COMPUTATION OF JACOBI SUMS OF ORDERS $l^2$ AND $2l^2$ WITH PRIME $l$

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Abstract. In this article, we present the fast computational algorithms for the Jacobi sums of orders $l^2$ and $2l^2$ with odd prime $l$ by formulating them in terms of the minimum number of cyclotomic numbers of the corresponding orders. We also implement two additional algorithms to validate these formulae, which are also useful for the demonstration of the minimality of cyclotomic numbers required.

1. Introduction

Jacobi has many remarkable contributions to the field of mathematics like formulations of Jacobi symbol, Jacobi triple product, Jacobian, Jacobi elliptic functions etc. Among these, Jacobi sums have appeared as one of the most important findings. Many people have attempted to calculate the Jacobi sums of certain order in terms of the solutions of the corresponding Diophantine system. It has also been observed that the level of complexity in dealing with a Diophantine system increases with the increment in the concerned order, which has been observed as a challenge for the computation of Jacobi sums of higher orders.

The objective of this paper is to develop some fast computational algorithms for calculation of Jacobi sums.

Let $p \in \mathbb{Z}$ be an odd prime and $q = p^r$ with $r \geq 1$ an integer. Let $e \geq 2$ be an integral divisor of $q - 1$, then $q = ek + 1$ for some positive integer $k$. Let $\gamma$ be a generator of the cyclic group $\mathbb{F}_q^*$. For $\zeta_e$ a primitive $e$-th root of unity, we define a multiplicative character $\chi_e$ of order $e$ on $\mathbb{F}_q^*$ by $\chi_e(\gamma) = \zeta_e$. We now extend $\chi_e$ to $\mathbb{F}_q$ by taking $\chi_e(0) = 0$. For $0 \leq i, j \leq e - 1$, the Jacobi sums of order $e$ is defined by

$$J_e(i, j) = \sum_{v \in \mathbb{F}_q} \chi_e^i(v)\chi_e^j(v + 1).$$

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For $0 \leq a, b \leq e - 1$, the cyclotomic numbers $(a, b)_e$ of order $e$ are defined as follows:

$$(a, b)_e := \# \left\{ v \in \mathbb{F}_q \mid \chi_e(v) = \zeta_e^a, \chi_e(v + 1) = \zeta_e^b \right\}$$

$$= \# \left\{ v \in \mathbb{F}_q \setminus \{0, -1\} \mid \text{ind}_\gamma v \equiv a \pmod{e}, \text{ind}_\gamma (v + 1) \equiv b \pmod{e} \right\}.$$ 

In view of the definitions of Jacobi sums and cyclotomic numbers, over a given finite field $\mathbb{F}_q$, Jacobi sums (resp. cyclotomic numbers) of order $e$ mainly depend on two parameters, therefore these values could be naturally assembled into a matrix of order $e$. Thus, the Jacobi sums $J_e(i, j)$ and the cyclotomic numbers $(a, b)_e$ are well connected by the following relations [22, 23]:

$$\sum_i \sum_j (a, b)_e \zeta_e^{ai+bj} = J_e(i, j),$$  \(1.1\)

and

$$\sum_i \sum_j \zeta_e^{-(ai+bj)} J_e(i, j) = e^2 (a, b)_e.$$  \(1.2\)

Jacobi [13] introduced the Jacobi sums of orders 3, 4 and 7. In the literature, problem concerning to computation of Jacobi sums of a particular order also has been studied as the possible minimization in relations between Jacobi sums and cyclotomic numbers of that order, like for $e \leq 6$, $e = 8$, 10 and 12 the relations were established by Dickson [9]. Later, he also studied the relations for $e = 9, 14, 15, 16, 18, 20, 22, 24$, which are recorded in [10, 11]. Muskat [18] established the relation of order 12 in terms of the fourth root of unity to resolve the sign ambiguity which was occurring in the Dickson’s relation and here he also studied for $e = 30$. He [19] also provided complete results for order 14. Complete methods of $e = 16$ and 20 exist in Whiteman [26] and Muskat [20] respectively. In fact Western [25] determined Jacobi sums of orders 8, 9, 16 earlier than Dickson. Baumert and Fredrickson [7] gave corrections and removed the sign ambiguity in the Dickson’s work for $e = 9, 18$. Zee [27, 28] found relations for orders 13, 22 and 60. Zee and Muskat [21] provided the relations for $e = 21, 28, 39, 55$ and 56. Berndt and Evans [5] obtained sums of orders 3, 4, 6, 8, 12, 20 and 24 and they also determined sums of orders 5, 10 and 16 in [6]. Parnami, Agarwal and Rajwade [22] obtained certain relationships for Jacobi sums of odd prime orders upto 19. Furthermore, Katre and Rajwade [14] extended their work for Jacobi sums of general odd prime orders.

Over the most recent couple of years, fast computation of Jacobi sums is one of the essential enthusiasm among researchers, in perspective of its applications to primality testing, cryptosystems and so forth [11, 15, 16, 17]. As illustrated in [12], Jacobi sums could be used for estimating the number of integral solutions to certain congruences such as $x^3 + y^3 \equiv 1 \pmod{p}$. These estimates played a key role in the advancement of Weil conjectures [24]. Jacobi sums could be used for the determination of a number of solutions of diagonal equations over finite fields [3].
In perspective of equations (1.1) and (1.2), to determine a Jacobi sum of order $e$, one needs to determine all the cyclotomic numbers of order $e$. The cyclotomic numbers of order $l^2$ have been formulated by Shirolkar and Katre [23], whereas Ahmed, Tanti and Hoque [2] have established the formula for $e = 2l^2$.

In this paper, we give algorithms for fast computation of Jacobi sums of orders $l^2$ and $2l^2$ with $l$ a prime $\geq 3$. The idea used behind this paper is to compute all the Jacobi sums of a particular order in terms of the minimal number of cyclotomic numbers of that order as hinted from [3]. Here we derive explicit expressions for Jacobi sums of orders $l^2$ and $2l^2$ in terms of the minimal numbers of cyclotomic numbers of orders $l^2$ and $2l^2$ respectively. We implement two algorithms (see algorithms 1 and 2) to validate these expressions for the minimality. The implementation of algorithms has been carried out at a high performance computing lab in Department of Computer Sciences and Technology, Central University of Jharkhand.

The paper is organized as follows: Section 2 presents some well known properties of cyclotomic numbers of order $e$. Section 3 presents algorithms for equality of cyclotomic numbers of orders $l^2$ and $2l^2$. Section 4 contains the expressions of Jacobi sums of orders $l^2$ & $2l^2$. The fast computational algorithms for Jacobi sums are in section 5. Finally, a brief conclusion is reflected in section 6.

2. SOME USEFUL EXPRESSIONS

It is clear that $(a, b)_e = (a', b')_e$ whenever $a \equiv a' \mod e$ and $b \equiv b' \mod e$ where as $(a, b)_e = (e-a, b-a)_e$. These imply the following identities:

$$(a, b)_e = \begin{cases} (b, a)_e & \text{if } k \text{ is even or } q = 2^r, \\ (b + \frac{e}{2}, a + \frac{e}{2})_e & \text{otherwise}. \end{cases} \tag{2.1}$$

Applying these facts, it is easy to see that

$$\sum_{a=0}^{e-1} \sum_{b=0}^{e-1} (a, b)_e = q - 2, \tag{2.2}$$

and

$$\sum_{b=0}^{e-1} (a, b)_e = k - n_a, \tag{2.3}$$

where $n_a$ is given by

$$n_a = \begin{cases} 1 & \text{if } a = 0, 2 \mid k \text{ or if } a = \frac{e}{2}, 2 \nmid k; \\ 0 & \text{otherwise}. \end{cases}$$

3. ALGORITHMS FOR EQUALITY OF CYCLOTOMIC NUMBERS

Solution of cyclotomy of order $e$, does not require to determine all the cyclotomic numbers of order $e$ [3]. The objective is to avoid the redundancy
in calculations by dividing the cyclotomic numbers into classes as hinted from [3], which certainly boost up the overall efficiency.

This section, presents two algorithms which shows the equality relations of cyclotomic numbers of orders \( l^2 \) and \( 2l^2 \) respectively. These algorithms exactly determine which cyclotomic numbers are enough for the determination of all the Jacobi sums of orders \( l^2 \) and \( 2l^2 \) respectively. Thus, it helps us to faster the computation of these Jacobi sums. Also, these algorithms play a major role to validate the expressions for Jacobi sums of orders \( l^2 \) and \( 2l^2 \) in terms of the minimum number of cyclotomic numbers of orders \( l^2 \) and \( 2l^2 \) respectively. The expressions in Theorem 4.1 gets validated by ‘Algorithm 1’, and those in Theorems 4.2 and 4.3 get validated by ‘Algorithm 2’.

**Algorithm 1** Determination of equality of cyclotomic numbers of order \( l^2 \).

1: START
2: Declare Integer variable \( p, q, r, l, i, j, i1, j1, a1, b1, flag \).
3: INPUT \( l \)
4: if \( l \) is not a prime or less than 3 then
5: goto 3
6: else
7: \( e = l^2 \)
8: end if
9: Declare an array of size \( e \times e \), where each element of array is 2 tuple structure (i.e. ordered pair of \( (a, b) \), where \( a \) and \( b \) are integers).
10: INPUT \( q \)
11: for \( p = \) all prime number within 2 to \( q \) do
12: for all \( r \) within 1 to \( q \) do
13: if \( q \) is not equal to \( p^r \) then
14: goto 10
15: else if \( (q - 1)\%e == 0 \) then
16: goto 20
17: else
18: goto 10
19: end if
20: end for
21: end for
for \(i = 0\) to \(e - 1\) do
  for \(j = 0\) to \(e - 1\) do
    int \(a_1\) = value of \(a\) at current array index i.e. \(a_1 = arr[i1][j1].a\)
    int \(b_1\) = value of \(b\) at current array index i.e. \(b_1 = arr[i1][j1].b\)
    set flag of current element of array to 0 i.e. lock the element which is updated once \(arr[i1][j1].flag = 0\)
  end for
end for
for \(i = 0\) to \(e - 1\) do
  for \(j = 0\) to \(e - 1\) do
    if flag is 1 i.e. element has not been updated yet then
      if \(a\) is equal to \(b_1 \% e\) (when \(b_1 \geq 0\)) and \(b_1 + e\) (when \(b_1 < 0\))
      AND \(b\) is equal to \(a_1 \% e\) (when \(a_1 \geq 0\)) and \(a_1 + e\) (when \(a_1 < 0\)) then
        \(a = a_1, b = b_1\) and \(flag = 0\)
      end if
      if \(a\) is equal to \((a_1 - b_1) \% e\) (when \((a_1 - b_1) \geq 0\)) and \((a_1 - b_1) + e\) (when \((a_1 - b_1) < 0\))
      AND \(b\) is equal to \((-b_1) \% e\) (when \((-b_1) \geq 0\))
      and \((-b_1) + e\) (when \((-b_1) < 0\)) then
        \(a = a_1, b = b_1\) and \(flag = 0\)
      end if
      if \(a\) is equal to \((b_1 - a_1) \% e\) (when \((b_1 - a_1) \geq 0\)) and \((b_1 - a_1) + e\) (when \((b_1 - a_1) < 0\))
      AND \(b\) is equal to \((-a_1) \% e\) (when \((-a_1) \geq 0\))
      and \((-a_1) + e\) (when \((-a_1) < 0\)) then
        \(a = a_1, b = b_1\) and \(flag = 0\)
      end if
      if \(a\) is equal to \((-a_1) \% e\) (when \((-a_1) \geq 0\)) and \((-a_1) + e\) (when \((-a_1) < 0\))
      AND \(b\) is equal to \((b_1 - a_1) \% e\) (when \((b_1 - a_1) \geq 0\))
      and \((b_1 - a_1) + e\) (when \((b_1 - a_1) < 0\)) then
        \(a = a_1, b = b_1\) and \(flag = 0\)
      end if
      if \(a\) is equal to \((-b_1) \% e\) (when \((-b_1) \geq 0\)) and \((-b_1) + e\) (when \((-b_1) < 0\))
      AND \(b\) is equal to \((a_1 - b_1) \% e\) (when \((a_1 - b_1) \geq 0\))
      and \((a_1 - b_1) + e\) (when \((a_1 - b_1) < 0\)) then
        \(a = a_1, b = b_1\) and \(flag = 0\)
      end if
    end if
  end for
end for
Algorithm 2 Determination of equality of cyclotomic numbers of order $2l^2$.

1: START
2: Declare Integer variables $p, q, r, i, j, k, l, i1, j1, a1, b1, flag.$
3: INPUT $l$
4: if $l$ is not a prime or less than 3 then goto 3
5: else
6: $e = 2l^2$
7: end if
8: Declare an array of size $e \times e$, where each element of the array is a 2-tuple structure (i.e., an ordered pair of $(a, b)$, where $a$ and $b$ are integers).
9: INPUT $q$
10: for $p = \text{all prime number within 2 to } q$ do
11: for all $r$ within 1 to $q$ do
12: if $q$ is not equal to $p^r$ then goto 10
13: else if $(q - 1) % e == 0$ then
14: $k = (q - 1) / e$
15: end if
16: end for
17: goto 10
18: else
19: if $k$ even then goto 25
20: end if
21: goto 53
22: end if
23: end for
24: end for
25: end for
26: end for
27: for $i1 = 0$ to $e - 1$ do
28: for $j1 = 0$ to $e - 1$ do
29: int $a1 =$ value of $a$ at current array index i.e. $a1 = arr[i1][j1].a$
30: int $b1 =$ value of $b$ at current array index i.e. $b1 = arr[i1][j1].b$
31: set flag of current element of array to 0 i.e. lock the element which is updated once $arr[i1][j1].flag = 0$
32: end for
33: end for
34: for $i = 0$ to $e - 1$ do
35: for $j = 0$ to $e - 1$ do
36: if flag is 1 i.e. element has not been updated yet then
37: if $a$ is equal to $b1 % e$ (when $b1 \geq 0$) and $b1 + e$ (when $b1 < 0$) AND $b$ is equal to $a1 % e$ (when $a1 \geq 0$) and $a1 + e$ (when $a1 < 0$) then
38: $a = a1, b = b1$ and $flag = 0$
39: end if
40: if $a$ is equal to $(-a1-b1) % e$ (when $a1-b1 \geq 0$) and $a1-b1 + e$ (when $a1-b1 < 0$) AND $b$ is equal to $(-b1) % e$ (when $-b1 \geq 0$) and $-b1 + e$ (when $-b1 < 0$) then
41: $a = a1, b = b1$ and $flag = 0$
42: end if
if $a$ is equal to $(b1 - a1)\% e$ (when $(b1 - a1) \geq 0$) and $(b1 - a1) + e$ (when $(b1 - a1) < 0$) AND $b$ is equal to $(-a1)\% e$ (when $(-a1) \geq 0$) and $(-a1) + e$ (when $(-a1) < 0$) then
  $a = a1$, $b = b1$ and $flag = 0$
end if

if $a$ is equal to $(-a1)\% e$ (when $(-a1) \geq 0$) and $(-a1) + e$ (when $(-a1) < 0$) AND $b$ is equal to $(b1 - a1)\% e$ (when $(b1 - a1) \geq 0$) and $(b1 - a1) + e$ (when $(b1 - a1) < 0$) then
  $a = a1$, $b = b1$ and $flag = 0$
end if

if $a$ is equal to $(-b1)\% e$ (when $(-b1) \geq 0$) and $(-b1) + e$ (when $(-b1) < 0$) AND $b$ is equal to $(a1 - b1)\% e$ (when $(a1 - b1) \geq 0$) and $(a1 - b1) + e$ (when $(a1 - b1) < 0$) then
  $a = a1$, $b = b1$ and $flag = 0$
end if

end for

for $i1 = 0$ to $e - 1$ do
  for $j1 = 0$ to $e - 1$ do
    int $a1 = $ value of $a$ at current array index i.e. $a1 = arr[i1][j1].a$
    int $b1 = $ value of $b$ at current array index i.e. $b1 = arr[i1][j1].b$
    set flag of current element of array to 0 i.e. lock the element which is updated once $arr[i1][j1].flag = 0$
  end for
end for

for $i = 0$ to $e - 1$ do
  for $j = 0$ to $e - 1$ do
    if flag is 1 i.e. element has not been updated yet then
      if $a$ is equal to $(b1 + l^2)\% e$ (when $(b1 + l^2) \geq 0$) and $(b1 + l^2) + e$ (when $(b1 + l^2) < 0$) AND $b$ is equal to $(a1 + l^2)\% e$ (when $(a1 + l^2) \geq 0$) and $(a1 + l^2) + e$ (when $(a1 + l^2) < 0$) then
        $a = a1$, $b = b1$ and $flag = 0$
      end if
    end if
    if $a$ is equal to $(l^2 + a1 - b1)\% e$ (when $(l^2 + a1 - b1) \geq 0$) and $(l^2 + a1 - b1) + e$ (when $(l^2 + a1 - b1) < 0$) AND $b$ is equal to $(-b1)\% e$ (when $(-b1) \geq 0$) and $(-b1) + e$ (when $(-b1) < 0$) then
      $a = a1$, $b = b1$ and $flag = 0$
    end if
    if $a$ is equal to $(l^2 + b1 - a1)\% e$ (when $(l^2 + b1 - a1) \geq 0$) and $(l^2 + b1 - a1) + e$ (when $(l^2 + b1 - a1) < 0$) AND $b$ is equal to $(l^2 - a1)\% e$ (when $(l^2 - a1) \geq 0$) and $(l^2 - a1) + e$ (when $(l^2 - a1) < 0$) then
      $a = a1$, $b = b1$ and $flag = 0$
    end if
    if $a$ is equal to $(-a1)\% e$ (when $(-a1) \geq 0$) and $(-a1) + e$ (when $(-a1) < 0$) AND $b$ is equal to $(b1 - a1)\% e$ (when $(b1 - a1) \geq 0$) and $(b1 - a1) + e$ (when $(b1 - a1) < 0$) then
      $a = a1$, $b = b1$ and $flag = 0$
    end if
  end for
end for

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77: if $a$ is equal to $(l^2 - b1)\%e$ (when $(l^2 - b1) \geq 0$) and $(l^2 - b1) + e$ (when $(l^2 - b1) < 0$) AND $b$ is equal to $(a1 - b1)\%e$ (when $(a1 - b1) \geq 0$) and $(a1 - b1) + e$ (when $(a1 - b1) < 0$) then
78: $a = a1$, $b = b1$ and flag = 0
79: end if
80: end if
81: end for
82: end for

The above algorithms demonstrate that for the determination of Jacobi sums of orders $2l^2$ and $l^2$ with prime $l \geq 5$, it is adequate to determine $2l^2 + (2l^2 - 1)(2l^2 - 2)/6$ and $l^2 + (l^2 - 1)(l^2 - 2)/6$ cyclotomic numbers of orders $2l^2$ and $l^2$ respectively. However for $l = 3$, it is sufficient to ascertain 64 and 19 cyclotomic numbers of orders $2l^2$ and $l^2$ respectively. Thus, it reduces the complexity of order $l^2$ to $l^4 - (l^2 + (l^2 - 1)(l^2 - 2)/6)$ for $l > 3$ and $l^4 - 19$ for $l = 3$ and of order $2l^2$ to $4l^4 - (2l^2 + (2l^2 - 1)(2l^2 - 2)/6)$ for $l > 3$ and $4l^4 - 64$ for $l = 3$. So, it could be easily observed that, for a large value of $l$, complexity for the determination of Jacobi sums reduces drastically.

### Table 1. Complexity comparison of order $l^2$

| Corresponding value of $l$ | Order $l^2$ |
|---------------------------|-------------|
|                           | Required number of cyclotomic numbers need to determine | Actual number of cyclotomic numbers need to determine | Number of reduced computations for Jacobi sums |
| $l = 3$                   | 81          | 19          | 62          |
| $l = 5$                   | 625         | 117         | 508         |
| $l = 7$                   | 2401        | 425         | 1976        |
| $l = 11$                  | 14641       | 2501        | 12140       |
| $l = 13$                  | 28561       | 4845        | 23716       |

### Table 2. Complexity comparison of order $2l^2$

| Corresponding value of $l$ | Order $2l^2$ |
|---------------------------|-------------|
|                           | Required number of cyclotomic numbers need to determine | Actual number of cyclotomic numbers need to determine | Number of reduced computations for Jacobi sums |
| $l = 3$                   | 324         | 64          | 260         |
| $l = 5$                   | 2500        | 442         | 2058        |
| $l = 7$                   | 9604        | 1650        | 7954        |
| $l = 11$                  | 58564       | 9882        | 48682       |
| $l = 13$                  | 114244      | 19210       | 95034       |
```
As illustrated in tables 1 and 2 for $l = 3$, naively summing the definition, we need to evaluate 81 and 324 numbers of cyclotomic numbers of orders $l^2$ and $2l^2$ respectively to determine the Jacobi sums of respective orders. But algorithms 1 and 2 validate that it is sufficient to evaluate only 19 and 64 numbers of cyclotomic numbers of order 9 and 18 respectively. Thus complexity of implementing algorithms for Jacobi sums reduces by 62 and 260 respectively. Also one can observe that, as the value of $l$ increases, the corresponding complexity reduces drastically. Consequently, efficiency of our implemented algorithms 3, 4, 5 increases as value of $l$ rises.

4. Expressions for Jacobi sums in terms of cyclotomic numbers

Here, we present the expressions for Jacobi sums of orders $l^2$ and $2l^2$, $l$ an odd prime, in terms of the minimum number of cyclotomic numbers of orders $l^2$ and $2l^2$ respectively.

**Theorem 4.1.** Let $l$ be an odd prime and $p$ a prime. For some positive integers $r$ and $k$, let $q = p^r = l^2k + 1$.

Then for $l \geq 5$

$$J_{l^2}(1, n) = \sum_{a=0}^{l^2-1-1} \sum_{b=0}^{l^2-1} (a, b)_{l^2} \zeta_{l^2}^{a+bn}$$

$$= (0, 0)_{l^2} + \sum_{b=1}^{l^2-1} (0, b)_{l^2} (\zeta_{l^2}^{bn} + \zeta_{l^2}^{b} + \zeta_{l^2}^{-b(n+1)}) + \left\{ \sum_{b=2}^{l^2-3} (1, b)_{l^2} + \sum_{b=4}^{l^2-3} (2, b)_{l^2} \right. $$

$$+ \sum_{b=6}^{l^2-4} (3, b)_{l^2} + \cdots + \sum_{b=(2l^2-2)/3}^{(2l^2-2)/3} ((l^2 - 1)/3, b)_{l^2} \right\} \left( \zeta_{l^2}^{an+b} + \zeta_{l^2}^{a+bn} 

+ \zeta_{l^2}^{a-b(n+1)} + \zeta_{l^2}^{an-b(n+1)} + \zeta_{l^2}^{bn-a(n+1)} + \zeta_{l^2}^{b-a(n+1)} \right). \quad (4.1)$$

and for $l = 3$

$$J_9(1, n) = \sum_{a=0}^{8} \sum_{b=0}^{8} (a, b)_{9} \zeta_{9}^{a+bn}$$

$$= (0, 0)_{9} + (3, 6)_{9} (\zeta_{9}^{3+6n} + \zeta_{9}^{6+3n}) + \sum_{b=1}^{8} (0, b)_{9} (\zeta_{9}^{bn} + \zeta_{9}^{b} + \zeta_{9}^{-b(n+1)})$$

$$+ \left\{ \sum_{b=2}^{7} (1, b)_{9} + \sum_{b=4}^{6} (2, b)_{9} \right\} \left( \zeta_{9}^{an+b} + \zeta_{9}^{a+bn} + \zeta_{9}^{a-b(n+1)} + \zeta_{9}^{an-b(n+1)}$$

$$+ \zeta_{9}^{bn-a(n+1)} + \zeta_{9}^{b-a(n+1)} \right). \quad (4.2)$$

**Proof.** Cyclotomic numbers $(a, b)_{l^2}$ of order $l^2$ over $\mathbb{F}_q$ is defined in [23] as:
\[ (a, b)_{l^2} := \#\{v \in \mathbb{F}_q \setminus \{0, -1\} \mid \text{ind}_p v \equiv a \pmod{l^2}, \text{ind}_q (v + 1) \equiv b \pmod{l^2}\}. \]

In [24], it was proved that \((a, b)_e\) has the following properties:

\[(a, b)_e = (e - a, b - a)_e, \quad (a, b)_e = (b, a)_e, \quad 2|k \text{ or } q = 2^r \quad (4.3)\]

and

\[(a, b)_e = (b + e/2, a + e/2)_e, \quad (a, b)_e = (e - a, b - a)_e, \quad \text{otherwise.} \quad (4.4)\]

For \(q = l^2k + 1\) for some positive integer \(k\), it is natural to see that \(k\) is always even. Now to permute only \((4.3)\), one gets

\[(a, b)_{l^2} = (b, a)_{l^2} = (a - b, -b)_{l^2} = (b - a, -a)_{l^2} = (-a, b - a)_{l^2} = (-b, a - b)_{l^2}. \quad (4.5)\]

We know that \((a, b)_{l^2}\) and \(J_{l^2}(1, n)\) are well connected by the following relations:

\[ J_{l^2}(1, n) = \sum_{a=0}^{l^2 - 1} \sum_{b=0}^{l^2 - 1} (a, b)_{l^2} \zeta_{l^2}^{a + bn} \quad (4.6) \]

Thus by \((4.5)\), cyclotomic numbers \((a, b)_{l^2}\) of order \(l^2\) partition into group of classes. For prime \(l \geq 5\), cyclotomic numbers \((a, b)_{l^2}\) of order \(l^2\) forms classes of singleton, three and six elements. \((0, 0)_{l^2}\) form singleton class, \((-a, 0)_{l^2}\), \((a, a)_{l^2}\), \((0, -a)_{l^2}\) forms a classes of three elements for every \(1 \leq a \leq l^2 - 1 \pmod{l^2}\) and rest of the cyclotomic numbers forms classes of six elements. For \(l = 3\) there are classes of singleton, second, three and six elements. The exception is \((6, 3)_{18} = (3, 6)_{18}\) which is grouped into a class of two elements.

Hence expression \((4.1)\) and \((4.2)\) directly follows by the relation \((4.6)\).

The following result gives an analogous expression for \(l = 3\).

**Theorem 4.2.** Let \(p\) be a prime. For some positive integers \(r\) and \(k\), let \(q = p^r = 18k + 1\).

Then for \(2|k\) or \(q = 2^r\),

\[ J_{18}(1, n) = \sum_{a=0}^{17} \sum_{b=0}^{17} (a, b)_{18} \zeta_{18}^{a + bn} \]

\[ = (0, 0)_{18} + (6, 12)_{18} (\zeta_{18}^{6+12n} + \zeta_{18}^{12+6n}) + \sum_{b=1}^{17} (0, b)_{18} (\zeta_{18}^{bn}) \]

\[ + \zeta_{18}^b + \zeta_{18}^{-b(n+1)} + \left\{ \sum_{b=2}^{16} (1, b)_{18} + \sum_{b=4}^{15} (2, b)_{18} + \sum_{b=6}^{14} (3, b)_{18} \right\} \]

\[ + \sum_{b=8}^{13} (4, b)_{18} + \sum_{b=9}^{12} (5, b)_{18} \left\{ \zeta_{18}^{an+b} + \zeta_{18}^{a+b(n+1)} + \zeta_{18}^{a-b(n+1)} \right\} \]

\[ + \zeta_{18}^{an-b(n+1)} + \zeta_{18}^{bn-a(n+1)} + \zeta_{18}^{b-a(n+1)} \] \quad (4.7)
and for $2 \nmid k$ and $q \neq 2^r$,

$$J_{18}(1, n) = \sum_{a=0}^{17} \sum_{b=0}^{17} (a, b)_{18} \zeta_{18}^{a+bn}$$

$$= (0, 9)_{18} \zeta_{18}^{9n} + (6, 3)_{18} (\zeta_{18}^{6+3n} + \zeta_{18}^{12+15n}) + \left\{ \sum_{b=0}^{8} (0, b)_{18} + \sum_{b=10}^{17} (0, b)_{18} \right\}$$

$$\{ \zeta_{18}^{n} + \zeta_{18}^{9(n+1)+b} + \zeta_{18}^{9-b(n+1)} \} + \left\{ \sum_{b=0}^{8} (1, b)_{18} + \sum_{b=12}^{17} (1, b)_{18} + \sum_{b=0}^{7} (2, b)_{18} \right\}$$

$$+ \sum_{b=14}^{17} (2, b)_{18} + \sum_{b=0}^{6} (3, b)_{18} + \sum_{b=16}^{17} (3, b)_{18} + \sum_{b=0}^{5} (4, b)_{18} + \sum_{b=1}^{3} (5, b)_{18}$$

$$\left\{ \zeta_{18}^{a+bn} + \zeta_{18}^{an+b+9(n+1)} + \zeta_{18}^{9+a-b(n+1)} + \zeta_{18}^{(n+1)9+b-a(n+1)} + \zeta_{18}^{bn-a(n+1)} \right\}.$$  \hspace{1cm} (4.8)

**Proof.** We recall the following from [2]:

$$(a, b)_{18} := \#\{ v \in \mathbb{F}_q \setminus \{0, -1\} \mid \text{ind}_v a \equiv (a, b)_{18}, \text{ind}_v (v+1) \equiv b \pmod{18} \}.$$ 

The following properties were derived in [2]:

$$(a, b)_{18} = (18 - a - b)_{18}, \quad (a, b)_{18} = (b, a)_{18}, \quad 2 \nmid k \text{ or } q = 2^r,$$ \hspace{1cm} (4.9)

and

$$(a, b)_{18} = (b+9, a+9)_{18}, \quad (a, b)_{18} = (18-a, b-a)_{18}, \quad \text{otherwise.}$$ \hspace{1cm} (4.10)

We permute (4.9) and (4.10) to get

$$(a, b)_{18} = (b, a)_{18} = (a-b, -b)_{18} = (b-a, -a)_{18} = (-a, -b-a)_{18} = (-b, a-b)_{18}$$ \hspace{1cm} (4.11)

and

$$(a, b)_{18} = (b+9, a+9)_{18} = (9+a-b, -b)_{18} = (9+b-a, 9-a)_{18} = (-a, b-a)_{18} = (9-b, a-b)_{18}$$ \hspace{1cm} (4.12)

respectively.

It is easy to see that $(a, b)_{18}$ and $J_{18}(1, n)$ are well-connected by the following:

$$J_{18}(1, n) = \sum_{a=0}^{17} \sum_{b=0}^{17} (a, b)_{18} \zeta_{18}^{a+bn}.$$ \hspace{1cm} (4.13)

Thus by (4.11) and (4.12), cyclotomic numbers $(a, b)_{18}$ of order 18 partition into classes. If $2 \mid k$ or $q = 2^r$, (4.11) gives classes of singleton, two, three and six elements. $(0, 0)_{18}$ forms a singleton class, $(-a, 0)_{18}, (a, a)_{18},$
(0, −a)_{18} forms classes of three elements for every 1 ≤ a ≤ 17 (mod 18), (6, 12)_{18} = (12, 6)_{18} which is grouped into a class of two elements and rest of the cyclotomic numbers forms classes of six elements. Hence expression (4.7) directly follows from the relation (4.13).

Now if neither 2 | k nor q = 2^r, (4.12) gives classes of singleton, two, three and six elements. (0, 9)_{18} forms a singleton class, (0, a)_{18}, (a + 9, 9)_{18}, (9 − a, −a)_{18} forms classes of three elements for every 0 ≤ a ≠ 9 ≤ 17 (mod 18), (6, 3)_{18} = (12, 15)_{18} which is grouped into a class of two elements and rest of the cyclotomic numbers forms classes of six elements. Hence expression (4.8) directly follows by the relation (4.13).

\[ \text{Remark 4.1.} \quad \# \left\{ (0, 0)_{18} + (6, 12)_{18} + \sum_{b=1}^{17} (0, b)_{18} + \sum_{b=2}^{16} (1, b)_{18} + \sum_{b=4}^{15} (2, b)_{18} + \sum_{b=6}^{14} (3, b)_{18} + \sum_{b=8}^{13} (4, b)_{18} + \sum_{b=10}^{12} (5, b)_{18} \right\} = \# \left\{ (0, 9)_{18} + (6, 3)_{18} + \sum_{b=0}^{8} (0, b)_{18} + \sum_{b=0}^{7} (2, b)_{18} + \sum_{b=4}^{17} (2, b)_{18} + \sum_{b=0}^{6} (3, b)_{18} + \sum_{b=10}^{17} (3, b)_{18} + \sum_{b=0}^{5} (4, b)_{18} + \sum_{b=1}^{3} (5, b)_{18} \right\}. \]

\[ \text{Remark 4.2. If} \quad 2 | k \text{ or } q = 2^r, \text{the sum of all } (a, b)_{18}, \quad 0 \leq a, b \leq 17 \text{ is equal to} \quad \left\{ (0, 0)_{18} + 2(6, 12)_{18} + 3 \sum_{b=1}^{17} (0, b)_{18} + 6 \sum_{b=2}^{16} (1, b)_{18} + \sum_{b=1}^{14} (2, b)_{18} + \sum_{b=6}^{13} (3, b)_{18} + \sum_{b=8}^{12} (4, b)_{18} + \sum_{b=10}^{12} (5, b)_{18} \right\} = q - 2. \]

\[ \text{Remark 4.3. If} \quad 2 \nmid k \text{ and } q \neq 2^r, \text{the sum of all } (a, b)_{18}, \quad 0 \leq a, b \leq 17 \text{ is equal to} \quad \left\{ (0, 9)_{18} + 2(6, 3)_{18} + 3 \left( \sum_{b=0}^{8} (0, b)_{18} + \sum_{b=10}^{17} (0, b)_{18} \right) + 6 \left( \sum_{b=0}^{8} (1, b)_{18} + \sum_{b=12}^{17} (1, b)_{18} + \sum_{b=0}^{7} (2, b)_{18} + \sum_{b=14}^{17} (2, b)_{18} + \sum_{b=0}^{6} (3, b)_{18} + \sum_{b=16}^{17} (3, b)_{18} + \sum_{b=0}^{5} (4, b)_{18} + \sum_{b=1}^{3} (5, b)_{18} \right) \right\} = q - 2. \]
Theorem 4.3. Let \( l \geq 5 \) and \( p \) be primes. For some positive integers \( r \) and \( k \), let \( q = p^r = 2l^2k + 1 \).

Then for \( 2|k \) or \( q = 2^r \),

\[
J_{2l^2}(1, n) = \sum_{a=0}^{2l^2-1} \sum_{b=0}^{2l^2-1} (a, b)_{2l^2} \zeta_{2l^2}^a \zeta_{2l^2}^b n
\]

\[
= (0, 0)_{2l^2} + \sum_{b=1}^{2l^2-1} (0, b)_{2l^2} (\zeta_{2l^2}^b + \zeta_{2l^2}^{-b} + \zeta_{2l^2}^{-b(n+1)}) + \left\{ \sum_{b=2}^{2l^2-2} (1, b)_{2l^2} \right\}
\]

\[
+ \sum_{b=1}^{2l^2-3} (2, b)_{2l^2} + \sum_{b=6}^{2l^2-4} (3, b)_{2l^2} + \cdots + \sum_{b=(4l^2-4)/3}^{(4l^2-4)/3+1} ((2l^2 - 2)/3, b)_{2l^2} \right\}
\]

\[
\left\{ \zeta_{2l^2}^{an+b} + \zeta_{2l^2}^{a+b(n+1)} + \zeta_{2l^2}^{an-b(n+1)} + \zeta_{2l^2}^{bn-a(n+1)} + \zeta_{2l^2}^{b-a(n+1)} \right\}.
(4.14)
\]

and for \( 2 \nmid k \) and \( q \neq 2^r \),

\[
J_{2l^2}(1, n) = \sum_{a=0}^{2l^2-1} \sum_{b=0}^{2l^2-1} (a, b)_{2l^2} \zeta_{2l^2}^a \zeta_{2l^2}^b n
\]

\[
= (0, l^2)_{2l^2} \zeta_{2l^2}^{l^2 n} + \left\{ \sum_{b=0}^{l^2-1} (0, b)_{2l^2} + \sum_{b=l^2+1}^{2l^2-1} (0, b)_{2l^2} \right\} \left\{ \zeta_{2l^2}^{bn} + \zeta_{2l^2}^{b(n+1)} \right\}
\]

\[
+ \zeta_{2l^2}^{l^2-b(n+1)} \right\} + \left\{ \sum_{b=0}^{l^2-1} (1, b)_{2l^2} + \sum_{b=l^2+3}^{2l^2-1} (1, b)_{2l^2} + \sum_{b=0}^{l^2-2} (2, b)_{2l^2} \right\}
\]

\[
+ \sum_{b=l^2+5}^{2l^2-1} (2, b)_{2l^2} + \sum_{b=l^2+1}^{l^2-3} (3, b)_{2l^2} + \sum_{b=l^2+1}^{2l^2-1} (3, b)_{2l^2} + \cdots
\]

\[
+ \sum_{b=0}^{(l^2-3)/2} (((l^2 - 1)/2) - 1, b)_{2l^2} + \sum_{b=2l^2-2}^{2l^2-1} (((l^2 - 1)/2) - 1, b)_{2l^2} \right\}
\]

\[
+ \sum_{b=1}^{(l^2-3)/2} (((l^2 - 1)/2) + 1, b)_{2l^2} + \sum_{b=3}^{((l^2-3)/2)-1} (((l^2 - 1)/2) + 2, b)_{2l^2} \right\}
\]

\[
+ \sum_{b=5}^{((l^2-3)/2)-2} (((l^2 - 1)/2) + 3, b)_{2l^2} + \sum_{b=7}^{((l^2-3)/2)-3} (((l^2 - 1)/2) + 4, b)_{2l^2} \right\}
\]

\[
+ \cdots + \sum_{b=((l^2-3)/2)-((l^2-7)/6)-1}^{((l^2-3)/2)-((l^2-7)/6)} (((l^2 - 1)/2) + ((l^2 - 1)/6), b)_{2l^2} \right\}.
\begin{equation}
\begin{aligned}
&+ \sum_{b=0}^{(l^2+1)/2} ((l^2 - 1)/2, b)_{2l^2} \left\{ \zeta_{2l^2}^{a+bn} + \zeta_{2l^2}^{b+(n+1)l^2+an} + \zeta_{2l^2}^{l^2+a-b(n+1)} + \zeta_{2l^2}^{b+an(b(n+1)} \right. \\
&+ \left. \zeta_{2l^2}^{b+a(n+1)} + \zeta_{2l^2}^{b-a(n+1)} + \zeta_{2l^2}^{l^2+a-n(b(n+1))} \right\}. 
\end{aligned}
\end{equation}

\textit{Proof.} The cyclotomic numbers \((a, b)_{2l^2}\) of order \(2l^2\) over \(\mathbb{F}_q\) is defined in \([2]\) as follows:
\((a, b)_{2l^2} := \# \{v \in \mathbb{F}_q \setminus \{0, -1\} \mid \text{ind}_v \equiv a \pmod{2l^2}, \text{ind}_v(v+1) \equiv b \pmod{2l^2} \}.
\)

We now recall the following properties of \((a, b)_{2l^2}\) from \([2]\):
\begin{equation}
(a, b)_{2l^2} = (2l^2-a, b-a)_{2l^2}, \quad (a, b)_{2l^2} = (b, a)_{2l^2}, \quad \text{if } k \text{ is even or } q = 2^r.
\end{equation}

and
\begin{equation}
(a, b)_{2l^2} = (b+l^2, a+l^2)_{2l^2}, \quad (a, b)_{2l^2} = (2l^2-a, b-a)_{2l^2}, \quad \text{otherwise}.
\end{equation}

By permuting \((4.16)\) and \((4.17)\), we obtain
\begin{equation}
(a, b)_{2l^2} = (b, a)_{2l^2} = (a-b, -b)_{2l^2} = (b-a, -a)_{2l^2} = (-a, b-a)_{2l^2} = (-b, a-b)_{2l^2}
\end{equation}

and
\begin{equation}
(a, b)_{2l^2} = (b+l^2, a+l^2)_{2l^2} = (l^2 + a - b, -b)_{2l^2} = (l^2 + b - a, l^2 - a)_{2l^2}
\end{equation}

\begin{equation}
= (-a, b-a)_{2l^2} = (l^2 - b, a-b)_{2l^2}.
\end{equation}

We know that \((a, b)_{2l^2}\) and \(J_{2l^2}(1, n)\) are well connected by the following:
\begin{equation}
J_{2l^2}(1, n) = \sum_{a=0}^{2l^2-1} \sum_{b=0}^{2l^2-1} (a, b)_{2l^2} \zeta_{2l^2}^{a+bn}
\end{equation}

Thus by \((4.18)\) and \((4.19)\), cyclotomic numbers \((a, b)_{2l^2}\) of order \(2l^2\) partition into group of classes. If \(2|k\) or \(q = 2^r\), \((4.18)\) gives classes of singleton, three and six elements. \((0, 0)_{2l^2}\) forms a singleton class, \((-a, 0)_{2l^2}\), \((a, a)_{2l^2}\), \((0, -a)_{2l^2}\) forms classes of three elements for every \(1 \leq a \leq 2l^2 - 1 \pmod{2l^2}\) and rest of the cyclotomic numbers forms classes of six elements. Hence expression \((4.14)\) directly follows by the relation \((4.20)\).

Now if neither \(2|k\) nor \(q = 2^r\), \((4.19)\) forms classes of singleton, three and six elements. \((0, l^2)_{2l^2}\) forms a singleton class, \((0, a)_{2l^2}\), \((a + l^2, l^2)_{2l^2}\), \((l^2 - a, -a)_{2l^2}\) forms classes of three elements for every \(0 \leq a \neq l^2 \leq 2l^2 - 1 \pmod{2l^2}\) and rest of the cyclotomic numbers forms classes of six elements. Hence expression \((4.15)\) directly follows by the relation \((4.20)\).

\textit{Remark 4.4.} \# \left\{ (0, 0)_{2l^2} + \sum_{b=1}^{2l^2-1} (0, b)_{2l^2} + \sum_{b=2}^{2l^2-2} (1, b)_{2l^2} + \sum_{b=4}^{2l^2-3} (2, b)_{2l^2} + \sum_{b=6}^{2l^2-4} (3, b)_{2l^2} + \cdots + \sum_{b=(4l^2-4)/3}^{(4l^2-4)/3+1} ((2l^2-2)/3, b)_{2l^2} \right\} = \# \left\{ (0, l^2)_{2l^2} + \sum_{b=0}^{l^2-1} (0, b)_{2l^2} + \cdots \right\}
\[ \sum_{b=0}^{2l-3}(3, b)_{2l^2} + \sum_{b=0}^{2l-1}(3, b)_{2l^2} + \sum_{b=0}^{2l^2-1}(3, b)_{2l^2} + \cdots + \sum_{b=0}^{2l^2-1}(l^2-3)/2 \left( \left( \left( l^2-1 \right)/2 \right) - 1, b \right)_{2l^2} + \sum_{b=0}^{2l^2-1}(l^2-3)/2 \left( \left( l^2+1 \right)/2, b \right)_{2l^2} + \sum_{b=3}^{(l^2-3)/2-1} \left( \left( l^2+1 \right)/2 + 2, b \right)_{2l^2} + \sum_{b=7}^{(l^2-3)/2-3} \left( \left( l^2+1 \right)/2 + 3, b \right)_{2l^2} + \cdots + \sum_{b=(l^2-3)/2-(l^2-7)/6}^{(l^2+1)/2-1} \left( \left( l^2+1 \right)/2 + \left( l^2-7 \right)/6, b \right)_{2l^2} + \sum_{b=0}^{(l^2+1)/2} \left( \left( l^2-1 \right)/2, b \right)_{2l^2} \right) . \]

Remark 4.5. If \( 2 \mid k \) or \( q = 2^r \), then the sum of all \( (a, b)_{2l^2} \), \( 0 \leq a, b \leq (2l^2-1) \) and \( l \geq 5 \) is equal to \( \left\{ (0, 0)_{2l^2} + 3 \sum_{b=1}^{2l^2-1}(0, b)_{2l^2} + 6 \left( \sum_{b=2}^{2l^2-2}(1, b)_{2l^2} + \sum_{b=1}^{2l^2-3}(2, b)_{2l^2} + \sum_{b=6}^{2l^2-4}(3, b)_{2l^2} + \cdots + \sum_{b=(4l^2-4)/3+1}^{(2l^2-3)/2}(3, b)_{2l^2} \right) \right\} = q - 2 . \)

Remark 4.6. If \( 2 \nmid k \) and \( q \neq 2^r \), then the sum of all \( (a, b)_{2l^2} \), \( 0 \leq a, b \leq (2l^2-1) \) and \( l \geq 5 \) is equal to \( \left\{ (0, 0)_{2l^2} + 3 \left( \sum_{b=1}^{l^2-1}(0, b)_{2l^2} + \sum_{b=l^2+1}^{2l^2-1}(0, b)_{2l^2} \right) + 6 \left( \sum_{b=1}^{l^2-1}(1, b)_{2l^2} + \sum_{b=l^2+3}^{2l^2-1}(1, b)_{2l^2} + \sum_{b=0}^{l^2-2}(2, b)_{2l^2} + \sum_{b=l^2+5}^{2l^2-1}(2, b)_{2l^2} + \sum_{b=0}^{2l^2-3}(3, b)_{2l^2} + \sum_{b=l^2+7}^{3l^2-1}(3, b)_{2l^2} + \cdots + \sum_{b=0}^{l^2-2}(l^2-3)/2 \left( \left( l^2-1 \right)/2 \right) - 1, b \right)_{2l^2} + \sum_{b=2l^2-2}^{2l^2-1}(l^2-1)/2 - 1, b \right)_{2l^2} + \sum_{b=1}^{l^2-3/2}(l^2+1)/2, b \right)_{2l^2} + \sum_{b=3}^{(l^2-3)/2-1}(l^2+1)/2 + 2, b \right)_{2l^2} + \sum_{b=7}^{(l^2-3)/2-3}(l^2+1)/2 + 3, b \right)_{2l^2} + \cdots + \sum_{b=(l^2-3)/2-(l^2-7)/6}^{(l^2+1)/2-1}(l^2+1)/2 + (l^2-7)/6, b \right)_{2l^2} + \sum_{b=0}^{(l^2+1)/2} \left( \left( l^2-1 \right)/2, b \right)_{2l^2} \right) \right\} = q - 2 . \)

5. Fast computational algorithms for Jacobi sums

In a given finite field \( \mathbb{F}_q \), Jacobi sums of order \( e \), mainly depend on two parameters. Therefore, these values could be naturally assembled into a matrix of order \( e \). For \( e = l^2 \) or \( 2l^2 \), we know that by knowing the Jacobi sums \( J_e(1, n), 0 \leq n \leq (e-1) \), one can readily determine all the Jacobi sums of the respective order \( e \). We implement algorithms for fast computation of \( J_{l^2}(1, n) \) and \( J_{2l^2}(1, n) \).

Throughout the algorithms, structure of individual term of a polynomial is by means of class structure “term” (which is of the form \( c \zeta_d^e ; e = l^2 \) or \( 2l^2 \)) and a different structure for a polynomial by means of a class structure “pol”. Further “pol * \( p_0 \)” is a variable pointing to the resulting polynomial or say the master polynomial, “pol * \( p_{l^2} \)” is again a variable pointing to
keep a polynomial temporarily. The function add_poly adds two polynomial expression or add a term with a polynomial.

Every time we declare a term, we need to assign the value of its coefficient and exponent. The function check_sign_of_exp() will check the sign of each of the exponent of input expression and if it has been found to be negative then add $2l^2$ (for input expression of order $2l^2$) or $l^2$ (for input expression of order $l^2$) to the corresponding exponent.

Further function check_break_replace(), first checks whether the term has exponent greater than or equals to $l(l-1)$, if so then breaks the exponent into a power of $l(l-1)$ and then replaces each of the polynomial whose exponent is equal to $l(l-1)$ by polynomial $*p_t$, where

$$*p_t = \begin{cases} 
1 - \zeta_{l^2}^l + \zeta_{l^2}^{2l} - \zeta_{l^2}^{3l} + \cdots - \zeta_{l^2}^{l(l-2)} & \text{if expression is of order } l^2, \\
-1 + \zeta_{l^2}^l - \zeta_{l^2}^{2l} + \zeta_{l^2}^{3l} + \cdots + \zeta_{l^2}^{l(l-2)} & \text{if expression is of order } 2l^2.
\end{cases}$$

**Algorithm 3** Determination of Jacobi sums of order $l^2$

1: START
2: Input expression 4.1 if $l > 3$; otherwise expression 4.2
3: class term
4:  int coff
5:  int exp
6:  check_sign_of_exp()
7:  check_break_replace()
8: class poly
9:  term *t
10:  int degree
11: int main()
12:  Declare integer variable itr=0, c, n, min=1, max=$l^2 - 1$, a=0, b, l
13:  poly *pa //Declare master polynomial
14:  poly *pt // Declare a temporary polynomial
15:  INPUT n and $l$
16:  /*------------------------------*/
17:  INPUT value of (0,0) $\rightarrow$ c
18:  term t1 //Declare a term
19:  t1.coff=c
20:  add_poly(pa, t1) // add a term in polynomial
21:  /* line number 22–32 required to evaluate particular case of order $l^2$, considering $l = 3$ */
22:  INPUT value of (3,6) $\rightarrow$ c
23:  term t2
24:  t2.coff=c
25:  t2.exp=3+6n
26:  t2.check_break_replace (pt)
27:  add_poly(pa, pt) // add two polynomial
As discussed in section 3, classes of cyclotomic numbers of order $l^2$ differ for different values of $l$. For the chosen value of $l \geq 5$, classes of cyclotomic numbers remain same but for $l = 3$, it forms an additional class, which is a class of two elements. Algorithm 3 determine all the Jacobi sums of order $l^2$. If $l = 3$, then initially line number 22-32 is required to evaluate and while loop would execute with a different conditional statement (which is $itr! = 3$).
The condition in While loop should be $itr != 3$ because it forms two different classes of six elements and one class of three elements.

**Algorithm 4** Determination of Jacobi sums of order $2l^2$, if either $2|k$ or $q = 2^r$

1: START  
2: Input expression 4.14 if $l > 3$; otherwise expression 4.7  
3: class term  
4: int coff  
5: int exp  
6: check_sign_ofexpo()  
7: check_break_replace()  
8: class poly  
9: term *t  
10: int degree  
11: int main()  
12: declare integer variable $itr=0$, $c$, $n$, $min=1$, $max=2l^2-1$, $a=0$, $b$, $l$  
13: poly *$p_a$/ Declare master polynomial  
14: poly *$p_t$/ Declare a temporary polynomial  
15: INPUT $n$ and $l$  
16: /*$\\\ldots$/INPUT value of $(0,0) \rightarrow c$/  
17: term t1 / Declare a term  
18: t1.coff=c  
19: add_poly($p_a$, t1) / add a term in polynomial  
20: /* line number 22–32 required to evaluate particular case of order $2l^2$; i.e. $l = 3$ */  
21: INPUT value of $(6,12) \rightarrow c$  
22: term t2  
23: t2.coff=c  
24: t2.exp=6+12$n$  
25: t2.check_break_replace ($p_t$)  
26: add_poly($p_a$, $p_t$) / add two polynomial  
27: term t3  
28: t3.coff=c  
29: t3.exp=12+6$n$  
30: t3. check_break_replace ($p_t$)  
31: add_poly($p_a$, $p_t$)  
32: /*$\\ldots$/ */
while $\text{itr}! = \frac{2l^2 - 2}{3} + 1$ do
  for $b = \text{min}$ to $\text{max}$ and $b++$ do
    INPUT value of $(a, b) \rightarrow c$
    Set limit = 3
    term $t[6]$ // Declaring a term array
    $t[0].coff = t[1].coff = t[2].coff = t[3].coff = t[4].coff = t[5].coff = c$
    $t[0].exp = an + b$
    $t[1].exp = a + bn$
    $t[2].exp = a - b(n+1)$
    if $\text{min} > 1$ then
      $t[3].exp = an - b(n+1)$
      $t[4].exp = bn - a(n+1)$
      $t[5].exp = b - a(n+1)$
      Set limit = 6
    end if
    for $i = 0$ to $(\text{limit}-1)$ and $i++$ do
      $t[i].check\_sign\_of\_exp()$
      $t[i].check\_break\_replace\ (p_t[i])$
      add\_poly\ $(p_a, p_t[i])$
    end for
  end for
  if $\text{min}$ is equal to 1 then
    $\text{min} \leftarrow \text{min} + 1$
  else
    $\text{min} \leftarrow \text{min} + 2$
  end if
  $\text{max} \leftarrow \text{max} - 1$
  $\text{itr} \leftarrow \text{itr} + 1$
  $a \leftarrow a + 1$
end while

Similarly, classes of cyclotomic numbers of order $2l^2$ differ for different values of $l$. For $l \geq 5$, classes of cyclotomic numbers remain same but for $l = 3$, it forms an additional class, which is a class of two elements. Algorithm 4 implemented to determine all the Jacobi sums of order $2l^2$ under the assumption either $2|k$ or $q = 2^r$. If $l = 3$, then initially line number 22-32 is required to evaluate and while loop would execute with a different conditional statement (which is $\text{itr}! = 6$). The condition in While loop should be $\text{itr}! = 6$ because it forms five different classes of six elements and one class of three elements.
Algorithm 5 Determination of Jacobi sums of order $2l^2$, if either $2 \nmid k$ or $q \neq 2^r$

1: START
2: Input expression 4.15 if $l > 3$; otherwise expression 4.8
3: class term
4: int coff
5: int exp
6: check_sign_of_exp
7: check_break_replace()
8: class poly
9: term *t
10: int degree
11: int main()
12: Declare integer variable count1=0, count2=0, c, n, min1=0, max1=$l^2-1$, min2=$l^2+1$, max2=2$l^2-1$, a=0, b, l
13: poly *pa // Declare master polynomial
14: poly *pt // Declare a temporary polynomial
15: INPUT n and l
16: /*////////////////////////////////////////////*/
17: INPUT value of $(0,l^2) \rightarrow c$
18: term t1 // Declare a term
19: t1.coff=c
20: t1.exp=$l^2n$
21: t1.check_break_replace(pt)
22: add_poly(pa, pt) // add two polynomial
23: /* line number 24–34 required to evaluate particular case of order $2l^2$;
   i.e. $l = 3$ */
24: INPUT value of $(6,3) \rightarrow c$
25: term t2
26: t2.coff=c
27: t2.exp=6+3n
28: t2.check_break_replace(pt)
29: add_poly(pa, pt) // add two polynomial
30: term t3
31: t3.coff=c
32: t3.exp=12+15n
33: t3.check_break_replace(pt)
34: add_poly(pa, pt)
35: /*////////////////////////////////////////////*/
36: while count1!=$(l^2-1)/2 + (l^2-1)/6 + 1$ do
37: if count1!=$(l^2-1)/2 + (l^2-1)/6 + 1$ then
38: for b=min1 to max1 and b++ do
39: INPUT value of (a,b) \rightarrow c
40: Set limit=3
41: term t[6] // Declaring a term array
42: t[0].coff=t[1].coff=t[2].coff=t[3].coff=t[4].coff=t[5].coff=c
43: t[0].exp=a+bn
44: t[1].exp=an+b+9(n+1)
45: t[2].exp=9+a-b(n+1)
if \ a > 1 \ then
\quad t[3].exp = 9(n+1) + b - a(n+1)
\quad t[4].exp = b(n-a(n+1))
\quad t[5].exp = 9 + a(n-b(n+1))
\quad \text{Set limit} = 6
\end{array}
\]
min2 = min2 + 2

Algorithm 4 implemented to determine all the Jacobi sums of order $2l^2$ under the assumption that neither $2|k$ nor $q = 2^r$. If $l = 3$, then initially line number 24-34 is required to evaluate and while loop would execute with a different conditional statement (which is count1! = 6). The condition in While loop should be count1! = 6 because it forms five different classes of six elements and one class of three elements.

6. Conclusion

In this article, we exhibited fast computational algorithms for determination of all the Jacobi sums of orders $l^2$ and $2l^2$ with $l \geq 3$ a prime. These algorithms were implemented in a High Performance Computing Lab. To increase the efficiency, we presented explicit expressions for Jacobi sums of orders $l^2$ and $2l^2$ in terms of the minimum number of cyclotomic numbers of respective orders, which has been utilized in implementing the algorithms. Also, we implemented two additional algorithms to validate the minimality of these expressions.

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