Sim-to-Real for Projective Dynamics on Soft Robots via Differentiability: Meshing, Damping, and Actuation

Mathieu Dubied*, Mike Y. Michelis*, Andrew Spielberg†, and Robert K. Katzschmann*

*ETH Zurich, Switzerland - Email: {mdubied, mmichelis, rkk}@ethz.ch
†MIT, USA - E-Mail: aespielberg@csail.mit.edu

Abstract—An accurate, physically-based model of soft robots can unlock downstream applications in optimal control. The Finite Element Method (FEM) is an expressive approach for modeling highly deformable structures such as dynamic, elastomeric soft robots. Recently, Projective Dynamics (PD) has been proposed as an explicit FEM; however, PD lacks rigorous benchmarking against reality. In this paper, we compare virtual robot models simulated using PD with measurements from their physical counterparts. In particular, we examine several soft structures with different morphologies: a clamped beam under external force, a pneumatically actuated soft robotic arm, and a soft robotic fish tail. We benchmark and analyze different mesh resolutions and elements (tetrahedra and hexahedra), numerical damping, and the differentiability of PD through a differentiable solution (DiffPD). We also advance PD in application to soft robotics by proposing a predictive model for pneumatic soft robot actuation. Through our case-studies, we provide strategies and algorithms for matching real-world physics in simulation, making PD useful for soft robots.

I. INTRODUCTION

Soft robotics promises the rise of a new breed of robots with built-in compliance, damping, elastic energy storage, continuous actuation, and other morphologically-encoded behavioral features. However, one of the key challenges in realizing the potential of this field is the description of a simulation model that is useful for analyzing these effects and developing real-world controllers. In this paper, we compare Projective Dynamics (PD) [1] as a means of faithfully simulating soft robots specifically for real-world applications. This approach, based on the Finite Element Method (FEM), holds particular promise for modeling real-world soft robots due to its speed and accuracy; furthermore, its differentiability [2] enables the fast and accurate computation of gradients for efficient optimization of a robot’s design and control.

PD, however, introduces a suite of challenges when applied to soft robot modeling. PD uses the implicit Euler method as an integration scheme. Although this method is unconditionally stable, it also yields significant numerical damping — a side-effect that causes simulated structures to exhibit mechanical damping, even when this damping is not an explicit feature of the material model. Moreover, despite PD being based on continuum mechanics, it attempts to approximate continuous phenomena via discretization. As with other FEM methods, the choice of this discretization can have serious ramifications on the model’s accuracy. In order for PD to be useful to the soft robotics community, these challenges must be addressed.

To date, PD lacks rigorous benchmarking against reality. In this paper, we evaluate the ability of PD to model real-world structures, in particular deformable soft robots with pneumatic actuation, relying on the differentiable nature of the Differentiable Projective Dynamics (DiffPD) framework [2] in order to overcome the sim-to-real gap. Our three contributions toward this goal are:

1) Benchmarking of PD against reality for different meshes: We perform characterizing experiments on three different soft robotic structures and compare the measurement data to the PD simulations. We consider different mesh resolutions and two different mesh element types (tetrahedra and hexahedra), and also benchmark the obtained results to a commercial FEM solution.

2) Characterization of the numerical damping and modeling of the material damping: We characterize the numerical damping originating from the implicit Euler scheme. We benchmark this numerical damping against an analytical solution we derive. We then propose a new method to adjust the numerical damping so that it matches the material damping.

3) FE Models for soft robotic actuation: We compare PD simulations to reality for geometrically complex soft robot actuators. As precise modeling of the pressure chambers leads to unsatisfactory results in terms of speed and accuracy, we present “Muscle Models”: an alternative means of modeling pneumatic chambers.

These contributions address the most challenging aspects of bringing robots simulated using PD, and provide a guide for future practitioners to quickly simulate and optimize real-world soft robots.

The remainder of this paper is as follows. First, we summarize related work in Section II and review necessary technical background in Section III. We study three model structures: a soft clamped beam under external force, a pneumatically actuated robotic arm, and a pneumatically actuated fish tail. We detail our modeling approach in Section IV and then describe our experimental setup in Section V. We present results and analysis in Section VI and conclude with discussions and future work in Section VII.

II. RELATED WORK

In this section, we first describe established methods for simulating soft robots, and then summarize related progress toward overcoming their sim-to-real gap.
A. Soft Robot Simulation

Many approaches for accurately simulating soft robots have been proposed over the years. One approach is to use a model based on simplifying assumptions; for example, the so-called “Augmented Rigid Body” model [3] that is based on the “Piecewise Constant Curvature” (PCC) model [4]. A second approach is to derive data-driven models that are based on measurements performed on the system [5], [6]. A third approach is to employ models which discretize system continuum mechanics and solve the equations of motion for every element of the discretization [7].

In this work, we focus on discretization approaches, specifically FEM. In the field of soft robotics, the Simulation Open Framework Architecture (SOFA) framework is widely used. SOFA accurately models highly deformable structures and can be used as a basis for fast simulations using both traditional and reduced order simulation methods [8], [9], [7]. Other promising simulation frameworks have recently emerged, such as the DiffPD framework [10], based on PD [11]. PD employs a projection-based implicit Euler method to solve the equations of motion and to achieve more stable and efficient simulation than explicit schemes (though at the expense of accuracy), and has been optimized for performance over recent years [12], [13], [14]. DiffPD, and other simulators [15], [16], can even be designed to handle contact both accurately and efficiently.

Notably, DiffPD extends PD to the family of simulators that are differentiable. Differentiable simulators allow for model-based computation of gradients of any variable in a system (state, model, or control) with respect to any other variable. Differentiable simulators have been used to efficiently solve control and design optimization tasks in the soft-body regime [17], [18], [16], [19].

B. Sim-to-Real for Soft Robotics

As the field of soft robotics has grown, so too has interest in translating simulated results to the physical world. In the case of linear manipulators such as arms, simplified models have been effective. For example, soft robots modeled as rigid links with torsional spring joints have been used to motion plan soft robot arms through real-world maze-like environments [20], [21]. The more complex PCC models have also been used to create dynamic, soft robot arms with optimal control [22], which translate faithfully to reality.

Finite element models have long garnered special attention, however, due to their ability to scale to more complex geometry. COMSOL, Abaqus, and other commercial and open-source FEM packages have been deployed for soft robot and soft structure modeling, and have been especially useful for designing physical soft robotic structures [23], [24]. SOFA [25] has successfully been deployed in a number of applications of kinematic control of real-world soft robots, including real-time control [26], [27], [28]. Similar work has demonstrated dynamic analysis and control of soft robots, relying on reduced order models for real-time tractability [9], [29]. Other relevant work includes demonstrations of trajectory optimization and control on physical soft walking robots and grippers [30], [31]. While the above listed work takes steps to faithfully model soft robots and compare physical and virtual performance, they do not investigate systematic approaches and analyses needed in order to put the virtual and physical models into correspondence; this step is especially important and difficult when dealing with dynamic robots and often not treated in-depth.

The research works most similar to ours which take systematic approaches to overcoming the sim-to-real gap include [32] and [33]. [32] takes a data-driven approach to building complex voxel-based soft robots, tuning parameters for simpler designs and scaling up complexity, using a spring-based simulator [34], [33] demonstrates how repeated application of system identification of model parameters in the DiffPD framework can improve the performance of swimmers, but does not address issues inherent to the model itself. Our work analyzes the important nuances of soft robot modeling, specifically in a PD framework, examining the effects of design choices such as meshing, damping (material and numerical), and actuation models.

III. THEORETICAL BACKGROUND

A. Soft Robot Finite Element Model

PD, like more classical FEM solvers, solves the PDEs specified by continuum mechanics through spatial and time discretization.

1) Spatial Discretization: Starting from a continuous body with an infinite number of DOFs, the spatial discretization in finite elements (FE) reduces the system dimensionality to a tractable, finite count. The shape of the body is approximated by a mesh of finite elements, each spanned by n nodes. For volumetric elements in 3D space, each node has 3 translational DOFs and is described by its position and velocity vectors through Newton’s second law:

$$M \ddot{q} = f_{\text{ext}} + f_{\text{int}}(q) \quad (1)$$

where \( q = q(t) \in \mathbb{R}^{3n} \) describes the position of every node, \( v = v(t) \in \mathbb{R}^{3n} \) describes their velocity, \( M \in \mathbb{R}^{3n \times 3n} \) is the mass matrix, \( f_{\text{ext}} = f_{\text{ext}}(t) \in \mathbb{R}^{3n} \) accounts for external forces and \( f_{\text{int}}(q) = f_{\text{int}}(q, t) \in \mathbb{R}^{3n} \) for internal forces.

2) Time Discretization: PD and classical FEM solvers consider (1) for discrete time steps, for example by using the implicit Euler method. The implicit Euler method approximates the derivative of a function \( u_i = u(x_i, t_i) \) as

$$\frac{\partial u}{\partial t}(x_i, t_i) = \dot{u}(x_i, t_i) \approx \frac{u_{i+1} - u_i}{h} \quad (2)$$

where \( h \) is the time step defined as \( h = t_{i+1} - t_i \) and \( i = 0, 1, 2, ..., \). Applying the implicit Euler method to (1), we obtain the following discretized equations of motion:

$$q_{i+1} = q_i + h v_{i+1}$$
$$v_{i+1} = v_i + h M^{-1}(q) [f_{\text{int}}(q_{i+1}) + f_{\text{ext}}] \quad (3)$$

B. Projective Dynamics

1) Optimization Formulation: PD formulates (3) as an optimization problem to solve this equation efficiently:

$$\min_{q_{i+1}} \left\{ \frac{1}{2h^2} \left\| M^{\frac{1}{2}}(q_{i+1} - y_i) \right\|^2 + E_{\text{int}}(q_{i+1}) \right\} \quad (4)$$
with \( f_{\text{int}}(q_{i+1}) = -\nabla E_{\text{int}}(q_{i+1}) \) and \( y_i = q_i + h\dot{v}_i + h^2 M^{-1}f_{\text{int}} \). The first term is the momentum potential energy and the second term \( E_{\text{int}} \) is the elastic potential energy.

2) Elastic Energy Formulation: In nonlinear continuum mechanics, the elastic energy \( E_{\text{int}}(q_{i+1}) \) is a nonlinear function that serves as a “restoring,” spring-like potential energy that attracts each state to its undeformed state. Such potential energies can yield linear or nonlinear spring-like forces. Some common potential energies for modeling are described in [35]. PD introduces an elastic energy in a form convenient for efficient computation. This form of the elastic energy is introduced in [1] and is used by DiffPD [2] as well.

C. Material Damping

Damping is a phenomenon that reduces the internal velocity of a system by dissipating energy, and is commonly associated with the amplitudes oscillatory responses.

With material damping, we understand all damping scenarios in which the source of damping is physical, and not associated with the amplitudes oscillatory responses.

1) Single DOF System and Damping Ratio: The prototypical equation of motion of a single DOF system undergoing free vibration can be expressed as

\[
\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0
\]

(5)

with the damping ratio \( \zeta \) and the undamped frequency \( \omega_0 \).

We can classify the solutions \( x = x(t) \) of (5) depending on the value of the damping ratio \( \zeta \). Important for this paper are underdamped vibrations, where \( 0 \leq \zeta < 1 \).

2) Underdamped Vibrations: In the case of underdamped vibrations, the solution of (5) can be written as

\[
x(t) = e^{-\zeta\omega_0 t} \left[ C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t) \right]
\]

(6)

where \( \omega_d = \omega_0 \sqrt{1 - \zeta^2} < \omega_0 \) is the damped natural frequency.

As an inverse problem, it is also possible to determine damping ratio \( \zeta \) based on data. We first need to compute the logarithmic decrement \( \delta \), which is defined as follows:

\[
\delta = \frac{1}{m} \cdot \ln \left( \frac{X_k}{X_{k+m}} \right)
\]

(7)

where \( X_k \) is the \( k \)-th peak amplitude, and \( X_{k+m} \) is the amplitude of the peak \( m \) periods afterward. In this paper, we consider \( m \in \{1, 2, 3, 4\} \) depending on the time step \( h \) of our simulations. From the logarithmic decrement \( \delta \), one can determine the damping ratio \( \zeta \):

\[
\zeta = \frac{\delta}{(4\pi^2 + \delta^2)^{1/2}}
\]

(8)

D. Numerical Stability and Damping

Unlike material damping which occurs due to physical phenomena (viscous drag, heat exchange, etc.), numerical damping arises because of the time discretization method we use. The implicit Euler method, used in PD, is known to be stable but also induces numerical damping [36], [37].

1) Numerical Stability: To show the linear stability of the implicit Euler method, we use the definition of (A-)stability for linear systems as introduced by [36]. We will here only present the 1-dimensional case, as it suffices for the purpose of this work. Let us first define the following linear system:

\[
\begin{align*}
\frac{dx}{dt} &= \lambda x(t), \quad t \in [0, +\infty) \\
x(0) &= x_0 \in \mathbb{R}
\end{align*}
\]

(9)

The system (9) is stable if \( \exists A_0 > 0 \) such that:

\[
|x(t)| \leq A_0 |x_0| \quad \forall t \in [0, +\infty)
\]

(10)

Let us use this definition of stability on the implicit Euler method as introduced in Section III-A.2. Replacing the left hand side of upper line in (9) leads to

\[
\frac{x_{j+1} - x_j}{h} = \lambda x_{j+1} \quad \Rightarrow \quad x_{j+1} = \frac{1}{1 - h\lambda} x_j, \quad j \in \mathbb{N}
\]

(11)

Using \( x(0) = x_0 \), we can rewrite this relation as

\[
x_j = \left( \frac{1}{1 - h\lambda} \right)^j \cdot x_0
\]

(12)

From this last equation, we can see that \( |x_j| < A_0 |x_0| \) if and only if \( \Re(\lambda) \leq 0 \), meaning that the real part of \( \lambda \) has to be smaller or equal to zero. If this condition is satisfied, the system is stable regardless of the step size \( h \). Therefore, the implicit Euler scheme is said to be unconditionally stable.

2) Numerical Damping: The implicit Euler method is known to introduce numerical damping. We illustrate this through a simple example. We consider system (9) once again, and choose \( \lambda = i \cdot \omega_1 \) and \( x(0) = x_0 = 1 \):

\[
\begin{align*}
\frac{dx}{dt} &= i\omega x(t), \quad \omega \in \mathbb{R}, \quad t \in [0, +\infty) \\
x(0) &= 1
\end{align*}
\]

(13)

The exact solution of this equation is an oscillation whose amplitude does not change over time:

\[
x(t) = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)
\]

(14)

The result from the implicit Euler integration (12) is in this case stable as \( \Re(i\omega) \leq 0 \). However, we can observe that for a finite value of \( h \) and \( \omega \in \mathbb{R}^+ \), the amplitude of the oscillation is decreasing over time:

\[
|x_j| = \left| \left( \frac{1}{1 - h\omega} \right)^j \right| \to 0 \quad \text{for} \; j \to \infty
\]

(15)

This phenomenon is called numerical damping, as the source of the damping is purely an artifact of the numerical method that is used for time discretization.

IV. MODELING: MESHING, DAMPING, AND ACTUATION

A. Meshing

In this work, we investigate two FE types for meshing: hexahedra and tetrahedra. Hexahedral meshes are traditionally locked to axis-aligned voxel grids in their rest configuration, leading to aliasing when attempting to fit curved or angled surfaces. However, while more geometrically accurate, meshes based on linear tetrahedra suffer from locking when
used to model (nearly) incompressible materials (\( \nu \rightarrow 0.5 \)) [38] or problems with highly nonlinear dynamical behavior [39]. These meshes yield a much stiffer behavior of the simulated structure than the real equivalent.

**B. Damping**

Numerical damping is inherently present when using the implicit Euler method (as in PD) and can be quite large (cf. Section VI-A). This can lead to large error in simulated dynamic motion. In order to compensate for this error and fit the dynamic behaviour of real structures, we model velocity-dependent state forces that correct for numerical damping:

\[
f^l_i = \Lambda \cdot v^l_{i-1}
\]

where \( \Lambda \in \mathbb{R} \) is a tuning parameter and \( l \) the vertex to which the force is applied. We note that, contrasted with traditional FEM, PD’s internal energy formulation does not typically include system velocities; thus, we model damping as a velocity-dependent external force. We use the simulated velocity \( v^l_{i-1} \) of the previous time step to compute the current state force \( f^l_i \). In general, as long as \(|f^l_i|\) is smaller than the magnitude of numerical damping, the injection of these forces will not cause a stable system to become unstable; in our experiments using \( f^l_i \), stability has not been an issue.

**C. Actuation**

We design “Muscle Models” in order to capture the behavior of elastic pneumatics. This will be shown to be a helpful alternative to the exact modeling of pressure chambers. We employ a muscle energy [10], [40] which adds an additional energy term to \( E_{\text{int}} \) in (3); this can be understood as a spring-like energy that is attributed to elements of the mesh:

\[
E(q) = \frac{w}{2} \| (1 - a) F m \|^2
\]

where \( w \) can be interpreted as the stiffness of the spring, \( a \in \mathbb{R} \) is the actuation signal, \( F \) the deformation gradient and \( m \) the direction of the actuation. If \( a > 1 \), extension occurs and if \( a < 1 \), contraction occurs. While springs-like models are not pneumatic models as derived from first principles, they are much simpler to model in a PD simulator, and, as we demonstrate in section Section VI-B can be tuned to provide closely matching dynamic behavior.

**V. EXPERIMENTAL AND SIMULATION SETUPS**

**A. Experimental Setup**

In this section, we discuss the three deformable structures that will serve as a testbed for our analysis.

1) **Clamped Beam under External Force**: To benchmark the accuracy of PD, we begin by considering a simple, yet important structure: a clamped beam made of a highly deformable silicone elastomer (Fig. 1).

We consider two scenarios: 1) we keep the beam at its horizontal starting position and then release it so it bends downwards under gravity; and 2) in addition to gravity, we apply an external force \( F \) along the edge of the tip. To track the two reference points (“Left” and “Right”), we use a Motion Capture System from Qualysis. See Fig. 1 for details.

**2) Soft Robotic Arm**: The second structure is a pneumatically actuated soft robotic arm made of a silicone elastomer.

Our arm is comprised of a single segment of the soft robotic arm described in [29]. It has 4 pressure chambers that have a complex geometry, as shown in Fig. 2. We place motion markers to track the tip’s center position.

3) **Soft Fish Tail**: The last structure we consider is a pneumatic soft fish tail (Fig. 2). We focus on the position of the last motion marker placed on the tail’s spine.

**B. Simulation Setup**

In this section we explain our simulation setup. For the remainder of this paper, if not specifically mentioned, our simulations use a time step \( h \) of 0.01 s.

1) **Material Parameters**: The silicone elastomer used for the beam is Smooth-On Dragon Skin 10 with Shore Hardness 10A. For silicones, we assume incompressibility, i.e., \( \nu = 0.5 \) \( (\nu = 0.499 \) to avoid numerical singularities). The material density specified by Smooth-On is \( \rho = 1070 \text{ kg m}^{-3} \).

A precise knowledge of the material properties is needed to achieve an accurate simulation, but it is a difficult task for elastomers which generally show large variability due to their fabrication process. We use COMSOL as a ground truth, but we will show in Section VI-A that DifPD can also be used to calibrate material parameters using its differentiability.

We simulate Case E (see Table 1) and tune the Young’s modulus so that the static solution from COMSOL matches the real data. We choose Case E because the large load \( F \) is dominant in the final deformation state, and the effects of setup inaccuracies are negligible. This method leads to a Young’s modulus of \( E = 263824 \text{ Pa} \). This value is similar to the values found in related literature [41], [42].

2) **Material Models**: We simulated our structures, experimenting with corotational linear elastic and Neo-Hookean material models [35]. In our experiments, both models yield similar results; thus only present results obtained with the corotational linear elastic model, which is numerically simpler to employ.

**Fig. 1**: Schematic with edge forces (left) and photo (right) of the setup, including motion markers, for the clamped beam scenario.

**Fig. 2**: (a) Arm segment with ribbed pressure chambers [29]. (b) Soft fish tail.
3) **Optimization algorithm:** When optimizing any variable \( \alpha \) in DiffPD, the following steps are taken. First, we choose an initial guess for \( \alpha \) randomly. Then, we choose a loss function \( \mathcal{L} \) that mathematically specifies our objective. We then optimize \( \alpha \) to match the target; specifically, we iteratively 1) simulate the trajectory of the soft structure given the current \( \alpha \), 2) use DiffPD to compute model-based gradients \( \frac{\partial \mathcal{L}}{\partial \alpha} \), and 3) perform an optimization step to update the guess for \( \alpha \) using L-BFGS. We repeat this process until convergence.

**VI. RESULTS**

**A. Clamped Beam under External Force**

Studying the clamped beam structure allows us to solve for certain variables and thus reduce the number of factors that influence more complex simulations. For this mechanical structure, we focus on the vertical deformation of the beam’s tip (z axis). We present here the results for the point “Left” as defined in Figure [1]. The results for point “Right” are similar and therefore not shown here.

1) **Meshing, Position Tracking and Steady State Comparison:** We experiment with three loading scenarios in this section (Table I). These scenarios were obtained by varying the edge force \( F \). Additionally, we distinguish cases by their number of Degrees of Freedom (DOFs) and the type mesh elements: hexahedra (“Hex Mesh”) or tethrahedra (“Tet Mesh”).

| Case | Load Scenario | DOFs Hex | DOFs Tet | DOFs COMSOL |
|------|---------------|----------|----------|-------------|
| A-1  | \( F = 0 \) N | 4608     | 3834     | 7110        |
| A-2  | \( F = 0 \) N | 4608     | 183516   | 7110        |
| B    | \( F = 0.510 \) N | 4608 | 3834 | 7110 |
| C    | \( F = 0.510 \) N | 9900 | 71688 | 7110 |
| D    | \( F = 0.991 \) N | 4608 | 3834 | 7110 |
| E    | \( F = 0.991 \) N | 9900 | 71688 | 7110 |

**TABLE I:** Summary of the loading cases for the clamped beam.

For Case A-1, we find close correspondence between the real data, the COMSOL solution and the Hex Mesh solution: all three steady state solutions are in a range of 1 mm from each other. On the other hand, the Tet Mesh response is too stiff with 6 mm absolute error (Fig. 3 Case A-1).

In Case A-2, we verify if the Tet Mesh behaviour is an artifact stemming from a poor choice of the mesh resolution. We perform the same simulation while increasing the number of DOFs for the Tet Mesh from 3834 to 183516. As shown in Fig. 4, Case A-2, only a very small difference (0.438 mm difference) can be observed.

Keeping the same number of DOFs for the two mesh types as in Case A-1, we add an edge force at the tip of the beam for Case B and D. Under this configuration, the simulation yields incorrect results for both mesh types. However, increasing the number of DOFs of the Hex Mesh resolves this issue and leads to more accurate results. For the Tet Mesh, this refining step is not enough to elicit a sufficient response to match real data. We only present the results for Case D and E in Fig. 3 as Case B and C exhibit similar results. The accurate DiffPD simulation of case E using the Hex Mesh allows us to use the differentiability of the framework to calibrate the Young’s Modulus. Following the steps described in Section V-B, with \( \mathcal{L} = \| q_{\text{sim,final}} - q_{\text{real,final}} \|^2 \)

and for different initial values, the average optimized Young’s Modulus is \( E = 283271 \text{ Pa}, \) which is close to the solution found using COMSOL (\( E = 263824 \text{ Pa}, \) Section V-B).

2) **Numerical Damping by the Implicit Euler Method:**

**Analytical Solution**

As introduced in Section III-C, we can describe the damping of an oscillation using the damping ratio \( \zeta \). Here, we show an analytical relationship between the implicit Euler time step \( h \) and the induced numerical damping, characterized by the damping ratio \( \zeta \).

We consider the system (13), whose exact solution is the free undamped vibration (14). We can reformulate the solution obtained using the implicit Euler method (12) as:

\[
x_n = \left( \frac{1}{1 - h \omega_i} \right)^n = \frac{(1 + \omega h)^n}{(1 + \omega^2 h^2)^n} = \frac{e^{n \cdot \angle(1 + \omega h)}}{1 + \omega^2 h^2} \label{eq:20}
\]

The values of the positive peaks of this oscillation can be found at step \( n^* \) as follows:

\[
n^* \cdot \angle(1 + \omega h) = k \cdot 2 \pi \Rightarrow n^* = \frac{k \cdot 2 \pi}{\angle(1 + \omega h)}, \quad k \in \mathbb{N} \quad \label{eq:19}
\]

The amplitude of the oscillation at \( n^* \) is

\[
X_k = \frac{1}{(1 + \omega^2 h^2)^{n^*/2}} = \frac{1}{(1 + \omega^2 h^2)^{k \pi}} \label{eq:21}
\]

To express the damping ratio \( \zeta \), we first need to find an expression for the logarithmic decrement \( \delta \) [7]:

\[
\delta(h) = \frac{1}{m} \cdot \ln \left( \frac{X_{k+m}}{X_k} \right) = \frac{1}{m} \cdot \ln \left( \frac{(1 + \omega^2 h^2)^{(k+m)/2}}{(1 + \omega^2 h^2)^{k/2}} \right) = \frac{\pi}{\angle(1 + \omega h)} \cdot \ln (1 + \omega^2 h^2) \label{eq:22}
\]

Knowing the logarithmic decrement \( \delta \), it is straightforward to write the damping ratio \( \zeta \) as a function of \( h \) using [8]:

\[
\zeta(h) = \frac{\delta(h)}{(4\pi^2 + \delta(h)^2)^{1/2}} \label{eq:23}
\]
**Numerical Damping in PD** To experimentally describe the relation between damping ratio $\zeta$ and time step $h$, we simulated the Hex Mesh beam for different $h$ (Fig. 4).

With the values we gathered from the simulations, we can determine $\zeta$ using (7) and (8). The vertical displacement of the tip points of the beam can be approximated as a single DOF vibration, and we can observe that our analytical solution closely matches the value computed from the measurement data (Fig. 5). We note that this result generalizes to other FEM simulators using implicit Euler solvers, as shown by simulation results using COMSOL with implicit Euler method.

3) **Material Damping:** As introduced in Section IV-B, we let the numerical damping be the only source of damping in the PD simulation and use the differentiability of DiffPD to find $\Lambda$ so that the simulated solution closely approximates the real-world physical response. We define the optimisation objective to be that the simulated dynamic oscillation should match the envelope of the real oscillation. Plotting the tuning parameter $\Lambda$ as a function of time step $h$, a similar relation as the one between $\zeta$ and $h$ can be observed (Fig. 6).

**B. Soft Robotic Arm**

In this section, we simulate a more complex structure that includes pneumatic actuation: a soft robotic arm.

![Fig. 4: Simulations results of Case A-1 for a selection of different time steps $h$.](image)

![Fig. 5: Comparison between the computed values of $\zeta(h)$ from the DiffPD and COMSOL (using implicit Euler) simulations and our analytical solution.](image)

![Fig. 6: Material damping modeling results using a velocity dependent state force.](image)

![Fig. 7: Simulation results for the soft robotic arm and soft fish tail.](image)
interpolated values for which we did not optimize. For these predicted actuation values there is high agreement between the physical arm and simulated models, with an average relative position error of 13.07%.

In the multi-actuation example, we optimize the similar loss $L = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} ||q_{\text{sim},i}(t) - q_{\text{real},i}(t)||^2$ (note the term $q_{\text{real},i}(t)$ of $q_{\text{real},i}(T)$); this promotes dynamic position tracking over time. In our experiments, the actuation frequency is 20Hz. Fig. 9 shows that the optimized actuation sequence accurately matches the tracked physical arm.

These results demonstrate that Muscle Models can be accurately and predictively mapped to real pressure actuation.

C. Pneumatically Actuated Fish Tail

To validate our Muscle Models on a separate geometry, we model the pneumatically actuated soft robotic fish tail introduced in Section V-A. As with the arm, our Muscle Models’ actuations are optimized to follow the dynamic behaviour of the real-world robot (Fig. 10). We optimized for actuation sequences at 5Hz. Despite the step-wise behavior of the simulation, our optimized solution is still able to very closely match the motions of this second robot morphology.

VII. DISCUSSION AND FUTURE WORK

In this paper, we investigated several traits of Projective Dynamics, and provided recommendations on how to better match physical reality and overcome the sim-to-real gap.

Our first recommendation resulting from the experiments is with regards to meshing. The linear tetrahedral mesh elements produced inaccurate behaviour in the incompressible regime. This problem, named locking problem, is well known in the FEM community. On the other hand, the steady state results of the DiffPD simulations using the Hex Mesh are satisfying and allow for parameter calibrations. Based on this, we only recommend using hexahedral elements.

Our second recommendation is with respect to damping. Although the steady state results are simple to match precisely once system identification of the Young’s modulus is achieved, the dynamic behaviour of vanilla PD simulation accumulates significant error. PD, using the implicit Euler method, suffers of numerical damping. This numerical damping follows the analytical solution we derived. Depending on the exact use of DiffPD simulations, it is crucial to be aware of numerical damping, and a linear damping model is recommended to fit the material’s response.

Our third recommendation is with respect to actuator modeling. Complex structures, such as the ribbed pressure chambers, require a fine mesh to be modeled correctly. These complex geometries can generally not be modeled with a Hex Mesh, due to the regular shape of the elements of this type of mesh. Although a linear Tet Mesh is able to represent the geometry of the pressure chambers of the arm correctly, it fails to model the deformation of the arm correctly and is computationally costly. Muscle Models, correctly designed, can lead to good results in accuracy and computational efficiency. The differentiability of DiffPD allows efficient optimization of the actuator behavior.

We note a few further avenues for investigation. First, in order to model more complex geometry without experiencing element locking, higher-order tetrahedral elements should be considered. Second, since the Muscle Models show promise as a means of modeling pneumatic actuation, it would be useful to develop more general, data-driven rules for pressure-to-actuation mapping for other geometries. Finally, investigating more nonlinear materials or dynamics where Neo-Hookean responses play a role and evaluating their performance would be helpful for future applications.

REFERENCES

[1] S. Bouaziz, S. Martin, T. Liu, L. Kavan, and M. Pauly, “Projective dynamics: Fusing constraint projections for fast simulation,” in ACM Transactions on Graphics, vol. 33, no. 4. Association for Computing Machinery, 7 2014.

[2] T. Du, K. Wu, P. Ma, S. Wah, A. Spielberg, D. Rus, and W. Matusik, “Diffpd: Differentiable projective dynamics,” ACM Transactions on Graphics, vol. 41, no. 2, nov 2021.

[3] C. Della Santina, A. Bicchi, and D. Rus, “On an Improved State Parametrization for Soft Robots with Piecewise Constant Curvature and Its Use in Model Based Control,” IEEE Robotics and Automation Letters, vol. 5, no. 2, pp. 1001–1008, 4 2020.

[4] I. Robert J. Webster and B. A. Jones, “Design and kinematic modeling of constant curvature continuum robots: A review,” The International Journal of Robotics Research, vol. 29, no. 13, pp. 1661–1683, 2010.

[5] D. A. Haggerty, M. J. Banks, P. C. Curtis, I. Mezić, and E. W. Hawkes, “Modeling, reduction, and control of a helically actuated inertial soft robotic arm via the koopman operator,” arXiv: 2011.07939, 2020.
[6] D. Bruder, B. Gillespie, C. David Remy, and R. Vasudevan, “Modeling and Control of Soft Robots Using the Koopmann Operator and Model Predictive Control,” arXiv: 1902.02827, 2019.
[7] S. Tonkens, J. Lorenzetti, and M. Pavone, “Soft robot optimal control via reduced order finite element models,” in 2021 IEEE International Conference on Robotics and Automation (ICRA), 2021, pp. 12010–12016.
[8] C. Duriez, E. Cooevoet, F. Largilliere, T. Morales-Biezee, Z. Zhang, M. Sanz-Lopez, B. Carrez, D. Marchal, O. Goury, and J. Dequidt, “A geometric framework for simulation of soft robots with optimization-based inverse model,” in 2016 IEEE International Conference on Simulation, Modeling, and Programming for Autonomous Robots, SIMPAR 2016, Institute of Electrical and Electronics Engineers Inc., 2 2017, pp. 111–118.
[9] O. Goury and C. Duriez, “Fest. Generic, and Reliable Control and Simulation of Soft Robots Using Model Order Reduction,” IEEE Transactions on Robotics, vol. 34, no. 6, pp. 1565–1576, 12 2018.
[10] T. Du, K. Wu, P. Ma, S. Wah, A. Spielberg, D. Rus, and W. Matusik, “DiffPD: Differentiable Projective Dynamics with Contact,” arXiv: 2101.05917, 1 2021.
[11] S. Bouaziz, S. Martin, T. Liu, L. Kavan, and M. Pauly, “Projective dynamics: Fusing constraint projections for fast simulation,” ACM Transactions on Graphics, vol. 33, no. 4, pp. 1–11, 2014.
[12] H. Wang, “A Chebychev semi-iterative approach for accelerating projective and position-based dynamics,” ACM Transactions on Graphics, vol. 34, no. 6, pp. 1–9, 11 2015.
[13] A. Wang and Y. Yang, “Descent Methods for Elastic Body Simulation on the GPU,” ACM Transactions on Graphics, vol. 35, no. 6, 2016.
[14] M. Fratarcangi, V. Tibuldo, and F. Pellacini, “Vivace: A Practical Gauss-Seidel Method for Stable Soft Body Dynamics,” ACM Transactions on Graphics, vol. 35, no. 6, 11 2016.
[15] D. Chen, D. I. Levin, W. Matusik, and D. M. Kaufman, “Dynamics-aware numerical coarsening for fabrication design,” ACM Transactions on Graphics, vol. 36, no. 4, 2017.
[16] Y. Hu, J. Liu, A. Spielberg, J. B. Tenenbaum, W. T. Freeman, J. Wu, D. Rus, and W. Matusik, “ChainQueen: A real-time differentiable physical simulator for soft robotics,” Proceedings - IEEE International Conference on Robotics and Automation, vol. 2019-May, pp. 6265–6271, 6 2019.
[17] D. Hahn, P. Banzet, J. M. Bern, and S. Coros, “Real2Sim: Visco-elastic parameter estimation from dynamic motion,” ACM Transactions on Graphics (TOG), vol. 38, no. 6, 11 2019.
[18] Y. Hu, L. Anderson, T.-M. Li, Q. Sun, N. Carr, J. Ragan-Kelley, and F. Durand, “DiffFace: Differentiable Programming for Physical Simulation,” arXiv: 1910.00925, 10 2019.
[19] M. Geißen, D. Hahn, J. Zehnder, M. Bächer, B. Thomaszewski, and S. Coros, “ADD: Analytically Differentiable Dynamics for Multi-Body Systems with Frictional Contact,” ACM Transactions on Graphics (TOG), vol. 39, no. 6, p. 15, 11 2020.
[20] A. D. Marchese and D. Rus, “Design, kinematics, and control of a soft spatial fluidic elastomer manipulator,” The International Journal of Robotics Research, vol. 35, no. 7, pp. 840–869, 2016.
[21] A. D. Marchese, R. Tedrake, and D. Rus, “Dynamics and trajectory optimization for a soft spatial fluidic elastomer manipulator,” The International Journal of Robotics Research, vol. 35, no. 8, pp. 1000–1019, 2016.
[22] C. Della Santina, R. K. Katzschmann, A. Biechi, and D. Rus, “Dynamic control of soft robots interacting with the environment,” in 2018 IEEE International Conference on Soft Robotics (RoboSoft). IEEE, 2018, pp. 46–53.
[23] B. Mosadegh, P. Polygerinos, C. Keplinger, S. Wennstedt, R. F. Shepherd, U. Gupta, J. Shim, K. Bertoldi, C. J. Walsh, and G. M. Whitesides, “Pneumatic networks for soft robotics that actuate rapidly,” Advanced functional materials, vol. 24, no. 15, pp. 2163–2170, 2014.
[24] E. B. Joyee and Y. Pan, “A fully three-dimensional printed inchworm-inspired soft robot with magnetic actuation,” Soft robotics, vol. 6, no. 3, pp. 333–345, 2019.
[25] J. Allard, S. Cotin, F. Faure, P.-J. Bensousan, F. Poyer, C. Duriez, H. Delingette, and L. Grisoni, “Sofa-an open source framework for medical simulation,” in MMVR 15-Medicine Meets Virtual Reality, vol. 125. IOP Press, 2007, pp. 13–18.
[26] C. Duriez, “Control of elastic soft robots based on real-time finite element method,” in 2013 IEEE international conference on robotics and automation. IEEE, 2013, pp. 3982–3987.
[27] Z. Zhang, J. Dequidt, A. Kruszewski, F. Largilliere, and C. Duriez, “Kinematic modeling and observer based control of soft robot using real-time finite element method,” in 2016 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS). IEEE, 2016, pp. 5509–5514.
[28] E. Coevoet, A. Escande, and C. Duriez, “Optimization-based inverse model of soft robots with contact handling,” IEEE Robotics and Automation Letters, vol. 2, no. 3, pp. 1413–1419, 2017.
[29] R. K. Katzschmann, M. Thieffry, O. Goury, A. Kruszewski, T. M. Guerra, C. Duriez, and D. Rus, “Dynamically closed-loop controlled soft robotic arm using a reduced order finite element model with state observer,” in RoboSoft 2019 - 2019 IEEE International Conference on Soft Robotics. Institute of Electrical and Electronics Engineers Inc., 5 2019, pp. 717–724.
[30] J. M. Bern, G. Kumagai, and S. Coros, “Fabrication, modeling, and control of plush robots,” in 2017 IEEE International Conference on Intelligent Robots and Systems (IROS). IEEE, 2017, pp. 3739–3746.
[31] J. M. Bern, P. Banzet, R. Poranne, and S. Coros, “Trajectory optimization for cable-driven soft robot locomotion,” in Robotics: Science and Systems, vol. 1, no. 3, 2019.
[32] S. Kriegman, A. M. Nasab, D. Shah, H. Steele, G. Branin, M. Levin, J. Bongard, and R. Kramer-Bottiglio, “Scalable sim-to-real transfer of soft robot designs,” in 2020 3rd IEEE International Conference on Soft Robotics (RoboSoft). IEEE, 2020, pp. 359–366.
[33] T. Du, J. Hughes, S. Wah, W. Matusik, and D. Rus, “Underwater soft robot modeling and control with differentiable simulation,” IEEE Robotics and Automation Letters, vol. 6, no. 3, pp. 4994–5001, 2021.
[34] J. Hiller and H. Lipson, “Dynamic simulation of soft multimaterial 3d-printed objects,” Soft robotics, vol. 1, no. 1, pp. 88–101, 2014.
[35] E. Sifakis, “FEM simulation of 3D deformable solids: A practitioner’s guide to theory, discretization and model reduction,” ACM SIGGRAPH 2012 Courses, SIGGRAPH '12, pp. 1–35, 2012.
[36] H. Ammari, W. Wei, and Y. Sanghyeon, “Numerical Methods for Ordinary Differential Equations,” ETH Zürich, Tech. Rep., 2018.
[37] Y. J. Chen, U. M. Ascher, and D. K. Pai, “Exponential Rosenbrock-Euler Integrators for Elastodynamic Simulation,” IEEE Transactions on Visualization and Computer Graphics, vol. 24, no. 10, pp. 2702–2713, 10 2018.
[38] I. Babuska and M. Suri, “Locking effects in the finite element approximation of elasticity problems,” Numerische Mathematik, vol. 62, pp. 439–463, 1992.
[39] M. A. Puso and J. Solberg, “A stabilized nodally integrated tetrahedral,” International Journal for Numerical Methods in Engineering, vol. 67, no. 6, pp. 841–867, 8 2006.
[40] S. Min, J. Won, S. Lee, J. Park, and J. Lee, “Softcon: Simulation and control of soft-bodied animals with biomimetic actuators,” ACM Transactions on Graphics (TOG), vol. 38, no. 6, pp. 1–12, 2019.
[41] P. Rothemund, A. Ainla, L. Belding, D. J. Preston, S. Kurihara, Z. Suo, and G. M. Whitesides, “A soft, bistable valve for autonomous control of soft actuators,” Science Robotics, vol. 3, no. 16, 3 2018.
[42] F. Connolly, J. C. Walsh, and K. Bertoldi, “Automatic design of fiber-reinforced soft actuators for trajectory matching,” Proceedings of the National Academy of Sciences of the United States of America, vol. 114, no. 1, pp. 51–56, 1 2017.