The small finitistic dimensions of commutative rings

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Abstract
Let \( R \) be a commutative ring with identity. The small finitistic dimension \( \text{fPD}(R) \) of \( R \) is defined to be the supremum of projective dimensions of \( R \)-modules with finite projective resolutions. In this paper, we characterize a ring \( R \) with \( \text{fPD}(R) \leq n \) using finitely generated semi-regular ideals, tilting modules, cotilting modules of cofinite type and vaguely associated prime ideals. As an application, we obtain that if \( R \) is a Noetherian ring, then \( \text{fPD}(R) = \sup\{\text{grade}(m, R) | m \in \text{Max}(R)\} \) where \( \text{grade}(m, R) \) is the grade of \( m \) on \( R \). We also show that a ring \( R \) satisfies \( \text{fPD}(R) \leq 1 \) if and only if \( R \) is a DW ring. As applications, we show that the small finitistic dimensions of strong Prüfer rings and LPVDs are at most one. Moreover, for any given \( n \in \mathbb{N} \), we obtain a total ring of quotients \( R \) satisfying \( \text{fPD}(R) = n \).

Key Words: small finitistic dimension; tilting module; Noetherian ring; DW ring; Prüfer ring.

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1. INTRODUCTION

Throughout this paper, \( R \) is a commutative ring with identity and \( \text{Mod}-R \) is the category of all \( R \)-modules. Let \( M \) be an \( R \)-module, we use \( \text{pd}_R M \) to denote the projective dimension of \( M \) over \( R \). The global dimension of \( R \), denoted by \( \text{gld}(R) \), is defined to be the supremum of the projective dimensions of all \( R \)-modules. It is well-known that a Noetherian local ring \( R \) has finite global dimension if and only if \( R \) is a regular local ring. So Noetherian local rings with nonzero zero divisors have infinite global dimensions. This motivates the definition of the so called finitistic dimension of a ring \( R \). The big finitistic dimension of \( R \), denoted by \( \text{FPD}(R) \), is defined to be the supremum of the projective dimensions of \( R \)-modules \( M \) with finite projective dimensions. Bass \cite{2} proved that a ring \( R \) is perfect if and only if \( \text{FPD}(R) = 0 \). For a Noetherian ring \( R \), Raynaud and Gruson \cite{19} showed that the
big finitistic dimension $\text{FPD}(R)$ of $R$ coincides with the Krull dimension $K.\dim(R)$ of $R$.

For a ring $R$, we denote $\text{mod-}R$ to be the full subcategory of all $R$-modules that have finitely generated projective resolutions, i.e. the $R$-modules $M$ fitting into an exact sequence

$$
\cdots \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0
$$

where each $P_i$ is a finitely generated projective $R$-module. Let $M$ be an $R$-module. Then $M$ is said to have a finite projective resolution, denoted by $M \in \mathcal{FP}R$, if there exist an integer $n$ and an exact sequence

$$
0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0
$$

with each $P_i$ finitely generated projective. We denote $\mathcal{P}^{\leq n}$ to be the class of $R$-modules with projective dimensions at most $n$ in $\mathcal{FP}R$. The small finitistic dimension of $R$, denoted by $\text{fPD}(R)$, is defined to be the supremum of the projective dimensions of $R$-modules in $\mathcal{FP}R$. Clearly, $\text{fPD}(R) \leq n$ if and only if $\mathcal{FP}R = \mathcal{P}^{\leq n}$, and $\text{fPD}(R) \leq \text{FPD}(R) \leq \text{gld}(R)$ for any ring $R$. In case $R$ is commutative Noetherian local, Auslander and Buchweitz showed that the small finitistic dimension $\text{fPD}(R)$ of $R$ coincides with the depth of $R$ (see [1]). In 2020, Wang et al. [25] proved that a ring $R$ is a DQ ring if and only if $\text{fPD}(R) = 0$. They also showed that if $\text{fPD}(R) \leq 1$, then $R$ is a DW ring and then gave examples of Prüfer rings with small finitistic dimensions larger than 1 getting a negative answer to the open question raised by Cahen et al. [6].

In the present paper, we characterize the small finitistic dimension of a ring $R$ using some special finitely generated semi-regular ideals, tilting modules, cotilting modules of cofinite type and the vaguely associated prime ideals of $R$ (see Theorem 3.1). As an application, we obtain that if $R$ is a Noetherian ring, then $\text{fPD}(R) = \sup\{\text{grade}(m, R)|m \in \text{Max}(R)\}$ where $\text{grade}(m, R)$ is the grade of $m$ on $R$ (see Proposition 3.2). For little small finitistic dimensions, we rediscover the rings with $\text{fPD}(R) = 0$ (see Corollary 3.6) and then show that a ring $R$ is a DW ring if and only if $\text{fPD}(R) \leq 1$ (see Corollary 3.7). As applications, we show that the small finitistic dimensions of strong Prüfer rings and LPVDs are at most one (see Corollary 3.8 and Corollary 3.9). Moreover, for any given $n \in \mathbb{N}$, we obtain examples of total rings of quotients with small finitistic dimensions $n$ which gives a deeper understanding of Cahen et al.’s open question [6] (see Example 3.10).
2. Preliminaries

In this section, we recall the classifications of infinitely generated tilting modules over commutative rings developed by Hrbek and Šťovíček (see [14, 15]). This theory is essential to our studies of the small finitistic dimensions of commutative rings.

Let \( R \) be a commutative ring, \( M \) an \( R \)-module. \( \Omega^{-i}(M) \) is denoted to be the \( i \)-th minimal cosyzygy of \( M \). For a class \( S \) of \( R \)-modules, we always use the following notations:

\[
S^\perp = \{ M \in \text{Mod-}R | \text{Ext}_R^n(S, M) = 0, \forall S \in S, \forall n \geq 1 \},
\]

\[
\perp S = \{ M \in \text{Mod-}R | \text{Ext}_R^n(M, S) = 0, \forall S \in S, \forall n \geq 1 \},
\]

\[
S^\top = \{ M \in \text{Mod-}R | \text{Tor}_R^n(S, M) = 0, \forall S \in S, \forall n \geq 1 \}.
\]

Following from [20], a filter \( G \) of ideals of \( R \) is called a Gabriel topology provided that:

1. if \( I \in G \) and \( t \in R \), then \((I : t) := \{ r \in R | tr \in I \} \in G\),
2. if \( J \) is an ideal and \( I \in G \) such that \((J : t) \in G \) for any \( t \in I \), then \( J \in G \).

A Gabriel topology is finitely generated if it has a basis of finitely generated ideals. We denote by \( G^f \) the set of all finitely generated ideals in \( G \). Obviously, a Gabriel topology is finitely generated if and only if it can be generated by \( G^f \).

Given a class \( C \) of \( R \)-modules, let \( \text{SubLim}(C) \) denote the smallest subclass of \( \text{Mod-}R \) containing \( C \) closed under direct limits and submodules. A prime ideal \( p \) of \( R \) is called vaguely associated to an \( R \)-module \( M \) if \( R/p \) is contained in \( \text{SubLim} \{ M \} \). Denote the spectrum of all vaguely associated primes of \( M \) by \( \text{VAss}(M) \). The set of vaguely associated prime ideals is a general version of that of associated prime ideals over Noetherian rings (see [14]).

An \( R \)-module \( T \) is said to be \( n \)-tilting for some \( n \geq 0 \), provided that it satisfies the following conditions:

1. \( \text{pd}_R T \leq n \),
2. \( \text{Ext}_R^i(T, T^{(\kappa)}) = 0 \) for all \( i > 0 \) and all cardinals \( \kappa \),
3. there are \( r \geq 0 \) and a long exact sequence

\[
0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_n \rightarrow 0
\]

where \( T_i \in \text{Add}(T) \) for all \( 0 \leq i \leq r \), where \( \text{Add}(T) \) denotes the class of all direct summands of arbitrary direct sums of copies of \( T \).

A cotorsion pair \((A, B)\) is said to be induced by an \( R \)-module \( M \) provided that \( B = M^\perp \). A cotorsion pair \((A, B)\) induced by an \( n \)-tilting module is said to be an \( n \)-tilting cotorsion pair, and then the class \( B \) is said to be the \( n \)-tilting class. We
suppress the index $n$- in the notation if we do not desire to specify the dimension bound on $T$. Two tilting modules are said to be equivalent provided that they induce the same tilting class. An $R$-module $M$ is called super finitely presented if it admits a projective resolution consisting of finitely generated projective modules. The following result, which can be implied by [4, Theorem 4.2] and [10, Theorem 13.26], shows that all tilting classes are induced by super finitely presented modules with bounded projective dimensions.

**Theorem 2.1.** A class $\mathcal{B}$ of $R$-modules is $n$-tilting if and only if there is a set $\mathcal{S}$ of super finitely presented modules of projective dimension at most $n$ such that $\mathcal{B} = \mathcal{S}^\perp$. 

An $R$-module $C$ is said to be $n$-cotilting for some $n \geq 0$, provided that it satisfies the following conditions:

1. $\text{id}_R C \leq n$,
2. $\text{Ext}_R^i(C^\kappa, C) = 0$ for all $i > 0$ and all cardinals $\kappa$,
3. there are $r \geq 0$ and a long exact sequence

$$0 \to C_r \to \cdots \to C_1 \to C_0 \to W \to 0$$

where $W$ is an injective cogenerator in $R$-Mod and $C_i \in \text{Prod}(C)$ for all $0 \leq i \leq r$, where $\text{Prod}(C)$ denotes the class of all direct summands of arbitrary direct products of copies of $C$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be co-induced by an $R$-module $C$ provided that $\mathcal{A} = \perp C$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ co-induced by an $n$-cotilting module is said to be an $n$-cotilting cotorsion pair, and then the class $\mathcal{A}$ is said to be the $n$-cotilting class. A class $\mathcal{C}$ of $R$-modules is of cofinite type if there exists $n < \infty$ and a set $\mathcal{S}$ of super finitely presented modules of projective dimensions at most $n$ such that $\mathcal{C} = \mathcal{S}^\perp$.

An $R$-module $C$ is of cofinite type if the class $\perp C$ of $R$-modules is of cofinite type. Recall that a subcategory $\mathcal{S}$ of $\text{mod-}R$ is said to be resolving, if all finitely generated projective modules are contained in $\mathcal{S}$, and $\mathcal{S}$ is closed under extensions, direct summands, and and $A \in \mathcal{S}$ where there is an exact sequence $0 \to A \to B \to C \to 0$ with $B, C \in \mathcal{S}$.

Over commutative rings, Hrbek and Šťovíček [15] classified the tilting classes in terms of sequences of Gabriel topologies, cotilting classes of cofinite type and resolving subcategories of $\text{mod-}R$.

**Theorem 2.2.** [15, Theorem 6.2] Let $R$ be a commutative ring and $n \geq 0$. There are 1-1 correspondences between the following collections:
(1) sequences \((\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1})\) of Gabriel topologies of finite type satisfying:
   
   (a) \(\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_{n-1}\),
   
   (b) \(\text{Ext}^j_R(R/I, R) = 0\) for all \(I \in \mathcal{G}_i\), all \(i = 0, 1, \ldots, n-1\), all \(j = 0, 1, \ldots, i\),

(2) \(n\)-cotilting classes \(\mathcal{C}\) in \(\text{Mod-}R\) of cofinite type,

(3) \(n\)-tilting classes \(\mathcal{T}\) in \(\text{Mod-}R\),

(4) resolving subcategories of \(\text{mod-}R\) consisting of modules of projective dimension at most \(n\).

The correspondences are given as follows:

\[
(1) \rightarrow (2): (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}) \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}^i_f} \text{Ker} \text{Ext}^i_R(R/I, -) = \bigcap_{i=0}^{n-1} (S_{I,i+1})^\top
\]

\[
(1) \rightarrow (3): (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}) \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}^i_f} \text{Ker} \text{Tor}^i_R(R/I, -) = \bigcap_{i=0}^{n-1} (S_{I,i+1})^\perp.
\]

\[
(1) \rightarrow (4): (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}) \mapsto \{M \in \text{mod-}R| M \text{ is isomorphic to a summand of a} \}
\]

finitely \(\{R\} \cup \{S_{I,i+1}| I \in \mathcal{G}^i_f, i < n\}\)-filtered module}.

3. Main Results

In this section, we give the main results of this paper and some applications.

**Theorem 3.1.** Let \(R\) be a commutative ring and \(n \geq 0\). The following conditions are equivalent:

1. \(\text{fPD}(R) \leq n\),
2. any tilting module is \(n\)-tilting,
3. any cotilting module of cofinite type is \(n\)-cotilting,
4. any finitely generated ideal \(I\) that satisfies \(\text{Ext}^i_R(R/I, R) = 0\) for each \(i = 0, \ldots, n\) is \(R\),
5. any finitely generated ideal \(I\) that satisfies \(\text{Vass}(\Omega^{-i}(R)) \cap V(I) = 0\) for each \(i = 0, \ldots, n\) is \(R\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(T\) be an \(m\)-tilting module with \(m \geq n\) and \((\mathcal{A}, \mathcal{B})\) be the cotorsion pair induced by \(T\). By [10] Theorem 13.46, there is an \(\mathcal{A}^{\infty}\)-filtered \(m\)-tilting module \(T_{\text{fin}}\) such that \(T_{\text{fin}}\) is equivalent to \(T\), where \(\mathcal{A}^{\infty}\) denotes the subclass \(\text{mod-}R \cap \mathcal{A}\) of \(\mathcal{A}\). By [10] Lemma 13.10, every \(R\)-module in \(\mathcal{A}\) has a projective dimension at most \(m\). It follows that \(T_{\text{fin}}\) is \(\mathcal{P}^{\leq m}\)-filtered. Since \(\text{fPD}(R) \leq n\) and
m \geq n$, we have $\mathcal{P}^{\leq m} = \mathcal{P}^{\leq n}$. Thus $T_{fin}$ is $\mathcal{P}^{\leq n}$-filtered. By Auslander Lemma, $\text{pd}_R T = \text{pd}_R T_{fin} \leq n$ and thus $T$ is $n$-tilting.

(2) $\Rightarrow$ (1): Let $M \in \mathcal{FP}\mathcal{R}$ and $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair induced by $M$. By Theorem 2.1, $(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair induced by a tilting module $T$. By (2), $T$ is $n$-tilting. Since $M \in \mathcal{A}$, $M$ has a projective dimension at most $n$ (see [10, Lemma 13.10]). Since $M \in \mathcal{FP}\mathcal{R}$, we have $M \in \mathcal{P}^{\leq n}$.

(2) $\Leftrightarrow$ (3): It follows by [10, Theorem 15.18].

(1) $\Rightarrow$ (4): Let $I = (a_1, \ldots, a_m)$ be a finitely generated ideal satisfying $\text{Ext}_R^i(R/I, R) = 0$ for all $i = 0, \ldots, n$. Let $K(I)$ be the Koszul complex induced by $\{a_1, \ldots, a_m\}$ as follows:

$$\cdots \xrightarrow{d_{n+2}} F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{0}.$$

Note that $F_0 = R$, $F_1 = R^m$ and $d_1(x_1, \ldots, x_m) = \sum_{i=1}^m a_i x_i$. Denote $S_{I,n+1}$ to be the cokernel of the map $d_{n+1}^* := \text{Hom}_R(d_{n+1}, R)$. Since $\text{Ext}_R^i(R/I, R) = 0$ for each $i = 0, \ldots, n$, by [15, Proposition 3.9], we have the following exact sequence:

$$0 \rightarrow F_0^* \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_n^*} F_n^* \xrightarrow{d_{n+1}^*} F_{n+1}^* \rightarrow S_{I,n+1} \rightarrow 0$$

where $F_i^* = \text{Hom}_R(F_i, R)$ and $d_i^* = \text{Hom}_R(d_i, R)$ for each $i$. Thus $\text{pd}_R(S_{I,n+1}) \leq n + 1$. Since $\text{fPD}(R) \leq n$, we have $\text{pd}_R(S_{I,n+1}) \leq n$. Thus $d_i^* : R^* \rightarrow (R^m)^*$ is a split monomorphism. Since the map $d_1 : R^m \rightarrow R$ satisfies $d_1(x_1, \ldots, x_m) = \sum_{i=1}^m x_i a_i$, we have $d_1^*(r) = (ra_1, \ldots, ra_m)$ under the natural isomorphisms $R^* \cong R$ and $(R^m)^* \cong R^m$. Then there exists a homomorphism $g : R^m \rightarrow R$ such that $gd_1^* = \text{Id}_R$. Thus $\sum_{i=1}^m g(e_i)a_i = 1$ where $\{e_1, \ldots, e_m\}$ is the standard basis for $R^m$. Consequently, we have $I = R$.

(4) $\Rightarrow$ (2): Let $\mathcal{T}$ be a tilting class induced by an $m$-tilting module $T$ and $(\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{m-1})$ the corresponding sequence of Gabriel topologies (see Theorem 2.2). If $m \leq n$, (2) holds obviously. Suppose $m > n$. Let $I$ be a finitely generated ideal in $\mathcal{G}_m$. We have $\text{Ext}_R^i(R/I, R) = 0$ for $i = 0, \ldots, n$, and then $I = R$ and thus $\mathcal{G}_{m-1} = \cdots = \mathcal{G}_n = \{R\}$. Consequently, the corresponding tilting class $\mathcal{T} = \bigcap_{i=0}^{m-1} \bigcap_{I \in \mathcal{G}_i} \text{KerTor}_i^R(R/I, -) = \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i} \text{KerTor}_i^R(R/I, -)$ is induced by $(\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1})$ and thus an $n$-tilting class by Theorem 2.2. Consequently, $T$ is $n$-tilting.

(4) $\Leftrightarrow$ (5): It follows from [15, Proposition 3.13].
Let $R$ be a Noetherian ring and $I$ a finitely generated proper ideal of $R$. Following [5], the common length of the maximal $R$-sequences in $I$, denoted by grade($I, R$), is said to be the grade of $I$ on $R$. A well-known result is that grade($I, R$) = min{$i \geq 0 | \text{Ext}^i_R(R/I, R) \neq 0$}. If $(R, m)$ is a Noetherian local ring, then grade($m, R$) is said to be the depth of $R$. Auslander and Buchsbaum [1] showed that the small finitistic dimension fPD($R$) of $R$ is equal to the depth of $R$. Now we extend this classical result to general Noetherian rings.

**Proposition 3.2.** Let $R$ be a Noetherian ring. Then fPD($R$) = sup{grade($m, R$) | $m \in \text{Max}(R)$}.

**Proof.** Suppose fPD($R$) is infinite. By Theorem 3.1 for any $n \geq 0$, there exists a proper ideal $I$ such that $\text{Ext}^i_R(R/I, R) = 0$ for each $i = 0, \ldots, n$. Let $m$ be the maximal ideal containing $I$. Then grade($m, R$) $\geq$ grade($I, R$) $\geq$ $n$. Thus sup{grade($m, R$) | $m \in \text{Max}(R)$} is infinite. Suppose fPD($R$) = $n$ for a non-negative integer $n$. Then, by Theorem 3.1 there exists a proper ideal $I$ of $R$ such that $\text{Ext}^i_R(R/I, R) = 0$ for each $i = 0, \ldots, n - 1$. Let $m$ be the maximal ideal containing $I$. Then we have grade($m, R$) $\geq$ grade($I, R$) $\geq$ $n$. If there is a maximal ideal $m'$ such that grade($m', R$) $>$ $n$, then $\text{Ext}^i_R(R/m', R) = 0$ for all $i = 0, \ldots, n$. Thus fPD($R$) $>$ $n$ by Theorem 3.1 which is a contradiction. \hfill $\square$

A Noetherian local ring $(R, m)$ is said to be a Cohen-Macaulay ring provided that grade($m, R$) = $K. \dim(R)$ where $K. \dim(R)$ denotes the Krull dimension of $R$. In general, a Noetherian ring $R$ is said to be a Cohen-Macaulay ring provided that $R_m$ is Cohen-Macaulay for each $m \in \text{Max}(R)$. For a Noetherian ring $R$, Raynaud and Gruson [19] proved that the big finitistic dimension FPD($R$) of $R$ coincides with the Krull dimension $K. \dim(R)$ of $R$. Thus by Proposition 3.2 we have the following result.

**Corollary 3.3.** Let $R$ be a Cohen-Macaulay ring. Then fPD($R$) = FPD($R$). In this case, fPD($R$) = FPD($R$) = $K. \dim(R)$.

If $R$ is a Noetherian local ring, the converse of Corollary 3.3 always holds. However, the converse is not always true for general Noetherian rings.

**Example 3.4.** Let $R = \mathbb{Z}[pX, X^2, X^3]$ where $p$ is a fixed prime. It is easy to check that $R$ is a Noetherian ring with $K. \dim(R) = 2$, and thus FPD($R$) = 2 (see [19 Theorem 3.2.6]). We denote $m_q = \langle q, pX, X^2, X^3 \rangle$ for each prime $q$, and then $m_q$ is a maximal ideal of $R$. One can show that grade($m_q, R$) = 1 if $q = p$, and grade($m_q, R$) = 2 if $q \neq p$. Thus fPD($R$) = 2 by Proposition 3.2. Note that
$K$. dim($R_{m_q}$) = 2 for each prime $q$. It follows that $R_{m_p}$ is not Cohen-Macaulay and thus $R$ is not a Cohen-Macaulay ring.

The following example shows that the small finitistic dimensions of Noetherian rings can be infinite.

**Example 3.5.** Let $R$ be the Nagata’s bad Noetherian ring given in [18, Appendix, Example 1]. Namely, let $D = k[x_1, \ldots, x_n, \ldots]$ be the polynomial ring with countably infinite variables over a field $k$. Let $p_i = \langle x_{2^{i-1}}, x_{2^{i-1}+1}, \ldots, x_{2^i-1} \rangle$ for $i \geq 1$ which is a prime ideal of $D$. Let $S$ be the multiplicative subset $S = \bigcap_{i \geq 1} (R - p_i)$.

Consider the ring $R = DS$. By [8, Section 02JC], $R$ is a Cohen-Macaulay ring with infinite Krull dimension. So $\text{fPD}(R) = \infty$ by Corollary 3.3.

Recall from [16], an ideal $I$ of $R$ is said to be dense if $\text{Ann}(I) := \{r \in R | Ir = 0\} = 0$, and semi-regular if it contains a finitely generated dense sub-ideal. Recall from [24], a ring $R$ is said to be a DQ ring provided that the only finitely generated semi-regular ideal of $R$ is $R$ itself. Note that a finitely generated ideal $I$ is semi-regular if and only if $\text{Hom}_R(R/I, R) = 0$. Thus a ring $R$ is DQ if and only if any finitely generated ideal $I$ satisfying $\text{Hom}_R(R/I, R) = 0$ is $R$. Thus the following result can be deduced from Theorem 3.1 in the case of $n = 0$.

**Corollary 3.6.** [25, Proposition 2.2] Let $R$ be a commutative ring. Then $\text{fPD}(R) = 0$ if and only if $R$ is a DQ ring.

Following from [21], a finitely generated ideal $J$ of $R$ is said to be a GV-ideal if and only if $\text{Hom}_R(R/J, R) = \text{Ext}^1_R(R/J, R) = 0$. The set of all GV-ideals of $R$ is denoted by $\text{GV}(R)$. A ring $R$ is said to be a DW ring provided that $\text{GV}(R) = \{R\}$. Examples of DW rings contain rings of Krull dimension equal to 0, integer domains of Krull dimension at most 1, Prüfer domains and so on. Wang et al. [25, Theorem 3.2.] showed that a commutative ring $R$ with $\text{fPD}(R) \leq 1$ is a DW ring. Obviously, Theorem 3.4 means that it is actually an equivalence.

**Corollary 3.7.** Let $R$ be a commutative ring. Then $\text{fPD}(R) \leq 1$ if and only if $R$ is a DW ring.

Recall that a commutative ring $R$ is said to be a strong Prüfer ring provided that every finitely generated semi-regular ideal is locally principal (see [16]). Recently, Wang et al. showed that the small finitistic dimensions of a connected strong Prüfer ring is at most 1 in [23, Theorem 2.4]. Now we extend this result to all strong Prüfer rings.
Corollary 3.8. Let $R$ be a strong Prüfer ring. Then $R$ is a DW ring. Consequently, $\text{fPD}(R) \leq 1$.

Proof. Let $J \in \text{GV}(R)$. Then $J$ is a finitely generated semi-regular ideal of $R$. Since $R$ is strong Prüfer, for any $p \in \text{Spec}(R)$, there exists a regular element $\frac{a}{b} \in J_p$ such that $J_p = \langle \frac{a}{b} \rangle$. Therefore $J_p$ is free over $R_p$ for any $p \in \text{Spec}(R)$. Thus $J$ is flat, we have $J = R$ (see [21, Theorem 6.7.24] and [21, Exercise 6.10(1)]).

Recall from [12] that a prime ideal $p$ of an integral domain $R$ is said to be strongly prime if whenever $xy \in p$, where $x, y \in K$ and $K$ is the quotient field of $R$, we have either $x \in p$ or $y \in p$. An integral domain $R$ is called a pseudo-valuation domain (PVD for short) if every prime ideal of $R$ is strongly prime. Wang et al. [22] computed the finitistic weak dimensions of PVDs using pullbacks of commutative rings. Recall from [9] that an integral domain $R$ is called a locally pseudo-valuation domain (LPVD for short) if $R_m$ is a PVD for each maximal ideal $m$ of $R$. Certainly, PVDs are LPVDs. Recently, Xie [26] pointed out that LPVDs are DW rings (see also [17, Theorem 2.9], [21, Theorem 11.8.5] and [27, Lemma 3.1.3]). Thus we have the following result.

Corollary 3.9. Let $R$ be an LPVD, then $\text{fPD}(R) \leq 1$.

Recall from [11] that a commutative ring $R$ is said to be a Prüfer ring provided that every finitely generated regular ideal is invertible. Obviously, every total ring of quotients (i.e. any regular element is invertible) is Prüfer. In [6, Problem 1], Cahen et al. posed the following two open questions:

- **Problem 1a:** Let $R$ be a Prüfer ring. Is $\text{fPD}(R) \leq 1$?
- **Problem 1b:** Let $R$ be a total ring of quotients. Is $\text{fPD}(R) = 0$?

Very recently, Wang et al. [24, 25] obtained a total ring of quotients $R$ but not a DW ring, and thus $\text{fPD}(R) > 1$ getting a negative answer to these two open questions. The next example shows that, for any $n \in \mathbb{N}$, there exists a total ring of quotients $R$ satisfying $\text{fPD}(R) = n$.

Example 3.10. Let $D = k[x_1, \ldots, x_n]$ be a polynomial rings with $n$ variables over a field $k$. Set $m = \langle x_1, \ldots, x_n \rangle$ be a maximal ideal of $D$ and $P = \text{Max}(R) - \{m\}$. Define $R = D(+)B$ to be the idealization constructed prior to [16, Theorem 11], where $B = \bigoplus_{p \in P} D/p$. Then $R$ is a total ring of quotients by [16, Theorem 11(a)]. By [16, Theorem 11(c)], the set of all semi-regular ideals of $R$ is $\{J(+)B \mid J$ is an ideal of $D$ and $\sqrt{J} = m\}$. Let $J$ be an ideal of $D$ satisfying $\sqrt{J} = m$ and $J = J(+)B$. Then, for any $i \geq 0$, $\text{Ext}^i_R(R/I, R) \cong \text{Ext}^i_R(D/J, R) \cong \text{Ext}^i_R(D/J \otimes_D R, R)$ since
\( J + p = R \) for any \( p \in \mathcal{P} \). By localizing at all maximal ideals, it is easy to verify \( \text{Tor}_i^D(D/J, R) = 0 \) for any \( i \geq 1 \). By Chapter VI, Proposition 4.1.3, we have \( \text{Ext}_R^i(D/J \otimes_D R, R) \cong \text{Ext}_D^i(D/J, D(+)B) \) for any \( i \geq 1 \). By localizing at all maximal ideals, one can also verify \( \text{Ext}_D^i(D/J, B) = 0 \) for any \( i \geq 0 \) (note that Hom\(_R(R/I, R) \cong \text{Hom}_D(D/J, D) = 0 \)). Thus we have \( \text{Ext}_R^i(R/I, R) \cong \text{Ext}_D^i(D/J, D) \) for any \( i \geq 0 \). By Theorem [5 and Proposition 1.2.10], we have \( \text{fPD}(R) = \text{grade}(J, D) = \text{grade}(m, D) = K \cdot \text{dim}(D_m) = n \) since \( D_m \) is a regular local ring.

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