The Volterra Integrable case

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Abstract

In this short note we reconsider the integrable case of the Hamiltonian $N$-species Volterra
system, as it has been introduced by Vito Volterra in 1937. In the first part, we discuss the
corresponding conserved quantities, and comment about the solutions of the equations of
motion. In the second part we focus our attention on the properties of the simplest model,
in particular on period and frequencies of the periodic orbits. The discussion and the results
presented here are to be viewed as a complement to a more general work, devoted to the
construction of a global stationary state model for a sustainable economy in the Hamiltonian
formalism.

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1 Introduction

The current twofold crisis, at a global level, of Economics and Environment has been deeply
investigated and reported by many authors, some of which have deeply criticized the deafness
of Economists with respect to the environmental crisis, that at most has been assessed for its
dramatic consequences on GDP (see the well-known “Stern’s Report” \cite{1}. The double crisis
and its entanglement would request models, even before than a theory, able to put together
economic and environmental variables in order to build a global stationary state to rule present
predicament in the perspective of a sustainable scenario. The latter theme is not a news, several
attempts having been realized starting from the Seventies for a definition of “steady state”\cite{2},
\cite{3}, \cite{4}) but it could be useful to face the problem with other scientific tools, as it has been
recently proposed in \cite{5}, \cite{6}, \cite{7}, where the leading idea is to put together pairs, each constituted
by an economic variable and an environmental one, that present a behavior “predator-prey”
type, as it is suggested by some of the most important pairs one can select for the model.
Already fifty years ago, a similar idea was applied to build the Goodwin model, but with a
pair of variables only economic to describe an economic cycle \cite{8}, the so called “class struggle”
model, where a variable linked to the wage rate assumes the role of predator and the one giving
employment rate is the prey.
2 The Model

As is well known, the original idea by Vito Volterra [9] was that of determining the evolution of a two species biological system, the so-called “predator-prey” model, answering a question raised by his son in law, the biologist Umberto d’Ancona [10], who was wondering why the total catch of selachians (mostly sharks) was considerably raising during World War 1, with respect to other more desirable kind of fishes, in correspondence with the decrease of fishing activity [11]. To answer that question, Vito Volterra constructed a dynamical system that enabled him to identify the essential features of what was going on, elucidating the properties entailing the existence of a stable equilibrium configuration (and of periodic orbits in its neighbourhood), and unveiling the asymptotic behaviour of the system under general initial conditions. He quickly realized that the predator-prey model was just the simplest example in a large class of biological, or rather ecological systems with pairwise interaction. He was soon interested in understanding the mathematical properties of the $N$ species pairwise interacting model, and devoted a considerable effort to find suitable Lagrangian and Hamiltonian formulations, with the final aim of achieving a description where the deep analogy with the well established theory of mechanical systems stemming from the Maupertuis minimal action principle be made transparent. We would say that not the whole “Biological-Mechanical” dictionary that he proposed in his famous paper (dating back to 1937), Principe de Biologie Mathematique [12], resisted the future developments of both disciplines, and some of the notions he tried to introduce look nowadays a bit artificial. But we believe that the core of his derivation is still alive, as it has been witnessed by a very wide spread applications over about a century in many scientific research subjects, such as Populations demography, Bio-physics, Biomedicine, Ecology and also Economics. We notice that in [12] his main aim was the formulation of this generalized model in a Hamiltonian language, with the purpose of elucidating the algebraic conditions leading to a completely integrable model. So we think that it could be worth recalling the key ingredients of Volterra’s approach, and even emphasizing the role that the very special case of completely integrable dynamics could play in the search for Stationary State models in the economic-ecological framework. His general Predator-Prey model reads:

\[
\frac{dN_r}{dt} = \epsilon_r N_r + \sum_{s \neq r=1}^{N} A_{rs} N_r N_s
\]  

In (2.1), we have set all the parameters introduced in [9] $\beta_r = 1 \forall r$; $\epsilon_r$ are the natural growth coefficients of each species and $A_{rs}$ are interaction coefficients between species $r$ and species $s$ that account for the probability of encountering between two individuals. Moreover, for the moment we assume that the matrix $A$, whose elements are $A_{rs}$, is nonsingular. The last requirement ensures that the system of equations defining the equilibrium configurations, namely

\[
0 = \epsilon_r + \sum_{s=1}^{N} A_{rs} N_s
\]  

has a unique solution, say $Q_r, r = 1, \cdots , N$. If, in addition, according to [9], we require $A$ to be skew-symmetric, $N$ has to be even, and the eigenvalues of $A$ must be purely imaginary and complex conjugate in pairs.

2.1 Lagrangian and Hamiltonian formulation

As many other researchers of his time, Volterra was feeling more assured if a phenomenon quantified by Mathematics could find an analogue with Mechanics, that moreover allowed resorting
to the powerful formalism and theorems of the latter. The quantity of life of the species \( r \), defined as \( X_r = \int_0^t N_r(t')dt' \) suggested to Volterra the introduction of a biological, or rather ecological, Lagrangian \( \Phi \), defined as the sum of three terms

\[
\Phi = X + \frac{1}{2}Z + P \tag{2.3}
\]

where

\[
X = \sum_r X'_r \log X'_r; \quad Z = \sum_{rs} A_{rs} X'_r X_s; \quad P = \sum_r \epsilon_r X_r \tag{2.4}
\]

In terms of \( \Phi \), \( \Phi \) can be written as Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial \Phi}{\partial X'_r} - \frac{\partial \Phi}{\partial X_r} = 0 \tag{2.5}
\]

yielding the ODEs

\[
X''_r = (\epsilon_r + \sum_s A_{sr} X'_s) X'_r \tag{2.6}
\]

which are just \( \Phi \), up to the substitution \( N_r = X'_r \), the \( t \) denoting time-derivative.

2.2 From Lagrange to Hamilton

The linear momenta, canonically conjugated to the quantities of life, are defined as

\[
p_r = \frac{\partial \Phi}{\partial X'_r} = \log X'_r + 1 + \frac{1}{2} \sum_s A_{rs} X_s \tag{2.7}
\]

whence

\[
X'_r = \exp(p_r - 1 - \frac{1}{2} \sum_s A_{rs} X_s) \tag{2.8}
\]

Through a transformation of Legendre type we define the Hamiltonian

\[
\mathcal{H} = \Phi - \sum_r X'_r p_r = \sum_r \epsilon_r X_r - X'_r = \sum_r \epsilon_r X_r - \exp(p_r - 1 - \frac{1}{2} \sum_s A_{rs} X_s) \tag{2.9}
\]

Volterra \cite{12} showed that \( \Phi \) can be written in the standard Hamiltonian form

\[
X'_r = -\frac{\partial \mathcal{H}}{\partial p_r} \tag{2.10}
\]

\[
p'_r = \frac{\partial \mathcal{H}}{\partial X_r} \tag{2.11}
\]

As the derivation is a bit tricky we prefer to sketch it.

One starts by writing

\[
\frac{\partial \mathcal{H}}{\partial X_r} = \epsilon_r - \frac{\partial}{\partial X_r} \left[ \sum_k \exp(p_k - 1 - \frac{1}{2} \sum_s A_{ks} X_s) \right] = \epsilon_r + \frac{1}{2} \sum_k A_{kr} \exp(p_k - 1 - \frac{1}{2} \sum_s A_{ks} X_s) = \epsilon_r + \frac{1}{2} \sum_k A_{kr} X'_k =
\]
\[ \frac{\partial \Phi}{\partial X_r} = \frac{d}{dt} \frac{\partial \Phi}{\partial X'_r} = p'_r \]

Notice however that (2.11) holds just if the Euler-Lagrange equations (2.5) are satisfied. It is easily seen (see again [4]) that the above Hamiltonian system has the following \( N \) independent non autonomous integrals of motion:

\[ H_r = p_r + \frac{1}{2} \sum_s A_{rs} X_s - t \quad r = 1, \ldots, N. \]  

whence one can select \( N - 1 \) autonomous integrals by taking for instance \( H_{1,r} = H_r - H_1 \), and have a complete set by adding the Volterra \( N - \) species Hamiltonian (2.9).

A more modern approach to the Hamiltonian structure underlying the generalised Volterra system can be be found in an elegant paper by R.Loja and W.Oliva [13], where a Poisson morphism is established between the original system, living in \( \mathbb{R}^N \) and equipped with a quadratic Poisson structure, and the one recast, after Volterra, in a canonical Hamiltonian form and thus living in \( \mathbb{R}^{2N} \).

However, in our opinion, a crucial question to ask is whether there exists a special form of the matrix elements \( A_{rs} \) that entails involutivity of that complete set of integrals of motion. It turns out that this form has been found by Volterra himself [12] and is the following:

\[ A_{rs} = \epsilon_r \epsilon_s (B_r - B_s) \quad r, s = 1, \ldots, N \]  

where \( N \) is the number of competing populations and the \( B_r \) are real (and positive) numbers. The matrix \( A \) can be written as:

\[ A = [B, \epsilon \otimes \epsilon] \]  

where \( B = \text{diag}(B_1, \ldots, B_N) \), and \( \epsilon \) is the vector \((\epsilon_1, \ldots, \epsilon_N)^t\), meaning that \( A \) is the commutator of a diagonal matrix with distinct entries and a rank one matrix. Coming back to the Lagrangian formulation, namely rewriting:

\[ X'_r = \exp(p_r - \frac{1}{2} \sum_s A_{rs} X_s) \]  

or

\[ p_r = \frac{1}{2} \sum_s A_{rs} X_s + 1 + \log X'_r \]

We see that (2.12) can be rewritten as

\[ H_r = \frac{\log X'_r + \sum_s A_{rs} X_s}{\epsilon_r} - t \quad r = 1, \ldots, N. \]

whence

\[ \exp(\epsilon_r H_r) = X'_r \exp(\sum_s A_{rs} X_s - \epsilon_r t) \]

that in the integrable case read

\[ \exp(\epsilon_r H_r) = X'_r \exp(\epsilon_r (B_r \tilde{X} - \langle \tilde{\epsilon} | B | X \rangle - t)) \]
where we have set

\[ \tilde{X} = \sum_s \epsilon_s X_s \]

### 2.3 A note about the equilibrium conditions

As a matter of fact, the highly degenerate structure of \( A \) in the integrable case entails that its kernel consists of all vectors which are orthogonal to both \( |\tilde{c}\rangle \) and \( B|\tilde{c}\rangle \), which are certainly linearly independent as by assumption the entries of the matrix \( B \) are all distinct. So, its Kernel \( \text{Ker}(A) \) is a linear space of codimension 2; by “Rouché-Capelli”, its Range \( R(A) \) is two-dimensional, and the equilibrium configuration is highly non-unique.

Indeed, in the integrable case the equilibrium condition reads:

\[ \sum_{s=1}^{N} \epsilon_s (B_s - B_r)Q_s = 1 \]

where we have denoted by \( Q_s (s = 1, \cdots, N) \), the equilibrium configuration. It follows (being \( \sum_{s=1}^{N} \epsilon_s Q_s = 0 \)) that they belong to the hyperplane:

\[ \sum_{s=1}^{N} \epsilon_s B_s Q_s = 1 \]  
\[ \text{(2.15)} \]

So, the equilibrium configurations are defined as the intersection of the two hyperplanes

\[ \langle \epsilon|Q \rangle = 0; \quad \langle B|\epsilon|Q \rangle = 1 \]

Incidentally, we notice that it does not seem that Volterra had paid a special attention to the singular nature of the matrix \( A \) in the integrable case for any \( N \) larger than 2.

So far, we have looked at the equilibrium in terms of the natural coordinates. The situation is unfortunately much less clear in the canonical setting. First of all, rephrased in terms of the quantities of life, the above equations take the form:

\[ 0 = \epsilon_r + \sum_{s=1}^{N} A_{rs} X'_s \]

which is a linear system in the variables \( X'_s \). In turn, by expressing \( X'_s \) in terms of the canonical variables we get:

\[ 0 = \epsilon_r + \sum_{s=1}^{N} A_{rs} \exp[p_s - 1 - \frac{1}{2} \sum k A_{sk} X_k] \]  
\[ \text{(2.16)} \]

On the other hand from the Hamilton’s equations we get:

\[ p' = 0 = -\frac{\partial \mathcal{H}}{\partial X_r} = \epsilon_r + \frac{1}{2} \sum_{s=1}^{N} A_{sr} X'_s = \epsilon_r + \frac{1}{2} \sum_{s=1}^{N} A_{sr} \exp(p_s - 1 - \frac{1}{2} \sum_{q=1}^{N} A_{sq} X_q) \]  
\[ \text{(2.17)} \]

Evidently formulas (2.16) and (2.17) do not match, because of the factor \( \frac{1}{2} \) in front of the exponential in (2.17), which in (2.16) is missing. This discrepancy has to be understood.
2.4 Back to the integrable case

However, once realized that integrability implies the non-uniqueness of the equilibrium configuration ($\forall N > 2$), what is really crucial to understand is whether, for the special form of the matrix $A$ given by (2.13), the integrals of motion are still independent. But a glance at (2.12) shows that the linear dependence of those integrals upon the momenta $p_r$ is in no-way affected by the specific form of the matrix $A$ (while the requirement that the $B_r$ be all distinct is mandatory!), so that the rank of the Jacobian matrix constructed with the gradients of the integrals of motion with respect to the canonical coordinates is maximal (namely $N$) whatever be that form. So, the integrable version of the $N$-species Volterra system is again a genuine hamiltonian system with $N$ degrees of freedom.

Here we write down explicitly the expression of the Hamiltonian and of the integrals of motion in the integrable case.

\[
H_{int} = \sum_{r=1}^{N} \epsilon_r X_r - \exp[p_r - (\epsilon_r/2)\sum_{s=1}^{N} \epsilon_s (B_r - B_s) X_s] \tag{2.18}
\]

and

\[
H_r = p_r/\epsilon_r + (1/2)\sum_{s=1}^{N} \epsilon_s (B_r - B_s) X_s - t \quad r = 1, \cdots, N \tag{2.19}
\]

so that

\[
H_{rl} \equiv H_r - H_l = p_r/\epsilon_r - p_l/\epsilon_l + \frac{1}{2} (B_r - B_l) \sum_{s=1}^{N} \epsilon_s X_s, \quad s = 1, \cdots, N \tag{2.20}
\]

The constants of motion (2.20) are mutually in involution. So we can take for instance $l = 1$ and get $N - 1$ independent integrals of motion in involution. The set can be completed by adding any function of the Hamiltonian, for instance the Hamiltonian itself.

The above formulas clearly show that, in the integrable case, both the Volterra Hamiltonian and the involutive family of integrals of motion depend on the quantities of life only through the inner products $\langle \epsilon|X \rangle$ and $\langle \epsilon|B|X \rangle$. Of course, if one comes back to the Lagrangian formulation, one will get expressions involving the variables $X_s$ and $X'_s$. Let us also notice that

\[
\exp(\epsilon_r H_r) = \exp[p_r + (\epsilon_r/2)\sum_{s=1}^{N} \epsilon_s (B_r - B_s) X_s - \epsilon_r t] \tag{2.21}
\]

Obviously, one could choose $\exp(H_{rl})$ as an alternative legitimate form for an involutive family of integrals of motion. In the simplest nontrivial case, $N = 2$, (2.18) reads:

\[
H_v = \epsilon_1 X_1 + \epsilon_2 X_2 - \exp[p_1 - (1/2)\epsilon_1 \epsilon_2 (B_1 - B_2) X_2] - \exp[p_2 + (1/2)\epsilon_1 \epsilon_2 (B_1 - B_2) X_1] \tag{2.22}
\]

The above formula can be slightly simplified through the canonical transformation (in fact, a rescaling):

\[
p_j \to \tilde{p}_j = p_j/\epsilon_j; \quad X_j \to \tilde{X}_j = \epsilon_j X_j \tag{2.23}
\]

that maps (2.20) into:

\[
H_{rl} = \tilde{p}_r - \tilde{p}_l + (1/2)(B_r - B_l) < 1|\tilde{X} > \tag{2.24}
\]
\[ H = \tilde{X}_1 + \tilde{X}_2 - \exp[\epsilon_1(\tilde{p}_1 - (1/2)(B_1 - B_2)\tilde{X}_2)] - \exp[\epsilon_2(\tilde{p}_2 + (1/2)(B_1 - B_2)\tilde{X}_1)] \quad (2.25) \]

For an arbitrary \( N \), the same canonical transformation yields:

\[ H_{\text{int}} = \sum_{r=1}^{N} \tilde{X}_r - \exp[\epsilon_r(\tilde{p}_r - 1/2 \sum_{s=1}^{N}(B_r - B_s)\tilde{X}_s)] \quad (2.26) \]

Always sticking on the integrable case, it looks a bit surprising that the reduction to quadratures was not performed by Volterra in the canonical setting, but rather in terms of the natural coordinates, namely those denoting the population numbers of the different species. In fact, Volterra defines the quantities \( N := \sum_s \epsilon_s N_s \) and \( M := 1 - \sum_s \epsilon_s B_s N_s \), then rewriting the original dynamical system (2.1) as:

\[ \dot{N}_r = \epsilon_r N_r (1 + \sum_s (B_r - B_s)\epsilon_s N_s) \quad (2.27) \]

or, in other terms:

\[ \dot{N}_r = \epsilon_r N_r (B_r N + M) \quad (2.28) \]

namely:

\[ (1/\epsilon_r) d/dt \log N_r = (B_r N + M) \quad (2.29) \]

Note that the previous equation implies \( \sum_r (\dot{N}_r - \epsilon_r N_r) = 0 \). In terms of the quantities of life \( X_r \) we have \( \sum_r (X''_r - \epsilon_r X'_r) = 0 \), yielding the conserved quantity \( C_0 = \sum_r (\epsilon_r X_r - X'_r) \), which is just the Hamiltonian (2.9). It is evident that the involutivity constraints on the coefficients \( A_{rs} \) entail a typical Mean Field dynamics. Each species interacts with the others through the collective variables \( N \) and \( M \). By taking two different values of the index \( r \) and subtracting, the variable \( M \) can be eliminated, resorting to:

\[ \frac{(1/\epsilon_r) d/dt \log N_r - (1/\epsilon_s) d/dt \log N_s}{B_r - B_s} = N \quad (2.30) \]

By setting \( F_k = N_k^{(1/\epsilon_k)} \), we see that the above formula implies that the time derivative of the differences \( g_k - g_l \), where \( g_m := \frac{\log(F_1/F_m)}{B_1 - B_m} \) vanishes. So, we immediately get \( N - 2 \) integrals of motion written in terms of the natural variables.

The remaining integrals can be obtained by taking the Hamiltonian itself and the quantity \( L := \sum_{r=1}^{N} \epsilon_r Q_r H_r \).

Note, however, that as the \( Q_r \) are not uniquely defined, this non-uniqueness affects \( L \) as well. Of course such “superabundance” of integrals of motion does not really take place, because the different \( L \) will not be functionally independent. It will be enough to consider a single representative element of the class of equilibrium configurations.

### 2.5 The Hamiltonian (2.25) is indeed integrable!

Let us slightly simplify the notations, by setting \( \mu := B_2 - B_1 \), and introducing the new canonical variables:

\[ P_1 = \frac{1}{\sqrt{\mu}}(\tilde{p}_1 + \frac{1}{2}\tilde{X}_2); \quad Q_1 = \frac{1}{\sqrt{\mu}}(-\tilde{p}_2 + \frac{1}{2}\tilde{X}_1) \quad (2.31) \]
\[ P_2 = \frac{1}{\sqrt{\mu}}(\tilde{p}_1 - \frac{1}{2}\tilde{X}_2); \quad Q_2 = \frac{1}{\sqrt{\mu}}(\tilde{p}_2 + \frac{1}{2}\tilde{X}_1) \quad (2.32) \]

In terms of these new variables, the first integral:
\[ \mathcal{H}_{12} = \tilde{p}_1 - \tilde{p}_2 + (1/2)(B_1 - B_2) < 1|\tilde{X} > \quad (2.33) \]

takes the form
\[ \mathcal{H}_{12} = \sqrt{\mu}(P_2 - Q_2) \quad (2.34) \]

while the two-particle Hamiltonian reads:
\[ \mathcal{H}_v = \frac{1}{\sqrt{\mu}}[Q_1 + Q_2 + P_1 - P_2] - \exp(\epsilon_1 P_1) - \exp(\epsilon_2 Q_1) \quad (2.35) \]

Inserting the first integral (2.34), on the level surface \( \mathcal{H}_{12} = C \), up to an irrelevant additive constant we can finally write:
\[ \mathcal{H}_v = \frac{1}{\sqrt{\mu}}(Q_1 + P_1) - \exp(\epsilon_1 P_1) - \exp(\epsilon_2 Q_1) \quad (2.36) \]

It follows that, in terms of these new coordinates, the Hamiltonian (2.25) becomes a “one-body” hamiltonian (integrable by definition), which is nothing but the traditional Volterra-Lotka Hamiltonian.

### 2.6 Suggesting a mixture

An interesting attempt to get a more realistic system could be performed by taking the Hamiltonian to be a linear combination of the sum of two non interacting predator-prey models, where prey and predator are taken as a conjugate symplectic pair, and the Hamiltonian version of the \( N = 2 \) original Volterra system.

Accordingly we will start with a non-interacting two-body system \((in the economic-echological interpretation each body is in fact a pair of conjugated quantities)\) described by the sum of two independent Hamiltonians \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \).

\[ \mathcal{H}_0 = \mathcal{H}_1 + \mathcal{H}_2 = \sum_{i=1}^{2} \epsilon_i q_i + \eta_i p_i - \alpha_i \exp(q_i) - \beta_i \exp(p_i) \quad (2.37) \]

The hamiltonian \((2.37)\) has a single equilibrium configuration corresponding to the point in \( \mathbb{R}^4 \) \( P = (\log(\eta_1/\beta_1), \log(\epsilon_1/\alpha_1), \log(\eta_2/\beta_2), \log(\epsilon_2/\alpha_2)) \). Of course the positivity requirements are fulfilled provided that \( \epsilon_i/\alpha_i, \eta_i/\beta_i > 1 \) \((i = 1, 2)\). Under the above hypotheses \( P \) will be a center, and the phase space will be foliated by two-dimensional tori, parametrized by the energy \( E_1 \) and \( E_2 \). We expect that the family of plane curves \( f(x, y) = C, C \) being a positive constant, with
\[ f(x, y) := \alpha \exp(x) + \beta \exp(y) - \epsilon x - \eta y \quad (2.38) \]

be bounded and contained in the first quadrant of the \((x, y)\) plane whenever \( \epsilon \) and \( \eta \) are both positive but smaller than \( \alpha \) and \( \beta \) respectively.
2.7 Linear stability and small oscillations for the decoupled system

The equations of motion for the Hamiltonian (2.37) read:

\[
\dot{q}_i = \eta_i - \beta_i \exp(p_i) \quad (i = 1, 2) \\
\dot{p}_i = -\epsilon_i + \alpha_i \exp(q_i) \quad (i = 1, 2)
\] (2.39) (2.40)

In terms of exponential functions, the equilibrium configuration is given by

\[
\exp(\tilde{q}_i) = \frac{\epsilon_i}{\alpha_i}; \quad \exp(\tilde{p}_i) = \frac{\eta_i}{\beta_i}
\]

However, it is worth noticing that in the natural variables (those originally introduced by Volterra) a discussion about equilibrium stability and small oscillations will be much easier. Accordingly, we set \( x_i = \exp(q_i), \) \( y_i = \exp(p_i), \) so that the Hamilton’s equations take the form:

\[
\dot{x}_i = -\beta_i (y_i - \tilde{y}_i) \quad \dot{y}_i = \alpha_i (x_i - \tilde{x}_i)
\] (2.41)

Introducing the differences:

\[
\xi_i = x_i - \tilde{x}_i; \quad \zeta_i = y_i - \tilde{y}_i
\]

which are clearly \( O(\epsilon) \) so that products like \( \xi_i \zeta_i \) will be \( O(\epsilon^2) \), we get:

\[
\dot{\xi}_i = -\beta_i \tilde{x}_i \zeta_i + O(\epsilon), \quad \dot{\zeta}_i = \alpha_i \tilde{y}_i \xi_i + O(\epsilon)
\] (2.42)

Consequently, the secular equation will be the following biquadratic equation:

\[
\lambda^4 + \lambda^2 (\alpha_1 \beta_1 \tilde{x}_1 \tilde{y}_1 + \alpha_2 \beta_2 \tilde{x}_2 \tilde{y}_2) + \alpha_1 \beta_1 \tilde{x}_1 \tilde{y}_1 \alpha_2 \beta_2 \tilde{x}_2 \tilde{y}_2 = 0
\]

whose solutions are:

\[
\lambda^2_i = -\alpha_i \beta_i \tilde{x}_i \tilde{y}_i
\]

Inserting the explicit formulas for the equilibrium positions, we get:

\[
\lambda^2_i = -\epsilon_i \eta_i
\]

meaning that, assuming the products \( \epsilon_i \eta_i \) to be all positive, the eigenvalues will be purely imaginary as expected. Of course, being decoupled from the very beginning, the system will keep its decoupled nature in the small oscillation regime as well. Investigating the small oscillation regime for the simplest coupled Volterra model will be of course more interesting. But a deeper look at the equations of motion of such model, possibly not written in Hamiltonian form, will be needed to identify its equilibrium positions.

The equation for the integral curve (conservation of energy) (2.38) can be written in a slightly more elegant way. Indeed, introducing the equilibrium position and conveniently rescaling the independent variables, \( f(x, y) = C \) acquires the form:

\[
\epsilon f(\xi) + \eta f(\zeta) = C - (\epsilon + \eta) := k^2
\] (2.43)

having set

\[
f(x) = \exp(x) - x - 1
\] (2.44)
we remark that the function \( f(x) \) defined by (2.44) is nonnegative on the whole real line, reaches its minimum at the origin where it vanishes, and is monotonically decreasing (resp. increasing) in the negative (resp. positive) semi-axis. Therefore there is a diffeomorphism mapping the curve \( \epsilon f(\xi) + \eta f(\zeta) = k^2 (\epsilon > 0, \eta > 0) \) to the curve \( aX^2 + bY^2 = 1 \) for some \( a > 0, b > 0 \).

One way to couple the two systems could be by adding to (2.37), through a dimensionless parameter \( \lambda \), the generalized Volterra predator-prey Hamiltonian (2.22), written in Volterra’s Hamiltonian variables, in the simplest nontrivial case \( (N = 2) \). Both (2.37) and (2.22) are completely integrable, but since they are not in involution the coupled system will lose the complete integrability property. However by acting on the control parameter \( \lambda \) one may think of following the behaviour of such non-integrable system, and see the corresponding merging of the two tori in a single 4-dimensional surface. The suggested Hamiltonian reads:

\[
\mathcal{H}_\lambda = \mathcal{H}_0 + \lambda \mathcal{H}_v
\]

where of course \( \mathcal{H}_v \) is given by (2.22). However, It would be reasonable to require that the coupled Hamiltonian \( \mathcal{H}_\lambda \) keeps one and only one equilibrium configuration, at least whenever \( \lambda \) is small enough. With respect to the direct sum of two single predator-prey models the full Hamiltonian exhibits of course modifications both in the linear part, where the coefficients of the \( q \) dependent terms are different (but this is really a minor modification, as those coefficients were arbitrary) and, more substantially, in the exponential part, where the arguments depend jointly upon \( p \) and \( q \) variables. From the theoretical point of view, one may of course resort to the KAM Theory (and theorem), having to do with Hamiltonian perturbations. Both Hamiltonians being completely integrable on their own right, deciding which one to take as a perturbation of the other is largely a matter of taste or of phenomenological motivations, although, in view of the previous observation on the equilibrium configuration, keeping \( \mathcal{H}_0 \) as the unperturbed term seems a natural choice. At a first glance, when looking at the "full" Hamiltonian

\[
\mathcal{H}_\lambda = \mathcal{H}_0 + \lambda \mathcal{H}_v
\]

one may get a bit confused, in view of the fact that \( \mathcal{H}_v \), in suitable canonical variables, takes the form of a one-body Hamiltonian as well. However, this does not entail that we are considering just a linear combination of one body systems, because the coordinates that separate \( \mathcal{H}_v \) are not the same as those working for \( \mathcal{H}_0 \)!

When \( \lambda = 0 \) we deal with a completely integrable Hamiltonian system with 2 degrees of freedom, already separated in the sum of two commuting Hamiltonians, whose level surfaces are compact (closed and bounded). So, in principle we have action-angle variables. But to our knowledge, in a generic case, there is no way to get an explicit expression for the frequencies, even in the case of a single degree of freedom, unless we resort to numerical computations. Let us now go back to the Hamiltonian \( \mathcal{H}_0 \), rewritten in terms of the variables \( \xi \) and \( \zeta \), that however, inspite of the possibly arising confusion, we denote again by \( x \) and \( y \), to simplify notations. Please remark that, being translated with respect to the original ones, those variables are no more constrained to the first quadrant, meaning that they can have both positive and negative values. In particular, the equilibrium position is now moved to the origin. So, we take as Hamiltonian function \( \mathcal{H} = af(x) + bf(y) \), whose level surfaces are given by the curve \( af(x) + bf(y) = E \), and have the Hamilton’s equation:

\[
\dot{x} = bf'(y) = b(1 - \exp(y)); \quad \dot{y} = -af'(x) = -a(\exp(x) - 1)
\] (2.45)
2.8 About the period of the simplest model

To get the period of the bounded motion described by (2.45), we write (for instance) the first of the above equations as

\[
\frac{dt}{dx} = \frac{1}{1 - \exp(y)} \tag{2.46}
\]

etailing that we have to express \(\exp(y) - 1\) as a function of \(E\) and \(x\), taking into account that

\[
E = af(x) + bf(y); \quad f(x) = \exp(x) - x - 1 \tag{2.47}
\]

Now \(x(y)\) varies on the whole real line, and correspondingly \(f(x)(f(y))\) decreases monotonically (from \(+\infty\) to 0) on the negative semiline, vanishes at the origin and increases monotonically to \(+\infty\) on the positive semiline. Hence for any positive \(E\) both the functions \(f(x)\) and \(f(y)\) intersect the horizontal line \(E = const\) in two points, a negative one and a positive one, implying that they are only piecewise invertible. Taking care of such warning, we write

\[
y = f^{-1}((E - af(x))/b) \tag{2.48}
\]

whence

\[
dt = \frac{dx}{b(1 - \exp(f^{-1}((E - af(x))/b))} \tag{2.49}
\]

In order to get the period, (2.41) has to be integrated from two subsequent critical points which are the negative and positive root of

\[
E/a = f(x) \tag{2.50}
\]

In the literature one can find several different approaches to the problem of determining the period of the Volterra-Lotka system. A rich (however, possibly not exhausting) survey can be found in [14]. There, 4 different methods are described, the first of them being due to Volterra himself. Here, we want to spend a few words on the one proposed by F. Rothe [15]. In that paper, the author writes the system in Hamiltonian form (the same as we have introduced above) for the Hamilton’s function

\[
\mathcal{H}(x, y) = af(x) + bf(y) \tag{2.51}
\]

With

\[
f(x) = \exp(x) - x - 1
\]

Rothe associates with (2.51) the partition function

\[
Z(\beta) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \exp(-\beta \mathcal{H}(x, y)) \tag{2.52}
\]

which turns out to be expressible in terms of the Euler’s \(\Gamma\) function as \(Z(\beta) = z(a\beta)z(b\beta)\), where

\[
z(\gamma) = \int_{-\infty}^{+\infty} dx \exp[-\gamma(\exp(x) - x - 1)] = \exp(\gamma)\gamma^{-\gamma}\Gamma(\gamma)
\]

On the other hand, under the ergodic hypothesis, the partition function can be considered as the expectation value of the energy-period function \(T(E)\), through:

\[
Z(\beta) = \int_{0}^{\infty} dET(E) \exp(-\beta E) \tag{2.53}
\]

entailing that the energy-period function \(T(E)\) is the inverse-Laplace transform of the partition function, and consequently is given by the following convolution:

\[
T(E) = \mathcal{L}^{-1}[Z(\beta)] = \mathcal{L}^{-1}[z(\beta a)] \ast \mathcal{L}^{-1}[z(\beta b)] \tag{2.54}
\]
To determine the inverse Laplace transform of \( z \), let us consider again \( f(x) = \exp(x) - x - 1 \), a real-analytic function mapping \( \mathbb{R} \) to \( \mathbb{R}_+ \), monotonically decreasing (increasing) in the negative (positive) semiline, and denote by \( x_-(x_+) \) the corresponding roots of the equation \( f(x) = E \). Define now \( \tau_+(E) = 1/|f'(x_+)| = \frac{1}{E+x_+} \) and and similarly \( \tau_-(E) = 1/|f'(x_-)| = \frac{1}{E-x_-} \) and set \( \tau(E) = \tau_+ + \tau_- \). Then:

\[
Z(\beta a) = \mathcal{L}(a^{-1} \tau(E/a)); \quad Z(\beta b) = \mathcal{L}(b^{-1} \tau(E/b))
\]

Whence it follows the result: The period of oscillations of the Volterra-Lotka system, parametrized as before, reads:

\[
T(E) = \frac{1}{ab} \int_0^E ds \tau(s/b) \tau((E - s)/a) \tag{2.55}
\]

On the other hand, we know from the textbooks on Hamiltonian mechanics (see for instance [10]), that in the case of a compact energy surface (meaning that all orbits are closed and bounded in the phase space) the frequencies, which are defined as the derivatives of the members of a set of commuting invariants with respect to the action integrals, coincide with the inverse of the corresponding periods. Here however, a problem arises. In a system with \( N \) degrees of freedom (take for simplicity a completely integrable one), there are, say, \( N \) (commuting) integrals of motion \( H_j \) and \( N \) action integrals \( J_k \). Moreover, in general, the integrals of motion will depend on several action integrals, so that typically we have matrices. The frequencies are defined as the partial derivatives

\[
\nu^{(k)}_j = \frac{\partial H_k}{\partial J_j} \tag{2.56}
\]

The quantities \( \frac{\partial H_j}{\partial J_j} \) will be the entries of the inverse matrix, which, unless everything is diagonal, won’t be of course just the inverse of the frequencies. So, a linear algebra operation, namely the inversion of a matrix, is needed to recover the frequencies in the case we are given the “action integrals” in terms of the “constants of motion”. However, if we restrict for a moment our attention to the simplest case, governed by the hamiltonian \( H_0 \), the constants of motion are nothing but the single pair hamiltonians \( H_1 = E_1 \) and \( H_2 = E_2 \). The equation defining the action is pretty similar to the one we have already encountered to calculate the period. Indeed, the expression of \( y \) as a function of \( x \) and of the constants of motion has been derived in (2.40), which entails:

\[
J_k = \oint dx_k (f^{-1}[(E_k - a_k f(x_k))/b_k]) \tag{2.57}
\]

whence:

\[
\nu_k = \oint dx_k \frac{b_k}{-1 + \exp[f^{-1}(E_k - a_k f(x_k))/b_k]} \tag{2.58}
\]

We emphasize again that we are considering the separated situation were \( H_1 = E_1 \) and \( H_2 = E_2 \), so that in the inversion procedure needed to express the linear momenta momenta \( y_k \) in terms of the conjugate coordinate \( x_k \) and of the corresponding energy level, we can attach to each degree of freedom it’s own energy value. In other words, the action integral \( J_k \) depends just upon \( E_k \), so that the matrix introduced above is diagonal and the frequency \( \nu_k^{(k)} \) is the reciprocal of the (total) derivative \( \frac{dE_k}{dx_k} \). Moreover, the procedure introduced in [15] to calculate the period in the original Lotka-Volterra has a trivial extension to the case of the direct sum of two systems. The partition function of the direct sum is in fact the product of those pertaining to each of them. Indeed, in that case the partition function will be given by

\[
Z^{(2)}(\beta) = \Pi_{i=1}^2 \int_{-\infty}^\infty dx_i \int_{-\infty}^\infty dy_i \exp(-\beta H_i(x_i, y_i)) \tag{2.59}
\]
the functions $H_i$ being the same for both systems, up to possible changes of the coefficients.

### 3 Concluding remarks

We stress that the results reported above are not complete. First of all, even in the completely integrable case, we lack an explicit reduction to quadratures of the equations of motion for an arbitrary $N$: here, we have performed it only up to two predator-pray pairs. Other issues, ubiquitous in the theory of integrable systems, such as the existence of a Lax representation and of a discrete (in time) integrable version of the continuous dynamics are at the moment out of our description. A further question, more important in view of the possible applications, is related to the construction of a richer model, still enjoying the complete integrability property, but involving a larger number of conjugated pairs of variables, all of them being significant in the economic-ecological approach. Work is progress in all the outlined directions.

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