Reductions in Distributed Computing
Part II: $k$-Threshold Agreement Tasks

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Abstract

We extend the results of Part I by considering a new class of agreement tasks, the so-called $k$-Threshold Agreement tasks (previously introduced by Charron-Bost and Le Fessant). These tasks naturally interpolate between Atomic Commitment and Consensus. Moreover, they constitute a valuable tool to derive irreducibility results between Consensus tasks only. In particular, they allow us to show that (A) for a fixed set of processes, the higher the resiliency degree is, the harder the Consensus task is, and (B) for a fixed resiliency degree, the smaller the set of processes is, the harder the Consensus task is.

The proofs of these results lead us to consider new oracle-based reductions, involving a weaker variant of the $C$-reduction introduced in Part I. We also discuss the relationship between our results and previous ones relating $f$-resiliency and wait-freedom.

1 Introduction

In Part I of this paper, we developed several formal definitions of reduction in distributed computing that allowed us to formalize in which sense some distributed task is easier to solve than another one. We applied this formalism for reduction to compare two fundamental classes of agreement tasks, namely Binary Consensus and Atomic Commitment: we showed that even if Consensus and Atomic Commitment are syntactically very close, these two types of task are incomparable in most cases in the sense that Consensus is not reducible to Atomic Commitment, and vice-versa.\footnote{More precisely, Consensus and Atomic Commitment are not comparable, except when the resiliency degree is 1 in which case Consensus is easier than Atomic Commitment.}

Here in Part II, we consider the new class of agreement tasks introduced by Charron-Bost and Le Fessant \cite{Charron-Bost2007}, the so-called $k$-Threshold Agreement tasks ($k$-TAg, for short). These tasks interpolate between Atomic Commitment and Consensus from a purely syntactic standpoint: for the lowest parameter value $k = 1$, $k$-TAg coincides with Atomic Commitment, and for the highest parameter value $k = n$ (where $n$ is the number of processes) $k$-TAg coincides with Consensus (see Section 2 in \cite{Charron-Bost2007} and infra).

We begin by comparing the various agreement tasks $k$-TAg($n, f$) when varying the parameter $k$, the number of processes $n$, and the resiliency degree $f$. We do that in generalizing each of the reducibility and irreducibility results established in Part I for Atomic Commitment and Consensus, and in extending their scope to the general class of $k$-Threshold Agreement tasks.

Then, by combining these results for the $k$-TAg tasks, we derive new irreducibility results for Consensus tasks only. Notably, we establish two irreducibility results both revealing that
“wait-freedom is harder to achieve than $f$-resiliency” in the case of Consensus tasks. More precisely, we show that (A) for a fixed set of processes, the higher the resiliency degree is, the harder the Consensus task is, and (B) for a fixed resiliency degree, the smaller the set of processes is, the harder the Consensus task is.

When the resiliency degree $f$ is less than $n/2$ (that is, when a majority of processes is correct), the fact that wait-free Consensus is harder than $f$-resilient Consensus is not very surprising. Indeed, Fischer, Lynch, and Paterson have shown that in the more benign failure model of initial crashes, $f$-resilient Consensus is solvable with a minority of faulty processes, whereas wait-free Consensus is not solvable with initial crashes. However, our two irreducibility results (A) and (B) are more surprising in view of prior work comparing Consensus tasks in the message passing and shared memory models.

Firstly, an immediate corollary of the main results established by Chandra, Hadzilacos, and Toueg is that for a fixed set of processes, the same information about failures is necessary and sufficient to solve all the Consensus tasks with a majority of correct processes. Using the notion of Failure Detectors, this can be rephrased by saying that the weakest Failure Detectors to solve the Consensus tasks with a majority of correct processes are identical. From this viewpoint, all these tasks are therefore equivalently hard to solve. On the contrary, we show that they are not equivalent with respect to any of the reductions defined in Part I – in particular with respect to the most natural and meaningful reduction in this context, namely the $C$-reduction. Compared with (A), the results in show that, if we introduce the Failure Information hierarchy (FI-hierarchy, for short) which assesses the hardness to solve a task only by the minimal information about failures that is required to solve it, then the $C$-hierarchy is strictly finer than the FI-hierarchy. In other words, the minimal information about failures – or equivalently the weakest Failure Detector – necessary for solving a task does not fully capture the hardness to solve it.

As for our second irreducibility result (B), it seems to contradict Borowsky and Gafni’s simulation, and more specifically its variants described by Lo and Hadzilacos and by Chandra, Hadzilacos, Jayanti, and Toueg in the case of Consensus tasks. Indeed, this simulation consists in a general algorithm for the shared memory model which allows a set of $f+1$ processes with at most $f$ crash failures to simulate any larger set of $n$ processes also with at most $f$ crashes. In the case of Consensus tasks, this simulation provides a general transformation of algorithms that solve the $f$-resilient Consensus task for $n$ processes using read/write registers into ones that solve the wait-free Consensus task for $f+1$ processes using read/write registers also. On the contrary, we show that in the message passing model, the wait-free Consensus task for $f+1$ processes is not $C$-reducible to the $f$-resilient Consensus task for a strict superset of $n$ processes. We could think that the discrepancy between the work in the papers cited above and our results comes from the fact that the message passing and the shared memory models are precisely not equivalent here (a majority of processes may fail in the wait-free case). However, a closer look at the transformations of algorithms reveals that the discrepancy actually results from a more basic point. Indeed, in our work, we use notions of reductions which rely on suitably defined distributed oracles, which are closed black boxes that cannot be opened. We have no access to their internal mechanisms, and are not allowed to dismantle them and to distribute them onto the different processes in the system. This is the reason why the transformations in have no translation in terms of oracle-based reductions. So the discrepancy between these prior results and ours reflects substantial differences between the underlying types of reduction. This highlights the need to give precise definitions to the reductions that we handle.

Another contribution of Part II is to introduce new oracle-based reductions. Namely,
when comparing the hardness to solve the two $f$-resilient tasks $f$-TAG($n, f$) and $(f + 1)$-TAG($n, f$), we have been led to consider various weakenings of the $C$-reduction in which the oracles are more powerful than those used for the $C$-reduction. The resulting new kinds of reductions differ from the $C$-reduction not in the way processes query oracles, but rather in the quality of the oracles that are used. More precisely, to each task $T = (P, f)$ we associate some $f$-resilient oracle suitable for $P$ which is more deterministic than the oracle $O.T$ defined in Part I, in the sense that the set of all its possible behaviors is smaller. Such an oracle is therefore more powerful than $O.T$, and so yields a weaker type of reduction à la Cook.

Part II of this paper is organized as follows. In Section 2, we introduce the $k$-Threshold Agreement tasks, and derive some simple reductions between these tasks from their specifications only. Then, in Sections 3 and 4, we generalize the $C$- and $C^*$-reducibility results of Part I to this new class of agreement tasks. In Section 5, we give generalizations of the irreducibility statements established in Part I, completing the picture of the $C$-hierarchy between the $k$-Threshold Agreement tasks. Then we derive irreducibility results between Consensus tasks only, and we show that wait-free Consensus is strictly harder to achieve than $f$-resilient Consensus in Section 6. This section proceeds with a discussion of these results comparing them with prior work relating wait-freedom and $f$-resiliency.

## 2 k-Threshold Agreement tasks

### 2.1 Definitions and notation

The main results in Part I show that $f$-resilient Consensus and Atomic Commitment tasks are generally not comparable from an algorithmical point of view: except in the case $f = 1$, there is no algorithm which converts a solution to Consensus into a solution to Atomic Commitment, and vice-versa.

However, Consensus and Atomic Commitment problems are very close in the sense that their specifications are identical, except the validity conditions which are slightly different. As a matter of fact, it is possible to link these two specifications from a purely syntactic standpoint, as was done in [7] by Charron-Bost and Le Fessant who introduced, for any set $\Pi$ of $n$ processes and any integer $k \in \{1, \cdots, n\}$, the $k$-Threshold Agreement problem for $\Pi$ (the $k$-TAG$_\Pi$ problem, for short). Parameter $k$ is called the threshold value of $k$-TAG$_\Pi$.

Formally, we have $\mathcal{V} = \{0, 1\}$, $\mathcal{V}_{k}$-TAG$_\Pi = \{0, 1\}^\Pi$, and for any $(F, \vec{V}) \in \mathcal{F}_\Pi \times \{0, 1\}^\Pi$, $\cdot \ k$-TAG$_\Pi(F, \vec{V}) = \{0\}$ if $|\{p \in \Pi : \vec{V}(p) = 0\}| \geq k$,

$\cdot \ k$-TAG$_\Pi(F, \vec{V}) = \{1\}$ if $\vec{V} = \vec{1}$ and $|Faulty(F)| \leq k - 1$,

$\cdot \ k$-TAG$_\Pi(F, \vec{V}) = \{0, 1\}$ otherwise.

In other words, the $k$-TAG validity condition expresses the fact that (1) if at least $k$ processes start with 0, then 0 is the only possible decision value, and (2) if all processes start with 1 and at most $k - 1$ failures occur, then 1 is the only possible decision value.

This new problem is a straightforward generalization of the Atomic Commitment problem ($1$-TAG coincides with Atomic Commitment), but turns out to be also a generalization of Consensus. Indeed, $n$-TAG actually corresponds to binary Consensus, since at least one process in $\Pi$ is correct. We have thereby defined a chain of problems which interpolates between Atomic Commitment and (binary) Consensus. The main motivation for this generalization is theoretical: it is interesting to connect two incomparable problems by exploiting
differences in validity conditions. But it is easy to imagine actual situations in which such a generalization arises naturally: for example, it might be desirable for processes to enforce them to decide 0 (\texttt{abort}) only if a majority of processes initially propose 0 (\texttt{no}), but to require that they decide 1 (\texttt{commit}) if all of them initially propose 1 (\texttt{yes}) as soon as a minority of processes are faulty. Indeed, the latter problem corresponds exactly to the $k$-TAG problem with the threshold value $k = \lfloor n/2 \rfloor + 1$.

For any integer $f$ such that $0 \leq f \leq n - 1$, we denote $k$-TAG($\Pi$, $f$) the distributed task defined by the problem $k$-TAG$_n$ and the resiliency degree $f$. Hence we have a chain of $n$ tasks 1-TAG($\Pi$, $f$), \ldots, $n$-TAG($\Pi$, $f$) which syntactically relates AC($\Pi$, $f$) = 1-TAG($\Pi$, $f$) to Cons($\Pi$, $f$) = $n$-TAG($\Pi$, $f$). Observe that two consecutive tasks in this chain are not comparable \textit{a priori}, since the two parts of the validity condition are entangled: the first part for $k$-TAG$_n$ enforces the first part for ($k+1$)-TAG$_n$, and the second part for $k$-TAG$_n$ is implied by the second part for ($k+1$)-TAG$_n$.

Recall that Fischer, Lynch, and Paterson [8] have established the impossibility of some agreement task with resiliency degree 1 that is attached to a problem weaker in its validity condition than all the above $k$-TAG problems.\footnote{Namely, the validity condition in [8] only specifies that for every value $v \in \mathcal{V}$, there is an execution in which some process decides $v$.}

Consequently, for any set $\Pi$ of $n$ processes and for any integers $k$ and $f$ such that $1 \leq k \leq n$ and $1 \leq f \leq n - 1$, the task $k$-TAG($\Pi$, $f$) is not solvable in an asynchronous system.

In the sequel, we denote by $k$-TAG($n$, $f$) the $f$-resilient task defined by the $k$-Threshold Agreement problem for the set of process names $\Pi = \{1, \cdots, n\}$, that is:

$$k\text{-TAG} (n, f) = k\text{-TAG} (\{1, \cdots, n\}, f).$$

Clearly, for any set $\Pi$ of $n$ processes and for any renaming $\Phi : \{1, \cdots, n\} \sim \Pi$, we have:

$$\Phi_k \text{-TAG} (n, f) = k\text{-TAG} (\Pi, f),$$

and consequently like Cons($\Pi$, $f$) and AC($\Pi$, $f$), $k$-TAG($\Pi$, $f$) is a symmetric task.

### 2.2 Oracles suitable for $k$-TAG problems

Following Section I.3.3, to each task $k$-TAG($\Pi$, $f$), we associate a unique oracle, denoted $O_{k\text{-TAG}} (\Pi, f)$, which is the most general $f$-resilient oracle suitable for the agreement problem $k$-TAG$_n$.

Several results in the sequel rely on the following claims about the $O_{k\text{-TAG}} (\Pi, f)$ oracles.

**Claim O1:** \textit{For any failure pattern $F$ for $\Pi$, and for any consultation in a history of $O_{k\text{-TAG}} (\Pi, f)(F)$, if at most $|\Pi| - k$ queries have value 1 (that is at least $k$ queries are either missing or have value 0), then the only possible response of the oracle is 0.}

**Proof:** Form the partial vector $\tilde{W}$ with the queries in the consultation of $O_{k\text{-TAG}} (\Pi, f)$. Since at most $|\Pi| - k$ components of $\tilde{W}$ are equal to 1, there is an extension $\tilde{V}$ of $\tilde{W}$ in $\{0, 1\}^{|\Pi|}$ with at least $k$ components equal to 0. By the first part in the validity condition of $k$-TAG$_n$, we have $k$-TAG$_n (F, \tilde{V}) = \{0\}$. Hence, 0 is the only possible response by $O_{k\text{-TAG}} (\Pi, f)(F)$ in the consultation with the query vector $\tilde{W}$. \hfill $\Box$

\footnote{Namely, the validity condition in [8] only specifies that for every value $v \in \mathcal{V}$, there is an execution in which some process decides $v$.}
Claim O2: For any failure pattern $F$ for $\Pi$ with less than $k$ faulty processes, and for any consultation in a history of $O$, $k$-TAg($\Pi$, $f$)($F$), if all the query values are 1, then the only possible response of the oracle is 1.

Proof: This is a straightforward consequence of the second part in the validity condition of $k$-TAg.$\Pi$.

Notice that the property $O_{\text{Cons}}$ of Consensus oracles (cf. Section I.3.3) coincides with the conjunction of Claim O1 and Claim O2 for the threshold value $k = |\Pi|$, and the property $O_{\text{AC}}$ of Atomic Commitment oracles (cf. Section I.3.3, too) coincides with Claim O1 for the threshold value $k = 1$.

2.3 Some k-TAg tasks are generalizations of Consensus

One trivial but useful kind of $K$-reduction is reduction by generalization. We say that task $T_2$ is a generalization of task $T_1$ when, informally, the resiliency degree of $T_2$ is not greater than the one of $T_1$, the inputs for $T_1$ are inputs for $T_2$, and for those inputs, any solution for $T_2$ is also a solution for $T_1$. Formally, $T_1 = (P_1, f_1)$ is a generalization of $T_2 = (P_2, f_2)$ if $f_1 \leq f_2$, $\mathcal{V}_{P_1} \subseteq \mathcal{V}_{P_2}$, and for any input vector $\vec{V} \in \mathcal{V}_{P_1}$ and any failure pattern $F$ such that $|\text{Faulty}(F)| \leq f_1$, we have $P_2(F, \vec{V}) \subseteq P_1(F, \vec{V})$. When these conditions are satisfied, we shall also say that $T_1$ is a special case of $T_2$. Notice that if $T_2$ is a generalization of $T_1$, then there is a trivial $K$-reduction from $T_1$ to $T_2$: just take $R$ to be the algorithm which does nothing.

A first example of reduction by generalization is the reduction from any task $T_1 = (P, f_1)$ to $T_2 = (P, f_2)$ with $f_1 \leq f_2$. To illustrate this notion with a less trivial example, consider the Weak Agreement problem for $\Pi$ introduced in [9], denoted $\text{WAg}_{\Pi}$ and defined by $\mathcal{V}_{\text{WAg}_{\Pi}} = \{0, 1\}^\Pi$ and for any $(F, \vec{V}) \in F_{\Pi} \times \{0, 1\}^\Pi$,

- $\text{WAg}_{\Pi}(F, \vec{V}) = \{0\}$ if $\vec{V} = \vec{0}$ and $\text{Faulty}(F) = \emptyset$;
- $\text{WAg}_{\Pi}(F, \vec{V}) = \{1\}$ if $\vec{V} = \vec{1}$ and $\text{Faulty}(F) = \emptyset$;
- $\text{WAg}_{\Pi}(F, \vec{V}) = \{0, 1\}$, otherwise.

Both Cons($\Pi, f$) and AC($\Pi, f$) are generalizations of WAg($\Pi, f$) = (WAg$\Pi$, f). More generally, WAg($\Pi, f$) is a special case of $k$-TAg($\Pi, f$) for any $k \in \{1, \ldots, n\}$.

As we shall show in Proposition 2.1, there is a chain of reductions by generalization that can be traced among the set of $f$-resilient tasks $\{1\text{-TAg}(n, f), \ldots, n\text{-TAg}(n, f)\}$ from threshold $k = f + 1$.

Proposition 2.1 If $n$ and $f$ are two integers such that $1 \leq f \leq n - 1$, then for any $k \in \{f + 1, \ldots, n - 1\}$, the task $k$-TAg($n, f$) is a generalization of $(k+1)$-TAg($n, f$), and so $(f+1)$-TAg($n, f$) is a generalization of Cons($n, f$).

Proof: Let $\Pi$ be any set of $n$ processes. We only have to prove that for any input vector $\vec{V} \in \{0, 1\}^\Pi$ and any failure pattern $F$ for $\Pi$ such that $|\text{Faulty}(F)| \leq f$, we have

\[ k\text{-TAg}_{\Pi}(F, \vec{V}) \subseteq (k+1)\text{-TAg}_{\Pi}(F, \vec{V}). \]

This inclusion is obvious when $(k+1)$-TAg$_{\Pi}(F, \vec{V}) = \{0, 1\}$. Therefore, we need to consider the following two non-trivial cases only:
1. \(|\{p \in \Pi : \vec{V}(p) = 0\}| \geq k+1\).

\text{A fortiori, } |\{p \in \Pi : \vec{V}(p) = 0\}| \geq k \text{ and thus we have}

\(k\text{-TA}_{\Pi}(F, \vec{V}) = (k+1)\text{-TA}_{\Pi}(F, \vec{V}) = \{0\}.\)

2. \(\vec{V} = \vec{1}\) and \(|Faulty(F)| \leq k\).

Since we only examine the failure patterns with at most \(f\) failures and \(k \geq f+1\), we actually have \(|Faulty(F)| \leq k - 1\). It follows that

\(k\text{-TA}_{\Pi}(F, \vec{1}) = (k+1)\text{-TA}_{\Pi}(F, \vec{1}) = \{1\}.\)

\(\Box\)

3 \ C-reductions between \(k\)-Threshold Agreement tasks

This section is devoted to several generalizations of the \(C\)-reductions established in Part I. First we shall show that \(k\text{-TA}(n, f)\) is \(C\)-reducible to \((k+1)\text{-TA}(n+1, f+1)\). Then we shall complete Proposition 2.1 and the comparison of the tasks \((f+1)\text{-TA}(n, f), \ldots, n\text{-TA}(n, f)\) by showing that all these tasks are actually equivalent – that is, of the same unsolvability degree – with respect to the \(C\)-reduction. Finally, we shall consider \(k\text{-TA}(n, f)\) when the threshold value \(k\) takes the preceding value \(f\), and shall compare the task \(f\text{-TA}(n, f)\) with \(n\text{-TA}(n, f) = \text{Cons}(n, f)\). For that, we shall introduce a slightly weaker notion of reduction a la Cook, the \(c\)\(C\)-reduction, which differs from the original \(C\)-reduction in the power of oracles (but not in the way oracles are queried). We shall show that when a majority of processes is correct, i.e., when \(n > 2f\), the degree of unsolvability – with respect to the \(c\)\(C\)-reduction – of \(f\text{-TA}(n, f)\) is higher or equal to the one of \(\text{Cons}(n, f)\).

3.1 \(C\)-reduction between \(k\)- and \((k+1)\text{-TA}\) tasks

Let \(\Pi\) be any set of \(n + 1\) processes, and let \(\Pi'\) be any subset of \(\Pi\) with \(n\) processes. The \(C\)-reduction from \(k\text{-TA}(\Pi', f)\) to \((k+1)\text{-TA}(\Pi, f+1)\) is simple: Each process in \(\Pi'\) just needs to query the oracle \(\mathcal{O} \text{ \ensuremath{(k+1)\text{-TA}(\Pi, f+1)}}\) with its initial value. The oracle eventually gives a response since it is consulted by at least \(n - f = (n+1) - (f+1)\) processes. Every process finally decides on the value provided by \(\mathcal{O} \text{ \ensuremath{(k+1)\text{-TA}(\Pi, f+1)}}\).

Thus termination and agreement are obviously guaranteed. For the \(k\)-validity condition, consider a run of this algorithm for \(\Pi'\) with at most \(f\) failures. Firstly, suppose that at least \(k\) processes start with 0; at least \(k+1\) queries in the consultation of \(\mathcal{O} \text{ \ensuremath{(k+1)\text{-TA}(\Pi, f+1)}}\) are either missing or with value 0 since the process in \(\Pi \setminus \Pi'\) does not query the oracle. By Claim O1, the oracle may not answer any value else than 0, and so the only possible decision value is 0. Now, suppose that all processes start with 1 and at most \(k-1\) processes fail in this run. With respect to \(\Pi\), at most \(k\) processes crash, and by Claim O2 applied to the oracle \(\mathcal{O} \text{ \ensuremath{(k+1)\text{-TA}(\Pi, f+1)}}\), the only possible answer value given by this oracle is 1. Therefore, 1 is the only possible decision value. This establishes:

**Theorem 3.1** If \(n, f\) and \(k\) are three integers such that \(1 \leq f \leq n - 1\) and \(1 \leq k \leq n\), then \(k\text{-TA}(n, f)\) is \(C\)-reducible to \((k+1)\text{-TA}(n+1, f+1)\).

Notice that in the particular case \(k = n\), Theorem 3.1 states that \(\text{Cons}(n, f)\) is \(C\)-reducible to \(\text{Cons}(n+1, f+1)\), and consequently it generalizes Proposition I.7.5.
Variables of process $p$:

$x_p \in V$, initially $v_p$

Algorithm for process $p$:

Send $\langle v_p \rangle$ to all
wait until $\{\text{Receive} \langle v_q \rangle \text{ from } n - f \text{ processes}\}$
$x_p := \min \{v_q : \text{received } v_q\}$
$\text{Query}(\mathcal{O}\text{-Cons}(\Pi, f))(x_p)$
$\text{Answer}(\mathcal{O}\text{-Cons}(\Pi, f))(d)$
$\text{Decide}(d)$

Figure 1: A $C$-reduction from $(f+1)$-TAg$(\Pi, f)$ to Cons$(\Pi, f)$.

3.2 Degree of unsolvability of Consensus tasks

As stated in Proposition 2.1, each task $k$-TAg$(n, f)$ with $k \in \{f+1, \ldots, n\}$ is a generalization of Cons$(n, f)$. Hence Cons$(n, f)$ trivially $C$-reduces to any of these $k$-Threshold Agreement tasks. We shall next show that conversely $(f+1)$-TAg$(\Pi, f)$ is $C$-reducible to Cons$(n, f)$. It will then follow that any $k$-TAg$(n, f)$ with $k \in \{f+1, \ldots, n\}$ is equivalent to Cons$(n, f)$ with respect to $C$-reducibility. In other words, $(f+1)$-TAg$(n, f), \ldots, n$-TAg$(n, f) = \text{Cons}(n, f)$ have the same unsolvability degree.

Let $\Pi$ be any set of $n$ processes. There is quite a simple $C$-reduction from $(f+1)$-TAg$(\Pi, f)$ to Cons$(\Pi, f)$: Firstly, processes make their initial values more uniform. For that, every process sends its initial value to all, waits until receiving initial values from $n - f$ processes, and then sets a local variable to the minimum value that it has received. Secondly, each process queries the oracle $\mathcal{O}\text{-Cons}(\Pi, f)$ with the value of this local variable, and then decides on the value answered by the oracle.

**Proposition 3.2** Let $n$ and $f$ be two integers such that $1 \leq f \leq n - 1$, and let $\Pi$ be a set of $n$ processes. The algorithm in Figure 1 that uses the oracle Cons$(\Pi, f)$, solves the task $\mathcal{O}\cdot(f+1)$-TAg$(\Pi, f)$, and so $(f+1)$-TAg$(n, f)$ is $C$-reducible to Cons$(n, f)$.

**Proof:** Let $\rho =< F, I, H >$ denote a run of the algorithm in Figure 1 with at most $f$ failures. Obviously, $\rho$ satisfies the termination, irrevocability, and agreement conditions. We are going to prove that $\rho$ also satisfies the two requirements of $(f+1)$-validity condition.

1. Suppose that at least $f+1$ processes start with value 0. Since at most $f$ processes fail, each process that is still alive receives at least one message with value 0, and so queries the $\mathcal{O}\cdot\text{Cons}(\Pi, f)$ oracle with value 0. From the property $O_{\text{Cons}}$ in Part I, it follows that $\mathcal{O}\cdot\text{Cons}(\Pi, f)$ definitely answers 0. Therefore every process that makes a decision decides 0.

2. Now suppose that all the processes start with value 1, all the query values of $\mathcal{O}\cdot\text{Cons}(\Pi, f)$ are equal to 1. Again by the property $O_{\text{Cons}}$, the only possible answer given by $\mathcal{O}\cdot\text{Cons}(n, f)$ is 1, and processes decide 1.

□
3.3 A reduction with a majority of correct processes

Propositions 3.1 and 3.2 establish that \((f+1)\text{-Tag}(n,f), \ldots, n\text{-Tag}(n,f) = \text{Cons}(n,f)\) are all of same degree of unsolvability with respect to \(\leq C\). In this section, we compare these equivalent tasks with \(f\text{-Tag}(n,f)\): we show that if \(n > 2f\), then \(f\text{-Tag}(n,f)\) is at least as hard to solve as \(\text{Cons}(n,f)\).

In the case \(f = 1\), this result will compare \(\text{Cons}(n,1)\) with \(\text{AC}(n,1)\) when \(n > 2\). However, the reduction that we shall describe does not coincide with the reduction from \(\text{Cons}(n,1)\) to \(\text{AC}(n,1)\) given in Proposition 1.8.4. Indeed, contrary to this latter reduction, our algorithm which uses an oracle for \(f\text{-Tag}(n,f)\) actually solves \(\text{Cons}(n,f)\) only if the oracle is consistent, namely if it satisfies the following condition: if the oracle answers 0 in some consultation in which all the query values are equal to 1, then it will also answer 0 in any subsequent consultation. In other words, a consistent oracle for \(f\text{-Tag}(n,f)\) which answers some value on the grounds of informations about future failures, does not forget these informations and takes them into account in its subsequent answers as it does in its previous answers.

We are going to define consistent oracles precisely. First, let us observe the following fact: if \(P\) is an agreement problem for some set \(\Pi\) of processes, and if \(\vec{W}_1\) and \(\vec{W}_2\) are two partial vectors in \(\vec{V}^{\Pi}\) such that \(\vec{W}_1\) is an extension of \(\vec{W}_2\) (denoted \(\vec{W}_1 \supseteq \vec{W}_2\), \(^3\) then for any failure pattern \(F\) for \(\Pi\) we have

\[
\bigcap_{\vec{V} \in \vec{V}^{\Pi} : \vec{V} \geq \vec{W}_2} P(F,\vec{V}) \subseteq \bigcap_{\vec{V} \in \vec{V}^{\Pi} : \vec{V} \geq \vec{W}_1} P(F,\vec{V}).
\]

**Definition 3.3** Let \(\sigma\) be an oracle whose set of consultants is \(\Pi\), and which is suitable for some agreement problem \(P\) for \(\Pi\). We say that \(\sigma\) is a consistent oracle if for any failure pattern \(F\) for \(\Pi\), any history \(H \in O_\sigma(F)\), and any two consultations of \(\sigma\) in \(H\) with query vectors \(\vec{W}_1\) and \(\vec{W}_2\) such that \(\vec{W}_1 \supseteq \vec{W}_2\), \(\sigma\) answers \(d\) in the consultation with query vector \(\vec{W}_2\) only if it answers \(d\) in the consultation with query vector \(\vec{W}_1\).

For any task \(T = (P,f)\), we restrain the set of histories of the oracle in order to get the most general oracle which is consistent, \(f\)-resilient, and suitable for \(P\). In this way, we obtain an oracle, denoted \(\text{\textit{c}}O.T\), which is at least as powerful as \(O.T\) in the sense that for any failure pattern \(F\) for \(\Pi\), \(\text{\textit{c}}O.T(F) \subseteq O.T(F)\).

This yields a new notion of reduction à la Cook, denoted \(\leq_{\text{\textit{c}}C}\), in which algorithms may only use the consistent versions of oracles. Formally, \(T_1 \leq_{\text{\textit{c}}C} T_2\) if there is an algorithm for \(T_1\) using the consistent oracle \(\text{\textit{c}}O.T_2\).

Since \(\text{\textit{c}}O.T\) is at least as powerful as \(O.T\), \(C\)-reducibility implies \(\text{\textit{c}}C\)-reducibility. Equivalently, \(\text{\textit{c}}C\)-irreducibility results yield the corresponding \(C\)-irreducibility results (see Section 6 infra).

Thanks to this new notion of reducibility, we shall be able to compare the two tasks \(f\text{-Tag}(\Pi,f)\) and \((f+1)\text{-Tag}(\Pi,f)\).

Let \(\Pi\) be any set of \(n\) processes, and let \(f\) be a positive integer such that \(n > 2f\). In Figure 2, we give an algorithm using the consistent oracle for \(f\text{-Tag}(\Pi,f)\) which solves the task \((f+1)\text{-Tag}(\Pi,f)\). Our algorithm uses the oracle \(\text{\textit{c}}O.f\text{-Tag}(\Pi,f)\) twice: the first time to achieve an “approximate \((f+1)\text{-Threshold Agreement}” on the initial values, and

\(^3\)Recall that \(\vec{W}_1\) is an extension of \(\vec{W}_2\) if the domain of definition \(\Pi_1\) of the mapping \(\vec{W}_1\) contains the one \(\Pi_2\) of \(\vec{W}_2\), and for any \(p \in \Pi_2\), we have \(\vec{W}_1(p) = \vec{W}_2(p)\).
Theorem 3.4 Let \( n \) and \( f \) be two integers such that \( 1 \leq 2f \leq n - 1 \), and let \( \Pi \) be a set of \( n \) processes. The algorithm in Figure 2 that uses \( ^{O.f}.f\text{-Tag}(\Pi, f) \) solves \( (f+1)\text{-Tag}(\Pi, f) \), and so \( (f+1)\text{-Tag}(n, f) \) is \(^{C}\)-reducible to \( f\text{-Tag}(n, f) \).

Proof: First notice that Claims O1 and O2 still hold for any consistent oracle \( ^{O.k}\text{-Tag}(\Pi, f) \).

Let \( \Pi \) be a set of \( n \) processes, and let \( \rho = \langle F, I, H \rangle \) denote a run of the algorithm in Figure 2 with at most \( f \) failures. By the \( f \)-resiliency property, the oracle \( ^{O.f}.f\text{-Tag}(\Pi, f) \) definitely answers in each of its two consultations in \( \rho \). Let \( d \) denote the second answer.

To prove that \( \rho \) satisfies the termination, irrevocability, agreement, and \((f+1)\)-validity conditions, we shall distinguish the cases \( d = 0 \) and \( d = 1 \).

Case \( d = 1 \). Irrevocability, termination and agreement are obvious.

Let us prove that \( \rho \) satisfies the \((f+1)\)-validity condition.

1. Suppose that at least \( f+1 \) processes start with 0. We consider the first consultation of \(^{O.f}\text{-Tag}(n, f)\): each of these processes that starts with 0 either does not query the oracle (because they crash) or queries it with value 0. By Claim O1, the first response given by \(^{O.f}\text{-Tag}(n, f)\) is necessarily equal to 0, and processes decide 0, as required.

2. Now assume that all the initial values are equal to 1. Since \(^{O.f}\text{-Tag}(n, f)\) is supposed to be consistent and \( d = 1 \), the oracle may not answer 0 in the first consultation. Therefore \( v = 1 \) and processes decide 1 in \( \rho \).

Case \( d = 0 \). First, we prove that \( \rho \) satisfies both the agreement and irrevocability conditions. By the rule which determines when a process proposes value \( v \in \{0, 1\} \) (i.e., sends \((P, v, r)\) to all), it is impossible for a process to propose 0 and for another one to propose 1 at the same round. Suppose now that some processes make a decision in \( \rho \); let \( r \) denote the first round at which a decision is made, and let \( p \) denote a process that decides at round \( r \). Process \( p \) has received at least \( f+1 \) propositions for its decision value \( v \) at round \( r \). Thus every process \( q \) receives at least one proposition for \( v \) at round \( r \), and so we have \( x_q = v \) at the end of round \( r \). Hence every process that is still alive decides \( v \) at the latest at round \( r + 1 \), and keeps deciding \( v \) in all subsequent rounds. In other words, \( \rho \) satisfies agreement and irrevocability.

For termination, we argue as for the reduction from Cons\((n, 1)\) to AC\((n, 1)\) (see Section I.8.3). Since every second query value of \(^{O.f}\text{-Tag}(\Pi, f)\) is 1 and \( d = 0 \), by Claim O2 exactly \( f \) failures occur in run \( \rho \). For every process \( p \), we consider the first round \( r_p \) process \( p \) executes after the last failure occurs in \( \rho \), and we let \( r_\rho = \max_{p \in \text{Correct}(F)} (r_p) \). If some process makes a decision by round \( r_\rho \), then the above argument for agreement shows that every process that is still alive at the end of round \( r_\rho \) has made a decision by the end of this round. Suppose no process has made a decision by the end of round \( r_\rho \). All correct processes receive the same set of \( n - f \) messages of the form \((R,-,r_\rho)\), and so they propose the same value \( w \in \{0,1,?\} \) at round \( r_\rho \).
Variables of process $p$:

$x_p \in V$, initially $v_p$
$r_p \in \mathbb{N}$, initially 1

Algorithm for process $p$:

\begin{verbatim}
Query("O.f-TAg(Π, f)\langle v_p \rangle")
Answer("O.f-TAg(Π, f)\langle v \rangle")
Query("O.f-TAg(Π, f)\langle 1 \rangle")
Answer("O.f-TAg(Π, f)\langle d \rangle")
if $d = 1$
then
  Decide(v)
else
  repeat forever
    Send\langle(R, x_p, r_p)\rangle to all
    wait until \[Receive\langle(R, *, r_p)\rangle from n - f processes\] (where * can be 0 or 1)
    if at least $f + 1$ of the \langle(R, *, r_p)\rangle’s received have value 0 in the second component
    then
      Send\langle(P, 0, r_p)\rangle to all
    else
      if all the \langle(R, *, r_p)\rangle’s received have value 1 in the second component
      then
        Send\langle(P, 1, r_p)\rangle to all
      else
        Send\langle(P, ?, r_p)\rangle to all
      wait until \[Receive\langle(P, *, r_p)\rangle from n - f processes\] (where * can be 0, 1, or ?)
    if at least $f + 1$ of the \langle(P, *, r_p)\rangle’s received have the same $w \in \{0, 1\}$ in the second component
    then
      $x_p := w$
      Decide(w)
    else
      if one of the \langle(P, *, r_p)\rangle’s received have $w \in \{0, 1\}$ in the second component
      then
        $x_p := w$
      else
        $x_p := 0$
    $r_p := r_p + 1$
\end{verbatim}

Figure 2: A $\mathcal{C}$-reduction from $(f+1)$-TAg(Π, f) to $f$-TAg(Π, f)
If \( w \neq \ast \), then every correct process decides \( w \) since it receives \( n - f \) propositions for \( w \) and \( n - f \geq f + 1 \). Otherwise, \( w = \ast \) and every correct process \( p \) sets its variable \( x_\rho \) to 0 in the end of round \( r_\rho \). Since \( n - f \geq f + 1 \), it is easy to see that every correct process proposes 0 at round \( r_\rho + 1 \), and so every correct process decides 0. This completes the proof of termination.

Finally, let us establish that \( \rho \) satisfies the \((f+1)\)-validity condition:

1. A simple inductive argument shows that if at least \( f + 1 \) processes start with value 0, then at any round, value 1 may not be proposed by any process. Therefore in this case, 0 is the only possible decision value.

2. Now suppose that all processes start with the same initial value 1. Every process proposes 1 at the first round, i.e., sends \((P, 1, 1)\) to all. As \( n - f \geq f + 1 \), it follows from the code that each process then decides 1.

\( \square \)

As mentioned above, Theorem 3.4 states a “\( C \)-reducibility result which is, in the particular case \( f = 1 \), slightly weaker than the \( C \)-reducibility result given by Theorem 1.8.4. An open question is whether \((f+1)\)-Tag\((n, f)\) is actually \( C \)-reducible to \( f \)-Tag\((n, f)\). If not, this would show in particular that \( C \)-reduction is strictly stronger than “\( C \)-reduction.

4 \( C^* \)-reductions between \( k \)-Threshold Agreement tasks

In this section, we establish two \( C^* \)-reducibility statements for the \( k \)-Threshold Agreement tasks which compare \( k \)-Tag\((n, f)\) with \((k+1)\)-Tag\((n+1, f)\) and \((k+1)\)-Tag\((n, f)\) respectively, and contain all the \( C^* \)-reducibility results established in Part I. Interestingly, the two \( C^* \)-reductions that we give here are similar, except in their final decision rule.

4.1 \( C^* \)-reduction when varying threshold value and number of processes

Since Cons\((n, f)\) is generally not \( C \)-reducible to AC\((n, f)\) (cf. Theorem 1.8.2), we cannot expect to extend the \( C \)-reducibility result in Proposition 2.1 from \((k+1)\)-Tag\((n, f)\) to \( k \)-Tag\((n, f)\) for all the threshold values \( k \) less than \( f + 1 \). Nevertheless, we are going to show that \((k+1)\)-Tag\((n+1, f)\) is always \( C^* \)-reducible to \( k \)-Tag\((n, f)\).

Let \( \Pi \) denote the set of \( n + 1 \) processes \( \{1, \ldots , n+1\} \). We consider the \( n + 1 \) subsets of \( \Pi \) of cardinality \( n \), and we denote \( \Pi_i = \Pi \setminus \{i\} \). We use \( \mathbf{7} \) as shorthand for \( k \)-Tag\((\Pi_i, f)\); hence \( \mathbf{7} \) is the sanctuary of the oracle \( \mathcal{O}.k \)-Tag\((\Pi_i, f)\) (cf. Section 1.3.3).

In Figure 4, we give the code of a simple \((k+1)\)-Threshold Agreement algorithm for \( \Pi \) using the oracles \( \mathcal{O}.k \)-Tag\((\Pi_1, f), \ldots, \mathcal{O}.k \)-Tag\((\Pi_{n+1}, f)\). Informally, every process \( i \) consults these oracles with its initial value \( v_i \), according to the order \( 1, \ldots , n + 1 \), except the oracle \( \mathcal{O}.k \)-Tag\((\Pi_i, f)\) since \( i \) is not a consultant of this oracle \((i \notin \Gamma(\mathbf{7}))\). As soon as process \( i \) gets a response from an oracle, \( i \) broadcasts it in \( \Pi \). In this way, it eventually knows all the values answered by the oracles (including the one by \( \mathcal{O}.k \)-Tag\((\Pi_i, f)\)), and then decides on the greatest value.

**Proposition 4.1** If \( n, f \) and \( k \) are integers such that \( 1 \leq f \leq n - 1 \) and \( 1 \leq k \leq n \), then the algorithm in Figure 4 solves the task \((k+1)\)-Tag\((n+1, f)\), and so \((k+1)\)-Tag\((n+1, f)\) is \( C^* \)-reducible to \( k \)-Tag\((n, f)\).
Algorithm for process $i$:

initialization:  
\[ d_i \in V \cup \{\bot\}, \text{ initially } \bot \]

for $l = 1$ to $n+1$ do:
  if $l \neq i$ then
    Query($O.k$-TAg($\Pi_l, f$))\langle$v_i$\rangle
    Answer($O.k$-TAg($\Pi_l, f$))\langle$w_l$\rangle
    Send\langle$(l, w_l)$\rangle to all

wait until $\text{Receive}\langle(l, w_l)\rangle$ for all $l \in \{1, \cdots, n+1\}$

\[ d_i := \max_{l=1, \cdots, n+1}(w_l) \]

Decide($d_i$)

Figure 3: A $C^*$-reduction from $(k+1)$-TAg($n+1, f$) to $k$-TAg($n, f$).

Proof: We first prove the termination property. By induction on $i$, we easily show that every oracle $O.k$-TAg($\Pi_i, f$) is consulted by at least $|\Pi_i| - f = n - f$ processes, and so no process is blocked in the sanctuary. Every correct process $p \in \Pi_i$ thus gets an answer from the oracle $O.k$-TAg($\Pi_i, f$), and then broadcasts it in $\Pi$. Since $n \geq f+1$, each subset $\Pi_i$ contains at least one correct process. Therefore every correct process eventually knows the $n+1$ values answered by the oracles $O.k$-TAg($\Pi_1, f$), $\ldots$, $O.k$-TAg($\Pi_{n+1}, f$), and then makes a decision.

Irreversibility is obvious. Agreement follows from the decision rule and the fact that every process which makes a decision knows all the values answered by the oracles.

For the validity condition, consider any run of the algorithm with at most $f$ failures. The proof of termination shows that each oracle $O.k$-TAg($\Pi_i, f$) answers to all its consultants.

1. Suppose that at least $k+1$ processes in $\Pi$ start with 0. In each subset $\Pi_i$, at least $k$ processes either do not query the oracle $O.k$-TAg($\Pi_i, f$) or query it with value 0. By Claim O1, every oracle necessarily answers 0, and so the decision value is 0.

2. Suppose now that all the all the initial values are 1 and at most $k$ (and so at most $\min(k, f)$) processes crash. Among the subsets $\Pi_1, \ldots, \Pi_{n+1}$ are at least $k$ with less than $k$ faulty processes. By Claim O2, the corresponding oracles are bound to answer 1. From the decision rule, it follows that the decision value is 1 since we have $k \geq 1$.

This shows the $(k+1)$-validity condition. \hfill $\square$

Note that for $k = n$, Proposition 4.1 yields

\[ \text{Cons}(n + 1, f) \leq C^* \text{Cons}(n, f), \]

which is also a consequence of the $C$-reduction from Cons($n + 1, f$) to Cons($n, f$) that we have shown in Part I (Proposition I.7.4).

More interestingly, by Proposition 4.1 applied $f$ times, we obtain

\[ (f+1)$-TAg(n + f, f) \leq C^* \text{ AC}(n, f). \]

Since the task $(f+1)$-TAg($n + f, f$) is equivalent to Cons($n + f, f$) with respect to $\leq C$, we have

\[ \text{Cons}(n + f, f) \leq C^* \text{ AC}(n, f), \]
which is the first $C^*$-reduction established in Theorem I.6.4.

Finally, observe that if we were able to strengthen Theorem 3.4 by proving that $(f+1)$-TAg$(n+f-1, f)$ is actually $C$-reducible (and not only $cC$-reducible) to $(f+1)$-TAg$(n+f-1, f)$ when $n+f-1 > 2f$, then we could stop one step before when applying Proposition 4.1. This would yield the better $C^*$-reducibility result

Cons$(n + f - 1, f) \leq C^* AC(n, f)$

when $f \leq n - 2$.

4.2 $C^*$-reduction when varying the number of processes only

From Proposition 4.1, we cannot derive that AC$(n+1, f)$ is $C^*$-reducible to AC$(n, f)$ (cf. Proposition I.7.1). A general statement for $k$-Threshold Agreement tasks which would extend this latter $C^*$-reducibility result would necessarily compare two tasks with the same threshold value.

It turns out that, by just substituting “min” for “max” in the decision rule in Figure 8, the resulting algorithm solves the task $k$-TAg$(n+1, f)$. This shows that $k$-TAg$(n+1, f)$ is $C^*$-reducible to $k$-TAg$(n, f)$. The proof is similar to the one of Proposition 4.1 and is therefore omitted.

**Proposition 4.2** If $n, f$ and $k$ are three integers such that $1 \leq k \leq n$ and $1 \leq k \leq n$, then the task $k$-TAg$(n+1, f)$ is $C^*$-reducible to $k$-TAg$(n, f)$.

Specializing $k$ to 1 in Proposition 4.2, we actually recover that AC$(n+1, f)$ is $C^*$-reducible to AC$(n, f)$. Note that when $k \geq f+1$, Proposition 4.1 can be derived from Propositions 2.1 and 4.2.

5 C-irreducibility results between $k$-Threshold Agreement tasks

In this section, we shall examine generalizations of the two $C$-irreducibility results between Consensus and Atomic Commitment tasks established in Part I. More precisely, we shall prove that for a fixed set $\Pi$ of $n$ processes, and a fixed resiliency degree $f$, $1 \leq f \leq n-1$, the task $k$-TAg$(\Pi, f)$ is incomparable with Cons$(\Pi, f)$ with respect to $\leq_C$ for any threshold value $k \in \{1, \ldots, f\}$.

By Propositions 2.1 and 3.1, we know that the tasks $(f+1)$-TAg$(\Pi, f), \ldots, n$-TAg$(\Pi, f) =$ Cons$(\Pi, f)$ are all of the same unsolvability degree. Thus it remains to compare the task $f$-TAg$(\Pi, f)$ with Cons$(\Pi, f)$ to get a complete picture of the relationships between the various $k$-Threshold Agreement tasks for a fixed set of processes and a fixed resiliency degree.

To some extent, Theorem 3.4 answers this question when a majority of processes is correct ($n > 2f$), since it establishes that Cons$(\Pi, f)$ is $cC$-reducible to $f$-TAg$(\Pi, f)$. We shall prove that this does not hold anymore with a minority of correct processes: if $n \leq 2f$, then Cons$(\Pi, f)$ is not $cC$-reducible (and so not $C$-reducible) to $f$-TAg$(\Pi, f)$.

Besides providing a better understanding of the connections between the various tasks $k$-TAg$(\Pi, f)$, the three irreducibility results that we establish in this section will play a key role in the proofs of our final results comparing wait-free and $f$-resilient Consensus tasks (see Section 6 infra).
5.1 C-irreducibility to wait-free Consensus tasks

**Theorem 5.1** For every integers \( n, f, k \) such that \( 1 \leq k \leq f \leq n - 1 \), the task \( k \)-TAg\((n, f)\) is not \( C \)-reducible to \( \text{Cons}(n, n - 1) \), and so is not \( C \)-reducible to \( \text{Cons}(n, f) \).

**Proof:** Let \( \Pi \) be a set of \( n \) process names. Suppose, for the sake of contradiction, that there is an algorithm \( R \) for the task \( k \)-TAg\((\Pi, f)\) which uses the oracle \( \mathcal{O}.\text{Cons}(\Pi, n - 1) \). Let \( \Pi_1 \) be any subset of \( \Pi \) of cardinality \( k \). Consider a run \( \rho = < F, I, H > \) of \( R \) such that, for any \( q \in \Pi \), \( I(q) = s^1_q \), and for any \( t \in T \), \( F(t) = \Pi_1 \). In other words, \( \rho \) is a run of \( R \) in which all processes start with initial value 1 and no process is faulty except the processes in \( \Pi_1 \), all of which initially crash. Since \( k \leq f \), every process in \( \Pi \setminus \Pi_1 \) eventually makes a decision in \( \rho \), and all the decision values are identical; let \( d \) denote this common decision value.

We now introduce the mapping \( I' \) which is identical to \( I \) over \( \Pi \setminus \Pi_1 \) and satisfies \( I'(p) = s^0_p \) for any process \( p \) in \( \Pi_1 \). Then we consider \( \rho' = < F, I', H > \); we claim that \( \rho' \) is a run of \( R \). Recall that the runs of \( R \) are defined by the compatibility rules R1–6 introduced in Section I.4.2.

Since \( \rho \) is a run of \( R \), it is straightforward that \( \rho' \) satisfies R1, R2, R3, R5, and R6. By an easy induction, we see that for any process \( q \), \( q \neq p \), the sequence of the local states reached by \( q \) are the same in \( \rho' \) as in \( \rho \). This ensures that every step in \( H \) is feasible from \( I' \), and so R4 holds in \( \rho' \). Thus, \( \rho' \) is a run of \( R \), and by the \( k \)-validity condition, the only possible decision value in \( \rho' \) is 0. This shows that \( d = 0 \).

Now from \( \rho \), we are going to construct a failure free run of \( R \) by using the asynchronous structure of computations. To achieve that, we need the following lemma, where \( F_0 \) denotes the failure pattern with no failure (defined formally by \( F_0(t) = \emptyset \), for any \( t \in T \)), and \( H[0, t] \) denotes the prefix in \( H \) of events with time less or equal to \( t \).

**Lemma 5.2** For any \( t_0 \in T \), there exists an extension \( H_0 \) of \( H[0, t_0] \) such that \( < F_0, I, H_0 > \) is a failure free run of \( R \).

**Proof:** The history \( H_0 \) is constructed in stages, starting from \( H[0, t_0] \); each stage consists in adding one event. A queue of the processes in \( \Pi \) is maintained, initially in an arbitrary order, and the messages in \( \beta \) are ordered according to the time the messages were sent, earliest first.

Suppose that the finite history \( H_0[0, t] \) extending \( H[0, t_0] \) is constructed. Let \( t^+ \) denote the successor of \( t \) in \( T \), and let \( q \) be the first process in the process queue. After \( H_0[0, t] \), \( q \) may achieve only one type \( T \) of event. There are three cases to consider:

1. \( T = S \) or \( T = Q \). The automaton \( R(q) \) entirely determines the event \( e = (\beta, q, t^+, S, m) \) or \( e = (\text{Cons}(\Pi, n - 1), q, t^+, Q, v) \) which \( q \) may achieve at time \( t^+ \).

2. \( T = R \). In this case, the message buffer \( \beta \) contains at least one message for \( q \). Then we let \( e = (\beta, q, t^+, R, m) \), where \( m \) denotes the earliest message for \( q \) in \( \beta \).

3. \( T = A \). Form the successive consultations of \( \mathcal{O}.\text{Cons}(\Pi, n - 1) \) in \( H_0[0, t] \), and focus on the latter consultation. Note that process \( q \) has necessarily queried \( \mathcal{O}.\text{Cons}(\Pi, n - 1) \) during this consultation; let \( v \) be the value of this query. There are three subcases:

   Case 1: \( \mathcal{O}.\text{Cons}(\Pi, n - 1) \) has already answered some value \( d \).

   In this case, we let \( e = (\text{Cons}(\Pi, n - 1), q, t^+, A, d) \).
Case 2: $O.\text{Cons}(\Pi, n - 1)$ has not answered yet.

We let $e = (\text{Cons}(\Pi, n - 1), q, t^+, A, v)$.

The above procedure determines a unique event $e$, and we let $H_0[0, t^+] = H_0[0, t]; e$ (where semicolon denotes concatenation). Process $q$ is then moved to the back of the process queue.

This inductively defines $H_0$. By construction, $\rho_0 = < F_0, I, H_0 >$ satisfies R1–6, and so is a failure free run of $R$.

We now instantiate $t_0$ to be the time when the last process makes a decision in $\rho$. The lemma provides an extension $H_0$ of $H[0, t_0]$ such that $\rho_0 = < F_0, I, H_0 >$ is a run of $R$. The decision value in $\rho_0$ is 0, which contradicts the fact that processes must decide on 1 in any failure free run of an algorithm solving $k$-TAG($\Pi, f$) in which all processes start with initial value 1.

In the case $f = 1$, Theorem 5.4 states that $AC(n, 1)$ is not $C$-reducible to $\text{Cons}(n, n - 1)$, and so reduces to Theorem 1.8.1.

### 5.2 C-irreducibility to wait-free $k$-TAG tasks

**Theorem 5.3** For every integers $n, k$ such that $2 \leq k \leq n - 1$, $\text{Cons}(n, k)$ is not $C$-reducible to $\text{TAG}(n, n - 1)$.

**Proof:** We proceed by contradiction: let $\Pi$ be a set of $n$ process names, and suppose that there is an algorithm $R$ for $\text{Cons}(\Pi, k)$ using the oracle $\mathcal{O}.\text{TAG}(\Pi, n - 1)$. Recall that the sanctuary of this oracle is $(k - 1)$-TAG($\Pi, n - 1$) itself (cf. Section I.3.3); to simplify notation, we let $\sigma = (k - 1)$-TAG($\Pi, n - 1$).

We fix some subset $\Pi_1 \subseteq \Pi$ of cardinality $k - 1$, and we denote $\Pi' = \Pi \setminus \Pi_1$. From $R$, we shall design an algorithm $A$ running on $\Pi'$, which uses no oracle. Then we shall prove that $A$ solves the task $\text{Cons}(\Pi', 1)$, which contradicts the impossibility of Consensus with one failure established by Fischer, Lynch, and Paterson [8] since $|\Pi'| = n - (k - 1) \geq 2$.

For each process $q$ in $\Pi'$, we define the automata $A(q)$ in the following way:

- the set of states of $A(q)$ is the same as the one of $R(q)$;
- the set of initial states of $A(q)$ is the same as the one of $R(q)$;
- each transition $(s_q, [q, m, \bot], s'_q)$ of $R(q)$ in which $q$ consults no oracle is also a transition of $A(q)$;
- each transition $(s_q, [q, m, 1], s'_q)$ of $R(q)$ in which the oracle answers 1 is removed;
- each transition $(s_q, [q, m, 0], s'_q)$ of $R(q)$ in which the oracle answers 0 is replaced by the transition $(s_q, [q, m, \bot], s'_q)$.

Note that all the steps in $A(q)$ are of the form $[q, m, \bot]$; in other words, the algorithm $A$ uses no oracle.

Let $\rho_A = < F, I, H >$ be any run of $A$. Since $A$ uses no oracle, each event in $H$ is of the form $e = (\beta, q, -, -, -)$ and is part of some transition $(s_q, [q, m, \bot], s'_q)$ of $A(q)$, where $m \in M \cup \{\text{null}\}$. In the construction of $A(q)$ described above, this transition results from some unique transition of $R(q)$, of the form $(s_q, [q, m, \bot], s'_q)$ or $(s_q, [q, m, 0], s'_q)$. In this
way, to each event in \( H \), we associate a unique transition of \( R(q) \) in which the oracle at sanctuary \( \sigma \) is not consulted or answers 0.

Now, to each run \( \rho_A = <F, I, H> \) of \( A \), we associate the triple \( \rho_R = <F', I', H'> \), where the failure pattern \( F' \) is defined by

\[
F' : t \in T \to F'(t) = F(t) \cup \Pi_1,
\]

the mapping \( I' \) by:

1. for any process \( q \in \Pi' \), \( I'(q) = I(q) \);
2. for any \( q \in \Pi_1 \), we let \( I'(q) = s^0_q \) if \( I(p) = s^0_p \) for some process \( p \in \Pi' \); otherwise we let \( I'(q) = s^1_q \);

and the sequence \( H' \) is constructed from \( H \) by the following rules:

1. any event in \( H \) that is associated to a transition of \( R \) in which the oracle is not consulted is left unchanged;
2. any event \( (\beta, q, t, R, m) \) in \( H \), even when associated to some transition in \( R(q) \) in which \( \sigma \) is consulted, is left unchanged;
3. an event \( (\beta, q, t, S, m) \) in \( H \) which is associated to some transition in \( R(q) \) of the form \( (s_q, [q, -, 0], s'_q) \), is replaced in \( H' \) by the three events series

\[
\langle (\sigma, q, t, Q, v), (\sigma, q, t, A, 0), (\beta, q, t, S, m) \rangle,
\]

where \( v \) is the query value determined by \( s_q \).

We claim that the triple \( \rho_R \) so defined is a run of \( R \). By construction of \( H' \), there is no event in \( H' \) whose process name is in \( \Pi_1 \), and each event in \( H' \) at time \( t \) corresponds to at least one event in \( H \) that also occurs at time \( t \). Since \( H \) is compatible with \( F \) and \( F'(t) = F(t) \cup \Pi_1 \), it follows that \( H' \) is compatible with \( F' \). For any process \( q \in \Pi' \), \( H'|q \) is well-formed, and so is \( H'|q \). This proves that \( H' \) satisfies R2.

From the R3, R4, and R6 conditions for \( H \), it is also immediate to prove that in turn \( H' \) satisfies R3, R4, and R6.

Now since \( F(t) \subseteq F'(t) \), every process \( q \) which is correct in \( F' \) is also correct in \( F \), and so takes an infinite number of steps in \( H \). By construction of \( H' \), it follows that \( q \) takes an infinite number of steps in \( H' \). Thus \( H' \) satisfies R5.

Finally, to show that \( \rho_R \) satisfies R1, we focus on a consultation of \( \sigma \) in \( H' \). By construction of \( H' \), the only value answered by the oracle at sanctuary \( \sigma \) is 0. This trivially enforces agreement. Since there are at least \( k - 1 \) faulty processes in \( F' \), the answer 0 is allowed for \( F' \) and any input vector \( \bar{V} \in \{0, 1\}^\Pi \) with regard to the \( (k-1) \)-validity condition. Besides, every step in \( H \) is complete (with a receipt and a state change), and so by construction of \( H' \), the oracle answers in each consultation of \( H' \). It follows that \( H'|\sigma \) is an history of the oracle \( \mathcal{O}.(k-1)-\text{TAg}(\Pi, n-1) \). This completes the proof that \( \rho_R = <F', I', H'> \) is a run of \( R \).

Let \( \rho_A \) be any run of \( A \) with at most one failure; in the corresponding run \( \rho_R \) of \( R \), at most \( 1 + (k - 1) = k \) processes fail. Since \( R \) is an algorithm that solves Cons(\( \Pi, k \)), \( \rho_R \) satisfies the termination, agreement, irrevocability and validity conditions of Consensus. It immediately follows that the run \( \rho_A \), which \( \rho_R \) stems from, also satisfies the termination, agreement, and irrevocability conditions. Moreover, by definition of \( I' \), if all processes start
with the same initial value \( v \) in \( \rho_A \), then they also have the same initial value \( v \) in \( \rho_R \); the only possible decision value in \( \rho_R \), and so in \( \rho_A \), is \( v \).

Consequently, \( A \) is an algorithm for Cons(\( \Pi' \), 1) using no oracle, a contradiction with \( \S \).

Notice that for \( k = 2 \), Theorem 5.3 states that Cons(\( \Pi \), 2) is not \( \CC \)-reducible to AC(\( n \), \( n - 1 \)), and so reduces to Theorem I.8.2.

Importantly, we may safely substitute the consistent oracle \( ^cO.(k-1)-Tag(\Pi, n-1) \) for \( O.(k-1)-Tag(\Pi, n-1) \) in the proof of Theorem 5.3. In this way, we prove a result slightly stronger than Theorem 5.3 by establishing that Cons(\( n \), \( k \)) is actually not \( ^c\CC \)-reducible to \( (k-1)-Tag(n, n-1) \).

**Corollary 5.4** For any integers \( k, f, n \) such that \( 1 \leq k \leq f - 1 \) and \( f \leq n - 1 \), Cons(\( n \), \( f \)) is not \( ^c\CC \)-reducible to \( k \)-Tag(\( n \), \( f \)).

**Proof:** Suppose, for the sake of contradiction, that Cons(\( n \), \( f \)) is \( ^c\CC \)-reducible to some \( k \)-Tag(\( n \), \( f \)) with \( 1 \leq k \leq f - 1 \). Since \( f \geq k + 1 \), Cons(\( n \), \( k+1 \)) is a special case of Cons(\( n \), \( f \)), and so

\[
\text{Cons}(n, k+1) \leq C \text{ Cons}(n, f).
\]

Similarly, \( k \)-Tag(\( n \), \( f \)) is a special case of \( k \)-Tag(\( n \), \( n - 1 \)), and we have

\[
k \text{-Tag}(n, f) \leq C k \text{-Tag}(n, n - 1).
\]

Using transitivity of the \( ^c\CC \)-reduction, we obtain that Cons(\( n \), \( k+1 \)) is \( ^c\CC \)-reducible to \( k \)-Tag(\( n \), \( n - 1 \)), a contradiction with the variant of Theorem 5.3 alluded above. \( \square \)

### 5.3 A \( \CC \)-irreducibility result when a majority of processes may fail

We now complete the comparison between the various \( k \)-Threshold Agreement tasks for a fixed set of processes \( \Pi \) and a fixed resiliency degree \( f \). We are going to prove that if a majority of processes may be faulty (\( |\Pi| \leq 2f \)), then Cons(\( \Pi \), \( f \)) is not \( \CC \)-reducible to \( f \)-Tag(\( \Pi \), \( f \)). Combining this latter irreducibility result with Theorem 5.1, we conclude that with respect to \( \leq \CC \), \( f \)-Tag(\( \Pi \), \( f \)) is incomparable with any of the equivalent tasks \( (f+1)-Tag(\Pi, f), \ldots, n \)-Tag(\( \Pi \), \( f \)) = Cons(\( \Pi \), \( f \)).

**Theorem 5.5** Let \( n \) and \( f \) be two integers such that \( 1 \leq f \leq n - 1 \). If \( n \leq 2f \), then the task \( (f+1)-Tag(n, f) \) is not \( \CC \)-reducible to \( f \)-Tag(\( n \), \( f \)), and so Cons(\( n \), \( f \)) is not \( \CC \)-reducible to \( f \)-Tag(\( n \), \( f \)).

**Proof:** Let \( \Pi \) be any set of \( n \) processes and let \( f \) denote an integer such that \( 1 \leq f \leq n - 1 \) and \( n \leq 2f \). Suppose, for the sake of contradiction that there is an algorithm \( R \) which solves \( (f+1)-Tag(\Pi, f) \) using the oracle \( O.f-Tag(\Pi, f) \).

We partition \( \Pi \) into two sets \( \Pi' \) and \( \Pi'' \) such that \( \Pi' \) contains \( f \) processes, and \( \Pi'' \) contains the remaining \( n - f \) processes. Since \( n > f \), we have \( f > 2f - n \); we fix any (possibly empty) strict subset \( \pi' \) of \( \Pi' \) with \( 2f - n \) processes.

Consider the triple \( \rho' = \langle F', I, H' \rangle \) where the failure pattern \( F' \) is defined by

\[
F' : t \in T \rightarrow F'(t) = \pi' \cup \Pi''.
\]

the mapping \( I \) by:
1. for any process \( p \in \Pi' \), \( I(p) = s^0_p \)
2. for any process \( p \in \Pi'' \), \( I(p) = s^1_p \)

and the sequence \( H' \) is constructed by induction on \( t \in T \), as follows.

First, \( H'[0,0] \) is defined to be the empty sequence. A queue of the processes in \( \Pi' \setminus \pi' \) is maintained, initially in an arbitrary order, and the messages in \( \beta \) are ordered according to the times the messages were sent, earliest first. Suppose that the finite history \( H'[0,t] \) is constructed. Let \( t^+ \) denote the successor of \( t \) in \( T \), and let \( q \) be the first process in the process queue. After \( H'[0,t] \), \( q \) may execute only one type \( T \) of event. There are three cases to consider:

1. \( T = S \) or \( T = Q \). The automaton \( R(q) \) entirely determines the event \( e = (\beta, q, t^+, S, m) \) or \( e = (f\text{-TAG}(\Pi, f), q, t^+, Q, v) \) that \( q \) may execute at time \( t^+ \).

2. \( T = R \). In this case, the message buffer \( \beta \) contains at least one message for \( q \). Then we let \( e = (\beta, q, t^+, R, m) \), where \( m \) denotes the earliest message for \( q \) in \( \beta \).

3. \( T = A \). We let \( e = (f\text{-TAG}(\Pi, f), q, t^+, A, 0) \).

The above procedure determines a unique event \( e \), and we let \( H'[0,t^+] = H'[0,t]; e \). Process \( q \) is then moved to the back of the process queue. This inductively defines \( H' \).

**Lemma 5.6** The triple \( \rho' = <F', I, H'> \) is a run of \( R \) in which every process in \( \Pi' \setminus \pi' \) decides 0.

**Proof:** By the definitions of \( F' \) and \( H' \), it is immediate that \( \rho' \) satisfies properties R2-6. For R1, the only non-trivial point is checking that \( H'|f\text{-TAG}(\Pi, f) \) satisfies the \( f \)-validity condition, or in other words that the oracle is always allowed to answer 0. For that, we just need to observe that any process in \( \pi' \cup \Pi'' \) takes no step in \( H' \), and never queries the oracle \( O\text{-TAG}(\Pi, f) \). Hence, \((n-f) + (2f-n) = f \) processes do not query the oracle, which is thus allowed to answer 0 with respect to the \( f\text{-TAG} \) specification. It follows that \( \rho' \) satisfies R1, and so is a run of \( R \).

Because \( R \) solves the task \((f+1)\text{-TAG}(\Pi, f) \) and the number of faulty processes in \( \rho' \) is \( f \), all the processes in \( \Pi' \setminus \pi' \) make the same decision in \( \rho' \). Let \( d' \) denote the common decision value in \( \rho' \).

Consider the mapping \( I_0 \) such that for any process \( p \in \Pi \), \( I_0(p) = s^0_p \). Since every process in \( \Pi'' \) initially crashes in the failure pattern \( F' \) and \( \rho' \) is a run of \( R \) with the decision value \( d' \), the triple \( <F', I_0, H'> \) is also a run of \( R \) in which \( f \) processes are faulty and the decision value is \( d' \). By the \((f+1)\)-validity condition, the decision value in this second run of \( R \) is equal to 0. Thus we derive that \( d' = 0 \). \( \square_{\text{Lemma 5.6}} \)

Let \( \theta' \) denote the first time when all processes in \( \Pi' \setminus \pi' \) have made a decision in \( \rho' \). Now, consider the triple \( \rho'' = <F'', I, H''> \) where the failure pattern \( F'' \) is defined by

\[ F'' : t \in T \rightarrow F''(t) = \Pi', \]

and the sequence \( H'' \) is constructed in the same way as \( H' \) with the additional requirement that the time of each event in \( H'' \) is greater than \( \theta' \). A proof similar to the one of Lemma 5.6 shows the following:
Lemma 5.7 The triple $\rho'' = < F'', I, H'' >$ is a run of $R$ in which every process in $\Pi''$ decides 1.

Let $\theta''$ denote the first time when all the processes in $\Pi''$ have made a decision in $\rho''$. For any $t \in T$, we let

$$F(t) = \emptyset \quad \text{when } 0 \leq t \leq \theta''$$

$$= \Pi' \quad \text{when } t > \theta''.$$

By construction, the time of each event in $H''$ is greater than $\theta'$, and so we may form the finite history $H'[0, \theta']; H''[0, \theta'']$.

Lemma 5.8 There exists an extension $H$ of $H'[0, \theta']; H''[0, \theta'']$ such that $< F, I, H >$ is a run of $R$.

Proof: The proof technique is similar to the one of Lemma 5.2. The history $H$ is constructed in stages, starting from $H'[0, \theta']; H''[0, \theta'']$; each stage consists in adding one event. A queue of the processes in $\Pi''$ is maintained, initially in an arbitrary order, and the messages in $\beta$ are ordered according to the times the messages were sent, earliest first.

Suppose that the finite history $H[0,t]$ extending $H'[0, \theta']; H''[0, \theta'']$ is constructed. Let $t^+$ denote the successor of $t$ in $T$, and let $q$ be the first process in the process queue. After $H[0, t]$, process $q$ may achieve only one type $T$ of event. There are three cases to consider:

1. $T = S$ or $T = Q$. The automaton $R(q)$ entirely determines the event $e = (\beta, q, t^+, S, m)$ or $e = (f$-TAG($\Pi$, $f), q, t^+, Q, v)$ which $q$ may achieve at time $t^+$.

2. $T = R$. In this case, the message buffer $\beta$ contains at least one message for $q$. Then we let $e = (\beta, q, t^+, R, m)$, where $m$ denotes the earliest message for $q$ in $\beta$.

3. $T = A$. In this latter case, we let $e = (f$-TAG($\Pi$, $f), q, t^+, A, 0)$.

The above procedure determines a unique event $e$, and we let $H[0, t^+] = H_0[0, t]; e$. Process $q$ is then moved to the back of the process queue.

This inductively defines $H$. By construction, the triple $\rho = < F, I, H >$ satisfies R2–6. Because every process in $\Pi'$ is faulty and $|\Pi'| = f$, the oracle $O.f$-TAG($\Pi$, $f$) is always allowed to answer 0 whatever query values are. This ensures that $\rho$ satisfies R1. Consequently $\rho$ is a run of $R$. □

So, we have just shown that any algorithm using $O.f$-TAG($\Pi$, $f$) for $(f + 1)$-TAG($\Pi$, $f$) would have a run in which processes in $\Pi' \setminus \pi'$ decide 0 and processes in $\Pi''$ decide 1. Since $f \leq n - 1$, both $\Pi' \setminus \pi'$ and $\Pi''$ are non-empty; we then conclude that this run violates the agreement property, a contradiction. □

Observe that the latter proof crucially relies on the fact that the oracle $O.f$-TAG($\Pi$, $f$) is allowed to answer on the grounds of informations concerning future failures: whatever the query values are, $O.f$-TAG($\Pi$, $f$) may answer 0 from the beginning in history $H$ whereas no process will crash before time $\theta''$. In the next section, we shall actually prove that this $C$-irreducibility result does not hold anymore when considering oracles that do not see into the future.
Algorithm for process \( p \):

\[
\begin{align*}
\text{Query} & (\ast \mathcal{O}, f\text{-TAg}(\Pi, f))\langle v_p \rangle \\
\text{Answer} & (\ast \mathcal{O}, f\text{-TAg}(\Pi, f))\langle v \rangle \\
\text{Query} & (\ast \mathcal{O}, f\text{-TAg}(\Pi, f))\langle 1 \rangle \\
\text{Answer} & (\ast \mathcal{O}, f\text{-TAg}(\Pi, f))\langle d \rangle \\
\text{if } d = 1 & \text{ then } \text{Decide}(v) \\
\text{else } & \text{Send} \langle v_p \rangle \text{ to all} \\
& \text{wait until } [\text{Receive} \langle \ast \rangle \text{ from } n - f \text{ processes}] \text{ (where } \ast \text{ can be 0 or 1)} \\
& \text{if at least one of the received values is 0} \\
& \text{then } \text{Decide}(0) \\
& \text{else } \text{Decide}(1)
\end{align*}
\]

Figure 4: A \( ^*C \)-reduction from \((f+1)\text{-TAg}(\Pi, f)\) to \(f\text{-TAg}(\Pi, f)\)

\section{5.4 Sham oracles}

We now describe an algorithm that solves \((f+1)\text{-TAg}(\Pi, f)\) with the help of some \( f \)-resilient oracle suitable for the agreement problem \( f\text{-TAg} \) for \( \Pi \) which does not see into the future. This reduction algorithm works for any resiliency degree, even when a majority of processes may fail, and so this mitigates the irreducibility result in Theorem 5.5 above.

First, we formally define such oracles. Let \( \Pi \) be a set of processes, and let \( F \) be any failure pattern for \( \Pi \). For any \( \theta \in T \), \( F_\theta \) denotes the failure pattern for \( \Pi \) defined by

\[
F_\theta : t \in T \rightarrow \begin{cases} 
F(t) & \text{if } 0 \leq t \leq \theta \\
F(\theta) & \text{otherwise.}
\end{cases}
\]

\textbf{Definition 5.9} Let \( \mathcal{O}_\sigma \) be an oracle whose set of consultants is \( \Pi \). We say that \( \mathcal{O}_\sigma \) is a sham oracle if for any failure pattern \( F \) for \( \Pi \), any history \( H \in \mathcal{O}_\sigma(F) \), and any time \( \theta \in T \), there exists an extension \( H' \) of \( H[0, \theta] \) such that \( H' \in \mathcal{O}_\sigma(F_\theta) \).

As for consistent oracles, for every task \( T = (P, f) \), we define the sham version of the oracle for \( T \), denoted \( ^*\mathcal{O}.T \), as the most general sham oracle which is \( f \)-resilient and suitable for \( P \). For any failure pattern \( F \) for \( \Pi \), we have \( ^*\mathcal{O}.T(F) \subseteq \mathcal{O}.T(F) \). In other words, \( ^*\mathcal{O}.T \) responses with less scope than \( \mathcal{O}.T \), and thus the answers given by \( ^*\mathcal{O}.T \) may be thought as more precise than the ones given by \( \mathcal{O}.T \). So, the sham oracle \( ^*\mathcal{O}.T \) is “at least as powerful as” \( \mathcal{O}.T \) in the sense that any algorithm using \( \mathcal{O}.T_2 \) for some task \( T_1 \) still solves \( T_1 \) when using \( ^*\mathcal{O}.T_2 \).

This leads to a new notion of reduction à la Cook, called \( ^*C \)-reduction and denoted \( \leq_{^*C} \), in which algorithms may only use sham oracles. Formally, \( T_1 \leq_{^*C} T_2 \) if there is an algorithm for \( T_1 \) using the sham oracle \( ^*\mathcal{O}.T_2 \). Since \( ^*\mathcal{O}.T \) is at least as powerful as \( \mathcal{O}.T \), \( C \)-reducibility implies \( ^*C \)-reducibility.
Theorem 5.10 Let \( n \) and \( f \) be two integers such that \( 1 \leq f \leq n - 1 \), and let \( \Pi \) be a set of \( n \) processes. The algorithm in Figure 4 that uses \( ^*O.f.TAg(\Pi, f) \) solves \((f+1).TAg(\Pi, f)\), and so \((f+1).TAg(n, f)\) is \(^cC\)-reducible to \( f.TAg(n, f)\).

Proof: First notice that Claims O1 and O2 also hold for any sham oracle \(^*O.k.TAg(\Pi, f)\).

Let \( \Pi \) be a set of \( n \) processes, and let \( \rho = <F, I, H> \) denote a run of the algorithm in Figure 4 with at most \( f \) failures. By the \( f \)-resiliency property, the oracle \(^*O.f.TAg(\Pi, f)\) definitely answers in each of its two consultations in \( \rho \). Let \( \theta_1 \) and \( \theta_2 \) denote the first time when \(^*O.f.TAg(\Pi, f)\) answers in the first and second consultation, respectively.

Termination and irrevocability are obvious. To prove that \( \rho \) satisfies the agreement and \((f+1)\)-validity conditions, we shall distinguish the cases \( d = 0 \) and \( d = 1 \) as we did for the \(^cC\)-reduction in Theorem 3.4

Case \( d = 1 \). In this case, it is immediate that \( \rho \) satisfies the agreement condition. Now we prove that \( \rho \) satisfies the \((f+1)\)-validity condition.

1. Suppose that at least \( f+1 \) processes start with 0. Concerning the first consultation of \(^*O.f.TAg(\Pi, f)\), each of these processes which starts with 0 either does not query the oracle (because it crashes) or queries it with value 0. By Claim O1, the first response given by \(^*O.f.TAg(\Pi, f)\) is necessarily equal to 0, and processes decide 0, as required.

2. Now assume that all the initial values are equal to 1. By Claim O1, at most \( f - 1 \) processes do not query \(^*O.f.TAg(\Pi, f)\) twice, since the second answer \( d \) given by the oracle is 1. From the definition of a well-formed oracle history, it follows that no process queries \(^*O.f.TAg(\Pi, f)\) for the second time before time \( \theta_1 \). Thus, at most \( f - 1 \) processes crash by time \( \theta_1 \) in \( F \), and so

\[
|\text{Faulty}(F_{\theta_1})| \leq f - 1.
\]

Because \(^*O.f.TAg(\Pi, f)\) is a sham oracle, it is not allowed to answer 0 in the first consultation. Therefore, \( v = 1 \) and processes decide 1 in \( \rho \).

Case \( d = 0 \). Since every query value is 1 in the second consultation of \(^*O.f.TAg(\Pi, f)\), and \(^*O.f.TAg(\Pi, f)\) is a sham oracle, we do know that in this case, at least \( f \) processes have crashed by time \( \theta_2 \). At most \( f \) processes are faulty in \( F \), and so exactly \( f \) processes have crashed by time \( \theta_2 \). Thus, every correct process receives the same set of \( n - f \) initial values. This ensures agreement. We easily check that \( \rho \) satisfies the \((f+1)\)-validity condition. \( \square \)

Combining Theorems 5.10 and 5.11 we derive that among agreement tasks, the \( C \)-reduction defines a strictly finer hierarchy than the \(^*C\)-reduction. In other words, the sham oracle \(^*O.T\) is in general more powerful than \( O.T\): to be unable to see into the future actually helps to make a decision!

Note that the two oracles \(^cO.T\) and \(^*O.T\) are generally not comparable. However, we easily check that the reduction from \((f+1).TAg(\Pi, f)\) to \( f.TAg(\Pi, f)\) described in Section 5.9 still works when the reduction algorithm uses \(^*O.f.TAg(\Pi, f)\) instead of \(^cO.f.TAg(\Pi, f)\): the fact that \(^*O.f.TAg(\Pi, f)\) does not see into the future makes it sufficiently consistent and ensures that it cannot answer 0 and then 1 with all the query values equal to 1. Conversely, note that the reduction described above does not work when substituting \(^cO.f.TAg(\Pi, f)\) for \(^*O.f.TAg(\Pi, f)\).
6 Wait-freedom vs. f-resiliency for Consensus tasks

In the previous section, we have established various irreducibility results between pairs of k-Threshold Agreement tasks, only one of which is a Consensus task. Relying on these results, we are now in position to establish C-irreducibility results between Consensus tasks only. More precisely, we shall derive two C-irreducibility results between wait-free and f-resilient Consensus tasks first for a fixed set of processes, and then for a fixed resiliency degree. In both cases, we shall show that with respect to C-reduction, wait-free Consensus is strictly harder to solve than (non wait-free) f-resilient Consensus. We shall discuss the relationship between our results and previous ones established in the message passing model (6, 5) and in the shared objects model (2, 1, 3, 4).

6.1 Wait-freedom and f-resiliency for a fixed set of processes

**Theorem 6.1** For any integers \( n \) and \( f \) such that \( 1 \leq f \leq n-2 \), Cons\((n, f+1)\) is not C-reducible to Cons\((n, f)\).

**Proof:** Suppose, for the sake of contradiction, that for some integers \( n, f \) such that \( 1 \leq f \leq n-2 \), we have Cons\((n, f+1)\) \(\leq_{C}\) Cons\((n, f)\). We distinguish the following two cases:

1. \( n \leq 2(f+1) \).
   
   The task Cons\((n, f)\) is trivially a special case of Cons\((n, f+1)\), and so Cons\((n, f)\) C-reduces to Cons\((n, f+1)\). By Proposition 2.1, the task \((f+1)\)-TAg\((n, f)\) is a generalization of Cons\((n, f)\); consequently Cons\((n, f)\) C-reduces to \((f+1)\)-TAg\((n, f)\).
   
   In turn, \((f+1)\)-TAg\((n, f)\) is a special case of \((f+1)\)-TAg\((n, f+1)\), and so \((f+1)\)-TAg\((n, f)\) C-reduces to \((f+1)\)-TAg\((n, f+1)\). By transitivity of \(\leq_{C}\), it follows that
   
   \[
   \text{Cons}(n, f+1) \leq_{C} (f+1)\text{-TAg}(n, f+1),
   \]
   
   which contradicts Theorem 5.5.

2. \( n > 2(f+1) \).
   
   A fortiori we have \( n > 2f \), and by Theorem 3.4 it follows that Cons\((n, f)\) is \(\leq_{C}\)-reducible to \(f\)-TAg\((n, f)\). This latter task is trivially \(\leq_{C}\)-reducible to \(f\)-TAg\((n, f+1)\) since it is a special case of \(f\)-TAg\((n, f+1)\). By transitivity of \(\leq_{C}\), it follows that
   
   \[
   \text{Cons}(n, f+1) \leq_{C} f\text{-TAg}(n, f+1),
   \]
   
   which contradicts Corollary 5.4.

\(\square\)

A straightforward consequence of Theorem 6.1 is the following.

**Corollary 6.2** For any integers \( n \) and \( f \) such that \( 1 \leq f \leq n-2 \), the wait-free Consensus task Cons\((n, n-1)\) is of higher degree of unsolvability than the f-resilient Consensus task Cons\((n, f)\) with respect to \(\leq_{C}\).
6.2 Failure-Information reduction

At this stage, it is worthy to compare Theorem 6.1 with a spinoff of the main results by Chandra, Hadzilacos and Toueg in [6, 5] concerning failure detectors solving Consensus tasks. In light of these two papers, it appears that all the Consensus tasks with a majority of correct processes require the same information about failures to be solved. Another way to say the same thing is that the weakest failure detectors for solving the various tasks Cons(n, 1), . . . , Cons(n, ⌈n/2⌉ − 1) are identical. From this standpoint, all these Consensus tasks are thus equivalent.

This can be formalized by introducing a new notion of reduction – quite different from the notions of reduction à la Karp and à la Cook that we have studied up to now – which will measure the hardness to solve a task in terms of the information about failures that is required for solving the task. Using the notation and the definitions of the formal model of failure detectors in [6], we formally capture this notion of Failure-Information reduction in the following definition.

**Definition 6.3** Let $T_1$ and $T_2$ be two tasks for a set $\Pi$ of processes. We say that $T_1$ is FI-reducible to $T_2$, and we note $T_1 \leq_{FI} T_2$, if any failure detector $D$ which can be used to solve $T_2$ can also be used to solve $T_1$.

Since a task is solvable iff it is solvable using the trivial failure detector $D_0$, we immediately derive the following proposition.

**Proposition 6.4** If $T_1$ FI-reduces to $T_2$ and $T_2$ is a solvable task, then $T_1$ is solvable.

Moreover, FI-reduction is reflexive and transitive. Consequently, as discussed in Part I for our previous notions of reduction, it makes sense to order tasks with respect to their “FI-difficulty”, that is with respect to $\leq_{FI}$.

In particular, we define two tasks $T_1$ and $T_2$ to be FI-equivalent ($T_1 \equiv_{FI} T_2$) when $T_1 \leq_{FI} T_2$ and $T_2 \leq_{FI} T_1$. As an immediate consequence of the reflexiveness and transitivity of $\leq_{FI}$, the relation $\equiv_{FI}$ is an equivalence relation.

Arguing as in the proof of Proposition I.6.2, we obtain that $C$-reducibility implies $FI$-reducibility: if $T_1 \leq_C T_2$ then $T_1 \leq_{FI} T_2$.

As a consequence of the main results in [6, 5], we are going to prove that for a fixed set of processes, all the Consensus tasks with a majority of correct processes have the same degree of unsolvability with respect to $\leq_{FI}$.

**Proposition 6.5** For every integer $n$, $n \geq 3$, the tasks Cons(n, 1), . . . , Cons(n, ⌈n/2⌉ − 1) are all FI-equivalent.

**Proof:** Since Cons(n, f + 1) trivially generalizes Cons(n, f), Cons(n, f) is FI-reducible to Cons(n, f + 1). In particular, we have

$$\text{Cons}(n, 1) \leq_{FI} \cdots \leq_{FI} \text{Cons}(n, \lceil n/2 \rceil − 1).$$

Conversely, suppose that some failure detector $D$ can be used to solve Cons(n, 1). By [5], we know that $D$ is at least as strong as the failure detector $\Omega$. (Recall that $\Omega$ is the most

---

4 The trivial failure detector $D_0$ is the function that maps each failure pattern $F$ to the singleton $\{H_0\}$, where $H_0$ is the failure detector history such that for any time $t \in T$ and any process $p \in \Pi$, $H_0(p, t) = \emptyset$. In other words, $D_0$ never suspects any process.

5 Here, we refer to the partial ordering on failure detectors defined in [6].
general failure detector such that eventually, all the correct processes always trust the same

correct process.) Moreover, Theorem 3 in [3] asserts that Cons\((n,\lceil n/2 \rceil - 1)\) is solvable

using \(\Omega\), and so using \(\mathcal{D}\). This shows that Cons\((n,\lceil n/2 \rceil - 1)\) is FI-reducible to Cons\((n,1)\),
i.e.,

\[
\text{Cons}(n, \lceil n/2 \rceil - 1) \leq_{FI} \text{Cons}(n,1).
\]

From [2] and [3], it follows that all the tasks Cons\((n,1),\ldots,\text{Cons}(n,\lceil n/2 \rceil - 1)\). are FI-
equivalent.

Together with Theorem 6.1, Proposition 6.5 shows that the C-hierarchy is strictly finer
than the FI-hierarchy. In other words, the minimal information about failures required to
solve a task – or equivalently the weakest failure detector needed to solve it (if it exists) –
does not fully capture the hardness to solve the task.

6.3 Wait-freedom and f-resiliency for a fixed resiliency degree

We now prove that for a fixed resiliency degree, the smaller the set of processes is, the
harder the Consensus tasks are. In particular, the wait-free Consensus task Cons\((f+1,f)\) is
of higher unsolvability degree than any non-wait free \(f\)-resilient Consensus task Cons\((n,f)\)
with respect to \(\leq_C\).

The proof is by a “meta-reduction” to the result in the previous section between wait-
free and \(f\)-resilient Consensus tasks for a fixed set of processes: we shall show that from
any hypothetical \(C\)-reduction from Cons\((n,f)\) to Cons\((n+1,f)\), we might construct a \(C\-
reduction from Cons\((n,f)\) to Cons\((n,f-1)\).

**Theorem 6.6** For any integers \(n\) and \(f\) such that \(1 \leq f \leq n - 1\), Cons\((n,f)\) is not \(C\-
reducible to Cons\((n+1,f)\).

**Proof:** Suppose, for the sake of contradiction, that there is an algorithm \(R^0\) for Cons\((n,f)\)
using the oracle \(\mathcal{O}\).Cons\((n+1,f)\). From \(R^0\), we shall construct an algorithm \(R\) that also
solves Cons\((n,f)\) but using the oracle \(\mathcal{O}\).Cons\((n,f-1)\), which contradicts Theorem 6.1.

Let us recall that the sanctuary of \(\mathcal{O}.T\) is \(T\) (cf. Section I.3.3). We denote

\[
\sigma^0 = \text{Cons}(n+1,f) \quad \text{and} \quad \sigma = \text{Cons}(n,f-1)
\]

the sanctuaries of \(\mathcal{O}\).Cons\((n+1,f)\) and \(\mathcal{O}\).Cons\((n,f-1)\), respectively. Let \(R\) be the algorithm
using the oracle of sanctuary \(\sigma\) such that, for any process \(p \in \{1,\ldots,n\}\), the automaton
\(R(p)\) coincides with \(R^0(p)\). We claim that \(R\) solves Cons\((n,f)\).

Let \(\rho = <F,I,H>\) be any run of \(R\). From \(H\), we construct a sequence \(H^0\) of events
as follows: \(H^0\) is identical to \(H\) except for events of type \((\sigma, p, t, Q, v)\) and \((\sigma, p, t, A, d)\)
which are replaced by \((\sigma_0, p, t, Q, v)\) and \((\sigma_0, p, t, A, d)\), respectively. In other words, \(H^0\) is
obtained from \(H\) by just substituting \(\sigma^0\) for \(\sigma\). Let us now consider \(\rho^0 = <F,I,H^0>\); we argue
that \(\rho^0\) is a run of \(R^0\).

Since \(\rho\) satisfies R2 and R3, the run \(\rho^0\) also satisfies R2 and R3 by construction of \(H^0\).
For every process \(p\), the automata \(R(p)\) and \(R^0(p)\) are identical, and so \(\rho^0\) satisfies R4 as \(\rho\)
does. From the definition of \(\rho^0\), we have Locked\((\rho^0) = \text{Locked}(\rho)\); it follows that \(\rho^0\) also
satisfies R5 and R6.

It remains to prove that \(\rho^0\) satisfies R1. We use the same notation as the one introduced
in Section I.4.2. In particular, we have

\[
F_{\sigma^0} = F \cup \{n+1\} \quad \text{and} \quad F_{\sigma} = F.
\]
Clearly, $H^0 | \sigma^0$ is well-formed and compatible with $F_{\sigma^0}$ as $H | \sigma$ is with $F_{\sigma}$. The more delicate point to prove is that $H^0 | \sigma^0$ is indeed a history of the oracle $\mathcal{O}$.Cons$(n + 1, f)$, i.e., $H^0 | \sigma^0 \in \mathcal{O}$.Cons$(n + 1, f)(F_{\sigma^0})$. For that, consider any consultation $H^0_k$ in $H^0$ of sanctuary $\sigma_0$; it naturally corresponds to a single consultation $H_k$ of $\sigma$ in $H$ with the same queries and responses as in $H^0_k$. Agreement in $H_k$ ensures agreement in $H^0_k$. For validity, form the query vector $\bar{W}$ for $H^0_k$ and let $\bar{V}^0$ be any extension of $\bar{W}$ in $\{0, 1\}^{\{1, \ldots, n + 1\}}$. The projection $\bar{V}$ of $\bar{V}^0$ onto $\{0, 1\}^{\{1, \ldots, n\}}$ is an extension of $\bar{W}$ in $\{0, 1\}^{\{1, \ldots, n\}}$. Since $\bar{W}$ is also the query vector in $H_k$, it follows that any decision $d$ in $H^0_k$ — which is also a decision value in $H_k$ — is allowed by the $\text{Cons}_{\{1, \ldots, n\}}$ specification, that is

$$d \in \text{Cons}_{\{1, \ldots, n\}}(F_{\sigma}, \bar{V}).$$

By definition of the Consensus mappings (cf. Section I.2.2), we have

$$\text{Cons}_{\{1, \ldots, n\}}(F_{\sigma}, \bar{V}) \subseteq \text{Cons}_{\{1, \ldots, n + 1\}}(F_{\sigma^0}, \bar{V}^0)$$

since $\bar{V}^0$ is an extension of $\bar{V}$. It follows that

$$d \in \text{Cons}_{\{1, \ldots, n + 1\}}(F_{\sigma^0}, \bar{V}^0).$$

This shows that $H^0_k$ satisfies the $\text{Cons}_{\{1, \ldots, n + 1\}}$-validity condition. Moreover, the number of queries in $H^0_k$ is the same as in $H_k$. Therefore if there are at least $(n + 1) - f$ queries in $H^0_k$, then there are at least $n - (f - 1)$ queries in $H_k$, and the oracle $\mathcal{O}$.Cons$(n, f - 1)$ necessarily answers in $H_k$. It follows that any consultation in $H^0_k$ with at least $(n + 1) - f$ queries contains a response to any correct process in $F_{\sigma^0}$. Hence, $H^0 | \sigma^0$ is a history of $\mathcal{O}$.Cons$(n + 1, f)$.

This shows that $\rho^0$ is a run of $R^0$. As $R^0$ solves the task $\text{Cons}(n, f)$, if at most $f$ processes are faulty in $F$, then $\rho^0$ satisfies the termination, irrevocability, agreement and $\text{Cons}_{\{1, \ldots, n\}}$-validity conditions. Since $\rho$ and $\rho^0$ are identical up to a renaming of $\sigma$ into $\sigma^0$, $\rho$ also satisfies these conditions. Therefore $R$ solves $\text{Cons}(n, f)$ using the oracle $\text{Cons}(n, f - 1)$, a contradiction with Theorem 5.5.

\textbf{Corollary 6.7} For any integers $n$ and $f$ such that $1 \leq f \leq n - 2$, the wait-free Consensus task $\text{Cons}(f + 1, f)$ is not $C$-reducible to the $f$-resilient Consensus task $\text{Cons}(n, f)$.

\textbf{Proof:} Let us assume, for the sake of contradiction, that $\text{Cons}(f + 1, f)$ is $C$-reducible to $\text{Cons}(n, f)$ for some integer $n$, $n \geq f + 2$. By repeated applications of Proposition I.7.4 and transitivity of $C$-reduction, we obtain that $\text{Cons}(n - 1, f)$ is $C$-reducible to $\text{Cons}(f + 1, f)$, and so $\text{Cons}(n - 1, f)$ is $C$-reducible to $\text{Cons}(n, f)$, which contradicts Theorem 6.6. \hfill \Box

\subsection{6.4 Related work: reducibility and unsolvability}

At first sight, Theorem 6.6 and Corollary 6.7 conflict with Borowsky and Gafni’s simulation [2, 11], and more specifically with prior work for Consensus tasks by Lo and Hadzilacos [10], and by Chandra, Hadzilacos, Jayanti, and Toueg [3, 4].

Recall that Borowsky and Gafni’s simulation consists in a general algorithm in the shared memory model which allows a set of $f + 1$ processes with at most $f$ crash failures to simulate any larger set of $n$ processes also with at most $f$ crashes. Its variant for Consensus tasks [3] provides a transformation of algorithms that solve the $f$-resilient Consensus task
for \( n \) processes using read/write registers into algorithms that solve the wait-free Consensus task for \( f+1 \) processes or using registers also.\(^6\) (The easily established unsolvability of Cons\((f+1, f)\) therefore entails the unsolvability of Cons\((n, f)\) in the shared memory model.)

We could think to explain the discrepancy between the existence of such an algorithm transformation and our irreducibility statement in Corollary \([14]\) by the fact that the message passing and the shared memory models are precisely not equivalent here (a majority of processes may fail in the wait-free case). However, a closer look at this transformation reveals that this discrepancy actually results from a more fundamental point which is worth being underlined.

Indeed the transformation works as follows. Consider any algorithm for the task Cons\((n, f)\) using registers, and let us fix a set of \( f+1 \) processes. The instructions in the \( n \) codes are distributed over the \( f+1 \) processes in a fair fashion way, and one by one. The key point is that the cooperation between processes that is necessary for a correct execution of the whole code for Cons\((n, f)\) can be achieved by the processes themselves using registers only. Translating this transformation in terms of oracle-based reductions would require that processes may access the internal mechanism of the oracle for Cons\((n, f)\) for sharing it between them. Basically, this is opposed to the notion of oracles which are closed black boxes that cannot be opened and dismantled.

The same argument explains the apparent contradiction between another prior work about Consensus tasks in the shared memory model and the results established in the previous section: In [10], Lo and Hadzilacos show how to convert any algorithm that solves the one-resilient Consensus task for \( n \) processes using some set of object types \( S \) into an algorithm that solves the one-resilient Consensus task for \( n-1 \) processes using the same set of types \( S \), when \( n \) is greater than 3. That contradicts an immediate spinoff of Theorem \([6.6]\) which states that Cons\((n-1, 1)\) is not C-reducible to Cons\((n, 1)\). The techniques used in [3] and [10] are similar, and the schemes of the two key transformations of Consensus algorithms are identical. As a matter of fact, Lo and Hadzilacos’s transformation, like the one in [3], corresponds to no oracle-based reduction in the asynchronous message passing model.

At that point, one might argue that the notion of oracle-based reducibility is too strong to capture such algorithm transformations, and so is not really useful. However, as in the classical theory of computation, oracles have been introduced for the purpose of classifying undecidable/unsolvable problems/tasks. Indeed, any reduction whose formal definition is a condition quantified over algorithms instead of oracles, of the type

\[
(\ast) \text{T}_1 \text{ is reducible to } \text{T}_2 \text{ if any algorithm solving } \text{T}_2 \text{ can be “transformed” into some algorithm solving } \text{T}_1, \\
\]

is trivial in the class of unsolvable tasks, since the above condition is tautologically satisfied by any task \( T_1 \) when the task \( T_2 \) is unsolvable. This observation may be applied to the pair of tasks \( T_1 = \text{Cons}(f+1, f) \) and \( T_2 = \text{Cons}(n, f) \), or to \( T_1 = \text{Cons}(n-1, 1) \) and \( T_2 = \text{Cons}(n, 1) \), and finally shows that the transformations in [3] [10], which actually lead to unsolvability results, however correspond to no meaningful reduction, oracle-based or of the type \((\ast)\).

This discussion illustrates the difficulty in introducing significant and well-defined notions of reducibility relating unsolvable distributed tasks. Above all, any such reducibility notion should correspond to a hierarchy on distributed tasks, on the model of the Turing (resp. the Cook) hierarchy on problems, the solvable tasks playing the role of decidable

\(^6\)Actually, Chandra et al. transformation works for any set of object types including read/write registers.
(resp. polynomial-time decidable) problems. The oracle-based notions of reducibility that we have introduced in this paper, especially the $C^*$- and $C^{**}$-reductions, give rise to non-trivial and sometimes unexpected results relating diverse classical distributed tasks, and qualify as appropriate counterparts of the Turing and Cook reductions in the framework of distributed computing.

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