Sequences of Orthogonal Polynomials related to Isotropy Orbits of Symmetric Spaces

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Abstract

Studying the isotropy orbits of compact symmetric spaces Reiswich [R] introduced a family of explicit polynomials in one variable in order to describe the unique minimal isotropy orbit of compact symmetric spaces with Dynkin diagram of type $D_m$. Based on this geometric interpretation he conjectured that these polynomials all have pairwise different real roots in the interval $[0,1]$. In this article the polynomials constructed by Reiswich will be identified as special cases of Jacobi polynomials thus proving the conjecture about minimal isotropy orbits of compact symmetric spaces with Dynkin diagram of type $D_m$.

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1 Introduction

Orthogonal polynomials are a classical topic of study and include many famous families of polynomials like the Hermite, Laguerre and Jacobi polynomials. Needless to say this article does not strive to make any serious contribution to this beautiful topic per se, in fact it will turn out eventually that the sequences of orthogonal polynomials studied in this article are special cases of Jacobi polynomials. Two of these sequences of orthogonal polynomials however appeared recently in a parametrization [R] of the unique minimal isotropy orbit of compact symmetric spaces with simply laced Dynkin diagram of type $D_m$, more precisely the roots of these polynomials determine the coefficients of the minimal isotropy orbit with respect to an orthonormal basis of a maximal flat $\mathbb{H}$.

Mandatory for this geometric interpretation recalled in Corollary 1.3 below is that all roots of the polynomials constructed by Reiswich are real, pairwise different and lie in the strict interior of the interval $[0,1]$. All these properties are quite suggestive for a link

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to sequences of orthogonal polynomials, in this article we will establish this link and thus prove the conjecture of Reiswich on minimal isotropy orbits of a certain family of compact symmetric spaces. The interesting question on whether this link extends to general symmetric spaces or not remains unanswered for the time being.

Throughout this article we use the notation \([ z ]_r\) for the falling factorial polynomial instead of the more customary \(( z )_r\), because the latter may be too easily confused with other constructs. Recall that the falling factorial polynomial is defined for all indices \( r \in \mathbb{Z} \) by setting \([ z ]_r := z(z - 1) \ldots (z - r + 1)\) for all positive \( r \) and \([ z ]_0 := 1\), while \([ z ]_r := 0\) in all other cases. The falling factorial polynomials make prominent appearance in the coefficients of a family of sequences of explicit polynomials \(( R^\tau_n )_{n \in \mathbb{N}_0}\) parametrized by \( \tau > -1\):

**Definition 1.1 (Family of Reiswich Polynomials)**

For \( n \in \mathbb{N}_0 \) the Reiswich polynomial \( R^\tau_n \in \mathbb{R}[ x ]\) with real parameter \( \tau > -1\) is defined by:

\[
R^\tau_n(x) := \sum_{r = 0}^{\lfloor \frac{n}{\tau} \rfloor - 1} (-1)^r \binom{n}{r} \frac{[n + \tau]_r}{[2n + \tau + 1]_r} x^{n-r}
\]

In his study of orbits of isotropy actions of compact symmetric spaces Reiswich [R] constructed a sequence of polynomials \(( P_m )_{m \geq 2}\) in order to characterize the unique minimal isotropy orbit for symmetric spaces with simply laced Dynkin diagram of type \( D_m\). More precisely the original definition of these polynomials in [R] can be rewritten in the form

\[
P_m(x) := \sum_{r = 0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^r \left( \prod_{d = 1}^{r} \frac{\sum_{\mu = d+1}^{\lfloor \frac{m}{2} \rfloor} (1 + 2m - 4\mu)}{\sum_{d=1}^{d} (1 + 2m - 4\mu)} \right) x^{\lfloor \frac{m}{2} \rfloor - 1-r}
\]

\[
= \sum_{r = 0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^r \prod_{d = 1}^{r} \left( \frac{\lfloor \frac{m}{2} \rfloor - d}{d} \frac{4m - 4\lfloor \frac{m}{2} \rfloor - 4d - 2}{4m - 4d - 2} \right) x^{\lfloor \frac{m}{2} \rfloor - 1-r}
\]

\[
= \sum_{r = 0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^r \left( \frac{\lfloor \frac{m}{2} \rfloor - 1}{r} \right) \left[ \frac{\lfloor \frac{m}{2} \rfloor - \frac{3}{2}}{m - \frac{3}{2}} \right]_r x^{\lfloor \frac{m}{2} \rfloor - 1-r}
\]

where the second line is simply young Gauß' formula \( a_1 + \ldots + a_k = \frac{k}{2}(a_1 + a_k) \) for arithmetic series. Changing the parameter \( m = 2n + 2\) or \( m = 2n + 3\) to the parameter \( n := \lfloor \frac{m}{2} \rfloor - 1\) we conclude \( P_{2n+2}(x) = R_{n}^{-\frac{1}{2}}(x)\) and \( P_{2n+3}(x) = R_{n}^{\frac{1}{2}}(x)\) respectively.

In order to study the more general Reiswich polynomials with arbitrary real parameter \( \tau > -1\) we consider the probability measure \( \mu^\tau(dx) = (\tau + 2) (\tau + 1) (1-x)^{x^\tau} dx\) on the interval \([0, 1] \subset \mathbb{R}\) with an integrable pole at \( x = 0\) for \( \tau \in ]-1, 0[\). The moments of the probability measure \( \mu^\tau(dx)\) are easily calculated via straightforward integration:

\[
\mu^\tau_n := (\tau + 2) (\tau + 1) \int_0^1 x^n (1-x) x^\tau dx \\
= (\tau + 2) (\tau + 1) \left( \frac{x^{n+\tau+1}}{n+\tau+1} - \frac{x^{n+\tau+2}}{n+\tau+2} \right)_{x=0}^{x=1} = \frac{(\tau + 2) (\tau + 1)}{(n+\tau+2) (n+\tau+1)}
\]
In particular the 0-th moment \( \mu_0^\tau = 1 \) tells us that \( \mu^\tau(dx) \) is in fact a probability measure on \([0,1]\). In turn the probability measure \( \mu^\tau(dx) \) gives rise to a positive definite scalar product \( \langle \cdot, \cdot \rangle \) on the space \( \mathbb{R}[x] \) of polynomials with real coefficients by integration against \( \mu^\tau(dx) \):

**Theorem 1.2 (Orthogonality Relation)**

The sequence \( (R_n^\tau)_{n \in \mathbb{N}_0} \) of Reiswich polynomials is a sequence of orthogonal polynomials with respect to the probability measure \( \mu^\tau(dx) := (\tau + 2)(\tau + 1)(1-x)^\tau dx \) on \([0,1]\):

\[
\int_0^1 R_n^\tau(x) R_m^\tau(x) \mu^\tau(dx) = (n+1)!n! \frac{[n+\tau+1]_n [n+\tau]_n}{[2n+\tau+2]_{2n} [2n+\tau+1]_{2n}} \delta_{n=m}
\]

Taking the vanishing orders \( \tau \) and 1 of the weight function \((1-x)^x\) of the probability measure \( \mu^\tau \) at the limits of the interval \([0,1]\) as the decisive clue we obtain as an immediate corollary of Theorem 1.2 that the Reiswich polynomials are special cases of Jacobi polynomials \( R_n^\tau(x) \sim P_n^{(1,\tau)}(2x-1) \) up to a linear change of variables and normalization \([S], [W]\). A classical consequence of an orthogonality relation like Theorem 1.2 between the polynomials in a sequence is that the \( n \)-th Reiswich polynomial \( R_n^\tau \) with parameter \( \tau > -1 \) has exactly \( n \) real roots in the interior of the support \([0,1]\) of the probability measure \( \mu^\tau(dx) \). Combining this statement with the main result of Reiswich \([R]\) we obtain:

**Corollary 1.3 (Minimal Isotropy Orbits [R])**

Recall that the root system of a compact symmetric space \( G/K \) equals the restriction \( t^* \rightarrow a^* \) of the root system of the Lie algebra \( g \) of the isometry group \( G \) with respect to a maximal torus \( t \) to a maximal flat \( a \subset t \). For symmetric spaces with Dynkin diagram of type \( D_m, m \geq 2 \), there exists an orthonormal basis \( \varepsilon_1, \ldots, \varepsilon_m \) of the dual space \( a^* \) such that the roots read:

\[
D_m \cong \{ \pm \varepsilon_\mu \pm \varepsilon_\nu \mid \text{all choices of signs and } 1 \leq \mu < \nu \leq m \}
\]

Without loss of generality we may assume that the maximal flat is contained \( a \subset p \) in the Cartan complement of the isotropy subalgebra \( t \subset g \) so that the exponential map is well-defined by \( \exp : a \rightarrow G/K, X \mapsto \exp(X)K \). The unique minimal orbit of \( K \) on the symmetric space \( G/K \) passes through the image \( \exp(X_{\min})K \) under \( \exp \) of the vector

\[
X_{\min} := \frac{\pi}{4} E_1 + \sum_{r=1}^n \left( \frac{\pi}{2} E_{r+1} + \frac{\arccos \sqrt{\xi_r}}{2} (E_{m-r} - E_{r+1}) \right) + \frac{\pi}{4} E_m
\]

where \( E_1, \ldots, E_m \) is the orthonormal basis of \( a \) dual to the basis \( \varepsilon_1, \ldots, \varepsilon_m \in a^* \) and \( 0 < \xi_1 < \ldots < \xi_n < 1 \) are the \( n := \lfloor \frac{m}{2} \rfloor - 1 \) different real zeroes of the polynomial \( P_m \).

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2 The Proof of the Orthogonality Relations

In order to prove the orthogonality of the Reiswich polynomials \(R_n^r\), formulated in Theorem 1.2, we prove a combinatorial identity, which may be of independent interest, between falling factorial polynomials in the first part of this section. In a second step we verify that the Reiswich polynomials satisfy a recursion formula, whose existence we should expect due to orthogonality. Combining both previous results we conclude this section with a proof of Theorem 1.2 and recall a standard result on orthogonal polynomials in Corollary 2.3.

Lemma 2.1 (Combinatorial Identity)
For every \(n \in \mathbb{N}_0\) and all \(u, v \in \mathbb{N}_0\) satisfying \(u + v \leq n\) the following polynomial identity holds true in the polynomial ring \(\mathbb{Z}[x, y]\) with indeterminates \(x, y\) and integer coefficients:

\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} [x-r]_u [y-r]_v = n! \delta_{u+v=n}
\]

**Proof:** In order to provide the base for an induction on \(n \in \mathbb{N}_0\) let us consider \(n = 0\) first: The only admissible choice for the additional parameters \(u, v\) equals \(u = 0 = v\) in this case leading to the trivial identity \(\binom{0}{0} [x]_0 [y]_0 = 1\). Assume now by induction hypothesis that the statement of the Lemma holds true for \(n \in \mathbb{N}_0\) and all \(u, v \in \mathbb{N}_0\) satisfying \(u + v \leq n\). Decomposing the binomial \(\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}\) as usual and shifting \(r - 1\) back to \(r\) we obtain for every choice \(u, v \in \mathbb{N}_0\) of the additional parameters satisfying \(u + v \leq n + 1\):

\[
\sum_{r=0}^{n+1} (-1)^r \binom{n+1}{r} [x-r]_u [y-r]_v
\]

\[
= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \left( [x-r]_u [y-r]_v - [x-r-1]_u [y-r-1]_v \right)
\]

\[
= \sum_{r=0}^{n} (-1)^r \binom{n}{r} [x-r-1]_{u-1} [y-r-1]_{v-1} \left( u(y-r) + v(x-r) - uv \right)
\]

\[
= u \sum_{r=0}^{n} (-1)^r \binom{n}{r} [x-r-1]_{u-1} [y-r]_v
\]

\[
+ v \sum_{r=0}^{n} (-1)^r \binom{n}{r} [x-r]_u [y-r-1]_{v-1}
\]

\[
- u v \sum_{r=0}^{n} (-1)^r \binom{n}{r} [x-r-1]_{u-1} [y-r-1]_{v-1}
\]

where we have used \((x-r)(y-r) - (x-r-u)(y-r-v) = u(y-r) + v(x-r) - uv\) in the third line; needless to say the cases \(u = 0\) and or \(v = 0\) care for themselves in this calculation. Recall now that \(u + v \leq n + 1\) holds true by assumption; if this inequality is strictly satisfied in the sense \(u + v - 1 < n\), then the three sums on the right hand side
all vanish according to our induction hypothesis. In the opposite case \( u + v = n + 1 \) and our induction hypothesis tells us that the first two sums both equal \( n! \) and the third sum vanishes as before so that the right hand side reduces to \( (u + v)n! = (n + 1)! \).

**Lemma 2.2 (Recursion Formulas for Reiswich Polynomials)**

Associated to every sequence \((R_n)_{n \in \mathbb{N}_0}\) of orthogonal polynomials is a recursion formula, which expresses \( R_{n+1} \) in terms of \( R_n \) and \( R_{n-1} \) for all \( n \in \mathbb{N}_0 \). The recursion formula for the sequence of Reiswich polynomials \((R_n^\tau)_{n \in \mathbb{N}_0}\) with real parameter \( \tau > -1 \) reads:

\[
R_{n+1}^\tau(x) = \left( x - \frac{2n^2 + 2(\tau + 2)n + (\tau + 1)^2}{(2n + \tau + 3)(2n + \tau + 1)} \right) R_n^\tau(x) - \frac{(n + \tau + 1)(n + \tau)(n + 1)n}{(2n + \tau + 2)(2n + \tau + 1)^2(2n + \tau)} R_{n-1}^\tau(x)
\]

**Proof:** Multiplying the coefficients of the Reiswich polynomials \( R_{n+1}^\tau \) as well as \( xR_n^\tau, R_n^\tau \) and \( R_{n-1}^\tau \) for \( x^{n+1-r} \) with given index \( r = 0, \ldots, n+1 \) by the lucky factor

\[
C := (-1)^r \frac{2n + \tau + 1}{\binom{n + \tau + 3}{r+1}} \binom{n + \tau}{r} \neq 0
\]

we obtain by straightforward, if slightly tedious calculation:

\[
\begin{align*}
C \left[ R_{n+1}^\tau(x) \right]_{n+1-r} &= + (n + 1)(n + \tau + 1)(2n + \tau + 1)(2n + \tau - r + 3)(2n + \tau - r + 2) \\
C \left[ xR_n^\tau(x) \right]_{n+1-r} &= + (n - r + 1)(n + \tau - r + 1)(2n + \tau + 3)(2n + \tau + 2)(2n + \tau + 1) \\
C \left[ R_n^\tau(x) \right]_{n+1-r} &= - r(2n + \tau + 3)(2n + \tau + 2)(2n + \tau + 1)(2n + \tau - r + 2) \\
C \left[ R_{n-1}^\tau(x) \right]_{n+1-r} &= + \frac{r(r - 1)(2n + \tau + 3)(2n + \tau + 2)(2n + \tau + 1)^2(2n + \tau)}{n(n + \tau)}
\end{align*}
\]

Inserting the stipulated coefficients we conclude that the proof of the lemma reduces to the finite time fun exercise to verify the following degree five polynomial identity in \( n, \tau \) and \( r \):

\[
(n + 1)(n + \tau + 1)(2n + \tau + 1)(2n + \tau - r + 3)(2n + \tau - r + 2)
\]

\[
- r(2n + \tau + 3)(2n + \tau + 2)(2n + \tau + 1)(2n + \tau - r + 2)
\]

\[
+ r(2n + \tau + 2)(2n + \tau - r + 2)(2n^2 + 2(\tau + 2)n + (\tau + 1)^2)
\]

\[
- (n - r + 1)(2n + \tau + 3)(n + \tau + 1)(n + 1)
\]

**Proof of Theorem 1.2:** Consider the special case \( u = s \) and \( v = n - s \) of the combinatorial identity of Lemma 2.1 for some pair of integers \( n, s \in \mathbb{N}_0 \) satisfying \( s \leq n \). Evaluating this identity at \( x = n + \tau + s \) and \( y = 2n + \tau + 1 \) for a real parameter \( \tau > -1 \) leads to

\[
n! = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{n + \tau + s - r}{s} \binom{2n + \tau - r + 1}{n-s}
\]
Dividing this identity by $\lfloor 2n + \tau + 1 \rfloor_{n+1} > 0$ is possible due to $\tau > -1$ and so we find
\[
\frac{n!}{\lfloor 2n + \tau + 1 \rfloor_{n+1}} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{[n + \tau + s - r]_s [2n + \tau - r + 1]_{n-s}}{\lfloor 2n + \tau + 1 \rfloor_{n+1}} \frac{1}{2n + \tau + s - r + 1}
\]
where $\lfloor 2n + \tau - r + 1 \rfloor_{n-s} (n + \tau + s - r + 1) [n + \tau + s - r]_s = \lfloor 2n + \tau - r + 1 \rfloor_{n+1}$ should suffice to explain the second equality. In the resulting identity the left hand side is independent of $s$, subtracting two successive instances for $s$, $s+1$ results in the key identity
\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{[n + \tau]_r}{\lfloor 2n + \tau + 1 \rfloor_{n+1}} \frac{1}{(n + \tau + s - r + 2)(n + \tau + s - r + 1)} = 0
\]
valid for all $n, s \in \mathbb{N}_0$ satisfying $s < n$; the inequality is strict now, because we need the previous identity for both $s$ and $s+1$. Calculating the scalar product of the Reischwitz polynomial $R^\tau_n$ with $x^s$ in light of this key identity we obtain directly
\[
\langle R^\tau_n, x^s \rangle = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{[n + \tau]_r}{\lfloor 2n + \tau + 1 \rfloor_{n+1}} \langle x^{n-r}, x^s \rangle
\]
\[
= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{[n + \tau]_r}{\lfloor 2n + \tau + 1 \rfloor_{n+1}} \frac{(\tau + 2)(\tau + 1)}{(n + \tau + s - r + 2)(n + \tau + s - r + 1)} = 0
\]
for all $n \in \mathbb{N}_0$ and $s = 0, \ldots, n-1$ by using the moments $\mu^\tau_n = \frac{(\tau+2)(\tau+1)}{(n+\tau+2)(n+\tau+1)}$ of the measure $\mu^\tau(dx) = (\tau+2)(\tau+1)(1-x) x^\tau dx$ calculated before in the auxiliary calculation:
\[
\langle x^{n-r}, x^s \rangle \overset{!}{=} \mu^\tau_{n+s-r} = \frac{(\tau + 2)(\tau + 1)}{(n + \tau + s - r + 2)(n + \tau + s - r + 1)}
\]
In consequence $R^\tau_n$ is orthogonal to the subspace of polynomials spanned by $1, \ldots, x^{n-1}$, which evidently contains the Reischwitz polynomials $R^\tau_0, \ldots, R^\tau_{n-1}$. Having thus proved the orthogonality of the Reischwitz polynomials we use the recursion formula of Lemma 2.2 to calculate their norm squares. Consider the two instances of the recursion formula
\[
R^\tau_{n+1}(x) = (x - *) R^\tau_n(x) + C R^\tau_{n-1}(x) \quad (1)
\]
\[
R^\tau_n(x) = (x - *) R^\tau_{n-1}(x) + * R^\tau_{n-2}(x) \quad (2)
\]
for $n$ and $n-1$ where $*$ denotes three irrelevant constants, not necessarily the same one. Taking the scalar product of instance (1) with $R^\tau_{n-1}$ we find $0 = \langle x R^\tau_n, R^\tau_{n-1} \rangle + C \langle R^\tau_{n-1}, R^\tau_{n-1} \rangle$, taking similarly the scalar product of instance (2) with $R^\tau_n$ we obtain the standard identity
\[
\langle R^\tau_n, R^\tau_{n} \rangle = \langle x R^\tau_{n-1}, R^\tau_{n} \rangle = \langle R^\tau_{n-1}, x R^\tau_{n} \rangle = -C \langle R^\tau_{n-1}, R^\tau_{n-1} \rangle
\]
due to the self–adjointness of the multiplication operator $p \mapsto xp$ with the polynomial $x$ under the integration scalar product $\langle \cdot, \cdot \rangle$ compare [S]. Inserting the explicit constant $C$ from Lemma 2.2 we find eventually the norm square of the Reiswich polynomial $R_\tau^n$:

$$\langle R_\tau^n, R_\tau^n \rangle = \frac{(n + \tau + 1)(n + \tau)(n + 1)n}{(2n + \tau + 2)(2n + \tau + 1)^2(2n + \tau)} \langle R_\tau^{n-1}, R_\tau^{n-1} \rangle = \ldots$$

$$= (n + 1)! n! \frac{(n + \tau + 1)_{n} (n + \tau)_{n}}{(2n + \tau + 2)_{2n} (2n + \tau + 1)_{2n}} \langle R_\tau^0, R_\tau^0 \rangle$$

For the application of the Reiswich polynomials to the characterization of the unique minimal isotropy orbit of a compact symmetric space with root diagram of type $D_m$ we recall the following classical statement [S] about polynomials in orthogonal sequences of polynomials:

**Corollary 2.3 (Zeroes of Reiswich Polynomials)**

According to a standard result about sequences of orthogonal polynomials the $n$–th polynomial $R_\tau^n$ in the Reiswich sequence $(R_\tau^n)_{n \in \mathbb{N}_0}$ of orthogonal polynomials with parameter $\tau > -1$ has exactly $n$ pairwise different real roots in the strict interior $]0, 1[$ of the interval $[0, 1]$.

**Proof:** Polynomial division with remainder provides us in every zero $\xi$ of a non–zero polynomial $R \neq 0$ with a natural number $o \in \mathbb{N}$ and a polynomial $p$ not vanishing in $\xi$ such that $R(x) = (x - \xi)^o p(x)$. Since $p$ considered as a function is continuous and $p(\xi) \neq 0$, the polynomial $R$ changes its sign in the zero $\xi$, exactly if the order $o \in \mathbb{N}$ of this zero is odd. Consider now the set $\{ \xi_1, \ldots, \xi_k \}$ of all real roots of $R_\tau^n \neq 0$ of odd order strictly in the interior of the interval $[0, 1]$. In the strict interior of $[0, 1]$ the auxiliary polynomial

$$p(x) := (x - \xi_1) \ldots (x - \xi_k)$$

which reduces to $p(x) = 1$ in case $k = 0$, changes sign in exactly the same points as the polynomial $R_\tau^n$. Replacing $p$ by $-p$ if necessary we may thus assume that for all $x \in [0, 1]$:

$$R_\tau^n(x) p(x) \geq 0$$

As a non–zero polynomial $R_\tau^n p \neq 0$ does not vanish identically on any open subset of $[0, 1]$ and so the positivity of the measure $\mu^\tau(dx) \sim (1 - x)^{\tau} dx$ ensures the stronger inequality:

$$\langle R_\tau^n, p \rangle := \int_0^1 R_\tau^n(x) p(x) \mu^\tau(dx) > 0$$

On the other hand $R_\tau^n$ is orthogonal to all polynomials of degree less than $n$ by assumption

$$R_\tau^n \in \text{span}_\mathbb{R}\{ R_\tau^0, \ldots, R_\tau^{n-1} \}^\perp = \text{span}_\mathbb{R}\{ 1, x, \ldots, x^{n-1} \}^\perp$$

so that the auxiliary polynomial $p$ is necessarily a polynomial of degree $k \geq n$. By construction however $p$ has degree $k \leq n$ at most equal to $n$ so that $R_\tau^n$ has $n$ pairwise different real roots in the interior of the interval $[0, 1]$, all of odd degree $o = 1$. □
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