Non-Spherical Models of Neutron Stars

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1 Introduction

Neutron stars are compact stellar objects with masses between around 1 and 2 solar masses and radii of around 10 to 15 km \cite{1, 2, 3}. They have magnetic fields up to around $10^{18}$ G \cite{4, 5, 6, 7}. Standard models for neutron stars traditionally assume that these objects are perfect spheres whose properties are described, in the framework of general relativity theory, by the well-known Tolman-Oppenheimer-Volkoff (TOV) equation \cite{8, 9}. The TOV equation is a simple first-order differential equation which can be solved with little numerical effort (see, for instance, Refs. \cite{10, 11}).

The assumption of perfect spherical symmetry may not be correct. It is known that magnetic fields are present in all neutron stars. In particular, if the magnetic field is strong (up to around $10^{18}$ Gauss in the core) such as for magnetars \cite{4, 5, 6, 7}, and/or the pressure of the matter in the cores of neutron stars is non-isotropic, as predicted by some models of color superconducting quark matter \cite{12}, then deformation of neutron stars can occur \cite{13, 14, 15, 16, 17, 18}. We also mention the recent work conducted by \cite{19} which shows that high magnetic fields in proto-quark stars modify quark star masses. The authors of this study conclude that using the TOV equation would be insufficient for numerical calculations of the properties of proto-quark stars.
The main goal of our study is to derive a TOV-like stellar-structure equation for deformed neutron stars whose mathematical form is similar to the standard TOV equation for spherical neutron stars. This equation will enable the user to explore the properties of deformed neutron stars from an equation that can be solved with rather little numerical effort, complementing more sophisticated numerical studies such as the one presented very recently in [20].

In contrast to the TOV stars that are composed of spherically symmetric mass shells, the stellar models considered in our paper are made of deformed mass shells which are either of oblate or prolate shape. Strategically, such a treatment is similar to the formalism developed by Hartle and Thorne [21], which is based on a quadrupole approximation of the metric of a rotating compact star. The oblate and prolate shapes are obtained by parametrizing the polar ($z$) direction of the metric in terms of the equatorial ($r$) direction along with a the deformation parameter $\gamma$, described as $z = \gamma r$, where we have assumed the symmetry to be axial symmetric. This parameter is normalized to $\gamma = 1$ for a perfect sphere. An object that is deformed in the equatorial direction (oblate spheroid) is obtained for $\gamma < 1$, while an object deformed in the polar direction (prolate spheroid) corresponds to $\gamma > 1$. Using this parametrization will allow us to keep the energy momentum tensor in spherical form, while maintaining deformation structure.

The parametrized metric allows us to derive the stellar structure equation of deformed neutron stars in analytic form. As already mentioned above, this equation constitutes a generalization of the well-known Tolman-Oppenheimer-Volkoff equation [8, 9], which describes the properties of perfect spheres in general relativity theory.

The paper is organized as follows. In Sect. 2 we discuss the derivation of the stellar structure equation of deformed neutron stars in analytic form. As already mentioned above, this equation constitutes a generalization of the well-known Tolman-Oppenheimer-Volkoff equation [8, 9], which describes the properties of perfect spheres in general relativity theory.

The nuclear equation of state used to solve this equation is introduced in Sect. 3. Our equation of state is based on a relativistic nuclear lagrangian which describes confined hadronic matter and a nonlocal Nambu-Jona-Lasinio lagrangian used to model quark deconfinement. The results are presented in Sect. 4. They are of generic nature and do not depend on the particular choice for the nuclear equation of state. Conclusions are drawn in Sect. 5.

2 Stellar Structure Equations

The properties of perfectly spherical stars are determined by the TOV equation, which is based on the Schwarzschild metric given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2,$$

where $t$ is the time coordinate, $r, \theta, \phi$ are the spatial coordinates, and $\Phi(r)$ and $\Lambda(r)$ denote metric functions which are determined from Einstein’s field equation of general
relativity theory. The deformed stellar models studied in this paper are based on a metric that is similar to the one of Eq. (1). However, instead of spherical mass shells the deformed stellar models are constructed from mass shells that are either of prolate or oblate shape. The mathematical form of the metric of such objects reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\Phi(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-\gamma} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2,$$

where $\gamma$ denotes a constant that determines the degree of deformation. To derive the hydrostatic equilibrium equation associated with Eq. (2), we start with Einstein’s field equation in the mixed tensor representation

$$G^\mu_{\ \nu} = R^\mu_{\ \nu} - \frac{1}{2} R g^\mu_{\ \nu} = -8\pi T^\mu_{\ \nu}.$$  

Here $G^\mu_{\ \nu}$ denotes the Einstein tensor, which is given in terms of the Ricci tensor $R^\mu_{\ \nu}$, the Ricci scalar $R$ and the metric tensor $g^\mu_{\ \nu} = \delta^\mu_{\ \nu}$. The energy-momentum tensor

$$T^\mu_{\ \nu} = (\epsilon + P) u^\mu u_\nu - g^\mu_{\ \nu} P$$

is given in terms of the stellar equation of state (pressure, $P$, as a function of energy density, $\epsilon$) and the matter’s four-velocity $u^\mu = dx^\mu/d\tau$ and $u_\nu = dx_\nu/d\tau$, with the proper time $\tau$ given by $d\tau^2 = ds^2$. Using Eqs. (2) through (4) along with the equations provided in the Appendix, one arrives at the stellar structure equation of a deformed neutron star,

$$\frac{dP}{dr} = \frac{(\epsilon + P) \left[\frac{1}{2} r + 4\pi r^3 P - \frac{1}{2} r \left(1 - \frac{2m}{r}\right)^\gamma\right]}{r^2 \left(1 - \frac{2m}{r}\right)^\gamma}.$$  

In the limiting case when $\gamma = 1$, Eq. (5) becomes the well-known Tolman-Oppenheimer-Volkoff equation

$$\frac{dP}{dr} = -\frac{(\epsilon + P) \left(m + 4\pi Pr^3\right)}{r^2 \left(1 - \frac{2m}{r}\right)},$$

which describes the structure of perfectly spherically symmetric objects. The gravitational mass of a deformed neutron star is given by

$$\frac{dm}{dr} = 4\pi r^2 \epsilon \gamma,$$

so that the total gravitational mass, $M$, of a deformed neutron star with an equatorial radius $R$ follows as

$$M = \gamma m(R).$$

In the spherical limit, the total mass is given by $M \equiv m(R) = 4\pi \gamma \int_0^R drr^2\epsilon$. The stellar radius $R$ is defined by the condition that pressure at the surface of a neutron stars vanishes, that is, $P(r = R) = 0$.  


It is important to investigate the space outside the star as well. For that we need to examine the $e^{2\Phi(r)}$ component of Eq. (2). Using the equations given in the Appendix, one finds

$$\frac{d\Phi}{dr} = \frac{\left[ \frac{1}{2} + 4\pi r^2 P - \frac{1}{2} \left( 1 - \frac{2m}{r} \right)^\gamma \right]}{r \left( 1 - \frac{2m}{r} \right) \gamma}.$$  

(9)

One can easily see from Eq. (9) that asymptotically $d\Phi/dr \to 0$, as required.

Now that we are equipped with the stellar structure equations (5) and (8), which are dependent on the deformation parameter $\gamma$, we solve them for a given equation of state. The model chosen here assumes that neutron stars are made of quark-hybrid matter. It is based on a relativistic nuclear lagrangian to describe confined hadronic matter and a nonlocal Nambu-Jona-Lasinio lagrangian to model quark matter [23, 24]. Phase equilibrium in the quark-hadron mixed phase is governed by the Gibbs condition. Section 3 briefly describes the key features of this equation of state. We stress, however, that the results presented in Sect. 4 are generic and do not depend on the particular choice for the nuclear equation of state.

3 Equation of State

3.1 Hadronic Matter

At densities higher than that of the inner neutron star crust, and lower than required for quark deconfinement, we model neutron star matter composed of baryons ($B = \{n, p, \Lambda, \Sigma, \Xi, \Delta, \Omega\}$) and leptons ($\lambda = \{e^-, \mu^-\}$) using the relativistic mean-field approximation. The Lagrangian is given by [10, 11, 25]

$$\mathcal{L} = \sum_B \bar{\psi}_B \left[ \gamma\mu(i\partial^\mu - g_{\omega}\omega^\mu - g_{\rho}\vec{\tau} \cdot \vec{\rho}^\mu) - (m_N - g_{\sigma}\sigma) \right] \psi_B + \frac{1}{2} (\partial^\mu\sigma\partial^\nu\sigma - m_N^2\sigma^2)$$

$$- \frac{1}{3} b_{\sigma} m_N (g_{\sigma}\sigma)^3 - \frac{1}{4} c_{\sigma} (g_{\sigma}\sigma)^4 - \frac{1}{4} \omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2} m_\rho^2 \vec{\rho}_{\mu\nu} \cdot \vec{\rho}^{\mu\nu}$$

$$- \frac{1}{4} \vec{\rho}_{\mu\nu} \cdot \vec{\rho}^{\mu\nu} + \sum_\lambda \bar{\psi}_\lambda (i\gamma\mu\partial^\mu - m_\lambda) \psi_\lambda.$$  

(10)

The interactions between baryons are described by the exchange of scalar, vector, and isovector mesons ($\sigma, \omega, \rho$) [26]. In the present work we employ the NL3 parametrization as given in Table 1 [27]. For further details, see Refs. [10, 11] and references therein.

3.2 Quark Matter

To determine the equation of state of the deconfined quark phase we use a nonlocal extension of the three-flavor Nambu-Jona-Lasinio model (see [23] and references
Table 1: Parametrization of hadronic matter, where the saturation properties are baryonic density $\rho_0$, energy per baryon $E/N$, nuclear incompressibility $K$, effective nucleon mass $m^*_N$, and asymmetry energy $a_{sy}$.

| Saturation Properties | NL3 Parametrization |
|-----------------------|----------------------|
| $\rho_0 \text{ (fm}^{-3}\text{)}$ | 0.148 |
| $E/N \text{ (MeV)}$ | -16.3 |
| $K \text{ (MeV)}$ | 272 |
| $m^*/m_N$ | 0.60 |
| $a_{sy} \text{ (MeV)}$ | 37.4 |

Therein). This model hosts numerous improvements over other models of deconfined quark matter, including but not limited to the treatment of vector interactions among quarks, reproduction of confinement for proper parametrization, lack of ultraviolet divergences with the introduction of the nonlocal form factor $g(\tilde{z})$, and momentum dependent dynamical quark masses. The Euclidean effective action is given by

$$S_E = \int d^4x \left\{ \bar{\psi}(x) \left[ -i \gamma_\mu \partial^\mu + \tilde{m} \right] \psi(x) - \frac{G_S}{2} \left[ j^S_a(x)j^S_a(x) - j^P_a(x)j^P_a(x) \right] ight. \right.$$  

$$\left. - \frac{H}{4} T_{abc} [j^S_a(x)j^S_b(x)j^S_c(x) - 3j^P_a(x)j^P_b(x)j^P_c(x)] - \frac{G_V}{2} j^\mu_{V,f}(x)j^\mu_{V,f}(x) \right\},$$

where $\psi = (uds)^T$, $\tilde{m} = \text{diag}(m_u, m_d, m_s)$, and $H$, $G_S$, and $G_V$ are coupling constants. For convenience we assume $m_u = m_d = \overline{m}$. The scalar, pseudo-scalar, and vector currents are respectively

$$j^S_a(x) = \int d^4z \tilde{g}(z) \bar{\psi} \left( x + \frac{z}{2} \right) \lambda_a \psi \left( x - \frac{z}{2} \right),$$

$$j^P_a(x) = \int d^4z \tilde{g}(z) \bar{\psi} \left( x + \frac{z}{2} \right) i\gamma_5 \lambda_a \psi \left( x - \frac{z}{2} \right),$$

$$j^\mu_{V}(x) = \int d^4z \tilde{g}(z) \bar{\psi} \left( x + \frac{z}{2} \right) \gamma^\mu \lambda_a \psi \left( x - \frac{z}{2} \right).$$

Applying standard bosonization to (11) we derive the thermodynamic potential in the mean-field approximation at zero temperature [23]. We use the same parametrization for the nonlocal NJL model as given in Ref. [23]. The vector coupling constant ($G_V$) is given in terms of the scalar coupling constant ($G_S$) and is chosen to be $G_V = 0.09 \ G_S$. 

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3.3 Quark-Hadron Mixed Phase

Phase equilibrium between the hadronic and quark phases of neutron star matter is governed by the Gibbs condition,

\[ p_H(\mu_n, \mu_e, \{\phi\}) = p_Q(\mu_n, \mu_e, \{\psi\}), \]  

(15)

where \(\mu_n\) and \(\mu_e\) are the neutron and electron chemical potentials, and \(\{\phi\}\) and \(\{\psi\}\) are the field variables and Fermi momenta associated with solutions of the equations of hadronic and quark matter, respectively. When this condition is initially met the first order phase transition from hadronic to quark matter begins. The relaxed condition of global charge neutrality allows the hadronic matter to become more isospin symmetric by transferring negative charge from the hadronic to the quark phase, lowering the asymmetry energy. This results in a mixed phase with coexisting regions of positively charged hadronic matter and negatively charged quark matter [10, 28, 29]. The equation of state for this phase is solved by combining the approaches for hadronic and quark matter under the Gibbs condition, baryon number conservation, and global electric charge neutrality.

4 Results

We first calculate the masses and radii of non-spherical neutron stars by solving Eqs. (5) and (7) numerically using the Runge Kutta method. The outcome is shown in Fig. 1. From the results shown in this figure, we see that the maximum mass of the spherical (\(\gamma = 1\)) neutron star obtained for the equation of state of this work is 2.3 \(M_\odot\). The equatorial radius of this star is close to 14 kilometers [23]. Oblate neutron stars are obtained for \(\gamma\) values less than one, since the polar coordinate obeys \(z = \gamma r\). We find that a decrease of \(\gamma\) by 10\% results in a \(\sim 15\%\) increase in gravitational mass and an increase in equatorial radius by a few kilometers (Fig. 1). If we continue decreasing \(\gamma\), the mass keeps increasing monotonically, ultimately extending into the mass region of solar-mass black holes. Prolate neutron stars are obtained for deformation parameters \(\gamma > 1\). In this case, as shown in Fig. 1, an increase of \(\gamma\) by 10\% leads to a \(\sim 12\%\) decrease in gravitational mass and a decrease in equatorial radius. If one keeps decreasing \(\gamma\) further, the maximum mass of a deformed neutron star drops down toward the 1.5 \(M_\odot\) region. The situation is graphically illustrated in Figs. 4 and 5 which show the deformations of the maximum-mass neutron stars of Fig. 1 for \(\gamma\) values ranging from 0.8 to 1.0 (oblate to spherical deformations) to 1.0 to 1.2 (spherical to prolate deformations). The pressure and energy-density profiles of the maximum-mass neutron stars of Fig. 1 are shown in Figs. 2 and 3.

The bottom line of all this is that there may be multiple maximum-mass neutron stars for one and for the same model for the nuclear equation of state, depending
on the type (oblate or prolate) of stellar deformation, which in the end is linked to the strengths of the magnetic fields of neutron stars and/or anisotropic pressure gradients in their cores. Moreover, as indicated by our calculations, the deformation does not need to be very large to appreciably change the bulk properties of neutron stars. This finding may be critical to better understand the ever-widening range of observed neutron star masses and to discriminate neutron stars from solar-mass black holes.

Next, we calculate the energy loss of photons emitted from the surface of a deformed neutron star. We consider a photon created at the surface of the star (emitter) and leaving its gravitational field toward a detector located at infinity, where spacetime is flat. The photon’s frequency at the emitter, $\nu_E$, is given as the inverse of the proper time between two wave crests, $d\tau_E$, that is, $\nu_E = 1/d\tau_E = \left(-g_{\mu\nu}dx^\mu dx^\nu\right)_E^{-1/2}$, where $dx^1 = dx^2 = dx^3 = 0$ because the emitter stays at a fixed position while emitting the photon. The same expression written down for the receiver at infinity reads $\nu_\infty = 1/d\tau_\infty = \left(-g_{\mu\nu}dx^\mu dx^\nu\right)_\infty^{-1/2}$. The ratio of these two frequencies is given by

$$\frac{\nu_\infty}{\nu_E} = \frac{\left[(g_{00})^{1/2}dx^0\right]_E}{\left[(g_{00})^{1/2}dx^0\right]_\infty}. \quad (16)$$

If we assume that the coordinate time $dx^0$ between two wave crests is the same as at

Figure 1: (Color online) Mass-radius relationships of deformed neutron stars ($\gamma < 1$: oblate neutron stars, $\gamma > 1$: prolate neutron stars, $\gamma = 1$: spherical neutron stars). The solid dots on each curve represent the maximum-mass star for each stellar sequence.
Figure 2: (Color online) (a) Pressure profiles and (b) energy-density profiles in equatorial direction for the maximum-mass stars shown Fig. 1.

Figure 3: (Color online) Same as Fig. 2 but in polar direction.
the star’s surface and the receiver, which is the case if the gravitational field is static so that whatever the world-line of one photon is from the star to the receiver, the next photon follows a congruent path, merely displaced by \( dx^0 \) at all points \([10, 11]\), this ratio simplifies to \( \nu_\infty / \nu_E = \left[ \frac{-g_{\mu\mu}}{\left(\frac{-g_{\mu\mu}}{2}\right)} \right]_E / \left[ \frac{-g_{\mu\mu}}{\left(\frac{-g_{\mu\mu}}{2}\right)} \right]_\infty \). Making use of the definition of the gravitational redshift, \( z = \left( \frac{\nu_E}{\nu_\infty} \right) - 1 \), we obtain

\[
z = \left( 1 - \frac{2M}{R} \right)^{-\gamma/2} - 1.
\]

Equation (17) shows that the gravitational redshift carries important information about the mass, radius, and the deformation of a neutron star. The \( z \) values at the equators of several deformed 1.5 \( M_\odot \) neutron stars are shown in Fig. [11]

The eccentricities \( e \),

\[
e \equiv \text{sign}(R_{eq} - R_p) \sqrt{1 - \left( \frac{R_<}{R_>} \right)^2},
\]

of the neutron stars shown in Figs. [4] and [5] are summarized in Table 2. For spherical neutron stars \( R_< \) (semi-minor axis) and \( R_> \) (semi-major axis) are equal, so that \( e = 0 \)
for such objects. Neutron stars whose $\gamma$ values differ by $\pm 10\%$ from the spherical case have eccentricities of $e = 0.43617$ if the deformation is oblate and $e = -0.41601$ if the deformation is prolate. Rapid rotation also deforms neutron stars away from spherical symmetry. For the neutron stars of this paper, we find eccentricities as low as 0.6 for rotation at the mass shedding frequency (which sets an absolute limit on rapid rotation), but not smaller. The metric of a rotating neutron star can also be used to study the structure of deformed non-rotating neutron stars. This has been done recently in Ref. [20]. The results of this paper cannot be directly compared with our results, however, because of specific assumptions about the energy-momentum tensor. We note, however, that the eccentricities of the oblate neutron stars of our study are compatible with those obtained in [20], depending on the degree of anisotropy generated by the magnetic field.

5 Conclusions

The goal of this work was to investigate the impact of deformation on the structure of non-rotating neutron stars in the framework of general relativity. For this purpose we first derived a stellar structure equation that describes deformed neutron stars. This equation constitutes a generalization of the well-known Tolman-Oppenheimer-Volkoff (TOV) equation, which describes the structure of non-rotating, perfectly spherically symmetric neutron stars. The mathematical structure of this generalized TOV equation is such that the deformation of a neutron star (or any other compact object, such as a hypothetical quark star) is expressed in terms of a deformation parameter, $\gamma$. By virtue of this parameter, models of deformed neutron stars can be built from non-spherical (prolate or oblate) mass shells rather than spherical mass shells. This leads to a stellar structure equation for deformed neutron stars which is of the same simple mathematical structure as the standard TOV equation and thus can be solved with little numerical effort.

The parametrization introduced in our paper allows one to use a model for the equation of state in the limiting case of isotropy while maintaining deformation structure. From the results shown in Fig. 4 one sees that modest deformations can lead to

| $\gamma$ | $\epsilon$ |
|----------|------------|
| 0.80     | 0.60000    |
| 0.90     | 0.43617    |
| 1.00     | 0          |

Table 2: Eccentricities, $\epsilon$, of the oblate and prolate neutron stars shown in Figs. 4 and 5, respectively.
appreciable changes in a neutron star’s gravitational mass and radius. In particular, we find that the mass of a neutron star increases with increasing oblateness, but decreases with increasing prolateness. This opens up the possibility that, depending on the degree of stellar deformation, there may exist multiple maximum-mass neutron stars for one and the same model for the nuclear equation of state, which is drastically different for spherically symmetric neutron stars whose mass-radius relationships are characterized by one and only one maximum-mass star. This finding may be critical to properly understand the ever widening range of observed neutron star masses and to discriminate neutron stars from solar-mass black holes.

Acknowledgments

This work is supported through the National Science Foundation under grants PHY-1411708 and DUE-1259951. A. Romero is supported by NIH through the Maximizing Access to Research Careers (MARC), grant number 5T34GM008303-25. Computing resources are provided by the Computational Science Research Center and the Department of Physics at San Diego State University. The authors would like to thank Vivian de la Incera and Efrain Ferrer (UTEP) for their insightful discussions and initial motivation on this work.

6 Appendix

Below, we outline the derivation of the stellar structure equation of deformed compact objects. For the metric given in Eq. (2), the non-vanishing Christoffel symbols are

\[
\Gamma^r_{tt} = \beta e^{2\Phi(r)} \Phi'(r), \quad \Gamma^t_{tr} = \Phi'(r), \quad \Gamma^r_{rr} = \frac{\gamma \left[ -m'(r)r + m(r) \right]}{r[r - 2m(r)]}, \quad \Gamma^r_{\theta\theta} = -\beta r,
\]

\[
\Gamma^\theta_{r\theta} = \Gamma^\Phi_{r\Phi} = \frac{1}{r}, \quad \Gamma^\Phi_{\Phi\Phi} = \cot(\theta), \quad \Gamma^r_{\Phi\Phi} = -\beta r \sin^2(\theta), \quad \Gamma^\theta_{\theta\theta} = -\sin(\theta) \cos(\theta),
\]

where primes denote derivatives with respect to the radial coordinate, \( r \), and

\[
\beta \equiv \left( \frac{r - 2m(r)}{r} \right) ^{\gamma}.
\]

The components of the Ricci tensor \( R^\mu_{\nu} \) for the metric of Eq. (2) are calculated to be

\[
R^t_{t} = \frac{1}{r(r - 2m(r))} \left[ \beta \Phi'(r) m'(r) \gamma r - \Phi'(r) m(r) \gamma - (\Phi'(r))^2 r m(r) - \Phi''(r) r^2 \\
+ 2\Phi''(r) r m(r) - 2\Phi'(r) r + 4\Phi'(r) m(r) \right],
\]

11
\[ R^r_r = \frac{1}{r^2(r-2m(r))} \left[ -\beta \Phi''(r)r^3 - 2\Phi''(r)r^2m(r) + (\Phi'(r))^2r^3 - 2(\Phi'(r))^2r^2m(r) - \gamma \Phi'(r)m'(r)r^2 + \gamma \Phi'(r)m(r) - 2\gamma m'(r)r + 2\gamma m(r) \right], \] \hspace{1cm} (22)

\[ R^\theta_\theta = \frac{1}{r^2(r-2m(r))} \left[ -\beta r^2\Phi'(r) + 2\beta \gamma r^2\Phi'(r)m(r) + \beta \gamma m'(r)r - \beta \gamma m(r) + r - 2m(r) - \beta \gamma + 2\beta m(r) \right], \] \hspace{1cm} (23)

and

\[ R^\phi_\phi = R^\theta_\theta. \] \hspace{1cm} (24)

The Ricci scalar, \( R \), is calculated to be

\[ R = \frac{2}{r^2(r-2m(r))} \left[ \beta \gamma \Phi'(r)m'(r)r^2 - \beta \gamma \Phi'(r)m(r)r - (\Phi'(r))^2r^3 + 2\beta(\Phi'(r))^2r^2m(r) - \beta \Phi''(r)r^3 + 2\beta \Phi''(r)r^2m(r) - 2m(r) - 2\beta \gamma^2\Phi'(r) + 4\beta \Phi'(r)m(r) + 2\beta \gamma m'(r)r - 2\beta \gamma m(r) + r - 2m(r) - \beta \gamma + 2\beta m(r) \right]. \] \hspace{1cm} (25)

Substituting Eqs. (21) to (24) along with Eq. (25) into Einstein’s field equation (3), one arrives at the general relativistic stellar structure equation (5) of deformed compact objects.

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