Application of Morse index in weak force $N$-body problem

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Abstract

Due to collision singularities, the Lagrange action functional of the $N$-body problem in general is not differentiable. Because of this, the usual critical point theory cannot be applied to this problem directly. Following ideas from Árriolo et al (2006 Commun. Math. Phys. 268 439–63); Bahri and Rabinowitz (1991 Ann. Inst. Henri Poincaré Anal. Non Linéaire 8 561–649); Tanaka (1993 Ann. Inst. Henri Poincaré Anal. Non Linéaire 10 215–38), we introduce a notion called weak critical point for such an action functional, as a generalization of the usual critical point. A corresponding definition of Morse index for such a weak critical point will also be given. Moreover it will be shown that the Morse index gives an upper bound of the number of possible binary collisions in a weak critical point of the $N$-body problem with weak force potentials including the Newtonian potential.

Keywords: $N$-body problem, variational method, Morse index

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1. Introduction

The motion of $N$ point masses, $m_i > 0$, $i \in \mathbb{N} := \{1, \ldots, N\}$, under the universal gravitational force is a classic problem that has been studied by many authors since the time of Newton. Let $q_i \in \mathbb{R}^d$ be the position of mass $m_i$, then it satisfies the following equation

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i} = -\alpha \sum_{j \in \mathbb{N} \setminus \{i\}} m_i m_j \frac{q_i - q_j}{|q_i - q_j|^3}, \quad \forall i \in \mathbb{N},$$

(1)

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where \( q = (q_i)_{i \in \mathbb{N}} \in \mathbb{R}^{dN} \) and \( U(q) = \sum_{(i<j)\subset \mathbb{N}} \frac{m_im_j}{|q_i-q_j|^s} \) is the potential function (the negative potential energy). \( \alpha \) here is a positive constant.

Traditionally \( U \) is called a strong force potential, when \( \alpha \geq 2 \), and a weak force potential, when \( 0 < \alpha < 2 \). The Newtonian potential is a weak force potential corresponding to \( \alpha = 1 \).

This is a singular Lagrange system with the Lagrangian
\[
L(q, \dot{q}) := K(\dot{q}) + U(q), \quad \text{where } K(\dot{q}) = \frac{1}{2} \sum_{i \in \mathbb{N}} m_i |\dot{q}_i|^2.
\]
The singularities are caused by collisions between two or more masses
\[
\Delta := \{ q \in \mathbb{R}^{dN} : q_i = q_j, \text{ for some } i \neq j \subset \mathbb{N} \}.
\]
For any \( I \subset \mathbb{N} \) with \(|I| \geq 2 \) (\(|I| \) is the cardinality of \( I \)), we say \( q \) has an \( I \)-cluster collision at the moment \( t \), when
\[
\forall i \in I, \quad \begin{cases} q_i(t) = q_j(t), & \text{if } j \in I, \\ q_i(t) \neq q_j(t), & \text{if } j \in \mathbb{N} \setminus I. \end{cases}
\]
An \( I \)-cluster collision is a binary collision, when \(|I| = 2 \).

Let \( H^1([T_1, T_2], \mathbb{R}^{dN}) \) be the space of Sobolev paths defined on \([T_1, T_2] \), and \( \mathbb{R}^{dN}_\Delta := \mathbb{R}^{dN} \setminus \Delta \) the set of collision-free configurations, we say a path \( q \in H^1([T_1, T_2], \mathbb{R}^{dN}_\Delta) \) is collision-free, if \( q(t) \in \mathbb{R}^{dN}_\Delta \), for any \( t \in [T_1, T_2] \). It is well known the Lagrange action functional
\[
A(q; T_1, T_2) := \int_{T_1}^{T_2} L(q, \dot{q}) \, dt, \quad \forall q \in H^1([T_1, T_2], \mathbb{R}^{dN}_\Delta),
\]
is \( C^2 \) on \( H^1([T_1, T_2], \mathbb{R}^{dN}_\Delta) \) (see [1]), and a critical point of \( A \) in \( H^1([T_1, T_2], \mathbb{R}^{dN}) \) is a classical solution of (1).

The solution of (1) is invariant under linear translations, in many cases it will be more convenient to fix the center of mass at the origin, so we set
\[
\mathcal{X} := \{ q \in \mathbb{R}^{dN} : \sum_{i \in \mathbb{N}} m_i q_i = 0 \}, \quad \mathcal{X}' := \mathcal{X} \setminus \Delta.
\]
As the action functional is also invariant under linear translation, a critical point of \( A \) in \( H^1([T_1, T_2], \mathcal{X}') \) will be a classical solution of (1) as well.

In general it is much easier to apply variational methods to the \( N \)-body, when the potential is a strong force, i.e. \( \alpha \geq 2 \), as in this case any path with a finite action value must be collision-free, see [9]. It is not so, when the potential is a weak force, i.e. \( \alpha \in (0, 2) \), as the attracting force between the masses are too weak and the action value of a path with collision may still be finite, see [13].

Because of this, for the strong force \( N \)-body problem, many results have been obtained by different authors using both minimization and non-minimization variational methods, see [1, 4, 9, 17, 18, 22] and the references within. However the problem is much more difficult for weak force potentials due to the possibility of collision. Set back by this, Bahri and Rabinowitz introduced the so called generalized solution in [3] and [4] (see definition 2.1), where such a solution is allowed to have a non-empty set of collision moments with zero Lebesgue measure. Here we are only interested in the weak force \( N \)-body problem, so we assume \( \alpha \in (0, 2) \) in the rest of the paper.

For action minimization methods, the breakthrough followed the proofs of the figure-eight solution of the three body by Chenciner and Montgomery [10] and the Hip-Hop solution...
of the four body by Chenciner and Venturelli [11], where both solutions are found as collision-free minimizers of the action functional under proper symmetric constraints. Since then action minimization methods have thrived in the study of the $N$-body problem with Newtonian potential as well as other weak force potentials. We refer the interested readers to [7, 8, 12, 27, 28] and the references within.

Now it is more or less well understood, when we can show an action minimizer is collision-free.

**Definition 1.1.** $q \in H^1([T_1, T_2], X)$ is a local action minimizer of $\mathcal{A}$ with fixed-ends in $H^1([T_1, T_2], X)$, if there is a $\delta > 0$ small enough, such that

$$\mathcal{A}(q; T_1, T_2) \leq \mathcal{A}(q + \tilde{q}; T_1, T_2), \forall \tilde{q} \in H^2_0([T_1, T_2], X) \text{ with } \|\tilde{q}\| \leq \delta,$$

where $H^2_0([T_1, T_2], X)$ is the space of Sobolev paths with compact support in $[T_1, T_2]$.

The following fundamental result is due to Marchal [19] and Chenciner [8], when $\alpha = 1$, and Ferrario and Terracini [12], when $\alpha \in (0, 2)$.

**Theorem 1.1.** For any $\alpha \in (0, 2)$, when $d \geq 2$, if $q$ is a local action minimizer of $\mathcal{A}$ with fixed-ends in $H^1([T_1, T_2], X)$, then $q(t)$ is collision-free, for any $t \in (T_1, T_2)$.

Despite the above progress, to our knowledge, for the $N$-body problem no result seems to be available regarding how to rule out collision when the corresponding path is obtained through non-minimization methods, like minimax or mountain-pass. Meanwhile for a special type of singular Lagrange systems with weak force potentials (essentially equivalent to perturbations of the $N$ center problem), using Morse index theory, in a series of papers ([23–26]), Tanaka showed how to rule out collisions when a critical point was obtained by the minimax approach of Bahri and Rabinowitz [3]. The main purpose of our paper is to generalize Tanaka’s idea to the $N$-body problem and show that the Morse index of a critical point can be used to give an upper bound of the number of binary collisions that could occur in it.

To give the precise statement, we recall the following definition according to [2].

**Definition 1.2.** Let $X$ be a Hilbert space and $\Omega$ an open and dense subspace of it, i.e. $\Omega = X$. If a functional $\mathcal{F}$ is lower semi-continuous in $X$ and $C^2$ in $\Omega$, then $c \in \mathbb{R}$ will be called a critical value of $\mathcal{F}$, if there is $q \in \Omega$, such that $\mathcal{F}(q) = c$ and the first derivative of $\mathcal{F}$ vanishes at $q$, i.e. $d\mathcal{F}(q) = 0$. Moreover such a $q$ will be called a critical point of $\mathcal{F}$.

If $q \in \Omega$ is a critical point of $\mathcal{F}$, we define its Morse index (with respect to $\mathcal{F}$), $m_\Omega(q, \mathcal{F})$, as the dimension of the largest subspace of $\Omega$, where the second derivative $d^2\mathcal{F}(q)$ is negative definite.

The above definition suits the study of the $N$-body problem, as $H^1([T_1, T_2], \tilde{X})$ is an open and dense subspace of $H^1([T_1, T_2], X)$ with the action functional $\mathcal{A}$ being $C^2$ in $H^1([T_1, T_2], \tilde{X})$ and lower semi-continuous in $H^1([T_1, T_2], X)$. Since $\mathcal{A}$ is generally not differentiable at a collision path, such a path cannot be a critical point and does not have a well-defined Morse index, although a collision path can still be a local minimizer.

To deal with the above problem, following ideas from [2, 3, 23], let us perturb the weak force $N$-body problem (1) by a strong force potential

$$\varepsilon \mathcal{A}(q) := \varepsilon \sum_{i<j \in N} \frac{m_i m_j}{|q_i - q_j|^2}, \text{ for } \varepsilon > 0 \text{ small enough.}$$

Then the motion of masses satisfies
\[ m \ddot{q}_i = -\alpha \sum_{j \in \mathbb{N}_i(t)} \frac{m m_j (q_i - q_j)}{|q_i - q_j|^{\alpha + 2}} - 2\varepsilon \sum_{j \in \mathbb{N}_i(t)} \frac{m m_j (q_i - q_j)}{|q_i - q_j|^4}, \quad \forall i \in \mathbb{N}, \]  
which is the Euler–Lagrange equation of the action functional
\[ \mathcal{A}^\varepsilon(q; T_1, T_2) := \int_{T_1}^{T_2} L^\varepsilon(q, \dot{q}) \, dt, \quad \text{where} \quad L^\varepsilon(q, \dot{q}) := L(q, \dot{q}) + \varepsilon M(q). \]

Furthermore we set \( \mathcal{A}^0(q; T_1, T_2) = \mathcal{A}(q; T_1, T_2). \) When \( \varepsilon > 0, \) any path with a finite value of \( \mathcal{A}^\varepsilon \) must be collision-free.

Given an arbitrary path \( q \in H^1([T_1, T_2], \mathcal{X}), \) set
\[ H^1(q) := \{ \bar{q} \in H^1([T_1, T_2], \mathcal{X}) : \bar{q}(T_i) = q(T_i), i = 1, 2 \}; \]
\[ \bar{H}^1(q) := H^1(q) \cap H^1([T_1, T_2], \mathcal{X}). \]

Then \( \bar{H}^1(q) \) is an open and dense subset of \( H^1(q), \) where \( \mathcal{A}^\varepsilon \) is lower semi-continuous in \( H^1(q) \) and \( C^2 \) in \( H^1(q). \) Hence if \( q \) is collision-free and \( dA^\varepsilon(q) = 0, \) then it is a critical point of \( \mathcal{A}^\varepsilon, \) and we will denote its Morse index in \( \bar{H}^1(q) \) by \( m_{T_1, T_2}^\varepsilon(q, \mathcal{A}^\varepsilon). \) Now we introduce a notion called weak critical points as a generalization of the usual critical points.

**Definition 1.3.** We say a path \( q \in H^1([T_1, T_2], \mathcal{X}) \) (which may contain collision) with finite action value, \( \mathcal{A}(q; T_1, T_2) < \infty, \) is a weak critical point of \( \mathcal{A}, \) if there exists a sequence of positive numbers \( \varepsilon_n \to 0 \) and a sequence of \( q^n \in H^1([T_1, T_2], \mathcal{X}) , \) such that
\begin{enumerate}[(i).]
    \item \( \mathcal{A}^{\varepsilon_n}(q^n; T_1, T_2) < C, \forall n, \) for some finite constant \( C; \)
    \item \( q^n \) is a critical point of \( \mathcal{A}^{\varepsilon_n}, \) for any \( n; \)
    \item \( q^n \to q \) weakly in \( H^1 \)-norm and strongly in \( L^\infty \)-norm.
\end{enumerate}

\[ c = \mathcal{A}(q; T_1, T_2) \] will be called a weak critical value of \( \mathcal{A}, \) and the Morse index of such a weak critical point \( q \) in \( H^1(q) \) (with respect to \( \mathcal{A} \)) will be defined as
\[ m_{T_1, T_2}^\varepsilon(q, \mathcal{A}) = \inf_{n \to \infty} \inf m_{T_1, T_2}^\varepsilon(q^n, \mathcal{A}^{\varepsilon_n}), \]
where the infimum is taken over all sequences \( \varepsilon_n \) and \( q^n \) satisfying the above conditions.

**Remark 1.1.** A similar notation was introduced in [2], where it was called generalized critical point.

With the above definition, we have the following result, which can be seen as a partial generalization of theorem 1.1.

**Theorem 1.2.** When \( d \geq 3, \) given a weak critical point \( q \in H^1([T_1, T_2], \mathcal{X}) \) of \( \mathcal{A}, \) let \( \mathcal{B}(q) \) represent the number of binary collisions occurring in \( q(t), t \in (T_1, T_2) \) (when there are more than one binary collision at a given moment, each of them should be counted separately), then
\[ (d - 2)i(\alpha)\mathcal{B}(q) \leq m_{T_1, T_2}^\varepsilon(q, \mathcal{A}), \]
where
\[ i(\alpha) = \max \{ k \in \mathbb{Z} : k < \frac{2}{2 - \alpha} \}. \]
In particular $q(t), t \in (T_1, T_2)$, is free of binary collision, i.e. $B(q) = 0$, if

$$m_{T_1, T_2}(q, A) < (d - 2)i(\alpha).$$

**Remark 1.2.** Notice that by (9), $i(\alpha) = 1$, if $\alpha \in (0, 1]$ and $i(\alpha) \geq 2$, if $\alpha \in (1, 2)$. Moreover $i(\alpha)$ goes to infinity, as $\alpha$ goes to 2.

**Remark 1.3.** It seems the above result is the best we can get based on Tanaka’s idea. In particular, we are unable to obtain any nontrivial result, when $d = 2$. For an explanation see remark 4.1.

The idea of using Morse index to rule out collision should work even when a collision cluster has more than two masses, although the technical difficulty seems very challenging. This is because when two masses approach to a binary collision, they behaves more and more like the two body problem, where the solutions are well understood and their Morse indices are relatively easy to compute. However when the collision cluster has more than two masses, as they approach to collision, the dynamics is much more complicate (see [20]) and the computation of Morse indices of the relevant solutions is also much more difficult. Despite of this, some progresses have been made recently in [5, 6, 15].

Theorem 1.2 has the following obvious corollaries.

**Corollary 1.1.** When $m_{T_1, T_2}(q, A) = 0$ and $d \geq 3$, $q(t), t \in (T_1, T_2)$, is free of binary collision.

**Corollary 1.2.** When $m_{T_1, T_2}(q, A) = 1$, the following results hold.

(a). If $d \geq 4$ and $\alpha \in (0, 2)$, then $q(t), t \in (T_1, T_2)$ is free of binary collision.
(b). If $d = 3$ and $\alpha \in (1, 2)$, then $q(t), t \in (T_1, T_2)$ is free of binary collision.
(c). If $d = 3$ and $\alpha \in (0, 1]$, then $q(t), t \in (T_1, T_2)$ has at most one binary collision, i.e. $B(q) \leq 1$.

Notice that in corollary 1.2, when $d = 3$ and $\alpha \in (0, 1]$, the weak critical point may very well contains a binary collision and in this case we have the following result.

**Theorem 1.3.** When $d = 3$ and $\alpha = 1$, let $q \in H^1([T_1, T_2], X)$ be a weak critical point of $A$ with $m_{T_1, T_2}(q, A) = 1$. If there is a binary collision between $m_0$ and $m_1$ at a moment $t_0 \in (T_1, T_2)$, then both limits $\lim_{t \to t_0^+} \frac{q_0(t) - q_0(t)}{\|q_0(t) - q_0(t)\|}$ exist and equal to each other.

**Remark 1.4.** The above result is interesting, because it is well-known if a solution of the spatial $N$-body problem has a single binary collision at a moment (no other partial collision exists at the same moment), then it can be regularized by Kustaanheimo–Stiefel regularization [16]. With the result from the above theorem, under the assumption that there is no other partial collisions, one can show the generalized solution corresponding to the weak critical point is actually a classical solution in the regularized system.

**Remark 1.5.** Results similar to theorem 1.3 can be obtained for potentials with $\alpha \neq 1$, see lemma 3.2. However the problem of regularizing a binary collision is more complicate for non-Newtonian potentials, see [21].

Since the Morse index of a critical point obtained by the mountain pass theorem must be less than or equal to one (see [14]), we believe corollaries 1.1 and 1.2 and theorem 1.3 could
be useful, when mountain pass methods are used in the study of the $N$-body problem. This shall be discussed in a forthcoming paper.

Our paper is organized as follows: in section 2 we show a weak critical point is a generalized solution, in section 3 the proofs of the main results will be given, and in sections 4 and 5 we give the proofs of some technical lemmas.

2. Generalized solutions

Consider the perturbed $N$-body problem (5), for any subset of indices $I \subset N$, we define the Lagrangian and energy of the $I$-cluster as

$$L_{I}^{ε}(q, \dot{q}) := K_{I}(\dot{q}) + U_{I}(q) + εU(q),$$

$$E_{I}^{ε}(q, \dot{q}) := K_{I}(\dot{q}) - U_{I}(q) - εU(q),$$

where

$$K_{I}(\dot{q}) := \frac{1}{2} \sum_{i \in I} m_{i} |\dot{q}_{i}|^{2}, \quad U_{I}(q) := \sum_{(i < j) \in I} \frac{m_{i} m_{j}}{|q_{i} - q_{j}|^{α}}, \quad U(q) := \sum_{(i < j) \in N} \frac{m_{i} m_{j}}{|q_{i} - q_{j}|^{α}}.$$

Let $I' := N \setminus I$ denote the complement of $I$ in $N$, then

$$L^{ε}(q, \dot{q}) = L_{I}^{ε}(q, \dot{q}) + L_{I'}^{ε}(q, \dot{q}) + U_{I'}(q) + εU_{I'}(q),$$

$$E^{ε}(q, \dot{q}) = E_{I}^{ε}(q, \dot{q}) + E_{I'}^{ε}(q, \dot{q}) - U_{I'}(q) - εU_{I'}(q),$$

where

$$U_{I'}(q) := \sum_{i \in I' j \in I} \frac{m_{i} m_{j}}{|q_{i} - q_{j}|^{α}}, \quad U_{I'}(q) := \sum_{i \in I' j \in I} \frac{m_{i} m_{j}}{|q_{i} - q_{j}|^{α}}.$$

**Definition 2.1.** $q \in H^{1}([T_{1}, T_{2}], \mathcal{X})$ is a generalized solution of (1), if it satisfies the following conditions:

(i) $Δ^{-1}(q) := \{ t \in [T_{1}, T_{2}] : q(t) \in Δ \}$ has measure 0 in $[T_{1}, T_{2}]$;

(ii) $q \in C^{2}$ on $[T_{1}, T_{2}] \setminus Δ^{-1}(q)$ and satisfies (1);

(iii) the total energy of $q(t)$, $E(t) = E(q(t), \dot{q}(t))$, is a constant, for all $t \in [T_{1}, T_{2}] \setminus Δ^{-1}(q)$;

(iv) for any subset $I \subset N$ and sub-interval $(t_{1}, t_{2}) \subset [T_{1}, T_{2}]$, if

$$q_{i}(t) \neq q_{j}(t), \quad \forall i \in I, \forall j \in I', \text{ and } \forall t \in (t_{1}, t_{2}),$$

then $E_{I}(t) = E_{I}(q(t), \dot{q}(t)) \in H^{1}((t_{1}, t_{2}), \mathbb{R})$. In particular, $E_{I}(t)$ is continuous in $(t_{1}, t_{2})$.

**Remark 2.1.** Condition (iv) in the above definition shows the energy of a $I$-cluster is continuous, as long as the masses from the $I$-cluster do not collide with masses outside of the cluster, even when there are collisions among the masses inside the cluster. This condition was not required by in the original definition of a generalized solution introduced by Bahri and Rabinowitz, see [1] and [4]. Our definition here is stronger and follows from [12, definition 4.6].

**Proposition 2.1.** A weak critical point $q \in H^{1}([T_{1}, T_{2}], \mathcal{X})$ of $\mathcal{A}$ is a generalized solution of (1).
Proof. Let $\varepsilon_n > 0$ and $q^n \in H^1(T_1, T_2, X)$ be two sequences satisfying the conditions given in definition 1.3. Then there is a finite constant $C > 0$, such that

$$\frac{1}{2} \int_{t_1}^{t_2} \sum_{i \in \mathbb{N}} m_i |q^n_i(t)|^2 \, dt \leqslant A^{\varepsilon_n}(q^n; T) \leqslant C, \quad \forall n.$$  \hfill (11)

The fact that $q$ satisfies the first three conditions given in definition 2.1 is a standard result, for details see [4] or [1]. In the following, we will show $q$ also satisfies condition (iv). Given an arbitrary $I \subset \mathbb{N}$, recall that

$$E_1^{\varepsilon_n}(t) = E_1^n(q^n(t), \dot{q}^n(t)) = K_1(\dot{q}^n(t)) - U_1(q^n) - \varepsilon_n \Delta_1(q^n).$$

By a direct computation,

$$\frac{dE_1^{\varepsilon_n}}{dt} = \sum_{i \in I} \left( \frac{\partial U_1(q^n)}{\partial q^n_i} + \varepsilon_n \frac{\partial \Delta_1(q^n)}{\partial q^n_i} \right).$$  \hfill (12)

Let us assume $q$ satisfies (10) for the above $I$ and an arbitrary sub-interval $(t_1, t_2) \subset [T_1, T_2]$. Since $q^n(t)$ converges to $q(t)$ uniformly on $[0, T]$,

$$\left| \frac{\partial U_1(q^n)}{\partial q^n_i} \right|, \left| \frac{\partial \Delta_1(q^n)}{\partial q^n_i} \right| \leqslant C_1, \quad \forall t \in [t_1, t_2], \quad \forall i \in I.$$  \hfill (13)

Here and in the rest of the proof $C_i, i \in \mathbb{Z}^+$, always represents some positive constant independent of $n$. With (12) and (13), the Cauchy–Schwarz inequality tells us

$$|E_1^{\varepsilon_n}(t)|^2 \leqslant C_2 \sum_{i \in I} m_i |q^n_i(t)|^2 \leqslant C_2 \sum_{i \in \mathbb{N}} m_i |q^n_i(t)|^2, \quad \forall t \in [t_1, t_2].$$

Then

$$\int_{t_1}^{t_2} |E_1^{\varepsilon_n}(t)|^2 \, dt \leqslant C_2 \int_{t_1}^{t_2} \sum_{i \in \mathbb{N}} m_i |q^n_i(t)|^2 \, dt \leqslant C_3.$$  \hfill (14)

Since $U_1, \Delta_1$ are always positive, $E_1^{\varepsilon_n}(t) \leqslant K_1(\dot{q}^n(t)), \forall t$. Then

$$\int_{t_1}^{t_2} E_1^{\varepsilon_n}(t) \, dt \leqslant \frac{1}{2} \int_{t_1}^{t_2} \sum_{i \in \mathbb{N}} m_i |q^n_i(t)|^2 \, dt \leqslant C_4.$$  \hfill (15)

Meanwhile by Poincaré inequality and (14),

$$\int_{t_1}^{t_2} |E_1^{\varepsilon_n}(t) - \int_{t_1}^{t_2} E_1^{\varepsilon_n}(s) \, ds|^2 \, dt \leqslant C_5 \int_{t_1}^{t_2} |E_1^{\varepsilon_n}|^2 \, dt \leqslant C_6.$$  \hfill (16)

Then (15) implies

$$\int_{t_1}^{t_2} |E_1^{\varepsilon_n}(t)|^2 \, dt \leqslant C_7.$$  

The above inequality and (14) implies $E_1^{\varepsilon_n}(t)$ is a bounded sequence in $H^1([t_1, t_2], \mathbb{R})$. After passing to a subsequence, it converges to a $\bar{E}_1(t) \in H^1([t_1, t_2], \mathbb{R})$ weakly in $H^1$ norm and strongly in $L^\infty$ norm.
Since $\dot{q}(t)$ and $E_q(t) = E_1(q(t), \dot{q}(t))$ are well defined for any $t \notin \Delta^{-1}(q)$, and

$$q^n(t) \rightarrow q(t), \quad \dot{q}^n(t) \rightarrow \dot{q}(t), \quad \text{as } n \rightarrow \infty, \quad \forall t \notin \Delta^{-1}(q),$$

we have $E_q^n(t) \rightarrow E_1(t)$, for any $t \notin \Delta^{-1}(q)$. As a result, $E_q(t) = E_1(t)$, for any $t \in [t_1, t_2) \setminus \Delta^{-1}(q)$. Since $\Delta^{-1}(q)$ is a set of measure zero, $E_q(t) = E_1(t)$ as a $H^1$-Sobolev function, and it is continuous in $(t_1, t_2)$.

**Definition 2.2.** Given a path $q \in H^1([T_1, T_2], \mathcal{X})$ with an $I$-cluster collision at a moment $t_0 \in (T_1, T_2)$, we say it is isolated, if there is a constant $a > 0$ small enough, such that for any $i \in I$,

$$q_i(t) \neq q_i(t_0), \quad \forall t \in [t_0 - a, t_0 + a] \setminus \{t_0\}, \quad \forall j \in N \setminus \{i\}.$$

**Proposition 2.2.** Given a weak critical point $q \in H^1([T_1, T_2], \mathcal{X})$, if there is a binary collision at the moment $t_0 \in (T_1, T_2)$, then it must be isolated.

**Proof.** By proposition 2.1, $q$ is a generalized solution of (1). In particular it satisfies condition (iv) in definition 2.1, then the desired result was already proven in [12, corollary 5.12]. Once the reader notices that every binary collision is a so called locally minimal collision defined in [12, definition 5.2].

**3. Proof of theorems 1.2 and 1.3**

To prove the main theorems, three technical lemmas will be needed. We present them as lemmas 3.1–3.3 in this section and postpone their proofs until the next two sections.

Let $q \in H^1([T_1, T_2], \mathcal{X})$ be a weak critical point of $\mathcal{A}$ with a binary collision at the moment $t_0 \in (T_1, T_2)$ and $q^n \in H^1([T_1, T_2], \mathcal{X})$ a sequence of critical points of $\mathcal{A}_n^+$ satisfying the conditions required in definition 1.3. Without loss of generality, we may assume such a binary collision is between $m_1$ and $m_2$, i.e.

$$q_1(t_0) = q_2(t_0) \neq q_i(t_0), \quad \forall i \in N \setminus \{1, 2\}.$$ 

By proposition 2.2, such an binary collision must be isolated, so we may choose an $a > 0$ small enough, such that $[t_0 - 2a, t_0 + 2a] \subset (T_1, T_2)$ and

$$q_1(t) \neq q_2(t), \quad \forall t \in [t_0 - 2a, t_0 + 2a] \setminus \{t_0\}; \quad (17)$$

$$q_1(t) \neq q_i(t), \quad \forall t \in [t_0 - 2a, t_0 + 2a], \quad \forall i \in \{1, 2\}, \quad \forall j \in N \setminus \{1, 2\}. \quad (18)$$

For each $n$, we can always find a $t_n \in [t_0 - 2a, t_0 + 2a]$ such that

$$\delta_n := |q_1^n(t_n) - q_2^n(t_n)| = \min \{ |q_1^n(t) - q_2^n(t)| : t \in [t_0 - 2a, t_0 + 2a] \}.$$ 

Obviously $\delta_n$ converges to $0$, as $n$ goes to infinity. After passing to a subsequence, we may assume the limit of $t_n$ exists. Since $q_1(t_0) = q_2(t_0)$ is an isolated binary collision at the moment $t_0$, $t_n$ must converge to $t_0$. Then for $n$ large enough, $[t_n - a, t_n + a] \subset [t_0 - 2a, t_0 + 2a]$. As a result,

$$\delta_n = \min \{ |q_1^n(t) - q_2^n(t)| : t \in [t_n - a, t_n + a] \}. \quad (19)$$
By definition 1.3, \( q^n(t) \) converges to \( q(t) \) uniformly on \([T_1, T_2]\). According to (18), there is constant \( C_1 > 0 \) independent of \( n \), such that
\[
|q^n_i(t) - q^n_j(t)| \geq C_1, \quad \forall t \in [a - a_n, a], \quad \forall i \in \{1, 2\}, \quad \forall j \in \mathbb{N} \setminus \{1, 2\}.
\]
(20)

Let
\[
\bar{U}(q^n) := U(q^n) - \frac{m_1 m_2}{|q^n_1 - q^n_2|^{\alpha}}, \quad \bar{\chi}(q^n) := \chi(q^n) - \frac{m_1 m_2}{|q^n_1 - q^n_2|^{\alpha}}.
\]
There are constants \( C_2, C_3 > 0 \) independent of \( n \), such that
\[
\left| \frac{\partial \bar{U}(q^n(t))}{\partial q^n_i} \right|, \quad \left| \frac{\partial \bar{\chi}(q^n(t))}{\partial q^n_i} \right| \leq C_2, \quad \forall t \in [a - a_n + a], \quad \forall 1 \leq i \leq 2,
\]
(21)
\[
\left| \frac{\partial^2 \bar{U}(q^n(t))}{\partial q^n_i \partial q^n_j} \right|, \quad \left| \frac{\partial^2 \bar{\chi}(q^n(t))}{\partial q^n_i \partial q^n_j} \right| \leq C_3, \quad \forall t \in [a - a_n + a], \quad \forall 1 \leq i, j \leq 2.
\]
(22)

We introduce a new function \( \eta^n(t) = (\eta^n_i(t))_{i=1}^{N} \) by
\[
\begin{cases}
\eta^n_1(t) = q^n_1(t) - q^n_i(t), \\
\eta^n_2(t) = m_1 \eta^n_1(t) + m_2 q^n_2(t), \\
\eta^n_i(t) = q^n_i(t), \quad \text{if} \quad i = 3, \ldots, N.
\end{cases}
\]
(23)

By a direct computation, \( \eta^n(t) \) is a solution of
\[
\frac{m_1 m_2}{m_1 + m_2} \eta^n_1 = -\alpha m_1 m_2 \frac{\eta^n_1}{|\eta^n_1|^{\alpha+2}} - 2m_1 m_2 \varepsilon \eta^n_1 - \frac{\partial \bar{U}(\eta^n)}{\eta^n_1} + \varepsilon \frac{\partial \bar{\chi}(\eta^n)}{\eta^n_1},
\]
(24)
\[
(m_1 + m_2) \eta^n_2 = \frac{\partial \bar{U}(\eta^n)}{\eta^n_2} + \varepsilon \frac{\partial \bar{\chi}(\eta^n)}{\eta^n_2};
\]
(25)
\[
m_1 \eta^n_3 = \frac{\partial \bar{U}(\eta^n)}{\eta^n_3} + \varepsilon \frac{\partial \bar{\chi}(\eta^n)}{\eta^n_3}, \quad i = 3, \ldots, N.
\]
(26)

This is the Euler–Lagrange equation of the following Lagrangian
\[
L(\eta^n, \dot{\eta}^n) = \frac{m_1 m_2}{2(m_1 + m_2)} |\dot{\eta}^n|^2 + \frac{m_1 + m_2}{2} |\eta^n_1|^2 + \frac{1}{2} \sum_{i=3}^{N} m_i |\eta^n_i|^2
\]
\[
+ \frac{m_1 m_2}{|\eta^n_1|^{\alpha}} + \frac{\varepsilon m_1 m_2}{|\eta^n_1|^{\alpha+2}} + \bar{U}(\eta^n) + \bar{\chi}(\eta^n).
\]
(27)

To study the behaviors of the solutions as they approach to the binary collision, Tanaka’s blow-up technique will be used. The precise argument depends on the limit of \( \varepsilon_n / \delta_n^{2-\alpha} \). After passing to subsequence, we may assume such a limit \( \lambda = \lim_{n \to \infty} \varepsilon_n / \delta_n^{2-\alpha} \) always exists. Then two different cases need to be considered: Case 1, \( \lambda \in [0, \infty) \); Case 2, \( \lambda = \infty \).

For Case 1, we blow up \( \eta^n(t) \) according to
\[
\xi^n(s) = (\xi^n_i(s))_{i=1}^{N} := (\delta_n^{1-\frac{\alpha}{2}} \eta^n_i(t(s)))_{i=1}^{N}, \quad \text{where} \quad t(s) = \delta_n^{1+\frac{\alpha}{2}} s + t_n.
\]
(28)
By changing the time parameter from $t$ to $s$, the time interval $[t_n - a, t_n + a]$ is mapped onto $[-a/\delta_n^{1+\alpha/2}, a/\delta_n^{1+\alpha/2}]$. Notice that the latter interval converges to $\mathbb{R}$, as $n$ goes to infinity. Let $\alpha$ denote derivatives with respect to $s$, then $\xi_1(s)$ satisfies

$$
\frac{1}{m_1 + m_2} (\xi_1^\alpha)^\prime = -\alpha \frac{\xi_1^n}{|\xi_1|^2} - 2 \frac{\varepsilon_n}{\delta_2^{\alpha}} \frac{\xi_1^n}{|\xi_1|^2} + \frac{\delta_1^{1+\alpha}}{m_1 m_2} \left( \frac{\partial U(n^\prime)}{\partial n^\prime} + \varepsilon_n \frac{\partial \tilde{U}(n^\prime)}{\partial n^\prime} \right). 
$$

(29)

**Lemma 3.1.** If $\lambda = \lim_{n \to \infty} \varepsilon_n / \delta_n^{2-\alpha}$ is finite, then the following results hold.

(a). After passing to a subsequence, $\xi_1^n(s)$ converges to a $\xi_1(s)$ in $C^2([-\ell, \ell], \mathbb{R}_d)$, for any $\ell > 0$, where $\xi_1(s)$ is a solution of

$$
\frac{1}{m_1 + m_2} \xi_1^\alpha = -\alpha \frac{\xi_1^n}{|\xi_1|^2} - \frac{\lambda}{|\xi_1|^2} \xi_1^n.
$$

(30)

$$
\frac{1}{2(m_1 + m_2)} |\xi_1|^2 - \frac{1}{|\xi_1|^2} = 0;
$$

(31)

$$
(\xi_1(0), \dot{\xi}_1(0)) = 0.
$$

(32)

(b). $\xi_1(t) \in W(\xi_1), \forall t \in \mathbb{R}$, where $W(\xi_1) = \text{span}\{\xi_1(0), \xi_1'(0)\}$ is a 2-dim subspace of $\mathbb{R}^d$. Moreover the following limits exist

$$
\lim_{t \to \pm \infty} |\xi_1(s)| = +\infty, \quad \lim_{t \to \pm \infty} \frac{\xi_1(s)}{|\xi_1(s)|} = u^\pm,
$$

and

$$
\angle(u^-, u^+) = 2\pi \frac{\sqrt{1 + \lambda}}{2 - \alpha}.
$$

For any two unit vectors $u, v \in \mathbb{R}^d$, $\angle(u, v)$ represents the angle between them.

(c). Let $W^\perp(\xi_1)$ be the orthogonal complement of $W(\xi_1)$ in $\mathbb{R}^d$ and $H(\xi_1)$ the largest subspace of $H_0^1(\mathbb{R}, W^\perp(\xi_1))$, such that

$$
d^2\mathcal{I}(\xi_1)(\phi, \phi) < 0, \quad \forall \phi \in H(\xi_1),
$$

where $\mathcal{I}$ is the Lagrange action functional corresponding to equation (30):

$$
\mathcal{I}(\xi_1) = \int \frac{1}{2(m_1 + m_2)} |\xi_1|^2 + \frac{1}{|\xi_1|^2} + \frac{\lambda}{|\xi_1|^2} \mathcal{L} dt.
$$

(33)

then $\dim(H(\xi_1)) \geq (d - 2)i(\alpha, \lambda)$, where

$$
i(\alpha, \lambda) := \max\{k \in \mathbb{Z} : k < \frac{2\sqrt{1 + \lambda}}{2 - \alpha} \}.
$$

(34)
Lemma 3.2. If \( \lambda = \lim_{n \to \infty} \varepsilon_n / \delta_n^{2-\alpha} \) is finite, then
\[
\lim_{t \to t_n^\pm} \frac{q_2(t) - q_1(t)}{|q_2(t) - q_1(t)|} = u^\pm,
\]
where \( u^\pm \) are the two unit vectors given in property (b), lemma 3.1.

For Case 2, we define a blow-up of \( \eta^n \) according to
\[
\zeta^n(t) = (\zeta^n_i(t))_{i=1}^N = (\delta_n^{-1} \eta^n_i(t(s)))_{i=1}^N, \quad \text{where} \quad t(s) = \varepsilon_n^{-1/2} \delta_n^2 s + t_n.
\]

Like the previous case, after changing the time parameter from \( t \) to \( s \), the time interval \([t_n - a, t_n + a]\) is mapped onto \([-a \varepsilon_n^3 / \delta_n^3, a \varepsilon_n^3 / \delta_n^3]\) which converges to \( \mathbb{R} \), as \( n \) goes to infinity.

Again if we let \( r \) represents derivatives with respect to \( s \), then \( \zeta^n_i(s) \) satisfies
\[
\frac{1}{m_1 + m_2} \zeta^n_i''(t) = -2 \frac{\zeta^n_i}{|\zeta^n_i|^4} - \frac{\delta_n^{2-\alpha}}{\varepsilon_n} |\zeta^n_i|^a + 1
\]
\[
+ \frac{\varepsilon_n}{\delta_n^4 m_1 m_2} \left( \frac{\partial U(\eta^n)}{\partial \eta^n_i} + \varepsilon_n \frac{\partial \tilde{A}(\eta^n)}{\partial \eta^n_i} \right).
\]

Lemma 3.3. If \( \lambda = \lim_{n \to \infty} \varepsilon_n / \delta_n^{2-\alpha} = \infty \), then the following results hold.

(a). After passing to a subsequence, \( \zeta^n_i(s) \) converges to a \( \zeta_1(s) \) in \( C^2([-\ell, \ell], \mathbb{R}^d) \), for any \( \ell > 0 \), where \( \zeta_1(s) \) is a solution of
\[
\frac{1}{m_1 + m_2} \dot{\zeta}_1'' = -2 \frac{\dot{\zeta}_1}{|\zeta_1|^4};
\]
\[
\frac{1}{2(m_1 + m_2)} |\dot{\zeta}_1|^2 - \frac{1}{|\zeta_1|^2} = 0;
\]
\[
\langle \dot{\zeta}_1(0), \dot{\zeta}_1(0) \rangle = 0.
\]
(b). \( \zeta_i(t) \in W(\zeta_1), \forall t \in \mathbb{R} \), where \( W(\zeta_1) = \text{span}\{\zeta_1(0), \dot{\zeta}_1(0)\} \) is a 2-dim subspace of \( \mathbb{R}^d \), and \( \lim_{n \to \pm \infty} |\zeta_1(s)| = +\infty \).
(c). Let \( W^+ (\zeta_1) \) be the orthogonal complement of \( W(\zeta_1) \) in \( \mathbb{R}^d \) and \( H(\zeta_1) \) the largest subspace of \( H_0^1(\mathbb{R}, W^+ (\zeta_1)) \), such that
\[
d^2 J(\zeta_1)(\phi, \phi) < 0, \quad \forall \phi \in H(\zeta_1),
\]
where \( J \) is the Lagrange action functional corresponding to equation (37),
\[
J(\dot{\zeta}_1) := \int \frac{1}{2(m_1 + m_2)} |\dot{\zeta}_1|^2 + \frac{1}{|\zeta_1|^2} ds.
\]
then \( \dim(\text{dim}(H(\zeta_1))) = +\infty \).

Proposition 3.1. Under the above notation,

(a). if \( \lambda = \lim_{n \to \infty} \varepsilon_n / \delta_n^{2-\alpha} \) is finite, then
Theorem 5.4. \( \liminf_{n \to \infty} m_{-a}^n(q^n, A^\alpha) \geq (d-2)i(\alpha, \lambda); \) \( (d-2)i(\alpha, \lambda); \)

(b) If \( \lambda = \lim_{n \to \infty} \varepsilon_n/\delta_n^{-\alpha} = +\infty, \) then

\[
\liminf_{n \to \infty} m_{-a}^n(q^n, A^\alpha) = +\infty.
\]

Proof.

(a) Given an arbitrary \( c > 0 \) and \( f = (f_i)_{i=1}^N \in H_0^1([t_n - a, t_n + a], \mathbb{R}^N) \) satisfying

\[
f_i(t) \equiv 0, \ \forall t, \quad \text{and} \quad \forall i \neq 1.
\]

If we keep each \( q_i^n(t), i = 3, \ldots, N, \) unchanged and modify \( q_1^n(t), i = 1, 2, \) as follows:

\[
q_1^n(t) = \frac{m_2 c f_i(t)}{m_1 + m_2}, \quad q_2^n(t) = \frac{m_1 c f_i(t)}{m_1 + m_2},
\]

then the center of mass is still at origin after the modification. Meanwhile according to (43), \( q_1^n(t) \) becomes \( \eta_1^n(t) + cf_i(t) \) after the above modification.

We shall compute the second variation of \( A^\alpha \) at \( \eta^n \) among all \( f \in H_0^1([t_n - a, t_n + a], \mathbb{R}^N) \) satisfying (43). By a direct computation,

\[
d^2 A^\alpha(\eta^n)[f, f] = \int_{t_n - a}^{t_n + a} \left( m_1 m_2 \left| \frac{f_1}{\eta_1^n} \right|^2 - c m_1 m_2 \frac{\|f_1\|^2}{|\eta_1^n|^\alpha} - 2 m_1 m_2 \varepsilon_n \frac{|f_1|^2}{|\eta_1^n|^\alpha} \right)
\]

\[
+ \alpha(\alpha + 2) m_1 m_2 \frac{\langle \eta_1^n, f_1 \rangle \langle \eta_1^n, f_1 \rangle}{|\eta_1^n|^{\alpha + 4}} + 8 m_1 m_2 \varepsilon_n \frac{\langle \eta_1^n, f_1 \rangle}{|\eta_1^n|^6}
\]

\[
+ \left( \frac{\partial^2 \tilde{U}(\eta^n)}{\partial \eta_1^n \partial \eta_1^n} f_1, f_1 \right) + \varepsilon_n \left( \frac{\partial^2 \tilde{U}(\eta^n)}{\partial \eta_1^n \partial \eta_1^n} f_1, f_1 \right) \, dt.
\]

Define the linear operators \( T_n : H_0^1(\mathbb{R}, W^1(\zeta_1)) \to H_0^1([t_n - a, t_n + a], \mathbb{R}^N) \) by \( f^n(t) = (f_i^n(t))_{i=1}^N \equiv (T_n \phi)(s(t)), \) where \( s(t) = \frac{a}{t_n - a} (t - t_n) \) and \( f_i^n(t) = \delta_n \phi(s(t)) \) and \( f_i^n(t) \equiv 0, \ \forall i \neq 1. \)

Then

\[
\delta_n \frac{d^n}{dt^n} d^2 A^\alpha(\eta^n)[f^n, f^n] = \int_{\text{supp}(\phi)} \frac{m_1 m_2}{m_1 + m_2} |\phi|^2 - c m_1 m_2 \frac{\|\phi\|^2}{|\xi_1^n|^{\alpha + 2}}
\]

\[
- 2 m_1 m_2 \varepsilon_n \frac{|\phi|^2}{|\xi_1^n|^{\alpha - 2}} + \alpha(\alpha + 2) m_1 m_2 \frac{\langle \xi_1^n, \phi \rangle \langle \xi_1^n, \phi \rangle}{|\xi_1^n|^{\alpha + 4}} + 8 m_1 m_2 \varepsilon_n \frac{\langle \xi_1^n, \phi \rangle}{|\xi_1^n|^6}
\]

\[
+ \left( \frac{\partial^2 \tilde{U}(\eta^n)}{\partial \eta_1^n \partial \eta_1^n} \phi, \phi \right) + \varepsilon_n \left( \frac{\partial^2 \tilde{U}(\eta^n)}{\partial \eta_1^n \partial \eta_1^n} \phi, \phi \right) \, ds.
\]

By (22), there is a constant \( C > 0 \) independent of \( n, \) such that

\[
\left| \frac{\partial \tilde{U}(\eta^n(t)))}{\partial \eta_1^n \partial \eta_1^n} \right|, \left| \frac{\partial^2 \tilde{U}(\eta^n(t)))}{\partial \eta_1^n \partial \eta_1^n} \right| \leq C, \ \forall s \in \left[ -a \delta_n^{1+\alpha/2}, -a \delta_n^{1+\alpha/2} \right], \ \forall 1 \leq i \leq 2.
\]
As $\xi_n^\alpha$ converges to $\xi_1$ in $C^2([\ell, \ell], \mathbb{R}^d)$, for any $\ell > 0$, the right hand side of (45) converges to
\[
m_{1,2} \left( \int_{\text{supp}(\phi)} \frac{1}{m_1 + m_2} |\phi'|^2 - \frac{|\phi|^2}{|\xi_1|^\alpha + 2} - \frac{2\lambda}{|\xi_1|^4} \right) = m_{1,2} (d^2 I(\xi_1)(\phi, \phi)).
\]

Then lemma 3.1 implies,
\[
\delta_n \rightarrow -\frac{2\alpha}{d^2 \mathcal{A}^{\alpha} (\eta^n)} |f^n, f^n| < 0, \text{ for } n \text{ large enough, if } \phi \in H(\xi_1).
\]

As $\dim(H(\xi_1)) \geq (d - 2)i(\alpha, \lambda)$, inequality (41) follows.

(b). If $\lambda$ is infinity, with lemma 3.3, the desired property follows from a similar argument. □

With the above results, we can now prove theorems 1.2 and 1.3.

Proof of theorem 1.2. By proposition 2.2, if $q(t)$ has a binary collision, then it is isolated. Then $B(q)$ must be finite.

Assume $q$ has a binary collision at the moment $t_0 \in (T_1, T_2)$, let the corresponding sequences $\varepsilon_n$, $q^n(t)$ and $\delta_n$ be defined as before. If $\lim_{n \to \infty} \varepsilon_n/\delta_n^{2-\alpha} = +\infty$, then property (b) in proposition 3.1 implies $m_{T_1, T_2}(q, \mathcal{A}) = +\infty$, which obviously implies (8).

If the corresponding limit of $\varepsilon_n/\delta_n^{2-\alpha}$ is finite for each binary collision, then (8) follows from the sub-additivity of the Morse index and property (a) in 3.1. Once we notice $i(\alpha, \lambda) \geq i(\alpha)$, for any $\lambda \geq 0$.

Proof of theorem 1.3. Without loss of generality let us assume the binary collision is between $m_1$ and $m_2$ with the sequences $\varepsilon_n$, $q^n(t)$ and $\delta_n$ defined as before, and $\lambda = \lim_{n \to \infty} \varepsilon_n/\delta_n^{2-\alpha}$. Obviously $\lambda$ must be finite, as otherwise by proposition 3.1, $m_{T_1, T_2}(q, \mathcal{A}) = +\infty$, which contradicts our assumption that $m_{T_1, T_2}(q, \mathcal{A}) = 1$.

Since $\lambda$ is finite, $\alpha = 1$ and $d = 3$, by proposition 3.1,
\[
1 = m_{T_1, T_2}(q, \mathcal{A}) \geq i(1, \lambda) = \max\{k \in \mathbb{Z} : k < 2\sqrt{1 + \lambda}\}.
\]

This implies $\lambda = 0$. Then by lemma 3.2, both limits $\lim_{t \to t_n^+} \frac{q^n(t) - q^n(t_0)}{|q^n(t) - q^n(t_0)|}$ exist and the angle between them is $2\pi$, as $\alpha = 1$ and $\lambda = 0$. □

4. Proof of lemmas 3.1 and 3.3

Proof of lemma 3.1.

(a). Recall that (19) and (23) imply,
\[
|\eta^n(t_n)| = \delta_n = \min\{|\eta^n(t)| : t \in [t_n - a, t_n + a]\}.
\]

Then by the definition of $\xi^n_\alpha(t)$,
\[
|\xi^n_\alpha(s)| \geq |\xi^n_\alpha(0)| = 1, \forall s \in [-a/\delta_n^{1+\frac{\alpha}{2}}, a/\delta_n^{1+\frac{\alpha}{2}}].
\]
As a result,
\[ \langle \xi_1^n(0), (\xi_1^n)'(0) \rangle = 0. \]

Let \( \xi_1(s) \) be the solution of (30), with initial condition
\[ \xi_1(0) = \lim_{n \to \infty} \xi_1^n(0), \quad \xi_1'(0) = \lim_{n \to \infty} (\xi_1^n)'(0). \]

(Upon passing to a subsequence, we may assume the above limits exist.) Then
\[ (\xi_1(0), (\xi_1)'(0)) = \lim_{n \to \infty} (\xi_1^n(0), (\xi_1^n)'(0)) = 0. \]

Meanwhile by (21) and (23), there is a constant \( C_1 > 0 \) independent of \( n \) with
\[ \left| \frac{\partial U(q^n(t))}{\partial \eta^n_1} \right|, \left| \frac{\partial^2 U(q^n(t))}{\partial \eta^n_1^2} \right| \leq C_1, \quad \forall t \in [t_n - a, t_n + a]. \]

This means the second line in equation (29) converges to zero, which gives us equation (30), as \( n \) goes to infinity. Then by the continuous dependence of solutions on initial conditions and coefficients of differential equations, we have \( \xi_1^n(s) \) converges to \( \xi_1(s) \) in \( C^1([-\ell, \ell], \mathbb{R}^d) \), for any \( \ell > 0 \).

To show that \( \xi_1(s) \) satisfies (31), consider the energy \( E_1^\varepsilon(t) = E_1^\varepsilon(q^n(t), \dot{q}^n(t)) \) and \( E(t) = E(q(t), \dot{q}(t)) \) of the I-cluster with \( I = \{1, 2, \ldots, n\} \) corresponding to \( q^n \) and \( q \). By the proof of proposition 2.1, \( E(t) \) is continuous on \([t_0 - a, t_0 + a]\) and \( E_1^\varepsilon(t) \) converges to it under the \( L^\infty \) norm, after passing to a subsequence. Therefore for \( n \) large enough, there is a constant \( C_2 > 0 \) independent of \( n \), such that
\[ |E_1^\varepsilon(t)| \leq C_2, \quad \forall t \in [t_n - a, t_n + a]. \]

Meanwhile with (23) and (28), a direct computation shows
\[ \frac{1}{2(m_1 + m_2)} |(\xi_1^n)'|^2 - \frac{1}{|\xi_1^n|^\alpha} - \frac{\varepsilon_n}{\delta_n^{2-\alpha}} - \frac{1}{|\xi_1^n|^2} \leq \frac{1}{2} |\dot{\eta}_2^n|^2. \]

To prove that \( \xi_1(s) \) satisfies (31), it is enough to show the right hand side of the above equation converges to zero, as \( n \) goes to infinity.

To see this, notice that \( \dot{\eta}_2^n(t) \) converges to \( \frac{m_2 q_1(t) + m_1 q_2(t)}{m_1 + m_2} \), which is the center of mass of \( m_1 \) and \( m_2 \). Although they collide at the moment \( t_0 \), the path of their center of mass, is actually \( C^1 \) on \([t_0 - a, t_0 + a]\) (see [12, remark 4.10]). Hence the convergence of \( \dot{\eta}_2^n(t) \) holds at least under the \( C^1 \) norm, and as a result, there is a constant \( C_3 > 0 \) independent of \( n \), such that
\[ |\dot{\eta}_2^n(t)| \leq C_3, \quad \forall t \in [t_n - a, t_n + a]. \]

Now our claim follows directly from (47) and (49).

(b). Notice that (30) describes the motion of a point mass under the attraction of an isotropic central force. As a result, \( (\xi_1(0), (\xi_1)'(0)) = 0 \) implies \( \xi_1(t) \in W(\xi_1) = \text{span}\{\xi_1(0), (\xi_1)'(0)\} \), for all \( t \in \mathbb{R} \). By property (a), \( \xi_1(s) \) is a collision-free zero energy solution of (30). Then the rest of the property is well known and a detailed proof can be found in [23, section 4].

(c). Let \( \{e_i : i = 1, \ldots, d\} \) be an orthogonal basis of \( \mathbb{R}^d \), such that \( W(\xi_1) = \text{span}\{e_1, e_2\} \).

Then for any \( \varphi \in H^1_0(\mathbb{R}, \mathbb{R}) \) and \( e_i, i = 3, \ldots, d \), a simple computation shows
\[ d^2I(\xi_1)[\varphi e_i, \varphi e_i] = \int_{-\infty}^{\infty} \frac{|\varphi|^2}{2(m_1 + m_2)} - \alpha \frac{|\varphi|^2}{|\xi_1|^\alpha + 2} - 2\lambda \frac{|\varphi|^2}{|\xi_1|^2} \, dr. \]
Let $\Lambda(\xi_1)$ be the largest subspace of $H^1_0(\mathbb{R}, \mathbb{R})$, such that the value in (50) is negative for any $\varphi \in \Lambda(\xi_1)$. Using the Sturm comparison theorem, in [23, section 4, [24, section 4] and [25, proposition 1.1], Tanaka showed the dimension of $\Lambda(\xi_1)$ is related to the winding number of $\xi_1(s), s \in \mathbb{R}$, in the plane $W(\xi_1)$ with respect to the origin. More precisely

$$\dim(\Lambda(\xi_1)) \geq i(\alpha, \lambda), \quad (51)$$

and this proves property (c).

**Remark 4.1.** Although Tanaka only state (51), a slight modification of his proof should show this is in fact an equality. As a result, if one wants to get a better estimate of the Morse index near a binary collision, one has to compute the Morse index of $\xi_1$ inside $W(\xi_1)$. However a result in [15, corollary 5.1] by Hu and the author shows this is actually zero. Because of this we believe with Tanaka’s approach, one can not get any nontrivial result for the planar $N$-body problem.

**Proof of lemma 3.3.** With the results given by Tanaka in [24, section 4], the lemma can be proven following the same argument given in lemma 3.1. The only difference is the blow up should follow (35), instead of (28), and correspondingly (48) needs to be replaced by

$$\frac{1}{2(m_1 + m_2)} |(\zeta_1^1)|^2 - \frac{1}{|\zeta_1^1|^2} \frac{\delta_n^{2-\alpha}}{\varepsilon_n} |\zeta_1^1|^\alpha = \frac{\delta_n^2}{\varepsilon_n m_1 m_2} (E_{I^*}^n - \frac{1}{2} |\eta_n^2|^2). \quad (52)$$

Nevertheless the right hand side of the equation still goes to zero, because $\lambda = +\infty$ implies $\delta_n^2/\varepsilon_n$ converge to zero, as $n$ goes to infinity.

5. Proof of lemma 3.2

Our proof follows the approach given by Tanaka in [25]. First we establish a lemma that corresponds to lemma 1.3 in [25].

**Lemma 5.1.**

(a) There is a constant $C_1 > 0$ independent of $n$, such that

$$|\eta_n^1(t)|^2 \leq \frac{2(m_1 + m_2)}{|n_n^1(t)|^\alpha} + \frac{2\varepsilon_n (m_1 + m_2)}{|n_n^1(t)|^2} + C_1, \quad \forall t \in [t_n - a, t_n + a]. \quad (53)$$

(b) There is a constant $\ell_0 > 0$, such that for $n$ large enough, if $t \in [t_n - a, t_n + a]$ and $n_n^1(t) \in B_{\ell_0}$ then $\frac{\delta}{\delta t} |n_n^1(t)|^2 > 0$, where $B_{\ell} = \{x \in \mathbb{R}^d : |x| \leq \ell\}$ for any $\ell > 0$.

**Proof.**

(a) Let $I = \{1, 2\}$. Recall that $E_{I^*}^n(t) = E_{I}^n(q^n(t), \dot{q}^n(t))$ is the energy of the $I$-cluster. By (23),

$$E_{I}^n(t) = \frac{m_1 m_2}{2(m_1 + m_2)} |\eta_n^1|^2 + \frac{m_1 + m_2}{2} |\eta_n^2|^2 - \frac{m_1 m_2}{\varepsilon_n |\eta_n^1|^2} - \frac{m_1 m_2}{\varepsilon_n |\eta_n^2|^2}. \quad (54)$$
As a result,
\[
|\ddot{\eta}_1(t)|^2 \leq \frac{2(m_1 + m_2)}{|\eta_1(t)|^{2\alpha}} + 2\varepsilon_{\eta_1} \frac{m_1 + m_2}{|\eta_1(t)|^2} - \frac{(m_1 + m_2)^2}{m_1 m_2} |\ddot{\eta}_2(t)|^2 + \frac{2(m_1 + m_2)}{m_1 m_2} C_1(t).
\]

Then property (a) following from (47) and (49).

(b). By a direct computation,
\[
\frac{1}{2} \frac{d^2}{dt^2} |\ddot{\eta}_1|^2 = |\dddot{\eta}_1|^2 + (\dddot{\eta}_1, \dddot{\eta}_2) = |\dddot{\eta}_1|^2 - \alpha \frac{m_1 + m_2}{|\eta_1|^{2\alpha}} - 2\varepsilon_{\eta_1} \frac{m_1 + m_2}{|\eta_1|^2}
\]
\[
+ \frac{m_1 + m_2}{m_1 m_2} \left( \eta_1, \frac{\partial \tilde{U}}{\partial \eta_1} + \varepsilon_n \frac{\partial \tilde{\Delta}_n}{\partial \eta_1} \right).
\]

Use (47) and (54) , we can find a positive constant $C_2$, such that
\[
\frac{1}{2} \frac{d^2}{dt^2} |\ddot{\eta}_1|^2 \geq (2 - \alpha) \frac{m_1 + m_2}{|\eta_1|^{2\alpha}} - \frac{m_1 + m_2}{2} |\eta_1|^2 + \frac{m_1 + m_2}{m_1 m_2} \langle \eta_1, \frac{\partial \tilde{U}}{\partial \eta_1} + \varepsilon_n \frac{\partial \tilde{\Delta}_n}{\partial \eta_1} \rangle - C_2,
\]

which clearly implies property (b). □

Again following [25], we introduce the following functions
\[ e_\omega(t) = \sqrt{|\ddot{\eta}_1(t)|^2 - (\dot{\eta}_1(t), \dot{\eta}_2(t))^2}, \quad \omega(t) = \frac{e_\omega(t)}{|\ddot{\eta}_1(t)|^2}. \]

Notice that $\omega_\omega(t)$ is well defined, when $\ddot{\eta}_1(t) \neq 0$ and $\dot{\eta}_2(t) \neq 0$. In particular, $\omega_\omega(t) = \sin(\angle(\ddot{\eta}_1(t), \eta_2(t), \ddot{\eta}_2(t)))$ and $|\omega_\omega(t)| \leq 1$. By (24),
\[
\frac{d e_\omega}{dt} = \frac{m_1 + m_2}{m_1 m_2} \left( |\ddot{\eta}_1|^2 |\ddot{\eta}_2|^2 - (\dot{\eta}_1, \dot{\eta}_2)^2, \frac{\partial \tilde{U}(\eta_2)}{\partial \eta_1} + \varepsilon_n \frac{\partial \tilde{\Delta}_n}{\partial \eta_1} \right).
\]

According to (21), there are positive constants $C_3, C_4$ independent of $n$, such that
\[
|e_\omega(t)| \leq C_3 |\ddot{\eta}_1(t)|, |\frac{\partial \tilde{U}(\eta_2)}{\partial \eta_1} + \varepsilon_n \frac{\partial \tilde{\Delta}_n}{\partial \eta_1}| \leq C_4 |\ddot{\eta}_1(t)|, \quad \forall t \in [t_n - a_n, t_n + a_n].
\]

Again by (24), a direct computation shows
\[
\frac{d \omega_\omega}{dt} = \frac{\dot{e}_\omega}{|\eta_1|^{2\alpha} + \varepsilon_n \frac{\partial \tilde{\Delta}_n}{\partial \eta_2} |\eta_1|^2} = \frac{\dot{e}_\omega}{|\eta_1|^{2\alpha} + \varepsilon_n \frac{\partial \tilde{\Delta}_n}{\partial \eta_2} |\eta_1|^2}
\]
\[
= \frac{m_1 + m_2}{m_1 m_2} \omega_\omega \left( |\ddot{\eta}_1|^2 - \frac{m_1 + m_2}{|\eta_1|^{2\alpha}} - 2\varepsilon_{\eta_1} \frac{m_1 + m_2}{|\eta_1|^2} \right).
\]

Then (53) and (56) implies,
\[
\dot{\omega}_n(t) \leq \frac{C_5}{|\eta|} \left( \frac{\partial \tilde{U}}{\partial \eta^*} + \varepsilon_n \frac{\partial \tilde{U}}{\partial \eta^*} \right) - \omega_n \left( \frac{\eta^*}{|\eta|^2} \right) \left( 2 - \alpha \right) \left( \frac{m_1 + m_2}{|\eta|^4} \right) - C_6
\]

where \( C_5 \) is positive constant and \( C_6 \) is a constant, whose sign depends on the sign of \( (\eta^*, \eta^*_0) \), independent of \( t \in [t_n - a, t_n + a] \) and \( n \).

With lemma 5.1 and (58) (this corresponds to (1.8) in [25]), the next result can be proven following the argument given in propositions 1.4 and 1.5 in [25] line by line, and we will not repeat it here.

**Lemma 5.2.** Let \( \ell_0 \) be the constant given in lemma 5.1, for any \( b > 0 \) small enough, there exist constants \( 0 < \ell_2 < \ell_0 < \ell_1 \), such that when \( n \) is large enough, for any

\[
t_n < t < t^* < t_n + a, \quad \text{or} \quad t_n - a < t < t^* < t_n,
\]

if \( \eta^*_t(t) \) and \( \eta^*_t(t^*) \in B_{\ell_2} \setminus B_{\delta_0, \ell_1} \), then

\[
|\eta^*_t(t)|/|\eta^*_t(t)| - |\eta^*_t(t^*)|/|\eta^*_t(t^*)| < b.
\]

Now we give a proof of lemma 3.2.

**Proof of lemma 3.2.** By property (c) in lemma 3.1, \( \lim_{s \to +\infty} \xi_1(s)/|\xi_1(s)| = u^+ \). Fix an arbitrary small \( b > 0 \), let \( \ell_1, \ell_2 \) be the constants given in lemma 5.2. We can choose an \( s^* > 0 \) large enough, such that for \( n \) large enough,

\[
\ell_1 < |\xi_1(s^*)| < \delta_n^{-1} \ell_2, \quad \text{and} \quad \left| \frac{\xi_1(s^*)}{|\xi_1(s^*)|} - u^+ \right| < b.
\]

By lemma 3.1, the same inequalities hold for \( \xi_1(s^*) \), when \( n \) is large enough.

Let \( t^* = t_n + \delta_n^{1/2} s^* \), by equation (28),

\[
\eta^*_t(t^*) \in B_{\ell_2} \setminus B_{\delta_0, \ell_1}, \quad \text{and} \quad |\eta^*_t(t^*)|/|\eta^*_t(t^*)| - u^+ | < b.
\]

Since \( \ell_2 < \ell_0 \), for any \( t \in (t^*, t_n + a) \), we claim if \( \eta^*_t(t) \in B_{\ell_2} \), then \( |\eta^*_t(t)| > \delta_0 \ell_1 \). Indeed this follows from the fact \( d|\eta^*_t(t)|/dt = 0 \) and property (b) in lemma 5.1.

As a result, for any \( t \in (t^*, t_n + a) \), \( \eta^*_t(t) \in B_{\ell_2} \) implies \( \eta^*_t(t) \in B_{\ell_2} \setminus B_{\delta_0, \ell_1} \). Then by lemma 5.2,

\[
|\eta^*_t(t)|/|\eta^*_t(t)| - u^+ \leq |\eta^*_t(t)/|\eta^*_t(t)| - \eta^*_t(t^*)/|\eta^*_t(t^*)| - u^+ | \leq 2b.
\]

This means

\[
\{ \eta^*_t(t) : t \in (t_n + \delta_n^{1/2} s^*, t_n + a) \} \cap B_{\ell_2} \subset \{ x \in B_{\ell_2} : |x/x| - u^+ | < 2b \} \cup B_{\delta_0, \ell_1}.
\]

Recall that when \( n \) goes to infinity, \( t_n \) converges to \( t_0 \), \( \delta_n \) converges to zero, and \( \eta^*_t(t) \) converges uniformly to \( q_2(t) - q_1(t) \). Then

\[
\{ q_2(t) - q_1(t) : t \in (t_0, t_0 + a) \} \cap B_{\ell_2} \subset \{ x \in B_{\ell_2} : |x/x| - u^+ | < 2b \}.
\]
Since the above result hold for any $b > 0$ small enough, we get

$$
\lim_{t \to +\infty} \frac{q_2(t) - q_1(t)}{|q_2(t) - q_1(t)|} = u^+.
$$

A similar argument shows

$$
\lim_{t \to -\infty} \frac{q_2(t) - q_1(t)}{|q_2(t) - q_1(t)|} = u^-.
$$

\[\square\]

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References

[1] Ambrosetti A and Coti Zelati V 1993 Periodic Solutions of Singular Lagrangian Systems (Progress in Nonlinear Differential Equations and their Applications vol 10) (Boston, MA: Birkhäuser)

[2] Arioli G, Barutello V and Terracini S 2006 A new branch of mountain pass solutions for the choreographical 3-body problem Commun. Math. Phys. 268 439–63

[3] Bahri A and Rabinowitz P H 1989 A minimax method for a class of Hamiltonian systems with singular potentials J. Funct. Anal. 82 412–28

[4] Bahri A and Rabinowitz P H 1991 Periodic solutions of Hamiltonian systems of 3-body type Ann. Inst. Henri Poincaré Anal. Non Linéaire 8 561–649

[5] Barutello V, Hu X, Portaluri A and Terracini S 2017 An index theory for asymptotic motions under singular potentials (arxiv:1705.01291)

[6] Barutello V and Secchi S 2008 Morse index properties of colliding solutions to the N-body problem Ann. Inst. Henri Poincaré Anal. Non Linéaire 25 539–65

[7] Chen K C 2008 Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses Ann. Math. 167 325–48

[8] Chenciner A 2002 Action minimizing solutions of the Newtonian n-body problem: from homology to symmetry Proc. of the Int. Congress of Mathematicians, vol 3 (Beijing, 2002) (Beijing: Higher Ed. Press) pp 279–94

[9] Chenciner A, Gerver J, Montgomery R and Simó C 2002 Simple choreographic motions of N bodies: a preliminary study Geometry, Mechanics, and Dynamics (New York: Springer) pp 287–308

[10] Chenciner A and Montgomery R 2000 A remarkable periodic solution of the three-body problem in the case of equal masses Ann. Math. 152 881–901

[11] Chenciner A and Venturelli A 2000 Minima de l’intégrale d’action du problème newtonien de 4 corps de masses égales dans $\mathbb{R}^3$: orbites ‘hip-hop’ Celest. Mech. Dyn. Astron. 77 139–52
[12] Ferrario D L and Terracini S 2004 On the existence of collisionless equivariant minimizers for the classical n-body problem Invent. Math. 155 305–62
[13] Gordon W B 1977 A minimizing property of Keplerian orbits Am. J. Math. 99 961–71
[14] Hofer H 1985 A geometric description of the neighbourhood of a critical point given by the mountain-pass theorem J. London Math. Soc. 31 566–70
[15] Hu X and Yu G 2018 Index theory for zero energy solutions of the planar anisotropic Kepler problem Comm. Math. Phys. 361 709–36
[16] Kustaanheimo P and Stiefel E 1965 Perturbation theory of Kepler motion based on spinor regularization J. Reine Angew. Math. 218 204–19
[17] Majer P and Terracini S 1993 Periodic solutions to some n-body type problems: the fixed energy case Duke Math. J. 69 683–97
[18] Majer P and Terracini S 1993b Periodic solutions to some problems of n-body type Arch. Ration. Mech. Anal. 124 381–404
[19] Marchal C 2002 How the method of minimization of action avoids singularities Celest. Mech. Dyn. Astron. 83 325–53 (Modern celestial mechanics: from theory to applications (Rome, 2001))
[20] McGehee R 1974 Triple collision in the collinear three-body problem Invent. Math. 27 191–227
[21] McGehee R 1981 Double collisions for a classical particle system with nongravitational interactions Comment. Math. Helv. 56 524–57
[22] Montgomery R 1998 The N-body problem, the braid group, and action-minimizing periodic solutions Nonlinearity 11 363–76
[23] Tanaka K 1993 Noncollision solutions for a second order singular Hamiltonian system with weak force Ann. Inst. Henri Poincaré Anal. Non Linéaire 10 215–38
[24] Tanaka K 1993 A prescribed energy problem for a singular Hamiltonian system with a weak force J. Funct. Anal. 113 351–90
[25] Tanaka K 1994 A note on generalized solutions of singular Hamiltonian systems Proc. Am. Math. Soc. 122 275–84
[26] Tanaka K 1994 A prescribed-energy problem for a conservative singular Hamiltonian system Arch. Ration. Mech. Anal. 128 127–64
[27] Yu G 2017 Shape space figure-eight solution of three body problem with two equal masses Nonlinearity 30 2279–307
[28] Yu G 2017 Simple choreographies of the planar Newtonian N-body problem Arch. Ration. Mech. Anal. 225 901–35