A COARSE GEOMETRIC EXPANSION OF A VARIANT OF ARTHUR’S TRUNCATED TRACES AND SOME APPLICATIONS

HONGJIE YU

Abstract

Let $F$ be a global function field with constant field $\mathbb{F}_q$. Let $G$ be a reductive group over $\mathbb{F}_q$. We establish a variant of Arthur’s truncated kernel for $G$ and for its Lie algebra which generalizes Arthur’s original construction. We establish a coarse geometric expansion for our variant truncation.

As applications, we consider some existence and uniqueness problems of some cuspidal automorphic representations for the functions field of the projective line $\mathbb{P}^1_{\mathbb{F}_q}$ with two points of ramifications.

Contents

1. Introduction 1
2. Notations 5
3. Characteristic of the base field and unipotent/nilpotent elements 6
4. Combinatorics of roots and coroots under field extension 8
5. Statements of the main theorems 12
6. Reduction theory and combinatoric lemmas 16
7. A relation between Jordan-Chevalley decomposition and Levi decomposition 20
8. Proof of the Theorem 5.1 and the Theorem 5.6 24
9. Another expression for $J_{\xi,X}^\xi$ 28
10. Some applications: non-existence or existence and uniqueness of some cuspidal automorphic representations 29

References 37

1. INTRODUCTION

In [Ar78], Arthur defines a truncated kernel, and establishes some of its basic properties. He then gives a coarse geometric expansion whose integrals give the geometric side of his non-invariant trace formula. Arthur’s work is over a number field. Later in [Laf97], Lafforgue shows that Arthur’s truncated kernel is well behaved over a function field for $GL_n$ as well as its inner forms. There is no restriction on the characteristic, while there is no coarse geometric expansion either, because of the lack of the Jordan-Chevalley decomposition. Lafforgue used the Weil’s dictionary between adeles and vector bundles. He shows that the truncated kernel over a function field has some nicer properties than that over a number field.

Let $F$ be a global function field with constant finite subfield $\mathbb{F}_q$. In this article, we generalize Lafforgue’s results to a reductive group over $\mathbb{F}_q$. We also consider a variant version of Arthur’s truncated kernel inspired by the $\xi$-stability of Hitchin pairs defined.
and studied by Chaudouard and Laumon ([CL10]). For some special test functions, this variant version will have nicer properties than the Arthur’s original version. We also establish a Lie algebra version of non-invariant trace formula following the number field case given by Chaudouard ([Ch02]).

We explain some phenomena that do not show up for a number field first. If $p \nmid |\pi_1(G^{\text{der}})|$, there exists a unipotent element which is not contained in any proper parabolic subgroup, for example the element

$$u = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$$

in $PGL_2(F)$ if $\text{char}(F) = 2$ and $c$ is not a square element. Fortunately, Gille [Gi02, Theorem 2] has shown that such phenomenon does not happen when $p \nmid |\pi_1(G^{\text{der}})|$ for a field such that $[F : F_p] \leq p$. We’ll extend Arthur’s constructions to function field under this hypothesis. Note that if $G^{\text{der}}$ is simply connected or the characteristic is very good for $G$ (in the sense of [SS70, I.4]), then the hypothesis is satisfied.

Another problem is the lack of Jordan-Chevalley decomposition. Recall that we say that an element $\gamma \in G(F)$ admits a Jordan decomposition if its unipotent part and semi-simple part of the Jordan decomposition in $G(F)$ lies in $G(F)$. To have a geometric expansion of the truncated trace, we need to define a partition on $G(F)$:

$$(1.1) \quad G(F) = \bigsqcup_{o \in E} o,$$

which should be compatible with conjugations. Over a number field, Arthur defines an equivalence relation on $G(F)$ so that two elements are equivalent if their semi-simple parts are conjugate. Such definitions do not work directly for a field in positive characteristic, but we can define the same equivalence relation only on those elements admitting a Jordan-Chevalley decomposition and putting those which do not admit Jordan-Chevalley decomposition in a single class. Then we give a coarse expansion under this partition.

To be precise, with notations commonly used in the literature which will be introduced in the Section 2, fixing two truncation parameters $X \in a_B$ and $\xi \in a_{B,\infty}$ (where Arthur uses $T$ for $X$ and $\xi$ is an additional parameter depending on a fixed place $\infty$), we construct following Arthur a linear function

$$J^{G,\xi,X} : C^\infty_c(G(\A)) \to \C,$$

which generalizes Arthur’s “truncated trace”. For each $o \in E$ as in (1.1), we also define a linear function

$$J^{G,\xi,X}_o : C^\infty_c(G(\A)) \to \C,$$

such that

$$J^{G,\xi,X} = \sum_{o \in E} J^{G,\xi,X}_o.$$

Note that we always assume $p \nmid |\pi_1(G^{\text{der}})|$. For $GL_n$, Lafforgue has given a very simple formula for Arthur’s truncated trace when the truncation parameter is taken to be deep enough in the positive chamber, we show that there is a similar formula for $J^{G,\xi,X}_o$ for any constant reductive group (cf. Theorem 5.1). In addition to the group version, we
also establish a coarse trace formula for the Lie algebra of $G$ under more restrictive hypothesis on the characteristics.

Finally, let me explain about the reason for introducing another parameter $\xi$ and a version for a Lie algebra.

Over a number field, the coarse geometric expansion is refined by Arthur into a sum of weighted orbital integrals, where the weight factor is given by a volume of convex envelope in a vector space. Over a function field, such volume will be replaced by a count of a lattice points bounded by a convex envelope, which is not so pleasant because of those lattice points lying on the boundary of the convex envelope. The additional parameter $\xi$ is aimed to translate the convex envelope so that there are no longer points on the boundary if $\xi$ is in general position. In fact, Chaudouard and Laumon ([CL10, 11.14.3]) showed that for regular semi-simple weighted orbital integrals, by introducing a $\xi$ in general position, one recovers Arthur’s weight factor.

Moreover, we showed in [Yu21, Appendix B] that $J^{g,\xi}(f)$ (where $X$ is taken to be 0) for some simple test function $f \in C^\infty_c(\mathfrak{g}(\mathbb{A}))$ gives the groupoid volume of semi-stable parabolic Hitchin bundles with a stability parameter corresponding to $\xi \in \mathfrak{a}_B$. Therefore a Lie algebra version could be useful to the study of Hitchin’s moduli spaces.

For the group $GL_n$, it has been studied by Chaudouard in [Ch15]. We study it for a more general group in [Yu21].

Remark 1.1. This work is motivated by an application in [Yu21]. In the preparation of these articles, there appears the work of Labesse and Lemaire [LL21] that, among many other results, also establishes a coarse geometric expansion of Arthur’s non-invariant trace formula. They give a coarse geometric expansion by introducing the notion of primitive pairs (cf. [LL21, 3.3]) which works in any characteristic. Note that under the hypothesis $p \nmid |\pi_1(G_{\text{der}})|$, the coarse expansion given by them coincides with the expansion in this article for those classes which admit a Jordan decomposition (which can be seen from the beginning of the proof of the Proposition [73]), and is finer otherwise. After these remarks, theirs works overlap the main results of the current paper and are more general than the cases treated here.

There still remain some novelties for the constructions in this article. First of all, the approach here is different from [LL21]. The key ingredient in our proof is that we use Behrend’s complementary polyhedra and semi-stable reductions of $G$-bundles as the source of reduction theory and some combinatoric lemmas. A direct advantage is that we do not need the truncation parameter $X$ to be “regular enough” to have integrability of the truncated kernel. Secondly, for $GL_n$, Lafforgue has given a very simple formula for Arthur’s truncation when the truncation parameter is taken to be deep enough in the positive chamber, we generalize it to any constant reductive group (cf. Theorem 5.4). Lastly, the introduction of an additional parameter $\xi$ and the trace formula for Lie algebras are new for a function field and they’re not useless as showed by the applications in the Section [10] and in [Yu21].

1.1. Some applications. The main theorems of this article are stated in the Section 5. To motivate the reader, we present here some applications that partially confirm a conjecture made to me personally by Harris. Note that the requirement that $\xi$ is chosen
in general position and our Lie algebra version trace formula are essentially used in the proofs.

We consider the case that $F = \mathbb{F}_q(t)$ is the function field of the projective line $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{F}_q}$. Let $G$ be a split reductive group defined over $\mathbb{F}_q$ such that the characteristic $p$ is very good for $G$ (cf. \cite{3}). Let $\lambda, \mu \in \mathbb{P}^1(\mathbb{F}_q)$ be two distinct $\mathbb{F}_q$-rational points, identified as two places of $F$. Let $T_\lambda$ and $T_\mu$ be two maximal torus of $G$ defined over $\mathbb{F}_q$. Suppose that $T_\lambda$ and $T_\mu$ are characters in general position of $T_\lambda(\mathbb{F}_q)$ and $T_\mu(\mathbb{F}_q)$ respectively. We obtain by Deligne-Lusztig induction (fixing any isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_l$) two irreducible representations

$$
\rho_\lambda = \epsilon_{T_\lambda} \epsilon_G R_{T_\lambda}^G (\theta_\lambda) \quad \text{and} \quad \rho_\mu = \epsilon_{T_\mu} \epsilon_G R_{T_\mu}^G (\theta_\mu)
$$

of $G(\mathbb{F}_q)$, where $\epsilon_H = (-1)^{rk_q H}$ is the sign according to the parity of the split rank of $H$. Let $O_\lambda$ and $O_\mu$ be the local ring in $\lambda$ and $\mu$ respectively. By inflation, we get irreducible representations of $G(O_\lambda)$ and $G(O_\mu)$ respectively, still denoted by $\rho_\lambda$ and $\rho_\mu$.

**Theorem 1.2** (Theorem 10.2). Suppose that $T_\lambda$ is split and $G$ is not a torus. Let $M_\mu$ be the minimal Levi subgroup containing $T_\mu$ defined over $\mathbb{F}_q$. Without loss of generality, we suppose that $M_\mu$ contains $T_\lambda$. Under the hypothesis that

$$
(1.2) \quad \theta^w \theta^\mu |_{Z_M(\mathbb{F}_q)} \neq 1,
$$

for any Weyl element $w \in W$ and for any maximal proper semi-standard Levi subgroup $M$ of $G$ that contains $M_\mu$. There is no cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that $\pi_\lambda$ contains $(G(O_\lambda), \rho_\lambda)$, $\pi_\mu$ contains $(G(O_\mu), \rho_\mu)$ and all other local components are unramified.

**Remark 1.3.** If the torus $T_\mu$ is maximally anisotropic over $\mathbb{F}_q$, then the hypothesis (1.2) holds automatically. It is expected (communicated to me by Harris) that the theorem holds under the weaker hypothesis that $\rho_\lambda$ is not contragredient to $\rho_\mu$ and we should have multiplicity one when $\rho_\lambda$ is contragredient to $\rho_\mu$ at least when $G$ is split semi-simple and simply connected. We verify this statement in a special case in the Theorem 1.4. Note that we have showed in \cite{Yn21} that if we allow at least three points of ramification, then there are “many” such automorphic cuspidal representations when $q$ is large.

**Theorem 1.4** (Theorem 10.3). Let $G = SL_l$ with $l$ being a prime number different from $p$. Suppose that $\rho_\lambda$ and $\rho_\mu$ constructed before are cuspidal. If $\rho_\lambda$ is contragredient to $\rho_\mu$, then there is exactly one cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ (with cuspidal multiplicity 1) such that $\pi_\lambda$ contains $(G(O_\lambda), \rho_\lambda)$, $\pi_\mu$ contains $(G(O_\mu), \rho_\mu)$ and all other local components are unramified. If $\rho_\lambda$ is not contragredient to $\rho_\mu$, then there is no such cuspidal automorphic representation.

1.2. Structure of the article. In Section 3 we discuss the restrictions on the characteristic. The main theorems are announced in Section 5. To state the constructions (mainly to introduce our variant parameter $\xi$), we need preparations in Section 4. Later sections are devoted to the proof of theorems in the Section 5. It is in Section 8 where the main theorems of this article are proved. In Section 9 we present another expression for the truncated trace which is relatively easier to use. In Section 10 we give some applications.
1.3. Acknowledgement. I’d like to thank Prof. Chaudouard for introducing me to this area. I’d like to thank Prof. Harris for asking me the question that makes Section 4 possible. I’m grateful for the support of Prof. Hausel and IST Austria. The author was funded by an ISTplus fellowship: This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 754411.

2. Notations

2.1. Notations for the groups. We use the language of group schemes, so a linear algebraic group $G$ over a ring $k$ is an affine group scheme $G \to \text{Spec}(k)$ of finite type. We do not require it to be smooth nor connected. Reductive groups are connected smooth group schemes with trivial geometric unipotent radical. Intersection and center are taken in the scheme theoretic sense.

$F, \overline{F}, \mathbb{F}_q$: Throughout the article $F$ is a global function field with finite constant field $\mathbb{F}_q$, i.e. a field of rational functions over a smooth, projective and geometrically connected curve $C$ defined over $\mathbb{F}_q$. We fix an algebraic closure $\overline{F}$ of $F$.

$G^0$, $Z_G$, $G^{\text{der}}$, $G^{\text{sc}}$: For a linear algebraic group $G$, we denote by $G^0$, $Z_G$, $G^{\text{der}}$ and $G^{\text{sc}}$ the connected component of the identity of $G$, the center of $G$, and the derived group of $G$, and the simply connected covering of $G^{\text{der}}$ respectively.

$\mathfrak{g}$: We denote by $\mathfrak{g}$ the Lie algebra of $G$. In general, if a group is denoted by a capital letter, we use its lowercase gothic letter for its Lie algebra.

$G_x$, $G_X$, $C_G(H)$, $N_G(H)$: When $G$ is defined over a field $F$ and $x \in G(F)$, and $X \in \mathfrak{g}(F)$, we denote by $G_x$ (resp. $G_X$) the centralizer of $x$ (resp. the adjoint centralizer of $X$) in $G$. If $H$ is a subgroup of $G$ defined over $F$, we denote by $C_G(H)$ and $N_G(H)$ the centralizer and normalizer of $H$ in $G$ respectively.

$G$, $B$, $T$, $A$, $\Xi_G$: Suppose that $G$ is a connected reductive group defined over $\mathbb{F}_q$. Let’s fix a Borel subgroup $B$ defined over $\mathbb{F}_q$, a maximal torus $T$ defined over $\mathbb{F}_q$ contained in $B$. Let $A$ be the maximal subtorus of $T$ that is defined and split over $\mathbb{F}_q$.

We fix a lattice $\Xi_G$ in $Z_G(F)\backslash Z_G(A)$, i.e. a finitely generated free abelian group so that $Z_G(F)\backslash Z_G(A)/\Xi_G$ is compact. For a field $k$ containing $\mathbb{F}_q$, we denote by $G_k$, or simply by $G$ itself if there is no harm, the base change of $G$ to $k$.

2.2. Arthur’s notations. Let $|F|$ be the set of places of $F$. For each $v \in |F|$, let $F_v$ be the completion of $F$ at $v$, $O_v$ be the ring of integers of $F_v$ and $\kappa_v$ be the residue field of $O_v$. Let $\mathbb{A}$ be the ring of adèles of $F$, and $\mathcal{O}$ be the ring of integral adèles.

We denote by $W(G,T) = N_G(T)/C_G(T)$ be the Weyl group of $G$ with respect to $T$, understood as a finite étale group scheme over $F$ and

$$W = W(G,T)(F) = W(G,T)(\mathbb{F}_q).$$

For each $s \in W$ we fix a representative $w_s \in N_G(T)(\mathbb{F}_q)$.

We will use notations introduced by Arthur which are now more or less standard in this area. An excellent source to notations and their properties is [LW11] Chapitre I. For reader’s convenience, instead of sending the reader to various articles of Arthur, we will prefer to cite this book if possible.

Throughout this article, we reserve letters $P,Q$ for parabolic subgroups, $M$ for a Levi subgroup of $G$ (i.e. Levi subgroup of a parabolic subgroup of $G$). When $P,Q$ are
semistandard, we denote by $M_P$ the semistandard Levi subgroup of $P$ and $N_P$ for the unipotent radical of $P$.

- $\mathcal{P}^Q(M)$ parabolic subgroups contained in $Q$ which admit $M$ as Levi subgroup;
- $\mathcal{P}(B)$ standard parabolic subgroups of $G$;
- $X^*(P)_F = \text{Hom}_F(P, \mathbb{G}_m/F)$;
- $a^*_P = X^*(P)_F \otimes \mathbb{R}$; $a_P = \text{Hom}_\mathbb{R}(a^*_P, \mathbb{R})$; $a_{P,Q} = \text{Hom}_\mathbb{R}(a^*_P, \mathbb{Q})$.

Let $A_P$ be the maximal split central torus defined over $F$ of $P$. We should think of $X^*(P) \subseteq X^*(A_P)$ as lattices in $a^*_P$ and think of $a^*_P$ as the space of linear functions on $a_P$.

Now suppose that $P, Q$ are standard parabolic subgroups of $G$. We will use the following notations.

- $\langle \cdot, \cdot \rangle$: the canonical pairing between $a^*_B$ and $a_B$;
- $a_P = a^*_P \oplus a_Q$ ($P \subseteq Q$), decomposition induced by $X^*(Q) \to X^*(P)$ and $X^*(A_P) \to X^*(A_Q)$;
- $\Phi(P, A)$: roots of the torus $A$ acting on the Lie algebra of $\mathfrak{p}$;
- $\Delta_B$ simple roots in $\Phi(B, A)$;
- $\Delta_B^P = \Delta_B \cap \Phi(M_P, A)$; $(\Delta_B^P)^\vee$ the set of corresponding coroots; $\hat{\Delta}_B^P$ the set of corresponding fundamental weights;
- $\Delta_B^Q = \{ \alpha | A_P | \alpha \in \Delta_B^Q - \Delta_B^P \}$, viewed as a set of linear functions on $a_B$ via the projection $a_B \to a_P^*$;
- $\hat{\Delta}_B^Q = \{ \varpi \in \hat{\Delta}_B^Q | \varpi|_{a_B^P} = 0 \}$;
- $\tilde{\tau}_B^Q$ the characteristic function of $H \in a_B$ such that $\langle \varpi, H \rangle > 0$ for all $\varpi \in \hat{\Delta}_B^Q$;
- $\tau^Q_B$ the characteristic function of $H \in a_B$ such that $\langle \alpha, H \rangle > 0$ for all $\alpha \in \Delta_B^Q$.

As a general rule of notations, for a given place $\infty$ of $F$, we will add an index $\infty$ if a relevant notation is defined using datum over $F_\infty$, for example we have

- $W_\infty$, $a_{P, \infty}$, $\Delta_{B, \infty}$, ...

2.3. Harish-Chandra’s map. For each semi-standard parabolic subgroup $P$ of $G$ defined over $F$, we define the Harish-Chandra’s map $H_P : P(\mathbb{A}) \to a_P$ by requiring

$$q^{(\alpha, H_P(p))} = |\alpha(p)|_{\mathbb{A}}, \quad \forall \alpha \in X^*(P)_F;$$

Using Iwasawa decomposition we may extend the definition of $H_P$ to $G(\mathbb{A})$ by asking it to be $G(\mathcal{O})$-right invariant.

2.4. Haar measures. The Haar measures are normalized in the following way. For any place $v$ of $F$ and for a Lie algebra $\mathfrak{g}$ defined over $\mathbb{F}_v$, the volumes of the sets $\mathfrak{g}(\mathcal{O}_v)$ and $\mathfrak{g}(\mathcal{O})$ are normalized to be 1; the volume of $G(\mathcal{O}_v)$ and $G(\mathcal{O})$ are normalized to be 1; for every semi-standard parabolic subgroup $P$ of $G$ defined over $\mathbb{F}_v$, the measure on $N_P(\mathbb{A})$ is normalized so that $\text{vol}(N_P(F) \setminus N_P(\mathbb{A})) = 1$.

3. Characteristic of the base field and unipotent/nilpotent elements

3.1. In this article, we will treat separately the case of a group and the case of a Lie algebra, according to that we suppose that the hypothesis $(*)_G$ or the hypothesis $(*)_\mathfrak{g}$ is satisfied.
(\textit{*})_G \text{ The characteristic does not divide } |\pi_1(G^{\text{der}})|.

(\textit{*})_\mathbb{F}_q \text{ The characteristic is very good for } G \text{ or } G = \text{GL}_n. \text{ The characteristic } p \text{ does not divide the degree of the minimal splitting field of } G.

We recall that if } G \text{ is simple, then the characteristic } p \text{ is said to be very good if } p \neq 2 \text{ when } G \text{ is one of type } B, C, D, p \neq 2, 3 \text{ if } G \text{ is of one of type } E, F, G, \text{ and } p \neq 2, 3, 5 \text{ if } G \text{ is of type } E_8. \text{ For type } A_{n-1}, \text{ we require that } p \mid n. \text{ In general, a prime is very good for } G \text{ if it’s very good for every simple factor of } G. \text{ For more discussion of “very good primes”, see } [\text{SS70, I.4}] \text{ or } [\text{McN04, 2.1}]. \text{ In particular, (\textit{*})_\mathbb{F}_q \text{ implies (\textit{*})_G}.

The following Proposition is well known to experts in algebraic groups over positive characteristic.

\textbf{Proposition 3.1.} \textit{Let } G \textit{ be a reductive group defined over } \mathbb{F}_q \textit{ so that } G. \textit{ Then }

\begin{enumerate}
  \item If (\textit{*})_G \textit{ is satisfied, the central isogeny } G^{\text{sc}} \rightarrow G^{\text{der}} \textit{ is smooth (equivalently is separable in older terminology).}
  \item If (\textit{*})_\mathbb{F}_q \textit{ is satisfied, the Lie algebra } g \textit{ admits a non-degenerate } G\text{-invariant bilinear form defined over } \mathbb{F}_q.
\end{enumerate}

\textbf{Proof.} \textit{The fundamental group } \pi_1(G^{\text{der}}) \textit{ is the kernel of } G^{\text{sc}} \rightarrow G^{\text{der}}. \textit{ It is étale since } p \textit{ does not divide the order of } \pi_1(G^{\text{der}}), \textit{ hence } G^{\text{sc}} \rightarrow G^{\text{der}} \textit{ is smooth.}

\textit{If } G \textit{ is semi-simple and simply connected, the assertion[2] is proved in } [\text{KV06, Lemma 1.8.12}] \textit{ (the hypothesis that the characteristic } p \textit{ does not divide the degree of the minimal splitting field of } G \textit{ ensures that the bilinear form in } \text{loc. cit.} \text{ is non-zero). \textit{In general, consider the central isogeny } Z \times G^{\text{sc}} \rightarrow G, \textit{ where } Z \textit{ is the maximal central torus of } G, \textit{ it is separable since } Z \cap G^{\text{sc}} \textit{ is a subgroup of } ZG^{\text{sc}} \textit{ which has order prime to } p. \textit{ Therefore, we have a } G\text{-equivariant isomorphism of Lie algebras } g \cong g^{\text{sc}} \oplus \mathfrak{z}. \textit{ We can extend a non-degenerate bilinear form on } g^{\text{sc}} \textit{ to } g \textit{ by choosing an arbitrary non-degenerate bilinear form on } \mathfrak{z}.\]

We know that every semi-simple element in } G(F) \textit{ is contained in the set of } F\text{-points of some maximal torus defined over } F. \textit{ We also know that a unipotent (resp. nilpotent) element in } G(F) \textit{ (resp. } g(F)) \textit{ is contained in the unipotent radical (resp. the Lie algebra of the unipotent radical) of a parabolic subgroup defined over } F. \textit{ In positive characteristic, it is not always true that one can choose such a parabolic subgroup defined over } F. \textit{ We will need the following result of P. Gille which is applicable for all fields } F \textit{ such that } |F : F^p| \leq p, \textit{ in particular for a global function field.}

\textbf{Proposition 3.2 (Gille, [Gi02]).} \textit{Let } G \textit{ be a reductive group so that the hypothesis (\textit{*})_G \textit{ in the[2.4] is satisfied. Then every unipotent element } u \in G(F) \textit{ is contained in the unipotent radical of some parabolic subgroup of } G \textit{ defined over } F.}\n
\textbf{Proof.} \textit{The theorem is proved by Gille when } |F : F^p| \leq p \textit{ for semi-simple and simply connected groups ([Gi02, Theorem 2, p. 313]).}

In general, let } G^{\text{sc}} \rightarrow G^{\text{der}} \textit{ be the simply connected covering of } G^{\text{der}} \textit{ which is of degree prime to } p \textit{ by our assumption, then any unipotent element in } G^{\text{der}}(F) \textit{ is contained in the unipotent radical of some parabolic subgroup defined over } F \textit{ by Proposition (ii) and (iii) of page 267 of } [\text{Ti87}]. \textit{ In fact, we have an exact sequence:}

\[1 \rightarrow \pi_1(G^{\text{der}})(F) \rightarrow G^{\text{sc}}(F) \rightarrow G^{\text{der}}(F) \rightarrow H^1(F, \pi_1(G^{\text{der}})).\]
Since \( p \nmid |\pi_1(G_{der})| \) and unipotent elements have \( p \)-power orders, there is a bijection between unipotent elements in \( G^{sc}(F) \) and \( G^{der}(F) \). The rest is clear.

Now consider the exact sequence:

\[
1 \rightarrow G^{der}(F) \rightarrow G(F) \rightarrow (G/G^{der})(F).
\]

The image of a unipotent element \( u \in G(F) \) in \( (G/G^{der})(F) \) is the identity as it’s a unipotent element in the torus \( G/G^{der} \). Hence \( u \) belongs to \( G^{der}(F) \). We know that \( P \mapsto P \cap G^{der} \) defines a bijection between the set of parabolic subgroups of \( G \) and that of \( G^{der} \) (Proposition 5.3.4 [Co14]), such that the unipotent radical \( N_{PG}^{der} \) of \( P \cap G^{der} \) in \( G^{der} \) equals to \( N_P \cap G^{der} \) (Proposition 4.1.10.(2) of [Co14]). As \( u \) is contained in the unipotent radical of an \( F \)-parabolic subgroup of \( G^{der} \), it is in the unipotent radical of the corresponding parabolic subgroup of \( G \).

\[\square\]

\textbf{Proposition 3.3} (McNinch, [McN04]). Let \( G \) be a reductive group so that the hypothesis \( (\ast)_q \) in the \( \mathbb{Z} \) is satisfied. Let \( X \in \mathfrak{g}(F) \) be a nilpotent element, then it’s contained in the Lie algebra of the unipotent radical of a parabolic subgroup defined over \( F \).

\textit{Proof}. Under our hypotheses, there is a cocharacter \( \phi \) associated to \( X \) which is defined over \( F \), i.e. \( \lambda : \mathbb{G}_m/F \rightarrow G \) such that \( \text{Ad}(\phi(t))X = t^\lambda X \) for all \( t \in \mathbb{G}_m \) ([McN04 Theorem 26 and Proposition 6]). This cocharacter defines the \( F \)-parabolic subgroup \( P_G(\lambda) := \{ g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \} \) of \( G \) such that \( X \) lies in the Lie algebra of the unipotent radical \( U_G(\lambda) := \{ g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = 1 \} \) of \( P(\lambda) \).

\[\square\]

4. COMBINATORICS OF ROOTS AND CORootS UNDER FIELD EXTENSION

4.1. When studying reduction theory of adeles, we need some results concerning the combinatorics of roots and coroots under field extensions.

Let \( \infty \) be a place of \( F \). Using the inclusion \( X^*(T)_F \hookrightarrow X^*(T)_{F,\infty} \), we may view \( \mathfrak{a}_B^* \) as a subspace of \( \mathfrak{a}_{B,\infty} \). Taking dual of this embedding, we get a surjection

\[(4.1) \quad p_\infty : \mathfrak{a}_{B,\infty} \twoheadrightarrow \mathfrak{a}_B.
\]

Let’s denote \( p_\infty(H) \) by \([H]\). We define the local Harish-Chandra’s function \( H_{B,\infty} : G(F_\infty) \rightarrow \mathfrak{a}_{B,\infty} \) by requiring that for any \( x = pk \) with \( p \in B(F_\infty) \) and \( k \in G(O_\infty) \), one has

\[(4.2) \quad |\kappa_\infty|^{(\alpha,H_{B,\infty}(x))}\infty = |\alpha(p)|_\infty, \quad \forall \alpha \in X^*(B)_{F,\infty}.
\]

\textbf{Proposition 4.1}. Using the embedding \( G(F_\infty) \hookrightarrow G(\mathbb{A}) \), the relation between global and local Harish-Chandra’s map are given by

\[(4.3) \quad H_B(g_\infty) = [\kappa_\infty : \mathbb{F}_q][H_{B,\infty}(g_\infty)], \quad \forall g_\infty \in G(F_\infty).
\]

\textit{Proof}. It follows directly from the definitions.

\[\square\]

Let \( A \) be the maximal \( F \)-split subtorus of \( T \) and \( A' \) be the maximal \( F_\infty \)-split subtorus of \( T_{F,\infty} \). Clearly \( A_{F,\infty} \subseteq A' \) and \( A' \) is in fact defined over the residual field \( \kappa_\infty \) of \( F_\infty \). As \( G \) is quasi-split over \( F \), equivalently the centralizer of \( A \) in \( G \) is the torus \( T \), a non-zero root \( \alpha \) for the adjoint action of \( A' \) on \( \mathfrak{g}_{F,\infty} \) is non-zero after restriction to \( A \). We thus have a surjective map of the root systems

\[(4.4) \quad \Phi(G, A')_{F,\infty} \twoheadrightarrow \Phi(G, A)_{F,\infty} \cong \Phi(G, A)_F.
\]
Theorem 4.2. Let $\alpha' \in \Phi(G, A')$. The restriction $\alpha'|_A$ is an element in $\Phi(G, A)$ and we have

$$[\alpha'^\vee] = c_{\alpha'}(\alpha'|_A)^\vee,$$

for some $1 \geq c_{\alpha'} > 0$.

Proof. There is no harm to choose $\kappa_\infty$ to be $\mathbb{F}_q$ by fixing an embedding $\kappa_\infty \hookrightarrow \mathbb{F}_q$. Suppose $\kappa_\infty \cong \mathbb{F}_q$. Let $\Gamma_d = \text{Gal}(\mathbb{F}_q^d/\mathbb{F}_q)$. It acts on $X^*(T)_{F_\infty} \cong X^*(T)_{\mathbb{F}_q^d}$. Note that $\Gamma_d$ preserves $\Phi' = \Phi(G, A')_{F_\infty}$ and is compatible with the natural action of $W(\Phi')$, the Weyl group of $\Phi'$, i.e. $\sigma(w\chi') = \sigma w^\sigma \chi'$, for any $\sigma \in \Gamma_d$, any $w \in W(\Phi')$, and any $\chi' \in X^*(A')_{F_\infty}$. Thus $a_{B, \infty}^*$ admits a positive definite symmetric bilinear $(,)_\infty$ which is invariant under the Weyl group $W(\Phi')$ and the Galois group $\Gamma_d$.

By definition, for any $\chi \in a_B^*$, one has $\langle \chi, [\alpha'^\vee] \rangle = \langle \chi, \alpha'^\vee \rangle_\infty$, hence

$$\langle \chi, [\alpha'^\vee] \rangle = \frac{2(\langle \alpha', \chi \rangle_\infty)}{(\alpha', \alpha')_\infty} \tag{4.5}$$

Since $A_{F_\infty}$ is a sub-torus of $A'$, we have a quotient map

$$X^*(A')_{F_\infty} \to X^*(A)_{F_\infty} \cong X^*(A)_F,$$

which defines a projection via the inclusions $X^*(T)_F \subseteq X^*(A)_F$ and $X^*(T)_{F_\infty} \subseteq X^*(A')_{F_\infty}$:

$$p^*_{\infty} : a_{B, \infty}^* \to a_B^*.$$

We leave it to the reader to verify that for any $\chi' \in X^*(A')_{F_\infty}$, we have

$$p^*_\infty(\chi') = \chi'|_A = \frac{1}{d} \sum_{\sigma \in \Gamma_d} \sigma \chi'.$$

Hence $p^*_\infty$ is an orthogonal projection and for any $\chi \in a_B^*$, we have

$$\langle \alpha', \chi \rangle_\infty = \langle p^*_\infty(\alpha'), \chi \rangle_\infty = \langle \alpha'|_A, \chi \rangle_\infty.$$

We may rewrite (4.5) as

$$\langle \chi, \frac{1}{c_{\alpha'}}[\alpha'^\vee] \rangle = \frac{2(\langle \alpha'|_A, \chi \rangle_\infty)}{(\alpha', \alpha')_\infty},$$

with $c_{\alpha'} = \frac{(\langle \alpha'|_A, \alpha' \rangle_\infty)}{(\alpha', \alpha')_\infty}$, which implies the assertion. \qed

4.2. Complementary polyhedra. The definition of complementary polyhedra is introduced by Behrend in [Be95] and by Arthur in different languages (see the remark below).

Definition 4.3. Let $(X_s)_{s \in \mathbb{W}}$ be a family of points in $a_B$ parametrized by elements of the Weyl group of $G$. It's called a complementary polyhedron (for the root system $\Phi(G, A)$) if for any $s = s_\alpha t$, where $s_\alpha$ is the symmetry with respect to the simple root $\alpha \in \Delta_B$, one has

$$X_t - X_s = b_\gamma(s, t) \gamma^\vee,$$

with $b_\gamma(s, t) \geq 0$ and $\gamma = s^{-1} \alpha = -t^{-1} \alpha$.

Remark 4.4. 1. If $(X_s)_{s \in \mathbb{W}}$ is a complementary polyhedron in $a_B$ for the root system $\Phi(G, A)$, then for any parabolic subgroup $Q \in \mathbb{P}(T)$, $(X_s)_{s \in \mathbb{W}_Q}$ is a complementary polyhedron in $a_B \cong a_{B \cap M_Q}$ for the root system $\Phi(M_Q, A)$. 2.
2. Arthur calls \((X_s)_{s\in W}\) a regular orthogonal family if the \(b_s(s,t)\) above are strictly negative (compare \cite{LW11} Section I.6).

Recall that \(W = W^{(G,T)}(F_q)\), and \(W_\infty \cong W^{(G,T)}(\kappa_\infty)\). Thus \(W\) is identified with a subgroup of \(W_\infty\).

**Proposition 4.5.** Let \((X_s)_{s\in W_\infty}\) be a complementary polyhedron in \(a_{B,\infty}\) for the root system \(\Phi(G,A')\), then \(((X_s))_{s\in W}\) is a complementary polyhedron in \(a_{B}\) for \(\Phi(G,A)\).

**Proof.** Let \(\alpha \in \Delta_B\) be a simple root, and \(s, t \in W\) such that \(s = s_{\alpha}t\). Then (by \cite{LW11} 1.5.1) there are non-negative numbers \(b'_{\beta}(s,t)\) depending on the choice of a reduced decomposition of \(s_{\alpha}\) into simple symmetries in \(W_\infty\) such that

\[
X_s - X_t = \sum_{\{\beta' \in \Phi(G,A) \mid \beta' > 0, s\beta < 0\}} b'_{\beta'}(s,t)\beta'\vee.
\]

Note that \(\beta' > 0\) if and only if \(\beta'|_\Delta > 0\), and the only root \(\beta \in \Phi(G,A)\) such that \(t\beta > 0\) but \(s\beta < 0\) is \(t^{-1}\alpha\). Hence those \(\beta'\) which show up in the sum above are those which satisfy \(\beta'|_\Delta = t^{-1}\alpha\). By Theorem 4.2,

\[
[X_s] - [X_t] = b_{t^{-1}\alpha}(s,t)(t^{-1}\alpha)\vee,
\]

where

\[
(4.6) \quad b_{t^{-1}\alpha}(s,t) = \sum_{\beta'|_\Delta = t^{-1}\alpha} c_{\beta'} b'_{\beta'}(s,t).
\]

\[
\square
\]

**Definition 4.6.** We say that a pair of vectors \((\xi, X) \in a_{B,\infty} \times a_B\) is admissible if \(d(X) \geq 0\), and

\[
-\frac{1}{[\kappa_\infty : \mathbb{F}_q]}d(X) \leq \langle \alpha, \xi \rangle_\infty \leq \frac{1}{[\kappa_\infty : \mathbb{F}_q]}d(X) + [\kappa_\infty : \mathbb{F}_q], \quad \forall \alpha \in \Phi(G_{F_\infty}, A')_+^\ast,
\]

where \(d(X) := \min_{\alpha \in \Delta_B} \alpha(X)\), and \(\Phi(G_{F_\infty}, A')_+^\ast\) is the set of positive roots in the reduced root system.

For any \(x \in G(A)\). We have an Iwasawa decomposition: \(x = pk\), with \(p \in B(A)\) and \(k \in G(O)\). The element \(k\) is unique up to left translation by \(B(A) \cap G(O) = B(O)\), hence determines an element in

\[
T \setminus G(O_\infty)/I \cong B(\kappa_\infty)\backslash G(\kappa_\infty)/B(\kappa_\infty) \cong W_\infty,
\]

where \(I\) is the Iwahori subgroup of \(G(F_\infty)\) with respect to \(B\), i.e. group of elements in \(G(O_\infty)\) whose images in \(G(\kappa_\infty)\) are contained in \(B(\kappa_\infty)\). We denote this Weyl element by \(s_x\).

We have the following proposition. In fact, for the proofs of our main theorems we only need it in the case when \(d(X) \gg 0\) depending on \(\xi\), which is a direct consequence of \cite{LW11} Lemme 3.3.2]. However, it’s interesting in itself to have an explicit bound since there might be some further applications that require \(X = 0\).

**Theorem 4.7.** Let \((\xi, X) \in a_{B,\infty} \times a_B\) be admissible. For any \(h \in G(A)\), we define for each \(s \in W\) a vector

\[
X_s = s^{-1}(H_B(w_s h) + [s_{w_s h} \xi] - X) \in a_B.
\]
Recall that \( w_s \) is a fixed representative in \( N_G(T)(\mathbb{F}_q) \) of \( s \in W \). Then the family \((X_s)_{s \in W}\) is a complementary polyhedron in \( a_B \).

**Proof.** The problem is local in nature. It suffices to prove that, for any place \( v \) of \( F \), any \( h \in G(F_v) \), the family \((s^{-1}H_B(w_s h))_{s \in W}\) is a complementary polyhedron in \( a_B \) and for any \( h \in G(F_\infty) \),

\[
(s^{-1}(H_B(w_s h) + [s_{w,h} \xi] - X))_{s \in W}
\]

is a complementary polyhedron in \( a_B \). Without loss of generality, we prove the second statement only.

Recall that \( A' \) is the maximal \( F_\infty \)-split subtorus of \( T \). For any \( \gamma \in \Phi(G, A') \), let \( N_\gamma \) be the root subgroup of \( G_{F_\infty} \) associated to \( \gamma \). Fix a valuation \((\varphi_\gamma)_{\gamma \in \Phi(G, A')}\) of the root datum (cf. [BT72, 6.2.1] and [BT84, 4.2]) which is centered at the hyperspecial point corresponding to \( G(\mathcal{O}_\infty) \), and \( \varphi_\gamma : N_\gamma(F_\infty) \to \mathbb{Z} \cup \{\infty\} \) is normalized to be surjective. Let

\[
N_{\gamma, n} = \varphi^{-1}_\gamma([n, \infty]) \subseteq N_\gamma(F_\infty).
\]

For any \( s \in W_\infty \), we can choose a representative in \( N_G(T)(\kappa_\infty) \). Let \( B_1 = w^{-1}_s B w_s \) and \( B_2 = w^{-1}_t B w_t \) for \( s, t \in W_\infty \), be a pair of adjacent Borel subgroups defined over \( F_\infty \) both containing \( T \). We may assume that \( s = s_\alpha t \) with \( \alpha \in \Delta_{B, \infty} \), then \( \Delta_{B_1, \infty} \cap (-\Delta_{B_2, \infty}) = \{\gamma\} \), where \( \gamma = s^{-1}(\alpha) = -t^{-1} \alpha \). We have

\[
N_{B_1} = (N_{B_1} \cap N_{B_2}) N_\gamma.
\]

By Iwasawa decomposition, we can write \( h = auxk \) with \( a \in T(F_\infty) \), \( u \in (N_{B_1} \cap N_{B_2})(F_\infty) \), \( x \in N_\gamma(F_\infty) \) and \( k \in G(\mathcal{O}_\infty) \). We claim that

\[
(4.7) \quad t^{-1}H_B(w_1 h) - s^{-1}H_B(w_s h) = \begin{cases} 0, & \text{if } x \in N_{\gamma, 0}; \\ -[\kappa_\infty : \mathbb{F}_q]\varphi_\gamma(x) \varphi_\gamma(x)^\vee, & \text{if } x \notin N_{\gamma, 0}; \end{cases}
\]

and that

\[
(4.8) \quad t^{-1}s_{w_1 h} \xi - s^{-1}s_{w_s h} \xi = \begin{cases} 0, & \text{if } x \in N_{\gamma, 0} \text{ and } s^{-1}_{w_1, \alpha} > 0; \\ 0 \text{ or } -\langle s^{-1}_{w_1, \alpha}, \xi \rangle_{\infty} \varphi_\gamma(x)^\vee, & \text{if } x \in N_{\gamma, 0} \text{ and } s^{-1}_{w_1, \alpha} < 0; \\ -\langle s^{-1}_{w_1, \alpha}, \xi \rangle_{\infty} \varphi_\gamma(x)^\vee, & \text{if } x \notin N_{\gamma, 0}. \end{cases}
\]

These assertions suffice to prove the theorem. In fact, if \( x \in N_{\gamma, 0} \), then the sum of \( (4.7) \) and \( (4.8) \) is either 0 or of the form \( c\gamma^\vee \) with \( c \geq \frac{1}{[\kappa_\infty : \mathbb{F}_q]} d(X) \). If \( x \notin N_{\gamma, 0} \), then since

\[
-\langle \kappa_\infty : \mathbb{F}_q \rangle \varphi_\gamma(x) \geq [\kappa_\infty : \mathbb{F}_q],
\]

and by the definition of admissible pairs (Definition 4.8), if \(-s^{-1}_{w_1, \alpha}\) is positive, we have

\[
\langle -s^{-1}_{w_1, \alpha}, \xi \rangle_{\infty} \geq -\frac{1}{[\kappa_\infty : \mathbb{F}_q]} d(X).
\]

Otherwise \(-s^{-1}_{w_1, \alpha}\) is positive, we have

\[
-\langle s^{-1}_{w_1, \alpha}, \xi \rangle_{\infty} \geq -\frac{1}{[\kappa_\infty : \mathbb{F}_q]} d(X) - [\kappa_\infty : \mathbb{F}_q].
\]

The sum of \( (4.7) \) and \( (4.8) \) is always of the form \( c\gamma^\vee \) with \( c \geq -\frac{1}{[\kappa_\infty : \mathbb{F}_q]} d(X) \). Note that for each \( \alpha_0 \in \Delta_B \), there are at most \([\kappa_\infty : \mathbb{F}_q]\) roots \( \alpha \in \Delta_{B, \infty} \) such that \( \alpha|_A = \alpha_0 \) (they
form one orbit for a Galois action). The statement of the theorem is then a corollary of (1.6), (1.7) and (1.8). We prove (1.7) and (1.8) in the following.

For (1.7), a proof for a split reductive group is written down in [CL10, 8.16.8]. The proof is similar in our case. For any \( h \in G(F_\infty) \) we have

\[
s^{-1}H_{B,\infty}(w_\gamma h) = H_{w_\gamma^{-1}Bw_\gamma,\infty}(h).
\]

We always have \( H_{B,\infty}(h) = H_{B,\infty}(a) \) and

\[
t^{-1}H_{B,\infty}(w_\gamma h) - s^{-1}H_{B,\infty}(w_\gamma h) = s^{-1}H_{B_1,\infty}(w_\gamma x).
\]

If \( x \in N_{\gamma,0} \), then clearly \( H_{B,\infty}(x) = 0 \). Otherwise let \( m = \varphi_\gamma(x) \) then \( m < 0 \). Let

\[
M_{\gamma,m} = M_\gamma \cap N_{\gamma}(F_\infty)\varphi_\gamma^{-1}(m)N_{\gamma}(F_\infty),
\]

where in our case \( M_\gamma = w_\gamma T(F_\infty) \). We have ([BT72, 6.2.2. (3)])

\[
(4.9) \quad w_\gamma \varphi_\gamma^{-1}(m) \subseteq w_\gamma N_{\gamma,-m}M_{\gamma,m}N_{\gamma,-m} = N_{\gamma,-m}(w_\gamma M_{\gamma,m})N_{\gamma,-m}.
\]

It follows that \( H_{B_1,\infty}(w_\gamma, x) = H_{B_1,\infty}(w, n) \) for some \( n \in M_{\gamma,m} \). As \( M_{\gamma,m} \subseteq C^0(F_\infty) \), we deduce from [BT72, 6.2.10(ii)] that \( H_{B_1,\infty}(w_\gamma M_{\gamma,m}) = m\gamma' \). The equality (1.7) then follows from the Proposition 4.1.

Now we prove (1.8). Note that we have

\[
w_t(N_{B_2} \cap N_{B_1})w_t^{-1} \subseteq B, \quad \text{and} \quad w_s(N_{B_2} \cap N_{B_1})w_s^{-1} \subseteq B,
\]

hence

\[
s_{w_t h} = s_{w_t z k}, \quad \text{and} \quad s_{w_s h} = s_{w_s z k}.
\]

Note that we always have

\[
Iw_\alpha Iw_k I \subseteq Iw_\alpha s_{w_t z k} I \cup I s_{w_t z k} I.
\]

If \( x \in N_{\gamma,0} \), then

\[
w_t x w_t^{-1} \in I.
\]

We have \( s_{w_t z k} = s_{w_t k} \). If moreover, \( s_{w_t k}^{-1} \alpha > 0 \), then \( l(s_\alpha s_{w_t k}) = l(s_{w_t k}) + 1 \) (cf. [LW11, 1.3.1]) hence we have (cf. Théorème 2, § 2, IV of [Bou68])

\[
Iw_\alpha Iw_k I = Iw_\alpha s_{w_k} I.
\]

Otherwise if \( s_{w_t k}^{-1} \alpha < 0 \), as \( Iw_\alpha Iw_t I \subseteq Iw_\alpha s_{w_k} I \cup I s_{w_k} I \), we deduce by definition that:

\[
s_{w_s z k} = \begin{cases} s_\alpha s_{w_t k}, & s_{w_t k}^{-1} \alpha > 0; \\ s_\alpha s_{w_t k} \text{ or } s_{w_t k}, & s_{w_t k}^{-1} \alpha < 0. \end{cases}
\]

If \( x \notin N_{\gamma,0} \), then there is an \( m < 0 \) such that \( x \in N_{\gamma,m} \). As \( N_{-\alpha,-m} \subseteq I \) whenever \( m < 0 \), by (1.9), we have \( s_{w_s a w_t z k} = s_{w_t k} \) and \( s_{w_t z k} = s_{w_s k} \).

The equality (1.8) follows from these calculations. \( \square \)

5. Statements of the main theorems

5.1. Let \( f \in C^\infty_c(G(\mathbb{A})) \) be a complex smooth function with compact support over \( G(\mathbb{A}) \). Let \( Q \) be a standard parabolic subgroup of \( G \) defined over \( F \), the kernel function of \( f \) acting on \( L^2(N_Q(\mathbb{A})M_\gamma(F) \backslash G(\mathbb{A})/\mathbb{Z}_G) \) is given by:

\[
k_Q(x, y) = \sum_{a \in \mathbb{Z}_G} \sum_{\gamma \in M_\gamma(F)} \int_{N_Q(\mathbb{A})} f(ay^{-1}\gamma u x) du.
\]
For any \( x \in N_Q(\mathbb{A})M_Q(F)\backslash G(\mathbb{A})/\Xi_G \), we will note
\( k_Q(x) = k_Q(x, x). \)

(5.2)

In the case of Lie algebras, a similar definition is given by analogy. Let \( f \in C_c(\mathfrak{g}(\mathbb{A})) \) and \( Q \) be a standard parabolic subgroup, we define \( \xi_Q(x) \) for any \( x \in G(\mathbb{A}) \) by:
\[
\xi_Q(x) = \sum_{\Gamma \in \text{m}_q(F)} \int_{\mathfrak{b}_Q(\mathbb{A})} f(\text{Ad}(x^{-1})(\Gamma + U))dU.
\]

(5.3)

We still fix a place \( \infty \) of \( F \). For any vector \( \xi \in \mathfrak{a}_{\mathbb{B}, \infty} \) and \( X \in \mathfrak{a}_B \), we define a variant of Arthur’s truncated kernel by:
\[
k^{\xi, X}(x) = \sum_{Q \in \mathcal{P}(\mathfrak{p})} (-1)^{\dim \mathfrak{a}_Q^+} \sum_{\delta \in Q(F) \backslash G(F)} \tilde{\tau}_Q(H_B(\delta x) + [s_{\delta x}\xi] - X) k_Q(\delta x),
\]
for any \( x \in G(\mathbb{A}) \) and \( (\xi, X) \in \mathfrak{a}_{\mathbb{B}, \infty} \times \mathfrak{a}_B \). One can see that, for every \( \delta' \in Q(F) \), the elements \( s_{\delta x} \) and \( s_{\delta' \delta x} \) represent the same coset in \( W_{\mathfrak{a}_Q} \backslash W_\infty \), hence \( s_{\delta x}\xi \) and \( s_{\delta' \delta x}\xi \) have the same projection in \( \mathfrak{a}_Q \). Besides, for each \( x \in G(\mathbb{A}) \) and each function \( f \) with compact support, the definition only involves a finite sum \((\text{LW11} \ 3.7.1))\). This shows that the definition makes sense.

As there is a strong analogy between two cases, we will only present one case if the other case follows by the same arguments.

5.2. We define two elements \( \gamma \) and \( \gamma' \) in \( G(F) \) (resp. in \( \mathfrak{g}(F) \)) to be equivalent if either both of them admit Jordan-Chevalley decomposition and their semi-simple parts \( \gamma_s \) and \( \gamma'_s \) are \( G(F) \)-conjugate, or neither of them admit Jordan-Chevalley decomposition. So an equivalent class is a union of elements which do not admit Jordan-Chevalley decomposition. Let \( \mathcal{E} = \mathcal{E}(G(F)) \) or \( \mathcal{E}(\mathfrak{g}(F)) \)

be the set of equivalent classes for either the group case or the Lie algebra case if it doesn’t cause a confusion.

Let \( o \in \mathcal{E} \) be an equivalence class. For any standard parabolic subgroup \( Q \) of \( G \) and \( x \in G(\mathbb{A}) \), let’s define
\[
k_{Q, o}(x) = \sum_{a \in \Xi_G} \sum_{\gamma \in M_Q(F) \gamma o} \int_{N_Q(\mathbb{A})} f(ax^{-1}\gamma nx)du,
\]
and
\[
k^{\xi, X}_o(x) = \sum_{Q \in \mathcal{P}(\mathfrak{p})} (-1)^{\dim \mathfrak{a}_Q^+} \sum_{\delta \in Q(F) \backslash G(F)} \tilde{\tau}_Q(H_B(\delta x) + [s_{\delta x}\xi] - X) k_{Q, o}(\delta x).
\]
It follows from definitions that
\begin{equation}
(5.8) \quad k^\xi,X(x) = \sum_{o \in \mathcal{E}} k^\xi,X_o(x).
\end{equation}

Again, for each \( x \in G(\mathbb{A}) \) and each compactly supported function \( f \in C_c^\infty(G(\mathbb{A})) \), the sum in (5.8) is in fact a finite sum. The same construction applies for the Lie algebra \( \mathfrak{g} \), so we can define \( J^\xi,X \).

Our definitions differ from that of Arthur only by the presence of \( \xi \). The following theorem generalizes [Laf97, Proposition 11, p.227] and [Ch15, Théorème 6.1.1. (2),(3)] whose proof will be given later in Section 8.

**Theorem 5.1.** Assume the hypothesis (*) in the Section 5 on the characteristic is satisfied. For any equivalence class \( o \in \mathcal{E} \) and any pair of vectors \( (\xi,X) \in \mathfrak{a}_B,\infty \times \mathfrak{a}_B \), the function \( x \mapsto k^\xi,X_o(x) \) (resp. \( x \mapsto \xi,X_o(x) \)) has compact support over \( G(F) \setminus G(\mathbb{A})/Z_G(\mathbb{A}) \).

Moreover, when \( (\xi,X) \in \mathfrak{a}_B,\infty \times \mathfrak{a}_B \) is admissible, there is a function \( F^\xi,X(\cdot) \) over \( G(F) \setminus G(\mathbb{A})/Z_G(\mathbb{A}) \) which is the characteristic function of a set which is compact in \( G(F) \setminus G(\mathbb{A})/Z_G(\mathbb{A}) \). For every function \( f \in C_c^\infty(G(\mathbb{A})) \) (resp. \( f \in C_c^\infty(\mathfrak{g}(\mathbb{A})) \)), if \( d(X) := \min_{\alpha \in \Delta_B} \alpha(X) \) is large enough depending on \( \xi \) and \( f \), we have for any \( x \in G(\mathbb{A}) \)
\[
(5.9) \quad k^\xi,X_o(x) = F^\xi,X(x)k_{G,o}(x)
= F^\xi,X(x) \sum_{\alpha \in \Xi_G} \sum_{\gamma \in o} f(ax^{-1}\gamma x).
\]

(resp. \( \xi,X_o(x) = F^\xi,X(x) \sum_{\Gamma \in \xi} f(ad(x^{-1})(\Gamma)) \).)

**Remark 5.2.** Unlike the above theorem which applies for each \( k^\xi,X_o(x) \) that requires an additional hypothesis on the characteristic, we can prove by the same proof that \( k^\xi,X(x) = \sum_{o \in \mathcal{E}} k^\xi,X_o(x) \) has compact support in \( G(F) \setminus G(\mathbb{A})/Z_G(\mathbb{A}) \) in any characteristic \( p \).

After this theorem, the following definitions make sense: for \( f \in C_c^\infty(G(\mathbb{A})) \) and any \( (\xi,X) \in \mathfrak{a}_B,\infty \times \mathfrak{a}_B \), we define the truncated trace by
\begin{equation}
(5.10) \quad J^{G,\xi,X}(f) := \int_{G(F) \setminus G(\mathbb{A})/\Xi_G} k^\xi,X(x)dx;
\end{equation}

and for any \( o \in \mathcal{E} \)
\begin{equation}
(5.11) \quad J^{o,\xi,X}(f) := \int_{G(F) \setminus G(\mathbb{A})/\Xi_G} k^\xi,X_o(x)dx;
\end{equation}

Similarly we define \( J^{\xi,X}(f) \) and \( J^{o,\xi,X}(f) \) in the Lie algebra case. We will omit the superscript \( G \) or \( \mathfrak{g} \) if it is clear from the context which case we’re dealing with. We will also omit \( X \) in the notation if it is set to be zero, which is also our main interests. We have the following corollary of the Theorem 5.1.

**Corollary 5.3.** In either the group case, or the Lie algebra case, we have the coarse geometric expansion of the truncated trace:
\begin{equation}
(5.11) \quad J^{\xi}(f) = \sum_{o \in \mathcal{E}} J^{\xi,o}(f),
\end{equation}
where there are only finitely many non-zero terms in the sum.

**Remark 5.4.** If the characteristic is not too large, but still satisfies \((\ast)_G\) of the Section 3.1, then the class \(o \in E\) consisting of elements which do not admit Jordan-Chevalley decomposition can be non-empty, hence the corresponding contribution \(J^G_o(f)\) can be non-zero.

### 5.3. Quasi-polynomial behaviour.

The definition below is taken from [Ch15, 4.5.3] (although slightly different).

**Definition 5.5.** Let \(\Phi : \mathfrak{g} \cdot \mathbb{Q} \to \mathbb{C}\) be a function. It’s called a quasi-polynomial, if for any lattice \(L_0 \subseteq \mathfrak{g} \cdot \mathbb{Q}\), there exist a finite set \(f \subseteq \frac{2\pi}{\log q} \mathfrak{g} \cdot \mathbb{Q}\), and a family of polynomials \((p_\nu)_{\nu \in f}\) such that for any \(X \in L_0\) we have

\[
\Phi(X) = \sum_{\nu \in f} p_\nu(X) q^{\langle \nu, X \rangle}.
\]

The following result is then an analogue of a theorem of Arthur over number fields and a generalization of the case \(G = \text{GL}_n\) and \(\xi = 0\) of a result of Chaudouard [Ch15, Théorème 6.1.1.(4)].

**Theorem 5.6.** For each \(o \in E\), the maps \(X \mapsto J^G_{G,\xi, X}(f)\) and \(X \mapsto J^G_{\text{Q},\xi, X}(f)\) are quasi-polynomials.

The proof will be given in the Section 8.

### 5.4. A trace formula for Lie algebras.

Suppose that our assumption \((\ast)_G\) on characteristic in the Section 3.1 is satisfied.

Let \(\langle \cdot, \cdot \rangle\) be a \(G\)-invariant bilinear form on \(\mathfrak{g}\) defined over \(\mathbb{F}_q\), which exists thanks to the assumption on characteristic (Proposition 3.1). Thus, it defines by taking respectively \(\mathbb{A}\)-points, a \(G(\mathbb{A})\)-invariant non-degenerate bilinear form on \(\mathfrak{g}(\mathbb{A})\).

We fix a non-trivial additive character: \(\psi : \mathbb{F} \setminus \mathbb{A} \to \mathbb{C}^\times\). For any \(f \in C^\infty_c(\mathfrak{g}(\mathbb{A}))\), the global Fourier transformation is defined by

\[
\hat{f}(X) := \int_{\mathfrak{g}(\mathbb{A})} f(X) \psi(\langle X, Y \rangle) dY.
\]

By Poisson summation formula, for any \(x \in G(\mathbb{A})\) we have

\[
\sum_{X \in \mathfrak{g}(F)} f(\text{ad}(x^{-1})X) = q^{(1-g) \dim \mathfrak{g}} \sum_{X \in \mathfrak{g}(F)} \hat{f}(\text{ad}(x^{-1})X).
\]

where the constant \(q^{(1-g) \dim \mathfrak{g}}\) is the inverse of volume of \(\mathfrak{g}(\mathbb{A})/\mathfrak{g}(F)\) since the measure is normalized so that \(\text{vol}(\mathfrak{g}(O)) = 1\).

Over a number field, when \(\xi\) is trivial, the following formula is obtained already in Ch02.

**Theorem 5.7.** Let \(J^\xi(f)\) be the truncated trace by taking \(X = 0\). For any \(f \in C^\infty_c(\mathfrak{g}(\mathbb{A}))\), we have

\[
J^\xi(f) = q^{(1-g) \dim \mathfrak{g}} f^\xi(\hat{f}).
\]

**Proof.** By Theorem 5.1, it follows that

\[
\mathfrak{t}^\xi_{X}(f)(x) = q^{(1-g) \dim \mathfrak{g}} \mathfrak{t}^\xi_{X}(\hat{f})(x),
\]
when $X$ is deep enough in the positive chamber. Hence for $X$ deep enough, we have
\[ J_{ξ,X}(f) = q^{(1-g)\dim g} J_{ξ,X}(\hat{f}). \]
As $J_{ξ,X}(f)$ and $J_{ξ,X}(\hat{f})$ are quasi-polynomials (the Theorem 5.6), the equality then extends to all $X ∈ a_{B, Q}$, in particular to $X = 0$.

6. Reduction theory and combinatorical lemmas

In this subsection, we study reduction theory using the notion of complementary polyhedron.

6.1. $(ξ, X)$-canonical parabolic subgroup. For any parabolic subgroup $R$ of $G$ defined over $F$, there is a unique standard parabolic subgroup $Q$, containing the fixed Borel subgroup $B$, and an element $η ∈ Q(F)\backslash G(F)$ such that $R = η^{-1}Qη$. We will take advantage of this to denote a parabolic subgroup defined over $F$ by a pair $(Q, η)$ if there is no confusion.

**Definition 6.1** (semi-stability and canonical refinement). Let $x ∈ G(𝔸)$. A parabolic subgroup $(Q, η)$ of $G$ defined over $F$ is called $(ξ, X)$-semi-stable for $x$ if for any $(P, δ)$ properly contained in $(Q, η)$ (i.e. $δ^{-1}Pδ ⊆ η^{-1}Qη$), we have
\[ \hat{r}_P^Q(H_B(δx) + [s_δξ] - X) = 0. \]
The element $x ∈ G(𝔸)$ itself is called $(ξ, X)$-semi-stable if $G$ is $(ξ, X)$-semi-stable for $x$.

A parabolic subgroup $(P, δ)$ contained in $(Q, η)$ is called a $(ξ, X)$-canonical refinement of $(Q, η)$ for $x ∈ G(𝔸)$ if
(1.) $(P, δ)$ is $(ξ, X)$-semi-stable for $x$;  
(2.) For any $α ∈ Δ_P$,
\[ \langle α, H_B(δx) + [s_δξ] - X \rangle > 0. \]

The following result is an adelic version of the existence and uniqueness of semi-stable reduction for $G$-bundles with parabolic structures over a curve [HS10] Theorem 4.3.2] based on the work of [Be95].

**Theorem 6.2.** Let $(ξ, X) ∈ a_{B, ∞} × a_B$ be an admissible pair of vectors. For any $x ∈ G(𝔸)$ and any parabolic subgroup $(Q, η)$, there exists a unique $(ξ, X)$-canonical refinement of $(Q, η)$ for $x$.

**Proof.** Let $P$ be a standard parabolic subgroup of $Q$ defined over $F$. For any $x ∈ G(𝔸)$, we define
\[ \deg^Q_x(P) = \sum_{α ∈ Φ(N_P ∩ M_Q, A)^r} \langle α, H_B(x) + [s_αξ] - X \rangle, \]
where $Φ(N_P ∩ M_Q, A)^r$ is the set of reduced roots of $A$ in $N_P ∩ M_Q$. If more generally $P$ is only supposed to be a semi-standard parabolic subgroup of $Q$, we extend the definition by choosing an $s ∈ W^Q$ so that $P_0 = w_sPw_s^{-1}$ is standard and we define $\deg^Q_{s_αx}(P)$ by requiring that
\[ \deg^Q_x(P) = \deg^Q_{s_αx}(P_0). \]
We have (see [Be95, Proposition 1.9])
\begin{equation}
\alpha \in \Phi(N_{P_0} \cap M_Q, A)^r \sum_{\delta \in \Delta_B^Q - \Delta_B^{P_0}} \alpha = \sum_{\varpi \in \Delta_B^Q - \Delta_B^{P_0}} n_{\varpi} \varpi,
\end{equation}
with $n_{\varpi} \geq 2$. As the restriction of an element in $\Delta_B^Q - \Delta_B^{P_0}$ to $a_{P_0}$ is trivial, it follows that for any standard parabolic subgroup $P_0$ and $x \in G(\mathbb{A})$, we have
\begin{equation}
\deg_{\delta x}^Q(P_0) = \deg_{\delta x}^Q(P_0), \quad \forall \delta \in P_0(F),
\end{equation}
and in particular the definition of $\deg_x(P)$ for a semi-standard parabolic subgroup $P$ is independent of the choice of $s$.

Given any complementary polyhedron for a root system, Behrend ([Be95, Definition 3.1]) has associated each facet with a degree. Specialised to our case, for any $P \in \mathcal{P}(T)$, the above defined $\deg_{\delta x}^Q(P)$ coincides with Behrend’s degree for the facet corresponding to $P \cap M_Q$ with respect to the following complementary polyhedron (Proposition [7.4] and Remark [4.4])
\begin{equation}
(s^{-1}H_B(w_\delta x) + s^{-1}[s_{\omega_\delta}x\xi] - s^{-1}X)_{s \in W \cdot \varrho},
\end{equation}
in $a_B \cong a_{B \cap M_Q}$ for the reduced root system $\Phi(M_Q, A)^r$. Behrend has proved ([Be95, Corollary 3.14, Corollary 3.16]) that for any complementary polyhedron, there is a unique facet (equivalently a semi-standard parabolic subgroup) which is the smallest for the partial order given by inclusions of the closure of the facets (equivalently the largest semi-standard parabolic subgroup) in the set of facets with maximal degree, this unique facet (parabolic subgroup) is called special.

Fixing an $x \in G(\mathbb{A})$, the set
\begin{equation}
\{\deg_{\delta x}^Q(P)\mid B \subseteq P \subseteq Q, \delta \in P(F) \setminus Q(F)\eta\}
\end{equation}
has an upper bound (because of equality (6.3) and [Ar05, (5.2) p.936] see also [LW11, 3.5.4]). By discreteness of degree, the upper bound can be achieved, say by a pair $(P_1, \delta_1)$. We suppose that $(P_1, \delta_1)$ is also a largest element with such property for the partial order defined by inclusion. Then the pair $(P_1, \delta_1)$ sharing these properties is unique by Behrend’s uniqueness theorem ([Be95, Corollary 3.14, Corollary 3.16]) on the special parabolic subgroup for the family of the complementary polyhedra
\begin{equation}
(s^{-1}H_B(w_\delta \delta x) + s^{-1}[s_{\omega_\delta}x\xi] - s^{-1}X)_{s \in W \cdot \varrho},
\end{equation}
when varying $\delta \in P_1(F)\delta_1$. In fact, for any two parabolic subgroups $(P_1, \delta_1)$ and $(P_2, \delta_2)$ contained in $(Q, \eta)$, with $P_1, P_2$ standard, $\delta_1 \in P_1(F) \setminus G(F)$ and $\delta_2 \in P_2(F) \setminus G(F)$, we must have $\delta_1 \delta_2^{-1} \in Q(F)$. Applying Bruhat decomposition of $Q(F)$ to $\delta_1 \delta_2^{-1}$, we can choose as representatives $\delta_1, \delta_2 \in G(F)$ such that $\delta_2 = w \delta_1$ for some Weyl element $w$ of $M_Q$. We see that $\deg_{\delta_1 x}^Q(P_1)$ and $\deg_{\delta_2 x}^Q(P_2)$ (defined by (6.11)) are respectively the degree of the parabolic subgroups $P_1 \cap M_Q$ and $w P_2 w^{-1} \cap M_Q$ for the complementary polyhedron
\begin{equation}
(s^{-1}H_B(w_\delta \delta_1 x) + s^{-1}[s_{\omega_\delta_1}x\xi] - s^{-1}X)_{s \in W \cdot \varrho}.
\end{equation}

Finally, observe that the statement of the theorem is a reformulation of the [Be95, 3.10] and the choice of $(P_1, \delta_1)$: a pair $(P_1, \delta_1)$ shares these properties if and only if it is a $(\xi, X)$-canonical refinement of $(Q, \eta)$. In fact [Be95, 3.2(ii)] and [Be95, 3.10.B2] imply the Definition [6.1] (1.). Let’s show that [Be95, 3.10.B1] is equivalent to the inequality in
In $\Delta_n$ we need to calculate $n(P_1, \lambda)$. Recall that $n(P_1, \lambda)$ is defined in the definition [Be95 3.7] by

$$n(P_1, \lambda) = \langle \sum_{\beta \in \Psi(P_1, \lambda)} \beta, H \rangle,$$

while [Be95] Lemma 3.6] says that $\sum_{\beta \in \Psi(P_1, \lambda)} \beta \in \mathbb{R}\Delta^Q_{P_1}$, hence we have

$$n(P_1, \lambda) = \langle \sum_{\beta \in \Psi(P_1, \lambda)} \beta, H_{P_1} \rangle,$$

where $H_{P_1}$ is the projection of $H$ in $a_{P_1}$. Note that by definition [Be95 3.5], an element in $\Psi(P_1, \lambda)$ is of the form $\alpha + \mu$ with $\mu \in a_{P_1}^*$, therefore

$$n(P_1, \lambda) = |\Psi(P_1, \lambda)|\langle \alpha, H_{P_1} \rangle.$$

Thus $n(P_1, \lambda) > 0$ is equivalent to the inequality in Definition 6.1 (2.) for the element in $\Delta^Q_{P_1}$ represented by $\alpha$. \qed

6.2. Functions $F^{Q, \xi, X}(\cdot)$.

**Definition 6.3.** Let $Q$ be a standard parabolic subgroup. Let $F^{Q, \xi, X}(x)$ be the function defined for $x \in Q(F) \setminus G(\mathbb{A})$ by

$$F^{Q, \xi, X}(x) = \sum_{\{P \mid B \subseteq P \subseteq Q\}} (-1)^{\dim a_P^Q} \sum_{\delta \in P(F) \setminus Q(F)} \tau^Q_P(H_B(\delta x) + [s_{\delta x} \xi] - X).$$

By Lemma 5.1. of [AY13], the sum is always finite, hence $F^{Q, \xi, X}(x)$ is well defined.

**Lemma 6.4.** For any $x \in G(\mathbb{A})$, $(\xi, X) \in a_{B, \infty} \times a_B$, and $Q$ a standard parabolic subgroup, one has

$$1 = \sum_{\{P \mid B \subseteq P \subseteq Q\}} \sum_{\delta \in P(F) \setminus Q(F)} \tau^Q_P(H_B(\delta x) + [s_{\delta x} \xi] - X) F^{P, \xi, X}(\delta x).$$

**Proof.** We can insert directly the definition of $F^{P, \xi, X}(\delta x)$ into the right-hand-side of the equation. After changing order of summation, the identity follows by the fact that the matrix $((-1)^{\dim a_P^Q} \tau^Q_P(H))$ (indexed by standard parabolic subgroups $P \subseteq Q$) is the inverse of $((-1)^{\dim a_P^Q} \tau^Q_P(H))$ (Proposition 1.7.2 of [LW11]). \qed

**Proposition 6.5.** For any admissible pair $(\xi, X) \in a_{B, \infty} \times a_B$, the function $F^{G, \xi, X}$ is the characteristic function of the set consisting of those $x \in G(\mathbb{A})$ which are $(\xi, X)$-semi-stable, i.e. the set of $x$ such that for any proper standard parabolic subgroup $P$ of $G$, any $\delta \in P(F) \setminus G(F)$, we have

$$\tau_P(H_B(\delta x) + [s_{\delta x} \xi] - X) = 0.$$ 

In particular the function $F^{G, \xi, X}$ is compactly supported over $G(F) \setminus G(\mathbb{A})/G(\mathbb{A})$ when $(\xi, X)$ is admissible.

**Proof.** The proof is similar to that of [Ch15 Lemme 2.4.2].

Note that $F^{G, \xi, X}(x)$ equals

$$\sum_{\{(Q, n)\}} (-1)^{\dim a_Q^G} \tau_Q(H_B(\eta x) + [s_{\eta x} \xi] - X),$$
where the sum is taken over the set of all parabolic subgroups of $G$ defined over $F$. By existence and uniqueness of $(\xi,X)$-canonical refinement (Theorem 6.2), we can organize the above sum by grouping together those $(Q,\eta)$ under which $x$ has the same $(\xi,X)$-canonical refinement. Hence $F^{G,\xi,X}(x)$ equals

$$
(6.8) \sum_{\{(P,\delta)\} \{(Q,\eta)|x(x)^{Q,\eta}=(P,\delta)\}} (-1)^{\dim a_\eta} \hat{\tau}_Q(H_B(\delta x) + [s_{\delta x} \xi] - X),
$$

where the first sum is taken over $(\xi,X)$-semi-stable parabolic subgroups for $x$, and the second sum is taken over the set of parabolic subgroups $(Q,\eta)$ such that the $(\xi,X)$-canonical refinement of $(Q,\eta)$ for $x$ is $(P,\delta)$. The inclusion $\delta^{-1}P\delta \subseteq \eta^{-1}Q\eta$ holds if and only if $P \subseteq Q$ and the image of $\delta$ in $Q(F) \setminus G(F)$ is $\eta$. Therefore, given a parabolic subgroup $(P,\delta)$ which is $(\xi,X)$-semi-stable for $x$ and a parabolic subgroup $(Q,\eta)$ containing $(P,\delta)$, $(P,\delta)$ is the $(\xi,X)$-canonical refinement of $(Q,\eta)$ for $x$ if and only if

$$\tau_P^Q(H_B(\delta x) + [s_{\delta x} \xi] - X) = 1.$$ 

Thus the inner sum of the expression (6.8) equals

$$
\sum_{\{Q|P \subseteq Q\}} (-1)^{\dim a_\eta} \hat{\tau}_Q(H_B(\delta x) + [s_{\delta x} \xi] - X) \tau_P^Q(H_B(\delta x) + [s_{\delta x} \xi] - X),
$$

which is zero except if $P = G$ ([LW11 Proposition 1.7.2]). This latter holds if and only if $x$ is $(\xi,X)$-semi-stable. We conclude that the sum (6.8) is 1 if $x$ is $(\xi,X)$-semi-stable and is 0 otherwise.

Finally, the assertion about compact support can be easily deduced from the definition of $(\xi,X)$-semi-stability and [LW11 Proposition 3.5.3] (applied for $Q = G$). \qed

**Proposition 6.6.** Let $Q$ be a standard parabolic subgroup of $G$ and $x \in G(\mathbb{A})$. Let $x = nmk$ be an Iwasawa decomposition of $x \in G(\mathbb{A})$ with $n \in N_Q(\mathbb{A})$, $m \in M_Q(\mathbb{A})$, and $k \in G(\mathbb{O})$ such that $s_k \in W$ has minimal length among all such Iwasawa decompositions of $x$. By identifying $\mathfrak{a}_B$ with $\mathfrak{a}_{B \cap M_Q}$, one has

$$F^{Q,\xi,X}(x) = F^{M_Q, s_k \xi, X}(m).$$

**Proof.** As $P(F) \setminus Q(F)$ is in bijection with $(P \cap M_Q)(F) \setminus M_Q(F)$, and the map $P \mapsto P \cap M_Q$ defines a bijection between the set $\{P|B \subseteq P \subseteq Q\}$ and the set of standard parabolic subgroups of $M_Q$ (with the fixed Borel subgroup being $B \cap M_Q$). By identifying $\mathfrak{a}_B$ with $\mathfrak{a}_{B \cap M_Q}$, we have

$$F^{Q,\xi,X}(x) = \sum_{\{P|B \subseteq P \subseteq Q\}} (-1)^{\dim a_P} \sum_{\delta \in P(F) \setminus Q(F)} \hat{\tau}_P^Q(H_B(\delta x) + [s_{\delta x} \xi] - X)$$

$$= \sum_{\{R \subseteq P(M_Q(B \cap M_Q))\}} (-1)^{\dim a_P} \sum_{\delta \in R(F) \setminus M_Q(F)} \hat{\tau}_R^M(H_B(\delta m) + [s_{\delta m} \xi] - X).$$

For any $R \in P(M_Q(B \cap M_Q)$ and $\delta \in M_Q(F)$, let $\delta m = b_0k_0$ with $b_0 \in (B \cap M_Q)(\mathbb{A})$ and $k_0 \in M_Q(\mathbb{O})$. Then $\delta x = \delta n x^{-1}b_0k_0k$. As $\delta n x^{-1}b_0 \in B(\mathbb{A})$, we have $s_{\delta x} = s_{b_0k_0}$ and $s_{\delta m} = s_{k_0}$. Note that $s_k$ has minimal length in $W^{Q}s_k$, so $s_{b_0k_0} = s_{k_0} s_k$ by Corollaire 1, Chapter IV §2 of [Bouro68] and Lemme 1.3.3 of [LW11]. By definition, we obtain $F^{Q,\xi,X}(x) = F^{M_P, s_k \xi, X}(m)$. \qed
7. A relation between Jordan-Chevalley decomposition and Levi decomposition

The hypothesis $(\ast)_G$ or $(\ast)_g$ in 3.1 is assumed to hold following the group case or the Lie algebra case respectively.

7.1. The following lemma is well known and will be used without further mention in this section.

**Lemma 7.1.** Let $P$ be a parabolic subgroup of $G$. Let $\gamma \in P(F)$ be a semi-simple element, then $N_{P,\gamma}$ is connected.

**Proof.** We can suppose that we’re over an algebraic closed field, and suppose $\gamma$ lies in a maximal torus $S$ of $P$, then $N_{P,\gamma}$ is generated by the connected groups $U_\alpha$ for all $\alpha \in \Phi(N_P,S)_F$ such that $\alpha(\gamma) = 1$. $\square$

The number field case of the following proposition is due to J. Arthur ([Ar86, Lemma 3.1]). Our statement and strategy of proof is adapted from [Ch02, Lemma 2.3.] where a Lie algebra version is proved. Note that our lemma 7.4 is used implicitly in the proof of [Ch02, Lemma 2.3.]. Over a function field, one needs to be more careful with unipotent elements, unipotent subgroups and smoothness properties. The third case of the following result is not needed in this article, but will be used in [Yu21]. We include it here as it does not increase too much the length of the proof.

**Proposition 7.2** (Arthur, Chaudouard). Suppose that $P$ is a standard parabolic subgroup of $G$ with standard Levi subgroup $M$ and unipotent radical $N$. Let $\sigma \in M(F)$. Suppose that $\sigma$ admits Jordan-Chevalley decomposition with semi-simple part $\sigma_s$ and unipotent part $\sigma_u$.

Let $A$ be one of the following cases:
(1) $A = F$;
(2) $A = \mathbb{A}$;
(3) In case that $\sigma_s \in M(\mathbb{F}_q)$, we also allow $A = \mathcal{O}$, the ring of integral adeles.

Fixing a system of representatives $\Delta$ for $N_{\sigma_s}(A)\backslash N(A)$ (note that in the third case $N_{\sigma_s}$ is defined over $\mathbb{F}_q$, hence $N_{\sigma_s}(A)$ makes sense), then for any $x \in N(A)$, there is a unique pair $(n,u) \in \Delta \times N_{\sigma_s}(A)$ such that

$$x = \sigma^{-1}n^{-1}\sigma un.$$

In the Lie algebra case, suppose that $\sigma \in \mathfrak{m}(F)$. Let $A$ be either the case (1) or (2) above or $A = \mathcal{O}$ if $\sigma_s \in \mathfrak{m}(\mathbb{F}_q)$. Fixing a system of representatives $\Delta$ for $N_{\sigma_s}(A)\backslash N(A)$, then for any $x \in \mathfrak{n}(A)$, there is a unique pair $(n,u) \in \Delta \times \mathfrak{n}_{\sigma_s}(A)$ such that

$$x = \text{Ad}(n^{-1})(\sigma + u) - \sigma.$$

**Proof.** Note that if the lemma is true for one $\sigma$, then it’s true for all elements in $M(F)$-conjugacy class of $\sigma$. Hence we can freely take an $M(F)$-conjugate of $\sigma$ if necessary.

After Proposition 3.2 we may suppose that $\sigma_u$ belongs to the unipotent radical of a parabolic subgroup $P'$ of $G$, hence it lies in $N_{P'}(F) \cap M(F)$. Then $\sigma_u$ is contained in the unipotent radical of a parabolic subgroup $P_1 = P' \cap M$ of $M$. We may suppose that $P_1$ is a minimal parabolic subgroup of $M$ defined over $F$. On the other hand,
there is a parabolic subgroup \( Q \) of \( M \) being minimal with the property that \( \sigma_s \in L(F) \) for a Levi subgroup \( L \) of \( Q \). After conjugation, we suppose that \( Q \) contains \( P_1 \).

We claim that \( \sigma_u \) belongs to the unipotent radical of \( Q \). This is because \( Q_{\sigma_s}^0 \) is a parabolic subgroup of \( M_{\sigma_s}^0 \) with a Levi factor \( M_{\sigma_s}^0 \) and with unipotent radical \( N_{Q,\sigma_s}^0 \).

As \( \sigma_u \in Q_{\sigma_s}^0(F) \), its projection to \( M_{Q,\sigma_s}^0(F) \) belongs to the unipotent radical of a parabolic subgroup of \( M_{Q,\sigma_s}^0 \) (cf. 3.2. Proposition (A) of [Ti87]). While the derived group of \( M_{Q,\sigma_s}^0 \) is anisotropic, hence no proper \( F \)-parabolic subgroup lies in it. Therefore the projection of \( \sigma_u \) in \( M_{Q,\sigma_s}^0 \) is trivial. It implies that \( \sigma_u \) belongs to the unipotent radical of \( Q_{\sigma_s}^0 \) and hence to that of \( Q \).

Let \( R = QN \) (which is a parabolic subgroup of \( G \), see 4.4 and 4.7 of [BoT65]), we have \( \sigma \in R(F) \), \( \sigma_u \in N_R(F) \) and \( R \subseteq P \). Let \( A_R \) be the maximal split torus in the center of \( M_R \). Then the Lie algebra of \( N \) can be decomposed into eigenspaces under the action of \( A_R \).

By Proposition 3.3.9 of [CGP15] and 3.3.5 to the semi-groups \( A_i \), we deduce that \( N_R \) admits a descending filtration \( N_R = \tilde{N}_1 \supseteq \tilde{N}_2 \supseteq \cdots \supseteq \tilde{N}_r = \{1\} \) by \( R \)-conjugate stable smooth closed \( F \)-subgroups such that:

\[
u u^{-1} u^{-1} \in \tilde{N}_{i+1}(F) \quad \text{for} \quad u_i \in \tilde{N}_i(F) \quad \text{and} \quad u_j \in \tilde{N}_j(F) \quad \text{(one sets} \quad \tilde{N}_k = \{1\} \quad \text{for} \quad k \geq r)\]

As \( N = N_P \subseteq N_R \) which is \( A_R \)-stable, we obtain a descending filtration

\[N = N_1 \supseteq N_2 \supseteq \cdots \supseteq N_r = \{1\},\]

with \( N_i = \tilde{N}_i \cap N \).

**Lemma 7.3.** The filtration constructed above satisfies the following properties:

1. each \( N_i \) is a normal subgroup of \( N \) and \( \sigma \) conjugate stable;
2. \( \sigma_n^{-1} n_i \sigma_i \in N_{i+1}(A) \) for any \( n_i \in N_i(A) \).
3. each \( N_i \) is a connected smooth closed subgroup of \( N \).

**Proof of the lemma.** (1) Because \( \tilde{N}_i \) is \( R \)-conjugate stable; (2) Since \( \sigma_u \in \tilde{N}_1(F) \). (3) By Proposition 3.3.9 of [CGP15].

Fix a system of representatives \( \Delta_k \) for \( N_{\sigma_s}(A)N_k(A) \setminus N(A) \) for each \( k = 1, 2, \cdots \).

Given \( x \in N(A) \), we proceed by recurrence on \( k \) for the following assertion: there exists a unique couple \( (n, u) \in \Delta_k \times N_{\sigma_s}(A)N_k(A) \) such that

\[x = \sigma^{-1} n^{-1} \sigma u n.\]

The case that \( k = 1 \) is trivial. Suppose that the assertion is proved for \( k \), i.e., there exists a unique pair \( (n_k, u_k) \in \Delta_k \times N_{\sigma_s}(A)N_k(A) \) such that \( x = \sigma^{-1} n_k^{-1} \sigma u_k n_k \). Let’s prove the case for \( k + 1 \).

For unicity, let \( (n, u) \in \Delta_{k+1} \times N_{\sigma_s}(A)N_{k+1}(A) \) be a couple such that \( x = \sigma^{-1} n^{-1} \sigma u n \).

Let \( \beta \in \Delta_k \) be the representative of the coset \( N_{\sigma_s}(A)N_k(A) \), let \( \alpha := n \beta^{-1} \in N_{\sigma_s}(A)N_k(A) \). Then \( n \) is uniquely determined by \( \beta \) and the coset \( N_{\sigma_s}(A)N_{k+1}(A) \alpha \), i.e. \( \{n\} = \Delta_{k+1} \cap (N_{\sigma_s}(A)N_{k+1}(A) \alpha) \), and \( u \) will be automatically unique (assuming its existence).

We have

\[x = \sigma^{-1} n^{-1} \sigma u n = \sigma^{-1} \beta^{-1} \sigma (\sigma^{-1} n \sigma) (u \alpha) \beta.\]
By our construction of $N_i$, we have $\sigma^{-1} \alpha^{-1} \sigma \in N_{\sigma}(A)N_k(A)$ and $u\alpha \in N_{\sigma}(A)N_k(A)$, hence $(\sigma^{-1} \alpha^{-1} \sigma)(u\alpha) \in N_{\sigma}(A)N_k(A)$. By unicity of $(n_k, u_k)$, we have

$$n_k = \beta,$$

$$u_k = (\sigma^{-1} \alpha^{-1} \sigma)(u\alpha).$$

Since $\sigma^{-1} \alpha^{-1} \sigma = (\sigma_u^{-1}(\sigma_u^{-1} \alpha^{-1} \sigma_u)\alpha)(\sigma_u^{-1} \alpha^{-1} \sigma_u)$ and both $\sigma_u^{-1}(\sigma_u^{-1} \alpha^{-1} \sigma_u)\alpha$ and $\alpha^{-1} \sigma_u \alpha$ belong to $N_{\sigma}(A)N_{k+1}(A)$, we have

$$N_{\sigma}(A)N_{k+1}(A)u_k = N_{\sigma}(A)N_{k+1}(A)\sigma_u^{-1} \alpha^{-1} \sigma_u \alpha.$$

We will show that this relation determines the coset $N_{\sigma}(A)N_{k+1}(A)\alpha$.

**Lemma 7.4.** Let $A$ be one of the cases in Proposition 7.3, then we have an isomorphism of abstract groups:

$$N_{\sigma}(A)N_{k+1}(A)/N_{\sigma}(A)N_k(A) \cong (N_{k, \sigma}N_{k+1}\setminus N_k)(A).$$

**Proof of the lemma.** We have an isomorphism of groups:

$$N_{\sigma}(A)N_{k+1}(A)/N_{\sigma}(A)N_k(A) \cong N_{k, \sigma}(A)N_{k+1}(A)/N_k(A).$$

By definition, $N_{k, \sigma} \cap N_{k+1} = N_{\sigma} \cap N_{k+1} = N_{k+1, \sigma}$. Due to Proposition 3.3.10 of [CGP15], $N_{k+1, \sigma}$ is connected and smooth. Moreover, as it admits an action by a split torus with no non-zero weight, it is $F$-split (Lemma 3.3.8 of loc. cit.), i.e. it admits a composition series over $F$ whose successive quotients are $F$-isomorphic to $\mathbb{G}_a$. Then one knows (by induction on the length of the composition series) that $H^1(F, N_{k+1, \sigma}) = 0$ and $H^1(F_v, N_{k+1, \sigma}) = 0$ for any place $v$ of $F$. Since $N_k$ and $N_{k, \sigma}$ are smooth, by short exact sequence $1 \to N_{k+1, \sigma} \to N_{k, \sigma} \times N_{k+1} \to N_{k, \sigma}N_{k+1} \to 1$ and the vanishing of $H^1$, we have ([Yu21] 2.3.6)]

$$(7.2) \quad N_{k, \sigma}(F)N_{k+1}(F) \cong N_{k+1, \sigma}(F)/N_{k, \sigma}(F) \times N_{k+1}(F) \cong (N_{k, \sigma}N_{k+1})(F),$$

and

$$(7.3) \quad N_{k, \sigma}(F_v)N_{k+1}(F_v) \cong (N_{k, \sigma}N_{k+1})(F_v),$$

for any place $v$ of $F$. Again as $N_{k, \sigma}N_{k+1}$ is connected, smooth, unipotent and $F$-split (as it’s a quotient of $N_{k, \sigma} \times N_{k+1}$), we deduce that

$$(7.4) \quad (N_{k, \sigma}N_{k+1})(F)/N_k(F) \cong (N_{k, \sigma}N_{k+1}\setminus N_k)(F),$$

and

$$(7.5) \quad (N_{k, \sigma}N_{k+1})(F_v)/N_k(F_v) \cong (N_{k, \sigma}N_{k+1}\setminus N_k)(F_v),$$

for any place $v$ of $F$. So the lemma is proved when $A = F$.

Consider the case that $A = \mathbb{A}$. The algebraic group $N_{k+1, \sigma}$ is smooth and connected, hence geometrically integral, we deduce by I. 3.6 of [Oe84], that the image of the group $N_{k, \sigma}(\mathbb{A})N_{k+1}(\mathbb{A})$ is open in $(N_{k, \sigma}N_{k+1})(\mathbb{A})$. While by relation (7.3), the image of $N_{k, \sigma}(\mathbb{A})N_{k+1}(\mathbb{A})$ is also dense in $(N_{k, \sigma}N_{k+1})(\mathbb{A})$. It follows that

$$N_{k, \sigma}(\mathbb{A})N_{k+1}(\mathbb{A}) \cong (N_{k, \sigma}N_{k+1})(\mathbb{A}).$$

Similarly,

$$N_{k, \sigma}(\mathbb{A})N_{k+1}(\mathbb{A}) \cong (N_{k, \sigma}N_{k+1}\setminus N_k)(F_v).$$

This finishes the proof of this case.
When $\sigma_s \in M(F_q)$, let’s treat the case that $A = O$. Now the exact sequence $1 \rightarrow N_{k+1,\sigma_s} \rightarrow N_{k,\sigma_s} \times N_{k+1} \rightarrow N_{k,\sigma_s}N_{k+1} \rightarrow 1$ is defined over $F_q$. By above arguments or by Lang’s theorem on vanishing of $H^1$ for connected algebraic group over finite fields, the homomorphism $N_{k,\sigma_s}(N) \rightarrow (N_{k,\sigma_s}N_{k+1})(N)$ is surjective for any place $v$ of $F$. Moreover, as the morphism $N_{k,\sigma_s} \times N_{k+1} \rightarrow N_{k,\sigma_s}N_{k+1}$ is smooth, in particular formally smooth, the homomorphism $N_{k,\sigma_s}(O_v) \times N_{k+1}(O_v) \rightarrow (N_{k,\sigma_s}N_{k+1})(O_v)$ is surjective too. Hence $N_{k,\sigma_s}(O_v)N_{k+1}(O_v) \cong (N_{k,\sigma_s}N_{k+1})(O_v)$, for any place $v$ of $F$. We obtain

$$N_{k,\sigma_s}(O)N_{k+1}(O) \cong (N_{k,\sigma_s}N_{k+1})(O).$$

Similarly,

$$(N_{k,\sigma_s}N_{k+1})(O)N_{k}(O) \cong (N_{k,\sigma_s}N_{k+1}\setminus N_{k})(O).$$

This finishes the case $A = O$. □

Now we’re going to finish the proof of Proposition 7.2.
Consider the morphism of schemes defined by:

$$\Phi_k : N_{k,\sigma_s}N_{k+1}\setminus N_k \rightarrow N_{k,\sigma_s}N_{k+1}\setminus N_k$$

$$y \mapsto (\sigma_s^{-1}y^{-1}\sigma_s) \cdot y$$

This is a morphism of algebraic groups as $N_{k,\sigma_s}N_{k+1}\setminus N_k$ is commutative. Let’s prove that $\Phi_k$ is an isomorphism. As $N_{k,\sigma_s}N_{k+1}\setminus N_k$ is smooth and connected, it’s sufficient to show that $\Phi_k$ induces an injection on geometric points and that the associated map of Lie algebra $\text{Lie}(\Phi_k)$ is surjective.

If $y \in (N_{k+1}\setminus N_k)(\bar{F})$ is an element such that

$$b := (\sigma_s^{-1}y^{-1}\sigma_s) \cdot y \in (N_{k+1}\setminus N_{k+1}N_{k,\sigma_s})(\bar{F}).$$

Then $y^{-1}\sigma_s y = \sigma_s b$ is the Jordan-Chevalley decomposition of $y^{-1}\sigma_s y$ (as an element in $(N_{k+1}\setminus R)(\bar{F})$). We must have $b = 1$ and $y \in (N_{k+1}\setminus N_{k+1}N_{k,\sigma_s})(\bar{F})$, which proves injectivity on geometric points.

Now $\Phi_k$ is induced by passing to quotient of the following morphism:

$$\tilde{\Phi}_k : N_{k+1}\setminus N_k \rightarrow N_{k+1}\setminus N_k :$$

$$y \mapsto (\sigma_s^{-1}y^{-1}\sigma_s) \cdot y.$$ 

We know that $\text{Lie}(\tilde{\Phi}_k) = \text{Id} - \text{Ad}(\sigma_s)$. Let $p : \text{Lie}(N_{k+1}\setminus N_k) \rightarrow \text{Lie}(N_{k+1}N_{k,\sigma_s}\setminus N_k)$ be the projection, then $\ker(p) = \text{Lie}(N_{k+1}\setminus N_{k+1}N_{k,\sigma_s})$. While we also have $\ker(\text{Lie}(\tilde{\Phi}_k)) \cong \text{Lie}(\ker(\tilde{\Phi}_k)) = \text{Lie}(N_{k+1}\setminus N_{k+1}N_{k,\sigma_s}) = \ker(p)$. Since $\text{Lie}(\tilde{\Phi}_k)$ is a semi-simple endomorphism,

$$\text{Lie}(N_{k+1}\setminus N_k) = \ker(\text{Lie}(\tilde{\Phi}_k)) \oplus \text{Im}(\text{Lie}(\tilde{\Phi}_k)).$$

It follows that the composition $p \circ \text{Lie}(\tilde{\Phi}_k)$ is surjective. We conclude then that $\text{Lie}(\Phi_k)$ is surjective.

Now $\Phi_k$ is known to be an isomorphism, combining with above lemma, we conclude that the coset $N_{\sigma_s}(A)N_{k+1}(A)\sigma_s^{-1}\alpha^{-1}\sigma_s\alpha$ is uniquely determined by equality (7.1). This finishes the proof of the uniqueness part of the induction.

For existence part, we see by the above argument that there is an $\alpha \in N_{\sigma_s}(A)N_k(A)$ such that the equality (7.1) is satisfied. Let $\beta = n_k$, $n$ be the unique element in $\Delta_{k+1} \cap (N_{\sigma_s}(A)N_{k+1}(A)\alpha)\beta$ and $u = \sigma^{-1}naxn^{-1}$. Using induction hypothesis, we have $u = \sigma^{-1}n_k^{-1}\sigma u_k n_k^{-1}$, we see that $u \in N_{\sigma_s}(A)N_k(A)$.
Corollary 7.5. An element $p \in P(F)$ (resp. $p(F)$), with $P$ a standard parabolic subgroup of $G$, admits Jordan-Chevalley decomposition if and only if its Levi factor does, and in this case their semi-simple parts are $N_P(F)$-conjugate.

In particular, let $o \in \mathcal{E}$, then for any pair of standard parabolic subgroup $P \subseteq Q$, we have

$$o \cap M_Q(F) \cap P(F) = (o \cap M_P(F))N_P^Q(F),$$
in the group case and

$$o \cap m_Q(F) \cap p(F) = (o \cap m_P(F)) + n_P^Q(F),$$
in the Lie algebra case.

Proof. We prove the group case as the Lie algebra case can be proved by the same arguments.

Let $p \in P(F)$, and $p = \sigma x$ be the Levi decomposition of $p$ with $\sigma \in M_P(F)$ and $x \in N_P(F)$. It amounts to prove that $\sigma$ lies in the equivalence class of $p$ in $\mathcal{E}$.

If $\sigma$ admits Jordan-Chevalley decomposition with semi-simple part $\sigma_s$, then by Proposition 7.2 we can write $p = n^{-1} \sigma un$ with $u \in N_{P,\sigma_s}(F)$ and $n \in N_P(F)$. Therefore $p$ must also have Jordan-decomposition with semi-simple part $n^{-1} \sigma_s n$ and we’re done.

If $\sigma$ does not admit Jordan-Chevalley decomposition, we need to prove that nor does $p$. But this is clear since $M_P$ is a Levi subgroup of $P$ means that $M_P \cong P/N_P$. If $p$ has a Jordan-Chevalley decomposition, then its image under the quotient map gives a Jordan-Chevalley decomposition of $\sigma$. $\square$

8. Proof of the Theorem 5.1 and the Theorem 5.6

8.1. Proof of the Theorem 5.1. The proof follows basically from the same strategy of [Laf97, Proposition 11, p.227]. We need a lemma first.

Lemma 8.1. Let $f \in C^\infty(G(\mathbb{A}))$ or $f \in C^\infty(g(\mathbb{A}))$ depending on the case we’re dealing with. There is a constant $c$ depending on $f$ and $\xi$, such that: for a couple of standard parabolic subgroups $P \subseteq Q$, a vector $X \in a_B$ with $d(X) > c$, then for any element $x \in G(\mathbb{A})$ satisfying

$$F^{\mathcal{P},X}(x)_{\tau_P^Q}(H_B(x) + [s_2 \xi] - X) = 1,$$

we have

$$k_{P,o}(x) = k_{Q,o}(x),$$

or for the Lie algebra case

$$\mathfrak{k}_{P,o}(x) = \mathfrak{k}_{Q,o}(x).$$

Proof. We prove the group case as the Lie algebra case is similar.

Note that we have

$$\sum_{\gamma \in M_Q(F)\gamma_0} \int_{N_Q(\mathbb{A})} f(x^{-1} \gamma n x)dn = \int_{N_Q(F) \setminus N_Q(\mathbb{A})} \sum_{\gamma \in (M_Q(F)\gamma_o)N_Q(F)} f(x^{-1} \gamma n x)dn.$$

After modifying $x$ from left by an element in $P(F)$, which doesn’t affect the statement of the lemma, we may write using Iwasawa decomposition that $x = bak$ with $b \in N_B(\mathbb{A})$ lying in a fixed compact subset, $a \in T(\mathbb{A})$, and $k \in G(\mathcal{O})$. The condition

$$F^{\mathcal{P},X}(x)_{\tau_P^Q}(H_B(x) + [s_2 \xi] - X) = 1$$

implies
implies that (\[\text{LW11, 1.2.7}\])
\[\alpha(H_B(a) - X) > c', \quad \forall \alpha \in \Delta_B^Q - \Delta_B^P,\]
where the constant \(c'\) depends only \(\xi\). Taking a \(X_G \in a_B\) such that \(-X_G\) is deep enough in the positive chamber (depending on the choice of a Siegel domain), then we may suppose, after changing \(x\) from left by an element in \(P(F)\) if necessary, that \(\alpha(H_B(a) - X_G) > 0\), for all \(\alpha \in \Delta_B^P\). Since \(\alpha(X - X_G) > 0\) for all \(\alpha \in \Delta_B\), we have
\[\alpha(H_B(a) - X) > 0, \quad \forall \alpha \in \Delta_B^Q.\]

If \(\gamma \in Q(F)\) satisfies that \(x^{-1}\gamma_n x\) lies in the support of \(f\) for some \(n \in N_Q(A)\) lying in a fixed fundamental domain of \(N_Q(F)\backslash N_Q(A)\), then \(a^{-1}b^{-1}\gamma ba\) lies in a compact subset depending only in the support of \(f\). Hence when \(d(X)\) is large enough, we must have \(\gamma \in P(F)\) (cf. \[\text{LW11, 3.6.6}\]). Note that by Corollary \[7.5\]
\[((M_Q(F) \cap o)N_Q(F)) \backslash P(F) = (M_P(F) \cap o)N_P(F).\]

It remains to show that, if \(X\) is deep enough in the positive chamber, then
\[
\int_{N_P(F) \backslash N_P(A)} \sum_{\gamma \in (M_P(F) \cap o)N_P(F)} f(x^{-1}\gamma_n x) d\gamma = \int_{N_Q(F) \backslash N_Q(A)} \sum_{\gamma \in (M_P(F) \cap o)N_P(F)} f(x^{-1}\gamma_n x) d\gamma.
\]
This follows by a similar proof of \[\text{MW94, I.2.8 Lemma}\] (there is a tiny omission in their proof as \(V'' \subseteq V\) can not be satisfied by only requiring \(a^a > c\)). For reader’s convenience, we complete the proof.

Let \(N_Q^Q = M_Q \cap N_P\). Then we have a decomposition \(N_P(A) = N_Q(A)N_P^Q(A)\). Note that we can moreover modify \(x\) from left by an element in \(N_Q(A)\), hence using the above decomposition of \(x\), we can write \(x = bma\) with \(b \in N_P^Q(A)\) and \(m \in M_P(A)\) are in fixed compact subsets, \(a \in T(A)\) and \(k \in G(O)\).

We can also decompose the integral over \(N_P(F) \backslash N_P(A)\) into a double integral over
\[N_Q^Q(F) \backslash N_P^Q(A) \times N_Q(F) \backslash N_Q(A)\].
For \(n \in N_P(A)\), let \(n = n_Qn_P^Q\) be the associated decomposition. We have
\[f(x^{-1}\gamma_n x) = f((x^{-1}\gamma_n x)(x^{-1}n_P^Q x)).\]
Since \(f\) is smooth with compact support, it is invariant from right by a compact open subgroup \(K\). Let \(K'\) be a compact open subgroup such that \(k^{-1}K'k \subseteq K\) for any \(k \in G(O)\). Choose a compact open subset \(V\) of \(N_P^Q(A)\), such that the projection \(V \to N_P^Q(F) \backslash N_P^Q(A)\) is bijective, then the integral over \(N_Q^Q(F) \backslash N_P^Q(A)\) is the same thing as the integral over \(V\). If we were in the local field case, then when \(X\) is deep enough in the positive chamber, \(a^{-1}m^{-1}b^{-1}Vbmaa\) would be contained in \(K'\). In our global case, whenever \(X\) is deep enough in the positive chamber, we conclude from the local case and weak approximation theorem that there is an \(a_f \in T(F)\) such that \(a_f^{-1}a^{-1}m^{-1}b^{-1}Vbmaa\) is contained in \(K'\). However as \(x\) can be modified from left by an element in \(P(F)\) without changing the statement, we can replace \(a\) by \(a_f a\). \(\square\)

Now we come back to the proof of the Theorem \[5.1\] We only prove the group case since the Lie algebra case can be proved by similar arguments.

For any \(\xi \in a_{B,\infty}\), let \(X' \in a_B\) be a vector such that the pair \((\xi, X')\) is admissible and \(d(X')\) is larger than the constant given in the lemma \[8.1\] Insert the identity \(6.7\)
into the expression of $k_0^{\xi,X}(x)$, after simplification, it equals the sum over all pairs of standard parabolic subgroups $P \subseteq Q$, followed by the sum over $\delta \in P(F)\backslash G(F)$ (where we have combined the sum over $Q(F)\backslash G(F)$ with the sum over $P(F)\backslash Q(F)$) of the following expression

\[(8.3) \quad (-1)^{\dim \mathcal{G}^0_Q} \tau_P(H_B(\delta x) + [s_\delta x \xi] - X) \tau_P^Q(H_B(\delta x) + [s_\delta x \xi] - X') F^{P,\xi,X}(\delta x) k_{Q,o}(\delta x).\]

Note that Lemma 8.1 says that the expression $\tau_P(H_B(\delta x) + [s_\delta x \xi] - X') F^{P,\xi,X}(\delta x) k_{Q,o}(\delta x)$ equals

\[\tau_P(H_B(\delta x) + [s_\delta x \xi] - X') F^{P,\xi,X}(\delta x) k_{P,o}(\delta x).\]

Hence, we can arrange the order of the sum in $k_0^{\xi,X}$ over the set of pairs of standard parabolic subgroups $P \subseteq Q$. We deduce that $k_0^{\xi,X}$ equals the sum over standard parabolic subgroups $P$ followed by the sum over $\delta \in P(F)\backslash G(F)$ of the expression $F^{P,\xi,X}(\delta x) k_{P,o}(\delta x)$ times

\[(8.4) \quad \sum_{\{Q \subseteq P \} Q \supset P} (-1)^{\dim \mathcal{G}^0_Q} \tau_P(H_B(\delta x) + [s_\delta x \xi] - X) \tau_P^Q(H_B(\delta x) + [s_\delta x \xi] - X').\]

We denote the sum $(8.4)$ by $\Gamma_P(H_B(\delta x) + s_\delta x \xi - X', X - X')$, where

\[\Gamma_P(H,X) = \sum_{\{Q \supset P \} Q \subset P} (-1)^{\dim \mathcal{G}^0_P} \tau_P^Q(H - X)\]

is a function on $\mathfrak{a}_B \times \mathfrak{a}_B$. One can find more details about it in [LW11, Section 1.8].

For us, we only need to know that

1. For any $X \in \mathfrak{a}_B$, the projection of the support of the function $H \mapsto \Gamma_P(H,X)$ in $\mathfrak{a}_B^G$ is compact ([LW11, 1.8.3]).
2. For any $H \in \mathfrak{a}_B$, $\Gamma_P(H,0) = 0$ if $P \neq G$ and $\Gamma_G(H,0) = 1$ ([LW11, 1.8.1, 1.8.3]).

In summary, for $X'$ deep enough, $k_0^{\xi,X}(x)$ equals

\[(8.5) \quad \sum_{P \in \mathcal{P}(B)} \sum_{\delta \in P(F)\backslash G(F)} \Gamma_P(H_B(\delta x) + [s_\delta x \xi] - X', X - X') F^{P,\xi,X'}(\delta x) k_{P,o}(\delta x).\]

We know that the function

\[x \mapsto \Gamma_P(H_B(x) + [s_\delta x \xi] - X', X - X') F^{P,\xi,X'}(x)\]

has compact support in $P(F)\backslash G(\mathfrak{a})/Z_G(\mathfrak{a})$ thanks to our Proposition 6.6 and Proposition 6.5. Therefore, we conclude that $k_0^{\xi,X}(x)$ always have compact support in $G(F)\backslash G(\mathfrak{a})/Z_G(\mathfrak{a})$.

When $d(X) \gg 0$, one can take simply $X' = X$. Since $\Gamma_P(H,0) = 0$ for any $H$ if $P \neq G$, we conclude that

\[k_0^{\xi,X}(x) = F^{G,\xi,X}(x) \sum_{s \in \mathcal{G}} \sum_{\gamma \in o} f(ax^{-1}\gamma x).\]

8.2. **Proof of the Theorem 5.6** We need a lemma first. Suppose $M$ is a standard Levi subgroup. Let’s choose a lattice $\Xi_M$ in $Z_M(F)\backslash Z_M(\mathfrak{a})$ containing $\Xi_G$. The following lemma is a variant of [Ch15, 4.5.5].
Lemma 8.2 (Chaudouard). For any standard parabolic subgroup $P$ with Levi subgroup $M = M_P$, any $H_0 \in a_P$, the function $X \in a_{B,Q} \mapsto \sum_{H \in H_P(\Xi_M)/H_P(\Xi_G)} \Gamma_Q(H + H_0, X)$ is a quasi-polynomial.

Proof. The proof is essentially the same as that of [Ch15, 4.5.5].

Let $L_0 \subseteq a_{B,Q}$ be any lattice. We can fix an integer $N$ that is divisible enough, such that for any parabolic subgroup $P \subseteq Q$, we have

$$H_P(\Xi_M) \subseteq \frac{1}{N}Z(\Delta_P^{Q,\vee}) \oplus \frac{1}{N}Z(\Delta_Q^{\vee}) \oplus \frac{1}{N}a_{G,Z},$$

and

$$L_0 \subseteq \frac{1}{N}Z(\Delta_P^{Q,\vee}) \oplus \frac{1}{N}Z(\Delta_Q^{\vee}) \oplus \frac{1}{N}a_{G,Z}.$$ 

For any $\lambda \in a_{P,C}^\ast$, let

$$f_{\lambda}(X) := \sum_{H \in H_P(\Xi_M)/H_P(\Xi_G)} \Gamma_P(H + H_0, X) q^{-(\lambda,H)}.$$

We need to prove that $f_{\lambda}(X)$ is a quasi-polynomial in $X$. As the sum above is finite (the support of $\Gamma_P(\cdot, X)$ is compact in $a_P^\ast$), for any $X \in a_B$, $f_{\lambda}(X)$ is holomorphic function in $\lambda$. While by definition of $\Gamma_P$, we have

$$f_{\lambda}(X) = \sum_{\{Q|P \subseteq Q\}} (-1)^{\dim a_{Q}^\ast} f_{\lambda}^Q(X),$$

where $f_{\lambda}^Q(X)$ is the series defined by

$$f_{\lambda}^Q(X) = \sum_{H \in H_P(\Xi_M)/H_P(\Xi_G)} \tau_P^Q(H + H_0) q^{-(\lambda+\nu,H)}.$$

Let $L_Q^\ast \subseteq a_P^\ast$ be the lattice dual to $L_Q = \frac{1}{N}Z(\Delta_P^{Q,\vee}) \oplus \frac{1}{N}Z(\Delta_Q^{\vee}) \oplus \frac{1}{N}a_{G,Z}$ and $H_P(\Xi_M)^\ast \subseteq a_P^\ast$ the lattice dual to $H_P(\Xi_M)$. Note that $L_Q \supseteq H_P(\Xi_M)$. Let $\hat{f}$ be a system of representatives for the finite abelian group $L_Q/H_P(\Xi_M)$, we have

$$f_{\lambda}^Q(X) = \left[\frac{H_P(\Xi_M)^\ast}{L_Q}\right]^{-1} \sum_{\nu \in \hat{f}} \sum_{H \in L_Q/H_P(\Xi_G)} \tau_P^Q(H + H_0) \hat{\tau}_Q(H + H_0 - X) q^{-(\lambda+\nu,H)}.$$

After a direct calculation, for any $X \in L_0$, $f_{\lambda}^Q(X)$ is equal to $\left[\frac{H_P(\Xi_M)^\ast}{L_Q}\right]^{-1} \frac{1}{N} a_{G,Z}$ times

$$\prod_{\nu \in \hat{f}} \frac{q^{-(\lambda+\nu, [\Sigma P^{\ast}]([N(a,H_0)]+1)/N)}}{1 - q^{-(\lambda+\nu, [\Sigma P^{\ast}])/N}} \prod_{\beta \in \Delta_Q^{\vee}} q^{-(\lambda+\nu, \beta^{\vee})(N(\varpi_\beta, X) + [N(\varpi_\beta, -H_0)]+1)/N},$$

where we use $[x]$ to denote the largest integer smaller than or equal to $x$.

As $f_{\lambda}(X) = \sum_{\{Q|P \subseteq Q\}} (-1)^{\dim a_{Q}^\ast} f_{\lambda}^Q(X)$ is holomorphic on $a_P^\ast$, for $\lambda = 0$, $f_0(X)$ equals the alternative sum of constant terms of Laurent expansions of $f_{\lambda}^Q(X)$ around 0, which shows that the result holds.

Now we come back to the proof of the Theorem 5.6.
Proof. As showed by the proof of lemma 8.1, if\( \text{support of } f \) (9.2) \( J \)
\( \int_{P(F)\backslash G(\mathbb{A})/\Xi_G} \Gamma_P(H_B(x) + [s_\xi] - X', X - X') F^{P,\xi,X'}(x) k_{P,o}(x) dx. \)

Let’s prove that for each standard parabolic subgroup \( P \), the expression (8.6) is a quasi-polynomial in \( X \). Using Iwasawa decomposition, we can decompose the integral into a double integral:\n\[
\int_{P(F)\backslash P(\mathbb{A})/\Xi_G} \int_{G(O)} \Gamma_P(H_B(p) + [s_\xi] - X', X - X') F^{P,\xi,X'}(pk) k_{P,o}(pk) dk dp.
\]

By definition, \( s_\xi \) depends on the coset in \( G(O)/I_\infty = G(O_v) \) represented by \( k \), and by Proposition 8.6, the same is true for \( F^{P,\xi,X'}(pk) \). Moreover, note that the function \( p \mapsto k_{P,o}(pk) \) is invariant under multiplication by \( Z_{MP}(k) \), in particular by \( \Xi_M \). The same is clearly true for \( p \mapsto F^{P,\xi,X'}(pk) \) too. Thus, the integral (8.6) equals the integral over \( u \in G(O_\infty)/I_\infty \) followed by the integral over \( p \in P(F)\backslash P(\mathbb{A})/\Xi_M \) of the product of
\[
F^{P,\xi,X'}(pu) \int_{I_\infty} \prod_{v \neq \infty} G(O_v) k_{P,o}(puk) dk,
\]
with
\[
\sum_{H \in H_F(\Xi_M)/H_F(\Xi_G)} \Gamma_P(H + H_B(p) + [s_\xi] - X', X - X').
\]

By lemma 8.2, this last expression is a quasi-polynomial in \( X \), hence \( X \mapsto J^\xi,X \) is a quasi-polynomial.

9. Another expression for \( J^{\xi,X} \)

Let \( Q \) be a standard parabolic subgroup of \( G \), set
\[
J_{Q,o}(x) = \sum_{a \in \Xi_G} \sum_{\gamma \in M_Q(F)\gamma_o} \sum_{n \in N_P(F)} f(ax^{-1}\gamma nx).
\]

We can easily verify that \( J_{Q,o}(x) \) is both \( M_Q(F) \) and \( N_Q(F) \) left invariant, hence \( Q(F) \) left invariant. Therefore, for any \( x \in G(\mathbb{A}) \), we can define a function \( J^\xi,X_{Q,o} \) using the same expression (8.6) as \( k^\xi,X_{o}(x) \) (page 13) but replacing \( k_{Q,o} \) by \( J_{Q,o} \).

In the Lie algebra case, when \( G = GL_n \) and \( f \) is some special test function, Chaudouard has showed (cf. [Ch15 3.9-3.11]) that the integral of \( J^{\xi,o,0} \) (the sum of \( J^{\xi,0,0} \) over all \( o \in E \) over \( G(F)\backslash G(\mathbb{A})/\Xi_G \) is closely related to the groupoid volume of semi-stable Higgs bundles. Moreover, he has showed in [Ch15 Corollaire 5.2.2] that the integral of \( J^{\xi,0,0} \) coincides with \( J^{\xi,0,0} \). Inspired by his results, we have the following proposition.

**Proposition 9.1.** Let \( (\xi, X) \in a_{B,\infty} \times a_{B,0}^* \) be any pair of vectors. The function \( J^\xi,X \)
has compact support in \( G(F)\backslash G(\mathbb{A})/Z_G(\mathbb{A}) \). Moreover, we have
\[
J^\xi,X(f) = \int_{G(F)\backslash G(\mathbb{A})/\Xi_G} J^\xi,X(x) dx.
\]

**Proof.** As showed by the proof of lemma 8.1 if \( X' \) is deep enough (depending on the support of \( f \)) in the positive chamber, then for any pair of standard parabolic subgroups
10.1. A non-existence theorem. Now let $F = \mathbb{F}_q(t)$ be the function field of the projective line $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{F}_q}$. Let $G$ be a split reductive group defined over $\mathbb{F}_q$. Let $\lambda, \mu \in \mathbb{P}^1(\mathbb{F}_q)$ be two distinct $\mathbb{F}_q$-rational points, identified as two places of $F$. Let $T_\lambda$ and $T_\mu$ be two maximal tori of $G$ defined over $\mathbb{F}_q$. Suppose that $\theta_\lambda$ and $\theta_\mu$ are generic in general position of $T_\lambda(\mathbb{F}_q)$ and $T_\mu(\mathbb{F}_q)$ respectively. We obtain by Deligne-Lusztig induction (fixing any isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_l$) two irreducible representations

$$\rho_\lambda = \epsilon_{T_\lambda} \epsilon_G R_{T_\lambda}^G (\theta_\lambda)$$

and

$$\rho_\mu = \epsilon_{T_\mu} \epsilon_G R_{T_\mu}^G (\theta_\mu)$$

of $G(\mathbb{F}_q)$, where $\epsilon_H = (-1)^{rk_H^H}$ is the sign according to the parity of the split rank of $H$ (the rank of a maximal split sub-torus). By inflation, we obtain irreducible representations of $G(\mathcal{O}_\lambda)$ and $G(\mathcal{O}_\mu)$ respectively, still denoted by $\rho_\lambda$ and $\rho_\mu$.

In [Yu21], we have defined the notion of cuspidal filter. Let $\rho$ be the representation of $G(\mathcal{O})$ as the tensor product of $\rho_\lambda$, $\rho_\mu$ and the trivial representation of $G(\mathcal{O}_v)$ for $v$ different from $\lambda, \mu$. Then $\rho$ is a cuspidal filter if for any $L^2$-automorphic representation $\pi$ of $G(\mathbb{A})$, the $\rho$-isotypic part $\pi_\rho$ is non zero implies that $\pi$ is cuspidal.

We have given in [Yu21] Proposition 5.1.2] a criterion for $\rho$ to be a cuspidal filter. If one of $T_\lambda$, $T_\mu$ is maximal anisotropic then $\rho$ is automatically a cuspidal filter. Suppose that $T_\lambda = T$ is split. Let $M_\mu$ be the minimal Levi subgroup containing $T_\mu$ defined over $\mathbb{F}_q$. Without loss of generality, we suppose that $M_\mu$ contains $T$. Then $\rho$ is a cuspidal filter if

$$\theta_\mu^w \theta_\mu | Z_M(\mathbb{F}_q) \neq 1,$$  

(10.1)
for any Weyl element \( w \in W \) and for any maximal semi-standard Levi subgroup \( M \) of \( G \) that contains \( M_\mu \).

**Example 10.1.** For \( SL_2 \), we choose \( T \) to be the group of diagonal matrix with determinant 1, so that \( \mathbb{F}_q^x \cong T(\mathbb{F}_q) \) via the map \( x \mapsto (x, x^{-1}) \). Let \( T_\sigma = T_\mu = T \), \( \theta_\mu \) and \( \theta_\lambda \) be two characters of \( \mathbb{F}_q^x \), then \( \rho \) is a cuspidal filter if
\[
\theta_\mu \neq \theta_\lambda^{-1}.
\]

We prove the following theorem using the coarse geometric expansion developed in this article by taking \( \xi \) to be in general position and using some results in [Yu21].

**Theorem 10.2.** Suppose that \( G \neq T \), \( T_\lambda = T \) is split and \( \rho \) is a cuspidal filter.

There is no cuspidal automorphic representation \( \pi = \otimes'_v \pi_v \) of \( G(\mathbb{A}) \) such that \( \pi_\lambda \) contains \( (G(\mathcal{O}_\lambda), \rho_\lambda) \), \( \pi_\mu \) contains \( (G(\mathcal{O}_\mu), \rho_\mu) \) and all other local components are unramified.

**Proof.** Let \( \mathcal{I}_\lambda \) be the standard Iwahori subgroup of \( G(F_\lambda) \) and \( \mathcal{O}^\lambda = \prod_{v \neq \lambda} \mathcal{O}_v \). We have a canonical morphism \( \mathcal{I}_\lambda \to B(\mathbb{F}_q) \). By inflation, the character \( \theta_\lambda \) of \( T(\mathbb{F}_q) \) defines a character of \( \mathcal{I}_\lambda \). Let \( \rho \) be the representation of \( G(\mathcal{O}^\lambda)\mathcal{I}_\lambda \) defined as the tensor product of \( \theta_\lambda \) of \( \mathcal{I}_\lambda \), \( \rho_\mu \) of \( G(\mathcal{O}_\mu) \) and the trivial representation of \( G(\mathcal{O}_v) \) for places \( v \) outside \( S = \{\lambda, \mu\} \). Let
\[
epsilon_\rho = \begin{cases} 
\frac{1}{\text{vol}(\mathcal{I}_\lambda)} \text{Tr}(x^{-1} | \rho), & x \in G(\mathcal{O}^\lambda)\mathcal{I}_\lambda; \\
0, & x \notin G(\mathcal{O}^\lambda)\mathcal{I}_\lambda.
\end{cases}
\]

After [Yu21] Proposition 5.2.6], we have
\[
J^{G,\xi}(\epsilon_\rho) = \sum_{\pi} m_\pi \text{dim} \text{Hom}_{G(\mathcal{O})}(\rho, \pi),
\]
where the sum is taken over the set of isomorphism classes of cuspidal automorphic representations of \( G(\mathbb{A}) \) and \( m_\pi \) is the multiplicity of \( \pi \) in the cuspidal automorphic spectrum, and we choose \( \lambda \) as the point \( \infty \), \( \xi \) is chosen to be in general position in the sense that the projection of \( \xi \) to \( a_P \) does not belong to \( X_*(P) + a_G \) for any semi-standard parabolic subgroup \( P \subseteq G \).

Using the coarse geometric expansion established earlier (Corollary 5.3], we have
\[
J^{G,\xi}(\epsilon_\rho) = \sum_{\sigma \in \Sigma} J^{G,\xi}_\sigma(\epsilon_\rho).
\]

We showed in [Yu21] Theorem 3.2.5] that \( J^{G,\xi}_\sigma(\epsilon_\rho) = 0 \), if \( \sigma \) is not represented by an elliptic element \( \sigma \in T(\mathbb{F}_q) \), i.e. \( \sigma \in \mathcal{T}(\mathbb{F}_q) \) and satisfies \( Z_{G,\mathcal{O}_v}^0 = Z_{G}^0 \). Note that the theorem loc. cit. is stated for split semi-simple reductive group but its proof works without semi-simplicity. For the situation we are discussing, we want to show further that
\[
J^{G,\xi}_\sigma(\epsilon_\rho) = 0,
\]
even for an elliptic element \( \sigma \in T(\mathbb{F}_q) \), then this completes the proof.

Using [Yu21] Theorem 3.2.5] again, it suffices to prove that \( J^{G,\xi}_\sigma(\epsilon_\rho) = 0 \), for any \( w \in W(G_\sigma^0, T) \setminus W \) which sends positive roots of \( (G_\sigma^0, T) \) to positive roots. Note that for such a \( w \), the group \( w\mathcal{I}_\lambda w^{-1} \cap G_\sigma^0(F_\lambda) \) is the standard Iwahori subgroup \( \mathcal{I}_{\lambda,\sigma} \) of
$G^0_u(F)$ and $w\xi$ is also in general position. Without loss of generality, let’s prove that

$$J^{G^0_u, \xi}_{[\sigma]}(\epsilon_p) = 0.$$  

Let $f = \otimes f_v \in C^\infty(G^0_u(\mathbb{A}))$ be a function defined as the tensor product of $f_\lambda = 1_{\mathbb{A}_\lambda}$,

$$f_\mu = \begin{cases} R^G_{T^\mu}(1)(x), & x \in G(O_\mu); \\ 0, & x \notin G(O_\mu). \end{cases}$$

where $T^\mu$ is any maximal torus of $G^0_u$ defined over $\mathbb{F}_q$, and $1_{G^0_u(O_v)}$ for places $v$ outside $\{\lambda, \mu\}$. After the character formula of Deligne-Lusztig ([DL76, Theorem 4.2]) which applies for every semi-simple element $\sigma \in G(\mathbb{F}_q)$ and every unipotent element $u \in G^0_u(\mathbb{F}_q)$:

$$R^G_{T^\mu}(\theta_\mu)(\sigma u) = \frac{1}{|G^0_u(\mathbb{F}_q)|} \sum_{\gamma \in G(\mathbb{F}_q) \sigma \in \gamma T^\mu \gamma^{-1}} \theta_\mu(\gamma^{-1} \sigma \gamma) R^G_{T^\mu}(1)(u),$$

it suffices to prove that

$$J^{G^0_u, \xi}_{unip}(f) = 0,$$

where $J^{G^0_u, \xi}_{unip} = J^{G^0_u, \xi}_{[1]}$. For this, we’re going to pass to Lie algebra. Let $g_\sigma$ be the Lie algebra of $G^0_u$ and $J_{\sigma, \lambda} \subseteq g_\sigma(O_\lambda)$ be the standard Iwahori Lie sub-algebra.

Let $\varphi = \otimes \varphi_v \in C^\infty(g_\sigma(\mathbb{A}))$, where for $v$ outside $\{\lambda, \mu\}$,

$$\varphi_v = 1_{g_\sigma(O_v)};$$

for $v = \lambda$,

$$\varphi_\lambda = 1_{J_{\sigma, \lambda}};$$

and for $v = \mu$,

$$\varphi_\mu = q^{\frac{1}{2}t^2 + \frac{1}{2}l_1 - 1} \bar{2}_{\Omega_{-t_\mu}},$$

where $t_\mu$ is a semi-simple regular element in $t^\mu(\mathbb{F}_q)$, and $\bar{2}_{\Omega_{-t_\mu}}$ is the Fourier transform of the characteristic function of the set $\Omega_{-t_\mu}$ consisting of elements $x \in g_\sigma(O_\mu)$ whose reduction mod-$\varphi_\mu$ lies in the $G(\mathbb{F}_q)$-conjugacy class of $-t_\mu$. The support of $\varphi$ is contained in $g_\sigma(O^\lambda)J_\lambda$. Therefore we have

$$J^{\theta_\sigma, \xi}(\varphi) = |g_\sigma(\mathbb{F}_q)| J^{\theta_\sigma, \xi}_{nilp}(\varphi).$$

Springer’s hypothesis proved by Kazhdan ([KV06, Theorem A.1]) tells us that

$$\varphi_\mu(u) = f_\mu(l(u)),$$

for any unipotent element $u \in G^0_u(O_\mu)$ and any Springer’s isomorphism (which exists since we’re in very good characteristic) $l : U_{G^0} \rightarrow N_{g_\sigma}$ defined over $\mathbb{F}_q$.

Let $\langle \cdot, \cdot \rangle$ be a $G$-invariant bilinear form on $g$ defined over $\mathbb{F}_q$. Thus, it defines by taking respectively $\mathbb{A}$-points, and $F_v$-points (for every place $v$), a bilinear form on $g(\mathbb{A})$ and $g(F_v)$ respectively. For $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{F}_q}$, the canonical line bundle is isomorphic to $O_{\mathbb{P}^1}(-2)$ and $K_{\mathbb{P}^1} = -\lambda - \mu$ is a canonical divisor. Let $\varphi_v$ be the maximal ideal of $O_v$, let $\varphi_v^{-K_{\mathbb{P}^1}} = \varphi_\lambda \otimes \varphi_\mu \prod_{v \neq \lambda, \mu} O_v$. We have

$$\mathbb{A}/(F + \varphi_v^{-K_{\mathbb{P}^1}}) \cong H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(-2)) \cong H^0(\mathbb{P}^1, O_{\mathbb{P}^1})^* \cong \mathbb{F}_q,$$

by Serre duality. Fix a non-trivial additive character $\psi$ of $\mathbb{F}_q$. Via above isomorphisms, $\psi$ can be viewed as a character of $\mathbb{A}/F$. We use this $\psi$ in the definition of Fourier
transform. By the trace formula of Lie algebra established earlier (Theorem 5.7), we have
\[ J_{\mathfrak{g}_0}(\mathbf{\hat{\varphi}}) = q^{1 - g} \dim \mathfrak{g}_0 \cdot J_{\mathfrak{g}_0}(\mathbf{\hat{\varphi}}). \]
The key point is that the Fourier transform $\mathbf{\hat{\varphi}}$ has support in $\mathfrak{g}_\sigma(O^{\lambda})I_\lambda$ as well ([Yu21, 5.3.2]). Therefore, by our vanishing theorem ([Yu21, Theorem 3.2.5] again
In fact, the function $\mathbf{\hat{\varphi}}$ is supported in the set of regular semi-simple elements. Therefore, $J_{\mathfrak{g}_0}(\mathbf{\hat{\varphi}}) = 0$.

It’s clear that for any $z \in \mathfrak{g}_\sigma(F_q)$, we have
\[ J_{\mathfrak{g}_0}(\mathbf{\hat{\varphi}}) = 0. \]
In fact, the function $\mathbf{\hat{\varphi}}_\mu$ is supported in the set of regular semi-simple elements. Therefore, $J_{\mathfrak{g}_0}(\mathbf{\hat{\varphi}}) = 0$, if $o$ is not represented by a regular semi-simple element in $\mathfrak{g}_\sigma(F)$. However $z$ is never regular if $G^0_{\sigma}$ is not a torus. Since $\sigma$ is an elliptic element contained in $T(F_q)$ and the torus $T$ is split, $G^0_{\sigma}$ is not a torus if $G$ is not a torus. \hfill \Box

10.2. $SL_l$ with $l$ being a prime. As in Section [10.1] we still suppose that $F = F_q(t)$ and $\lambda, \mu$ are two distinct places of $F$ of degree 1.

Let $l$ be a prime number. Let $G = SL_l$. Recall that $p$ is a very good prime for $SL_l$ if $p \neq l$. Let $T_\lambda = T_\mu$ be an anisotropic maximal torus defined over $F_q$. Recall that $\rho_\lambda$ is cuspidal if and only if $T_\lambda$ is anisotropic (since $\theta_\lambda$ is in general position).

**Theorem 10.3.** If $\rho_\lambda$ is contragredient to $\rho_\mu$, then there is exactly one cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $G(A)$ (with cuspidal multiplicity 1) such that $\pi_\lambda$ contains $(G(O_\lambda), \rho_\lambda)$, $\pi_\mu$ contains $(G(O_\mu), \rho_\mu)$ and all other local components are unramified. If $\rho_\lambda$ is not contragredient to $\rho_\mu$, then there is no such cuspidal automorphic representation.

**Proof.** In $SL_l$, there is exactly one conjugacy class of anisotropic tori, which are isomorphic to the norm 1 torus $R^1_{\mathbb{G}_m, |F_q|} \mathbb{G}_m$. Hence $T_\lambda(F_q) = T_\mu(F_q)$ is the group of norm 1 elements in $\mathbb{F}_q^\times$ for the norm map of $F_q^n/\mathbb{F}_q$. Since Deligne-Lusztig induction preserves the property of being contragredient ([DM91, Proposition 11.4]), $\rho_\lambda$ is contragredient to $\rho_\mu$ if and only if $(T_\lambda, \theta_\lambda)$ is conjugate to $(T_\mu, \theta_\mu^{-1})$. Using the identification $W = \mathfrak{S}_l$, the torus $T_\lambda$ corresponds to the conjugacy class of a cyclic permutation $w$ of length $l$. Moreover, the group $W(G, T_\lambda)(F_q)$ is isomorphic to the cyclic group generated by $w$. We have
\[ T_\lambda(F_q) \cong (\mathbb{F}_q^\times)^{N_{\text{Nm}}} := \{ s \in \mathbb{F}_q^\times | s^{q - 1 + q^2 - 2 + \cdots + 1} = 1 \}, \]
and, under the above isomorphisms, the generator $w$ sends $s \in (\mathbb{F}_q^\times)^{N_{\text{Nm}}}$ to $s^q$. Therefore, $\theta_\lambda$ being in general position implies that
\[ \theta_\lambda^l \neq \theta_\lambda, \quad \text{if } l \nmid i. \]
If $\rho_\lambda$ is contragredient to $\rho_\mu$, we can assume that $\theta_\lambda = \theta_\mu^{-1}$. If $\rho_\lambda$ is not contragredient to $\rho_\mu$ we have $\theta_\lambda \neq \theta_\mu^{-1}$. For all $i \in \mathbb{Z}$, hence
\[ \theta_\lambda^i \theta_\mu^j \neq 1, \quad \forall i, j \in \mathbb{Z}. \]
Moreover, we clearly have \( \theta_\lambda^{q^{-1}} |_{Z_G(\mathbb{F}_q)} = 1 \) for all \( i \in \mathbb{N} \) since the order of \( Z_G(\mathbb{F}_q) \) divides \( q - 1 \). A final remark is that we may assume
\[
(10.4) \quad (\theta_\lambda \theta_\mu)|_{Z_G(\mathbb{F}_q)} = 1,
\]
otherwise there is no cuspidal automorphic representation with the desired properties as we can see by checking its central character which must be trivial on \( Z_G(\mathbb{F}_q) \subseteq Z_G(F) \).

For any automorphic representation \( \pi = \otimes' \pi_v \) of \( SL_l(\mathbb{A}) \). Since \( \rho_\lambda \) is cuspidal, if \( \pi_\lambda \) contains \( \rho_\lambda \), then \( \pi_\lambda \) is supercuspidal and hence \( \pi \) must be cuspidal automorphic. We will do an explicit calculation with our coarse geometric expansion by setting \( \xi = 0 \).

Let \( \rho \) be the representation of \( G(\mathcal{O}) \) defined as the tensor product of \( \rho_\lambda \) (resp. \( \rho_\mu \)) of \( G(\mathcal{O}_\lambda) \) (resp. of \( G(\mathcal{O}_\mu) \)) and the trivial representation of \( G(\mathcal{O}_v) \) for places \( v \) outside \( S = \{\lambda, \mu\} \). Let
\[
e_\rho = \begin{cases} 
\text{Tr}(x^{-1} | \rho), & x \in G(\mathcal{O}); \\
0, & x \notin G(\mathcal{O}).
\end{cases}
\]

After [Yu21] Proposition 5.2.6], we have
\[
J^G(e_\rho) = \sum_\pi m_\pi \dim \text{Hom}_{G(\mathcal{O})}(\rho, \pi).
\]

It suffices to prove that \( J^G(e_\rho) \) equals 1 or 0 depending on whether or not \( \rho_\lambda \) and \( \rho_\mu \) are contragredient.

Using our coarse geometric expansion (Corollary 5.3):
\[
J^G(e_\rho) = \sum_{o \in \mathcal{E}} J^G_o(e_\rho).
\]

First of all, suppose \( o \in \mathcal{E} \) is a class such that \( J^G_o(e_\rho) \neq 0 \). As \( X \mapsto J^G_{o,X}(e_\rho) \) is a quasi-polynomial (Theorem 5.6), there is an \( X \in \mathfrak{a}_B \) which we can assume to be deep enough in the positive chamber, so that \( J^G_{o,X}(e_\rho) \neq 0 \). Since the support of \( e_\rho \) is contained in \( G(\mathcal{O}) \), the Theorem 5.1 implies that there is an element \( \gamma \in o \) and \( x \in G(\mathbb{A}) \) such that \( x^{-1} \gamma x \in G(\mathcal{O}) \). However, this implies that \( \gamma^n \in x G(\mathcal{O}) x^{-1} \cap G(F) \) for any \( n \). Note that \( x G(\mathcal{O}) x^{-1} \cap G(F) \) is finite, so the element \( \gamma \) is a torsion element. By [Yu21] Proposition 2.3.2], \( \gamma \) admits Jordan-Chevalley decomposition. Moreover, [Yu21] Theorem 2.4.1 implies that we can take as representative a semi-simple element \( \sigma \in G(\mathbb{F}_q) \). Furthermore, the Deligne-Lusztig induced character \( R^G_{\lambda \sigma}(\theta_\lambda) \) is supported in the set of elements whose semi-simple part can be conjugate to an element in \( T_\lambda(\mathbb{F}_q) \).

Therefore, we conclude that
\[
J^G(e_\rho) = \sum_{s \in T_\lambda(\mathbb{F}_q)/\sim_{\text{conj}}} J^G_{[s]}(e_\rho).
\]

Note that two elements in \( s_1, s_2 \in T_\lambda(\mathbb{F}_q) \) are conjugate if there is \( 1 \leq i \leq l \) such that \( s_1^i = s_2 \). The fact that \( l \) is a prime implies that any element in \( T_\lambda(\mathbb{F}_q) \) is either regular and elliptic or lies in center of \( G = SL_l \). If \( s \in T_\lambda(\mathbb{F}_q) \) is regular, then there are \( l \) elements in \( T_\lambda(\mathbb{F}_q) \) that are conjugate to \( s \). Therefore we have
\[
(10.5) \quad J^G(e_\rho) = \frac{1}{l} \sum_{s \in T_\lambda(\mathbb{F}_q)/Z_G(\mathbb{F}_q)} J^G_{[s]}(e_\rho) + |Z_G(\mathbb{F}_q)| J^G_{\text{unip}}(e_\rho).
\]
Suppose that $s \in T_\lambda(\mathbb{F}_q)$ is regular and elliptic, there is no proper Levi subgroup of $G$ containing $s$. We have by definition

$$J^G_{[s]}(e_\rho) = \int_{G(F) \backslash G(\mathbb{A})} \sum_{\gamma \in T_\lambda(F) \backslash G(F)} e_\rho(x^{-1} \gamma^{-1} s \gamma x) dx.$$ 

Therefore

$$J^G_{[s]}(e_\rho) = \text{vol}(T_\lambda(F) \backslash T_\lambda(\mathbb{A})) O_s(e_\rho),$$

where

$$O_s(e_\rho) = \int_{T_\lambda(\mathbb{A}) \backslash G(\mathbb{A})} e_\rho(x^{-1} s x) dx.$$

If $x \in G(\mathbb{A})$ is an element such that $x^{-1} s x \in G(\mathcal{O})$, then by [Ko86, Proposition 7.1], $x^{-1} s x$ and $s$ are conjugates by an element in $G(\mathcal{O})$. In this case, we have

$$e_\rho(x^{-1} s x) = (\sum_{i=1}^l \theta_\lambda(s)^{\eta_i})(\sum_{i=1}^l \theta_\mu(s)^{\eta_i}).$$

If $x^{-1} s x \notin G(\mathcal{O})$, then $e_\rho(x^{-1} s x) = 0$. We obtain that

$$O_s(e_\rho) = (\sum_{i=1}^l \theta_\lambda(s)^{\eta_i})(\sum_{i=1}^l \theta_\mu(s)^{\eta_i}) O_s(1_{G(\mathcal{O})}).$$

To calculate $O_s(1_{G(\mathcal{O})})$, we can decompose it into product of local orbital integrals:

$$O_s(1_{G(\mathcal{O})}) = \prod_v \int_{T_\lambda(F_v) \backslash G(F_v)} 1(x^{-1} s x) dx.$$

We use the above result of Kottwitz again, then we have

$$\int_{T_\lambda(F_v) \backslash G(F_v)} 1(x^{-1} s x) dx = \frac{\text{vol}(G(\mathcal{O}_v))}{\text{vol}(T_\lambda(\mathcal{O}_v))}.$$ 

We obtain that for $s \in T_\lambda(\mathbb{F}_q) - Z_G(\mathbb{F}_q)$,

$$(10.6) \quad J^G_{[s]}(e_\rho) = \frac{\text{vol}(T_\lambda(F) \backslash T_\lambda(\mathbb{A}))}{\text{vol}(T_\lambda(\mathcal{O}))} (\sum_{i=1}^l \theta_\lambda(s)^{\eta_i})(\sum_{i=1}^l \theta_\mu(s)^{\eta_i}).$$

For any non-trivial character $\theta$ of $T_\lambda(\mathbb{F}_q)$ that is trivial on $Z_G(\mathbb{F}_q)$ we have,

$$\sum_{s \in T_\lambda(\mathbb{F}_q) - Z_G(\mathbb{F}_q)} \theta(s) = -|Z_G(\mathbb{F}_q)|.$$

Therefore, if $\rho_\mu$ is not contragredient to $\rho_\lambda$, then applying (10.3) and (10.4), we have

$$\frac{l}{1} (\sum_{i=1}^l \theta_\lambda(s)^{\eta_i})(\sum_{i=1}^l \theta_\mu(s)^{\eta_i}) = -|Z_G(\mathbb{F}_q)|l^2.$$

If $\rho_\mu$ is contragredient to $\rho_\lambda$, then there are $l$ trivial characters among $\theta_\mu^{\eta_j} \theta_\lambda^{\eta_i}$ ($1 \leq i, j \leq l$), we deduce similarly that

$$\frac{l}{1} (\sum_{i=1}^l \theta_\lambda(s)^{\eta_i})(\sum_{i=1}^l \theta_\mu(s)^{\eta_i}) = -|Z_G(\mathbb{F}_q)|(l^2 - l) + \frac{q^l - 1}{q - 1} - |Z_G(\mathbb{F}_q)|l.$$

Here we have used the fact that $|T_\lambda(\mathbb{F}_q)| = \frac{q^l - 1}{q - 1}$.
To calculate $\text{vol}(T_\lambda(F) \backslash T_\lambda(A))/\text{vol}(T_\lambda(O))$, we may use the short exact sequence:

$$1 \rightarrow T_\lambda(O)/T_\lambda(F) \rightarrow T_\lambda(A)/T_\lambda(F) \rightarrow T_\lambda(A)/T_\lambda(O) \rightarrow 1.$$ 

It’s easy to see that $T_\lambda(A)/T_\lambda(F)T_\lambda(O)$ is trivial since $F$ is the function field of the projective line $\mathbb{P}^1_F$ which has trivial Jacobian variety. We obtain

$$\frac{\text{vol}(T_\lambda(F) \backslash T_\lambda(A))}{\text{vol}(T_\lambda(O))} = \frac{1}{|T_\lambda(F)|} = \frac{q-1}{q^d-1}. \tag{10.7}$$

To summarize, we have

$$J^G(e_\rho) = \begin{cases} |Z_G(F)|J^G_{\text{unip}}(e_\rho) - (\frac{q^d-1}{q-1})|Z_G(F)|l + 1, & \rho_\lambda \text{ is contragredient to } \rho_\mu; \\ |Z_G(F)|J^G_{\text{unip}}(e_\rho) - (\frac{q-1}{q}-1)|Z_G(F)|l, & \rho_\lambda \text{ is not contragredient to } \rho_\mu. \end{cases} \tag{10.8}$$

In the following, we calculate $J^G_{\text{unip}}(e_\rho)$. We will derive it from calculations of a trace formula of Lie algebra.

As in the proof of the Theorem 10.2 we fix a $G$-equivariant, non-degenerate bilinear form $\langle , \rangle$ defined over $F_q$. We also use the same additive character $\psi$ of $\mathbb{A}/F$ for our Fourier transform.

Let’s fix two regular elements $x$ and $y$ in $t_\lambda(F_q)$ which are not $G(F_q)$-conjugate (equivalently not $G(F_q)$-conjugate). Let $\Omega_x$ (resp. $\Omega_y$) be the characteristic function over $g(F_q)$ of the conjugacy class of $x$ (resp. of $y$). We have a character defined by

$$\chi_\lambda: t_\lambda(F_q) \rightarrow \mathbb{C}^\times; \quad z \mapsto \psi((z, x)).$$

Similarly we define $\chi_\mu$.

Since the bilinear form $\langle , \rangle$ is non-degenerate on $t_\lambda(F_q)$ (McN03 Lemma 5), the fact $x$ is not conjugate to $y$ implies that $\chi_\lambda$ is not conjugate to $\chi_\mu$. We know that for the torus $T_\lambda$, the Frobenius element acts by an element of Weyl group, therefore the conjugates of $\chi_\lambda$ (resp. $\chi_\mu$) are $\chi^{q^i}_\lambda$ (resp. $\chi^{q^i}_\mu$), $i = 1, \ldots, l$. We deduce that $\chi^{q^i}_\lambda \chi^{q^j}_\mu$ is non-trivial for any $1 \leq i, j \leq l$. By a result of Letellier (Le05, 7.3.3) (note that his Fourier transform is normalized differently from the one used here), the Fourier transform of the characteristic function $\mathbb{1}_{\Omega_x}$ is given by a Lie algebra analogue of the Deligne-Lusztig induced character: for two commuting elements $s, n \in g(F_q)$ with $s$ semi-simple and $n$ nilpotent:

$$\mathbb{1}_{\Omega_x}(s+n) = \frac{1}{|G_s(F_q)|} \sum_{\gamma \in G(F_q) \in \text{Ad}(\gamma)(T_\lambda)} \chi_\lambda(\text{Ad}(\gamma^{-1})s)R_{\gamma}^{G_s}(1)(l^{-1}(n)), \tag{10.9}$$

where $l$ is any Springer’s isomorphism $l: U_G \rightarrow N_q$ defined over $F_q$. Let $\varphi = \otimes_v \varphi_v \in C_c^\infty(g(\mathbb{A}))$ with $\varphi_v = \mathbb{1}_{G(O_v)}$ if $v \neq \lambda, \mu$, and $\varphi_\lambda$ (resp. $\varphi_\mu$) is a function supported in $g(O_\lambda)$ (resp. $g(O_\mu)$) so that for any element $x \in g(O_\lambda)$, the value $\varphi_\lambda(x)$ (resp. $\varphi_\mu(x)$) is given by the right-hand side of (10.9) with $s, n$ being respectively the semi-simple part and nilpotent part of $\mathfrak{p} \in g(F_q)$. Using the quasi-polynomial behaviour (Theorem 5.5) of $X \mapsto J^G_{\text{nilp}}(\varphi)$ and $X \mapsto J^G_{\text{unip}}(e_\rho)$ and the Theorem 5.1 we deduce that

$$J^G_{\text{nilp}}(\varphi) = J^G_{\text{unip}}(e_\rho).$$
We know that the Fourier transform of $\varphi$ (by a direct calculation, see [Yu21 5.3.2]) is $\bigotimes_v \hat{\varphi}_v$ with $\hat{\varphi}_v = 1_{g(O_v)}$ for $v \neq \lambda, \mu,$

$$\hat{\varphi}_\lambda(z) = \begin{cases} q^{-\frac{1}{2}((2-l)z \bar{z})}1_{\Omega_\lambda}(-z), & z \in g(O_\lambda); \\ 0, & z \notin g(O_\lambda); \end{cases}$$

where $\overline{z}$ is the reduction mod $\varphi_\lambda$ of $z,$ and $\hat{\varphi}_\mu$ is given by a similar formula. It’s then easy to see that

$$J^g(\hat{\varphi}) = 0.$$ 

In fact, $\hat{\varphi}$ is supported in $g(O),$ therefore $J^g(\hat{\varphi})$ vanishes except if $o$ is represented by a semi-simple element in $s \in G(F_q)$ ([Yu21 2.4.2]). If $J^g_{s}(\hat{\varphi}) \neq 0,$ by [Ko86 Proposition 7.1], we deduce that $s$ is conjugate both to $-x$ and to $-y.$ However, by the choices of $x$ and $y,$ this is impossible.

By our trace formula of the Lie algebra (Theorem 5.7), we get that

$$J^g(\varphi) = 0.$$ 

As in the group case,

$$J^g_{nilp}(\varphi) = \sum_{\ell \in \ell(F_q)-\{0\}} J^g(\varphi) = -\frac{1}{l} \sum_{s \in \ell(F_q)-\{0\}} J^g(\varphi),$$

where the last equality follows from the fact that every element in $\ell(F_q)-\{0\}$ is regular hence is conjugate to $l$ elements in $\ell(F_q)-\{0\}.$ Moreover every element in $\ell(F_q)-\{0\}$ is also elliptic. As in the group case again, we find that for $s \in \ell(F_q)-\{0\}:

$$J^g_{s}(\varphi) = \frac{\text{vol}(T_\lambda(F)/T_\lambda(A))}{\text{vol}(T_\lambda(O))} \left( \sum_{i=1}^{l} \chi_\lambda(s)^{q^i} \right) \left( \sum_{i=1}^{l} \chi_\mu(s)^{q^i} \right).$$

Therefore,

$$J^g_{nilp}(\varphi) = -\frac{1}{l} \sum_{s \in \ell(F_q)-\{0\}} \frac{\text{vol}(T_\lambda(F)/T_\lambda(A))}{\text{vol}(T_\lambda(O))} \left( \sum_{i=1}^{l} \chi_\lambda(s)^{q^i} \right) \left( \sum_{i=1}^{l} \chi_\mu(s)^{q^i} \right).$$

Recall that for any $1 \leq i, j \leq l,$ the character $\chi_\lambda^{q^i} \chi_\mu^{q^j}$ is non-trivial on $\ell(F_q).$ Therefore,

$$\sum_{s \in \ell(F_q)-\{0\}} \chi_\lambda^{q^i}(s) \chi_\mu^{q^j}(s) = -1.$$

We deduce that

$$J^g_{nilp}(\varphi) = \frac{q-1}{q^l-1}. \tag{10.10}$$

Finally, we conclude from (10.10) and (10.8) that

$$J^G(e_\rho) = \begin{cases} 1, & \rho_\lambda \text{ is contragredient to } \rho_\mu; \\ 0, & \rho_\lambda \text{ is not contragredient to } \rho_\mu. \end{cases} \tag{10.11}$$

As we have explained in the beginning of the proof, this suffices to prove the theorem. \hfill \Box
References

[Ar78] Arthur, J. A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbb{Q})$. Duke Math. J. 45 (1978), no. 4, 911-952.

[Ar80] Arthur, J. On a family of distributions obtained from orbits. Canad. J. Math. 38 (1986), no. 1, 179-214.

[Ar05] Arthur, J. An introduction to the trace formula. Harmonic analysis, the trace formula, and Shimura varieties, 1-263, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.

[Be95] Behrend, K. A. Semi-stability of reductive group schemes over curves. Math. Ann. volume 301, p.281-305(1995)

[Bou68] Bourbaki, N. Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines. Actualités Scientifiques et Industrielles, No. 1337 Hermann, Paris 1968 288 pp.

[BoT65] Borel, A.; Tits, J. Groupes réductifs. Inst. Hautes Études Sci. Publ. Math. No. 27 (1965), 55-150.

[BT72] Bruhat, F.; Tits, J. Groupes réductifs sur un corps local. Inst. Hautes Études Sci. Publ. Math. No. 41 (1972), 5-251.

[BT84] Bruhat, A.; Tits, J. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée. Inst. Hautes Études Sci. Publ. Math. No. 60 (1984), 197-376.

[Co14] Conrad, B. Reductive group schemes. Autour des schémas en groupes. Vol. I, 93-444, Panor. Synthèses, 42/43, Soc. Math. France, Paris, 2014.

[CGP15] Conrad, B.; Gabber, O.; Prasad, G. Pseudo-reductive groups. Second edition. New Mathematical Monographs, 26. Cambridge University Press, Cambridge, 2015. xxiv+665 pp.

[Ch02] Chaudouard, P.-H. La formule des traces pour les algèbres de Lie. Math. Ann. 322 (2002), no. 2, 347-382.

[Ch15] Chaudouard, P.-H. Sur le comptage des fibrés de Hitchin. Astérisque No. 369 (2015), 223-284.

[CL10] Chaudouard, P.-H.; Laumon, G. Le lemme fondamental pondéré. I. Constructions géométriques. Compos. Math. 146 (2010), no. 6, 1416-1506.

[DL76] Deligne, P.; Lusztig, G. Representations of reductive groups over finite fields. Ann. of Math. (2) 103 (1976), no. 1, 103-161.

[DM91] Digne, F.; Michel, J. Representations of finite groups of Lie type. London Mathematical Society Student Texts, 21. Cambridge University Press, Cambridge, 1991. iv+159 pp.

[Gi02] Gille, P. Unipotent subgroups of reductive groups in characteristic $p > 0$. Duke Math. J. 114 (2002), no. 2, 307-328.

[HS10] Heinloth, J.; Schmitt, A. The cohomology rings of moduli stacks of principal bundles over curves. Doc. Math. 15 (2010), 423-488.

[KV06] Kazhdan, D.; Varshavsky, Y. Endoscopic decomposition of certain depth zero representations. Studies in Lie theory, 223-301, Progr. Math., 243, Birkhäuser Boston, Boston, MA, 2006.

[Ko86] Kottwitz, R. Stable trace formula: elliptic singular terms. Math. Ann. 275 (1986), no. 3, 365-399.

[La97] Lafforgue, L. Chiroucas de Drinfeld et conjecture de Ramanujan-Petersson. Astérisque No. 243 (1997), ii+329 pp.

[LL21] Labesse, J.-P., Lemaire, B. La formule des traces tordue pour les corps de fonctions. preprint. https://arxiv.org/abs/2102.02517

[LW11] Labesse, J.-P.; Waldspurger, J.-L. La formule des traces tordue d’après le Friday Morning Seminar. With a foreword by Robert Langlands. CRM Monograph Series, 31. American Mathematical Society, Providence, RI, 2013. xxvi+234 pp.

[Le05] Letellier, E. Fourier transforms of invariant functions on finite reductive Lie algebras. Lecture Notes in Mathematics, 1859. Springer-Verlag, Berlin, 2005. xii+165 pp.

[McN04] McNinch, G. J. Nilpotent orbits over ground fields of good characteristic. Math. Ann. 329 (2004) 49-85
[MW94] Moeglin, C.; Waldspurger, J.-L. Spectral decomposition and Eisenstein series. Une paraphrase de l’Écriture [A paraphrase of Scripture]. Cambridge Tracts in Mathematics, 113. Cambridge University Press, Cambridge, 1995. xxviii+338 pp.

[Oe84] J. Oesterlé, Nombres de Tamagawa et groupes unipotents en caractéristique $p$. Inventiones Math. 78 (1984), pp. 13-88

[SS70] Springer, T.A., Steinberg, R.: Conjugacy classes, Seminar on algebraic groups and related finite groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Springer, Berlin, 1970, pp. 167-266, Lecture Notes in Mathematics, Vol. 131

[Ti87] Tits, J. Unipotent elements and parabolic subgroups of reductive groups. II. Algebraic groups Utrecht 1986, 265-284, Lecture Notes in Math., 1271, Springer, Berlin, 1987.

[Yu21] Yu, H. Number of cuspidal automorphic representations and Hitchin’s moduli spaces preprint. arXiv:2110.13858v1. https://arxiv.org/abs/2110.13858

IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria. Current address: Weizmann Institute of Science, Herzl St 234, Rehovot, Israel. Email: hongjie.yu@weizmann.ac.il