The field of quantum noise in mesoscopic systems has been exploded during the last decade, most achievements being summarized in a recent review article. Measurement of fractional charge in Quantum Hall regime \[1\], noise measurements in atomic-size junctions \[2\] and superconductors \[3\] are milestones of the field and demonstrate the importance of quantum noise as a unique tool to study electron correlations and entanglements of different kinds. A very important step has been made in \[4\] where an elegant theory of full counting statistics (FCS) has been presented. This theory encompasses not only noise, but all higher momenta of the charge transfer.

Starting from pioneering work of Büttiker \[5\], a special attention has been paid to noise and statistics of electron transfer in multi-terminal circuits. The correlations of currents flowing to different terminals reveal Fermi statistics of electrons. These cross-correlations have been recently observed. \[6\] Although the noise correlations for several relevant layouts have been understood \[7\], the evaluation of FCS still encountered difficulties. For instance, an attempt to build up FCS with "minimal correlation approach" \[8\] has lead to contradictions \[9\] This is unfortunate, since higher-order current correlations supply information about higher-order electron correlations and multi-particle interference. This information is of fundamental importance and can be hardly obtained by any other means.

In this letter, we present a calculational scheme that allows for easy evaluation of FCS in a multi-terminal mesoscopic system. It is of a great intellectual enjoyment that this scheme is a simple and universal one. The theory appears to be a circuit theory of 2 \times 2 matrices associated with Keldysh Green functions. We illustrate the theory by considering the big fluctuations of currents in various three-terminal circuits.

We start by introducing current operators \(\hat{I}_i\), each being associated with the current to a certain terminal \(i\). Extending the method of \[10\] we introduce a Keldysh-type Green function defined by

\[
\left(i\frac{\partial}{\partial t} - \hat{H} + \frac{1}{2} \tau_3 \sum_i \chi_i(t) \hat{I}_i\right) \otimes G(t, t') = \delta(1 - 1')
\]  

(1)

Here we follow notations of a comprehensive review \[11\], \(\chi_i\) are time-dependent parameters, \(\tau_3\) is a 2 \times 2 matrix in Keldysh space, \(\hat{H}\) is the one-particle Hamiltonian that incorporates all information about the system layout, including boundaries, defects and all kinds of elastic scattering. We use "hat", "bar" and "check" to denote operators in coordinate space, matrices in Keldysh space and operators in direct product of these spaces respectively. The Eq. (1) defines the Green function unambiguously provided boundary conditions are satisfied: \(\bar{G}(t, t') = G(x, x', t, t')\) approaches the common equilibrium Keldysh Green functions \(\bar{G}_i(t - t')\) provided \(x, x'\) are sufficiently far in the terminal \(i\). These \(\bar{G}_i(t - t')\) incorporate information about the state of the terminals: their voltages \(V_i\) and temperatures \(T_i\).

One can easily see by traditional diagrammatic methods \[11\] that the expansion of \(\bar{G}\) in \(\chi_i(t)\) generates all possible diagrams for higher order correlators of \(\hat{I}_i(t)\) and thereby incorporates all the information about statistics of charge transfer. If we limit our attention to low-frequency limit of current correlations, we can keep time-independent \(\chi_i\). In this case, the Green functions are functions of time difference only and the Eq. 1 separates in energy representation. It is convenient to introduce the following \(\chi_i\)-dependent action defined as a sum of closed diagrams:

\[
\frac{\partial S}{\partial \chi_i} = -i\tilde{t}_0 \int \frac{d\varepsilon}{2\pi} \text{Tr} \left( \tilde{\tau}_3 \hat{I}_i \bar{G}(\varepsilon) \right)
\]  

(2)

This allows us to express the probability for \(N_i\) electrons to be transferred to the terminal \(i\) during time interval \(t_0\)

\[
P(\{N_i\}) = \int_{-\pi}^{\pi} d\chi_i \frac{1}{2\pi} e^{-S(\chi_i) - \frac{i}{2} \sum_i N_i \chi_i}
\]  

(3)

(Higher-order) derivatives of \(S\) with respect to \(\chi_i\) give (higher-order) moments of \(P(\{N_i\})\). First derivatives yield average currents to terminals, second derivatives correspond to the noises and noise correlations.

Using special properties of current operators, \(\chi\)-dependent terms in Eq. (1) can be gauged away \[10\] \[12\]. The \(\chi\) dependence of \(\bar{G}\) is thereby transferred to the boundary conditions: the gauged Green function far in
Each terminal shall approach \( \tilde{G}_i(\epsilon) \) defined as

\[
\tilde{G}_i(\epsilon) = \exp(i\chi_i r_3/2)\hat{G}^{(0)}_i(\epsilon) \exp(-i\chi_i r_3/2)
\]  

In the present form, the Eq. (4) with relations (4,3) solves the problem of determination of the FCS for any arbitrary system layout: one just has to find exact quantum-mechanical solution of a Green function problem. This is hardly constructive, and we proceed further to derive a simplified semiclassical approach. First, we note that even in its exact quantum-mechanical form the Green function \( G \) is fixed in the terminals and yet to be determined in the nodes. The \( \tilde{G} \) in nodes are determined from Kirchoff rules reflecting the conservation law (3): sum of the matrix currents from the node over all connectors should equal zero at each energy. For this, we should be able to express the matrix current via each connector as a function of two matrices \( \tilde{G}_{ij} \) at its ends.

The connector \((i, j)\) can be quite generally characterized by a set of transmission eigenvalues \( T^{(ij)}_n \). The problem to solve is to express matrix current via the connector in terms of \( \tilde{G}_{ij} \). This problem shall be addressed by a more microscopic approach and was solved in [13] for Keldysh-Nambu matrix structure of \( \tilde{G} \). It is a good news that the derivation made in [13] does not depend on concrete matrix structure and can be used for the present problem without any modification yielding

\[
\tilde{I}_{ij} = \frac{1}{2\pi} \sum_n \int dE \frac{T^{(ij)}_n [\tilde{G}_i, \tilde{G}_j]}{4 + T^{(ij)}_n ([\tilde{G}_i, \tilde{G}_j]) - 2}.
\]  

Each connector \((i, j)\) in the layout contributes to the total \( \chi_i \)-dependent action (6). The corresponding \( S_{ij} \) contribution reads (6,2)

\[
S_{ij}(\chi) = -\frac{t_0}{2\pi} \sum_n \int dE \ln \left[ 1 + \frac{1}{4} T^{(ij)}_n ([\tilde{G}_i, \tilde{G}_j]) - 2 \right].
\]  

Now we are ready to present a set of circuit theory rules that enables us to evaluate the FCS for an arbitrary mesoscopic layout. i. The layout is separated onto terminals, nodes, and connectors. ii. The \( \tilde{G}_i \) in each terminal \( j \) is fixed by relation (3) thus incorporating information about voltage, temperature and counting field \( \chi \) in each node. iii. For each node \( k \), the matrix current conservation yields a Kirchoff equation \( \sum_i I_{ik} = 0 \), where the summation is going over all connectors \((i, k)\) attached to node \( k \), and \( I_{ik} \) are expressed with (6) in terms of \( \tilde{G}_{ik} \). iv. The solution of resulting equations with condition \( \tilde{G}_k^2 = 1 \) fixes \( \tilde{G}_k \) in each node. v. The total action \( S(\chi) \) is obtained by summing up the contributions \( S_{ij}(\{\chi_i\}) \) of individual connectors, those are given by (6,3): \( S(\{\chi_i\}) = \sum_{(i,j)} S_{ij}(\{\chi_i\}) \) vi. The statistics of electron transfer is obtained from the relation (6)

It is time to discuss the limits of applicability of the whole scheme. By virtue of semiclassical approach, the mesoscopic fluctuations coming from interference of electrons penetrating different connectors are disregarded. So that, we assume that conductivities of all connectors are much bigger than conductance quantum \( e^2/\pi h \). The same condition provides the absence of Coulomb blockade effects in the system. Besides, we have disregarded the possible processes of inelastic relaxation in the system. The latter can be eventually taken into account but it would considerably complicate the scheme. The point is that the inelastic scattering would mix up the \( \tilde{G} \) at

![Diagram](image_url)
different energies, so that one can not solve the circuit theory equations separately at each energy.

As an illustration of the presented scheme, we will consider in the rest of the paper the FCS of the 3-terminal chaotic quantum dot. The system is sketched in the inset of Fig. 2. The heuristic circuit, associated with this mesoscopic system is shown by dashed lines. It includes only 3 external vertices, corresponding to terminals, 3 arbitrary connectors, associated with the contacts, and the node (4), representing the quantum dot itself. This separation is valid provided the cavity is in the quantum chaotic regime. (See [16] for definition). This regime corresponds to full isorropization of the Green function \( \hat{G}(x, x', \epsilon) \) within the dot, so that \( \hat{G}_d(\epsilon) \) can be regarded as a constant at a given energy.

Since the normalization \( \hat{G}_k^2 = 1 \) holds for each vertex, we use the parametrization \( \hat{G}_k = g_k \cdot \tau, g_k \cdot g_k = 1 \). Here \( g_k \) is a 3-D vector, and \( \tau = (\hat{t}_1, \hat{t}_2, \hat{t}_3) \). In the absence of counting fields the Green functions in the terminals are given by a zero condition \( \hat{G}_k(0) = \left( \begin{array}{cc} 1 - 2f_k & -2f_k \\ -2(1 - f_k) & 2f_k - 1 \end{array} \right) \), where Fermi distribution function \( f_k(E) = \{ \exp[(E - \epsilon V_k)/T_k] + 1 \}^{-1} \) accounts for the bias voltages \( V_k \) and the temperatures \( T_k \) in the terminals. The \( \chi_i \)-dependence of \( \hat{G}_k(\chi) \) is then given by Eq. 3.

We see that \( \hat{G}_d(\chi) = g_4 \cdot \tau \) is in fact the only function to be found. For that, we proceed by applying the current conservation law, \( \sum_{k=1}^{3} \hat{I}_{k,4} = 0 \), inside the dot. We present the currents \( \hat{I}_{k,4} \) given by Eq. 3 in the form \( \hat{I}_{k,4} = \frac{1}{2} Z_k(g_k \cdot g_4) [\hat{G}_k, \hat{G}_4] \), the scalar function \( Z_k(x) \) incorporating the information about transmission eigenvalues in each connector \( k \): \( Z_k(x) = \sum_n T_n^{(k,4)}/[2 + T_n^{(k,4)}(x - 1)] \). It can be evaluated for any particular distribution \( \rho(T) \) of transmission eigenvalues in the given connector and completely defines its scattering properties. For a example, if we denote \( R_0 = \pi h/e^2 \), then \( R_k^{-1} = 2R_0^{-1} Z_k(1) \) is an inverse resistance of the connector. One can also express the Fano factor \( F = \langle T(1 - T)/\langle T \rangle \rangle / \langle T \rangle = 1 - 2(d/dx)\log Z(x)|_{x=1} \). With the use of \( Z_k(x) \) the conservation law can be efficiently rewritten as \( \sum_{k=1}^{3} p^k \hat{G}_k, \hat{G}_4 = 0 \), where \( p^k = Z_k(g_k \cdot g_4) \).

The latter enables one to look for the vector \( g_4 \) in the form \( g_4 = M^{-1} \sum_{k=1}^{3} p^k g_k \), with \( M(\chi) \) being an unknown normalization constant. Using the condition \( g_4 \cdot g_4 = 1 \) we obtain the set of equations

\[
p_i = Z_i \left( M^{-1} \sum_{j=1}^{3} g_{ij}(\chi) p_j^i \right), \quad M^2 = \sum_{i,j=1}^{3} g_{ij}(\chi)p_i^i p_j^j \tag{8}
\]

where

\[
g_{ij}(\chi) = g_i(\chi) \cdot g_j(\chi) = (1 - 2f_i)(1 - 2f_j) + 2 e^{i(\chi_i - \chi_j)} f_i(1 - f_j) + 2 e^{-i(\chi_i - \chi_j)} f_j(1 - f_i)
\]

The fixed point \( \{ p_i^\ast(\chi), M(\chi) \} \) of the mapping \( \tilde{G}_4 \) finally yields \( \hat{G}_4 \) in question.

The total action can be found by applying the rule \( (v) \) at page \( \tilde{G}_4 \) and reads

\[
S(\chi) = \frac{t_0}{\pi} \sum_{i=1}^{3} \int dx S_i \left( M_i^{-1}(\chi) \sum_{j=1}^{3} g_{ij}(\chi) p_j^i(\chi) \right) \tag{9}
\]

The partial contributions \( S_i(x) \) in the above equation should be determined from the relation \( \frac{d}{d\chi} S_i(x) = -Z_i(x), S_i(1) = 0 \).

We specifically consider three particular types of connectors: tunnel(T), ballistic(B) and diffusive(D). The corresponding contributions to action are: \( S_T(x) = -\frac{1}{2}(R_0/R)(x - 1), S_B(x) = -(R_0/R)\log[(1 + x)/2], S_D(x) = -\frac{1}{2}(R_0/R)\log^2(x + \sqrt{x^2 - 1}) \) in \( R \) being a resistance of the connector. For tunnel connector \( T_n \ll 1 \) for all \( n \). For ballistic connector \( N \) channels are opened (\( T_n \approx 1 \) for \( n \leq N \)), and the other are closed. In the diffusive connector the transmission eigenvalues are distributed according to universal law \( \rho(T) = R_0/2RT\sqrt{1 - T} \).

Analytical results for FCS \( \tilde{G}_4 \) are plausible only for the system with tunnel connectors. To assess general situation we solve the Eqs. \( \tilde{G}_4 \) for given \( \chi_i \) numerically. To find the probability distribution, we evaluated the integral \( \tilde{G}_4 \) in the saddle point approximation, assuming \( \chi_i \) to be complex numbers. Saddle point approximation is always valid in the low frequency limit we consider, since
in this case both action $S$ and number of transmitted particles $N_f = I_f t_0/e \gg 1$. Due to the current conservation law only two of three counting fields $\chi_i$ are independent, and one can set $\chi_3 = 0$. The relevant saddle point of the function $\Omega(\chi) = S(\chi) + i\chi_1 I_1 t_0/e + i\chi_2 I_2 t_0/e$ always corresponds to purely imaginary numbers $\{\chi_1^*, \chi_2^*\}$ in the upper half plane. The probability reads $P(I_1, I_2) \approx \exp[-\Omega(\chi^*)]$. Evidently, $\Omega(\chi^*)$ is the Legendre transform of the action, and it can be regarded as implicit function on $I(\chi^*)$.

In the following we assume the short noise regime $eV \gg kT$ when the thermal fluctuations can be disregarded. The energy integration in (8) becomes trivial, since $f_i(\epsilon) = 0$ or 1, and it is sufficient to consider only the case $V_1 = V_2 = 0$, $V_3 = V$. Any other possible setup can be reduced to the number of previous ones by appropriate subdividing a relevant energy strip. The results of these calculations are shown in Fig.2 and 3. We see that the maximum of probability occurs at $I_1 = I_2 = -V/3F$, $I_3 = 2V/3R$ that simply reflects the usual Kirchoff rules. The current distribution $P(I_1, I_2)$ for a ballistic system is bounded. It is due to the fact, that $Z_B(x)$ contains the finite number of open channels, contrary to the tunnel or diffusive type configurations, where it is not the case. From (8) and (8) we can also find a zero noise and noise correlations matrix $\tilde{S}_{ij} = eR^{-1}V F_{1j}$, $F_{11} = F_{22} = (4 + 3F)/27$, $F_{33} = (4 + 6F)/27$, $F_{12} = -2/27$, $F_{13} = F_{23} = -(2 + 3F)/27$, where $F$ is a Fano factor. Since $F_B = 0$, $F_D = 1/3$ and $F_T = 1$ one concludes, that for a fixed average currents through connectors the Gaussian’s currents fluctuations will increase in the sequence ballistic→diffusive→tunnel. Fig.2 and 3 show, that the similar behavior is also traced in the regime of the large current fluctuations. The essential point here is that the cross-correlations always persist regardless the concrete construction of the connectors. For the case of multilead chaotic cavities the results for shot noise in our theory coincides with those, obtained by means of random matrix theory [17], and with use of “minimal correlation approach” [8].

In conclusion, we present a constructive theory for counting statistics for electron transfer in mesoscopic systems. With this theory, one can easily make theoretical predictions for all FCS, thereby facilitating experimental activities in this direction. Up to now, only the noise has been measured. In our opinion, the measurements of FCS can be easily performed with *threshold detectors* that produce a signal provided the current in a certain terminal exceeds the threshold value. If the threshold value exceeds the average current, the detector will be switched by relatively improbable fluctuations of the current described by FCS, its signal being proportional to the probability $P(I_1, I_2)$ of these fluctuations.

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FIG. 3: The contour maps of the current distribution $\log[P(I_1, I_2)]$ in the 3-terminal chaotic quantum dot for different configurations of connectors. (a) - ballistic connectors, (b) - diffusive connectors, (c) - tunnel connectors.