THE DIRICHLET-TO-NEUMANN OPERATOR FOR FUNCTIONS OF LEAST GRADIENT AND ASSOCIATED DIFFUSION PROBLEMS

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ABSTRACT. Our aim is to study the Dirichlet-to-Neumann operator associated with the 1-Laplacian operator and to study the diffusion problem involving this operator. As an application we obtain well-posedness and long-time stability of solutions of a singular coupled elliptic-parabolic initial boundary-value problem.

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1. INTRODUCTION

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d, d \geq 2 \), with a Lipschitz boundary \( \partial \Omega \). Then (cf. [34]) for every \( u \in L^1(\partial \Omega) \), there is at least one weak solution to the singular Dirichlet problem

\[
\begin{aligned}
- \text{div} \left( \frac{\partial \hat{u}}{|D\hat{u}|} \right) &= 0 &\text{in } \Omega, \\
\hat{u} &= u &\text{on } \partial \Omega.
\end{aligned}
\]

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Our goal in this paper is to present new insights on the operator \( \Lambda \) assigning Dirichlet data \( u \) on \( \partial \Omega \) to the co-normal derivative \( \frac{D\hat{u}}{|D\hat{u}|} \cdot v \) on \( \partial \Omega \) of an extension \( \hat{u} \) of \( u \) which is a weak solution \( \hat{u} \) of Dirichlet problem (1.1), and to study diffusion problems involving this operator. Here, \( v \) denotes the outward pointing unit normal vector on \( \partial \Omega \). We emphasize that for general boundary data \( u \) on \( \partial \Omega \), problem (1.1) might have infinitely many solutions \( \hat{u} \) extending \( u \) on \( \Omega \) (see Remark 3.5 below). Thus the operator

\[
\Lambda : u|_{\partial \Omega} \mapsto \frac{D\hat{u}}{|D\hat{u}|} \cdot v|_{\partial \Omega} \quad \text{(for a weak solution \( \hat{u} \) of (1.1))}
\]

might be multi-valued. Due to its construction, \( \Lambda \) is called the Dirichlet-to-Neumann operator (D-t-N operator) associated with the 1-Laplace operator

\[
\Delta_1 \hat{u} := \text{div} \left( \frac{D\hat{u}}{|D\hat{u}|} \right)
\]

or, equivalently, the D-t-N operator for functions of least gradient. Our first results reads as follows.

**Theorem 1.1.** The D-t-N operator \( \Lambda \) associated with the 1-Laplace operator \( \Delta_1 \) is \( m \)-completely accretive in \( L^1(\partial \Omega) \) with dense domain. In addition, the D-t-N operator \( \Lambda \) can be realized as the sub-differential operator in \( L^2(\partial \Omega) \) of a proper, convex and lower semicontinuous function \( \varphi_{L^2} : L^2(\partial \Omega) \to (-\infty, +\infty] \) and thereby, \( \Lambda \) is maximal monotone.

The property that an operator is \( m \)-accretive is sufficient for the well-posedness of the (in the sense of mild solutions) of Cauchy problem (1.5) (below). For more details in this direction, we refere the interested reader to Section 2.2. The statement that \( \Lambda \) is maximal monotone in \( L^2(\partial \Omega) \) means that the Cauchy problem (1.5) associated with the D-t-N operator \( \Lambda \) enjoys a regularizing effect; for initial data \( u_0 \) in \( L^2(\partial \Omega) \) and forcing term \( g \in L^2(0, T; L^2(\partial \Omega)) \), the mild solution \( u \) of (1.5) is strong (cf [17]).

The link between Dirichlet problem (1.1) and functions of least gradient was established by Rossi, Segura and the second author in [34] where they showed that for given \( u \in L^1(\partial \Omega) \), every solution \( \hat{u} \) of the constrained least gradient problem

\[
\min \left\{ \int_{\Omega} |D\hat{v}| \bigg| \hat{v} \in BV(\Omega), \right. \left. \text{tr}(\hat{v}) = u \text{ on } \partial \Omega \right\}
\]

satisfies the inclusion of the first variation

\[
0 \in \partial \Phi_u(\hat{u}) \quad \text{in } L^{d.\tau}(\Omega) \times L^d(\Omega)
\]

of the energy functional

\[
\Phi_u : L^{d.\tau}(\Omega) \to (-\infty, +\infty]
\]

given by

\[
\Phi_u(v) = \begin{cases} 
\int_{\Omega} |D\hat{v}| + \int_{\partial \Omega} |\hat{v} - u| \, d\mathcal{H}^{d-1} & \text{if } v \in BV(\Omega), \\
+\infty & \text{if } \hat{v} \in L^{d.\tau}(\Omega) \setminus BV(\Omega).
\end{cases}
\]
Existence of solutions to the minimizing problem (1.2) was obtained by Parks [40, 41] under the hypotheses Ω is strictly convex and the boundary data u satisfies the bounded slope condition. Sternberg, Williams and Ziemer [47] improved this result by establishing existence and uniqueness of a minimizer \( \hat{u} \in BV(\Omega) \cap C(\overline{\Omega}) \) for boundary data \( u \in C(\partial \Omega) \) on bounded domains \( \Omega \) with a Lipschitz boundary \( \partial \Omega \) of non-negative mean curvature (in the weak sense) and not being locally area-minimizing. Existence and properties of least gradient functions were studied also by many other authors, including Miranda [38], Parks and Ziemer [42], Bombieri, De Giorgi, Giusti [15], or more recently, Jerrard, Moradifam, and Nachman [32], Górný [28, 29, 26], and Rybka and Sabra [29]. The characterization of the sub-differential \( \partial\Phi_u \) led to the notion of weak solutions to Dirichlet problem (1.1) (see Definition 3.2 in Section 3) and a better understanding of the phenomenon of non-uniqueness in problem (1.1). In fact, examples were given in [34] and [27] demonstrating well that even though the domain \( \Omega \) has nice geometric properties (as, for instance, convexity or \( \partial \Omega \) satisfies a uniform exterior cone condition), for discontinuous boundary data \( u \in L^\infty(\partial \Omega) \) the Dirichlet problem (1.1) may have infinitely many solutions \( \hat{u} \). This justifies the notation of differential inclusion used in (1.3) and makes the D-t-N operator \( \Lambda \) more appealing.

The 1-Laplace operator \( \Delta_1 \) is not only interesting from his geometric perspectives and its applications to engineering sciences, but also by his mathematical challenges. For a given \( \hat{u} \in BV(\Omega), \Delta_1 \hat{u} \) is the scalar mean curvature of the level sets of \( \hat{u} \). Thus, every level surface \( \{ \hat{u} = t \} \) of a function \( \hat{u} \) of least gradient has mean curvature zero; a necessary condition for functions \( \hat{u} \) whose superlevel sets \( \{ \hat{u} \geq t \} \) are area-minimizing. Functions of least gradient do not have too much regularity, in the sense, that even though \( \hat{u} \) might be essentially bounded, necessarily, \( \hat{u} \) need not admit a continuous representative on \( \overline{\Omega} \). In fact, in some applications, this property of functions of least gradient is strongly desired, for example, in image processing (see [6] and the references therein); if the nonlinear diffusion process associated with \( \Delta_1 \) is used to deblur a given picture \( u_0 : \Omega \to [0, 1], (\Omega \subseteq \mathbb{R}^2) \), then the contours in \( u_0 \) are maintained and not smoothen as compared to diffusion processes involving linear or degenerate differential operators. But the operator \( \Delta_1 \) also appears in other engineering fields. For example in free material design (see [29]), or conductivity imaging (see [32]).

If \( \Omega \) represents, for example, an electricity conducting medium, then the operator \( \Lambda \) associated with the classical Laplace operator \( \Delta \hat{u} := \sum_{i=1}^d D_i \hat{u} \) appears in a natural way in measuring the current through the boundary for given voltages on the boundary. Thus the operator \( \Lambda \) is the main object in Calderón’s inverse problem [19]. The D-t-N operator \( \Lambda \) can be constructed with various kind of differential operators (linear, nonlinear, singular, or degenerate) provided the corresponding Dirichlet problem admits a solution; for \( 1 < p < \infty \), the D-t-N operator \( \Lambda \) associated with the p-Laplace operator \( \Delta_p \hat{u} := \text{div} (|D\hat{u}|^{p-2}D\hat{u}) \) is also referred to as the interior capacity operator (cf [22]) and was studied intensively by many authors including by Díaz and Jiménez [23], Ammar, Andreu and Toledo [2], Arendt
and Ter Elst [9], Salo and Zhong [46], Brander [16], the first author [30], and with co-authors [20, 21, 8].

As an application of our main result (Theorem 1.1), we obtain well-posedness and long-time stability of solutions to the singular coupled elliptic-parabolic initial boundary-value problem

\begin{equation}
\tag{1.4}
\begin{aligned}
- \text{div} \left( \frac{\partial u}{\partial t} \right) &= 0 & \text{in } \Omega \times (0, T), \\
\partial_t u + \frac{\partial u}{\partial t} \cdot v + f(\cdot, u) &\ni g(t) & \text{on } \partial \Omega \times (0, T), \\
u & = u_0 & \text{on } \partial \Omega \times \{t = 0\}.
\end{aligned}
\end{equation}

Here, \( f : \partial \Omega \times \mathbb{R} \to \mathbb{R} \) is a Lipschitz-Carathéodory function satisfying \( f(x, 0) = 0 \), that is, for a.e. \( x \in \Omega \), \( f(x, \cdot) : \mathbb{R} \to \mathbb{R} \) be Lipschitz continuous (with constant \( \omega > 0 \)) uniformly for a.e. \( x \in \partial \Omega \), and \( f(\cdot, u) : \partial \Omega \to \mathbb{R} \) is measurable on \( \partial \Omega \) for every \( u \in \partial \Omega \), \( g \) is a force in \( L^q(0, T; L^r(\partial \Omega)) \) for appropriate \( 1 \leq q, r \leq \infty \), and \( u_0 \in L^q(\partial \Omega) \).

By using the D-t-N operator \( \Lambda \) associated with least gradient functions realized in \( L^q(\Omega) \) and if \( F \) is the Nemytskii operator of \( f \), then the elliptic-parabolic initial boundary-value problem (1.4) can be rewritten as the parabolic initial problem

\begin{equation}
\tag{1.5}
\begin{aligned}
\partial_t u(t) + \Lambda u(t) + F(u(t)) &\ni g(t) & \text{on } \partial \Omega \times (0, T), \\
u(0) & = u_0 & \text{on } \partial \Omega.
\end{aligned}
\end{equation}

As a consequence of Theorem 1.1, we obtain the following well-posedness result and comparison principles.

**Corollary 1.2.** For every \( u_0 \in L^q(\partial \Omega) \) and \( g \in L^1(0, T; L^q(\partial \Omega)) \), \( 1 \leq q < \infty \), there is a unique mild solution of (1.5) in \( L^q(\partial \Omega) \). Moreover, if \( u \) and \( v \) are two mild solutions of (1.5) with initial datum \( u_0 \) and \( v_0 \in L^1(\partial \Omega) \) and \( g_1, g_2 \in L^1(0, T; L^q(\partial \Omega)) \), then

\begin{equation}
\tag{1.6}
\| (u(t) - v(t))^+ \|_q \leq \| (u(s) - v(s))^+ \|_q + \int_s^t \| g_1(r) - g_2(r) \|_q \, dr
\end{equation}

for every \( 0 \leq s < t \leq T \), and (1.6), in particular, holds for \( q = \infty \) if \( u_0 \in L^1(\partial \Omega) \cap L^\infty(\partial \Omega) \).

To prove that the mild solution, obtained in Corollary 1.2, is a strong solution we use the regularizing effect due to the homogeneity of the \( m \)-completely accretive operators we have obtained recently in [31].

In the case \( g \equiv 0 \) and \( F \equiv 0 \), we have the following regularity and decay estimates.

**Theorem 1.3.** For every initial datum \( u_0 \in L^1(\partial \Omega) \) there is a unique strong solution of the Cauchy problem

\begin{equation}
\tag{1.7}
\begin{aligned}
u'(t) + \Lambda u(t) &\ni 0 & t \geq 0 \\
u(0) & = u_0.
\end{aligned}
\end{equation}
Moreover, if \( \{T_t\}_{t \geq 0} \) is the semigroup generated by \( \Lambda \) in \( L^1(\partial \Omega) \), that is, if for any \( u_0 \in L^1(\partial \Omega) \), \( u(t) = T(t)u_0 \) is the unique strong solution of the Cauchy problem (1.7), the we have the estimates:

\[
|\Lambda^\ast T_t u_0| \leq 2e^{\omega t} \frac{|u_0|}{t}, \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial \Omega, \quad \text{for every } \omega > 0
\]

and

\[
\frac{d T_t u_0}{dt} \leq \frac{T_t u_0}{t}, \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial \Omega, \quad \text{for every } t > 0 \text{ if } u_0 \geq 0.
\]

The paper is organized as follows. In Section 3 and Section 4 we briefly review existence and uniqueness of Dirichlet problem (1.1) and the Neumann problem for the operator \( \Delta_1 \). Section 5 is dedicated to proof our first main result (Theorem 1.1). Subsection 5.2 deals with asymptotic behaviour of the solution and Subsection 5.3 with the application of the theory of \( j \)-elliptic functionals to show that the D-t-N operator is a maximal monotone operator in \( L^2(\partial \Omega) \). In Section 6 we apply the result of Section 5 to get the the well-posedness of the elliptic-parabolic boundary value problem (1.5). Finally in Appendix A we developed the generalisation of the theory of \( \tau_{w_\ast}j \)-elliptic functionals.

2. Preliminaries.

We begin by summarizing some fundamental notions, definitions, and results which we will apply later in this paper.

2.1. Functions of bounded variation. We begin by recalling some fundamental facts about functions of bounded variation. For more details on this topic, we refer the interested reader to [1], or [49].

Let \( \Omega \) an open subset of \( \mathbb{R}^d \), \( d \geq 1 \). Then, a function \( u \in L^1(\Omega) \) is said to be a function of bounded variation in \( \Omega \), if the distributional partial derivatives \( D_1 u := \frac{\partial u}{\partial x_1}, \ldots, D_d u := \frac{\partial u}{\partial x_d} \) are finite Radon measures in \( \Omega \), that is, if

\[
\int_\Omega u D_i \varphi \, dx = -\int_\Omega \varphi \, dD_i u
\]

for all \( \varphi \in C^\infty_0(\Omega), i = 1, \ldots, d \). The linear vector space of functions \( u \in L^1(\Omega) \) of bounded variation in \( \Omega \) is denoted by \( BV(\Omega) \). Further, we set \( Du = (D_1 u, \ldots, D_d u) \) for the distributional gradient of \( u \). Then, \( Du \) belongs to the class \( M^b(\Omega, \mathbb{R}^d) \) of \( \mathbb{R}^d \)-valued bounded Radon measure on \( \Omega \), and throughout this paper, we either write \( |Du|_1(\Omega) \) or \( \|Du\|_1(\Omega) \) to denote the total variation measure of \( Du \). The space \( BV(\Omega) \) equipped with the norm

\[
||u||_{BV(\Omega)} := ||u||_{L^1(\Omega)} + ||Du||_1(\Omega),
\]

forms a Banach space. For \( u \in L^1_{loc}(\Omega) \), the variation of \( u \) in \( \Omega \) is defined by (2.1)

\[
V(u, \Omega) := \sup \left\{ \int_\Omega u \text{ div } z \, dx \mid z \in C^\infty_0(\Omega, \mathbb{R}^d), |z(x)| \leq 1 \text{ for } x \in \Omega \right\}
\]

and if \( u \) is continuously differentiable, then an integration by parts shows that \( V(u, \Omega) = \int_\Omega |\nabla u| \, dx \). The variation \( V(\cdot, \Omega) \) is directly related to \( BV(\Omega) \) via the property (cf [1, Proposition 3.6]), that a function \( u \in L^1(\Omega) \)
belongs to $BV(\Omega)$ if and only if $V(u, \Omega)$ is finite. In addition, it is worth noting that $V(u, \Omega) = |Du|(\Omega)$ for $u \in BV(\Omega)$ and $u \mapsto V(u, \Omega)$ is lower semicontinuous with respect to the $L^1_{\text{loc}}(\Omega)$-topology.

By Riesz’s theorem (cf [45, Theorem 6.19]), the dual space $(C_0(\Omega))^*$ is isometrically isomorphic with the space $M^b(\Omega)$ of bounded Radon-measures. Thus, for a sequence $(\mu_n)_n$ and $\mu$ in $M^b(\Omega)$, $(\mu_n)_n$ is said to be weakly*-convergent to $\mu$ in $M^b(\Omega)$ if
\[
\int_\Omega \xi d\mu_n \to \int_\Omega \xi d\mu \quad \text{for every } \xi \in C_0(\Omega).
\]

Following this definition, one calls a sequence $(u_n)_{n \geq 1}$ in $BV(\Omega)$ to be weakly*-convergent to $u$ in $BV(\Omega)$ if $u_n \to u$ in $L^1(\Omega)$ as $n \to +\infty$ and $Du_n$ weakly*-converges to $Du$ in $M^b(\Omega; \mathbb{R}^d)$ as $n \to +\infty$. By [1, Proposition 3.13], we have that $(u_n)_{n \geq 1}$ in $BV(\Omega)$ weakly*-convergent to $u$ if and only if $(u_n)_{n \geq 1}$ is bounded in $BV(\Omega)$ and converges to $u$ in $L^1(\Omega)$.

**Definition 2.1.** Let $u_n, u \in BV(\Omega)$. We say that $(u_n)$ strictly converges to $u$ in $BV$ if $(u_n)$ converges to $u$ in $L^1(\Omega)$ and $|Du_n|(\Omega) \to |Du|(\Omega)$.

Further, according to [24, Theorem 5.3.1] and [1, Theorem 3.87], if $\Omega$ is an open and bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial \Omega$, then there is a bounded linear mapping $\text{tr} : BV(\Omega) \to L^1(\partial \Omega)$ assigning to each $u \in BV(\Omega)$ an element $\text{tr}(u) \in L^1(\partial \Omega)$ such that for $\mathcal{H}^{d-1}$-almost every $x \in \partial \Omega$, one has that $\text{tr}(u)(x) \in \mathbb{R}$ and
\[
\lim_{\rho \downarrow 0} \rho^{-N} \int_{\partial \Omega \cap B_\rho(x)} |u(y) - \text{tr}(u)(x)| dy = 0.
\]

Moreover, for every $u \in BV(\Omega)$,
\[
(2.2) \quad \int_\Omega u \div \xi \, dx = -\int_\Omega \xi \cdot D u + \int_{\partial \Omega} (\xi \cdot v) \text{tr}(u) \, d\mathcal{H}^{d-1}
\]
for all $\xi \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, where $v$ denotes the outer unit normal vector on $\partial \Omega$. We call $\text{tr}(u)$ the trace of $u$ and call $\text{tr}$ the trace operator on $BV(\Omega)$. Note, if there is no danger of confusion, we sometimes also write simply $u$. In particular, we have the following useful result.

**Proposition 2.2** ([1, Theorem 3.88]). Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial \Omega$. Then, the trace operator $\text{tr} : BV(\Omega) \to L^1(\partial \Omega)$ is continuous from $BV(\Omega)$ equipped with the strict topology to $L^1(\partial \Omega)$.

Moreover, there exists a constant $C > 0$ such that
\[
(2.3) \quad \|\text{tr}(u)\|_1 \leq |Du|(\Omega) \quad \forall u \in BV(\Omega).
\]

Next, we recall the following embedding theorems as stated in [36, Theorem 6.5.7/1, Theorem 9.5.7] and [44].

**Theorem 2.3.** Suppose that $\Omega \subset \mathbb{R}^d$ is an open bounded set with Lipschitz boundary. Then for every function $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, there is a constant $C_{p,d} > 0$ such that
\[
(2.4) \quad \|u\|_{L^p(\Omega)} \leq C_{p,d} \left[ \|\nabla u\|_p(\Omega) + \|u\|_{L^p(\partial \Omega)} \right].
\]
Moreover, if $u \in BV(\Omega)$ then
\begin{equation}
\|u\|_{L^\infty_T(\Omega)} \leq C_d \left[ \|Du\|_1(\Omega) + \|u\|_{L^1(\partial \Omega)} \right].
\end{equation}

2.1.1. A generalized Green’s formula. In this subsection, we recall several results from [7] (see also cf [6]). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary $\partial \Omega$.

For $1 \leq p \leq d$ and $d/(d-1) \leq p' \leq \infty$ satisfying $1 = \frac{1}{p} + \frac{1}{p'}$, we introduce the following spaces
\begin{align*}
X_p(\Omega) &:= \left\{ z \in L^\infty(\Omega, \mathbb{R}^d) : \text{div}(z) \in L^p(\Omega) \right\}, \text{ and} \\
BV(\Omega)_{p'} &:= BV(\Omega) \cap L^{p'}(\Omega).
\end{align*}

Then, motivated by the integral
\begin{equation}
\int_{\Omega} z \cdot \nabla w \, dx \quad \text{for } w \in C^1(\Omega) \text{ and } z \in L^\infty(\Omega; \mathbb{R}^d),
\end{equation}

one can define a bilinear mapping $(\cdot, D \cdot) : X_p(\Omega) \times BV(\Omega)_{p'} \to M^p(\Omega)$ by
\begin{equation}
\langle (z, Dw), \varphi \rangle = -\int_{\Omega} w \varphi \text{div}(z) \, dx - \int_{\Omega} w \cdot \nabla \varphi \, dx
\end{equation}
for all $\varphi \in C^\infty_0(\Omega)$, $z \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$. The, for given $z \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$, the linear functional $(z, Dw) : C^\infty_0(\Omega) \to \mathbb{R}$ is a signed Radon measure on $\Omega$ with total variation measure $\|z, Dw\|$ and provides a generalization of (2.6). More precisely, for given $z \in X_p(\Omega)$, one has that
\begin{equation}
\int_{\Omega} (z, Dw) = \int_{\Omega} z \cdot \nabla w \, dx \quad \text{for every } w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)
\end{equation}
and
\begin{equation}
\left| \int_{\mathcal{B}} (z, Dw) \right| \leq \int_{\mathcal{B}} |(z, Dw)| \leq \|z\|_{\infty} \int_{\mathcal{B}} |Dw|
\end{equation}
for every Borel set $\mathcal{B} \subseteq \Omega$. Thus, $(z, Dw)$ is absolutely continuous with respect to the total variation $\mu := |Dw|$ and so, there is a function $\theta(z, Dw, \cdot) \in L^1(\Omega, \mu)$ satisfying
\begin{equation}
\theta(z, Dw, \cdot) = \frac{d(z, Dw)}{d|Dw|} \quad \text{and} \quad |\theta(z, Dw, x)| = 1 \text{ for } |Dw|-a.e. \ x \in \Omega.
\end{equation}
The function $\theta(z, Dw, \cdot)$ is called the Radon–Nikodým derivative of $(z, Dw)$ with respect to $|Dw|$. Moreover, the following results holds.

**Proposition 2.4** ([7], Chain Rule). For $1 \leq p \leq N$ and $N/(N-1) \leq p' \leq \infty$ satisfying $1 = \frac{1}{p} + \frac{1}{p'}$, let $z \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$. Then, for every Lipschitz continuous, monotonically increasing function $q : \mathbb{R} \to \mathbb{R}$, one has that
\begin{equation}
\theta(z, D(q \circ w), x) = \theta(z, Dw, x) \quad \text{for } |Dw|-a.e. \ x \in \Omega.
\end{equation}

Further, there is a unique linear extension $\gamma : X_p(\Omega) \to L^\infty(\partial \Omega)$ satisfying $\|\gamma(z)\|_{\infty} \leq \|z\|_{\infty}$ and
\begin{equation}
\gamma(z)(x) = z(x) \cdot v(x) \quad \text{for every } x \in \partial \Omega \text{ and } z \in C^1(\overline{\Omega}, \mathbb{R}^d).
\end{equation}
Definition 2.5 ([7]). For every $z \in X_p(\Omega)$, we write $[z, v]$ for $\gamma(z)$ and call $[z, v]$ the weak trace of the normal component of $z$.

With these notions, we can now state the generalised Green formula for functions $w \in BV(\Omega)$.

**Proposition 2.6 ([7], Generalised Green Formula).** Let $1 \leq p \leq N$ and $N/(N-1) \leq p' \leq \infty$ satisfying $1 = \frac{1}{p} + \frac{1}{p'}$. Then

$$
\int_{\Omega} w \div(z) \, dx + \int_{\Omega} (z, Dw) = \int_{\partial\Omega} [z, v]w \, d\mathcal{H}^{d-1}.
$$

for every $z \in X_p(\Omega)$ and $w \in X_{p'}$.

2.2. Completely accretive operators. Here, we recall the notion of completely accretive operators introduced in [13] and further developed in [21].

We begin by introducing the framework of completely accretive operators. Let $(\Sigma, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space, and $M(\Sigma, \mu)$ the space of $\mu$-a.e. equivalent classes of measurable functions $u : \Sigma \to \mathbb{R}$. For $u \in M(\Sigma, \mu)$, we write $[u]^+ \sigma$-to denote $\max\{u, 0\}$ and $[u]^− = -\min\{u, 0\}$. We denote by $L^q(\Sigma, \mu), 1 \leq q \leq \infty$, the corresponding standard Lebesgue space with norm

$$
\|u\|_q = \begin{cases} 
\left(\int_\Sigma |u|^q \, d\mu\right)^{1/q} & \text{if } 1 \leq q < \infty, \\
\inf \left\{ k \in [0, +\infty] \mid |u| \leq k \text{ \(\mu\)-a.e. on } \Sigma \right\} & \text{if } q = \infty.
\end{cases}
$$

For $1 \leq q < \infty$, we identify the dual space $(L^q(\Sigma, \mu))^\prime$ with $L^{q'}(\Sigma, \mu)$, where $q'$ is the conjugate exponent of $q$ given by $1 = \frac{1}{q} + \frac{1}{q'}$.

Now, let

$$
J_0 := \left\{ j : \mathbb{R} \to [0, +\infty] \mid j \text{ is convex, lower semicontinuous, } j(0) = 0 \right\}.
$$

Then, for every $u, v \in M(\Sigma, \mu)$, we write

$$
uu u \ll v \text{ if and only if } \int_\Sigma j(u) \, d\mu \leq \int_\Sigma j(v) \, d\mu \text{ for all } j \in J_0.
$$

With these preliminaries in mind, we can now state the following definitions.

**Definition 2.7.** A mapping $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ is called a complete contraction if

$$
Su - S\hat{u} \ll u - \hat{u} \quad \text{for every } u, \hat{u} \in D(S).
$$

Now, we can state the definition of completely accretive operators.

**Definition 2.8.** An operator $A$ on $M(\Sigma, \mu)$ is called completely accretive if for every $\lambda > 0$, the resolvent operator $I_\lambda$ of $A$ is a complete contraction, or equivalently, if for every $(u_1, v_1), (u_2, v_2) \in A$ and $\lambda > 0$, one has that

$$
\|u_1 - u_2\| \ll \|u_1 - u_2 + \lambda(v_1 - v_2)\|.
$$
If \( X \) is a linear subspace of \( M(\Sigma, \mu) \) and \( A \) an operator on \( X \), then \( A \) is \( m \)-completely accretive on \( X \) if \( A \) is completely accretive and satisfies the range condition
\[
\operatorname{Rg}(I + \lambda A) = X \quad \text{for some (or equivalently, for all) } \lambda > 0.
\]
Further, for \( \omega \in \mathbb{R} \), an operator \( A \) on a linear subspace \( X \subseteq M(\Sigma, \mu) \) is called \( \omega \)-quasi \( (m) \)-completely accretive in \( X \) if \( A + \omega I \) is \( (m) \)-completely accretive in \( X \). Finally, an operator \( A \) on a linear subspace \( X \subseteq M(\Sigma, \mu) \) is called quasi \( m \)-completely accretive if there is some \( \omega \in \mathbb{R} \) such that \( A + \omega I \) is \( m \)-completely accretive in \( X \).

Before stating a useful characterization of completely accretive operators, we first need to introducing the following function spaces. Let \( L^{1+\infty}(\Sigma, \mu) := L^1(\Sigma, \mu) + L^\infty(\Sigma, \mu) \) and \( L^{1\cap\infty}(\Sigma, \mu) := L^1(\Sigma, \mu) \cap L^\infty(\Sigma, \mu) \) be the sum and the intersection space of \( L^1(\Sigma, \mu) \) and \( L^\infty(\Sigma, \mu) \), which respectively equipped with the norms
\[
\|u\|_{1+\infty} := \inf \left\{ \|u_1\|_1 + \|u_2\|_\infty : u = u_1 + u_2, \ u_1 \in L^1(\Sigma, \mu), \ u_2 \in L^\infty(\Sigma, \mu) \right\},
\]
\[
\|u\|_{1\cap\infty} := \max \left\{ \|u\|_1, \|u\|_\infty \right\}
\]
are Banach spaces. In fact, \( L^{1+\infty}(\Sigma, \mu) \) and \( L^{1\cap\infty}(\Sigma, \mu) \) are respectively the largest and the smallest of the rearrangement-invariant Banach function spaces (cf [14, Chapter 3.1]). If \( \mu(\Sigma) \) is finite, then \( L^{1+\infty}(\Sigma, \mu) = L^1(\Sigma, \mu) \) with equivalent norms, but if \( \mu(\Sigma) = \infty \) then \( L^{1+\infty}(\Sigma, \mu) \) contains \( \bigcup_{1 \leq q \leq \infty} L^q(\Sigma, \mu) \).

Further, we will employ the space
\[
L_0(\Sigma, \mu) := \left\{ u \in M(\Sigma, \mu) \mid \int_\Sigma |u|^{-k} \, d\mu < \infty \text{ forall } k > 0 \right\},
\]
which equipped with the \( L^{1+\infty} \)-norm is a closed subspace of \( L^{1+\infty}(\Sigma, \mu) \). In fact, one has (cf [13]) that \( L_0(\Sigma, \mu) = L^1(\Sigma, \mu) \cap L^\infty(\Sigma, \mu)^{+\infty} \). Since for every \( k \geq 0 \), \( T_\epsilon(s) := |s|^{-k} \) is a Lipschitz mapping \( T_\epsilon : \mathbb{R} \to \mathbb{R} \) and by Chebyshov’s inequality, one see that \( L^q(\Sigma, \mu) \hookrightarrow L_0(\Sigma, \mu) \) for every \( 1 \leq q < \infty \) (and \( q = \infty \) if \( \mu(\Sigma) < +\infty \).

**Proposition 2.9** ([13], [21]). Let \( P_0 \) denote the set of all functions \( T \in C^\infty(\mathbb{R}) \) satisfying \( 0 \leq T' \leq 1 \), \( T' \) is compactly supported, and \( x = 0 \) is not contained in the support \( \operatorname{supp}(T) \) of \( T \). Then for \( \omega \in \mathbb{R} \), an operator \( A \subseteq L_0(\Sigma, \mu) \times L_0(\Sigma, \mu) \) is \( \omega \)-quasi completely accretive if and only if
\[
\int_\Sigma T(u - \hat{u})(v - \hat{v}) \, d\mu + \omega \int_\Sigma T(u - \hat{u})(u - \hat{u}) \, d\mu \geq 0
\]
for every \( T \in P_0 \) and every \( (u, v), (\hat{u}, \hat{v}) \in A \).

**2.2.1. Completely accretive operators of homogeneous order zero.** Here, we are concerned with the following class of operators.

**Definition 2.10.** An operator \( A \) on a vector space \( X \) is said to be homogeneous of order zero if for every \( u \in D(A) \) and \( \lambda \geq 0 \), one has that \( \lambda u \in D(A) \) and
\[
A(\lambda u) = A(u) \quad \text{for all } \lambda \geq 0 \text{ and } u \in D(A).
\]
With this definition in mind, we can now state the regularization effect of the semigroup \( \{ T_t \}_{t \geq 0} \) generated by a \( \omega \)-quasi \( m \)-completely accretive operator of homogeneous order zero.

**Theorem 2.11** ([31, Theorem 4.13]). Let \( X \) be a normal Banach space with \( X \subseteq L_0(\Sigma, \mu) \), for \( \omega \in \mathbb{R} \), \( A \) be \( \omega \)-quasi \( m \)-completely accretive in \( X \), and \( \{ T_t \}_{t \geq 0} \) be the semigroup generated by \( -A \) on \( D(A) \). Suppose that \((0, 0) \in A \) and \( A \) is homogeneous of order zero. Then for every \( u_0 \in D(A) \) and \( t > 0 \), \( \frac{d}{dt} T_t u_0 \) exists in \( X \) and

\[
|A^0 T_t u_0| \leq 2 e^{\omega t} \frac{|u_0|}{t} \quad \mu\text{-a.e. on } \Sigma.
\]

In particular, one has that

\[
\left\| \frac{d}{dt} T_t u_0 \right\| \leq 2 e^{\omega t} \frac{\| u_0 \|}{t} \quad \text{for every } t > 0,
\]

and every \( u_0 \in D(A)^\times \) with \( \| \cdot \| \) being the norm on \( X \), respectively, for every \( u_0 \in D(A)^\times \cap L^{1,\infty}(\Sigma, \mu)^p \) and \( \| \cdot \| \) denoting the \( L^p \)-norm.

3. THE DIRICHLET PROBLEM ASSOCIATED WITH THE 1-LAPLACIAN

Here, we consider the Dirichlet problem for the 1–Laplacian:

\[
(1.1) \begin{cases} 
- \text{div} \left( \frac{Du}{|Du|} \right) = 0 & \text{in } \Omega, \\
 u = h & \text{on } \partial \Omega.
\end{cases}
\]

Note that in regions where \( u \) is smooth and \( Du \) does not vanish, \( \text{div} \left( \frac{Du}{|Du|} \right) \) is the scalar mean curvature of the level sets of \( u \). So this PDE asserts that each level surface of \( u \) has mean curvature zero. From now on we will denote

\[
\Delta_1 u := \text{div} \left( \frac{Du}{|Du|} \right).
\]

It is well known (cf [25] and [3]) that for given \( h \in L^1(\partial \Omega) \), the relaxed energy functional associated with Dirichlet problem \((1.1)\) is the functional

\[
\Phi_h : L^{d-1}(\Omega) \to (-\infty, +\infty]
\]

given by

\[
\Phi_h(u) = \begin{cases} 
\int_\Omega |Du| + \int_{\partial \Omega} |u - h| \, d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega), \\
+\infty & \text{if } u \in L^{\frac{d}{d-1}}(\Omega) \setminus BV(\Omega).
\end{cases}
\]

The functional \( \Phi_h \) is convex, lower semicontinuous on \( L^{d-1}(\Omega) \), and thanks to the Sobolev inequality \((2.5)\), \( \Phi_h \) is coercive. Thus, there is an \( u \in BV(\Omega) \) solving the variational problem

\[
(3.1) \min_{w \in BV(\Omega)} \Phi_h(w) = \Phi_h(u).
\]

Recall that \((3.1)\) is equivalent to

\[
(3.2) \quad 0 \in \partial L \Phi_h(u),
\]
where
\[ \partial_{L^d} \Phi_h := \begin{cases} (u, f) \in L^{\frac{d}{d-1}} \times L^d(\Omega) & \text{if } u \in BV(\Omega) \text{ and for every } v \in L^{\frac{d}{d-1}}(\Omega), \\
\Phi_h(v) - \Phi_h(u) \geq \int_\Omega f(v - u) \, dx \end{cases} \]
is the sub-differential operator of \( \Phi_h \). Furthermore, the following important characterisation of (3.2) is known.

**Proposition 3.1** ([34, Theorem 2.5]). For \( h \in L^1(\partial \Omega) \) and \( u \in BV(\Omega) \), the following statements are equivalent:

(i) \( 0 \in \partial \Phi_h(u) \).

(ii) there exists a vector field \( z \in L^\infty(\Omega; \mathbb{R}^d) \) satisfying

\[
\|z\|_\infty \leq 1, \quad -\text{div}(z) = 0 \quad \text{in } D'(\Omega),
\]

\[
(z, Du) = |Du| \quad \text{as measures},
\]

\[
[z, v] \in \text{sign}(h - u) \quad \mathcal{H}^{d-1}-\text{a.e. on } \partial \Omega.
\]

Thanks to this characterization, the following notion of solutions of solution of Dirichlet problem (1.1) has sense (cf [34, Definition 2.10] and [34, Definition 2.3]).

**Definition 3.2.** For \( h \in L^1(\partial \Omega) \), a function \( u \in BV(\Omega) \) is called a weak solution to Dirichlet problem (1.1) if there is a vector field \( z \in L^\infty(\Omega; \mathbb{R}^d) \) satisfying (3.3-3.6).

**Remark 3.3.** The proof of [34, Theorem 2.4] provides a non-variational method to the existence of solution of Dirichlet problem (1.1) for given \( h \in L^1(\partial \Omega) \).

By following the method of [3, Proposition 2], one obtains the following characterisation of the notion of solutions of Dirichlet problem (1.1).

**Proposition 3.4.** For given \( h \in L^1(\partial \Omega) \) and \( u \in BV(\Omega) \) the following statements are equivalent:

1. \( u \) is a weak solution of Dirichlet problem (1.1).
2. there exists a vector field \( z \in L^\infty(\Omega; \mathbb{R}^d) \) satisfying

\[
\|z\|_\infty \leq 1, \quad -\text{div}(z) = 0 \quad \text{in } D'(\Omega),
\]

and for all \( w \in BV(\Omega),
\[
\int_\Omega |Du| + \int_{\partial \Omega} |u - h|d\mathcal{H}^{d-1} \leq \int_\Omega (z, Dw) - \int_{\partial \Omega} [z, v](w - h)d\mathcal{H}^{d-1}
\]

3. there exists a vector field \( z \in L^\infty(\Omega; \mathbb{R}^d) \) satisfying \( \|z\|_\infty \leq 1, \)

\[
-\text{div}(z) = 0 \quad \text{in } D'(\Omega),
\]

and for all \( w \in BV(\Omega),
\[
0 \leq \int_\Omega (z, Dw) - \int_\Omega |Du| + \int_{\partial \Omega} [w - h] - |u - h|d\mathcal{H}.
\]
Remark 3.5. Sternberg, Williams and Ziemer in [47] (see also [48]) established uniqueness of solutions of Dirichlet problem (1.1) under several geometrical assumptions on \(\partial \Omega\) and provided \(h \in C(\partial \Omega)\). In addition, then the unique solution \(u\) of (1.1) belongs to \(C(\overline{\Omega})\) and satisfies \(u = h\) pointwise on \(\partial \Omega\). On the other hand, non-uniqueness was shown in [34] if the boundary condition \(h\) admits discontinuous boundary values. More precisely, the counter example is as follows. Let \(\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}\) and

\[ f(\cos \theta, \sin \theta) := \begin{cases} 
\cos(2\theta) + 1, & \text{if } \cos(2\theta) > 0; \\
\cos(2\theta) - 1, & \text{if } \cos(2\theta) < 0;
\end{cases} \]

for \((\cos \theta, \sin \theta) \in \partial \Omega\). For \(-1 \leq \lambda \leq 1\), define the function \(u^\lambda\) by

\[ u^\lambda(x, y) = \begin{cases} 
2x^2, & \text{if } |x| > \frac{\sqrt{2}}{2}, |y| < \frac{\sqrt{2}}{2}; \\
\lambda, & \text{if } |x| < \frac{\sqrt{2}}{2}, |y| < \frac{\sqrt{2}}{2}; \\
-2y^2, & \text{if } |x| < \frac{\sqrt{2}}{2}, |y| > \frac{\sqrt{2}}{2}.
\end{cases} \]

In [34], it was shown that the functions \(u^\lambda\) are functions of least gradient for every \(-1 \leq \lambda \leq 1\).

We finish this part with the following interesting observation.

Proposition 3.6. For given \(h \in L^1(\partial \Omega)\), let \(u\) be a solution of Dirichlet problem (1.1) and \(z\) be a vector field satisfying (3.4–3.6) with respect to \(u\). If \(\hat{u}\) is another solution of (1.1) and \(\hat{z}\) a vector field satisfy (3.4–3.6) with respect to \(\hat{u}\), then \(\hat{z}\) also satisfies (3.4–3.6) with respect to \(u\).

Proof. Let \(u\) and \(\hat{u}\) be two solutions of Dirichlet problem (1.1) for the same given boundary function \(h \in L^1(\partial \Omega)\). Then, there exist two bounded vector fields \(z\) and \(\hat{z}\) satisfying (3.4–3.6). Multiplying equation (3.4) by \((u - \hat{u})\) and applying the generalized Green’s formula (2.10), one finds

\[ \int\limits_{\Omega} (z, D(u - \hat{u})) - \int\limits_{\partial \Omega} [z, v](u - \hat{u}) \, d\mathcal{H}^{N-1} = 0. \]

Similarly, multiplying (3.4) with \(z = \hat{z}\) by \((u - \hat{u})\), gives

\[ \int\limits_{\Omega} (\hat{z}, D(u - \hat{u})) - \int\limits_{\partial \Omega} [\hat{z}, v](u - \hat{u}) \, d\mathcal{H}^{N-1} = 0. \]

Subtracting these two equations from each other and using the fact that the pairing \((z, Dw)\) is bilinear in \(z \in X_p(\Omega)\) and \(w \in BV(\Omega)_{pr}\) yields

\[ \int\limits_{\Omega} (z - \hat{z}, D(u - \hat{u})) \, d\mathcal{H}^{N-1} + \int\limits_{\partial \Omega} ([z, v] - [\hat{z}, v]) (h - u - (h - \hat{u})) \, d\mathcal{H}^{d-1} = 0. \]

By (3.5), (2.8), since \(\|z\|_\infty \leq 1\), \(\|\hat{z}\|_\infty \leq 1\), and by the bilinearity of \((z, Dw)\) in \(z \in X_p(\Omega)\) and \(w \in BV(\Omega)_{pr}\), one has that

\[ (z - \hat{z}, D(u - \hat{u})) = |Du| + |D\hat{u}| - (z, D\hat{u}) - (\hat{z}, Du) \geq 0 \]

and hence, (3.7) implies that

\[ (z - \hat{z}, D(u - \hat{u})) = 0 \quad \text{a.e. on } \Omega. \]
From this, it follows that \((z, D\hat{u}) = |D\hat{u}|\) and \((\hat{z}, Du) = |Du|\) as measures. On the other hand, by (3.6), we have that
\[
(z, v) - (\hat{z}, v) = |h - u - (h - \hat{u})| = |h - u| + |h - \hat{u}| - (z, v)(h - \hat{u}) - (\hat{z}, v)(h - u) \geq 0,
\]
which by (3.7) implies that
\[
(z, v) - (\hat{z}, v) = |h - u - (h - \hat{u})| = 0 \quad \text{a.e. on } \partial \Omega,
\]
or equivalently, \([\hat{z}, v](h - u) = |h - u|\) and \([z, v](h - \hat{u}) = |h - \hat{u}|\) a.e. on \(\partial \Omega\).
From this, we can conclude that \([\hat{z}, v] \in \text{sign}(h - u)\) and \([z, v] \in \text{sign}(h - \hat{u})\)
a.e. on \(\partial \Omega\). Thereby, we have shown that \(\hat{z}\) also satisfies (3.4–3.6) with respect to \(u\).

4. THE NEUMAN PROBLEM FOR THE 1-LAPLACIAN

Throughout this section, let \(\Omega\) be a bounded domain in \(\mathbb{R}^d, (d \geq 2)\), with a Lipschitz continuous boundary \(\partial \Omega\).

The main object of this section is the Neumann problem associated with the 1–Laplacian:
\[
(N_f) \begin{cases} 
-\Delta_1 u = 0 & \text{in } \Omega, \\
\frac{Du}{|Du|} \cdot v = f & \text{on } \partial \Omega,
\end{cases}
\]
for given \(f \in L^\infty(\partial \Omega)\) satisfying \(\|f\|_\infty \leq 1\).

To derive the correct notion of solutions of Neumann problem \((N_f)\), we introduce the linear vector space
\[
V := \left\{ u \in BV(\Omega) \mid \text{tr}(u) \in L^2(\partial \Omega) \right\},
\]
where \(\text{tr}\) is the trace operator on \(BV(\Omega)\). Then, for given \(f \in L^2(\partial \Omega)\), we defined the functional \(\Psi_f : L^2(\Omega) \to [-\infty, +\infty)\) by
\[
(4.1) \quad \Psi_f(u) = \begin{cases} 
\int_\Omega |Du| - \int_{\partial \Omega} f u \, d\mathcal{H}^{d-1} & \text{if } u \in V, \\
+\infty & \text{if } u \in L^2(\Omega) \setminus V.
\end{cases}
\]
The functional \(\Psi_f\) is convex and since \(\|f\|_\infty \leq 1\), by [39, Proposition 1.2], \(\Psi_f\) is lower semicontinuous on \(L^2(\Omega)\). Therefore, the sub-differential
\[
\partial \Psi_f = \left\{ (u, h) \in L^2(\Omega) \times L^2(\Omega) \right\} \quad \text{such that for all } v \in V
\]
\[
\int_{\Omega} |Dv| - \int_{\Omega} |Du| - \int_{\partial \Omega} f(v - u) \, d\mathcal{H}^{d-1} \geq \int_{\Omega} h(v - u) \, dx
\]
is maximal monotone in \(L^2(\Omega)\).

To characterize \(\partial \Psi_f\) we introduce the operator \(A_f\) defined as
\[
(u, v) \in A_f \iff u, v \in L^2(\Omega), u \in V \text{ and there exists a vector field } z \in L^\infty(\mathbb{R}^d), \text{ with } \|z\|_\infty \leq 1 \text{ satisfying}
\]

\[
\n
\]
Theorem 4.2. The operator \( A_f \) is m-completely accretive and \( \partial \Psi_f = A_f \).

To prove this theorem we need to introduce the following operator which is related to the \( p \)-Laplacian operator with Neumann boundary conditions. For \( p > 1 \) we define the operator \( A_{p,f} \) in \( L^p(\Omega) \) as

\[
(u, v) \in A_{p,f} \iff u \in W^{1,p}(\Omega) \cap L^\infty(\Omega), v \in L^1(\Omega) \quad \text{and} \quad 
\int_\Omega (w - u)vdx \leq \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (w - u) - \int_{\partial \Omega} f(w - u)H^{d-1}
\]
for all \( w \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \).

Working as [5, Theorem 2.1] we have the following result.
Theorem 4.3. The operator $A_{p,f}$ is completely accreting and $L^\infty(\Omega) \subset R(I + A_{p,f})$.

Proof of Theorem 4.2. First let us see that the operator $A_f$ is completely accreting. For that we need to prove that given $(u_i, v_i) \in A_f$ ($i = 1, 2$) and $q \in P_0$, then
\begin{equation}
\int_\Omega (v_1 - v_2)q(u_1 - u_2)dx \geq 0.
\end{equation}

We have that there exists vector fields $z_i \in L^\infty(\Omega; \mathbb{R}^d)$, with $\|z_i\|_\infty \leq 1$ satisfying
\begin{equation}
\int_\Omega (w - u)v_i dx \leq \int_\Omega (z_i, Dw) - \int_\Omega |Du_i| - \int_{\partial \Omega} f(w - u_i) \mathcal{H}^{d-1} \quad \forall w \in V.
\end{equation}
Then, taken as test functions $w = u_1 - q(u_1 - u_2)$, we get
\begin{equation}
- \int_\Omega q(u_1 - u_2)v_i dx \leq - \int_\Omega (z_i, Dq(u_1 - u_2)) + \int_\Omega f(q(u_1 - u_2)) \mathcal{H}^{d-1}.
\end{equation}
Thus, by (2.9), we have
\begin{equation}
\int_\Omega (v_1 - v_2)q(u_1 - u_2)dx \geq \int_\Omega ((z_1 - z_2), Dq(u_1 - u_2))
\end{equation}
\begin{equation}
= \int_\Omega \theta(z_1 - z_2, D(q \circ (u_1 - u_2)), x) |Dq(u_1 - u_2)|
\end{equation}
\begin{equation}
= \int_\Omega \theta(z_1 - z_2, D(u_1 - u_2), x) |Dq(u_1 - u_2)|
\end{equation}
Now,
\begin{equation}
(z_1 - z_2, D(u_1 - u_2)) = |Du_1| + |Du_2| - (z_1, Du_2) - (z_2, Du_1) \geq 0.
\end{equation}
Hence, the Radon-Nikodým derivative $\theta(z_1 - z_2, D(u_1 - u_2), x)$ of $(z_1 - z_2, D(u_1 - u_2))$ is positive and (4.7) holds.

We claim now that
\begin{equation}
A_f \text{ is closed in } L^2(\Omega).
\end{equation}
In fact: let $(u_n, v_n) \in A_f$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in $L^2(\Omega) \times L^2(\Omega)$. Then, there exists vector fields $z_n \in L^\infty(\Omega; \mathbb{R}^d)$, with $\|z_n\|_\infty \leq 1$ satisfying
\begin{equation}
\int_\Omega (w - u_n)v_n dx \leq \int_\Omega (z_n, Dw) - \int_\Omega |Du_n| - \int_{\partial \Omega} f(w - u_n) \mathcal{H}^{d-1}
\end{equation}
for all $w \in V$. Now, since $\|z_n\|_\infty \leq 1$ we may assume that there exists $z \in L^\infty(\Omega; \mathbb{R}^d)$ such that $z_n$ weakly$^*$-covers $z$ in $L^\infty(\Omega; \mathbb{R}^d)$ with $\|z\|_\infty \leq 1$. Moreover, since $v_n \rightharpoonup v$ in $L^2(\Omega)$ and $v_n = \text{div}(z_n)$, we get $v = \text{div}(z)$ in $D'(\Omega)$ and
\begin{equation}
\lim_{n \to \infty} \int_\Omega (z_n, Dw) = \int_\Omega (z, Dw).
\end{equation}
Then, having in mind the lower semicontinuity of $\Psi_f$, letting $n \to \infty$ in (4.10), we get
\begin{equation}
\int_\Omega (w - u)v dx \leq \int_\Omega (z, Dw) - \int_\Omega |Du| - \int_{\partial \Omega} f(w - u) \mathcal{H}^{d-1},
\end{equation}
consequently $(u, v) \in A_f$ and $A_f$ is closed in $L^2(\Omega)$.
Let us see that
\begin{equation}
L^\infty(\Omega) \subset R(I + A_f).
\end{equation}
Let \( v \in L^\infty(\Omega) \). We need to find \( u \in V \) such that \( (u, v - u) \in A_f \); i.e., there exists a vector field \( z \in L^\infty(\Omega; \mathbb{R}^d) \), with \( \|z\|_\infty \leq 1 \) such that \( v - u = -\text{div}(z) \) and
\begin{equation}
\int_\Omega (w - u)(v - u)dx \leq \int_\Omega (z, Dw) - \int_\Omega |Du| - \int_{\partial \Omega} f(w - u)H^{d-1}
\end{equation}
for all \( w \in V \).

For every \( 1 < p \leq 2 \), applying Theorem 4.3, there exists \( u_p \in D(A_{p,f}) \) such that \( (u_p, v - u_p) \in A_{p,f} \), i.e., \( u_p \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \) and
\begin{equation}
\int_\Omega (w - u_p)(v - u_p)dx \leq \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla (w - u_p) - \int_{\partial \Omega} f(w - u_p)H^{d-1}
\end{equation}
for all \( w \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \). Moreover, since \( A_{p,f} \) is completely accretive, we also get \( \|u_p\|_\infty \leq \|v\|_\infty \).

Taking \( w = 0 \) in (4.13), we get
\[-\int_\Omega u_p(v - u_p)dx \leq -\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla u_p + \int_{\partial \Omega} f u_p H^{d-1},\]
and consequently,
\[\int_\Omega |\nabla u_p|^p + \int_\Omega |u_p|^2 \leq \int_\Omega uu_p + \int_{\partial \Omega} f u_p H^{d-1} \quad \text{for every } 1 < p \leq 2.\]
Thus,
\begin{equation}
\int_\Omega |\nabla u_p|^p \leq M_1 \quad \text{for every } 1 < p \leq 2,
\end{equation}
where \( M_1 \) does not depend on \( p \). Hence, applying Young’s inequality we also have the boundness of \( |\nabla u_p| \) in \( L^1(\Omega) \) and so \( \{u_p\}_{p>1} \) is bounded in \( W^{1,1}(\Omega) \) and then we may extract a subsequence such that \( u_p \) converges in \( L^1(\Omega) \) and almost everywhere to some \( u \in L^1(\Omega) \) as \( p \to 1^+ \). From the estimates we also get \( u_p \to u \) in \( L^2(\Omega) \) and \( u \in BV(\Omega) \cap L^\infty(\Omega) \).

By (4.14), applying Hölder’s inequality, we have
\[\int_\Omega |\nabla u_p|^{p-1} \leq \left( \int_\Omega |\nabla u_p|^p \right)^{\frac{p-1}{p}} |\Omega|^\frac{1}{p} \leq M_2,\]
where \( M_1 \) does not depend on \( p \). On the other hand, for any measurable subset \( E \subset \Omega \) such that \( |E| < 1 \), we have
\[\int_E |\nabla u_p|^{p-2}\nabla u_p | \leq \int_E |\nabla u_p|^{p-1} \leq M_1^{\frac{p-1}{p}} |\Omega|^\frac{1}{p} \leq M_3 |\Omega|^\frac{1}{p}.\]
Thus, \( \{\nabla u_p|^{p-2}\nabla u_p\}_{p>1} \) being bounded and equiintegrable in \( L^1(\Omega, \mathbb{R}^d) \), is weakly relatively compact \( L^1(\Omega, \mathbb{R}^d) \). Hence, we may assume that
\[|\nabla u_p|^{p-2}\nabla u_p \to z \quad \text{as } p \to 1^+ \quad \text{weakly in } L^1(\Omega, \mathbb{R}^d).\]
Given \( \varphi \in D'(\Omega) \), taking \( w = u_p \pm \varphi \) as test functions in (4.13) and letting \( p \to 1^+ \), we obtain
\[
\int_\Omega (v - u_p) \varphi dx = \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx,
\]
that is \( v - u = - \text{div}(z) \) in \( D'(\Omega) \). Moreover, we also get \( \|z\|_\infty \leq 1 \) (see the proof of [4, Lemma 1]).

For every \( w \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \), by (4.13), applying Young’s inequality we get
\[
p \int_\Omega |\nabla u_p| - \int_\partial \Omega f u_p d\mathcal{H}^{d-1}
\leq (p - 1)|\Omega| - \int_\Omega (w - u_p)(v - u_p)dx
\]
\[
+ \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w - \int_\partial \Omega f w d\mathcal{H}^{d-1}.
\]
Then, using the semicontinuity of the functional \( \Psi_f \) and letting \( p \to 1^+ \) we obtain
\[
\int_\Omega (w - u)(v - u)dx \leq \int_\Omega (z, Dw) - \int_\Omega |Du| - \int_\partial \Omega f(w - u) d\mathcal{H}^{d-1}
\]
for all \( w \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \), from where, by approximation we can conclude that \( (u, v - u) \in \mathcal{A}_f \) and, therefore (4.12) holds.

Since \( \mathcal{A}_f \) is completely accretive, by (4.9) and (4.11), we get that the operator \( \mathcal{A}_f \) is \( m \)-completely accretive. Finally, if \( (u, v) \in \mathcal{A}_f \), for every \( w \in D(\Psi_f) = V \), we have
\[
\Psi_f(w) - \Psi_f(u) = \int_\Omega |Dw| - \int_\partial \Omega f w d\mathcal{H}^{d-1} - \int_\Omega |Du| - \int_\partial \Omega f (w - u) d\mathcal{H}^{d-1}
\geq \int_\Omega (z, Dw) - \int_\partial \Omega |Du| - \int_\partial \Omega f(w - u) d\mathcal{H}^{d-1} \geq \int_\Omega (w - u)v dx.
\]
Therefore, \( \mathcal{A}_f \subset \partial \Psi_f \), and consequently, \( \mathcal{A}_f = \partial \Psi_f \) and we concludes the proof. \( \square \)

**Definition 4.4.** We say that \( u \) is a solution of the Neumann problem \((N_f)\) if \( 0 \in \partial \Psi_f \), in other words if \( u \in V \cap L^2(\Omega) \) and there exists a vector field \( z \in L^\infty(\Omega; \mathbb{R}^d) \), with \( \|z\|_\infty \leq 1 \) satisfying:
\[
\begin{align*}
- \text{div}(z) &= 0, \quad \text{in } D'(\Omega), \\
(z, Du) &= |Du|, \\
[z, v] &= f, \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial \Omega.
\end{align*}
\]

**Remark 4.5.** Obviously to have solution of the Neumann problem \((N_f)\), we need that \( \int_\partial \Omega f d\mathcal{H}^{d-1} = 0 \) and also that \( \|f\|_\infty \leq 1 \). In [37] authors studied the existence of solutions of the Neumann problem \((N_f)\) by analyzing the behavior of solutions of the \( p \)-Laplacian problem
\[
\begin{cases}
-\Delta_p u = 0 & \text{in } \Omega, \\
|\nabla u|^{p-2} \nabla u \cdot \frac{\partial u}{\partial \nu} = f & \text{on } \partial \Omega,
\end{cases}
\]
as $p \to 1$ and showed that if $\|f\|_* = 1$, then solutions of \((4.18)\) converge to a solution of \((N_f)\), where

$$\|f\|_* := \sup_{u \in S_1 \setminus \{0\}} \frac{\int_{\partial \Omega} fu \, d^d - 1}{\int_{\Omega} |u| \, dx}.$$ 

and 

$$S_1 := \left\{ u \in W^{1,1}(\Omega) : \int_{\partial \Omega} u \, d^d - 1 = 0 \right\}.$$

If $\|f\|_* < 1$ or $\|f\|_* > 1$, then solutions of \((4.18)\) converge to $u = 0$, or $\infty$ on a set of positive measure, respectively [37].

5. MAIN RESULTS

In this section, we present the construction of the Dirichlet-to-Neumann operator $\Lambda$ associated with the 1-Laplacian $\Delta_1$. We begin by introducing this operator $\Lambda$ in $L^1(\partial \Omega)$, and show that $\Lambda$ is completely accretive.

5.1. The DtN-map $\Lambda$ in $L^1(\partial \Omega)$.

**Definition 5.1.** We define the Dirichlet-to-Neumann map in $L^1(\partial \Omega)$ associated with the 1-Laplace operator as the multivalued operator

$$\Lambda = \left\{ (h, v) \in L^1(\partial \Omega) \times L^1(\partial \Omega) \ : \ \exists \text{ solution } u \in BV(\Omega) \text{ of } (1.1) \right.$$ 

$&$ $\& \ z \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying $||z||_\infty \leq 1$, $v = [z, v]$ $\right\}$

We come to our first theorem.

**Theorem 5.2.** The operator $\Lambda$ is completely accretive and closed.

*Proof.* For $i = 1, 2$, let $(h_i, v_i) \in \Lambda$ and $q \in P_0$. To see that $\Lambda$ is completely accretive, we need to show that

$$\int_{\partial \Omega} (v_1 - v_2)q(h_1 - h_2) \, d\mathcal{H}^{d-1} \geq 0.$$

Then for $i = 1, 2$, there exist $u_i \in BV(\Omega)$, and vector fields $z_i \in L^\infty(\Omega; \mathbb{R}^d)$, with $||z_i||_\infty \leq 1$, satisfying

\begin{align*}
-\text{div}(z_i) &= 0, \quad \text{in } D'(\Omega), \\
(z_i, Du_i) &= |Du_i|, \\
[z_i, v] &\in \text{sign}(h_i - u_i), \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial \Omega,
\end{align*}

and such that $v_i = [z_i, v]$. From (5.1), multiplying by $q(u_1 - u_2)$ and applying Green formula (2.10), we get

$$\int_{\partial \Omega} [z_i, v]q(u_1 - u_2) \, d\mathcal{H}^{d-1} = \int_{\Omega} (z_i, D(q(u_1 - u_2)))$$

and hence

$$\int_{\partial \Omega} ([z_1, v] - [z_2, v])q(u_1 - u_2) \, d\mathcal{H}^{d-1} = \int_{\Omega} (z_1 - z_2, D(q(u_1 - u_2)))$$
By (2.9),
\[
\int_{\Omega} (z_1 - z_2, D(q(u_1 - u_2))) = \int_{\Omega} \theta(z_1 - z_2, D(q \circ (u_1 - u_2)), x) |Dq(u_1 - u_2)| \]
\[
= \int_{\Omega} \theta(z_1 - z_2, D(u_1 - u_2), x) |Dq(u_1 - u_2)|.
\]
Thus and since the Radon-Nikodým derivative $\theta(z_1 - z_2, D(u_1 - u_2), x)$ of $(z_1 - z_2, D(u_1 - u_2))$ is positive by (3.8), it follows that
\[
\int_{\partial \Omega} (|z_1, v| - |z_2, v|) q(u_1 - u_2) d\mathcal{H}^{d-1} \geq 0.
\]
Using this inequality, we see that
\[
\int_{\partial \Omega} (v_1 - v_2) q(h_1 - h_2) d\mathcal{H}^{d-1}
\]
\[
= \int_{\partial \Omega} (|z_1, v| - |z_2, v|) q(h_1 - h_2) d\mathcal{H}^{d-1}
\]
\[
\geq \int_{\partial \Omega} (|z_1, v| - |z_2, v|) [q(h_1 - h_2) - q(u_1 - u_2)] d\mathcal{H}^{d-1}.
\]
Now, by (5.3),
\[
(|z_1, v| - |z_2, v|) ((h_1 - u_1) - (h_2 - u_2))
\]
\[
= |h_1 - u_1| + |h_2 - u_2| - |z_1, v| (h_2 - u_2) - |z_2, v| (h_1 - u_1) \geq 0
\]
\[H^{d-1}\text{-a.e. on } \partial \Omega.\]
Then, since $q \in C^\infty(\mathbb{R})$ with $0 \leq q' \leq 1$, we find that
\[
\int_{\partial \Omega} (|z_1, v| - |z_2, v|) [q(h_1 - h_2) - q(u_1 - u_2)] d\mathcal{H}^{d-1}
\]
\[
\geq \int_{\partial \Omega} (|z_1, v| - |z_2, v|) q'(\xi) [((h_1 - u_1) - (h_2 - u_2)] d\mathcal{H}^{d-1} \geq 0,
\]
where $\xi(s)$ is some function between $h_1(s) - h_2(s)$ and $u_1(s) - u_2(s)$ for $H^{d-1}$-a.e. $s \in \partial \Omega$. Therefore, $\Lambda$ is completely accretive.

Let us see now that $\Lambda$ is closed in $L^1(\partial \Omega) \times L^1(\partial \Omega)$. Let $(f_n, g_n) \in \Lambda$ such that $(f_n, g_n) \rightarrow (f, g)$ in $L^1(\partial \Omega) \times L^1(\partial \Omega)$. We have that there exists $u_n \in BV(\Omega)$ and $z_n \in L^\infty(\Omega; \mathbb{R}^d)$, with $\|z_n\|_\infty \leq 1$, satisfying
\[
(5.4) \quad -\text{div}(z_n) = 0, \quad \text{in } \mathcal{D}'(\Omega),
\]
\[
(5.5) \quad (z_n, Du_n) = |Du_n|,
\]
\[
(5.6) \quad [z_n, v] \in \text{sign}(f_n - u_n), \quad H^{d-1}\text{-a.e. on } \partial \Omega,
\]
such that $g_n = [z_n, v]$. Since $\|z_n\|_\infty \leq 1$, we can assume by taking a subsequence if necessary, that
\[
(z_n \rightharpoonup z) \quad \text{weakly}^* \text{ in } L^\infty(\Omega, \mathbb{R}^d).
\]
Moreover, $\|z\|_\infty \leq 1$, and by (5.4), we also have $-\text{div}(z) = 0$ in $\mathcal{D}'(\Omega)$.

By [7] (see also [6, Proposition C.12]), we have that
\[
(5.7) \quad (z_n, Dw) \rightarrow (z, Dw) \quad \text{weak}^* \text{ in } M^b(\Omega, \mathbb{R}).
\]
By using the limit in (5.7), we can conclude that
\begin{equation}
\lim_{n \to \infty} \int_{\Omega} (z_n, Dw) = \int_{\Omega} (z, Dw).
\end{equation}

In fact, given \( \varepsilon > 0 \), we take an open set \( U \subset \subset \Omega \) such that
\[ \int_{\Omega \setminus U} |Dw| \leq \frac{\varepsilon}{2}. \]

Let \( \varphi \in \mathcal{D}(\Omega) \) be such that \( \varphi \equiv 1 \) on \( U \) and \( 0 \leq \varphi \leq 1 \) on \( \Omega \). Then
\[ \left| \int_{\Omega} (z_n, Dw) - \int_{\Omega} (z, Dw) \right| \leq \int_{\Omega} \left| (z_n, Dw) - (z, Dw) \right| \varphi + \int_{\Omega \setminus U} \left| (z_n, Dw) - (z, Dw) \right| (1 - \varphi) \]

The limit (5.7) implies that the first term on the right hand side goes to zero as \( n \to \infty \). By (2.8) and since \( \|z_n\|_\infty \leq 1 \) and \( \|z\|_\infty \leq 1 \), the second term is bounded by
\[ \|z_n\|_\infty \int_{\Omega \setminus U} |Dw| + \|z\|_\infty \int_{\Omega \setminus U} |Dw| \leq \varepsilon. \]

As \( \varepsilon > 0 \) was arbitrary, we have thereby shown that limit (5.8) holds.

Since every \( g_n \) satisfies \( |g_n(x)| \leq 1 \) for a.e. \( x \in \partial \Omega \) and by assumption, \( g_n \to g \) in \( L^1(\partial \Omega) \), we can pass to a subsequence if necessary, in order to conclude that \( |g(x)| \leq 1 \) for a.e. \( x \in \partial \Omega \).

Next, let \( h \in L^1(\partial \Omega) \) and \( w \in BV(\Omega) \) such that \( \text{tr}(w) = h \). Then by the generalized Green formula (2.10), by (5.1), since \( -\text{div}(z) = 0 \) and by limit (5.8), we see
\[ \int_{\partial \Omega} g h \, d\mathcal{H}^{d-1} = \lim_{n \to \infty} \int_{\partial \Omega} g_n h \, d\mathcal{H}^{d-1} = \lim_{n \to \infty} \int_{\partial \Omega} [z_n, v] w \, d\mathcal{H}^{d-1} \]

This shows that \( g = [z, v] \) in \( L^\infty(\partial \Omega) \) and hence, to complete this proof, it remains to show that there exists a function \( u \in BV(\Omega) \) such that \( (z, Du) = |Du| \) and \( [z, v] \in \text{sign}(f - u) \mathcal{H}^{N-1} \) a.e. on \( \partial \Omega \). In fact, we have
\[ \int_{\partial \Omega} |u_n| d\mathcal{H}^{d-1} \leq \int_{\partial \Omega} |u_n - f_n| d\mathcal{H}^{d-1} + \int_{\partial \Omega} |f_n| d\mathcal{H}^{d-1} \]
\[ = \int_{\partial \Omega} |f_n| d\mathcal{H}^{d-1} + \int_{\partial \Omega} [z_n, v](f_n - u_n) d\mathcal{H}^{d-1} \]
\[ = \int_{\partial \Omega} |f_n| d\mathcal{H}^{d-1} + \int_{\partial \Omega} [z_n, v] f_n d\mathcal{H}^{d-1} - \int_{\partial \Omega} \text{div}(z_n) u_n dx - \int_{\partial \Omega} (z_n, Du_n) \]
\[ = \int_{\partial \Omega} |f_n| d\mathcal{H}^{d-1} + \int_{\partial \Omega} g_n f_n d\mathcal{H}^{d-1} - \int_{\partial \Omega} |Du_n|, \]
Therefore, we can assume, taking a subsequence if necessary, that there exists \( u \in BV(\Omega) \), such that
\[
(5.9) \quad u_n \to u \quad \text{in } L^1(\Omega).
\]
On the other hand, for \( \varphi \in D(\Omega) \), since \( z_n \nabla \varphi \to z \nabla \varphi \) and \( u_n \to u \) strongly, we have
\[
\begin{align*}
\lim_{n \to \infty} \langle (z_n, Du_n), \varphi \rangle &= \lim_{n \to \infty} \left[ -\int_{\Omega} \div (z_n) \varphi u_n - \int_{\Omega} z_n \nabla \varphi u_n \right] \\
&= -\lim_{n \to \infty} \int_{\Omega} z_n \nabla \varphi u_n = \int_{\Omega} z \nabla \varphi u \\
&= \langle (z, Du), \varphi \rangle.
\end{align*}
\]
Therefore,
\[
(z_n, Du_n) \to (z, Du) \quad \text{weakly}^* \quad \text{in } M^b(\Omega, \mathbb{R}).
\]
Then, working as in the proof of (5.8), we get
\[
(5.10) \quad \lim_{n \to \infty} \int_{\Omega} (z_n, Du_n) = \int_{\Omega} (z, Du)
\]
By (5.10), applying the lower semicontinuity of the total variation and using (5.10), we obtain that
\[
\begin{align*}
\int_{\Omega} |Du| &\leq \liminf_{n \to \infty} \int_{\Omega} |Du_n| \leq \limsup_{n \to \infty} \int_{\Omega} |Du_n| = \limsup_{n \to \infty} \int_{\Omega} (z_n, Du_n) \\
&= \int_{\Omega} (z, Du) \leq \|z\| \int_{\Omega} |Du| \leq \int_{\Omega} |Du|.
\end{align*}
\]
Then,
\[
\int_{\Omega} (z, Du) = \int_{\Omega} |Du| = \lim_{n \to \infty} |Du_n|.
\]
Consequently, \( u_n \) strictly converges to \( u \) in \( BV(\Omega) \) and so, by Proposition 2.2, we have that \( tr(u_n) \to tr(u) \) in \( L^1(\partial \Omega) \). Thus, and since \( z_n, v \in \text{sign}(f_n - u_n) \mathcal{H}^{d-1} \) a.e. on \( \partial \Omega \), we have that \( z, v \in \text{sign}(f - u) \mathcal{H}^{d-1} \) a.e. on \( \partial \Omega \).

It remains to show that the DtN-map \( \Lambda \) in \( L^1(\partial \Omega) \) satisfies the range condition
\[
(5.11) \quad R(I + \Lambda) = L^1(\partial \Omega).
\]
To obtain (5.11) we need to recall some results from [35], in which it was studied the 1-Laplacian elliptic equation with inhomogeneous Robin boundary conditions
\[
(5.12) \quad \begin{cases} \\
-\Delta_1 u = 0 & \text{in } \Omega, \\
u + \frac{Du}{|Du|} \cdot v = g & \text{in } \partial \Omega.
\end{cases}
\]
The concept of solution of this problem is the following

**Definition 5.3.** Given \( g \in L^2(\partial \Omega) \), we say that \( u \in BV(\Omega) \cap L^2(\partial \Omega) \) is a weak solution to (5.12) if there exists a vector field \( z \in L^\infty(\Omega; \mathbb{R}^d) \) satisfying

\[
\begin{align*}
|z| &\leq 1, \\
div(z) & = 0, \quad \text{in } D'(\Omega), \\
(z, Du) & = |Du|, \\
-|z,v| & = T_1(u - g), \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial \Omega.
\end{align*}
\]

where
\[
T_k(r) := [k - (k - |r|)^+] \text{sign}(r), \quad r \in \mathbb{R}, \, k > 0,
\]
In [35, Theorem 1.1.] it is given the following result

**Theorem 5.4.** For every \( g \in L^2(\partial \Omega) \) there exists a weak solution to (5.12).

We also need the following result given in [35, Proposition 2.13.]

**Proposition 5.5.** Let \( g \in L^2(\partial \Omega) \) and \( u \in BV(\Omega) \). If \( u \) is a solution to the Robin problem (5.12), then \( u \) is a solution to the Dirichlet problem

\[
\begin{align*}
-\Delta u & = 0, \quad \text{in } \Omega, \\
u & = (g - z,v) \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( z \) is any vector field associated with the solution \( u \). Moreover, if \( \hat{z} \) is another vector field associated with the solution \( u \), we have \( \|z,v\| = \|\hat{z},v\| \) \( \mathcal{H}^{d-1}\text{-a.e. on } \partial \Omega \).

**Theorem 5.6.** The operator \( \Lambda \) is \( m \)-completely accretive and \( D(\Lambda) \) is dense in \( L^1(\partial \Omega) \).

**Proof.** By Theorem 5.2, we only need to show that \( \Lambda \) verifies the range condition (5.11). Now, since \( \Lambda \) is closed it is enough to prove that

\[
L^2(\partial \Omega) \subset R(I + \Lambda).
\]
By Theorem 5.4, given \( g \in L^2(\partial \Omega) \) there exists a weak solution to problem (5.12). Now, by Proposition 5.17, \( u \) is a solution to the Dirichlet problem (5.17). Then, there exists a vector field \( \hat{z} \in L^\infty(\Omega; \mathbb{R}^d) \) satisfying

\[
\begin{align*}
|\hat{z}| &\leq 1, \\
div(\hat{z}) & = 0, \quad \text{in } D'(\Omega), \\
(\hat{z}, Du) & = |Du|, \\
|\hat{z},v| & \in \text{sign}(g - [z,v] - u), \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial \Omega.
\end{align*}
\]
Therefore, if we define \( w := g - [z,v] \), we have \( (w, g - w) \in \Lambda \) and consequently \( g \in R(I + \Lambda) \).

Next, let us show that \( D(\Lambda) \) is dense in \( L^1(\partial \Omega) \). By the accretivity of \( \Lambda \) and by (5.18), one has that

\[
L^\infty(\partial \Omega) \subset R \left( I + \frac{1}{n} \Lambda \right) \quad \text{for all } n \geq 1.
\]
Hence, for given \( g \in L^\infty(\partial \Omega) \) and every \( n \geq 1 \), there exists \( h_n \in L^\infty(\partial \Omega) \cap D(\Lambda) \) such that \( (h_n, n(g - h_n)) \in \Lambda \). Thus, having in mind Proposition
3.4, there exists \( u_n \in BV(\Omega) \) and \( z_n \in L^\infty(\Omega; \mathbb{R}^d) \) with \( \|z_n\|_\infty \leq 1 \), with 
\[- \mathrm{div}(z_n) = 0 \text{ and satisfying} \]
\[ \int_\Omega |Du_n| + \int_{\partial \Omega} |u_n - h_n| \leq \int_\Omega (z_n, Dw) - n \int_{\partial \Omega} (g - h_n)(w - h_n) \quad \forall w \in BV(\Omega). \]

Then, taking \( w \in BV(\Omega) \) such that \( tr(w) = g \), we get
\[ \int_{\partial \Omega} (g - h_n)^2 d\mathcal{H}^{d-1} \leq \frac{1}{n} \int_\Omega (z_n, Dw) \leq \frac{1}{n} \int_\Omega |Dw|, \]
from where it follows that \( h_n \to g \) in \( L^1(\partial \Omega) \), so \( L^\infty(\partial \Omega) \subset \overline{D(\Lambda)}^{L^1(\partial \Omega)} \),
and consequently, \( D(\Lambda) \) is dense in \( L^1(\partial \Omega) \).

□

Let us see that the operator \( \Lambda \) is positively homogeneous of degree 0. In fact, we need to show that
(5.23) \[ \text{If } (h, v) \in \Lambda \text{ and } \lambda > 0 \text{ then } (\lambda h, v) \in \Lambda. \]

Given \( (h, v) \in \Lambda \), there exists a solution \( u \in BV(\Omega) \) of (1.1) and \( z \in L^\infty(\Omega, \mathbb{R}^N) \) satisfying \( \|z\|_\infty \leq 1 \), such that \( v = [z, v] \). Then, if \( \hat{u} := \lambda u \), we have
\[ (z, D\hat{u}) = \lambda (z, Du) = \lambda |Du| = |D\hat{u}| \]
and
\[ [z, v] \in \text{sign}(h - u) = \frac{1}{\lambda} \text{sign}(\lambda h - \hat{u}), \]
which imply that \( v \in \text{sign}(\lambda h - \hat{u}) \). Thus, \( (\lambda h, v) \in \Lambda \), and (5.23) holds.

Therefore, by Theorem 2.11 ([31, Theorem 4.13]), we have proved Theorem 1.3.

5.2. Asymptotic behaviour. Let \( (T(t))_{t \geq 0} \) be the semigroup in \( L^1(\partial \Omega) \) generated by the operator \( \Lambda \). In this section we will study the asymptotic behaviour of \( (T(t))_{t \geq 0} \). For a function \( w \in L^1(\partial \Omega) \), we denote by \( \overline{w} \) its media, i.e.,
\[ \overline{w} := \frac{1}{\mathcal{H}^{d-1}(\partial \Omega)} \int_{\partial \Omega}wd\mathcal{H}^{d-1}. \]

We have that conserve the mass.

**Lemma 5.7.** For every \( u_0 \in L^1(\partial \Omega) \), if \( u(t) = T(t)u_0 \), we have
\[ \overline{u(t)} = \overline{u_0} \text{ for all } t \geq 0. \]

**Proof.** We have that there exists a function \( \hat{u}(t) \in BV(\Omega) \) and \( z(t) \in L^\infty(\Omega, \mathbb{R}^d) \), \( \|z(t)\|_\infty \leq 1 \) verifying (6.5) and (6.6). Hence, since
\[ - \mathrm{div}(z(t)) = 0, \quad \text{in } D'(\Omega), \]
we have
\[ \frac{d}{dt} \int_{\partial \Omega} u(t)d\mathcal{H}^{d-1} = - \int_{\partial \Omega} [z(t), v] d\mathcal{H}^{d-1} = 0, \]
and the result follows. □
Theorem 5.8. For every $u_0 \in L^2(\partial \Omega)$, we have there exists $C > 0$ such that

$$\|T(t)u_0 - \bar{u}_0\|_1 \leq \frac{1}{2Ct}\|u_0\|^2_2 \quad \text{for all } t \geq 0. \tag{5.24}$$

Proof. Since the semigroup $\{T(t): t \geq 0\}$ preserves the mass (Lemma 5.8), we have

$$v(t) := T(t)u_0 - \frac{T(t)u_0}{T(0)} = T(t)u_0 - \bar{u}_0.$$

The complete accretivity of the operator $\Lambda$ (Theorem 5.2) implies that

$$\mathcal{L}(v) := \|v - \bar{u}_0\|_{L^1(\partial \Omega)}$$

is a Liapunov functional for the semigroup $\{T(t): t \geq 0\}$, which implies that

$$\|v(t)\|_{L^1(\partial \Omega)} \leq \|v(s)\|_{L^1(\partial \Omega)} \quad \text{if } t \geq s. \tag{5.25}$$

By (2.3), there exists a constant $C > 0$ such that

$$C\|v(s)\|_{L^1(\partial \Omega)} \leq |Dv(s)|(|\Omega|). \tag{5.26}$$

From (5.25) and (5.26) we obtain that

$$t\|v(t)\|_{L^1(\partial \Omega)} \leq \int_0^t \|v(s)\|_{L^1(\partial \Omega)} ds \leq \frac{1}{C} \int_0^t |Dv(s)|(|\Omega|) ds. \tag{5.27}$$

Let $u(t) := T(t)u_0$, then, $(u(t), -u'(t)) \in \Lambda$ for almost all $t \geq 0$. Now, by the complete accretivity of $\Lambda$, we have $u(t) \in L^2(\partial \Omega)$. Then, by Theorem 5.12, we have $(u(t), -u'(t)) \in \partial_M \phi$. Hence by Theorem 5.11, we have there exists $\hat{u}(t) \in V$, with $tr(\hat{u}(t)) = w(t)$ such that $(\hat{u}(t), 0) \in \partial_M -u'(t)$. Therefore, for all $v \in V$, we have

$$\int_\Omega |Dv| - \int_\Omega |D\hat{u}(t)| - \int_{\partial \Omega} (-u'(t))(v - \hat{u}(t)) d\mathcal{H}d-1 \geq 0.$$

So, taking $v = 0$, we arrive to

$$\int_\Omega |D\hat{u}(t)| \leq - \int_{\partial \Omega} u'(t)u(t) d\mathcal{H}d-1 = -\frac{d}{dt}\|u(t)\|^2_2.$$

Then,

$$\frac{1}{2}\|u(t)\|^2_2 - \frac{1}{2}\|u_0\|^2_2 = -\int_0^t \int_\Omega |D\hat{u}(s)| ds = -\int_0^t \int_\Omega |Dv(s)| ds,$$

which implies

$$\int_0^t \int_\Omega |Dv(s)| ds \leq \frac{1}{2}\|u_0\|^2_2.$$

Hence, by (5.27)

$$\|v(t)\|_{L^1(\partial \Omega)} \leq \frac{1}{2C} \frac{\|u_0\|^2_{L^2(\partial \Omega)}}{t},$$

which concludes the proof. \qed

Remark 5.9. Suppose we have the inequality

$$\|w - \bar{w}\|_{L^2(\partial \Omega)} \leq C|Dw|(|\Omega|) \quad \forall w \in V. \tag{5.28}$$

Then, there exists $T^* > 0$ such that

$$T(t)u_0 = \bar{u}_0 \quad \text{for all } t \geq T^*. \tag{5.29}$$
In fact, as in the above proof let
\[ v(t) := T(t)u_0 - \overline{T(t)u_0} = T(t)u_0 - \overline{u_0}. \]
We have \((v(t), -v'(t)) \in \Lambda\) for almost all \(t \geq 0\). Then, working as in the above proof, we get
\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 \leq -\int_\Omega |Du(t)|. \]
Now, by (5.28), we have
\[ \|v(t)\|_{L^2(\partial\Omega)} \leq C\|u(t)\|_2(\Omega), \]
and consequently
\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 + \frac{1}{C}\|v(t)\|_{L^2(\partial\Omega)} \leq 0, \]
from where (5.29) follows.

5.3. The DtN map \(\Lambda\) as a tr-sub-differential in \(L^2(\partial\Omega)\). In this section, we apply the theory of \(\tau_{w-f}\)-elliptic functionals developed in Section A to show that the DtN map \(\Lambda\) in \(L^2(\partial\Omega)\) can be seen as the subdifferential of an energy functional.

Throughout this section, \(\Omega\) is a bounded domain in \(\mathbb{R}^d\), \((d \geq 2)\), with a Lipschitz continuous boundary \(\partial\Omega\). Then, consider the linear vector space
\[ V := \left\{ u \in BV(\Omega) \left| \operatorname{tr}(u) \in L^2(\partial\Omega) \right. \right\}, \]
where \(\operatorname{tr}(u)\) is the trace of \(u\) on \(\partial\Omega\), and set
\[ \|u\|_V := |Du|(\Omega) + \|u\|_{L^2(\partial\Omega)} \quad \text{for every } u \in V. \]
Then, thanks to Sobolev inequality (2.5) and the continuous embedding of \(L^2(\partial\Omega)\) into \(L^1(\partial\Omega)\), one has that
\[ \|u\|_{L^{\infty}(\partial\Omega)} \leq C \|u\|_V \quad \text{for all } u \in V, \]
and \(V\) equipped with the norm \(\cdot\|_V\) becomes a Banach space and the linear operator \(j = \operatorname{tr}_V : V \to L^2(\partial\Omega)\) assigning each \(u \in V\) to the unique trace \(\operatorname{tr}(u) \in L^2(\partial\Omega)\) is continuous. We introduce the follow topology in \(V\).

Let \(\tau_w\) be the coarsest topology on \(V\) such that
\[ \text{the operator } \operatorname{tr} \text{ is } \tau_{w\to\sigma}(L^2(\partial\Omega), (L^2(\partial\Omega))^*) \text{ continuous.} \]

By [18, Proposition 3.1], for the locally convex topology \(\tau_w\), we have that if \(\{u_n\}\) is a sequence in \(V\), then \(u_n \to u\) in \((V, \tau_w)\) if and only if \(\operatorname{Tr}(u_n) \to \operatorname{tr}(u)\) respect to \(\sigma(L^2(\partial\Omega), (L^2(\partial\Omega))^*)\).

It is clear that \(\tau_w\) is weaker than the topology in \(V\) induced by the norm \(\cdot\|_V\), but \(\tau_w\) is certainly not the weak topology of \((V, \cdot\|_V)\). The advantage of the topology \(\tau_w\) on \(V\) is that the following compactness result, which might be of independent interest, holds.
Theorem 5.10. Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, ($d \geq 2$), with a Lipschitz continuous boundary $\partial \Omega$. Then for every $\nu$-bounded sequence $(u_n)_{n \geq 1}$ in $V$, there is a subsequence $(u_{k_n})_{n \geq 1}$ of $(u_n)_{n \geq 1}$ and a $u \in BV(\Omega)$ such that $u_{k_n} \rightharpoonup u$ in $\tau_w$ and
\begin{equation}
|Du|(\Omega) \leq \liminf_{n \to +\infty} |Du|(u_{k_n}).
\end{equation}

Proof. If $(u_n)_{n \geq 1}$ is a bounded sequence in $(V, \| \cdot \|)$, then there is a constant $C > 0$ such that
\begin{equation}
|Du_n|(\Omega) + \|u_n\|_{L^2(\partial \Omega)} \leq C \quad \text{for every } n \geq 1,
\end{equation}
and by (5.30),
\begin{equation}
\|u_n\|_{L^{d/(d-1)}(\Omega)} \leq C \quad \text{for every } n \geq 1.
\end{equation}
Thus, $(u_n)_{n \geq 1}$ is bounded in $BV(\Omega)$ and so, by [1, Theorem 3.23], there is a subsequence $(u_{k_n})_{n \geq 1}$ of $(u_n)_{n \geq 1}$ and a $u \in BV(\Omega)$ such that $u_{k_n}$ weakly* to $u$ in $BV(\Omega)$. Thus and since the map $u \mapsto |Du|(\Omega)$ of total variational measures is lower semicontinuous on $L^1(\Omega)$, we have that (5.32) holds.

Since $(tr(u_{k_n}))_{n \geq 1}$ is bounded in $L^2(\partial \Omega)$, the weak-compactness of $L^2(\partial \Omega)$, there is a $\bar{u} \in L^2(\partial \Omega)$ such that after possibly passing again to another subsequence of $(u_{k_n})_{n \geq 1}$, $tr(u_{k_n}) \rightharpoonup \bar{u}$ weakly in $L^2(\partial \Omega)$. Thus and since by (6.4),
\begin{equation}
\int_{\Omega} u_{k_n} \nabla \cdot \xi \, dx + \int_{\Omega} \xi \cdot Du_{k_n} = \int_{\partial \Omega} (\xi \cdot v) \, tr(u_{k_n}) \, dH^{d-1}
\end{equation}
for every $\xi \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$. If we prove that
\begin{equation}
\lim_{n \to +\infty} \int_{\Omega} u \nabla \cdot \xi \, dx + \int_{\Omega} \xi \cdot Du = \int_{\partial \Omega} (\xi \cdot v) \, \bar{u} \, dH^{d-1},
\end{equation}
sending $n \to +\infty$ in (5.35), one obtains that
\begin{equation}
\int_{\partial \Omega} (\xi \cdot v) \, tr(u) \, dH^{d-1} = \int_{\partial \Omega} (\xi \cdot v) \, \bar{u} \, dH^{d-1} \quad \text{for every } \xi \in C^\infty(\mathbb{R}^d; \mathbb{R}^d).
\end{equation}
Then, for every $i = 1, \ldots, d$, one has
\begin{equation}
\int_{\partial \Omega} \xi_i \, tr(u) \, v_i \, dH^{d-1} = \int_{\partial \Omega} \xi_i \, \bar{u} \, v_i \, dH^{d-1} \quad \text{for every } \xi \in C^\infty(\mathbb{R}^d).
\end{equation}
We can choose an open covering $(U_j)_{j=1}^m$ of $\partial \Omega$ such that for every $j \in \{1, \ldots, m\}$, there is at least one $i \in \{1, \ldots, d\}$ and $c_i > 0$ such that $|v_i(z)| \geq c_i > 0$ for a.e. $z \in U_j \cap \partial \Omega$. The existence of such an open covering follows from the fact that locally
\begin{equation}
\nu(z) := (\nabla g(z'), -1)/\sqrt{\|\nabla g(z')\|^2 + 1}, \quad z = (z', z_d), \quad z' = (z_1, \ldots, z_{d-1}),
\end{equation}
being $g$ a Lipschitz function. By [30, Lemma 2.1], $\{\xi | \partial \Omega \, | \xi \in C^\infty(\mathbb{R}^d)\}$ is dense in $L^2(\partial \Omega)$. For every $j \in \{1, \ldots, m\}$, we apply this result to $L^2(U_j \cap \partial \Omega)$. Then, we obtain $\bar{u} = tr(u)$. This completes the proof of showing that $(u_n)_{n \geq 1}$ admits a $\tau_w$-convergent subsequence.
To finish the proof, we need to show claim (5.36). Let \( v_n \) be the function
\[
v_n(x) := \begin{cases} 
    u_n(x) & \text{if } x \in \Omega \\
    0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}.
\end{cases}
\]
By [1, Corollary 3.89], we have that
\[
\nabla v_n = \nabla u_n + \text{tr}(u_n) \otimes \nu \mathcal{H}^{d-1} \setminus \partial \Omega.
\]
Then
\[
|\nabla v_n|(\mathbb{R}^d) = |\nabla u_n|(\Omega) + \int_{\partial \Omega} |\text{tr}(u_n)|d\mathcal{H}^{d-1}.
\]
Hence, by (5.33) and Hölder’s inequality, we have \( \{v_n\} \) is bounded in \( BV(\mathbb{R}^d) \).
Therefore, we know (cf [1]), there is a positive measure \( \nu \in M^b(\mathbb{R}^d) \) and a subsequence \( (v_{n_k})_{k\geq 1} \) of \( (v_n)_{n\geq 1} \) (we can assume is this one), such that \( |\nabla v_{n_k}| \rightharpoonup \nu \) weakly* in \( M^b(\mathbb{R}^d) \). Then, by the inner regularity of \( |\nabla u| \) and \( \nu \), for a given \( \varepsilon > 0 \), there is an open set \( U \subset \subset \Omega \) such that
\[
(5.37) \quad |\nabla u|(\Omega \setminus U) < \varepsilon \quad \text{and} \quad \mu(\Omega \setminus U) < \varepsilon.
\]
Let \( \xi \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) \) and \( f \in D(\Omega) \) such that \( f(x) = 1 \) on \( U \) and \( 0 \leq f \leq 1 \) in \( \Omega \). Then,
\[
\left| \int_\Omega \xi \cdot \nabla u_{n_k} - \int_\Omega \xi \cdot \nabla u \right| \leq \|\langle \xi, \nabla u_{n_k} \rangle, f \rangle - \langle \xi, \nabla u \rangle, f \rangle \|
\]
\[
+ \int_\Omega |\langle \xi, \nabla u_{n_k} \rangle|\mathcal{L}_\Omega - f \rangle + \int_\Omega |\langle \xi, \nabla u \rangle|\mathcal{L}_\Omega - f \rangle.
\]
Now, since \( u_{n_k} \rightharpoonup u \) in \( L^1(\Omega) \),
\[
\lim_{n \to \infty} \langle \xi, \nabla u_{n_k} \rangle, f \rangle = \langle \xi, \nabla u \rangle, f \rangle,
\]
and
\[
\int_\Omega |\langle \xi, \nabla u \rangle|\mathcal{L}_\Omega - f \rangle \leq \|\xi\|_\infty |\nabla u|(\Omega \setminus U) < \varepsilon \|\xi\|_\infty.
\]
Moreover since \( |\nabla v_{n_k}| \rightharpoonup \nu \) weakly* in \( M^b(\mathbb{R}^d) \) and \( \Omega \setminus U \) is compact, applying [1, Example 1.36] we have
\[
\limsup_{n \to \infty} \int_\Omega |\langle \xi, \nabla v_{n_k} \rangle|\mathcal{L}_\Omega - f \rangle \leq \|\xi\|_\infty \limsup_{n \to \infty} |\nabla v_{n_k}|(\Omega \setminus U)
\]
\[
\leq \|\xi\|_\infty \limsup_{n \to \infty} |\nabla v_{n_k}|(\Omega \setminus U) \leq \|\xi\|_\infty \mu(\Omega \setminus U) < \varepsilon \|\xi\|_\infty.
\]
Then, since \( \varepsilon \) is arbitrary, the claim (5.36) holds. \( \square \)

Next, let \( \varphi : V \to \mathbb{R} \) be defined by
\[
\varphi(\hat{u}) = \int_\Omega |\nabla \hat{u}| \quad \text{for every } \hat{u} \in V.
\]
Then, \( \varphi \) is convex on \( V \) and by Theorem 5.10, for every \( \omega > 0 \), the shifted functional
\[
\varphi_\omega(\hat{u}) := \varphi(\hat{u}) + \frac{\omega}{2} \|\text{tr}(\hat{u})\|^2_{L^2(\partial \Omega)}
\]
has sequentially-relatively compact sublevel sets \( E_c \) with respect to the topology \( \tau_{0} \) for all \( c \in \mathbb{R} \). Thus, \( \varphi \) is \( \tau_{0} - \text{tr-elliptic} \). In addition, since the map \( u \mapsto |\nabla u|(\Omega) \) for \( u \in V \) is lower semicontinuous with respect to the \( L^1(\Omega) \) topology, one also has that \( \varphi \) is sequentially-\( \tau \)-lower semicontinuous on \( V \).
Therefore, by Theorem A.4, the tr-subdifferential operator $\partial_{tr} \varphi$ is maximal monotone in $L^2(\partial \Omega)$. Further, by the definition of $\partial_{tr} \varphi$, Theorem 4.2 and Lemma 4.1, one has that $(u, f) \in \partial_{tr} \varphi$ if and only if there is an $\hat{u} \in V$ such that $tr(\hat{u}) = u$ and $(\hat{u}, 0) \in \partial \Psi_f$. We summarize this in the following theorem.

**Theorem 5.11.** The tr-subdifferential operator

$$\partial_{tr} \varphi = \left\{ (u, f) \in L^2(\partial \Omega) \times L^2(\partial \Omega) \ \bigg| \ \exists \hat{u} \in V \ s.t. \ tr(\hat{u}) = u \ \text{and for every} \ \hat{v} \in V, \ \int_{\Omega} |D(\hat{v})| - \int_{\Omega} |D\hat{u}| \geq (f, tr(\hat{v}) - u)_{L^2(\partial \Omega)} \right\}$$

is maximal monotone in $L^2(\partial \Omega)$. Furthermore,

$$\tag{5.38} (u, f) \in \partial_{tr} \varphi \quad \text{if and only if} \quad \begin{cases} \text{there is an} \ \hat{u} \in V \ s.t. \ tr(\hat{u}) = u \ \text{and} \ (\hat{u}, 0) \in \partial \Psi_f. \end{cases}$$

Moreover, in $L^2(\partial \Omega)$, we have the following characterization of the Dirichlet-to-Neumann operator $\Lambda$ associated with the 1-Laplacian with the tr-subdifferential.

**Theorem 5.12.** One has that

$$\Lambda \cap (L^2(\partial \Omega) \times L^2(\partial \Omega)) = \partial_{tr} \varphi. \tag{5.39}$$

**Proof.** Since by Theorem 5.2, $\Lambda \cap (L^2(\partial \Omega) \times L^2(\partial \Omega))$ is monotone and by the previous theorem, $\partial_{tr} \varphi$ is maximal monotone in $L^2(\partial \Omega)$, for proving this theorem, it is sufficient to show that the inclusion $\partial_{tr} \varphi \subseteq \Lambda \cap (L^2(\partial \Omega) \times L^2(\partial \Omega))$ holds. Thus, let $(u, f) \in \partial_{tr} \varphi$. Then, by (5.38), $(\hat{u}, 0) \in \partial \Psi_f$ and so, by Theorem 4.2, $(\hat{u}, 0) \in \mathcal{A}_f$. Thus, there exists a vector field $z \in L^\infty(\Omega; \mathbb{R}^d)$, with $\|z\|_\infty \leq 1$ satisfying

$$\tag{5.40} -\text{div}(z) = 0, \quad \text{in} \ D'(\Omega),$$

$$\tag{5.41} (z, D\hat{u}) = |D\hat{u}|,$$

$$\tag{5.42} [z, v] = f, \quad \mathcal{H}^{d-1}\text{-a.e. on} \ \partial \Omega,$$

from where it follows that $(u, f) \in \Lambda \cap (L^2(\partial \Omega) \times L^2(\partial \Omega))$. \hfill \Box

**Remark 5.13.** Note that, since $\partial_{tr} \varphi$ is a maximal monotone in $L^2(\partial \Omega)$, as consequence Theorem 5.12, we get another proof of the fact that the Dirichlet-to-Neumann operator $\Lambda$ verifies the range condition

$$L^2(\partial \Omega) \subset R(I + \Lambda).$$

As a consequence of Theorem 5.11, we have the following characterization of $\text{DTr-map} \ \partial_{tr} \varphi$ in $L^2(\partial \Omega)$ and the Neumann problem ($N_f$).

**Corollary 5.14.** For $u, f \in L^2(\partial \Omega)$ with $\|f\|_\infty \leq 1$, the following statements are equivalent.

1. $(u, f) \in \partial_{tr} \varphi$;
2. there exists an $\hat{u} \in V$ such that $tr(\hat{u}) = u$ and $(\hat{u}, 0) \in \partial \Psi_f$;
3. there is a solution $\hat{u} \in V$ of Neumann problem ($N_f$) such that $tr(\hat{u}) = u$. 
In this section, we revisit the problem of well-posedness of the elliptic-parabolic boundary value problem

\[
-\Delta_t \hat{u}(t) = 0 \quad \text{in } (0, T) \times \Omega,
\]

\[
\hat{u}(t) = u(t) \quad \text{on } (0, T) \times \partial \Omega,
\]

\[
\partial_t u(t) + \frac{Du(t)}{|Du(t)|} \cdot v + f(\cdot, u(t)) \ni g(t, \cdot) \quad \text{on } (0, T) \times \partial \Omega,
\]

\[
u(0) = u_0 \quad \text{on } \partial \Omega.
\]

Let \( g \in L^q(0, T; L'((\partial \Omega))) \) and suppose \( f : \partial \Omega \times \mathbb{R} \to \mathbb{R} \) is a Lipschitz-continuous Carathéodory function, that is, \( f \) satisfies the following three properties:

\[(6.2) \quad f(\cdot, u) : \partial \Omega \to \mathbb{R} \text{ is measurable on } \partial \Omega \text{ for every } u \in \mathbb{R},\]

\[(6.3) \quad f(x, 0) = 0 \text{ for a.e. } x \in \partial \Omega, \text{ and}\]

\[(6.4) \quad |f(x, u) - f(x, \hat{u})| \leq \omega |u - \hat{u}| \quad \text{for all } u, \hat{u} \in \mathbb{R}, \text{ a.e. } x \in \partial \Omega.\]

Then, for every \( 1 \leq q \leq \infty, F : L^q(\partial \Omega) \to L^q(\partial \Omega) \) defined by

\[F(u)(x) := f(x, u(x)) \quad \text{for every } u \in L^q(\partial \Omega)\]

is the associated Nemytskii operator on \( L^q(\partial \Omega) \). Moreover, by (6.4), \( F \) is globally Lipschitz continuous on \( L^q(\partial \Omega) \) with constant \( \omega > 0 \) and \( F(0)(x) = 0 \) for a.e. \( x \in \partial \Omega \).

Since \( g \in L^q(0, T; L'((\partial \Omega))) \) and \( F \) is globally Lipschitz continuous, by the general theory of Nonlinear Semigroups we will assume that \( g \equiv 0 \) and \( f \equiv 0 \).

By Theorem 1.3, given \( u_0 \in L^1(\partial \Omega) \) there is a unique \( u \in C(0, \infty; L^1(\partial \Omega)) \cap W^{1,1}_{loc}(0, \infty; L^1(\partial \Omega)) \), functions \( \hat{u}(t) \in BV(\Omega) \) and \( z(t) \in L^\infty(\Omega, \mathbb{R}^d), \| z(t) \|_\infty \leq 1 \) such that for almost all \( t \geq 0 \), we have

\[
-\text{div}(z(t)) = 0, \quad \text{in } \mathcal{D}'(\Omega)
\]

\[
(z(t), D\hat{u}(t)) = |Du(t)|
\]

\[[z(t), v] \in \text{sign}(u(t) - \hat{u}(t)), \quad \mathcal{H}^{d-1}-\text{a.e. on } \partial \Omega\]

and

\[
\frac{\partial u}{\partial t}(t) + [z(t), v] = 0 \quad \mathcal{H}^{d-1}-\text{a.e. on } \partial \Omega.
\]

Note that (6.5) means that \( \hat{u}(t) \) is a solution of the Dirichlet

\[
-\Delta \hat{u}(t) = 0 \quad \text{in } \Omega,
\]

\[
\hat{u}(t) = u(t) \quad \text{on } \partial \Omega.
\]
Consequently, if we consider the following problem, which consists in an elliptic equation involving the 1-Laplacian and a dynamical boundary condition, namely

\[
\begin{aligned}
&-\Delta_1 \hat{u} = 0 \quad \text{in } \Omega \times (0, \infty), \\
u_t + \frac{\partial \hat{u}}{\|\hat{u}\|}, v = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
&\hat{u} = u \quad \text{on } \partial \Omega \times (0, \infty), \\
u(x, 0) = u_0(x) \quad \text{on } \partial \Omega,
\end{aligned}
\] (6.8)

we have that for every initial data \(u_0 \in L^1(\partial \Omega)\), the problem (6.8) has a unique strong solution.

**Remark 6.1.** Existence and uniqueness of strong solution for a similar problem to problem (6.8) has been obtained in [33]. More precisely the problem studied in [33] is the problem (6.8) but where the first equation is

\[\lambda \hat{u} - \Delta_1 \hat{u} = 0, \quad \text{with } \lambda > 0.\]

Let us point out that since \(\lambda > 0\), we have uniqueness for the Dirichlet problem associated with the above equation, which does not happen in our case where \(\lambda = 0\) and this lack of uniqueness is one of the difficulties of our problem.

**APPENDIX A. APPENDIX: \(j\)-ELLIPTIC FUNCTIONALS**

In this section, we develop an important generalisation of the theory of \(j\)-elliptic functionals developed in [20] by replacing the condition that for a given locally convex topological vector space \((V, \tau)\) and Hilbert space \(H\), the linear map \(j : V \to H\) is weak-to-weak continuous, by the assumption that \(j : V \to H\) is \(\tau_{w}-\text{to-weak continuous}\) for a topology \(\tau_{w}\) on \(V\), which one the one side, is weaker than the initial topology \(\tau\), but on the other side, \(\tau_{w}\) is not necessarily the weak topology associated with \(\tau\).

The advantage of this functional analytic tools is that it can be used to obtain that the negative Dirichlet-to-Neumann operator \(\Lambda\) generates a semi-group in \(L^2(\partial \Omega)\).

Throughout this section, let \((V, \tau_{w})\) be a real topological vector space, and \(H\) be a real Hilbert space equipped with inner product \((\cdot, \cdot)_H\) and \(\sigma(H, H^*)\) denote the weak topology on \(H\). Suppose, there is a \(\tau_{w}\)-to-\(\sigma(H, H^*)\) continuous, linear operator \(j : V \to H\).

By following the same notation as in [20], for a functional \(\varphi : V \to (-\infty, +\infty]\), we call the set \(D(\varphi) := \{\varphi < +\infty\}\) its effective domain, and we say that \(\varphi\) is proper if \(D(\varphi) \neq \emptyset\). Its \(j\)-sub-differential is the (possibly) multi-valued operator

\[\partial_j \varphi := \left\{(u, f) \in H \times H \mid \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{\vartheta} \in V, \liminf_{t \searrow 0} \frac{\varphi(\hat{u} + t\hat{\vartheta}) - \varphi(\hat{u})}{t} \geq (f, j(\hat{\vartheta}))_H \right\}.\]
Next, for $\omega \in \mathbb{R}$, a functional $\varphi : V \to (-\infty, +\infty]$ is called $\omega$-quasi $j$-convex if the “shifted” functional $\varphi_\omega : V \to (-\infty, +\infty]$ defined by
\[
\varphi_\omega(\hat{u}) := \varphi(\hat{u}) + \frac{\omega}{2} \|j(\hat{u})\|_H^2
\]
is convex, and $\varphi$ is simply called quasi $j$-convex if there is an $\omega \in \mathbb{R}$ such that $\varphi$ is $\omega$-quasi $j$-convex. In the case that $\varphi_\omega$ is convex for some $\omega \in \mathbb{R}$, then
\[
\partial_j \varphi = \left\{ (u, f) \in H \times H \left| \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in V, \right. \varphi_\omega(\hat{v}) - \varphi_\omega(\hat{u}) \geq (f + \omega j(\hat{u}), j(\hat{v} - \hat{u}))_H \right\}.
\]

**Remark A.1.** We note that our notion of quasi $j$-convex coincides with the notion $j$-semiconvex used in [20]. Here, we chose the name quasi $j$-convex in order to be consistent with the standard notion quasi contractive and quasi accretive used in nonlinear semigroup theory (cf [11, 12]).

Given a topological space $(V, \tau)$, a functional $\varphi : V \to (-\infty, +\infty]$ is said to be sequentially-$\tau$-lower semicontinuous if for every sequence $(\hat{u}_n)_{n \geq 1}$ in $V$ $\tau$-converging to an element $\hat{u} \in V$ as $n \to \infty$, one has
\[
\varphi(\hat{u}) \leq \liminf_{n \to \infty} \varphi(\hat{u}_n).
\]
Further, a functional $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ is called $\tau$-inf-compact if for any $c \in \mathbb{R}$, the sub-level set $E_c := \{ \hat{u} \in V \mid \varphi(\hat{u}) \leq c \}$ is sequentially relative compact for the topology $\tau$.

Our next definition generalizes the notion of $j$-elliptic functionals in [20, p 418].

**Definition A.2.** A functional $\varphi : V \to (-\infty, +\infty]$ is called $\tau$-$j$-elliptic if there are $\omega \in \mathbb{R}$ such that $\varphi_\omega$ is convex and $\tau$-inf-compact.

**Remark A.3 (The classical notion of sub-differential operators).** (a) If $V$ is a real locally convex topological vector space and $\tau$ the weak topology induced by the topology on $V$, then $\tau$-$j$-ellipticity is equivalent to the notion of $j$-ellipticity as introduced in [20].

(b) There exists a well-established classical setting of sub-differential operators of functionals $\varphi : H \to (-\infty, +\infty]$. This is the setting $V = H$ and $j = I$ the identity operator. Then the $j$-sub-differential $\partial_j \varphi$ coincides with the classical sub-differential operator $\partial \varphi$ defined in the literature; for instance, see Brezis [17] or Rockafellar [43]. In this classical situation, we call $\tau$-$j$-elliptic functionals (where $\tau$ is chosen to be the weak topology $\sigma(H, H^*)$), simply, elliptic functionals, we call the $j$-sub-differential operator, simply, sub-differential operator, and we write $\partial \varphi$ instead of $\partial_j \varphi$.

The next theorem is the main result of this section.

**Theorem A.4.** Let $(V, \tau_\omega)$ be a topological real vector space. If $\varphi : V \to (-\infty, +\infty]$ is convex, proper, sequentially-$\tau_\omega$-lower semicontinuous and $\tau_\omega$-$j$-elliptic. Then the $j$-sub-differential $\partial_j \varphi$ is maximal monotone.

**Proof.** The proof follows the same idea as the one of [20, Theorem 2.6] with the subtle difference that one does not apply Hahn-Banach’s theorem but in stead a topological minimization theorem. For convenience of the reader,
we give the details of the proof. By [20, Lemma 2.4], the \( j \)-sub-differential \( \partial_j \varphi \) is monotone and so, it remains to show that for some \( \hat{\omega} > 0 \), the operator \( \hat{\omega} I + \partial_j \varphi \) is surjective. Then, by taking \( \hat{\omega} > \omega \) (where one can always assume that \( \omega \geq 0 \)) and \( f \in H \), the latter is equivalent to show that the functional \( \psi : V \to \mathbb{R} \cup \{+\infty\} \) defined by
\[
\psi(\hat{u}) := \varphi_{\hat{\omega}}(\hat{u}) - (f, j(\hat{u}))_H \\
\text{for all } \hat{u} \in V,
\]
attains a minimum (cf [20, p 420f]). By hypothesis, \( \psi \) is convex, \( \tau_\omega \)-lower semicontinuous and \( \tau_\omega \)-inf-compact. Thus, by Weierstrass’ minimization theorem in general vector spaces (see [10, Theorem 3.2.1]), there exists some \( \hat{u} \in D(\varphi) \) minimizing \( \psi \) on \( V \).

From Theorem A.4, we can immediately conclude the following consequence (cf [20, Corollary 2.7]).

**Corollary A.5.** Let \((V, \tau_\omega)\) be a topological real vector space and assume \( \varphi \) is quasi \( j \)-convex. Then there exists a proper, lower semicontinuous, elliptic functional \( \varphi^H : H \to (-\infty, +\infty] \) such that \( \partial_j \varphi \subseteq \partial \varphi^H \). If, in addition, \( \varphi \) is proper, sequentially-\( \tau_\omega \)-lower semicontinuous and \( \tau_\omega \)-j-elliptic, then \( \partial_j \varphi = \partial \varphi^H \), and \( \partial_j \varphi_{\omega} = \omega I + \partial_j \varphi \) is maximal monotone for some \( \omega \geq 0 \).

From Theorem A.4 and Corollary A.5 and the classical theory of evolution equations governed by sub-differential operators in Hilbert spaces (see [17] and also [11]) imply the following well-posedness of Cauchy problem (A.1) or also-called generation theorem. Here, \( \mathbb{R}_+ \) denotes the half-closed positive real line \([0, +\infty)\), and for an operator \( A \subseteq H \times H \), the operator \( A^\circ : D(A) \to H \) defined by \( A^\circ u := \arg\min\{\|v\|_H | v \in Au\} \) denotes the minimal selection of \( A \).

**Theorem A.6.** Let \((V, \tau_\omega)\) be a topological real vector space, \( H \) a real Hilbert space and \( j : V \to H \) a \( \tau_\omega \)-to-\( \sigma \)(\( H, H^* \)) continuous linear operator. Let \( \varphi : V \to \mathbb{R} \cup \{+\infty\} \) be proper, sequentially-\( \tau_\omega \)-lower semicontinuous and \( \tau_\omega \)-j-elliptic. Then for every initial value \( u_0 \in D(\varphi^H)^\prime = j(D(\varphi))^\prime \) the Cauchy problem

\[
\begin{align*}
\frac{du}{dt} + \partial_j \varphi(u) &\geq 0 \quad \text{on } (0, +\infty) \\
u(0) &= u_0
\end{align*}
\]

admits a unique solution
\[
u \in C(\mathbb{R}_+, H) \cap W^{1,\infty}_{\text{loc}}((0, \infty); H)
\]
satisfying \(-\frac{du}{dt}(t) \in \partial_j \varphi(u(t)) \) for almost every \( t > 0 \), \( u(t) \in D(\partial_j \varphi) \) and \( u \) is right differentiable at every \( t > 0 \), and

\[
\frac{du}{dt}(t) + (\partial_j \varphi)^\prime u(t) = 0 \quad \text{for all } t > 0.
\]

Denoting by \( u \) the unique solution corresponding to the initial value \( u_0 \), setting \( T_t u_0 := u(t), (t \geq 0) \), defines a strongly continuous semigroup \( \{T_t\}_{t \geq 0} \) of mappings \( T_t : j(D(\varphi))^\prime \to j(D(\varphi))^\prime \) satisfying

\[
T_t D(\partial_j \varphi) \subseteq D(\partial_j \varphi) \quad \text{for all } t > 0,
\]
and
\[
\| T_t u_0 - T_t u_1 \|_H \leq e^{\omega t} \| u_0 - u_1 \|_H \quad \text{for every } t \geq 0, u_0, u_1 \in \overline{j(D(\phi))}^H,
\]
where \( \omega \in \mathbb{R} \) is the minimal among all \( \omega \in \mathbb{R} \) such that \( \varphi_{\omega} \) is convex.

Proof of Theorem A.6. The well-posedness of Cauchy problem (A.1) and the fact that the semigroup \( \{ T_t \}_{t \geq 0} \) satisfies (A.4) is shown in [20, Theorem 3.1]. It remains to show that for every \( 0 \in \overline{j(D(\phi))}^H \), the unique solution \( u \) of (A.1) satisfies \( u(t) \in D(\partial_\phi) \), is right differentiable and (A.2) holds. By hypothesis and Corollary A.5, there is an \( \omega \in \mathbb{R} \) and a proper, convex, lower semicontinuous functional \( \varphi^H_{\omega} : H \to (-\infty, +\infty] \) such that \( \partial \varphi^H_{\omega} = \partial_\phi = \omega I_H + \partial_\phi \) is maximal monotone. Thus, taking \( f(t) = \omega u(t) \), it follows from [11, Corollary 4.4] that for every \( 0 \in \overline{j(D(\phi))}^H \), the unique solution \( u \) of (A.1) satisfies \( u(t) \in D(\partial_\phi) \), is right differentiable and (A.2) holds. \( \square \)

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