Homogenization of Boundary Value Problems in Perforated Lipschitz Domains

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Abstract

This paper is concerned with boundary regularity estimates in the homogenization of elliptic equations with rapidly oscillating and high-contrast coefficients. We establish uniform nontangential-maximal-function estimates for the Dirichlet, regularity, and Neumann problems with $L^2$ boundary data in a periodically perforated Lipschitz domain.

Keywords: Homogenization; Perforated Domain; Boundary Estimate.

MR (2020) Subject Classification: 35B27; 35J25.

1 Introduction

This paper is concerned with boundary regularity estimates in the homogenization of elliptic equations with rapidly oscillating and high-contrast coefficients in perforated Lipschitz domains. Let $A = A(y)$ be a real-valued $d \times d$ matrix satisfying the ellipticity condition,

$$\mu |\xi|^2 \leq (A\xi) \cdot \xi \quad \text{and} \quad \|A\|_{\infty} \leq \mu^{-1},$$

for any $\xi \in \mathbb{R}^d$ and some $\mu > 0$, and the periodicity condition,

$$A(y + z) = A(y) \quad \text{for any } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d.$$  \hfill (1.2)

Let $\omega$ be a connected and unbounded open set in $\mathbb{R}^d$. Assume that $\omega$ is 1-periodic; i.e., its characteristic function is periodic with respect to $\mathbb{Z}^d$. We also assume that each of connected components of $\mathbb{R}^d \setminus \omega$ is the closure of a bounded open set $F_k$ with Lipschitz boundary, and that

$$\min_{k \neq \ell} \operatorname{dist}(F_k, F_\ell) > 0.$$  \hfill (1.3)

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ and

$$\Omega^\varepsilon = \Omega \cap \varepsilon \omega = \Omega \setminus \varepsilon F,$$  \hfill (1.4)

where $0 < \varepsilon \leq 1$ and $F = \cup_k F_k$. We will assume that

$$\operatorname{dist}(\partial \Omega, \varepsilon F) \geq \kappa \varepsilon$$  \hfill (1.5)

for some $\kappa \in (0, 1)$. As a result, $\partial \Omega^\varepsilon = \partial \Omega \cup \Gamma^\varepsilon$, where $\Gamma^\varepsilon = \Omega \cap \partial(\varepsilon \omega)$, and $\operatorname{dist}(\partial \Omega, \Gamma^\varepsilon) \geq \kappa \varepsilon$. For $0 \leq \delta \leq 1$, define

$$\Lambda_\delta(x) = \begin{cases} 1 & \text{if } x \in \omega, \\ \delta & \text{if } x \in F. \end{cases}$$  \hfill (1.6)

*Supported in part by NSF grant DMS-1856235.
We are interested in the regularity estimates, which are uniform in $0 < \varepsilon \leq 1$ and $0 \leq \delta \leq 1$, for the elliptic operator,

$$L_{\varepsilon, \delta} = -\text{div}(A_{\delta}^\varepsilon(x) \nabla)$$

in $\Omega$, where

$$A_{\delta}^\varepsilon(x) = [A_\delta(x/\varepsilon)]^2 A(x/\varepsilon).$$

The operator $L_{\varepsilon, \delta}$ arises naturally in the modeling of acoustic propagation in porous media, periodic electromagnetic structures, and soft inclusions in composite materials [14, 26].

In the case $\delta = 1$, the regularity estimates for $L_\varepsilon = -\text{div}(A(x/\varepsilon) \nabla)$ have been studied extensively in recent years. Using a compactness method, the interior Lipschitz estimate and the boundary Lipschitz estimate for the Dirichlet problem in a $C^{1,\alpha}$ domain were established by M. Avellaneda and F. Lin in a seminal work [5]. The boundary Lipschitz estimate for the Neumann problem in a $C^{1,\alpha}$ domain was obtained in [10]. In [11, 12], C. Kenig and the present author investigated the $L^p$ Dirichlet, regularity, and Neumann problems in Lipschitz domains. We obtain nontangential-maximal-function estimates for the sharp ranges of $p$’s in the scalar case [11], and for $p$ close to 2 in the case of elliptic systems [12]. The results in [11, 12] extend an earlier work of B. Dahlberg (unpublished) on the $L^p$ Dirichlet problem in the scalar case. They also extend the classical work of B. Dahlberg, E. Fabes, D. Jerison, C. Kenig, J. Pipher, G. Verchota, and many others on the $L^p$ boundary value problems for elliptic equations and systems in Lipschitz domains to the periodic homogenization setting. We refer the reader to [8] for references on boundary value problems in nonsmooth domains, to [19] for further references on periodic homogenization, and to [2] for related work on large-scale regularity in stochastic homogenization.

In this paper we will be concerned with the case $0 \leq \delta < 1$ in perforated domains, where $\delta^2$ represents the conductivity ratio of the disconnected matrix block subset $\Omega \cap \varepsilon F$ to the connected subregion $\Omega'$. Notice that the operator $L_{\varepsilon, \delta}$ is not elliptic uniformly in $\delta$. In the case where $A = I$, $\Omega$ is $C^{1,\alpha}$ and $\omega$ sufficiently smooth, using the compactness method in [5], the $W^{1,p}$ and Lipschitz estimates were obtained by L.-M. Yeh [22, 23, 24, 25]. Also see earlier work in [17, 13] for uniform estimates in the case $\delta = 0$. In [16, 15], B. Russell established the large-scale interior Lipschitz estimates for $0 \leq \delta < 1$, using an approximation method originated in [3]. In the stochastic setting with $\delta = 0$, S. Armstrong and P. Dario [1] obtained large-scale regularity results for the random conductance model on a supercritical percolation. In this paper we shall be mainly interested in the nontangential-maximal-function estimates, which are uniform in $\varepsilon$ and $\delta$, for the Dirichlet, regularity, and Neumann problems, under the assumption that both $\Omega$ and $\omega$ are domains with Lipschitz boundary.

More precisely, we consider the Dirichlet problem,

$$\begin{cases}
\text{div}(A_{\delta}^\varepsilon(x) \nabla u_{\varepsilon, \delta}) = 0 & \text{in } \Omega, \\
u_{\varepsilon, \delta} = f & \text{on } \partial \Omega,
\end{cases}$$

and the Neumann problem,

$$\begin{cases}
\text{div}(A_{\delta}^\varepsilon(x) \nabla u_{\varepsilon, \delta}) = 0 & \text{in } \Omega, \\
\frac{\partial u_{\varepsilon, \delta}}{\partial \nu} = g & \text{on } \partial \Omega,
\end{cases}$$

where $\frac{\partial u_{\varepsilon, \delta}}{\partial \nu} = n(x) \cdot A(x/\varepsilon) \nabla u_{\varepsilon, \delta}$ and $n$ denotes the outward unit normal to $\partial \Omega$. Our main results in this paper give nontangential-maximal-function estimates, which are uniform in both $\varepsilon \in (0,1]$ and $\delta \in [0,1]$, for (1.9) and (1.10). These estimates are new even for the case where $A = I$ and $\omega$, $\Omega$ are smooth.
For a function $u \in L^2(\Omega)$, let $N(u)$ denote the (generalized) nontangential maximal function of $u$, defined by

$$N(u)(x) = \sup \left\{ \left( \frac{1}{|B(y, d(y)/4)|} \int_{B(y, d(y)/4)} |u|^2 \right)^{1/2} : y \in \Omega \text{ and } |y - x| < C_0 d(y) \right\}$$

for $x \in \partial \Omega$, where $d(y) = \text{dist}(y, \partial \Omega)$ and $C_0 = C_0(\Omega) > 1$ is sufficiently large. We impose a Hölder continuity condition on $A = A(y)$ in $\omega$,

$$\|A\|_{C^{\alpha, \sigma}(\omega)} \leq M \quad \text{for some } \sigma \in (0, 1) \text{ and } M > 0. \tag{1.12}$$

No smoothness condition is needed for $A$ in $F$.

**Theorem 1.1.** Let $0 < \varepsilon \leq 1$ and $0 \leq \delta \leq 1$. Assume that $A$ satisfies conditions (1.1), (1.2), (1.13), and is symmetric. Let $\omega$ be a connected, unbounded and 1-periodic open set with Lipschitz boundary and satisfying (1.3). Let $\Omega$ be a bounded Lipschitz domain satisfying (1.5). Then, for any $f \in L^2(\partial \Omega)$, the unique solution $u_{\varepsilon, \delta}$ of (1.9) with $N(u_{\varepsilon, \delta}) \in L^2(\partial \Omega)$ satisfies the estimate,

$$\|N(u_{\varepsilon, \delta})\|_{L^2(\partial \Omega)} \leq C \|f\|_{L^2(\partial \Omega)}, \tag{1.13}$$

where $C$ depend only on $d$, $\mu$, $(M, \sigma)$ in (1.12), $\omega$, $\kappa$, and the Lipschitz character of $\Omega$.

**Theorem 1.2.** Let $0 < \varepsilon \leq 1$ and $0 \leq \delta \leq 1$. Let $A$, $\omega$ and $\Omega$ satisfy the same conditions as in Theorem 1.1. Suppose $f \in H^1(\partial \Omega)$. Then the unique weak solution $u_{\varepsilon, \delta}$ in $H^1(\Omega)$ of (1.9) satisfies the estimate,

$$\|N(\nabla u_{\varepsilon, \delta})\|_{L^2(\partial \Omega)} \leq C \|f\|_{H^1(\partial \Omega)}, \tag{1.14}$$

where $C$ depends only on $d$, $\mu$, $(M, \sigma)$, $\omega$, $\kappa$, and the Lipschitz character of $\Omega$.

**Theorem 1.3.** Let $0 < \varepsilon \leq 1$ and $0 \leq \delta \leq 1$. Suppose $A$, $\omega$ and $\Omega$ satisfy the same conditions as in Theorem 1.1. Then, for any $g \in L^2(\partial \Omega)$ with $\int_{\partial \Omega} g \, d\sigma = 0$, the weak solutions in $H^1(\Omega)$ of (1.10) satisfy the estimate

$$\|N(\nabla u_{\varepsilon, \delta})\|_{L^2(\partial \Omega)} \leq C \|g\|_{L^2(\partial \Omega)}, \tag{1.15}$$

where $C$ depends only on $d$, $\mu$, $(M, \sigma)$, $\omega$, $\kappa$, and the Lipschitz character of $\Omega$.

A few remarks are in order.

**Remark 1.4.** By (1.5), the matrix $A_\delta(x)$ is Hölder continuous near $\partial \Omega$. As a result, the existence and uniqueness of solutions $u_{\varepsilon, \delta}$ with $N(u_{\varepsilon, \delta}) \in L^2(\partial \Omega)$ in Theorem 1.1 and with $N(\nabla u_{\varepsilon, \delta}) \in L^2(\partial \Omega)$ in Theorems 1.2 and 1.3 are more or less well known [8]. The main contribution of this paper is that the constants $C$ in (1.13), (1.14), and (1.15) do not depend on $\varepsilon \in (0, 1)$ and $\delta \in [0, 1]$.

**Remark 1.5.** The boundary data in Theorem 1.1 is taken in the sense of nontangential convergence. The boundary data in Theorems 1.2 and 1.3 may also be taken in the sense of nontangential convergence. In (1.13), the nontangential maximal function $N(u_{\varepsilon, \delta})$ may be replaced by $\tilde{N}(u_{\varepsilon, \delta})$, defined by

$$\tilde{N}(u)(x) = \sup \{ |u(y)| : y \in \Omega^c \text{ and } |y - x| < \tilde{C}_0 \text{dist}(y, \partial \Omega) \}.$$

This follows from the proof of Theorem 1.1.
Remark 1.6. Let $\delta > 0$ and $u_{\varepsilon,\delta}$ be a weak solution of $\text{div}(A^\varepsilon_\delta \nabla u_{\varepsilon,\delta}) = 0$ in $\Omega$. Then $\text{div}(A(x/\varepsilon)\nabla u_{\varepsilon,\delta}) = 0$ in $\Omega^\varepsilon$ and $\Omega \setminus \overline{\Omega^\varepsilon}$. Moreover,

$$
\left( \frac{\partial u_{\varepsilon,\delta}}{\partial \nu_\varepsilon} \right)_+ = \delta^2 \left( \frac{\partial u_{\varepsilon,\delta}}{\partial \nu_\varepsilon} \right)_- \quad \text{and} \quad (u_{\varepsilon,\delta})_+ = (u_{\varepsilon,\delta})_- \quad \text{on } \Gamma^\varepsilon = \Omega \cap \partial \Omega^\varepsilon, \tag{1.16}
$$

where $\pm$ indicates the traces taken from $\Omega^\varepsilon$ and $\Omega \setminus \overline{\Omega^\varepsilon}$, respectively. In the case $\delta = 0$, the Dirichlet problem (1.9) is reduced to the mixed boundary value problem in $\Omega^\varepsilon$,

$$
\begin{cases}
\text{div}(A(x/\varepsilon)\nabla u_{\varepsilon,0}) = 0 & \text{in } \Omega^\varepsilon, \\
n \cdot A(x/\varepsilon)\nabla u_{\varepsilon,0} = 0 & \text{on } \Gamma^\varepsilon, \\
 u_{\varepsilon,0} = f & \text{on } \partial \Omega,
\end{cases} \tag{1.17}
$$

while (1.10) is reduced to the Neumann problem in $\Omega^\varepsilon$,

$$
\begin{cases}
\text{div}(A(x/\varepsilon)\nabla u_{\varepsilon,0}) = 0 & \text{in } \Omega^\varepsilon, \\
n \cdot A(x/\varepsilon)\nabla u_{\varepsilon,0} = 0 & \text{on } \Gamma^\varepsilon, \\
 n \cdot A(x/\varepsilon)\nabla u_{\varepsilon,0} = g & \text{on } \partial \Omega.
\end{cases} \tag{1.18}
$$

In this case we shall extend the solution $u_{\varepsilon,0}$ in $\Omega^\varepsilon$ to $\Omega$ by solving the Dirichlet problem,

$$
\text{div}(A(x/\varepsilon)\nabla u_{\varepsilon,0}) = 0 \quad \text{in } \varepsilon F_k \quad \text{and} \quad (u_{\varepsilon,0})_- = (u_{\varepsilon,0})_+ \quad \text{on } \partial(\varepsilon F_k), \tag{1.19}
$$

for each $\varepsilon F_k$ contained in $\Omega$. Consequently, (1.16) continues to hold in the case $\delta = 0$. As we pointed out earlier, the estimates in Theorems 1.1, 1.2, and 1.3 for the boundary value problems (1.17) and (1.18) are new even in the case of Laplace’s equation $\Delta u = 0$ in $\Omega^\varepsilon$.

We now describe our approaches to Theorems 1.1, 1.2, and 1.3. Our main tool is the Rellich estimates,

$$
\left\| \frac{\partial u_{\varepsilon,\delta}}{\partial \nu_\varepsilon} \right\|_{L^2(\partial \Omega)} \approx \| \nabla_{\text{tan}} u_{\varepsilon,\delta} \|_{L^2(\partial \Omega)}, \tag{1.20}
$$

for weak solutions of $L_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Omega$. To prove (1.20), we first use local Rellich estimates for the elliptic operator $\text{div}(A\nabla)$ and condition (1.5) as well as a covering argument to reduce the problem to the boundary layer estimates,

$$
\frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} |\nabla u_{\varepsilon,\delta}|^2 dx \leq C \int_{\partial \Omega} |\nabla_{\text{tan}} u_{\varepsilon,\delta}|^2 d\sigma + C \int_{\Omega} |\nabla u_{\varepsilon,\delta}|^2 dx, \tag{1.21}
$$

$$
\frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} |\nabla u_{\varepsilon,\delta}|^2 dx \leq C \int_{\partial \Omega} \left| \frac{\partial u_{\varepsilon,\delta}}{\partial \nu_\varepsilon} \right|^2 d\sigma + C \int_{\Omega} |\nabla u_{\varepsilon,\delta}|^2 dx,
$$

where $\Sigma_\varepsilon = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \}$. The estimates in (1.21) are then proved by establishing a sharp convergence rate in $H^1(\Omega)$ for a two-scale expansion. We remark that a similar approach has been used in [13, 6] in the case $\delta = 1$ for elliptic systems of elasticity. To deal with the fact that the operator $L_{\varepsilon,\delta}$ is not uniformly elliptic in $\delta$, techniques of extension are used to treat the block regions $\varepsilon F_k$ with small ellipticity constants.

With the Rellich estimates (1.20) at our disposal, we follow an approach used in [11] for scalar elliptic equations with periodic coefficients. To prove Theorem 1.1, we show that

$$
N(u_{\varepsilon,\delta}) \leq C \left[ M_{\partial \Omega}(|f|^{p_0}) \right]^{1/p_0} \quad \text{on } \partial \Omega \quad \tag{1.22}
$$
for some $p_0 < 2$, where $u_{\varepsilon, \delta}$ is a solution of (1.9) and $\mathcal{M}_{\partial \Omega}$ denotes the Hardy-Littlewood maximal operator on $\partial \Omega$. This is done by applying a localized version of (1.20) to the Green function $G_{\varepsilon, \delta}(x, y)$ and using the estimate,

$$|G_{\varepsilon, \delta}(x, y)| \leq \frac{C[\text{dist}(x, \partial \Omega)]^\sigma}{|x - y|^{d - 2 + \sigma}},$$

(1.23)

which holds if either $x, y \in \Omega^\varepsilon$ or $x, y \in \Omega$ with $|x - y| \geq c_d \varepsilon$. Estimate (1.23) follows from the boundary H"{o}lder estimate for the operator $\mathcal{L}_{\varepsilon, \delta}$ in Lipschitz domains, which is proved by an approximation argument. A great amount of efforts, involving a reverse H"{o}lder argument above the scale $\varepsilon$, is also needed to lower the exponent in (1.22) to some $p_0 < 2$.

To prove Theorem 1.2, we exploit the fact that the operator $\mathcal{L}_{\varepsilon, \delta}$ is invariant under the translation $x \rightarrow x + \varepsilon z$, where $z \in \mathbb{Z}^d$. As in [11], this allows us to dominate $|\nabla u_{\varepsilon, \delta}(x)|$ for $\text{dist}(x, \partial \Omega) \geq C \varepsilon$ by $M_{\partial \Omega}\{|\nabla \tan f| + M_{\text{rad}}(Q_{\varepsilon}(u_{\varepsilon, \delta}))\}$, where $M_{\text{rad}}$ is a radial maximal operator and

$$Q_{\varepsilon}(u) = \varepsilon^{-1}\{u(x + \varepsilon e_d) - u(x)\},$$

with $e_d = (0, \ldots, 0, 1)$. Since $Q_{\varepsilon}(u_{\varepsilon, \delta})$ is a solution, the term involving $Q_{\varepsilon}(u_{\varepsilon, \delta})$ may be handled by using a localized version of Theorem 1.1 and the boundary layer estimates in (1.21). Finally, Theorem 1.3 follows from Theorem 1.2 and (1.21).

Our proof of Theorem 1.1 yields the estimate $\|N(u_{\varepsilon, \delta})\|_{L^p(\partial \Omega)} \leq C_p \|f\|_{L^p(\partial \Omega)}$ for $2 - \gamma < p \leq \infty$, where $\gamma > 0$ depends on $\Omega$, $\omega$, and $A$. In view of the results in [11], it would be interesting to establish the $L^p$ estimates of $N(\nabla u_{\varepsilon, \delta})$ for $1 < p < 2$, uniform in $\varepsilon > 0$ and $\delta \in [0, 1]$, for the regularity and Neumann problems considered in Theorems 1.2 and 1.3. Another interesting problem would be the study of the uniform boundary regularity estimates for the case $1 < \delta \leq \infty$, the so-called stiff problem [26]. The interior Lipschitz estimates have been established recently in [20].

2 Preliminaries

Throughout this paper we assume that the matrix $A = A(y)$ satisfies conditions (1.1)-(1.2), $\omega$ is a connected, unbounded and 1-periodic open set in $\mathbb{R}^d$, and that

$$\mathbb{R}^d \setminus \omega = \bigcup_k \overline{F_k},$$

(2.1)

where each $\overline{F_k}$ is the closure of a bounded Lipschitz domain $F_k$ with connected boundary. We assume that $\overline{F_k}$’s are mutually disjoint and satisfy the condition (1.3). This allows us to construct a sequence of mutually disjoint open sets $\{F_k\}$ with connected smooth boundary such that $\overline{F_k} \subset \overline{F_k}$,

$$\begin{cases}
(1/100)\kappa \leq \text{dist}(F_k, \partial \overline{F_k}), \\
(1/100)\kappa \leq \text{dist}(\overline{F_k}, \overline{F_\ell}) \text{ for } k \neq \ell.
\end{cases}$$

(2.2)

Since $\text{diam}(F_k) \leq \text{diam}(Y) = \sqrt{d}$, we may assume $\text{diam}(\overline{F_k}) \leq d$. By the periodicity of $\omega$, $F_k$’s are the shifts of a finite number of bounded Lipschitz domains. As a result, we may assume that $\overline{F_k}$’s are the shifts of a finite number of bounded smooth domains.
2.1 Extension operators

Since $\tilde{F}_k \setminus F_k$ is a bounded Lipschitz domain, there exists an extension operator $T_k$ from $\tilde{F}_k \setminus F_k$ to $\tilde{F}_k$ such that

$$
\|T_k(u)\|_{L^p(\tilde{F}_k)} \leq C_p \|u\|_{L^p(\tilde{F}_k \setminus F_k)},
$$

$$
\|T_k(u)\|_{W^{1,p}(\tilde{F}_k)} \leq C_p \|u\|_{W^{1,p}(\tilde{F}_k \setminus F_k)},
$$

for $1 < p < \infty$. Let

$$
\tilde{T}_k(u) = \int_{\tilde{F}_k \setminus F_k} u + T_k\left(u - \int_{\tilde{F}_k \setminus F_k} u\right).
$$

Then $\tilde{T}_k$ is an extension operator from $\tilde{F}_k \setminus F_k$ to $\tilde{F}_k$,

$$
\|\tilde{T}_k(u)\|_{L^p(\tilde{F}_k)} \leq C_p \|u\|_{L^p(\tilde{F}_k \setminus F_k)},
$$

and

$$
\|\nabla \tilde{T}_k(u)\|_{L^p(\tilde{F}_k)} \leq C_p \|u - \int_{\tilde{F}_k \setminus F_k} u\|_{W^{1,p}(\tilde{F}_k \setminus F_k)} \leq C_p \|\nabla u\|_{L^p(\tilde{F}_k \setminus F_k)},
$$

where we have used a Poincaré inequality. By dilation there exist extension operators $E_{\epsilon,k}$ from $\epsilon \tilde{F}_k \setminus \epsilon F_k$ to $\epsilon \tilde{F}_k$ such that

$$
\|E_{\epsilon,k}(u)\|_{L^p(\epsilon \tilde{F}_k)} \leq C_p \|u\|_{L^p(\epsilon \tilde{F}_k \setminus \epsilon F_k)},
$$

$$
\|\nabla E_{\epsilon,k}(u)\|_{L^p(\epsilon \tilde{F}_k)} \leq C_p \|\nabla u\|_{L^p(\epsilon \tilde{F}_k \setminus \epsilon F_k)},
$$

(2.3)

for $1 < p < \infty$ and $\epsilon > 0$, where $C_p$ depends only on $d$, $p$, and $\omega$. As a result, we obtain the following.

**Lemma 2.1.** Let $\Omega$ be a bounded Lipschitz domain satisfying (1.5). Let $1 < p < \infty$. Then, for any $u \in W^{1,p}(\Omega^\epsilon)$, there exists $\tilde{u} \in W^{1,p}(\Omega)$ such that $\tilde{u} = u$ in $\Omega^\epsilon$,

$$
\|\tilde{u}\|_{L^p(\Omega^\epsilon)} \leq C_p \|u\|_{L^p(\Omega^\epsilon)} \quad \text{and} \quad \|\nabla \tilde{u}\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)},
$$

(2.4)

where $C_p$ depends only on $d$, $p$, $\kappa$, and $\omega$.

The next lemma will be used to treat regions where the ellipticity constant is small.

**Lemma 2.2.** Suppose $u \in H^1(\epsilon \tilde{F}_k)$ and $\text{div}(A(x/\epsilon)\nabla u) = 0$ in $\epsilon F_k$. Then

$$
\|\nabla u\|_{L^2(\epsilon \tilde{F}_k)} \leq C \|\nabla u\|_{L^2(\epsilon \tilde{F}_k \setminus \epsilon F_k)},
$$

(2.5)

where $C$ depends only on $d$, $\mu$, and $\omega$.

**Proof.** By dilation we may assume $\epsilon = 1$. Let $\tilde{u} \in H^1(\tilde{F}_k)$ be an extension of $u|_{\tilde{F}_k \setminus F_k}$ such that $\|\nabla \tilde{u}\|_{L^2(\tilde{F}_k)} \leq C \|\nabla u\|_{L^2(\tilde{F}_k \setminus F_k)}$. Since $\tilde{u} - u \in H^1_0(F_k)$ and $\text{div}(A \nabla (\tilde{u} - u)) = \text{div}(A \nabla \tilde{u})$ in $F_k$, by energy estimates,

$$
\|\nabla (\tilde{u} - u)\|_{L^2(F_k)} \leq C \|\nabla \tilde{u}\|_{L^2(F_k)},
$$

from which the inequality (2.5) with $\epsilon = 1$ follows.

We now give the energy estimates for (1.9) and (1.10).
Lemma 2.3. Let \( \Omega \) be a bounded Lipschitz domain satisfying (1.5). Let \( u = u_{\varepsilon, \delta} \) be a weak solution of (1.9) with \( f \in H^{1/2}(\partial \Omega) \). Then

\[
\|u\|_{H^{1}(\Omega)} \leq C\|f\|_{H^{1/2}(\partial \Omega)},
\]

(2.6)

where \( C \) depends only on \( d, \mu, \kappa, \omega, \) and \( \Omega \).

Proof. Let \( v \in H^{1}(\Omega) \) be a function such that \( v = f \) on \( \partial \Omega \) and \( \|v\|_{H^{1}(\Omega)} \leq 2\|f\|_{H^{1/2}(\partial \Omega)} \). Since \( u - v \in H^{1}_0(\Omega) \) and \( \text{div}(A^\delta \nabla (u - v)) = -\text{div}(A^\delta \nabla v) \) in \( \Omega \), it follows that

\[
\int_{\Omega} A^\delta \nabla (u - v) \cdot \nabla \psi \, dx = -\int_{\Omega} A^\delta \nabla v \cdot \nabla \psi \, dx
\]

(2.7)

for any \( \psi \in H^1_0(\Omega) \). Let \( \psi = u - v \) in (2.7). By using (1.11) and the Cauchy inequality,

\[\int_{\Omega} |A^\delta \nabla (u - v)|^2 \, dx \leq C \int_{\Omega} |\nabla \psi|^2 \, dx,\]

where \( A^\delta(x) = A^\delta(x/\varepsilon) \) and \( C \) depends only on \( \mu \). It follows that \( \|\nabla u\|_{L^2(\Omega^\varepsilon)} \leq C\|f\|_{H^{1/2}(\partial \Omega)}. \) Using (1.3) and Lemma 2.2 we obtain

\[\|\nabla u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega^\varepsilon)} \leq C\|f\|_{H^{1/2}(\partial \Omega)}. \]

This, together with Poincaré’s inequality \( \|u - v\|_{L^2(\Omega)} \leq C\|\nabla (u - v)\|_{L^2(\Omega)} \), gives (2.6). \( \square \)

Lemma 2.4. Let \( \Omega \) be a bounded Lipschitz domain satisfying (1.5). Let \( u = u_{\varepsilon, \delta} \) be a weak solution of (1.10) with \( g \in H^{-1/2}(\partial \Omega) \) and \( \langle g, 1 \rangle = 0. \) Then

\[\|\nabla u\|_{L^2(\Omega)} \leq C\|g\|_{H^{-1/2}(\partial \Omega)}, \]

(2.8)

where \( C \) depends only on \( d, \mu, \kappa, \omega, \) and \( \Omega \).

Proof. Note that for any \( \psi \in H^1(\Omega), \)

\[\int_{\Omega} A^\delta \nabla u \cdot \nabla \psi \, dx = \langle g, \psi \rangle_{H^{-1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)}, \]

(2.9)

By letting \( \psi = u \) in (2.7), we obtain

\[\|A^\delta \nabla u\|_{L^2(\Omega)} \leq C\|g\|_{H^{-1/2}(\partial \Omega)} \|\nabla u\|_{L^2(\Omega)}. \]

(2.10)

By Lemma 2.2 \( \|\nabla u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega^\varepsilon)}. \) This, together with (2.10), yields (2.8). \( \square \)

The next lemma gives a Caccioppoli inequality for the operator \( \text{div}(A^\delta \nabla) \).

Lemma 2.5. Let \( u = u_{\varepsilon, \delta} \in H^1(\Omega) \) be a weak solution of \( \text{div}(A^\delta \nabla u) = 0 \) in a bounded Lipschitz domain \( \Omega \). Then

\[\int_{\Omega} |A^\delta \nabla (u \varphi)|^2 \, dx \leq C \int_{\Omega} |A^\delta u|^2 |\nabla \varphi|^2 \, dx\]

(2.11)

for any \( \varphi \in C^0_0(\Omega) \), where \( C \) depends only on \( d \) and \( \mu \).

Proof. This may be proved by using the test function \( u \varphi^2 \) in the weak formulation of (4.1), as in the proof of the standard Caccioppoli inequality for \( \delta = 1. \) \( \square \)
2.2 Correctors

For $0 \leq \delta \leq 1$, let $A_\delta(y) = \Lambda_{\delta^2}(y)A(y) = [\Lambda_{\delta}(y)]^2A(y)$, where $\Lambda_{\delta}$ is defined by (1.6). Observe that the $d \times d$ matrix $A_\delta(y)$ is 1-periodic. Let $\chi_{\delta}(y) = (\chi_{\delta,1}(y), \ldots, \chi_{\delta,d}(y))$ be the corrector for the operator $-\text{div}(A\nabla)$, where, for $1 \leq j \leq d$, the function $\chi_{\delta,j} \in H^1_{\text{loc}}(\mathbb{R}^d)$ is a weak solution of

\[
\begin{cases}
- \text{div}(A_\delta \nabla \chi_{\delta,j}) = \text{div}(A_\delta \nabla y_j) & \text{in } \mathbb{R}^d, \\
\chi_{\delta,j} & \text{is 1-periodic,}
\end{cases}
\]

(2.12)

with

\[
\begin{align*}
\int_Y \chi_{\delta,j} \, dy &= 0 & \text{if } \delta > 0, \\
\int_{Y \cap \omega} \chi_{\delta,j} \, dy &= 0 & \text{if } \delta = 0,
\end{align*}
\]

(2.13)

and $Y = [0, 1]^d$. If $\delta > 0$, the existence and uniqueness of correctors follow readily from the Lax-Milgram Theorem by using the bilinear form,

\[
\int_Y A_\delta \nabla \phi \cdot \nabla \psi \, dy,
\]

on the Hilbert space $H^1_{\text{per}}(Y)$, the closure of 1-periodic $C^\infty$ functions in $H^1(Y)$. In the case $\delta = 0$, one uses the bilinear form

\[
\int_{Y \cap \omega} A \nabla \phi \cdot \nabla \psi \, dy
\]

on the Hilbert space $H^1_{\text{per}}(Y \cap \omega)$, the closure of 1-periodic $C^\infty$ functions in $H^1(Y \cap \omega)$. This gives the definition of $\chi_{0,j}$ on $\omega$. Recall that $\mathbb{R}^d \setminus \overline{\omega} = F = \bigcup_k F_k$. We extend $\chi_{0,j}$ to each $F_k$ by using the weak solution in $H^1(F_k)$ of $-\text{div}(A\nabla u) = \text{div}(A\nabla y_j)$ in $F_k$, with Dirichlet data $u = \chi_{0,j}$ on $\partial F_k$.

**Lemma 2.6.** Let $0 \leq \delta \leq 1$. Then

\[
\int_Y \left( |\nabla \chi_{\delta}|^2 + |\chi_{\delta}|^2 \right) \, dy \leq C,
\]

(2.14)

where $C$ depends only on $d$, $\mu$, and $\omega$.

**Proof.** Let $0 < \delta \leq 1$. By energy estimates,

\[
\int_Y |\Lambda_{\delta} \nabla \chi_{\delta}|^2 \, dy \leq C \int_Y |\Lambda_{\delta} A|^2 \, dy \leq C.
\]

This gives $\|\nabla \chi_{\delta}\|_{L^2(Y \cap \omega)} \leq C$. Next, note that $\text{div}(A\nabla(y_{\delta,j} + y_j)) = 0$ in $F_k$. It follows by Lemma [2.2] that

\[
\|\nabla \chi_{\delta}\|_{L^2(F_k)} \leq C + C\|\nabla \chi_{\delta}\|_{L^2(Y \cap \omega)} \leq C.
\]

Since $Y \cap F_k \neq \emptyset$ only for a finite number of $k$’s, we obtain $\|\nabla \chi_{\delta}\|_{L^2(Y \cap \omega)} \leq C$. As a result, we have proved that $\|\nabla \chi_{\delta}\|_{L^2(Y)} \leq C$. In view of (2.13), the estimate $\|\chi_{\delta}\|_{L^2(Y)} \leq C$ follows by Poincaré’s inequality.

If $\delta = 0$, the energy estimate gives $\|\nabla \chi_{0}\|_{L^2(Y \cap \omega)} \leq C$. By Poincaré’s inequality and (2.13), we obtain $\|\chi_{0}\|_{H^1(Y \cap \omega)} \leq C$. In view of the definition of $\chi_{0}$ on $F_k$, we have

\[
\|\chi_{0}\|_{H^1(F_k)} \leq C + C\|\chi_{0}\|_{H^{1/2}(\partial F_k)} \leq C + C\|\chi_{0}\|_{H^1(F_k \setminus F_k)} \leq C + C\|\chi_{0}\|_{H^1(Y \cap \omega)} \leq C.
\]

It follows that $\|\chi_{0}\|_{H^1(Y \setminus \omega)} \leq C$. \qed

8
2.3 Homogenized operator

The homogenized matrix for the operator $-\text{div}(A\nabla)$ is given by

$$\hat{A}_\delta = \int_Y \left\{ A_\delta + A_\delta \nabla \chi_\delta \right\} dy.$$  \hspace{1cm} (2.15)

**Lemma 2.7.** Let $0 < \delta \leq 1$. Then

$$|\hat{A}_\delta - \hat{A}_0| \leq C\delta^2,$$  \hspace{1cm} (2.16)

where $C$ depends only on $d$, $\mu$, and $\omega$.

**Proof.** We provide a proof, which also may be found in [15], for the reader’s convenience. Note that

$$|\hat{A}_\delta - \hat{A}_0| \leq \int_Y |A_\delta^2 - A_0| |A| dy + \int_Y |A_\delta^2 A\nabla \chi_\delta - A_0 A\nabla \chi_0| dy$$

$$\leq C\delta^2 + C \int_{Y\cap\omega} |\nabla (\chi_\delta - \chi_0)| dy + C\delta^2 \int_{Y\setminus\omega} |\nabla \chi_\delta| dy$$

$$\leq C\delta^2 + C \int_{Y\cap\omega} |\nabla (\chi_\delta - \chi_0)| dy,$$

where we have used (2.14) for the last inequality. To bound $|\nabla (\chi_\delta - \chi_0)|$, we observe that

$$-\text{div}(\Lambda_\delta^2 A\nabla (\chi_\delta - \chi_0)) = \text{div}((\Lambda_\delta^2 - \Lambda_0) A\nabla (y + \chi_0)).$$

Thus, for any $\psi \in H^1_{\text{per}}(Y)$,

$$\int_Y A_\delta^2 A\nabla (\chi_\delta - \chi_0) \cdot \nabla \psi dy = -\delta^2 \int_{Y\setminus\omega} A\nabla (y + \chi_0) \cdot \nabla \psi dy.$$  \hspace{1cm} (2.17)

We now choose $\psi \in H^1_{\text{per}}(Y)$ such that $\psi = \chi_\delta - \chi_0$ on $Y\cap\omega$ and

$$\|\nabla \psi\|_{L^2(Y)} \leq C\|\nabla (\chi_\delta - \chi_0)\|_{L^2(Y\cap\omega)}.$$  \hspace{1cm} (2.18)

It follows from (2.17) and (2.18) that

$$\int_{Y\cap\omega} |\nabla (\chi_\delta - \chi_0)|^2 dy \leq C\delta^2 \int_{Y\setminus\omega} |\nabla (\chi_\delta - \chi_0)||\nabla \psi| dy + C\delta^2 \int_{Y\setminus\omega} |\nabla (y + \chi_0)||\nabla \psi| dy$$

$$\leq C\delta^2 \|\nabla (\chi_\delta - \chi_0)\|_{L^2(Y\cap\omega)}.$$

Hence,

$$\|\nabla (\chi_\delta - \chi_0)\|_{L^1(Y\cap\omega)} \leq \|\nabla (\chi_\delta - \chi_0)\|_{L^2(Y\cap\omega)} \leq C\delta^2.$$

\[Q.E.D.\]

Note that for $\xi \in \mathbb{R}^d$,

$$(\hat{A}_\delta \xi) \cdot \xi = \int_Y A_\delta \nabla ((y + \chi_\delta) \cdot \xi) \cdot \nabla ((y + \chi_\delta) \cdot \xi) dy.$$  \hspace{1cm} (2.19)

It follows that $\hat{A}_\delta$ is symmetric if $A$ is symmetric.

**Theorem 2.8.** Let $0 \leq \delta \leq 1$. Then for any $\xi \in \mathbb{R}^d$,

$$\mu_0|\xi|^2 \leq (\hat{A}_\delta \xi) \cdot \xi \quad \text{and} \quad |\hat{A}_\delta| \leq \mu_0^{-1},$$  \hspace{1cm} (2.20)

where $\mu_0 > 0$ depends only on $d$, $\mu$, and $\omega$.  \hspace{1cm}
Proof. The second inequality in (2.20) follows readily from (2.14). To see the first, note that
\[
(\hat{A}_0 \xi) \cdot \xi \geq \mu \min(1, \delta^2) \int_Y |\nabla ((y + \chi_0) \cdot \xi)|^2 \, dy
\]
\[
= \mu \min(1, \delta^2) \int_Y \left\{ |\nabla (\chi_0 \cdot \xi)|^2 + |\xi|^2 + 2(\nabla \chi_0) \cdot \xi \right\} \, dy
\]
\[
\geq \mu \min(1, \delta^2) |\xi|^2.
\]
Thus, in view of (2.16), it suffices to consider the case \( \delta = 0 \). To this end, suppose there exists a sequence \( \{A^\ell\} \) of 1-periodic matrices satisfying (1.11) and a sequence \( \{\xi^\ell\} \subset \mathbb{R}^d \) with \( |\xi^\ell| = 1 \) such that \( (\hat{A}_0^\ell \xi^\ell) \cdot \xi^\ell \to 0 \), as \( \ell \to \infty \). By passing to a subsequence we may assume \( \xi^\ell \to \xi \). It follows that \( (\hat{A}_0^\ell \xi) \cdot \xi \to 0 \), as \( \ell \to \infty \), where \( |\xi| = 1 \). Let \( \chi_0 \) denote the corrector for the matrix \( A_0 \). Then
\[
\int_{Y \cap \omega} |\nabla ((\chi_0 + y) \cdot \xi)|^2 \, dy \to 0,
\]
as \( \ell \to \infty \). Let \( E_\ell \) denote the average of \( (\chi_0 + y) \cdot \xi \) over \( Y \cap \omega \). Then, \( (\chi_0 + y) \cdot \xi - E_\ell \) converges to zero in \( H^1(Y \cap \omega) \). Since the sequence \( \{\chi_0 \cdot \xi - E_\ell\} \) is bounded in \( H^1_{\text{per}}(Y \cap \omega) \), by passing to a subsequence, we may assume it converges weakly in \( H^1_{\text{per}}(Y \cap \omega) \). This implies that \( y \cdot \xi \in H^1_{\text{per}}(Y \cap \omega) \). However, since all connected components of \( \mathbb{R}^d \setminus \omega \) are bounded, for each \( 1 \leq j \leq d \), there exists \( y \in \partial Y \cap \omega \) such that \( y + e_j \in \partial Y \cap \omega \). It follows that \( (y + e_j) \cdot \xi = y \cdot \xi \). Consequently, we obtain \( \xi = 0 \), which contradicts with the fact \( |\xi| = 1 \). The argument above shows that \( (\hat{A}_0 \xi) \cdot \xi \geq \mu_0 |\xi|^2 \) for some \( \mu_0 > 0 \) depending only on \( d, \mu, \) and \( \omega \) (not directly on \( A \)). \( \square \)

2.4 Flux correctors

Let
\[
B_{\delta} = A_{\delta} + A_{\delta} \nabla \chi_{\delta} - \hat{A}_{\delta}.
\]
Write \( B_\delta = (b_{\delta,ij})_{d \times d} \). By the definitions of \( \chi_\delta \) and \( \hat{A}_\delta \),
\[
\frac{\partial}{\partial y_i} b_{\delta,ij} = 0 \quad \text{and} \quad \int_Y b_{\delta,ij} \, dy = 0,
\]
where the repeated index is summed.

Lemma 2.9. There exist \( \phi_{\delta,ki,j} \in H^1_{\text{per}}(Y) \), where \( 1 \leq i, j, k \leq d \), such that \( \int_Y \phi_{\delta,ki,j} \, dy = 0 \),
\[
b_{\delta,ij} = \frac{\partial}{\partial y_k} \phi_{\delta,ki,j} \quad \text{and} \quad \phi_{\delta,ki,j} = -\phi_{\delta,ik,j}.
\]
Moreover, for \( 0 \leq \delta \leq 1 \),
\[
\int_Y \left( |\nabla \phi_{\delta,ki,j}|^2 + |\phi_{\delta,ki,j}|^2 \right) \, dy \leq C,
\]
where \( C \) depends only on \( d, \mu, \) and \( \omega \).

Proof. The proof is similar to the case \( \delta = 1 \). Since \( \int_Y b_{\delta,ij} \, dy = 0 \), there exists \( f_{ij} \in H^2_{\text{per}}(Y) \) such that
\[
\Delta f_{ij} = b_{\delta,ij} \quad \text{in} \ Y \quad \text{and} \quad \|f_{ij}\|_{H^2_{\text{per}}(Y)} \leq C\|b_{\delta,ij}\|_{L^2(Y)}.
\]
Let
\[
\phi_{\delta,ki,j} = \frac{\partial}{\partial y_k} f_{ij} - \frac{\partial}{\partial y_i} f_{kj}.
\]
The second equation in (2.23) is obvious, while the first follows from the first equation in (2.22).

Finally, note that if $0 \leq \delta \leq 1$,
\begin{align*}
\|\phi_{\delta,ij}\|_{H^1(Y)} &\leq C \left\{ \|f_{ij}\|_{H^2(Y)} + \|f_{kj}\|_{H^2(Y)} \right\} \\
&\leq C \left\{ \|b_{\delta,ij}\|_{L^2(Y)} + \|b_{\delta,kj}\|_{L^2(Y)} \right\} \\
&\leq C,
\end{align*}
where we have used (2.14) for the last inequality.

### 2.5 Small-scale Hölder estimates in Lipschitz domains

Let $B_r = B(0, r)$,
\[ B^+_r = B_r \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \psi(x')\}, \quad B^-_r = B_r \cap \{(x', x_d) \in \mathbb{R}^d : x_d < \psi(x')\}, \]
and $\Delta_r = B_r \cap \{(x', x_d) : x_d = \psi(x')\}$, where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function with $\psi(0) = 0$ and $\|\nabla \psi\|_{\infty} \leq M_1$. Consider the weak solution of $\text{div}(A_\delta \nabla u) = 0$ in $B_1$, where $A_\delta(x) = A(x)$ for $x \in B^+_1$ and $A_\delta(x) = \delta^2 A(x)$ for $x \in B^-_1$. In the case $\delta = 0$ we also assume $\text{div}(A \nabla u) = 0$ in $B^-_1$.

**Lemma 2.10.** Suppose $A$ satisfies the ellipticity condition (1.1). Let $u \in H^1(B_1)$ be a weak solution of $\text{div}(A_\delta \nabla u) = 0$ in $B_1$. Then
\begin{align*}
\|u\|_{C^0(B^+_1)} &\leq C \left\{ \left( \int_{B^+_1} |u|^2 \right)^{1/2} + \delta \left( \int_{B^-_1} |u|^2 \right)^{1/2} \right\}, \quad (2.25) \\
\|u\|_{C^0(B^-_1)} &\leq C \left( \int_{B^-_1} |u|^2 \right)^{1/2}, \quad (2.26)
\end{align*}
where $C > 0$ and $\sigma \in (0, 1)$ depend only on $d$, $\mu$, and $M_1$.

**Proof.** We first note that if $0 < \delta_0 \leq \delta \leq 1$, the estimates (2.25)–(2.26) follow directly from the De Giorgi - Nash estimates, with $C$ and $\sigma$ depending on $d$, $\mu$, and $\delta_0$. Next, consider the case $\delta = 0$. Since $(\frac{\partial u}{\partial r})^+ = 0$ on $\Delta_1$, it follows from the De Giorgi - Nash theory by a reflection argument that there exists $\rho > 0$, depending only on $d$, $\mu$, and $M_1$, such that
\begin{align*}
\|u\|_{C^0(B^+_1)} &\leq C \left( \int_{B^+_1} |u|^2 \right)^{1/2}, \quad (2.27)
\end{align*}
Using Theorem 8.29 in [7] and (2.27), we obtain
\begin{align*}
\|u\|_{C^{\rho/2}(B^+_1)} &\leq C \|u\|_{C^0(\Delta_{3/4})} + C \left( \int_{B^-_1} |u|^2 \right)^{1/2} \\
&\leq C \left( \int_{B^-_1} |u|^2 \right)^{1/2}.
\end{align*}
To treat the case $0 < \delta < \delta_0$, we use a perturbation argument. Define
\[ \Phi(r; u) = \inf_k \left\{ \left( \int_{B^+_1} |u - k|^2 \right)^{1/2} + \delta \left( \int_{B^-_1} |u - k|^2 \right)^{1/2} \right\}. \]
Let $v \in H^1(B_R)$ be a solution of $\text{div}(A\nabla v) = 0$ in $B_R^+$ and $B_R^-$ such that $v = u$ on $\partial B_R$, $(\frac{\partial u}{\partial \nu})_+ = 0$ on $\Delta_R$, and $v_+ = v_-$ on $\Delta_R$. By the estimates for the case $\delta = 0$,

$$\Phi(r; v) \leq C \left(\frac{r}{R}\right)^{\rho/2} \Phi(R; v)$$

(2.29)

for $0 < r < R$. Also, note that for any $\varphi \in H^1_0(B_R)$,

$$\int_{B_R^+} A\nabla (u - v) \cdot \nabla \varphi \, dx = -\delta^2 \int_{B_R^-} A\nabla u \cdot \nabla \varphi \, dx.$$  

(2.30)

We now choose $\varphi \in H^1_0(B_R)$ in (2.30) to be an extension of $(u - v)|_{B_R^+}$ such that

$$\|\nabla \varphi\|_{L^2(B_R)} \leq C\|\nabla (u - v)\|_{L^2(B_R^+)}.$$  

This gives

$$\|\nabla (u - v)\|_{L^2(B_R^+)} \leq C\delta^2\|\nabla u\|_{L^2(B_R)}.$$  

(2.31)

Since $\varphi = u - v$ on $\partial B_R^-$ and $\text{div}(A\nabla (u - v)) = 0$ in $B_R^-$, by the energy estimate,

$$\|\nabla (u - v)\|_{L^2(B_R^-)} \leq C\|\nabla \varphi\|_{L^2(B_R^-)} \leq C\|\nabla (u - v)\|_{L^2(B_R^-)}.$$  

As a consequence, we have proved that

$$\left(\int_{B_R^+} \|\nabla (u - v)\|^2 \right)^{1/2} \leq C\delta^2 \left(\int_{B_R} \|\nabla u\|^2 \right)^{1/2} \leq \frac{C\delta}{R} \left\{ \left(\int_{B_{2R}^+} \|u\|^2 \right)^{1/2} + \delta \left(\int_{B_{2R}^-} \|u\|^2 \right)^{1/2} \right\},$$

where we have used Caccioppoli’s inequality (2.11) for the last step. By Poincaré’s inequality,

$$\left(\int_{B_R} \|u - v\|^2 \right)^{1/2} \leq C\delta \left\{ \left(\int_{B_{2R}^+} \|u\|^2 \right)^{1/2} + \delta \left(\int_{B_{2R}^-} \|u\|^2 \right)^{1/2} \right\}.$$  

It follows that

$$\Phi(\theta R; u) \leq \Phi(\theta R; v) + C_\theta \left(\int_{B_{3R}} \|u - v\|^2 \right)^{1/2} \leq C_0 \theta^{\rho/2} \Phi(R; v) + C_\theta \left(\int_{B_{3R}} \|u - v\|^2 \right)^{1/2} \leq C_0 \theta^{\rho/2} \Phi(R; u) + C_\theta \delta \left\{ \left(\int_{B_{2R}^+} \|u\|^2 \right)^{1/2} + \delta \left(\int_{B_{2R}^-} \|u\|^2 \right)^{1/2} \right\}.$$  

Since $u - k$ is also a solution, we obtain

$$\Phi(\theta R; u) \leq \left\{ C_0 \theta^{\rho/2} + C_\theta \delta \right\} \Phi(2R; u).$$
Choose $\theta \in (0, 1/4)$ so small that $C_0 \theta^{1/2} \leq (1/4)$. With $\theta$ chosen, we then choose $\delta_0$ so small that $C_0 \delta_0 \leq (1/4)$. This yields that if $0 < \delta < \delta_0$, then

$$\Phi(\theta R; u) \leq \frac{1}{2} \Phi(2R; u).$$

It follows that for $0 < r < R \leq 1$,

$$\Phi(r; u) \leq C \left(\frac{r}{R}\right)^{\sigma} \Phi(R; u), \quad (2.32)$$

where $\sigma$ depends only on $d$, $\mu$ and $M_1$. The inequality (2.32), together with the interior Hölder estimates for $u$, gives (2.25). As in (2.28), since $\text{div}(A \nabla u) = 0$ in $B_{1}^-$, the estimate (2.26) follows from (2.25). \hfill \square

**Theorem 2.11.** Suppose $A$ satisfies (1.1). Let $u = u_{\varepsilon, \delta}$ be a weak solution of $\text{div}(A_{\varepsilon}^\delta \nabla u) = 0$ in $B_{2r} = B(x_0, 2r)$, where $A_{\varepsilon}^\delta(x) = [\Lambda_{\delta}(x/\varepsilon)]^2 A(x/\varepsilon)$ and $0 < r < (8d)\varepsilon$. Then

$$\|u\|_{L^\infty(B_r \cap \omega)} \leq C \left(\int_{B_{2r}} |\Lambda_{\delta}^\varepsilon u|^2 \right)^{1/2}, \quad (2.33)$$

$$\|u\|_{L^\infty(B_r \cap F)} \leq C \left(\int_{B_{2r}} |u|^2 \right)^{1/2}, \quad (2.34)$$

$$\|u\|_{C^{0,\sigma}(B_r \cap \omega)} \leq Cr^{-\sigma} \left(\int_{B_{2r}} |\Lambda_{\delta}^\varepsilon u|^2 \right)^{1/2}, \quad (2.35)$$

$$\|u\|_{C^{0,\sigma}(B_r \cap F)} \leq Cr^{-\sigma} \left(\int_{B_{2r}} |u|^2 \right)^{1/2}, \quad (2.36)$$

where $\Lambda_{\delta}^\varepsilon(x) = \Lambda_{\delta}(x/\varepsilon)$ and $C > 0$, $\sigma \in (0, 1)$ depend only on $d$, $\mu$, and $\omega$.

**Proof.** By rescaling we may assume $\varepsilon = 1$. There are three cases: (1) $B_{3r/2} \subset \omega$; (2) $B_{3r/2} \subset \mathbb{R}^d \setminus \omega$; and (3) $B_{3r/2} \cap \partial \omega \neq \emptyset$. The first two cases follow readily from the interior Hölder estimates for the elliptic operator $-\text{div}(A(x)\nabla)$. To treat the third case, without loss of generality, we assume that $r$ is small and $x_0 \in \partial \omega$. By a change of variables, the desired estimates follow from Lemma 2.10. \hfill \square

### 3 Convergence rates in $H^1$

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. In this section we establish a sub-optimal convergence rate in $H^1(\Omega)$, without the condition (1.5). Let $u_{\varepsilon, \delta} \in H^1(\Omega)$ and $v_{\delta} \in H^1(\Omega) \cap H^2_{\text{loc}}(\Omega)$. Suppose that

$$\begin{align*}
\text{div}(A_{\varepsilon}^\delta \nabla u_{\varepsilon, \delta}) &= \text{div}(\widehat{A}_{\delta} \nabla v_{\delta}) \quad \text{in } \Omega, \\
u_{\varepsilon, \delta} &= v_{\delta} \quad \text{on } \partial \Omega,
\end{align*} \quad (3.1)$$

where $\widehat{A}_{\delta}$ is the homogenized matrix given by (2.19). Let

$$\Sigma_t = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < t \}. \quad (3.2)$$

Consider the function

$$w_{\varepsilon, \delta} = u_{\varepsilon, \delta} - v_{\delta} - \varepsilon \chi_{\delta}(x/\varepsilon) S_{\varepsilon}(\eta_{\varepsilon}(\nabla v_{\delta})), \quad (3.3)$$
\[ \chi_\delta \text{ is the corrector defined by (2.12) and } \eta_\varepsilon \in C_0^\infty(\Omega) \text{ is a cut-off function satisfying } 0 \leq \eta_\varepsilon \leq 1, \ |\nabla \eta_\varepsilon| \leq C/\varepsilon, \text{ and} \]
\[
\begin{cases} 
\eta_\varepsilon(x) = 1 & \text{ if } x \in \Omega \setminus \Sigma_{4\varepsilon}, \\
\eta_\varepsilon(x) = 0 & \text{ if } x \in \Sigma_{3\varepsilon}.
\end{cases} \tag{3.4}
\]

The smoothing operator \( S_\varepsilon \) in (3.3) is defined by
\[
S_\varepsilon(f)(x) = \int_{\mathbb{R}^d} f(x-y)\varphi_\varepsilon(y) \, dy,
\tag{3.5}
\]
where \( \varphi_\varepsilon(y) = \varepsilon^{-d}\varphi(y/\varepsilon) \) and \( \varphi \) is a (fixed) function in \( C_0^\infty(B(0,1/2)) \) such that \( \varphi \geq 0 \) and \( \int_{\mathbb{R}^d} \varphi \, dx = 1 \).

**Lemma 3.1.** Let \( S_\varepsilon \) be the operator given by (3.3). Let \( 1 \leq p < \infty \). Then
\[
\|g^\varepsilon S_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} + \varepsilon \|g^\varepsilon \nabla S_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq C \sup_{x \in \mathbb{R}^d} \left( \int_{B(x,1/2)} |g|^p \right)^{1/p} \|f\|_{L^p(\mathbb{R}^d)},
\tag{3.6}
\]
where \( g^\varepsilon(x) = g(x/\varepsilon) \), \( C \) depends only on \( d \) and \( p \), and
\[
\|f - S_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq \varepsilon \|\nabla f\|_{L^p(\mathbb{R}^d)}.
\tag{3.7}
\]

**Proof.** See e.g. [19] pp.37-38. \( \square \)

**Lemma 3.2.** Let \( \Omega \) be a bounded Lipschitz domain. For \( 0 < \varepsilon \leq 1 \) and \( 0 \leq \delta \leq 1 \), let \( u_{\varepsilon,\delta}, v_\delta, w_{\varepsilon,\delta} \) be given as above. Then, for any \( \psi \in H^1_0(\Omega) \),
\[
\left| \int_{\Omega} A_\delta^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla \psi \, dx \right| \leq C \|\nabla \psi\|_{L^2(\Omega)} \left\{ \|\nabla v_\delta\|_{L^2(\Sigma_{4\varepsilon})} + \varepsilon \|\nabla^2 v_\delta\|_{L^2(\Omega \setminus \Sigma_{3\varepsilon})} \right\},
\tag{3.8}
\]
where \( C \) depends only on \( d, \mu, \omega \), and the Lipschitz character of \( \Omega \).

**Proof.** The proof is similar to that for the case \( \delta = 1 \) (see e.g. [19]). Let \( \psi \in H^1_0(\Omega) \). Using
\[
\int_{\Omega} A_\delta^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla \psi \, dx = \int_{\Omega} \hat{A}_\delta \nabla \psi \, dx,
\tag{3.9}
\]
we obtain
\[
\begin{align*}
\int_{\Omega} A_\delta^\varepsilon \nabla w_{\varepsilon,\delta} \cdot \nabla \psi \, dx &= \int_{\Omega} \left( \hat{A}_\delta - A_\delta^\varepsilon \right) \nabla v_\delta \cdot \nabla \psi \, dx \\
& \quad - \int_{\Omega} A_\delta^\varepsilon \nabla \chi_\delta(x/\varepsilon) \eta_\varepsilon(\nabla v_\delta) \cdot \nabla \psi \, dx \\
& \quad - \varepsilon \int_{\Omega} A_\delta^\varepsilon \nabla S_\varepsilon(\eta_\varepsilon(\nabla v_\delta)) \cdot \nabla \psi \, dx \\
& = \int_{\Omega} \left( \hat{A}_\delta - A_\delta^\varepsilon \right) \nabla v_\delta - S_\varepsilon(\eta_\varepsilon(\nabla v_\delta)) \cdot \nabla \psi \, dx \\
& \quad - \int_{\Omega} \left( A_\delta^\varepsilon + A_\delta^\varepsilon \nabla \chi_\delta(x/\varepsilon) - \hat{A}_\delta \right) \eta_\varepsilon(\nabla v_\delta) \cdot \nabla \psi \, dx \\
& \quad - \varepsilon \int_{\Omega} A_\delta^\varepsilon \nabla S_\varepsilon(\eta_\varepsilon(\nabla v_\delta)) \cdot \nabla \psi \, dx \\
& = I_1 + I_2 + I_3.
\end{align*}
\tag{3.10}
\]
By the Cauchy inequality and (3.7),
\[ |I_1| \leq C \| \nabla v_\delta - S_\epsilon (\eta_\epsilon (\nabla v_\delta)) \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)} \]
\[ \leq C \varepsilon \| \nabla (\eta_\epsilon (\nabla v_\delta)) \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)} + C \| \nabla v_\delta \|_{L^2(\Sigma_{4\delta \epsilon})} \| \nabla \psi \|_{L^2(\Omega)} \]
\[ \leq C \| \nabla \psi \|_{L^2(\Omega)} \{ \| \nabla v_\delta \|_{L^2(\Sigma_{4\delta \epsilon})} + \varepsilon \| \nabla^2 v_\delta \|_{L^2(\Omega \backslash \Sigma_{3\delta \epsilon})} \}. \tag{3.11} \]

It follows from the Cauchy inequality, (3.6) and (2.12) that
\[ I_3 \leq C \varepsilon \| \chi_\delta (x/\epsilon) \nabla S_\epsilon (\eta_\epsilon (\nabla v_\delta)) \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)} \]
\[ \leq C \varepsilon \| \nabla (\eta_\epsilon (\nabla v_\delta)) \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)} \]
\[ \leq C \| \nabla \psi \|_{L^2(\Omega)} \{ \| \nabla v_\delta \|_{L^2(\Sigma_{4\delta \epsilon})} + \varepsilon \| \nabla^2 v_\delta \|_{L^2(\Omega \backslash \Sigma_{3\delta \epsilon})} \}. \tag{3.12} \]

Finally, to bound $I_2$, we note that
\[ A^\delta_\epsilon + A^\delta_\epsilon \nabla \chi_\delta (x/\epsilon) - \widehat{A}_\delta = B_\delta (x/\epsilon), \]
where the matrix $B_\delta$ is given by (2.21). Since supp$(S_\epsilon (\eta_\epsilon (\nabla v_\delta))) \subset \Omega \backslash \Sigma_{2\delta \epsilon}$, it follows by Lemma 2.9 that
\[ I_2 = - \int_{\Omega} b_{\delta,ij} (x/\epsilon) S_\epsilon (\eta_\epsilon (\nabla v_\delta)) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx \]
\[ = - \varepsilon \int_{\Omega} \frac{\partial}{\partial x_k} (\phi_{\delta,ki} (x/\epsilon)) S_\epsilon (\eta_\epsilon (\nabla v_\delta)) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx \]
\[ = \varepsilon \int_{\Omega} \phi_{\delta,ki} (x/\epsilon) \frac{\partial}{\partial x_k} S_\epsilon (\eta_\epsilon (\nabla v_\delta)) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx. \]

Thus, as in the case of $I_3$,
\[ |I_2| \leq C \varepsilon \| \nabla (\eta_\epsilon (\nabla v_\delta)) \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)} \]
\[ \leq C \| \nabla \psi \|_{L^2(\Omega)} \{ \| \nabla v_\delta \|_{L^2(\Sigma_{4\delta \epsilon})} + \varepsilon \| \nabla^2 v_\delta \|_{L^2(\Omega \backslash \Sigma_{3\delta \epsilon})} \}. \tag{3.12} \]

This, together with (3.11) and (3.12), completes the proof. \hfill \square

**Lemma 3.3.** Let $\Omega$ be a bounded Lipschitz domain with connected boundary. Suppose $v_\delta \in H^1(\Omega)$ is a weak solution of $\text{div}(\widehat{A}_\delta \nabla v_\delta) = 0$ in $\Omega$ with $v_\delta = f \in H^1(\partial \Omega)$. Then, for $0 < t < \text{diam}(\Omega)$,
\[ \| \nabla v_\delta \|_{L^2(\Sigma_t)} + t \| \nabla^2 v_\delta \|_{L^2(\Omega \backslash \Sigma_t)} \leq C t^{1/2} \| \nabla \tan f \|_{L^2(\partial \Omega)}, \tag{3.13} \]
where $C$ depends only on $d, \mu$, and the Lipschitz character of $\Omega$.

**Proof.** This follows from the interior estimates and the nontangential-maximal-function estimate for second-order elliptic equations with constant coefficients,
\[ \| N(\nabla v_\delta) \|_{L^2(\partial \Omega)} \leq C \| \nabla \tan f \|_{L^2(\partial \Omega)}. \tag{3.14} \]

See [3] for references. \hfill \square

**Theorem 3.4.** Let $\Omega$ be a bounded Lipschitz domain with connected boundary. Assume $1 \leq \text{diam}(\Omega) \leq 10$. Suppose that $u_{\epsilon,\delta} \in H^1(\Omega)$ is a weak solution of $\text{div}(\Lambda_\delta \nabla u_{\epsilon,\delta}) = 0$ in $\Omega$ with $u_{\epsilon,\delta} = f \in H^1(\partial \Omega)$ on $\partial \Omega$. Let $v_\delta$ be the solution of $\text{div}(\widehat{A}_\delta \nabla v_\delta) = 0$ in $\Omega$ with $v_\delta = f$ on $\partial \Omega$.

Then, for $0 < \epsilon < 1$ and $0 \leq \delta \leq 1$,
\[ \| \Lambda_\delta^\varepsilon \nabla w_{\epsilon,\delta} \|_{L^2(\Omega)} \leq C \varepsilon^{1/4} \| \nabla \tan f \|_{L^2(\partial \Omega)}^{1/2} \left\{ \| \nabla \tan f \|_{L^2(\partial \Omega)}^{1/2} + \| \nabla u_{\epsilon,\delta} \|_{L^2(\Omega)}^{1/2} \right\}, \tag{3.15} \]
where $\Lambda_\delta^\varepsilon (x) = \Lambda_\delta (x/\epsilon)$, $w_{\epsilon,\delta}$ is given by (3.14) and $C$ depends only on $d, \mu, \omega$, and the Lipschitz character of $\Omega$. 

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Lemma 4.2. Suppose $B$ in Theorem 4.1. Then, for $0 < F \leq B$ the rescaling property will be used repeatedly. Let
\[ \Lambda \|v_{\delta,\epsilon}\|_{L^2(\Omega)} \leq C \|\nabla \tan f\|_{L^2(\partial\Omega)} \|\nabla v_{\delta,\epsilon}\|_{L^2(\Omega)}. \] (3.16)
Using (3.6), it is not hard to see that
\[ \|v_{\delta,\epsilon}\|_{L^2(\Omega)} \leq C \left\{ \|v_{\delta,\epsilon}\|_{L^2(\Omega)} + \|v_f\|_{L^2(\Omega)} \right\} \]
This, together with (3.16), gives (3.15). \qed

4 Large-scale interior estimates

In this section we investigate large-scale interior estimates for weak solutions of
\[ \text{div}(\Lambda_\delta \nabla u_{\delta,\epsilon}) = 0 \] (4.1)
in $\Omega$, where $A_\delta^\epsilon(x) = A_\delta(x/\epsilon) = [\Lambda_\delta(x/\epsilon)]^2 A(x/\epsilon)$ and $A$ satisfies conditions (1.1)-(1.2). No smoothness condition for $A$ is needed. In the case $\delta = 0$, it is assumed that $u_{\epsilon,0} \in H^1(\Omega)$ and $\text{div}(A(x/\epsilon) \nabla u_{\epsilon,0}) = 0$ in $\Omega \setminus \omega$. The main results in this section were already established in [16, 15]. We present a unified and simplified proof here for the reader’s convenience. A different approach, which also works for the case $2 < \delta \leq \infty$, may be found in [20].

Observe that if $u_{\epsilon,\delta}$ is a solution of (4.1) and $v(x) = u_{\epsilon,\delta}(rx)$, then $\text{div}(A_\delta^\epsilon(r) \nabla v) = 0$. This rescaling property will be used repeatedly. Let $B_r = B(x_0, r)$ for some fixed $x_0 \in \mathbb{R}^d$. The goal of this section is to prove the following.

Theorem 4.1. Suppose $A$ satisfies (1.1) and (1.2). Let $u_{\epsilon,\delta} \in H^1(B_R)$ be a weak solution of (4.1) in $B_R$ for some $R > (10d)\epsilon$. Then, for $0 \leq \delta \leq 1$ and $\epsilon \leq r < R/2$,
\[ \left( \int_{B_r} |\nabla u_{\epsilon,\delta}|^2 \right)^{1/2} \leq C \left( \int_{B_{R/\delta}\cap\omega} |\nabla u_{\epsilon,\delta}|^2 \right)^{1/2}, \] (4.2)
where $C$ depends only on $d$, $\mu$, and $\omega$.

Lemma 4.2. Suppose $A$ satisfies (1.1). Let $u_{\epsilon,\delta}$ be a weak solution of (4.1) in $B_{2r}$ and $r \geq 4\epsilon$. Then, for $0 \leq \delta \leq 1$,
\[ \int_{B_r} |\nabla u_{\epsilon,\delta}|^2 dx \leq C \int_{B_{3r/2}\cap\omega} |\nabla u_{\epsilon,\delta}|^2 dx, \] (4.3)
\[ \int_{B_r} |\nabla u_{\epsilon,\delta}|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}\cap\omega} |u_{\epsilon,\delta}|^2 dx, \] (4.4)
\[ \int_{B_r} |u_{\epsilon,\delta}|^2 dx \leq C \int_{B_{2r}\cap\omega} |u_{\epsilon,\delta}|^2 dx, \] (4.5)
where $C$ depends only on $d$, $\mu$, and $\omega$.

Proof. To show (4.3), by rescaling, we may assume $\epsilon = 1$. Recall that $\mathbb{R}^d \setminus \omega = \cup_k \bar{F}_k$, where $\bar{F}_k$’s are connected components of $\mathbb{R}^d \setminus \omega$. Suppose $F_k \cap B_r \neq \emptyset$. Since $\text{diam}(\bar{F}_k) \leq d$ and $r \geq 4d$, $\bar{F}_k \subset B_{3r/2}$. Note that $\text{div}(A \nabla u_{1,\delta}) = 0$ in $F_k$. By Lemma 2.2 we obtain
\[ \|\nabla u_{1,\delta}\|_{L^2(F_k)} \leq C \|\nabla u_{1,\delta}\|_{L^2(\bar{F}_k \setminus F_k)}, \] (4.6)
which yields (4.3) for $\varepsilon = 1$, using the fact that $\tilde{F}_k$’s are disjoint.

To prove (4.4), we first show that
\[
\delta^2 \int_{B_{3r/2} \setminus \varepsilon \omega} |u_{\varepsilon, \delta}|^2 \, dx \leq C \int_{B_{2r} \cap \varepsilon \omega} |u_{\varepsilon, \delta}|^2 \, dx. \tag{4.7}
\]

By rescaling we may assume $\varepsilon = 1$. Suppose $B_{3r/2} \cap F_k \neq \emptyset$ for some $k$. It follows from (2.11) that
\[
\int_{\tilde{F}_k} |\Lambda_\delta \nabla (u_{1, \delta} \varphi)|^2 \, dx \leq C \int_{\tilde{F}_k} |\Lambda_\delta u_{1, \delta}|^2 |\nabla \varphi|^2 \, dx
\]
for any $\varphi \in C_0^1(\tilde{F}_k)$. Since $0 \leq \delta \leq 1$, we obtain
\[
\delta^2 \int_{\tilde{F}_k} |\nabla (u_{1, \delta} \varphi)|^2 \, dx \leq C \int_{\tilde{F}_k} |u_{1, \delta}|^2 |\nabla \varphi|^2 \, dx.
\]

Choose $\varphi \in C_0^1(\tilde{F}_k)$ such that $\varphi = 1$ on $F_k$. By Poincaré’s inequality,
\[
\delta^2 \int_{F_k} |u_{1, \delta}|^2 \, dx \leq C \int_{\tilde{F}_k \setminus F_k} |u_{1, \delta}|^2 \, dx. \tag{4.8}
\]

Hence,
\[
\delta^2 \int_{B_{3r/2} \setminus \varepsilon \omega} |u_{1, \delta}|^2 \, dx \leq C \int_{B_{3r/2} \setminus \varepsilon \omega} |u_{1, \delta}|^2 \, dx, \tag{4.9}
\]

which gives (4.7) for $\varepsilon = 1$, since $3r/2 + 2d \leq 2r$. It follows from (2.11) and (4.7) that
\[
\int_{B_{3r/4} \setminus \varepsilon \omega} |\nabla u_{\varepsilon, \delta}|^2 \, dx \leq \frac{C}{r^2} \left\{ \int_{B_{3r/2} \setminus \varepsilon \omega} |u_{\varepsilon, \delta}|^2 \, dx + \delta^2 \int_{B_{3r/2} \setminus \varepsilon \omega} |u_{\varepsilon, \delta}|^2 \, dx \right\} \leq \frac{C}{r^2} \int_{B_{2r} \cap \varepsilon \omega} |u_{\varepsilon, \delta}|^2 \, dx. \tag{4.10}
\]

Finally, to prove (4.5), we again assume $\varepsilon = 1$. Suppose $F_k \cap B_r \neq \emptyset$. By Poincaré’s inequality,
\[
\int_{F_k} |u_{1, \delta}|^2 \, dx \leq C \int_{\tilde{F}_k \setminus F_k} |u_{1, \delta}|^2 \, dx + C \int_{\tilde{F}_k} |\nabla u_{1, \delta}|^2 \, dx.
\]

It follows that
\[
\int_{B_r} |u_{1, \delta}|^2 \, dx \leq C \int_{B_{3r/4} \cap \omega} |u_{1, \delta}|^2 \, dx + C \int_{B_{3r/4}} |\nabla u_{1, \delta}|^2 \, dx
\]

\[
\leq C \int_{B_{2r} \cap \omega} |u_{1, \delta}|^2 \, dx,
\]

where we have used (4.10) as well as the fact $r \geq 1$ for the last inequality. \hfill \square

The next lemma provides an approximation of $u_{\varepsilon, \delta}$ by solutions of elliptic equations with constant coefficients.

**Lemma 4.3.** Suppose $A$ satisfies (1.1) and (1.2). Let $u_{\varepsilon, \delta} \in H^1(B_2)$ be a weak solution of $\text{div} (A_{\varepsilon} \nabla u_{\varepsilon, \delta}) = 0$ in $B_2$. Then there exists $v \in H^1(B_1)$ such that $\text{div} (\hat{A}_{\varepsilon} \nabla v) = 0$ in $B_1$ and
\[
\left( \int_{B_1 \cap \omega} |u_{\varepsilon, \delta} - v|^2 \right)^{1/2} \leq C \varepsilon^{1/4} \left( \int_{B_2 \cap \omega} |u_{\varepsilon, \delta}|^2 \right)^{1/2}, \tag{4.11}
\]

where $0 < \varepsilon < 1$, $0 \leq \delta \leq 1$, and $C > 0$ depends only on $d$, $\mu$, and $\omega$. 17
Proof. We assume $\varepsilon > 0$ is sufficiently small; for otherwise the estimate is trivial with $v = 0$. By Lemma 4.2 (or rather its proof),

$$\int_{B_{T/4}} |\nabla u_{\varepsilon, \delta}|^2 \, dx \leq C \int_{B_{2\varepsilon, \delta}} |u_{\varepsilon, \delta}|^2 \, dx. \quad (4.12)$$

It follows that there exists some $t \in (3/2, 7/4)$ such that

$$\int_{\partial B_t} |\nabla u_{\varepsilon, \delta}|^2 \, d\sigma \leq C \int_{B_{2\varepsilon, \delta}} |u_{\varepsilon, \delta}|^2 \, dx. \quad (4.13)$$

Let $v \in H^1(B_t)$ be the weak solution of $\text{div}(A_{\delta} \nabla v) = 0$ in $B_t$ with $v = u_{\varepsilon, \delta}$ on $\partial B_t$. Let $w_{\varepsilon, \delta}$ be defined by (3.3), with $v$ in the place of $u_{\delta}$. In view of Theorem 3.1 we have

$$\|\nabla w_{\varepsilon, \delta}\|_{L^2(B_t \cap \varepsilon\omega)} \leq \|\nabla w_{\varepsilon, \delta}\|_{L^2(B_t)} \leq C \varepsilon^{1/4} \left\{ \|\nabla u_{\varepsilon, \delta}\|_{L^2(\partial B_t)}^{1/2} \left\| \nabla u_{\varepsilon, \delta} \right\|_{L^2(B_t)}^{1/2} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(B_t)}^{1/2} \right\} \quad (4.14)$$

where we have used (4.12) and (4.13) for the last inequality. Since $w_{\varepsilon, \delta} = 0$ on $\partial B_t$, by Poincaré’s inequality

$$\|\phi\|_{L^2(\varepsilon\omega \cap \Omega)} \leq C\|\nabla \phi\|_{L^2(\varepsilon\omega \cap \Omega)} \quad (4.15)$$

for any $\phi \in H^1_0(\Omega)$, and (4.14),

$$\|w_{\varepsilon, \delta}\|_{L^2(B_t \cap \varepsilon\omega)} \leq C \varepsilon^{1/4} \|u_{\varepsilon, \delta}\|_{L^2(B_{2\varepsilon, \delta} \cap \varepsilon\omega)}. \quad (4.16)$$

It follows that

$$\|u_{\varepsilon, \delta} - v\|_{L^2(B_t \cap \varepsilon\omega)} \leq \|w_{\varepsilon, \delta}\|_{L^2(B_t \cap \varepsilon\omega)} + C \varepsilon \|\nabla v\|_{L^2(B_t)} \leq C \varepsilon^{1/4}\|u_{\varepsilon, \delta}\|_{L^2(B_{2\varepsilon, \delta} \cap \varepsilon\omega)}.$$

Finally, to see (4.15), we first extend $\phi$ from $\Omega$ to $\mathbb{R}^d$ by zero. Let $\tilde{\phi} = \phi|_{\varepsilon\omega}$. We then extend $\tilde{\phi}$ from $\varepsilon\omega$ to $\mathbb{R}^d$ such that for each $k$,

$$\|\nabla \tilde{\phi}\|_{L^2(\varepsilon F_k)} \leq C\|\nabla \phi\|_{L^2(\varepsilon F)};$$

where $C$ depends on $\kappa$ and the Lipschitz character of $F_k$. It follows that $\|\nabla \tilde{\phi}\|_{L^2(\mathbb{R}^d)} \leq C\|\nabla \phi\|_{L^2(\varepsilon\omega \cap \Omega)}.$

Note that $\tilde{\phi}$ has compact support. Hence,

$$\|\phi\|_{L^2(\varepsilon\omega \cap \Omega)} \leq \|\tilde{\phi}\|_{L^2(\Omega)} \leq C\|\nabla \tilde{\phi}\|_{L^2(\mathbb{R}^d)} \leq C\|\nabla \phi\|_{L^2(\varepsilon\omega \cap \Omega)},$$

where we have used the Poincaré inequality for the second inequality. \qed

Lemma 4.4. Suppose $A$ satisfies (1.1)-(1.2) and $0 \leq \delta \leq 1$. Let $v \in H^1(B_1)$ be a solution of $\text{div}(A_{\delta} \nabla v) = 0$ in $B_1$. Then, for any $0 < \theta, \varepsilon < 1/(8d)$,

$$\inf_{P} \frac{1}{\theta} \left( \int_{B_{\rho}} |v - P|^2 \right)^{1/2} \leq C_0 \theta \inf_{P} \left( \int_{B_{1\varepsilon, \delta}} |v - P|^2 \right)^{1/2}, \quad (4.16)$$

where $P$ is a linear function and $C_0$ depends only on $d$, $\mu$, and $\omega$. 

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Proof. It follows from (4.5) with $\delta = 1$ and $A$ being a constant matrix that
\[
\int_{B_{1/2}} |v|^2 \, dx \leq C \int_{B_1 \cap \omega} |v|^2 \, dx.
\]
Thus, by using the $C^2$ estimates for second-order elliptic equations with constant coefficients,
\[
\inf_{P} \frac{1}{\theta} \left( \int_{B_\theta} |v - P|^2 \right)^{1/2} \leq \theta \| \nabla^2 v \|_{L^\infty(B_\theta)} \leq \theta \| \nabla^2 v \|_{L^\infty(B_{1/4})}
\]
\[
\leq C \theta \left( \int_{B_{1/2}} |v|^2 \right)^{1/2}
\]
\[
\leq C \theta \left( \int_{B_1 \cap \omega} |v|^2 \right)^{1/2},
\]
from which the estimate (4.16) follows, as $v - P$ is also a solution. \hfill \Box

For $u \in L^2(B_r)$, define
\[
H(r; u) = \inf_{P} \frac{1}{r} \left( \int_{B_r \cap \omega} |u - P|^2 \right)^{1/2},
\]
where the infimum is taken over all linear functions $P$.

**Lemma 4.5.** Suppose $A$ satisfies (1.1)-(1.2). Let $u_{\varepsilon, \delta} \in H^1(B_{2r})$ be a weak solution of $\text{div}(A_{\varepsilon} \nabla u_{\varepsilon, \delta}) = 0$ in $B_{2r}$ for some $r > \varepsilon$. Then there exists $\theta \in (0, 1/4)$, depending only on $d$, $\mu$ and $\omega$, such that
\[
H(\theta r; u_{\varepsilon, \delta}) \leq \frac{1}{2} H(r; u_{\varepsilon, \delta}) + C \left( \frac{\varepsilon}{r} \right)^{1/4} \inf_{q \in \mathbb{R}} \left( \int_{B_{2r} \cap \omega} |u_{\varepsilon, \delta} - q|^2 \right)^{1/2},
\]
where $C$ depends only on $d$, $\mu$, and $\omega$.

**Proof.** By rescaling we may assume $r = 1$. We may further assume that $\varepsilon > 0$ is sufficiently small; for otherwise the estimate is trivial. Given $u_{\varepsilon, \delta} \in B_2$, let $v$ be the solution of $\text{div}(\hat{A}_{\delta} \nabla v) = 0$ in $B_1$, given by Lemma 4.3. Note that
\[
H(\theta; u_{\varepsilon, \delta}) \leq H(\theta; v) + \frac{1}{\theta} \left( \int_{B_\theta \cap \omega} |u_{\varepsilon, \delta} - v|^2 \right)^{1/2}
\]
\[
\leq C_0 \theta H(1; v) + \frac{1}{\theta} \left( \int_{B_\theta \cap \omega} |u_{\varepsilon, \delta} - v|^2 \right)^{1/2}
\]
\[
\leq C_0 \theta H(1; u_{\varepsilon, \delta}) + C_\theta \left( \int_{B_1 \cap \omega} |u_{\varepsilon, \delta} - v|^2 \right)^{1/2}
\]
\[
\leq C_0 \theta H(1; u_{\varepsilon, \delta}) + C_\theta \varepsilon^{1/4} \left( \int_{B_{2} \cap \omega} |u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]
where we have used Lemma 4.4 for the second inequality and Lemma 4.3 for the last. Since $u_{\varepsilon, \delta} - q$ is also a solution of (4.1) for any $q \in \mathbb{R}$, we see that
\[
H(\theta; u_{\varepsilon, \delta}) \leq C_0 \theta H(1; u_{\varepsilon, \delta}) + C_\theta \varepsilon^{1/4} \inf_{q \in \mathbb{R}} \left( \int_{B_{2} \cap \omega} |u_{\varepsilon, \delta} - q|^2 \right)^{1/2}.
\]
The proof is complete by choosing $\theta$ so small that $C_0 \theta \leq (1/2)$. \hfill \Box
The proof of the following lemma may be found in [19, pp.157-158].

**Lemma 4.6.** Let $H(r)$ and $h(r)$ be two nonnegative, continuous functions on $(0,1]$. Let $0 < \varepsilon < (1/4)$. Suppose that there exists a constant $C_0$ such that

\[
\max_{r \leq t,s \leq 2r} H(t) \leq C_0 H(2r) \quad \text{and} \quad \max_{r \leq t,s \leq 2r} |h(t) - h(s)| \leq C_0 H(2r) \tag{4.19}
\]

for any $r \in [\varepsilon, 1/2]$. Suppose further that

\[
H(\theta r) \leq \frac{1}{2} H(r) + C_0 \beta(\varepsilon/r) \left\{ H(2r) + h(2r) \right\} \tag{4.20}
\]

for any $r \in [\varepsilon, 1/2]$, where $\theta \in (0, 1/4)$ and $\beta(t)$ is a nonnegative, nondecreasing function on $[0,1]$ such that $\beta(0) = 0$ and

\[
\int_0^1 \beta(t) \, dt < \infty. \tag{4.21}
\]

Then

\[
\max_{\varepsilon \leq r \leq 1} \{ H(r) + h(r) \} \leq C \{ H(1) + h(1) \}, \tag{4.22}
\]

where $C$ depends only on $C_0$, $\theta$, and the function $\beta(t)$.

We are now in a position to give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** By rescaling we may assume $R = 2$. We may also assume that $\varepsilon > 0$ is sufficiently small; for otherwise, the estimate is trivial. Let $\text{div}(A(x) \nabla u_{\varepsilon, \delta}) = 0$ in $B_2$. We shall apply Lemma 4.6 with the function $H(t) = H(t; u_{\varepsilon, \delta})$ and $h(t) = |E_t|$ for $(10d) \varepsilon < t < 1$, where $H(t; u_{\varepsilon, \delta})$ is defined by (4.17) and $E_t$ is a vector in $\mathbb{R}^d$ such that

\[
H(t; u_{\varepsilon, \delta}) = \inf_{q \in \mathbb{R}} \frac{1}{t} \left( \int_{B_t \cap \varepsilon \omega} |u_{\varepsilon, \delta} - E_t \cdot x - q|^2 \right)^{1/2}.
\]

The first inequality in (4.19) follows from the observation $|B_r \cap \varepsilon \omega| \approx r^d$ if $r \geq (10d) \varepsilon$. To see the second, we note that estimate (4.14) gives

\[
|E_t - E_\delta| \leq C \inf_{q \in \mathbb{R}} \frac{1}{r} \left( \int_{B_r \cap \varepsilon \omega} |(E_t - E_\delta) \cdot x - q|^2 \right)^{1/2}
\]

if $r \geq (8d) \varepsilon$. Furthermore, the condition (4.20) is given by Lemma 4.5 with $\beta(t) = t^{1/4}$. Consequently, by Lemma 4.6 we obtain

\[
\inf_{q \in \mathbb{R}} \frac{1}{r} \left( \int_{B_r \cap \varepsilon \omega} |u_{\varepsilon, \delta} - q|^2 \right)^{1/2} \leq H(r) + h(r) \leq C \{ H(1) + h(1) \} \leq C \left( \int_{B_1} |u_{\varepsilon, \delta}|^2 \right)^{1/2}
\]

for any $r \in ((10d) \varepsilon, 1)$. By replacing $u_{\varepsilon, \delta}$ with $u_{\varepsilon, \delta} - q$, where $q$ is the average of $u_{\varepsilon, \delta}$ over $B_1$, and using Poincaré’s inequality, we see that

\[
\inf_{q \in \mathbb{R}} \frac{1}{r} \left( \int_{B_r \cap \varepsilon \omega} |u_{\varepsilon, \delta} - q|^2 \right)^{1/2} \leq C \left( \int_{B_1} |\nabla u_{\varepsilon, \delta}|^2 \right)^{1/2}.
\]
for any \( r \in ((10d)\varepsilon, 1) \). This, together with (4.4), yields
\[
\left( \int_{B_r} |\nabla u_{\varepsilon, \delta}|^2 \right)^{1/2} \leq C \left( \int_{B_1} |\nabla u_{\varepsilon, \delta}|^2 \right)^{1/2} \leq C \left( \int_{B_{2\delta} \cap \omega} |\nabla u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]
for \( \varepsilon \leq r < 1 \), where we have used (4.3) for the last inequality.

As a corollary of Theorem 4.1, we obtain the large-scale \( L^\infty \) estimate.

**Theorem 4.7.** Suppose \( A \) satisfies (1.1) and (1.2). Let \( u = u_{\varepsilon, \delta} \in H^1(B_R) \) be a weak solution of \( \text{div}(A_{\delta} \nabla u) = 0 \) in \( B_R \) for some \( R > (10d)\varepsilon \). Then, for \( 0 \leq \delta \leq 1 \) and \( \varepsilon \leq r < R/2 \),
\[
\left( \int_{B_r} |u|^2 \right)^{1/2} \leq C \left( \int_{B_{R/2} \cap \omega} |u|^2 \right)^{1/2},
\]
(4.23) where \( C \) depends only on \( d, \mu, \) and \( \omega \).

**Proof.** We may assume \( r < R/4 \). By Theorem 4.1
\[
\left( \int_{B_r} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{B_{R/2}} |\nabla u|^2 \right)^{1/2}
\]
Let \( J \geq 1 \) be the integer such that \( 2^J r \leq R/2 < 2^{J+1} r \). Note that for \( 1 \leq j \leq J \),
\[
\left| \int_{B_{2^j-1}r} u - \int_{B_{2^j}r} u \right| \leq C2^j r \left( \int_{B_{2^j}r} |\nabla u|^2 \right)^{1/2}
\]
\[
\leq C2^j r \left( \int_{B_{R/2}} |\nabla u|^2 \right)^{1/2},
\]
where we have used Poincaré’s inequality for the first inequality and Theorem 4.1 for the second. It follows by summation that
\[
\left| \int_{B_r} u - \int_{B_{2^j}r} u \right| \leq CR \left( \int_{B_{R/2}} |\nabla u|^2 \right)^{1/2}.
\]
Hence, by Poincaré’s inequality and Theorem 4.1
\[
\left( \int_{B_r} |u|^2 \right)^{1/2} \leq Cr \left( \int_{B_r} |\nabla u|^2 \right)^{1/2} + \left| \int_{B_r} u \right|
\]
\[
\leq CR \left( \int_{B_{R/2}} |\nabla u|^2 \right)^{1/2} + C \left( \int_{B_{R/2}} |u|^2 \right)^{1/2}
\]
\[
\leq C \left( \int_{B_{R/2} \cap \omega} |u|^2 \right)^{1/2},
\]
where we have used Lemma 4.2 for the last inequality. 
\[\square\]
Remark 4.8. Let $u$ be a weak solution of $\text{div}(A^s \nabla u) = 0$ in $B_{2r}$ for some $r > 0$. Then
\[
\|A^s u\|_{L^\infty(B_r)} \leq C \left( \int_{B_{2r}} |A^s u|^2 \right)^{1/2},
\]
where $C$ depends only on $d$, $\mu$, and $\omega$. The small-scale case $0 < r < 100d\varepsilon$ is contained in Theorem 2.11. To prove the estimate for the large-scale case $r \geq 100d\varepsilon$, let $x \in B_r = B(x_0, r)$. Then
\[
|\Lambda^s u(x)| \leq C \left( \int_{B(x, 100d\varepsilon)} |u|^2 \right)^{1/2} \leq C \left( \int_{B(x, r) \cap \varepsilon \omega} |u|^2 \right)^{1/2}
\leq C \left( \int_{B(x_0, 2r) \cap \varepsilon \omega} |\Lambda^s u|^2 \right)^{1/2},
\]
where we have used Theorem 2.11 for the first inequality and Theorem 4.7 for the second. By a logarithmic convexity argument, one may deduce from (4.24) that
\[
\|\Lambda^s u\|_{L^\infty(B_r)} \leq C_p \left( \int_{B_{2r}} |\Lambda^s u|^p \right)^{1/p}
\]
for any $p > 0$, where $C_p$ depends only on $d$, $\mu$, $p$, and $\omega$.

5 Boundary estimates

In this section we study the boundary regularity for solutions of (4.11). Let $\Omega$ be a bounded Lipschitz domain satisfying (1.3) and $\Omega^\varepsilon = \Omega \setminus \overline{F^\varepsilon}$, where $F^\varepsilon = \cup_k F_k$. Fix $x_0 \in \partial \Omega$. Let
\[
D_r = B(x_0, r) \cap \Omega \quad \text{and} \quad \Delta_r = B(x_0, r) \cap \partial \Omega,
\]
for $0 < r < r_0 = c_0 \text{diam}(\Omega)$. We assume $c_0 > 0$ is sufficiently small so that in a suitable coordinated system, obtained from the standard one through translation and rotation, $x_0 = 0$ and
\[
B(0, (100d)r_0) \cap \Omega = B(0, (100d)r_0) \cap \{ (x', x_d) \in \mathbb{R}^d : x_d > \psi(x') \},
\]
\[
B(0, (100d)r_0) \cap \partial \Omega = B(0, (100d)r_0) \cap \{ (x', x_d) \in \mathbb{R}^d : x_d = \psi(x') \},
\]
where $\psi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a Lipschitz function with $\psi(0) = 0$ and $\|\nabla \psi\|_\infty \leq M_1$.

Let
\[
D^\varepsilon_r = B(x_0, r) \cap \Omega^\varepsilon.
\]

Lemma 5.1. Suppose $A$ satisfies (1.1)-(1.2). Let $4d\varepsilon \leq r < r_0$. Let $u_{\varepsilon, \delta} \in H^1(D_{2r})$ be a weak solution of (4.11) in $D_{2r}$ with $u_{\varepsilon, \delta} = 0$ on $\Delta_{2r}$. Then, for $0 \leq \delta \leq 1$,
\[
\int_{D_r} |\nabla u_{\varepsilon, \delta}|^2 \, dx \leq C \int_{D^\varepsilon_{2r}} |\nabla u_{\varepsilon, \delta}|^2 \, dx,
\]
\[
\int_{D_r} |\nabla u_{\varepsilon, \delta}|^2 \, dx \leq C \frac{1}{r^2} \int_{D^\varepsilon_{2r}} |u_{\varepsilon, \delta}|^2 \, dx,
\]
\[
\int_{D_r} |u_{\varepsilon, \delta}|^2 \, dx \leq C \int_{D^\varepsilon_{2r}} |u_{\varepsilon, \delta}|^2 \, dx,
\]
where $C$ depends only on $d$, $\mu$, $\omega$, $\kappa$, and $M_1$. 22
Lemma 5.3. Suppose \( D \) where \( c > \kappa \epsilon \).

\[
\int_{D_{2r}} |\Lambda_0^2 \nabla (u_{\epsilon, \delta} \varphi)|^2 \, dx \leq C \int_{D_{2r}} |\Lambda_0^2 u_{\epsilon, \delta}|^2 |\nabla \varphi|^2 \, dx, \tag{5.7}
\]

holds for any \( \varphi \in C_0^1(B_{2r}) \). Also observe that by (1.3), if \( D_r \cap \Omega \neq \emptyset \), then \( \epsilon \hat{F}_k \subset D_{r+\delta \epsilon} \). We omit the details.

Remark 5.2. Since \( \Sigma_{\kappa \epsilon} \subset \Omega \), it follows from (3.1) that (5.3) also holds for \( 0 < r \leq \kappa \epsilon / 2 \). If \( \kappa \epsilon / 2 < r < 4d \epsilon \), we note that

\[
\int_{D_r} |\nabla u_{\epsilon, \delta}|^2 \, dx \leq \int_{D_{4d \epsilon}} |\nabla u_{\epsilon, \delta}|^2 \, dx \leq C \left( \frac{r}{\epsilon} \right)^{1/4} \left( \int_{D_{5r}} |u_{\epsilon, \delta}|^2 \, dx \right)^{1/2}, \tag{5.8}
\]

where \( C \) depends only on \( d, \mu, \omega, \kappa, \) and \( M_1 \).

Lemma 5.3. Suppose \( A \) satisfies (1.1)-(1.2). Let \( u_{\epsilon, \delta} \in H^1(D_{2r}) \) be a weak solution of (4.1) in \( D_{2r} \) with \( u_{\epsilon, \delta} = 0 \) on \( \Delta_{2r} \), where (10d) \( \epsilon < r < r_0 \). Then there exists \( v \in H^1(D_r) \) such that \( \text{div}(\hat{A}_{\delta} \nabla v) = 0 \) in \( D_r \), \( v = 0 \) on \( \Delta_r \), and

\[
\left( \frac{1}{D_{5r}} \int_{D_{5r}} |u_{\epsilon, \delta} - v|^2 \right)^{1/2} \leq C \left( \frac{r}{\epsilon} \right)^{1/4} \left( \int_{D_{5r}} |u_{\epsilon, \delta}|^2 \, dx \right)^{1/2},
\]

where \( C \) depends only on \( d, \mu, \omega, \kappa, \) and \( M_1 \).

Proof. By rescaling we may assume \( r = 1 \) and \( \epsilon > 0 \) is sufficiently small. The argument is similar to that for Lemma 4.3 with \( D_r \) in the place of \( B_r \). By the proof of Lemma 5.1

\[
\int_{D_{7/4}} |\nabla u_{\epsilon, \delta}|^2 \, dx \leq C \int_{D_2} |u_{\epsilon, \delta}|^2 \, dx.
\]

It follows that there exists some \( t \in (3/2, 7/4) \) such that

\[
\int_{\partial D_t} |\nabla u_{\epsilon, \delta}|^2 \, d\sigma \leq C \int_{D_2} |u_{\epsilon, \delta}|^2 \, dx.
\]

Let \( v \in H^1(D_t) \) be the weak solution of \( \text{div}(\hat{A}_{\delta} \nabla v) = 0 \) in \( D_t \) with \( v = u_{\epsilon, \delta} \) on \( \partial D_t \). Note that \( v = 0 \) on \( \Delta_1 \). Let \( w_{\epsilon, \delta} \) be defined by (3.3), with \( v \) in the place of \( v_{\delta} \). By Theorem 3.4

\[
\|\nabla w_{\epsilon, \delta}\|_{L^2(D_{5/2})} \leq C \epsilon^{1/4} \|u_{\epsilon, \delta}\|_{L^2(D_2)} \tag{5.9}
\]

We claim that

\[
\|w_{\epsilon, \delta}\|_{L^2(D_{5/2})} \leq C \|\nabla w_{\epsilon, \delta}\|_{L^2(D_{5/2})} \tag{5.10}
\]

This, together with (5.9), gives

\[
\|u_{\epsilon, \delta} - v\|_{L^2(D_1)} \leq \|w_{\epsilon, \delta}\|_{L^2(D_1)} + C \epsilon \|\nabla v\|_{L^2(D_t)} \leq C \epsilon^{1/4} \|u_{\epsilon, \delta}\|_{L^2(D_2)}.
\]
Finally, to prove the Poincaré-type inequality (5.10), we use an extension argument. Observe that if \( \varepsilon F_k \) is one of the holes in \( F^{\varepsilon} \) such that \( \varepsilon F_k \cap D_1 \neq \emptyset \), then \( \varepsilon F_k \subset D_{3/2} \). We introduce a new function \( \tilde{w}_{\varepsilon,\delta} \) by replacing \( w_{\varepsilon,\delta} \) in such \( \varepsilon F_k \) by the harmonic extension of \( w_{\varepsilon,\delta}|_{\partial(\varepsilon F_k)} \). Note that

\[
\|\nabla \tilde{w}_{\varepsilon,\delta}\|_{L^2(\varepsilon F_k)} \leq C \|\nabla w_{\varepsilon,\delta}\|_{L^2(\varepsilon F_k \setminus F_k)}.
\]

Since \( \tilde{w}_{\varepsilon,\delta} = w_{\varepsilon,\delta} = 0 \) on \( \Delta_1 \), it follows by Poincaré’s inequality that

\[
\|w_{\varepsilon,\delta}\|_{L^2(D_1)} \leq C \|\nabla w_{\varepsilon,\delta}\|_{L^2(D_1)} \leq C \|\nabla w_{\varepsilon,\delta}\|_{L^2(D_{3/2})},
\]

Since \( \tilde{w}_{\varepsilon,\delta} = w_{\varepsilon,\delta} \) on \( D_1 \), this gives (5.10).

With the approximation result in Lemma 5.3 at our disposal, we establish the large-scale Hölder estimate on Lipschitz domains as well as the Lipschitz estimate on \( C^{1,\alpha} \) domains.

**Theorem 5.4.** Suppose \( A \) satisfies (1.1) - (1.2) and \( 0 \leq \delta \leq 1 \). Let \( \Omega \) be a bounded Lipschitz domain satisfying (1.5). Let \( u_{\varepsilon,\delta} \in H^1(D_R) \) be a weak solution of \( \text{div}(A_{\varepsilon} \nabla u_{\varepsilon,\delta}) = 0 \) in \( D_R \) with \( u_{\varepsilon,\delta} = 0 \) on \( \Delta_R \) for some \( (10d)\varepsilon < R < r_0 \). Then, for \( \varepsilon < r < R/2 \),

\[
\left( \int_{D_\varepsilon} |u_{\varepsilon,\delta}|^2 \right)^{1/2} \leq C \left( \frac{r}{R} \right)^\sigma \left( \int_{D_R^\varepsilon} |u_{\varepsilon,\delta}|^2 \right)^{1/2},
\]

where \( \sigma > 0 \) and \( C > 0 \) depend only on \( d, \mu, \omega, \kappa, \) and \( M_1 \).

**Proof.** By rescaling we may assume \( R = 2 \). For \( (10d)\varepsilon < r < 1 \), let

\[
\Phi(r) = \frac{1}{r^\sigma} \left( \int_{D_r^\varepsilon} |u_{\varepsilon,\delta}|^2 \right)^{1/2}.
\]

Let \( v \in H^1(D_r) \) be a solution of \( \text{div}(A_{\varepsilon} \nabla v) = 0 \) in \( D_r \) satisfying \( v = 0 \) on \( \Delta_r \) and (5.8). Since \( \Omega \) is a Lipschitz domain, by the boundary Hölder estimate for second-order elliptic operators with constant coefficients and (5.6),

\[
\left( \int_{D_\varepsilon^r} |v|^2 \right)^{1/2} \leq C \theta^\lambda \left( \int_{D_\varepsilon^r} |v|^2 \right)^{1/2}
\]

for any \( \theta \in (0, 1/4) \) such that \( \theta r \geq d\varepsilon \), where \( \lambda > 0 \) and \( C > 0 \) depend only on \( d, \mu, \omega, \kappa, \) and \( M_1 \). It follows that

\[
\Phi(\theta r) \leq \frac{1}{(\theta r)^\sigma} \left( \int_{D_{\theta r}} |v|^2 \right)^{1/2} + \frac{1}{(\theta r)^\sigma} \left( \int_{D_{\theta r}} |u_{\varepsilon,\delta} - v|^2 \right)^{1/2}
\]

\[
\leq C \theta^{\lambda-\sigma} \frac{1}{r^\sigma} \left( \int_{D_\varepsilon} |v|^2 \right)^{1/2} + C_\theta^\sigma \left( \int_{D_\varepsilon} |u_{\varepsilon,\delta} - v|^2 \right)^{1/2}
\]

\[
\leq \left\{ C \theta^{\lambda-\sigma} + C_\theta \left( \frac{\varepsilon}{r} \right)^{1/4} \right\} \Phi(2r).
\]

Fix \( \sigma \in (0, \lambda) \). Choose \( \theta \in (0, 1/4) \) so small that \( C \theta^{\lambda-\sigma} \leq (1/4) \). With \( \theta \) fixed, we see that if \( r \geq N\varepsilon \) and \( N > 1 \) is large,

\[
\Phi(\theta r) \leq (1/2)\Phi(2r).
\]

By an iteration argument this implies that \( \Phi(r) \leq C\Phi(1) \) for any \( r \in (\varepsilon, 1) \).
**Theorem 5.5.** Suppose that \( \text{div}(A^\varepsilon \nabla u_{\varepsilon, \delta}) = 0 \) in \( B(x_0, r) \cap \Omega \) and \( u_{\varepsilon, \delta} = 0 \) on \( B(x_0, r) \cap \partial \Omega \) for some \( x_0 \in \Omega \). Assume that \( r \geq (4d)\varepsilon \). Then

\[
|u_{\varepsilon, \delta}(x_0)| \leq C \min \left( \left( \frac{d(x_0)}{r} \right)^\sigma, 1 \right) \left( \int_{B(x_0, r) \cap \Omega^\varepsilon} |u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]

(5.13)

where \( d(x_0) = \text{dist}(x_0, \partial \Omega) \), and \( C > 0, \sigma \in (0, 1) \) depend only on \( d \), \( \mu \), \( \omega \), \( \kappa \), and the Lipschitz character of \( \Omega \).

**Proof.** In the case \( d(x_0) \geq r \), we use the large-scale interior estimate in Theorem 4.7

\[
\left( \int_{B(x_0, 2d\varepsilon)} |u_{\varepsilon, \delta}|^2 \right)^{12} \leq C \left( \int_{B(x_0, r) \cap \omega} |u_{\varepsilon, \delta}|^2 \right)^{12},
\]

and the small-scale estimate,

\[
|u_{\varepsilon, \delta}(x_0)| \leq \left( \int_{B(x_0, 2d\varepsilon)} |u_{\varepsilon, \delta}|^2 \right)^{12},
\]

which follows from Theorem 2.11. Next, suppose \( d(x_0) \leq \kappa\varepsilon/2 \). Since \( \Sigma \cap F^\varepsilon = \emptyset \), it follows from the small-scale boundary Hölder estimate that

\[
|u_{\varepsilon, \delta}(x_0)| \leq C \left( \frac{d(x_0)}{\varepsilon} \right)^\sigma \left( \int_{B(x_0, \varepsilon) \cap \Omega} |u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]

(5.14)

\[
\leq C \left( \frac{d(x_0)}{r} \right)^\sigma \left( \int_{B(x_0, r) \cap \partial \Omega} |u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]

where we have used (5.11) for the last inequality. Finally, suppose that \( \kappa\varepsilon/2 < d(x_0) < r \). Then

\[
|u_{\varepsilon, \delta}(x_0)| \leq C \left( \int_{B(x_0, d(x_0))} |u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]

\[
\leq C \left( \frac{d(x_0)}{r} \right)^\sigma \left( \int_{B(x_0, r) \cap \partial \Omega} |u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]

where we have used (5.11) for the last inequality.

**Remark 5.6.** If \( \Omega \) is a \( C^1 \) domain, then the estimate (5.12) holds for any \( \lambda \in (0, 1) \). It follows from the proof of Theorem 5.4 that the large-scale boundary Hölder estimate (5.11) holds for any \( \sigma \in (0, 1) \).

If \( \Omega \) is \( C^{1, \alpha} \) for some \( \alpha > 0 \), we obtain the large-scale boundary Lipschitz estimate.

**Theorem 5.7.** Suppose \( A \) satisfies (1.1)-(1.2) and \( 0 \leq \delta \leq 1 \). Let \( \Omega \) be a bounded \( C^{1, \alpha} \) domain satisfying (1.5). Let \( u_{\varepsilon, \delta} \in H^1(D_R) \) be a weak solution of \( \text{div}(A^\varepsilon \nabla u_{\varepsilon, \delta}) = 0 \) in \( D_R \) with \( u_{\varepsilon, \delta} = 0 \) on \( \Delta_R \) for some \( (10d)\varepsilon < R < r_0 \). Then for \( \varepsilon < r < R/2 \),

\[
\left( \int_{D_r} |\nabla u_{\varepsilon, \delta}|^2 \right)^{1/2} \leq C \left( \int_{D_R} |\nabla u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]

(5.14)

where \( C \) depends only on \( d \), \( \mu \), \( \omega \), \( \kappa \), and \( \Omega \).
Proof. The proof uses the boundary $C^{1,\alpha}$ estimate for second-order elliptic equations with constant coefficients in $C^{1,\alpha}$ domains. With the approximation result in Lemma 5.3 at our disposal, the proof is similar to the case $\delta = 1$ in [9].

6 Estimates of Green’s functions

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, satisfying (1.5). For $0 < \delta \leq 1$, let $G_{\varepsilon,\delta}(x,y)$ denote the Green’s function for the operator $L_{\varepsilon,\delta} = -\text{div}(A_{\varepsilon}^T \nabla)$ in $\Omega$, where $A$ satisfies (1.1)-(1.2). No smoothness condition on $A$ is needed. The goal of this section is to prove the following.

Theorem 6.1. Suppose $d \geq 3$ and $0 < \delta \leq 1$. Then, for $x, y \in \Omega$ and $x \neq y$,

$$|G_{\varepsilon,\delta}(x,y)| \leq \frac{C}{|x - y|^{d-2}} \quad \text{if } x, y \in \Omega^e \text{ or } |x - y| \geq 4\varepsilon,$$

(6.1)

where $C$ depends only on $d$, $\mu$, $\omega$, $\kappa$, and the Lipschitz character of $\Omega$.

We begin with some energy estimates.

Lemma 6.2. Suppose $d \geq 3$. Let $u_{\varepsilon,\delta} \in H^1_0(\Omega)$ be a weak solution of $-\text{div}(A_{\varepsilon}^T \nabla u_{\varepsilon,\delta}) = f$ in $\Omega$, where $f \in L^2(\Omega)$ with support in $B(y_0, r) \cap \Omega$, where $y_0 \in \overline{\Omega}$ and $0 < r < \text{diam}(\Omega)$. Then

$$\|\Lambda_{\varepsilon}^T \nabla u_{\varepsilon,\delta}\|_{L^2(\Omega)} \leq Cr\|f\|_{L^2(\Omega)},$$

(6.2)

where $C$ depends only on $d$, $\mu$, $\omega$, $\kappa$, and the Lipschitz character of $\Omega$.

Proof. Let $\varphi \in H^1_0(\Omega)$. Then

$$\int_\Omega [\Lambda_{\varepsilon}(x/\varepsilon)]^T 2A(x/\varepsilon) \nabla u_{\varepsilon,\delta} \cdot \nabla \varphi \, dx = \int_{B(y_0, r) \cap \Omega} f \cdot \varphi \, dx.$$  

(6.3)

By letting $\varphi = u_{\varepsilon,\delta}$ in (6.3) and using

$$\|\varphi\|_{L^2(B(y_0, r) \cap \Omega)} \leq Cr\|\varphi\|_{L^p(\Omega)} \leq C\|\nabla \varphi\|_{L^2(\Omega)},$$

where $p = \frac{2d}{d-2}$, we obtain

$$\delta^2\|\nabla u_{\varepsilon,\delta}\|_{L^2(\Omega)} \leq Cr\|f\|_{L^2(\Omega)}.$$  

(6.4)

Next, in (6.3), we let $\varphi \in H^1_0(\Omega)$ be an extension of $u_{\varepsilon,\delta}|_{\Omega^e}$ such that

$$\|\nabla \varphi\|_{L^2(\Omega)} \leq C\|\nabla u_{\varepsilon,\delta}\|_{L^2(\Omega^e)}.$$

(6.5)

(see Lemma 2.1). It follows that

$$\|\nabla u_{\varepsilon,\delta}\|_{L^2(\Omega^e)}^2 \leq C\delta^2\|\nabla u_{\varepsilon,\delta}\|_{L^2(\Omega)\setminus\Omega^e}\|\nabla \varphi\|_{L^2(\Omega)\setminus\Omega^e} + Cr\|f\|_{L^2(\Omega)}\|\nabla \varphi\|_{L^2(\Omega)}.$$

(6.6)

where we have used (6.4) and (6.5) for the last inequality. This, together with (6.4), gives (6.2). □

Lemma 6.3. Suppose $d \geq 3$. Let $u_{\varepsilon,\delta} \in H^1_0(\Omega)$ be a weak solution of $-\text{div}(A_{\varepsilon}^T \nabla u_{\varepsilon,\delta}) = \Lambda_{\varepsilon} f$ in $\Omega$, where $f \in L^2(B(y_0, r) \cap \Omega)$. Then

$$\|\Lambda_{\varepsilon}^T \nabla u_{\varepsilon,\delta}\|_{L^2(\Omega)} \leq Cr\|f\|_{L^2(\Omega)},$$

(6.7)

where $C$ depends only on $d$, $\mu$, $\omega$, $\kappa$, and the Lipschitz character of $\Omega$. 

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Proof. By using the test function $u_{\epsilon,\delta}$ and Cauchy inequality, we obtain
\[
\|A_\delta^s \nabla u_{\epsilon,\delta}\|_{L^2(\Omega)}^2 \leq C \|f\|_{L^2(\Omega)} \|A_\delta^s u_{\epsilon,\delta}\|_{L^2(B(y_0,r)\cap\Omega)}.
\] (6.7)

Note that
\[
\|A_\delta^s u_{\epsilon,\delta}\|_{L^2(B(y_0,r)\cap\Omega)} \leq Cr \|A_\delta^s u_{\epsilon,\delta}\|_{L^P(\Omega)} \leq Cr \|A_\delta^s \nabla u_{\epsilon,\delta}\|_{L^2(\Omega)},
\]
where we have used Hölder’s inequality as well as the Sobolev type inequality,
\[
\|A_\delta^s \varphi\|_{L^P(\Omega)} \leq C \|A_\delta^s \nabla \varphi\|_{L^2(\Omega)},
\] (6.8)
with $p = \frac{2d}{d-2}$, for $\varphi \in H^1_0(\Omega)$. This, together with (6.7), gives (6.6). Finally, to see (6.8) we let $\tilde{\varphi} \in H^1_0(\Omega)$ be an extension of $\varphi|_{\Omega^c}$ such that $\|\nabla \tilde{\varphi}\|_{L^2(\Omega)} \leq C \|\nabla \varphi\|_{L^2(\Omega^c)}$. Then
\[
\|A_\delta^s \varphi\|_{L^P(\Omega)} \leq \|\tilde{\varphi}\|_{L^P(\Omega)} + \delta \|\tilde{\varphi}\|_{L^P(\Omega)} \leq C \|\nabla \tilde{\varphi}\|_{L^2(\Omega)} + C\delta \|\nabla \tilde{\varphi}\|_{L^2(\Omega)} \leq C \|\nabla \varphi\|_{L^2(\Omega)},
\]
where we have used the usual Sobolev inequality for the second inequality. $\square$

Proof of Theorem 6.1. Fix $x_0, y_0 \in \Omega$ and let $r = |x_0 - y_0|/2$. We first consider the case $r \geq 2\epsilon$. Let $f \in L^2(B(y_0, r) \cap \Omega)$ and
\[
u_{\epsilon,\delta}(x, y) = \int_{\Omega} G_{\epsilon,\delta}(x, y) f(y) dy.
\]
Then $-\text{div}(A_\delta^s \nabla u_{\epsilon,\delta}) = f$ in $\Omega$ and $u_{\epsilon,\delta} = 0$ on $\partial \Omega$. It follows from Lemma 6.2 that
\[
\|\nabla u_{\epsilon,\delta}\|_{L^2(\Omega^c)} \leq Cr \|f\|_{L^2(\Omega)}.
\]
Let $\tilde{u}_{\epsilon,\delta} \in H^1_0(\Omega)$ be an extension of $u_{\epsilon,\delta}|_{\Omega^c}$ such that $\|\nabla \tilde{u}_{\epsilon,\delta}\|_{L^2(\Omega)} \leq C \|\nabla u_{\epsilon,\delta}\|_{L^2(\Omega^c)}$. By (5.6),
\[
\|u_{\epsilon,\delta}\|_{L^2(\Omega \cap B(x_0, r))} \leq C \|u_{\epsilon,\delta}\|_{L^2(\Omega \cap B(x_0, 2r))} \leq C r \|u_{\epsilon,\delta}\|_{L^P(\Omega)} \leq C r^2 \|f\|_{L^2(\Omega)}.
\]
where $p = \frac{2d}{d-2}$, and we have used a Sobolev inequality for $\tilde{u}_{\epsilon,\delta}$. Since $\text{div}(A_\delta^s \nabla u_{\epsilon,\delta}) = 0$ in $B(x_0, r) \cap \Omega$, $u_{\epsilon,\delta} = 0$ on $\partial \Omega$, and $r > 2\epsilon$, it follows from (5.13) that
\[
|u_{\epsilon,\delta}(x_0)| \leq C \left( \int_{B(x_0, r) \cap \Omega} |u_{\epsilon,\delta}|^2 \right)^{1/2} \leq C r^{2-d/4} \|f\|_{L^2(B(y_0, r) \cap \Omega)}.
\]
By duality this implies that
\[
\left( \int_{B(y_0, r) \cap \Omega} |G_{\epsilon,\delta}(x_0, y)|^2 dy \right)^{1/2} \leq C r^{2-d/4}.
\]
Since $\text{div}(A_\delta^s \nabla G_{\epsilon,\delta}(x_0, \cdot)) = 0$ in $B(y_0, r)$ and $G_{\epsilon,\delta}(x_0, \cdot) = 0$ on $\partial \Omega$, we obtain
\[
|G_{\epsilon,\delta}(x_0, y_0)| \leq C \left( \int_{B(y_0, r) \cap \Omega} |G_{\epsilon,\delta}(x_0, y)|^2 dy \right)^{1/2} \leq C r^{2-d}.
\]
We now consider the case where \( r = (1/2)|x_0 - y_0| < 2\varepsilon \). Let \( g \in L^2(B(y_0, r) \cap \Omega) \) and
\[
w_{\varepsilon, \delta} = \int_{\Omega} G_{\varepsilon, \delta}(x, y) \Lambda_{\delta}(y/\varepsilon) g(y) \, dy.
\]
Then \(-\text{div}(A_{\delta}^\varepsilon \nabla w_{\varepsilon, \delta}) = \Lambda_{\delta}^\varepsilon g\) in \( \Omega \) and \( w_{\varepsilon, \delta} = 0 \) on \( \partial \Omega \). By Lemma 6.3,
\[
\|\Lambda_{\delta}^\varepsilon \nabla w_{\varepsilon, \delta}\|_{L^2(\Omega)} \leq Cr\|g\|_{L^2(\Omega)}.
\]
It follows that
\[
\|\Lambda_{\delta}^\varepsilon w_{\varepsilon, \delta}\|_{L^2(B(x_0, r) \cap \Omega)} \leq Cr\|\Lambda_{\delta}^\varepsilon w_{\varepsilon, \delta}\|_{L^p(\Omega)} \leq Cr\|\Lambda_{\delta}^\varepsilon \nabla w_{\varepsilon, \delta}\|_{L^2(\Omega)} \leq Cr^2\|g\|_{L^2(\Omega)},
\]
where we have used (6.8) for the second inequality. Hence, by Theorem 2.11
\[
\Lambda_{\delta}^\varepsilon(x_0)|w_{\varepsilon, \delta}(x_0)| \leq C \left( \int_{B(x_0, r) \cap \Omega} |\Lambda_{\delta}^\varepsilon w_{\varepsilon, \delta}| \right)^{1/2} \leq C r^{2-d} \|g\|_{L^2(\Omega)}.
\]
By duality we obtain
\[
\left( \int_{B(y_0, r) \cap \Omega} |\Lambda_{\delta}^\varepsilon(x_0)G_{\varepsilon, \delta}(x_0, y)\Lambda_{\delta}^\varepsilon(y)|^2 \, dy \right)^{1/2} \leq C r^{2-d},
\]
which, by Theorem 2.11 again, leads to
\[
\Lambda_{\delta}^\varepsilon(x_0)\Lambda_{\delta}^\varepsilon(y_0)G_{\varepsilon, \delta}(x_0, y_0) \leq C r^{2-d},
\]
since \( \text{div}(A_{\delta}^\varepsilon \nabla G_{\varepsilon, \delta}(x_0, \cdot)) = 0 \) in \( \Omega \setminus \{x_0\} \) and \( G_{\varepsilon, \delta}(x_0, \cdot) = 0 \) on \( \partial \Omega \).

\[\square\]

**Remark 6.4.** Suppose \( d \geq 3 \). It follows from (5.13) and (6.1) that if \( x, y \in \Omega \) and \( |x - y| \geq 4\varepsilon \), then
\[
|G_{\varepsilon, \delta}(x, y)| \leq C \frac{|d(x)|^\sigma |d(y)|^\sigma}{|x - y|^{d-2+2\sigma}} \quad (6.9)
\]
for some \( \sigma \in (0, 1) \), where \( d(x) = \text{dist}(x, \partial \Omega) \). The estimate (6.9) also holds if \( |x - y| < 4\varepsilon \) and \( x, y \in \Omega^\varepsilon \).

**Remark 6.5.** In the case \( d = 2 \), the inequality \( \|\varphi\|_{L^\infty(\Omega)} \leq C \|\nabla \varphi\|_{L^2(\Omega)} \) fails for \( \varphi \in H^1_0(\Omega) \). However, an inspection of the proof of Lemma 6.2 as well as the proof of (5.4) shows that one only needs
\[
\|\varphi\|_{L^2(B(y_0, r) \cap \Omega)} \leq C r \|\nabla \varphi\|_{L^2(\Omega)} \quad \text{for} \ \varphi \in H^1_0(\Omega).
\]
This Poincaré-type inequality holds if \( r \geq c_0 d(y_0) \) for \( d \geq 2 \). As a result, the estimate (6.9) continues to hold in the case \( d = 2 \), if \( |x - y| \geq 8\varepsilon \) and \( |x - y| \geq c_0 \max(d(x), d(y)) \). The estimate also holds if \( c_0 \max(d(x), d(y)) < |x - y| < 8\varepsilon \) and \( x, y \in \Omega^\varepsilon \).

**Remark 6.6.** In the case where \( \omega, \Omega \) are smooth, estimates of Green functions may be found [22] [25].
7 Boundary layer estimates

Throughout this section we assume that \( \Omega \) is a bounded Lipschitz domain satisfying \( (1.5) \), \( A \) satisfies \( (1.1)-(1.2) \), and \( A^\varepsilon_{\delta} \) is given by \( (1.8) \). We use the scale-invariant \( H^1(\partial \Omega) \) norm,

\[
\|f\|_{H^1(\partial \Omega)} = \|\nabla_{\text{tan}} f\|_{L^2(\partial \Omega)} + [\text{diam}(\partial \Omega)]^{-1} \|f\|_{L^2(\partial \Omega)}.
\]

**Theorem 7.1.** Let \( u_{\varepsilon, \delta} \in H^1(\Omega) \) be a weak solution of \( (1.9) \) with \( f \in H^1(\partial \Omega) \). Then, for \( 0 < \varepsilon \leq 1 \) and \( 0 \leq \delta \leq 1 \),

\[
\|\nabla u_{\varepsilon, \delta}\|_{L^2(\Sigma_{\alpha_\varepsilon})} \leq C_{\alpha} \varepsilon^{1/2} \|f\|_{H^1(\partial \Omega)}, \tag{7.1}
\]

where \( \alpha \geq d \) and \( C_{\alpha} \) depends only on \( d, \mu, \omega, \kappa, \alpha, \) and the Lipschitz character of \( \Omega \).

**Proof.** Let \( v \) be the solution of the homogenized problem: \( \text{div}(\widehat{A}_0 \nabla v) = 0 \) in \( \Omega \) and \( v = f \) on \( \partial \Omega \). Recall that for \( 0 < t < 1 \),

\[
\|\nabla v\|_{L^2(\Sigma_t)} + t\|\nabla^2 v\|_{L^2(\partial \Omega)} \leq C t^{1/2} \|\nabla v\|_{L^2(\partial \Omega)} \leq C t^{1/2} \|f\|_{H^1(\partial \Omega)}, \tag{7.2}
\]

where \( C \) depends only on \( d, \mu, \) and the Lipschitz character of \( \Omega \). Let \( w_{\varepsilon, \delta} \) be defined by \( (3.3) \), with \( v \) in the place of \( v_\delta \). The cut-off function \( \eta_\varepsilon \) in \( (3.3) \) is chosen so that \( \eta_\varepsilon = 1 \) in \( \Omega \setminus \Sigma_{3\alpha_\varepsilon} \) and \( \eta_\varepsilon = 0 \) in \( \Sigma_{2\alpha_\varepsilon} \), and \( |\nabla \eta_\varepsilon| \leq C_{\alpha} \varepsilon^{-1} \). It follows from \( (3.8) \) and \( (7.2) \) that

\[
\left| \int_{\Omega} A_{\delta} \nabla w_{\varepsilon, \delta} \cdot \nabla \psi dx \right| \leq C \varepsilon^{1/2} \|\nabla \psi\|_{L^2(\Omega)} \|f\|_{H^1(\partial \Omega)} \tag{7.3}
\]

for any \( \psi \in H^1_0(\Omega) \). Let \( \psi = w_{\varepsilon, \delta} \) in \( (7.3) \). Using the ellipticity condition \( (1.1) \) and \( A_{\delta} \geq \delta \), we obtain

\[
\delta^2 \|\nabla w_{\varepsilon, \delta}\|_{L^2(\Omega)} \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial \Omega)}. \tag{7.4}
\]

By Lemma 2.7.1 there exists \( \tilde{w}_{\varepsilon, \delta} \in H^1_0(\Omega) \) such that \( \tilde{w}_{\varepsilon, \delta} = w_{\varepsilon, \delta} \) in \( \Omega^\varepsilon \) and

\[
\|\nabla \tilde{w}_{\varepsilon, \delta}\|_{L^2(\Omega^\varepsilon)} \leq C \|\nabla w_{\varepsilon, \delta}\|_{L^2(\Omega^\varepsilon)}. \tag{7.5}
\]

Let \( \psi = \tilde{w}_{\varepsilon, \delta} \) in \( (7.3) \). Using \( (7.4) \) and \( (7.5) \), we see that

\[
\|\nabla w_{\varepsilon, \delta}\|_{L^2(\Omega^\varepsilon)} \leq C \varepsilon^{1/2} \|\nabla w_{\varepsilon, \delta}\|_{L^2(\Omega)} + C \varepsilon^{1/2} \|\nabla \tilde{w}_{\varepsilon, \delta}\|_{L^2(\Omega)} \|f\|_{H^1(\partial \Omega)} \leq C \varepsilon^{1/2} \|\nabla w_{\varepsilon, \delta}\|_{L^2(\Omega^\varepsilon)} \|f\|_{H^1(\partial \Omega)},
\]

which yields \( \|\nabla u_{\varepsilon, \delta}\|_{L^2(\Omega^\varepsilon)} \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial \Omega)} \). Since \( w_{\varepsilon, \delta} = u_{\varepsilon, \delta} - v \) on \( \Sigma_{2\alpha_\varepsilon} \), we obtain

\[
\|\nabla u_{\varepsilon, \delta}\|_{L^2(\Omega \cap \Sigma_{2\alpha_\varepsilon})} \leq \|v\|_{L^2(\Omega \cap \Sigma_{2\alpha_\varepsilon})} + \|\nabla w_{\varepsilon, \delta}\|_{L^2(\Omega \cap \Sigma_{2\alpha_\varepsilon})} \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial \Omega)},
\]

where we also used \( (7.2) \) for the last inequality. Finally, we note that by Lemma 2.7.2 \( \|\nabla u_{\varepsilon, \delta}\|_{L^2(\Sigma_{\alpha_\varepsilon})} \leq C \|\nabla u_{\varepsilon, \delta}\|_{L^2(\Omega \cap \Sigma_{2\alpha_\varepsilon})} \). This completes the proof. \( \square \)

Next we consider the Neumann problem \( (1.10) \).

**Theorem 7.2.** Assume that \( \widehat{A}_0 \) is symmetric. Let \( u_{\varepsilon, \delta} \in H^1(\Omega) \) be a weak solution of \( (1.10) \) with \( g \in L^2(\partial \Omega) \) and \( \int_{\partial \Omega} g = 0 \). Then, for \( 0 < \varepsilon \leq 1 \) and \( 0 \leq \delta \leq 1 \),

\[
\|\nabla u_{\varepsilon, \delta}\|_{L^2(\Sigma_{\alpha_\varepsilon})} \leq C_{\alpha} \varepsilon^{1/2} \|g\|_{L^2(\partial \Omega)}, \tag{7.6}
\]

where \( \alpha \geq d \) and \( C \) depends only on \( d, \mu, \omega, \kappa, \alpha, \) and the Lipschitz character of \( \Omega \).
Proof. The proof is similar to that of Theorem 7.1. Let $v$ be a solution of the homogenized problem:

$$\text{div}(\hat{A}_\delta \nabla v) = 0 \text{ in } \Omega \text{ and } \frac{\partial v}{\partial \nu} = g \text{ on } \partial \Omega, \text{ where } \partial u / \partial v = n \cdot \hat{A}_\delta \nabla v.$$ 

Since $\hat{A}_\delta$ is symmetric, it is known that $\|\nabla(Nv)\|_{L^2(\partial \Omega)} \leq C\|g\|_{L^2(\partial \Omega)}$ (see [3] for references). Thus, for $0 < t < 1$,

$$\|\nabla v\|_{L^2(\Sigma_t)} + t\|\nabla^2 v\|_{L^2(\Omega \setminus \Sigma_t)} \leq Ct^{1/2}\|\nabla(Nv)\|_{L^2(\partial \Omega)} \leq Ct^{1/2}\|g\|_{L^2(\partial \Omega)}.$$  

(7.7)

Let $w_{\varepsilon, \delta}$ be defined as in (3.3), with $v$ in the place of $v_\delta$. An inspection of the proof of Theorem 3.2 shows that the inequality (3.3) holds for any $\psi \in H^1(\Omega)$. In fact, the only place that the boundary condition plays a role is in (3.3). It follows that (7.7) holds for any $\psi \in H^1(\Omega)$. The rest of the proof is exactly the same as that of Theorem 7.1. \hfill \square

We end this section with a localized version of Theorem 7.1.

**Theorem 7.3.** Let $u_{\varepsilon, \delta} \in H^1(D_{2r})$ be a weak solution of $\text{div}(\hat{A}_\delta \nabla u_{\varepsilon, \delta}) = 0$ in $D_{2r}$ for some $r > (10d)\varepsilon$. Then

$$\frac{1}{\varepsilon} \int_{\Sigma_{\varepsilon, r}} \|\nabla u_{\varepsilon, \delta}\|^2 \, dx \leq C \int_{\Delta_{2r}} \|\nabla \tan u_{\varepsilon, \delta}\|^2 \, d\sigma + \frac{C}{r} \int_{D_{2r}} \|\nabla u_{\varepsilon, \delta}\|^2 \, dx,$$

where $\Sigma_{\varepsilon, r} = \Sigma_{\varepsilon, \delta} \cap D_r$ and $C$ depends only on $d, \mu, \omega, \kappa$, and the Lipschitz character of $\Omega$.

**Proof.** By rescaling we may assume $r = 1$ and $\varepsilon > 0$ is sufficiently small. By Fubini’s Theorem and the co-area formula we see that

$$\int_{3/2}^2 \left( \frac{1}{\varepsilon} \int_{D_{t \setminus D_{t-(2d)\varepsilon}}} \|\nabla u_{\varepsilon, \delta}\|^p \, dx + \int_{\Omega \cap \partial B_t} \|\nabla u_{\varepsilon, \delta}\|^p \, d\sigma \right) \, dt \leq C \int_{D_2} \|\nabla u_{\varepsilon, \delta}\|^p \, dx,$$

where $2 \leq p < \infty$. It follows that there exists $t \in (3/2, 2)$ such that

$$\frac{1}{\varepsilon} \int_{D_{t \setminus D_{t-(2d)\varepsilon}}} \|\nabla u_{\varepsilon, \delta}\|^p \, dx + \int_{\Omega \cap \partial B_t} \|\nabla u_{\varepsilon, \delta}\|^p \, d\sigma \leq C \int_{D_2} \|\nabla u_{\varepsilon, \delta}\|^p \, dx.$$  

(7.9)

Let $v$ be the solution of $\text{div}(\hat{A}_\delta \nabla v) = 0$ in $D_t$ with $v = u_{\varepsilon, \delta}$ on $\partial D_t$. Note that

$$\|\nabla(Nv)\|_{L^2(\partial D_t)} \leq C\|\nabla \tan v\|_{L^2(\partial D_t)} \leq C\{\|\nabla \tan u_{\varepsilon, \delta}\|_{L^2(\Delta_2)} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_2)}\},$$

(7.10)

where we used (7.9) with $p = 2$ for the last inequality. Let $w_{\varepsilon, \delta}$ be defined by (3.3). It follows from (3.8) and (7.10) that

$$\int_{D_t} A_\delta^\varepsilon \nabla w_{\varepsilon, \delta} \cdot \nabla \psi \, dx \leq C\varepsilon^{1/2}\|\nabla \psi\|_{L^2(D_t)}\{\|\nabla \tan u_{\varepsilon, \delta}\|_{L^2(\Delta_2)} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_2)}\}$$

(7.11)

for any $\psi \in H^1_0(D_t)$. By letting $\psi = w_{\varepsilon, \delta}$ in (7.11) we obtain

$$\delta^2\|\nabla w_{\varepsilon, \delta}\|_{L^2(D_t)} \leq C\varepsilon^{1/2}\{\|\nabla \tan u_{\varepsilon, \delta}\|_{L^2(\Delta_2)} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_2)}\}.$$  

(7.12)

To proceed, we construct a new function $\tilde{w}_{\varepsilon, \delta}$ in $D_t$ as follows. For each $\varepsilon F_k \subset \Omega$ with $\varepsilon \tilde{F}_k \subset D_t$, we replace $w_{\varepsilon, \delta}|_{\partial \varepsilon F_k}$ by the harmonic extension of $\tilde{w}_{\varepsilon, \delta}|_{\partial (\varepsilon \tilde{F}_k)}$. This gives us a function $\tilde{w}_{\varepsilon, \delta}$ in $H^1_0(D_t)$ with the property that

$$\|\nabla \tilde{w}_{\varepsilon, \delta}\|_{L^2(D_{t-d\varepsilon})} \leq C\|\nabla w_{\varepsilon, \delta}\|_{L^2(D_t)},$$

(7.13)
Also note that
\[
\|\nabla \tilde{w}_{\varepsilon, \delta}\|_{L^2(D_t)} \leq C \left\{ \|\nabla w_{\varepsilon, \delta}\|_{L^2(D_t \setminus D_{t-\varepsilon})} + \|\nabla w_{\varepsilon, \delta}\|_{L^2(D_t^c)} \right\}.
\] (7.14)

Let \( \psi = \tilde{w}_{\varepsilon, \delta} \) in (7.11). We obtain
\[
\|\nabla w_{\varepsilon, \delta}\|_{L^2(D_t^c)}^2 \leq C \delta^2 \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_t)} \|\nabla \tilde{w}_{\varepsilon, \delta}\|_{L^2(D_t)} + C \varepsilon \|\nabla \tilde{w}_{\varepsilon, \delta}\|_{L^2(D_t)} \left\{ \|\nabla \tan u_{\varepsilon, \delta}\|_{L^2(\Delta_2)} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_2)} \right\}
\]
where we have used (7.12) for the last inequality. This, together with (7.14) and (7.9) for \( p = 2 \), gives
\[
\|\nabla w_{\varepsilon, \delta}\|_{L^2(D_t^c)} \leq C \varepsilon^{1/2} \left\{ \|\nabla \tan u_{\varepsilon, \delta}\|_{L^2(\Delta_2)} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_2)} \right\}.
\]
which, combining with (7.12), leads to
\[
\|\Lambda^2_{\delta} \nabla w_{\varepsilon, \delta}\|_{L^2(D_1)} \leq C \varepsilon^{1/2} \left\{ \|\nabla \tan u_{\varepsilon, \delta}\|_{L^2(\Delta_2)} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_2)} \right\}.
\] (7.15)

Observe that \( \Sigma_{\kappa_1, 1} \subset D_1^c \), and \( w_{\varepsilon, \delta} = u_{\varepsilon, \delta} - v \) on \( \Sigma_{\kappa_1, 1} \). We obtain
\[
\|\nabla u_{\varepsilon, \delta}\|_{L^2(\Sigma_{\kappa_1, 1})} \leq \|\nabla v\|_{L^2(\Sigma_{\kappa_1, 1})} + \|\nabla w_{\varepsilon, \delta}\|_{L^2(\Sigma_{\kappa_1, 1})}
\]
\[
\leq C \varepsilon^{1/2} \left\{ \|\nabla \tan u_{\varepsilon, \delta}\|_{L^2(\Delta_2)} + \|\nabla u_{\varepsilon, \delta}\|_{L^2(D_2)} \right\},
\]
where we also used (7.10) for the last inequality.

\[\square\]

8 Reverse Hölder inequalities

Recall that \( \Lambda^+_{\delta}(x) = \Lambda_{\delta}(x/\varepsilon) \). The goal of this section is to prove the following.

**Theorem 8.1.** Assume \( A \) satisfies (1.1) and (1.2). Let \( u_{\varepsilon, \delta} \in H^1(B_{6r}) \) be a weak solution of \( \text{div}(A^+_{\delta} \nabla u_{\varepsilon, \delta}) = 0 \) in \( B_{6r} \). Then
\[
\left( \int_{B_r} |\Lambda^+_{\delta} \nabla u_{\varepsilon, \delta}|^p \right)^{1/p} \leq C \left( \int_{B_{2r}} |\Lambda^+_{\delta} \nabla u_{\varepsilon, \delta}|^2 \right)^{1/2},
\] (8.1)

where \( p > 2 \) and \( C > 1 \) depend only on \( d, \mu, \) and \( \omega \). Moreover, if \( r \geq 2d\varepsilon \), then
\[
\left( \int_{B_r} |\nabla u_{\varepsilon, \delta}|^p \right)^{1/p} \leq C \left( \int_{B_{6r} \cap \omega} |\nabla u_{\varepsilon, \delta}|^2 \right)^{1/2}.
\] (8.2)

By the self-improving property of the (weak) reverse Hölder inequalities, to show (8.1), it suffices to prove that if \( \text{div}(A^+_{\delta} \nabla u_{\varepsilon, \delta}) = 0 \) in \( B_{4r} \), then
\[
\left( \int_{B_r} |\Lambda^+_{\delta} \nabla u_{\varepsilon, \delta}|^2 \right)^{1/2} \leq C \left( \int_{B_{4r}} |\Lambda^+_{\delta} \nabla u_{\varepsilon, \delta}|^q \right)^{1/q},
\] (8.3)

for some \( q < 2 \) and \( C > 1 \), depending at most on \( d, \mu, \) and \( \omega \).

We first treat the case \( 0 < r < 10d\varepsilon \).

**Lemma 8.2.** Let \( u_{\varepsilon, \delta} \in H^1(B_{4r}) \) be a weak solution of \( \text{div}(A^+_{\delta} \nabla u_{\varepsilon, \delta}) = 0 \) in \( B_{4r} \). Then (8.3) holds if \( 0 < r < 10d\varepsilon \).
Proof. By dilation we may assume $\varepsilon = 1$. We may also assume $0 < r \leq c_0$ and $c_0 > 0$ is small. The case $c_0 < r < 10d$ follows from the case $r = c_0$ by a simple covering argument. There are three cases: (1) $B_{3r/2} \cap \omega$; (2) $B_{3r/2} \subset F = \mathbb{R}^d \setminus \varpi$; and (3) $B_{3r/2} \cap \partial \omega \neq \emptyset$. Since $\text{div}(A \nabla u_{1,\delta}) = 0$ in $B_{4r} \cap \omega$ and $B_{4r} \cap F$, the first two cases follow directly from the well-know reverse Hölder inequalities for the elliptic operator $-\text{div}(A \nabla)$. The third case may be reduced to the case where $B_{4r} = B(x_0, 4r)$ for some $x_0 \in \partial \omega$. By Caccioppoli’s inequality (2.11),

$$\left( \int_{B_r} |A_\delta \nabla u_{1,\delta}|^2 \, dx \right)^{1/2} \leq C \inf_{\beta \in \mathbb{R}} \left( \int_{B_{2r}} |A_\delta (u_{1,\delta} - \beta)|^2 \, dx \right)^{1/2}. \tag{8.4}$$

Let $B_{2r}^+ = B_{2r} \cap \omega$ and $B_{2r}^- = B_{2r} \cap F$. Choose $\beta$ to be the average of $u_{1,\delta}$ over $B_{3r}^+$. By Sobolev inequality, the right-hand side of (8.4) is bounded by

$$C \left( \int_{B_{3r}^-} |\nabla u_{1,\delta}|^q \right)^{1/q} + \frac{C \delta}{r} \left( \int_{B_{3r}^+} |u_{1,\delta} - \beta|^2 \right)^{1/2}, \tag{8.5}$$

where $q = \frac{2d}{d-2}$ for $d \geq 3$, and $1 < q < 2$ for $d = 2$. To bound the second term in (8.5), we apply the inequality,

$$\frac{1}{r} \left( \int_{B_{2r}^-} |v|^2 \right)^{1/2} \leq C \left( \int_{B_{3r}^-} |\nabla v|^q \right)^{1/q} + C \frac{1}{r} \left( \int_{B_{3r}^+} |v|^2 \right)^{1/2}, \tag{8.6}$$

to $v = u_{1,\delta} - \beta$. It follows that (8.5) is bounded by the right-hand side of (8.3).

Finally, to see (8.3), by dilation we may assume $r = 1$. Since $\omega$ is a Lipschitz domain, by a change of variables, we may also assume $B_{3r}^+ = B_3 \cap \{x_d > 0\}$ and $B_3^- = B_3 \cap \{x_d < 0\}$. In this case, (8.6) follows by a compactness argument. \hfill \Box

Lemma 8.3. Let $u \in H^1(B_{2r})$ for some $r \geq 2d\varepsilon$. Then

$$\inf_{\beta \in \mathbb{R}} \frac{1}{r} \left( \int_{B_r} |A_\delta^+(u - \beta)|^2 \right)^{1/2} \leq C \left( \int_{B_{2r}} |A_\delta^+ \nabla u|^q \right)^{1/q}, \tag{8.7}$$

where $q = \frac{2d}{d-2}$ for $d \geq 3$, $1 < q < 2$ for $d = 2$, and $C$ depends only on $d$ and $\omega$.

Proof. We consider the case $d \geq 3$. The case $d = 2$ is similar. By rescaling we may assume $\varepsilon = 1$. Let $\tilde{u} \in H^q(B_{3r/2})$ be an extension of $u|_{B_{3r/2} \cap \omega}$ such that

$$\|
abla \tilde{u}\|_{L^q(B_{3r/2})} \leq C \|
abla u\|_{L^q(B_{2r} \cap \omega)}.$$

Note that

$$\left( \int_{B_{3r/2} \cap \omega} |u - \beta|^2 \right)^{1/2} \leq \left( \int_{B_{3r/2}} |\tilde{u} - \beta|^2 \right)^{1/2} \leq C \left( \int_{B_{3r/2}} |\nabla \tilde{u}|^q \right)^{1/q}, \tag{8.8}$$

where $\beta = \int_{B_{3r/2}} \tilde{u}$. Also observe that if $F_k \cap B_r \neq \emptyset$, then $\tilde{F}_k \subset B_{3r/2}$ and

$$\int_{F_k} |u - \beta|^2 \, dx \leq C \left( \int_{\tilde{F}_k} |\nabla \tilde{u}|^q \, dx \right)^{2/q} + C \int_{\tilde{F}_k \setminus F_k} |u - \beta|^2 \, dx.$$
It follows that
\[
\int_{B_r \cap F} |u - \beta|^2 \leq C \sum_k \left( \int_{\tilde{F}_k} |\nabla u|^q \, dx \right)^{\frac{2}{q}} + C \int_{B_{3r/2} \cap \omega} |u - \beta|^2 \, dx \\
\leq C \left( \int_{B_{2r}} |\nabla u|^q \, dx \right)^{\frac{2}{q}},
\]
where we have used the Minkowski inequality as well as \(8.8\) for the last inequality. This, together with \(8.8\), gives
\[
\left( \int_{B_r} |\Lambda_\delta (u - \beta)|^2 \, dx \right)^{\frac{1}{2}} \leq C \left( \int_{B_{2r}} |\Lambda_\delta \nabla u|^q \, dx \right)^{1/q},
\]
from which the inequality \(8.7\) follows, as \(\frac{d}{q} = \frac{d}{2} + 1\). \(\square\)

**Proof of Theorem 8.1.** As we indicated before, to prove \(8.1\), it suffices to show that \(8.3\) holds for some \(q < 2\). The case \(0 < r < 10d\varepsilon\) is treated in Lemma 8.2. To handle the case \(r \geq 10d\varepsilon\), we use \(8.4\) and Lemma 8.3.

To prove \(8.2\), we assume \(\varepsilon = 1\) and use the fact that there exists some \(p_0 > 2\), depending only on \(d, \mu, \) and \(\omega\), such that if \(u \in W^{1,p}(\tilde{F}_k)\) for some \(2 < p < p_0\) and \(\text{div}(\Lambda_{\delta} u) = 0\) in \(\tilde{F}_k\), then
\[
\|\nabla u\|_{L^p(\tilde{F}_k)} \leq C \|\nabla u\|_{L^p(\tilde{F}_k \setminus \tilde{F})},
\]
where \(C > 0\) depends only on \(d, \mu, \) and \(\omega\). It follows that
\[
\left( \int_{B_r} |\nabla u_{1,\delta}|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{B_{3r/2} \cap \omega} |\nabla u_{1,\delta}|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{B_{3r}} |\nabla u_{1,\delta}|^2 \, dx \right)^{\frac{1}{2}},
\]
where we have used \(8.1\) for the second inequality and \(4.4\) for the last. To see \(8.9\), let \(\tilde{u}\) be an extension of \(u_{\tilde{F}_k} \cap \tilde{F}_k\) to \(\tilde{F}_k\) such that \(\|\nabla \tilde{u}\|_{L^p(\tilde{F}_k)} \leq C \|\nabla u\|_{L^p(\tilde{F}_k \setminus \tilde{F})}\) for \(p > 2\). Since \(\text{div}(\Lambda_{\delta} (u - \tilde{u})) = -\text{div}(\Lambda_{\delta} \nabla \tilde{u})\) in \(F_k\) and \(u - \tilde{u} = 0\) on \(\partial F_k\), by Meyers’ estimates, there exists some \(p_0 > 2\), depending only on \(d, \mu, \) and \(\omega\), such that for \(2 < p < p_0\),
\[
\|\nabla (u - \tilde{u})\|_{L^p(\tilde{F}_k)} \leq C \|\nabla \tilde{u}\|_{L^p(\tilde{F}_k)},
\]
from which the inequality \(8.9\) follows. \(\square\)

Recall that \(D_r = B(x_0, r) \cap \Omega\) and \(\Delta_r = B(x_0, r) \cap \partial \Omega\), where \(x_0 \in \partial \Omega\).

**Theorem 8.4.** Assume that \(A\) satisfies \(1.1\) and \(1.2\). Let \(\Omega\) be a bounded Lipschitz domain satisfying \(1.5\). Suppose \(u_{\delta,\varepsilon} \in H^1(D_{6r})\) is a weak solution of \(\text{div}(A_{\delta}^{\varepsilon} \nabla u_{\delta,\varepsilon}) = 0\) in \(D_{6r}\) with \(u_{\delta,\varepsilon} = 0\) on \(\Delta_{6r}\). Then
\[
\left( \int_{D_r} |A_{\delta}^{\varepsilon} \nabla u_{\delta,\varepsilon}|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{D_{2r}} |A_{\delta}^{\varepsilon} \nabla u_{\delta,\varepsilon}|^2 \, dx \right)^{1/2},
\]
where \(p > 2\) and \(C > 1\) depend only on \(d, \mu, \kappa, \omega,\) and the Lipschitz character of \(\Omega\). Moreover, if \(r \geq 2d\varepsilon\),
\[
\left( \int_{D_r} |\nabla u_{\delta,\varepsilon}|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{D_{6r} \cap \omega} |\nabla u_{\delta,\varepsilon}|^2 \, dx \right)^{1/2}.
\]
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Proof. The proof is similar to that of Theorem 8.1. To show (8.10), by (8.3), it suffices to prove that if \( \text{div}(A_\delta^x \nabla u_{\varepsilon, \delta}) = 0 \) in \( D_{4r} \) and \( u_{\varepsilon, \delta} = 0 \) on \( \Delta_{4r} \), then
\[
\left( \int_{D_{4r}} |A_\delta^x \nabla u_{\varepsilon, \delta}|^q \right)^{1/q} \leq C \left( \int_{D_{4r}} |A_\delta^x \nabla u_{\varepsilon, \delta}|^r \right)^{1/r},
\]
for some \( q < 2 \) and \( C > 1 \), depending at most on \( d, \mu, \kappa, \omega \), and the Lipschitz character of \( \Omega \). By dilation we may assume \( \varepsilon = 1 \). In view of the condition (1.5) as well as the interior estimate (8.3), the case \( 0 < r < 10d \) follows from the reverse H"older inequalities for solutions of \( \text{div}(A \nabla u) = 0 \), where \( q = \frac{2d}{d+2} \) for \( d \geq 3 \) and \( 1 < q < 2 \) for \( d = 2 \). Suppose \( r \geq 10d \) and \( d \geq 3 \) (the case \( d = 2 \) is similar). Let \( \tilde{u} \in H^q(D_{2r}) \) be an extension of \( u_{1, \delta}|_{\Delta_{2r}} \) such that \( \tilde{u} = 0 \) on \( \Delta_{2r} \) and \( \|
abla \tilde{u}\|_{L^q(D_r)} \leq C \|
abla u_{1, \delta}\|_{L^q(D_{2r} \cap \omega)} \),
where \( q = \frac{2d}{d+2} \). Then
\[
\left( \int_{D_{r} \cap \omega} |u|^2 \right)^{1/2} \leq C \left( \int_{D_r} |	ilde{u}|^2 \right)^{1/2} \leq C \left( \int_{D_{2r} \cap \omega} |
abla \tilde{u}|^q \right)^{1/q} \leq C \left( \int_{D_{2r} \cap \omega} |u|^q \right)^{1/q}.
\]
This, together with the observation,
\[
\|u\|_{L^2(D_r \cap F)} \leq \|u\|_{L^2(D_r)} \leq C \|
abla u\|_{L^q(D_r)}
\]
gives the Sobolev inequality,
\[
\frac{1}{r} \left( \int_{D_r} |\Lambda_\delta u|^2 \right)^{1/2} \leq C \left( \int_{D_{2r}} |
abla u|^q \right)^{1/q}.
\]
The desired estimate (8.12) follows from (8.13) and the Caccioppoli inequality (5.7).

To show (8.11), we assume \( \varepsilon = 1 \). It follows from (8.9) that
\[
\left( \int_{D_r} |
abla u_{1, \delta}|^p \right)^{1/p} \leq C \left( \int_{D_{3r}} |
abla u_{1, \delta}|^p \right)^{1/p} \leq C \left( \int_{D_{3r} \cap \omega} |
abla u_{1, \delta}|^2 \right)^{1/2} \leq C \left( \int_{D_{3r} \cap \omega} |\Lambda_\delta u_{1, \delta}|^2 \right)^{1/2},
\]
where \( 2 < p < p_0 \) and we have used (8.10) for the second inequality and (5.4) for the last.

\[\Box\]

9 Proof of Theorem 1.1

Throughout this section we assume that \( A \) satisfies the ellipticity condition (1.1), the periodicity condition (1.2) and the H"older continuity condition (1.12). We also assume \( A \) is symmetric.

Lemma 9.1. Let \( u_{\varepsilon, \delta} \in H^1(D_{3r}) \) be a weak solution of \( -\text{div}(A_\delta^x \nabla u_{\varepsilon, \delta}) = \text{div}(h) \) in \( D_{3r} \) with \( u_{\varepsilon, \delta} = 0 \) on \( \Delta_{3r} \). Assume that \( r \geq 4d\varepsilon \). Let
\[
s(x) = \left( \int_{B(x, d\varepsilon) \cap \Omega} |\Lambda_\delta^x \nabla u_{\varepsilon, \delta}|^2 \right)^{1/2}.
\]

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Then
\[ \left( \int_{D_r} |s|^p \right)^{1/p} \leq C \left( \int_{D_{2r}} |s|^2 \right)^{1/2} + C \left( \int_{D_{2r}} |h|^p \right)^{1/p}, \]  
where \( p > 2 \) and \( C > 0 \) depend only on \( d, \mu, \omega, \kappa, \) and the Lipschitz character of \( \Omega. \)

**Proof.** By the self-improving property of the reverse Hölder inequality, it suffices to prove that if \( y_0 \in D_r \) and \( 0 < t < cr, \) then
\[ \left( \int_{\Omega(y_0,t)} |s|^q \right)^{1/q} \leq C \left( \int_{\Omega(y_0,2t)} |s|^2 \right)^{1/2} + \theta \left( \int_{\Omega(y_0,2t)} |s|^2 \right)^{1/2} \]
for some \( 1 < q < 2 \) and \( \theta < 1, \) where \( \Omega(y_0,t) = \Omega(y_0) \cap \Omega. \) In view of the definition of \( s(x), \) the inequality \((9.2)\) is trivial for \( 0 < t \leq 4\varepsilon. \) For the case \( 4\varepsilon < t < r, \) by Fubini’s Theorem and Hölder’s inequality, it is enough to show that
\[ \left( \int_{\Omega(y_0,t)} |A_\delta^2 \nabla u_{\varepsilon,\delta}|^2 \right)^{1/2} \leq C \left( \int_{\Omega(y_0,ct)} |A_\delta^2 \nabla u_{\varepsilon,\delta}|^q \right)^{1/q} + C \left( \int_{\Omega(y_0,ct)} |h|^2 \right)^{1/2} + \theta \left( \int_{\Omega(y_0,ct)} |A_\delta^2 \nabla u_{\varepsilon,\delta}|^2 \right)^{1/2}, \]
for some \( 1 < q < 2 \) and \( \theta \in (0,1) \) sufficiently small. We will only consider the case \( y_0 = x_0 \in \Delta_r. \)
The other case may be reduced either to this case or to the case where \( B(y_0,2t) \subset D_{2r}, \) which may be handled by using \((8.7).\)

To this end, we note that for \( \varphi \in H_0^1(D_{3t}), \)
\[ \int_{D_{3t}} A_\delta^2 \nabla u_{\varepsilon,\delta} \cdot \nabla \varphi \, dx = - \int_{D_{2t}} h \cdot \nabla \varphi \, dx. \]  
By letting \( \varphi = u_{\varepsilon,\delta} \xi^2, \) where \( \xi \in C^1_0(B(x_0,2t)), \) in \((9.4)\) and using the Cauchy inequality, we obtain
\[ \delta^4 \int_{D_{2t}} |\nabla u_{\varepsilon,\delta}|^2 \xi^2 \, dx \leq C \delta^2 \int_{D_{2t}} |A_\delta^2 u_{\varepsilon,\delta}|^2 |\nabla \xi|^2 \, dx + C \int_{D_{2t}} |h|^2 |\xi|^2 \, dx. \]  
It follows that
\[ \delta^2 \|\nabla u_{\varepsilon,\delta}\|_{L^2(D_{3t/2})} \leq C t^{-1} \|A_\delta^2 u_{\varepsilon,\delta}\|_{L^2(D_{2t})} + C \|h\|_{L^2(D_{2t}).} \]
Next, let \( \varphi = \tilde{u} \xi^2 \) in \((9.4), \) where \( \tilde{u} \) is an extension of \( u_{\varepsilon,\delta}|_{D_{3t/2}} \) such that \( \tilde{u} = 0 \) on \( \Delta_{2t} \) and
\[ \|\nabla \tilde{u}\|_{L^q(D_{3t/2})} \leq C_q \|\nabla u_{\varepsilon,\delta}\|_{L^q(D_{2t})}. \]
for all \( 1 < q < 2. \) This gives
\[ \int_{D_{2t}} \xi^2 A_\delta^2 \nabla u_{\varepsilon,\delta} \cdot \nabla \tilde{u} \, dx + 2 \int_{D_{2t}} \xi (A_\delta^2 \nabla u_{\varepsilon,\delta} \cdot \nabla \xi) \tilde{u} \, dx = - \int_{D_{2t}} (h \cdot \nabla \tilde{u}) \xi^2 \, dx - 2 \int_{D_{2t}} (h \cdot \nabla \xi) \tilde{u} \xi \, dx. \]
It follows that for \( \xi \in C^1_0(B(x_0,3t/2)), \)
\[ \int_{D_{2t}} |\nabla u_{\varepsilon,\delta}|^2 \xi^2 \, dx \leq C \delta^2 \int_{D_{2t}} |\nabla u_{\varepsilon,\delta}| |\nabla \tilde{u}|^2 \xi^2 \, dx + C \int_{D_{2t}} |A_\delta^2 \nabla u_{\varepsilon,\delta}| |A_\delta^2 \tilde{u}| |\xi| |\nabla \xi| \, dx \]
\[ + C \int_{D_{2t}} |h||\nabla \tilde{u}|^2 \xi^2 \, dx + C \int_{D_{2t}} |h||\tilde{u}||\xi||\nabla \xi| \, dx. \]
Choose $\xi \in C_0^0(B(x_0,3t/2))$ such that $\xi = 1$ on $B(x_0,t)$ and $|\nabla \xi| \leq Ct^{-1}$. By splitting the second term in the right-hand side into integrals over $D_{2t}^c$ and $D_{2t} \setminus D_{2t}^c$ and using the Cauchy inequality on the integral over $D_{2t}^c$, we see that
\[
\int_{D_{2t}^c} |\nabla u_{\epsilon,\delta}|^2 \, dx \leq C \delta^2 \int_{D_{3t/2}} |\nabla u_{\epsilon,\delta}| |\nabla \tilde{u}| \, dx + C t^{-2} \int_{D_{3t/2}} |\tilde{u}|^2 \, dx
+ C \delta^4 \int_{D_{3t/2}} |\nabla u_{\epsilon,\delta}|^2 \, dx + C \int_{D_{3t/2}} |h| |\nabla \tilde{u}| \, dx + C t^{-1} \int_{D_{3t/2}} |h| |\tilde{u}| \, dx
\leq C \delta^4 \int_{D_{3t/2}} |\nabla u_{\epsilon,\delta}|^2 \, dx + \theta \int_{D_{3t/2}} |\nabla \tilde{u}|^2 \, dx
+ C t^{-2} \int_{D_{3t/2}} |\tilde{u}|^2 \, dx + C \int_{D_{3t/2}} |h|^2 \, dx.
\]
This, together with (9.6), (9.7), and (8.13), gives
\[
\left(\int_{D_t} |[\Lambda_\delta]^{-2} \nabla u_{\epsilon,\delta}|^2 \, dx \right)^{1/2} \leq C \delta \left(\int_{D_{2r}} |[\Lambda_\delta]^{-2} \nabla u_{\epsilon,\delta}|^q \, dx \right)^{1/q} + C \theta \left(\int_{D_{2r}} |[\Lambda_\delta]^{-2} \nabla u_{\epsilon,\delta}|^2 \, dx \right)^{1/2}
+ C \int_{D_{2r}} |h|^2 \, dx,
\]
for some $q < 2$ and any $\theta \in (0,1)$. As a result, we have proved (9.3) for the case $y_0 \in \Delta_r$. 

**Theorem 9.2.** Let $u_{\epsilon,\delta} \in H^1(D_{2r})$ be a weak solution of $\text{div}(A_\delta \nabla u_{\epsilon,\delta}) = 0$ in $D_{2r}$ with $u_{\epsilon,\delta} = 0$ on $\Delta_{2r}$. Then
\[
\left(\int_{\Delta_r} |\nabla u_{\epsilon,\delta}|^p \, d\sigma \right)^{1/p} \leq C \left(\int_{D_{2r}} |\nabla u_{\epsilon,\delta}|^2 \, dx \right)^{1/2},
\]
for some $p > 2$ and $C > 1$ depending only on $d$, $\mu$, $\omega$, $\kappa$, $(M,\sigma)$ in (1.12), and the Lipschitz character of $\Omega$.

**Proof.** Since $A_\delta(x) = A(x/\epsilon)$ for $x \in \Omega$ with $\text{dist}(x,\partial \Omega) < \epsilon \varepsilon$, the inequality (9.8) for $0 < r < C \varepsilon$ follows from the case $\varepsilon = 1$, by a rescaling argument. The Hölder continuity condition and the symmetry condition are used. See [8] for references for the case $\varepsilon = 1$ and $0 < r < C$. To treat the large-scale case $r \geq C \varepsilon$, by a rescaling argument, we may assume $r = 1$. By using the estimate (9.8) for the case $r = c \varepsilon$ and a simple covering argument, we see that
\[
\int_{\Delta_{c}} |\nabla u_{\epsilon,\delta}|^p \, d\sigma \leq C \epsilon \int_{\Sigma_{c\epsilon,5/4}} T([\Lambda_\delta]^{-2} \nabla u_{\epsilon,\delta})^p \, dx,
\]
where $T(G)(x)$ denotes the $L^2$ average of $|G|$ over $B(x,d\varepsilon) \cap \Omega$, and
\[
\Sigma_{c\epsilon,5/4} = \{ x \in D_{5/4} : \text{dist}(x,\partial \Omega) < c \epsilon \}.
\]
To bound the right-hand side of (9.9), we let $t \in (3/2,2)$, $v$ and $w_{\epsilon,\delta}$ be the same as in the proof of Theorem 7.3. It follows from the proof of Lemma 8.2 that
\[
-\text{div}(A_\delta \nabla w_{\epsilon,\delta}) = \text{div}(h) \quad \text{in} \ D_t,
\]
and
\[
h = -(A_\delta - A_\delta^0)(\nabla v - S_\epsilon(\eta_\epsilon(\nabla v))) + \varepsilon A_\delta \chi_\delta(x/\epsilon) \nabla S_\epsilon(\eta_\epsilon(\nabla v))
- \varepsilon \phi_\epsilon(x/\epsilon) \nabla S_\epsilon(\eta_\epsilon(\nabla v)).
\]

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Observe that
\[ \|h\|_{L^p(D_t)} \leq C\|\nabla v - S_\varepsilon(\nabla v)\|_{L^p(D_t)} + C\varepsilon\|\nabla(\nabla_\varepsilon(\nabla v))\|_{L^p(D_t)} \]
\[ \leq C\|\nabla v\|_{L^p(D_t(\varepsilon \delta))} + C\varepsilon\|\nabla^2 v\|_{L^p(D_t \setminus D_t(\varepsilon \delta))}, \]
where \( D_t(s) = \{ x \in D_t : \text{dist}(x, \partial D_t) < s \} \). By the \( L^p \) estimate for the regularity problem for the operator \( \text{div}(A_\varepsilon \nabla) \), and (7.9),
\[ \|h\|_{L^p(D_t)} \leq C\varepsilon^{-1/p}\|N(\nabla u_\delta)\|_{L^p(\partial D_t)} \leq C\varepsilon^{1/p}\|\nabla u_\delta\|_{L^p(\partial D_t)} \]
\[ \leq C\varepsilon^{1/p}\|\nabla u_{\varepsilon,\delta}\|_{L^p(D_2)}, \]
where \( p > 2 \) depends only on \( d, \mu, \omega, \kappa \), and the Lipschitz character of \( \Omega \). We now use Lemma 9.1 to obtain
\[ \left( \frac{1}{\varepsilon} \int_{\Sigma_{\varepsilon,5/4}} |T([A_\delta^\varepsilon]^{1/2}\nabla u_{\varepsilon,\delta})|^p \right)^{1/p} \]
\[ \leq \left( \frac{1}{\varepsilon} \int_{\Sigma_{\varepsilon,5/4}} |T([A_\delta^\varepsilon]^{1/2}\nabla w_{\varepsilon,\delta})|^p \right)^{1/p} + \left( \frac{1}{\varepsilon} \int_{\Sigma_{\varepsilon,5/4}} |T(\nabla v)|^p \right)^{1/p} \]
\[ \leq C\|\nabla u_{\varepsilon,\delta}\|_{L^p(D_2)} + C\|N(\nabla v)\|_{L^p(\partial D_t)} \]
\[ \leq C\|\nabla u_{\varepsilon,\delta}\|_{L^2(D_3)}, \]
where we have used \( (7.15) \) for the last inequality. This completes the proof.

**Proof of Theorem 1.1 for the case \( 0 < \delta \leq 1 \).** By dilation we may assume \( \text{diam}(\Omega) = 1 \). Let \( u_{\varepsilon,\delta} \) be a weak solution of \( \text{div}(A_\delta^\varepsilon \nabla u_{\varepsilon,\delta}) = 0 \) in \( \Omega \) with \( u_{\varepsilon,\delta} = f \) on \( \partial \Omega \), where \( f \in H^1(\partial \Omega) \). Let \( \mathcal{M}_{\partial \Omega} \) denote the Hardy-Littlewood maximal operator on \( \partial \Omega \). Define
\[ \tilde{N}(u)(y) = \sup \left\{ |u(x)| : x \in \Omega^\varepsilon \text{ and } |x - y| < \tilde{C}_0 \text{dist}(x, \partial \Omega) \right\} \]
for \( y \in \partial \Omega \). Using the fact \( \Sigma_{\kappa \varepsilon} \subset \Omega^\varepsilon \) and the inequality (15), we see that
\[ N(u_{\varepsilon,\delta}) \leq C\tilde{N}(u_{\varepsilon,\delta}) \text{ on } \partial \Omega. \]
We will show that
\[ \tilde{N}(u_{\varepsilon,\delta}) \leq C \left[ \mathcal{M}_{\partial \Omega}(|f|^q) \right]^{1/q}, \]
(9.11)
where $1 < q < 2$ and $C > 0$ depend only on $d$, $\mu$, $\omega$, $\kappa$, and the Lipschitz character of $\Omega$. This would imply that $\|N(u_{\epsilon, \delta})\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}$ for any $q < p \leq \infty$. The Hardy-Littlewood maximal operator $M_{\partial\Omega}$ in (9.11) is defined by

$$M_{\partial\Omega}(h)(y) = \sup \left\{ \frac{\int_{B(y,r) \cap \partial\Omega} |h| \, d\sigma} {r^{d-1}} : 0 < r < \text{diam}(\partial\Omega) \right\}$$

for $y \in \partial\Omega$, and it is well known that $\|M_{\partial\Omega}(h)\|_{L^p(\partial\Omega)} \leq C_p \|h\|_{L^p(\partial\Omega)}$ for $1 < p < \infty$.

To prove (9.11), we fix $y_0 \in \partial\Omega$ and $x_0 \in \Omega^\varepsilon$ with $r = |x_0 - y_0| < \tilde{C}_0 \text{dist}(x_0, \partial\Omega)$. Using the Green function representation,

$$u_{\epsilon, \delta}(x_0) = -\int_{\partial\Omega} \frac{\partial} {\partial \nu_\epsilon(x)} \{G_{\epsilon, \delta}(x_0, y)\} f(y) \, d\sigma(y),$$

it is not hard to see that (9.11) follows from the estimates,

$$\left( \int_{B(y_0, cr) \cap \partial\Omega} |\nabla G_{\epsilon, \delta}(x_0, y)|^p \, d\sigma(y) \right)^{1/p} \leq \frac{C} {r^{d-1}}, \quad (9.12)$$

$$\left( \int_{B(z,R) \cap \partial\Omega} |\nabla G_{\epsilon, \delta}(x_0, y)|^p \, d\sigma(y) \right)^{1/p} \leq \frac{C} {R^{d-1}} \left( \frac{r} {R} \right) \sigma, \quad (9.13)$$

for some $p > 2$ and $\sigma \in (0, 1)$, where $z \in \partial\Omega$, $|z - y_0| \geq cr$, and $R \approx |z - y_0|$. Moreover, since $\text{div}(A_\delta \nabla G_{\epsilon, \delta}(x_0, y)) = 0$ in $\Omega \setminus \{x_0\}$ and $G_{\epsilon, \delta}(x_0, y) = 0$ for $y \in \partial\Omega$, in view of (9.8), we only need to show that

$$\left( \int_{B(y_0, cr) \cap \Omega} |\nabla G_{\epsilon, \delta}(x_0, y)|^2 \, dy \right)^{1/2} \leq \frac{C} {r^{d-1}}, \quad (9.14)$$

$$\left( \int_{B(z,R) \cap \Omega} |\nabla G_{\epsilon, \delta}(x_0, y)|^2 \, dy \right)^{1/2} \leq \frac{C} {R^{d-1}} \left( \frac{r} {R} \right) \sigma. \quad (9.15)$$

Furthermore, by Caccioppoli’s inequality in Remark 5.2, we reduce (9.14)-(9.15) to

$$\left( \int_{B(y_0, cr) \cap \Omega^\varepsilon} |G_{\epsilon, \delta}(x_0, y)|^2 \, dy \right)^{1/2} \leq \frac{C} {r^{d-2}}, \quad (9.16)$$

$$\left( \int_{B(z,R) \cap \Omega^\varepsilon} |G_{\epsilon, \delta}(x_0, y)|^2 \, dy \right)^{1/2} \leq \frac{C} {R^{d-2}} \left( \frac{r} {R} \right) \sigma. \quad (9.17)$$

Finally, we point out that since $x_0 \in \Omega^\varepsilon$, estimates (9.16)-(9.17) follow from (6.9) (see Remark 5.3 for the case $d = 2$).

**Proof of Theorem 1.1 for the case $\delta = 0$.** Let $f \in H^1(\partial\Omega)$. By the classical theory for mixed boundary value problems, there exists a unique $u_{\epsilon, 0} \in H^1(\Omega^\varepsilon)$ satisfying (1.17). We extend $u_{\epsilon, 0}$ from $\Omega^\varepsilon$ to $\Omega$ by solving the Dirichlet problem (1.17) for each $\varepsilon F_k \subset \Omega$. Let $0 < \delta \leq 1$ and $u_{\epsilon, \delta} \in H^1(\Omega)$ be the unique solution of (1.19). Then

$$\int_{\Omega} A_\delta^\varepsilon \nabla (u_{\epsilon, \delta} - u_{\epsilon, 0}) \cdot \nabla \psi \, dx = -\delta^2 \int_{\Omega \cap \varepsilon F} A(x/\varepsilon) \nabla u_{\epsilon, 0} \cdot \nabla \psi \, dx \quad (9.18)$$
for any \( \psi \in H^1_0(\Omega) \). By letting \( \psi = u_{\varepsilon, \delta} - u_{\varepsilon, 0} \) in (9.18) and using the Cauchy inequality we obtain

\[
\int_{\Omega^c} |\nabla (u_{\varepsilon, \delta} - u_{\varepsilon, 0})|^2 \, dx \leq C \delta^2 \int_{\Omega \cap F} |\nabla u_{\varepsilon, 0}|^2 \, dx,
\]

where \( C \) depends only on \( \mu \). Note that

\[
\int_{\Omega \cap F} |\nabla u_{\varepsilon, 0}|^2 \, dx \leq C \int_{\Omega^c} |\nabla u_{\varepsilon, 0}|^2 \, dx \leq C \|f\|_{H^{1/2}(\partial \Omega)}^2.
\]

This, together with (9.19) and Lemma 2.2, shows that

\[
\|u_{\varepsilon, \delta} - u_{\varepsilon, 0}\|_{H^1(\Omega)} \leq C \|\nabla (u_{\varepsilon, \delta} - u_{\varepsilon, 0})\|_{L^2(\Omega)}
\]

\[
\leq C \|\nabla (u_{\varepsilon, \delta} - u_{\varepsilon, 0})\|_{L^2(\Omega^c)} \to 0 \quad \text{as} \quad \delta \to 0.
\]

Using \( \|N(u_{\varepsilon, \delta})\|_{L^2(\partial \Omega)} \) in (9.21) and Lemma 2.2, by a limiting argument, we obtain

\[
\|N(u_{\varepsilon, 0})\|_{L^2(\partial \Omega)} \leq C \|f\|_{L^2(\partial \Omega)}.
\]

A density argument gives (9.21) for solutions with any boundary data \( f \in L^2(\partial \Omega) \).

We end this section with a localized version of Theorem 1.1. For \( y \in \partial \Omega \), define

\[
N_t(u)(y) = \sup \left( \int_{B(y, \delta(y)/4)} |u|^2 \right)^{1/2},
\]

where the supremum is taken over all \( x \in \Omega \) with \( d(x) < t \) and \( |x - y| < C_0 d(x) \).

**Theorem 9.3.** Assume \( A \) and \( \Omega \) satisfy the same conditions as in Theorem 1.1. Let \( u = u_{\varepsilon, \delta} \in H^1(D_{4r}) \) be a weak solution of \( \text{div}(A^\varepsilon \nabla u) = 0 \) in \( D_{4r} \), where \( r > 4\varepsilon \).

Then

\[
\int_{D_4} |N_r(u)|^2 \, d\sigma \leq C \int_{D_{4r}} |u|^2 \, dx + \frac{C}{r} \int_{D_{4r}} |u|^2 \, dx
\]

where \( C \) depends only on \( d, \mu, \kappa, \), and the Lipschitz character of \( \Omega \).

**Proof.** By dilation we may assume \( r = 1 \) and \( \varepsilon > 0 \) is sufficiently small. We may also assume \( 0 \in \partial \Omega \) and

\[
D_4 = B(0, 4) \cap \Omega = B(0, 4) \cap \{(x', x_d) : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi(x')\},
\]

\[
\Delta_4 = B(0, 4) \cap \partial \Omega = B(0, 4) \cap \{(x', \psi(x')) : x' \in \mathbb{R}^{d-1}\}.
\]

Using conditions (1.3) and (1.5) we may construct a family \( \{\Omega_t\} \), where \( t \in J \subset (3, 4) \), of Lipschitz domains with uniform Lipschitz character and satisfying the following conditions:

- \( D_2 \subset \Omega_t \subset D_4 \) and \( \Delta_2 \subset \partial \Omega_t \);
- dist(\( \partial \Omega_t ; \varepsilon F \)) \( \geq c_1 \varepsilon \),

\[
\int_{t \in J} \int_{\partial \Omega_t \setminus \Delta_4} |v|^2 \, d\sigma \, dt \leq C_1 \int_{D_4} |v|^2 \, dx
\]

for any \( v \in H^1(D_4) \), and \( |J| \geq c_1 \), where \( C_1 > 0, c_1 > 0 \) depend only on \( d, \kappa, \omega, \) and the Lipschitz character of \( \Omega \). Note that

\[
\int_{\Delta_4} |N_1(w)|^2 \, d\sigma \leq C \int_{\partial \Omega_t} |N(w)|^2 \, d\sigma \leq C \int_{\partial \Omega_t} |u|^2 \, d\sigma
\]

\[
\leq C \int_{\Delta_4} |u|^2 \, d\sigma + C \int_{\partial \Omega_t \setminus \Delta_4} |u|^2 \, d\sigma,
\]

where we have used Theorem 1.1 for the second inequality. By integrating the inequalities above in \( t \) over \( J \), we obtain (9.23) for \( r = 1 \).

\[\square\]
10 Proof of Theorem 1.2

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ satisfying (1.5). By dilation we assume $\text{diam} (\Omega) = 1$. Let $u = u_{\varepsilon, \delta}$ be a weak solution of $\text{div}(A_{\varepsilon} \nabla u) = 0$ in $\Omega$ with $u = f$ on $\partial \Omega$, where $f \in H^1(\partial \Omega)$. Since $A_{\varepsilon}(x) = A(x/\varepsilon)$ if $d(x) < \kappa \varepsilon$, we obtain

$$
\int_{B(y, c_1 \varepsilon) \cap \partial \Omega} |N_{c_1 \varepsilon}(\nabla u)|^2 d\sigma \leq C \int_{B(y, \varepsilon) \cap \partial \Omega} |\nabla_{\text{tan}}f|^2 d\sigma + \frac{C}{\varepsilon} \int_{B(y, \varepsilon) \cap \Omega} |\nabla u|^2 dx
$$

(10.1)

for any $y \in \partial \Omega$, where $c_1 > 0$ is sufficiently small. We remark that the estimate (10.1) is known for $\varepsilon = 1$ under the conditions that $A$ is elliptic, symmetric, and Hölder continuous [8]. The case $0 < \varepsilon < 1$ follows from the case $\varepsilon = 1$ by a rescaling argument. Note that if $c_1 \varepsilon \leq d(x) < \alpha \varepsilon$ and $\alpha > 1$ is large, the $L^2$ average of $|\nabla u_{\varepsilon, \delta}|$ over $B(x, d(x)/4)$ is controlled by its average over $B(y, \alpha \varepsilon) \cap \Omega$, where $y \in \partial \Omega$ and $d(x) < C_0 d(x)$. It follows that

$$
\int_{B(y, c_1 \varepsilon) \cap \partial \Omega} |N_{\alpha \varepsilon}(\nabla u)|^2 d\sigma \leq C \int_{B(y, \varepsilon) \cap \partial \Omega} |\nabla_{\text{tan}}f|^2 d\sigma + \frac{C}{\varepsilon} \int_{B(y, \varepsilon) \cap \Omega} |\nabla u|^2 dx,
$$

where $C$ depends on $\alpha$. By covering $\partial \Omega$ with balls with radius $c_1 \varepsilon$, we see that

$$
\int_{\partial \Omega} |N_{\alpha \varepsilon}(\nabla u)|^2 d\sigma \leq C \int_{\partial \Omega} |\nabla_{\text{tan}}f|^2 d\sigma + \frac{C}{\varepsilon} \int_{\Omega} |\nabla u|^2 dx
$$

(10.2)

$$
\leq C \|f\|_{H^1(\partial \Omega)}^2,
$$

where $\Sigma_{C_\varepsilon} = \{x \in \Omega : d(x) < C\varepsilon\}$ and we have used the boundary layer estimate (7.1) for the last inequality.

To treat the case where $d(x) \geq \alpha \varepsilon$ and $\alpha > 1$ is large, we will use the estimate in Theorem 9.3. Without loss of generality we assume that $0 \in \partial \Omega$ and

$$
B(0, r_0) \cap \Omega = B(0, r_0) \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \psi(x')\}.
$$

We will show that if $y = (y', \psi(y')) \in \partial \Omega$ and $|y| \leq c r_0$, then

$$
N(\nabla u)(y) \leq C M_{\partial \Omega}\left\{ M_{\text{rad}}(Q_{\varepsilon}(u)) + |\nabla_{\text{tan}}f| + N_{\alpha \varepsilon}(\nabla u)\right\}(y),
$$

(10.3)

where

$$
Q_{\varepsilon}(u)(x', x_d) = \left( u(x', x_d + \varepsilon) - u(x', x_d) \right)/\varepsilon
$$

(10.4)

is a difference operator, and

$$
M_{\text{rad}}(u)(y', \psi(y')) = \sup_{(\alpha - 1) \varepsilon < s < c r_0} \int_{B((y', \psi(y'))+s, d\varepsilon)} |u|
$$

(10.5)

a radial maximal function. To see (10.3), we fix $y \in \partial \Omega$ with $|y| \leq c r_0$. Let $z \in \Omega$ with $d(z) \geq \alpha \varepsilon$ and $r = |z - y| < C_0 d(z)$. Note that by (4.4) and (4.25)

$$
\left( \int_{B(z, d(z)/4)} |\nabla u|^2 \right)^{1/2} \leq \frac{C}{r^{d+1}} \int_{|x_d - z_d| < d(z)/2} \int_{|x' - z'| < d(z)/2} |u - E| dx' dx_d.
$$
where $E \in \mathbb{R}$, and that for $x = (x', x_d)$,
\[
|u(x) - E| \leq |u(x', x_d) - u(x', x_d - \varepsilon)| + \cdots + |u(x', x_d - k\varepsilon) - u(x', \psi(x'))| + |u(x', \psi(x')) - E| \\
= \varepsilon|Q_\varepsilon(u)(x', x_d - \varepsilon)| + \cdots + \varepsilon|Q_\varepsilon(u)(x', x_d - (k - 1)\varepsilon)| \\
+ |u(x', x_d - k\varepsilon) - u(x', \psi(x'))| + |u(x', \psi(x')) - E|
\]
where $k = k(x) \approx \varepsilon^{-1}r$ is chosen so that $\psi(x') \leq x_d - k\varepsilon < \psi(x') + \varepsilon$. Let
\[
E = \int_{|x' - z'| < d(\varepsilon)/2} u(x', \psi(x')) \, dx'.
\]
Using
\[
|u(x', x_d - k\varepsilon) - u(x', \psi(x'))| \leq \int_0^{k\varepsilon} |\nabla u(x', \psi(x') + s)| \, ds
\]
and Poincaré’s inequality for the term $|u(x', \psi(x')) - E|$, we see that
\[
\left( \int_{B(z, d(\varepsilon)/4)} |\nabla u|^2 \right)^{1/2} \leq CM_{\partial\Omega} \left\{ M_{\text{rad}}(Q_\varepsilon(u)) + |\nabla_{\text{tan}} f| + N_{\alpha\varepsilon}(\nabla u) \right\}(y),
\]
which yields (10.3). It follows from (10.3) and (10.2) that
\[
\int_{B(0, cr_0) \cap \Omega} |N(\nabla u)|^2 \, d\sigma \leq C \int_{B(0, 2cr_0) \cap \partial\Omega} |M_{\text{rad}}(Q_\varepsilon(u))|^2 \, d\sigma + C\|f\|^2_{H^1(\partial\Omega)}. \tag{10.6}
\]
Finally, to handle $M_{\text{rad}}(Q_\varepsilon(u))$, we note that
\[
\text{div}(A_\delta^T \nabla Q_\varepsilon(u)) = 0
\]
in $B(0, 100cr_0) \cap \Omega$. By applying Theorem 9.3 to $Q_\varepsilon(u)$, we obtain
\[
\int_{B(0, 2cr_0) \cap \partial\Omega} |M_{\text{rad}}(Q_\varepsilon(u))|^2 \, d\sigma \leq C \int_{B(0, 3cr_0) \cap \partial\Omega} |N_{3cr_0}(Q_\varepsilon(u))|^2 \, d\sigma \\
\leq C \int_{B(0, 12cr_0) \cap \partial\Omega} |Q_\varepsilon(u)|^2 \, d\sigma + C \int_{B(0, 12cr_0) \cap \Omega} |Q_\varepsilon(u)|^2 \, dx \\
\leq C \varepsilon \int_{\Sigma_{C\varepsilon}} |\nabla u|^2 \, dx + C \int_{\Omega} |\nabla u|^2 \, dx,
\]
where, for the last step, we have used the inequality,
\[
|Q_\varepsilon(u)(x', x_d)| \leq \left( \frac{1}{\varepsilon} \int_{x_d}^{x_d + \varepsilon} |\nabla u(x', s)|^2 \, ds \right)^{1/2}.
\]
This, together with (7.1), (10.6), (2.6), and a simple covering argument, gives (1.14).
11 Proof of Theorem 1.3

By dilation we may assume diam(Ω) = 1. Let $u = u_{\varepsilon, \delta} \in H^1(\Omega)$ be a weak solution of the Neumann problem (1.10) with $g \in L^2(\partial\Omega)$. It follows from the proof of Theorem 1.2 that

$$\int_{\partial\Omega} |N(\nabla u)|^2 \, d\sigma \leq C \int_{\partial\Omega} |\nabla u|^2 \, d\sigma + C \int_{\Omega} |\nabla u|^2 \, dx. \quad (11.1)$$

Since $A^\delta_\varepsilon(x) = A(x/\varepsilon)$ for $d(x) < \kappa\varepsilon$, we obtain

$$\int_{B(y, c\varepsilon) \cap \partial\Omega} |\nabla u|^2 \, d\sigma \leq C \int_{B(y, \kappa\varepsilon) \cap \partial\Omega} |g|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{B(y, \kappa\varepsilon) \cap \Omega} |\nabla u|^2 \, dx \quad (11.2)$$

for any $y \in \partial\Omega$, where $c > 0$ is sufficiently small. We point out that the estimate (11.2) is known for $\varepsilon = 1$, under the conditions that $A$ is elliptic, symmetric, and Hölder continuous [8]. The case $0 < \varepsilon < 1$ follows from the case $\varepsilon = 1$ by a rescaling argument. By a covering argument it follows from (11.2) that

$$\int_{\partial\Omega} |\nabla u|^2 \, d\sigma \leq C \int_{\partial\Omega} |g|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{\Sigma_{\kappa\varepsilon}} |\nabla u|^2 \, dx$$

$$\leq C \|g\|^2_{L^2(\partial\Omega)};$$

where we have used (7.6) for the last inequality. This, together with (11.1) and the energy estimate (2.8), gives (1.15).

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