Unique expansions and intersections of Cantor sets

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Abstract

To each $\alpha \in (1/3, 1/2)$ we associate the Cantor set

$$\Gamma_\alpha := \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\}, \ i \geq 1 \right\}.$$ 

In this paper we consider the intersection $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ for any translation $t \in \mathbb{R}$. We pay special attention to those $t$ with a unique $\{-1, 0, 1\}$-$\alpha$-expansion, and study the set

$$D_\alpha := \{ \dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \ has \ a \ unique \ {-1, 0, 1} \ \alpha\text{-expansion} \}.$$ 

We prove that there exists a transcendental number $\alpha_{KL} \approx 0.39433 \ldots$ such that: $D_\alpha$ is finite for $\alpha \in (\alpha_{KL}, 1/2)$, $D_{\alpha_{KL}}$ is infinitely countable, and $D_\alpha$ contains an interval for $\alpha \in (1/3, \alpha_{KL})$. We also prove that $D_\alpha$ equals $[0, \log 2 - \log \alpha]$ if and only if $\alpha \in (1/3, 3/2 - \sqrt{5}/2]$. As a consequence of our investigation we prove some results on the possible values of $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$ when $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a self-similar set. We also give examples of $t$ with a continuum of $\{-1, 0, 1\}$-$\alpha$-expansions for which we can explicitly calculate $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$, and for which $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a self-similar set. We also construct $\alpha$ and $t$ for which $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains only transcendental numbers.

Our approach makes use of digit frequency arguments and a lexicographic characterisation of those $t$ with a unique $\{-1, 0, 1\}$-$\alpha$-expansion.
1. Introduction

To each $\alpha \in (0, 1/2)$ we associate the contracting similarities $f_0(x) = \alpha x$ and $f_1(x) = \alpha(x + 1)$. The middle $(1 - 2\alpha)$ Cantor set $\Gamma_\alpha$ is defined to be the unique compact non-empty set satisfying the equation

$$\Gamma_\alpha = f_0(\Gamma_\alpha) + f_1(\Gamma_\alpha).$$

It is easy to see that the maps $\{f_0, f_1\}$ satisfy the strong separation condition. Thus $\dim_H(\Gamma_\alpha) = \frac{-\log 2}{\log \alpha}$, where $\dim_H$ and $\dim_B$ denote the Hausdorff dimension and box dimension respectively.

A natural and well studied question is ‘What are the properties of the intersection $\Gamma_\alpha \cap (\Gamma_\alpha + t)$?’ This question has been studied by many authors. We refer the reader to [10, 15–18] and the references therein for more information. As we now go on to explain, when $\alpha \in (0, 1/3]$ the set $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is well understood, however when $\alpha \in (1/3, 1/2)$ additional difficulties arise.

Note that $\Gamma_\alpha \cap (\Gamma_\alpha + t) \neq \emptyset$ if and only if $t \in \Gamma_\alpha - \Gamma_\alpha$. Thus it is natural to investigate the difference set $\Gamma_\alpha - \Gamma_\alpha$, which is the self-similar set generated by the iterated function system $\{f_{-1}, f_0, f_1\}$, where $f_{-1}(x) = \alpha(x - 1)$. Alternatively, one can write

$$\Gamma_\alpha - \Gamma_\alpha := \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{-1, 0, 1\}, i \geq 1 \right\}.$$

Importantly, for $\alpha \in (0, 1/3)$ each $t \in \Gamma_\alpha - \Gamma_\alpha$ has a unique $\alpha$-expansion with alphabet $\{-1, 0, 1\}$, i.e. there exists a unique sequence $(t_i) \in \{-1, 0, 1\}^\mathbb{N}$ such that $t = \sum t_i \alpha^i$. When $\alpha = 1/3$ there is a countable set of $t$ with precisely two $\alpha$-expansions. These $t$ are well understood and do not pose any real difficulties, thus in what follows we suppress the case where $t$ has two $\alpha$-expansions.

For $\alpha \in (0, 1/3]$ let $t \in \Gamma_\alpha - \Gamma_\alpha$ have a unique $\alpha$-expansion $(t_i)$. Then the sequence $(t_i)$ provides a useful description of the set $\Gamma_\alpha \cap (\Gamma_\alpha + t)$. Indeed, we can write (see [18])

$$\Gamma_\alpha \cap (\Gamma_\alpha + t) = \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\} \cap \{0, 1\} + t_i \right\}.$$  

(1)

With this new interpretation many questions regarding the set $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ can be reinterpreted and successfully answered using combinatorial properties of the $\alpha$-expansion $(t_i)$.

The straightforward description of $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ provided by (1) does not exist for $\alpha \in (1/3, 1/2)$ and a generic $t \in \Gamma_\alpha - \Gamma_\alpha$. The set $\Gamma_\alpha - \Gamma_\alpha$ is still a self-similar set generated by the transformations $\{f_{-1}, f_0, f_1\}$, however this set is now equal to the interval $[\frac{-\alpha}{1-\alpha}, \frac{1}{1-\alpha}]$ and the good separation properties that were present in the case where $\alpha \in (0, 1/3]$ no longer exist. It is possible that a $t \in \Gamma_\alpha - \Gamma_\alpha$ could have many $\alpha$-expansions. In fact it can be shown that Lebesgue almost every $t \in \Gamma_\alpha - \Gamma_\alpha$ has a continuum of $\alpha$-expansions (see [4, 21, 22]). Thus
within the parameter space (1/3, 1/2) we are forced to have the following more complicated interpretation of $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ (see [18, lemma 3.3])

$$\Gamma_\alpha \cap (\Gamma_\alpha + t) = \bigcup_{t} \left\{ \sum_{i=1}^{\infty} c_i \alpha^i : c_i \in \{0, 1\} \cap (\{0, 1\} + \tilde{t}) \right\}, \quad (2)$$

where the union is over all $\alpha$-expansions $\tilde{t} = (\tilde{t}_i)$ of $t$. As stated above, for a generic $t$ this union is uncountable, this makes many questions regarding the set $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ intractable. In what follows we focus on the case where $t$ has a unique $\alpha$-expansion. For these $t$ the description of $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ given by (2) simplifies to that given by (1).

We now introduce some notation. For $\alpha \in (0, 1/2)$ let

$$\mathcal{U}_\alpha := \{ t \in \Gamma_\alpha - \Gamma_\alpha : t \ has \ a \ unique \ \alpha \text{-expansion w.r.t. } \{-1, 0, 1\}\}.$$ 

Within this paper one of our main objects of study is the following set $\mathcal{U}_\alpha := \{ \dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in \mathcal{U}_\alpha \}$. In particular we will prove the following theorems.

**Theorem 1.1.** There exists a transcendental number $\alpha_{KL} \approx 0.39433 \ldots$ such that:

(i) For $\alpha \in (\alpha_{KL}, 1/2)$ there exists $n^* \in \mathbb{N}$ such that

$$D_\alpha = \left\{ 0, \frac{\log 2}{-\log \alpha} \right\} \cup \left\{ \frac{\log 2}{\log \alpha} \sum_{i=1}^{n} \left( \frac{-1}{2} \right)^i : 1 \leq n \leq n^* \right\}.$$ 

(ii) $D_{\alpha_{KL}} = \left\{ 0, \frac{\log 2}{-\log \alpha_{KL}}, \frac{\log 2}{3 \log \alpha_{KL}} \right\}$

$$\cup \left\{ \frac{\log 2}{\log \alpha_{KL}} \sum_{i=1}^{n} \left( \frac{-1}{2} \right)^i : 1 \leq n < \infty \right\}.$$ 

(iii) $D_\alpha$ contains an interval if $\alpha \in (1/3, 1/2]$.

In [18] it was asked ‘When $\alpha \in (1/3, 1/2]$ what are the possible values of $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$ for $t \in \Gamma_\alpha - \Gamma_\alpha$?’ The following theorem provides a partial solution to this problem.

**Theorem 1.2.**

(i) If $\alpha \in (1/3, \frac{3 - \sqrt{5}}{2}]$ then $D_\alpha = \left[ 0, \frac{\log 2}{-\log \alpha} \right].$

(ii) If $\alpha \in (\frac{1 - \sqrt{5}}{2}, 1/2) \text{ then } D_\alpha \text{ is a proper subset of } \left[ 0, \frac{\log 2}{-\log \alpha} \right].$

Amongst $\Gamma_\alpha - \Gamma_\alpha$ a special class of $t$ are those for which $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a self-similar set. Determining whether $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a self-similar set is a difficult problem for a generic $t$ with many $\alpha$-expansions, thus we consider only those $t \in \mathcal{U}_\alpha$. Let

$$S_\alpha := \{ t \in \mathcal{U}_\alpha : \Gamma_\alpha \cap (\Gamma_\alpha + t) \text{ is a self - similar set } \}.$$ 

We prove the following result.

**Theorem 1.3.**

(i) If $\alpha \in (1/3, \frac{3 - \sqrt{5}}{2}]$ then $\{ \dim(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in S_\alpha \}$ is dense in $[0, \frac{\log 2}{-\log \alpha}].$

(ii) If $\alpha \in (\frac{1 - \sqrt{5}}{2}, 1/2) \text{ then } \{ \dim(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in S_\alpha \}$ is not dense in $[0, \frac{\log 2}{-\log \alpha}].$
What remains of this paper is arranged as follows. In section 2 we recall the necessary preliminaries from expansions in non-integer bases, and recall an important result of [18] that connects the dimension of $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ with the frequency of 0’s in the $\alpha$-expansion ($t$). In section 3 we prove theorem 1.1, and in section 4 we prove theorems 1.2 and 1.3. In section 5 we include some examples. We give two examples of an $\alpha \in (1/3, 1/2)$, and $t \in \Gamma_\alpha - \Gamma_\alpha$, with a continuum of $\alpha$-expansions, for which we can explicitly calculate $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$. The techniques used in our first example can be applied to the more general case where $\alpha$ is the reciprocal of a Pisot number and $t \in \mathbb{Q}(\alpha)$. Our second example demonstrates that it is possible for $t$ to have a continuum of $\alpha$-expansions and for $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ to be a self-similar set. Moreover, both of these examples show that it is possible to have

$$\dim_H(\Gamma_\alpha + (\Gamma_\alpha + t)) > \sup_t \dim_H \left( \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\} \cap ((0, 1] + t) \right\} \right).$$

Our final example demonstrates the existence of $\alpha \in (1/3, 1/2)$ and $t \in \Gamma_\alpha - \Gamma_\alpha$ for which $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains only transcendental numbers.

2. Preliminaries

Let $M \in \mathbb{N}$ and $\alpha \in [-1/\sum_{i=1}^{\infty} \epsilon_i \alpha^i]$. Given $x \in I_{\alpha, M} := [0, M\alpha_1/1-\alpha]$, we call a sequence $(\epsilon_i) \in \{0, \ldots, M\}^{\mathbb{N}}$ an $\alpha$-expansion for $x$ with alphabet $\{0, \ldots, M\}$ if

$$x = \sum_{i=1}^{\infty} \epsilon_i \alpha^i.$$ 

This method of representing real numbers was pioneered in the early 1960’s in the papers of Rényi [20] and Parry [19]. One aspect of these representations that makes them interesting is that for $\alpha \in (1/M, 1)$ a generic $x \in I_{\alpha, M}$ has many $\alpha$-expansions (see [4, 21, 22]). This naturally leads researchers to study the set of $x \in I_{\alpha, M}$ with a unique $\alpha$-expansion, the so called univoque set. We define this set as follows

$$U_{\alpha, M} := \{ x \in I_{\alpha, M} : x \text{ has a unique } \alpha \text{-expansion w.r.t. } \{0, \cdots, M\} \}.$$ 

Accordingly, let $\widehat{U}_{\alpha, M}$ denote the set of corresponding expansions, i.e.

$$\widehat{U}_{\alpha, M} := \left\{ (\epsilon_i) \in \{0, \ldots, M\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \epsilon_i \alpha^i \in U_{\alpha, M} \right\}.$$ 

The sets $U_{\alpha, M}$ and $\widehat{U}_{\alpha, M}$ have been studied by many authors. For more information on these sets we refer the reader to [5–7, 9, 11, 12] and the references therein. Before continuing with our discussion of the sets $U_{\alpha, M}$ and $\widehat{U}_{\alpha, M}$ we make a brief remark. In the introduction we were concerned with $\alpha$-expansions with digit set $\{-1, 0, 1\}$, not with a digit set $\{0, \ldots, M\}$. However, all of the result that are stated below for a digit set $\{0, \ldots, M\}$ also hold for any digit set of $M + 1$ consecutive integers $\{s, \ldots, s + M\}$. In particular, statements that are true for the digit set $\{0, 1, 2\}$ translate to results for the digit set $\{-1, 0, 1\}$ by performing the substitutions $0 \rightarrow -1, 1 \rightarrow 0, 2 \rightarrow 1$.

We now define the lexicographic order and introduce some notations. Given two finite sequences $\omega = (\omega_1, \ldots, \omega_s), \omega' = (\omega'_1, \ldots, \omega'_t) \in \{0, \ldots, M\}^n$, we say that $\omega$ is less than $\omega'$ with respect to the lexicographic order, or simply write $\omega < \omega'$, if $\omega_1 < \omega'_1$ or if there exists $1 \leq j < n$...
such that $\omega_i = \omega'_i$ for $1 \leq i \leq j$ and $\omega_{j+1} < \omega'_{j+1}$. One can also define the relations $\leq, >, \geq$ in the natural way, and we can extend the lexicographic order to infinite sequences. We define the reflection of a finite/infinite sequence $(\epsilon_i)$ to be $(M - \epsilon_i)$, where the underlying $M$ should be obvious from our context. For a finite sequence $\omega = (\omega_1, \ldots, \omega_n)$ we define the finite sequence $\omega^-$ to be $(\omega_1, \ldots, \omega_n - 1)$. Moreover, we denote the concatenation of $\omega$ with itself $n$ times by $\omega^n$, we also let $\omega^\infty$ denote the infinite sequence obtained by indefinitely concatenating $\omega$ with itself.

Given $x \in I_{\alpha, M}$ we define the greedy $\alpha$-expansion of $x$ to be the lexicographically largest sequence amongst the $\alpha$-expansions of $x$. We define the quasi-greedy $\alpha$-expansion of $x$ to be the lexicographically largest infinite sequence amongst the $\alpha$-expansions of $x$. Here we call a sequence $(\epsilon_i)$ infinite if $\epsilon_i \neq 0$ for infinitely many $i$. When studying the sets $\alpha M$ and $\sim \alpha M$, a pivotal role is played by the quasi-greedy $\alpha$-expansion of $1$. In what follows we will denote the quasi-greedy $\alpha$-expansion of $1$ by $\delta_\alpha$.

**Lemma 2.1.** A sequence $(\epsilon_i)$ belongs to $\sim \alpha M$ if and only if the following two conditions are satisfied:

\[ (\epsilon_{n+1}) \prec (\delta_\alpha) \text{ whenever } \epsilon_1 \ldots \epsilon_n = M^n \]

\[ (\overline{\epsilon_{n+1}}) \prec (\delta_\alpha) \text{ whenever } \epsilon_1 \ldots \epsilon_n = 0^n \]

Lemma 2.1 provides a useful characterisation of the set $\sim \alpha M$ in terms of the sequence $\delta_\alpha$. The following lemma describes the sequences $\delta_\alpha$.

**Lemma 2.2.** Let $M \in \mathbb{N}$, $\alpha \in [\frac{1}{M+1}, 1)$ and $(\delta_\alpha)$ be the quasi-greedy $\alpha$-expansion of $1$. The map $\alpha \mapsto (\delta_\alpha)$ is a strictly decreasing bijection from the interval $[\frac{1}{M+1}, 1)$ onto the set of all infinite sequences $(\delta_i) \in \{0, \ldots, M\}^\mathbb{N}$ satisfying

\[ \delta_{k+1} \delta_{k+2} \ldots \leq \delta_1 \delta_2 \ldots \text{ for all } k \geq 0. \]

The following technical result was proved in [18, theorem 3.4] for $\alpha \in (0, 1/3]$, where importantly every $t$ has a unique $\alpha$-expansion, except for $\alpha = 1/3$ where countably many $t$ have two $\alpha$-expansions. The proof translates over to the more general case where $\alpha \in (1/3, 1/2)$ and $t \in U_\alpha$.

**Lemma 2.3.** Let $\alpha \in (1/3, 1/2)$ and $t \in U_\alpha$, then

\[ \text{dim}_0(\Gamma_\alpha \cap (\Gamma_\alpha + t)) = \frac{\log 2}{-\log \alpha} d(t_i), \]

where

\[ d(t_i) := \lim \inf_{n \to \infty} \frac{\#\{1 \leq i \leq n : t_i = 0\}}{n}. \]

Lemma 2.3 will be a vital tool in proving theorems 1.1 and 1.2. This result allows us to reinterpret theorems 1.1 and 1.2 in terms of statements regarding the frequency of $0$’s that can occur within an element of $\tilde{U}_\alpha$.

In what follows, for an infinite sequence $(t_i) \in \{-1, 0, 1\}^\mathbb{N}$ we will use the notation

\[ \overline{d}(t_i) := \lim \sup_{n \to \infty} \frac{\#\{1 \leq i \leq n : t_i = 0\}}{n}. \]
When this limit exists, i.e. \( d((t_i)) = \bar{d}((t_i)) \), we simply use \( d(t) \). For a finite sequence \( t_1, \ldots, t_n \in \{-1, 0, 1\}^n \) we will use the notation
\[
d(t_1 \cdots t_n) := \frac{\#\{1 \leq i \leq n : t_i = 0\}}{n}.
\]

3. Proof of theorem 1.1

In this section we prove theorem 1.1. We start by defining the Thue-Morse sequence and its natural generalisation.

Let \((\tau_i)_{i=0}^\infty \in \{0, 1\}^\mathbb{N}\) denote the classical Thue-Morse sequence. This sequence is defined iteratively as follows. Let \(\tau_0 = 0\) and if \(\tau_i\) is defined for some \(i \geq 0\), set \(\tau_{2i} = \tau_i\) and \(\tau_{2i+1} = 1 - \tau_i\). Then the sequence \((\tau_i)_{i=0}^\infty\) begins with
\[
01101001011010010110010110 \ldots
\]

For more on this sequence we refer the reader to [1]. Within expansions in non-integer bases the sequence \((\tau_i)_{i=0}^\infty\) is important for many reasons. In [13] Komornik and Loreti proved that the unique \(\alpha\) for which \(\delta_\alpha \tau_i \equiv \infty\) is the largest \(\alpha \in (1/2, 1)\) for which 1 has a unique \(\alpha\)-expansion. This \(\alpha\) has since become known as the Komornik–Loreti constant. Interesting connections between the size of \(\alpha\) and the Komornik Loreti constant were made in [9]. Using the Thue-Morse sequence we define a new sequence \((\lambda_i)_{i=0}^\infty \in \{-1, 0, 1\}^\mathbb{N}\) as follows
\[
(\lambda_i)_{i=0}^\infty = (\tau_i - \tau_{i-1})_{i=1}^\infty.
\]

We denote the unique \(\alpha \in (1/2, 1)\) for which \(\sum_{i=1}^\infty (1 + \lambda_i)\alpha^i = 1\) by \(\alpha_{KL}\). Our choice of subscript is because the constant \(\alpha_{KL}\) is a type of generalised Komornik–Loreti constant. This number is transcendental (see [14]) and is approximately 0.39433. This sequence satisfies the property
\[
\begin{align*}
\lambda_i &= 1, \\
\lambda_{i+2^n} &= 1 - \lambda_i, \\
\lambda_{i+2^n+1} &= -\lambda_i \\
\text{for any } 1 \leq i < 2^n.
\end{align*}
\]

This property can be deduced directly from [14, lemma 5.2]. So, the sequence \((\lambda_i)_{i=0}^\infty\) starts at
\[
10 (-1)1 (-1)010 (-1)01(-1) 10(-1)1 \ldots
\]

It will be useful when it comes to determining the frequency of zeros within certain sequences.

To each \(n \in \mathbb{N}\) we associate the finite sequence \(w_n = \lambda_1 \cdots \lambda_{2^n}\). By (3) the following property of \(w_n\) can be verified.
\[
w_{n+1}^+ = w_n w_n^-
\]

Here the reflection of \(w_n\) w.r.t. the digit set \{-1, 0, 1\} is defined by \(w_n^- := (-\lambda_1)(-\lambda_2) \cdots (-\lambda_{2^n})\), which is the concatenation of the digits \((-\lambda_1), (-\lambda_2), \cdots, (-\lambda_{2^n})\).

We now prove two lemmas that allow us to prove statements (i) and (ii) from theorem 1.1.

**Lemma 3.1.** For \(n \geq 2\) the following inequalities hold:
\[
\begin{align*}
\#\{1 \leq i \leq 2^n : \lambda_i = 0\} &= 2\#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} - 1 \text{ if } n \text{ is even;} \\
\#\{1 \leq i \leq 2^n : \lambda_i = 0\} &= 2\#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} + 1 \text{ if } n \text{ is odd.}
\end{align*}
\]
Moreover

\[ d(w_n) = -\sum_{i=1}^{n} \left( \frac{-1}{2} \right)^i \]  \hspace{1cm} (7)

for all \( n \in \mathbb{N} \).

**Proof.** We begin by observing that \( w_1 = 10, \) so \( d(w_1) = 1/2 \) and (7) holds for \( n = 1 \). We now show that (5) and (6) imply (7) via an inductive argument. Let us assume (7) is true for odd \( N \in \mathbb{N} \). Then

\[
\sum_{\lambda = 0}^{N} \sum_{i=1}^{2N+1} (-1)^i \lambda = -\sum_{i=1}^{N+1} \left( \frac{-1}{2} \right)^i .
\]

In our second equality we used (5). The case where \( N \) is even is done similarly. Proceeding inductively we may conclude that (7) holds assuming (5) and (6).

It remains to show (5) and (6) hold. For \( n = 1 \) we know that \( w_1 = 10, \) (4) therefore implies that the last digit of \( w_2 \) equals 1. What is more, repeatedly applying (4) we see that the last digit of \( w_n \) equals 0 if \( n \) is odd, and equals 1 if \( n \) is even. Property (3) implies that \( \lambda_{2i+1} = 0 \) if \( \lambda_i = 0 \) for any \( 1 \leq i < 2^n \). Therefore, when \( n \) is even we see that

\[
\#\{1 \leq i \leq 2^n : \lambda_i = 0\} = \#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} + \#\{2^n-1 + 1 \leq i \leq 2^n : \lambda_i = 0\}
\]

Thus (5) is proved. Equation (6) is proved similarly. \( \square \)

Lemma 3.1 determines the frequency of 0’s within the finite sequences \( w_n \). For our proof of theorem 1.1 we also need to know the frequency of 0’s within the sequence \( (\lambda_n)_{n=1}^\infty \).

**Lemma 3.2.**

\[ d((\lambda_n)) = -\sum_{i=1}^{\infty} \left( \frac{-1}{2} \right)^i = \frac{1}{3}. \]

**Proof.** Let us begin by fixing \( \varepsilon > 0 \). Let \( N \in \mathbb{N} \) be sufficiently large such that

\[
\left| -\sum_{i=1}^{N} (-1/2)^i - 1 \right| < \varepsilon
\]

for all \( n \geq N \). Now let us pick \( N' \in \mathbb{N} \) large enough such that
\[
\frac{\sum_{j=0}^{N-1} 2^j}{N'} < \epsilon \tag{9}
\]

Let \( n \gg N' \) be arbitrary and write \( n = \sum_{j=0}^{k} \epsilon_j 2^j \), where we assume \( \epsilon_k = 1 \). By splitting \((\lambda_i)_{i=1}^n\) into its first \(2^k\) digits, then the next \(2^{k-1}\) digits, then the next \(2^{k-2}\) digits, etc we obtain:

\[
\frac{\# \{1 \leq i \leq n : \lambda_i = 0\}}{n} = \frac{\# \{1 \leq i \leq 2^k : \lambda_i = 0\}}{n} + \sum_{l=0}^{k-1} \frac{\# \{1 \leq i \leq \sum_{j=k-l-1}^{k-1} \epsilon_j 2^j : \lambda_i = 0\}}{n} \tag{10}
\]

By repeatedly applying (3) we see

\[
\frac{\# \{1 \leq i \leq \epsilon_k \cdots \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\}}{n} = \frac{\# \{1 \leq i \leq \epsilon_k \cdots \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\}}{n} = \cdots = \frac{\# \left\{ \sum_{j=k-l}^{k} \epsilon_j 2^j + 1 \leq i \leq \sum_{j=k-l-1}^{k} \epsilon_j 2^j : \lambda_i = 0 \right\}}{n} \tag{11}
\]

Substituing (11) into (10) we obtain

\[
\frac{\# \{1 \leq i \leq n : \lambda_i = 0\}}{n} = \frac{\# \{1 \leq i \leq 2^k : \lambda_i = 0\}}{n} + \sum_{l=0}^{k-1} \frac{\# \{1 \leq i \leq \epsilon_k \cdots \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\}}{n},
\]

By ignoring lower order terms and applying lemmas 3.1, (8) and (9) we obtain the lower bound

\[
\frac{\# \{1 \leq i \leq n : \lambda_i = 0\}}{n} \geq \frac{\# \{1 \leq i \leq 2^k : \lambda_i = 0\}}{n} + \frac{k-n-1}{n} \sum_{l=0}^{k} \frac{\# \{1 \leq i \leq \epsilon_k \cdots \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\}}{n} \geq \frac{(1-\epsilon)}{3} \left( \frac{2^k}{n} + \frac{k-n-1}{n} \sum_{l=0}^{k} \frac{\epsilon_{k-l-l} 2^{k-l-1}}{n} \right) = \frac{(1-\epsilon)}{3} \left( \frac{\sum_{j=0}^{k} \epsilon_j 2^j}{n} - \sum_{j=0}^{k} \epsilon_j 2^j \right) \geq \frac{(1-\epsilon)}{3} \left( 1 - \frac{\sum_{j=0}^{k} \epsilon_j 2^j}{n} \right) \geq \frac{(1-\epsilon)^2}{3}.
\]
As $\varepsilon > 0$ was arbitrary this implies $d((\lambda_i)) \geq 1/3$. By a similar argument it can be shown that $\bar{d}(\lambda_i) \leq 1/3$. Thus $d((\lambda_i)) = 1/3$.

Statements (i) and (ii) of theorem 1.1 follow from lemmas 2.3, 3.1, and 3.2, when combined with the following results from [15, lemma 4.12].

**Lemma 3.3.** Let $\alpha \in (\alpha_{KL}, 1/2)$, then there exists $n^* \in \mathbb{N}$ such that every element of $\widetilde{U}_n \setminus \{(-1)^\infty, F^\infty\}$ ends with one of 

$$(0)^\infty, (w_1 \pi_1)^\infty, \ldots, (w_n \pi_n)^\infty.$$ 

**Lemma 3.4.** Each element of $\widetilde{U}_{\alpha_{KL}} \setminus \{(-1)^\infty, F^\infty\}$ is either eventually periodic with period contained in 

$$(0)^\infty, (w_1 \pi_1)^\infty, (w_2 \pi_2)^\infty, \ldots,$$

or ends with a sequence of the form 

$$(w_0 \pi_0)^{k_0}(w_0 \pi_0)^{k_1}(w_0 \pi_0)^{k_2}(w_0 \pi_0)^{k_3} \cdots (w_0 \pi_0)^{k_n}(w_0 \pi_0)^{k_1} \cdots,$$

and its reflection, where $k_n \geq 0$, $k_1 \in \{0, 1\}$ and 

$$0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots.$$

By lemmas 2.3, 3.1 and 3.3 we may conclude 

$$D(\alpha) = \left\{ 0, \frac{\log 2}{-\log \alpha} \cup \left\{ \frac{\log 2}{\log \alpha} \sum_{i=1}^{n} \frac{-1}{2} \right\} : 1 \leq n \leq n^* \right\}$$

for some $n^* \in \mathbb{N}$ for $\alpha \in (\alpha_{KL}, 1/2)$. Whilst at the constant $\alpha_{KL}$ by lemmas 2.3, 3.1, 3.2 and 3.4 we have 

$$D(\alpha_{KL}) = \left\{ 0, \frac{\log 2}{-\log \alpha_{KL}}, \frac{\log 2}{-3 \log \alpha_{KL}} \right\} \cup \left\{ \frac{\log 2}{\log \alpha_{KL}} \sum_{i=1}^{n} \frac{-1}{2} : 1 \leq n < \infty \right\}.$$ 

Thus statements (i) and (ii) from theorem 1.1 hold. It remains to prove statement (iii).

We start by introducing the following finite sequences. Let 

$\zeta_0 = 0\lambda_1 \cdots \lambda_{2^n-1}$ and $\eta_n = (-1)^{\lambda_1} \cdots \lambda_{2^n-1}$. 

(12)

The following result was proved in [15].

**Lemma 3.5.** Let $\alpha \in (1/3, \alpha_{KL})$, then there exists $n \in \mathbb{N}$ such that $\widetilde{U}_n$ contains the subshift of finite type over the alphabet $A = \{ \zeta_0, \eta_n, \pi_0, \pi_n \}$ with transition matrix 

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$ 

**Proof of theorem 1.1 (iii).** Let $\alpha \in (1/3, \alpha_{KL})$ and let $n$ be as in lemma 3.5. So $\widetilde{U}_n$ contains the subshift of finite type determined by the alphabet $A$ and the transition matrix $A$. On closer
examination we see that this subshift of finite type allows the free concatenation of the words \( \omega_1 = \zeta n_1 \zeta \) and \( \omega_2 = \zeta n_2 \zeta \). Importantly \( d(\omega_1) < d(\omega_2) \) by \((12)\). For any \( c \in [d(\omega_1), d(\omega_2)] \) we can pick a sequence of integers \( k_1, k_2, \ldots \) such that the sequence \( (\epsilon_i) = \omega_1^{k_1} \omega_2^{k_2} \omega_1^{k_3} \omega_2^{k_4} \ldots \) satisfies \( d((\epsilon_i)) = c \). Thus by lemma 2.3 the set \( D(\alpha) \) contains the interval \( \left[ \frac{\log 2}{-\log \alpha} d(\omega_1), \frac{\log 2}{-\log \alpha} d(\omega_2) \right] \) and our proof is complete.

Appealing to standard arguments from multifractal analysis we could in fact show that for any \( \omega \in \mathbb{C}_{-1,2} \), there exists a set of positive Hausdorff dimension within \( \mathbb{U}_\alpha \) with frequency \( c \).

### 4. Proof of theorems 1.2 and 1.3

We start this section by proving theorem 1.2. Theorem 1.3 will follow almost immediately as a consequence of the arguments used in the proof of theorem 1.2. To prove theorem 1.2 we rely on the lexicographic description of \( \mathbb{U}_\alpha \) and \( (\delta(\alpha)) \) given in section 2. We take this opportunity to again emphasise that the preliminary results that hold in section 2 for the alphabet \( \{0, 1, 2\} \) have an obvious analogue that holds for the digit set \( \{-1, 0, 1\} \).

It is instructive here to state our analogue of the quasi greedy \( \alpha \)-expansion of 1 when \( \alpha = \frac{3-\sqrt{5}}{2} \) and our digit set is \( \{-1, 0, 1\} \). A straightforward calculation proves that this analogue satisfies

\[
\left( \delta \left( \frac{3-\sqrt{5}}{2} \right) \right) = 1(0)^\infty.
\]

We split our proof of theorem 1.2 into two lemmas.

**Lemma 4.1.** Let \( \alpha \in \left( \frac{3-\sqrt{5}}{2}, 1/2 \right) \), then there exists \( n \in \mathbb{N} \) such that any element of \( \widetilde{\mathbb{U}}_\alpha \) cannot contain the sequence \( 1(0)^n \) or \( (-1)(0)^n \) infinitely often.  

**Proof.** Suppose \( \alpha \in \left( \frac{3-\sqrt{5}}{2}, 1/2 \right) \). Then by lemmas 2.2 and (13) we have

\[
(\delta(\alpha)) \prec (1(0)^\infty).
\]

For any \( \alpha \in (1/3, 1/2) \) we have \( \delta(\alpha) = 1 \). Therefore by \((14)\) there exists \( k \geq 0 \) such that \( (\delta(\alpha)) \) begins with the word \( 1(0)^k (-1) \). If a sequence \( (\epsilon_i) \in \widetilde{\mathbb{U}}_\alpha \) contained the sequence \( 1(0)^{k+1} \) infinitely often, then it is a consequence of lemma 2.1 for the digit set \( \{-1, 0, 1\} \) that the following lexicographic inequalities would have to hold

\[
(-1)(0)^k 1 \leq 1(0)^{k+1} \leq 1(0)^k (-1).
\]

Clearly the right hand side of \((15)\) does not hold, therefore \( 1(0)^{k+1} \) cannot occur infinitely often. Similarly, one can show that \( (-1)(0)^{k+1} \) cannot occur infinitely often by considering the left hand side of \((15)\).  

**Lemma 4.2.** If \( \alpha \in (1/3, \frac{3-\sqrt{5}}{2}] \) then for any sequence of natural numbers \( (n_i) \) the sequence

\[
1(-1)^{n_i} 0^{n_i} (1(-1)^{n_i} 0^{n_i}) \ldots
\]

is contained in \( \widetilde{\mathbb{U}}_\alpha \).
Proof. Fix a sequence of natural numbers \((n_i)\). It is a consequence of lemmas 2.1 and 2.2 that \(\tilde{u}_{\alpha - 3/2} \subseteq \tilde{u}_\alpha\) for all \(\alpha \in (1/3, 3/2)\). Therefore it suffices to show that the sequence

\[
(e_i)_{i=1}^\infty := ((1(-1))^n_0 0^n_0 (1(-1))^n_0 0^n_0 \ldots
\]

is contained in \(\tilde{u}_{\alpha - 3/2}\). For all \(n \geq 0\) the following lexicographic inequalities hold

\[
(-1)(0)^n \prec (e_i)_{i=n+1}^\infty \prec 1(0)^\infty.
\]

Applying lemma 2.1 we see that \((e_i) \in \tilde{u}_{\alpha - 3/2}\) and our proof is complete.

Proof of theorem 1.2. Let \(\alpha \in (3\sqrt{2}/2, 1/2)\) and let \(n \in \mathbb{N}\) be as in lemma 4.1. Now let us pick \(a \in (\frac{n}{n+1}, \frac{n}{n+2})\). Any \((t_i) \in \tilde{u}_\alpha\) with \(d((t_i)) = a\) must contain either the sequence \(1(0)^n\) infinitely often or \((-1)(0)^n\) infinitely often. By lemma 4.1 this is not possible. Thus by lemma 2.3 the set \(D_\alpha\) is a proper subset of \([0, \frac{\log 2}{-\log \alpha}]\) and statement (ii) of theorem 1.2 holds.

By lemma 2.3 it remains to show that for any \(\alpha \in (1/3, 3\sqrt{2}/2)\) and \(a \in [0, 1]\) there exists \((t_i) \in \tilde{u}_\alpha\) such that \(d((t_i)) = a\). The existence of such a \((t_i)\) now follows from lemma 4.2 by making an appropriate choice of \((n_i)\).

We now prove theorem 1.3. To prove this theorem we require the following technical characterisation of \(S_\alpha\) from \([15, \text{theorem 3.2}].\) We recall that an infinite sequence \(\omega \in \{0, 1\}^\infty\) is called strongly eventually periodic if \((\omega_i) = \hat{I}^\infty\), where \(I, J\) are two finite words of the same length and \(I \neq J\). Clearly, a periodic sequence is strongly eventually periodic.

Proposition 4.3. \(t \in S_\alpha\) if and only if \((1-|t|)^\infty_{i=1} \in \text{strongly eventually periodic}.

Proof of theorem 1.3. Statement (ii) of theorem 1.3 follows from the proof of theorem 1.2. It is a consequence of our proof that for \(\alpha \in (3\sqrt{2}/2, 1/2)\) there exists \(\varepsilon > 0\) such that \(d((t_i)) \not\in (1-\varepsilon, 1)\) for all \((t_i) \in \tilde{u}_\alpha\). This statement when combined with lemma 2.3 implies statement (ii) of theorem 1.3.

To prove statement (i) we remark that for any \(\alpha \in (1/3, 3\sqrt{2}/2)\) and \(n, \ldots, n \in \mathbb{N}\), the sequence

\[
(t_i) = ((1(-1))^n_0 0^n_0 (1(-1))^n_0 0^n_0 \ldots
\]

is contained in \(\tilde{u}_\alpha\). The sequence \((1-|t|)\) is strongly eventually periodic, therefore by proposition 4.3 the corresponding \(t\) is contained in \(S_\alpha\). For any \(a \in [0, 1]\) and \(\varepsilon > 0\), we can pick \(n, \ldots, n \in \mathbb{N}\) such that \(|d((t_i)) - a| < \varepsilon\). Applying lemma 2.3 we may conclude that statement (i) of theorem 1.3 holds.

In terms of proposition 4.3 it is clear that the set \(\{\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in S_\alpha\}\) is at most countable. Furthermore, by lemma 4.1 and the proof of theorem 1.3 it follows that when \(\alpha \in (3\sqrt{2}/2, 1/2)\) the topological closure of \(\{\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in S_\alpha\}\) is of the form \([0, c]\) with \(c \in (0, \frac{\log 2}{-\log \alpha}]\).
5. Examples

We end our paper with some examples. We start with two examples of an \( \alpha \in \mathbb{Q} \), and a \( t \in \Gamma_\alpha - \Gamma_\alpha \) with a continuum of \( \alpha \)-expansions for which the Hausdorff dimension of \( \Gamma_\alpha \cap (\Gamma_\alpha + t) \) is explicitly calculable. The approach given in the first example applies more generally to \( \alpha \) the reciprocal of a Pisot number and \( \alpha \in \mathbb{Q} \).

Our second example demonstrates that it is possible for \( t \) to have a continuum of \( \alpha \)-expansions and for \( \Gamma_\alpha \cap (\Gamma_\alpha + t) \) to be a self-similar set.

**Example 5.1.** Let \( \alpha = 0.449 \ldots \) be the unique real root of \( 2x^3 + 2x^2 + x - 1 = 0 \). Then Consider \( t = \sum_{i=1}^{\infty} (- \alpha)^i \). For this choice of \( \alpha \) the set of \( \alpha \)-expansions of \( t \) is equal to the allowable sequences of edges in figure 1 that start at the point \( ((-1)1)^\infty \).

Using (2) we see that \( \Gamma_\alpha \cap (\Gamma_\alpha + t) \) coincides with those numbers \( \sum_{i=1}^{\infty} \epsilon_i \alpha^i \) where \( (\epsilon_i) \) is a sequence of allowable edges in figure 2 that start at \( ((-1)1)^\infty \).

We let

\[
C_\alpha := \left\{ (\epsilon_i)_{i=1}^n \in \{0, 1\}^n : \left[ \sum_{i=1}^{n} \epsilon_i \alpha^i + \frac{\alpha^{n+1}}{1 - \alpha} \right] \cap (\Gamma_\alpha \cap (\Gamma_\alpha + t)) \neq \emptyset \right\}.
\]

Given \( \delta_1 \ldots \delta_n \in C_\alpha \) we let

\[
C_\alpha(\delta_1 \ldots \delta_n) := \{ (\epsilon_i)_{i=1}^{m+n} \in C_{m+n} : (\epsilon_1, \ldots, \epsilon_m) = (\delta_1, \ldots, \delta_m) \}.
\]

Making use of standard arguments for transition matrices it can be shown that there exists \( c > 0 \) such that

\[
\frac{\lambda^n}{c} \leq \#C_n \leq c \lambda^n \quad \text{and} \quad \frac{\lambda^n}{c} \leq \#C_\alpha(\delta_1 \ldots \delta_m) \leq c \lambda^n,
\]

for any \( \delta_1 \ldots \delta_m \in C_m \). Here \( \lambda \approx 1.695 \) is the unique maximal eigenvalue of the matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
In the following we will show that
\[ \lambda \alpha \Gamma \cap \Gamma^+ = - \approx \alpha \alpha t \dim \log 0.644 297. \]

In fact we show that \( \langle \Gamma \cap \Gamma^+ \rangle < \infty \). By (16) the upper bound follows from the following straightforward argument:

\[
\mathcal{H}^{-\log \lambda} (\Gamma \cap (\Gamma^+ + t)) \\
\leq \lim \inf_{n \to \infty} \sum_{(\alpha) \in C_n} \text{Diam} \left( \left[ \sum_{j=1}^{n} \epsilon_j \alpha^j, \sum_{j=1}^{n} \epsilon_j \alpha^j + \frac{\alpha^{n+1}}{1-\alpha} \right] \right) \left( \log \lambda \frac{\log \alpha}{\log \alpha} \right) \\
\leq c \lambda^\alpha \left( \frac{\alpha^{n+1}}{1-\alpha} \right) \log \lambda \\
< \infty
\]

In what follows we use the notation \( \mathcal{I}_n \) to denote the basic intervals corresponding to the elements of \( C_n \), and \( \mathcal{I}_n(b_1 \cdots b_m) \) to denote the basic intervals corresponding to elements of \( C_n(b_1 \cdots b_m) \).

The proof that \( \mathcal{H}^{-\log \alpha}(\Gamma \cap (\Gamma + t)) > 0 \) is based upon arguments given in [2] and example 2.7 from [8]. Let \( \{ U_j \}_{j=1}^\infty \) be an arbitrary cover of \( \Gamma \cap (\Gamma^+ + t) \). Since \( \Gamma \cap (\Gamma^+ + t) \) is compact we can assume that \( \{ U_j \}_{j=1}^t \) is a finite cover. For each \( U_j \) there exists \( \ell(j) \in \mathbb{N} \) such that

\[ \alpha^{\ell(j)+1} < \text{Diam}(U_j) \leq \alpha^{0.0} \frac{1}{1-\alpha}. \]

This implies that \( U_j \) intersects at most two elements of \( \mathcal{I}_{\ell(j)} \). This means that for each \( j \) there exists at most two codes \((\epsilon_1, \ldots, \epsilon_{\ell(j)}), (\epsilon'_1, \ldots, \epsilon'_{\ell(j)}) \in C_{\ell(j)} \) such that

\[ U_j \cap \left[ \sum_{i=1}^{\ell(j)} \epsilon_i \alpha^i, \sum_{i=1}^{\ell(j)} \epsilon_i \alpha^i + \frac{\alpha^{\ell(j)+1}}{1-\alpha} \right] \neq \emptyset \quad \text{and} \quad U_j \cap \left[ \sum_{i=1}^{\ell(j)} \epsilon'_i \alpha^i, \sum_{i=1}^{\ell(j)} \epsilon'_i \alpha^i + \frac{\alpha^{\ell(j)+1}}{1-\alpha} \right] \neq \emptyset. \]

Without loss of generality we may assume that \( U_j \) always intersects at least one element of \( \mathcal{I}_{\ell(j)} \). Since \( \{ U_j \}_{j=1}^\infty \) is a finite cover there exists \( J \in \mathbb{N} \) such that \( \alpha^J < \text{Diam}(U_j) \) for all \( j \). By (16) the following inequalities hold by counting arguments:
\[
\frac{\lambda^j}{c} \leq \#C_j \leq \sum_{j=1}^{p} \left\{ (\epsilon_j) \in C_j : \left[ \frac{\sum_{i=1}^{j} \epsilon_i \alpha^i + \frac{\alpha^{j+1}}{1-\alpha}}{} \right] \cap U_j \neq \emptyset \right\} \\
\leq \sum_{j=1}^{p} \#C_{j-H}(\epsilon_{1} \cdots \epsilon_{H(j)}) + \sum_{j=1}^{m} \#C_{j-H}(\epsilon'_{1} \cdots \epsilon'_{H(j)}) \\
\leq 2c \sum_{j=1}^{p} \lambda^{j-H(j)} \\
\leq 2c \sum_{j=1}^{p} \lambda^{j} \cdot \alpha^{-H(j)} \frac{\log \lambda}{\log \alpha}.
\]

Cancelling through by \( \lambda^j \) we obtain \((2c^2)^{-1} \leq \sum_{j=1}^{p} \alpha^{-H(j)} \frac{\log \lambda}{\log \alpha} \). Since \( \text{Diam}(U_j) \) is \( \alpha^{(j)} \) up to a constant term we may deduce that \( \sum_{j=1}^{p} \text{Diam}(U_j) \frac{\log \lambda}{\log \alpha} \) can be bounded below by a strictly positive constant that does not depend on our choice of cover. This in turn implies \( \mathcal{H}^{-\log \alpha}(\Gamma_\alpha \cap (\Gamma_\alpha + 2)) > 0 \).

By (2) we know that

\[
\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + 2)) \geq \sup \dim_H \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\} \cap \{0, 1\} \right\},
\]

(18)

where the supremum is over all \( \alpha \)-expansions of \( t \). If \( t \) has countably many \( \alpha \)-expansions, then by the countable stability of the Hausdorff dimension we would have equality in (18). In the case where \( t \) has a continuum of \( \alpha \)-expansions it is natural to ask whether equality persists. This example shows that this is not the case. Upon examination of figure 1 we see that any \( \alpha \)-expansion of \(((-1)^{\infty})\) satisfies \( d((t_i)) \leq 1/3 \). In which case the right hand side of (18) can be bounded above by \( \frac{1}{3} \cdot \frac{\log 2}{\log \alpha} \approx 0.281 \). However by (17) this quantity is strictly less than our calculated dimension \( \frac{\log \lambda}{-\log \alpha} \approx 0.644 \).

**Example 5.2.** Let \( \alpha = \sqrt{2} - 1 \) and \( t = \frac{-\alpha + \alpha^2}{1-\alpha} \). Then a simple calculation demonstrates that the set of \( \alpha \)-expansions of \( t \) is precisely the set \( \{0(-1)(-1), (-1)10\} \). Applying (2) we see that

\[
(\Gamma_\alpha \cap (\Gamma_\alpha + t)) = \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : (\epsilon_i) \in \{100, 000, 010, 011\} \right\}.
\]

This last set is clearly a self-similar set generated by four contracting similitudes of the order \( \alpha^3 \). This self-similar set satisfies the strong separation condition. So

\[
\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) = \frac{\log 4}{3 \log \alpha}.
\]

Each \( \alpha \)-expansion of \( t \) satisfies \( d((t_i)) = 1/3 \). Thus the right hand side of (18) can be bounded above by \( \frac{\log 2}{-3 \log \alpha} \). Thus this choice of \( \alpha \) and \( t \) gives another example where we have strict inequality within (18).
We now give an example of an \( \alpha \in (1/3, 1/2) \) and \( t \in \Gamma - \Gamma_\alpha \) for which \( \Gamma_\alpha \cap (\Gamma_\alpha + t) \) contains only transcendental numbers. For \( \alpha \in (0, 1/3) \) examples are easier to construct, however, when \( \alpha \in (1/3, 1/2) \) the problem of multiple codings arises and a more delicate approach is required. Our examples arise from our proof of theorem 1.2 and make use of ideas from the well known construction of Liouville.

We call a number \( x \in \mathbb{R} \) a Liouville number if for every \( \delta > 0 \) the inequality
\[
|x - p/q| \leq q^{-(2+\delta)}
\]
has infinitely many solutions. An important result states that every Liouville number is a transcendental number \cite{3}. This result will be critical in what follows.

**Example 5.3.** Let \( p/q \in (1/3, 1/2) \). Then there exists \( t \in \mathbb{U}_{p/q} \) such that \( \Gamma_\alpha \cap (\Gamma_\alpha + t) \) only contains Liouville numbers. For any sequence of integers \( (n_k)_{k=1}^{\infty} \) the sequence
\[
(\epsilon_i) = (1(-1))^{n_0} 0 (1(-1))^{n_1} 0 \cdots
\]
is contained in \( \mathbb{U}_{p/q} \). Now let \( (n_k) \) be a rapidly increasing sequence of integers such that
\[
\left( \frac{q}{p} \right)^{2n_1 + \cdots + 2n_{i+1} + k + 1} \geq q^{(2n_1 + 2n_{i+2} + k + 3)}
\]
(19)

Let \( x \in \Gamma_{p/q} \cap (\Gamma_{p/q} + t) \), then \( x = \sum_{i=1}^{\infty} \epsilon_i \left( \frac{p}{q} \right)^i \) where \( \epsilon_i = 1 \) if \( t_i = 1 \), \( \epsilon_i = 0 \) if \( t_i = -1 \), and \( \epsilon_i \in \{0, 1\} \) if \( t_i = 0 \). It follows from our choice of \( (t_i) \) that
\[
(\epsilon_i) = (10)^{n_0} 0 (10)^{n_1} 0 (10)^{n_2} 0 \cdots.
\]

For each \( k \in \mathbb{N} \) we consider the rational
\[
\frac{p_k}{q_k} := \sum_{i=1}^{2n_i + \cdots + 2n_{k+1} + k} \epsilon_i \left( \frac{p}{q} \right)^i + \sum_{i=0}^{\infty} (p/q)^{2i},
\]
(20)

where \( p_k \) and \( q_k \) are coprime. Either the block 00 or 11 occurs infinitely often within \( (\epsilon_i) \). So \( p_k/q_k \approx x \). Importantly, if we expand the right hand side of (20) we can bound the denominator by
\[
q_k \leq q^{2n_1 + \cdots + 2n_{k+1} + k + 3}.
\]
(21)

The \( p/q \)-expansion on \( p_k/q_k \) agrees with that of \( x \) up to the first \( (2n_1 + \cdots + 2n_{k+1} + k) \) position. Therefore
\[
|x - p_k/q_k| \leq c \left( \frac{p}{q} \right)^{2n_1 + \cdots + 2n_{k+1} + k + 1}
\]
(22)

for some constant \( c \). Combining (19), (21), and (22) we see that for each \( k \in \mathbb{N} \)
\[
|x - p_k/q_k| \leq cq_k^{-k}.
\]
Therefore $x$ is a Liouville number. Since $x$ was arbitrary, every $x \in \Gamma_{plq} \cap (\Gamma_{plq} + t)$ is Liouville.

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