ON THE DISTRIBUTION OF MAXIMUM OF A BROWNIAN SHEET
RESTRICTED TO A LOWER-DIMENSIONAL SET

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1. INTRODUCTION

We consider the distribution of an $n$-dimensional Brownian sheet (Chentsov field) $X(s_1, \ldots, s_n)$ on a set $S$ of a dimension lower than the dimension of the field. The Brownian sheet was first described by Chentsov [7] and Yeh [6]. In Russian-language literature it was previously known as Chenstov field, whereas in English-language literature a commonly used name was Wiener-Yeh field. Now, it is usually referred to as a Brownian sheet. It is known that random functions such as Brownian sheet occur in modeling of external effects influencing a system at a random moment in time and at a random location. For instance, this is a common scenario in the problem of modeling small transverse vibrations of a string under influence of random external forces or in the problem of heat transfer in a rod with presence of random heating/cooling sources [1], [2] as well as in filtration problems [3].

The problem of finding distribution on a set of a dimension lower than the dimension of the field occurs, for instance, in percolation theory [4].

Although some research on the distribution of maximum of a Brownian sheet on a unit square has been conducted, no exact results have been obtained yet.

For instance, a lot of results regarding the distribution of supremum of a field restricted to curves and polylines have been obtained. For the case of 2-dimensional field on the unit square, Park, Paranjape [9], [10] have obtained the distribution of supremum of the field restricted to a polyline with a single vertex. Also, by considering limit of the polyline they have obtained distribution on the boundary of a square. Generalization of these results for a polyline with $n$ vertices was considered by Klesov and Kruglova [13], [12]. However, the obtained exact distribution of supremum is given in terms of quite cumbersome integrals direct estimation of which may become problematic. That is why Kruglova and Dykhovychnyi [15] suggested an empirical approach to finding the distribution by modeling corresponding restriction to polylines. The suggested method is based on Doob’s Transformation Theorem [5].
Also, the problem of finding the following probability is of a special interest:

\[
P \left\{ \sup_{0 \leq t \leq T} w(t) - f(t) < b \right\},
\]

where \(w(t)\) is a Wiener process, and \(f(t)\) is a deterministic function (drift).

For a multidimensional Brownian sheet \(X(s_1, \ldots, s_n)\), this problem can be formulated as follows:

\[
P \left\{ \sup_{S} \left( X(s_1, \ldots, s_n) - g(s_1, \ldots, s_n) \right) < 0 \right\},
\]

where \(S\) is a set of dimension lower than \(n\), and \(g(\cdot)\) is a continuous function.

2. Definitions and preliminaries

To prove our results the following definitions, lemmas and theorems are needed. Let \(K = [0, 1]^n\) denote an \(n\)-dimensional unit hypercube.

**Definition 2.1.** A real-valued separable Gaussian stochastic process \(\{X(s_1, \ldots, s_n), s = (s_1, \ldots, s_n) \in K\}\) is called a Brownian sheet with \(n\) parameters, if \(X(s_1, \ldots, s_n)\) is such that:

1. \(X(s_1, \ldots, s_n) = 0\), if \(s_1 \cdot \ldots \cdot s_n = 0\);
2. \(E[X(s_1, \ldots, s_n)] = 0\), for all \((s_1, \ldots, s_n) \in K\);
3. \(E[X(s_1, \ldots, s_n)X(t_1, \ldots, t_n)] = \prod_{i=1}^{n} \min\{s_i, t_i\}\), for all \((s_1, \ldots, s_n) \in K\) and \((t_1, \ldots, t_n) \in K\).

**Definition 2.2.** Processes \(\xi(t)\) and \(\xi^*(t)\) are said to be stochastically equivalent if, for all \(t \in T\),

\[
P \{\xi(t) = \xi^*(t)\} = 1.
\]

**Theorem 2.1.** (Doob’s Transformation Theorem [3]) Let \(Y(t)\) be a Gaussian process with \(E[Y(t)] = 0\), for all \(t \geq 0\), and covariance function

\[
R(s, t) = u(s)v(t), \quad s \leq t.
\]

If the function \(a(t) = u(t)/v(t)\) is continuous and strictly increasing with the inverse \(a^{-1}(t)\), then \(w(t)\) and \(Y(a^{-1}(t))/v(a^{-1}(t))\) are stochastically equivalent processes.

3. Main results

3.1. Generalization of Doob’s Transformation Theorem. In order to solve [1], we need the following analogue of Theorem [2.1]

**Lemma 3.1.** Let \(Y(s_1, \ldots, s_n)\) be a Gaussian field with \(E[Y(s_1, \ldots, s_n)] = 0\), for all \(\overrightarrow{s} = (s_1, \ldots, s_n) \in [0, \infty)^n\), and covariance function

\[
E[Y(s_1, \ldots, s_n)Y(t_1, \ldots, t_n)] = \prod_{i=1}^{n} a_i(s_i \wedge t_i) v_i(s_i \vee t_i)
\]

for all \(\overrightarrow{s}, \overrightarrow{t} \in [0, \infty)^n\) (where \(\wedge\) and \(\vee\) denote maximum and minimum, respectively).

If the functions \(a_i(t) = \frac{u_i(t)}{v_i(t)}\), \(i = 1, n\), are increasing and continuous, then the field

\[
Z(t_1, \ldots, t_n) = \frac{Y(a_1^{-1}(t_1), \ldots, a_n^{-1}(t_n))}{v_1(a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(t_n))}
\]

is stochastically equivalent to \(X(t_1, \ldots, t_n)\), where \(a_i^{-1}(\cdot)\), \(i = 1, n\), denotes the inverse of \(a_i(\cdot)\).
Due to the fact that the Brownian sheet and Hasting process are both Gaussian, the field $Z(t_1, \ldots, t_n)$ is stochastically equivalent to $X(t_1, \ldots, t_n)$.

**Lemma 3.2.** Let $Y(s_1, \ldots, s_n)$ be a Gaussian field with $E[Y(s_1, \ldots, s_n)] = 0$, for all $s = (s_1, \ldots, s_n) \in [0, \infty)^n$, and covariance function $E^{(2)}$. If the functions $a_i(t) = \frac{w_i(t)}{w_{i-1}(t)}$, $i = 1, n$, are decreasing and continuous, then the field

$$Z(t_1, \ldots, t_n) = \frac{Y(a_1^{-1}(t_1), \ldots, a_n^{-1}(t_n))}{u_1(a_1^{-1}(t_1)) \cdots u_n(a_n^{-1}(t_n))}$$

is stochastically equivalent to $X(t_1, \ldots, t_n)$, where $a_i^{-1}()$, $i = 1, n$, denotes the inverse of $a_i()$.

**Доведення.**

Let us calculate the expectation and the covariance function of $Z(t_1, \ldots, t_n)$. It is clear that $E[Z(t_1, \ldots, t_n)] = 0$. Then, using (3), we get

$$E[Z(s_1, \ldots, s_n)Z(t_1, \ldots, t_n)] =$$

$$= u_1(a_1^{-1}(s_1) \land a_1^{-1}(t_1)) \cdots u_n(a_n^{-1}(s_n) \land a_n^{-1}(t_n))$$

$$\times u_1(a_1^{-1}(s_1) \lor a_1^{-1}(t_1)) \cdots u_n(a_n^{-1}(s_n) \lor a_n^{-1}(t_n))$$

$$= a_1(a_1^{-1}(s_1 \land t_1)) \cdots a_n(a_n^{-1}(s_n \land t_n))$$

The penultimate equality holds because the functions $a_i^{-1}()$, $i = 1, n$, are strictly increasing. Thus,

$$a_i^{-1}(s_1) \land a_i^{-1}(t_1) = a_i^{-1}(s_1 \land t_1), \quad i = 1, n.$$
= a_1 \left( a_1^{-1}(s_1 \land t_1) \right) \cdot \ldots \cdot a_n \left( a_n^{-1}(s_n \land t_n) \right) = (s_1 \land t_1) \cdot \ldots \cdot (s_n \land t_n).\]

The above holds because the functions \( a_i^{-1}(\cdot) \), \( i = \overline{1,n} \), are strictly decreasing. Therefore,

\[
a_i^{-1}(s_i) \lor a_i^{-1}(t_i) = a_i^{-1}(s_i \lor t_i), \quad i = \overline{1,n}.\]

Since

\[
E[X(t_1, \ldots, t_n)X(s_1, \ldots, s_n)] = (s_1 \land t_1) \cdot \ldots \cdot (s_n \land t_n)
\]

and

\[
E[X(t_1, \ldots, t_n)] = 0,
\]

the field \( Z(t_1, \ldots, t_n) \) is stochastically equivalent to \( X(t_1, \ldots, t_n) \).

3.2. Main Theorem. Let \( G = [0, y_1] \times \ldots \times [0, y_d] \), where \( y_i \in [0,1], i = \overline{1,d} \).

Suppose that \( d < n \), and let us consider a \( d \)-dimensional \( S \subset K \) given by

\[
s_d+1 = f_{d+1}(s_{j_d}), \quad \ldots, \quad s_n = f_n(s_{j_d}),
\]

where \( j_k \in \{1, \ldots, d\}, k = \overline{1,d}, 0 \leq s_i \leq y_i, i = \overline{1,n} \).

Suppose also that there exist decreasing functions \( z_i(\cdot), i = \overline{1,d} \), such that the functions \( f_{i+d}(\cdot), i = \overline{1,n-d} \), are decreasing and

\[
f_{d+1}(s_{j_d}) \cdot \ldots \cdot f_n(s_{j_d}) = \prod_{i=1}^d z_i(s_{j_i}).
\]

Let \( a_i(t) = \frac{1}{z_i(t)} \) and \( x_i = a_i(y_i), i = \overline{1,d} \), and \( D = I_1 \times \ldots \times I_d \), where \( I_i = [0, x_i] \).

Theorem 3.1. Let \( X(s_1, \ldots, s_n) \) be an \( n \)-parameter Brownian sheet on \( G \). Let \( g_S(s_1, \ldots, s_d) \) denote the restriction of \( g(s_1, \ldots, s_n) \) to \( S \). If, for all \( i = \overline{1,d} \), there exists \( a_i^{-1} : I_i \to [0, y_i] \), the inverse of \( a_i(\cdot) \), then

\[
\mathbb{P} \left\{ \sup_S (X(s_1, \ldots, s_n) - g(s_1, \ldots, s_n)) < 0 \right\} = \mathbb{P} \left\{ \sup_D \left( X(t_1, \ldots, t_d) - \frac{g_S(a_1^{-1}(t_1) \cdot \ldots \cdot a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1)) \cdot \ldots \cdot z_d(a_d^{-1}(t_d))} \right) < 0 \right\}.
\]

Доведення.

Let \( X_S(s_1, \ldots, s_d) \) denote the restriction of \( X(s_1, \ldots, s_n) \) to \( S \). The expectation and the covariance function of the field \( X_S(s_1, \ldots, s_d) \) are

\[
E[X_S(s_1, \ldots, s_d)] = E[X(s_1, \ldots, s_d, f_{d+1}(s_{j_1}), \ldots, f_n(s_{j_d}))] = 0.
\]

Then,

\[
E[X_S(s_1, \ldots, s_d)X_S(t_1, \ldots, t_d)] = E[X(s_1, \ldots, s_d, f_{d+1}(s_{j_1}), \ldots, f_n(s_{j_d}))X(t_1, \ldots, t_d, f_{d+1}(t_{j_1}), \ldots, f_n(t_{j_d}))]
\]

\[
= (s_1 \land t_1) \cdot \ldots \cdot (s_d \land t_d) \cdot (f_{d+1}(s_{j_1}) \land f_{d+1}(t_{j_1})) \cdot \ldots \cdot (f_n(s_{j_d}) \land f_n(t_{j_d}))
\]

\[
= (s_1 \land t_1) \cdot \ldots \cdot (s_d \land t_d) \cdot (f_{d+1}(s_{j_1} \lor t_{j_1})) \cdot \ldots \cdot (f_n(s_{j_d} \lor t_{j_d}))
\]

\[
= \prod_{i=1}^d (s_i \land t_i) z_i(s_i \lor t_i).
\]

Since \( z_i(t), i = \overline{1,d} \), are decreasing, \( \frac{1}{z_i(t)}, i = \overline{1,d} \) are increasing. Hence, the assumptions of the theorem allow us to apply Lemma 3.3

\[
\mathbb{P} \left\{ \sup_S (X(s_1, \ldots, s_n) - g(s_1, \ldots, s_n)) < 0 \right\} = \mathbb{P} \left\{ \sup_{[0,y_1] \times \ldots \times [0,y_d]} (X_S(s_1, \ldots, s_d) - g_S(s_1, \ldots, s_d)) < 0 \right\}
\]
\[
\begin{align*}
&= P \left\{ \sup_D \left( \frac{X_S(a_1^{-1}(t_1), \ldots, a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1)) \cdot \ldots \cdot z_d(a_d^{-1}(t_d))} - \frac{g_S(a_1^{-1}(t_1), \ldots, a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1)) \cdot \ldots \cdot z_d(a_d^{-1}(t_d))} \right) < 0 \right\} \\
&= P \left\{ \sup_D \left( X(t_1, \ldots, t_d) - \frac{g_S(a_1^{-1}(t_1), \ldots, a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1)) \cdot \ldots \cdot z_d(a_d^{-1}(t_d))} \right) < 0 \right\}.
\end{align*}
\]

Therefore, by using Theorem 3.1, one can reduce the problem of finding the distribution of maximum of \(X(t_1, \ldots, t_d)\) on \(S\) to the problem of finding its distribution on the parallelepiped \(D\), which may be easier to deal with. For instance, this may be the case if one wants to study the asymptotic behavior of this distribution.

### 3.3. Examples

Consider the two following applications of Theorem 3.1.

**Example 3.1.** Let \(X(s_1, s_2, s_3)\) be a 3-dimensional Brownian sheet \((n = 3 \text{ and } d = 1)\), and let \(S\) be a curve given by
\[
\begin{align*}
  s_2 &= \sqrt{1 - s_1}, \\
  s_3 &= \sqrt{1 - s_1}.
\end{align*}
\]
Suppose \(g(s_1, s_2, s_3) = s_1 + s_2s_3\). Let us show that
\[
P \left\{ \sup_S (X(s_1, s_2, s_3) - g(s_1, s_2, s_3)) < 0 \right\} = 1 - e^{-2}.
\]

**Solution.** From the definition of \(S\) we have
\[
z_1(s_1) = 1 - s_1,
\]
\[
a_1(s_1) = \frac{s_1}{1 - s_1}.
\]
The inverse of \(a_1(\cdot)\) is
\[
a_1^{-1}(s_1) = \frac{s_1}{1 + s_1}.
\]
Then,
\[
z_1(a_1^{-1}(s_1)) = \frac{1}{1 + s_1}
\]
and
\[
g_S(s_1) = s_1 + 1 - s_1 = 1.
\]
Hence,
\[
\frac{g_S(a_1^{-1}(s_1))}{z_1(a_1^{-1}(s_1))} = 1 + s_1.
\]

\[
P \left\{ \sup_S (X(s_1, s_2, s_3) - s_1 - s_2s_3) < 0 \right\} =
\]
\[
P \left\{ \sup_{t \in [0, \infty)} (w(t) - 1 - t) < 0 \right\} = 1 - e^{-2}.
\]
The above identity is due to the following result [5]:

\[
(4) \quad P \left\{ \sup_{0 \leq t < \infty} (w(t) - at - b) < 0 \right\} = 1 - e^{-2ab}.
\]

**Example 3.2.** Now, suppose \(g(s_1, s_2, s_3) = s_1 + s_2^2 + s_3^2\), and let us show that
\[
P \left\{ \sup_S (X(s_1, s_2, s_3) - g(s_1, s_2, s_3)) < 0 \right\} = 1 - e^{-4}.
\]
Solution. Since \( g(s_1, s_2, s_3) = s_1 + s_2^2 + s_3^2 \), we have
\[
g_S(s_1) = s_1 + 1 - s_1 + 1 - s_1 = 2 - s_1.
\]
Finally, using Theorem 3.1, we get
\[
P\left\{ \sup_S (X(s_1, s_2, s_3) - s_1 - s_2^2 - s_3^2) < 0 \right\} = P\left\{ \sup_{t \in [0, \infty)} (w(t) - 2 - t) < 0 \right\} = 1 - e^{-4}.
\]
It is reasonable to compare obtained probabilities with corresponding empirical estimates resulting from simulation of the 3-dimensional Brownian sheet. Let us use the algorithm for simulation of a Gaussian processes with special covariance function suggested in [15].

The following is R code simulates the Gaussian processes provided in Examples 3.1 and 3.2 and computes both empirical and theoretical distributions of their maximums.

```r
> m <- numeric(10^4)
> n <- numeric(10^4)
> for (i in 1:10^4) {
+ t <- seq(0, 1-1/1000, length.out = 1000)
+ vt <- 1-t
+ at <- t/(1-t)
+ D <- diff(at)
+ y <- vt * c(0, cumsum(rnorm(999, 0, sqrt(D))))) # the field
+ m[i] <- max(y) # compute maximum (Example 3.1)
+ n[i] <- max(y-2+t) # compute maximum (Example 3.2)
+ }
> m1 <- m[m<1]
> length(m1)/10^4
[1] 0.8683 # empirical probability (Example 3.1)
> 1-exp(-2)
[1] 0.8646647 # theoretical probability (Example 3.1)
> n1 <- n[n<0]
> length(n1)/10^4
[1] 0.9833 # empirical probability (Example 3.2)
> 1-exp(-4)
[1] 0.9816844 # theoretical probability (Example 3.2)
```

It could be seen that the theoretical results are quite close to the empirical ones in both cases (increasing the number of points in the mesh will clearly increase accuracy of the estimate).

Let us consider another example for a 4-dimensional Brownian sheet.

Example 3.3. Let \( X(s_1, s_2, s_3, s_4) \) be a 4-parameter Brownian sheet. For \( \lambda > 0 \), let us consider the following probability:

\[
P_S(X) = P\left\{ \sup_{(s_1, s_2, s_3, s_4) \in S} \left( X(s_1, s_2, s_3, s_4) - \frac{\lambda s_3 s_4}{(s_1 + s_3)(s_2 + s_4)} \right) < 0 \right\},
\]
where \( S = \{ (s_1, s_2, s_3, s_4) : 0 \leq s_i \leq \frac{1}{2}, i = 1, 4, s_3 = 1 - s_1, s_4 = 1 - s_2 \} \).
Solution. Let $X_S$ denote the restriction of the field $X(s_1, s_2, s_3, s_4)$ to $S$. It is clear that $X_S(s_1, s_2) = X(s_1, s_2, 1 - s_1, 1 - s_2)$ and

$$R(\bar{s}, \bar{t}) = E[X_S(\bar{s})X_S(\bar{t})] = \prod_{i=1}^{2}(s_i \lor t_i)(1 - s_i \land t_i).$$

Using the notation of Theorem 3.1 we have

$$z_1(s_1) = 1 - s_1, z_2(s_2) = 1 - s_2.$$

Therefore,

$$a_1(s_1) = \frac{s_1}{1 - s_1}, a_2(s_2) = \frac{s_2}{1 - s_2}$$

and

$$a_1^{-1}(s_1) = \frac{s_1}{1 + s_1}, a_2^{-1}(s_2) = \frac{s_2}{1 + s_2}; y_1 = \frac{1}{2}, y_2 = \frac{1}{2}, x_1 = a_1(y_1) = 1, x_2 = a_1(y_1) = 1.$$

Using Theorem 3.1 we can rewrite the probability (13) in the following form:

$$P_S(X) = P \left\{ \sup_{(s_1, s_2) \in [0, 1/2]^2} (X_S(s_1, s_2) - \lambda (1 - s_1)(1 - s_2)) < 0 \right\} =$$

$$= P \left\{ \sup_{(s_1, s_2) \in [0, 1]^2} (1 + s_1)(1 + s_2)X_S(s_1 + s_2) < \lambda \right\} =$$

$$= P \left\{ \sup_{(s_1, s_2) \in [0, 1]^2} X_{s_1, s_2} < \lambda \right\}.$$

where $X(s_1, s_2)$ is a 2-dimensional Brownian sheet. In this way, the initial problem is reduced to the problem of finding the distribution of supremum of the Brownian sheet on a square.

Conclusion

In this paper, we obtained a generalization of Doob’s Transformation Theorem for multi-dimensional Gaussian random field.

We used this generalization to reduce the problem of finding the distribution of supremum of a $n$-dimensional Brownian sheet restricted to a lower-dimensional set to the problem of finding its distribution on simpler sets (parallelepipeds).

With the help of a special algorithm we performed modeling of a Brownian sheet restrictions to various curves and obtained their empirical distribution which appeared to coincide with the theoretical ones.
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