Flows in Graphs and Homology of Free Categories *

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Abstract

We introduce the notion of a generalized flow on a graph with coefficients in a $R$-representation and show that the module of flows is isomorphic to the first derived functor of the colimit. We generalize Kirchhoff’s laws and build an exact sequence for calculating the module of flows on the union of graphs.

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1 Preliminaries

This work is devoted to the $R$-module of generalized flows in a graph. We apply the homology of small categories in the sense of \cite{2}. Our approach is distinguished with the theory described in Wagner’s work \cite{10} where groups of flows are considered as a new invariant of graphs.

Let $C$ be a small category, $R$ a ring with identity. Denote by $Mod_R$ the category of left $R$-modules and $R$-homomorphisms, $Mod^C_R$ the category of functors $C \to Mod_R$, $\text{colim}^C: Mod^C_R \to Mod_R$ the colimit functor. The category $Mod^C_R$ has enough projectives \cite{2}. The functor $\text{colim}^C$ is right exact. Hence for every integer $n \geq 0$ it is defined the $n$-th right derived functor $\text{colim}^n_C: Mod^C_R \to Mod_R$ of the colimit. Let $F: C \to Mod_R$ be a functor. For arbitrary family $\{a_i\}_{i \in I}$ we will say that almost all $a_i$ are zeros.

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if there exists a finite subset \( J \subseteq I \) such that \( a_i = 0 \) for all \( i \in I \setminus J \). Denote \( C_n(C, F) = \sum_{c_0 \rightarrow \cdots \rightarrow c_n} F(c_0) \) and write elements of \( C_n(C, F) \) as sums

\[
\sum_{c_0 \rightarrow \cdots \rightarrow c_n} f_{c_0 \rightarrow \cdots \rightarrow c_n}[c_0 \rightarrow \cdots \rightarrow c_n] \text{ with } f_{c_0 \rightarrow \cdots \rightarrow c_n} \in F(c_0) \text{ where almost all } f_{c_0 \rightarrow \cdots \rightarrow c_n} \text{ are zeros. For every } c_0 \rightarrow \cdots \rightarrow c_{n+1} \text{ and } f \in F(c_0) \text{ we let }
\]

\[
d_{n}(f[c_0 \rightarrow \cdots \rightarrow c_{n+1}]) = F(c_0 \rightarrow c_1)(f[c_1 \rightarrow \cdots \rightarrow c_{n+1}]) + \sum_{i=1}^{n+1} (-1)^i f[c_0 \rightarrow \cdots \rightarrow \hat{c}_i \rightarrow \cdots \rightarrow c_{n+1}]
\]
and define homomorphisms \( d_n : C_{n+1}(C, F) \to C_n(C, F) \) by

\[
d_n \left( \sum f_{c_0 \rightarrow \cdots \rightarrow c_{n+1}}[c_0 \rightarrow \cdots \rightarrow c_{n+1}] \right) = \sum_{c_0 \rightarrow \cdots \rightarrow c_{n+1}} d_n \left( f_{c_0 \rightarrow \cdots \rightarrow c_{n+1}}[c_0 \rightarrow \cdots \rightarrow c_{n+1}] \right)
\]

Here

\[
[c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} \hat{c}_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}] =
\begin{cases}
[c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} c_{i-1} \xrightarrow{\alpha_{i+1}} c_i \xrightarrow{\alpha_{i+2}} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}], & \text{if } 0 < i < n+1, \\
[c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_q} c_n], & \text{if } i = n+1.
\end{cases}
\]

It is well-known [2, Application 2] that \( R \)-modules \( \text{colim}_n C \) are isomorphic to homologies of the complex

\[
0 \xleftarrow{d_{-1}} C_0(C, F) \xleftarrow{d_0} C_1(C, F) \xleftarrow{d_1} \cdots \xleftarrow{d_{n-1}} C_n(C, F) \xleftarrow{d_n} \cdots
\]
in the sense of \( \text{colim}_n C \approx \text{Ker} d_{n-1}/\text{Im} d_n, n \geq 0 \).

If \( \alpha \circ \beta = \text{id} \) in \( C \) implies \( \alpha = \text{id} \) and \( \beta = \text{id} \), then \( C \) is called to be without retractions. If \( C \) is a small category without retractions, then the complex \( \{ C_n(C, F), d_n \} \) includes the subcomplex \( \{ C^+_n(C, F), d_n \} \) where

\[
C^+_n(C, F) = \sum_{c_0 \rightarrow \cdots \rightarrow c_n} F(c_0)
\]
is the submodule in which sequences \( c_0 \rightarrow \cdots \rightarrow c_n \) do not contain identity morphisms if \( n > 0 \), with \( C^+_0(C, F) = C_0(C, F) \). As the dual affirmation [3, Proposition 2.2] we can prove the following.
Lemma 1.1 Let $C$ be a small category without retractions, $R$ a ring with identity. Then for each functor $F : C \to \text{Mod}_R$ the $R$-modules $\text{colim}_C^n F$ are isomorphic to $n$-th homology modules of the complex $\{C_n^+(C, F), d_n\}$.

Let $\Delta_n C \mathbb{Z}$ be a functor from a small category to the category of Abelian groups and homomorphisms which assign to every $c \in C$ the group of integers $\mathbb{Z}$ and to every $\alpha \in \text{Mor}_C$ the identity homomorphism $\text{id}_\mathbb{Z} : \mathbb{Z} \to \mathbb{Z}$. We denote by $H_n(C)$ the groups $\text{colim}_C^n \Delta_n C \mathbb{Z}$ for all $n \geq 0$. Let $S : C \to D$ be a functor between small categories, $d$ an object in $D$. The comma-category $d/S$ is defined as the following category:

Objects of $d/S$ are pairs $(c, \alpha)$ with $c \in \text{Ob} C$ and $\alpha \in D(d, S(c))$, morphisms $(c_1, \alpha_1) \to (c_2, \alpha_2)$ in $d/S$ consist of the triples $(\beta \in C(c_1, c_2), \alpha_1, \alpha_2)$ satisfying $S(\beta) \circ \alpha_1 = \alpha_2$. A functor $S : C \to D$ is called strong cofinal if $H_n(d/S) \cong H_n(pt)$ for all $n \geq 0$. Here $pt = \{\ast\}$ is the discrete category with one object, thus $S$ is strong cofinal if and only if for all $d \in \text{Ob} D$ the groups $H_n(d/S)$ are zeros for all $n > 0$ and $d/S$ are connected.

By Oberst’s Theorem [7, Theorem 2.3] if $S$ is strong cofinal, then for every functor $F : D \to \text{Mod}_R$ the canonical homomorphisms $\text{colim}_D^n F \to \text{colim}_C^n (F \circ S)$ are isomorphisms.

Lemma 1.2 The functor $s : (\mathcal{FC})^{\text{op}} \to C$ which assign to any $f \in \text{Ob} \mathcal{FC}$ its domain $s(f)$ and to any morphism $(\alpha, \beta)$ the morphism $\alpha$ is strong cofinal.

Proof. Objects of $c/s$ for $c \in \text{Ob} C$ are pairs $(x, \alpha)$ of morphisms $x \xrightarrow{\alpha} s(\alpha)$, and morphisms $(x, \alpha) \to (y, \beta)$ are commutative diagrams

$$
\begin{array}{ccc}
c & \xrightarrow{x} & s(\alpha) \\
\downarrow \text{id}_c & & \downarrow \alpha \\
c & \xrightarrow{y} & s(\beta)
\end{array}
\Rightarrow
\begin{array}{ccc}
t(\alpha) & \xrightarrow{\alpha} & t(\alpha) \\
\uparrow & & \uparrow \\
t(\beta) & \xrightarrow{\beta} & t(\beta)
\end{array}
$$

For every $c \in \text{Ob} C$ the category $c/s$ includes the full subcategory consisting of all the objects $(\text{id}_c, \alpha)$. This subcategory is isomorphic to $(c/C)^{\text{op}}$. For every object $(x, \alpha)$ there is a morphism $(\text{id}_c, \alpha \circ x) \to (x, \alpha)$ such that for
each morphism \((id_c, \beta) \rightarrow (x, \alpha)\) there exists the unique morphism \((id_c, \beta) \rightarrow (id_c, \alpha \circ x)\) for which the following diagram is commutative

\[
\begin{array}{ccc}
(id_c, \alpha \circ x) & \rightarrow & (x, \alpha) \\
\uparrow \exists! & & \uparrow \\
(id_c, \beta) & \rightarrow & (id_c, \beta)
\end{array}
\]

It follows that there exists a right adjoint functor to the inclusion \((c/C)^{op} \subseteq c/s\). A right adjoint is strong cofinal, hence

\[
\text{colim}_n^{c/s} \Delta \mathbb{Z} \cong \text{colim}_n^{(c/C)^{op}} \Delta \mathbb{Z}.
\]

But \((c/C)^{op}\) has a terminal object. Thus \(H_n(c/s) \cong H_n(pt)\) for all \(n \geq 0\). \(\square\)

2 Generalized Flows

By a \((directed)\) graph we mean a pair of sets \((A, V)\) and a pair of functions \(A \xrightarrow{s,t} V\). The elements of \(A\) are called \textit{arrows}, \(V\) is the set of \textit{vertexes}, \(s(\alpha)\) and \(t(\alpha)\) are called the \textit{source} and the \textit{target} of \(\alpha \in A\) respectively.

Let \(\Gamma = (A, V, s, t)\) be a graph, \(R\) a ring with identity. A \(R\)-\textit{representation} of \(\Gamma\) is a family of \(R\)-modules \(\{F(v)\}_{v \in V}\) with a family of homomorphisms \(\{F(\alpha) : F(s(\alpha)) \rightarrow F(t(\alpha))\}_{\alpha \in A}\). A \textit{path} in \(\Gamma\) from \(u \in V\) to \(v \in V\) is an arbitrary word \(\alpha_1 \alpha_2 \cdots \alpha_n\) with \(t(\alpha_1) = v\) and \(s(\alpha_n) = u\) such that \(s(\alpha_i) = t(\alpha_{i+1})\) for all \(1 \leq i \leq n-1\). For each vertex \(v \in V\) define \(id_v\) as the empty path from \(v\) to \(v\). Objects of the \textit{category of paths in} \(\Gamma\) are vertexes \(v \in V\), and morphisms are paths in \(\Gamma\) with the composition law

\[
\alpha_1 \cdots \alpha_n \circ \beta_1 \cdots \beta_m = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m,
\]

for \(s(\alpha_n) = t(\beta_1)\). Let \(WT\) be the category of paths in \(\Gamma\). For every \(R\)-representation \(F\) there is the unique functor \(\tilde{F} : WT \rightarrow \text{Mod}_R\) such that \(\tilde{F}|_\Gamma = F\). It is defined by \(\tilde{F}(\alpha_1 \cdots \alpha_n) = F(\alpha_1) \cdots F(\alpha_n)\) if \(n > 0\), and \(\tilde{F}(id_v) = id_{F(v)}\).

**Definition 2.1** Let \(\Gamma\) be a graph, \(F\) a \(R\)-representation of \(\Gamma\). A flow on \(\Gamma\) with coefficients in \(F\) is a family \(\{f_\gamma\}_{\gamma \in A}\) of elements \(f_\gamma \in F(s(\gamma))\) such that almost all of \(f_\gamma\) are zeros and for each \(v \in V\) the following equality holds:

\[
\sum_{s(\gamma) = v} f_\gamma = \sum_{t(\gamma) = v} F(\gamma)(f_\gamma)
\]
We denote by $\Phi(\Gamma; F)$ the $R$-module of flows on $\Gamma$ with coefficients in $F$. We have the following exact sequence:

$$0 \to \Phi(\Gamma; F) \to \sum_{\gamma \in A} F(s(\gamma)) \xrightarrow{d} \sum_{v \in V} F(v) \to \text{colim}^{\text{WT}} \tilde{F} \to 0, \quad (1)$$

where $d(\sum_{\gamma \in A} f_\gamma \cdot [\gamma])_v = \sum_{t(\gamma) = v} F(\gamma)(f_\gamma) - \sum_{s(\gamma) = v} F(\gamma)$.

Let $S : C \to D$ be a functor between small categories, $d \in \text{Ob} D$ an object. We denote by $S/d$ the category which objects are pairs $(c, \alpha)$ with $c \in \text{Ob} C$, $\alpha \in D(S(c), d))$; morphisms $(c_1, \alpha_1) \to (c_2, \alpha_2)$ in $S/d$ are triples $(\beta \in C(c_1, c_2), \alpha_1, \alpha_2)$ satisfying $\alpha_2 \circ S(\beta) = \alpha_1$. It is clear that $d/(S^{\text{op}}) \cong (S/d)^{\text{op}}$. Denote $\mathcal{FW} \Gamma = \mathcal{F}(W \Gamma)$.

**Lemma 2.2** Let $\Gamma = (A, V, s, t)$ be a graph, $\Gamma'$ the full subcategory of $\mathcal{FW} \Gamma$ such that $\text{Ob} \Gamma' = A \coprod V$. Then $\Gamma'^{\text{op}}$ is strong cofinal in $(\mathcal{FW} \Gamma)^{\text{op}}$.

**Proof.** We denote by $S$ the inclusion $\Gamma' \subseteq \mathcal{FW} \Gamma$. The objects of $S/w$ are pairs $(x, (\alpha, \beta))$ where $\alpha$ and $\beta$ are paths in $\Gamma$ satisfying $\beta \circ x \circ \alpha = w$ with either $x = \text{id}$ or $x \in A$. Hence for $w = (v_0 \xrightarrow{\alpha_1} v_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} v_n)$ the category $S/w$ is the following:

```
   e_0   e_1   \cdots   e_n

/          /          /          /
f_1         f_2         f_n
```

The category $S/w$

with the objects $e_0 = (\text{id}_{v_0}, (\text{id}_{v_0}, \alpha_n \cdots \alpha_1))$, $f_1 = (\alpha_1, (\text{id}_{v_0}, \alpha_n \cdots \alpha_2))$, $e_1 = (\text{id}_{v_1}, (\alpha_1, \alpha_n \cdots \alpha_2))$, $\cdots$, $f_n = (\alpha_n, (\alpha_{n-1} \cdots \alpha_1, \text{id}_{v_n}))$, $e_n = (\text{id}_{v_n}, (\alpha_n \cdots \alpha_1, \text{id}_{v_n}))$, where all morphisms excluding $e_{i-1} \to f_i$ and $e_i \to f_i$, for $i \in \{1, 2, \cdots, n\}$, are identities.

But $w/S^{\text{op}}$ is isomorphic to $(S/w)^{\text{op}}$, hence $H_n(w/S^{\text{op}}) = 0$ for $n > 0$ and $w/S^{\text{op}}$ is connected. □
Lemma 2.3  Let $\Gamma = (A, V, s, t)$ be a graph, $F : (\mathcal{F} \mathcal{W} \mathcal{T})^{\text{op}} \to \text{Mod}_R$ a functor. Then $\text{colim}_{1}^{(\mathcal{F} \mathcal{W} \mathcal{T})^{\text{op}}} F$ is isomorphic to the submodule in $\sum_{\gamma \in A} F(\gamma)$ consisting of families $\{g_{\gamma}\}_{\gamma \in A}$ for which the following equality holds for every $v \in V$:

$$\sum_{v = s(\gamma)} F(v \xrightarrow{\text{id}} \gamma) g_{\gamma} = \sum_{v = t(\gamma)} F(v \xrightarrow{(\gamma, \text{id})} \gamma) g_{\gamma}$$

(2)

Here $\gamma$ runs the set $A$.

PROOF. We have by Lemma 2.2 from the Oberst theorem that $\text{colim}_{1}^{(\mathcal{F} \mathcal{W} \mathcal{T})^{\text{op}}} F$ is isomorphic to $\text{colim}_{1}^{\Gamma^{\text{op}}} F|_{\Gamma^{\text{op}}}$. The category $\Gamma^{\text{op}}$ has no retractions except identities. Hence, by Lemma 1.1 $R$-modules $\text{colim}_{1}^{(\mathcal{F} \mathcal{W} \mathcal{T})^{\text{op}}} F$ are isomorphic to the homology groups of the chain complex

$$0 \to C_{1}^{+}(\Gamma^{\text{op}}, F) \to C_{0}^{+}(\Gamma^{\text{op}}, F) \to 0$$

which is equal to

$$0 \to \sum_{\gamma \in A} F(\gamma) \xrightarrow{d} \sum_{\gamma \in A} F(\gamma) \oplus \sum_{v \in V} F(v) \to 0$$

with

$$d(\sum_{v \to \gamma} f_{v \to \gamma}[v \to \gamma]) = \sum_{v \to \gamma} d(f_{v \to \gamma}[v \to \gamma]) = \sum_{v \to \gamma} (F(v \to \gamma)(f_{v \to \gamma}[v] - f_{v \to \gamma}[\gamma])).$$

(Here we consider the homology of the category which is opposite to $\Gamma'$, in particular $f_{v \to \gamma} \in F(\gamma)$ and a homomorphism $F(v \to \gamma)$ acts from $F(\gamma)$ into $F(v)$.) Therefore $\text{colim}_{1}^{\Gamma^{\text{op}}} F$ is isomorphic to the $R$-module of families $f_{v \to \gamma} \in F(\gamma)$ satisfying $f_{s(\gamma) \to \gamma} + f_{t(\gamma) \to \gamma} = 0$ for each $\gamma \in A$, and $\sum_{v \to \gamma} F(v \to \gamma)(f_{v \to \gamma}) = 0$ for each $v \in V$.

We denote $g_{\gamma} = f_{s(\gamma) \to \gamma}$ for a such family. Then $f_{t(\gamma) \to \gamma} = -g_{\gamma}$, and $\sum_{v = s(\gamma)} F(v \to \gamma)(g_{\gamma}) = \sum_{v = t(\gamma)} F(v \to \gamma)(g_{\gamma})$. Thus $\text{colim}_{1}^{\mathcal{F} \mathcal{W} \mathcal{T}^{\text{op}}} F$ is isomorphic to the submodule of $\sum_{\gamma \in A} F(\gamma)$ consisting of $\{g_{\gamma}\}_{\gamma \in A}$ for which the equation (2) holds. $\square$

Theorem 2.4  Let $\Gamma = (A, V, s, t)$ be a graph, $R$ a ring with identity, $F$ a $R$-representation of $\Gamma$. Then $\Phi(\Gamma; F) \cong \text{colim}_{1}^{\mathcal{W} \mathcal{T}} \tilde{F}$. 

6
By Lemma 1.2 the functor $s : (\mathcal{FC})^{op} \to \mathcal{C}$ is strong cofinal for an arbitrary small category. Hence $\text{colim}^{WT}_1 \tilde{F} \cong \text{colim}^{FWT\cdot op}_1 (\tilde{F} \circ s)$. The substitution of $\tilde{F} \circ s$ instead of $F$ in Lemma 2.3 leads to the concluding that $\text{colim}^{WT}_1 \tilde{F}$ is isomorphic to the submodule of $\sum_{\gamma \in A} F(\gamma)$ which consists of families $f_\gamma \in F(s(\gamma))$ satisfying $\sum_{v = s(\gamma)} f_\gamma = \sum_{v = t(\gamma)} F(\gamma)(f_\gamma)$, for each $v \in V$. □

### 3 The First Kirchhoff Law

Let $\Gamma = (A, V, s, t)$ be a graph, $F$ a $R$-representation of $\Gamma$. Elements of $\sum_{v \in V} F(v)$ and $\sum_{\gamma \in A} F(s(\gamma))$ are called 0-chains and 1-chains respectively.

Let $\varepsilon : \sum_{v \in V} F(v) \to \text{colim}^{WT}_1 \tilde{F}$ be the canonical $R$-homomorphism. It follows from the exact sequence (1) that the equation $d f = \varphi$ has a solution for $\varphi \in \sum_{\gamma \in A} F(s(\gamma))$ if and only if $\varepsilon(\varphi) = 0$. Hence $\text{colim}^{WT}_1 \tilde{F}$ can be interpreted as the $R$-module of ”obstructions”. Denote it by $\Phi_0(\Gamma; F)$. We have the exact sequence

$$0 \to \Phi(\Gamma; F) \xrightarrow{\varepsilon} \sum_{\gamma \in A} F(s(\gamma)) \xrightarrow{d} \sum_{v \in V} F(v) \xrightarrow{\xi} \Phi_0(\Gamma; F) \to 0$$

with $\Phi(\Gamma; F) \cong \text{colim}^{WT}_1 \tilde{F}$ and $\Phi_0(\Gamma; F) \cong \text{colim}^{WT}_1 \tilde{F}$.

A network $(\Gamma, E, F)$ consists of the following data:
1) a graph $\Gamma = (A, V, s, t)$;
2) an arbitrary subset $E \subseteq V$ which elements are called external;
3) a $R$-representation $F$ of $\Gamma$.

We say that 1-chain $\{f_\gamma\}_{\gamma \in A}$ satisfies to the first Kirchhoff law if $d(\{f_\gamma\}_{\gamma \in A})_v = 0$, $\forall v \notin E$.

Let $\Phi(\Gamma, E; F)$ be a $R$-module of all 1-chains satisfying to the first Kirchhoff law in the network $(\Gamma, E, F)$. For $E = \emptyset$ an 1-chain satisfies to the first Kirchhoff law if and only if it is a flow on $\Gamma$ with coefficients in $F$. Thus, $\Phi(\Gamma, \emptyset; F) = \Phi(\Gamma; F)$.

A vertex $v \in V$ is called to be attractive if there are not arrows with $s(\gamma) = v$. Let $E \subseteq V$ be any subset such that all $e \in E$ are attractive vertexes, $F_E$ the $R$-representation of $\Gamma$ with $F_E(v) = 0$ for all $v \notin E$, and $F_E(v) = F(v)$ for all $v \in E$. We have by Theorem 2.4 the following
Lemma 3.1 Let \((\Gamma, E, F)\) be a network. If all vertexes in \(E\) are attractive then \(\Phi(\Gamma, E; F) \cong \text{colim}_1^{W}(F/F_E)\).

To a description the \(R\)-module of 1-chains satisfying to the first Kirchhoff law in any network \((\Gamma, E, F)\) we add to the graph \(\Gamma\) the vertex \(*\) and the arrows \(\gamma_e\) for all \(e \in E\) with \(s(\gamma_e) = e\) and \(t(\gamma_e) = *\). Denote by \(\Gamma \cup E pt\) the obtained graph. Let \(F \oplus E 0\) be the \(R\)-representation of \(\Gamma \cup E pt\) such that \((F \oplus E 0)|_E = F\) and \((F \oplus E 0)(* ) = 0\).

Theorem 3.2 For any network \((\Gamma, E, F)\) the \(R\)-module \(\Phi(\Gamma, E; F)\) is isomorphic to \(\text{colim}_1^{W}(\Gamma \cup E pt)(F \oplus E 0)\).

Proof. Consider the network \((\Gamma \cup E pt, pt, F \oplus E 0)\). The vertex \(*\) is attractive in \(\Gamma \cup E pt\). It follows from the previous lemma that \(\Phi(\Gamma \cup E pt, pt; F \oplus E 0) \cong \text{colim}_1^{W}(\Gamma \cup E pt)(F \oplus E 0)\). But \(\Phi(\Gamma \cup E pt, pt; F \oplus E 0) = \Phi(\Gamma, E; F)\). \(\square\)

4 Flows on the Union of Graphs

Let \((I, \leq)\) be a partially ordered set. A covering \(X = \bigcup_{i \in I} X_i\) of a set is called to be locally filtered if \(i < j\) in \(I\) implies \(X_i \subseteq X_j\) and if for each \(x \in X_i \cap X_j\) there exists \(k \in I\) such that \(k < i, k < j,\) and \(x \in X_k \subseteq X_i \cap X_j\).

A graph \(\Gamma = (A, V, s, t)\) is called to be locally filtered covered by graphs \(\{\Gamma_i\} = \{A_i, V_i, s_i, t_i\}\) if \(A = \bigcup_{i \in I} A_i\) and \(V = \bigcup_{i \in I} V_i\) are locally filtered coverings and the following diagrams are commutative

\[
\begin{array}{ccc}
A_i & \subseteq & A_j \\
s_i \downarrow & & \downarrow s_j \\
V_i & \subseteq & V_j
\end{array}
\quad
\begin{array}{ccc}
A_i & \subseteq & A_j \\
& \downarrow t_i & \downarrow t_j \\
V_i & \subseteq & V_j
\end{array}
\]

for all \(i \leq j\) in \(I\).

Theorem 4.1 Let \(\Gamma = (A, V, s, t)\) be a graph which is locally filtered covered by graphs \(\{\Gamma_i\}_{i \in I} = \{A_i, V_i, s_i, t_i\}\), \(E \subseteq V\) a subset such that \(E = \bigcup_{i \in I} E_i\) is a locally filtered covering by \(E_i \subseteq V_i\). Then for each \(R\)-representation \(F\) of \(\Gamma\) there exists an exact sequence

\[
0 \to \text{colim}_I^{J}\{\Phi_0(\Gamma_i, E_i; F_i)\} \to \text{colim}_I^{J}\{\Phi(\Gamma_i, E_i; F_i)\}
\]
\[ \Phi(\Gamma, E; F) \to \text{colim}_0^f \{ \Phi_0(\Gamma_i; E_i; F_i) \} \to 0 \]

where \( F_i = F|_{\Gamma_i} \).

**Proof.** At first we consider the case \( E = \emptyset \). Let \( \Gamma'_i \subseteq \Gamma' \) are the categories defined for the graphs \( \Gamma_i \subseteq \Gamma \) in Lemma 2.2. The covering \( \{ \Gamma'_i \}_{i \in I} \) satisfies to the conditions of [4, Corollary 3.2]. The functors \( s : (\mathcal{F} \mathcal{W} \Gamma)'_o \to \mathcal{W} \Gamma \) and \( S^o : \Gamma'^o \to (\mathcal{F} \mathcal{W} \Gamma)'_o \) are strong cofinal by Lemma 1.2 and Lemma 2.2. Hence, \( \text{colim}_i^1 (\Gamma_\mathcal{W} F) \) is isomorphic to \( \text{colim}_i^1 (\tilde{F} \circ S^o) \). By [4, Corollary 3.2] there is the spectral sequence with \( E^2_{p,q} = \text{colim}_p^I (\text{colim}_q^I \tilde{F} \circ S^o) \) which converges to \( \text{colim}_i^1 \tilde{F} \circ S^o \). The substitution \( \Gamma_i \) instead \( \Gamma \) leads to the functors \( s_i : (\mathcal{F} \mathcal{W} \Gamma)_i'^o \to \mathcal{W} \Gamma_i \) and \( S^o_i : \Gamma'^o_i \to (\mathcal{F} \mathcal{W} \Gamma)_i'^o \) which are strong cofinal by Lemmas 1.2 and 2.2. It follows from \( \tilde{F} \circ S^o \) is isomorphic to \( \text{colim}_i^1 (\Gamma_\mathcal{W} F) \) \( \Rightarrow \text{colim}_i^1 \tilde{F} \). Then the exact sequence of terms of low degree [4, P.332] gives the exact sequence

\[ 0 \to \text{colim}_2^I \{ \Phi_0(\Gamma_i; F_i) \} \to \text{colim}_1^I \{ \Phi_0(\Gamma_i; F_i) \} \to 0 \]

For \( E \neq \emptyset \) we consider the locally filtered covering \( \Gamma \cup_E pt = \bigcup_{i \in I} (\Gamma_i \cup_E pt) \).

There is an exact sequence

\[ 0 \to \text{colim}_2^I \{ \Phi_0(\Gamma_i; \cup_E pt; F_i \oplus_E), 0 \} \to \text{colim}_1^I \{ \Phi_0(\Gamma_i; \cup_E pt; F_i \oplus_E), 0 \} \]

\[ \to \Phi(\Gamma \cup_E pt; F \oplus_E) \to \text{colim}_1^I \{ \Phi_0(\Gamma_i \cup_E pt; F_i \oplus_E), 0 \} \to 0 \]

The equalities \( \Phi_0(\Gamma \cup_E pt; F \oplus_E) = \Phi_0(\Gamma, E; F) \) and \( \Phi(\Gamma \cup_E pt; F \oplus_E) = \Phi(\Gamma, E; F) \) give looking. □

5 The Second Kirchhoff Law

Let \( R \) be a field. The *internal product* on a vector space \( T \) is a bilinear map \( \langle, \rangle : T \times T \to R \) such that \( \langle a, b \rangle = \langle b, a \rangle \) for all \( a, b \in T \). A network \((\Gamma, E, F)\) together with an internal product \( \langle, \rangle \) on \( \sum_{\gamma \in A} F(s(\gamma)) \) is called to be *Euclidian* if the implication \( \langle f, f \rangle = 0 \Rightarrow f = 0 \) is true. We say that an 1-chain \( f = \{ f_\gamma \} \) of an Euclidian network satisfies to the *second Kirchhoff*
law if the linear map \(< f, - \>: \sum_{\gamma \in A} F(s(\gamma)) \rightarrow R\) has zero values on \(\Phi(\Gamma, F)\) in the sense that
\[ < f, - > |_{\Phi(\Gamma, F)} = 0. \]

**Theorem 5.1** Let \((\Gamma, E, F)\) be an Euclidian network in which \(R\) is a field, \(\Gamma\) a finite graph, and \(F(v)\) finite dimensional vector spaces for all \(v \in V\). Then for each \(\varphi \in \sum_{v \in E} F(v)\) satisfying \(\varepsilon(\varphi) = 0\) there is the unique 1-chain \(f = \{f_\gamma\}\) such that \(df = \varphi\) and \(< f, - > |_{\Phi(\Gamma, F)} = 0\).

**Proof.** If \(df = 0\) then \(f \in \Phi(\Gamma, F)\), in this case \(< f, - > |_{\Phi(\Gamma, F)} = 0\) implies \(< f, f > = 0\) and \(f = 0\). Considering \(f_i\), with \(i \in \{1, 2\}\), for which \(df_i = \varphi\) and \(< f_i, - > |_{\Phi(\Gamma, F)} = 0\), we obtain \(f_1 - f_2 = 0\). Hence, the solution is unique.

Consider a map \(\sum_{\gamma \in A} F(s(\gamma)) \rightarrow Mod_R(\Phi(\Gamma, F), R) \oplus \varepsilon^{-1}(0)\) which assigns to any \(g \in \sum_{\gamma \in A} F(s(\gamma))\) the pair \(< g, - > |_{\Phi(\Gamma, F)} \oplus dg\). It is easy to verify that this map is the isomorphism. We let \(g = \eta^{-1}(0 \oplus \varphi)\). Then \(\eta(g) = 0 \oplus \varphi\). Hence \(dg = \varphi\) and \(< g, - > |_{\Phi(\Gamma, F)} = 0\). Thus there exists the solution. \(\square\)

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