Discrete skew selfadjoint canonical systems and the isotropic Heisenberg magnet model

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Abstract

A discrete analog of a skew selfadjoint canonical (Zakharov-Shabat or AKNS) system with a pseudo-exponential potential is introduced. For the corresponding Weyl function the direct and inverse problem are solved explicitly in terms of three parameter matrices. As an application explicit solutions are obtained for the discrete integrable nonlinear equation corresponding to the isotropic Heisenberg magnet model. State space techniques from mathematical system theory play an important role in the proofs.

0 Introduction

In this paper we shall treat a discrete analog of the well-known skew selfadjoint canonical (Dirac type, Zakharov-Shabat or AKNS) system:

\[-iJ \frac{dY}{dx}(x, z) = zY(x, z) + V(x)Y(x, z), \quad J = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, \quad x \geq 0. \tag{0.1}\]

Here \(z\) is a spectral variable, \(Y\) and \(V\) are \(2m \times 2m\) matrix functions on the half line, and \(V\) is skew selfadjoint, that is, \(V(x)^* = JV(x)\) with \(V(x)^*\) being the matrix adjoint of \(V(x)\). To obtain the discrete analog of (0.1) let \(U\) be the unique solution of the initial value problem

\[\frac{dU}{dx}(x) = -iU(x)JV(x), \quad x \geq 0, \quad U(0) = I_{2m}. \tag{0.2}\]
Since $JV(x)$ is selfadjoint, we get from (0.2) that $U(x)$ is unitary for each $x \geq 0$. Now put $S(x) = U(x)JU(x)^*$ and $W(x, z) = U(x)Y(x, z)$. Then
\[
\frac{dW}{dx}(x, z) = izS(x)W(x, z), \quad S(x) = S(x)^* = S(x)^{-1}, \quad x \geq 0. \quad (0.3)
\]
It is now immediate that
\[
W_{n+1}(\lambda) - W_n(\lambda) = -\frac{i}{\lambda}S_nW_n(\lambda), \quad S_n = S_n^* = S_n^{-1}, \quad n = 0, 1, \ldots \quad (0.4)
\]
is a natural discrete analog of (0.1). This discrete analog of the continuous pseudo-canonical system is very important. In fact, when $m = 1$, then the system (0.4) turns out to be an auxiliary system for the nonlinear isotropic Heisenberg magnet (IHM) model [36] (see also the detailed discussion after Theorem 0.5 below and the historical remarks in [12]). Motivated by the IHM model we shall use the term spin sequence to denote any sequence of $N \times N$ matrices $\{S_n\}$ satisfying
\[
S_n = S_n^* = S_n^{-1}, \quad n = 0, 1, 2, \ldots \quad (0.5)
\]
As for the skew selfadjoint continuous case [29] (see also [16]), one can associate with (0.4) an $m \times m$ matrix function $\varphi(\lambda)$, meromorphic on $\mathbb{C} \setminus (-\delta, \infty)$, such that
\[
\sum_{n=0}^{\infty} \begin{bmatrix} \varphi(\lambda)^* & I_m \end{bmatrix} W_n(\lambda)^* W_n(\lambda) \begin{bmatrix} \varphi(\lambda) \\ I_m \end{bmatrix} < \infty, \quad (0.6)
\]
where $W_n(\lambda)$, $n \geq 0$, is the fundamental solution of (0.3), i.e., the $2m \times 2m$ matrix solution $W_n(\lambda)$ of (0.4) normalized by the condition $W_0(\lambda) = I_{2m}$. One refers to $\varphi$ as the Weyl function of (0.3). When the Weyl function is rational and strictly proper we shall recover the system (0.4) explicitly from its Weyl function. For this purpose we need to introduce spin sequences that are the discrete analogs of the pseudo-exponential potentials from [16, 17] (see also the references therein).

The spin sequences from this special class are defined in terms of three parameter matrices with the following properties. First fix an integer $N > 0$, and consider an $N \times N$ matrix $\alpha$ with $\det \alpha \neq 0$, an $N \times N$ matrix $\Sigma_0$ such
that $\Sigma_0 = \Sigma_0^*$, and an $N \times 2m$ matrix $\Lambda_0$. These matrices should satisfy the following matrix identity

$$\alpha \Sigma_0 - \Sigma_0 \alpha^* = i \Lambda_0 \Lambda_0^*. \quad (0.7)$$

Given these three matrices $\alpha$, $\Sigma_0$, and $\Lambda_0$ we define for $n = 1, 2, \ldots$ the $N \times 2m$ matrix $\Lambda_n$ and the $N \times N$ matrix $\Sigma_n$ via recursion:

$$\Lambda_{n+1} = \Lambda_n + i \alpha^{-1} \Lambda_n J, \quad \Sigma_{n+1} = \Sigma_n + \alpha^{-1} \Sigma_n (\alpha^*)^{-1} + \alpha^{-1} \Lambda_n J \Lambda_n^*(\alpha^*)^{-1}. \quad (0.8)$$

Next assume that the matrices $\Sigma_n$, $n = 0, 1, 2, \ldots$, are non-singular. Then we say that the sequence of matrices $\{S_n\}$ defined by

$$S_n = J + \Lambda_n^* \Sigma_n^{-1} \Lambda_n - \Lambda_{n+1}^* \Sigma_{n+1}^{-1} \Lambda_{n+1}, \quad n = 0, 1, 2, \ldots, \quad (0.9)$$

is the spin sequence determined by the parameter matrices $\alpha$, $\Sigma_0$ and $\Lambda_0$. Notice that this requires the invertibility of the matrices $\Sigma_n$.

For spin sequences defined in this way our first theorem presents an formula for the fundamental solution $W_n(\lambda)$ of (0.4).

**Theorem 0.1** Let $\alpha$ (det $\alpha \neq 0$), $\Sigma_0$ ($\Sigma_0 = \Sigma_0^*$) and $\Lambda_0$ satisfy (0.7), and assume that det $\Sigma_n \neq 0$ for $0 \leq n \leq M$, where $\Sigma_n$ is given by (0.8). For $0 \leq n \leq M - 1$ let $S_n$ be the matrices determined by $\alpha$, $\Sigma_0$ and $\Lambda_0$ via (0.9) and (0.8). Then $S_n = S_n^* = S_n^{-1}$ for $0 \leq n \leq M - 1$, and for $0 \leq n \leq M$ the fundamental solution $W_n(\lambda)$ of the discrete system (0.4) can be represented in the form

$$W_n(\lambda) = W_{\alpha, \Lambda}(n, \lambda) \left( I_{2m} - \frac{i}{\lambda} J \right)^n W_{\alpha, \Lambda}(0, \lambda)^{-1}, \quad (0.10)$$

where $W_{\alpha, \Lambda}(n, \lambda)$ is defined by

$$W_{\alpha, \Lambda}(n, \lambda) = I_{2m} + i \Lambda_n^* \Sigma_n^{-1}(\lambda I_N - \alpha)^{-1} \Lambda_n. \quad (0.11)$$

When $\Sigma_0 > 0$, there exist simple conditions on $\alpha$ and $\Lambda_0$ to guarantee that det $\Sigma_n \neq 0$. First, if $\Sigma_0 > 0$, then without loss of generality we can assume that $\Sigma_0 = I_N$. Indeed, it is easy to see that the sequence of matrices
\{S_n\} defined by (0.9) and (0.8) does not change if we substitute $\alpha$, $\Sigma_0$ and $\Lambda_0$ by $\Sigma_{0}^{-\frac{1}{2}}\alpha\Sigma_{0}^{-\frac{1}{2}}$, $I_N$ and $\Sigma_{0}^{-\frac{1}{2}}\Lambda_0$. So let us assume that $\Sigma_0 = I_N$. Next, we partition $\Lambda_0$ into two $N \times m$ blocks $\theta_1$ and $\theta_2$ as follows: $\Lambda_0 = [\theta_1 \theta_2]$. This together with $\Sigma_0 = I_N$ allows us to rewrite (0.7) in the form

$$\alpha - \alpha^* = i(\theta_1\theta_1^* + \theta_2\theta_2^*). \quad (0.12)$$

Furthermore, in this case $\Lambda_n$ is given by

$$\Lambda_n = [(I_N + i\alpha^{-1})^n\theta_1 \quad (I_N - i\alpha^{-1})^n\theta_2]. \quad (0.13)$$

Finally, we shall assume that the pair $\{\alpha, \theta_1\}$ is full range which means that

$$\mathbb{C}^N = \text{span}\{\alpha^k\theta_1 \mathbb{C}^m \mid k = 0, 1, 2, \ldots, N - 1\}.$$ 

The following proposition shows that under these conditions automatically \(\det \alpha \neq 0\) and \(\det \Sigma_n \neq 0\) for \(n = 0, 1, 2, \ldots\).

**Proposition 0.2** Let $\alpha$ be a square matrix of order $N$, and $\theta_1$ and $\theta_2$ be $N \times m$ matrices satisfying (0.12). Assume that the pair $\{\alpha, \theta_1\}$ is full range. Then all the eigenvalues of $\alpha$ are in the open upper half plane $\mathbb{C}_+$, and for $n = 1, 2, \ldots$ the matrices $\Sigma_n$ defined by (0.8), with $\Sigma_0 = I_N$ and $\Lambda_n$ given by (0.13), are positive definite and satisfy the identity

$$\alpha\Sigma_n - \Sigma_n\alpha^* = i\Lambda_n\Lambda_n^*. \quad (0.14)$$

**Definition 0.3** A triple of matrices $\alpha$, $\theta_1$ and $\theta_2$, with $\alpha$ square of order $N$ and $\theta_1$ and $\theta_2$ of size $N \times m$, is called admissible if the pairs $\{\alpha, \theta_1\}$ and $\{\alpha, \theta_2\}$ are full range and satisfy the identity (0.12) holds.

We denote by the acronym FG (finitely generated) the class of spin sequences $\{S_n\}$ determined by the matrices $\alpha$, $\Sigma_0 = I_N$ and $\Lambda_0 = [\theta_1 \theta_2]$, where $\alpha$, $\theta_1$ and $\theta_2$ form an admissible triple. In this case we also say that these spin sequences are determined by the corresponding admissible triples. The next two theorems present the solutions of the direct and inverse problem in terms of the Weyl function.
Theorem 0.4 Assume that the spin sequence \( \{S_n\}_{n \geq 0} \) of the discrete pseudo-canonical system (0.4) belongs to the class FG and is determined by the admissible triple \( \alpha, \theta_1, \) and \( \theta_2 \). Then the system (0.4) has a unique Weyl function \( \varphi \), which satisfies (0.6) on the half plane \( \Im \lambda < -\frac{1}{2} \), a finite number of poles excluded, and this function is given by the formula

\[
\varphi(\lambda) = i\theta_1^*(\lambda I_N - \beta)^{-1}\theta_2,
\]
where

\[
\beta = \alpha - i\theta_2\theta_2^*.
\]

Notice that the function \( \varphi \) in (0.15) is a strictly proper \( m \times m \) rational matrix function. Conversely, if \( \varphi \) is a strictly proper \( m \times m \) rational matrix function, then it admits a representation of the form

\[
\varphi(\lambda) = i\vartheta_1^*(\lambda I_n - \gamma)^{-1}\vartheta_2,
\]
where \( \gamma \) is a square matrix and \( \vartheta_1, \vartheta_2 \) are matrices of size \( n \times m \). We refer to the right hand side of (0.17) as a minimal realization of \( \varphi \) if among all possible representations (0.17) of \( \varphi \) the order \( n \) of the matrix \( \gamma \) is as small as possible. This terminology is taken from mathematical system theory. We can now state the solution of the inverse problem.

Theorem 0.5 Let \( \varphi \) be a strictly proper rational \( m \times m \) matrix function, given by the minimal realization (0.17). There is a unique positive definite \( n \times n \) matrix solution \( X \) of the algebraic Riccati equation

\[
\gamma X - X\gamma^* = i(X\vartheta_1\vartheta_1^*X - \vartheta_2\vartheta_2^*).
\]

Using \( X \) define matrices \( \theta_1, \theta_2, \) and \( \alpha = \beta + i\theta_2\theta_2^* \) by

\[
\theta_1 = X^{\frac{1}{2}}\vartheta_1, \quad \theta_2 = X^{-\frac{1}{2}}\vartheta_2, \quad \beta = X^{-\frac{1}{2}}\gamma X^{\frac{1}{2}}.
\]

Then \( \alpha, \theta_1, \) and \( \theta_2 \) form an admissible triple, and the given matrix function \( \varphi \) is the Weyl function of a system (0.4) of which the spin sequence \( \{S_n\} \in FG \) and is uniquely determined by the admissible triple \( \alpha, \theta_1, \) and \( \theta_2 \).
Next, we describe connections with the nonlinear IHM equation. For this purpose consider the zero curvature representation \[36\] of the IHM model:

\[
\frac{d}{dt}G_n(t, \lambda) = F_{n+1}(t, \lambda)G_n(t, \lambda) - G_n(t, \lambda)F_n(t, \lambda),
\]

(0.20)

where

\[
G_n(t, \lambda) = I_2 - \frac{i}{\lambda}S_n(t), \quad F_n(t, \lambda) = \frac{V_n^+(t)}{\lambda - i} + \frac{V_n^-(t)}{\lambda + i},
\]

(0.21)

\[
V_r^\pm(t) := (1 + \overrightarrow{S}_{r-1}(t) \cdot \overrightarrow{S}_r(t))^{-1}(I_2 \pm S_r(t))(I_2 \pm S_{r-1}(t)).
\]

(0.22)

Here the vectors \(\overrightarrow{S}_r = [S_r^1 \ S_r^2 \ S_r^3]\) belong to \(\mathbb{R}^3\), \(\mathbb{R}\) is real axis, the dot \(\cdot\) denotes the scalar product in \(\mathbb{R}^3\), and the correspondence between the spin matrix \(S_r\) and the spin vector \(\overrightarrow{S}_r\) is given by the equality

\[
S_r = \begin{bmatrix}
S_r^3 \\
S_r^1 + iS_r^2 \\
-S_r^3
\end{bmatrix}.
\]

(0.23)

In other words the IHM equation

\[
\frac{d\overrightarrow{S}_n}{dt} = 2\overrightarrow{S}_n \wedge \left( \frac{\overrightarrow{S}_{n+1}}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1}} + \frac{\overrightarrow{S}_{n-1}}{1 + \overrightarrow{S}_{n-1} \cdot \overrightarrow{S}_n} \right),
\]

(0.24)

where \(\wedge\) stands for the vector product in \(\mathbb{R}^3\), is equivalent to the compatibility condition (0.20) of the systems

\[
\frac{d}{dt}W_{n+1}(t, \lambda) = G_n(t, \lambda)W_n(t, \lambda),
\]

\[
\frac{d}{dt}W_n(t, \lambda) = F_n(t, \lambda)W_n(t, \lambda) \quad (n \geq 0).
\]

(0.25)

In (0.24) it is required that \(\|\overrightarrow{S}_n\| = 1\). Now one can see easily that the representation (0.23), where \(\overrightarrow{S}_n \cdot \overrightarrow{S}_n = 1\), is equivalent to the equalities (0.5) with \(S_n \neq \pm I_2\). Thus the first system in (0.25) coincides with system (0.4), where \(m = 2\), and \(S_n \neq \pm I_2\). We use these connections to obtain explicit solutions of the IHM model.

The literature on continuous canonical systems is very rich, especially for the selfadjoint case; see, for instance, the books [10, 12, 14, 26, 35].
Selfadjoint continuous canonical systems with pseudo-exponential potentials have been introduced in [15], and for this class of potentials various direct and inverse problems have been solved; see [17] and the references therein. The subclass of strictly pseudo-exponential potentials has been treated in [3, 4, 5]. Interesting recent results on the spectral theory of selfadjoint discrete systems and various useful references on this subject can be found in [2, 7, 9, 13, 28, 37]. Mainly Jacobi matrices (or block Jacobi matrices as in [2]) that are related to Toda chain problems have been studied. For the skew selfadjoint discrete case some references can be found in [11, 12, 15].

Theorem 0.1 is the discrete analog of Theorem 1.2 in [16]. The right hand side of (0.11) can be viewed as the transfer function of a linear input-output system (see [6]). Transfer functions of the special form given by (0.11) were introduced in [33], and also used for the representation of the fundamental solutions of continuous canonical systems [34, 35]. In Theorem 0.1 we are closer to [16] (see also [30, 32]), where the dependence on the parameter \( x \) differs from the one in [34, 35]. The condition on an admissible triple \( \alpha, \theta_1, \theta_2 \) that the pairs \( \{ \alpha, \theta_1 \} \) and \( \{ \alpha, \theta_2 \} \) are full range pairs is specific for the discrete case. Nevertheless Theorems 0.4, 0.5 and parts of their proofs are analogous to results and proofs in Section 2 of [16].

This paper consists of four sections not counting this introduction. Since elements from mathematical system theory play an important role in this paper, we present in the first section the necessary preliminaries from that area. In the second section we prove Theorem 0.1 and present some auxiliary results that will be used in the application to the IHM model. Theorems 0.4, 0.5 and Proposition 0.2 are proved in Section 3. In Section 4 we construct solutions of the IHM equation (0.24), describe the evolution of the Weyl function and consider a simple example.

1 Preliminaries from mathematical system theory

The material from the state space theory of rational matrix functions, that is used in this paper, has its roots in the Kalman theory of input-output
systems [21], and can be found in books, see, e.g., [19, 8]. In general the rational matrix functions appearing in this paper are proper; that is, analytic at infinity, and they are square, of size $m \times m$, say. Such a function $F$ can be represented in the form

$$ F(\lambda) = D + C(\lambda I_N - A)^{-1}B, \quad (1.1) $$

where $A$ is a square matrix (of which the order $N$ may be much larger than $m$), the matrices $B$ and $C$ are of sizes $N \times m$ and $m \times N$, respectively, and $D = F(\infty)$. In this paper $D$ is often a zero matrix, and in that case $F$ is called strictly proper. The representation (1.1) is called a realization or a transfer matrix representation of $F$, and the number $N = \text{ord}(A)$, is called the state space dimension of the realization. Here $\text{ord}(A)$ denotes the order of the matrix $A$.

Realizations of a fixed $F$ are not unique. The realization (1.1) is said to be minimal if its state space dimension $N$ is minimal among all possible realizations of $F$. The state space dimension of a minimal realization of $F$ is called the McMillan degree of $F$ and is denoted by $\deg F$. Notice that $\deg F = 0$ corresponds to the case when $\text{ord}(A) = 0$, and this occurs if and only if $F(\lambda) \equiv D$. The realization (1.1) of $F$ is minimal if and only if

$$ \text{span} \bigcup_{k=0}^{N-1} \text{Im} A^kB = \mathbb{C}^N, \quad \bigcap_{k=0}^{N-1} \text{Ker} CA^k = \{0\}, \quad N = \text{ord}(A). \quad (1.2) $$

If for a pair of matrices $\{A, B\}$ the first equality in (1.2) holds, then $\{A, B\}$ is called controllable or a full range pair. If the second equality in (1.2) is fulfilled, then $\{C, A\}$ is said to be observable or a zero kernel pair. If a pair $\{A, B\}$ is full range, and $K$ is an $m \times N$ matrix, where $N$ is the order of $A$ and $m$ is the number of columns for $B$, then the pair $\{A - BK, B\}$ is also full range. An analogous result holds true for zero kernel pairs.

Minimal realizations are unique up to a basis transformation, that is, if (1.1) is a minimal realization of $F$ and $F(\lambda) = D + \tilde{C}(\lambda I_N - \tilde{A})^{-1}\tilde{B}$ is a second minimal realization of $F$, then there exists an invertible matrix $S$ such that

$$ \tilde{A} = SAS^{-1}, \quad \tilde{B} = SB, \quad \tilde{C} = CS^{-1}. $$
In this case $S$ is called a *state space similarity*.

Finally if in (1.1) we have $D = I_m$, then $F(\lambda)$ is invertible whenever $\lambda$ is not an eigenvalue of $A - BC$ and in that case

$$F(\lambda)^{-1} = I_m - C(\lambda I_N - A^\times)^{-1}B, \quad A^\times = A - BC. \quad (1.3)$$

### 2 The fundamental solution

In this section we prove Theorem 0.1 and present some results that will be used in Section 4 (a result on the invertibility of matrices $\Sigma_n$, in particular).

**Proof of Theorem 0.1.** First we shall show that equalities (0.7) and (0.8) yield the identity (0.14) for all $n \geq 0$. The statement is proved by induction. Indeed, for $n = 0$ it is true by assumption. Suppose (0.14) is true for $n = r$. Then using the expression for $\Sigma_{r+1}$ from (0.8) and identity (0.14) for $n = r$ we get

$$\alpha \Sigma_{r+1} - \Sigma_{r+1}\alpha^* =$$

$$= i\Lambda_r \Lambda_r^* + i\alpha^{-1}\Lambda_r \Lambda_r^*(\alpha^*)^{-1} + \Lambda_r J \Lambda_r^* (\alpha^*)^{-1} - \alpha^{-1}\Lambda_r J \Lambda_r^*. \quad (2.1)$$

The first relation in (0.8) and formula (2.1) yield (0.14) for $r = n + 1$ and thus for all $n \geq 0$.

The next equality will be crucial for our proof. Namely we shall show that for $0 \leq n \leq M - 1$ we have

$$W_{\alpha,\Lambda}(n + 1, \lambda)(I_{2m} - \frac{i}{\lambda} J) = (I_{2m} - \frac{i}{\lambda} S_n)W_{\alpha,\Lambda}(n, \lambda). \quad (2.2)$$

By (0.11) formula (2.2) is equivalent to the formula

$$\frac{1}{\lambda}(S_n - J) = (I_{2m} - \frac{i}{\lambda} S_n)\Lambda_n^* \Sigma_n^{-1}(\lambda I_N - \alpha)^{-1} \Lambda_n -$$

$$-\Lambda_{n+1}^* \Sigma_{n+1}^{-1}(\lambda I_N - \alpha)^{-1} \Lambda_{n+1}(I_{2m} - \frac{i}{\lambda} J). \quad (2.3)$$

Using the Taylor expansion of $(\lambda I_N - \alpha)^{-1}$ at infinity one shows that (2.3)
is in its turn equivalent to the set of equalities:

\[ S_n - J = \Lambda_n^* \Sigma_n^{-1} \Lambda_n - \Lambda_{n+1}^* \Sigma_{n+1}^{-1} \Lambda_{n+1}, \quad (2.4) \]

\[ \Lambda_{n+1}^* \Sigma_{n+1}^{-1} \alpha^p \Lambda_{n+1} - i \Lambda_{n+1}^* \Sigma_{n+1}^{-1} \alpha^{p-1} \Lambda_{n+1} J = \]

\[ = \Lambda_n^* \Sigma_n^{-1} \alpha^p \Lambda_n - i S_n \Lambda_n^* \Sigma_n^{-1} \alpha^{p-1} \Lambda_n \quad (p > 0). \quad (2.5) \]

Equality (2.4) is equivalent to (0.9). Taking into account the first relation in (0.8) we have \( \alpha \Lambda_{n+1} - i \Lambda_{n+1} J = \alpha \Lambda_n + \alpha^{-1} \Lambda_n \). Thus the equalities in (2.6) can be rewritten in the form \( K_n \alpha^{p-2} \Lambda_n = 0 \), where

\[ \Lambda_n = \Lambda_{n+1}^* \Sigma_{n+1}^{-1} (\alpha^2 + I_N) - \Lambda_n^* \Sigma_n^{-1} \alpha^2 + i S_n \Lambda_n^* \Sigma_n^{-1} \alpha. \quad (2.6) \]

Therefore, if we prove that \( K_n = 0 \), then equalities (2.5) will be proved, and so formula (2.2) will be proved too. Substitute (0.9) into (2.6), and again use the first relation in (0.8) to obtain

\[ K_n = \Lambda_{n+1}^* \Sigma_{n+1}^{-1} (\alpha^2 + I_N) - \Lambda_n^* \Sigma_n^{-1} \alpha^2 + i J \Lambda_n^* \Sigma_n^{-1} \alpha + \]

\[ + i \Lambda_n^* \Sigma_n^{-1} \Lambda_n \Lambda_n^* \Sigma_n^{-1} \alpha - i \Lambda_{n+1}^* \Sigma_{n+1}^{-1} (\Lambda_n + i \alpha^{-1} \Lambda_n J) \Lambda_n^* \Sigma_n^{-1} \alpha. \quad (2.7) \]

Now we use (0.14) to obtain \( i \Lambda_n \Lambda_n^* \Sigma_n^{-1} = \alpha - \Sigma_n \alpha^* \Sigma_n^{-1} \) and substitute this relation into (2.7). After easy transformations it follows that

\[ K_n = \Lambda_{n+1}^* \Sigma_{n+1}^{-1} \left( \alpha^{-1} \Sigma_n (\alpha^*)^{-1} + \Sigma_n + \alpha^{-1} \Lambda_n J \Lambda_n^* (\alpha^*)^{-1} \right) \alpha^* \Sigma_n^{-1} \alpha + \]

\[ + i J \Lambda_n^* \Sigma_n^{-1} \alpha - \Lambda_n^* \alpha^* \Sigma_n^{-1} \alpha. \quad (2.8) \]

In view of the second relation in (0.8) the first term on the right hand side of (2.8) equals \( \Lambda_{n+1}^* \alpha^* \Sigma_n^{-1} \alpha \) and we have

\[ K_n = (\Lambda_{n+1}^* + i J \Lambda_n^* (\alpha^*)^{-1} - \Lambda_n^*) \alpha^* \Sigma_n^{-1} \alpha. \quad (2.9) \]

By the first relation in (0.8) the equality \( K_n = 0 \) is now immediate, i.e., (2.2) is true.

Notice that equality (0.10) is valid for \( n = 0 \). Suppose that it is valid for \( n = r \). Then (0.4) and (0.10) yield

\[ W_{r+1}(\lambda) = (I_{2m} - \frac{i}{\lambda} S_r) W_{\alpha, \Lambda}(r, \lambda) \left( I_{2m} - \frac{i}{\lambda} J \right)^r W_{\alpha, \Lambda}(0, \lambda)^{-1}. \quad (2.10) \]
By (2.2) and (2.10) the validity of (0.10) for \( n = r + 1 \) easily follows, i.e., (0.10) is proved by induction.

Consider now the matrices \( S_n \) given by (0.9). It is easy to see that \( S_n = S_n^* \). Notice also that in view of (0.14) we have

\[
W_{\alpha, \Lambda}(r, \lambda)W_{\alpha, \Lambda}(r, \overline{\lambda})^* = I_{2m} \quad (r \geq 0),
\]

(2.11)

where \( \overline{\lambda} \) stands for complex conjugate for \( \lambda \). From (2.2) and (2.11) it follows that

\[
(I_{2m} - i\lambda^{-1}S_n)(I_{2m} + i\lambda^{-1}S_n) = \lambda^{-2}(\lambda^2 + 1)I_{2m}.
\]

Thus the equality \( S_n^* = S_n^{-1} \) holds, which finishes the proof of the theorem. □

The case when \( \pm i \not\in \sigma(\alpha) \) (\( \sigma \) means spectrum) is important for the study of the IHM model. Assume this condition is fulfilled, and put

\[
R_n = (I_N - i\alpha^{-1})^{-n}\Sigma_n(I_N + i(\alpha^*)^{-1})^{-n},
\]

(2.12)

\[
Q_n = (I_N + i\alpha^{-1})^{-n}\Sigma_n(I_N - i(\alpha^*)^{-1})^{-n}.
\]

(2.13)

The following proposition will be useful for formulating the conditions of invertibility of \( \Sigma_n \) in a somewhat different form then those in Proposition 0.2 (see Corollary 2.2 below).

**Proposition 2.1** Let the matrices \( \alpha \) (\( \det \alpha \neq 0 \)), \( \Sigma_0 = \Sigma_0^* \), and \( \Lambda_0 \) satisfy (0.7), and let the matrices \( \Sigma_n \) be given by (0.8). If \( i \not\in \sigma(\alpha) \), then the sequence of matrices \( \{R_n\} \) is well defined and non-decreasing. If \( -i \not\in \sigma(\alpha) \), then the sequence of matrices \( \{Q_n\} \) is well defined and non-increasing.

**Proof.** To prove that the sequence \( \{R_n\} \) is non-decreasing it will suffice to show that

\[
\Sigma_{n+1} - (I_N - i\alpha^{-1})\Sigma_n(I_N + i(\alpha^*)^{-1}) \geq 0.
\]

(2.14)

For this purpose notice that

\[
\Sigma_{n+1} - (I_N - i\alpha^{-1})\Sigma_n(I_N + i(\alpha^*)^{-1}) = \\
= \Sigma_{n+1} - \Sigma_n - \alpha^{-1}\Sigma_n(\alpha^*)^{-1} - i\alpha^{-1}(\alpha\Sigma_n - \Sigma_n\alpha^*)(\alpha^*)^{-1}.
\]

Hence, in view of (0.8) and (0.14) we get

\[
\Sigma_{n+1} - (I_N - i\alpha^{-1})\Sigma_n(I_N + i(\alpha^*)^{-1}) = \\
= \alpha^{-1}(\Lambda_n J\Lambda_n^* + \Lambda_n \Lambda_n^*)(\alpha^*)^{-1}.
\]

(2.15)
Since $J + I_{2m} \geq 0$, the inequality (2.14) is immediate from (2.13).

Similarly, from (0.8) and (0.14) we get

$$\Sigma_{n+1} - (I_N + i\alpha^{-1})\Sigma_n (I_N - i(\alpha^*)^{-1}) =$$

$$= \alpha^{-1}(\Lambda_n J\Lambda_n^* - \Lambda_n^*\Lambda_n)(\alpha^*)^{-1} \leq 0,$$  (2.16)

and so the sequence of matrices $\{Q_n\}$ is non-increasing. □

According to Proposition 2.1, when $i \notin \sigma(\alpha)$ and $\Sigma_0 > 0$, we have $R_n > 0$.

**Corollary 2.2** Let the conditions of Proposition 2.1 hold, and assume that $i \notin \sigma(\alpha)$ and $\Sigma_0 > 0$. Then we get $\Sigma_n > 0$ for all $n > 0$.

Partition the matrices $W_{\alpha,\Lambda}(r, \lambda)$ and $\Lambda_r$ into two $m$-column blocks each:

$$W_{\alpha,\Lambda}(r, \lambda) = [(W_{\alpha,\Lambda}(r, \lambda))_1, (W_{\alpha,\Lambda}(r, \lambda))_2], \quad \Lambda_r = [(\Lambda_r)_1, (\Lambda_r)_2].$$

The next lemma will be used in Section 4.

**Lemma 2.3** Let the matrices $\alpha (0, \pm i \notin \sigma(\alpha))$, $\Sigma_0 = \Sigma_0^*$, and $\Lambda_0$ satisfy (0.1), and let the matrices $\Sigma_n$ be given by (0.3). Then for $n \geq 0$ the following relations hold:

$$\begin{align*}
(W_{\alpha,\Lambda}(n, i))_1 &= (W_{\alpha,\Lambda}(n + 1, -i))_1 \times \\
&\times \left(I_m + 2(W_{\alpha,\Lambda}(n, i))^*\Lambda_n^*\Sigma_n^{-1}(\alpha^2 + I_N)^{-1}(\Lambda_n)_1\right), \quad (2.17) \\
(W_{\alpha,\Lambda}(n, -i))_2 &= (W_{\alpha,\Lambda}(n + 1, i))_2 \times \\
&\times \left(I_m - 2(W_{\alpha,\Lambda}(n, -i))^*\Lambda_n^*\Sigma_n^{-1}(\alpha^2 + I_N)^{-1}(\Lambda_n)_2\right). \quad (2.18)
\end{align*}$$

**Proof.** From the proof of Theorem 0.1, we know that $K_n = 0$, where $K_n$ is given by (2.6). In particular, we get

$$K_n\alpha^{-1}(\alpha^2 + I_N)^{-1}(\Lambda_n)_1 = 0, \quad K_n\alpha^{-1}(\alpha^2 + I_N)^{-1}(\Lambda_n)_2 = 0.$$  (2.19)

To prove (2.17) notice that $(\Lambda_{n+1})_1 = \alpha^{-1}(\alpha + iI_N)(\Lambda_n)_1$ and rewrite the first equality in (2.19) as

$$\Lambda_{n+1}^*\Sigma_{n+1}^{-1}\alpha + iI_N^{-1}(\Lambda_{n+1})_1 - \Lambda_n^*\Sigma_n^{-1}(\alpha - iI_N)^{-1}(\Lambda_n)_1 +$$

$$+ i(I_{2m} + S_n)\Lambda_n^*\Sigma_n^{-1}(\alpha^2 + I_N)^{-1}(\Lambda_n)_1 = 0.$$  (2.20)
Put $\lambda = -i$ in (2.2) and take into account (2.11) to derive

$$I_{2m} + S_n = 2(W_{\alpha, \Lambda}(n + 1, -i))_1 (W_{\alpha, \Lambda}(n, i))_1^*.$$  \hfill (2.21)

In view of definition (0.11) of $W_{\alpha, \Lambda}$, equality (2.17) follows from (2.20) and (2.21).

Putting $\lambda = i$ in (2.2) we get

$$I_{2m} - S_n = 2(W_{\alpha, \Lambda}(n + 1, i))_2 (W_{\alpha, \Lambda}(n, -i))_2^*.$$  \hfill (2.22)

Analogously to the proof of (2.17) we derive (2.18) from (2.22) and the second equality in (2.19). \hfill $\square$

**Remark 2.4** According to (2.21) and (2.22) the rank of the matrices $I_{2m} \pm S_n$ is less than or equal to $m$. Together with the formula (0.5) this implies that under the conditions of Lemma 2.3 we have $S_n = U_n J U_n^*$, where $U_n$ are unitary matrices and $J$ is defined in (0.1).

### 3 Weyl functions: direct and inverse problems

In this section we prove Theorems 0.4 and 0.5 and Proposition 0.2. At the end of the section a lemma on the case $i \not\in \sigma(\alpha)$ is treated too.

**Proof of Proposition 0.2** Suppose $f$ is an eigenvector of $\alpha$, that is, $\alpha f = cf$, $f \neq 0$. Then formula (0.12) yields the equality

$$i(\overline{c} - c) f^* f = f^* (\theta_1^* \theta_1 + \theta_2^* \theta_2) f \geq 0.$$  \hfill (3.1)

So $c \in \mathbb{C}_+$. Moreover, if $c = \overline{c}$, then according to (3.1) we have $\theta_1^* f = \theta_2^* f = 0$, and therefore $\alpha f = \alpha^* f$. It follows that

$$f^* \theta_1 = 0, \quad f^* (\alpha - c I_N) = 0 \quad (f \neq 0).$$  \hfill (3.2)

As $\{\alpha, \theta_1\}$ is a full range pair, so the pair $\{\alpha - c I_N, \theta_1\}$ is full range, which contradicts (3.2). This implies that $c \in \mathbb{C}_+$, i.e., $\sigma(\alpha) \subset \mathbb{C}_+$. 

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Recall that identity (0.14) was deduced in the proof of Theorem 0.1. Taking into account that \( \sigma(\alpha) \subset \mathbb{C}^+ \), identity (0.14) yields

\[
\Sigma_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha - \lambda I_N)^{-1}\Lambda_n \Lambda_n^*(\alpha^* - \lambda I_N)^{-1}d\lambda. \tag{3.3}
\]

Notice now that the pair \( \{\alpha, (I_N + i\alpha^{-1})^n\theta_1\} \) is full range and use (0.13), (3.3) to obtain \( \Sigma_n > 0 \) for all \( n \geq 0 \). \( \square \)

**Remark 3.1** In the same way as in the proof of Proposition 0.2 above the inclusion \( \sigma(\alpha) \subset \mathbb{C}^+ \) follows from the weaker condition that the pair \( \{\alpha, \Lambda_0\} \) is full range. However, the example \( N = 1, \alpha = i, \theta_1 = 0, \theta_2 \theta_2^* = 2 \), which yields \( \Sigma_n \equiv 0 \) for \( n > 0 \), shows that we have to require that the pair \( \{\alpha, \theta_1\} \) is full range in order to get \( \Sigma_n > 0 \). The full range condition on the pair \( \{\alpha, \Lambda_0\} \) is not enough for this conclusion.

Recall now Definition 0.3 of the admissible triple. Proposition 0.2 implies, in particular, that \( \det \alpha \neq 0 \) for the admissible triple and the spin sequences \( \{S_n\} \) determined by it are well defined for all \( n \geq 0 \). In other words the class \( \text{FG} \) is well defined.

**Proof of Theorem 0.4**. Let \( W_{\alpha,\Lambda}(n, \lambda) \) be given by (0.11). Write \( W_{\alpha,\Lambda}(0, \lambda) \) as

\[
W_{\alpha,\Lambda}(0, \lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}, \tag{3.4}
\]

where the \( m \times m \) matrix functions \( b(\lambda) \) and \( d(\lambda) \) are given by

\[
b(\lambda) = i\theta_1^*(\lambda I_N - \alpha)^{-1}\theta_2, \quad d(\lambda) = I_m + i\theta_2^*(\lambda I_N - \alpha)^{-1}\theta_2. \tag{3.5}
\]

We first prove that

\[
b(\lambda)d(\lambda)^{-1} = i\theta_1^*(\lambda I_N - \beta)^{-1}\theta_2. \tag{3.6}
\]

Using (1.3) in preliminaries, from (0.16) and (3.5) we obtain

\[
d(\lambda)^{-1} = I_m - i\theta_2^*(\lambda I_N - \beta)^{-1}\theta_2. \tag{3.7}
\]

The equalities (0.16), (3.5) and (3.7) yield

\[
b(\lambda)d(\lambda)^{-1} = i\theta_1^*(\lambda I_N - \alpha)^{-1}\theta_2 + i\theta_1^*(\lambda I_N - \alpha)^{-1}(\beta - \alpha)(\lambda I_N - \beta)^{-1}\theta_2. \tag{3.8}
\]
From (3.8) formula (3.6) follows.

Let $\varphi$ be defined by (0.15), and thus by virtue of (3.6) we have

$$\varphi(\lambda) = b(\lambda)d(\lambda)^{-1}. \quad (3.9)$$

By (3.4), (3.9) and the representation (0.10) of the fundamental solution we get

$$W_n(\lambda) \left[ \begin{array}{c} \varphi(\lambda) \\ I_m \end{array} \right] = \left(\frac{\lambda + i}{\lambda}\right)^n W_{n,\Lambda}(n, \lambda) \left[ \begin{array}{c} 0 \\ d(\lambda)^{-1} \end{array} \right]. \quad (3.10)$$

Notice also that (0.14) yields a more general formula than (2.11), namely:

$$W_{n,\Lambda}(n, \lambda)^* W_{n,\Lambda}(n, \lambda) = I_{2m} - i(\lambda - \bar{\lambda})\Lambda_n^*(\lambda I_N - \alpha)^{-1}\Sigma_n^{-1}(\lambda I_N - \alpha)^{-1}\Lambda_n. \quad (3.11)$$

As the second term in the right hand side of (3.11) is nonpositive, it follows from formula (3.11) that

$$W_{n,\Lambda}(n, \lambda)^* W_{n,\Lambda}(n, \lambda) \leq I_{2m} \quad (\lambda \in \mathbb{C}_-). \quad (3.12)$$

Taking into account (3.10) and (3.12) we obtain (0.6), i.e., $\varphi$ is a Weyl function.

It remains only to prove the uniqueness of the Weyl function. Let us first show that for some $M > 0$ and all $n \geq 0$ we have the inequality

$$f^*(I_N - i(\alpha)^{-1})^n\Sigma_n^{-1}(I_N + i\alpha^{-1})^n f \leq Mf^*f, \quad f \in L, \quad (3.13)$$

where

$$L := \text{span}_{\lambda \notin \sigma(\alpha)} \text{Im}(\lambda I_N - \alpha)^{-1}\theta_1.$$ 

In view of (0.13) formula (3.11) yields

$$\theta_1^*(\lambda I_N - \alpha)^{-1} \left( I_N - i(\alpha)^{-1})^n\Sigma_n^{-1} \right. \times \left. (I_N + i\alpha^{-1})^n(\lambda I_N - \alpha)^{-1}\theta_1 \leq \frac{i}{\lambda - \lambda}I_m. \quad (3.14)$$

Now to get (3.13) from (3.14) we note that $\text{span}_{\lambda \notin \sigma(\alpha)} \text{Im}(\lambda I_N - \alpha)^{-1}\theta_1$ coincides with the same span when $\lambda$ runs over an $\varepsilon$-neighbourhood $O_\varepsilon$ of any $\lambda_0 \notin \sigma_\alpha$, for any sufficiently small $\varepsilon > 0$. 

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By (3.11) and (3.13) we can choose $M_1 > 0$ such that we have
\[
[I_m 0]W_{\alpha,\Lambda}(n, \lambda)^*W_{\alpha,\Lambda}(n, \lambda) \begin{bmatrix} I_m \\ 0 \end{bmatrix} \geq \frac{1}{2} \quad \text{for all } |\lambda| > M_1.
\]
Without loss of generality we may assume that $M_1$ is large enough in order that $M_1 > ||\alpha||$ and $a(\lambda)$ is invertible for $|\lambda| > M_1$. Then, taking into account (0.10) and (3.15), we obtain
\[
\sum_{n=0}^{\infty} [I_m (c(\lambda)a(\lambda)^{-1})^*] \times \times W_{\alpha}(\lambda)^*W_{\alpha}(\lambda) \begin{bmatrix} I_m \\ c(\lambda)a(\lambda)^{-1} \end{bmatrix} > \frac{r}{2}(a(\lambda)^{-1})^*a(\lambda)^{-1}
\]
for all $\lambda$ in the domain
\[
D = \{\lambda : |\lambda| > M_1, \exists \lambda < -\frac{1}{2}\}.
\]
In other words, for $\lambda \in D$ we have
\[
\sum_{n=0}^{\infty} f^*W_{\alpha}(\lambda)^*W_{\alpha}(\lambda)f < \infty \quad (f \in L_2), \quad (3.17)
\]
where
\[
L_1 := \text{Im} \begin{bmatrix} I_m \\ c(\lambda)a(\lambda)^{-1} \end{bmatrix}.
\]
Suppose now that $\varphi$ and $\tilde{\varphi}$ are Weyl functions of (0.4) and that for some fixed $\lambda_0 \in D$ we have $\tilde{\varphi}(\lambda_0) \neq \varphi(\lambda_0)$. Put
\[
L_2 = \text{Im} \begin{bmatrix} \varphi(\lambda_0) \\ I_m \end{bmatrix} + \text{Im} \begin{bmatrix} \tilde{\varphi}(\lambda_0) \\ I_m \end{bmatrix}.
\]
According to the definition of the Weyl function we have
\[
\sum_{n=0}^{\infty} f^*W_{\alpha}(\lambda)^*W_{\alpha}(\lambda)f < \infty \quad (f \in L_2). \quad (3.18)
\]
As dim $L_1 = m$ and dim $L_2 > m$, there is a non-zero vector $f$ such that $f \in (L_1 \cap L_2)$, which contradicts (3.17) and (3.18). □

For the proof of Theorem 0.5 we shall use the following lemma which is of independent interest.
Lemma 3.2 A strictly proper rational $m \times m$ matrix function $\varphi$ admits a minimal realization of the form

$$\varphi(\lambda) = i\theta_1^*(\lambda I_N - \beta)^{-1}\theta_2,$$  \hspace{1cm} (3.19)

such that

$$\beta - \beta^* = i(\theta_1\theta_1^* - \theta_2\theta_2^*).$$  \hspace{1cm} (3.20)

Proof. We may assume that $\varphi$ is given by the minimal realization (0.17). First let us show that equation (0.18) has a unique solution $X > 0$.

The minimality of the realization (0.17) means that the pair $\{\theta_1^*, \gamma\}$ is observable and the pair $\{\gamma, \theta_2\}$ is controllable. Notice that $\text{Im} \vartheta \supseteq \text{Im} \vartheta \vartheta^*$ and $f^*\vartheta \vartheta^* = 0$ yields $f^*\vartheta = 0$, i.e., $\text{Im} \vartheta = \text{Im} \vartheta \vartheta^*$. Hence the pair $\{\gamma, \theta_2\}$ is controllable too. Therefore the pair $\{\vartheta_2\vartheta_2^*, i\gamma^*\}$ is observable. The pair $\{i\gamma^*, \vartheta_1\}$ is controllable and hence c-stabilizable, that is, there exists a matrix $K$ such that $i\gamma^* + \vartheta_1K$ has all its eigenvalues in the open left half-plane. But then we can use Theorem 16.3.3 [24] (see also [20]) to show that the equation (0.18) has a unique non-negative solution $X$ and that this solution $X$ is positive definite.

Next, let $\theta_1, \theta_2, \beta$ be defined by (0.19). From (0.18) and (0.19) it follows that (3.20) is satisfied.

According to (0.17) and (0.19) the function $\varphi$ is also given by the realization (3.19). Moreover, as the realization (0.17) is minimal, the same is true for the realization (3.19).\Box

Proof of Theorem 0.5. Let $\varphi$ be a strictly proper rational $m \times m$ matrix function. Let $\theta_1, \theta_2, \beta$ be as in Lemma 3.2 and put $\alpha = \beta + i\theta_2\theta_2^*$. Then the triple $\alpha, \theta_1$, and $\theta_2$ satisfies (0.12). Furthermore, the pairs $\{\beta, \theta_2\}$ and $\{\beta^*, \theta_1\}$ are full range. Since $\alpha = \beta + i\theta_2\theta_2^*$, it is immediate that the pair $\{\alpha, \theta_2\}$ is full range (see Section 11). By (0.12) we have $\beta^* = \alpha^* + i\theta_2\theta_2^* = \alpha - i\theta_1\theta_1^*$. Hence, as $\{\beta^*, \theta_1\}$ is a full range pair, so the pair $\{\alpha, \theta_1\}$ is full range too. Therefore the triple $\alpha, \theta_1$, and $\theta_2$ is admissible. From Theorem 0.4 it follows now that the function $\{S_n\}$ determined by the admissible triple $\alpha, \theta_1$ and $\theta_2$ is indeed a solution of the inverse problem.

Let us prove now the uniqueness of the solution of the inverse problem. Suppose that there is system (0.4) with another spin sequence $\{\tilde{S}_n\}$, given by
the admissible triple $\tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2$, and with the same Weyl function $\varphi$. According to Theorem 0.4 we have another realization for $\varphi$, namely

$$
\varphi(\lambda) = i\tilde{\theta}_1^*(\lambda I_N - \tilde{\beta})^{-1}\tilde{\theta}_2, \quad \tilde{\beta} = \tilde{\alpha} - i\tilde{\theta}_2\tilde{\theta}_2^*. \quad (3.21)
$$

As the pairs $\{\tilde{\alpha}, \tilde{\theta}_1\}$ and $\{\tilde{\alpha}, \tilde{\theta}_2\}$ are controllable, and $\tilde{\beta} = \tilde{\alpha} - i\tilde{\theta}_2\tilde{\theta}_2^*$, it follows that the pairs $\{\tilde{\beta}, \tilde{\theta}_2\}$ and $\{\tilde{\beta}^*, \tilde{\theta}_1\}$ are also controllable. Thus the realization (3.21) is minimal and $\tilde{N} = N$. Moreover, there is (see Section 1) a state space similarity transforming the realization (0.15) into (3.21), that is, there exists an invertible $N \times N$ matrix $S$ such that

$$
\tilde{\beta} = S\beta S^{-1}, \quad \tilde{\theta}_2 = S\theta_2, \quad \tilde{\theta}_1^* = \theta_1^*S^{-1}. \quad (3.22)
$$

Recall that $\tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2$ is an admissible triple, and therefore we have

$$
\tilde{\beta} - \tilde{\beta}^* = i(\tilde{\theta}_1^* - \tilde{\theta}_2^*). \quad (3.23)
$$

From (3.22) and (3.23) it follows that $Z = S^{-1}(S^*)^{-1}$ satisfies

$$
\beta Z - Z\beta^* = i(\theta_1^*Z - \theta_2^*Z), \quad Z > 0. \quad (3.24)
$$

Completely similar to the uniqueness of $X > 0$ in (0.18) one obtains the uniqueness of the solution $Z > 0$ of (3.24). Compare now (3.20) and (3.24) to see that $Z = I_N$ and thus $S$ is unitary. In view of this we have

$$
\tilde{\alpha} = S\alpha S^*, \quad \tilde{\theta}_2 = S\theta_2, \quad \tilde{\theta}_1 = S\theta_1. \quad (3.25)
$$

The last transformation does not change the spin sequence, i.e., $\{S_n\} = \{\tilde{S}_n\}$. □

The case when $i \notin \sigma(\alpha)$ is important in the next section. We shall use the acronym $\tilde{\text{FG}}$ to denote the class of spin sequences $\{S_n\}$ determined by the triples $\alpha, \theta_1, \theta_2$, with $\alpha$ an $N \times N$ non-singular matrix and $\theta_1, \theta_2$ of size $N \times m$, satisfying the identity (0.12) and the additional special condition $i \notin \sigma(\alpha)$. Notice (see the beginning of the proof of Proposition 0.2) that (0.12) implies that $\sigma(\alpha) \subset \mathbb{C}_{\text{+}}$. The next lemma shows that without loss of generality we can also require that $\{\alpha, \theta_1\}$ and $\{\alpha, \theta_2\}$ are full range pairs, i.e., $\tilde{\text{FG}} \subseteq \text{FG}$. 18
Lemma 3.3 Assume that the spin sequence \( \{S_n\} \) belongs \( \widetilde{FG} \). Then it can be determined by a triple \( \alpha, \theta_1, \theta_2 \) such that \( \alpha \) is non-singular, (0.12) holds, \( i \not\in \sigma(\alpha) \), and the pairs \( \{\alpha, \theta_1\} \) and \( \{\alpha, \theta_2\} \) are full range.

Proof. Let \( N \) denote the minimal order of \( \alpha \) \( (0, i \not\in \sigma(\alpha)) \) in the set of triples that satisfy (0.12) and determine the given spin sequence \( \{S_n\} \). Suppose the \( N \times N \) matrix \( \tilde{\alpha} \) and the \( N \times m \) matrices \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) form such a triple but the pair \( \{\tilde{\alpha}, \tilde{\theta}_2\} \) is not full range. Put

\[
\tilde{L}_0 := \operatorname{span} \bigcup_{k=0}^{\infty} \operatorname{Im} \tilde{\alpha}^k \tilde{\theta}_2, \quad N_0 := \dim \tilde{L}_0
\]

and choose a unitary matrix \( q \) that maps \( \tilde{L}_0 \) onto the \( L_0 := \operatorname{Im} [I_{N_0} \ 0]^* \).

Then we have

\[
\alpha := q \tilde{\alpha} q^*, \quad \theta_1 := q \tilde{\theta}_1 = \begin{bmatrix} \tilde{\theta}_1 \\ \kappa \end{bmatrix}, \quad \theta_2 := q \tilde{\theta}_2 = \begin{bmatrix} \tilde{\theta}_2 \\ 0 \end{bmatrix},
\]

where the \( N_0 \times N_0 \) matrix \( \tilde{\alpha} \) \( (0, i \not\in \sigma(\tilde{\alpha})) \) and the \( N_0 \times m \) matrices \( \tilde{\theta}_1, \tilde{\theta}_2 \) form a triple which satisfies (0.12) and determines \( \{S_n\} \). To show this we need to make some preparations. As \( \tilde{L}_0 \) is an invariant subspace of \( \tilde{\alpha} \), so \( L_0 \) is an invariant subspace of \( \alpha \), and thus \( \alpha \) has the block triangular form given in (3.25). Moreover in view of the inclusion \( \operatorname{Im} \theta_2 \subseteq \tilde{L}_0 \), we have \( \operatorname{Im} \theta_2 \subseteq L_0 \), i.e., \( \theta_2 \) has the block form given in (3.25). Taking into account that \( q \) is unitary, \( 0, i \not\in \sigma(\tilde{\alpha}) \) and that \( \tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2 \) satisfy (0.12) and determine \( \{S_n\} \), we see that \( 0, i \not\in \sigma(\alpha) \) and that \( \alpha, \theta_1, \theta_2 \) satisfy (0.12) and determine \( \{S_n\} \) too. So in view of (3.25) we have \( 0, i \not\in \sigma(\alpha) \) and the triple \( \tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2 \) satisfies (0.12) as well. Now use the matrices \( Q_n \) defined in (2.13) to rewrite \( \Lambda_n^* \Sigma_n^{-1} \Lambda_n \) as a \( 2 \times 2 \) block matrix with block of size \( m \times m \). This yields

\[
\Lambda_n^* \Sigma_n^{-1} \Lambda_n = \{(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{kj}\}_{k,j=1}^2
\]
with blocks

\[
\begin{align*}
(A_n^*\Sigma_n^{-1}A_n)_{11} &= \theta_1^*Q_n^{-1}\theta_1, \\
(A_n^*\Sigma_n^{-1}A_n)_{12} &= \theta_1^*Q_n^{-1}(I_N + i\alpha^{-1})^{-n}(I_N - i\alpha^{-1})^{n}\theta_2, \\
(A_n^*\Sigma_n^{-1}A_n)_{21} &= \theta_2^*(I_N + i(\alpha^*)^{-1})^{n}(I_N - i(\alpha^*)^{-1})^{-n}Q_n^{-1}\theta_1, \\
(A_n^*\Sigma_n^{-1}A_n)_{22} &= \theta_2^*(I_N + i(\alpha^*)^{-1})^{n}(I_N - i(\alpha^*)^{-1})^{-n}Q_n^{-1} \\
&\times(I_N + i\alpha^{-1})^{-n}(I_N - i\alpha^{-1})^{n}\theta_2.
\end{align*}
\]

Partition \(Q_{n+1} - Q_n\) into four blocks: \(Q_{n+1} - Q_n = \{\chi_{kj}\}_{k,j=1}^{2}\), where \(\chi_{11}\) is an \(N_0 \times N_0\) block. In view of (3.25), (3.26), and (3.27) we obtain

\[
\begin{align*}
\chi_{21} = 0, \quad \chi_{12} = 0, \quad \chi_{22} = 0, \quad \chi_{11} = -2\alpha^{-1}(I_{N_0} + i\alpha^{-1})^{-n-1} \\
\times(I_{N_0} - i\alpha^{-1})^{n}\theta_2\theta_2^*(I_{N_0} + i(\alpha^*)^{-1})^{n}(I_{N_0} - i(\alpha^*)^{-1})^{-n-1}(\alpha^*)^{-1}.
\end{align*}
\]

Denote by \(\tilde{\Lambda}_n, \tilde{\Sigma}_n, \tilde{Q}_n, \tilde{S}_n\) et cetera the matrices generated by the triple \(\tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2\). One can see that \(\chi_{11} = \tilde{Q}_{n+1} - \tilde{Q}_n\). Taking into account \(Q_0 = I_N\) we obtain that the matrices \(Q_n\) are block diagonal: \(Q_n = \text{diag} \{\tilde{Q}_n, I_{N-N_0}\}\).

Now according to (3.25), (3.26) it follows that

\[
\Lambda_n^*\Sigma_n^{-1}\Lambda_n - \tilde{\Lambda}_n^*\tilde{\Sigma}_n^{-1}\tilde{\Lambda}_n = \begin{bmatrix} \kappa^*\kappa & 0 \\
0 & 0 \end{bmatrix}.
\]

By (3.27) and (3.28) we get \(S_n = \tilde{S}_n\), i.e., the triple \(\tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2\) determines the spin sequence \(\{S_n\}\). This contradicts the assumption that \(N\) is minimal and therefore the pair \(\{\tilde{\alpha}, \tilde{\theta}_2\}\) should be full range.

In the same way we shall show that the pair \(\{\tilde{\alpha}, \tilde{\theta}_1\}\) is full range too. Indeed, suppose \(\{\tilde{\alpha}, \tilde{\theta}_1\}\) is not full range. Put now

\[
\hat{L}_0 := \text{span} \bigcup_{k=0}^{\infty} \text{Im} \tilde{\alpha}^k\tilde{\theta}_1, \quad N_0 := \dim \hat{L}_0
\]

and choose a unitary matrix \(q\) that maps \(\hat{L}_0\) onto the \(L_0 := \text{Im} [I_{N_0} 0]^*\). Then similar to the previous case we obtain

\[
\alpha := q\tilde{\alpha}q^* = \begin{bmatrix} \tilde{\alpha} & \alpha_{12} \\
0 & \alpha_{22} \end{bmatrix}, \quad \theta_1 := q\tilde{\theta}_1 = \begin{bmatrix} \tilde{\theta}_1 \\
0 \end{bmatrix}, \quad \theta_2 := q\tilde{\theta}_2 = \begin{bmatrix} \tilde{\theta}_2 \\
\kappa \end{bmatrix},
\]

(3.28)
where the \( N_0 \times N_0 \) matrix \( \tilde{\alpha} (0, i \not\in \sigma(\tilde{\alpha})) \) and the \( N_0 \times m \) matrices \( \tilde{\theta}_1, \tilde{\theta}_2 \) form a triple which satisfies \((1.12)\). To show that the triple \( \tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2 \) determines \( S_n \) we shall use the fact that \( \alpha, \theta_1, \theta_2 \) determines \( S_n \) and rewrite \( \Lambda_n^* \Sigma_n^{-1} \Lambda_n \) as a \( 2 \times 2 \) block matrix with blocks of size \( m \times m \) as follows:

\[
\Lambda_n^* \Sigma_n^{-1} \Lambda_n = \{(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{kj}\}_{k,j=1}^2
\]

with

\[
\begin{align*}
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{22} &= \theta_2^* R_n^{-1} \theta_2, \\
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{21} &= \theta_2^* R_n^{-1} (I_N - i \alpha^{-1})^{-n} (I_N + i \alpha^{-1})^n \theta_1, \\
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{12} &= \theta_1^* (I_N - i (\alpha^*)^{-1})^{n} (I_N + i (\alpha^*)^{-1})^{-n} R_n^{-1} \theta_2, \\
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{11} &= \theta_1^* (I_N - i (\alpha^*)^{-1})^{n} (I_N + i (\alpha^*)^{-1})^{-n} R_n^{-1} \times \\
&\quad (I_N - i \alpha^{-1})^{-n} (I_N + i \alpha^{-1})^n \theta_1.
\end{align*}
\]

Here the matrices \( R_n \) are given by \((2.12)\). Partition now \( R_{n+1} - R_n = \{\chi_{kj}\}_{k,j=1}^2 \), where \( \chi_{11} \) is an \( N_0 \times N_0 \) block. In view of \((2.15)\) and \((3.28)\) we obtain

\[
\begin{align*}
\chi_{21} &= 0, \quad \chi_{12} = 0, \quad \chi_{22} = 0, \quad \chi_{11} = 2 \tilde{\alpha}^{-1} (I_{N_0} - i \tilde{\alpha}^{-1})^{-n-1} \times \\
&\quad (I_{N_0} + i \tilde{\alpha}^{-1})^n \tilde{\theta}_1^* \tilde{\theta}_1 (I_{N_0} - i (\tilde{\alpha}^*)^{-1})^{n} (I_{N_0} + i (\tilde{\alpha}^*)^{-1})^{-n-1} (\tilde{\alpha}^*)^{-1}.
\end{align*}
\]

Denote by \( \tilde{\Lambda}_n, \tilde{\Sigma}_n, \tilde{R}_n, \tilde{S}_n \) et cetera the matrices generated by the triple \( \tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2 \). One can see that \( \chi_{11} = \tilde{R}_{n+1} - \tilde{R}_n \). Taking into account \( R_0 = I_N \) we obtain that the matrices \( R_n \) are block diagonal: \( R_n = \text{diag} \{\tilde{R}_n, I_{N - N_0}\} \).

Now according to \((3.28)\), \((3.29)\) it follows that

\[
\Lambda_n^* \Sigma_n^{-1} \Lambda_n - \Lambda_n^* \tilde{\Sigma}_n^{-1} \tilde{\Lambda}_n = \begin{bmatrix} 0 & 0 \\ 0 & \kappa^* \kappa \end{bmatrix}.
\]

By \((0.9)\) and \((3.30)\) we get \( S_n = \tilde{S}_n \), i.e., the triple \( \tilde{\alpha}, \tilde{\theta}_1, \tilde{\theta}_2 \) determines the spin sequence \( \{S_n\} \). So the pair \( \{\tilde{\alpha}, \tilde{\theta}_1\} \) should also be full range. \( \square \)

Finally, from the proof of Theorem \((0.4)\) we have the following corollary.

**Corollary 3.4** Let the parameter matrices \( \alpha (0, i \not\in \sigma(\alpha)) \), \( \Sigma_0(0) > 0 \) and \( \Lambda_0(0) \) satisfy the identity \((0.7)\). Then the Weyl function \( \varphi \) of the system determined by these matrices is given by the formula

\[
\varphi(\lambda) = i \theta_1^* \Sigma_0^{-1} (\lambda I_N - \tilde{\beta})^{-1} \theta_2, \quad \tilde{\beta} = \alpha - i \theta_2 \theta_2^* \Sigma_0^{-1}.
\]
This corollary is proved by transforming the matrices $\alpha$, $\Sigma_0(0)$ and $\Lambda_0(0)$ into the equivalent set $\Sigma_0^{-\frac{1}{2}}\alpha\Sigma_0^\frac{1}{2}$, $I_N$, $\Sigma_0^{-\frac{1}{2}}\Lambda_0$.

4 Isotropic Heisenberg magnet

Explicit solutions of the discrete integrable nonlinear equations form an interesting and actively studied domain (see references in [11 11 12 22 23 25 27 37]). To study the IHM model we insert an additional variable $t$ in our notations: $\Lambda_n(t)$, $\Sigma_n(t)$, $S_n(t)$, $W_{\alpha,\Lambda}(n, t, \lambda)$, $\varphi(t, \lambda)$ and so on. Notice that the order $N$ and the parameter matrix $\alpha$ do not depend on $t$, and the dependence on $t$ of the other matrix functions is defined by the equations

$$\frac{d\Lambda_0}{dt} = -2\left((\alpha - iI_N)^{-1}\Lambda_0 P_+ + (\alpha + iI_N)^{-1}\Lambda_0 P_-\right),$$

$$P_\pm = \frac{1}{2}(I_{2m} \pm J),$$

$$\frac{d\Sigma_0}{dt} = -\left((\alpha - iI_N)^{-1}\Sigma_0(t) + (\alpha + iI_N)^{-1}\Sigma_0(t) + \Sigma_0(t)(\alpha^* + iI_N)^{-1} + \Sigma_0(t)(\alpha^* - iI_N)^{-1} + 2(\alpha^2 + I_N)^{-1}(\alpha\Lambda_0(t)J\Lambda_0(t)^* + \Lambda_0(t)J\Lambda_0(t)^*\alpha^*)(\alpha^2 + I_N)^{-1}\right).$$

We assume that the parameter matrices $\alpha$, $\Sigma_0(0)$ and $\Lambda_0(0)$ satisfy the identity

$$\alpha\Sigma_0(t) - \Sigma_0(t)\alpha^* = i\Lambda_0(t)\Lambda_0(t)^*$$

at $t = 0$. Then according to [11 11 12] the identity [13] holds for all $t$. (This result can be obtained by differentiating both sides of [13].)

**Theorem 4.1** Assume the parameter matrices $\alpha$ ($0, i \not\in \sigma(\alpha)$), $\Sigma_0(0) > 0$ and $\Lambda_0(0)$ satisfy the identity

$$\alpha\Sigma_0(0) - \Sigma_0(0)\alpha^* = i\Lambda_0(0)\Lambda_0(0)^*.$$

Define $\Sigma_0(t)$ and $\Lambda_0(t)$ by equations [4.1] and [4.2]. Then $\Sigma_0(t) > 0$ on some interval $-\varepsilon < t < \varepsilon$, and the sequence $\{S_n(t)\}$ given by [0.9] and [0.8] belongs to $\tilde{FG}$ for each $t$ from this interval. Moreover, $\{S_n(t)\}$ ($-\varepsilon < t < \varepsilon$) satisfies the IHM equations [0.23], [0.24].
Proof. As $\alpha$ does not depend on $t$ and (4.3) is true, it is immediate that 
\[ \{ S_n(t) \} \subset \mathbb{F} G. \]

Similar to the cases treated in [31, 32] we shall successively obtain the 
derivatives \( \frac{d}{dt} \Lambda_n, \frac{d}{dt} \Sigma_n, \frac{d}{dt} (\Lambda^* \Sigma_n^{-1}) \), and \( \frac{d}{dt} W_{\alpha, \Lambda}(n, t, \lambda) \), and use the expressions 
for these derivatives to derive the zero curvature equation (4.20), which is 
equivalent to (4.23), (4.24); see [12]. In view of (4.13) and (4.1) we have
\[ \Lambda_n(t) = [(I_N + i\alpha^{-1})^n e^{-2t(\alpha-iI_N)^{-1}} \theta_1 (I_N - i\alpha^{-1})^n e^{-2t(\alpha+iI_N)^{-1}} \theta_2], \quad (4.4) \]
where \( \theta_p := (\Lambda_0(0))_p \) \( (p = 1, 2) \). Hence it follows that
\[ \frac{d\Lambda_n}{dt} = -2((\alpha - iI_N)^{-1} \Lambda_n P_+ + (\alpha + iI_N)^{-1} \Lambda_n P_-). \quad (4.5) \]

Now we shall show by induction that
\[ \frac{d \Sigma_n}{dt} = -\left( (\alpha - iI_N)^{-1} \Sigma_n(t) + (\alpha + iI_N)^{-1} \Sigma_n(t) + \Sigma_n(t)(\alpha^* + iI_N)^{-1} + \Sigma_n(t)(\alpha^* - iI_N)^{-1} + 2(\alpha^2 + I_N)^{-1} (\alpha \Lambda_n(t) J \Lambda_n(t)^* + \Lambda_n(t) J \Lambda_n(t)^* \alpha^*) ((\alpha^*)^2 + I_N)^{-1} \right). \quad (4.6) \]

By (4.2) formula (4.6) is true for \( n = 0 \). Suppose it is true for \( n = r \). Then, 
taking into account the second relation in (0.8) we obtain
\[ \frac{d \Sigma_{r+1}}{dt} = -\left( (\alpha - iI_N)^{-1} \Sigma_{r+1}(t) - (\alpha - iI_N)^{-1} \alpha^{-1} \Lambda_r(t) J \Lambda_r(t)^* (\alpha^*)^{-1} + + (\alpha + iI_N)^{-1} \Sigma_{r+1}(t) - (\alpha + iI_N)^{-1} \alpha^{-1} \Lambda_r(t) J \Lambda_r(t)^* (\alpha^*)^{-1} + + \Sigma_{r+1}(t)(\alpha^* + iI_N)^{-1} - \alpha^{-1} \Lambda_r(t) J \Lambda_r(t)^* (\alpha^*)^{-1} (\alpha^* + iI_N)^{-1} + + \Sigma_{r+1}(t)(\alpha^* - iI_N)^{-1} - \alpha^{-1} \Lambda_r(t) J \Lambda_r(t)^* (\alpha^*)^{-1} (\alpha^* - iI_N)^{-1} + + 2(\alpha^2 + I_N)^{-1} ((\alpha + \alpha^{-1}) \Lambda_r(t) J \Lambda_r(t)^* + \Lambda_r(t) J \Lambda_r(t)^* \times \times (\alpha^* + (\alpha^*)^{-1}) ((\alpha^*)^2 + I_N)^{-1} \right) + \frac{d}{dt} (\alpha^{-1} \Lambda_r(t) J \Lambda_r(t)^* (\alpha^*)^{-1}). \quad (4.7) \]
In view of \((4.5)\) we easily calculate that
\[
C_1(t) := \frac{d}{dt} \left( \alpha^{-1} \Lambda_{r}(t) J \Lambda_{r}(t)^* (\alpha^*)^{-1} \right) + (4.8)
\]
\[
+ (\alpha - i I_N)^{-1} \alpha^{-1} \Lambda_{r}(t) J \Lambda_{r}(t)^* (\alpha^*)^{-1} + (\alpha + i I_N)^{-1} \alpha^{-1} \Lambda_{r}(t) J \times \\
\times \Lambda_{r}(t)^*(\alpha^*)^{-1} + \alpha^{-1} \Lambda_{r}(t) J \Lambda_{r}(t)^*(\alpha^*)^{-1} (\alpha^* + i I_N)^{-1} + \alpha^{-1} \times \\
\times \Lambda_{r}(t) J \Lambda_{r}(t)^*(\alpha^*)^{-1} (\alpha^* - i I_N)^{-1} = -(\alpha - i I_N)^{-1} \alpha^{-1} \Lambda_{r}(t) \times \\
\times \Lambda_{r}(t)^*(\alpha^*)^{-1} + (\alpha + i I_N)^{-1} \alpha^{-1} \Lambda_{r}(t) \Lambda_{r}(t)^*(\alpha^*)^{-1} - \alpha^{-1} \Lambda_{r}(t) \times \\
\times \Lambda_{r}(t)^*(\alpha^*)^{-1} (\alpha^* + i I_N)^{-1} + \alpha^{-1} \Lambda_{r}(t) \Lambda_{r}(t)^*(\alpha^*)^{-1} (\alpha^* - i I_N)^{-1}.
\]

Notice that \((\alpha - i I_N)^{-1} - (\alpha + i I_N)^{-1} = 2i(\alpha^2 + I_N)^{-1}\). Therefore we can rewrite \((4.8)\) as
\[
C_1(t) = \begin{align*}
2i & \left( \alpha^{-1} \Lambda_{r}(t) \Lambda_{r}(t)^*(\alpha^*)^{-1} (\alpha^*)^2 + I_N \right)^{-1} - \\
& - (\alpha^2 + I_N)^{-1} \alpha^{-1} \Lambda_{r}(t) \Lambda_{r}(t)^*(\alpha^*)^{-1}.
\end{align*}
\]
Notice also that
\[
C_2(t) := 2(\alpha^2 + I_N)^{-1} \left( (\alpha + \alpha^{-1}) \Lambda_{r}(t) J \Lambda_{r}(t)^* + \Lambda_{r}(t) J \Lambda_{r}(t)^* \times \\
\times (\alpha^* + (\alpha^*)^{-1}) (\alpha^*)^2 + I_N \right)^{-1} = 2(\alpha^{-1} \Lambda_{r}(t) J \Lambda_{r}(t)^* \times \\
\times (\alpha^*)^2 + I_N)^{-1} + (\alpha^2 + I_N)^{-1} \Lambda_{r}(t) \Lambda_{r}(t)^*(\alpha^*)^{-1}.
\]
Using the first relation in \((4.8)\) and equalities \((4.3), (4.10)\) we get
\[
C_2(t) - C_1(t) = 2(\alpha^{-1} \Lambda_{r}(t) J \Lambda_{r+1}(t)^* (\alpha^*)^2 + I_N)^{-1} + \\
+ (\alpha^2 + I_N)^{-1} \Lambda_{r+1}(t) J \Lambda_{r}(t)^*(\alpha^*)^{-1}) = 2(\alpha^2 + I_N)^{-1} \times \\
\times (\alpha \Lambda_{r+1}(t) J \Lambda_{r+1}(t)^* + \Lambda_{r+1}(t) J \Lambda_{r+1}(t)^* \alpha^*) (\alpha^*)^2 + I_N)^{-1}.
\]
From \((4.7)\) and \((4.11)\) it follows that \((4.6)\) is valid for \(n = r + 1\) and so for all \(n > 0\).

Taking into account \((4.5)\) and \((4.6)\) we can obtain the equation
\[
\frac{d}{dt} (\Lambda_n(t)^* \Sigma_n(t)^{-1}) = H_n^+(t) \Lambda_n(t)^* \Sigma_n(t)^{-1} (\alpha - i I_N)^{-1} + \\
+ H_n^-(t) \Lambda_n(t)^* \Sigma_n(t)^{-1} (\alpha + i I_N)^{-1},
\]
where
\[
H_n^+(t) = 2W_{\alpha,\Lambda}(n, t, i) P_+ W_{\alpha,\Lambda}(n, t, -i)^*,
\]
\[
H_n^-(t) = 2W_{\alpha,\Lambda}(n, t, -i) P_- W_{\alpha,\Lambda}(n, t, i)^*.
\]
Indeed, by (4.5) we have
\[
\frac{d}{dt} \left( \Lambda_n(t) \Sigma_n(t)^{-1} \right) = - \left( 2P_+ \Lambda_n(t)^* (\alpha^* + iI_N)^{-1} + 
+ 2P_- \Lambda_n(t)^* (\alpha^* - iI_N)^{-1} + \Sigma_n(t)^{-1} \frac{d\Sigma_n}{dt}(t) \right) \Sigma_n(t)^{-1}.
\] (4.15)

Identity (0.14) yields
\[
((\alpha^*) \pm iI_N)^{-1} \Sigma_n(t)^{-1} = \Sigma_n(t)^{-1} (\alpha \pm iI_N)^{-1} + 
+ i((\alpha^*) \pm iI_N)^{-1} \Sigma_n(t)^{-1} \Lambda_n(t) \Lambda_n(t)^* \Sigma_n(t)^{-1} (\alpha \pm iI_N)^{-1}.
\] (4.16)

Finally notice that
\[
2(\alpha^2 + I_N)^{-1} \alpha = (\alpha - iI_N)^{-1} + (\alpha + iI_N)^{-1}.
\] (4.17)

By using (4.6), (4.16) and (4.17), and after some calculations, we rewrite (4.15) in the form (4.12), where
\[
H_n^+(t) = I_{2m} + J - i\Lambda_n(t)^* \Sigma_n(t)^{-1} (\alpha - iI_N)^{-1} \Lambda_n(t)J + 
+ iJ\Lambda_n(t)^*(\alpha^* - iI_N)^{-1} \Sigma_n(t)^{-1} \Lambda_n(t) + \Lambda_n(t)^* \Sigma_n(t)^{-1} \Lambda_n(t)J + 
\times (\alpha - iI_N)^{-1} \Lambda_n(t)J\Lambda_n(t)^*(\alpha^* - iI_N)^{-1} \Sigma_n(t)^{-1} \Lambda_n(t),
\] (4.18)

\[
H_n^-(t) = I_{2m} - J + i\Lambda_n(t)^* \Sigma_n(t)^{-1} (\alpha + iI_N)^{-1} \Lambda_n(t)J - 
- iJ\Lambda_n(t)^*(\alpha^* + iI_N)^{-1} \Sigma_n(t)^{-1} \Lambda_n(t) - \Lambda_n(t)^* \Sigma_n(t)^{-1} \Lambda_n(t)J 
\times (\alpha + iI_N)^{-1} \Lambda_n(t)J\Lambda_n(t)^*(\alpha^* + iI_N)^{-1} \Sigma_n(t)^{-1} \Lambda_n(t).
\] (4.19)

From (0.11) and (4.18) it follows that
\[
H_n^+(t) = I_{2m} + W_{\alpha,\Lambda}(n,t,i) JW_{\alpha,\Lambda}(n,t,-i)^*,
\]
and so, taking into account (2.11), we derive (4.13). Equality (4.14) follows from (4.19) in a similar way.
Recall now that $m = 2$ and (2.11) holds. Then according to (2.17), (2.21) and (4.13) we get

$$H^+_n(t) = c^+_n(t)(I_2 + S_n(t))(I_2 + S_{n-1}(t)), \quad (4.20)$$

and according to (2.18), (2.22) and (4.14) we get

$$H^-_n(t) = c^-_n(t)(I_2 - S_n(t))(I_2 - S_{n-1}(t)), \quad (4.21)$$

where $c^+_n(t)$ are scalar functions. In view of (4.13) and (4.14) we obtain also

$$\text{Tr} \ H^\pm_n(t) = 2. \quad (4.22)$$

By Remark 2.4 and formula (0.23) we derive

$$\text{Tr} (I_2 \pm S_n(t))(I_2 \pm S_{n-1}(t)) = \text{Tr} (I_2 + S_n(t)S_{n-1}(t)) = 2(1 + \overrightarrow{S}_{n-1}(t) \cdot \overrightarrow{S}_n(t)). \quad (4.23)$$

From (4.20)-(4.23) it follows that $1 + \overrightarrow{S}_{n-1}(t) \cdot \overrightarrow{S}_n(t) \neq 0$. Now formulas (0.22), (4.23) and (4.22) yield that

$$\text{Tr} V^\pm_n(t) = 2 \equiv \text{Tr} H^\pm_n(t). \quad (4.24)$$

Taking into account (0.22), (4.20) and (4.21) we see that $V^\pm_n = \hat{c}^\pm H^\pm_n$, and so (4.24) yields equalities $V^\pm_n \equiv H^\pm_n$. Hence we have

$$\frac{d}{dt}(\Lambda_n(t)^*\Sigma_n(t)^{-1}) = V^+_n(t)\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha - iI_N)^{-1} + V^-_n(t)\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha + iI_N)^{-1}. \quad (4.25)$$

From $V^\pm_n \equiv H^\pm_n$, (2.11) and definitions (4.13) and (4.14) we also get:

$$V^\pm_n(t)W_{\alpha,\Lambda}(n,t,\pm i) = 2W_{\alpha,\Lambda}(n,t,\pm i)P_\pm. \quad (4.26)$$

Let us differentiate now $W_{\alpha,\Lambda}$. For this purpose notice that

$$(\alpha \pm iI_N)^{-1}(\alpha - \lambda I_N)^{-1} = \frac{(\alpha - \lambda I_N)^{-1} - (\alpha \pm iI_N)^{-1}}{\lambda \pm i}. \quad (4.27)$$
Using (4.5), (4.25) and (4.27) we derive
\[
\frac{d}{dt} W_{\alpha, \Lambda}(n, t, \lambda) = V_+ + n(t) W_{\alpha, \Lambda}(n, t, \lambda) - W_{\alpha, \Lambda}(n, t, i) \lambda - i - 2 W_{\alpha, \Lambda}(n, t, \lambda - i) P_+ - 2 W_{\alpha, \Lambda}(n, t, \lambda - i) P_-. \tag{4.28}
\]

In view of (4.26) we can rewrite (4.28) as
\[
\frac{d}{dt} W_{\alpha, \Lambda}(n, t, \lambda) = F_n(t, \lambda) W_{\alpha, \Lambda}(n, t, \lambda) - W_{\alpha, \Lambda}(n, t, \lambda) \hat{F}(t, \lambda), \tag{4.29}
\]
where \(F_n\) is given by the second relation in (0.21) and
\[
\hat{F} = 2((\lambda - i)^{-1} P_+ + (\lambda + i)^{-1} P_-).
\]

Thus in view of Theorem 0.1 and formula (4.29) the non-degenerate matrix functions \(\{\hat{W}_n\}\) given by
\[
\hat{W}_n(t, \lambda) = \lambda^{-n} W_{\alpha, \Lambda}(n, t, \lambda) \begin{pmatrix} (\lambda - i)^n e^{2t(\lambda - i)^{-1}} & 0 \\ 0 & (\lambda + i)^n e^{2t(\lambda + i)^{-1}} \end{pmatrix}
\]
satisfy equations (0.25), i.e., the compatibility condition (0.20) is valid. As equation (0.20) is equivalent to (0.21), the theorem is proved. \(\Box\)

Theorem 4.1 together with Corollary 3.4 yields the following result.

**Corollary 4.2** Under the conditions of Theorem 4.1 the evolution of the Weyl function \(\varphi\) of the system \(W_{n+1}(t, \lambda) = G_n(t, \lambda) W_n(t, \lambda)\) is given by the formula
\[
\varphi(t, \lambda) = i\theta_1^* e^{-2i(\alpha^* + iI_N)^{-1} \Sigma_0(t)^{-1}(\lambda I_N - \tilde{\beta}(t))^{-1} e^{-2i(\alpha + iI_N)^{-1} \theta_2} (\alpha + iI_N)^{-1} \Sigma_0(t)^{-1} \theta_2)^* e^{-2i(\alpha^* - iI_N)^{-1} \Sigma_0(t)^{-1}} \tag{4.30}
\]

As an illustration let us consider a simple example.

**Example 4.3** Put \(m = n = 1\) and \(\alpha = ih\) \((h > 0, h \neq 1)\), and choose scalars \(\theta_1, \theta_2\) such that \(|\theta_1|^2 + |\theta_2|^2 = 2h\). Then \(\alpha, \theta_1, \theta_2\) form an admissible
triple and $\alpha$, $\Sigma_0(0) = 1$, $\Lambda_0(0) = [\theta_1 \ \theta_2]$ satisfy the conditions of Theorem 4.1 and Corollary 4.2. Therefore by (4.4) we have

$$\Lambda_n(t) = h^{-n} \left[ (h + 1)^n \theta_1 \exp \left\{ \frac{2it}{h-1} \right\} - (h - 1)^n \theta_2 \exp \left\{ \frac{2it}{h+1} \right\} \right]. \quad (4.31)$$

From (0.14) and (4.31) it follows that

$$\Sigma_n(t) \equiv \frac{c_n(h)}{2h^{2n+1}}, \quad c_n(h) := (h + 1)^{2n} |\theta_1|^2 + (h - 1)^{2n} |\theta_2|^2. \quad (4.32)$$

According to (0.9), (4.31), and (4.32) we get now

$$(S_n(t))_{11} = 1 - \frac{8h^2|\theta_1\theta_2|^2(h^2 - 1)^{2n}}{c_n(h)c_{n+1}(h)}, \quad (S_n(t))_{22} = -(S_n(t))_{11},$$

$$(S_n(t))_{12} = (S_n(t))_{21} = \frac{4h\bar{\theta_1}\theta_2}{c_n(h)c_{n+1}(h)} \exp \left\{ \frac{4it}{1-h^2} \right\} \times \left( h^2 - 1 \right)^n \left( (h + 1)^{2n+1} |\theta_1|^2 - (h - 1)^{2n+1} |\theta_2|^2 \right).$$

Finally Corollary 4.2 yields:

$$\varphi(t, \lambda) = \exp \left\{ \frac{4it}{1-h^2} \right\} \frac{i\theta_1\theta_2}{\lambda + i(|\theta_2|^2 - \bar{\lambda})}.$$

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