A new Quantum Mechanics in Phase Space

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Abstract

For each bounded operator $A$ on the Hilbert space $L^2(\mathbb{R}^m)$ we define a function \langle A \rangle : \mathbb{R}^{2m}_{qp} \to \mathbb{C}$ taking on $(q,p)$ the expected values of $A$ on a suitable state $\vartheta_{qp}$. For $A \geq B$ we have $\langle A \rangle \geq \langle B \rangle$. Dually for each couple $\varphi, \psi$ we define a function $S_{\varphi \psi}$ on the $\langle A \rangle$ in such a way to have $S_{\varphi \psi}(\langle A \rangle) = \langle \varphi, A \psi \rangle$. 

1 Introduction

This paper developes a new version of the Quantum Mechanics on Phase Space (cfr: [CZ], [dG], [Gr], [K], [M] or [P]) associating a (restricted) expected value function $\langle A \rangle$ to each bounded self-adjoint operator $A$ and a "distribution" $S_{\varphi \psi}$ on $\mathbb{R}^{2m}$ to each couple of vector states in such a way to have:

The new symbol $\langle A \rangle$ introduced here has the advantage to be positive when $A$ is positive.

The terms $\langle A \rangle$ and $S_{\varphi \psi}$ are connected with the Wigner and Husimi transforms (cfr. Remark [17]). We give explicit formulas for $\langle A \rangle (q,p), S_{\varphi \psi}(\langle A \rangle)$ and for the product in $\mathcal{M}$.

2 Symbols

As usual $\mathcal{E}(\mathbb{R}^m)$ will denote the space of all differentiable functions on $\mathbb{R}^m$. On the space of square integrable functions $L^2(\mathbb{R}^m)$ we will use the convolution operation $\ast : L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ given by: $(f \ast g)(x) = \frac{1}{(2\pi)^{m/2}} \int f(a) \cdot g(x-a) \cdot da$. With these positions we have exactly $\mathcal{F}(f \ast g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$. On the space $L^2(\mathbb{R}_x^m \times \mathbb{R}_y^m)$ we will
consider also the partial Fourier transform $\mathcal{F}_I : L^2(\mathbb{R}^{2m}) \to L^2(\mathbb{R}^{2m})$ given by $\mathcal{F}_I(F)(u,y) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} F(a,y) e^{-i a \cdot u} da$, analogously we define $\mathcal{F}_{II}$. In this context we will meet the change of variables $\tau : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ given by $\tau(x,y) = (y + \frac{x}{2}, y - \frac{x}{2})$, its inverse $\tau^{-1}(u,v) = (u - v, \frac{u + v}{2})$ and the exchange map $\Xi : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ given by $\Xi(x,y) = (y,x)$. We will also reserve the symbol $\omega$ to the function $\omega : \mathbb{R}_+^m \times \mathbb{R}_+^m \to \mathbb{R}_+$ given by $\omega(x,y) = \frac{1}{\pi} e^{-\frac{1}{2}(\|x\|^2 + \|y\|^2)}$.

We will denote by $SK : L(H) \to S(\mathbb{R}_+^{2m})$ the (injective) Schwartz kernel map characterized by $\langle \varphi, A\psi \rangle = SK(A)(\overline{\varphi} \otimes \psi)$ whenever $\varphi, \psi \in S(\mathbb{R}_+^m)$ (cfr. [T]). When $\psi$ is a unitary vector, we will denote by $E_\psi$ its associated projector on the line $\mathbb{C} \cdot \psi$.

3 A functional representation for bounded operators

Definition 1 Taken $(q,p)$ in $\mathbb{R}^{2m}$ the fundamental state centered in $(q,p)$ is the unitary vector $\vartheta_{qp} : \mathbb{R}^m \to \mathbb{C}$ given by:

$$\vartheta_{qp}(x) = \frac{1}{\pi^{m/4}} e^{ip \cdot (x - \frac{q}{2})} e^{-\frac{1}{4}\|x - q\|^2}$$

Remark 2 Note that $\mathcal{F} \vartheta_{qp} = \vartheta_{p,-q}$. We will write: $\vartheta$ for $\vartheta_{oo}$.

Remark 3 If we introduce the map $\mathcal{U} : \mathbb{C}^m \to \text{Unit}(H)$ defined by $\mathcal{U}_{q+ip}(\psi)(x) = e^{ip \cdot (x - \frac{q}{2})} \psi(x - q)$ we have $\vartheta_{qp} = \mathcal{U}_{q+ip} \vartheta$. Note that $\mathcal{U}_{q+ip} \psi \sim \psi$ only for $(q,p) = (0,0)$.

Definition 4 The expected value on the fundamental states map is the map $\langle \cdot \rangle : L_{sa}(H) \to \mathcal{E}(\mathbb{R}_+^{2m})$ given, for each self-adjoint bounded operator $A$ on $H$, by $\langle A \rangle (q,p) = \langle \vartheta_{qp}, A \vartheta_{qp} \rangle = \langle A \rangle \vartheta_{qp}$.

Remark 5 Whenever $A \geq 0$ we have $\langle A \rangle \geq 0$ and if $A \geq B$ we have $\langle A \rangle \geq \langle B \rangle$; if $\{E^A_{(-\infty, r]} \}_{r \in \mathbb{R}}$ is the spectral family of the bounded operator $A$ then the function $\langle E^A_{(-\infty, r]} \rangle (q,p)$ is, for each $(q,p)$, a monotone non-decreasing function in $r$, right continuous, with $\inf = 0$ and $\sup = 1$. The map $\langle \cdot \rangle$ extends to $\langle \cdot \rangle : L(H) \to \mathcal{E}_c(\mathbb{R}^{2m})$ as a $C^\infty$-linear map.

Exercise 6

- $\langle g(Q_k) \rangle (q,p) = \sqrt{2\pi} \cdot (g * \vartheta^2)(q_k)$ for each $g$ bounded
- $\langle f(P_k) \rangle (q,p) = \sqrt{2\pi} \cdot (f * \vartheta^2)(p_k)$ for each $f$ bounded
- Note that: $\langle Q^2 - \frac{1}{2} I \rangle (q,p) = \frac{1}{\sqrt{\pi}} \|q\|^2 \geq 0$ but $Q^2 - \frac{1}{2} I \neq 0$.
- $\langle E_\psi \rangle (q,p) = |\langle \psi, \vartheta_{qp} \rangle|^2$ for each unitary $\psi$.
Let \( \{ E_{\psi_1}, \ldots, E_{\psi_n}, \ldots \} \) be a sequence of pairwise orthogonal projectors in \( H \) such that \( I = \sum_{k \geq 0} E_{\psi_k} \); let \( \lambda_0, \ldots, \lambda_n, \ldots \) be a sequence of real numbers giving a bounded operator \( A = \sum_{k \geq 0} \lambda_k \cdot E_{\psi_k} \). We have: \( \langle A \rangle(q,p) = \sum_{k \geq 0} \lambda_k \cdot |\langle \psi_k, \theta_qp \rangle|^2 \)

**Notation 7** Denoted by \( M \) the image of the linear map \( \langle \cdot \rangle : L(H) \to \mathcal{E}(\mathbb{R}^{2m}) \), we will consider the following isomorphisms (as restricted maps): \( SK : L(H) \to SK(L(H)) \subset S'((\mathbb{R}^{2m})^\prime), \gamma : (\tau^* \circ SK)(L(H)) \to (\tau^* \circ SK)(L(H)) \subset S'((\mathbb{R}^{2m})^\prime), F_I : (\tau^* \circ SK)(L(H)) \to (F_I \circ \tau^* \circ SK)(L(H)) \subset S'((\mathbb{R}^{2m})^\prime) \) and \( \gamma : (F_I \circ \tau^* \circ SK)(L(H)) \to M \subset \mathcal{E}(\mathbb{R}^{2m}) \) defined by \( \gamma(T)(q,p) = (\omega \ast T)(p,q) = [(\omega \ast T) \circ \Xi](q,p) \). Note that \( M \) is contained in the space of slowly increasing differentiable functions (cfr. [V] ch.I, par.5.6.c). Since \( \langle A^\ast \rangle = \langle A \rangle \), the image \( \langle \cdot \rangle \) \((L_{sa}(H))\) is the real part \( \mathcal{M}_{\mathbb{R}} \) of \( M \).

**Theorem 8** For every bounded operator \( A \) and every \( (q,p) \) in \( \mathbb{R}^{2m} \) we have:

\[ \langle A \rangle(q,p) = (2\pi)^{3m/2} \cdot \{ \omega \ast F_I [SK(A) \circ \tau] \} (p,q) \text{ and if } SK(A) \text{ is regular} \]

\[ \langle A \rangle(q,p) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^{3m}} SK(A)(b + \frac{t}{2}, b - \frac{t}{2}) \cdot e^{-\frac{1}{2}||p-a||^2 + \frac{1}{2}||q-b||^2} \cdot e^{-it \cdot a} \cdot dt \cdot da \cdot db \]

**Proof.** \( \langle \hat{A}_{\omega q}, A \hat{q} \rangle = (\pi)^{-m} \cdot \hat{SK}(A)_{rs} \left( \int_{\mathbb{R}m} e^{-||q - \frac{r}{2} - \frac{s}{2}||^2} \cdot e^{-i(r-s)(p-a)} \cdot da \right) \)

\[ \text{and} \quad (2\pi)^{3m/2} \cdot \{ \omega \ast F_I [SK(A) \circ \tau] \} (p,q) = \]

\[ = (\pi)^{-m} \cdot \hat{SK}(A)_{rs} \left( \int_{\mathbb{R}m} e^{-||q - \frac{r}{2} - \frac{s}{2}||^2} \cdot e^{-i(r-s) \gamma^{-1}} \cdot dt \right) \]

**Corollary 9** The map \( \langle \cdot \rangle = (2\pi)^{3m/2} \cdot \gamma \circ F_I \circ \tau^* \circ SK : L(H) \to \mathcal{M} \) is a \( C^\ast \)-linear isomorphism with \( \langle \cdot \rangle^{-1} = (2\pi)^{-3m/2} \cdot SK^{-1}(\tau^{-1})^* \circ F^{-1}_{I} \circ \gamma^{-1} \)

**Notation 10** To avoid to deal with the "exotic" Fourier transform \( \tilde{F} \in L^1([\mathbb{R}^{2m}]) \) we introduce a cut-off. For each positive integer \( N \) let’s choose, once for all, a "hat" function \( h_N \in \mathcal{D}(\mathbb{R}^{2m}) \) always between 0 and 1 with value 1 on \( \prod_{1}^{2m} [-N, N] \) and value 0 outside of \( \prod_{1}^{2m} [-N - 1, N + 1] \) and moreover pair and invariant under the exchange of \( x \) with \( y \).

**Theorem 11** For each \( G \in \mathcal{M} \) we have:

\[ \gamma^{-1}(G)(F) = (2\pi)^m \cdot \lim_{N \to \infty} G \left( \mathcal{F}[h_N \cdot e^{x \cdot ||\cdot||^2 + y \cdot ||\cdot||^2} \ast (F \circ \Xi) \right) \]

on every \( F \) in \( S(\mathbb{R}^{2m}) \)

**Proof.** Computation.

**Definition 12** For each \( \varphi \) and \( \psi \) in \( H \) let’s define \( S_{\varphi \psi} : \mathcal{M} \to \mathcal{C} \) as \( S_{\varphi \psi}(G) = \langle \varphi, (\gamma^{-1}(G)) \psi \rangle \) (when \( \varphi = \psi \) we will write \( S_{\varphi} \) instead of \( S_{\varphi \varphi} \)).

**Remark 13** Obviously \( S_{\varphi \psi} \) is defined in such a way to have:

\[ S_{\varphi \psi}(A) = \langle \varphi, A\psi \rangle \]
Theorem 14 If $\varphi$ and $\psi$ are in $S(\mathbb{R}^m)$ we have, for every $G$ in $\mathcal{M}$:

$$S_{\varphi\psi}(G) = (2\pi)^{-m/2} \lim_{N} [\mathcal{F}(h_N \cdot e^{\frac{1}{4}\|\cdot\|^2 + \|\cdot\|^2}) \ast \{\mathcal{F}_I[(\psi \otimes \overline{\psi}) \circ \tau] \circ \Xi\}](G)$$

Proof. It is a straightforward application of the definition of $S_{\varphi\psi}$ and the formula for $\gamma^{-1}(G)$.

Exercise 15. 1. For each $(q_0, p_0)$ in $\mathbb{R}^{2m}$ we have $S_{q_0 p_0} = \delta_{q_0 p_0}$

2. For each non-zero polynomial $P(x_1, ..., x_m)$ the map $S_{p}$ is a distribution with support in $(0, 0)$ (if $\psi$ is in $\theta \cdot \mathbb{C}[x]$ then $\mathcal{F}_I^{-1}[(\psi \otimes \overline{\psi}) \circ \tau] \circ \Xi$ is in $e^{-\frac{1}{4}\|\cdot\|^2 \cdot \|\cdot\|^2} \cdot \mathbb{C}[a, b]$).

3. $S_{\sqrt{\tau}(G)} = \delta_{00}(G) + \frac{1}{2}(\partial^2_{y_i y_i} + \partial^2_{x_j x_j}) |_{00}(G)$ and it is not a non-negative distribution or a signed measure.

4. For every $\varphi$ and $\psi$ in $L^2(\mathbb{R}^m)$ and $N \geq 1$ the function:

$$S_{\varphi\psi N} = (2\pi)^{-m/2} \mathcal{F}(h_N \cdot e^{\frac{1}{4}\|\cdot\|^2 + \|\cdot\|^2}) \ast \{\mathcal{F}_I[(\psi \otimes \overline{\psi}) \circ \tau] \circ \Xi\}$$

is a well defined differentiable function and a multiplier on $\mathbb{R}^{2m}$.

Theorem 16 Given $\varphi$ and $\psi$ in $L^2(\mathbb{R}^m)$ for every bounded operator $A$ we have

$$\langle \varphi, A\psi \rangle = \lim_N S_{\varphi\psi N}(\langle A \rangle) = \lim_N \int_{\mathbb{R}^{2m}} S_{\varphi\psi N} \cdot \langle A \rangle \cdot d\lambda.$$ 

Remark 17 The terms $(A)$ and $S_{\varphi\psi}$ are connected with the Wigner and Husimi transforms: since the expression $(2\pi)^{-m/2} \mathcal{F}_I[(\psi \otimes \overline{\psi}) \circ \tau] \circ \Xi$ corresponds to $W^1(\psi, \varphi)$ and $H^1(\psi, \varphi) = 2^m \cdot W^1(\psi, \varphi) \ast e^{-\|\cdot\|^2 + \|\cdot\|^2}$ (cfr. [K] when $\varepsilon = 1$) we have:

$W^1(\psi, \varphi) = 2^m \cdot e^{-\|\cdot\|^2 + \|\cdot\|^2} \ast S_{\varphi\psi}$ and $H^1(\psi, \varphi) = e^{-\frac{1}{4}\|\cdot\|^2 + \|\cdot\|^2} \ast S_{\varphi\psi}$.

Note also the equality: $\langle A \psi \rangle = (\pi/2)^m \cdot H^1(\psi, \varphi)$. Moreover it is not difficult to prove that the Weyl operator $Op^W(a)$ associated to the symbol $a$ has:

$\langle Op^W(a) \rangle = (2\pi)^m \cdot \omega \ast a.$

Notation 18 Sometimes we will find useful to identify $(x, y)$ in $\mathbb{R}^{2m}$ with $z = x + iy$ in $\mathbb{C}^m$, $(q, p)$ with $w = q + ip$ etc. We will need in the following the functions: $\Omega_N : \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{C}$ and $\Delta : \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{R}$ given by:

$$\Omega_N(w, w', w'') = \frac{4}{(2\pi)^m} \cdot e^{-\|w\|^2} \cdot \mathcal{F}(h_N \cdot e^{\frac{1}{4}\|\cdot\|^2})(w') \cdot \mathcal{F}(h_N \cdot e^{\frac{1}{4}\|\cdot\|^2})(w'')$$

and

$$\Delta(w, w', w'') = \det \begin{bmatrix} 1 & RE(w) & IM(w) \\ 1 & RE(w') & IM(w') \\ 1 & RE(w'') & IM(w'') \end{bmatrix}$$

Theorem 19 For every couple $A$ and $B$ of bounded operators we have:

$$\langle A \cdot B \rangle (z) = \lim_N \int_{\mathbb{C}^m \times \mathbb{C}^m} \langle A \rangle (z') \cdot \langle B \rangle (z'') \cdot (\Omega_N \ast e^{-2\Delta})(z, z', z'') \cdot dz' \cdot dz''.$$
Proof. It is a lengthy but not difficult calculation of $\langle A \cdot B \rangle(z)$. □

Notation 20 If we denote by $G \times H$ the function

$$(G \times H)(z) = \lim_{N} \int_{\mathbb{C}^m \times \mathbb{C}^m} G(z') \cdot H(z'') \cdot (\Omega_N \ast e^{-2i\Delta})(z, z', z'') \cdot dz' \cdot dz''$$

we have an associative product in $\mathcal{M}$: such that: $\langle A \cdot B \rangle = \langle A \rangle \times \langle B \rangle$ (that is the map $\langle \cdot \rangle : (L(H), +, \cdot) \rightarrow (\mathcal{M}, +, \times)$ is an isomorphism of algebras).

We will denote by $\{H, G\}$ the expression $i \cdot (H \times G - G \times H)$ given by the function

$$\langle \{H, G\} \rangle(z) = 2 \cdot \lim_{N} \int_{\mathbb{C}^m \times \mathbb{C}^m} H(z') \cdot G(z'') \cdot [\Omega_N \ast \sin(2 \cdot \Delta)] \cdot (z, z', z'') \cdot dz' \cdot dz''$$

4 Bibliography

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