Test for entanglement using physically observable witness operators and positive maps

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Motivated by the Peres-Horodecki criterion and the realignment criterion we develop a more powerful method to identify entangled states for any bipartite system through a universal construction of the witness operator. The method also gives a new family of positive but non-completely positive maps of arbitrary high dimensions, which provide a much better test than the witness operators themselves. Moreover, we find that there are two types of positive maps that can detect $2 \times N$ and $4 \times N$ bound entangled states. Since entanglement witnesses are physical observables and may be measured locally our construction could be of great significance for future experiments.

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I. INTRODUCTION

Quantum entangled states lie at the heart of the rapidly developing field of quantum information science, which encompasses important potential applications such as quantum communication, quantum computation and quantum lithography \[1, 2, 3\]. However, the fundamental nature of entangled states has tantalized physicists since the earliest days of quantum mechanics, and even today the nature of entangled states has tantalized physicists since the earliest days of quantum mechanics, and in the 20th century the realization of quantum entangled states paved the way for the development of quantum information science.

One of the most basic problems is how can one tell if a quantum state is entangled?, and how entangled is it still after some noisy quantum process (e.g. long distance quantum communication)?

A pure entangled state is a quantum state which cannot be factorized, i.e., $|\Psi\rangle_{AB} \neq |\psi\rangle_A |\phi\rangle_B$, and shows remarkable nonlocal quantum correlations. From a practical point of view, the state of a composite quantum system which usually becomes a mixed state after a noisy process, is called unentangled or separable if it can be prepared in a “local” or “classical” way. It can then be expressed as an ensemble realization of pure product states $|\psi\rangle_A |\phi\rangle_B$ occurring with a certain probability $p_i$ and the density matrix $\rho_{AB} = \sum_i p_i |\psi_i\rangle_A \langle \psi_i| \otimes |\phi_i\rangle_B \langle \phi_i|$, where $\rho^A_i = |\psi_i\rangle_A \langle \psi_i|$, $\rho^B_i = |\phi_i\rangle_B \langle \phi_i|$, $\sum_i p_i = 1$, and $|\psi\rangle_A$, $|\phi\rangle_B$ are normalized pure states of subsystems A and B, respectively $\[4\]$. If no convex linear combination of $\rho^A_i \otimes \rho^B_i$ exists for a given $\rho_{AB}$, then the state is called “entangled”.

However, for a generic mixed state $\rho_{AB}$, finding a separable decomposition or proving that it does not exist is a non-trivial task (see the recent good reviews $\[3, 8, 9, 10\]$ and references therein). There have been many efforts in recent years to analyze the separability and quantitative character of quantum entanglement. The Bell inequalities satisfied by a separable system give the first necessary condition for separability $\[6\]$. In 1996, Peres made an important step towards proving that, for a separable state, the partial transposition with respect to one subsystem of a bipartite density matrix is positive, $\rho^{T_A} \geq 0$. This is known as the PPT (positive partial transposition) criterion or Peres-Horodecki criterion $\[10\]$. By establishing a close connection between positive map theory and separability, Horodecki et al. promptly showed that this is a sufficient condition for separability for bipartite systems of $2 \times 2$ and $2 \times 3$ $\[11\]$. Regarding the quantitative character of entanglement, Wootters succeeded in giving an elegant formula to compute the “entanglement of formation” $\[12\]$ of $2 \times 2$ mixtures, thus giving also a separability criterion $\[13\]$.

Very recently, Rudolph and other authors $\[14, 15, 16, 17\]$ proposed a new operational criterion for separability, the realignment criterion (named, thus, following the suggestion of Ref. $\[18\]$ which is equivalent to the computational cross norm criterion of Ref. $\[14\]$). The criterion is very simple to apply and shows a dramatic ability to detect bound entangled states (BESs) $\[19\]$ in any high dimension $\[15\]$. It is even strong enough to detect the true tripartite entanglement shown in Ref. $\[13\]$.

An alternative method to detect entanglement is to construct so-called entanglement witnesses (EWs) $\[11, 20, 21\]$ and positive maps (PMs) $\[22\]$. Entanglement witnesses $\[11, 20\]$ are physical observables that can “detect” the presence of entanglement. Starting from the witness operators one can also obtain PMs $\[22\]$ that detect more entanglement. Although there are constructions of EWs related to the unextendible bases of product vectors in Ref. $\[20\]$ and to the existence of “edge” positive partial transpose entangled states (PPTES) in Ref. $\[21\]$, a universal construction of EWs and PMs for a general bipartite quantum state has still to be discovered.

The aim of this paper is to introduce a new powerful technique for universal construction of EWs and PMs for any bipartite density matrix and to obtain a stronger operational test for identifying entanglement. Our starting point will be the PPT criterion and the realignment criterion for separability. The universal construction is
given in Sec. III and several typical examples of entangled states which can be recognized by the corresponding EWs and PMs are presented in Sec. IV. We show that many of the recognized bound entangled states cannot be detected by the realignment criterion or any of the EWs and PMs constructed previously in the literature. Moreover, we demonstrate in Sec. IV that there are two types of positive maps that can detect systematically the $2 \times N$ and $4 \times N$ bound entangled states. A brief summary and discussion are given in the last section.

II. UNIVERSAL CONSTRUCTION OF ENTANGLED WITNESSES AND POSITIVE MAPS FOR IDENTIFYING ENTANGLEMENT

In this section we will give two universal constructions for EWs and PMs for any bipartite density matrix that provide stronger operational tests for identifying entanglement. The starting points are the PPT criterion [10, 11] and the realignment criterion [10, 11] for separability [12–14, 16–17]. The main tools that we shall use are drawn from the general theory of matrix analysis [24, 25].

A. Some notation

The various matrix operations from [24, 25] that we need will employ the following notation.

Definition: For each $m \times n$ matrix $A = [a_{ij}]$, where $a_{ij}$ is the matrix entry of $A$, we define the vector $\text{vec}(A)$ as

$$\text{vec}(A) = [a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}]^T.$$ 

Here the superscript “$T$” means standard transposition. Let $Z$ be an $m \times m$ block matrix with block size $n \times n$. We define the following “realignment” operation $\mathcal{R}$ to change $Z$ to a realigned matrix $\tilde{Z}$ of size $n^2 \times n^2$ that contains the same elements as $Z$ but in different positions as follows:

$$\mathcal{R}(Z) \equiv \tilde{Z} \equiv \begin{bmatrix} \text{vec}(Z_{1,1})^T \\ \vdots \\ \text{vec}(Z_{m,1})^T \\ \vdots \\ \text{vec}(Z_{1,m})^T \\ \vdots \\ \text{vec}(Z_{m,m})^T \end{bmatrix}. \quad (1)$$

For example, a $2 \times 2$ bipartite density matrix $\rho$ can be transformed as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \rightarrow \mathcal{R}(\rho) = \begin{pmatrix} \rho_{11} & \rho_{21} & \rho_{12} & \rho_{22} \\ \rho_{31} & \rho_{41} & \rho_{32} & \rho_{42} \\ \rho_{13} & \rho_{23} & \rho_{14} & \rho_{24} \\ \rho_{33} & \rho_{43} & \rho_{34} & \rho_{44} \end{pmatrix}. \quad (2)$$

B. The realignment criterion

Motivated by the Kronecker product approximation technique for a matrix [26, 27], we developed a very simple method to obtain the realignment criterion in Ref. [15] (called the cross norm criterion in [14]). To recollect, the criterion says that, for any separable $m \times n$ bipartite density matrix $\rho_{A,B}$, the $m^2 \times n^2$ matrix $\mathcal{R}(\rho_{A,B})$ should satisfy $||\mathcal{R}(\rho_{A,B})|| \leq 1$, where $||\cdot||$ means the trace norm defined as $||G|| = Tr((G^G)^{1/2})$. Thus $||\mathcal{R}(\rho_{A,B})|| > 1$ implies the presence of entanglement in $\rho_{A,B}$.

This criterion is strong enough to detect most of the bound entangled states in the literature, as shown in Ref. [11], and holds even for genuine multipartite systems, as shown in Ref. [18].

C. Entanglement witnesses and positive maps

Entanglement witness: an entanglement witness is a Hermitian operator $W = W^\dagger$ acting on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ that satisfies $Tr(W \rho_A \otimes \rho_B) \geq 0$ for any pure separable state $\rho_A \otimes \rho_B$, and has at least one negative eigenvalue. If a density matrix $\rho$ satisfies $Tr(W \rho) < 0$, then $\rho$ is an entangled state and we say that $W$ “detects” entanglement in $\rho$ [11, 20, 21]. It has been shown in Ref. [11] that a given density matrix is entangled if and only if there exists an EW that detects it.

Positive map: it was shown in Ref. [11] that $\rho$ is separable iff for any positive map $\Lambda$ the inequality

$$\langle \Lambda \rangle \rho \geq 0$$

holds where $\Lambda$ means a identity matrix with respect to the $A$ subsystem. In practice, detecting entanglement only involves finding those maps which are positive but not completely positive (non-CP), since a CP map will satisfy Eq. (3) for any given separable $\rho$ [11].

It was shown in [22] that there is a close connection between a positive map and the entanglement witness, i.e., the Jamiołkowski isomorphism

$$W = (\Lambda \otimes \Lambda) P^m_+,$$  \quad (4)

where $P^m_+ = |\Phi\rangle\langle\Phi|$ and $\Phi = {\sqrt m \over n} \sum_{i=1}^m |ii\rangle$ is the maximally entangled state in $\mathcal{H}_A \otimes \mathcal{H}_A$. 


D. Universal construction of EWs

With the above-mentioned notation and concepts in mind we will now derive the main result of this paper: two universal constructions for EWs and PMs to identify entanglement of bipartite quantum systems in arbitrary dimensions.

**Theorem 1:** For any density matrix $\rho$, we can associate with it an EW defined as

$$W = Id - (R^{-1}(U^*V^T))^T, \quad (5)$$

where $U,V$ are the unitary matrices that yield the singular value decomposition (SVD) of $R(\rho)$, i.e., $R(\rho) = U\Sigma V^\dagger$.

**Proof:** Using a result of matrix analysis (see chapter 3 of [23]), we have

$$||R(\rho)|| = \max\{ |Tr(X^\dagger R(\rho)Y)| : X \in M_{m^2,q}, Y \in M_{n^2,q}, X^\dagger X = Id = Y^\dagger Y \}, \quad (6)$$

where $q = \min\{m^2, n^2\}$. We thus find that the maximum value for $|Tr(X^\dagger R(\rho)Y)|$ occurs at $X = U$ and $Y = V$. This is because

$$|Tr(U^\dagger R(\rho)V)| = |Tr(U^\dagger U\Sigma V^\dagger V)| = |Tr\Sigma| = Tr\Sigma = ||R(\rho)||.$$ 

In the same way, for a separable state $\rho_{sep}$, $|Tr(X^\dagger R(\rho_{sep})Y)|$ has its maximum value at $X = U'$ and $Y = V'$ where $R(\rho_{sep}) = U'\Sigma' V'^\dagger$ is the SVD of $R(\rho_{sep})$. From the realignment criterion for separability we have $||R(\rho_{sep})|| \leq 1$, thus

$$|Tr(U^\dagger R(\rho_{sep})V)| \leq |Tr(U^\dagger R(\rho_{sep})V')| = ||R(\rho_{sep})|| \leq 1.$$ 

Since $R(\rho')$ is just a rearrangement of entries in $\rho'$, we find by direct observation that $W_2 = (R^{-1}(W_1^T))^T$ if we require $|Tr(W_1 R(\rho'))| = |Tr(W_2 \rho')|$ to hold for all $\rho'$. Here $R^{-1}(W_1^T)$ means the inverse of $R$, realigning the entries of $W_1^T$ according to Eq. (11). Letting $W_1 = VU$, since $Tr(W_1 \rho_{sep}) = 1 - Tr(W_2 \rho_{sep}) \geq 1 - |Tr(W_2 \rho_{sep})| = 1 - |Tr(W_1 R(\rho_{sep}))| \geq 0$, we have the EW

$$W = Id - W_2 = Id - (R^{-1}(W_1^T))^T = Id - (R^{-1}(U^*V^T))^T. \quad (7)$$

Whenever we have an entangled state $\rho$ which can be detected by the realignment criterion, i.e., $||R(\rho)|| > 1$, it can also be detected by the EW in Theorem 1, since $Tr(W_1) = 1 - Tr(VU^\dagger R(\rho)) = 1 - ||R(\rho)|| < 0$. It should be remarked that for an $m \times m$ system, we have $(R^{-1}(U^*V^T))^T \equiv R(VU^\dagger)$ which is a simpler expression for practical operation by direct observation.

As for the PPT criterion, we can also have a universal construction for EWs as follows:

**Theorem 2:** For any density matrix $\rho$, we can associate with it another EW defined as

$$W = Id - (VU^\dagger)T_A, \quad (8)$$

where $U,V$ are unitary matrices that yield the SVD of $\rho^{T_A}$, i.e., $\rho^{T_A} = U\Sigma V^\dagger$.

**Proof:** For any separable density matrix $\rho_{sep}$, we have $||\rho_{sep}^{T_A}|| = 1$ due to positivity of $\rho_{sep}^{T_A}$. Similar to the procedure in the proof of Theorem 1, we have

$$\begin{align*}
 Tr(W_{sep}) &= 1 - Tr((VU^\dagger)T_A \rho_{sep}) \\
 &\geq 1 - |Tr((VU^\dagger)T_A \rho_{sep})| \\
 &= 1 - |Tr(VU^\dagger \rho_{sep})| \\
 &\geq 1 - ||\rho_{sep}^{T_A}|| = 0. \quad (9)
\end{align*}$$

Thus $W$ is an EW.

Whenever we have an entangled state $\rho$ which can be detected by the PPT criterion, i.e., $||\rho^{T_A}|| > 1$, it can also be detected by the EW in Theorem 2, since $Tr(W_1) = 1 - Tr((VU^\dagger)T_A \rho) = 1 - ||\rho^{T_A}|| < 0$.

E. Optimization of EWs and universal construction of PMs

The universal EWs that we have constructed from Theorem 1 and Theorem 2 are no weaker than the PPT criterion and the realignment criterion. We anticipate that better tests should exist. Motivated by the idea developed in Ref. [21], we now derive a better witness $W'$ from Theorems 1 and 2:

$$W' = W - \epsilon Id, \quad (10)$$

where $\epsilon = \min Tr(W_{\rho_A} \otimes \rho_B)$ for all possible pure states $\rho_A$ and $\rho_B$. We observe that $Tr(W_{\rho_{sep}}) = Tr(W_{\rho_{sep}}) - \epsilon \geq 0$ since $Tr(W \sum_i p_i \rho_A^{\otimes i} \otimes \rho_B^{\otimes i}) = \sum_i p_i Tr(W \rho_A^{\otimes i} \otimes \rho_B^{\otimes i}) \geq \sum_i p_i \epsilon = \epsilon$. Thus $W'$ is a reasonable EW.

Let us see how the minimum of $Tr(W_{\rho_A} \otimes \rho_B)$ for a given $W$ is obtained. Partitioning $W$ to be an $m \times m$ block matrix $W_{ij}$ $(i,j = 1, \ldots m)$ with block size $n \times n$, we have

$$\epsilon = \min Tr(W_{\rho_A} \otimes \rho_B) = \min Tr \left\{ \sum_{i,j} W_{ij}(\rho_A)_{ij} \rho_B \right\}. \quad (11)$$

Here $W_{ij}$ is an $n \times n$ matrix while $(\rho_A)_{ij}$ is a single entry of $\rho_A$. Using a known result of matrix analysis:

$$\lambda_{\min}(G) = \min Tr(U^\dagger GU) = \min Tr(GU^\dagger),$$

where $U^\dagger U = 1$ and $\lambda_{\min}(G)$ is the minimum eigenvalue of $G$, we deduce that $\epsilon$ is in fact the minimum eigenvalue...
of $G = \sum_{i,j} W_{i,j}(\rho_A)_{j|i}$. Thus the problem changes to that of finding $\lambda_{\min}(G)$ for all possible $\rho_A$, which can be done with common numerical optimization software. This is much simpler for the low dimensional cases of the first subsystem such as $m = 2, 3$. 

However, the ability of the optimized $W'$ to detect entanglement is still not ideal, and some stronger entanglement tests than the EWs should be found. The Jamiołkowski isomorphism Eq. 11 between a PM and an EW serves us a good candidate. A positive map can depict this character. The constructed EWs, their optimal versions and corresponding PM are always much more powerful than the realignment criterion. It is straightforward to verify that we can obtain a single EW

$$\Lambda(|i\rangle\langle j|) = (i|W|j),$$

where $\{|i\rangle\}_{i=1}^m$ is an orthogonal basis for $\mathcal{H}^A$.

Using Theorems 1 and 2 we now have a universal EW construction to detect entanglement for a given quantum state, which is not weaker than the two criteria for separability. We can also detect all other possible entangled states (in particular the same class of states) using the constructed EW. Moreover, we can use the optimizing procedure of Eq. 10 to get a better witness, while the best detection can be obtained with the positive map $\Lambda$ given by Eq. 12. We say that our constructions are good detection can be obtained with the positive map $\Lambda$ corresponding to a given quantum state, which is not weaker than the two criteria for separability. We can also detect all other possible entangled states, in particular, the bound entangled states. Example 1 is a simple $2 \times 2$ Werner state ([29]) is such an example as shown in the following section. To our surprise, we find that even for some prior knowledge of a quantum state, we can even detect its entanglement with just a single EW and a PM from the constructions of Theorem 1 and 2, without having to apply full quantum state tomography. The $2 \times 2$ Werner state ([29]) is such an example as shown in the following section. To our surprise, we find that even for some prior knowledge of a quantum state, we can even detect its entanglement with just a single EW.

**III. APPLICATION OF EWS AND PMS TO ENTANGLED STATES**

In this section we give several typical examples to display the power of our constructions to distinguish entangled states, in particular, the bound entangled states from separable states. **Example 1** is a simple $2 \times 2$ Werner state, which is to show that we can still obtain good EWs and PMs even with a separable states. Since the $PPT$ criterion is strong enough to detect all non-$PPT$ entangled states we shall just examine PPTES in the following two examples. We have tested most of the bipartite BESs in the literature [14, 30, 31] and found that the optimized EW $W'$ and corresponding PM are always much more powerful than the realignment criterion.

**Example 1**: $2 \times 2$ Werner state.

Consider the family of two-dimensional Werner states $\rho$,

$$\rho = \frac{1}{6}((2 - f) I_4 + (2f - 1)V),$$

where $-1 \leq f \leq 1$, $V(\alpha \otimes \beta) = \beta \otimes \alpha$, the operator $V$ has the representation $V = \sum_{i,j=0}^3 |ij\rangle\langle ji|$, and $\rho$ is inseparable if and only if $-\frac{1}{2} \leq f < 0$. As is well known, the entanglement in a $2 \times 2$ Werner state can be detected completely using the $PPT$ criterion and the realignment criteria. It is straightforward to verify that we can obtain a single EW

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

from Theorem 1 whenever $-1 \leq f < 1/2$. From $Tr(W\rho) = f < 0$, we see that this EW and corresponding PM can detect all of the entangled $2 \times 2$ Werner states. This is a surprising result since we still obtain a good witness operator associated with a separable state ($0 \leq f < 1/2$) to detect the whole family of Werner states.

**Example 2**: $3 \times 3$ BES constructed from unextendible product bases In Ref. [31], Bennett et al introduced a $3 \times 3$ inseparable BES from the following bases:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle(|0\rangle - |1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|1\rangle, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle, \quad |\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)$$

from which a bound entangled density matrix can be constructed as

$$\rho = \frac{1}{4}(\text{Id} - \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i|).$$

Consider a mixture of this state and the maximally mixed state $\rho_p = p\rho + (1 - p)\text{Id}/9$. Direct calculation using the realignment criterion gives $p > 88.97\%$ where $\rho_p$ still has entanglement. This is much stronger than the optimal witness given in Ref. [20, 32] where they give $p > 94.88\%$ for the existence of entanglement in $\rho_p$. Using Theorem 1 for $\rho$, we obtain an EW (not optimized) and further a positive map to detect entanglement for $\rho_p$ when $p > 88.41\%$. A surprising result is that we can still obtain a good EW to detect entanglement in $\rho$ whenever we generate an EW from $\rho_p$ using Theorem 1 for all values of $0 < p \leq 1$. Numerical calculation shows that the best PM is the one corresponding to the EW generated from
\( \rho_p \) when \( p \geq 0.3 \), which is, to a large degree, a separable state (we have no operational separability criterion to guarantee this but we can estimate that it is so). The best PM detects entanglement for \( \rho_p \) when \( p > 87.44 \% \), which, to our knowledge, is the strongest test up to now.

**Example 3: 3 × 3 chessboard BES**

Bruß and Peres constructed a seven parameter family of PPTES in Ref. [31]. Using the above mentioned constructions we perform a systematic test for these states. Choosing a state with a relatively large \( |\langle R | \rho \rangle| = 1.164 \) in the family and constructing a PM from the EW corresponding to \( \rho \) we can detect about 9.48 \% of \( 10^4 \) randomly chosen density matrix \( \sigma \) satisfying \( \sigma = \sigma^T \). Half of these detected BESs cannot be detected by the realignment criterion, and it should also be noted that we only used one PM here. For every state in this family (including those that cannot be detected by the realignment criterion) we can almost always obtain an EW and a PM which can detect some BESs in this family, many of which cannot be recognized by the realignment criterion by a direct numerical calculation.

Actually, we have a third choice in constructing an EW with

\[
W = \epsilon Id - \rho,
\]

where \( \epsilon = \max \text{Tr}(\rho (\rho_A \otimes \rho_B)) \) for a given density matrix \( \rho \), as first proposed in Ref. [6]. We can calculate \( \epsilon \) following the same procedure as for optimizing the EW, and find that \( \epsilon \) is actually the maximum eigenvalue of \( G = \sum_{i,j} W_{i,j}(\rho_A)_{j,i} \) for all possible \( \rho_A \). The optimal witness given in Ref. [32] for states constructed from unextendible product bases is in fact equivalent to this method.

Since some PMs can detect BES they cannot be decomposed to the form \( \Lambda_1 + T \otimes \Lambda_1 \) where \( \Lambda_1 \) and \( \Lambda_2 \) are completely positive maps, and \( T \) is the standard transposition [11]. Thus our construction gives a universal method to find the indecomposable positive linear map in any dimension.

**IV. TWO INDECOMPOSABLE POSITIVE MAPS**

Here we also show that two indecomposable positive maps, which were first given in Ref. [33], can systematically detect the 2 × 4 BES given by Horodecki [19]. One map \( \Lambda : M_2 \rightarrow M_2 \) is defined as \( \Lambda : [a_{ij}]_{i,j=1}^{4} \rightarrow \)

\[
\begin{pmatrix}
(1-\epsilon)a_{11} + a_{22} & -2a_{23} - 2a_{34} \\
+2a_{31} + a_{44} & +ua_{31} - a_{12} \\
-2a_{32} - 2a_{43} & u^2a_{11} - ua_{14} + 2a_{22} \\
+ua_{13} - a_{21} & -ua_{11} + a_{44}
\end{pmatrix},
\]

where \( 0 < u < 1 \) and \( 0 < \epsilon < \pi^2/6 \). The density matrix \( \rho \) of the 2 × 4 BES in Ref. [19] is real and symmetric, and has the form:

\[
\rho = \frac{1}{7b+1} \begin{pmatrix}
b & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & 0 & b
\end{pmatrix}, \quad (18)
\]

where \( 0 < b < 1 \). Assuming \( \varepsilon = u^2/6 \), we see that \( (Id_A \otimes \Lambda)\rho \) can detect all the entanglement in \( \rho \) for \( 0 < b < 1 \), as shown in Fig. 1 where we have plotted \( f = \min \{0, \lambda_{\text{min}}[(Id_A \otimes \Lambda)\rho]\} \), and \( \lambda_{\text{min}} \) means the minimum eigenvalue. It is straightforward to verify that the dual map \( \Lambda' : M_2 \rightarrow M_2 \) to \( \Lambda \) in Ref. [33] can also detect \( \rho \) by the action of \( (\Lambda' \otimes Id_B)\rho \). If we assume \( \rho_p = \rho \) with \( (1-p)Id/8 \), the PM \( \Lambda \) gives \( p > 99.26 \% \) for existence of entanglement in \( \rho_p \) with \( u = 0.849 \) and \( b = 0.218 \), which is a stronger test than the PM constructed from an optimal EW shown in Ref. [21] to give \( p > 99.65 \% \). For any 2 × 4 or 4 × 4 system, the maps \( \Lambda \) and \( \Lambda' \) are expected to give a very strong test for recognizing BES.

**FIG. 1:** Detection of a Horodecki 2 × 4 bound entangled state with the indecomposable positive map \( \Lambda \) of Eq. (17).

**V. CONCLUSION**

Summarizing, we have presented a universal construction for entanglement witnesses and positive maps that can detect entanglement systematically and operationally. They are stronger than both the PPT and the realignment criteria, and provide a powerful method to detect entanglement since entanglement witnesses are physical observables and may be measured locally [32]. The construction gives us a new subtle way to enhance significantly our ability to detect entanglement beyond
previously known methods. Even when associated with some separable states, the construction gives EWs and PMs to identify entanglement. If we have some prior knowledge of a quantum state, we can even detect its entanglement with just a single EW and a PM. In addition, our construction gives a systematic way to obtain positive but non-CP maps, which may also be of interest to the mathematics community. Moreover, we find that two types of positive maps can detect completely a $2 \times 4$ bound entangled state and promise to give a very strong test for any $2 \times N$ and $4 \times N$ systems. We hope that our method can shed some light on the final solution of the separability problem, as well as motivate new interdisciplinary studies connected with mathematics.

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