Some identities of symmetry for the generalized Bernoulli numbers and polynomials

By

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Abstract. In this paper, by the properties of $p$-adic invariant integral on $\mathbb{Z}_p$, we establish various identities concerning the generalized Bernoulli numbers and polynomials. From the symmetric properties of $p$-adic invariant integral on $\mathbb{Z}_p$, we give some interesting relationship between the power sums and the generalized Bernoulli polynomials.

2000 Mathematics Subject Classification: 11B68, 11M38, 11S80.

Key Words and Phrases: $p$-adic invariant integral, Bernoulli numbers, Bernoulli polynomials.

§1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p,$ and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [6])}. \tag{1}$$

From the definition (1), we have

$$I_1(f_1) = I_1(f) + f'(0), \quad \text{where} \quad f'(0) = \left. \frac{df}{dx} \right|_{x=0} \quad \text{and} \quad f_1(x) = f(x+1). \tag{2}$$

Let $f_n(x) = f(x+n), \ (n \in \mathbb{N})$. Then we can derive the following equation (3) from (2).

$$I(f_n) = I(f) + \sum_{i=0}^{n} f'(i), \quad \text{(see [6])}. \tag{3}$$
It is well known that the ordinary Bernoulli polynomials $B_n(x)$ are defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [1-25]}),$$

and the Bernoulli number $B_n$ are defined as $B_n = B_n(0)$.

Let $d$ a fixed positive integer. For $n \in \mathbb{N}$, we set

$$X = X_d = \lim_{N \to \infty} \left( \mathbb{Z}/d^N \mathbb{Z} \right), \quad X_1 = \mathbb{Z}_p;$$

$$X^* = \bigcup_{0 < a < dp, (a,p) = 1} (a + dp \mathbb{Z}_p);$$

$$a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. In [6], it is known that

$$\int_X f(x) dx = \int_{\mathbb{Z}_p} f(x) dx, \quad \text{for } f \in UD(\mathbb{Z}_p).$$

Let us take $f(x) = e^{tx}$. Then we have

$$\int_{\mathbb{Z}_p} e^{tx} dx = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Thus, we note that

$$\int_{\mathbb{Z}_p} x^n dx = B_n, \quad n \in \mathbb{Z}_+, \quad (\text{see [1-25]}).$$

Let $\chi$ be the Dirichlet’s character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli polynomials attached to $\chi$ are defined as

$$\sum_{a=1}^{d} \frac{\chi(a) e^{at}}{e^{at} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \quad (\text{see [22]}),$$

and the generalized Bernoulli numbers attached to $\chi$, $B_{n,\chi}$ are defined as $B_{n,\chi} = B_{n,\chi}(0)$.

In this paper, we investigate the interesting identities of symmetry for the generalized Bernoulli numbers and polynomials attached to $\chi$ by using the properties of $p$-adic invariant integral on $\mathbb{Z}_p$. Finally, we will give relationship between the power sum polynomials and the generalized Bernoulli numbers attached to $\chi$. 
§2. Symmetry of power sum and the generalized Bernoulli polynomials

Let \( \chi \) be the Dirichlet character with conductor \( d \in \mathbb{N} \). From (3), we note that

\[
\int_X \chi(x)e^{xt}dx = \frac{t \sum_{i=0}^{d-1} \chi(i)e^{it}}{e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!},
\]

(5)

where \( B_{n,\chi}(x) \) are \( n \)-th generalized Bernoulli numbers attached to \( \chi \). Now, we also see that the generalized Bernoulli polynomials attached to \( \chi \) are given by

\[
\int_X \chi(y)e^{(x+y)t}dy = \frac{\sum_{i=0}^{d-1} \chi(i)e^{it}}{e^{dt} - 1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}.
\]

(6)

By (5) and (6), we easily see that

\[
\int_X \chi(x)x^n dx = B_{n,\chi}, \quad \text{and} \quad \int_X \chi(y)(x+y)^n dy = B_{n,\chi}(x).
\]

(7)

From (6), we have

\[
B_{n,\chi}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} B_{\ell,\chi}x^{n-\ell}.
\]

(8)

From (6), we can also derive

\[
\int_X \chi(x)e^{xt}dx = \sum_{i=0}^{d-1} \chi(i) \frac{t}{e^{dt} - 1}e^{(\frac{i}{d})dt} = \sum_{n=0}^{\infty} \left( d^n \sum_{i=0}^{d-1} \chi(i)B_n \left( \frac{i}{d} \right) \right) \frac{t^n}{n!}.
\]

Therefore, we obtain the following lemma.

**LEMMA1.** For \( n \in \mathbb{Z}_+ \), we have

\[
\int_X \chi(x)x^n dx = B_{n,\chi} = d^n \sum_{i=0}^{d-1} \chi(i)B_i \left( \frac{i}{d} \right).
\]

We observe that

\[
\frac{1}{t} \left( \int_X \chi(x)e^{(nd+x)t}dx - \int_X e^{xt} \chi(x)dx \right) = \frac{nd \int_X \chi(x)e^{xt}dx}{\int_X e^{ndxt}dx} = \frac{e^{ndt} - 1}{e^{dt} - 1} \left( \sum_{i=0}^{d-1} \chi(i)e^{it} \right). \tag{9}
\]

Thus, we have

\[
\frac{1}{t} \left( \int_X \chi(x)e^{(nd+x)t}dx - \int_X e^{xt} dx \right) = \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{d-1} \chi(\ell)e^{\ell k} \right) \frac{t^k}{k!}. \tag{10}
\]
Let us define the $p$-adic functional $T_k(\chi, n)$ as follows:

$$T_k(\chi, n) = \sum_{\ell=0}^{n} \chi(\ell) \ell^k, \quad \text{for } k \in \mathbb{Z}_+.$$ (11)

By (10) and (11), we see that

$$\frac{1}{t} \left( \int_X \chi(x)e^{(nd+x)t}dx - \int_X e^{xt}dx \right) = \sum_{n=0}^{\infty} \left( T_k(\chi, nd - 1) \right) \frac{t^k}{k!}. \quad (12)$$

By using Taylor expansion in (12), we have

$$\int_X \chi(x)(dn + x)^kdx - \int_X \chi(x)x^kdx = kT_{k-1}(\chi, nd - 1), \quad \text{for } k, n, d \in \mathbb{N}. \quad (13)$$

That is,

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd - 1).$$

Let $w_1, w_2, d \in \mathbb{N}$. Then we consider the following integral equation

$$\frac{d}{X} \int_X \chi(x_1)\chi(x_2)e^{(w_1x_1+w_2x_2)t}dx_1dx_2 \int_X e^{dw_1x_2t}dx$$

$$= \frac{t(e^{dw_1x_2t} - 1)}{(e^{w_1dt} - 1)(e^{w_2dt} - 1)} \left( \sum_{a=0}^{d-1} \chi(a)e^{w_1at} \right) \left( \sum_{b=0}^{d-1} \chi(b)e^{w_2bt} \right). \quad (14)$$

From (9) and (12), we note that

$$\frac{d}{X} \int_X \chi(x)e^{xt}dx \int_X e^{dw_1x_2t}dx = \sum_{k=0}^{\infty} \left( T_k(\chi, dw_1 - 1) \right) \frac{t^k}{k!}. \quad (15)$$

Let us consider the $p$-adic functional $T_\chi(w_1, w_2)$ as follows:

$$T_\chi(w_1, w_2) = \frac{d}{X} \int_X \chi(x_1)\chi(x_2)e^{(w_1x_1+w_2x_2+w_1w_2x)\mu}dx_1dx_2 \int_X e^{dw_1x_2t}dx_3 \int_X e^{dw_1x_2t}dx_3. \quad (16)$$

Then we see that $T_\chi(w_1, w_2)$ is symmetric in $w_1$ and $w_2$, and

$$T_\chi(w_1, w_2) = \frac{t(e^{dw_1x_2t} - 1)e^{w_1w_2t}}{(e^{w_1dt} - 1)(e^{w_2dt} - 1)} \left( \sum_{a=0}^{d-1} \chi(a)e^{w_1at} \right) \left( \sum_{b=0}^{d-1} \chi(b)e^{w_2bt} \right). \quad (17)$$
By (16) and (17), we have

\[
T_\chi(w_1, w_2) = \left( \frac{1}{w_1} \int_X \chi(x_1)e^{w_1(x_1+w_2x)}dx_1 \right) \left( \frac{dw_2 \int_X \chi(x_1)e^{w_1x_1}dx_1}{\int_X e^{w_1x_1}dx} \right)
\]

\[
= \left( \frac{1}{w_1} \sum_{i=0}^{\infty} B_{i,\chi}(w_2x) \frac{w_1^i i!}{i!} \right) \left( \sum_{k=0}^{\infty} T_k(\chi, dw_1-1) \frac{w_2^k k!}{k!} \right)
\]

\[
= \frac{1}{w_1} \left( \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} B_{i,\chi}(w_2x) T_{\ell-i}(\chi, dw_1-1) \frac{w_1^i w_2^{\ell-i} i!}{i!(\ell-i)!} \right) \frac{t^\ell}{\ell!}
\]

(18)

From the symmetric property of \( T_\chi(w_1, w_2) \) in \( w_1 \) and \( w_2 \), we note that

\[
T_\chi(w_1, w_2) = \left( \frac{1}{w_2} \int_X \chi(x_2)e^{w_2(x_2+w_1x)}dx_2 \right) \left( \frac{dw_1 \int_X \chi(x_1)e^{w_2x_2}dx_2}{\int_X e^{w_2x_2}dx} \right)
\]

\[
= \left( \frac{1}{w_2} \sum_{i=0}^{\infty} B_{i,\chi}(w_1x) \frac{w_2^i i!}{i!} \right) \left( \sum_{k=0}^{\infty} T_k(\chi, dw_2-1) \frac{w_1^k k!}{k!} \right)
\]

\[
= \frac{1}{w_2} \left( \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} B_{i,\chi}(w_1x) w_2^i T_{\ell-i}(\chi, dw_2-1) \frac{w_1^{\ell-i} (\ell-i)!}{(\ell-i)!} \right) \frac{t^\ell}{\ell!}
\]

(19)

\[
= \sum_{\ell=0}^{\infty} \left( \sum_{i=0}^{\ell} \frac{\ell!}{i!} w_2^{i-1} w_1^{\ell-i} B_{i,\chi}(w_1x) T_{\ell-i}(\chi, dw_2-1) \right) \frac{t^\ell}{\ell!}
\]

By comparing the coefficients on the both sides of (18) and (19), we obtain the following theorem.

**Theorem 2.** For \( w_1, w_2, d \in \mathbb{N} \), we have

\[
\sum_{i=0}^{\ell} \left( \frac{\ell!}{i!} B_{i,\chi}(w_2x) T_{\ell-i}(\chi, dw_1-1) w_1^{i-1} w_2^{\ell-i} \right)
\]

\[
= \sum_{i=0}^{\ell} \left( \frac{\ell!}{i!} B_{i,\chi}(w_1x) T_{\ell-i}(\chi, dw_2-1) w_2^{i-1} w_1^{\ell-i} \right).
\]

Let \( x = 0 \) in Theorem 2. Then we have

\[
\sum_{i=0}^{\ell} \left( \frac{\ell!}{i!} B_{i,\chi} T_{\ell-i}(\chi, dw_1-1) w_1^{i-1} w_2^{\ell-i} \right)
\]

\[
= \sum_{i=0}^{\ell} \left( \frac{\ell!}{i!} B_{i,\chi} T_{\ell-i}(\chi, dw_2-1) w_2^{i-1} w_1^{\ell-i} \right).
\]
By (15) and (17), we also see that

$$T(w_1, w_2) = \left( \frac{e^{w_1 w_2 x}}{w_1} \right) \int_X \chi(x_1) e^{w_1 x_1 t} dx_1 \left( \frac{d}{dx} \int_X \chi(x_2) e^{w_2 x_2 t} dx_2 \right)$$

$$= \left( \frac{e^{w_1 w_2 x}}{w_1} \right) \int_X \chi(x_1) e^{w_1 x_1 t} dx_1 \left( \frac{d}{dx} \int_X \chi(x_2) e^{w_2 x_2 t} dx_2 \right)$$

$$= \left( \frac{e^{w_1 w_2 x}}{w_1} \right) \int_X \chi(x_1) e^{w_1 x_1 t} dx_1 \left( \frac{d}{dx} \int_X \chi(x_2) e^{w_2 x_2 t} dx_2 \right)$$

$$= \left( \frac{e^{w_1 w_2 x}}{w_1} \right) \int_X \chi(x_1) e^{w_1 x_1 t} dx_1 \left( \frac{d}{dx} \int_X \chi(x_2) e^{w_2 x_2 t} dx_2 \right)$$

$$= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{w_2-1} \chi(i) B_{k,\chi}(w_1 x + \frac{w_1 i}{w_2}) \frac{w_2^{k+1}}{k!} \right) t^k.$$
Theorem 3. For $w_1, w_2, d \in \mathbb{N}$, we have
\[
\sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(w_2x + \frac{w_2}{w_1}i)w_1^{k-1} = \sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(w_1x + \frac{w_1}{w_2}i)w_2^{k-1}.
\]

Remark. Let $x = 0$ in Theorem 3. Then we see that
\[
\sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(\frac{w_2}{w_1}i)w_1^{k-1} = \sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(\frac{w_1}{w_2}i)w_2^{k-1}.
\]
If we take $w_2 = 1$, then we have
\[
\sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(\frac{i}{w_1})w_1^{k-1} = \sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(w_1i).
\]

References

[1] L. C. Carlitz, $q$-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948), 987-1000.

[2] M. Cenkci, Y. Sisek, V. Kurt, Further remarks on multiple $p$-adic $q$-$L$-function of two variables, Adv. Stud. Contemp. Math. 14 (2007), 49-68.

[3] M. Cenkci, M. Can, V. Kurt, Multiple two-variable $q$-$L$-function and its behavior at $s = 0$, Russ. J. Math. Phys. 15(4) (2008), 447-459.

[4] T. Ernst, Example of a $q$-umbral calculus, Adv. Stud. Contemp. Math. 16(1) (2008), 1-22.

[5] A. S. Hegazi, M. Mansour, A note on $q$-Bernoulli numbers and polynomials, J. Nonlinear Math. Phys. 13 (2006), 9-18.

[6] T. Kim, $q$-Volkenborn Integration, Russ. J. Math. Phys. 9 (2002), 288-299.

[7] T. Kim, Non-archimedean $q$-integrals associated with multiple Changhee $q$-Bernoulli polynomials, Russ. J. Math. Phys. 10 (2003), 91-98.

[8] T. Kim, Power series and asymptotic series associated with the $q$-analog of the two-variable $p$-adic $L$-function, Russ. J. Math. Phys. 12(2) (2005), 186-196.
T. Kim, *Multiple p-adic L-function* Russ. J. Math. Phys. **13**(2) (2006), 151-157.

T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integral*, J. Nonlinear Math. Phys. **14**(1) (2007), 15-27.

T. Kim, *A note on p-adic q-integral on \( \mathbb{Z}_p \)*, Adv. Stud. Contemp. Math. **15** (2007), 133-138.

T. Kim, *q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients*, Russ. J. Math. Phys. **15**(1) (2008), 51-57.

T. Kim, *On the symmetry of the q-Bernoulli polynomials*, Abstr. Appl. Anal. **2008** (2008), Article ID914367, 7 pages.

T. Kim, *Symmetries p-adic invariant integral on \( \mathbb{Z}_p \) for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. **14**(12) (2008), 1267-1277.

T. Kim, *Note on q-Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. **17**(1) (2008), 9-15.

T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on \( \mathbb{Z}_p \)*, Russ. J. Math. Phys. **16**(1) (2009), 51-54.

Y. -H. Kim, W. Kim, L.-C. Jang, *On the q-extension of Apostol-Euler numbers and polynomials*, Abstr. Appl. Anal. **2008** (2008), Article ID296159, 10 pages.

B. A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials*, J. Nonlinear Math. Phys. **12** (2005), 412-422.

H. Ozden, Y. Simsek, S. -H. Rim, I. N. Cangul *A note on p-adic q-Euler measure*, Adv. Stud. Contemp. Math. **14** (2007), 233-239.

K. H. Park, Y.-H. Kim *On some arithmetical properties of the Genocchi numbers and polynomials*, Advances in Difference Equations. [http://www.hindawi.com/journals/ade/aip.195049.html](http://www.hindawi.com/journals/ade/aip.195049.html).

M. Schork, *A representation of the q-fermionic commutation relations and the limit q = 1*, Russ. J. Math. Phys. **12**(3) (2005), 394-399.

Y. Simsek, *Theorems on twisted L-function and twisted Bernoulli numbers*, Adv. Stud. Contemp. Math. **11** (2005), 205-218.

Y. Simsek, *On p-adic twisted q-L-functions related to generalized twisted Bernoulli numbers*, Russ. J. Math. Phys. **13**(3) (2006), 340-348.
[24] Y. Simsek, *Complete sums of $(h, q)$-extension of the Euler polynomials and numbers*, arXiv:0707.2849v1[math.NT].

[25] Y.-H. Kim, K.-W. Hwang. *Symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math. **18** (2) (2009), 105-113.
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