Mathematical Representation of Cerebral Palsy Students in Constructing the Concept of Plane Geometry Based on APOS Theory

E D Wulandari, E Hidayanto, Subanji, and R Rahardi
Department of Mathematics Education, Postgraduate, Malang State University, East Java, 0341-551312 Fax: 0341-551921, Indonesia.
elis.dwi.1703118@students.um.ac.id, erry.hidayanto.fmipa@um.ac.id, subanji.fmipa@um.ac.id, rustanto.rahardi.fmipa@um.ac.id

Abstract. Mathematical representation is needed in conceptual understanding, solving mathematical problems, and communicating mathematical ideas by students. This research describes the form of Cerebral Palsy students’ mathematical representation which appear when solving a problem related to the concept of plane geometry. The emerging representation consists of four stages of mental construction namely action, process, object and scheme (APOS). Student work is influenced by physical and psychological conditions in which Cerebral Palsy has limitations in writing, speaking, or performing other acts in the learning process. The work result of the students studied, it appears that there are 3 types form of representation used, they are verbal representation, representation of mathematical expression, and visual representation. In verbal representations, there are complete mental constructs, namely actions, processes, objects and schemes. In the representation of mathematical expressions, and visual representations, mental construction only reaches the object stage. Stage action, the subject mentions what is known and asked from the question. Process stage, shows the ability of the subject to explain the way used in solving the plane geometry problem. Stage object, subject using verbal representation, visual as well as mathematical expression. The scheme stage, the subject associates the multiplication concept with the plane geometry area.

1. Introduction

Representation is the way used to communicate mathematical ideas and solutions to the problems encountered. The term representation refers to processing and generation – in other words, for the action of capturing mathematical concepts or relationships in some form and for the form itself [1].

1.1. Mathematical Representation

Mathematical representation is needed in the understanding of concepts, solving mathematical problems and communicating mathematical ideas by students. In communicating mathematical ideas, it takes external representations in the form of action (contextual), verbal, symbolic, visual, and the real object (physical) [1]. External representation serves to: (1) inform teachers about how students think about mathematical content or ideas, (2) provide information about student patterns and trends, and (3) tools in the learning process. Student representation is not only useful for describing and communicating mathematical objects but also for working with mathematics, for example solving the problems in
mathematics and with mathematics. When students construct knowledge, it is important if they can represent it in order to know how much knowledge they have. According to [2], the success of students in solving the problems can not be separated from the role of representation. It was also stated by [3] that mathematical representation is the key to success in understanding mathematics concept and problem solving. The mathematical representation consists of symbolic representation and verbal representation [4].

By using mathematical symbols students can perform summary calculations and facilitate students in solving the problems. The forms of verbal representation correspond to what students understand based on known information. Verbal and written representations, can help teachers understand how their students learn and use the language of mathematics [5]. Related research in mathematics, among others done by [2], [5], [4], and [6].

Visual representation in mathematics is a highly recommended practice in education [6]. [2] also explains some type of visual representation that students use to solve mathematical problems. Visual representation often helps in understanding the problem. Using visual representation leads to better understanding and enhances specific mathematical reasoning [7].

Each student has different representations in solving mathematical problems. In general schools, knowing the interpretation of the students' thinking in terms of how the student represents and solve the given mathematical problems, can support the selection of strategies or methods used in the learning. If appropriate in choosing a strategy or method then the learning process will run smoothly and effectively so that the learning objectives can be achieved. It shows that it is necessary to know how the mathematical representation of students in solving mathematical problems. But not everyone know the interpretation results of students with special needs in solving mathematical problems.

### 1.2. Cerebral Palsy (CP)

One of the students with special needs is a Cerebral Palsy student. Cerebral Palsy is a disorder caused by brain damage resulting in abnormalities in motion and coordination, psychological and cognitive functions that affect the teaching and learning process [8]. Cerebral Palsy (CP) is a generic term for a group of chronic diseases that specifically affect the control of body movement and its segments. CP is a major cause of disability in children and is a group of complex and heterogeneous conditions with varying etiologies and types and has a severity of damage. Abnormal development or damage to the motor area in the brain after hypoxia destroys the brain's ability to control and coordinate movement and posture. Body balance is a complex human function to realize the position of the body and its segment and control it. The child's body can affect how the child builds knowledge [9].

Limitations possessed by Cerebral Palsy students enable the influence in representing the students' thought. Limitations are like the ability to move, think, etc. In learning, it is also important to note how Cerebral Palsy students can learn and develop their ability to move. Developing movable abilities in children with Cerebral Palsy can be through special materials, special tools that help to move, special methods made by physiotherapist, occupational therapist, recreational therapist, and special teacher [8]. Specific mathematical materials that can provide information on how student representation one of which is plane geometry. Knowing the mathematical representation in the concept of mathematical conception and pay attention to how the ability to move owned, is necessary for the achievement of learning objectives. According to [10], an understanding of a mathematical concept is the result of construction or reconstruction of mathematical objects.

### 1.3. APOS

Construction or reconstruction is done through the activities of actions, processes, and objects of mathematics organized in a scheme to solve mathematical problems. To know the understanding of mathematical concept through mathematical representation, is necessary basic theory or the appropriate method.

Ongoing achievement or learning of a mathematical concept and principle, which can be used as an elaboration of mental construction, there are actions, processes, objects, and schemes, or APOS theory.
abbreviated. APOS theory illustrates a construction of individual mathematical knowledge. One's mathematical knowledge is the tendency to respond to situations of perceived mathematical problems by reflecting in the social context and by constructing actions, processes and mathematical objects and arranging them in schemes for use in handling situations [10]. The concept of action can be a concept of process through mental construction called meaning. Then, students can think about the outcome of the process without actually doing it and in particular, can imagine reversing the process. A student who has the concept of an object of mathematical ideas can imagine that as totality and in particular, can act on it with a higher level, action or process. The process can be packaged into useful objects so students can encapsulate objects to focus on the underlying process. Schemes coordinate actions, processes, objects, and other schemes that are packaged into objects by themselves.

The APOS theory arose from attempts to understand the mechanism of the reflective abstraction introduced by Piaget to illustrate the development of logical thinkers in children, and to expand this idea to improve mathematical concepts that have been studied. In research [11] used the theory of APOS to analyze basic steps in constructing the principle of mathematical induction. Another study by [12] on the application of APOS theory related to the approach of the circle to construct the sine and consinus functions as well as its inverse. The research supported by [13] who examines the neighboring representation of geometry in the learning function of two variables also describe it based on APOS theory.

2. Research purposes
Based on the above explanation, it takes research on "How Mathematical Representation of Cerebral Palsy Students in Constructing the Concept of Plane geometry is reviewed from APOS Theory" to be useful as a consideration in determining the learning strategies that should be done by educators. So this study aims to describe the form of mathematical representation Cerebral Palsy (CP) students who appear when finishing the problem related to the concept of plane geometry (constructing the concept of plane geometry).

3. Method

3.1. Research design
This study is descriptive qualitative research. This study illustrates how the representation used by students in constructing a plane geometry. According to [14] qualitative research is a research procedure that produces descriptive data in the form of words written or spoken of the observed behavior.

3.2. Research subject
The study was conducted on three subjects: Cerebral Palsy students in 2 schools. One subject is a student of SMALBN Banyuwangi, and two others are students of SLB YPAC Malang. Subject selection is based on many students at the school with high ability at the CP level. The three subjects in the study were given S1, S2 and S3 codes to facilitate the analysis process.

3.3. Research instruments
Instruments used in this study in the form of questions on the Student Worksheet which contains questions related a plane geometry. The questions are answered by the students where the answers are the data to be analyzed to trace how the process of representation conducted by students in terms of the theory of APOS. The results of the student's work will be reinforced by interviews conducted after the students have worked on the given Student Worksheet. The representation indicators used are as presented in Table 1 below:
Table 1. Indicators of representation

| No | Component Representation | Indicators                                                                                                                                 |
|----|---------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|
| 1  | Verbal representation     | • Students use writing or greeting in pouring what they perceive in regard to widespread nonstandard issues on a flat stand.                |
|    |                           | • Students write information that is known in a language they understand.                                                                     |
|    |                           | • Students plan and perform calculations using both written and spoken sentences. (Adopted from [4])                                         |
| 2  | Representation of          | • Students are able to identify problems well so that students can create symbols to assist in solving problems. (Adopted from [4])          |
|    | mathematical expressions   |                                                                                                                                              |
| 3  | Visual representation     | • Students can describe, pour problems that have been identified by creating a form or model as a tool to present answers and help solve problems. (Adopted from [3]) |

4. Results and discussion

Based on student test results and interview results, overall there is an external representation produced by the three subjects (S1, S2 and S3). Some forms of mathematical representation that are representations of students' visual thoughts such as verbal, images, models, numerical, algebraic symbols, tables, charts and graphs are an integral part of mathematical learning [15]. In each type of representation generated in this study, there is a mental construction occurring in solving the geometric problem (the raw material size is not standard). Mental construction according to [10], i.e. action, process, object, and scheme or called APOS. Here is the exposure of research results based on the analysis of researchers in accordance with the types of representation found.

4.1. Verbal Representation

Verbal representation is used by all three subjects. The subjects use verbal representation in the form of words and words directly. Once analyzed by the researchers, on the verbal representation found, there is a mental construction of action, process, object and scheme.

The S1 subject indicates that he uses the description of the sentence to convey the results of his representation. The result of representation of S1 subject in the form of writing and direct greeting on some answers which according to him needs an explanation. The results of representations in the form of writing indicate that the construction of the object which begins with stage of action and process. The results of S1 work as shown in Figure 1.

![Figure 1. The results of S1 work](image-url)
looks at the pictures and counts the small squares present in the mass of each image. As in the exposure of the subject S1 and with the researcher (P), which is as follows.

**P:** "What did you do first after got the question sheet?"
**S1:** "I saw the picture on the questionnaire."

**P:** "Do you understand the problem when first asked about?"
**S1:** "Yes, but hesitate, keep trying to mother what it really means like this."

**P:** "After understanding the meaning of the matter, then what is known from this matter?"
**S1:** "What's known are the pictures that each have small squares."

**P:** "Then what is asked of this question?"
**S1:** "Asked to find the contents of the image asked that how many times the image is known."

At the stage of the process, the subject explains how to answer the given question. Students complete by imagining what is commanded and linking some understanding that influences the resulting conditions. Students are aware of the many small boxes that compile each picture. Students count small squares in order to get numbers on each image. Then the students link between the numbers obtained by multiplication. This is in accordance with the explanation of the exposure of S1 subject, i.e.

**P:** "How can you do this?"
**S1:** "After I see it, I just calculate the small box on each picture."

**P:** "Then what do you think?"
**S1:** "Yes it turns out that the little squares in each shape are a lot different. The shapes of the drawings are similar."

**P:** "Then what is the method do you use to get that answer?"
**S1:** "I use multiplication."

**P:** "How can that be?"
**S1:** "... that's what's being asked how many times, keep it if the little squares in picture V, if compiled, it becomes another picture."

In the representation of writing, there is a stage object, where the subject can identify the purpose of the problem, and successfully identify the images presented. After identifying the problem, the student counts many small squares on each picture according to what he thinks. Student marks the drawing, to count many small squares on each presented image.

Here is a scheme that occurs on the subject S1 in constructing the plane geometry concept.

![Figure 2](image)

**Figure 2**

| A1: knowing the pictures presented in the question, where each picture is composed of small boxes |
| A2: has the task of finding how many squares on each picture according to the question |
| P1: calculate the small boxes contained in each image |
| P2: compile images that are known and associate with the number of boxes in other images (linking with multiplication / multiples) |
| Q1: The W image takes 3 times the V image |
| Q2: Figure X is 2 times V image |
| Q3: Figure Y takes 8 times the V image |
| Q4: Figure Y takes 4 times the X image |
S1: The area of $W$ is three times the area of $V$, $V$ image there are 2 boxes. So that area $Y$ is $3 \times 2 = 6$ boxes

S2: The area of $X$ is two times the area of $V$, $V$ image there are 2 boxes. So that area $X$ is $2 \times 2 = 4$ boxes

S3: The area of $Y$ is eight times the area of $V$, $V$ image there are 2 boxes. So that area $Y$ is $8 \times 2 = 16$ boxes

S4: The area of $Y$ is 4 times the area of $X$, $X$ image there are 4 boxes. So that area $Y$ is $4 \times 4 = 16$ boxes

The blue area is the action area, green is the process area, red is the object area and yellow is the scheme area. Overall, in Figure 2 are complete mental construction stages that occur in verbal representations by S1.

The results of the interviews presented below show that the student experiences a scheme stage in which he is able to relate what he has learned to the widespread flat matter, the following results:

P: "What did you get after doing all this questions?"

S1: "This was learning about plane geometry and look for an area of plane geometry."

P: "Try explaining what that means? What is the relationship between each picture?"

S1: "This one, the extent of $Y$ is 8 times the area of $V$, the $V$ image is 2 boxes, it means that the area of $Y$ is two times eight equals 16 ", (pointing at the picture)

Verbal representation is also used by S2 subject and S3 subject. The images on Figure 3 and Figure 4 presented each subject.

Figure 3. Work Results of S2 Subject

Figure 4. Work Results of S3 Subject
Verbal representation can also be used in conjunction with the representation of mathematical expressions. Mathematical expression in the form of writing, mathematical symbols, the terms used in representing the subject's understanding of the not standard size of plane geometry area.

4.2. Mathematical Expression Representation
The subject of S2 is representative of two other subjects who use the representation of mathematical expressions. After being analyzed by the researcher, on the result of representation by the S2 subject, it was found that there was stage of action, process, object and scheme. This is supported by the statement spoken by the subject S2 directly as Figure 5.

4.3. Visual Representation
Visual representations can be symbols, drawings, diagrams, models, charts, or using physical members. The S1 subject is representative of 2 other subjects who use visual representation to represent their understanding of the plane geometry problems. In the research result obtained the understanding of subject S1 on the matter describes the shape of the rectangle to the action, process and object. In accordance with the study by [3] that although the visual representations shown are inaccurate for any reason, the most important thing is that students have made the image as the subject of its representation.

Figure 5. Work Results of S2 Subject
The student writes a number next to the picture showing many small squares present on each image. It can be said that the process has become an object. It shows that the object stage also occurs in the representation of mathematical expression. As has been disclosed by subject S1 as follows

P : “What is the initial steps do you make after understanding the problem?”
S2 : “I count the number of small squares. Then I write beside each picture.”
P : “Then what method do you use to get that answer?”
S2 : “I use multiplication method.”
At the stage of action, the subject mentions what is known and asked from the questions given. As in the exposure of the subject S1, which is as follows.

\[ P: \text{“What did you do first after got the question sheet?”} \]
\[ S1: \text{“I saw the picture on the questionnaire.”} \]
\[ P: \text{“Do you understand when it was first asked?”} \]
\[ S1: \text{“At first not, but after I ask, then I understand the meaning of it.”} \]

This is consistent with [11] who mentions that a student is said to have taken action if he acted upon receiving the matter and without understanding the intent of the problem.

At the stage of the process, the subject explains how to answer the given question. Students complete by imagining what is commanded and linking some understanding that influences the resulting conditions. This is in accordance with the explanation of the exposure of subject S1, i.e.

\[ P: \text{“In question number 4, how can you draw like that?”} \]
\[ S1: \text{“Because the requested is drawing a shape the three times of the Q image, so I draw a square three times.”} \]

At the stage of the object, the subject S1 has not been able to identify the rectangular shape requested from the rectangular features. This means that students can only identify some intentions about the problem. After identifying the problem, then the student describes from what is he thinks. This is in accordance with [10] opinion that if a process can be transformed by an action, it is said the process has become an object. The following interview results as a result of student visual representation

\[ P: \text{“What does this picture mean?”} \]
\[ S1: \text{“This is the first square Q image, then this second Q square image, and this third Q image.”} \]

(Pointing to the S1 subject work ie Figure 6)

This is in accordance with research [16] which states that the image representation will be followed by a verbal representation or mathematical expression to support the created image.

5. Conclusion

From the results of this study, it is found that there are three representations used by Cerebral Palsy students in constructing the concept of a plane geometry, namely verbal representation, mathematical expression representation, and visual representation. In verbal representations, there are complete
mental constructs, namely actions, processes, objects and schemes or called APOS. In the representation of mathematical expressions, and visual representations, mental constructions only arrive at the stage of the object. At the action stage, the subject mentions what is known and asked about the questions that have been given, their representations are verbal, visual and symbolic (mathematical expressions). At the process stage, representations in the form of written and oral verbal that is the ability of the subject in explaining the method used in solving the plane geometry problem. Stage object, subject using verbal representation, visual as well as mathematical expression. The scheme stage is a verbal representation in which the subject relates the concept of multiplication to the area of a plane geometry. This shows that the mathematical representation that appears in constructing a flat building is different and shows the occurrence of four stages of mental construction (APOS).

Cerebral Palsy students have difficulty in describing the plane geometry. By content, the resulting image does not conform to the question command, i.e. it does not meet the characteristics of the intended wake-up. This shows that there is a need for follow-up to handle visual representation condition of CP students. The follow-up can be used as an idea for further research.

Overall, Cerebral Palsy students are able to represent what they understand regarding the concept of plane geometry. Create mathematical symbols to help solve problems, create drawings or shapes as plane geometry representatives in question, and explain with writing and direct oral of what they have in mind.

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