Asymptotic dynamics of Hamiltonian polymatrix replicators

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Received 25 October 2021; revised 28 March 2023
Accepted for publication 25 April 2023
Published 10 May 2023

Abstract
In a previous paper (Alishah et al 2019 Nonlinearity 33 469) we have studied flows defined on polytopes, presenting a method to encapsulate its asymptotic dynamics along the edge-vertex heteroclinic network. Using this result we study here the Hamiltonian character of the asymptotic dynamics of conservative polymatrix replicator systems. Our main result states that for such conservative polymatrix replicator systems the map describing its asymptotic dynamics is Hamiltonian with respect to some appropriate Poisson structure.

Keywords: Hamiltonian polymatrix replicator system, Poisson structure, Poincaré map, asymptotic dynamics, heteroclinic network.
Mathematics Subject Classification numbers: 34D05, 37J06, 37J46, 53D17, 91A22

(Some figures may appear in colour only in the online journal)

1. Introduction
A new method to study the asymptotic dynamics of flows defined on polytopes was presented in [6]. This method allows us to analyse the asymptotic dynamics of flows defined on
polytopes along the edge-vertex heteroclinic network. Examples of such dynamical systems arise naturally in the context of evolutionary game theory developed by J. Maynard Smith and G. R. Price [19]. One such example is the polymatrix replicator, introduced in [4, 5], which extends the classes of replicator and bimatrix replicators in [17, 18, 20].

The polymatrix replicator is a system of ordinary differential equations that models the time evolution of behavioural strategies of individuals in a stratified population, whose flows evolve on prisms (products of simplices). See for instance the polymatrix replicator system used in [1] to model social corruption.

Several other works on polymatrix replicators have recently appeared in the literature. Namely, in [13] the author extends to these systems the concept of permanence initially studied in replicator systems, in [14] the authors study the occurrence of strange attractors for 3D polymatrix replicators, and in [15] a new method to determine the asymptotic stability of cycles in a heteroclinic network is presented. The method presented in [15] is based on the procedure developed in [6]. We believe that the study of dynamical systems in other works (e.g. [2, 11, 16]) may benefit from the method in [6] and results of this work for conservative models.

In [4] the authors have introduced the subclass of conservative polymatrix replicators (see definition 5.1) which are Hamiltonian systems with respect to appropriate Poisson structures.

For Hamiltonian vector fields on symplectic manifolds it is well known that the Poincaré map preserves the induced symplectic structure on any transversal section. In this paper we extend this fact to Hamiltonian systems on Poisson manifolds, showing that any transversal section inherits a Poisson structure that is preserved by the Poincaré map.

Notice that to apply the method in [6] to conservative polymatrix replicators it is useful to know in advance that the conservative character of a polymatrix replicator flow has a reflection on the corresponding asymptotic dynamics, e.g. existence of invariant quantities and eigenvalue symmetries. Our main result states that for conservative polymatrix replicators, the asymptotic dynamics is also Hamiltonian with respect to some appropriate Poisson structure (see theorem 7.15). This a priori knowledge allows one to compute the Casimirs of the asymptotic Poisson structure, which via dimension reduction facilitates the analysis of the asymptotic dynamics.

The paper is organized as follows. In section 2 we provide a short introduction to polymatrix replicators. In section 3 we recall the method in [6], outlining the construction of the asymptotic dynamics for a large class of flows on polytopes that includes the polymatrix replicators. At the end we state our main results. In section 4 we define Poincaré maps for Hamiltonian systems on Poisson manifolds. In section 5 we state the definition and some basic results for the class of conservative polymatrix replicators, that we also designate as Hamiltonian polymatrix replicators. In section 6 we review the technique developed in [6] to analyse the asymptotic dynamics of a flow on a polytope along its edge-vertex heteroclinic network. In particular we review the main definitions and results for the polymatrix replicator $X_{\alpha,\gamma}$ on the polytope $\Gamma_{\alpha,\gamma}$. In section 7 we analyse the Hamiltonian character of Poincaré maps in the case of Hamiltonian polymatrix replicators. Finally, in section 8 we present an example of a five-dimensional Hamiltonian polymatrix replicator to illustrate the main concepts and results of this paper. The graphics of this section were produced with Wolfram Mathematica and Geogebra software.

2. Polymatrix replicator

In this section we define the system of ordinary differential equations that we designate as polymatrix replicator.
Consider a population divided in \( p \) groups where each group is labelled by an integer \( \alpha \in \{1, \ldots, p\} \), and the individuals of each group \( \alpha \) have exactly \( n_\alpha \) strategies to interact with other members of the population (including of the same group). In total we have \( n = \sum_{\alpha=1}^p n_\alpha \) strategies that we label by the integers \( i \in \{1, \ldots, n\} \), denoting by

\[
[a] := \{n_1 + \cdots + n_{\alpha-1} + 1, \ldots, n_1 + \cdots + n_\alpha\} \subset \mathbb{N}
\]

the set (interval) of strategies of group \( \alpha \).

Given \( \alpha, \beta \in \{1, \ldots, p\} \), consider a real \( n_\alpha \times n_\beta \) matrix, say \( A^{\alpha, \beta} \), whose entries \( a^{\alpha, \beta}_{ij} \), with \( i \in [\alpha] \) and \( j \in [\beta] \), represent the payoff of an individual of the group \( \alpha \) using the \( i \)th strategy when interacting with an individual of the group \( \beta \) using the \( j \)th strategy. Thus the matrix \( A \) with entries \( a^{\alpha, \beta}_{ij} \), where \( \alpha, \beta \in \{1, \ldots, p\} \), \( i \in [\alpha] \) and \( j \in [\beta] \), is a square matrix of order \( n = n_1 + \cdots + n_p \), consisting of the block matrices \( A^{\alpha, \beta} \).

Let \( \mathfrak{a} = (n_1, \ldots, n_p) \). The state of the population is described by a point \( x = (x^\alpha)_{1 \leq \alpha \leq p} \) in the polytope

\[
\Gamma_\mathfrak{a} := \Delta^{n_1-1} \times \cdots \times \Delta^{n_p-1} \subset \mathbb{R}^n,
\]

where \( \Delta^{n_\alpha-1} = \{x \in [\alpha] : \sum_{i \in [\alpha]} x_i^{\alpha} = 1\} \), \( x^\alpha = (x_i^\alpha)_{i \in [\alpha]} \) and the entry \( x_i^\alpha \) represents the usage frequency of the \( i \)th strategy within the group \( \alpha \). We denote by \( \partial \Gamma_\mathfrak{a} \) the boundary of \( \Gamma_\mathfrak{a} \).

Assuming random encounters between individuals, for each group \( \alpha \in \{1, \ldots, p\} \), the average payoff of a strategy \( i \in [\alpha] \) within a population with state \( x \) is given by

\[
(Ax)_i = \sum_{\beta=1}^p (A^{\alpha, \beta})_{i \beta} x^\beta = \sum_{\beta=1}^p \sum_k a^{\alpha, \beta}_{ik} x^\beta_k,
\]

where the overall average payoff of group \( \alpha \) is given by

\[
\sum_{i \in [\alpha]} x_i^\alpha (Ax)_i.
\]

Demanding that the logarithmic growth rate of the frequency of each strategy \( i \in [\alpha], \alpha \in \{1, \ldots, p\} \), is equal to the payoff difference between strategy \( i \) and the overall average payoff of group \( \alpha \) yields the system of ordinary differential equations defined on the polytope \( \Gamma_\mathfrak{a} \).

\[
\frac{dx_i^\alpha}{dt} = x_i^\alpha \left( (Ax)_i - \sum_{i \in [\alpha]} x_i^\alpha (Ax)_i \right), \quad \alpha \in \{1, \ldots, p\}, i \in [\alpha],
\]

that we designate as polymatrix replicator.

If \( p = 1 \) equation (2.1) becomes the usual replicator equation with payoff matrix \( A \). When \( p = 2 \) and \( A^{11} = A^{22} = 0 \) are null matrices, equation (2.1) becomes the bimatrix replicator equation with payoff matrices \( A^{12} \) and \( (A^{21})^\top \).

The flow \( \phi^\mathfrak{a}_{\Gamma_\mathfrak{a}} \) of this equation leaves the polytope \( \Gamma_\mathfrak{a} \) invariant. The proof of this fact is analogous to that for the bimatrix replicator equation, see [12, section 10.3]. Hence, by compactness of \( \Gamma_\mathfrak{a} \), the flow \( \phi^\mathfrak{a}_{\Gamma_\mathfrak{a}} \) is complete. From now on the term polymatrix replicator will also refer to the flow \( \phi^\mathfrak{a}_{\Gamma_\mathfrak{a}} \) and the underlying vector field on \( \Gamma_\mathfrak{a} \) denoted by \( X^\mathfrak{a}_{\Gamma_\mathfrak{a}} \).

Given \( \mathfrak{a} = (n_1, \ldots, n_p) \), let

\[
\mathcal{I}_\mathfrak{a} := \{I \subseteq \{1, \ldots, n\} : \#(I \cap [\alpha]) \geq 1, \forall \alpha = 1, \ldots, p\}.
\]

A set \( I \in \mathcal{I}_\mathfrak{a} \) determines the facet \( \sigma_I := \{x \in \Gamma_\mathfrak{a} : x_I = 0, \forall j \notin I\} \) of \( \Gamma_\mathfrak{a} \). The correspondence between labels in \( \mathcal{I}_\mathfrak{a} \) and facets of \( \Gamma_\mathfrak{a} \) is bijective.
Remark 2.1. The partition of $\Gamma_n$ into the interiors $\sigma_i^\circ := \text{int}(\sigma_i)$, with $I \in \mathcal{I}_n$, is a smooth stratification of $\Gamma_n$ with strata $\sigma_i^\circ$. Every stratum $\sigma_i^\circ$ is a connected open submanifold and for any pair $\sigma_i^\circ, \sigma_j^\circ$ if $\sigma_i^\circ \cap \sigma_j^\circ \neq \emptyset$ then $\sigma_i \subset \sigma_j$. For more on smooth stratification see [10] and references therein.

For a set $I \in \mathcal{I}_n$ consider the pair $(n_I^\circ ; A_I)$, where $n_I^\circ = (n_I^1, \ldots, n_I^p)$ with $n_{i}^\circ = \#(I \cap [\alpha])$, and $A_I = [a_{\alpha}]_{\alpha \in I}$. The following result says that the restriction of a polymatrix replicator to a face is still a polymatrix replicator. This result was first stated in [4, proposition 3].

Proposition 2.2. Given $I \in \mathcal{I}_n$, the facet $\sigma_I$ of $\Gamma_n$ is invariant under the flow of $X_{\alpha ; A}$ and the restriction of (2.1) to $\sigma_I$ is the polymatrix replicator $X_{\alpha ; A_I}$.

For a fixed $a = (n_1, \ldots, n_p)$ the correspondence $A \mapsto X_{\alpha ; A}$ is linear with a non-trivial kernel that is characterized in [4, proposition 1].

In [4, proposition 2] we characterize interior equilibria as follows:

Proposition 2.3. given a polymatrix replicator $X_{\alpha ; A}$, a point $q \in \text{int}(\Gamma_n)$ is an equilibrium of $X_{\alpha ; A}$ iff $(Aq)_i = (Aq)_j$ for all $i, j \in [\alpha]$ and $\alpha = 1, \ldots, p$. In particular, the set of interior equilibria of $X_{\alpha ; A}$ is the intersection of some affine subspace with $\text{int}(\Gamma_n)$.

3. Outline of the construction and main results

We now outline the construction of the asymptotic dynamics for a large class of flows on polytopes that includes the polymatrix replicators. We state the main results at the end of this section.

A polytope is a compact convex set in some Euclidean space obtained as the intersection of finitely many half-spaces. A polytope is called simple if the number of edges (or faces) incident with each vertex equals the polytope’s dimension. The phase space of polymatrix replicators, that are prisms given by a finite product of simplices, are examples of simple polytopes. In [6] we consider analytic vector fields defined on simple polytopes which have the property of being tangent to every face of the polytope. Such vector fields induce complete flows on the polytope which leave all faces invariant. Vertices of the polytope are singularities of the vector field, while edges without singularities, called flowing edges, consist of single orbits flowing between two endpoint vertices. The vertices and flowing edges form a heteroclinic network of the vector field. The purpose of this construction is to analyse the asymptotic dynamics of the vector field along this one-dimensional skeleton. Throughout the text we assume that every vector field is non-degenerate. This means that the transversal derivative of the vector field is never identically zero along any facet of the polytope.

The analysis of the vector field’s dynamics along its edge-vertex heteroclinic network makes use of Poincaré maps between cross sections transversal to the flowing edges. Any Poincaré map along a heteroclinic or homoclinic orbit is a composition of two types of maps, global and local Poincaré maps. A global map, denoted by $P_{\gamma}$, is defined in a tubular neighbourhood of any flowing-edge $\gamma$. It maps points between two cross sections $\Sigma_{-\gamma}^-$ and $\Sigma_{+\gamma}^+$ transversal to the flow along the edge $\gamma$. A local map, denoted by $P_{\gamma}$, is defined in a neighbourhood of any vertex $v$. For any pair of flowing-edges $\gamma, \gamma'$ such that $v$ is both the ending point of $\gamma'$ and the starting point of $\gamma$, the local map $P_{\gamma}$ takes points from $\Sigma_{+\gamma'}^+$ to $\Sigma_{-\gamma}^-$. See figure 1 (6, figure 1).

Asymptotically, the nonlinear character of the global Poincaré maps fade away as we approach a heteroclinic orbit. This means that these nonlinearities are irrelevant for the
asymptotic analysis. For regular\textsuperscript{4} vector fields, the skeleton character at a vertex, defined as the set of eigenvalues of the tangent map along the edge eigen-directions, completely determines the asymptotic behaviour of the local Poincaré map at that vertex.

To describe the limit dynamical behaviour we introduce the dual cone of a polytope where the asymptotic piecewise linear dynamics unfolds. This space lies inside $\mathbb{R}^F$, where $F$ is the set of the polytope’s facets. The dual cone of a $d$-dimensional simple polytope $\Gamma$ is the union

$$C^*(\Gamma) := \bigcup_{v \in V} \Pi_v,$$

where for each vertex $v$, $\Pi_v$ is the $d$-dimensional sector consisting of points $y \in \mathbb{R}^F$ with non-negative coordinates such that $y_{\sigma} = 0$ for every facet that does not contain $v$. See figure 2 in [6].

Given a vector field $X$ on a $d$-dimensional polytope $\Gamma \subset \mathbb{R}^d$, we now describe a rescaling change of coordinates $\Psi^X_\epsilon$, depending on a blow up parameter $\epsilon$.

This change of coordinates maps tubular neighbourhoods of edges and vertices to the dual cone $C^*(\Gamma)$. For instance, the tubular neighbourhood $N_v$ of a vertex $v$ is defined as follows. Consider a system $(x_1, \ldots, x_d)$ of affine coordinates around $v$, which assigns coordinates $(0, \ldots, 0)$ to $v$ and such that the hyperplanes $x_j = 0$ are precisely the facets of the polytope through $v$. Then $N_v$ is defined by

$$N_v := \{ p \in \Gamma^d : 0 \leq x_j(p) \leq 1 \text{ for } 1 \leq j \leq d \}.$$

The sets $\{ x_j = 0 \} \cap N_v$ are called the outer facets of $N_v$. The remaining facets of $N_v$, defined by equations like $x_1 = 1$, are called the inner facets of $N_v$. The previous cross sections $\Sigma_{+}^\gamma$ can be chosen to match these inner facets of the neighbourhoods $N_v$.

The rescaling change of coordinates $\Psi^X_\epsilon$ maps $N_v$ to the sector $\Pi_v$. Enumerating $F$ so that the facets through $v$ are precisely $\sigma_1, \ldots, \sigma_d$, the map $\Psi^X_\epsilon$ is defined on the neighbourhood $N_v$ by

$$\Psi^X_\epsilon(q) := (-\epsilon^2 \log x_1(q), \ldots, -\epsilon^2 \log x_d(q), 0, \ldots, 0).$$

Similarly, given an edge $\gamma$, $\Psi^X_\epsilon$ maps a tubular neighbourhood $N_\gamma$ of $\gamma$ to the facet sector $\Pi_\gamma := \Pi_v \cap \Pi_{v'}$ of $\Pi_v$ where $v'$ is the other endpoint of $\gamma$. The map $\Psi^X_\epsilon$ sends interior facets of

\textsuperscript{4} As we will see, for non-degenerate conservative polymatrix replicators regularity is automatic. See definition 6.7 and proposition 6.8.
asymptotic dynamics of the flow of $X$. We consider a subset $S$ of flowing-edges with the property that every heteroclinic cycle goes through at least one edge in $S$. Such sets are called structural sets. The flow of $X$ induces a Poincaré map $P_S$ on the system of cross sections $S := \cup_{\gamma \in S} \Sigma_\gamma$. Each branch of the Poincaré map $P_S$ follows a heteroclinic path starting with an edge in $S$ and ending at its first return to another edge in $S$. Consider next the corresponding system of cross sections $S := \cup_{\gamma \in S} \Sigma_\gamma$ of $C^*({\Gamma})$ which are transversal to the skeleton vector field $\chi$. The flow of $\chi$ induces a piecewise linear first return map $\pi_S: D_S \subset S \rightarrow S$, that we call the skeleton flow map. The domain $D_S$ of $\pi_S$ is a finite union of open convex cones, each associated to a different heteroclinic path which starts and ends in $S$ but otherwise consists of flowing edges not in $S$. In some cases, see proposition 6.14, the map $\pi_S$ becomes a closed dynamical system.

We can now recall the main result in [6] which says that under the rescaling change of coordinates $\Psi^\epsilon$, the Poincaré map $P_S$ converges in the $C^\infty$ topology to the skeleton flow map $\pi_S$, in the sense that the following limit holds

$$\lim_{\epsilon \rightarrow 0} \Psi^\epsilon \circ P_S \circ (\Psi^\epsilon)^{-1} = \pi_S$$

with uniform convergence of the map and its derivatives over any compact set contained in the domain $D_S \subset S$.

For each facet $\sigma$ of the polytope, fix an affine function $R^d \ni q \mapsto x_\sigma(q) \in R$ such that $x_\sigma \geq 0$ on $\Gamma^d$ and $\sigma = \Gamma^d \cap \{x_\sigma = 0\}$. Representing the polytope as $\Gamma^d = \cap_{\sigma \in F} \{x_\sigma \geq 0\}$, consider the (finite dimensional) space of analytic functions $h : \text{int}(\Gamma^d) \rightarrow R$ of the form

$$h(q) = \sum_{\sigma \in F} c_\sigma \log x_\sigma(q) \quad (c_\sigma \in R) \quad (3.1)$$

which rescale to piecewise linear functions $\eta : C^*({\Gamma^d}) \rightarrow R$ of the form

$$\eta(y) := \sum_{\sigma \in F} c_\sigma y_\sigma$$

in the sense that $\eta = \lim_{\epsilon \rightarrow 0} \epsilon^{-2}(h \circ (\Psi^\epsilon)^{-1})$, see [6, proposition 8.2]. Notice that if all $c_\sigma$ have the same sign then $h : \text{int}(\Gamma^d) \rightarrow R$ and $\eta : C^*({\Gamma^d}) \rightarrow R$ are proper functions and all their levels sets are compact. If $h$ is invariant under the flow of $X$, i.e. $h \circ P_S = h$, then $\eta$ is also invariant under the skeleton flow, i.e. $\eta \circ \pi_S = \eta$. Thus any integral of motion of the form (3.1) for the vector field $X$ determines a piecewise linear integral of motion for the skeleton flow.

The following are the main results of this manuscript.
Theorem A. Given a polymatrix replicator vector field \( X \) on a prism \( \Gamma^d \) assume that:

(1) \( X \) is conservative (see definition 5.1, which means that \( X \) is Hamiltonian w.r.t. to a Poisson structure introduced in [4]);
(2) \( X \) is non-degenerate (see definition 6.2);
(3) \( S \) is a set of flowing edges which is a structural set for the skeleton vector field \( \chi \) of \( X \) (see definition 6.12). Moreover, \( \pi_S : D_S \subseteq \Pi_S \rightarrow \Pi_S \) is the corresponding skeleton flow map.

Then there exists a Poisson bracket \( \{\cdot,\cdot\}_S \) on \( \Pi_S \) for which the map \( \pi_S : D_S \rightarrow \Pi_S \) is Poisson.

Proof. Follows from theorem 7.18.

Non trivial structural sets may not exist, e.g. the second example in section 4 of [4] (see figure 1(b) there). There are also examples where the \( \pi_S \)-orbit of Lebesgue almost every point in \( D_S \) eventually leaves \( D_S \), e.g. the Cartesian product of the previous model in \( \Delta^2 \times \Delta^1 \) with a Hamiltonian replicator in \( \Delta^2 \) whose boundary consists of a single heteroclinic cycle. The following result shows however that there are plenty of conservative polymatrix replicators where the dynamics of the skeleton flow map \( \pi_S \) is not trivial as above.

Theorem B. Given a polymatrix replicator vector field \( X \) on a prism \( \Gamma^d \) under the assumptions of theorem A, if \( X \) admits an equilibrium point in \( \text{int}(\Gamma^d) \) and all diagonal entries of the matrix \( D \) in definition 5.1 have the same sign then \( \hat{D}_S := \bigcap_{n \in \mathbb{Z}} \pi^{-n}_S(D_S) \) has full Lebesgue measure in \( \Pi_S \).

In section 8 we give an example which satisfies the assumptions and conclusions of theorems A and B. This example exhibits chaotic behaviour, something that can be rigorously proven using the method in [6]. The advantage of theorems A and B here is that they give us the a priori knowledge that the skeleton flow map is Poisson and defined almost everywhere. Moreover, the explicit form of the Poisson structure on \( \Pi_S \) (see section 7) allows one to compute its Casimirs. In the example this provides an extra invariant of motion for the skeleton flow map which combined with the initial Hamiltonian reduces the dimension of the phase space by 2, in fact by 3 if we consider the skeleton flow map. Hence we know in advance that the restriction of the skeleton flow map to an appropriate level set is an area preserving piecewise affine map on some convex quadrilateral.

4. Poisson Poincaré maps

In this section we will define Poincaré map for Hamiltonian systems on Poisson manifolds. For Hamiltonian vector fields on symplectic manifolds it is well known that the Poincaré map preserves the induced symplectic structure on any transversal section (see [21, theorem 1.8]). We extend this fact to Hamiltonian systems on Poisson manifolds, showing that any transversal section inherits a Poisson structure and the Poincaré map preserves this structure.

A Poisson manifold is a pair \((M,\pi)\) where \( M \) is a smooth manifold without boundary and \( \pi \) a Poisson structure on \( M \). Recall that a Poisson structure is a smooth bivector field \( \pi \) with the property that \( [\pi,\pi] = 0 \), where \([\cdot,\cdot]\) is the Schouten bracket (cf e.g. [9]). The bivector field \( \pi \) defines a vector bundle map

\[
\pi^\sharp : T^*M \rightarrow TM \quad \text{by} \quad \xi \mapsto \pi(\xi,\cdot).
\]
The image of this map is an integrable singular distribution which integrates to a symplectic foliation, i.e. a foliation whose leaves have a symplectic structure induced by the Poisson structure.

Notice that a Poisson structure can also be defined as a Lie bracket \{\cdot,\cdot\} on \(C^\infty(M)\times C^\infty(M)\) satisfying the Leibniz rule

\[ \{f,gh\} = \{f,g\}h + g\{f,h\}, \quad f,g,h \in C^\infty(M). \]

These two descriptions are related by \(\pi(df, dg) = \{f, g\}\). In a local coordinate chart \((U, x_1, \ldots, x_n)\), or equivalently when \(M = \mathbb{R}^n\), a Poisson bracket takes the form

\[ \{f,g\}(x) = (d_x f)' [\pi(y)(x)]_{ij} d_x g, \]

where \(\pi(x) = [\pi(y)(x)]_{ij} = [x_i x_j](x)\) is a skew symmetric matrix valued smooth function satisfying

\[ \sum_{i=1}^n \frac{\partial \pi_{ij}}{\partial x_i} \pi_{ik} + \frac{\partial \pi_{jk}}{\partial x_i} \pi_{ij} + \frac{\partial \pi_{ij}}{\partial x_i} \pi_{kj} = 0 \quad \forall i,j,k. \]

**Definition 4.1.** Let \((M, \{\cdot,\cdot\}_M)\) and \((N, \{\cdot,\cdot\}_N)\) be two Poisson manifolds. A smooth map \(\psi : M \to N\) will be called a Poisson map if

\[ \{f \circ \psi, h \circ \psi\}_M = \{f, h\}_N \circ \psi \quad \forall f, h \in C^\infty(N). \tag{4.2} \]

Using the map \(\pi^\sharp\), defined in (4.1), this condition reads as

\[ (d\psi)^* \pi^\sharp_M (d\psi)^* = \pi^\sharp_N \circ \psi, \tag{4.3} \]

where we use the notation \((d\psi)^*\) to denote the adjoint operator of \(d\psi\). Notice that, if \(d\psi\) is the Jacobian matrix of \(\psi\) in local coordinates, then the matrix representative of the pullback is \((d\psi)^\sharp\).

**Remark 4.2.** When \(\psi\) is a diffeomorphism and only one of the manifolds \(M\) or \(N\) is Poisson manifold, definition 4.1 can be used to push-forward or pullback the Poisson structure to the other manifold.

**Definition 4.3.** Let \((M, \pi)\) be a Poisson manifold. The Hamiltonian vector field associated to a given function \(H : M \to \mathbb{R}\) is defined by the derivation \(X_H(f) := \{H, f\}\), for \(f \in C^\infty(M)\), or equivalently \(X_H := \pi^\sharp(dH)\).

As in the symplectic case, to define the Poincaré map we will consider the transversal sections inside the level set of the Hamiltonian. We will show that such a transversal section is a Poisson transversal of the ambient Poisson manifold and naturally inherits a Poisson structure.

**Definition 4.4.** [8, definition 5.2] An embedded submanifold \(\Sigma \subset (M, \pi)\) is a Poisson transversal if it satisfies the condition

\[ \pi^\sharp(T_x \Sigma^\circ) + T_x \Sigma = T_x M, \quad \forall x \in N, \tag{4.4} \]

where \(T_x \Sigma^\circ \subset T_x^* M\) denotes the annihilator of \(T_x \Sigma^\circ\).

As noticed in [8], (4.4) is equivalent to its direct sum version

\[ \pi^\sharp(T_x \Sigma^\circ) \oplus T_x \Sigma = T_x M, \quad \forall x \in \Sigma. \]

In definition 4.4, instead of requiring that \(\Sigma\) is an embedded submanifold satisfying (4.4), one may equivalently ask that for any given symplectic leaf \(S \subset M\) the intersection \(\Sigma \cap S\) is a symplectic submanifold of \(S\) (see [8, proposition 5.6]). This furnishes \(\Sigma\) with a symplectic
foliation and consequently a Poisson structure. This Poisson bracket \( \{ \ldots \}_\Sigma \) can be calculated by
\[
\{f,g\}_\Sigma = \{\tilde{f},\tilde{g}\}|_\Sigma, \quad \forall f,g \in C^\infty(\Sigma),
\]
where \( \tilde{f}, \tilde{g} \) are extensions of \( f, g \) to \( M \) such that their differentials vanish on \( \pi^\sharp(T\Sigma) \).

**Remark 4.5.** Poisson transversals also appear in the literature under the name of *cosymplectic submanifolds*. We prefer to use the name Poisson transversals for two reasons. First, we will be working with transversal sections equipped with a Poisson structure so the name suits perfectly. Second, nowadays the term cosymplectic submanifold is widely used for another type of submanifolds.

We consider a Hamiltonian \( H \) on the \( m \)-dimensional Poisson manifold \( (M, \pi) \) and its associated Hamiltonian vector field defined by \( X_H = \{H, .\} = \pi^\sharp(dH) \). For a given point \( x_0 \in M \) let \( U \) be a neighbourhood around it such that \( X_H(x) \neq 0 \quad \forall x \in U \), and \( E_m \) be the energy surface passing through \( x_0 \), i.e. the connected component of \( H^{-1}(H(x_0)) \) containing \( x_0 \). We call *level transversal section* to \( X_H \) at a regular point \( x_0 \in M \) any \( (m-2) \)-dimensional transversal section \( \Sigma \subset E_m \cap U \) through \( x_0 \).

The following lemma shows that \( \Sigma \) is a Poisson transversal.

**Lemma 4.6.** Every level transversal section \( \Sigma \) is a Poisson transversal of \( M \).

**Proof.** Since \( d_{x_0}H \neq 0 \), there exists a function \( G \) locally defined in \( U \) (shrink \( U \) if necessary) and linearly independent of \( H \) such that
\[
\Sigma = E_m \cap U \cap G^{-1}(G(x_0)).
\]
Submanifold \( \Sigma \) being a level set means it is an embedded submanifold of \( M \) so we only need to verify (4.4). Clearly,
\[
\pi^\sharp_0((T\Sigma)^\circ) = \pi^\sharp_0(\text{R}dH \oplus \text{R}dG) = \text{R}X_H(x_0) \oplus \text{R}X_G(x_0).
\]
Consequently, (4.4) holds at \( x_0 \). Since (4.4) is an open condition, it holds in sufficiently small neighbourhood of \( x_0 \). \( \square \)

**Remark 4.7.** By transversality \( \pi_0(dH,dG) = d_{x_0}G(X_H) \neq 0 \). In other words the matrix
\[
\begin{pmatrix}
0 & \{H,G\} \\
\{G,H\} & 0
\end{pmatrix},
\]
is invertible. In general, if \( \Sigma = \bigcap_{i=1}^{2k} G_i^{-1}(0), \) where \( \{G_1, \ldots, G_{2k}\} \) are functions such that \( \{[G_i,G_j]\}(x)_{i,j} \) is an invertible matrix at all points \( x \in \Sigma \), then \( \Sigma \) is a Poisson transversal. For the Poisson transversal \( \Sigma \) defined in this way, the induced Poisson bracket can be calculated by
\[
\{f,g\}_\Sigma = \{f,g\} - [\{f,G_i\}']^\prime [\{G_i,G_j\}]^{-1} [\{G_i,g\}],
\]
where \( \{.,G_i\} \) is the column matrix with components \( \{.,G_i\} \), \( i = 1, \ldots, 2k \). Functions \( f \) and \( g \) in the right hand side of (4.6) are, with an abuse of notation, arbitrary extensions of \( f, g \) to \( M \). This abuse of notation is justified by the fact that (4.6) defines a Poisson bracket \( \{.,.\}_\text{Dirac} \) on \( M \), known as Dirac bracket. The restriction of a Dirac bracket to the Poisson transversal \( \Sigma = \bigcap_{i=1}^{2k} G_i^{-1}(0) \) coincides with the induced Poisson bracket of \( \Sigma \). We will use (4.6) to explicitly calculate the induced Poisson bracket, and use the notation \( \{.,.\}_\text{Dirac} \) for the Dirac bracket defined on \( M \) (as well as its restriction to the Poisson transversal \( \Sigma \)).
For a fixed time $t_0$, let $x_1 = \phi_H(t_0, x_0)$, where $\phi_H$ is the flow of the Hamiltonian vector field $X_H$, and let $\Sigma_0, \Sigma_1$ be level transversal sections at $x_0$ and $x_1$, respectively. As usual, a Poincaré map $P = \phi_H(\tau(x), x)$ can be defined from an appropriate neighbourhood of $x_0$ in $\Sigma_0$ to a neighbourhood of $x_1$ in $\Sigma_1$. The existence of the smooth function $\tau(x)$ is guaranteed by the Implicit Function theorem. We replace $\Sigma_0$ and $\Sigma_1$ by the domain and the image of the Poincaré map $P$.

By lemma 4.6 both $\Sigma_i$, $i = 0, 1$, are Poisson transversals equipped with Dirac brackets $\{ \ldots \}_{\text{Dirac}}$, $i = 0, 1$. We will show that the Poincaré map $P$ is a Poisson map (see definition 4.1).

**Proposition 4.8.** The Poincaré map

$$P : (\Sigma_0, \{ \ldots \}_{\text{Dirac}}) \rightarrow (\Sigma_1, \{ \ldots \}_{\text{Dirac}})$$

is a Poisson map.

**Proof.** For two given functions $f, g \in C^\infty(\Sigma_0)$ we need to show that

$$\{ f, g \}_{\text{Dirac}} \circ P = \{ f \circ P, g \circ P \}_{\text{Dirac}}.$$

Let us rewrite (4.6) using the notations $\bar{f}, \bar{g}$ for the extended functions.

$$\{ f, g \}_{\text{Dirac}} = \{ \bar{f}, \bar{g} \} - \begin{bmatrix} \{ f, H \} \\ \{ g, H \} \end{bmatrix} \begin{bmatrix} 0 & \{ H, G \} \\ \{ G, H \} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \{ H, \bar{g} \} \\ \{ \bar{f}, \bar{g} \} \end{bmatrix}.$$

If the differentials of the extended functions $\bar{f}, \bar{g}$ vanish on $X_H$, then $\{ f, g \}_{\text{Dirac}}$ is simply equal to $\{ \bar{f}, \bar{g} \}$. The Flow-box theorem permits to extend $f, g$ by

$$\bar{f}(\phi^\lambda_{x_0}(x)) = f(x), \quad \bar{g}(\phi^\lambda_{x_0}(x)) = g(x), \forall x \in \Sigma_0,$$

to an appropriate neighbourhood of the orbit connecting $x_0$ to $x_1$ which contains both $\Sigma_0$ and $\Sigma_1$. By definition, these extensions are constant along the orbits of $X_H$, i.e. $\{ \bar{f}, H \} = \{ \bar{g}, H \} = 0$. The proposition is proven if we show that $\{ \bar{f}, \bar{g} \}$ is also constant along the orbits of $X_H$, i.e. $\{ \{ \bar{f}, \bar{g} \}, H \} = 0$. Indeed, by the Jacobi identity we have

$$\{ \{ \bar{f}, \bar{g} \}, H \} = \{ \{ \bar{f}, H \}, \bar{g} \} + \{ \bar{f}, \{ \bar{g}, H \} \},$$

which finishes the proof.

5. Hamiltonian polymatrix replicators

In this section, for later reference, we state some definitions and basic properties from [4] about the class of conservative polymatrix replicators that we refer here as Hamiltonian polymatrix replicators.

**Definition 5.1.** [4, definition 3.7] A polymatrix replicator $X_{A,A}$ is said to be conservative if there exists:

(a) A point $q \in \mathbb{R}^n$, called formal equilibrium, such that $(Aq)_i = (Aq)_j$ for all $i, j \in [n]$, and all $\alpha = 1, \ldots, p$ and $\sum_{i \in [\alpha]} q_i = 1$;

(b) Matrices $A_0, D \in \text{Mat}_{n \times n}(\mathbb{R})$ such that

(i) $X_{A_0,D} = X_{A,A}$,

(ii) $A_0$ is a skew symmetric, and

(iii) $D = \text{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_p I_{n_p})$ with $\lambda_{\alpha} \neq 0$ for all $\alpha \in \{1, \ldots, p\}$.
The matrix $A_0$ will be referred to as a skew symmetric model for $X_{2^A}$, and $(\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}^*)^p$ as a scaling co-vector.

In [5], another characterization of conservative polymatrix replicators, using quadratic forms, is provided. Furthermore, in [3] the concept of conservative replicator equations (where $p = 1$) is generalized using Dirac structures.

In what follows, the vectors in $\mathbb{R}^n$, or $\mathbb{R}^{[\alpha]}$, are identified with column vectors. Let $I_n = (1, \ldots, 1)^T \in \mathbb{R}^n$. We will omit the subscript $n$ whenever the dimension of this vector is clear from the context. Similarly, we write $I = I_n$ for the $n \times n$ identity matrix. Given $x \in \mathbb{R}^n$, we denote by $D_x$ the $n \times n$ diagonal matrix $D_x := \text{diag}(x_1, \ldots, x_n)$. For each $\alpha \in \{1, \ldots, p\}$ we define the $n_\alpha \times n_\alpha$ matrix

$$T^\alpha_x := x^\alpha I^\prime - I,$$

and $T_x$ the $n \times n$ block diagonal matrix $T_x := \text{diag}(T^1_x, \ldots, T^p_x)$.

Given an anti-symmetric matrix $A_0$, we define the skew symmetric matrix valued mapping $\pi_{A_0} : \mathbb{R}^n \to \text{Mat}_{n \times n}(\mathbb{R})$

$$\pi_{A_0}(x) := \alpha \in (-1) T_x D_A A_0 D_A T^\alpha_x.$$

The interior of the polytope $\Gamma_n$, denoted by $\text{int}(\Gamma_n)$, equipped with $\pi_{A_0}$ is a Poisson manifold, see [4, theorem 3.5]. Furthermore, we have proved in [4, theorem 3.7] that given a conservative polymatrix replicator $X_{2^A}$ with formal equilibrium $q$, skew symmetric model $A_0$ and scaling co-vector $(\lambda_1, \ldots, \lambda_p)$, the vector field $X_{2^A}$ restricted to $\text{int}(\Gamma_n)$ is Hamiltonian with Hamiltonian function

$$h(x) = \sum_{\beta=1}^p \lambda_\beta \sum_{j \in [\beta]} q^\beta_j \log s^\beta_j.$$

(5.1)

6. Asymptotic dynamics

Given a polymatrix replicator $X_{2^A}$, the edges and vertices of the polytope $\Gamma_n$ form a (edge-vertex) heteroclinic network for the associated flow. In this section we recall the technique developed in [6] to analyse the asymptotic dynamics of a flow on a polytope along its edge-vertex heteroclinic network. In particular we review the main definitions and results for the polymatrix replicator $X_{2^A}$ on $\Gamma_n$.

Notice that we need to conveniently label vertices, edges and facets of the prism $\Gamma_n$, which forces us to merge notations from [4, 6].

The affine support of $\Gamma_n$ is the smallest affine subspace of $\mathbb{R}^n$ that contains $\Gamma_n$. It is the subspace $E = E_1 \times \ldots \times E_p$ where for $\alpha = 1, \ldots, p$,

$$E_\alpha := \left\{ x^\alpha \in \mathbb{R}^{[\alpha]} : \sum_{i \in [\alpha]} x^\alpha_i = 1 \right\}.$$

Following [6, definition 3.1] we introduce a defining family for the polytope $\Gamma_n$. The affine functions $\{f_i : \mathbb{R} \to \mathbb{R}\}_{1 \leq i \leq n}$ where $f_i(x) = x_i$, form a defining family for $\Gamma_n$ because they satisfy:

(a) $\Gamma_n = \bigcap_{1 \leq i \leq n} f^{-1}_i([0, +\infty])$,

(b) $\Gamma_n \cap f^{-1}_i(0) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$, and
(c) Given \( J \subseteq \{1, \ldots, n\} \) such that \( \Gamma_\mathbb{R} \cap \left( \bigcap_{i \in J} f_i^{-1}(0) \right) \neq \emptyset \), the linear 1-forms \((df)_p\) are linearly independent at every point \( p \in \bigcap_{i \in J} f_i^{-1}(0) \).

Next we introduce convenient labels for vertices, facets and edges of \( \Gamma_\mathbb{R} \). Let \((e_1, \ldots, e_n)\) be the canonical basis of \( \mathbb{R}^n \) and denote by \( \mathcal{V}_\mathbb{R} \) the Cartesian product \( \mathcal{V}_\mathbb{R} := \prod_{\alpha=1}^p [\alpha] \) which contains \( \prod_{\alpha=1}^p n_{\alpha} \) elements. Each label \( j = (j_1, \ldots, j_p) \in \mathcal{V}_\mathbb{R} \) determines the vertex \( v_j := (e_{j_1}, \ldots, e_{j_p}) \) of \( \Gamma_\mathbb{R} \). This labelling is one-to-one. The set \( \mathcal{F}_\mathbb{R} := \{1, 2, \ldots, n\} \) can be used to label the \( n \) facets of \( \Gamma_\mathbb{R} \). Each integer \( i \in \mathcal{F}_\mathbb{R} \) labels the facet \( \sigma_i := \Gamma_\mathbb{R} \cap \{ x_i = 0 \} \) of \( \Gamma_\mathbb{R} \). Edges can be labelled by the set \( \mathcal{E}_\mathbb{R} := \{ J \in \mathcal{F}_\mathbb{R} : \#J = p + 1 \} \). Given \( J \in \mathcal{E}_\mathbb{R} \) there exists a unique (unordered) pair of labels \( j_1, j_2 \in \mathcal{V}_\mathbb{R} \) such that \( J \) is the union of the strategies in \( j_1 \) and \( j_2 \). The label \( J \) determines the edge \( \gamma_J := \{ t v_{j_1} + (1-t)v_{j_2} : 0 \leq t \leq 1 \} \). Again the correspondence \( J \mapsto \gamma_J \) between labels \( J \in \mathcal{E}_\mathbb{R} \) and edges of \( \Gamma_\mathbb{R} \) is one-to-one.

Given a vertex \( v \) of \( \Gamma_\mathbb{R} \), we denote by \( F_v \) and \( E_v \), respectively the sets of facets and edges of \( \Gamma_\mathbb{R} \) that contain \( v \). Given \( j = (j_1, \ldots, j_p) \in \mathcal{V}_\mathbb{R} \)
\[
F_v = \{ \sigma_i : i \in \mathcal{F}_\mathbb{R} \setminus \{ j_1, \ldots, j_p \} \}
\]
and this set of facets contains exactly \( n - p = \dim(\Gamma_\mathbb{R}) \) elements.

Triples in
\[
C := \{ (v, \gamma, \sigma) \in V \times E \times F : \gamma \cap \sigma = \{ v \} \},
\]
are called corners. Any pair of elements in a corner uniquely determines the third one. Therefore, sometimes we will shortly refer to a corner \((v, \gamma, \sigma)\) as \((v, \gamma)\) or \((v, \sigma)\). An edge \( \gamma \) with endpoints \( v, v' \) determines two corners \((v, \gamma, \sigma)\) and \((v', \gamma, \sigma')\), called the end corners of \( \gamma \). The facets \( \sigma, \sigma' \) are referred to as the opposite facets of \( \gamma \).

**Remark 6.1.** In a small neighbourhood of a given vertex \( v = v_j \), where \( j = (j_1, \ldots, j_p) \in \mathcal{V}_\mathbb{R} \), the affine functions \( f_k : \Gamma_\mathbb{R} \to \mathbb{R}, f_k(x) := x_k, \) with \( k \in \mathcal{F}_\mathbb{R} \setminus \{ j_1, \ldots, j_p \} \), can be used as a coordinate system for \( \Gamma_\mathbb{R} \).

Given a polymatrix replicator \( X_{\mathbb{R}A} \) and a facet \( \sigma_i \) with \( i \in [\alpha], \alpha \in \{1, \ldots, p\} \), the \( i \)th component of \( X_{\mathbb{R}A} \) is given by
\[
df_i(X_{\mathbb{R}A}) = x_i \left( (Ax)_i - \sum_{\beta=1}^p (x^\alpha)^T A^{\alpha,\beta} x^\beta \right).
\]

As defined in section 3, a vector field on a polytope is non-degenerate if its transversal derivative is never identically zero along any facet of the polytope.

**Definition 6.2.** A polymatrix replicator \( X_{\mathbb{R}A} \) is said to be non-degenerate if for any \( i \in \mathcal{F}_\mathbb{R} \), the function \( H_i : \Gamma_\mathbb{R} \to \mathbb{R}, \)
\[
H_i(x) := f_i(x)^{-1} \df_i(X_{\mathbb{R}A})(x) = (Ax)_i - \sum_{\beta=1}^p (x^\alpha)^T A^{\alpha,\beta} x^\beta
\]
is not identically zero along \( \sigma_i \).

Clearly generic polymatrix replicators are non-degenerate. Using the concept of order of a vector field along a facet [6, definition 4.2], \( X_{\mathbb{R}A} \) is non-degenerate if and only if all facets of \( \Gamma_\mathbb{R} \) have order 1. From now on we will only consider non-degenerate polymatrix replicators.
Definition 6.3. The skeleton character of a polymatrix replicator $X_{A^n}$ is defined to be the matrix $\chi := (\chi_\sigma^v)(v, \sigma, j) \in V \times F$ where

$$
\chi_\sigma^v := \begin{cases} 
-H_\sigma(v), & v \in \sigma \\
0 & \text{otherwise}
\end{cases}
$$

where $H_\sigma$ stands for $H_i$ when $\sigma = \sigma_i$ with $i \in F$. For a fixed vertex $v$, the vector $\chi_\sigma^v := (\chi_\sigma^v)_{\sigma \in F}$ is referred to as the skeleton character at $v$.

Remark 6.4. Given a corner $(v, \gamma, \sigma)$ of $\Gamma_{A^n}$, $H_\sigma(v)$ is the eigenvalue of the tangent map $(dX_{A^n})_v$ along the eigen-direction parallel to $\gamma$.

Proposition 6.5. If $X_{A^n}$ is a non-degenerate polymatrix replicator for every vertex $v = v_i$ with label $j = (j_1, \ldots, j_p) \in V_{A^n}$ and every facet $\sigma = \sigma_i$ with $i \in F$, and $i \in [\alpha]$ the skeleton character of $X_{A^n}$ is given by

$$
\chi_\sigma^v = \begin{cases} 
\sum_{j=1}^p (a_{j_0, j_1} - a_{j_1, j_0}) & \text{if } v \in \sigma \\
0 & \text{otherwise}
\end{cases}
$$

Proof. Apply proposition 2.2 to an edge (1-dimensional face).

Remark 6.6. For a given corner $(v, \gamma, \sigma)$ of $\Gamma_{A^n}$,

- If $\chi_\sigma^v < 0$ then $v$ is the $\omega$-limit of an orbit in $\gamma$, and
- If $\chi_\sigma^v > 0$ then $v$ is the $\alpha$-limit of an orbit in $\gamma$.

Let $\gamma$ be an edge with endpoints $v$ and $v'$ and opposite facets $\sigma$ and $\sigma'$, respectively. This means that $(v, \gamma, \sigma)$ and $(v', \gamma, \sigma')$ are corners of $\Gamma_{A^n}$. If $X_{A^n}$ does not have singularities in int$(\gamma)$, then int$(\gamma)$ consists of a single heteroclinic orbit with $\alpha$-limit $v$ and $\omega$-limit $v'$ if and only if $\chi_\sigma^v < 0$ and $\chi_{\sigma'}^{v'} > 0$. This type of edges will be referred to as flowing edges. The vertices $v = s(\gamma)$ and $v' = t(\gamma)$ are respectively called the source and target of the flowing edge $\gamma$ and we will write $\gamma \xrightarrow{v} \gamma'$ to express it. When the two characters $\chi_\sigma^v = \chi_{\sigma'}^{v'} = 0$ the edge $\gamma$ is called neutral.

Definition 6.7. A polymatrix replicator $X_{A^n}$ is called regular\footnote{This concept of regularity is more restrictive than the one in [6, definition 6.3]. This stronger regularity is needed to prove the main theorem.} if it is non-degenerate and moreover every edge is either neutral or a flowing edge.

Proposition 6.8. Given a polymatrix replicator vector field $X_{A^n}$, if it is conservative and non-degenerate then $X_{A^n}$ is regular.

Proof. Because $X_{A^n}$ is conservative, we can assume that $A = BD$ for some skew symmetric matrix $B$ and some invertible diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with the property that $\lambda_i = \lambda_j$ for $i, j \in [\alpha]$ and any group $\alpha$. Notice that $a_{ii} = 0$ and $a_{ij} = b_{ij} \lambda_j$ with $b_{ij} = -b_{ji}$ for $i, j = 1, \ldots, n$.

Next consider an edge $\gamma$ with endpoints $v_j$ and $v_{j'}$, where $j = (j_1, \ldots, j_p)$ and $j' = (j'_1, \ldots, j'_p)$ are labels in $V_{A^n}$. Without loss of generality we may assume that $j_i \neq j'_i$ for $i = 2, \ldots, p$. Then $(v_j, \gamma, \sigma_{j_i})$ and $(v_{j'}, \gamma, \sigma_{j'_i})$ are the end corners of $\gamma$ and by proposition 6.5,
consider the vertex neighbourhood
\[ N_v := (6.2) := \left( \sum_{i=2}^{p} a_{j_ih} - a_{j'_ih} \right) \]
while using that \( \lambda_{j_i} = \lambda_{j'_i} \).

Hence \( \chi^{\nu}_{j_i} + \chi^{\nu'}_{j'_i} = 0 \), which implies that \( \gamma \) is either a neutral edge or a flowing edge. \( \square \)

Finally we define the edge skeleton’s tubular neighbourhood.

Given \( v = v_i \) with \( j = (j_1, \ldots, j_p) \in V_2 \) consider the vertex neighbourhood
\[ N_v := \{ q \in \Gamma_2 : 0 \leq f_k(q) \leq 1, \forall k \in \mathcal{F}_2 \setminus \{j_1, \ldots, j_p\} \} \].

Rescaling the defining functions \( f_k \) we may assume these neighbourhoods are pairwise disjoint. See remark 6.1.

For any edge \( \gamma \) with endpoints \( v \) and \( v' \) we define a tubular neighbourhood connecting \( N_v \) to \( N_{v'} \) by
\[ N_{\gamma} := \{ q \in \Gamma_2 \setminus (N_v \cup N_{v'}) : 0 \leq f_k(q) \leq 1, \forall k \in \mathcal{F}_2 \text{ with } \gamma \subset \sigma_k \} \].

Again we may assume that these neighbourhoods are pairwise disjoint between themselves.

Finally we define the edge skeleton’s tubular neighbourhood of \( \Gamma_2 \) to be
\[ N_{\Gamma_2} := (\cup_{v \in \mathcal{V}} N_v) \cup (\cup_{\gamma \in \mathcal{E}} N_{\gamma}). \] (6.1)

The next step is to define the rescaling map \( \Psi^{A}_{\epsilon} \) on \( N_{\Gamma_2} \setminus \partial \Gamma_2 \). See [6, definition 5.2]. We will write \( f_{\sigma} \) to denote the affine function \( f_k \) associated with the facet \( \sigma = \sigma_k \) with \( k \in \mathcal{F}_2 \).

**Definition 6.9.** Let \( \epsilon > 0 \) be a small parameter. The \( \epsilon \)-rescaling coordinate system
\[ \Psi^{A}_{\epsilon} : N_{\Gamma_2} \setminus \partial \Gamma_2 \to \mathbb{R}^F \]
maps \( q \in N_{\Gamma_2} \) to \( y := (y_{\sigma})_{\sigma \in \mathcal{F}} \) where

- If \( q \in N_v \) for some vertex \( v \):
  \[ y_{\sigma} = \begin{cases} -\epsilon^2 \log f_{\sigma}(q) & \text{if } v \in \sigma \\ 0 & \text{if } v \not\in \sigma \end{cases} \]

- If \( q \in N_{\gamma} \) for some edge \( \gamma \):
  \[ y_{\sigma} = \begin{cases} -\epsilon^2 \log f_{\sigma}(q) & \text{if } \gamma \subset \sigma \\ 0 & \text{if } \gamma \not\subset \sigma \end{cases} \]

We now turn to the space where these rescaling coordinates take values. For a given vertex \( v \in \mathcal{V} \) we define
\[ \Pi_v := \{ (y_{\sigma})_{\sigma \in \mathcal{F}} : y_{\sigma} = 0, \forall \sigma \not\in \mathcal{F}_v \} \]. (6.2)
where $\mathbb{R}_+ = [0, +\infty)$. Since $\{f_\sigma\}_{\sigma \in F_\gamma}$ is a coordinate system over $N_\gamma$ and the function $h : (0, 1] \to [0, +\infty)$, $h(x) := -\log x$, is a diffeomorphism, the restriction $\Psi^A_{\tau} : N_\gamma \setminus \partial \Gamma^\epsilon_\gamma \to \Pi_\gamma$ is also a diffeomorphism denoted by $\Psi^A_{\tau, \epsilon}$. If $\gamma$ is an edge connecting two corners $(\nu, \sigma)$ and $(\nu', \sigma')$, $F_\gamma \cap F_{\nu'} = \{\sigma \in F : \gamma \subset \sigma\}$ and we define

$$\Pi_\gamma := \{(y_\sigma)_{\sigma \in F} \in \mathbb{R}^F : y_\sigma = 0 \quad \text{when} \quad \gamma \not\subset \sigma\}. \quad (6.3)$$

Then $\Psi^A_{\tau} (N_\gamma \setminus \partial \Gamma^\epsilon_\gamma) = \Pi_\gamma = \Pi_\nu \cap \Pi_{\nu'}$ has dimension $d - 1$ while $\Pi_\nu = \Psi^A_{\tau} (N_\nu \setminus \partial \Gamma^\epsilon_\nu)$ has dimension $d$. In particular the map $\Psi^A_{\tau, \epsilon}$ is not injective over $N_\gamma$. See figure 9 in [6].

**Definition 6.10.** The dual cone of $\Gamma^\epsilon_\gamma$ is defined to be

$$C^*(\Gamma^\epsilon_\gamma) := \bigcup_{\nu \in \mathcal{D}} \Pi_\nu,$$

where $\Pi_\nu$ is the sector in (6.2).

Hence $\Psi^A_{\tau} : N_\nu \setminus \partial \Gamma^\epsilon_\nu \to C^*(\Gamma^\epsilon_\gamma)$.

Denote by $\{\varphi^{\epsilon}_{d, \nu} : \Gamma^\epsilon_\nu \to \Gamma^\epsilon_\gamma\}_{\epsilon \in \mathbb{R}}$ the flow of the vector field $X_{\nu A}$. Given a flowing edge $\gamma$ with source $\nu = s(\gamma)$ and target $\nu' = t(\gamma)$ we introduce the cross-sections

$$\Sigma^- := (\Psi^A_{\tau, \epsilon})^{-1} (\text{int}(\Pi_\gamma)) \quad \text{and} \quad \Sigma^+ := (\Psi^A_{\tau, \epsilon})^{-1} (\text{int}(\Pi_\gamma))$$

transversal to the flow $\varphi^{\epsilon}_{d, \nu}$. The sets $\Sigma^-_\gamma$ and $\Sigma^+_\gamma$ are inner facets of the tubular neighbourhoods $N_\nu$ and $N_{\nu'}$ respectively. Let $\mathcal{D}_\gamma$ be the set of points $x \in \Sigma^-_\gamma$ such that the forward orbit $\{\varphi^{\epsilon}_{d, \nu}(x) : t > 0\}$ has a first transversal intersection with $\Sigma^+_\gamma$. The global Poincaré map

$$P_{\gamma} : \mathcal{D}_\gamma \subset \Sigma^-_\gamma \to \Sigma^+_\gamma$$

is defined by $P_{\gamma} (x) := \varphi^{\tau(x)}_{d, \nu} (x)$, where

$$\tau(x) = \min\{ t > 0 : \varphi^{\epsilon}_{d, \nu}(x) \in \Sigma^+_\gamma \}.$$

Given a family of smooth functions $\{F_\gamma : \mathcal{U}_\gamma \to \mathbb{R}^n\}_{\gamma \in \mathcal{D}}$ with varying open domains $\mathcal{U}_\gamma$, and a smooth map $F : \mathcal{U} \to \mathbb{R}^n$ defined on an open set $\mathcal{U}$, we say that $\lim_{\gamma \to 0^+} F_\gamma = F$ in the $C^k$ topology if for every compact subset $K \subseteq \mathcal{U}$, we have $K \subseteq \mathcal{U}_\gamma$ for every small enough $\epsilon > 0$, and $F_\gamma$ converges uniformly to $F$ over the compact set $K$ with all its derivatives up to order $k$. As usual, $C^\infty$ convergence means convergence in the $C^k$ topology for all $k \geq 1$. It will be implicitly assumed that the domain of a map $F_\gamma$ which is expressed as a composition of two or more mappings is the composition domain. See [6, definition 5.5].

Let now

$$\Pi_\gamma (\epsilon) := \{ y \in \Pi_\gamma : y_\sigma \geq \epsilon \quad \text{whenever} \quad \gamma \subset \sigma\}, \quad (6.4)$$

and define

$$F^\epsilon_{\gamma} := \Psi^{A_{\tau, \epsilon}}_{\gamma} \circ P_{\gamma} \circ (\Psi^{A_{\tau, \epsilon}}_{\gamma})^{-1}.$$

Notice that $\lim_{\epsilon \to 0^+} \Pi_\gamma (\epsilon) = \text{int}(\Pi_\gamma)$.

In [6, lemma 7.2] we prove that global Poincaré maps converge towards the identity map as we approach the heteroclinic orbit along a flowing edge. More precisely we show that for any $k \geq 1$, there exists a number $r$ such that

$$\lim_{\epsilon \to 0^+} F^\epsilon_{\gamma} |_{\mathcal{U}_{\gamma}^r} = \text{id}_{\Pi_\gamma} \quad \text{in the} \quad C^k \text{topology}, \quad (6.5)$$
where $\mathcal{U}' \subset \Pi_v(\epsilon')$ is the domain of $F^c_{\gamma'}$. Hence, the asymptotic behaviour of the flow is solely determined by local Poincaré maps.

From definition 6.3, for any vertex $v$, the vector $\chi^v$ is tangent to $\Pi_v$, in the sense that $\chi^v$ belongs to the linear span of the local Poincaré maps. Let

$$
\Pi_v(\epsilon) := \{ y \in \Pi_v : y_\sigma \geq \epsilon \quad \text{for all} \quad \sigma \in F_v \}. 
$$

Using the notation of definition 6.3, the rescaled vector field $\tilde{X}_v := \frac{1}{\epsilon}(\psi_{A,v}^A, X_{A,v})$ has components

$$
\tilde{X}_{v,\sigma}(y) := \begin{cases} 
-H_\sigma \left( (\psi_{A,v}^A)^{-1}(y) \right) & \text{if } \sigma \in F_v \\
0 & \text{if } \sigma \not\in F_v
\end{cases}.
$$

Then in [6, lemma 5.6] we prove that for any $k \geq 1$ there exists $r > 0$ such that

$$
\lim_{\epsilon \to 0^+} (\tilde{X}_v)|_{\Pi_v(\epsilon)} = \chi^v \text{ in the } C^k \text{ topology}. 
$$

Consider a vertex $v$ with an incoming flowing-edge $v \xrightarrow{\gamma} v'$, and an outgoing flowing-edge $v \xrightarrow{\gamma'} v''$. We define the sector

$$
\Pi_{\gamma,\gamma'} := \left\{ y \in \text{int}(\Pi_v) : y_\sigma = \frac{x_\sigma}{\lambda_\sigma} > 0, \forall \sigma \in F_v, \sigma \neq \sigma^* \right\}
$$

and the linear map $L_{\gamma,\gamma'} : \Pi_{\gamma,\gamma'} \to \Pi_v$ by

$$
L_{\gamma,\gamma'}(y) := \left( \frac{y_\sigma - x_\sigma}{\lambda_\sigma} \right)_{\sigma \in F}. 
$$

Notice that $\Pi_{\gamma,\gamma'} = \{ y \in \Pi_v : y_{\sigma^*} = 0 \}$ as well as $\Pi_{\gamma,\gamma'}$ are facets to $\Pi_v$.

Given flowing-edges $\gamma$ and $\gamma'$ such that $t(\gamma) = s(\gamma') = v$ we denote by $\mathcal{D}_{\gamma,\gamma'}$ the set of points $x \in \Sigma_{\gamma,\gamma'}$ such that the forward orbit $\{ x_{i+1} : i \geq 0 \}$ has a first transversal intersection with $\Sigma_{\gamma,\gamma'}$. The local Poincaré map

$$
P_{\gamma,\gamma'} : \mathcal{D}_{\gamma,\gamma'} \subset \Sigma_{\gamma,\gamma'} \to \Sigma_{\gamma,\gamma'}
$$

is defined by $P_{\gamma,\gamma'}(x) := \varphi_{\gamma,\gamma'}^{\tau(x)}(x)$, where

$$
\tau(x) := \min \{ t > 0 : x_{i+1} \in \Sigma_{\gamma,\gamma'} \}. 
$$

The asymptotic behaviour of the local Poincaré maps is addressed in [6, lemma 7.5] where it is proven that given $k \geq 1$ there exist $r > 0$ such that

$$
\lim_{\epsilon \to 0^+} (F^c_{\gamma,\gamma'})_{|_{\mathcal{U}_{\gamma,\gamma'}}} = L_{\gamma,\gamma'} \text{ in the } C^k \text{ topology},
$$

where $\mathcal{U}_{\gamma,\gamma'} \subset \Pi(\epsilon')$ denotes the composition domain of the map

$$
F^c_{\gamma,\gamma'} := \psi_{A,v}^A \circ P_{\gamma,\gamma'} \circ (\psi_{A,v}^A)^{-1}. 
$$

Given a chain of flowing-edges

$$
v_0 \xrightarrow{\gamma_0} v_1 \xrightarrow{\gamma_1} v_2 \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_m} v_m \xrightarrow{\gamma_m} v_{m+1}
$$

the sequence $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$ is called a heteroclinic path, or a heteroclinic cycle when $\gamma_m = \gamma_0$. 

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Definition 6.11. Given a heteroclinic path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$:

1. The Poincaré map of a polymatrix replicator $X_{nA}$ along $\xi$ is the composition
   \[ P_\xi := (P_{\gamma_m} \circ P_{\gamma_{m-1}}, \gamma_m) \circ \ldots \circ (P_{\gamma_1} \circ P_{\gamma_0}, \gamma_1), \]
   whose domain is denoted by $U_\xi$.

2. The skeleton flow map (of $\chi$) along $\xi$ is the composition map $\pi_\xi : \Pi_{\gamma_m} \rightarrow \Pi_{\gamma_m}$ defined by
   \[ \pi_\xi := L_{\gamma_m, \gamma_m} \circ \ldots \circ L_{\gamma_1, \gamma_1}, \]
   whose domain is
   \[ \Pi_\xi := \text{int}(\Pi_{\gamma_0}) \cap \bigcap_{j=1}^m (L_{\gamma_j, \gamma_j} \circ \ldots \circ L_{\gamma_0, \gamma_1})^{-1} \text{int}(\Pi_{\gamma_0}). \]

The previous asymptotics (6.5) and (6.10) imply that given a heteroclinic path $\xi$, the asymptotic behaviour of the Poincaré map $P_\xi$ along $\xi$ is given by the Poincaré map $\pi_\xi$ of $\chi$. See [6], proposition 7.7. More precisely we have
\[ \lim_{\epsilon \to 0^+} \left( F_\xi^\epsilon \right)_{|U_\xi^\epsilon} = \pi_\xi \text{ in the } C^k \text{ topology}, \]
where $U_\xi^\epsilon$ is the domain of the map
\[ F_\xi^\epsilon := \Psi_{\gamma_0, \epsilon} \circ P_\xi \circ \left( \Psi_{\gamma_0, \epsilon}^{-1} \right)^{-1} : \Pi_{\gamma_0}(\epsilon') \rightarrow \Pi_{\gamma_0}(\epsilon'). \]

To analyse the dynamics of the flow of the skeleton vector field $\chi$ we have introduced the concept of structural set and its associated skeleton flow map. See [6, definition 6.8].

Definition 6.12. A non-empty set of flowing-edges $S$ is said to be a structural set for $\chi$ if every heteroclinic cycle contains an edge in $S$.

Structural sets are in general not unique. We say that a heteroclinic path $\xi = (\gamma_0, \ldots, \gamma_m)$ is an $S$-branch if

1. $\gamma_0, \gamma_m \in S$,
2. $\gamma_j \notin S$ for all $j = 1, \ldots, m - 1$.

Denote by $\mathcal{B}_S(\chi)$ the set of all $S$-branches.

Definition 6.13. The skeleton flow map $\pi_S : D_S \rightarrow \Pi_\xi$ is defined by
\[ \pi_S(y) := \pi_\xi(y) \quad \text{for all } y \in \Pi_\xi, \]
where
\[ D_S := \bigcup_{\xi \in \mathcal{B}_S(\chi)} \Pi_\xi \quad \text{and} \quad \Pi_S := \bigcup_{\gamma \in S} \Pi_{\gamma_\xi}. \]

The reader should picture $\pi_S : D_S \rightarrow \Pi_\xi$ as the first return map of the piecewise linear flow of $\chi$ on $C^*(\Gamma_\xi)$ to the system of cross-sections $\Pi_\xi$. The following result provides a sufficient condition for the skeleton flow map $\pi_S$ to be a (closed) dynamical system.
Proposition 6.14. Given a skeleton vector field $\chi$ on $C^*$ ($\Gamma_\mathcal{A})$ with a non-empty structural set $S$, assume

(1) Every edge of $\Gamma_\mathcal{A}$ is either neutral or a flowing-edge,

(2) Every vertex $v$ is either of saddle type, i.e. $\chi^\alpha_v, \chi^\beta_v < 0$ for some $\alpha, \beta \in F_v$, or else neutral, i.e. $\chi^\alpha_v = 0$ for every $\sigma \in F_v$.

Then

$$\tilde{D}_S := \bigcap_{n \in \mathbb{Z}} (\pi_S)^{-n}(D_S)$$

is a Baire space with full Lebesgue measure in $\Pi_\mathcal{A}$ and $\pi_S : \tilde{D}_S \rightarrow D_S$ is a homeomorphism.

Proof. This is a reformulation of [6, proposition 6.10]. The difference is that here assumption (2) is slightly weaker because it admits the possibility of some vertices being neutral. Notice that by (2) the endpoints of any flowing edge must be of saddle type. Neutral vertices are not visited by heteroclinic cycles. The proof here is the same as in [6] because whenever assumption (2) is invoked there at some vertex, that vertex is an endpoint of some flowing edge. \hfill \square

Given a structural set $S$ any orbit of the flow $\varphi^t_{\mathcal{A}}(S)$ that shadows some heteroclinic cycle must intersect the cross-sections $\bigcup_{v \in S} \Sigma^v_{S}$ recurrently. The following map encapsulates the semi-global dynamics of these orbits.

Definition 6.15. Given $X_{\mathcal{A}}$, let $S$ be a structural set of its skeleton vector field. We define $P_S : U_S \subset \Sigma_S \rightarrow \Sigma_S$ setting $\Sigma_S := \bigcup_{v \in S} \Sigma^v_S$, $U_S := \bigcup_{x \in \partial^0(S)} U_x$ and $P_S(p) := P_S^t(\chi_S(p))$ for all $p \in U_S$. The domain components $U_x$ and $U_x^*$ are disjoint for branches $\xi \neq \xi'$ in $\mathcal{B}(\chi)$. Up to a time reparametrization, the map $P_S : D_S \subset \Sigma_S \rightarrow \Sigma_S$ embeds in the flow $\varphi^t_{\mathcal{A}}$. In this sense the dynamics of $P_S$ encapsulates the qualitative behaviour of the flow $\varphi^t_{\mathcal{A}}$ of $X_{\mathcal{A}}$ along the edges of $\Gamma_\mathcal{A}$. Finally we recall [6, theorem 7.9] which states that given a regular polymatrix replicator vector field $X_{\mathcal{A}}$ with skeleton vector field $\chi$, if $S$ is a structural set of $\chi$ then

$$\lim_{\epsilon \rightarrow 0^+} \Psi_\epsilon \circ P_S \circ (\Psi_\epsilon)^{-1} = \pi_S$$

in the $C^\infty$ topology.

7. Hamiltonian character of the asymptotic dynamics

In this section we discuss the Poisson geometric properties of the Poincaré maps $\pi_\xi$ in the case of Hamiltonian polymatrix replicator equations. Given a generic Hamiltonian polymatrix replicator, $X_{\mathcal{A},\mathcal{V}}$, we study its asymptotic Poincaré maps, proving that they are Poisson maps. Let $X_{\mathcal{A}}$ be a conservative polymatrix replicator, $q$ a formal equilibrium, $A_0$ and $D$ as in definition 5.1. Its Hamiltonian function $h$ defined in (5.2) belongs to a class of prospective constants of motion for vector fields on polytopes discussed in [6, section 8]. Since the polymatrix replicator is fixed we drop superscript $'\mathcal{A}$ and use $\Psi_{t,\epsilon}$ for the rescaling coordinate systems defined in definition 6.9. For later reference we restate here, as in [6, proposition 8.2], the asymptotic constant of motion on the dual cone associated with $h$.

Proposition 7.1. Given $\eta : C^*(\Gamma_{\mathcal{A}}) \rightarrow \mathbb{R}$ defined by

$$\eta(y) := \sum_{\beta=1}^p \sum_{j \in [\beta]} \lambda_\beta y_j^\beta,$$

(7.1)
(1) \( \eta = \lim_{\epsilon \to 0^+} c^2 h \circ (\Psi_{\epsilon, \alpha})^{-1} \) over \( \text{int}(\Pi_v) \) for any vertex \( v \), with convergence in the \( C^\infty \) topology;
(2) \( d\eta = \lim_{\epsilon \to 0^+} c^2 \left[ (\Psi_{\epsilon, \alpha})^{-1}\right]^* (dh) \) over \( \text{int}(\Pi_v) \) for any vertex \( v \), with convergence in the \( C^\infty \) topology;
(3) Since \( h \) is invariant under the flow of \( X_{2A} \), i.e. \( dh(X_{2A}) \equiv 0 \), the function \( \eta \) is invariant under the skeleton flow of \( \chi \), i.e. \( d\eta(\chi) \equiv 0 \).

We are now ready to prove our second main result, theorem B in section 3.

**Proof of theorem B.** Follows from proposition 6.14, for which we need to prove that (1) every edge is either neutral or a flowing-edge, and (2) every vertex \( v \) is either of saddle or neutral type. Item (1) follows from proposition 6.8. To check item (2) notice that being conservative, by [4, theorem 3.20], \( X \) admits the Hamiltonian function (5.2). From proposition 7.1, see also [6, proposition 8.2], the skeleton vector at any vertex \( \chi^0 = (\chi^0_k) \in \mathbb{R}^p \) (see definition 6.3) satisfies
\[
\langle \eta, \chi^0 \rangle = \sum_{\beta=1}^p \sum_{i \in [\beta]} \lambda_{\beta} q_{\beta i} \chi^0_i = 0.
\]

Therefore, since by assumption, all \( \lambda_{\beta} \) have the same sign and \( q_{\beta i} > 0 \) for all \( i \in [\beta] \) and \( \beta = 1, \ldots, p \), either \( \chi_{\beta i} = 0 \) for all \( i \) and \( v_j \) is neutral, or else two of this characters have opposite signs and \( v_j \) is of saddle type. This proves that item (2) of proposition 6.14 holds. \( \square \)

We will use the following family of coordinate charts for the Poisson manifold \( (\text{int}(\Gamma_n), \pi_{A_0}) \) where \( \pi_{A_0} \) is defined in (5.1).

**Definition 7.2.** Given a vertex \( v = (e_1, \ldots, e_p) \) of \( \Gamma_n \) we set \( \hat{x}_\alpha := (x^\alpha_k)_{k \in [\alpha] \setminus \{j_\alpha\}} \) and \( \hat{x} := (\hat{x}_\alpha)_{\alpha} \), and define the projection map
\[
P_v : \text{int}(N_v) \to (\mathbb{R}^{n-1} \times \cdots \times \mathbb{R}^{n-1}), \quad P_v(x) := \hat{x}.
\]

\( P_v \) is a diffeomorphism onto its image \((0, 1)^{n-p} \times \mathbb{R} \) and the inverse map \( \psi_v := P_v^{-1} \) can be regarded as a local chart for the manifold \( \text{int}(\Gamma_n) \).

**Remark 7.3.** The projection map \( P_v \) extends linearly to \( \mathbb{R}^n \) and it is represented by the \((n-p) \times n\) block diagonal matrix
\[
P_v = \text{diag}(P_1, \ldots, P_p),
\]
where \( P_\alpha \), for \( \alpha = 1, \ldots, p \), is the \((n_\alpha - 1) \times n_\alpha \) constant matrix obtained from the identity matrix by removing its row \( j_\alpha \).

Using the definitions of \( D_x \) and \( T_x \) given in section 5 we can state the following lemma.

**Lemma 7.4.** Consider the Poisson manifold \( (\text{int}(\Gamma_n), \pi_{A_0}) \) where \( \pi_{A_0} \) is defined in (5.1). Then for any vertex \( v \), the matrix that represents \( \pi_{A_0} \) in the local chart \( \psi_v \) is
\[
\pi_{A_0}^\psi(\hat{x}) = (-1)^{P_v T_v D_v A_0 D_v T_v^T P_v^T}.
\]

**Proof.** Notice that \( \pi_{A_0}^\psi(\hat{x}) := \{(x^\alpha_k, x^\beta_l)\} \) with \( \alpha, \beta = 1, \ldots, p \) and \( k \in [\alpha] \setminus \{j_\alpha\}, l \in [\beta] \setminus \{j_\beta\} \). \( \square \)

We used the notation \( \hat{x} \) instead of \( \hat{\chi} \) to make it clear that the representing matrix is with respect to the local chart \( \psi_v \). The following trivial lemma gives us the differential
of the $\epsilon$-rescaling map $\Psi_{v,\epsilon}$ (in definition 6.9) for the coordinate chart $\psi_v$. Given a vertex $v = (e_1,\ldots,e_p)$ and using the notation introduced in definition 7.2 we write $D_{e_k} = \text{diag}(x_1^{\alpha},\ldots,x_{n_\alpha}^{\alpha}-1,x_{n_\alpha}^{\alpha}+1,\ldots,x_n)$ and denote by $D_\epsilon$ the diagonal matrix $\text{diag}(D_{e_1},\ldots,D_{e_p})$.

**Lemma 7.5.** The differential of the diffeomorphism

$$\Psi_{v,\epsilon} \circ \psi_v : \text{int}(N_v) \to \text{int}(\Pi_v)$$

is given by

$$d_\epsilon(\Psi_{v,\epsilon} \circ \psi_v) = -\epsilon^2 D_\epsilon^{-1}.$$

We push forward, by the diffeomorphism $\Psi_{v,\epsilon} \circ \psi_v$, the Poisson structure $\pi^v_{\lambda_0}$ defined on $P_v(\text{int}(N_v))$ to $\text{int}(\Pi_v)$. The following lemma provides the matrix representative of the push forwarded Poisson structure. In order to simplify the notation we set

$$\mathbb{J}(\hat{x}) := -\epsilon^2 D_\epsilon^{-1} P_\lambda T_\epsilon D_\epsilon$$

and for every $\alpha = 1,\ldots,p$

$$\mathbb{J}_\alpha(\hat{x}^\alpha) := -\epsilon^2 D_\epsilon^{-1} P^\alpha_\lambda T^\alpha_\epsilon D_\epsilon.$$

Notice that $\mathbb{J}(\hat{x}) = \text{diag}(\mathbb{J}_1(\hat{x}^1),\ldots,\mathbb{J}_p(\hat{x}^p))$.

**Lemma 7.6.** The diffeomorphism $\Psi_{v,\epsilon} \circ \psi_v$ pushes forward the Poisson structure $\pi^v_{\lambda_0}$ to the Poisson structure $\pi^v_{A_0,\epsilon}$ on $\text{int}(\Pi_v)$ where

$$\pi^v_{A_0,\epsilon}(y) = (-1)(\mathbb{J}_1(\hat{x}^1) \circ (\Psi_{v,\epsilon} \circ \psi_v)^{-1}(y).$$

**Proof.** See definition 4.1 and remark 4.2.

The Poisson structure $\pi^v_{A_0,\epsilon}$ is asymptotically equivalent to a linear Poisson structure. Let

$$E_v = \text{diag}(E^1_v,\ldots,E^p_v),$$

be the $(n-p) \times n$ matrix defined by diagonal blocks $E^\alpha_v$, for $\alpha = 1,\ldots,p$, where the $\alpha$th block is the $(n_\alpha - 1) \times n_\alpha$ matrix in which the column $j_\alpha$ is equal to $1_{n_\alpha-1}$ and every other column $k_\alpha \neq j_\alpha$ is equal to $-\epsilon e_\alpha \in \mathbb{R}^{n_\alpha-1}$.

**Lemma 7.7.** Given a vertex $v = (e_1,\ldots,e_p)$, if $E_v$ is the matrix in (7.6) and $B_\epsilon := E_v A_0 E_v^T$, then

$$\lim_{\epsilon \to 0^+} -\frac{1}{\epsilon^2} \mathbb{J} \circ (\Psi_{v,\epsilon} \circ \psi_v)^{-1}(y) = E_v,$$

over $\text{int}(\Pi_v)$ with convergence in $C^\infty$ topology. Consequently,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^2} \pi^v_{A_0,\epsilon}(y) = B_\epsilon,$$

over $\text{int}(\Pi_v)$ with convergence in $C^\infty$ topology.
Proof. A simple calculation shows that for every $\alpha = 1, \ldots, p$

$$
\begin{array}{c}
\frac{-1}{\epsilon^2} \mathbb{J}_\alpha = \\
\left( \begin{array}{cccc}
(x_1^\alpha - 1) & \cdots & x_{j_\alpha}^\alpha - 1 & \cdots & x_{n_\alpha}^\alpha \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
x_1^\alpha & \cdots & x_{j_\alpha}^\alpha - 1 & \cdots & x_{n_\alpha}^\alpha \\
x_1^\alpha & \cdots & x_{j_\alpha}^\alpha - 1 & \cdots & (x_{j_\alpha + 1}^\alpha - 1) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
x_1^\alpha & \cdots & x_{j_\alpha}^\alpha - 1 & \cdots & x_{n_\alpha}^\alpha \\
\end{array} \right).
\end{array}
$$

For every $\sigma \in F_v$ and $k \in [\alpha] \setminus \{j_\alpha\}$ we have

$$
\lim_{\epsilon \to 0^+} x_k^\alpha \circ (\Psi_{v, \epsilon} \circ \psi_v)^{-1}(y) = \lim_{\epsilon \to 0^+} e^{-\frac{y_k^\alpha}{\epsilon}} = 0.
$$

Considering that $x_{j_\alpha}^\alpha = 1 - \sum_{k \in [\alpha] \setminus \{j_\alpha\}} x_k^\alpha$, we get the first claim of the lemma and the second claim is an immediate consequence. \(\square\)

Figure 2 illustrates the case $\Gamma = \Delta^2$.

Remark 7.8. The same linear Poisson structure $B_v : = E_v A_0 E_v^t$ appears in [4, theorem 3.5].

Lemma 7.9. For a given vertex $v = (e_{j_1}, \ldots, e_{j_p})$, let $\chi^v$ be the skeleton character of $X_{e(A)}$, as in definition 6.3. Then

$$
\chi^v = B_v d\eta_v,
$$

where $\eta_v$ is the restriction of the function $\eta$ (defined in (7.1)) to $\text{int}(\Pi_v)$.  

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Proof. We use the notation $X_{(\partial\Sigma_0)}^r(\hat{x}) := (dP_v)X_{\Sigma_0}(x)$ for the local expression of the replicator vector field $X_{\Sigma_0}$ in the local chart $\psi_v$. If we write the function $h(x)$, defined in (5.2), as $h(x) = h(\psi_v \circ P_v(\hat{x}))$ then

$$d_xh = (P_v)'d_x(\psi_v \circ h)(\hat{x}).$$

Notice that $dP_v = P_v$. By (5.2), $X_{\partial\Sigma_0} = \pi_{\partial\Sigma}dh$. Locally,

$$X_{(\partial\Sigma_0)}^r(\hat{x}) = P_vX_{\Sigma_0}(x) = P_v\pi_{\partial\Sigma}P_v'(d_xh \circ \psi_v).$$

Similarly, writing $h \circ \psi_v(\hat{x}) = h \circ \psi_v \circ (\Psi_{v,e} \circ \psi_v)^{-1} \circ (\Psi_{v,e} \circ \psi_v)(\hat{x})$ we have

$$d_x(h \circ \psi_v) = (d_x(\Psi_{v,e} \circ \psi_v))'d_x(h \circ (\Psi_{v,e})^{-1}).$$

The vector field $\tilde{\chi}_v$ defined in lemma 6.7 is

$$\tilde{\chi}_v = \frac{1}{e_v}(d_i(\Psi_{v,e})X_{\Sigma_0}) = \frac{1}{e_v}(d_i(\Psi_{v,e} \circ \psi_v)X_{(\partial\Sigma_0)}^r)$$

$$= \frac{1}{e_v}(d_i(\Psi_{v,e} \circ \psi_v)P_v\pi_{\partial\Sigma}P_v'(d_xh \circ \psi_v))d_jh \circ (\Psi_{v,e})^{-1}$$

$$= \frac{1}{e_v}\pi_{\partial\Sigma}^{\nu}(e^2(\Psi_{v,e})^{-1})^*dh,$$

where in the second equality we use $\psi_v \circ P_v = \text{Id}$. Then, applying lemma 6.7, lemma 7.7, and proposition 7.1, the result follows. Notice that $\Pi_i(e') \subset \text{int}(\Pi_i)$.}

In other words, this previous lemma says that $\chi^v$ restricted to int$(\Pi_i)$ is Hamiltonian with respect to the constant Poisson structure $B_v$ having $\eta_i$ as a Hamiltonian function.

Our aim is to show that for a given heteroclinic path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$, the skeleton flow map of $\chi$ along $\xi$ (see definition 6.11),

$$\pi_\xi := L_{\gamma_{m+1}, \gamma_m} \circ \cdots \circ L_{\gamma_1, \gamma_0},$$

restricted to the level set of $\eta_i$ is a Poisson map. Notice that the Poisson structure $B_v$ is only defined in int$(\Pi_i)$ and neither $\Pi_i$ nor $\Pi_i'$ are submanifolds of int$(\Pi_i)$. So we need to define Poisson structures on the sections $\Pi_{\gamma_i, \gamma_{i+1}}$ for all $i = 0, \ldots, m$.

For the heteroclinic path

$$\xi : v_0 \xrightarrow{\gamma_0} v_1 \xrightarrow{\gamma_1} v_2 \xrightarrow{\ldots} v_m \xrightarrow{\gamma_m} v_{m+1}, \tag{7.7}$$

we store in the $i$th column of the matrix

$$J_\xi = \begin{bmatrix} j_{01} & j_{11} & \cdots & j_{(m+1)1} \\ j_{02} & j_{12} & \cdots & j_{(m+1)2} \\ \vdots & \vdots & \ddots & \vdots \\ j_{0p} & j_{1p} & \cdots & j_{(m+1)p} \end{bmatrix},$$

the indices of the non zero components of the vertex $v_i = (e_{j_{0i}}, e_{j_{1i}}, \ldots, e_{j_{pi}})$. By definition of $\Gamma_2^v$, there exists $(\xi_0, \xi_1, \ldots, \xi_m) \in \{1, 2, \ldots, r\}^{m+1}$ such that $j_{(i-1)p} = j_{ip}$ for $i \neq \xi_{i-1}$ and $j_{(i-1)p} \neq j_{\xi_{i-1}}$, i.e. $\xi_{i-1}$ is the group containing the nonzero component that differ between the end points of the edge $\gamma_{i-1}$. In order to simplify notations for every vertex $v_i$ in $\xi$ we denote

$$r_i = j_{(i-1)p} \xi_{(i-1)} \; \text{and} \; s_i = j_{(i+1)p} \xi_i.$$
First we consider the vertex $v_i$ with incoming and outgoing edges

$$
\gamma_i^{-1} : v_{i-1} + t(0, \ldots, e_{i-1}, -e_n, \ldots, 0),
$$
$$
\gamma_i : v_i + t(0, \ldots, e_i, -e_i, \ldots, 0),
$$

respectively, i.e. $\frac{\gamma_i^{-1}}{v_i} \rightarrow \gamma_i$, where $t \in [0, 1]$. Notice that

$$
\Pi_{v_{i-1}} = \{ y \in \mathbb{R}_+^n : y_{i-1} = 0, \ l = 1, \ldots, p \},
$$
$$
\Pi_{v_i} = \{ y \in \mathbb{R}_+^n : y_i = 0, \ l = 1, \ldots, p \}.
$$

Since $j_{i(k-1)} = j_k$ for $l \neq \xi_i - 1$ and $\Pi_{v_{i-1}} = \Pi_{v_{i-1}} \cap \Pi_{v_i}$ we have

$$
\Pi_{v_{i-1}} = \{ y \in \mathbb{R}_+^n : y_{i-1} = y_{j_{i(k-1)}} = 0, \ l = 1, \ldots, p \}.
$$

The opposite facet to $\gamma_i$ at $v_i$ is then $\sigma_i := \{ y_{i-1} = 0 \}$ where we omitted the superscript $i_l$ from $y$ since it is evident that $j_k \in \xi_l$. We keep omitting the superscript whenever there is no ambiguity. The sector defined in (6.8) is

$$
\Pi_{\gamma_{i-1}, \gamma_i} = \left\{ y \in \text{int}(\Pi_{v_{i-1}}) : y_{i-1} - \frac{\chi_i}{\chi_{i-1}} y_{i-1} > 0, \ \forall \sigma \neq \xi_1, \ldots, \xi_{n(m+1)} \right\}. \quad (7.8)
$$

The skeleton flow map of $\chi$ at vertex $v_i$ is the linear map $L_{\gamma_{i-1}, \gamma_i} : \Pi_{\gamma_{i-1}, \gamma_i} \rightarrow \Pi_{v_i}$ defined by

$$
L_{\gamma_{i-1}, \gamma_i} (y) := \left( y_{\sigma} - \frac{\chi_i}{\chi_{i-1}} y_{i-1} \right)_{\sigma} \quad (7.9)
$$

Notice that $L_{\gamma_{i-1}, \gamma_i} (y) = \phi_{\chi_i} (\tau(y), y)$ where $\phi_{\chi_i} (\tau, y) = y + \tau \chi_i$, is the flow of the skeleton vector field $\chi_i$ and $\tau(y) := -\frac{\chi_i}{\chi_{i-1}}$. We denote $L^t_{\gamma_{i-1}, \gamma_i} (y) := \phi_{\chi_i} (\tau(y), y)$ where $t \in (0, 1)$. More precisely

$$
L^t_{\gamma_{i-1}, \gamma_i} (y) := \left( y_{\sigma} - \frac{\chi_i}{\chi_{i-1}} y_{i-1} \right)_{\sigma} \quad .
$$

**Definition 7.10.** We define by

$$
T_{\gamma_{i-1}, \gamma_i} := \bigcup_{0 < t < 1} L^t_{\gamma_{i-1}, \gamma_i} \left( \Pi_{\gamma_{i-1}, \gamma_i} \right), \quad (7.10)
$$

the convex cone containing the line segments of the flow of $\chi_i$ connecting the points in the domain of $L_{\gamma_{i-1}, \gamma_i}$ to their images.

We consider two Poisson-Transversal foliations⁶ interior to each sector $\Pi_{\gamma_j}$ in order to use the techniques introduced in section 4. In the following lemma, we describe the Poisson structures on $\Pi_{\gamma_{i-1}, \gamma_i}$ and $L_{\gamma_{i-1}, \gamma_i} (\Pi_{\gamma_{i-1}, \gamma_i})$.

**Lemma 7.11.** With the notation adopted in lemma 7.9, let $\eta_i$ be the restriction of function $\eta_i$ defined in (7.4), to $\text{int}(\Pi_{\gamma_i})$. Consider two functions $G^i_\gamma, G^o_\gamma : T_{\gamma_{i-1}, \gamma_i} \rightarrow \mathbb{R}$ defined by $G^i_\gamma (y) = y_i$ and $G^o_\gamma (y) = y_{i-1}$ then:

1. Level sets of $(\eta_i, G^i_\gamma)$, $(\eta_i, G^o_\gamma) : T_{\gamma_{i-1}, \gamma_i} \rightarrow \mathbb{R}^2$ partition $T_{\gamma_{i-1}, \gamma_i}$ into a Poisson-Transversal foliation $\mathcal{F}^i_\gamma$ and $\mathcal{F}^o_\gamma$, i.e. every leaf of these foliations is a Poisson transversal of $(T_{\gamma_{i-1}, \gamma_i}, \mathcal{B}_\gamma)$. Furthermore, every leaf $\Sigma$ of these foliations is a level transversal section to $\chi_i$ at every point $x \in \Sigma$.

---

⁶ By Poisson-Transversal foliation we mean that every leaf of the foliation is a Poisson transversal.
Given two leaves $\Sigma^l_i = (\eta^l_i, G^\psi_i)^{-1}(c, d_i), l = 1, 2, \text{ of } \mathcal{F}^\eta_i$ and two leaves $\Sigma^r_i = (\eta^r_i, G^\psi_i)^{-1}(c, d'_i), l = 1, 2, \text{ of } \mathcal{F}^\eta_i$, then the Poincaré map between any pair of these four leaves is a Poisson map.

**Proof.** Clearly,

$$\{ \eta^l_i, G^\psi_i \} = X_{\eta^l_i} (dG^\psi_i) = \chi^\psi_i (dy_{\eta^l_i}) = \chi^\psi_i,$$

and similarly $\{ \eta^r_i, G^\psi_i \} = \chi^\psi_i$. As before, using the notation $\sigma_{j_{l-1}}$ for the facet $\{ y_{j_{l-1}} = 0 \}$, we see that $\gamma_{j_{l-1}}$ is a flowing edge from the corner $(\eta_{j_{l-1}}, \sigma_{j_{l-1}})$ to the corner $(\eta_l, \sigma_l)$. So $\chi^\psi_{j_{l-1}} > 0$. In a similar way we have $\chi^\psi_{j_l} < 0$, i.e. both $\{ \eta^l_i, G^\psi_i \}$ and $\{ \eta^r_i, G^\psi_i \}$ are nonzero. Then the level sets of both $\{ \eta^l_i, G^\psi_i \}$ and $\{ \eta^r_i, G^\psi_i \}$ are Poisson transversals (see remark 4.7). The fact that $\Sigma$ is a level transversal section is clear.

The Poincaré map between $\Sigma^l_i, \Sigma^r_i$ is the translation

$$P(y) = \phi_{\chi^\psi_i} \left( \frac{d_2 - d_1}{\chi^\psi_i}, y \right) = \left( \frac{d_2 - d_1}{\chi^\psi_i} \right) \chi^\psi_i + y,$$

and a similar translation for $\Sigma^l_i, \Sigma^r_i$. Clearly, these translations are Poisson maps.

The Poincaré map between two level sets $\Sigma^l_i$ and $\Sigma^r_i$ is

$$P(y) = \phi_{\chi^\psi_i} \left( \frac{d_2 - y_{j_k}}{\chi^\psi_i}, y \right).$$

By proposition 4.8 this map is a Poisson map as well.

**Remark 7.12.** Note that $y_{j_k}$ is not constant on $\Sigma^l_i$, so the map (7.12) is not a fixed time map of the flow $\phi_{\chi^\psi_i}$. Therefore, being Poisson is not a direct consequence of the flow being Hamiltonian. Furthermore, proving that this map is Poisson by direct calculation is not straightforward. This makes the contents of section 4 inevitable.

Let $\widetilde{\mathcal{F}}^\psi_i$ and $\widetilde{\mathcal{F}}^\psi_j$ be the foliations consisting of level sets of the functions $G^\psi_i$ and $G^\psi_j$, respectively. Every leaf of $\widetilde{\mathcal{F}}^\psi_i, \ast = r, s$, is equipped with a Poisson structure, $\pi^\psi_{i, \ast}$, which has $\eta^l_i$ as a Casimir, and the leaf sets of this Casimir are the leaves of the Poisson-Transversal foliation $\tilde{\mathcal{F}}^\psi_i$. The leaves of $\tilde{\mathcal{F}}^\psi_i$ can be identified (as Poisson manifolds) through translations of type (7.11). By $(\tilde{\Sigma}^l_i, \tilde{\pi}^\psi_i), \ast = r, s$, we denote a typical leaf of the Poisson foliation $\tilde{\mathcal{F}}^\psi_i$.

Ignoring (for a moment) the fact that the function $G^\psi_i$ is only defined on $T_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$, we may consider $T_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$ as the zero level set of $G^\psi_i$. Hence a typical leaf $\tilde{\Sigma}^l_i$ is diffeomorphic to $T_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$ through a translation of type (7.11). Through this, diffeomorphism $T_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$ secures a Poisson structure. Similarly, $L_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}} (\Pi_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}})$ gains a Poisson structure from $(\tilde{\Sigma}^l_i, \tilde{\pi}^\psi_i)$.

**Proposition 7.13.** Let $\Pi_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$ be equipped with the Poisson structure induced from $(\tilde{\Sigma}^l_i, \tilde{\pi}^\psi_i)$ via a translation of type (7.11), and $L_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}} (\Pi_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}})$ with the one induced from $(\tilde{\Sigma}^l_i, \tilde{\pi}^\psi_i)$ in a similar way. Then $L_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$ is Poisson map (see figure 3).

**Proof.** We decompose $L_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$ into three maps $P^l_1, P^l_2$ and $P^l_3$, where $P^l_1$ and $P^l_2$ are the translations used to define the Poisson structures on $\Pi_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}}$ and $L_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}} (\Pi_{\eta_{j_{l-1}}, \gamma_{j_{l-1}}})$, respectively, and $P^l_3$ is the Poincaré map from $(\tilde{\Sigma}^l_i, \tilde{\pi}^\psi_i)$ to $(\tilde{\Sigma}^l_i, \tilde{\pi}^\psi_i)$. By the construction of these two sections, together with lemma 7.11, $P^l_2$ is a Poisson map, which ends the proof.

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Notice that the flow of $\chi^\psi_i = X_{\eta^l_i}$ preserves $\eta^l_i$. 

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7 Notice that the flow of $\chi^\psi_i = X_{\eta^l_i}$ preserves $\eta^l_i$. 

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Figure 3. Illustration of proposition 7.13.

Notice that \((\tilde{\Sigma}^v_i, \tilde{\pi}^v_i)\), \(* = r, s\) is a union of Poisson submanifolds equipped with Dirac bracket. We describe now the matrix representative of this Dirac bracket.

**Lemma 7.14.** The matrix representative, in the coordinate system \((y_l)_{l \in [\alpha]} \setminus \{j_i\}\), of the Dirac bracket in \(\text{int}(\Pi_{v_i})\) generated by \((\eta_{v_i}, G_{v_i}^s)\), is given by

\[
(\pi_{v_i}^s)^\# = B_{v_i} - C_{v_i,s},
\]

where \(C_{v_i,s} = [C_{v_i,s}^{\alpha,\beta}]_{\alpha,\beta}\) with

\[
C_{v_i,s}^{\alpha,\beta} := \left[ c_{y_l}^{(\alpha,\beta, v_i, s)}(l, f) \in ([\alpha] \setminus \{j_i\}) \times ([\beta] \setminus \{j_i\}) \right]
\]

and

\[
c_{y_l}^{(\alpha,\beta, v_i, s)} := \frac{1}{\chi_{v_i}^s} \left( (\chi_{v_i}^s)^{\alpha} b_{v_i,1}^{\beta} + b_{v_i,1}^{\alpha} (\chi_{v_i}^s)^{\beta} \right).
\]

The \(i\)th row and column of the matrix \((\pi_{v_i}^s)^\#\) are both null. Removing these row and column one obtains the matrix representative of the Poisson structure \(\tilde{\pi}_{v_i}^s\) on \(\tilde{\Sigma}^v_i\). Similarly, for \((\eta_{v_i}, G_{v_i}^r)\) we have

\[
(\pi_{v_i}^r)^\# = B_{v_i} - C_{v_i,r},
\]

where

\[
c_{y_l}^{(\alpha,\beta, v_i, r)} := \frac{1}{\chi_{v_i}^r} \left( (\chi_{v_i}^r)^{\alpha} b_{v_i,1}^{\beta} + b_{v_i,1}^{\alpha} (\chi_{v_i}^r)^{\beta} \right),
\]

and removing the \(s\)th row and column yields the matrix representative of the Poisson structure \(\tilde{\pi}_{v_i}^r\) on \(\tilde{\Sigma}^v_i\).

**Proof.** By definition,

\[
(\pi_{v_i}^s)^\# = \left[ \{y_l^\alpha, y_l^\beta\} \right]_{(l, f) \in ([\alpha] \setminus \{j_i\}) \times ([\beta] \setminus \{j_i\})},
\]
So we need to calculate
\[
\begin{bmatrix}
    \{y_\alpha^i, \eta_\beta\} & \{y_\alpha^i, G^\gamma_\nu\}
\end{bmatrix}
\begin{bmatrix}
    0 & \{\eta_\beta, G^\gamma_\nu\}
\end{bmatrix}
^{-1}
\begin{bmatrix}
    \{\eta_\beta, y_\alpha^i\} \\
    \{G^\gamma_\nu, y_\alpha^i\}
\end{bmatrix},
\]
(7.15)
see the definition of \(\{\ldots\}^{\text{Dirac}}\) in (4.6). Reminding that
\[
\{\eta_\beta, y_\alpha^i\} = (x^\nu)^i_\beta \quad \text{and} \quad \{y_\alpha^i, y_\beta^j\} = b^\alpha_\beta,
\]
together with a simple calculation, yields (7.13). The \(r^\alpha_i\) row and column are zero simply because, by definition, \(G^\gamma_\nu = \eta_\beta\) is a Casimir of the Dirac bracket. Note that the representative matrix \((\pi^\gamma_i)^2\) is with respect to the coordinate system as \(\{y_\gamma\}_{\gamma \in \{\alpha\} \setminus \{\eta_\beta\}}\) of \(\Pi_\eta\), and by omitting the component \(y_\eta\), from this coordinate system one obtains a coordinate system on \(\tilde{\Sigma}_r\). Therefore, removing the null \(r^\alpha_i\) row and column yields the representative matrix of \(\tilde{\pi}^\gamma_i\) with respect to the obtained coordinate. The same reasoning holds for \((\pi^\gamma_i)^2\).

We now extend proposition 7.13 to the whole heteroclinic path \(\xi\). Our main result is the following.

**Theorem 7.15.** Let
\[
\xi : v_0 \xrightarrow{\gamma_0} v_1 \xrightarrow{\gamma_1} v_2 \longrightarrow \ldots \longrightarrow v_m \xrightarrow{\gamma_m} v_{m+1}
\]
(7.16)
be a heteroclinic path. Then for every \(i = 1, \ldots, m\), the Poisson structures induced on the intersection
\[
L_{r_1} (\Pi_{\gamma_0} \cap \Pi_{\gamma_1} \cap \Pi_{\gamma_2}) \cap \Pi_{\gamma_3} \cap \cdots \cap \Pi_{\gamma_{m-1}} \cap \Pi_{\gamma_m},
\]
(7.17)
from Poisson submanifolds \((\tilde{\Sigma}_{\gamma_{i-1}}^{\Gamma_{\gamma_0}}, \tilde{\pi}^{\Gamma_{\gamma_0}}_{\gamma_{i-1}})\) and \((\tilde{\Sigma}_{\gamma_{i}}^{\Gamma_{\gamma_0}}, \tilde{\pi}^{\Gamma_{\gamma_0}}_{\gamma_{i}})\) is the same. Consequently, the skeleton flow map of \(\chi\) along \(\xi\) (see definition 6.11),
\[
\pi_{\xi} := L_{\gamma_{m-1}} \circ \ldots \circ L_{\gamma_0},
\]
is a Poisson map w.r.t. the Poisson structures induced by \((\tilde{\Sigma}_{\gamma_{0}}^{\Gamma_{\gamma_0}}, \tilde{\pi}^{\Gamma_{\gamma_0}}_{\gamma_{0}})\) and \((\tilde{\Sigma}_{\gamma_{m}}^{\Gamma_{\gamma_0}}, \tilde{\pi}^{\Gamma_{\gamma_0}}_{\gamma_{m}})\) on its domain and range, respectively.

Considering the segment
\[
\ldots \longrightarrow v_{i-2} \xrightarrow{\gamma_{i-2}} v_{i-1} \xrightarrow{\gamma_{i-1}} v_i \xrightarrow{\gamma_i} v_{i+1} \longrightarrow \ldots,
\]
the key point is to show that the Poisson structure induced from \((\tilde{\Sigma}_{\gamma_{i-1}}^{\Gamma_{\gamma_0}}, \tilde{\pi}^{\Gamma_{\gamma_0}}_{\gamma_{i-1}})\) on \(L_{\gamma_{i-2}} \cap \Pi_{\gamma_{i-1}} \cap \Pi_{\gamma_{i}} \cap \Pi_{\gamma_{i+1}}\) and the one induced from \((\tilde{\Sigma}_{\gamma_{i}}^{\Gamma_{\gamma_0}}, \tilde{\pi}^{\Gamma_{\gamma_0}}_{\gamma_{i}})\) on \(\Pi_{\gamma_{i}} \cap \Pi_{\gamma_{i+1}}\), match on the intersection (7.17) (see figure 3). To prove theorem 7.15 we need to state and prove two preliminary lemmas regarding this key point. The two sectors \(\Pi_{v_{i-1}}\) and \(\Pi_{v_{i}}\) are only different in the group \(\xi_{i-1}\), where \(y_{\gamma_i} = 0\) for the elements of \(\Pi_{v_{i-1}}\) and \(y_{\gamma_{i-1}} = 0\) for the elements \(\Pi_{v_{i}}\). Let \(P_{v_{i-1}, v_{i}} : \text{int}(\Pi_{v_{i-1}}) \to \text{int}(\Pi_{v_{i}})\) be the diffeomorphism of the form
\[
T_{v_{i-1}} \circ (P_{v_{i-1}, v_{i}}^1 \times \ldots \times P_{v_{i-1}, v_{i}}^p),
\]
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where:

(1) For $\beta \neq \xi_{i-1}$ the associated component $P_{\gamma_{i-1}}^{(\beta)}$ is the identity map;

(2) For any $l \in [\xi_{i-1}] \setminus \{s_{i-1}\}$

$$
(P_{\gamma_{i-1}}^{(\beta)}(y))_l^{\xi_{i-1}} = \begin{cases} 
y_l^{\xi_{i-1}} - y_{s_{i-1}} & \text{if } l \neq r_i \\
y_l^{\xi_{i-1}} - y_{s_{i-1}} & \text{if } l = r_i
\end{cases};
$$

(3) For the following notation to be consistent, without loss of generality we assume that $s_{i-1} = j_{\xi_{i-1}} < r_i = j_{(i-1)\xi_{i-1}}$. Notice that for any given point $y \in \tilde{\Sigma}_{i-1}^\gamma = (G_{i-1}^\gamma)^{-1}(c)$ the map $P_{\gamma_{i-1}}^{(\beta)}$ acts on the component $\xi_{i-1}$ as

$$
y_{i-1} = (y_1^{i-1}, \ldots, c, \ldots, y_m^{i-1})
\rightarrow (y_1^{i-1} - c, \ldots, y_{m-1}^{i-1} - c, y_m^{i-1} - c),
$$

where the notation $\gamma^c_i$ means that the entry $y_i$ is missing in the corresponding vector.

The image point is not in $\tilde{\Sigma}_{i-1}^\gamma = (G_{i-1}^\gamma)^{-1}(c') \subset \Pi_i$. However composing with the translation

$$
T_{i-1,i}(y) := y + (\tilde{0}, \ldots, (0, \ldots, t_{(i-1)r}, \ldots, 0), \ldots, 0),
$$

we get

$$
P_{\gamma_{i-1},i}(\tilde{\Sigma}_{i-1}^\gamma) = \tilde{\Sigma}_i^\gamma.
$$

We restrict the diffeomorphism $P_{\gamma_{i-1},i}$ to an open set $U_r^{i-1}$ around $\tilde{\Sigma}_i^\gamma$ to get

$$
P_{\gamma_{i-1},i} : U_s^{i-1} \rightarrow U_r^{i},
$$

where $U_r^{i}$ is an open set around $\tilde{\Sigma}_i^\gamma$.

**Lemma 7.16.** The diffeomorphism

$$
P_{\gamma_{i-1},i} : (U_r^{i-1}, B_{\gamma_{i-1}}) \rightarrow (U_r^{i}, B_{\gamma_i})
$$

is Poisson, i.e. $P_{\gamma_{i-1},i}$ preserves the ambient Poisson structure.

**Proof.** A simple calculation shows that $(dP_{\gamma_{i-1},i})_{E_{\gamma_{i-1}}}^{E_{\gamma_{i}}} = E_{\gamma_{i}}^{E_{\gamma_{i}}}$. To give the reader an idea, let $n_{\xi_{i-1}} = 5, s_{i-1} = 2$ and $r_i = 4$ then

$$
P_{\gamma_{i-1},i}^{(\xi_{i-1})}(y_1, y_2, y_3, y_5) = (y_1 - y_2, y_3 - y_2, -y_2, y_5 - y_2)
$$

and

$$
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 \\
0 & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}
= 
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & -1
\end{bmatrix}.
$$
Since for $\beta \neq \xi_{i-1}$ the component $P_{i-1,v_i}^\beta$ is the identity map we get $(dP_{i-1,v_i})E_{v_{i-1}} = E_{v_i}$. This fact together with (4.3) and the definitions of $B_{v_{i-1}}, B_i$ (see lemma 7.7) finishes the proof.

Lemma 7.17. For the diffeomorphism $P_{v_{i-1},v_i}$ we have that:

1. $G_i^{v_i} \circ P_{v_{i-1},v_i} = -G_{v_{i-1}} \circ T_{i-1};$
2. $\eta_i \circ P_{v_{i-1},v_i} = \eta_{v_{i-1}} - \lambda \xi_{i-1} q_i^{v_i} - \lambda \xi_{i-1} q_i^{v_{i-1}} T_{i-1,i}.$

Proof. The first equality is trivial since, for any $y \in \Pi_{v_{i-1}},$ the $r^\beta$ component of $G_i^{v_i} \circ P_{v_{i-1},v_i}(y)$ is $-y_{i-1} + t_{i-1,i}.$ For the second equality we have

$$\eta_i \circ P_{v_{i-1},v_i}(y) = \left( \sum_{\beta \in \xi_{i-1}, \beta \neq \xi_{i-1}} \lambda \beta y_i \right) + \left( \sum_{\beta \in \xi_{i-1}, \beta \neq \xi_{i-1}} \lambda \beta q_i^{v_i} y_i - q_i^{v_i} (-y_{i-1} + t_{i-1,i}) + \lambda \xi_{i-1} q_i^{v_i} (-y_{i-1} + t_{i-1,i}) + \lambda \xi_{i-1} q_i^{v_{i-1}} T_{i-1,i} \right)$$

Then, using the fact that $\sum_{\beta \in \xi_{i-1}, \beta \neq \xi_{i-1}} q_i^{v_i} = q_{v_{i-1}} - 1$ we get

$$\eta_i \circ P_{v_{i-1},v_i}(y) = \left( \sum_{\beta \in \xi_{i-1}, \beta \neq \xi_{i-1}} \lambda \beta q_i^{v_i} y_i + \lambda \xi_{i-1} (q_i^{v_i} t_{i-1,i} - q_i^{v_{i-1}} y_{i-1}) \right) = \eta_{v_{i-1}}(y) - \lambda \xi_{i-1} q_i^{v_{i-1}} y_{i-1} + \lambda \xi_{i-1} q_i^{v_{i-1}} T_{i-1,i}.$$

Proof of theorem 7.15: By lemma 7.16

$$\{\eta_i \circ P_{v_{i-1},v_i}, G_i^{v_i} \circ P_{v_{i-1},v_i}\}_{\Pi_{v_{i-1}}} = \{\eta_i, G_i^{v_i}\}_{\Pi_{v_{i-1}}}.$$

As shown in the proof of lemma 7.11, $\{G_i^{v_i}, \eta_i\} \neq 0,$ so

$$\{\eta_i \circ P_{v_{i-1},v_i}, G_i^{v_i} \circ P_{v_{i-1},v_i}\} \neq 0,$$

and consequently the level sets of $(\eta_i \circ P_{v_{i-1},v_i}, G_i^{v_i} \circ P_{v_{i-1},v_i})$ are Poisson transversals. Furthermore, the Dirac structure on $\Pi_{v_{i-1}}$ generated by $(\eta_{v_{i-1}}, G_{v_{i-1}})$ is the same as the one generated by $(\eta_i \circ P_{v_{i-1},v_i}, G_i^{v_i} \circ P_{v_{i-1},v_i}).$ To see this, note that the foliation constituted from the level sets of $(\eta_{v_{i-1}}, G_{v_{i-1}})$ is the same as the one made up from the level set of $(\eta_i \circ P_{v_{i-1},v_i}, G_i^{v_i} \circ P_{v_{i-1},v_i}).$ Also, Dirac bracket (see (4.6)) defined by them is the same, since the second term in definition (4.6) is the same whether it is computed using $(\eta_{v_{i-1}}, G_{v_{i-1}})$ or $(\eta_i \circ P_{v_{i-1},v_i}, G_i^{v_i} \circ P_{v_{i-1},v_i}).$ Simply following the above equations

$$\left[ \frac{f, \eta_{v_{i-1}}}{G_{v_{i-1}}, \eta_{v_{i-1}}} \right] \left[ \frac{0}{G_{v_{i-1}}, \eta_{v_{i-1}}} \right]^{-1} \left[ \frac{\eta_{v_{i-1}}, G_{v_{i-1}}^{-1}}{G_{v_{i-1}}, G_{v_{i-1}}^{-1}} \right] = \left[ \frac{0}{G_{v_{i-1}}, \eta_{v_{i-1}}} \right]^{-1} \left[ \frac{\eta_{v_{i-1}}, G_{v_{i-1}}^{-1}}{G_{v_{i-1}}, G_{v_{i-1}}^{-1}} \right].$$
where $a = \lambda \xi_i \xi_{i-1}^1$. The constant terms are ignored and we used the fact that $\{G_{i}^{1}, aG_{i}^{1} \} = 0$ to simplify the middle term in the second equation.

We conclude that the diffeomorphism $P_{v_{i-1}, v_{i}}$, in addition to preserving the ambient Poisson structures, preserves the Dirac brackets as well, and consequently

$$
(P_{v_{i-1}, v_{i}}) |_{\Sigma_{v_{i-1}}^{v_{i}}} : (\Sigma_{v_{i-1}}^{v_{i}}, \overline{\pi}_{v_{i-1}}) \to (\Sigma_{v_{i}}^{v_{i}}, \overline{\pi}_{v_{i}}).
$$

Let $P_{\gamma_{i-1}}^{v_{i-1}}$ and $P_{\gamma_{i}}^{v_{i}}$ be the translations as defined in the proof of proposition 7.13. The restriction map $(P_{v_{i-1}, v_{i}}) |_{\Sigma_{v_{i-1}}^{v_{i}}}$ is also a translation, so there exists a vector $K_{(i-1), i}$ such that the following diagram is commutative.

This shows that the Poisson structures coming from different sides of $\Pi_{\gamma_{i-1}, \gamma_{i}}$ match and we can compose the Poisson map $L_{\gamma_{l-1}, \gamma_{l}}$ for $l = 1, \ldots, m + 1$. This finishes the proof.

For a given edge $v_{i-1} \gamma_{i-1} \gamma_{i}$ if there are more than one edge going out from the vertex $v_{i}$, say $\gamma_{k}$, with $k = 1, 2$, the $\Pi_{\gamma_{i-1}, \gamma_{i}}$ are disjoint open subsets of $\Pi_{\gamma_{i-1}}$. Considering all these disjoint Poisson submanifold all together we can state the following result whose proof is immediate from the previous results.

**Theorem 7.18.** Let $\mathcal{B}_S(\chi)$ denote the set of all $S$-branches of the skeleton vector field $\xi$ (see definition 6.12) and set $D_S := \bigcup_{\xi \in \mathcal{B}_S(\chi)} \Pi_{\xi}$ to be the open submanifold of

$$(\Pi_S, \{\cdot, \cdot\}_S) := \bigcup_{\gamma \in S} (\Pi_{\gamma}, \{\cdot, \cdot\}_\gamma),$$

with the same Poisson structure. Then the skeleton flow map $\pi_S : (D_S, \{\cdot, \cdot\}_S) \to (\Pi_S, \{\cdot, \cdot\}_S)$ is Poisson.

### 8. Example

We will now present an example of a Hamiltonian polymatrix replicator system with a non trivial dimension. This example was chosen to provide an illustration of the concepts and main results of this paper and at the same time to exhibit chaotic behaviour. The example was tuned to meet the following criteria:

1. After reduction to a level set of the Hamiltonian, the skeleton flow map becomes a piecewise affine area preserving map.
2. It has a small structural set with a simple but non trivial heteroclinic network.
3. It has a relatively small number (five in the example) of not too long branches (heteroclinic cycles).
4. The domains of all branches of the skeleton flow map are computationally visible (see figure 5).
8.1. The fish example

Consider the polymatrix replicator system defined by matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}.
\]

This example models a population divided in two groups with 5 and 2 strategies each. In the first group strategies are linearly ordered in a way the first wins the second, which wins the third and so on until the fifth. There are no interactions among individuals of the second group. The following relations hold regarding interactions between distinct groups: the second strategy of group 2 fosters the last and thrives on the first strategies of group 1. Simultaneously, the second strategy of group 2 inhibits the first strategy and declines with the fifth strategy of that group.

We denote by \(X_A\) the vector field associated to this polymatrix replicator that is defined on the polytope \(\Gamma_{(5,2)} := \Delta^4 \times \Delta^1\).

The point \(q = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)\) satisfies

(1) \(Aq = (0, 0, 0, 0, 0, 0)\);
(2) \(q_1 + q_2 + q_3 + q_4 + q_5 = 1\) and \(q_6 + q_7 = 1\),

where \(q_i\) stands for the \(i\)th component of \(q\), and hence is an equilibrium of \(X_A\) (see proposition 2.3). Since matrix \(A\) is skew-symmetric, the associated polymatrix replicator is conservative (see definition 5.1).

The polytope \(\Gamma_{(5,2)}\) has seven facets labelled by an index \(j\) ranging from 1 to 7, and designated by \(\sigma_1, \ldots, \sigma_7\). The vertices of the phase space \(\Gamma_{(5,2)}\) are also labelled by \(i \in \{1, \ldots, 10\}\), and designated by \(v_1, \ldots, v_{10}\), as described in table 1.

The skeleton character \(\chi_A\) of \(X_A\) is displayed in table 2. (See definition 6.3 and proposition 6.5.)

The edges of \(\Gamma_{(5,2)}\) are designated by \(\gamma_1, \ldots, \gamma_{25}\), according to table 3, where we write \(\gamma = (i,j)\) to mean that \(\gamma\) is an edge connecting the vertices \(v_i\) and \(v_j\). This model has 25 edges: 12 neutral edges,

\(\gamma_2, \gamma_3, \gamma_4, \gamma_7, \gamma_8, \gamma_10, \gamma_12, \gamma_16, \gamma_17, \gamma_18, \gamma_16, \gamma_22, \gamma_25,\)

and 13 flowing-edges,

\(\gamma_1, \gamma_5, \gamma_6, \gamma_9, \gamma_{11}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{19}, \gamma_{20}, \gamma_{21}, \gamma_{23}, \gamma_{24} .\)

The flowing-edge directed graph of \(\chi_A\) is depicted in figure 4.

From this graph we can see that

\(S = \{\gamma_1 = (1, 2)\}\)
is a structural set for $\chi_A$ (see definition 6.12) whose $S$-branches denoted by $\xi_1, \ldots, \xi_5$ are displayed in table 4, where we write $\xi_j = (j,k,l,\ldots)$ to indicate that $\xi_j$ is a path from vertex $v_j$ passing along vertices $v_k, v_l, \ldots$.

Considering the vertex $v_1$, which has the incoming edge $v_1 \xrightarrow{\gamma_1} v_2$ and the outgoing edge $v_1 \xrightarrow{\gamma_2} v_2$, we will now illustrate proposition 7.13.
Figure 4. The oriented graph of $\chi_A$.

Table 4. $S$-branches of $\chi_A$.

| From \(\gamma_i\) | To \(\gamma_j\) |
|-----------------|-----------------|
| $\gamma_i = (1,2)$ | $\xi_1 = (1,2,10,9,7,5,3,1,2)$ |
| $\gamma_i = (1,2)$ | $\xi_2 = (1,2,6,10,9,7,5,3,1,2)$ |
| $\gamma_i = (1,2)$ | $\xi_3 = (1,2,4,10,9,7,5,3,1,2)$ |
| $\gamma_i = (1,2)$ | $\xi_4 = (1,2,8,6,10,9,7,5,3,1,2)$ |
| $\gamma_i = (1,2)$ | $\xi_5 = (1,2,8,6,4,10,9,7,5,3,1,2)$ |

For $i = 1, 2, 3$, the constant Poisson structures $B_{v_i}$ induced by asymptotic rescaling on each $\Pi_{v_i}$ (see lemma 7.7) can be easily calculated:

$$B_{v_1} = \begin{pmatrix}
0 & 2 & 1 & 1 & 1 \\
-2 & 0 & 1 & 0 & 1 \\
-1 & -1 & 0 & 1 & 1 \\
-1 & 0 & -1 & 0 & 2 \\
-1 & -1 & -1 & -2 & 0
\end{pmatrix}, \quad B_{v_2} = \begin{pmatrix}
0 & 2 & 1 & 1 & -1 \\
-2 & 0 & 1 & 0 & -1 \\
-1 & -1 & 0 & 1 & -1 \\
-1 & 0 & -1 & 0 & -2 \\
1 & 1 & 1 & 2 & 0
\end{pmatrix}$$

and

$$B_{v_3} = \begin{pmatrix}
0 & -2 & -1 & -1 & -1 \\
2 & 0 & 2 & 1 & 0 \\
1 & -2 & 0 & 1 & 0 \\
1 & -1 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 & 0
\end{pmatrix},$$

and by (7.14) we get

$$(\pi^{v_1}_{\text{Dirac},2})^2 \pi^{v_2}_{\text{Dirac},0} = \begin{pmatrix}
0 & 1 & 0 & -1 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
The matrix \((\pi_{\text{Dirac},0}^\gamma)^2\) represents the Poisson structure on \(\Pi_\gamma\) in the coordinates \((y_2, y_3, y_4, y_5, y_7)\). Notice that \(y_2 = 0\) on \(\Pi_\gamma\). Similarly, the matrix \((\pi_{\text{Dirac},1}^\gamma)^2\) represents the Poisson structure on \(\Pi_\gamma\) in the same coordinates \((y_2, y_3, y_4, y_5, y_7)\). Notice again that \(y_7 = 0\) on \(\Pi_\gamma\). Now the matrix representative of \(L_{\gamma_6\gamma_1}\) in the coordinates \((y_2, y_3, y_4, y_5, y_7)\) is

\[
L_{\gamma_6\gamma_1} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

A simple calculation shows that

\[
L_{\gamma_6\gamma_1} (\pi_{\text{Dirac},0}^\gamma)^2 (L_{\gamma_6\gamma_1})^t = (\pi_{\text{Dirac},1}^\gamma)^2,
\]

confirming the fact that the asymptotic Poincaré map \(L_{\gamma_6\gamma_1}\) is Poisson (see (4.3) in definition 4.1).

Consider now the subspaces of \(\mathbb{R}^7\)

\[
H = \left\{ (x_1, \ldots, x_7) \in \mathbb{R}^7 : \sum_{i=1}^{5} x_i = 1, \sum_{i=6}^{7} x_i = 1 \right\}
\]

and

\[
H_0 = \left\{ (x_1, \ldots, x_7) \in \mathbb{R}^7 : \sum_{i=1}^{5} x_i = 0, \sum_{i=6}^{7} x_i = 0 \right\}.
\]

For the given matrix \(A\), its null space \(\text{Ker}(A)\) has dimension 3. Take a non-zero vector \(w \in \text{Ker}(A) \cap H_0\). For example,

\[
w = (-2, 3, -2, 3, -2, -3, 3).
\]

The set of equilibria of the natural extension of \(X_A\) to the affine hyperplane \(H\) is

\[
\text{Eq}(X_A) = \text{Ker}(A) \cap H = \{ q + tw : t \in \mathbb{R} \}.
\]

By (5.2) the Hamiltonian of \(X_A\) is the function \(h_q : \Gamma_{(5,2)} \to \mathbb{R}\)

\[
h_q(x) := \sum_{i=1}^{7} q_i \log x_i,
\]

where \(q_i\) is the \(i\)th component of the equilibrium point \(q\). Another integral of motion of \(X_A\) is the function \(h_w : \Gamma_{(5,2)} \to \mathbb{R}\)

\[
h_w(x) := \sum_{i=1}^{7} w_i \log x_i,
\]

where \(w_i\) is the \(i\)th component of \(w\), which is a Casimir of the underlying Poisson structure.
The skeletons of \( h_q \) and \( h_w \) are respectively \( \eta_q, \eta_w : C^*(\Gamma_{(5,2)}) \to \mathbb{R}, \)

\[
\eta_q(y) := \sum_{i=1}^7 q_i y_i \quad \text{and} \quad \eta_w(y) := \sum_{i=1}^7 w_i y_i,
\]

(see proposition 7.1), which we use to define \( \eta : C^*(\Gamma_{(5,2)}) \to \mathbb{R}^2, \)

\[
\eta(y) := (\eta_q(y), \eta_w(y)).
\]

Consider the skeleton flow map \( \pi_{\mathcal{S}} : \Pi_{\mathcal{S}} \to \Pi_{\mathcal{S}} \) of \( \chi_A \) (see definition 6.13). Notice that \( \Pi_{\mathcal{S}} = \Pi_{\mathcal{S}_1}, \) where by proposition 6.14, \( \Pi_{\mathcal{S}_i} = \bigcup_{i=1}^5 \Pi_{\Xi_i} \) (mod 0). By proposition 7.1 the function \( \eta \) is invariant under \( \pi_{\mathcal{S}}. \) Moreover, the skeleton flow map \( \pi_{\mathcal{S}} \) is Hamiltonian with respect to a Poisson structure on the system of cross sections \( \Pi_{\mathcal{S}} \) (see theorem 7.15).

For all \( i = 1, \ldots, 5, \) the polyhedral cone \( \Pi_{\Xi_i} \) has dimension 4. Hence, each polytope \( \Delta_{\Xi_i,c} := \Pi_{\Xi_i} \cap \eta^{-1}(c) \) is a 2-dimensional polygon.

**Remark 8.1.** We came from dimension 5 to 2. This will happen for any other conservative polymatrix replicator with the same number of groups and the same number of strategies per group. In fact when \( n - p \) is odd, where \( n \) is the total number of strategies in the population and \( p \) is the number of groups, we will have a minimum drop of three dimensions. The reason is that a Poisson manifold with odd dimension (in this example is 5) has at least one Casimir, and considering the transversal section we drop two dimensions from the symplectic part (not from the Casimir). If the original Poisson structure has more Casimirs, the invariant submanifolds yielded geometrically, are going to have even less dimensions, which is good as long as it is not zero. In the case of an even dimension, the drop will be at least of two dimensions.

By invariance of \( \eta, \) the set \( \Delta_{\mathcal{S},c} \) is also invariant under \( \pi_{\mathcal{S}}. \) Consider now the restriction \( \pi_{\mathcal{S},\Delta_{\mathcal{S},c}} \) of \( \pi_{\mathcal{S}} \) to \( \Delta_{\mathcal{S},c}. \) This is a piecewise affine area preserving map. Figure 5 shows the domain \( \Delta_{\mathcal{S},c} \) and 20000 iterates by \( \pi_{\mathcal{S}} \) of a random point in \( \Delta_{\mathcal{S},c}. \) Following the itinerary of the random point we have picked the following heteroclinic cycle consisting of 4S-branches

\[
\xi := (\xi_4, \xi_1, \xi_3, \xi_4).
\]

The map \( \pi_{\xi} \) is represented by the matrix

\[
M_{\xi} = \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -13 & 2 & -\frac{3}{2} & 0 \\
1 & 0 & 1 & -1 & 1 & 2 & 0 \\
-1 & 2 & -1 & \frac{15}{2} & -2 & \frac{5}{2} & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The eigenvalues of \( M_{\xi}, \) besides 0 and 1 (with geometric multiplicity 3 and 2, respectively), are

\[
\lambda_u = 5.31174 \ldots, \quad \text{and} \quad \lambda_i = \lambda_u^{-1}.
\]
Figure 5. Plot of 20000 iterates (in orange) by $\pi_S$ of a random point in $\Delta_{S,c}$, with $c = \left(\frac{1}{3}, -\frac{1}{2}\right)$, the iterates of the periodic point $p_0$ (in green) of the skeleton flow map $\pi_S$ with period 4, and the iterates of another periodic point of the skeleton flow map $\pi_S$ with period 14 (in blue).

Remark 8.2. The determinant of $(\pi^{-1}_{\text{Dirac},0})^e$ is zero which means that the Poisson structure on $\Pi_{\gamma_6}$ is non-degenerate. So, $\Pi_6$ has a two dimensional symplectic foliation invariant under the asymptotic Poincaré map. The leaf of this foliation are affine spaces parallel to the kernel of $$(\pi^{-1}_{\text{Dirac},0})^e|_{\Pi_{\gamma_6}} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix},$$ i.e. the set of the form $$\{(q_3, q_4, q_5, q_7) + (s, t, -t, -s) \mid (q_3, q_4, q_5, q_7) \in \Pi_{\gamma_6}, s, t \in \mathbb{R}\} \cap \Pi_{\gamma_6}.$$ The restriction of the asymptotic Poincaré map to these leaves is a symplectic map. One important consequence is that its eigenvalues are of the form $\lambda$ and $\frac{1}{\lambda}$.

An eigenvector associated to the eigenvalue 1 is $$p_0 = \left(0, 1, 0, 0, 0, 0\right).$$ We have chosen $c := (c_1, c_2) = \left(\frac{1}{3}, -\frac{1}{2}\right)$ so that $\eta(p_0) = c$, i.e., $p_0 \in \Delta_{S,c}$. In fact we have $p_0 \in \Delta_{S,c} \subset \Delta_{\gamma_1,c}$. Hence $p_0$ is a periodic point of the skeleton flow map $\pi_S$ with period 4 (whose iterates are represented by the green dots in figure 5).
Figure 5 also depicts the polygons $\Delta_{\xi_1, c}, \Delta_{\xi_2, c}, \Delta_{\xi_3, c}, \Delta_{\xi_4, c}, \Delta_{\xi_5, c}$ contained in $\Delta_{\gamma_1}$, and the orbit of another periodic point of the skeleton flow map $\pi_S$ with period 14 (represented by the blue dots in figure 5).

Following the procedure to analyse the dynamics in [6, section 9] and using theorem 8.7 also in [6] we could deduce the existence of chaotic behaviour for the flow of $X_A$ in some level set $h_q^{-1}(c_1/\epsilon) \cap h_w^{-1}(c_2/\epsilon)$, with the $c$ chosen above and for all small enough $\epsilon > 0$.

Data availability statement

No new data were created or analysed in this study.

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