On Exponential Splitting Methods for Semilinear Abstract Cauchy problems

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Abstract. Due to the seminal works of Hochbruck and Ostermann (Appl Numer Math 53(2–4):323–339, 2005, Acta Numer 19:209–286, 2010) exponential splittings are well established numerical methods utilizing operator semigroup theory for the treatment of semilinear evolution equations whose principal linear part involves a sectorial operator with angle greater than \( \frac{\pi}{2} \) (meaning essentially the holomorphy of the underlying semigroup). The present paper contributes to this subject by relaxing the sectoriality condition, but in turn requiring that the semigroup operators act consistently on an interpolation couple (or on a scale of Banach spaces). Our conditions (on the semigroup and on the semilinearity) are inspired by the approach of Kato (Math Z 187(4):471–480, 1984) to the local solvability of the Navier–Stokes equation, where the \( L^p - L^r \)-smoothing of the Stokes semigroup was fundamental. The present abstract operator theoretic result is applicable for this latter problem (as was already the result of Hochbruck and Ostermann), or more generally in the setting of Hochbruck and Ostermann (2005), but also allows the consideration of examples, such as non-analytic Ornstein–Uhlenbeck semigroups or the Navier–Stokes flow around rotating bodies.

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1. Introduction

We investigate the semilinear Cauchy problem

\[
\dot{u}(t) = Au(t) + g(t, u(t)), \quad u(0) = u_0, \quad t \in [0, T],
\]

where \([0, T]\) is a given time interval, \( A: D(A) \subset X \to X \) generates a \( C_0 \)-semigroup \((e^{tA})_{t \geq 0}\) on a Banach space \( X \), \( u_0 \in D(A) \) and \( g: [0, T] \times V \to X \) is a locally Lipschitz continuous function, where \( V \) is a Banach space as well. The precise conditions on \( A \) and \( g \) are described below. As \( A \) generates a \( C_0 \)-semigroup, the linear problem \( \dot{u}(t) = Au(t) \) is well-posed and we suppose that
the solutions can be calculated effectively with high precision (or are explicitly known). In this article we prove convergence estimates for the so-called exponential splitting methods. This is a particular operator splitting method, that is, a general procedure for finding (numerical) solutions of complicated evolution equations by reduction to subproblems, whose solutions are then to be combined in order to recover the (approximate) solution of the compound problem. The literature both on the functional and the numerical analysis sides are extremely extensive, see, e.g., the surveys [4,18,19,34]. The decomposition of the compound problem can be based on various things, such as: on physical grounds (say, separating advection and diffusion phenomena, e.g., [26]), by mathematical-structural reasons (separating linear and nonlinear parts, see e.g., [21,23,24]; separating the history and present in case of delay equations, see [2]), etc. The starting point for exponential splitting methods is the definition of the mild solution of problem (1), that is the variation-of-constants formula:

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-\tau)A}g(\tau, u(\tau)) \, d\tau.$$  

(2)

First, we describe the method introduced by Hochbruck and Ostermann in [23,24] in the case when $(e^{tA})_{t\geq 0}$ is an analytic semigroup, see also [22]. An exponential integrator is a time stepping method and it approximates the convolution term on the right-hand side by a suitable quadrature rule in a given time step, where the effect of the linear propagator is not approximated but inserted precisely. Thus, for given time step $h > 0$ and $t_n:=nh$ with $n \in \mathbb{N}$ and $nh \leq T = N\, h$, the solution $u(t_n)$ of the semilinear Eq. (1), given recursively by

$$u(t_{n+1}) = e^{hA}u(t_n) + \int_0^h e^{(h-\tau)A}g(t_n + \tau, u(t_n + \tau)) \, d\tau,$$

(3)

is approximated by the $s$-stage Runge–Kutta approximation $u_n$, which is subject to the recursion

$$u_{n+1} = e^{hA}u_n + \int_0^h e^{(h-\tau)A}\sum_{j=1}^s \ell_j(\tau)g(t_n + c_jh, U_{n,j}) \, d\tau$$

(the initial value $u_0 = u(0)$ is known). Here $s$ is a positive integer, $c_1, \ldots, c_s \in [0,1]$ are pairwise distinct and $(U_{n,i})_{i=1,\ldots,s}$ are defined as the solution of the integral equation

$$U_{n,i} = e^{c_i hA}u_n + \int_0^{c_i h} e^{(c_i h-\tau)A}\sum_{j=1}^s \ell_j(\tau)g(t_n + c_jh, U_{n,j}) \, d\tau.$$

(4)

For $n \in \{0, \ldots, N-1\}$ the values $U_{n,i}$ provide an approximation of the solution $u(t_n + c_i h)$ at internal steps and $\ell_1, \ldots, \ell_s$ are Lagrange interpolation polynomials with nodes $c_1 h, \ldots, c_s h \in [0, h]$, thus $\sum_{j=1}^s \ell_j(\tau)g(t_n + c_jh, U_{n,j})$ yields an approximation of $g(t_n + \tau, u(t_n + \tau))$ for $\tau \in [0, h]$. We require the following conditions on the semilinear Cauchy problem (1).
Assumption 1.1. (The linear setting)

1. \( A : D(A) \subset X \to X \) generates a \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) on a Banach space \( X \) and \( u_0 \in X \).

2. \( (X, V) \) is an interpolation couple, that is, also \( V \) is a Banach space, \( V \) and \( X \) are (continuously) embedded in a topological vector space \( X \).

We assume moreover that for each \( t > 0 \) the linear operator \( e^{tA} \) leaves \( V \cap X \) invariant and extends to a linear operator \( e^{tA} \in L(V) \cap L(X, V) \) with \( M := \max_{t \in [0, T]} \{ \| e^{tA} \|_{L(X)}, \| e^{tA} \|_{L(V)} \} < \infty \).

3. \( W \) satisfies the same condition as \( V \) above. (Interesting will be the case \( W \in \{ X, V \} \).)

4. There is a continuous, non-increasing function \( \rho_X : (0, \infty) \to [0, \infty) \) with \( \rho_X \in L^1(0, T) \) such that

\[
\| e^{tA} \|_{L(X, V)} \leq \rho_X(t) \quad \text{for every } t \in (0, T].
\]

And similarly, there is a continuous, non-increasing function \( \rho_W : (0, \infty) \to [0, \infty) \) with \( \rho_W \in L^1(0, T) \) such that

\[
\| e^{tA} \|_{L(W, V)} \leq \rho_W(t) \quad \text{for every } t \in (0, T].
\]

This set of conditions, together with the ones about the non-linearity (see Assumption 1.3 below), is inspired by T. Kato’s iteration scheme in his operator theoretic approach to the Navier–Stokes equations, see [27] and Example 3.2 below. He used the \( L^p - L^r \)-smoothing of the linear Stokes semigroup to “compensate the unboundedness” of the non-linearity, and thus could apply Banach’s fixed point theorem, just as it is required by the exponential splitting in the internal steps, see also [15] for some further information in the abstract setting.

In the setting of Assumption 1.1 we clearly have \( M \geq 1 \), and if we set

\[
\Omega_X(h) := \int_0^h \rho_X(\tau) \, d\tau \quad \text{and} \quad \Omega_W(h) := \int_0^h \rho_W(\tau) \, d\tau,
\]

then \( \Omega_X, \Omega_W \) is a monotone increasing, continuous function from \([0, T]\) to \([0, \infty)\) with \( \Omega_X(0) = \Omega_W(0) = 0 \). Thus \( \Omega_X \) and \( \Omega_W \) are so-called \( K \)-functions. Moreover, we abbreviate \( \rho := \rho_X \) and \( \Omega := \Omega_X \), and set

\[
C_\Omega := \sup \left\{ h \sum_{k=1}^n \| e^{khA} \|_{L(X, V)} \left| 0 < nh \leq T \right. \right\} \leq \sup \left\{ h \sum_{k=1}^n \rho(kh) \left| 0 < nh \leq T \right. \right\} \leq \| \rho \|_{L^1(0, T)} < \infty.
\]

The following example is motivated by the framework of the paper [23] by Hochbruck and Ostermann.

Example 1.2. (Bounded, analytic semigroups) Let \( A \) generate a bounded, analytic \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) on \( X \) and suppose (without loss of generality) that \( 0 \in \rho(A) \). For a fixed \( \alpha \in [0, 1) \) we set \( V := D((-A)^\alpha) \), the domain of the fractional power of \(-A\), and equip it with the norm \( \| v \|_V := \| (-A)^\alpha v \|_X \).

Since the semigroup operators commute with the powers of the generator we
have \( \|e^{hA}\|_{L(V)} = \|e^{hA}\|_{L(X)} \). Moreover, there exists a constant \( C_A > 0 \) such that for \( h > 0 \) we have

\[
\|e^{hA}\|_{L(X,V)} \leq C_A h^{-\alpha}.
\]  

(5)

(We refer to [7, Ch. 9], [16, Ch. 3], or [29, Ch. 4] for details concerning fractional powers of sectorial operators.) Thus for this example \( \rho(h) \leq C_A h^{-\alpha} \) and \( \rho \in L^1(0,1) \). Further, \( \Omega(h) \leq C_A h^{1-\alpha} \) and \( C_\Omega \leq C_A^{\frac{1}{1-\alpha}} \).

We remark that the fractional powers for negative generators of not necessarily analytic \( C_0 \)-semigroups can be also defined, see, [28] and e.g., [9, Sec. II.5.c], but the validity of an estimate as in (5) for some \( \alpha \in (0,1) \) implies analyticity of the semigroup, see [28, Thm. 12.2].

More examples, also for non-analytic semigroups, are provided in Sect. 3 below.

Assumption 1.3. (Properties of the solution)

1. The semilinear Cauchy problem (1) has a unique mild solution \( u \), that is, \( u: [0,T] \to X \) and \( u: (0,T] \to V \) are continuous and \( u \) satisfies the integral equation (2).

2. Let \( r > 0 \) and \( g: [0,T] \times V \to X \) be bounded on the strip

\[
S_r := \{(t,v) \in (0,T) \times V \mid \|v - u(t)\|_V \leq r \}
\]

around the solution \( u \) and Lipschitz continuous on \( S_r \) in the second variable, i.e., there exists a real number \( L > 0 \) such that for all \( t \in (0,T) \) and \( (t,v),(t,w) \in S_r \):

\[
\|g(t,v) - g(t,w)\|_X \leq L\|v - w\|_V.
\]  

(6)

3. The composition \( f: [0,T] \to X \), with \( f(t):=g(t,u(t)) \) satisfies \( f \in W^{s,1}([0,T], W) \) for a given natural number \( s \geq 1 \). Note that \( W^{1,1}([0,T], W) \) equals the set of absolutely continuous functions.

Remark 1.4. That \( f \in W^{s,1}([0,T], W) \) is a requirement whose validity is not easily established in the infinite dimensional situation. Classical theory, see, e.g., [38, Thm 6.1.6], tells that if \( g \) is continuously differentiable (and \( W = V = X \)), then there is a (local) classical solution to the semilinear equation and the regularity condition is fulfilled with \( s = 1 \). This abstract smoothness condition can be relaxed if \( A \) generates a (bounded) analytic semigroup (\( W = X, V = D((-A)^\alpha) \)): The Lipschitz continuity of \( g \) is sufficient to have Assumption 1.3.3 with \( s = 1 \), cf Theorem 6.3.1 in [38] and Corollary 6.3.2 afterwards. More recent results are described, e.g., in Chapter 7 of [29]. In case of particular equations such regularity conditions (along with the existence of solution at all) are to be investigated with specific techniques, we indicate such cases in Sect. 3 below.

Remark 1.5. If \( g: [0,T] \times V \to X \) is uniformly Lipschitz continuous and bounded in the second variable on bounded sets in \( V \), i.e., for each \( B \subseteq V \) bounded there is \( L_B \geq 0 \) such that for all \( t \in [0,T] \) and \( (v,w) \in B \) one has

\[
\|g(t,v) - g(t,w)\|_X \leq L_B\|v - w\|_V,
\]
and the solution $u: [0, T] \to V$ is bounded, then Assumption 1.3.2 is satisfied.

The main result of this paper reads as follows.

**Theorem 1.6.** Suppose Assumption 1.3 and let the initial value problem (1) satisfy Assumption 1.1. Then there exist constants $C > 0$ and $h_0 > 0$ that only depend on $T, s, \ell, S, r, g$, the space $V$ and the semigroup $(e^{tA})_{t \geq 0}$, such that for $h \in (0, h_0)$ and $0 \leq t_n = nh \leq T$, the approximation $u_n$ is well-defined, that is equation (4) has a unique solution $U_{n,1}, \ldots, U_{n,s} \in V$ satisfying $(t_n + c_j h, U_{n,j}) \in S_r$ for $j = 1, \ldots, s$, and its error satisfies

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s-1} \Omega_W(h) \|f(s)\|_{L^1([0,t_n],W)}.$$  \hfill (7)

It is worth formulating the previous error estimate for the two special cases $W \in \{X, V\}$: For $W = V$, we can choose $\Omega_V(h) = Mh$ and (7) takes the form

$$\|u_n - u(t_n)\|_V \leq C \cdot h^s \|f(s)\|_{L^1([0,t_n],V)},$$  \hfill (8)

whereas for $W = X$ one relaxes the condition on $f$ and arrives at

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s-1} \Omega(h) \|f(s)\|_{L^1([0,t_n],X)}.$$  \hfill (9)

Suppose that $V$ equals the domain of the fractional power $(-A)^\alpha$ of the negative generator $-A$, where $\alpha \in (0, 1)$ and $A$ is assumed to be the generator of an analytic semigroup if $\alpha > 0$, see Example 1.2 above. This is setting of the paper [23] by Hochbruck and Ostermann. In this case we can take $\rho(h) = ch^{1-\alpha}$ and hence $\Omega(h) = ch^{1-\alpha}$. So the error estimate from Theorem 1.6 takes the form

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s-\alpha} \|f(s)\|_{L^1([0,t_n],X)}.$$  \hfill (10)

The paper [23] states the estimate (see (22) therein)

$$\|u_n - u(t_n)\|_V \leq C \cdot h^s \sup_{s \in [0,T]} \|f(s)\|.$$  \hfill (11)

(Note that if $\alpha = 0$, i.e., $X = V$ the proof in [23] works also for non-analytic semigroups and the order of the two bounds in (9) and (11) coincide.) Our abstract approach does not recover the result in [23] as a special case, but we can remark the main novelty here: We do not require $V$ being a subspace of $X$, this allows for a larger flexibility. Note also that [23] considers also $s$-stage methods an with additional order condition on the underlying Runge–Kutta method and the authors prove an improved, $(s+1)$-order error estimate under extra regularity assumptions on the solutions. In this paper, we do not cover such $s$-stage methods and leave the study of them in the present framework to future research.

Problems that fit into this setting, beside the case of analytic semigroups, include non-analytic Ornstein–Uhlenbeck semigroups perturbed by non-linear potentials, Navier–Stokes equations in 3D, incompressible 3D flows around rotation obstacles, wave equation with a non-linear damping, see Sect. 3.

The structure of the paper is as follows. The proof of Theorem 1.6. takes up the next section. To make the paper as self-contained as possible, some
auxiliary results concerning Lagrange interpolation and Gronwall’s lemma, are recalled in the Appendix. Finally, in Sect. 3 various examples, mentioned above, are presented.

2. Proof of Theorem 1.6

This section is devoted to the proof of Theorem 1.6. We remark that for $s = 1$ many of the sums in the following proof are empty, so equal 0, and that the Lagrange “interpolation” polynomial $\ell_1 \equiv 1$. Let the initial value problem (1) satisfy Assumptions 1.1 and 1.3 with constants $M$, $r$ and $L$. Further, let $C_\ell > 0$ be given by Lemma 4.2, i.e., for all $h > 0$

$$|\ell_i(\tau)| \leq C_\ell \quad \text{for all } i \in \{1, \ldots, s\} \text{ and for all } \tau \in [0, h].$$

For $n \in \mathbb{N}$, $h > 0$ we define $C_{f,W}(n, h)$ by

$$C_{f,W}(n, h) := \int_0^h \|e^{tA}\|_{L^1(W,V)} dt \|f^{(s)}\|_{L^1([0,t_n],W)} \leq \Omega_W(h)\|f^{(s)}\|_{L^1([0,t_n],W)} \leq \Omega_W(h)\|f^{(s)}\|_{L^1([0,T],W)}.$$

For the proof of Theorem 1.6 the following lemmas are needed. Recall that $\Omega_W(h), \Omega(h) \to 0$ for $h \to 0$, so $C_{f,W}(n, h) \to 0$ for $h \to 0$ uniformly in $n \leq T/h$.

Lemma 2.1. Let $C > 0$, $\delta > 0$ with

$$MC\delta^{s-1}C_{f,W}(n, \delta) + \Omega(\delta)(sC_\ell + 1) \max_{(t,y) \in S_r} \|g(t, y)\|_X \leq r$$

and $\Omega(\delta)C_\ell sL < 1$.

Suppose that for fixed $n \in \mathbb{N}$, $u_n \in V$ and $h \in (0, \delta)$ with $(n+1)h \leq T$ we have

$$\|u_n - u(t_n)\|_V \leq Ch^{s-1}C_{f,W}(n, h).$$

(12)

Then, the equation (4) has a unique solution $U_{n,1}, \ldots, U_{n,s} \in V$ satisfying $(t_n + c_1h, U_{n,j}) \in S_r$ for $j = 1, \ldots, s$.

Proof. The main idea is to show existence of $U_{n,1}, \ldots, U_{n,s} \in V$ by means of Banach’s fixed point theorem. We equip $V^s$ with the maximum norm over the norms of its $s$ components. For $i = 1, \ldots, s$ we define

$$Y^i_h := \{v \in V \mid \|u(t_n + c_ih) - v\|_V \leq r\},$$

and $Y_h := Y^1_h \times \cdots \times Y^s_h \subset V^s$. Further, let $\Phi_h : Y_h \to Y_h$ defined by

$$(\Phi_h(x_1, x_2, \ldots, x_s))_i = e^{c_i h A} u_n + \int_0^{c_i h} e^{(c_i h - \tau)A} \sum_{j=1}^s \ell_j(\tau)g(t_n + c_jh, x_j) d\tau.$$
First, we show that $\Phi_h(x) \in Y_h$ for $x \in Y_h$. Indeed this follows from the calculation
\[
\|\Phi_h(x) - u(t_n + c_i h)\|_V
\]
\[
= \left\| e^{c_i h A} u_n + \int_0^{c_i h} e^{(c_i h - \tau) A} \sum_{j=1}^{s} \ell_j(\tau) g(t_n + c_j h, x_j) d\tau - e^{c_i h A} u(t_n) - \int_0^{c_i h} e^{(c_i h - \tau) A} g(t_n + \tau, u(t_n + \tau)) d\tau \right\|_V
\]
\[
\leq M\|u_n - u(t_n)\|_V + \Omega(h)(sC_\ell + 1) \max_{(s,y) \in S_r} \|g(s, y)\|_X
\]
\[
\leq MCh^{s-1}C_{f,W}(n, h) + \Omega(h)(sC_\ell + 1) \max_{(s,y) \in S_r} \|g(s, y)\|_X
\]
\[
\leq r.
\]
Next, we show that $\Phi_h$ is a strict contraction. Let $x = (x_1, \ldots, x_s)$, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_s) \in Y_h$. Since $g$ is Lipschitz continuous on $S_r$, we obtain
\[
\|\Phi_h(x) - \Phi_h(\tilde{x})\|_V
\]
\[
= \max_{i \in \{1, \ldots, s\}} \left\| (\Phi_h(x))_i - (\Phi_h(\tilde{x}))_i \right\|_V
\]
\[
= \max_{i \in \{1, \ldots, s\}} \left\| \int_0^{c_i h} e^{(c_i h - \tau) A} \sum_{j=1}^{s} \ell_j(\tau) (g(t_n + c_j h, x_j) - g(t_n + c_j h, \tilde{x}_j)) d\tau \right\|_V
\]
\[
\leq \max_{i \in \{1, \ldots, s\}} \Omega(h)C_{\ell} s \max_{j \in \{1, \ldots, s\}} \|g(t_n + c_j h, x_j) - g(t_n + c_j h, \tilde{x}_j)\|_X
\]
\[
\leq \Omega(h)C_{\ell} s L \|x - \tilde{x}\|_V.
\]
Thus the statement follows by Banach’s fixed point theorem and the definition of $Y_h$. \hfill \Box

**Proof of Theorem 1.6.** Let $h_0 > 0$ such that $\Omega(h_0)C_{\ell} s L \leq \frac{1}{2}$ (possible by $\Omega(h) \to 0$ for $h \to 0$).

Plainly, there exists a constant $C_F > 0$ such that for $n \in \mathbb{N}$, $\xi \in [t_n, t_{n+1}]$ and $i = 1, \ldots, s$ we have
\[
\left| \frac{(t_n + c_i h - \xi)^{s-1}}{(s-1)!} \right| \leq C_F h^{s-1}.
\]

We define
\[
C_{G,1} := 2sM^2C_{\ell} s L \max\{M, \Omega(h_0)\},
\]
\[
C_{G,2} := 2 \max\{2C_\ell^2 L s^2C_F(\Omega(h) + MC_\Omega), MC_\ell s C_F\},
\]
\[
C := C_{G,2} \exp(C_{G,1} C_\Omega + C_{G,1}).
\]

Let $h \in (0, h_0)$ with
\[
MCh^{s-1}C_{f,W}(n, h) + (sC_\ell + 1) \max_{(s,y) \in S_r} \|g(s, y)\|_X \Omega(h) \leq r.
\]

It suffices to show the following statement by induction over $n \in \mathbb{N}$, $n \geq 0$:
For $h \in (0,b)$ and $n \in \mathbb{N}$ with $(n+1)h \leq T$ equation (4) has a unique solution $U_{n,1}, \ldots, U_{n,s} \in V$ satisfying $(t_n + c_j h, U_{n,j}) \in S_r$ for $j = 1, \ldots, s$, and

$$
\|u_n - u(t_n)\|_V \leq C h^{s-1} C_{f,W}(n,h).
$$

(14)

If $n = 0$, then $u_0 = u(t_0)$. Thus the norm estimate of $\|u_0 - u(t_0)\|_V$ is trivial and the unique existence of $U_{0,1}, \ldots, U_{0,s} \in V$ satisfying $(t_0 + c_j h, U_{0,j}) \in S_r$ follows from Lemma 2.1.

Next, we assume that the statement holds for $0, \ldots, n$, for some $n \in \mathbb{N}$, and we aim to show the statement for $n+1$ with $(n+1)h \leq T$. For $k = 0, \ldots, n$ let

$$
e_k := u_k - u(t_k), \quad E_{k,i} := U_{k,i} - u(t_k + c_i h)
$$

(recall $t_k = kh$). We divide the proof in several steps.

**Step 1.** We show

$$
\sum_{j=1}^s \ell_j(\tau) g(t_n + c_j h, U_{n,j}) - g(t_n + \tau, u(t_n + \tau))
$$

$$
= \sum_{j=1}^s \ell_j(\tau) (g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h))
$$

$$
+ \sum_{j=1}^s \ell_j(\tau) \int_{t_n + \tau}^{t_n + c_j h} \frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, d\xi.
$$

We can write

$$
\sum_{j=1}^s \ell_j(\tau) g(t_n + c_j h, U_{n,j}) - g(t_n + \tau, u(t_n + \tau))
$$

$$
= \sum_{j=1}^s \ell_j(\tau) \left( g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h) + f(t_n + c_j h) - f(t_n + \tau) \right)
$$

as $\sum_{j=1}^s \ell_j(\tau) = 1$ (see Lemma 4.1). Taylor expansion yields

$$
f(t_n + c_j h) - f(t_n + \tau) = \sum_{k=1}^{s-1} \frac{f^{(k)}(t_n + \tau)}{k!} (c_j h - \tau)^k
$$

$$
+ \int_{t_n + \tau}^{t_n + c_j h} \frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, d\xi.
$$

Recall the following property of Lagrange interpolation polynomials, see (29) in Lemma 4.1: For $k \leq s - 1$

$$
\sum_{j=1}^s \ell_j(\tau) (c_j h - \tau)^k = 0.
$$

(16)
Inserting this into the Eq. (15) above finishes Step 1 as

\[ \sum_{j=1}^{s} \ell_j(\tau)g(t_n + c_j h, U_{n,j}) - g(t_n + \tau, u(t_n + \tau)) \]
\[ = \sum_{j=1}^{s} \ell_j(\tau)(g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h)) \]
\[ + \sum_{j=1}^{s} \ell_j(\tau) \left( \sum_{k=1}^{s-1} \frac{f(k)(t_n + \tau)}{k!} (c_j h - \tau)^k \right) \]
\[ + \int_{t_n + \tau}^{t_n + c_j h} \frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, d\xi \]
\[ = \sum_{j=1}^{s} \ell_j(\tau)(g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h)) \]
\[ + \sum_{k=1}^{s-1} \frac{f(k)(t_n + \tau)}{k!} \sum_{j=1}^{s} \ell_j(\tau)(c_j h - \tau)^k \]
\[ = 0 \text{ by (16)} \]
\[ + \sum_{j=1}^{s} \ell_j(\tau) \int_{t_n + \tau}^{t_n + c_j h} \frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, d\xi. \]

**Step 2.** Let \( \tilde{\delta}_{k+1} \) and \( \delta_{k+1}, k = 0, \ldots, n, \) be given by

\[ \tilde{\delta}_{k+1} := \int_0^{h} e^{(h-\tau)A} \sum_{i=1}^{s} \ell_i(\tau)(g(t_k + c_i h, U_{k,i}) - f(t_k + c_i h)) \, d\tau, \quad (17) \]
\[ \delta_{k+1} := \int_0^{h} e^{(h-\tau)A} \sum_{i=1}^{s} \ell_i(\tau) \int_{t_k + \tau}^{t_k + c_i h} \frac{(t_k + c_i h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, d\xi \, d\tau. \quad (18) \]

Then Step 1 implies

\[ e_{n+1} = u_{n+1} - u(t_{n+1}) \]
\[ = e^{hA} e_n + \int_0^{h} e^{(h-\tau)A} \sum_{i=1}^{s} \ell_i(\tau) \left( g(t_n + c_i h, U_{n,i}) - f(t_n + \tau) \right) \, d\tau \]
\[ = e^{hA} e_n + \int_0^{h} e^{(h-\tau)A} \left( \sum_{i=1}^{s} \ell_i(\tau) \left( g(t_n + c_i h, U_{n,i}) - f(t_n + c_i h) \right) \right) \, d\tau \]
\[ + \int_0^{h} e^{(h-\tau)A} \left( \sum_{i=1}^{s} \ell_i(\tau) \int_{t_n + \tau}^{t_n + c_i h} \frac{(t_n + c_i h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, d\xi \right) \, d\tau \]
\[ = e^{hA} e_n + \tilde{\delta}_{n+1} + \delta_{n+1}. \]
Solving the recursion yields:

\[
e_{n+1} = e^{hA}e_n + \tilde{\delta}_{n+1} + \delta_{n+1}
\]

\[
e_{n+1} = e^{hA} \left( e^{hA}e_{n-1} + \tilde{\delta}_n + \delta_{n+1} \right) + \tilde{\delta}_{n+1} + \delta_{n+1}
\]

\[
e_{n+1} = \sum_{k=0}^{n} e^{khA} \left( \tilde{\delta}_{n+1-k} + \delta_{n+1-k} \right),
\]

as \( e_0 = 0 \). Hence

\[
\|e_{n+1}\|_V \\
\leq \sum_{k=0}^{n} \left( \|e^{khA}\tilde{\delta}_{n+1-k}\|_V + \|e^{khA}\delta_{n+1-k}\|_V \right)
\]

\[
= \sum_{k=0}^{n} \left( \|e^{(n-k)hA}\tilde{\delta}_{k+1}\|_V + \|e^{(n-k)hA}\delta_{k+1}\|_V \right).
\]

(19)

We will estimate the norms on the right-hand side in (19) separately.

**Step 3.** We start by bounding the Taylor-remainders for each fixed \( i \).

For \( k = 0, \ldots, n \) and \( i = 1, \ldots, s \) we define

\[
\Delta_{k,i} := \int_0^{c_i h} e^{(c_i h - \tau)A} \sum_{j=1}^{s} \ell_j(\tau) \int_{t_k+\tau}^{t_k+c_j h} \frac{(t_k + c_j h - \xi)^{s-1}}{(s-1)!} f(s)(\xi) \, d\xi \, d\tau,
\]

and estimate

\[
\|\Delta_{k,i}\|_V \\
\leq \int_0^{c_i h} \|e^{(c_i h - \tau)A}\|_{L(W,V)} \sum_{j=1}^{s} |\ell_j(\tau)| \\
\times \left\| \int_{t_k+\tau}^{t_k+c_j h} \frac{(t_k + c_j h - \xi)^{s-1}}{(s-1)!} f(s)(\xi) \, d\xi \right\|_W \, d\tau
\]

\[
\leq C_{\ell} \left( \int_0^{h} \|e^{\tau A}\|_{L(W,V)} \, d\tau \right)^s \sum_{j=1}^{s} \int_{t_k}^{t_k+h} \left| \frac{(t_k + c_j h - \xi)^{s-1}}{(s-1)!} \right| \|f(s)(\xi)\|_W \, d\xi
\]

(13)

\[
\leq C_{\ell} s C_F h^{s-1} \left( \int_0^{h} \|e^{\tau A}\|_{L(W,V)} \, d\tau \right)^s \|f(s)\|_{L^1([t_k,t_{k+1}], W)}. \]

(21)

**Step 4.** We now consider the norm of \( \delta_{k+1} \) (see (18)). We estimate as in Step 3

\[
\|\delta_{k+1}\|_V \leq C_{\ell} s C_F h^{s-1} \left( \int_0^{h} \|e^{\tau A}\|_{L(W,V)} \, d\tau \right)^s \|f(s)\|_{L^1([t_k,t_{k+1}], W)},
\]
and obtain
\[
\sum_{k=0}^{n} \| e^{(n-k)hA} \delta_{k+1} \|_V \\
\leq \sum_{k=0}^{n} MC\ell s C_F h^{s-1} \left( \int_0^h \| e^{\tau A} \|_{L(W,V)} \, d\tau \right) \| f(s) \|_{L^1([t_k,t_{k+1}],W)} \\
\leq \frac{1}{2} C_{G,2} h^{s-1} C_f, W(n + 1, h).
\] (22)

Step 5. We prove
\[
\sum_{i=1}^{s} \| E_{k,i} \|_V \leq 2 s M \| e_k \|_V + 2 C\ell s^2 C_F h^{s-1} \\
\cdot \left( \int_0^h \| e^{\tau A} \|_{L(W,V)} \, d\tau \right) \| f(s) \|_{L^1([t_k,t_{k+1}],W)}. (23)
\]

Thanks to Step 1 we can calculate
\[
E_{k,i} = U_{k,i} - u(t_k + c_i h) \\
= e^{c_i h A} u_k + \int_0^{c_i h} e^{(c_i h - \tau) A} \sum_{j=1}^{s} \ell_j(\tau) g(t_k + c_j h, U_{k,j}) \, d\tau \\
- \left( e^{c_i h A} u(t_k) + \int_0^{c_i h} e^{(c_i h - \tau) A} g(t_k + \tau, u(t_k + \tau)) \, d\tau \right) \\
= e^{c_i h A} e_k + \int_0^{c_i h} e^{(c_i h - \tau) A} \sum_{j=1}^{s} \ell_j(\tau) (g(t_k + c_j h, U_{k,j}) - f(t_k + c_j h)) \, d\tau \\
+ \Delta_{k,i},
\]
where \( \Delta_{k,i} \) is given by (20). Using the Lipschitz continuity of \( g \) on \( S_r \), we obtain
\[
\| E_{k,i} \|_V \leq M \| e_k \|_V + \int_0^{c_i h} \left\| e^{(c_i h - \tau) A} \right\|_{L(W,V)} \sum_{j=1}^{s} |\ell_j(\tau)| \\
\cdot \| g(t_k + c_j h, U_{k,j}) - g(t_k + c_j h, u(t_k + c_j h)) \|_W \, d\tau + \| \Delta_{k,i} \|_V \\
\leq L \| E_{k,j} \|_V + \Delta_{k,i} \|_V.
\]

Thus, using \( \Omega(h_0)C\ell sL \leq \frac{1}{2} \) we conclude
\[
\sum_{i=1}^{s} \| E_{k,i} \|_V \leq s M \| e_k \|_V + \frac{1}{2} \sum_{j=1}^{s} \| E_{k,j} \|_V + \sum_{i=1}^{s} \| \Delta_{k,i} \|_V.
\]
This together with (21) implies (23).
Step 6. For \( k = 0, \ldots, n \), we now investigate the norm of \( \tilde{\delta}_{k+1} \) (see (17)). Using (23), we obtain

\[
\|\tilde{\delta}_{k+1}\|_V \leq \Omega(h) C_t L \sum_{i=1}^{s} \|E_{k,i}\|_V \\
\leq \Omega(h) C_t L \left( 2sM \|e_k\|_V \\
+ 2C_\ell s^2 C_F h^{s-1} \left( \int_0^h \|e^{\tau A}\|_{L(W,V)} \, d\tau \right) \|f^{(s)}\|_{L^1([t_k, t_{k+1}], W)} \right),
\]

and

\[
\|\tilde{\delta}_{k+1}\|_X \leq MhC_t L \sum_{i=1}^{s} \|E_{k,i}\|_V \\
\leq MhC_t L \left( 2sM \|e_k\|_V \\
+ 2C_\ell s^2 C_F h^{s-1} \left( \int_0^h \|e^{\tau A}\|_{L(W,V)} \, d\tau \right) \|f^{(s)}\|_{L^1([t_k, t_{k+1}], W)} \right).
\]

Thus, using \( \|e^{(n-k)hA}\|_{L(X,V)}h \leq C_\Omega \) we have

\[
\sum_{k=0}^{n} \|e^{(n-k)hA}\tilde{\delta}_{k+1}\|_V \\
\leq \|\tilde{\delta}_{n+1}\|_V + \sum_{k=0}^{n-1} \|e^{(n-k)hA}\|_{L(X,V)} \|\tilde{\delta}_{k+1}\|_X \\
\leq \Omega(h) C_t L \left( 2sM \|e_n\|_V \\
+ 2C_\ell s^2 C_F h^{s-1} \left( \int_0^h \|e^{\tau A}\|_{L(W,V)} \, d\tau \right) \|f^{(s)}\|_{L^1([t_{n+1}, t_n], W)} \right) \\
+ MhC_t L 2sM \sum_{k=0}^{n-1} \|e^{(n-k)hA}\|_{L(X,V)} \|e_k\|_V + 2MhC_t LC_\ell s^2 C_F h^{s-1} \\
\cdot \|e^{(n-k)hA}\|_{L(X,V)} \left( \int_0^h \|e^{\tau A}\|_{L(W,V)} \, d\tau \right) \|f^{(s)}\|_{L^1([t_{n+1}, t_n], W)} \\
\leq C_{G,1} \sum_{k=0}^{n-1} \|e^{(n-k)hA}\|_{L(X,V)} \|e_k\|_V + C_{G,1} \|e_n\|_V \\
+ \frac{1}{2} C_{G,2} h^{s-1} C_{f,W} (n+1, h).
\]
Step 7. In the final step we estimate $e_{n+1}$ and complete the proof. Using (19), (22) and (24) we obtain

\[
\|e_{n+1}\|_V \leq \sum_{k=0}^{n} (\|e^{(n-k)hA} \delta_k + 1\|_V + \|e^{(n-k)hA} \delta_k + 1\|_V)
\]

\[
\leq C_{G,1} \sum_{k=0}^{n-1} \|e^{(n-k)hA} \|_{L(X,V)} h \|e_k\|_V + C_{G,1} \|e_n\|_V
\]

\[
+ C_{G,2} h^{s-1} C_{f, W}(n+1, h),
\]

and Gronwall’s Lemma (see Theorem 4.3) implies

\[
\|e_{n+1}\|_V \leq C_{G,2} \exp \left( C_{G,1} \sum_{k=0}^{n-1} \|e^{(n-k)hA} \|_{L(X,V)} h \|e_k\|_V + C_{G,1} \right) h^{s-1} C_{f, W}(n+1, h)
\]

By Lemma 2.1 applied to $n+1$, for $h \in (0, h)$ if $(n+1)h \leq T$ equation (4) has a unique solution $U_{n+1, 1}, \ldots, U_{n+1, s} \in V$ with $(t_{n+1} + c_j h, U_{n+1, j}) \in S_r$.

\[\square\]

3. Examples

We present here examples for the situation described in Assumption 1.1 and also for some admissible non-linearities satisfying Assumption 1.3.

Example 3.1. (Gaussian heat semigroup) Consider the Gaussian heat semigroup $(e^{tA})_{t \geq 0}$ on $L^2(\mathbb{R}^d)$, for $t > 0$ given as

\[e^{tA}f := g_t \ast f,\]

where $g_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}$, $x \in \mathbb{R}^d$, is the Gaussian kernel. Then, actually, $(e^{tA})_{t \geq 0}$ yields a consistent family of analytic $C_0$-semigroups on the whole $L^p(\mathbb{R}^d)$-scale, $p \in [1, \infty)$. A short calculation using the Young convolution inequality yields for $1 < p < r < \infty$ and $f \in L^p(\mathbb{R}^d)$ that

\[
\|e^{tA} f\|_r \leq c_{p, r} t^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{r} \right)} \|f\|_p
\]

with an absolute constant $c_{p, r}$ (whose optimal value can be determined, cf. [3]). As usual, we shall refer to this phenomenon as $L^p$-$L^r$-smoothing. We conclude that the choices $X = L^p(\mathbb{R}^d)$, $V = L^r(\mathbb{R}^d)$ and

\[\Omega(h) = c_{p, r} h^{1-\alpha}\]
with $\alpha = \frac{d}{2} (\frac{1}{p} - \frac{1}{r})$ are admissible choices in Assumption 1.1, if $\frac{d}{2} (\frac{1}{p} - \frac{1}{r}) < 1$.

For similar estimates in case of symmetric, Markov semigroups we refer, e.g., to [8, Ch 2].

**Example 3.2.** (Stokes semigroup) Similarly to the foregoing examples, $L^p$-$L^r$-smoothing is valid for the Stokes semigroup on the divergence free space $L^p_\sigma(\mathbb{R}^d)^d$, so $X = L^p_\sigma(\mathbb{R}^d)^d$, $V = L^r_\sigma(\mathbb{R}^d)^d$ and the $\Omega$ from Example 3.1 are admissible in Assumption 1.1, see [27].

**Example 3.3.** (Ornstein–Uhlenbeck semigroups) Let $Q \in \mathbb{R}^{d \times d}$ be a positive semidefinite matrix and $B \in \mathbb{R}^{d \times d}$. Suppose that the positive semidefinite matrix

$$Q_t := \int_0^t e^{\sigma B} Q e^{\sigma^* B^*} d\sigma$$

is invertible for some $t > 0$ (for this, a sufficient but not necessary, assumption is that $Q$ itself is invertible). Then $Q_t$ is invertible for all $t > 0$, see [41, Ch.1]. Consider the Kolmogorov kernel

$$k_t(x) := \frac{1}{(4\pi)^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}} e^{-\frac{1}{2} \langle Q_t^{-1} x, x \rangle},$$

and for $t > 0$ the operator $S(t)$ defined by

$$S(t)f(x) := (k_t \ast f)(e^{tB}x) = \frac{1}{(4\pi)^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \langle Q_t^{-1} y, y \rangle} f(e^{tB}x - y) dy.$$

Then by Young's convolution inequality, we see that $S(t)$ acts indeed on $L^p(\mathbb{R}^d)$ for each fixed $p \in [1, \infty)$, it is linear and bounded with

$$\|S(t)\|_{L(L^p)} \leq e^{-\frac{\text{tr}(B)}{p} t}.$$ Setting $S(0) = I$, we obtain a $C_0$-semigroup on $L^p(\mathbb{R}^d)$, called the Ornstein–Uhlenbeck semigroup, see, e.g., [33] for details. The Ornstein–Uhlenbeck semigroup is, in general, not analytic on $L^p(\mathbb{R}^d)$ (see [11] and [35]). Similarly to Example 3.1 for $r \geq p$ and $f \in L^p(\mathbb{R}^d)$ we have for $t > 0$ that

$$\|S(t)f\|_r \leq e^{-\frac{\text{tr}(B)}{r} t} \| k_t \ast f \|_r \leq e^{-\frac{\text{tr}(B)}{r} t} \| k_t \|_q \| f \|_p,$$

with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. We can calculate

$$\| k_t \|_q^q = \frac{1}{(4\pi)^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \langle Q_t^{-1} y, y \rangle} dy = \frac{-\frac{1}{2}}{q \pi^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \langle Q_t^{-1} y, y \rangle} dy$$

$$= \frac{(4\pi)^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}}{q \pi^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}} = c_q \det(Q_t)^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{q})} = c_q \det(Q_t)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{r})}.$$

If $Q$ is invertible, then we have $\| Q_t^{-\frac{1}{2}} \| \leq Ct^{-\frac{1}{2}}$, see, e.g., [30] (but also below). Since $\det(Q_t^{-1}) \leq C\| Q_t^{-1} \|^d$, we obtain $\det(Q_t) \geq C't^{-d}$, which, when inserted into (25), yields

$$\| k_t \|_q \leq c_{p,r,Q} t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{r})}.$$
for some constant $c_{p,r,Q}$ depending on $p,r,Q$. This result, for invertible $Q$ is essentially contained in [20] (or [17] in an even more general situation of evolution families, see also [14]). It follows that $X = L^p(\mathbb{R}^d)$, $V = L^r(\mathbb{R}^d)$ and

$$\Omega(h) = c_{p,r,Q} h^{1 - \frac{d}{2}(\frac{1}{p} - \frac{1}{r})},$$

are admissible choices in Assumption 1.1 provided $r \geq p$, $\frac{d}{2}(\frac{1}{p} - \frac{1}{r}) < 1$.

Now if $Q$ is not necessarily invertible, but for some/all $t > 0$ the matrix $Q_t$ is non-singular, then there is a minimal integer $n > 0$ such that

$$[Q^{\frac{1}{2}}, BQ^{\frac{1}{2}}, B^2Q^{\frac{1}{2}}, \ldots, B^{n-1}Q^{\frac{1}{2}}]$$

has rank $d$, see, for example, [41, Ch.1] (if $Q$ is invertible, then $n = 1$). One can show that in this case $\|Q_t^{-\frac{1}{2}}\| \leq C t^{-\frac{1}{2} - n}$ (for $t$ near 0), see, [31, Lemma 3.1] (and, e.g., [10,39]), hence

$$\|k_t\|_q \leq c t^{-\frac{d(2n-1)}{2}(\frac{1}{p} - \frac{1}{r})},$$

i.e., $X = L^p(\mathbb{R}^d)$, $V = L^r(\mathbb{R}^d)$ and

$$\Omega(h) = c_{p,r,Q} h^{1 - \alpha}$$

with $\alpha = \frac{d(2n-1)}{2}(\frac{1}{p} - \frac{1}{r})$ are admissible choices in Assumption 1.1 provided $r \geq p$ and $r$ is near to $p$. Similar results hold for (strongly elliptic) Ornstein–Uhlenbeck operators on $L^p(\Omega)$, $\Omega$ an exterior domain with smooth boundary, see [13].

**Example 3.4.** (Ornstein–Uhlenbeck semigroups on Sobolev space) Consider again the Ornstein–Uhlenbeck semigroup $S$ from the foregoing example, given by $S(t)f(x) := (k_t \ast f)(e^{tB}x)$ for $t \geq 0$, $f \in L^p(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

We have $\partial_x S(t)f(x) = (k_t \ast \partial_x f)(e^{tB}x)e^{tB}$, hence $S(t)$ leaves $W^{1,r}(\mathbb{R}^d)$ invariant, and is locally uniformly bounded thereon. On the other hand $\partial_x S(t)f(x) = (\partial_x k_t \ast f)(e^{tB}x)e^{tB}$, and analogously to Example 3.3 one can prove that for $t > 0$, $1 \leq p \leq r$ and $f \in L^p(\mathbb{R}^d)$

$$\|\partial_x S(t)f\|_r \leq c t^{-\frac{d(2n-1)}{2}(\frac{1}{p} - \frac{1}{r}) + \frac{d}{2} - n}\|f\|_p.$$ 

So if $n = 1$, i.e., in the elliptic case, we obtain that $V = W^{1,r}(\mathbb{R}^d)$ and

$$\Omega(h) = c_{p,r,Q} h^{1 - \alpha}$$

with $\alpha = \frac{d}{2}(\frac{1}{p} - \frac{1}{r}) + \frac{1}{2}$ are admissible choices in Assumption 1.1 provided $r \geq p$ and $r$ is near to $p$ (if $\frac{1}{p} - \frac{1}{r} < \frac{1}{d}$).

**Example 3.5.** (Stokes operator with a drift) Examples 3.2 and 3.3 can be combined. The operator $A$, defined by $Au(x) = \Delta u(x) + Mx \cdot \nabla u(x) - Mu(x)$ (with appropriate domain) generates a (in general, non-analytic) $C_0$-semigroup on the divergence free spaces $L^p_\sigma(\Omega)^d$, $1 < p \leq r < \infty$ subject to $L^p$-$L^r$-smoothing, with $\Omega = \mathbb{R}^d$, see [20], $\Omega$ a bounded or an exterior domain, see [12]. Thus $X = L^p(\mathbb{R}^d)^d$, $V = L^r(\mathbb{R}^d)^d$ and the $\Omega$ from Example 3.1 are admissible in Assumption 1.1.
Example 3.6. Consider the Stokes-semigroup $S$ generated by

$$Au(x) = \Delta u(x) + Mx \cdot \nabla u(x) - Mu(x)$$

(with appropriate domain) on $L^p_\sigma(\Omega)^d$ $(1 < p < \infty)$, where $\Omega = \mathbb{R}^d$ or $\Omega$ is a bounded or an exterior domain, see [12]. We then have for $t > 0$, $1 \leq p \leq r$ and $L^p_\sigma(\Omega)^d$

$$\|\nabla S(t)f\|_r \leq ct^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{r}\right)} - \frac{1}{2}\|f\|_p,$$

where $1 < p \leq r < \infty$.

If $\frac{1}{p} = \frac{1}{s} + \frac{1}{r}$, $d < s$ and $\frac{1}{p} - \frac{1}{s} < \frac{2}{d}$, then $V = L^s(\Omega)^d \cap W^{1,r}(\Omega)^d$ with

$$\Omega(h) = c_{p,r,Q}h^{1-\alpha}$$

and $\alpha = \max\{\frac{d}{2r} + \frac{1}{2}, \frac{d}{2s}\}$ is an admissible choice in Assumption 1.1 and the non-linearity $g(u) = u \cdot \nabla u$ also satisfies the required Lipschitz conditions (see Example 3.10).

Example 3.7. (Ornstein–Uhlenbeck semigroups on spaces with invariant measures) Similar results as in Example 3.3 are valid for the Ornstein–Uhlenbeck semigroup $(S(t))_{t \geq 0}$ on spaces $L^p(\mathbb{R}^d, \mu)$ with invariant measures $\mu$ (cf., e.g., [10,30]). Note however that $(S(t))_{t \geq 0}$ is analytic in this case (provided $p > 1$), see, e.g., [5,36,37] also for further details.

Example 3.8. (Interpolation spaces vs. growth function) A special case of a theorem of Lunardi, [32, Thm 2.5] yields some information, when the “growth function” $\Omega$ can be taken to be of the form $\Omega(h) = ch^{1-\alpha}$ for some $\alpha \in (0, 1)$. Let $(S(t))_{t \geq 0}$ be a $C_0$-semigroup on the Banach space $X$ with generator $A$. Let $V \subseteq X$ be a further Banach space, and suppose that for some constants $\beta \in (0, 1)$, $\omega \in \mathbb{R}$, $c > 0$ one has

$$\|S(t)\|_{L(X,V)} \leq \frac{ce^{\omega t}}{t^\beta}$$

for $t > 0$,

and that for each $x \in X$ the function $(0, \infty) \ni t \mapsto S(t)x \in V$ is measurable. Then for the real interpolation spaces

$$(X, D(A))_{\theta,p} \hookrightarrow (X, V)_{\theta/\beta,p} \quad \text{for all } \theta \in (0, \beta) \text{ and } 1 \leq p \leq \infty$$

(with continuous embedding).

Example 3.9. Let $\alpha > 1$, $p \in [\alpha, \infty)$ and $U \subseteq \mathbb{R}^d$ be open. Then the map

$$F: L^p(U) \to L^\frac{p}{\alpha}(U), \quad F(u) = |u|^\alpha - 1u,$$

is Lipschitz continuous on bounded sets. Furthermore, $F$ is real continuously differentiable with derivative

$$F'(u)v = |u|^\alpha - 1v + (\alpha - 1)|u|^\alpha - 3u \Re(u \overline{v}) \quad \text{for } u, v \in L^p(U).$$

For a proof see [25, Cor. 9.3].
Example 3.10. The function $g: L^s(\mathbb{R}^d)^d \cap W^{1,r}(\mathbb{R}^d)^d \rightarrow L^p(\mathbb{R}^d)^d$ defined by $g(u) = u \cdot \nabla u$ is Lipschitz continuous on bounded sets if $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Indeed, for $u, v \in L^s(\mathbb{R}^d)^d \cap W^{1,r}(\mathbb{R}^d)^d$ we can write by Hölder inequality that
\[
\|u \cdot \nabla u - v \cdot \nabla v\|_{L^p} \leq \|u \cdot \nabla u - u \cdot \nabla v\|_{L^p} + \|u \cdot \nabla v - v \cdot \nabla v\|_{L^p} \\
\leq C(\|u\|_{L^s} \|u - v\|_{W^{1,r}} + \|u - v\|_s \|v\|_{W^{1,r}}),
\]
proving the asserted Lipschitz continuity.

Example 3.11. (Second order systems) The result can be applied to second order problems via the following technique.

Consider the second order Cauchy problem
\[
\begin{align*}
\ddot{w}(t) &= A\dot{w}(t) + g(t, w(t), \dot{w}(t)), & t \in [0, T] \\
w(0) &= w_0 \in X, & \dot{w}(0) = w_1 \in X \quad (26)
\end{align*}
\]
on a Banach space $X$, where $A: D(A) \subset X \rightarrow X$ generates a Cosine function $(\cos(t))_{t \in \mathbb{R}}$. The associated Sine function $\sin: \mathbb{R} \rightarrow L(X)$ is given by
\[
\sin(t) := \int_0^t \cos(s) \, ds.
\]
We assume that the system (26) has a classical solution $w \in C^2$. Sufficient conditions for the existence of solutions can be found in [40].

Choosing $u = (\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix})$ we can rewrite (26) as a first order problem
\[
\dot{u}(t) = Au(t) + \tilde{g}(t, u(t)), \quad u(0) = u_0,
\]
where $\mathcal{A} := (\begin{smallmatrix} 0 & 1 \\ A & 0 \end{smallmatrix})$, $\tilde{g}(t, u(t)) = (g(t, w(t), \dot{w}(t)))$ and $u_0 = (\begin{smallmatrix} w_0 \\ \dot{w}_0 \end{smallmatrix})$.

As stated in [1, Thm. 3.14.11] there exists a Banach space $V$ such that $D(A) \hookrightarrow V \hookrightarrow X$ and such that the part $\mathcal{A}$ of $(\begin{smallmatrix} 0 & 1 \\ A & 0 \end{smallmatrix})$ in $V \times X$ generates a $C_0$-semigroup given by
\[
T(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ A \sin(t) & \cos(t) \end{pmatrix}, \quad t \in \mathbb{R}.
\]
We will illustrate this procedure with a short example which is presented in [25, Ch. 9].

Consider the non-linear wave equation with Dirichlet boundary conditions on a bounded open set $\emptyset \neq U \subseteq \mathbb{R}^3$ described by the system
\[
\begin{align*}
\ddot{w}(t) &= \Delta_D w(t) - \alpha w(t)|w(t)|^2, & t \in [0, T], \\
w(0) &= w_0, & \dot{w}(0) = w_1, \quad (27)
\end{align*}
\]
where $w_0 \in D(\Delta_D)$, $D(\Delta_D) := \{u \in H^1_0(U) \mid \exists f \in L^2(U) \forall v \in H^1_0(U) : \int_U \nabla u \cdot \nabla v \, dx = \langle f, v \rangle_{L^2} \}$, $w_1 \in H^1_0(U)$ and $\alpha \in \mathbb{R}$ are given. We can reformulate this as the semilinear system
\[
\dot{u}(t) = Au(t) + F(u(t)), \quad t \in [0, T], \quad u(0) = u_0
\]
on the Hilbert space $X = H^1_0(U) \times L^2(U)$ endowed with the norm given by $\|(u_1,u_2)\|^2 = \|\nabla u_1\|^2_2 + \|u_2\|^2_2$. Choose $u_0 = (w_0, w_1)$ and

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix}$$

with $D(A) = D(\Delta_D) \times H^1_0(U)$,

$$F(u) = (0, -\alpha u_1|u_1|^2) = (0, F_0(u_1))$$

for $u = (u_1, u_2) \in X$.

As mentioned in Example 3.9, $F_0: L^6(U) \to L^2(U)$ is real continuously differentiable and Lipschitz continuous on bounded sets. Since $U \subseteq \mathbb{R}^3$, Sobolev’s embedding yields $H^1_0(U) \hookrightarrow L^6(U)$ thus $F: X \to X$ has the same properties. Now Duhamel’s formula or the variation of constants in combination with the semigroup given above yield that the mild solution of (27) satisfies

$$w(t) = \cos(t)w_0 + \sin(t)w_1 + \int_0^t \sin(t-s)F_0(w(s))\,ds, \quad t \in [0,T],$$

on $H^1_0(U)$. Thus the main theorem is applicable with $s = 1$.

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A. Lagrange interpolation polynomials

In this section we summarize some useful facts on Lagrange interpolation polynomials.

For fixed $s \in \mathbb{N}$ and different $c_1, \ldots, c_s \in [0, 1]$ denote by $\ell_i$, $i = 1, \ldots, s$ the Lagrange basis polynomials with nodes $c_1, \ldots, c_s$, i.e.

$$\ell_j(\tau) = \prod_{m=1}^s \frac{\tau - c_m}{c_j - c_m}.$$  

(This if $s = 1$, then $\ell_1 \equiv 1$.) Thus we have $\ell_i(c_j) = 1$ if $i = j$, and $\ell_i(c_j) = 0$ if $i \neq j$.

**Lemma 4.1.** The Lagrange basis polynomials are of degree $s - 1$ and satisfy

$$\sum_{i=1}^s \ell_i(\tau) = 1 \quad \text{for all } \tau \in \mathbb{R},$$  

(28)

and for every integer $k$ with $1 \leq k \leq s - 1$

$$\sum_{i=1}^s \ell_i(\tau)c_i^k = \tau^k \quad \text{and} \quad \sum_{i=1}^s \ell_i(\tau)(c_i - \tau)^k = 0 \quad \text{for all } \tau \in \mathbb{R}. \quad (29)$$

(Note that for $s = 1$ the foregoing statement is vacuously true.)

**Proof.** The polynomial $\sum_{i=1}^s \ell_i(\tau)$ is of degree at most $s - 1$ and equal 1 at $s$ points $c_1, \ldots, c_s$. So equation (28) follows by the identity theorem.

We show equation (29) by induction over $k$. For $k = 1$ we can write by (28)

$$q(\tau) := \sum_{i=1}^s \ell_i(\tau)(c_i - \tau) = \sum_{i=1}^s \ell_i(\tau)c_i - \tau \sum_{i=1}^s \ell_i(\tau) = \sum_{i=1}^s \ell_i(\tau)c_i - \tau.$$

Since each $\ell_i$ is of degree $s - 1 \geq 1$, the first expression on the right-hand side is of degree at most $s - 1$. So $q$ is polynomial of degree at most $s - 1$ with the $s$ zeroes $c_1, \ldots, c_s$. By the identity theorem $q \equiv 0$, i.e.,

$$\sum_{i=1}^s \ell_i(\tau)c_i = \tau.$$  

Let $k < s - 1$ and suppose that

$$\sum_{i=1}^s \ell_i(\tau)(c_i - \tau)^m = 0 \quad \text{and} \quad \sum_{i=1}^s \ell_i(\tau)c_i^m = \tau^m \quad \text{if } 1 \leq m \leq k.$$
We need to show that these hold also for \( m = k + 1 \). By the binomial theorem

\[
p(\tau) := \sum_{i=1}^{s} \ell_i(\tau)(c_i - \tau)^{k+1} = \sum_{i=1}^{s} \ell_i(\tau) \sum_{j=0}^{k+1} \binom{k+1}{j} c_i^{k+1-j} (-\tau)^j
\]

\[
= \sum_{i=1}^{s} \ell_i(\tau) c_i^{k+1} + \sum_{i=1}^{s} \ell_i(\tau) \sum_{j=1}^{k} \binom{k+1}{j} c_i^{k+1-j} (-\tau)^j + (\tau)^{k+1} \sum_{i=1}^{s} \ell_i(\tau)
\]

by (28) we can write

\[
= \sum_{i=1}^{s} \ell_i(\tau) c_i^{k+1} + \sum_{j=1}^{k} \binom{k+1}{j} (-\tau)^j \sum_{i=1}^{s} \ell_i(\tau) c_i^{k+1-j} + (\tau)^{k+1}
\]

by the induction hypothesis we conclude

\[
= \sum_{i=1}^{s} \ell_i(\tau) c_i^{k+1} + \sum_{j=1}^{k} \binom{k+1}{j} (-\tau)^j \tau^{k+1-j} + (\tau)^{k+1}
\]

\[
= \sum_{i=1}^{s} \ell_i(\tau) c_i^{k+1} + \tau^{k+1} \sum_{j=1}^{k} \binom{k+1}{j} (-1)^j + (\tau)^{k+1}
\]

\[
= \sum_{i=1}^{s} \ell_i(\tau) c_i^{k+1} - \tau^{k+1}.
\]

Since \( k < s - 1 \), the polynomial \( p(\tau) \) has degree at most \( s - 1 \) but \( s \) zeros \( c_1, \ldots, c_s \). So that the identity theorem again yields that \( p(\tau) \equiv 0 \). By induction equality (29) holds for all \( k \leq s - 1 \).

\[\square\]

**Lemma 4.2.** Let \( 0 \leq c_1 < \cdots < c_s \leq 1 \). There is a constant \( C_\ell \) such that for all \( h > 0 \) and for the Lagrange basis polynomials \( \ell_1, \ldots, \ell_s \) with nodes \( c_1h, \ldots, c_s h \) one has

\[
|\ell_i(\tau)| \leq C_\ell \quad \text{for all } i \in \{1, \ldots, s\} \text{ and for all } \tau \in [0, h].
\]

Note that the constant \( C_\ell \) depends only on the nodes \( c_1, \ldots, c_s \) but not on \( h \).

**Proof.** The assertion follows directly from the definition, since \( |\tau/h - c_m| \leq 1 \) for \( \tau \in [0, h] \) and \( \min\{|c_m - c_j| : m \neq j\} > 0 \).

\[\square\]

**B. Discrete Gronwall Inequality**

The discrete Gronwall inequality will be crucial for the proof of our main result. We refer to [6] for a short proof, even in a more general case.

**Theorem 4.3.** (Discrete Gronwall Inequality) For \( N \in \mathbb{N} \) let \( a_0, \ldots, a_N \geq 0 \), \( b_0, \ldots, b_N \geq 0 \) and \( z_0, \ldots, z_N \in \mathbb{R} \) be given. Suppose that for each \( n \in \{1, \ldots, N\} \)

\[
z_n \leq a_n + \sum_{j=0}^{n-1} b_j z_j.
\]
Then for each \( n \in \{1,\ldots,N\} \)

\[
z_n \leq \left( \max_{j=0,\ldots,n} a_j \right) \prod_{j=0}^{n-1} (1 + b_j).
\]

Proof. Here we recall the proof of the discrete Gronwall inequality from [6]: For \( 0 \leq j \leq k \leq N \) set

\[
B_{k,j} := \prod_{i=j}^{k-1} (1 + b_i)^{-1}
\]

and notice that

\[
B_{k,j+1} - B_{k,j} = \prod_{i=j+1}^{k-1} (1 + b_i)^{-1} \left( 1 - \frac{1}{1 + b_j} \right) = B_{k,j} b_j
\]

and for \( i \leq j \leq k \)

\[
B_{k,j} B_{j,i} = B_{k,i}.
\]

Let \( m \in \{0,\ldots,N\} \) such that \( z_m B_{m,0} = \max\{z_j B_{j,0} : j = 0,\ldots,N\} \). By the assumption we have

\[
z_m B_{m,0} \leq a B_{m,0} + B_{m,0} \sum_{j=0}^{m-1} b_j z_j = a B_{m,0} \sum_{j=0}^{m-1} B_{m,j} b_j z_j
\]

\[
\leq a B_{m,0} + z_m B_{m,0} \sum_{j=0}^{m-1} B_{m,j} b_j
\]

\[
= a B_{m,0} + z_m B_{m,0} \sum_{j=0}^{m-1} (B_{m,j+1} - B_{m,j})
\]

\[
= a B_{m,0} + z_m B_{m,0} (B_{m,m} - B_{m,0}) = a B_{m,0} + z_m B_{m,0} - z_m B_{m,0}^2.
\]

By rearranging we arrive at

\[
z_m B_{m,0} \leq a.
\]

Since for each \( n \in \{0,\ldots,N\} \) one has \( z_n B_{n,0} \leq z_m B_{m,0} \leq a \), the assertion is proved. \( \square \)

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