STABLE REDUCTIVE VARIETIES II: PROJECTIVE CASE

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0. Introduction

This is the second part of our work on stable reductive varieties in which we extend the results of the affine part \cite{SRV-I} to the projective settings. The main aim of this paper is to construct a compactified moduli of stable higher–dimensional projective varieties, in a special case when there is a nontrivial reductive group action. Several examples of such moduli spaces are known. The most familiar must be Deligne–Mumford–Knudsen moduli space \(\overline{M}_{g,n}\) of \(n\)--pointed genus \(g\) curves and Kontsevich’s relative version for stable maps. The space \(\overline{M}_{g,n}\) was extended to the case of dimension two, for surfaces \(X\) of general type, and pairs \((X, D)\) of log general type by
Kollár, Shepherd–Barron, the first author, and others, using the methods of log Minimal Program [KSB88, Ale94, Ale96].

Finally, in [Ale02] it was extended to the case of pairs \((X, D)\) in which the variety \(X\) has arbitrary dimension but comes with a nontrivial semiabelian group action, for example, an abelian or toric variety. In the abelian case, this gives a moduli compactification of \(A_g\), the moduli space of principally polarized abelian varieties, extending previous work of Namikawa and others [Nam76].

On the other hand, the toric case is closely related to, and influenced by the work of Gelfand, Kapranov, Zelevinsky, Sturmfels and others on \(A\)–hypergeometric functions. Each stable toric pair \((X, D)\) comes with a canonical map to \(\mathbb{P}^n\) and the image is its “shadow cycle”. The works [KSZ92, GKZ94] describe the toric Chow variety parameterizing these cycles. Another related moduli space is the toric Hilbert scheme of Peeva and Stillman [PS02].

In addition to the work in the toric case, a starting point for our investigation was M. Kapranov’s paper [Kap98] which studies in the case of reductive group action some of the varieties analogous to the shadow cycles.

There are several reasons why we consider the moduli problem for pairs of varieties with divisors, rather than simply for polarized projective varieties. The most obvious is that without the additional structure provided by the divisor there are only finitely many isomorphism classes of varieties of a fixed numerical type (just as for projective toric varieties), and the varieties have infinite automorphism groups. Hence, even though the moduli stack can still be defined and studied, it is rather exotic, à la Lafforgue’s “toric stacks”. With the divisor added, the automorphism groups become finite. Also, just as in the case of toric or abelian varieties, a one–parameter family may have several “stable” limits. With the divisor added, the limit becomes unique, and it always exists, possibly after a finite base change. The last, but not the least, reason comes from the log Minimal Program. One of its lessons is that varieties and pairs that have good moduli spaces must have an ample log canonical class \(K_X + D\), and semi log canonical singularities. Since reductive varieties are rational, a nonzero divisor is necessary.

Here is the structure of the present paper. In Section 1 we define polarized stable varieties and pairs. In the next two sections we give their complete combinatorial description, in terms of discrete data – complexes of polytopes invariant under the action of Weyl group, and continuous data – certain cohomology groups.

In Section 4 we prove that, in the case of multiplicity–free support, there exists a coarse moduli space of stable reductive pairs (Theorem 4.8) and that it is a disjoint union of projective schemes (Theorem 4.11). Our method of proof is different from that in the toric case [Ale02]. Further, in the case of conjugation–invariant pairs we show that the moduli space is a union of projective toric varieties corresponding to special fiber polytopes, which we define.
In Section 5 we prove that, as predicted by the log Minimal Program, our varieties have semi log canonical singularities (Theorem 5.14).

Finally, the last Section is devoted to several generalizations.

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1. Main definitions

We freely use notations and basic definitions from [SRV-I]. In particular, $G$ will always denote a connected reductive algebraic group over an algebraically closed field $k$ of characteristic zero, $B, B^-$ a pair of opposite Borel subgroups, $T$ their common torus, and $W$ the corresponding Weyl group. Because of its fundamental importance, let us recall the following:

**Definition 1.1.** An affine stable reductive variety (resp. an affine reductive variety) for $G$ is a connected (resp. irreducible) affine $G \times G$–scheme $X$ satisfying the following conditions:

1. (on singularities) $X$ is seminormal,
2. (on stabilizers) for any $x \in X$, the stabilizer $\text{Stab}_{G \times G}(x)$ is connected,
3. (on orbits) $X$ contains only finitely many $G \times G$–orbits,
4. (group–like condition) $(B^- \times B)X^\text{diag}T$ contains a dense subset of every $G \times G$-orbit.

(Here $X^\text{diag}T$ denotes the fixed point set of $\text{diag}T \subset G \times G$.)

We now adapt this definition to the projective setting. We will consider polarized projective $G \times G$–varieties, that is, pairs $(X, L)$ where $X$ is a projective scheme with $G \times G$–action, and $L$ is an ample invertible sheaf on $X$. Recall that $L$ is linearized if we have a lift of the action of $G \times G$ from $X$ to $L$, linear on fibers.

**Definition 1.2.** A linearized stable reductive variety (resp. linearized reductive variety) for $G$ consists of a connected (resp. irreducible) projective scheme $X$ equipped with a $G \times G$–action and with an ample, $G \times G$–linearized invertible sheaf $L$, satisfying conditions (1), (2), (3) and

4. (group–like condition) $(B^- \times B)X^\text{diag}T$ contains a dense subset of every $G \times G$-orbit.

(4') for any $G \times G$–orbit $O$, there exists a $T \times T$–orbit $O' \subseteq O^\text{diag}T$ such that: $(B^- \times B)O'$ is dense in $O$, and $\text{diag}T$ acts trivially on the fibers of $L$ at all points of $O'$.

Equivalently, $(X, L) = (\text{Proj} R, O(1))$ for a graded algebra $R$ such that $\tilde{X} = \text{Spec} R$ is an affine (stable) reductive variety for the action of the group $\tilde{G} = G_m \times G$ (see Theorem 2.6).

As an example, let $X = G/B \times G/B^-$ and let $L$ be the $G \times G$–linearized invertible sheaf associated with characters $\lambda$ of $B$, and $\mu$ of $B^-$. Then $(X, L)$ is a polarized stable reductive variety if and only if: $\lambda$ is regular dominant, and $\mu = -\lambda$. Other examples include DeConcini–Procesi’s wonderful compactifications of adjoint semisimple groups; more generally, any
normal projective $G \times G$–equivariant compactification of $G$, endowed with
an ample $G \times G$-linearized invertible sheaf, is a linearized reductive variety.

**Definition 1.3.** A stable reductive pair (for $G$) consists of a linearized stable reductive variety $(X, L)$ and an effective ample Cartier divisor $D$ such that $L = \mathcal{O}_X(D)$, satisfying the crucial

(transversality condition) Supp $D$ does not contain any $G \times G$–orbits.

**Remark 1.4.** If $X$ is normal, any effective Weil divisor satisfying the transversality condition is automatically Cartier, cf. [Kno94] Lemma 2.2.

**Definition 1.5.** We say that a stable pair $(X, D)$ is conjugation–invariant if $D$ is, i.e., one has $(g, g)D = D$ for all $g \in G$.

By a basic result of GIT [Mum94] Props. 1.4 and 1.5 if $X$ is normal then some positive power $L^n$ can be linearized. The difference between assuming linearization of $L$ and of $L^n$ is a minor technical detail which can be treated by a simple trick, see Section 6. When studying stable pairs $(X, D)$ we can simply replace them by pairs $(X, nD)$.

A more serious situation is when no positive power of $L$ admits a $G \times G$–linearization. This is only possible if $X$ is not normal and the group $G$ has a nontrivial character. In this case, by [Ale02] Sec.4] there exists a cover $\pi : \tilde{X} \to X$ by a connected scheme, locally of finite type but with infinitely many irreducible components, such that $X = \tilde{X}/\mathbb{Z}^r$ and the pullback $\pi^*L^n$ is linearizable. This construction is absolutely crucial for the study of degenerations of abelian varieties, but not so for reductive groups. Hence, we also treat it as an aside in Section 6. Until that Section, we always assume that $L$ has a fixed linearization.

**Definition 1.6.** A family of stable reductive pairs over a $k$–scheme $S$ is a proper, flat morphism $\pi : X \to S$ from a scheme $X$ equipped with an action of the constant group scheme $G \times G$ over $S$, and with a relatively ample Cartier divisor $D$, such that: $L = \mathcal{O}_X(D)$ is $G \times G$–linearized, and the fiber $(X_{\bar{s}}, D_{\bar{s}})$ over every geometric point $\bar{s}$ of $S$ is a stable reductive pair.

The moduli functor $\mathcal{M}$ of stable reductive pairs associates to any scheme $S$ the set $\mathcal{M}(S)$ of families over $S$ modulo isomorphisms. We will assume our schemes to be separated and locally Noetherian $k$–schemes.

**2. Polarized stable reductive varieties**

**2.1. Classification.** Let $(X, L)$ be a polarized projective scheme, $R(X, L)$ be the ring $\bigoplus_{n \geq 0} H^0(X, L^n)$ and $\tilde{X} = \text{Spec} R(X, L)$ be the affine cone. Then $\tilde{X}$ is obtained by contracting the zero section of the line bundle over $X$ corresponding to the sheaf $L^{-1}$, to the vertex $0 \in \tilde{X}$. Denote this contraction $\pi : \mathbb{P}^{-1} \to \tilde{X}$, it is a proper birational morphism. We have $R(X, L) = \Gamma(\mathbb{P}^{-1}, \mathcal{O}_{\mathbb{P}^{-1}})$, and hence $\pi_*\mathcal{O}_{\mathbb{P}^{-1}} = \mathcal{O}_{\tilde{X}}$. The multiplicative group $\mathbb{G}_m$ acts on $\tilde{X}$, and the map $\tilde{\pi} : \tilde{X} \setminus 0 \to X$ is the geometric quotient; it is an affine morphism with fibers the $\mathbb{G}_m$–orbits, and $\mathcal{O}_X = (\tilde{\pi}_*\mathcal{O}_{\tilde{X}\setminus 0})^{\mathbb{G}_m}$.
Lemma 2.1. $X$ is normal (resp. seminormal) if and only if so is $\tilde{X}$.

Proof. The direction from $\tilde{X}$ to $X$ follows from [SRV-I, Lem.2.1(2)].

If $X$ is normal then $R(X, L)$ is the ring of regular functions on the normal variety $L^{-1}$, and it is normal since it is the intersection of the integrally closed subrings $O_{L^{-1}, z}$ of the function field of $L^{-1}$.

Now, assume that $X$, and hence $L^{-1}$ are seminormal and let $p : Z \to \tilde{X}$ be the seminormalization. The morphism $\pi : L^{-1} \to \tilde{X}$ factors through $Z$ by the universal property of seminormalization, see e.g. [Kol96, I.7.2.3.3]. In the resulting exact sequence

$$O_{\tilde{X}} \xrightarrow{f} p_* O_Z \to \pi_* O_{L^{-1}}$$

the composition $g \circ f$ is an isomorphism, and $g$ is injective. Hence, $g$ and $f$ are isomorphisms. Since $p$ is finite, the variety $Z$ is affine, and it follows that $p$ is an isomorphism. \qed

Now, let $X$ be a $G \times G$–variety and assume that the sheaf $L$ is $G \times G$–linearized. Then the varieties $L^{-1}$ and $\tilde{X}$ have a natural $\tilde{G} \times \tilde{G}$–action, where

$$\tilde{G} = G_m \times G,$$

with $(t_1, t_2) \in G_m \times G_m$ acting on the fibers by multiplying by $t_1 t_2^{-1}$. Note that

$$\tilde{T} = G_m \times T$$

is a maximal torus of $\tilde{G}$, with character group

$$\tilde{\Lambda} = Z \oplus \Lambda^+$$

and positive Weyl chamber

$$\tilde{\Lambda}^+ = Z \times \Lambda^+.$$

Lemma 2.2. The affine cone $\tilde{X}$ of a polarized stable reductive variety $(X, L)$ for the group $G$ is an affine stable reductive variety for the group $\tilde{G}$.

Proof. The conditions (1) for $X$ and $\tilde{X}$ are equivalent by Lemma 2.1. Since the morphism $\tilde{X} \setminus \{0\} \to X$ is the geometric quotient by $G_m$, the $\tilde{G} \times \tilde{G}$–orbits in $\tilde{X}$ are in bijection with the $G \times G$–orbits in $X$, except for the 1–point orbit $0 \in \tilde{X}$. So, the equivalence of conditions (2),(3) for $\tilde{X}$ and $X$ is immediate. Conditions (4) for $\tilde{X}$ and (4′) for $X$ are equivalent because $\text{diag} G_m$ acts trivially on $\tilde{X}$. \qed

The paper [SRV-I] gives a complete classification of affine stable reductive varieties for the group $\tilde{G}$ as follows:

1. Every affine reductive variety corresponds to an admissible cone in $\mathbb{R} \oplus \Lambda_\mathbb{R}$, i.e. a rational polyhedral cone $\sigma$ such that the relative interior $\sigma^0$ meets $\mathbb{R} \oplus \Lambda_\mathbb{R}^+$ and the distinct $w\sigma^0$ ($w \in W$) are disjoint. The associated algebra $R_\sigma$ is isomorphic as a $\tilde{G} \times \tilde{G}$–module
to $\bigoplus_{\lambda \in \sigma \cap \tilde{\Lambda}^+} \text{End} V_{\lambda}$, where $V_{\lambda}$ is a simple $\tilde{G}$-module with highest weight $\lambda$.

(2) An affine stable reductive variety corresponds to a complex $\Sigma = \{\sigma\}$ of admissible cones together with a 1-cocycle $t$. The associated algebra $R_{\Sigma,t}$ is the inverse limit of the algebras $R_{\sigma}$ twisted by the cocycle $t$.

In the case of the algebra $R = R(X, L)$, because $R_0 = k$ and $R_n = 0$ for $n < 0$, each $\sigma$ is the cone over a rational polytope $(1, \delta)$, $\delta \subset \Lambda_R$.

Lemma 2.3. Each $\delta$ is a lattice polytope, i.e. its vertices are all in $\Lambda$.

Proof. By [SRV-I] Proposition 4.4, it suffices to prove integrality of those vertices of $\delta$ that are in $\Lambda^+_R$. These correspond to those orbit closures in $\tilde{X}$ that are cones over some closed orbit $Y$ in $X$. Since $H^0(Y, L)$ is a simple $G \times G$-module, and $L$ is globally generated (more generally, any nef invertible sheaf on a complete spherical variety is globally generated, see e.g. [B194, Thm. 4]), the restriction map $H^0(X, L) \rightarrow H^0(Y, L)$ is surjective. The corresponding highest weight in $H^0(X, L)$ is of the form $(\lambda, -\lambda)$, where $\lambda$ is the vertex of $\delta$ associated with $Y$. So this vertex is integral. □

Let us give the corresponding basic definitions for polytopes.

Definition 2.4. A $W$-admissible polytope is a polytope $\delta$ in $\Lambda_R$ with vertices in $\Lambda$, satisfying the following conditions:

1. The relative interior $\delta^0$ meets $\Lambda^+_R$.
2. The distinct $w\delta^0$ ($w \in W$) are disjoint.

We will put $\delta^+ = \delta \cap \Lambda^+_R$.

Let $\sigma = \text{Cone} \delta$ be the cone over $(1, \delta)$ in $\mathbb{R} \oplus \Lambda_R$. Associated with this data, we have a ring $R_{\delta} := R_{\text{Cone} \delta}$ and the corresponding affine reductive variety.

Definition 2.5. A $W$-complex of polytopes $\Delta$ referenced by $\Lambda$ is a topological space $|\Delta|$ represented as a finite union of distinct closed subsets $\delta$ ($\delta \in \Delta$), together with a map $\rho : |\Delta| \rightarrow \Lambda_R$ such that:

1. $\rho$ identifies each $\delta \in \Delta$ with a polytope in $\Lambda_R$ with vertices in $\Lambda$.
2. If $\delta \in \Delta$, then each face $\gamma < \delta$ is in $\Delta$.
3. If $\delta, \gamma$ in $\Delta$, then their intersection in $|\Delta|$ is a union of faces of both.
4. $W$ acts on $|\Delta|$, the reference map $\rho$ is $W$-equivariant, and its restriction to any subset $\bigcup_{w \in W} w\delta$ is injective.

(In particular, $W$ permutes the subsets $\delta$.) We will put $|\Delta|^+ = |\Delta| \cap \Lambda^+_R$.

As in the affine case, we have the set $\Delta/W$, partially ordered by inclusion, and the complex of diagonalizable groups

$$0 \rightarrow \bigoplus_{\delta \in \Delta/W} \text{Aut} \tilde{G} \times \tilde{G}(\tilde{X}_{\text{Cone} \delta}) \rightarrow \bigoplus_{\gamma < \delta} \text{Aut} \tilde{G} \times \tilde{G}(\tilde{X}_{\text{Cone} \gamma}) \rightarrow \cdots$$
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with the obvious differential. We denote this complex by $C^*(\Delta/W, \text{Aut})$, with cocycle groups $Z^i(\Delta/W, \text{Aut})$ and cohomology groups $H^i(\Delta/W, \text{Aut})$. For each $t \in Z^1(\Delta/W, \text{Aut})$ we can define the associated algebra $R_{\Delta,t}$ and the corresponding affine stable reductive variety, as follows. For each $\gamma \prec \delta$, we have a natural surjective homomorphism $R_\delta \to R_\gamma$ sending all the $\text{End}_V^{\tilde{\lambda}}$-blocks not in $\gamma$ to zero. We twist these homomorphisms by the cocycle $t$, and define $R_{\Delta,t} = \lim_{\Delta/W} R_\delta$.

We can now prove the following:

**Theorem 2.6.** (1) For any polarized stable reductive variety $(X, L)$, the graded algebra $R(X, L)$ coincides with the algebra $R_{\Delta,t}$ for some $W$–admissible complex of polytopes $\Delta$ and a 1-cocycle $t$. In particular, all $\delta \in \Delta$ are lattice polytopes.

(2) Vice versa, for any $W$–admissible complex of polytopes $\Delta$ and any 1–cocycle $t$, $(X, L) := (R_{\Delta,t}, \mathcal{O}(1))$ is a polarized stable reductive variety. In particular, the sheaf $\mathcal{O}(1)$ is invertible. Moreover, $R(X, L) = R_{\Delta,t}$.

(3) For any polarized stable reductive variety $(X, L)$, the sheaf $L$ is isomorphic to $\mathcal{O}_{\text{Proj} R(X,L)}(1)$.

**Proof.** (1) holds by Lemmas 2.2 and 2.3.

(2) Since all vertices of polytopes $\delta$ are integral, the proof of Lemma 2.3 shows that the zero set of $(R_{\Delta,t})_1$ is empty. By [EGA2, 2.5.8] this implies that $\mathcal{O}(1)$ is invertible. The algebras $R(X, L)$ and $R_{\Delta,t}$ coincide in degrees $n = 0$ and $n \geq n_0$, they are both seminormal, and the corresponding affine varieties map bijectively to $\text{Spec}(R_0 \oplus \oplus_{n \geq n_0} R_n)$. By uniqueness of seminormalization, they coincide.

(3) On $\text{Proj} R_{\Delta,t}$ one has $\mathcal{O}(n) = \mathcal{O}(1)^n$. Since $L^n = \mathcal{O}(n)$ for two consecutive sufficiently large $n$, it follows that $L = \mathcal{O}(1)$. \qed

We can now translate the classification results from the affine case to the polarized case. [SRV-I, Prop.4.2,4.4] gives

**Theorem 2.7.** (1) The polarized reductive varieties are precisely the polarized varieties $(X_\delta, L_\delta)$, each for a unique $W$–admissible polytope $\delta$.

(2) The group $\text{Aut}^{G \times G}(X_\delta, L_\delta)$ is diagonalizable with character group $\left( \Lambda \cap \text{lin} \delta \right) / \mathbb{Z}K_\delta = \mathbb{Z} \oplus (\Lambda \cap \text{lin} \delta) / \mathbb{Z}K_\delta$, where $K_\delta$ denotes the set of those simple roots $\alpha$ such that $\delta^0$ meets the hyperplane $(\alpha = 0)$.

(3) The assignment $(X, L) \mapsto (X^{\text{diag} T}, L|_{X^{\text{diag} T}})$ defines a bijection from the polarized reductive varieties for $G$, to the polarized stable toric varieties for $T$ with a compatible $W$–action such that the quotient by $W$ is irreducible. Moreover, the $G \times G$–orbits in $X$ are in bijection with
the WT–orbits in $X^\text{diag} T$, and $\text{Aut}^{G\times G}(X, L)$ is isomorphic to the automorphism group of the polarized WT–variety $(X^\text{diag} T, L|_{X^\text{diag} T})$. 

[SRV4] Prop.5.3 and Thm.5.4] translate to

Theorem 2.8. (1) The polarized stable reductive varieties are precisely the $(X_{\Delta,t}, L_{\Delta,t})$, each for a uniquely determined $W$–admissible complex of polytopes $\Delta$ and a $1$–cohomology class $[t] \in H^1(\Delta/W, \text{Aut})$. The irreducible components of $X_{\Delta,t}$ are the varieties $X_\delta$ where $\delta \in \Delta$ is a maximal polytope such that $\delta^0 \cap \Lambda^+_R \neq \emptyset$.

(2) The $G\times G$–orbits in $X_{\Delta,t}$ are in bijection with the $W$–orbits of polytopes in $\Delta$.

(3) $\text{Aut}^{G\times G}(X_{\Delta,t}, L_{\Delta,t}) = H^0(\Delta/W, \text{Aut})$.

(4) The assignment $(X, L) \mapsto (X^\text{diag} T, L|_{X^\text{diag} T})$ defines a bijective correspondence from the polarized stable reductive varieties (for $G$), to the polarized stable toric varieties (for $T$) with a compatible action of $W$. This correspondence preserves orbits and automorphism groups.

2.2. Cohomology groups.

Theorem 2.9. Let $(X, L)$ be a polarized stable reductive variety. Then $H^i(X, L^n) = 0$ for all $i, n > 0$.

Proof. For irreducible $X$, the assertion follows for example from [Bri97, Cor.5.2]. We prove the general case by induction on $\text{dim}|\Delta|$, where $(\Delta, t)$ are the combinatorial data of $X$.

Let $\delta_1, \ldots, \delta_m$ be the maximal polytopes in $\Delta$ such that $\delta^0_1 \cap \Lambda^+_R \neq \emptyset$. Consider the following resolution

$$0 \to \mathcal{O}_{X_{\Delta,t}} \to \bigoplus \mathcal{O}_{\delta_0} \to \bigoplus \mathcal{O}_{\delta_0 \cap \delta_1} \to \cdots$$

with the differential corresponding to $t$. Here, $\Delta_{i_0 \ldots i_p} = \delta_{i_0} \cap \cdots \cap \delta_{i_p}$ is a complex of polytopes. It may consist of more than one polytope if the reference map $\rho$ is not injective. The existence of the resolution is immediate in the affine case, for the affine cones $\tilde{Y} = \tilde{X}$, $\tilde{X}_{\delta_0 \cap \delta_1}$ etc. over $Y = X$, $X_{\delta_0 \cap \delta_1}$ etc. For each of these varieties, $\pi : \tilde{Y} \setminus \{0\} \to Y$ is the geometric quotient by the $\mathbb{G}_m$–action; this implies the existence of the resolution in the projective case.

By twisting this resolution by $L^n$ and taking the hypercohomology, we see that each $H^i(X, L^n)$ is computed by a spectral sequence with first terms $H^q(X_{\Delta_{i_0 \ldots i_p}}, L^n)$. Among these, all the groups with $q > 0$ are zero either by induction assumption or by the irreducible case. Moreover, the sequence in degree 0 splits into a direct sum over the $\tilde{\lambda} \in |\text{Cone} \Delta| \cap (\text{id}, \rho)^{-1}(|\tilde{\lambda}|)$. For each such $\tilde{\lambda}$ we get an elementary inclusion–exclusion complex whose cohomology is $\text{End} V_{\tilde{\lambda}}$ in degree zero, and 0 in higher degrees. This completes the proof. □
3. Pairs

By Theorem 2.9, each Cartier divisor \( D \) with \( O_X(D) = L \) is given by a section
\[
s = \sum s_\lambda \in H^0(X, L) = \bigoplus \text{End} V_\lambda, \quad \lambda \in |\Delta| \cap \rho^{-1}(\Lambda^+).
\]
The transversality condition of Definition 1.3 plainly means that \( s_\lambda \neq 0 \) for all \( \lambda \) corresponding to the closed \( G \times G \)–orbits, i.e. for the vertices of all polytopes \( \delta \in \Delta \).

**Definition 3.1.** The type of a stable reductive pair \((X, D)\) is the pair \((\Delta, C)\), where \( \Delta \) is a \( W \)–admissible complex of polytopes, and \( C \) is the \( W \)–invariant subset of \(|\Delta| \cap \rho^{-1}(\Lambda)\) defined by
\[
\lambda \in C \cap \rho^{-1}(\Lambda^+) \iff s_\lambda \neq 0.
\]
Then \( C \) contains all vertices of polytopes \( \delta \in \Delta \). We will put \( C_\delta = C \cap \delta \), \( C^+ = C \cap \rho^{-1}(\Lambda^+) \), and \( C^+_\delta = C_\delta \cap C^+ \).

It is obvious that \( D \) is conjugation–invariant if and only if each \( s_\lambda \in \text{End} V_\lambda \) is a scalar matrix. In this case, \( s \) can be considered to be a section of the restriction of \( L \) to the stable toric variety \( X^{\text{diag} T} \), since the restriction map
\[
H^0(X, L)^{\text{diag} G} \to H^0(X^{\text{diag} T}, L)^W
\]
is an isomorphism by 2.1 and [SRV-I, 4.5, 5.4]. The following theorem now readily follows:

**Theorem 3.2.** The assignment \((X, D = (s)) \mapsto (X^{\text{diag} T}, D' = (s))\) defines a bijective correspondence from the conjugation–invariant stable reductive pairs (for \( G \)), to the stable toric pairs (for \( T \)) with a compatible \( W \)–action such that the equation of \( D' \) is \( W \)–invariant. This correspondence preserves automorphism groups.

**Definition 3.3.** Denote \( \text{Fun}_{\delta, C} = \text{Fun}(C^+_\delta, \mathbb{Z}) \), the free abelian group of \( \mathbb{Z} \)–valued functions on the finite set \( C^+_\delta \). We have an exact sequence
\[
0 \to L_{\delta, C} \to \text{Fun}_{\delta, C} \xrightarrow{p} (\widetilde{\Lambda} \cap \text{lin} \delta)/\mathbb{Z}K_\delta \to K_{\delta, C} \to 0
\]
Here the homomorphism \( p \) in the middle maps \( 1_\lambda \) to \((1, \lambda) \mod \mathbb{Z}K_\delta \), and the first and the last groups are its kernel and cokernel respectively.

Since \( C^+_\delta \) contains all the vertices of \( \delta \) in \( \Lambda^+ \), the group \( K_{\delta, C} \) is finite.

**Theorem 3.4.** The set of isomorphism classes of pairs of type \((\delta, C)\) is the torus with character group \( L_{\delta, C} \). The automorphism group of any pair of type \((\delta, C)\) is the finite diagonalizable group \( \mathbb{K}_{\delta, C} \) with character group \( K_{\delta, C} \).

**Proof.** Indeed, the set of all pairs of type \((\delta, C)\) is a torus with character group \( \text{Fun}_{\delta, C} \); we just have to divide this set by the action of
\[
\text{Aut}^{G \times G}(X_\delta, L_\delta) = \text{Hom}((\widetilde{\Lambda} \cap \text{lin} \delta)/\mathbb{Z}K_\delta, \mathbb{C}_m).
\]
\( \square \)
For a more general type \((\Delta, C)\) which corresponds to a variety with several irreducible components, the groups \(\text{Fun}_{\delta,C}\) define a cosheaf \(\hat{\text{Fun}}_{\Delta,C}\), and the tori \(\text{Hom}(\text{Fun}_{\delta,C}, \mathbb{G}_m)\) – a sheaf \(\hat{\text{Fun}}_{\Delta,C}\). Entirely analogously to the toric case \([\text{Ale02}, \text{Sec.2}]\), one can explicitly write down a complex \(\hat{M}^*\), whose 0th (resp. 1st) cohomology groups compute the automorphism group (resp. the set of isomorphism classes) of stable reductive pairs of type \((\Delta, C)\). This complex is the cone of the homomorphism \(C^*(\Delta/W, \text{Aut}) \to C^*(\Delta/W, \hat{\text{Fun}}_{C})\). We note one important corollary:

**Corollary 3.5.** The automorphism group of any stable reductive pair is finite.

Explicitly, this is the diagonalizable group \(H^0(\Delta/W, \hat{\mathbb{K}}_{\Delta,C})\).

### 4. Moduli of stable reductive pairs

#### 4.1. General remarks on families.

Let \(S\) be a scheme, \(H\) a reductive group, \(\pi : X \to S\) a proper morphism with an action \(\sigma : H_S \times X \to X\) of the constant group scheme \(H_S = H \times_k S\), and \(F\) an \(H\)-linearized coherent sheaf on \(X\). For any affine open subset \(S_0\) of \(S\), the \(\Gamma(S_0, \mathcal{O}_{S_0})\)-module \(M = \Gamma(S_0, \pi_* F)\) comes with a coaction of the group \(H\). By the basic Lemma \([\text{Mum94}, \text{p.25}]\), \(M\) is a union of finite–dimensional invariant subspaces. Since \(\text{char}\ k = 0\) and \(H\) is reductive, \(M\) splits into a direct sum over the finite–dimensional irreducible representations \(V_\lambda\) of \(H\):

\[
M = \bigoplus \lambda M_\lambda \otimes_k V_\lambda,
\]

for uniquely defined \(\Gamma(S_0, \mathcal{O}_{S_0})\)-modules \(M_\lambda = \text{Hom}^H(V_\lambda, M)\). Hence the coherent sheaf \(\pi_* F\) uniquely splits into a direct sum over irreducible representations.

If \(H = G \times G\) and \(L\) is a \(G \times G\)-linearized sheaf on \(X\), we consider the \(G \times G\)-action on it, where \(G = \mathbb{G}_m \times G\), and \(\mathbb{G}_m \times \mathbb{G}_m\) acts by dilation \(t_1 t_2^{-1}\) on fibers. The weight lattice for \(G\) is \(\Lambda = \mathbb{Z} \oplus \Lambda\) and each sheaf \(\pi_* L^n\) has weight \(n\) in the first variable.

**Theorem 4.1.** Let \(\pi : (X, L) \to S\) be a family of polarized stable reductive varieties over a scheme \(S\). Then the \(\mathcal{O}_S\)-algebra \(R(X, L) = \bigoplus_{n \geq 0} \pi_* L^n\) can be written uniquely in the form

\[
R(X, L) = \bigoplus \lambda F_\lambda \otimes_k \text{End} V_\lambda
\]

where each \(F_\lambda\) is a locally free \(\mathcal{O}_S\)-module of finite rank.

**Proof.** By Theorem \([\text{Har77}, \text{Theorem III.12.11}]\) each sheaf \(\pi_* L^n\) is a locally free \(\mathcal{O}_S\)-module of finite rank. By
the remark above, with $H = \tilde{G} \times \tilde{G}$ and $M = \pi_* L^n$, we have a canonical splitting

$$M = \bigoplus_{\lambda, \mu \in \Lambda^+} M_{\lambda \mu} \otimes_k (V^*_\mu \otimes_k V_\lambda)$$

Every direct summand must be a locally free sheaf of finite rank, and by looking at any fiber we see that it is zero unless $\lambda = \mu$.

**Definition 4.2.** The support of a polarized stable reductive variety is the corresponding topological space $|\Delta|$ with its reference map $\rho$ to $\Lambda_R$. The variety $(X, L)$ is multiplicity–free if each $\lambda$–block $\text{End} V_\lambda$ appears in $R(X, L)$ with multiplicity at most one. Clearly, this is equivalent to the condition that $\rho$ is injective.

**Theorem 4.3.** Let $\pi : (X, L) \to S$ be a family of polarized stable reductive varieties over a connected scheme $S$. If one geometric fiber $(X_s, L_s)$ is multiplicity–free then any other geometric fiber $(X_t, L_t)$ is multiplicity–free and has the same support $\rho : |\Delta| \to \Lambda_R$.

**Proof.** If the rank of the locally free module $F_\lambda$ equals 1 at one $s \in S$, then the same is true at any $t \in S$. □

We will consider the moduli problem only for multiplicity–free varieties. As a consequence of the last theorem, we can work with varieties that have a fixed multiplicity–free support $|\Delta| \subseteq \Lambda_R$. This will give us an open and closed subscheme in our moduli space.

**4.2. One-parameter degenerations.** We can work in two fairly similar situations:

1. $R_0$ is a discrete valuation ring with maximal ideal $(z)$, residue field $k$, and quotient field $K$. $S = \text{Spec} R_0$ with generic point $\eta = \text{Spec} K$ and special point $s = \text{Spec} k$; or

2. $S = \text{Spec} R_0$ is a smooth curve, $z \in R_0$ has a unique zero at the closed point $s \in S$, and $\mathcal{K} = R_0[1/z]$.

Given a family of stable reductive pairs over $S - \{s\}$, we first extend it, after a finite base change $(S', s') \to (S, s)$, to a family over the whole $S'$. Next, we show that such an extension is unique. We will give the arguments for the first situation, and leave making the obvious changes to adapt it for the second one to the reader.

We begin with a pair $(X_\eta, D_\eta)$ where the geometric generic fiber $X_\eta \otimes_{k(\eta)} \overline{k(\eta)}$ is irreducible. Let $L_\eta = \mathcal{O}_{X_\eta}(D_\eta)$. After a finite base change, $(X_\eta, L_\eta)$ becomes the standard polarized reductive variety $(X_\delta, L_\delta)$ over $k(\eta)$ for some $W$–admissible lattice polytope $\delta \subset \Lambda_R$. The divisor $D_\eta$ is given by a section

$$s = \sum_{\lambda \in \Lambda^+ \cap \delta} s_\lambda, \quad s_\lambda \in \mathcal{K} \otimes_k \text{End} V_{1, \lambda}.$$
Let \( m_\lambda = \text{val}_z s_\lambda \), so that \( s_\lambda = z^{m_\lambda} s'_\lambda \) with \( s'_\lambda(0) \neq 0 \). The lower convex envelope of the points \((w, m_\lambda) \in \Lambda \oplus \mathbb{Z}, w \in \text{Stab}_W(\delta)\), defines a \( W \)-admissible height function \( h : \text{Cone} \delta \to \mathbb{R} \), as in [SRV-1, Sec.7.4]. It is piecewise linear, \( W \)-invariant, and takes rational values at all \( \lambda \in \Lambda \). After the base change \( z = (z')^n \), we can assume that \( h|_{\overline{\Lambda}} \) is in fact integer–valued. Consider the subspace

\[
\mathcal{R} = \bigoplus_{\lambda \in \overline{\Lambda} + \text{Cone} \delta} z^{h(\lambda)} \mathcal{R}_0 \otimes_k \text{End} V_{\overline{\lambda}} \subseteq R(X_\eta, L_\eta) = \bigoplus_{\lambda \in \overline{\Lambda} + \text{Cone} \delta} \mathcal{K} \otimes_k \text{End} V_{\overline{\lambda}}.
\]

By [SRV-1, Sec.7.4], \( \mathcal{R} \) is a subalgebra, and \( \overline{X} = \text{Spec} \mathcal{R} \) is a family of affine stable reductive varieties over \( S \). Hence, \((X, L) = (\text{Proj}_{\mathcal{R}_0} \mathcal{R}, \mathcal{O}(1))\) is a family of polarized stable reductive varieties over \( S \). Since by definition \( m_\lambda \geq h(1, \lambda) \), \( s \) is in \( \mathcal{R} \), so it extends to a regular section of \( L \) and gives a compactification \( D \) of the divisor \( D_\eta \). Moreover, since for the lower convex envelope we have \( m_\lambda = h(1, \lambda) \) for every vertex \( \lambda \) of \( \delta \), the corresponding \( \lambda \)-components of \( s'(0) \), the residue of \( s \) modulo \( (z) \), are all nonzero. Hence, the divisor \( D_0 \) on the special fiber satisfies the transversality condition, and we have constructed an extended family of stable reductive pairs over \( S \).

Now consider the general case, with \( X_\eta \) not necessarily geometrically irreducible. After a finite base change, \((X_\eta, L_\eta)\) becomes the standard polarized variety \((X_{\Delta, t}, L_{\Delta, t})\) for some \( W \)-admissible complex of polytopes and \( t \in Z^1(\Delta, \text{Aut}) \) with coefficients in \( k(\eta) \). By taking the valuations at \( z \), we can write \( t = z^\gamma t' \) with \( t'(0) \neq 0 \). Explicitly, \( \gamma \) is a collection of homomorphisms \( \gamma_\varphi \in \text{Hom}(\Lambda \cap \text{lin} \gamma/\mathbb{Z}K_\varphi, \mathbb{Z}) \) for all \( \varphi \subseteq \delta_1 \cap \delta_2 \) which satisfy the 1–cocycle condition on triple intersections. Note that for any two maximal polytopes \( \delta_1, \delta_2 \in \Delta \) and a face \( \varphi \subseteq \delta_1 \cap \delta_2 \), we have

\[
s_{\delta_1}|_{X_\varphi} = t_\varphi(s_{\delta_2}|_{X_\varphi}).
\]

Therefore, for all faces \( \varphi \subseteq \delta_1 \cap \delta_2 \) the following condition holds:

\[
(h_{\delta_1} - h_{\delta_2})|_{\varphi} = \gamma_\varphi.
\]

After a common base change, we can now extend each irreducible component \((X_\delta)_\eta\) of \( X_\eta \) and glue them along \( t \). This proves the existence part of the following theorem

**Theorem 4.4.** Every stable reductive pair \((X_\eta, D_\eta)\) over \( S - \{s\} \) can be extended to a family of stable reductive pairs over \( (S, s) \), possibly after a finite base change \((S', s') \to (S, s)\). Such an extension is unique.

To prove the uniqueness part, we are free to make further finite base changes. So, let \( \pi : (X, D) \to S \) be a family of stable reductive pairs over \( S \). After a base change, the generic fiber becomes a standard pair \((X, D)_{\Delta, t}\) over \( \mathcal{K} \). We have

\[
(X, L) = (\text{Proj}_{\mathcal{R}_0} R(X, L), \mathcal{O}(1)), \quad R(X, L) = \bigoplus \pi_\ast L^n.
\]
The family $\tilde{X} = \text{Spec } R(X, L)$ is a family of affine stable reductive pairs over $S$. By the classification in [SRV-I, 7.11], every such family corresponds to a system of integral–valued height functions $h_\delta, \delta \in \Delta$. All we have to show is that each of these height functions is defined by the lower envelope of points $(\lambda, m_\lambda = \text{val}_{\delta}(s_\delta)_{\lambda})$, as in our construction. But among all functions that take integral values on $\tilde{\Lambda}$, this one is the only function that satisfies the following two conditions:

1. $s_\delta$ remains regular, i.e. $h(1, \lambda) \geq m_\lambda$ for $\lambda \in \delta \cap \Lambda^+$; and
2. on the special fiber, the divisor $D_0$ satisfies the transversality condition, i.e. $h(1, \lambda) = m_\lambda$ for all vertices of $\delta$.

This completes the proof of the Theorem.

**Definition 4.5.** A subdivision $\Delta_h$ of a $W$–admissible polytope $\delta$ (resp. subdivision $\Delta_h$ of a $W$–complex of polytopes) corresponding to a $W$–admissible height function (resp. system of height functions) is called coherent.

For ordinary polytopes, in absence of the Weyl group $W$, such subdivisions are also sometimes called regular, or convex. As a corollary of the proof of Theorem 4.4, we have

**Corollary 4.6.** Combinatorially, one–parameter degenerations of stable reductive varieties correspond to coherent subdivisions. In the conjugation–invariant case, one–parameter degenerations of stable reductive pairs are in bijection with one–parameter degenerations of associated stable toric pairs with $W$–action.

4.3. **Construction of the moduli space of pairs.** Recall that the moduli functor $\mathcal{M} : (\text{Schemes})^o \rightarrow (\text{Sets})$ associates to $S$ the set of isomorphism classes of families of stable reductive pairs over $S$. We will also consider a version of this functor with conjugation–invariant pairs, for which the equation $s = \sum s_\lambda$ is a collection of $O_S$–scalar matrices. We will denote the latter functor $\mathcal{M}^{ci}$. We are going to prove that both of these moduli functors have coarse moduli spaces. For this, we will use the following basic tool from our affine paper [SRV-I].

Let $H$ be a connected reductive group with set of dominant weights $\Lambda^+_H$. Fix a function $h : \Lambda^+_H \rightarrow \mathbb{N}$. Consider a flat family $\pi : \tilde{X} \rightarrow S$ of affine $H$–varieties. Then we have a decomposition

$$\pi_* O_{\tilde{X}} = \bigoplus_{\lambda \in \Lambda^+_H} F_\lambda \otimes_k V_\lambda$$

where the $F_\lambda$ are flat $O_S$–modules. We say that $h$ is the Hilbert function of $\tilde{X}$, if each $F_\lambda$ is locally free of rank $h(\lambda)$.

Fix, in addition, a finite–dimensional $H$–module $V$ and define the moduli functor

$$\mathcal{M}_{h,V} : (\text{Schemes})^o \rightarrow (\text{Sets})$$

by associating to $S$ the set of all closed $H$–invariant subfamilies $\tilde{X} \subseteq V \times S$ with Hilbert function $h$. 
Theorem 4.7 ([SRV-I], Thm.7.6). The functor $\mathcal{M}_{h,V}$ is representable by a scheme $M_{h,V}$ which is quasiprojective over $k$.

Here is our main existence theorem.

Theorem 4.8. The functor $\mathcal{M}$ (resp. $\mathcal{M}^{ci}$) is coarsely represented by a scheme $M$ (resp. $M^{ci}$) which is proper over $k$.

Proof. Fix a type $(\Delta, C)$ and put $D = |\Delta|$. There is a natural partial order on types: we say that $(\Delta', C') \geq (\Delta, C)$ if $\Delta$ is a subdivision of $\Delta'$ and $C \subseteq C'$. The pairs of type $\geq (\Delta, C)$ clearly form an open subfunctor.

If we prove that the latter functor is coarsely representable, then $M$ will be obtained by the obvious gluing of these open subschemes. The idea is to represent $M_{\geq (\Delta, C)}$ as a finite quotient of an appropriate locally closed subscheme of some $M_{h,V}$, where the Hilbert function $h$ takes value 1 at points of $D$, and 0 else. We will prove the result in several steps, starting with the simplest case and then adding layers of complexity.

Case 1. The pairs are conjugation–invariant, and for each $\delta \in \Delta$ the set $C \cap \delta^+$ generates the semigroup $\text{Cone} \delta^+ \cap \tilde{\Lambda}^+$.

Let $\pi : (X, D) \to S$ be a family of stable reductive pairs and let $L = \mathcal{O}_X(D)$. We have

$$R(X, L) = \bigoplus_{\tilde{\lambda} \in \tilde{\Lambda}^+ \cap \text{Cone} \ D} F_{\tilde{\lambda}} \otimes_k \text{End} V_{\tilde{\lambda}}$$

for some invertible sheaves $F_{\tilde{\lambda}}$, and this ring comes with an element of degree one, the equation $s = \sum s_c$ of $D$. The rule $1_c \to s_c$, $c \in C^+$, defines a $\tilde{G} \times \tilde{G}$–homomorphism of $\mathcal{O}_S$–algebras

$$\mathcal{O}_S \otimes_k \text{Sym}^* \left( \bigoplus_{c \in C^+} \text{End} V_{1,c} \right) \to R(X, L)$$

(4.1)

We claim that his homomorphism is surjective. Indeed, it is surjective on the $\mathcal{O}_S$–submodules corresponding to highest weight vectors: when restricted to each subset $\text{Cone} \delta^+$ this is just the condition that the set $C \cap \delta^+$ is generating. Since the image is $\tilde{G} \times \tilde{G}$–invariant and contains the highest weight vectors, it is everything. This gives a closed $\tilde{G} \times \tilde{G}$–invariant embedding $\tilde{X} \subseteq V \times S$, where $V$ is the $\tilde{G} \times \tilde{G}$–module $\bigoplus_{c \in C^+} \text{End} V_{1,c}$.

We have just constructed, in a canonical way, a morphism $S \to M_{h,V}$ which must factor through the open subscheme $M^{0}_{h,V}$ parametrizing reduced schemes. Vice versa, any morphism $S \to M_{h,V}^{0}$ gives an $\mathcal{O}_S$–algebra $\mathcal{R}$ with an element $s$ of degree one which defines $(X, L) = (\text{Proj} \mathcal{R}, \mathcal{O}(1))$ and $D = (s)$. Hence, $M_{h,V}^{0}$ represents $\mathcal{M}_{\geq (\Delta, C)}$.

Case 2. Conjugation–invariant pairs, arbitrary $(\Delta, C)$.

Starting with the sections $s_c$ of $F_{1,c}$, we would like to construct nonzero $\mathcal{O}_S$–scalar matrices $t_{\tilde{\lambda}} \in \Gamma(S, F_{\tilde{\lambda}})$ for all $\tilde{\lambda} \in \text{Cone} |\Delta| \cap N\tilde{\Lambda}^+$, for some
fixed positive integer \( N \). We would like to do it in a canonical way, that is, depending only on the type \((\Delta, C)\).

Let \( p_\lambda : k\lambda \to (\text{End} V_\lambda)^{\hat{\lambda}} \) be the canonical isomorphism from the line of scalar matrices to the highest weight line (of weight \((-\hat{\lambda}, \hat{\lambda})\)). Let \( q_\lambda \) be its inverse. These morphisms extend uniquely to the \( \mathcal{O}_S \)-module \( F_\lambda \otimes_k \text{End} V_\lambda \). For any element \( u = \sum u_\lambda \) of \( R(X, L) \) define \( r_\lambda(u) = q_\lambda((u_\lambda)^{\hat{\lambda}}) \).

This is an \( \mathcal{O}_S \)-scalar matrix in the \( \hat{\lambda} \)-block \( \text{End} V_\lambda \) of \( R(X, L) \).

Fix one polytope \( \delta \in \Delta \). We are going to describe a universal method for obtaining nowhere–vanishing scalar matrices \( t_\lambda \) in our \( \mathcal{O}_S \)-algebra for every \( \hat{\lambda} \in N\Lambda \cap \text{Cone} \delta^+ \).

Let us start with the group algebra \( k[G] \). Suppose we have a scalar matrix \( 1_\lambda \in \text{End} V_\lambda \subset k[G] \). First of all, we want to produce scalar matrices \( t_\mu \) for all vertices \( \mu \) of the polytope \( Q = \text{Conv}(W\lambda) \cap \Lambda_+^+ \), possibly after replacing it with the dilated polytope \( NQ \) so that these vertices become lattice points.

We want to do it in a canonical fashion, considering only the powers of \( 1_\lambda \) in \( k[G] \) and its components in various \( \text{End} V_\mu \). Now every vertex \( \mu \) of \( Q \) is the centroid of a face of \( \text{Conv}(W\lambda) \) containing \( \lambda \). And by Lemma 4.9 below, the \( N\mu \)-component of \( 1_\lambda^N \) is nonzero for any multiple \( N \) of some positive integer \( N_0 \).

Now, let \( R = R_\sigma \) be the algebra of the affine reductive variety for \( G \) corresponding to a cone \( \sigma \) and let \( K \) be the set of simple roots \( \alpha \) such that \((\alpha = 0)\) meets \( \sigma^0 \). Given \( \lambda \in \sigma \cap \Lambda^+ \), let \( Q = \text{Conv}(W_K\lambda) \cap \Lambda_+^+ \) and let \( \mu \) be a vertex of \( Q \). Then the previous construction still gives a nonzero scalar matrix in the \( N\mu \)-block of \( R \), by the description of the multiplication in \( R \) \cite[Sec.3.2]{SRV-II}]. This implies that the construction still works when \( R = R_{\Sigma \cap \delta} \) is the algebra of an affine stable reductive variety if \( \Sigma \) is multiplicity-free and has a cone containing \( \sigma \).

Finally, apply this construction to the graded \( k \)-algebra coming from a pair of type \( \geq (\Delta, C) \). Starting with elements \( s_c \) in the \((1, c)\)-blocks \((c \in \text{Vert} \delta) \) we produce elements \( t_{(N_0, N_0c')} \) for all \( c' \in \text{Vert} \delta^+ \) and some \( N_0 \) \( > 0 \). Once we have these elements, taking the components of highest weight in their products gives us nonzero scalar matrices for every \( \hat{\lambda} \in N\Lambda \cap \text{Cone} \delta^+ \) for an even larger \( N \). Denote the latter matrices by \( f_\lambda(1_c) \) \((c \in C^+) \).

The functions \( f_\lambda \) involve only multiplication in the algebra \( R \) and taking the components in a canonical splitting. Therefore, for any \( \mathcal{O}_S \)-algebra \( R \) associated to a family of pairs of type \( \geq \Delta \) we can define

\[
(4.2) \quad t_\lambda = r_\lambda(f_\lambda(s_c)), \quad \hat{\lambda} \in N\Lambda \cap \text{Cone} \delta^+.
\]

These are scalar \( \mathcal{O}_S \)-matrices, and by the above they do not vanish on any fiber.

Now, for each cone \( \delta \in \Delta \) choose a finite set \( \{v = v_i, \delta\} \) generating the semigroup \( \text{Cone} \delta \cap \Lambda^+ \). For each \( Nv \) we have a section \( t_{Nv} \). Let us choose
a scalar section $s_v$ in each $v$–block of $R(X, L)$ so that

$$r_Nv(s_v^N) = t_Nv$$

(4.3)

It is always possible to find these $N$th roots $s_v$ after a finite base change $S' \to S$. Any two choices of $s_v$'s differ by $N$th roots of unity. After the $s_v$'s were chosen for all $\{v_{i,\delta}\}$, as in the first case, we have a canonical surjection

$$O_S \otimes_k \text{Sym}^*(\bigoplus_{v_{i,\delta}} \text{End} V_{i,\delta}) \to R(X, L)$$

which encodes our pair. The equations (4.2, 4.3) define a closed subscheme $M'$ of a certain moduli scheme $M_{0h,V}$, and we have constructed a morphism $S' \to M'$. It is canonical up to rescaling by $N$th roots of unity. Clearly, the functor $M_{ci} \geq (\Delta, C)$ is coarsely representable by the quotient of $M'$ by a product of several groups of $N$th roots of unity.

**Case 3.** General pairs, not necessarily conjugation–invariant.

In this case, the $s_c$ are some nonzero $n_c \times n_c$–matrices, not scalar in general. For any of $\prod n_c^2$ choices of matrix coefficients there is an open subfunctor in $M$ where the corresponding coefficients $(s_c)_{ij}$ are invertible. For any such choice, $(s_c)_{ij}$ defines a nonzero scalar matrix, so we can argue as in the previous case. The $\prod n_c^2$ thus constructed schemes glue in an obvious way (just the way a projective space is glued from affine spaces) to a coarse moduli space $M$.

The fact that space $M$ is proper is a consequence of Theorem 4.4 and the valuative criterion of properness. □

**Lemma 4.9.** (see also [Tim02] Lemma 1). Let $\lambda$ be a dominant weight and let $\mu$ be the centroid of a face of the polytope $\text{Conv}(W\lambda)$ containing $\lambda$. Then there exists a positive integer $N_0$ such that the decomposition of $V_{\lambda}^{\otimes N}$ into irreducible $G$–representations contains $V_{N\mu}$, for any multiple $N$ of $N_0$.

**Proof.** Consider first the case where $G$ is semisimple, and $\mu = 0$. Then we may take $N_0 = \dim V_{\lambda}$, since the top exterior power $\wedge^{N_0} V_{\lambda}$ is a copy of the trivial representation in $V_{\lambda}^{\otimes N_0}$.

The case where $G$ is arbitrary and $\mu$ extends to a character of $G$ follows by an obvious shifting argument.

In the general case, write $\mu = \lambda - \sum_{i \in I} n_i \alpha_i$ for uniquely defined positive roots $\alpha_i$, and positive rational numbers $n_i$. Let $P \supset B$ be the parabolic subgroup associated to the subset $I$ of simple roots, and let $L \supset T$ be its Levi subgroup. Then the invariant subspace $V_{\lambda}^{R_u(P)}$ is a simple $L$–module with highest weight $\lambda$, and $\mu \in \text{Conv}(W_L \lambda)$ is invariant under $W_L$. Hence it extends to a character of $L$, so that our assertion holds for $V_{\lambda}^{R_u(P)}$. By considering highest weight vectors, it also holds for $V_{\lambda}$. □
4.4. Projectivity of the moduli space. The following fact is well known:

**Lemma 4.10.** Let \( f : M_1 \to M_2 \) be a quasi–finite morphism of proper schemes. If \( M_2 \) is projective then so is \( M_1 \). If \( f \) is surjective (on points) and \( M_1 \) is projective then so is \( M_2 \).

**Theorem 4.11.** The moduli spaces \( M \) and \( M^{ci} \) are projective.

**Proof.** For the proof, we will construct two Chow morphisms from \( M^{ci}_{\text{red}} \) (resp. \( M_{\text{red}} \)) to appropriate Chow varieties parametrizing cycles of a certain, easily computable, degree in some fixed projective space \( \mathbb{P} \). Then the previous lemma, applied to the morphisms \( M_{\text{red}} \to \text{Chow} \) and \( M_{\text{red}} \to M \), will imply projectivity.

Let \( (X, D) \) be a family of conjugation–invariant pairs over \( S \). Consider the homomorphism of equation (4.1). In general, it is not surjective. However, on the fiber at any \( \overline{s} \), the image is a subalgebra \( R' \) of \( R = R(X, L) \) over which \( R \) is finite. Indeed, the image contains all \( \tilde{\lambda} \)–blocks with \( \tilde{\lambda} \in N \Lambda^+ \cap |\Delta| \) for some fixed large positive integer \( N \) depending only on \( |\Delta| \). This gives us a finite morphism

\[
\varphi : X \to \mathbb{P} \times S = \text{Proj} \text{Sym}^* \left( \bigoplus_{c \in C^+} \text{End} V_{1,c} \right) \times S
\]

The image is not well defined as a family of subschemes of \( \mathbb{P} \) over \( S \). However, it gives a well–defined family of cycles over \( S_{\text{red}} \). By the universal property of the Chow variety (see e.g. [Kol96, I.3.21]), we obtain a morphism \( f : M^{ci}_{\text{red}} \to \text{Chow} \). To show that \( f \) is quasifinite, we have to prove that a variety \( X \) over \( k \) is defined by its image in \( \mathbb{P} \) up to finitely many choices. \( X \) is determined by the associated stable toric variety \( Y = X_{\text{diag}} T \). Since \( \varphi \) is \( G \times G \)–equivariant, \( \varphi|_Y \) is the canonical morphism from \( Y \) to \( \mathbb{P} \) given by the equation \( s \) of \( D \), and by [Ale02, 2.11.11], the variety \( Y \) is determined by its image up to finitely many choices. This completes the proof for \( M^{ci} \).

For a general pair \( (X, D) \) over \( S \), we will construct, analogously to (4.1), a \( \tilde{G} \times G \)-equivariant homomorphism from a bigger algebra:

\[
\Phi : \mathcal{O}_S \otimes_k \text{Sym}^* \left( \bigoplus_{c \in C^+} \text{End} V_{1,c} \otimes_k (\text{End} V_{1,c})^* \right) \to R(X, L).
\]

For each \( \tilde{\lambda} \in \text{Cone} |\Delta^+| \), the \( \tilde{\lambda} \)–block \( B_{\tilde{\lambda}} \subset R(X, L) \) differs from \( \mathcal{O}_S \otimes_k \text{End} V_{\tilde{\lambda}} \) by an invertible sheaf. This implies that the sheaves \( B_{\tilde{\lambda}} \otimes B_{\tilde{\lambda}}^* \) and \( \mathcal{O}_S \otimes_k (\text{End} V_{\tilde{\lambda}} \otimes_k (\text{End} V_{\tilde{\lambda}})^*) \) are canonically isomorphic. The element \( s_c \in B_{1,c} \) defines a linear map

\[
\mathcal{O}_S \otimes_k (\text{End} V_{1,c} \otimes_k (\text{End} V_{1,c})^*) = B_{1,c} \otimes \mathcal{O}_S B_{1,c}^* \xrightarrow{(1,s_c)} B_{1,c}^* B_{1,c} \to R(X, L),
\]

and that gives a homomorphism \( \Phi \) such that the algebra \( R(X, L) \) is once again finite over its image. This homomorphism is equivariant if we let \( G \times G \) act trivially on \( (\text{End} V_{1,c})^* \). So, this time we get a family of cycles in a bigger projective space and a quasifinite morphism to a Chow variety. \( \square \)
4.5. Structure of the moduli space. From Theorems 2.8 and 3.2 we know that over an algebraically closed field there exists a bijective correspondence between stable reductive varieties, resp. pairs and $W$-equivariant stable toric varieties, resp. pairs. This toric correspondence extends to families over reduced schemes:

**Lemma 4.12.** Let $(X, L) \rightarrow S$ be a flat family of polarized stable reductive varieties over a reduced scheme $S$. Then $X^{\text{diag}T}$, considered with the reduced scheme structure, is flat over $S$ and $(X^{\text{diag}T}, L|_{X^{\text{diag}T}})$ is a family of polarized stable toric varieties with a compatible $W$-action.

**Proof.** We can assume $S$ to be connected. By the toric correspondence over an algebraically closed field, on every geometric fiber we obtain a polarized stable toric variety with the same Hilbert polynomial. The flatness follows by applying a well-known criterion of flatness for projective morphisms over reduced Noetherian bases. □

As above, let $\rho : D \rightarrow \Lambda_R$ be a multiplicity–free support.

**Corollary 4.13.** There exists a natural bijective morphism $M^{\text{ci}}_{\text{red}} \rightarrow \text{SP}^W_{\text{red}}$ from the moduli space of stable reductive pairs with support $D$ to the moduli space of stable toric pairs with compatible $W$–action and $W$–invariant equation, and with support $D$.

We can now apply [Ale02, Theorem 1.2.15] to deduce the structure of $M^{\text{ci}}$. First, we need a few definitions. Given a $W$–admissible polytope $\delta$, consider the simplex $\alpha_{\delta,C}$ with vertices $1_c$ in the lattice $\text{Fun}_{\delta,C}$ of Definition 3.3. The homomorphism $p$ maps $\alpha_{\delta,C}$ to some polytope $q_{\delta,C}$.

**Definition 4.14.** $\Sigma(\delta/W)$ will denote the fiber polytope of $p : \alpha_{\delta,C} \rightarrow q_{\delta,C}$ (see [BS92] or [GKZ94] for the definition of fiber polytopes). It is a lattice polytope in $\mathbb{L}_{\delta,C}$. Similarly, the polytope $\Sigma(\Delta/W, C)$ is defined as the projection of the polytope

$$\prod_{\delta \in \Delta} \Sigma(\delta/W, C) \subset C_0(\text{Fun}_{\Delta,C})$$

to the space $H_0(\text{Fun}_{\Delta,C}) = C_0/\partial C_1$.

Now, [Ale02] yields the following

**Theorem 4.15.** The normalization of $M^{\text{ci}}_{\text{red}}$ is a union of the projective toric varieties corresponding to the polytopes $\Sigma(\Delta/W, C)$ for all $W$–invariant subdivisions $(\Delta, C)$ of $(D, D \cap \Lambda)$. The component containing reductive varieties is the toric variety for the polytope $\Sigma(D/W, D \cap \Lambda)$.

Note that the moduli space $M$ of general pairs also has a natural stratification by types $(\Delta, C)$ with the same inclusions as for $M^{\text{ci}}$, but each stratum has a larger dimension $\sum_{c \in C^+} (n_c^2 - 1)$, where $n_c = \dim V_{1,c}$. 
5. Connection with the log Minimal Model Program

5.1. Arbitrary spherical varieties. Let $X$ be a variety with an action of a connected reductive group $G$. We will assume that $X$ is spherical, that is, $X$ is normal and contains an open orbit of a Borel subgroup $B$ of $G$. By compactifying, we will also assume $X$ to be complete, this will not affect our results on singularities below. Since the open $B$–orbit is affine, its complement has pure codimension one in $X$. Thus, unlike in the toric case, $X$ has not one but two kinds of boundaries:

(1) $\partial G X$, the codimension one part of the complement of the open $G$–orbit in $X$,
(2) $\partial B X$, the complement of the open $B$–orbit minus $\partial G X$ (which it contains).

The following are well–known facts in the theory of spherical varieties [BI94]:

Lemma 5.1. (1) A canonical divisor for $X$ is $-\Delta_G - \Delta_B$, where $\Delta_G$ is the reduced divisor $\partial G X$, and $\Delta_B$ is a unique effective divisor with support $\partial B X$ (it need not be reduced).
(2) The linear system $|\Delta_B|$ has no fixed components.
(3) There exists a $G$–equivariant resolution of singularities $\pi : X' \to X$ such that $X'$ is a projective (spherical) variety satisfying:
   (a) $\Delta'_G$ is a divisor with normal crossings,
   (b) $\Delta'_B$ is the strict preimage of $\Delta_B$,
   (c) (transversality condition) $\Delta'_B$ does not contain any $G$–orbits.
   Then the linear system $|\Delta'_B|$ is free.
(4) Assuming $X$ is projective, there exists an effective ample divisor $H$ such that $\text{Supp } H = \partial G X \cup \partial B X = \text{Supp}(\Delta_G + \Delta_B)$.

Recall the following

Definition 5.2. A pair $(X, D)$ has log canonical, resp. klt (Kawamata log terminal) singularities if
(1) $X$ is normal and $D$ is an effective $\mathbb{Q}$–Weil divisor on $X$,
(2) $K_X + D$ is $\mathbb{Q}$–Cartier,
(3) for one (and then any) log-resolution of singularities $\pi : X' \to X$ (i.e. $X'$ is smooth, and the exceptional locus + the strict preimage $\pi^{-1}D$ is a divisor with normal crossings), in the formula
   $$\pi^*(K_X + D) = K_{X'} + \pi^{-1}D + \sum b_j E_j$$
   all the coefficients of irreducible divisors appearing in $\pi^{-1}D + \sum b_j E_j$ are $\leq 1$ (resp. $< 1$).

The coefficients $a_j = -b_j$ are called the discrepancies of the pair $(X, D)$.

Theorem 5.3. On any spherical variety $X$, the pair $(X, \Delta_G + |\Delta_B|)$ has log canonical singularities, i.e. for a generic choice of a divisor $D_B \in |\Delta_B|$, the pair $(X, \Delta_G + D_B)$ has log canonical singularities.
Proof. Consider the resolution of singularities $\pi : X' \to X$ as above. Then
\[ \pi^*(K_X + \Delta_G + D_B) = 0 = K_{X'} + \Delta_G' + D_B'. \]
Since $|D_B'|$ is free, for a generic choice of $D_B$, $\Delta_G' + D_B'$ is a reduced divisor with normal crossings. \qed

Lemma 5.4. For any spherical variety $X$, on some spherical resolution $X'$ there exists a $\mathbb{Q}$-divisor $A'$ such that $-(K_{X'} + A')$ is ample and the pair $(X', A')$ has klt singularities.

Proof. For a generic $D_B' \in |\Delta_G'|$, $H'$ as in 5.1(4) and $0 < \delta \ll \varepsilon \ll 1$, the divisor $A' = \Delta_G' + (1 - \varepsilon)D_B' + \varepsilon \Delta_B' - \delta H'$ will do. Indeed, $K_{X'} + A' \sim -\delta H'$. By the transversality condition 5.1(3c) the divisor $\varepsilon \Delta_B'$ does not affect the discrepancies of the previous divisor that were equal to $-1$ (on some further log resolution $X''$), and by continuity in $\varepsilon$ (we are dealing with finitely many linear functions, after all) the new discrepancies are also $>-1$. Finally, subtracting $\delta H'$ strictly increases all $-1$ discrepancies because their locus is inside $\operatorname{Supp} \Delta_G'$. \qed

The next several statements are well known, see f.e. [BI94, Bri97]. However, the usual proofs are more complicated, that of [BI94] using Frobenius splitting. Here, we give one–line proofs using the well-known theorem, see f.e. [KMM85, Thm.1-2-5]:

Theorem 5.5 (Kawamata-Viehweg Vanishing Theorem). Let $\pi : X \to Y$ be a projective morphism from a normal variety $X$. Assume that the pair $(X, D)$ has klt singularities and let $L$ be an invertible sheaf on $X$ such that $L(-K_X - D)$ is $\pi$–ample. Then $R^i\pi_*L = 0$ for $i > 0$.

Corollary 5.6. For any nef line bundle $L'$ on a spherical resolution $\pi : X' \to X$, one has $R^i\pi_*L' = 0$ for $i \geq 1$.

Proof. Follows immediately from Lemma 5.4 and Theorem 5.5. \qed

Corollary 5.7. Any spherical variety $X$ has rational singularities.

Proof. Applying the previous corollary with $L' = \mathcal{O}_{X'}$ gives $R^i\pi_*\mathcal{O}_{X'} = 0$. Moreover, $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$, since $X$ is normal. Thus, $X$ has rational singularities. \qed

Corollary 5.8. For any nef line bundle $L$ on a complete spherical variety $X$, one has $H^i(X, L) = 0$ for $i \geq 1$.

Proof. For $\pi : X' \to X$ as above, we set $L' = \pi^*L$ and apply the Leray spectral sequence $H^p(X, R^q\pi_*L') \Rightarrow H^{p+q}(X', L')$. By Corollary 5.6 all $R^q\pi_*L' = 0$ for $q > 0$, so the spectral sequence degenerates in the first term and we get $H^p(X, L) = H^p(X', L')$, which are zero for $p > 0$ by Lemma 5.4 and Theorem 5.5. \qed
5.2. Reductive varieties.

**Theorem 5.9.** Let \((X, D)\) be a conjugation–invariant reductive pair. Then for \(0 < \varepsilon \ll 1\) the pair \((X, \Delta_{G \times G} + |\Delta_{B \times B^-}| + \varepsilon D)\) has log canonical singularities.

**Proof.** Without \(\varepsilon D\), this is Theorem 5.3. To prove it with \(\varepsilon D\), we must check that \(D\) does not contain a center of log canonical singularities of \((X, \Delta_{G \times G} + |\Delta_{B \times B^-}|)\), i.e. the image of a divisor on a resolution \(X' \to X\) that has discrepancy \(-1\). But any such center must be \(G \times G\)-invariant, so it must contain a \(G \times G\)-orbit. However, by the transversality condition the divisor \(D\) does not contain any \(G \times G\)-orbits. \(\square\)

Let \(X = X_\delta\) be a projective reductive variety for \(G\), such that \(\delta\) meets the interior of \(\Lambda^+\). (This holds, for example, if \(X\) has maximal dimension i.e. \(\text{dim } G\).) The choice of opposite Borel subgroups \(B, B^-\) in \(G\) defines a Borel subgroup \(B \times B^-\) in \(G \times G\). In this case, all coefficients in \(\Delta_{B \times B^-}\) equal 2. (This follows from [Bri97]; alternatively, this can be seen by looking at the maximal irreducible degeneration of \(X\), where the statement reduces to the case of \(G/B\).) Hence, we have \(\Delta_{B \times B^-} = 2E\), where \(E = \partial_{B \times B^-} X\). Similarly, \(\Delta_{B^- \times B} = 2E^-\). The \(T \times T\)-invariant divisors \(E, E^-\) differ by a conjugation: \(E^- = (w_0, w_0)E\), where \(w_0 \in W\) is the element exchanging \(B\) and \(B^-\) (that is, the longest element of \(W\)). Hence, in this case we have a canonical representative \(E + E^-\) in the linear system \(|\Delta_{B \times B^-}|\). Without the divisor \(D\), we can now prove a much stronger statement than Theorem 5.3 (it no longer holds with \(\varepsilon D\) added):

**Theorem 5.10.** Let \(X = X_\delta\) be a reductive variety such that \(\delta\) meets the interior of \(\Lambda^+\). Then the pair \((X, \Delta_{G \times G} + E + E^-)\) has log canonical singularities.

**Proof.** It is enough to prove the statement in the most degenerate irreducible case: when \(R(X, L)\) is a subalgebra of \(R_{K, A} = \text{gr } k[\tilde{G}]\) (with notation as in [SRV-1 4.1]). Indeed, the most degenerate variety is the special fiber of the Vinberg family, and the original variety \(X\) can be found as the fiber over a point in any open neighbourhood. By using the same trick as in the proof of Lemma 5.4 \(\Delta_{G \times G} + E + E^-\) is log canonical iff \(\Delta_{G \times G} + (1 - \varepsilon)(E + E^-) + \varepsilon \Delta_{B \times B^-} - \delta H\) is klt. But the property of being klt is open in families (this follows easily from [K+92 17.6], for example), and we are done.

In the most degenerate case, since the stabilizer of a generic point is contained in \(B^- \times B\), there exists a rational map from \(X\) to the projective variety \(G/B^- \times G/B\). The morphism \(X_1 \to G/B^- \times G/B\) from the graph of this rational map is locally trivial, with fibers being toric varieties. Therefore, the singularities of \(X_1\) can be resolved equivariantly to obtain a morphism \(\pi : \tilde{X} \to G/B^- \times G/B\).

Combinatorially, the first step – degenerating – corresponds to intersecting \(\delta\) with the positive chamber; the second – taking the graph – to cutting
faces so that the new polytope is in the interior of $\Lambda_+^1$; and third – desingularization – to cutting faces so that cones at the vertices become nonsingular.

Let $f : \tilde{X} \to X$ be the resolution of singularities obtained this way. Since $\tilde{X}$ is also a spherical variety, one has

$$K_{\tilde{X}} + \tilde{\Delta}_{G \times G} + \tilde{E} + \tilde{E}^- = 0 = f^*(K_X + \Delta_{G \times G} + E + E^-)$$

and the divisors $\tilde{E}, \tilde{E}^-$ are the strict preimages of $E, E^-$. Hence, it is sufficient to prove that the singularities of $(\tilde{X}, \tilde{\Delta}_{G \times G} + \tilde{E} + \tilde{E}^-)$ are log canonical.

At this point we have a normally crossing divisor $\tilde{\Delta}_{G \times G} = \partial_{G \times G} \tilde{X}$. The divisor $\tilde{E}$ is the closure of $(\text{open } G \times G \text{–orbit}) \setminus (\text{open } B \times B \text{–orbit})$, similarly for $\tilde{E}^-$ with the opposite orbits. Hence, both are pullbacks of divisors from $G/B^- \times G/B$:

$$\tilde{E} = \pi^*(D^- \times G/B + G/B^- \times D),$$

$$\tilde{E}^- = \pi^*(w_0D^- \times G/B + G/B^- \times w_0D),$$

where $D$ is the complement in $G/B$ of the open subset $B^- B/B$, and likewise for $D^-$. Thus, we are reduced to a problem about flag varieties.

Let $Y = G/B$. The theorem would be proved if we could show that the pair $(Y, D + w_0D)$ has log canonical singularities. For this purpose, we are going to use the “Bott-Samelson resolution of $G/B$”, that we now recall. Let $w_0 = s_1 s_2 \cdots s_N$ be a shortest decomposition of $w_0$ as a product of simple reflections; then $N = \dim G/B$. Let $P_i = B \cup Bs_i B$ be the corresponding minimal parabolic subgroups, then $P_i/B \simeq \mathbb{P}^1$. The Bott-Samelson resolution is the iterated $\mathbb{P}^1$–bundle

$$Z = P_1^B \times P_2^B \times \cdots \times P_N^B / B \to G/B$$

In $G/B$, the divisor $w_0D$ is the complement of $w_0B^- B/B = Bw_0B/B$. Thus, $w_0D = \bigcup_i P_1^B P_2^B \cdots \hat{P}_i \cdots P_N^B / B$, and its pullback to $Z$ is

$$\bigcup_i P_1^B P_2^B \cdots \hat{P}_i \cdots P_N^B / B.$$ 

This boundary of $Z$ is a reduced divisor with normal crossings. To complete the proof, we notice that the fibered product of two opposite Bott–Samelson resolutions

$$\tilde{Z} = Z \times_{G/B} Z^- = Z \times_{G/B} w_0Z$$

is nonsingular, and its boundary is a normal crossing divisor. Indeed, by Kleiman’s transversality theorem [Kle74], there exists $g \in G$ such that this property holds for the variety $Z \times_{G/B} gZ$. It is also clear that the set of all such $g \in G$ is open and invariant under left and right multiplication by $B$. Hence, it contains the unique open $B \times B$–coset $Bw_0B$; so it contains $w_0$. This proves that the pair $(X, \Delta_{G \times G} + E + E^-)$ has log canonical singularities.
5.3. **Stable reductive varieties.** Recall the following

**Definition 5.11.** The pair \((X, D)\) has semi log canonical singularities if

1. \(X\) is reduced and satisfies Serre’s condition \(S_2\),
2. in codimension one, \(X\) has simple crossings only,
3. for the normalization \(\nu: \tilde{X} \to X\), the pair \((\tilde{X}, \nu^{-1}D + \text{double locus})\) has log canonical singularities.

Let \((X, D)\) be a stable reductive pair with support \(\rho: |\Delta| \to \Lambda_R\) such that:

1. \(|\Delta|^+ = |\Delta| \cap \rho^{-1}\Lambda_R^+\) is homeomorphic to a disc,
2. \(D\) is conjugation–invariant.

In particular, any stable reductive variety in the connected component of \(M^c\) that contains a maximal–dimensional reductive variety is such: in this case, \(|\Delta|^+ = \delta^+\) for some \(W\)–admissible polytope \(\delta\).

Define \(\Delta_{\Delta^G}^+\) to be the closure of union of the codimension one orbits which correspond to the boundary of \(|\Delta|\). Hence, in the irreducible case we get the same definition as before. For a degeneration of an irreducible variety, we are only including the limit of the previous boundary, and excluding the simple normal singularities corresponding to the codimension one polytopes lying inside \(|\Delta|\).

We may now generalize Theorem 5.9 to stable reductive varieties:

**Theorem 5.12.** Let \((X = X_{\Delta^G}, D)\) be a conjugation-invariant stable reductive pair. Then for \(0 < \varepsilon \ll 1\) the pair \((X, \Delta_{\Delta^G}^+ + |\Delta_{B^xB^y}^-| + \varepsilon D)\) has semi log canonical singularities.

**Proof.** By Lemma 5.15 below, \(X\) is Cohen-Macaulay, and therefore \(S_2\). The normalization of \(X\) is a disjoint union of reductive varieties \(X_i\), corresponding to the polytopes in \(|\Delta| \cap \Lambda_R^+\). The divisorial intersections of two irreducible components correspond to codimension one faces not lying in the supporting hyperplanes of \(\Lambda_R^+\), and by an easy local computation they are generically normal crossings. By the transversality condition, \(D\) does not contain any of these intersections. So, we have

\[
(\text{double locus}) + \nu^*(\Delta_{\Delta^G}^+ + |\Delta_{B^xB^y}^-| + \varepsilon D) = \sum (\Delta_{\Delta^G,i}^+ + |\Delta_{B^xB^y}^-| + \varepsilon D_i),
\]

and we are done by the irreducible case. \(\square\)

**Remark 5.13.** This result confirms a prediction of the log Minimal Program that there should be a nice projective moduli space for pairs \((X, D)\) such that \((X, D)\) has semi log canonical singularities and \(\omega_X(D)\) is ample, cf. [Ale96]. Of course, the log Minimal Program remains conjectural in dimension larger than three.

In the same way, we obtain:
Theorem 5.14. Assume that $X$ is equidimensional of maximal dimension. Then the pair $(X, \Delta_{G \times G} + E + E^-)$ has semi log canonical singularities.

Here $E, E^-$ are defined by gluing the divisors $E_i, E_i^-$ on the irreducible components.

To prove that stable reductive varieties are CM, we will prove the stronger statement that their affine cone is CM. As for stable toric varieties and Stanley-Reisner rings, this is a topological property:

Lemma 5.15. Let $X = X_{\Sigma, t}$ be an affine stable reductive variety. Assume that the support $|\Sigma'| = \Sigma \cap \Lambda^+_R$ is homeomorphic to a convex cone. Then $X$ is Cohen-Macaulay.

Proof. Consider the Vinberg family for $X$, as in [SRV-I, Sec.7.5]. Since the property of being CM is open for fibers of this family, we can replace $X$ with its maximal degeneration, and assume that every cone $\sigma \in \Sigma$ is contained in $\Lambda^+_R$. Then we have: $X^{\text{diag}} = WX'$ where $X' = X'_{\Sigma', t'}$ is the affine stable toric variety associated with a complex of cones with support $|\Sigma'|$. Since $|\Sigma'|$ is Cohen-Macaulay as a topological set, $X'$ is CM by [Ale02, Thm.2.3.19]. Moreover, $X'$ is fixed pointwise by $U \times U^-$, since we are dealing with the maximal degeneration. So the left and right stabilizers of $X'$ in $G$ are opposite parabolic subgroups $P$ (containing $B$) and $P^-$ (containing $B^-$). The corresponding set of simple roots consists of those orthogonal to $|\Sigma'|$.

Let $\tilde{X}$ be the fiber product $G \times G \times P \times P^- X'$. Then $\tilde{X}$ is CM and comes with a proper birational morphism

$$\pi: \tilde{X} \to X$$

(eg, $\pi$ is an isomorphism on all open $G \times G$-orbits). By a result of Kempf [Kem73 pp.49-51] (consequence of the duality theorem for $\pi$, see e.g. [BP00 Lemma 15] for details), $X$ is CM if the following three conditions hold:

1. $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.
2. $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$ for $i \geq 1$.
3. $R^i \pi_* \omega_{\tilde{X}} = 0$ for $i \geq 1$, where $\omega_{\tilde{X}}$ denotes the dualizing sheaf.

To check these, we consider the projection

$$p: \tilde{X} \to G/P \times G/P^-.$$ 

This is an affine morphism, and $p_* \mathcal{O}_{\tilde{X}}$ is the $G \times G$-linearized sheaf on $G/P \times G/P^-$ associated with the $P \times P^-$-module $k[X']$. The latter decomposes as a sum of lines with weights $(\lambda, -\lambda)$ where $\lambda \in |\Sigma'|$. Let $L(\lambda, -\lambda)$ be the corresponding line bundles on $G/P \times G/P^-$, then

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^i(G/P \times G/P^-, \pi_* \mathcal{O}_{\tilde{X}}) = \bigoplus_{\lambda \in |\Sigma'|} H^i(G/P \times G/P^-, L(\lambda, -\lambda)).$$

Using the Borel–Weil–Bott theorem, it follows that $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = k[X]$ and $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for $i \geq 1$. Since $X$ is affine, this implies (1) and (2).
For (3), note that

\[ \omega_n = p^* \omega_{G/P \times G/P} \otimes \omega_p, \]

where \( \omega_p \) denotes the relative dualizing sheaf. Therefore,

\[ p_* \omega_n = \omega_{G/P \times G/P} \otimes p_* \omega_p. \]

This is an equality of \( G \times G \)-linearized sheaves, and \( p_* \omega_p \) is the \( G \times G \)-linearized sheaf on \( G/P \times G/P \), associated with the \( P \times P \)-module \( H^0(X', \omega_{X'}) \).

We claim that this module decomposes as a sum of lines with weights \( (\lambda, -\lambda) \) where \( \lambda \in |\Sigma'|_0 \) (the relative interior of \( |\Sigma'| \)). Such \( \lambda \)'s correspond to ample line bundles on \( G/P \times G/P \), so that \( H^0(G/P \times G/P, p_* \omega_n) = 0 \) for \( i \geq 1 \), by the Kodaira vanishing theorem. Hence the claim completes the proof.

To prove the claim, we have to show that \( H^0(X', \omega_{X'}) \) is the ideal \( I_{X'} \) of \( \Delta_{G \times G} \) in \( k[X'] \). Indeed, the codimension–1 singularities of \( X' \) correspond to codimension–1 cones \( \sigma' \in |\Sigma'|_0 \) that intersect \( |\Sigma'|_0 \). By an easy local computation they are generically normal crossings. Hence, if \( Z \) is the closed subset corresponding to the union of codimension–2 cones then \( X'_0 = X' \setminus Z \) is Gorenstein. Let \( i : X'_0 \to X' \) be the inclusion. We have \( \omega_{X'_0} = I_{X'_0} \), as a combination of two results: for toric varieties, and for the curve \( \text{Spec} k[x, y]/(xy) \), with \( \deg x = 1, \deg y = -1 \).

Since the sheaf \( \omega_{X'} \) is torsion free, there is an exact sequence

\[ 0 \to \omega_{X'} \to i_* \omega_{X'_0} = i_* I_{X'_0} = I_{X'} \to Q \to 0 \]

and the quotient \( Q \) has support in codimension \( \geq 2 \). Since \( X' \) is Cohen-Macaulay, the sheaf \( \omega_{X'} \) is Cohen-Macaulay, and in particular \( S_2 \). Hence, \( \text{Ext}^1(Q, \omega_{X'}) = 0 \) and the sequence splits. But the sheaf in the middle is torsion free, so \( Q = 0 \).

\[ \square \]

6. Generalizations

6.1. Polarized varieties and pairs without linearization. We call a polarized \( G \times G \)-variety \( (X, L) \) potentially \( G \times G \)-linearizable if some positive power of \( L \) is linearizable. Equivalently, \( L \) is \( G_1 \times G_1 \)-linearizable for a finite central extension \( G_1 \) of \( G \). The affine cone \( \tilde{X} \) of such a variety has an action of a group \( \tilde{G} \) which is a central \( G_m \)-extension \( \tilde{G} \) of \( G \) (as opposed to the trivial extension \( G_m \times G \)). That is the only change we need to make, after which our classification results and construction of the moduli go through.

For a non–normal but seminormal \( X \) the sheaf \( L \) is \( G_1 \times G_1 \)-linearizable if the action of the connected center is linearizable. That may fail, the simplest example being a plane nodal cubic curve with the action of \( C^* \) and \( L = O(1) \). However, by the result of [Ale02, Sec.4] there exists an infinite \( \mathbb{Z}^r \)-cover \( \tilde{X} \) of \( X \) so that the pull–back of \( L \) is linearizable. This cover is connected but is only locally of finite type (in the simplest example above,
it is an infinite chain of $\mathbb{P}^1$’s). In this case, we can classify the varieties and pairs, and construct their moduli, by classifying the infinite covers first, and then adding the $\mathbb{Z}^r$-action. Formally, the answers are the same but the $W$–complexes get replaced by $W \times \mathbb{Z}^r$–complexes of polytopes. We did not have a compelling reason to write down the gory details, since the bookkeeping gets very tedious. But this could be easily done, following the toric case of [Ale02].

6.2. Pairs $(X, D_1, \ldots, D_n)$ with several divisors. In complete analogy with $n$-pointed genus $g$ stable curves and their moduli space $M_{g,n}$, one can consider pairs with $n$ divisors:

**Definition 6.1.** A stable pair with $n$ divisors is a polarized stable reductive variety $(X, L)$ together with $n$ Cartier divisors $D_1 \ldots D_n$ such that $L = \mathcal{O}_X(\sum D_i)$ and the pair $(X, \sum D_i)$ is a stable pair in our previous definition.

**Theorem 6.2.** For each $n$, there exists a coarse moduli space of stable pairs with $n$ divisors, having multiplicity-free support. Every connected component of this space is projective.

**Proof.** From the transversality condition on $D$ and the group action, it follows that each $D_i$ is nef. Hence, for a fixed numerical type of $D$ there are only finitely many possibilities for the $D_i$’s. The moduli space is now constructed as a closed subscheme in the moduli space of pairs $(X, D)$ consisting of those pairs that split into $n$ parts. □

**Remark 6.3.** One can give a combinatorial description of this space, similar to Theorem 4.15. Instead of the fiber polytope $\Sigma(\alpha_\delta \to q_\delta)$, the component containing reductive varieties will be the toric variety for the polytope $\Sigma(\prod_{i=1}^n \alpha_{\delta_i} \to q_\delta)$, etc.

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