that its restriction to \((\mathbb{P}^1)^n\) is isomorphic to
\[
\bigotimes_{i=1}^n p_i^* \mathcal{O}(-1) = \mathcal{O}(-1, -1, \ldots, -1).
\]
This proves the first statement. The second statement is obvious. □

Consider a divisor of multidegree \((1, 1, \ldots, 1)\) on \(\mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_n)\), where \(K_1, K_2, \ldots, K_n\) are two-dimensional vector spaces. Such a divisor is given by a multilinear form \(s \in K_1^\vee \otimes K_2^\vee \otimes \cdots \otimes K_n^\vee\), which may also be regarded as a linear map
\[
s : K_n \to K_1^\vee \otimes K_2^\vee \otimes \cdots \otimes K_{n-1}^\vee.
\]
It gives a pencil of sections of the line bundle \(\mathcal{O}(1, 1, \ldots, 1)\) on \(\mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_{n-1})\). Let \(Z \subset \mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_{n-1})\) be the intersection of the corresponding divisors.

**Lemma 3.4.** *The projection*

\[
\mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_n) \to \mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_{n-1})
\]

*identifies the zero locus \(X\) of the section \(s\) of \(\mathcal{O}(1, 1, \ldots, 1)\) with the blow-up of \(\mathbb{P}(K_1) \times \mathbb{P}(K_2) \times \cdots \times \mathbb{P}(K_{n-1})\) in \(Z\). In particular, \(X\) is rational.***

**Proof.** This is obvious. □

Taking \(n = 5\), we obtain a description of K"uchle 4-folds of type (d3).

**Corollary 3.5.** *Every K"uchle fourfold \(X_{d3}\) is isomorphic to the blow-up of \((\mathbb{P}^1)^4\) in a K3-surface \(Z\) defined as the intersection of two divisors of multidegree \((1, 1, 1, 1)\).*

**Proof.** The only thing to check is that \(Z\) (defined as above) is a K3 surface. But this follows immediately from the adjunction formula and Lefschetz theorem. □

Let \(h_i\) be the pullback to \(X_{d3}\) of the ample generator of the Picard group of \(\mathbb{P}(K_i)\).

**Corollary 3.6.** *There is a semiorthogonal decomposition*

\[
\mathcal{D}(X_{d3}) = \left\langle \{\mathcal{O}_X(k_1 h_1 + k_2 h_2 + k_3 h_3 + k_4 h_4)\}_{0 \leq k_i \leq 1}, \mathcal{D}(Z) \right\rangle
\]

*consisting of 16 exceptional line bundles and the derived category of the K3-surface \(Z\).*

**Proof.** This follows from Orlov’s blow-up formula for the standard exceptional collection on a product of projective lines. □

**Remark 3.7.** One can also predict the existence of a K3-category in \(\mathcal{D}(X_{d3})\) using Theorem 5.3 in [9]. Indeed, \(\mathcal{D}(M)\) clearly has a rectangular Lefschetz decomposition of length 2, whence the conclusion.
§ 4. Varieties of type (c7)

By definition, a Küchle fourfold of type (c7) is the zero locus of a generic section of the vector bundle $\Lambda^2 U^\perp(1) \oplus O(1)$ on $\Gr(3,8)$. As in the case of type (d3), we first consider a 5-fold $M$ defined as the zero locus of a section of $\Lambda^2 U^\perp(1)$. For convenience we replace $\Gr(3,8)$ by $\Gr(5,8)$. Since the bundle $U^\perp$ on the first Grassmannian is isomorphic to the bundle $U$ on the second, $M$ is the zero locus of a generic section of $\Lambda^2 U(1) \cong \Lambda^3 U^\vee$ on $\Gr(5,8)$.

Let $\lambda \in \Lambda^3 V^\vee$ be a generic 3-form on a vector space $V$ of dimension 8. Let $M \subset \Gr(5, V)$ be the zero locus of $\lambda$ regarded as a section of $\Lambda^3 U^\vee$. In other words, $M$ is the locus of all five-dimensional subspaces $U \subset V$ such that the restriction of $\lambda$ to $U$ is equal to zero (one could call such $U$ isotropic). We shall prove that $M$ is isomorphic to the blow-up of $\mathbb{P}^5$ in a Veronese surface.

4.1. The blow-up of $\mathbb{P}^5$ in a Veronese surface. Let $W$ be a vector space of dimension 3. Consider the Veronese embedding $\mathbb{P}(W) \to \mathbb{P}(S^2 W)$. Let $Y$ be the blow-up of $\mathbb{P}(S^2 W)$ with centre at $\mathbb{P}(W)$.

**Lemma 4.1.** There is an embedding $Y \subset \mathbb{P}(S^2 W) \times \mathbb{P}(S^2 W^\vee)$ such that 

$$Y = \left\{(C, C') \in \mathbb{P}(S^2 W) \times \mathbb{P}(S^2 W^\vee) \mid C \cdot C' = t \cdot 1_W \text{ for some } t \in \mathbb{C}\right\},$$

where $C \cdot C' \in \mathbb{P}(W \otimes W^\vee)$ is the image of $C \otimes C'$ under the natural map $S^2 W \otimes S^2 W^\vee \to W \otimes W^\vee$. The projections $\pi: Y \to \mathbb{P}(S^2 W)$ and $\pi': Y \to \mathbb{P}(S^2 W^\vee)$ are blow-ups of the Veronese surfaces $\mathbb{P}(W) \subset \mathbb{P}(S^2 W)$ and $\mathbb{P}(W^\vee) \subset \mathbb{P}(S^2 W^\vee)$.

The Picard group of $Y$ is generated by the pullbacks $H$ and $H'$ of the positive generators of $\text{Pic}(\mathbb{P}(S^2 W))$ and $\text{Pic}(\mathbb{P}(S^2 W^\vee))$ respectively. The exceptional divisors $E$ and $E'$ of $\pi$ and $\pi'$ are given by the formulae 

$$E = 2H - H', \quad E' = 2H' - H,$$

and the canonical class is 

$$K_Y = 2E - 6H = 2E' - 6H' = -2(H + H').$$

**Proof.** Define a rational map 

$$\mathbb{P}(S^2 W) \to \mathbb{P}(S^2 W^\vee)$$

by the formula $C \mapsto \hat{C}$, where $\hat{C}$ is the adjoint matrix of $C$. One can check that $Y$ is its graph. Alternatively, $Y$ may be identified with the space of complete quadrics in $\mathbb{P}(W)$. All the properties of $Y$ are well known. □

Consider the natural action of $\GL(W)$ on $Y$.

**Lemma 4.2.** The action of $\GL(W)$ on $Y$ has precisely four orbits:

$$Y_0 = \left\{(C, C') \mid r(C) = 3, r(C') = 3, C' = \hat{C}\right\},$$

$$y_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
\[ Y_1 = \{(C, C') \mid r(C) = 2, \ r(C') = 1, \ C' = \hat{C}\}, \]
\[
y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ Y_2 = \{(C, C') \mid r(C) = 1, \ r(C') = 2, \ C = \hat{C}'\}, \]
\[
y_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ Y_3 = \{(C, C') \mid r(C) = 1, \ r(C') = 1, \ C \cdot C' = 0\}, \]
\[
y_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

the points \( y_0, y_1, y_2, y_3 \in Y \) being typical representatives of the orbits.

\textbf{Proof.} A straightforward computation. \( \square \)

Consider the maps
\[ c: \Lambda^2 W^\vee \otimes O_Y(\mathcal{H}) \to W \otimes W^\vee \otimes O_Y, \quad \xi' \mapsto C \cdot \xi', \]
\[ c': \Lambda^2 W \otimes O_Y(\mathcal{H}') \to W \otimes W^\vee \otimes O_Y, \quad \xi \mapsto \xi \cdot C'. \]

\textbf{Lemma 4.3.} The sum of the maps \( c \) and \( c' \)
\[ \Lambda^2 W^\vee \otimes O_Y(\mathcal{H}) \oplus \Lambda^2 W \otimes O_Y(\mathcal{H}') \xrightarrow{c+c'} W \otimes W^\vee \otimes O_Y \]
has constant rank 3. The image \( \mathfrak{g} := \text{Im}(c + c') \) is a vector subbundle of rank 3 in \( \mathfrak{sl}(W) \otimes O_X \), all of whose fibres are Lie subalgebras.

\textbf{Proof.} Clearly, the map is \( \text{GL}(W) \)-equivariant. Hence it suffices to compute it at the points \( y_0, y_1, y_2, y_3 \). A direct computation shows that the images at these points are the following spaces of \( 3 \times 3 \) matrices:

\[ \mathfrak{g}_0 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \quad \mathfrak{g}_1 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \]
\[ \mathfrak{g}_2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}, \quad \mathfrak{g}_3 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \]

respectively. Each space is 3-dimensional and consists of matrices with trace 0, whence the first claim. Moreover, each space is a Lie subalgebra of \( \mathfrak{sl}(W) \), whence the second claim. \( \square \)

\textbf{Remark 4.4.} Note that \( \mathfrak{g}_0 = \mathfrak{so}(W, C') \) is the special orthogonal Lie algebra (with respect to the form \( C' \)).
Remark 4.5. Each of the Lie subalgebras which are fibres of $\mathfrak{g}$ possesses the following strange property. For every vector in $W$ (resp. covector in $W^\vee$) the subalgebra contains a non-zero element $\xi$ annihilating this vector (resp. covector).

4.2. Isotropic five-dimensional subspaces for a 3-form on an eight-dimensional space. We can now relate the variety $Y$ studied in §4.1 to the scheme $M \subset \text{Gr}(5, V)$ of five-dimensional subspaces (in a vector space $V$ of dimension 8) that are isotropic with respect to a 3-form $\lambda \in \Lambda^3 V^\vee$. Indeed, we can identify $V$ with the Lie algebra $\mathfrak{sl}(W)$. Since $\mathfrak{sl}(W)$ has a natural non-degenerate scalar product (the Killing form), we can consider the orthogonal complement $\mathfrak{g}^\perp$ to the subbundle $\mathfrak{g}$ defined in Lemma 4.3. Then $\mathfrak{g}^\perp \subset \mathfrak{sl}(W) \otimes O_Y = V \otimes O_Y$ is a vector subbundle of rank 5. Hence it induces a morphism

$$\varphi : Y \to \text{Gr}(5, V).$$

It remains to construct a 3-form on $V$ and show that this map identifies $Y$ with the submanifold $M \subset \text{Gr}(5, V)$.

Lemma 4.6. There is a unique $\text{SL}(W)$-invariant 3-form $\lambda$ on the adjoint representation $\mathfrak{sl}(W)$ of the group $\text{SL}(W)$. It is given by the formula

$$\lambda(\xi_1, \xi_2, \xi_3) = \text{Tr}([\xi_1, \xi_2] \xi_3).$$

Proof. We claim that the right-hand side a skew-form. Indeed, since $\lambda$ is clearly skew-symmetric in the first two arguments, it remains to show that it is invariant under a cyclic permutation. But this follows from the invariance of the trace under cyclic permutations. Finally, a direct computation by the Littlewood–Richardson rule yields an isomorphism

$$\Lambda^3(\mathfrak{sl}(W)) \cong \Sigma^{2,0, -2} W \oplus S^3 W^\vee \oplus S^3 W \oplus \mathfrak{sl}(W) \oplus \mathbb{C}$$

of $\text{SL}(W)$-modules. We see that the trivial module has multiplicity 1, whence the invariant 3-form is unique. □

Proposition 4.7. The form $\lambda$ is a generic 3-form on a vector space of dimension 8.

Proof. Let $V$ be a vector space of dimension 8. We recall from Table 1 in [12] that the action of $\text{SL}(V)$ on the space $\mathbb{P}(\Lambda^3 V^\vee)$ of 3-forms has a dense orbit, the Lie algebra of the stabilizer of a generic 3-form is isomorphic to $\mathfrak{sl}_3 \subset \mathfrak{sl}(V)$, and the restriction of $V$ to $\mathfrak{sl}_3$ is isomorphic to the adjoint representation. Thus we can identify

$$V \cong \mathfrak{sl}(W)$$

in such a way that the generic 3-form on $V$ is identified with the $\text{SL}(W)$-invariant 3-form on $\mathfrak{sl}(W)$. Since we have already shown that such a form $\lambda$ is unique, the proposition follows. □

Remark 4.8. The form $\lambda$ can be written explicitly in an appropriate basis as

$$\lambda = x_{238} + x_{167} - x_{247} - x_{356} - x_{148} - x_{158}.$$
Proposition 4.9. The map \( \varphi : Y \rightarrow \text{Gr}(5, V) \) is a closed embedding and its image coincides with the zero locus \( M \) of the global section \( \lambda \) of the bundle \( \Lambda^3 U^\vee \).

Proof. We first verify that the image of \( \varphi \) lies in \( M \), that is, it consists of five-dimensional subspaces isotropic with respect to \( \lambda \). Since \( Y \) is irreducible and the condition of being isotropic is closed, it suffices to check that the image of the open \( \text{GL}(W) \)-orbit \( Y_0 \subset Y \) consists of isotropic subspaces. Moreover, since the map \( \varphi \) and the form \( \lambda \) are \( \text{GL}(W) \)-invariant, it suffices to check that the image of \( y_0 \) is isotropic. This can be done by a straightforward computation. Indeed, \( \varphi(y_0) = g_0^\perp \) is the space of symmetric \( 3 \times 3 \) matrices with trace 0. Take any \( \xi_1, \xi_2, \xi_3 \in g_0^\perp \). Then it is clear that \( [\xi_1, \xi_2] \) is a skew-symmetric matrix. Hence it lies in \( g_0 \) and, therefore, \( \text{Tr}([\xi_1, \xi_2]) = 0 \).

We now check that \( \varphi \) is an embedding of the open orbit \( Y_0 \). Indeed, this is clear since a non-degenerate quadratic form \( C' \) can be recovered from the Lie algebra \( g = \mathfrak{so}(W, C') \subset \mathfrak{sl}(W) \) as the unique \( g \)-invariant quadratic form in \( S^2 W^\vee \).

Thus we see that \( \varphi \) is a morphism \( Y \rightarrow M \subset \text{Gr}(5, V) \), and this morphism is injective on the open subset \( Y_0 \subset Y \). Since \( Y \) and \( M \) are five-dimensional, the morphism is birational. Since both varieties are smooth and have Picard number 2 (this is clear by definition for \( Y \) and was proved in [1] for \( M \)), the morphism is an isomorphism. \( \square \)

Thus we have proved the following theorem.

Theorem 4.10. The zero locus \( M \) of a generic section of the vector bundle \( \Lambda^3 U^\vee \) on \( \text{Gr}(5, 8) \) is isomorphic to the blow-up of \( \mathbb{P}^5 \) in the Veronese surface \( \nu_2(\mathbb{P}^2) \).

4.3. A hyperplane section. We recall that a Küchle fourfold of type (c7) is by definition a half-anticanonical section of the variety \( M \). By Lemma 4.1, the corresponding linear system on \( M \) is \( 3H - E \), where \( H \) is the pullback to \( M \) of the generator of \( \text{Pic}(\mathbb{P}^5) \) and \( E \) is the exceptional divisor of the blow-up \( M \rightarrow \mathbb{P}^5 \).

Corollary 4.11. Let \( X \) be a Küchle fourfold of type (c7). Then there is a four-dimensional cubic \( Z \) containing the Veronese surface \( S = \nu_2(\mathbb{P}^2) \) such that \( X \) is isomorphic to the blow-up of \( Z \) in \( S \).

Proof. This is obvious. \( \square \)

We can now describe the derived category. Let \( E_X \) be the exceptional divisor of the blow-up \( X \rightarrow Z \), and let \( i : E_X \rightarrow X \) be its embedding. Note that \( E_X \) is a \( \mathbb{P}^1 \)-bundle over \( S \). We write \( h \) for the positive generator of \( \text{Pic}(S) \) and for its pullback to \( E_X \).

Corollary 4.12. There is a semiorthogonal decomposition

\[
\mathcal{D}(X_{c7}) = \langle \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H), i_* \mathcal{O}_{E_X}, i_* \mathcal{O}_{E_X}(h), i_* \mathcal{O}_{E_X}(2h), \mathcal{A}_X \rangle,
\]

where \( \mathcal{A}_X \) is a non-commutative K3-category equivalent to the non-trivial part of the derived category of the cubic fourfold \( Z \).

Proof. The derived category of \( Z \) has a semiorthogonal decomposition

\[
\mathcal{D}(Z) = \langle \mathcal{O}_Z, \mathcal{O}_Z(H), \mathcal{O}_Z(2H), \mathcal{A}_Z \rangle,
\]
where $\mathcal{A}_Z$ is a non-commutative K3-category (see [4]). Since $X$ is the blow-up of $Z$ in $S$, we have $D(X) = \langle \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H), \mathcal{A}_Z, D(S) \rangle$ by Orlov’s blow-up formula. Mutating $\mathcal{A}_Z$ to the right and replacing $D(S) = D(\mathbb{P}^2)$ by the standard exceptional collection, we get the result. □

**Remark 4.13.** One can also predict the presence of a K3-category using Theorem 5.3 in [9]. Indeed, the blow-up $M$ of $\mathbb{P}^5$ in the Veronese surface has an exceptional collection consisting of six line bundles pulled back from $\mathbb{P}^5$ and six line bundles on the exceptional divisor. Mutating, one can arrange them into a rectangular Lefschetz decomposition consisting of two blocks and then apply Theorem 5.3 in [9].

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