\section{Introduction}

Recently Hu, Lin and Wu constructed $H^m$-conforming finite elements of degree $k$ on simplices in $\mathbb{R}^n$ with $k \geq 2^n(m - 1) + 1$ and $m, n \geq 1$ in a unified way \cite{29}, which generalizes the finite elements in two dimensions in \cite{14, 37, 8} and the finite elements in three dimensions in \cite{38, 46, 47}. The simplical lattice is used in \cite{24} to show the geometric decomposition of smooth finite elements. The work \cite{29} is theoretically important, and is a significant progress in the field of construction of $H^m$-conforming elements in $\mathbb{R}^n$. Since polynomial shape functions are infinitely differentiable, the $2^{n-1}(m - 1)$th order derivatives of shape functions at vertices are included in the degrees of freedom (DoFs), which results in the very high polynomial degree $k \geq 2^n(m - 1) + 1$ for $H^m$-conforming finite elements. In \cite{45}, Xu devised $H^m$-conforming piecewise polynomials based on the artificial neural network with $k \geq m$ and then developed a finite neuron method, whose practical value is also limited since solving the underlying non-linear and non-convex optimization problem is challenging. We refer to \cite{30} for $H^m$-conforming finite elements on macro-hypercubes and \cite{27} for $H^2$-conforming finite elements on macro-simplices in arbitrary dimension.

Alternatively, in \cite{23, 34} we devised $H^m$-nonconforming virtual elements of any degree $k$ on any shape of polytope $K$ in $\mathbb{R}^n$ with $k \geq m$ in a universal way by employing a generalized Green’s identity. When $K$ is a simplex, $1 \leq m \leq n$ and
$k = m$, the virtual elements in [23] are exactly the nonconforming finite elements in [41, 40]. And when $K$ is a simplex, $m = n + 1$ and $k = m$, the DoFs of the virtual elements in [34] are same as those of the nonconforming finite elements in [44]. We refer to [43, 31, 32] for more $H^m$-nonconforming finite elements and [48, 49, 6] for more $H^m$-nonconforming virtual elements.

We shall construct $H^m$-conforming virtual elements of any degree $k$ of polynomials on a very general polytope $K \subset \mathbb{R}^n$ in arbitrary dimension $n$ and any derivative order $m$ with $k \geq m$ and $m, n \geq 1$ in this paper. The $H^1$-conforming virtual elements were initially developed in [11, 12] in two and three dimensions. The $H^m$-conforming virtual elements of degree $k$ for $k \geq m$ and $m \geq 1$ in two dimensions have been designed in a series of works [13, 7, 5, 21]. In three dimensions, the $H^2$-conforming virtual elements for $k \geq 2$ were devised in [9]. When $K$ is a tetrahedron in three dimensions, by using the Argyris element [8, 17] and Hermite element [26] on faces, $H^2$-conforming virtual elements for $k \geq 5$ were advanced in [25]. A different approach is adopted in [19] to construct $H^2$-conforming virtual elements on tetrahedrons. We intend to extend these works to arbitrary spacial dimension $n$, any order $m$ of Sobolev spaces and any polynomial degree $k \geq m$.

We construct $H^m$-conforming virtual elements $(K, \mathcal{N}_k^m(K), V^m_k(K))$ by gluing conforming virtual elements on faces recursively. The virtual element space is defined as

$$V^m_k(K) := \{ v \in H^m(K) : (-\Delta)^m v \in \mathbb{P}_k(K), (v - \Pi^K v, q)_K = 0 \ \forall \ q \in \mathbb{P}^1_{k-2m}(K), \left(\nabla^j v\right)|_{S_K} \in H^j(S^K_\alpha; S_n(j)) \text{ for } j = 0, 1, \cdots, m-1, \left.\frac{\partial^{\alpha} v}{\partial v_{F}^{\alpha}}\right|_{F} \in V^{m-|\alpha|}_k(F) \ \forall \ F \in F^*(K), \quad r = 1, \cdots, n-1, \alpha \in A_r, \text{ and } |\alpha| \leq m-1\}$$

with $V^{m-|\alpha|}_k(e) := \mathbb{P}_{\max(k-|\alpha|, 2(m-|\alpha|)-1)}(e)$ for each one-dimensional edge $e \in F^{n-1}(K)$, where the local $H^m$-projection operator $\Pi^K K$ is introduced to ensure the $L^2$-orthogonal projection $Q^K K v$ is computable using only the DoFs in $\mathcal{N}^m_k(K)$ for any virtual function $v \in V^m_k(K)$ following the idea in [4]. When $n \geq 2$, $\mathbb{P}_k(K) \subseteq V^m_k(K)$ but $\mathbb{P}_{k+1}(K) \not\subseteq V^m_k(K)$. The DoFs in $\mathcal{N}^m_k(K)$ are motivated by $\frac{\partial^{\alpha} v}{\partial v_{F}^{\alpha}} \in V^{m-|\alpha|}_k(F)$ in the definition of $V^m_k(K)$. With the help of the concepts of data spaces and Whitney arrays [36], the dimension of $V^m_k(K)$ is exactly counted by using the inverse trace theorem of $H^m(K)$ and the well-posedness of the $m$th harmonic equation with Dirichlet boundary conditions.

For the lowest degree case $k = m$, the set of DoFs $\mathcal{N}^m_m(K)$ is very simple, only involving function values and derivatives up to order $m-1$ at the vertices of polytope $K$, i.e.

$$h^j_K \nabla^j v(\delta) \quad \forall \ \delta \in F^m(K), \quad j = 0, 1, \cdots, m-1.$$

Here the scaling $h^j_K$ is used so that all the DoFs share the same order of magnitude. These DoFs are even simpler than those of non-conforming virtual elements in [23, 34]. If furthermore $K \subset \mathbb{R}^n$ is a simplex, $\dim V^m_k(K) = (n+1) \dim \mathbb{P}_{m-1}(K)$, which is much smaller than the dimension $\dim \mathbb{P}_{2^{m}(m-1)+1}(K)$ of the lowest degree $H^m$-conforming finite element in [29]. And there are no super-smooth DoFs included.
in \( N^m_k(K) \), i.e., all the orders of the derivatives involved in the DoFs are less than \( m \). This is one of the attractive features of virtual elements.

Another contribution of this paper is establishing the inverse inequality and norm equivalences for the \( H^m \)-conforming virtual elements \( (K, N^m_k(K), V^m_k(K)) \) under the assumption that the polytope \( K \) is star-shaped and all the diameters of all faces of \( K \) are equivalent to the diameter of \( K \). The inverse inequality for \( V^m_k(K) \) is derived from the multiplicative trace inequality, the inverse trace theorem, the inverse inequality for polynomials and the mathematical induction. Employing the inverse inequality, the trace inequality and the Poincaré-Friedrichs inequality, we arrive at several norm equivalences on virtual element spaces \( V^m_k(K) \) and \( \ker(Q^m_k) \cap V^m_k(K) \), where \( \ker(T) \cap V^m_k(K) := \{ v \in V^m_k(K) : Tv = 0 \} \) with operator \( T = Q^m_k \) or \( \Pi^m_k \). Especially we acquire the classical \( L^2 \) norm equivalence as finite elements

\[
\|v\|_{0,K}^2 \simeq \|Q^m_k v\|_{0,K}^2 + \sum_{\delta \in F^2(K)} h_n^{n+2i} \|\nabla^i v(\delta)\|^2 + \sum_{r=1}^{n-1} \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in A_r, |\alpha| \leq m-1} h_F^{r+2|\alpha|} \|Q^m_{k-2m+|\alpha|} \frac{\partial^{|\alpha|} v}{\partial v_F^{|\alpha|}}\|_{0,F}^2 \quad \forall \ v \in V^m_k(K),
\]

in which all terms in the right hand side completely coincide with all the DoFs in \( N^m_k(K) \). This extends the stability analysis of virtual elements in [15, 10, 22, 18, 33].

The constructed conforming virtual elements are then applied to discretize a polyharmonic equation with a lower order term. To analyze the conforming virtual element method, we construct a quasi-interpolation operator and derive the interpolation error estimate with the help of the norm equivalence on \( V^m_k(K) \). Finally the optimal error estimates are presented for the conforming virtual element method. This paper is motivated by the theoretical purposes. We also present numerical results for a fourth-order elliptic problem and a sixth-order elliptic problem in two dimensions.

The rest of this paper is organized as follows. Some notations and mesh conditions are shown in Section 2. In Section 3 \( H^m \)-conforming virtual elements are constructed. The inverse equality and several norm equivalences are proved in Section 4. In Section 5 the \( H^m \)-conforming virtual elements are applied to discretize a polyharmonic equation with a lower order term. And numerical results are provided in Section 6.

2. Preliminaries

2.1. Notation.

In this paper we will adopt the same notations as in [23, 34]. For any non-negative integer \( r \) and \( 1 \leq \ell \leq n \), notation \( \mathbb{T}_\ell(r) := \mathbb{R}^{\ell} \otimes \cdots \otimes \mathbb{R}^{\ell} \) stands for the set of \( r \)-tensor spaces over \( \mathbb{R}^{\ell} \). Introduce the symmetric \( r \)-tensor space

\[
\mathbb{S}_\ell(r) := \{ \tau = (\tau_{i_1i_2\cdots i_r}) \in \mathbb{T}_\ell(r) : \tau_{i_{\sigma(1)}i_{\sigma(2)}\cdots i_{\sigma(r)}} = \tau_{i_1i_2\cdots i_r} \text{ for any } \sigma \in \mathfrak{S}_r \},
\]

where \( \mathfrak{S}_r \) is the set of all permutations of \( (1, 2, \cdots, r) \). For tensor \( \tau \in \mathbb{T}_\ell(r) \), the symmetric part of \( \tau \) is a symmetric tensor in \( \mathbb{S}_\ell(r) \) defined by

\[
(\text{sym } \tau)_{i_1i_2\cdots i_r} := \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \tau_{i_{\sigma(1)}i_{\sigma(2)}\cdots i_{\sigma(r)}} \quad \text{for } 1 \leq i_1, i_2, \cdots, i_r \leq \ell.
\]
Given $r$-tensors $\tau, \varsigma \in T_\ell(r)$, define the scalar product $\tau : \varsigma \in \mathbb{R}$ by

$$\tau : \varsigma := \sum_{i_1=1}^\ell \cdots \sum_{i_r=1}^\ell \tau_{i_1,\ldots,i_r}s_{i_1,\ldots,i_r}.$$

Denote by $\mathbb{N}$ the set of all non-negative integers. For an $n$-dimensional multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \in \mathbb{N}$, define $|\alpha| := \sum_{i=1}^n \alpha_i$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. For $0 \leq j \leq n$, let $A_j$ be the set consisting of all multi-indexes $\alpha$ with $\sum_{i=j+1}^n \alpha_i = 0$, i.e., non-zero index only exists for $1 \leq i \leq j$.

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded polytope with positive integer $n$. Given a bounded domain $G \subset \mathbb{R}^n$ and a non-negative integer $k$, let $H^k(G; \mathbb{X})$ be the usual Sobolev space of functions over $G$ taking values in the tensor space $\mathbb{X}$ for $\mathbb{X} = T_\ell(r), S_\ell(r)$, whose norm and semi-norm are denoted by $\| \cdot \|_{k,G}$ and $| \cdot |_{k,G}$ respectively. Set $H^k(\Gamma) := H^k(G; \Gamma(0))$. Define $H^0_0(G)$ as the closure of $C_0^\infty(G)$ with respect to the norm $\| \cdot \|_{0,G}$. Let $(\cdot, \cdot)_G$ be the standard inner product on $L^2(G; \mathbb{X})$. If $G$ is $\Omega$, we abbreviate $\| \cdot \|_{k,G}, | \cdot |_{k,G}$ and $(\cdot, \cdot)_G$ by $\| \cdot \|_k, | \cdot |_k$ and $(\cdot, \cdot)$, respectively. Denote by $h_G$ the diameter of $G$. Let $\mathbb{P}_k(G)$ be the set of all polynomials over $G$ with the total degree no more than $k$, whose tensorial version space is denoted by $\mathbb{P}_k(G; \mathbb{X})$. Let $\mathbb{P}_k(G) := \{0\}$ if $k < 0$. Let $Q_k^0$ be the $L^2$-orthogonal projection onto $\mathbb{P}_k(G; \mathbb{X})$. For a function $v$, $Q_k^0 v$ is understood as $v|_G$ when $G$ is a point, whether $k$ is non-negative or negative. For non-negative integers $k$ and $m$, let $\mathbb{P}_k^{m-2m}(G) \subset \mathbb{P}_k(G)$ be the orthogonal complement space of $\mathbb{P}_k(m-2m)(G)$ of $\mathbb{P}_k(G)$ with respect to the inner product $(\cdot, \cdot)_G$. Denote by $\# S$ the number of elements in a finite set $S$.

Let $\{T_h\}$ be a family of partitions of $\Omega$ into nonoverlapping simple polytopal elements with $h := \max_{K \in T} h_K$. Let $F^r_h$ be the set of all $(n-r)$-dimensional faces of the partition $T_h$ for $r = 1, 2, \cdots, n$. For simplicity, let $F^r_h := T_h$. Moreover, we set for each $K \in T_h$

$$F^s(K) := \{F \in F^r_h : F \subset K\}.$$

The superscript $r$ in $F^r_h$ represents the co-dimension of an $(n-r)$-dimensional face $F$. Similarly, we define

$$F^s(F) := \{e \in F^{r+s}_h : e \subset F\}.$$

Here $s$ is the co-dimension relative to the face $F$. For any $F \in F^r_h$ with $r = 0, 1, \cdots, n-2$, let the $(n-r-s)$-dimensional skeleton $S^s_F$ be the union of all faces in $F^s(F)$ for $s = 1, \cdots, n-r-1$.

For any $F \in F^r_h$ with $1 \leq r \leq n-1$, let $\nu_{F,1}, \cdots, \nu_{F,r}$ be its mutually perpendicular unit normal vectors, and $t_{F,1}, \cdots, t_{F,n-r}$ be its mutually perpendicular unit tangential vectors. We abbreviate $\nu_{F,1}$ as $\nu_F$ when $r = 1$, and $t_{F,1}$ as $t_F$ when $r = n-1$. We refer to Fig. 1 for an example of normal vectors and tangential vectors. Define the surface gradient on $F$ as

$$\nabla_F v := \nabla v - \sum_{i=1}^r \frac{\partial v}{\partial \nu_{F,i}} \nu_{F,i} = \sum_{i=1}^{n-r} \frac{\partial v}{\partial t_{F,i}} t_{F,i},$$

namely the projection of $\nabla v$ to the face $F$, which is independent of the choice of the normal vectors. And denote by $\text{div}_F$ the corresponding surface divergence. For any $\delta \in F^0_h$ and $i = 1, \cdots, n$, let $\nu_{\delta,i} := e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ be the $n$-tuple
with all entries equal to 0, except the \( i \)th, which is 1. For any \( F \in \mathcal{F}_h \), \( \alpha \in A_r \) and \( \beta \in A_{n-r} \) with \( r = 1, \ldots, n \), set
\[
\nu^\alpha_F := \nu^\alpha_{F,1} \otimes \cdots \otimes \nu^\alpha_{F,r}, \quad t^\beta_F := t^\beta_{F,1} \otimes \cdots \otimes t^\beta_{F,r},
\]

where \( \nu^\alpha_{F,i} := \nu_{F,i} \otimes \cdots \otimes \nu_{F,i} \) and \( t^\beta_{F,i} := t_{F,i} \otimes \cdots \otimes t_{F,i} \). For any \( e \in \mathcal{F}^s(F) \) with \( 1 \leq s < n - r \), let \( \nu_{F,e,1}, \ldots, \nu_{F,e,s} \) be its mutually perpendicular unit normal vectors parallel to \( F \), and abbreviate \( \nu_{F,e} \) as \( \nu_{F,e,1} \). For any \( \delta \in \mathcal{F}^n(K) \), and any function \( v \) defined on \( K \), we will rewrite \( v(x_{\delta}) \) as \( v(\delta) \) for simplicity, where \( x_{\delta} \) is the position of the point \( \delta \).

### 2.2. Mesh conditions

We impose the following conditions on the mesh \( \mathcal{T}_h \).

(A1) Each element \( K \in \mathcal{T}_h \) and each face \( F \in \mathcal{F}_h \) for \( 1 \leq r \leq n - 1 \) is star-shaped with a uniformly bounded chunkiness parameter. For a domain \( D \), the chunkiness parameter \( \gamma_D := h_D/\rho_D \), where \( \rho_D \) is the radius of the largest ball contained in \( D \).

(A2) There exists a real number \( \eta > 0 \) such that for each \( K \in \mathcal{T}_h \), \( h_K \leq \eta h_F \) for all \( F \in \mathcal{F}^r(K) \) with \( r = 1, \ldots, n - 1 \).

Throughout this paper, we also use \( \approx \) to mean that \( \leq C \cdots \), where \( C \) is a generic positive constant independent of mesh size \( h \), but may depend on the chunkiness parameter of the polytope, constant \( \eta \), the degree of polynomials \( k \), the order of differentiation \( m \), and the dimension of space \( n \), which may take different values at different appearances. And \( A \approx B \) means \( A \approx B \) and \( B \approx A \). Hereafter, we always assume \( k \geq m \).
We conclude (7) by applying the multinomial theorem to (8).

\[\Box\]

Proof.

(1) \[\parallel v \parallel^2_{0, \partial D} \lesssim h D^{-1} \parallel v \parallel_{0, D} (\parallel v \parallel_{0, D} + h D \parallel v \parallel_{1, D}) \quad \forall v \in H^1(D).\]

This implies the trace inequality (cf. [18, (2.18)])

(2) \[\parallel v \parallel_{0, \partial D}^2 \lesssim h D^{-1} \parallel v \parallel_{0, D}^2 + h D \parallel v \parallel_{1, D}^2 \quad \forall v \in H^1(D).\]

When \(D\) is a set of a finite number of points, the notation \(\parallel v \parallel_{0, D}\) means \(\parallel v \parallel_{L^\infty(D)}\).

We also have the Poincaré-Friedrichs inequality [18, (2.15)]

(3) \[\parallel v \parallel_{0, D} \lesssim h D \parallel v \parallel_{1, D} \quad \forall v \in H^1_0(D),\]

and the inverse inequality for polynomials [34, Lemma 10]

(4) \[\parallel q \parallel_{0, D} \lesssim h D^{-1} \parallel q \parallel_{i-i, D} \quad \forall q \in P_i(D)\]

for any non-negative integers \(\ell\) and \(i\). As a result of (4), the Bramble-Hilbert lemma [16, Lemma 4.3.8] and (2), it holds the estimate of the \(L^2\)-orthogonal projection

(5) \[h D^2 \parallel v - Q^D v \parallel_{i, D} + h D \parallel v - Q^D v \parallel_{0, \partial D} \lesssim h D^2 \parallel v \parallel_{j, D} \quad \forall v \in H^j(D)\]

for \(0 \leq i \leq j \leq k + 1\) with \(i, j, k\) being non-negative integers. The hidden constants in (1)-(5) depend on the chunkiness parameter \(\eta\) and the spatial dimension \(n\).

3. \(H^m\)-Conforming Virtual Elements

We will construct \(H^m\)-conforming virtual elements \((K, \mathcal{N}_k^m(K), V_k^m(K))\) for any integers \(m, n \geq 1, k \geq m\) and \(n\)-dimensional polytope \(K \subset \mathbb{R}^n\) by gluing conforming virtual elements on faces recursively.

We first list a Green’s identity for later uses.

Lemma 3.1. For any \(v \in H^m(K)\) and \(q \in H^2m(K)\),

(6) \[\nabla^m v, \nabla^m q)_K = (v, (-\Delta)^m q)_K + \sum_{i=0}^{m-1} (\nabla^i v, \nabla^i(-\Delta)^{m-i-1} \partial_v q)_{\partial K},\]

where \(\partial_v q|_F := \frac{\partial q}{\partial n_F}\) for each face \(F \in F^1(K)\).

Proof. For \(i = 0, 1, \cdots, m - 1\), applying the integration by parts, it follows

\[(\nabla^{i+1} v, \nabla^{i+1}(\Delta)^{m-i-1} q)_K = (\nabla^{i+1} v, \nabla^i(-\Delta)^{m-i} q)_K + (\nabla^{i+1} v, \nabla^i(-\Delta)^{m-i-1} \partial_v q)_{\partial K}.

Thus (6) holds from the sum of the last identity from \(i = 0\) to \(m - 1\). \(\square\)

Lemma 3.2. Let \(F \in F^r(K)\) with \(1 \leq r \leq n - 1\), and integer \(j > 0\). It holds for any smooth function \(v\) that

(7) \[\nabla^j v = \sum_{0 \leq |\alpha| + |\beta| = j} \frac{j!}{\alpha! \beta!} \text{sym}(\nu_F^\alpha \otimes t_F^\beta) \frac{\partial^j v}{\partial t_F^\alpha \partial \nu_F^\beta}.

Proof. Recalling that \(\nabla^j v\) is a symmetric \(j\)-tensor, apparently it follows

(8) \[\nabla^j v = \text{sym}(\nabla^j v) = \text{sym} \left( \sum_{i=1}^r \nu_{F,i} \frac{\partial}{\partial \nu_{F,i}} + \sum_{i=1}^{n-r} t_{F,i} \frac{\partial}{\partial t_{F,i}} \right)^j v.

We conclude (7) by applying the multinomial theorem to (8). \(\square\)
3.1. $H^m$-conforming virtual elements in one dimension. We start from one dimension, i.e. $n = 1$. Now the polytope $K$ is an interval. The DoFs $\mathcal{N}^m(K)$ are chosen as

\[
(9) \quad h^j_k v^{(j)}(\delta) \quad \forall \delta \in \mathcal{F}^1(K), \quad j = 0, 1, \ldots, m - 1,
\]

\[
(10) \quad \frac{1}{|K|} (v, q)_K \quad \forall \ q \in \mathbb{P}_{k-2m}(K),
\]

where $v^{(j)}$ is the $j$th order derivative of $v$. And take the space of shape functions

\[
V^m_k(K) := \{ v \in H^m(K) : v^{(2m)} \in \mathbb{P}_{k-2m}(K) \}.
\]

Clearly we have

\[
V^m_k(K) = \begin{cases} \mathbb{P}_k(K), & k \geq 2m - 1, \\ \mathbb{P}_{2m-1}(K), & k < 2m - 1. \end{cases}
\]

Hence the $H^m$-conforming virtual element of degree $k$ in one dimension is exactly the $C^{m-1}$-continuous finite element, whose shape functions are polynomials of degree $\max\{k, 2m-1\}$. And the $H^m$-conforming virtual elements $(K, \mathcal{N}^m_k(K), V^m_k(K))$ coincide with the nonconforming ones in [34, Remark 1].

3.2. $H^m$-conforming virtual elements in two dimensions. Then we consider the construction of the $H^m$-conforming virtual elements in two dimensions, i.e. $n = 2$, where the polytope $K$ is a polygon. The $H^m$-conforming virtual elements $(K, \mathcal{N}^m_k(K), V^m_k(K))$ in two dimensions have been designed in [13, 7, 5, 21]. Here we review them to motivate the construction of $H^m$-conforming virtual elements in higher dimensions.

The space of shape functions in the virtual elements is defined through local partial differential equations [11, 7]. To ensure the $L^2$ projection $Q^k \delta v$ is computable for any virtual element function $v$ by using the DoFs, following the idea in [4] we first define a preliminary virtual element space with the help of the conforming virtual elements in one dimension

\[
\tilde{V}^m_k(K) := \{ v \in H^m(K) : (-\Delta)^m v \in \mathbb{P}_k(K),
\]

\[
\frac{\partial^j v}{\partial n_e} \in V^{m-j}_k(e) \quad \forall e \in \mathcal{F}^1(K), \ j = 0, 1, \ldots, m - 1 \}.
\]

Clearly $\mathbb{P}_k(K) \subseteq \tilde{V}^m_k(K)$. On the other hand, $\frac{\partial^{m-1} v}{\partial n_e} \in V^1_{k-(m-1)}(e) = \mathbb{P}_{k-m+1}(e)$, thus $\mathbb{P}_{k+1}(K) \nsubseteq \tilde{V}^m_k(K)$.

**Lemma 3.3.** For any $v \in \tilde{V}^m_k(K)$, $\nabla^j v$ is continuous on $\partial K$, and $(\nabla^j v)|_{\partial K} \in H^1(\partial K; S_2(j))$ for $j = 0, 1, \ldots, m - 1$.

**Proof.** On each edge $e \in \mathcal{F}^1(K)$, it holds from (7) that

\[
(11) \quad \nabla^j v = \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!} \text{sym}(t^\ell_e \otimes \nu^j_e) \frac{\partial}{\partial \ell_e} \left( \frac{\partial^{j-\ell} v}{\partial \nu^j_e} \right).
\]

By the definition of $\tilde{V}^m_k(K)$, $\frac{\partial^{j-\ell} v}{\partial \nu^j_e} \in V^{m-j+\ell}_k(e) = \mathbb{P}_{\max(k-j+\ell,2m-1-2j+2\ell)}(e)$ is a polynomial. Hence it follows from (11) that $\nabla^j v|_e \in \mathbb{P}_{\max(k-j,2m-1-j)}(e, S_2(j))$ is a tensor with components being polynomials. Finally we acquire from the fact
Notice that (17) \( \Pi_K \) is well-posed, and it holds that the projection \( \Pi_K \) for any \( v \in H^m(K) \) that \( \nabla^j v \) is continuous on \( \partial K \) for \( j = 0, 1, \cdots, m - 1 \) (cf. comments after Theorem 1.5.2.3 in [28]), which also means \( (\nabla^j v)|_{\partial K} \in H^1(\partial K; \mathbb{S}_2(j)) \). \( \square \)

In the definition of \( \tilde{V}_k^m(K) \), let \( \Pi_K \) be the DoFs (12)-(13) for any \( v \in \tilde{V}_k^m(K) \). Then we get from the Green’s identity (6) that the projection \( \Pi_K \) in two dimensions should cover the following ones

\[
\frac{h_e^j}{|e|} \left( \frac{\partial^j v}{\partial \nu^j e} \right) (\delta) \quad \forall \, \delta \in F^1(e), \, i = 0, 1, \cdots, m - j - 1, \\
\frac{1}{|e|} \left( \frac{\partial^j v}{\partial \nu^j e} \right) (q) \quad \forall \, q \in \mathbb{P}_{k-2(m-j)}(e).
\]

Noting that \( \nabla^j v \) is continuous on \( \partial K \) for \( j = 0, 1, \cdots, m - 1 \), we propose the following DoFs \( N^m_k(K) \) for the \( H^m \)-conforming virtual elements in two dimensions

\[
(12) \quad h_K^j \nabla^j v(\delta) \quad \forall \, \delta \in F^2(K), \, j = 0, 1, \cdots, m - 1, \\
(13) \quad \frac{1}{|e|} \left( \frac{\partial^j v}{\partial \nu^j e} \right) (q) \quad \forall \, q \in \mathbb{P}_{k-2m+j}(e), \, e \in F^1(K), \, j = 0, 1, \cdots, m - 1, \\
(14) \quad \frac{1}{|K|} (v, q)_K \quad \forall \, q \in \mathbb{P}_{k-2m}(K).
\]

To define the space of shape functions \( V_k^m(K) \), we also need a local \( H^m \) projection operator \( \Pi^K \) : \( H^m(K) \to \mathbb{P}_k(K) \): given \( v \in H^m(K) \), let \( \Pi^K v \in \mathbb{P}_k(K) \) be the solution of the problem

\[
(15) \quad (\nabla^m \Pi^K v, \nabla^m q)_K = (\nabla^m v, \nabla^m q)_K \quad \forall \, q \in \mathbb{P}_k(K), \\
(16) \quad \sum_{\delta \in F^j(K)} (\nabla^j \Pi^K v)(\delta) = \sum_{\delta \in F^j(K)} (\nabla^j v)(\delta), \quad j = 0, 1, \cdots, m - 1.
\]

The number of equations in (16) is

\[
\sum_{j=0}^{m-1} (j + 1) = \frac{1}{2} m(m + 1) = \dim \mathbb{P}_{m-1}(K).
\]

Applying the argument in [23, Section 3.3 and Lemma 3.5], the local problem (15)-(16) is well-posed, and it holds

\[
(17) \quad \Pi^K q = q \quad \forall \, q \in \mathbb{P}_k(K).
\]

Notice that \( \nabla^j v|_e \in \mathbb{P}_{\max(k-j,2m-1-j)}(e, \mathbb{S}_2(j)) \) is a tensor with polynomial components for each \( e \in F^1(K) \) and \( j = 0, 1, \cdots, m - 1 \), which is computable using the DoFs (12)-(13) for any \( v \in \tilde{V}_k^m(K) \). Then we get from the Green’s identity (6) that the projection \( \Pi^K v \) is computable using only the DoFs (12)-(14) for any \( v \in \tilde{V}_k^m(K) \).

Following the ideas in [4, 23], define the space of shape functions

\[
V_k^m(K) := \{ v \in \tilde{V}_k^m(K) : (v - \Pi_K v, q)_K = 0 \quad \forall \, q \in \mathbb{P}_{k-2m}(K) \}.
\]

Due to (17), it holds \( \mathbb{P}_k(K) \subseteq V_k^m(K) \) and \( \mathbb{P}_{k+1}(K) \not\subseteq V_k^m(K) \). Therefore we arrive at the \( H^m \)-conforming virtual elements \( (K, N^m_k(K), V_k^m(K)) \) in two dimensions. The uni-solvence of \( (K, N^m_k(K), V_k^m(K)) \) will be covered in the arbitrary dimension in Subsections 3.5 and 3.6.
For any \( v \in V^m_k(K) \), since \((v - \Pi^K v, q - Q^K_{k-2m} q)_K = 0 \) for each \( q \in \mathbb{P}_k(K) \), we have
\[
(Q^K_k - Q^K_{k-2m})(v - \Pi^K v) = Q^K_k (I - Q^K_{k-2m})(v - \Pi^K v) = 0.
\]
This yields
\[
Q^K_k v = \Pi^K v + Q^K_{k-2m} v - Q^K_{k-2m} \Pi^K v.
\]
Hence \( Q^K_k v \) is computable using only the DoFs (12)-(14) for any \( v \in V^m_k(K) \). This combined with the integration by parts implies that \( Q^K_{k+j} (\nabla^j v) \) is computable using only the DoFs (12)-(14) for any \( v \in V^m_k(K) \) and \( j = 1, \ldots, m \).

3.3. \( H^m \)-conforming virtual elements in three dimensions. Next we construct the \( H^m \)-conforming virtual elements for \( k \geq m \) and \( m \geq 1 \) in three dimensions. Several \( H^2 \)-conforming virtual elements in three dimensions are devised in [9, 25, 19].

Let polyhedron \( K \subset \mathbb{R}^3 \). Similarly as the two dimensions, we first define a preliminary virtual element space
\[
\tilde{V}^m_k(K) := \{ v \in H^m(K) : (-\Delta)^m v \in \mathbb{P}_k(K), \quad (\nabla^j v)|_{S^*_K} \in H^1(S^*_K; S_3(j)) \text{ for } j = 0, 1, \ldots, m - 1 \text{ and } r = 1, 2, \quad \frac{\partial^r v}{\partial v_F} \in V^{m-r}_{k-j}(F) \text{ for } F \in \mathcal{F}^1(K), j = 0, 1, \ldots, m - 1, \quad \frac{\partial^2 v}{\partial v_{e,1} \partial v_{e,2}} \in V^{m-j}_k(e) \text{ for } e \in \mathcal{F}^2(K), 0 \leq i \leq j, j = 0, 1, \ldots, m - 1 \}.
\]

The requirement \( (\nabla^j v)|_{S^*_K} \in H^1(S^*_K; S_3(j)) \) in the definition of \( \tilde{V}^m_k(K) \) is motivated by Lemma 3.3. Since \( \mathbb{P}_{k-j}(F) \subset V^{m-j}_{k-j}(F) \) and \( \mathbb{P}_{k-j}(e) \subset V^{m-j}_{k-j}(e) \), we have \( \mathbb{P}_k(K) \subset \tilde{V}^m_k(K) \). Take \( v \in \tilde{V}^m_k(K) \). By the definition of \( \tilde{V}^m_k(K) \), \( \frac{\partial^2 v}{\partial v_{e,1} \partial v_{e,2}} \in V^{m-j}_k(e) = \mathbb{P}_{\max(k-j,2m-1-2j+2i)}(e) \) is a polynomial for each edge \( e \in \mathcal{F}^2(K) \). It follows from (7) that \( \nabla^j v|_e \in \mathbb{P}_{\max(k-j,2m-1-j)}(e, S_3(j)) \) is a tensor with components being polynomials. By \( (\nabla^j v)|_{S^*_K} \in H^1(S^*_K; S_3(j)) \), \( \nabla^j v \) is continuous on the one-dimensional skeleton \( S^*_K \). For \( F \in \mathcal{F}^1(K) \) and \( e \in \mathcal{F}^2(K) \), applying (7), we have \( \nabla^j v|_F \in H^{m-j}(F; S_3(j)) \) and \( \nabla^j v|_e \in H^{m-j}(e; S_3(j)) \).

Inspired by \( \frac{\partial^2 v}{\partial v_{e,1} \partial v_{e,2}} \in V^{m-j}_k(e), \frac{\partial v}{\partial n}|_F, \in V^{m-j}_{k-j}(F) \), the DoFs (9)-(10) and the DoFs (12)-(14), we propose the following DoFs \( \mathcal{N}^m_k(K) \) for the \( H^m \)-conforming virtual elements in three dimensions
\[
(18) \quad h^j_K \nabla^j v(\delta) \quad \forall \ \delta \in \mathcal{F}^3(K), \ j = 0, 1, \ldots, m - 1,
\]
\[
(19) \quad h^j_K \left| \frac{\partial v}{\partial v_{e,1} \partial v_{e,2}} \right|_e \quad \forall \ q \in \mathbb{P}_{k-2m+j}(e), e \in \mathcal{F}^2(K), \ 0 \leq i \leq j, j = 0, 1, \ldots, m - 1,
\]
\[
(20) \quad \frac{h^j_K}{|F|} \frac{\partial v}{\partial n}|_F \quad \forall \ q \in \mathbb{P}_{k-2m+j}(F), F \in \mathcal{F}^1(K), \ j = 0, 1, \ldots, m - 1,
\]
\[
(21) \quad \frac{1}{|K|} (v, q)_K \quad \forall \ q \in \mathbb{P}_{k-2m}(K).
\]
To define the space of shape functions \( V^m_k(K) \), we introduce a local \( H^m \)-projector \( \Pi^K_k : H^m(K) \to \mathbb{P}_k(K) \): given \( v \in H^m(K) \), let \( \Pi^K_k v \in \mathbb{P}_k(K) \) be the solution of the problem
\[
(22) \quad (\nabla^m \Pi^K_k v, \nabla^m q)_K = (\nabla^m v, \nabla^m q)_K \quad \forall \ q \in \mathbb{P}_k(K),
\]
\[
(23) \quad \sum_{\delta \in \mathcal{F}^3(K)} (\nabla^j \Pi^K_k v)(\delta) = \sum_{\delta \in \mathcal{F}^3(K)} (\nabla^j v)(\delta), \quad j = 0, 1, \cdots, m - 1.
\]
The number of equations in \( (23) \) is
\[
\sum_{j=0}^{m-1} C_{j+2}^2 = C_{m+2}^3 = \dim \mathbb{P}_{m-1}(K).
\]
The problem \( (22)-(23) \) is well-posed, and it holds the identity
\[
(24) \quad \Pi^K_k q = q \quad \forall \ q \in \mathbb{P}_k(K).
\]
For \( v \in \tilde{V}_k^m(K) \), \( \nabla^j v|_e \) on edge \( e \in \mathcal{F}^2(K) \) is clearly computable by using the DoFs \( (18)-(19) \) for \( j = 0, 1, \cdots, m - 1 \), since \( \nabla^j v|_e \) is a tensor-valued polynomial. By \( (7) \), \( Q^m_j(\nabla^j v) \) is computable by using the DoFs \( (18)-(20) \) for \( F \in \mathcal{F}^1(K) \) and \( j = 0, 1, \cdots, m - 1 \). Therefore it follows from \( (6) \) that the projection \( \Pi^K_k v \) is computable using only the DoFs \( (18)-(21) \) for any \( v \in \tilde{V}_k^m(K) \).

Define the space of shape functions
\[
V^m_k(K) := \{ v \in \tilde{V}_k^m(K) : (v - \Pi^K_k v, q)_K = 0 \quad \forall \ q \in \mathbb{P}_{k-2m}(K) \}.
\]
Due to \( (24) \), it holds \( \mathbb{P}_k(K) \subseteq V^m_k(K) \). We finish the construction of the \( H^m \)-conforming virtual elements \( (K, \mathcal{N}^m_k(K), V^m_k(K)) \) in three dimensions.

### 3.4. \( H^m \)-conforming virtual elements in arbitrary dimension.

Now we construct the \( H^m \)-conforming virtual elements for \( k \geq m \) and \( m \geq 1 \) in arbitrary dimension recursively.

Let polytope \( K \subset \mathbb{R}^n \) with \( n \geq 2 \). Assume \( H^\ell \)-conforming virtual elements \( (F, \mathcal{N}^\ell_k(F), V^\ell_k(F)) \) for \( \ell = 1, \cdots, m \) and \( k_\ell \geq \ell \) have been constructed for each \( F \in \mathcal{F}^\ell(K) \) with \( r = 1, \cdots, n - 1 \). The DoFs \( \mathcal{N}^\ell_k(F) \) are given by
\[
(25) \quad h^j_{\ell} \nabla^j v(\delta) \quad \forall \ \delta \in \mathcal{F}^{n-r}(F), j = 0, 1, \cdots, \ell - 1,
\]
\[
(26) \quad \frac{h^j_{\ell}}{|e|} \left( \frac{\partial^{\alpha|v|}}{\partial v^\alpha_{F,e}} q \right) \quad \forall \ q \in \mathbb{P}_{k_{\ell}-2|\alpha|}(e), e \in \mathcal{F}^s(F), s = 1, \cdots, n - r - 1,
\]
\[
\alpha \in A_s \text{, and } |\alpha| \leq \ell - 1,
\]
\[
(27) \quad \frac{1}{|F|} (v, q)_{F} \quad \forall \ q \in \mathbb{P}_{k_{\ell}-2r}(F).
\]
And assume
\begin{itemize}
  \item [(i)] \( \mathbb{P}_{k_\ell}(F) \subseteq V^\ell_{k_\ell}(F) \subset H^\ell(F) \);
  \item [(ii)] for any \( v \in V^\ell_{k_\ell}(F) \), we have \( (\nabla^j v)|_e \in H^1(S^1_e; S_{n-r}(j)) \) and \( (\nabla^j v)|_e \in H^{\ell-j}(e; S_{n-r}(j)) \) for \( e \in \mathcal{F}^\ell(F), j = 0, 1, \cdots, \ell - 1 \) and \( s = 1, \cdots, n - r - 1 \);
  \item [(iii)] for any \( v \in V^\ell_{k_\ell}(F) \), \( \frac{\partial^{\beta|v|}}{\partial v^\beta_{F,e}} q \in V^\ell_{k_{\ell}-|\beta|}(e) \) for each \( e \in \mathcal{F}^s(F), \beta \in A_s, |\beta| \leq \ell - 1 \) and \( s = 1, \cdots, n - r - 1 \);
  \item [(iv)] \( Q^j_{k_\ell}(\nabla^j v) \) is computable using only the DoFs \( (25)-(27) \) for any \( v \in V^\ell_{k_\ell}(F) \) and \( j = 0, 1, \cdots, \ell \).
\end{itemize}
The assumption (ii) is inspired by Lemma 3.3.

First define a preliminary virtual element space

\[ V_k^m(K) := \{ v \in H^m(K) : (-\Delta)^m v \in \mathbb{P}_k(K), \quad (\nabla^j v)|_{S_K^r} \in H^1(S_K^r; S_n(j)) \text{ for } j = 0, 1, \ldots, m-1, \quad \frac{\partial^{|\alpha|} v}{\partial \nu_F^\alpha} |_{F} \in V_{k-|\alpha|}^{m-|\alpha|} (F) \quad \forall F \in \mathcal{F}^r(K), \quad r = 1, \ldots, n-1, \alpha \in A_r, \text{ and } |\alpha| \leq m-1 \}. \]

By the assumption (i), we have \( \mathbb{P}_k(K) \subseteq \tilde{V}_k^m(K) \). Take \( v \in \tilde{V}_k^m(K) \). Applying the same argument as in Lemma 3.3, \( \nabla^j v|_e \in \mathbb{P}_{\max\{k-j,2m_1-j\}}(e,S_n(j)) \) for each edge \( e \in \mathcal{F}^{n-1}(K) \) and \( j = 0, 1, \ldots, m-1 \), then it follows from \( (\nabla^j v)|_{S_k^{n-1}} \in H^1(S_k^{n-1}; S_n(j)) \) that \( \nabla^j v \) is continuous on the one-dimensional skeleton \( S_K^{n-1} \). For any \( F \in \mathcal{F}^r(K) \) with \( 1 \leq r \leq n-1 \), we get from the definition of \( \tilde{V}_k^m(K) \) and (7) that \( \nabla^j v|_{F} \in H^{m-j}(F; S_n(j)) \).

Inspired by \( \frac{\partial^{|\alpha|} v}{\partial \nu_F^\alpha} |_{F} \in V_{k-|\alpha|}^{m-|\alpha|} (F) \) and the DoFs (25)-(27), we propose the following degrees of freedom (DoFs) \( N_k^m(K) \) for the \( H^m \)-conforming virtual elements in arbitrary dimension

\begin{align}
(28) \quad & h_K^{|\alpha|} \nabla^j v(\delta) \quad \forall \delta \in \mathcal{F}^n(K), \quad j = 0, 1, \ldots, m-1, \\
(29) \quad & \frac{h_K^{|\alpha|}}{|F|} \left( \frac{\partial^{|\alpha|} v}{\partial \nu_F^\alpha}, q \right)_{F} \quad \forall q \in \mathbb{P}_{k-2m+|\alpha|}(F), \quad F \in \mathcal{F}^r(K), \quad r = 1, \ldots, n-1, \\
& \quad \alpha \in A_r, \text{ and } |\alpha| \leq m-1, \\
(30) \quad & \frac{1}{|K|} (v, q)_K \quad \forall q \in \mathbb{P}_{k-2m}(K). 
\end{align}

To define the space of shape functions \( V_k^m(K) \), we introduce a local \( H^m \)-projector \( \Pi_k^F : H^m(K) \to \mathbb{P}_k(K) \): given \( v \in H^m(K) \), let \( \Pi_k^F v \in \mathbb{P}_k(K) \) be the solution of the problem

\begin{align}
(31) \quad & (\nabla^m \Pi_k^F v, \nabla^m q)_K = (\nabla^m v, \nabla^m q)_K \quad \forall q \in \mathbb{P}_k(K), \\
(32) \quad & \sum_{\delta \in \mathcal{F}^n(K)} (\nabla^j \Pi_k^F v)(\delta) = \sum_{\delta \in \mathcal{F}^n(K)} (\nabla^j v)(\delta), \quad j = 0, 1, \ldots, m-1. 
\end{align}

The number of equations in (32) is

\[ \sum_{j=0}^{m-1} C_{n+1}^{n} = C_{n+m-1}^{n} = \dim \mathbb{P}_{m-1}(K). \]

We refer to [23, Section 3.3 and Lemma 3.5] for the well-posedness of (31)-(32), and the identity

\[ \Pi_k^F q = q \quad \forall q \in \mathbb{P}_k(K). \]

By the assumption (iv) of conforming virtual elements on faces, \( Q_k^{F_{k-|\alpha|}}(\nabla^j \frac{\partial^{|\alpha|}}{\partial \nu_F^\alpha} v) \) is computable by using the DoFs (28)-(29) for any \( v \in \tilde{V}_k^m(K), \ F \in \mathcal{F}^r(K), \ \alpha \in A_r, \ |\alpha| \leq m-1, \ r = 1, \ldots, n-1, \text{ and } \ell = 0, \ldots, m - |\alpha|. \) This together with (7) implies \( Q_k^{F_{k-|\alpha|}}(\nabla^j v) \) is computable by using the DoFs (28)-(29) for any \( v \in \tilde{V}_k^m(K) \).
Following the ideas in [4, 23], define the space of shape functions

\[ V^m_k(K) := \{ v \in \mathring{V}^m_k(K) : (v - \Pi^K v, q)_K = 0 \quad \forall \ q \in \mathbb{P}^1_{k-2m}(K) \} \]

Due to (33), it holds \( \mathbb{P}_k(K) \subseteq V^m_k(K) \). Finally we finish the construction of the \( H^m \)-conforming virtual elements \((K, \mathcal{N}^m_k(K), V^m_k(K))\) in arbitrary dimension.

### 3.5. Data spaces and trace

From now on in this section we will show that the DoFs (28)-(30) are uni-solvent for the local virtual element space \( V^m_k(K) \). The main difficulty is to count the dimension of \( V^m_k(K) \). To this end, we introduce data spaces

\[
\mathcal{D}(\partial K) := \prod_{\delta \in \mathcal{F}^n(K)} \prod_{j=0}^{m-1} \mathcal{S}_n(j) \times \prod_{r=1}^{n-1} \prod_{F \in \mathcal{F}^r(K)} \mathbb{P}_{k-2m+|\alpha|}(F),
\]

\[
\mathcal{D}(K) := \mathcal{D}(\partial K) \times \mathbb{P}_{k-2m}(K), \quad \tilde{\mathcal{D}}(K) := \mathcal{D}(\partial K) \times \mathbb{P}_k(K).
\]

Clearly we have \( \dim \mathcal{D}(K) = \# \mathcal{N}^m_k(K) \). For simplicity, let notation \((d^\delta_{n,j}, d^r_{F,\alpha}) \in \mathcal{D}(\partial K)\) mean

- \(d^\delta_{n,j} \in \mathcal{S}_n(j)\) for each \(\delta \in \mathcal{F}^n(K)\) and \(j = 0, 1, \ldots, m - 1\);
- \(d^r_{F,\alpha} \in \mathbb{P}_{k-2m+|\alpha|}(F)\) for each \(F \in \mathcal{F}^r(K)\) with \(r = 1, \ldots, n - 1\), \(\alpha \in \mathcal{A}_r\), and \(|\alpha| \leq m - 1\).

Notation \((d^\delta_{n,j}, d^r_{F,\alpha}, d_0) \in \mathcal{D}(K)\) means \((d^\delta_{n,j}, d^r_{F,\alpha}) \in \mathcal{D}(\partial K)\) and \(d_0 \in \mathbb{P}_{k-2m}(K)\), and notation \((d^\delta_{n,j}, d^r_{F,\alpha}, d_0) \in \tilde{\mathcal{D}}(K)\) is understood similarly. We will show that both the mapping \(\mathcal{D}_K : V^m_k(K) \rightarrow \mathcal{D}(K)\) given by

\[
\mathcal{D}_K v := \left( \nabla^j v(\delta), Q^r_{k-2m+|\alpha|}(F), \frac{\partial |\alpha| v}{\partial \nu^r_F}, (-\Delta)^m v \right) \in \mathcal{D}(K) \quad \text{with} \quad v \in V^m_k(K),
\]

and the mapping \(\tilde{\mathcal{D}}_K : V^m_k(K) \rightarrow \tilde{\mathcal{D}}(K)\) given by

\[
\tilde{\mathcal{D}}_K v := \left( \nabla^j v(\delta), Q^r_{k-2m+|\alpha|}(F), \frac{\partial |\alpha| v}{\partial \nu^r_F}, (-\Delta)^m v \right) \in \tilde{\mathcal{D}}(K) \quad \text{with} \quad v \in V^m_k(K),
\]

are bijective. The idea of introducing data spaces can be found in [25], and similar idea, i.e. degrees of freedom tuple, is advanced in [5].

For a function \(v \in H^m(K)\), the trace \(\text{Tr} \ v := \left( v|_{\partial K}, \left. \frac{\partial v}{\partial \nu^r_F} \right|_{\partial K}, \cdots, \left. \frac{\partial^{m-1} v}{\partial \nu^r_F} \right|_{\partial K} \right) \in H^{m-1/2}(\mathcal{F}^1(K)) \times \cdots \times H^{1/2}(\mathcal{F}^1(K))\), where

\[ H^s(\mathcal{F}^1(K)) := \{ v \in L^2(\partial K) : v|_F \in H^s(F) \quad \forall \ F \in \mathcal{F}^1(K) \} \quad \text{for} \ s > 0. \]

The trace space \(\text{Tr} H^m(K) \neq H^{m-1/2}(\mathcal{F}^1(K)) \times \cdots \times H^{1/2}(\mathcal{F}^1(K))\), since there exist some compatibility conditions among the components of \(\text{Tr} v\) [35]. To present the characterization of the trace space \(\text{Tr} H^m(K)\) in [2, 3, 36], we first define the space of Whitney arrays

\[ \mathcal{W}(\partial K) := \{ g_\alpha : \alpha \in \mathcal{A}_n, |\alpha| \leq m - 1 : g_\alpha \in H^1(\partial K) \quad \forall \ \alpha \in \mathcal{A}_n \ \text{with} \ |\alpha| \leq m - 2, \]

\[ g_\alpha \in H^{1/2}(\partial K) \quad \forall \ \alpha \in \mathcal{A}_n \ \text{with} \ |\alpha| = m - 1, \]

and the compatibility conditions (34) for \(g_\alpha\) are satisfied}.
where the compatibility conditions are

\[(\nu_F)_j \partial g_\alpha - (\nu_F)_j \partial g_\alpha - (\nu_F)_j g_\alpha + \epsilon_i - (\nu_F)_i g_\alpha + \epsilon_i \text{ on each } F \in \mathcal{F}^1(K)\]

for each \(\alpha \in A_n, |\alpha| \leq m - 2\) and \(1 \leq i \neq j \leq n\), \((\nu_F)_i = e_i\), \(\nu_F\) and \(\partial_i g_\alpha = e_i \cdot \nabla g_\alpha\).

For \(v \in H^m(K)\), clearly we have the array \(\{\partial^\alpha v|_{\partial K}\}_{\alpha \in A_n, |\alpha| \leq m-1} \in WA(\partial K)\), where \(\partial^\alpha v := e^\alpha : \nabla |\alpha| v\) with \(e^\alpha := e_1^{\alpha_1} \otimes \cdots \otimes e_n^{\alpha_n}\). And in this case, expressions in both sides of (34) are two representations of some tangential derivative of the trace of \(\partial^\alpha v\). Moreover, such a trace mapping is onto, which is listed in the following lemma.

**Lemma 3.4** (Theorem 5 in [2], Theorem 4 in [3] and Theorem R(m) in [36]). Let \(K \in \mathbb{R}^n\) be a polytope. For each Whitney array \(\{g_\alpha\}_{\alpha \in A_n, |\alpha| \leq m-1} \in WA(\partial K)\), there exists a function \(v \in H^m(K)\) such that

\[\partial^\alpha v|_{\partial K} = g_\alpha \quad \forall \alpha \in A_n, |\alpha| \leq m - 1.\]

Moreover, there exists a linear and bounded operator from \(WA(\partial K)\) to \(H^m(K)\).

In the next two lemmas, we will construct a Whitney array for each data in \(\mathcal{D}(\partial K)\).

**Lemma 3.5.** Given data \((d^\delta_j, d^F_j) \in \mathcal{D}(\partial K)\), there exist \(g^\delta_j \in S_n(j)\) and \(g^F_j \in H^{m-j}(F; S_n(j))\) for any \(\delta \in \mathcal{F}^n(K), F \in \mathcal{F}^r(K)\), \(r = 1, \cdots, n-1\), and \(j = 0, 1, \cdots, m-1\) such that

(i) \(g^\delta_j = d^\delta_j\) for each \(\delta \in \mathcal{F}^n(K)\) and \(j = 0, 1, \cdots, m-1\);

(ii) \(g^F_j|_F : \nu^F_F \in V^{m-|\alpha|}(F)\) and \(Q_k^{F-2m+|\alpha|}(g^F_j|_F : \nu^F_F) = d^F_F\) for each \(F \in \mathcal{F}^r(K), \alpha \in A_r, |\alpha| \leq m - 1, r = 1, \cdots, n - 1, j = 0, 1, \cdots, m-1, (35)\)

\[g^F_j = \sum_{\alpha \in A_r, |\beta| = m-r} \alpha! |\beta|! \frac{j}{\alpha! \beta!} \left| \frac{\partial^{\beta}}{\partial \nu^F_F} \right| \left( g^F_j|_F : \nu^F_F \right); \]

(iv) \(g^F_j|_e = g^F_j|_e\) for each \(F \in \mathcal{F}^r(K), e \in \mathcal{F}^s(F), r = 1, \cdots, n - 1, s = 1, \cdots, n - r, j = 0, 1, \cdots, m-1.\)

**Remark 1.** We will see in the proof of Lemma 3.7 that \(g^F_j = \nabla^j v^b|_F\) for some \(v^b \in H^m(K)\). In the colusems of Lemma 3.5, \(g^F_j|_F : \nu^F_F \in V^{m-|\alpha|}(F)\) is motivated by \((\nabla |\alpha| v^b)|_F : \nu^F_F = \frac{\partial^{\alpha+a+b}}{\partial \nu^F_F}|_F \in V^{m-|\alpha|}(F)\) in the definition of \(V^m_{k-|\alpha|}(K)\), \(Q_{k-2m+|\alpha|}(g^F_j|_F : \nu^F_F) = d^F_F\) is motivated by the DoFs (29), and equation (35) is motivated by (7). It follows from (35) that

\[g^F_j : \text{sym}(\nu^F_F \otimes t^F_F) = \frac{\partial^{\beta}}{\partial \nu^F_F \partial t^F_F} (g^F_j|_F : \nu^F_F).\]

**Proof of Lemma 3.5.** First take \(g^\delta_j = d^\delta_j\) for each \(\delta \in \mathcal{F}^n(K)\) and \(j = 0, 1, \cdots, m-1\). For each \(e \in \mathcal{F}^{n-1}(K), \alpha \in A_{n-1}\) and \(|\alpha| \leq m - 1\), take \(v^e = \nu^e \in V^{m-|\alpha|}(e)\) satisfying

\[\left\{ \begin{array}{l}
(\nabla |\alpha| v^e)|_e : \text{sym}(\nu^e \otimes t^e_F) = (\nabla |\alpha| v^e)|_e : \text{sym}(\nu^e \otimes t^e_F) \quad \forall \delta \in \mathcal{F}^1(e), j = 0, 1, \cdots, m - |\alpha| - 1, \\
Q_k^{F-2m+|\alpha|+(e)} v^e = d^e |_{n-1} \end{array} \right.\]
For $j = 0, 1, \cdots, m - 1$, inspired by (7), let

$$
\gamma_{n-1}^{e,j} = \sum_{\alpha \in A_{n-1}, |\alpha| \leq j} \frac{j!}{\alpha!(j-|\alpha|)!} \text{sym}(\nu_e^\alpha \otimes t_e^{-|\alpha|})(\partial_{e}^{j-|\alpha|} \gamma_{n-1}^{e,\alpha}).
$$

Then we have for any $\delta \in F^1(e)$ that

$$
\gamma_{n-1}^{e,j}(\delta) = \sum_{\alpha \in A_{n-1}, |\alpha| \leq j} \frac{j!}{\alpha!(j-|\alpha|)!} \text{sym}(\nu_e^\alpha \otimes t_e^{-|\alpha|})(\partial_{e}^{j-|\alpha|} \gamma_{n-1}^{e,\alpha} : \text{sym}(\nu_e^\alpha \otimes t_e^{-|\alpha|}))(\delta) = \gamma_{n-1}^{e,j}.
$$

And it follows

$$
\gamma_{n-1}^{e,|\alpha|} : \nu_e^\alpha = \sum_{\beta \in A_{n-1}, |\beta| \leq |\alpha|} \frac{|\alpha|!}{\beta!(|\alpha| - |\beta|)!} \text{sym}(\nu_e^\beta \otimes t_e^{-|\alpha|-|\beta|})(\partial_{e}^{|\alpha|-|\beta|} \gamma_{n-1}^{e,\beta} : \nu_e^\alpha).
$$

In turn we have

$$
Q_{k-2m+\alpha}^{e,|\alpha|} (\gamma_{n-1}^{e,|\alpha|} : \nu_e^\alpha) = \delta_{n-1}^{e,\alpha},
$$

$$
g_{n-1}^{e,j} = \sum_{\alpha \in A_{n-1}, |\alpha| \leq j} \frac{j!}{\alpha!(j-|\alpha|)!} \text{sym}(\nu_e^\alpha \otimes t_e^{-|\alpha|})(\partial_{e}^{j-|\alpha|} \gamma_{n-1}^{e,\alpha} : \nu_e^\alpha).
$$

Assume we have found $g_{s,j}^{e} \in H^{m-j}(e; S_n(j))$ for any $e \in F^s(K)$, $s = r + 1, \cdots, n - 1$, and $j = 0, 1, \cdots, m - 1$ for $1 \leq r \leq n - 2$ satisfying

1. $g_{s,j}^{e,|\alpha|} : \nu_e^\alpha \in V_{k-|\alpha|}^{m-|\alpha|}(e)$ and $Q_{k-2m+\alpha}^{e,|\alpha|} (g_{s,j}^{e,|\alpha|} : \nu_e^\alpha) = d_{s+1}^{e,\alpha}$ for each $e \in F^s(K)$, $\alpha \in A_s$, $|\alpha| \leq m - 1$, $s = r + 1, \cdots, n - 1$;
2. for each $e \in F^s(K)$, $s = r + 1, \cdots, n - 1$ and $j = 0, 1, \cdots, m - 1$,

$$
g_{s,j}^{e,j} = \sum_{\alpha \in A_{s}, \beta \in A_{n-s}} \frac{j!}{\alpha!(j-|\alpha|)!} \text{sym}(\nu_e^\beta \otimes t_e^{-|\alpha|})(\partial_{e}^{j-|\alpha|} (g_{s+1}^{e,|\alpha|} : \nu_e^\alpha));
$$

$$
\gamma_{s,j}^{e,j} \mid_{e'} = g_{s',j}^{e,j} \quad \text{for each } e' \in F^s(K), e' \in F^{s'}(e), s = r + 1, \cdots, n - 1, s' = 1, \cdots, n - s.
$$

Now consider the construction of $g_{r}^{e,j}$ for each $F \in F^r(K)$ and $j = 1, \cdots, m - 1$. To this end, for any $\alpha \in A_r$ and $|\alpha| \leq m - 1$, by (25)-(27) let $v_{r}^{e,\alpha} \in V_{k-|\alpha|}^{m-|\alpha|}(F)$ be determined by

$$
\left\{ \begin{array}{ll}
\left( \frac{\partial_{F}^{\beta}}{\partial v_{r}^{\alpha}} \right)(\delta) = \gamma_{n-1}^{e,|\beta|+|\alpha|} : \text{sym}(\nu_F^\alpha \otimes \nu_F^\beta) & \forall \delta \in F^{n-r}(F), \beta \in A_{n-r}, \\
Q_{k-2m+\alpha+|\beta|}^{e,|\alpha|} (\frac{\partial_{F}^{\beta}}{\partial v_{r}^{\alpha}}) \mid_{e} - g_{r+s}^{e,|\alpha|+|\beta|} : \text{sym}(\nu_F^\alpha \otimes \nu_F^\beta) = 0 & \forall e \in F^s(F), \\
Q_{k-2m+\alpha+|\beta|}^{F} \mid_{e} = d_{r}^{e,\alpha} & \text{in } F,
\end{array} \right.
$$

where $s = 1, \cdots, n - r - 1, \beta \in A_s, |\beta| \leq m - |\alpha| - 1,
Noting that \( \frac{\partial^{\beta}[F,e]}{\partial \nu^{\beta}_{F,e}} : \text{sym}(\nu^\beta_F \otimes \nu^\beta_{F,e}) \in V^{m-|\alpha|-|\beta|}_{k-|\alpha|-|\beta|}(e) \) for each \( e \in \mathcal{F}^s(F) \), and they share the same values of the DoFs, it follows

\[
\frac{\partial^{\beta}[F,e]}{\partial \nu^{\beta}_{F,e}} = g_{r+s}^{\epsilon,|\beta|+|\alpha|} : \text{sym}(\nu^\beta_F \otimes \nu^\beta_{F,e}) \quad \forall \ e \in \mathcal{F}^s(F).
\]

For \( j = 0, 1, \cdots, m - 1 \), let

\[
g_{r}^{F,j} = \sum_{\alpha \in A_r, \beta \in A_{n-r}} \frac{j!}{\alpha!\beta!} \text{sym}(\nu^\alpha_F \otimes \nu^\beta_{F,e} \otimes t^\gamma_e) \left( \frac{\partial^{\beta}[F,e]}{\partial \nu^{\beta}_{F,e}} \right).
\]

Then

\[
g_{r}^{F,j} : \nu^\alpha_F = \nu^\alpha_F \in V^{m-|\alpha|}_{k-|\alpha|}(F),
\]

which yields

\[
Q^F_{k-2m+|\alpha|}(g_{r}^{F,|\alpha|} : \nu^\alpha_F) = d_{r}^{F,\alpha},
\]

\[
g_{r}^{F,j} = \sum_{\alpha \in A_r, \beta \in A_{n-r}} \frac{j!}{\alpha!\beta!} \text{sym}(\nu^\alpha_F \otimes \nu^\beta_{F,e} \otimes t^\gamma_e) \left( \frac{\partial^{\beta}[F,e]}{\partial \nu^{\beta}_{F,e}} \right) (g_{r+s}^{\epsilon,|\beta|+|\alpha|})
\]

For each \( e \in \mathcal{F}^s(F) \) with \( s = 1, \cdots, n - r \), it follows from (36) that

\[
g_{r+s}^{F,j} |_e = \sum_{\alpha \in A_r, \beta \in A_{n-r}, \gamma \in A_{n-r-s}} \frac{j!}{\alpha!\beta!\gamma!} \text{sym}(\nu^\alpha_F \otimes \nu^\beta_{F,e} \otimes t^\gamma_e) \left( \frac{\partial^{\beta}[F,e]}{\partial \nu^{\beta}_{F,e}} \right) (g_{r+s}^{\epsilon,|\beta|+|\alpha|})
\]

Finally we finish the proof by the mathematical induction.

\[\square\]

**Lemma 3.6.** Given data \( (d^s_{\alpha}, \varphi^F_{r,\alpha}) \in \mathcal{D}(\partial K) \), let \( g_{1}^{F,j} \in H^{m-1}(\mathcal{F}; S_n(j)) \) for any \( F \in \mathcal{F}^1(K) \) and \( j = 0, 1, \cdots, m - 1 \) be defined in Lemma 3.5. For each \( \alpha \in A_n, |\alpha| \leq m - 1 \), define \( g_{\alpha} | F = g_{1}^{F,|\alpha|} : \varphi^F_{1,\alpha} \) \( \forall F \in \mathcal{F}^1(K) \). Then \( \{ g_{\alpha} |_{A_n, |\alpha| \leq m - 1} \in WA(\partial K) \).

**Proof.** By (ii) and (iv) in Lemma 3.5, we have \( g_{\alpha} | F \in H^{m-|\alpha|}(F) \) and \( g_{\alpha} \in H^{1}(\partial K) \) for each \( F \in \mathcal{F}^1(K) \), and \( \alpha \in A_n, |\alpha| \leq m - 1 \).

Next we check the compatibility conditions in (34). Noting that

\[
\partial_i g_{\alpha} = (\nu_F) \frac{\partial g_{\alpha}}{\partial \nu_F} + \sum_{\ell=1}^{n-1} (t_{F,\ell}) i \frac{\partial g_{\alpha}}{\partial t_{F,\ell}}, \quad \partial_j g_{\alpha} = (\nu_F) j \frac{\partial g_{\alpha}}{\partial \nu_F} + \sum_{\ell=1}^{n-1} (t_{F,\ell}) j \frac{\partial g_{\alpha}}{\partial t_{F,\ell}},
\]

we get

\[
(\nu_F) i \partial_i g_{\alpha} = (\nu_F) j \partial_j g_{\alpha} = \sum_{\ell=1}^{n-1} ((\nu_F) i(t_{F,\ell}) - (\nu_F) j(t_{F,\ell})) \frac{\partial g_{\alpha}}{\partial t_{F,\ell}}.
\]
On the other side, it follows from (37) that
\[
g_1^{F,|\alpha|+1} = \sum_{\beta \in \mathcal{A}_{n-1} \atop \beta \leq |\alpha|} \frac{(|\alpha| + 1)!}{(|\alpha| + 1 - |\beta|)!|\beta|!} \text{sym} (\nu^{[|\alpha|+1-|\beta|]}) t_F^\beta \frac{\partial |\beta| F,^{[|\alpha|+1-|\beta|]} t_F}{\partial t_F},
\]
\[
g_1^{F,|\alpha|} = \sum_{\beta \in \mathcal{A}_{n-1} \atop \beta \leq |\alpha|} \frac{|\alpha|!}{(|\alpha| - |\beta|)!|\beta|!} \text{sym} (\nu^{[|\alpha|-|\beta|]}) t_F^\beta \frac{\partial |\beta| F,^{[|\alpha|-|\beta|]} t_F}{\partial t_F}.
\]
Hence for \( \ell = 1, \cdots, n - 1 \), we get
\[
g_1^{F,|\alpha|+1} : (e_\alpha \otimes t_{F,\ell})
= \sum_{\beta \in \mathcal{A}_{n-1} \atop \beta \leq |\alpha|} \frac{(|\alpha| + 1)!}{(|\alpha| - |\beta|)!|\beta|!} \text{sym} (\nu^{[|\alpha|-|\beta|]}) t_F^\beta \frac{\partial |\beta| F,^{[|\alpha|-|\beta|]} t_F}{\partial t_F} : (e_\alpha \otimes t_{F,\ell})
= \sum_{\beta \in \mathcal{A}_{n-1} \atop \beta \leq |\alpha|} \frac{(|\alpha| + 1)!}{(|\alpha| - |\beta|)!|\beta|!} \frac{\beta}{|\beta|!} \text{sym} (\nu^{[|\alpha|-|\beta|]}) t_F^\beta \frac{\partial |\beta| F,^{[|\alpha|-|\beta|]} t_F}{\partial t_F} : (e_\alpha \otimes t_{F,\ell})
= \sum_{\beta \in \mathcal{A}_{n-1} \atop \beta \leq |\alpha|} \frac{|\alpha|!}{(|\alpha| - |\beta|)!|\beta|!} \frac{\beta}{|\beta|!} \text{sym} (\nu^{[|\alpha|-|\beta|]}) t_F^\beta \frac{\partial |\beta| F,^{[|\alpha|-|\beta|]} t_F}{\partial t_F} : (e_\alpha \otimes t_{F,\ell})
= \frac{\partial (g_1^{F,|\alpha|+1} : e_\alpha)}{\partial t_{F,\ell}} = \frac{\partial g_\alpha}{\partial t_{F,\ell}}.
\]
By \( e_i = (\nu_F)_i \nu_F + \sum_{\ell=1}^{n-1} (t_{F,\ell})_i t_{F,\ell} \), we get
\[
g_\alpha + e_i = g_1^{F,|\alpha|+1} : (e_\alpha \otimes e_i)
= (\nu_F) g_1^{F,|\alpha|+1} : (e_\alpha \otimes \nu_F) + \sum_{\ell=1}^{n-1} (t_{F,\ell})_i g_1^{F,|\alpha|+1} : (e_\alpha \otimes t_{F,\ell}).
\]
Thus it holds
\[
(\nu_F)_j g_\alpha + e_i - (\nu_F)_i g_\alpha + e_j = \sum_{\ell=1}^{n-1} ((\nu_F)_j (t_{F,\ell})_i - (\nu_F)_i (t_{F,\ell})_j) g_1^{F,|\alpha|+1} : (e_\alpha \otimes t_{F,\ell})
= \sum_{\ell=1}^{n-1} ((\nu_F)_j (t_{F,\ell})_i - (\nu_F)_i (t_{F,\ell})_j) \frac{\partial g_\alpha}{\partial t_{F,\ell}}.
\]
Therefore we conclude the compatibility conditions in (34) from (38). □

3.6. Uni-solvence of virtual elements. With previous preparations, we will prove the uni-solvence of the \( H^m \)-conforming virtual elements \((K, N^m_k(K), V^m_k(K))\) in arbitrary dimension in this subsection.

Lemma 3.7. The mapping \( \tilde{D}_K : \tilde{V}^m_k(K) \to \tilde{D}(K) \) is onto. Consequently
\[
\dim \tilde{V}^m_k(K) \geq \dim \tilde{D}(K).
\]
Proof. Take any data \((d_n^j, d_F^\alpha, d_0) \in \tilde{D}(K)\). Due to \((d_n^j, d_F^\alpha) \in D(\partial K)\), let \(\{g_\alpha\}_{\alpha \in A_n, |\alpha| \leq m-1} \in WA(\partial K)\) be defined in Lemma 3.6. By Lemma 3.4, there exists \(v^b \in H^m(K)\) such that
\[
\partial^\alpha v^b|_{\partial K} = g_\alpha \quad \forall \alpha \in A_n, |\alpha| \leq m - 1.
\]
Then
\[
\partial^\alpha v^b|_F = g_1^F|\alpha| : e^\alpha \quad \forall F \in F^1(K), \alpha \in A_n, |\alpha| \leq m - 1,
\]
which implies
\[
\nabla^j v^b|_F = g_1^F : e^\alpha \quad \forall F \in F^1(K), j = 1, \ldots, m - 1.
\]
And we get from (iv) in Lemma 3.5 that
\[
\nabla^j v^b|_F = g_r^F : e^\alpha \quad \forall F \in F^r(K), r = 1, \ldots, n, j = 1, \ldots, m - 1.
\]
On the other side, there exists unique \(v^0 \in H^m_0(K)\) determined by
\[
(-\Delta)^m v^0 = d_0 - (-\Delta)^m v^b.
\]
Take \(v = v^0 + v^b \in H^m(K)\). We have \((-\Delta)^m v = d_0 \in \mathbb{P}_k(K)\), and
\[
(39) \quad \nabla^j v|_F = \nabla^j v^b|_F = g_r^F : e^\alpha \quad \forall F \in F^r(K), r = 1, \ldots, n, j = 1, \ldots, m - 1.
\]
It follows from the last identity and (iv) in Lemma 3.5 that \((\nabla^j v)|_{S^r_K} \in H^1(S^r_K; \mathbb{S}_n(j))\) for \(r = 1, \ldots, n, j = 0, \ldots, m - 1\). And thanks to (ii) in Lemma 3.5,
\[
(40) \quad \frac{\partial^{|\alpha|} v^b}{\partial \nu^F} _| F = \frac{\partial^{|\alpha|} v}{\partial \nu^F} _| F = g_r^F : e^\alpha \quad \forall F \in F^r(K), r = 1, \ldots, n - 1, \alpha \in A_r, \text{ and } |\alpha| \leq m - 1.
\]
for any \(F \in F^r(K), r = 1, \ldots, n - 1, \alpha \in A_r, \text{ and } |\alpha| \leq m - 1\). Thus \(v \in \tilde{V}^m_k(K)\). And it follows from (39)-(40), (i) and (ii) in Lemma 3.5 that \(\tilde{D}_K v = (d_n^j, d_F^\alpha, d_0)\).

\[\square\]

Lemma 3.8. The following DoFs \(\tilde{N}^m_k(K)\)
\[
h^j \nabla^j v(\delta) \quad \forall \delta \in \mathcal{F}^n(K), j = 0, 1, \ldots, m - 1,
\]
\[
\frac{h^{|\alpha|}}{|F|} \left( \frac{\partial^{|\alpha|} v}{\partial \nu^F} _| F, q \right) _| F \quad \forall q \in \mathbb{P}_{k-2m+|\alpha|}(F), F \in F^r(K), r = 1, \ldots, n - 1,
\]
\[
\alpha \in A_r, \text{ and } |\alpha| \leq m - 1
\]
are uni-solvent for the local virtual element space \(\tilde{V}^m_k(K)\). Consequently the mapping \(\tilde{D}_K : \tilde{V}^m_k(K) \to \tilde{D}(K)\) is bijective.

Proof. Due to Lemma 3.7, we have \(\dim \tilde{V}^m_k(K) \geq \#\tilde{N}^m_k(K)\). Assume \(v \in \tilde{V}^m_k(K)\) and all the DoFs in \(\tilde{N}^m_k(K)\) vanish. By the recursive definition of \(\tilde{V}^m_k(K)\), it follows from the vanishing DoFs on the boundary in \(\tilde{N}^m_k(K)\) that \(v \in H^m_0(K)\). Employing the integration by parts, we get from \((\nabla^m v)|_{\partial K} \in \mathbb{P}_k(K)\) that
\[
\|\nabla^m v\|^2_{0,K} = \langle \nabla^m v, \nabla^m v \rangle_K = \langle v, (-\Delta)^m v \rangle_K = 0.
\]
Thus \(v = 0\). \(\square\)
Lemma 3.9. The DoFs (28)-(30), i.e. $\mathcal{N}^m_k(K)$, are uni-solvent for the local virtual element space $V^m_k(K)$. Consequently the mapping $\mathcal{D}_K : V^m_k(K) \to \mathcal{D}(K)$ is bijective.

Proof. By the definition of $\dim V^m_k(K)$, it holds $\dim V^m_k(K) \geq \# \mathcal{N}^m_k(K)$. Assume $v \in V^m_k(K)$ and the DoFs (28)-(30) vanish. Notice that $\Pi^K v = 0$. Hence

$$ (v, q)_K = 0 \quad \forall \ q \in \mathbb{P}_k(K), $$

which together with Lemma 3.8 yields $v = 0$. $\square$

As the two dimensional case, it holds

$$ Q^K v = \Pi^K v + Q^K_{k-2m} v - Q^K_{k-2m} \Pi^K v \quad \forall \ v \in V^m_k(K). \quad (41) $$

Hence $Q^K v$ is computable using only the DoFs (28)-(30) for any $v \in V^m_k(K)$. And then $Q^K (\nabla^j v)$ is computable using only the DoFs (28)-(30) for any $v \in V^m_k(K)$ and $j = 1, \ldots, m$. As a result of (41), we have

$$ Q^K v = Q^K_{k-2m} v \quad \forall \ v \in \ker(\Pi^K) \cap V^m_k(K), \quad (42) $$

where $\ker(\Pi^K) \cap V^m_k(K) := \{ v \in V^m_k(K) : \Pi^K v = 0 \}$.

The $H^2$-conforming virtual elements in three dimensions have been constructed in [9].

Remark 2. For the lowest degree case $k = m$, the DoFs (29)-(30) disappear, and $\mathcal{N}^m_k(K)$ will reduce to

$$ h^K_0 \nabla^j v(\delta) \quad \forall \ \delta \in \mathcal{F}^n(K), \ j = 0, 1, \ldots, m - 1. $$

Remark 3. When $k = m$ and $K \subset \mathbb{R}^n$ is a simplex,

$$ \dim V^m_k(K) = (n + 1) \dim \mathbb{P}_{m-1}(K). $$

As a comparison, the dimension of the lowest degree $H^m$-conforming finite element in [29] is $\dim \mathbb{P}_{2^n(m-1)+1}(K)$, which is much larger than $\dim V^m_k(K)$.

4. Inverse inequality and norm equivalences

We will derive the inverse inequality and several norm equivalences of the virtual elements $(K, \mathcal{N}^m_k(K), V^m_k(K))$ by the mathematical induction in this section, which are vitally important in the error analysis of virtual element methods. Henceforth, we always assume mesh conditions (A1) and (A2) hold, and polytope $K \in \mathcal{T}_h$.

4.1. Inverse inequality. We first employ the multiplicative trace inequality, the inverse trace theorem and the inverse inequality for polynomials to prove the inverse inequality for $V^m_k(K)$ through the mathematical induction.

Lemma 4.1. Let $F \in \mathcal{F}^r(K)$ with $r = 0, 1, \ldots, n - 1$, $\ell = 2, \ldots, m$ and integer $k_\ell \geq \ell$. Let $v \in V^r_k(F)$ and positive integer $j \leq \ell - 1$. If $|v|_{j+1,F} \lesssim h^{-1}_F |v|_{j,F}$, then

$$ |v|_{j,F} \lesssim h^{-1}_F |v|_{j-1,F}. $$
Thus we end the proof by applying the Young’s inequality to the last inequality.

□

(43)

Let

Proof.

(44)

Then we have for any positive integer \( \ell \leq m \) and integer \( k_\ell \geq \ell \) that

Thus we end the proof by applying the Young’s inequality to the last inequality.

Lemma 4.2. Let \( F \in \mathcal{F}^r(K) \) with \( r = 0, 1, \cdots, n - 2 \). Assume on each \( e \in \mathcal{F}^1(F) \), it holds for any positive integer \( \ell \leq m \) and integer \( k_\ell \geq \ell \) that

Then we have for any positive integer \( \ell \leq m \) and integer \( k_\ell \geq \ell \) that

Proof. Let \( v^b \in H^\ell(F) \) be the solution of the polyharmonic equation with nonhomogeneous Dirichlet boundary condition

By (7) we have \( \nabla^j v^b |_{\partial F} = \nabla^j v |_{\partial F} \) for \( j = 0, 1, \cdots, \ell - 1 \). It is easy to check that

On the other side, due to Lemma 3.4 and the Lipschitz isomorphism [18], there exists \( \phi \in H^\ell(F) \) such that

and

Hence

By the space interpolation theory [1, 16],

|\( v^b |_{\ell, F} \| \lesssim \| \nabla^{\ell - 1} v |_{1/2, \partial F} \).
By (7) and (43) with $w = \frac{\partial^j v}{\partial x_j^j} \in V_{k-l}^j(e)$,

$$|\nabla F^{-1} v|^2_{1,\partial F} = \sum_{e \in F^1(F)} \left| \sum_{\alpha \in A_{n-r-1} \in \mathbb{N}} \alpha! \text{sym}(v^j_{F,e} \otimes t^n_{F,e}) \frac{\partial^{j-1} v}{\partial x_i \partial x_j^j} \right|^2_{1,e} \leq h^{-2} \sum_{e \in F^1(F)} \left| \sum_{\alpha \in A_{n-r-1} \in \mathbb{N}} \alpha! \text{sym}(v^j_{F,e} \otimes t^n_{F,e}) \frac{\partial^{j-1} v}{\partial x_i \partial x_j^j} \right|^2_{1,e} \leq h^{-2} \|\nabla F^{-1} v\|^2_{0,\partial F}.$$  

Thus

$$|v|^2_{\ell,F} \lesssim h^{-1/2} \|\nabla F^{-1} v\|_{0,\partial F}. \hspace{1cm} (45)$$

Applying the multiplicative trace inequality (1), we get

$$|v|^2_{\ell,F} \lesssim h^{-1} |v|^2_{\ell-1,F} (|v|^2_{\ell-1,F} + h^{-1} |v|^2_{\ell,F}) \leq h^{-1} |v|^2_{\ell-1,F} + h^{-1} |v|^2_{\ell,F}. \hspace{1cm} (46)$$

Notice that $v - \nu \in H^1_0(F)$. It follows from the inverse inequality for polynomials (4), the fact $(-\Delta F)^{j} v = 0$ and the Poincaré-Friedrichs inequality (3) that

$$|v|_{\ell,F} \lesssim |v|^2_{\ell,F} = \langle \nabla F(v - \nu^j), \nabla F^\ell(v - \nu^j) \rangle_F = (-\Delta F)^{j} (v - \nu^j, v - \nu^j)_F \leq \|(-\Delta F)^{j} v\|_{0,F} \|v - \nu^j\|_{0,F} \leq h^{-1} \|(-\Delta F)^{j-1} v\|_{\ell-1,F} \|v - \nu^j\|_{0,F} \leq h^{-1} \|v\|_{\ell,F} \leq h^{-1} \|v - \nu^j\|_{\ell,F},$$

which means $|v|_{\ell,F} \lesssim h^{-1} |v|_{\ell,F}$. Together with (46), we obtain

$$|v|_{\ell,F} \leq |v|_{\ell,F} + |v - \nu^j|_{\ell,F} \lesssim h^{-1} |v|_{\ell-1,F} + h^{-1} |v|^2_{\ell-1,F}.$$ 

Thus

$$|v|_{\ell,F} \lesssim h^{-1} |v|_{\ell-1,F}. \hspace{1cm} \square$$

Finally we achieve (44) from Lemma 4.1.

**Lemma 4.3** (Inverse inequality). For each $F \in F^r(K)$ with $r = 0, 1, \cdots, n - 1$, it holds the inverse inequality

$$|v|_{i,F} \lesssim h^{-j} |v|_{i,F} \quad \forall \, v \in V^m_{k_i}(K), \, 0 \leq i < j \leq m. \hspace{1cm} (48)$$

**Proof.** On each $e \in F^{n-1}(K)$, since $V^j_k(e) = \mathbb{P}_{\max(k_l,2\ell-1)}(e)$ for any positive integer $\ell \leq m$ and integer $k_l \geq \ell$, applying the inverse inequality for polynomials (4),

$$|w|_{i,e} \lesssim h^{-j} |w|_{i,e} \quad \forall \, w \in V^j_k(e), \, 0 \leq i < j \leq \ell.$$

Thus (48) follows from Lemma 4.2 by applying the mathematical induction. \hspace{1cm} \square
4.2. Norm equivalences. Next we show several norm equivalences on the finite dimensional spaces $V^m_k(K)$, $\ker(Q^K_k) \cap V^m_k(K)$ and $\ker(\Pi^k) \cap V^m_k(K)$, where $\ker(Q^k_K) \cap V^m_k(K) := \{ v \in V^m_k(K) : Q^k_K v = 0 \}$.

**Lemma 4.4.** Let $F \in \mathcal{F}^r(K)$ with $r = 0, 1, \ldots, n - 1$. It holds for any positive integer $\ell \leq m$, integer $k_{\ell} \geq \ell$ and non-negative integer $j \leq \ell - 1$ that

\[
|v|_{j,F} \simeq h_F|v|_{j+1,F} + h_F^{(n-r)/2} \sum_{\delta \in \mathcal{F}^{n-r}(F)} |(\nabla^j_F v)(\delta)| \quad \forall \, v \in V^j_k(F).
\]

**Proof.** Thanks to the trace inequality (2) and the inverse inequality (48), it is sufficient to prove

\[
|v|_{j,F} \lesssim h_F|v|_{j+1,F} + h_F^{(n-r)/2} \sum_{\delta \in \mathcal{F}^{n-r}(F)} |(\nabla^j_F v)(\delta)| \quad \forall \, v \in V^j_k(F).
\]

Noting that $\nabla^j_F(Q^j_F v)$ is constant,

\[
|Q^j_F v|_{j,F} \equiv \|\nabla^j_F(Q^j_F v)\|_{0,F} \simeq h_F^{(n-r)/2} |\nabla^j_F(Q^j_F v)|
\]

\[
\leq h_F^{(n-r)/2} \left| \nabla^j_F(Q^j_F v) - \frac{1}{\#\mathcal{F}^{n-r}(F)} \sum_{\delta \in \mathcal{F}^{n-r}(F)} (\nabla^j_F v)(\delta) \right|
\]

\[
+ \frac{1}{\#\mathcal{F}^{n-r}(F)} h_F^{(n-r)/2} \sum_{\delta \in \mathcal{F}^{n-r}(F)} |(\nabla^j_F v)(\delta)|
\]

\[
\lesssim h_F^{(n-r)/2} \sum_{\delta \in \mathcal{F}^{n-r}(F)} |(\nabla^j_F(Q^j_F v) - (\nabla^j_F v))(\delta)|
\]

\[
+ h_F^{(n-r)/2} \left| \sum_{\delta \in \mathcal{F}^{n-r}(F)} (\nabla^j_F v)(\delta) \right|.
\]

Applying the trace inequality (2) and the inverse inequality (48) recursively,

\[
h_F^{(n-r)/2} \sum_{\delta \in \mathcal{F}^{n-r}(F)} |(\nabla^j_F(Q^j_F v) - v)(\delta)|
\]

\[
\lesssim h_F^{(n-r-1)/2} \sum_{e \in \mathcal{F}^{n-r-1}(F)} (\|\nabla^j_F(Q^j_F v - v)\|_{0,e} + h_e |\nabla^j_F(Q^j_F v - v)|_{1,e})
\]

\[
\lesssim h_F^{(n-r-1)/2} \sum_{e \in \mathcal{F}^{n-r-1}(F)} \|\nabla^j_F(Q^j_F v - v)\|_{0,e}
\]

\[
\lesssim \cdots \lesssim h_F^{1/2} \sum_{e \in \mathcal{F}^1(F)} \|\nabla^j_F(Q^j_F v - v)\|_{0,e} \lesssim |Q^j_F v - \nabla^j_F v|_{j,F}.
\]

Then we get from the last two inequalities that

\[
|v|_{j,F} \leq |v - Q^j_F v|_{j,F} + |Q^j_F v|_{j,F} \lesssim |v - Q^j_F v|_{j,F} + h_F^{(n-r)/2} \left| \sum_{\delta \in \mathcal{F}^{n-r}(F)} (\nabla^j_F v)(\delta) \right|,
\]

which together with (5) gives (50).
Lemma 4.5. Let \( F \in \mathcal{F}^r(K) \) with \( r = 0, 1, \cdots, n-2 \). Assume on each \( e \in \mathcal{F}^1(F) \), it holds for any positive integer \( \ell \leq m \) and integer \( k_\ell \geq \ell \) that

\[
\|w\|_{0,e}^2 \lesssim \|Q_{k_\ell-2\ell}w\|_{0,e}^2 + \sum_{\delta \in \mathcal{F}^{n-r-1}(e)} \sum_{i=0}^{\ell-1} h_e^{n-r-1+2i} |\nabla^i w(\delta)|^2
\]

(51)

\[
+ \sum_{s=1}^{n-r-2} \sum_{e' \in \mathcal{F}^r(e)} \sum_{\alpha \in A_s, |\alpha| \leq \ell-1} h_e^{s+2|\alpha|} \left\| Q_{k_\ell-2\ell+|\alpha|}^e \frac{\partial^{|\alpha|} w}{\partial \nu_{e',e'}^{\alpha}} \right\|_{0,e'}^2 \forall w \in V_{k_\ell}(e).
\]

Then we have for any positive integer \( \ell \leq m \), integer \( k_\ell \geq \ell \), non-negative integer \( j \leq \ell \) and \( v \in V_{k_\ell}(F) \) that

\[
h_F^{2j} |\Pi_{k_\ell}^F v|_{j,F}^2 \lesssim \|Q_{k_\ell-2\ell}v\|_{0,F}^2 + \sum_{\delta \in \mathcal{F}^{n-r}(F)} \sum_{i=0}^{\ell-1} h_F^{n-r+2i} |\nabla^i v(\delta)|^2
\]

(52)

\[
+ \sum_{s=1}^{n-r-1} \sum_{e' \in \mathcal{F}^s(F)} \sum_{\alpha \in A_s, |\alpha| \leq \ell-1} h_F^{s+2|\alpha|} \left\| Q_{k_\ell-2\ell+|\alpha|}^{e'} \frac{\partial^{|\alpha|} v}{\partial \nu_{e',e'}^{\alpha}} \right\|_{0,e'}^2.
\]

Proof. Due to (49), it is sufficient to prove (52) with \( j = \ell \). It follows from (6) and the inverse inequality for polynomials (4) that

\[
|\Pi_{k_\ell}^F v|_{\ell,F}^2 = (\nabla_F v, \nabla_F^\ell \Pi_{k_\ell}^F v)_F
\]

\[
= (v, (-\Delta_F)^\ell \Pi_{k_\ell}^F v)_F + \sum_{i=0}^{\ell-1} \sum_{e \in \mathcal{F}^1(F)} (\nabla_F v, \nabla_F^i (-\Delta_F)^{\ell-i-1} \frac{\partial \Pi_{k_\ell}^F v}{\partial \nu_{e,F}})_e
\]

\[
= (Q_{k_\ell-2\ell}v, (-\Delta_F)^\ell \Pi_{k_\ell}^F v)_F + \sum_{i=0}^{\ell-1} \sum_{e \in \mathcal{F}^1(F)} (\nabla_F v, \nabla_F^i (-\Delta_F)^{\ell-i-1} \frac{\partial \Pi_{k_\ell}^F v}{\partial \nu_{e,F}})_e
\]

\[
\lesssim \|Q_{k_\ell-2\ell}v\|_{0,F} \|\Pi_{k_\ell}^F v\|_{2\ell,F} + \sum_{i=0}^{\ell-1} \sum_{e \in \mathcal{F}^1(F)} \|\nabla_F v\|_{0,e} \|\nabla_F^{2\ell-i-1} \Pi_{k_\ell}^F v\|_{0,e}
\]

\[
\lesssim h_F^{-\ell} \|Q_{k_\ell-2\ell}v\|_{0,F} \|\Pi_{k_\ell}^F v\|_{\ell,F} + \sum_{i=0}^{\ell-1} \sum_{e \in \mathcal{F}^1(F)} h_F^{-\ell+i+1/2} \|\nabla_F v\|_{0,e} \|\Pi_{k_\ell}^F v\|_{\ell,F}.
\]

Hence we have

(53)

\[
h_F^{2j} |\Pi_{k_\ell}^F v|_{j,F}^2 \lesssim \|Q_{k_\ell-2\ell}v\|_{0,F}^2 + \sum_{i=0}^{\ell-1} \sum_{e \in \mathcal{F}^1(F)} h_F^{2i+1} \|\nabla_F v\|_{0,e}^2.
\]

For \( i = 0, \cdots, \ell-1 \) and each \( e \in \mathcal{F}^1(F) \), it follows from (7) and the inverse inequality (48) that

\[
h_F^{2i+1} \|\nabla_F v\|_{0,e}^2 = \|\nabla_F v\|_{0,e}^2 \left\| \sum_{\alpha \in A_{n-r-1}, |\alpha| + j = i} \frac{\tilde{d}}{j! \tilde{\alpha}!} \text{sym}(\nu_{e,F}^j \otimes \nu_{e,F}^j) \frac{\partial^j v}{\partial \nu_{e,e}^j} \right\|_{0,e}^2
\]

\[
\lesssim \sum_{\alpha \in A_{n-r-1}, |\alpha| + j = i} h_F^{2i+1} \left\| \frac{\partial^j v}{\partial \nu_{e,e}^j} \right\|_{0,e}^2 \lesssim \sum_{j=0}^{\ell} h_F^{2j+1} \left\| \frac{\partial^j v}{\partial \nu_{e,e}^j} \right\|_{0,e}^2.
\]
Noting that $\frac{\partial w}{\partial v_{F,e}}|_e \in V_{k_{\ell-1}}(e)$, we get from (51) with $w = \frac{\partial v}{\partial v_{F,e}}|_e$ that

\[ h_F^{2j + 1} \| \nabla_F v \|^2_{0,e} \lesssim \sum_{j=0}^{\ell-1} h_e^{2j+1} \left\| Q_{k_{\ell-2}+j} \frac{\partial^j v}{\partial v_{F,e}^j} \right\|^2_{0,e}. \]

(54)

Similarly as (54), by $\frac{\partial^j v}{\partial v_{F,e}^j}|_e \in V_{k_{\ell-1}}(e)$, we get from (51) with $w = \frac{\partial w}{\partial v_{F,e}}|_e$ that

\[ h_F^{2j + 1} \| \nabla_F v \|^2_{0,e} \lesssim \sum_{j=0}^{\ell-1} h_e^{2j+1} \left\| Q_{k_{\ell-2}+j} \frac{\partial^j v}{\partial v_{F,e}^j} \right\|^2_{0,e}. \]

(55)

Thus (52) with $j = \ell$ follows from (53).

Lemma 4.6. Let $F \in \mathcal{F}^r(K)$ with $r = 0, 1, \cdots, n-2$. Assume (51) holds on each $e \in \mathcal{F}^1(F)$ for any positive integer $\ell \leq m$, integer $k_{\ell} \geq \ell$. Then we have for any positive integer $\ell \leq m$, integer $k_{\ell} \geq \ell$, $j \leq \ell$ and $v \in V_k^F(F)$ that

\[ h_F^{2j} \| v \|^2_{0,F} \lesssim \left\| Q_{k_{\ell-2}+j} \frac{\partial^j v}{\partial v_{F,e}^j} \right\|^2_{0,e} \]

(56)

Proof. Let $v^0 \in H^\ell(F)$ be defined as in Lemma 4.2. By (45) and (7), we have

\[ |v^0|_{0,F} \lesssim h_F^{-1/2} \| \nabla_F v \|_{0,F} \lesssim h_F^{-1/2} \sum_{e \in \mathcal{F}^1(F)} \sum_{e' \in \mathcal{F}^1(F)} \sum_{|\alpha| = j} \left\| \frac{\partial^j v}{\partial v_{F,e}^j} \right\|_{0,e}, \]

which together with the inverse inequality (48) yields

\[ h_F^{2j} \| v^0 \|^2_{0,F} \lesssim \sum_{e \in \mathcal{F}^1(F)} \sum_{j=0}^{\ell-1} h_e^{2j+1} \left\| \frac{\partial^j v}{\partial v_{F,e}^j} \right\|^2_{0,e}. \]

(56)
Notice that $\nabla_F^j v^b_{j,F} = \nabla_F^j v_{j,F}$ for $i = 0, 1, \ldots, \ell - 1$. For $j = 1, \ldots, \ell - 1$,

$$
|v^b_{j,F} = (\nabla_F^j v^b, \nabla_F^j v^b)_F = (\nabla_F^j v^b, \nabla_F^j v^b - Q_0^F \nabla_F^j v^b)_F
= -(\Delta_F \nabla_F^{j-1} v^b, \nabla_F^{j-1} v^b - Q_0^F \nabla_F^{j-1} v^b)_F
+ \sum_{e \in F^{(1)}(F)} \left( \frac{\partial}{\partial v_{F,e}} \nabla_F^{j-1} v^b, \nabla_F^{j-1} v^b - Q_0^F \nabla_F^{j-1} v^b)_e \right)
\lesssim |v^b_{j+1,F} \nabla_F^{j-1} v^b - Q_0^F \nabla_F^{j-1} v^b|_{0,F}
+ \sum_{e \in F^{(1)}(F)} |\nabla_F^j v^b|_{0,e} \nabla_F^{j-1} v^b - Q_0^F \nabla_F^{j-1} v^b|_{0,e}.
$$

It follows from the last inequality and (5) that

$$h_F^j |v^b_{j,F} \lesssim h_F^{j+1} |v^b_{j+1,F} + h_F^{j+1/2} \nabla_F^j v^b|_{0,F} \quad \text{for } j = 1, \ldots, \ell - 1.$$  

Hence we achieve from the last inequality, (51) and (56) that

$$
(57) \quad h_F^{2j} |v^b_{j,F}|^2 \lesssim \sum_{s=1}^{n-r} \sum_{e' \in F^{(s)}(F)} \sum_{\alpha \in A_{s-r}^{(s)}} h_F^{s+2j+2|\alpha|} \left\| Q_e^e |v^b_{e,0}|^2 \right\|^2_{0,e'}
$$

for $j = 1, \ldots, \ell$. Take some $\delta \in F^{n-r}(F)$. Applying the trace inequality (2) recursively, (5) and the inverse inequality (48),

$$
|(v^b - Q_0^F v^b)(\delta)|^2 \lesssim \sum_{e \in F^{n-r-1}(F)} h_e^{-1} |v^b - Q_0^F v^b|^2_{0,e} + \sum_{e \in F^{n-r-1}(F)} h_e |v^b|^2_{1,e}
\lesssim \cdots \lesssim h_F^{r-n} |v^b - Q_0^F v^b|^2_{0,F} + \sum_{s=0}^{n-r-1} \sum_{e \in F^{(s)}(F)} h_e^{r-n+s+2} |v^b|^2_{1,e}
\lesssim h_F^{r-n+2} |v^b|^2_{1,F} + \sum_{s=1}^{n-r-1} \sum_{e \in F^{(s)}(F)} h_e^{r-n+s} |v|^2_{0,e}.
$$

This implies

$$
|Q_0^F v^b|^2 = |(Q_0^F v^b)(\delta)|^2 \lesssim |v^b(\delta)|^2 + |(v^b - Q_0^F v^b)(\delta)|^2
\lesssim |v(\delta)|^2 + h_F^{r-n+2} |v|^2_{1,F} + \sum_{s=1}^{n-r-1} \sum_{e \in F^{(s)}(F)} h_e^{r-n+s} |v|^2_{0,e},
$$

which together with (5), the trace inequality (2) and the inverse inequality (48) means

$$
|v^b|^2_{0,F} \lesssim |v^b - Q_0^F v^b|^2_{0,F} + |Q_0^F v^b|^2_{0,F} \lesssim h_F^{2} |v^b|^2_{1,F} + h_F^{n-r} |Q_0^F v^b|^2
\lesssim h_F^{n-r} |v(\delta)|^2 + h_F^{r} |v|^2_{1,F} + \sum_{s=1}^{n-r-1} \sum_{e \in F^{(s)}(F)} h_e^{n} |v|^2_{0,e}
\lesssim h_F^{n-r} |v(\delta)|^2 + h_F^{r} |v|^2_{1,F} + \sum_{e \in F^{(s)}(F)} h_e |v|^2_{0,e}.
$$
Then we drive from (57) and (51) that
\[
\|v^b\|_0,F \lesssim \sum_{s=1}^{n-r} \sum_{e \in \mathcal{E}^s(F)} \sum_{\alpha \in \mathcal{A}_r} h_\mathcal{E}^{s+2|\alpha|} \left\| Q_{k-2\ell+|\alpha|}^{F} \frac{\partial |\alpha| v}{\partial \nu_F} \right\|^2_{0,F}.
\]

On the other side, it follows from (47), the inverse inequality for polynomials (4) and the fact \((-\Delta_F)^{\ell} v^b = 0\) that
\[
|v - v^b|^2_{j,F} = ((-\Delta_F)^{\ell}(v - v^b), v - v^b)_F = (-\Delta_F)^{\ell}(v - v^b), Q_{k_i}^F(v - v^b)_F
\]
\[
\leq \|(-\Delta_F)^{\ell}(v - v^b)\|_{0,F} \|Q_{k_i}^F(v - v^b)\|_{0,F}
\]
\[
\lesssim h_\mathcal{E}^{-\ell} \|(-\Delta_F)^{\ell}(v - v^b)\|_{-\ell,F} \|Q_{k_i}^F(v)_0,F + \|v^b\|_{0,F}
\]
\[
\lesssim h_\mathcal{E}^{-\ell} \|v - v^b|_{\ell,F} \|Q_{k_i}^F(v)|_{0,F} + \|v^b\|_{0,F},
\]
which yields
\[
h_\mathcal{E}^{2\ell} \|v - v^b|^2_{j,F} \lesssim \|Q_{k_i}^F(v)|_{0,F} + \|v^b\|^2_{0,F}.
\]

Combined with the fact \(v - v^b \in H_0^1(F)\) and the Poincaré-Friedrichs inequality (3), we get
\[
h_\mathcal{E}^{2\ell} \|v^b|^2_{j,F} \lesssim h_\mathcal{E}^{2\ell} \|v - v^b|_{j,F} + h_\mathcal{E}^{2\ell} \|v^b|_{j,F} \lesssim \|Q_{k_i}^F(v)|_{0,F} + \|v^b\|^2_{0,F} + h_\mathcal{E}^{2\ell} \|v^b|_{j,F}
\]
for \(j = 0, \cdots, \ell\). By (41), \(Q_{k_i}^F v = Q_{k_i}^F\delta - 2\delta v + (I - Q_{k_i}^{k_i - 2\ell}) \Pi_{k_i}^F v, \) hence
\[
h_\mathcal{E}^{2\ell} \|v^b|_{j,F} \lesssim \|Q_{k_i}^F\delta - 2\delta v\|^2_{0,F} + \|\Pi_{k_i}^F v\|^2_{0,F} + \|v^b\|^2_{0,F} + h_\mathcal{E}^{2\ell} \|v^b|_{j,F}.
\]
Finally we conclude (55) from the last inequality, (52), (58) and (57).

\begin{lemma}
\textbf{Norm equivalence of virtual element spaces.} For any \(v \in V_k^m(K)\) and \(j = 0, \cdots, m\), we have
\[
h_\mathcal{E}^{2j} \|v|_{j,K}^2 \lesssim \|Q_{k-2m}^K v\|^2_{0,K} + \sum_{\delta \in \mathcal{F}^n(K)} \sum_{i=0}^{m-1} h_\mathcal{E}^{n+2i} \|\nabla v(\delta)\|_{i,K}^2
\]
\[
+ \sum_{r=1}^{n-1} \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in \mathcal{A}_r} \sum_{|\alpha| \leq m-1} h_\mathcal{E}^{r+2|\alpha|} \left\| Q_{k-2m+|\alpha|}^F \frac{\partial |\alpha| v}{\partial \nu_F} \right\|^2_{0,F},
\]
and
\[
\|v\|^2_{0,K} \lesssim \|Q_{k-2m}^K v\|^2_{0,K} + \sum_{\delta \in \mathcal{F}^n(K)} \sum_{i=0}^{m-1} h_\mathcal{E}^{n+2i} \|\nabla v(\delta)\|_{i,K}^2
\]
\[
+ \sum_{r=1}^{n-1} \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in \mathcal{A}_r} \sum_{|\alpha| \leq m-1} h_\mathcal{E}^{r+2|\alpha|} \left\| Q_{k-2m+|\alpha|}^F \frac{\partial |\alpha| v}{\partial \nu_F} \right\|^2_{0,F}.
\]
\end{lemma}

\textbf{Proof.} Clearly the inequality (59) holds for \(n = 1\) since \(V_k^m(K) = P_{\max(k,2m-1)}(K)\). Then we obtain (59) for general \(n\) from Lemma 4.6 and the mathematical induction.

For the norm equivalence (60), due to (59) with \(j = 0\), it is sufficient to prove
\[
\|Q_{k-2m}^K v\|^2_{0,K} + \sum_{\delta \in \mathcal{F}^n(K)} \sum_{i=0}^{m-1} h_\mathcal{E}^{n+2i} \|\nabla v(\delta)\|_{i,K}^2
\]
\[
+ \sum_{r=1}^{n-1} \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in \mathcal{A}_r} \sum_{|\alpha| \leq m-1} h_\mathcal{E}^{r+2|\alpha|} \left\| Q_{k-2m+|\alpha|}^F \frac{\partial |\alpha| v}{\partial \nu_F} \right\|^2_{0,F} \lesssim \|v\|^2_{0,K}.
\]
Applying the trace inequality \((2)\) and the inverse inequality \((48)\) recursively,
\[
\sum_{\delta \in \mathcal{F}^n(K)} \sum_{i=0}^{m-1} h_K^{n+2i} |\nabla^i v(\delta)|^2 \lesssim \sum_{e \in \mathcal{F}^{n-1}(K)} \sum_{i=0}^{m-1} h_K^{n+2i} (h_e^{-1} \|\nabla^i v\|_{0,e}^2 + h_e |\nabla^i v|_{1,e}^2)
\lesssim \sum_{e \in \mathcal{F}^{n-1}(K)} \sum_{i=0}^{m-1} h_K^{n-1+2i} |\nabla^i v|_{0,e}^2
\lesssim \cdots \lesssim \sum_{F \in \mathcal{F}^1(K)} \sum_{i=0}^{m-1} h_K^{1+2i} |\nabla^i v|_{0,F}^2
\lesssim \sum_{i=0}^{m-1} h_K^{2i} |\nabla^i v|_{0,K}^2 \lesssim \|v\|_{0,K}^2.
\]

Similarly we have
\[
\sum_{r=1}^{n-1} \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in \mathcal{A}, |\alpha| \leq m-1} h_K^{r+2|\alpha|} \left| \partial^{2|\alpha|} v \right|_{0,F}^2 \lesssim \|v\|_{0,K}^2.
\]

Thus \((61)\) follows from the last two inequalities and \(\|Q_{k-2m}^K v\|_{0,K} \leq \|v\|_{0,K}.\)

Now we present the norm equivalences of the kernel space of the local \(L^2\)-projector \(Q_k^K\) and the local \(H^m\)-projector \(\Pi_k^K\), which only involve the boundary DoFs.

**Lemma 4.8** (Norm equivalence of the kernel space of \(Q_k^K\)). For any \(v \in \ker(Q_k^K) \cap V_k^m(K)\) and \(j = 0, \cdots, m\), we have
\[
\tag{62}
\|Q_{k-2m}^K v\|_{0,K} \approx n \sum_{r=1}^{n} \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in \mathcal{A}, |\alpha| \leq m-1} h_K^{r+2|\alpha|} \left| \partial^{2|\alpha|} v \right|_{0,F}^2.
\]

**Proof.** Noting that \(Q_{k-2m}^K v = 0\), we achieve from \((60)\) that
\[
\|v\|_{0,K}^2 \approx n \sum_{r=1}^{n} \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in \mathcal{A}, |\alpha| \leq m-1} h_K^{r+2|\alpha|} \left| \partial^{2|\alpha|} v \right|_{0,F}^2.
\]

On the other side, we get from the inverse inequality \((48)\) and \((5)\) that
\[
h_K^{2j} |v|_{j,K} \approx \|v\|_{0,K} \quad \forall j = 0, \cdots, m.
\]
Therefore \((62)\) holds. 

Noting that \(\sum_{\delta \in \mathcal{F}^n(K)} (\nabla^j v)(\delta) = 0\) for any \(v \in \ker(\Pi_k^K) \cap V_k^m(K)\) and \(j = 0, 1, \cdots, m\), it holds from \((49)\) that
\[
\tag{63}
h_K^{j} |v|_{j,K} \approx h_K^{j+1} |v|_{j+1,K} \quad \forall v \in \ker(\Pi_k^K) \cap V_k^m(K), j = 0, 1, \cdots, m - 1.
\]
Lemma 4.9 (Norm equivalence of the kernel space of $\Pi^K_v$). For any $v \in \ker(\Pi^K_v) \cap V^m_k(K)$ and $j = 0, \ldots, m$, we have

$$h^2_j |v|_{j,K}^2 \geq \sum_{r=1}^{n} \sum_{F \in \mathcal{F}(K)} \sum_{\alpha \in \mathcal{A}_r, |\alpha| \leq m-1} h^2_{k+2|\alpha|} \left\| Q_{k-2m+|\alpha|}^F \frac{\partial |\alpha| v}{\partial n^F} \right\|_{0,F}^2.$$  

Proof. Due to (60) and (63), it is sufficient to prove

$$h^2_{0,m} |v|_{m,K}^2 \geq \sum_{r=1}^{n} \sum_{F \in \mathcal{F}(K)} \sum_{\alpha \in \mathcal{A}_r, |\alpha| \leq m-1} h^2_{k+2|\alpha|} \left\| Q_{k-2m+|\alpha|}^F \frac{\partial |\alpha| v}{\partial n^F} \right\|_{0,F}^2.$$  

Let $v^b \in H^m(K)$ be defined as in Lemma 4.2 with $r = 0$, $\ell = m$ and $k_\ell = k$. By (57) and (58), we have

$$h^2_{0,m} |v^b|_{m,K}^2 \geq \sum_{r=1}^{n} \sum_{F \in \mathcal{F}(K)} \sum_{\alpha \in \mathcal{A}_r, |\alpha| \leq m-1} h^2_{k+2|\alpha|} \left\| Q_{k-2m+|\alpha|}^F \frac{\partial |\alpha| v}{\partial n^F} \right\|_{0,F}^2$$

for $j = 0, \ldots, m$. On the other hand, according to Lemma 4.8 in [34], there exists $p \in \mathbb{P}_k(K)$ satisfying

$$(-\Delta)^m p = Q^K_{k-2m}((-\Delta)^m v),$$

$$|p|_{m,K} \lesssim h^K_\ell \| Q^K_{k-2m}((-\Delta)^m v) \|_{0,K} \lesssim h^K_\ell \| (-\Delta)^m v \|_{0,K} \lesssim |v|_{m,K}.$$  

Noting that $(\nabla^m p, \nabla^m v)_K = 0$, $v - v^b \in H^0_0(K)$ and $(-\Delta)^m v^b = 0$, $|v - v^b|_{m,K}^2 = (\nabla^m (v - v^b), \nabla^m (v - v^b))_K$

$$= (\nabla^m v - \nabla^m v^b, \nabla^m (v - v^b))_K = ((-\Delta)^m (v - v^b - p), (\nabla^m p, \nabla^m v^b))_K$$

$$= ((-\Delta)^m (v - v^b - p), (\nabla^m p, \nabla^m v^b))_K$$

For the first term in the right hand side of the last equation, it follows from (42) and (67) that

$$((-\Delta)^m (v - p), v)_K = ((-\Delta)^m (v - p), Q^K_{k-2m} v)_K = ((-\Delta)^m (v - p), Q^K_{k-2m} v)_K$$

$$= (Q^K_{k-2m}((-\Delta)^m (v - p)), v)_K = 0.$$  

Then we acquire from the inverse inequality for polynomials (4) and (68) that

$$|v - v^b|_{m,K}^2 \lesssim \| (-\Delta)^m (v - v^b - p) \|_{0,K} \| v^b \|_{0,K} + |p|_{m,K} \| v^b \|_{m,K}$$

$$\lesssim h^K_\ell (|v - v^b|_{m,K} + |p|_{m,K}) \| v^b \|_{0,K} + |p|_{m,K} \| v^b \|_{m,K}$$

$$\lesssim h^K_\ell (|v - v^b|_{m,K} \| v^b \|_{0,K} + |v|_{m,K} \| v^b \|_{m,K}$$

$$\lesssim h^K_\ell (|v - v^b|_{m,K} \| v^b \|_{0,K} + |v|_{m,K} (h^{-m} v^b \| v^b \|_{0,K} + |v^b|_{m,K}),$$

which gives

$$|v - v^b|_{m,K}^2 \lesssim h^{-2m} \| v^b \|_{0,K}^2 + |v|_{m,K} (h^{-m} \| v^b \|_{0,K} + |v^b|_{m,K}).$$  

Hence

$$h^2_{2m} |v|_{m,K}^2 \lesssim h^2_{2m} |v^b|_{0,K}^2 + h^2_{2m} |v^b|_{m,K}^2$$

$$\lesssim h^2_{2m} |v|_{m,K}^2 (\| v^b \|_{0,K} + h^{-m} |v^b|_{m,K}) + |v^b|_{0,K}^2 + h^2_{2m} |v^b|_{m,K}^2.$$
which means
\[ h^{2m}_K |v|^2_{m,K} \lesssim \|v\|_{0,K}^2 + h^{2m}_K |\partial^\beta v|^2_{m,K}. \]

Therefore (65) holds from (66).

5. Conforming Virtual Element Method for a Polyharmonic Equation

In this section we will adopt the constructed conforming virtual elements to discretize the following polyharmonic equation with a lower order term: find \( u \in H^m_0(\Omega) \) such that
\[
(\nabla^m u, \nabla^m v) + c(u, v) = (f, v) \quad \forall \ v \in H^m_0(\Omega),
\]
where \( f \in L^2(\Omega) \) and constant \( c \geq 0 \).

5.1. Conforming virtual element method. Let the global virtual element space

\[
V_h := \{ v_h \in H^m_0(\Omega) : v_h|_K \in V^m_K(K) \text{ for each } K \in T_h \}.
\]

To define the discrete bilinear form, we first introduce the stabilization term
\[
S_K(w, v) := \sum_{r=1}^n \sum_{F \in F(r)(K)} \sum_{a \in A_r} h^{r+2|\alpha| - 2m}_K \left( Q^{E-2m+|\alpha|} K \frac{\partial^{\alpha} w}{\partial \nu_F}, Q^{E-2m+|\alpha|} K \frac{\partial^{\alpha} v}{\partial \nu_F} \right)_F
\]
for any \( w, v \in V^m_K(K) \). When \( k = m \), the stabilization term will reduce to
\[
S_K(w, v) = \sum_{\delta \in F^n(K)} \sum_{j=0}^{m-1} h^{n+2j-2m}_K (\nabla^j w)(\delta) : (\nabla^j v)(\delta).
\]

By (64), we acquire the norm equivalence of the stabilization term
\[
S_K(v - \Pi^K_K v, v - \Pi^K_K v) \approx |v - \Pi^K_K v|^2_{m,K} \quad \forall \ v \in V^m_K(K).
\]

Next define the linear form \( a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R} \) as
\[
a_h(w_h, v_h) := \sum_{K \in T_h} a_{h,K}(w_h, v_h),
\]
where
\[
a_{h,K}(w_h, v_h) := (\nabla^m \Pi^K_K w_h, \nabla^m \Pi^K_K v_h)_K + S_K(w_h - \Pi^K_K w_h, v_h - \Pi^K_K v_h) + c(Q_h^m w_h, Q_h^m v_h).
\]

Clearly we obtain from (33) and (31) that
\[
a_{h,K}(v, q) = (\nabla^m v, \nabla^m q)_K + c(v, q)_K \quad \forall \ v \in V^m_K(K), q \in P_k(K).
\]

Lemma 5.1. We have
\[
a_{h,K}(w, v) \lesssim (|w|_{m,K} + \|w\|_{0,K})(|v|_{m,K} + \|v\|_{0,K}) \quad \forall \ w, v \in V^m_K(K),
\]
\[
a_{h}(v_h, v_h) \approx |v_h|^2_{m,K} \quad \forall \ v_h \in V_h.
\]

Proof. Let \( v \in V^m_K(K) \). By (70), we get
\[
a_{h,K}(v, v) \approx |v|^2_{m,K} + c|Q^K_K v|_{0,K}^2,
\]
which implies (72), and (73) by the Poincaré inequality. \( \square \)
With previous preparations, we propose the following conforming virtual element method for the polyharmonic equation (69): find $u_h \in V_h$ such that
\begin{equation}
    a_h(u_h, v_h) = \langle f, v_h \rangle \quad \forall \ v_h \in V_h,
\end{equation}
where $\langle f, v_h \rangle := \sum_{K \in \mathcal{T}_h} (f, Q^K v_h)_K$. Due to (73), the virtual element method (74) is well-posed.

5.2. Interpolation error estimate. To derive the error estimate of the virtual element method (74), we define an interpolation operator $I_h : H_0^m(\Omega) \rightarrow V_h$ based on the DoFs (28)-(30): for any $v \in H_0^m(\Omega)$, $I_h v \in V_h$ is determined by
\begin{equation}
    Q^K_{k-2m}(I_h v) = Q^K_{k-2m} v,
\end{equation}
\begin{equation}
    Q^F_{k-2m+|\alpha|} \frac{\partial^{(|\alpha|)}}{\partial v^\alpha_F} = \frac{1}{|F|} \sum_{K \in F} Q^K_{k-2m+|\alpha|} \frac{\partial^{(|\alpha|)}}{\partial v^\alpha_F} \quad \forall \ n \in \mathbb{R}, \ |\alpha| \leq m - 1,
\end{equation}
for each $K \in \mathcal{T}_h$, interior $F \in \mathcal{F}_h$, $r = 1, \cdots, n$, $\alpha \in A_r$, and $|\alpha| \leq m - 1$, where $\mathcal{T}_F$ is the set of all $n$-dimensional polytopes in $\mathcal{T}_h$ sharing face $F$.

Lemma 5.2. Let $s \geq m$. It holds
\begin{equation}
    \sum_{j=0}^m h^j |I_h v|_j \lesssim h^m|{v}|_m \quad \forall \ \v \in H^s(\Omega) \cap H_0^m(\Omega).
\end{equation}

Proof. Since $Q^K_{k-2m}(Q^K v - I_h v) = Q^K_{k-2m} v - Q^K_{k-2m}(I_h v) = 0$, it follows from (59) and the definition of $I_h v$ that
\begin{equation}
    \sum_{K \in \mathcal{T}_h} |Q^K v - I_h v|_{j,K}^2 \lesssim \sum_{K \in \mathcal{T}_h} \sum_{r=1}^m \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in A_r, |\alpha| \leq m - 1} h^{r+2|\alpha|-2j} \left\| Q^F_{k-2m+|\alpha|} \frac{\partial^{(|\alpha|)}}{\partial v^\alpha_F} \right\|_{0,F}^2 \lesssim \sum_{K \in \mathcal{T}_h} \sum_{r=1}^m \sum_{F \in \mathcal{F}^r(K)} \sum_{\alpha \in A_r, |\alpha| \leq m - 1} \sum_{K' \in \mathcal{T}_F} h^{r+2|\alpha|-2j} \left\| \frac{\partial^{(|\alpha|)}}{\partial v^\alpha_F} - \frac{\partial^{(|\alpha|)}}{\partial v^\alpha_{K'}} \right\|_{0,F}^2 \lesssim \sum_{K \in \mathcal{T}_h} \sum_{r=1}^m \sum_{F \in \mathcal{F}^r(K)} \sum_{i=0}^{m-1} \sum_{K' \in \mathcal{T}_F} h^{r+2i-2j} \left\| \nabla^i (Q^K v) - \nabla^i (Q^K_{K'}) \right\|_{0,F}^2.
\end{equation}

By the inverse inequality for polynomials (4), the similar argument as in [39, Lemma 3.3] and [20, Lemma 2.1], and the trace inequality (2),
\begin{equation}
    \sum_{K \in \mathcal{T}_h} |Q^K v - I_h v|_{j,K}^2 \lesssim \sum_{K \in \mathcal{T}_h} \sum_{r=1}^m \sum_{F \in \mathcal{F}^r(K)} \sum_{i=0}^{m-1} h^{r+2i-2j} \left\| \nabla^i (Q^K v) - \nabla^i (Q^K_{K'}) \right\|_{0,F}^2 \lesssim \sum_{K \in \mathcal{T}_h} \sum_{r=1}^{m-1} h^{r+2i-2j} \left\| \nabla^i v - \nabla^i (Q^K v) \right\|_{0,\partial K}^2 \lesssim \sum_{K \in \mathcal{T}_h} \sum_{i=0}^{m-1} h^{2i-2j} \left( |v - Q^K v|_{i,K}^2 + h^2 |v - Q^K v|_{i+1,K}^2 \right).
\end{equation}
Hence
\[
\sum_{K \in T_h} |Q^K v - I_h v|_{j, K}^2 \lesssim \sum_{K \in T_h} \sum_{i=0}^m h_K^{2i-2j} |v - Q^K v|_{i, K}^2.
\]

Then we achieve from the triangle inequality that
\[
|v - I_h v|_{j}^2 \lesssim \sum_{K \in T_h} \sum_{i=0}^m h_K^{2i-2j} |v - Q^K v|_{i, K}^2.
\]

Thus (75) holds from (5). \(\square\)

5.3. **Error estimate.** With the interpolation error estimate (75), we can present the a priori error estimate of the conforming virtual element method (74). Define a global operator \(\Pi_h : V_h \to L^2(\Omega)\) by \((\Pi_h v_h)|_K := \Pi^K(v_h)|_K\) for each \(K \in T_h\). For an element-wise smooth function \(v\), let the usual squared broken semi-norm
\[
|v|^2_{m, h} := \sum_{K \in T_h} |v|^2_{m, K}.
\]

**Theorem 5.3.** Let \(u \in H^s(\Omega) \cap H_0^m(\Omega)\) with \(s \geq m\) be the solution of the polyharmonic equation (69), and \(u_h \in V_h\) be the solution of the conforming virtual element method (74). Assume the mesh \(T_h\) satisfies conditions (A1) and (A2). Assume \(f \in H^m(T_h)\). Then we have
\[
|u - u_h|_{m} \lesssim h^{m(s,k+1) - m}|u|_{\min\{s,k+1\}} + \text{osc}_h(f),
\]
\[
|u - \Pi_h u_h|_{m, h} \lesssim h^{m(s,k+1) - m}|u|_{\min\{s,k+1\}} + \text{osc}_h(f),
\]
where \(\text{osc}_h^2(f) := \sum_{K \in T_h} h_K^{2m} \|f - Q^K f\|_{0,K}^2\).

**Proof.** Let \(v_h = I_h u - u_h \in H_0^m(\Omega)\) for simplicity. Thanks to (71) and (72),
\[
a_h(K, I_h u, v_h) - (\nabla^m u, \nabla^m v_h)_K - c(u, v_h)_K
= a_h(K, I_h u - Q^K u, v_h) - (\nabla^m (u - Q^K u), \nabla^m v_h)_K - c(u - Q^K u, v_h)_K
\lesssim (|I_h u - Q^K u|_{m,K} + |I_h u - Q^K u|_{0,K})(|v_h|_{m,K} + |v_h|_{0,K})
+ (|u - Q^K u|_{m,K} + |u - Q^K u|_{0,K})(|v_h|_{m,K} + |v_h|_{0,K})
\lesssim (|u - I_h u|_{m,K} + |u - I_h u|_{0,K})(|v_h|_{m,K} + |v_h|_{0,K})
+ (|u - Q^K u|_{m,K} + |u - Q^K u|_{0,K})(|v_h|_{m,K} + |v_h|_{0,K}).
\]

By (5), it holds
\[
(f, v_h) - (f, v_h) = \sum_{K \in T_h} (f, v_h - Q^K v_h)_K = \sum_{K \in T_h} (f - Q^K f, v_h - Q^K v_h)_K
\lesssim \sum_{K \in T_h} h_K^{m} \|f - Q^K f\|_{0,K} |v_h|_{m,K} \lesssim \text{osc}_h(f) |v_h|_{m}.
\]

Combining the last two inequalities, we get from (69), (75), (5) and the Poincaré inequality that
\[
a_h(I_h u, v_h) - (f, v_h) = a_h(I_h u, v_h) - (\nabla^m u, \nabla^m v_h) - c(u, v_h) + (f, v_h) - (f, v_h)
\lesssim (h^{m(s,k+1) - m}|u|_{\min\{s,k+1\}} + \text{osc}_h(f)) |v_h|_{m}.
\]
Then we acquire from (73) and (74) that
\[
\|v_h\|_m^2 \approx a_h(I_h u - u_h, v_h) = a_h(I_h u, v_h) - \langle f, v_h \rangle \lesssim (h^{\min\{s,k+1\}} - m|u|_{\min\{s,k+1\}} + \text{osc}_h(f))|v_h|_m.
\]

As a result,
\[
|I_h u - u_h|_m \lesssim h^{\min\{s,k+1\}} - m|u|_{\min\{s,k+1\}} + \text{osc}_h(f).
\]

Therefore (76) holds from the last inequality and (75).

Next we prove (77). By (33), on each \(K \in \mathcal{T}_h\) we have
\[
u - \Pi^K u_h = u - u_h + \Pi^K (Q^K u - u_h) - (Q^K u - u_h).
\]

Then it follows from (63) and the triangle inequality that
\[
|u - \Pi_h u_h|_m \lesssim |u - u_h|_m + \sum_{K \in \mathcal{T}_h} |\Pi^K (Q^K u - u_h) - (Q^K u - u_h)|_m,K \lesssim |u - u_h|_m + \sum_{K \in \mathcal{T}_h} |u - Q^K u|_m,K.
\]

Finally we arrive at (77) from (76) and (5). \(\Box\)

Under the assumption that the partition \(\mathcal{T}_h\) is quasi-uniform and \(h\) is sufficiently small, we can show that the condition number of the resulting coefficient matrix of the conforming virtual element method (74) is \(O(h^{2m})\), whose order is only related to the order of the differential operator. See also Section 3.4 in [9].

6. Numerical results

In this section, we provide two examples to numerically verify the convergence of the \(H^m\)-conforming virtual element method (74) with \(c = 1\). Let \(\Omega = (0,1) \times (0,1)\). And the rectangular domain \(\Omega\) is partitioned by the convex polygonal mesh \(\mathcal{T}_0\) and non-convex polygonal mesh \(\mathcal{T}_1\) respectively, shown in Figure 2. The numerical examples are implemented by using the FEALPy package [42].

![Convex polygon mesh \(\mathcal{T}_0\)(left) and non-convex polygon mesh \(\mathcal{T}_1\)(right).](image)

**Example 6.1.** Consider polyharmonic equation (69) with \(m = 2\). Take the exact solution \(u = \sin^2(\pi x)\sin^2(\pi y)\), and the right-hand side \(f\) is computed from polyharmonic equation (69).
Choose $k = 2, 3, 4, 5$ for the virtual element method (74). The numerical results are listed in Figure 3. We can see that $|u - \Pi_h u_h|_{2,h} = O(h^{k-1})$, which coincides with Theorem 5.3.

![Figure 3](image1.png)

**Figure 3.** Error $|u - \Pi_h u_h|_{2,h}$ of Example 6.1 with $m = 2$ on convex polygon mesh $T_0$ (left) and non-convex polygon mesh $T_1$ (right) with $k = 2, 3, 4, 5$.

**Example 6.2.** Consider polyharmonic equation (69) with $m = 3$. Take the exact solution $u = \sin^3(\pi x)\sin^3(\pi y)$, and the right-hand side $f$ is computed from polyharmonic equation (69).

In this example we set $k = 3, 4, 5, 6$, and present numerical results in Figure 4. We observe from Figure 4 that $|u - \Pi_h u_h|_{3,h} = O(h^{k-2})$, which again agrees with Theorem 5.3.

![Figure 4](image2.png)

**Figure 4.** Error $|u - \Pi_h u_h|_{3,h}$ of Example 6.2 with $m = 3$ on convex polygon mesh $T_0$ (left) and non-convex polygon mesh $T_1$ (right) with $k = 3, 4, 5, 6$.

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