CONCORDANCE SURGERY AND THE OZSVÁTH–SZABÓ 4-MANIFOLD INVARIANT

ANDRÁS JUHÁSZ AND IAN ZEMKE

ABSTRACT. We compute the effect of concordance surgery, a generalization of knot surgery defined using a self-concordance of a knot, on the Ozsváth–Szabó 4-manifold invariant. The formula involves the graded Lefschetz number of the concordance map on knot Floer homology. The proof uses the sutured Floer TQFT, and a version of sutured Floer homology perturbed by a 2-form.

1. Introduction

Let $X$ be a smooth oriented 4-manifold with $b_2^+(X) \geq 2$. Suppose that $T \subseteq X$ is a smoothly embedded homologically essential torus with trivial self-intersection, and let $K \subseteq S^3$ be a knot. Fintushel and Stern [FS98] defined the knot surgery operation on $X$ along $K$, resulting in the 4-manifold $X_K$. This is obtained by gluing $X \setminus N(T)$ and $S^1 \times (S^3 \setminus N(K))$ via an orientation reversing diffeomorphism of their boundaries that maps a meridian of $T$ to a longitude of $K$. They showed that

\begin{equation}
SW(X_K) = SW(X) \cdot \Delta_K(t),
\end{equation}

where $SW$ denotes the Seiberg–Witten invariant, $\Delta_K(t)$ is the symmetrized Alexander polynomial of $K$, and the variable $t$ corresponds to the homology class induced by $T$ in $H_2(X_K)$.

When $\pi_1(X) = 1$ and $\pi_1(X \setminus T) = 1$, then $X$ and $X_K$ are homeomorphic by Freedman’s theorem. Note that every symmetric integral Laurent polynomial $p(x)$ satisfying $p(1) = \pm 1$ is the Alexander polynomial of a knot in $S^3$. Consequently, if $SW(X) \neq 0$, then we obtain a different exotic smooth structure on $X$ for every such Laurent polynomial.

Mark [Mar13, Theorem 3.1] obtained a result analogous to equation (1) for the Ozsváth–Szabó 4-manifold invariant [OS06], which is expected to coincide with the Seiberg–Witten invariant. For a closed 4-manifold $X$ with $b_2^+(X) \geq 2$, Ozsváth and Szabó define a mixed invariant of $X$, which is a map

$$
\Phi_X : \text{Spin}^c(X) \to \mathbb{F}_2.
$$

We write $\Phi_{X,s}$ for the value of $\Phi_X$ on $s$. It is convenient to organize the mixed invariants of different Spin$^c$ structures into a single polynomial. Recall that Spin$^c$($X$) is an affine space over $H^2(X)$, so the difference of two Spin$^c$ structures is a well-defined cohomology class. If $\omega = (\omega_1, \ldots, \omega_n)$ is an $n$-tuple of closed 2-forms that induce a basis of $H^2(X; \mathbb{R})$, we can arrange the mixed invariants into the polynomial

$$
\Phi_{X: \omega} := \sum_{s \in \text{Spin}^c(X)} \Phi_{X,s} \cdot t_1^{(s-s_0) \cup \omega_1}[X] \cdots t_n^{(s-s_0) \cup \omega_n}[X],
$$

where $s_0$ is some choice of base Spin$^c$ structure on $X$. If $H^2(X)$ is torsion-free, then $\Phi_{X: \omega}$ completely encodes the map $s \mapsto \Phi_{X,s}$.

Concordance surgery is a generalization of knot surgery due to Fintushel and Stern; see Akbulut [Akbu02, Section 2]. Let $K$ be a knot in the homology 3-sphere $Y$ (note that Akbulut only considered the case $Y = S^3$). Given a self-concordance $C = (I \times Y, A)$ from $(Y, K)$ to itself, we can construct a 4-manifold $X_C$, as follows. We glue the ends of $A$ together to form a 2-torus $T_C$ embedded in $S^1 \times Y$. After removing a neighborhood of $T_C$, we get a 4-manifold $W_C$ with boundary.
equal to $\mathbb{T}^3$. We pick any orientation preserving diffeomorphism $\phi: \partial (X \setminus N(T)) \to \partial N(T_C)$ that sends $[\partial D^2 \times \{p\}]$ to $[(0) \times \ell_K]$, where $\ell_K$ is a longitude of $K$. We write $X_C$ for any manifold constructed as the union

$$X_C := (X \setminus N(T)) \cup_\phi W_C.$$ 

Fintushel and Stern asked in the late 90s whether a formula similar to equation (1) relates $SW(X)$ and $SW(X_C)$; see Akbulut [Akb02, Remark 2.2].

Our main result gives a formula relating the Ozsváth–Szabó 4-manifold invariants of $X$ and $X_C$ in terms of the graded Lefschetz number of the concordance map

$$\hat{F}_C: \hat{HFK}(Y, K) \to \hat{HFK}(Y, K)$$

defined by the first author [Juh16]. This map preserves the Alexander and Maslov gradings [JM18, Theorem 5.18]. The graded Lefschetz number is the polynomial

$$\text{Lef}_i(C) := \sum_{i \in \mathbb{Z}} \text{Lef}_{\hat{F}_C|_{\text{HFK}(Y,K,i)}}(\hat{HFK}(Y, K, i) \to \hat{HFK}(Y, K, i)) \cdot t^i.$$ 

We note that the concordance map $\hat{F}_C$ on knot Floer homology depends on some extra decorations that we are suppressing from the notation. Nonetheless, we will see that the graded Lefschetz number is independent of these decorations.

If $[T] \neq 0 \in H_2(X; \mathbb{R})$, then we can pick a collection of closed 2-forms $\omega = (\omega_1, \ldots, \omega_n)$ that induce a basis for $H^2(X; \mathbb{R})$, such that

$$\int_T \omega_1 = 1 \quad \text{and} \quad \int_T \omega_i = 0 \quad \text{for} \quad i > 1.$$ 

We can now state our main result:

**Theorem 1.1.** Let $X$ be a closed 4-manifold such that $b_2^+(X) \geq 2$. Suppose that $T$ is a smoothly embedded 2-torus in $X$ with trivial self-intersection, such that $[T] \neq 0 \in H_2(X; \mathbb{R})$. Furthermore, let $\omega = (\omega_1, \ldots, \omega_n)$ be a collection of closed 2-forms satisfying equation (2). If $C$ is a self-concordance of $(Y, K)$, where $Y$ is a homology 3-sphere, then

$$\Phi_{X_C; \omega} = \text{Lef}_{i_1}(C) \cdot \Phi_{X; \omega}.$$ 

If $C$ is the product concordance $(I \times Y, I \times K)$, then $\hat{F}_C$ is the identity of $\hat{HFK}(Y, K)$, so $\text{Lef}_{i_1}(C)$ is the graded Euler characteristic of $\hat{HFK}(Y, K)$, which is $\Delta_K(t)$. Hence, as a special case, we recover the formula of Mark [Mar13, Theorem 3.1]; i.e., the Heegaard Floer version of the Fintushel–Stern knot surgery formula.

When $\pi_1(X) = \pi_1(X \setminus T) = 1$ and $Y = S^3$, the manifold $X_C$ is homeomorphic to $X$. Hence, concordance surgery yields an exotic copy of $X$ whenever $\Phi_X \neq 0$ and $\text{Lef}_{i_1}(C) \neq 1$. However, $\text{Lef}_{i_1}(C)$ is always symmetric and satisfies $\text{Lef}_{i_1}(C)(1) = \pm 1$, so it is unclear whether we obtain any smooth structures not arising from knot surgery. We note that the proofs of the knot surgery formula (1) due to Fintushel and Stern for the Seiberg–Witten invariant, and to Mark for the Ozsváth–Szabó invariant, are based on the skein relation for the Alexander polynomial, and hence are only well-suited to knots in $S^3$. Our Theorem 1.1 could be used to construct exotic smooth structures on 4-manifolds with non-trivial fundamental group. If $\Phi_X \neq 0$, and $K$ and $K'$ are knots in a homology 3-sphere $Y$ such that $X_K$ and $X_{K'}$ are homeomorphic and $\Delta_K(t) \neq \Delta_{K'}(t)$, then $X_K$ and $X_{K'}$ are non-diffeomorphic 4-manifolds with fundamental group $\pi_1(Y)$.

Our proof uses a version of sutured Floer homology perturbed by a 2-form, extending a construction of Ozsváth and Szabó [OS04a, Section 3.1], and our computation of the trace and cotrace sutured manifold cobordism maps [JZ18, Theorem 1.1].
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2. Perturbing SFH by a 2-form

Ozsváth and Szabó [OS04a, Section 3.1] defined a version of Heegaard Floer homology for closed 3-manifolds perturbed by a second cohomology class, which we now extend to sutured manifolds. Let Λ denote the Novikov ring over $\mathbb{F}_2$ in a single variable $t$. Its elements are formal sums $\sum_{x \in \mathbb{R}} n_x t^x$, where $n_x \in \mathbb{F}_2$, and the set

$$\{ x \in (-\infty, c] : n_x \neq 0 \}$$

is finite for every $c \in \mathbb{R}$. Note that Λ is a field.

Suppose that $(M, \gamma)$ is a balanced sutured manifold, and $\omega$ is a closed 2-form on $M$. Then $\omega$ induces an action of $H^1(M, \partial M) \cong H_2(M)$ on Λ, as follows. If $a \in H_2(M)$, then the action of $a$ on Λ is given by

$$a \cdot t^x = t^{x + \int_a \omega}$$

for $x \in \mathbb{R}$. We denote by $\Lambda_\omega$ the ring Λ viewed as a module over $\mathbb{Z}[H^1(M, \partial M)]$.

To define $SFH(M, \gamma; \Lambda_\omega)$, we pick an admissible sutured diagram $(\Sigma, \alpha, \beta)$ for $(M, \gamma)$, as well as a collection of compressing disks $D_{\alpha, \beta} := D_{\alpha} \cup D_{\beta}$ for the $\alpha$ and $\beta$ curves in the sutured compression bodies $U_{\alpha}$ and $U_{\beta}$, respectively, each identified with the unit disk in $\mathbb{C}$. For a class $\phi \in \pi_2(x, x)$, the domain $D(\phi)$ is a 2-chain in $\Sigma$. As in [OS04a, Section 3.1], we extend $D(\phi)$ to a 2-chain $\tilde{D}(\phi)$ inside $M$ by coning off the $\alpha$-boundary edges of $D(\phi)$ using the compressing disks $D_{\alpha}$, and coning off the $\beta$-boundary edges of $D(\phi)$ using the compressing disks $D_{\beta}$. This yields a well-defined 2-chain $\tilde{D}(\phi)$, which, in general, will not be closed. We define

$$A_{\omega}(\phi) := \int_{\tilde{D}(\phi)} \omega.$$

When the choice of $\omega$ is clear from the context, we just write $A(\phi)$. Note that there is a map $H : \pi_2(x, x) \rightarrow H_2(M)$, obtained by coning off the periodic domain $D(\phi)$ for $\phi \in \pi_2(x, x)$; see [Juh06, Definition 3.9]. In particular,

$$H(\phi) = \left[ \tilde{D}(\phi) \right].$$

We define $CF(\Sigma, \alpha, \beta, D_{\alpha, \beta}; \Lambda_\omega)$ to be the free Λ-module generated by $T_\alpha \cap T_\beta$. The differential $\partial$ on $CF(\Sigma, \alpha, \beta, D; \Lambda_\omega)$ is given by the formula

$$\partial \mathbf{x} := \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y)} \left( |\tilde{M}(\phi)| \mod 2 \right) \cdot t^{A(\phi)} \cdot \mathbf{y}$$

for $\mathbf{x} \in T_\alpha \cap T_\beta$. The fact that $\partial^2 = 0$ follows from a standard argument by analyzing the ends of 1-dimensional moduli spaces $\tilde{M}(\phi)$. We then set

$$SFH(\Sigma, \alpha, \beta, D_{\alpha, \beta}; \Lambda_\omega) := H_\ast(CF(\Sigma, \alpha, \beta, D_{\alpha, \beta}; \Lambda_\omega), \partial).$$

As in the unperturbed case, this is graded by relative $\text{Spin}^c$ structures on $(M, \gamma)$:

$$CF(\Sigma, \alpha, \beta, D_{\alpha, \beta}; \Lambda_\omega) = \bigoplus_{\xi \in \text{Spin}^c(M, \gamma)} CF(\Sigma, \alpha, \beta, D_{\alpha, \beta}; \xi, \Lambda_\omega),$$

and we write

$$SFH(\Sigma, \alpha, \beta, D_{\alpha, \beta}; \xi, \Lambda_\omega) := H_\ast(CF(\Sigma, \alpha, \beta, D_{\alpha, \beta}; \xi, \Lambda_\omega)).$$

Suppose that $(\Sigma, \alpha, \beta, \gamma)$ is an admissible sutured triple diagram, and let

$$W_{\alpha, \beta, \gamma} = (W_{\alpha, \beta, \gamma}, Z_{\alpha, \beta, \gamma}, [\xi_{\alpha, \beta, \gamma}])$$
be the associated sutured manifold cobordism from \((M_{\alpha,\beta}, \gamma_{\alpha,\beta}) \sqcup (M_{\beta,\gamma}, \gamma_{\beta,\gamma})\) to \((M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})\), as in [JZ18, Section 7], where \(M_{i,j} = U_i \cup -U_j\) for \(i, j \in \{\alpha, \beta, \gamma\}\), and \(U_i\) is the sutured compression body obtained from \(\Sigma \times I\) by attaching 3-dimensional 2-handles along \(i \times \{0\} \subseteq \Sigma \times \{0\}\). Recall that

\[
W_{\alpha,\beta,\gamma} = (\Delta \times \Sigma) \cup (e_\alpha \times U_\alpha) \cup (e_\beta \times U_\beta) \cup (e_\gamma \times U_\gamma),
\]

where \(\Delta\) is a regular triangle in \(\mathbb{C}\) with edges labeled \(e_\alpha, e_\beta,\) and \(e_\gamma\), clockwise. Let \(x \in T_\alpha \cap T_\beta, y \in T_\beta \cap T_\gamma,\) and \(z \in T_\alpha \cap T_\gamma\). Furthermore, let \(\psi \in \pi_2(x, y, z)\) be a homotopy class, \(u\) a topological Whitney triangle representing \(\psi\), and \(\omega\) a closed 2-form on \(W_{\alpha,\beta,\gamma}\). Analogously to [OS04a, Section 3.1] and [GW10, Proposition 3.7], the Whitney triangle \(u\) determines a 2-chain \(\tilde{D}(u)\) in \(W_{\alpha,\beta,\gamma}\): Choose sets of compressing disks \(D_\alpha, D_\beta,\) and \(D_\gamma\) with centers \(C_\alpha, C_\beta,\) and \(C_\gamma\) in each of the sutured compression bodies \(U_\alpha, U_\beta,\) and \(U_\gamma,\) respectively. For every \(x \in \Delta\), we let \(\tilde{D}(u) \cap \{x\} = \{x\} \times u(x)\). For every \(i \in \{\alpha, \beta, \gamma\}\) and \(x \in e_i\), we connect the points in \(\{x\} \times u(x)\) to the centers \(\{x\} \times C_i\) radially in \(\{x\} \times D_i \subseteq \{x\} \times U_i\). Note that

\[
\partial \tilde{D}(u) = \gamma_x \cup \gamma_y \cup \gamma_z \cup \bigcup_{i \in \{\alpha, \beta, \gamma\}} (e_i \times C_i),
\]

where the multi-trajectory \(\gamma_x\) is obtained by radially connecting the points of \(x\) with \(C_\alpha \cup C_\beta\) in \(M_{\alpha,\beta}\), and similarly \(\gamma_y\) is a multi-trajectory in \(M_{\beta,\gamma}\), and \(\gamma_z\) is a multi-trajectory in \(M_{\alpha,\gamma}\). We write

\[
A(\psi) = A_\omega(\psi) := \int_{\tilde{D}(u)} \omega.
\]

This is independent of the representative \(u\) since \(\omega\) is a cocycle, and, given another representative \(u'\) of \(\psi\), the difference \(\tilde{D}(u) - \tilde{D}(u')\) is a 0-homologous cycle.

Let \(\omega_{\alpha,\beta} = \omega|_{M_{\alpha,\beta}}, \omega_{\beta,\gamma} = \omega|_{M_{\beta,\gamma}},\) and \(\omega_{\alpha,\gamma} = \omega|_{M_{\alpha,\gamma}}\). We define the triangle map

\[
F_{\alpha,\beta,\gamma};: CF(\Sigma, \alpha, \beta, D_{\alpha,\beta}; \lambda_{\alpha,\beta,\gamma}) \rightarrow CF(\Sigma, \beta, \gamma, D_{\beta,\gamma}; \lambda_{\beta,\gamma}),
\]

by the formula

\[
F_{\alpha,\beta,\gamma}\omega(x \otimes y) := \sum_{\substack{z \in T_\alpha \cap T_\beta \\ \psi \in \pi_2(x, y, z) \\ \mu(\psi) = 0}} \left(\lvert M(\psi)\rvert \mod 2\right) \cdot t^{A(\psi)} \cdot z
\]

for \(x \in T_\alpha \cap T_\beta\) and \(y \in T_\beta \cap T_\gamma\). It is straightforward to see that \(F_{\alpha,\beta,\gamma}\omega\) is a chain map. The associativity relations are easily seen to be satisfied for these triangle maps. The triangle map splits over relative Spin\(^c\) structures:

\[
F_{\alpha,\beta,\gamma}\omega = \bigoplus_{\lambda \in \text{Spin}^c(W_{\alpha,\beta,\gamma})} F_{\alpha,\beta,\gamma,\lambda}\omega,
\]

where \(\text{Spin}^c(W_{\alpha,\beta,\gamma})\) was defined in [Juh16, p. 18], and is an affine space over \(H^2(W_{\alpha,\beta,\gamma}, Z_{\alpha,\beta,\gamma})\). We now address the naturality of the invariants \(SFH(\Sigma, \alpha, \beta, D; \Lambda_\omega)\).

**Proposition 2.1.** Let \((M, \gamma)\) be a balanced sutured manifold, and \(\lambda \in \text{Spin}^c(M, \gamma)\) a relative Spin\(^c\) structure. Suppose that \(\mathcal{H}\) and \(\mathcal{H}'\) are admissible sutured diagrams for \((M, \gamma)\), and \(D\) and \(D'\) are corresponding collections of compressing disks. Furthermore, let \(\omega\) and \(\omega'\) be cohomologous closed 2-forms on \(M\). Then there is an \(H^1(M, \partial M)\)-equivariant chain homotopy equivalence

\[
\Phi_{(\mathcal{H}, \omega) \to (\mathcal{H}', \omega')} : CF(\mathcal{H}, D, \lambda; \Lambda_\omega) \rightarrow CF(\mathcal{H}', D', \lambda; \Lambda_{\omega'}),
\]

well-defined up to chain homotopy and multiplication by an element \(t^x\) for \(x \in \mathbb{R}\). The maps \(\Phi_{(\mathcal{H}, \omega) \to (\mathcal{H}', \omega')}\) form a transitive system up to chain homotopy and multiplication by an element \(t^x\) for \(x \in \mathbb{R}\). If \(\omega = \omega' = 0\), then the transition maps can be defined with no ambiguity.

**Proof.** Let us first consider the case when \(\mathcal{H} = \mathcal{H}'\) and \(D = D'\). Then \(\omega' = \omega + \eta\) for some \(\eta \in \Omega^1(M)\). We now describe the map \(\Phi = \Phi_{(\mathcal{H}, \omega) \to (\mathcal{H}', \omega')}\). Let \(x_0, \ldots, x_n\) be an enumeration of the intersection points representing \(\lambda\). For each \(i \in \{0, \ldots, n\}\), pick a homology class \(\phi_i \in \pi_2(x_0, x_i)\). For \(x \in \mathbb{R}\), let

\[
\Phi(t^x \cdot x_i) = t^{x + \int_{\mathcal{H}(\phi_i)} \eta} \cdot x_i.
\]
It is straightforward to see that this is a chain isomorphism. It is independent of the choice of \( \phi_0, \ldots, \phi_n \), since \( \int_H d\eta = 0 \) for any 2-cycle \( H \). If we replace \( x_0 \) with \( x_j \), then the resulting map gets multiplied by \( f_{\partial(x_0)}d\eta \). Furthermore, the map \( \Phi \) is canonical, since it only depends on \( d\eta = \omega' - \omega \). If \( \omega, \omega', \) and \( \omega'' \) are homologous closed 2-forms, then

\[
\Phi(H, \omega) \mapsto (H, \omega'') \circ \Phi(H, \omega) \mapsto (H, \omega') = \Phi(H, \omega) \mapsto (H, \omega'').
\]

We now address invariance of \( CF(H, D, \mathbb{Z}; \Lambda_\Theta) \) under ambient isotopies of \( H \) and \( D \) in \( (M, \gamma) \). If \( \phi : (M, \gamma) \to (M', \gamma') \) is a diffeomorphism isotopic to \( \text{id}_{(M, \gamma)} \), then we note that the complexes \( CF(H, D, \mathbb{Z}; \Lambda_\Theta) \) and \( CF(\phi(H), (\phi(D), \mathbb{Z}; \Lambda_{\phi(\omega)}) \) are canonically chain isomorphic. Since \( \omega \) and \( \phi_\ast \omega \) are homologous, combining with the first step shows that \( CF(H, D, \mathbb{Z}; \Lambda_\Theta) \) and \( CF(\phi(H), (\phi(D), \mathbb{Z}; \Lambda_{\omega}) \) are chain isomorphic. Since the choice of \( D \) is unique up to ambient isotopy, this shows invariance under \( D \). We henceforth write \( CF(H, \mathbb{Z}; \Lambda_\Theta) \).

The proof of stabilization invariance is essentially the same as in the unperturbed case, with the following modifications. Suppose that \( \omega = \omega' \). For simplicity, we can assume that stabilizations occur near \( \partial \Sigma \). If \( H = (\Sigma, \alpha, \beta) \), then \( H' = (\Sigma \# T^2, \alpha \cup \{\alpha\} \cup \{\beta\}) \), where the connected sum is taken near \( \partial \Sigma \), and \( \alpha \) and \( \beta \) intersect in a single point \( c \). If \( x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) and \( \phi \in \pi_2(x, y) \) is a homology class, then we write \( \phi' \in \pi_2(x \times c, y \times c) \) for its stabilization. The 2-chains \( D(\phi) \) and \( D(\phi') \) differ by a boundary, and hence \( A(\phi) = A(\phi') \). It follows that the map \( x \mapsto x \times c \) is a chain isomorphism; we let this be \( \Phi_t(H, \omega) \mapsto (H', \omega) \).

Now suppose that \( H = (\Sigma, \alpha, \beta) \) and \( H' = (\Sigma', \alpha', \beta) \) are Heegaard diagrams for \( (M, \gamma) \) with the same Heegaard surface, and \( \omega = \omega' \). Note that \( (M'_{\alpha', \gamma_{\alpha', \alpha}}) \) is a product sutured manifold surgered along \( \{\alpha\} \) framed 0-spheres. Let \( \mathbb{Z}_f \in \text{Spin}^c(M'_{\alpha', \gamma_{\alpha', \alpha}}) \) be the unique Spin^c structure that is represented by a vertical vector field on the product part, and such that if \( S \) is the belt sphere of a 4-dimensional 1-handle attached along one of the framed 0-spheres, then \( \langle c_1(\mathbb{Z}^e), [S] \rangle = 0 \).

Let \( W_\alpha = (W_\alpha, Z_\alpha, [\xi_\alpha]) \) be the sutured manifold cobordism from \( (M_{\alpha, \gamma_{\alpha, \alpha}}) \) to \( \emptyset \), obtained from the sutured monodiant \( (\Sigma, \alpha) \). If we glue this to the sutured manifold cobordism \( W_{\alpha', \gamma_{\alpha', \alpha}} = (W_{\alpha', \beta, Z_{\alpha', \beta, \beta}, [\xi_{\alpha', \alpha}, \beta]) \) along \( (M_{\alpha', \gamma_{\alpha', \gamma_{\alpha', \alpha}}}) = (M_{\alpha, \gamma_{\alpha, \alpha}}, \alpha) \), then we obtain the product cobordism

\[
\text{id}_{(M, \gamma)} = (I \times M, I \times \partial M, [\zeta]),
\]

where \( \zeta \) is an \( I \)-invariant contact structure such that \( \{x\} \times \partial M \) is convex with dividing set \( \{x\} \times \gamma \) for every \( x \in I \). Hence, there is a unique relative Spin^c structure \( \mathbb{Z}^{e} \in \text{Spin}^c(W_{\alpha', \beta, \beta, \beta}) \) such that \( \mathbb{Z}^{e}(M_{\alpha', \gamma_{\alpha', \alpha}}) = \mathbb{Z}^{e} \), \( \mathbb{Z}^{e}(M_{\alpha', \alpha'}) = \mathbb{Z}^{e} \), and \( \mathbb{Z}^{e}(\cap \{\alpha\}, \alpha) = \mathbb{Z}^{e} \). Furthermore, if \( \pi : I \times M \to M \) denotes the projection and \( \omega_{\alpha', \alpha} : = \pi^*\omega |_{M_{\alpha', \alpha}} \), then

\[
[\omega_{\alpha', \alpha}] = 0 \in H^2(M_{\alpha', \alpha}).
\]

Indeed, \( H_2(M_{\alpha', \alpha}) \) is generated by the spherical classes \([S]\). As \( S \) bounds a ball in \( W_\alpha \) and \( d(\pi^*\omega) = 0 \), we have \( \int_S \pi^*\omega = 0 \) by Stokes' theorem. We set \( \eta : = \pi^*\omega |_{W_{\alpha', \beta, \beta}} \). Similarly, the Spin^c structure \( \mathbb{Z}^{e} : = \pi^*\mathbb{Z}^{e} \).

Let \( \theta_{\alpha', \alpha} \in CF(\Sigma, \alpha', \alpha, \mathbb{Z}) \) be a class representing the top-graded generator \( \Theta_{\alpha', \alpha} \) of the group \( SFH(\Sigma, \alpha', \alpha) \). By a slight abuse of notation, we also denote by \( \theta_{\alpha', \alpha} \in CF(\Sigma, \alpha', \alpha, \mathbb{Z}); \Lambda_\Theta \) the image of \( \theta_{\alpha', \alpha} \) under the composition

\[
CF(\Sigma, \alpha', \alpha, \mathbb{Z}) \to CF(\Sigma, \alpha', \alpha, \mathbb{Z}) \otimes \Lambda = CF(\Sigma, \alpha', \alpha, \mathbb{Z}; \Lambda_\Theta) \to CF(\Sigma, \alpha', \alpha, \mathbb{Z}; \Lambda_\Theta|_{\omega_{\alpha', \alpha}}),
\]

where the first map is \( a \mapsto a \otimes 1 \), and the second map – well-defined up to multiplication by \( t^x \) for some \( x \in \mathbb{R} \) – is \( \Phi(\Sigma, \alpha', \alpha, \mathbb{Z}; \omega_{\alpha', \alpha}) \), constructed in the first paragraph, using the fact that \( [\omega_{\alpha', \alpha}] = 0 \). Then, for \( x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) such that \( \mathbb{Z}(x) = \mathbb{Z} \), we set

\[
\Phi(H, \omega) \mapsto (H', \omega), \mathbb{Z}(x) := F_{\alpha, \beta, \beta} \circ \eta(\theta_{\alpha', \alpha} \otimes x).
\]

As \( \theta_{\alpha', \alpha} \) is only well-defined up to multiplication by \( t^x \), so is \( \Phi(H, \omega) \mapsto (H', \omega), \mathbb{Z} \). Analogously, if \( H = (\Sigma, \alpha, \beta), H' = (\Sigma, \alpha, \beta') \), and \( \omega = \omega' \), then

\[
\Phi(H, \omega) \mapsto (H', \omega), \mathbb{Z}(x) := F_{\alpha, \beta, \beta, \beta} \circ \eta(x \otimes \theta_{\beta, \beta'}).
\]
In general, given \((\mathcal{H}, \omega)\) and \((\mathcal{H}', \omega')\), and collections of compressing disks \(D\) and \(D'\), we can connect them by a sequence of the above moves. We define \(\Phi_{(\mathcal{H}, \omega) \to (\mathcal{H}', \omega')} : \Lambda \to \Lambda\) by composing the corresponding chain homotopy equivalences. The resulting map is independent of the choice of moves as in [JT12, Section 9].

When \(\omega = \omega' = 0\), we recover unperturbed sutured Floer homology with coefficients in \(\Lambda\), and hence the canonical isomorphisms are well-defined up to chain homotopy by [JT12].

\section*{Remark 2.2}
Note that we only obtain naturality, up to a factor of \(t^x\) for \(x \in \mathbb{R}\), once we fix a Spin\(^c\) structure, and not for the whole group \(SFH(M, \gamma; \Lambda_\omega)\). When \(\omega = 0\), we recover the untwisted version, in which case \(SFH(M, \gamma; \Lambda_0)\) is natural without any ambiguity.

\section*{Remark 2.3}
If \(\omega = (\omega_1, \ldots, \omega_n)\) is an \(n\)-tuple of closed 2-forms on \(M\), then the \(n\)-variable Novikov ring \(\Lambda_n\) over \(\mathbb{F}_2\) in the variables \(t_1, \ldots, t_n\) obtains the structure of an \(H^1(M, \partial M) \cong H_2(M)\)-module. If \(a \in H_2(M)\), then the action of \(a\) on \(\Lambda_n\) is given by

\[ a \cdot t_i^x = t_i^{x + f_i \omega_i} \]

for \(x \in \mathbb{R}\). We will write \(\Lambda_\omega\) for \(\Lambda_n\) with this module structure. The construction of the perturbed sutured Floer complexes can be adapted to construct \(H^1(M, \partial M)\)-modules \(SFH(M, \gamma; \Lambda_\omega)\).

\section{Perturbed sutured cobordism maps}

In this section, we describe a perturbed version of the sutured manifold cobordism maps due to the first author [Juh16]. If \(W = (W, Z, [\xi])\) is a cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\), and \(\omega\) is a closed 2-form on \(W\), then let \(\omega_i = \omega|_{M_i}\) for \(i \in \{0, 1\}\). We will define a chain map

\[ F_{W, \omega} : SFH(M_0, \gamma_0; \Lambda_\omega_0) \to SFH(M_1, \gamma_1; \Lambda_\omega_1), \]

up to an ambiguity described in Proposition 3.1. The construction is an adaptation of the coning construction sketched by Ozsváth and Szabó [OS04a].

\subsection*{Construction of the perturbed cobordism maps}
The twisted cobordism maps are defined similarly to the maps from [Juh16]. We construct them by defining perturbed versions of the 4-dimensional handle attachment maps, as well as a perturbed version of the Honda–Kazez–Matić contact gluing map [HKM08].

We first describe a perturbed version of the Honda–Kazez–Matić contact gluing map. If \((M, \gamma)\) is a sutured submanifold of \((M', \gamma')\), and \(\omega\) and \(\omega'\) are closed 2-forms on \(M\) and \(M'\), respectively, such that \(\omega = \omega'|_M\), then we can define a gluing map

\[ \Phi_{\xi} : SFH(-M, -\gamma; \Lambda_\omega) \to SFH(-M', -\gamma'; \Lambda_{\omega'}), \]

using the same formula as for the untwisted map. The key observation is that if we write

\[ \Phi_{\xi}(x) = x \times c, \]

where \(c\) is the canonical intersection point, then all disks counted by \(\partial(x \times c)\) have homology class of the form \(\phi \times c\), where \(\phi \in \pi_2(x, y)\) is a homology class, and \(c\) is the constant class at \(c\). However, \(A_{\omega'}(\phi \times c) = A_\omega(\phi)\), so \(\Phi_\xi\) is a chain map on the perturbed complexes.

We now describe the 4-dimensional handle attachment maps. Consider first 1-handle and 3-handle maps. Let \(S_0 \subseteq M\) denote a framed 0-sphere. Note that \(M(S_0)\) is obtained by removing two balls from \(M\), and gluing together the resulting boundary spheres. This leaves a distinguished 2-sphere \(S\) in \(M(S_0)\). If \(\omega\) is a closed 2-form on \(M\), then \(\omega\) up to addition of a coboundary there is a unique closed 2-form \(\omega'\) on \(M(S_0)\) such that \(\omega'|_S = 0\), and which restricts to \(\omega\) on \(M(S_0) \setminus N(S) \approx M \setminus S_0\). In this case, we can define the 1-handle map

\[ SFH(M, \gamma; \Lambda_\omega) \to SFH(M(S_0), \gamma; \Lambda_{\omega'}) \]

via the usual formula

\[ x \mapsto x \times \theta^+. \]
The standard proof that this map is a chain map carries over; though, we have to check that the evaluation of \( \omega \) and \( \omega' \) on the domains are compatible. This is ensured since \( \omega'|_{M(\Sigma_0) \setminus S} = \omega|_{M \setminus \Sigma_0} \) and \( \omega'|_S = 0 \). The 3-handle maps can be similarly defined.

Finally, to describe the 2-handle maps, suppose that \( S \) is a framed link in \((M, \gamma)\), and \( \omega \) is a 2-form and \( 1 \) is a Spin\(^c\) structure on \( W(S) \), the trace of the surgery along \( S \). We pick an admissible Heegaard triple \((\Sigma, \alpha, \beta, \beta')\) subordinate to a bouquet for \( S \). The triple determines a 4-manifold \( W_{\alpha, \beta, \beta'} \), which embeds into \( W(S) \) by [Juh16, Proposition 6.6]. Furthermore, \( 1 \) restricts to the torsion Spin\(^c\) structure \( \tilde{\omega}_0 \) on \((M_{\beta, \beta'}, \gamma)\), and \( \omega_{\beta, \beta'} = \omega|_{M_{\beta, \beta'}} \) is null-cohomologous. Let \( \theta_{\beta, \beta'} \in CF(\Sigma, \beta, \beta', \tilde{\omega}_0; \Lambda_\omega) \) be the image of a representative of the top-graded generator \( \Theta_{\beta, \beta'} \) of \( SFH(\Sigma, \beta, \beta', \tilde{\omega}_0) \) under the composition

\[
CF(\Sigma, \beta, \beta', \tilde{\omega}_0) \rightarrow CF(\Sigma, \beta, \beta'; \tilde{\omega}_0; \Lambda_0) \rightarrow CF(\Sigma, \beta, \beta'; \tilde{\omega}_0; \Lambda_{\beta, \beta'}),
\]

where the first map is \( x \mapsto x \otimes 1 \) and the second chain map is given by Proposition 2.1 and the fact that \( [\omega_{\beta, \beta'}] = 0 \), and is well-defined up to multiplication by \( t^x \) for \( x \in \mathbb{R} \). The perturbed 2-handle map is then defined by counting holomorphic triangles via the formula

\[
F_{W(S;\omega)}(x) := \sum_{y \in \tau_\alpha \cap \tau_{\beta'}} \sum_{\psi \in \tau_2(x, y, \theta)} (|\mathcal{M}(\psi)| \mod 2) \cdot t^{A_\omega(\psi)} \cdot y.
\]

If \( W = (W, Z, [\xi]) \) is an arbitrary sutured cobordism, then, after possibly removing some 3-balls along \( Z \), the map \( F_{W, \omega} \) is defined as a composition of the above maps. Viewing

\[
SFH(M, \gamma; \Lambda_\omega) \cong \bigoplus_{\tilde{\omega} \in \text{Spin}^c(M, \gamma)} SFH(M, \gamma; \tilde{\omega}, \Lambda_\omega),
\]

let us write \( \pi_{\tilde{\omega}}: SFH(M, \gamma; \Lambda_\omega) \rightarrow SFH(M, \gamma; \tilde{\omega}, \Lambda_\omega) \) for projection and \( i_{\tilde{\omega}}: SFH(M, \gamma; \tilde{\omega}, \Lambda_\omega) \rightarrow SFH(M, \gamma; \Lambda_\omega) \) for the direct summand map. We have the following:

**Proposition 3.1.** If \( W = (W, Z, [\xi]) \): \((M, \gamma) \rightarrow (M', \gamma')\) is a sutured manifold cobordism, \( \omega \) is a closed 2-form on \( W \), and \( \tilde{\omega} \in \text{Spin}^c(M, \gamma) \) and \( \tilde{\omega}' \in \text{Spin}^c(M', \gamma') \), then the map

\[
\pi_{\tilde{\omega}'} \circ F_{W, \omega} \circ i_{\tilde{\omega}}: SFH(M, \gamma; \tilde{\omega}, \Lambda_\omega|_{\tilde{\omega}}) \rightarrow SFH(M', \gamma'; \tilde{\omega}', \Lambda_\omega|_{\tilde{\omega}'})
\]

is well-defined up to an overall factor of \( t^x \) for \( x \in \mathbb{R} \). If \( \omega \) restricts trivially to \( M \) and \( M' \), then the total map \( F_{W, \omega} \) is also well-defined up to an overall factor of \( t^x \).

Proof. The groups \( SFH(M, \gamma; \tilde{\omega}, \Lambda_\omega|_{\tilde{\omega}}) \) and \( SFH(M', \gamma'; \tilde{\omega}', \Lambda_\omega|_{\tilde{\omega}'}) \) are both well-defined up to multiplication by \( t^x \) for some \( x \in \mathbb{R} \) by Proposition 2.1. There is no ambiguity for the contact gluing map and for the 1- and 3-handle maps. The 2-handle maps are only well-defined up to a factor of \( t^x \), since so is \( \theta_{\beta, \beta'} \). \( \square \)

We now state the following version of the composition law, which follows from an argument similar to [Juh16, Theorem 11.3], using the associativity of the perturbed triangle maps:

**Proposition 3.2.** Suppose \( W = (W, Z, [\xi]) \) decomposes as \( W_2 \circ W_1 \), where \( W_i = (W_i, Z_i, [\xi_i]) \) for \( i \in \{1, 2\} \). Let \( \omega \) be a closed 2-form on \( W \), and write \( \omega_1 = \omega|_{W_1} \) and \( \omega_2 = \omega|_{W_2} \). Then there are representatives of \( F_{W_2, \omega} \) and \( F_{W_1, \omega_1} \) such that

\[
F_{W_2, \omega} = F_{W_2, \omega_2} \circ F_{W_1, \omega_1}.
\]

3.2. Relative Spin\(^c\) structures and twisted coefficients. If \((\Sigma, \alpha, \beta, \gamma)\) is a sutured triple diagram, Grigsby and Wehrli [GW10] defined a map

\[
\tilde{\omega}: \tau_2(x, y, z) \rightarrow \text{Spin}^c(W_{\alpha, \beta, \gamma}).
\]

The following property of the map \( \tilde{\omega} \) will be useful:

**Proposition 3.3.** Let \((\Sigma, \alpha, \beta, \gamma)\) be an admissible sutured triple diagram, and let \( \omega \in \Omega^2(W_{\alpha, \beta, \gamma}) \) be a closed 2-form that restricts trivially to \( M_{\alpha, \beta}, M_{\beta, \gamma}, \) and \( M_{\alpha, \gamma} \). Then \( \omega \) represents a class

\[
[\omega] \in H^2(W, M_{\alpha, \beta} \cup M_{\beta, \gamma} \cup M_{\alpha, \gamma}; \mathbb{R}).
\]
Proof. Let $u$ and $u'$ be topological Whitney triangles representing $\psi$ and $\psi'$, respectively. By definition, the left-hand side is
\[
\int_{\overline{D}(u') - \overline{D}(u)} \omega.
\]
Note that $\overline{D}(u') - \overline{D}(u)$ is a relative 2-cycle in $H_2(W, M_{\alpha, \beta} \cup M_{\beta, \gamma} \cup M_{\alpha, \gamma})$ with boundary
\[
(\gamma_\psi - \gamma_x) + (\gamma_\psi' - \gamma_y) + (\gamma_x - \gamma_y).
\]
On the other hand, by [GW10, Proposition 3.7], the relative Spin$^c$ structure $\underline{g}(\psi)$ is obtained by extending a canonical 2-plane field on $W_{\alpha, \beta, \gamma} \setminus N(\overline{D}(u))$ to the complement of finitely many balls in $W_{\alpha, \beta, \gamma}$, and choosing an almost complex structure for which the extended 2-plane field consists of complex lines. The Spin$^c$ structure $\underline{g}(\psi')$ is constructed similarly. As the 2-plane fields for $\underline{g}(\psi)$ and $\underline{g}(\psi')$ agree in the complement of $\overline{D}(u') - \overline{D}(u)$, the difference
\[
\underline{g}(\psi') - \underline{g}(\psi) = n \cdot PD[\overline{D}(u') - \overline{D}(u)]
\]
for some $n \in \mathbb{Z}$. By [Juh06, Lemma 4.7], $\underline{g}(x') - \underline{g}(x) = PD[\gamma_x - \gamma_\psi]$. As $\underline{g}(\psi)|_{M_{\alpha, \beta}} = \underline{g}(x)$ and $\underline{g}(\psi')|_{M_{\alpha, \beta}} = \underline{g}(y)$, we must have $n = 1$. Hence, by Poincaré-Lefschetz duality,
\[
\int_{\overline{D}(u') - \overline{D}(u)} \omega = (i_* PD[\overline{D}(u') - \overline{D}(u)] \cup [\omega]) \cup [W, \partial W]) = (i_*(\underline{g}(\psi') - \underline{g}(\psi)) \cup [\omega]) \cup [W, \partial W])
\]
and the result follows. □

Recall that a sutured manifold cobordism $W = (W, Z, [\xi])$ from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$ is special if $Z \cong -I \times \partial M_0$, and $\xi$ is an $I$-invariant contact structure such that $\{x\} \times \partial M_0$ is convex for every $x \in I$ with dividing set $\{x\} \times \gamma_0$ for every $x \in I$ (in particular, $\gamma_0 = \gamma_1$). Let $J'$ be an almost complex structure on $TW|Z$ such that $\xi$ consists of complex lines. Then, according to [Juh16, Remark 3.8], we have
\[
\text{Spin}^c(W) \cong \text{Spin}^c(W, J').
\]

Corollary 3.4. Suppose that $W = (W, Z, [\xi])$ is a special cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$. Furthermore, let $\omega$ be a closed 2-form on $W$ whose restrictions to $M_0$ and $M_1$ vanish, and let $\underline{s}_0 \in \text{Spin}^c(W, J')$ be an arbitrary Spin$^c$ structure. Then
\[
F_{W, \omega} = \sum_{\underline{s} \in \text{Spin}^c(W, J')} t^{(i_*(\underline{s} - \underline{s}_0)) \cup [\omega], [W, \partial W]} F_{W, \underline{s}, \omega},
\]
up to an overall factor of $t^e$ for some $x \in \mathbb{R}$.

4. Background on the Ozsváth–Szabó mixed invariants

We review some background on Heegaard Floer homology, due to Ozsváth and Szabó [OS04b] [OS06]. To a closed 3-manifold $Y$ with a Spin$^c$ structure $s$, Ozsváth and Szabó assign $\mathbb{F}_2[U]$-modules $HF^-(Y, s)$, $HF^\infty(Y, s)$, and $HF^+(Y, s)$ that fit into a long exact sequence
\[
\ldots \rightarrow HF^-(Y, s) \rightarrow HF^\infty(Y, s) \rightarrow HF^+(Y, s) \rightarrow HF^-(Y, s) \rightarrow \ldots
\]
The modules are functorial with respect to cobordisms. There is also an $\mathbb{F}_2$-vector space $\widehat{HF}(Y, s)$. If $W$ is a cobordism from $Y_0$ to $Y_1$, and $s \in \text{Spin}^c(W)$ restricts to $s_0$ on $Y_0$ and to $s_1$ on $Y_1$, then there are maps
\[
F_{W, s}: HF^\circ(Y_0, s_0) \rightarrow HF^\circ(Y_1, s_1)
\]
for $\circ \in \{-, \infty, +, \land\}$ that commute with the maps in the long exact sequence in equation (3).
If $\omega$ is a closed 2-form on $Y$, then an $H^1(Y)$-module $HF^0(Y, s; \Lambda_\omega)$ can be defined, using the same coning off procedure we described in Section 2. Similarly, if $\omega = (\omega_1, \ldots, \omega_n)$ is an $n$-tuple of closed forms, we can define the $H^1(Y)$-module $HF^0(Y, s; \Lambda_\omega)$, which is also a $\Lambda_n[U]$-module.

Ozsváth and Szabó [OS04a] defined perturbed versions of their cobordism maps (and more generally, fully twisted versions in [OS06]). Let $\omega$ be a closed 2-form on the cobordism $W$ from $Y_0$ to $Y_1$, and write $\omega_i = \omega_i|_{X_i}$ for $i \in \{0, 1\}$. Then there is a map

$$F_{W, s; \omega}^0 : HF^0(Y, s_0; \Lambda_{\omega_0}) \to HF^0(Y, s_1; \Lambda_{\omega_1}),$$

well-defined up to multiplication by $t^x$ for $x \in \mathbb{R}$.

When $[\omega]$ vanishes on $\partial W$, the map $F_{W, s; \omega}^0$ agrees with the unperturbed map $F_{W, s}^0$ up to an overall factor of $t^x$. We can normalize the perturbed maps by picking a base Spin$^c$ structure $s_0 \in Spin^c(W)$, and defining

$$F_{W, s; \omega}^0 := t^{\langle i(s-s_0) \cup [\omega], [\partial W] \rangle} \cdot F_{W, s}^0.$$ 

With this convention, we have that

$$F_{W, s}^+ = \sum_{s \in Spin^c(W)} F_{W, s; \omega}^0 = \sum_{s \in Spin^c(W)} t^{\langle i(s-s_0) \cup [\omega], [\partial W] \rangle} \cdot F_{W, s}^0,$$

and similarly on $HF$.

For a closed 4-manifold $X$ with $b_2^+(X) \geq 2$, Ozsváth and Szabó define a mixed invariant of $X$, which is a map

$$\Phi_X : Spin^c(X) \to \mathbb{F}_2.$$ 

We write $\Phi_{X, s}$ for the value of $\Phi_X$ on $s$.

The map $\Phi_X$ is defined by picking a codimension one submanifold $N \subset X$ that cuts $X$ into two pieces, $W_1$ and $W_2$, such that $b_2^+(W_1) > 0$, and such that the restriction map

$$H^2(X) \to H^2(W_1) \oplus H^2(W_2)$$

is an injection. Such a cut is called admissible. The invariant $\Phi_{X, s}$ is defined as the coefficient of the bottom-graded generator of $HF^+(S^3)$ in the expression

$$\left( F_{W_2, s|_{W_2}}^+ \circ \delta^{-1} \circ F_{W_1, s|_{W_1}}^- \right)(1),$$

where 1 denotes top-graded generator of $HF^-(S^3) \cong \mathbb{F}_2[U]$, and we view $W_1$ as a cobordism from $S^3$ to $N$, and $W_2$ as a cobordism from $N$ to $S^3$. Ozsváth and Szabó prove that this is independent of the admissible cut $N$.

More generally, if $W : Y_1 \to Y_2$ is a cobordism with $b_2^+(W) \geq 2$, there is a mixed map

$$F_{W, s}^{\text{mix}} : HF^-(Y_1, s|_{Y_1}) \to HF^+(Y_2, s|_{Y_2})$$

defined by picking an admissible cut $N$ of $W$, and factoring through $HF_{\text{red}}(N, s|_N)$ using the inverse of the boundary map $\delta$.

For our purposes, it is convenient to organize the mixed invariants of different Spin$^c$ structures into a single polynomial. If $\omega = (\omega_1, \ldots, \omega_n)$ is an $n$-tuple of closed 2-forms that induce a basis of $H^2(X; \mathbb{R})$, we can arrange the mixed invariants into the polynomial

$$\Phi_{X, \omega} := \sum_{s \in Spin^c(X)} \Phi_{X, s} \cdot t^{\langle i(s-s_0) \cup [\omega_1], [X] \rangle} \cdots t^{\langle i(s-s_0) \cup [\omega_n], [X] \rangle},$$

where $s_0$ is some choice of base Spin$^c$ structure. If $H^2(X)$ is torsion-free, then $\Phi_{X, \omega}$ completely encodes the map $s \mapsto \Phi_{X, s}$. 


5. Fintushel–Stern knot surgery and concordance surgery

Fintushel and Stern [FS98] described a knot surgery operation on a 4-manifold $X$. Given a knot $K$ in $S^3$ and an embedded torus $T$ in $X$ with zero self-intersection, we define the 4-manifold

$$X_0 := X \setminus N(T)$$

with boundary $T^3$. A neighborhood of $T$ can be identified with $T \times D^2$. We pick any orientation preserving diffeomorphism $\phi: \partial(T \times D^2) \to S^1 \times \partial N(K)$ such that $\phi_*([\{p\} \times \partial D^2]) = [\{q\} \times \ell_K]$, where $\ell_K$ is the curve on $\partial N(K)$ induced by a Seifert surface for $K$, while $p \in T$ and $q \in S^1$. We let

$$X_K := X_0 \cup_\phi (S^1 \times (S^3 \setminus N(K)))$$

be the result of knot surgery on $X$ using $K$ and $T$. Note that there is some ambiguity in the choice of $\phi$, so we write $X_K$ for any 4-manifold constructed in this way. It is straightforward to see that $H^*(X_K)$ and $H^*(X)$ are canonically isomorphic.

Fintushel and Stern described a generalization of this operation called concordance surgery; see Akbulut [Akb02]. Let $K$ be a knot in a homology 3-sphere $Y$ (note that Akbulut only considered $Y = S^3$). Given a self-concordance $C = (I \times Y, A)$ from $(Y, K)$ to itself, we can construct a 4-manifold $X_C$, as follows. We take the annulus $A$, and glue its ends together to form a 2-torus $T_C$ embedded in $S^1 \times Y$. After removing a neighborhood of $D_0 \cup D_1 \cup A$ we get a 4-manifold $W_C$ with boundary equal to $T^3$. We pick any orientation preserving diffeomorphism $\phi: \partial X_0 \to \partial N(T_C)$ that sends $[\partial D^2 \times \{p\}]$ to $[\{0\} \times \ell_K]$. We write $X_C$ for any manifold constructed as the union

$$X_C := X_0 \cup_\phi W_C.$$  

It is easy to see that $H^*(X_C)$ and $H^*(X)$ are canonically isomorphic.

If $C = (I \times Y, A)$ is a self-concordance of the knot $K$ in $Y$, and $a$ is an arc on $A$ connecting the two components of $\partial A$, then there is an induced map on knot Floer homology

$$\widehat{F}_{C,a}: \widehat{HFK}(Y, K) \to \widehat{HFK}(Y, K),$$

described by the first author [Juh16]. The map $\widehat{F}_{C,a}$ preserves the Alexander and Maslov gradings according to Marengon and the first author [JM18, Theorem 5.18], and is non-vanishing when $Y = S^3$ by [JM16, Theorem 1.2]. Note that the group $\widehat{HFK}(Y, K)$ only becomes natural once we choose a decoration $P$ of $K$, which we suppress in our notation, and we require $\partial a$ to be disjoint from $P$. We define $\text{Lef}_i(C)$ to be the polynomial

$$\text{Lef}_i(C) := \sum_{i \in \mathbb{Z}} \text{Lef} \left( \left. \widehat{F}_{C,a} \right|_{\widehat{HFK}(Y, K, i)} : \widehat{HFK}(Y, K, i) \to \widehat{HFK}(Y, K, i) \right) \cdot t^i$$

for any arc $a$ connecting the two boundary components of $C$. Although the map $\widehat{F}_{C,a}$ depends on the arc $a$, we have the following:

**Lemma 5.1.** The graded Lefschetz number of $\widehat{F}_{C,a}$ is independent of the arc $a$.

**Proof.** Up to isotopy, any two arcs $a$ and $a'$ differ by a sequence of Dehn twist along one of the boundary components of the annulus $A$. The action of a Dehn twist on $\widehat{HFK}(Y, K)$ was computed by Sarkar [Sar15] when $Y = S^3$, and by the second author [Zem16, Theorem B] for a null-homologous knot in a general 3-manifold $Y$. Hence, if $a'$ differs from $a$ by a single Dehn twist along one of the ends of the annulus, then

$$\widehat{F}_{C,a'} = \widehat{F}_{C,a} \circ (\text{id} + \Phi \Psi),$$

where $\Phi$ and $\Psi$ are two endomorphisms of $\widehat{HFK}(Y, K)$ that satisfy

$$\Phi^2 = \Psi^2 = 0, \Phi \Psi = \Psi \Phi,$$

and also both commute with $\widehat{F}_{C,a}$. In particular, the map $\widehat{F}_{C,a} \circ (\Phi \Psi)$ is nilpotent, so has Lefschetz number 0 in each Alexander grading. \qed
If $X$ is a closed, oriented 4-manifold with a smoothly embedded 2-torus $T$ such that $[T] \neq 0 \in H_2(X; \mathbb{R})$, then we can pick a collection of closed 2-forms $\omega = (\omega_1, \ldots, \omega_n)$ that induce a basis for $H^2(X; \mathbb{R})$ such that
\begin{equation}
\int_T \omega_1 = 1 \quad \text{and} \quad \int_T \omega_i = 0 \quad \text{for } i > 1.
\end{equation}

We restate our main theorem.

**Theorem 1.1.** Let $X$ be a closed 4-manifold such that $b^+_4(X) \geq 2$. Suppose that $T$ is a smoothly embedded 2-torus in $X$ with trivial self-intersection, such that $[T] \neq 0 \in H_2(X; \mathbb{R})$. Furthermore, let $\omega = (\omega_1, \ldots, \omega_n)$ be a collection of closed 2-forms satisfying equation (4). If $C$ is a self-concordance of $(Y,K)$, where $Y$ is a homology 3-sphere, then
\[ \Phi_{X;\omega} = \text{Lef}_t(C) \cdot \Phi_{X;\omega}. \]

In order to prove Theorem 1.1, we need to perform several computations. Let $C$ be a self-concordance of a knot $K$ in the homology 3-sphere $Y$. On the torus $T_C \subseteq S^1 \times Y$, we pick a pair of dividing curves, each intersecting $\{1\} \times K$ exactly once. The dividing set specifies an isotopically unique $S^1$-invariant contact structure $\xi_C$ on $-\partial N(T_C)$. Note that this contact structure is positive with respect to the boundary orientation from $W_C$.

**Proposition 5.2.** Let $\omega_C$ be a closed 2-form on the 4-manifold $W_C$ Poincaré dual to $\{1\} \times S_K$, where $S_K$ is a Seifert surface for the knot $K$. If we view $W_C$ as cobordisms from $T^3$ to $\emptyset$, then
\[ \hat{F}_{W_C;\omega_C}(\partial(-T^3, \xi_C)) = \text{Lef}_t(C), \]

as an element of $\overline{HF}(\emptyset) \otimes_{\mathbb{F}_2} A$, up to an overall factor of $t^x$ for $x \in \mathbb{R}$.

**Proof.** We consider the sutured manifold cobordism $W_C := (W_C, T^3, [\xi_C])$ from the empty sutured manifold to itself. By construction, the perturbed sutured cobordism map $F_{W_C;\omega_C}$ satisfies
\[ F_{W_C;\omega_C}(1) = \hat{F}_{W_C;\omega_C}(\partial(-T^3, \xi_C)). \]

Let us write $S^3(K)$ for the sutured manifold obtained by adding two meridional sutures to $S^3 \setminus N(K)$. We decompose $W_C$ as $W_1 \circ W_3 \circ W_2 \circ W_1$, where
- $W_1$ is the cotrace cobordism from $\emptyset$ to $Y(K) \cup -Y(K)$,
- $W_2$ is the sutured manifold cobordism from $Y(K) \cup -Y(K)$ to itself complementary to the link cobordism $C \cup (I \times Y, I \times K)$,
- $W_3$ is the identity cobordism of $Y(K) \cup -Y(K)$, and
- $W_4$ is the trace cobordism from $Y(K) \cup -Y(K)$ to $\emptyset$.

The 2-form $\omega_3$ restricts trivially to $W_1$, $W_2$, and $W_4$. Its restriction $\omega_3$ to $W_3$ is Poincaré dual to the Seifert surface $S_K \subseteq Y(K)$.

By [JZ18, Theorem 1.1], we know that $W_1$ and $W_3$ induce the canonical cotrace and trace maps, respectively. By Corollary 3.4, and since $\text{Spin}^c(W_3, J') \cong \text{Spin}^c(Y(K) \cup -Y(K))$ as $W_3$ is a product cobordism, we have
\[ F_{W_3;\omega_3}(x \otimes y) = t^{(i_x[(x, y) - x, y]) \cup PD[S_K, [Y(K), \partial N(K)]]} \cdot (x \otimes y), \]
\[ \hat{F}_{C,\omega_0} \text{ on } \overline{HF}(Y,K). \]

The special case of the unknot $U$ and the trivial concordance $(I \times S^3, I \times U)$ is important. In this case, the dividing set on the torus $S^1 \times U \subseteq S^1 \times S^3$ determines the contact structure $\xi_0$ on $-\partial N(S^1 \times U)$. Consider the 4-manifold
\[ W_0 = S^1 \times (S^3 \setminus N(U)) \cong S^1 \times S^1 \times D^2. \]
Corollary 5.3. Let $\omega_0$ be a closed 2-form on the 4-manifold $W_0$, such that $[\omega_0]$ is Poincaré dual to $\{1, 1\} \times D^2$. If we view $W_0$ as a cobordism from $\mathbb{T}^3$ to $\emptyset$, then

$$\hat{F}^{1}_{W_0; \omega_0} (\mathfrak{c}(-\mathbb{T}^3, \xi_0)) = 1,$$

as an element of $\widehat{HF} (\emptyset) \otimes \mathbb{F}_2 \Lambda$, up to an overall factor of $t^x$ for $x \in \mathbb{R}$.

Note that a choice of dividing sets of $S^1 \times U$ and $T_C$ in $S^1 \times Y$ induces a diffeomorphism between $S^1 \times X$ and $T_C$ that maps $\{1\} \times U$ to $\{1\} \times K$, well-defined up to isotopy. We can extend this diffeomorphism to a $D^2$-bundle map from $(S^1 \times U) \times D^2$ to $T_C \times D^2$. We will write $\mathbb{T}^3$ for the boundaries of both $N(S^1 \times U)$ and $N(T_C)$, identified via the restriction of such a diffeomorphism. Furthermore, the contact structures $\xi_0$ and $\xi_C$ are identified by this diffeomorphism, and hence we will write $\xi$ for both. Similarly, the restrictions of $\omega_0$ and $\omega_C$ to $\mathbb{T}^3$ are identified, so we will write $\eta \in \Omega^2 (\mathbb{T}^3)$ for both.

Corollary 5.4. For $k \in \mathbb{Z}$, let $t_k$ be the Spin$^c$ structure on $W_0$ such that $\langle c_1 (t_k), [S^1 \times U] \rangle = 2k$, and let $\tau_k \in \text{Spin}^c (W_C)$ be such that $\langle c_1 (\tau_k), [T_C] \rangle = 2k$. As maps from $HF^+ (\mathbb{T}^3; \Lambda_\eta)$ to $HF^+ (\emptyset) \otimes \Lambda$, we have

$$F^+_{W_C, t_k, \omega_C} (\mathfrak{c}^\tau (-\mathbb{T}^3, \xi)) = \text{Lef}_t (\mathcal{C}) \cdot F^+_{W_0, t_0, \omega_0},$$

up to an overall factor of $t^x$. Furthermore, the maps $F^+_{W_0, t_k, \omega_0}$ and $F^+_{W_C, \tau_k, \omega_C}$ vanish for every $k \in \mathbb{Z} \setminus \{0\}$.

**Proof.** The contact element

$$c^\tau (-\mathbb{T}^3, \xi) \in HF^+ (\mathbb{T}^3; \Lambda_\eta)$$

was defined by Ozsváth and Szabó [OS05] as the image of $\mathfrak{c}(-\mathbb{T}^3, \xi)$ under the natural map

$$\iota_* : \widehat{HF} (\mathbb{T}^3; \Lambda_\eta) \to HF^+ (\mathbb{T}^3; \Lambda_\eta).$$

Since $\iota_*$ commutes with the perturbed cobordism maps for $W_0$ and $W_C$ on $\widehat{HF}$ and $HF^+$, we have

$$F^+_{W_C, \omega_C} (c^\tau (-\mathbb{T}^3, \xi)) = \text{Lef}_t (\mathcal{C})$$

by Proposition 5.2, and

$$F^+_{W_0, \omega_0} (c^\tau (-\mathbb{T}^3, \xi)) = 1$$

by Corollary 5.3. Hence $c^\tau (-\mathbb{T}^3, \xi) \neq 0$, and

$$F^+_{W_C, \omega_C} (c^\tau (-\mathbb{T}^3, \xi)) = \text{Lef}_t (\mathcal{C}) \cdot F^+_{W_0, \omega_0} (c^\tau (-\mathbb{T}^3, \xi)).$$

Next, we use the well-known fact that if $\eta$ is any non-vanishing, closed 2-form on $\mathbb{T}^3$, then

$$HF^+ (\mathbb{T}^3; \Lambda_\eta) \cong \Lambda,$$

and, furthermore, $HF^+ (\mathbb{T}^3; \Lambda_\eta)$ is supported in the torsion Spin$^c$ structure on $\mathbb{T}^3$; see Ai and Peters [AP10, Theorem 1.3], Lekili [Lek13, Theorem 14], and Wu [Wu09]. It follows that $F^+_{W_C, \omega_C}$ and $F^+_{W_0, \omega_0}$, whose domains are thus rank 1 over $\Lambda$, must be constant multiples of each other. Equation (5) and the fact that $c^\tau (-\mathbb{T}^3, \xi) \neq 0$ now establish that the ratio is $\text{Lef}_t (\mathcal{C})$.

Finally, we note that $t_k \mid_{\mathbb{T}^3}$ or $\tau_k \mid_{\mathbb{T}^3}$ is the torsion Spin$^c$ structure if and only if $k = 0$. Hence the maps in the Spin$^c$ structures $t_k$ and $\tau_k$ for $k \in \mathbb{Z} \setminus \{0\}$ vanish because they have trivial domain. In particular,

$$F^+_{W_C, \omega_C} = F^+_{W_C, \tau_k, \omega_C} \text{ and } F^+_{W_0, \omega_0} = F^+_{W_0, t_0, \omega_0},$$

completing the proof.

□

Corollary 5.5. If $\omega = (\omega_1, \ldots, \omega_n)$ is a collection of closed 2-forms on $X$ satisfying equation (4), and $\omega' = (\omega'_1, \ldots, \omega'_n)$ is the induced collection on $X_C$ under the canonical isomorphism $H^2 (X_C; \mathbb{R}) \cong H^2 (X; \mathbb{R})$, then

$$F^+_{W_C, \omega'} = \text{Lef}_t (\mathcal{C}) \cdot F^+_{W_0, \omega_0},$$

and both maps vanish for all other Spin$^c$ structures.
Proof. The result follows from Corollary 5.4, since $[\omega_2]_\ast, \ldots, [\omega_n]_\ast$ vanish on $W_0$ and $[\omega_2]_\ast, \ldots, [\omega_n]_\ast$ vanish on $W_C$, so the cobordism maps $F_{W_C, t^i_C, \omega}^+$ and $F_{W_0, t^i_C, \omega}^+$ are induced by the maps $F_{W_C, t^i_C, \omega}^+$ and $F_{W_0, t^i_C, \omega}^+$, respectively, for every $k \in \mathbb{Z}$.

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Recall that $X_0 = X \setminus N(T)$. Since $b_2^+(X) \geq 2$, by analyzing the Mayer–Vietoris sequence for $X = (X \setminus T) \cup N(T)$, it is easy to see that $b_2^+(X_0) \geq 1$. Hence, there is a surface $Q$ of positive self-intersection in the complement of $T$. Let $N$ denote the boundary of a tubular neighborhood of $Q$. For $u \in \text{Spin}^c(X_0)$, we define a perturbed mixed map

$$F_{X_0, u, \omega}^{\text{mix}}: HF^- (\emptyset) \otimes \Lambda_n \to HF^+ (T^3, u|_{T^1}; \Lambda|_{T^1}) ,$$

by factoring through $HF_{\text{red}}(N, u|_N)$ using the inverse of the boundary map

$$\delta: HF^+ (N, u|_N) \to HF^- (N, u|_N) .$$

The Spin$^c$ composition law for the cobordism maps implies that

$$(F_{W_C, t^i_C, \omega}^+ \circ F_{X_0, u, \omega}^{\text{mix}})(1) = \sum_{s' \in \text{Spin}^c(X_C)} \sum_{s'|_{W_C} = t} \sum_{s'_{X_0} = u} (1, (s'-s_0'; u_0)', (1, (s'-s_0'; u_0)', X_C)),$$

where $s_0' \in \text{Spin}^c(X_C)$ is some choice of base Spin$^c$ structure. A similar formula holds for the composition $(F_{W_0, t^i_C, \omega}^+ \circ F_{X_0, u, \omega}^{\text{mix}})(1)$.

We sum equation (6) over $(t', u) \in \text{Spin}^c(W_C) \times \text{Spin}^c(X_0)$, and sum the analogous equation for the composition $(F_{W_0, t^i_C, \omega}^+ \circ F_{X_0, u, \omega}^{\text{mix}})(1)$ over $(t, u) \in \text{Spin}^c(W_0) \times \text{Spin}^c(X_0)$. Noting as well that $X_0 \cup W_0$ is diffeomorphic to $X$, by applying Corollary 5.5 we see that

$$\Phi_{X; C; \omega} = \text{Lef}_1(C) \cdot \Phi_{X; \omega},$$

completing the proof. \qed

References

[Akb02] Selman Akbulut, Variations on Fintushel-Stern knot surgery on 4-manifolds, Turkish J. Math. 26 (2002), no. 1, 81–92.

[AP10] Yinghua Ai and Thomas Peters, The twisted Floer homology of torus bundles, Algebr. Geom. Topol. 10 (2010), 679–695.

[FS98] Ronald Fintushel and Ronald Stern, Knots, links, and 4-manifolds, Invent. Math. 134 (1998), 365–400.

[GW10] Elisenda Grigsby and Stephan Wehrli, On the colored Jones polynomial, sutured Floer homology, and knot Floer homology, Adv. Math. 223 (2010), no. 6, 2114–2165.

[HKM08] Ko Honda, William Kazez, and Gordana Matić, Contact structures, sutured Floer homology and TQFT, 2008. e-print, arXiv:0807.2431.

[JM16] András Juhász and Marco Marengon, Concordance maps in knot Floer homology, Geom. Topol. 20 (2016), no. 6, 3623–3673.

[JL18] András Juhász and Dylan Thurston, Naturality and mapping class groups in Heegaard Floer homology, 2012. e-print, arXiv:1210.4996.

[Juh06] András Juhász, Holomorphic discs and sutured manifolds, Algebr. Geom. Topol. 6 (2006), 1429–1457.

[Juh16] , Cobordisms of sutured manifolds and the functoriality of link Floer homology, Adv. Math. 299 (2016), 940–1038.

[JZ18] András Juhász and Ian Zemke, Contact handles, duality, and sutured Floer homology, 2018. e-print, arXiv:1803.04401.

[Lek13] Yanki Lekili, Heegaard-Floer homology of broken fibrations over the circle, Adv. Math. 244 (2013), 268 –302.

[Mar13] Thomas Mark, Knotted surfaces in 4-manifolds, Forum Math. 25 (2013), no. 3, 597–637.

[OS04a] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and genus bounds, Geom. Topol. 8 (2004), 311–334.

[OS04b] , Holomorphic disks and topological invariants for closed three-manifolds, Ann. of Math. (2) 159 (2004), no. 3, 1027–1158.

[OS05] , Heegaard Floer homology and contact structures, Duke Math. J. 129 (2005), no. 1, 39–61.
[O806] Holomorphic triangles and invariants for smooth four-manifolds, Adv. Math. 202 (2006), no. 2, 326–400.

[Sar15] Sucharit Sarkar, Moving basepoints and the induced automorphisms of link Floer homology, Algebr. Geom. Topol. 15 (2015), no. 5, 2479–2515.

[Wu09] Zhongtao Wu, Perturbed Floer homology of some fibered three-manifolds, Algebr. Geom. Topol. 9 (2009), 337–350.

[Zem16] Ian Zemke, Quasi-stabilization and basepoint moving maps in link Floer homology, 2016. e-print, arXiv:1604.04316.