ON NMHV FORM FACTORS IN $\mathcal{N}=4$ SYM THEORY FROM GENERALIZED UNITARITY

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Abstract

In this paper a supersymmetric version of a generalized unitarity cut method in application to MHV and NMHV for form factors of operators from the $\mathcal{N}=4$ SYM stress-tensor current supermultiplet $T^{AB}$ at one loop is discussed. The explicit answers for 3 and 4 point NMHV form factors at tree and one loop level are obtained. The general structure of n-point NMHV form factor at one loop is discussed as well as the relation between form factor with super momentum equal to zero and the logarithmic derivative of the superamplitude with respect to the coupling constant.

Keywords: Super Yang-Mills Theory, amplitudes, form factors, superspace.
1 Introduction

Much attention in the past decade has been paid to the study of the scattering amplitudes (the S-matrix) in four dimensional gauge theories, especially in the planar limit of $\mathcal{N} = 4$ SYM theory.

It is believed that the hidden symmetries of the $\mathcal{N} = 4$ SYM theory which are responsible for its integrability properties will completely fix the structure of the amplitudes. The hints that the S-matrix for the $\mathcal{N} = 4$ SYM theory can be fixed by some underlying integrable structure were found at weak [1, 2, 3] and strong [3, 4] coupling regimes.

There is another class of objects of interest in the $\mathcal{N} = 4$ SYM theory which resemble the amplitudes – the form factors which are the matrix elements of the form

$$ \langle p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} | \mathcal{O} | 0 \rangle, $$

(1.1)

where $\mathcal{O}$ is some gauge invariant operator which acts on the vacuum and produces some state $| p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} \rangle$ with momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$. The S-matrix

\footnote{Note that scattering amplitudes in "all going" notation can schematically be written as $\langle p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} | 0 \rangle$.}
operator is assumed in both cases. One can think about this object as an amplitude of the proses where classical current coupled through gauge invariant operator produces quantum state. The example of such process is $\gamma^* \to \text{Jet}'s$ in perturbative QCD [5] (see also [6,7] for recent results) where we take into account all orders in $\alpha_s$ but the first order in $\alpha_{em}$. The amplitude of such process is given by the matrix element of the following form:

$$\langle p_{1}^{\lambda_{1}} \cdots p_{n}^{\lambda_{n}} | j_{em}^{QCD} | 0 \rangle,$$

where $j_{em}^{QCD}$ is the QCD quark electromagnetic current operator.

The two-point form factors in $\mathcal{N} = 4$ SYM were studied long time ago in [8] and recently in [10]. Using the $\mathcal{N} = 3$ superfield formalism the form factors of none gauge invariant operators (off-shell currents) at tree level were derived in [9]. Recently, the strong coupling limit of form factors has been studied in [11] and the weak coupling regime in [12,13,14,15,16]. Also different regularizations for form factors were discussed in [17].

The motivations for the systematic study of form factors in $\mathcal{N} = 4$ SYM are

- it might help in understanding of the symmetry properties of the amplitudes \cite{1,2}. It is believed that the symmetries completely fix the amplitudes of the $\mathcal{N} = 4$ SYM theory and it is interesting to see whether they fix/restrict the form factors as well;
- the form factors are the intermediate objects between the fully on-shell quantities such as the amplitudes and the fully off-shell quantities such as the correlation functions (which are one of the central objects in AdS/CFT). Since the powerful computational methods have appeared recently for the amplitudes in $\mathcal{N} = 4$ SYM (see, for example, [18,19] and [20,21]), it would be desirable to have some analog of them for the correlation functions [22]. The understanding of the structure of form factors and the development of computational methods might shed light on the correlation functions;
- also, it might be useful for understanding of the relation between the conventional description of the gauge theory in terms of local operators and its (possible) description in terms of Wilson loops. The latter fact is the so-called amplitude/Wilson loop duality which originated for the case of $\mathcal{N} = 4$ SYM in [23,24,25]. This duality was intensively studied in the weak and strong coupling regimes and tested in different cases, and its generalizations to the non-MHV amplitudes were proposed in [26]. Moreover such dual description for amplitudes of $\mathcal{N} = 4$ SYM together with the developments of OPE technique for Wilson loops [27,28] led to the proposal of the equation [29] which in principle should define the whole $\mathcal{N} = 4$ SYM S-matrix for any value of coupling constant. Note that similar equation for the amplitudes may be derived from ”twistor space” (see for example [30]) point of view [31]. It is interesting to investigate whether such dual description for form factors in $\mathcal{N} = 4$ SYM exists and if it is possible to formulate this equation for form factors.

To make progress in the above-mentioned directions, the perturbative computations at several first orders of perturbative theory are likely required.
The aim of this paper is to continue investigations of the form factor [14] of the operators from stress tensor supermultiplet in $\mathcal{N} = 4$ SYM. We will use the developed in [14] formulation of form factors in $\mathcal{N} = 4$ on-shell momentum superspace, which allows us to consider form factors with different types of particles in $\langle p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} |$ external state in $\mathcal{N} = 4$ covariant manner. We will use the generalized unitarity technique to study the structure of NMHV sector at one loop. First we are going to discuss how the generalized unitarity technique works for form factors for MHV sector at one loop in $\mathcal{N} = 4$ on-shell momentum superspace. Then we will continue with the NMHV sector. We will perform explicit computations of 3 and 4 point NMHV form factors at one loop and will discuss the structure of the general n point situation. We make a brief comment on the relation between form factors with operator insertion with zero momentum and amplitudes.

2 Amplitudes and form factors in on-shell momentum superspace

2.1 Super form factors of chiral truncation of $\mathcal{N} = 4$ stress tensor supermultiplet

To describe the $\mathcal{N} = 4$ SYM stress tensor supermultiplet it is convenient to use standard $\mathcal{N} = 4$ coordinate superspace

$$\mathcal{N} = 4\text{ coordinate superspace } = \{x^{\alpha\dot{\alpha}}, \theta^A_\alpha, \bar{\theta}^\dot{A}^{\dot{\alpha}}\},$$

where $x^{\alpha\dot{\alpha}}$ are bosonic coordinates and $\theta$’s, which are $SU(4)_R$ vectors and Lorentz $SL(2,C)$ spinors, are fermionic ones. The $\mathcal{N} = 4$ supermultiplet of fields (containing $\phi^{AB}$ scalars (anti-symmetric in $SU(4)_R$ indices $AB$), $\psi^A_\alpha$, $\bar{\psi}_\dot{A}^{\dot{\alpha}}$ fermions and $F^{\mu\nu}$– the gauge field strength tensor, all in the adjoint representation of $SU(N_c)$ gauge group) is realized in $\mathcal{N} = 4$ coordinate superspace as a constrained superfield $W^{AB}(x, \theta, \bar{\theta})$ with the lowest component $W^{AB}(x, 0, 0) = \phi^{AB}(x)$. $W^{AB}$ in general is not a chiral object and satisfies several constraints [32] [14]: a self-duality constraint

$$W^{AB}(x, \theta, \bar{\theta}) = \overline{W_{AB}(x, \theta, \bar{\theta})} = \frac{1}{2} \varepsilon^{ABCD} W_{CD}(x, \theta, \bar{\theta}),$$

which implies $\phi^{AB} = \overline{\phi^{AB}} = \frac{1}{2} \varepsilon^{ABCD} \phi_{CD}$ and two additional constraints

$$D_C^a W^{AB}(x, \theta, \bar{\theta}) = \frac{2}{3} \delta_C^A D_C^a W^{B[L}(x, \theta, \bar{\theta}),$$

$$\bar{D}^{\dot{A}}(C) W^{A)B}(x, \theta, \bar{\theta}) = 0,$$

$^2[*, *]$ denotes antisymmetrization in indices, while $(*, *)$ denotes symmetrization in indices.
where $D^A_\alpha$ is a standard coordinate superspace derivative\(^3\). Note that in this formulation the full $\mathcal{N} = 4$ supermultiplet of fields is on-shell in the sense that the algebra of the generators $Q_A^\alpha, \bar{Q}_B^{\dot{\alpha}}$ of the supersymmetric transformation of the fields in this supermultiplet is closed only if the fields obey their equations of motion. The $\mathcal{N} = 4$ SYM stress tensor supermultiplet $T^{AB}$ is given then by

$$T^{AB} = \text{Tr} \left( W^{AB} W^{AB} \right). \tag{2.5}$$

We will consider in this article the chiral truncation of the $\mathcal{N} = 4$ SYM stress tensor supermultiplet (which contains only self-dual part of full multiplet) rather than the supermultiplet itself \([14]\). The main reason for this is that the chiral truncation has the off-shell description in terms of superfields on $\mathcal{N} = 4$ superspace i.e. the component fields in such truncated multiplet are arbitrary and the chiral part of the algebra of supersymmetric transformations of the component fields can be still closed without any constraints on the component fields. Note that the off-shell description for the full $\mathcal{N} = 4$ supermultiplet in any superspace is unknown.

To describe this truncated supermultiplet one has to break $SU(4)_R$ group into two $SU(2)$ and $U(1)$

$$SU(4)_R \rightarrow SU(2) \times SU(2)' \times U(1), \tag{2.6}$$

so that the index $A$ of $R$-symmetry group $SU(4)_R$ splits into

$$A \rightarrow (+a | -a'), \tag{2.7}$$

where $+a$ and $-a'$ correspond to two copies of $SU(2)$ and $\pm$ correspond to the $U(1)$ charge. We will not write the $U(1)$ factor explicitly hereafter, and will use the notation

$$(+a | -a') \equiv (a|\dot{a}). \tag{2.8}$$

After that one has to take the particular $(ab)$ projection of $W^{AB}$ that depends on half of the Grassmann coordinates \([13]\) (this can be seen from the (2.4)): $W^{ab}(x, \theta^c, \bar{\theta}^{\dot{c}})$. The truncated stress tensor supermultiplet is then given by

$$T^{ab} = Tr \left( W^{ab} W^{ab} \right) \bigg|_{\bar{\theta} = 0}. \tag{2.9}$$

To describe external states in $\mathcal{N} = 4$ covariant manner it is convenient to use $\mathcal{N} = 4$ on-shell momentum superspace \([33]\). This superspace is parameterized in terms of $SL(2,\mathbb{C})$ spinors $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \alpha, \dot{\alpha} = 1, 2$ and Grassmannian coordinates $\eta^A, A = 1, \ldots, 4$ which are Lorentz scalars and $SU(4)_R$ vectors

$$\text{On-shell } \mathcal{N} = 4 \text{ momentum superspace} = \{\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \eta^A\}. \tag{2.10}$$

\(^3\)which is $D^A_\alpha = \partial/\partial \theta^A_\alpha + i \bar{\theta}^{A\dot{\alpha}} \partial/\partial x^{\alpha\dot{\alpha}}$. 

5
In this superspace the creation/annihilation operators
\[ \{ g^-, \Gamma^A, \phi^{AB}, \bar{\Gamma}^A, g^+ \}, \]

of the \( \mathcal{N} = 4 \) supermultiplet, for the on-shell states which are two physical polarizations of gluons \(|g^-), |g^+\rangle\), four fermions \(|\Gamma^A\rangle\) with positive and four fermions \(|\bar{\Gamma}^A\rangle\) with negative helicity, and three complex scalars \(|\phi^{AB}\rangle\) (anti-symmetric in \(SU(4)_R\) indices \(AB\)) can be combined together into one \( \mathcal{N} = 4 \) invariant superstate ("superwave-function") \( |\Omega_i\rangle = \Omega_i |0\rangle \):

\[ |\Omega_i\rangle = \left( g_i^+ + \eta^A \Gamma_{i,A} + \frac{1}{2!} \eta^A \eta^B \phi_{i,AB} + \frac{1}{3!} \eta^A \eta^B \eta^C \varepsilon_{ABCD} \bar{\Gamma}_i^D + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \varepsilon_{ABCD} g_i^+ \right) |0\rangle, \tag{2.11} \]

where \( i \) corresponds to the on-shell momentum \( p_{\alpha \dot{\alpha}} = \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i \), \( p_i^2 = 0 \) carried by the particle. The \( n \) particle external state \(|\Omega_n\rangle\) is then given by \(|\Omega_n\rangle = \prod_{i=1}^n |\Omega_i\rangle \).

The form factor \( \mathcal{F}_n \) of the truncated stress tensor supermultiplet for general \( n \) particle external state is then given by:

\[ \mathcal{F}_n(\{ \lambda, \tilde{\lambda}, \eta \}, q, \theta^a) = \langle \Omega_n | \mathcal{T}_{ab}(x, \theta^a) |0\rangle, \tag{2.12} \]

where \( \{ \lambda, \tilde{\lambda}, \eta \} \) is short notation for \( (\lambda_1, \tilde{\lambda}_1, \eta_1 \ldots \lambda_n, \tilde{\lambda}_n, \eta_n) \). Here we are considering colour ordered object \( \mathcal{F}_n \). The physical form factor \( \mathcal{F}_n^{phys} \) in the planar limit\(^4 \) should be obtained from \( \mathcal{F}_n \) as:

\[ \mathcal{F}_n^{phys}(\{ \lambda, \tilde{\lambda}, \eta \}, q, \theta^a) = (2\pi)^4 g^{n-2} n^{n/2} \sum_{\sigma \in S_n/Z_n} Tr(t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}) \mathcal{F}_n(\sigma(\{ \lambda, \tilde{\lambda}, \eta \}), q, \theta^a), \tag{2.13} \]

where the sum runs over all possible none-cyclic permutations \( \sigma \) of the set \( \{ \lambda, \tilde{\lambda}, \eta \} \) and the trace involves \( SU(N_c) \) generators \( t^a \) in the fundamental representations. The normalization \( Tr(t^{a} t^{b}) = 1/2 \) is used.

Performing Fourier transform for bosonic coordinate \( x^{a \dot{\alpha}} \rightarrow q_{a \dot{\alpha}} \) and taking into account that \( \mathcal{F}_n \) is chiral, translationally invariant and \( \mathcal{T}^{ab} \) is \( 1/2 \)-BPS we see that \( \mathcal{F}_n \) should satisfy the following conditions \([14]\):

\[ P_{a \dot{\alpha}} \mathcal{F}_n = Q_{a \dot{\alpha}}^a \mathcal{F}_n = \bar{Q}_{a \dot{\alpha}}^a \mathcal{F}_n = 0, \tag{2.14} \]

where generators of supersymmetry algebra \((P_{a \dot{\alpha}}, Q_{a \dot{\alpha}}^a, \bar{Q}_{a \dot{\alpha}}^a, \bar{Q}_{a \dot{\alpha}})\) acting on \( \mathcal{F}_n \) are given by

\[ \text{4 translations } P_{a \dot{\alpha}} = - \sum_{i=1}^n \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i + q_{a \dot{\alpha}}, \]

\[^{4}g \rightarrow 0 \text{ and } N_c \rightarrow \infty \text{ of } SU(N_c) \text{ gauge group so that } \lambda = g^2 N_c = \text{fixed.} \]
4 supercharges $Q_\alpha^n = -\sum_{i=1}^n \lambda^i_\alpha \eta_i^n + \frac{\partial}{\partial \theta_a^n}$;

4 supercharges $Q_{\dot\alpha}^a = -\sum_{i=1}^n \lambda_{\dot\alpha}^i \eta_i^a + \frac{\partial}{\partial \theta_{\dot\alpha}^a}$;

4 conjugated supercharges $\bar{Q}_{a\dot\alpha} = -\sum_{i=1}^n \bar{\lambda}_{a\dot\alpha}^i \frac{\partial}{\partial \bar{\eta}_{a\dot\alpha}^i} + \theta_{a\dot\alpha} q_{a\dot\alpha}$;

4 conjugated supercharges $\bar{Q}_{\dot{a}\alpha} = -\sum_{i=1}^n \bar{\lambda}_{\dot{a}\alpha}^i \frac{\partial}{\partial \bar{\eta}_{\dot{a}\alpha}^i} + \theta_{\dot{a}\alpha} q_{\dot{a}\alpha}$. \hfill (2.15)

This relations imply that $\mathcal{F}_n$ takes the following form \cite{14}:

$$
\mathcal{F}_n(\{\lambda, \tilde{\lambda}, \eta, q, \theta_a\}) = \delta^4(\sum_{i=1}^n \lambda^i_\alpha \tilde{\lambda}^{\dot{i}}_{\dot{\alpha}} - q_{a\dot{\alpha}}) e^{\theta_a^\alpha q^a_{\dot{a}\alpha}} \delta^4_{GR}(q^a_{\dot{a}\alpha}) \mathcal{X}_n(\{\lambda, \tilde{\lambda}, \eta, q\})
$$

$$
\mathcal{X}_n = \mathcal{X}_n^{(0)} + \mathcal{X}_n^{(4)} + \ldots + \mathcal{X}_n^{(4n-8)}, \hfill (2.16)
$$

where

$$
q^a_{\dot{a}\alpha} = \sum_{i=1}^n \lambda^i_\alpha \eta_i^a, \quad q^\dot{a}_a = \sum_{i=1}^n \lambda_{\dot{a}}^i \eta_{i\dot{a}}^a, \quad \delta^4_{GR}(q^a_{\dot{a}\alpha}) = \prod_{a, \dot{a}, b, \dot{b}=1}^2 e^{\alpha_\beta q^a_{\dot{a}\alpha} q_b^\dot{b}}. \hfill (2.17)
$$

and $\mathcal{X}_n^{(4m)}$ are the homogenous $SU(4)_R$ invariant (more accurately $SU(2) \times SU(2)' \times U(1)$ invariant, they can be written in $SU(4)_R$ covariant form in $\mathcal{N}=4$ harmonic superspace formulation) polynomials of the order of $4m$.

Assigning helicity $\lambda = +1$ to $|\Omega_\lambda\rangle$ and $\lambda = +1/2$ to $\eta$ and $\lambda = -1/2$ to $\theta_a^n$, one sees that $\mathcal{F}_n$ has an overall helicity $\lambda_\Sigma = n$, $\delta^4_{GR}$ has $\lambda_\Sigma = 2$, exponential factor has $\lambda_\Sigma = 0$ so that $\mathcal{X}_n^{(0)}$ has $\lambda_\Sigma = n - 2$, $\mathcal{X}_n^{(4)}$ has $\lambda_\Sigma = n - 4$, etc. $\mathcal{X}_n^{(0)}$, $\mathcal{X}_n^{(4)}$ etc. are understood as analogs \cite{33} of the MHV, NMHV etc. parts of superamplitude i.e. part of super form factor proportional to $\mathcal{X}_n^{(0)}$ will contain component form factors with overall helicity $n-2$ which we will call MHV form factors, part of super form factor proportional to $\mathcal{X}_n^{(4)}$ will contain component form factors with overall helicity $n-4$ which we will call NMHV etc. up to $\mathcal{X}_n^{(4n-8)}$ overall helicity $2-n$ which we will call MHV.

It is convenient to perform transformation from $\theta_a^n$ to the set of axillary variables $\{\lambda', \eta'^a, \lambda'', \eta''^a\}$:

$$
\tilde{T}[\ldots] = \int d^4\theta_a^n \exp(\theta_a^n \sum_{i=1}^2 \lambda'^i \eta'^a_i) \ldots \hfill (2.18)
$$

After such transformation we can write $\tilde{T}[\mathcal{F}_n]$ as (let’s use the notation $\lambda'^i \eta'^a_i + \lambda''^i \eta''^a_i = \gamma^a_{\alpha}$):

$$
Z_n(\{\lambda, \tilde{\lambda}, \eta, \{\gamma^a_{\alpha}\}) = \tilde{T}[\mathcal{F}_n] = \delta^4(\sum_{i=1}^n \lambda'^i \tilde{\lambda}^{\dot{i}}_{\dot{\alpha}} - q_{a\dot{\alpha}}) e^{\theta_a^\alpha q^a_{\dot{a}\alpha} + \gamma^a_{\alpha} q^a_{\dot{a}\alpha}} \delta^4_{GR}(q^a_{\dot{a}\alpha}) \mathcal{X}_n. \hfill (2.19)
$$
The algorithm of obtaining component form factors from this supersymmetric expression was discussed in \[14\].

Let’s make a comment about total $U(1)$ charge of $\mathcal{F}_n$ and $\mathcal{T}[\mathcal{F}_n]$. The $\mathcal{T}_{ab}$ operator carries $(-4)$ charge, the $\langle \Omega_n \rangle$ carries charge $(0)$ so the $\mathcal{F}_n$ form factor should carry $(-4)$ charge, which indeed true and can be seen from (2.16): $\delta_{GR} (q_\alpha)$ carries $(-4)$ charge, while $e^{\theta q_\alpha}$ and $\lambda_{\alpha (4m)}$ are neutral. The $\mathcal{T}$ transformation (the integration measure $d^4 \theta_\alpha$) also carries $(+4)$ charge, so that $Z_n = \mathcal{T}[\mathcal{F}_n]$ is neutral with respect to $U(1)$. Note that this will be no longer true for the form factors of operators from different supermultiplets.

The formulation of form factors discussed so far lacks of explicit $SU(4)_R$ covariance. $SU(4)_R$ covariance can be restored in the $\mathcal{N} = 4$ harmonic superspace formulation. However such formulation does not give us any computational benefits for the purpose of our computation and all results obtained in our none covariant formulation can be easily translated to $SU(4)_R$ covariant formulation. We will discuss such formulation briefly in appendix.

### 2.2 Generalized unitarity for form factors at one loop

In general at one loop\(^5\) level $Z^{(1)}_n$ can be decomposed as combination of all possible scalar boxes ($B^{4m}, B^{3m}, B^{2mh}, B^{2me}, B^{1m}$) and triangles ($T^{3m}, T^{2m}, T^{1m}$) integrals:

\[
Z^{(1)}_n = \sum_i C^{4m}_i B^{4m}_i + C^{3m}_i B^{3m}_i + C^{2mh}_i B^{2mh}_i + C^{2me}_i B^{2me}_i + C^{1m}_i B^{1m}_i + \sum_j C^{3m}_j T^{3m}_j + C^{2m}_j T^{2m}_j + C^{1m}_j T^{1m}_j + \text{perm.,}
\]

(2.20)

where the sum runs over all possible distributions of the ordered set $(p_1, \ldots, p_n)$ of individual momenta between vertexes of the scalar integrals, while the position of the momentum $q$ carried by the operator is fixed. \text{perm.} corresponds to the cyclic permutations of the $(p_1, \ldots, p_n)$ set of the momenta of external particles. The latter is necessary due to the fact that while we are considering object that is colour ordered in the colour space of external particles the operator is colour singlet hence the momentum $q$ carried by the operator can be incepted at any position in the colour ordering \[13\]. This is equivalent to the consideration of all possible permutations of external momenta, while the position of the operator momenta $q$ is fixed. In general we can write scalar box integral as:

\[
B_{K_1^i, K_2^i, K_3^i, K_4^i} = \int \frac{d^Dl}{(2\pi)^D} \frac{1}{l^2 (K_1 + l)^2 (K_2 + l)^2 (l - K_4)^2},
\]

(2.21)

where $\sum_{i=1}^{4} K_1^i = \sum_{i=1}^{n} \lambda_\alpha T_\alpha - q_{\alpha\dot{\alpha}}$. The particular scalar box integrals $B^{4m}, B^{3m}, B^{2mh}, B^{2me}, B^{1m}$ are defined then as: for $B^{4m}_{K_1^i, K_2^i, K_3^i, K_4^i}$ all $K_2^2 \neq 0$, for $B^{3m}_{K_2^i, K_3^i, K_4^i}$ $K_1^2 = 0$, for

\(^5\)Hereafter we do not write common one loop factor $\lambda i \pi^{D/2} r_1^D/(2\pi)^D$ explicitly. See appendix.
\( B_{K_1^2,K_2^2}^{2m} K_1^2 = K_2^2 = 0, \) for \( B_{K_2^2,K_1^2}^{2m} K_1^2 = K_3^2 = 0, \) for \( B_{K_2^2,K_1^2}^{1m} K_1^2 = K_2^2 = K_3^2 = 0. \) For triangle integrals we use similar notations:

\[
T_{K_1^2,K_2^2,K_3^2} = \int \frac{d^D l}{(2\pi)^D l^2(K_1 + l)^2(l - K_3)^2},
\]

(2.22)

where \( \sum_{i=1}^3 K_{i\dot{a}} = \sum_{i=1}^n \lambda_i \dot{\lambda}_i - q_{\alpha\dot{a}}, \) for \( T_{K_1^2,K_2^2,K_3^2}^{3m} \) all \( K_i^2 \neq 0, \) for \( T_{K_2^2,K_3^2}^{2m} K_1^2 = 0, \) for \( T_{K_3^2,K_1^2}^{1m} K_1^2 = K_2^2 = 0. \)

The dependence on the helicities of the external particles as well as the type of operator are encoded in the \( C_k \) coefficients. The \( C_k \) coefficients are Grassmann polynomials and in general, as was explained earlier, should have the form:

\[
C_k \sim \delta_{GR}^{4} (q_4^a + \gamma^a_\alpha) \delta_{GR}^{4} (q_4^a) \left( C_k^{(0)} + C_k^{(4)} + \ldots + C_k^{(4m-8)} \right),
\]

(2.23)

where \( C_k^{(4m)} \) are the homogenous \( SU(4)_R \) invariant polynomials of the order of \( 4m. \) For example coefficients before scalar integrals for NMHV form factors will be proportional to \( \delta_{GR}^{4} (q_4^a + \gamma^a_\alpha) \delta_{GR}^{4} (q_4^a) C_k^{(4)} \). The analytical answers for all types of one loop triangles and boxes are known (see [33] for review) and therefore the problem of computation of \( Z_n^{(1)} \) reduces to the determination of \( C_k \) coefficients. The latter are computed in the unitarity based methods by comparing the analytical properties of both sides of the relation (2.20) viewed as the functions of Mandelstam kinematical invariants of momenta of external particles.

To obtain the values of the coefficients \( C_i \) before box integrals it is very convenient to consider quadruple cuts [35, 36]. Such cuts are unique to each box integral\(^6\) and stop the loop momenta flow. Therefore the quadruple cut completely determines the value of the \( C_i \) coefficient for the chosen box integral. One can schematically write that

\[
C_i = \frac{1}{2} \sum_{\pm S} \prod_{i}^4 d^2 \eta_i d^2 \eta_i \hat{Z}_{tree} \times \hat{A}_{tree} \times \hat{A}_{tree},
\]

(2.24)

where \( \sum_{\pm S} \) corresponds to the summation over the solutions of the on-shell and momentum conservation conditions [36, 37]. We will not write \( \sum_{\pm S} \) sums explicitly in the most cases in the next chapters. Note that as in the case of amplitudes [35] for the MHV and NMHV form factors we will not need the explicit form of the solution. Also note that the summation over the states (types of particles) that runs through the cuts is ”hidden” in the Grassmann integration (”supersums”). Since we are working with \( \mathcal{N} = 4 \) SYM at one loop all cuts can be evaluated in \( D = 4 \) with \( O(\epsilon) \) accuracy. \( \hat{Z}_n \) and \( \hat{A}_n \) correspond to form factors and amplitudes stripped from the overall delta function \( \delta^4(\sum_{i=1}^n \lambda_i \dot{\lambda}_i - q_{\alpha\dot{a}}) \) of momentum conservation. The exact types of form factors and amplitudes entering the

\(^6\)Each scalar box can be uniquely specified by its leading singularity. The latter are obtained by cutting all four scalar propagators in the integral [35].
expression are determined by what type of one-loop form factor (MHV, NMHV, etc.) and what particular cut we are considering.

The situation with the coefficients before triangle scalar integrals is a little more involved. Though triple cuts are unique to each triangle integral the triple cut do not stop the flow of the loop momenta completely and leaves one parameter integral \( \int dt \). However one can construct algorithm that allows one to fix the corresponding coefficient using the form of the triple cut intergrand \[37, 48\]. One can parametrise the momenta of particles \( l_i^{\alpha} \) which crosses the cuts and associated spinors \( \lambda_i, \tilde{\lambda}_i \) in terms of combinations of external momenta and the \( t \) parameter which is the remainder of loop integral (see appendix and \[37\] for details). Then one can obtain the following relation

\[
C_j = \text{Inf}_t \left[ \frac{1}{2} \sum_{\pm S} \int_0^1 d^2 \eta_i^a d^2 \eta_i^{\dot{\alpha}} (\tilde{Z}_{\text{tree}} \times \tilde{A}_{\text{tree}} \times \tilde{A}_{\text{tree}}) \right](t) \bigg|_{t=0},
\]

where \( \text{Inf}_t \) means that one takes expansion in \( t \) at \( t \to \infty \) and separate term proportional to \( t^0 \). In other words the coefficient before scalar triangle integral is given by the first term in the series expansion of the corresponding triple cut integrand in \( t \) at infinity \[37, 48\]. In the case of MHV form factors we will not need the explicit form of the solutions \( \pm S \), while the case of NMHV form factor is more involved. We will use IR properties of the form factors (see below) in some cases to adjust the value of the coefficients before triangle integrals instead of the direct computations.

### 2.3 Grassmann delta functions

Throughout this article we will need different types of the Grassmann valued delta functions. The following notations will be used\[7\] for \( \delta_{GR}^{4} \) type delta functions introduced earlier:

\[
\delta^8(X^A_\alpha) \equiv \delta_{GR}^4(X^a_\alpha)\delta_{GR}^4(X^{\dot{a}}_\alpha),
\]

while for general \( N \) we have:

\[
\delta^{2N}(X^A_\alpha) = \prod_{A,B=1}^{N} \epsilon^{\alpha\beta} X^A_\beta X^B_\alpha.
\]

We also will use the Grassmann delta functions of another type:

\[
\tilde{\delta}^N(\sum_i \eta_i^A C_i) = \prod_{A=1}^{N} (\sum_i \eta_i^A C_i),
\]

\[\text{The relevant for us cases are } N = 2, N = 4.\]
where $A$ index runs from 1 to $N$, $\eta^A_i$ are Grassmann variables and $C_i$ are bosonic ones. For such delta functions we will use the following notation:

$$\delta^4\left(\sum_i \eta^A_i C_i\right) \equiv \delta^2\left(\sum_i \eta^A_i\right) \delta^2\left(\sum_i \eta^A_i C_i\right).$$

We will also use for saving space the notation $d^2\eta^a d^2\eta^\dot{a} \equiv d^4\eta^A$ for the integration measure. In computation of the corresponding coefficients before scalar boxes and triangles we will be performing multiple Grassmann integrals with Grassmann delta functions. The following relation is extremely useful: for some $q^A_\alpha$ ($A$ runs from 1 to $N$) $q^A_\alpha = \sum_{i=1}^n \lambda_i^\alpha \eta^A_i$ one can write the following expansion over some basis

$$q^A_\alpha = \lambda^l_\alpha \langle ml \rangle + \lambda^m_\alpha \langle lq \rangle, \quad 1 \leq l \leq n, \quad 1 \leq m \leq n, \quad m \neq l,$$

where $\lambda_l, \lambda_m$ should be linear independent. The latter relation implies that

$$\delta^{2N}(q^A_\alpha) = \langle ml \rangle^{N} \delta^{N} \left(\eta^A_i + \sum_{i=1}^n \langle ml \rangle \eta^A_i\right) \delta^{N} \left(\eta^A_m + \sum_{i=1}^n \langle lm \rangle \eta^A_i\right), \quad i \neq l, \ i \neq m. \quad (2.30)$$

For example using this relation one can immediately show that [14]

$$\int d^N \eta^A_i d^N \eta^\dot{a}_i \delta^{2N} \left(\lambda^1_\alpha \eta^A_1 + \lambda^2_\alpha \eta^A_2 + Q^A_\alpha\right) \delta^{2N} \left(\lambda^1_\alpha \eta^\dot{a}_1 + \lambda^2_\alpha \eta^\dot{a}_2 - P^A_\alpha\right)$$

$$= \langle l_1 l_2 \rangle^N \delta^{2N} \left(P^A_\alpha + Q^A_\alpha\right), \quad (2.31)$$

which is important relation for the two particle supersums.

We also want to note that the computation of integrals over $\int d^2\eta^a_i$ and $\int d^2\eta^\dot{a}_i$ in the quadruple and triple cuts may be different in details, but can be performed in such a way that leads to the same bosonic coefficient and slightly different Grassmann delta functions. One can formulate the following rule to simplify computations: one takes integrals over $\int d^2\eta^a_i d^2\eta^\dot{a}_i$ formally replacing

$$\delta^A_{GR}(X^A_\alpha + \gamma^A_\alpha) \delta^A_{GR}(X^\dot{A}_\alpha) \rightarrow \delta^8(X^A_\alpha + \gamma^A_\alpha)$$

and integrating over $\int d^4\eta^A_i$. After the integration we have to put $\gamma^A_\alpha = 0$.

### 2.4 Tree level MHV and $\mathbf{\overline{MHV}}$ amplitudes and form factors

We will need as the building blocks in the computation of the coefficients before scalar integrals in the MHV and NMHV case at one loop level several explicit expressions for the tree level form factors [14] and amplitudes [36]:

$$Z^{tree, MHV}_n = \delta^4\left(\sum_{i=1}^n \lambda^A_i \bar{\lambda}^\dot{a}_i + q^{\dot{a}_i} \right) \frac{\delta^4_{GR} \left(q^\dot{a}_i + \gamma^\dot{a}_i\right) \delta^4_{GR} \left(q_i\right)}{\langle 12 \rangle \ldots \langle n1 \rangle}. \quad (2.33)$$
This expression is valid for any $n \geq 2$ without any kinematical constraints on $\lambda_i, \tilde{\lambda}_i$ variables. For the MHV amplitudes we have:

$$A_{\text{tree, MHV}}^n = \delta^4 \left( \sum_{i=1}^{n} \lambda_i^a \tilde{\lambda}_i^\dot{a} \right) \frac{\delta^8(q_i^4)}{\langle 12 \rangle \ldots \langle n1 \rangle}.$$  \hspace{1cm} (2.34)

This expression is valid for any $n \geq 4$ without any kinematical constraints. While for $n = 3$ this expression exists only for the complex values of the momenta in $(+ - - -)$ signature. This implies the following kinematical constraints on $\lambda_i, \tilde{\lambda}_i$ variables:

$$\text{MHV}_3 : \tilde{\lambda}_i \langle ik \rangle = -\tilde{\lambda}_j \langle jk \rangle, \ [ij] = 0, \ \text{any } i, j, k \text{ at the same vertex.}$$  \hspace{1cm} (2.35)

We will also need the expression for the three point $\overline{\text{MHV}}_3$ amplitude:

$$A_{\text{tree, } \overline{\text{MHV}}}^3 = \delta^4 \left( \sum_{i=1}^{3} \lambda_i^a \tilde{\lambda}_i^\dot{a} \right) \frac{\delta^8(q_i^4)[23] + \text{cycl. perm.}}{[12][23][31]}.$$  \hspace{1cm} (2.36)

This expression also exists only for the complex values of the momenta, that implies the following constraints:

$$\overline{\text{MHV}}_3 : \lambda_i [ik] = -\lambda_j [jk], \ \langle ij \rangle = 0, \ \text{any } i, j, k \text{ at the same vertex.}$$

Note that this expression is 4’th degree in $\eta$’s while all MHV amplitudes and form factors are of the 8’th degree. We also will need the form of $Z_{\text{tree, } \overline{\text{MHV}}}$ which is NMHV form factor at the same time in full analogy with $n = 5$ point amplitude. We will discuss the structure of it in separate section later on.

The discussed above kinematical constrains on $\lambda_i, \tilde{\lambda}_i$ spinors immediately lead to the fact that some configurations of MHV and $\overline{\text{MHV}}$ vertexes give vanishing result. See fig.1.

### 3 MHV warm-up

As a warm-up before computations of NMHV form factors we will discuss how generalized unitarity works in MHV case. MHV form factors should be the lowest components in $\eta$’s expansion of $Z_n$ of Grassmann degree 8. This implies the following configurations of MHV and $\overline{\text{MHV}}$ vertexes

- for the quadruple cut integrand, and
- for the triple cut integrand. Taking into account that configurations of vertexes depicted on fig.1 vanish we conclude that the only contributing cuts are those depicted on fig.2 and fig.3. They correspond to the $B^{2\text{me}}$ (and $B^{1m}$ as the limiting case) and $T^{2m}$ scalar

---

8One can make analytical continuation to the real values of the momenta of external particles in the final expressions.
integrals. So the MHV form factor should have the form

$$Z_{n, MHV}^{(1)} = \sum_i C_i^{2me} B_i^{2me} + \sum_j C_j^{2m} T_j^{2m} + \text{perm.}, \quad (3.37)$$

($B^{3m}$ scalar integrals can also appear as the limiting case of $B^{2me}$). Let’s start with the quadruple cuts for $B^{2me}$ integral. I.e. we are going to compute the corresponding $C^{2me}$ coefficient before such integral. This computation is very similar to those in [36]. We will treat the configurations where the vertex $\hat{Z}_{tree, MHV}$ is present and configurations with $\hat{Z}_{tree, MHV}$, $n \geq 3$ separately. The usefulness of such treatment will be clear later on.

We have for the configuration $A$ which corresponds to $B^{2me}_{s_2...s_1...s}$ scalar integral $9$ (the notation $s_{r...s} = \left(\sum_{i=r}^{s-1} p_i\right)^2$)

$$s_{r...s-1} = \left(\sum_{i=r}^{s-1} p_i\right)^2 = p_{r...s-1}^2 \quad (3.38)$$
is used:

$$C_{s_2...s_1...s}^{2me} = \int \prod_{i=1}^{4} d^4\eta_i^A \frac{\delta^4(\eta_1[l_2l_4] - \eta_2[l_11] + \eta_4[l_31]) \delta^4(\eta_2[l_4l_3] - \eta_4[l_3s] + \eta_3[l_4s])}{[l_2][l_4][l_3][l_1][l_3s]} \times \frac{\delta^4(\sum_{i=s+1}^{n} \lambda_i \eta_i^a + \gamma^a + \lambda_i \eta_i^a - \lambda_i \eta_i^a)}{\langle l_1l_4 \rangle \ldots \langle nl_1 \rangle} \times \frac{\delta^4(\sum_{i=2}^{s-1} \lambda_i \eta_i^A + \lambda_i \eta_i^A - \lambda_i \eta_i^A)}{\langle l_2l_3 \rangle \ldots \langle 2l_2 \rangle} \quad (3.39)$$

$9$ We will suppress $\alpha$ $SL(2,\mathbb{C})$ indexes in arguments of Grassmann delta functions in some cases.
Figure 2: All possible cuts for the box scalar integrals to MHV form factor. Dark grey vertex is MHV form factor, grey vertex is MHV amplitude, white vertex is $\overline{\text{MHV}}_3$ amplitude.

Figure 3: All possible cuts for the triangle scalar integrals to MHV form factor.

Performing Grassmann integration with first two $\hat{\delta}^4$ and then integrating the remaining delta functions with the use of $\eqref{2.30}$ and kinematical constraints of the $\overline{\text{MHV}}$ vertexes ($\eqref{2.32}$ also can be used to simplify algebraic manipulations) we obtain

$$C_{s_2...s_{-1}s_1...s}^{2\text{me}} = Z_{n}^{\text{tree, MHV}} \frac{\langle 12 \rangle \langle 2s-1s \rangle \langle ss+1 \rangle \langle n1 \rangle [1][l_1l_3][s][s][s][s][s][s]}{[1][l_1l_4][s][1][l_2l_3][s][l_2][l_4][l_3][s]} \langle s-1l_3 \rangle \langle 2l_2 \rangle \langle l_3l_4 \rangle \langle n_1 \rangle \langle s+1l_4 \rangle \langle n1 \rangle. \quad \eqref{3.40}$$

This expression can be transformed with the use of momentum conservation conditions in each vertex and $\overline{\text{MHV}}$ kinematical constraints to the form

$$C_{s_2...s_{-1}s_1...s}^{2\text{me}} = Z_{n}^{\text{tree, MHV}} \frac{1}{2} Tr(l_2l_1l_4l_3) = Z_{n}^{\text{tree, MHV}} \frac{1}{2} \left( (l_1 - l_4)^2(l_3 - l_2)^2 - (l_1 - l_3)^2(l_4 - l_2)^2 \right). \quad \eqref{3.41}$$

14
This result is identical to the MHV amplitude case \[36\]. Note also that a $s$ was in the amplitude case the $\sum_{\pm s}$ can be evaluated without the use of explicit solutions for $l_i$. So we have that the coefficient before $B_{s_2,...,s_1,...}^{2me}$ scalar integral takes the form:

$$C_{s_2,...,s_1,...}^{2me} = Z_n^{tree,MHV} \frac{1}{2} \Delta_{s_2,...,s_1,...}^{2me}$$

The case of configuration $B$ which corresponds to the $B_{s_2,...,q^2}$ scalar integral is slightly different in computational details:

$$C_{s_2,...,q^2}^{2me} = \int \prod_{i=1}^4 d^4 \eta_i A_i \frac{\delta^4(\eta_1[l_2l_1] - \eta_2[l_1] + \eta_1[1l_2])}{[1l_2][l_2][l_1][1]} \frac{\delta^4(\eta_1[l_3l_2] - \eta_3[l_2] + \eta_1[1l_3])}{[n_4][l_3][l_2][n]}$$

$$\times \frac{\delta^4(\sum_{-l_1,l_4} \lambda_i \eta_i^2 + \gamma^2) \delta^4(\sum_{-l_1,l_4} \lambda_i \eta_i^2) \delta^8(\sum_{i=2}^{n-1} \lambda_i \eta_i^2 + \lambda_2 \eta_3^2 - \lambda_3 \eta_2^2)}{\langle l_1 l_4 \rangle^2}$$

but leads to essentially the same final result:

$$C_{s_2,...,q^2}^{2me} = Z_n^{tree,MHV} \frac{\langle 12 \rangle \langle n - 1n \rangle \langle 1n1 \rangle[1][l_1 l_1][n]}{\langle 11 l_1 l_1 | n \rangle} \langle 11 l_2 l_2 \rangle \langle n - 1n \rangle \langle 11 l_2 l_2 \rangle \langle 12 l_2 \rangle \langle 2 l_2 \rangle \langle l_3 l_2 \rangle \langle l_3 n \rangle \langle 1 l_4 \rangle$$

which, as in the previous case, can be simplified so that the coefficient before $B_{s_2,...,q^2}$ scalar integral takes the form:

$$C_{s_2,...,q^2}^{2me} = Z_n^{tree,MHV} \frac{1}{2} \Delta_{s_2,...,q^2}^{2me}$$

$$\Delta_{s_2,...,q^2}^{2me} = s_{2,...,q^2} - s_{1,...,1}$$

Now let’s consider triple cuts. Let’s consider the $A$ configuration which corresponds to the coefficient before $T_{s_2,...,q^2}$ scalar integral. We have for the corresponding triple cut integrand

$$\int \prod_{i=1}^3 d^4 \eta_i A_i \frac{\delta^4(\eta_1[l_2l_1] - \eta_2[l_1] + \eta_1[1l_2])}{[1l_2][l_2][l_1][1]} \frac{\delta^4(\sum_{l_2,l_3} \lambda_i \eta_i^2 + \gamma^2) \delta^4(\sum_{l_2,l_3} \lambda_i \eta_i^2)}{\langle l_2 l_3 \rangle^2}$$

$$\times \frac{\delta^8(\sum_{i=2}^{n-1} \lambda_i \eta_i^2 + \lambda_2 \eta_3^2 - \lambda_3 \eta_2^2)}{\langle 1 l_1 \rangle \ldots \langle n l_1 \rangle}.$$
This expression can be further simplified with the use of the momentum conservation conditions associated with each vertex to the form:

\[
Z_{n}^{\text{tree,MHV}} \frac{\text{Tr}(2l_{2}l_{3})}{(l_{3}2)},
\]

where the trace can be evaluated, with the use of momentum conservation conditions and kinematical constraints of the MHV vertex to the form

\[
Z_{n}^{\text{tree,MHV}} \frac{1}{4} \left( 2(s_{2...n} - q^2) + \sum_{\pm S} \frac{(q^2)s_{2...n} - (2,1+q)q^2}{(l_{3}2)} \right).
\]

Now applying the following parametrization for \( l_{i}^{\alpha} \): \( l_{i}^{\alpha} = A_{i}^{\alpha} + tA_{i}^{\alpha} + 1/tA_{i}^{\alpha} \), where \( A_{1,2,3}^{\alpha} \) are some constants that depend on external momenta (see [37] and appendix for details), we see that

\[
C_{s_{2...n},q^{2}}^{2m} = Z_{n}^{\text{tree,MHV}} \frac{1}{4} \text{Inf}_{l}[2(s_{2...n} - q^2) + \sum_{\pm S} \frac{(q^2)s_{2...n} - (2,1+q)q^2}{(l_{3}2)}]_{l=0}
\]

so that the coefficient before \( T_{s_{2...n},q^{2}}^{2m} \) triangle scalar integral takes the form:

\[
C_{s_{2...n},q^{2}}^{2m} = Z_{n}^{\text{tree,MHV}} \frac{1}{2} \Delta_{s_{2...n},q^{2}}^{2m}
\]

so that the coefficient before \( T_{s_{2...n},q^{2}}^{2m} \) triangle scalar integral takes the form:

\[
\Delta_{s_{2...n},q^{2}}^{2m} = s_{2...n} - q^2.
\]

Now let’s consider the \( B \) configuration which corresponds to the coefficient before scalar integral \( T_{s_{2...n},q^{2}}^{2m} \). We have for the corresponding triple cut integrand

\[
\int \prod_{i=1}^{3} q^{i} \eta_{i}^{A} \delta^{4}(\eta_{i}[l_{2}l_{1}] - \eta_{i}[l_{1}l_{2}]) \delta^{8}(\sum_{i=s+1}^{n} \lambda_{i}^{A} + \lambda_{i}^{A} - \lambda_{i}^{A}) \frac{[l_{2}[l_{2}l_{1}][l_{1}]}{[l_{1}l_{2}]...[n]} \lambda_{i}^{A} - \lambda_{i}^{A}) \delta_{GR}^{4}(\sum_{i=2}^{s} \lambda_{i}^{A} + \lambda_{i}^{A} - \lambda_{i}^{A}) \frac{[l_{2}[l_{2}l_{1}][l_{1}]}{[l_{1}l_{2}]...[n]} \lambda_{i}^{A} - \lambda_{i}^{A})
\]

Performing the Grassmann integration with delta functions we obtain:

\[
Z_{n}^{\text{tree,MHV}} \frac{\langle l_{2}2 \rangle \langle s_{3}l_{3} \rangle \langle l_{3}d_{2} \rangle \langle l_{2}l_{2}[l_{2}l_{1}][l_{1}]}{[l_{1}l_{2}]...[n]} \lambda_{i}^{A} - \lambda_{i}^{A}) \frac{[l_{2}[l_{2}l_{1}][l_{1}]}{[l_{1}l_{2}]...[n]} \lambda_{i}^{A} - \lambda_{i}^{A})
\]

\[
\text{We use the following notations for the scalar products of momenta of external particles} \ (p_{i...k}) = (i + ... + l, j + ... + k).
\]

10
This expression can be simplified with the use of the momentum conservation conditions associated with each vertex to the form

\[
Z_{\text{tree}, \text{MHV}} n \frac{1}{4} \sum_{\pm S} \left( \frac{\text{Tr}(l_1 l_2 s + 1 l_3)}{(l_3 s + 1)} - \frac{\text{Tr}(l_1 l_2 s l_3)}{(l_3 s)} \right). \tag{3.54}
\]

The traces can be evaluated as in the previous case and we get

\[
Z_{\text{tree}, \text{MHV}} n \frac{1}{4} \sum_{\pm S} \left( \frac{D_{1s+1}}{(l_3 s + 1)} + \frac{D_{1s}}{(l_3 s)} \right),
\]

\[
D_{1j} = (1, 1 + \sum_{k=s+1}^{n} k)(j, 1 + \sum_{k=s+1}^{n} k) - (1 j)(1 + \sum_{k=s+1}^{n} k)^2. \tag{3.55}
\]

Note that \(D_{1j}\) depends only on the external momenta. So using the parametrization for \(l_i^a\): \(l_i^a = A_i^a + t A_i^a + 1/t A_i^a\), where \(A_i^a, a, 2, 3\) are some constants that depend on external momenta, we see that

\[
C_{s_1 + \ldots + s_n} = Z_{\text{tree}, \text{MHV}} n \frac{1}{4} \sum_{\pm S} \left. \right| \frac{\text{Tr}(l_1 l_2 s + 1 l_3)}{(l_3 s + 1)} + \frac{\text{Tr}(l_1 l_2 s l_3)}{(l_3 s)} \bigg|_{t=0} = 0. \tag{3.56}
\]

So we conclude that potentially contributing integrals \(T_{2m} s_{s+1, n}, s_{1n}\) does not actually appear. This fact was first established in [13].

The latter fact may be puzzling. The resolution of this puzzle comes from the observation that \(Z_{\text{tree}, \text{MHV}}^2\) and \(Z_{\text{tree}, \text{MHV}}^n\), \(n > 2\) vertexes that enter \(A\) and \(B\) triple cuts have different number of \(\lambda_i\) spinors in denominators, so that the \(Z_{\text{tree}, \text{MHV}}^2\) vertex is in some sense singled out (that’s why we treated \(A\) and \(B\) cases separately for the quadruple cuts)

\[
Z_{\text{tree}, \text{MHV}}^2 \sim \frac{1}{(l_2 l_3)^2}, \quad Z_{\text{tree}, \text{MHV}}^n \sim \frac{1}{(l_2)(l_2 l_3)(j l_3)}. \]

Using explicit solutions for \(\lambda_i\) in terms of \(t\) and external momenta [37] for \(T_{2m}\) triangles one sees that

\[
\frac{1}{(l_2 l_3)^2} \sim t^0,
\]

so if we take into account the whole expression we have none vanishing contribution in \(t \to \infty\) limit, while

\[
\frac{1}{(l_2)(l_2 l_3)(j l_3)} \sim t^{-1},
\]

so in the \(t \to \infty\) limit we get zero. We verified that this is the general pattern for MHV and NMHV cases for the \(T_{2m}\) and \(T_{3m}\) integrals, so on general grounds we can conclude that the only allowed triangle integrals should necessary contain one massive \(q^2\) leg in MHV.
and NMHV sectors. This fact reduces the number of necessary triple cuts significantly. Note also that the properties of $Z_{\text{tree, MHV}}^2$ vertex in the $t \to \infty$ limit resemble those of the $z \to \infty$ limit in the BCFW recursion. One also can use only $Z_{\text{tree, MHV}}^n$, $n > 2$ but not $Z_{\text{tree, MHV}}^2$ in the BCFW recursion \[13\].

We see now that the contributing type of scalar integrals in the MHV case are $B_{s_2, s_3, \ldots, s_{n-1}, s_{1\ldots s_i}}^2$, $T_{s_2 \ldots s_n, q^2}^{2m}$ (and $B_{s_2, s_3, \ldots, s_{n-1}, s_{1\ldots s_i}}^{2m}$ as the limiting case of $B_{s_2, s_3, \ldots, s_{n-1}, s_{1\ldots s_i}}^2$). The coefficients before these integrals are given correspondingly by (3.42) and (3.50).

Note also that the combinations of scalar products of external momenta $\Delta$’s in the coefficients before corresponding scalar integrals (boxes and triangles) match the scalar integral in such way that the coefficients before the $1/\epsilon^2$ IR pole (the use of the dimensional regularization is implied) will have the form $\sim \Delta^{-1}/\epsilon^2$ (see appendix). This allows us to define dimensionless functions $B^i$ and $T^i$, just as in the case of amplitudes \[36\], for all types of boxes and triangles as the result of evaluation of the corresponding scalar integral through $O(\epsilon)$ multiplied by the corresponding $\Delta$ coefficient.

As an example, for three point MHV form factor $Z_{\text{MHV}}^{(1)}$ one can obtain, in precise agreement with \[13, 14\]:

$$Z_{\text{MHV}}^{(1)} = \frac{1}{2} B^1 (1, 2, 3|q^2) + \frac{1}{2} B^1 (1, 3, 2|q^2) + \frac{1}{2} B^1 (2, 1, 3|q^2) + T^2 (1|q^2, (2 + 3)^2) + T^2 (3|q^2, (1 + 2)^2).$$

Here we write the ordering of massive/massless legs explicitly, using the convention:

$$G^i(\text{massless legs}|(\text{massive legs})^2),$$

where $G^i$ is dimensionless function based on the type of scalar integral under consideration. The IR divergent part of this result is given by:

$$Z_{\text{MHV}}^{(1)} / Z_{\text{tree, MHV}}^{(1)} \bigg|_{\text{IR}} = \frac{1}{\epsilon^2} \sum_{i=1}^{3} \left( \frac{s_{ii} + 1}{\mu^2} \right)^\epsilon,$$

while finite part is:

$$Z_{\text{MHV}}^{(1)} / Z_{\text{tree, MHV}}^{(1)} \bigg|_{\text{fin}} = -\text{Li}_2 \left( 1 - \frac{q^2}{s_{12}} \right) - \text{Li}_2 \left( 1 - \frac{q^2}{s_{23}} \right) - \text{Li}_2 \left( 1 - \frac{q^2}{s_{31}} \right) + \frac{1}{2} \log \left( \frac{s_{12}}{s_{23}} \right) - \frac{1}{2} \log \left( \frac{s_{23}}{s_{31}} \right) - \frac{\pi^2}{2}.$$  

Note that the IR divergences in the sum of $B_{s_2, s_3, \ldots, s_{n-1}, s_{1\ldots s_i}}^{2m}$, $T_{s_2 \ldots s_n, q^2}^{2m}$ will always \[13\] combine in such a way that

$$Z_{\text{MHV}}^{(1)} \bigg|_{\text{IR}} = Z_{\text{tree, MHV}}^{(1)} \frac{1}{\epsilon^2} \sum_{i=1}^{n} \left( \frac{s_{ii} + 1}{\mu^2} \right)^\epsilon.$$
This is in fact the consequence of the fact that IR poles should cancel in IR finite observables \[38\] such as inclusive cross sections \[39, 40, 41, 42, 43\], energy flow functions \[44, 45, 46\], etc. based on form factors \[8, 47\] and we expect that similar behavior will take place for all types (MHV, NMHV, etc) of form factors.

\[
Z_n^{(1)} \bigg|_{IR} = Z_{n,\text{tree}}^1 \sum_{i=1}^n \left( \frac{s_{ii+1}}{\mu^2} \right)^\epsilon.
\]

(3.61)

It was noticed in \[13\] that it is likely possible to find the perturbative description of the form factor in terms of the periodic Wilson loops. This fact suggests that the use of the dual variables \[33\] as in the amplitude case will be useful though the dual conformal properties \[12, 14\] of form factors remain obscure. We will use dual variables as compact notations in some cases. One introduces dual coordinates \(x_{i\alpha}^\alpha\) as

\[
x_{i\alpha}^\alpha = \sum_{i=r}^{s-1} p_i^{\alpha \dot{\alpha}}.
\]

(3.62)

Note that in the case of the periodic contour within one period there are \(n\) labels for the momenta of external particles but \(n+1\) label for the dual \(x_i\) points. For example we can write the momentum carried by the operator \(q\) as \(q = \sum_{i=1}^n p_i = x_{1n+1}\). See fig.4 for the \(n = 3\) case. Note also that for some kinematical invariants we will need labels from different periods. For example \(s_{31} = (1 + 3)^2 = x_{35}^2\). Moreover one kinematical invariant can have different representations from \(x\)'s from different periods. For example the following identity holds: \(s_{31} = (1 + 3)^2 = x_{35}^2 = x_{02}^2\).

In such notations we have for the coefficients before \(B_{s_{2\ldots s-1},s_{1\ldots s}^1}^{2\text{me}}, T_{s_{2\ldots n},q^2}^{2\text{m}}\) scalar integrals:

\[
C_{s_{2\ldots s-1},s_{1\ldots s}^1}^{2\text{me}} = Z_{n,\text{tree},MHV}^1 \Delta_{s_{2\ldots s-1},s_{1\ldots s}^1}^{2\text{me}}
\]

\[
\Delta_{s_{2\ldots s-1},s_{1\ldots s}^1}^{2\text{me}} = x_{2s}^2 x_{1s+1}^2 - x_{0s}^2 x_{2s+1}^2.
\]

(3.63)

\[
C_{s_{2\ldots n},q^2}^{2\text{m}} = Z_n^1 \Delta_{s_{2\ldots n},q^2}^{2\text{m}}
\]
Figure 5: Diagrammatical representation of the $R_{rst}^{(1)}$.

\[ \Delta_{s_2...n,q^2}^{2m} = x_{2n+1}^2 - x_{1n+1}^2. \]  

(3.64)

For example, the finite part of three point MHV form factor can be written in terms of dual variables as:

\[ Z_{3, MHV}^{(1), MHV} |_{\text{fin}} = -(1 + \mathbb{P} + \mathbb{P}^2) \left( \text{Li}_2 \left( 1 - \frac{x_{14}^2}{x_{13}^2} \right) + \frac{1}{2} \log^2 \left( \frac{x_{13}^2}{x_{24}^2} \right) + \frac{\pi^2}{6} \right), \]

(3.65)

Here $\mathbb{P}$ is permutation operator which acts on the momenta or dual variables labels. Note that there are no periodicity conditions on indices of $x$ dual coordinates. We will use such variables for the coefficients before scalar integrals also in the NMHV case which we are going to discuss now.

### 4 NMHV form factor

For NMHV form factors in general we have the following expansion in terms of scalar integrals:

\[ Z_{n, NMHV}^{(1), NMHV} = \sum_i C_{i}^{2me} B_{i}^{2me} + C_{i}^{2mh} B_{i}^{2mh} + C_{i}^{3m} B_{i}^{3m} + \sum_j C_{j}^{2m} T_{j}^{2m} + C_{j}^{3m} T_{j}^{3m} + \text{perm.}, \]

(4.66)

($B_{i}^{1m}$ scalar integrals can also appear as the limiting case of $B_{i}^{2me}$). NMHV form factors should be the next-to-the lowest components in $\eta$’s expansion of $Z_n$ of Grassmann degree $8+4$, so the Grassmann degree of the $C$ coefficients are 12. This implies the following configurations of MHV and $\overline{\text{MHV}}$ vertexes

\[ \text{MHV} \times \text{MHV} \times \text{MHV} \times \overline{\text{MHV}}_3, \]
and

\[ \text{NMHV} \times \text{MHV} \times \overline{\text{MHV}}_3 \times \overline{\text{MHV}}_3, \]

for the quadruple cut integrands which defines the coefficients before box type scalar integrals. The configuration involving NMHV vertex (which can be amplitude or form factor) can be treated recursively. To compute \( n \) point NMHV form factor at one loop one will need \( n - 1 \) point NMHV tree form factor (all the tree NMHV amplitudes are known at least in principle in the form which can be used in our computations \([33,36]\)). One can extract \( n - 1 \) point NMHV tree form factor from \( n - 1 \) point NMHV one loop form factor using (3.61). The recursion starts with the \( \text{MHV}_3 \) form factor which is also NMHV_3 one. In the NMHV_3 case there are no contributions from NMHV \( \times \text{MHV} \times \overline{\text{MHV}}_3 \times \overline{\text{MHV}}_3 \) configuration. So one can extract from (3.61) the form of NMHV_3 at tree level and then use it in the computations of NMHV_4 from which using (3.61) one can extract NMHV_4 at tree level and then use it in the computations of NMHV_5 etc. One can also obtain the form of NMHV_3 at tree level using the representation of NMHV_3 as MHV vertex in conjugated \( \bar{\eta} \) variables \([36]\). We will use both methods as consistency check and will discuss the structure of \( \overline{\text{MHV}}_3 = \text{NMHV}_3 \) form factor in the next sub section at tree and one loop level.

For the triple cut integrand situation is the similar and one have to consider the following type of integrands:

\[ \text{MHV} \times \text{MHV} \times \text{MHV}, \]

and

\[ \text{NMHV} \times \text{MHV} \times \overline{\text{MHV}}_3. \]

The last case can be considered in full analogy with the quadruple cut case. The case of triple MHV cut is more complicated and one have to use explicit kinematical solutions of
However in the case of $T^{2m}$ triangles at least in the case of $n = 3, 4$ one can take a short cut and fix the coefficient using the (3.61) without any direct computations.

Let’s now discuss the general structure of MHV $\times$ MHV $\times$ MHV $\times$ MHV $^3$ quadruple cuts which correspond in general to the coefficients $C^{3m}$ before $B^{3m}$ integrals (the coefficients before $B^{3m_{\text{ih}}}$ and $B^{3m}$ in general can be obtained using different combinations of $C^{3m}$ coefficients and cuts involving NMHV vertexes just as in the amplitude case [36]). There are two types of configuration of vertexes which we will call 1) and 2) (see fig 5 and 6) and the special limiting case of 1) (see fig 7). We will discuss this last case in detail other cases can be considered in the same fashion. The coefficient $C^{3m}_{s_r+1,\ldots,t-1,s_t,\ldots,r-1,q^2}$ before $B^{3m}_{s_r+1,\ldots,t-1,s_t,\ldots,r-1,q^2}$ scalar box integral is given by the following expression:

$$
C^{3m}_{s_r+1,\ldots,t-1,s_t,\ldots,r-1,q^2} = \int \prod_{i=1}^{4} d^4\eta_i^A \frac{\delta^4_{\text{GR}}(\sum_{i=t+1}^{4} \lambda_i \eta_i^a + \gamma^a) \delta^4_{\text{GR}}(\sum_{i=t}^{4} \lambda_i \eta_i^a)}{\langle l_3 l_4 \rangle^2} \times \frac{\delta^8(\eta_r[l_2 l_1] + \eta_l[l_1 r] - \eta_l[l_2 l]) \delta^8(\sum_{i=r+1}^{s-1} \lambda_i \eta_i^A + \lambda_l \eta_l^A - \lambda_{l_2} \eta_{l_2}^A)}{\langle l_2 l_1 | l_1 r \rangle \langle l_3 l_2 \rangle \cdots \langle s - 1 l_3 \rangle} \times \frac{\delta^8(\sum_{i=s}^{r-1} \lambda_i \eta_i^A + \lambda_l \eta_l^A - \lambda_l \eta_l^A)}{\langle l_1 l_4 \rangle \cdots \langle r - 1 l_1 \rangle}.
$$

(4.67)

This expression can be simplified by using the following strategy similar to those used in [36]: one has to split the third and the fourth $\delta^8$ functions into products of two $\delta^4_{\text{GR}}$ (or use (2.32) prescription) and add their arguments to the arguments of first two $\delta^4_{\text{GR}}$’s. After that the dependence on $\eta_i$ cancels out and we obtain the $\delta^4_{\text{GR}}$’s with the total super momentum as their argument. The remaining delta functions can be integrated using...
which after some simplifications gives the following result:

\[
C_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2}^{3m} = Z_n^{tree, MHV} \frac{\langle 12 \ldots n \rangle}{\text{den.}} \left( \frac{[l_{1}r]}{[l_{3}l_{4}]} \right)^4 \times \hat{\delta}^4 \left( \sum_{i=t}^{r-1} \eta_3 \langle i|l_{3}l_{4}|l_{1} \rangle + \sum_{i=r}^{s-1} \eta_3 \langle i|l_{3}l_{4}|l_{1} \rangle \right),
\]

(4.68)

where \( \text{den.} \) is the product of all denominators in (4.67). The argument of the \( \hat{\delta}^4 \) can be simplified using the kinematical constraints of MHV vertex which implies relation

\[
\langle i|X|l_{1} \rangle = \langle i|X|r \rangle \frac{[r_{l_{2}}]}{[l_{1}l_{2}]}, \quad X = l_{4}l_{3}, l_{3}l_{4}, \quad \lambda_i \text{ is arbitrary.}
\]

(4.69)

So using momentum conservation conditions associated with the MHV and \( \overline{\text{MHV}}_3 \) vertexes we obtain:

\[
C_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2}^{3m} = Z_n^{tree, MHV} \frac{\langle 12 \ldots n \rangle}{\text{den.}} \left( \frac{[l_{1}r]}{[l_{3}l_{4}]} \right)^4 \times \hat{\delta}^4 \left( \sum_{i=t}^{r-1} \eta_3 \langle i|q_{r...t-1}|r \rangle + \sum_{i=r}^{s-1} \eta_3 \langle i|q_{r...r-1}|r \rangle \right),
\]

(4.70)

which finally can be written as:

\[
C_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2}^{3m} = Z_n^{tree, MHV} \hat{R}_{rtt}^{(1)} \frac{1}{2} \Delta_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2}
\]

\[
\hat{R}_{rtt}^{(1)} = \frac{\langle tt - 1 \rangle \hat{\delta}^4 \left( \sum_{i=t}^{r-1} \eta_3 \langle i|q_{r...t-1}|r \rangle + \sum_{i=r}^{s-1} \eta_3 \langle i|q_{r...r-1}|r \rangle \right)}{q^2 \langle r|p_{r...t-1}q|t \rangle \langle r|p_{r...r}q|t - 1 \rangle \langle r|x_{t...r-1}|q \rangle},
\]

\[
\Delta_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2} = s_{r...t-1}s_{t...r-1} - s_{t...r-1}s_{r+1...t-1}.
\]

(4.71)

This expression can be further simplified and written in a compact form if we introduce dual Grassmann coordinates:

\[
\langle \theta_{rt} \rangle = \sum_{i=r}^{t-1} \eta_3 \langle i \rangle = \sum_{i=r}^{t-1} \eta_3 \lambda_i,
\]

(4.72)

\[
C_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2}^{3m} = Z_n^{tree, MHV} \hat{R}_{rtt}^{(1)} \frac{1}{2} \Delta_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2}
\]

\[
\hat{R}_{rtt}^{(1)} = \frac{\langle tt - 1 \rangle \hat{\delta}^4 \left( \langle \theta_{rt}|x_{1n+1}x_{rt}|r \rangle + \langle \theta_{rt}|x_{1n+1}x_{rt}|r \rangle \right)}{x_{1n+1}^2 \langle r|x_{rt}x_{1n+1}|t \rangle \langle r|x_{rt}x_{1n+1}|t - 1 \rangle \langle r|x_{rt}x_{1n+1}|r \rangle},
\]

\[
\Delta_{s_{r+1}\ldots t-1,s_{t}...r-1,q^2} = x_{rt}^{2}x_{rt+1} - x_{rt}^{2}x_{rt+1}.
\]

(4.73)
Note, that the labels $r, s, t$ when we are using standard helicity spinor notations belong to the corresponding momenta, and obeys periodicity conditions $i + n \equiv i$, $i \leq n$. But when we use dual coordinates $x, \theta$ which lives on the infinite periodic contour there are no periodicity conditions on the $r, s, t$ labels any more. Throughout this paper if not mentioned otherwise we will think of $r, s, t$ as labels belonging to the corresponding momenta, and so obeys periodicity conditions $i + n \equiv i$, $i \leq n$.

Similar computations in the 1) and 2) cases give the results:

$$C_{s_1...s_n}^{3m} = Z^{\text{tree}, \text{MHV}} R_{rst}^{(1)} \frac{1}{2} \Delta_{s_1...s_n},$$

$$R_{rst}^{(1)} = \frac{\langle s - 1 \rangle \langle t - 1 \rangle \frac{1}{2} \delta^4 (\langle \theta_{tr} | x_{ts} x_{tr} | r \rangle + \langle \theta_{rs} | x_{ts} x_{tr} | r \rangle)}{x_{ts}^2 \langle r | x_{ts} x_{ts} | t - 1 \rangle \langle r | x_{ts} x_{ts} | t \rangle \langle r | x_{ts} x_{ts} | s - 1 \rangle \langle r | x_{ts} x_{ts} | s \rangle},$$

$$\Delta_{s_1...s_n} = s_{r...s} - s_{t...r} - s_{t...s} = x_{ts}^2 x_{rt} + x_{ts} x_{rt}.$$ (4.74)

$$C_{s_1...s_n}^{3m} = Z^{\text{tree}, \text{MHV}} R_{rst}^{(2)} \frac{1}{2} \Delta_{s_1...s_n},$$

$$R_{rst}^{(2)} = \frac{\langle s - 1 \rangle \langle t - 1 \rangle \frac{1}{2} \delta^4 (\langle \theta_{tr} | x_{st} x_{tr} | r \rangle + \langle \theta_{rs} | x_{st} x_{tr} | r \rangle)}{x_{st}^2 \langle r | x_{st} x_{st} | t - 1 \rangle \langle r | x_{st} x_{st} | t \rangle \langle r | x_{st} x_{st} | s - 1 \rangle \langle r | x_{st} x_{st} | s \rangle},$$

$$\Delta_{s_1...s_n} = s_{r...s} - s_{t...r} - s_{t...s} = x_{st}^2 x_{rt} + x_{st} x_{rt}.$$ (4.75)

Note that these expressions are very similar to those of the coefficients $R_{rst}$ in the NMHV amplitudes [36]. The structures of $R_{rst}^{(1)}$, $R_{rst}^{(2)}$ and $R_{rrt}^{(1)}$ are in fact identical and the only difference is the rearrangements of the sums in $x$ and $|\theta|$ dual coordinates which are made in such a way that to avoid the dependence on $q$ and $\gamma$ axillary variables which parameterize the dependence on the (super)momentum of the operator. Such rearrangements are always possible due to the total (super)momentum conservation conditions in $Z^{\text{tree}, \text{MHV}}$.

In the case of amplitudes there are large sets of relations between different combinations of $R_{rst}$ [33]. For example

$$R_{r,r+2,t} = R_{r+2,t,r+1}.$$ (4.76)

For the case of form factors one can show that for $n = 4$ such relation gives

$$R_{r,r+2,t}^{(1)} = R_{r+2,t,r+1}^{(2)}.$$ (4.77)

To see this one has to consider relation $R_{r,r+2,t} = R_{r+2,t,r+1}$ for $n = 6$ and then put $q_6^a = \lambda^i \eta^a_i$, $q_6^a = \lambda^i \eta^a_i$ and $q_6^a = q_6^a = 0$. The clockwise cyclic order of external legs is implied. Such relations in the case of amplitudes can be easily proved using momentum twistor representation [49]. It is interesting to note that obtained here $R_{rst}^{(1)}$ and $R_{rst}^{(2)}$ coefficients can be rewritten in momentum twistor notations as well, and are equal to special cases of $[abcde]$ momentum twistor invariants [49]. We are going to discuss this in more details in separate publication.

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In the NMHV computations one will also encounter the \( T_{s_1,s_2,...,t,q^2}^{3m} \) three mass triangles. The coefficients \( C_{s_1,s_2,...,t,q^2}^{3m} \) before such triangles should be fixed by the triple MHV cuts. In such case the explicit solutions for \( \lambda_i \) and \( l_i^{a,i} \) should be used, so it is problematic to obtain representation for such coefficients in terms of only \( \lambda_i \) spinors, which corresponds to external momenta. For the integrand of such cut one has:

\[
\int \prod_{i=1}^{3} d^4 \eta_i^{a,i} \delta^4(\sum l_{i}l_{j} \lambda_i \eta_i^{a,i} + \alpha^i) \delta^4(\sum l_{i}l_{j} \lambda_i \eta_i^{a,i}) \delta^8(\sum_{i=r}^{s} \lambda_i \eta_i^{a,i} + \lambda_i \eta_i^{a,i} - \lambda_i \eta_i^{a,i})
\]

\[
\times \delta^8(\sum_{i=s+1}^{t} \lambda_i \eta_i^{a,i} + \lambda_i \eta_i^{a,i} - \lambda_i \eta_i^{a,i})
\]

\[
\langle l_{1}l_{2}\rangle \ldots \langle s+1l_{1}\rangle
\]

(4.78)

After the integration over Grassmann variables one obtains:

\[
Z_{n}^{MHV,tree} \frac{\langle rt \rangle \langle ss + 1 \rangle \delta^4(\sum_{i=r}^{s} \eta_i(l_{1}l_{3})l_{2i}) + \sum_{i=s+1}^{t} \eta_i(l_{1}l_{3})l_{3i})}{\langle r,l_{2}\rangle \langle l_{1},l_{3}\rangle \langle s + 1l_{1}\rangle \langle l_{1},l_{3}\rangle \langle l_{3},l_{1}\rangle \langle s + 1l_{1}\rangle ^{2}}.
\]

(4.79)

At this moment one has to use explicit solutions for \( \lambda_i \). After substitution of these solutions one can take \( t \to \infty \) limit and obtain:

\[
C_{s_1,s_2,...,t,q^2}^{3m} = Z_{n}^{MHV,tree} \frac{1}{2} R_{rst} \Delta_{s_1,s_2,...,t,q^2},
\]

\[
R_{rst} = \sum_{\pm S} \gamma K_{1}^{2} K_{2}^{2} \left( \frac{K_{1}^{2}}{\gamma} - 1 \right)^{-3} \frac{\langle rt \rangle \langle ss + 1 \rangle}{\prod_{i=1}^{n} \langle iK_{1}^{2} \rangle}
\]

\[
\times \delta^4 \left( K_{1}^{2}/\gamma \sum_{i=r}^{s} \eta_i \langle K_{1}^{2}i \rangle + (K_{1}^{2}/\gamma - 1) \sum_{i=s+1}^{t} \eta_i \langle K_{1}^{2}i \rangle \right),
\]

\[
\Delta_{s_1,s_2,...,t,q^2} = q^2 = \frac{2 \lambda_1}{\lambda_1 + 1}.
\]

(4.80)
Figure 9: Contributing cuts for the box $B^{1m}$ and triangle $T^{2m}$ scalar integrals coefficients to NMHV$_3$ form factor. Permutations of external momenta are not shown.

Here $K_1 = p_{s+1...t} = x_{s+1...t-1}$, $K_2 = q = x_{1n+1}$ and $K_♭$ - are the massless projections of one massive leg in the direction of another masslessly projected leg (see appendix).

Let’s now discuss the beginning of the recursive procedure discussed above for the NMHV tree level form factors which one will need in general for computation of quadruple cuts. We will discuss one step of this procedure and obtain answers for $n = 3$ and $n = 4$ point NMHV form factors at tree and one loop level.

### 4.1 3 point NMHV form factor at tree and one loop level

Let’s consider the representation for MHV$_3$ form factor at tree level in terms of $\tilde{\eta}$ variables \[36\]:

$$\langle \Omega_3 | T_{\hat{a}\hat{b}}(0) | 0 \rangle_{\text{MHV}} = \bar{F}_{3\text{MHV}}(\{\lambda, \tilde{\lambda}, \tilde{\eta}\}) = \frac{\delta^{4}_{GR} \left( \sum_{i=1}^{3} \tilde{\lambda}_{\hat{a}} \tilde{\eta}^{\hat{a}} \right)}{[12][23][31]},$$  \hspace{0.5cm} (4.81)

where

$$T_{\hat{a}\hat{b}} = (T^{ab})^{2} \bigg|_{\theta = \tilde{\theta} = 0} = (\phi^{ab})^{2} = (\phi_{ab})^{2} = (\phi^{\hat{a}\hat{b}})^{2}. \hspace{0.5cm} (4.82)$$

We can perform the Fourier transformation from $\tilde{\eta}$ to $\eta$’s \[36\] and write representation $\bar{F}_{3\text{MHV}}$ for MHV$_3$ form factor at tree level in terms of $\eta$’s as:

$$\bar{F}_{3\text{MHV}}(\{\lambda, \tilde{\lambda}, \eta\}) = \int \prod_{i=1}^{3} d^{4}\eta_{i} \exp(\eta_{i}^{A} \tilde{\eta}_{i}^{\hat{A}}) \bar{F}_{3\text{MHV}}(\{\lambda, \tilde{\lambda}, \eta\}).$$  \hspace{0.5cm} (4.83)

This integral can be evaluated exactly and for $Z_{3\text{MHV}}$ which is connected with $\bar{F}_{3\text{MHV}}$ as

$$F_{n\text{tree,MHV}} \exp(\theta_{a}^{r} q_{a}^{r}) F_{n\text{tree,MHV}}^{*} = \tilde{T}[F_{n\text{tree,MHV}}], \hspace{0.5cm} (4.84)$$
we can obtain, noticing that $\overline{\text{MHV}}_3$ form factor is also NMHV one:

$$Z_{\text{tree},\overline{\text{MHV}}}^3 = Z_{\text{tree},\text{NMHV}}^3 = \delta_{\text{GR}}^4 \sum_{i=1}^{n} \lambda_i \eta_i^\alpha + \gamma_i^\alpha \prod_{i=1}^{3} \delta^2(\eta_i^\alpha) \delta^2(\eta_i^{\alpha}[23] + \text{cycl.perm.}) \frac{12[23][31]}{}	ag{4.85}$$

This expression is explicitly cyclically invariant. Using identities:

$$\langle \theta_{12}|x_{14}x_{34}|2\rangle + \langle \theta_{24}|x_{14}x_{12}|2\rangle = \langle 12 \rangle \langle 23 \rangle (\eta_1[23] + \eta_2[31] + \eta_3[12]) \tag{4.86}$$

and

$$x_{14}^4 \prod_{i=1}^{3} \hat{\delta}^2(\eta_i) = \delta_{\text{GR}}^4 \sum_{i=1}^{3} \lambda_i \eta_i^\alpha \delta^2(\eta_i^{\alpha}[23] + \eta_2^{\alpha}[31] + \eta_3^{\alpha}[12]), \tag{4.87}$$

one can write $Z_{\text{tree},\text{NMHV}}^3$ as:

$$Z_{\text{tree},\text{NMHV}}^3 = Z_{\text{tree},\text{MHV}}^3 \tilde{R}_{211}^{(1)}, \tag{4.88}$$

where $\tilde{R}_{211}^{(1)}$ is given by (4.71) for $n = 3$. Note also that in this case ($n = 3$) the following identity holds: $\tilde{R}_{211}^{(1)} = \tilde{R}_{322}^{(1)} = \tilde{R}_{133}^{(1)}$ which is consequence of the cyclical symmetry of the initial expression for NMHV three point form factor at tree level. Using this we can write $Z_{\text{tree},\text{NMHV}}^3$ in manifestly cyclically invariant form using $\tilde{R}_{rtt}^{(1)}$ coefficients:

$$Z_{\text{tree},\text{NMHV}}^3 = Z_{\text{tree},\text{MHV}}^3 \text{tree} \frac{1}{3} (1 + \mathbb{P} + \mathbb{P}^2) \tilde{R}_{211}^{(1)}, \tag{4.89}$$

where $\mathbb{P}$ is permutation operator. We can now use $Z_{\text{tree},\text{NMHV}}^3$ in the computations of $Z_{4}^{(1),\text{NMHV}}$, but before that let’s perform one loop computation also for the $Z_{3}^{(1),\text{NMHV}}$, which is essentially trivial. The contributing scalar integrals are the same as in the MHV case: $B_{1m}$ and $T_{2m}$. Moreover because $\text{NMHV}_3 = \overline{\text{MHV}}_3$ the ratio of the one loop correction over tree result will be the same as in MHV. The coefficients $C_{1m}$ before $B_{1m}$ integrals are fixed by the quadruple cuts of the type (4.71). For example for $B_{1m}(1,2,3|q^2)$ integral we obtain:

$$C_{1m}^{(1)} = Z_{\text{tree},\text{MHV}}^3 \frac{1}{2} \tilde{R}_{211}^{(1)} x_{24}^2 x_{13}^2.$$

The computation of the $C_{2m}$ coefficients before scalar integrals $T_{2m}$ is more complicated. The integrand of corresponding cut is given by the triple product of MHV vertexes, and there are not enough kinematical conditions to express $l_i^{\alpha}$ momenta and $\lambda_i$ spinors in terms of the external momenta and spinors associated with them not using exact form of $\pm S$ solutions. However there is only one type of such coefficients and one can fix their value by requiring the (3.61) condition must be satisfied. We will use this approach from now on.
One can see that in the case under consideration the (3.61) condition satisfied if and only if (let’s consider coefficient before $T_{m}^{2m(1|q^2,(2+3)^2)}$ integral):

$$C_{s_{23},q^2}^{2m} = Z_{3}^{tree,NMHV} \frac{1}{2} \tilde{R}_{211}^{(1)}(x_{24}^2 - x_{14}^2).$$

Coefficients before other integrals can be obtained by action of permutation operator $P$. Combining all contributions together we can arrange the final result as:

$$Z_{3}^{(1),NMHV}/Z_{3}^{tree,NMHV} = \frac{1}{2} B_{1m}^{1m(1,2,3|q^2)} + \frac{1}{2} B_{1m}^{1m(1,3,2|q^2)} + \frac{1}{2} B_{1m}^{1m(2,1,3|q^2)} + T_{2m}^{2m(1|q^2,(2+3)^2)} + T_{2m}^{2m(2|q^2,(1+3)^2)} + T_{2m}^{2m(3|q^2,(1+2)^2)}. \quad (4.90)$$

Substituting the expansions in $\epsilon$ of $B_{1m}^{1m}$ and $T_{2m}^{2m}$ functions we obtain the (3.65) result.

### 4.2 4 point NMHV form factor at one loop

Now we are ready for the computation of NMHV form factor at one loop. The contributing scalar integrals in this case are $B_{2m}^{2m_{h}}$, $B_{1m}^{1m}$, $T_{2m}^{2m}$ and $T_{3m}^{3m}$. The first three types of integrals are IR divergent, while the last one (tree mass triangle) is IR finite. Let us remind the reader that there are no other contributing scalar triangle integrals for the reasons discussed in the previous section. Let’s concentrate first on the IR divergent part

\cite{36} The $B_{2mc}^{2mc}$ scalar boxes are absent by the same reasons as in the 6 point NMHV amplitude \cite{36}.
of the answer. We will label the coefficients before cyclically inequivalent type of integrals as:

\[ C^{2\text{m}_{\text{q}}}_{q^2,(1+2)^2}, C^{2\text{m}_{\text{q}}}_{q^2,(4+3)^2}, C^{1\text{m}_{\text{q}}}_{(q-4)^2}, C^{2\text{m}_{\text{q}}}_{q^2,(1+2+3)^2}. \]  

(4.91)

For \( C^{2\text{m}_{\text{q}}}_{q^2,(1+2)^2} \) we have two types of cuts contributing (see fig.11):

\[ C^{2\text{m}_{\text{q}}}_{q^2,(1+2)^2} = A) + B). \]  

(4.92)

These cuts give us:

\[ C^{2\text{m}_{\text{q}}}_{q^2,(1+2)^2} = Z_{\text{tree}, \text{MHV}}^{\text{1}} \frac{1}{2} (\tilde{R}_{311}^{(1)} + R_{413}^{(2)}) x_{35} x_{14}. \]  

(4.93)

For \( C^{1\text{m}_{\text{q}}}_{(q-4)^2} \) we have two types of contributions:

\[ C^{1\text{m}_{\text{q}}}_{(q-4)^2} = C) + D). \]  

(4.94)

These cuts give us:

\[ C^{1\text{m}_{\text{q}}}_{(q-4)^2} = Z_{\text{tree}, \text{MHV}}^{\text{1}} \frac{1}{2} (\tilde{R}_{144}^{(1)} + R_{241}^{(2)}) x_{25} x_{12}. \]  

(4.95)

Note that C) cut contains vertex which is NMHV 3-point form factor. To evaluate such cuts we use the same technique as in [36]: we substitute explicit expression for NMHV 3-point form factor and then use the kinematical constraints attached to the neighboring MHV3 vertexes which imply \( \lambda_{l2} \sim \lambda_1 \) and \( \lambda_{l3} \sim \lambda_3 \). Using the cyclical symmetry of NMHV 3-point form factor we can obtain the following relations for \( \tilde{R}^{(1)} \):

\[ \tilde{R}_{144}^{(1)} = \tilde{R}_{311}^{(1)}, \tilde{R}_{241}^{(2)} = \tilde{R}_{211}^{(1)}. \]  

(4.96)

For \( C^{2\text{m}_{\text{q}}}_{q^2,(4+3)^2} \) similar to \( C^{2\text{m}_{\text{q}}}_{q^2,(1+2)^2} \) we have:

\[ C^{2\text{m}_{\text{q}}}_{q^2,(4+3)^2} = Z_{\text{tree}, \text{MHV}}^{\text{1}} \frac{1}{2} (\tilde{R}_{241}^{(1)} + R_{142}^{(2)}) x_{13} x_{25}. \]  

(4.97)

For \( C^{2\text{m}_{\text{q}}}_{q^2,(1+2+3)^2} \) we have two contributing cuts. Cut involving NMHV 5 point tree amplitude can be evaluated using explicit expression for NMHV 5 point form factor and kinematical constraints associated with the neighboring MHV3 vertexes which imply \( \lambda_{l2} \sim \lambda_1 \). We will adjust the value of the other triple MHV cut using the (3.61) just as in the previous NMHV3 case. Finally we have:

\[ C^{\text{T}_{2\text{m}_{\text{q}}}}_{q^2,(1+2+3)^2} = Z_{\text{tree}, \text{MHV}}^{\text{1}} \frac{1}{2} (\tilde{R}_{311}^{(1)} + R_{413}^{(2)}) (x_{14}^2 - x_{15}^2). \]  

(4.98)

One can notice that \( C^{2\text{m}_{\text{q}}}_{q^2,(4+2+3)^2} = \mathbb{P} C^{2\text{m}_{\text{q}}}_{q^2,(1+2+3)^2} \), so we can write all contributions from the IR divergent triangle integrals in the following form:

\[ C^{2\text{m}_{\text{q}}}_{q^2,(1+2+3)^2} \mathcal{T}_{q^2,(1+2+3)^2}^{2\text{m}_{\text{q}}} + C^{2\text{m}_{\text{q}}}_{q^2,(4+2+3)^2} \mathcal{T}_{q^2,(4+2+3)^2}^{2\text{m}_{\text{q}}} + \text{perm.} = 2 C^{2\text{m}_{\text{q}}}_{q^2,(1+2+3)^2} \mathcal{T}_{q^2,(1+2+3)^2}^{2\text{m}_{\text{q}}} + \text{perm.} \]  

(4.99)
So, finally gathering all IR divergent contributions together we obtain

\[
Z_{NMHV,1}^{4}/Z_{tree,MHV}^{4} \bigg|_{IR} = \frac{1}{2}(\tilde{R}_{311}^{(1)} + R_{141}^{(1)})\mathcal{B}^{2mh}(3, 4|q^2, (1 + 2)^2) \bigg|_{IR} + \frac{1}{2}(\tilde{R}_{411}^{(1)} + R_{221}^{(1)})\mathcal{B}^{1m}(1, 2, 3|(q - 4)^2) \bigg|_{IR} + \frac{1}{2}(\tilde{R}_{241}^{(1)} + R_{142}^{(1)})\mathcal{B}^{2mh}(1, 2|q^2, (3 + 4)^2) \bigg|_{IR} + \frac{1}{2}(\tilde{R}_{311}^{(1)} + R_{413}^{(1)})\mathcal{T}^{2m}(4|q^2, (1 + 2 + 3)^2) \bigg|_{IR} + \text{perm.}
\]

(4.100)

Using equation

\[
R_{241}^{(2)} = R_{241}^{(1)}
\]

(4.101)

which is consequence of \(R_{r,r+2,t} = R_{r+2,t,r+1}\) relation for amplitudes, eq.(4.96) and

\[
\tilde{R}_{241}^{(1)} = \tilde{R}_{211}^{(1)} = \mathbb{P}\tilde{R}_{141}^{(1)}
\]

also noticing that \(R_{312}^{(1)} = \mathbb{P}R_{241}^{(1)}\) we can write:

\[
Z_{NMHV,1}^{4}/Z_{tree,MHV}^{4} \bigg|_{IR} = \frac{1}{2}(\tilde{R}_{311}^{(1)} + R_{141}^{(1)})\{\mathcal{B}^{2mh}(3, 4|q^2, (1 + 2)^2) + \mathcal{B}^{2mh}(4, 1|q^2, (3 + 2)^2) + \mathcal{B}^{1m}(1, 2, 3|(q - 4)^2) + 2\mathcal{T}^{2m}(4|q^2, (1 + 2 + 3)^2)\} \bigg|_{IR} + \text{perm.}
\]

(4.102)

Using explicit expressions for the IR divergent parts of integrals

\[
\mathcal{B}^{2mh}(3, 4|q^2, (1 + 2)^2) \bigg|_{IR} = \frac{\mu^{-2\epsilon}}{\epsilon^2} \left( s_{43}^{\epsilon} + 2(4 - q)^{2\epsilon} - q^{2\epsilon} - s_{12}^{\epsilon} \right),
\]

\[
\mathcal{B}^{2mh}(4, 1|q^2, (3 + 2)^2) \bigg|_{IR} = \frac{\mu^{-2\epsilon}}{\epsilon^2} \left( s_{14}^{\epsilon} + 2(4 - q)^{2\epsilon} - q^{2\epsilon} - s_{23}^{\epsilon} \right),
\]

\[
\mathcal{B}^{1m}(1, 2, 3|(q - 4)^2) \bigg|_{IR} = \frac{\mu^{-2\epsilon}}{\epsilon^2} \left( 2s_{32}^{\epsilon} + 2s_{21}^{\epsilon} - 2(4 - q)^{2\epsilon} \right),
\]

\[
\mathcal{T}^{2m}(4|q^2, (1 + 2 + 3)^2) \bigg|_{IR} = \frac{\mu^{-2\epsilon}}{\epsilon^2} \left( q^{2\epsilon} - (4 - q)^{2\epsilon} \right),
\]

(4.103)

one can see that IR divergent part of the NMHV four point form factor indeed has the form:

\[
Z_{NMHV,1}^{4}/Z_{tree,MHV}^{4} \bigg|_{IR} = \frac{1}{2}(1 + \mathbb{P} + \mathbb{P}^2 + \mathbb{P}^3)(\tilde{R}_{311}^{(1)} + R_{141}^{(1)})\sum_{i=1}^{4} \frac{1}{\epsilon^2} \left( \frac{s_{i+1}}{\mu^2} \right)^{\epsilon}.
\]

(4.104)
From this we conclude that

$$Z_{\text{NMHV,tree}}^4 = Z_{\text{MHV,tree}}^4 \frac{1}{2} (1 + P + P^2 + P^3) (\tilde{R}_{311}^{(1)} + R_{241}^{(1)}).$$

(4.105)

Let’s now consider the contribution of IR finite $$T^{3n}((1+2)^2, (3+4)^2, q^2)$$ scalar integral. For the corresponding coefficient $$C^{3n}_{(1+2)^2, (3+4)^2, q^2}$$ using previous results we immediately obtain:

$$C^{3n}_{(1+2)^2, (3+4)^2, q^2} = Z_{\text{MHV,tree}}^4 R_{124} q^2.$$ (4.106)

The finite part of four point NMHV form factor then can be written in the following form:

$$Z_{\text{NMHV,tree}}^4 / Z_{\text{tree,MHV}}^4 \bigg|_{\text{fin}} = \frac{1}{2} (1 + P + P^2 + P^3) (\tilde{R}_{311}^{(1)} + R_{241}^{(1)}) V_4$$

$$+ \frac{1}{2} (1 + P + P^2 + P^3) R_{124} W_4.$$ (4.107)

Where $$(\log(x) \equiv L(x))$$:

$$V_4 = -2 \sum_{i=1}^{2} \left( \text{Li}_2 \left( 1 - \frac{x_{i+2}^2}{x_{14}^2} \right) + \text{Li}_2 \left( 1 - \frac{x_{i+2}^2}{x_{14}^2} \right) - 4 \text{Li}_2 \left( 1 - \frac{x_{i+2}^2}{x_{14}^2} \right) \right)$$

$$+ \sum_{i=1}^{2} \left( -L^2 \left( \frac{x_{i+2}^2}{x_{14}^2} \right) + L \left( \frac{x_{i+2}^2}{x_{14}^2} \right) L \left( \frac{x_{15}^2}{x_{i+2i+4}^2} \right) \right) - L^2 \left( \frac{x_{13}^2}{x_{24}^2} \right) - \frac{\pi^2}{3}.$$ (4.108)

while $$W_4$$ is given by Davdychev function [50], which in our case has the form:

$$W_4 = \frac{1}{Q} \left( 2 \text{Li}_2(-xR) + 2 \text{Li}_2(-yR) + L(xR)L(yR) + L \left( \frac{y}{x} \right) L \left( \frac{1 + yR}{1 + xR} \right) + \frac{\pi^2}{3} \right),$$

$$Q = ((1 - x - y)^2 - 4xy)^{1/2}, \quad R = 2(1 - x - y + R)^{-1}, \quad x = \frac{x_{33}^2}{x_{15}^2}, \quad y = \frac{x_{35}^2}{x_{15}^2}. \quad (4.109)$$

This is the end of computation of the four point NMHV form factor. Using results obtained in previous section for MHV sector one can in principle define analog of $$R_{n}^{\text{NMHV}}$$ ratio function as in the amplitude case [36]. One also can use (4.105) as an input in the computation of the five point NMHV form factor. The structure of such computation should be essentially similar to the four point case. The IR divergent part of the answer will be determined by the IR divergent parts of box and two mass triangle scalar integrals, while in the finite part there will be additional contributions from IR finite three mass integrals.

It is interesting to note that the number of $$R_{n}^{(i)}$$ coefficients in tree level 3 and 4 point form factors exactly matches the number of diagrams which one will have for the corresponding super Wilson loop [26] if one will use the prescriptions of [15] about selection.
of diagrams. It will be interesting to clarify by explicit computations the status of the Wilson loop/form factor duality in the none-MHV case.

In [47, 14] it was observed that the following relation between form factors and amplitudes likely holds

\[ Z_n(\{\lambda, \tilde{\lambda}, \eta\}, 0, \{0\}) = \hat{T}[F_n^{MHV}](\{\lambda, \tilde{\lambda}, \eta\}, 0, 0) = g \frac{\partial A_n(\lambda, \tilde{\lambda}, \eta)}{\partial g}. \] (4.110)

This relation was verified at tree and one loop level for the MHV sector. While for the component answer for the 4 point NMHV form factor the limit \( q \to 0 \) was in general singular. If one considers the coefficients before corresponding scalar integrals in the case of five point form factor at one loop one can observe the following: in the limit \( q \to 0, \gamma^a \to 0 \) the coefficients before triangle integrals vanish and so do most of the coefficients before box scalar integrals. The only coefficients that survive are the coefficients before \( B^{lm} \) box integrals which in this limit reproduce the coefficients for the five point NMHV amplitude. It is likely the general pattern for the n point case. One may think that relation (4.110) holds for such numbers of external legs for which the objects on both sides of the equality exists. I.e. equality should not hold for example for the NMHV sector for \( n = 4 \) because there are no \( n = 4 \) NMHV amplitude.

5 Conclusions

In this paper the systematic study of form factors in the \( \mathcal{N} = 4 \) SYM theory in on-shell momentum superspace formalism is performed for the MHV and NMHV sectors at one loop order of PT by means of generalized unitarity technique. The use of \( \mathcal{N} = 4 \) covariant methods allows us to obtain answers for any type of operator from stress tensor multiplet and arbitrary external states in these sectors. The explicit answers for the 3 and 4 point NMHV form factors were obtained as well as the n point situation was discussed. As the byproduct of this investigation the representation for 3 and 4 point form factors that does not depend on any kind of "reference spinors" at tree level were obtained.

The application of the generalized unitarity methods to form factors clarifies several issues: the structure of the basis of scalar integrals at one loop and the relations between form factors with operator insertion with zero momenta and the amplitudes.

The recent studies of the structure of the amplitudes and their relations to Wilson loops in \( \mathcal{N} = 4 \) SYM led to the formulation of the equation which at least in principle should define full S-matrix of the theory [29] at any value of the coupling constant. The conjectured derivation of this equation is based on the amplitudes/Wilson loops duality. There is also the conjecture [13] that similar duality between form factors and Wilson loops also holds. It is interesting to investigate whether such duality survives for the NMHV and other sectors in some form and if it is possible to formulate similar equation for the form factors. The results obtained in this paper may be considered as starting point in such investigation.
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A $\mathcal{N} = 4$ harmonic superspace

We discuss here the reformulation of (2.19) in the $\mathcal{N} = 4$ harmonic superspace. Our discussion is based mostly on section 3 of [32]. The $\mathcal{N} = 4$ harmonic superspace is obtained by adding additional bosonic coordinates (harmonic variables) to the $\mathcal{N} = 4$ coordinate superspace or on-shell momentum superspace. These additional bosonic coordinates parameterize the coset

$$SU(4) / SU(2) \times SU(2) \times U(1)$$

and carry the $SU(4)$ index $A$, two copies of $SU(2)$ indices $a, \hat{a}$ and $U(1)$ charge $\pm (u_A^+, u^-_{\hat{a}})$.

Using these variables one presents all the Grassmannian objects with $SU(4)_R$ indices. For example, for Grassmannian coordinates in the original $\mathcal{N} = 4$ coordinate superspace

$$\theta^{\alpha a} = u_A^{+a} \theta^A_\alpha, \quad \theta^{-\hat{a}} = u_{\hat{a}}^- \theta^A_\alpha,$$

and in the opposite direction

$$\theta^A_\alpha = \theta^{\alpha a}, \tilde{u}_+^a + \theta^{\alpha -\hat{a}}, \tilde{u}_{-\hat{a}}^-.$$

The same can be done with supercharges etc.. Note that harmonic variable projection leaves helicity properties of the objects unmodified. Also, similar projections can be performed for Grassmannian coordinates $\eta^A$ and supercharges $q^A_a, \tilde{q}_{\hat{a}A}$ of on-shell momentum superspace.

So the $\mathcal{N} = 4$ harmonic superspace is parameterized with the following set of coordinates

$$\mathcal{N} = 4 \text{ harmonic superspace} = \{ x^{\rho \hat{\rho}}, \theta^{\alpha a}, \theta^{-\hat{a}} \tilde{u}_{\hat{a}}^+, \tilde{u}_-^a, u \}$$

or

$$\{ \lambda, \tilde{\lambda}_{\hat{a}}, \eta^{\alpha a}, \eta^{-\hat{a}}, u \}. \quad (A.113)$$

Using $u$ harmonic variables one can project the $W^{AB}$ superfield as

$$W^{AB} \to W^{AB} u^{+a}_A u^{-b}_B = \epsilon^{ab} W^{++},$$

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where \( \epsilon^{ab} \) is an \( SU(2) \) totally antisymmetric tensor and the Grassmannian analyticity conditions \[32\] such that

\[
D^-_{-a} W^{++} = 0, \quad \bar{D}^-_{+a} W^{++} = 0.
\]

Thus, the superfield \( W^{++} \) contains the dependence on half of the Grassmannian variables \( \theta \)'s and \( \bar{\theta} \)'s.

\[
W^{++} = W^{++}(x, \theta_+^a, \bar{\theta}^{-\dot{a}}, u),
\]

Performing the expansion of \( W^{++} \) in \( u \) all the projections like (2.8) in \( SU(4)_R \) covariant fashion can be obtained. This is the main purpose of introduction of the harmonic superspace. The component expansion of \( W^{++} \) in \( \theta \)'s and \( \bar{\theta} \)'s can be found in [32]. The lowest component of the \( W^{++} \) expansion is

\[
W^{++}(x, 0, 0, u) = \phi^{++}, \quad \phi^{++} = \frac{1}{2} \epsilon_{a b} u^{+a} u^{+b} \phi^{AB},
\]

where according to [32]

\[
Q_{-a}^a \phi^{++} = 0. \quad (A.114)
\]

Using this condition and the translation invariance we can write the expression for MHV ”super state -super form factor” at tree level in harmonic superspace [11]:

\[
\hat{Z}_{\text{tree}, \text{MHV}}^n(\{\lambda, \bar{\lambda}, \eta\}, q, u, \gamma^a) = \delta^{+4}(q^a + \gamma^a)\delta^{-4}(q^a) \frac{\langle 12 \rangle \cdots \langle n1 \rangle}{\langle 12 \rangle \cdots \langle n1 \rangle}, \quad (A.115)
\]

where \( \delta^{+4} \) is the Grassmannian delta function; \( \delta^{\pm 4} \) are defined as

\[
\delta^{\pm 4}(q^a/\dot{a}) = \sum_{i,j=1}^n \prod_{a/\dot{a}, b/\dot{b}}^2 \langle ij \rangle^{\pm a/\dot{a}} \eta_{i}^{\pm b/\dot{b}}. \quad (A.116)
\]

We can also define \( \delta^{\pm 2} \) as usual Grassmann delta functions:

\[
\hat{\delta}^{\pm 2}(X^a/\dot{a}) = \prod_{a/\dot{a}, b/\dot{b}=1}^2 X^{a/\dot{a}}. \quad (A.117)
\]

The obtained expression for form factor looks just like (2.19), but now both the Grassmannian delta functions \( \delta^{\pm 4} \) are \( SU(4)_R \) covariant. One can write also the MHV part of a superamplitude in a similar manner. Projecting the condition of superamplitude invariance under \( q^A_{\alpha} \) supersymmetry transformations we have

\[
q^A_{\alpha} \hat{A}_{\text{tree}, \text{MHV}}^n = 0 \rightarrow (q^+_{\alpha} + q^-_{\alpha}) \hat{A}_{\text{tree}, \text{MHV}}^n = 0,
\]

\[\text{\footnotesize{\textsuperscript{12}}}\text{Strictly speaking this is true only in the free theory (g = 0), in the interacting theory one has to replace D}^A_{\alpha}, D^-_{\alpha} \text{ by their gauge covariant analogs, which contain superconnection, but the final result is the same [32].}\]
and taking into account that the helicity properties of projected supercharges are not modified we get

\[ \hat{A}_{n,\text{tree, MHV}}^{\text{tree, MHV}} = \frac{\delta^{+4}(q_{\alpha}^{n})\delta^{-4}(q_{\dot{\alpha}}^{n})}{\langle 12 \rangle \ldots \langle n1 \rangle}. \]  

(A.118)

Now both \( \hat{A}_{n,\text{tree, MHV}}^{\text{tree, MHV}} \) and \( \hat{Z}_{n,\text{tree, MHV}}^{\text{tree, MHV}} \) are \( SU(4)_{R} \) invariant and one can use them in unitarity based computations, where Grassmann integration (super summation) should be performed separately for \( d^{+2}\eta \) and \( d^{-2}\eta \).

We see now that all results obtained in previous sections can be simply generalized to \( SU(4)_{R} \) covariant harmonic superspace version. We have to replace common \( \hat{Z}_{\text{tree, MHV}}^{\text{tree, MHV}} \) prefactor by its harmonic superspace generalization and replace all \( \delta^{4}(X_{\alpha}^{A}) \) functions in \( R^{(i)}_{rst} \) by combination of \( \delta^{-2}(X_{n}^{-a})\delta^{+2}(X_{n}^{a}) \).

\[ K_{1}^{\alpha\dot{\alpha}} = \frac{K_{1}^{\alpha\dot{\alpha}} - K_{2}^{\alpha\dot{\alpha}}K_{2}/\gamma}{1 - K_{1}^{2}K_{2}^{2}/\gamma^{2}}, \quad K_{2}^{\alpha\dot{\alpha}} = \frac{K_{2}^{\alpha\dot{\alpha}} - K_{1}^{\alpha\dot{\alpha}}K_{2}/\gamma}{1 - K_{1}^{2}K_{2}^{2}/\gamma^{2}}. \]  

(B.119)

Here \( \gamma = (K_{1}, K_{2}) \pm ((K_{1}, K_{2})^{2} - K_{1}^{2}K_{2}^{2})^{1/2} \) which corresponds to two possible kinematical solutions \( \pm S \). In general \( K_{j}^{2} \neq 0, \ j = 1...3 \).

Using these massless projections one can define corresponding spinors. The notations of \( \lambda_{K_{j}}^{\alpha} = |K_{j}^{\alpha}| \equiv \langle K_{j}^{\alpha} \rangle \) and \( \lambda_{K_{j}}^{\dot{\alpha}} = [K_{j}^{\dot{\alpha}}] \equiv \langle K_{j}^{\dot{\alpha}} \rangle \) are used in the sense that:

\[ \langle ij \rangle = \langle i|j^{+} \rangle = \bar{\omega}_{+}(k_{i})u_{-}(k_{j}), \quad [ij] = \langle i^{+}|j \rangle = \bar{\omega}_{+}(k_{i})u_{+}(k_{j}), \]

where \( u_{\pm} \) are four component Weyl spinors. Then we have:

\[ \langle l_{i}^{+} \rangle = t\langle K_{1}^{+} \rangle + \alpha_{i1}\langle K_{2}^{-} \rangle, \quad \langle l_{i}^{-} \rangle = \frac{\alpha_{i2}}{t}\langle K_{1}^{+} \rangle + \alpha_{i1}\langle K_{2}^{-} \rangle, \]  

(B.120)

while for \( l_{i}^{\alpha\dot{\alpha}} = \sigma_{\mu}^{\alpha\dot{\alpha}}l_{i}^{\mu} \) one can write:

\[ l_{i}^{\mu} = \alpha_{i2}K_{1}^{\mu} + \alpha_{i1}K_{2}^{\mu} + \frac{t}{2}\langle K_{1}^{-}\gamma^{\mu}|K_{2}^{-} \rangle + \frac{\alpha_{i1}\alpha_{i2}}{2t}\langle K_{2}^{-}\gamma^{\mu}|K_{1}^{-} \rangle. \]  

(B.121)

Here the explicit expressions in terms of \( K_{i}^{2} \) and \( \gamma \) for \( \alpha_{ij} \) can be found in appendix A of [37].

\[ \alpha_{01} = \frac{K_{1}^{2}(\gamma - K_{2}^{2})}{\gamma^{2} - K_{1}^{2}K_{2}^{2}}, \quad \alpha_{02} = \frac{K_{2}^{2}(\gamma - K_{1}^{2})}{\gamma^{2} - K_{1}^{2}K_{2}^{2}}. \]
\[
\begin{align*}
\alpha_{11} &= \alpha_{01} - \frac{K_1^2}{\gamma}, \quad \alpha_{12} = \alpha_{02} - 1, \\
\alpha_{21} &= \alpha_{01} - 1, \quad \alpha_{22} = \alpha_{02} - \frac{K_2^2}{\gamma}.
\end{align*}
\] (B.122)

Using these expressions one has the following relations for the spinor products of \(\langle l_k l_j \rangle\):

\[
\begin{align*}
\langle l_1 l_2 \rangle &= -t \left(1 - \frac{K_1^2}{\gamma}\right) \langle K_1^2 K_2^\gamma \rangle, \\
\langle l_1 l_3 \rangle &= \frac{tK_1^2}{\gamma} \langle K_1^2 K_2^\gamma \rangle, \\
\langle l_2 l_3 \rangle &= t \langle K_1^2 K_2^\gamma \rangle.
\end{align*}
\] (B.123)

In the case when \(K_j^2 = 0\) for some \(j\) (T\textsuperscript{2m} scalar integral) the explicit solution is different. If the massless leg is attached to the MHV vertex then the solution takes the form (here we assume that \(K_2^2 = 0\)):

\[
\begin{align*}
\langle l_1^\gamma \rangle &= \frac{t}{K_2 K_1^2} \langle K_1^\gamma | K_1 + \frac{K_2^2}{\gamma} \langle \chi | \rangle, \quad \langle l_1^\gamma \rangle = \frac{[\chi K_1^2]}{K_2 K_1^2} \langle K_1^\gamma | K_2^\gamma \rangle, \\
\langle l_2^\gamma \rangle &= \frac{1}{[K_1 K_3^2]} \langle K_2^\gamma | K_3 + \frac{t}{[K_2 K_1^2]} \langle K_2^\gamma | K_1, \quad \langle l_2^\gamma \rangle = \langle l_3^\gamma \rangle, \\
\langle l_3^\gamma \rangle &= \frac{1}{[\chi K_1^2]} \langle K_1^\gamma | K_3 + \frac{t}{[K_2 K_1^2]} \langle K_1^\gamma | K_2^\gamma \rangle.
\end{align*}
\] (B.124)

where \(\langle \chi \rangle\) is arbitrary spinor. Using these solutions we have the following relevant for our computations results:

\[
\begin{align*}
\langle l_1 l_2 \rangle &= -\frac{K_1^2}{\gamma} \langle \chi K_1^\gamma \rangle, \\
\langle l_1 l_3 \rangle &= -\frac{K_1^2}{\gamma} \langle \chi K_1^\gamma \rangle, \\
\langle l_2 l_3 \rangle &= \frac{[K_1 K_2^2]}{[\chi K_1^2]} \langle l_3 K_2 \rangle.
\end{align*}
\] (B.125)

### C Scalar integrals

For the Box type integral:

\[
B_{K_1^2, K_2^2, K_3^2, K_4^2} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (K_1 + l)^2 (K_1 + K_2 + l)^2 (l - K_4)^4}.
\] (C.126)

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where \( D = 4 - 2\epsilon \) we define dimensionless function \( B_{K_1^2, K_2^2, K_3^2, K_4^2} \) as:

\[
B_{K_1^2, K_2^2, K_3^2, K_4^2} = \left( i\pi^{D/2} r_\Gamma \right)^{-1} \Delta (2\pi)^D B_{K_1^2, K_2^2, K_3^2, K_4^2},
\]

where

\[
r_\Gamma = \frac{\Gamma(1 - \epsilon) \Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)},
\]

while for \( \Delta \) for \( B^{3m}, B^{2mh}, B^{2me} \) and \( B^{1m} \) box scalar integrals we have \((s_{ij} = (K_i + K_j)^2)\)

\[
\begin{align*}
\Delta^{3m} &= s_{12}s_{23} - K_2^2 K_4^2, \\
\Delta^{2mh} &= s_{12}s_{23}, \\
\Delta^{2me} &= s_{12}s_{23} - K_2^2 K_4^2, \\
\Delta^{1m} &= s_{12}s_{23}.
\end{align*}
\]

For the \( B \) functions we have [34] (note that we rearranged IR divergent part of \( B^{2mh} \) in comparison with [34]):

\[
\begin{align*}
B^{2mh}(1, 2|K_3^2, K_4^2) &= \frac{\mu^{-2\epsilon}}{\epsilon^2} \left( s_{12}^2 + 2s_{23}^2 - K_3^2 - K_4^2 \right) + \log \left( \frac{K_3^2}{s_{12}} \right) \log \left( \frac{K_4^2}{s_{12}} \right) - \\
&\quad -2\text{Li}_2 \left( 1 - \frac{K_3^2}{s_{23}} \right) - 2\text{Li}_2 \left( 1 - \frac{K_4^2}{s_{23}} \right) - \log^2 \left( \frac{s_{12}}{s_{23}} \right) + O(\epsilon), \\
B^{2me}(1, 3|K_2^2, K_4^2) &= \frac{\mu^{-2\epsilon}}{\epsilon^2} \left( 2s_{12}^2 + 2s_{23}^2 - 2K_2^2 - 2K_4^2 \right) - 2\text{Li}_2 \left( 1 - \frac{K_3^2}{s_{12}} \right) - \\
&\quad -2\text{Li}_2 \left( 1 - \frac{K_2^2}{s_{12}} \right) - 2\text{Li}_2 \left( 1 - \frac{K_4^2}{s_{12}} \right) - 2\text{Li}_2 \left( 1 - \frac{K_3^2}{s_{23}} \right) - 2\text{Li}_2 \left( 1 - \frac{K_2^2}{s_{23}} \right) - 2\text{Li}_2 \left( 1 - \frac{K_4^2}{s_{23}} \right) - \\
&\quad +2\text{Li}_2 \left( 1 - \frac{K_2^2 K_4^2}{s_{12}s_{23}} \right) - \log^2 \left( \frac{s_{12}}{s_{23}} \right) + O(\epsilon), \\
B^{1m}(1, 2, 3|K_4^2) &= \frac{\mu^{-2\epsilon}}{\epsilon^2} \left( 2s_{12}^2 + 2s_{23}^2 - 2K_4^2 \right) - 2\text{Li}_2 \left( 1 - \frac{K_3^2}{s_{12}} \right) - 2\text{Li}_2 \left( 1 - \frac{K_2^2}{s_{12}} \right) - \\
&\quad -2\text{Li}_2 \left( 1 - \frac{K_4^2}{s_{12}} \right) - \log^2 \left( \frac{s_{12}}{s_{23}} \right) - \frac{\pi^2}{3} + O(\epsilon).
\end{align*}
\]

For triangle scalar integrals

\[
T_{K_1^2, K_2^2, K_3^2} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2(K_1 + l)^2(l - K_3)^2},
\]

we have similar definitions:

\[
T_{K_1^2, K_2^2, K_3^2} = \left( i\pi^{D/2} r_\Gamma \right)^{-1} \Delta (2\pi)^D T_{K_1^2, K_2^2, K_3^2}.
\]
While for $\Delta$ coefficients for the relevant for our discussion cases ($T^{2m}$ and $T^{3m}$ scalar triangles with $q^2$ massive leg) we have:

$$\Delta^{3m} = q^2,$$
$$\Delta^{2m} = K^2 - q^2.$$  \hfill (C.132)

$$T^{2m}(1|K^2, q^2) = \frac{\mu^{-2e}}{2\epsilon^2}(K^2 - q^2).$$  \hfill (C.133)

The $T^{3m}$ triangle is IR finite and the answer for it is given in terms of Davydychev function $\mathcal{T}^{3m}(K^2_1, K^2_2, q^2) = \mathcal{T}^{3m}(K^2_1/q^2, K^2_2/q^2)$:

$$\mathcal{T}^{3m} = \frac{2\text{Li}_2(-xR) + 2\text{Li}_2(-yR) + \text{Log}(xR)\text{Log}(yR) + \text{Log}(\frac{y}{x})\text{Log}(\frac{1+yR}{1+xR})}{Q} + \frac{\pi^2}{3Q},$$

$$Q = ((1 - x - y)^2 - 4xy)^{1/2}, \quad R = 2(1 - x - y + R)^{-1}, \quad x = \frac{K^2_1}{q^2}, \quad y = \frac{K^2_2}{q^2}.  \hfill (C.134)$$
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Figure 11: List of contributing quadruple and all possible triple cuts for NMHV four point form factor. B), G), D), F) triple cuts give vanishing results for corresponding coefficients. Permutations of external momenta are not shown.