Vector bundles with numerically flat reduction on rigid analytic varieties and $p$-adic local systems

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Abstract

We show how to functorially attach continuous $p$-adic representations of the profinite fundamental group to vector bundles with numerically flat reduction on a proper rigid analytic variety over $\mathbb{C}_p$. This generalizes results by Deninger and Werner for vector bundles on smooth algebraic varieties. Our approach uses fundamental results on the pro-étale site of a rigid analytic variety introduced by Scholze. This enables us to get rid of the smoothness condition and to work in the analytic category. Moreover, under some mild conditions, the functor we construct gives a full embedding of the category of vector bundles with numerically flat reduction into the category of continuous $\mathbb{C}_p$-representations. This provides new insights into the $p$-adic Simpson correspondence in the case of a vanishing Higgs-field.

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1 Introduction

In complex geometry the Corlette-Simpson correspondence is a very elaborate theory relating $\mathbb{C}$-local systems on a compact Kähler manifold to semistable Higgs bundles with
vanishing Chern classes (see [Sim92]). The case of a vanishing Higgs field goes back to a theorem of Narasimhan-Seshadri which gives an equivalence of irreducible unitary representations of the topological fundamental group and stable vector bundles of degree 0 on a compact Riemann surface (see [NS65]). This was generalized by Mehta-Ramanathan to general projective manifolds ([MR84]) and then by Uhlenbeck-Yau to compact Kähler manifolds (see [UY86, §8]). The results by Uhlenbeck-Yau give an equivalence of irreducible unitary representations and stable vector bundles $E$ satisfying $\int \omega^{\dim X - 1} \wedge c_1(E) = \int \omega^{\dim X - 2} \wedge c_2(E) = 0$, where $\omega$ is a Kähler-class.

For $p$-adic varieties Faltings in [Fal05] has proposed a theory relating (small) Higgs bundles and so called (small) generalized representations (see in particular [Fal05, Theorem 6]). This is expanded in the work [ACT16]. Moreover, in [LZ16], using the methods from [Sch13a] and [KL16], Liu and Zhu have constructed a functor from the category of $\mathbb{Q}_p$-local systems on a smooth rigid analytic variety $X$ over a finite extension $K$ of $\mathbb{Q}_p$, to the category of Higgs bundles on $X_{\mathbb{K}}^\hat{}$ (see [LZ16 Theorem 2.1]). What is still missing from these approaches is the other direction. In Faltings’s approach the problem is to determine the Higgs-bundles whose associated generalized representations come from actual representations. One may expect that, similarly to the complex situation, this should be related to a semistability condition on the Higgs bundle. In the zero Higgs field case, such a semistability condition was found by Deninger and Werner (cf. [DW05b], [DW17]) for vector bundles on proper smooth algebraic varieties over $\bar{\mathbb{Q}}_p$. Namely, they construct a functor from the category of vector bundles, which possess an integral model whose special fiber is a numerically flat vector bundle, to the category of continuous $\mathbb{C}_p$-representations of the étale fundamental group. We remark here that on a complex compact Kähler manifold numerically flat bundles are precisely the semistable bundles satisfying $\int \omega^{\dim X - 1} \wedge c_1(E) = \int \omega^{\dim X - 2} \wedge c_2(E) = 0$ (see [DPS94, Theorem 1.18]). So the condition one is familiar with from complex geometry shows up here as a condition on the special fiber of a model.

The main goal of this article is to develop a new approach to the Deninger-Werner correspondence via the pro-étale site introduced by Scholze in [Sch13a]. Using this approach we can get rid of many assumptions in the results of [DW17]. In particular we do not need any smoothness of the variety $X$. Also, we don’t need to assume that it is defined over a finite extension of $\mathbb{Q}_p$. Moreover our construction now takes place in the analytic category, so that we can assume $X$ to be any proper connected rigid analytic variety over $\mathbb{C}_p$. We consider vector bundles $E$ on $X$, for which there exists a proper flat formal scheme $\mathcal{X}$ with generic fiber $X$ over $\text{Spf}(\mathcal{O}_{\mathbb{C}_p})$ and a vector bundle $\mathcal{E}$ on $\mathcal{X}$ with generic fiber $E$, such that $\mathcal{E} \otimes \mathbb{F}_p$ is a numerically flat vector bundle. These kinds of vector bundles form a tensor category $\mathcal{B}^\ast(X)$. Our main result is the following

**Theorem 1.1.** Let $X$ be a proper connected seminormal rigid analytic variety over $\mathbb{C}_p$. Then there is a fully faithful functor

$$DW : \mathcal{B}^\ast(X) \rightarrow \text{Rep}_{\pi_1(X)}(\mathbb{C}_p)$$

which is exact and compatible with tensor products, duals, internal homs and exterior products.

Here $\text{Rep}_{\pi_1(X)}(\mathbb{C}_p)$ denotes the category of continuous representations of the étale fundamental group $\pi_1(X) = \pi_1^\text{ét}(X, x)$ (for a fixed base point $x$) on finite dimensional $\mathbb{C}_p$-vector spaces.

Using the methods developed in [Sch13a] we are able to work for the most part on the generic fiber, by which we can avoid the complications arising in [DW17] in the study of
integral models. At the same time the results from [Sch13a] and [KL16] allow us to derive the full faithfulness of the Deninger-Werner functor, which could not be seen from the construction in [DW17]. We note that this last point is close in spirit to the article [Xu17], where the constructions by Deninger and Werner are analyzed via the Faltings topos and full faithfulness is established in the curve case.

We want to remark that one of the main open problems is still to find out which vector bundles admit a model with numerically flat reduction. We have nothing to say on this, except that we are able to extend the results on line bundles from [DW17, §10] to our more general setup. We wish to draw the reader’s attention to the recent preprint [HW19] which reports on progress on this problem.

Let us sketch how we go about proving the above theorem. The category $Rep_{\pi_1(X)}(\mathcal{O}_{C_p})$ of continuous representations on finite free $\mathcal{O}_{C_p}$-modules is equivalent the category of locally free $\hat{\mathcal{O}}_{X}$-modules, where $\hat{\mathcal{O}}_{X}$ is the completed integral structure sheaf on the pro-étale site of $X$, given by $\mathbb{L} \mapsto \mathbb{L} \otimes \hat{\mathcal{O}}_{X}$. We note here that the analogous statement on the Faltings topos is also the starting point of Faltings’s $p$-adic Simpson correspondence. One can then show (see corollary 3.12) that the essential image of the above functor is given by the $\hat{\mathcal{O}}_{X}$-modules which become trivial on a profinite étale covering of $X$. If $E$ is a vector bundle on a formal scheme $X$ over $Spf(\mathcal{O}_{C_p})$, with generic fiber $X$, we can form its pullback $E_{\pi}^+$ to the pro-étale site of $X$. Its $p$-adic completion $\hat{E}_{\pi}^+$ is an $\hat{\mathcal{O}}_{X}$-module. We will then show the following:

**Theorem 1.2.** Let $X$ be a proper flat connected formal scheme over $Spf(\mathcal{O}_{C_p})$ and $E$ a vector bundle on $X$ with numerically flat reduction. Then $\hat{E}_{\pi}^+$ is trivialized by a profinite étale cover.

In the terminology of [Xu17] (see definition 3.14) this proves that all vector bundles with numerically flat reduction are Weil-Tate. This was shown in loc. cit. for the case of curves using the constructions from [DW05b]. This theorem will also be used to construct étale parallel transport on $E$, as in [DW17]. The proof of the theorem follows very much the path laid out in [DW17]. In particular one shows that a vector bundle $E$ with numerically flat reduction is trivialized modulo $p$ after pullback along a composition of a finite étale cover and some power of the absolute Frobenius map. One is then faced with two problems: One is dealing with the Frobenius pullback and the other is to inductively get rid of obstructions preventing the bundle in question to be trivial modulo $p^n$. Both problems become much simpler after pulling back to the pro-étale site (theorem 3.19).

Let us make some remarks on the contents of the individual sections. Section 2 is a recollection of the results from [Sch13a] which are needed in this article. In section 3 we first show how to attach continuous $\mathcal{O}_C$-representations to integral vector bundles on the pro-étale site whose $p$-adic completion is trivialized by a finite étale cover (theorem 3.9). These constructions work for proper rigid analytic varieties over any complete algebraically closed perfectoid field $C$. We then specialize to the case $C = C_p$ where we have the alternative

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1We remark that locally free (almost) $\hat{\mathcal{O}}_{X}$-modules are analogous to the objects called generalized representations studied in [Fal05] and [AGT16].
viewpoint of local systems. Finally, we show that any $O^+$-module which is trivialized modulo $p$ by a Frobenius pullback will give rise to a representation (Theorem 3.19).

Section 4 then mostly deals with vector bundles with numerically flat reduction. We first generalize the results from [DW17, §2] on numerically flat vector bundles on projective schemes over finite fields to the non-projective case (Theorem 4.5). The proof of this is an application of $v$-descent for vector bundles on perfect schemes as established in [BS17]. Then the results from section 3 are used to construct the Deninger-Werner functor for vector bundles with numerically flat reduction. We then show that the discussion from section 3 can be improved to construct an étale parallel transport functor for the given bundle and that our construction indeed recovers the results from [DW17] (Theorem 4.28).

We finish by showing that the cohomology of the constructed local systems come with a Hodge-Tate spectral sequence. Note that this has also been treated in [Xu17] via the Faltings topos building on the work of Abbes, Gros and Tsuji (see in particular [Xu17, Proposition 11.7] and [Xu17, Proposition 11.8]).

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2 Preliminaries

2.1 Notation

Throughout this paper we will work with proper rigid analytic varieties (viewed as a full subcategory of the category of adic spaces (see [Hub03]) over an algebraically closed perfectoid field $C$ of characteristic 0. We denote by $O_C \subset C$ its ring of integers. If $\Gamma$ is the value group of $C$, we denote by $log\Gamma \subset \mathbb{R}$ the induced subset obtained by taking the logarithm with base $|p|$. Then for any $\epsilon \in log\Gamma$ we choose an element $p^\epsilon \in C$, which satisfies $|p^\epsilon| = |p|^\epsilon$.

Moreover we fix a pseudouniformizer $t$ in the tilt $O_C^\flat$ such that $t^\# = p$.

Whenever we speak about almost mathematics we mean almost mathematics with respect to the maximal ideal $m \subset O_C$.

If $X$ is a rigid analytic space over a non-archimedean field $K$, then by formal model we mean an admissible formal scheme over $O_K$ with generic fiber $X$.

2.2 The pro-étale site of a rigid analytic space

Let $X$ be a locally noetherian adic space over $Spa(Q_p, Z_p)$. The pro-étale site $X_{pro\acute{e}t}$ of $X$ was introduced in [Sch13a]. The idea is that one wants to extend the usual étale site to allow inverse limits along finite étale morphisms. Since inverse limits along affinoid morphism may not be well behaved in the category of adic spaces (as there is no canonical way to put a topology on a direct limit of affinoid algebras) one simply considers formal filtered pro-systems $\lim_{i \in I} Y_i \rightarrow X$ of étale maps, such that there exists some $i_0$ such that the transition maps $Y_i \rightarrow Y_{i'}$ are all finite étale for $i' \geq i_0$. We refer to [BMS18] §5 for the
Hence we can define the pullback of a finite locally free $\hat{\mathcal{O}}_X$.

We have the following structure sheaves

$\mathsf{Sch}_{13a}$, Definition 4.1

Definition 2.1.

We have the following structure sheaves

- $\mathcal{O}_X^+ := \nu^{-1}\mathcal{O}_{X^{\text{et}}}^+\mathcal{O}_X = \nu^{-1}\mathcal{O}_{X^{\text{et}}}$

- $\hat{\mathcal{O}}_X^+ = \lim_{\leftarrow i} \mathcal{O}_X^+/p^n$, $\hat{\mathcal{O}}_X = \hat{\mathcal{O}}_X^+[1/p]$ (completed structure sheaves)

- $\hat{\mathcal{O}}_X^+ := \lim_{\leftarrow \phi} \hat{\mathcal{O}}_X^+/p$ (tilted structure sheaf)

where $\phi$ denotes the (surjective) Frobenius on $\hat{\mathcal{O}}_X^+/p$.

Remark 2.2. Let $f : X \to Y$ be a morphism of adic spaces over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Then the map $f^{-1}_{\text{proet}}\mathcal{O}_Y^+ \to \mathcal{O}_X^+$ extends to the $p$-adic completion, so we get a map $f^{-1}_{\text{proet}}\hat{\mathcal{O}}_Y^+ \to \hat{\mathcal{O}}_X^+$. Hence we can define the pullback of a finite locally free $\hat{\mathcal{O}}_Y^+$-module $\mathcal{F}$ along $f$ in the usual way.

Now let $\mathcal{X}$ be an admissible formal scheme over $\text{Spf}(\mathcal{O}_C)$. By $\underline{\text{Hub96}}$, Proposition 1.9.1 there is an adic space $X$ over $\text{Spa}(\mathcal{C}, \mathcal{O}_C)$, which comes with a specialization map $\text{sp} : (X, \mathcal{O}_X^+) \to (\mathcal{X}, \mathcal{O}_X)$ of locally ringed spaces, and is such that for any morphism of locally ringed spaces $f : (Z, \mathcal{O}_Z^+) \to (\mathcal{X}, \mathcal{O}_X)$ where $Z$ is an adic space over $\text{Spa}(\mathcal{C}, \mathcal{O}_C)$, there is a unique morphism $g : \hat{Z} \to X$ such that $f = \text{sp} \circ g$. Moreover, $X$ is canonically isomorphic to the adic space associated to Raynaud’s rigid analytic generic fiber of $\mathcal{X}$. We call $X$ the generic fiber of $\mathcal{X}$.

We again have a canonical projection $\mu = \text{sp} \circ \lambda : X^{\text{proet}} \to X^{\text{Zar}}$. There is a natural map $\mu^{-1}\mathcal{O}_X \to \mathcal{O}_{\hat{X}}$. Hence for any $\mathcal{O}_X$-module $\mathcal{E}$, we can define the associated $\mathcal{O}_{\hat{X}}$-module $\mathcal{E}^+ := \mu^{-1}\mathcal{E} \otimes \mu^{-1}\mathcal{O}_X \mathcal{O}_{\hat{X}}$. One easily proves the following:

Lemma 2.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of admissible formal schemes over $\text{Spf}(\mathcal{O}_C)$. Then for any $\mathcal{O}_Y$-module $\mathcal{E}$ there is a canonical isomorphism $(f_C)^{(\mathcal{F})}_{\text{proet}}(\mathcal{E}^+) \cong (f^*\mathcal{E})^+$.

The main result from $\underline{\text{Sch13a}}$ that we need is the so called primitive comparison theorem:

Theorem 2.4. $\underline{\text{Sch13a}}$, Theorem 5.1. Let $X$ be a proper rigid analytic space over $\text{Spa}(\mathcal{C}, \mathcal{O}_C)$, where $\mathcal{C}$ is an algebraically closed perfectoid field of characteristic 0 with ring of integers $\mathcal{O}_C$. Then the canonical maps

$$H^i(X^{\text{et}}, \mathbb{Z}_p/p^n) \otimes \mathcal{O}_C \to H^i(X, \mathcal{O}_X^+/p^n)$$

$$H^i(X^{\text{et}}, \mathbb{Z}_p) \otimes \mathcal{O}_C \to H^i(X, \hat{\mathcal{O}}_X^+)$$

are almost isomorphisms for all $i \geq 0$.

If $X$ is in addition smooth or the analytification of a proper scheme over $\text{Spec}(\mathcal{C})$, then the canonical maps
for any lisse \( \mathbb{Z}_p \)-sheaf \( \mathbb{L} \).

**Proof.** The mod \( p \) statements can be found in [Sch13a, Theorem 5.1] (smooth case), [Sch13b, Theorem 3.13] (algebraic case) and [Sch13b, Theorem 3.17] (for general proper rigid analytic varieties). The full statements all then follow by induction combined with [Sch13a, Lemma 3.18].

The categories of locally free sheaves on the pro-étale, étale and analytic sites all agree:

**Lemma 2.5.** [Sch13a, Lemma 7.3] Pullback along the natural projections

\[ X_{\text{pro}} \xrightarrow{\nu} X_{\text{ét}} \rightarrow X_{\text{an}} \]

induces equivalences of categories between the categories of finite locally free modules over \( \mathcal{O}_X \), resp. \( \mathcal{O}_{X_{\text{ét}}} \), resp. \( \mathcal{O}_{X_{\text{an}}} \).

Note that in [Sch13a] \( X \) is assumed to be smooth, but the proof given there works also for the general case (see also [KL15, Theorem 8.2.22]).

By a vector bundle on \( X \) we usually mean a locally free \( \mathcal{O}_{X_{\text{an}}} \)-module. We will however often use the above lemma to freely switch between the topologies. We hope that this will not be a source of confusion.

**Remark 2.6.** If \( X \) is quasi-compact and quasi-separated pullback along \( \nu \) also induces an equivalence of categories between finite locally free \( \mathcal{O}_{X_{\text{ét}}} \)-modules and finite locally free modules over \( \mathcal{O}^+_X \): As \( \nu_* \mathcal{O}^+_X = \mathcal{O}^+_X \) (by [Sch13a, Corollary 3.17 (i)]) one sees that \( \nu^* \) is fully faithful. Now assume \( \mathcal{E}^+ \) is a finite locally free \( \mathcal{O}^+_X \)-module given by a gluing datum on \( \tilde{Y} \times_X Y \) for some pro-étale cover \( \tilde{Y} \to X \), which we can assume to be qcqs. By [Sch13a, Lemma 3.16] one has \( \mathcal{O}^+_X(V) = \lim \nu^{-1} \mathcal{O}^+_X(V_j) \) for any qcqs \( V \in X_{\text{pro}} \). So the gluing datum descends to some \( Y_i \times_X Y_i \), which shows that \( \mathcal{E}^+ \) lies in the essential image of \( \nu^* \).

### 2.3 Local systems with coefficients in \( \mathcal{O}_{\mathbb{C}_p} \)

Let \( X \) be a connected locally noetherian adic space which is quasi-compact and quasi-separated. We fix a geometric point \( \bar{x} \) of \( X \) and denote by \( \pi_1(X) := \pi_1^\text{\text{pro}}(X, \bar{x}) \) the profinite fundamental group with respect to the base point \( \bar{x} \). Then by [Sch13a, Proposition 3.5] the category of profinite étale covers of \( X \) is equivalent to the category of profinite sets with a continuous \( \pi_1(X) \)-action. For any topological ring \( R \) we denote by \( \text{Rep}_{\pi_1(X)}(R) \) the category of continuous representations of \( \pi_1(X) \) on finite rank free \( R \)-modules.

**Definition 2.7** (cf. [Sch13a, 8.1]). We define \( \hat{\mathcal{O}}_{\mathbb{C}_p} := \varprojlim_n \mathcal{O}_{\mathbb{C}_p}/p^n \), where \( \mathcal{O}_{\mathbb{C}_p}/p^n \) denotes the constant sheaf on \( X_{\text{pro}} \). An \( \mathcal{O}_{\mathbb{C}_p} \)-local system is a finite locally free \( \hat{\mathcal{O}}_{\mathbb{C}_p} \)-module.

We denote the category of \( \hat{\mathcal{O}}_{\mathbb{C}_p} \)-local systems by \( \text{LS}_{\hat{\mathcal{O}}_{\mathbb{C}_p}}(X) \).

Similarly an \( \mathcal{O}_{\mathbb{C}_p}/p^n \)-local system is a finite locally free module over the constant sheaf \( \mathcal{O}_{\mathbb{C}_p}/p^n \) on \( X_{\text{pro}} \). We write \( \text{LS}_{\mathcal{O}_{\mathbb{C}_p}/p^n}(X) \) for the category of \( \mathcal{O}_{\mathbb{C}_p}/p^n \)-local systems.

**Remark 2.8.** \( \bullet \) As \( X \) is qcqs one has \( \text{LS}_{\mathcal{O}_{\mathbb{C}_p}/p^n}(X) = \text{colim}_{K/\mathbb{Q}_p} \text{LS}_{\mathcal{O}_K/p^n}(X) \), where \( K \) runs over finite extensions of \( \mathbb{Q}_p \). This is because \( \mathcal{O}_{\mathbb{C}_p}/p^n = \varprojlim_{K/\mathbb{Q}_p} \mathcal{O}_K/p^n \).
Let $LS_{O_C/p^n}(X)$ be equivalent to the category of finite locally free $O_{C_p}/p^n$-modules on the étale site, i.e. every $L \in LS_{O_{C_p}/p^n}(X)$ is in the essential image of $\nu^*$, where $\nu : X_{\text{proet}} \to X_{\text{ét}}$ is again the natural projection. For this note that if $A$ is any abelian group and $A_{\text{proet}}, A_{\text{ét}}$ the constant sheaves with values in $A$ on $X_{\text{proet}}, \text{resp.}$ on $X_{\text{ét}}$, we get $\nu_! A_{\text{proet}} = \nu_* \nu^* A_{\text{ét}} = A_{\text{ét}}$ by \cite{Sch13a} Corollary 3.17.

Since $O_K \subset O_{C_p}$ is flat, for any finite extension $K/\mathbb{Q}_p$, for any $O_K/p^n$-local system $L$ we have $H^i(X_{\text{proet}}, L \otimes O_{C_p}/p^n) = H^i(X_{\text{proet}}, L) \otimes O_{C_p}/p^n$, for all $i \geq 0$.

Consider the category $LS_{O_{C_p}^\bullet}(X)$ of inverse systems $\{L_n\}$ of finite free $O_{C_p}/p^n$-modules on $X_{\text{proet}}$, where $\{L_n\}$ is isomorphic to an inverse system $\{L'_n\}$ satisfying $L_{n+1}/p^n \cong L'_n$. Then there is a functor $LS_{O_{C_p}^\bullet}(X) \to LS_{O_{C_p}}(X)$, taking $\{L_n\}$ to $\lim L_n$. The proof of \cite{Sch13a} Theorem 4.9 shows that the inverse system $\{L_n\}$ satisfies the conditions from \cite{Sch13a} Lemma 3.18], which gives the following:

**Proposition 2.9** (cf. \cite{Sch13a} Proposition 8.2). The functor $LS_{O_{C_p}^\bullet}(X) \to LS_{O_{C_p}}(X)$ is an equivalence of categories.

**Proposition 2.10.** There is an equivalence of categories

$$LS_{O_{C_p}}(X) \leftrightarrow \text{Rep}_{\pi_1^G(X, \mathbb{Z})}(O_{C_p})$$

*Proof.* The following arguments are well known. First fix $n \geq 1$. As usual, finite $\pi_1(X)$-sets correspond to finite étale covers, and $L \to \mathbb{Z}/n$ gives an equivalence of categories

$$LS_{O_K/p^n}(X) \leftrightarrow \text{Rep}_{\pi_1^G(X, \mathbb{Z}/n)}(O_K/p^n)$$

for all finite extensions $K/\mathbb{Q}_p$. Clearly, these equivalences are compatible with base extensions $O_K/p^n \to O_K'/p^n$, for $K \subset K'$. So we get

$$\text{colim}_{K \subset \mathbb{Q}_p} LS_{O_K/p^n}(X) \leftrightarrow \text{colim}_{K \subset \mathbb{Q}_p} \text{Rep}_{\pi_1^G(X, \mathbb{Z})}(O_K/p^n).$$

Now by remark 2.8, we know that $\text{colim}_{K \subset \mathbb{Q}_p} LS_{O_K/p^n}$ is equivalent to $LS_{O_{C_p}/p^n}(X)$. On the other hand, that $\text{colim}_{K \subset \mathbb{Q}_p} \text{Rep}_{\pi_1^G(X, \mathbb{Z})}(O_K/p^n)$ follows from the fact that $\pi_1(X)$ is compact: since $GL_n(O_{C_p}/p^n)$ carries the discrete topology, the image $\rho(\pi_1(X))$ will be finite for any continuous representation $\rho : \pi_1(X) \to GL_n(O_{C_p}/p^n)$. Now passing to the $p$-adic completion, using proposition 2.9 gives the claim.\hfill $\Box$

One can then also generalize the primitive comparison theorem to the case of $O_{C_p}$-coefficients.

**Theorem 2.11.** Let $X$ be a proper smooth rigid analytic space over Spa($C_p, O_{C_p}$). And let $L$ be an $O_{C_p}$-local system on $X$. The canonical map

$$H^i(X_{\text{proet}}, L) \to H^i(X_{\text{proet}}, L \otimes O_{C_p}^+)$$

is an almost isomorphism, for all $i \geq 0$.

*Proof.* By the above remark we see that $L/p \cong L'/p$ where $L'$ is defined over $O_K/p$ for $K/\mathbb{Q}_p$, a finite extension. Let $\pi$ be a uniformizer of $K$. Then $L'/\pi$ is an $F_q$-local system, for some $q = p^n$. Now, if we replace the Frobenius occurring in the proof of Theorem 5.1 in \cite{Sch13a} everywhere by its $m$-th power $x \mapsto x^m$, the proof goes through for $F_q$-local systems and we get an almost isomorphism $H^i(X_{\text{ét}}, L'/\pi) \otimes O_{C_p}/p \cong H^i(X_{\text{ét}}, L'/\pi \otimes O^+/p)$. But then by induction along the exact sequences
we find that
\[ H^i(\mathbb{L}/p) = H^i(\mathbb{L}') \otimes_{\mathcal{O}_K/p} \mathcal{O}_\mathbb{C}_p/p \rightarrow H^i(\mathbb{L}' \otimes_{\mathcal{O}_K/p} \hat{\mathcal{O}}^+/p) = H^i(\mathbb{L} \otimes_{\mathcal{O}_\mathbb{C}_p} \hat{\mathcal{O}}^+/p) \]
is an almost isomorphism. But then the full statement follows again by induction and using \cite[Lemma 3.18]{Sch13a} as in the case of \( \mathbb{Z}_p \)-local systems (see the proof of \cite[Theorem 8.4]{Sch13a}).

Remark 2.12. There is a functor
\[ LS_{\mathcal{O}_\mathbb{C}_p}(X) \rightarrow LF(\hat{\mathcal{O}}_X^+) \]
which takes \( \mathbb{L} \) to \( \mathbb{L} \otimes \hat{\mathcal{O}}_X^+ \). Assume now that \( X \) is connected and proper smooth over \( Spa(\mathbb{C}_p, \mathcal{O}_\mathbb{C}_p) \). Then one immediately gets from theorem 2.11 that the induced functor
\[ Rep_{\pi_1(X)}(\mathcal{O}_\mathbb{C}_p) \otimes \mathbb{Q} \cong LS_{\mathcal{O}_\mathbb{C}_p}(X) \otimes \mathbb{Q} \rightarrow LF(\hat{\mathcal{O}}_X), \]
taking \( \rho \) to \( \mathbb{L}_p \otimes \hat{\mathcal{O}}_X \), is fully faithful. Here for a representation \( \rho \), we denote by \( \mathbb{L}_p \) the associated local system. At the integral level theorem 2.11 shows that one has a fully faithful embedding of \( \mathcal{O}_\mathbb{C}_p \)-local systems into the category of finite locally free almost \( \hat{\mathcal{O}}_X^+ \)-modules. Note however, that the discussions in the following section will show that full faithfulness holds also at the integral level (without passing to almost modules) and without any restrictions on \( X \) (see corollary 3.12).

3 Representations attached to \( \hat{\mathcal{O}}_X^+ \)-modules

3.1 Trivializable \( \hat{\mathcal{O}}_X^+ \)-modules and representations

Fix a \( Spa(\mathbf{C}, \mathcal{O}_\mathbf{C}) \)-valued point \( x \) of \( X \). We will show how to attach a continuous \( \mathcal{O}_\mathbf{C} \)-representation of \( \pi_1(X) := \pi_1^{et}(X, x) \) to certain \( \hat{\mathcal{O}}_X^+ \)-modules \( \mathcal{E}^+ \) for which the \( p \)-adic completion \( \hat{\mathcal{E}}^+ \) is trivialized on a profinite étale cover, i.e. an inverse limit of finite étale surjective maps.

Let \( X \) be a proper connected rigid analytic space over \( Spa(\mathbf{C}, \mathcal{O}_\mathbf{C}) \). Then the only global functions are the constant ones, i.e. \( \Gamma(X, \mathcal{O}_X) = \mathbf{C} \). As \( \Gamma(X, \hat{\mathcal{O}}_X^+) \) consists of the functions \( f \) for which \( |f(x)| \leq 1 \) for all \( x \in X \), we see that \( \Gamma(X, \hat{\mathcal{O}}_X^+) = \mathcal{O}_\mathbf{C} \). Similarly one has \( \Gamma(X, \hat{\mathcal{O}}_X^+) \subset \Gamma(X, \hat{\mathcal{O}}_X) = \mathcal{O}_\mathbf{C} \). And hence \( \Gamma(X, \hat{\mathcal{O}}_X^+) = \mathcal{O}_\mathbf{C} \).

We first record the following:

Lemma 3.1. Let \( \hat{Y} = \lim_i Y_i \rightarrow X \) be a profinite étale cover and let \( \mathcal{E}^+ \) be a locally free \( \hat{\mathcal{O}}_X^+ \)-module, such that \( \hat{\mathcal{E}}^+|_{\hat{Y}} \) is trivial. Then for any \( n \geq 1 \) there exists some \( i \), such that \( \mathcal{E}^+/p^n \) becomes trivial on \( Y_i \).

Proof. Let \( \nu : X_{proet} \rightarrow X_{et} \) denote the canonical projection. There exists a locally free \( \hat{\mathcal{O}}_{X_{et}}^+ \)-module \( \mathcal{F} \), such that \( \nu^* \mathcal{F} = \mathcal{E}^+ \). Hence we also have \( \hat{\mathcal{E}}^+/p^n = \nu^*(\mathcal{F}/p^n) \). But then by \cite[Lemma 3.16]{Sch13a} \( \hat{\mathcal{E}}^+/p^n \) is the sheaf given by \( \hat{\mathcal{E}}^+/p^n(V) = \lim_{\rightarrow}^{\mathcal{F}/p^n(V_j)} \) for any qcqs object \( V = \lim_{\leftarrow}^{\mathcal{F}/p^n(V_j)} \in X_{proet} \).

As all \( Y_i \) are quasi-compact and quasi-separated we have that \( \hat{Y} \) is qcqs by \cite[Lemma 3.12]{Sch13a} (v). Now note that if \( \mathcal{E}, \mathcal{E}' \) are locally free \( \hat{\mathcal{O}}_X^+/p^n \)-modules, then by what we said above...
\[ \text{Hom}(\mathcal{E}|_{\tilde{Y}}, \mathcal{E}'|_{\tilde{Y}}) = \text{Hom}(\mathcal{E}, \mathcal{E}')(\tilde{Y}) = \varinjlim \text{Hom}(\mathcal{E}, \mathcal{E}'(Y_j)) = \varinjlim \text{Hom}(\mathcal{E}|_{Y_j}, \mathcal{E}'|_{Y_j}). \]

From this we see that the isomorphism \((\hat{\mathcal{E}}^+ / p^n)|_{\tilde{Y}} \cong (\hat{\mathcal{O}}_{\tilde{Y}}^+ / p^n)^r\) descends to an isomorphism \((\hat{\mathcal{E}}^+ / p^n)|_{Y} \cong \hat{\mathcal{O}}_{Y}^+ / p^n\) for some large enough \(i\).

Let \(X\) be proper connected and \(\tilde{Y} = \varprojlim Y_i \rightarrow X\) be a profinite étale cover where each \(Y_i\) is connected. Then one checks that \(\Gamma(\tilde{Y}, \hat{\mathcal{O}}_{\tilde{Y}}^+) = \mathcal{O}_X\).

\textbf{Remark 3.2.} Actually if the \(Y_i\) are connected, then so is \(\tilde{Y}\). For this note that \(\tilde{Y}\) is quasi-compact by \cite[3.12 (v)]{Sch13a}. Now suppose that \(|\tilde{Y}| = V_1 \cup V_2\) for some open and closed \(V_1, V_2\). Then \(V_1\) and \(V_2\) descend to closed subsets \(V_{i_0}, V_{i_0}^{\circ} \subset Y_{i_0}\), for some \(i_0\), which cover \(|Y_{i_0}|\). But then, since all \(Y_i\) are connected, \(f_{j_{i_0}} V_{1}^{\circ} \cap f_{j_{i_0}} V_{2}^{\circ}\) is non-empty for all \(j \geq i_0\). But \(V_1 \cap V_2 = \varprojlim j_{\geq i_0} f_{j_{i_0}} V_{1}^{\circ} \cap f_{j_{i_0}} V_{2}^{\circ}\) and an inverse limit of non-empty spectral spaces along spectral maps is non-empty.

Assume now that \(\mathcal{E}^+\) is as above with \(p\)-adic completion \(\hat{\mathcal{E}}^+\) trivialized on some connected profinite étale cover \(f : \tilde{Y} = \varprojlim Y_i \rightarrow X\), so we have \(\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \cong (\mathcal{O}_X)^r\).

We will now adjust the exposition in \cite[§4]{DW17} to our setting to define an action of \(\pi_1^\text{ét}(X, x)\) on the fiber \(\hat{\mathcal{E}}^+_x = \Gamma(x^* \hat{\mathcal{E}}^+)\):

Pick a point \(y : \text{Spa}(\mathbb{C}, \mathcal{O}_\mathbb{C}) \rightarrow \tilde{Y}\) lying over \(x\). As \(\hat{\mathcal{E}}^+\) is trivial on \(\tilde{Y}\) we have an isomorphism \(y^* : \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \xrightarrow{\cong} \Gamma(x^* \hat{\mathcal{E}}^+)\) by pullback (and using the natural identification \(y^* f^* \cong (f \circ y)^* = x^*\)). For any \(g \in \pi_1(X)\) we get another point \(gy\) lying above \(x\). We can then define an automorphism on \(\hat{\mathcal{E}}^+_x\) by

\[ \hat{\mathcal{E}}^+_x (y^*)^{-1} \xrightarrow{\phi^*} \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) (gy)^* \xrightarrow{\gamma^*} \hat{\mathcal{E}}^+_x. \]

By this we get a map

\[ \rho_{\mathcal{E}^+}^{(y, y)} : \pi_1(X) \rightarrow GL_r(\mathcal{E}^+_x). \]

\textbf{Remark 3.3.} What we mean by pullback along \(y\) is the following: For any point \(x \in X\), valued in \(\text{Spa}(\mathbb{C}, \mathcal{O}_\mathbb{C})\), a point \(y \in \tilde{Y}\) over \(x\) is given by a compatible system of points \(y_i : \text{Spa}(\mathbb{C}) \rightarrow Y_i\) over \(x_i\). By this one gets compatible morphisms of (ringed) sites

\[ \xymatrix{ (Y_i)_{\text{proét}} \cong X_{\text{proét}}/Y_i \ar[d] \ar[r]^-{(y_i)_{\text{proét}}} & \text{Spa}(\mathbb{C})_{\text{proét}} \ar[d]^-{x_{\text{proét}}} & X_{\text{proét}} \ar[l]_-{(y)_{\text{proét}}}} \]

This then gives a morphism \(y_{\text{proét}} : \text{Spa}(\mathbb{C})_{\text{proét}} \rightarrow 2 - \varprojlim X_{\text{proét}}/Y_i \cong X_{\text{proét}}/\tilde{Y}\), lying over \(x_{\text{proét}}\). We write \(y^*\) for the pullback on global sections along \(y_{\text{proét}}\). Alternatively, one may carry out the construction within the category of diamonds.

We need to show that this defines a continuous representation and is independent of the choices.

\textbf{Lemma 3.4.} Let \(\tilde{Y}, y\) be as above. Let \(\phi : \tilde{Z} \rightarrow \tilde{Y}\) be a morphism of connected objects in \(X_{\text{proét}}\), and let \(z\) be a point lying above \(y\). Then \(\rho_{\mathcal{E}^+}^{(Z, z)} = \rho_{\mathcal{E}^+}^{(\tilde{Z}, z)}\).

\textbf{Proof.} For any \(g \in \pi_1(X)\), \(gz\) lies above \(gy\). Then there is a commutative diagram

\[ \xymatrix{ \hat{\mathcal{E}}^+_x (z^*)^{-1} \ar[r]^-{\phi^*} & \Gamma(\tilde{Z}, \hat{\mathcal{E}}^+) (gz)^* \ar[r]^-{(z^*)^*} & \hat{\mathcal{E}}^+_x. } \]
which gives the claim.

**Proposition 3.5.** The map $\rho_{\tilde{\mathcal{E}}^+}$ mod $p^n$ has finite image for all $n \geq 0$.

**Proof.** Let $\tilde{Y} = \lim Y_i$ be a presentation where each $Y_i \to X$ is connected finite étale. Then $y$ corresponds to a compatible system $y_i$ of points of $Y_i$. By lemma 3.4, $\hat{\mathcal{E}}^+/p^n$ is trivialized on some $Y_i \to X$. On the almost level, we then get $\Gamma(\hat{Y}, \mathcal{E}^+/p^n)_0 \cong \Gamma(Y_i, \mathcal{E}^+/p^n)_0 \cong (\mathcal{O}_C/p^n)^r$, where $r$ denotes the rank of $\mathcal{E}^+$. The reason here is that $\mathcal{O}_{Y_i}^+ / p^n(Y_j) \to \mathcal{O}_{Y_i}^+ / p^n(Y_i)$ is an isomorphism for any transition map $Y_j \to Y_j$. We thus get an action $\alpha_{n}(Y_i, y_i)$ of $\pi_1(X)$ on the $\mathcal{O}_C/p^n$-module of almost elements $(\hat{\mathcal{E}}^+/p^n)_*$ via

$$(\hat{\mathcal{E}}^+/p^n)_* \xrightarrow{(\gamma_i^{-1})} \Gamma(Y_i, \hat{\mathcal{E}}^+/p^n)_* \xrightarrow{(g_{y_i})}\Gamma((\hat{\mathcal{E}}^+/p^n)_*).$$

But now the natural map $\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+)/p^n \to \Gamma(\tilde{Y}, \mathcal{E}^+/p^n)_*$ is injective (after fixing a basis it is just given by the embedding $(\mathcal{O}_C/p^n)_r \to (\mathcal{O}_C/p^n)_s$).

This realizes $\rho_{\tilde{\mathcal{E}}^+}(Y_i, y_i) \otimes \mathcal{O}_C/p^n$ as a subrepresentation of $\alpha_{n}(Y_i, y_i)$. But now there are of course only finitely many points of $Y_i$ lying over $x$, so $\alpha_{n}(Y_i, y_i)$ has finite image, hence so has $\rho_{\tilde{\mathcal{E}}^+}(Y_i, y_i) \otimes \mathcal{O}_C/p^n$. □

**Lemma 3.6.** The map $\rho_{\tilde{\mathcal{E}}^+}$ does not depend on $(\tilde{Y}, y)$.

**Proof.** We only need to show the independence of the point $y$, the rest then follows from lemma 3.4. Moreover it is enough to show that $\rho_{\tilde{\mathcal{E}}^+}(Y_i, y_i) \otimes \mathcal{O}_C/p^n$ is independent of $y$ for any $n \geq 1$. Let $\tilde{Y} = \lim Y_i$ be a presentation as before, and $y$ correspond to a compatible system of points $y_i$ of $Y_i$. Assume that $\hat{\mathcal{E}}^+/p^n$ becomes trivial on $Y_i$. Then by the proof of the previous proposition, $\rho_{\tilde{\mathcal{E}}^+}(Y_i, y_i) \otimes \mathcal{O}_C/p^n$ is a subrepresentation of $\alpha_{n}(Y_i, y_i)$, so it is enough to show that $\alpha_{n}(Y_i, y_i)$ is independent of the point $y_i$ above $x$. Now consider the Galois closure $Y_i' \to X$ of $Y_i$. Again, as any point $y_i$ has a point $y_i'$ of $Y_i'$ lying above it, by lemma 3.4 it is enough to show that $\alpha_{n}(Y_i', y_i')$ is independent of the point $y_i'$ above $x$. But now the Galois group $G = Aut(X(Y'_i))$ of $Y_i' \to X$ acts simply transitively on the points lying above $x$, and one can immediately check as in the proof of [DW17] lemma 4.5 that the representation is independent of the point $y_i'$. Remark for this that $\alpha_{n}(Y_i', y_i')$ is given by transporting the $G$-action given by

$$\Gamma(f^*\hat{\mathcal{E}}^+/p^n)_* \xrightarrow{f_i^*} \Gamma(f_{y_i}^*f^*\hat{\mathcal{E}}^+/p^n)_* \xrightarrow{can} \Gamma(f^*\hat{\mathcal{E}}^+/p^n)_*$$

to $(\hat{\mathcal{E}}^+/p^n)_*$ via $(y_i')^*$ (see also [DW05], Proposition 23). Here $f$ denotes the map $Y_i' \to X$, $f_i$ is the automorphism associated to $g$ and $can$ is induced by the natural identification $f_i^*f^* \cong (f \circ f_i)^*$ using that the construction is independent of the chosen point above $x$. We thus see that we get a well defined continuous representation $\rho_{\tilde{\mathcal{E}}^+}$ associated to $\mathcal{E}^+$. Note that one does indeed get a representation, since if $g, h \in \pi_1(X)$ and $y$ is any point above $x$ one can write $\rho_{\tilde{\mathcal{E}}^+}(g) = (gy)^* \circ (y)^{-1}$ and $\rho_{\tilde{\mathcal{E}}^+}(h) = (hgy)^* \circ ((gy)^*)^{-1}$ using that the construction is independent of the chosen point above $x$. From this one then gets

$$\rho_{\tilde{\mathcal{E}}^+}(gh) = \rho_{\tilde{\mathcal{E}}^+}(g)\rho_{\tilde{\mathcal{E}}^+}(h).$$
We denote by $\mathcal{B}^\text{proet}(\mathcal{O}_X^+)$ the category of locally free $\mathcal{O}_X^+$-modules, whose $p$-adic completion is trivialized on a profinite étale cover of $X$. One easily checks:

**Lemma 3.7.** The category $\mathcal{B}^\text{proet}(\mathcal{O}_X^+)$ is closed under taking tensor products, duals, internal homs, exterior products and extensions.

For any $\mathcal{E}^+ \in \mathcal{B}^\text{proet}(\mathcal{O}_X^+)$ we can always find a trivializing cover which is an inverse limit along connected finite étale maps:

**Lemma 3.8.** Let $\mathcal{E}^+ \in \mathcal{B}^\text{proet}(\mathcal{O}_X^+)$. Then there is a profinite étale cover with presentation $\hat{C} = \varprojlim C_i$ such that each $C_i$ is connected and $\hat{\mathcal{E}}^+$ is trivial on $\hat{C}$.

**Proof.** First we can assume that $\hat{\mathcal{E}}^+$ becomes trivial on $\hat{Y}$ which has a presentation $\varprojlim I_i Y_i$ along a countable index set $I$(use for this that $\mathcal{E}^+/p^n$ is trivial on some finite étale cover). Then pick a connected component $C_1 \rightarrow Y_1$. Pull back $C_1$ along $Y_2 \rightarrow Y_1$, to get a finite étale map $\overline{C}_2 \rightarrow C_1$ and again choose a connected component $C_2 \rightarrow \overline{C}_2 \rightarrow C_1$. Then continuing like this we get a connected profinite étale cover $\varprojlim C_i$ with a map $\varprojlim C_i \rightarrow \hat{Y}$, so that $\hat{\mathcal{E}}^+$ is trivial on $\hat{C}$.

**Theorem 3.9.** The association $\mathcal{E}^+ \mapsto \rho_{\mathcal{E}^+}$ defines an exact functor

$$
\rho_\mathcal{O} : \mathcal{B}^\text{proet}(\mathcal{O}_X^+) \rightarrow \text{Rep}_{\pi_1}(\mathcal{O}_C).
$$

Moreover $\rho_\mathcal{O}$ is compatible with tensor products, duals, inner homs and exterior products, and for every morphism $f : X' \rightarrow X$ of connected proper rigid analytic spaces over $\text{Spa}(\mathcal{C}, \mathcal{O}_C)$ we have $\rho_{f^* \mathcal{E}^+} = f^* \rho_{\mathcal{E}^+}$, where $f^* \rho_{\mathcal{E}^+}$, denotes the composition

$$
\pi_1(X', x') \overset{f^*}{\rightarrow} \pi_1(X, f(x')) \overset{\mathcal{E}^+}{\rightarrow} GL_r(\mathcal{O}_C)
$$

and $x' : \text{Spa}(\mathcal{C}, \mathcal{O}_C) \rightarrow X'$ is a point of $X'$.

Define by $\mathcal{B}^\text{proet}(\mathcal{O}_X) := \mathcal{B}^\text{proet}(\mathcal{O}_X^+) \otimes \mathbb{Q}$ the category of finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ for which there exists a locally free $\mathcal{O}_X^+$-module $\mathcal{E}^+$ with $\mathcal{E}^+[1/p] = \mathcal{E}$, and $\mathcal{E}^+ \in \mathcal{B}^\text{proet}(\mathcal{O}_X^+)$. Passing to isogeny classes induces a functor

$$
\rho : \mathcal{B}^\text{proet}(\mathcal{O}_X) \rightarrow \text{Rep}_{\pi_1(X)}(\mathcal{C}),
$$

which is compatible with tensor products, duals, inner homs and exterior products.

**Proof.** The functoriality of the construction and the compatibility with the operations follows immediately using the previous lemma. For compatibility with pullbacks, note that if $\hat{\mathcal{E}}^+$ is trivial on a connected profinite étale cover $\hat{Y} \rightarrow X$, then $(f^* \mathcal{E}^+)$ becomes trivial on the pullback $\hat{Y}_{X'} \in X'_{\text{proet}}$, which we first assume to be connected. There is a commutative diagram of ringed sites

$$
\begin{array}{ccc}
X'_{\text{proet}}/\hat{Y}_{X'} & \overset{pr_{\hat{Y}}}{\longrightarrow} & X_{\text{proet}}/\hat{Y} \\
\downarrow & & \downarrow \\
X'_{\text{proet}} & \overset{f_{\text{proet}}}{\longrightarrow} & X_{\text{proet}}
\end{array}
$$

By pullback one has an isomorphism

$$
\Gamma(\hat{Y}, \hat{\mathcal{E}}^+) \cong \Gamma(\hat{Y}_{X'}, f^* \mathcal{E}^+).
$$
And for any \( g \in \pi_1(X', x') \) and point \( y' \) of \( \tilde{Y}_{X'} \) above \( x' \), one has \( pr_{g'}(gy') = f'_*(g)y' \) and there is a commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{E}}^+_{f(x')} & \xrightarrow{(pr_{g'}(y'))^{-1}} & \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \\
\downarrow f^* & & \downarrow pr_{g'} \\
(f^*\hat{\mathcal{E}}^+)_{x'} & \xrightarrow{(y')^{-1}} & \Gamma(\tilde{Y}_{X'}, f^*\hat{\mathcal{E}}^+) \\
\end{array}
\]

Now if \( \tilde{Y}_{X'} \) is not connected we can again construct a connected cover \( \tilde{C} \) as above such that \( f^*\hat{\mathcal{E}}^+ \) is trivial on \( \tilde{C} \) and there is a morphism \( \tilde{C} \to \tilde{Y}_{X'} \) of objects in \( X'_{pro\ell} \) which gives a commutative diagram

\[
\begin{array}{ccc}
X'_{pro\ell}/\tilde{C} & \xrightarrow{pr} & X'_{pro\ell}/\tilde{Y} \\
\downarrow & & \downarrow \\
X'_{pro\ell} & \xrightarrow{f_{pro\ell}} & X_{pro\ell}
\end{array}
\]

and one can now go on as before.

\[\square\]

### 3.2 Weil-Tate local systems

For this section we assume that \( (\mathbb{C}, \mathcal{O}_\mathbb{C}) = (\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \). Let again \( X \) denote a proper connected rigid analytic variety over \( Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \). Recall that we have a functor \( \mathbb{L} \to \mathbb{L} \otimes \hat{\mathcal{O}}^+_X \) from \( LS_{\mathcal{O}_{\mathbb{C}_p}}(X) \) to \( LF(\hat{\mathcal{O}}^+_X) \).

**Remark 3.10.** Recall from proposition 2.10 that there is an equivalence of categories \( LS_{\mathcal{O}_{\mathbb{C}_p}}(X) \leftrightarrow \text{Rep}_{\pi_1(X, x)}(\mathcal{O}_{\mathbb{C}_p}) \). This equivalence can be realized in the following way: For any \( \mathbb{L} \in LS_{\mathcal{O}_{\mathbb{C}_p}}(X) \) we can find a profinite étale cover \( \tilde{Y} = \lim \mathbb{L}_i \), where each \( \mathbb{L}_i \) is connected, and such that \( \mathbb{L} \) is trivial on \( \tilde{Y} \). Then as above we may define the associated representation as

\[
\mathbb{L}_x \xrightarrow{(y)^{-1}} \Gamma(\tilde{Y}, \mathbb{L}) \xrightarrow{(y')^*} \mathbb{L}_x.
\]

where \( \mathbb{L}_x = \Gamma(x^*_{pro\ell}, \mathbb{L}) = \lim \mathbb{L}_n \), where \( \mathbb{L}_n \) is the étale local system with \( \nu^*\mathbb{L}_n = \mathbb{L} \otimes \mathbb{L}^n \), and \( y \) is some point above \( x \). This is independent of the cover and chosen point as above. Moreover, modulo \( p^n \) we get the representation

\[
(\mathbb{L}/\mathbb{L}^n) \otimes (y)^{-1}_x \mathbb{L}_{Y, x} \xrightarrow{(y')^*} (\mathbb{L}/\mathbb{L}^n)_x,
\]

where now we can choose \( Y_i \to X \) to be Galois with Galois group \( G \). Then this action is the left action of \( G \) on \( \Gamma(Y_i, \mathbb{L}/\mathbb{L}^n) \) transported to \( (\mathbb{L}/\mathbb{L}^n)_x \), via \( y'_i \).

Conversely, if \( V \) is a finite free \( \mathcal{O}_{\mathbb{C}_p} \)-module with a continuous \( \pi_1(X) \)-action, we define an inverse system \( \{ \mathbb{L}_n \} \) as follows: Let \( \pi_1(X) \) act on \( \mathbb{L}/\mathbb{L}^n \) through the finite quotient \( G \). Let \( Y_n \to X \) be the finite étale Galois cover with group \( G \). Then the action of \( G \) on \( \mathbb{L}/\mathbb{L}^n \) defines a Galois descent datum on the constant sheaf \( \mathbb{L}/\mathbb{L}^n \) on \( Y_n \), by letting \( \mathbb{L}/\mathbb{L}^n \to f_g^*\mathbb{L}/\mathbb{L}^n \) be the map defined by \( g^{-1} \) for any \( g \in G \), where \( f_g \in Aut_X(Y_i) \) is \( C^\text{opp} \) denotes the automorphism of \( Y_n \) associated to \( g \). This gives rise to a local system \( \mathbb{L}_n \) on \( X \).

**Lemma 3.11.** Let \( \hat{\mathcal{E}}^+ \in B^{\text{et}}(\mathcal{O}_X^+) \) and let \( \mathbb{L} \) be the \( \hat{\mathcal{O}}_{\mathbb{C}_p} \)-local system corresponding to \( \rho_{\mathcal{E}^+} \). Then \( \hat{\mathcal{E}}^+ \cong \hat{\mathcal{O}}_X^+ \otimes_{\hat{\mathcal{O}}_{\mathbb{C}_p}} \mathbb{L} \).

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Proof. We can compare the gluing data: Assume that $\mathcal{E}^+$ and $L$ are trivialized on $\tilde{Y} \to X$, a connected profinite étale cover. By the discussion above we then have a $\pi_1(X, x)$-equivariant isomorphism $\Gamma(\tilde{Y}, \mathcal{E}^+) \xrightarrow{=} \tilde{\mathcal{E}}_x^+ = L|_{\tilde{Y}}$, where $\Gamma(\tilde{Y}, \mathcal{E}^+) := \varprojlim \Gamma(Y, \mathcal{E}^+)/p^n$, where $\Gamma(Y, \mathcal{E}^+)/p^n$ denotes the constant sheaf on $X_{\text{proet}}/Y$ associated to $\Gamma(Y, \mathcal{E}^+)/p^n$ and similarly $\mathcal{E}^+_x := \varprojlim \tilde{\mathcal{E}}_x^+/p^n$. From this one also gets a $\pi_1(X)$-equivariant isomorphism

$$f : \tilde{\mathcal{E}}^+_x|_{\tilde{Y}} = \Gamma(\tilde{Y}, \mathcal{E}^+) \otimes \hat{O}_Y^+ \xrightarrow{=} (L \otimes \hat{O}_Y^+)|_{\tilde{Y}}.$$  

Then modulo $p^n$ this equivariant isomorphism descends to some $Y_i$, which we can assume to be a finite étale Galois cover (if $Y_i$ is not Galois we can pull back to the Galois closure $Y'_i$, and then extend $Y'_i$ to a connected profinite étale cover $\tilde{Y}' \to Y$ and pull the whole discussion back to $\tilde{Y}'$). We then get an equivariant isomorphism

$$\hat{\mathcal{E}}^+/p^n|_{Y_i} \cong (\hat{O}^+ \otimes L)/p^n|_{Y_i}.$$  

But here the $\pi_1(X)$-action is through the quotient $G$, where $G = \text{Aut}_X(Y_i)^{\text{pp}}$ denotes the Galois group of $Y_i \to X$. But then the isomorphism descends to an isomorphism $\hat{\mathcal{E}}^+/p^n \cong (\hat{O}^+ \otimes L)/p^n$.

To be a bit more precise we claim that, if $Y_i$ is such that $\mathcal{E}^+/p^n|_{Y_i}$ and $L/p^n|_{Y_i}$ are trivial, there is a canonical isomorphism

$$\mathcal{E}^+/p^n|_{Y_i} \cong \Gamma(\tilde{Y}, \mathcal{E}^+)/p^n \otimes \hat{O}_{Y_i}^+/p^n$$

which takes the canonical gluing datum on the left to the gluing datum obtained from the action on $\Gamma(\tilde{Y}, \mathcal{E}^+)/p^n$, and such that

$$\mathcal{E}^+/p^n|_{Y_i} \cong \Gamma(\tilde{Y}, \mathcal{E}^+)/p^n \otimes \hat{O}_{Y_i}^+/p^n \to (\hat{O}^+ \otimes L)/p^n|_{Y_i}$$

descends $f$ mod $p^n$ to $Y_i$. For this, note that there is a commutative diagram

$$\begin{array}{ccc}
\Gamma(Y_i, \mathcal{E}^+/p^n) & \longrightarrow & \Gamma(Y_i, \mathcal{E}^+/p^n)_* \\
\downarrow & & \downarrow \cong \\
\Gamma(\tilde{Y}, \mathcal{E}^+)/p^n & \longrightarrow & \Gamma(\tilde{Y}, \mathcal{E}^+/p^n)_*
\end{array}$$

where the vertical maps are just given by pulling back sections. This proves that the image of $\Gamma(\tilde{Y}, \mathcal{E}^+)/p^n$ in $\Gamma(Y_i, \mathcal{E}^+/p^n)_*$ is contained in the image of $\Gamma(Y_i, \mathcal{E}^+/p^n)$. For this note that $\Gamma(Y_i, \hat{O}_{Y_i}^+/p^n)$ always contains a canonical copy of $O_{C^p}/p^n$ coming from the base.

Fixing a trivialization of $\mathcal{E}^+_x|_{\tilde{Y}}$, one sees that then the canonical copy of $(O_{C^p}/p^n)^r$ (r is the rank of $\mathcal{E}^+$) in $\Gamma(Y_i, \mathcal{E}^+/p^n)$ is identified with $\Gamma(\tilde{Y}, \mathcal{E}^+)/p^n$ through the diagram above. Moreover, as $\mathcal{E}^+/p^n$ is trivial on $Y_i$, the canonical map $\Gamma(Y_i, \mathcal{E}^+/p^n) \otimes \hat{O}_{Y_i}^+/p^n \to \mathcal{E}^+/p^n|_{Y_i}$ is surjective and its kernel is almost zero (as it becomes an isomorphism after passing to almost modules). This produces a unique (injective) map $g$ fitting into the commutative diagram

$$\begin{array}{ccc}
\Gamma(Y_i, \mathcal{E}^+/p^n) \otimes \hat{O}_{Y_i}^+/p^n & \longrightarrow & \mathcal{E}^+/p^n|_{Y_i} \\
\downarrow & & \downarrow g \\
\Gamma(Y_i, \mathcal{E}^+/p^n)_* \otimes \hat{O}_{Y_i}^+/p^n.
\end{array}$$

From the first diagram one sees that one then gets an injective map

$$\phi : \Gamma(\tilde{Y}, \mathcal{E}^+)/p^n \otimes \hat{O}_{Y_i}^+/p^n \to \mathcal{E}^+/p^n|_{Y_i},$$

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of submodules of $\Gamma(\bar{Y}, \hat{\mathcal{E}}^+/p^n) \otimes \mathcal{O}_Y^+/p^n$. But now, as $\phi$ becomes an isomorphism after passing to almost modules, it must also already be surjective itself (as the image is finitely generated the cokernel could not be almost zero otherwise). Thus we have a canonical isomorphism

$$\phi^{-1} : \mathcal{E}^+/p^n|_{Y_i} \xrightarrow{\sim} \Gamma(\bar{Y}, \hat{\mathcal{E}}^+)/p^n \otimes \mathcal{O}_{Y_i}^+/p^n$$

and one can now check that the canonical gluing datum on $\mathcal{E}^+/p^n|_{Y_i}$ is transported to the one coming from the action on $\Gamma(\bar{Y}, \hat{\mathcal{E}}^+)/p^n$ (recall that the $G$-action on $\Gamma(\bar{Y}, \hat{\mathcal{E}}^+)/p^n$ comes from the action on $\Gamma(Y_i, \hat{\mathcal{E}}^+/p^n)_x$ which is induced by the left action of $G^{opp}$ on $Y_i$).

So all in all, $f \mod p^n$ descends to $X$ for all $n$, hence so does $f$ itself. \hfill \Box

Denote by $LF^{pet}(\mathcal{O}_X^+)$ the category of profinite étale trivializable modules over $\mathcal{O}_X^+$ (we remark that by theorem 3.19 this category is equivalent to the category of $\mathcal{O}_X^+$-modules $\hat{\mathcal{E}}^+$ for which $\hat{\mathcal{E}}^+/p^n$ is trivial on a finite étale cover for some $n \geq 0$). The discussion from the previous section gives a functor

$$\hat{\rho}_O : LF^{pet}(\mathcal{O}_X^+) \to \text{Rep}_{\pi_1(X)}(\mathcal{O}_{\mathbb{C}_p}) \to LSO_{\mathbb{C}_p}(X)$$

(where the second functor is the equivalence from proposition 2.10).

**Corollary 3.12.** The functor $LSO_{\mathbb{C}_p}(X) \to LF^{pet}(\mathcal{O}_X^+)$, taking $\mathbb{L}$ to $\mathcal{O}_X^+ \otimes \mathbb{L}$, is an equivalence of categories with quasi-inverse given by $\hat{\rho}_O$.

**Proof.** First note that any $\mathcal{O}_{\mathbb{C}_p}$-local system $\mathbb{L}$ becomes trivial on a profinite étale cover, so that $\mathbb{L} \otimes \mathcal{O}_X^+ \in LF^{pet}(\mathcal{O}_X^+)$. Using the lemma above one can then check that $\hat{\rho}_O$ is a quasi-inverse.

From the remark above we know that if $\bar{Y}$ is a connected profinite étale cover trivializing $\mathbb{L}$, then the representation associating to $g \in \pi_1(X, x)$ the automorphism

$$\mathbb{L}_x (y^*)^{-1} \Gamma(\bar{Y}, \mathbb{L}) \xrightarrow{(gy)^*} \mathbb{L}_x$$

is the representation associated to $\mathbb{L}$ in proposition 2.10. But then the commutative diagram

$$\begin{array}{ccc}
\mathbb{L}_x & \xrightarrow{(y^*)^{-1}} & \Gamma(\bar{Y}, \mathbb{L}) \\
\cong & & \cong \\
(\mathcal{O}_X^+ \otimes \mathbb{L})_x & \xrightarrow{(y^*)^{-1}} & \Gamma(\bar{Y}, \mathcal{O}_X^+ \otimes \mathbb{L}) \\
\end{array}$$

shows that $\hat{\rho}(\mathbb{L} \otimes \mathcal{O}_X^+) = \mathbb{L}$. \hfill \Box

Using results by Kedlaya-Liu we can then also show that $\rho$ is fully faithful.

**Theorem 3.13.** Assume that $X$ is seminormal. Then the functor

$$\rho : \mathcal{B}^{pet}(\mathcal{O}_X) \to \text{Rep}_{\pi_1(X)}(\mathbb{C}_p)$$

is fully faithful.

**Proof.** The functor $\rho$ is given by composing $E \to E \otimes \hat{\mathcal{O}}$ with the functor $\hat{\rho}_O \otimes \mathbb{Q}$. The latter is fully faithful by the previous corollary, while the full faithfulness of the functor $E \to E \otimes \hat{\mathcal{O}}$ follows from [KL16, Corollary 8.2.4], since $X$ is seminormal. \hfill \Box
We see that the vector bundles $E$ in $B^{pet}(\mathcal{O}_X)$ are precisely the vector bundles for which there exists an $\mathcal{O}_{\mathcal{C}_p}$-local system $\mathbb{L}$ such that $E \otimes \hat{\mathcal{O}}_X \cong \hat{\mathcal{O}}_X \otimes \mathbb{L}$. We borrow the following terminology from [Xu17, Definition 10.3].

**Definition 3.14.** A vector bundle $E$ is called Weil-Tate if there exists an $\hat{\mathcal{O}}_{\mathcal{C}_p}$-local system $\mathbb{L}$ such that $E \otimes \hat{\mathcal{O}}_X \cong \hat{\mathcal{O}}_X \otimes \mathbb{L}$.

Similarly, the $\hat{\mathcal{O}}_{\mathcal{C}_p}$-local systems $\mathbb{L}$ that are associated to vector bundles in this sense are also called Weil-Tate.

**Remark 3.15.** Let $X$ be a smooth rigid analytic variety over $Spa(K, \mathcal{O}_K)$ for $K$ a finite extension of $\mathbb{Q}_p$. In [LZ16] Liu and Zhu have introduced a functor from the category of arithmetic $\mathbb{Z}_p$-local systems on $X$ to the category of nilpotent Higgs bundles on $X_{\mathbb{K}}$. Recall that a Higgs bundle is a vector bundle $E$ on $X_{\mathbb{K}}$ together with an endomorphism valued 1-form, i.e. a section $\phi \in H^0(End(E) \otimes \Omega^1)$, satisfying the integrability condition $\phi \wedge \phi = 0$.

We give a quick recollection of their construction. Let $\mathcal{O}_{B^{dR}}$ be the de Rham structure sheaf defined in [Sch16]. It carries a flat connection $\nabla$ (acting trivially on $\mathcal{O}_{B^{dR}}$) and a filtration coming from the usual filtration on $\mathcal{B}_{dR}$. Then define $\mathcal{O}_C = gr^0(\mathcal{O}_{B^{dR}})$.

Let $\nu' : X_{proet} / X_{\mathbb{K}} \rightarrow (X_{\mathbb{K}})_{\text{ét}}$ denote the natural projection. Then for any $\mathbb{Z}_p$-local system $\mathbb{L}$ the associated Higgs bundle is defined to be $\mathcal{H}(\mathbb{L}) = \nu'_*(\mathcal{O}_C \otimes_{\hat{\mathcal{O}}_p} \mathbb{L})$, with Higgs field $\theta_\mathbb{L}$ coming from the Higgs field on $\mathcal{O}_C$. Note that $\mathcal{O}_C$ carries a Higgs field coming from the associated graded of $\nabla$.

One may try to write down this functor for geometric local systems, i.e. we will consider the functor $\mathcal{H}(\mathbb{L}) = \nu'_*(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathcal{C}_p}} \mathbb{L})$ for $\hat{\mathcal{O}}_{\mathcal{C}_p}$-local systems on $X_{\mathbb{K}}$.

We wish to show that our constructions are compatible with this functor. In general, proving that this functor actually gives a Higgs bundle is a complicated endeavour, and is what occupies the large part of §2 in [LZ16] (for arithmetic local systems). In our case however this is immediate, as the local system is already attached to a vector bundle.

More precisely we have the following result (compare also with [Xu17, Proposition 11.7]):

**Proposition 3.16.** Let $X$ be a smooth proper rigid analytic variety over $Spa(K, \mathcal{O}_K)$. Let $E$ be a Weil-Tate vector bundle on $X_{\mathbb{K}}$, viewed as a sheaf on the étale site. Let $\mathbb{L}$ be the $\hat{\mathcal{O}}_{\mathcal{C}_p}$-local system associated to $E$.

Then $\nu'_*(\mathcal{O}_C \otimes \mathbb{L}) \cong E$, with vanishing Higgs field $\theta_\mathbb{L} = 0$.

**Proof.** We have $(\nu'^* \mathcal{E}) \cong \hat{\mathcal{O}}_X \otimes \mathbb{L}$. We have $gr^0 \mathcal{B}_{dR} = \hat{\mathcal{O}}_{X_{\mathbb{K}}}$, hence $\hat{\mathcal{O}}_{X_{\mathbb{K}}} \subset \mathcal{O}_C$ (see [Sch13a, Prop. 6.7]). Then $\mathcal{O}_C \otimes \mathbb{L} = (\nu'^* \mathcal{E}) \otimes_{\hat{\mathcal{O}}_{X_{\mathbb{K}}}} \mathcal{O}_C$. Since $\nu_*= \mathcal{O}_{X_{\text{ét}}}$, we get $\nu'_*(\mathcal{O}_C \otimes \mathbb{L}) \cong E$.

For the claim $\theta_\mathbb{L} = 0$, note that the Higgs field on $\mathcal{O}_C$ is trivial on $\hat{\mathcal{O}}_{X_{\mathbb{K}}}$ (as it comes from the connection $\nabla$ on $\mathcal{O}_{B^{dR}}$ which is trivial on $\mathcal{B}_{dR}$). But then the induced Higgs field on $\mathcal{H}(\mathbb{L}) = \nu'_*((\nu'^* \mathcal{E}) \otimes_{\hat{\mathcal{O}}_{X_{\mathbb{K}}}} \mathcal{O}_C)$ is simply obtained by trivially extending the Higgs field from $\nu'_*(\mathcal{O}_C = \mathcal{O}_{X_{\text{ét}}}$. But this is the zero Higgs field. \hfill $\square$

### 3.3 Frobenius-trivial modules

In this section we will deal with $\mathcal{O}_{X_{\mathbb{K}}}^+$/modules whose mod $p$ reduction is trivialized by some Frobenious pullback. Let $\mathcal{E}$ be an $\mathcal{O}_{X_{\mathbb{K}}}^+$/module. Then $\mathcal{E}$ is called $F^m$-trivial if $\Phi_{m*} \mathcal{E} \cong (\mathcal{O}_{X_{\mathbb{K}}}^+/p)^\vee$, where $\Phi$ denotes the Frobenious on $\mathcal{O}_{X_{\mathbb{K}}}^+$. The goal of this section is to show that all such modules lie in $B^{pet}(\mathcal{O}_{X_{\mathbb{K}}}^+)$.

We have the following (Recall that $t \in \mathcal{O}_C$, such that $t^p = p$):
Lemma 3.17. The Frobenius induces an equivalence of categories
\[ \{ F^m\text{-trivial locally free } \mathcal{O}_X^+/p\text{-modules } \} \]
\[ \longleftrightarrow \]
\[ \{ \text{locally free } \hat{\mathcal{O}}_X^+/t^m\text{-modules trivial mod } t \}. \]

Proof. Denote the Frobenius on \( \hat{\mathcal{O}}_X^+ \) by \( \hat{\Phi} \). There is a commutative diagram
\[
\begin{array}{ccc}
\hat{\mathcal{O}}_X^+ & \xrightarrow{\hat{\Phi}} & \hat{\mathcal{O}}_X^+ \\
\theta \downarrow & & \theta \\
\hat{\mathcal{O}}_X^+/t & \xrightarrow{\hat{\Phi}} & \hat{\mathcal{O}}_X^+/t.
\end{array}
\]

As \( \theta \) is surjective, we see that \( \theta^*\mathcal{M} = \mathcal{M} \) for any \( \hat{\mathcal{O}}_X^+/t \text{-module } \mathcal{M} \). Here \( \theta_* \) denotes the restriction of scalars. But then using the commutativity of the diagram above we see that \( (\hat{\Phi}^m)^* \) defines the desired functor with quasi-inverse given by \( (\hat{\Phi}^{-m})^* \).

Remark 3.18. One also sees in the same manner that we also have an equivalence of categories
\[ \{ \text{mod } p \text{ } F^m\text{-trivial locally free } \hat{\mathcal{O}}_X^+ \text{-modules } \} \]
\[ \longleftrightarrow \]
\[ \{ \text{locally free } \mathcal{A}_{inf,X}/\hat{\Phi}(\xi)\text{-modules trivial mod } (\xi,p) \} \]

where \( \mathcal{A}_{inf,X} = W(\hat{\mathcal{O}}_{X^+}) \) and now \( \hat{\Phi} \) denotes the Frobenius on \( \mathcal{A}_{inf,X} \).

Theorem 3.19. Let \( \mathcal{E}^+ \) be a locally free \( \mathcal{O}_X^+ \text{-module of rank } r \) such that \( \mathcal{E}^+/p \) is \( F^m\text{-trivial} \). Then there exists a profinite étale cover \( \tilde{Y} = \lim_{\leftarrow n} Y_n \to X \) such that \( \mathcal{E}^+|_{\tilde{Y}} \cong (\hat{\mathcal{O}}_{\tilde{Y}}^+|_{\tilde{Y}})^r \).

Proof. Assume first that \( m = 0 \), so that \( \mathcal{E}^+/p \) is trivial. We want to show that \( \mathcal{E}^+/p^n \) becomes trivial after passing to a further finite étale cover. We have an exact sequence:
\[ 0 \to M_n(\mathcal{I}) \to GL_r(\mathcal{O}_X^+/p^2) \to GL_r(\mathcal{O}_X^+/p) \to 1 \]
where \( \mathcal{I} = (p)\mathcal{O}_X^+/p^2 \) and the first map is given by \( A \mapsto 1 + A \). Taking cohomology we get an exact sequence
\[ H^1(M_n(\mathcal{I})) \to H^1(GL_r(\mathcal{O}_X^+/p^2)) \to H^1(GL_r(\mathcal{O}_X^+/p)). \]

Pick a pro-étale cover \( \{ U_i \} \) of \( X \) on which \( \mathcal{E}^+ \) becomes trivial. Using the exact sequence above plus the fact that \( \mathcal{E}^+/p \) is trivial we see that \( \mathcal{E}^+/p^2 \) is (after possibly refining the cover \( \{ U_i \} \)) defined by a cocycle of the form \( (id + g_{ij})_{ij} \) on the overlaps \( U_i \times_X U_j \), where \( g_{ij} \) defines a class in \( H^1(M_n(\mathcal{I})) \). Since \( \mathcal{O}_X^+ \) is \( p \)-torsion free we have an isomorphism of pro-étale sheaves \( \mathcal{I} \cong \mathcal{O}_X^+/p \). Hence by the primitive comparison theorem \[ 2.24 \] we see that \( H^1(M_n(\mathcal{I})) = M_n(H^1(\mathcal{I})) \) is almost isomorphic to \( M_n(H^1(X_{\et}, \mathbb{F}_p) \otimes \mathcal{O}_C/p) \). But the classes in the latter cohomology group become zero on suitable finite étale covers. Hence we can assume that the class defined by \( g_{ij} \) is almost trivial, which means that \( p^n g_{ij} \) becomes a coboundary for any \( n \in \log \Gamma \). Write \( p^n g_{ij} = p(\gamma_j - \gamma_i) \), where the \( \gamma_i \) are matrices with entries in \( \mathcal{O}^+/p^2(U_i) \). Then \( p^{1-\epsilon}(\gamma_j - \gamma_i) - g_{ij} \) is divisible by \( p^{2-\epsilon} \). Hence \( g_{ij} \) is given by a coboundary modulo \( p^{2-\epsilon} \), so \( \mathcal{E}^+/p^{2-\epsilon} \) is trivial. Inductively we see that \( \mathcal{E}^+ \) can be trivialized on suitable finite étale covers modulo \( p^{n-\epsilon n} \) where we can choose the sequence \( \epsilon_n \) in such a way that \( \epsilon_n \to 0 \) as \( n \) goes to infinity, hence giving the claim.
Now assume that $\Phi^m_*(\mathcal{E}^+/p)$ is trivial for some $m > 0$. Using lemma 3.17 we see that $\mathcal{F} = \Phi^m_*(\mathcal{E}^+/p)$ is a locally free $\mathcal{O}_{X^+}/t^m$-module trivial mod $t$. The obstruction for triviality of $\mathcal{F}/t^2$ lies again in $M_n(H^1(\mathcal{O}_{X^+}/t^2)) = M_n(H^1(\mathcal{O}_{X}^+/p))$. Now applying the same arguments as in the first part of the proof, we see that $\mathcal{F}$ becomes trivial on a finite étale cover $Y \to X$. But then, applying $(\Phi^{-m})^*$, we see that $\mathcal{E}^+/p$ becomes trivial on $Y$ as well.

We found out that the idea for the first part of the proof of the last theorem is essentially already contained in [Fal05, §5].

For generalities on non-abelian cohomology on sites we refer to [Gir71] (see in particular [Gir71, III Proposition 3.3.1]).

4 The Deninger-Werner correspondence for rigid analytic varieties

We wish to use the results from the previous section to give a new approach to the Deninger-Werner correspondence, which works for general (seminormal) proper rigid analytic varieties.

4.1 Numerically flat vector bundles

We first need to generalize the results from [DW17, §2] on numerically flat vector bundles over finite fields to the non-projective case. For any $\mathbb{F}_p$-scheme $Y$ we will denote by $F_Y$ the absolute Frobenius morphism of $Y$. For a vector bundle $E$ on $Y$ we denote its dual by $E^\vee$. Recall that a vector bundle $E$ on a smooth projective curve $C$ is called semistable if for all subbundles $E' \subset E$ we have $\mu(F') \leq \mu(F)$, where $\mu = \frac{\text{deg} \text{rk}}{\text{rk}}$ denotes the slope.

Proposition 4.1. Let $k$ be a perfect field. Let $Y$ be a proper, connected $k$-scheme and $E$ a vector bundle on $Y$. Then the following are equivalent:

1. For all morphisms $f : C \to Y$ from a smooth projective curve $C$, we have that $f^*E$ is semistable of degree 0. (This is also called Nori-semistability)

2. The canonical line bundles $\mathcal{O}_{\mathbb{P}(E)}(1)$, resp. $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ on the associated projective bundle $\mathbb{P}(E)$, resp. $\mathbb{P}(E^\vee)$ are numerically effective.

Definition 4.2. A vector bundle $E$ satisfying the equivalent conditions in proposition 4.1 is called numerically flat.

Remark 4.3. Assume for simplicity that $Y$ is a smooth, projective curve. Over a characteristic 0 field one can check that numerically flat vector bundles are simply the semistable vector bundles of degree 0.

One of the main complications in the theory of vector bundles over a field of positive characteristic is the fact that a semistable bundle might become unstable after pullback along an inseparable morphism (in contrast to this, semistability is preserved under pullback along any separable map). In characteristic $p$ the numerically flat vector bundles coincide with so called strongly semistable vector bundles of degree 0, i.e. bundles $E$ for which $F^{n*}E$ is semistable of degree 0 for all $n \geq 0$.

In contrast to the category of degree 0 semistable vector bundles, the category of numerically flat bundles is still well behaved over a field of positive characteristic. In particular it is a neutral Tannakian category, which has been extensively studied in [Lan11].
A standard example of a semistable vector bundle in positive characteristic which becomes unstable after Frobenius pullback is given by $F_* C^* O_C$ for a smooth projective curve $C$ of genus $\geq 2$. In this case a direct computation of the degree shows that $F_* C^* O_C \to O_C$ destabilizing.

We record the following

**Lemma 4.4.** Let $Y$ be a proper connected scheme over a perfect field $k$. Then a vector bundle $E$ is numerically flat on $Y$ if and only if $f^* E$ is numerically flat for any proper surjective morphism $f : Z \to Y$. Moreover $E$ is numerically flat if and only if $E_{k'}$ is numerically flat for any field extension $k'/k$.

**Proof.** Using the characterization of numerically flat bundles via Nori-semistability we see that we are reduced to show that a vector bundle $E$ on a smooth projective curve $C$ is strongly semistable of degree 0 if and only if $f^* E$ is strongly semistable for some finite map of smooth curves $f : C' \to C$. But this is a standard result in the theory of vector bundles on curves.

The second statement follows similarly from the invariance of semistability with respect to arbitrary field extensions (see [HL10, Theorem 1.3.7]).

In particular we see that if $E$ is numerically flat, $F^n_* E$ is also numerically flat for all $n \geq 0$.

The main goal of this section is to generalize a structure theorem for numerically flat bundles ([DW17, Theorem 2.2]) to non-projective proper schemes. As we will later study formal models over $\text{Spec}(\mathcal{O}_{\mathbb{C}_p}/p)$ we will actually immediately deal with the situation of a proper scheme over $\mathcal{O}_{\mathbb{C}_p}/p$.

**Theorem 4.5.** Let $Y$ be a proper connected scheme over $\text{Spec}(\mathcal{O}_{\mathbb{C}_p}/p)$ and let $E$ be a vector bundle on $Y$.

Then $E \otimes \bar{\mathbb{F}}_p$ is numerically flat on $Y \times_{\text{Spec}(\mathcal{O}_{\mathbb{C}_p}/p)} \text{Spec}(\bar{\mathbb{F}}_p)$ if and only if there exists a finite étale cover $f : Y' \to Y$, and an $e \geq 0$ such that $F_e^* f^* E \cong O_{Y'}$.

The proof for projective schemes over $\bar{\mathbb{F}}_p$ in [DW17] relies on Langer’s boundedness theorem for semistable sheaves (see [Lan04]). Using these results Deninger-Werner show the following

**Proposition 4.6.** [DW17, Theorem 2.4] Let $Y$ be a projective connected scheme over $\mathbb{F}_q$. Then the set of isomorphism classes of numerically flat vector bundles of fixed rank $r$ on $Y$ is finite.

From this it follows that for any numerically flat bundle $E$ there exist numbers $r > s \geq 0$, such that $F_r^* E \cong F_s^* E$. One then concludes with the following

**Theorem 4.7.** [LS77, Satz 1.4] [Kat73, Proposition 4.1] Let $Z$ be any $\mathbb{F}_p$-scheme and $G$ a vector bundle on $Z$ for which there exists an isomorphism $F_n^* G \cong G$ for some $n > 0$. Then there is a finite étale cover of $Z$ on which $G$ becomes trivial.

As the author knows of no way to bound vector bundles on non-projective schemes, we are not able to show finiteness of isomorphism classes as in proposition 4.6. The proof of theorem 4.5 will instead be an application of $v$-descent for perfect schemes as established in [BS17]. For this we will briefly recall the necessary ingredients.

**Definition 4.8.** An $\mathbb{F}_p$-scheme $Z$ is called perfect if $F_Z$ is an automorphism. The category of perfect $\mathbb{F}_p$-schemes will be denoted by $\text{Perf}$. 

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Remark 4.9.  • As explained in [BST17, §3] there is a functor $Z \mapsto Z_{\text{perf}}$ from $\mathbb{F}_p$-schemes to the category of perfect schemes, where $Z_{\text{perf}} = \lim_{\to} F_Z^r Z$ denotes the inverse limit along the absolute Frobenius morphism.

• The topology on $\text{Perf}$ studied in loc. cit. is the so called $v$-topology, which is a non-noetherian version of the $h$-topology. For its definition we refer to [BST17, §2].

The only thing we need here, is that for any proper surjective cover $f : Z' \to Z$ of $\mathbb{F}_p$-schemes, $f_{\text{perf}}$ is a $v$-cover.

The main result for us is then:

Theorem 4.10. [BST17, Theorem 4.1] Let $\text{Vect}_{r}(\_)$ denote the groupoid of vector bundles of rank $r$. The association $Z \mapsto \text{Vect}_{r}(Z)$ is a $v$-stack on $\text{Perf}$.

We are now ready to prove the main result of this section:

Proof of theorem 4.3. Let $Y$ be proper connected over $\mathcal{O}_{C_{\eta}}/p$ and $E$ a vector bundle on $Y$. By standard descent results for finitely presented modules, we can assume that $Y$ and $E$ descend to $(Y', E')$ over $\mathcal{O}_K/p$ for some finite extension $K/\mathbb{Q}_p$. Let $\pi \in \mathcal{O}_K$ be a uniformizer and $\mathcal{O}_K/\pi = \mathbb{F}_q$. Assume first that $Y'$ is projective. Then the following argument is essentially already contained in the proof of Theorem 2.2 in [DW17]:

By proposition 4.6 there are only finitely many numerically flat bundles on $Y' \times \text{Spec}(\mathbb{F}_q)$ up to isomorphism. But $Y'$ is an infinitesimal thickening of $Y' \times \text{Spec}(\mathbb{F}_q)$ (as $\mathcal{O}_K/p$ is an Artin ring). But then the lifts of a fixed vector bundle $G$ on $Y' \times \text{Spec}(\mathbb{F}_q)$ to $Y'$ are parametrized by a finite dimensional vector space over $\mathbb{F}_q$. This means that there are only finitely many vector bundles of rank $r$ on $Y'$ whose reduction mod $\pi$ is numerically flat. As $F_{Y'}^{s*} E'$ lies in this set for all $n \geq 0$, we find some natural numbers $r > s \geq 0$ such that $F_{Y'}^{s*} E' \cong F_{Y'}^{r*} E'$.

Now assume that $Y'$ is proper but not projective. By Chow’s lemma we can find a proper surjective cover $f : Z \to Y'$ where $Z$ is projective over $\mathcal{O}_K/p$. By lemma 4.3 $f^* E'$ is a numerically flat vector bundle. We have the canonical gluing datum $\phi_{\text{can}} : pr_1^* (f^* E') \to pr_2^* (f^* E')$ where $pr_1, pr_2 : Z \times Y' \to Z$ denote the canonical projections.

Claim. The set $M = \{ \text{descent data } (G, \phi) \text{ wrt } f \text{ where } G \text{ is numerically flat of rk } r \}/\text{Iso is finite.}

The claim follows from the fact that $G$ runs through finitely many isomorphism classes and $\phi$ lies in the finite $\mathbb{F}_q$-vector space $\text{Hom}_{Z \times Y', Z}(pr_i^* G, pr_i^* G)$.

As Frobenius commutes with all maps, $F_Z$ acts on $M$. Hence we get an isomorphism

$$\Psi : F_Z^{s*} (f^* E', \phi_{\text{can}}) \cong F_Z^{s*} (f^* E', \phi_{\text{can}})$$

for some natural numbers $r > s \geq 0$. Now of course $\Psi$ will in general not descend to an isomorphism between $F_Y^{s*} E'$ and $F_Y^{r*} E'$. But by theorem 4.10 after passing to the perfection, we see that $f_{\text{perf}} : Z_{\text{perf}} \to Y'_{\text{perf}}$ satisfies effective descent for vector bundles. Hence $\pi_Z^{s*} (\Psi)$ descends to an isomorphism $\pi_Y^{s*} (F_Y^{s*} E') \cong \pi_Y^{s*} (F_Y^{r*} E')$ where $\pi_Z : Z_{\text{perf}} \to Z$ and $\pi_Y : Y_{\text{perf}} \to Y'$ denote the canonical projections.

But the category of (descent data of) vector bundles on $Z_{\text{perf}}$ is the colimit of the categories of (descent data of) vector bundles on copies of $Z$ along Frobenius pullbacks. But then we see that $\Psi$ already becomes effective after a high enough Frobenius pullback. This gives an isomorphism $F_Y^{s*} F_Y^{r*} E' \cong F_Y^{s*} F_Y^{r*} E'$ for some $n >> 0$. 

\[\square\]
4.2 The Deninger-Werner correspondence

In this section we will prove our main result, which is the construction of $p$-adic representations attached to vector bundles with numerically flat reduction on an arbitrary proper (seminormal) rigid analytic variety $X$. Moreover we will later show that our representations coincide with the ones constructed in [DW17] whenever $X$ is the analytification of a smooth algebraic variety over $\mathbb{Q}_p$.

We will treat the integral and rational case simultaneously. So let $\mathcal{X}$ be a proper connected admissible formal scheme over $\text{Spf}(\mathcal{O}_{\mathbb{C}_p})$ with generic fiber $X$.

**Definition 4.11.** Define $\mathcal{B}^s(\mathcal{X})$ to be the category of vector bundles $\mathcal{E}$ on $\mathcal{X}$ for which $\mathcal{E} \otimes \mathcal{F}$ is a numerically flat vector bundle.

Similarly, we define $\mathcal{B}^s(X)$ to be the category of vector bundles on $X$ for which there exists an integral formal model with numerically flat reduction.

We remark the following (compare with [DW17, §9]):

**Lemma 4.12.** The categories $\mathcal{B}^s(\mathcal{X})$ and $\mathcal{B}^s(X)$ are closed under tensor products, extensions, duals, internal homs and exterior powers.

**Proof.** The claim for $\mathcal{B}^s(\mathcal{X})$ follows from the analogous statement for numerically flat vector bundles on a proper scheme, which is well known (alternatively it can be deduced from theorem 4.5).

For $\mathcal{B}^s(X)$ we just note that by the fundamental results of Raynaud, for any two formal models $\mathcal{X}, \mathcal{Y}$ of $X$, there exists an admissible blowup $\mathcal{X}' \to \mathcal{X}$, together with a morphism $\mathcal{X}' \to \mathcal{Y}$ (which can actually also be assumed to be an admissible blowup), which also induces an isomorphism on the generic fiber.

Using this the claim follows. \qed

Recall that we have a canonical projection $\mu : X_{\text{pro\acute et}} \to X_{\text{Zar}}$. For any $\mathcal{E} \in \text{Vect}(\mathcal{X})$ we again denote by $\mathcal{E}^+ := \mu^{-1}\mathcal{E} \otimes_{\mu^{-1}\mathcal{O}_X} \mathcal{O}_X^+$ its pullback to the pro-étale site.

**Proposition 4.13.** For any $\mathcal{E} \in \mathcal{B}^s(\mathcal{X})$, the pullback to the pro-étale site $\mathcal{E}^+$ is contained in $\mathcal{B}^{\text{p\acute et}}(\mathcal{O}_X^+)$.\footnote{We get our main result:}

**Theorem 4.14.** The composition $\mathcal{B}^s(\mathcal{X}) \xrightarrow{\mu^*} \mathcal{B}^{\text{p\acute et}}(\mathcal{O}_X^+) \xrightarrow{\rho_{\mathcal{O}_X}} \text{Rep}_{\pi_1}(\mathcal{O}_{\mathbb{C}_p})$ is an exact functor of tensor categories

$$DW : \mathcal{B}^s(\mathcal{X}) \to \text{Rep}_{\pi_1}(\mathcal{O}_{\mathbb{C}_p})$$

compatible with duals, internal homs and exterior products. Moreover, for any morphism $f : \mathcal{Y} \to \mathcal{X}$ of proper connected admissible formal schemes over $\text{Spf}(\mathcal{O}_{\mathbb{C}_p})$, with generic fiber $f_{\mathbb{C}_p} : Y \to X$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{B}^s(\mathcal{X}) & \xrightarrow{DW} & \text{Rep}_{\pi_1}(\mathcal{O}_{\mathbb{C}_p}) \\
\downarrow f_{\mathbb{C}_p} & & \downarrow f_{\mathbb{C}_p} \\
\mathcal{B}^s(\mathcal{Y}) & \xrightarrow{DW} & \text{Rep}_{\pi_1}(Y)(\mathcal{O}_{\mathbb{C}_p}).
\end{array}$$
Proof. Everything follows from theorem 3.9 and lemma 2.3.

Using theorem 3.13 we see that:

**Corollary 4.15.** There is an exact tensor functor

\[ DW_Q : B^*(X) \rightarrow \text{Rep}_{\pi_1(X)}(\mathbb{C}_p) \]

which is compatible with duals, internal homs and exterior products. It is again compatible with pullback along any morphism \( f : X \rightarrow Y \) of proper connected rigid analytic varieties. If \( X \) is seminormal, then \( DW_Q \) is fully faithful.

In the language of \([Xu17]\), proposition 4.13 implies that all vector bundles with numerically flat reduction are Weil-Tate (compare definition 3.14). The case of curves has already been dealt with in \([Xu17, \text{Corollaire 14.5}]\) (using the Faltings topos). All Weil-Tate vector bundles are semistable in the following sense (compare with \([DW17, \text{Theorem 9.7}]\)):

**Proposition 4.16.** Let \( E \) be a vector bundle on \( X \) such that \( \lambda^*E \otimes \hat{O}_X \) is trivialized by a profinite étale cover. Then \( f^*E \) is semistable of degree 0 for every morphism \( f : C \rightarrow X \) where \( C \) is a smooth projective curve.

**Proof.** We have to show that any vector bundle \( E \) on a smooth projective curve for which \( \lambda^*E \otimes \hat{O}_X \) is profinite étale trivializable, is semistable of degree 0. So assume that \( X \) is a curve of genus \( g \), and that \( \hat{Y} = \lim_{\longrightarrow} Y_i \rightarrow X \) is a profinite étale cover trivializing \( \lambda^*E \otimes \hat{O}_X \). If \( \hat{Y} \) stabilizes, i.e. if \( \lambda^*E \otimes \hat{O}_X \) becomes trivial on some finite étale cover \( f_i : Y_i \rightarrow X \), then the pullback \( f_i^*E \) to \( Y_i \) is also trivial (as \( \lambda_{Y_i*}\hat{O}_{Y_i} = \hat{O}_{(Y_i)_{\text{an}}} \), so in particular semistable of degree 0. But then \( E \) is also semistable of degree 0. So assume that \( \hat{Y} \) does not stabilize. We can assume that \( \text{deg}(E) \geq 0 \) (otherwise pass to the dual bundle). Let \( r \) be the rank of \( E \). Assume that \( L \subset E \) is destabilizing. By passing to exterior powers we can assume that \( L \) is a line bundle, and by passing to tensor powers we can assume that \( \text{deg}(L) \geq g \). Then by Riemann-Roch \( \dim_{\mathbb{C}_p} \Gamma(Y_i, f_i^*L) \geq \text{deg}(f_i) \), where \( f_i \) is the finite étale map \( Y_i \rightarrow X \). As \( \hat{Y} \) does not stabilize, \( \text{deg}(f_i) \) must grow to infinity.

But \( \lim_{\longrightarrow} \Gamma(Y_i, f_i^*L) = \Gamma(\hat{Y}, L) \subset \Gamma(\hat{Y}, E) \subset \Gamma(\hat{Y}, E \otimes \hat{O}_X) \subset \mathbb{C}_p^r \).

In the same way one may check that the degree of \( E \) must be 0.

If \( X \) is an algebraic variety the proposition says that \( E \) is numerically flat. In general, if \( X \) is not algebraic, one may expect that \( E \) is semistable with respect to any polarization on the special fiber of a formal model in the sense of \([Li17]\). We do not pursue this question here.

### 4.2.1 The case of line bundles

We will show that line bundles on a rigid analytic variety \( X \) over \( \mathbb{C}_p \) which are deformations of the trivial bundle over a connected quasi-compact base are Weil-Tate. In case the Picard functor is representable and its identity component \( Pic^0_{\pi_1}(X) \) is bounded, this implies that all line bundles in \( Pic^0_{\pi_1}(\mathbb{C}_p) \) are Weil-Tate. General representability results for the Picard functor have been announced in \([War17]\) (see in particular Theorem 1.0.2 in loc. cit.). If \( Pic_{\pi_1}(X) \) is representable and in addition \( X \) is proper smooth and admits a formal model whose special fiber is projective, then the properness of \( Pic^0_{\pi_1}(X) \) is ensured by \([Li17, \text{Theorem 1.1}]\).

More generally we will prove that any line bundle \( L \) for which some tensor power is a qc connected deformation of the trivial bundle gives rise to a local system.
Definition 4.17. Let $X$ be a proper connected rigid analytic variety over $\mathbb{C}_p$. Then a line bundle $L$ on $X$ is said to be analytically equivalent to $\mathcal{O}_X$ over $T$, where $T$ is a connected rigid-analytic variety over $\mathbb{C}_p$, if there exists a line bundle $\tilde{L}$ on $X \times T$, such that both $L$ and $\mathcal{O}_X$ occur as fibers of $\tilde{L}$.

We call a line bundle $L$ $\tau$-equivalent to $\mathcal{O}_X$ if some tensor-power of $L$ is analytically equivalent to $\mathcal{O}_X$. 

Remark 4.18. If $\text{Pic}_X$ is representable, one may check that the line bundles which are analytically equivalent to the trivial bundle are precisely the ones in $\text{Pic}_X^0(\mathbb{C}_p)$.

In this case let $\phi_n : \text{Pic}_X \to \text{Pic}_X$ denote the $n$-th power map. The group adic space $\text{Pic}_X^\tau := \bigcup_{n \leq 1} \phi_n^{-1}(\text{Pic}_X^0(\mathbb{C}_p)) \subset \text{Pic}_X$ is called the torsion component of the identity. As in the case of schemes (see [Kle14, §22]) one sees that, whenever $\text{Pic}_X$ is representable, $\text{Pic}_X^\tau$ is also representable by an open subgroup adic space.

We do not need this here.

Lemma 4.19. Let $Z$ be a proper reduced connected rigid analytic variety over $\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$, and let $L$ be a line bundle on $Z$ which is analytically equivalent to $\mathcal{O}_Z$ over a quasi-compact base $T$, then $L$ has a formal model $L$ on some $Z$, such that $L_s$ is numerically flat.

In case $\text{Pic}_Z^0$ exists and is proper the lemma is equivalent to the claim that all $L \in \text{Pic}_Z^0(\mathbb{C}_p)$ have numerically flat reduction.

Proof. Let $P$ be said family on $Z \times T$. By assumption there exist points $v : \text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to T$ and $v_0 : \text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to T$ such that $v^* P = L$ and $v_0^* P \cong \mathcal{O}_Z$. Using the reduced fiber theorem from [BLR95] we can find a connected formal model $\tilde{Z} \times T$ of $Z \times T$, such that $\tilde{Z}$ is proper connected and has reduced special fiber, together with a locally free model $\mathcal{P}$ of $P$ (using the flattening techniques in [BL93, Theorem 4.1]).

Let $\tilde{v}, \tilde{v}_0 : \text{Spf}(\mathcal{O}_{\mathbb{C}_p}) \to T$ be the specializations of $v$ and $v_0$. Moreover we can replace $\mathcal{P}$ by another model $\tilde{\mathcal{P}}$ for which some isomorphism $(\tilde{v}_0^* \mathcal{P})|_{\mathbb{C}_p} \cong \mathcal{O}_Z$ extends to an isomorphism $\tilde{v}_0^* \mathcal{P} \cong \mathcal{O}_Z$: Namely, simply consider the pullback $pr_2^\tau((\tilde{v}_0^* \mathcal{P})^{-1})$. Then $\mathcal{P}' = pr_2^\tau((\tilde{v}_0^* \mathcal{P})^{-1}) \otimes \mathcal{P}$ does the job.

But then the special fiber of $L = \tilde{v}^* \mathcal{P}$ is a deformation of the trivial line bundle over a connected base. Using Chow’s lemma there is a proper surjective map $\pi : Y \to Z_s$ where $Y$ is a normal projective variety. Then $\text{Pic}_Y^\tau$ exists by classical results, and all line bundles in $\text{Pic}_Y^\tau(\mathbb{F}_p)$ are numerically flat. Pulling back the family $\mathcal{P}_s$ along $\pi$ shows that $\pi^* \mathcal{L}_s$ lies in $\text{Pic}_Y^\tau(\mathbb{F}_p)$ and hence is numerically flat. But then $\mathcal{L}_s$ is numerically flat as well. 

We define $\hat{\mathcal{C}}_p := \hat{\mathcal{O}}_{\mathbb{C}_p}[\frac{1}{p}]$, as sheaves on the pro-étale site, and say that a vector bundle $E$ is associated to a $\mathbb{C}_p$-local system, if there exists a locally free $\hat{\mathcal{C}}_p$-module $L$ such that $L \otimes \hat{\mathcal{O}}_X \cong \hat{\mathcal{O}}_X \otimes \hat{\lambda}^* E$.

Proposition 4.20. Let $L$ be a line bundle on $X$ such that $L^{\otimes n}$ is Weil-Tate. Then $L$ is associated to a $\hat{\mathcal{C}}_p$-local system $\mathbb{M}$.

Proof. We use Kummer-theory on the pro-étale site. Let $\hat{\mathcal{L}}$ be the $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system associated to $L^{\otimes n}$.

Claim. There is a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mu_n \longrightarrow \hat{\mathcal{O}}^\times_{\mathbb{C}_p} \longrightarrow \hat{\mathcal{O}}^\times_{\mathbb{C}_p} \longrightarrow 0 \\
0 & \longrightarrow & \mu_n \longrightarrow \hat{\mathcal{O}}^\times_X \longrightarrow \hat{\mathcal{O}}^\times_X \longrightarrow 0 \\
\end{array}
\]
of Kummer exact sequences.

The commutativity of the diagram is clear. Moreover so is exactness of the upper sequence. For the lower sequence, note that as usual the only non-trivial part is showing right exactness. This follows in the same way as exactness of the Artin-Schreier sequence (for the tilted structure sheaf) proved in the proof of [Sch13a, Theorem 5.1]: if $U = \lim Spa(A_i, A_i^+)$, then $\hat{\mathcal{O}}_X(U) = A$ and so any equation $x^n - a = 0$ for fixed $a \in \hat{\mathcal{O}}_X(U)$ can be solved by passing to a finite étale extension of $A$, which gives a finite étale map of affinoid perfectoid spaces $\hat{V} \to \hat{U}$, say $\hat{V} = Spa(B, B^+)$. But now by [Sch12, Lemma 7.5] any finite étale cover of $\hat{U}$ comes from a finite étale cover $V \to U$ in $X_{pro\acute{e}t}$ where then $V$ is affinoid perfectoid (as an object in $X_{pro\acute{e}t}$) with $\hat{V} = \hat{V}$. Hence $\hat{\mathcal{O}}_X(V) = B$, so we can find an $n$-th root of $a$ by passing to the étale cover $V \to U$.

Now, taking pro-étale cohomology gives a commutative diagram

$$
\begin{array}{c}
H^1(\mu_n) \to H^1(\mathcal{O}_{C_p}) \to H^1(\hat{\mathcal{O}}_X^\times) \to H^2(\mu_n) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H^1(\mu_n) \to H^1(\hat{\mathcal{O}}_X^\times) \to H^1(\hat{\mathcal{O}}_X^\times) \to H^2(\mu_n),
\end{array}
$$

where the map $H^1(\hat{\mathcal{O}}_X^\times) \to H^1(\hat{\mathcal{O}}_X^\times)$ is given by taking a rank 1 local system $L$ to $\hat{\mathcal{O}}_X \otimes L$. By some diagram chasing one gets from this that there exists an $\mathcal{O}_{C_p}$-local system $\mathbb{H}$ such that $\mathbb{H}^{\otimes n} \cong L$. From this one sees that

$$(\mathbb{H} \otimes \hat{\mathcal{O}}_X)^{\otimes n} \cong L \otimes \hat{\mathcal{O}}_X \cong (\lambda^*L \otimes \hat{\mathcal{O}}_X)^{\otimes n}.$$

But then these sheaves become isomorphic on some finite étale Kummer cover $\pi : X' \to X$, so that $\lambda_X^* (\pi^*L) \otimes \hat{\mathcal{O}}_{X'} = \pi_{pro\acute{e}t}^*(\lambda_X^*L \otimes \hat{\mathcal{O}}_X) \cong \pi_{pro\acute{e}t}^*\mathbb{H} \otimes \hat{\mathcal{O}}_{X'}$. In particular $\pi^*L$ is Weil-Tate.

We conclude with the following lemma:

**Lemma 4.21.** Let $\pi : X' \to X$ be a finite étale cover and let $E$ be a vector bundle on $X$ such that $\pi^*E$ is associated to an $\mathcal{O}_{C_p}$-local system $\mathbb{K}$. Then $E$ is associated to a $\hat{\mathcal{C}}_p$-local system.

For this assume that $\pi$ is Galois with Galois group $G$. We then have a canonical Galois descent datum on $\pi_{pro\acute{e}t}^*(E \otimes \hat{\mathcal{O}}_X) \cong \mathbb{K} \otimes \hat{\mathcal{O}}_{X'}$. This induces a descent datum on $\mathbb{K}$, as $\mathbb{K} \otimes \hat{\mathcal{O}}_{X'}$ is fully faithful (by corollary 4.12), hence it descends to some $M$ on $X$ and then, as the glueing datum is compatible with the one on $\pi_{pro\acute{e}t}^*(E \otimes \hat{\mathcal{O}}_X)$, one has $M \otimes \hat{\mathcal{O}}_X \cong E \otimes \hat{\mathcal{O}}_X$.

**Corollary 4.22.** If $L$ is a line bundle on $X$ which is $\tau$-equivalent to $\mathcal{O}_X$ over a quasicompact base, then $L$ is associated to a $\hat{\mathcal{C}}_p$-local system.

If $X$ is a normal algebraic variety, this $C_p$-local system admits a lattice, in which case $L$ is Weil-Tate.

We recall that if $X$ is a projective scheme, then the line bundles in $\text{Pic}^\tau(X)$ are precisely the numerically flat line bundles.
4.3 Étale parallel transport

In this section we wish to show that the discussion in section 3.1 can be upgraded to construct étale parallel transport on pro-étale trivializable vector bundles. We will then compare our construction to the one from [DW17]. This is in some sense close to the discussion in [Xu17] §8, where the results from [DW05b] for the curve case are recast in light of the Faltings topos. Note that even though we need to pass through almost mathematics our final statement (theorem 4.28) will be an honest isomorphism even at the integral level.

We will first recall the concept of étale parallel transport.

Definition 4.23. [DW05b] §3 Let $X$ be a proper, connected rigid analytic variety over $\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. The étale fundamental groupoid $\Pi_1(X)$ of $X$ is defined to be the category whose objects are given by $X(\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}))$ and for any two points $x, y \in X(\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}))$ we set $\text{Mor}(x,y) = \text{Isom}(F_x,F_y)$. Here $F_x$ denotes the étale fiber functor with respect to the point $x$. $\text{Mor}(x,y)$ carries the profinite topology, making $\Pi_1(X)$ into a topological groupoid.

Remark 4.24. Assume that $X$ is a finite type scheme over $\text{Spec}(\mathbb{C}_p)$, and denote by $X^{an}$ its analytification. By [Liu93] Theorem 3.1], every finite étale cover of $X^{an}$ is algebraizable. In particular one gets an equivalence $\Pi_1(X) \cong \Pi_1(X^{an})$ of fundamental groupoids.

Let $\text{Free}_r(\mathcal{O}_{\mathbb{C}_p})$ (resp. $\text{Free}_r(\mathbb{C}_p)$) denote the topological groupoid of free rank $r$ modules over $\mathcal{O}_{\mathbb{C}_p}$ (resp. $\mathbb{C}_p$). Let $E$ be a vector bundle on $X$. We say that $E$ has étale parallel transport if the association $x \mapsto E_x$, can be extended to a functor $\Pi_1(X) \to \text{Free}_r(\mathbb{C}_p)$. Similarly, for a locally free $\mathcal{O}_{X^{an}}^+$-module $E^+$, we say that $E^+$ has étale parallel transport if $x \mapsto \Gamma(x^*E^+)$ can be extended to a functor $\Pi_1(X) \to \text{Free}_r(\mathcal{O}_{\mathbb{C}_p})$.

Let $\mathcal{E}^+ \in \mathcal{B}^{\mathit{proet}}(\mathcal{O}_X^+)$, such that $\hat{\mathcal{E}}^+$ is trivial on $\hat{Y}$. Let $r$ denote the rank of $\mathcal{E}^+$. We can define a functor

$$\alpha_{\mathcal{E}^+}: \Pi_1(X) \to \text{Free}_r(\mathcal{O}_{\mathbb{C}_p})$$

of pro-étale parallel transport on $\hat{\mathcal{E}}^+$ as in [DW17] §4 (see also section 3): On objects $\alpha_{\mathcal{E}^+}$ takes $x \in X(\mathbb{C}_p)$ to $\hat{\mathcal{E}}^+_x$. On morphisms, if we have an étale path $\gamma \in \text{Mor}(\Pi_1(X),x,x')$, we can pick a $\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$-valued point $y$ of $\hat{Y}$ lying over $x$, then $\gamma$ induces a point $y'$ over $x'$. Using triviality of $\mathcal{E}^+$ on $\hat{Y}$ pulling back global sections along $y, y'$ will then give isomorphisms

$$\hat{\mathcal{E}}^+_x \xleftarrow{y^*} \Gamma(\hat{Y}, \hat{\mathcal{E}}^+) \xrightarrow{y'^*} \hat{\mathcal{E}}^+_{x'}.$$ 

and so we let $\gamma$ map to $y'^* \circ (y^*)^{-1}$.

As in section 3.1 one can check that this is independent of the trivializing cover $\hat{Y}$ and the chosen point $y$. Furthermore one can check in a similar fashion as in section 3.1 that this is a continuous functor of topological groupoids. Fixing a base point then gives back the representation from theorem 3.9.

Now, if $E$ is a vector bundle on $X$ such that the pullback $x^*E$ to the pro-étale site lies in $\mathcal{B}^{\mathit{proet}}(\mathcal{O}_X)$ we can also define étale parallel transport on $E$: For any point $x: \text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to X$ there is a canonical isomorphism

$$\Gamma(x^*E) \cong \Gamma(x^*(\Lambda^*E)) \cong \Gamma(x^*(\Lambda^*E \otimes \mathcal{O}_X)).$$
Hence, using étale parallel transport on \( E \otimes \mathcal{O}_X \), we get an isomorphism \( E_x \cong_{E_y} \) for any étale path \( x \mapsto y \). Moreover this construction is compatible with composition of étale paths.

The same construction also works for locally free \( \mathcal{O}_{X_{\text{an}}}^+ \)-modules \( E^+ \) whose pullback to the pro-étale site lies in \( \mathcal{B}^{\text{pro-ét}}(\mathcal{O}_X^+) \). We arrive at:

**Proposition 4.25.** Let \( E \) be a vector bundle on \( X \) (resp. let \( E^+ \) be a locally free \( \mathcal{O}_{X_{\text{an}}}^+ \)-module) for which the pullback to the pro-étale site \( \lambda^*E \) (resp. \( \lambda^*E^+ \)) lies in \( \mathcal{B}^{\text{pro-ét}}(\mathcal{O}_X^+) \). Then \( E \) (resp. \( E^+ \)) has étale parallel transport.

In particular we get functors

\[
\alpha : \mathcal{B}^{\text{pro-ét}}(\mathcal{O}_{X_{\text{an}}}) \to \text{Rep}_{\Pi_1(X)}(\mathbb{C}_p)
\]

\[
\alpha_{\mathcal{O}_p} : \mathcal{B}^{\text{pro-ét}}(\mathcal{O}_{X_{\text{an}}}^+) \to \text{Rep}_{\Pi_1(X)}(\mathcal{O}_p).
\]

**Remark 4.26.** Here \( \text{Rep}_{\Pi_1(X)}(\mathcal{O}_p) \) denotes the category of continuous functors from \( \Pi_1(X) \) to \( \text{Free}_e(\mathcal{O}_p) \). A functor \( F : \Pi_1(X) \to \text{Free}_e(\mathcal{O}_p) \) is called continuous if the induced maps on morphisms \( \text{Mor}(x, x') \to \text{Mor}(F(x), F(x')) \) are continuous maps for all \( x, x' \in \Pi_1(X) \).

Note that whenever \( E^+ = \mathcal{O}_{X_{\text{an}}}^+ \otimes_{\mathcal{O}_X} \mathbb{C}_p \) comes from an integral model \((\mathcal{X}, \mathcal{E})\) of \((X, E)\) the canonical isomorphisms \( \Gamma(\mathcal{O}_X^+) \cong \Gamma(\mathcal{X}) \) also allow us to define parallel transport on \( \mathcal{E} \).

For the comparison with the Deninger-Werner construction we need to work modulo \( p^n \). As we have less control here, we need to pass through the almost setting. So denote by \( \text{Free}_e((\mathcal{O}_p/p^n)_\ast) \) the groupoid (endowed with the discrete topology) of free \((\mathcal{O}_p/p^n)_\ast\)-modules of rank \( r \), where \((\mathcal{O}_p/p^n)_\ast\) again denotes the almost elements. We can then similarly define a mod \( p^n \) almost parallel transport

\[
\alpha_{\mathcal{O}_p}^n(\mathcal{E}^+/p^n) : \Pi_1(X) \to \text{Free}_e((\mathcal{O}_p/p^n)_\ast).
\]

Now as in the proof of proposition 3.3 we see that \((\mathcal{O}_p/p^n)^\ast = \Gamma(\mathcal{Y}^\ast) \cong \Gamma(\mathcal{Y}^\ast/p^n)_\ast \) realizes \( \alpha(\mathcal{E})(\gamma)/p^n \) as a subobject of \( \alpha_{\mathcal{O}_p}^n(\mathcal{E}^+/p^n)(\gamma) \) for any étale path. Here by subobject we mean that there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}_x^+/p^n & \xrightarrow{\alpha(\mathcal{E})(\gamma)/p^n} & \mathcal{E}_x^+/p^n \\
\downarrow \alpha_{\mathcal{O}_p}^n(\mathcal{E}^+/p^n) & & \downarrow \\
(\mathcal{E}_x^+/p^n)_\ast & \xrightarrow{\alpha_{\mathcal{O}_p}^n(\mathcal{E}^+/p^n)(\gamma)} & (\mathcal{E}_{x'}^+/p^n)_\ast
\end{array}
\]

(4.1)

for each étale path \( \gamma \) from \( x \) to \( x' \), and this association is compatible with composition of paths.

Assume now that \( X \) is a proper smooth algebraic variety over \( \mathbb{Q}_p \) with a flat proper integral model \( \mathcal{X} \) over \( \mathbb{Z}_p \). Assume further, that we have a vector bundle \( \mathcal{E} \) on \( \mathcal{X}_{\mathbb{C}_p} \) with numerically flat reduction. Then one of the main results in [DW17] is the following:

**Proposition 4.27.** [DW17, Theorem 7.1] Fix \( n \geq 1 \). Then there exists an open cover \( \{U_i\} \) of \( X \), such that for every \( i \) there is a proper surjective map \( f_i : \mathcal{Y}_i \to \mathcal{X} \) which is finite étale over \( U_i \) and is such that \( f_i^* \mathcal{E} \) is trivial mod \( p^n \).

One can then moreover assume that \( \mathcal{Y}_i \) is a good model in the sense of [DW17, Definition 3.5]. Using this result they construct a parallel transport functor \( \alpha_{\mathcal{O}_{n,1}}^D(\mathcal{E}) : \Pi_1(U_i) \to \text{Free}_e(\mathcal{O}_p/p^n) \) for every \( i \), which is then shown to glue to a functor \( \alpha_{n,1}^D(\mathcal{E}) \) from \( \Pi_1(X) \),
which is again functorial in $\mathcal{E}$.

The construction of $\alpha_{n,i}^{\text{DW}}(\mathcal{E})$ is of course as above: For any étale path $\gamma$ from $x$ to $x' \in U_i(\mathcal{O}_p)$ one gets an isomorphism

$$\Gamma(\mathcal{F}/\mathcal{E}/p^n) = \Gamma(\mathcal{F}/\mathcal{E}/p^n) \xrightarrow{\alpha} \Gamma(f_i^*\mathcal{E}/p^n) \xrightarrow{\alpha} \Gamma((\gamma y)^*f_i^*\mathcal{E}/p^n) = \Gamma(\mathcal{F}/\mathcal{E}/p^n)$$

where $\mathcal{F}, \cdots$ denotes the specialization with respect to the integral model $\mathcal{X}$ (resp. $\mathcal{Y}$). Taking the projective limit one gets $\alpha_{\mathcal{O}_C}^{\text{DW}}(\mathcal{E}) : \Pi_1(X) \to \text{Free}_r(\mathcal{O}_{C_p})$. Now let $\hat{\mathcal{X}}$ be the admissible formal scheme obtained by completing $\mathcal{X}_{C_p}$ along its special fiber. We denote by $X^{an}$ the adic space generic fiber of $\hat{\mathcal{X}}$. $X^{an}$ then coincides with the analytification of $X_{C_p}$. We have $\Pi_1(X) \cong \Pi_1(X^{an})$ by remark 4.23. Let $\mathcal{E} \in B^s(\mathcal{X}_{C_p})$. Pulling back $\mathcal{E}$ to $\hat{\mathcal{X}}$ gives an object $\hat{\mathcal{E}}$ in $B^s(\hat{\mathcal{X}})$.

Pulling back $\hat{\mathcal{E}}$ further to the pro-étale site gives an object $\mathcal{E}^+$ of $B^p_{\text{et}}(\mathcal{X}_{C_p}^{\text{an}})$.

Denote by $\tilde{\alpha}_{\mathcal{O}_C}$ the composition

$$\tilde{\alpha}_{\mathcal{O}_C} : B^s(\mathcal{X}_{C_p}) \to B^s(\hat{\mathcal{X}}) \xrightarrow{sp^*} B^p_{\text{et}}(\mathcal{X}_{C_p}^{\text{an}}) \xrightarrow{\alpha_{\mathcal{O}_C}} \text{Rep}_{\Pi_1(X)}(\mathcal{O}_{C_p})$$

where $sp^*(\mathcal{E}) = sp^{-1}(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{X}_{C_p}^{\text{an}}}$, and $sp : (\mathcal{X}_{C_p}^{\text{an}})(\gamma) \to \hat{\mathcal{X}}^{\text{zar}}$ denotes the specialization.

**Theorem 4.28.** The functors $\tilde{\alpha}_{\mathcal{O}_C}$ and $\alpha_{\mathcal{O}_C}^{\text{DW}}$ are naturally isomorphic.

**Proof.** We will show that $\tilde{\alpha}_{\mathcal{O}_C}(\mathcal{E})$ is isomorphic to $\alpha_{\mathcal{O}_C}^{\text{DW}}(\mathcal{E})$. Functoriality in $\mathcal{E}$ will be left to the reader.

First fix $n \geq 1$, and let $\{U_i\}$ be an open cover of $X$ and $\{\mathcal{Y}_i\}$ the associated proper, surjective covers trivializing $\mathcal{E}/p^n$ as in proposition 4.27. Denote by $Y_i$ the generic fiber of $\mathcal{Y}_i$. Let $Z_n \to X_{C_p}$ be a finite étale cover trivializing $\mathcal{E}^+/p^n$. Let $Z_n \to \mathcal{X}_{C_p}$ be a good model for $Z_n$ in the sense of [DW17] Definition 3.5). The cover $f_i : Z_n \times X_{C_p} \mathcal{Y}_i \mathcal{O}_C \to \mathcal{X}_{C_p}$ is still a trivializing covering for $\mathcal{E}/p^n$ and finite étale over $U_i$. We can moreover assume that it is connected; otherwise one uses [DW17] Lemma 3.12 to find a connected cover dominating $Z_n \times X_{C_p} \mathcal{Y}_i \mathcal{O}_C$ which is still finite étale over $U_i$. Denote by $f^{\text{an}}_i$ the analytification of $f_i \otimes \mathcal{O}_p$. Then $f^{\text{an}}_i$ trivializes $\mathcal{E}^+/p^n$ and $(\tilde{\alpha}_{\mathcal{O}_C}(\mathcal{E})|_{U_i})$ can be realized on $f^{\text{an}}_i\mathcal{E}^+/p^n$.

Let $x, x' \in U_i(\mathcal{O}_p)$ and let $\gamma \in \text{Mor}(x, x')$ be an étale path. Consider the following commutative diagram of $\mathcal{O}_{C_p}/p^n$-modules:

$$
\begin{array}{ccc}
(\mathcal{E}/p^n)_{\mathcal{X}_{C_p}^{\text{an}}} & \xrightarrow{=} & \Gamma(\mathcal{F}/\mathcal{E}/p^n) \\
(\mathcal{E}^+_x/p^n) & \xrightarrow{=} & \Gamma(f_i^*(\mathcal{E}/p^n)) \\
(\mathcal{E}^+_x/p^n) & \xrightarrow{=} & \Gamma((\gamma y)^*(f_i^*(\mathcal{E}/p^n))) \\
\end{array}
\text{where the upper row is } (\alpha_{n,i}^{\text{DW}}(\mathcal{E})(\gamma)) \text{ and the lower row is } \alpha_{n,i}(\mathcal{E}^+/p^n)(\gamma) \text{ and } y \text{ is a point of } (Z_n \times X_{\mathcal{Y}_i})_{C_p} \text{ above } x, \text{ and } \mathcal{F} \in \mathcal{X}(\mathcal{O}_{C_p}) \text{ is the specialization of } x.
\end{array}
$$

For the construction of the vertical maps note that if $\mathcal{Z}$ is a proper scheme with associated formal scheme $\hat{\mathcal{Z}}$, one has a composition of morphisms of ringed sites

$$(\mathcal{Z}_{\text{pro-ét}}, \mathcal{O}_{\mathcal{Z}}^{\text{an}}) \to (\hat{\mathcal{Z}}_{\text{zar}}, \mathcal{O}_2) \to (\mathcal{Z}_{\text{zar}}, \mathcal{O}_2).$$

The vertical maps are then given by pulling back global sections along this and embedding into almost elements at the end. As everything is functorial, the diagram commutes. All vertical arrows are canonical almost isomorphisms.

More precisely, going through all identifications, one checks that this takes $\alpha_{n,i}^{\text{DW}}(\mathcal{E})(\gamma)$
isomorphically to \((\tilde{\alpha}/p^n)(\mathcal{E})(\gamma) \rightarrow \alpha_n^\alpha(\mathcal{E}^+/p^n)(\gamma)\) (see diagram [4.1]): Indeed, the map on the fiber \((\mathcal{E}/p^n)^{\mathfrak{r}} \rightarrow (\mathcal{E}^+/p^n)^{\mathfrak{r}}\) factors of course through \((\mathcal{E}^+/p^n)^{\mathfrak{r}} \rightarrow (\mathcal{E}^+/p^n)^{\mathfrak{r}}\). Also, everything is compatible with the composition of paths. From this we get an isomorphism \(\alpha_n^{DW}(\mathcal{E})(\gamma) \cong (\tilde{\alpha}/p^n)|_{\mathcal{U}}\). But then, using the Seifert-van-Kampen result from [DW17, Theorem 4.1], we see that \((\alpha_n^{DW}) \cong (\tilde{\alpha}/p^n)\) are isomorphic.

Passing to the \(p\)-adic completion we get the desired result. \(\Box\)

### 4.4 The Hodge-Tate spectral sequence

Keep assuming that \(X\) is proper smooth over \(\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})\).

**Theorem 4.29.** [Sch13b, Theorem 3.20] Let \(E\) be a vector bundle on \(X\) associated to an \(\hat{\mathcal{O}}_{\mathbb{C}_p}\)-local system \(\mathbb{L}\). Then there is a Hodge-Tate spectral sequence.

\[
E_2^{ij} = H^i(X, E \otimes \Omega^j(-j)) \Rightarrow H^{i+j}(X_{\text{pro-et}}, \mathbb{L})[\frac{1}{p}]\]

Recall that \(\Omega^j(-j) := \Omega^j \otimes \hat{\mathbb{Z}}_p(-1)^{\otimes j}\), where \(\hat{\mathbb{Z}}_p(1) := \lim_{\to \mathbb{Z}/p^n}\) as pro-étale sheaves, and \(\hat{\mathbb{Z}}_p(-1)\) denotes the dual of \(\hat{\mathbb{Z}}_p(1)\).

**Proof.** By assumption \(\hat{E} = \hat{\mathcal{O}}_X \otimes \mathbb{L}\). Let again \(\nu : X_{\text{pro-et}} \rightarrow X_\text{ét}\) denote the canonical projection. Then the Cartan-Leray spectral sequence reads

\[
H^i(X_\text{ét}, R^j\nu_*(\hat{\mathcal{O}}_X \otimes \mathbb{L})) \Rightarrow H^{i+j}(X_{\text{pro-et}}, \hat{\mathcal{O}}_X \otimes \mathbb{L}).
\]

By theorem 2.11 we have \(H^{i+j}(X_{\text{pro-et}}, \hat{\mathcal{O}}_X \otimes \mathbb{L}) \cong H^{i+j}(X_{\text{pro-et}}, \mathbb{L})[\frac{1}{p}]\).

But now, by [Sch13b, Proposition 3.23] there is an isomorphism \(R^j\nu_*\hat{\mathcal{O}}_X \cong \Omega^j(-j)\), and hence

\[
R^j\nu_*\hat{\mathcal{O}}_X \otimes \mathbb{L}) = R^j\nu_*(\nu^*E \otimes \mathcal{O}_X \hat{\mathcal{O}}_X) \cong E \otimes R^j\nu_*(\hat{\mathcal{O}}_X) \cong E \otimes \Omega^j(-j).
\]

\(\Box\)

As usual one does not get a canonical splitting of this spectral sequence in general. However one does have the following:

**Proposition 4.30.** Assume that \(X\) is a proper smooth rigid analytic space over \(\text{Spa}(K, \mathcal{O}_K)\) where \(K/\mathbb{Q}_p\) is a finite extension. Let further \(E\) be a vector bundle on \(X\) such that \(E_{\hat{K}}\) is associated to an \(\hat{\mathcal{O}}_{\mathbb{C}_p}\)-local system \(\mathbb{L}\). Then the Hodge-Tate spectral sequence degenerates canonically at \(E_2\).

Note that the local system must not be defined over a finite extension of \(K\), as indeed will generally not be the case for Deninger-Werner local systems.

**Proof.** Now there is a \(G_K := \text{Gal}(\hat{K}/K)\)-action on the cohomology groups in theorem 4.29 and in particular the differentials in the Cartan-Leray spectral sequence will be invariant under this action. But then, as

\[
H^i(X_{\hat{K}}, E_{\hat{K}} \otimes \Omega^j_{X_{\hat{K}}}(-j)) = H^i(X, E \otimes \Omega^j_X) \otimes_K \mathbb{C}_p(-j)
\]

by base change for cohomology, one gets that all differentials are zero, as \(H\text{om}_{G_K}(\mathbb{C}_p(-j), \mathbb{C}_p(-j')) = 0\) for \(j \neq j'\) by Tate’s theorem. \(\Box\)
We do not know whether there is non-canonical degeneration in general (i.e. when $X$ is not defined over a finite extension of $\mathbb{Q}_p$).

For the constant local system, consider the map $\alpha : H^1(X, \mathcal{O}_X) \to H^1_{\text{ét}}(X, \mathbb{Z}_p) \otimes \hat{\mathbb{K}}$ from the Hodge-Tate decomposition. In [DW05a] Deninger and Werner showed that, if $X = A$ is an abelian variety with good reduction, there is a commutative diagram

$$
\begin{array}{ccc}
H^1(A, \mathcal{O}_A) & \xrightarrow{\alpha} & H^1_{\text{ét}}(A, \mathbb{Z}_p) \otimes \hat{\mathbb{K}} \\
\cong & & \cong \\
\text{Ext}^1(\mathcal{O}_A, \mathcal{O}_A) & \xrightarrow{\text{DW}} & \text{Ext}^1(\hat{\mathcal{C}}_p, \hat{\mathcal{C}}_p)
\end{array}
$$

where the map below means applying the Deninger-Werner functor to a unipotent rank 2 vector bundle, which gives a unipotent rank 2 local system.

We can show that this generalizes to arbitrary extensions on any proper smooth rigid analytic variety. Namely, let $X$ be any proper smooth rigid analytic variety over $\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$.

Then for any $E \in \mathcal{B}(X)$ the Hodge-Tate spectral sequence gives an injection $H^1(X, E) \to H^1_{\text{ét}}(X, L_E)$. We then get the following

**Proposition 4.31.** For any $E, E' \in \mathcal{B}(X)$ there is a commutative diagram

$$
\begin{array}{ccc}
H^1(X, \mathcal{H}om(E, E')) & \xrightarrow{\alpha} & H^1_{\text{ét}}(X, \mathcal{H}om(L_E, L_{E'})) \\
\cong & & \cong \\
\text{Ext}^1(E, E') & \xrightarrow{\text{DW}} & \text{Ext}^1(L_E, L_{E'})
\end{array}
$$

where for any $F \in \mathcal{B}(X)$ we denote by $L_F$ the local system obtained by applying the functor $\text{DW}$, and $\alpha$ denotes the map coming from the Hodge-Tate spectral sequence.

**Proof.** The map $\alpha$ is given by the composition

$$
H^1(X, \mathcal{H}om(E, E')) \xrightarrow{\beta} H^1(X, \mathcal{H}om(E, E') \otimes \mathcal{O}_X) \xrightarrow{\text{com}} H^1_{\text{ét}}(X, \mathcal{H}om(L_E, L_{E'}))
$$

where $\text{com}$ is the comparison isomorphism from theorem 2.11 and $\beta$ is simply given as the map on $H^1$ associated to the injection of pro-étale sheaves

$$
\nu^* \nu_* (\mathcal{O}_X \otimes \mathcal{H}om(L_E, L_{E'})) \hookrightarrow \mathcal{O}_X \otimes \mathcal{H}om(L_E, L_{E'}).$$

Now take an extension

$$
e = (0 \to E' \to E \to E'' \to 0)
$$

of vector bundles on $X$. By lemma 3.11 the functors $\text{DW}(-) \otimes \mathcal{O}_X$ and $\nu^*(-) \otimes \mathcal{O}_X$ are canonically isomorphic. So one only has to check that the map $\beta$ takes the extension $e$ to

$$
0 \to \nu^* E' \otimes \mathcal{O}_X \to \nu^* E \otimes \mathcal{O}_X \to \nu^* E'' \otimes \mathcal{O}_X \to 0.
$$

But this is clear. 

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