The dynamics of hyperbolic rational maps with Cantor Julia sets

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Abstract

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a hyperbolic rational map of degree $d \geq 2$ on the Riemann sphere. We give several conditions which are equivalent to the condition for the Julia set $J_f$ to be a Cantor set. It has been known that $J_f$ is a Cantor set if and only if there exists a positive integer $n > 0$ such that $f^{-n}(U) \subset U$ for some open topological disc $U$ containing no critical values. Let $n_f$ denote the minimal positive integer satisfying the above. The problem is whether $n_f = 1$ or not.

Let $S_d$ denote the shift locus of rational maps of degree $d$. We show that $n_f = 1$ for generic $f \in S_d$ and that there is a rational map $\bar{f} \in S_4$ with $n_{\bar{f}} = 2$. We also prove that $S_d$ is connected using the generic case result. In particular, generic hyperbolic rational maps of degree $d$ with Cantor Julia sets are qc-conjugate to each other.

1 Introduction

In this paper, we investigate the dynamics of rational maps with Cantor Julia sets. Specifically, we are interested in rational maps whose dynamics on the Julia sets are the full shift. Throughout this paper, we denote by $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational map of the Riemann sphere to itself of degree $d \geq 2$, by $J = J_f$ its Julia set, and by $F = F_f$ its Fatou set.

In general, Cantor sets are often observed in spaces on which several contracting mappings act. For example, suppose that $U$ is an open subset of a complete metric space $(X,d)$, and that $f_i : U \to \hat{\mathbb{C}}, i = 1, 2, \ldots, N$ are injective mappings with disjoint images such that $d(f_i(x), f_i(y)) \leq cd(x, y)$ for some $0 < c < 1$. Then there exists a Cantor set $K \subset \bigcup_{i=1}^{N} f_i(K)$. The mapping $g : K \to K$ defined by $g|_{f_i(K)} = f_i^{-1}$ is topologically conjugate to the one-sided $N$-full shift.

In the context of complex dynamics, the appearance of Cantor Julia sets for the quadratic family $q_c(z) = z^2 + c$ is well-known. If $c$ does not belong to the Mandelbrot set, then the Julia set $J_c$ is a Cantor set, and $q_c|_{J_c}$ is topologically conjugate to the one-sided $2$-full shift. Let $G$ be the Green’s function for such a $q_c$, and take $x$ with $G(0) < x < G(c)$. Then $U = \{z | G(z) < x\}$ is connected and
simply connected, and $q_c$ has two inverse branches $f_1, f_2$ on $U$. They satisfy the general setting above for the Poincaré metric on $U$. If $c'$ is another point outside the Mandelbrot set, then clearly $q_c$ and $q_{c'}$ are topologically conjugate to each other on the Julia sets. It is known that the conjugacy is extended to a $q_c$-conjugacy on the whole sphere. However this is not the case with general polynomials of degree more than two.

As to rational maps, one of known sufficient conditions to have Cantor Julia set is as follows:

There exists a (super)attracting fixed point whose immediate basin of attraction contains all the critical points.

This is a classical result, due to Fatou [8] and Julia [10]. We call this condition $\text{Cond C}$.

For general rational maps, $\text{Cond C}$ is not equivalent to having Cantor Julia set; one of the equivalent conditions is known as the Branner-Hubbard condition ([5],[21],[22]). However, we use $\text{Cond C}$ in this study because we are interested in the situation where the dynamics on $J$ is topologically conjugate to the full shift. If $J$ is Cantor, then there exists a semi-conjugacy from the full shift (Proposition 3), which is not necessarily homeomorphic.

**Definition 1.** Let $(\sigma, \Sigma_d)$ be the one-sided $d$-full shift, namely, $\Sigma_d = \{i_1i_2\cdots : i_k = 1, 2, \ldots, d\}$ is the set of infinite words of symbols $\{1, 2, \ldots, d\}$ and the shift map $\sigma : \Sigma_d \to \Sigma_d$ carries $i_1i_2i_3\cdots$ to $i_2i_3\cdots$. The set $\Sigma_d$ is endowed with the product topology of the discrete space.

**Definition 2.** We say

1. $f$ is Cantor if there exists a homeomorphism $h : \Sigma_d \to J$.

2. $f$ is $d$-Cantor (dynamically Cantor) if $(f, J)$ is topologically conjugate to $(\sigma, \Sigma_d)$, namely there exists a homeomorphism $h : \Sigma_d \to J$ such that $f \circ h = h \circ \sigma$.

3. $f$ is $s$-Cantor (strongly Cantor) if there exists a closed topological disc $\overline{D} \subset \hat{\mathbb{C}}$ containing no critical value such that $f^{-1}(\overline{D}) \subset \overline{D}$.

We raise three questions. Let $f$ and $g$ be rational maps of degree $d$ satisfying $\text{Cond C}$.

Q1 Is $f$ $d$-Cantor?

Q2 Is $f$ $s$-Cantor?

Q3 Are $f$ and $g$ topologically conjugate on $\hat{\mathbb{C}}$ to each other?

The answers will be given in the present paper. To sum up,

**Answer to Q1** Yes. We show this result in two ways. In Theorem 1, we choose a collection of $d$ inverse branches of $f$ on $J_f$ which are contracting in some metric. The other proof is a consequence of the next answer.
For two hyperbolic rational maps with Cond C, we have a topological
conjugacy between the Julia sets, and moreover it can be quasiconformally
extended to their neighborhoods (Corollary 2).

Answer to Q2 No, in general. But yes, generically. We give an example of
a non-s-Cantor rational map \( f \) satisfying Cond C. In fact, we show the
(possibly) stronger statement that any radial (Definition 6) for this \( f \)
duces a coding map which is not one-to-one (Theorem 7). On the other
hand, if \( f \) with Cond C satisfies that no two critical orbits meet and that
no critical orbit lands at a (super)attracting fixed point, then \( f \) is s-Cantor
(Theorem 2). This implies that \( f \) with Cond C is topologically conjugate
on \( J_f \) to the full shift, since the assumption of Theorem 2 is generic in the
shift locus of rational maps.

Answer to Q3 No, in general. But yes, generically. It is trivial that two rational
maps of the same degree with Cond C are in general not topologically
conjugate on the whole sphere to each other. Meanwhile we show that
if such rational maps satisfy the assumption of Theorem 2 and a generic
condition on the critical points, then they are qc-conjugate on \( \hat{C} \) (Theorem
6). Moreover we see that the shift locus is connected (Theorem 5). These
are consequences of Theorem 2.

There exists a lot of research on the shift locus for polynomials. Among them,
[4] and [2] are the pioneer works. The shift locus for rational maps of degree
2 is investigated in [9]. The connectedness of the shift locus for polynomials is
known. A proof of this fact appears in [7]. However the connectedness of the
shift locus for general rational maps has not been known so far.

Our main results are stated as follows.

In Section 3, we restrict \( f \) to be geometrically finite or hyperbolic. We give
many equivalent conditions for \( f \) to be d-Cantor. Among them,

**Theorem A.** (Partial statements of Theorem 1) Suppose that \( f \) is a hyperbolic
rational map. Then the following are equivalent:

- \( f \) is Cantor.
- \( f \) is d-Cantor.
- \( f^n \) is s-Cantor for some integer \( n > 0 \).
- There exists a (super)attracting fixed point such that the immediate basin
  of attraction includes at least \( 2d - 4 \) critical values counted with multi-
  plicity.
- Any ‘bounded iterated monodromy group’ is a finite group.

In Section 4 and 5, we consider a sufficient condition to be s-Cantor, and
deduce some knowledge about the shift locus of rational maps.
**Theorem 2.** If a Cantor hyperbolic rational map $f$ satisfies that no two critical orbits meet and that the orbit of any non-fixed critical point contains no fixed point, then $f$ is s-Cantor.

**Definition 3.** The shift locus $S_d$ is the set of d-Cantor hyperbolic rational maps of degree $d$. The strong shift locus $T_d$ is the set of s-Cantor hyperbolic rational maps.

Clearly, $T_d \subset S_d$.

Let Pol denote the set of polynomial maps.

**Theorem B.** (Theorem 3, 4, 5, and 6)

1. $S_d$ ($d \geq 2$) is connected.

2. $T_d \cap \text{Pol} = S_d \cap \text{Pol}$ ($d \geq 2$), $T_2 = S_2$, and $T_d \subset S_d$ ($d \geq 4$).

3. If $f$ and $g$ satisfy the assumption of Theorem 2 and a generic condition on the critical points, then they are qc-conjugate on $\hat{\mathbb{C}}$.

The proof of the last statement of 2 is deferred to Section 5. There we prove:

**Theorem 7.** There exists a hyperbolic rational map $\bar{f}$ of degree four which is Cantor but not s-Cantor.

There we show that $\bar{f}$ is not s-Cantor and $\bar{f}^2$ is s-Cantor.

The following problems are open:

**Unsolved Problem 1**

- Does there exist a Cantor hyperbolic rational map of degree three which is not s-Cantor?

- For any integer $n > 0$, does there exist a Cantor hyperbolic rational map $f$ such that $f^n$ is not s-Cantor and $f^{n+1}$ is s-Cantor?

## 2 Preliminary

We give several notations, definitions, and basic facts. Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map of degree $d$ with Julia set $J = J_f$.

**Definition 4.** For $z \in \hat{\mathbb{C}}$, the orbit of $z$ is $\text{Orb}(z) := \{f^k(z) : k \geq 0\}$. Let $\text{Cr} = \text{Cr}_f$ denote the set of critical points of $f$, and set $P = P_f := \{f^k(c) : c \in \text{Cr}, k > 0\} = \bigcup_{c \in \text{Cr}} \text{Orb}(f(c))$. The closure of $P$ is said to be the postcritical set. For $c \in \text{Cr}$, the orbit $\text{Orb}(f(c))$ is called a critical orbit.

Let $\text{At} = \text{At}_f$ denote the set of attracting periodic points of $f$, and $\text{Pa} = \text{Pa}_f$ the set of parabolic periodic points.

We say that $f$ is geometrically finite if $J \cap \overline{P}$ is finite. In other words, each critical orbit is either eventually periodic, or converges to an attracting or parabolic periodic orbit.
Definition 5. We use the symbol \( \sigma \) also as the shift map on the set of finite words \( \text{Word}(k) = \{i_1i_2\cdots i_k : i_m = 1,2,\ldots,d \} \). Each finite word \( w = i_1i_2\cdots i_k \in \text{Word}(k) = \{i_1i_2\cdots i_k : i_m = 1,2,\ldots,d \} \) can be considered as the branch of \( \sigma^{-k} \), namely, \( w : \text{Word}(n) \to \text{Word}(n+k), j_1\cdots j_n \mapsto wj_1\cdots j_n \). The empty word is denoted by \( \emptyset \), namely \( \text{Word}(0) = \{ \emptyset \} \). We write \( \text{Word} = \bigcup_{k=0}^{\infty} \text{Word}(k) \)

Definition 6. Let \( l_i : [0,1] \to \mathcal{C} - P \) (\( i = 1,2,\ldots,d \)) be continuous paths with the initial point \( \bar{x} = l_i(0) \) in common. We say that a collection of paths \( r = (l_i)_{i=1,2,\ldots,d} \) is a radial if \( f^{-1}(\bar{x}) \supset \{l_i(1) : i = 1,2,\ldots,d \} \). Set \( R = \bigsqcup_{i=1}^{d} [0,1], \sim, \) where \([0,1] \) are copies of the unit interval \([0,1] \) and we glue them at 0. (Recall that \( R \) is called the complete bipartite graph \( K_{1,d} \) in graph theory.) It is convenient to consider a radial to be the mapping \( r : R \to \mathcal{C} - P \) with \( r(t) = l_i(t) \) for \( t \in [0,1] \). Note that a continuous mapping \( r : R \to \mathcal{C} - P \) is a radial if and only if \( f^{-1}(r(0)) \supset \{r(i_1) : i = 1,2,\ldots,d \} \).

In the situation above, we call \( \bar{x} \in \mathcal{C} - P \) the basepoint.

For a path \( \gamma : [0,1] \to \mathcal{C} - P \) with \( \gamma(0) = \bar{x} \), let \( L_i(\gamma) \) be the lift of \( \gamma \) by \( f \) (i.e. \( f \circ L_i(\gamma) = \gamma \)) with \( L_i(\gamma)(0) = l_i(1) \). For \( w \in \text{Word}(k) \), we inductively define paths \( l_w \) such that \( l_{iw} = l_i L_i(l_w) \). Naturally \( l_w \) is defined for any \( w \in \text{Word} \), and \( l_{iw} = l_w L_i(l_w) \). For \( \omega = i_1i_2\cdots \in \Sigma_d \), we write \( \phi_r(\omega) = \lim_{t \to \infty} l_{i_1i_2\cdots i_n}(1) \) if exists. We can naturally define an arc \( l_w \) connecting \( \bar{x} \) and \( \phi_r(\omega) \) up to a change of parameter. We say that \( \phi_r : \Sigma' \to J \) is the coding map associated with \( r \), where \( \Sigma' \) is the set of \( \omega \in \Sigma_d \) for which \( \phi_r(\omega) \) exists. This method of coding is originated by F. Przytycki \([19],[20]\), and is called geometric coding tree. In many cases, \( \phi_r \) is continuous and \( \Sigma' = \Sigma_d \), in particular whenever \( f \) is geometrically finite (Lemma 2) or \( f \) is Cantor (Proposition 3).

We may consider deformation of radials. Let \( r = (l_i), r' = (l'_i) \) be two radials which do not necessarily satisfy \( r(0) = r'(0) \). We say that \( r \) and \( r' \) are homotopic if there exists a homotopy \( h : \bigsqcup_{i=1}^{d} [0,1] \times [0,1] \to \mathcal{C} - P \) between \( r \) and \( r' \) such that \( h(\cdot, s) : \bigsqcup_{i=1}^{d} [0,1] \to \mathcal{C} - P \) is a radial for every \( s \in [0,1] \).

Definition 7. We say

1. \( f \) is \( t \)-Cantor (tree Cantor) if there exists a radial \( r \) such that the coding map \( \phi_r : \Sigma_d \to J \) is a homeomorphism.

2. \( f \) is \( ss \)-Cantor (semi-strongly Cantor) if there exists a path-connected set \( X \subset \mathcal{C} \) such that \( f^{-1}(X) \) has \( d \) connected components \( X_1, X_2,\ldots, X_d \subset X \), and such that \( X_i \cap X_j \) has no point of \( J \) for any \( i \neq j \).

Now we have five notions of Cantor Julia sets. It is easy to see that ‘s-Cantor’ \( \Rightarrow \) ‘ss-Cantor’ \( \Rightarrow \) ‘t-Cantor’ \( \Rightarrow \) ‘d-Cantor’ \( \Rightarrow \) ‘Cantor’ (Proposition 1). It will be proved that the last three are different each other. If \( f \) is d-Cantor, then \( f \) is geometrically finite (Proposition 2).

Unsolved Problem 2 Does it hold that ‘ss-Cantor’ \( \Rightarrow \) ‘s-Cantor’ or that ‘t-Cantor’ \( \Rightarrow \) ‘ss-Cantor’?
If $f$ is Cantor, the Fatou set $F$ is connected (see Theorem 13.4 [17]). Thus a Cantor rational map has neither Siegel disc nor Herman ring, and $\#(\hat{A} \cup Pa) = 1$. The unique fixed point in $At \cup Pa$ is (super)attracting or parabolic with the local form $z \mapsto z + az^2 + \cdots$.

**Definition 8.** Let $p \in At \cup Pa$, let $F_0$ be a connected component of the Fatou set $F$ with $p \in F_0$, and let $n$ be the period of $F_0$. A connected and simply connected open set $U$ included in $F_0$ satisfying the following condition is called a *simple domain* for $p$ and $F_0$: $p \in \overline{U}$, $\partial U \cap \overline{F} \subset \{p\} \cap Pa$, $\partial U$ is a piecewise smooth simple closed curve, $f^n(U) \subset U \cup \{p\}$, $\{f^{kn}(U)\}_{k>0}$ converges to $p$ uniformly, and for any $x \in F_0$ there exists $k > 0$ such that $f^{kn}(x) \in U$. If $p$ is parabolic, we take $U$ to be a connected component of an attracting petal. Remark that $p \in U$ if $p \in At$, and $p \notin U$ if $p \in Pa$.

We denote by $U_k$ the connected component of $f^{-kn}(U)$ including $U$.

**Lemma 1.** Let $p \in At \cup Pa$, and let $F_0$ be a connected component of $F$ with $p \in F_0$, and $U$ a simple domain for $p$ and $F_0$. Take $U_k$ as above. Then $F_0 = \bigcup_{k=1}^{\infty} U_k$. In particular, $P \cap F_0 \subset U_k$ for some $k > 0$.

*Proof.* Assume that $F_0 \neq \bigcup_{k=1}^{\infty} U_k$. Then $F_0 \cap \left(\bigcup_{k=1}^{\infty} U_k - \bigcup_{k=1}^{\infty} U_k\right) \neq \emptyset$. Indeed, otherwise, $\bigcup_{k=1}^{\infty} U_k$ is open and closed in $F_0$. Existence of $x \in F_0 \cap \left(\bigcup_{k=1}^{\infty} U_k - \bigcup_{k=1}^{\infty} U_k\right) \neq \emptyset$ contradicts the fact $f^{kn}(x) \in U$ for some $k > 0$. \hfill \square

**Lemma 2.** If $f$ is geometrically finite, then $\phi_r(\omega)$ converges for any radial $r : R \to \hat{C} - \overline{F}$ and any $\omega \in \Sigma_d$. Moreover, $\phi_r = \phi_{r'}$ if $r$ and $r'$ are homotopic.

*Proof.* This is a consequence of expansiveness of $f$ on $\Omega$, where $\Omega \subset \hat{C}$ is a subset defined below. The crucial part of the proof is due to [14]. We show the outline.

Let $r = (l_i) : R \to \hat{C} - \overline{F}$ be a radial with the basepoint $\bar{x}$. Let $W$ be a union of simple domains of every $p \in At \cup Pa$ and every connected component $F_0 \subset F$ with $p \in F_0$ such that $f(W) \subset W$ and $r(R) \cap W = \emptyset$. Then the compact set $\Omega = \hat{C} - W$ satisfies the following: $J \subset \Omega$, $f^{-1}(\Omega) \subset \Omega$, and $\Omega$ is a connected and finitely connected domain which has the boundary expressed as a union of finitely many smooth curves.

Let $\nu : \Omega \to \mathbb{N}$ be the ramification function for $f|_{\Omega}$, namely $\nu$ is the minimal function such that $\nu(z) = 1$ for $z \notin P$ and $\nu(z)$ is a multiple of $\nu(w) \deg_w f$ for $z \in P, w \in f^{-1}(z)$. Let $p : S \to (\text{int} \Omega, \nu)$ be the universal covering for the Riemann surface orbifold $(\text{int} \Omega, \nu)$, where $S$ is $D = \{|z| < 1\}$ or $\hat{C}$ according to the Euler characteristic of $(\text{int} \Omega, \nu)$.

Fix a point $\hat{x} \in p^{-1}(\bar{x}) \subset S$. Let $\hat{l}_i$ be the lift of $l_i$ by $p$ with $\hat{l}_i(0) = \hat{x}$. There exist holomorhic maps $g_i : S \to S$ such that $f \circ p \circ g_i = p$ and $g_i(\hat{x}) = \hat{l}_i(1)$ for $i = 1, 2, \ldots, d$. If $Pa = \emptyset$, then there exists $0 < \alpha < 1$ such that $\rho(g_i(x), g_i(y)) < \alpha \rho(x, y)$, where $\rho$ is the distance determined by the Poincare or Euclidean metric on $S$. If $Pa \neq \emptyset$, then $S = D$ and we extend $p : D \to \text{int} \Omega$ to a surjection $p : D \to \Omega$, where $\hat{D} \subset \overline{F}$ and any continuous arc $c : [0, 1] \to \hat{D}$ can be lifted to $\hat{c} : [0, 1] \to D$. The maps $g_i : D \to D$ are also extended to

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\(g_i : \hat{D} \to \hat{D}\). Modifying the Poincare metric, we can construct a metric on \(\hat{D}\) which determines the distance \(\rho\) such that there exists an increasing function \(h : [0, \infty) \to [0, \infty)\) with \(h(t) < t\) for \(t > 0\) and \(\rho(g_i(x), g_i(y)) < h(\rho(x, y))\).

Then we have a compact set \(K \subset \hat{S} (\hat{S} = S \text{ or } \hat{D})\) such that

\[
K = \bigcup_{i=1, \ldots, d} g_i(K).
\]

Namely, \(K\) is the attractor of the iterated function system \((\{g_i\}, \hat{S})\). By construction, \(p(K) \subset J\). Moreover, \(p \circ g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_k} = l_{i_1i_2 \cdots i_k}(1)\) for \(i_1i_2 \cdots i_k \in \text{Word}(k)\). Canonically, we have a coding map \(\pi' : \Sigma_d \to K\) and \(\phi_r = p \circ \pi' : \Sigma_d \to J\).

Let \(r' = (l'_i)\) be another radial which is homotopic to \(r\). Then there exists a homotopy \(h\) between \(r\) and \(r'\). Let \(c = h(0, \cdot)\) be the path between the basepoints of \(r\) and \(r'\), and \(\hat{c}\) the lift of \(c\) with \(\hat{c}(0) = \hat{x}\). Then \(p(\hat{c}(1)) = r'(0)\). We have the lifts \(\hat{I}_i'\) of \(I_i'\), \(\hat{I}_i'(0) = \hat{c}(1)\), and then \(g_i(\hat{c}(1)) = \hat{I}_i'(1)\) for \(i = 1, 2, \ldots, d\). Thus we obtain the same iterated function system \((\{g_i\}, \hat{S})\), and so \(\phi_{r'} = \phi_r\).

**Definition 9.** For a geometrically finite rational map \(f\), we say that \(\Omega\) in the proof is an expanding domain for \(f\). Note that (i) for any \(z \in F\) there exists \(n > 0\) such that \(f^n(z) \notin \Omega\), and (ii) for any compact set \(K \subset \hat{\mathbb{C}} - (\hat{A} \cup \hat{P}a)\) there exists \(n > 0\) such that \(f^{-n}(K) \subset \Omega\).

**Proposition 1.** If \(f\) is ss-Cantor, then it is \(t\)-Cantor.

**Proof.** Let \(X\) and \(X_i\) be path-connected sets as in the definition. Let us take a basepoint \(\hat{x} \in X\). We have a radial \(r = (l_i)\) with \(l_i \subset X\) joining \(\hat{x}\) and \(x_i \in f^{-1}(\hat{x}) \cap X_i\). Then \(\phi_r\) is one-to-one since \(\phi_r(i \cdots) \in X_i\).

**Proposition 2.** Suppose that \(f\) is \(d\)-Cantor. Then \(J \cap \mathbb{C}r = \emptyset\). In particular, \(f\) is geometrically finite and \(P \cap \mathbb{C}a = \emptyset\).

**Proof.** Suppose that \(J \cap \mathbb{C}r\) is not empty. Let \(c \in J\) be a critical value. Then \#\(f^{-1}(c)\) < \(d\). Thus \(f\) cannot be topologically conjugate to a \(d\)-to-1 map.

**Proposition 3.** Suppose that \(f\) is Cantor. Then there exists a continuous semi-conjugacy \(\phi : \Sigma_d \to J\).

**Proof.** It is easily seen that \(P\) has a path-connected compliment. Thus we can take a radial \(r = (l_i)\).

For \(\omega = i_1i_2 \cdots \in \Sigma_d\), the set of accumulation points

\[
\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} L_{i_1i_2 \cdots i_{m-1}}(l_{i_m}l_{i_{m+1}} \cdots i_n)
\]

is a connected subset of \(J\), so it is a singleton. Here \(L_w(\gamma) (w \in \text{Word}(k))\) is the lift of the curve \(\gamma : [0, 1] \to \hat{\mathbb{C}} - \hat{P}\) by \(f^k\) with \(L_w(\gamma)(0) = l_w(1)\). Moreover the diameter of \(\bigcup_{m=n}^{\infty} L_{i_1i_2 \cdots i_{m-1}}(l_{i_m}l_{i_{m+1}} \cdots i_n)\) converges to zero as \(m\) tends to \(\infty\). Thus the coding \(\phi_r(\omega)\) is well-defined and continuous.
3 Equivalent conditions

Suppose that \( f \) is a geometrically finite rational map with \( J \cap C = \emptyset \).

**Definition 10.** Let \( X \) be a topological space. We say that a continuous mapping \( a : X \to \hat{C} \) is homotopically trivial if there exists a homotopy \( H : X \times [0,1] \to \hat{C} - P \) between \( a \) and a constant mapping. In particular, if \( a(X) \cap P \neq \emptyset \), then \( a \) is homotopically nontrivial.

A subset \( X \subset \hat{C} \) is said to be homotopically trivial if the inclusion map is homotopically trivial.

**Remark 1.** For \( Y \subset X \), if \( X \) is homotopically trivial, then \( Y \) is also homotopically trivial. If a simple closed curve \( \gamma \) is homotopically trivial, then one of the connected components of \( \hat{C} - \gamma \) is also homotopically trivial.

**Definition 11.** We denote by \( M_n : S^1 \to S^1 \), the map \( s \mapsto ns \mod 1 \) of \( S^1 = \mathbb{R}/\mathbb{Z} \) to itself. Let \( \gamma : S^1 \to \hat{C} \) be a closed curve. We say a closed curve \( \gamma' : S^1 \to \hat{C} \) is a lifted loop of \( \gamma \) by \( f \) if \( \gamma \circ M_n = f \circ \gamma' \) for some \( n > 0 \).

**Definition 12.** Let \( \Omega \) be an expanding domain.

A closed curve \( \gamma : S^1 \to \Omega - P \) is homotopically invariant if \( \gamma \) is homotopically nontrivial, and if there exists a lifted loop \( \gamma' : S^1 \to \Omega - P \) of \( \gamma \) by \( f^k \) for some \( k > 0 \) and there exists a homotopy \( H : S^1 \times [0,1] \to \Omega - P \) between \( \gamma \) and \( \gamma' \).

**Lemma 3.** Let \( p \in \mathcal{A}_t \cup \mathcal{P}_a \), and let \( F_0 \) be a connected component of the Fatou set \( F \) with period \( n \) satisfying \( p \in \mathcal{F}_0 \), and \( U \) a simply connected domain for \( p \) and \( F_0 \). Take \( U_k \) as in Definition 8. Let \( V_k \) be the set of homotopically nontrivial connected components of \( \hat{C} - U_k \).

Then for large enough \( k \), \( V_k \) is constant up to homotopy relative to \( P \). Moreover, there is a \( x \in P \) which is not contained in \( F_0 \) \iff \( V_k \neq \emptyset \) for any \( k \iff \) there exists \( V \in V_k \) such that \( \partial V \) is homotopically invariant.

**Proof.** Note that each \( V \in V_k \) is a topological disc containing a point in \( P \). Let \( k_0 \) be an integer such that \( P \cap F_0 \subset U_{k_0} \). Then \#\( V_k \leq \#V_{k+1} \) for \( k \geq k_0 \).

Thus there exists \( k_0 \leq k_1 \) such that \#\( V_k \) is constant for \( k \geq k_1 \), since \#\( V_k \leq \#\{ \text{components of } F \text{ containing a point of } P \} < \infty \).

Suppose \( k \geq k_1 \). It is evident that \( P \subset F_0 \) if and only if \( V_k \) is empty. Suppose \( V_k \neq \emptyset \). Then for \( V \in V_k \), there exists unique \( V' \in V_{k+1} \) such that \( V' \subset V \) and \( (V - V') \cap P = \emptyset \). It is easy to see that there exist continuous maps \( \gamma : S^1 \to \partial V' \) and \( \gamma' : S^1 \to \partial V' \) which are homotopic in \( \Omega - P \).

On the other hand, for \( V \in V_{k+1} \), there exists \( V' \in V_k \) such that \( \partial V' = f^n(\partial V) \) (recall that \( n \) is the period of \( F_0 \)). Indeed, there exists \( V' \), a connected component of \( \hat{C} - U_k \) such that \( \partial V' = f^n(\partial V) \). If \( V' \notin V_k \), then \( V' \) is homotopically trivial. Then every connected component of \( f^{-n}(V') \) is also homotopically trivial and simply connected. This implies that \( V \) is a connected component of \( f^{-n}(V') \), and we arrived at a contradiction.

Now we consider \( V' \mapsto V' \mapsto V'' \) as a selfmap on the finite set \( V_k \). It is easy to see that there exists \( V \in V_k \) such that \( \partial V \) is homotopically invariant. \( \square \)
Corollary 1. Let $\Omega$ be an expanding domain. Then $f^{-n}(\Omega)$ is homotopically nontrivial for any $n$ if and only if there exists a homotopically invariant curve.

Proof. We use the notation of Lemma 3. Suppose that there exists no homotopically invariant curve. By Lemma 3, $P \subset U_k$ for some $k$. Then $\mathbb{C} - U_k$ is homotopically trivial, and so is $f^{-k}(\Omega)$.

Conversely, suppose that $f^{-k}(\Omega)$ is homotopically trivial for some $k$. For any curve $\gamma \subset \Omega$, any connected component of $f^{-k}(\gamma)$ is homotopically trivial. Thus there exists no homotopically invariant curve.

Lemma 4. Let $\Omega$ be an expanding domain, and $\gamma \subset \Omega$ a homotopically invariant curve. Then there exists a closed curve $\tilde{\gamma} \subset J$ which is a uniform limit of a sequence of curves homotopic to $\gamma$ in $\Omega - P$. The curve is nonconstant and $f^k : \tilde{\gamma} \to f(\tilde{\gamma})$ is of degree more than one, where $k$ is the period in the Definition 12.

Proof. We have a closed curve $\gamma_1 = \gamma' : S^1 \to \Omega - P$ and a homotopy $H_0 = H : S^1 \times [0,1] \to \Omega$ as in Definition 12. We inductively obtain curves $\gamma_m$ such that $\gamma_{m-1} \circ M_n = f^k \circ \gamma_m$ and homotopies $H_m : S^1 \times [0,1] \to \Omega - P$ ($m = 1, 2, \ldots$) between $\gamma_{m-1}$ and $\gamma_m$. Since $\gamma_m$ is included in an expanding domain, it is easy to see that $\gamma_m$ uniformly converges to a continuous curve in $J$. The limit $\tilde{\gamma}$ is nonconstant since it is homotopically nontrivial. Thus $f^k : \tilde{\gamma} \to f(\tilde{\gamma})$ is of degree more than one.

Definition 13. We say that a fixed point $p$ is solitary if $p$ is either (super)attracting or parabolic with the local form $z \mapsto z + az^2 + \cdots$ around $p$ with $a \neq 0$. Then the unique Fatou component $A_0(p)$ whose closure contains $p$ is the immediate basin of attraction of $p$.

Definition 14. Fix a basepoint $\tilde{x} \in \mathbb{C} - P$. For a set $X$, the symmetric group of $X$ is denoted by $\mathfrak{S}(X)$. The monodromy $\alpha_f : \pi_1(\mathbb{C} - P, \tilde{x}) \to \mathfrak{S}(f^{-1}(\tilde{x}))$ is an automorphism defined by $\alpha_f([\gamma])(x) = y \iff L_x(\gamma)(1) = y$, where $\gamma : [0,1] \to \mathbb{C} - P$ is a loop with $\gamma(0) = \gamma(1) = \tilde{x}$, and $L_x(\gamma)$ is the lift of $\gamma$ by $f$ satisfying $L_x(\gamma)(0) = x$.

Note that $[\gamma] \in \ker \alpha_f \iff L_x(\gamma)$ is a closed curve for every $x \in f^{-1}(\tilde{x})$. Thus $\ker \alpha_{f^n} \subset \ker \alpha_{f^m}$ if $n > m$. Denote

$$\alpha_\infty := \prod_{n > 0} \alpha_{f^n} : \pi_1(\mathbb{C} - P, \tilde{x}) \to \prod_{n > 0} \mathfrak{S}(f^{-n}(\tilde{x})).$$

Then $\ker \alpha_\infty = \bigcap_{n > 0} \ker \alpha_{f^n}$. The image of $\alpha_\infty$ is called the iterated monodromy group (see [1],[18]).

Let $\Omega$ be an expanding domain. The image of

$$\pi_1(\Omega - P, \tilde{x}) \hookrightarrow \pi_1(\mathbb{C} - P, \tilde{x}) \xrightarrow{\alpha_\infty} \prod_{n > 0} \mathfrak{S}(f^{-n}(\tilde{x}))$$

is called the bounded iterated monodromy group for $\Omega$.
Theorem 1. Let \( f \) be a geometrically finite rational map of degree \( d \) with \( J \cap \mathbb{C}r = \emptyset \). Take an expanding domain \( \Omega \). The following are equivalent.

1. \( f \) is Cantor.
2. \( f \) is \( d \)-Cantor.
3. There exist \( n > 0 \) such that \( f^n \) is \( s \)-Cantor.
4. The Fatou set is connected.
5. Every connected component of \( J \) is simply connected.
6. \( J \) is homotopically trivial in \( \hat{\mathbb{C}} - P \).
7. There is no homotopically invariant curve.
8. There exists \( n > 0 \) such that for any closed curve \( \gamma \subset \Omega \), any lifted loop of \( \gamma \) by \( f^{-n} \) is homotopically trivial.
9. There exists \( n > 0 \) such that \( f^{-n}(\Omega) \) is homotopically trivial in \( \hat{\mathbb{C}} - P \).
10. There exists a solitary fixed point \( p \) such that the immediate attracting basin \( A_0(p) \) includes \( \mathbb{C}r \) (or \( A_0(p) \) includes \( P \)).
11. There exists a solitary fixed point \( p \) such that the immediate attracting basin \( A_0(p) \) contains at least \( 2d-4 \) critical values counted with multiplicity, and \( \lim_{k \to \infty} f^k(c) = p \) for any critical point \( c \).
12. The bounded iterated monodromy group \( \pi_1(\Omega - P, \bar{x})/\ker \alpha_\infty \) is a finite group.

Proof. (2 or 3) \( \Rightarrow \) 1 \( \Rightarrow \) 5 and 9 \( \Rightarrow \) 8 \( \Rightarrow \) 7 and 4 \( \Rightarrow \) 10 \( \Rightarrow \) 11 are trivial.

7 \( \Rightarrow \) 9 by Corollary 1.

7 \( \Rightarrow \) 10 by Lemma 3.

10 \( \Rightarrow \) 9. Using the notation of Lemma 3, each connected component of \( \hat{\mathbb{C}} - U_k \) is homotopically trivial for large \( k \). By Remark 1, \( f^{-k}(\Omega) \) is homotopically trivial.

3 \( \Rightarrow \) 6. Let \( r \) be a radial for \( f^n \). For \( \omega \in \Sigma_{d^n} \), we have a path \( l_\omega : [0,1] \to \hat{\mathbb{C}} - P \) such that \( l_\omega(0) = \bar{x}, l_\omega(1) = \phi_r(\omega) \). Thus if \( \phi_r \) is homeomorphic, then the map \( J \times [0,1] \to \hat{\mathbb{C}} - P \) defined by \( (z,t) \mapsto l_{\phi_r^{-1}(z)}(1-t) \) is a homotopy between the inclusion map of \( J \) and a constant map.

10 \( \Rightarrow \) 2 and 3. Suppose \( P \subset A_0(p) \). Let \( U \) be a simple domain of \( p \), and take \( U_k \) as in Definition 8. Then \( P \subset U_n \) for some \( n \). The simply connected domain \( V = \hat{\mathbb{C}} - \overline{U} \) satisfies \( f^{-n}(V) \subset V \) and that any connected component of \( V - f^{-n}(V) \) other than \( U_n - \overline{U} \) contains no postcritical point.

There exists a topological tree \( \Gamma \subset U_n - U \subset V - f^{-n}(V) \) satisfying:

1. \( V' := V - \Gamma \) is connected and simply connected,
2. \( \Gamma \) includes \( P \cap V \),

10.
3. For $k = 1, 2, \ldots, n-1$ and every connected component $W$ of $f^{-k}(V)$, $W - \Gamma$ is connected.

Indeed, let $C_i$, $i = 1, 2, \ldots, m$ be the connected components of $\bigcup_{k=0}^{n-1} f^{-k}(\partial V) \cap U_n$, and $W_j$, $j = 1, 2, \ldots, s$ be the connected components of $(U_n \cap V) - \bigcup_i C_i$. Take points $a_i \in C_i$ and $b_j \in W_j$ for each $i, j$. Construct a tree $\Gamma$ with vertex set $\{a_i, b_j\} \cup P \cap V$ and edge set $\{e_{ij} : C_i \subset \partial W_j\} \cup \{e_{qj} : q \in P \cap W_j\}$, where $e_{ij}$ joins $a_i$ and $b_j$ in $W_j$, and $e_{qj}$ joins $q$ and $b_j$ in $W_j$.

Then $f^{-1}(V')$ has exactly $d$ connected components, and so the inverse of $f : f^{-1}(V') \rightarrow V'$ has $d$ branches. Let $g_i : J \rightarrow J, i = 1, 2, \ldots, d$ be the branches restricted on the Julia set.

The following fact is crucial. For any $w = i_1 i_2 \cdots i_k \in \text{Word}(k)$, some connected component of $f^{-k}(V)$ includes $g_w(J)$, where $g_w = g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_k}$. Indeed, first $J \subset V$, and by induction suppose that some connected component $E$ of $f^{-k}(V)$ includes $g_w(J)$. Then the connected component of $f^{-k}(V) - \Gamma$ included in $E$ includes $g_w(J)$ by the third condition of $\Gamma$. Hence for $i \in \text{Word}(1)$, $g_i(g_w(J))$ is included in a connected component of $f^{-k-1}(V)$.

Let $\rho(x, y)$ be the distance introduced in Lemma 2. If $k > n + 1$, then $f$ is injective on each connected component of $f^{-k}(V)$. Thus the diameters in $\rho$ of connected components of $f^{-k}(V)$ uniformly converge to zero as $k \rightarrow \infty$, and consequently so do the diameters of $g_w(J), w \in \text{Word}(k)$. Therefore there exists a conjugacy between $(f, J)$ and $(\sigma, \Sigma_d)$. Thus 2 is verified.

It is easier to show 3. Indeed, take a topological disc $\overline{D}$ with $f^{-n}(V') \subset \overline{D} \subset V'$.

(5 or 6) $\Rightarrow$ 7. Assume that there exists a homotopically invariant curve $\gamma$. By Lemma 4, we have a closed curve $\tilde{\gamma}$ in $J$ homotopic to $\gamma$, which is not homotopically trivial. The connected component of $J$ including $\tilde{\gamma}$ is neither simply connected nor homotopically trivial.

11 $\Rightarrow$ 10. Suppose that 11 is satisfied. Let $U$ be a simple domain of $p$, and take $U_k$ as in Definition 8. By Lemma 1, there exists $n > 0$ such that $(P \cup f^{-1}(p)) \cap A_0(p) \subset U_n$. Let $U_n^{0} = U_{n+1}$, and let $U^{0}_n, i = 1, 2, \ldots, s$ be the connected components of $f^{-1}(U_n)$ other than $U_{n+1}$. Note that $U^{0}_{n+1} \not\subset A_0(p)$ for $i = 1, 2, \ldots, s$ since $f^{-1}(p) \cap A_0(p) \subset U_n$. It is sufficient to show $s = 0$. Indeed, if $s = 0$, then $f^{-1}(A_0(p)) = A_0(p)$, and so $A(p) = A_0(p)$, and every critical value is contained in $A_0(p)$.

Let $W = U_n - \bigcup_{a \in a_i \text{ critical value}} B_a$ and $W_i = U_n^{i} - \bigcup_{a \in a_i \text{ critical value}} (B_a)$, the connected component of $f^{-i}(W)$ included in $U_{n+1}^{i}$, where $B_a$ is a small disc around $a$. Let $m_i$ be the number of critical points in $W_i$ counted with multiplicity, and $d_i = \text{deg}(f : W^{i} \rightarrow W)$.

We show

$$\sum_{\gamma : \text{ connected component of } \partial W^{i}} (\text{deg}(f : \gamma \rightarrow f(\gamma)) - 1) = 2d_i - 2 - m_i \quad (1)$$

for $s = 0, 1, \ldots, s$. To this end, first note that $\chi(W^{i}) = d_i \chi(W)$, where $\chi$
denotes the Euler characteristic. We have
\[
\#\{\text{connected components of } \partial W\} = -\chi(W) + 2 = -d_i\chi(W) + 2,
\]
\[
\#\{\text{connected components of } \partial W\} = -\chi(W) + 2.
\]

This implies that
\[
\sum_{a \in \mathcal{C}(U_{n+1})} (\deg(f : \partial B'_a \to \partial B_{f(a)}) - 1)
\]
\[
+ \sum_{\gamma : \text{connected component of } \partial U^i_{n+1}} (\deg(f : \gamma \to f(\gamma)) - 1)
\]
\[
= \sum_{\gamma : \text{connected component of } \partial W^i} (\deg(f : \gamma \to f(\gamma)) - 1)
\]
\[
= d_i(-\chi(W) + 2) - (-d_i\chi(W) + 2) = 2d_i - 2,
\]
where $B'_a$ is a connected component of $f^{-1}(B_{f(a)})$ including $a$. Note that
\[
\sum_{a \in \mathcal{C}(U_{n+1})} (\deg(f : \partial B'_a \to \partial B_{f(a)}) - 1) = m_i.
\]

Suppose $\sum_{i=0}^s m_i \geq 2d - 3.$ From (1), we have $0 \leq \sum_{i=0}^s (2d_i - 2 - m_i) \leq 2d - 2(s + 1) - (2d - 3) = 1 - 2s$. Thus $s = 0$.

Suppose $\sum_{i=0}^s m_i = 2d - 4.$ Then by the similar argument to the above, we have $s \leq 1$. Assume $s = 1$. Then we have an invariant curve $\tilde{\gamma}$ which is a connected component of $\partial U_N$ for large $N > 0$ by Lemma 3. Moreover, \(\deg(f^k : \tilde{\gamma}' \to \tilde{\gamma}) > 1\) by Lemma 4.

From $0 \leq 2d_i - 2 - m_i$, $i = 0, 1$, and $d_0 + d_1 = d, m_0 + m_1 = 2d - 4$, we have
\[
0 = 2d_i - 2 - m_i = \sum_{\gamma : \text{connected component of } \partial U^i_{n+1}} (\deg(f : \gamma \to f(\gamma)) - 1)
\]
for $i = 0, 1$. Thus for any connected component $\gamma \subset \partial U_n$ and any connected component $\gamma' \subset f^{-1}(\gamma)$, the degree of $f : \gamma' \to \gamma$ is one. This is a contradiction.

9 $\Rightarrow$ 12. Suppose that every connected component of $f^{-n}(\Omega)$ is homotopically trivial. To see $\alpha_{\infty}(\pi_1(\Omega - P, \bar{x})) \subset \prod_{k=0}^\infty \mathcal{G}(f^{-k}(\bar{x}))$ is a finite group, we show that the projection $\alpha_{\infty}(\pi_1(\Omega - P, \bar{x})) \to \mathcal{G}(f^{-0}(\bar{x}))$ is injective. To this end, let $\gamma : [0, 1] \to \Omega - P$ be a closed curve such that $g = [\gamma] \in \pi_1(\Omega - P, \bar{x})$ satisfies $\alpha_n(g) = 1$. We denote by $L_{x,\gamma}(\bar{x})$ the lift of $\gamma$ by $f^k$ with $L_{x,\gamma}(\bar{x})(0) = x$. Then $L_{x,\gamma}(\bar{x})$ is a closed curve for any $x \in f^{-n}(\bar{x})$. This implies that $L_{x,\gamma}(\bar{x})$ is a closed curve for any $0 \leq k \leq n - 1$ and any $x \in f^{-k}(\bar{x})$. Moreover, for any $x \in f^{-n}(\bar{x})$, the closed curve $L_{x,\gamma}(\bar{x})$ is homotopically trivial, since it lies in $f^{-n}(\Omega)$. Thus every $L_{x,\gamma}(\bar{x}), x \in f^{-k}(\bar{x}), k > 0$ is a closed curve. Consequently, $\alpha_{\infty}(\pi_1(\Omega - P, \bar{x})) \to \mathcal{G}(f^{-n}(\bar{x}))$ is injective, and so $|\alpha_{\infty}(\pi_1(\Omega - P, \bar{x}))| \leq |\mathcal{G}(f^{-n}(\bar{x}))|$.

12 $\Rightarrow$ 7. Suppose that there exists an invariant curve $\gamma$. Let $\gamma'$ and $k$ be as in Definition 12. Then $\deg(f^k : \gamma' \to \gamma) > 1$. Thus for any $m > 0$, there exist $n > 0$ and $x \in f^{-n}(\bar{x})$ such that $L_{n,\gamma}(\gamma^m)$ is not a closed curve. Therefore for the element $g \in \pi_1(K - P, \bar{x})$ corresponding to $\gamma$, $\alpha_{\infty}(g)$ is not of finite order. \(\square\)
Corollary 2. Let $f$ and $g$ be Cantor hyperbolic rational maps of the same degree $d$. Then there exist open sets $U_f, U_g$ and a quasiconformal map $h : U_f \to U_g$ such that $J_f \subset U_f, J_g \subset U_g$ and $g \circ h = h \circ f$ on $f^{-1}(U_f)$.

Proof. This assertion is a consequence of Theorem 5 we will prove, since two rational maps in the same hyperbolic components are qc-conjugate on neighborhoods of the Julia sets [15]. But here we show this assertion as a corollary of $10 \Rightarrow 2$ of Theorem 1.

We can assume that $g$ is s-Cantor without loss of generality. There exists an open topological disc $D$ such that $f^{-1}(\overline{D}) \subset D$ and $D \cap P_g = \emptyset$. We have a conjugacy $\phi_g : J_g \to \Sigma_d$. On the other hand, by Theorem 1 there is a conjugacy $\phi_f : J_f \to \Sigma_d$ and an open set $U_f := \mathbb{C} - \overline{U_n}$ such that $f^{-1}(\overline{U_f}) \subset U_f$ and $U_f \cap P_f = \emptyset$. The open set $U_f$ is a disjoint union of finite topological discs $D_1, D_2, \ldots, D_k$. We can construct topological discs $D'_1, D'_2, \ldots, D'_k \subset D$ such that $\phi_f(D_i \cap J_f) = \phi_g(D'_i \cap J_g)$ and $g^{-1}(\overline{D'_i}) \subset \bigcup_i D'_i$. Now it is easy to find a required map $h$.

Remark 2. The number $2d - 4$ in 11 of Theorem 1 is the best possible. Namely, for $d \geq 3$ there exists a rational map of degree $d$ with solitary fixed point $p$ such that $A(p)$ contains all critical points, and such that $A_0(p)$ contains exactly $2d - 5$ critical points counted with multiplicity.

Remark 3. In Definition 12, the condition $\gamma \subset \Omega - P$ is important. The following examples explain the difference between $\Omega - P$ and $\Omega - \overline{P}, \hat{\mathbb{C}} - \overline{P}$, or $\hat{\mathbb{C}} - P$.

(a) There exists a geometrically finite rational map that does not satisfy the condition of Theorem 1 and has no homotopically invariant curve in $\Omega - \overline{P} \subset \hat{\mathbb{C}} - \overline{P}$.

(b) There exists a hyperbolic rational map that satisfies the condition of Theorem 1 and has a homotopically invariant curve in $\hat{\mathbb{C}} - P$.

Set $f_\lambda(z) = \lambda z/(z^2 + 1)$ for $\lambda \in \mathbb{R}$. We have $\text{Cr} = \{\pm 1\}$ and $P \subset \mathbb{R} - \{0\}$. It is easy to see that 0 is a fixed point with multiplier $\lambda$. The curve $\gamma = \{ia : a \in \mathbb{R}\} \cup \{\infty\}$ is a completely invariant set for $f_\lambda$. If $\lambda = \pm 1$, then $f_\lambda$ does not satisfy the condition of Theorem 1, and $0 \in \Omega - P$ but $0 \notin \Omega - \overline{P}$ (the case (a)). If $-1 < \lambda < 1$, then $f_\lambda$ satisfies the condition of Theorem 1, and $0 \notin \Omega - P$ but $0 \in \hat{\mathbb{C}} - P$ (the case (b)).

4 Strongly Cantor rational maps

In this section, we give a sufficient condition for $f$ to be s-Cantor.

Theorem 2. Let $f$ be a Cantor rational map with a (super)attracting fixed point $\infty$. Suppose that the critical orbits $\text{Orb}(f(c)), c \in \text{Cr}$ are distinct, and that $\infty$ is outside other critical orbits (i.e. $\infty \notin \text{Orb}(f(c))$ for $c \in \text{Cr}_0 := \text{Cr} - \{\infty\}$). Then $f$ is s-Cantor.
Proof. For a connected open set $W$ containing $\infty$ and an integer $k \geq 0$, let us denote by $W(k)$ the connected component of $f^{-k}(W)$ containing $\infty$. We say a connected open set $W$ containing $\infty$ is *appropriate* if $\partial W$ is a simple closed curve, $W \subset f^{-1}(W)$ (i.e., $f(W) \subset W$), and for any $c \in \mathfrak{c}r_0$, $W(1) - W(0)$ contains $f^m(c)$ for some $m > 0$. We define $\kappa = \kappa(W)$ to be the maximal integer such that for any $c \in \mathfrak{c}r_0$, $W(\kappa) - W(\kappa - 1)$ contains $f^m(c)$ for some $m > 0$.

We define $\lambda = \lambda(W)$ to be the maximal integer such that $W(\lambda)$ contains no critical point other than $\infty$. Clearly, $\kappa \leq \lambda$. If $W$ is appropriate, then $W(i)$ for $1 \leq i \leq \lambda$ is a topological disc, and $W(i)(j) = W(i + j)$ if $i + j \leq \lambda + 1$.

In the sequel, we will construct an appropriate domain $W$ such that $\kappa = \lambda$ and $\mathfrak{c}r_0 \subset W(\kappa + 1) - W(\kappa)$. To this end, we take an initial appropriate domain $W_0$ near $\infty$ and deform it in several steps. If $\infty$ is a superattracting fixed point of degree $\delta$, then take a Böttcher coordinate $\varphi : U \to \mathbb{D}$ for the superattracting fixed point $\infty$. Here $U$ is a connected and simply connected neighborhood of $\infty$. There exists $r > 0$ such that for any $c \in \mathfrak{c}r_0$, $\varphi^{-1}(\{0 < |z| \leq r\})$ contains $f^m(c)$ for some $m > 0$, since we have assumed that the orbit of $c$ does not land at $\infty$.

We also let $r$ satisfy $\varphi^{-1}(\{|z| = r\}) \cap P = \emptyset$. For a large enough integer $N$, we define $W_0 = \varphi^{-1}(\{|z| < r^N\})$. Then $W_0$ is appropriate with $\kappa(W_0) \geq N$. If $\infty$ is attracting, then take a linealizing coordinate $\varphi : U \to \mathbb{C}$ for $\infty$, and define $W_0$ similarly such that $\kappa(W_0) = N$ is large enough. The integer $N$ will be determined later.

Let $W$ be an appropriate domain. We use the notation $A_i = A_W(i) = W(i) - W(i - 1)$ for $i = 1, \ldots, \lambda$. Note that $A_i$ is an annulus and is the connected component of $f^{-1}(A_{i-1})$ such that $A_{i-1}$ and $A_i$ share a boundary component. The restriction $f : A_i \to A_{i-1}$ is a covering of degree $\delta$. We set $A_{\lambda+1}$ to be the connected component of $f^{-1}(A_\lambda)$ such that $A_\lambda$ and $A_{\lambda+1}$ share a boundary component. Note that $A_{\lambda+1} \subset W(\lambda+1) - W(\lambda)$ is not an annulus. Let us denote $C_i = C_W(i) = \partial W(i)$ for $i = 0, 1, \ldots, \lambda$. It equals $A_{i+1} \cap A_i$ for $i = 1, 2, \ldots, \lambda$. For $c \in \mathfrak{c}r_0$, we define an integer $\mu = \mu(c) = \mu_W(c) \geq \kappa$ satisfying $f^\mu(c) \in A_1$. Then $f^{\mu-\kappa-1}(c) \in A_\kappa$.

Let $\mathcal{B}_m$ be the set of connected components of $f^{-m}(A_1)$, and

$$ \mathcal{B} = \mathcal{B}_W = \bigcup_{m=0}^{\infty} \mathcal{B}_m. \quad (2) $$

Note that $A_i \in \mathcal{B}_{i-1}$ for $i = 1, 2, \ldots, \lambda + 1$. We say distinct $X, X' \in \mathcal{B}$ are *adjacent* if they share a boundary component. If $X, X' \in \mathcal{B} - \{A_1\}$ are adjacent, then $f(X), f(X') \in \mathcal{B}$ are also adjacent. Define an order $\prec$ on $\mathcal{B}$ by $X \prec X'$ if $X'$ separates $X$ and $W$. It is easily seen that for any $X \in \mathcal{B} - \{A_1\}$, there uniquely exists $X' \in \mathcal{B}$ such that $X, X'$ are adjacent and $X \prec X'$. We use the notation $X + 1$ for this $X'$, and define inductively $X + m = (X + m - 1) + 1$ for $m = 2, 3, \ldots$. If $X' = X + m$, we write

$$ L(X, X') := m. $$

Claim 1. If $X \prec X'$, then there exists $m > 0$ such that $X + m = X'$.
Proof of Claim 1. There is a path $\gamma$ in the Fatou set $F$ which connect $X$ and $X'$. By the compactness of $\gamma$, we have $\gamma \subset \bigcup_{0 \leq k \leq k_0} f^{-k}(W)$ for some $k_0$. We have $F - \bigcup_{X \in B} X = W \cup \bigcup_{m=0}^{\infty} f^{-m}(f^{-1}(W) - W(1))$. The connected components of $f^{-1}(W)$ other than $W(1)$ are topological disc with no point of $P$. Thus all the connected components of $F - \bigcup_{X \in B} X$ are topological discs. Since only finite of them intersect $\gamma$, we can retake $\gamma$ such that $\gamma \subset \bigcup_{0 \leq k \leq k_0} f^{-k}(A_1)$. Hence $\# \{ X \in B : X \cap \gamma \neq \emptyset \} < \infty$, and the claim is true.

Claim 2. If $X < X'$, then $f^k(X') \neq X$ for any integer $k > 0$.

Proof of Claim 2. Let $X_0 = X$ and $X_i = X_0 + i$ for $i = 1, 2, \ldots, s, \ldots, m$, where $X_s = X'$ and $X_m = A_1$. Let $k > 0$ be an integers such that $f^k(X_s) \in B$. We have $\{ f^k(X_{j}) : 0 \leq j \leq m - 1 \} \cap B \supset \{ f^k(X_{s}), f^k(X_{s}) + 1, f^k(X_{s}) + 2, \ldots, A_1 \}$, since $f^k(X_j)$ and $f^k(X_{j+1})$ are adjacent. Hence $L(f^k(X_s), A_1) \leq m$. This fact implies the claim.

Suppose $f^k(X) = A_1$. We write $X \triangleleft X + 1$ if $f^k(X + 1) = f(A_1)$; $X \triangleright X + 1$ if $f^k(X + 1) = A_2$.

For $z \in \bigcup_{X \in B} X$, let $X(z) = X_W(z) \in B$ be the connected component containing $z$.

Let $c$ be a critical point other than $\infty$. Suppose

$$c \notin A_{\lambda+1}.$$

Let $\gamma : [0, 1] \to \overline{A_1}$ be a simple path such that $\gamma(0) = f_\mu(c)$, $\gamma(1) \in \partial A_1$, and $\gamma(t) \in A_1 - P$ for $0 < t < 1$. We say $\gamma$ is of V-type or of $\Lambda$-type for $(c, W)$ depending on if $\gamma(1) \in C_0 = \partial W$ or $\gamma(1) \in C_1 = \partial W(1)$.

The following lemma will be proved later.

Lemma 5. Suppose that $X(c)$ and $X \in B$ are adjacent. Then there exists a simple path $\gamma$ of V-type or $\Lambda$-type for $(c, W)$ such that the connected component $\gamma'$ of $f^{-\mu}(\gamma)$ containing $c$ has an endpoint in $E_0 = \overline{X(c)} \cap X$.

In particular, if $X(c) \triangleleft X(c) + 1 = X$ (resp. $X(c) \triangleright X(c) + 1 = X$), then $\gamma$ is of V-type (resp. $\Lambda$-type).

By this lemma, we have a simple path $\gamma$ of V-type or $\Lambda$-type such that the connected component $\gamma'$ of $f^{-\mu}(\gamma)$ containing $c$ has an endpoint in $\overline{X(c)} \cap X(c) + 1$. Let $\gamma_j : [0, 1] \to f^{-j}(\overline{A_1})$ for $1 \leq j \leq \mu - 1$ be the lift of $\gamma$ by $f^j$ with

$$\gamma_j(0) = f^{\mu-j}(c).$$

Note that $\gamma_j(0) = f^{\mu-j}(c) \in A_{j+1}$ for $1 \leq j \leq \mu - 1$.

We take a Jordan domain $U_\gamma$ including $\gamma$ which is so close to $\gamma$ that $U_\gamma$ contains no point of $P$ other than $\gamma(0) = f_\mu(c)$, that

$$U_\gamma \subset A_1 \cup C_0 \cup f(A_1) \text{ } \text{(V-type)} \text{ or } U_\gamma \subset A_1 \cup C_1 \cup A_2 \text{ } \text{(\Lambda-type)},$$

that $U_\gamma - A_1$ is connected, and that $U_\gamma, f^{-1}(U_\gamma)$ are disjoint. Let $U_{\gamma_j}$ be the connected component of $f^{-j}(U_\gamma)$ including $\gamma_j \ni f^{\mu-j}(c)$ for $1 \leq j \leq \mu - 1$.

Let $U_j$ be the set of connected components $U$ of $f^{-j}(U_\gamma)$ such that the restriction $f^j|_U : U \to U_\gamma$ is a homeomorphism. Then $\bigcup_{U \in U_j} U = f^{-j}(U_\gamma)$ for
Let us denote $\mathcal{W}$ for $\mathcal{W} / \mathcal{U}$ defined and $\mathcal{C} / \mathcal{U}$.

In particular, we have

$$
\hat{U} = \bigcup_{j=0}^{\infty} \bigcup_{i \in \mathcal{U}_j} U, \quad T = \bigcup_{j=0}^{\infty} f^{-j} (U) - \hat{U} = \bigcup_{j=0}^{\infty} f^{-j} (U_{\gamma'}).$$

Since $c \notin \mathcal{A}_{\lambda+1}$,

$$
T \cap \mathcal{A}_{i} = \emptyset, \quad (i = 1, 2, \ldots, \lambda)
$$

and

$$(T - U_{\gamma'}) \cap \mathcal{A}_{\lambda+1} = \emptyset.$$  \hspace{1cm} (5)

Take a point $a \in U_{\gamma} - \mathcal{A}_1$. Then $a \in f(A_1)$ (V-type) or $a \in A_2$ (A-type).

We take a diffeomorphism $h_0$ of the Fatou set $F$ to itself satisfying $h_0(z) = z$ for $z \notin U_{\gamma}$, $h_0(a) = \gamma(0)$, and $h_0(A_1) \subset A_1$. We define a homeomorphism $h : F \to F$ by $h = (f)\Gamma^{-1} \circ h_0 \circ f)$ on $U \in \mathcal{U}_j, j = 1, 2, \ldots$ and $h(z) = z$ if $z \notin \hat{U}$. Then $h$ is homotopic to the identity on $F - \mathcal{C}_\mathcal{R}$, and

$$
f \circ h = h \circ f \text{ on } F - (U_{\gamma} \cup U_{\gamma'}),
$$

$$
f \circ h(U) = h \circ f(U) \text{ for } U = U_{\gamma}, U_{\gamma'}.
$$

Set

$$
\sigma = \begin{cases} 
0 & \text{(V-type)} \\
1 & \text{(A-type)}.
\end{cases}
$$

Let us denote $W_{\gamma} = h(W(\sigma))$.

Claim 3. Suppose $\kappa(W) \geq 2$. The domain $W_{\gamma}$ is an appropriate domain with $\kappa(W) - 1 \leq \kappa(W_{\gamma}) \leq \kappa(W)$ and $\lambda(W_{\gamma}) = \lambda(W) - \sigma$. Let

$$
\mathcal{B}_0 = \mathcal{B}_{W}^0 = \{X \in \mathcal{B}_W : X \cap T = \emptyset \text{ and } f^m(X) = A_W(\sigma + 1) \text{ for some } m \geq 0\},
$$

and

$$
\mathcal{B}_1 = \mathcal{B}_{W,\gamma}^1 = \{X \in \mathcal{B}_{W,\gamma} : X \cap T = \emptyset\}.
$$

Then we have a one-to-one correspondence $\Psi = \Psi_{\gamma}$ between $\mathcal{B}_0$ and $\mathcal{B}_1$ such that $X \cap \text{orb}(c') = \Psi(X) \cap \text{orb}(c')$ for every $c' \in \mathcal{C}_\mathcal{R} - \{c, \infty\}$, which preserves the order $\prec$, the +1 operation, and the relation $\prec$. The mapping is given by $\Psi(X) = h(X)$. In particular, $\Psi(A_W(\lambda(W))) = A_{W,\gamma}(\lambda(W) - \sigma) = A_{W,\gamma}(\lambda(W_{\gamma})) \in \mathcal{B}_1$.

Proof of Claim 3. We have $W \subset W_{\gamma} \subset \overline{W}_\gamma \subset W(1)$ (V-type) or $\overline{W} \subset W_{\gamma} \subset W(1)$ (A-type). Hence $f(W_{\gamma}) \subset W_{\gamma}$. From $W_{\gamma} \cap P \subset W(1) \cap P$ and $\kappa \geq 2$, we can see that $W_{\gamma}$ is appropriate.

By (4) and (7), we have $f \circ h(X) = h \circ f(X)$ for $X \in \mathcal{B}_0 - \{A_W(\sigma + 1)\}$. Since $f(X)$ and $h \circ f(X)$ are homotopic in $\hat{U}$ minus the critical values of $f|_{\hat{U}}$, we see $h(X)$ is a connected component of $f^{-1}(h \circ f(X))$. Hence $\Psi : \mathcal{B}_0 \to \mathcal{B}_1$ is well-defined and $X \cap \text{orb}(c') = \Psi(X) \cap \text{orb}(c')$ for every $X \in \mathcal{B}_0$ and $c' \in \mathcal{C}_\mathcal{R} - \{c, \infty\}$. In particular, we have $f \circ h(A_W(i)) = h \circ f(A_W(i + 1)) = h(A_W(i))$ for
\[ \sigma + 1 \leq i \leq \lambda(W) - 1 \] by (5) and (7). This means \( A_{W_i}(i) = h(A_{W_i}(i + \sigma)) \) for \( 1 \leq i \leq \lambda(W) - \sigma \). Clearly,

\[ A_{W_i}(i) \cap (P \cup \text{cr} - \text{orb}(c)) = A_{W_i}(i + \sigma) \cap (P \cup \text{cr} - \text{orb}(c)) \]

and

\[ A_{W_i}(i) \cap \text{orb}(c) = A_{W_i}(i + 1 - \sigma) \cap \text{orb}(c) \]

for \( 1 \leq i \leq \lambda(W) - \sigma \). Thus \( \kappa(W_\gamma) \geq \kappa(W) - 1 \) and \( \lambda(W_\gamma) \geq \lambda(W) - \sigma \). Since \( A_{W_\gamma}(\lambda(W) + 1 - \sigma) \cap \text{cr} \supset A_{W_i}(\lambda(W) + 1) \cap \text{cr} \), we have \( \lambda(W_\gamma) = \lambda(W) - \sigma \).

Case 1. Suppose \( \gamma \) is V-type. If \( A_{W}(\kappa(W) + 1) \cap \text{orb}(c) \neq \emptyset \), then \( \kappa(W_\gamma) = \kappa(W) \); otherwise \( \kappa(W_\gamma) = \kappa(W) - 1 \).

Case 2-i. Suppose \( \kappa(W) < \lambda(W) \) and \( \gamma \) is A-type. If \( A_{W}(\kappa(W) + 1) \cap \text{orb}(c') \neq \emptyset \) for every \( c' \in \text{cr} - \{c, \infty\} \), then \( \kappa(W_\gamma) = \kappa(W) \); otherwise \( \kappa(W_\gamma) = \kappa(W) - 1 \).

Case 2-ii. Suppose \( \kappa(W) = \lambda(W) \) and \( \gamma \) is A-type. Then \( \kappa(W_\gamma) = \kappa(W) - 1 \).

It is clear that \( \Psi \) preserves the order \(<\), the +1 operation, and the relation \( \prec \). This completes the proof of Claim 3. We call this operation a \( \gamma \)-deformation of an appropriate domain.

It is easily seen that if \( X \in B_W \) intersects \( U_\gamma \), then either \( X = X_W(c) \) or \( X \) is adjacent to \( X_W(c) \). By Claim 2, \( X_W(c) + 1 \cap (T - U_\gamma) = \emptyset \) and there is no \( X \in B_W \) such that \( X_W(c) + 1 < X \) and \( X \cap T \neq \emptyset \). Hence we have \( X_W(c) \supset h(X_W(c) + 1) \cup U_\gamma \) and \( X_W(c) + 1 = \Psi(X_W(c) + 2) \). This means that \( L(X_W(c), A_W(\lambda(W))) = L(X_W(c), A_W(\lambda(W))) + 1 \).

We show that if \( \kappa(W) \geq L(X_W(c), A_W(\lambda(W))) = n \), then by repeating \( \gamma \)-deformation \( n - 1 \) times, we get an appropriate domain \( W' \) such that \( c \in A_{W_\gamma}(\lambda(W') + 1) \) and \( A_{W'}(\lambda(W') + 1) \cap (P \cup \text{cr} - \text{orb}(c)) = A_{W'}(\lambda(W') + 1) \cap (P \cup \text{cr} - \text{orb}(c)) \). Indeed, we take a sequence of paths \( \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \) which satisfies the following: \( \gamma_k \) is of V-type or A-type for \( (c, W^{k-1}) \) obtained by Lemma 5, where inductively we write \( W^0 = W \) and \( W^k = W^{k-1}_k \) \( (k = 1, 2, \ldots, n - 1) \). Then \( X_{W_{n-1}}(c + 1) = \Psi_{J_n} \circ \cdots \circ \Psi_{\gamma_1}(A_W(\lambda(W))) = X \). We have \( X = A_{W_{n-1}}(\lambda(W^{n-1})) \), since \( \Psi_{\gamma_n} \circ \cdots \circ \Psi_{\gamma_1}(A_W(\lambda(W))) \in B_{W_k} \) for \( k = 1, 2, \ldots, n - 2 \).

Taking \( N \) large enough, we can apply this procedure to all the critical points other than \( \infty \), and we have an appropriate domain \( W \) such that \( W(\gamma) = W(\lambda) \) contains all the critical values.

**Proof of Lemma 5.** We assume \( f^\mu(X) = f(A_1) \) without loss of generality. Then \( E_0 \) is a connected component of \( f^{-\mu}(C_0) \). Let \( P_0 \subset A_1 \) be the set of critical values of \( f^\mu(X(c) \to A_1) \). Let \( m = |P_0| \) and take \( m \) points \( a_0, a_1, \ldots, a_{m-1} \in C_0 \) arranged counterclockwise. Take \( L = \{ L_p \}_{p \in P_0} \), a collection of disjoint simple paths \( L_p \)'s in \( A_1 \) joining \( p \in P_0 \) and some \( a_{j(p)} \) such that \( j(p) \neq j(p') \) if \( p \neq p' \), and such that \( L_p \) does not intersect \( P \) except at the endpoint. We consider that \( j : P_0 \to \mathbb{Z}/(m) \) is a bijection which depends on \( L \). We may write the inverse \( p = j^{-1} : \mathbb{Z}/(m) \to P_0 \). Write \( L = \bigcup_{p \in P_0} L_p \). Since each connected component of \( f^{-\mu}(A_1 - L) \) in \( X(c) \) is an annulus, \( f^{-\mu}(C_0 \cup L) \cap X(c) \) is connected.
For $z \in \overline{X(c)}$, let us denote by $I(z)$ the connected component of $f^{-\mu}(L)$ containing $z$ if it exists, and similarly by $\overline{I(z)}$ the connected component of $f^{-\mu}(L - P_0)$ containing $z$. We say $C = \{E_i\}_{i=0}^k, \{c_i\}_{i=0}^k$ is a chain between $E_0$ and $c$ for $L$ if $E_0, E_1, \ldots, E_k$ are distinct connected components of $f^{-\mu}(C_0)$ in $\overline{X(c)}$ and $c_0, c_1, \ldots, c_k$ are distinct critical points of $f^{\mu} | \overline{X(c)}$ satisfying the following: $I_i = I(c_i)$ joins $E_i$ and $E_{i+1}$ ($i = 0, 1, \ldots, k - 1$), $I_k = I(c_k)$ has an endpoint in $E_k$, and $c_k = c$. The existence of such a chain is guaranteed by the fact mentioned at the end of the previous paragraph. We say $k$ is the length of the chain, and denote by $\mathcal{L}(C)$. If there exists a chain with $\operatorname{Len}(C) = 0$, then $L f^{\mu}(c)$ is a required path $\gamma$.

If $m = 1$, then $c$ is the only critical point of $f^{\mu} : X(c) \to A_1$. In this case, any chain between $E_0$ and $c$ has length zero. Therefore we can assume $m > 1$.

Take a chain $C = \{\{E_i\}, \{c_i\}\}$ between $E_0$ and $c$ for $L$ minimizing $k = \operatorname{Len}(C)$. Moreover, if there are several candidates for $E_k$, then choose one minimizing $\mathcal{D}(C)$ defined below. Assume $k > 0$. We will show that we can deform $L$ such that there exists a chain of length 0. To this end, we use the deformation by half Dehn twists several times. At each step, either the length $\operatorname{Len}(C)$ of the chain or the “distance” $\mathcal{D}(C)$ between $I_k$ and $I_{k-1}$ reduces by at least one.

Let $p_0 = f^{\mu}(c)$ and $p_1 = p(j(p_0) + 1)$. For simplicity, we set $j(p_0) = 0, j(p_1) = 1$. Let us denote by $[a_0, a_1]$ the counterclockwise arc in $C_0$ from $a_0$ to $a_1$. Take a simple path $l$ between $p_0$ and $p_1$ in $A_1$ which does not intersect $L$ except at the endpoints such that the closed curve $L_{p_0} \cup l \cup L_{p_1} \cup [a_0, a_1]$ is homotopically trivial in $A_2$ (i.e. it does not enclose $A_2$). Note that $l$ is unique up to homotopy. Take a Jordan domain $U_l \subset (A_1 - L) \cup (L_{p_0} \cup L_{p_1})$ which includes $l$ and does not include any point of $P$ other than $p_0, p_1$.

We define a half Dehn twist on $U_l$ as follows. Let $\tau : \mathbb{D} \to \mathbb{D}$ be the homeomorphism defined by $\tau(z) = -z$ for $|z| \leq 1/2$ and $\tau(z) = e^{2\pi i |z|/1} z$ for $1/2 < |z| \leq 1$. Note that $\tau((1/2, 1))$ is a path joining $-1/2$ and 1, and $\tau((-1, -1/2))$ is a path joining $1$ and $1/2$. Choose an orientation-preserving diffeomorphism $h : \overline{U_l} - l \to \overline{D} - \{0\}$ with $h(L_{p_0} \cap \overline{U_l} - \{p_0\}) = (0, 1)$, and $h(L_{p_1} \cap \overline{U_l} - \{p_1\}) = [-1, 0)$. Let $T = h^{-1} \circ \tau \circ h$. Extend $T$ to a self-homeomorphism of $A_1$ by the identity outside $U_l$.

We take a counterclockwise universal covering $\zeta : \mathbb{R} \to C_0$ such that $\zeta(i + n/m) = a_n$ for $i, n \in \mathbb{Z}$. For a connected component $E$ of $f^{-\mu}(C_0)$, and $\tilde{a} \in f^{-\mu}(\{a_0, a_1, \ldots, a_{m-1}\}) \cap E$, we define $k = d(\tilde{a}, \tilde{a})$ to be the smallest positive integer such that $\tilde{\zeta}_{\tilde{a}}(k/m) = \tilde{a}$, where $\tilde{\zeta}_{\tilde{a}} : \mathbb{R} \to E$ is the lift of $\zeta$ by $f^{\mu}$ with $\tilde{\zeta}(0) = \tilde{a}$.

Let
\[
\mathcal{D}(C) = \mathcal{D}_l(C) = \min\{d(\tilde{a}_0, \tilde{a}) : \tilde{a}_0 \in E_k \cap I_k, \tilde{a} \in E_k \cap I_{k-1}\}.
\]

We can consider $\mathcal{D}(C)$ to be a counterclockwise distance from $I_k$ to $I_{k-1}$ along $E_k$.

The following lemma is easy to see.

**Lemma 6.** Let $E$ be a connected component of $f^{-\mu}(C_0)$, let $\tilde{a}_0 \in f^{-\mu}(a_0) \cap E, \tilde{a}_1 \in f^{-\mu}(a_1) \cap E$ such that $d(\tilde{a}_0, \tilde{a}_1) = 1$. Let $\alpha : \mathbb{R} \times (0, 1) \to \overline{U_l}$ be the
Lemma 7. Let \( \sim \) of \( \tilde{f} \) and \( \tilde{l} \) be the connected component of \( \tilde{f} \) containing \( z \). Then the following statements hold:

- \( \tilde{a}(1/2, (0, 1)) \subset I(\tilde{a}_1) \).
- For \( z \in f^{-\mu}(p_0) \), if \( z \in I(\tilde{a}_0(n, 1)) \), then \( z \in f^{2n-1}(\tilde{a}_1) \).
- For \( z \in f^{-\mu}(p_1) \), if \( z \in I(\tilde{a}_0(n + 1/2, 1)) \), then \( z \in f^{2n+1}(\tilde{a}_0) \).

**Lemma 7.** Let \( E \) be a connected component of \( f^{-\mu}(C_0) \). Let \( \tilde{a}_0 \in f^{-\mu}(a_0) \cap E \) and \( \tilde{a}_1 \in f^{-\mu}(a_1) \cap E \) such that \( d(\tilde{a}_0, \tilde{a}_1) = 1 \). Let \( z_1 \in f^{-\mu}(p_1) \cap I(\tilde{a}_1) \) and let \( \tau' \) be the connected component of \( f^{-\mu}(l) \) containing \( z_1 \). Then the following hold:

1. If \( c \in \tau' \), then there exist a connected component \( E' \) of \( f^{-\mu}(C_0) \) and \( \tilde{a}_0 \in I(c) \cap E', \tilde{a}_1 \in I(z_1) \cap E' \) such that \( d(\tilde{a}_0, \tilde{a}_1) = 1 \).
2. If \( c \notin \tau' \), then there exists an odd integer \( n \) such that \( \tilde{a}_1 \in I^n(c) \).
3. If \( c \notin \tau' \), then \( \tilde{a}_1 \in I^n(z_1) \) for any odd integer \( n \).

**Proof of Lemma 7.** We may consider a connected component \( \tau' \) to be a graph with vertex set \( V = f^{-\mu}(\{p_0, p_1\}) \cap \tau' \). Note that every edge \( e \) of \( \tau' \) is an inverse image of \( l \), that is, \( f^{-\mu} : e \rightarrow l \) is bijective.

The first key to the proof is that \( e \) is the only critical point of \( f^{-\mu} \) in \( f^{-\mu}(p_1) \). The second key is that if \( z \in f^{-\mu}(\{p_0, p_1\}) \cap \tau' \) is not a critical point of \( f^{-\mu} \), then \( z \) is an endpoint of \( \tau' \).

1. For any edge \( e \) of the graph \( \tau' \), there uniquely exist an connected component \( E' \) and \( \tilde{a}_0 \in f^{-\mu}(a_0) \cap E', \tilde{a}_1 \in f^{-\mu}(a_1) \cap E' \) such that \( d(\tilde{a}_0, \tilde{a}_1) = 1 \) and \( I(\tilde{a}_0) \cap e \neq \emptyset \), \( I(\tilde{a}_1) \cap e \neq \emptyset \). Indeed, let \( \Gamma \) be the connected component of \( f^{-\mu}(L_{p_0} \cup L_{p_1} \cup a_0, a_1) \) including \( e \). Then the required \( \tilde{a}_0, \tilde{a}_1 \) are the points of \( f^{-\mu}(a_0) \cap \Gamma, f^{-\mu}(a_1) \cap \Gamma \).

We have to show that the graph \( \tau' \) has an edge with endpoints \( z_1 \) and \( c \). If the graph \( \tau' \) has no edge containing both \( c \) and \( z_1 \), then each edge containing \( z_1 \) has an endpoint which is not a critical point of \( f^{-\mu} \), and \( \tau' \) does not contain \( c \). Thus we have an edge \( e \) with endpoints \( z_1 \) and \( c \).

2. Let \( \tilde{a}_0 : \mathbb{R} \times (0, 1) \rightarrow S \) be as in Lemma 6. Note that \( S \) is a connected component of \( f^{-\mu}(\mathbb{R} \times \{0\}) \). We can see that \( l_0 := \mathbb{R} - S \) is a subgraph of \( \tau' \). We show that \( c \in l_0 \). To the contrary, assume that \( c \notin l_0 \). Every point of \( f^{-\mu}(p_0) \cap l_0 \) is not a critical point and is an endpoint of \( \tau' \). Therefore by the monodromy theorem, the covering \( f^{-\mu} : S \rightarrow \mathbb{R} \) is extended to \( f^{-\mu} : S' \rightarrow \mathbb{R} \) as a covering. Then \( S' = S \cup l_0 - \{z_1\} \), which is a punctured disc. This means \( l' = l_0 \), and a contradiction to the fact \( c \notin \tau' \). Thus \( c \in l_0 \), and \( \tilde{a}_1 \in I^n(c) \) for some \( n \in \mathbb{Z} \) by Lemma 6.

3. By a similar argument to the above, we can see that \( z_1 \) is the only point of \( f^{-\mu}(p_1) \cap \tau' \). Thus the claim is true by Lemma 6. \( \square \)
We define a new system of paths by $\mathcal{L}' = \{T^n(L_p)\}_{p \in P_0}$ for some non-zero integer $n$. The integer $n$ will be determined below. Set $I'_i = I'(c_i)$ for $\mathcal{L}'$ similarly to $I_i$. We show that $I'_i$ joins $E_i$ and $E_{i+1}$ if $i < k - 1$ or $\mathcal{D}_L(C) > 1$. Indeed, $I'_i = I_i$ if $f^{\mu}(c_i) \notin \{p_0, p_1\}$. We know that $c = c_k$ is the only point in $\{c_i\}$ satisfying $f^{\mu}(c_i) = p_0$. Suppose $f^{\mu}(c_i) = p_1$. If $c_i$ and $c$ belong to the same connected component of $f^{-\mu}(l)$, then $i = k - 1$ by the minimality of $\text{Len}(C)$ and $\mathcal{D}_L(C) = 1$ by the minimality of $D(C)$ by 1 of Lemma 7. Equivalently, if $i < k - 1$ or $\mathcal{D}_L(C) > 1$, then the connected component of $f^{-\mu}(l)$ containing $c_i$ does not contain $c$, and so we can see that $I'_i$ joins $E_i$ and $E_{i+1}$ for any $n$ by 3 of Lemma 7.

Let $\tilde{a}_0 \in E_k \cap I_k, \tilde{a} = \zeta_{\tilde{a}_0}(\mathcal{D}_L(C)/m) \in I_{k-1} \cap E_k$ be the points we have taken to define $\mathcal{D}_L(C)$. Let $\tilde{a}_1 = \zeta_{\tilde{a}_0}(1/m) \in f^{-\mu}(a_1) \cap E_k, \tilde{b} = \zeta_{\tilde{a}_0}(\mathcal{D}_L(C)/m - 1/m)$. Let $\tilde{a}'_1 \in E_{k-1} \cap I_{k-1}$.

Case 1. Suppose $\mathcal{D}_L(C) = 1$. Using 2 of Lemma 7, we have an odd integer $n$ such that $\tilde{a}'_1 \in I'(c)$. Then $\mathcal{L}' = \{E_0, E_1, \ldots, E_{k-1}\}, \{c_0, c_1, \ldots, c_{k-2}, c\}$ is a chain for $\mathcal{L}' = (T^n(L_p))_{p \in P_0}$, and so $\text{Len}(\mathcal{L}') = \text{Len}(C) - 1$.

Case 2. Suppose $\mathcal{D}_L(C) > 1$. We set $n = -1$. As we have seen above, $\mathcal{L}$ is a chain for $\mathcal{L}'$.

- If $c_{k-1} \in f^{-\mu}(p_{-1})$, then $\tilde{b} \in I'(c_{k-1})$ and $\tilde{a}_1 \in I'(c)$ by 3 of Lemma 7 and Lemma 6. Thus $\mathcal{D}_L(C) = \mathcal{D}_L(C) - 2$.

- If $c_{k-1} \notin f^{-\mu}(p_{1})$, then $\tilde{b} \in I'(c_{k-1})$ and $\tilde{a}_1 \in I'(c)$ by Lemma 6. Thus $\mathcal{D}_L(C) = \mathcal{D}_L(C) - 1$.

This completes the proof of Lemma 5. 

\section{The shift locus}

In this section, we prove several results on the parameter spaces. Hereafter we assume $f$ has the unique solitary fixed point $\infty$ without loss of generality.

\begin{theorem}
$T_d \cap \text{Pol} = S_d \cap \text{Pol}$.
\end{theorem}

\begin{proof}
Let $f \in S_d \cap \text{Pol}$. Then all critical points belong to the attracting basin of $\infty$. Let $U$ be a simple domain of $\infty$ such that $U$ contains no critical value other than $\infty$. For every $p \in (f^{-1}(U) - U) \cap P$, we take a simple arc $\gamma_p \subset f^{-1}(U) - U$ joining $p$ and a point of $\partial U$ such that $\gamma_p \cap \gamma_q = \emptyset$ if $p \neq q$. Note that $f^{-n}(U)$ is connected for every $n > 0$ since $f^{-1}(\infty) = \{\infty\}$. We inductively take a simple arc $\gamma_p \subset f^{-n}(U) - U$ for $p \in (f^{-n}(U) - f^{-n+1}(U)) \cap P$ joining $p$ and a point of $\partial U$ such that $\gamma_{f(p)} = f(\gamma_p \cap (f^{-n}(U) - f^{-1}(U)))$ and $\gamma_p \cap \gamma_q = \emptyset$ if $p \neq q$. Thus we obtain a simply connected open set $V = \hat{\mathcal{C}} - (\overline{\cup_{p \in P-U} \gamma_p})$ which satisfies $f^{-1}(V) \subset V$. Then it is easy to see that $f$ is s-Cantor.
\end{proof}

\begin{theorem}
$T_d \subseteq S_d$ for $d \geq 4$, and $T_2 = S_2$.
\end{theorem}

The $d = 2$ case is a consequence of the following.

\begin{lemma}
Let $f \in S_d$ which has exactly two critical points. Then $f \in T_d$.
\end{lemma}

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Proof. Note that both of the critical points have local degree \(d\). Take \(U\) a simple domain of \(\infty\). We take open sets \(U_k\) as in Definition 8. Suppose that all critical points belong to the immediate basin of attraction for \(\infty\). If \(U_k\) contains at most one critical value, then \(U_{k+1}\) is simply connected. Let \(n\) be the smallest integer such that \(U_n\) contains two critical values. Then \(f\) satisfies the condition of \(s\)-Cantor rational map with \(\mathcal{D} = \hat{\mathbb{C}} - U_n\).

To prove the other cases, it is sufficient to show that there exists a rational map \(f \in S_d - T_d\). We will do this in section 6.

**Corollary 3.** \(T_d\) is open and dense in \(S_d\).

**Theorem 5.** \(S_d\) is connected.

**Theorem 6.** The set of rational maps \(f \in T_d\) satisfying the assumption of Theorem 2 with an attracting (not superattracting) fixed point \(\infty\) such that all the critical points have the local degree two is connected, and all the maps in the set are \(qc\)-conjugate to each other.

To prove these three statements, crucial is Theorem 2. Additionally, we use \(qc\)-surgery (quasiconformal surgery). For the basics of \(qc\)-surgery, see [3].

**Proof of Corollary 3.** It is easy to see that \(S_d\) and \(T_d\) are open in the space of rational maps of degree \(d\). If \(f \in S_d - T_d\), then we add an arbitrary small perturbation to \(f\) by \(qc\)-surgery that makes the map satisfy the assumption of Theorem 2.

**Proof of Theorem 5.** We can deduce this theorem from Theorem 6. However we give another proof here.

Let \(f \in S_d\). We construct a path in \(S_d\) which connects \(f\) and some map in \(S_d \cap \text{Pol}\). First by a perturbation, we obtain \(f_0\) satisfying the condition of Theorem 2, and moreover we can assume that any critical point other than \(\infty\) has local degree two.

For simplicity, we omit the subscript of \(f_0\). By the proof of Theorem 2, we have a topological disc \(W\) containing \(\infty\) such that \(\overline{W} \subset f^{-1}(W)\), and such that \(A = W' - W\) contains all the critical values other than \(\infty\), where \(W'\) is the connected component of \(f^{-1}(W)\) including \(W\). If \(f\) is not a polynomial, then \(f^{-1}(W)\) is not connected.

Claim. There exists a topological disc \(\tilde{W}\) with smooth boundary such that \(W \subset \tilde{W} \subset W'\) and such that \(f^{-1}(\tilde{W})\) is a topological disc.

**Proof of Claim.** Let \(P_0 \subset A\) be the set of critical values other than \(\infty\). Take distinct points \(a_p \in \partial W\) for \(p \in P_0\), and disjoint simple paths \(L_p \subset A \cup \partial W\) joining \(p \in P_0\) and \(a_p\). Then \(f^{-1}(\bigcup_{p \in P_0} L_p \cup \partial W)\) is a connected set including \(\partial W'\). Take a minimal subset \(P'_0 \subset P_0\) such that \(f^{-1}(\bigcup_{p \in P'_0} L_p \cup \partial W)\) is connected. Then \(f^{-1}(\bigcup_{p \in P'_0} L_p \cup W)\) is connected. We take a topological disc \(\tilde{W}\) with smooth boundary such that \(W' \supset \tilde{W} \supset \bigcup_{p \in P'_0} L_p \cup W\) and \(\tilde{W} \cap P_0 = P'_0\). This completes the proof of Claim.

From Lemma 9, we have a path \(f_t, 0 \leq t \leq 1\) between \(f_0 = f\) and a rational map \(f_1\) such that \(f_t(\infty) = \infty\), such that there is a continuous family of
topological discs between $U_0 = f^{-1}(W)$ and $U_1$ with $f_t(U_i) \subset U_i$, such that all of $f_t : \mathbb{C} - f_t^{-1}(U_t) \to \mathbb{C} - U_t$ are topologically conjugate to each other, and such that $U_1$ contains no critical point other than $\infty$. It is easy to see that $f_t \in S_d$ and $f_1$ is a polynomial. The proof is completed, since the shift locus of $d$-dimensional polynomial is connected ([7]).

Lemma 9. Let $f_0$ be a rational map with a (super)attracting fixed point $\infty$. Suppose that there exists a topological disc $U_0$ containing $\infty$ with smooth boundary $\partial U_0$ such that $f_0|_{\partial U_0} : U_0 \to f(U_0) \subset U_0$ is a proper map of degree $m \geq 2$ and $\partial U_0$ contains no critical point. Then there exists a continuous family $f_t, 0 \leq t \leq 1$ of rational maps with a (super)attracting fixed point $\infty$ and a continuous family of topological discs $U_t$ containing $\infty$ such that $f_t|_{U_t} : U_t \to f_t(U_t) \subset U_t$, such that all of $f_t : \mathbb{C} - f_t^{-1}(U_t) \to \mathbb{C} - U_t$ are topologically conjugate to each other, and such that $U_1$ contains no critical point of $f_1$ other than $\infty$.

Proof. Take a topological disc $V \subset f(U_0)$ with smooth boundary such that $A := f(U_0) - V$ is an annulus containing no critical value. Set $V' := U_0 \cap f^{-1}(V)$ and $A' := U_0 \cap f^{-1}(A)$. Then $f : A' \to A$ is a degree $m$ covering.

Let $\phi : V \to \mathbb{D}$ and $\phi' : V' \to \mathbb{D}$ be Riemann maps with $\phi(\infty) = \phi'(\infty) = 0$. Then $h = \phi \circ f \circ \phi'^{-1} : \mathbb{D} \to \mathbb{D}$ is a proper holomorphic map with $h(0) = 0$. Thus $h$ is a Blaschke product $e^{\theta} \prod_{k=1}^{m} \frac{z - a_k}{1 - t a_k z}$, where $\{a_k\} = \phi' \circ f^{-1}(\infty)$.

Set
\[ h_t(z) = e^{t \theta} \prod_{k=1}^{m} \frac{z - (1-t)a_k}{1 - (1-t)a_k z} \]
for $0 \leq t \leq 1$. Define a continuous family of smooth branched coverings $g_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfying $g_t(z) = \phi^{-1} \circ h_t \circ \phi'$ for $z \in V'$, $g_t(z) = f_0(z)$ for $z \in \mathbb{C} - U_0$, and that $g_t : A' \to A$ is a degree $m$ covering map.

Since for any point $z \in \hat{\mathbb{C}}$ its orbit passes through $A'$ at most once, by Shishikura principle we obtain the required $f_t$.

Proof of Theorem 6. Let $f_1, f_2$ be rational maps satisfying the assumption of the theorem. We construct a qc-conjugacy $h$ between $f_1$ and $f_2$ in the sequel. From this, we can construct a path of qc-equivalent rational maps which connects $f_1$ and $f_2$ (cf. Theorem 2.9 of [16]).

By Theorem 2, we can assume that for $i = 1, 2$, there is a topological disc $W_i$ containing $\infty$ with smooth boundary such that $A_i := W_i - W_i$ contains all the critical values, where $W_i'$ is the connected component of $f_i^{-1}(W_i)$ containing $\infty$.

To show the existence of $h$, it is sufficient to construct a diffeomorphism $m : A_1 \to A_2$ such that there exists a lift $\hat{m} : \hat{A}_1 \to \hat{A}_2$ with $f_2 \circ \hat{m} = m \circ f_1$ on $\hat{A}_1$, where each $A_i', i = 1, 2$ is the connected component of $f_i^{-1}(A_i)$ adjacent to $A_i$. Recall that $A_i'$ contains all the critical points. If we have such a diffeomorphism $m$, then we define a qc map $h$ as follows: First define $h = f_2^k \circ m \circ f_1^{-k}$ on $f_1^k(A_1), k = 1, 2, \ldots$. Then we have $h : \hat{W}_1 \to \hat{W}_2$. Secondly extend $h$ on $f^{-k}(\hat{W}_1), k = 1, 2, \ldots$ to satisfy $f_2 \circ h = h \circ f_1$. This extension is guaranteed by
the existence of \( \bar{m} \), and we have \( h \) on the Fatou set. Finally it is evident that \( h \) can be continuously extended to the Julia set by construction.

The proof is completed by Lemma 11. Note that the homeomorphism \( m \) in Lemma 11 can be modified to be a diffeomorphism.

**Lemma 10.** Let \( A \subset \mathbb{C} \) be a topological annulus with smooth boundary, \( A' \subset \mathbb{C} \) a 2d-connected domain with smooth boundary, and \( f : A' \to A \) a branched covering of degree \( d \) with \( 2d - 2 \) critical point \( p_1, p_2, \ldots, p_{2d-2} \in A' \) of degree two. We may consider the subscripts \( i = 1, 2, \ldots, 2d - 2 \) to be elements of \( \mathbb{Z}/(2d - 2) \). Suppose \( q_i = f(p_i), i = 1, 2, \ldots, 2d - 2 \) are distinct. Take \( x \in A', y = f(x) \in A \). Then renumbering \( p_i \)'s if necessary, there exist simple paths \( l_i \in A, i = 1, 2, \ldots, 2d - 2 \) satisfying:

1. \( l_i \) joins \( y \) and \( q_i \) \((i = 1, 2, \ldots, 2d - 2) \), and they are disjoint except at \( y \),
2. \( l_i, i = 1, 2, \ldots, 2d - 2 \) are arranged counter-clockwise around \( y \),
3. \( p_i \in l_i' \), where \( l_i' \) is the connected component of \( f^{-1}(l_i) \) containing \( x \), and
4. each of \( l_{2k-1}' \cup l_{2k}' \), \( k = 1, 2, \ldots, d \) is a closed curve.

**Proof of Lemma 10.** First we take a system of paths \( \mathcal{L} = \{ l_i : i = 1, 2, \ldots, 2d - 2 \} \) satisfying 1. Set \( L = \bigcup_{i=1}^{2d-2} l_i \). Modifying it step by step, we obtain required paths.

Observe that \( f \) is one-to-one on each boundary component of \( A' \). We can show that if \( L = \{ l_i \} \) satisfies 1 and 2, then \( f^{-1}(L) \) is connected and \( f^{-1}(A - L) \) has exactly \( d \) connected components. Indeed, if \( f \) is of degree \( q \) on some connected component \( D \subset f^{-1}(A - L) \) and \( p \) is the number of connected components of \( \partial D \cap f^{-1}(L) \), then the Euler characteristic fulfills \( 2 - p - 2q = \chi(D) = q\chi(A - L) = -q \), and so \( q = 1 \) and \( p = 1 \).

Moreover, observe that if there are distinct connected components \( D, D' \in f^{-1}(A - L) \) such that \( \overline{D} \cap \overline{D'} \) includes a curve, then \( \overline{D} \cap \overline{D'} \) contains a critical point.

In the following, we denote by \( D, D' \) connected components of \( f^{-1}(A - L) \).

Step 1. Let \( \mathcal{L} = \{ l_i \} \) be a system of paths satisfying 1. We modify \( \mathcal{L} \) to satisfy 1 and 3. Suppose that there exist a \( D \) with \( x \in \partial D \) and a critical point \( p_i \in \partial D \) with \( p_i \not\in l_i' \). Then we take a simple path \( l_i' \subset \overline{D} \) between \( x \) and \( p_i \), and replace \( l_i \) with \( f(l_i') \). By repeating this procedure, we obtain \( \mathcal{L} \) such that \( p_i \in l_i' \) for any \( p_i \in D \) with \( x, p_i \in \partial D \). This means that for any \( D \) with \( x \in \partial D \) and for any \( i \), either \( p_i \in \partial D \cap f^{-1}(l_i) = l_i' \) or \( p_i \not\in \partial D \cap f^{-1}(l_i) \). Therefore any \( D \) satisfies \( x \in \partial D \). Indeed, if \( x \in \partial D \) and \( \partial D \cap \partial D' \neq \emptyset \), then there exists \( p_i \) such that \( p_i \in l_i' \subset \partial D \cap \partial D' \), and so \( x \in \partial D' \). Hence we have \( \mathcal{L} \) with 1 and 3.

Step 2. By renumbering, we have \( \mathcal{L} \) satisfying 1, 2, and 3. We modify \( \mathcal{L} \) to satisfy 1, 2, 3, and 4. We can show that for each \( i \), there exists \( j \neq i \) such that \( l_i' \cup l_j' \) is a closed curve. Indeed, it is sufficient to show that for each \( x' \in f^{-1}(y) - \{ x \} \), there exist two integers \( i, j \in \{ 1, 2, \ldots, 2d - 2 \} \) such that \( l_i' \) and \( l_j' \) joins \( x \) and \( x' \). Since \( f^{-1}(L) \) is connected, every \( x' \in f^{-1}(y) - \{ x \} \) has at least one \( i \) such that \( l_i' \) joins \( x \) and \( x' \). If \( l_i' \) is the only one which joins \( x \) and \( x' \),

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then there exists a $D$ such that a neighborhood of $p_i$ is included in the interior of $\overline{D}$, which contradicts the injectivity of $f : D \rightarrow A - L$.

Let $i, j$ be such that $i - j \neq 0, \pm 1$ and such that $l'_i \cup l'_j$ is a closed curve. Then there exists a $D$ such that $l'_i, l'_j \subset \partial D$. We can take a simple path $\tilde{l}_j$ in $D$ joining $x$ and $p_j$ such that $f(\tilde{l}_j)$ is either between $l_i$ and $l_{i+1}$ or between $l_{i-1}$ and $l_i$ in the cyclic order around $y$. We replace $l_j$ with $f(\tilde{l}_j)$. Then the new $l'_i \cup l'_j$ is also a closed curve by the fact proved in the previous paragraph. Renumber $L$. By repeating this procedure, we obtain a required $L$.

This completes the proof of Lemma 10. \hfill \Box

Lemma 11. Let $f_1 : A'_1 \rightarrow A_1$ and $f_2 : A'_2 \rightarrow A_2$ be branched coverings satisfying the assumption of Lemma 10. Then they are isomorphic to each other, that is, there exist homeomorphisms $m : A_1 \rightarrow A_2$ and $\tilde{m} : A'_1 \rightarrow A'_2$ such that $f_2 \circ \tilde{m} = m \circ f_1$.

Proof of Lemma 11. By Lemma 10, we have a system of paths $L_1, L_2$ for $f_1, f_2$ respectively. Take a homeomorphism $m : A_1 \rightarrow A_2$ which carries $L_1$ to $L_2$. Then the image of $(m \circ f_1)_* : \pi_1(A'_1 - f_1^{-1}(Q_1), x_1) \rightarrow \pi_1(A - Q_2, y_2)$ coincides with the image of $(f_2)_* : \pi_1(A'_2 - f_2^{-1}(Q_2), x_2) \rightarrow \pi_1(A - Q_2, y_2)$, where $Q_i, i = 1, 2$ are the sets of critical values for $f_i$, and $x_i, y_i, i = 1, 2$ are the basepoints in $A'_i$ and $A_i$. Indeed, the images are determined only the graph structures of $(m \circ f_1)^{-1}(L_2) = f_1^{-1}(L_1)$ and $f_2^{-1}(L_2)$.

Now the existence of $\tilde{m}$ is assured by the theory of covering spaces. \hfill \Box

Remark 4.

1. The limitation “renumbering $p_i$’s” in Lemma 10 can be deleted. In fact, any permutation $\tau : \{q_i\} \rightarrow \{q_i\}$ can be extended to a homeomorphism $m : A \rightarrow A$ with a homeomorphism $\tilde{m} : A' \rightarrow A'$ such that $m \circ f = f \circ \tilde{m}$. The isotopy class of $m$ relative to $\{q_i\}$ is not unique.

2. Take a generating set $\{q_i : i = 1, 2, \ldots, 2d - 2\}$ of $\pi_1(A - \{q_i\}, y)$ in Lemma 10 as follows: $q_i = [\gamma_i], \gamma_i$ is a loop with basepoint $y$ which is the boundary of a topological disc $U_i \subset A$ such that $\overline{U_i} \cap L = \emptyset$. Then the image of $f_* : \pi_1(A' - f^{-1}(\{q_i\}), x) \rightarrow \pi_1(A - \{q_i\}, y)$ is generated by $\{\gamma_i^2, 2q_{2k-1}g_{2k}, 2q_{2k}-1g_{2k}\}$, where $i, k, l$ run over integers such that $1 \leq i \leq 2d - 2, 1 \leq k \leq d + 1$, and $l \neq 2k - 1, 2k$.

6 An example of a rational map which is Cantor but not strongly Cantor

In this section, we complete the proof of Theorem 4.
6.1 Basic facts

Definition 15. Let \( G, T \) be groups with a homomorphism \( \varphi : T \to \text{Aut}G \). Then we have the semidirect product \( G \rtimes_\varphi T \), namely, \( G \rtimes_\varphi T = \{(g, \tau) \mid g \in G, \tau \in T\} \), and for \((g, \tau), (g', \tau') \in G \rtimes_\varphi T \), the product \((g'', \tau'') = (g, \tau)(g', \tau')\) is defined by \( g'' = g\varphi(\tau)(g') \), \( \tau'' = \tau \tau' \). We have an inclusion \( i_1 : G \to G \rtimes_\varphi T, i_1(g) = (g, 1) \) and a projection \( p_2 : G \rtimes_\varphi T \to T, p_2(g, \tau) = \tau \), and an exact sequence \( 1 \to G \to G \rtimes_\varphi T \to T \to 1 \).

For a set \( X \), let \( \mathfrak{S}(X)^{\text{op}} \) be the opposite group of the symmetric group \( \mathfrak{S}(X) \). Namely, \( \mathfrak{S}(X)^{\text{op}} \) consists of all bijective self-maps on \( X \), and for \( \tau, \tau' \in \mathfrak{S}(X)^{\text{op}} \), we set \( \tau \tau' = \tau' \circ \tau \). For a group \( G \) and a set \( X \), we have a homomorphism \( \varphi = \varphi_G, X : \mathfrak{S}(X)^{\text{op}} \to \text{Aut}(G^X) \) defined by \( \varphi(\tau)(a) = a \circ \tau \). We write \( M(G, X) = G^X \rtimes_\varphi \mathfrak{S}(X)^{\text{op}} \).

For convenience of calculation, an element \((a, \tau) \in M(G, X)\) is expressed in the form of formal series
\[
\sum_{x \in X} a(x) \cdot (x, \tau(x)).
\]

Define the product of two terms of such a series by
\[
[g \cdot (x, y)][g' \cdot (x', y')] = \begin{cases} gg' \cdot (x, y') & \text{if } y = x', \\ 0 & \text{if } y \neq x', \end{cases}
\]

where \( g, g' \in G, x, y, x', y' \in X \). Then we recover the operation of \( M(G, X) \) using distributive law.

For subgroups \( Q \subset G^X \) and \( T \subset \mathfrak{S}(X) \) such that \( Q \) is \( T \)-invariant, the subgroup \( G \rtimes T = (i_1(Q), p_2^{-1}(T)) \subset M(G, X) \) is well-defined.

Definition 16. Let \( G_0 = \pi_1(\mathcal{C} - P, \bar{x}) \) be the fundamental group, and \( \hat{G}_0 = \pi_1(\tilde{\mathcal{C}} - \mathcal{P}, \bar{x}) = \pi_1(\tilde{\mathcal{C}} - \mathcal{P}, \bar{x})/\ker \alpha_f \) the iterated monodromy group (see Definition 14). The monodromy \( \alpha = \alpha_f \) descend to the right action of \( \hat{G}_0 \) on \( f^{-1}(\bar{x}) \), which is denoted by the same notation \( \alpha \). Therefore \( \alpha : \hat{G}_0 \to \mathfrak{S}(f^{-1}(\bar{x}))^{\text{op}} \) is a homomorphism. Let \( G_k = M(G_0, f^{-k}(\bar{x})) \) and \( \hat{G}_k = M(\hat{G}_0, f^{-k}(\bar{x})) \), \( k = 1, 2, \ldots \). For \((g, \tau) \in G_k \), we say \( g \in f^{-k}(\bar{x}) \) is the loop part of \((g, \tau)\). This group and the homomorphism \( f^*_\gamma \) below are used by the author in [12],[11].

Let \( r = (l_i) \) be a radial, and \( \chi_r : \text{Word}(k) \to f^{-k}(\bar{x}) \) the bijection determined by \( w \mapsto l_w(1) \) using the notation in Definition 6. We identify \( f^{-k}(\bar{x}) \) with \( \text{Word}(k) \) by \( \chi_r \), and also \( G_k \) with \( M(G_0, \text{Word}(k)) \). We consider \( \alpha \) to be a right action of \( G_0 \) on \( \text{Word}(1) = \{1, 2, \ldots, d\} \). This action depends on \( r \).

For a loop \( \gamma : ([0,1], \{0,1\}) \to (\mathcal{C} - \mathcal{P}, \bar{x}) \), let denote by \([\gamma] \in G_0\) the equivalence class including \( \gamma \). We obtain (non-homomorphic) mappings \( \beta_{r,i} : G_0 \to G_0, i = 1, 2, \ldots, d \) defined by \( \beta_{r,i}([\gamma]) = [l_i L_i(\gamma) l_{\alpha(i)}^{-1}] \), where \( L_i(\gamma) \) is the lift of \( \gamma \) defined in Definition 6. Note that
\[
\beta_{r,i}([\gamma']) = [l_i L_i(\gamma') l_j^{-1}] = [l_i L_i(\gamma) l_j^{-1} l_j L_j(\gamma') l_j^{-1}] = \beta_{r,j}([\gamma]) \beta_{r,j}([\gamma']),
\]
where \( j = \alpha(i) \) and \( j' = \alpha([\gamma']) \). From this, we have the
homomorphism $f^*_r : G_k \to G_{k+1}$ determined by
\[
f^*_r : \sum_{w \in \text{Word}(k)} a(w) \cdot (w, \tau(w)) \mapsto \sum_{i=1}^d \sum_{w \in \text{Word}(k)} \beta_{r,i}(a(w)) \cdot (iw, \alpha(a(w))(i)\tau(w)).
\]
Particularly, for $k = 0$,
\[
f^*_r : g \mapsto \sum_{i=1}^d \beta_{r,i}(g) \cdot (i, \alpha(g)(i)).
\]
We write $\beta_{r,w} = \beta_{r,i_1} \circ \beta_{r,i_2} \circ \cdots \circ \beta_{r,i_k}$ for $w = i_1i_2\cdots i_k \in \text{Word}(k)$. It is easy to see that $\beta_{r,w}([\gamma]) = [l_w L_w(\gamma)] l_{1}^{\gamma^{-1}}([\gamma])(w)$. The action $\alpha$ on $\text{Word}(k)$ is inductively expressed as
\[
\alpha(g)(i_1i_2\cdots i_k) = \alpha(\beta_{r,i_2\cdots i_k}(g))(i_1)\alpha(g)(i_2\cdots i_k).
\]
It is easily seen that $\beta_{r,i} : \hat{G}_0 \to \hat{G}_0$ is well-defined and the homomorphism $f^*_r : G_k \to G_{k+1}$ descends to $f^*_r : \hat{G}_k \to \hat{G}_{k+1}$ (we use the same symbol) as the diagram
\[
\begin{array}{ccc}
\hat{G}_k & \xrightarrow{f^*_r} & \hat{G}_{k+1} \\
\end{array}
\]
commutes, where $\pi$ is the projection.

Let $r' = (l'_1)$ be a radial other than $r$ with the same numbering of $f^{-1}(x)$, i.e. $l_i(1) = l'_i(1)$. Let $h_i = [l'_i l_i^{-1}] \in G_0$. Then $\beta_{r',i}([\gamma]) = h_i \beta_{r,i}([\gamma]) h_i^{-1}([\gamma])(i)$. Setting $h = ((h_1, h_2, \ldots, h_d), \text{id}) = \sum_{i=1}^d h_i \cdot (i, i) \in G_1$, we can describe $f^*_{r'} : G_0 \to G_1$ as
\[
f^*_{r'} : g \mapsto h f^*_r(g) h^{-1}.
\]
However, since $\chi_r \neq \chi_{r'}$ on $\text{Word}(k), k \geq 2$, the relation between $f^*_r$ and $f^*_{r'}$ on $G_k, k \geq 1$ is complicated.

**Remark 5.** Let $\gamma : ([0, 1], \{0, 1\}) \to (\hat{C} - P, \bar{x})$ be a loop. There is a one-to-one correspondence between the lifted loops of $\gamma$ and the periodic cycles of $\alpha([\gamma])$. Namely, $(i_1, i_2, \ldots, i_m)$ is a periodic cycle of $\alpha([\gamma])$, if and only if $\bar{\gamma} = L_{i_1}(\gamma)L_{i_2}(\gamma)\cdots L_{i_m}(\gamma)$ is a loop and $f \circ \bar{\gamma} = \gamma^m$.

**Proposition 4.** Let $r$ be a radial. Then for a loop $\gamma : ([0, 1], \{0, 1\}) \to (\hat{C} - P, \bar{x})$, we have
\[
f^*_r \circ f^*_r \circ \cdots \circ f^*_r([\gamma]) = \sum_{w \in \text{Word}(k)} \beta_{r,w}([\gamma]) \cdot (w, \alpha f_k([\gamma])(w)).
\]
Proof. By the definition of $f^*_r$, it is sufficient to show
\[
\alpha_{f^{k+1}}(\omega)(i)w) = \alpha(\beta_r(\omega)(i)\alpha_{f^k}(\omega)(w)
\]
for $i \in \text{Word}(1)$ and $w \in \text{Word}(k)$. Suppose $\alpha_{f^{k+1}}(\omega)(i)w) = ju$, $j \in \text{Word}(1)$, $u \in \text{Word}(k)$. Then $L_{iu}(\gamma)(0) = l_{iu}(1)$ and $L_{iu}(\gamma)(1) = l_{iu}(1)$. Composing $f$, we have $L_{iu}(\gamma)(0) = l_{iu}(1)$ and $L_{iu}(\gamma)(1) = l_{iu}(1)$. Thus $\alpha_{f^k}(\omega)(w) = u$. On the other hand, $L_{iu}(\gamma)(1) = l_{iu}(1)$ implies that the endpoint of $L_{iu}(\gamma(l_{iu}(\gamma)l_{iu}^{-1})$ is $l_{iu}(1)$. Thus $\alpha(\beta_r(\omega)(\gamma)(i)) = j$.

Let us denote by $e$ the identity element of $\hat{G}_n$.

Proposition 5. The homomorphism $f^*_r : \hat{G}_k \to \hat{G}_{k+1}$ is injective.

Proof. Suppose $\sum_{i=1}^d \sum_{w \in \text{Word}(k)} \beta_{r,i}(a(w)) : (iw, a(w)) = e$. Then $\tau = \text{id}$, and $\beta_{r,i}(a(w)) = e, a(w)) = \text{id}$ for any $w \in \text{Word}(k)$ and any $i \in \text{Word}(1)$. Thus we have $\beta_{\tau,w}(a(w)) = e$ for any $u \in \text{Word}$ and $w \in \text{Word}(k)$, and so $\alpha_{\tau}(a(w)) = 1$.

Lemma 12. Let $f$ be a Cantor hyperbolic rational map of degree $d$, and $r = (l_i)$ a radial for $f$. Then the coding map $\phi_r : \Sigma_d \to J$ is one-to-one if and only if for any $g \in G_0$, there exists $k > 0$ such that $(f^*_r)^k(g) \in \{e\} \times G(\text{Word}(k))$.

Proof. Let $\Omega$ be an expanding domain including $r$, and set $\hat{G}_\Omega = \pi_1(\Omega - P, \bar{x}) \times \ker \alpha_\infty \subset \hat{G}_0$. Recall that $\hat{G}_\Omega$ is a finite group (Theorem 1). We have $\beta_{r,w}(\hat{G}_\Omega) \subset \hat{G}_\Omega$ for any $w \in \text{Word}$; for any $g \in G_0$, there exists $w \in \text{Word}$ such that $\beta_{r,w}(g) \in \hat{G}_\Omega$.

Let $Q = \{(\omega, \omega') \in \Sigma_d \times \Sigma_d : \phi_r(\omega) = \phi_r(\omega')\}$. For $q = (\omega, \omega') \in Q$, let us denote $\sigma(q) = (\sigma(\omega), \sigma(\omega'))$, $\phi(q) = \phi_r(\omega) = \phi_r(\omega')$, and $[\gamma_q] = [l_w l_{\omega}^{-1}] \in \hat{G}_\Omega$, where $\sigma : \Sigma_d \to \Sigma_d$ is the shift map. Note that $l_w \subset \Omega$ for any $w \in \text{Word}$, and $\{l_w \in \hat{G}_\Omega$ for any $q \in Q$.

We show $(\omega, j\omega) \notin Q$ if $j \neq 1$. Indeed, since the Julia set $J$ contains no critical point, $f^{-1}(z)$ consists of exactly $d$ points for $z \in J$. Hence for $z = \phi_r(\omega)$, the two sets of lifts of $l_w$ coincide:

$$\{L_i(l_\omega) : i \in \text{Word}(1)\} = \{L'_y(l_\omega) : y \in f^{-1}(z)\},$$

where by $L'_y(l)$ we denotes the lift of $l$ by $f$ such that $y$ is one of the endpoints. This means that $\phi_r(i\omega), i \in \text{Word}(1)$ are all distinct.

For $q = (\omega, \omega') \in Q$, we have $\sigma(q) \in Q$ and $(i\omega, j\omega') \in Q$ for every $i \in \text{Word}(1)$ and $j = \alpha([\gamma_q])$. Clearly, $\beta_{r,i}([\gamma_q]) = [\gamma_{(i\omega, j\omega')}]$. From this fact, if $q = (\omega, \omega') \in Q$ and $[\gamma_q] = e$, then $wq = (w\omega, w\omega') \in Q$ and $[\omega] = e$ for any $w \in \text{Word}$, and $\gamma_{(\omega, \omega')} = e$ for any $k > 0$.

We show that the following are equivalent:

1. $\phi_r : \Sigma_d \to J$ is one-to-one.
2. $\{[\gamma_q] \in \hat{G}_\Omega : q \in Q\} = \{e\}$. 

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3. \( \bigcap_{k>0} \bigcup_{w \in \text{Word}(k)} \beta_{r,w}(\hat{G}_\Omega) = \{e\} \).

4. For any \( g \in \hat{G}_0 \), there exists \( k > 0 \) such that \( (f^k)^*(g) \in \{e\} \times \mathcal{S}(\text{Word}(k)) \).

5. There does not exist a non-trivial \( g \in \hat{G}_0 \) such that \( \beta_{r,w}(g) = g \) for some \( w \in \text{Word} \).

4 \( \iff \) 3 \( \implies \) 5 is easy.

5 \( \implies \) 3 is obtained by the finiteness of \( G_\Omega \).

2 \( \implies \) 1. Suppose \( \{ [\gamma_q] \in G_\Omega : q \in Q \} = \{e\} \). First we show that if \( q = (i\omega, j\omega') \in Q \), then \( i = j \). Indeed, by the fact mentioned above, \( \sigma(q) = (\omega, \omega') \in Q \) and \( (i\omega, i\omega') \in Q \) since \( i = \alpha([\gamma_q])(i) \). Hence \( (j\omega', i\omega') \in Q \), and so \( i = j \).

Applying this observation to \( \sigma^k(q), k = 1, 2, \ldots \), we have that \( Q \) is the diagonal set, and \( \phi_r \) is one-to-one.

5 \( \implies \) 2. Suppose \( [\gamma_q] \neq e \) for some \( q \in Q \). Then \( [\gamma_{\sigma^k(q)}] \neq e \) for \( k \geq 0 \). There exist \( k, k' \in \mathbb{N} \) such that \( k > k' \) and \( [\gamma_{\sigma^k(q)}] = [\gamma_{\sigma^{k'}(q)}] \), since \( \hat{G}_\Omega \) is a finite group. Hence \( g = [\gamma_{\sigma^k(q)}] \neq e \) satisfies \( \beta_{r,w}(g) = g \) for some \( w \in \text{Word}(k-k') \).

1 \( \implies \) 5. Suppose that there exists a non-trivial \( g \in \hat{G}_0 \) such that \( \beta_{r,w}(g) = g \) for some \( w \in \text{Word}(k) \). Set \( w' := \alpha_{f^k}(g)(w) \). Take a loop \( \gamma \) such that \( g = [\gamma] \). We know that the length of \( L_{w^n}(\gamma) \) tends to zero as \( n \to \infty \), and that \( [l_{w^n}L_{w^n}(\gamma)l_{w^{-n}}^\ast] = g \) is a non-trivial loop. Hence \( w' \neq w \), and \( (ww\cdots,w'w'\cdots) \in Q \). \( \square \)

**Remark 6.** In Lemma 12, the assumption that \( f \) is Cantor is not necessary. In fact, we have used only the finiteness of \( G_\Omega \). Instead, we may use the finiteness of \( \hat{G}_{0,L} = \{ [\gamma] \in \hat{G}_0 : |\gamma| < L \} \) for \( L > 0 \), where \( |\gamma| \) is the length of a path. Moreover, for a general hyperbolic rational map \( f \), we can show \( \{ [\gamma_q] : q \in Q \} = \bigcap_{k>0} \bigcup_{w \in \text{Word}(k)} \beta_{r,w}(\hat{G}_{0,L}) \) for some \( L > 0 \).

For another version of this lemma, see Section 7 of [13].

### 6.2 Proof

Let

\[
 f(z) = \frac{az^4 - 2az^2 + a + \frac{1}{4a}}{z^2 - 1} = a(z^2 - 1) + \frac{1}{4a(z^2 - 1)}.
\]

Then \( \mathcal{C}_r = \{0, \infty, \pm \sqrt{1+1/2a}, \pm \sqrt{1-1/2a}\} \). \( \pm \sqrt{1+1/2a} \to 1 \to \infty \to \infty \), \( \pm \sqrt{1-1/2a} \to -1 \to \infty \).

**Lemma 13.** If \( |a| \) is large enough, then \( f \) is \( d \)-Cantor and \( f^2 \) is \( s \)-Cantor.

**Proof.** Suppose \( |a| > 2 \). Then it is easily seen that \( |f(z)| > |a||z| \) provided \( |z| > 5/3 \). Hence \( \{ |z| > 5/3 \} \) is included in \( A_0(\infty) \), the immediate basin of attraction of \( \infty \). Thus \( f(0) = -a - \frac{1}{4a} \in A_0(\infty) \).

Note that \( f = g \circ h \) with \( g(z) = \frac{z+1}{z}, h(z) = 2a(z^2 - 1) \). It is easy to see that \( g^{-1}([-1,1]) = \{|z| = 1\} \). Hence \( f^{-1}([-1,1]) \) is a union of two
encircle two critical values \( \pm x \) and \( \pm x' \). There exists a topological disc \( D \). Note that the parameter is not important. We fix the parameter to avoid complicated notation.

Suppose that there are \( \Gamma \) simple closed curves, say \( \Gamma_1, \Gamma_2 \), which encircle \(-1\) and \(1\) separately. Observe that \(-\sqrt{1+1/2a} \in \Gamma_1, \sqrt{1+1/2a} \in \Gamma_2.\)

Set \( D = \{ |z| < 5/3 \} \). Then \( f^{-1}(D) \subset D \) is the union of two annuli, which encircle two critical values \( \pm 1 \) separately; \( f^{-2}(D) \subset f^{-1}(D) \) is the union of four annuli, which encircle four critical points \( \pm \sqrt{1+1/2a} \) separately. Two critical values \( \pm 1 \) are included in the unbounded component of \( \mathbb{C} - f^{-2}(D) \). Thus there exists a topological disc \( D' \subset D \) such that \( \pm 1 \notin \overline{D} \) and \( f^{-2}(D) \subset D' \), which satisfies the property of being \( s \)-Cantor. This completes the proof. \( \blacksquare \)

**Theorem 7.** Suppose that \( |a| \) is large enough for \( f^2 \) to be \( s \)-Cantor. Then \( f \) is not \( t \)-Cantor, namely, for any radial \( r \) for \( f \), \( \phi_r: \Sigma \to J \) is not one-to-one.

**Proof.** First we show that for some radial \( r \) and a generating set \( \{ A, B, C_k \ (k = 0, 1, 2, \ldots) \} \subset G_0 \), the homomorphism \( f_r: G_0 \to G_1 \) has the following description:

\[
A \mapsto A \cdot (1, 2) + A^{-1} \cdot (2, 1) + (3, 4) + (4, 3) \quad (9)
\]

\[
B \mapsto (1, 2) + (2, 1) + B \cdot (3, 4) + B^{-1} \cdot (4, 3) \quad (10)
\]

\[
C_0 \mapsto B \cdot (1, 4) + (2, 2) + (3, 3) + B^{-1} \cdot (4, 1) \quad (11)
\]

\[
C_k \mapsto (1, 1) + (2, 2) + (3, 3) + C_{k-1} \cdot (4, 4) , \ (k \geq 1), \quad (12)
\]

where we simply write \( (i, j) \) instead of \( e \cdot (i, j) \).

To this end, let \( a = \sqrt{-1}c \) with \( c > 0 \) large enough. The choice of the parameter is not important. We fix the parameter to avoid complicated notation. Note that \( f^k(0) \in \sqrt{-\mathbb{R}_{\leq 0}}, k = 1, 2, \ldots \) with \( 0 > \text{Im} f(0) > \text{Im} f^2(0) > \cdots \) and

\[
f^{-1}(\sqrt{-\mathbb{R}}) = \mathbb{R} \cup \sqrt{-\mathbb{R}}, \quad f^{-1}(\sqrt{-\mathbb{R}_{\geq 0}}) = J_1 \cup J_2 \cup J_3 \cup J_4 \subset \mathbb{R},
\]

where

\[
J_1 = (-\infty, -\sqrt{1+1/2c}), \quad J_2 = (-1, -\sqrt{1+1/2c}],
\]

\[
J_3 = (\sqrt{1+1/2c}, 1), \quad J_4 = [\sqrt{1+1/2c}, \infty).
\]

We also set \( \bar{x} = \sqrt{-1} \). Then we have \( f^{-1}(\bar{x}) = \{ x_1, x_2, x_3, x_4 \} \subset \mathbb{R} \) with \( x_i \in J_i, i = 1, 2, 3, 4 \). Recall that we have two loops \( \Gamma_1, \Gamma_2 \), which are the inverse image of \([-1, 1]\) encircling \(-1\) and \(1\) respectively. Observe that the graph \( L := \bigcup_{i=1}^{4} J_i \cup \bigcup_{i=1}^{2} \Gamma_i = f^{-1}(\sqrt{-\mathbb{R}_{\geq 0}} \cup [-1, 1]) \) divides \( \mathbb{C} \) into three connected components; one is unbounded and the other two are bounded.

Take a radial \( r = (l_i)_{i=1}^{2} \) such that \( l_i \)'s are arcs joining \( \bar{x} \) and \( x_i \) in the upper half-plane. The group \( G_0 \) is generated by \( A = [\gamma_A], B = [\gamma_B], C_k = [\gamma_{C_k}], k = 0, 1, \ldots \) where \( \gamma_A, \gamma_B, \gamma_{C_k} \) are loops in \( \mathbb{C} - P \) counterclockwise encircling \(-1, 1, f^{k+1}(0)\) respectively. Specifically, we define \( \gamma_A \) resp. \( \gamma_B \) as a loop obtained by going around the segment \( I_1 \) (resp. \( I_2 \)) joining \( \bar{x} \) and \(-1\) (resp. \( 1 \)), and \( \gamma_{C_k} \) as a loop obtained by going around a simple arc \( I_3 \) joining \( \bar{x} \) and \( f(0) \) in \( \mathbb{C} - \{(x \leq 1) \cup \{\sqrt{-1}y : y \leq \text{Im} f^2(0)\} \). Finally we set \( \gamma_{C_{k+1}} = f(l^{-1}_{k+1} \gamma_{C_k} l_k) \). Observe that \( f^{-1}(I_1) \) has two connected components \( I_{11} \subset \{ \text{Im} z \leq 0 \} \) joining \( x_1 \) and \( x_2 \), and \( I_{12} \subset \{ \text{Im} z \geq 0 \} \) joining \( x_3 \) and \( x_4 \), and that \( I_1 \cup I_{11} \cup I_{12} \).
is homotopic to $\gamma_A$ in $\hat{C} - P$, and $I_3 \cup I_{12} \cup I_4$ is a trivial loop in $\hat{C} - P$. A similar argument is true for $I_2$. Now we obtain (9) and (10). The inverse image $f^{-1}(I_3)$ has three connected components, one of which, say $I_{31}$, joins $x_1$ and $x_4$ included in the unbounded connected component of $C - L$. It is easily seen that $I_1 \cup I_{31} \cup I_4$ is homotopic to $\gamma_B$. Thus we have (11).

Let $T$ be the image of $\alpha : \pi_1(\hat{C} - P, \bar{x}) \to \mathbb{S}(\text{Word}(1))$ defined in Definition 16. Recall $\text{Word}(1) = \{1, 2, 3, 4\}$. Then

$$T = ((1, 4), (1, 2)(3, 4)) = \{\text{id}, (1, 4), (2, 3), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 2, 3, 4), (1, 3, 4, 2)\}$$

(by write a permutation in the form of a product of cycles). We have the left action of $T$ on $\mathbb{Z}_2^4 = \mathbb{Z}_2^4 \text{Word}(1)$ defined by $(\tau \cdot q)(i) = q(\tau(i))$, $q \in \mathbb{Z}_2^4$, $\tau \in T$, $i \in \text{Word}(1)$. The action keeps the following two subgroups unchanged:

$$Q := \{q \in \mathbb{Z}_2^4 \mid \sum_i q_i = 0\},$$

$$S := \ker((q_1, q_2, q_3, q_4) \mapsto (q_1 + q_2, q_3 + q_4)).$$

The group $T$ acts on the quotient groups $\mathbb{Z}_2^4 / Q$ and $Q / S$ trivially (not on $\mathbb{Z}_2^4 / S$), and the group $M(\mathbb{Z}_2^4, \text{Word}(1))$ has subgroups $\mathbb{Z}_2^4 \times T, Q \times T, S \times T$ with $\mathbb{Z}_2^4 \times T \triangleright Q \times T$ and $Q \times T \triangleright S \times T$. Notice the index $|\mathbb{Z}_2^4 \times T : S \times T| = |Q \times T : S \times T| = 2$.

The permutation group $T$ is characterized as the subgroup of $\mathbb{S}(\text{Word}(1))$ which preserves the partition $\mathcal{P} = \{\{1, 4\}, \{2, 3\}\}$, namely,

$$T = \{\tau \in \mathbb{S}(\text{Word}(1)) : \tau(P) \in \mathcal{P} \text{ for } P \in \mathcal{P}\}.$$

Let $L$ be the subgroup fixing each block of $\mathcal{P}$, namely,

$$L = \{\tau \in \mathbb{S}(\text{Word}(1)) : \tau(P) = P \text{ for } P \in \mathcal{P}\} = \{\text{id}, (1, 4), (2, 3), (1, 4)(2, 3)\}.$$

If $(q, \tau) \in \mathbb{Z}_2^4 \times T, (s, \sigma) \in S \times L$, then we have $(q, \tau)(s, \sigma)(q \circ \tau^{-1}, \tau^{-1}) = (q + s \circ \tau + q \circ \tau^{-1} \circ \sigma, \tau^{-1} \circ \sigma \circ \tau) \in S \times L$ from $s \circ \tau \in S$, $\tau^{-1} \circ \sigma \circ \tau \in L$, and the fact $q + q \circ \sigma \in S$ for any $\sigma \in L$. Thus $\mathbb{Z}_2^4 \times T \triangleright S \times L$. A similar argument implies $\mathbb{Z}_2^4 \times T \triangleright Q \times L$. Remark that $\mathbb{Z}_2^4 \times T / S \times L \cong D_4$ is the dihedral group of order 8, and $Q \times T / S \times L \cong D_2$ is the Klein four-group.

The normalizer of $S \times T$ is $Q \times T$ in $\mathbb{Z}_2^4 \times T$, namely, the normalizer of $S \times T / S \times L$ is $Q \times T / S \times L$ in $\mathbb{Z}_2^4 \times T / S \times L$. Indeed, since $\mathbb{Z}_2^4 \times T / S \times L$ itself is the only subgroup of $\mathbb{Z}_2^4 \times T / S \times L$ greater than $Q \times T / S \times L$, it is sufficient to see that there exist $\zeta \in S \times T$ and $h \in \mathbb{Z}_2^4 \times T - Q \times T$ such that $h\zeta h^{-1} \not\in S \times T$. Take $h = (q, \text{id}) \in \mathbb{Z}_2^4 \times T - Q \times T$ and $\zeta = (0, (12)(34)) \in S \times T - S \times L$, where $q = (q_1, q_2, q_3, q_1 + q_2 + q_3 + 1)$. Then $q + q \circ (12)(34) = (q_1 + q_2, q_1 + q_2, q_1 + q_2 + 1, q_1 + q_2 + 1) \not\in S$. Thus $h\zeta h^{-1} = (q + q \circ (12)(34), (12)(34)) \not\in S \times T$. 

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Moreover, we have proved the uniqueness of a conjugate subgroup of $S \times T/S \times L$ other than itself. We define $e, \zeta_1, \zeta_2, \zeta_3 = \zeta_1 \zeta_2$ the elements of $Q \times T/S \times L \cong D_2$ by

$$e = S \times L, \zeta_1 = S \times T - S \times L, \zeta_2 = Q \times L - S \times L, \zeta_3 = Q \times T - Q \times L - S \times T.$$ 

Then $h \zeta_1 h^{-1} = \zeta_1$ for any $h \in Q \times T/S \times L$ and $h \zeta_2 h^{-1} = \zeta_3$ for any $h \in \mathbb{Z}_2^2 \times T/S \times L - Q \times T/S \times L$.

An element $g \in G_0$ is expressed as

$$g = t_1^{n_1} t_2^{n_2} \cdots t_m^{n_m}, t_i \in \{A, B, C_k (k = 0, 1, 2 \ldots)\}, n_i \in \mathbb{Z}.$$ 

We write $|g|_A = \sum_{t_i = A} n_i$ and $|g|_B = \sum_{t_i = B} n_i$. Consider homomorphisms $\mu_A : G_0 \rightarrow \mathbb{Z}_2$ and $\mu_B : G_0 \rightarrow \mathbb{Z}_2$ by $\mu_A(g) = |g|_A \mod 2$ and $\mu_B(g) = |g|_B \mod 2$.

The homomorphisms $\mu_A$ and $\mu_B$ are naturally extended to $\mu_A, \mu_B : G_k \rightarrow M(\mathbb{Z}_2, \text{Word}(k))$. It is easily seen that the image of $f^*_r : G_0 \rightarrow G_1$ is included in $G_0^4 \times T$, and that the images of $\mu_A \circ f^*_r, \mu_B \circ f^*_r : G_0 \rightarrow \mathbb{Z}_2^4 \times T$ are included in $Q \times T$. Consider $\mu_A \times \mu_B : f^*_r(G_0) \rightarrow (Q \times T) \times (Q \times T)$, and let $\pi : (Q \times T) \times (Q \times T) \rightarrow D_2 \times D_2$ be the projection. Then $\pi \circ (\mu_A \times \mu_B) \circ f^*_r$ sends

- $A \mapsto ((1, 1, 0, 0), (12)(34)) \times ((0, 0, 0, 0), (12)(34)) \mapsto (\zeta_3, \zeta_1)$
- $B \mapsto ((0, 0, 0, 0), (12)(34)) \times ((0, 0, 1, 1), (12)(34)) \mapsto (\zeta_1, \zeta_3)$
- $C_0 \mapsto ((0, 0, 0, 0), (14)) \times ((1, 0, 0, 1), (14)) \mapsto (e, e)$
- $C_k \mapsto ((0, 0, 0, 0), id) \times ((0, 0, 0, 0), id) \mapsto (e, e), k = 1, 2, \ldots$

Since the kernel of $\mu_A \oplus \mu_B : G_0 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the normal subgroup of $G_0$ generated by

$$\{A^2, B^2, (AB)^2, C_k, |k = 0, 1, \ldots\},$$

we obtain the homomorphism $\tilde{f}_r : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow (\mathbb{Z}_2^4 \times T/S \times L) \times (\mathbb{Z}_2^4 \times \mathbb{Z}_2^4 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong D_4 \times D_4$ reduced from $f^*_r : G_0 \rightarrow G_0^4 \times T \subset G_1$:

$$\tilde{f}_r : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow (\mathbb{Z}_2^4 \times T) \times (\mathbb{Z}_2^4 \times T)$$

The image of $\tilde{f}_r$ is included in $(Q \times T/S \times L) \times (Q \times T/S \times L) \cong D_2 \times D_2$, and $\tilde{f}_r$ sends $(1, 0)$ to $(\zeta_3, \zeta_1), (0, 1)$ to $(\zeta_1, \zeta_3)$, and so $(1, 1)$ to $(\zeta_2, \zeta_2)$. Note that if $g \in \ker \alpha_r$, then $\pi \circ (\mu_A \times \mu_B) \circ f^*_r(g) \neq (\zeta_1, \zeta_3, \zeta_3, \zeta_1)$, and so $(\mu_A \oplus \mu_B)(g) = (0, 0)$ or $(1, 1)$.

The homomorphism $\mu_A \oplus \mu_B : G_0 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$ breaks up into $G_0 \rightarrow \tilde{G}_0 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Indeed, we show $\ker \alpha_{\infty} \subset \ker (\mu_A \oplus \mu_B)$. Take $g \in \ker \alpha_{\infty}$. Then
$g_w := \beta_{r,w}(g) \in \ker \alpha_f$ for any $w \in \text{Word}(k)$. Therefore $(\mu_A \oplus \mu_B)(g_w) = (0, 0)$ or $(1, 1)$ for any $w$. By (9) to (12), for $w \in \text{Word}(k)$ with large $k$, it is easily seen that $g_w \in \mathcal{H} := \langle A, B \rangle$, and we have

$$g_w \in \mathcal{H}_0 := \langle A^2, B^2, AB \rangle = \mathcal{H} \cap (\mu_A \oplus \mu_B)^{-1}(\{(0, 0), (1, 1)\}).$$

Let $\mathcal{H}_1$ be the normal subgroup of $\mathcal{H}$ generated by $\{A^2, B^2\}$. Then $\mu_A(\mathcal{H}_1) = \mu_B(\mathcal{H}_1) = 0$, and $f_\ast(\mathcal{H}_1) = \{e\}$. The quotient group $\mathcal{H}_0/\mathcal{H}_1$ is the cyclic group generated by $\{AB\mathcal{H}_1\}$. Observe $f_\ast(AB) = A(1,1) + A^{-1}(2,2) + B(3,3) + B^{-1}(4,4)$. Thus if $g_w \in (AB)^m \mathcal{H}_1$ with $m$ odd, then $g_{wi} \not\in \mathcal{H}_0$ for $i \in \text{Word}(1)$. This means that $g_w \in \mathcal{H}_0$ implies $g_w \in (AB)^m \mathcal{H}_1 \subset \ker(\mu_A \oplus \mu_B)$. Assume that $(\mu_A \oplus \mu_B)(g) = (1, 1)$. Since $f_r : (1,1) \mapsto (\zeta_2, \zeta_2)$, we can inductively see that for any $k$, there exists $w \in \text{Word}(k)$ such that $(\mu_A \oplus \mu_B)(g_w) = (1, 1)$. This contradicts the fact we have just proved.

We have the commutative diagram:

$\mathcal{G}_0 \quad \mathcal{G}_0^1 \times T$

$(\mathbb{Z}_2^4 \times T) \times (\mathbb{Z}_2^4 \times T)$

$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad D_4 \times D_4$

Let $r' = (l'_r)$ be a radial for $f$ other than $r$. Then $f_\ast(r'(g)) = hf_\ast(r(g))h^{-1}$ for some $h \in G_0^4 \times \{\text{id}\}$ as seen in Definition 16. From $\mathbb{Z}_2^4 \times T \triangleright \mathbb{Q} \times T$, we also have a similar commutative diagram as above for $f_\ast$, namely, we reduce $f_\ast$ to $\tilde{f}_r' : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow D_4 \times D_4$, which is obtained by

$$\tilde{f}_r'(x) = (\mu_A(h)k_1\mu_A(h^{-1}), \mu_B(h)k_2\mu_B(h^{-1})).$$

where $\tilde{f}_r(x) = (k_1, k_2)$. Therefore we have only four possibilities of the behavior of $\tilde{f}_r'$:

Case 1. $(1, 0) \mapsto (\zeta_1, \zeta_3)$, $(0, 1) \mapsto (\zeta_3, \zeta_1)$

Case 2. $(1, 0) \mapsto (\zeta_3, \zeta_1)$, $(0, 1) \mapsto (\zeta_1, \zeta_3)$

Case 3. $(1, 0) \mapsto (\zeta_1, \zeta_1)$, $(0, 1) \mapsto (\zeta_3, \zeta_3)$

Case 4. $(1, 0) \mapsto (\zeta_3, \zeta_1)$, $(0, 1) \mapsto (\zeta_1, \zeta_3)$

In each case, we can find $g \in G_0$ such that for any $k > 0$ there exists $w \in \text{Word}(k)$ satisfying $\beta_{r',w}(g) \not\in \ker \alpha_\infty$. We know that $\ker \alpha_\infty \subset \ker(\mu_A \oplus \mu_B)$.
Hence any $g$ and so for some $i$

This is verified as follows. From $\tilde{f}_{r'}(1,0) = (\zeta_1, \zeta_3)$, we have

$$\mu_A(f_{r'}^i(g)) \in S \times T - S \times L, \mu_B(f_{r'}^i(g)) \in Q \times T - S \times T - Q \times L.$$ 

Thus

$$p_1 \circ \mu_A(f_{r'}^i(g)) = (0,0,0,0), (1,0,0,1), (0,1,1,0), \text{ or } (1,1,1,1)$$

and so for some $i \in \text{Word}(1)$, $(\mu_A \oplus \mu_B)(\beta_{r',i}(g)) \in \{ (1,0), (0,1), (1,1) \}$. Similarly, we have

$$(\mu_A \oplus \mu_B)(g) = (0,1) \Rightarrow \exists i \in \text{Word}(1), \beta_{r',i}(g) \notin \ker(\mu_A \oplus \mu_B)$$

$$(\mu_A \oplus \mu_B)(g) = (1,1) \Rightarrow \exists i \in \text{Word}(1), \beta_{r',i}(g) \notin \ker(\mu_A \oplus \mu_B).$$

Hence any $g \notin \ker(\mu_A \oplus \mu_B)$ satisfies the required property.

Case 2.

$$(\mu_A \oplus \mu_B)(g) = (0,1) \Rightarrow \exists i \in \text{Word}(1), \beta_{r',i}(g) \in \{ (1,0), (1,1) \}$$

$$(\mu_A \oplus \mu_B)(g) = (1,1) \Rightarrow \exists i \in \text{Word}(1), \beta_{r',i}(g) \in \{ (0,1), (1,1) \}.$$ 

Hence any $g \in (\mu_A \oplus \mu_B)^{-1}(\{(0,1), (1,1)\})$ satisfies the required property.

The remaining two cases are similar. From Lemma 12, $\phi_{r'}$ is not one-to-one.

\[ \square \]

**Theorem 8.** There exists $\tilde{f} \in S_d - T_d$ for $d \geq 4$.

We use the Thurston type theorem for a characterization of subhyperbolic rational maps.

**Definition 17.** Let $g : \hat{C} \to \hat{C}$ be a topological branched covering. Let $P_g$ be the set of the points in the critical orbits. We say $g$ is a subhyperbolic semi-rational map if $\overline{P_g} - P_g$ is a finite set, $g$ is holomorphic in a neighborhood of $\overline{P_g} - P_g$, and every periodic points in $\overline{P_g} - P_g$ is (super)attracting.

We say two subhyperbolic semi-rational maps $g_1, g_2$ are $c$-equivalent to each other if there exist two homeomorphisms $\phi, \psi : \hat{C} \to \hat{C}$ satisfying: (1) $\phi \circ g_1 = g_2 \circ \psi$, (2) $\phi = \psi$ and is holomorphic on a neighborhood $U$ of $\overline{P_{g_1}} - P_{g_1}$, (3) $\phi = \psi$ on $P_{g_1}$, and (4) $\phi$ and $\psi$ are isotopic to each other rel $P_{g_1} \cup U$.

**Theorem 9.** (Zhang-Jiang [22], Cui-Tan [6]) Let $g$ be a subhyperbolic semi-rational map with $\#P_g = \infty$. The map $g$ is $c$-equivalent to a rational map if and only if $g$ has no Thurston obstruction.

**Corollary 4.** If $g$ is a subhyperbolic semi-rational map without homotopically invariant curve, then $g$ is $c$-equivalent to a hyperbolic rational map whose Julia set is Cantor.

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Proof of Theorem 8. We construct $\tilde{f}$ by modifying $f$ above. Let $a = ic$ with $c > 0$ large again. Then $|\sqrt{1 + 1/2y}|, \infty)$ is included in $A_0(\infty)$ the attracting basin of $\infty$. Take two points $z_1, z_2 \in \mathbb{R}$ with $\sqrt{1 + 1/2y} < z_1 < z_2$ and $d - 3$ smooth functions $h_i : [z_1, z_2] \to \mathbb{R}$, $i = 1, 2, \ldots, d - 3$ such that $h_i(z_1) = h_i(z_2) = 0$ $(i = 1, 2, \ldots, d - 3)$, $h_i(x) < h_{i+1}(x)$ $(z_1 < x < z_2, i = 1, 2, \ldots, d - 4)$, and $\{x + iy : z_1 < x < z_2, h_1(x) < y \leq h_{d-3}(x)\} \subset A_0(\infty)$. Let $E_i = \{x + iy : z_1 \leq x < z_2, h_i(x) \leq y \leq h_{i+1}(x)\}$ $(i = 1, 2, \ldots, d - 4)$. Define a branched covering $g$ of degree $d$ with critical set $C_{r_g} = C_{r_f} \cup \{z_1, z_2\}$ satisfying $g(\partial E_i) = f([z_1, z_2])$, $g|_{\text{int} E_i} : \text{int} E_i \to \mathbb{C} - f([z_1, z_2])$ is homeomorphic, and $g|_{\mathbb{C} - \bigcup E_i} = f \circ \beta$, where $\beta : \mathbb{C} - \bigcup E_i \to \mathbb{C} - \{z_1, z_2\}$ is a homeomorphism which is the identity outside a small neighborhood of $\bigcup E_i$.

By Corollary 4, there exists a rational map $\tilde{f}$ c-equivalent to $g$ with $J_f$ Cantor. Let $r = (l_i)_{i=1,2,\ldots,d}$ be a radial for $\tilde{f}$. If the coding map $\phi_r : \Sigma_d \to J_f$ is one-to-one, then we have a radial $r'$ for $f$ with $\phi_r' : \Sigma_4 \to J_f$ is one-to-one. Indeed, we can assume that $x_i = l_i(1) \in \mathbb{C} - \bigcup E_i$ for $i = 1, 2, 3, 4$ and $x_i \in E_{i-4}$ for $i = 5, 6, \ldots, d$. Moreover we can assume that $l_i \subset \mathbb{C} - \bigcup E_i$ for $i = 1, 2, 3, 4$, since $\bigcup E_i$ does not intersect $P_f$. Define a radial $r' = (l_i')_{i=1,2,3,4}$ for $f$ which is derived from $(l_i)_{i=1,2,3,4}$. It is easy to see that if $\phi_r$ is one-to-one, so is $\phi_r'$.

Appendix

In Appendix, we describe a generalization of Section 3. We state a version of Theorem 1 without the assumption $J \cap Cr = \emptyset$. We omit the proof, most of which are parallel to Theorem 1. The detail is left to the reader.

Definition 18. Let $X$ be a topological space. We say that a continuous mapping $a : X \to \mathbb{C}$ is tiny if there exist $p \in J$ and a homotopy $H : X \times [0, 1] \to \mathbb{C} - (P - \{p\})$ between $a$ and a constant mapping $p$. In particular, a homotopically trivial mapping is tiny. We say that $p$ is the center of $a$. If $a$ is tiny and homotopically nontrivial, then the choice of $p$ is unique and $p \in P$.

A subset $X \subset \mathbb{C}$ is said to be tiny if the inclusion map is tiny.

Definition 19. Set $P' = P - \bigcup_{k=0}^{\infty} f^{-k}(P_a)$. Let $\Omega$ be an expanding domain.

A closed curve $\gamma : S^1 \to \Omega - P' \equiv \text{homotopically invariant}$ if $\gamma$ is homotopically nontrivial, and there exists a lifted loop $\gamma' : S^1 \to \Omega - P'$ of $\gamma$ by $f^k$ for some $k > 0$ and there exists a homotopy $H : S^1 \times [0, 1] \to \Omega - P'$ between $\gamma$ and $\gamma'$ such that for $s \in S^1$, if $H(s, t_0) \in P$ for some $t_0 \in [0, 1)$, then $H(s, t) = H(s, t_0)$ for any $t \in [0, 1]$.

Definition 20. For a subset $U \subset \mathbb{C}$ and $p \in P$, we say that $p$ is weakly contained in $U$ if either $p \in U$ or $p$ is the center of some tiny topological disc $D$ satisfying $p \in D$ and $\partial D \subset U$.

Theorem 10. Let $f$ be a geometrically finite rational map of degree $d$. Take an expanding domain $\Omega$. The following are equivalent.

1. $f$ is Cantor.
2. Every connected component of $J$ is tiny.

3. $F \neq \emptyset$, and every homotopically invariant curve is tiny.

4. There exists $n > 0$ such that for any closed curve $\gamma \subset \Omega - P$, any lifted loop of $\gamma$ by $f^{-n}$ is tiny.

5. There exists $n > 0$ such that any connected component of $f^{-n}(\Omega)$ is tiny.

6. There exists a solitary fixed point $p$ such that every postcritical value is weakly contained in $A_0(p)$.

7. There exists a solitary fixed point $p$, and every critical value is weakly contained in the basin of attraction $A(p)$, and at least $2d - 4$ critical values counted with multiplicity are weakly contained in the immediate basin of attraction $A_0(p)$.

**Theorem 11.** Let $f$ be a geometrically finite rational map with $\mathcal{P}a \cap P = \emptyset$. Take an expanding domain $\Omega$. The following are equivalent.

1. $f$ is Cantor.

2. The quotient group $\pi_1(\Omega - P, \bar{x})/\ker \alpha_\infty$ is a finite group.

3. There exists a neighborhood $K$ of $J$ such that $K - P$ is arcwise-connected and $\pi_1(K - P, \bar{x})/\ker \alpha_\infty$ is a finite group.

**Remark 7.** Let $f$ be a geometrically finite rational map. If $J$ is Cantor, then $F$ contains at least two critical values. Indeed, for a solitary fixed point $p$, take $U_k$ as in Definition 8. If every $U_k$ has at most one critical value, then every $U_k$ is simply connected, and so $J$ is not Cantor.

Conversely, for $d \geq 2$, there exists a rational map of degree $d$ such that $J$ is Cantor and $F$ contains exactly two critical values of degree two.

**Remark 8.** In 3 of Theorem 10, the condition $F \neq \emptyset$ is necessary. We give an example of a subhyperbolic rational map with $F = \emptyset$ such that every homotopically invariant curve $\gamma$ in $\hat{\mathbb{C}} - P$ is tiny.

Let $\alpha = 1 + 2i$ and define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \alpha z$. Consider the quotient space $S = \mathbb{C}/\Gamma$, where $\Gamma = \langle z \mapsto -z, z \mapsto z + 1, z \mapsto z + i \rangle$. Then $g$ descends to $\hat{f} : S \to \hat{S}$ by $[g(z)] = \hat{f}([z])$. We have a bijection $h : S \to \hat{\mathbb{C}}$ with singularity at $[0], [1/2], [i/2], [(1 + i)/2]$ such that $f = h \circ \hat{f} \circ h^{-1}$ is a rational map. The map $h$ is induced from the Weierstrass $\wp$ function for the lattice $(1, i)$. For example, we have $f(z) = z(z^2 - \alpha)^2/(\alpha z^2 - 1)^2$. We can see that $P = \{0, \pm 1, \infty\}$ consists of fixed points with multiplier $\alpha^2$, and $J = \hat{\mathbb{C}}$. Then every homotopically invariant curve $\gamma$ in $\hat{\mathbb{C}} - P$ is tiny.
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