Chaotic Amplification of Neutrino Chemical Potentials by Neutrino Oscillations in Big Bang Nucleosynthesis

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ABSTRACT

We investigate in detail the parameter space of active-sterile neutrino oscillations that amplifies neutrino chemical potentials at the epoch of Big Bang Nucleosynthesis. We calculate the magnitude of the amplification and show evidences of chaos in the amplification process. We also discuss the implications of the neutrino chemical potential amplification in the Big Bang Nucleosynthesis. It is shown that with a $\sim 1 \text{ eV } \nu_e$, the amplification of its chemical potential by active-sterile neutrino oscillations can lower the effective number of neutrino species at Big Bang Nucleosynthesis to significantly below 3.

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I. INTRODUCTION

Neutrino oscillations have been suggested to explain several experimental results and if proven true, they will represent a significant step toward physics beyond the standard model of particle physics [1]. Mixings between active neutrinos ($\nu_e$, $\nu_\mu$ or $\nu_\tau$) and sterile neutrinos (hypothetical neutrinos that do not interact with known particles via the strong, weak or electromagnetic interactions) are one possible source of neutrino oscillations. The most stringent constraints on the parameters of active-sterile neutrino mixings come from cosmological and astrophysical considerations [2–6]. In particular, the active-sterile neutrino oscillations at the epoch of Big Bang Nucleosynthesis (BBN) have been investigated extensively and tight constraints have been obtained based on the primordial $^4$He abundance in our universe. Interestingly, it was recently pointed out by Foot, Thomson and Volkas [7] that some parameter space of the active-sterile neutrino mixings can amplify neutrino asymmetries (neutrino chemical potentials) so that the previous constraints on active-sterile neutrino mixings based on BBN can be alleviated [7,8]. Also an asymmetry in the electron neutrino sector as large as $\sim 0.1$ at the time of BBN can change significantly the BBN prediction of the primordial $^4$He abundance.

In section 2 of this paper we expand the original investigation of Foot, Thomson and Volkas [7], by calculating the parameter space that amplifies neutrino chemical potentials and the magnitude of the amplification. But instead of relying on a simplified equation that only applies outside the resonant regime and when neutrinos are incoherent, we analyze the problem based on the original equations in the density matrix formalism, both analytically and numerically. Our analyses reveal many interesting features of the amplification process that cannot be revealed by the simplified approach. For example, the neutrino asymmetry can be oscillatory long after the initial resonant crossing. There are evidences which suggest that the oscillatory asymmetry is chaotic. As a result, although the order of magnitude of the final neutrino chemical potential is readily predictable, the sign of the chemical potential is very sensitive to the mixing parameters and the input parameters of numerical calculations.
This oscillatory behavior and evidences of a chaotic amplification are probed in section 2.

In section 3, we discuss two implications of our results in section 2 and show how active neutrinos as dark matter candidates—having $\sim 1\text{eV}$ mass—can lower the effective number of neutrino species in BBN to significantly below 3 by mixing with a lighter sterile neutrino.

II. FORMALISM AND CALCULATIONS

Throughout the paper, we adopt a unit in which $\hbar = c = k = 1$. We also use a convention to denote the number density of a particle $i$ by $N_i$, and the number density relative to its equilibrium value ($2\zeta(3)T^3/\pi^2$ for photons, $3/4$ of that for electron-positrons, and $3/8$ of that for neutrinos) by $n_i$. The Hubble expansion rate is $H = 7T^2/M_p$ where $M_p$ is the Planck mass and $T$ the temperature of the universe. Finally, $T_6$ denotes $T$ in the unit of MeV. Since we are only concerned with the era of BBN, we limit our discussion to $1 \lesssim T_6 \lesssim 100$.

Mixtures of an active neutrino $\nu_\alpha$ and a sterile neutrino $\nu_s$ can be described by a density matrix

$$\rho_{\nu} = \begin{pmatrix} \rho_{\alpha\alpha} & \rho_{\alpha s} \\ \rho_{\alpha s} & \rho_{ss} \end{pmatrix} = \frac{P_0 I + \mathbf{P} \cdot \vec{\sigma}}{2},$$

(1)

where $\vec{\sigma}$ are the Pauli matrices. The number densities of $\nu_\alpha$ and $\nu_s$ in the mixture, relative to their equilibrium values, are respectively

$$n_{\nu_\alpha} = \frac{P_0 + P_z}{2}, \quad n_{\nu_s} = \frac{P_0 - P_z}{2}.$$  

(2)

The evolution of the total relative number density of the neutrino mixture at the epoch of BBN is

$$\dot{P}_0 = \sum_{i=e,\nu; \beta \neq \alpha} \langle \Gamma(\nu_\alpha \bar{\nu}_\alpha \rightarrow i\bar{i}) \rangle (n_i n_{\bar{i}} - n_{\nu_\alpha} n_{\bar{\nu}_\alpha}),$$

(3)

where $\langle \Gamma \rangle$ are reaction rates averaged over a thermal spectrum. Values of $\langle \Gamma \rangle$ are listed in table 1 of ref. [3] or ref. [5]. The evolution of $\mathbf{P}$ is
\[
\dot{\mathbf{P}} = \mathbf{V} \times \mathbf{P} + \dot{P}_0 \hat{z} - D \mathbf{P}_\perp \tag{4}
\]

where \( \mathbf{V} \) represents the frequency and the axis of the oscillation in the \( \mathbf{P} \)-space, and \( \mathbf{P}_\perp = P_x \hat{x} + P_y \hat{y} \). The \( D \)-term represents the damping of \( \mathbf{P}_\perp \) due to neutrino interactions which constantly reduce a mixed neutrino state into an eigenstate of either \( \nu_\alpha \) or \( \nu_s \).

At the epoch of BBN,

\[
V_x = \frac{\delta m^2}{2E} \sin 2\theta, \quad V_y = 0, \quad V_z = -\frac{\delta m^2}{2E} \cos 2\theta + V^L_\alpha + V^T_\alpha, \tag{5}
\]

where \( \delta m^2 \) and \( \theta \) are the usual vacuum mixing parameters, and \( E \) is the energy of the neutrinos. \( V^L_\alpha \) is the contribution of the matter effect from asymmetries in the background plasma \cite{9}:

\[
V^L_\alpha = \sqrt{2} G_F N_\gamma \left[ L_0 + 0.375 \left( 2(n_\nu_\alpha - n_\bar{\nu}_\alpha) + \sum_{\nu_\beta \neq \nu_\alpha} (n_\nu_\beta - n_\bar{\nu}_\beta) \right) \right]
\approx 0.13 G_F T^3 \left[ 8L_0/3 + 2(n_\nu_\alpha - n_\bar{\nu}_\alpha) + \sum_{\nu_\beta \neq \nu_\alpha} (n_\nu_\beta - n_\bar{\nu}_\beta) \right] \tag{6}
\]

where \( L_0 \) represents the contributions from the baryonic asymmetry as well as the asymmetry in electron-positions, and is \( \sim 10^{-9} \). \( N_\gamma \) is the photon number density. The \( n_\nu - n_\bar{\nu} \) terms represent the asymmetries in active neutrinos and thus their non-zero chemical potentials. If \( \xi_\nu \)-the chemical potential of \( \nu \) divided by \( kT \)-is much smaller than 1, \( n_\nu - n_\bar{\nu} \approx 1.8 \xi_\nu \).

\( V^T_\alpha \) is the contribution of the matter effect due to a finite temperature \cite{9}:

\[
V^T_\alpha = -\sqrt{2} G_F N_\gamma \left[ 12.61 ET (n_\nu_\alpha + n_\bar{\nu}_\alpha)/4M_Z^2 + 12.61 ET / M_W^2 \right], \quad \alpha = e; \quad \alpha = \mu, \tau. \tag{7}
\]

It has been shown that eqs. (3)–(7) give a good description of neutrino oscillations in BBN if the average neutrino energy \( E \approx 3.151T \) is inserted in \( \mathbf{V} \) and if \( D \) is thermally averaged \cite{3}. Therefore, numerically,

\[
V^T_\alpha \approx -250 G_F^2 T^5, \quad \alpha = e; \quad \approx -70 G_F^2 T^5, \quad \alpha = \mu, \tau. \tag{8}
\]
The damping coefficient $D$, consisting of contributions from both elastic scatterings and inelastic scatterings of $\nu_\alpha$, is \[3,5\]

\[ D \approx (1.3 + 0.4n_\nu + 0.5n_{\bar{\nu}_\alpha})G_F^2T^5, \quad \alpha = e; \]

\[ \approx (0.8 + 0.4n_\nu + 0.5n_{\bar{\nu}_\alpha})G_F^2T^5, \quad \alpha = \mu, \tau. \]

The reason we leave out $n_\nu$ and $n_{\bar{\nu}_\alpha}$ explicitly without approximating them to 1 is for the convenience of calculating the difference in the coefficient between the $\nu_\alpha$-$\nu_s$ and $\bar{\nu}_\alpha$-$\bar{\nu}_s$ oscillations. Since we often compare $D$ to the Hubble expansion rate $H$, we note

\[ D \approx 0.5T^3H. \]

The initial condition for eq. (3) and (4) is usually chosen to be $P_0 = P_z = 1$ and $P_x = P_y = 0$, at $T_{\text{init}}$ when $V_\alpha^T$ dominates over $\delta m^2/2E$, i.e.,

\[ T_{\text{init}} \gg 15 \left| \frac{\delta m^2 \cos 2\theta}{eV^2} \right|^{1/6} \]

(for the moment we assume any neutrino asymmetry is negligible). That is, the neutrino ensemble consists purely of $\nu_\alpha$, which is a good approximation because $V_z \gg V_x$ so that $\mathbf{V}$ is almost aligned with the $\hat{z}$-axis.

As the universe expands and its temperature drops, $|V_x/V_z|$ becomes larger, the amplitude of the oscillation consequently increases. Eventually, if there is no amplification of neutrino asymmetries, $V_\alpha$ becomes negligible, and $\mathbf{V}$ settles down into its vacuum value. During the process, if the mixing has $\delta m^2 < 0$ ($\nu_\alpha$ heavier than $\nu_s$), a resonance can occur when $\mathbf{V}$ crosses the $\hat{x}$-axis at a temperature

\[ T_{\text{res}} \approx 13(16) \cdot \left| \frac{\delta m^2 \cos 2\theta}{1eV^2} \right|^{1/6} \text{ MeV} \quad \text{for } \alpha = e, (\mu, \tau). \]

$\mathbf{P}$ and $\mathbf{V}$ before and after the resonance are illustrated in figure 1.

During the oscillation, the interactions between $\nu_\alpha$ and the background plasma play two roles. First, the interactions (including both elastic and inelastic ones, represented by the $D$-term) reduce mixed neutrino states into either $\nu_\alpha$ or $\nu_s$, which effectively damp the
amplitude of $\mathbf{P}_\perp$ and at the same time randomize the phase of the neutrino oscillation. When the regenerated $\nu_\alpha$ (and $\nu_s$ but mostly $\nu_\alpha$) oscillate into $\nu_s$ (and $\nu_\alpha$) again, the portion of $\nu_\alpha$ in excess of $\nu_s$, $P_z$, decreases toward 0. Secondly, the inelastic process--$\nu_\alpha\bar{\nu}_\alpha$ pair productions (the $\dot{P}_0\mathbf{z}$ term)--constantly replenishes the number of $\nu_\alpha$ that is being depleted by oscillation, maintaining its population as a full relativistic species as long as such pair productions are potent ($T \gtrsim 3$ MeV for $\nu_e$ and $\gtrsim 5$ MeV for $\nu_\mu$ or $\nu_\tau$).

Eq. (1)–(11) can be equally applied to the anti-neutrino sector, with notations for particles and anti-particles switched and $L_0$ replaced by $-L_0$. Apparently, since $\nu_\alpha$ and $\bar{\nu}_\alpha$ can only be produced as pairs, $\dot{\bar{P}}_0 = \dot{\bar{P}}_0$.

The relevance to BBN comes at $T \sim 1$ MeV when the neutron to proton ratio freezes out. If a significant population of $\nu_s$ is produced, or a significant asymmetry in $\nu_e\bar{\nu}_e$ is generated through the active-sterile neutrino oscillations, the neutron to proton ratio can be affected and the resultant $^4\text{He}$ primordial abundance altered from the standard BBN predictions. When neutrino asymmetries are negligible, to be consistent with the observed primordial $^4\text{He}$ abundance requires [3,5]

$$\delta m^2 \sin^4 2\theta \lesssim 10^{-9}\text{eV}^2$$  \hspace{1cm} (13)

(The bound on the $\nu_e$-$\nu_s$ mixing is tighter on the low $\delta m^2$ end. See refs. [3] and [5] for precise constraints.)

Under conditions that

$$|V| \gg D \gg |\mathbf{V}|/|V|,$$  \hspace{1cm} (14)

i.e., the damping of $\mathbf{P}$ and the change in $\mathbf{V}$ are negligible within one cycle of oscillation of $\mathbf{P}$, eq. (14) can be simplified to the lowest order to

$$P_x = V_x P_z / V_z, \quad P_y = 0, \quad \dot{P}_z = -D V_x^2 P_z / (V_x^2 + V_z^2) + \dot{P}_0.$$  \hspace{1cm} (15)

In the epoch of our concern, and in the absence of an amplification of neutrino asymmetries, eq. (14) is satisfied except near the resonance region where $|V_z| \sim |V_x|$. This is because
\( V^T_\alpha = \mathcal{O}(10^2)D \) and \( |\dot{V}_\alpha/V_\alpha| \sim H \ll D \) at \( T_6 \geq 1 \). We will discuss the case of amplified asymmetries later in the section.

Similarly for anti-neutrinos, the approximate equations are

\[
\begin{align*}
\dot{P}_x &= V_x \dot{P}_x/V_z, \quad \dot{P}_y = 0, \quad \dot{P}_z = -DV^2_x \dot{P}_x/(V_x^2 + V_z^2) + \dot{P}_0 \quad (16)
\end{align*}
\]

under conditions that \( |\nabla| \gg \bar{D} \gg |\dot{\nabla}|/|\nabla| \). Assuming there is no asymmetries in neutrinos other than the oscillating \( \nu_\alpha \bar{\nu}_\alpha \) sector,

\[
\dot{V} = V - 2\beta(\Delta P + 8L_0/3) = V_0 - \beta(\Delta P + 8L_0/3) \quad (17)
\]

where \( \beta = 0.375\sqrt{2}G_FN_\gamma \approx 0.13G_FT^3, V_0 = -\delta m^2 \cos2\theta/2E + V^T_\alpha \) and \( \Delta P_p = P_p - \bar{P}_z = 2(n_{\nu_\alpha} - n_{\bar{\nu}_\alpha}) \). It is also noted that \( \bar{D} - D \approx 0.05\Delta P_3G^2_FT^5 \ll D\Delta P_3 \).

The asymmetry in the \( \nu_\alpha \bar{\nu}_\alpha \) sector can then be described by \( \Delta P_z \) which to its lowest order satisfies

\[
\frac{\Delta P_z}{V_z} = \frac{DP_zV^2_z}{V_x^2 + V_z^2} \left( \frac{1}{V_x^2 + V_z^2} - \frac{1}{V_x^2 + V_z^2} \right) - D \frac{V^2_x}{V_x^2 + V_z^2} \Delta P_z + \frac{V^2_x}{V_x^2 + V_z^2} \Delta P_z + (\bar{D} - D) \frac{V^2_x}{V_x^2 + V_z^2} \Delta P_z
\]

\[
\approx \frac{DV^2_z}{V_x^2 + [V_0 - \beta(\Delta P_z + 8L_0/3)]^2} \left\{ \frac{4V_0\beta(\Delta P_z + 8L_0/3)P_z}{[V_x^2 + [V_0 + \beta(\Delta P_z + 8L_0/3)]^2]} \right\} \Delta P_z \quad (18)
\]

The equation resembles equation (15) of Foot, Thomson and Volkas [7] except that their equation omitted the second term and \( L_0 \), and assumed \( P_z \) to be 1.

When \( V_0 < 0 \), eq. (18) is a damping equation for \( |\Delta P_z| > 8L_0/3 \), and no amplification of \( |\Delta P_z| \) to \( \gg 10^{-9} \) can occur. This rules out \( \nu_\alpha-\nu_s \) mixings with \( \delta m^2 > 0 \) (\( \nu_s \) heavier than \( \nu_\alpha \)). Only when \( \delta m^2 < 0 \) and \( V_0 \) switches to a positive value (resonance crossing) before \( T \sim 1 \) MeV does an amplification of \( \Delta P_z \) become plausible.

For mixings with \( \delta m^2 < 0 \), \( V_0 \) still starts out negative at high temperatures \( \gg T_{res} \). Any \( |\Delta P_z| > 8L_0/3 \) will be damped toward an asymptotic value such that \( \Delta P_z \rightarrow 0 \). Thus

\[
\Delta P_z \rightarrow -\frac{8}{3}L_0 + \frac{8V^2_xL_0/3}{4V_0\beta P_z} \approx -\frac{8}{3}L_0
\]

when \( |V_0| \gg |V_x| \).
When the system enters the resonant regime, eq. (18) does not apply. We have to go back to the original equation (4) and its anti-neutrino counterpart, which give

\[
\begin{align*}
\dot{\Delta P}_x &= -V_0 \Delta P_y - \beta(\Delta P_z + 8L_0/3)(P_y + \bar{P}_y) - D\Delta P_x, \\
\dot{\Delta P}_y &= V_0 \Delta P_x + \beta(\Delta P_z + 8L_0/3)(P_x + \bar{P}_x) - V_x \Delta P_z - D\Delta P_y, \\
\dot{\Delta P}_z &= V_x \Delta P_y.
\end{align*}
\]

(20)

where \(\Delta P_x = P_x - \bar{P}_x\) and \(\Delta P_y = P_y - \bar{P}_y\). If we don’t want \(\nu_s\) to be brought into equilibrium, we have to restrict our discussions to the parameter space that satisfies eq. (13). Then: (1) the resonance crossing is non-adiabatic; (2) \(D\) has to be small enough compared to the time scale of the resonance crossing so that most of the neutrinos do not scatter during the crossing. The non-adiabatic change in \(V\) leads to a coherent oscillation of \(P\) around the new \(V\) (see figure 1(b)). The impotency of scatterings enables this coherency to be maintained throughout the resonant regime and beyond until \(D > \sim |\dot{V}|/|V|\) so that the interactions have enough time to randomize the phases of neutrinos again. It also implies that the \(D\)-term in eq. (20) can be dropped in resonance.

When the temperature was high above \(T_{\text{res}}\) so that eq. (18) applies, \(\Delta P_x \approx V_z \Delta P_z/V_z\), \(\Delta P_y\) and \(\dot{\Delta P}_y\) are approximately 0. But in and right after resonance, the approximation breaks down because of the rapid change of \(V_z\) and \(\bar{V}_z\). Instead \(\Delta P_y\) becomes oscillatory with a frequency of \(\sim V_z\) (figure 1(b)). \(|\Delta P_z + 8L_0/3|\) will quickly be of order \(10^{-9}\), but a more important question is whether \(|\Delta P_z + 8L_0/3|\) (and thus \(|\Delta P_z|\)) can be amplified to \(\gg 10^{-9}\). Assuming that \(|\beta \Delta P_z| \ll V_0\), the amplitude of the oscillating \(\Delta P_y\) will be of order \(\sim |P_z V_x(V_z^{-1} - \bar{V}_z^{-1})| \approx |2V_x^2 \beta \Delta P_z/V_0^2|\) (from now on \(P_z \sim 1\) and \(L_0\) is dropped for simplicity because \(|\Delta P_z + 8L_0/3| \sim |\Delta P_z|\)). The amplification of \(|\Delta P_z|\) depends on whether \(\dot{\Delta P}_z\) has enough time to change \(\Delta P_z\) by a factor of more than 1, i.e.,

\[
\left|\frac{2V_x^2 \beta}{V_0^2}\right| \gtrsim V_0.
\]

(21)

Since \(V_0\) is a changing quantity, the condition of amplification depends on which \(V_0\) to choose. For a crude estimate, a reasonable choice is the \(V_0\) at the time when \(\Delta P_y\) oscillates one cycle since the resonance (so that eq (21) is meaningful). So
\[ V_0 \sim \dot{V}_0 \cdot V_0^{-1} \sim \frac{H}{V_0} \cdot \frac{\delta m^2}{2E}. \]  

Solving the equation assuming \( E = 3.151T \) yields \( V_0 \sim 10^{-2} |\delta m^2|^{-0.25} \cdot |\delta m^2/2E|. \) Thus the condition of amplifying \( \Delta P_z \), i.e., eq. (21), is
\[
|\delta m^2|^{-5/12} \sin^2 2\theta \gtrsim 10^{-12}. \tag{23}
\]

The growth of \( |\Delta P_z| \) is limited once \( |\beta \Delta P_z| \gg V_0 \) (when \( |\Delta P_z| \gg |\delta m^2/9eV^2|T_6^{-4} \)). Because at this moment, the amplitude of \( \Delta P_y \) becomes \( \sim |2V_x/\beta \Delta P_z| \), and the amplitude of \( \Delta P_z/\Delta P_z \) becomes \( \sim 2V_x^2/\beta \Delta P_z^2 \), proportional to \( \Delta P_z^{-2} \).

An interesting feature of \( \Delta P_z \), shown both from eq. (21) and numerical calculations, is that it keeps oscillating below the resonant temperature (figure 2). This is a direct consequence of the coherent oscillation of \( P \) and \( \bar{P} \). As a result of the oscillation, the sign of \( \Delta P_z \) flips (so does \( V_z \) and \( \bar{V}_z \)) unless the change of \( \Delta P_z \) in each cycle is smaller than the amplitude of \( \Delta P_z \) itself, i.e.,
\[
|\Delta P_z| > \frac{V_x^2}{|\beta \Delta P_z|} \cdot |\beta \Delta P_z|^{-1}. \tag{24}
\]

Since for parameters that satisfy eq. (23), the amplitude of \( \Delta P_z \gtrsim (|\delta m^2|/9eV^2)T_6^{-4} \), a rough realization of the condition that \( \Delta P_z \) will not be oscillatory between positive and negative values is
\[
\frac{|\delta m^2|}{\sin^2 2\theta} \gtrsim 9T_6^4 eV^2. \tag{25}
\]

At \( T_6 = 1 \), \( \Delta P_z \) will not be oscillatory if \( |\delta m^2|/\sin^2 2\theta \gtrsim 9eV^2 \).

Once \( \Delta P_z \) stops flipping its sign, \( |\dot{V}|/|V| \) and \( |\dot{\bar{V}}|/|\bar{V}| \) decrease dramatically so that the damping \( D \)-term becomes more and more important. Eventually eq. (18) reappears. Assuming \( P_z \sim 1 \) and neglecting \( L_0 \) and \( V_x \) in the denominator eq. (18) yields
\[
\Delta \dot{P}_z \approx D \frac{V_x^2}{(V_0 - \beta \Delta P_z)^2} \left[ \frac{4V_0\beta}{(V_0 + \beta \Delta P_z)^2} - 1 \right] \Delta P_z. \tag{26}
\]

Since \( V_0 > 0 \) after the initial resonance, this is an amplification equation if
\[
\frac{4V_0\beta}{(V_0 + \beta \Delta P_z)^2} > 1, \quad \text{or} \quad \delta m^2 \lesssim 10^2 \text{eV}^2.
\] (27)

The amplification will not stop until
\[
D \frac{V_z^2}{(V_0 + \beta \Delta P_z)^2} \frac{4V_0\beta}{(V_0 - \beta \Delta P_z)^2} \lesssim H,
\] (28)

which occurs at \( |\beta \Delta P_z| \gtrsim V_0 \sim |\delta m^2/2E| \) and
\[
\left( \frac{18T_6^7 \sin^2 2\theta}{|\delta m^2|} \right) \left( \frac{\delta m^2/2E}{\beta \Delta P_z} \right)^4 \lesssim 1.
\] (29)

At \( T_6 \sim 1 \), nearly all the mixing parameters that show no oscillatory \( \Delta P_z \) have \( 18T_6^7 \sin^2 2\theta/|\delta m^2| < 1 \), so \( |\beta \Delta P_z| \) is limited to \( \sim |\delta m^2/2E| \) and
\[
|\Delta P_z| \sim \frac{|\delta m^2|}{9T_6^4}
\] (30)

at a temperature of \( \sim 1 \) MeV. This limit is confirmed by our numerical calculations and

is similar to that of Foot, Thomson and Volkas \[7\] based on their simplified equation. For
\( |\delta m^2| \gtrsim 10 \text{eV}^2 \), however, since \( \Delta P_z \) has to be much smaller than \( P_z \sim 1 \), \( \Delta P_z \sim 0.1 \).

The oscillatory behavior of \( \Delta P_z \) after resonance is illustrated in figure 2 (a), (b) and (c),
for three different \( \nu_e-\nu_s \) mixing parameters. Each graph is the result of several millions steps
of integrations of eq. (3) and eq. (4) by adaptive Runge-Kutta method, with an error of less
than \( 10^{-10} \) in each step. In figure 2 (a), the mixing parameters do not satisfy equation (25)
at \( T_6 = 1 \), so \( \Delta P_z \) is still oscillatory at 1 MeV. In figure 2 (b) and (c), \( |\Delta P_z| \) settles down to
\( \sim |\delta m^2/9\text{eV}^2|T_6^{-4} \) at \( T_6 \gtrsim 1 \), in line with our estimate eq. (30). Our numerical calculations
also show that although the final settle-down value of \( |\Delta P_z| \) is predictable, the sign of \( \Delta P_z \)
seems random among different parameter choices. For example, in case (b), \( \delta m^2 = -10^{-2} \)
\( \text{eV}^2 \) and \( \sin^2 2\theta = 10^{-4.25} \), a small change of \( \sin^2 2\theta \) to \( 10^{-4.1875} \) yields an opposite sign of
the final \( \Delta P_z \). The sign of \( \Delta P_z \) can also be flipped by slight changes in the initial \( L_0 \) (as
tiny as 0.01%) and calculational parameters, such as a different error control (from \( 10^{-10} \)
to \( 5 \times 10^{-10} \) in our example), or a step size, or even a slightly different relation between
the average neutrino energy and the temperature (from \( E = 3.151T \) to \( E = 3.150T \) in
our example). This is due to the large number of oscillations of $\Delta P_z$ before it approaches one of the two possible values, so that the final $\Delta P_z$ is very sensitive to the input and calculational parameters. Such behavior is not so significant in the case of $\delta m^2 = -1 \text{ eV}^2$ and $\sin^2 2\theta = 10^{-8}$, because the number of oscillations of $\Delta P_z$ is small (figure 2 (c)). In this case, using $E = 3.150T$ the evolution of $\mathbf{P}$ and $\mathbf{\bar{P}}$ traces the evolution of $\mathbf{P}$ and $\mathbf{\bar{P}}$ using $E = 3.151T$ very well (in a sense that their difference is obviously still a perturbation at lower temperatures). This is also true if we change the initial $L_0$ by 0.01% (although the resultant perturbation can be several percent in the oscillatory epoch of $\Delta P_z$). Nevertheless, a small change of $\sin^2 2\theta$ to $10^{-8.0625}$ still flips the sign of the final $\Delta P_z$.

Figure 2 (a) and (b) suggest a chaotic behavior in the epoch of oscillating $\Delta P_z$. Figure 2 (c) might be intrinsically chaotic too but the time scale of the oscillatory epoch may be too short for such behavior to show up. To find more evidences of chaos, we try to determine the Lyapunov exponents of the system [10]. Assuming $\vec{\phi}(t) = (\Delta P_x, \Delta P_y, \Delta P_z)$ represents a solution of eq (20), we investigate the behavior of a nearby solution $\vec{\phi}(t) + \delta \vec{\phi}(t) = (\Delta P_x, \Delta P_y, \Delta P_z) + (\delta P_x, \delta P_y, \delta P_z)$ ($\delta P_x$ and $\delta P_y$ are assumed to arise from $P_x$ and $P_y$ only so that we can carry out our analysis). The evolution of the $\delta \vec{\phi}(t)$ is obtained by linearizing eq. (20):

$$\dot{\delta \vec{\phi}} = M \delta \vec{\phi}$$

where matrix

$$M = \begin{pmatrix}
-D & -V_0 - \beta \Delta P_z & -\beta (P_y + \bar{P}_y) \\
-V_0 - \beta \Delta P_z & -D & -V_x + \beta (P_y + P_y) \\
0 & V_x & 0
\end{pmatrix}.$$  \hspace{1cm} (32)

The eigenvalue of $M$, called the Lyapunov exponent, is

$$\lambda = \frac{\beta (P_y + \bar{P}_y)(V_0 + \beta \Delta P_z)}{\beta (P_x + \bar{P}_x) - V_x} - D.$$  \hspace{1cm} (33)

If $\lambda$ is positive, nearby solutions will depart exponentially in phase space within a timescale of $\lambda^{-1}$. In our problem, since $P_x + \bar{P}_x$, $P_y + \bar{P}_y$ and $\Delta P_z$ are all oscillatory, so is $\lambda$. A crude
analysis is to plug in the amplitudes of $P_x + P_x$, $P_y + P_y$ and $\Delta P_z$, to see whether $\lambda$ can be sometimes positive. In the limiting case of $|\beta \Delta P_z| \ll V_0$, $P_x + \bar{P}_x \sim P_y + \bar{P}_y \sim V_x/V_0$, the resultant $\lambda$ is $\sim \beta V_0/(\beta - V_0) \Delta P_z$ which can certainly be positive if $V_0 > 0$ and $|\delta m^2| \ll 9 T_e^4 \text{eV}^2$, conditions satisfied after the initial resonance. In the limiting case of $|\beta \Delta P_z| \gg V_0$, $P_x + \bar{P}_x \sim P_y + \bar{P}_y \sim V_x/\beta \Delta P_z$, the resultant $\lambda$ is $\sim \beta \Delta P_z/(1 - \Delta P_z) - D$, which can again be positive (remember $\Delta P_z \ll 1$ and $\beta \Delta P_z \gg V_0 \gg D$). Therefore, our crude analysis shows that $\lambda$ can at least be positive within a timescale of order $\lambda^{-1}$ intermittently.\footnote{In the analysis, we have chosen a particular set of nearby solutions, namely those having deviations in $P_x$ and $P_y$ but not in $\bar{P}_x$ and $\bar{P}_y$. The other extreme choices, in which $\delta P_x$ or $\delta P_y$ arises only in one particle population but not in its anti-particles, merely change the sign of $\beta \Delta P_z$ in the nominator of eq. (33), thus do not affect the general behavior of $\lambda$.} In other words, the system is not a classical textbook example of a chaotic system.

We are ultimately interested in the region of mixing parameters that amplifies neutrino chemical potentials, and the size of the amplification. In figure 3 (a) and (b), we plot the parameter space allowed by BBN that amplifies neutrino chemical potentials. The boundary to the right which excludes parameters that bring $\nu_s$ into equilibrium is adopted from Shi, Schramm and Fields \footnote{In the analysis, we have chosen a particular set of nearby solutions, namely those having deviations in $P_x$ and $P_y$ but not in $\bar{P}_x$ and $\bar{P}_y$. The other extreme choices, in which $\delta P_x$ or $\delta P_y$ arises only in one particle population but not in its anti-particles, merely change the sign of $\beta \Delta P_z$ in the nominator of eq. (33), thus do not affect the general behavior of $\lambda$.}. The boundary to the left that distinguishes parameters that amplify neutrino chemical potentials from those that do not is based on our numerical calculations (smoothed), and agrees with our analytical estimate eq. (23) within an order of magnitude. The lower cut on $|\delta m^2|$ is determined by requiring $T_{\text{res}} \geq 1 \text{MeV}$. The upper cut on $|\delta m^2|$ is dictated by laboratory bounds on $\nu_e$ mass in the $\nu_e$-\nu_s mixing case, and by eq. (24) as well as cosmological considerations \footnote{In the analysis, we have chosen a particular set of nearby solutions, namely those having deviations in $P_x$ and $P_y$ but not in $\bar{P}_x$ and $\bar{P}_y$. The other extreme choices, in which $\delta P_x$ or $\delta P_y$ arises only in one particle population but not in its anti-particles, merely change the sign of $\beta \Delta P_z$ in the nominator of eq. (33), thus do not affect the general behavior of $\lambda$.} in the $\nu_\mu$-$\nu_\tau$-$\nu_s$ mixing case. The boundary that singles out parameters that have oscillatory $\Delta P_z$ at 1 MeV is plotted according to eq. (25) which is confirmed by our numerical calculations.

We note that our numerical calculation of $\delta m^2 = -1 \text{eV}^2$, $\sin^2 2\theta = 10^{-8}$ $\nu_e$-\nu_s mixing
yields an opposite sign of $\Delta P_z$ from that of Foot, Thomson and Volkas \[7\]. But this may not be surprising due to the chaotic feature of the system, that different signs of $\Delta P_z$ may arise from different initial $L_0$, or even different choices of integrators, different errors or step sizes. It is also noted that the simplified equation in ref. \[7\] (corresponding to eq. (18) without the second term and $L_0$) is not suited for investigating the behavior of $\Delta P_z$ in the resonant regime and in the epoch of oscillatory $\Delta P_z$ thereafter. Finally, we note that the first calculation of the neutrino asymmetry done by Enqvist et al. \[12\] shows an oscillatory asymmetry down to $T_6 \sim 1$, because their parameter choice, $\Delta m^2 = -10^{-5} \text{ eV}^2$ and $\sin^2 2\theta = 10^{-2}$, does not satisfy eq. (25).

III. IMPLICATIONS

We concentrate on two implications of a neutrino asymmetry as large as eq. (30) in BBN.

The first is on other active-sterile neutrino oscillations in BBN \[7,8\]. When neglecting neutrino asymmetries, large areas of parameter space of active-sterile neutrino oscillation are ruled out based on the argument that the sterile neutrino cannot be significantly populated so as to violate the primordial $^4\text{He}$ abundance observation. The forbidden areas include the large angle $\nu_\mu-\nu_s$ mixing with $\delta m^2 \sim 10^{-2} \text{ eV}^2$ which can solve the atmospheric neutrino problem, and the large angle $\nu_e-\nu_s$ mixing with $\delta m^2 \sim 10^{-5} \text{ eV}^2$ which can solve the solar neutrino problem. This argument, however, no longer stands when a neutrino asymmetry as large as in eq. (30) is in place. For example, if $\nu_\tau$ mixes with a lighter $\nu_s$ with $|\delta m^2| \gg 10^{-2}\text{eV}^2$ and an angle in the shaded region of figure 3 (b), the $\nu_\tau\bar{\nu}_\tau$ asymmetry amplified by the $\nu_\tau-\nu_s$ oscillation can be large enough to suppress the $\nu_\mu-\nu_s$ oscillation from the $\nu_\mu-\nu_s$ mixing solution to the atmospheric problem \[8\]. Similarly, if $\nu_\tau$ or $\nu_\mu$ mixes with a lighter $\nu_s$ with $|\delta m^2| \gg 10^{-5}\text{eV}^2$ and an angle in the shaded region of figure 3 (b), the amplified neutrino asymmetry can be large enough to suppress the $\nu_e-\nu_s$ oscillation originating from the large angle $\nu_e-\nu_s$ mixing solution to the solar neutrino problem. The suppression is in place even in the epoch of oscillating $\Delta P_z$, because although an oscillating $\Delta P_z$ constantly
drives the other active-sterile neutrino oscillations through resonance, the time of resonance crossing is too short to allow any significant oscillation. Thus both these two solutions to the atmospheric neutrino problem and the solar neutrino problem ruled out previously may still be viable if an active neutrino (more massive than \( \nu_\mu \) or \( \nu_e \) respectively) mixes with a lighter sterile neutrino with parameters in the shaded region of figure 3 (b).

The second implication is on the primordial \(^4\text{He}\) abundance itself. Besides the number of neutrino species, the primordial \(^4\text{He}\) abundance is also affected by a non-zero chemical potential in the \( \nu_e \bar{\nu}_e \) sector at \( T \sim 1 \text{ MeV} \). The mechanism is that the asymmetry in \( \nu_e \bar{\nu}_e \) changes the neutron/proton conversion rates, thereby changes the freeze-out time of the neutron to proton ratio as well as the ratio itself. A \( \xi_{\nu_e} \) (the \( \nu_e \) chemical potential divided by \( kT \)) of order 0.1 can induce an appreciable change in the prediction of the primordial \(^4\text{He}\) abundance [13]. When \( \xi_{\nu_e} \ll 1 \), \( Y \approx Y(\xi_{\nu_e} = 0) - 0.234\xi_{\nu_e} \) [14] and \( \Delta P_z = 2(n_{\nu_e} - n_{\bar{\nu}_e}) \approx 3.6\xi_{\nu_e} \). So

\[
Y = Y_0 - 0.065\Delta P_z. \tag{34}
\]

The comparison with equation \( Y = Y(N_\nu = 3) + 0.012(N_\nu - 3) \) (where \( N_\nu \) is the effective number of neutrino species in BBN) [13] indicates that \( \Delta P_z \sim 0.1 \) in the \( \nu_e \bar{\nu}_e \) sector corresponds to roughly \(-0.55\) neutrino species, and therefore has a significant impact on the predicted \(^4\text{He}\) abundance.

There are two ways to generate a \( \xi_{\nu_e} \) of order \( \pm 0.1 \) by active-sterile neutrino oscillations. The direct way is to have a \( \sim 1 \text{ eV} \) \( \nu_e \) mix with a lighter \( \nu_s \) (figure 3 (a)). If the atmospheric neutrino problem and the solar neutrino problem are to be solved by active neutrino oscillations, this implies that all three active neutrinos are almost degenerate with a mass of order 1 eV. This will be consistent with supernovae nucleosynthesis constraints [16,17] and compatible with the controversial LSND result [18,19] if the claimed detection of \( \nu_\mu - \nu_e \) oscillation solves the atmospheric neutrino problem [20]. Laboratory experiments limit the mass of \( \nu_e \) to less than 5 eV [21]. If \( \nu_e \) is a majorana neutrino, its mass is further limited to less than about 1 eV [22].
The indirect way of generating a significant $\xi_{\nu e}$ is to have $\nu_\tau$ (or $\nu_\mu$) mix with a lighter $\nu_s$ with $10^2 \text{eV}^2 \gtrsim |\delta m^2| \gtrsim 1 \text{eV}^2$ and a desired angle, and transfer the asymmetry in the $\nu_\tau \bar{\nu}_\tau$ ($\nu_\mu \bar{\nu}_\mu$) sector into $\nu_e \bar{\nu}_e$ by a $\nu_e - \nu_\tau$ ($\nu_\mu$) mixing. But to yield an asymmetry of order 0.1 as well in $\nu_e \bar{\nu}_e$, the transfer has to be efficient. Take the $\nu_e - \nu_\tau$ oscillation as an example, the mixing has to satisfy

$$D' \left( \frac{V_{\nu e}'}{V_{\nu_\tau}'} \right)^2 \gtrsim H,$$

where $D'$, $V'_{\nu e}$ and $V'_{\nu_\tau}$ denote the counterparts of $D$, $V_{\nu e}$ and $V_{\nu_\tau}$ in the $\nu_\tau - \nu_s$ oscillation. Approximately

$$D' \sim 5G_F^2 T^5 \approx T^3_6 H,$$

$$V'_{\nu e} \approx \frac{\delta M^2}{6.3 T} \sin 2\theta',$$

$$V'_{\nu_\tau} \approx -\frac{\delta M^2}{6.3 T} \cos 2\theta' - 180G_F^2 T^5 + 0.13G_F T^3(n_{\nu_e} - n_{\bar{\nu}_e}) - 0.13G_F T^3(n_{\nu_\tau} - n_{\bar{\nu}_\tau}),$$

where $\delta M^2$ and $\theta'$ are the vacuum mixing parameters of the $\nu_e - \nu_\tau$ mixing. An efficient transfer of asymmetry means that $(n_{\nu_e} - n_{\bar{\nu}_e}) \sim (n_{\nu_\tau} - n_{\bar{\nu}_\tau}) \sim |\delta m^2| / 18 T^4_6$. If the mass of $\nu_e$ is much lighter than $\nu_\tau$, $|\delta M^2| \approx |\delta m^2|$. Then $(D/H)(V'_{\nu e}/V'_{\nu_\tau})^2 \sim T^3_6 \sin^2 2\theta'$ at temperatures approaching 1 MeV. So if $\nu_\tau$ has a cosmologically interesting mass, $\Delta m^2 \sim 20 - 100 \text{ eV}^2$, $|\Delta P_\nu| \approx 0.1$ in $\nu_\tau \bar{\nu}_\tau$ can be reached at $T \approx 2 - 3$ MeV by the $\nu_\tau - \nu_s$ mixing according to eq. (30), which can efficiently transfer into an asymmetry of similar order in $\nu_e \bar{\nu}_e$ if $\nu_\tau$ mixes with $\nu_e$ with $\sin^2 2\theta' \sim 0.1$ and $|\delta M^2| \sim |\delta m^2|$. This required mixing between $\nu_e$ and $\nu_\tau$ lies near the edge of current lab limits on the $\bar{\nu}_e - \bar{\nu}_\tau$ mixing [23], and may be testable in the near future. Based on supernovae nucleosynthesis arguments, however, the required $\nu_e - \nu_\tau$ mixing is ruled out (although supernova models are uncertain to some extent) [16,17]. The above analyses applies similarly to the $\nu_\mu - \nu_s$ and $\nu_\mu - \nu_e$ mixings, but the required mixing between $\nu_e$ and $\nu_\mu$ to transfer asymmetries efficiently has already been ruled out by laboratory experiments [21].

Of course, the indirect way of transferring asymmetries from $\nu_\tau \bar{\nu}_\tau$ (or $\nu_\mu \bar{\nu}_\mu$) into $\nu_e \bar{\nu}_e$ works if they have almost degenerate masses, $\sim 1\text{eV}$. In this case the required $\nu_e - \nu_\mu$ ($\nu_\tau$)
mixing will not be ruled out by laboratory experiments or astrophysical considerations.

Recently, Hata et al. [24] suggested a possible “crisis” in BBN because the predicted $^4$He abundance from the standard BBN, coupled with predictions of $^3$He and D from generic chemical evolution models, is too high to be consistent with the observed abundance, unless the number of effective neutrino species $N_\nu$ at the time of BBN is less than 2.6. This is at odds with popular beliefs and most theoretical models which assume all $\nu_e$, $\nu_\mu$ and $\nu_\tau$ to be lighter than $\sim 1$ MeV and therefore $N_\nu \geq 3$. But as seen above, if an asymmetry of order $\Delta P_z \sim 0.1$ arises in the $\nu_e\bar{\nu}_e$ sector from active-sterile neutrino oscillations, this potential “crisis” can be solved. Of course, the mixing parameters of the active-sterile oscillations have to be right to yield a positive $\Delta P_z$ instead of a negative one. An interesting note is that the neutrino mass required to solve this “crisis” is $\sim 1$ eV, which qualifies neutrinos as dark matter candidates. A $\nu_e$ mass of $\sim 1$ eV is within a factor of 5 below the current limit on $\nu_e$ mass, and right on the edge of detection limit if $\nu_e$ are majorana neutrinos. If not introducing more sterile neutrinos, the solar neutrino problem and the atmospheric neutrinos have to be solved by mixings among the three active neutrinos, therefore requiring their masses to be almost degenerate. A model of three neutrinos almost degenerate in mass is favorable in forming structures in the universe [25], but may not be the most natural theoretical model so far.

IV. SUMMARY

In summary, we have calculated the parameter space of active-sterile neutrino mixings that amplifies neutrino chemical potentials, and the size of the amplification. Results are summarized in figure 3. By exploring the sensitivity of the amplification to the initial condition, the mixing parameters and the calculational parameters, and by analyzing the Lyapunov exponent of the system, we showed evidences that the amplification process is chaotic. We have also discussed the implications of our results on BBN. It was shown that a $\nu_e$ chemical potential of order $0.1kT$ could be achieved by either a mixing between $\sim 1$
eV $\nu_e$ and a lighter sterile neutrino, or a mixing between a $\sim 1$ eV $\nu_\mu$ (or $\nu_\tau$) and a lighter sterile neutrino coupled with a mixing between almost degenerate $\nu_e$ and $\nu_\mu$ (or $\nu_\tau$). Such a chemical potential in $\nu_e$ can lower the effective number of neutrino species to significantly below 3.

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Figure Captions:

Figure 1. Illustrations of evolution of $P$ and $V$ for $\delta m^2 < 0$. (a) before resonance; (b) right after resonance.

Figure 2. The solid lines show the evolution of $\Delta P_z$ vs. the temperature of the universe, for $\nu_e-\nu_s$ mixing. Asymmetries less than $10^{-9}$ are ignored. The dash line shows $|\Delta P_z| = |\delta m^2/9eV^2|T_0^{-4}$. $L_0 = 10^{-9}$. (a) $\delta m^2 = -10^{-4}eV^2$, $\sin^2 2\theta = 10^{-5}$; (b) $\delta m^2 = -10^{-2}eV^2$, $\sin^2 2\theta = 10^{-4.25}$; (c) $\delta m^2 = -1eV^2$, $\sin^2 2\theta = 10^{-8}$.

Figure 3. Regions in between the two thick lines represent allowed mixing parameters that amplify neutrino chemical potential. The magnitude of $\Delta P_z$ at 1 MeV is shown. The region at the lower right noted with “$\Delta P_z$ Osc.” has an oscillating $\Delta P_z$ at 1 MeV. For $|\delta m^2| \gtrsim 10$ eV$^2$, $\Delta P_z$ is limited to $\sim 0.1$. (a) $\nu_e-\nu_s$ mixing ($\nu_e$ heavier than $\nu_s$); (b) $\nu_\mu (\nu_\tau)-\nu_s$ mixing ($\nu_\mu$ or $\nu_\tau$ heavier than $\nu_s$).
Figure 1 (a)
Figure 1(b)
(a) $\delta m^2 = -10^{-4} \text{eV}^2$, $\sin^2 2\theta = 10^{-5}$
(b) $\delta m^2 = -10^{-2}\text{eV}^2$, $\sin^22\theta = 10^{-4.25}$
(c) $\delta m^2 = -1\text{eV}^2$, $\sin^2 \vartheta = 10^{-8}$
$|\Delta P_z| \sim \frac{|\delta m^2|}{9}$
\[ |\Delta P_z| \sim \frac{|\delta m^2|}{9} \]

\[ |\Delta P_z| \sim 0.1 \]

Figure 3(b)