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COMPRESSED DRINFELD ASSOCIATORS

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Abstract. Drinfeld associator is a key tool in computing the Kontsevich integral of knots. A Drinfeld associator is a series in two non-commuting variables, satisfying highly complicated algebraic equations — hexagon and pentagon. The logarithm of a Drinfeld associator lives in the Lie algebra $L$ generated by the symbols $a, b, c$ modulo $[a, b] = [b, c] = [c, a]$. The main result is a description of compressed associators that satisfy the compressed pentagon and hexagon in the quotient $L/[[L, L], [L, L]]$. The key ingredient is an explicit form of Campbell-Baker-Hausdorff formula in the case when all commutators commute.

1. Introduction

1.1. Motivation and the previous results. Let $A$ be a quasi-Hopf algebra with a non-commutative non-associative coproduct $\Delta$. Roughly speaking, an associator is an element $\Phi \in A^\otimes 3$ controlling non-coassociativity of the coproduct $\Delta$. Another element $R \in A^\otimes 2$ measures non-cocommutativity of $\Delta$. For the representations of $A$ to form a tensor category, $R$ and $\Phi$ have to obey the so-called ”pentagon” and ”hexagon” equations. V. Drinfeld found a ”universal” formula $(R_{KZ}, \Phi_{KZ})$ by using analytic methods — differential equations and iterated integrals. Also Drinfeld proved that there is an iterative algebraic procedure for finding a universal formula for an associator over the rationals. Although this procedure is constructive, it does not give a close explicit formula.

The main motivation is the construction of the Kontsevich integral of knots via associators, investigated by T. Q. T. Le, J. Murakami, and D. Bar-Natan. Another combinatorial constructions of the universal Vassiliev invariant are in [19, 7]. Recall that the Kontsevich integral takes values in the algebra $A$ of chord diagrams. The LM-BN construction gives an isotopy invariant of parenthesized framed tangles expressed via a Drinfeld associator that is a solution of rather complicated equations — hexagon and pentagon (the same as mentioned above). Any solution of these equations gives rise to a knot invariant. Le and Murakami have proved that the resulting invariant is independent of a particularly chosen associator and coincides with the Kontsevich integral from [13].

In other words, if one would know all coefficients of at least one associator, then one can calculate the whole Kontsevich integral for any knot. Another approach of Bar-Natan, Le, and D. Thurston has led to a formula for the Kontsevich integral of the unknot and all torus knots in the space of Jacobi diagrams [3].

One of non-even associators was expressed via multiple zeta values [14], i.e. via transcendental numbers. Drinfeld computed the logarithm of the same associator in the case when all commutators commute by using classical zeta values [10]. This result was the starting point of the present researches. Bar-Natan calculated one of even rational Drinfeld associators up to degree 7 in [3]. J. Lieberum determined explicitly a rational even associator in a completion of the universal enveloping algebra of the Lie superalgebra $gl(1|1)^{\otimes 3}$. Up to now a close formula of a rational associator is still unknown [18, p. 433, Problem 3.13].

Extreme coefficients of all Drinfeld associators will be calculated in Theorem 1.5c below. It turns out that they are rational and expressed via classical Bernoulli numbers $B_n$.

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1.2. Basic definitions.

Definition 1.1 (associative algebra $A_n$, algebra of chord diagrams $A(X)$).
(a) For each $n \geq 2$, let the associative algebra $A_n$ over the field $\mathbb{C}$ be generated by the symbols $t^{ij} = t^{ji}$ with $1 \leq i \neq j \leq n$ and the relations
$$[t^{ij}, t^{kl}] = 0 \text{ if } i, j, k, l \text{ are pairwise disjoint}, \quad [t^{ij}, t^{jk} + t^{ki}] = 0 \text{ if } i, j, k \text{ are pairwise disjoint},$$
where the bracket $[,] : A_n \oplus A_n \to A_n$ is defined by $[a, b] := ab - ba$. Observe that the relations $[t^{ij}, t^{jk} + t^{ki}] = 0$ of $A_n$ are equivalent to
$$[t^{ij}, t^{jk}] = [t^{jk}, t^{ki}] = [t^{ki}, t^{ij}]$$
for all pairwise disjoint $i, j, k \in \{1, \ldots, n\}$.

The associative algebra $A_n$ is graded by the degree defined by $\deg(t^{ij}) = 1$.

(b) Let us define the same object $A(X)$.

Definition 1.2 (Lie algebra $L_n$, quotient $\hat{L}_n$, long commutators $[a_1 \ldots a_k]$).
(a) The Lie algebra $L_n$ is generated by the same generators and relations as the associative algebra $A_n$ of Definition 1.1. The Lie algebra $L_n$ is graded with respect to $\deg(t^{ij}) = 1$.

(b) By $[L_n, L_n]$ denote the Lie subalgebra of $L_n$, generated by all commutators $[a, b]$ with $a, b \in L_n$. Introduce the compressed quotient $\hat{L}_n = L_n[[L_n, L_n], [L_n, L_n]]$. Let $\hat{A}_n, \hat{L}_n$, and $\bar{L}_n$ be the algebras of formal series of elements from $A_n$, $L_n$, and $L_n$, respectively.

(c) For elements $a_1, \ldots, a_n$ of a Lie algebra $L$, set $[a_1 a_2 \ldots a_k] = \{a_1, [a_2, \ldots, a_k] \ldots\}$. For example, the algebras $\hat{A}_2$ and $\bar{L}_3$ contain the series $\exp(t^{12})$ and $\sum_{k=1}^{\infty} (t^{12})^k t^{23}$, respectively.

Definition 1.3 (operators $\Delta_k$ and $\varepsilon_k$, Drinfeld associators and compressed associators).
(a) Let $t^{ij}$ be the generators of $A_n$. Let $\Delta_k : A_n \to A_{n+1}$ for $0 \leq k \leq n+1$ and $\varepsilon_k : A_n \to A_{n-1}$ for $1 \leq k \leq n$ be the algebra morphisms defined by their action on $t^{ij}$ (here $i < j$):

$$\Delta_k(t^{ij}) = \begin{cases}
  t^{ij}, & \text{if } i < j < k, \\
  t^{i,j+1}, & \text{if } i < k < j, \\
  t^{i+1,j+1}, & \text{if } k < i < j, \\
  t^{ij} + t^{i,j+1}, & \text{if } i < j = k, \\
  t^{i,j+1} + t^{i+1,j+1}, & \text{if } i = k < j;
\end{cases}$$

$$\varepsilon_k(t^{ij}) = \begin{cases}
  t^{ij}, & \text{if } i < j < k, \\
  t^{ij-1}, & \text{if } i < k < j, \\
  t^{i-1,j-1}, & \text{if } k < i < j, \\
  0, & \text{if } i = j = k, \\
  0, & \text{if } i = k < j.
\end{cases}$$

The dotted arcs represent parts of the diagrams that are not shown in the figure. These parts are assumed to be the same in all four diagrams.

If $X = X_n$ is the disjoint union of $n$ oriented segments (strands), then $A(X_n)$ can be equipped with a natural product. If in the definition of $A(X_n)$ one allows only horizontal chords with endpoints on $n$ vertical strands, then the resulting algebra $A^{\text{hor}}(X_n)$ is isomorphic to the algebra $A_n$. Indeed, thinking of $t^{ij}$ as a horizontal chord connecting the $i$th and $j$th vertical strands, the relations between the $t^{ij}$ become the 4T relations:

$$[t^{12}, t^{23}] = [t^{12}, t^{23}] = [t^{23}, t^{13}] = [t^{12}, t^{23}]$$
\[ \Delta_0 (\Delta_{n+1}) \text{ acts by adding a strand on the left (right), } \Delta_k \text{ for } 1 \leq k \leq n \text{ acts by doubling the } \text{ith strand and summing up all the possible ways of lifting the chords that were connected to the } \text{ith strand to the two daughter strands. The operator } \varepsilon_k \text{ acts by deleting the } \text{ith strand and mapping the chord diagram to 0, if any chord in it was connected to the } \text{ith strand.} \]

(b) A horizontal Drinfeld associator (briefly, a Drinfeld associator) is an element \( \Phi \in \hat{A}_3 \) satisfying the following equations (here set \( \Phi^{ijk} := \Phi(t^{ij}, t^{jk}) \) and \( \Phi := \Phi^{123} \))

(symmetry) \[ \Phi \cdot \Phi^{321} = 1 \text{ in } \hat{A}_3, \]

(hexagon) \[ \Delta_1(\exp(t^{12})) = \Phi^{312} \cdot \exp(t^{13}) \cdot (\Phi^{-1})^{132} \cdot \exp(t^{23}) \cdot \Phi^{123} \text{ in } \hat{A}_3, \]

(pentagon) \[ \Delta_0(\Phi) \cdot \Delta_2(\Phi) \cdot \Delta_4(\Phi) = \Delta_3(\Phi) \cdot \Delta_1(\Phi) \text{ in } \hat{A}_4, \]

(non-degeneracy) \[ \varepsilon_1 \Phi = \varepsilon_2 \Phi = \varepsilon_3 \Phi = 1 \text{ in } \hat{A}_2, \]

(group-like) \[ \Phi = \exp(\varphi) \text{ in } \hat{A}_3 \text{ for some element } \varphi \in \hat{L}_3. \]

A geometric interpretation of the hexagon and pentagon is shown below:

\[ \exp(t^{12}) \leftrightarrow \]

\[ \Phi^{123} \leftrightarrow \]

(c) If an associator \( \Phi \in \hat{A}_3 \) vanishes in all odd degrees, then \( \Phi \) is said to be \textit{even}. Note that the symmetry (1.3a) implies \( \varphi(a, b) = -\varphi(b, a) \) in \( \hat{L}_3 \). By taking the logarithms of (1.3b) and (1.3c) and projecting them under \( \hat{L}_3 \rightarrow \hat{L}_3 \) and \( \hat{L}_4 \rightarrow \hat{L}_4 \) one gets \textit{the compressed hexagon (1.3b) and pentagon (1.3c)}, respectively. A \textit{compressed associator} \( \bar{\varphi} \in \hat{L}_3 \) is a solution of (1.3b) and (1.3c), satisfying \( \bar{\varphi}(a, b) = -\bar{\varphi}(b, a) \) and \( \bar{\varphi}(a, 0) = \bar{\varphi}(0, b) = 0. \)

Definition 1.3b uses a non-classical normalization. Drinfeld considered the two hexagons

\[ [9]: \Delta_1 \left( \exp \left( \pm \frac{t^{12}}{2} \right) \right) = \Phi^{312} \cdot \exp \left( \pm \frac{t^{13}}{2} \right) \cdot (\Phi^{-1})^{132} \cdot \exp \left( \pm \frac{t^{23}}{2} \right) \cdot \Phi^{123}. \]

To avoid huge denominators in future the change of the variables \( t^{ij} \mapsto 2t^{ij} \) was made. Moreover, Bar-Natan has proved that both above hexagons are equivalent to the positive hexagon (with the sign “+”) and the symmetry (1.3a), see [3] Proposition 3.7. The logarithm \( \varphi = \log(\Phi) \) of any Drinfeld associator projects under \( \hat{L}_3 \rightarrow \hat{L}_3 \) onto a compressed associator.

**Example 1.4.** Bar-Natan calculated\(^1\) one of even Drinfeld associators up to degree 7

\[ \varphi^B(a, b) = \left( \frac{[ab]}{12} - \frac{8[a^2b] + [abab]}{720} + \frac{96[a^5b] + 4[a^3bab] + 65[a^2b^2ab] + 68[aba^3b] + 4[(ab)^3]}{90720} \right) - \]

-(interchange of \(a \leftrightarrow b\), where \(a = t^{12}, b = t^{23}\). Degree 7 is the maximal achievement of Bar-Natan’s computer programme. Then in \( \hat{L}_3 \) one gets:

\[ \bar{\varphi}^B(a, b) = \frac{[ab]}{6} - \frac{4[a^3b] + [abab] + 4[b^2ab]}{360} + \frac{[a^5b] + [b^4ab]}{945} + \frac{[a^3bab] + [ab^3ab]}{1260} + \frac{23[a^2b^2ab]}{30240} + \cdots \]

\(^1\)By the above normalization one needs to divide the denominators at all terms of the degree \(n\) by \(2^n\).

**1.3. Main results.** For a series \(f(\lambda, \mu)\), let us introduce its \textit{even} and \textit{odd} parts:

\[ \text{Even}(f(\lambda, \mu)) = \frac{f(\lambda, \mu) + f(-\lambda, -\mu)}{2}, \quad \text{Odd}(f(\lambda, \mu)) = \frac{f(\lambda, \mu) - f(-\lambda, -\mu)}{2}. \]

A simple reformulation of Theorem 1.5 will be given in Corollary 1.6c.
Theorem 1.5. (a) Any compressed Drinfeld associator \( \varphi \in \hat{L}_3 \) from Definition 1.3c is

\[
\varphi(a, b) = \sum_{k,l \geq 0} \alpha_{k,l}[a^k b^l ab], \quad a = i^{12}, \ b = i^{23}, \ \alpha_{k,l} \in \mathbb{C}.
\]

Moreover, for all \( k, l \geq 1 \), the coefficients \( \alpha_{k,l} \) are symmetric \( \alpha_{k,l} = \alpha_{l,k} \) and could be expressed linearly in terms of \( \alpha_{ij} \) with \( i + j = k + l \).

(b) Let \( f(\lambda, \mu) = \sum_{k,l \geq 0} \alpha_{k,l} \lambda^k \mu^l \) be the generating function of the coefficients \( \alpha_{k,l} = \alpha_{l,k} \). Then the compressed hexagon \( (1.3b) \) from Definition 1.3c is equivalent to the equation

\[
(1.5b) \quad f(\lambda, \mu) + e^\mu f(\mu, -\lambda - \mu) + e^{-\lambda} f(\lambda, -\lambda - \mu) = \frac{1}{\lambda + \mu} \left( \frac{e^\mu - 1}{\mu} + \frac{e^{-\lambda} - 1}{\lambda} \right).
\]

Moreover, the compressed pentagon \( (1.3c) \) for \( \varphi \in \hat{L}_3 \) follows from the symmetry \( \alpha_{k,l} = \alpha_{l,k} \).

(c) The general solution of \( (1.5b) \) is \( f(\lambda, \mu) = \text{Even}(f(\lambda, \mu)) + \text{Odd}(f(\lambda, \mu)) \), where

\[
(1.5c) \quad \left\{ \begin{array}{l}
1 + \lambda \mu \cdot \text{Even}(f(\lambda, \mu)) = \frac{e^{\lambda + \mu} - e^{-\lambda - \mu}}{2(\lambda + \mu)} \left( \frac{2\omega}{e^\omega - e^{-\omega}} + \sum_{n=3}^{\infty} h_n(\lambda, \mu) \right) , \\
\text{Odd}(f(\lambda, \mu)) = \frac{e^{\lambda + \mu} - e^{-\lambda - \mu}}{2} \left( \sum_{n=0}^{\infty} \tilde{h}_n(\lambda, \lambda, \mu) + \sum_{n=3}^{\infty} h_n(\lambda, \mu) \right) ,
\end{array} \right.
\]

\[
h_n(\lambda, \mu) = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \beta_{nk} \lambda^k \mu^{2k} (\lambda + \mu)^{2k} \omega^{2n-6k} \quad \text{for} \quad n \geq 3, \quad \beta_{nk} \in \mathbb{C}, \quad \omega = \sqrt{\lambda^2 + \lambda \mu + \mu^2}.
\]

In particular, any honest Drinfeld associator has the extreme coefficients

\[
\alpha_{2k,0} = \frac{2^{2k+1} B_{2k+2}}{(2k + 2)!}
\]

for every \( k \geq 0 \), where \( B_n \) are Bernoulli numbers. The polynomials \( h_n(\lambda, \mu) \) are defined by the same formula as \( h_n(\lambda, \mu) \), except the coefficients \( \tilde{h}_n \in \mathbb{C} \) are substituted for \( \beta_{nk} \). The coefficients \( \beta_{n0} \) (for \( n \geq 0 \)), \( \beta_{nk} \), and \( \tilde{\beta}_{nk} \) are free parameters for \( 1 \leq k \leq \lfloor \frac{n}{3} \rfloor, \ n \geq 3 \).

By Theorem 1.5c the differences of all compressed associators form a linear space generated by \( \beta_{nk}, \tilde{\beta}_{nk} \). The projection of any Drinfeld associator is in \( (1.5c) \), see Problem 6.10a.

Corollary 1.6. (a,b) There are two distinguished even compressed Drinfeld associators

\[
(1.6a) \text{ the first series: } \quad 1 + \lambda \mu f^I(\lambda, \mu) = \frac{e^{\lambda + \mu} - e^{-\lambda - \mu}}{2(\lambda + \mu)} \cdot \frac{2\omega}{e^\omega - e^{-\omega}}, \quad \omega = \sqrt{\lambda^2 + \lambda \mu + \mu^2};
\]

\[
(1.6b) \text{ the second series: } \quad 1 + 2\lambda \mu f^{II}(\lambda, \mu) = \frac{e^{\lambda + \mu} - e^{-\lambda - \mu}}{2(\lambda + \mu)} \left( \frac{2\lambda}{e^\lambda - e^{-\lambda}} + \frac{2\mu}{e^\mu - e^{-\mu}} - 1 \right).
\]

(c) Any compressed Drinfeld associator \( \varphi \in \hat{L}_3 \) can be defined by the Drinfeld series

\[
1 + \lambda \mu f^D(\lambda, \mu) = \exp \left( \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} \cdot \frac{\lambda^n + \mu^n - (\lambda + \mu)^n}{(\pi \sqrt{-1})^n} \right),
\]

where after rewriting the above exponent as a series in \( \lambda, \mu \) one substitutes a free parameter for each monomial consisting of odd zeta numbers like \( \zeta(3) \zeta(5) \zeta(7) \cdots \) (see Definition 6.1).
In particular, one gets the third compressed associator

\[(1.6c) \text{ the third series: } 1 + \lambda \mu f^{III}(\lambda, \mu) = \exp \left( \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{4n(2n)!} \left( (\lambda + \mu)^{2n} - \lambda^{2n} - \mu^{2n} \right) \right). \]

1.4. Scheme of proofs. Key points of proofs are listed below.

First key point: a behavior of the Bernoulli numbers \((B_n = 0 \text{ for each odd } n \geq 3)\).

Second key point: the Bernoulli numbers \(B_n\) can be extended in a natural way, this extension gives a compressed variant of CBH formula (Definition 2.4 and Proposition 2.8).

Third key point: properties of the extended Bernoulli numbers \(C_{mn}\) and their generating function \(C(\lambda, \mu)\): a non-trivial symmetry \(C(\lambda, \mu) = C(-\mu, -\lambda)\) (Lemma 2.10) and an explicit expression of \(C(\lambda, \mu)\) (Proposition 2.12).

4th key point: the original hexagon equation \((1.3b)\) can be simplified in such a way that it remains to apply CBH formula in an essential way exactly once (Lemma 3.1).

5th key point: the quotient \(\bar{L}_3 = L_3/[[L_3, L_3], [L_3, L_3]]\), where a compressed associator lives, is isomorphic to a Lie algebra with a small basis of commutators (Proposition 3.4).

6th key point: the compressed hexagon equation \((1.3b)\) is equivalent to a recursive linear system for the coefficients \(\alpha_{kl}\) (Proposition 3.9 and Lemma 4.1).

7th key point: the extreme coefficients \(\alpha_{2k,0}\) of the exact logarithm of any Drinfeld associator (not only the compressed one) are expressed via Bernoulli numbers \(B_{2n}\) (Lemma 4.2).

8th key point: for any compressed associator, the compressed hexagon \((1.3b)\) can be split into two equations for the even and odd parts of this associator (Lemma 4.5).

9th key point: up to certain factor the general solution of the compressed hexagon \((1.3b)\) is a series \(h(\lambda, \mu)\) with the symmetry \(h(\lambda, \mu) = h(\lambda, -\lambda - \mu)\) (Lemmas 4.6 and 4.12).

10th key point: non-uniqueness of compressed associators is closely related with non-uniqueness of associator polynomials (Definition 4.7 and Lemma 4.9).

11th key point: all associator polynomials can be described explicitly: in each degree \(2n\) the family of all associator polynomials depends on \([\frac{n}{2}]\) free parameters (Proposition 4.10).

12th key point: for any compressed associator, the compressed pentagon equation \((1.3c)\) follows from the symmetry condition \(\alpha_{kl} = \alpha_{lk}\) (Proposition 5.10).

All the above results are the positive ones. But there is also the negative one.

13th key point: the Drinfeld series (a compressed associator expressed via zeta values) does not lead to non-trivial polynomial relations between odd zeta values (Proposition 6.9).

The 13th key point means that odd zeta values are too complicated numbers.

The paper is organized as follows. In Section 2 the extended Bernoulli numbers \(C_{mn}\) are introduced, one deduces a compressed variant of CBH formula in the case when all commutators commute with each other. In Sections 3 and 4 the compressed hexagon is written explicitly and solved. Section 5 is devoted to checking the compressed pentagon. In Section 6 one explains, why the Drinfeld series does not lead to polynomial relations between odd zeta values. Theorem 1.5a is proved at the end of Subsection 5.3. The hexagon and pentagon parts of Theorem 1.5b are checked in Subsections 4.1 and 5.3, respectively. Theorem 1.5c, Corollaries 1.6a-b and 1.6c are verified in Subsections 4.3 and 6.2, respectively. In Subsection 6.3 open problems and suggestions for future researches are formulated. Moreover, Appendix contains a lot of explicit formulae discussed in the paper in their general forms.

The following diagram shows the scheme for the proof of Theorem 1.5. The most important steps are called propositions, they are of independent interest, especially Propositions 2.8 and 2.12 together. Lemmas are of less importance. Claims are technical assertions.
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2. Campbell-Baker-Hausdorff formula (CBH)

This section is devoted to an explicit form of CBH formula in the case when all commutators commute with each other, see Propositions 2.8 and 2.12.

2.1. Classical recursive CBH formula. Recall a classical recursive CBH formula (Theorem 2.3) originally proved by J. Campbell [6], H. Baker [1], and F. Hausdorff [11].

Definition 2.1 (Hausdorff series $H$, Bernoulli numbers $B_n$, derivative $D = H_1 \frac{\partial}{\partial Q}$).

(a) Let $L$ be the free Lie algebra generated by the symbols $P, Q$. By $\hat{L}$ denote the algebra of formal series of elements from $L$. The Hausdorff series is $H = \log(\exp(P) \cdot \exp(Q)) \in \hat{L}$.

(b) The classical Bernoulli numbers $B_n$ are defined by the exponential generating function: $\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}$. One can verify that $B_0 = 1$, $B_1 = -\frac{1}{2}$, and that $\frac{x}{e^x - 1} + \frac{x}{2}$ is an even function, which shows that $B_n = 0$ for odd $n \geq 3$. The first key point: the function $\frac{x}{e^x - 1}$ vanishes in almost all odd degrees, while $\frac{e^x - 1}{x}$ does not. The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = \frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, $B_7 = 0$.

(c) A derivative of a Lie algebra $L$ is a linear function $D : \hat{L} \to \hat{L}$ such that $D[x,y] = [Dx,y] + [x,Dy]$ for all $x, y \in \hat{L}$. For an element $H_1 \in \hat{L}$, denote by $D = H_1 \frac{\partial}{\partial Q}$ the derivative of $L$, which maps $P$ onto 0 and $Q$ onto $H_1$.

The first key point implies recursive formulae for the Bernoulli numbers $B_n$.

Lemma 2.2. For each $m \geq 1$, the Bernoulli numbers $B_n$ satisfy the relations

(a) $\sum_{n=1}^{m} \binom{m+1}{n} B_n = -1$,  
(b) $\sum_{k=1}^{\lceil \frac{m}{2} \rceil} \binom{m+1}{2k} B_{2k} = \frac{m-1}{2}$,  
(c) $\sum_{n=1}^{m} (-1)^n \binom{m+1}{n} B_n = m$. 


In particular, the first four relations from the item (a) are
\[2B_1 = -1, \quad 3B_1 + 3B_2 = -1, \quad 4B_1 + 6B_2 + 4B_3 = -1, \quad 5B_1 + 10B_2 + 10B_3 + 5B_4 = -1.\]

Proof. (a) One obtains:
\[1 = \frac{e^x - 1}{x} \cdot \frac{x}{e^x - 1} = \left(\sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{B_n x^n}{n!}\right) = 1 + \sum_{m=1}^{\infty} \left(\frac{1}{(m+1)!} + \sum_{k+n=m+1}^{\infty} \frac{1}{k!} \cdot \frac{B_n}{n!}\right) x^m,
\]
i.e.
\[\sum_{n=1}^{m} \frac{1}{(m-n+1)!} \cdot \frac{B_n}{n!} = -\frac{1}{(m+1)!} \text{ as required.}
\]
(b) Since \(B_{2k+1} = 0\) for every \(k \geq 1\), then
\[-1 = \sum_{n=1}^{\infty} \binom{m+1}{n} B_n = (m+1)B_1 + \sum_{k=1}^{\infty} \binom{m+1}{2k} B_{2k},
\]
hence
\[\sum_{k=1}^{\infty} \binom{m+1}{2k} B_{2k} = -1 + \frac{1}{2}(m+1) = \frac{m-1}{2}.
\]
(c) The required formula is equivalent to the item (b), since \(B_{2k+1} = 0\) for each \(k \geq 1\). \(\square\)

The following theorem is quoted from [20 Corollaries 3.24–3.25, p. 77–79].

**Theorem 2.3.** [6, 11] The Hausdorff series \(H = \log(\exp(P) \cdot \exp(Q))\) is \(H = \sum_{m=0}^{\infty} H_m,\)
\[H_0 = Q, \quad H_1 = P - \frac{1}{2}[Q,P] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}[Q^{2n}P], \quad H_m = \frac{1}{m} \left(H_1 \frac{\partial}{\partial Q}\right)^m (H_{m-1}) \text{ for } m \geq 2. \quad \square
\]

2.2. **Extended Bernoulli numbers** \(C_{mn}\) and compressed variant of CBH formula.

*The compressed variant* is the case when all commutators commute. To get an explicit form of CBH formula in this setting, one needs extended Bernoulli numbers.

**Definition 2.4** (extended Bernoulli numbers \(C_{mn}\), generating function \(C(\lambda, \mu)\)).

(a) Introduce the extended Bernoulli numbers \(C_{mn}\) in terms of the classical ones:

\[C_{1n} = B_n, \quad C_{m+1,n} = \frac{n}{n+1} C_{m,n+1} - \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} B_k C_{m,n-k+1} \text{ for } m,n \geq 1.
\]

The numbers \(C_{mn}\) are calculated in Table A.2 of Appendix for \(m+n \leq 12\).

(b) Let us introduce the generating function
\[C(\lambda, \mu) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{m!n!} \lambda^{n-1} \mu^{m-1}. \quad (2.4b) \quad \blacktriangle
\]

The formula (2.4a) does not look very naturally. But there is a more natural definition of \(C_{mn}\) equivalent to (2.4a), see Proposition 2.8.

**Example 2.5.** The first few values of the extended Bernoulli numbers are
\[C_{21} = -\frac{1}{6}, \quad C_{22} = \frac{1}{6}, \quad C_{23} = -\frac{1}{15}, \quad C_{31} = 0, \quad C_{32} = \frac{1}{15}, \quad C_{41} = \frac{1}{30}.
\]
Then the generating function \(C(\lambda, \mu)\) starts with
\[C(\lambda, \mu) = -\frac{1}{2} + \frac{1}{12} (\lambda - \mu) + \frac{1}{24} \lambda \mu + \frac{1}{720} (\mu^3 + 4\lambda \mu^2 - 4\lambda^2 \mu - \lambda^3) + \cdots
\]
Up to degree 10 the function \(C(\lambda, \mu)\) is computed in Example A.3 of Appendix. \blacktriangle
If \( W \) is a word in \( P, Q \), then the expression \([W]\) in a long commutator is regarded as a formal symbol, i.e. \([PQ][W] := [P, [Q, [W]]]\). But the symbol \( W \) in a long commutator is considered as the word in \( P, Q \), for \( W = Q^n P^m \) one gets \([WQP] := [Q^n P^m QP]\).

**Claim 2.6.** Let \( P, Q \) be two elements of a Lie algebra \( L \), \( W \) be a word in the letters \( P, Q \).

(a) In the quotient \( \hat{L} = L/[(L, L), [L, L]] \), for any word \( W \) containing at least one letter \( P \) and at least one letter \( Q \), one has \([PQ][W] = [QP][W]\).

(b) In the quotient \( \hat{L} \), for any word \( W \) containing exactly \( m \) letters \( P \) and exactly \( n \) letters \( Q \), one has \([WQP] = [Q^n P^m QP]\).

**Proof.** (a) Since the element \([W]\) contains at least one commutator, then \([PQ][W], [W] = 0 \) in the quotient \( \hat{L} \). The following calculations in \( \hat{L} \) imply (a): \([PQ][W] - [QP][W] = \]

\[ = [P, Q[W] - W][Q] - [Q, P[W] - W][P] = (PQ[W] - P[W]Q - Q[W]P + [W]QP) - (QP[W] - Q[W]P - P[W][Q + W][PQ]) = (PQ - QP)[W] - [W]Q(2Q - PQP) = [[P, Q], [W]] = 0. \]

(b) By the item (a) one can permute the letters of \( W \), i.e. one may assume \( W = Q^n P^m \). \( \square \)

Let \( L \) be the Lie algebra freely generated by the symbols \( P, Q \). Recall that the series \( H_1 \in \hat{L} \) was introduced in Theorem 2.3. Put \( \hat{L} = L/[(L, L), [L, L]] \). As usual by \( \hat{L} \) denote the algebra of formal series of elements from \( L \).

**Claim 2.7.** In the algebra \( \hat{L} \), for the derivative \( D = H_1 \frac{\partial}{\partial Q} \) and all \( m, n \geq 1 \), one has

\[
(2.7a) \quad [H_1, P] = - \sum \frac{B_k}{k!} [PQ^k P];
\]

\[
(2.7b) \quad D[Q^n P] = (n - 1)[Q^{n-2} PQP] - \sum \frac{B_k}{k!} [Q^{n-1} PQ^k P];
\]

\[
(2.7c) \quad D[Q^{n-1} P^{m-1} QP] = (n - 1)[Q^{n-2} P^{m} QP] - \sum \frac{B_k}{k!} [Q^{k+n-2} P^{m} QP].
\]

**Proof.** (a) It suffices to rewrite the formula of \( H_1 \) from Theorem 2.3 as follows:

\[
H_1 \overset{(2.3)}{=} P + \sum \frac{B_k}{k!} [Q^n P] \quad \Rightarrow \quad [H_1, P] = - \left[ P, \sum \frac{B_k}{k!} [Q^n P] \right] = - \sum \frac{B_k}{k!} [PQ^k P].
\]

Observe that here the first key point \((B_n = 0 \text{ for odd } n \geq 3)\) was used.

(b) Induction on \( n \). The base \( n = 1 \) follows from (a): \( D[QP] = \overset{(2.1c)}{=} [DQ, P] + [Q, DP] = [H_1, P] \overset{(2.7a)}{=} - \sum \frac{B_k}{k!} [PQ^k P]. \) Induction step (from \( n \) to \( n + 1 \)): \( D[Q^{n+1} P] = \overset{(2.1c)}{=} \)

\[
\overset{(2.1c)}{=} [H_1, [Q^n P]] + [Q, D[Q^n P]] = [PQ^n P] + (n - 1)[Q^{n-1} PQP] - \left[ Q, \sum \frac{B_k}{k!} [Q^{n-1} PQ^k P] \right].
\]

It remains to apply Claim 2.6a: \( [PQ^n P] + (n - 1)[Q^{n-1} PQP] = n[Q^n PQP] \). The item (b) and \( D[Q^{n-1} P^{m-1} QP] = \overset{(2.6a)}{=} D[P^{m-1} Q^n P] = \overset{(2.1c)}{=} [P, [\ldots, D[Q^n P] \ldots]] \) imply (c). \( \square \)

The following result gives a natural definition of the extended Bernoulli numbers \( C_{mn} \): they give rise to an explicit compressed CBH formula (the second key point).
Proposition 2.8 (compressed variant of CBH). Let $L$ be the Lie algebra freely generated by the symbols $P, Q$. Under the natural projection $\hat{L} \to \tilde{L}$, where $\tilde{L} = L/[[L, L], [L, L]]$, the Hausdorff series $H = \log(\exp(P) \cdot \exp(Q))$ maps onto the series

$$\bar{H} = P + Q + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{m!n!} [Q^{n-1}P^{m-1}Q].$$

Proof. By Theorem 2.3 the series $H$ maps onto the series $\bar{H} = \sum_{m=0}^{\infty} \bar{H}_m$, where

$$\bar{H}_0 = Q, \quad \bar{H}_1 = H_1 = P + \sum_{n=1}^{\infty} \frac{B_n}{n!} [Q^n P], \quad \bar{H}_{m+1} = \frac{1}{m!} D^m(H_1) \text{ for } m \geq 1 \text{ and } D = H_1 \frac{\partial}{\partial Q}.$$ 

It remains to prove the following formula:

$$D^m(H_1) = \sum_{n=1}^{\infty} \frac{C_{m+1,n}}{n!} [Q^{n-1}P^m Q] \text{ for each } m \geq 1.$$ 

The base $m = 1$ is completely analogous to the inductive step (from $m - 1$ to $m$):

$$D^m(H_1) = D \left( \sum_{n=1}^{\infty} \frac{C_{mn}}{n!} [Q^{n-1}P^{m-1}Q] \right) = \sum_{n=1}^{\infty} \frac{C_{mn}}{n!} D[Q^{n-1}P^{m-1}Q] = \sum_{n=1}^{\infty} \frac{C_{mn}}{n!} \left( (n-1)[Q^{n-2}P^m Q] - \sum_{k=1}^{\infty} \frac{B_k}{k!} [Q^{k+n-2}P^m Q] \right) = \sum_{n=0}^{\infty} \frac{C_{m,n+1}}{(n+1)!} [Q^{n-1}P^{m}Q] - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C_{mn} B_k}{n! k!} [Q^{k+n-2}P^m Q] = \sum_{n=1}^{\infty} \left( \frac{C_{m,n+1}}{(n+1)!} - \sum_{k=1}^{n} \frac{C_{m,n-k+1} B_k}{(n-k+1)! k!} \right) [Q^{n-1}P^m Q] = \sum_{n=1}^{\infty} \frac{C_{m+1,n}}{n!} [Q^{n-1}P^m Q]. \quad \square$$

The series $\bar{H}$ will be calculated up to degree 10 in Proposition A.4 of Appendix.

2.3. Properties of the extended Bernoulli numbers and $C(\lambda, \mu)$.

Claim 2.9. Under the natural projection $\hat{L} \to \tilde{L}$, where $\tilde{L} = L/[[L, L], [L, L]]$, the Hausdorff series $H = \log(\exp(P) \cdot \exp(Q))$ maps onto the series

$$\bar{H} = P + Q + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C'_{mn}}{m!n!} [P^{n-1}Q^{m-1}PQ], \text{ where } C'_{11} = \frac{1}{2}. \quad C'_{1n} = B_n,$$

$$C'_{m+1,n} = \frac{n}{n+1} C'_{m,n+1} - \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} C'_{1k} C'_{m,n-k+1} \text{ for } m \geq 1, \ n \geq 2.$$ 

Proof is completely analogous to the proof of Proposition 2.8. One can use the following analog of Theorem 2.3 [20] the remark after Corollary 3.25 on p. 80:

the Hausdorff series $H = \log(\exp(P) \cdot \exp(Q))$ is equal to $H = \sum_{m=0}^{\infty} H'_m$, where

$$H'_0 = P, \quad H'_1 = Q + \frac{1}{2} [P, Q] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} [P^{2n} Q], \quad H'_m = \frac{1}{m} \left( H'_1 \frac{\partial}{\partial P} \right) (H'_{m-1}) \text{ for } m \geq 2.$$

Note that $B_1 = -\frac{1}{2}$ but one sets $C'_{11} = \frac{1}{2}$. 
Lemma 2.10. The extended Bernoulli numbers are symmetric in the following sense:

\[ C_{mn} = (-1)^{m+n}C_{nm} \text{ for all } m, n \geq 1. \]

Hence the generating function \( C(\lambda, \mu) \) obeys the symmetry \( C(\lambda, \mu) = C(-\mu, -\lambda) \).

Proof. Let us rewrite the recursive formula (2.9b) in a more explicit form:

\[ C'_{m+1,n} = \frac{n}{n+1} C'_{m,n+1} - \frac{1}{2} C_{mn} \quad \text{and} \quad -\frac{1}{n+1} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{n+1}{2k} \right) B_{2k} C_{m,n-2k+1}. \]

In the same form the formula (2.4a) looks like

\[ C_{m+1,n} = \frac{n}{n+1} C_{m,n+1} + \frac{1}{2} C_{mn} - \frac{1}{n+1} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{n+1}{2k} \right) B_{2k} C_{m,n-2k+1}. \]

If the latter equation is multiplied by \((-1)^{m+n}\), then one obtains

\[ (-1)^{m+n} C_{m+1,n} = \frac{n}{n+1} (-1)^{m+n} C_{m,n+1} - \frac{1}{2} (-1)^{m+n-1} C_{mn} - \frac{1}{n+1} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{n+1}{2k} \right) B_{2k} (-1)^{m+n-2k} C_{m,n-2k+1}. \]

Hence the numbers \( C'_{mn} \) and \((-1)^{m+n-1} C_{mn}\) obey the same recursive relation. Since \( C'_{1n} = (-1)^n B_n = (-1)^n C_{1n} \) (the first key point) for each \( n \geq 1 \), then \( C'_{mn} = (-1)^{m+n-1} C_{mn} \) for all \( m, n \geq 1 \). Then the formula (2.9a) converts to

\[ H - P - Q = \sum_{m,n \geq 1} (-1)^{m+n-1} \frac{C_{mn}}{m!n!} [P^{n-1} Q^{m-1} P Q] = \sum_{m,n \geq 1} (-1)^{m+n} \frac{C_{mn}}{m!n!} [Q^{m-1} P^{n-1} Q P]. \]

By comparing the above formula with (2.8), one gets \( C_{mn} = (-1)^{m+n} C_{nm} \) as required. \( \Box \)

The extended Bernoulli numbers show that the extended Bernoulli numbers \( C_{mn} \) are not too complicated, contrary they can be expressed via binoms and classical Bernoulli numbers.

Lemma 2.11. The extended Bernoulli numbers can be expressed via the classical ones:

\[ C_{mn} = \sum_{k=0}^{m-1} \binom{m}{k} B_{n+k} \text{ for each } m \geq 1. \]

In particular, one gets
\[ C_{1n} = B_n, \quad C_{2n} = B_n + 2B_{n+1}, \quad C_{3n} = B_n + 3B_{n+1} + 3B_{n+2}, \quad C_{4n} = B_n + 4B_{n+1} + 6B_{n+2} + 4B_{n+3}. \]

**Proof.** Multiplying (2.4a) by \( \frac{n+1}{n} \), one has \( \frac{n+1}{n} C_{m+1,n} = C_{m,n+1} - \frac{1}{n} \sum_{k=1}^{n} \binom{n+1}{k} B_k C_{m,n-k+1} \), i.e.

\[ (2.4a') \quad C_{m,n+1} = \frac{n+1}{n} C_{m+1,n} + \frac{1}{n} \sum_{k=1}^{n} \binom{n+1}{k} B_k C_{m,n-k+1} \text{ for all } m, n \geq 1. \]

The equation (2.11) will be checked by induction on \( m \). The base \( m = 1 \) follows from Definition 2.4a. Suppose that the formula (2.11) holds for \( m \), let us prove it for \( m + 1 \).

\[ C_{m+1,n} \overset{(2.10)}{=} (-1)^m n+1 C_{m+1,n+1} \overset{(2.4a')}{=} \frac{(-1)^{m+n+1}}{m} \left( (m+1)C_{n+1,m+1} + \sum_{k=1}^{m} \binom{m+1}{k} B_k C_{n,m-k+1} \right) \]

(by hypothesis) \[ = \frac{m+1}{m} \sum_{k=0}^{m-1} \binom{m}{k} B_{n+k+1} + \frac{1}{m} \sum_{k=1}^{m} (-1)^k \binom{m+1}{k} B_k \sum_{l=0}^{m-k} \binom{m-k+1}{l} B_{n+l} \]

\[ = \frac{m+1}{m} \sum_{l=1}^{m} \binom{m}{l-1} B_{n+l} + \frac{1}{m} \sum_{k=1}^{m} (-1)^k \binom{m+1}{k} B_k B_{n+l} + \frac{1}{m} \sum_{k=1}^{m} \binom{m+1}{k} B_k \sum_{l=1}^{m-k} \binom{m-k+1}{l} B_{n+l} \]

\[ B_n \sum_{k=1}^{m} (-1)^k \binom{m+1}{k} B_k + \sum_{l=1}^{m} \frac{B_{n+l}}{m} \left( \frac{m+1}{l+1} \right) + \sum_{l=1}^{m} \frac{(-1)^k \binom{m+1}{k} B_k \left( \frac{m-k+1}{l} \right) B_{n+l}}{m} \right). \]

Since by Lemma 2.2c the first term is equal to \( B_n \), then it remains to check the formula

\[ \frac{m+1}{m} \left( \frac{m}{l-1} \right) + \frac{1}{m} \sum_{k=1}^{m-l} (-1)^k \binom{m+1}{k} B_k \left( \frac{m-k+1}{l} \right) = \left( \frac{m+1}{l} \right) \text{ for all } m \geq l \geq 1. \]

The left hand side of the above equation is

\[ \frac{(m+1) \cdot (m-1)!}{(l-1)!(m-l+1)!} + \sum_{k=1}^{m-l} (-1)^k B_k \cdot \frac{(m+1) \cdot (m-1)!}{k!(m-k+1)!} \cdot \frac{(m-k+1)!}{l!(m-k-l+1)!} = \]

\[ = \frac{(m+1) \cdot (m-1)!}{(l-1)!(m-l+1)!} + \frac{(m+1) \cdot (m-1)!}{l!(m-l+1)!} \cdot \sum_{k=1}^{m-l} (-1)^k B_k \left( \frac{m-l+1}{k} \right) \overset{(2.2c)}{=} \]

\[ \overset{(2.2c)}{=} \frac{(m+1) \cdot (m-1)!}{(l-1)!(m-l+1)!} \left( 1 + \frac{m-l}{l} \right) = \frac{(m+1)!}{l!(m-l+1)!} = \left( \frac{m+1}{l} \right) \] as required. \( \square \)

Now one can get an explicit expression of \( C(\lambda, \mu) \), which is not followed immediately from Definition 2.4 or Proposition 2.8 (the third key point).

**Proposition 2.12.** The generating function \( C(\lambda, \mu) \) from Definition 2.4b is equal to

\[ C(\lambda, \mu) = \frac{e^\mu - 1}{\lambda \mu} \cdot \frac{\lambda + \mu}{e^{\lambda+\mu} - 1 - \frac{\mu}{e^\mu - 1}}. \]
Proof. The next trick is applied: to get a non-trivial relation, one destroys a symmetry.

\[
\lambda \mu C(\lambda, \mu) = \sum_{m,n \geq 1} \frac{C_{mn}}{m!n!} \lambda^m \mu^n = (\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{k=0}^{m-1} \binom{m}{k} B_{n+k} \right) \frac{\lambda^n}{n!} \cdot \frac{\mu^m}{m!}) = 
\]

\[
\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \sum_{m=k+1}^{\infty} \frac{B_{n+k} \lambda^m \mu^m}{n!(m-k)!k!} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{n+k} \frac{\lambda^m \mu^k}{n!m!k!} = \left( \sum_{m=1}^{\infty} \frac{\mu^m}{m!} \cdot \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} B_{n+k} \frac{\lambda^m \mu^k}{n!k!} \right) = (e^\mu - 1) \cdot \left( \sum_{k,n \geq 1} B_n \frac{\lambda^n}{n!} + \sum_{k,n \geq 1} B_{n+k} \frac{\lambda^n \mu^k}{n!k!} \right) \overset{(2.10)}{=} \left( e^\mu - 1 \right) \cdot \left( \frac{\lambda}{e^\lambda - 1} - 1 + \tilde{C}(\lambda, \mu) \right),
\]

where

\[
\tilde{C}(\lambda, \mu) = \sum_{k,n \geq 1} B_{n+k} \frac{\lambda^n \mu^k}{n!k!}. Since (-1)^{n+k} B_{n+k} = B_{n+k} for n, k \geq 1 (the first key point),
\]

then \( \tilde{C}(-\mu, -\lambda) = \sum_{k,n \geq 1} (-1)^k B_{n+k} \frac{\lambda^k \mu^n}{k!n!} = \sum_{k,n \geq 1} B_{n+k} \frac{\lambda^k \mu^n}{k!n!} = \tilde{C}(\lambda, \mu). \]

One obtains

\[
(e^\mu - 1) \cdot \left( \frac{\lambda}{e^\lambda - 1} - 1 + \tilde{C}(\lambda, \mu) \right) = \lambda \mu C(\lambda, \mu) \overset{(2.10)}{=} \lambda \mu C(-\mu, -\lambda) =
\]

\[
= (e^{-\lambda} - 1) \cdot \left( \frac{-\mu}{e^{-\mu} - 1} - 1 + \tilde{C}(-\mu, -\lambda) \right) = (e^{-\lambda} - 1) \cdot \left( \frac{-\mu}{e^{-\mu} - 1} - 1 + \tilde{C}(\lambda, \mu) \right),
\]

hence

\[
(e^{-\lambda} - e^\mu) \tilde{C}(\lambda, \mu) = (e^\mu - 1) \left( \frac{\lambda}{e^\lambda - 1} - 1 \right) - (e^{-\lambda} - 1) \left( \frac{-\mu}{e^{-\mu} - 1} - 1 \right) = \lambda \frac{e^\mu - 1}{e^\lambda - 1} + \mu \frac{e^{-\lambda} - 1}{e^{-\mu} - 1} + e^{-\lambda} - e^\mu.
\]

By substituting the above formula into the expression of \( \lambda \mu C(\lambda, \mu) \) via \( \tilde{C}(\lambda, \mu) \), one gets

\[
C(\lambda, \mu) = \frac{e^\mu - 1}{\lambda \mu} \cdot \left( \frac{\lambda}{e^\lambda - 1} + \left( \frac{\mu}{e^\mu - 1} \right. \right. + \left. \left. \mu \frac{e^{-\lambda} - 1}{e^{-\mu} - 1} \right) \cdot (e^{-\lambda} - e^\mu) \right).
\]

Note that after dividing by \( \lambda + \mu \) both series \( \frac{e^\mu - 1}{e^\lambda - 1} + \frac{e^{-\lambda} - 1}{e^{-\mu} - 1} \) and \( e^{-\lambda} - e^\mu \) begin with 1. It remains to prove the following equation:

\[
\lambda \frac{e^\mu - 1}{e^\lambda - 1} + \mu \frac{e^{-\lambda} - 1}{e^{-\mu} - 1} = (e^{-\lambda} - e^\mu) \cdot \left( \frac{\lambda + \mu}{e^\lambda + \mu - 1} - \frac{\lambda}{e^\lambda - 1} - \frac{\mu}{e^\mu - 1} \right),
\]

or

\[
\lambda e^{-\lambda} + \mu e^\mu \frac{1 - e^{-\lambda}}{1 - e^\mu} = (1 - e^{\lambda+\mu}) \cdot \left( \frac{\lambda + \mu}{e^\lambda + \mu - 1} - \frac{\lambda}{e^\lambda - 1} - \frac{\mu}{e^\mu - 1} \right),
\]

or

\[
(e^\mu - 1) \frac{\lambda e^\lambda}{e^\lambda - 1} + (e^\lambda - 1) \frac{\mu e^\mu}{e^\mu - 1} = -\lambda - \mu - (1 - e^{\lambda+\mu}) \cdot \left( \frac{\lambda}{e^\lambda - 1} + \frac{\mu}{e^\mu - 1} \right),
\]

or

\[
(e^\mu - 1) \left( \frac{\lambda}{e^\lambda - 1} + \lambda \right) + (e^\lambda - 1) \left( \frac{\mu}{e^\mu - 1} + \mu \right) = -\lambda - \mu + (e^{\lambda+\mu} - 1) \cdot \left( \frac{\lambda}{e^\lambda - 1} + \frac{\mu}{e^\mu - 1} \right),
\]

or

\[
\lambda e^\mu + \mu e^\lambda = (e^{\lambda+\mu} - e^\mu) \frac{\lambda}{e^\lambda - 1} + (e^{\lambda+\mu} - e^\lambda) \frac{\mu}{e^\mu - 1},
\]

that is clear. □

3. Compressed hexagon equation

In this section one shall write explicitly the compressed hexagon and prove Theorem 1.5a.
3.1. First simplification of the hexagon. Lemma 3.1 gives a first simplification of the hexagon (1.3b). Due to this 4th key point it remains to apply CBH formula exactly once.

**Lemma 3.1.** In $\hat{A}_3$ the hexagon equation (1.3b) is equivalent to the following equation:

\[
\exp(a + b + c) = \exp(\psi(c, a)) \cdot \exp(\psi(b, c)) \cdot \exp(\psi(a, b)),
\]

where $\psi(a, b) := \log(\exp(\varphi(a, b)) \cdot \exp(a)) \in \hat{L}_3$, $a := t^{12}$, $b := t^{23}$, $c := t^{13}$.

**Proof.** Let us rewrite the original hexagon (1.3b) in a more explicit form:

\[
\exp(t^{13} + t^{23}) = \exp(\varphi(t^{13}, t^{23})) \cdot \exp(t^{13}) \cdot \exp(-\varphi(t^{13}, t^{23})) \cdot \exp(t^{23}) \cdot \exp(\varphi(t^{12}, t^{23})).
\]

Introduce the symbols $a = t^{12}$, $b = t^{23}$, $c = t^{13}$ and apply the symmetry $\varphi(b, c) = -\varphi(c, b)$.

\[
\exp(b + c) = \exp(\varphi(c, a)) \cdot (\exp(c) \cdot \exp(\varphi(b, c)) \cdot \exp(b) \cdot \exp(\varphi(a, b))).
\]

It remains to multiply the above equation by $\exp(a)$ from the right. Note the element $a$ commutes with $b + c$ in $L_3$ by definition. Moreover, $a + b + c$ is a central element of $L_3$. □

Recall $\hat{L}_3$ is the algebra of formal series of elements from $\hat{L}_3 = L_3/[[L_3, L_3], [L_3, L_3]]$.

**Claim 3.2.** In $\hat{L}_3$, for the compressed series $\tilde{\psi}(a, b) = \log(\exp(\tilde{\varphi}(a, b)) \cdot \exp(a))$, one has

\[
\tilde{\psi}(a, b) = a + \sum_{n=0}^{\infty} \frac{B_n}{n!} [a^n \tilde{\varphi}(a, b)] = a + \varphi(a, b) - \frac{1}{2} [a, \varphi(a, b)] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} [a^{2n} \varphi(a, b)].
\]

**Proof.** Apply Proposition 2.8 for the Hausdorff series $\tilde{\psi}(a, b) = \log(\exp(\tilde{\varphi}(a, b)) \cdot \exp(a))$ and $Q = a, P = \tilde{\varphi}(a, b)$. Since all commutators including in $\tilde{\varphi}(a, b)$ commute with each other in $L_3$, then $[Q^n P] = [a^n \varphi(a, b)]$ and $[Q^{n-1} P^{m-1} Q P] = 0$ for all $m \geq 2, n \geq 1$. □

In particular, by Claim 3.2 the series $\tilde{\psi}(a, b)$ starts with

\[
\tilde{\psi}(a, b) = a + \varphi(a, b) - \frac{1}{2} [a, \varphi(a, b)] + \frac{1}{12} [a, [a, \varphi(a, b)]] - \frac{1}{720} [a, [a, [a, \varphi(a, b)]]] + \cdots
\]

3.2. The Lie algebra $L_3(\lambda, \mu)$. Recall that $\bar{L}_3 = L_3/[[L_3, L_3], [L_3, L_3]]$ by definition.

**Definition 3.3.** (Lie algebra $L_3(\lambda, \mu)$). Let us introduce the linear space $L_3(\lambda, \mu)$ generated by $a, b$, and $\lambda^k \mu^l [ab]$ for $k, l \geq 0$, where $\lambda, \mu \in \mathbb{C}$ are formal parameters. At this moment the expression $[a \lambda] + [a \lambda] + [a \lambda] + [a \lambda]$ means a formal symbol, which denotes an element of $L_3(\lambda, \mu)$. Define in $L_3(\lambda, \mu)$ the bracket by $[a, b] := [ab], [a, \lambda^k \mu^l [ab]] := \lambda^{k+1} \mu^l [ab], [b, \lambda^k \mu^l [ab]] := \lambda^k \mu^{l+1} [ab]$, the other brackets are zero. One can check that this bracket satisfies the Jacobi identity. Actually, only the following identities (up to permutation $a \leftrightarrow b$) contain non-zero terms:

\[
[a, [a, \lambda^k \mu^l [ab]]] + [a, [\lambda^k \mu^l [ab], a]] = 0, [a, [b, \lambda^k \mu^l [ab]]] + [b, [\lambda^k \mu^l [ab], a]] + [\lambda^k \mu^l [ab], [ab]] = 0.
\]

Hence the space $L_3(\lambda, \mu)$ becomes a Lie algebra. Now the formal symbol $[ab]$ is the true commutator $[a, b] \in L_3(\lambda, \mu)$. Note that in $L_3(\lambda, \mu)$ one has $[a^k b^l ab] = \lambda^k \mu^l [ab, a]$. By $L_3(\lambda, \mu)$ denote the algebra of formal series of elements from $L_3(\lambda, \mu)$. Observe that in $\hat{L}_3(\lambda, \mu)$ any series of commutators $[a^k b^l ab]$ is (a series in $\lambda, \mu$) $\times [a, b]$. □

Remind the notation $[a_1 a_2 \ldots a_k] := [a_1, [a_2, \ldots, a_k]]$ from Definition 1.2c.

**Proposition 3.4.** (a) Let $L(a + b + c) \subset L_3$ be the 1-dimensional Lie subalgebra generated by the element $a + b + c$, $L(a, b) \subset L_3$ be the Lie subalgebra freely generated by $a, b$. Then the Lie algebra $L_3$ is isomorphic to the direct sum $L(a + b + c) \oplus L(a, b)$.

(b) The quotient $\bar{L}(a, b) := L(a, b)/[[L(a, b), L(a, b)], [L(a, b), L(a, b)]]$ is isomorphic to the Lie algebra $L_3(\lambda, \mu)$ introduced in Definition 3.3.

(c) The quotient $\bar{L}_3 = L_3/[[L_3, L_3], [L_3, L_3]]$ is isomorphic to $L(a + b + c) \oplus L_3(\lambda, \mu)$. □
Proof. (a) In the setting \( a := t^{12}, b := t^{23}, c = t^{13} \), the element \( a + b + c \) is a central element of \( L_3 \). Moreover, the defining relations of the Lie algebra \( L_3 \) become \([a, a + b + c] = [b, a + b + c] = 0\). Now the isomorphism \( L_3 \cong L(a + b + c) \oplus L(a, b) \) is obvious.

(b) By definition the Lie algebra \( L(a, b) \) is linearly generated by \( a, b \) and all commutators \([wab]\), where \( w \) is a word in \( a, b \). By Claim 2.6b the quotient \( L(a, b) \) is linearly generated by \( a, b \) and commutators \([a^kb^lab]\) with \( k, l \geq 0 \). The only non-zero brackets of these elements in \( \hat{L}(a, b) \) are \([a, b], [a, a^kb^l] = [a^{k+1}b^l]ab, [b, a^kb^l] = [a^{k+l}b^l]ab \). One has \([L_3(\lambda, \mu), L_3(\lambda, \mu)], [L_3(\lambda, \mu), L_3(\lambda, \mu)] = 0\). The required isomorphism \( \hat{L}(a, b) \to L_3(\lambda, \mu) \) is defined by \([a^kb^l]ab \mapsto \lambda^k\mu^l[a, b] \). The item (c) follows from (a) and (b). \( \square \)

Due to Proposition 3.4 the image of the logarithm of a Drinfeld associator is \([a, b] \times \) (a series in the commuting parameters \( \lambda, \mu \)). Hence the compressed hexagon lives in \( \hat{L}_3(\lambda, \mu) \). This simplification (the 5th key point) allows to solve the compressed hexagon completely.

Definition 3.5 (generating series \( f(\lambda, \mu) \) and \( g(\lambda, \mu) \); symmetric coefficients \( \alpha_{kl} \)). Let \( \mathbb{C}[[\lambda, \mu]] \) be the set of formal power series with complex coefficients in the commuting arguments \( \lambda, \mu \). Introduce series \( f, g \in \mathbb{C}[[\lambda, \mu]] \) by the formulae in the algebra \( \hat{L}_3(\lambda, \mu) \)

\[
\varphi(a, b) = \sum_{k,l \geq 0} \alpha_{kl} [a^kb^l]ab = f(\lambda, \mu) \cdot [a, b], \quad \psi(a, b) = a + g(\lambda, \mu) \cdot [a, b],
\]

respectively.

The formula for \( \varphi(a, b) \) is a general form of a series in \( \hat{L}_3(\lambda, \mu) \). A series \( g(\lambda, \mu) \) exists due to Claim 3.2. Since \( \varphi(a, b) = -\varphi(b, a) \), then the series \( f(\lambda, \mu) = \sum_{k,l \geq 0} \alpha_{kl} \lambda^k\mu^l \) (but not \( g(\lambda, \mu) \)) is symmetric in the arguments \( \lambda, \mu \), i.e. \( \alpha_{kl} = \alpha_{lk} \) are symmetric coefficients.

Claim 3.6. (a) The series \( f(\lambda, \mu) \) and \( g(\lambda, \mu) \) are related as follows

\[
(3.6) \quad g(\lambda, \mu) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \lambda^k \cdot f(\lambda, \mu) = \left( 1 - \frac{\lambda}{2} + \frac{\lambda^2}{12} - \frac{\lambda^4}{720} + \cdots \right) f(\lambda, \mu) \stackrel{(3.1)}{=} \frac{\lambda f(\lambda, \mu)}{e^\lambda - 1},
\]

(b) In the algebra \( \hat{L}_3(\lambda, \mu) \) one has

\[
(3.6) \quad \psi(b, c) = b + g(\mu, \rho) \cdot [a, b], \quad \psi(c, a) = c + g(\rho, \lambda) \cdot [a, b], \text{ where } \rho = -\lambda - \mu.
\]

Proof. The following calculations imply the item (a): \( g(\lambda, \mu)[ab] \stackrel{(3.5)}{=} \varphi(a, b) - a \stackrel{(3.2)}{=} \sum_{k=0}^{\infty} \frac{B_k}{k!} [a^k \varphi(a, b)] \stackrel{(3.2)}{=} \sum_{k=0}^{\infty} \frac{B_k}{k!} [a^k f(\lambda, \mu)[ab]] \stackrel{(3.3)}{=} \sum_{k=0}^{\infty} \frac{B_k}{k!} \lambda^k f(\lambda, \mu) \cdot [ab].
\]

The item (b) follows from Definition 3.5 and \( \varphi(b, c) = f(\mu, \rho)[ab], \varphi(c, a) = f(\rho, \lambda)[ab] \). The latter formula is proved similarly to the former one by using (1.2a) \([ab] = [bc] = [ca] \):

\[
\varphi(b, c) \stackrel{(3.5)}{=} \sum_{k,l \geq 0} \alpha_{kl} [b^k c^l bc] \stackrel{(1.2a)}{=} \sum_{k,l \geq 0} \alpha_{kl} [b^k (-a - b)^l ab] \stackrel{(3.3)}{=} \sum_{k,l \geq 0} \alpha_{kl} \lambda^k (-\lambda - \mu)^l [ab] \stackrel{(3.5)}{=} f(\mu, \rho)[ab]. \quad \square
\]

Example 3.7. For the series \( \varphi^B(a, b) = f^B(\lambda, \mu) \cdot [a, b] \) from Example (1.4), one has

\[
\alpha_{00} = \frac{1}{6}; \quad \alpha_{10} = \alpha_{01} = 0; \quad \alpha_{20} = \alpha_{02} = -\frac{1}{90}; \quad \alpha_{11} = -\frac{1}{360}; \quad \alpha_{30} = \alpha_{21} = \alpha_{12} = \alpha_{03} = 0.
\]

Then up to degree 3 the corresponding series \( f^B(\lambda, \mu) \) and \( g^B(\lambda, \mu) \) are

\[
f^B(\lambda, \mu) = \frac{1}{6} - \frac{4 \lambda^2 + \lambda \mu + 4 \mu^2}{360}, \quad g^B(\lambda, \mu) = \frac{1}{6} - \frac{\lambda}{12} + \frac{\lambda^2 - \lambda \mu - 4 \mu^2}{360} + \frac{4 \lambda^3 + \lambda^2 \mu + 4 \lambda \mu^2}{720}.
\]
Moreover, one can compute the series from Claim 3.6b (they will be needed later):

\[ g^B(\mu, \rho) = \frac{1}{6} - \frac{\mu}{12} - \frac{4\lambda^2 + 7\lambda\mu + 2\mu^2}{360} + \frac{4\lambda^2\mu + 7\lambda\mu^2 + 7\mu^3}{720} + \cdots, \]

\[ g^B(\rho, \lambda) = \frac{1}{6} + \frac{\lambda + \mu}{12} - \frac{2\lambda^2 + 3\lambda\mu + \mu^2}{360} - \frac{7\lambda^3 + 14\lambda^2\mu + 11\lambda\mu^2 + 4\mu^3}{720} + \cdots, \]

hence

\[ G^B(\lambda, \mu) := g^B(\lambda, \mu) + g^B(\mu, \rho) + g^B(\rho, \lambda) = \frac{1}{2} - \frac{\lambda^2 + \lambda\mu + \mu^2}{72} + \frac{\mu^3 - 3\lambda^2\mu - \lambda^3}{240} + \cdots, \]

\[ T^B(\lambda, \mu) := 1 + \lambda g^B(\mu, \rho) - \mu g^B(\lambda, \mu) = 1 + \frac{\lambda - \mu}{6} + \frac{4\mu^3 - \lambda\mu^2 - 8\lambda^2\mu - 4\lambda^3}{360} + \cdots. \]

3.3. **Explicit form of the compressed hexagon** \((1.3b)\).

**Claim 3.8.** For the generators \(a = t^{12}, b = t^{23}, c = t^{33}\) of the Lie algebra \(L_3\), set \(P := \bar{\psi}(b, c), Q := \bar{\psi}(a, b)\). Then in the algebra \(\hat{L}_3(\lambda, \mu)\) one has

\[ [Q, P] = T(\lambda, \mu) \cdot [a, b], \text{ where } T(\lambda, \mu) = 1 + \lambda g(\mu, -\lambda - \mu) - \mu g(\lambda, \mu). \]

**Proof.** By Definition 3.4a and Claim 3.6 one gets

\[ [Q, P] = [a + g(\lambda, \mu)[ab], b + g(\mu, -\lambda - \mu)[ab]] = [ab] + g(\mu, -\lambda - \mu)[aab] - g(\lambda, \mu)[bab]. \]

It remains to use the relations \([aab] = \lambda[ab]\) and \([bab] = \mu[ab]\) of \(L_3(\lambda, \mu)\).

**Proposition 3.9.** Let \(\bar{\varphi} = \sum_{k,l \geq 0} \alpha_{kl}[a^k b^l ab]\) be a compressed Drinfeld associator, \(\alpha_{kl} \in \mathbb{C}\), \(f(\lambda, \mu) = \sum_{k,l \geq 0} \alpha_{kl} \lambda^k \mu^l\) be the generating function of the coefficients \(\alpha_{kl}\). Then the compressed hexagon \((1.3b)\) is equivalent to the following equation in the algebra \(\mathbb{C}[[\lambda, \mu]]\)

\[ G(\lambda, \mu) + C(\lambda, \mu) \cdot T(\lambda, \mu) = 0, \]

where \(G(\lambda, \mu) := g(\lambda, \mu) + g(\mu, \rho) + g(\rho, \lambda), \quad T(\lambda, \mu) := 1 + \lambda g(\mu, -\lambda - \mu) - \mu g(\lambda, \mu), \quad \rho := -\lambda - \mu. \)

**Proof.** Let us apply Proposition 2.8 for the Hausdorff series \(\bar{H} = \log(\exp(P) \cdot \exp(Q))\), where \(P = \bar{\psi}(b, c) = b + g(\mu, \rho)[a, b], \quad Q = \bar{\psi}(a, b) = a + g(\lambda, \mu)[a, b]\). One has

\[ \bar{H} = P + Q + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{m! n!} [Q^{n-1}P^{m-1}Q]. \]

Since in the quotient \(\bar{L}_3\) all commutators commute, then the formula \((3.8)\) implies

\[ [Q^{n-1}P^{m-1}Q] = T(\lambda, \mu) \cdot [a^{n-1}b^{m-1}ab] = \lambda^{n-1} \mu^{m-1}T(\lambda, \mu) \cdot [a, b], \]

hence

\[ \bar{H} = a + g(\lambda, \mu)[a, b] + b + g(\mu, \rho)[a, b] + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{m! n!} \lambda^{n-1} \mu^{m-1} T(\lambda, \mu) \cdot [a, b] = \]

\[ = (\text{by Definition 2.4b}) = a + b + (g(\lambda, \mu) + g(\mu, \rho)) \cdot [a, b] + C(\lambda, \mu) T(\lambda, \mu) \cdot [a, b]. \]

On the other hand, by Lemma 3.1 one has \(\bar{H} = \log(\exp(-\bar{\psi}(c, a)) \cdot \exp(a + b + c))\). But \(a + b + c\) is a central element of \(L_3\), hence \(\bar{H} = a + b + c - \bar{\psi}(c, a)\).

Let us take together both above expressions for \(\bar{H}\) and use Claim 3.6b for \(\bar{\psi}(c, a)\)

\[ a + b + (g(\lambda, \mu) + g(\mu, \rho))[ab] + C(\lambda, \mu) T(\lambda, \mu)[ab] = a + b + c - (c + g(\rho, \lambda)[ab]) \Leftrightarrow (3.9). \]
Example 3.10. By $G^B_k, C_k, T^B_k$ denote the degree $k$ parts of the functions $G^B(\lambda, \mu), C(\lambda, \mu), T^B(\lambda, \mu)$, respectively. Due to Examples 2.5 and 3.7 one can calculate

$$G^B_0(\lambda, \mu) = \frac{1}{2}, \quad G^B_1(\lambda, \mu) = 0, \quad G^B_2(\lambda, \mu) = -\frac{\lambda^2 + \lambda \mu + \mu^2}{72}, \quad G^B_3(\lambda, \mu) = \frac{\mu^3 - 3\lambda^2 \mu - \lambda^3}{240};$$

$$C_0(\lambda, \mu) = -\frac{1}{2}, \quad C_1(\lambda, \mu) = \frac{\lambda - \mu}{12}, \quad C_2(\lambda, \mu) = \frac{\lambda \mu}{24}, \quad C_3(\lambda, \mu) = \frac{\mu^3 + 4\lambda \mu^2 - 4\lambda^2 \mu - \lambda^3}{720};$$

$$T^B_0(\lambda, \mu) = 1, \quad T^B_1(\lambda, \mu) = \frac{\lambda - \mu}{6}, \quad T^B_2(\lambda, \mu) = 0, \quad T^B_3(\lambda, \mu) = \frac{4\mu^3 - \lambda \mu^2 - 8\lambda^2 \mu - 4\lambda^3}{360}. $$

Now one can check by hands the first four compressed hexagons:

$$G^B_0 + C_0 = 0, \quad G^B_1 + C_0 T^B_1 + C_1 = 0, \quad G^B_2 + C_1 T^B_1 + C_2 = 0, \quad G^B_3 + C_0 T^B_2 + C_2 T^B_1 + C_3 = 0. \quad \blacksquare$$

Proof of Theorem 1.5a. The expression $\check{\varphi}(a, b) = \sum_{k,l \geq 0} \alpha_{kl} [a^k b^l a b]$ follows from Definition 3.5. The condition $\check{\varphi}(a, b) = -\check{\varphi}(b, a)$ is the symmetry $\alpha_{kl} = \alpha_{lk}$. The non-degeneracy $\check{\varphi}(a, 0) = \check{\varphi}(0, b) = 0$ holds trivially. Let us rewrite (3.9) as follows:

$$(3.9') \quad g(\lambda, \mu) \cdot (\lambda - \mu C(\lambda, \mu)) + g(\mu, \rho) \cdot (\lambda + \mu C(\lambda, \mu)) + g(\rho, \lambda) + C(\lambda, \mu) = 0. $$

Then, for each $n \geq 1$, the degree $n$ parts of the functions $g(\lambda, \mu), g(\mu, \rho), g(\rho, \lambda)$ can be expressed linearly via the degree $k < n$ parts of the same functions. Due to Claim 3.6a the degree $n$ parts of these functions contain the coefficients $\alpha_{kl}$ only for $k + l = n$. Hence the coefficients $\alpha_{kl}$ could be expressed linearly in terms of $\alpha_{ij}$ with $i + j < k + l$. \qed

4. Solving the compressed hexagon equation

Here one solves completely the compressed hexagon (1.3b) = (3.9) from Proposition 3.9.

4.1. Further simplifications of the compressed hexagon.

Lemma 4.1. Let $f(\lambda, \mu) = \sum_{k,l \geq 0} \alpha_{kl} \lambda^k \mu^l$ be the generating function of the coefficients $\alpha_{kl}$ of a compressed associator $\check{\varphi} \in \tilde{L}_3$. Then the compressed hexagon (1.3b) = (3.9) is equivalent to the equation (4.1a) and to the equation (4.1b) in the algebra $\mathbb{C}[[\lambda, \mu]]$

$$(4.1a) \quad \frac{\lambda f(\lambda, \mu)}{e^\lambda - 1} (1 - \mu C(\lambda, \mu)) + \frac{\mu f(\mu, \rho)}{e^\mu - 1} (1 + \lambda C(\lambda, \mu)) + \frac{\rho f(\rho, \lambda)}{e^\rho - 1} + C(\lambda, \mu) = 0, \quad \text{where}$$

$$\rho = -\lambda - \mu, \quad C(\lambda, \mu) = \frac{e^\mu - 1}{\lambda \mu} \cdot \left( \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} - \frac{\mu}{e^\mu - 1} \right);$$

$$(4.1b) \quad f(\lambda, \mu) + e^\mu f(\mu, -\lambda - \mu) + e^{-\lambda} f(\lambda, -\lambda - \mu) = \frac{1}{\lambda + \mu} \left( \frac{e^\mu - 1}{\mu} + \frac{e^{-\lambda} - 1}{\lambda} \right).$$

Proof. (a) The equation (4.1a) follows from the formula (3.9') and Claim 3.6a.

(b) By using the formula (2.12) in the form $C(\lambda, \mu) = \frac{e^\mu - 1}{\lambda \mu} \cdot \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} - \frac{1}{\lambda}$, one gets

$$\frac{\mu}{e^\mu - 1} (1 + \lambda C(\lambda, \mu)) = \frac{\mu}{e^\mu - 1} \left( \frac{e^\mu - 1}{\mu} - \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} \right) = \frac{\lambda + \mu}{e^{\lambda + \mu} - 1}. $$

Now apply the symmetry (the third key point)

$$C(\lambda, \mu) = C(-\mu, -\lambda) = \frac{e^{-\lambda} - 1}{\lambda \mu} \cdot \frac{-\lambda - \mu}{e^{-\lambda - \mu} - 1} + \frac{1}{\mu}, \quad \text{hence}$$

$$\frac{\lambda}{e^\lambda - 1} (1 - \mu C(\lambda, \mu)) = \frac{\lambda}{e^\lambda - 1} \left( \frac{e^{-\lambda} - 1}{\lambda} \cdot \frac{\lambda + \mu}{e^{-\lambda - \mu} - 1} \right) = \frac{1 - e^\lambda}{e^\lambda - 1} \cdot e^\mu \frac{\lambda + \mu}{1 - e^{\lambda + \mu}} = \frac{e^\mu}{e^{\lambda + \mu} - 1} \cdot \frac{\lambda + \mu}{e^{\lambda + \mu} - 1}.$$
Then the equation (4.1a) converts to
\[
e^\mu \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} f(\lambda, \mu) + \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} f(\mu, \rho) - \frac{\lambda + \mu}{e^{-\lambda - \mu} - 1} f(\rho, \lambda) + e^\mu \frac{1}{\lambda} \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} = 0, \quad \text{or}
\]
e^\mu \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} f(\lambda, \mu) + \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} f(\mu, \rho) + \frac{\lambda + \mu}{e^{\lambda + \mu} - 1} f(\rho, \lambda) = \frac{1}{\lambda} \left(1 - \frac{e^\mu - 1}{\mu} \cdot \frac{\lambda + \mu}{e^{\lambda + \mu} - 1}\right).

One can multiply both sides by \(\frac{e^{\lambda + \mu} - 1}{\lambda + \mu}\) since this series begins with 1, i.e.
\[
e^\mu f(\lambda, \mu) + f(\mu, \rho) + e^{\lambda + \mu} f(\rho, \lambda) = \frac{1}{\lambda} \left(\frac{e^\mu - 1}{\mu} - \frac{e^\mu - 1}{\mu}\right).
\]

Let us swap \(\lambda\) and \(\rho = -\lambda - \mu\) (in other words one substitutes \((-\lambda - \mu)\) for \(\lambda\)):
\[
e^\mu f(-\lambda - \mu, \mu) + f(\mu, \lambda) + e^{-\lambda} f(\lambda, -\lambda - \mu) = \frac{1}{\lambda + \mu} \left(\frac{e^{-\lambda} - 1}{\lambda} + \frac{e^\mu - 1}{\mu}\right).
\]

To get the equation (4.1b) it remains to use the symmetry \(f(\lambda, \mu) = f(\mu, \lambda)\). □

**The hexagon part of Theorem 1.5b** follows from Lemma 4.1b: the compressed hexagon equation (1.3b) is equivalent to (4.1b) = (1.5b).

An explicit form of the compressed hexagon (1.3b) is the 6th key point.

**Lemma 4.2.** For the generating function \(f(\lambda, \mu)\) of any compressed associator, one has

\[
(4.2) \quad \text{Even}(f(\lambda, 0)) = \frac{1}{2\lambda^2} \left(\frac{2\lambda}{e^{2\lambda} - 1} + \lambda - 1\right), \quad f(\lambda, -\lambda) = \frac{1}{\lambda^2} - \frac{2}{\lambda(e^\lambda - e^{-\lambda})}.
\]

In particular, one obtains the extreme coefficients \(\alpha_{2k,0} = \frac{2^{2k+1}B_{2k+2}}{(2k+2)!}\) for every \(k \geq 0\).

**Proof.** Let us solve the equation (4.1b) explicitly for \(\mu = 0\) (then set \(e^\mu - 1\bigg|_{\mu=0} = 1\)).

One has
\[
\text{Even}(f(\lambda, 0)) + f(0, -\lambda) + e^{-\lambda} f(\lambda, -\lambda) = \frac{1}{\lambda} \left(1 + \frac{e^{-\lambda} - 1}{\lambda}\right) = \frac{e^{-\lambda} + \lambda - 1}{\lambda^2}.
\]

Since \(\text{Odd}(f(\lambda, -\lambda)) = 0, \text{Odd}(f(\lambda, 0) + f(0, -\lambda)) = 0\), and
\[
\text{Even}(f(0, -\lambda)) = \text{Even}(f(\lambda, 0)) = \sum_{k=0}^{\infty} \alpha_{2k,0} \lambda^{2k}, \quad \text{then one obtains}
\]
\[
2 \sum_{k=0}^{\infty} \alpha_{2k,0} \lambda^{2k} + e^{-\lambda} \text{Even}(f(\lambda, -\lambda)) = \frac{e^{-\lambda} + \lambda - 1}{\lambda^2}.
\]

By substituting \((-\lambda)\) for \(\lambda\) and using \(\text{Even}(f(-\lambda, \lambda)) = \text{Even}(f(\lambda, -\lambda))\), one gets
\[
2 \sum_{k=0}^{\infty} \alpha_{2k,0} \lambda^{2k} + e^\lambda \text{Even}(f(\lambda, -\lambda)) = \frac{e^\lambda - \lambda - 1}{\lambda^2}.
\]

From the two above equations one deduces
\[
f(\lambda, -\lambda) = \text{Even}(f(\lambda, -\lambda)) = \frac{1}{\lambda^2} - \frac{2}{\lambda(e^\lambda - e^{-\lambda})} \quad \text{and}
\]
\[
\text{Even}(f(\lambda, 0)) = \sum_{k=0}^{\infty} \alpha_{2k,0} \lambda^{2k} = \frac{1}{2\lambda^2} \left(\frac{2\lambda}{e^{2\lambda} - 1} + \lambda - 1\right) = \frac{1}{2\lambda^2} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (2\lambda)^{2n}. \quad \square
\]
Example 4.3. Note that $\alpha_{2k,0} = \frac{2^{2k+1} B_{2k+2}}{(2k + 2)!}$ are correct coefficients of the logarithm of any honest Drinfeld associator $\Phi(a, b) = \exp(\varphi(a, b)) \in \hat{L}_3$, not only the compressed one.

This form of the extreme coefficients $\alpha_{2k,0}$ is the 7th key point:

$$\alpha_{00} = \frac{2B_2}{2!} = \frac{1}{6}, \quad \alpha_{20} = \frac{2^3 B_4}{4!} = -\frac{1}{90}, \quad \alpha_{40} = \frac{2^5 B_6}{6!} = \frac{1}{945}, \quad \alpha_{60} = \frac{2^7 B_8}{8!} = -\frac{1}{9450}.$$  

For the coefficients $\gamma_k$ of the series $\frac{2\lambda}{e^\lambda - e^{-\lambda}} = \sum_{k=0}^\infty \gamma_k \lambda^{2k}$, similarly to Lemma 2.2 one can find the recursive formula $\sum_{k=0}^n \frac{\gamma_{n-k}}{(2k+1)!} = 0$ for each $n > 1$, $\gamma_0 = 0$. Hence

$$2 \lambda e^\lambda - e^{-\lambda} = 1 - \frac{1}{6} \lambda^2 + \frac{7}{360} \lambda^4 - \frac{31}{3 \cdot 7!} \lambda^6 + \frac{127}{15 \cdot 8!} \lambda^8 + \cdots \quad (4.3)$$

For the generating function of any compressed Drinfeld associator $\hat{\varphi} \in \hat{L}_3$, one gets

$$f(\lambda, -\lambda) = \frac{1}{6} - \frac{7}{360} \lambda^2 + \frac{31}{15120} \lambda^4 - \frac{127}{604800} \lambda^6 + \cdots \quad \square$$

Lemma 4.4. Under the conditions of Lemma 4.1, set $\tilde{f}(\lambda, \mu) := 1 + \lambda \mu f(\lambda, \mu)$.

(a) The new function $\tilde{f}(\lambda, \mu)$ obeys the same symmetry: $\tilde{f}(\lambda, \mu) = \tilde{f}(\mu, \lambda)$. Moreover, the function $f(\lambda, \mu)$ is even (i.e. $f(\lambda, \mu) = f(-\lambda, -\mu)$) if and only if $\tilde{f}(\lambda, \mu)$ is even.

(b) The compressed hexagon $(1.3 b) = (4.1 b)$ is equivalent to the following equation in $\mathbb{C}[\lambda, \mu]$

$$\left(\lambda + \mu\right) \tilde{f}(\lambda, \mu) = \lambda e^\mu \tilde{f}(\mu, -\lambda - \mu) + \mu e^{-\lambda} \tilde{f}(\lambda, -\lambda - \mu). \quad (4.4)$$

Proof. The item (a) follows from the definition of $\tilde{f}(\lambda, \mu)$.

(b) The equation $(4.1 b)$ can be rewritten as follows:

$$\left( f(\lambda, \mu) + \frac{1}{\lambda \mu} \right) + e^\mu \left( f(\mu, -\lambda - \mu) - \frac{1}{\mu (\lambda + \mu)} \right) + e^{-\lambda} \left( f(\lambda, -\lambda - \mu) - \frac{1}{\lambda (\lambda + \mu)} \right) = 0. \quad (4.1 b)$$

To get $(4.4)$ it remains to multiply by $\lambda \mu (\lambda + \mu)$. \square

Recall that the even and odd parts of a series were introduced before Theorem 1.5.

Lemma 4.5. In the notations of Lemma 4.4, the compressed hexagon $(1.3 b) = (4.4)$ can be split into the two following equations in the algebra $\mathbb{C}[\lambda, \mu]$

$$\left(\lambda + \mu\right) \text{Even}(\tilde{f}(\lambda, \mu)) = \lambda e^\mu \text{Even}(\tilde{f}(\mu, -\lambda - \mu)) + \mu e^{-\lambda} \text{Even}(\tilde{f}(\lambda, -\lambda - \mu)); \quad (4.5a)$$

$$\text{Odd}(\tilde{f}(\lambda, \mu)) + e^\mu \text{Odd}(f(\mu, -\lambda - \mu)) + e^{-\lambda} \text{Odd}(f(\lambda, -\lambda - \mu)) = 0. \quad (4.5b)$$

Proof. Let us substitute the pair $(-\lambda, -\mu)$ for $(\lambda, \mu)$ into the equation $(4.4)$

$$-\left(\lambda + \mu\right) \tilde{f}(-\lambda, -\mu) = -\lambda e^{-\mu} \tilde{f}(-\mu, \lambda + \mu) - \mu e^\lambda \tilde{f}(-\lambda, \lambda + \mu).$$

Now swap the arguments $\lambda, \mu$ and use the symmetry $\tilde{f}(\lambda, \mu) = \tilde{f}(\mu, \lambda)$

$$-\left(\lambda + \mu\right) \tilde{f}(-\lambda, -\mu) = -\mu e^{-\lambda} \tilde{f}(-\lambda, \lambda + \mu) - \lambda e^\mu \tilde{f}(-\mu, \lambda + \mu).$$

If one subtracts the latter equation from $(4.4)$, then after dividing by 2 one obtains the equation $(4.5a)$. If one adds the latter equation to $(4.4)$, then one has

$$\left(\lambda + \mu\right) \text{Odd}(\tilde{f}(\lambda, \mu)) = \lambda e^\mu \text{Odd}(\tilde{f}(\mu, -\lambda - \mu)) + \mu e^{-\lambda} \text{Odd}(\tilde{f}(\lambda, -\lambda - \mu)).$$

Since $\text{Odd}(\tilde{f}(\lambda, \mu)) = \lambda \mu \text{Odd}(f(\lambda, \mu))$, after dividing by $\lambda \mu (\lambda + \mu)$ one gets $(4.5b)$. \square

The splitting of Lemma 4.5 is the 8th key point.
4.2. Explicit description of all even compressed associators.

**Lemma 4.6.** The general solution of the equation (4.5a) is

\[
\text{Even}(\tilde{f}(\lambda, \mu)) = 1 + \lambda \mu \cdot \text{Even}(f(\lambda, \mu)) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)} h(\lambda, \mu),
\]

where \( h(\lambda, \mu) \) is a function satisfying the boundary condition \( h(\lambda, 0) = \frac{2\lambda}{e^{\lambda} - e^{-\lambda}} \) and the symmetry relations \( h(\lambda, \mu) = h(\mu, \lambda) = h(-\lambda, -\mu) = h(\lambda, -\lambda - \mu) \).

**Proof.** Let us swap the arguments \( \lambda, \mu \) in (4.5a). Due to \( \tilde{f}(\lambda, \mu) = \tilde{f}(\mu, \lambda) \) one gets

\[
(\lambda + \mu) \text{Even}(\tilde{f}(\lambda, \mu)) = \mu e^\lambda \text{Even}(\tilde{f}(\lambda, -\lambda - \mu)) + \lambda e^{-\lambda} \text{Even}(\tilde{f}(\mu, -\lambda - \mu)).
\]

By subtracting the above equation from (4.5a) one obtains

\[
\lambda(e^\mu - e^{-\mu}) \cdot \text{Even}(\tilde{f}(\mu, -\lambda - \mu)) + \mu(e^{-\lambda} - e^\lambda) \cdot \text{Even}(\tilde{f}(\lambda, -\lambda - \mu)) = 0,
\]

or

\[
\text{Even}(\tilde{f}(\mu, -\lambda - \mu)) : \left(\frac{e^\lambda - e^{-\lambda}}{2\lambda}\right) = \text{Even}(\tilde{f}(\lambda, -\lambda - \mu)) : \left(\frac{e^\mu - e^{-\mu}}{2\mu}\right).
\]

Introduce the function

(4.6') \( h(\lambda, \mu) := \text{Even}(\tilde{f}(\mu, -\lambda - \mu)) : \left(\frac{e^\lambda - e^{-\lambda}}{2\lambda}\right) = \text{Even}(\tilde{f}(\lambda, -\lambda - \mu)) : \left(\frac{e^\mu - e^{-\mu}}{2\mu}\right) \).

Then the new function is even and symmetric: \( h(\lambda, \mu) = h(\mu, \lambda) = h(-\lambda, -\mu) \). Let us substitute the expressions

\[
\text{Even}(\tilde{f}(\mu, -\lambda - \mu)) = \left(\frac{e^\lambda - e^{-\lambda}}{2\lambda}\right) \cdot h(\lambda, \mu), \quad \text{Even}(\tilde{f}(\lambda, -\lambda - \mu)) = \left(\frac{e^\mu - e^{-\mu}}{2\mu}\right) \cdot h(\lambda, \mu)
\]

into the equation (4.5a). One has

\[
(\lambda + \mu) \text{Even}(\tilde{f}(\lambda, \mu)) = \left(\lambda e^\mu e^{\lambda-\lambda} + \mu e^{-\lambda} e^{\mu-\mu}\right) h(\lambda, \mu) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2} h(\lambda, \mu).
\]

So, (4.6) is proved. Let us substitute \((-\lambda - \mu)\) for \( \mu \) into the above equation:

\[
\text{Even}(\tilde{f}(\lambda, -\lambda - \mu)) = \frac{e^{-\mu} - e^{\mu}}{2(-\mu)} h(\lambda, -\lambda - \mu) = \frac{e^{-\mu} - e^{\mu}}{2\mu} h(\lambda, -\lambda - \mu).
\]

By comparing the latter formula with the equation (4.6'), one gets \( h(\lambda, \mu) = h(\lambda, -\lambda - \mu) \). To obtain the condition \( h(\lambda, 0) = \frac{2\lambda}{e^{\lambda} - e^{-\lambda}} \) it remains to substitute \( \mu = 0 \) into (4.6). \( \square \)

Observe that the conditions of Lemma 4.6 imply also \( h(\lambda, -\lambda) = h(\lambda, 0) = \frac{2\lambda}{e^{\lambda} - e^{-\lambda}} \). This agrees with the formula (4.2): \( h(\lambda, -\lambda) = 1 - \lambda^2 f(\lambda, -\lambda) = \frac{2\lambda}{e^{\lambda} - e^{-\lambda}} \). So, the compressed hexagon (1.3b) was reduced to \( h(\lambda, \mu) = h(\lambda, -\lambda - \mu) \), this is the 9th key point. To describe all series \( h(\lambda, \mu) \) with this symmetry one needs associator polynomials.

**Definition 4.7** (associator polynomials). For each \( n \geq 0 \), a homogeneous polynomial \( F_n(\lambda, \mu) = \sum_{k=0}^{n} \delta_k \lambda^k \mu^{n-k} \) with \( \delta_k \in \mathbb{C} \) is called an associator polynomial, if \( F_n(\lambda, \mu) = F_n(\mu, \lambda) = F_n(\lambda, -\lambda - \mu) \) hold for all \( \lambda, \mu \in \mathbb{C} \). The Drinfeld series \( s(\lambda, \mu) \) from Definition 6.1b contains associator polynomials \( F_{2n+1}^D(\lambda, \mu) = (\lambda + \mu)^{2n+1} - \lambda^{2n+1} - \mu^{2n+1} \). \( \blacksquare \)
Example 4.8. In degrees 0, 2, 4 an associator polynomial is unique up to factor:
\[ F_0(\lambda, \mu) = 1, \quad F_2(\lambda, \mu) = \lambda^2 + \lambda \mu + \mu^2, \quad F_4(\lambda, \mu) = \lambda^4 + 2\lambda^3 \mu + 3\lambda^2 \mu^2 + 2\lambda \mu^3 + \mu^4. \]
In each odd degree \( n = 1, 3, 5 \) there is also exactly one associator polynomial up to factor:
\[ F_1(\lambda, \mu) = 0, \quad F_3(\lambda, \mu) = \lambda^2 \mu + \lambda \mu^2, \quad F_5(\lambda, \mu) = \lambda^4 \mu + 2\lambda^3 \mu^2 + 2\lambda^2 \mu^3 + \lambda \mu^4 + \mu^5. \]
Rather surprisingly, that an associator polynomial of the degree \( n \) is not unique in general, for instance, in degree 6. One can check by hands that in degree 6 there is a 1-parametric family of associator polynomials:
\[ F_6(\lambda, \mu) = \lambda^6 + 3\lambda^5 \mu + 3\lambda^4 \mu^2 + (2\delta - 5)\lambda^3 \mu^3 + \delta \lambda^2 \mu^4 + 3\lambda \mu^5 + \mu^6, \quad \delta \in \mathbb{C}. \]
However, in degree 7 there is a unique associator polynomial up to factor:
\[ F_7(\lambda, \mu) = \lambda^6 \mu + 3\lambda^5 \mu^2 + 5\lambda^4 \mu^3 + 5\lambda^3 \mu^4 + 3\lambda^2 \mu^5 + \lambda \mu^6. \]
\[ \Box \]

Lemma 4.9. Any function \( \tilde{h}(\lambda, \mu) \) satisfying the boundary conditions \( \tilde{h}(\lambda, 0) = \sum_{n=0}^{\infty} \gamma_n \lambda^{2n} \)
and the symmetry relations \( \tilde{h}(\lambda, \mu) = \tilde{h}(\mu, \lambda) = \tilde{h}(\lambda, -\lambda - \mu) \) is \( \tilde{h}(\lambda, \mu) = \sum_{n=0}^{\infty} \gamma_n F_n(\lambda, \mu), \)
where \( F_n(\lambda, \mu) \) is an associator polynomial with the extreme coefficient 1 (respectively, 0) for each even (respectively, odd) \( n \geq 0 \).

Proof follows from Definition 4.7. The relation \( F_{2n+1}(\lambda, \mu) = F_{2n+1}(\lambda, -\lambda - \mu) \) implies that the extreme coefficient of any associator polynomial \( F_{2n+1}(\lambda, \mu) \) is always 0. □

Non-uniqueness of even compressed associators will follow from non-uniqueness of associator polynomials (the 10th key point), see the proof of Theorem 1.5c.

Proposition 4.10. (a) For each \( n \geq 0 \), any associator polynomial of the degree \( 2n \) is
\[ F_{2n}(\lambda, \mu) = \sum_{k=0}^{[n/2]} \beta_{nk} \lambda^{2k} \mu^{2k} (\lambda + \mu)^{2k} (\lambda^2 + \lambda \mu + \mu^2)^{n-2k}, \]
where \( \beta_{nk} \in \mathbb{C} \) are free parameters for \( 0 \leq k \leq [n/2] \).
(b) For every \( n \geq 1 \), any associator polynomial \( F_{2n+1}(\lambda, \mu) \) has the following form:
\[ F_{2n+1}(\lambda, \mu) = \sum_{k=0}^{[n-1]/2} \tilde{\beta}_{nk} \lambda^{2k+1} \mu^{2k+1} (\lambda + \mu)^{2k+1} (\lambda^2 + \lambda \mu + \mu^2)^{n-2k-1}, \]
where \( \tilde{\beta}_{nk} \in \mathbb{C} \) are free parameters for \( 0 \leq k \leq [n-1]/3 \).

Proof. (a) Induction on \( n \). The bases \( n = 0, 1, 2 \) are in Example 4.8.

Induction step goes down from \( n \) to \( n - 3 \).
Suppose that \( F_{2n}(\lambda, \mu) \) is an associator polynomial of the degree \( 2n \) with the extreme coefficient \( \beta_{n0} \). Then the polynomial \( \tilde{F}_{2n}(\lambda, \mu) = F_{2n}(\lambda, \mu) - \beta_{n0} (\lambda^2 + \lambda \mu + \mu^2)^n \) satisfies Definition 4.7. The relations \( F_{2n}(\lambda, \mu) = F_{2n}(\mu, \lambda) = F_{2n}(\lambda, -\lambda - \mu) \) imply
\[ F_{2n}(\lambda, \mu) = \beta_{n0} \lambda^{2n} + n \beta_{n0} \lambda^{2n-1} \mu + \ldots + n \beta_{n0} \lambda^{2n-1} + \beta_{n0} \mu^{2n}. \]
The polynomial \( \beta_{n0} (\lambda^2 + \lambda \mu + \mu^2)^n \) has the same form. Hence the first two (and the last two) coefficients of \( \tilde{F}_{2n}(\lambda, \mu) \) are always zero.
One gets \( \tilde{F}_{2n}(\lambda, \mu) = \lambda^2 \mu^2 F_{2n-4}(\lambda, \mu) \) for a polynomial \( F_{2n-4}(\lambda, \mu) \) of the degree \( 2n - 4 \) such that \( F_{2n-4}(\lambda, \mu) = F_{2n-4}(\mu, \lambda) \) and \( \lambda^2 \mu^2 F_{2n-4}(\lambda, \mu) = \lambda^2 (-\lambda - \mu)^2 F_{2n-4}(\lambda, -\lambda - \mu) \).
The equation \( \mu^2 F_{2n-4}(\lambda, \mu) = (\lambda + \mu)^2 F_{2n-4}(\lambda, -\lambda - \mu) \) implies that there is a polynomial \( F_{2n-6}(\lambda, \mu) \) of the degree \( 2n - 6 \) such that \( F_{2n-6}(\lambda, \mu) = (\lambda + \mu)^2 F_{2n-6}(\lambda, \mu) \).
Moreover, the new polynomial $\overline{F}_{2n-6}(\lambda, \mu)$ satisfies all the conditions of Definition 4.7, i.e. $\overline{F}_{2n-6}(\lambda, \mu)$ is equal to an associator polynomial $F_{2n-6}(\lambda, \mu)$. Hence

$$F_{2n}(\lambda, \mu) - \beta_{n0}(\lambda^2 + \lambda \mu + \lambda \mu)^n = \lambda^2 \mu^2(\lambda + \mu)^2 F_{2n-6}(\lambda, \mu).$$

The induction hypothesis

$$F_{2n-6}(\lambda, \mu) = \sum_{k=0}^{\left[\frac{n}{3}\right]-1} \beta_{n-3,k} \lambda^{2k} \mu^{2k} (\lambda + \mu)^{2k} (\lambda^2 + \lambda \mu + \mu^2)^{n-3k-3}$$

implies

$$F_{2n}(\lambda, \mu) = \beta_{n0}(\lambda^2 + \lambda \mu + \lambda \mu)^n + \lambda^2 \mu^2(\lambda + \mu)^2 \sum_{k=0}^{\left[\frac{n}{3}\right]-1} \beta_{n-3,k} \lambda^{2k} \mu^{2k} (\lambda + \mu)^{2k} (\lambda^2 + \lambda \mu + \mu^2)^{n-3k-3}.$$ 

To get (4.10a) it remains to set $\beta_{n,k+1} := \beta_{n-3,k}$ for $0 \leq k \leq \left[\frac{n}{3}\right] - 1$.

(b) The proof is analogous to the item (a). If $F_{2n+1}(\lambda, \mu)$ is an associator polynomial, then its extreme coefficient is zero, i.e. $F_{2n+1}(\lambda, \mu) = \lambda \mu F_{2n-1}(\lambda, \mu)$ for some polynomial $F_{2n-1}(\lambda, \mu)$ of the degree $2n - 1$ with the properties

$$\overline{F}_{2n-1}(\lambda, \mu) = F_{2n-1}(\lambda, \mu) \quad \text{and} \quad \lambda \mu F_{2n-1}(\lambda, \mu) = (\lambda - \mu) F_{2n-1}(-\lambda, -\mu).$$

The equation $\mu \overline{F}_{2n-1}(\lambda, \mu) = - (\lambda + \mu) \overline{F}_{2n-1}(\lambda, -\lambda - \mu)$ implies that there is a polynomial $F_{2n-2}(\lambda, \mu)$ of the degree $2n - 2$ such that $\overline{F}_{2n-1}(\lambda, \mu) = (\lambda + \mu) F_{2n-2}(\lambda, \mu)$. Moreover, the new polynomial $F_{2n-2}(\lambda, \mu)$ satisfies all the conditions of Definition 4.7, i.e. $\overline{F}_{2n-2}(\lambda, \mu)$ is equal to an associator polynomial $F_{2n-2}(\lambda, \mu)$.

To get (4.10b) it remains to apply the formula (4.10a) for this polynomial $F_{2n-2}(\lambda, \mu)$ and to set $\beta_{n,k} := \beta_{n-1,k}$ for $0 \leq k \leq \left[\frac{n-1}{3}\right]$.

The 11th key point: the symmetry $F_n(\lambda, \mu) = F_n(\lambda, -\lambda - \mu)$ has led to (4.10a) and (4.10b).

**Example 4.11.** By Proposition 4.10 the number of free parameters, on which the family $F_n(\lambda, \mu)$ depends, increases by 1 when $n$ increases by 3. The first six (starting with 0) associator polynomials are unique up to factor, but not the seventh $F_6(\lambda, \mu)$. One gets

- $F_6(\lambda, \mu) = 3 \beta_{30}(\lambda^2 + \lambda \mu + \mu^2)^3 + 3 \beta_{31} \lambda^2 \mu^2 (\lambda + \mu)^2$,
- $F_8(\lambda, \mu) = 5 \beta_{40}(\lambda^2 + \lambda \mu + \mu^2)^4 + 4 \beta_{41} \lambda^2 \mu^2 (\lambda + \mu)^2 (\lambda^2 + \lambda \mu + \mu^2)$,
- $F_9(\lambda, \mu) = \beta_{50}(\lambda^2 + \lambda \mu + \mu^2)^5 + 3 \beta_{51} \lambda^2 \mu^2 (\lambda + \mu)^2 (\lambda^2 + \lambda \mu + \mu^2)^2$,
- $F_10(\lambda, \mu) = \beta_{60}(\lambda^2 + \lambda \mu + \mu^2)^6 + 6 \beta_{61} \lambda^2 \mu^2 (\lambda + \mu)^2 (\lambda^2 + \lambda \mu + \mu^2)^3 + \beta_{62} \lambda^4 \mu^4 (\lambda + \mu)^4$.
- $F_7(\lambda, \mu) = \beta_{30} \lambda^2 \mu (\lambda + \mu)^2 (\lambda^2 + \lambda \mu + \mu^2)^2$,
- $F_9(\lambda, \mu) = \beta_{30} \lambda \mu (\lambda + \mu)^2 (\lambda^2 + \lambda \mu + \mu^2)^3 + 3 \beta_{41} \lambda^2 \mu^3 (\lambda + \mu)^3$,
- $F_{11}(\lambda, \mu) = \beta_{50} (\lambda + \mu)^2 (\lambda^2 + \lambda \mu + \mu^2)^4 + 4 \beta_{51} \lambda^3 \mu^3 (\lambda + \mu)^3 (\lambda^2 + \lambda \mu + \mu^2)$,
- $F_{13}(\lambda, \mu) = \beta_{60} \lambda^2 \mu (\lambda + \mu)^2 (\lambda^2 + \lambda \mu + \mu^2)^5 + 6 \beta_{61} \lambda^3 \mu^3 (\lambda + \mu)^3 (\lambda^2 + \lambda \mu + \mu^2)^2$.

One can check that the parameter $\beta_{31}$ is related with $\delta$ from Example 4.8 as $\delta = \beta_{31} + 6$.

Up to degree 10 the function $h(\lambda, \mu)$ from Lemma 4.6 is

$$h(\lambda, \mu) = 1 - \frac{1}{6} F_2(\lambda, \mu) + \frac{7}{360} F_4(\lambda, \mu) - \frac{31}{3 \cdot 7!} F_6(\lambda, \mu) + \frac{127}{15 \cdot 8!} F_8(\lambda, \mu) + \cdots$$

$\blacksquare$
4.3. Description of all odd compressed associators.

Lemma 4.12. The general solution of the equation (4.5b) is

\[ \text{Odd}(f(\lambda, \mu)) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2} \tilde{h}(\lambda, \mu), \]  

where \( \tilde{h}(\lambda, \mu) \) is a function satisfying the relations \( \tilde{h}(\lambda, \mu) = \tilde{h}(\mu, \lambda) = \tilde{h}(-\lambda, -\mu) = \tilde{h}(\lambda, -\lambda - \mu) \).

Proof. is analogous to the proof of Lemma 4.6. Let us swap the arguments \( \lambda, \mu \) in the equation (4.5b). Due to the symmetry \( f(\lambda, \mu) = f(\mu, \lambda) \) one gets

\[ \text{Odd}(f(\lambda, \mu)) + e^{\lambda} \text{Odd}(f(\lambda, -\lambda - \mu)) + e^{-\mu} \text{Odd}(f(\mu, -\lambda - \mu)) = 0. \]

By subtracting the above equation from (4.5b) one obtains

\[ (e^\mu - e^{-\mu}) \cdot \text{Odd}(f(\mu, -\lambda - \mu)) + (e^{-\lambda} - e^{\lambda}) \cdot \text{Odd}(f(\lambda, -\lambda - \mu)) = 0, \text{ or} \]

\[ \text{Odd}(f(\mu, -\lambda - \mu)) : \left( \frac{e^\lambda - e^{-\lambda}}{2} \right) = \text{Odd}(f(\lambda, -\lambda - \mu)) : \left( \frac{e^{\mu} - e^{-\mu}}{2} \right). \]

Introduce the function

\[ (4.12') \]

\[ \tilde{h}(\lambda, \mu) := -\text{Odd}(f(\mu, -\lambda - \mu)) : \left( \frac{e^\lambda - e^{-\lambda}}{2} \right) = -\text{Odd}(f(\lambda, -\lambda - \mu)) : \left( \frac{e^{\mu} - e^{-\mu}}{2} \right). \]

Then the new function is even and symmetric: \( \tilde{h}(\lambda, \mu) = \tilde{h}(\mu, \lambda) = \tilde{h}(-\lambda, -\mu) \). Let us substitute the expressions

\[ \text{Odd}(f(\mu, -\lambda - \mu)) = -\left( \frac{e^\lambda - e^{-\lambda}}{2} \right) \tilde{h}(\lambda, \mu), \quad \text{Odd}(f(\lambda, -\lambda - \mu)) = -\left( \frac{e^{\mu} - e^{-\mu}}{2} \right) \tilde{h}(\lambda, \mu) \]

into the equation (4.5b). One has

\[ \text{Odd}(f(\lambda, \mu)) = \left( e^{\mu} \frac{e^\lambda - e^{-\lambda}}{2} + e^{-\lambda} e^{\mu} - e^{-\mu} \right) \tilde{h}(\lambda, \mu) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2} \tilde{h}(\lambda, \mu). \]

So, (4.12) is proved. Let us substitute \((-\lambda - \mu)\) for \(\mu\) into the above equation:

\[ \text{Odd}(f(\lambda, -\lambda - \mu)) = -\frac{e^\mu - e^{-\mu}}{2} \tilde{h}(\lambda, -\lambda - \mu). \]

By comparing the latter formula with (4.12'), one gets \( \tilde{h}(\lambda, \mu) = \tilde{h}(\lambda, -\lambda, -\mu) \).

Proof of Theorem 1.5c. By Lemmas 4.6, 4.9, 4.12, and Proposition 4.10a, the general solution of the equation (1.5b)=(4.1b) is \( f(\lambda, \mu) = \text{Even}(f(\lambda, \mu)) + \text{Odd}(f(\lambda, \mu)) \), where

\[ 1 + \lambda \mu \cdot \text{Even}(f(\lambda, \mu)) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)} \left( \sum_{n=0}^{\infty} \sum_{k=0}^{[k/2]} \beta_{nk} \lambda^{2k} \mu^{2k}(\lambda + \mu)^{2k}\omega^{2n-6k} \right), \]

\[ \text{Odd}(f(\lambda, \mu)) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2} \left( \sum_{n=0}^{\infty} \sum_{k=0}^{[k/2]} \beta_{nk} \lambda^{2k} \mu^{2k}(\lambda + \mu)^{2k}\omega^{2n-6k} \right), \quad \omega = \sqrt{\lambda^2 + (\lambda + \mu)^2}. \]

Let us substitute \( \mu = -\lambda \) (then \( \omega = \lambda \)) into the former equation and apply Lemma 4.2:

\[ \sum_{n=0}^{\infty} \beta_{n0} \lambda^{2n} = 1 - \lambda^2 \text{Even}(f(\lambda, -\lambda)) = \frac{2\lambda}{e^\lambda - e^{-\lambda}}, \text{ hence} \sum_{n=0}^{\infty} \beta_{n0} \omega^{2n} = \frac{2\omega}{e^\omega - e^{-\omega}}. \]
as required. Let us relate $\alpha_{2n+1,0}$ with $\tilde{\beta}_{n0}$. One has $\text{Odd}(f(\lambda, \mu)) = \sum_{n=0}^{\infty} \sum_{k=0}^{2n+1} \alpha_{2n+1,k} \lambda^{2n-k+1} \mu^k$. Hence one obtains $\text{Odd}(f(\lambda, 0)) = \sum_{n=0}^{\infty} \alpha_{2n+1,0} \lambda^{2n+1}$. On the other hand, for $\mu = 0$, one gets

$$\frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2} - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \tilde{\beta}_{nk} \lambda^{2k} \mu^k (\lambda + \mu)^{2k} \omega^{2n-6k} = \frac{e^{\lambda} - e^{-\lambda}}{2} - \sum_{n=0}^{\infty} \tilde{\beta}_{n0} \lambda^{2n} = \sum_{n=0}^{\infty} \alpha_{2n+1,0} \lambda^{2n+1}. \quad \square$$

**Proof of Corollaries 1.6a-b.** (a) The first series $f^I(\lambda, \mu)$ is obtained from the general formula (1.5c) by taking $\beta_{nk} = \tilde{\beta}_{nk} = 0$. This solution is related with the associator polynomials $F_{2n}^I(\lambda, \mu) = (\lambda^2 + \lambda \mu + \mu^2)^n$. (b) The second series $f^{II}(\lambda, \mu)$ appears due to the associator polynomials $F_{2n}^{II}(\lambda, \mu) = \frac{(\lambda + \mu)^{2n} + \lambda^{2n} + \mu^{2n}}{2}$, $n > 1$. Actually, let $\gamma_k$ be the coefficients of the series $\frac{2\lambda}{e^\lambda - e^{-\lambda}} = \sum_{k=0}^{\infty} \gamma_k \lambda^{2k}$, see (4.3). Then the series $1 + \sum_{k=1}^{\infty} \gamma_k F_{2k}^{II}(\lambda, \mu)$ plays the role of the function $h(\lambda, \mu)$ from Lemma 4.6. It remains to compute

$$1 + \lambda \mu \cdot f^{II}(\lambda, \mu) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)} \left( 1 + \sum_{k=1}^{\infty} \gamma_k \frac{(\lambda + \mu)^{2n} + \lambda^{2n} + \mu^{2n}}{2} \right) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)} \cdot \frac{1}{2} \left( \frac{2(\lambda + \mu)}{e^{\lambda+\mu} - e^{-\lambda-\mu}} + \frac{2\lambda}{e^\lambda - e^{-\lambda}} + \frac{2\mu}{e^\mu - e^{-\mu}} - 1 \right) = \frac{1}{2} + \frac{1}{2} \cdot \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)} \left( \frac{2\lambda}{e^\lambda - e^{-\lambda}} + \frac{2\mu}{e^\mu - e^{-\mu}} - 1 \right) \Leftrightarrow (1.6b). \quad \square$$

5. **Compressed pentagon equation**

This section is devoted to the proof of the pentagon part of Theorem 1.5b. It turns out that if all commutators commute, then the compressed pentagon (1.3c) follows from the symmetry $\alpha_{kl} = \alpha_{lk}$ (the 12th key point), see Proposition 5.10.

5.1. **Generators and relations of the quotient $\bar{L}_4$.**

Here one studies the quotient $\bar{L}_4$, where the compressed pentagon (1.3c) = (5.9) lives.

**Definition 5.1** (alphabet $\mathbb{L}$, algebras $L_4$ and $\bar{L}_4$, simple and non-simple commutators).

(a) Let the Lie algebra $L_4$ be generated by the letters of the alphabet

$$\mathbb{L} = \{ a := t^{12}, b := t^{23}, c := t^{13}, d := t^{24}, e := t^{34}, v := t^{14} \}$$

and the relations

$$[a, c] = [b, v] = [c, d] = 0 \quad \text{and} \quad \{ x := [a, b] = [b, c] = [c, a], \quad y := [a, d] = [d, v] = [v, a], \quad z := [b, e] = [e, d] = [d, b], \quad u := [c, e] = [e, v] = [v, c] \}. $$

The Lie algebra $L_4$ is graded by $\text{deg}(s) = 1$ for any letter $s \in \mathbb{L}$. Put $\bar{L}_4 = L_4/[L_4, L_4, L_4]$. By $\bar{L}_4$ denote the algebra of formal series of elements from $\bar{L}_4$.

(b) Let $w$ be a word in the alphabet $\mathbb{L} = \{ t^{ij}, 1 \leq i < j \leq 4 \}$. Let $I_w$ be the set forming by the upper indices of the letters $t^{ij}$ including in $w$. If the set $I_w$ contains at most three different indices, then the commutator $[w] \in \bar{L}_4$ is called simple, otherwise $[w]$ is non-simple. For example, the commutators $[aab]$ and $[vad]$ are simple, but $[dab]$ is not simple. \[\blacksquare\]
Claim 5.2. The following relations hold in the quotient \( \tilde{L}_4 \)

(a) \([a + b + c, x] = 0, \ [a + d + v, y] = 0, \ [b + e + d, z] = 0, \ [c + e + v, u] = 0;\)

(b) \([da] + [ea] + [va] = 0, \ [db] + [eb] + [vb] = 0, \ [dx] + [ex] + [vx] = 0;\)

(c) \([dx] = -[cy] = -[cz] = [du], \ [ex] = -[ey] = -[az] = [au], \ [vx] = -[by] = -[vz] = [bu].\)

Proof. (a) By Definition 5.1a one gets \([a + b + c, a] = [c, a] - [a, b] \ (5.1a) = 0\) and similarly \([a + b + c, b] \ (5.1a) = 0,\) hence \([a + b + c, x] = 0.\) The other relations are proved analogously.

(b) By Definition 5.1a one has \([d + e + v, a] = (-[a, d] + [v, a]) - [a, e] \ (5.1a) = 0\) and similarly \([d + e + v, b] \ (5.1a) = 0,\) hence \([d + e + v, x] = 0.\)

(c) Definition 5.1a, the Jacobi identity \((J),\) and \([cd] = 0\) are used below.

\[
\begin{align*}
[dx] &= [5.1a] \ [d] = [5.1a] \ [dbc] \ (J) = -[bc] - [cda] \ (5.1a) = -[cz], \ [dx] = [5.1a] \ [dab] \ (J) = -[abd] - [bda] \ (5.1a) = [az] + [by], \\
[dx] &= [5.1a] \ [dca] \ (J) = -[adc] - [cad] \ (5.1a) = -[cy], \ [cy] = [5.1a] \ [cva] \ (J) = -[vac] - [acv] \ (5.1a) = [vx] + [au], \\
[cy] &= [5.1a] \ [cdv] \ (J) = -[dvc] - [vcd] \ (5.1a) = -[du], \ [cz] = [5.1a] \ [cbe] \ (J) = -[bec] - [ceb] \ (5.1a) = [bu] + [ex].
\end{align*}
\]

One gets \([dx] = -[cy] = -[cz] = [du] = [az] + [by] = -[bu] - [ex] = -[vz] - [au],\) i.e. \([bu] = -[dx] - [ex] \ (5.2b) = [vx] \) and \([au] = -[dx] - [vx] \ (5.2b) = [ex].\) Similarly, by \([ae] = 0\) one has \([ex] = [5.1a] \ [eab] \ (J) = -[abe] - [bca] \ (5.1a) = -[az], \ [az] = [5.1a] \ [aed] \ (J) = -[eda] - [dae] \ (5.1a) = [ey].\)

Then \([ex] = -[az] = -[ey] \) and \([by] = [dx] - [az] = [dx] + [ex] = -[vz].\) Finally, one has \([du] = [5.1a] \ [dev] \ (J) = -[vde] - [vde] \ (5.1a) = [vz] + [ey] \ (5.1a) = [vz] \) and \([ex] = [du] - [ey] = [dx] + [ex] = -[vz].\)

Claim 5.3. (a) Every simple (resp., non-simple) commutator \([w] \in \tilde{L}_4\) of degree 3 can be expressed linearly via \([ax], \ [bx], \ [ay], \ [dy], \ [bz], \ [ez], \ [cu], \ [eu]\) (resp., via \([dx]\) and \([ex]\)).

(b) The degree 3 part of \(\tilde{L}_4\) is linearly generated by 8 simple commutators \([ax], \ [bx], \ [ay], \ [dy], \ [bz], \ [ez], \ [cu], \ [eu]\) and 2 non-simple ones \([dx], \ [ex]\).

Proof. (a) The degree 3 part of \(\tilde{L}_4\) contains exactly

12 simple commutators \([ax], \ [bx], \ [ex], \ [ay], \ [dy], \ [vy], \ [bz], \ [ez], \ [dz], \ [cu], \ [eu], \ [vu]\) and
12 non-simple ones \([dx], \ [ex], \ [vx], \ [by], \ [cy], \ [ey], \ [az], \ [cz], \ [vz], \ [au], \ [bu], \ [du]\).

Due to the relations \((5.2a)\) one can throw out the four simple commutators \([cx], \ [vy], \ [dz],\) and \([vu].\) By the relations \((5.2c)\) any non-simple commutator reduces to one of \([dx], \ [ex],\) and \([vx].\) To eliminate \([vx]\) it remains to apply the relation \((5.2b):\) \([vx] = -[dx] - [ex].\)

(b) Any element of \(\tilde{L}_4\) is a sum of simple and non-simple commutators, i.e. \((a) \Rightarrow (b).\)

Lemma 5.4. (a) For every word \(w\) in the alphabet \(\mathbb{L} = \{a, b, c, d, e, v\},\) containing at least two letters, and for any letters \(s, s' \in \mathbb{L}, \ [ss'w] = [s'sw]\) holds in the quotient \(\tilde{L}_4.\)

(b) Let \(w\) be any word in the alphabet \(\mathbb{L},\) containing at least one letter \(d\) or \(e.\) Then in the quotient \(\tilde{L}_4\) the following relations hold:

\[
\begin{align*}
[adx] &= [edx], \quad [aex] = [e^2x], \quad [awx] = [ewx]; \\
[bdx] &= [(-d - e)dx], \quad [bex] = [(-d - e)ex], \quad [bwx] = [(-d - e)wx]; \\
[cdx] &= [d^2x], \quad [cex] = [dex], \quad [cwx] = [dwx].
\end{align*}
\]
Proof. (a) This is an analog of Claim 2.6a.

(b) Apply the item (a) and Claim 5.2 as follows
\[[adx] (5.2c) = [adu] (5.4a) = [dau] (5.2c) = [dex] (5.4a) = [edx], \quad [aex] (5.4a) = [eax] (5.2a) = -[ebx] - [ecx] = [e^2x];\]
\[[bdx] (5.2c) = [bdy] (5.4a) = [dbu] (5.2c) = [dwx] (5.2b) = [d(-d - e)x] (5.4a) \equiv [(-d - e)dx];\]
\[[bcx] (5.2c) = -[bey] (5.4a) = -[eby] (5.2c) = [evx] (5.2b) = [e(-d - e)x] (5.4a) \equiv [(-d - e)ex];\]
\[[cx] (5.4a) = [dax] - [dx] \equiv [d^2x], \quad [ecx] (5.2c) \equiv [-cey] (5.4a) \equiv [-cey] (5.2c) = [edax].\]

If a word \(w\) in \(L\) contains at least one letter \(d\) or \(e\), then by the item (a) there is a word \(w'\) in \(L\) such that \([wx] = [w'dx]\) (without loss of generality) or \([wx] = [w'ex]\). Then
\[ [awx] = [aw'dx] \equiv [w''adx] \equiv [w'edx] \equiv [w'edx] = [ew'dx] = [ewx].\]

The proof in the other cases is similar. \(\square\)

Lemma 5.5. (a) In each degree \(n \geq 2\) every simple commutator \([w] \in \bar{L}_4\) can be expressed linearly via \(4(n - 1)\) simple ones \([a^kb^lx], [a^kd^ly], [b^ke^lz], [c^ke^lu], k + l = n - 2, k, l \geq 0\).

(b) In each degree \(n \geq 3\) any non-simple commutator \([w] \in \bar{L}_4\) can be expressed linearly via \(n - 1\) non-simple commutators \([d^ke^lx], \text{ where } k + l = n - 2, k, l \geq 0\).

(c) In any degree \(n \geq 3\) the quotient \(L_4\) is linearly generated by \(4(n - 1)\) simple commutators \([a^kb^lab], [a^kd^lad], [b^ke^lbe], [c^ke^lce]\) and \(n - 1\) non-simple ones \([d^ke^lba], k + l = n - 2, k, l \geq 0\).

Proof. (a) Let us consider a simple commutator \([w]\) not containing upper index 4. Hence the commutator \([w]\) contains only the letters \(a, b, c\). By the relations (5.2a) \([cx] = [-ax] - [bx]\), one can express \([w]\) via commutators \([a^kb^lx], k, l \geq 0\). The proof is analogous for another three types of simple commutators.

(b) Let \(w = w''s_1s_2s_3\) be the given word, where \(s_1, s_2, s_3\) are the three last letters of \(w\). Since \([w]\) is non-simple, then by Lemma 5.4a one can permute the letters of \(w\) in such a way that one may assume the commutator \([s_1s_2s_3]\) is non-simple. By Claim 5.3b the commutator \([s_1s_2s_3]\) is expressed via \([dx]\) and \([cx]\). Then by Lemma 5.4a \([w] = [w''s_1s_2s_3]\) can be written in terms of \([dw''x]\) and \([ew''x]\). Hence one can apply the induction on the length of \(w\).

The item (c) follows from (a) and (b). \(\square\)

5.2. Calculations in the quotient \(L_4\).

Claim 5.6. For all \(k \geq 0, l \geq 1\), in the quotient \(L_4\) the following relations hold:

(a) \([b^kd^lx] = [(-d - e)^kdx], \quad [d^ky] = -[d^k(-d - e)^l x];\]

(b) \([b^ke^lx] = [(-d - e)^kdx], \quad [e^kuy] = [d^k(-d - e)^l x];\]

(c) \([e^kux] = [d^ke^lx], \quad [e^kd^ly] = [e^kdx];\]

(d) \([b^kax] = [d^ke^lx], \quad [c^kux] = [d^ke^lx].\]

Proof. (a) By Lemma 5.4b one has \([bd^lx] = [(-d - e)d^lx], \text{ i.e. } [b^2d^lx] = [(-d - e)bd^lx] = [(-d - e)^2d^lx]\) and so on. \([b^kd^lx] = [(-d - e)^kd^lx]\) holds for all \(k \geq 0, l \geq 1\). Similarly,
\[[by] (5.2c) = -[vwx] (5.2b) = -[(-d - e)x] \Rightarrow [b^2y] = -[b(-d - e)x] (5.4b) \equiv -[(-d - e)^2x]\]
\[\Rightarrow [b^ky] = -[(-d - e)^lx] \Rightarrow [d^kby] = -[d^k(-d - e)^l x] \text{ for all } k \geq 0, l \geq 1.\]

The items (b), (c), and (d) are proved analogously to (a). Apply the following formulæ:

\[[c^2z] (5.2c) = -[c^2dx] (5.4b) = -[d^2x] \Rightarrow [c^2z] = -[d^2x] \Rightarrow [b^k c^l z] = -[b^k d^l x] (5.6a) \equiv -[(-d - e)^k d^lx] \text{ for all } k \geq 0, l \geq 1.\]
\[ [bu]^{(5.2c)} = [vx]^{(5.2b)} = [(d-e)x] \Rightarrow [b^2u] = b(-d-e)x \quad (5.4b) \Rightarrow [(d-e)x] \Rightarrow [b^l u] = [(-d-e)^l x] \Rightarrow [cb^l u] = [c(-d-e)^l x] \quad (5.4b) \Rightarrow [d(-d-e)^l x] \Rightarrow [c^{b^l} u] = [d^k(-d-e)^l x] \text{ for all } k \geq 0, \ l \geq 1. \]

\[ [ey]^{(5.2c)} \Rightarrow [e^2y] = [e^2x] \Rightarrow [e^l y] = [e^lx] \Rightarrow [d^k e^l y] = [d^k e^lx] = 0. \]

\[ [du]^{(5.2c)} = [dx] \Rightarrow [d^l u] = [d^lx] \Rightarrow [e^k d^l u] = [e^k d^lx] \text{ for all } k \geq 0, \ l \geq 1. \]

\[ [cy]^{(5.2c)} = [dx] \Rightarrow [c^2y] = [c^2x] \Rightarrow [c^y] = [c^x] \Rightarrow [d^lx] \Rightarrow [ac^y] = [a^x] \Rightarrow [a^l y] = [a^lx] \Rightarrow [c^k a^l y] = [c^k a^lx]. \]

Finally, one has \[ [au]^{(5.2c)} = [ex] \Rightarrow [a^2u] = [aex] \Rightarrow [e^2x] \Rightarrow [a^l u] = [e^lx] \Rightarrow [ca^l u] = [ce^lx] \Rightarrow [d^l x] \Rightarrow [c^k a^l u] = [d^k e^lx]. \]

Claim 5.7. For any \( k \geq 0 \), in the quotient \( \mathbb{L}_4 \) one has

(a) \( [(b + d)^k x] = [b^k x] - [(-d-e)^k x] + [(e)^k x], \) \( [(b + d)^k y] = [d^k y] + [d^k x] - [(-e)^k x]; \)

(b) \( [(b + c)^k z] = [b^k z] + [(-d-e)^k x] - [(e)^k x], \) \( [(b + c)^k u] = [c^k u] - [d^k x] + [(-e)^k x]; \)

(c) \( [(d + e)^k y] = [d^k y] + [d^k x] - [(d + e)^k x], \) \( [(d + e)^k u] = [e^k u] - [e^k x] + [(d + e)^k x]; \)

(d) \( [(a + c)^k y] = [a^k y] + [c^k x] - [(d + e)^k x], \) \( [(a + c)^k u] = [c^k u] - [d^k x] + [(d + e)^k x]. \)

Proof. (a) For each \( k \geq 0 \), one obtains

\[ [(b + d)^k x] = [b^k x] + \sum_{j=1}^{k} \binom{k}{j} b^{k-j} d^j x \quad (5.6a) \]

\[ = [b^k x] - [(-d-e)^k x] + \sum_{j=0}^{k} \binom{k}{j} (-d-e)^{k-j} d^j x = [b^k x] - [(-d-e)^k x] + [(d - d + e)^k x]. \]

\[ [(b + d)^k y] = [d^k y] + \sum_{j=0}^{k} \binom{k}{j} (-d-e)^{k-j} d^j x \quad (5.6a) \]

\[ = [d^k y] + [d^k x] - \sum_{j=0}^{k} \binom{k}{j} (-d-e)^{k-j} d^j x = [d^k y] + [d^k x] - [(-d - e + d)^k x]. \]

The items (b), (c), (d) are proved analogously to (a). The following formulae are used:

\[ [(b+c)^k z] - [b^k z] = \sum_{j=1}^{k} \binom{k}{j} [b^{k-j} c^j z] = \sum_{j=1}^{k} \binom{k}{j} [(-d-e)^{k-j} d^j x] = [(-d-e)^k x] - [(e)^k x], \]

\[ [(b + c)^k u] - [c^k u] = \sum_{j=0}^{k-1} \binom{k}{j} [b^{k-j} c^j u] = \sum_{j=0}^{k-1} \binom{k}{j} [(-d-e)^{k-j} d^j x] = [d^k x] + [(-e)^k x], \]

\[ [(d + e)^k y] - [d^k y] = \sum_{j=1}^{k} \binom{k}{j} [d^{k-j} e^j y] = \sum_{j=1}^{k} \binom{k}{j} [d^{k-j} e^j x] = [d^k x] - [(d + e)^k x]. \]
\[
((d + e)^k u) - [e^k u] = \sum_{j=1}^{k} \binom{k}{j} [e^{k-j} d^j u] \stackrel{(5.6c)}{=} \sum_{j=1}^{k} \binom{k}{j} [e^{k-j} d^j x] = -[e^k x] + [(d + e)^k x],
\]

\[
([a + c]^k y) - [a^k y] = \sum_{j=1}^{k} \binom{k}{j} [a^{k-j} c^j y] \stackrel{(5.6d)}{=} -\sum_{j=1}^{k} \binom{k}{j} [e^{k-j} d^j x] = [e^k x] - [(d + e)^k x],
\]

\[
([a + c]^k u) - [c^k u] = \sum_{j=1}^{k} \binom{k}{j} [c^{k-j} a^j y] \stackrel{(5.6d)}{=} -\sum_{j=1}^{k} \binom{k}{j} [d^{k-j} e^j x] = -[d^k x] + [(d + e)^k x].
\]

Lemma 5.8. (a) For all \(k, l \geq 0\), in the quotient \(\bar{L}_4\) one gets

\[
\begin{align*}
(a + c)^k (d + e)^l & (a + c)(d + e) = ([a^k d^j y] + [c^k e^l u]) + [e^k d^j x] - [d^k e^j x]. \\
(b) & \text{For all } k, l \geq 0, \text{ in the quotient } \bar{L}_4 \text{ one has}
\end{align*}
\]

\[
([a + c]^k (d + e)^l (a + c)(d + e)) = ([a^k d^j y] + [c^k e^l u]) + [e^k d^j x] - [d^k e^j x].
\]

Proof. (a) One has \([a, b + d] \stackrel{(5.1a)}{=} x + y, [b + c, e] \stackrel{(5.1a)}{=} z + u\). For all \(k \geq 0, l \geq 1\), one gets

\[
[a^k (b + d) d^l a(b + d)] = [a^k (b + d) d^l x] + [a^k (b + d)^l y] \stackrel{(5.7a)}{=} ([a^k b^l x] - [a^k (-d - e)^l x]) + ([a^k b^l z] + [c^k e^l u]) - [d^k e^l x] - [e^k (-d - e)^l x].
\]

(b) Analogously to the item (a), for all \(k, l \geq 0\), one obtains

\[
\begin{align*}
(a + c)^k (d + e)^l (a + c)(d + e) & \stackrel{(5.1a)}{=} ([a^k (d + e)^l y] + [(a + c)^k (d + e)^l u]) \\
& \stackrel{(5.7c)}{=} ([a + c]^k d^j y] + [(a + c)^k d^j x] - [(a + c)^k (d + e)^l x] + [(a + c)^k e^l u] - [(a + c)^k e^l x] + [(a + c)^k (d + e)^l x] \\
& \stackrel{(5.2a),(5.4a)}{=} [d^l (a + c) k^y] + [d^l (-b)^k x] + [e^l (a + c) k^y] - [e^l (-b)^k x] \stackrel{(5.7d),(5.4a)}{=} \\
& ([d^l a^k y] + [d^l e^k x] - [d^l (d + e)^k x]) + ([e^l c^k u] - [e^l d^k x] + [e^l (d + e)^k x]) + [(-b)^l (d^l - e^l)x] \\
& \stackrel{(5.4a),(5.4b)}{=} ([a^k d^j y] + [c^k e^l u]) + [e^k d^j x] - ([e^l d^j x] + [e^l (d + e)^k x]) + ([d + e]^k (d^l - e^l)x] \\
& \stackrel{(5.4b)}{=} [(-b)^k (d^l - e^l)x] \text{ was used} \quad = ([a^k d^j y] + [c^k e^l u]) + [e^k d^j x] - [d^k e^l x].
\end{align*}
\]

Note that the relation \([(-b)^k (d^l - e^l)x] \stackrel{(5.4b)}{=} [(d + e)^k (d^l - e^l)x]\) holds for any \(l \geq 0\). □
5.3. Checking the compressed pentagon \((1.3c)\).

**Lemma 5.9.** For any compressed associator \(\bar{\varphi} \in \hat{L}_3\), the compressed pentagon \((1.3c)\) is equivalent to the following equation in the algebra \(\hat{L}_4\):

\[
\bar{\varphi}(b, e) + \bar{\varphi}(a + c, d + e) + \bar{\varphi}(a, b) = \bar{\varphi}(a, b + d) + \bar{\varphi}(b + c, e).
\]

**Proof.** Let us rewrite explicitly the pentagon \((1.3c)\) for a compressed associator \(\bar{\varphi} \in \hat{L}_3\)

\[
\exp(\bar{\varphi}(b, e)) \cdot \exp(\bar{\varphi}(a + c, d + e)) \cdot \exp(\bar{\varphi}(a, b)) = \exp(\bar{\varphi}(a, b + d)) \cdot \exp(\bar{\varphi}(b + c, e)).
\]

Since in the quotient \(\hat{L}_4\) all commutators commute, then the series \(\bar{\varphi}(b, e), \bar{\varphi}(a + c, d + e), \bar{\varphi}(a, b), \bar{\varphi}(a, b + d), \text{ and } \bar{\varphi}(b + c, e)\) commute with each other in \(\hat{L}_4\). Hence, taking the logarithm of both sides of the above pentagon, one needs to apply the simplest case of CBH formula \((2.3)\): \(\log(\exp(P) \cdot \exp(Q)) = P + Q\) provided that \(P, Q\) commute. 

**Proposition 5.10.** Let \(f(\lambda, \mu) = \sum_{k,l \geq 0} \alpha_{kl} k^k \mu^l\) be the generating function of the coefficients \(\alpha_{kl} = \alpha_{lk}\) of a compressed Drinfeld associator \(\bar{\varphi} \in \hat{L}_3\). Then in the algebra \(\hat{L}_4\) the compressed pentagon equation \((1.3c)\) = \((5.9)\) follows from the symmetry \(\alpha_{kl} = \alpha_{lk}\).

**Proof.** By Lemma 5.8b the left hand side of \((5.9)\) is

\[
\sum_{k,l \geq 0} \alpha_{kl} \left( [b^k c^l d] + [(a + c)^k(d + e)^l(a + c)(d + e)] + [a^k b^l a] \right) \quad (5.8b)
\]

\[
= \sum_{k,l \geq 0} \alpha_{kl} \left( [b^k c^l d] + [(a^k d)^l + [c^k e^l u] + [e^k d^l x] - [d^k e^l x]] + [a^k b^l x] \right). \quad (5.8b)
\]

Similarly by Lemma 5.8a the right hand side of \((5.9)\) is

\[
\sum_{k,l \geq 0} \alpha_{kl} \left( [a^k (b + d)^k a(b + d)] + [(b + c)^k e(b + c) e] \right) \quad (5.8a)
\]

\[
= \sum_{k,l \geq 0} \alpha_{kl} \left( [(a^k b^l x) + [a^k d] + [c^k e^l x] - [c^k d^l x] - [d^k e^l x]] \right) + \left( [(b^k c^l d) + [c^k e^l u] - [d^k e^l x]] \right). \quad (5.8a)
\]

The difference is

\[
\sum_{k,l \geq 0} \alpha_{kl} \left( [e^k (d - e)^l x] - [(d - e)^k e^l x] \right) \quad (5.4a)
\]

is \(0\) if \(\alpha_{kl} = \alpha_{lk}\). 

6. Drinfeld series, zeta values, and problems

In this section one shall check that the Drinfeld series from Definition 6.1b (a compressed associator expressed via zeta values) is contained in the general family \((1.5c)\).

6.1. Riemann zeta-function of even integers.

**Definition 6.1** (Riemann zeta function \(\zeta(n)\), the Drinfeld series \(s(\lambda, \mu)\)).

(a) Let \(\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}\) be the classical Riemann zeta function. Put \(\theta_n := \frac{\zeta(n)}{n(\pi \sqrt{-1})^n}\).

(b) Introduce \(S(\lambda) = \sum_{n=2}^{\infty} \theta_n \lambda^n = \sum_{n=2}^{\infty} \frac{\zeta(n) \lambda^n}{n(\pi \sqrt{-1})^n}\). The Drinfeld series is

\[
s(\lambda, \mu) = S(\lambda) + S(\mu) - S(\lambda + \mu) = \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} \frac{\lambda^n + \mu^n - (\lambda + \mu)^n}{(\pi \sqrt{-1})^n}.
\]

Due to the change \(t^{ij} \mapsto 2t^{ij}\) in the hexagon \((1.3b)\), one has missed \(2^n\) in the denominators of \(s(\lambda, \mu)\). The following theorem is quoted from [12, Chapter XIX, remark 6.6(b), p. 468].
Theorem 6.2. There is a compressed Drinfeld associator defined by
\[
\tilde{f}^D(\lambda, \mu) = 1 + \lambda \mu f^D(\lambda, \mu) = \exp(s(\lambda, \mu)).
\]

The Drinfeld series \( s(\lambda, \mu) \) leads to the well-known formula for even zeta values.

Lemma 6.3. For each \( n \geq 1 \), one has \( 2n\theta_{2n} = -\frac{2^{2n}B_{2n}}{2(2n)!} \) and \( \zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n} \).

**Proof.** Theorem 1.5c will be used for \( \mu = -\lambda \). Lemma 4.2 says that \( f(\lambda, -\lambda) = \frac{1}{\lambda^2} - \frac{2}{\lambda(e^\lambda - e^{-\lambda})} \). By Definition 6.1b one has \( s(\lambda, -\lambda) = 2 \sum_{n=1}^{\infty} \theta_{2n}\lambda^{2n} \). Let us substitute \( \mu = -\lambda \) in the formula (6.2). Then by using the formula (1.5c) one gets
\[
\log((\tilde{f}(\lambda, -\lambda))) = \log(1 - \lambda^2 f(\lambda, -\lambda)) = \log \left( \frac{2\lambda}{e^\lambda - e^{-\lambda}} \right) = 2 \sum_{n=1}^{\infty} \theta_{2n}\lambda^{2n}.
\]

Taking the first derivative of the above equation by \( \lambda \), one obtains
\[
\left( \frac{e^\lambda - e^{-\lambda} - \lambda(e^\lambda + e^{-\lambda})}{(e^\lambda - e^{-\lambda})^2} \right) : \left( \frac{\lambda}{e^\lambda - e^{-\lambda}} \right) = \frac{1}{\lambda} - 1 - \frac{2}{e^{2\lambda} - 1} = 2 \sum_{n=1}^{\infty} (2n\theta_{2n})\lambda^{2n-1}.
\]

Multiply the resulting equation by \( \lambda \) and use the definition of the Bernoulli numbers:
\[
1 - \lambda - \frac{2\lambda}{e^{2\lambda} - 1} = -\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2\lambda)^{2n} = 2 \sum_{n=1}^{\infty} (2n\theta_{2n})\lambda^{2n}, \text{ hence } 2n\theta_{2n} = -\frac{B_{2n}}{2(2n)!}2^{2n}.
\]

It remains to use the formula from Definition 6.1a: \( \theta_{2n} = \frac{\zeta(2n)}{2n(\pi\sqrt{-1})^{2n}} = (-1)^n \frac{\zeta(2n)}{2n\pi^{2n}} \).

Example 6.4. By Lemma 6.3 and Table A.2 of Appendix one can easily calculate
\[
\theta_2 = -\frac{1}{12}, \quad \theta_4 = \frac{1}{360}, \quad \theta_6 = -\frac{1}{5670}, \quad \theta_8 = \frac{1}{75600}, \quad \theta_{10} = -\frac{1}{935550};
\]
\[
\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}. \quad \blacktriangleleft \]

6.2. Substitution of the Drinfeld series \( s(\lambda, \mu) \).

Claim 6.5. The Drinfeld series \( s(\lambda, \mu) \) satisfies the following equations:
\[
(6.5a) \quad \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)} \left( \frac{2\omega}{e^\omega - e^{-\omega}} + \sum_{n=3}^{\infty} h_n(\lambda, \mu) \right) = \text{Even} \left( \exp(s(\lambda, \mu)) \right), \quad \text{and}
\]
\[
(6.5b) \quad \lambda \mu \cdot \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2} \left( \sum_{n=0}^{\infty} \tilde{h}_n(\lambda, \mu) + \sum_{n=3}^{\infty} \bar{h}_n(\lambda, \mu) \right) = \text{Odd} \left( \exp(s(\lambda, \mu)) \right),
\]
where \( h_n(\lambda, \mu) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \beta_{nk}\lambda^{2k}\mu^{2k}(\lambda + \mu)^{2k}\omega^{2n-6k} \) for \( n \geq 3 \), \( \omega = \sqrt{\lambda^2 + \lambda\mu + \mu^2} \).

The polynomials \( \bar{h}_n(\lambda, \mu) \) are defined by the same formula as \( h_n(\lambda, \mu) \), except the coefficients \( \beta_{nk} \in \mathbb{C} \) are substituted for \( \beta_{nk} \in \mathbb{C} \).

**Proof** follows from Theorems 1.5c, 6.2, and \( \text{Odd}(1 + \lambda \mu f^D(\lambda, \mu)) = \lambda \mu \text{Odd}(f^D(\lambda, \mu)) \).
Proof. Actually, one has $\exp(\text{Even}(s(\lambda, \mu))) = \frac{\exp(s(\lambda, \mu)) + \exp(-s, -\mu)}{2}$

$= \exp\left(\text{Even}(s(\lambda, \mu)) + \text{Odd}(s(\lambda, \mu))\right) + \exp\left(\text{Even}(s(\lambda, \mu)) - \text{Odd}(s(\lambda, \mu))\right)

= \exp\left(\text{Even}(s(\lambda, \mu))\right) \cdot \frac{\exp(\text{Odd}(s(\lambda, \mu)) + \exp(-\text{Odd}(s(\lambda, \mu)))}{2} \Leftrightarrow (6.6a).

The formula (6.6b) is proved absolutely analogously. \qed

Claim 6.7. (a) For the series $S(\rho)$, one has $\exp\left(-2\text{Even}(S(\rho))\right) = \frac{e^\rho - e^{-\rho}}{2\rho}$.

(b) For the Drinfeld series $s(\lambda, \mu)$, one has

\begin{equation}
(6.7) \quad \exp\left(\text{Even}(s(\lambda, \mu))\right) = \sqrt{\frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)}} \cdot \frac{2\lambda}{e^\lambda - e^{-\lambda}} \cdot \frac{2\mu}{e^\mu - e^{-\mu}}.
\end{equation}

Proof. (a) By Definition 6.1b, one obtains $\text{Even}(S(\rho)) = \sum_{n=1}^{\infty} \theta_{2n}\rho^{2n}$. So, one needs to get

$-2 \sum_{n=1}^{\infty} \theta_{2n}\rho^{2n} = \log \left(\frac{e^\rho - e^{-\rho}}{2\rho}\right)$. This formula holds for $\rho = 0$. Then it suffices to check that the first derivatives of both hand sides are equal. For the left hand side, apply the formula for $2n\theta_{2n}$ from Lemma 6.3:

\begin{equation}
-2 \sum_{n=1}^{\infty} 2n\theta_{2n}\rho^{2n-1} = \frac{1}{\rho} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2\rho)^{2n} \overset{(2.1b)}{=} \frac{1}{\rho} \left(\frac{2\rho}{e^{2\rho} - 1} - 1 + \rho\right) = 1 - \frac{1}{\rho} + \frac{2}{e^{2\rho} - 1}.
\end{equation}

For the right hand side, one gets

$$\left(\frac{e^\rho - e^{-\rho}}{2\rho}\right)^{'} = \frac{(e^\rho + e^{-\rho})\rho - (e^\rho - e^{-\rho})}{\rho^2} \cdot \frac{\rho}{e^\rho - e^{-\rho}} = 1 - \frac{1}{\rho} + \frac{2}{e^{2\rho} - 1}.$$

(b) Apply the item (a) and Definition 6.1b:

\begin{equation}
\exp\left(\text{Even}(s(\lambda, \mu))\right) \overset{(6.1b)}{=} \exp\left(\text{Even}(S(\lambda))\right) \cdot \exp\left(\text{Even}(S(\mu))\right) \cdot \exp\left(-\text{Even}(S(\lambda+\mu))\right) \overset{(6.7a)}{=}
\end{equation}

\begin{equation}
\sqrt{\frac{2\lambda}{e^\lambda - e^{-\lambda}}} \cdot \sqrt{\frac{2\mu}{e^\mu - e^{-\mu}}} \cdot \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)} = \sqrt{\frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)}} \cdot \frac{2\lambda}{e^\lambda - e^{-\lambda}} \cdot \frac{2\mu}{e^\mu - e^{-\mu}}. \qed
\end{equation}

Claim 6.8. The equations (6.5a) and (6.5b) are equivalent to (6.8a) and (6.8b), respectively

\begin{equation}
(6.8a) \quad \sum_{k=0}^{\infty} \frac{(\theta(\lambda, \mu))^{2k}}{(2k)!} = h(\lambda, \mu) \sqrt{\frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)}} \cdot \frac{e^\lambda - e^{-\lambda}}{2\lambda} \cdot \frac{e^\mu - e^{-\mu}}{2\mu},
\end{equation}

\begin{equation}
(6.8b) \quad \sum_{k=0}^{\infty} \frac{(\theta(\lambda, \mu))^{2k+1}}{(2k+1)!} = h(\lambda, \mu)\lambda(\mu + \mu) \sqrt{\frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)}} \cdot \frac{e^\lambda - e^{-\lambda}}{2\lambda} \cdot \frac{e^\mu - e^{-\mu}}{2\mu},
\end{equation}

where $h(\lambda, \mu)$ is given by (6.3).
where \( \theta(\lambda, \mu) = -\sum_{n=1}^{\infty} \theta_{2n+1} F_{2n+1}^{D}(\lambda, \mu) = \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n+1)}{(2n+1)\pi^{2n+1}} \left( \lambda^{2n+1} + \mu^{2n+1} - (\lambda + \mu)^{2n+1} \right) \),

\[ h(\lambda, \mu) = \frac{2\omega}{e^\omega - e^{-\omega}} + \sum_{n=3}^{\infty} h_n(\lambda, \mu), \quad \tilde{h}(\lambda, \mu) = \sum_{n=1}^{\infty} \beta_n \omega^{2n} + \sum_{n=3}^{\infty} \tilde{h}_n(\lambda, \mu). \]

The polynomials \( h_n(\lambda, \mu) \) and \( \tilde{h}_n(\lambda, \mu) \) are the same as in Claim 6.5.

**Proof** follows from Claims 6.6–6.7, the formula \( \text{Odd}(s(\lambda, \mu)) = \theta(\lambda, \mu) \), and

\[
\text{Even} \left( \exp \left( \text{Odd}(s(\lambda, \mu)) \right) \right) = \sum_{k=0}^{\infty} \frac{\theta^{2k}(\lambda, \mu)}{(2k)!}, \quad \text{Odd} \left( \exp \left( \text{Odd}(s(\lambda, \mu)) \right) \right) = \sum_{k=0}^{\infty} \frac{\theta^{2k+1}(\lambda, \mu)}{(2k+1)!},
\]

**Proposition 6.9.** There exist parameters \( \beta_{nk}, \tilde{\beta}_{nk} \in \mathbb{C} \), expressed linearly via monomials of odd zeta values, such that the equations \((6.8a)\) and \((6.8b)\) hold identically. Hence the Drinfeld series \( s(\lambda, \mu) \) does not lead to polynomial relations between odd zeta values.

**Proof.** The left hand side of \((6.8a)\) is unchanged under the transformation \( \mu \leftrightarrow (\lambda - \mu) \). By Lemma 4.9 such a series is a sum of associator polynomials of even degrees. In other words, there exist parameters \( \beta'_{nk} \in \mathbb{C} \), expressed in terms of \( \theta_{2n+1} \), such that

\[
\text{the left hand side of } (6.8a) = 1 + \sum \sum \beta'_{nk} \lambda^{2k} \mu^{2k}(\lambda + \mu)^{2k} (\lambda^2 + \lambda \mu + \mu^2)^{n-3k}.
\]

Actually, the left hand side of \((6.8a)\) minus 1 is divided by \( \lambda^2 \mu^2 (\lambda + \mu)^2 \), hence \( \beta'_{n0} = 0 \) for \( n \geq 1 \). The right hand side of \((6.8a)\) has the same form: it equals to 1 when \( \lambda + \mu = 0 \). Then one can find the parameters \( \beta_{nk} \) via \( \beta'_{nk} \) (hence via odd zeta values) in such a way that the equation \((6.8a)\) holds identically. The proof for the equation \((6.8b)\) is similar. \( \square \)

So, Proposition 6.9 supports the long standing conjecture in the number theory: odd zeta values are algebraically independent over the rationals \( \mathbb{Q} \).

Proposition 6.9 will be reproved explicitly up to degree 7 in Claim A.6 of Appendix.

**Proof of Corollary 1.6c** follows from Proposition 6.9 since all monomials consisting of odd zeta values can be considered as free parameters. The third associator \((1.6c)\) is obtained from the Drinfeld series \( f^D(\lambda, \mu) \) by taking the free parameters \( \zeta(2k + 1) = 0 \) for each \( k \geq 1 \). \( \square \)

### 6.3. Conjectures and open problems

Theorem 1.5 describes only compressed associators. This is a first step in the general problem to find a complete rational associator. Theorem 1.5c gives a hope to describe Drinfeld associators up to triple commutators.

**Problem 6.10.** (a) Is it true that any compressed Drinfeld associator is the projection under \( \hat{L}_3 \to \left[ [ \hat{L}_3, \hat{L}_3], [\hat{L}_3, \hat{L}_3] \right] \) of the logarithm \( \varphi(a, b) \) of an honest one from Definition 1.3b? (b) Describe all Drinfeld associators up to triple commutators. In other words, solve the hexagon and pentagon in the quotient \( L_3/[L_3, [L_3, L_3']] \), where \( L_3' = [L_3, L_3] \).

A compressed associator \( \hat{\varphi}(a, b) \) already contains a lot of information. One may try to pass through the LM-BN construction [16] [3] to get a well-defined invariant of knots in a quotient of the algebra \( A \) of chord diagrams. Let \( \Delta \varphi \) be the difference between the exact logarithm of a Drinfeld associator \( \varphi(a, b) \in \hat{L}_3 \) and its compressed image \( \hat{\varphi}(a, b) \in \hat{L}_3 \). Then

\[
\Delta \Phi = \exp(\hat{\varphi} + \Delta \varphi) - \exp(\hat{\varphi}) = \Delta \varphi + \frac{1}{2} (\Delta \varphi)^2 + \Delta \varphi \hat{\varphi} + \hat{\varphi} \Delta \varphi + \cdots
\]
is the error of a Drinfeld associator in the completion $\hat{A}_3$. By the definition of the quotient $L_3$ the error $\Delta \varphi$ is expressed via differences $[w_{kl}ab] - [a^kb^l ab]$, where $w_{kl}$ is a word containing exactly $k$ letters $a$ and $l$ letters $b$.

One wants to factorize the algebra $A$ in such a way that the error $\Delta \Phi$ disappears and the LM-BN construction leads to a knot invariant in this quotient. Since all terms of $\Delta \Phi$ contain $\Delta \varphi$ as a factor, then it suffices to kill chord diagrams containing $[w_{kl}ab] - [a^kb^l ab]$. Roughly, the LM-BN construction maps chord diagrams on $n$ vertical strands onto chord diagrams on the circle. Let $\Sigma$ be a sum of commutators containing the symbols $a, b, c$. Define the closure $\tilde{\Sigma} = 0$ as the relation $\Sigma = 0$ considered in the algebra $\hat{A}$ of chord diagrams. Formally, one draws the relation $\Sigma = 0$ on 3 vertical strands and assume that these strands are three arcs of a circle. For instance, the 4T relation from Definition 1.1b is the closure of $\Sigma = [a, b] - [b, c]$. Since $[abab] = [baab]$ holds in $L_3$, then at the stage of chord diagrams the first non-trivial relation is the closure of $[[ab], [aab]] = [ab] \cdot [aab] - [aab] \cdot [ab]$.

In the above figure the relation was drawn briefly by using STU relations from \cite{2}.

After Vassilev’s paper \cite{21} one usually uses another definition of Vassiliev invariants via chord diagrams \cite{2}. Roughly, a Vassiliev invariant of framed knots is the composition of the Kontsevich integral and a weight function on $A$, i.e. a linear function on chord diagrams, satisfying the 4T relations, see Definition 1.1b.

**Definition 6.11** (compressed algebra $\hat{A}$ of chord diagrams, compressed Vassiliev invariants).

(a) For a word $w_{kl}$ containing exactly $k$ letters $a$ and exactly $l$ letters $b$, put $\Sigma(w_{kl}) := [w_{kl}ab] - [a^kb^l ab]$, $k, l \geq 1$. Let $\hat{A}$ be the quotient of the classical algebra $A$ of chord diagrams on the circle by the ideal generated by the relations $\Sigma(w_{kl}) = 0$ for all $k, l \geq 1$.

(b) A compressed weight function is a linear function on the compressed algebra $\hat{A}$. In other words, a Vassiliev invariant is compressed, if the corresponding weight function satisfies $\Sigma(w_{kl}) = 0$ for all $k, l \geq 1$. The compressed Kontsevich integral $Z_{\hat{K}}$ of a knot $K$ is the image of the classical Kontsevich integral $Z_K$ under the natural projection $A \rightarrow \hat{A}$.

Vassiliev invariants of degrees 2, 3, 4 are compressed ones, i.e. the theory is not empty.

**Problem 6.12.** (a) Check carefully that the LM-BN construction \cite{16,3}, for a compressed Drinfeld associator, gives rise to a well-defined knot invariant in the compressed algebra $\hat{A}$. Does the resulting invariant depend on a particularly chosen compressed associator?

(b) Which quantum invariants are compressed Vassiliev invariants?

(c) Describe all compressed Vassiliev invariants (as linear functions on the algebra $\hat{A}$).

(d) Compute the compressed Kontsevich integral for non-trivial knots, e.g. torus knots.

(e) Which knots can be classified via compressed Vassiliev invariants?
Claim A.1. For each \( n \geq 1 \), the following formulae hold:
\[
\begin{align*}
C_{2,2n} &= C_{2n,2} = B_{2n}, & C_{4,2n+1} &= -C_{2n+1,4} = 4B_{2n+4} + 4B_{2n+2}, \\
C_{2,2n+1} &= -C_{2n+1,2} = 2B_{2n+2}, & C_{5,2n} &= -C_{2n,5} = 5B_{2n+4} + 10B_{2n+2} + B_{2n}, \\
C_{3,2n} &= -C_{2n,3} = 3B_{2n+2} + B_{2n}, & C_{5,2n+1} &= C_{2n+1,5} = 10B_{2n+4} + 5B_{2n+2}, \\
C_{3,2n+1} &= C_{2n+1,3} = 3B_{2n+2}, & C_{6,2n} &= C_{2n,6} = 15B_{2n+4} + 15B_{2n+2} + B_{2n}, \\
C_{4,2n} &= C_{2n,4} = 6B_{2n+2} + B_{2n}, & C_{6,2n+1} &= -C_{2n+1,6} = 6B_{2n+6} + 20B_{2n+4} + 6B_{2n+2}.
\end{align*}
\]

Proof follows from Lemma 2.11 and the first key point: \( B_{2n+1} = 0 \) for each \( n \geq 1 \). \( \square \)

By Claim A.1 one can easily compute the numbers \( C_{mn} \) for \( m + n \leq 12 \).

**Table A.2** of the extended Bernoulli numbers \( C_{mn} \).

| \( m \setminus n \) | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    |
|---------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1                   | -\( \frac{1}{2} \) | \( \frac{1}{6} \) | 0     | -\( \frac{1}{30} \) | 0     | \( \frac{1}{42} \) | 0     | \( -\frac{1}{30} \) | 0     | \( \frac{5}{66} \) | 0     |
| 2                   | \( \frac{1}{30} \) | \( \frac{1}{6} \) | \( \frac{1}{15} \) | \( \frac{1}{30} \) | \( \frac{1}{21} \) | \( \frac{1}{15} \) | \( -\frac{1}{15} \) | \( \frac{5}{30} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) |
| 3                   | 0     | \( \frac{1}{15} \) | \( \frac{1}{10} \) | \( \frac{1}{105} \) | \( \frac{1}{14} \) | \( -\frac{1}{105} \) | \( -\frac{1}{10} \) | \( \frac{165}{165} \) | \( \frac{22}{330} \) | \( \frac{22}{330} \) |
| 4                   | \( \frac{1}{30} \) | \( \frac{1}{30} \) | \( \frac{1}{10} \) | \( \frac{1}{210} \) | \( \frac{1}{105} \) | \( \frac{1}{210} \) | \( \frac{1}{165} \) | \( \frac{3}{139} \) | \( \frac{13}{330} \) | \( \frac{13}{330} \) |
| 5                   | 0     | \( \frac{1}{21} \) | \( \frac{1}{14} \) | \( \frac{1}{105} \) | \( \frac{1}{14} \) | \( \frac{1}{231} \) | \( \frac{1}{22} \) | \( \frac{5}{30} \) | \( \frac{16}{13} \) | \( \frac{16}{13} \) |
| 6                   | \( \frac{1}{42} \) | \( \frac{1}{42} \) | \( \frac{1}{105} \) | \( \frac{1}{210} \) | \( \frac{1}{231} \) | \( \frac{1}{462} \) | \( \frac{1}{13} \) | \( \frac{5}{30} \) | \( \frac{5}{16} \) | \( \frac{5}{16} \) |
| 7                   | 0     | \( \frac{1}{15} \) | \( \frac{1}{10} \) | \( \frac{1}{165} \) | \( \frac{1}{13} \) | \( \frac{1}{22} \) | \( \frac{5}{30} \) | \( \frac{16}{13} \) | \( \frac{16}{13} \) | \( \frac{16}{13} \) |
| 8                   | \( \frac{1}{30} \) | \( \frac{1}{30} \) | \( \frac{1}{165} \) | \( \frac{1}{139} \) | \( \frac{1}{330} \) | \( \frac{1}{330} \) | \( \frac{1}{330} \) | \( \frac{1}{330} \) | \( \frac{1}{330} \) | \( \frac{1}{330} \) |
| 9                   | 0     | \( \frac{5}{33} \) | \( \frac{5}{22} \) | \( \frac{5}{22} \) | \( \frac{5}{22} \) | \( \frac{5}{22} \) | \( \frac{5}{22} \) | \( \frac{5}{22} \) | \( \frac{5}{22} \) | \( \frac{5}{22} \) |
| 10                  | \( -\frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) |
| 11                  | 0     | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) | \( \frac{5}{66} \) |

By Table A.2 one can calculate the function \( C(\lambda, \mu) \) up to degree 10, see Definition 2.4b.

**Example A.3.** Up to degree 10 one has
\[
C(\lambda, \mu) = -\frac{1}{2} + \frac{1}{12}(\lambda - \mu) + \frac{1}{24}\lambda\mu + \left(\frac{1}{720}\lambda^3 - \frac{1}{180}\lambda^2\mu - \frac{1}{180}\lambda^2\mu - \frac{1}{180}\lambda^3\mu\right) + \left(\frac{1}{5\lambda^3} + \frac{4\lambda^2\mu}{3} - \frac{2\lambda^2\mu}{3} - \frac{1}{6}\lambda^3\mu\right) + \left(\frac{1}{240}\lambda^7 + \frac{1}{30}\lambda^6\mu + \frac{4\lambda^5\mu}{15} - \frac{4\lambda^5\mu}{15} - \frac{4\lambda^5\mu}{15} - \frac{1}{15}\lambda^4\mu^2 - \frac{1}{15}\lambda^4\mu^2 - \frac{1}{15}\lambda^4\mu^2 - \frac{1}{15}\lambda^4\mu^2 - \frac{1}{15}\lambda^4\mu^2 - \frac{1}{15}\lambda^4\mu^2\right).
\]
\[-\frac{1}{8!}\left(\frac{\lambda^7\mu}{60} + \frac{2\lambda^6\mu^2}{15} + \frac{37\lambda^5\mu^3}{90} + \frac{3\lambda^4\mu^4}{5} + \frac{37\lambda^3\mu^5}{90} + \frac{2\lambda^2\mu^6}{15} + \frac{\lambda\mu^7}{60}\right) + \\
+ \frac{1}{11!}\left(\frac{\lambda^9}{6} + \frac{25\lambda^8\mu}{3} + \frac{32\lambda^7\mu^2}{56} + \frac{56\lambda^6\mu^3}{3} + \frac{32(\lambda^5\mu^4 - \lambda^4\mu^5)}{6} - \frac{56\lambda^3\mu^6}{3} - \frac{32\lambda^2\mu^7}{5} - \frac{25\lambda\mu^8}{3} - \frac{\mu^9}{6}\right) + \\
+ \frac{\lambda\mu}{11!}\left(\frac{5\lambda^8 + \mu^8}{12} + \frac{\lambda^7\mu + \lambda^6\mu^7}{24} + \frac{139}{8}(\lambda^6\mu^2 + \lambda^5\mu^6) + 39(\lambda^5\mu^3 + \lambda^4\mu^5) + \frac{305}{6}\lambda^4\mu^4\right). \]

Proposition A.4. Let \( L \) be the Lie algebra freely generated by the symbols \( P, Q \). Under \( L \to \mathcal{L}/[[L, L], [L, L]] \) the Hausdorff series \( \bar{H} = \log(\exp(P) \cdot \exp(Q)) \) maps onto

\[
\bar{H} = (P + Q) + \frac{1}{2} [PQ] + \left( \frac{1}{12} [P^2Q] - \frac{1}{12} [QPQ] \right) + \frac{1}{24} [PQ^2PQ] + \\
+ \left( \frac{[Q^3PQ] - [P^4Q]}{720} + \frac{[PQ^2PQ] - [P^2QPQ]}{180} \right) + \left( \frac{[P^3QPQ] + [PQ^3PQ]}{1440} + \frac{[P^2QP^2PQ]}{360} \right) + \\
+ \frac{1}{7!} \left( \frac{[P^6Q] + [P^4QPQ] + \frac{4}{3}[P^3Q^2PQ] - \frac{4}{3}[P^2Q^3PQ] - [P^4QPQ] - \frac{1}{6}[Q^5PQ]\right) + \\
+ \frac{1}{7} \left( \frac{1}{12} [P^5QPQ] + \frac{1}{2} [P^4Q^2PQ] + \frac{23}{24} [P^3Q^3PQ] + \frac{1}{2} [P^2Q^4PQ] + \frac{1}{12} [P^5P^5PQ] \right) + \\
+ \frac{1}{7!} \left( \frac{[P^8Q] - [Q^7PQ]}{240} + \frac{[P^6QPQ] - [PQ^6PQ]}{30} + \frac{4}{45} ([P^5Q^2PQ] - [P^2Q^5PQ]) \right) + \\
+ \frac{[P^4Q^3PQ] - [P^3Q^4PQ]}{15 \cdot 7!} - \frac{1}{8} \left( \frac{[P^7QPQ] + [PQ^7PQ]}{60} + \frac{2}{15} ([P^6Q^2PQ] + [P^2Q^6PQ]) \right) + \\
- \frac{1}{8} \left( \frac{37}{90} ([P^5Q^3PQ] + [P^3Q^5PQ]) - \frac{3}{5} [P^4Q^4PQ] \right) + \text{(higher degree terms)}. \]

Proof. The coefficient of the term \([P^{n-1}Q^{n-1}PQ]\) in \( \bar{H} \) coincides with the coefficient of the term \(\lambda^{n-1}\mu^{m-1}\) in \( C(\lambda, \mu) \). Hence Proposition A.4 follows from Example A.3. \( \square \)

Proposition A.5. (a) The even part of any compressed Drinfeld associator \( \varphi(a, b) \) is

\[
\text{Even}(\varphi(a, b)) = \frac{ab}{6} - \frac{4[a^3b] + [abab] + 4[b^2ab]}{360} + \frac{[a^5b] + [b^4ab]}{945} + \\
+ \left( \beta_{31} + \frac{20}{3 \cdot 7!} \right) ([a^3bab] + [ab^3ab]) + \left( \frac{2\beta_{31} + 13}{7!} \right) [a^2b^2ab] - \frac{[a^7b] + [b^6ab]}{9450} + \\
+ \left( \frac{\beta_{31}}{6} + \beta_{41} - \frac{1}{4200} \right) ([a^5bab] + [ab^5ab]) + \left( \frac{2}{3} \beta_{31} + 3\beta_{41} - \frac{113}{45 \cdot 7!} \right) ([a^4b^2ab] + [a^2b^4ab]) + \\
+ \left( \beta_{31} + 4\beta_{41} - \frac{947}{5 \cdot 9!} \right) [a^3b^3ab] + \frac{[a^9b] + [b^8ab]}{93555} + \cdots 
\]

(b) Up to degree 9 the odd part of any compressed Drinfeld associator \( \varphi(a, b) \in \hat{L}_3 \) is

\[
\text{Odd}(\varphi(a, b)) = \tilde{\beta}_{00} ([a^2b] + [bab]) + \left( \tilde{\beta}_{10} + \frac{\tilde{\beta}_{00}}{6} \right) ([a^4b] + [b^3ab]) + \\
+ \left( 2\tilde{\beta}_{10} + \frac{\tilde{\beta}_{00}}{2} \right) ([a^2bab] + [ab^2ab]) + \left( \tilde{\beta}_{20} + \frac{\tilde{\beta}_{10}}{6} + \frac{\tilde{\beta}_{00}}{120} \right) ([a^6b] + [b^5ab]) + \\
+ \left( 3\tilde{\beta}_{20} + \frac{2}{3}\tilde{\beta}_{10} + \frac{\tilde{\beta}_{00}}{24} \right) ([a^4bab] + [ab^4ab]) + \left( 5\tilde{\beta}_{20} + \frac{7}{6}\tilde{\beta}_{10} + \frac{\tilde{\beta}_{00}}{12} \right) ([a^2b^2ab] + [a^2b^3ab]) + 
\]
that the Drinfeld series follows from Theorem 1.5c by using routine computations. The degree 6 part of the above formulae reprove explicitly Proposition 6.9 up to degree 7.

\[ \lambda^2 + \mu^2 - \frac{\lambda \mu}{360} + \frac{\lambda^4 + \mu^4}{945} + \left( \frac{9}{2} \theta_3 + \frac{1}{1260} \right) \lambda^2 \mu^2 - \frac{\lambda^6 + \mu^6}{9450} + \left( 15 \theta_3 \theta_5 - \frac{2}{3 \cdot 7!} \right) \lambda^5 \mu + \mu^5 \]

(b) Up to degree 7 the odd part of the Drinfeld series \( f^D(\lambda, \mu) \) from Theorem 6.2 is

\[ \text{Odd}(f^D(\lambda, \mu)) = -3 \theta_3 (\lambda + \mu) - 5 \theta_5 (\lambda^3 + \mu^3) - \left( 10 \theta_5 + \frac{\theta_3}{2} \right) (\lambda^2 \mu + \lambda \mu^2) - 7 \theta_7 (\lambda^5 + \mu^5) \]

\[ - \left( 21 \theta_7 + \frac{5}{6} \theta_6 - \frac{3 \theta_5}{30} \right) (\lambda^4 \mu + \lambda \mu^4) - \left( 35 \theta_7 + \frac{5}{3} \theta_5 - \frac{3 \theta_3}{24} \right) (\lambda^3 \mu^2 + \lambda^2 \mu^3) - 9 \theta_9 (\lambda^7 + \mu^7) - \left( 36 \theta_9 + \frac{7}{6} \theta_7 - \frac{5 \theta_5}{18} + \frac{\theta_3}{315} \right) (\lambda^6 \mu + \mu^6) - \left( \frac{9}{2} \theta_3 + 84 \theta_9 + \frac{7}{2} \theta_7 - \frac{\theta_5}{8} + \frac{\theta_3}{180} \right) (\lambda^5 \mu^2 + \lambda^4 \mu^3) - \left( \frac{27}{2} \theta_3 + 126 \theta_9 + \frac{35}{6} \theta_7 - \frac{7}{36} \theta_5 + \frac{\theta_3}{144} \right) (\lambda^4 \mu^3 + \lambda^3 \mu^4) + \text{(higher degree terms)} \]

Proof. Rewrite the formula (6.2) in a more explicit form:

\[ f^D(\lambda, \mu) = -2 \theta_2 - 3 \theta_3 (\lambda + \mu) - 4 \theta_4 (\lambda^2 + \mu^2) - \left( 2 \theta_5^2 - 6 \theta_4 \right) \lambda \mu - 5 \theta_5 (\lambda^3 + \mu^3) + \left( 6 \theta_2 \theta_3 - 10 \theta_5 \right) (\lambda^2 \mu + \lambda \mu^2) - 6 \theta_6 (\lambda^4 + \mu^4) + \left( \frac{9}{2} \theta_3^2 + 8 \theta_2 \theta_4 - 15 \theta_6 \right) (\lambda^3 \mu + \lambda \mu^3) + \left( - \frac{4}{3} \theta_2^3 + 9 \theta_3^2 + 12 \theta_2 \theta_4 - 20 \theta_6 \right) (\lambda^2 \mu^2 - 7 \theta_7 (\lambda^5 + \mu^5) + \left( 10 \theta_2 \theta_5 + 12 \theta_3 \theta_4 - 21 \theta_7 \right) (\lambda^4 \mu + \lambda \mu^4) + \left( - 6 \theta_2 \theta_3 + 20 \theta_2 \theta_5 + 30 \theta_3 \theta_4 - 35 \theta_7 \right) (\lambda^3 \mu^2 + \lambda^2 \mu^3) - 8 \theta_9 (\lambda^6 + \mu^6) + \left( 12 \theta_2 \theta_6 + 15 \theta_3 \theta_5 + 8 \theta_4^2 - 28 \theta_8 \right) (\lambda^5 \mu + \lambda \mu^5) + \left( 30 \theta_2 \theta_6 + 45 \theta_3 \theta_5 + 24 \theta_2^2 - 9 \theta_2 \theta_3^2 - 8 \theta_2 \theta_4^2 - 58 \theta_8 \right) (\lambda^4 \mu^2 + \lambda^2 \mu^4) + \left( \frac{2}{3} \theta_2^4 - 18 \theta_2 \theta_3^2 - 12 \theta_2 \theta_4 + 40 \theta_2 \theta_6 + 60 \theta_3 \theta_5 + 34 \theta_2^2 - 70 \theta_8 \right) \lambda^3 \mu^3 + \left( 14 \theta_2 \theta_7 + 18 \theta_3 \theta_6 + 20 \theta_4 \theta_5 - 36 \theta_9 \right) (\lambda^6 \mu + \lambda \mu^6) - 9 \theta_9 (\lambda^7 + \mu^7) + \left( -24 \theta_2 \theta_3 - 9 \theta_2^3 - 100 \theta_2 \theta_5 + 42 \theta_2 \theta_7 + 63 \theta_3 \theta_6 + 70 \theta_4 \theta_5 - 84 \theta_9 \right) (\lambda^5 \mu^2 + \lambda^2 \mu^5) + \left( 4 \theta_3^2 \theta_4 - 6 \theta_3 \theta_5 \theta_4 - \frac{27}{2} \theta_3^3 - 200 \theta_2 \theta_5 + 70 \theta_2 \theta_7 + 105 \theta_3 \theta_6 + 120 \theta_4 \theta_5 - 126 \theta_9 \right) (\lambda^4 \mu^3 + \lambda^3 \mu^4) + \cdots \]

It remains to substitute \( \theta_2n \) for Example 6.4 and split the even and odd parts. Observe that the Drinfeld series \( f^D(\lambda, \mu) \) is obtained from Proposition A.5 for the parameters

\[ \beta_{31} = \frac{9}{2} \theta_3^2 - \frac{8}{3 \cdot 7!}, \quad \beta_{41} = 15 \theta_3 \theta_5 - \frac{3 \theta_3^2}{4} + \frac{44}{45 \cdot 7!}, \quad \beta_{00} = -3 \theta_3, \quad \beta_{10} = -5 \theta_5 + \frac{\theta_3}{2}, \]

\[ \beta_{20} = -7 \theta_7 + \frac{5}{6} \theta_5 - \frac{7}{120} \theta_3, \quad \beta_{30} = -9 \theta_9 + \frac{7}{6} \theta_7 - \frac{7}{72} \theta_5 + \frac{31}{7} \theta_3, \quad \beta_{31} = -\frac{9}{2} \theta_3^2 - 3 \theta_9 + \frac{\theta_3}{630}. \]

The above formulae reprove explicitly Proposition 6.9 up to degree 7.
Example A.7. The first and second distinguished even compressed Drinfeld associators from Corollary 1.6a–b of Subsection 1.3 are defined by the following series:

\[
(a) \quad f^I(\lambda, \mu) = \frac{1}{6} - \frac{4\lambda^2 + \lambda \mu + 4\mu^2}{360} + \frac{\lambda^4 + \mu^4}{945} + \frac{20}{4 \cdot 7!}(\lambda^3 \mu + \lambda \mu^3) + \frac{13}{7!}\lambda^2 \mu^2 - \frac{\lambda^6 + \mu^6}{9450} - \frac{\lambda^5 \mu + \mu^5}{4200} - \frac{113}{45 \cdot 7!}(\lambda^4 \mu^2 + \lambda^2 \mu^4) - \frac{947}{5 \cdot 9!}\lambda^3 \mu^3 + \frac{\lambda^8 + \mu^8}{93555} + \ldots
\]

\[
(b) \quad f^{II}(\lambda, \mu) = \frac{1}{6} - \frac{4\lambda^2 + \lambda \mu + 4\mu^2}{360} + \frac{\lambda^4 + \mu^4}{945} - \frac{53}{6 \cdot 7!}(\lambda^3 \mu + \lambda \mu^3) - \frac{18}{7!}\lambda^2 \mu^2 - \frac{\lambda^6 + \mu^6}{9450} + \frac{\lambda^5 \mu + \mu^5}{11200} - \frac{13}{90 \cdot 7!}(\lambda^4 \mu^2 + \lambda^2 \mu^4) + \frac{431}{5 \cdot 9!}\lambda^3 \mu^3 + \frac{\lambda^8 + \mu^8}{93555} + \ldots
\]

The proof is a computation following from Corollary 1.6. Both series are obtained from Proposition A.5 for \(\beta_{31} = \beta_{41} = 0\) and \(\beta_{31} = -\frac{31}{2 \cdot 7!}, \beta_{41} = \frac{127}{30 \cdot 7!}\), respectively. ▲

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