THE ENERGY OF CRUMPLED SHEETS IN FÖPPL-VON KÁRMÁN PLATE THEORY

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Abstract. We study investigate a long, thin rectangular elastic membrane that is bent through an angle $2\alpha$, using the Föppl–von Kármán ansatz in a geometrically linear setting. We study the associated variational problem, and show the existence of a minimizer for the elastic energy. We also prove rigorous upper and lower bounds for the minimum energy of this configuration in terms of the plate thickness $\sigma$ and the bending angle, and we also obtain results for the structure of the elastic ridge along its length.

1. Introduction

Crumpled elastic sheets can be thought of as minimum energy configurations for the Föppl–von Kármán (FvK) energy. Using this approach of elastic energy minimization, the crumpling response is understood as a result of the elastic energy of the sheet concentrating on a small subset of the entire sheet [2, 27, 25]. The energy in a crumpled sheet is concentrated on a network of thin line-like creases (ridges) that meet in point-like vertices. Recent work has resulted in quantitative understanding of both the vertices [6, 7, 8, 5, 31] and the ridges [28, 29, 14, 15]. Scaling laws governing the behavior of crumpled sheets have been obtained [2, 27, 28] using scaling arguments.

Minimum energy configurations for the FvK energy have also been studied in the context of the blistering problem, viz. the buckling of membranes as a result of isotropic compression along the boundary [32, 19].

There is a considerable body of mathematical work focused on the blistering problem [32, 21, 11, 22, 23, 3, 4]. Upper and lower bounds have been obtained for approximations to the elastic energy [32, 21, 23], for the FvK energy [22, 3] and for full three dimensional nonlinear elasticity [4]. The FvK energy and full three dimensional nonlinear elasticity give the same scaling for the upper and the lower bounds.

Our goal in this paper is to prove corresponding rigorous results for the energy in a minimal ridge – a single crease in a crumpled sheet. In addition to scaling results for the energy, we also investigate the structure of the ridge by obtaining pointwise bounds for its “width”. Our results for the ridge show an interesting contrast with the corresponding results for the blistering problem [3, 4]. In particular, the scaling of the energy with the thickness of the sheet has a different exponent. This implies that the boundary conditions play an important role in determining the $\Gamma$–limit of...
the FvK energy in the limit the thickness goes to zero. We discuss this issue further in Sec. 7.

This paper is organized as follows – In Sec. 2 we describe the problem of interest, set up the relevant energy functional and determine the appropriate boundary conditions. We also rescale the various quantities to a form that is suitable for further analysis, and recast the problem in terms of the rescaled quantities. In Sec. 4 we prove a lower bound for the elastic energy for our boundary conditions. In Sec. 5 we prove the corresponding upper bounds by explicit construction of a test solution. In Sec. 6 we investigate the structure of a single ridge, and we present a concluding discussion in Sec. 7.

2. THE VARIATIONAL PROBLEM

We are interested in a minimal ridge, i.e., the single crease that is formed when a long rectangular elastic strip is bent through an angle by clamping the lateral boundaries to a bent frame. This situation is depicted in Figure 1.

As we will see below, the idealized boundary conditions with a sharp corner are not appropriate, since they lead to an infinite energy for a sheet with a finite thickness. If we make the corner extremely sharp, all the energy (asymptotically) will be at the corners, and this obscures the interesting physics in the problem, namely the energy and the structure of the ridge. Thus we have to incorporate the smoothness of the corners into our boundary conditions. In general, curvatures on scale smaller than the thickness cannot occur for a real sheet, and our model energy is not appropriate for
this situation. For a crease in a real crumpled sheet, the corner is definitely smooth on the scale of the thickness of the sheet.

We will consider the situation where the sheet is clamped to a frame, that is much like the idealized situation depicted in Fig. 1. The sheet is a rectangular strip $|x| \leq L, |y| \leq L'$. We generally consider the situation $L \lesssim L'$. The sheet is clamped to the frame at $x = \pm L$ and at $y = \pm L'$. We will place the following requirements on the frame –

1. The boundary conditions at the frame are non-stretching, i.e., the strain $\gamma_{yy}$ is identically zero on the boundaries $x = \pm L$, and the strains $\gamma_{xx} = \gamma_{xy} = 0$ at $y = \pm L'$.
2. The bending at the boundaries is localized. If $s$ is an arclength parameter along the boundary, we will require that both the boundaries are straight on a set of the form $|s| \geq k$, where $k \ll L$.
3. We will assume that the two “bent” boundaries on the frame ($x = \pm L$) are given by planar curves, and that the angle between the straight sections on these boundaries for $y > k$ and $y < -k$ is $2\alpha$ for both boundaries.
4. We will require that the straight segments for both the “bent” boundaries are parallel and a distance $L$ apart. This implies that the two boundaries are not twisted with respect to one another.
5. We will also require that the two boundaries have the same asymptotic shift. This is necessary to make $\gamma_{xy} = 0$ at $y = \pm L'$. To define this precisely, we need to introduce appropriate coordinates, and we will do this below.

2.1. Coordinates. We will use (the material) coordinates $(x, y)$ on the reference half strip $|x| \leq L, |y| \leq L'$. The planes containing the parallel straight portions of the two boundaries pick out preferred in-plane directions for $y \gg k$ and $y \ll -k$. We introduce two sets of Cartesian coordinate systems in the ambient space. The coordinate direction $u$ is perpendicular to the straight portion of the boundary at $x = -L$, and is directed toward the boundary at $x = L$. The coordinate directions $v^+$ and $v^-$ are along the straight portions of the boundaries for $s > k$ and $s < -k$ respectively. The coordinate directions $w^+$ and $w^-$ give the out of plane directions, and are chosen so that $(u, v^+, w^+)$ and $(u, v^-, w^-)$ are right handed orthogonal triads. Finally, the origins of both the coordinate systems coincide, and they are chosen such that the straight portions of the boundaries lie in the plane $w^\pm = 0$, and the frame boundaries are in the planes $u = \pm L$. Note that this completely specifies the definition of the coordinate systems, and in particular, we do not have a freedom to translate $v^\pm$.

The various coordinate are represented schematically in Figure 2. The grid in the figure is generated by the lines $x = \text{constant}$ and $y = \text{constant}$ that are straight in the reference (material) coordinates. We will use the coordinate system $(u, v^+, w^+)$ for the portion of the sheet with $y \geq 0$ and $(u, v^-, w^-)$ for $y \leq 0$. At $y = 0$, we have the matching conditions

$$
\begin{pmatrix}
  w^- \\
  v^- 
\end{pmatrix} = \begin{pmatrix}
  \cos 2\alpha & -\sin 2\alpha \\
  \sin 2\alpha & \cos 2\alpha 
\end{pmatrix} \begin{pmatrix}
  w^+ \\
  v^+
\end{pmatrix}
$$
Also, in the straight portion of the boundaries $|s| \geq k$, since $\gamma_{yy} = 0$, it follows that $v^\pm(\pm L, y) - y$ is a constant for sufficiently large (small) $y$. For the boundary at $x = -L$, we define the asymptotic shifts by

$$\delta_1^ \pm = v^\pm(-L, y) - y, \quad y \geq k \quad \text{(respectively } y \leq -k).$$

Similarly, the asymptotic shifts for the boundary at $x = L$ are constant if $|y|$ is sufficiently large. For the boundary at $x = -L$, we define the asymptotic shifts by

$$\delta_2^ \pm = v^\pm(L, y) - y, \quad y \geq k \quad \text{(respectively } y \leq -k).$$

We will say that the two boundaries are compatible, if $\delta_1^+ = \delta_2^+$ and $\delta_1^- = \delta_2^-$. This clearly a necessary condition for the existence of a configuration of the sheet that satisfies the boundary conditions, and is asymptotically strain free, i.e., the strain is identically zero for $|y| \geq l$ for a sufficiently large $l$. In particular, we can take $l = k$.

Assuming that the two boundaries are compatible, we will set $\delta^+ = \delta_1^+ = \delta_2^+$, $\delta^- = \delta_1^- = \delta_2^-$, and $\delta = \delta^+ - \delta^-$. The quantities $\delta^+$ and $\delta^-$ change under translations of the coordinate $y$, but $\delta$ is an invariant under these translations, and is purely a geometrical property of the frame. Below, we will give an expression for $\delta$ in terms of functions specifying the boundary conditions. W.L.O.G, we can, and henceforth will, translate the coordinate $y$ such that $\delta^+ = -\delta^- = \frac{1}{2}\delta$.

**Figure 2.** A schematic representation of our coordinate system and the boundary conditions imposed on the sheet. The thick lines depict the “frame”, and the “corners” are smooth on a scale $a$. $u^\pm$ and $v^\pm$ are the in-plane directions and $w^\pm$ are the out-of-plane directions. The grid is given by the lines $x = \text{constant}$ and $y = \text{constant}$.

2.2. **The Elastic energy.** A mathematically justified way to obtain the elastic energy of the deformed sheet is to treat the sheet as a three dimensional (albeit thin) object and use a full nonlinear three dimensional elastic energy functional for the energy density. The sheet is now a three dimensional object $S \times [-\frac{h}{2}, \frac{h}{2}]$ with thickness
Let \( x, y \) denote the in plane coordinates as above, and \( z \) denote the coordinate in the thin direction. If the configuration of the sheet is given by a mapping \( \phi: S \times \left[-\frac{h}{2}, \frac{h}{2}\right] \to \mathbb{R}^3 \) the elastic energy is given by

\[
I_{3D} = \int_S dx dy \int_{-\frac{h}{2}}^{\frac{h}{2}} dz W_{3D}(\nabla \phi)
\]

This approach however does not take advantage of the “thinness” of the sheet. In particular, we would like to treat the thin sheet as a two dimensional object. The derivation of reduced dimensional descriptions of thin sheets has a long history. There is a classical theory for thin elastic sheets built on the work of Euler, Cauchy, Kirchoff, Föppl and Von Kármán [30, 26, 9].

In the classical Föppl–von Kármán ansatz, the behavior of the deformation \( \phi \) is completely determined by the behavior of the center-plane \( z = 0 \). An asymptotic expansion with this ansatz [9] yields an effective 2-D elastic energy

\[
I = h \left[ \int_S dx dy W_{2D}(\phi_x, \phi_y) + qh^2 \int_S dx dy |\nabla \nu|^2 \right],
\]

where \( q \) is a nondimensional \( O(1) \) factor, \( \nu = \phi_x \times \phi_y / \|\phi_x \times \phi_y\| \) is the normal to the center surface, and \( W_{2D} \) is an effective two dimensional energy. This is the geometrically nonlinear Föppl–von Kármán energy of the thin sheet.

The functional \( W_{2D} \) is zero if \( (\phi_x, \phi_y) \in O(2, 3) \), the set of matrices that give isometric linear mappings \( \mathbb{R}^2 \to \mathbb{R}^3 \). We will demand that

\[
W_{2D}(\phi_x, \phi_y) \geq c \text{dist}^2((\phi_x, \phi_y), O(2, 3)).
\]

and that \( W_{2D}(\phi_x, \phi_y) \leq \|\phi_x\|^4 + \|\phi_y\|^4 \) for large \( \phi_x, \phi_y \). These conditions are identical to the conditions on the energy in Ref. [4].

A typical (or canonical) energy functional which satisfies these conditions, and has the natural invariances for the problem, viz., the action of \( O(3) \) on \((u, v, w)\) and \( O(2) \) on \((x, y)\) is

\[
W_{2D} = \text{dist}^2((\phi_x, \phi_y), O(2, 3))
\]

\[
= (u_x^2 + v_x^2 + w_x^2 - 1)^2 + 2(u_x v_x + u_y v_y + w_x w_y)^2 + (u_y^2 + v_y^2 + w_y^2 - 1)^2
\]

For the most part, we will work with the energy that we get by linearizing the above expression, only in the in plane deformation \((u, v)\) about the reference state \( u_x = 1, v_y = 1 \) and all the other derivatives are zero. This yields the linearized energy

\[
\hat{W}_{2D} = \left[2(u_x - 1) + u_x^2\right]^2 + \left[2(v_y - 1) + u_y^2\right]^2 + 2 \left[u_x + v_y + w_x w_y\right]^2.
\]

Note that, by linearizing the energy functional, we have destroyed the natural invariances of the energy, and have picked out preferred in–plane and out of plane directions in the ambient space [9]. We will also linearize the normal vector so that

\[
\nu \approx -w_x \mathbf{e}_u - w_y \mathbf{e}_v + \mathbf{e}_w
\]
Using this expression for the normal vector, gives the \textit{geometrically linear} Föppl – von Kármán energy
\[ I_{lin} = h \left[ \iint_S \left( 2(u_x - 1) + w_x^2 \right)^2 + \left( 2(v_y - 1) + w_y^2 \right)^2 \right. \\
+ 2 [u_x + v_y + w_x w_y]^2 + q h^2 \left( w_{xx}^2 + 2w_{xy}^2 + w_{yy}^2 \right) \] 

We will henceforth normalize the energy by \( 4h \), and also define the scaled thickness \( \sigma = \sqrt{2}h \). The linearized FvK energy for our problem is
\[ I = I^+ + I^- , \]
where we have suppressed the subscript + on \( v^+ \) and \( w^+ \). \( I^- \) is given by the corresponding expression for \( y \leq 0 \).

As defined above \( x \) and \( y \) are reference coordinates on the sheet, \( u \) and \( v \) are in-plane coordinates, \( w \) is the out of plane displacement and \( \sigma \) is the scaled thickness of the sheet. The integrand includes the squares of the linearized strains,
\[ \gamma_{xx} = u_x + \frac{1}{2} w_x^2 - 1, \quad \gamma_{xy} = \gamma_{yx} = \frac{1}{2} (v_x + u_y + w_x w_y), \quad \gamma_{yy} = v_y + \frac{1}{2} w_y^2 - 1. \]

2.3. Boundary and matching conditions. The blistering of thin films is also described by the elastic energy in \cite{2}. A similar energy also describes multiple scale buckling in \textit{free} elastic sheets (\textit{i.e.} sheets that are not forced through the boundary conditions) that are not intrinsically flat \cite{33, 34}.

The difference between the blistering problem and a minimal ridge is in the boundary conditions, which we describe below. If the bending half-angle \( \alpha \ll 1, \tan \alpha \approx \sin \alpha \approx \alpha \). In this case, the deformations and the linearized strains are small.

If \( s \) is an arclength parameter along the boundary, with our choice of coordinate systems, the frame is given by functions \((v^+ (s), w^+ (s))\) for \( s \geq 0 \) (resp. \( s \leq 0 \)) at \( x = -L \), and functions \((v^\pm (s), w^\pm (s))\) for \( s \geq 0 \) (resp. \( s \leq 0 \)) at \( x = L \). We define the \textit{curvature} \( \kappa_1 \) of the boundary at \( x = -L \) by
\[ \kappa_1 = \frac{1}{a_1} = \int_0^{L'} (w^+_{ss})^2 ds + \int_{-L'}^0 (w^-_{ss})^2 ds \]

\( a_1 \) is a length scale associated with the boundary at \( x = -L \). We define \( k_1 \) by demanding that the boundary at \( x = -L \) is straight for \(|s| \geq k_1 \). We will require that \( k_1 \) be on the same scale as \( a_1 \), so that \( k_1 = Ka_1 \), with \( K \) staying order 1 as we change the parameters in the problem. We can similarly define \( a_2 \) and \( k_2 \) using the boundary at \( x = L \). We will assume that the length scales \( a_1 \) and \( a_2 \) are comparable, \textit{i.e} \( a_2/a_1 \sim O(1) \) and likewise for \( k_1 \) and \( k_2 \). In fact, we will typically assume that they are equal \( a_1 = a_2 = a \), and \( k_1 = k_2 = Ka \).

This length scale \( a \) sets the natural length scale for the boundaries. We will assume that the boundary conditions are compatible, and further, the asymptotic shift is on
the scale $aa^2$, i.e., $\delta = \Delta aa^2$, where $\Delta$ stays order 1 as we change the parameters of the problem.

Since the linearized strain $\gamma_{yy}$ is zero at the boundaries, we have $v_y^\pm = 1 - \frac{1}{2}(w_y^\pm)^2$. Outside $|y| \leq k$, $w_y^\pm = 0$. Since the asymptotic shift is $\delta$, we get $v^\pm(\pm L, y) - y = \pm \frac{1}{2}\delta$ for $|y| > k$.

The elastic energy penalizes the square of the curvature. Consequently, all finite energy configurations of the sheet have a continuous tangent plane a.e. This yields the matching condition

$$\left( \begin{array}{c} \nabla w^- \\ \nabla v^- \end{array} \right) = \left( \begin{array}{cc} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{array} \right) \left( \begin{array}{c} \nabla w^+ \\ \nabla v^+ \end{array} \right)$$

at $y = 0$, where $\nabla$ denotes the 2 dimensional gradient. Since (4) holds for all $x$ at $y = 0$, it automatically implies that

$$\left( \begin{array}{c} w_x^- \\ v_x^- \end{array} \right) = \left( \begin{array}{cc} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{array} \right) \left( \begin{array}{c} w_x^+ \\ v_x^+ \end{array} \right)$$

We therefore have an independent matching condition

$$\left( \begin{array}{c} w_y^- \\ v_y^- \end{array} \right) = \left( \begin{array}{cc} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{array} \right) \left( \begin{array}{c} w_y^+ \\ v_y^+ \end{array} \right) \quad y = 0$$

For small $\alpha$, we can linearize the matching conditions, and to first order in $\alpha$, we obtain

$$w^-(x, 0) = w^+(x, 0)$$
$$w^-y(x, 0) = w^y+(x, 0) - 2\alpha$$
$$v^-(x, 0) = v^+(x, 0) + 2\alpha w^+(x, 0)$$

If we assume that $w_y^\pm(x, 0) \sim O(\alpha)$, then for small strains, $v_y^\pm(x, 0) \approx 1$, and the matching condition for $v_y$ is satisfied as an identity to first order in $\alpha$.

Since $a$ is the natural length scale at the boundaries, $w_y^\pm \sim O(\alpha)$ near $y = 0, v^\pm = y \pm \frac{1}{2}\delta$ and $v_y^\pm + \frac{1}{2}(w_y^\pm)^2 - 1 = 0$, we define the boundary conditions in terms of scaling functions

$$w_{1,2}^\pm(s) = a\alpha \sqrt{2} \Phi_{1,2}^\pm \left( \frac{s}{a} \right), \quad v_{1,2}^\pm(s) = \alpha^2 a \Psi_{1,2}^\pm \left( \frac{s}{a} \right) + s \pm aa^2 \Delta$$

The scaling functions $(\Phi_{1,2}^\pm(\eta), \Psi_{1,2}^\pm(\eta))$ are normalized (have curvature 1)

$$\int_0^{L'/a} [\Phi_{1,2}^\pm](\eta)^2 d\eta + \int_{-L'/a}^0 [\Phi_{1,2}^\pm](\eta)^2 d\eta = 1, \quad \text{for } i = 1, 2,$$

are strain free,

$$[\Phi_{1,2}^\pm](\eta)^2 + \Psi_{1,2}^\pm(\eta)^2 = 0, \quad \text{for } i = 1, 2,$$

and satisfy the matching conditions

$$\Phi_i^+(0) = \Phi_i^-(0) \quad \text{for } i = 1, 2,$$
$$\Phi_i'(0) = \Phi_i''(0) - \sqrt{2} \quad \text{for } i = 1, 2,$$
$$\Psi_i(0) = \Psi_i^+(0) + 2\sqrt{2} \Phi_i^+(0) + \Delta \quad \text{for } i = 1, 2,$$
Finally, we also have the conditions $v = \pm \frac{1}{2} \delta, u = w = 0$ at $y = \pm L'$, where $\delta$ is the asymptotic shift determined by the (compatible) frame boundaries at $x = \pm L$. We will explore the relation between the frame $(\Phi^{\pm}, \Psi^{\pm})$ and the value of $\Delta$ (and consequently the value $\delta$) now. Using $w^{\pm} = 0$ for $|y| \geq k$, and the definition of $\Delta$, we see that

$\Phi^{\pm}_{i}(\eta) = \Psi^{\pm}_{i}(\eta) = 0$ for $|\eta| \geq K = k/a, i = 1, 2$.

Using $\gamma_{yy} = 0$, we get $\Psi^{\pm}_{i}(\eta) = -\int_{\pm L'/a}^{0} [\Phi^{\pm}_{i}'(\eta)]^{2}d\eta$ for $\eta \geq 0$ (resp. $\eta \leq 0$).

Using this relation in the matching condition, we obtain

$\Delta = - \left[ 2\sqrt{2}\Phi_{i}^{\pm}(0) + \int_{-L'/a}^{0} [\Phi_{i}^{\pm}'(\eta)]^{2}d\eta + \int_{0}^{L'/a} [\Phi_{i}^{\pm}'(\eta)]^{2}d\eta \right]$.

This gives the explicit relation between $\Delta$ and the out of plane displacement of a no-stretching profile.

2.4. Rescalings. From Lobkovsky’s results [28] and the analysis in Re. [36], we know that the dominant energy balance is between the longitudinal curvature $w_{yy}^{2}$ and the transverse strain $w_{x}^{4}$. The natural length scale in $X$ is $L$, and as we saw above, the natural scale for $w_{y}$ is $\alpha$, the bending angle.

Let $L_{q}$ denote the natural scale for the quantity $q$. For example, we have $L_{x} = L$. The above considerations lead to the conclusions

$w_{y} \sim \frac{L_{w}}{L_{y}} \sim \alpha \quad \sigma^{2} w_{yy}^{2} \sim \frac{\sigma^{2} L_{w}^{2}}{L_{y}^{4}} \sim w_{x}^{4} \sim \frac{L_{w}^{4}}{L_{x}^{4}}$.

This motivates the introduction of rescaled coordinates and displacements by

$x = LX, \quad y = \sigma^{1/3} L^{2/3} \alpha^{-1/3} Y \equiv L_{y} Y,$

and

$w^{\pm} = \sqrt{2}\sigma^{1/3} L^{2/3} \alpha^{2/3} W^{\pm} \equiv L_{w} W^{\pm},$

$v^{\pm} = y \pm \frac{\delta}{2} + \sigma^{1/3} L^{2/3} \alpha^{5/3} V^{\pm}$

$u = x + \sigma^{2/3} L^{1/3} \alpha^{4/3} U.$

Here $\delta$ is the asymptotic shift, and is determined by the frame boundaries. Since $\sigma, L, x, y, u, v, w, \delta$ all have dimensions of a length, and $\alpha$ is dimensionless, it is clear that the rescaled quantities $X, Y, U, V, W, U^{\mp}$ are all dimensionless. Note that these rescalings are different from the rescalings in Ref. [36].

With these rescalings, the dimensionless energy $I^{\pm} = \sigma^{-5/3} L^{-1/3} \alpha^{-7/3} I^{\pm}$ is given by

$I^{\pm}(U, V, W) = \iint \left[ (U_{X} + W_{X}^{2})^{2} + \frac{1}{2} \epsilon^{-2/3} (V_{X} + U_{Y} + 2W_{X}W_{Y})^{2} \right. \\
+ \epsilon^{-4/3} (V_{Y} + W_{Y}^{2})^{2} \left. + \frac{1}{8} W_{Y}^{2} + 2\epsilon^{2/3} W_{XY}^{2} + \epsilon^{4/3} W_{XX}^{2} \right] dX dY$.

(5)
where we have to use the appropriate $V^\pm$ and $W^\pm$. We have suppressed the super-
scripts $\pm$ on $V$ and $W$ for clarity. $\epsilon = \sigma/(La)$ is the natural small dimensionless parameter in the problem. We think of $\epsilon$ as a dimensionless thickness.

Our quest for rigorous scaling results for the energy $I$ reduces to the following –
Show that the rescaled energy $I^\ast = I^\ast + I^\ast$, of a minimizer $(\U^\ast, V^\ast, W^\ast)$, is bounded above and below by positive constants uniform in the dimensionless thickness parameter $\epsilon$, as $\epsilon \to 0$.

With this rescaling, we have the matching conditions

\begin{align*}
W^+(X,0) &= W^-(X,0) \\
W^+_1(X,0) &= W^-_1(X,0) + \sqrt{2} \\
V^-(X,0) &= V^+(X,0) + 2\sqrt{2}W^+(X,0) + \frac{\delta}{L_v} \\
&= V^+(X,0) + 2\sqrt{2}W^+(X,0) + A\Delta
\end{align*}

(6)

and the boundary conditions

\begin{align*}
W^\pm(-1,Y) &= A\Phi^\pm_1 \left( \frac{Y}{A} \right), \quad U(-1,Y) = 0, \quad V^\pm(-1,Y) = A\Psi^\pm_1 \left( \frac{Y}{A} \right), \\
W^\pm(1,Y) &= A\Phi^\pm_2 \left( \frac{Y}{A} \right), \quad U(1,Y) = 0, \quad V^\pm(1,Y) = A\Psi^\pm_2 \left( \frac{Y}{A} \right), \\
W^\pm(X, \pm L'/L_y) &= 0, \quad U(X, \pm L'/L_y) = 0, \quad V^\pm(X, \pm L'/L_y) = 0,
\end{align*}

(7)

where $A = a/L_y$ is the nondimensionalized inverse curvature, and $\Phi_i^\pm, \Psi_i^\pm$ are the scaling functions from above.

3. Existence of a Minimizer

From this point forward, we will use $c, C, C', C_1, C_2$, etc. to denote constants whose precise numerical values are not important. These constants can change from one line to the next. By doing this, we can focus on the scalings of the various quantities, without worry about the numerical values of the constants in the scaling relations. If these constants depend on a parameter $q$, we show this dependence by writing $C(q)$ or $C_q$. Also, we will suppress the superscripts $\pm$ whenever this does not cause any confusion.

**Lemma 3.1.** $u, v^\pm : S^\pm = [-L, L] \times [0, \pm L'] \to \mathbb{R}$ are $H^1$ functions such that $u = 0$ at $x = \pm L$ and $u = 0$ at $y = \pm L'$. The functional

\[ J(u, v^+, v^-) = \iint_{S^+} u_x^2 + (v_y^+)^2 + \frac{1}{2}(v_x^+ + u_y)^2 + \int_{S^-} u_x^2 + (v_y^-)^2 + \frac{1}{2}(v_x^- + u_y)^2 \]

is coercive, in the sense $\exists c, C > 0$ such that

\[ J(u, v^+, v^-) \geq c\left[ \|Du\|^2_{L^2} + \|Dv^+\|^2_{L^2} + \|Dv^-\|^2_{L^2} \right] - C \left[ \int [v^+ - v^-]^2 dx \right] \]
Proof. We first consider the integral over $S^+$. Using the boundary conditions, and repeated integration by parts yields,

$$\int \int u(y, x)v^+_x(x, y)dxdy = - \int _{-L} ^L u(x, 0)v^+_x(x, 0)dx$$

$$- \int \int u(x, y)v^+_x(x, y)dxdy$$

$$= - \int _{-L} ^L u(x, 0)v^+_x(x, 0)dx$$

(8)

Adding a similar result for $S^-$, we obtain

$$J = \frac{1}{2}(\|Du\|_{L^2}^2 + \|Dv^+\|_{L^2}^2 + \|Dv^-\|_{L^2}^2) + \int _{-L} ^L u(x, 0) [v^-_x(x, 0) - v^+_x(x, 0)] dx$$

$$+ \frac{1}{2} \int _{S^+} (u_x(x, y) + v^+_x(x, y))^2dxdy + \frac{1}{2} \int _{S^-} (u_x(x, y) + v^-_x(x, y))^2dxdy$$

$$\geq \frac{1}{2}(\|Du\|_{L^2}^2 + \|Dv^+\|_{L^2}^2 + \|Dv^-\|_{L^2}^2)$$

$$- \int \left[ \theta u^2(x, 0) + C_0 |v^-_x(x, 0) - v^+_x(x, 0)|^2 \right] dx$$

(9)

where $\theta > 0$ can be arbitrarily small. By the trace theorem \[38\], and the boundary conditions $u = 0$ at $x = \pm L$ and $y = \pm L'$, we have

$$\int _{-L} ^L u^2(x, 0)dx \leq C\|Du\|_{L^2}^2.$$  

Using this inequality in the estimate (9) for a sufficiently small $\theta$ we get

$$J(u, v^+, v^-) \geq c(\|Du\|_{L^2}^2 + \|Dv^+\|_{L^2}^2 + \|Dv^-\|_{L^2}^2) - C\alpha^2 \left[ \int [v^+_x(x, 0) - v^-_x(x, 0)]^2dx \right],$$

and we can take $c = 1/4$ if we so choose. \hfill \Box

We are now in a position to prove the existence of a minimizer for the function $I$ with $\sigma > 0$, for our “no stretch” boundary conditions.

**Theorem 3.2.** If $\sigma > 0$, every minimizing sequence $(u_j, v^+_j, v^-_j)$ for the energy functional $I$, that satisfies the boundary and the matching conditions, has a subsequence that converges in $H^1(\mathcal{S}) \times H^1(\mathcal{S}) \times H^2(\mathcal{S})$ to a global minimizer of $I$.

**Proof.** This conclusion follows easily from the direct method in the Calculus of variations \[18, 37, 11\].

$(u_j, v^+_j, v^-_j)$ is a minimizing sequence for $I$ that satisfies all the boundary conditions and the matching conditions at $y = 0$. For all $j$, we have the boundary conditions $w^+_j = \sqrt{2\alpha a}\Phi^\pm_{1,2}(y/a) at x = \pm L$ and $w^-_j = 0 at y = \pm L'$. These boundary conditions, along with $I \geq \sigma^2\|D^2w\|_{L^2}^2$, imply that $w_j$ is a bounded sequence in $H^2$. 


Consequently, up to extraction of a subsequence, \( w_j^\pm \to w^\pm \) in \( H^2 \). The compactness of the embedding \( H^2 \to W^{1,4} \) implies that \( (w_j^\pm)^2, (w_j^y)^2 \) converge strongly to \((w^\pm)^2, (w_y^*)^2\) respectively in \( L^2 \).

The standard trace theorems \([38]\) imply that
\[
\|Hx\|_{L^2} \leq C \alpha^2 a^3, \quad \int_L^L (w_j^+(x, L'))^2 dx = 0,
\]

Consequently, \((u_j, v_j^\pm, w^\pm)\) satisfy the no-stretch boundary conditions for all \( j \). In general, they do not satisfy the matching conditions at \( y = 0 \). Rather, we have the relation
\[
v_j^+(x, 0) - v_j^-(x, 0) = 2\alpha w_j^+(x, 0).
\]

We set \( \tilde{u}_j = u_j - x \). We will henceforth drop the subscript \( j \) where it won’t cause any confusion. Since \((u_j, v_j^\pm, w^\pm)\) satisfy the no-stretch boundary conditions, \( \tilde{u} = 0 \) at \( x = \pm L \) and at \( y = \pm L' \). Also, from the matching conditions for \( v_j^\pm \), we have
\[
v_j^+(x, 0) - v_j^-(x, 0) = 2\alpha w_j^+(x, 0).
\]

Since \( \|D^2 w_j^\pm\| \) is bounded, and
\[
\int_0^L (w_j^+(\pm L, y))^2 dy \leq C \alpha^2 a^3, \quad \int_{-L}^L (w_j^+(x, L'))^2 dx = 0,
\]

it follows from the trace theorem that \( w_j^+(x, 0) \) is bounded in \( L^2 \).

With this definition of \( \tilde{u} \), and suppressing the subscripts \( j \) and the superscripts \( \pm \), we have
\[
I(u, v, w) \geq \int \left[ \left( \tilde{u}_x + \frac{1}{2} w_x^2 \right)^2 + \frac{1}{2} (v_x + \tilde{u}_y + w_x w_y)^2 + (v_y + \frac{1}{2} w_y^2)^2 \right] dxdy
\]

Given any \( \epsilon > 0 \), up to extraction of a further subsequence, we get
\[
I \geq \int \int (\tilde{u}_x)^2 + (v_y)^2 + \frac{1}{2} (v_x + \tilde{u}_y)^2 - C(\|w^*\|_{W^{1,4}}^2 + \|D\tilde{u}\| + \|Dv\|).
\]

The argument from above shows that \( v_j^+ - v_j^- = 2\alpha w_j^+ \) is bounded in \( L^2 \). Combining this with the previous lemma, we see that \( \|D\tilde{u}\| + \|Dv\| \) is bounded in \( L^2 \).

\( \tilde{u} \) satisfies the boundary conditions \( \tilde{u} = 0 \) at \( x = \pm L \), \( y = \pm L' \). \( v \) satisfies the boundary conditions
\[
v^\pm(\pm L, y) = y + \frac{\delta}{2} + \alpha^2 a \varphi_{1,2} \left( \frac{y}{a} \right), \quad v^\pm(x, \pm L') = \pm L' + \frac{\delta}{2}.
\]

By the boundedness of \( \|Dv\| \), it follows that \( v^\pm(x, 0) \) exist in the sense of traces, and further are in \( L^2(dx) \). Consequently both \( \tilde{u} \) and \( v^\pm \) are bounded in \( L^2(\partial S^\pm) \). Combining this with the boundedness of \( \|D\tilde{u}\| + \|Dv\| \), it follows that a further subsequence \((\tilde{u}_j, v_j)\) converges weakly to \((\tilde{u}^*, v^*)\) in \( H^1 \times H^1 \), and consequently \((u_j, v_j) \to (u^*, v^*)\) where
\[
\tilde{u}^* = u^* + x\]

Since \( H^1_0 \) is weakly closed, it follows that \( u^* \) and \( v^{**} \) satisfy the boundary conditions in the sense of traces (See \([38]\) for a detailed argument). We now observe that the
The existence of a minimizer opens the door to a direct analysis of the Euler-Lagrange equations for the energy functional $I$. We will pursue this approach elsewhere. In this paper, we restrict ourselves to arguments that do not involve forming the first variation of $I$. □

4. Lower Bound

In this section, we prove a lower bound for the linearized Elastic energy $I$ in Eq. (2), by proving a corresponding result in terms of the scaled energy $I$ in Eq. (5). In the remainder of this section, we will mostly work with the half sheet $S^+$, although the same arguments also hold on $S^-$. With this understanding, we will drop the superscripts $\pm$.

As we show in [36], it follows from the boundary condition $U = 0$ at $X = \pm 1$, that

$$I(U, V, W) \geq \int [(U_X + W_X^2)^2 + W_{YY}^2] dXdY$$

(10)

$$\geq \int_0^\infty \left[ \frac{1}{2} \left( \int_{-1}^1 W_X^2 dX \right)^2 + \int_{-1}^1 W_{YY}^2 dX \right] dY$$

So, it suffices to prove a lower bound for the functional

$$E(W) = \int_0^\infty \left[ \frac{1}{2} \left( \int_{-1}^1 W_X^2 dX \right)^2 + \int_{-1}^1 W_{YY}^2 dX \right] dY$$

As in [36], we let $E_b$ and $E_s$ denote the quantities

$$E_b = \int W_{YY}^2 dX dY, \quad E_s = \int_0^\infty \frac{1}{2} \left( \int_{-1}^1 W_X^2 dX \right)^2 dY.$$  

(11)

which we will call the (lower bounds for) the bending and stretching energies respectively.

For every $X$, we define

$$\rho(X) = \left[ \int_0^\infty W_{YY}^2(X, Y) dY \right]^{-1}.$$

$\rho(X)$ is a “local” (in $X$) measure of the bending energy, and $[\rho(X)]^{-1}$ can be thought of as the bending energy density in $X$ that is obtained by integrating out the $Y$ dependence.

What we will see below it that $\rho(X)$ is the natural length scale associated with the ridge as a function of $X$, viz., $W_0(X) \sim \rho(X)$, and the bending energy density in $Y$ decays rapidly for $Y/\rho(X) \gg 1$ (See Fig. 1).

Before we begin the proof of the lower bound, we prove the following elementary, but very useful result.
Lemma 4.1. \( f \in H^2 \), \( \int (f'')^2 = \rho^{-1} < \infty \), \( f(0) = f_0 \), \( f'(0) = \beta \). Then we have
\[
\int_0^Y f^2(\eta) d\eta \geq \max_{Z \leq Y, \theta \in (0,1)} (1 - \theta) \left[ Z \left( f_0 + \frac{\beta Z}{2} \right)^2 + \frac{Z^3}{12} \left( \beta^2 - \frac{Z}{\theta \rho} \right) \right]
\]

Proof. By the Sobolev Embedding theorem, \( f \) is a \( C^1 \) function and \( f(\xi) = f_0 + \beta \xi + \int_0^\xi f''(\eta)(\xi - \eta) d\eta \).

Defining \( T(\xi) \) by \( T(\xi) = \int_0^\xi f''(\eta)(\xi - \eta) d\eta \), we have
\[
|T(\xi)|^2 \leq \int_0^\xi |f''(\eta)|^2 d\eta \int_0^\xi (\xi - \eta)^2 d\eta \leq \frac{\xi^3}{3\rho}
\]
and integrating this equation in \( \xi \) yields
\[
\int_0^Z |T(\xi)|^2 d\xi \leq \frac{Z^4}{12\rho}.
\]
If \( \tau(Z) = \int_0^Z f^2(\eta) d\eta \), \( \tau \) is nondecreasing in \( Z \). Using \( f(Z) = f_0 + \beta Z + T(Z) \) and the elementary inequality
\[
|a + b|^2 \geq (1 - \theta)|a|^2 - \frac{1 - \theta}{\theta}|b|^2,
\]
for all \( 0 < \theta < 1 \), we see that
\[
\tau(Z) \geq (1 - \theta) \int_0^Z (f_0 + \beta \eta)^2 d\eta - \frac{1 - \theta}{\theta} \int_0^Z T^2(\eta) d\eta
\geq (1 - \theta) \left[ f_0^2 Z + \beta f_0 Z^2 + \frac{\beta^2}{3} Z^3 \right] - \frac{1 - \theta}{\theta} \left[ Z^4 \frac{1}{12\rho} \right]
\geq (1 - \theta) \left[ Z \left( f_0 + \frac{\beta Z}{2} \right)^2 + \frac{Z^3}{12} (\beta^2 - \frac{Z}{\theta \rho}) \right]
\]
The result follows by observing that \( \tau \) is nondecreasing, and optimizing the choice of \( \theta \). \( \square \)

Theorem 4.2. \( \mathcal{I}(u, v, w) \) is as defined in Eq. (2). For all \( u \in H^1 \), \( v^\pm \in H^1 \) and \( w^\pm \in H^2 \cap W^{1,4} \) satisfying the no-stretch boundary conditions
\[
u = x, \ v^\pm = \Psi_{1,2}^\pm(y/a), \ w = \sqrt{2} \Phi_{1,2}^\pm(y/a), \text{ at } x = \pm L,
\]
and the matching condition \( w_y^+(x, 0) = w_y^-(x, 0) + 2\alpha \), we have the lower bound
\[
\mathcal{I}(u, v^\pm, w^\pm) \geq \min \left( C_1 \sigma^{7/3} \sigma^{5/3} L^{1/3}, C_2 \alpha^2 \sigma^2 \frac{L}{a} \right).
\]

Remark. Note that we do not need all the matching conditions in Eq. (4).
Remark. The form of the lower bound gives a crossover scale \( L_a \sim \sigma^{1/3} L_y^{2/3} \alpha^{-1/3} = L_y \) for \( a \). This scale is the same as the one we obtained in our earlier rescalings.

We will prove the theorem by doing the cases \( a \ll L_a \) and \( a \gg L_a \) separately. The result for \( a \ll L_a \) is obtained by proving the scaled version of the statement, viz.,

\[
I \geq C_1 \quad \text{for } A < A^*,
\]

for a constant \( A^* \) that will be determined below. The case \( A > A^* \) is much easier, and follows immediately as a corollary.

Remark. In earlier work [36], we proved the same scaling result for \( a \ll L_a \), but with extra assumptions on the behavior of \( v \) and \( w \) at \( y = 0 \). We do not know \textit{a priori} that these assumptions are satisfied for a real crumpled sheet. In this theorem, we have removed these hypothesis, and this result is directly applicable to crumpled sheets.

We now begin our proof of the theorem for \( a \ll L_a \). As in [36], the idea behind the proof is to show that the stretching energy \( E_s \) can be bounded from below by a negative power of the bending energy \( E_b \), so that the total energy \( E_s + E_b \) tends to \( +\infty \) as \( E_b \to 0 \) and \( E_b \to \infty \). This ensures the existence of a positive lower bound for \( E \) (and consequently also for \( I \)).

We set \( \beta^\pm(X) = W^\pm(Y, 0) \). The matching condition therefore is \( \beta^+(X) = \beta^-(X) + \sqrt{2} \). Before we prove the theorem, we collect a few useful results in the following lemmas.

**Lemma 4.3.**

\[
\int_{1}^{1} \int_{0}^{Y} W^2 dY dX \geq C_{\max}^{\max} \left[ Z^3 \int_{1}^{1} \beta(Y)^2 dX - 2Z^4 E_b \right] - C'A^3.
\]

**Proof.** Using Lemma 4.1 with \( f(\xi) = W(X, \xi) \), taking \( \theta = 1/2 \) and integrating the result in \( X \), we see that

\[
\int_{1}^{1} \int_{0}^{Y} W^2 dY dX \geq \frac{1}{24} \max_{Z \leq Y} \left[ Z^3 \int_{1}^{1} \beta(X)^2 dX - 2Z^4 E_b \right].
\]

The Poincare inequality now yields,

\[
\int_{1}^{1} \int_{0}^{Y} W^2 dY dX \geq C \int_{1}^{1} \int_{0}^{Y} W^2 dY dX - C' \int_{0}^{Y} \left[ W^2(-1, \xi) + W^2(1, \xi) \right] d\xi.
\]

The result follows from the observation

\[
\int_{0}^{Y} \left[ W^2(\pm 1, \xi) \right] d\xi \leq A^3 \int_{0}^{\infty} \Phi_{1,2}(\eta) d\eta \leq C'A^3.
\]

Our proof is based on demonstrating that a small bending energy \( E_b \) will lead to a large stretching energy. For \( A < \mu \), this idea is quantified by the following lemma.
Lemma 4.4. Let $B = \int_{-1}^{1} \beta(X)^2 dX$, and let

$$\mu = \frac{1}{B} \left( \frac{2AE_b}{B} \right)^3$$

There is a constant $\mu^* > 0$ such that, if $\mu < \mu^*$, the stretching energy $E_s$ and the total energy satisfy lower bounds

$$E_s \geq \frac{CB^7}{E_b^5}, \quad \text{and} \quad E \geq E_0 = (5CB^7)^{1/6}$$

Proof. By Jensen’s inequality, we have

$$E_s = \int_{0}^{\infty} \left( \int_{-1}^{1} W_X^2 dX \right)^2 dY \geq \frac{1}{2} \int_{0}^{Y} \left( \int_{-1}^{1} W_X^2 dX \right)^2 dY \geq \frac{1}{2Y} \left[ \int_{0}^{Y} \int_{-1}^{1} W_X^2 dXdY \right]^2$$

Lemma 4.3 now implies that

$$\sqrt{E_s} \geq \max_{Y \in \mathbb{R}} 2^{-1/2} \left[ C(BY^{5/2} - 2E_0Y^{7/2}) - C'A^3Y^{-1/2} \right].$$

Setting $BY^{5/2} = 2Y^{7/2}E_b$, we deduce that a characteristic scale $\tilde{Y}$ for $Y$ is given by

$$\tilde{Y} = \frac{B}{2E_b}.$$

Rescaling $Y$ in terms of $\tilde{Y}$, we obtain

$$\sqrt{E_s} \geq \frac{CB^{7/2}}{8E_b^{5/2}} \left[ \left( \frac{Y}{\tilde{Y}} \right)^{5/2} \left( 1 - \frac{Y}{\tilde{Y}} \right) - C'\mu \sqrt{\frac{\tilde{Y}}{Y}} \right],$$

where $\mu$ is as defined above, i.e.

$$\mu = \frac{1}{B} \left( \frac{2AE_b}{B} \right)^3.$$

Observe that $z^{5/2}(1-z)$ has a positive maximum at $z = 5/7 > 0$. The lower bound for the stretching energy $E_s$ follows by continuity of the function $z^{5/2}(1-z) - C'\mu z^{-1/2}$ with respect to $\mu$ at $z = 5/7$.

Minimizing $E_s + E_b$, we see that

$$E = E_s + E_b \geq E_0 \equiv (5CB^7)^{1/6}$$

We can now prove the theorem.

Proof. Set $A^* = (\mu^*)^{1/3}/(2E_0)$, where $\mu^*$ and $E_0$ are as in Lemma 4.4.

The precise statement we will prove is

$$E \geq E_0 \min(1, A^*/A)$$

If $E_b \geq E_0$, there is nothing to prove. Therefore, we can assume that $E_b < E_0$. It follows that $E_b < E_0$ for each half-sheet.
We first consider the case \( A \leq A^* \). As we argued before, the sheet has a well defined tangent vector at \( y = 0 \), and this gives the matching condition \( \beta_+(X) = \beta_-(X) + \sqrt{2} \). This, along with the convexity of the map \( \beta(X) \mapsto B = \int_{-1}^{1} \beta^2(X) dX \), implies that \( B^+ + B^- \) is minimized when \( \beta^+(X) = -\beta^-(X) = \frac{1}{\sqrt{2}} \). Therefore, W.L.O.G. \( B^+ \geq 1 \).

For the half sheet the half-sheet \( y \geq 0 \), we have \( \dot{A} \leq A^* \), \( E < E_0 \) and \( B \geq 2 \) implies that \( \mu = 1 \) \( B \left( \frac{2E_bA}{B} \right)^{3/2} < \mu^* \).

Lemma 4.4 now implies that \( E \geq E_0 \), for the half-sheet \( y \geq 0 \), and consequently for the whole sheet.

If \( A \geq A^* \), we still obtain the conclusion \( E \geq E_0 \) by the preceding argument if \( E_b \) is so small that

\[
\mu = \frac{1}{B} \left( \frac{2E_bA}{B} \right)^{3/2} < \mu^*.
\]

Therefore, we only need to consider the case \( \mu > \mu^* \). W.L.O.G \( B (= B^+) \geq 1 \), so that \( \mu > \mu^* \) implies

\[
E_b > \frac{(\mu^*)^{1/3}}{2A}
\]

and this gives the desired conclusion.

\[\square\]

5. Upper bounds

Our goal is to obtain upper bounds for the functional \( I \) that scale in the same way as the lower bound from the previous section as a function of the nondimensional parameters in the problem, \( \nuiz. \epsilon, \alpha \) and \( A \). This will show that we have captured the optimal scaling behavior of the elastic energy for a single ridge in a crumpled sheet.

In particular, we want an upper bound that is a constant (independent of \( \epsilon, \alpha \) and \( A \)) if \( A < A^* \), and an upper bound that scales as \( 1/A \) for \( A > A^* \). Also, we want upper bounds that are independent of \( \epsilon \) and \( \alpha \).

The existence of such upper bounds can be motivated as follows. The energy in the rescaled variables is given by

\[
I(U, V, W) = \iint \left[ (U_X + W_X^2)^2 + \frac{\epsilon^{-2/3}}{2} (V_X + U_Y + 2W_XW_Y)^2 + \epsilon^{-4/3} (V_Y + W_Y^2)^2 \right] dXdY.
\]

We would like to show the existence of \( (U, V, W) \) satisfying the boundary conditions such that \( I(U, V, W) \leq C(A) < \infty \) uniformly in \( \epsilon \) for \( A > 0 \). The idea behind the construction of an appropriate \( (U, V, W) \) is as follows. We first pick a smooth \( W \) satisfying all the boundary conditions. For this \( W \), we will pick an \( V \) such that \( V_Y = -W_Y^2 \). This equation can (we hope) be solved for every \( X \), along with the appropriate boundary conditions \( V(X, Y) \rightarrow 0 \) as \( Y \rightarrow \pm \infty \). Once we have \( V \), we
determine $U$ by $U_Y = -V_X - 2W_XW_Y$, again with the appropriate initial condition for $U$. With such a choice for $U, V$ and $W$, the energy becomes

$$\int \int \left[ (U_X + W_X^2)^2 + W_{YY}^2 + 2\epsilon^{2/3}W_X^2 + \epsilon^{4/3}W_{XX}^2 \right] dX dY,$$

and since $U$, $V$ and $W$ are assumed smooth, it easily follows that there is a finite upper bound, uniform in $\epsilon$ as $\epsilon \to 0$. Of course, we are not guaranteed that we have the right dependence on $A$. Also, we are not guaranteed to get the right asymptotic shifts in $V^\pm(X, \cdot)$.

In the remainder of this section, we will deduce the upper bound by using ideas similar to the simple argument from above to explicitly construct smooth functions $(u, v^\pm, w^\pm)$ satisfying all the boundary conditions. With these functions, we can show

$$I(u, v, w) \leq \min \left[ C\sigma^{5/3}L^{1/3}\alpha^{7/3} + C'\sigma^2\alpha^2\log^+ \left( \frac{a}{\sigma} \right), C'\sigma^2\alpha^2\frac{L}{a} \right],$$

where $\log^+ x = \max(\log x, 0)$. This is not exactly the scaling that we obtained for the lower bounds. In particular, the upper bound indicates that we are missing some of the relevant physics in our lower bound if $a \ll \sigma \exp(-\epsilon^{-1/3})$, i.e $A \ll \epsilon^{1/3} \exp(\epsilon^{-1/3})$.

### 5.1. Self similar test solutions

In this section, we show that, for identical, no-stretch boundary conditions at $x = \pm L$, with zero asymptotic shift $(\Delta = 0)$, we can construct “self-similar” test solutions that yield the “correct” upper bound.

Let $\Phi^\pm$ and $\Psi^\pm$ be as in the definition of the boundary conditions, so that $\Phi^\pm(\eta), \Psi^\pm(\eta)$ are smooth functions, that are supported in $|\eta| \leq K$.

We will choose $W$ and $V$ in the following “self-similar” form

$$W^\pm(X, Y) = \rho(X)\Phi^\pm \left( \frac{Y}{\rho(X)} \right), \quad V^\pm(X, Y) = \rho(X)\Psi^\pm \left( \frac{Y}{\rho(X)} \right),$$

where $\rho(X)$ is a function smooth that will be chosen later satisfying $\rho(X) > 0$ for all $X$. The boundary conditions at $X = \pm 1$ require that $\rho(1) = \rho(-1) = A$.

Let $\eta$ denote $Y/\rho(X)$, so that $W^\pm(X, Y) = \rho(X)\Phi^\pm(\eta)$. From this we obtain.

$$W^\pm_Y(X, Y) = \Phi^\pm'(\eta), \quad V^\pm_Y(X, Y) = \Psi^\pm'(\eta)$$

From the matching conditions on $\Phi^\pm$ and $\Psi^\pm$ at $\eta = 0$, it is clear that the functions $V^\pm$ and $W^\pm$ from above satisfy the appropriate matching conditions (6) at $Y = 0$, only if $\Delta = 0$.

Differentiating in $X$ we obtain,

$$W^\pm_X = \rho'(X) \left[ \Phi^\pm(\eta) - \eta\Phi^\pm'(\eta) \right], \quad V^\pm_X = \rho'(X) \left[ \Psi^\pm(\eta) - \eta\Psi^\pm'(\eta) \right].$$
Differentiating once more, we get
\[ W_{YY}^{\pm}(X,Y) = \frac{1}{\rho(X)} \Phi^{\pm''}(\eta), \]
\[ W_{XY}^{\pm}(X,Y) = -\frac{\rho'(X)}{\rho(X)} \left[ \eta \Phi^{\pm''}(\eta) \right], \]
\[ W_{XX}^{\pm}(X,Y) = \frac{1}{\rho(X)} \left( \rho(X) \rho''(X) \left[ \Phi^{\pm}(\eta) - \eta \Phi^{\pm'}(\eta) \right] + [\rho'(X)]^2 \eta^2 \Phi^{\pm''}(\eta) \right). \]

From the no-stretch boundary condition \( \Psi^{\pm'} + (\Phi^{\pm'})^2 = 0 \), it follows that \( V_Y^{\pm} + [W_Y^{\pm}]^2 \equiv 0 \).

We would also like \( U_Y^{\pm} = -V_X^{\pm} - 2W_X^{\pm}W_Y^{\pm} \). From the above scalings, we see that
\[ V_X^{\pm} + 2W_X^{\pm}W_Y^{\pm} = \rho'(X) \left[ \Psi^{\pm}(\eta) - \eta \Psi^{\pm'}(\eta) + 2\Phi^{\pm'}(\eta) \left( \Phi^{\pm}(\eta) - \eta \Phi^{\pm'}(\eta) \right) \right]. \]

In order that \( U_Y^{\pm} \) have this scaling behavior, we will set
\[ U^{\pm}(X,Y) = \rho'(X)\rho(X)\Xi^{\pm} \left( \frac{Y}{\rho(X)} \right). \]

If we choose \( \Xi \) such that
\[ \Xi^{\pm'} = - \left[ \Psi^{\pm}(\eta) - \eta \Psi^{\pm'}(\eta) + 2\Phi^{\pm'}(\eta) \left( \Phi^{\pm}(\eta) - \eta \Phi^{\pm'}(\eta) \right) \right] \]

Then, we will have \( U_Y^{\pm} = -V_X^{\pm} - 2W_X^{\pm}W_Y^{\pm} \).

In this approach, we have first order ODEs for \( \Xi^{\pm} \), with boundary conditions \( \Xi^{\pm}(\eta) \rightarrow 0 \) as \( \eta \rightarrow \pm \infty \), and these ODEs can be solved (in principle) to yield the functions \( \Xi^{\pm} \). These functions are also required to satisfy the matching condition \( U^{\pm}(X,0) = U^{-}(X,0) \) for all \( X \), i.e., the condition \( \Xi^{\pm}(0) = \Xi^{-}(0) \).

Since the unique solutions for the ODEs determining \( \Xi^{\pm} \) are also supported in \( |\eta| \leq K \).

Using the no-stretch condition \( \Psi^{\pm'} + (\Phi^{\pm'})^2 = 0 \) in the ODEs for \( \Xi^{\pm} \), we get
\[ \Xi^{\pm'} = - \left[ \Psi^{\pm}(\eta) + \eta \Psi^{\pm'}(\eta) + 2\Phi^{\pm'}(\eta) \Phi^{\pm}(\eta) \right] = - \left[ \eta \Psi^{\pm}(\eta) + [\Phi^{\pm}(\eta)]^2 \right]'. \]

Integrating the above equation, using the fact that \( \Phi^{\pm} \) and \( \Psi^{\pm} \) are supported in \( |\eta| \leq K \), we deduce that \( \Xi^{\pm} \) is supported in \( [-K,K] \) and \( \Xi^{\pm}(0) = [\Phi^{\pm}(0)]^2 \). The matching condition for \( \Phi^{\pm} \) now implies that \( \Xi^{\pm}(0) = \Xi^{-}(0) \). Consequently,
\[ U^{\pm}(X,Y) = \rho'(X)\rho(X)\Xi^{\pm} \left( \frac{Y}{\rho(X)} \right). \]

satisfies the boundary conditions \( U^{\pm} \rightarrow 0 \) as \( Y \rightarrow \pm \infty \), and the matching conditions
\( U^{\pm}(X,0) = U^{-}(X,0) = 0 \). We also require that \( U = 0 \) at \( X = \pm 1 \). since \( \rho(\pm 1) = A \), we will now require that \( \rho'(\pm 1) = 0 \).

We will henceforth restrict ourselves to considering the half-sheet \( Y \geq 0 \), since the same arguments will also apply to the half-sheet \( Y \leq 0 \), and we can now drop the subscripts \( \pm \). The above procedure yields an appropriate test configuration \( (U,V,W) \).
for boundary conditions that are identical at \( x = \pm L \) and satisfy \( \Delta = 0 \). We will henceforth refer to this situation as the \textit{self-similar case}. In the remainder of this section, we will consider this special case, and we will consider the general case in Sec. 5.3.

From the above arguments we see that, for appropriate boundary conditions, it is indeed possible to choose \((U, V, W)\) in the \textit{self-similar form}

\[
W(X, Y) = \rho(X) \Phi \left( \frac{Y}{\rho(X)} \right),
\]

\[
V(X, Y) = \rho(X) \Psi \left( \frac{Y}{\rho(X)} \right),
\]

\[
U(X, Y) = \rho'(X) \rho(X) \Xi \left( \frac{Y}{\rho(X)} \right).
\]

such that \( V_Y + W^2_Y = 0, U_Y + V_X + 2W_X W_Y = 0 \). With these choices, we have

\[
I(U, V, W) = \iint \left[ (U_X + W^2_X)^2 + W^2_{YY} + 2\epsilon^{2/3}W^2_{XY} + \epsilon^{4/3}W^2_{XX} \right] dXdY.
\]

A straightforward calculation allows us to estimate the various terms in this expression. We obtain,

\[
\iint (U_X + W^2_X)^2 dXdY \leq C \int \left( [\rho''(X)\rho(X)]^2 + [\rho'(X)]^4 \right) \rho(X) dX.
\]

\[
\iint W^2_{YY} dXdY \leq C \int \frac{1}{\rho(X)} dX
\]

\[
\iint W^2_{XY} dXdY \leq C \int \frac{[\rho'(X)]^2}{\rho(X)} dX
\]

\[
\iint W^2_{XX} dXdY \leq C \int \left( [\rho''(X)]^2 \rho(X) + \frac{[\rho'(X)]^4}{\rho(X)} \right) dX
\]

(12)

where \( C \) is a constant that only depends on \( \Phi \), and \( \Psi \).

5.2. \textbf{Construction of the upper bound.} In order to prove the claimed upper bound, we need to show the existence of a smooth \( \rho(X) \) such that \( \rho(\pm 1) = A \), \( \rho'(\pm 1) = 0 \), and all the terms in \( I \) are bounded uniformly in \( \epsilon \) as \( \epsilon \to 0 \) with the appropriate dependence \( A \).

Lobkovsky’s \cite{28} analysis motivates the choice \( \rho(X) \sim (1 - X)^{2/3} \) near \( X = 1 \). For this choice however, the contribution of \( W^2_{XX} \) is given by

\[
[\rho''(X)]^2 \rho(X) \sim \frac{[\rho'(X)]^4}{\rho(X)} \sim (1 - X)^{-2},
\]

and is not integrable near \( X = 1 \). Therefore the choice \( \rho(X) \sim (1 - |X|)^{2/3} \) will not yield an upper bound for \( I \).

In our analysis of the lower bound, we ignored the contribution of \( W_{XX} \) to the energy, and obtained results that agree with Lobkovsky’s boundary layer analysis. This suggests that the scaling \( \rho(X) \sim (1 - |X|)^{2/3} \) might still be appropriate in
regions where the contribution of the $W_{XX}$ and the $W_{XY}$ terms are small. However, close to the boundaries near $X = \pm 1$, the dominant energy balance is different, and we need to modify the behavior of $\rho$ to account for this.

For small $X$, we expect that $\epsilon^{4/3}W_{XX} \sim W_{YY}$ is the leading order balance for the energy in $I$. Since $Y \sim \rho(X)$, it follows that $\rho(X) \sim \epsilon^{-1/3}(1 - X)$ near $X = 1$, and similarly $\rho(X) \sim \epsilon^{-1/2}(1 + X)$ near $X = -1$. Note that these behaviors match $\rho(X) \sim (1 - X)^{2/3}$ (respectively $(1 + X)^{2/3}$) when $1 - X \sim \epsilon$ (respectively $1 + X \sim \epsilon$).

We also have the boundary conditions $\rho(X) \approx A$ and $\rho'(\pm 1) = 0$. This suggests $\rho(X) \approx A$ for $(1 - |X|) < A\epsilon^{1/3}$ if $A \ll \epsilon^{2/3}$. Therefore we will choose $\rho(X)$ with the following behavior.

- In the case $A \ll \epsilon^{2/3} \ll 1$,
  
  $\rho(X) \sim \begin{cases} 
  A & (1 - |X|) \lesssim A\epsilon^{1/3} \\
  \epsilon^{-1/3}(1 - |X|) & A\epsilon^{1/3} \ll (1 - |X|) \ll \epsilon \\
  (1 - X^2)^{2/3} & \epsilon \ll (1 - |X|) \sim 1 
  \end{cases}$

- In the case $\epsilon^{2/3} \lesssim A \ll 1$

  $\rho(X) \sim \begin{cases} 
  A & (1 - |X|) \lesssim A \\
  (1 - X^2)^{2/3} & \epsilon \ll (1 - |X|) \sim 1 
  \end{cases}$

- In the case $A \sim 1$, we set $\rho(X) = A$.

We will now make the above considerations precise. Let $\varphi \in C_c^\infty$ be a smooth, nonnegative, non-increasing function, that is identically one on $(-\infty, 1/2]$ and zero on $[2, \infty)$. An example of such a function is illustrated in Fig. Also, $\bar{\varphi}$ will denote the complementary function $1 - \varphi$.

**Lemma 5.1.** For $A > 0$ and $\epsilon > 0$, let

$$
g(z) = A\varphi \left(\frac{z}{A\epsilon^{1/3}}\right) + z\epsilon^{-1/3}\bar{\varphi} \left(\frac{z}{A\epsilon^{1/3}}\right) \varphi \left(\frac{z}{\epsilon}\right) + z^{2/3}\bar{\varphi} \left(\frac{z}{\epsilon}\right),
$$

$$
h(z) = z\epsilon^{-1/3}\chi \left[1/4, 2A\epsilon^{1/3}\right] + z^{2/3}\chi \left[1/4, \infty\right),
$$

where $\chi$ denotes the characteristic function. Then $g$ is a smooth function. Also, there exist constants $c$ and $C$ such that $g$ satisfies the following inequalities $\forall z$

$$
g(z) \geq c \left[A\chi_{[0, 2A\epsilon^{1/3}]} + h(z)\right]
$$

$$
g(z) \leq C \left[A\chi_{[0, 2A\epsilon^{1/3}]} + h(z)\right]
$$

$$
g'(z) \leq Ch(z) \left[1 + \frac{1}{z}\right]
$$

$$
g''(z) \leq Ch(z) \left[1 + \frac{1}{z^2}\right]
$$

**Proof.** We begin with an elementary observation. Near $z \sim A\epsilon^{1/3}$, the functions $g = A$ and $g = z\epsilon^{-1/3}$ are comparable, i.e., there exist constants $\theta, \Theta$ independent of $\epsilon, A$, such that $\theta A \lesssim \epsilon^{-1/3} \leq \Theta A$ for $1/2 A\epsilon^{1/3} \leq z \leq 2A\epsilon^{1/3}$. Similarly, near $z \sim \epsilon$, we have $\theta z\epsilon^{-1/3} \leq z^{2/3} \leq \Theta z\epsilon^{-1/3}$ for $1/2 \epsilon \leq 2\epsilon$. 

20
This observation implies that \( g(z) \geq c [A \chi_{[0,2A^{1/3}]} + h(z)] \). The inequality \( g(z) \leq C [A \chi_{[0,2A^{1/3}]} + h(z)] \) is elementary and follows from the boundedness of \( \varphi \).

We also observe that, for all \( l > 0 \) and \( n = 1, 2, 3, \ldots \),

\[
\left| \frac{d^n}{dz^n} \varphi \left( \frac{z}{l} \right) \right| \leq \frac{C_{n}}{l^{n}} \chi_{[\frac{1}{2}l, 2l]},
\]

and the same inequality also holds for the complementary function \( \bar{\varphi} \). We also have the elementary inequality

\[
\left| \frac{d^n}{dz^n} z^{\alpha} \right| \leq \frac{C_{n,\alpha}}{z^n}, \quad z > 0
\]

Differentiating \( g \) twice, using all of the above observations, and recognizing that \( z^{-1} \leq 1 + z^{-2} \), we get the inequalities

\[
|g'(z)| \leq Ch(z) \left[ 1 + \frac{1}{z} \right]
\]

\[
|g''(z)| \leq Ch(z) \left[ 1 + \frac{1}{z^2} \right]
\]

This proves the lemma.

\[\square\]

**Lemma 5.2.** For boundary conditions that support a self-similar test function, we have an upper bound

\[
I(U^*, V^*, W^*) \leq C \left[ 1 + \epsilon^{1/3} \log^{+} \left( \frac{\epsilon^{2/3}}{A} \right) \right].
\]

**Proof.** We set \( \rho(X) = g(1 - X^2) \), where \( g \) is as defined in lemma 5.1. Note that \( g(0) = A, g'(0) = 0 \) implies \( \rho(\pm 1) = A, \rho'(\pm 1) = 0 \).

We begin with a few observations.

(1) Our definition makes \( \rho \) an even function of \( X \), and by lemma 5.1, \( \rho \) is smooth.

(2) \( z'(X) = -2X \) and \( z''(X) = -2 \) are bounded for \( X \in [-1, 1] \), so that the inequalities in lemma 5.1 also hold in terms of \( \rho \) with the understanding \( z = 1 - X^2 \).

(3) If \( A < \epsilon^{2/3} < 1 \), the sets \( [0, A^{1/3} \epsilon), [A^{1/3} \epsilon, \epsilon) \) and \( [\epsilon, 1] \) give a disjoint partition of \( [0, 1] \). If \( A > \epsilon^{2/3} \), the sets \( [0, A^{1/3} \epsilon), \) and \( [\epsilon, 1] \) cover \( [0, 1] \).

We can now estimate the various terms in (12) using the results of lemma 5.1 and the observations from above.
We first estimate the \((U_X+W_X^2)^2\) term. Using \((1+z^{-1})^4 \leq C(1+z^{-2})^2 \leq C(1+z^{-4})\), we obtain

\[
\int \left( [\rho''(X)\rho(X)]^2 + [\rho'(X)]^4 \right) \rho(X) \, dX \leq C \int_0^1 [h(z)]^5 \left( 1 + \frac{1}{z^4} \right) \, dz,
\]

\[
\leq C \epsilon^{-5/3} \int_{\frac{1}{2} \epsilon \alpha^{1/3}}^{2 \epsilon} z^5 \left( 1 + \frac{1}{z^4} \right) \, dz
\]

\[
+ C \int_{\frac{1}{2} \epsilon}^{1} z^{10/3} \left( 1 + \frac{1}{z^4} \right) \, dz,
\]

\[
\leq C \left[ \epsilon^{1/3} + 1 \right]
\]

The \(W_Y^2\) term yields

\[
\int \frac{1}{\rho(X)} \, dX \leq C \int_0^{2 \alpha \epsilon^{1/3}} \frac{dz}{A} + C \int_{\frac{1}{2} \epsilon \alpha^{1/3}}^{2 \epsilon} \frac{dz}{z \epsilon^{-1/3}} + C \int_{\frac{1}{2} \epsilon}^{1} \frac{dz}{z^{2/3}}
\]

\[
\leq C \left[ \epsilon^{1/3} \left( \log \left( \frac{\epsilon^{2/3}}{A} \right) + 1 \right) + 1 \right]
\]

We will now estimate the \(W_{XX}\) term, since a bound for the \(W_{XY}\) term can be obtained from the bounds for the \(W_{YY}\) and the \(W_{XX}\) terms. We have

\[
\int \left( [\rho''(X)]^2 \rho(X) + \frac{[\rho'(X)]^4}{\rho(X)} \right) \, dX \leq C \int_0^1 [h(z)]^3 \left( 1 + \frac{1}{z^4} \right) \, dz
\]

\[
\leq C \epsilon^{-1} \int_{\frac{1}{2} \epsilon \alpha^{1/3}}^{2 \epsilon} z^3 \left( 1 + \frac{1}{z^4} \right) \, dz
\]

\[
+ C \int_{\frac{1}{2} \epsilon}^{1} z^2 \left( 1 + \frac{1}{z^4} \right) \, dz,
\]

\[
\leq C \left( \log \left( \frac{\epsilon^{2/3}}{A} \right) + 1 \right)
\]

Using

\[
\frac{[\rho'(X)]^4}{\rho^2(X)} = \frac{[\rho'(X)]^4}{\rho(X)} \cdot \frac{1}{\rho(X)}
\]

and the Cauchy-Schwarz inequality, the above estimates yield

\[
\int \frac{[\rho'(X)]^2}{\rho(X)} \, dX \leq C \left[ \epsilon^{-1/3} \left( \log \left( \frac{\epsilon^{2/3}}{A} \right) + 1 \right) + \epsilon^{-2/3} \right]
\]

We have thus bounded the \(W_{XY}\) term. Using these estimates in

\[
I(U,V,W) = \iint \left[ (U_X + W_X^2)^2 + W_Y^2 + 2 \epsilon^{2/3} W_{XY}^2 + \epsilon^{4/3} W_{XX}^2 \right] \, dXdY,
\]

we see that, for \(A < \epsilon^{2/3}\) the self-similar test function has

\[
I \leq C \left[ 1 + \log \left( \frac{\epsilon^{2/3}}{A} \right) \right].
\]
If $\epsilon^{2/3} < A$, the sets $[0, A\epsilon^{1/3})$ and $[\epsilon, 1]$ cover $[0, 1]$. In this case, the same analysis as above gives $I \leq C$. Combining this with the above result proves the lemma. \[ \square \]

Setting $\rho(X) = A$, a direct calculation using (12), shows that $I \leq C/A$. Combining this result with the lemma 5.2 and “unscaling” these results, we get

**Theorem 5.3.** If the no stretch boundary conditions are identical at $x = \pm L$, and have $\Delta = 0$, we have the upper bound

$$I(U^*, V^*, W^*) \leq C \min \left( 1 + \epsilon^{1/3} \log^+ \left( \frac{\epsilon^{2/3}}{A} \right), \frac{C'}{A} \right).$$

5.3. **Non-self similar test solutions.** We will now consider upper bounds for the general case, i.e., for the situation where the boundary conditions at $x = \pm L$ are not identical and/or $\Delta \neq 0$.

The strategy of the proof will be the following:

1. We introduce boundary layers (in $x$) near the boundaries $x = \pm L$ of width $x = b$. In these boundary layers, we connect the boundary condition at $x = \pm L$ with identical profiles that have $\Delta = 0$, at a distance $a$ from the boundaries.

2. In the region, $L - |x| \geq b, |y| \leq KL_y$, we use our self similar construction from the last section.

3. In the remaining region, we introduce a small, uniform strain $\gamma_{yy} \sim \delta/L'$, to get the appropriate asymptotic shift $\delta$.

The various regions are illustrated in Fig. 3. The idea behind this construction is from matched asymptotic expansions. The self-similar solutions from the last section play the role of the outer solutions in $x$, but the inner solutions in $y$!

Before we rigorously construct a test solution that gives the upper bound, we present a heuristic scaling argument that motivates our choices for the length scales $b$ and $l$ in our test solution.

The boundary conditions at $x = \pm L$ are given by no-stretch profiles, whose (curvature) length scale is $a$. We assume that we can find a no-stretch profile with zero asymptotic shift, and curvature $a^{-1}$. This will be the profile of the test solution at $x = \pm (L - b)$. Also, we set $u = x$ in these layers, so that $u_x = 1, u_y = 0$.

In the boundary layer(s) $L - b \leq |x| \leq L$, $w(x, y)$ is supported in $|y| \leq Ka$. Also, $w \sim O(aa)$ so that

$$w_x \sim \frac{\alpha a}{b}, \quad w_{xx} \sim \frac{\alpha a}{b^2}, w_{yy} \sim \frac{\alpha}{a},$$

and all of these derivatives are supported in $|y| \leq Ka$. This gives $\gamma_{xx} \sim w_x^2 \sim \alpha^2 a^2/b^2$

If we assume that there is no stretching in the $y$ direction, so that $\gamma_{yy} = 0$, we see that $w_y \sim 1 - \frac{1}{2}w_y^2$. Consequently, we see that the asymptotic shift $\delta$ is a function of $x$, and by integrating $w_y^2$ we see that $\delta(x) \sim O(\alpha^2 a)$, and this gives $v_x \sim \delta_x \sim \alpha^2 a/b$. Also, since this $v_x$ arises from the difference in the asymptotic shifts, it is supported in $|y| \leq l$, and not only in $|y| \leq a$, as for $w_y$. This gives $\gamma_{xy} \sim v_x \sim \alpha^2 a/b$. 

23
Putting all of this together, the elastic energy in the two boundary layers comprising Region I is

\[ E_I \sim \gamma_{xx}^2 ab + \gamma_{xy}^2 lb + \sigma^2 (w_{xx}^2 + w_{yy}^2) ab \]
\[ \sim \frac{\alpha^4 a^5}{b^3} + \frac{\alpha^4 a^2 l}{b} + \frac{\sigma^2}{a} + \frac{\alpha^2}{b^3} \]

In Region II, we use the self similar construction from above, i.e., the dominant energy balance is between the curvature \( w_{yy} \) and the strain \( \gamma_{xx} \). Following the scaling argument in Ref. [27], we get

\[ E_{II} \sim \frac{\alpha^4 l^5}{L^3} + \frac{\alpha^2 \sigma^2 L}{l} \]

In region III, we are connecting \( v = y \) (zero asymptotic shift) with \( v = y \pm \delta \). We can do this with a profile that has a uniform strain \( \gamma_{yy} \sim \delta / (L' - l) \). Consequently,

\[ E_{III} \sim \left( \frac{\delta}{L' - l} \right)^2 L(L' - l) \sim \frac{\alpha^4 a^2 L}{L' - l}. \]

Note that \( E_{III} \) is completely determined by the boundary conditions, and is independent of any choice we make for \( b \) and \( l \) provided that \( l \ll L' \).

For any given \( a, \sigma, L, L' \), \( \mathcal{E} = E_I + E_{II} + E_{III} \) diverges as \( b, l \to 0, \infty \) independently. Consequently there are optimal finite, nonzero choice \( b = b^* \) and \( l = l^* \).
There are two scaling regimes of interest. In physically realistic situations, $\sigma \lesssim a$. If $a = C\sigma$, with $C$ staying $O(1)$ as $\sigma \to 0$, optimizing $l$ and $b$ gives

$$l^* \sim \alpha^{-1/3}\sigma^{1/3}L^{2/3}, \quad b^* \sim \alpha^{5/6}\epsilon^{2/3}L^{1/3}.$$ 

The energies in the three regions are

$$\mathcal{E}_I \sim \alpha^{17/6}\sigma^{5/3}L^{1/3}, \quad \mathcal{E}_{II} \sim \alpha^{7/3}\sigma^{5/3}L^{1/3}, \quad \mathcal{E}_{III} \sim \alpha^4\sigma^2\left(\frac{\sigma}{L}\right)^{1/3}\frac{\sigma}{L'}.$$ 

For $\sigma \ll L, L'$, the energy $\mathcal{E}_{III}$ is asymptotically negligible. $\mathcal{E}_I \sim \alpha^{1/2}\mathcal{E}_{II}$, so the two energies scale in the same way for $\alpha \sim O(1)$, but the energy in the boundary layers is asymptotically negligible in the small angle limit.

We can also consider the situation $a \ll \sigma$. In this case, the natural scaling regime is $a \sim C\sigma^{5/3}L^{-2/3}$, with $C$ staying $O(1)$ as $\sigma \to 0$. Optimizing $l$ and $b$ gives

$$l^* \sim \alpha^{-1/3}\sigma^{1/3}L^{2/3}, \quad b^* \sim \epsilon^{5/3}L^{-2/3} \sim a.$$ 

The energies in the various regions are

$$\mathcal{E}_I \sim \alpha^2\sigma^2, \quad \mathcal{E}_{II} \sim \alpha^{7/3}\sigma^{5/3}L^{1/3}, \quad \mathcal{E}_{III} \sim \alpha^4\sigma^2\left(\frac{\sigma}{L}\right)^{1/3}\frac{\sigma}{L'}.$$ 

For $\sigma \ll L, L'$, the energies $\mathcal{E}_I$ and $\mathcal{E}_{III}$ are asymptotically negligible, and the energy is determined, essentially by the the self-similar solution in Region II, in the limit $\epsilon = (\sigma/L\alpha) \to 0$.

We will now use these scaling results as motivation, and rigorously construct test solutions that give the appropriate upper bound in situations where the boundary conditions are non-identical, or have nonzero asymptotic shift. Before we begin our construction, we first show that there exist no-stretch profiles with zero asymptotic shift.

**Lemma 5.4.** $\exists K_0$, such that $\forall K \geq K_0$, there exist smooth $\phi, \psi$ supported on $[0, K]$ such that

$$\phi'(0) = 1, \quad \int_0^K [\phi''(\eta)]^2d\eta = \frac{1}{2},$$ 

and

$$\int_0^K [\phi'(\eta)]^2d\eta + \phi(0) = 0$$

*Proof.* Let $\zeta$ be a smooth nonnegative function such that $0 \leq \zeta \leq 1$, $\zeta(x) \equiv 1$ for $x \leq -1$ and $\zeta(x) \equiv 0$ for $x \geq 0$. Let $\varpi$ be a smooth function supported in $[-1, 0]$, that is odd about $-\frac{1}{2}$, and is not identically zero.

We set $\varsigma(x) = \zeta(x) + q\varpi(x)$. The map

$$q \mapsto \Delta(q) \equiv \int_{-1}^0 \zeta(\xi) [1 - \varsigma(\xi)] d\xi$$

is clearly continuous. The reason for calling this map $\Delta$ will become clear below.

We have

$$\Delta(0) = \int_{-1}^0 \zeta(\xi) [1 - \varsigma(\xi)] d\xi \geq 0$$
since \( 0 \leq \zeta \leq 1 \). Also,

\[
\Delta(q) = \int_{-1}^{0} \zeta(\xi) d\xi + q \int_{-1}^{0} \varpi(\xi) d\xi - \int_{-1}^{0} \left[ \zeta(\xi) + q\varpi(\xi) \right]^2 d\xi
\]

\[
\leq \int_{-1}^{0} \zeta(\xi) d\xi + 2 \int_{-1}^{0} \zeta^2(\xi) d\xi - \frac{q^2}{2} \int_{-1}^{0} \varpi^2(\xi) d\xi
\]

\[
\leq 3 - Cq^2
\]

where we have used \(|\zeta| \leq 1\) and \(\varpi\) is not identically zero in passing to the last line.

Since \(\Delta(0) \geq 0\), and \(\Delta(q) \to -\infty\) as \(q \to -\infty\); \(\exists q^* < \infty\) such that \(\Delta(q^*) = 0\).

We set \(\varsigma = \zeta + q^* \varpi\), and define \(l\) by

\[
\int_{-1}^{0} [\varsigma'(x)]^2 dx = \frac{l}{2}.
\]

Let \(K_0 = 2l\). For a given \(K\), we will set

\[
\phi(x) = \int_{-\frac{x}{l}}^{x} \zeta \left( \frac{\xi - K}{l} \right) d\xi.
\]

Then, \(\phi'(x) = \zeta \left( \frac{x-K}{l} \right)\), so that \(\phi'(x) \equiv 1\) for \(x \leq l \leq K - l\). Also,

\[
\int_{0}^{K} [\phi''(x)]^2 dx = \frac{1}{l^2} \int_{0}^{K} \left[ \zeta' \left( \frac{x-K}{l} \right) \right]^2 dx = \frac{1}{2}.
\]

Since \(\phi\) is supported on \([0, K]\), it follows that

\[
\phi(0) = -\int_{0}^{K} \phi'(x) dx.
\]

Since \(\phi(x) = 1\) for \(0 \leq x \leq K - l\), it follows that

\[
\int_{0}^{K} [\phi'(\eta)]^2 d\eta + \phi(0) = \int_{K-l}^{K} [\phi'(\eta)]^2 d\eta = \Delta(q^*) l = 0.
\]

We now set

\[
\psi(x) = \int_{x}^{K} [\phi'(\xi)]^2 d\xi.
\]

We can extend the functions \(\phi, \psi\) to a no-stretch profile on \(\mathbb{R}\) by setting

\[
\Phi^+(\eta) = \Phi^-(\eta) = \phi(\eta), \quad \Psi^+(\eta) = -\Psi^-(\eta) = \psi(\eta), \quad \eta \in [0, K]
\]

and \(\Phi^\pm = \Psi^\pm = 0\) otherwise. This profile satisfies the matching conditions and has zero asymptotic shift.

We now have to prove that, by introducing a thin boundary layer near the boundaries, we can “connect” the prescribed boundary conditions to the profile we constructed in lemma 5.4 without incurring a large energy penalty. We begin with the following lemma which estimates norms of the first derivatives of a no-stretch profile \((\Phi^\pm, \Psi^\pm)\) in terms of the curvature of the profile, and it’s support.
Lemma 5.5. \((\Phi^\pm, \Psi^\pm)\) is a normalized no-stretch profile with support \(K\), i.e., \(\Psi^\pm + [\Phi^\pm]^2 = 0\), \(\Phi^\pm(\eta) = \Psi^\pm(\eta) = 0\) if \(|\eta| \geq K\),

\[\Phi^-(0) = \Phi^+(0) - \sqrt{2}\]

and

\[\int_0^\infty [\Phi''^+(\eta)]^2 d\eta + \int_{-\infty}^0 [\Phi''^-(\eta)]^2 d\eta = 1.\]

It then follows \(K \geq 1/2\) and

\[\frac{1}{K^3} \sup_{0 \leq \eta \leq K} [\Phi^+(\eta)]^2 + \frac{1}{K} \int_0^K [\Phi^+(\eta)]^2 d\eta + \frac{1}{K} \int_0^K [\Phi^+(\eta)]^2 d\eta \leq C\]

(13)

\[\frac{1}{K^2} \left[ \int_0^K [\Phi^+(\eta)]^2 d\eta + \sqrt{2}\Phi^+(0) \right] \leq C\]

where \(C\) is a universal constant (independent of \(K\)). A similar result also hold for \(\Phi^+\).

Proof. This follows immediately from the Sobolev embedding theorem in \(\mathbb{R}^1\) [17, 38]. Since \(\Phi^+(K) = 0\), we have

\[|\Phi^+(\eta)| = \left| \int_\eta^K \Phi''^+(\xi) d\xi \right| \leq \left[ \int_\eta^K [\Phi''^+(\xi)]^2 d\xi \cdot \int_\eta^K 1 d\xi \right]^{1/2} \leq \sqrt{K - \eta}\]

Using \(\Phi^-')(0) = \Phi^+(0) - \sqrt{2}\), it follows that \(2\sqrt{K} \geq \sqrt{2}\), which implies that \(K \geq 1/2\).

The remaining inequalities follows from integrating \(|\Phi^+'(\eta)| \leq \sqrt{K - \eta}\) and using \(\Phi^+(K) = 0\), and \(K \geq 1/2\) so that \(K^2 > \frac{1}{2}K^{3/2}\).

\[\square\]

Lemma 5.6. \(\Phi_{1,2}^\pm\) and \(\Psi_{1,2}^\pm\) are smooth on \(\mathbb{R}^+\) and supported on \([0, \pm K]\). Further, for \(i = 1, 2\), \(\Psi_i^\pm + [\Phi_i^\pm]^2 = 0\), and

\[\int_0^\infty [\Phi''^+(\eta)]^2 d\eta + \int_{-\infty}^0 [\Phi''^-(\eta)]^2 d\eta = 1.\]

Also, \(\Phi_{1,2}^\pm\) and \(\Psi_{1,2}^\pm\), satisfy the matching conditions (6), where the asymptotic shifts of the two profiles are given by

\[-\left[ \int_{-\infty}^0 [\Phi_i^-'(\eta)]^2 d\eta + \sqrt{2}\Phi_i^-'(0) + \int_0^\infty [\Phi_i^+(\eta)]^2 d\eta + \sqrt{2}\Phi_i^+(0) \right] = \Delta_i.\]
\( \zeta \) is a nonnegative, nondecreasing \( C^\infty \) function such that \( \zeta(x) \equiv 0 \) for \( x \leq 0 \) and \( \zeta(x) = 1 \) for \( x \geq 1 \). Let

\[
U(X, Y) = 0
\]
\[
W^\pm(X, Y) = \zeta \left( \frac{X}{B} \right) \Phi^\pm_2(Y) + \left[ 1 - \zeta \left( \frac{X}{B} \right) \right] \Phi^\pm_1(Y)
\]
\[
V^\pm(X, Y) = -\int_{\pm K}^Y \left[ W^\pm_Y(X, Y) \right]^2 dY + Y \pm \frac{1}{2} \Delta(X)
\]

where

\[
\Delta(X) = -\left[ \int_{-\infty}^0 \left[ W^-_Y(X, \eta) \right]^2 d\eta + \int_0^\infty \left[ W^+_Y(X, \eta) \right]^2 d\eta + 2\sqrt{2} W^+(X, 0) \right]
\]

Then, \((U, V^\pm, W^\pm)\) satisfies the matching conditions at \( Y = 0 \), and we have

\[
\int_{Y=0}^M \int_{X=0}^B (U_X + W^2_X)^2 dXdY \leq \frac{C}{B^3}
\]
\[
\int_{Y=0}^M \int_{X=0}^B (V_X + U_Y + 2W_XW_Y)^2 dXdY \leq \frac{CM}{B}
\]
\[
\int_{Y=0}^M \int_{X=0}^B (V_Y + W^2_Y)^2 dXdY = 0
\]
\[
\int_{Y=0}^M \int_{X=0}^B W^2_{YY} dXdY \leq CB
\]
\[
\int_{Y=0}^M \int_{X=0}^B W^2_{XX} dXdY \leq \frac{C}{B^3}
\]

where \( C \) is a constant that only depends on \( K \).

**Proof.** Most of the inequalities follow by direct integration using the definition of \((U, V^\pm, W^\pm)\), and using Lemma 5.5. The only result that needs proof is

\[
\int_{Y=0}^M \int_{X=0}^B (V_X + U_Y + 2W_XW_Y)^2 dXdY \leq \frac{CM}{B}.
\]

Since \( W(X, \cdot) \) is supported in \( [0, K] \), and \( V^\pm(X, Y) = Y \pm \frac{1}{2} \Delta(X) \) for \( |Y| \geq K \), we have \( V_X = \Delta_X \). From the definition of \( \Delta(X) \), we see that

\[
\frac{d}{dX} \Delta(X) = -\frac{2}{B} \zeta' \left( \frac{X}{B} \right) \left[ \int_{-K}^0 W^-_Y(X, \eta) [\Phi^-_2(\eta) - \Phi^-_1(\eta)] d\eta \right.
\]
\[
+ \left. \int_0^K W^+_Y(X, \eta) [\Phi^+_2(\eta) - \Phi^+_1(\eta)] d\eta + \sqrt{2}(\Phi^-_2(0) - \Phi^-_1(0)) \right]
\]

Estimating the integrals by the Cauchy-Schwarz inequality, and using Lemma 5.5 we see that

\[
|\Delta_X| \leq \frac{C}{B} \left| \zeta' \left( \frac{X}{B} \right) \right| \quad \Rightarrow \quad \int_0^B [\Delta_X(\xi)]^2 d\xi \leq \frac{C}{B}.
\]
We also have,
\[
\frac{d}{dY}(V_X + U_Y + 2W_X W_Y) = V_{XY} + 2W_{XY} Y + 2W_X Y_Y
\]
\[
= \frac{d}{dX}(V_Y + W_Y^2) + 2W_X Y_Y = 2W_X Y_Y
\]
Using \( U \equiv 0 \), and \( V_X = \Delta_X, W_X = W_Y = 0 \) for \( Y \geq K \), we get
\[
\int_0^M (V_X + U_Y + 2W_X W_Y)^2 d\eta = (M - K)\Delta_X^2 + \int_0^K (V_X + U_Y + 2W_X W_Y)^2 d\eta
\]
\[
\leq CM\Delta_X^2 + CK^2 \int_0^K W_X^2 W_Y^2 d\eta.
\]
Using the Poincare inequality. By the convexity of the map,
\[
W(X, \cdot) \mapsto \int_0^K W_Y^2 d\eta,
\]
and the normalization of \( \Phi_{1,2}^\pm \), it follows that \( \int_0^K W_Y^2 d\eta \leq 1 \). Lemma 5.5 along with
\[
W_X(X, \eta) = \frac{1}{B} \zeta'(\frac{X}{B}) \left[ \Phi_2(\eta) - \Phi_1(\eta) \right],
\]
implies that
\[
\sup_{X, \eta} |W_X(X, \eta)| \leq \frac{C}{B}.
\]
Combining this with the earlier estimate, we see that
\[
\int_0^M (V_X + U_Y + 2W_X W_Y)^2 d\eta \leq CM\Delta_X^2 + \frac{CK^2}{B^2}
\]
Integrating this inequality in \( X \), and using \( M \geq K \), we obtain the desired result. \( \square \)

This Lemma provides a rigorous basis for our heuristic calculation for the energies of the three regions at the beginning of this section. Using this lemma, and the ideas form the heuristic calculation, we have

**Theorem 5.7.** The minimizer of the elastic energy functional \( \mathcal{I} \) in (2) subject to compatible, no-stretch boundary conditions

\[
u = x, \quad v^\pm = \alpha^2 a \Psi_{1,2}^\pm \left( \frac{y}{a_{1,2}} \right) + y \pm \frac{1}{2} \delta, \quad w^\pm = \sqrt{2\alpha a} \Phi_{1,2}^\pm \left( \frac{y}{a_{1,2}} \right), \quad \text{at } x = \pm L,
\]

\[
u = x, \quad v^\pm = \pm L' \pm \frac{\delta}{2}, \quad w^\pm = 0, \quad \text{at } y = \pm L',
\]
satisfies the upper bound
\[
\mathcal{I}^* \leq \min \left( C_1 \alpha^{7/3} \sigma^{5/3} L^{1/3}, C_2 \alpha^2 \sigma^2 \frac{L}{a} \right)
\]
where \( a = \min(a_1, a_2) \) and the constants \( C_1 \) and \( C_2 \) depend only on \( K \), where \( K \) is the larger of the supports of \( \Phi_{1,2}^\pm \) divided by \( a \). The constants are independent of \( a, \sigma, L, L' \).
Proof. We will consider two test solutions, and our upper bound will be the minimum of the energies of the two test solutions.

One test solution is obtained by interpolating between the two boundary conditions in the region $|y| \leq Ka$, and connecting this solution to the boundary conditions at $y = \pm L'$ by a solution with a nearly uniform strain.

More precisely, we set $u = x$ and

$$w^\pm(x, y) = \sqrt{2}a \left[ \zeta \left( \frac{1}{2} + \frac{x}{2L} \right) \Phi^\pm \left( \frac{y}{a} \right) + \left[ 1 - \zeta \left( \frac{1}{2} + \frac{x}{2L} \right) \right] \Phi^\pm \left( \frac{y}{a} \right) \right]$$

where $\zeta$ is as defined in Lemma 5.6. Defining $\delta(x)$ by

$$\delta(x) = -\left[ \frac{1}{2} \int_{-\infty}^{0} [w^-_y(x, \eta)^2 d\eta + \frac{1}{2} \int_{0}^{\infty} [w^+_y(x, \eta)]^2 d\eta + 2\alpha w^+(x, 0) \right]$$

we set $v(x, y) = v_1(x, y) + v_2(x, y)$ where

$$v_1^\pm(x, y) = \vartheta \left( \frac{y}{L'} \right) \left[ -\frac{1}{2} \int_{x}^{y} [w^\pm_y(x, \eta)]^2 d\eta + y \pm \frac{1}{2} \delta(x) \right]$$

$$v_2^\pm(x, y) = \left[ 1 - \vartheta \left( \frac{y}{L'} \right) \right] \left( y \pm \frac{1}{2} \delta \right)$$

where $\vartheta$ is a smooth function such that $0 \leq \vartheta \leq 1$, $\vartheta(x) \equiv 1$ for $x \leq \frac{1}{3}$, and $\vartheta(x) \equiv 0$ for $x \geq \frac{2}{3}$.

We can estimate the energy of this test solution noting that $\delta(x)$ is bounded by $C\alpha^4 a^2$ by lemma 5.5, and using the appropriate rescaling of lemma 5.6 to bound the energy due to $v_1$. A straightforward calculation gives

$$I \leq C \left[ \frac{\alpha^4 a^5}{L^3} + \frac{\alpha^4 a^2 L'}{L} + \frac{\sigma^2 \alpha^2 L}{a} + \frac{\sigma^2 \alpha^2 a^3}{L^3} + \frac{\alpha^4 a^2 L'}{L^3} \right].$$

This energy corresponds to $E_1 + E_{III}$ in our heuristic calculation, and this is of course reasonable, since we do not have a region corresponding to the self-similar solution in this test function. It is also clear that for $a \ll L \sim L'$, the dominant term in the energy is $\sigma^2 \alpha^2 L/a$, and this will be larger than the other terms provided that $\alpha^4 a^2 \ll \sigma^2 \alpha^2 L/a$, i.e. in the asymptotic regime $a \ll \alpha^{-2/3} \sigma^{2/3} L^{1/3}$.

Another test solution that we will consider is the following. Let $\phi^\pm$, $\psi^\pm$ be the zero asymptotic shift profile with support $K_0$ that we constructed in Lemma 5.4.

We will first assume that $a_1 = a_2 = a$. Let $(\tilde{u}, \tilde{v}^\pm, \tilde{w}^\pm)$ denote the self-similar solution with profile $\phi^\pm$, $\psi^\pm$, constructed in the same manner as in Theorem. 5.3 with $\rho(-1) = \rho(1) = a$.

We define the test solution by

$$u(x, y) = \zeta \left( \frac{L - |x|}{b} \right) \tilde{u} + \left[ 1 - \zeta \left( \frac{L - |x|}{b} \right) \right] x$$

$$w^\pm(x, y) = \zeta \left( \frac{L - |x|}{b} \right) \tilde{w} + \sqrt{2} \left[ 1 - \zeta \left( \frac{L - x}{B} \right) \right] \Phi^\pm \left( \frac{y}{a} \right) + \left[ 1 - \zeta \left( \frac{L + x}{B} \right) \right] \Phi^\pm \left( \frac{y}{a} \right)$$
where $\zeta$ is as defined in Lemma 5.6 and $b < L$ is a length scale we will choose below. Defining $\delta(x)$ by

$$\delta(x) = -\frac{1}{2} \left[ \int_{-\infty}^{0} [w^-_y(x, \eta)]^2 \, d\eta + \int_{0}^{\infty} [w^+_y(x, \eta)]^2 \, d\eta + 4w^+(x, 0) \right]$$

it follows from the construction of $\tilde{w}$, and of $w$ that $\delta(x) = 0$ if $L - |x| > b$.

We set $v(x, y) = v_1(x, y) + v_2(x, y)$ where

$$v_1^\pm(x, y) = \vartheta \left( \frac{|y|}{L'} \right) \left[ - \int_{\pm\infty}^{y} [w_y^\pm(x, \eta)]^2 \, d\eta + y \pm \frac{1}{2} \delta(x) \right]$$

$$v_2^\pm(x, y) = \left[ 1 - \vartheta \left( \frac{|y|}{L} \right) \right] \left( y \pm \frac{1}{2} \delta \right)$$

where $\vartheta$ is a smooth function such that $0 \leq \vartheta \leq 1$, $\vartheta(x) \equiv 1$ for $x \leq \frac{1}{3}$, and $\vartheta(x) \equiv 0$ for $x \geq \frac{2}{3}$. Note that $v(x, y) = \tilde{v}(x, y)$ for $|y| \leq K_0 \alpha^{-1/3} \sigma^{1/3} L^{2/3}$ and $L - |x| \geq b$.

We can estimate the energy of this test solution as above and we obtain,

$$\mathcal{I} \leq C(K) \left[ \frac{\alpha^4 a^5}{b^2} + \frac{\alpha^{1/3} a^2 \sigma^{1/3} L^{2/3}}{b} + \frac{\sigma^2 \alpha^2 b}{a} + \frac{\sigma^2 \alpha^2 a^3}{b^3} + \frac{\alpha^4 a^2 L}{L'} \right] + D \alpha^{7/3} \sigma^{5/3} L^{1/3}$$

**Remark.** The constant $C$ depends on the boundary conditions through the support $K$, but the constant $D$ is independent of $K$.

This energy corresponds exactly to the energy $\mathcal{E}$ in our heuristic calculation. Consequently, we get

$$\mathcal{I} \leq C \left( K, \frac{a}{\sigma} \right) \alpha^{7/3} \sigma^{5/3} L^{1/3}$$

in the scaling regime $a/\sigma \sim O(1)$ for the choice $b = \alpha^{5/6} \sigma^{2/3} L^{1/3}$.

$$\mathcal{I} \leq D \alpha^{7/3} \sigma^{5/3} L^{1/3} + C \left( K, \frac{a L^{2/3}}{\sigma^{5/3}} \right) \alpha^2 \sigma^2,$$

in the scaling regime $a L^{2/3} \sigma^{-5/3} \sim O(1)$ for the choice $b = a \sim \epsilon^{5/3} L^{-2/3}$. $D$ is a universal constant, independent of the scaling functions $\Phi$ and $\Psi$ determining the boundary conditions.

6. **Structure of the Minimal Ridge**

In this section we will derive pointwise and integrated bounds for the ridge width $\rho(X)$ and for the ridge sag $W(X, 0)$. Lobkovsky’s analysis predicts that these two quantities should scale in the same way, and further, the associated length scale is not uniform along the ridge [28].

Given a test solution $(u, v^\pm, w^\pm)$ that satisfies the boundary conditions, we can naturally construct three different $x$ dependent length scales from the solution. They are, the inverse curvature

$$a(x) = \kappa^{-1}(x) = \left[ \int_{-\infty}^{0} [w_{yy}^-(x, \eta)]^2 \, d\eta + \int_{0}^{\infty} [w_{yy}^+(x, \eta)]^2 \, d\eta \right]^{-1}$$
the ridge sag $w^+(x, 0) = w^-(x, 0)$, and the support

$$k(x) = \sup \{ y > 0 | [w^+(x, \pm y)]^2 + [v^+(x, \pm y) - y \mp \delta/2]^2 + [u(x, \pm y)]^2 > 0 \}$$

In our construction for the self-similar test solutions yielding the upper bound, we see that $a(x), w(x, 0)$ and $k(x)$ all scale in the same way as $\rho(X) = \rho(x/L)$. This suggests that the structure of the ridge in the $y$-direction is given by a single length scale, which depends on $x$, and further, this length scale is given by our assumed scaling for $\rho(X)$ in the construction for the upper bound.

Our basic tool will be bounding the stretching energy using only the length scales $a(x)$ and $w(x, 0)$ at a given point $x$. Combining these estimates with our estimates for the energy, we obtain pointwise bounds for $a(x)$ and $w(x, 0)$. This estimate for the stretching energy is obtained in the following lemma.

**Lemma 6.1.** Given the profile $W(X, \cdot)$ at a point $X \in [-1, 1]$, the stretching energy $E_s$ is bounded from below by

$$\sqrt{E_s} \geq \max_{X \in [-1,1], Y \in R, \delta \in (0,1)} C(1 - \delta) \left[ \sqrt{Y} \left( W(X, 0) + \frac{Y}{\sqrt{2}} \right)^2 \right.$$

$$+ \frac{Y^{5/2}}{6} \left( 1 - \frac{Y}{2\delta \rho(X)} - \frac{C'A^3}{\sqrt{Y}} \right)$$

where

$$\rho(X) = \left[ \int_{-\infty}^{\infty} W_{YY}^+ (X, Y) dY + \int_{-\infty}^{0} W_{YY}^- (X, Y) dY \right]^{-1}.$$

**Proof.** The stretching energy is given by

$$E_s = \int_{-\infty}^{\infty} \frac{1}{2} \left( \int_{-1}^{1} W_X^2 dX \right)^2 dY$$

Since the integrand is non-negative, we have

$$E_s \geq \max_Y \int_{-Y}^{Y} \frac{1}{2} \left( \int_{-1}^{1} W_X^2 dX \right)^2 dY$$

$$\geq \max_Y \frac{1}{2Y} \left[ \int_{-Y}^{Y} \int_{-1}^{1} W_X^2 dX dY \right]^2,$$

by Jensen’s inequality. For each $\eta \in [-Y, Y]$, and for all $X \in (-1, 1)$, we have the elementary inequality

$$\int_{-1}^{1} W_X^2(\xi, \eta) d\xi \geq \frac{(W(X, \eta) - W(-1, \eta))^2}{1 + X} + \frac{(W(X, \eta) - W(1, \eta))^2}{1 - X}$$

$$\geq \frac{W^2(X, \eta)}{2} \left[ \frac{1}{1 + X} + \frac{1}{1 - X} \right] - 2[W^2(-1, \eta) + W^2(1, \eta)]$$

$$= \frac{W^2(X, \eta)}{1 - X^2} - 2[W^2(-1, \eta) + W^2(1, \eta)].$$

(15)
\( W(\pm 1, \eta) \) are given by the boundary conditions on the frame. Using the scaling form for the boundary conditions, we see that

\[
(16) 
2 \left[ \int_{-Y}^{Y} W^2(-1, \eta) d\eta + \int_{-Y}^{Y} W^2(1, \eta) d\eta \right] \leq CA^3
\]

All we now need is an estimate for \( \int \int W(\xi, \eta) d\xi d\eta \). Let \( \rho^\pm(\xi) \) be given by

\[
\rho^\pm(\xi) = \pm \left[ \int_0^{\pm \infty} W^\pm_{YY}(X, Y) dY \right]^{-1},
\]

so that \( \rho(\xi) = \rho^+ + \rho^- \). By Lemma 4.1, we obtain

\[
\int_{-Y}^{Y} \left[ W^+ (\xi, \eta) \right]^2 d\eta \geq \max_{Z \leq Y, \delta \in (0, 1)} (1 - \delta) \left[ Z \left( \frac{W(0, 0) + \beta^+(X) Z}{2} \right)^2 + \frac{Z^3}{12} \left( [\beta^+(X)]^2 - \frac{Z}{\delta \rho^+(X)} \right) \right].
\]

Adding the corresponding result for \( W^- \), we get

\[
\int_{-Y}^{Y} \left[ W(\xi, \eta) \right]^2 d\eta \geq \max_{Z \leq Y, \delta \in (0, 1)} (1 - \delta) \left[ Z_1 \left( \frac{W(0, 0) - \beta^-(X) Z_1}{2} \right)^2 + \frac{Z_1^3}{12} \left( [\beta^-(X)]^2 - \frac{Z_1}{\delta \rho^+(X)} \right) \right].
\]

By the matching condition, \( \beta^+(X) = \beta^-(X) + \sqrt{2} \). This also implies that \( [\beta^-(X)]^2 + [\beta^+(X)]^2 \geq 1 \). Replacing the separate maximization over \( Z_1 \leq Y \) and \( Z_2 \leq Y \), by a single maximization over \( Z = Z_1 = Z_2 \leq Y \), we obtain

\[
\int_{-Y}^{Y} \left[ W(\xi, \eta) \right]^2 d\eta \geq \max_{Z \leq Y, \delta \in (0, 1)} 2(1 - \delta) \left[ Z \left( \frac{W(0, 0) + Z}{\sqrt{2}} \right)^2 + \frac{Z^3}{12} \left( 2 - \frac{Z}{\delta \rho(X)} \right) \right].
\]

Combining this with inequalities (13), (14) and (15) proves the lemma. \( \square \)

We can now prove pointwise upper bounds for \( \rho(X) \) and \( W(X, 0) \).

**Theorem 6.2.** \((U, V^\pm, W^\pm)\) is a test solution that satisfies the boundary and the matching conditions. Also, \( I(U, V^\pm, W^\pm) \leq \bar{I} \). Then, \( \exists C_1, C_2, C_1', C_2' \) and \( C_3' \) that only depend on \( \bar{I} \) such that

\[
\rho(X) \leq C_1(1 - X^2)^{2/5} + C_1'A.
\]
Also, the ridge sag satisfies the pointwise bound
\[ W^2(X, 0) \leq \max \left( C'_1 \rho^2(X), C'_2 A^2 \left( \frac{A}{\rho(X)} \right)^{1/4}, C'_3 \left( \frac{(1 - X^2)^6}{\rho(X)} \right)^{1/7} \right). \]

**Proof.** Setting \( \delta = \frac{1}{2}, Y = \frac{5}{7} \rho(X) \), Lemma 6.1 yields
\[ \sqrt{\bar{Y}} \geq \sqrt{E_s} \geq \frac{C}{\sqrt{\rho(X)(1 - X^2)}} \{[\rho(X)]^3 - C' A^3\}. \]

If \( \rho(X) \geq (2C')^{1/3} A \), it follows that \( \rho^3(X) - C'A^3 \geq \frac{1}{2} \rho^3(X) \). Using this in the above inequality, we see that either \( \rho(X) < (2C')^{1/3} A \), or
\[ \sqrt{\bar{Y}} \geq \frac{C}{2(1 - X^2)} [\rho(X)]^{5/2}. \]

Combining these two inequalities, we see that \( \exists C_1, C'_1 \) only depending on \( \bar{I} \), such that
\[ \rho(X) \leq C_1 (1 - X^2)^{2/5} + C'A. \]

This finishes the first part of the proof. To illustrate the pointwise bounds for \( W^2(X, 0) \), we begin with a heuristic calculation. If \( W^2(X, 0) \ll \rho^2(X) \), the dominant balance in Lemma 6.1 is
\[ Y^{5/2} \sim \frac{Y^{7/2}}{\rho}, \]
and this gives the characteristic scale \( \tilde{Y} \sim \rho \). This is the same calculation as above, and this gives \( W^2 \ll \rho^2 \leq (1 - X^2)^{2/5} \) as in the previous part. If \( W^2(X, 0) \gg \rho^2(X) \), after ignoring the boundary term \( C'A^3 Y^{-1/2} \), the four remaining terms in Lemma 6.1 are of orders
\[ W^2(X, 0) \sqrt{\tilde{Y}}, \ W(X, 0) \tilde{Y}^{3/2}, \ \tilde{Y}^{5/2}, \ \text{and} \ \frac{\tilde{Y}^{7/2}}{\rho(X)}, \]
respectively. The dominant balance is between the first and the last terms, and this gives the characteristic scale \( \tilde{Y} \sim [W^2(X, 0) \rho(X)]^{1/3} \). Also, this gives
\[ \sqrt{E_s} \sim \frac{C}{1 - X^2} W^2(X, 0) \sqrt{\tilde{Y}} \sim \frac{C}{1 - X^2} W^{7/3}(X, 0) \rho^{1/6}(X). \]

Rearranging, we get
\[ W^2(X, 0) \lesssim \left( \frac{(1 - X^2)^6}{\rho(X)} \right)^{1/7}. \]

We will now make these considerations precise.

**Lemma 6.3.** \( \exists C < \infty, C' > 0 \) such that \( |W(X, 0)| \geq C \rho(X) \) implies that
\[ \sqrt{\tilde{Y}} \left( W(X, 0) + \frac{Y}{2} \right)^2 + \frac{Y^{5/2}}{6} \left( 1 - \frac{Y}{\rho(X)} \right) \geq C' W^{7/3}(X, 0) \rho^{1/3}(X), \]
where
\[ Y = \left( \frac{6}{7} \right)^{1/3} \left[ W^2(X, 0) \rho(X) \right]^{1/3}. \]

**Proof.** Let \( W(X, 0) = C_1 \rho(X) \), and \( Y \) be as defined above. A direct calculation shows that
\[
\sqrt{Y \left( W(X, 0) + \frac{Y}{2} \right)^2 + \frac{Y^{5/2}}{6} \left( 1 - \frac{Y}{\rho(X)} \right)} \geq \sqrt{Y \left[ W^2 - |W|Y - \frac{Y^3}{6\rho(X)} \right]}
\]
\[
= \left[ \left( \frac{6}{7} \right)^{7/6} - C_1^{-1/3} \right] W^{7/3} \rho^{1/3}
\]
By taking \( C_1 \) sufficiently large (> \(C\)), we obtain the required inequality. In particular, we can take \( C = 8 \) and \( C' = 1/3 \). \( \square \)

We can now finish the proof of Theorem 6.2.

**Proof.** If \( W^2(X, 0) \geq C^2 \rho(X, 0) \), combining Lemma 6.1 and Lemma 6.3, we obtain
\[
\sqrt{E_s} \geq \frac{C}{1 - X^2} \left[ (W^{14}(X, 0) \rho(X))^{1/6} - \frac{C' A^3}{(W^2(X, 0) \rho(X))^{1/6}} \right].
\]
If
\[
(W^{14}(X, 0) \rho(X))^{1/6} \geq \frac{2C' A^3}{(W^2(X, 0) \rho(X))^{1/6}},
\]
it follows that
\[
\sqrt{I} \geq \sqrt{E_s} \geq \frac{C}{2(1 - X^2)} (W^{14}(X, 0) \rho(X))^{1/6}.
\]
Combining all the above considerations, we see that one of the following inequalities has to hold
\[
W^2(X, 0) \leq C'_1 \rho^2(X),
\]
\[
W^2(X, 0) \leq C'_2 A^2 \left( \frac{A}{\rho(X)} \right)^{1/4},
\]
\[
W^2(X, 0) \leq C'_3 \left( \frac{(1 - X^2)^6}{\rho(X)} \right)^{1/7}.
\]
This proves the theorem. \( \square \)

**Remark.** Our pointwise bounds are not optimal for \( \rho(X) \). In particular, our construction for the upper bound shows that we can have test solutions with
\[
\rho(X) \leq C \max \left( A, \epsilon^{-1/3}(1 - |X|), (1 - X^2)^{2/3} \right),
\]
that have a uniformly bounded energy.

For \( A \ll (1 - |X|) \ll 1 \), this function is asymptotically (in \( \epsilon \)) much smaller than the upper bound from Theorem 6.2. In the remaining range, \( i.e \), for \( 1 - |X| \lesssim A \), of for \( C \lesssim 1 - |X| \), where \( C \) is an \( O(1) \) constant, our pointwise upper bound captures the behavior of \( \rho(X) \) in the self-similar construction.
Remark. If \( \rho(X) \) did scale like the pointwise upper bound in Theorem 6.2, i.e. \( \rho(X) \sim C_1 A + C_2 (1 - X^2)^{2/5} \), then the pointwise bounds for the ridge sag imply that
\[
|W(X,0)| \leq C'_1 A + C'_2 (1 - X^2)^{2/5} \sim \rho(X).
\]

We cannot obtain lower bounds for \( \rho(X) \) or \( W(X,0) \) purely by energetic arguments as the following “pinching” argument shows. For any given point \( x \in (-L,L) \), we can consider the test solution obtained by pinching at \( x \), i.e., we set
\[
u(x,s) = 0, \quad w(x,s) = \sqrt{2} \alpha a \Phi \left( \frac{s}{a} \right), \quad v(x,s) = \alpha^2 a \Psi \left( \frac{s}{a} \right) + s,
\]
where \( a \) is a length scale that we are free to choose subject to \( a > \sigma \exp(-e^{-1/3}) \) and \((\Phi, \Psi)\) is a no-stretch profile with zero asymptotic shift that we constructed in lemma 5.4. We use our construction for the upper bound to construct minimal ridge solutions in \([-L,x]\) and \([x,L]\).

These solutions connect smoothly at \( x \), since our constructions have \( u(x',s) = u(x,s), v(x',s) = v(x,s) \) and \( w(x',s) = w(x,s) \) for sufficiently small \( |x' - x| \). Also, \((L + x)^p + (L - x)^p \leq 2^{p+1} L^p\), for all \( p > 0 \). Combining this with our upper bounds, we see that the energy of the “pinched” solution scales in the same manner as the upper bound. Since the length scale \( a \) can be chosen (essentially) arbitrarily small, it follows that the energetics are not enough to give us a pointwise lower bound on the ridge-sag \( W(X,0) \) or the the ridge-width \( \rho(X) \).

We need these pointwise lower bounds to prove rigorous scaling results for \( \rho(X) \) and \( W(X,0) \). To obtain such results, we need to use the fact that the solution of interest is a minimizer, i.e., the first variations vanish. This type of analysis is carried out for the structure of an Austenite-Martensite boundary in [24, 10]. A similar analysis is possible for the minimal ridge, and we will present the details elsewhere.

7. Discussion

We conclude our discussion by indicating some of the issues/open problems relating to thin elastic sheets in general and to minimal ridges in particular, and in the process we point out the relevance of our results to some of these questions.

We have proved rigorous scaling laws for the energy of a single minimal ridge with a geometrically linear FvK ansatz. A natural question is the extension of these results to the Nonlinear FvK energy, and also to full three dimensional elasticity, as in [4]. It is easy to extend this to a mixed energy functional where the bending energy is treated in a geometrically linear fashion, but we use the “full” energy \( W_{2D} \) for the in-plane stretching. Extending this analysis to the Geometrically nonlinear functional, or to full three dimensional elasticity will require some new techniques [4].

Another problem is to show that the scaling laws also hold for a real crumpled sheet, where the forcing is not through clamping the boundaries to a frame, but rather through the confinement in a small volume. In this case, there are interesting global geometric and topological considerations, some of which are explored in Refs. [35] and [16]. As Lobkovsky and Witten [29] argue, the boundary condition that the deformation goes to zero far away from the ridge implies that the ridges do not
interact with each other significantly. The ridges can be considered the elementary excitations of a crumpled sheet.

More precisely, we have constructed ridge solutions with zero asymptotic shift, that are exactly strain free on the boundaries. These solutions give “non-interacting” ridges and patching these solutions together, it is possible to construct test solutions for a sheet confined inside a sphere. This gives us upper bounds which scale in the same way as the energy of a single ridge.

To show the corresponding lower bound, we have to show that confinement actually leads to the formation of ridges, and that the competition between the bending and the stretching energy for this situation has the same form as in lemma 4.4. In this context, we expect that global topological considerations, as well as the non self-intersection of the sheet will play a key role in the analysis, as they do in the analysis of elastic rods (one dimensional objects) [20].

As we note above, the blistering problem is described by the same elastic energy (Eq. (2)), but with different boundary conditions. Our results show an interesting contrast with results for the blistering problem. Ben Belgacem et al. have shown that [3], for an isotropically compressed thin film, the energy of the minimizer satisfies

$$c\lambda^{3/2}\sigma L \leq \mathcal{I} \leq C\lambda^{3/2}\sigma L,$$

where $L$ is a typical length scale of the domain, and $\lambda$ is the compression factor. A construction for the upper bound strongly suggests that the minimizers develop an infinitely branched network with oscillations on increasingly finer scales as $\sigma \to 0$. In contrast, our results indicate that the energy of a minimal ridge satisfies

$$c\sigma^{5/3}L^{1/3} \leq \mathcal{I} \leq C\sigma^{5/3}L^{1/3},$$

and the energy concentrates in a region of width $\sigma^{1/3}L^{2/3}$. This shows that the nature of the solution of the variational problem for the elastic energy in (2) depends very strongly on the boundary conditions. In particular the very nature of the energy minimizers is different for the two problems – For the blistering problems, as $\sigma \to 0$ the minimizers develop a branched network of folds refining towards the boundary.

Finally, one would like to prove $\Gamma$–convergence and find the $\Gamma$–limit [12] [13] for the elastic energy as $\sigma \to 0$. The difference in the scaling of the energy minimum for the minimal ridge, and the blistering problem shows that the $\Gamma$–limit of the elastic energy depends crucially on the imposed boundary conditions.

The analysis in this paper only pertains to situations where the configuration of the sheet is either smooth, or consists of a finite number of minimal ridges. More precisely, the sheet configurations $\phi : S \to \mathbb{R}^3$ is piecewise smooth, strain-free a.e., has gradient $D\phi$ in $BV$, and the singular support of $D^2\phi$ lives on a finite union of straight line segments. For the boundary conditions that admit such configurations, our analysis suggests that the asymptotic energy is on the scale $\sigma^{5/3}$ as $\sigma \to 0$. Further, if the following limit exists, the is necessarily follows that

$$\mathcal{I}[\phi] = \Gamma - \lim_{\sigma \to 0} \sigma^{-5/3}\mathcal{I} = C \sum \alpha_j^{7/3}l_j^{1/3},$$
where \( l_j \) is the length of the \( j \)th segment in the singular support of \( D^2 \phi \), and \( \alpha_j \) is the jump in \( D \phi \) across the segment \( \alpha_j \). Note that, because of the \( l_j^{\frac{1}{3}} \) dependence, the \( \Gamma \)–limit cannot be written as the integral with respect to the \( \mathcal{H}^1 \) measure on the singular support of \( D^2 \phi \), of a local energy density, which only depends on \( \alpha_j \).

The reason we get a \( l_j^{\frac{1}{3}} \) dependence instead of a linear dependence in \( l_j \) is that the ridges have a nonuniform structure along their length for any \( \sigma > 0 \). Any formulation of the \( \Gamma \)–limit should therefore incorporate a “hidden” variable which reflects the nonuniform structure, although, this non-uniformity is no longer detectable in the limiting \( \sigma \to 0 \) configuration. A natural candidate for this variable is the scaled ridge width \( \rho(x) \) (= the inverse curvature \( a(x) \) defined as in Sec. 6), where \( x \) is a coordinate along a ridge. We expect that the \( \Gamma \)–limit can be written as an integral of an energy density with respect to \( \mathcal{H}^1 \) measure on the defect set, where the energy density depends on \( \alpha \) and also on \( \rho(x) \) and derivatives \( \rho'(x) \) and \( \rho''(x) \). In fact, Eq. (12) strongly suggest that the \( \Gamma \)-limit for the energy of a single ridge can be written as

\[
\mathcal{I}[\phi] = C \sum \alpha_j \inf_{\rho} \int_0^{l_j} \left[ (\rho''(x)\rho(x))^2 + [\rho'(x)]^4 \right] \rho(x) + \frac{1}{\rho(x)} \right] dx
\]

where \( x \) is a coordinate along the ridge, and the infimum is over all smooth functions \( \rho \) that vanish at both the endpoints of the ridge.

We hope this paper spurs further investigation of the question of the \( \Gamma \)–limit of the elastic energy. This question is very much open, there are no proofs for either \( \Gamma \)–convergence, or of our conjectured structure for the \( \Gamma \)–limit.

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