Endomorphism algebras of Kuga-Satake varieties.

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Abstract

We compute endomorphism algebras of Kuga-Satake varieties associated to $K3$ surfaces.

1 Preliminary remarks.

Let $V$ be a $\mathbb{Q}$-lattice of transcendental cycles on a $K3$ surface $X$, $\phi: V \otimes_{\mathbb{Q}} V \to \mathbb{Q}$ the polarization of the weight 2 Hodge structure on $V$, $E = \text{End}_{Hdg}(V)$, $\Phi: V \otimes_{E} V \to E$ the hermitian or bilinear form constructed in [15], $\phi = \text{tr} \circ \Phi$.

Let $C(V)$ be the Clifford algebra of the quadratic space $(V, \phi)$ over $\mathbb{Q}$, $C^+(V)$ the even Clifford algebra and $KS(X)$ the Kuga-Satake variety of $X$. Here we define $KS(X)$ from the weight 2 Hodge structure on the lattice of transcendental cycles $V$ rather than on the whole lattice of primitive cycles $H^2(X, \mathbb{Q})_{\text{prim}}$. In particular, the Kuga-Satake variety defined here is isogenous to a power of the Kuga-Satake variety defined using the whole lattice of primitive cycles (see [7], [10], §4).

We want to compute the endomorphism algebra $\text{End}(KS(X))_{\mathbb{Q}} = \text{End}_{Hdg}(C^+(V))$.

Let $Z(\Phi)$ be the $\mathbb{Q}$-algebraic group $\text{Res}_{E/\mathbb{Q}}(SO(V, \Phi))$, if $E$ is a totally real field, or $\text{Res}_{E_0/\mathbb{Q}}(U(V, \Phi))$, if $E = E_0(\theta)$ is a CM-field (with the totally real subfield $E_0$). Recall, that according to [15], $Z(\Phi)$ is the Hodge group of the Hodge structure on $V$.

Let $CSpin(\phi): = \{g \in C^+(V)^* \mid gVg^{-1} \subset V\}$. Consider the vector representation $\rho: CSpin(\phi) \to GL(V)$, $g \mapsto (v \mapsto gvg^{-1})$ and the spin representation $\sigma: CSpin(\phi) \to GL(C^+(V))$, $g \mapsto (x \mapsto gx)$. Let $ZSpin(\Phi): = \{g \in CSpin(\phi) \mid \rho(g) \in Z(\Phi)\} = \rho^{-1}(Z(\Phi)) \subset CSpin(\phi)$. Note that $\rho(ZSpin(\Phi)) = Z(\Phi)$.

**Lemma 1.** The Mumford-Tate group of the weight 1 Hodge structure on $C^+(V)$ is the preimage with respect to $\rho$ of the Mumford-Tate group of the weight 2 Hodge structure on $V$.

**Proof:** The same as Proposition 6.3 in [12]. If $h_X: S^1 \to GL(V)$ and $h_{KS(X)}: S^1 \to GL(C^+(V))$ denote the corresponding Hodge structures, then $h_X = \rho \circ \sigma^{-1} \circ h_{KS(X)}$ (as...
shown in [12]. \textit{QED}

**Corollary.** \(\text{End}(KS(X))_Q \cong \text{End}_{Z\text{Spin}(\Phi)}(C^+(V))\), where \(Z\text{Spin}(\Phi)\) acts on \(C^+(V)\) via the spin representation \(\sigma|_{Z\text{Spin}(\Phi)}\).

So, if \(C^+(V) = \bigoplus_j T_j^{m_j}\) is the decomposition of \(\sigma|_{Z\text{Spin}(\Phi)}\) into a direct sum of irreducible (mutually non-isomorphic) representations \(T_j\), then \(\text{End}(KS(X))_Q \cong \prod_j \text{Mat}_{m_j \times m_j}(D_j)\) as \(Q\)-algebras, where \(D_j = \text{End}_{C\text{Spin}(\Phi)}(T_j)\).

Let us assume that \(m = \text{dim}_EV \geq 3\), if \(E = E_0\) is totally real, and \(m = \text{dim}_EV \geq 2\), if \(E = E_0(\theta)\) is a CM-field. In the totally real case condition \(m \geq 3\) is automatically satisfied for any \(K3\) surface \(X\) (see [9] and [14]). In what follows we will often denote the field of rational numbers \(Q\) by \(k\) and \(E_0\) by \(L\). Our approach is not invariant in the sense that we choose a basis in \(V\) which diagonalizes \(\Phi\) right from the start (see Section 2).

Consider the epimorphism \(\pi : C\text{Spin}(\phi) \to SO(\phi)\) of algebraic groups over \(Q\) (induced by the vector representation \(\rho\) above) with fiber \(\ker(\pi) = \mathbb{G}_m \subset C\text{Spin}(\phi)\) and its restriction \(\pi_0 : \text{Spin}(\phi) \to SO(\phi)\) to the subgroup \(\text{Spin}(\phi) \subset C\text{Spin}(\phi)\). Then \(\pi_0\) is a double etale covering [3].

The argument above shows that the Hodge group \(Hdg\) of the Kuga-Satake structure on \(C^+(V)\) satisfies inclusions:

\[Hdg \subset (\pi_0^{-1}(Z(\Phi)))^0 \cdot \mathbb{G}_m \quad \text{and} \quad (\pi_0^{-1}(Z(\Phi)))^0 \subset Hdg\]

(hereafter for an algebraic group \(G\) we let \(G^0\) denote the connected component of the identity and \(\text{Lie}(G)\) the Lie algebra of \(G\)).

Hence the \(Q\)-algebra

\[\text{End}_{Hdg}(C^+(V)) = \text{End}_{(\pi_0^{-1}(Z(\Phi)))^0}(C^+(V)) = \text{End}_{\text{Lie}(\pi_0^{-1}(Z(\Phi)))}(C^+(V)) = \text{End}_{\text{Lie}(Z(\Phi))}(C^+(V)).\]

Let \(g = \text{Lie}(Z(\Phi))\). Then \(g = \text{Res}_{E/k}(\mathfrak{so}(\Phi))\), if \(E\) is totally real, or \(g = \text{Res}_{E_0/k}(\mathfrak{u}(\Phi))\), if \(E = E_0(\theta)\) is a CM-field (\(\theta^2 \in E_0\)), where \(k = Q\).

Hence what we are looking for is the algebra of intertwining operators \(\text{End}_g(C^+(V))\) of the \(Q\)-linear representation of the Lie algebra \(g\) over \(Q\) induced by the spin representation of \(\mathfrak{so}(\phi)\) in \(C^+(V)\) via the inclusion of Lie algebras \(g \subset \mathfrak{so}(\phi)\) corresponding to the inclusion of the \(Q\)-algebraic groups \(Z(\Phi) \subset SO(\phi)\) above.

The problem of computing endomorphism algebras of Kuga-Satake varieties was addressed earlier by Bert van Geemen in papers [13] and [14]. In particular, in [13] he considered the case of the CM-field, which is quadratic over \(Q\) and in [14] he considered the case of the totally real field, computed the endomorphism algebra in several special cases and made some general remarks. A different computation of the endomorphism algebra of the
Kuga-Satake variety in the totally real case was done by Ulrich Schlickewei [11].

Our solution uses the same ideas as (some of the ideas) in papers [13] and [14]. We compute the decomposition of the restriction to $g$ of the spin representation of $\mathfrak{so}(\phi)$ into irreducible subrepresentations over a splitting field of $g$, and then apply Galois descent.

Our main result is Theorem 1 in Section 4 complemented by the computation of primary representations (which are the multiples of irreducible representations $T_j$ above) and division algebras (which are the endomorphism algebras of $T_j$) in subsequent sections. In this text a ‘primary representation’ means a multiple of an irreducible representation. Some general observations regarding representations over arbitrary fields are collected in the Section 2. In Section 3 we introduce Galois-invariant Cartan subalgebras. In Section 4 we compute decompositions of representations over a splitting field. In Section 5 we construct primary representations over $\mathbb{Q}$ whose irreducible components appear in Theorem 1. In Section 6 we compute the division algebras which are the endomorphism algebras of those irreducible components. Section 7 is devoted to examples.

2 Some remarks on Galois theory of representations.

Let $F/k = \mathbb{Q}$ be a finite Galois extension, $g = c \oplus g'$ be a reductive Lie algebra over $k$, $c \subset g$ be its center and $g' \subset g$ be its derived subalgebra. Let $S = Gal(F/k)$ and $h \subset g \otimes_k F$ be a Galois-invariant (i.e. such that $g(h) = h$ for any $g \in S$) splitting Cartan subalgebra. Let $B$ be a basis of the root system $R$ of $(g \otimes_k F, h)$. In what follows we assume that all the representations of $g$ we are dealing with are finite-dimensional and can be integrated to representations of a reductive algebraic group with Lie algebra $g$ (in order to guarantee their complete reducibility).

Let $\rho : g \to End_k(W)$ be a representation of $g$ over $k$ and $W \otimes_k F = \oplus_\alpha V_\alpha$ its decomposition into irreducible subrepresentations over $F$. Let $\rho_\alpha = \rho|_{V_\alpha}$ be an irreducible representation of $g \otimes_k F$ with primitive element $v_\alpha \in W \otimes_k F$ with highest weight $\omega_\alpha \in Hom_F(h, F)$ (with respect to $B$). Then for any $g \in S$, $\rho_\alpha : = \rho|_{g(V_\alpha)}$ is an irreducible representation of $g \otimes_k F$ with primitive element $g(v_\alpha) \in W \otimes_k F$ with highest weight $g \circ \omega_\alpha \circ g^{-1} \in Hom_F(h, F)$ with respect to the basis $g \circ B \circ g^{-1}$ of $R$. Since the Weyl group $W_R$ of $R$ acts simply transitively on the set of bases of $R$, for any $g \in S$ there exists unique $w(g) \in W_R$ such that $g \circ B \circ g^{-1} = w(g)(B)$. Hence $\rho_\alpha^g$ is an irreducible representation of $g \otimes_k F$ with primitive element $g(v_\alpha) \in W \otimes_k F$ with highest weight $\omega_\alpha^g : = w(g)^{-1}(g \circ \omega_\alpha \circ g^{-1}) \in Hom_F(h, F)$ (with respect to $B$).

Lemma 3. Suppose that $\rho_1 : g \to End_k(W_1)$ and $\rho_2 : g \to End_k(W_2)$ are two irreducible representations of $g$ over $k$, $V_\alpha \subset W_1 \otimes_k F$ and $V_\beta \subset W_2 \otimes_k F$ are two irreducible subrepresentations of $g \otimes_k F$ over $F$. Then $W_1 \cong W_2$ as $g$-modules over $k$, if and only if there exist $\sigma, \tau \in S$ such that $(\rho_1|_{V_\alpha})^\sigma \cong (\rho_2|_{V_\beta})^\tau$ as $g \otimes_k F$-modules over $F$. 

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Proof: Schur’s lemma. QED

Corollary. If $ρ_α : \mathfrak{g} \otimes_k F \to End_F(V_α)$ is an irreducible representation of $\mathfrak{g} \otimes_k F$ over $F$, then there exists at most one irreducible representation $\rho : \mathfrak{g} \to End_k(W)$ of $\mathfrak{g}$ over $k$, such that $ρ_α$ is a subrepresentation of $ρ \otimes_k F$.

Using the notation of the remark preceding Lemma 3, let $W = \oplus \gamma W_γ$ be a decomposition of $ρ$ into irreducible subrepresentations over $k$. Then for any $γ$ such that $V_α \subset W_γ \otimes_k F$, by Galois descent we have:

$$\bigoplus_{γ' : W_γ \cong W_γ} W_γ' = \left( \bigoplus_{α' : ρ_α \cong ρ_α'} V_{α'} \right)^S.$$

Hence $\dim_k \left( \bigoplus_{γ' : W_γ \cong W_γ} W_γ' \right) = \dim_F \left( \bigoplus_{α' : ρ_α \cong ρ_α'} V_{α'} \right) = \sum_{α'} : \exists ω_α \in S : \dim_F(V_{α'}) = m_α \cdot \dim_k(W_γ)$, where $m_α$ is the multiplicity of $W_γ$ in the decomposition above.

So, if $W_{γ_1}, ..., W_{γ_p}$ are pairwise nonisomorphic (as $g$-modules) irreducible $\mathfrak{g}$-submodules of $W$ over $k$ (with the corresponding $\mathfrak{g} \otimes_k F$-submodules $V_{α_1} \subset W \otimes_k F$) appearing in the decomposition above, then $W = \oplus_i W_γ \oplus m_{α_i}$ and

$$End_{\mathfrak{g}}(W) \cong \prod_i Mat_{m_{α_i} \times m_{α_i}}(D_i)$$

as $k$-algebras,

where $D_i = End_{\mathfrak{g}}(W_γ)$, $W_γ$ is the unique irreducible $\mathfrak{g}$-module over $k$ such that $W_γ \otimes_k F$ contains $V_{α_i}$ as a $\mathfrak{g} \otimes_k F$-submodule over $F$ and $m_{α_i} = (\sum_{α'} : \exists ω_α \in S : \dim_F(V_{α'}) / \dim_k(W_γ))$. We can also write:

$$m_{α_i} = \frac{\dim_F(V_{α_i}) \cdot \sum_{σ \in S} \text{mult}(ω_{α_i})}{n_{ω_{α_i}} \cdot \dim_k(W_γ)}$$

where $\text{milt}(ω)$ is the multiplicity of the irreducible representation of $\mathfrak{g} \otimes_k \mathbb{C}$ with highest weight $ω$ (relative to the chosen $h$ and $B$) in $W \otimes_k \mathbb{C}$ and $n_ω$ is the stabilizer of $ω$ under the action of the Galois group $S = Gal(F/k)$ on weights. Note that $\{ω_{α_i}\}$ is a set of representatives of the orbits of the action of $S$ on the set of highest weights of irreducible representations of $\mathfrak{g} \otimes_k \mathbb{C}$ appearing as irreducible components of $W \otimes_k \mathbb{C}$.

This reduces the study of $End_{\mathfrak{g}}(W)$ to the study of the (uniquely determined) $(k = \mathbb{Q})$-forms of irreducible $\mathfrak{g} \otimes_k \mathbb{C}$-submodules of $W \otimes_k \mathbb{C}$ (i.e. $D_i = End_{\mathfrak{g}}(W_γ)$ and $\dim_k(W_γ))$ and the description of the Galois action (of the finite group $Gal(F/k)$) on the weights of $\mathfrak{g} \otimes_k \mathbb{C}$ over $\mathbb{C}$. 

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3 Description of the Galois action, Cartan subalgebras and bases of the root systems.

According to Section 2, we need to specify a splitting field $F$ of $g$ (which should be a Galois extension of $k$), a Galois-invariant splitting Cartan subalgebra $\mathfrak{h} \subset g \otimes_k F$ (i.e. $\mathfrak{h}$ should be $Gal(F/k)$-stable) and a basis $B$ of the root system $R$ of the split reductive Lie algebra $(g \otimes_k F, \mathfrak{h})$.

Let us assume that $\Phi = d_1 \cdot X_1^2 + \ldots + d_m \cdot X_m^2$ (if $E = E_0 = L$ is totally real) or $\Phi = d_1 \cdot X_1 X_1 + \ldots + d_m \cdot X_m X_m$ (if $E = E_0(\theta)$, $\theta^2 \in E_0 = L$ is a CM-field), where $d_i \in L$ for any $i$. In other words, we reduce the Hermitian (or quadratic) form $\Phi$ to a diagonal form, i.e. choose an orthogonal (with respect to $\Phi$) basis of $V$.

Let $k = \mathbb{Q}$ and $F/k$ be a finite Galois extension such that $F$ contains $L$, $\sqrt{a_i}$ for any $i$, $\sqrt{-1}$ and $\theta$ (if $E = E_0(\theta)$ is a CM-field, $\theta^2 \in E_0$).

Let $r = [L: k]$ and $\sigma_1, \ldots, \sigma_r : L \to F$ be the list of all field embeddings of $L$ into $F$.

3.1 Case of the totally real field.

Let us consider first the case $g = Res_{L/k}(so(\Phi)) \subset so(\phi)$ (i.e. $E = E_0$ is totally real). We will denote by $E_{i,j}$ a matrix with all entries equal to 0 except for the entry $(i, j)$ which is equal to 1.

Let $\mathfrak{h}_0 = Span_L(A_1, \ldots, A_l)$, where $l = \left\lfloor \frac{m}{2} \right\rfloor$ and $A_i = d_{m-i+1} \cdot E_{m-i+1,i} - d_i \cdot E_{i,m-i+1}$, $1 \leq i \leq l$. Let $\mathfrak{h}_i = \mathfrak{h}_0 \otimes_{L,\sigma_i} F \subset so(\Phi) \otimes_{L,\sigma_i} F$ and $\mathfrak{h} = \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_r \subset \oplus_{i=1}^r (so(\Phi) \otimes_{L,\sigma_i} F) \cong Res_{L/k}(so(\Phi)) \otimes_k F = g \otimes_k F$. Then $\mathfrak{h} \subset g \otimes_k F$ is a splitting Cartan subalgebra.

Note that over $F$ we have $\Phi = d_1 \cdot X_1^2 + \ldots + d_m \cdot X_m^2 = \sum_{i=1}^l Y_i \cdot Y_i + \epsilon Y_0^2$, where $\epsilon = 0$, if $m$ is even, $\epsilon = 1$, if $m$ is odd, $Y_i = \sqrt{a_i} \cdot X_i + \sqrt{-d_{m-i+1} \cdot X_{m-i+1}}$, $Y_i = \sqrt{d_i} \cdot X_i - \sqrt{-d_{m-i+1} \cdot X_{m-i+1}}$ and $Y_0 = \sqrt{d_{l+1}} \cdot X_{l+1}$.

This implies that for any $i,j$ we have $A_j \otimes_{L,\sigma_i} 1 = \Gamma_j \cdot H_j$, where $\Gamma_j = -\sqrt{\sigma_i(d_j)} \cdot \sqrt{-\sigma_i(d_{m-j+1})} \in F$ (1 $\leq j \leq l$) (in future we will be writing $d_j$ instead of $\sigma_i(d_j)$) and $H_j = E_{j,j} - E_{j,-j}$ (using notation form [I], §13). Hence for any $i$ subalgebra $\mathfrak{h}_i \subset so(\Phi) \otimes_{L,\sigma_i} F$ is the same splitting Cartan subalgebra as in [I], §13. By construction $\mathfrak{h} \subset g \otimes_k F$ is Galois-invariant.

Let $R_0$ be the root system of type $B_l$, if $m = 2l + 1$ (respectively, of type $D_l$, if $m = 2l$) from [I], §13, i.e. $R_0 = \{\pm \epsilon_p, \pm \epsilon_p \pm \epsilon_q\}$ (respectively, $R_0 = \{\pm \epsilon_p \pm \epsilon_q\}$) with basis $B_0 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{l-1} - \epsilon_l, \epsilon_l\}$ (respectively, $B_0 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{l-1} - \epsilon_l, \epsilon_l, \epsilon_{l-1} + \epsilon_l\}$) (using notation from [I], §13).
Then for any $i$ the root system of $(so(\Phi) \otimes_{L,\sigma_i} F, h_0 \otimes_{L,\sigma_i} F)$ is $R_i = \{ \pm \epsilon_p \otimes_{L,\sigma_i} \Gamma_p, \pm \epsilon_p \otimes_{L,\sigma_i} \Gamma_p \pm \epsilon_q \otimes_{L,\sigma_i} \Gamma_q \}$ with basis

$$B_i = \{ \epsilon_1 \otimes_{L,\sigma_i} \Gamma_1 - \epsilon_2 \otimes_{L,\sigma_i} \Gamma_2, \epsilon_2 \otimes_{L,\sigma_i} \Gamma_2 - \epsilon_3 \otimes_{L,\sigma_i} \Gamma_3, ..., \epsilon_{l-1} \otimes_{L,\sigma_i} \Gamma_{l-1} - \epsilon_l \otimes_{L,\sigma_i} \Gamma_l, \epsilon_l \otimes_{L,\sigma_i} \Gamma_l \}$$

(respectively, $R_i = \{ \pm \epsilon_p \otimes_{L,\sigma_i} \Gamma_p \pm \epsilon_q \otimes_{L,\sigma_i} \Gamma_q \}$ with basis

$$B_i = \{ \epsilon_1 \otimes_{L,\sigma_i} \Gamma_1 - \epsilon_2 \otimes_{L,\sigma_i} \Gamma_2, \epsilon_2 \otimes_{L,\sigma_i} \Gamma_2 - \epsilon_3 \otimes_{L,\sigma_i} \Gamma_3, ..., \epsilon_{l-1} \otimes_{L,\sigma_i} \Gamma_{l-1} - \epsilon_l \otimes_{L,\sigma_i} \Gamma_l, \epsilon_l \otimes_{L,\sigma_i} \Gamma_l \})$$. Then $R = R_1 \sqcup \ldots \sqcup R_r$ is the root system of $(g \otimes_k F, h)$ and as a basis we can take $B = B_1 \sqcup \ldots \sqcup B_r \subset R$.

The action of the Galois group $S = Gal(F/k)$ on weights reduces to its action by permutation on factors of $R_1 \times \ldots \times R_r$ (or on the left cosets $Gal(F/k)/Gal(F/L)$) and to switching signs in front of various $\Gamma_p$.

Note the isomorphism of root systems $R \cong R_0 \sqcup \ldots \sqcup R_0$ ($r$ factors) under which basis $B$ is identified with $B_0 \sqcup \ldots \sqcup B_0$ ($r$ factors).

Let $w_p \in \mathcal{W}_{R_0}$ (where $\mathcal{W}_R$ denotes the Weyl group of a root system $R$) be the element of the Weyl group such that $w_p(B_0) = \sigma_p(B_0)$, where $\sigma_p$ is a linear transformation of the $\mathbb{Q}$-vector space generated by the roots of $R_0$ which switches the sign in front of $\epsilon_p$ and does not change other $\epsilon_q$'s. Then in the notation of Section 2 for any $g \in S$, $\omega^\alpha_\sigma = (\prod_{p \in P_\tau(g)} w_p^{-1}) \sqcup \ldots \sqcup (\prod_{p \in P_\tau(g)} w_p^{-1})(g \circ \omega_\sigma \circ g^{-1}) \in Hom_F(h, F)$, where $P_\tau(g) = \{ p \mid g^{-1}(\epsilon_p \otimes_{L,\sigma_i} \Gamma_p) = -\epsilon_p \otimes_{L,\sigma_i} \Gamma_p \}$.

### 3.2 Case of the CM-field.

Now let us consider the case $g = Res_{L/k}(u(\Phi)) \subset so(\phi)$ (i.e. $E = E_0(\theta)$ is a CM-field, $\theta^2 \in E_0 = L$).

Let $h_0 = Span_L(A_1, \ldots, A_m)$, where $A_i = \theta \cdot E_{i,i}$, $h_i = h_0 \otimes_{L,\sigma_i} F \subset u(\Phi) \otimes_{L,\sigma_i} F \cong gl(m, F)$ and $h = h_1 \times \ldots \times h_r \subset \bigoplus \epsilon_i^r(u(\Phi) \otimes_{L,\sigma_i} F) \cong gl(m, F)^{\otimes r} \cong Res_{L/k}(u(\Phi)) \otimes_k F = g \otimes_k F$. Then $h \subset g \otimes_k F$ is a splitting Cartan subalgebra.

Note that over $F$ we have $\Phi = d_1 \cdot X_1 \hat{X}_1 + \ldots + d_m \cdot X_m \hat{X}_m = Y_1 \hat{Y}_1 + \ldots + Y_m \hat{Y}_m$, where $Y_i = \sqrt{d_i} \cdot X_i$. Hence for any $i, j$ we have $A_j \otimes_{L,\sigma_i} 1 = \theta \cdot E_{j,j}$ (more precisely we have to write $\sqrt{\sigma_i(\theta^2)}$ instead of $\theta$ here) in $u(\Phi) \otimes_{L,\sigma_i} F \cong gl(m, F)$ and so for any subalgebra $h_i \subset u(\Phi) \otimes_{L,\sigma_i} F \cong gl(m, F)$ is the same splitting Cartan subalgebra as in [P], §13. By construction $h \subset g \otimes_k F$ is Galois-invariant.

Let $R_0$ be the root system of type $A_{m-1}$ (for the reductive Lie algebra $gl(m) = \mathfrak{c} \oplus sl(m)$, where $\mathfrak{c} \subset gl(m)$ is the center), i.e. $R_0 = \{ \epsilon_p - \epsilon_q \}_{p \neq q}$ with basis $B_0 = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m \}$. 

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Then for any $i$ the root system of $(\mathfrak{u}(\Phi) \otimes_{L,\sigma_i} F, \mathfrak{h}_i) \cong (\mathfrak{gl}(m), \text{diagonal matrices})$ is $R_i = \{\epsilon_p \otimes_{L,\sigma_i} \theta - \epsilon_q \otimes_{L,\sigma_i} \theta\}$ with basis $B_i = \{\epsilon_1 \otimes_{L,\sigma_i} \theta - \epsilon_2 \otimes_{L,\sigma_i} \theta, \ldots, \epsilon_{m-1} \otimes_{L,\sigma_i} \theta - \epsilon_m \otimes_{L,\sigma_i} \theta\}$. Then $R = R_1 \sqcup \ldots \sqcup R_r$ is the root system of $(\mathfrak{g} \otimes_k F, \mathfrak{h})$ and as a basis we can take $B = B_1 \sqcup \ldots \sqcup B_r \subset R$.

The action of the Galois group $S = \text{Gal}(F/k)$ on weights reduces to its action by permutation on factors of $R_1 \sqcup \ldots \sqcup R_r$ (or on the left cosets $\text{Gal}(F/k)/\text{Gal}(F/L)$) and to multiplication of various $\theta$ by $-1$.

Note the isomorphism of root systems $R \cong R_0 \sqcup \ldots \sqcup R_0$ ($r$ factors) under which basis $B$ is identified with $B_0 \sqcup \ldots \sqcup B_0$ ($r$ factors).

Let $w_0 \in \mathcal{W}_{R_0}$ be such that $w_0(B_0) = -B_0$. Then in the notation of Section 2 for any $g \in S$, $\omega^g_\alpha = (w_0)^{-P_i(g)} \sqcup \ldots \sqcup (w_0)^{-P_i(g)}(g \circ \omega_\alpha \circ g^{-1}) \in \text{Hom}_F(\mathfrak{h}, F)$, where $P_i(g) = 1$, if $g^{-1}(\epsilon_p \otimes_{L,\sigma_i} \theta) = -\epsilon_p \otimes_{L,g^{-1}\sigma_i} \theta$ and $P_i(g) = 0$ otherwise (the action of $w_0$ is extended to the center of $\mathfrak{gl}(m, F)$ as multiplication by $-1$).

4 Decomposition of the restriction of the spin representation over a splitting field.

In order to apply the general statements of Section 2, we need to decompose the $F$-linear extension of the restriction of the spin representation of $\mathfrak{so}(\phi)$ in $C^+(V)$ to $\mathfrak{g} \subset \mathfrak{so}(\phi)$ over $F$. For this we need to describe the embedding of Cartan subalgebras induced by the embedding of Lie algebras $\mathfrak{g} \otimes_k F \subset \mathfrak{so}(\phi) \otimes_k F$.

**Lemma 2.** If $E$ is totally real, then the Lie algebra homomorphism $\oplus_{i=1}^r \mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F \subset \mathfrak{so}(\oplus_{i=1}^r (\Phi \otimes_{L,\sigma_i} F)) = \mathfrak{so}(\phi) \otimes_k F$ sends $(M_1, \ldots, M_r)$ to diag$(M_1, \ldots, M_r)$.

If $E = E_0(\theta)$ is a CM-field (and $\theta^2 \in E_0 = L$ as usual), then the Lie algebra homomorphism $\oplus_{i=1}^r \mathfrak{gl}(m, F) \cong \oplus_{i=1}^r \mathfrak{u}(\Phi) \otimes_{L,\sigma_i} F \subset \mathfrak{so}(\oplus_{i=1}^r ((\Phi \otimes_{E,\sigma_i} F) \oplus (\Phi \otimes_{E,\sigma_i} F))) = \mathfrak{so}(\phi) \otimes_k F$ (where in the last formula $\sigma_i$ and $\tilde{\sigma}_i$ denote the two extensions of $\sigma_i$ to an embedding of $E/k$ into $F/k$) sends $(M_1, \ldots, M_r)$ to diag$(M_1, -\Phi \cdot M_1^T \cdot \Phi^{-1}, \ldots, M_r, -\Phi \cdot M_r^T \cdot \Phi^{-1})$.

**Proof:** One should notice that $\text{Res}_{L/k}$ on vector spaces over $L$ is the forgetful functor to the vector spaces over $k$. Hence on the $\text{Res}_{L/k}(\mathfrak{so}(\Phi))$ (respectively, $\text{Res}_{L/k}(\mathfrak{u}(\Phi))$), which is the Galois-invariant subspace of the source, our homomorphisms have exactly the form needed. Extending scalars to $F$ gives the result. See also Proposition 3.8 in [13] and [14].

QED
4.1 Case of the totally real field.

Let $E = L$ be a totally real field.

For any $i = 1, \ldots, r$, $j = 1, \ldots, l$ (where $l = \left[ \frac{m}{2} \right]$) let $\hat{H}_j^i = \sigma_i(d_{m-j+1})E_{m-j+1+m(i-1),j+m(i-1)} - \sigma_i(d_j) \cdot E_{j+m(i-1),m-j+1+m(i-1)} \in \mathfrak{so}(\hat{\phi}) \otimes_k F$. Then $\hat{H}_j^i$ are linearly independent elements of the splitting Cartan subalgebra $\hat{\mathfrak{h}} \subset \mathfrak{so}(\hat{\phi}) \otimes_k F$ described in [1], §13. They form a basis of $\hat{\mathfrak{h}}$, if $m$ is even or $r = 1$. If $m$ is odd and $r \geq 2$, then $\hat{H}_j^i$ together with $\hat{H}_1^1, \ldots, \hat{H}_{r-1}^1$ form a basis of $\hat{\mathfrak{h}}$, if we take $\hat{H}_j^i$ as $\sigma_i(d_{m-j+1})E_{(l+1)(r-i+1),j+m(i-1)} - \sigma_i(d_j) \cdot E_{j+m(i-1),m-j+1+m(i-1)}$, $1 \leq i \leq \left[ \frac{r}{2} \right]$.

Let us denote by $\{\hat{e}_j^i\}$ the corresponding dual basis of $\hat{\mathfrak{h}}^* = \text{Hom}_F(\hat{\mathfrak{h}}, F)$. Its elements differ from the elements of the corresponding basis of the dual Cartan subalgebra considered in [1], §13 by scalar factors of the form $-\sqrt{\sigma_i(d_j)} \cdot \sqrt{-\sigma_i(d_{m-j+1})}$.

Lemma 2 above implies that the restriction of $\hat{e}_j^i$ to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ is zero, while for any $j \leq l$ the restriction of $\hat{e}_j^i$ to $\mathfrak{h}$ is the corresponding element of the dual basis of $\mathfrak{h}^*$ of the basis $\{A_j \otimes L, \sigma_i | 1 \leq i \leq r, 1 \leq j \leq l\}$ of $\mathfrak{h}$.

If $m \cdot r = \text{dim}_k(V) \geq 5$, then according to [1], §13 the weights of the spin representation of $\mathfrak{so}(\hat{\phi}) \otimes_k F$ in $C^+(V) \otimes_k F$ (V is considered as a vector space over $k$) are $\frac{1}{2} \sum_{i,j} \hat{e}_j^i - \sum_{i,j} \hat{e}_j^i$, where $I$ runs over the subsets of the set of parameters $i$ and $j$ (i.e. $\{i, j \} | 1 \leq j \leq l$ and $1 \leq i \leq r$ or (if $m$ is odd and $r \geq 2$) $j = l + 1$ and $1 \leq i \leq \left[ \frac{r}{2} \right]$) and each weight has multiplicity $\frac{\text{dim}_k(C^+(V))}{2m}$.

As it was remarked in [14], Lemma 5.5, this implies (if $m \geq 5$) that the restrictions of these weights to $h \subset \hat{\mathfrak{h}}$ are exactly the weights of the exterior tensor product of the spin representations of $\mathfrak{so}(\hat{\phi}) \otimes_{L, \sigma_i} F$ in $C^+(V) \otimes_{L, \sigma_i} F$ (V is considered as a vector space over $L$), $1 \leq i \leq r$, taken with multiplicity $\frac{\text{dim}_k(C^+(V))}{2m-1-[mr/2]} = 2^{r-1}$, if $m$ is even, or with multiplicity $\frac{\text{dim}_k(C^+(V))}{2m-1-[mr/2]} \cdot 2^{[r/2]} = 2^{r-1}$, if $m$ is odd.

Corollary 1. If $E = E_0 = L$ is totally real, then the restriction of the spin representation $\rho: \mathfrak{so}(\phi) \otimes_k F \to \text{End}_F(C^+(V \otimes_k F))$ to $\mathfrak{g} \otimes_k F = \bigoplus_{r=1}^r (\mathfrak{so}(\phi) \otimes_{L, \sigma_i} F) \subset \mathfrak{so}(\phi) \otimes_k F$ is the exterior tensor product $\Gamma \cdot (\rho_1 \boxtimes \ldots \boxtimes \rho_r)$ of spin representations $\rho_i: \mathfrak{so}(\phi) \otimes_{L, \sigma_i} F \to \text{End}_F(C^+(V \otimes_{L, \sigma_i} F))$ with multiplicity $\Gamma = 2^{r-1}$.

4.2 Case of the CM-field.

Let $E = E_0(\theta), \theta^2 \in E_0 = L$ be a CM-field.

For any $i = 1, \ldots, r$, $j = 1, \ldots, m$ let $\hat{H}_j^i = E_{j+2m(i-1),j+2m(i-1)} - E_{j+m+2m(i-1),j+m+2m(i-1)} \in \mathfrak{so}(\phi) \otimes_k F$. Then $\hat{H}_j^i$ form a basis of the splitting Cartan subalgebra $\hat{\mathfrak{h}} \subset \mathfrak{so}(\phi) \otimes_k F$ described in [1], §13. Let us denote by $\{\hat{e}_j^i\}$ the corresponding dual basis of $\hat{\mathfrak{h}}^* = \text{Hom}_F(\hat{\mathfrak{h}}, F)$.
This is the same Cartan subalgebra and the same basis as considered in \[\text{[1]}, \S13.\]

Lemma 2 above implies that the restriction of \(\hat{e}_j^i\) to the Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{g} \otimes_k F\) is the element \((0, \ldots, \hat{e}_j, \ldots, 0)\) with \(0\) outside of the \(i\)-th spot of the Cartan subalgebra (consisting of diagonal matrices) of \(\mathfrak{gl}(m, F)\) consisting of diagonal matrices, where \(\hat{e}_j^i \cong E_{j,j} \in \mathfrak{gl}(m, F)\) is the \(j\)-th element of the dual basis of the Cartan subalgebra of \(\mathfrak{gl}(m, F)\) considered in \[\text{[1]}, \S13.\]

If \(m \cdot r = \frac{1}{2} \cdot \dim_k(V) \geq 3\), then according to \[\text{[1]}, \S13\] the weights of the spin representation of \(\mathfrak{so}(\phi) \otimes_k F\) in \(\mathfrak{gl}(m, F)^{\otimes r}\) are \(\frac{1}{2} \sum_{i,j} \hat{e}_j^i - \sum_{(i,j) \in I} \hat{e}_j^i\). Here \(I\) runs over the subsets of \([1, \ldots, r]\) \(\times [1, \ldots, m]\). Each weight has multiplicity \(\frac{\dim_k(C^+(V))}{2^{mr-1}}\) \([\text{[1]}, \S13\].

Suppose \(m \geq 2\). Then the restrictions of these weights to \(h \subset \hat{h}\) are exactly the weights of the exterior tensor product of the exterior algebra representations of \(u(\Phi) \otimes_{L,\sigma} F \cong \mathfrak{gl}(m, F)\) in \(\wedge^r_E(V) \otimes_{E,\sigma} F\) \((V\text{ is considered as a vector space over } k)\) twisted by \(D^{-1/2}\), \(1 \leq i \leq r\). Here \(D^c\), \(c \in \mathbb{Q}\) denotes the representation of \(u(\Phi) \otimes_{L,\sigma} F \cong \mathfrak{gl}(m, F)\) in \(\wedge^r_E(V) \otimes_{E,\sigma} F \cong \wedge^r_E(V \otimes_{E,\sigma} F)\) such that \(\mathfrak{sl}(m, F)\) acts trivially, while \(1 \in F \cong \mathbb{C}\) acts as \(c \cdot Id\). In other words, \(D^c\): \(\mathfrak{gl}(m, F) \to \text{End}_F(\wedge^r_E(V) \otimes_{E,\sigma} F), M \mapsto c \cdot Tr(M) \cdot Id\).

Indeed, for any \(i\), \(\sum_j \hat{e}_j^i\) restricts to \(0\) to the Cartan subalgebra of the semi-simple part \(\mathfrak{sl}(m, F) \subset \mathfrak{gl}(m, F)\) \(\cong u(\Phi) \otimes_{L,\sigma} F\) and to \(m \cdot Id\) to the center \(F \cong \mathbb{C} \subset \mathfrak{gl}(m, F) \cong u(\Phi) \otimes_{L,\sigma} F\).

The exterior tensor product above has multiplicity \(\Gamma = 2^{mr-1}\). Indeed, \(\dim_F(C^+(V) \otimes_k F) = 2^{2mr-1}\) and \(\dim_F(\wedge^r_E(V) \otimes_{E,\sigma} F) = 2^m\). Hence the dimension of the exterior tensor product is \(\dim_F(\wedge^r_E(V) \otimes_{E,\sigma} F)^\Gamma = 2^{mr}\) and so the multiplicity is \(2^{2mr-1}/2^{mr} = 2^{mr-1}\).

**Corollary 2.** If \(E = E_0(\theta), \theta^2 \in E_0 = L\) is a CM-field, then the restriction of the spin representation \(\rho: \mathfrak{so}(\phi) \otimes_k F \to \text{End}_F(C^+(V \otimes_k F))\) to \(\mathfrak{g} \otimes_k F = \bigoplus_{i=1}^r (u(\Phi) \otimes_{L,\sigma} F) \cong \mathfrak{gl}(m, F)^{\otimes r} \subset \mathfrak{so}(\phi) \otimes_k F\) is the exterior tensor product \(\Gamma \cdot (\rho_1 \boxtimes \ldots \boxtimes \rho_r)\) of \(\mathfrak{so}\) representations \(\rho_i\): \(\mathfrak{gl}(m, F) \to \text{End}_F(\wedge^r_E(V \otimes_{E,\sigma} F) \otimes F)\) twisted by one-dimensional representations \(D^{-1/2}\): \(\mathfrak{gl}(m, F) \to \text{End}_F(F) \cong F, M \mapsto (-1)^{\frac{1}{2m}} \cdot Tr(M)\) with multiplicity \(\Gamma = 2^{mr-1}\).

**Remark.** \(\rho_i\) is a double-valued 'spin' representation of \(GL(m, F)\).

From these Corollaries one can deduce the highest weights of irreducible subrepresentations over \(F\) of the restriction to \(\mathfrak{g} \otimes_k F \subset \mathfrak{so}(\phi) \otimes_k F\) of the spin representation \(\rho: \mathfrak{so}(\phi) \to \text{End}_k(C^+(V))\). Then one can use the description of the Galois action of \(S = \text{Gal}(F/k)\) on weights of \(\mathfrak{g} \otimes_k F\) given above in order to break down the highest weights into orbits \(\{S \cdot \omega_1, \ldots, S \cdot \omega_t\}\). Let us denote the dimension of the irreducible representation of \(\mathfrak{g} \otimes_k F\) with highest weight \(\omega_i\) by \(d_i\). Let \(\hat{\rho}_i: \mathfrak{g} \to \text{End}_k(W_i)\) be the (unique) irreducible representation of \(\mathfrak{g} \otimes_k F\) with highest weight \(\omega_i\) as a \((\mathfrak{g} \otimes_k F)\)-submodule. Then our analysis in Section 2 implies:
Theorem 1.
\[ \text{End}(KS(X))_Q \cong \text{End}_g(W) \cong \prod_i \text{Mat}_{m_i \times m_i}(D_i) \text{ as } Q \text{- algebras,} \]
where \( D_i = \text{End}_g(W_i), m_i = (d_i/\text{dim}_k(W_i)) \cdot \sum_{\omega \in \mathcal{S}} \text{mult}(\omega) \) and \( \text{mult}(\omega) \) is the multiplicity of the irreducible subrepresentation of the representation of \( g \otimes_k F \) on \( C^+(V \otimes_k F) \) with highest weight \( \omega \).

Remark. In the analysis above we assumed that \( m = \text{dim}_E V \geq 5 \) (if \( E \) is totally real) or \( m \geq 2 \) (if \( E \) is a CM-field and \( r = [E : k]/2 \geq 2 \)) or \( m \geq 3 \) (if \( E \) is a CM-field and \( r = [E : k]/2 = 1 \)). In the case of small \( m \) Lie algebras we consider ’degenerate’ and require a separate consideration.

5 Q-forms of spin representations.

Let us describe more explicitly \( Q \)-forms \( W_i \) above or at least the corresponding primary representations. We will use corestriction of algebraic structures, as in [14], §6 and (in the case of totally real fields) representation spaces which we are going to construct in the following subsection.

5.1 Galois-invariant sums of ideals of Clifford algebra.

Let \( k = Q, E = L \) be a totally real number field, \( r = [L : k] \). Let \( \Phi = d_1 \cdot X_1^2 + \ldots + d_m \cdot X_m^2 \) with respect to basis \( \{e_1, \ldots, e_m\} \) of \( V, m = \text{dim}_LV \). Let \( F/k \) be a finite Galois extension containing \( L, \sqrt{-1} \) and \( \sqrt{d_i} \) for all \( i \). Let \( \sigma_1, \ldots, \sigma_r : L \rightarrow F \) be all the field embeddings over \( k \).

Let \( f_i = \frac{1}{\sqrt{d_i}} \cdot e_i + \frac{1}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1}, f_{-i} = \frac{1}{\sqrt{d_i}} \cdot e_i - \frac{1}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1}, 1 \leq i \leq l = \left[ \frac{m}{2} \right] \) and \( f_0 = \frac{1}{\sqrt{d_{l+1}}} \cdot e_{l+1} \). Then \( \{f_i, f_{-i} \mid 1 \leq i \leq l\} \) (if \( m \) is even) or \( \{f_0, f_i, f_{-i} \mid 1 \leq i \leq l\} \) (if \( m \) is odd) is a basis of \( V \otimes_{L, \sigma_i} F \), where we denote \( \sigma_i(d_j) \) by \( d_j \). With respect to this basis \( \Phi = 2 \sum_{i=1}^l Y_i \cdot Y_{-i} + \epsilon Y_0^2 \), where \( \epsilon = (1 - (-1)^m)/2 \).

5.1.1 Even dimension.

Assume that \( m \) is even. Let \( f_{\alpha_1, \ldots, \alpha_l} = f_{\alpha_1 \cdot 1} \cdot \ldots \cdot f_{\alpha_l \cdot l} \in C(V \otimes_{L, \sigma_i} F) \) for various \( \alpha_i \in \{\pm 1\} \) and \( I_{\alpha_1, \ldots, \alpha_l}^i = C(V \otimes_{L, \sigma_i} F) \cdot f_{\alpha_1, \ldots, \alpha_l}^i, 1 \leq i \leq r \). \( I_{\alpha_1, \ldots, \alpha_l}^i \) are left ideals of the Clifford algebra \( C(V \otimes_{L, \sigma_i} F) \) viewed as \( F \)-vector subspaces.

Consider the direct sum of \( F \)-vector spaces
\[ \tilde{C}(V \otimes_{L, \sigma_i} F) = \tilde{C}(V) \otimes_{L, \sigma_i} F = \bigoplus_{\alpha_1, \ldots, \alpha_l \in \{\pm 1\}} I_{\alpha_1, \ldots, \alpha_l}^i. \]
Note that \( g(f_i) \in \{ \pm f_i, \pm f_{-i} \} \) for any \( i \) and \( g \in S \). Hence the Galois group \( S = Gal(F/k) \) acts on \( \tilde{C}(V \otimes_{L,\sigma_i} F) \) (by sending an element of the summand \( I_{\alpha_1,\ldots,\alpha_l} \) to its image under the action of \( S \) on \( C(V \otimes_{L,\sigma} F) \) viewed as an element of the summand \( I_{\beta_1,\ldots,\beta_l} \), where \( f_{\beta_1,\ldots,\beta_l} \) is up to a scalar factor the image of \( f_{\alpha_1,\ldots,\alpha_l} \)).

It follows from the construction that \( F \)-vector subspaces \( \bigoplus_{r=1}^r I_{\alpha_1,\ldots,\alpha_l}^i \subset \bigoplus_{r=1}^r \tilde{C}(V) \otimes_{L,\sigma_i} F \) for various choices of \( \alpha_i^j \in \{ \pm 1 \} \) are permuted among themselves under the action of the Galois group \( S = Gal(F/k) \).

**Remark.** For any \( \alpha_1,\ldots,\alpha_l \) the left ideal \( I_{\alpha_1,\ldots,\alpha_l}^i \subset C(V \otimes_{L,\sigma} F) \) is an \((\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F)\)-subrepresentation of the spin representation, which is either irreducible (if \( m \) is odd) or is the sum of two irreducible and non-isomorphic (semi-spin) representations \([2, 3]\). In the latter case, let us write \( I_{\alpha_1,\ldots,\alpha_l}^i = I_{\alpha_1,\ldots,\alpha_l}^{i,+} \oplus I_{\alpha_1,\ldots,\alpha_l}^{i,-} \) for the corresponding (unique) decomposition.

### 5.1.2 Odd dimension.

Assume that \( m \) is odd. Let \( f_{\alpha_1,\ldots,\alpha_l,\gamma}^i = f_{\alpha_1} \cdot \cdots \cdot f_{\alpha_l} \cdot (1 + \gamma \cdot f_0) \in C(V \otimes_{L,\sigma} F) \) for various \( \alpha_i, \gamma \in \{ \pm 1 \} \) and \( I_{\alpha_1,\ldots,\alpha_l,\gamma}^i = C(V \otimes_{L,\sigma} F) \cdot f_{\alpha_1,\ldots,\alpha_l,\gamma}^i, 1 \leq i \leq r \). \( I_{\alpha_1,\ldots,\alpha_l,\gamma}^i \) are left ideals of the Clifford algebra \( C(V \otimes_{L,\sigma} F) \) viewed as \( F \)-vector subspaces.

Consider the direct sum of \( F \)-vector spaces

\[
\tilde{C}(V \otimes_{L,\sigma} F) = \tilde{C}(V) \otimes_{L,\sigma} F = \bigoplus_{\alpha_1,\ldots,\alpha_l,\gamma \in \{ \pm 1 \}} I_{\alpha_1,\ldots,\alpha_l,\gamma}^i.
\]

Note that \( g(1 + \gamma \cdot f_0) = (1 \pm \gamma \cdot f_0) \) for any \( g \in S \). Hence the Galois group \( S = Gal(F/k) \) acts on \( \tilde{C}(V \otimes_{L,\sigma} F) \) (by sending an element of the summand \( I_{\alpha_1,\ldots,\alpha_l,\gamma} \) to its image under the action of \( S \) on \( C(V \otimes_{L,\sigma} F) \) viewed as an element of the summand \( I_{\beta_1,\ldots,\beta_l,\gamma'} \), where \( f_{\beta_1,\ldots,\beta_l,\gamma'} \) is up to a scalar factor the image of \( f_{\alpha_1,\ldots,\alpha_l,\gamma} \)).

It follows from the construction that \( F \)-vector subspaces \( \bigoplus_{i=1}^r I_{\alpha_1,\ldots,\alpha_l,\gamma}^i \subset \bigoplus_{i=1}^r \tilde{C}(V) \otimes_{L,\sigma_i} F \) for various choices of \( \alpha_i^j, \gamma_i^j \in \{ \pm 1 \} \) are permuted among themselves under the action of the Galois group \( S = Gal(F/k) \).

**Remark.** For any \( \alpha_1,\ldots,\alpha_l, \gamma \) the left ideal \( I_{\alpha_1,\ldots,\alpha_l,\gamma}^i \subset C(V \otimes_{L,\sigma} F) \) is an irreducible \((\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F)\)-subrepresentation of the spin representation (since \( m \) is odd by assumption) \([2, 3]\).

We will use \( \tilde{C}(V \otimes_{L,\sigma} F) \) as representation spaces of \((\mathfrak{so}(\Phi) \otimes_{L,\sigma} F)\) (the direct sum of its representations on the left ideals of the Clifford algebra) in order to construct primary \( \mathbb{Q} \)-forms of spin representations.
5.2 Case of the totally real field and odd dimension.

Let $E = E_0 = L$ be totally real and $m = \dim_L V$ odd. Let $\Sigma_i \subset C^+(V \otimes_{L,\sigma_i} F)$, $1 \leq i \leq r$ be the irreducible subrepresentation of the spin representation of $\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F$. Then $\Sigma_1 \otimes_F ... \otimes_F \Sigma_r$ is an irreducible representation of $\bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F) = \mathfrak{g} \otimes_k F$.

Let $\tilde{C}(V \otimes_{L,\sigma_i} F) = \oplus_p S_p^1$ be a decomposition into irreducible components of the representation of $\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F$ considered above. Let $\Omega'$ be the finite set of $F$-vector subspaces of $\tilde{C}(V \otimes_{L,\sigma_i} F) \otimes_F ... \otimes_F \tilde{C}(V \otimes_{L,\sigma_i} F)$ (or of $C(V \otimes_{L,\sigma_i} F) \otimes_F ... \otimes_F C(V \otimes_{L,\sigma_i} F)$) of the form $S_{p_1}^1 \otimes_F ... \otimes_F S_{p_r}^r$ for various $p_1, ..., p_r$. These subspaces are irreducible subrepresentations of the exterior tensor product of spin representations as a representation of $\bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F)$.

Galois group $S = \text{Gal}(F/k)$ acts on $\Omega'$. Take any element $S_{p_1}^1 \otimes_F ... \otimes_F S_{p_r}^r$ of $\Omega'$. Let $U \subset \tilde{C}(V \otimes_{L,\sigma_i} F) \otimes_F ... \otimes_F \tilde{C}(V \otimes_{L,\sigma_i} F)$ be the sum of the elements of $\Omega'$ (as subspaces of $\tilde{C}(V \otimes_{L,\sigma_i} F) \otimes_F ... \otimes_F \tilde{C}(V \otimes_{L,\sigma_i} F)$) lying in the $S$-orbit of $S_{p_1}^1 \otimes_F ... \otimes_F S_{p_r}^r$. Then $U \subset \tilde{C}(V \otimes_{L,\sigma_i} F) \otimes_F ... \otimes_F \tilde{C}(V \otimes_{L,\sigma_i} F)$ is an $S$-submodule.

Since the actions of $\mathfrak{g} \subset \mathfrak{g} \otimes_k F$ and $S = \text{Gal}(F/k)$ commute, by Galois descent

$$(U)^S \cong ((\Sigma_1 \otimes_F ... \otimes_F \Sigma_r)^{\oplus n_0})^S$$

is a primary representation of $\mathfrak{g}$ over $k$ of dimension $n_0 \cdot 2^{l \! r}$, which contains $\Sigma_1 \otimes_F ... \otimes_F \Sigma_r$ after extending scalars to $F$.

Multiplicity $n_0$ is the length of the $S$-orbit in $\Omega'$ of the chosen element $S_{p_1}^1 \otimes_F ... \otimes_F S_{p_r}^r$ of $\Omega'$.

**Remark.** We will use notation introduced above. Consider the action of $S = \text{Gal}(F/k)$ on $2^{l \! r + 1}$ elements (or more precisely on the lines generated by them) $f_{\beta_1, ..., \beta_i, \gamma}$ of $C(V \otimes_L F)$ for various $\beta_1, ..., \beta_i, \gamma$ by sign changes in front of $\sqrt{d_i}$'s and $\sqrt{-d_{m-i+1}}$'s in the definition of $f_i$ in terms of $e_j$ (see notation above). Then (if we choose all $S_{p_1}^1$ to be the same)

$$n_0 = \frac{\text{order of } S = \text{Gal}(F/k)}{\text{order of the stabilizer of } f_{1, ..., 1, 1}}.$$

5.3 Case of the totally real field and even dimension.

Let $E = E_0 = L$ be a totally real field and $m = \dim_L V$ even. Let $\Sigma_i^+, \Sigma_i^- \subset C^+(V \otimes_{L,\sigma_i} F)$, $1 \leq i \leq r$ be irreducible (semi-spin) subrepresentations of the spin representation of $\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F$.

Consider the finite set $\Omega$ of $F$-vector spaces of the form $\Sigma_{\alpha_1}^1 \otimes_F ... \otimes_F \Sigma_{\alpha_r}^r$ for various $\alpha_i \in \{+,-\}$. They are exactly the irreducible components of the exterior tensor product of spin representations $\Sigma_i = \Sigma_i^+ \oplus \Sigma_i^- \subset C^+(V \otimes_{L,\sigma_i} F)$ of $\bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F)$ (see [Pi], §13, [2], [3]). They are also the isomorphism classes of simple $\bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F)$-submodules of $C(V \otimes_{L,\sigma_i} F) \otimes_F ... \otimes_F C(V \otimes_{L,\sigma_i} F)$. Let $\tilde{C}(V \otimes_{L,\sigma_i} F) = \oplus_p S_p^i$ be a decomposition
into irreducible components of the representation of \( \mathfrak{so}(\Phi) \otimes_{L,\sigma} F \) considered above. Let \( \Omega' \) be the finite set of \( F \)-vector subspaces of \( C(V \otimes_{L,\sigma_1} F) \otimes_F \ldots \otimes_F C(V \otimes_{L,\sigma_r} F) \) (or of \( \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \)) of the form \( S_{p_1}^1 \otimes_F \ldots \otimes_F S_{p_r}^r \) for various \( p_1, \ldots, p_r \). These subspaces are irreducible subrepresentations of the exterior tensor product of spin representations as a representation of \( \bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F) \).

Galois group \( S = Gal(F/k) \) acts naturally on both \( \Omega \) and \( \Omega' \). Let \( \Omega_1, \ldots, \Omega_u \) be the orbits of \( S \) on \( \Omega \). For any \( i \) choose \( (a_1, \ldots, a_r) \in \Omega_i \) and define \( U_i \subset \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \) to be the sum of the elements of \( \Omega' \) (as subspaces of \( \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \)) lying in the \( S \)-orbit of any \( S_{p_1}^1 \otimes_F \ldots \otimes_F S_{p_r}^r \), which is isomorphic to \( \Sigma_{a_1}^1 \otimes_F \ldots \otimes_F \Sigma_{a_r}^r \) as an \( \bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F) \)-module.

Then \( U_i \subset \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \) is an \( S \)-submodule and

\[
(U_i)^S \cong \left( \bigoplus_{(a_1, \ldots, a_r) \in \Omega_i} (\Sigma_{a_1}^1 \otimes_F \ldots \otimes_F \Sigma_{a_r}^r)^{\oplus n_{a_1, \ldots, a_r}} \right)^S
\]

is a primary representation of \( \mathfrak{g} \) over \( k \) of dimension \( \sum_{(a_1, \ldots, a_r) \in \Omega_i} n_{a_1, \ldots, a_r} \cdot 2^{r \cdot (l-1)} \). These representations \( (U_i)^S \), \( 1 \leq i \leq u \) contain all representations of \( \mathfrak{g} \otimes_k F \) of the form \( \Sigma_{a_1}^1 \otimes_F \ldots \otimes_F \Sigma_{a_r}^r \) after extending scalars to \( F \).

Multiplicities \( n_{a_1, \ldots, a_r} \) can be computed as follows:

\[
n_{a_1, \ldots, a_r} = \frac{\text{order of the stabilizer of } (a_1, \ldots, a_r) \in \Omega}{\text{order of the stabilizer of } (p_1, \ldots, p_r) \in \Omega'}.
\]

**Remark.** We will use notation introduced above. Consider the action of \( S = Gal(F/k) \) on \( 2^l \) elements (or more precisely on the lines generated by them) \( f_{\beta_1, \ldots, \beta_l} \) of \( C(V \otimes L F) \) for various \( \beta_1, \ldots, \beta_l \) by sign changes in front of \( \sqrt{d_i} \)'s and \( \sqrt{-d_{m-i+1}} \)'s in the definition of \( f_i \) in terms of \( e_j \) (see notation above). Then (if we choose all \( S_{p_i} \) to be the same)

\[
(\text{stabilizer of } (p_1, \ldots, p_r) \in \Omega') = (\text{stabilizer of } (a_1, \ldots, a_r) \in \Omega) \cap (\text{stabilizer of } f_{1, \ldots, 1}).
\]

**Remark.** Instead of \( \tilde{C}(V \otimes L F) \) one can also consider the Clifford algebra \( C(V \otimes L F) \) (or its even part \( C^+(V \otimes L F) \)). Then the corestriction of \( C(V) \) (or of \( C^+(V) \)) (with \( V \) viewed as a vector space over \( L \)) from \( L \) to \( k = \mathbb{Q} \) (or Galois-fixed subspaces of sums (inside of tensor products of \( C(V) \otimes L F \) of tensor products of \( \mathfrak{g} \otimes_k F \))-invariant \( F \)-vector subspaces (or ideals used above)) of \( C(V) \otimes L F \), which form a single Galois orbit) would be a representation of \( \mathfrak{g} \) over \( \mathbb{Q} = k \), whose extension of scalars to \( F \) contains all the irreducible representations (and only them) of \( \mathfrak{g} \otimes_k F \) over \( F \) which we need. In particular, in the case of odd \( m \) it would be another primary representation of \( \mathfrak{g} \) over \( k \).

### 5.4 Case of the CM-field.

Let \( E = E_0(\theta), \theta^2 \in E_0 = L \) be a CM-field.
Note that the tautological representation of $u(\Phi) \otimes_{L, \sigma_i} F$ in $V \otimes_{L, \sigma_i} F$ splits into the direct sum of two representations of $\mathfrak{gl}(m, F) \cong u(\Phi) \otimes_{L, \sigma_i} F$

$$V \otimes_{L, \sigma_i} F = \left( V \otimes_{E, \sigma_i} F \right) \oplus \left( V \otimes_{E, \sigma_i} F \right),$$

where $\sigma_i$ and $\bar{\sigma}_i$ are the two extensions of $\sigma_i : E_0 \to F$ to embeddings $E \to F$.

Since the exterior power representations $\wedge^p_F(V \otimes_{E, \sigma_i} F)$ and $\wedge^p_F(V \otimes_{E, \sigma_i} F)$ of $u(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ are identified by the Lie algebra automorphism $\mathfrak{gl}(m, F) \to \mathfrak{gl}(m, F), \ M \mapsto -\Phi \cdot M^T \cdot \Phi^{-1}$, we have isomorphisms

$$\wedge^p_F(V \otimes_{E, \sigma_i} F) \to \wedge^{m-p}_F(V \otimes_{E, \sigma_i} F) \otimes_F D^{-1}$$

and hence also isomorphisms

$$\tau_p : \wedge^p_F(V \otimes_{E, \sigma_i} F) \otimes_F (E \otimes_{E, \sigma_i} F) \to \wedge^{m-p}_F(V \otimes_{E, \sigma_i} F) \otimes_F D^{-1/2}, \ 1 \leq p \leq m$$

of representations of $\mathfrak{gl}(m, F) \cong u(\Phi) \otimes_{L, \sigma_i} F$.

Let $\wedge^j_1 \subset \wedge^r_F(V \otimes_{E, \sigma_i} F) \otimes_F F$, $1 \leq i \leq r, 1 \leq j \leq m$ be the irreducible representation of $\mathfrak{gl}(m, F)$ on the $F$-vector space $\wedge^j_1 F \otimes_{E, \sigma_i} F$ twisted by $D^{-1/2}$. We define an $E_0$-linear representation $D^c, c \in \mathbb{Q}$ of $u(\Phi)$ in the $E_0$-vector space $E$ in exactly the same way as for $\mathfrak{gl}(m, F)$ above, i.e. by taking the trace of a matrix and multiplying it by $\frac{c}{m}$.

Consider the finite set $\Omega$ of $F$-vector spaces of the form $\wedge^{j_1}_1 \otimes_F ... \otimes_F \wedge^{j_r}_r$ for various $j_i \in \{1, ..., m\}$. They are exactly the isomorphism classes of irreducible subrepresentations of the exterior product of (twisted by $D^{-1/2}$ and extended to $F$) exterior algebra representations $\wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F)$ of $\oplus_{i=1}^r u(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)^{\oplus r}$.

Let $\wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F) = \oplus_{p_r} S_{p_r}$ be the decomposition into irreducible components of the representation of $u(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ obtained from the decompositions $E \otimes_{L, \sigma_i} F = (E \otimes_{E, \sigma_i} F) \oplus (E \otimes_{E, \sigma_i} F) \cong D^{-1/2} \oplus D^{1/2} \cong F \oplus F$ and $V \otimes_{L, \sigma_i} F = (V \otimes_{E, \sigma_i} F) \oplus (V \otimes_{E, \sigma_i} F)$ above.

Let $\Omega'$ be the finite set of $F$-vector subspaces of $(\wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F)) \otimes_F \wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F)$ of the form $S_{p_1} \otimes_F ... \otimes_F S_{p_r}$ for various $p_1, ..., p_r$. These subspaces are irreducible subrepresentations of the exterior product of exterior algebra representations as a representation of $\oplus_{i=1}^r u(\Phi) \otimes_{L, \sigma_i} F$.

Galois group $S = Gal(F/k)$ acts on $\Omega$ by permuting factors in tensor products. It also acts on $\Omega'$. Let $\Omega_1, ..., \Omega_\omega$ be the orbits of $S$ on $\Omega$. For any $i$ choose $(j_1, ..., j_r) \in \Omega_i$ and define $U_i = (\wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F)) \otimes_F (\wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F))$ to be the sum of the elements of $\Omega'$ (as subspaces of $(\wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F)) \otimes_F (\wedge^r_F(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F))$) lying in the $S$-orbit of any $S_{p_1} \otimes_F ... \otimes_S S_{p_r}$, which is isomorphic to $\wedge^{j_1}_1 \otimes_F ... \otimes_F \wedge^{j_r}_r$ as a $\oplus_{i=1}^r u(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)^{\oplus r}$-module.
Then $U_i \subset (\wedge^r_F (V \otimes_{L,\sigma_i} F) \otimes_F (E \otimes_{L,\sigma_i} F)) \otimes_F \ldots \otimes_F (\wedge^r_F (V \otimes_{L,\sigma_i} F) \otimes_F (E \otimes_{L,\sigma_i} F))$ is an $S$-submodule and
\[
(U_i)^S \cong \left( \bigoplus_{(j_1, \ldots, j_r) \in \Omega_i} (\wedge_{i}^{j_1} \otimes_F \ldots \otimes_F \wedge_{s}^{j_r}) \otimes \otimes \right)_{S}^{n_{j_1, \ldots, j_r}}
\]
is a primary representation of $\mathfrak{g}$ over $k$ of dimension $\sum_{(j_1, \ldots, j_r) \in \Omega_i} n_{j_1, \ldots, j_r} \cdot \binom{m}{j_1} \cdot \ldots \cdot \binom{m}{j_r}$. These representations $(U_i)^S$, $1 \leq i \leq u$ contain all representations of $\mathfrak{g} \otimes_k F$ of the form $\wedge_{i}^{j_1} \otimes_F \ldots \otimes_F \wedge_{s}^{j_r}$ after extending scalars to $F$.

The reason why nontrivial multiplicities may appear is exactly the doubling $V \otimes_{L,\sigma_i} F = (V \otimes_{E,\sigma_i} F) \oplus (V \otimes_{E,\sigma_i} F)$ described above. Hence one can compute multiplicities $n_{j_1, \ldots, j_r}$ as follows. Consider the finite set $\Omega''$ of $r$-tuples of signs + and –, i.e. $\Omega'' = \{ (\alpha_1, \ldots, \alpha_r) \mid \alpha_i = \pm \}$. Note that the $i$-th sign corresponds to the $i$-th embedding $\sigma_i : L \to F$ over $k$. Consider the action of $S = Gal(F/k)$ on $\Omega''$ such that $g \in S$ acts on entries of $r$-tuples by the same permutations as on the set of left cosets $S/H$ (where $H = \{ g \in S \mid g \circ \sigma_1 = \sigma_1 \}$) and $g$ changes the sign in the $i$-th entry to the opposite sign (in the $j$-th entry, where $\sigma_j = g \circ \sigma_i$) if and only if $g(\theta) = -\theta$. Then
\[
n_{j_1, \ldots, j_r} = \frac{\text{order of the stabilizer of } (j_1, \ldots, j_r) \in \Omega}{\text{order of the intersection of stabilizers of } (+, \ldots, +) \in \Omega'' \text{ and of } (j_1, \ldots, j_r) \in \Omega''}
\]
This gives a description of some multiples of $(k = \mathbb{Q})$-linear irreducible representations $W_i$ of $\mathfrak{g}$ mentioned in the Theorem above (as well as formulas for their dimensions - some multiples of $\text{dim}_k(W_i)$) in terms of the Galois action.

### 6 Cohomology classes of division algebras.

In this section we compute division algebras $D_i$ as elements of the Brauer group $Br(F/C_j) \cong H^2(Gal(F/C_j), F^*)$ as well as their centers $C_j$.

#### 6.1 Case of the totally real field and odd dimension.

Let $E = E_0 = L$ be totally real and $m = \text{dim}_E V$ odd. We saw above how to construct a primary representation $W = U^S$ of $\mathfrak{g}$ over $k = \mathbb{Q}$, which contains irreducible representation $\rho^0 \boxtimes \ldots \boxtimes \rho^0$ (the exterior tensor product of irreducible spin representations) of $\mathfrak{g} \otimes_k F \cong \mathfrak{so}(\Phi) \otimes_{L,\sigma_i} F$ after extending scalars to $F$. This means that $W \cong W_{0}^{m}$, where $W_0$ is an irreducible representation of $\mathfrak{g}$ over $k$ and $W_0 \otimes_k F \cong \bigotimes_{\mu = \text{dim}_{\rho^0}}^{ \text{dim}_k W} \rho^0 \boxtimes \ldots \boxtimes \rho^0$. Since we are interested only in the endomorphism algebra $D_0 = \text{End}_{\mathfrak{g}}(W_0)$ which is a central division algebra over $k$ split over $F$, we can describe it by computing the Galois cohomology invariant of the central simple algebra $A = \text{End}_{\mathfrak{g}}(W) \cong \text{Mat}_{\mu \times \mu}(D_0)$, i.e. its Brauer invariant in $Br(F/k) \cong H^2(S, F^*)$, where $S = Gal(F/k)$. Then $\mu = \frac{\text{deg}(A)}{\text{deg}(D_0)} = \frac{\text{dim}_k W}{\text{deg}(D_0)}$. 

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We will use the same notation as above with the following exceptions:

\[ f_{\alpha_1,\ldots,\alpha_l;\gamma} = (1 + \gamma \cdot f_0) \cdot f_{\alpha_1;1} \cdot \ldots \cdot f_{\alpha_l;l}, \]

\[ f_{\alpha;i} = \left( e_i + \alpha \cdot \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right). \]

Some parts of our construction (in particular, the construction of the generators of endomorphism algebras) may be viewed as a generalization of some constructions of van Geemen \[13, \S3.\]

Consider \( F \)-linear homomorphisms

\[ r_{((\alpha_1),\gamma),((\beta_1),\tilde{\gamma})}: \tilde{C}(V \otimes_L F) \rightarrow I_{\beta_1,\ldots,\beta_l;\tilde{\gamma}}, \quad \xi \mapsto \tau^{\delta(\gamma,\tilde{\gamma})}(\xi \cdot R_{((\alpha_1),\gamma),((\beta_1),\tilde{\gamma})}), \]

where \( \tau: C(V \otimes_L F) \rightarrow C(V \otimes_L F) \) is the algebra homomorphism induced by multiplication by \((-1)\) on \( V \), \( \delta(\gamma,\tilde{\gamma}) = 1 \), if \( \gamma \neq \tilde{\gamma} \). \( P(\alpha,\beta) = \text{card} \{ i \mid \alpha_i \neq \beta_i \} \) and 0 otherwise, and

\[ R_{((\alpha_1),\gamma),((\beta_1),\tilde{\gamma})} = \frac{(-1)^{c(\alpha,\beta)}}{\prod_{i: \alpha_i = \beta_i} \Phi(f_i, f_{-i}) \cdot \prod_{i: \alpha_i \neq \beta_i} (f_{-\alpha,i} \cdot f_{\alpha,i}) \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta,i}}, \]

where \( c(\alpha,\beta) \) is the number of transpositions of factors needed to transform the product \( \prod_i f_{\alpha,i} \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta,i} \) into the product \( q \cdot \prod_i f_{\beta,i} \) with some coefficient \( q \in C(V \otimes_L F) \). Then \( r_{((\alpha_1),\gamma),((\beta_1),\tilde{\gamma})} \) is nonzero only on the factor \( I_{\alpha_1,\ldots,\alpha_l;\gamma} \) of \( \tilde{C}(V \otimes_L F) \) and induces an isomorphism \( I_{\alpha_1,\ldots,\alpha_l;\gamma} \rightarrow I_{\beta_1,\ldots,\beta_l;\tilde{\gamma}} \) which commutes with the action of \( \mathfrak{so}(\Phi) \otimes_L F \).

In order to simplify notation we will denote index \(((\alpha_i), \gamma)\) by \( \alpha \).

One can choose coefficients \( \lambda_{\alpha,\beta} \in F^* \) such that under an isomorphism of \( F \)-algebras \( \text{Mat}(F) \cong \text{End}_{\mathfrak{so}(\Phi) \otimes_L F}(\tilde{C}(V \otimes_L F)) \) matrices of the form \( E_{ij} \) (in the notation of \[13, \S13\]) correspond to endomorphisms \( \lambda_{\alpha,\beta} \cdot r_{\alpha,\beta} \). In order to do this, one can choose and fix index \( \alpha^0 = ((\alpha_i^0), \gamma^0) \) and take

\[ \lambda_{\alpha^0,\beta} = 1, \quad \lambda_{\beta,\alpha^0} = (-1)^{P(\alpha,\beta) \cdot \delta(\gamma^0,\tilde{\gamma}) + P(\alpha,\beta) \cdot (P(\alpha,\beta) - 1)/2} \cdot \prod_{i: \alpha_i^0 \neq \beta_i} \frac{1}{\Phi(f_i, f_{-i})}, \]

and

\[ \lambda_{\alpha,\beta} = \lambda_{\alpha,\alpha^0} \cdot (-1)^{e(\alpha,\beta) + \delta(\gamma,\tilde{\gamma}) \cdot (1 + P(\alpha,\beta)) + \delta(\gamma,\gamma^0) \cdot (1 + P(\alpha,\alpha^0)) + \delta(\gamma,\tilde{\gamma}) \cdot (1 + P(\alpha,\beta))} \cdot \prod_{i: \alpha_i = \beta_i \neq \alpha_i^0} \Phi(f_i, f_{-i}), \]

where \( \alpha = ((\alpha_i), \gamma), \beta = ((\beta_i), \tilde{\gamma}), e(\alpha,\beta) \) is the number of transpositions of factors needed in order to transform the product \( \prod_i: \alpha_i^0 \neq \beta_i \cdot f_{\alpha,i} \cdot \prod_i: \alpha_i \neq \beta_i \cdot f_{-\beta,i} \) into the product \( \prod_i: \alpha_i \neq \beta_i \cdot f_{\alpha,i} \cdot \prod_i: \alpha_i = \beta_i \neq \alpha_i^0 (f_{\beta,i} \cdot f_{-\beta,i}) \). Note that in this construction \( \lambda_{\alpha,\beta} \in L^* \).
Then we construct endomorphisms

\[ r_{(\alpha^i, (\beta^i))} = r_{\alpha^i, \beta^i}^1 \circ \ldots \circ r_{\alpha^p, \beta^p} : \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F) \to \]
\[ I_{\beta^1} \otimes_F \ldots \otimes_F I_{\beta^r} \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F) \]

which commute with \( g \otimes_k F \), where \( \alpha^p = ((\alpha_1^p, \ldots, \alpha_l^p), \gamma^p) \), \( \beta^p = ((\beta_1^p, \ldots, \beta_l^p), \tilde{\gamma}^p) \) and

\[ r_{\alpha^p, \beta^p}^p = 1 \otimes_F \ldots \otimes_F (r_{\alpha^p, \beta^p}) \circ 1 : \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F) \to \]
\[ \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F) \]

(with 1 outside of the \( p \)-th spot).

As in [13], Proposition 3.6 \( F \)-algebra \( \text{End}_{g \otimes_k F}(W \otimes_k F) = A \otimes_k F \) is generated by elements \( r_{(\alpha^i, (\beta^i))} \) (more precisely, by those of them which correspond to the summands of \( \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F) \) included in \( W \otimes_k F = U \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F) \)) or by elements \( r_{\alpha^i, \beta^i}^p \), while \( k \)-algebra \( A = \text{End}_q(W) = (A \otimes_k F)^S \) is generated by elements \( r_{\alpha^i, \beta^i}^p \), \( \gamma \) is the class of a 2-cocycle \( (\alpha_1, \ldots, \alpha_l) \) is given by the rule \( g(\alpha) = (c_1(g) \cdot \alpha_1, \ldots, c_l(g) \cdot \alpha_l, c_0(g) \cdot \gamma) \), where \( c_i(g) \in \{ \pm 1 \} \) and \( g(f_1, \ldots, f_l) = f_1(c_1(g) \cdot \alpha_1, \ldots, c_l(g) \cdot \alpha_l, c_0(g) \cdot \gamma) \).

Hence the matrix of \( m(g) \in \text{GL}(W \otimes_k F) \) is such that

\[ m(g) \cdot E_{i,j} \cdot m(g)^{-1} = \phi_g(E_{i,j}) = \left( \prod_{i=1}^{r} \frac{g(\lambda_{i,1}, \ldots, \lambda_{i,\beta_i})}{\lambda_{g(\alpha^i, (\beta^i))}} \right) \cdot E_{g(i), g(j)}, \]

where \( E_{i,j} \) denotes a matrix from \( \text{Mat}(F) \cong \text{End}_{g \otimes_k F}(W \otimes_k F) \) corresponding to \( r_{(\alpha^i, (\beta^i))} \), i.e. up to a scalar multiple conjugation by \( m(g) \) acts on matrices as the (same) permutation of columns and rows induced by \( g \) on indices \( ((\alpha^i_j), (\beta^i_j)) \).

Then the element of \( H^2(S, F^*) \) corresponding to the central division algebra \( D_0 = \text{End}_q(W_0) \) is the class of a 2-cocycle \( \lambda : S \times S \to F^* \cong F^* \cdot \text{Id} \subset \text{Mat}(F) \), \( (g_1, g_2) \mapsto m(g_1 g_2) \cdot (g_1(m(g_2)))^{-1} \cdot m(g_1)^{-1} \) [8], [2].
6.2 Case of the totally real field and even dimension.

Let $E = E_0 = L$ be totally real and $m = \text{dim}_E V$ even. We saw above how to construct a primary representation $W = (U_i)^S$ of $\mathfrak{g}$ over $k = \mathbb{Q}$, which contains irreducible representation $\rho^\alpha \boxtimes \ldots \boxtimes \rho^\beta$ (the exterior tensor product of irreducible semi-spin representations) of $\mathfrak{g} \otimes F \cong \bigoplus_{i=1}^r \mathfrak{so}(\Phi) \otimes L_{\omega_i} F$ after extending scalars to $F$ (as well as its Galois conjugates). This means that $W \cong W_0^{\oplus \mu}$, where $W_0$ is an irreducible representation of $\mathfrak{g}$ over $k$, $W \otimes_k F \cong \bigoplus_i W_i$ and $W_i \cong (\text{dim} \rho_{\omega_i})^* \cdot \rho_{\omega_i} \boxtimes \ldots \boxtimes \rho_{\omega_r}$ are the isotypical components (over $F$).

Since we are interested only in the endomorphism algebra $D_0 = \text{End}_G(W_0)$ which is a division algebra over $k$ (and over its center $C$) split over $F$, we can describe it by computing the Galois cohomology invariant of the central simple algebra $A = \text{End}_G(W) \cong \text{Mat}_{\mu \times \mu}(D_0)$ (over $C$), i.e. its Brauer invariant in $Br(F/C) \cong H^2(S', F^*)$, where $S' = \text{Gal}(F/C)$. Then

$$\mu = \frac{\text{deg}(A)}{\text{deg}(D_0)} = \frac{n_{\alpha_1 \cdots \alpha_r}}{\text{deg}(D_0)}.$$  

We will use the same notation as above with the following exceptions:

$$f_{\alpha_1, \ldots, \alpha_l} = f_{\alpha_1, i} \cdot \ldots \cdot f_{\alpha_l, i}.$$  

$$f_{\alpha, i} = \left(e_i + \alpha \cdot \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1}\right).$$

Some parts of our construction (in particular, the construction of the generators of endomorphism algebras) may be viewed as a generalization of some constructions of van Geemen [13], §3.

Consider $F$-linear homomorphisms

$$r_{(\alpha), (\beta)} : \tilde{C}(V \otimes_L F) \to I_{\beta_1, \ldots, \beta_l}, \; \xi \mapsto \xi \cdot R_{(\alpha), (\beta)},$$

where $P(\alpha, \beta) = \text{card}\{i \mid \alpha_i \neq \beta_i\}$ and

$$R_{(\alpha), (\beta)} = \frac{(-1)^c(\alpha, \beta)}{\prod_{i: \alpha_i = \beta_i} \Phi(f_i, f_i)} \cdot \prod_{i: \alpha_i = \beta_i} (f_{\alpha, i} \cdot f_{\alpha, i}) \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta, i},$$

where $c(\alpha, \beta)$ is the number of transpositions of factors needed to transform the product $\prod_i f_{\alpha, i} \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta, i}$ into the product $q \cdot \prod_i f_{\beta, i}$ with some coefficient $q \in C(V \otimes_L F)$. Then $r_{(\alpha), (\beta)}$ is nonzero only on the factor $I_{\alpha_1, \ldots, \alpha_l}$ of $\tilde{C}(V \otimes_L F)$ and induces an isomorphism $I_{\alpha_1, \ldots, \alpha_l} \to I_{\beta_1, \ldots, \beta_l}$ which commutes with action of $\mathfrak{so}(\Phi) \otimes_L F$. Without mentioning this explicitly, we will be restricting all our endomorphisms to the factors of $\tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ contributing to an isotypical component $W_i \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$.

In order to simplify notation we will denote $\text{index}(\alpha_i)$ by $\alpha$.

One can choose coefficients $\lambda_{\alpha, \beta} \in F^*$ such that under an isomorphism of $F$-algebras $\text{Mat}(F) \cong \text{End}_{\mathfrak{so}(\Phi) \otimes_L F}(W_i)$ (note that $W_i \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \ldots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ and see

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the remark above) matrices of the form $E_{ij}$ correspond to endomorphisms $\lambda_{\alpha,\beta} \cdot r_{\alpha,\beta}$. In order to do this, one can choose and fix index $\alpha^0 = (\alpha^0_i)$ and take

$$\lambda_{\alpha,\beta} = 1, \quad \lambda_{\alpha,\beta} = (-1)^{P(\alpha,\beta)}(P(\alpha,\beta)-1)/2 \cdot \prod_{i: \alpha^0_i \neq \beta_i} \Phi(f_i, f_{-i})$$

and

$$\lambda_{\alpha,\beta} = \lambda_{\alpha,\alpha^0} \cdot (-1)^{e(\alpha,\beta)} \cdot \prod_{i: \alpha^0_i \neq \alpha^0_i} \Phi(f_i, f_{-i}),$$

where $\alpha = (\alpha_i), \beta = (\beta_i), e(\alpha,\beta)$ is the number of transpositions of factors needed in order to transform the product $\prod_{i: \alpha^0_i \neq \alpha^0_i} f_{\alpha,\beta} \cdot \prod_{i: \alpha^0_i \neq \alpha^0_i} f_{\beta,\beta}$ into the product $\prod_{i: \alpha^0_i \neq \alpha^0_i} f_{\alpha,\beta} \cdot \prod_{i: \alpha^0_i \neq \alpha^0_i} f_{\beta,\beta}$. Note that in this construction $\lambda_{\alpha,\beta} \in L^*.$

Then we construct endomorphisms

$$r_{(\alpha^0), (\beta^0)} = r_{\alpha^0,\beta^0}^1 \circ \cdots \circ r_{\alpha^0,\beta^0}^p : C(V \otimes_{L,\sigma_1} F) \otimes_F \cdots \otimes_F C(V \otimes_{L,\sigma_r} F) \to$$

$$\to I_{\beta^0} \otimes_F \cdots \otimes_F I_{\beta^0} \subset C(V \otimes_{L,\sigma_1} F) \otimes_F \cdots \otimes_F C(V \otimes_{L,\sigma_r} F)$$

which commute with $g \otimes_k F,$ where $\alpha^p = (\alpha^p_1, \ldots, \alpha^p_l), \beta^p = (\beta^p_1, \ldots, \beta^p_l)$ and

$$r_{(\alpha^0), (\beta^0)}^p = 1 \otimes_F \cdots \otimes_F (r_{\alpha^0,\beta^0}^p) \otimes_F \cdots \otimes_F 1 : C(V \otimes_{L,\sigma_1} F) \otimes_F \cdots \otimes_F C(V \otimes_{L,\sigma_r} F) \to$$

$$\to C(V \otimes_{L,\sigma_1} F) \otimes_F \cdots \otimes_F C(V \otimes_{L,\sigma_r} F)$$

(with 1 outside of the $p$-th spot).

As in [13], Proposition 3.6 $F$-algebra $End_{g \otimes_k F}(W \otimes_k F) = A \otimes_k F$ is generated by elements $r_{(\alpha^0), (\beta^0)}$ (more precisely, by those of them which correspond to the summands of $\tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \cdots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F)$ included in various isotypical components $W_{i'} \otimes_k F \subset U_i \subset \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \cdots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F)$) or by elements $r_{\alpha,\beta}^p,$ while $k$-algebra $A = End_{g}(W) = (A \otimes_k F)^S$ is generated by elements $r_{\alpha,\beta}^p = \sum_{g \in S} g(e_q) \cdot g \circ r_{\alpha,\beta}^p,$ where $\{e_q\}$ is a basis of $F/k.$

The center $C$ of $A$ (and of $D_0$) consists of Galois averages (as above) of $F$-linear combinations of sums $C_{i'} = \sum_{(\alpha^0) \in I_{i'}} (\prod_{i=1}^{l} \sigma_i(\lambda_{\alpha^0_i,\alpha_i})) \cdot r_{(\alpha^0), (\alpha^0)}$ (over the sets $I_{i'}$ of indices $\alpha^0_i$) corresponding to irreducible subrepresentations over $F$ of $W \otimes_k F$ contained in various isotypical components $W_{i'}.$ Each of the coefficients of these $F$-linear combinations gives a field embedding $C \to F$ over $k = \mathbb{Q}$. Note that $A \otimes_k F \cong \prod A \otimes_C F,$ where the product is taken over these embeddings (which are numbered by the isotypical components $W_{i'}$ of $W \otimes_k F$ over $F$) and $A \otimes_C F \cong End_{g \otimes_k F}(W_{i'}).$ Moreover, the projection $A \otimes_k F \to A \otimes_C F$ is given by annihilating endomorphisms between irreducible subrepresentations of isotypical components $W_{i'}$ different from $W_{i''}$. More explicitly the subfield $C \subset F$ under the embedding corresponding to an isotypical component $W_{i'}$ is the fixed subfield of the subgroup $S'' \subset S$ consisting of those $g \in S$ which preserve the isotypical component: $g(W_{i'}) = W_{i'}.$ Let us choose one such embedding $C \to F$ (which corresponds to a choice of an isotypical
component $W'_i$ of $W \otimes_k F$).

Let us denote by $(c_{q,g})$ the inverse matrix of the matrix $(g(e_q))$. Then \( r^p_{\alpha,\beta} = \sum_q c_{q,Id} r^p_{\alpha,\beta} \) and for any $g \in S' = \text{Gal}(F/C) \subset S = \text{Gal}(F/k)$ if we denote by $\phi_g : A \otimes_C F \rightarrow A \otimes_C F$ the conjugation by $g$: $a \otimes f \mapsto a \otimes g(f)$, then

\[
\phi_g(r_{(\alpha'),(\beta')}) = g \circ r_{(\alpha'),(\beta')} = r_{g(\alpha'),g(\beta')},
\]

where the action of $S' \subset S$ on upper indices $i$ (which number embeddings $\sigma_i : L \hookrightarrow F$) coincides with its action on left cosets $S/\bar{H}$, where $\bar{H} = \{ g \in S \mid g|_{\sigma_i(L)} = Id_{\sigma_i(L)} \}$ and the action of $g \in S' \subset S$ on indices $\alpha = (\alpha_1, ..., \alpha_l)$ is given by the rule $g(\alpha) = (c_1(g) \cdot \alpha_1, ..., c_l(g) \cdot \alpha_l)$, where $c_i(g) \in \{ \pm 1 \}$ and $g(f_{\alpha_1, ..., \alpha_l}) = f_{c_1(g) \cdot \alpha_1, ..., c_l(g) \cdot \alpha_l}$.

Hence the matrix of $m(g) \in GL(W'_i)$ is such that

\[
m(g) \cdot E_{i,j} \cdot m(g)^{-1} = \phi_g(E_{i,j}) = \left( \prod_{i=1}^r g \left( \lambda_{g(\alpha^i),g(\beta^i)} \right) \right) \cdot E_{g(i),g(j)},
\]

where $E_{i,j}$ denotes a matrix from $\text{Mat}(F) \cong \text{End}_{g \otimes_k F}(W'_i)$ corresponding to $r_{(\alpha^i),(\beta^i)}$, i.e. up to a scalar multiple conjugation by $m(g)$ acts on matrices as the (same) permutation of columns and rows induced by $g$ on indices $(\alpha^i_j)$.

Then the element of $H^2(S', F^*)$ corresponding to the central division algebra $D_0 = \text{End}_g(W_0)$ (over $C$) is the class of a 2-cocycle $\lambda : S' \times S' \rightarrow F^* \cong F^* \cdot Id \subset \text{Mat}(F)$, $(g_1, g_2) \mapsto m(g_1g_2) \cdot (g_1(m(g_2)))^{-1} \cdot m(g_1)^{-1}$ [8], [5].

### 6.3 Case of the CM-field.

Let $E = E_0(\theta), \theta^2 \in E_0 = L$ be a CM-field and $m = \dim_E V$. We saw above how to construct a primary representation $W = (U_i)^S$ of $\mathfrak{g}$ over $k = \mathbb{Q}$, which contains the irreducible representation $\rho^{\alpha_1}_{i_1} \boxtimes ... \boxtimes \rho^{\alpha_r}_{i_r}$ of $\mathfrak{g} \otimes_k F \cong \bigoplus_{i=1}^r \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m,F)^{\otimes r}$ after extending scalars to $F$ (as well as its Galois conjugates). Here $\alpha_i \in \{ \pm \}, 1 \leq j_i \leq m$ and

\[
\rho^{\alpha_i}_{j_i} : \mathfrak{gl}(m,F) \rightarrow \text{End}_F(\bigotimes_{\sigma_i}^\mu (V \otimes_{E,\alpha_i,\sigma} F) \otimes_F F)
\]

is the exterior product representation twisted by $D^{\alpha_i/2}$, where $\pm \sigma : E \rightarrow F$ are the two embeddings extending $\sigma : L \rightarrow F$. This means that $W \cong W_0^{\otimes \mu}$, where $W_0$ is an irreducible representation of $\mathfrak{g}$ over $k$, $W \otimes_k F \cong \bigoplus W_i$ and $W_i \cong \bigotimes_{\dim_F (\rho^{\alpha_i}_{j_i})}^{\dim_F (\rho^{\alpha_i}_{j_i})} \rho^{\alpha_i}_{j_i} \boxtimes ... \boxtimes \rho^{\alpha_i}_{j_i}$ are the isotypical components (over $F$). Since we are interested only in the endomorphism algebra $D_0 = \text{End}_g(W_0)$ which is a division algebra over $k$ (and over its center $C$) split over $F$, we can describe it by computing the Galois cohomology invariant of the central simple algebra $A = \text{End}_g(W) \cong \text{Mat}_{\mu \times \mu}(D_0)$ (over $C$), i.e. its Brauer invariant in $\text{Br}(F/C) \cong H^2(S', F^*)$, where $S' = \text{Gal}(F/C)$. Then $\mu = \frac{\deg(A)}{\deg(D_0)} = \frac{\alpha_{j_1, ..., j_r}}{\deg(D_0)}$. 

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Our computation is analogous to the case of a totally real field considered above.

Consider $F$-linear homomorphisms

$$r_{\alpha, \beta} : \wedge^p_F(V \otimes_{E, \alpha} F) \otimes F \to \wedge^*_{F}(V \otimes_{E, \beta} F) \otimes F, \xi \mapsto (\tau_s)^{p(\alpha, \beta)}(\xi),$$

where $P(-1, +1) = 1$, $P(+1, -1) = -1$, $P(\alpha, \alpha) = 0$ and $\tau_s = \oplus_p \tau_p$ is the direct sum of isomorphisms of $\mathfrak{gl}(m, F)$-modules

$$\wedge^p_{F}(V \otimes_{E, \alpha} F) \otimes F \to \wedge^{m-p}_{F}(V \otimes_{E, \alpha} F) \otimes F \rightarrow D^{-1/2}$$

introduced above. Then $r_{\alpha, \beta}$ induces an isomorphism

$$\wedge^p_{F}(V \otimes_{E, \alpha} F) \otimes F \to \wedge^*_{F}(V \otimes_{E, \alpha} F) \otimes F$$

which commutes with the action of $\mathfrak{u}(\Phi) \otimes_{L} F \cong \mathfrak{gl}(m, F)$. Without mentioning this explicitly, we will be restricting all our endomorphisms to the factors of $(\wedge^p_{F}(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \otimes \cdots \otimes F (\wedge^p_{F}(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F))$ contributing to an isotypical component $W_i \subset (\wedge^p_{F}(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \otimes \cdots \otimes F (\wedge^p_{F}(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F))$.

Then we construct endomorphisms

$$r_{(\alpha^i), (\beta^i)} = r^1_{\alpha^i, \beta^i} \circ \cdots \circ r^{r^i}_{\alpha^i, \beta^i} : (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \to (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \to \cdots$$

$$\cdots \otimes F (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \to (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \to \cdots$$

which commute with $\mathfrak{g} \otimes_{k} F$, where $(\alpha^i) = (\alpha^1, \ldots, \alpha^r)$, $(\beta^i) = (\beta^1, \ldots, \beta^r)$ and

$$r^p_{\alpha^p, \beta^p} = 1 \otimes F \cdots \otimes F (r^p_{\alpha^p, \beta^p}) \otimes F \cdots \otimes F (r^p_{\alpha^p, \beta^p}) \otimes F \to \cdots$$

$$\cdots \otimes F (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \to (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \cdots$$

(with 1 outside of the $p$-th spot).

As in the case of a totally real field $E$, $F$-algebra $\text{End}_{\mathfrak{g} \otimes_{k} F}(W \otimes_{k} F) = A \otimes_{k} F$ is generated by elements $r_{(\alpha^i), (\beta^i)}$ (more precisely, by those of them which correspond to the summands of $(\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \cdots \otimes F (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F))$ included in various isotypical components $W_i \otimes_{k} F \subset U_i \subset (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F)) \otimes F \otimes \cdots \otimes F (\wedge^*_F(V \otimes_{L, \sigma} F) \otimes F (E \otimes_{L, \sigma} F))$) or by elements $r^p_{\alpha^p, \beta^p}$, while $k$-algebra $A = \text{End}_{\mathfrak{g}}(W) = (A \otimes_{k} F)^{S}$ is generated by elements $r^p_{\alpha^p, \beta^p} = \sum_{g \in S} g(e_q) \cdot g \circ r^p_{\alpha^p, \beta^p}$, where $\{e_q\}$ is a basis of $F/k$.

The center $C$ of $A$ (and of $D_0$) can be computed exactly as in the case of a totally real field. In particular, field embeddings $C \to F$ correspond to the isotypical components $W'_i$ of $W \otimes_{k} F$ over $F$, $A \otimes_{C} F \cong \text{End}_{\mathfrak{g} \otimes_{C} F}(W'_i)$, the projection $\text{Proj} A \otimes_{C} F \cong \prod A \otimes_{C} F \to A \otimes_{C} F$ is given by annihilating endomorphisms between irreducible subrepresentations of isotypical
components $W_{i'}$ different from $W_i$ and the subfield $C \subset F$ under the embedding corresponding to an isotypical component $W_i$ is the fixed subfield of the subgroup $S' \subset S$ consisting of those $g \in S$ which preserve the isotypical component: $g(W_i) = W_{i'}$. Let us choose one such embedding $C \to F$ (which corresponds to a choice of an isotypical component $W_i$ of $W \otimes_k F$).

Let us denote by $(c_{q,g})$ the inverse matrix of the matrix $(g(e_q))$. Then $r^p_{\alpha,\beta} = \sum_q c_{q,1a}r^p_{\alpha,\beta}$ and for any $g \in S' = Gal(F/C) \subset S = Gal(F/k)$ if we denote by $\phi_g: A \otimes_C F \to A \otimes_C F$ the conjugation by $g$: $a \otimes f \mapsto a \otimes g(f)$, then

$$\phi_g(r_{(\alpha'),(\beta')}) = g \circ r_{(\alpha'),(\beta')} = \left( \prod_k \lambda_{\alpha^k,\beta^k}(g) \right) \cdot r_{g(\alpha'),g(\beta')}$$

where the action of $S' \subset S$ on upper indices $i$ (which number embeddings $\sigma_i: L \to F$) coincides with its action on the left cosets $S/H$, where $H = \{g \in S \mid g|_{\sigma_i(L)} = Id_{\sigma_i(L)}\}$ and moreover $g \in S' \subset S$ multiplies the $i$-th index $\alpha^i$ in the $r$-tuple $(\alpha^i) = (\alpha^1, ..., \alpha^r)$ by $g(\theta)/\theta = \pm 1$.

Here $\lambda_{\alpha^k,\beta^k}(g) \in F^*$ are suitable constants. In order to compute them, note that isomorphisms

$$\tau_p: \wedge^p_F(V \otimes_{E,\bar{\sigma}} F) \otimes_F (E \otimes_{E,\sigma} F) \to \wedge^p_F(V \otimes_{E,\bar{\sigma}} F) \otimes_F (E \otimes_{E,\sigma} F) \cong$$

$$\cong \wedge^p_F((V \otimes_{E,\bar{\sigma}} F)^*) \otimes_F (E \otimes_{E,\sigma} F)^* \to \wedge^{m-p}_F(V \otimes_{E,\bar{\sigma}} F) \otimes_F (E \otimes_{E,\sigma} F)$$

(where the first arrow is the isomorphism determined by the matrix of $\Phi^{-1}$) are defined over $E$. If we assume that the isomorphism $\wedge^p_E(V)^* \to \wedge^{m-p}_E(V) \otimes E$ is defined via the pairing

$$\wedge^p_E(V) \otimes_E \wedge^{m-p}_E(V) \to \wedge^p_E(V) \cong E, \ x \otimes y \mapsto x \wedge y,$$

then we find that $\lambda_{\alpha^k,\beta^k}(g) = 1$, if $g(\theta) = \theta$ or $\alpha^k = \beta^k$ and $\lambda_{\alpha^k,\beta^k}(g) = (-1)^{p(m-p)} \cdot (g(\sigma_k(disc(\Phi))))^{-p(\alpha^k,\beta^k)}$ otherwise.

Hence the matrix of $m(g) \in GL(W_{i'})$ is such that

$$m(g) \cdot E_{i,j} \cdot m(g)^{-1} = \phi_g(E_{i,j}) = \left( \prod_k \lambda_{\alpha^k,\beta^k}(g) \right) E_{g(i),g(j)},$$

where $E_{i,j}$ denotes a matrix from $Mat(F) \cong End_{\Phi \otimes k F}(W_{i'})$ corresponding to $r_{(\alpha'),(\beta')}$, i.e. conjugation by $m(g)$ acts on matrices up to a constant as the (same) permutation of columns and rows induced by $g$ on indices $(\alpha^i)$.

Then the element of $H^2(S', F^*)$ corresponding to the central division algebra $D_0 = End_{\Phi}(W_0)$ (over $C$) is the class of a 2-cocycle $\lambda: S' \times S' \to F^* \cong F^* \cdot Id \subset Mat(F)$,

$$(g_1, g_2) \mapsto m(g_1 g_2) \cdot (g_1(m(g_2)))^{-1} \cdot m(g_1)^{-1} \text{ [8], [5].}$$

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7 Example.

Let \( k = \mathbb{Q} \), \( r = 3 \) and \( 5 \leq m \leq 6 \). Let \( \rho < 0 \) be the negative root of the cubic polynomial \( f(t) = t^3 - 3t + 1 \). Then \( \frac{1}{\sqrt{\rho}} \) and \( 1 - \frac{1}{\rho} \) are the other two roots of \( f(t) \) and \( E = L = k(\rho) \) is a totally real cyclic cubic Galois number field [6].

Let \( \Phi = -\rho \cdot X_1^2 - \rho \cdot X_2^2 - X_3^2 - \cdots - X_n^2 \). Then by [9] there is a K3 surface \( X \) such that \( \text{End}_{Hdg}(V) \cong E \) (where \( V \) is the \( \mathbb{Q} \)-lattice of transcendental cycles on \( X \)), \( \dim_E V = m \) and \( \Phi: V \otimes_E V \to E \) is the quadratic form constructed in [15].

Let \( F = k(\sqrt[3]{\rho}, \sqrt[3]{1 - \frac{1}{\rho}}) \) be our choice of a splitting field. Note that \( L \subset F \) and \( \sqrt{-1} = \sqrt{\rho} \cdot \sqrt[3]{1 - \frac{1}{\rho}} \in F \). Then

\[
S = \text{Gal}(F/k) \cong (\mathbb{Z}/2\mathbb{Z})^\oplus 3 \rtimes \mathbb{Z}/3\mathbb{Z}
\]

is a nonabelian extension of \( \mathbb{Z}/3\mathbb{Z} \cong \text{Gal}(L/k) \) with generator \( g \) by \((\mathbb{Z}/2\mathbb{Z})^\oplus 3\) with generators \( h_1, h_2, h_3 \), where \( g \) acts on the generators \((h_1, h_2, h_3)\) by the permutation \((123)\).

We also denote by \( g \) the element of \( S \) such that \( g(\sqrt{\rho}) = \sqrt[3]{1 - \frac{1}{\rho}}, g(\sqrt[3]{\rho}) = \sqrt[3]{1 + \frac{1}{\rho}} \).

There are 3 field embeddings \( L \hookrightarrow F \): \( \sigma_1 = Id \), \( \sigma_2 = g|_L \) and \( \sigma_3 = g^2|_L \). Then \( \sqrt{\sigma_1(d_1)} = \sqrt{\sigma_1(d_2)} = \sqrt{-1} \cdot \sqrt[3]{\rho}, \sqrt{\sigma_2(d_1)} = \sqrt{\sigma_2(d_2)} = \sqrt{-1} \cdot \sqrt[3]{1 - \frac{1}{\rho}}, \sqrt{\sigma_3(d_1)} = \sqrt{\sigma_3(d_2)} = -\sqrt{-1} \cdot \sqrt[3]{1 - \frac{1}{\rho}} \) for any \( i = 1, 2, 3 \), \( 1 \leq j \leq l = [m \over 2] \). Hence \( \otimes_{L, \sigma_1} \Gamma_1 = \otimes_{L, \sigma_1} \Gamma_2 = \sqrt{-1} \cdot \sqrt[3]{\rho}, \otimes_{L, \sigma_2} \Gamma_1 = \otimes_{L, \sigma_2} \Gamma_2 = \sqrt{-1} \cdot \sqrt[3]{1 - \frac{1}{\rho}}, \otimes_{L, \sigma_3} \Gamma_3 = \sqrt{-1} \) for all \( i \) (if \( m = 6 \)).

(1) Let us consider first the case \( m = 5 \). The root system is of type \( B_2 \): \( R_0 = \{ \pm e_p, \pm e_p \pm e_q \mid p, q = 1, 2 \} \) with basis \( B_0 = \{ e_1 - e_2, e_2 \} \). Hence \( B_1 = \{ e_1 \otimes_{L, \sigma_1} \Gamma_1 - e_2 \otimes_{L, \sigma_1} \Gamma_2, e_2 \otimes_{L, \sigma_1} \Gamma_2 \} \), \( 1 \leq i \leq 3 \). The restriction of the spin representation of \( \mathfrak{s}\mathfrak{o}(\Phi) \otimes_k F \) in \( C^+(V \otimes_k F) \) to \( \mathfrak{g} \otimes_k F = \text{Res}_{L/k}(\mathfrak{s}\mathfrak{o}(\Phi)) \otimes_k F \) is isomorphic over \( F \) to \( 2^8 \) copies of the exterior tensor product \( \rho^0 \otimes \rho^0 \otimes \rho^0 \) of the irreducible spin representation of \( \mathfrak{s}\mathfrak{o}(\Phi) \otimes_L F \). Hence over \( k = \mathbb{Q} \) the restriction of the spin representation of \( \mathfrak{s}\mathfrak{o}(\Phi) \) in \( C^+(V) \) to \( \mathfrak{g} = \text{Res}_{L/k}(\mathfrak{s}\mathfrak{o}(\Phi)) \subset \mathfrak{s}\mathfrak{o}(\Phi) \) is one single irreducible representation with multiplicity \( \mu \) which splits over \( F \) into \( \frac{2^8}{\mu} \) copies of \( \rho^0 \otimes \rho^0 \otimes \rho^0 \); \( C^+(V) \cong U^{\otimes \mu} \).

In order to estimate \( \frac{2^8}{\mu} \) (which divides \( n_0 \)), let us consider

\[
f_{1, \ldots, 1, l} = f_1 \cdots f_l \cdot (1 + f_0) = q \cdot \prod_{i=1}^{l} \left( e_i + \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right) \cdot \left( 1 + \frac{1}{\sqrt{d_{l+1}}} \cdot e_{l+1} \right)
\]
(we use notation as above), where \( q \in F \) is such that \( \sigma(q) = \pm q \) for any \( \sigma \in S = Gal(F/k) \). In our case

\[
f_{1,\ldots,1,1} = q \cdot (e_1 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_3) \cdot (e_2 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_4) \cdot (1 - \sqrt{-1} \cdot e_3).
\]

Hence the stabilizer of (the line in \( C(V \otimes L F) \) generated by) \( f_{1,\ldots,1,1} \) consists of the elements \( g_k \), i.e. has order 3. Since \( Gal(F/k) \) has 24 elements total, we find that \( n_0 = 8 \). Hence either \( \frac{2\sigma}{\rho} = 1 \) or \( \frac{2\sigma}{\rho} = 2 \) or \( \frac{2\sigma}{\rho} = 4 \) or \( \frac{2\sigma}{\rho} = 8 \). In the first case, \( \rho^0 \otimes \rho^1 \otimes \rho^0 \) is already defined over \( \mathbb{Q} \) and \( \mu = 2^8 \), while in the other cases \( \mu = 2^7, \mu = 2^6 \) and \( \mu = 2^5 \) respectively.

Hence in this case \( End(KS(X))_Q \cong Mat_{\mu \times \mu}(D) \), where \( D = End_\sigma(U) \) is a division algebra. Let us check that \( D \cong \mathbb{Q} \).

Let us compute the cohomological invariant of \( D \). In our case

\[
W \otimes_k F = V(1,1,1) \oplus V(1,1',1') \oplus V(1,1',1) \oplus V(1',1',1) \oplus V(1',2,2) \oplus V(2,2,2) \oplus V(2,2',2) \oplus V(2',2',2'),
\]

where \( V(p_1,p_2,p_3) = S_{p_1}^1 \otimes_F S_{p_2}^2 \otimes_F S_{p_3}^3 \) in the notation of Section 5.2 and the values 1, 1', 2, 2' of \( p_i \) correspond to the indices \((\alpha_1, \alpha_2, \gamma)\) of ideals \( I_{\alpha_1,\alpha_2,\gamma} \) as follows: \( 1 = (+ + +), 1' = (- - -), 2 = (- + +), 2' = (+ + -) \).

Let us denote \( \tilde{1} = 2, \tilde{1}' = 2', \tilde{2} = 1, \tilde{2}' = 1' \) and \( \bar{1} = 2', \bar{1}' = 2, \bar{2} = 1', \bar{2}' = 1. \) Then \( g(V(p_1,p_2,p_3)) = V(p_1,p_1,p_2) \) and \( h_i(V(p_1,p_2,p_3)) = V(q_i,q_2,q_3) \), where \( q_i = \tilde{p}_i \) and \( q_j = \bar{p}_j \) for \( j \neq i \).

Let us denote \( a = (1,1',1'), b = (1',1,1'), c = (1',1',1), d = (1,1,1), p = (2',2,2), q = (2,2',2), r = (2,2,2'), s = (2',2',2'). \) Then using formulas from Section 6 we can choose coefficients \( \lambda_{\alpha,\beta} = \prod_{i=1}^r \lambda_{\alpha^i,\beta^i} \in F^\times \) as follows:

- \( \lambda_{\alpha,\beta} = 1 \) for \( (\alpha, \beta) \in \{(d, -), (s, -), (a, a), (b, b), (c, c), (p, p), (q, q), (r, r)\} \),
- \( \lambda_{\alpha,\beta} = 1 \) for \( (\alpha, \beta) \in \{(b, q), (a, p), (c, r), (q, b), (p, a), (r, c)\} \),
- \( \lambda_{\alpha,\beta} = c_1 \) for \( (\alpha, \beta) \in \{(b, a), (b, p), (c, a), (c, p), (q, a), (q, p), (r, a), (r, p)\} \),
- \( \lambda_{\alpha,\beta} = c_2 \) for \( (\alpha, \beta) \in \{(a, b), (a, q), (c, b), (c, q), (p, b), (p, q), (r, b), (r, q)\} \),
- \( \lambda_{\alpha,\beta} = c_3 \) for \( (\alpha, \beta) \in \{(b, c), (b, r), (a, c), (a, r), (p, c), (p, r), (q, c), (q, r)\} \),
- \( \lambda_{\alpha,\beta} = c_1 c_2 \) for \( (\alpha, \beta) \in \{(c, d), (c, s), (r, d), (r, s)\} \),
- \( \lambda_{\alpha,\beta} = c_1 c_3 \) for \( (\alpha, \beta) \in \{(b, d), (b, s), (q, d), (q, s)\} \),
- \( \lambda_{\alpha,\beta} = c_2 c_3 \) for \( (\alpha, \beta) \in \{(a, d), (a, s), (p, d), (p, s)\} \).

Here we denoted \( c_i \sigma_i \left( \frac{-1}{\Phi(f_{1,1,1})} \Phi(f_{2,2,2}) \right) = \sigma_i \left( \frac{-1}{4\rho^2} \right) \).

Then in the formulas in Section 6 we can take:
• \( m(g) = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix} \) is an 8 \times 8 matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} : (dabcspqr) \),

• \( m(h_1) = \begin{pmatrix} 1 & X_1^{-1} \\ c_{2c_1} & 0 \end{pmatrix} \) is an 8 \times 8 matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} : (dabcpsrq) \),

• \( m(h_2) = \begin{pmatrix} 1 & X_2^{-1} \\ c_{1c_2} & 0 \end{pmatrix} \) is an 8 \times 8 matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} : (dabcqrsp) \),

• \( m(h_3) = \begin{pmatrix} 1 & X_3^{-1} \\ c_{1c_2} & 0 \end{pmatrix} \) is an 8 \times 8 matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} : (dabcrqps) \),

• \( m(g^k \cdot h_1^{a_1} h_2^{a_2} h_3^{a_3}) = m(g)^k \cdot g^k (m(h_1)^{a_1} \cdot m(h_2)^{a_2} \cdot m(h_3)^{a_3}) \), where \( 0 \leq a_i \leq 1, k \geq 0 \).

Here we denoted \( G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \), \( X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2c_3 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_2 \end{pmatrix} \), \( X_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & c_1c_3 & 0 \\ 0 & 0 & 0 & c_1 \end{pmatrix} \), and \( X_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & c_1c_2 & 0 & 0 \end{pmatrix} \).

Note that \( m(h_i) \cdot m(h_j) = m(h_j) \cdot m(h_i) \), \( m(h_i)^2 = \frac{c_{i}}{c_{1c_2c_3}} \), \( m(g)^3 = 1 \) and \( m(gk_ig^{-1}) = m(g) \cdot g(m(h_i)) \cdot m(g)^{-1} \).

This implies that the class of \( D \) in \( H^2(S,F^*) \) is represented by the 2-cocycle \( \lambda : S \times S \to F^* \) such that \( \lambda(h_1^{a_1} h_2^{a_2} h_3^{a_3}, h_1^{b_1} h_2^{b_2} h_3^{b_3}) = (c_{2c_3})^{x_1} \cdot (c_{1c_3})^{x_2} \cdot (c_{1c_2})^{x_3} \) and \( \lambda(g^k h, g^l h') = g^{k+l}(\lambda(g^{-1}hg^l, h')) \), where \( 0 \leq a_i \leq 1, 0 \leq b_i \leq 1, x_i = 1 \) if \( a_i = b_i = 1 \) and 0 otherwise, and \( h, h' \) are elements of the subgroup \((\mathbb{Z}/2\mathbb{Z})^\oplus 3 \subset S \) generated by \( h_1, h_2, h_3 \).

Since \( c_{i}c_{j} = \left( \frac{1}{\prod_{\sigma_i,\sigma_j}} \right)^2 \) is a square in \( L^* \), we conclude that \( \lambda \) is a coboundary. Namely, the required morphism \( c : S \to F^* \) (whose coboundary is \( \lambda \)) can be defined as follows:

\[
c(g^k \cdot h_1^{a_1} h_2^{a_2} h_3^{a_3}) = g^k \left( (\sqrt{c_{2c_3}})^{a_1} \cdot (\sqrt{c_{1c_3}})^{a_2} \cdot (\sqrt{c_{1c_2}})^{a_3} \right),
\]

where \( 0 \leq a_i \leq 1, k \geq 0 \). Note that \( c(gk_ig^{-1}) = g(c(h_i)) \). So, the class of \( D \) in \( H^2(S,F^*) \) vanishes. Hence \( D \cong \mathbb{Q} \).

(2) Now let us consider the case \( m = 6 \). The root system is of type \( D_3 \): \( R_0 = \{ \pm \varepsilon_p \pm \varepsilon_q \mid p, q = 1, 2, 3 \} \) with basis \( B_0 = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_2 + \varepsilon_3 \} \). Hence \( B_1 = \)
\{e_1 \otimes_{L, \sigma} \Gamma_1 - e_2 \otimes_{L, \sigma} \Gamma_2, e_2 \otimes_{L, \sigma} \Gamma_2 - e_3 \otimes_{L, \sigma} \Gamma_3, e_2 \otimes_{L, \sigma} \Gamma_2 + e_3 \otimes_{L, \sigma} \Gamma_3\}, \ 1 \leq i \leq 3,$
and the Weyl group is generated by sign inversions in front of two of \(e_1, e_2, e_3\) and by all possible permutations of \(e_1, e_2, e_3\).

The restriction of the spin representation of \(\mathfrak{so}(\phi) \otimes_k F\) in \(C^+ (V \otimes_k F)\) to \(g \otimes_k F = Res_{L/k}(\mathfrak{so}(\Phi)) \otimes_k F\) is isomorphic over \(F\) to the sum of the exterior tensor products of semispin representations (in all possible combinations) each with multiplicity \(2^8\): \(C^+ (V \otimes_k F) \cong \bigoplus_{\alpha_1, \alpha_2, \alpha_3 \in \{\pm\}} 2^8 \cdot (\rho^{\alpha_1} \otimes \rho^{\alpha_2} \otimes \rho^{\alpha_3}).\) Hence the set \(\Omega\) of highest weights consists of the elements \(\omega_{\alpha_1, \alpha_2, \alpha_3} = \frac{1}{2} \cdot \sum_{i=1}^{3} (e_1 \otimes_{L, \sigma} \Gamma_1 + e_2 \otimes_{L, \sigma} \Gamma_2 + \alpha_1 \cdot e_3 \otimes_{L, \sigma} \Gamma_3)\) for various \(\alpha_i \in \{\pm 1\}\).

Note that \(g(\omega_{\alpha_1, \alpha_2, \alpha_3}) = \omega_{\alpha_3, \alpha_1, \alpha_2}\) and \(h_i(\omega_{\alpha_1, \alpha_2, \alpha_3}) = \omega_{-\alpha_1, -\alpha_2, -\alpha_3} \cdot \Omega = \Omega_1 \cup \Omega_2\) has two \(S\)-orbits: \(\Omega_1 = \{\omega_{+, ++}, \omega_{-, -}\}\) and \(\Omega_2 = \{\omega_{+, +, -}, \omega_{+, -, +}, \omega_{-, +, -}, \omega_{-, -,-}\}\).

Hence over \(k = \mathbb{Q}\) we have: \(C^+ (V) \cong U^{\otimes \mu} \otimes V^{\otimes \nu}\) as \(g\)-modules, where \(U\) and \(V\) are not isomorphic as representations of \(g = Res_{L/k}(\mathfrak{so}(\Phi))\). \(U \otimes_k F\) splits into \(2^\mu\) copies of \(\rho^+ \otimes \rho^+ \otimes \rho^+\) and \(2^\nu\) copies of \(\rho^- \otimes \rho^- \otimes \rho^-\), while \(V \otimes_k F\) splits into \(2^\nu\) copies of \(\rho^{\alpha_1} \otimes \rho^{\alpha_2} \otimes \rho^{\alpha_3}\) with other \(\alpha_i\)'s.

In order to estimate multiplicities \(\mu\) and \(\nu\), let us consider
\[
 f_{1, \ldots, l} = f_1 \cdot \ldots \cdot f_l = q \cdot \prod_{i=1}^{l} \left( e_i + \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right)
\]
(we use notation introduced above), where \(q \in F\) is such that \(\sigma(q) = \pm q\) for any \(\sigma \in S = Gal(F/k)\). In our case
\[
 f_{1, \ldots, l} = q \cdot (e_1 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_6) \cdot (e_2 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_5) \cdot (e_3 + \sqrt{-1} \cdot e_4).
\]
Hence the stabilizer of (the line in \(C(V \otimes_L F)\) generated by) \(f_{1, \ldots, l}\) consists of the elements \(g^k\). Since the stabilizer of \(\omega_{+, +, +} \in \Omega\) as a subgroup of \(S\) is generated by elements \(g, h_1 h_2, h_1 h_3, h_2 h_3\), we conclude that \(n_{+, +, +} = 4\). Since the stabilizer of \(\omega_{+, +, -} \in \Omega\) has 4 elements: \(Id\) and \(h_1 h_2, h_1 h_3, h_2 h_3\), we conclude that \(n_{+, +, -} = 4\) as well. The same computation as in the case \(m = 5\) above shows that \(\frac{2^\mu}{\nu} = \frac{2^8}{8} = 1\), i.e. \(\mu = \nu = 256\), and the division algebras \(D_1 = \text{End}_g(U)\) and \(D_2 = \text{End}_g(V)\) are fields, i.e. coincide with their centers.

According to Section 6.2, the center \(C_1\) of \(D_1\) is the subfield of \(F\) fixed by the stabilizer of \(\omega_{-, -}\) \(\in \Omega\), i.e. \(D_1 = C_1 \cong k(\sqrt{-1})\). Similarly, the center \(C_2\) of \(D_2\) is the subfield of \(F\) fixed by the stabilizer of \(\omega_{-, +}\) \(\in \Omega\), i.e. \(D_2 = C_2 \cong k(\sqrt{-1}, \rho)\).

So, in this example \(\text{End}(KS(X))_Q \cong \text{Mat}_{256 \times 256}(Q(\sqrt{-1})) \times \text{Mat}_{256 \times 256}(Q(\sqrt{-1}, \rho))\).

(3) Let us modify the first example above. Consider the same number \(\rho\) and the same totally real cubic field \(E = L = k(\rho)\), but a different quadratic form
\[
\Phi = -(a + \rho) \cdot X_1^2 - (a + \rho) \cdot X_2^2 - X_3^2 - X_4^2 - X_5^2.
\]
where $a$ is a fixed rational number between 0 and $-\rho$: $0 < a < -\rho$. As above, these quadratic form and totally real field correspond to a $K3$ surface $X$ ([9]). Assume that $1 + 3a - a^3 > 0$ is not a square of a rational number.

Let $F = k \left( \sqrt{a + \rho}, \sqrt{a + \frac{1}{1 - \rho}}, \sqrt{a + 1 - \frac{1}{\rho}}, \sqrt{-1} \right)$ be our choice of a splitting field. Note that $L \subset F$ and $\sqrt{a + \rho} \cdot \sqrt{a + \frac{1}{1 - \rho}} \cdot \sqrt{a + 1 - \frac{1}{\rho}} = \sqrt{-1} - 3a + a^3$. Then

$$S = \text{Gal}(F/k) \cong \mathbb{Z}/2\mathbb{Z} \oplus G,$$

where $G$ is the group isomorphic to the Galois group of the splitting field from the first example above, i.e. $G$ is a noncommutative group extension of $\mathbb{Z}/3\mathbb{Z} \cong \text{Gal}(L/k)$ by $(\mathbb{Z}/2\mathbb{Z})^\oplus 3$. Let $g$ be a generator of $\mathbb{Z}/3\mathbb{Z}$ such that $g(\sqrt{a + \rho}) = \sqrt{a + \frac{1}{1 - \rho}}, g(\sqrt{a + 1 - \frac{1}{\rho}}) = \sqrt{a + 1 - \frac{1}{\rho}}, g(\sqrt{-1}) = \sqrt{-1}$. Let $h_1, h_2, h_3$ be the generators of $(\mathbb{Z}/2\mathbb{Z})^\oplus 3$ and $h_0$ be the generator of the first factor $\mathbb{Z}/2\mathbb{Z}$ in $S$ above such that each $h_i$, $0 \leq i \leq 3$ multiplies by $-1$ the $i$-th square root among $\sqrt{-1}, \sqrt{a + \rho}, \sqrt{a + \frac{1}{1 - \rho}}, \sqrt{a + 1 - \frac{1}{\rho}}$ and does not change the others. We also assume that $h_i|_L = Id$, $0 \leq i \leq 3$.

There are 3 field embeddings $L \hookrightarrow F$: $\sigma_1 = Id$, $\sigma_2 = g|_L$ and $\sigma_3 = g^2|_L$. Then $\sqrt{\sigma_1(d_1)} = \sqrt{\sigma_1(d_2)} = \sqrt{1} \cdot \sqrt{a + \rho}$, $\sqrt{\sigma_2(d_1)} = \sqrt{\sigma_2(d_2)} = \sqrt{1} \cdot \sqrt{a + \frac{1}{1 - \rho}}$, $\sqrt{\sigma_3(d_1)} = \sqrt{\sigma_3(d_2)} = \sqrt{1} \cdot \sqrt{a + 1 - \frac{1}{\rho}}$, $\sqrt{\sigma_1(d_3)} = \sqrt{\sigma_2(d_3)} = \sqrt{1} \cdot \sqrt{a + \rho}$, $\sqrt{\sigma_3(d_3)}$ $= \sqrt{1} \cdot \sqrt{a + 1 - \frac{1}{\rho}}$ for any $i = 1, 2, 3$. Hence $\otimes_{L,\sigma_1} \Gamma_1 = \otimes_{L,\sigma_1} \Gamma_2 = \sqrt{1} \cdot \sqrt{a + \rho}$, $\otimes_{L,\sigma_2} \Gamma_1 = \otimes_{L,\sigma_2} \Gamma_2 = \sqrt{1} \cdot \sqrt{a + 1 - \frac{1}{\rho}}$.

As in the first example above, the root system is of type $B_3$: $R_0 = \{ \pm \epsilon_i, \pm \epsilon_i \pm \epsilon_j | p, q = 1, 2 \}$ with basis $B_0 = \{ \epsilon_1 - \epsilon_2, \epsilon_2 \}$. Hence $B_1 = \{ \epsilon_1 \otimes_{L,\sigma_1} \Gamma_1 - \epsilon_2 \otimes_{L,\sigma_1} \Gamma_2, \epsilon_2 \otimes_{L,\sigma_1} \Gamma_2 \}$, $1 \leq i \leq 3$. The restriction of the spin representation of $\mathfrak{so}(\phi) \otimes_k F$ in $C^+(V \otimes_k F)$ to $\mathfrak{g} \otimes_k F = \text{Res}_L/k(\mathfrak{so}(\Phi)) \otimes_k F$ is isomorphic over $F$ to $2^8$ copies of the exterior tensor product $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$ of the irreducible spin representation of $\mathfrak{so}(\Phi) \otimes L F$. Hence over $k = \mathbb{Q}$ the restriction of the spin representation of $\mathfrak{so}(\phi)$ in $C^+(V)$ to $\mathfrak{g} = \text{Res}_L/k(\mathfrak{so}(\Phi)) \subset \mathfrak{so}(\phi)$ is one single irreducible representation with multiplicity $\mu$ which splits over $F$ into $\frac{2^8}{\mu}$ copies of $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$: $C^+(V) \cong U^\oplus \mu$.

In order to estimate $\frac{2^8}{\mu}$ (which divides $n_0$), let us consider

$$f_{1,...,1,1} = f_1 \cdot \ldots \cdot f_1 \cdot (1 + f_0) = q \prod_{i=1}^l \left( e_i + \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right) \cdot \left( 1 + \frac{1}{\sqrt{d_{l+1}}} \cdot e_{l+1} \right)$$

(we use notation as above), where $q \in F$ is such that $\sigma(q) = \pm q$ for any $\sigma \in S = \text{Gal}(F/k)$. In our case

$$f_{1,...,1,1} = q \cdot (e_1 + \sqrt{1} \cdot \sqrt{a + \rho} \cdot e_5) \cdot (e_2 + \sqrt{1} \cdot \sqrt{a + \rho} \cdot e_4) \cdot (1 - \sqrt{-1} \cdot e_3).$$
Hence the stabilizer of (the line in \(C(V \otimes L F)\) generated by) \(f_{1,\ldots,1,1}\) consists of the elements 
\(g^k\), i.e. has order 3. Since \(Gal(F/k)\) has 48 elements total, we find that \(n_0 = 16\). Hence either \(\frac{2^8}{\mu} = 1\) or \(\frac{2^8}{\mu} = 2\) or \(\frac{2^8}{\mu} = 4\) or \(\frac{2^8}{\mu} = 8\) or \(\frac{2^8}{\mu} = 16\). In the first case, \(\rho^0 \otimes \rho^0 \otimes \rho^0\) is already defined over \(\mathbb{Q}\) and \(\mu = 2^8\), while in the other cases \(\mu = 2^7, \mu = 2^6, \mu = 2^5\) and \(\mu = 2^4\) respectively.

Hence in this case \(End(KS(X))_\mathbb{Q} \cong \text{Mat}_{\mu \times \mu}(D)\), where \(D = \text{End}_g(U)\) is a division algebra. Let us compute the cohomological invariant of \(D\). In our case

\[
W \otimes_k F = V_{(1,1,1)} \oplus V_{(1',1,1)} \oplus V_{(1,1',1)} \oplus V_{(1,1,1')} \oplus V_{(1',1,1')} \oplus V_{(1',1',1)} \oplus \\
\oplus V_{(2,2,2)} \oplus V_{(2',2,2)} \oplus V_{(2,2',2)} \oplus V_{(2,2',2')} \oplus V_{(2',2',2')} \oplus V_{(2',2',2')},
\]

where \(V_{(p_1,p_2,p_3)} = S^1_{p_1} \otimes_F S^2_{p_2} \otimes_F S^3_{p_3}\) in the notation of Section 5.2 and the values 1, 1', 2, 2' of \(p_i\) correspond to the indices \((\alpha_1, \alpha_2, \gamma)\) of ideals \(I_{\alpha_1, \alpha_2, \gamma}\) as follows: \(1 = (+ + +)\), \(1' = (- - +)\), 
\(2 = (+ + -)\), \(2' = (- - -)\).

Let us denote \(\bar{I} = 1', \bar{I}' = 1, \bar{2} = 2', \bar{2}' = 2\) and \(\bar{I} = 2', \bar{I}' = 2, \bar{2} = 1', \bar{2}' = 1\). Then 
\(g(V_{(p_1,p_2,p_3)}) = V_{(p_3,p_1,p_2)}\), \(h_0(V_{(p_1,p_2,p_3)}) = V_{(p_1,p_2,p_3)}\) and \(h_i(V_{(p_1,p_2,p_3)}) = V_{(q_1,q_2,q_3)}, 1 \leq i \leq 3\), where \(q_i = p_i\) and \(q_j = p_j\) for \(j \neq i\).

Let us denote \(a = (1', 1, 1), b = (1, 1', 1), c = (1, 1, 1'), d = (1, 1, 1'), a' = (1', 1, 1'), b' = (1', 1, 1'), c' = (1', 1', 1'), d' = (1', 1', 1'), p = (2', 2, 2), q = (2, 2', 2), r = (2, 2', 2'), s = (2, 2, 2), \(p' = (2', 2', 2'), q' = (2', 2, 2'), r' = (2', 2', 2'), s' = (2', 2', 2')\). Consider the set of indices \(T = \{d,a,b,c,d',a',b',c',s,p,q,r,s',p',q',r',s'\}\) and the morphism \(t: T \to T, s \mapsto d, p \mapsto a, q \mapsto b, r \mapsto c, s' \mapsto d', p' \mapsto a', q' \mapsto b', r' \mapsto c'\) and \(x \mapsto x\) for all other \(x \in T\).

Then using formulas from Section 6 we can choose coefficients \(\lambda_{\alpha, \beta} = \prod_{i=1}^{\alpha*} \lambda_{\alpha^i, \beta^i} \in F^*\) as follows:

- \(\lambda_{d,x} = \lambda_{s,x} = \lambda_{x,x} = 1\), \(\lambda_{d,d} = \lambda_{s,s} = \lambda_{x,x} = \lambda_{t(x),t(y)}\) for any \(x, y \in T\),
- \(\lambda_{\alpha, \beta} = 1\) for \((\alpha, \beta) \in \{(a, d'), (a, b'), (a, c'), (b, d'), (b, a'), (b, c')\}\),
- \(\lambda_{\alpha, \beta} = 1\) for \((\alpha, \beta) \in \{(c, d'), (c, b'), (c, a'), (a', d'), (b', d'), (c', d')\}\),
- \(\lambda_{\alpha, \beta} = c_1\) for \((\alpha, \beta) \in \{(a, a'), (a, d), (a, b), (a, c), (d, a'), (b', c), (b', a'), (c', b), (c', a')\}\),
- \(\lambda_{\alpha, \beta} = c_2\) for \((\alpha, \beta) \in \{(b, b'), (b, d), (b, a), (b, c), (d', b'), (a', c), (a', b'), (c', a), (c', b')\}\),
- \(\lambda_{\alpha, \beta} = c_3\) for \((\alpha, \beta) \in \{(c, c'), (c, d), (c, b), (c, a), (d', c'), (a', c'), (a', b), (a', c'), (b', a), (b', c')\}\),
- \(\lambda_{\alpha, \beta} = c_1 c_2\) for \((\alpha, \beta) \in \{(d', c), (c', d), (c', d)\}\),
- \(\lambda_{\alpha, \beta} = c_1 c_3\) for \((\alpha, \beta) \in \{(d', b), (b', b), (b', b)\}\),
- \(\lambda_{\alpha, \beta} = c_2 c_3\) for \((\alpha, \beta) \in \{(d', a), (a', a), (a', d')\}\).
Here we denoted \( c_i = \sigma_i \left( \frac{-1}{\Phi(f_j,f_{j-1}) \Phi(f_{j-2},f_{j-3})} \right) = \sigma_i \left( \frac{-1}{4(a+b)^2} \right) \).

Then in the formulas in Section 6 we can take:

- \( m(g) = \begin{pmatrix} G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G \end{pmatrix} \) is a \( 16 \times 16 \) matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} \): \( (dabcd'a'b'c'spqr's'p'q'r') \),

- \( m(h_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \) is a \( 16 \times 16 \) matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} \): \( (da'bcad'c'b'spqr's'q'r') \),

- \( m(h_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \) is a \( 16 \times 16 \) matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} \): \( (dab'c'bc'd'spqr's'q'r') \),

- \( m(h_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \) is a \( 16 \times 16 \) matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} \): \( (dabc'e'b'c'd'spqr's'q'r') \),

- \( m(h_0) = \begin{pmatrix} 0 & X_0^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X_0^{-1} & 0 \\ 0 & 0 & 0 & X_0^{-1} \end{pmatrix} \) is a \( 16 \times 16 \) matrix whose rows and columns are numbered according to the following sequence of indices of \( V_{(p_1,p_2,p_3)} \): \( (dabc's'p'q'r'd'a'b'c'spqr) \),

\[ m(g^k \cdot h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}) = m(g)^k \cdot g^k (m(h_0)^{a_0} \cdot m(h_1)^{a_1} \cdot m(h_2)^{a_2} \cdot m(h_3)^{a_3}) \], where \( 0 \leq a_i \leq 1 \), \( k \geq 0 \).

Here we denoted \( G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \), \( 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) (in the definitions of \( m(h_i) \))

and \( X_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \end{pmatrix} \).

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Note that $m(h_i) \cdot m(h_j) = m(h_j) \cdot m(h_i)$, $m(h_i)^2 = \frac{1}{c_i}$, $1 \leq i \leq 3$, $m(h_0)^2 = \frac{1}{c_1 c_2 c_3}$, $m(g)^3 = 1$ and $m(g h_i g^{-1}) = m(g) \cdot m(h_i) \cdot m(g)^{-1}$.

This implies that the class of $D$ in $H^2(S, F^*)$ is represented by the 2-cocycle $\lambda: S \times S \to F^*$ such that

$$\lambda(h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}, h_0^{b_0} h_1^{b_1} h_2^{b_2} h_3^{b_3}) = (c_1 c_2 c_3)^{x_0} \cdot (c_2 c_3)^{x_1} \cdot (c_1 c_3)^{x_2} \cdot (c_1 c_2)^{x_3}$$

and $\lambda(g^k h, g^l h') = g^{k+l}(\lambda(g^{-1} h g', h'))$, where $0 \leq a_i \leq 1$, $0 \leq b_i \leq 1$, $x_i = 1$ if $a_i = b_i = 1$ and 0 otherwise, and $h, h'$ are elements of the subgroup $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \subset S$ generated by $h_0, h_1, h_2, h_3$.

Let us multiply $\lambda$ by the inverse of the coboundary of the 1-cochain given by the morphism $c: S \to F^*$ such that

$$c(g^k \cdot h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}) = g^k ((\sqrt{c_1 c_2 c_3})^{a_0} \cdot (\sqrt{c_1})^{a_1} \cdot (\sqrt{c_2})^{a_2} \cdot (\sqrt{c_3})^{a_3}),$$

where $0 \leq a_i \leq 1$, $k \geq 0$. Note that $c(g h_i g^{-1}) = g(c(h_i))$.

This changes $\lambda$ to a 2-cocycle $\lambda': S \times S \to F^*$ such that

$$\lambda'(g^k \cdot h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}, g^l \cdot h_0^{b_0} h_1^{b_1} h_2^{b_2} h_3^{b_3}) = (-1)^{a_0 b_0 + b_1 + b_2 + b_3},$$

where $0 \leq a_i \leq 1$, $0 \leq b_i \leq 1$.

Let $H \subset S$ be the subgroup generated by $g, h_1 h_2, h_1 h_3, h_2 h_3$ and

$$F^H = k \left( \sqrt{-1}, \sqrt{(a + \rho)(a + \frac{1}{1 - \rho})(a + 1 - \frac{1}{\rho})} \right) = k(\sqrt{-1}, \sqrt{-1 - 3a + a^2})$$

be the corresponding fixed subfield of $F$. Denote the generators of $Gal(F^H/k) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ by $h_0$ and $h = h_1 h_2 h_3$.

We see that the class of $D$ in $H^2(S, F^*)$ is the image under the inflation homomorphism $H^2(Gal(F^H/k), (F^H)^*) \to H^2(S, F^*)$ of a class represented by the 2-cocycle $\lambda'': Gal(F^H/k) \times Gal(F^H/k) \to k(\sqrt{-1}, \sqrt{-1 - 3a + a^2})^*$ such that $\lambda''(h_0, h_0) = \lambda''(h_0, h) = -\lambda''(h, h_0) = -\lambda''(h, h) = -1$. Multiplying it by the coboundary of the 1-cochain given by the morphism $c: Gal(F^H/k) \to (F^H)^*$ such that $c(h) = c(h_0) = \sqrt{-1}$, $c(h h_0) = 1$, we obtain a 2-cocycle (also denoted by $\lambda''$) with the property $\lambda''(h_0 h, -) = \lambda''(-, h_0 h) = 1$ and $\lambda''(h_0, h_0) = 1$. Note that $(F^H)^{<h_0 h>} = k(\sqrt{-1 + 3a - a^2})$ is a totally real quadratic field with Galois group $\mathbb{Z}/2\mathbb{Z}$ with generator 1.

This means that the cohomological class of $D$ can be obtained via the inflation homomorphism from the class in $H^2(Gal(k(\sqrt{-1 + 3a - a^2})/k), k(\sqrt{-1 + 3a - a^2})^*)$ of the 2-cocycle $\lambda_0: \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to k(\sqrt{-1 + 3a - a^2})^*$ such that $\lambda_0(1, 1) = -1$. 30
Hence $D$ is a quaternion algebra over $\mathbb{Q} = k$ of degree $\deg(D) = 2$ split over $\mathbb{Q}(\sqrt{1 + 3a - a^3})$ with 4 generators over $\mathbb{Q}$: $1, i, j, k$ such that $i^2 = j^2 = 1 + 3a - a^3, k = ij = -ji$. In other words, $D = (1 + 3a - a^3, 1 + 3a - a^3)_{\mathbb{Q}}$.

So, in this example $\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{128 \times 128}((1 + 3a - a^3, 1 + 3a - a^3)_{\mathbb{Q}})$.

(4) If in the previous example we take

$$\Phi = -(b \cdot \rho) \cdot X_1^2 - (b \cdot \rho) \cdot X_2^2 - X_3^2 - X_4^2 - X_5^2,$$

where $b > 0$ is a rational number which is not a square of another rational number, then the same computation as above gives:

$$\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{128 \times 128}((b, b)_{\mathbb{Q}})$$

for the corresponding $K3$ surface $X$.

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