Determinants of Dirac operators with local boundary conditions. *

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Abstract

We study functional determinants for Dirac operators on manifolds with boundary. We give, for local boundary conditions, an explicit formula relating these determinants to the corresponding Green functions. We finally apply this result to the case of a bidimensional disk under bag-like conditions.

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I Introduction

It is well known that functional determinants have wide application in Quantum and Statistical Physics. Typically, one faces the necessity of defining a regularized determinant for elliptic differential operators. In this context, the Dirac first order differential operator plays a central role.

Seeley’s construction of complex powers of elliptic differential operators provides a powerful tool to regularize such determinants: the so called ζ-function method [1].

In the case of boundaryless manifolds, this construction has been largely studied and applied (see, for instance, [2] and references therein).

For manifolds with boundary, the study of complex powers was performed in [3, 4] for the case of local boundary conditions, while for the case of nonlocal conditions, this task is still in progress (see, for example, [5]).

In general, the regularized determinant turns out to be nonlocal and so, it cannot be expressed in terms of just a finite number of Seeley’s coefficients. However, such determinant can always be obtained from the Green function in a finite number of steps involving these coefficients. For boundaryless manifolds this was proved in [6], while for a particular type of local boundary conditions the procedure was introduced in [7].

The aim of this paper is to give the explicit relationship between determinants and the corresponding Green functions of Dirac operators under general local elliptic boundary conditions.

Dirac operators defined on manifolds with boundaries have been the subject of a vast literature (see, for instance, [8, 9] and references therein), mainly concerning anomalies and index theorems. But, in these papers, the emphasis was put on nonlocal boundary conditions of the type introduced in [10]. We leave for a forthcoming publication the treatment of such conditions.

The outline of this paper is as follows:

In Section II we introduce some general definitions and conventions, concerning elliptic boundary problems for Dirac operators.

In Section III we present a formula relating the determinant of the Dirac operator with its Green function, for the case of local boundary conditions.

In Section IV, an explicit computation of the determinant of a Dirac operator in a bidimensional disk with bag-like boundary conditions is performed, making use of the results in section III.
II Elliptic boundary problems, complex powers and regularized determinants

Throughout this paper we will be concerned with boundary value problems associated to first order elliptic operators

\[ D : C^\infty(M, E) \to C^\infty(M, F), \]  

where \( M \) is a bounded closed domain in \( \mathbb{R}^\nu \) with smooth boundary \( \partial M \), and \( E \) and \( F \) are \( k \)-dimensional complex vector bundles over \( M \).

In a collar neighborhood of \( \partial M \) in \( M \), we will take coordinates \( \bar{x} = (x, t) \), with \( t \) the inward normal coordinate and \( x \) local coordinates for \( \partial M \) (that is, \( t > 0 \) for points in \( M \setminus \partial M \) and \( t = 0 \) on \( \partial M \) ), and conjugated variables \( \bar{\xi} = (\xi, \tau) \).

As stated in the Introduction, we will mainly consider the Euclidean Dirac operator. Let us recall that the free Euclidean Dirac operator \( i \partial \) is defined as

\[ i \partial = \nu^{-1} \sum_{\mu=0}^{\nu-1} i \gamma_\mu \frac{\partial}{\partial x_\mu}, \]  

where the matrices \( \gamma_\mu \) satisfy

\[ \gamma_\mu \gamma_\alpha + \gamma_\alpha \gamma_\mu = 2 \delta_{\mu \alpha}, \]  

and that, given a gauge potential \( A = \{ A_\mu, \mu = 0, \ldots, \nu - 1 \} \) on \( M \), the coupled Dirac operator is defined as

\[ D(A) = i \partial + A \]  

with \( A = \sum_{\mu=0}^{\nu-1} \gamma_\mu A_\mu \).

One of the most suitable tools for studying boundary problems is the Calderón projector \( Q \) \[11, 12\]. For the case we are interested in, \( D \) of order 1 as in (1), \( Q \) is a (not necessarily orthogonal) projection from \( L^2(\partial M, E/\partial M) \) onto the subspace \( \{ (T \varphi / \varphi \in \ker(D)) \} \), being \( T : C^\infty(M, E) \to C^\infty(\partial M, E/\partial M) \) the trace map.

As shown in \[11\], \( Q \) is a zero-th order pseudodifferential operator and its principal symbol \( q(x; \xi) \), that depends only on the principal symbol of
\( D, \sigma_1(D) = a_1(x, t; \xi, \tau) \), turns out to be the \( k \times k \) matrix

\[
q(x; \xi) = \frac{1}{2\pi i} \int_{\Gamma} \left( a_1^{-1}(x, 0; 0, 1) a_1(x, 0; \xi, 0) - z \right)^{-1} dz,
\]

where \( \Gamma \) is any simple closed contour oriented clockwise and enclosing all poles of the integrand in \( \text{Im}(z) < 0 \).

\( Q \) is not unique, since it can be constructed from any fundamental solution of \( D \), but its principal symbol \( q(x; \xi) \) is uniquely determined \(^{[1]}\).

According to Calderón \(^{[1]}\) and Seeley \(^{[12]}\), elliptic boundary conditions can be defined in terms of \( q(x; \xi) \).

**Definition 1:**

Let us assume that the rank of \( q(x; \xi) \) is a constant \( r \) (as is always the case for \( \nu \geq 3 \) \(^{[1]}\)).

A zero order pseudodifferential operator \( B : [L^2(\partial M, E_{/\partial M})] \rightarrow [L^2(\partial M, G)] \), with \( G \) an \( r \) dimensional complex vector bundle over \( \partial M \), gives rise to an *elliptic boundary condition* for a first order operator \( D \) as in \(^{[1]}\) if, \( \forall \xi : |\xi| \geq 1 \),

\[
\text{rank}(b(x; \xi) \ q(x; \xi)) = \text{rank}(q(x; \xi)) = r,
\]

where \( b(x; \xi) \) coincides with the principal symbol of \( B \) for \( |\xi| \geq 1 \).

In this case we say that

\[
\left\{ \begin{array}{l}
D \varphi = \chi \text{ in } M \\
BT \varphi = f \text{ on } \partial M
\end{array} \right.
\]

is an *elliptic boundary problem*, and denote by \( D_B \) the closure of \( D \) acting on the sections \( \varphi \in C^\infty(M, E) \) satisfying \( B(T\varphi) = 0 \).

An elliptic boundary problem as \(^{[7]}\) has a solution \( \varphi \in H^1(M, E) \) for any \((\chi, f)\) in a subspace of \( L^2(M, E) \times H^{1/2}(\partial M, G) \) of finite codimension. Moreover, this solution is unique up to a finite dimensional kernel \(^{[1]}\). In other words, the operator

\[
(D, BT) : H^1(M, E) \rightarrow L^2(M, E) \times H^{1/2}(\partial M, G)
\]

is Fredholm.

When \( B \) is a local operator, Definition 1 yields the classical local elliptic boundary conditions, also called Lopatinsky-Shapiro conditions (see for instance \(^{[13]}\)).
For Euclidean Dirac operators on $\mathbb{R}^n$, $E_{\partial M} = \partial M \times \mathbb{C}^k$, local boundary conditions arise when the action of $B$ is given by the multiplication by a $\frac{k}{2} \times k$ matrix of functions defined on $\partial M$.

Owing to topological obstructions, chiral Dirac operators in even dimensions, $D$, do not admit local elliptic boundary conditions (see for example [14]). Nevertheless, it is easy to see from Definition 1 that local boundary conditions can be defined for the full, either free or coupled, Euclidean Dirac operator

$$D(A) = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$$

on $M$.

We now sketch Seeley’s construction of the complex powers of the operator $D$ under local elliptic boundary condition $B$ [3, 4, 15].

**Definition 2:**

The elliptic boundary problem (7) admits a cone of Agmon’s directions if there is a cone $\Lambda$ in the $\lambda$-complex plane such that

1) $\forall \bar{x} \in M$, $\forall \bar{\xi} \neq 0$, $\Lambda$ contains no eigenvalues of the matrix $\sigma_1(D)(\bar{x}, \bar{\xi})$ .

2) $\forall \xi : |\xi| \geq 1$, $\text{rank}(b(x; \xi) \ q(\lambda)(x; \xi)) = \text{rank}(q(\lambda)(x; \xi))$, $\forall \lambda \in \Lambda$,

where $q(\lambda)$ denotes the principal symbol of the Calderón projector $Q(\lambda)$ associated to $D - \lambda I$, with $\lambda$ included in $\sigma_1(D - \lambda I)$ (i.e. considering $\lambda$ of degree one in the expansion of $\sigma(D - \lambda I)$ in homogeneous functions) [3, 15].

An expression for $q(\lambda)(x; \xi)$ is obtained from (5):

$$q(\lambda)(x; \xi) = \frac{1}{2\pi i} \int_{\Gamma} (a_1^{-1}(x, 0; 0, 1; 0) a_1(x, 0; \xi, 0; \lambda) - z)^{-1} dz, \quad (9)$$

where $a_1(x, t; \xi, \tau; \lambda) = \sigma_1(D - \lambda I)$, with $\lambda$ considered of degree one as stated above.

Henceforth, we assume the existence of an Agmon’s cone $\Lambda$. Moreover, we will consider only boundary conditions $B$ giving rise to a discrete spectrum $sp(D_B)$. Note that this is always the case for elliptic boundary problems unless $sp(D_B)$ is the whole complex plane (see, for instance,[13]). Now, for $|\lambda|$ large enough, $sp(D_B) \cap \Lambda$ is empty, since there is no $\lambda$ in $sp(\sigma_1(D_B)) \cap \Lambda$. Then, $sp(D_B) \cap \Lambda$ is a finite set.
The usual definition of elliptic boundary conditions through ordinary differential equations in the normal variable can be recovered from Def. 1 by introducing the “partial symbol” at the boundary [3]: Let us write

\[
\sigma(D - \lambda I) = a_0(x, t; \xi, \tau; \lambda) + a_1(x, t; \xi, \tau; \lambda),
\]

with \(a_l\) homogeneous of degree \(l\) in \((\xi, \lambda)\). We replace the coefficients of \(D\) by their Taylor expansions in powers of \(t\), and group the resulting terms according to their degree of homogeneity in \((1/t, \xi, -i\partial_t, \lambda)\). More precisely, we set

\[
a^{(j)} = a^{(j)}(x, t, \xi, -i\partial_t, \lambda) = \sum_{l-k=j} \frac{t^k}{k!} a^j_l(x, 0, \xi, -i\partial_t, \lambda),
\]

with \(a^j_l = \partial^l_t a_l\).

Let us denote \(\sigma'(D - \lambda I) = \sum_j a^{(j)}\) the partial symbol of \(D - \lambda I\) at the boundary.

Now, condition 2 is equivalent to the following:

2') \(\forall \lambda \in \Lambda, \forall x \in \partial M, \forall g \in C^r\), the initial value problem

\[
\sigma'_1(D)(x; \xi) u(t) = \lambda u(t)
\]

\[
b(x; \xi) u(t)|_{t=0} = g
\]

has, for each \(\xi \neq 0\), a unique solution satisfying \(\lim_{t \to \infty} u(t) = 0\). This is the form under which this condition is stated in [3].

For \(\lambda \in \Lambda\) not in \(\text{sp}(D_B)\), an asymptotic expansion of the symbol of \(R(\lambda) = (D_B - \lambda I)^{-1}\) can be explicitly given [3]:

\[
\sigma(R(\lambda)) \sim \sum_{j=o}^{\infty} c_{-1-j} - \sum_{j=o}^{\infty} d_{-1-j}
\]

(12)

where the Seeley coefficients \(c_{-1-j}\) and \(d_{-1-j}\) satisfy

\[
\sum_{j=o}^{1} a_{1-j} \circ \sum_{j=0}^{\infty} c_{-1-j} = I
\]

(13)
with \( a_{1-j} \) as in (10), \( \circ \) denoting the usual composition of homogeneous symbols, and

\[
\begin{align*}
\sigma'(D - \lambda) \circ \sum_{j=0}^{\infty} d_{-1-j} &= 0 \\
\sigma'(B) \circ \sum_{j=0}^{\infty} d_{-1-j} &= \sigma(B) \circ \sum_{j=0}^{\infty} c_{-1-j} \quad \text{at } t = 0 \\
\lim_{t \to \infty} d_{-1-j} &= 0.
\end{align*}
\]

(14)

Note that condition 2') implies the existence and unicity of the solution of (14).

The coefficients \( c_{-1-j} \) \((x,t;\xi,\tau;\lambda)\) and \( d_{-1-j} \) \((x,t;\xi,\tau;\lambda)\) are meromorphic functions of \( \lambda \) with poles at those points where \( \det[\sigma_1(D - \lambda)(x,t;\xi,\tau)] \) vanishes. The \( c_{-1-j} \)'s are homogeneous of degree \(-1 - j\) in \((\xi,\tau,\lambda)\); the \( d_{-1-j} \)'s are also homogeneous of degree \(-1 - j\), but in \((\frac{1}{t},\xi,\tau,\lambda)\).

This gives an approximation to \((D_B - \lambda)^{-1}\), a parametrix constructed as

\[
P_K(\lambda) = \sum_{\varphi} \psi \left[ \sum_{j=0}^{K} Op(\theta_2 \ c_{-1-j}) - \sum_{j=0}^{K} Op'(\theta_1 \ d_{-1-j}) \right] \varphi,
\]

(15)

where \( \varphi \) is a partition of the unity, \( \psi \equiv 1 \) in \( \text{Supp}(\varphi) \),

\[
\theta_2(\xi,\tau,\lambda) = \chi(|\xi|^2 + |\tau|^2 + |\lambda|^2)
\]

(16)

\[
\theta_1(\xi,\lambda) = \chi(|\xi|^2 + |\lambda|^2),
\]

with

\[
\chi(t) = \begin{cases} 
0 & t \leq 1/2 \\
1 & t \geq 1 
\end{cases},
\]

(17)

and
\[ Op(\sigma)h(x,t) = \int \sigma(x,t;\xi,\tau) \hat{h}(\xi,\tau) e^{i(x\xi + t\tau)} \frac{d\xi}{(2\pi)^{\nu-1}} \frac{d\tau}{2\pi}, \quad (18) \]

\[ Op'(\sigma)h(x,t) = \int \int \tilde{\sigma}(x,t;\xi,s) \tilde{h}(\xi,s) e^{ix\xi} \frac{d\xi}{(2\pi)^{\nu-1}} \frac{ds}{2\pi}, \quad (19) \]

where \( \hat{h}(\xi,\tau) \) is defined in (30) and

\[ \tilde{h}(\xi,s) = \int h(x,s) e^{-ix\xi} \, dx. \]

Moreover, it can be proved from (12) that, for \( \lambda \in \Lambda \),

\[ ||R(\lambda)||_{L^2} \leq C|\lambda|^{-1} \quad (20) \]

with \( C \) a constant \[3, 15\].

The estimate (20) allows for expressing the complex powers of \( D_B \) as

\[ D^z_B = \frac{i}{2\pi} \int_{\Gamma} \lambda^z R(\lambda) \, d\lambda \quad (21) \]

for \( Re \, z < 0 \), where \( \Gamma \) is a closed path lying in \( \Lambda \), enclosing the spectrum of \( D_B \) \[4\]. Note that such a curve \( \Gamma \) always exists for \( sp(D_B) \cap \Lambda \) finite.

For \( Re \, z \geq 0 \), one defines

\[ D^z_B = D^l \circ D_B^{z-l}, \quad (22) \]

for \( l \) a positive integer such that \( Re \, (z-l) < 0 \).

If \( Re(z) < -\nu \), the power \( D^z_B \) is an integral operator with continuous kernel \( J_z(x,t;y,s) \) and, consequently, it is trace class. As a function of \( z \), \( Tr(D^z_B) \) can be extended to a meromorphic function in the whole complex plane \( C \), with only simple poles at \( z = j - \nu, \ j = 0,1,2,... \) and vanishing residues when \( z = 0,1,2,... \) \[4\]. Throughout this paper, analytic functions and their meromorphic extensions will be given the same name.

The function \( Tr(D^z_B) \) is usually called \( \zeta_D(z) \) because of its similarity with the classical Riemann \( \zeta \)-function: if \( \{\lambda_j\} \) are the eigenvalues of \( D_B \),
\{\lambda_j\} are the eigenvalues of \(D^*_B\); so \(\text{Tr}(D^*_B) = \sum \lambda_j\) when \(D^*_B\) is a trace class operator.

A regularized determinant of \(D_B\) can then be defined as

\[
\det (D_B) = \exp\left[ -\frac{d}{dz} \text{Tr} (D^*_B) \right]_{z=0}.
\] (23)

Now, let \(D(\alpha)\) be a family of elliptic differential operators on \(M\) sharing their principal symbol and analytically depending on \(\alpha\). Let \(B\) give rise to an elliptic boundary condition for all of them, in such a way that \(D(\alpha)_B\) is invertible and the boundary problems they define have a common Agmon’s cone. Then, the variation of \(\det D(\alpha)_B\) with respect to \(\alpha\) is given by (see, for example, [10, 16])

\[
\frac{d}{d\alpha} \ln \det D(\alpha)_B = \frac{d}{dz} \left[ z \text{Tr} \{ \frac{d}{d\alpha} (D(\alpha)_B) D(\alpha)_B^{-1} \} \right]_{z=0}.
\] (24)

Note that, under the assumptions made, \(\frac{d}{d\alpha} (D(\alpha)_B)\) is a multiplication operator.

Given \(\alpha_0\) and \(\alpha_1\), the quotient \(\frac{\det(D(\alpha_1)_B)}{\det(D(\alpha_0)_B)}\) can be obtained by integrating the variation in (24) along a path from \(\alpha_0\) to \(\alpha_1\).

Although \(J_z(x,t;x,t;\alpha)\), the kernel of \(D(\alpha)_B\) evaluated at the diagonal, can be extended to the whole \(z\)-complex plane as a meromorphic function, the r.h.s. in (24) cannot be simply written as the integral over \(M\) of the finite part of

\[
\text{tr} \left\{ \frac{d}{d\alpha} (D(\alpha)_B) J_{z-1}(x,t;x,t;\alpha) \right\}
\] (25)

at \(z = 0\) (where \(\text{tr}\) means matrix trace). In fact, \(J_{z-1}(x,t;x,t;\alpha)\) is in general non integrable in the variable \(t\) near \(\partial M\) for \(z \approx 0\).

Nevertheless, an integral expression for the r.h.s. in (24) will be constructed in Section III, from the integral expression for \(\text{Tr}(D(\alpha)_B^{-1})\) holding in a neighborhood of \(z = 0\) and obtained in the following way [4]:

if \(T > 0\) is small enough, the function \(j_z(x;\alpha)\) defined as

\[
j_z(x;\alpha) = \int_0^T J_z(x,t;x,t;\alpha) \, dt
\] (26)

for \(\text{Re } z < 1 - \nu\), admits a meromorphic extension to \(C\) as a function of \(z\). So, if \(V\) is a neighborhood of \(\partial M\) defined by \(t < \epsilon\), with \(\epsilon\) small enough,
\[ Tr(D(\alpha)_{B}^{-1}) \] can be written as the finite part of
\[
\int_{M/V} \text{tr} \ J_{z-1}(x,t; x, t; \alpha) \ dx dt + \int_{\partial M} \text{tr} \ j_{z-1}(x; \alpha) \ dx ,
\] (27)
where a suitable partition of the unity is understood.

III Green functions and determinants

In this section, we will give an expression for \( \frac{d}{d\alpha} \ln \ Det[D(\alpha)_{B}] \) in terms of \( G_{B}(x, t; y, s; \alpha) \), the Green function of \( D(\alpha)_{B} \) (i.e., the kernel of the operator \( D(\alpha)_{B}^{-1} \)).

With the notation of the previous Section, (24) can be rewritten as:
\[
\frac{d}{d\alpha} \ln \ Det \ D(\alpha)_{B} = \text{F.P.} \int_{M} \text{tr} \left[ \frac{d}{d\alpha}(D(\alpha)_{B}) J_{z-1}(x, t; x, t; \alpha) \right] d\bar{x} ,
\] (28)
where the r.h.s. must be understood as the finite part of the meromorphic extension of the integral at \( z = 0 \).

The finite part of \( J_{z-1}(x, t; x, t; \alpha) \) at \( z = 0 \) does not coincide with the regular part of \( G_{B}(x, t; y, s; \alpha) \) at the diagonal, since the former is defined through an analytic extension.

However, it can be shown that there exists a relation between them, involving a finite number of Seeley’s coefficients. In fact, for boundaryless manifolds this problem has been studied in [4], by comparing the iterated limits \( \text{F.P.} \lim_{z \to -1} \{ \lim_{y \to \bar{x}} J_{z}(x, t; y, s; \alpha) \} \) and \( \text{R.P.} \lim_{y \to \bar{x}} \lim_{z \to -1} J_{z}(x, t; y, s; \alpha) \).

In the case of manifolds with boundary, the situation is more involved owing to the fact that the finite part of the extension of \( J_{z}(x, t; x, t; \alpha) \) at \( z = -1 \) is not integrable near \( \partial M \). (A first approach to this problem appears in [7]). Nevertheless, as mentioned in Section 2, a meromorphic extension of \( \int_{0}^{T} J_{z}(x, t; x, t; \alpha) dt \), with \( T \) small enough can be performed and its finite part at \( z = -1 \) turns to be integrable in the tangential variables. A similar result holds, \textit{a fortiori}, for \( \int_{0}^{T} t^{n} J_{z}(x, t; x, t; \alpha) dt \), with \( n = 1, 2, 3, \ldots \) Then, near the boundary, the Taylor expansion of the function \( A_{\alpha} = \frac{d}{d\alpha} D(\alpha)_{B} \) will naturally appear, and the limits to be compared are
F.P. \( \lim \{ \lim_{y \to x} \int_0^T t^n J_z(x, t; y, s; \alpha) dt \} \) and R.P. \( \lim \{ \lim_{z \to -1} \int_0^T t^n J_z(x, t; y, s; \alpha) dt \} \) \\
\( = R.P. \lim_{y \to x} \int_0^T t^n G_B(x, t; y, s; \alpha) dt. \)

The starting point for this comparison is to carry out asymptotic expansions and to analyze the terms for which the iterated limits do not coincide (or do not even exist).

An expansion of \( G_B(x, t, y, s) \) in \( M \setminus \partial M \) in homogeneous and logarithmic functions of \( (\bar{x} - \bar{y}) \) can be obtained from (12) for \( \lambda = 0 \):

\[
G_B(x, t, y, s) = \sum_{j=1}^n h_j(x, t, x - y, t - s) + M(x, t) \log |(x, t) - (y, s)| + R(x, t, y, s),
\]

with \( h_j \) the Fourier transform \( \mathcal{F}^{-1}(c_{-\nu} - j) \) of \( c_{-\nu} - j \) for \( j > 0 \) and \( h_0 = \mathcal{F}^{-1}(c_{-\nu}) - M(x, t) \log |(x, t) - (y, s)| \). The function \( M(x, t) \) will be explicitly defined below (see (35)). Our convention for the Fourier transform is

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int f(\bar{x}) e^{-i\bar{x} \cdot \xi} d\bar{x},
\]

\[
\mathcal{F}^{-1}(\hat{f})(\bar{x}) = f(\bar{x}) = \frac{1}{(2\pi)^\nu} \int \hat{f}(\xi) e^{i\bar{x} \cdot \xi} d\xi.
\]

For \( t > 0, R(x, t, y, s) \) is continuous even at the diagonal \( (y, s) = (x, t) \). Nevertheless, \( R(x, t, y, s)|_{(y, s) = (x, t)} \) is not integrable because of its singularities at \( t = 0 \). On the other hand, the functions \( t^n R(x, t, y, t) \) are integrable with respect to the variable \( t \) for \( y \neq x \) and \( n = 0, 1, 2, \ldots \).

An expansion of \( \int_0^\infty t^n R(x, t, y, t) dt \) in homogeneous and logarithmic functions of \( (x - y) \) can also be obtained from (12):

\[
\int_0^\infty t^n R(x, t, y, t) dt = \sum_{j=n+2-\nu}^0 g_{j,j+n+2-\nu}(x, x-y) + M_n(x) \log |x-y| + R_n(x, y)
\]

where \( R_n(x, y) \) is continuous even at \( y = x \), and \( g_{j,j+n+2-\nu} \) is the Fourier transform of the (homogeneous extension of) \( \int_0^\infty t^n d_{-j}(x, t, \xi, t, 0) dt \), with

\[
d_{-j}(x, t, \xi, s, \lambda) = -\int_{\Gamma^-} e^{-i\tau \tau} d_{-j}(x, t, \xi, \tau, \lambda) d\tau
\]

for \( \Gamma^- \) a closed path enclosing the poles of \( d_{-j}(x, t, \xi, \tau, \lambda) \) lying in \( \{ Im \tau > 0 \} \).
Since $d_{-1-j}$ is homogeneous of degree $-j$ in $(1/t, \xi, 1/s, \lambda)$, \(g_{j+2-n-\nu} \) turns out to be homogeneous of degree $j + n + 2 - \nu$ in \(x - y\).

From the forementioned comparison, the following Theorem can be shown to hold (the proof will be given in [17]):

**Theorem 1:** Let \(M\) be a bounded closed domain in \(\mathbb{R}^{\nu}\) with smooth boundary \(\partial M\) and \(E\) a \(k\)-dimensional complex vector bundle over \(M\).

Let \((D_\alpha)_B\) be a family of elliptic differential operators of first order, acting on the sections of \(E\), with a fixed local boundary condition \(B\) on \(\partial M\), and denote by \(J_z(x,t;x,t;\alpha)\) the meromorphic extension of the evaluation at the diagonal of the kernel of \((D_\alpha)_B\).

Let us assume that, for each \(\alpha\), \((D_\alpha)_B\) is invertible, the family is differentiable with respect to \(\alpha\), and \(\frac{\partial}{\partial \alpha} (D_\alpha)_B f = A_\alpha f\), with \(A_\alpha\) a differentiable function.

If \(V\) is a neighborhood of \(\partial M\) defined by \(t < \epsilon\) and \(T > 0\) small enough, then:

a) \[
\frac{\partial}{\partial \alpha} \ln \text{Det}(D_\alpha)_B = \text{F.P.} \int_{\partial M} \int_0^T \text{tr} \{ A_\alpha(x,t) J_z(x,t;x,t;\alpha) \} \, dt \, dx \tag{33}
\]

where a suitable partition of the unity is understood. (This expression must be understood as the finite part at \(z = -1\) of the meromorphic extension).

b) For every \(\alpha\), the integral \(\int_0^T A_\alpha(x,t) J_z(x,t;x,t;\alpha) \, dt\) is a meromorphic function of \(z\), for each \(x \in \partial M\), with a simple pole at \(z = -1\). Its finite part
(dropping, from now on, the index α for the sake of simplicity) is given by

\[
F.P. \int_0^T A(x, t) J_\nu(x, t) dt = - \int_0^T A(x, t) \int_{(\xi, \tau)} \frac{i}{2\pi} \int \ln \lambda c_{-\nu}(x, t; \xi, \tau; \lambda) d\lambda \frac{d\sigma_{\xi, \tau}}{(2\pi)^\nu} dt
\]

\[
+ \sum_{l=0}^{\nu-2} \frac{\partial_l^A(x, 0)}{l!} \int_{[\xi]=1} \int_0^\infty \int_{[\xi]=1} \frac{i}{2\pi} \int \ln \lambda c_{-\nu}(x, t; \xi, \tau; \lambda) d\lambda \frac{d\sigma_{\xi, \tau}}{(2\pi)^\nu} dt
\]

\[
+ \lim_{y \to x} \left\{ \int_0^T A(x, t) \left[ G_B(x, t; y, t) - \sum_{l=-\nu}^0 h_l(x, t; x - y, 0) - M(x, t) \frac{\Omega_{\nu}}{(2\pi)^\nu} \left( \ln |x - y|^{-1} + K_{\nu} \right) \right] dt
\]

\[
+ \sum_{(j=0}^{\nu-2} \sum_{l=0}^{\nu-2-j} \frac{\partial_l^A(x, 0)}{l!} g_{j-l}(x, x - y)
\]

\[
+ \sum_{l=0}^{\nu-2} \frac{\partial_l^A(x, 0)}{l!} M_{\nu-2-l}(x) \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}} \left( \ln |x - y|^{-1} + K_{\nu-1} \right) \right\},
\]

with

\[
M(x, t) = \frac{1}{\Omega_{\nu}} \int_{(\xi, \tau)} c_{-\nu}(x, t; \xi, \tau; 0) d\sigma_{\xi, \tau}
\]

\[
M_j(x) = \frac{1}{\Omega_{\nu-1}} \int_{[\xi]=1} \int_0^\infty \int_{[\xi]=1} \frac{i^{\nu-2-j}}{2\pi} \tilde{a}_{-1-j}(x, t; \xi, \tau; 0) dt d\sigma_{\xi, \tau}
\]

where \( \Omega_n = \text{Area}(S^{n-1}) \), \( K_{\nu} = \ln 2 - \frac{1}{2} \gamma + \frac{\Gamma'(\nu/2)}{2 \Gamma(\nu/2)} \) with \( \gamma \) the Euler’s constant.
and where $h_l$ and $g_l$ are related to the Green function $G_B$ as in (29) and (31)

$$h_{1-\nu+j}(x,t;w,u) = F_{(\xi,\tau)}^{-1} \left[ c_{1-\nu+j}(x,t;((\xi,\tau))/|\xi,\tau|); 0) |(\xi,\tau)|^{-1-j} \right](w,u),$$

$$h_0(x,t;w,u) = F_{(\xi,\tau)}^{-1} \left[ P.V. \left\{ (c_{-\nu}(x,t;((\xi,\tau))/|\xi,\tau|); 0) - M(x,t) \right\} |(\xi,\tau)|^{-\nu} \right](w,u),$$

$$g_{j,l}(x,w) = \mathcal{F}_{\xi}^{-1} \left[ \int_0^\infty t^n \tilde{d}_{1-j}(x,t;\xi/|\xi|,t;0) dt |\xi|^{-1-j-n} \right](w),$$

with

$$l = j + n - \nu + 2,$$

and

$$g_{j,0}(x,w) = \mathcal{F}_{\xi}^{-1} \left[ P.V. \left[ \int_0^\infty t^{\nu-j-2} \tilde{d}_{1-j}(x,t;\xi/|\xi|,t;0) dt - M_j(x) \right] |\xi|^{-\nu-1} \right](w).$$

(36)

c) The integral $\int_{M\setminus V} \text{tr} [A(\bar{x}) J_z(\bar{x};\bar{x})] \, d\bar{x}$ in the second term in the r.h.s. of (33), is a meromorphic function of $z$ with a simple pole at $z = -1$. Its finite part is given by

$$F.P. \int_{\bar{x}} \text{tr} [A(\bar{x}) J_z(\bar{x};\bar{x})] \, d\bar{x}$$

$$= \int_{M\setminus V} A(\bar{x}) \int_{|\xi|=1} i \frac{\ln \lambda}{\lambda} c_{-\nu}(\bar{x},\bar{\xi};\lambda) \, d\lambda \, \frac{d\xi}{(2\pi)^\nu}$$

$$+ \int_{M\setminus V} \lim_{y \to \bar{x}} A(\bar{x}) [G_B(\bar{x},\bar{y}) - \sum_{l=1-\nu}^0 h_l(\bar{x},\bar{x} - \bar{y})$$

$$- M(\bar{x}) \frac{\Omega_{\nu}}{(2\pi)^\nu} (\ln |\bar{x} - \bar{y}|^{-1} + K_{\nu})] \, d\bar{x}. $$

This Theorem gives a closed expression for the evaluation of the determinant when the associated Green function is known.

Awful as it looks, (34) is not so complicated: In the first place, all terms can be systematically evaluated. Moreover, the terms containing $h_l$ subtract the singular part of the Green function in the interior of the manifold (see (29)) and can, thus, be easily identified from the knowledge of
$G_B$. $R(x,t,y,t)$, the regular part so obtained, is still nonintegrable near the boundary. Those terms containing $g_{j,l}$ subtract the singular part of the integrals $\int_0^T t^n R(x,t,y,t) \, dt$ (see (31)). Finally, the terms containing $c_{-\nu}$ and $d_{-\nu+1}$ arise as a consequence of having replaced an analytic regularization by a point splitting one.

Even though Seeley’s coefficients $c$ and $\tilde{d}$ are to be obtained through an iterative procedure, which can make their evaluation a tedious task, in the cases of physical interest only the few first of them are needed. In fact, for the two dimensional example in the next section we will only need two such coefficients.

IV Two dimensional Dirac operator on a disk.

In this section, we will use the method previously discussed to evaluate the determinant of the operator $D = i\partial + A$ acting on functions defined on a two dimensional disk of radius $R$. A family of local bag-like elliptic boundary conditions will be assumed.

We take $A_{\mu}$ to be an Abelian field in the Lorentz gauge; as it is well known, it can be written as $A_{\mu} = \epsilon_{\mu
u} \partial_{\nu} \phi$ ($\epsilon_{01} = -\epsilon_{10} = 1$). For $\phi$ we choose a smooth bounded function $\phi = \phi(r)$. Notice that, with these assumptions, $A_r = 0$ and $A_\theta(r) = -\partial_r \phi(r)$. We call

$$\Phi = \oint_{r=R} A_\theta R \, d\theta = -2\pi R \partial_r \phi(r) \big|_{r=R}. \quad (38)$$

The free Dirac operator in polar coordinates is:

$$i \not\partial = i(\gamma_r \partial_r + \frac{1}{r} \gamma_\theta \partial_\theta), \quad (39)$$

with

$$\gamma_r = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \quad \gamma_\theta = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}. \quad (40)$$

With these conventions, the full Dirac operator can be written as:

$$D = e^{-\gamma_5 \phi(r)} i \not\partial e^{-\gamma_5 \phi(r)}. \quad (41)$$
Now, in order to perform our calculations, we consider the family of operators:

\[ D_\alpha = i \not\!\partial + \alpha \mathcal{A} = e^{-\alpha \gamma_5 \phi(r)} \not\!\partial e^{-\alpha \gamma_5 \phi(r)}, \quad 0 \leq \alpha \leq 1, \]  

which will allow us to go smoothly from the free to the full Dirac operator. If we call

\[ W(\alpha) = \ln \text{Det}(D_\alpha)_B, \]  

where \( B \) represents the elliptic boundary condition, we have

\[ \frac{\partial}{\partial \alpha} W(\alpha) = F.P. \left[ \text{Tr} \left( \mathcal{A} \left( D_\alpha \right) \right)^{-z} \right]. \]  

From the Theorem in the previous section we get:

\[ \frac{\partial}{\partial \alpha} W(\alpha) = \frac{1}{(2\pi)^2} \text{tr} \left\{ \int \lim_{y \to x} \left[ \int \left[ \mathcal{A}(t) \left( 4\pi^2 G_B(x, t, y, t) \right) \right] \right] \right. \]

\[ \left. - \frac{1}{|x - y|} \int e^{i \xi (x - y)} c_{-1}(x, t; \frac{(\xi, \tau)}{|(\xi, \tau)|}; 0) d\xi d\tau \right. \]

\[ - \int_{|\xi, \tau| \geq 1} e^{i \xi (x - y)} c_{-1}(x, t; \xi, \tau; 0) d\xi d\tau \]

\[ - \int \frac{i}{2\pi} \int_{\Gamma} \frac{\ln \lambda}{\lambda} c_{-2}(x, t; \frac{(\xi, \tau)}{|(\xi, \tau)|}; \lambda) d\lambda d\sigma_{\xi, \tau} \]

\[ + \mathcal{A}(0) \left( \int_{|\xi| \geq 1} e^{i \xi (x - y)} \tilde{d}_{-1}(x, t; \xi, t; 0) d\xi \right) \]

\[ + \int \frac{i}{2\pi} \int_{\Gamma} \frac{\ln \lambda}{\lambda} \tilde{d}_{-1}(x, t; \frac{\xi}{|\xi|}, t; \lambda) d\lambda d\sigma_{\xi} \right] dx \right\}, \]

where the Fourier transforms of \( c_{-2} \) and \( \tilde{d}_{-1} \) have been left explicitly indicated.

Now, the coefficients \( c \) and \( \tilde{d} \) in the previous equation are those appearing in the asymptotic expansion of the resolvent \((D_\alpha - \lambda I)^{-1}\).
From (41), the symbol of \((D_\alpha - \lambda I)\) is:

\[
\sigma(D_\alpha - \lambda I) = (-\xi - \lambda I) + \alpha A
\]

where

\[
a_1 = -\xi - \lambda I,
\]

\[
a_0 = \alpha A.
\]

The required Seeley’s \(c\)-coefficients are given by [2] :

\[
c_{-1} = \frac{1}{(\lambda^2 - \xi^2 - \tau^2)}(\xi - \lambda I),
\]

\[
c_{-2} = \frac{\alpha}{(\lambda^2 - \xi^2)^2}(2\lambda\xi\mu A_\mu I - (\lambda^2 - \xi^2) A - 2\xi\mu A_\mu \xi),
\]

where \(\xi = \xi_\theta + \tau_\gamma t\).

As regards the boundary contributors to the parametrix, i.e., the coefficients \(d_{-1-j}\) are the solutions of (14). In our case, the equation to be solved is

\[
(-\lambda I - \xi_\gamma_\theta + i\gamma_\gamma t \partial_t) d_{-1} = 0,
\]

with boundary conditions

\[
b_0 \ d_{-1} = b_0 \ c_{-1} \quad \text{at} \ t = 0,
\]

plus the vanishing of \(d_{-1}\) as \(t \to +\infty\). (49) can be recast in the form

\[
\partial_t d_{-1} = -M \ d_{-1},
\]

where \(M = \xi_\gamma_5 + i\lambda_\gamma t\). It can be easily verified that

\[
tr(M) = 0,
\]

\[
M^2 = (\xi^2 - \lambda^2)I.
\]

So, \(M\) has eigenvalues \(\pm\sqrt{\xi^2 - \lambda^2}\), corresponding to the eigenvectors

\[
u_{\pm} = \left(\frac{ie^{-i\theta}(\xi \pm \sqrt{\xi^2 - \lambda^2})}{\lambda}\right).
\]
Since $d_{-1} \to 0$ for $t \to \infty$, we get

$$d_{-1}(x, t; \xi, \tau; \lambda) = e^{-t\sqrt{\xi^2 - \lambda^2}} u_+ \otimes \left( \begin{array}{c} f \\ g \end{array} \right)^\dagger,$$

(54)

where the vector $\left( \begin{array}{c} f \\ g \end{array} \right)$ must be determined from the boundary condition at $t = 0$ ($r = R$), given by (50).

We now consider a parametric family of bag-like local boundary conditions leading to an elliptic boundary problem,

$$b_0 = \left( 1, w e^{-i\theta} \right),$$

(55)

with $w$ a nonzero complex constant. (Notice that these boundary conditions reduce to those of an MIT bag [18] when $w = \pm 1$.)

We define the operator $(D_\alpha)_{B}$ as the differential operator in (41), acting on the dense subspace of functions satisfying

$$B \psi \equiv b_0 \psi|_{t=0} = 0.$$

(56)

It is easy to verify that this operator has no normalizable zero modes (Notice that these are not the most general local elliptic boundary conditions. In fact, zero modes would in general arise if one allowed $w$ to depend on $\theta$).

Now, from (50) and the expression for $c_{-1}$ given in (18), it turns out that:

$$\left( \begin{array}{c} f \\ g \end{array} \right)^\dagger = \frac{e^{i\theta}}{(\xi^2 + \tau^2 - \lambda^2)(\lambda w + i\xi + i\sqrt{\xi^2 - \lambda^2})} \times \left( \begin{array}{cc} \lambda + w & -i\xi + \tau + \lambda w \\ -i\xi + \tau - i\lambda & e^{-i\theta}(i\xi + \tau + \lambda w) \end{array} \right).$$

(57)

Replacing this expression into (54), and taking into account (32), we finally get:

$$\tilde{d}_{-1} = \pi i \frac{e^{-(u+t)\sqrt{\xi^2 - \lambda^2}}}{\sqrt{\xi^2 - \lambda^2}(iw\lambda - \xi - \sqrt{\xi^2 - \lambda^2})} \times \left( \begin{array}{cc} (\xi + \sqrt{\xi^2 - \lambda^2}) & e^{-i\theta}(\xi + \sqrt{\xi^2 - \lambda^2}) \\ -i\lambda e^{i\theta}(i\lambda - w(\xi + \sqrt{\xi^2 - \lambda^2})) & -i\lambda \end{array} \right).$$

(58)
In order to apply (45), we look for the function \( G_B(x, y) \) satisfying:

\[
D_\alpha G_B(x, y) = \delta(x, y),
\]

\[
B G_B(x, y)|_{x \in \partial\Omega} = 0,
\]

where \( D_\alpha \) and \( B \), are given by equations (41) and (56) respectively. Now, with the notation

\[
x = (x_0, x_1) = (r \cos \theta, r \sin \theta),
\]

\[
X = x_0 + i x_1 = r e^{i \theta},
\]

\[
y = (y_0, y_1) = (\rho \cos \varphi, \rho \sin \varphi),
\]

\[
Y = y_0 + i y_1 = \rho e^{i \varphi},
\]

it is easy to see that \( G_B(x, y) \) is given by

\[
G_B(x, y) = \frac{1}{2\pi i} \left( \begin{array}{cc}
\frac{R w e^{\alpha(\phi(x) + \phi(y) - 2\phi(R))}}{XY^* - R^2} & \frac{e^{\alpha(\phi(x) - \phi(y))}}{X - Y} \\
\frac{e^{-\alpha(\phi(x) - \phi(y))}}{(X - Y)^*} & \frac{R e^{-\alpha(\phi(x) + \phi(y) - 2\phi(R))} w}{(XY^* - R^2)^*}
\end{array} \right).
\]

With these elements at hand, we now perform the calculation of the determinant.

From (61), one can see that

\[
G_B(\theta, r, \varphi, r) \sim \text{diagonal matrix} + \frac{1}{2\pi i r (\theta - \varphi)} \gamma_\theta.
\]

When replaced into (45), we get for the first term in the r.h.s.

\[
\text{tr} \left\{ A_\theta \gamma_\theta G_B(\theta, r, \varphi, r) \right\} \sim \frac{A_\theta}{\pi i r (\theta - \varphi)}.
\]

For the second term in (45)

\[
- \frac{1}{4\pi^2 |x - y|} \int d\xi \, d\tau \, e^{i \frac{x-y}{|x-y|} \cdot c_{-1}(x, t; \frac{(\xi, \tau)}{|(\xi, \tau)|}; \lambda = 0)} \sim \frac{-1}{2\pi i r (\theta - \varphi)} \gamma_\theta,
\]

(64)
which exactly cancels the singularity of the Green function. Therefore, the contribution of the first two terms in (45) vanishes.

As regards the third term,

\[
\frac{-1}{(2\pi)^2} \text{tr} \int \lim_{y \to x} A(t) \int_{[(\xi,\tau)] \geq 1} e^{i\xi(x-y)} c_{-2}(x, t; \xi, \tau; 0) \, d\xi \, d\tau \, dx \, dt
\]

\[
= \frac{-\alpha}{2\pi} \lim_{y \to x} \int A_\theta^2 \, d^2x \int_{[(\xi,\tau)] \geq 1} e^{i\xi(x-y)} \frac{(\tau^2 - \xi^2)}{(\xi^2 + \tau^2)^2} \, d\xi \, d\tau
\]

\[
= \frac{-\alpha}{\pi} \int A_\theta^2 \, d^2x \lim_{y \to x} \int_{|x-y|}^\infty J_2(u) \frac{du}{u} = \frac{-\alpha}{2\pi} \int A_\nu A_\nu \, d^2x.
\]

where \(J_2(u)\) is the Bessel function of order two.

Now, the fourth term in (45) is:

\[
\frac{-1}{(2\pi)^2} \text{tr} \int \mathcal{A}(t) \int \frac{i}{2\pi} \int_\Gamma \ln \lambda \ c_{-2}(x, t; \xi, \tau; \lambda) \frac{d\lambda}{\lambda} \, d\sigma_{\xi,\tau} \, dx \, dt
\]

\[
= \frac{-i\alpha}{4\pi^3} \int A_\theta^2 \, d^2x \int_\Gamma \ln \lambda \frac{1}{(\lambda^2 - 1)^2} \int (1 - \lambda^2 - 2\xi^2) \, d\sigma_{\xi,\tau} \, d\lambda
\]

\[
= \frac{i\alpha}{2\pi^2} \int A_\theta^2 \, d^2x \ 2\pi i \int_0^\infty \frac{\mu \, d\mu}{(\mu^2 + 1)^2} = \frac{-\alpha}{2\pi} \int A_\nu A_\nu \, d^2x.
\]

This term gives rise to a contribution identical to that of (65).

The last term in (45) is:

\[
\frac{i}{(2\pi)^3} \text{tr} \int \mathcal{A}(0) \sum_{\xi \pm 1} \int_\Gamma \ln \lambda \ \tilde{a}_{-1}(x, t; \xi, t; \lambda) \frac{d\lambda}{\lambda} \, dx \, dt
\]

\[
= \frac{i \Phi}{(2\pi)^2} \int_\Gamma \frac{u \ln \lambda}{(1 + u^2 \lambda^2)} [\lambda \sqrt{1 + u^2} - i \sqrt{1 - \lambda^2}] \frac{d\lambda}{\sqrt{1 - \lambda^2}},
\]

where \(u = (1 - w^2) / 2w\). We choose the curve \(\Gamma\) as in Fig. 1.
Therefore, (67) reads
\[-\frac{\Phi}{2\pi} u \int_0^\infty \frac{1}{(1 - u^2 \mu^2)} \left[ \mu \frac{\sqrt{1 + u^2}}{\sqrt{1 + \mu^2}} - 1 \right] d\mu \]
\[= -\frac{\Phi}{4\pi} \ln w^2. \tag{68} \]

Putting all pieces together ((65), (66) and (68)), we finally find:
\[
\ln \det(D_B) - \ln \det(\hat{\partial})_B = -\frac{1}{2\pi} \int_\Omega A_\nu A_\nu d^2x - \frac{\Phi}{4\pi} \ln w^2
\]
\[= -\frac{1}{2\pi} \int_\Omega A_\nu A_\nu d^2x - \frac{1}{4\pi} \ln w^2 \int_{\partial\Omega} A_\nu dx_\nu. \tag{69} \]

The first term is the integral, restricted to the region \(\Omega\), of the same density appearing in the well known case of the whole plane [19]. The second term is well-defined for every \(w \neq 0\), and vanishes for a null total flux, \(\Phi = 0\). For \(w = 0\), \(b_0\) in (55) does not define an elliptic boundary problem. It is also interesting to notice that this term vanishes in the case of MIT bag boundary conditions, i.e., \(w = \pm 1\).

This calculation is to be compared with the case of the compactified plane [3], where the determinant can be expressed in terms of just the kernel of
the $z$-power of the operator analytically extended to $z = 0$, which is a local quantity. The presence of boundaries makes the evaluation more involved, since even in simple cases as the present (or the half plane treated in [4]), the knowledge of the Green function of the problem is needed.

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References

[1] S. W. Hawking, Comm. Math. Phys. 55, 133 (1977).

[2] R.E. Gamboa Saraví, M.A. Muschietti, F.A. Schaposnik and J.E. Solomin, Ann. Phys. 157, 360 (1984).

[3] R.T. Seeley, Am. J. Math. 91, 889 (1969).

[4] R.T. Seeley, Am. J. Math. 91, 963 (1969).

[5] G. Grubb and R.T. Seeley, C.R. Acad. Sci. Paris 317, serie I, 1123 (1993); Zeta and eta functions for Atiyah-Patodi-Singer operators, Copenh. Univ. Math. Dept. Preprint Ser., #11 (1993); Weakly parametric pseudodifferentials operators and Atiyah-Patodi-Singer boundary problems, Copenh. Univ. Math. Dept. Preprint Ser. #5 (1993).

[6] R.E. Gamboa Saraví, M.A. Muschietti, F.A. Schaposnik and J.E. Solomin, J. Math. Phys. 26, 2045 (1985).

[7] H. Falomir, M. A. Muschietti and E.M. Santangelo, J. Math. Phys. 31, 989 (1990).

[8] E. F. Moreno, Phys. Rev. D48, 921 (1993).

[9] M. Ninomiya and C.I. Tan, Nucl. Phys. B257, 199 (1985).

[10] M. F. Atiyah, V.K. Patodi and I. M. Singer, Math. Proc. Camb. Phil. Soc. 77, 43 (1975); 78, 405 (1975); 79, 71 (1976).

[11] A.P. Calderón, Lectures notes on pseudodifferential operators and elliptic boundary value problems, I (Publicaciones del I.A.M., Buenos Aires, 1976).

[12] R.T. Seeley, Am. J. Math. 88, 781 (1966).

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Pseudo-Differential Operators. (Springer-Verlag, Berlin Heidelberg, 1985).
[14] B. Booss and D.D. Bleecker, *Topology and Analysis. The Atiyah-Singer Index Formula and Gauge-Theoretic Physics.* (Springer-Verlag, New York, 1985).

[15] P. B. Gilkey and L. Smith, Communications on Pure and Applied Mathematics, Vol. XXXVI, 85 (1983).

[16] R. Forman, Invent. math. **88**, 447 (1987).

[17] H. Falomir, R.E. Gamboa Saraví, M.A. Muschietti, E. M. Santangelo and J.E. Solomin, to be published.

[18] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn and V. F. Weisskopf, Phys. Rev. **D9**, 3471 (1974); A. Chodos, R.L. Jaffe, K. Johnson and C.B. Thorn, Phys. Rev. **D10**, 2599 (1974).

[19] J. Schwinger, Phys. Rev. **82**, 664 (1951); **128**, 2425 (1962).