Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
A novel mathematics model of covid-19 with fractional derivative. Stability and numerical analysis

Badr Saad T. Alkhahtani*, Sara Salem Alzaid

Department of Mathematics, College of Science, King Saud University, P.O. Box 1142, Riyadh 11989, Saudi Arabia

ARTICLE INFO

Article history:
Received 31 May 2020
Revised 5 June 2020
Accepted 12 June 2020
Available online 17 June 2020

Keywords:
Covid-19 model
Non-local operators
Reproductivity numbers
Lagrange polynomial

ABSTRACT

A mathematical model depicting the spread of covid-19 epidemic and implementation of population covid-19 intervention in Italy. The model has 8 components leading to system of 8 ordinary differential equations. In this paper, we investigate the model using the concept of fractional differential operator. A numerical method based on the Lagrange polynomial was used to solve the system equations depicting the spread of COVID-19. A detailed investigation of stability including reproductive number using the next generation matrix, and the Lyapunov were presented in detail. Numerical simulations are depicted for various fractional orders.

© 2020 Elsevier Ltd. All rights reserved.

1. Introduction

Uncertainties around the spread of Covid-19 have lead many researchers to understand investigation in many field of technology, science and engineering in the last five months since its appearance in Wuhan-China last December-2019 Many mathematical models were suggested in the last five months with the aim to understand the dynamics spread of the novel deadly disease [10]. Many journal have launch special issues on Covid-19, in field of science, technology and engineering, with the aim together all novel results changing from theoretical to practical point of view. Of course so far many new results have been collected many many data are ready although they are still being collected. Mathematician while they do not provide a cure nor a vaccine for any infectious disease however the mathematical models can help in many ways [11]. For example, their result are very useful to predict the future behavior of the spread and even control it. Technique like Markov chain, Fuzzy, Stochastic, Monte-Carlo approach and many others are very useful in this process [5–8]. On the other hand fractional differential operators are used to include into mathematical models the effect of non locality often divide by power process, fading memory process and cross-over [12,15]. In this paper we consider, the model suggested in [9].

2. Preliminaries

In this section, we recall some basic definitions and properties of fractional calculus theory which are useful in the next sections.

Definition 2.1. Let \( u \) be a function not necessarily differentiable, and \( \varphi \) be a real number such that \( \varphi > 1 \), then the Caputo derivative with \( \varphi \) order with power law is given as [13]

\[
\begin{align*}
\psi D_0^\varphi u(t) &= \frac{1}{\Gamma(\varphi)} \int_0^t (t-y)^{\varphi-1} u(y) dy, \quad (2.1)
\end{align*}
\]

Definition 2.2. Let \( u \in H^1(a, b) \), \( b > a, \varphi \in [0, 1] \) then the new Caputo derivative of fractional order is given by:

\[
\begin{align*}
D_0^\varphi u(t) &= \frac{M(\varphi)}{(1-\varphi)} \int_a^t u'(x) \exp\left[-\varphi \frac{t-x}{1-\varphi}\right] dx. \quad (2.2)
\end{align*}
\]

where \( M(\varphi) \) is a normalization function such that \( M(0) = M(1) = 1 \) [4]. But, if the function \( u \neq H_1(a, b) \) then, new derivative called the Caputo-Fabrizio fractional derivative can be defined as

\[
\begin{align*}
D_0^\varphi u(t) &= \frac{M(\varphi)}{(1-\varphi)} \int_a^t (u(t) - u(x)) \exp\left[-\varphi \frac{t-x}{1-\varphi}\right] dx. \quad (2.3)
\end{align*}
\]

Remark 2.1. The authors remarked that, if \( \psi = \frac{1-\varphi}{\varphi} \in [0, \infty) \), \( \varphi = \frac{1}{1-\psi} \in [0, 1] \), then Eq. (2.1) assumes the form

\[
\begin{align*}
D_0^\varphi u(t) &= \frac{N(\psi)}{\psi} \int_a^t u'(x) \exp\left[-\frac{t-x}{\psi}\right] dx, \quad N(0) = N(\infty) = 1 \quad (2.4)
\end{align*}
\]
In Addition,
\[ \lim_{\psi \to 0} \psi \exp \left[ -\frac{t-x}{\psi} \right] = \omega(x-t) \]  
(2.5)

Now after the introduction of a new derivative, the associate antiderivative becomes important, the associated integral of the new Caputo derivative with fractional order was proposed by Losada and Nieto [14].

**Definition 2.3.** \( u \in H^1(x, y), y > x, \delta \in [0, 1] \) with the function \( f \) differentiable then, the definition of the new fractional derivative (Atangana-Baleanu derivative in Caputo sense) is given as
\[ \alpha^D_t^\alpha[u(t)] = \frac{M(\delta)}{1 - \delta} \int_0^t u(s)E_{\alpha} \left[ -\frac{\theta(t-s)^\alpha}{1-\delta} \right] \, ds, \]  
(2.6)

where \( M(\delta) \) has the same properties as in the case of the Caputo-Fabrizio fractional derivative.

It should be noted that we do not recover the original function when \( \delta = 0 \) except when at the origin the function vanishes. To avoid this kind of problem, the following definition is proposed.

**Definition 2.4.** Let \( u \in H^1(x, y), y > x, \delta \in [0, 1] \) and not necessary differentiable then, the definition of the new fractional derivative (Atangana-Baleanu fractional derivative in Riemann-Liouville sense) is given as [1]:
\[ \alpha^D_t^\alpha[u(t)] = \frac{B(\delta)}{1 - \delta} \int_0^t u(s)E_{\alpha} \left[ -\frac{\theta(t-s)^\alpha}{1-\delta} \right] \, ds. \]  
(2.7)

**Definition 2.5.** The fractional integral associate to the new fractional derivative with nonlocal kernel (Atangana-Baleanu fractional integral) is given as [1]:
\[ \alpha^D_0^\alpha[u(t)] = \frac{1 - \delta}{B(\delta)} u(t) + \frac{\theta(t)}{B(\delta) 1^{-\delta}} \int_0^t (t-j)^{\delta-1} \, dj. \]  
(2.8)

When alpha is zero we recover the initial function and if also alpha is 1, we obtain the ordinary integral.

3. Mathematical model

In this section we consider the model suggested in [9].
\[ \dot{S}(t) = -S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) \]
\[ \dot{I}(t) = S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) - (\epsilon + \zeta + \lambda)I(t) \]
\[ \dot{D}(t) = \epsilon I(t) - (\eta + \rho)D(t) \]
\[ \dot{A}(t) = \xi I(t) - (\theta + \mu + \kappa)A(t) \]
\[ \dot{R}(t) = \eta D(t) + \theta A(t) - (\nu + \xi)R(t) \]
\[ \dot{T}(t) = \mu A(t) + \nu R(t) - (\sigma + \tau)T(t) \]
\[ H(t) = \lambda I(t) + \rho D(t) + \kappa A(t) + \xi R(t) + \sigma T(t) \]
\[ E(t) = \tau T(t) \]  
(3.1)

Here
\( S(t) \) is the class of susceptible,
\( I(t) \) is the class of infected asymptomatic infected undetected,
\( D(t) \) is the class of asymptomatic infected, detected,
\( H(t) \) is the healed class,
\( A(t) \) is ailing symptomatic infected, undetected,
\( R(t) \) is recognized symptomatic infected, detected
\( T(t) \) is the class of acutely symptomatic infected detected
\( E(t) \) is the death class. parameters therein and their physical meaning and interpretation can be found in [9].

4. Well-Poseness and stability analysis

Since all parameters used in the model are positive. If the initial assumptions are positive then all the classes are positive, for the models with classical and non-local operators. We start with classical cases
\[ \dot{I}(t) = S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) - (\epsilon + \zeta + \lambda)I(t) \]  
(4.1)

However the product \( \beta D(t) + \gamma A(t) + \delta R(t) \) is positive. Since all classes should have same sign, thus
\[ \dot{I}(t) > - (\epsilon + \zeta + \lambda)I(t) \]  
(4.2)
\[ I(t) \geq I(0) \exp[-(\epsilon + \zeta + \lambda)t] \]  
(4.3)
Since \( I(t) \) is positive then
\[ \dot{D}(t) = \epsilon I(t) - (\eta + \rho)D(t) \]
\[ \geq - (\eta + \rho)D(t) \]  
(4.4)
\[ D(t) \geq D(0) \exp[-(\eta + \rho)t] \]  
also
\[ A(t) \geq A(0) \exp[(\theta + \mu + \kappa)] \]  
(4.5)
with \( D(t) \) and \( A(t) \) being positive \( \forall t \geq 0 \), we have that
\[ \dot{R}(t) = \eta D(t) + \theta A(t) - (\nu + \xi)R(t) \]
\[ \geq - (\nu + \xi)R(t) \]  
(4.6)
\[ R(t) \geq R(0) \exp[(\nu + \xi)t] \]  
(4.7)
\[ T(t) = \mu A(t) + \nu R(t) - (\sigma + \tau)T(t) \]
\[ \geq - (\sigma + \tau)T(t) \]  
(4.8)
\[ T(t) \geq T(0) \exp[-(\sigma + \tau)t] \]
\[ E(t) = E(0) + \tau \int_0^t T(t) \, dt \]  
\[ \forall t \geq 0 \]

Since \( E(0) \) is positive or zero \( \tau \) and \( T(t) \) are positive, then \( E(t) \geq 0 \) \( \forall t \geq 0 \).

To prove for classes \( \delta(t) \) and \( A(t) \), we define the following norm
\[ \forall f \in \mathbb{C}[a, b] \]
\[ \|f\|_\infty = \sup_{t \in [a, b], |f(t)|} \]
\[ \|f\|_\infty \geq - S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) \]
\[ \geq - S(t)(\alpha |I(t)| + \beta |D(t)| + \gamma |A(t)| + \delta |R(t)|) \]
\[ \geq - S(t)(\alpha \sup_{t \in [a, b]} |I(t)| + \beta \sup_{t \in [a, b]} |D(t)| + \gamma \sup_{t \in [a, b]} |A(t)| + \delta \sup_{t \in [a, b]} |R(t)|) \]
\[ \geq - S(t)(\alpha \|I(t)\|_\infty + \beta \|D(t)\|_\infty + \gamma \|A(t)\|_\infty + \delta \|R(t)\|_\infty) \]  
(4.9)
\[ \delta(t) \geq \delta(0) \exp[-(\alpha \|I(t)\|_\infty + \beta \|D(t)\|_\infty + \gamma \|A(t)\|_\infty + \delta \|R(t)\|_\infty)] \]
\[ + \|A(t)\|_\infty + \delta \|R(t)\|_\infty \]  
(4.10)
\[ H(t) = H(0) + \tau \int_0^t (\lambda I(t) + \rho D(t) + \kappa A(t) + \xi R(t) + \sigma T(t)) \, dt \]  
(4.11)

since all the other classes are positive then \( \forall t > 0 \)
\[ H(t) > 0 \]

with the non local operator, we only show the positiveness for Caputo derivative
\[ \dot{S}(t)* \frac{t-\alpha}{\Gamma(1-\alpha)} = -S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) \]
\[ \dot{D}(t)* \frac{t-\alpha}{\Gamma(1-\alpha)} = \epsilon I(t) - (\eta + \rho)D(t) \]
\[ \dot{I}(t)* \frac{t-\alpha}{\Gamma(1-\alpha)} = S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) \]
\[ - (\epsilon + \zeta + \lambda)I(t) \]
Thus, following the procedure suggested before

\[
I(t) \geq E_0[(-\epsilon + \xi + \lambda)\eta t^\alpha]I(0)
\]

\[
(A(t) \geq A(0)E_0[\xi I(t) - (\theta + \mu + \kappa)t^\alpha]
\]

\[
R(t) \geq R(0)E_0[(-\sigma + \tau)t^\alpha]
\]

\[
E(t) \geq E(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1}(\lambda I(t) + \rho D(t) + \kappa A(t) + \xi R(t) + \sigma T(t))d\tau
\]

\[
S(t) \geq S(0)E_0[-\eta(\alpha\|I(t)\|_\infty + \beta\|D(t)\|_\infty)]
\]

\[
T(t) \geq T(0)E_0[-(\sigma + \tau)t^\alpha]
\]

\[
E(t) \geq H(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1}(\lambda I(t) + \rho D(t) + \kappa A(t) + \xi R(t) + \sigma T(t))d\tau
\]

The disease equilibrium is

\[
(\xi + \epsilon + \lambda, 0, 0, 0, 0, 0)
\]

\[
B = \alpha + \frac{\beta \epsilon}{(\eta + \rho)} + \frac{\nu \zeta}{\theta + \mu + \kappa} + \frac{\delta \epsilon \eta}{(\eta \rho)(v + \zeta)} + \frac{\delta \epsilon \eta}{(\eta \rho)(v + \zeta)}
\]

Although the reproductive number was given in [9], we only present the next generation matrix associated to the model. We choose the 5 classes of infected.

\[
\hat{I}(t) = S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) - (\epsilon + \xi + \lambda)I(t)
\]

\[
\hat{D}(t) = \epsilon I(t) - (\eta + \rho)D(t)
\]

\[
\hat{A}(t) = \xi I(t) - (\theta + \mu + \kappa)A(t)
\]

\[
\hat{R}(t) = \eta D(t) + \theta A(t) - (v + \xi)R(t)
\]

\[
\hat{T}(t) = \mu A(t) + v R(t) - (\sigma + \tau)T(t)
\]

(4.23)

From the above, the matrix

\[
F = \begin{pmatrix}
\alpha S & \gamma S & \beta S & \delta S & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(4.24)

\[
V = \begin{pmatrix}
\epsilon + \xi + \lambda & 0 & 0 & 0 & 0 \\
-\epsilon & \theta + \mu + \lambda & 0 & 0 & 0 \\
0 & -\theta & -\eta & 0 & 0 \\
0 & -\mu & 0 & -v & \sigma + \tau \\
\end{pmatrix}
\]

(4.25)

Thus

\[
R_0 = \rho(FV^{-1}) = \frac{S}{\epsilon + \xi + \lambda} \begin{pmatrix}
\alpha + \frac{\beta \epsilon}{(\eta + \rho)} + \frac{\nu \zeta}{\theta + \mu + \kappa} + \frac{\delta \epsilon \eta}{(\eta \rho)(v + \zeta)} + \frac{\delta \epsilon \eta}{(\eta \rho)(v + \zeta)}
\end{pmatrix}
\]

The model suggested will lead to endemic situations if

\[
I(t), A(t), D(t), R(t), T(t) > 0 \quad \forall t \geq 0
\]

that is to say

\[
S(\alpha I + \beta D + \gamma A + \delta R) - (\epsilon + \xi + \lambda)I > 0
\]

\[
\epsilon I - (\eta + \rho)D > 0
\]

\[
\xi I - (\theta + \mu + \kappa)A > 0
\]

\[
\eta D + \theta A - (v + \xi)R > 0
\]

\[
\mu A + \gamma R - (\sigma + \tau)T > 0
\]

First we have that

\[
S(\alpha I + \beta D + \gamma A + \delta R) > \frac{\epsilon I}{(\epsilon + \xi + \lambda)}
\]

\[
D < \frac{\epsilon}{\eta + \rho}I \quad R < \frac{\eta D}{v + \rho + \frac{\delta \epsilon \eta}{(\eta \rho)(v + \zeta)}} \quad A < \frac{\xi I}{(\theta + \mu + \kappa)(v + \zeta)}
\]

\[
R < \frac{\eta D}{(\eta \rho)(v + \zeta)} + \frac{\theta \zeta}{(\theta + \mu + \kappa)(v + \zeta)}
\]

From

\[
S(\alpha I + \beta D + \gamma A + \delta R) = 0, \quad I = 0
\]

Thus

\[
H^* = 0
\]
\[ T < \left\{ \frac{\mu \zeta}{\eta} \left[ \frac{1}{(\theta + \mu + \lambda)} \right] + \frac{\mu \zeta}{(\theta + \mu + \lambda)^2} \right\} I \] (4.26)

\[ \frac{S}{(e + \zeta + \lambda)} \left\{ \alpha l + \beta e l + \eta + \rho \cdot \frac{\gamma \zeta}{\theta + \mu + \lambda} + \frac{\delta \eta e l}{(\theta + \mu + \lambda) (v + \zeta)} \right\} > I \] simplifying I, then we get

\[ \frac{S}{(e + \zeta + \lambda)} \left\{ \alpha l + \beta e l + \eta + \rho \cdot \frac{\gamma \zeta}{\theta + \mu + \lambda} + \frac{\delta \eta e l}{(\theta + \mu + \lambda) (v + \zeta)} \right\} > 1 \] (4.27)

\[ R_0 > 1 \]

**Theorem 4.1.** The disease free equilibrium are asymptotically globally stable within the acceptable interval if \( R_0 < 1 \) and unstable if \( R_0 > 1 \).

**Proof.** The proof will be achieved the use of the Lyapunov function defined by

\[ L = \frac{1}{\lambda_1} I + \frac{1}{\lambda_2} A + \frac{1}{\lambda_3} D + \frac{1}{\lambda_4} R + \frac{1}{\lambda_5} T \] (4.28)

\[ \frac{dL}{dt} = \frac{1}{\lambda_1} I + \frac{1}{\lambda_2} A + \frac{1}{\lambda_3} D + \frac{1}{\lambda_4} R + \frac{1}{\lambda_5} T \]

\[ = \frac{1}{\lambda_2} (S(\alpha l + \beta D + \gamma A + \delta R) - \lambda_1 I + \frac{1}{\lambda_2} (\zeta I - \lambda_2 A) + \frac{1}{\lambda_3} (\eta \zeta l - \lambda_3 D) + \frac{1}{\lambda_4} (\mu D - \theta A - \lambda_4 R) + \frac{1}{\lambda_5} (\mu A + \gamma R - \lambda_5 T) \]

\[ = \frac{1}{\lambda_1} (S(\alpha l + \beta D + \gamma A + \delta R) + \frac{1}{\lambda_2} (\zeta I - \lambda_2 A) + \frac{1}{\lambda_3} (\eta \zeta l - \lambda_3 D) + \frac{1}{\lambda_4} (\mu D - \theta A) + \frac{1}{\lambda_5} (\mu A + \gamma R)) \]

\[ \geq (R_0 - 1)(I + A + D + R + T) \leq 0 \] (4.30)

If \( R_0 \leq 0 \) if \( R_0 > 1 \) and zero if \( I + A + D + R + T = 0 \).

We present the Lyapunov associate to the model

\[ L(S^*, T^*, A^*, D^*, R^*, T^*, H^*) = \left( S - S^* + S^* \log \frac{S}{S^*} \right) + \left( I - I^* + I^* \log \frac{I}{I^*} \right) + \left( A - A^* + A^* \log \frac{A}{A^*} \right) + \left( D - D^* + D^* \log \frac{D}{D^*} \right) \]

\[ + \left( R - R^* + R^* \log \frac{R}{R^*} \right) + \left( T - T^* + T^* \log \frac{T}{T^*} \right) + \left( H - H^* + H^* \log \frac{H}{H^*} \right) \] (4.31)

\[ \frac{dL}{dt} = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \frac{\left( S - S^* \right)}{S} \left( (I - I^*) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \right) \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]

\[ = \left( S - S^* \right) \frac{dS}{dt} + \left( I - I^* \right) \frac{dI}{dt} + \left( A - A^* \right) \frac{dA}{dt} + \left( D - D^* \right) \frac{dD}{dt} + \left( R - R^* \right) \frac{dR}{dt} + \left( T - T^* \right) \frac{dT}{dt} + \left( H - H^* \right) \frac{dH}{dt} \]
\[ + \lambda I + \xi D + \kappa A + \zeta R + \sigma T - (\lambda I^* + \xi D^* + \kappa A^* + \zeta R^* + \sigma T^*) + \frac{H^*}{H}(\lambda I + \xi D + \kappa A + \zeta R + \sigma T) \]

Thus
\[ \frac{dL}{dt} = a_1 - a_2 \]
this implies
\[ \frac{dL}{dt} > 0 \quad \text{if} \quad a_1 > a_2 \]
\[ \frac{dL}{dt} < 0 \quad \text{if} \quad a_1 < a_2 \]
\[ \frac{dL}{dt} = 0 \quad \text{if} \quad a_1 = a_2 \]
\[ \square \]

5. Covid-19 model with fractional derivative

In this section, we obtain alternative representations of the cancer model considering the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives, for the discretization of fractional model equations by Adams-Bashforth method [2,3].

5.1. Covid-19 model with Caputo-Fabrizio fractional derivative

Considering Eq. (3.1), the modified cancer model with Caputo Derivative is given as

\[ \begin{align*}
\frac{D^\alpha CF}{D\tau} S(t) &= -S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) \\
\frac{D^\alpha CF}{D\tau} I(t) &= S(t)(\alpha I(t) + \beta D(t) + \gamma A(t) + \delta R(t)) \\
\frac{D^\alpha CF}{D\tau} A(t) &= (\epsilon + \zeta + \lambda) I(t) \\
\frac{D^\alpha CF}{D\tau} D(t) &= \epsilon I(t) - (\eta + \rho) D(t) \\
\frac{D^\alpha CF}{D\tau} H(t) &= \lambda I(t) + \rho D(t) + \kappa A(t) + \xi R(t) + \sigma T(t) \\
\frac{D^\alpha CF}{D\tau} E(t) &= \tau T(t)
\end{align*} \]

(5.1)

5.1.1. Numerical scheme for Caputo-Fabrizio fractional derivative

We consider the following general fractional differential equation with fading memory included via the Caputo-Fabrizio fractional derivative. That is,

\[ \begin{cases}
\frac{D^\alpha CF}{D\tau} x(t) = f(t, x(t)), \\
x(0) = x_0
\end{cases} \]

(5.2)

or

\[ M(\vartheta) \int_0^t \chi(\varphi) \exp \left[ -\vartheta \frac{t - \varphi}{1 - \vartheta} \right] \, d\varphi = f(t, x(t)). \]

(5.3)

Using the fundamental theorem of calculus, we convert the above to

\[ x(t) - x(0) = \frac{1}{\vartheta M(\vartheta)} f(t, x(t)) + \frac{\vartheta}{1 - \vartheta} \int_0^t f(\varphi, x(\varphi)) \, d\varphi \]

(5.4)

so that

\[ x(t_{n+1}) - x(0) = \frac{1}{\vartheta M(\vartheta)} f(t_n, x(t_n)) + \frac{\vartheta}{1 - \vartheta} \int_0^{t_n} f(t, x(t)) \, dt \]

(5.5)

and

\[ x(t_n) - x(0) = \frac{1}{\vartheta M(\vartheta)} f(t_{n-1}, x(t_{n-1})) + \frac{\vartheta}{1 - \vartheta} \int_0^{t_{n-1}} f(t, x(t)) \, dt \]

(5.6)

by subtracting (5.6) from (5.5), we get

\[ x(t_{n+1}) - x(t_n) = \frac{1}{\vartheta M(\vartheta)} [f(t_n, x_n) - f(t_{n-1}, x_{n-1})] + \frac{\vartheta}{1 - \vartheta} \int_{t_{n-1}}^{t_n} f(t, x(t)) \, dt \]

(5.7)

so the solution is,

\[ x_{n+1} = x_n + \left( 1 - \frac{\vartheta}{M(\vartheta)} \right) f(t_n, x_n) + \left( \frac{1 - \vartheta}{M(\vartheta)} \right) f(t_{n-1}, x_{n-1}) \]

(5.8)
so for the Eq. (3.1) the solution is

\[
S_{n+1} - S_n = \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left[ -S_n(t_n)[\alpha I_n(t_n) + \beta D_n(t_n) + \gamma A_n(t_n) + \delta R_n(t_n)] + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left[ -S_{n-1}(t_{n-1})[\alpha I_{n-1}(t_{n-1}) + \beta D_{n-1}(t_{n-1}) + \gamma A_{n-1}(t_{n-1}) + \delta R_{n-1}(t_{n-1})] \right] \right]
\]

\[ (5.9) \]

\[
I_{n+1} = I_n + \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left( S_n(t_n)[\alpha I_n(t_n) + \beta D_n(t_n) + \gamma A_n(t_n) + \delta R_n(t_n)] + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left( S_{n-1}(t_{n-1})[\alpha I_{n-1}(t_{n-1}) + \beta D_{n-1}(t_{n-1}) + \gamma A_{n-1}(t_{n-1}) + \delta R_{n-1}(t_{n-1})] \right] \right)
\]

\[ (5.10) \]

\[
D_{n+1} = D_n + \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left( \epsilon I_n(t_n) - (\eta + \rho)D_n(t_n) + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left( \epsilon I_{n-1}(t_{n-1}) - (\eta + \rho)D_{n-1}(t_{n-1}) \right] \right)
\]

\[ (5.11) \]

\[
A_{n+1} = A_n + \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left( \xi I_n(t_n) - (\theta + \mu + \kappa)A_n(t_n) + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left( \xi I_{n-1}(t_{n-1}) - (\theta + \mu + \kappa)A_{n-1}(t_{n-1}) \right] \right)
\]

\[ (5.12) \]

\[
R_{n+1} = R_n + \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left( \eta I_n(t_n) + \theta A_n(t_n) + (\nu + \xi)R_n(t_n) + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left( \eta I_{n-1}(t_{n-1}) + \theta A_{n-1}(t_{n-1}) + (\nu + \xi)R_{n-1}(t_{n-1}) \right] \right)
\]

\[ (5.13) \]

\[
T_{n+1} = T_n + \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left( \mu I_n(t_n) + \nu R_n(t_n) + (\sigma + \tau)T_n(t_n) + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left( \mu I_{n-1}(t_{n-1}) + \nu R_{n-1}(t_{n-1}) + (\sigma + \tau)T_{n-1}(t_{n-1}) \right] \right)
\]

\[ (5.14) \]

\[
H_{n+1} = H_n + \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left( \lambda I_n(t_n) + \rho D_n(t_n) + \kappa A_n(t_n) + (\xi + \sigma)H_n(t_n) + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left( \lambda I_{n-1}(t_{n-1}) + \rho D_{n-1}(t_{n-1}) + \kappa A_{n-1}(t_{n-1}) + (\xi + \sigma)H_{n-1}(t_{n-1}) \right] \right)
\]

\[ (5.15) \]

\[
E_{n+1} = E_n + \left( 1 - \frac{\partial}{M(\partial)} + \frac{3 \partial h}{2M(\partial)} \right) \left( \tau T_n(t_n) + \left( 1 - \frac{\partial}{M(\partial)} + \frac{\partial h}{2M(\partial)} \right) \left( \tau T_{n-1}(t_{n-1}) \right] \right)
\]

\[ (5.16) \]
5.2. Covid-19 model with Atangana-Baleanu derivative

Considering Eq. (3.1), the modified cancer model with Atangana-Baleanu Derivative Derivative is given as

\[
\begin{align*}
_0^{ABC}D_t^\alpha S(t) &= -S(t)(\alpha_l(t) + \beta D(t) + \gamma A(t) + \delta R(t)), \\
_0^{ABC}D_t^\alpha D(t) &= S(t)(\alpha_l(t) + \beta D(t) + \gamma A(t) + \delta R(t)) - (\epsilon + \zeta + \lambda) I(t), \\
_0^{ABC}D_t^\alpha A(t) &= t I(t) - (\eta + \rho) D(t), \\
_0^{ABC}D_t^\alpha R(t) &= \eta D(t) + \lambda I(t) - (\nu + \xi) R(t), \\
_0^{ABC}D_t^\alpha T(t) &= \mu A(t) + \nu R(t) - (\sigma + \tau) T(t), \\
_0^{ABC}D_t^\alpha E(t) &= \tau T(t).
\end{align*}
\]  

(5.17)

5.2.1. Numerical scheme for Atangana-Baleanu derivative

We consider the following fractional differential equation Let us consider the following fractional differential equation

\[
\begin{align*}
_0^{ABC}D_t^\alpha x(t) &= f(t, x(t)), \\
x(0) &= x_0.
\end{align*}
\]  

(5.18)

The above equation can be converted to a fractional integral equation by applying the fundamental theorem of fractional calculus:

\[
x(t) - x(0) = \frac{(1 - \theta)}{\Gamma(\theta)} \int_0^t f(x(t)) (t - \varpi)^{\theta - 1} d\varpi.
\]  

(5.19)

At a given point \( t_{n+1}, n = 1, 2, 3, \ldots \) the above equation is reformulated as

\[
x(t_{n+1}) - x(0) = \frac{(1 - \theta)}{\Gamma(\theta)} \int_0^{t_{n+1}} f(x(t)) (t_{n+1} - \varpi)^{\theta - 1} d\varpi
\]  

(5.20)

and at the point \( t = t_n, n = 1, 2, 3, \ldots \), we have

\[
x(t_n) - x(0) = \frac{(1 - \theta)}{\Gamma(\theta)} \int_0^{t_n} f(x(t)) (t_n - \varpi)^{\theta - 1} d\varpi
\]  

(5.21)

which on subtraction yields

\[
x(t_{n+1}) - x(t_n) = \frac{(1 - \theta)}{\Gamma(\theta)} \left\{ f(t, x(t_n)) - f(t_{n+1}, x(t_{n+1})) \right\} + \frac{\theta}{\Gamma(\theta)} \int_0^{t_{n+1}} f(x(\varpi)) (t_{n+1} - \varpi)^{\theta - 1} d\varpi
\]  

(5.22)

the solution is

\[
x_{n+1} - x_n = f(t_n, x(t_n)) \left\{ \frac{(1 - \theta)}{\Gamma(\theta)} + \frac{\theta}{\Gamma(\theta)} h \left( \frac{2h t_n^{\alpha+1}}{\theta} - \frac{t_n^{\alpha+1}}{\theta + 1} \right) \right\}
\]  

(5.23)

so for the Eq. (5.17) the solution is

\[
S_{n+1} - S_n = [-S_n(t_n)[\alpha_l(t_n) + \beta D(t_n) + \gamma A(t_n) + \delta R(t_n)]]
\]  

(5.24)

\[
\times \left\{ \frac{(1 - \theta)}{\Gamma(\theta)} + \frac{\theta}{\Gamma(\theta)} h \left( \frac{2h t_n^{\alpha+1}}{\theta} - \frac{t_n^{\alpha+1}}{\theta + 1} \right) \right\} + [-S_{n+1}(t_{n+1})[\alpha_l(t_{n+1}) + \beta D(t_{n+1}) + \gamma A(t_{n+1}) + \delta R(t_{n+1})]]
\]  

(5.25)

\[
\times \left\{ \frac{(1 - \theta)}{\Gamma(\theta)} + \frac{\theta}{\Gamma(\theta)} h \left( \frac{2h t_n^{\alpha+1}}{\theta} - \frac{t_n^{\alpha+1}}{\theta + 1} \right) \right\}
\]  

\[
I_{n+1} - I_n = [S_n(t_n)[\alpha_l(t_n) + \beta D(t_n) + \gamma A(t_n) + \delta R(t_n)] - [\epsilon + \zeta + \lambda] I_n(t_n)]
\]  

(5.25)

\[
\times \left\{ \frac{(1 - \theta)}{\Gamma(\theta)} + \frac{\theta}{\Gamma(\theta)} h \left( \frac{2h t_n^{\alpha+1}}{\theta} - \frac{t_n^{\alpha+1}}{\theta + 1} \right) \right\} + [S_{n+1}(t_{n+1})[\alpha_l(t_{n+1}) + \beta D(t_{n+1}) + \gamma A(t_{n+1}) + \delta R(t_{n+1})] - [\epsilon + \zeta + \lambda] I_n(t_{n+1})]
\]  

(5.25)

\[
\times \left\{ \frac{(1 - \theta)}{\Gamma(\theta)} + \frac{\theta}{\Gamma(\theta)} h \left( \frac{2h t_n^{\alpha+1}}{\theta} - \frac{t_n^{\alpha+1}}{\theta + 1} \right) \right\}
\]  

\[
D_{n+1} - D_n = [\epsilon I_n(t_n) - (\eta + \rho) D_n(t_n)]
\]  

(5.25)
\[ +\{I_{n-1}(t_{n-1}) - (\eta + \rho)D_{n-1}(t_{n-1}) \} \times \left\{ \frac{(1 - \vartheta)}{ABC(\vartheta)} \times \frac{\vartheta}{ABC(\vartheta)} + \left( \frac{2h_{n+1}^\vartheta}{\vartheta} - \frac{t_{n+1}^\vartheta}{\vartheta + 1} \right) - \frac{\vartheta}{\Gamma(\vartheta) \times ABC(\vartheta)} \left( \frac{h_{n}^\vartheta}{\vartheta} - \frac{t_{n}^\vartheta}{\vartheta + 1} \right) \right\} \] 

\[ A_{n+1} - A_n = \zeta I_n(t_n) - (\theta + \mu + \kappa)A_n(t_n) \times \left\{ \frac{(1 - \vartheta)}{ABC(\vartheta)} \times \frac{\vartheta}{ABC(\vartheta)} + \left( \frac{2h_{n+1}^\vartheta}{\vartheta} - \frac{t_{n+1}^\vartheta}{\vartheta + 1} \right) - \frac{\vartheta}{\Gamma(\vartheta) \times ABC(\vartheta)} \left( \frac{h_{n}^\vartheta}{\vartheta} - \frac{t_{n}^\vartheta}{\vartheta + 1} \right) \right\} \] 

\[ R_{n+1} - R_n = f(t_n, x(t_n)) \times \left\{ \frac{(1 - \vartheta)}{ABC(\vartheta)} + \left( \frac{2h_{n+1}^\vartheta}{\vartheta} - \frac{t_{n+1}^\vartheta}{\vartheta + 1} \right) - \frac{\vartheta}{\Gamma(\vartheta) \times ABC(\vartheta)} \left( \frac{h_{n}^\vartheta}{\vartheta} - \frac{t_{n}^\vartheta}{\vartheta + 1} \right) \right\} \] 

\[ T_{n+1} - T_n = \mu A_n(t_n) + vR_n(t_n) - (\sigma + \tau)T_n(t_n) \times \left\{ \frac{(1 - \vartheta)}{ABC(\vartheta)} + \left( \frac{2h_{n+1}^\vartheta}{\vartheta} - \frac{t_{n+1}^\vartheta}{\vartheta + 1} \right) - \frac{\vartheta}{\Gamma(\vartheta) \times ABC(\vartheta)} \left( \frac{h_{n}^\vartheta}{\vartheta} - \frac{t_{n}^\vartheta}{\vartheta + 1} \right) \right\} \] 

\[ H_{n+1} - H_n = \lambda I_n(t_n) + \rho D_n(t_n) + \kappa A_n(t_n) + \xi R_n(t_n) + \sigma T_n(t_n) \times \left\{ \frac{(1 - \vartheta)}{ABC(\vartheta)} + \left( \frac{2h_{n+1}^\vartheta}{\vartheta} - \frac{t_{n+1}^\vartheta}{\vartheta + 1} \right) - \frac{\vartheta}{\Gamma(\vartheta) \times ABC(\vartheta)} \left( \frac{h_{n}^\vartheta}{\vartheta} - \frac{t_{n}^\vartheta}{\vartheta + 1} \right) \right\} \] 

\[ E_{n+1} - E_n = \tau T_n(t_n) \times \left\{ \frac{(1 - \vartheta)}{ABC(\vartheta)} \times \frac{\vartheta}{ABC(\vartheta)} + \left( \frac{2h_{n+1}^\vartheta}{\vartheta} - \frac{t_{n+1}^\vartheta}{\vartheta + 1} \right) - \frac{\vartheta}{\Gamma(\vartheta) \times ABC(\vartheta)} \left( \frac{h_{n}^\vartheta}{\vartheta} - \frac{t_{n}^\vartheta}{\vartheta + 1} \right) \right\} \] 

6. Numerical simulation

In this section, we present numerical simulation for different values of fractional \( \alpha \). The numerical simulation are presented in Figs. 1, 2, 3, 4, 5, 6, 7 and 8. We observed that with this model all classes are increasing exponentially.

**Fig. 1.** Numerical simulation of susceptible class for different value of fractional order.
Fig. 2. Numerical simulation of infected asymptomatic infected undetected class for different value of fractional order.

Fig. 3. Numerical simulation of asymptomatic infected class for different value of fractional order.

Fig. 4. Numerical simulation of healed class for different value of fractional order.
Fig. 5. Numerical simulation of ailing symptomatic infected class for different value of fractional order.

Fig. 6. Numerical simulation of recognized symptomatic infected class for different value of fractional order.

Fig. 7. Numerical simulation of acutely symptomatic infected detected class for different value of fractional order.
7. Conclusion

In this paper, we considered a set of 8 nonlinear ordinary differential equations to model the spread of COVID-19 in a given population. The model is comprised of susceptible class, 5 sub-classes of infected, recovered and death. We presented the positivity of each class as function of time, for classical and fractional case. We used the concept of next generation matrix to derive the reproductive number, we presented a detailed study of stability of equilibrium points. Numerical simulations are presented for different values of fractional orders.

Author statement

Dear editor, this is to confirm that both authors have done the work. We have done equal work in this paper. We confirm that we are both aware of the submission in this special issue in chaos solitons and fractal

Declaration of Competing Interest

Both authors declare there is no conflict of interest for this paper.

Acknowledgement

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this group no. RG-1437-017.

References

[1] Atangana A, Dumitru B. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Thermal Sci 2016;18. doi: 10.2298/TS160111018A.
[2] Atangana A, Jain S. The role of power decay, exponential decay and Mittag–Leffler functions waiting time distributions: application of cancer spread. Physica A 2019;512:330–51. doi: 10.1016/j.physa.2018.08.033.
[3] Atangana A, Owolabi K.M. New numerical approach for fractional differential equations. 2017. Preprint, arXiv, 1707.08177.
[4] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Prog Fract Differ Appl 2015;1:73–85.
[5] Chen W, Zhang XD, Korosak D. Investigation on fractional and fractal derivative relaxation-oscillation models. Int J Nonlin Sci Num 2010;11(2):3–9.
[6] Chen T, Rui J, Wang Q, Zhao Z, Cui J.A., Yin L. A mathematical model for simulating the transmission of Wuhan novel coronavirus. 2020. BioRxiv.
[7] Cheng ZJ, Shan J. 2019 novel coronavirus: where we are and what we know. Infection 2020;1–9.
[8] Chan JWM, Ng CK, Chan YH, et al. Short term outcome and risk factors for adverse clinical outcomes in adults with severe acute respiratory syndrome (SARS). Thorax 2003;58: 686–89
[9] Giordano G, Blanchini F, Bruno R, Colaneri P, Di Filippo A, Di Matteo A, et al. Modelling the COVID-19 epidemic and implementation of population-wide interventions in Italy. Nat Med 2020. doi: 10.1038/s41591-020-0883-7.
[10] Giordano G, Blanchini F, Bruno R, et al. Modelling the COVID-19 epidemic and implementation of population-wide interventions in Italy. Nat Med 2020.
[11] Kizito M, Tumwine J. A mathematical model of treatment and vaccination interventions of pneumococcal pneumonia infection dynamics. Journal of Applied Mathematics 2018;2018: 2539465); 16. 2018
[12] Khan MA, Atangana A. Modeling the dynamics of novel coronavirus (2019-nCoV) with fractional derivative. Alexandria Eng J 2020.
[13] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives, Theory and applications, Edited and with a foreword by S. M. Nikolski, Translated from the 1987 Russian original, Revised by the authors. Gordon and Breach Science Publishers; 1993. Yverdon
[14] Losada J, Nieto JJ. Properties of the new fractional derivative without singular kernel. Prog Fract Differ Appl 2015;1:87–92.
[15] Jain S. Numerical analysis for the fractional diffusion and fractional Buckmaster’s equation by two step Adam- Bashforth method. Eur Phys J Plus 2018;133(19). doi:10.1140/epjp/i2018-11854-x.