GROUP ACTIONS ON CONTRACTIBLE 2-COMPLEXES II

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ABSTRACT. In this second part we prove that, if $G$ is one of the groups $PSL_2(q)$ with $q > 5$ and $q \equiv 5 \pmod{24}$ or $q \equiv 13 \pmod{24}$, then the fundamental group of every acyclic 2-dimensional, fixed point free and finite $G$-complex admits a nontrivial representation in a unitary group $U(m)$. This completes the proof of the following result: every action of a finite group on a finite and contractible 2-complex has a fixed point.

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1. INTRODUCTION

In this second part we prove the following:

**Theorem C.** Let $G$ be one of the groups $PSL_2(q)$ with $q > 5$ and $q \equiv 5 \pmod{24}$ or $q \equiv 13 \pmod{24}$. Then the fundamental group of every 2-dimensional, fixed point free, finite and acyclic $G$-complex admits a nontrivial representation in a unitary group $U(m)$.

This completes the proof of the following result: every action of a finite group $G$ on a finite and contractible 2-complex $X$ has a fixed point.

The groups $G$ considered in [SC21, Theorem B] share a key property: they admit a nontrivial representation $\rho_0$ which restricts to an irreducible representation on the Borel subgroup. The moduli $M_k$ of representations of $\Gamma_k = X_1^{OS+k}(G) : G$ constructed in the proof of [SC21, Theorem B] is built from a representation with this property. When $q \equiv 1 \pmod{4}$ no nontrivial representation of $PSL_2(q)$ restricts to an irreducible representation on the Borel subgroup. To prove Theorem C, we circumvent this difficulty by instead considering the action of $\hat{G} = SL_2(q)$ on $X_1^{OS+k}(G)$.

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2. More representation theory

We denote the set of eigenvalues of a square matrix $M$ by $\Lambda(M)$.

**Lemma 2.1.** Let $G$ be a finite group, $g_1, g_2 \in G$ and $H_1 = \langle g_i \rangle$. Let $\rho: G \to U(m)$ be a unitary representation and let $k_i = \# \Lambda(\rho(g_i))$. Then there are matrices $A_1, A_2 \in U(m)$ such that $A_i \rho(g_i) A_i^{-1}$ is diagonal (for $i = 1, 2$), $A_1^{-1} A_2$ commutes with $C_{U(m)}(\rho(G))$ and

$$\dim \left( (A_1 C_{U(m)}(\rho(H_1)) A_1^{-1}) \cap (A_2 C_{U(m)}(\rho(H_2)) A_2^{-1}) \right) \geq \frac{m^2}{k_1 k_2}.$$ 

**Proof.** By [SC21, Theorems 4.1 and 4.2], we can take $T \in U(n)$ and irreducible representations $\rho_j: G \to U(m_j)$ with $j = 1, \ldots, k$ such that $T \rho T^{-1} = \rho_1 \oplus \ldots \oplus \rho_k$. Moreover, we can do this so that whenever $\rho_j$ and $\rho_j'$ are isomorphic we have $\rho_j = \rho_j'$. For each $i = 1, 2$ we take matrices $D_{i,1}, \ldots, D_{i,k}$ with $D_{i,j} \in U(m_j)$ such that

$$D_{i,j} \rho_j(g_i) D_{i,j}^{-1}$$

is diagonal. We choose the $D_{i,j}$ so that $\rho_j = \rho_j'$ implies $D_{i,j} = D_{i,j'}$. Let $D_i = D_{i,1} \oplus \ldots \oplus D_{i,k}$. Then, by [SC21, Proposition 4.3 and Remark 4.4], $D_i$ commutes with $C_{U(m)}(T \rho(G) T^{-1})$ and letting $A_i = D_i T$ we have that $A_i \rho(g_i) A_i^{-1}$ is diagonal and $A_1^{-1} A_2$ commutes with $C_{U(m)}(\rho(G))$.

Now for $\lambda_1 \in \Lambda(\rho(g_1))$ and $\lambda_2 \in \Lambda(\rho(g_2))$ we define

$$n(\lambda_1, \lambda_2) = \# \{ j : 1 \leq j \leq m \text{ and } (A_1 \rho(g_1) A_1^{-1})_{j,j} = \lambda_1 \text{ and } (A_2 \rho(g_2) A_2^{-1})_{j,j} = \lambda_2 \}.$$ 

Note that

$$(A_1 C_{U(m)}(\rho(H_1)) A_1^{-1}) \cap (A_2 C_{U(m)}(\rho(H_2)) A_2^{-1})$$

has a subgroup isomorphic to

$$\prod_{\lambda_1 \in \Lambda(\rho(g_1)), \lambda_2 \in \Lambda(\rho(g_2))} U(n(\lambda_1, \lambda_2))$$

and therefore has dimension at least $\sum_{\lambda_1 \in \Lambda(\rho(g_1)), \lambda_2 \in \Lambda(\rho(g_2))} n(\lambda_1, \lambda_2)^2$. The AM-QM inequality gives

$$\frac{m}{k_1 k_2} = \frac{\sum_{\lambda_1 \in \Lambda(\rho(g_1)), \lambda_2 \in \Lambda(\rho(g_2))} n(\lambda_1, \lambda_2)}{k_1 k_2} \leq \sqrt{\frac{\sum_{\lambda_1 \in \Lambda(\rho(g_1)), \lambda_2 \in \Lambda(\rho(g_2))} n(\lambda_1, \lambda_2)^2}{k_1 k_2}},$$

and we obtain the desired inequality.

3. The $\hat{G}$-graph $X_1^{O_S}(G)$

Let $G = \text{PSL}_2(q)$ with $q \equiv 5 \pmod{24}$ or $q \equiv 13 \pmod{24}$. Let $\hat{G} = \text{SL}_2(q)$, so that $Z(\hat{G}) = \{1, -1\}$ and $\hat{G}/Z(\hat{G}) = G$. In what follows we denote $z = -1 \in Z(\hat{G})$.

We consider a construction of $X_1^{O_S}(G)$ as in [SC21, Proposition 5.5]. Recall that for any $k \geq 0$, we can also consider the $G$-graph $X_1^{O_S+k}(G)$ obtained from $X_1^{O_S}(G)$ by attaching $k$ free orbits of 1-cells (note that by [SC20, Proposition 3.10] the $G$-homotopy type of $X_1^{O_S+k}(G)$ does not depend on the particular way these free orbits are attached).

We consider the action of $\hat{G} = \text{SL}_2(q)$ on $X_1^{O_S+k}(G)$ defined using the projection $\pi: \hat{G} \to G$. The stabilizer of a vertex (resp. edge) for the action of $\hat{G}$ is a central extension, by $Z(\hat{G})$, of the
stabilizer for the action of $G$. Then the $\hat{G}$-orbits are connected as in Figure 1. The group $B$ denotes the Borel subgroup of $\hat{G}$ and $Q_8$ denotes the quaternion group.

\[
\begin{array}{c|c}
B & \text{SL}_2(3) \\
\hline
C_q - 1 & C_6 \\
\hline
2D_q - 1 & \text{SL}_2(3) \\
\end{array}
\]

\[
\begin{array}{c|c}
B & 2D_{q+1} \\
\hline
C_q - 1 & C_6 \\
\hline
2D_q - 1 & Q_8 \\
\end{array}
\]

$\hat{G} = \text{SL}_2(q)$, $q \equiv 13 \pmod{24}$. 

$\hat{G} = \text{SL}_2(q)$, $q \equiv 5 \pmod{24}$.

**Figure 1.** The $\hat{G}$-graph $X^O_{1S}(G)$.

We now apply Brown’s result [SC21, Theorem 3.1]. The choices in each case are the following. Note that in each case the stabilizers are given in Tables 1 and 2.

- For $\hat{G} = \text{SL}_2(q)$ with $q \equiv 13 \pmod{24}$ we take $V = \{v_0, v_1, v_2, v_3\}$, $E = \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_1', \ldots, \eta_k'\}$, $T = \{\eta_0, \eta_1, \eta_2\}$, with $v_0 \overset{\eta_0}{\rightarrow} v_1$, $v_1 \overset{\eta_2}{\rightarrow} v_2$, $v_2 \overset{\eta_3}{\rightarrow} v_3$, $v_3 \overset{\eta_k}{\rightarrow} g_{\eta_k}v_0$ and $v_0 \overset{\eta_i'}{\rightarrow} v_0$ for $i = 1, \ldots, k$.
- For $\hat{G} = \text{SL}_2(q)$ with $q \equiv 5 \pmod{24}$ we take $V = \{v_0, v_1, v_2, v_3\}$, $E = \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_1', \ldots, \eta_k\}$, $T = \{\eta_0, \eta_2, \eta_3\}$, with $v_0 \overset{\eta_0}{\rightarrow} v_1$, $v_1 \overset{\eta_2}{\rightarrow} v_2$, $v_2 \overset{\eta_3}{\rightarrow} v_3$, $v_3 \overset{\eta_k}{\rightarrow} g_{\eta_k}v_0$ and $v_0 \overset{\eta_i}{\rightarrow} v_0$ for $i = 1, \ldots, k$.

| $\hat{G}$ | $q$ | $\hat{G}_{v_0}$ | $\hat{G}_{v_1}$ | $\hat{G}_{v_2}$ | $\hat{G}_{v_3}$ |
|-----------|-----|----------------|----------------|----------------|----------------|
| $\text{SL}_2(q)$ | $q$ odd | $B = \mathbb{F}_q \rtimes C_{q - 1}$ | $2D_{q - 1}$ | $2D_{q + 1}$ | $\text{SL}_2(3)$ |

**Table 1.** Stabilizers of vertices for the graph $X^O_{1S}(G)$.

| $\hat{G}$ | $q$ | $\hat{G}_{\eta_0}$ | $\hat{G}_{\eta_1}$ | $\hat{G}_{\eta_2}$ | $\hat{G}_{\eta_3}$ | $\hat{G}_{\eta_k}$ |
|-----------|-----|----------------|----------------|----------------|----------------|----------------|
| $\text{SL}_2(q)$ | $q$ odd | $C_{q - 1}$ | $C_4$ | $Q_8$ | $C_6$ | $\mathbb{Z}(\hat{G})$ |

**Table 2.** Stabilizers of edges for the graph $X^O_{1S}(G)$.

In what follows $\Gamma_k$ is the group obtained by applying Brown’s result to the action of $G$ on $X^O_{1S+k}(G)$ with these choices. The following lemma is an extension of [SC21, Lemma 5.6] for the action of $\hat{G}$ on $X^O_{1S}(G)$.

**Lemma 3.1.** Let $G$ be one of the groups in Theorem C. Let $E$ be a set of representatives of the orbits of edges in $X^O_{1S}(G)$. Let $X$ be an acyclic 2-complex obtained from $X^O_{1S}(G)$ by attaching a free orbit of 2-cells along the $G$-orbit of a closed edge path $\xi = (a_1e_1, \ldots, a_ne_n)$ with $e_i \in E$, $a_i \in \hat{G}$ and $\varepsilon_i \in \{-1, 1\}$. Then it is possible to choose, for each $e \in E$ an element $x_e \in \mathbb{C}[\hat{G}]$ and an element $\delta \in \mathbb{C}[\hat{G}]$ so that

$$1 = (1 - z)\delta + \sum_{i=1}^{n} \varepsilon_i a_i N(\hat{G}_{e_i})x_{e_i}.$$ 

Therefore, for any complex representation $V$ of $\hat{G}$ we have $V = (1 - z)V + \sum_{e \in E} s_e V^{G_e}$, where $s_e = \sum_{i \in I_e} \varepsilon_i a_i$ and $I_e = \{i : e_i = e\}$. 
more concretely, \( B \) of the subgroups appearing in the above list and by [SC21, Theorem A.5.1]. For example, note [Dor71, p.234]). The above computation follows then from the structure description of each one.

**Proof.**

Consider the ring homomorphism \( \pi: \mathbb{C}[\hat{G}] \to \mathbb{C}[G] \). By [SC21, Lemma 5.6], there are elements \( \hat{x}_e \in \mathbb{C}[\hat{G}] \) such that \( 1 = \sum_{i=1}^{n} \epsilon_i \pi(a_i) N(G_{e_i}) \pi(\hat{x}_{e_i}) \). Let \( x_e = \frac{1}{2} \hat{x}_e \). Note that \( \pi(N(G_x)) = 2 \cdot N(G_x) \) and then \( \pi(\sum_{i=1}^{n} \epsilon_i a_i N(G_{e_i}) x_{e_i}) = 1 \). Therefore, since the kernel of \( \pi \) is the ideal generated by \( 1 - z \), there is an element \( \delta \in \mathbb{C}[\hat{G}] \) such that \( 1 = (1-z)\delta + \sum_{i=1}^{n} \epsilon_i a_i N(G_{e_i}) x_{e_i} \). \( \square \)

4. **Representations and centralizers**

In this section we fix a suitable irreducible representation \( \rho_0: \hat{G} \to \mathbb{G} \) and compute the dimension of the centralizers \( \mathbb{C}_G(\rho_0(\hat{G}_h)) \) and \( \mathbb{C}_G(\rho_0(\hat{G}_{h})) \). To perform these computations, we first need to know how many elements of each conjugacy class of \( \hat{G} \) appear in each of the subgroups \( \hat{G}_h \) and \( \hat{G}_{h} \). Recall that if \( x \) is an element of \( \hat{G} \), then \( (x) \) denotes its conjugacy class. For the structure and conjugacy classes of the groups \( \text{SL}_2(q) \) we refer to [SC21, Appendix A.5].

**Proposition 4.1.** Let \( \hat{G} = \text{SL}_2(q) \), with \( q \equiv \pm 1 \) (mod 3) and \( q \equiv 1 \) (mod 4). Then

(i) \( C_{q-1} \) contains 1 and \( z \); and 2 elements of each class \( \langle a^l \rangle \), for \( l = 1, \ldots, \frac{q-3}{2} \).

(ii) \( C_4 \) contains 1 and \( z \); and 2 elements of the class \( \langle a^\frac{q-1}{4} \rangle \).

(iii) \( Q_8 \) contains 1 and \( z \); and 6 elements of the class \( \langle a^\frac{q-3}{2} \rangle \).

(iv) If \( q \equiv 1 \) (mod 3) then \( C_6 \) contains 1 and \( z \); and 2 elements of each class \( \langle a^l \rangle \) for \( l = \frac{q-1}{3}, \frac{q-1}{6} \). If \( q \equiv 2 \) (mod 3) then \( C_6 \) contains 1 and \( z \); and 2 elements of each class \( \langle b^m \rangle \) for \( m = \frac{q+1}{3}, \frac{q+1}{6} \).

(v) The Borel subgroup \( B \) contains 1 and \( z \); \( \frac{q-1}{2} \) elements of each of the classes \( \langle a \rangle, \langle d \rangle, \langle zc \rangle \) and \( \langle zd \rangle \); and \( 2q \) elements of each class \( \langle a^l \rangle \), for \( l = 1, \ldots, \frac{q-3}{2} \).

(vi) If \( q \equiv 1 \) (mod 3) then \( 2D_{q-1} \) contains 1 and \( z \); 2 elements of each class \( \langle a^l \rangle \), for \( l = 1, \ldots, \frac{q-3}{2} \); and \( q-1 \) extra elements of the class \( \langle a^\frac{q-1}{4} \rangle \).

(vii) \( 2D_{q+1} \) contains 1 and \( z \); 2 elements of each class \( \langle b^m \rangle \), for \( m = 1, \ldots, \frac{q-1}{2} \); and \( q+1 \) elements of the class \( \langle a^\frac{q+1}{4} \rangle \).

(viii) If \( q \equiv 1 \) (mod 3) then \( \text{SL}_2(3) \) contains 1 and \( z \); 6 elements of the class \( \langle a^\frac{q+1}{4} \rangle \); and 8 elements of each class \( \langle a^l \rangle \) for \( l = \frac{q-1}{3}, \frac{q-1}{6} \). If \( q \equiv 2 \) (mod 3) then \( \text{SL}_2(3) \) contains 1 and \( z \); 6 elements of the class \( \langle a^\frac{q+1}{4} \rangle \); and 8 elements of each class \( \langle b^m \rangle \) for \( m = \frac{q+1}{3}, \frac{q+1}{6} \).

**Proof.** Note that if \( q \equiv 1 \) (mod 4), then every element of \( \text{SL}_2(q) \) is conjugate to its inverse (cf. [Dor71, p.234]). The above computation follows then from the structure description of each one of the subgroups appearing in the above list and by [SC21, Theorem A.5.1]. For example, note that \( B = \mathbb{F}_q \rtimes C_{q-1} \), and that \( 2D_{q-1} \) and \( 2D_{q+1} \) can be described as follows:

\[ 2D_{q-1} \simeq C_{(q-1)/4} \rtimes Q_8, \quad 2D_{q+1} \simeq C_{(q+1)/2} \rtimes C_4. \]

More concretely, \( 2D_{q-1} = \langle a, \alpha \rangle \) (resp. \( 2D_{q+1} = \langle b, \alpha \rangle \)), where \( a \) (resp. \( b \)) has order \( q-1 \) (resp. \( q+1 \)), \( \alpha \) has order 4, \( a^\alpha = a^{-1} \) (resp. \( b^\alpha = b^{-1} \)) and \( \alpha^2 = z \). \( \square \)

For \( i = 1, 3 \) and each of the groups \( G \) in Theorem C, we fix a generator \( \hat{g}_i \) of \( \hat{G}_i \).

**Proposition 4.2.** Let \( \hat{G} = \text{SL}_2(q) \) where \( q \equiv 5 \) (mod 8) and let \( G = U \left( \frac{q-1}{2} \right) \). There is an irreducible representation \( \rho_0: \hat{G} \to \mathbb{G} \) satisfying the following properties:

(i) The centralizer \( \mathbb{C}_G(\rho_0(\hat{G}_m)) \) has dimension \( \frac{q-1}{2} \).

(ii) The eigenvalues of \( \rho_0(\hat{g}_1) \) are \( i \) and \( -i \). The centralizer \( \mathbb{C}_G(\rho_0(\hat{G}_m)) \) has dimension \( \frac{(q-1)^2}{8} \).
(iii) The centralizer \( C_G(\rho_0(\hat{G}_{v_0})) \) has dimension \( \frac{(q-1)^2}{18} \).
(iv) The eigenvalues of \( \rho_0(\hat{g}_3) \) are \( \omega, \omega^5 \) and \(-1\), where \( \omega = e^{2\pi i/6} \). The dimension of \( C_G(\rho_0(\hat{G}_{v_3})) \) is given by
\[
\dim C_G(\rho_0(\hat{G}_{v_3})) = \begin{cases} 
\frac{(q-1)^2}{12} & \text{if } q \equiv 1 \pmod{3} \\
\frac{q^2-2q+9}{12} & \text{if } q \equiv 2 \pmod{3}.
\end{cases}
\]
(v) The restriction of \( \rho_0 \) to the Borel subgroup \( \hat{G}_{v_0} \) is irreducible.
(vi) The centralizer \( C_G(\rho_0(\hat{G}_{v_0})) \) has dimension \( \frac{q-1}{4} \).
(vii) The centralizer \( C_G(\rho_0(\hat{G}_{v_2})) \) has dimension \( \frac{q-1}{4} \).
(viii) The dimension of \( C_G(\rho_0(\hat{G}_{v_3})) \) is given by
\[
\dim C_G(\rho_0(\hat{G}_{v_3})) = \begin{cases} 
\frac{(q-1)^2}{48} & \text{if } q \equiv 1 \pmod{3} \\
\frac{(q-1)^2+32}{48} & \text{if } q \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** We take \( \rho_0 \) a representation realizing the degree \( \frac{q-1}{2} \) character \( \eta_1 \) of [SC21, Theorem A.5.1]. By [SC21, Theorem 4.1], we can take \( \rho_0 \) to be unitary. By [SC21, Lemma A.1.1] and [SC21, Lemma 4.5] we can prove parts (i) to (viii) by computing inner products of the restrictions of \( \eta_1 \). These restrictions are computed using Proposition 4.1. \( \square \)

5. The proof of Theorem C

For each of the groups \( G \) in Theorem C, we consider a closed edge path \( \pi \) in \( X_1^{OS}(G) \) such that attaching a free orbit of 2-cells along this path gives an acyclic 2-complex. We define \( x_0 = i(\xi) \), where \( i: \pi_1(X_1^{OS}(G), v_0) \to \Gamma_0 \) is the inclusion given by Brown’s theorem. We set \( x_i = x_{i\eta} \) for \( i = 1, \ldots, k \). Let \( \tilde{\eta} \) be the unique edge of \( X_1^{OS}(G) \) which lies in \( E - T \). We define \( y_0 = x_{\tilde{\eta}} \) and \( y_i = x_{i\eta} \) for \( i = 1, \ldots, k \).

Let \( M_k \) be the moduli of representations of \( \Gamma_k \) obtained from the representation \( \rho_0: \hat{G} \to U(m) \) of Proposition 4.2 using [SC21, Theorem 3.2]. Let \( \overline{M}_k \) be the corresponding quotient obtained using [SC21, Proposition 3.3]. Note that the equalities \( M_k = M_0 \times G^k \) and \( \overline{M}_k = \overline{M}_0 \times G^k \) still hold, because \( \rho_0(z) \in Z(G) \).

In what follows we consider the induced maps \( X_i(\tau) = \rho_\tau(x_i), Y_i(\tau) = \rho_\tau(y_i) \).

**Proof of Theorem C.** By [SC21, Corollary 3.4], \( M_k \) and \( \overline{M}_k \) are connected and orientable. A computation using Proposition 4.2 shows \( \dim \overline{M}_k = \dim G^{k+1} \) (alternatively, note that this also follows from [SC21, Lemma 7.2]). By Lemma 6.2, \( 1 \) is a regular point of \( X_0 \). By Propositions 7.1 and 7.2, \( \overline{Y}_0: \overline{M}_0 \to G \) has degree 0. The rest of the proof now continues in exactly the same way as the proof of [SC21, Theorem B]. See [SC21, Sections 8, 9 and 10] for more details. \( \square \)

6. The differential of \( X_0 \) at \( 1 \)

**Proposition 6.1.** The representation \( \rho_0 \) satisfies \( \text{Ad} \circ \rho_0(z) = 1 \), where \( \text{Ad}: G \to GL(T_1 G) \) is the adjoint representation.

**Proof.** This is immediate, for \( \text{Ad}(g) \) is the differential of the map \( x \mapsto gxg^{-1} \) and \( \rho_0(z) = -1 \) is central. \( \square \)

**Lemma 6.2.** For each of the groups in Theorem C, \( 1 \) is a regular point of \( X_0: M_0 \to G \).
Proof. Consider the representation $\text{Ad} \circ \rho_0 : \hat{G} \to \text{GL}(T_1 \mathbb{G})$ which is given by $g \cdot v = d_{\rho_0(g)}^{-1} L_{\rho_0(g)} \circ d_1 R_{\rho_0(g)}^{-1}(v)$. By [SC21, Proposition 2.4], we have $T_1 C_G(\rho_0(H)) = (T_1 \mathbb{G})^H$. By Proposition 6.1 we have $(1 - z) \cdot T_1 \mathbb{G} = 0$ and then Lemma 3.1 gives $T_1 \mathbb{G} = \sum_{e \in E} s_e \cdot T_1 C_G(\rho_0(\hat{G}_e))$. Then the result follows from [SC21, Theorem 3.7].

7. The degree of $\mathbf{Y}_0$

We now prove the degree of $\mathbf{Y}_0$ is 0 for each of the groups in Theorem C. When $q \equiv 13 \pmod{24}$, the approach is similar to that of [SC21, Propositions 9.1 and 9.2]. When $q \equiv 5 \pmod{24}$ the approach needs to be modified. Table 3 gives the value of $Y_0$ in the different cases we consider.

| $\hat{G}$ | $q \equiv 13 \pmod{24}$ | $q \equiv 5 \pmod{24}$ |
|-----------|------------------|------------------|
| $\text{SL}_2(\mathbb{Q})$ | $\tau_{\eta_1}^{-1} \tau_{\eta_2}^{-1} \rho_0(\eta_{31})$ | $\tau_{\eta_1}^{-1} \tau_{\eta_2}^{-1} \tau_{\eta_3}^{-1} \rho_0(\eta_{31})$ |

**Table 3.** The map $Y_0$, for each of the groups $G$ that we consider.

**Proposition 7.1.** In the case of $q \equiv 13 \pmod{24}$ the degree of $\mathbf{Y}_0 : \mathcal{M}_0 \to \mathbb{G}$ is 0.

**Proof.** Consider the manifold $M = C_G(\rho_0(\hat{G}_{\eta_1})) \times C_G(\rho_0(\hat{G}_{\eta_2})) \times C_G(\rho_0(\hat{G}_{\eta_3}))$, the group $H = C_G(\rho_0(\hat{G}_{v_1})) \times C_G(\rho_0(\hat{G}_{v_2}))$ and the free right action $M \acts H$ given by $(\tau_{\eta_1}, \tau_{\eta_2}, \tau_{\eta_3}) \cdot (\alpha_{v_1}, \alpha_{v_2}) = (\alpha_{v_1}^{-1} \tau_{\eta_0}, \alpha_{v_2}^{-1} \tau_{\eta_2} \alpha_{v_1}, \tau_{\eta_3} \alpha_{v_2})$.

For $q > 3$ we have

\[
\dim \frac{M}{H} = \dim M - \dim H = \dim \mathcal{M}_0 - \dim H - \dim C_G(\rho_0(\hat{G}_{\eta_1})) - \dim C_G(\rho_0(\hat{G}_{v_2})) = \dim \mathbb{G} - \frac{(q - 1)^2}{8} + \frac{q - 1}{4} < \dim \mathbb{G}.
\]

Note that the image of $\mathbf{Y}_0$ is the image of the map $M/H \to \mathbb{G}$ given by $(\tau_{\eta_0}, \tau_{\eta_2}, \tau_{\eta_3}) \mapsto \tau_{\eta_0}^{-1} \tau_{\eta_2}^{-1} \rho_0(\eta_{31})$. Since this map is differentiable we conclude that $\mathbf{Y}_0$ is not surjective and therefore has degree 0.

**Proposition 7.2.** In the case of $q \equiv 5 \pmod{24}$ and $q > 5$ the degree of $\mathbf{Y}_0 : \mathcal{M}_0 \to \mathbb{G}$ is 0.

**Proof.** By Lemma 2.1 (and parts (ii) and (iv) of Proposition 4.2) there are matrices $A_{\eta_1}, A_{\eta_3} \in \mathbb{G}$ such that $A_{\eta_3} = A_{\eta_1} A_{\eta_3}$ commutes with $C_G(\rho_0(\hat{G}_{v_2}))$ and letting

\[
K = \left(A_{\eta_3} C_G(\rho_0(\hat{G}_{\eta_1})) A_{\eta_3}^{-1}\right) \cap \left(A_{\eta_1} C_G(\rho_0(\hat{G}_{\eta_1})) A_{\eta_1}^{-1}\right)
\]

we have $\dim K \geq \frac{(q - 1)^2}{24}$. Consider the $H$-equivariant map $Z : \mathcal{M}_0 \to \mathbb{G}$ defined by

$\tau \mapsto A_{\eta_3} \tau_{\eta_0}^{-1} \tau_{\eta_2}^{-1} \tau_{\eta_3}^{-1} A_{\eta_3}^{-1} \cdot A_{\eta_1} \tau_{\eta_1}^{-1} A_{\eta_1}^{-1} \rho_0(\eta_{31})$. 

By [SC21, Proposition 3.11], the induced maps \( Y_0, Z : M_0 \to \mathcal{G} \) are homotopic. To conclude, we will prove that \( Z \) is not surjective. Let
\[
M = \left( A_{\eta_3} C_{\mathcal{G}}(\rho_0(\hat{G}_{\eta_3})) A_{\eta_3}^{-1} \right) \times \left( A_{\eta_1} C_{\mathcal{G}}(\rho_0(\hat{G}_{\eta_1})) A_{\eta_1}^{-1} \right)
\times \left( A_{\eta_5} C_{\mathcal{G}}(\rho_0(\hat{G}_{\eta_5})) A_{\eta_5}^{-1} \right)
\times \left( A_{\eta_3} C_{\mathcal{G}}(\rho_0(\hat{G}_{\eta_3})) A_{\eta_3}^{-1} \right) \times K
\]
and consider the free right action \( M \curvearrowright H \) given by
\[
(\tau_0, \tau_1, \tau_2, \tau_3) \cdot (\alpha_1, \alpha_3, \alpha_K) = (\alpha_1^{-1} \tau_0, \tau_1 \alpha_K, \alpha_3^{-1} \tau_2 \alpha_1, \alpha_K^{-1} \tau_3 \alpha_3).
\]
Finally, note that the image of \( Z \) is the image of the \( H \)-equivariant map \( T : M \to \mathcal{G} \) given by
\[
(\tau_0, \tau_1, \tau_2, \tau_3) \mapsto \tau_0^{-1} \tau_2^{-1} \tau_3^{-1} \tau_1^{-1} \rho(g_{\eta_1})
\]
which cannot be surjective since we have
\[
\dim M / H = \dim M - \dim H
\]
\[
= \dim M_0 - \dim \mathcal{H} + \dim C_{\mathcal{G}}(\rho_0(\hat{G}_{\eta_2})) - \dim K
\]
\[
\leq \dim \mathcal{G} + \frac{q - 1}{4} - \frac{(q - 1)^2}{24}
\]
(since \( q > 7 \))
\[
< \dim \mathcal{G}.
\]

References

[Dor71] Larry Dornhoff. Group representation theory. Part A: Ordinary representation theory. Marcel Dekker, Inc., New York, 1971. Pure and Applied Mathematics, 7.

[SC20] Iván Sadofschi Costa. Group actions of \( A_5 \) on contractible 2-complexes. Preprint, arXiv:2009.01755, 2020.

[SC21] Iván Sadofschi Costa. Group actions on contractible 2-complexes I. With an appendix by Kevin I. Piterman. Preprint, 2021.