A Degree Condition for a Graph to have \((a, b)\)-Parity Factors

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Abstract

Let \(a, b, n\) be three positive integers such that \(a \equiv b \pmod{2}\) and \(n \geq b(a + b)/(a + b + 2)/2a\). Let \(G\) be a graph of order \(n\) with minimum degree at least \(a + b/a - 1\). We show that \(G\) has an \((a, b)\)-parity factor, if \(\max\{d_G(u), d_G(v)\} \geq an\) for any two nonadjacent vertices \(u, v\) of \(G\). It is an extension of Nishimura’s results for the existence of \(k\)-factors \((J. Graph Theory, 16\) (1992), 141–151) and generalizes Li and Cai’s result in some senses \((J. Graph Theory, 27\) (1998), 1–6). These conditions are tight.

Keywords: degree condition, parity factor

1 Introduction

In this paper we consider only simple graphs. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). Given \(v \in V(G)\), let \(N_G(v)\) denote the set of vertices adjacent to \(v\) in \(G\) and \(d_G(v) = |N_G(v)|\). The minimum vertex degree in graph \(G\) is denoted by \(\delta(G)\). We write \(N_G[v] = N_G(v) \cup \{v\}\). Given \(D \subseteq V(G)\), let \(N_D(V) = N_G[v] \cap D\). For \(X \subseteq V(G)\), the subgraph of \(G\) whose vertex set is \(X\) and whose edge set consists of the edges of \(G\) joining vertices of \(X\) is called the subgraph of \(G\) induced by \(X\) and is denoted by \(G[X]\).

Let \(g, f\) be two non-negative integer-valued function such that \(g(v) \leq f(v)\) and \(g(v) \equiv f(v) \pmod{2}\) for all \(v \in V(G)\). A spanning subgraph \(F\) of \(G\) is called \((g, f)\)-parity factor if \(d_F(v) \equiv f(v) \pmod{2}\) and \(g(v) \leq d_F(v) \leq f(v)\) for all \(v \in V(G)\). A \((g, f)\)-parity factor is called \(f\)-factor if \(f(v) = g(v)\) for all \(v \in V(G)\). If \(f(v) = k\) for all \(v \in V(G)\), then an \(f\)-factor is called a \(k\)-factor. Let \(a, b\) be two integers such that \(a \leq b\) and \(a \equiv b \pmod{2}\). If \(f(v) = b\) and \(g(v) = a\) for all \(v \in V(G)\), then a \((g, f)\)-parity factor is called an \((a, b)\)-parity factor.

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Lovász [6] gave a characterization of graphs having \((g, f)\)-parity factors. Amahashi [1] found a Tutte’s type characterization for \((1, k)\)-odd factors, which was generalized to \((1, f)\)-odd factors by Cui and Kano [2].

**Theorem 1.1 (Lovász, [6])** A graph \(G\) has a \((g, f)\)-parity factor if and only if for any two disjoint subsets \(S, T\) of \(V(G)\),

\[
\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - q(S, T) \geq 0,
\]

where \(q(S, T)\) denotes the number of components \(C\) of \(G - S - T\), called \(g\)-odd components, such that \(g(V(C)) + e_G(V(C), T) \equiv 1 \pmod{2}\).

**Theorem 1.2 (Amahashi, [1])** Let \(k \geq 1\) be an odd integer. A graph \(G\) contains an \((1, k)\)-parity factor if and only if for any subset \(S \subseteq V(G)\),

\[
c_0(G - S) \leq k|S|,
\]

where \(c_0(G - S)\) denotes the number of odd components of \(G - S\).

Nishimura [3] gave a degree conditions for a graph to have a \(k\)-factor.

**Theorem 1.3** Let \(k\) be an integer such that \(k \geq 3\), and let \(G\) be a connected graph of order \(n\) with \(n \geq 4k - 3\), \(kn\) even, and minimum degree at least \(k\). If \(G\) satisfies

\[
\max\{d_G(u), d_G(v)\} \geq n/2
\]

for each pair of nonadjacent vertices \(u, v\) in \(G\), then \(G\) has a \(k\)-factor.

Li and Cai [4] give a degree conditions for a graph to have an \((a, b)\)-factor, which extended Nishimura’s result.

**Theorem 1.4** Let \(G\) be a graph of order \(n\), and let \(a\) and \(b\) be integers such that \(1 \leq a < b\). Then \(G\) has an \([a, b]\)-factor if \(\delta(G) \geq a, n \geq 2a + b + \frac{a^2-a}{b}\) and

\[
\max\{d_G(u), d_G(v)\} \geq \frac{an}{a+b}
\]

for any two nonadjacent vertices \(u\) and \(v\) in \(G\).

In this paper we give a sufficient condition for a graph to have an \((a, b)\)-parity factor in term of the minimum degree of graph \(G\). Our main result generalizes Nishimura’s result and improves Li and Cai’s result in some sense.
Theorem 1.5 Let $a, b, n$ be three integers such that $a \equiv b \pmod{2}$, $na$ is even and $n \geq b(a + b)(a + b + 2)/(2a)$. Let $G$ be a graph of order $n$. If $\delta(G) \geq a + \frac{b-a}{a}$ and

$$\max\{d_G(u), d_G(v)\} \geq \frac{an}{a+b}$$

(2)

for any two nonadjacent vertices, then $G$ has an $(a, b)$-parity factor.

2 Proof of Theorem 1.5

Firstly, we show that Theorem 1.5 holds for $a = 1$.

Lemma 2.1 Let $k, n$ be two positive integers such that $k$ is odd, $n$ is even and $n \geq k + 1$. Let $G$ be a connected graph with order $n$. If $G$ satisfies

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{1+k}$$

(3)

for each pair of nonadjacent vertices, then $G$ has a $(1, k)$-odd factor.

Proof. Suppose that $G$ contains no $(1, k)$-parity factors. By Theorem 1.2, there exists a subset $S \subset V(G)$ such that

$$c_o(G - S) > k|S|.$$ 

Let $C_1, \ldots, C_q$ be these odd components of $G - S$ such that $|C_1| \leq \cdots \leq |C_q|$. Note that $n$ is even and $G$ is connected. By parity, one can see that

$$q = c_o(G - S) \geq k|S| + 2 \text{ and } S \neq \emptyset.$$ 

(4)

Let $u \in V(C_1)$ and $v \in V(C_2)$. By (4), we infer that

$$\max\{d_G(u), d_G(v)\} \leq \{|C_1| - 1 + |S|, |C_2| - 1 + |S|\}$$

$$\leq |C_2| - 1 + |S|$$

$$\leq \frac{n - |S| - 1}{q - 1} - 1 + |S|$$

$$\leq \frac{n}{k|S| + 1} + |S| - 1 - \frac{1}{k} \quad (\text{since } |S| \geq 1 \text{ and } n \geq (k + 1)|S| + 2)$$

$$\leq \frac{n}{k+1},$$

contradicts with (3) since $uv \notin E(G)$. \qed
Proof of Theorem 1.5. By Lemma 2.1 we may assume that \( a \geq 2 \). By Theorem 1.3 we may assume that \( a \leq b - 2 \). Suppose that \( G \) contains no \((a, b)\)-parity factors. By Theorem 1.1 there exist two disjoint vertex sets \( S \) and \( T \) such that

\[
\eta(S, T) = b|S| - a|T| + \sum_{x \in T} d_{G-S}(x) - q(S, T) \leq -2,
\]

where \( q(S, T) \) denotes the number of components \( C \) of \( G - S - T \), called \( a \)-odd components, such that \( g(V(C)) + e_G(V(C), T) \equiv 1 \pmod{2} \). We write \( s = |S|, t = |T| \) and \( w = q(S, T) \).

From (5), one can see that

\[
\eta(S, T) = bs - at + \sum_{x \in T} d_{G-S}(x) - w \leq -2.
\]

If \( S \cup T = \emptyset \), we have \( w \geq 2 \) by (1.1), which implies that \( G \) consists of at least two components. However, this contradicts the connectedness of \( G \). So we may assume that

\[
S \cup T \neq \emptyset.
\]

If \( w \geq 1 \), let \( C_1, C_2, \ldots, C_w \) denote these \( a \)-odd components of \( G - S - T \), and \( m_i = |V(C_i)| \) for \( 1 \leq i \leq w \). Put \( U = \bigcup_{1 \leq i \leq w} V(C_i) \).

We pick \( S \) and \( T \) such that \( U \) is minimal and \( V(G) - S - T - U \) is maximal.

Claim 1. \( d_{G-S}(u) \geq a + 1 \) and \( e_G(u, T) \leq b - 1 \) for every vertex \( u \in V(U) \).

Firstly, suppose that there exists \( u \in U \) such that

\[
d_{G-S}(u) \leq a.
\]

Let \( T' = T \cup \{u\} \). One can see that

\[
\eta(S, T') = bs - a|T'| + \sum_{x \in T'} d_{G-S}(x) - q(S, T')
\]

\[
= bs - at - a + \sum_{x \in T} d_{G-S}(x) + d_{G-S}(u) - q(S, T')
\]

\[
\leq bs - at + \sum_{x \in T} d_{G-S}(x) - (q(S, T) - 1)
\]

\[
\leq -1,
\]

which implies by parity

\[
\eta(S, T') = bs - a|T'| + \sum_{x \in T'} d_{G-S}(x) - q(S, T') \leq -2,
\]

\[
\begin{align*}
\eta(S, T) &= bs - at + \sum_{x \in T} d_{G-S}(x) - w \\
&\leq -2.
\end{align*}
\]
contradicting the minimality of $U$. Secondly, suppose that there exists $u \in U$ such that 

$$e_G(u, T) \geq b.$$ 

Let $S' = S \cup \{u\}$. One can see that 

$$\eta(S', T) = b|S'| - at + \sum_{x \in T} d_{G-S'}(x) - q(S', T)$$

$$= bs + b - at + \sum_{x \in T} d_{G-S}(x) - e_G(u, T) - q(S', T)$$

$$\leq bs - at + \sum_{x \in T} d_{G-S}(x) - (q(S, T) - 1)$$

$$\leq -1,$$

which implies by parity

$$\eta(S', T) = b|S'| - at + \sum_{x \in T} d_{G-S'}(x) - q(S', T) \leq -2,$$

contradicting the minimality of $U$ again. This completes Claim 1. 

\[ \square \]

**Claim 2.** Let $C_{i_1}, \ldots, C_{i_\tau}$ be any $\tau$ components of $G[U]$ and let $U' = \bigcup_{j=1}^\tau V(C_{i_j})$. 

$d_{G[T \cup U']}(u) \leq a - 1 + \tau$ for every vertex $u \in T$.

Suppose that there exists $u \in T$ such that $d_{G[T \cup U']}(u) \geq a + \tau$. Let $T' = T - u$. One may see that 

$$\eta(S, T') = bs - a|T'| + \sum_{x \in T'} d_{G-S}(x) - q(S, T')$$

$$= bs - at + a + \sum_{x \in T} d_{G-S}(x) - d_{G-S}(u) - q(S, T')$$

$$\leq bs - at + a + \sum_{x \in T} d_{G-S}(x) - (a + \tau) - (q(S, T) - \tau)$$

$$= bs - at + \sum_{x \in T} d_{G-S}(x) - q(S, T) \leq -2,$$

contradicting to the maximality of $V(G) - S - T - U$. This completes Claim 2. 

\[ \square \]

From the definition of $U$, we have

$$|U| \geq m_1 + m_2(w - 1). \quad (8)$$

By Claim 1, one can see that for every $u \in C_j$ ($1 \leq j \leq w$),

$$d_G(u) \leq (m_j - 1) + s + r \quad (9)$$
where \( r = \min\{b, t\} \). Let \( u_1 \in V(C_1) \) and \( u_2 \in V(C_2) \). It follow from (9) that
\[
\max\{d_G(u_1), d_G(u_2)\} \leq (m_2 - 1) + s + r \tag{10}
\]

**Claim 3.** \( S \neq \emptyset \).

Suppose that \( S = \emptyset \). By (7), one may see that \( t \geq 1 \). Note that \( \delta(G) \geq a + \frac{b-a}{a} \). So we have \( d_G(x_1) \geq a + \frac{b-a}{a} \). From Theorem (1.1), one can see that
\[
w = q(S, T) \geq \sum_{v \in T} d_G(v) - at + 2 \geq \frac{b-a}{a}t + 2. \tag{11}
\]
If \( w > \frac{b}{a} + 2 \), since \( b \geq a + 2 \), then it follows that
\[
w \geq \frac{b}{a} + 2. \tag{12}
\]
Combining (10) and (12), one can see that
\[
\max\{d_G(u_1), d_G(u_2)\} \leq m_2 - 1 + s + r \leq \frac{n-t-1}{w-1} + b < \frac{an}{a+b} + b < \frac{an}{a+b},
\]
contradicting to (1). So we may assume that \( w \leq \frac{b}{a} + 2 \). From (11), we infer that
\[
\frac{b-a}{a}t + 2 \leq \frac{b}{a} + 2,
\]
i.e.,
\[
t \leq \frac{b}{b-a}. \tag{13}
\]
From (11), we have
\[
m_1 \leq \frac{a(n-t)}{a+b}.
\]
Consider \( H = G[V(C_1) \cup T] \). By Claim 2, for every \( y \in T \), one can see that
\[
d_G(y) \leq a - 1 + w \leq \frac{b}{a} + 1 + a < \frac{an}{a+b}.
\]
By Claims 1 and 2, \( d_{G-S}(u) = d_H(u) \geq a + 1 \) for every \( u \in V(C_1) \) and \( d_H(v) \leq a \) for every \( v \in T \). Thus there exists two non-adjacent vertices \( u \in V(C_1) \) and \( v \in T \). If \( m_1 \leq \frac{an}{a+b} - \frac{b}{b-a} \), then one can see that
\[
\max\{d_G(u), d_G(v)\} \leq m_1 - 1 + t < \frac{an}{a+b},
\]

a contradiction. Thus we may assume that $m_1 \geq \frac{an}{a+b} - \frac{b}{b-a}$. We claim that there exists $u \in V(C_1)$ such that $e_G(u,T) = 0$, otherwise, by Claim 2 and (13), we have

$$\frac{ab}{b-a} \geq at \geq \sum_{x \in T} d_G[V(C_1) \cup T](x) \geq m_1 \geq \frac{an}{a+b} - \frac{b}{b-a},$$

i.e.,

$$n \leq \frac{(a+1)b(a+b)}{a(b-a)},$$

a contradiction. It follows

$$\max\{d_G(u), d_G(v)\} \leq m \leq m_1 - 1 + s = m_1 - 1 < \frac{an}{b+a},$$

a contradiction. This completes Claim 3.

$\square$

**Claim 4.** $T \neq \emptyset$.

Suppose that $T = \emptyset$. By Theorem 1.1, then we have

$$w \geq bs + 2.$$

By Claim 1, we have $|V(C_i)| \geq a + 1$ for $1 \leq i \leq w$. Thus we infer that

$$n \geq (a+1)w + s \geq (a+1)(bs + 2) + s > (a+1)bs,$$

which implies that

$$s \leq \frac{n}{(a+1)b}.$$ 

Hence we have

$$m_2 - 1 + s \leq \frac{n - s}{w - 1} + s \leq \frac{n}{bs + 1} + s < \frac{an}{a+b},$$

a contradiction. This completes Claim 4.

$\square$

Put $h_1 := \min\{d_G-S(v) \mid v \in T\}$, and let $x_1 \in T$ be a vertex satisfying $d_G-S(x_1) = h_1$. We write $p = |N_T[x_1]|$. Further, if $T - N_T[x_1] \neq \emptyset$, let $h_2 := \min\{d_G-S(v) \mid v \in T - N_T[x_1]\}$ and let $x_2 \in T - N_T[x_1]$ such that $d_G-S(x_2) = h_2$. By the definition of $x_i$, we have

$$\max\{d_G(x_1), d_G(x_2)\} \leq \max\{h_1 + s, h_2 + s\} \leq h_2 + s. \tag{14}$$

Now we discuss four cases.

**Case 1.** $h_1 \geq a$.

By Theorem 1.1, one can see that

$$w \geq bs - at + \sum_{v \in T} d_G-S(v) + 2$$

$$\geq bs + (h_1 - a)t + 2$$

$$\geq bs + 2,$$
i.e.,
\[ w \geq bs + 2. \quad (15) \]

Note that \( n \geq w + s + t \). From (15), we infer that
\[ s < \frac{n}{b + 1}. \]

Hence we have
\[
m_2 - 1 + s + r \leq \frac{n - t}{bs + 1} + s + b \\
\leq \frac{n}{bs + 1} + s + b - 1 \\
< \frac{an}{a + b},
\]
a contradiction.

So we may assume that \( h_1 < a \).

**Case 2.** \( T = N_T[x_1] \).

We write \( t = |N_T[x_1]| \). Since \( h_1 < a \), we have \( t \leq a \). By Claim 1, one can see that for every \( u \in V(C_1) \), \( d_{G-S}(u) \geq a + 1 > h_1 \). Thus we infer that \( V(C_1) - N_G(x_1) \neq \emptyset \), i.e., there exists a vertex \( v \in V(C_1) \) such that \( x_1v \notin E(G) \). By Theorem 1.1,
\[
w \geq bs + (h_1 - a)t + 2 \\
\geq bs + (h_1 - a)(h_1 + 1) + 2 \\
\geq b(a + \frac{b}{a} - 1 - h_1) + (h_1 - a)(h_1 + 1) + 2 \quad \text{(since } s + h_1 \geq \delta(G) \geq a + \frac{b}{a} - 1) \\
= h_1^2 - (a + b - 1)h_1 + ab + \frac{b^2}{a} - a - b + 2 \quad \text{(since } 1 \leq h_1 \leq a \text{ and } a < b) \\
\geq \frac{b^2}{a} - b + 2 > 0,
\]
i.e.,
\[ w \geq bs + (h_1 - a)(h_1 + 1) + 2 > 2. \quad (16) \]
One can see that
\[
\max\{d_G(v), d_G(x_1)\} \leq m_1 + s + t - 1 \\
\leq \frac{n - s - t}{w} + s + t - 1 \\
\leq \frac{n - s - t}{bs + (h_1 - a)(h_1 + 1) + 2} + s + t - 1 \\
\leq \frac{n - s - h_1 - 1}{bs + (h_1 - a)(h_1 + 1) + 2} + s + h_1 \\
= \frac{n - h_1 - 1 + \frac{1}{b}(h_1 - a)(h_1 + 1)}{bs + (h_1 - a)(h_1 + 1) + 2} - \frac{1}{b} + s + h_1,
\]
i.e.,
\[
\max\{d_G(v), d_G(x_1)\} \leq \frac{n - h_1 - 1 + \frac{1}{b}(h_1 - a)(h_1 + 1)}{bs + (h_1 - a)(h_1 + 1) + 2} - \frac{1}{b} + s + h_1.
\]
We write
\[
f(s) = \frac{n - h_1 - 1 + \frac{1}{b}(h_1 - a)(h_1 + 1)}{bs + (h_1 - a)(h_1 + 1) + 2} - \frac{1}{b} + s + h_1.
\] (17)
So we have
\[
f'(s) = -\frac{b(n - h_1 - 1) + (h_1 - a)(h_1 + 1)}{(bs + (h_1 - a)(h_1 + 1) + 2)^2} + 1.
\] (18)

Now we discuss two subcases.

**Case 2.1.** \(s \leq \frac{an}{a+b} - h_1 - 1\).

By (16) and (18), we infer that
\[
f(s) \leq \max\{f(a + \frac{b}{a} - h_1 - 1), f(\frac{an}{a+b} - h_1 - 1)\}.
\] (19)
Hence one can see that
\[
\max\{d_G(x_1), d_G(x_2)\} \leq m_1 - 1 + s + t \\
\leq \max\{f(a + \frac{b}{a} - h_1 - 1), f(\frac{an}{a+b} - h_1 - 1)\} \\
< \frac{an}{a+b},
\]
contradicting to the degree condition.

**Case 2.2.** \(s > \frac{an}{a+b} - h_1 - 1\).
One can see that

\[ n \geq s + t + w \]
\[ \geq s + t + bs + (h_1 - a)t + 2 \]
\[ \geq (b + 1) \frac{an}{a + b} - (b + 1)h_1 - (b + 1) + (h_1 - a)(h_1 + 1) + 2 \]
\[ = (b + 1) \frac{an}{a + b} + h_1^2 - (a + b)h_1 - a - b + 1 \quad \text{(since } 0 \leq h_1 \leq a) \]
\[ \geq \frac{abn}{a + b} + \frac{an}{a + b} - ab - a - b + 1 > n \quad \text{(since } a \geq 2), \]

a contradiction.

**Case 3.** \( h_2 \geq a. \)

By Lovász Theorem 1.1,

\[ w \geq bs + \sum_{v \in T} d_{G-S}(v) - at + 2 \]
\[ \geq bs + (h_1 - a)p + (h_2 - p)(t - p) + 2 \]
\[ \geq bs + (h_1 - a)p + 2 \]

Now we discuss two subcases.

**Subcase 3.1.** \( h_2 \leq \frac{1}{4}(a^2 + 6a + 5). \)

By Lovász Theorem 1.1, we find

\[ s \geq \frac{an}{a + b} - h_2 \geq \frac{an}{a + b} - \frac{1}{4}(a^2 + 6a + 5). \]

Hence one can see that

\[ n \geq w + s + t \]
\[ \geq (b + 1)s + (h_1 - a)p + 2 + t \]
\[ \geq (b + 1)s + (h_1 - a + 1)p + 2 \]
\[ \geq (b + 1)(\frac{an}{a + b} - \frac{1}{4}(a^2 + 6a + 5) + 1) + (h_1 - a + 1)(h_1 + 1) + 2 \]
\[ > n, \]

a contradiction.

**Subcase 3.2.** \( h_2 \geq \frac{1}{4}(a^2 + 6a + 5). \)
By Lovász Theorem 1.1

\[ w \geq bs + \sum_{v \in T} d_{G-S}(v) - at + 2 \]

\[ \geq bs + (h_1 - a)p + (h_2 - a)(t - p) + 2 \]

\[ \geq bs + (h_1 - a)(h_1 + 1) + h_2 - a + 2 \]

\[ \geq bs - \frac{1}{4}(a + 1)^2 + h_2 - a + 2 \]

\[ \geq bs + 3. \]

We find

\[ n \geq s + t + w \geq (b + 1)s + 2, \]

which implies that

\[ s \leq \frac{n - 2}{b + 1}. \]

Thus by Claim 1, we infer that

\[ m_2 - 1 + s + b \leq \frac{n - s - t}{bs + 2} + s + b - 1 \]

\[ \leq \frac{n - s}{bs + 2} + s + b - 1 \]

\[ < \frac{n - 2}{bs + 1} + s + b \]

\[ < \frac{an}{a + b}, \]

a contradiction.

**Case 4.** \( 0 \leq h_1 \leq h_2 \leq a - 1. \)

By (1), we infer that

\[ s \geq \frac{an}{a + b} - h_2. \]  

(20)

By Lovasz Theorem 1.1

\[ w \geq bs + \sum_{v \in T} d_{G-S}(v) - at + 2 \]

\[ \geq bs + (h_1 - a)p + (h_2 - a)(t - p) + 2, \]

where \( p = |N_T[x_1]| \) and naturally there is \( p \leq a, \)

\[ w \geq bs + (h_1 - a)p + (h_2 - a)(t - p) + 2. \]  

(21)
Thus we get
\[ n \geq s + t + w \]
\[ \geq (b + 1)s + (h_1 - a)p + (h_2 - a)(t - p) + 2 + t \]
\[ = (b + 1)s + (h_1 - h_2)p + (h_2 + 1 - a)t + 2 \]
\[ \geq (b + 1)\left(\frac{an}{a + b} - h_2\right) + (h_1 - h_2)(h_1 + 1) + (h_2 - a + 1)\left(\frac{bn}{a + b} + h_2\right) + 2 \]
\[ \geq (b + 1)\frac{an}{a + b} + (-a + 1)\left(\frac{bn}{a + b} + h_2\right) + 2 + h_2\left(\frac{bn}{a + b} + h_2 - b - 1\right) + (h_1 - h_2)(h_1 + 1) \]
\[ \geq n + 2, \]
a contradiction. This completes the proof. \( \square \)

Remark : These minimum degree conditions are sharp. Let \( a, b, m \) be three integers, such that \( m \) is sufficiently large. Consider graph \( K_{ma,mb+1} \). Denote \( K_{ma,mb+1} \) by \( G \). Li and Cai \[4\] show that \( K_{ma,mb+1} \) contains no \([a, b]-factors\). One can see that
\[ \frac{an}{a + b} > \delta(G) \geq ma > \frac{a|V(G)|}{a + b} - 1. \]

Other hand. Let \( C_1, \ldots, C_q \) be \( q \) copies of \( K_m \), where \( q = a + \lceil \frac{b-a}{a} \rceil - 1 \). Let \( G' \) be a graph obtained form \( C_1, \ldots, C_q \) by adding a new vertex \( v \) connecting one of vertices of each copy. Clearly, \( G' \) is connected and \( \delta(G') = a + \lceil \frac{b-a}{a} \rceil - 1 \). By taking \( S = \emptyset \) and \( T = \{v\} \), we infer that \( G' \) contains no \((a, b)\)-parity factor by Lovasz's Theorem.\[11\]

References

[1] A. Amahashi, On factors with all degree odd, Graph and Combin., 1 (1985), 111–114.
[2] Y. Cui and M. Kano, Some results on odd factors of graphs, J. Graph Theory, 12 (1988), 327–333.
[3] T. Nishimura, A degree condition for the existence of \( k \)-factors. J. Graph Theory, 16 (1992), 141–151.
[4] Y. Li and M. Cai, A degree condition for a graph to have \([a, b]\)-factors, J. Graph Theory, 27 (1998), 1–6.
[5] L. Lovász, Subgraphs with prescribed valencies, J. Combin. Theory, 8 (1970), 391–416.
[6] L. Lovász, The factorization of graphs. II, Acta Math. Acad. Sci. Hungar., 23 (1972), 223–246.
[7] W. Tutte, The factors of graphs, Canad. J. Math, 4 (1952), 314-328.