Attractors for weak and strong solutions of the three-dimensional Navier-Stokes equations with damping

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Abstract

In this paper we obtain the existence of global attractors for the dynamical systems generated by weak solution of the three-dimensional Navier-Stokes equations with damping.

We consider two cases, depending on the values of the parameters β, α controlling the damping term and the viscosity µ. First, for β ≥ 3 we define a multivalued dynamical systems and prove the existence of the global attractor as well. Second, for either β > 3 or β = 3, 4αµ > 1 the weak solutions are unique and we prove that the global attractor for the corresponding semigroup is more regular. Also, we prove in this case that it is the global attractor for the semigroup generated by the strong solutions.

Finally, some numerical simulations are performed.

Keywords: Three-dimensional Navier-Stokes equations with damping, global attractors, set-valued dynamical systems, asymptotic behaviour

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1 Introduction

The three-dimensional Navier-Stokes equations with damping have been studied intensively over the last years. They describe the situation where there exists resistance to the motion of a flow. One outstanding model in which a damping term appears comes from the flow of cerebrospinal fluid inside the porous brain tissues [14]. Such dissipative damping is also common in many different models. For example, compressible Euler equations with damping describe the flow of a compressible gas through a porous medium [8], whereas Saint-Venant equations are used in oceanography to describe the flow of viscous shallow water with friction [1].

In this paper we study the asymptotic behaviour of weak solutions of the following equation

\[ u_t - \mu \Delta u + (u \cdot \nabla) u + \alpha |u|^{\beta-1} u + \nabla p = f, \quad (x,t) \in \Omega \times (0,T), \]

where \( \Omega \subset \mathbb{R}^3, \beta \geq 1, \mu, \alpha > 0, \mu > 0 \) is the kinematic viscosity and \( u \) is the velocity vector of an incompressible fluid satisfying Dirichlet boundary conditions.

We would like to point out that the damping term is very helpful from the mathematical point of view, as it allows us to obtain solutions more regular than in the standard Navier-Stokes equations without damping (that is, when \( \alpha = 0 \)). For this reason it is possible to prove the existence of global attractors for \( \Omega \), at least in a given range of the parameter \( \beta \), whereas to date this problem remains open for the standard three-dimensional Navier-Stokes equations.

Existence of weak solutions for problem [1] with initial condition in the space of free-divergence square integrable functions was established at first in [2, Theorem 1] for \( \beta \geq 1 \) and \( \Omega = \mathbb{R}^3 \) and in [18, Theorem 2.1] for bounded domains \( \Omega \). The uniqueness of weak solutions was established in [16] for \( \beta \geq 4 \) and this result was extended in [10] for \( \beta > 3 \) and \( \beta = 3, \alpha \mu \geq \frac{1}{4} \). A conditional result about smoothness of weak solutions is given in [26].
Concerning global strong solutions of (1) with \( \Omega \) bounded and more regular initial conditions, its existence was established in [9, Theorem 1.1] for either \( \beta \in (3, 5) \) or \( \beta = 3, \ \alpha \mu > \frac{1}{4} \). Recently, this result has been extended to \( \beta = 5 \) [11]. Previously, the existence of strong solutions was stated in [18] and [20] for \( \beta > 3 \). However, as pointed out in [9], [11] it is unclear whether the proof of this result is correct, because the function \( -\Delta u \) is used as a test function, which it seems cannot be done when \( \Omega \neq \mathbb{R}^3 \). Instead, the function \( Au \), where \( A \) is the Stokes operator, has to be used (see [9, Theorem 1.1]).

Concerning global strong solutions of (1) with \( \Omega = \mathbb{R}^3 \) and more regular initial conditions, its existence was established in [2], [27] for \( \beta > 3 \), and in [29] for \( \beta = 3, \ \alpha = \mu = 1 \). Also, in [29] uniqueness of strong solutions was proved to be true for all \( \beta \geq 1 \). If either the initial condition \( u_0 \) is small enough or the viscosity \( \mu \) is large, existence of global strong solutions was stated in [28] for \( 1 \leq \beta < 3 \). It is pointed out in [11] that these results on existence of strong solutions are conditional in the sense that it is necessary to prove previously the existence of local strong solutions, because Galerkin approximations cannot be used when \( \Omega = \mathbb{R}^3 \). This is done in [11].

The asymptotic behaviour of strong solutions when \( 3 < \beta \leq 5 \) was studied in [18], [19] and [20] in the autonomous and nonautonomous situation, proving the existence of the global attractor. Again, the proof of these results is at least unclear as the test function \( -\Delta u \) has been used to obtain the suitable estimates of the solutions. In this paper, we give an alternative proof of the existence of the global strong attractor for either \( 3 < \beta < 5 \) or \( \beta = 3, \ \alpha \mu > \frac{1}{4} \). For \( \beta \geq 5 \) the problem remains open.

The present paper is an improvement of [16]. In [10] the existence of the global attractor when \( \beta \geq 3 \) was proved for weak solutions, which was the first result of this kind in the literature. It is worth mentioning that it was necessary to use the theory of multivalued semiflows for some values of the parameter \( \beta \) due to the absence of uniqueness. On top of that, the solutions and the attractor were shown to be more regular for \( \beta > 4 \). However, the proof uses the function \( -\Delta u \) a test function, so as said before the proof is unclear. In this paper, we prove the regularity results for weak solutions and the global attractor for either \( 3 < \beta < 5 \) or \( \beta = 3, \ \alpha \mu > \frac{1}{4} \). For \( \beta \geq 5 \) the problem remains open.

This paper is organized as follows. In Section 2 we prove first suitable estimates for weak solutions. In Section 3 we prove, for \( \beta \geq 3 \), the existence of the global attractor for the multivalued semiflow generated by the weak solutions. When either \( \beta > 3 \) or \( \beta = 3, \ \alpha \mu \geq \frac{1}{4} \), this semiflow is a semigroup of operators and the attractor is proved to be connected. If either \( 3 < \beta < 5 \) or \( \beta = 3, \ \alpha \mu > \frac{1}{4} \), we obtain more regularity of the attractor. Finally, when either \( 3 < \beta < 5 \) or \( \beta = 3, \ \alpha \mu > \frac{1}{4} \), we establish that the global attractor for the weak solutions is also the global attractor for the semigroup generated by the strong solutions.

## 2 Estimates of weak solutions

Consider a bounded open set \( \Omega \subset \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \). We study the three-dimensional Navier-Stokes equations with damping

\[
\begin{align*}
\partial_t u &- \mu \Delta u + (u \cdot \nabla) u + \alpha |u|^{\beta-1} u + \nabla p = f, \quad (x, t) \in \Omega \times (0, T), \\
\text{div} u & = 0, \quad (x, t) \in \Omega \times (0, T), \\
u |_{\partial \Omega} & = 0, \quad t \in (0, T), \\
u |_{t=0} & = u_0, \quad x \in \Omega,
\end{align*}
\]

where \( \mu > 0 \) is the kinematic viscosity and \( f \) is an external force. Also, \( \beta \geq 1 \) and \( \alpha > 0 \) are given constants. The functions \( u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \), \( p(x, t) \) stand for the velocity field and the pressure, respectively. Here and further, \(|\cdot|\) denotes in general the norm in \( \mathbb{R}^d \) for any \( d \geq 1 \).

We define the usual function spaces

\[
\begin{align*}
V & = \{ u \in (C_0^\infty(\Omega))^3 : \text{div} u = 0 \}, \\
H & = cl( L^2(\Omega))^3 V, \\
V & = cl( H^1_0(\Omega))^3 V,
\end{align*}
\]

where \( cl_X \) denotes the closure in the space \( X \). It is well known that \( H, V \) are separable Hilbert spaces and identifying \( H \) and its dual we have \( V \subset H \subset V' \) with dense and continuous injections. We denote by \( (\cdot, \cdot) \), \(|\cdot|\) and \( (\cdot, \cdot) \), \(|\cdot|\), \(||\cdot||\) the inner product and norm in \( H \) and \( V \), respectively, and by \( (\cdot, \cdot) \) duality between \( V' \) and
Let \( H_w \) be the space \( H \) endowed with the weak topology. As usual, we define the continuous trilinear form \( b : V \times V \times V \to \mathbb{R} \) by
\[
b(u, v, w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.
\]

It is well-known that \( b(u, v, v) = 0 \), if \( u \in V, v \in (H^1_0(\Omega))^3 \). For \( u, v \in V \) we denote by \( B(u, v) \) the element of \( V' \) defined by \( \langle B(u, v), w \rangle = b(u, v, w) \), for all \( w \in V \).

The norm in the spaces \( L^p(\Omega), (L^p(\Omega))^3, p \geq 1 \), will be denoted indistinctly by \( |.|_p \).

Let \( P \) be the orthogonal projection from \((L^2(\Omega))^3\) onto \( H \) and \( Au = -P\Delta u \) be the Stokes operator, defined by \( \langle Au, v \rangle = \langle (u, v) \rangle \) for \( u, v \in V \). Since the boundary \( \partial \Omega \) is smooth, \( D(A) = (H^2(\Omega))^3 \cap V \) and \( \|Au\|_2 \) defines a norm in \( D(A) \) which is equivalent to the norm in \((H^2(\Omega))^3\).

For \( u_0 \in H \), \( f \in H \) the function
\[
u \in L^\infty (0, T; H) \cap L^2 (0, T; V) \cap L^{\beta+1} \left( 0, T; \left( L^{\beta+1}(\Omega) \right)^3 \right)
\]
is said to be a weak solution to problem \( \text{(2)} \) on \( (0, T) \) if \( u(0) = u_0 \) and
\[
\frac{d}{dt} (u, v) + \mu((u, v)) + b(u, u, v) + \alpha \left(|u|^{\beta-1} u, v \right) = (f, v),
\]
for any \( v \in V \cap \left( L^{\beta+1}(\Omega) \right)^3 \), in the sense of scalar distributions.

We recall the following well-known result on existence of weak solutions.

**Theorem 1** [15] **Theorem 2.1** For any \( u_0 \in H, f \in H, \beta \geq 1 \) there exists at least one weak solution \( u \) to problem \( \text{(2)} \).

Let \( Y = V' + \left( L^{\frac{\beta+1}{\beta}}(\Omega) \right)^3 \), the dual space of \( V \cap \left( L^{\beta+1}(\Omega) \right)^3 \). We note that by standard estimates on \( B \) for any weak solution we have that
\[
Au \in L^2(0, T; V'),
\]
\[
B(u, u) \in L^\frac{4}{3}(0, T; V'),
\]
\[
|u|^{\beta-1} u \in L^\frac{\beta+1}{\beta} \left( 0, T; \left( L^{\beta+1}(\Omega) \right)^3 \right),
\]
which implies in particular that
\[
-\mu Au - B(u, u) - \alpha |u|^{\beta-1} u + f \in L^\frac{4}{3} (0, T; V') + L^\frac{\beta+1}{\beta} \left( 0, T; \left( L^{\frac{\beta+1}{\beta}}(\Omega) \right)^3 \right) \subset L^1 (0, T; Y).
\]
It follows from equality \( \text{(4)} \) and a standard result [21] p.250, Lemma 1.1] that
\[
\frac{du}{dt} = -\mu Au - B(u, u) - \alpha |u|^{\beta-1} u + f
\]
in the sense of \( Y \)-valued distributions. Hence, the derivate \( \frac{du}{dt} \) belongs to the space
\[
L^\frac{4}{3} (0, T; V') + L^\frac{\beta+1}{\beta} \left( 0, T; \left( L^{\frac{\beta+1}{\beta}}(\Omega) \right)^3 \right)
\]
and equality \( \text{(5)} \) is satisfied in the space \( Y \) for a.a. \( t \in (0, T) \).

In order to obtain good estimates of weak solutions we need \( \frac{du}{dt} \) to be more regular. We can obtain such a result for \( \beta \geq 3 \).
Lemma 2 Let $u$ be a weak solution to (2) such that $u \in L^q \left(0,T; \left(L^q(\Omega)\right)^3\right)$ with $q \geq 4$. Then

$$
\frac{d u}{dt} \in L^2 \left(0,T; V'\right) + L^{\frac{q+1}{q}} \left(0,T; \left(L^{\frac{q+1}{3}}(\Omega)\right)^3\right),
$$

$$u \in C \left([0,T], H\right),$$

the map $t \mapsto \|u(t)\|_H^2$ is absolutely continuous and

$$
\frac{d}{dt} \|u(t)\|^2_H = 2 \left\langle u, \frac{d u}{dt}\right\rangle \text{ for a.a. } t \in (0,T).
$$

Proof. Using the well-known inequality (see [21, p.297])

$$|b(u,u,v)| \leq C \|u\|_4^2 \|v\|, \forall u,v \in V,$$

and $u \in L^q \left(0,T; \left(L^q(\Omega)\right)^3\right) \subset L^4 \left(0,T; \left(L^4(\Omega)\right)^3\right)$, we have

$$B(u,u) \in L^2 \left(0,T; V'\right)$$

so (6) follows.

Properties (7)-(8) follow from [3, Chapter II, Theorem 1.8].

Corollary 3 If $\beta \geq 3$, then any weak solution to (2) satisfies (6)-(8).

Proof. Since $u \in L^{\beta+1} \left(0,T; \left(L^{\beta+1}(\Omega)\right)^3\right)$ and $\beta \geq 3$, we obtain that $u$ belongs to $L^q \left(0,T; \left(L^q(\Omega)\right)^3\right)$ with $q \geq 4$.

Lemma 4 If $\beta \geq 3$, then any weak solution satisfies the estimates

$$|u(t)|^2 \leq e^{-\mu \lambda_1 t} |u_0|^2 + \frac{|f|^2}{\mu^2 \lambda_1^2},$$

$$\mu \int_s^t \|u\|^2 \, dt + 2 \alpha \int_s^t |u|^\beta + 1 \|u\|^\beta + 1 \, dt \leq |u_0|^2 + \frac{|f|^2}{\mu^2 \lambda_1^2} + \frac{1}{\mu \lambda_1} |f|^2 (t-s),$$

for any $t \geq s \geq 0$.

Proof. Multiplying equality (5) by $u$ and using (8) and $b(u,u,u) = 0$ we have

$$\frac{1}{2} \frac{d}{dt} |u|^2 + \mu \|u\|^2 + \alpha |u|^\beta + 1 = (f,u) \leq \frac{\mu \lambda_1}{2} |u|^2 + \frac{1}{2 \mu \lambda_1} |f|^2.$$

As $\mu \|u\|^2 \geq \mu \lambda_1 |u|^2$, we deduce that

$$\frac{d}{dt} |u|^2 + \mu \lambda_1 |u|^2 \leq \frac{1}{\mu \lambda_1} |f|^2$$

and Gronwall’s lemma yields

$$|u(t)|^2 \leq e^{-\mu \lambda_1 t} |u_0|^2 + \frac{|f|^2}{\mu^2 \lambda_1^2}.$$

By $\mu \|u\|^2 \geq \frac{\mu}{4} \|u\|^2 + \frac{\mu \lambda_1}{2} |u|^2$, integrating over the interval $(s,t)$ in (11) it follows that

$$\mu \int_s^t \|u\|^2 \, dt + 2 \alpha \int_s^t |u|^\beta + 1 \|u\|^\beta + 1 \, dt \leq |u(s)|^2 + \frac{1}{\mu \lambda_1} |f|^2 (t-s)$$

$$\leq e^{-\mu \lambda_1 s} |u_0|^2 + \frac{|f|^2}{\mu^2 \lambda_1^2} + \frac{1}{\mu \lambda_1} |f|^2 (t-s),$$

so the lemma is proved.

The uniqueness of weak solution was established at first in [16, Theorem 2.4] for $\beta \geq 4$. Later on, in [10, Corollary 2.1] this result was extended for $\beta > 3$ and $\beta = 3$. $4 \alpha \mu \geq 1$.

Theorem 5 Let either $\beta > 3$ or $\beta = 3$ and $4 \alpha \mu \geq 1$. Then for any $u_0 \in H$ there exists a unique weak solution $u(\cdot)$ to problem (2), which is continuous with respect to the initial datum $u_0$. 

3 Global attractor for weak and strong solutions

Our aim now is to prove the existence of the global attractor for the weak and strong solutions of problem 2.

We shall divide this section into two cases: 1) $\beta \geq 3$; 2) $\beta > 3$ or $\beta = 3$, $4\alpha\mu \geq 1$. In the first one, as in general more than one solution can possibly exist for a given initial datum, we make use of the theory of attractors for multivalued semiflows to prove the existence of a global attractor. In the second one, uniqueness of weak solutions implies that we can define a semigroup of operators, to which we can apply the classical theory of attractors for semigroups, proving the existence of a global connected attractor. More regularity of the attractor is obtained if either $\beta > 3$ or $\beta = 3$, $4\alpha\mu > 1$. The attractor is shown to be the global attractor for the strong solutions as well.

3.1 Case 1: $\beta \geq 3$

Let us define the set

$$D_T (u_0) = \{ u (\cdot) \text{ is a weak solution of } 2 \text{ in the interval } (0, T) \}.$$  

We know by Theorem 1 that for any $u_0 \in H$ and $T > 0$ the set $D_T (u_0)$ is non-empty.

We observe that as $q = \frac{\beta+1}{\beta} \leq \frac{4}{3}$, we have the time derivative of a weak solution satisfies

$$\frac{du}{dt} \in L^4 (0, T; V') + L^q (0, T; (L^q (\Omega))^3) \subset L^q (0, T; V' + (L^q (\Omega))^3).$$

**Lemma 6** Let $\beta \geq 3$. If $u (\cdot) \in D_T (u_0)$, then for any $s \in (0, T)$, the function $w (\cdot) = u (\cdot + s)$ belongs to $D_{T-s} (u(s))$.

If $u (\cdot) \in D_s (u_0)$ and $w (\cdot) \in D_{T-s} (u(s))$, then the function

$$z(t) = \begin{cases} 
  u(t) & \text{if } t \in [0,s], \\
  w(t-s) & \text{if } t \in [s,T], 
\end{cases}$$

belongs to $D_T (u_0)$.

**Proof.** Let $u (\cdot) \in D_T (u_0)$. Then it is obvious that

$$w (\cdot) = u (\cdot + s) \in L^\infty (0, T-s; H) \cap L^2 (0, T-s; V) \cap L^{\beta+1} (0, T-s; L^{\beta+1} (\Omega)).$$

Also, [4] implies that for any $v \in V \cap (L^{\beta+1} (\Omega))^3$, $\phi \in C_0^\infty (0, T-s)$ one has

$$w (\cdot) \phi (\cdot) \in L^2 (0, T-s; L^2 (\Omega)).$$

so $w$ satisfies [4] in the interval $(0, T-s)$. We infer that $w \in D_{T-s} (u(s))$.

Let now $u (\cdot) \in D_s (u_0)$ and $w (\cdot) \in D_{T-s} (u(s))$. Arguing as in the previous case we obtain that $w(t-s)$ satisfies equality [4] in the interval $(s,T)$. As the time derivative of a weak solution belongs to $L^q \left(0, T; V' + (L^q (\Omega))^3\right)$, by [21], p.250, Lemma 1.1 equality [4] is equivalent to saying that

$$\int_0^T \left( \frac{du}{dt}, \xi \right) + \mu (u, \xi) + B(u, u, \xi) \right) dt + \alpha \int_0^T \int_\Omega |u|^{\beta-1} u \xi dx dt = \int_0^T (f, \xi) dt,$$
for any $\xi \in L^{3+1} \left(0, T; V \cap (L^{3+1} (\Omega))^3 \right)$. The function $z$ satisfies (3) in the interval $(0, T)$ and this equality as well. Indeed, denoting $h(t) = w(t - s)$ we have

$$\int_0^T \left( \frac{dz}{dt}, \xi \right) + \mu ((z, \xi)) + \langle B (z, z), \xi \rangle dt + \alpha \int_0^T \int_\Omega |z|^{3-1} z \xi dx dt$$

$$= \int_0^T \left( \frac{du}{dt}, \xi \right) + \mu ((u, \xi)) + \langle B (u, u), \xi \rangle dt + \alpha \int_0^T \int_\Omega |u|^{3-1} u \xi dx dt$$

$$+ \int_s^T \left( \frac{dh}{dt}, \xi \right) + \mu ((h, \xi)) + \langle B (h, h), \xi \rangle dt + \alpha \int_s^T \int_\Omega |h|^{3-1} h \xi dx dt$$

$$= \int_s^T (f, \xi) dt + \int_s^T (f, \xi) dt = \int_0^T (f, \xi) dt,$$

proving that $z$ is really a weak solution. ■

In view of this lemma every solution can be extended to a globally defined one, that is, a solution which exists for $t \in [0, +\infty)$. In this situation we denote by $D (u_0)$ the set of all globally defined solutions with initial condition $u_0$ and observe that for any $t \geq 0$ the following equality holds:

$$\{ u(t) : u \in D (u_0) \} = \{ u(t) : u \in \cup_{T > 0} D_T (u_0) \}.$$

Denote by $P (H)$ the set of all non-empty subsets of $H$. Let us define the following (possibly multivalued) family of operators $G : \mathbb{R}^+ \times H \to P (H)$:

$$G (t, u_0) = \{ y \in H : y = u (t), u (\cdot) \in D (u_0) \}.$$

Using Lemma 8 we can easily prove that $G$ is a strict multivalued semiflow, that is, the following two properties hold:

- $G (0, u_0) = u_0$ for all $u_0 \in H$;
- $G (t + s, u_0) = G (t, G (s, u_0))$, for all $u_0 \in H$, $t, s \geq 0$.

The set $A$ is a global attractor for $G$ if:

- $A$ is negatively invariant, i.e., $A \subset G (t, A)$ for all $t \geq 0$;
- $A$ attracts every bounded set of $H$, that is,

$$\text{dist} (G (t, B), A) \to 0 \text{ as } t \to +\infty.$$

It is invariant if, moreover, $A = G (t, A)$ for all $t \geq 0$.

The next lemma is crucial for proving the existence of a global attractor.

**Lemma 7** Assume that $\beta \geq 3$. Let $u_0^0 \to u_0$ weakly in $H$ and let $u_n (\cdot) \in D (u_0^n)$. Then there exists a weak solution $u (\cdot)$ to (2) with $u (0) = u_0$ and a subsequence $u_{n_k} (\cdot)$ such that $u_{n_k} \to u$ in $C ([\varepsilon, T], H)$ for all $0 < \varepsilon < T$.

If, moreover, $u_0^n \to u_0$ strongly in $H$, then $u_{n_k} \to u$ in $C ([0, T], H)$ for all $T > 0$.

**Proof.** We fix $T > 0$. We deduce from Lemma 1 that the sequence $u_n$ is bounded in

$$L^\infty (0, T; H) \cap L^2 (0, T; V) \cap L^{3+1} (0, T; (L^{3+1} (\Omega))^3).$$

Also, using (5) and standard estimates (see [21, p.297]) we obtain that $\frac{du_n}{dt}$ is bounded in the space $L^q \left(0, T; V' + (L^q (\Omega))^3 \right).$
Thus, making use of the compactness theorem \cite{13} we obtain a function \( u ( \cdot ) \) and a subsequence (denoted again by \( u_n \)) such that

\[
    u_n \to u \text{ weakly star in } L^\infty (0, T; H),
\]
\[
    u_n \to u \text{ weakly in } L^2 (0, T; V),
\]
\[
    u_n \to u \text{ weakly in } L^{\beta +1} \left(0, T; (L^{\beta +1}(\Omega))^3\right),
\]
\[
    \frac{du_n}{dt} \to \frac{du}{dt} \text{ weakly in } L^q \left(0, T; V' + (L^q(\Omega))^3\right),
\]
\[
    u_n \to u \text{ strongly in } L^2 (0, T; H),
\]
\[
    u_n (t, x) \to u (t, x) \text{ for a.a. } (t, x).
\]

Let us prove that

\[
    u_n (t_n) \to u (t_0) \text{ weakly in } H
\]

for any sequence \( \{t_n\} \) such that \( t_n \to t_0 \), where \( t_n, t_0 \in [0, T] \). The time derivatives are bounded in the space \( L^q \left(0, T; V' + (L^q(\Omega))^3\right) \), which implies readily that the sequence \( u_n (\cdot) \) is equicontinuous in the space \( V' + (L^q(\Omega))^3 \). Moreover, \( u_n (t_n) \) is bounded in \( H \), and then the compact embedding \( H \subset V' \) yields that it is relatively compact in \( V' + (L^q(\Omega))^3 \). Hence, by Ascoli-Arzelà’s theorem we have \( u_n \to u \) in \( C \left([0, T], V' + (L^q(\Omega))^3\right) \). Thus, by a contradiction argument we obtain that \( u_n (t_n) \to u (t_0) \) weakly in \( H \).

In particular, we have that \( u (0) = u_0 \).

Further, we need to check that \( u (\cdot) \) is a weak solution to problem \( (2) \). The sequence \( h (u_n (\cdot)) = |u_n (\cdot)|^{\beta -1} u_n (\cdot) \) is bounded in \( L^q \left(0, T; (L^q(\Omega))^3\right) \) and \( h (u_n (t, x)) \to h (u (t, x)) \) for a.a. \( (t, x) \). Hence, \( h (u_n (\cdot)) \to h (u (\cdot)) \) weakly in \( L^q \left(0, T; (L^q(\Omega))^3\right) \) \cite{17} Lemma 8.3].

In order to show that \( u \) is a weak solution it remains to pass to the limit in the term \( B \). Since \( u_n \to u \) in \( L^2(0, T; H) \) implies that \( u_{ni} u_{nj} \to u_i u_j \) in \( L^1(0, T; L^1(\Omega)) \), for any \( \zeta \in \mathcal{V}, \phi \in C^0_0(0, T) \) we have

\[
    \int_0^T (b(u_n, u_n, \zeta) - b(u, u, \zeta)) \phi dt = - \int_0^T (b(u_n, \zeta, u_n) - b(u, \zeta, u)) \phi dt
\]

\[
    = - \sum_{i, j=1}^3 \int_0^T \int_\Omega (u_{ni} u_{nj} - u_i u_j) \frac{\partial \zeta_j}{\partial x_i} \phi dx dt \to 0,
\]

as \( n \to \infty \).

We conclude that equality \( (4) \) is satisfied for the function \( u \) for all \( \zeta \in \mathcal{V} \), and by density of \( \mathcal{V} \) in \( V \) we obtain that \( (4) \) holds true. Thus, \( u \) is a weak solution.

Finally, we will prove that \( u_n \to u \) in \( C ([\varepsilon, T], H) \) for all \( 0 < \varepsilon < T \). From \( (12) \) we get

\[
    |u_n (t)|^2 \leq |u_n (s)|^2 + \frac{1}{\mu \Lambda_1} |f|^2 (t - s), \text{ for any } s \leq t,
\]

and the same inequality is true for \( u \). Hence, the functions \( J_n (t) = |u_n (t)|^2 - \frac{1}{\mu \Lambda_1} |f|^2 t, J (t) = |u (t)|^2 - \frac{1}{\mu \Lambda_1} |f|^2 t \) are non-increasing and continuous. Take a sequence \( t_n \to t_0 \) with \( t_n, t_0 \in [\varepsilon, T] \). We know that \( u_n (t_n) \to u (t_0) \) weakly in \( H \), so

\[
    |u (t_0)| \leq \lim \inf |u (t_n)|.
\]

It is a consequence of \( (14) \) that \( J_n (t) \to J (t) \) for a.a. \( t \). Then we can choose \( t_k < t_0 \) as close to \( t_0 \) as we wish such that \( J_n (t_k) \to J (t_k) \), and we can assume without loss of generality that \( t_k < t_n \). Therefore,

\[
    J_n (t_n) - J (t_0) = J_n (t_n) - J_n (t_k) + J_n (t_k) - J (t_k) + J (t_k) - J (t_0)
\]

\[
    \leq |J_n (t_k) - J (t_k)| + |J (t_k) - J (t_0)|.
\]

Since \( u (\cdot) \) is continuous, for any \( \delta > 0 \) there exists \( t_k \) and \( N (t_k) \) such that \( |J (t_k) - J (t_0)| \leq \delta /2 \) and \( |J_n (t_k) - J (t_k)| \leq \delta /2 \) for all \( n \geq N \). This implies that

\[
    \lim \sup |u (t_n)| \leq |u (t_0)|.
\]
Joining (16) and (17) we deduce that \(|u(t_n)| \to |u(t_0)|\) and then \(u(t_n) \to u(t_0)\) in \(H\).

Since \(T > 0\) is arbitrary, by a diagonal arguments we obtain a common subsequence on an arbitrary interval \([\varepsilon, T]\).

The first part of the lemma is proved.

For the second part, we need to prove only that \(u(t_n) \to u(0)\) if \(t_n \to 0\), \(t_n \geq 0\). For this aim we repeat the above argument with \(t_k = 0 = t_0\). Hence,

\[
J_n(t_n) - J(t_0) = J_n(t_n) - J_n(0) + J_n(0) - J(0) \leq |J_n(0) - J(0)| \to 0\text{ as } n \to \infty,
\]

because \(u_n(0) \to u(0)\) in \(H\). Then we obtain the result arguing in the same way as above.

\[\square\]

**Corollary 8** Assume that \(\beta \geq 3\). For any \(t \geq 0\) the map \(u_0 \mapsto G(t, u_0)\) has compact values and closed graph. In addition, for any \(t_0 > 0\) the map \(G(t_0, \cdot)\) is compact.

The map \(u_0 \mapsto G(t, u_0)\) is said to be upper semicontinuous if for all \(u_0 \in H\) and any neighborhood \(O\) of \(u_0\) in \(H\) there exists \(\delta > 0\) such that \(G(t, u) \subset O\) for all \(u\) satisfying \(|u - u_0| < \delta\).

**Lemma 9** Assume that \(\beta \geq 3\). For any \(t \geq 0\) the map \(u_0 \mapsto G(t, u_0)\) is upper semicontinuous.

**Proof.** If not, there exist \(u_0 \in H\), \(t > 0\), sequences \(u_n^0 \to u_0\), \(y_n \in G(t, u_n^0)\) and a neighborhood \(O\) of \(G(t, u_0)\) such that \(y_n \notin O\). Let \(y_n = u_n(t)\), where \(u_n(\cdot) \in D(u_0^0)\). Then by Lemma 7 there is a subsequence \(y_{n_k}\) satisfying \(y_{n_k} \to y \in G(t, u_0)\), which is a contradiction.

\[\square\]

A set \(B_0\) is called absorbing if for any bounded set \(B\) there exists a time \(T(B)\) such that

\[
G(t, B) \subset B_0 \text{ for any } t \geq T(B).
\]

The semiflow \(G\) is said to be asymptotically compact if for any bounded subset \(B\) every sequence \(y_k \in G(t_k, B)\), where \(t_k \to +\infty\), is relatively compact in \(H\).

The following conditions are sufficient in order to obtain a global compact invariant minimal attractor \(\mathcal{A}\) for a strict multivalued semiflow \(G\) [15] Theorem 3 and Remark 8]:

1. \(G\) possesses a bounded absorbing set \(B_0\);
2. \(G\) is asymptotically compact;
3. \(G\) has closed values;
4. the map \(u_0 \mapsto G(t, u_0)\) is upper semicontinuous.

**Theorem 10** Assume that \(\beta \geq 3\). Then \(G\) has a global invariant compact attractor \(\mathcal{A}\), which is minimal among all closed attracting sets.

**Proof.** We need to check the four aforementioned conditions.

It follows from (9) that the ball \(B_0 = \{u \in H : \|u\|_H^2 \leq 1 + \frac{12}{\mu^2\kappa^2}\}\) is absorbing.

In view of Corollary 8 and Lemma 9 \(G\) has compact (and then closed) values and the map \(u_0 \mapsto G(t, u_0)\) is upper semicontinuous.

Finally, again by Corollary 8 the operator \(G(1, \cdot)\) is compact. Hence, for any bounded set \(B\) an arbitrary sequence \(y_n \in G(t_n, B)\), which belongs to

\[
G(1, G(t_n - 1, B)) \subset G(1, B_0), \text{ for all } n \geq N,
\]

is relatively compact in \(H\), so \(G\) is asymptotically compact.

\[\square\]

We can give also some information about the structure of the global attractor in terms of bounded complete trajectories, which are continuous functions \(\gamma : \mathbb{R} \to H\) such that \(u(\cdot) = \gamma(\cdot + s)\) belongs to \(D(u_0)\) for all \(s \in \mathbb{R}\) and satisfying that \(\cup_{t \in \mathbb{R}} \gamma(t)\) is a bounded set. Indeed, by Lemmas 6, 7 we can apply Theorems 9, 10 from [6] and obtain that

\[
\mathcal{A} = \{\gamma(0) : \gamma \in \mathcal{K}\}, \quad (18)
\]
where is $\mathbb{K}$ the set of all bounded complete trajectories.

Finally, in a similar way as in [7] let us prove that the global attractor is stable, which means that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$G(t, O_\delta(A)) \subset O_\varepsilon(A)$$

for all $t \geq 0$, (19)

where $O_\eta(A) = \{z \in H : \text{dist}(z, A) < \eta\}$.

**Lemma 11** Assume that $\beta \geq 3$. The global attractor $A$ given in Theorem [10] is stable.

**Proof.** By contradiction if (19) does not hold, then there exist $\varepsilon > 0$ and sequences $\delta_k \to 0$, $x_k \in O_{\delta_k}(A)$, $t_k \geq 0$, $y_k \in G(t_k, x_k)$ such that

$$\text{dist} (y_k, A) \geq \varepsilon. \quad (20)$$

We consider two cases: 1) $t_k \to +\infty$ for some subsequence; 2) $t_k \leq C$.

In the first situation, as the sequence $\{x_k\}$ belongs to a bounded set, by the definition of global attractor we get that $\text{dist} (y_k, A) \to 0$, which contradicts (20).

In the second one, up to a subsequence $t_k \to t_0$, $x_k \to x_0 \in A$, so by Lemma [7] and the invariance of $A$ we obtain that

$$y_k \to y \in G(t_0, x_0) \subset G(t_0, A) \subset A,$$

which is again a contradiction. ■

### 3.2 Case 2: $\beta > 3$ or $\beta = 3$, $4\alpha\mu \geq 1$

In view of Theorem [5] we can define the semigroup of operators $S : \mathbb{R}^+ \times H \to H$ by

$$S(t, u_0) = u(t),$$

where $u(\cdot)$ is the unique solution to problem [2] with initial condition $u_0$. It is straightforward to see that $S$ satisfies the semigroup properties: $S(0, u_0) = u_0$, for any $u_0 \in H$, and $S(t + s, u_0) = S(t, S(s, u_0))$, for any $u_0 \in H$, $t, s \geq 0$. Also, making use again of Theorem [5] we obtain that $S(t, u_0)$ is continuous with respect to the initial condition $u_0$ for fixed $t \geq 0$.

We recall that the set $A$ is said to be a global attractor for $S$ if it is invariant, i.e. $S(t, A) = A$, for all $t \geq 0$, and it attracts every bounded subset $B$ of the phase space $H$, which means that

$$\text{dist}_H (S(t, B), A) \to 0 \text{ as } t \to +\infty,$$

where $\text{dist}_X (C, A) = \sup_{x \in C} \inf_{y \in A} \|x - y\|_X$ is the Hausdorff semidistance between subsets of the Banach space $X$.

Usually in the literature a global attractor is supposed to be compact as well. However, we prefer to use this more general definition and add compactness as an additional property, as generally speaking a global attractor does not have to be bounded (see [23] for a non-trivial example of an unbounded non-locally compact attractor).

The existence of the global compact attractor follows directly from Theorem [10] as a particular case. Nevertheless, we will explain this result using the theory of attractors for semigroups as well.

For a semigroup $S$ the concepts of absorbing set and asymptotically compactness are given in exactly the same way as for semiflows. It follows from [9] that the ball

$$B_0 = \left\{ u \in H : \|u\|^2 \leq 1 + \frac{|f|^2}{\mu^2 \lambda_f^2} \right\}$$

is absorbing for the semigroup $S$. Also, Lemma [7] implies, arguing as in the proof of Theorem [10] that the semigroup $S$ is asymptotically compact.

The existence of a bounded absorbing set and the asymptotic compactness ensure the existence of the global compact attractor [12]. Also, as the space $H$ is connected, the attractor is connected [5] p.4]. Hence, we have obtained the following result.
Theorem 12 If $\beta > 3$ or $\beta = 3$, $4 \alpha \mu > 1$, the semigroup $S$ possesses the global compact connected attractor $A$.

It is possible to prove that the global attractor is more regular if $\beta \in (3, 5)$ or $\beta = 3$ and $\mu a > 1/4$. Indeed, let us check that $A$ is in fact bounded in the space $(H^2(\Omega))^3$, and then compact in $V$ and $(L^{\beta+1}(\Omega))^3$.

Lemma 13 Let $\beta \in (3, 5)$ or $\beta = 3$ and $\mu a \geq 1/4$. Then any weak solution of (2) with initial data such that $|u_0| \leq R$ satisfies the estimate

$$u(t) + |u(t)|^\beta \leq 3 \frac{1}{\mu} |Au|^2 + C_2|u|^2 + C_3|u|^{2\beta},$$

for any $r > 0$ and $\bar{t} \geq 0$, where $D(R, r)$ is such that $D(R, r) \to \infty$ if $r \to 0^+$ or $R \to +\infty$.

Proof. Since we have uniqueness of the Cauchy problem, the following formal calculations can be justified via Galerkin Approximations.

We prove first the result for $\bar{t} = 0$. Multiplying the equation by $Au$ we obtain that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \mu |Au|^2 = -b(u, u, Au) - \alpha \left( |u|^\beta u, Au \right) + (f, Au).$$

By

$$|b(u, u, Au)| \leq C_1 \|u\|^2 |Au|^2 \leq \frac{\mu}{8} |Au|^2 + C_2 \|u\|^6,$$

$$|(f, Au)| \leq \frac{\mu}{4} |Au|^2 + \frac{1}{\mu} |f|^2,$$

$$\alpha \left( |u|^\beta u, Au \right) \leq \frac{\mu}{8} |Au|^2 + C_3 |u|^{2\beta},$$

we have used inequality (9,27) in [17], we get

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{\mu}{2} |Au|^2 \leq \frac{1}{\mu} |f|^2 + C_2 \|u\|^6 + C_3 |u|^{2\beta}.$$  

Also, $|u|^2 \leq C_4 \|u\||Au|$ [22] gives

$$|u|^{2\beta} \leq \int_\Omega |u|^{2\beta-6} |u|^6 \, dx \leq |u|^2 |u|_6^6 \leq C_4 \|u\|^{2\beta} \leq \frac{\mu}{4C_3} |Au|^2 + C_5 \|u\|^{2\beta+3}.$$  

Thus,

$$\frac{d}{dt} \|u\|^2 + \frac{\mu}{2} |Au|^2 \leq C_0(1 + \|u\|^{2q}),$$

where $q = (\beta + 3)/(5 - \beta)$. The function $y(t) = 1 + \|u(t)\|^2$ satisfies

$$y' \leq C_0 \left(1 + (y(t) - 1)^q\right) \leq C_0 y^q,$$

where we have used that $1 + (y - 1)^q \leq y^q$ for any $y \geq 1$. Hence,

$$(y(t))^{1-q} \geq ((y(t_0))^{1-q} + (1-q) C_0 (t-t_0),$$

$$y(t) \leq \frac{y(t_0)}{(1 + (y(t_0))^{q-1} (1-q) C_0 (t-t_0))^{1/(q-1)},$$

which is finite if

$$t < t_0 + \frac{1}{(y(t_0))^{q-1}(q-1)C_0}. $$
Take $T' = t_0 + 1/(2(g(t_0))^{q-1}(q-1)C_0)$. Then

$$
\|u(t)\|^2 \leq 2^{1/(q-1)} \left( 1 + \|u(t_0)\|^2 \right) \quad \text{for } t \in [t_0, T'].
$$

(23)

By Lemma 4 we have

$$
\mu \int_0^t \|u(s)\|^2 ds \leq |u_0|^2 + \frac{1 + r \mu \lambda_1}{\mu^2 \lambda_1^2} |f|^2,
$$

for $r > 0$, which implies the existence of $t_0 \in (0, r)$ such that

$$
\|u(t_0)\|^2 \leq \frac{1}{\mu r} \left( |u_0|^2 + \frac{1 + r \mu \lambda_1}{\mu^2 \lambda_1^2} |f|^2 \right) \leq D_1(R, r).
$$

Hence, (23) implies

$$
\|u(t)\|^2 \leq 2^{1/(q-1)} \left( 1 + \frac{1}{\mu} \left( |u_0|^2 + \frac{1 + r \mu \lambda_1}{\mu^2 \lambda_1^2} |f|^2 \right) \right) \leq D_2(R, r) \quad \forall t_0 \leq t \leq T'.
$$

Let $\gamma = \min\{T', r\}$. From (22) we have

$$
\sup_{t_0 \leq t \leq \gamma} \|u(t)\|^2 + \frac{\mu}{2} \int_{t_0}^{\gamma} |Au|^2 dt \leq D_1(R, r) + C_0(1 + (2D_2(R, r)) q) r = D_3(R, r).
$$

(24)

Multiplying the equation by $u_t$ and using again inequality (9.27) in [17] we obtain

$$
|u_t|^2 + \frac{\mu}{2} \frac{d}{dt} \|u\|^2 + \frac{\alpha}{\beta + 1} \frac{d}{dt} |u|^{\beta+1}
$$

$$
\leq -b(u, u, u_t) + (f, u_t) \leq C_7 \|u\|^2 |Au|^2 |u_t| + |f|^2 + \frac{1}{4} |u_t|^2
$$

$$
\leq \frac{1}{2} |u_t|^2 + C_8(|f|^2 + |Au|^2 + \|u\|^6).
$$

(25)

Integrating over $(t_0, \gamma)$ and using (24) and the embedding $V \subset (L^{\beta+1}(\Omega))^3$ we have

$$
\int_{t_0}^{\gamma} |u_t|^2 dt \leq \mu \|u(t_0)\|^2 + \frac{2\alpha}{\beta + 1} |u(t_0)|^{\beta+1} + 2C_8 \left( |f|^2 r + \frac{2}{\mu} D_3(R, r) + (D_3(R, r))^3 r \right)
$$

$$
\leq \mu D_1(R, r) + C_9 \left( D_1(R, r) \right)^{\frac{\beta+1}{2}} + 2C_8 \left( |f|^2 r + \frac{2}{\mu} D_3(R, r) + (D_3(R, r))^3 r \right) = D_4(R, r).
$$

Hence, there is $t_1 \in (t_0, \gamma)$ such that

$$
|u_t(t_1)|^2 \leq \frac{D_4(R, r)}{\gamma - t_0} \leq \frac{D_4(R, r)}{T' - t_0} \leq D_4(R, r) 2(1 + D_1(R, r))^{q-1} (q - 1) C_0 = D_5(R, r).
$$

(26)

Further, we differentiate the equation with respect to $t$ and multiply by $u_t$:

$$
\frac{1}{2} \frac{d}{dt} |u_t|^2 + \mu \|u_t\|^2 + \alpha \left( \left( |u|^{\beta-1} u \right)_t, u_t \right)
$$

$$
= -b(u, u, u_t) - b(u, u_t, u) = b(u_t, u, u) \leq \varepsilon_1 \mu \|u_t\|^2 + \frac{1}{4 \varepsilon_1 \mu} \|u| \|u_t\|^2,
$$

for $\varepsilon_1 > 0$. Using

$$
\left( \left( |u|^{\beta-1} u \right)_t, u_t \right) = (|u|^{\beta-1} u_t, u_t) + (\beta - 1) \int_{\Omega} |u|^{\beta-1} |u_t|^2 dx \geq (|u|^{\beta-1} u_t, u_t)
$$

we obtain

$$
\frac{1}{2} \frac{d}{dt} |u_t|^2 + \mu (1 - \varepsilon_1) \|u_t\|^2 + \alpha \left( |u|^{\beta-1} |u_t| \right)^2 \leq \frac{1}{4 \varepsilon_1 \mu} \|u| \|u_t\|^2.
$$

(27)
Thus, in particular, using Gronwall lemma, the embedding $H \subset V$ gives (21).

Corollary 14 For any $r > 0$ the map $u_0 \mapsto S(r, u_0)$ maps bounded subsets of $H$ onto bounded subsets of $V \cap L^{\beta + 1}(\Omega)$. Hence, $S(r)$ is a compact operator, i.e., it maps bounded subsets of $H$ onto relatively compact ones.
Lemma 15 Let $\beta \in (3,5)$ or $\beta = 3$ and $\mu a > 1/4$. Then any weak solution of (2) with initial data such that $|u_0| \leq R$ satisfies the estimate

$$|Au(r)| \leq K(R,r),$$

for any $r > 0$, where $K(R,r)$ is such that $K(R,r) \to \infty$ if $r \to 0^+$ or $R \to +\infty$.

Proof. By Proposition 9.2 in [17] we have

$$|B(u,u)| \leq d_1 \|u\|_2^2 |Au|^2 \leq \frac{\mu}{4} |Au| + d_2 \|u\|^3.$$

Using the Gagliardo-Nirenberg inequality and $\beta < 5$ we find that

$$\alpha |u|^{\beta-1} u = \alpha \|u\|_{2}\beta \leq d_3 |Au|^\frac{3(\beta-1)}{\beta+1} |u|^\frac{\beta^2+4\beta+3}{\beta+1} \leq \frac{\mu}{4} |Au| + d_4 \|u\|_{\beta+1}^\frac{\beta^2+4\beta+3}{\beta+2\beta} + |f|,$$

Hence,

$$\frac{\mu}{2} |Au(r)| \leq |u_t(r)| + d_2 \|u(r)\|^3 + d_4 |u(r)|_{\beta+1}^\frac{\beta^2+4\beta+3}{\beta+2\beta} + |f|,$$

so the result follows by applying Lemma 13. $lacksquare$

We are now in position of proving the regularity of the global attractor.

Theorem 16 Let $\beta \in (3,5)$ or $\beta = 3$ and $\mu a > 1/4$. Then the global attractor $A$ is bounded in $(H^2(\Omega))^3$, and then compact in $V$ and $(L^{\beta+1}(\Omega))^3$. Moreover,

$$\text{dist}_V(S(t,B),A) \to 0 \text{ as } t \to +\infty,$$

(29)

for any $B$ bounded in $H$.

Proof. Since the global attractor is invariant, $A = S(r,A)$ for $r > 0$, so $A$ is bounded in $(H^2(\Omega))^3$ by Lemma 15. The compact embeddings $H^2(\Omega) \subset H^1(\Omega)$, $H^2(\Omega) \subset L^{\beta+1}(\Omega)$ imply the compactness of the attractor in $V$ and $(L^{\beta+1}(\Omega))^3$.

Let $B_0$ be an absorbing ball. By Lemma 15 the set $B_1 = S(r,B_0)$ is bounded in $(H^2(\Omega))^3$ and

$$S(t,B) = S(r,S(t-r,B)) \subset B_1 \text{ for } t \geq t_0(B).$$

From here it is easy to deduce (29). $lacksquare$

In the paper [16] the results of Theorem 13, Lemma 15 and Theorem 16 were stated for any $\beta \geq 4$. However, as mentioned in the introduction, the proof is not correct because in the case of bounded domains we cannot multiply the equation by $-\Delta u$ to obtain the regularity of weak solutions. Hence, these results remain open for $\beta \geq 5$.

We recall that a function $u : [0,T] \to V \cap (L^{\beta+1}(\Omega))^3$ is called a strong solution of (2) if $u$ is a weak solution and

$$u \in L^\infty(0,T;V) \cap L^2(0,T;D(A)) \cap L^\infty(0,T;(L^{\beta+1}(\Omega))^3).$$

Theorem 17 [3, Theorem 1.1] Let $u_0 \in V \cap (L^{\beta+1}(\Omega))^3$, $f \in H$ and either $\beta \in (3,5)$ or $\beta = 3$ and $\mu a > 1/4$. Then there exists a unique strong solution of (3).

For $\beta \in (3,5)$ or $\beta = 3$ and $\mu a > 1/4$ let $S_V : \mathbb{R}^+ \times V \to V$ be the semigroup defined by the strong solutions of (2).

Corollary 18 Let $\beta \in (3,5)$ or $\beta = 3$ and $\mu a > 1/4$. Then $A$ is also the global attractor for $S_V$.

We have given an alternative proof of the existence of the strong attractor, as the proof given in the papers [18], [19] and [20] for $3 < \beta \leq 5$ is unclear. For $\beta \geq 5$ the problem remains open. Also, in [19] a conditional result was stated about the strong attractor for $\beta > 5$. For the same reason as before, the proof is also unclear.
4 Numerical simulations

We shall now focus on solving numerically the equation (2) employing computational fluid dynamics (CFD) to visualize scenarios in which the evolution of the fluid flow converges to a steady state. It is important to stress that the examples reported below are only intended for showing the asymptotic behaviour of the fluid flow numerically when taking different values of the parameters $\alpha$ and $\beta$ in the momentum equation of (2), but no conclusive results should be deduced from the numerical simulations.

The geometry of the flow domain used in all our numerical experiments is a sphere $\Omega$ of radius 6 centered at the origin. We also take the source term $f$ in (2) as

$$f(x) = \begin{cases} 
(0, 2, 0) & \text{if } x \in C \\
(0, 0, 0) & \text{if } x \in \Omega \setminus C
\end{cases}$$

where $C$ is a cylinder, with both radius and height of 4, within the flow domain symmetrically located at the center of the sphere $\Omega$ in such a way that the base of the cylinder is parallel to the $xz$-plane as in Figure 1. Observe that $f(x, t)$ can be seen as a constant source force within the cylinder $C$ propelling the fluid flow upwards.

Numerical simulations were all performed by using the CFD package OpenFOAM®️, which is the acronym of Open Source Field Operation and Manipulation. OpenFOAM®️ is an open-source CFD software based on C++ that contains a toolbox for tailored numerical solvers for a wide variety of problems relevant to the industry and scientific community. The solvers implemented in OpenFOAM®️ uses the Finite Volume Method (FVM) to discretize the governing equations on unstructured meshes (see [1, 25]). The solver used to integrate our model numerically was pimpleFoam, which combines the two most common algorithms for solving the Navier-Stokes equations, namely, SIMPLE and PISO algorithms. The pimpleFoam code is inherently transient, requiring an initial condition and boundary conditions. OpenFOAM®️ includes pre-processing and post-processing capabilities such as snappyHexMesh and ParaFoam for meshing and visualization, respectively.

Figures 2, 3 and 4 show the steady state for the numerical solution of the equations (2) for experiments with different values of the parameters $\alpha$ and $\beta$. In those experiments we have set the initial condition $u_0(x) = (0, 0, 0)$, and the images represent the velocity vector field in the $xy$-section at $z = 0$. The darker areas in the images are those where the magnitude for the velocity vector $u$ is smaller, while the lighter ones represent the areas where the velocity is higher. According to the results of the experiments, when the magnitude of the velocity vector is greater than 1 and the parameter $\beta$ becomes bigger, the medium provides increased resistance to movement, so the fluid flow slows down more quickly. On the contrary, when the magnitude of the velocity vector is less than 1 and the parameter $\beta$ becomes smaller, then medium provides decreased resistance to movement, so the fluid flow spreads further through the medium. The effect of the parameter $\alpha$ does not depend of the $u$ magnitude, acting proportionally, i.e., the larger the value of $\alpha$, the higher resistance to motion of the fluid flow. It has also been observed that convergence speed to the steady state is higher as $\alpha$ and $\beta$ increase. Therefore, when $\alpha$ and $\beta$ are small, a higher period of time to get convergence to the steady state is required. In fact, we have also performed simulations (not shown here) for values of $\alpha$ close to 0 (also for $\alpha = 0$) and a low value of $\beta$ (for instance, $\beta = 1$), but we did not achieve convergence to a steady state for an approachable (from a computational point of view) time value. It is likely that for such values of the parameters the global attractor (if it exists!) is more complex than a fixed point.

Lastly, an experiment with a non-vanishing initial condition was carried out. The performance was made by taking $u_0(x) = (1, 0, 0)$, $\alpha = 0.2$ and $\beta = 1$, and the results of the model is shown in Figure 5. As one might expect, the steady state does not depend of the initial condition, hence, it is the same as taking $u_0$ equal to zero (compare the right panel of Figure 5 to the left panel of Figure 2).

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Figure 1: Flow domain

Figure 2: Flow velocity $u$ for the steady state in the $xy$-section at $z = 0$. The darker areas mean lower fluid flow speed. The initial velocity $u_0$ is identically zero. Left panel parameters: $\alpha = 0.2; \beta = 1$. Right panel parameters: $\alpha = 0.5; \beta = 1$. 
Figure 3: Flow velocity $u$ for the steady state in the $xy$-section at $z = 0$. The darker areas mean lower fluid flow speed. The initial velocity $u_0$ is identically zero. Left panel parameters: $\alpha = 0.2; \beta = 2$. Right panel parameters: $\alpha = 0.5; \beta = 2$.

Figure 4: Flow velocity $u$ for the steady state in the $xy$-section at $z = 0$. The darker areas mean lower fluid flow speed. The initial velocity $u_0$ is identically zero. Left panel parameters: $\alpha = 0.2; \beta = 4$. Right panel parameters: $\alpha = 0.5; \beta = 4$. 
Figure 5: Flow velocity $u$ in the $xy$-section at $z = 0$ for $\alpha = 0.2$ and $\beta = 1$. The darker areas mean lower fluid flow speed. The initial velocity $u_0 = (1, 0, 0)$. Left panel: state when $t = 0.1$. Right panel: steady state ($t$ large enough).