Subreducts and Subvarieties of PBZ ∗–lattices *

Claudia Mureșan 1

Abstract

PBZ ∗–lattices are bounded lattice–ordered structures endowed with two complements, called Kleene and Brouwer; by definition, they are the paraorthomodular Brouwer–Zadeh lattices in which the pairs of elements with their Kleene complements satisfy the Strong De Morgan condition. These algebras arise in the study of Quantum Logics and they form a variety PBZL ∗ which includes orthomodular lattices with an extended signature (with the two complements coinciding), as well as antiortholattices (whose Brouwer complements are trivial).

We establish a lattice isomorphism between the lattice of subvarieties of the variety SAOL generated by the antiortholattices with the Strong De Morgan property and the ordinal sum of the three–element chain with the lattice of subvarieties of the variety PKA of pseudo–Kleene algebras, which proves that SAOL and thus PBZL ∗ has uncountably many subvarieties since PKA does and also gives us axiomatizations for all subvarieties of SAOL from those of the subvarieties of PKA; furthermore, it proves that the variety PKA is generated by the class of the bounded involution lattice reducts of the members of SAOL and thus of those of any subvariety of PBZL ∗ that includes SAOL, hence neither of these classes is a variety. We also obtain an infinity of pairwise disjoint infinite ascending chains of subvarieties of SAOL.

Keywords Pseudo–Kleene algebra · Orthomodular lattice · PBZ ∗–lattice, (anti)ortholattice, (lattice of, relative axiomatizations for) subvarieties

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1 Introduction

PBZ ∗–lattices are bounded lattices endowed with two unary operations: an involution ′, called Kleene complement, which satisfies the Kleene condition: \( x \land x' \leq y \lor y' \), as well...
as paraorthomodularity, and the \textit{Brouwer complement}, which reverses order, is smaller than the Kleene complement, and satisfies only one of the \textit{De Morgan laws}, along with condition (\textit{*}), which is a weakening of the other \textit{De Morgan law} (called \textit{Strong De Morgan}), obtained from it by replacing one of the variables with the Kleene complement of the other.

The study of PBZ$^*$--lattices originates in the foundations of quantum mechanics; they have been introduced in [7] as abstractions for the sets of effects of complex separable Hilbert spaces endowed with the spectral order and two kinds of complements [3, 13, 17], and, from the previous such abstractions used in the unsharp approach to Quantum Logics, which include \textit{effect algebras} [5], \textit{quantum MV--algebras} [6] and \textit{Brouwer--Zadeh posets} [2], they present the advantage of forming a variety, which we denote by \textit{PBZL}$^*$, as paraorthomodularity becomes an equational property under the other axioms of PBZ$^*$--lattices.

PBZ$^*$ includes the variety \textit{OML} of \textit{orthomodular lattices} considered with an extended signature, by endowing each orthomodular lattice with a second complement equalling their Kleene complement, and it also includes the proper universal class \textit{AOL} of \textit{antiortholattices}, which are, by definition, the PBZ$^*$--lattices with all nonzero elements \textit{dense}, i.e. having 0 as their Brouwer complement. Antiortholattices are exactly the PBZ$^*$--lattices in which 0 and 1 are the only elements whose Kleene complements are bounded lattice complements.

The PBZ$^*$--lattices with the 0 meet--irreducible are exactly the antiortholattices that satisfy the \textit{Strong De Morgan condition}; we prove that the variety \textit{SAOL} they generate is also generated by the class of the antiortholattices with the 0 strictly meet--irreducible, namely the ordinal sums of the two--element chain with pseudo--Kleene algebras and again the two--element chain, while the variety \textit{PKA} of pseudo--Kleene algebras is generated by the bounded involution lattice reducts of the members of \textit{SAOL}, hence also by those of any variety of PBZ$^*$--lattices that includes it, therefore neither of these classes of reducts is a variety, in contrast with the class of bounded involution lattice reducts of the members of any subvariety of \textit{OML}. As a reinterpretation and strengthening of [7, Corollary 5.2], we point out that the lattice reducts of antiortholattices satisfying the \textit{Strong de Morgan law} generate the whole variety \textit{L} of lattices, and we have further results for subvarieties of \textit{L}. Not all paraorthomodular pseudo--Kleene algebras are bounded involution lattice reducts of PBZ$^*$--lattices, and not even all bounded lattices that are reducts of paraorthomodular pseudo--Kleene algebras are also bounded lattice reducts of PBZ$^*$--lattices.

Our \textbf{main theorem} (Theorem 3.1) states that the operator that takes subvarieties \textit{V} of the variety \textit{PKA} of pseudo--Kleene algebras to the subvarieties of \textit{SAOL} generated by the ordinal sums of the two--element chain with members of \textit{V} and again the two--element chain is a lattice isomorphism between the lattice of subvarieties of \textit{PKA} and the principal filter of the lattice of subvarieties of \textit{SAOL} generated by the smallest non--orthomodular subvariety of PBZ$^*$, and thus the lattice of subvarieties of \textit{SAOL} is isomorphic to the glued sum of the three--element chain with the lattice of subvarieties of \textit{PKA}. Hence the splitting pair in the lattice of subvarieties of \textit{PKA} formed of the variety \textit{OL} of ortholattices and the variety of Kleene algebras is taken by this lattice isomorphism into a splitting pair in the lattice of subvarieties of \textit{SAOL}. From the axiomatizations of the subvarieties of \textit{PKA} we construct axiomatizations for the subvarieties \textit{V} of \textit{SAOL} in two different ways, depending on whether \textit{V} belongs to the principal ideal or the principal filter determined by this splitting pair (Theorem 3.2). Since \textit{OL} and thus \textit{PKA} has uncountably many subvarieties, it follows that so does \textit{SAOL} and thus \textit{PBZL}$^*$. We also determine an infinity of pairwise disjoint infinite ascending chains of subvarieties of \textit{SAOL}.
2 Preliminaries

In this paper, we are using the notations from [7–10], along with conventions such as denoting, for any algebra $A$, by $A$ the sub reduct of $A$.

We denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For any set $M$, $|M|$ will denote the cardinality of $M$ and $\mathcal{P}(M)$ will be the set of the subsets of $M$.

For any (bounded) lattice $L$, $\prec$ denotes the cover relation of $L$, $L^d$ will be the dual of $L$, $M_i(L)$ and $Sm_i(L)$ denote the sets of the meet–irreducible elements of $L$ and the strictly (i.e. completely) meet–irreducible elements of $L$, respectively. For any $a, b \in L$, we denote by $[a, b]_L = [a]_L \cap [b]_L$ the interval of $L$ bounded by $a$ and $b$, as well as any algebraic structure we consider on it; the index $L$ will be eliminated from this notation when $L$ is $\mathbb{N}$ with the natural order. Recall that $(a, b)$ is called a splitting pair in $L$ iff $b \not\leq a$ and $L = (a]_L \cup [b)_L$. For all $n \in \mathbb{N}^*$, we denote by $D_n$ the $n$–element chain, regardless of the bounded lattice–ordered structure we consider on it.

Let $\mathbb{V}$ be a variety of algebras of a similarity type $\tau$, $\mathbb{C} \subseteq \mathbb{V}$, $\mathbb{W}$ a variety of similar algebras with reducts in $\mathbb{V}$, $\mathbb{D}$ a subclass of $\mathbb{W}$ and $\mathbb{A}$ and $\mathbb{B}$ members of $\mathbb{W}$. Then $Si(\mathbb{C})$ will be the class of the members of $\mathbb{C}$ which are subdirectly irreducible in $\mathbb{V}$, where the singleton algebra will be considered subdirectly irreducible. We denote by $\mathbb{T}$ the trivial subvariety of $\mathbb{V}$, consisting solely of its singleton members, by $I_V(\mathbb{D})$, $H_V(\mathbb{D})$, $S_V(\mathbb{D})$ and $P_V(\mathbb{D})$ the class of the isomorphic images, homomorphic images, subalgebras and direct products of the $\tau$–reducts of the members of $\mathbb{D}$, respectively, and by $V(\mathbb{D}) = H_VS_VP_V(\mathbb{D})$ the subvariety of $\mathbb{V}$ generated by the $\tau$–reducts of the members of $\mathbb{D}$. For any class operator $O_V$ and any $M \in \mathbb{D}$, $O_V([M])$ will be streamlined to $O_V(M)$. We will denote by $\mathbb{A} \cong_{\mathbb{V}} \mathbb{B}$ the fact that the $\tau$–reducts of $\mathbb{A}$ and $\mathbb{B}$ are isomorphic. $(Con_V(\mathbb{A}), \cap, \lor, \Delta_A, \nabla_A)$ will be the bounded lattice of the congruences of the $\tau$–reduct of $\mathbb{A}$, and, for any $n \in \mathbb{N}^*$ and any constants $\kappa_1, \ldots, \kappa_n$ from $\tau$, we denote by $Con_{\tau \kappa_1, \ldots, \kappa_n}(\mathbb{A}) = \{ \theta \in Con_V(\mathbb{A}) : (\forall i \in [1, n]) (\kappa_i^A/\theta = \kappa_i^A) \}$. If $\mathbb{V}$ is the variety of (bounded) lattices, then the index $V$ will be eliminated from the notations above, unless it’s necessary for clarity. $\Lambda(\mathbb{V})$ will be the lattice of subvarieties of $\mathbb{V}$.

A straightforward consequence of [12, Corollary 2, p.51] is the fact that $Con_V(\mathbb{A})$ is a bounded complete sublattice of $Con_V(\mathbb{A})$, while $Con_{\tau \kappa_1, \ldots, \kappa_n}(\mathbb{A})$ is a complete sublattice of $Con_V(\mathbb{A})$ and thus a bounded lattice [10, Lemma 2.(iii)]. An obvious consequence of the first of these two properties and the well–known fact that the $\tau$–reduct of $\mathbb{A}$ is subdirectly irreducible iff either $\mathbb{A}$ is singleton or $\Delta_A$ has a unique cover in $Con_{\tau}(\mathbb{A})$ [12] is the fact that, if the $\tau$–reduct of $\mathbb{A}$ is subdirectly irreducible (in $\mathbb{V}$), then $\mathbb{A}$ is subdirectly irreducible (in $\mathbb{W}$). Hence, if we denote, for any $\mathbb{C} \subseteq \mathbb{W}$, by $\mathbb{C}_V$ the class of the $\tau$–reducts of the members of $\mathbb{C}$, then $Si(\mathbb{C}_V) \subseteq Si(\mathbb{C})$. Furthermore, $H_V(\mathbb{C}) \subseteq H_V(\mathbb{C}_V)$, $S_V(\mathbb{C}) \subseteq S_V(\mathbb{C}_V)$ and $P_V(\mathbb{C}_V) = P_V(\mathbb{C})$, hence $V(\mathbb{C}_V) \subseteq V(\mathbb{C})$ and, if $\mathbb{C}$ is closed w.r.t. direct products, in particular if $\mathbb{C}$ is a subvariety of $\mathbb{W}$, then $V(\mathbb{C}_V) = H_VS_VP_V(\mathbb{C}_V) = H_VS_V(\mathbb{C}_V)$.

If $t$ and $u$ are $n$–ary terms over $\tau$ for some $n \in \mathbb{N}^*$ and $A_1, \ldots, A_n$ are subsets of $A$, then we denote by $A = A_1 \ldots A_n$ $t(x_1, \ldots, x_n) \approx u(x_1, \ldots, x_n)$ the fact that $t^A(a_1, \ldots, a_n) \approx u^A(a_1, \ldots, a_n)$ for all $a_1 \in A_1, \ldots, a_n \in A_n$, where $x_1, \ldots, x_n$ are the variables in their order of appearance in the equation $t \approx u$.

If $\mathbb{L}$ is a lattice with top element $1_\mathbb{L}$ and $\mathbb{M}$ is a lattice with bottom element $0_\mathbb{M}$, then we denote by $\mathbb{L} \oplus \mathbb{M}$ the ordinal sum (also called glued sum) of $\mathbb{L}$ with $\mathbb{M}$ and by $\mathbb{L} \oplus M$ its underlying set. So $\mathbb{L} \oplus \mathbb{M}$ will be the lattice obtained by stacking $\mathbb{M}$ on top of $\mathbb{L}$ and glueing the elements $1_\mathbb{L}$ and $0_\mathbb{M}$ together; see its rigorous definition in [10, 18–20]. Note that, for any $\alpha \in Con(\mathbb{L})$ and any $\beta \in Con(\mathbb{M})$, if we denote by $\alpha \oplus \beta$ the equivalence on $\mathbb{L} \oplus \mathbb{M}$ whose restrictions to $\mathbb{L}$ and $\mathbb{M}$ are $\alpha$ and $\beta$, respectively, then the map $(\alpha, \beta) \mapsto \alpha \oplus \beta$ is a
lattice isomorphism from $\text{Con}(L \times M) \cong \text{Con}(L) \times \text{Con}(M)$ to $\text{Con}(L \oplus M)$ and that the operations $\oplus$ on bounded lattices and their congruences are associative.

Now let us take a look at the algebras and varieties studied in the following sections. See [7–10] for more details on the following notions.

Recall that a bounded involutive lattice (brief, BI–lattice) is an algebra $L = (L, \wedge, \vee, \cdot, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, \cdot)$ is a lattice and $\cdot : L \to L$ is an order–reversing operation that satisfies $a'' = a$ for all $a \in L$. This makes $\cdot$ a dual lattice automorphism of $L$, called involu- tion.

For any (bounded) lattice–ordered algebra $A$, we denote by $A_l$ the (bounded) lattice reduct of $A$. For any algebra $A$ having a BI–lattice reduct, we denote that reduct by $A_{bi}$. If $C$ is a class of (bounded) lattice–ordered algebras and $D$ is a class of algebras having BI–lattice reducts, then we denote by $C_L = \{L_d : L \in C\}$ and by $D_{BI} = \{L_{bi} : L \in D\}$, rather than using the general notation above, to avoid confusion in several instances of formulas with many indexes.

A pseudo–Kleene algebra is a BI–lattice $K$ satisfying $a \wedge a' \leq b \vee b'$ for all $a, b \in K$. The involution of a pseudo–Kleene algebra is called Kleene complement. Distributive pseudo–Kleene algebras are called Kleene algebras or Kleene lattices.

Until mentioned otherwise, let $M$ be a bounded lattice and $K$ a BI–lattice.

We denote by $S(K) = \{x \in K : x \vee x' = 1\}$ and call the elements of $S(K)$ sharp elements of $K$. $K$ is called an ortholattice iff all its elements are sharp, and it is called an orthomodular lattice iff, for all $a, b \in K$, $a \leq b$ implies $b = (b \wedge a') \vee a$. Note that Boolean algebras are exactly the distributive ortholattices, any modular ortholattice is an orthomodular lattice, any orthomodular lattice is an ortholattice and any ortholattice is a pseudo–Kleene algebra.

If $f$ is a dual lattice automorphism of $M$, then the ordinal sum $M \oplus K_i \oplus M^d$ becomes a BI–lattice denoted $M \oplus K \oplus M^d$ when endowed with the involution $\cdot^{M\oplus K\oplus M^d}$ defined by $\cdot^{M\oplus K\oplus M^d} |_M = f$, $\cdot^{M\oplus K\oplus M^d} |_K = \cdot^K$ and $\cdot^{M\oplus K\oplus M^d} |_{M^d} = f^{-1}$. The BI–lattice $M \oplus D_1 \oplus M^d$ and its bounded lattice reduct will be denoted by $M \oplus M^d$.

$K$ is said to be paraorthomodular iff, for all $a, b \in K$, if $a \leq b$ and $a' \wedge b = 0$, then $a = b$. Algebras with BI–lattice reducts will be said to be orthomodular, respectively paraorthomodular iff their BI–lattice reducts are such. Note that any orthomodular lattice is a paraorthomodular BI–lattice and any paraorthomodular ortholattice is orthomodular. For any nonzero cardinality $\kappa$, $\text{MO}_\kappa$ denotes the modular ortholattice of length three with $2\kappa$ atoms. The smallest ortholattice which is not orthomodular is the Benzene ring $B_6$, with the Kleene complement defined as in the middle Hasse diagram below. To avoid confusion with our notation for chains, for every $n \in \mathbb{N} \setminus \{0, 1\}$, we denote by $\text{Ch}_n$ the non–orthomodular ortholattice $\mathcal{D}_{2n+1}$ from [16], represented in the rightmost diagram below, where $a'_i = b_{(n+i)} \mod (2n+1)$ for all $i \in [1, 2n+1]$.

A Brouwer–Zadeh lattice (brief, BZ–lattice) is an algebra $L = (L, \wedge, \vee, \cdot, \sim, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(L, \wedge, \vee, \cdot, \sim, 0, 1)$ is a pseudo–Kleene algebra and $\sim : L \to L$ is an order–reversing operation, called Brouwer complement, that satisfies: $a \wedge a\sim = 0$ and

\begin{center}
\begin{tikzpicture}

\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (0.5,1) {$a'$};
\node (d) at (1.5,1) {$a'' = a$};
\node (e) at (0,2) {$1$};

\draw (a) -- (b);
\draw (a) -- (c);
\draw (b) -- (c);
\draw (c) -- (d);
\draw (d) -- (e);
\draw (e) -- (a);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}

\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (0.5,1) {$a'$};
\node (d) at (1.5,1) {$a'' = a$};
\node (e) at (0,2) {$1$};
\node (f) at (2,2) {$\cdots$};
\node (g) at (3,2) {$a_{2n}$};
\node (h) at (4,2) {$a_{2n+1}$};

\draw (a) -- (b);
\draw (a) -- (c);
\draw (b) -- (c);
\draw (c) -- (d);
\draw (d) -- (e);
\draw (e) -- (a);
\draw (a) -- (f);
\draw (b) -- (f);
\draw (c) -- (f);
\draw (d) -- (f);
\draw (e) -- (f);
\draw (f) -- (g);
\draw (g) -- (h);
\end{tikzpicture}
\end{center}
a ≤ a^\sim\sim = a^\sim\sim\sim = a^\sim for all a \in L. Note that, in any BZ–lattice L, we have, for all a, b ∈ L:

a^\sim\sim\sim = a^\sim ≤ a', (a ∨ b)^\sim = a^\sim ∨ b^\sim and (a ∧ b)^\sim ≥ a^\sim ∧ b^\sim.

We consider the modularity and the distributivity laws for lattices, along with the following equations over the type of BZ–lattices, out of which SDM clearly implies (*) and WSDM, while J0 implies J1:

|        | MOD       | DIST       | (*)        | SDM (Strong De Morgan) | WSDM (weak SDM) | SK        | J0        | J1        |
|--------|-----------|------------|------------|------------------------|-----------------|----------|----------|----------|
|        | x ∨ (y ∧ (x ∨ z)) ≈ (x ∨ y) ∧ (x ∨ z) | x ∨ (y ∧ z) ≈ (x ∨ y) ∧ (x ∨ z) | (x ∧ x')^\sim ≈ x^\sim ∨ x'^\sim | (x ∧ y)^\sim ≈ x^\sim ∨ y^\sim | (x ∧ y^\sim) ≈ x^\sim ∨ y^\sim\sim | x ∧ y^\sim\sim ≤ x^\sim ∨ y | (x ∧ y^\sim) ∨ (x ∧ y^\sim\sim) ≈ x | (x ∧ (x ∧ y)^\sim) ∨ (x ∧ (x ∧ y)^\sim\sim) ≈ x |

A PBZ*-lattice is a paraorthomodular BZ–lattice that satisfies condition (*). PBZ*-lattices form a variety. In any PBZ*-lattice L, S(L) = {a ∈ L : a' = a^\sim} = {a^\sim : a ∈ L} and S(L) is the universe of the largest orthomodular subalgebra of L, so that L is orthomodular iff S(L) = L if L \models x' ≈ x^\sim.

We denote by L, BA, MOL, OML, OL, KA, PKA and BI the varieties of lattices, Boolean algebras, modular ortholattices, orthomodular lattices, ortholattices, Kleene algebras, pseudo–Kleene algebras and BI–lattices, respectively, and by POML the quasivariety of the paraorthomodular pseudo–Kleene algebras. We denote by BZL and PBZL* the varieties of BZ–lattices and PBZ*-lattices, respectively, by DIST, MOD, SDM and WSDM the varieties of the PBZ*-lattices that satisfy DIST, MOD, SDM and WSDM, respectively, and by PBZL the quasivariety of the paraorthomodular BZ–lattices.

By the above, OML can be identified with the subvariety \{L ∈ PBZL* : L \models x' ≈ x^\sim\} of PBZL*, by endowing each orthomodular lattice, in particular every Boolean algebra, with a Brouwer complement equaling its Kleene complement. In the same way, we can identify OL with the subvariety \{L ∈ BZL : L \models x' ≈ x^\sim\} of BZL. With this extended signature, BA ⊆ MOL ⊆ OML ⊆ SDM ⊆ WSDM.

An antiortholattice is a PBZ*-lattice L with S(L) = \{0, 1\}, or, equivalently, a PBZ*-lattice L whose Brouwer complement is trivial, that is a^\sim = 0 for all a ∈ L \− \{0\} (and, of course, 0^\sim = 1, as in every BZ–lattice). Note that any pseudo–Kleene algebra L with S(L) = \{0, 1\} (which implies paraorthomodularity), in particular any pseudo–Kleene algebra with the 0 meet–irreducible, in particular any Kleene chain, becomes an antiortholattice when endowed with the trivial Brouwer complement. Moreover, clearly, in any BZ–lattice L with the 0 meet–irreducible (which implies (*)), the Brouwer complement is trivial, so L is an antiortholattice. Furthermore, if M is a nonsingleton bounded lattice and K is a pseudo–Kleene algebra, then the BI–lattice M ⊕ K ⊕ M^d, endowed with the trivial Brouwer complement, becomes an antiortholattice, that we denote by M ⊕ K ⊕ M^d, as well; the antiortholattice M ⊕ D_j ⊕ M^d will be denoted M ⊕ M^d, as its bounded lattice reduct. We denote by AOL the proper universal class of antiortholattices, which is included in WSDM, and by SAOL = SDM ∩ VBZL(AOL).

A relative axiomatization w.r.t. PBZL* for VBZL(AOL), respectively OML ∨ VBZL(AOL), is given by J0, respectively (J1, WSDM), hence one for SAOL, respectively OML ∨ SAOL, is given by [J0, SDM], respectively [J1, SDM] [9].
3 The Lattice of Subvarieties of SAOL

Clearly, for any bounded lattices or BI–lattices $K$ and $L$, $D_2 \oplus K \oplus D_2 = D_2 \oplus L \oplus D_2$ if $K = L$, hence the map $C \mapsto D_2 \oplus C \oplus D_2$ is a poset embedding of $P(PK_A)$ into $P(AOL \cap SDM)$. Note that $AOL \cap SDM = \{L \in BZL : 0 \in Mi(L)\} = \{L \in PBZL^* : 0 \in Mi(L)\}$, so $D_2 \oplus PKA \oplus D_2 = \{L \in BZL : 0 \in Smi(L)\} = \{L \in PBZL^* : 0 \in Smi(L)\}$, since, clearly, $D_2 \oplus K \oplus D_2 \in PKA$ if $K \in PKA$. Of course, a similar property holds for modularity or distributivity instead of the Kleene condition.

**Lemma 3.1** $AOL \cap SDM = S_{BZL}(D_2 \oplus PKA \oplus D_2)$.

**Proof** Any nonsingleton antiortholattice $L$ in which $0^L$ is meet–irreducible is a subalgebra of the antiortholattice $A = D_2 \oplus Lbi \oplus D_2$, because the map $f : L \rightarrow A$ defined by $f(0^L) = 0^A$, $f(1^L) = 1^A$ and $f(x) = x$ for all $x \in L \setminus \{0^L, 1^L\}$ is an embedding of BZ–lattices. So $AOL \cap SDM = \{L \in AOL : 0^L \in Mi(L)\} \subseteq S_{BZL}(D_2 \oplus PKA \oplus D_2) \subseteq AOL \cap SDM$ since $AOL$ and thus $AOL \cap SDM$ is closed w.r.t. subalgebras. $\square$

**Remark 3.1** By [8, Lemma 3.3.1(1)], all subdirectly irreducible members of $V_{BZL}(AOL)$ are antiortholattices, hence every subvariety $V$ of $V_{BZL}(AOL)$ satisfies $Si(V) = Si(V \cap V_{BZL}(AOL)) = Si(V) \cap Si(V_{BZL}(AOL)) = Si(V \cap AOL) = Si(V \cap AOL)$ and thus $V$ is generated by the (subdirectly irreducible) antiortholattices it contains.

We start by investigating the relations between the class operators applied to a subclass $C$ of $BI$ or $PKA$ and these operators applied to the class $D_2 \oplus C \oplus D_2$, and, in the process, obtain an independent proof of the result from [8] stating that $V_{BZL}(D_3) = SAOL \cap DIST = SDM \cap DIST$, hence $V_{BZL}(D_3)$ is relatively axiomatized by $(SDM, DIST)$ w.r.t. $PBZL^*$. Note that, for any antiortholattice $L$, any proper congruence of $L$ has singleton classes of 0 and 1 and any lattice congruence of $L$ that preserves the Kleene complement and has singleton classes of 0 and 1 also preserves the trivial Brouwer complement, that is: $\text{Con}_{BZL}(L) = \text{Con}_{BZL01}(L) \cup \{\nabla_L\} = \text{Con}_{BZL01}(L) \cup \{\nabla_L\} [10]$. Clearly, $\text{Con}_{BI}(D_2 \oplus K \oplus D_2) = \{\Delta_{D_2} \oplus \theta \oplus \Delta_{D_2}, \nabla_{D_2} \oplus \theta \oplus \nabla_{D_2} : \theta \in \text{Con}_{BI}(K)\}$ for any $K \in BI$, thus, if $K \in PKA$, then $\text{Con}_{BZL}(D_2 \oplus K \oplus D_2) = \{\Delta_{D_2} \oplus \beta \oplus \Delta_{D_2}, \beta \in \text{Con}_{BI}(K)\} \cup \{\nabla_{D_2} \oplus K \oplus D_2\} \subseteq \text{Con}_{BI}(K) \oplus \Delta_{D_2}$, hence this particular case of a property in [10]:

**Remark 3.2** For any class $D \subseteq PKA$, we have, in the variety $BZL$: $Si(D_2 \oplus D \oplus D_2) = D_2 \oplus Si(D) \oplus D_2$.

**Lemma 3.2** Let $C \subseteq BI$ and $D \subseteq PKA$. Then:

(i) $D_2 \oplus PB_{BI}(C) \oplus D_2 \subseteq S_{BZL}(D_2 \oplus C \oplus D_2)$ and $D_2 \oplus PB_{BI}(D) \oplus D_2 \subseteq S_{BZL}(D_2 \oplus D \oplus D_2)$;

(ii) if $D \subseteq OL$, then $D_2 \oplus S_{BI}(D) \oplus D_2 = S_{BI}(D_2 \oplus D \oplus D_2) \setminus \{D_2\}$ and $D_2 \oplus S_{BI}(D) \oplus D_2 = S_{BZL}(D_2 \oplus D \oplus D_2) \setminus \{D_2\}$;

(iii) $D_2 \oplus S_{BI}PB_{BI}(C) \oplus D_2 \subseteq S_{BZL}(D_2 \oplus C \oplus D_2)$ and $D_2 \oplus S_{BI}PB_{BI}(D) \oplus D_2 \subseteq S_{BZL}(D_2 \oplus D \oplus D_2)$;

(iv) $HB_{BI}(D_2 \oplus C \oplus D_2) = (D_2 \oplus HB_{BI}(C) \oplus D_2) \cup HB_{BI}(C) \cup I_BZL(D_2)$; if $C \not\subseteq T$, then $HB_{BI}(D_2 \oplus C \oplus D_2) = (D_2 \oplus HB_{BI}(C) \oplus D_2) \cup HB_{BI}(C) \cup HB_{ZL}(D_2 \oplus D \oplus D_2) = (D_2 \oplus HB_{BI}(D) \oplus D_2) \cup I_BZL(D_1, D_2)$;

(v) $D_2 \oplus HB_{BI}(C) \oplus D_2 \subseteq HB_{BI}(D_2 \oplus C \oplus D_2) \setminus I_{BZL}((D_1, D_2))$ and $D_2 \oplus HB_{BI}(D) \oplus D_2 = HB_{ZL}(D_2 \oplus D \oplus D_2) \setminus I_BZL((D_1, D_2))$. 

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**Proof** We will be using the fact that, for any PBZ\(^*-\)lattice \(L\), \(\text{Con}_{\text{BZL}}(L)\) is a sublattice of \(\text{Con}_{\text{B}}(L)\). If \(I\) is a non–empty set, then, for any families \((L_i)_{i \in I} \subseteq \mathbb{B}\) and \((K_i)_{i \in I} \subseteq \text{PKA}\), we have: 

\[ D_2 \oplus \left( \bigcap_{i \in I} L_i \right) \oplus D_2 \in S_{\text{BI}}(\bigcap_{i \in I} (D_2 \oplus L_i \oplus D_2)) \text{ and } D_2 \oplus \left( \bigcap_{i \in I} K_i \right) \oplus D_2 \in S_{\text{BZL}}(\bigcap_{i \in I} (D_2 \oplus K_i \oplus D_2)), \]

since the map from \(D_2 \oplus \left( \bigcap_{i \in I} L_i \right) \oplus D_2\) to \(\bigcap_{i \in I} (D_2 \oplus L_i \oplus D_2)\), that preserves the 0 and 1 and restricts to the set inclusion on \(\bigcap_{i \in I} L_i\), respectively \(\bigcap_{i \in I} K_i\), is an embedding of BI–lattices, respectively BZ–lattices. Hence (i). If \(L \in \mathbb{B}\), \(K \in \text{PKA}\), \(M \in \text{S}_{\text{BI}}(L)\) and \(N \in \text{S}_{\text{BI}}(K)\), then \(D_2 \oplus M \oplus D_2 \in \text{S}_{\text{BI}}(D_2 \oplus L \oplus D_2)\) and \(D_2 \oplus N \oplus D_2 \in \text{S}_{\text{BZL}}(D_2 \oplus K \oplus D_2)\), since the map from \(D_2 \oplus M \oplus D_2\) to \(D_2 \oplus L \oplus D_2\), respectively \(D_2 \oplus N \oplus D_2\) to \(D_2 \oplus K \oplus D_2\), that preserves the 0 and 1 and restricts to a BI–lattice embedding of \(M\) into \(L\), respectively of \(N\) into \(K\), is an embedding of BI–lattices, respectively BZ–lattices. By (i), (iii) follows. If \(K \in \mathbb{O}_L\), then \(\text{S}_{\text{BI}}(D_2 \oplus K \oplus D_2) = \{D_2\} \cup \{D_2 \oplus \text{S}_{\text{BI}}(K) \oplus D_2\}\) and \(\text{S}_{\text{BZL}}(D_2 \oplus K \oplus D_2) = \{D_2\} \cup \{D_2 \oplus \text{S}_{\text{BI}}(K) \oplus D_2\}\). Indeed, \(D_2 \in \text{S}_{\text{BI}}(D_2 \oplus K \oplus D_2)\) and \(D_2 \in \text{S}_{\text{BZL}}(D_2 \oplus K \oplus D_2)\) and, by the above, \(\text{S}_{\text{BI}}(K) \subseteq \text{S}_{\text{BZL}}(D_2 \oplus K \oplus D_2)\). Now, if \(A \in \text{S}_{\text{BI}}(D_2 \oplus K \oplus D_2) \setminus \text{S}_{\text{BZL}}(D_2 \oplus K \oplus D_2)\), then there exists an \(a \in A \setminus \{0, 1\} \subseteq K\), and for each such \(a\) we have \(a' \in A \setminus \{0, 1\} \subseteq K\) as well, so that \(a \wedge a' = 1^K\) and \(a \wedge a' = 0^K\). Of course, for all \(x, y \in A \setminus \{0, 1\} \subseteq K\), we have \(x \wedge y, x \wedge y \in A \setminus \{0, 1\} \subseteq K\). Hence \(A \setminus \{0, 1\} \subseteq \text{S}_{\text{BI}}(K)\), therefore \(A \subseteq \text{S}_{\text{BI}}(K) \subseteq \text{S}_{\text{BZL}}(D_2 \oplus K \oplus D_2)\). Hence (ii). If \(L \in \mathbb{B}\), \(K \in \text{PKA}\), \(\theta \in \text{Con}_{\text{B}}(L)\) and \(\zeta \in \text{Con}_{\text{B}}(K)\), then \(D_2 \oplus L/\theta \oplus D_2 \simeq \text{BZL}(D_2 \oplus L \oplus D_2)/(\Delta_{D_2} \oplus \theta \oplus \Delta_{D_2})\), \(L/\theta \simeq \text{BZL}(D_2 \oplus L \oplus D_2)/(\Delta_{D_2} \oplus \theta \oplus \Delta_{D_2})\), since \(\Delta_{D_2} \oplus \zeta \oplus \Delta_{D_2} \simeq \text{Con}_{\text{B}}(D_2 \oplus K \oplus D_2)\) by an observation above. Hence (iv), which implies (v).

**Proposition 3.1** Let \(C \subseteq \mathbb{B}\) and \(D \subseteq \text{PKA}\). Then:

- \(D_2 \oplus \text{V}_{\text{BI}}(C) \oplus D_2 \subseteq \text{V}_{\text{BI}}(D_2 \oplus C \oplus D_2) = \text{V}_{\text{BI}}(D_2 \oplus \text{V}_{\text{BI}}(C) \oplus D_2);\)
- \(D_2 \oplus \text{V}_{\text{BI}}(D) \oplus D_2 \subseteq \text{V}_{\text{BZL}}(D_2 \oplus D \oplus D_2) = \text{V}_{\text{BZL}}(D_2 \oplus \text{V}_{\text{BI}}(D) \oplus D_2).\)

**Proof** The inclusions follow from Lemma 3.2 and their strictness from the clear fact that any bounded lattice–ordered algebra with the 0 strictly meet–irreducible is directly irreducible. These inclusions also prove the nontrivial right–to–left inclusions from the equalities between varieties. Trivially, \(\text{V}_{\text{BZL}}(D_3) = \text{V}_{\text{BZL}}(D_2 \oplus D_1 \oplus D_2) = \text{V}_{\text{BZL}}(D_2 \oplus \text{V}_{\text{BI}}(D_1) \oplus D_2) = \text{V}_{\text{BZL}}(D_2 \oplus T \oplus D_2)\). But, furthermore:

**Corollary 3.1**

- \(\text{V}_{\text{BZL}}(D_4) = \text{V}_{\text{BZL}}(D_2 \oplus \text{B}A \oplus D_2).\)
- \(\text{V}_{\text{BZL}}(D_5) = \text{V}_{\text{BZL}}(D_2 \oplus \text{K}A \oplus D_2) = \text{SDM} \cap \text{DIST} = \text{SAOL} \cap \text{DIST}, \) so \(\text{V}_{\text{BZL}}(D_5)\) contains all antiortholattice chains.

**Proof** By Proposition 3.1, \(\text{V}_{\text{BZL}}(D_4) = \text{V}_{\text{BZL}}(D_2 \oplus D_2 \oplus D_2) = \text{V}_{\text{BZL}}(D_2 \oplus \text{V}_{\text{BI}}(D_2) \oplus D_2) = \text{V}_{\text{BZL}}(D_2 \oplus \text{B}A \oplus D_2),\) while \(\text{V}_{\text{BZL}}(D_5) = \text{V}_{\text{BZL}}(D_2 \oplus D_3 \oplus D_2) = \text{V}_{\text{BZL}}(D_2 \oplus \text{V}_{\text{BI}}(D_3) \oplus D_2) = \text{V}_{\text{BZL}}(D_2 \oplus \text{K}A \oplus D_2)\) since \(\text{K}A = \text{V}_{\text{BI}}(D_3)\) [15]. By Lemma 3.1 and the observation above it, the subdirectly irreducible members of \(\text{SAOL} \cap \text{DIST}\) are distributive antiortholattices that satisfy SDM, that is distributive antiortholattices with the 0 meet–irreducible, and any such antiortholattice \(L\) is a subalgebra of the distributive antiortholattice.
\(D_2 \oplus L \oplus D_2 \subseteq D_2 \oplus KA \oplus D_2\), hence \(SAOL \cap \text{DIST} = V_{BZL}(D_2 \oplus KA \oplus D_2) = V_{BZL}(D_5)\) by the above. Since all antiortholattice chains are distributive and satisfy SDM, the latter statement follows.

As pointed out in Section 2, for any class \(D\) of PBZ*-lattices, \(Si(D_{B1}) \subseteq Si(D_{B1})\), thus \(D_2 \oplus Si(D_{B1}) \oplus D_2 \subseteq D_2 \oplus Si(D_{B1}) \oplus D_2\), hence \(V_{BZL}(D_2 \oplus Si(D_{B1}) \oplus D_2) \subseteq V_{BZL}(D_2 \oplus Si(D_{B1}) \oplus D_2)\) and, for any subvariety \(V\) of \(BZL\), \(V_{B1} \subseteq V_{B1}(V_{B1}) = H_{B1}(V_{B1})\).

**Proposition 3.2** For any subvariety \(V\) of SAOL:

- \(\text{Si}(V) = T \cup \text{Si}(S_{BZL}((D_2 \oplus PKA \oplus D_2) \cap V)) \subseteq T \cup S_{BZL}(D_2 \oplus \text{Si}(V_{B1}) \oplus D_2) \subseteq T \cup S_{BZL}(D_2 \oplus V_{B1} \oplus D_2)\);
- \(V \subseteq V_{BZL}(D_2 \oplus \text{Si}(V_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus \text{Si}(V_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus V_{B1} \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(\text{Si}(V_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2)\).

In particular:

- \(\text{Si}(SAOL) = T \cup \text{Si}(S_{BZL}(D_2 \oplus PKA \oplus D_2))\);
- \(SAOL = V_{BZL}(D_2 \oplus \text{Si}(SAOL_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus \text{Si}(SAOL_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(\text{Si}(SAOL_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2)\).

**Proof** By Lemma 3.1 and Remark 3.1, any \(A \in \text{Si}(V)\) satisfies \(A \in S_{BZL}(D_2 \oplus A_{B1} \oplus D_2) \subseteq S_{BZL}(D_2 \oplus \text{Si}(V_{B1}) \oplus D_2)\), hence \(\text{Si}(V) \subseteq S_{BZL}(D_2 \oplus \text{Si}(V_{B1}) \oplus D_2) \subseteq S_{BZL}(D_2 \oplus V_{B1} \oplus D_2)\), therefore \(V = V_{BZL}(\text{Si}(V)) \subseteq V_{BZL}(D_2 \oplus \text{Si}(V_{B1}) \oplus D_2) \subseteq V_{BZL}(D_2 \oplus V_{B1} \oplus D_2)\).

By Remark 3.2 and Proposition 3.1, it follows that \(\text{Si}(V_{B1} \oplus D_2) = D_2 \oplus \text{Si}(V_{B1}) \oplus D_2\), hence \(V_{BZL}(D_2 \oplus V_{B1} \oplus D_2) \supseteq V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2) \supseteq V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2) \supseteq \text{Si}(V)\), hence \(V \subseteq V_{BZL}(D_2 \oplus V_{B1} \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2) = V_{BZL}(D_2 \oplus V_{Bi}(V_{B1}) \oplus D_2)\).

Hence \(\text{Si}(V) = \text{Si}(SAOL) \cap V = \text{Si}(V) \cup \text{Si}(V_{B1}) \cup V = T \cup \text{Si}(S_{BZL}(D_2 \oplus PKA \oplus D_2)) \cup V = T \cup \text{Si}(S_{BZL}(D_2 \oplus PKA \oplus D_2) \cap V) = T \cup \text{Si}(S_{BZL}(D_2 \oplus PKA \oplus D_2) \cap V) = T \cup \text{Si}(S_{BZL}(D_2 \oplus PKA \oplus D_2) \cap V)\).

If a variety \(V\) of PBZ*-lattices is such that \(V = V_{BZL}(D_2 \oplus V_{B1} \oplus D_2)\), then, since \(D_3 = D_2 \oplus D_1 \oplus D_2 \subseteq D_2 \oplus V_{B1} \oplus D_2 \subseteq D_2 \oplus PKA \oplus D_2\), it follows that \(V = V_{BZL}(D_2 \oplus V_{B1} \oplus D_2) \subseteq \text{SAOL}\) by Proposition 3.2, and \(D_3 \subseteq V_{B1}\), thus \(D_5 = D_2 \oplus D_3 \oplus D_2 \subseteq D_2 \oplus V_{B1} \oplus D_2\), hence \(D_5 \subseteq V_{BZL}(D_5) \subseteq V \subseteq \text{SAOL}\). For any \(k, n \in N^*\) with \(k \leq n\), we have \(D_k \subseteq S_{B1}H_{B1}(D_k) \subseteq V_{B1}(D_k)\) and \(D_k \subseteq S_{B1}H_{B1}(D_k) \subseteq V_{B1}(D_k)\); more precisely \(D_k\) is a quotient of \(D_n\) if \(k\) is odd and \(n\) is even, and \(D_k\) is a subalgebra of \(D_n\) in all the other cases.

**Proposition 3.3** \(I_{BZL}((D_1, D_2, D_3)) = \{L \in AOL : L \vdash \text{SK}\} = V_{BZL}(D_3) \cap AOL = Si(V_{BZL}(D_3))\), hence \(V_{BZL}(D_3)\) is relatively axiomatized by SK w.r.t. \(V_{BZL}(AOL)\), thus by \(\{0, \text{SK}\}\) w.r.t. PBZL*.

**Proof** For any antiortholattice \(L\) and any \(a \in L\), clearly \(L \models_{[a], [0]} \text{SK}\) and \(L \vdash_{[1], [a]} \text{SK}\), hence \(L \vdash \text{SK}\) iff \(L \models_{L \setminus [1], L \setminus [0]} \text{SK}\) iff \(x \leq y\) for all \(x \in L \setminus \{1\}\) and all \(y \in L \setminus \{0\}\) iff
| \( L \setminus \{0, 1\} \) | ≤ 1 iff \( L \) | ≤ 3 iff \( L \in \text{I}_{\text{BZL}}(\{D_1, D_2, D_3\}) \). Clearly \( D_1, D_2, D_3 \in \text{V}_{\text{BZL}}(D_3) \), thus \( \text{I}_{\text{BZL}}(\{D_1, D_2, D_3\}) \subseteq \text{V}_{\text{BZL}}(D_3) \cap \text{AOL} \). By the above, \( \text{V}_{\text{BZL}}(D_3) \equiv \text{SK} \) and thus, if an antiortholattice \( L \) belongs to \( \text{V}_{\text{BZL}}(D_3) \), then \( L \equiv \text{SK} \), hence \( L \in \text{I}_{\text{BZL}}(\{D_1, D_2, D_3\}) \).

The antiortholattices \( D_1, D_2 \) and \( D_3 \) are simple, thus subdirectly irreducible, hence \( \text{I}_{\text{BZL}}(\{D_1, D_2, D_3\}) \subseteq \text{Si}(\text{V}_{\text{BZL}}(D_3)) \subseteq \text{V}_{\text{BZL}}(D_3) \cap \text{AOL} = \text{I}_{\text{BZL}}(\{D_1, D_2, D_3\}) \) by Remark 3.1 and the above, therefore \( \text{Si}(\text{V}_{\text{BZL}}(D_3)) = \text{I}_{\text{BZL}}(\{D_1, D_2, D_3\}) \).

Part of Proposition 3.3 can be obtained by applying the fact that, if a finite algebra \( A \) belongs to a congruence–distributive variety, \( \text{Si}(V_V(A)) \subseteq \text{H}_V(S_V(A)) \). An easy consequence of Proposition 3.3 is the fact that \( \text{V}_{\text{BZL}}(D_3) \subseteq \text{V}_{\text{BZL}}(D_4) \), since \( D_3 \in \text{H}_{\text{BZL}}(D_4) \), but the antiortholattice \( D_4 \notin \text{I}_{\text{BZL}}(\{D_1, D_2, D_3\}) = \text{V}_{\text{BZL}}(D_3) \cap \text{AOL} \), thus \( D_4 \neq \text{V}_{\text{BZL}}(D_3) \).

**Lemma 3.3** Let \( L \) be a BI–lattice. Then:

(i) \( L \in \text{H}_{\text{BI}}(D_2 \oplus L \oplus D_2) \subseteq \text{V}_{\text{BI}}(D_2 \oplus L \oplus D_2) \), so \( \text{V}_{\text{BI}}(L) \subseteq \text{V}_{\text{BI}}(D_2 \oplus L \oplus D_2) \);

(ii) \( D_2 \oplus L \oplus D_2 \in \text{S}_{\text{BI}}(D_3 \times L) \subseteq \text{V}_{\text{BI}}(D_3 \times L) \), so \( \text{V}_{\text{BI}}(D_2 \oplus L \oplus D_2) \subseteq \text{V}_{\text{BI}}(D_3 \times L) \);

(iii) if \( D_3 \in \text{V}_{\text{BI}}(L) \), then \( \text{V}_{\text{BI}}(L) = \text{V}_{\text{BI}}(D_2 \oplus L \oplus D_2) \);

(iv) \( D_3 \neq \text{V}_{\text{BI}}(L) \) iff \( L \) is an ortholattice.

**Proof** (i) \( L \cong_{\text{BI}} (D_2 \oplus L \oplus D_2)/(\nabla V_2 \oplus \Delta L \oplus \nabla V_2) \in \text{H}_{\text{BI}}(D_2 \oplus L \oplus D_2) \), as already noticed in [7, Theorem 3.2]. (ii) If we denote \( D_3 = \{0^{D_3}, 1^{D_3}\} \) and by \( A = D_2 \oplus L \oplus D_2 \in \text{BI} \), then the map \( \varphi : D_2 \oplus L \oplus D_2 \rightarrow D_3 \times L \) defined by: \( \varphi(0^A) = (0^{D_3}, 0^L), \varphi(1^A) = (1^{D_3}, 1^L) \) and \( \varphi(x) = (c, x) \) for all \( x \in L = A \setminus \{0^A, 1^A\} \) is a BI–lattice embedding of \( A \) into \( D_3 \times L \), thus \( A \in \text{S}_{\text{BI}}(D_3 \times L) \). (iii) By (ii). (iv) Since \( D_3 \notin \text{OL} \), we have the right–to–left implication. If \( D_3 \notin \text{V}_{\text{BI}}(L) \) and hence, by the fact that \( D_3 \in \text{H}_{\text{BI}}(D_4) \subseteq \text{V}_{\text{BI}}(D_4) \), it follows that also \( D_4 \notin \text{V}_{\text{BI}}(L) \), then:

- \( D_3 \notin \text{S}_{\text{BI}}(L) \), so there exists no \( x \in L \) with \( x = x' \),
- \( D_4 \notin \text{S}_{\text{BI}}(L) \), so there exists no \( x \in L \setminus \{0\} \) with \( x < x' \),

hence there exists no \( x \in L \setminus \{0\} \) with \( x < x' \). But, for every \( u \in L \), we have \( u \vee u' \leq u \wedge u' = (u \wedge u')' \). Therefore \( u \vee u' = 0 \) for all \( u \in L \), which means that \( L \in \text{OL} \). 

**Remark 3.3** If \( V \) is a subvariety of \( \text{BI} \), then:

- \( D_3 \in V \) iff \( KA \subseteq V \), since \( KA = \text{V}_{\text{BI}}(D_3) \) [15];
- \( D_3 \notin V \) iff \( V \subseteq \text{OL} \); indeed, the left–to–right implication is trivial and the converse follows from Lemma 3.3.(iv). \( KA \not\subseteq \text{OL} \), thus \( (\text{OL}, KA) \) is a splitting pair in \( \Lambda(\text{BI}) \) and in \( \Lambda(PKA) \).

For any \( k, n, p \in \mathbb{N} \) and any pair \((t, u)\), where \( t(x_1, \ldots, x_k, z_1, \ldots, z_p) \) and \( u(y_1, \ldots, y_n, z_1, \ldots, z_p) \) are terms in the language of \( \mathbb{B} \) having the arities \( k + p \), respectively \( n + p \) and \( p \) common variables \( z_1, \ldots, z_p \), we consider the \((k + n)\)-ary term \( m(t, u) \) in the language of \( \mathbb{B} \), defined as follows:

\[
m(t, u)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) = \bigvee_{i=1}^{k} (x_i \wedge x_i') \vee \bigwedge_{j=1}^{n} (y_j \wedge y_j') \vee \bigvee_{h=1}^{p} (z_h \wedge z_h')
\]

Note that:

\[
m(u, t)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) = \bigvee_{i=1}^{k} (x_i \wedge x_i') \vee \bigwedge_{j=1}^{n} (y_j \wedge y_j') \vee \bigvee_{h=1}^{p} (z_h \wedge z_h') \vee u(y_1, \ldots, y_n, z_1, \ldots, z_p).
\]
Remark 3.4 If $L \in \mathbb{B}$ is such that $D_3 \not\in V_{\mathbb{B}}(L)$, then $L \in \mathbb{OL}$ by Lemma 3.3.(iv), so that $L \vDash x \land x' \approx 0$ and $L \vDash x \lor x' \approx 1$, therefore, for any terms $t$ and $u$ in the language of $\mathbb{B}$, there exist terms $r$ and $s$ in the language of $\mathbb{B}$ having nonzero arities such that: $L \vDash t \approx u$ iff $L \vDash r \approx s$.

Lemma 3.4 Let $L \in \mathbb{B}$ and $t$ and $u$ be terms in the language of $\mathbb{B}$. Then:

- if $D_3 \in V_{\mathbb{B}}(L)$, then: $L \vDash t \approx u$ iff $D_2 \oplus L \oplus D_2 \vDash t \approx u$;
- if $t$ and $u$ have nonzero arities and $L \in \mathbb{PKA}$, so that $D_2 \oplus L \oplus D_2 \in \mathbb{OL}$, then: $L \vDash t \approx u$ iff $D_2 \oplus L \oplus D_2 \vDash m(t, u) \approx m(u, t)$.

Proof Lemma 3.3.(iii) implies the first equivalence. Now let us assume that $L \in \mathbb{PKA}$ and denote by $A = D_2 \oplus L \oplus D_2 \in \mathbb{OL}$. Let $k, n, p$ be as in the notation above, and assume that $k + p, n + p \in \mathbb{N}$. Then, for any $a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p \in A$, we have, in $A$:

- if all the elements $a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p \in L$, then $m(t, u)^A(a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p) = t^L(a_1, \ldots, a_k, c_1, \ldots, c_p)$ and $m(u, t)^A(a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p) = u^L(b_1, \ldots, b_n, c_1, \ldots, c_p)$;
- if at least one of the elements $a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p$ belongs to $A \setminus L = \{0^A, 1^A\}$, then $m(t, u)^A(a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p) = m(u, t)^A(a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p)$ for all $a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p \in A$ iff $t^L(a_1, \ldots, a_k, c_1, \ldots, c_p) = u^L(b_1, \ldots, b_n, c_1, \ldots, c_p)$ for all $a_1, \ldots, a_k, b_1, \ldots, b_n, c_1, \ldots, c_p \in L$, that is $A \vDash m(t, u) \approx m(u, t)$ iff $L \vDash t \approx u$. \hfill \Box

Proposition 3.4 For any subclass $C \subseteq \mathbb{B}$, we have:

- $D_3 \in V_{\mathbb{B}}(C)$ iff $KA \subseteq V_{\mathbb{B}}(C)$ iff $V_{\mathbb{B}}(C) = V_{\mathbb{B}}(D_2 \oplus K \oplus D_2)$;
- $D_3 \not\in V_{\mathbb{B}}(C)$ iff $V_{\mathbb{B}}(C) \subseteq \mathbb{OL}$ iff $V_{\mathbb{B}}(C) \not\subseteq V_{\mathbb{B}}(D_2 \oplus K \oplus D_2)$.

For any subvariety $\mathbb{V}$ of $\mathbb{B}$, we have:

(i) $D_3 \in \mathbb{V}$ iff $\mathbb{V} = H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2)$ is a variety;
(ii) $D_3 \not\in \mathbb{V}$ iff $\mathbb{V} \not\subseteq H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2)$ is not closed w.r.t. direct products iff $H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2) \not\subseteq V_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2)$.

Proof By Lemma 3.3.(i), $C \subseteq H_{\mathbb{B}}(D_2 \oplus \mathbb{C} \oplus D_2) \subseteq V_{\mathbb{B}}(D_2 \oplus \mathbb{C} \oplus D_2)$, hence $V_{\mathbb{B}}(C) \subseteq V_{\mathbb{B}}(D_2 \oplus \mathbb{C} \oplus D_2)$. $D_3 = D_2 \oplus D_1 \oplus D_2 \in D_2 \oplus \mathbb{T} \oplus D_2 \subseteq D_2 \oplus \mathbb{V} \oplus D_2 \subseteq V_{\mathbb{B}}(D_2 \oplus \mathbb{C} \oplus D_2)$, thus, if $V_{\mathbb{B}}(C) = V_{\mathbb{B}}(D_2 \oplus \mathbb{C} \oplus D_2)$, then $D_3 \in V_{\mathbb{B}}(C)$. By Lemma 3.4, if $D_3 \in V_{\mathbb{B}}(C)$, then $V_{\mathbb{B}}(D_2 \oplus \mathbb{C} \oplus D_2)$, hence $V_{\mathbb{B}}(C) = V_{\mathbb{B}}(D_2 \oplus \mathbb{C} \oplus D_2)$. Now apply Remark 3.3. The above and Lemma 3.3.(i), which shows that $\mathbb{V} \subseteq H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2) \subseteq V_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2)$, prove the first equivalence in each of (i) and (ii), along with the right-to-left implication in the second equivalence from (i). By Lemma 3.2.(iv), $H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2) = (D_2 \oplus \mathbb{V} \oplus D_2) \cup \mathbb{V}$. We have $D_3 \in D_2 \oplus \mathbb{V} \oplus D_2 \subseteq H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2) \subseteq H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2)$, $D_3 \not\in \mathbb{V}$ since $D_3 \not\in \mathbb{V}$, and $D_3 \not\in D_2 \oplus \mathbb{V} \oplus D_2$ since $D_3 \not\in \mathbb{L}$ is directly reducible, thus $D_3 \not\in H_{\mathbb{B}}(D_2 \oplus \mathbb{V} \oplus D_2)$. Hence the rest of the implications. \hfill \Box

Proposition 3.5 For any subvariety $\mathbb{V}$ of $\mathbb{SAOL}$ such that $D_3 \in \mathbb{V}$, if $\mathbb{D} = \{K \in \mathbb{PKA} : D_2 \oplus K \oplus D_2 \in \mathbb{V}\}$, then:

(i) $\text{Si}(\mathbb{V}) = I_{\mathbb{B}}(D_1, D_2) \cup \text{Si}(S_{\mathbb{B}}(D_2 \oplus \mathbb{D} \oplus D_2))$;

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(ii) if $D \subseteq \mathbb{O}L$, then $Si(V) = I_{3BZL}(\{D_1, D_2\}) \cup (D_2 \oplus Si(S_{BI}(D)) \oplus D_2)$;
(iii) $D$ is a subvariety of $PKA$ and $V = V_{3BZL}(D_2 \oplus D \oplus D_2) = V_{3BZL}(D_2 \oplus PKA \oplus D_2) \cap V$.

**Proof** (i) If $D$ is as in the enunciation, then $D \supseteq T$ and $D_2 \oplus D \oplus D_2 = (D_2 \oplus PKA \oplus D_2) \cap V$, hence the equality in the enunciation by Proposition 3.2. (ii) By (i), using Lemma 3.2.(ii) and Remark 3.2. (iii) From (i), keeping in mind that $D_2 \oplus D \oplus D_2 \subseteq V$, we obtain that $V = V_{3BZL}(Si(V)) = V_{3BZL}(Si(S_{3BZL}(D_2 \oplus D \oplus D_2))) = V_{3BZL}(S_{3BZL}(D_2 \oplus D \oplus D_2)) = V_{3BZL}(D_2 \oplus PKA \oplus D_2) \cap V$. By Proposition 3.1, $D_2 \oplus V_{RI}(D) \oplus D_2 \subseteq V_{3BZL}(D_2 \oplus D \oplus D_2) = V$, thus $D_2 \oplus V_{RI}(D) \oplus D_2 \subseteq (D_2 \oplus PKA \oplus D_2) \cap V = D_2 \oplus D \oplus D_2$, hence $V_{RI}(D) \subseteq D$, so $V_{RI}(D) = D$. □

**Theorem 3.1** The operator $V \mapsto V_{3BZL}(D_2 \oplus V \oplus D_2)$ from the lattice of subvarieties of $PKA$ to the principal filter generated by $V_{3BZL}(D_3)$ in the lattice of subvarieties of $SAOL$ is a lattice isomorphism whose inverse is defined by $W \mapsto \{K \in PKA : D_2 \oplus K \oplus D_2 \in W\}$, so $A(SAOL) \cong D_3 \oplus \Lambda(PKA)$.

**Proof** Let $\nu : \Lambda(PKA) \to [V_{3BZL}(D_3), SAOL]_{A(PBZL^*)}$ be the map in the enunciation: $\nu(V) = V_{3BZL}(D_2 \oplus V \oplus D_2)$ for all $V \in \Lambda(PKA)$. Recall that $\nu(T) = V_{3BZL}(D_3)$ and, by Proposition 3.2, $\nu(PKA) = SAOL$. Clearly, for any subclasses $C$ and $D$ of $B_1$, we have: $C \subseteq D$ if $D_2 \oplus C \oplus D_2 \subseteq D_2 \oplus D \oplus D_2$, which implies $V_{3BZL}(D_2 \oplus C \oplus D_2) \subseteq V_{3BZL}(D_2 \oplus D \oplus D_2)$, so $\nu$ is order–preserving. Now let $V$ and $W$ be subvarieties of $PKA$ such that $V \neq W$. Then $V \not\subseteq W$ or $W \not\subseteq V$. W.l.g. we may assume that $W \not\subseteq V$, so that, for some terms $t$, $u$ in the language of $BI$–lattices, $V \models t \approx u$, but $W \not\models t \approx u$, hence $L \neq t \approx u$ for some $L \in W$. By Remarks 3.3 and 3.4 and Lemma 3.4, if $V \subseteq \mathbb{O}L$, then $t$ and $u$ can be chosen to have nonzero arities and hence $\nu(V) = V_{3BZL}(D_2 \oplus V \oplus D_2) \models m(t, u) \approx m(u, t)$ and $\nu(W) = V_{3BZL}(D_2 \oplus W \oplus D_2) \not\models m(t, u) \approx m(u, t)$, therefore $\nu(W) \not\subseteq \nu(V)$. By Remark 3.3, Lemma 3.4 and Lemma 3.3.(ii), if $KA \subseteq V$, then $\nu(V) \models t \approx u$ and, since $L \in H_{3BZL}(D_2 \oplus L \oplus D_2)$, we have $D_2 \oplus L \oplus D_2 \not\models t \approx u$, thus $D_2 \oplus L \oplus D_2 \not\in \nu(W) \setminus \nu(V)$. By Remark 3.3, it follows that, whenever $W \not\subseteq V$, we have $\nu(W) \not\subseteq \nu(V)$, in particular $\nu(W) \neq \nu(V)$, hence $\nu$ is injective. So $\nu$ is a poset embedding and it preserves non–inclusion, hence it also reflects order. By Proposition 3.5.(iii), $\nu$ is also surjective, thus it is a lattice isomorphism, and its inverse maps each $W \in [V_{3BZL}(D_3), SAOL]_{A(PBZL^*)}$ to the variety $\nu^{-1}(W) = \{K \in PKA : D_2 \oplus K \oplus D_2 \in W\} \in \Lambda(PKA)$. Since the subvarieties of $SAOL$ are $T$, $IBA$ and those including $V_{3BZL}(D_3)$, it follows that $\Lambda(SAOL) \cong D_3 \oplus \Lambda(PKA)$. □

An easy consequence of Theorem 3.1 and Proposition 3.1 is that the map $C \mapsto V_{3BZL}(D_2 \oplus C \oplus D_2)$ from $P(PKA)$ to $P(SAOL)$ preserves arbitrary unions. Here is another direct consequence of Theorem 3.1, in which we restate the result on the second splitting pair from [11], where a typo appeared; note that condition $\odot$ featured in [11, Subsection 3.1] is equational in $\mathbb{O}L$:

**Corollary 3.2**
- $(V_{3BZL}(D_2 \oplus \mathbb{O}L \oplus D_2), V_{3BZL}(D_5)) = V_{3BZL}(D_2 \oplus KA \oplus D_2)$ is a splitting pair in the lattice of subvarieties of $SAOL$.
- $(V_{3BZL}(D_2 \oplus \mathbb{O}M \oplus D_2), V_{3BZL}(D_2 \oplus B_6 \oplus D_2))$ is a splitting pair in $\Lambda(V_{3BZL}(D_2 \oplus \mathbb{O}L \oplus D_2))$, but not in $\Lambda(SAOL)$.
- If $CH = \{Ch_n : n \in N \setminus \{0, 1\}\}$ and $C, D$ are subsets of $CH$ such that $C \neq D$, then $V_{3BZL}(D_2 \oplus C \oplus D_2) \neq V_{3BZL}(D_2 \oplus D \oplus D_2)$.
- $SAOL$, thus also $PBZL^*$, has continuum many subvarieties.
- For any permutation $\pi$ of $N \setminus \{0, 1\}$, $CH_{\pi} = (V_{3BZL}(D_2 \oplus \{Ch_{\pi(i)} : i \in [2, n]\} \oplus D_2))_{n \in N \setminus \{0, 1\}}$ is an infinite ascending chain in the interval $[V_{3BZL}(D_2 \oplus B_6 \oplus \odot}) Springer
\( D_2 \)), \( V_{\text{BZL}}(D_2 \oplus \text{OL} \oplus D_2) \) of \( \Lambda(\text{SAOL}) \), and, if \( \rho \) is a permutation of \( \mathbb{N} \setminus \{0, 1\} \) such that \( \pi(\{2, n\}) \neq \rho(\{2, n\}) \) for any \( n \in \mathbb{N} \setminus \{0, 1\} \), then the chains \( \text{Ch}_\pi \) and \( \text{Ch}_\rho \) are disjoint.

- \( T = V_{\text{BZL}}(D_1) < \text{BA} = V_{\text{BZL}}(D_2) < V_{\text{BZL}}(D_3) < V_{\text{BZL}}(D_4) = V_{\text{BZL}}(D_2 \oplus D_2 \oplus D_2) = V_{\text{BZL}}(D_2 \oplus \text{MO}_1 \oplus D_2) < V_{\text{BZL}}(D_2 \oplus \text{MO}_2 \oplus D_2) < \ldots < V_{\text{BZL}}(D_2 \oplus \text{MO}_n \oplus D_2) < \ldots \subset V_{\text{BZL}}(D_2 \oplus \text{MO}_{n_0} \oplus D_2) \subset V_{\text{BZL}}(D_2 \oplus \text{OL} \oplus D_2) = \text{MOD} \cap V_{\text{BZL}}(D_2 \oplus \text{OL} \oplus D_2) \) \((n \in \mathbb{N}, n \geq 3)\) is an infinite ascending chain of subvarieties of \( \text{MOD} \cap V_{\text{BZL}}(D_2 \oplus \text{OL} \oplus D_2) \), disjoints from each of the infinite ascending chains \( \text{Ch}_\pi \) above;

- \( T = V_{\text{BZL}}(D_1) < V_{\text{BZL}}(D_2) < V_{\text{BZL}}(D_3) < V_{\text{BZL}}(D_4) < V_{\text{BZL}}(D_5) \subset \text{SAOL} \).

**Proof** By Remark 3.3 and Theorem 3.1, the first pair is splitting in the interval \([V_{\text{BZL}}(D_3), \text{SAOL}]\) of \( \Lambda(\text{PBZL}^*) \) and \( \Lambda(\text{SAOL}) = \{T, \text{BA}\} \cup [V_{\text{BZL}}(D_3), \text{SAOL}] \cong D_3 \oplus [V_{\text{BZL}}(D_3), \text{SAOL}] \), hence the first statement in the enunciation. By the well–known fact that \( \text{OMIL} \cap V_{\text{BI}}(B_0) \) is a splitting pair in \( \Lambda(\text{OL}) \) and Theorem 3.1, which also ensures us that \( V_{\text{BZL}}(D_3) \) is incompatible to each of the varieties \( V_{\text{BZL}}(D_2 \oplus \text{OMIL} \oplus D_2) \) and \( V_{\text{BZL}}(D_2 \oplus B_6 \oplus D_2) \), we get the second statement. We have \( \text{CH} \cap \text{OL} \subset \text{PKA} \) and \( \text{CH} \cap \text{OMIL} = \emptyset \). By [16, Theorem 4.7], \( \text{OL} \) has continuum many subvarieties, specifically, if \( C, D \in \mathcal{P}(\text{CH}) \) such that \( C \neq D \), then \( V_{\text{BI}}(C) \neq V_{\text{BI}}(D) \), so, that for any ascending chain \((S_n)_{n \in \mathbb{N}\setminus\{0,1\}}\) of subsets of \( \text{CH} \), \((V_{\text{BI}}(S_n))_{n \in \mathbb{N}\setminus\{0,1\}}\) is an ascending chain of subvarieties of \( \text{OL} \). \( \text{PBZL}^* \) has finite type and thus at most continuum many subvarieties, hence so does \( \text{SAOL} \). By Theorem 3.1 and Proposition 3.1, it follows that \( \text{SAOL} \), thus also \( \text{PBZL}^* \), has at least continuum many subvarieties, specifically, if \( C, D \in \mathcal{P}(\text{CH}) \) such that \( C \neq D \), then \( V_{\text{BZL}}(D_2 \oplus C \oplus D_2) \neq V_{\text{BZL}}(D_2 \oplus D \oplus D_2) \), so, furthermore, for any ascending chain \((S_n)_{n \in \mathbb{N}\setminus\{0,1\}}\) of subsets of \( \text{CH} \), \((V_{\text{BZL}}(D_2 \oplus S_n \oplus D_2))_{n \in \mathbb{N}\setminus\{0,1\}}\) is an ascending chain of subvarieties of \( V_{\text{BZL}}(D_2 \oplus \text{OL} \oplus D_2) \) disjoint from \( \Lambda(V_{\text{BZL}}(D_2 \oplus \text{OMIL} \oplus D_2)) \), as well as from the chain \((V_{\text{BZL}}(D_2 \oplus R_n \oplus D_2))_{n \in \mathbb{N}\setminus\{0,1\}}\) for any ascending chain \((R_n)_{n \in \mathbb{N}\setminus\{0,1\}}\) in \( \mathcal{P}(\text{CH}) \setminus \{S_n : n \in \mathbb{N}\setminus\{0,1\}\} \). By Corollary 3.1, Proposition 3.1, Theorem 3.1 and the well–known fact that \( T = V_{\text{BI}}(D_1) < \text{BA} = V_{\text{BI}}(D_2) = V_{\text{BI}}(MO_1) < V_{\text{BI}}(MO_2) < \ldots < V_{\text{BI}}(MO_n) < \ldots \subset V_{\text{BI}}(MO_{n_0}) \subset \text{MOL} \) \((n \in \mathbb{N}, n \geq 3)\) is an infinite ascending chain of subvarieties of \( \text{MOL} \), in which the cover relations are consequences of the property from [14] mentioned above, we get the other infinite ascending chain, included in \( V_{\text{BZL}}(D_2 \oplus \text{OMIL} \oplus D_2) \subset V_{\text{BZL}}(D_2 \oplus \text{OMIL} \oplus D_2) \), thus disjoint from any of the previous ones, as well as the chain of varieties generated by antiortholattice chains.

\[ \square \]

Note that \( \text{OL} = \{L \in \text{BI} : L \vdash x \land x' \equiv y \land y' \} = \{L \in \text{BI} : L \vdash x \land x' \equiv y \land y' \} \), and recall that \( \text{OMIL} = \{L \in \text{BI} : L \vdash x \land (x \lor y) \equiv x \lor y \} \). Let us denote by \( \text{D2OL} \land \), \( \text{D2OL} \lor \) and \( \text{D2OML} \) the equations \( m(x \land x', y \land y') \approx m(y \land y', x \land x'), m(x \lor x', y \land y') \approx m(y \land y', x \lor x') \) and \( m(x \lor (x' \land (x \lor y)), x \lor y) \approx m(x \lor y, x \lor (x' \land (x \lor y))) \), respectively:

| Equation | Description |
|----------|-------------|
| \( \text{D2OL} \land \) | \( (x \land x') \lor (y \land y') \lor (x \land x') \lor (y \land y') \) |
| \( \text{D2OL} \lor \) | \( (x \land x') \lor (y \land y') \lor x \land x' \lor y \land y' \) |
| \( \text{D2OML} \) | \( (x \land x') \land (y \land y') \land x \land x' \land y \land y' \approx (x \land x') \land (y \land y') \land x \lor y \lor y' \) |
Proposition 3.6 (i) \( \{ A \in AOL : A \models D2OL \land \} = \{ A \in AOL : A \models D2OL \lor \} = (D_2 \oplus 
abla \otimes A_2) \cup I_{BZL}(D_1, D_2)). \)

(ii) \( Si(V_{BZL}(D_2 \oplus D_2)) = Si((D_2 \oplus D_2) \cup I_{BZL}(D_1, D_2)) = (D_2 \oplus Si(OL) \oplus D_2) \cup I_{BZL}(D_1, D_2)). \)

(iii) \( V_{BZL}(D_2 \oplus D_2) \) is relatively axiomatized by \( D2OL \lor \) or, equivalently, by \( D2OL \land \) w.r.t. \( V_{BZL}(AOL). \)

Proof (i) The antiortholattices \( D_1 \) and \( D_2 \) trivially satisfy \( D2OL \land \) and, for every \( L \in OL \), the antiortholattice \( L \cap D_2 \) fulfills \( D2OL \land \), according to Lemma 3.4. Also, \( D_4 = D_2 \oplus D_2 \oplus D_2 \in D_2 \oplus OL \oplus D_2 \). Now let \( A \in AOL \setminus I_{BZL}(D_1, D_2)) \) such that \( A \models D2OL \land \), \( a \in A \setminus [0, 1] = A \setminus S(A) \) and \( c = a \land a' \). Then \( c \leq a \lor a' = c' \) and, for all \( x \in A \setminus [0, 1] = A \setminus S(A) \), \( x \land x' = c \) and \( x \lor x' = (x \land x')' = c' \), in particular \( c \leq x \leq c' \), therefore the interval \( [c, c'] \) of \( A \) is an involution sublattice of \( A_{bi} \), thus a BI–lattice since it is bounded, and fulfills: \( [c, c'] = A \setminus [0, 1] \) and, as a BI–lattice, \( [c, c'] \in OL \). Therefore \( A = D_2 \oplus [c, c'] \oplus D_2 \in D_2 \oplus OL \oplus D_2 \). Similarly for \( D2OL \lor \). (ii) By (i), Remarks 3.2 and 3.1 or directly from Proposition 3.5.(iii). (iii) Let \( W = V_{BZL}(D_2 \oplus D_2) \) and \( U = \{ L \in V_{BZL}(AOL) : L \models D2OL \land \}. \) By Lemma 3.4, \( W \subseteq U \). By (ii), any \( A \in Si(U) \) belongs to \( (D_2 \oplus OL \oplus D_2) \cup I_{BZL}(D_1, D_2) \), hence \( U \subseteq V_{BZL}(D_2 \oplus OL \oplus D_2) \cup I_{BZL}(D_1, D_2)) \) \( = V_{BZL}(D_2 \oplus OL \oplus D_2) \cup (D_1, D_2) \). Therefore \( W \subseteq U = U \). Similarly for \( D2OL \lor \). \( \square \)

As shown by Remark 3.4, the following theorem provides us with a way to axiomatize any subvariety of \( SAOL \).

Theorem 3.2 Let \( V \) be a subvariety of \( PKA \), \( A \) a (not necessarily nonempty) set and, for all \( i \in I, t_i \) and \( u_i \) terms in the language of \( BI \).

(i) If \( D_3 \in V \), then: \( V \) is relatively axiomatized by \( \{ t_i \approx u_i : i \in I \} \) w.r.t. \( PKA \) iff \( V_{BZL}(D_2 \oplus V \oplus D_2) \) is relatively axiomatized by \( \{ t_i \approx u_i : i \in I \} \) w.r.t. \( SAOL \).

(ii) If \( D_3 \notin V \) and, for all \( i \in I, t_i \) and \( u_i \) have nonzero arities, then: \( V \) is relatively axiomatized by \( \{ t_i \approx u_i : i \in I \} \) w.r.t. \( PKA \) iff \( V_{BZL}(D_2 \oplus V \oplus D_2) \) is relatively axiomatized by \( \{ m(t_i, u_i) \approx m(u_i, t_i) : i \in I \} \) w.r.t. \( V_{BZL}(D_2 \oplus D_2) \) or, equivalently, by \( \{ D2OL \lor \} \cup \{ m(t_i, u_i) \approx m(u_i, t_i) : i \in I \} \) w.r.t. \( V_{BZL}(AOL) \) (in particular w.r.t. \( SAOL \)).

Proof Let us denote by \( W = V_{BZL}(D_2 \oplus V \oplus D_2) \subseteq SAOL \). (i) Assume that \( D_3 \in V \). Proposition 3.2 gives us the statement for \( I = \emptyset \). Now assume that \( I \) is nonempty, and let us denote by \( K = \{ K \in PKA : K \models \{ t_i \approx u_i : i \in I \} \} \) and \( U = \{ L \in SAOL : L \models \{ t_i \approx u_i : i \in I \} \} \), so that \( U_{BI} \subseteq K \), thus \( D_2 \oplus U_{BI} \oplus D_2 \subseteq D_2 \oplus K \oplus D_2 \), and, by Lemma 3.4, if \( D_3 \in K \), then \( D_2 \oplus U_{BI} \oplus D_2 \subseteq D_2 \oplus K \oplus D_2 \subseteq U \), so that \( U = V_{BZL}(D_2 \oplus U_{BI} \oplus D_2) = V_{BZL}(D_2 \oplus K \oplus D_2) \) by Proposition 3.2. Assume that \( V = K \) and let us prove that \( W = U \). Since \( D_3 \in V = K \), by the above it follows that \( D_2 \oplus V \oplus D_2 = D_2 \oplus K \oplus D_2 \subseteq U \), hence \( W = V_{BZL}(D_2 \oplus V \oplus D_2) \subseteq U \). On the other hand, by Lemma 3.1, for any \( A \in Si(U) \), we have \( A \in S_{BZL}(D_2 \oplus A_{bi} \oplus D_2) \subseteq W \) since \( D_2 \oplus A_{bi} \oplus D_2 \subseteq D_2 \oplus U_{BI} \oplus D_2 \subseteq D_2 \oplus K \oplus D_2 = D_2 \oplus V \oplus D_2 \subseteq W \), therefore \( U \subseteq W \), as well. Now assume that \( W = U \) and let us prove that \( V = K \). Since \( V_{BZL}(D_2 \oplus V \oplus D_2) = W = U \subseteq V_{BZL}(D_2 \oplus U_{BI} \oplus D_2) \subseteq V_{BZL}(D_2 \oplus K \oplus D_2) \) by Proposition 3.2 and the above, it follows that \( V \subseteq K \) by Theorem 3.1, so that \( D_3 \in V \subseteq K \), thus \( D_3 \in K \), so, by the above, \( V_{BZL}(D_2 \oplus K \oplus D_2) = U = W = V_{BZL}(D_2 \oplus V \oplus D_2) \), hence \( V = K \), again by Theorem 3.1. (ii) Assume that \( D_3 \notin V \), so that \( V \subseteq OL \) by Remark 3.3
and thus \( W \subseteq V_{BZL}(D_2 \oplus O \cup D_2) \). Proposition 3.6.(iii) gives us the statement for \( I = \emptyset \), as well as the last equivalence. Now assume that \( I \) is nonempty and that \( t_i \) and \( u_i \) are nonnullary for each \( i \in I \), and let us denote by \( K = (K \in O \cup: K \ni \{ t_i \approx u_i: i \in I \}) \) and by \( U = (L \in V_{BZL}(D_2 \oplus O \cup D_2) \colon L \ni \{ m(t_i, u_i) \approx m(u_i, t_i): i \in I \}) \), so that \( V_{BZL}(D_2 \oplus K \cup D_2) \subseteq U \) by Lemma 3.4. Assume that \( V = K \) and let us prove that \( W = U \). By Lemma 3.4 and the fact that \( W \subseteq V_{BZL}(D_2 \oplus O \cup D_2) \), it follows that \( W \subseteq U \). Now let \( A \in \text{Si}(U) \subseteq \text{Si}(V_{BZL}(D_2 \oplus O \cup D_2)) \), so that, by Proposition 3.6.(i), either \( A \in I_{BZL}(\{D_1, D_2\}) \) \( \subseteq W \) or \( A = D_2 \oplus K \cup D_2 \) for some \( K \in O \cup \), and, since \( A \in U \), by Lemma 3.4 it follows that \( K \subseteq K \in V \), thus \( A \in D_2 \oplus V \cup D_2 \subseteq W \), hence \( U \subseteq W \), as well. Now assume that \( W = U \) and let us prove that \( V = K \). We have \( V_{BZL}(D_2 \oplus \bigvee D_2) = W = U \supseteq V_{BZL}(D_2 \oplus K \cup D_2) \), thus \( V \supseteq K \) by Theorem 3.1. But \( D_2 \oplus V \cup D_2 \subseteq V_{BZL}(D_2 \oplus \bigvee D_2) = W = U \supseteq V_{BZL}(D_2 \oplus O \cup D_2) \), hence, by Lemma 3.4 and Theorem 3.1, \( V \subseteq K \), as well.

\[ \square \]

**Corollary 3.3** If \( V \) is a subvariety of \( O \cup \), \( I \) (a necessarily nonempty) set and, for all \( i \in I, t_i, u_i, v_i \) and \( w_i \) terms in the language of \( B \cup \) such that either \( t_i \) and \( u_i \) are nonnullary, \( v_i = m(t_i, u_i) \) and \( w_i = m(u_i, t_i) \), or, more generally, for any \( L \in O \cup, L \ni \{ t_i \approx u_i: i \in I \} \) \( if \ D_2 \oplus L \cup D_2 \ni \{ v_i \approx w_i: i \in I \} \), then:

- \( V \) is axiomatized by \( \{ t_i \approx u_i: i \in I \} \) w.r.t. \( O \cup \) iff \( V_{BZL}(D_2 \oplus \bigvee D_2) \) is relatively axiomatized by \( \{ v_i \approx w_i: i \in I \} \) w.r.t. \( V_{BZL}(D_2 \oplus O \cup D_2) \), thus \( by \ \{ D_2 \oplus D_2 \} \cup \{ v_i \approx w_i: i \in I \} \) or, equivalently, \( by \ \{ D_2 \oplus O \cup D_2 \} \cup \{ v_i \approx w_i: i \in I \} \) w.r.t. \( V_{BZL}(A \cup) \), and, in this case: \( A \cup \cap V_{BZL}(D_2 \oplus \bigvee D_2) = \{ A \in A \cup: A \ni D_2 \cup \{ V_i \approx w_i: i \in I \} \} = \{ A \in A \cup: A \ni D_2 \cup \{ V_i \approx w_i: i \in I \} \} \) \( = (D_2 \oplus \bigvee D_2 \cup I_{BZL}(\{D_1, D_2\})) \).

- \( \text{Si}(V_{BZL}(D_2 \oplus \bigvee D_2)) = \text{Si}(D_2 \oplus \bigvee D_2 \cup I_{BZL}(\{D_1, D_2\})) = (D_2 \oplus \bigvee D_2 \cup I_{BZL}(\{D_1, D_2\})) \).

Hence \( A \cup \cap V_{BZL}(D_2 \oplus O \cup D_2) = \{ A \in A \cup: A \ni D_2 \cup \{ V_i \approx w_i: i \in I \} \} \) \( = (D_2 \oplus O \cup D_2 \cup I_{BZL}(\{D_1, D_2\})) \)

Also, \( A \cup \cap V_{BZL}(D_4) = \{ A \in A \cup: A \ni D_2 \cup \{ V_i \approx w_i: i \in I \} \} \) \( = (D_2 \oplus O \cup D_2 \cup I_{BZL}(\{D_1, D_2\})) \).

The Brouwer complement of any \( PBZ^* \)-lattice is unique [11], in other words any paraorthomodular pseudo–Keleene algebra \( (L, \land, \lor, \ast, 0, 1) \) can be endowed with at most one unary operation \( \sim \) such that \( (L, \land, \lor, \ast, \sim, 0, 1) \) is a \( PBZ^* \)-lattice, so, for any \( PBZ^* \)-lattices \( A \) and \( B, A = B \iff A_{ab} = B_{ab} \), hence the map \( \mathbb{D} \mapsto D_{B1} \) is a poset embedding of \( \mathcal{P}(PBZ^*) \) into \( \mathcal{P}(PKA) \). Thus, for instance, \( SAOL_{B1} \subseteq PBZ^*_{B1} \).

Let us notice that:

- \( PBZ^*_{B1} \subseteq POML \).

\[ \square \]
Indeed, any pseudo–Kleene algebra endowed with the trivial Brouwer complement becomes a BZ–lattice and, in particular, any paraorthomodular pseudo–Kleene algebra endowed with the trivial Brouwer complement becomes a paraorthomodular BZ–lattice, so $\text{BZL}_{B1} = \text{PKA}$ and thus $\text{PBZL}^*_B \subseteq \text{PBZL}_{B1} = \text{POML}$. As for the strictness of the inclusion, the lattice $D^3$ can be endowed with two involutions, both of which are Kleene complements; out of these Kleene algebras (which are distributive, thus modular, thus paraorthomodular), the direct product of the three–element Kleene chain with itself is the BI–lattice reduct of the direct product of the three–element antiortholattice chain with itself, while the other is the paraorthomodular Kleene algebra in the following leftmost diagram, which can not be endowed with any Brouwer complement $\sim$ that would make it a PBZ$^*$–lattice, since then the element $a$ would be sharp, thus $b \leq a$ would imply $a' = a^* \leq b^* \leq b'$, thus $b^* \in [a') \cap (b')] = \emptyset$, a contradiction, hence this BI–lattice belongs to $\text{KA} \setminus \text{DIST}_{B1}$, which also shows that $\text{DIST}_{B1}$ is strictly included in the class of paraorthomodular Kleene algebras.

We can even find lattices that can be organized as paraorthomodular pseudo–Kleene algebras, but not as PBZ$^*$–lattices. Indeed, the modular lattice in the rightmost diagram above has two involutions, and the unique involution above up to a BI–lattice isomorphism. This involution makes it a modular, thus paraorthomodular pseudo–Kleene algebra whose set of sharp elements is $\{0, u, u', w, w', 1\}$, which is not even a sublattice of this lattice, hence we have no Brouwer complement w.r.t. which this would be the underlying set of a subalgebra of the resulting BZ–lattice. Therefore the lattice in this rightmost diagram is not the bounded lattice reduct of any PBZ$^*$–lattice, so it does not belong to $\text{MOD}_L$ or $\text{PBZL}^*_B$, nor does this modular pseudo–Kleene algebra belong to $\text{MOD}_{B1}$ or $\text{PBZL}^*_B$, which also shows that $\text{MOD}_{B1}$ is strictly included in the variety of modular pseudo–Kleene algebras.

Any lattice $L = (L, \leq_L)$ is a sublattice of the bounded lattice $B(L) = (L \cup \{0, 1\}, \leq_L \cup \{(0, x), (x, 1) : x \in L\})$, where $0, 1 \notin L$ and $0 \neq 1$, which, in turn, as pointed out in the proof of [7, Lemma 5.3], is a sublattice of the lattice reduct of the antiortholattice $B(L) \oplus B(L)^d$, so that $L \in S_L(B(L) \oplus B(L)^d)$. The antiortholattice $B(L) \oplus B(L)^d$ has the 0 meet–irreducible, thus it belongs to $\text{SAOL}$, it is modular iff $L$ is modular, and distributive iff $L$ is distributive. Hence, in addition to [7, Corollary 5.3], we get that $V_L(\text{PBZL}^*_L) = V_L(\text{PBZL}(\text{AOL})_L) = V_L(\text{SAOL}_L) = V_L(\text{AOL}_L) = V_L(\text{AOL} \cap \text{SDM}_L) = L$. Also, $V_L(\text{MOD}_L) = V_L((\text{MOD} \cap \text{PBZL}(\text{AOL}))_L) = V_L((\text{MOD} \cap \text{SAOL})_L) = V_L((\text{MOD} \cap \text{AOL})_L) = V_L((\text{MOD} \cap \text{AOL} \cap \text{SDM})_L)$ is the variety of modular lattices. By these facts and the second example above, neither of the following classes of lattices is a variety: $\text{PBZL}^*_L, V_K(\text{PBZL}(\text{AOL})_L, \text{SAOL}_L, \text{AOL}_L, (\text{AOL} \cap \text{SDM})_L, \text{MOD}_L, (\text{MOD} \cap \text{PBZL}(\text{AOL}))_L, (\text{MOD} \cap \text{SAOL})_L, (\text{MOD} \cap \text{AOL})_L, (\text{MOD} \cap \text{AOL} \cap \text{SDM})_L$. As expected, the variety of distributive lattices is $V_L(\text{DIST}) = V_L((\text{DIST} \cap \text{SAOL})_L) = V_L((\text{DIST} \cap \text{SAOL})_L) = V_L((\text{DIST} \cap \text{AOL})_L)$, which is the variety generated by self–dual bounded chains; and, of course, we have similar results for any equation in the language of lattices which is satisfied by $L$ iff it is satisfied by $B(L) \oplus B(L)^d$.

Now let $W, X \in \Lambda(\text{PBZL}^*)$ such that $W \subseteq \text{SAOL} \subseteq X$. Then:
\[ V_{BZL}(\text{Si}(X_{B1})) = V_{BZL}(\text{Si}(X_{B1})) = V_{BZL}(X_{B1}) = \text{PKA}, \text{ so } X_{B1} \text{ is not a variety, by Proposition 3.2 and Theorem 3.1.} \]

\[ \text{If } D_3 \notin W, \text{ then: } V_{BZL}(W_{B1}) = \{ K \in \text{PKA} : D_2 \oplus K \oplus D_2 \in W \} \text{ and } W = V_{BZL}(D_2 \oplus W_{B1} \oplus D_2) = V_{BZL}(D_2 \oplus V_{BZL}(W_{B1}) \oplus D_2). \text{ Indeed, } V = \{ K \in \text{PKA} : D_2 \oplus K \oplus D_2 \in W \}, \text{ then } D_3 \in V \text{ and, by Theorem 3.1, } V \text{ is a subvariety of PKA such that } W = V_{BZL}(D_2 \oplus V \oplus D_2). \text{ Thus, by Theorem 3.2, } \text{if a set } \Gamma \text{ of equations over } B \text{ axiomatizes } V \text{ w.r.t. PKA, } \Gamma \text{ also axiomatizes } W \text{ w.r.t. } \text{SAOL, thus } W_{B1} = \Gamma, \text{ so } V_{BZL}(W_{B1}) = \Gamma, \text{ hence } V_{BZL}(W_{B1}) \subseteq V. \text{ But then, by Proposition 3.2, it follows that } W \subseteq V_{BZL}(D_2 \oplus W_{B1} \oplus D_2) = V_{BZL}(D_2 \oplus V_{BZL}(W_{B1}) \oplus D_2) \subseteq V_{BZL}(D_2 \oplus V \oplus D_2) = W, \text{ so } W = V_{BZL}(D_2 \oplus W_{B1} \oplus D_2) = V_{BZL}(D_2 \oplus V_{BZL}(W_{B1}) \oplus D_2), \text{ from which, again by Theorem 3.1, it follows that } V_{BZL}(W_{B1}) = V. \]

\[ \text{If } D_3 \notin W \text{ or, equivalently, } W \subseteq V_{BZL}(D_2 \oplus \text{OL} \oplus D_2) \text{ (see Corollary 3.2), then } W \subseteq V_{BZL}(D_2 \oplus W_{B1} \oplus D_2). \text{ Indeed, } T \subseteq V_{BZL}(D_2 \oplus T \oplus D_2) = V_{BZL}(D_3) \text{ and } \text{BA} \subseteq V_{BZL}(D_2 \oplus \text{BA} \oplus D_2) = V_{BZL}(D_4) \text{ by Proposition 3.1 and Corollary 3.1. If } W \text{ is a subvariety of } \text{SAOL other than } T \text{ and } \text{BA}, \text{ then } D_3 \in W, \text{ thus } D_3 \in W \oplus W_{B1} \oplus D_2, \text{ so, if } D_3 \notin W, \text{ then } W \neq V_{BZL}(D_2 \oplus W_{B1} \oplus D_2), \text{ thus } W \subseteq V_{BZL}(D_2 \oplus W_{B1} \oplus D_2) \text{ by Proposition 3.2.} \]

\[ \text{If } W \subseteq \text{SAOL and } W \text{ is comparable to } V_{BZL}(D_3) \text{ in } \Lambda(P_{BZL}), \text{ then } V_{BZL}(W_{B1}) \subseteq \text{PKA, by Theorem 3.1 and the above, proving the case when } D_3 \in W, \text{ in particular for } W = V_{BZL}(D_3), \text{ hence also the case when } W \subseteq V_{BZL}(D_3). \]

\[ \text{If } V_{BZL}(D_3) \subseteq W \subseteq \text{DIST, then } V_{BZL}(W_{B1}) = \text{KA} \text{ and } W_{B1} \text{ is not a variety, by the above.} \]

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