Representations of the $SU(N)$ $T$-algebra and the loop representation in $1 + 1$-dimensions

Joakim Hallin

Abstract

We consider the phase-space of Yang-Mills on a cylindrical space-time ($S^1 \times \mathbb{R}$) and the associated algebra of gauge-invariant functions, the $T$-variables. We solve the Mandelstam identities both classically and quantum-mechanically by considering the $T$-variables as functions of the eigenvalues of the holonomy and their associated momenta. It is shown that there are two inequivalent representations of the quantum $T$-algebra. Then we compare this reduced phase space approach to Dirac quantization and find it to give essentially equivalent results. We proceed to define a loop representation in each of these two cases. One of these loop representations (for $N = 2$) is more or less equivalent to the usual loop representation.

1Email address: tfejh@fy.chalmers.se


1 Introduction

In \cite{1, 2} Ashtekar showed that the phase-space of pure gravity can be embedded in the phase-space of \( SL(2, \mathbb{C}) \) Yang-Mills theory. This was used in \cite{3} where a number of gauge-invariant observables for Yang-Mills theory, the \( T \)-variables, were used in an attempt to quantize gravity non-perturbatively. The \( T^0 \)-variable is the trace of the holonomy of the connection around a loop i.e. what is more commonly known as the Wilson loop variable. The higher \( T \)-variables are phase-space generalizations of \( T^0 \) also containing momenta, being essentially time derivatives of \( T^0 \). This quantization was performed by means of the so called loop representation where states are functionals of loops and the \( T \)-variables act in a specified way on such states. States satisfying all constraints were found in this formalism. However, in \cite{4} quantum general relativity in \( 2 + 1 \)-dimensions, which is embedded in the phase space of \( SL(2, \mathbb{R}) \) Yang-Mills theory, was investigated and it was found that the loop representation inadequately described the theory. The reason for this was essentially the non-compactness of the gauge group. Also in \cite{5} it was argued that the loop representation is incomplete due to the appearance of certain non-linear inequalities being satisfied by the classical \( T \)-variables which seemingly where ignored by the loop representation. A similar scenario is expected to take place in \( 3 + 1 \)-dimensions although much less is known there, partly because of the complicated reality conditions one needs to impose on the phase space variables. One possible conclusion one might draw from this is to forget about \( T \)-variables. This is however not a very useful conclusion since after all the \( T \)-variables form a large class of gauge-invariant functions on phase space. In this paper we will, as a toy model for the higher dimensional cases, investigate the phase space of \( SU(N) \) Yang-Mills theory in \( 1 + 1 \)-dimensions and its corresponding \( T \)-variables.

The motivation for doing the analysis in \( 1 + 1 \)-dimensions is that everything is completely and explicitly doable. Also, we investigate \( SU(N) \) instead of just \( SU(2) \) to see whether \( SU(2) \) is special in any way. It turns out that it is not. We will however not choose the loop representation as a starting point. Instead we quantize the \( T \)-algebra by means of a reduced phase space approach as well as by Dirac quantization. Then by using the eigenstates of the Yang-Mills Hamiltonian we can proceed to define loop representations. These loop representations are more or less equivalent to the usual loop representation. This is to be expected since \( SU(N) \) is a compact group. In a forthcoming paper we hope to do the same analysis for \( SL(2, \mathbb{R}) \) as we do here for \( SU(N) \).

2 Classical theory

Our starting point is the gauge-invariant part of the \( SU(N) \) Yang-Mills Hamiltonian,

\[
H = \frac{1}{2} \int_0^L dx \, tr(E^2(x)),
\]

where \( L \) is the length of the circle. The basic Poisson bracket is,

\[
\{ A^a(x), E^b(y) \} = \delta^{ab} \delta(x - y),
\]
and \( A(x) = A^a(x)t^a \), \( E(x) = E^a(x)t^a \) where \( t^a \) are the \( N \times N \)-matrix generators of the group \((a = 1, \ldots, \text{dim}(SU(N)))\). Here we have chosen

\[
\text{tr}(t^at^b) = \delta^{ab}
\]

which implies the identity,

\[
(t^a)_{ij}(t^a)_{kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}, \quad \text{(3)}
\]

where \( i, j, k, l \) denotes matrix indices. We assume that the connection and electric field are periodic fields on the circle i.e. \( A(x) = A(x + L), E(x) = E(x + L) \). We also have the first class constraint (Gauss’ law),

\[
D_xE(x) = \partial_xE(x) + ig[A(x), E(x)] \approx 0, \quad \text{(4)}
\]

where \( g \) is the coupling constant. Let us define parallel transport by

\[
U(x, y) = \mathcal{P}\exp(ig\int_x^y dx' A(x')),
\]

where \( \mathcal{P} \) denotes path ordering, i.e. \( U(x, y) \) is the solution to the integral equation

\[
U(x, y) = 1 + ig\int_x^y dx' U(x, x')A(x').
\]

\( U(x, y) \) is an element of the group \( SU(N) \). The holonomy \( h(x) \) is defined by

\[
h(x) = U(x, x + L)
\]

i.e. parallel transport once around the whole circle. Note also that \( U(x, x + nL) = h^n(x) \) where \( n \) is an integer which follows from the basic sewing property of path ordered exponentials,

\[
U(x, x')U(x', y) = U(x, y).
\]

Let \( \Lambda(x) \) be a (finite) \( SU(N) \) gauge-transformation (generated by (4)). Then

\[
\begin{align*}
A'(x) &= \Lambda(x)A(x)\Lambda^{-1}(x) + \frac{1}{ig}\Lambda(x)\partial_x\Lambda^{-1}(x) \\
E'(x) &= \Lambda(x)E(x)\Lambda^{-1}(x).
\end{align*}
\]

This implies that \( U(x, y) \) transforms homogeneously i.e.

\[
U''(x, y) = \Lambda(x)U(x, y)\Lambda^{-1}(y),
\]

and in particular,

\[
h'(x) = \Lambda(x)h(x)\Lambda^{-1}(x),
\]

\[2\]
since $\Lambda(x)$ is periodic. We might allow for non-periodic gauge-transformations (these are not generated by the constraint) still keeping $A$ periodic. These satisfy,

$$\Lambda(x + L) = z_N \Lambda(x), \tag{10}$$

where $z_N$ is any element in the center of the group i.e. an $N$:th root of unity $\xi$, $\xi^N = 1$. We will call such non-periodic gauge-transformations $z_N$ transformations. Under such a transformation, the holonomy transforms as,

$$h'(x) = \xi \Lambda(x) h(x) \Lambda^{-1}(x). \tag{11}$$

There is no reason to demand invariance under such transformations unless one is really interested in one of the corresponding groups like e.g. $SU(2)/\mathbb{Z}_2 \approx SO(3)$. Furthermore, as soon as we couple fermions these transformations are not allowed any longer.

### 2.1 Loop variables

Following [3] we introduce the following functions on phase space, the loop variables,

$$T^0(n) = \text{tr}(h^n(x)) \tag{12}$$

$$T^1(x; n) = \text{tr}(E(x) h^n(x)) \tag{13}$$

$$T^2(x; n, y; m) = \text{tr}(E(x) U(x, y + nL) E(y) U(y, x + mL)). \tag{14}$$

They are easily seen to be gauge-invariant. Furthermore, $T^0(nN)$ etc, is $z_N$ invariant. Note also that $T^0(n)$ is independent of $x$, motivating the notation. In fact, on the constraint surface $T^1(x; n)$ is also independent of $x$ since

$$\partial_x T^1(x; n) = \text{tr}((D_x E(x)) h^n(x)) \approx 0. \tag{15}$$

Similarly, $T^2(x; n, y; m)$ is independent of $x$ and $y$. Using the identity,

$$T^2(x + n'L; n, y; m) = T^2(x; n - n', y; m + n'),$$

it also follows that $T^2(x; n, y; m)$ is independent of $n - m$ on the constraint surface, i.e. $T^2(x; n, y; m) = T^2(n + m)$. Analogously, one may consider loop variables of higher order in $E$ i.e.

$$T^p(x_1; n_1, \ldots, x_p; n_p).$$

On the constraint surface $T^p$ will only depend on $n_1 + \cdots + n_p$. To calculate Poisson brackets we need,

$$\frac{\delta U(x, y)}{\delta A^a(x')} = ig \theta(x, y, x') U(x, x') t^a U(x', y), \tag{16}$$

where

$$\theta(x, y, x') = \int_x^y dx'' \delta(x'' - x').$$
In particular,
\[
\frac{\delta h(x)}{\delta A^a(x')} = igU(x, x')t^aU(x', x)h(x).
\] (17)

In what follows, all brackets will be evaluated on the constraint surface, where they simplify. Using (17) and (3) we obtain,
\[
\{T^0(n), T^0(m)\} = 0
\] (18)
\[
\{T^1(n), T^0(m)\} = -igm(T^0(n + m) - \frac{1}{N}T^0(n)T^0(m))
\] (19)
\[
\{T^2(n), T^0(m)\} = -2igm(T^1(n + m) - \frac{1}{N}T^1(n)T^0(m))
\] (20)
\[
\{T^2(n), T^1(m)\} = ig(n - 2m)T^2(n + m) + \frac{ig}{N}(2mT^1(n)T^1(m) - nT^2(n)T^0(m))
\] (21)
\[
\{H, T^p(n)\} = -igmT^p(n) - 1
\] (22)

The last identity follows since \(H = \frac{1}{2}T^2(0)\). Note the central importance of \(H\) (or \(T^2(0)\)).

Even if we’re not interested in the Yang-Mills time-evolution, \(H\) still acts as a generator of \(T^p\):s through (22). Since \(T^p(n)\) is a continuous function of \(n\), (22) defines \(T^{p+1}(n)\) even for \(n = 0\). We also have the reality conditions,
\[
(T^p(n))^* = T^p(-n)
\]
\[
H^* = H
\] (23)

where * denotes complex conjugation.

### 2.2 Mandelstam identities

The Mandelstam identities were first discussed for a special case in [6]. In [7] and [8] they are discussed in general. We will not need to know their general form, we will only illustrate them by examples. For \(N = 2\) the identities are,
\[
T^0(n) = T^0(-n),
\] (24)
\[
T^0(n)T^0(m) = T^0(n + m) + T^0(n - m),
\] (25)

while for \(N = 3\) one of the identities is,
\[
T^0(n)^3 - 3T^0(n)T^0(2n) + 2T^0(3n) = 6.
\] (26)

The other identity for \(N = 3\) is much more complicated. It is only for \(N = 2\) that these identities have a simple form. There are also analogous identities for the higher \(T\)-variables. Let us illustrate this for \(N = 2\). Calculating the bracket of both the left- and right-hand side of (24) with \(H\) using (22) one finds,
\[
T^1(n) = -T^1(-n).
\] (27)
Doing the same operation on (25) and using (27) one obtains,

$$nf(n, m) + mf(m, n) = 0,$$

(28)

where

$$f(n, m) = T^1(n)T^0(m) - T^1(n + m) - T^1(n - m).$$

(29)

Using (24) and (27) we get

$$f(n, m) = -f(-n, -m).$$

Having found this it is easy to see that (28) implies

$$f(n, m) = 0 \text{ i.e.,}$$

$$T^1(n)T^0(m) = T^1(n + m) + T^1(n - m).$$

(30)

One can then repeat this construction for the $T^1$-identities and construct identities involving $T^2$ etc. Analogously, having the explicit form of the identities for $T^0$ for any $N$, one can find identities involving the higher $T$-variables.

2.3 Conjugacy classes

As seen by (9), the holonomy transforms under gauge-transformations by conjugation in $SU(N)$. Gauge-invariant functions of the holonomy are therefore class functions $f$,

$$f(h) = f(ghg^{-1}), \ \forall g \in SU(N).$$

A particular example of a class function is $T^0(n)$. Let us note some properties of the conjugacy classes of $SU(N)$, the classic source of information being [11]. Any $SU(N)$ matrix is conjugate to a diagonal matrix $D$. Two diagonal matrices are conjugate if and only if their eigenvalues are related by permutation. Let $D = \text{diag}(\lambda_1, \ldots, \lambda_N)$. Since det $D = 1$ we have,

$$\lambda_N = \lambda_1^{-1} \cdots \lambda_{N-1}^{-1}. \quad (31)$$

Furthermore, since $D$ is unitary the eigenvalues all have modulus 1 i.e. $\lambda_i = e^{i\varphi_i}$, ($\varphi_i$ real $i = 1, \ldots, N - 1$). Any class function $f$ is therefore a function of $N - 1$ eigenvalues, symmetric under permutations

$$\lambda_i \leftrightarrow \lambda_j, \ \ i, j = 1, \ldots, N$$

where $\lambda_N$ is given by (31), e.g. for $N = 2$, $f(\lambda_1) = f(\lambda_1^{-1})$. From now on, permutations will always mean permutations of all $N$ eigenvalues, $\lambda_N$ being given by (31). We can express $T^0(n)$ in terms of the eigenvalues of $h(x)$ (which are independent of $x$),

$$T^0(n) = \lambda_1^n + \ldots + \lambda_{N-1}^n + \lambda_1^{-n} \cdots \lambda_{N-1}^{-n}. \quad (32)$$

When we’ve expressed $T^0(n)$ in terms of $\lambda$:s it is evident that all the Mandelstam identities are satisfied.
### 2.4 Momenta

Having expressed $T^0(n)$ in terms of eigenvalues, it is natural to look for some variables “conjugate” to the eigenvalues. We do this by postulating the brackets,

\[
\{\lambda_i, p_j\} = i\delta_{ij}\lambda_i \tag{33}
\]

\[
\{p_i, p_j\} = 0. \tag{34}
\]

This defines the $N - 1$ momenta $p_i$. We obviously already have the bracket,

\[
\{\lambda_i, \lambda_j\} = 0.
\]

Furthermore, from (33) follows $\{\lambda^n_i, p_j\} = i\delta_{ij}\lambda^n_i$. Now assume that $T^1(n)$ is purely first order in momenta (with $\lambda$-dependent coefficients). By inserting this ansatz into (18) using (32) one finds (uniquely),

\[
T^1(n) = g \sum_{i=1}^{N-1} (\lambda^n_i - \frac{1}{N}T^0(n))p_i. \tag{35}
\]

A check shows that this expression for $T^1(n)$ satisfies (19). Now having found this, assume that $H$ is purely second order in momenta. Inserting this ansatz into (22) (for $p = 0$) one obtains the unique expression,

\[
H = \frac{g^2 L}{2} \left( \sum_{i=1}^{N-1} p_i^2 - \frac{1}{N} (\sum_{i=1}^{N-1} p_i)^2 \right). \tag{36}
\]

Having found (36) one can generate any $T^p(n)$ using (22). In particular one finds,

\[
T^2(n) = g^2 \sum_{i=1}^{N-1} p_i^2 \lambda^n_i - \frac{g}{N} T^1(n) \sum_{i=1}^{N-1} p_i - \frac{g^2}{N} \sum_{i=1}^{N-1} p_i \sum_{i=1}^{N-1} p_i \lambda^n_i. \tag{37}
\]

Checking, one finds that this expression for $T^2(n)$ satisfies (20) and (21). Let us finally see how gauge-transformations affect the momenta. A gauge-transformation permutes the eigenvalues. In particular exchanging $\lambda_i$ and $\lambda_j$ exchanges $p_i$ and $p_j$ for (33) to hold after the transformation. Changing $\lambda_i$ into $\lambda_N = \lambda_1^{-1} \cdots \lambda_{N-1}^{-1}$ one finds $p_i \rightarrow -p_i$ and $p_j \rightarrow p_j - p_i$ ($j \neq i$). As a consistency check one finds that indeed $T^3(n)$ and $T^2(n)$ are gauge-invariant.

### 3 Quantization

Poisson brackets without ordering problems will go over into commutators unchanged,

\[
[\hat{T}^0(n), \hat{T}^0(m)] = 0 \tag{38}
\]

\[
[\hat{T}^1(n), \hat{T}^0(m)] = g\hbar m(\hat{T}^0(n + m) - \frac{1}{N} \hat{T}^0(n)\hat{T}^0(m)) \tag{39}
\]

\[
[\hat{H}, \hat{T}^p(n)] = g\hbar L n \hat{T}^{p+1}(n) \tag{40}
\]

\[
[\hat{\lambda}_i, \hat{p}_j] = -\hbar \delta_{ij}\hat{\lambda}_i \tag{41}
\]
Let these operators act on wavefunctions that are class functions, i.e. symmetric functions of the \( N - 1 \) eigenvalues of the holonomy \( h(x) \). Hence let \( \hat{\lambda}_i \) and \( \hat{p}_i \) act as,

\[
\hat{\lambda}_i \Psi(\lambda_1, \ldots, \lambda_{N-1}) = \lambda_i \Psi(\lambda_1, \ldots, \lambda_{N-1}) \quad (42)
\]

\[
\hat{p}_i \Psi(\lambda_1, \ldots, \lambda_{N-1}) = \hbar \lambda_i \partial_{\lambda_i} \Psi(\lambda_1, \ldots, \lambda_{N-1}) . \quad (43)
\]

Under a \( \mathbb{Z}_N \) transformation, \( \Psi(\lambda_1, \ldots, \lambda_{N-1}) \) transforms into \( \Psi(\xi \lambda_1, \ldots, \xi \lambda_{N-1}) \), where \( \xi^N = 1 \). Phasespace functions without ordering problems will turn into operators unchanged i.e.,

\[
\hat{T}^0(n) = \hat{\lambda}_1^n + \ldots + \hat{\lambda}_{N-1}^n + \hat{\lambda}_1^{-n} \cdots \hat{\lambda}_{N-1}^{-n} \quad (44)
\]

\[
\hat{H} = \frac{g^2 L}{2} \left( \sum_{i=1}^{N-1} \hat{p}_i^2 - \frac{1}{N} \left( \sum \hat{p}_i \right)^2 \right) . \quad (45)
\]

\( \hat{T}^0(n) \) will obviously satisfy the Mandelstam identities now. Having defined \( \hat{T}^0 \) and \( \hat{H} \) we define \( \hat{T}^p \) for \( p \geq 1 \) by (44). Hence one e.g. finds,

\[
\hat{T}^1(n) = g \sum_{i=1}^{N-1} \left( \hat{\lambda}_i^n - \frac{1}{N} \hat{T}^0(n) \right) \hat{p}_i + g \hbar n \frac{N - 1}{2N} \hat{T}^0(n) . \quad (46)
\]

Comparing with the classical expression (44) one sees that \( \hat{T}^1(n) \) has acquired a quantum correction. Checking (45) it is found to be satisfied. One also finds,

\[
[\hat{T}^1(n), \hat{T}^1(m)] = g \hbar (m - n) \hat{T}^1(n + m) + \frac{g \hbar}{N} (n \hat{T}^0(m) \hat{T}^1(n) - m \hat{T}^0(n) \hat{T}^1(m)) . \quad (47)
\]

Let us note some properties of \( \hat{H} \). Introduce

\[
\Xi_{\{n_1, \ldots, n_{N-1}\}}(\{\lambda\}) = \lambda_1^{n_1} \cdots \lambda_{N-1}^{n_{N-1}} , \quad (48)
\]

where \( n_1, \ldots, n_{N-1} \) are integers and \( \{\lambda\} = (\lambda_1, \ldots, \lambda_{N-1}) \). \( \Xi \) is an eigenvector of \( \hat{H} \), i.e.

\[
\hat{H} \Xi_{\{n_1, \ldots, n_{N-1}\}}(\{\lambda\}) = \frac{(g \hbar)^2 L}{2N} P_N(n_1, \ldots, n_{N-1}) \Xi_{\{n_1, \ldots, n_{N-1}\}}(\{\lambda\}) , \quad (49)
\]

where

\[
P_N(\{n\}) = (N - 1) \sum_{i=1}^{N-1} n_i^2 - 2 \sum_{j>i=1}^{N-1} n_i n_j . \quad (50)
\]

### 3.1 Symmetric representation

Let us investigate \( \hat{H} \). This is in fact, in our formalism, the Hamiltonian derived in [9]. The eigenstates are totally symmetric linear combinations of \( \Xi_{\{n\}} \) (remember that physical states are class functions), i.e.

\[
\Psi_{S(n_1, \ldots, n_{N-1})}(\{\lambda\}) = \sum_{\text{perms}} \Xi_{\{n_1, \ldots, n_{N-1}\}}(\{\lambda_1\}, \ldots, \{\lambda_{N-1}\}) ,
\]

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where \( \pi \) permutes all \( \lambda \)'s including \( \lambda_N \). Evidently, not all indices \((n_1, \ldots, n_{N-1})\) correspond to different eigenstates. If we want these states to be \( \mathbb{Z}_N \) invariant we have to require \( \sum_{i=1}^{N-1} n_i \) to be a multiple of \( N \). The eigenenergies are given by (49). The action of the loop variables is very simple on the eigenstates, e.g.

\[
\hat{T}^0(n) \Psi_S(n_1, \ldots, n_{N-1})(\{\lambda\}) = \sum_{i=1}^{N-1} \Psi_S(n_1, \ldots, n_i + n_i, n_{N-1})(\{\lambda\}) + \Psi_S(n_1, \ldots, n_{N-1} - n)(\{\lambda\}).
\]

An inner product is determined by requiring \( (\hat{T}^0(n))^{\dagger} = \hat{T}^0(-n) \) and \( \hat{H}^{\dagger} = \hat{H} \). Then (40) implies that all the classical reality conditions, (23), are quantized exactly i.e. \( (\hat{T}^p(n))^{\dagger} = \hat{T}^p(-n) \). Hence, (up to an overall factor),

\[
<\Phi_S, \Psi_S> = \int d\varphi_1 \cdots d\varphi_{N-1} \Phi_S^*(\{\varphi\}) \Psi_S(\{\varphi\}). \tag{51}
\]

Here all integrals are taken from \(-\pi\) to \(\pi\) in the angles. Alternatively we can integrate over the eigenvalues,

\[
d\lambda_i / i\lambda_i = d\varphi_i.
\]

Different eigenstates are orthogonal using this inner product. The groundstate is \( \Psi_S(0, \ldots, 0) \) and it has zero energy.

### 3.2 Antisymmetric representation

Let’s make a quantum canonical transformation using \( C = \Delta^{-1} \) where \( \Delta \) is,

\[
\Delta = \prod_{j>i=1}^N (\lambda_i - \lambda_j).
\]

For a general discussion of such transformations see [13]. An arbitrary operator \( \hat{O} \) will be mapped into

\[
\hat{O}' = C \hat{O} C^{-1} = \Delta^{-1} \hat{O} \Delta.
\]

We note that it is a well-defined canonical transformation mapping gauge-invariant operators into gauge-invariant operators. Under this transformation \( T^0 \) is invariant while \( H' = \Delta^{-1} H \Delta \neq H \). This is (up to a constant) the radial part of the Laplacian on \( SU(N) \), [12], which is the Hamiltonian considered in [10]. \( \Delta \) is totally antisymmetric under permutations of eigenvalues. Hence eigenstates of \( H' \) are given as,

\[
\Psi_A(n_1, \ldots, n_{N-1})(\{\lambda\}) = \Delta^{-1} \sum_{\text{perms}} \text{sgn}(\pi) \Xi_{(n_1, \ldots, n_{N-1})}(\pi(\lambda_1), \ldots, \pi(\lambda_{N-1})).
\]

These are the characters of \( SU(N) \). Eigenenergies are still given by (49). The groundstate is \( \Psi_A(1, \ldots, N-1) \) with energy

\[
(gh)^2 L N^2 24 (N^2 - 1).
\]
The spectrum of \( \hat{H}' \) is a proper subset of that of \( \hat{H} \). Hence these Hamiltonians are clearly physically inequivalent. The action of loop variables on eigenstates is the same as for the symmetric representation. The inner product is,

\[
< \Phi_A, \Psi_A > = \int d\varphi_1 \cdots d\varphi_{N-1} \Delta \Delta^* \Phi_A^*(\{\varphi\}) \Psi_A(\{\varphi\}).
\]

The measure density \( \Delta \Delta^* \) is the measure density induced by the Haar-measure on the group. Note how utterly sensible it is from the point of view of the group, e.g. the conjugacy class \( \lambda_1 = \ldots = \lambda_{N-1} = 1 \) consists of a single group element, the unit matrix, in contrast to a generic conjugacy class having all eigenvalues distinct which consists of a set of group elements forming a submanifold of the group with non-zero dimension. Thinking about the group it is natural to give a larger weight to this generic conjugacy class than the unit element class. \( \Delta \) does just this as it vanishes on the unit element class. In general, the so called singular set which is the set of conjugacy classes having not all eigenvalues distinct, has Haar-measure zero (\( \Delta \) is zero on this set).

### 3.3 Generalities

Having found that a canonical transformation can map us into an inequivalent representation of the algebra of gauge-invariant operators we might wonder if we can construct other inequivalent transformations by the same means. Hence let \( \hat{C}(\{\lambda\}) \) be some canonical transformation. Now an arbitrary operator \( \hat{O} \) will transform as,

\[
\hat{O}' = \hat{C} \hat{O} \hat{C}^{-1}.
\]

But we require gauge-invariant operators to be mapped into gauge-invariant operators. Hence \( \hat{C}(\{\lambda\}) \) must either be totally symmetric or totally antisymmetric, and eigenstates of the transformed Hamiltonian will clearly be equivalent to either the symmetric or the antisymmetric representation. We can in principle allow for momentum dependent canonical transformations as well, as long as we’re careful with their (possible) kernels, but this will not lead to anything new. This quantization ambiguity is also discussed in \[\text{[14]}\] from a completely different point of view.

### 3.4 Dirac quantization

What we have done is more or less reduced phase-space quantization since we consider (almost) only gauge-invariant functions on the constraint surface when quantizing. It would be interesting to compare this to Dirac quantization where one solves the constraint on the quantum level. It turns out that in the cases we have worked out (\( N = 2 \) and \( N = 3 \)), this approach leads to results that are equivalent to the reduced approach. Let us define the smeared classical constraint,

\[
C_\omega = \frac{1}{g} \int_0^L dx \operatorname{tr}(\omega(x) D_x E(x)),
\]

(52)
where \( D_x E(x) \) is given by (3) and \( \omega(x) = \omega^a(x)t^a \). Gauge-transformations are generated by \( C_\omega \) and,

\[
\{ A(x), C_\omega \} = -\frac{1}{g} + i[\omega(x), A(x)],
\]

\[
\{ C_\omega, C_{\omega'} \} = iC_{[\omega, \omega']},
\]

Upon quantization (2) turns into,

\[
[\hat{A}^a(x), \hat{E}^b(x)] = i\hbar\delta^{ab}\delta(x-y).
\]

Now when quantizing the constraint \( C_\omega \) we choose to order the \( \hat{E}^a \):s to the right (it is actually ordering independent). Hence define \( \hat{C}_\omega \) as,

\[
\hat{C}_\omega = \int_0^L dx \omega^a(x)(\partial_x \hat{E}^a(x) + ig[t^a, t^b]\hat{A}^a(x)\hat{E}^b(x)).
\]

Then one obtains,

\[
[\hat{A}(x), \frac{1}{i\hbar}\hat{C}_\omega] = -\frac{1}{g}\partial_x \omega(x) + i[\omega(x), \hat{A}(x)],
\]

\[
[\frac{1}{i\hbar}\hat{C}_{\omega_1}, \frac{1}{i\hbar}\hat{C}_{\omega_2}] = i\frac{1}{i\hbar}\hat{C}_{[\omega_1, \omega_2]},
\]

as one should. Finite gauge-transformations are given by,

\[
\hat{\Lambda}_\omega = e^{-\frac{i}{\hbar}\hat{C}_\omega}.
\]

There are no ordering problems in the expressions for \( T_0(n) \) and \( H \). Hence,

\[
\hat{T}_0(n) = \text{tr}(\hat{h}^n(x)),
\]

\[
\hat{H} = \frac{1}{2} \int_0^L dx \text{tr}(\hat{E}^2(x)).
\]

Then we define the higher order loop variables when acting on physical states by \( \Psi(A) \). This corresponds to choosing some particular ordering of the classical expressions. Let us now choose the connection representation i.e. states are wavefunctionals of connections \( \Psi(A) \) and,

\[
\hat{A}^a(x)\Psi(A) = A^a(x)\Psi(A),
\]

\[
\hat{E}^a(x)\Psi(A) = \frac{\hbar}{i} \frac{\delta\Psi(A)}{\delta A^a(x)}.
\]

This is a representation of (55). Now act upon \( \Psi(A) \) with a finite gauge-transformation. One then finds,

\[
\hat{\Lambda}_\omega \Psi(A) = \Psi(A'),
\]
where $A'(x)$ is given by (3) and $\Lambda(x) = \exp(i\omega(x))$. The physical states should be gauge-invariant i.e. $\Psi(A) = \Psi(A')$. But we already know of solutions to this equation, the class functions of the holonomy of $A$. Hence physical states are symmetric functions of $N - 1$ eigenvalues of the holonomy $h(x)$ of $A(x)$,

$$\Psi_{\text{phys}}(A) = \Psi(\lambda_1, \ldots, \lambda_{N-1}).$$

It is necessary but not sufficient (in this case) for the physical states to be annihilated by the constraint i.e. $\hat{C}_\omega \Psi(A) = 0$. This is because two different such states (satisfying $\hat{C}_\omega \Psi(A) = 0$) might be related by a finite gauge-transformation and the path in the space of connections goes via states not being annihilated by the constraint. If one then restricts ones attention only to states being annihilated by the constraint, this effect will never be seen. We obviously have,

$$\hat{T}^0(n)\Psi(\{\lambda\}) = T^0(n)\Psi(\{\lambda\}), \quad (65)$$

where $T^0(n)$ is given by (32). Let us work out everything explicitly for $N = 2$. Then we have $\Psi_{\text{phys}}(A) = \Psi(\lambda_1)$ where $\Psi(\lambda_1) = \Psi(\lambda_1^{-1})$ and,

$$\lambda_1^2 - \lambda_1 \text{tr}(h(x)) + 1 = 0. \quad (66)$$

Differentiating this equation with respect to $A^a(x)$ and using (17) we obtain,

$$\frac{\delta \lambda_1}{\delta A^a(x)} = ig \frac{\lambda_1}{\lambda_1 - \lambda_1^{-1}} \text{tr}(t^a h(x)). \quad (67)$$

Hence,

$$\hat{E}^a(x)\Psi(\lambda_1) = \frac{\hbar}{i} \frac{\delta \Psi(\lambda_1)}{\delta A^a(x)} = \frac{\hbar}{i} \frac{\delta \lambda_1}{\delta A^a(x)} \partial_1 \Psi(\lambda_1) = gh \text{tr}(t^a h(x)) \frac{\lambda_1}{\lambda_1 - \lambda_1^{-1}} \partial_1 \Psi(\lambda_1). \quad (68)$$

We can now see explicitly that $\Psi(\lambda_1)$ is annihilated by the constraint i.e.,

$$\hat{C}_\omega \Psi(\lambda_1) = 0.$$

Proceeding, we find,

$$\hat{H} \Psi(\lambda_1) = \frac{(gh)^2 L}{4} (\lambda_1^2 \partial_1^2 + \frac{3\lambda_1 + \lambda_1^{-1}}{\lambda_1 - \lambda_1^{-1}} \partial_1 \Psi(\lambda_1)), \quad (69)$$

having used,

$$\frac{\delta \text{tr}(t^a h(x))}{\delta A^a(x)} = ig \frac{3}{2} (\lambda_1 + \lambda_1^{-1}),$$

and

$$\text{tr}(t^a h(x)) \text{tr}(t^a h(x)) = \frac{1}{2} (\lambda_1 - \lambda_1^{-1})^2.$$
The expression (69) doesn’t look very transparent but in fact,
\[
\hat{H}\Psi(\lambda_1) = \left( \frac{gh}{2} \right)^2 (\Delta^{-1}(\lambda_1 \partial_{\lambda_1})^2 \Delta - 1)\Psi(\lambda_1),
\] (70)
where \(\Delta = \lambda_1 - \lambda_1^{-1}\) i.e. \(\hat{H}\) is identical to the Hamiltonian in the antisymmetric representation up to a constant (unobservable) term. Add this term for complete equivalence i.e. let
\[
\hat{H} = \frac{1}{2} \int_0^L dx \text{tr}(\hat{E}^2(x)) + \left( \frac{gh}{2} \right)^2.
\]
This ensures that \(T^2(n)\), being defined by (40), is continuous in \(n = 0\). Hence if we do a canonical transformation \(C(\lambda_1) = \Delta\) we can map all operators into the symmetric representation. In particular \(\hat{E}^a(x)\) transforms as,
\[
\hat{E}'^a(x) = C(\lambda_1) \hat{E}^a(x) C(\lambda_1)^{-1} = \hat{E}^a(x) - gh \frac{\lambda_1 + \lambda_1^{-1}}{(\lambda_1 - \lambda_1^{-1})^2} \text{tr}(t^a h(x)).
\]
This seems to be a very contrived representation for the electric field operator and we cannot help getting a feeling that somehow the antisymmetric representation is the “correct” representation. We have also checked \(N = 3\). Then we again find that the Hamiltonian is given by the Hamiltonian in the antisymmetric representation except for the ground state energy but the calculations involved are much longer than for \(N = 2\). We are completely convinced that the same thing will happen for a general \(N\), but we haven’t proved this.

### 3.5 Loop representations

Let us now try to establish a link between our representation of the \(T\)-algebra and a loop representation. It has been suggested [3] that one can go from the connection representation (wavefunctionals of connections) to the loop representation using the loop transform. Denoting a state in the connection representation by \(\Psi(A)\) it looks like (in any dimension),
\[
\tilde{\Psi}(\gamma) = \int \mathcal{D}A T^0(\gamma, A)\Psi(A),
\]
where \(\gamma\) is a loop and \(T^0(\gamma, A) = \text{trP}\exp(\oint \gamma A)\). In our case this would translate into something like,
\[
\tilde{\Psi}(n) = \int \prod_{i=1}^{N-1} d\lambda_i \mu(\{\lambda\}) T^0(n) \Psi(\{\lambda\}),
\]
where \(\mu(\{\lambda\})\) is some kind of measure. This is clearly inadequate (at least when \(N > 2\)). \(\tilde{\Psi}(n)\) takes values on a single integer \(n\) while \(\Psi(\{\lambda\})\) takes values on \(N - 1\) eigenvalues. Let instead \(\Psi_{(n_1,\ldots,n_{N-1})}(\{\lambda\})\) be the (complete) set of eigenstates of \(\hat{H}\) (i.e. \(\hat{T}^2(0)\)), which are either equivalent to the eigenstates \(\Psi_S\) in the symmetric representation or the eigenstates \(\Psi_A\) in the antisymmetric representation. Then define the loop representation by means of the transform,
\[
\tilde{\Phi}(n_1,\ldots,n_{N-1}) = \langle \Psi_{(n_1,\ldots,n_{N-1})}(\{\lambda\}), \Phi(\{\lambda\}) \rangle,
\] (71)
\[ \tilde{\Phi}(n_1, \ldots, n_{N-1}) = \langle n_1, \ldots, n_{N-1} | \Phi \rangle, \]

where \( \langle \{ \lambda \}|n_1, \ldots, n_{N-1} \rangle = \Psi_{(n_1, \ldots, n_{N-1})}(\{ \lambda \}) \). Furthermore define,

\[
\hat{O} \tilde{\Phi}(n_1, \ldots, n_{N-1}) = \langle \Psi_{(n_1, \ldots, n_{N-1})}(\{ \lambda \}) + \hat{O} \Psi_{(n_1, \ldots, n_{N-1})}(\{ \lambda \}), \Phi(\{ \lambda \}) \rangle,
\]

where \( \hat{O} \) is any (gauge-invariant) operator. Hence we get,

\[
\hat{T}^0(n) \tilde{\Phi}(n_1, \ldots, n_{N-1}) = \langle \hat{O} \tilde{\Phi}(n_1, \ldots, n_{N-1}) \rangle = \sum_{i=1}^{N-1} \tilde{\Phi}(n_1, \ldots, n_i - n, \ldots, n_{N-1}) - \tilde{\Phi}(n_1 + n, \ldots, n_{N-1} + n),
\]

since

\[
\hat{T}^0(n) \Psi_{(n_1, \ldots, n_{N-1})}(\{ \lambda \}) = \sum_{i=1}^{N-1} \Psi_{(n_1, \ldots, n_i + n, \ldots, n_{N-1})}(\{ \lambda \}) + \Psi_{(n_1 - n, \ldots, n_{N-1} - n)}(\{ \lambda \}),
\]

independent of representation. In particular, for \( N = 2 \),

\[
\hat{T}^0(n) \tilde{\Phi}(n_1) = \tilde{\Phi}(n_1 + n) + \tilde{\Phi}(n_1 - n),
\]

and we have the even and odd loop representation, arising from the symmetric and anti-symmetric representation respectively,

\[
\tilde{\Phi}_E(-n_1) = \tilde{\Phi}_E(n_1) \quad \tilde{\Phi}_O(-n_1) = -\tilde{\Phi}_O(n_1).
\]

The action of \( \hat{T}^0(n) \) is exactly what one expects and we see that the even loop representation corresponds to the ordinary loop representation while the odd one is something new. Let us also see how \( T^1(n) \) acts in the loop representation. Using (40) for \( p = 0 \),

\[
T^1(n) = \frac{1}{g h L n} [\hat{H}, \hat{T}^0(n)], \quad (n \neq 0),
\]

and \( T^1(0) = 0 \). Hence, using (43),

\[
\hat{T}^1(n) \tilde{\Phi}(n_1, \ldots, n_{N-1}) =
\]

\[
= \frac{1}{g h L n} \langle [\hat{T}^0(-n), \hat{H}] \Psi_{(n_1, \ldots, n_{N-1})}(\{ \lambda \}), \Phi(\{ \lambda \}) \rangle =
\]

\[
-\frac{g h}{N} \sum_{i=1}^{N-1} (Nn_i - s) \tilde{\Phi}(n_1, \ldots, n_i - n, \ldots, n_{N-1}) - s \tilde{\Phi}(n_1 + n, \ldots, n_{N-1} + n) +
\]

\[
\frac{g h n (N - 1)}{2N} \hat{T}^0(n) \tilde{\Phi}(n_1, \ldots, n_{N-1}),
\]

(74)
where \( s = \sum_{i=1}^{N-1} n_i \). In the special case \( N = 2 \) this expression becomes,

\[
\hat{T}_1(n)\tilde{\Phi}(n_1) = -\frac{g\hbar}{2}(n_1\tilde{\Phi}(n_1 - n) - n_1\tilde{\Phi}(n_1 + n)) + \frac{g\hbar n}{4}\hat{T}_0(n)\tilde{\Phi}(n_1).
\] (75)

This is what to be expected from the ordinary loop representation except for the last term which contains a \( \hat{T}_0 \) and an overall sign. This term has arisen since we are doing “minimal quantization” i.e. if there aren’t any ordering problems in a classical Poisson bracket we let this bracket go into a commutator without any modifications. This is not the philosophy adopted in [3], giving the reason for the discrepancy. The sign is there because contrary to the ordinary loop representation, operators act directly on loop states instead of first acting on a state and then evaluating that state on a particular loop. It should be noted that it is important for applications to general relativity exactly what one chooses as \( \hat{T}_1 \) and \( \hat{T}_2 \) since the diffeomorphism and hamiltonian constraint respectively are defined as limits of those operators.

## 4 Conclusion

We investigate the Poisson bracket algebra of \( T \)-variables and show that on the constraint surface the \( T \)-variables can be expressed as functions of the eigenvalues of the holonomy and their associated momenta. The essential structure of the \( T \)-algebra is seen to be very simple, with \( T^2(0) \) acting as a kind of generating function. Quantization then proceeds without problems, the only surprise being that a canonical transformation can map us into an inequivalent representation of the quantum \( T \)-algebra. We then compare this reduced phase space approach to Dirac quantization and find it to give essentially equivalent results. Dirac quantization seems however to “naturally” prefer one of the inequivalent representations. Having the complete set of eigenstates of the Yang-Mills Hamiltonian (essentially \( \hat{T}^2(0) \)) one can define a loop transform and hence loop representations. The main conclusion is then that the loop representation isn’t essential for the quantum \( T \)-algebra. Rather, by using a more solid starting point, the loop representation might or might not be derived from that formalism depending on whether it is “complete” or not. There is really just one point which isn’t clear, should the symmetric or the antisymmetric representation of the quantum \( T \)-algebra be preferred somehow? As we have seen, the antisymmetric representation seems much more natural from the point of view of Dirac quantization and as long as we’re only considering pure gauge theory we can say no more. Coupling fermions to the theory might give further insight into this problem.

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## References

[1] A. Ashtekar, *New variables for classical and quantum gravity*, Phys. Rev. Lett. 57 (1986) 2244-2247
[2] A. Ashtekar, *New Hamiltonian formulation of general relativity*, Phys. Rev. **D36** (1987) 1587-1602

[3] C. Rovelli and L. Smolin, *Loop space representation of quantum general relativity*, Nucl. Phys. **B331** (1990) 80-152

[4] D.M. Marolf, *Loop representations for 2+1 gravity on a torus*, preprint Syracuse SU-GP-93/3-1 gr-qc-9303019 (1993)

[5] R. Loll, *Loop variable inequalities in gravity and gauge theory*, Class. Quant. Grav. **10** (1993) 1471-1476

[6] S. Mandelstam, *Charge-Monopole duality and the phases of nonabelian gauge theories*, Phys. Rev. **D19** (1979) 2391-2409

[7] R. Giles, *The reconstruction of gauge potentials from Wilson loops*, Phys. Rev. **D24** (1981) 2160-2168

[8] R. Gambini and A. Trias, *Gauge dynamics in the C-representation*, Nucl. Phys. **B278** (1986) 436-448

[9] J.E. Hetrick and Y. Hosotani, *Yang-Mills theory on a circle*, Phys. Lett. **B** (1989) 88-92

[10] S.G. Rajeev, *Yang-Mills theory on a cylinder*, Phys. Lett. **B** (1988) 203-205

[11] H. Weyl, *The classical groups*, Princeton U.P. (1946)

[12] S. Helgason, *Groups and geometric analysis*, Academic Press (1984)

[13] A. Anderson, *Canonical transformations in quantum mechanics*, preprint Imperial-TP-92-93-31/hep-th-9305054 (1993)

[14] J.E. Hetrick, *Canonical quantization of two dimensional gauge fields*, preprint Amsterdam/hep-lat-9305020 (1993)