The geometric meaning of the complex dilatation

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Introduction

The classical complex dilatation. As usual, we will identify $\mathbb{C}$ with $\mathbb{R}^2$ by the map
$x + iy \mapsto (x, y)$. Let $U$ be an open subset of $\mathbb{C}$ and let $f: U \rightarrow \mathbb{C}$ be a map
differentiable as a map from $U \subset \mathbb{C} = \mathbb{R}^2$ to $\mathbb{R}^2$. The complex dilatation $\mu_f$ of $f$
is a measure of the distortion of the conformal structure of $\mathbb{C}$ by $f$. Let us recall its
definition following Ahlfors [Ah]. Let $z = x + iy$ and let

$$f(z) = u(x, y) + iv(x, y).$$

Let $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$ be the usual partial derivatives and let

$$f_z = (f_x - if_y)/2, \quad f_\overline{z} = (f_x + if_y)/2.$$

The complex dilatation of $f$ is defined as the quotient $\mu_f = f_{\overline{z}}/f_z$.  

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It is assumed that \( f \) has non-zero Jacobian and is orientation-preserving. This ensures that \( f_z \neq 0 \) and, moreover, that \( |f_z| > |f_{\bar{z}}| \) and hence \( |\mu_f| < 1 \). In other words, \( \mu_f \) belongs to the open unit disc \( \mathbb{U} \) in the complex plane. See [Ah], Section 1.A. We will sometimes refer to \( \mu_f \) defined as above as the classical complex dilatation.

Ahlfors explains the geometric meaning of \( \mu_f \) as follows. At any given point of \( \mathbb{U} \) the tangent map \( T_f \) of \( f \) is a real linear map. Since \( f \) has non-zero Jacobian, \( T_f \) is invertible, and therefore takes circles with the center at the origin into ellipses. The ratio of the major axis to the minor axis of every such ellipse is equal to the dilatation 

\[
D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}.
\]

of \( f \). It is related to the absolute value \( d_f = |\mu_f| \) of \( \mu_f \) by the formulas

\[
(1) \quad D_f = \frac{1 + d_f}{1 - d_f}, \quad d_f = \frac{D_f - 1}{D_f + 1}.
\]

The ratio \( |T_f(z)|/|z| \) is maximal when \( \arg z = \arg \mu_f/2 \).

Therefore, the complex dilatation \( \mu_f \) is characterized by the following two properties.

(i) \( d_f = |\mu_f| \) is related to the ratio \( D_f \) of the axes of an image ellipse by (1).

(ii) \( \arg \mu_f = 2\alpha \), where \( \alpha \) is the direction of the maximal distortion of \( T_f \).

This characterization is usually considered as the explanation of the geometric meaning of \( \mu_f \). Indeed, the absolute value \( |\mu_f| \) encodes the shape of the ellipses to which \( T_f \) takes the circles with the center at the origin. These ellipses are circles if and only if \( \mu_f = 0 \), and if \( \mu_f \neq 0 \), then the argument \( \arg \mu_f \) together with \( T_f \) determines the directions of the axes of such an ellipse. Namely, the direction of the major axis is equal to the image under \( T_f \) of the direction with the angle \( \arg \mu_f/2 \).

Still, the most natural way to encode the shape of an ellipse is to take the ratio of its axes, and \( D_f \) seems to be a much more natural measure of the distortion of shapes by \( f \) than the absolute value of the complex dilatation. The replacement of \( D_f \) by \( d_f \) is not justified geometrically at all.

The main goal of this paper is to propose a geometric framework in which the complex dilatation \( \mu_f \) and, in particular, its absolute value \( d_f \) appear naturally. This requires a radical change of the point of view: instead of considering the individual ellipses, one should consider the space of all ellipses with the center at the origin up to homothetic transformations centered at the origin. In other terms, one needs to consider the space of all conformal structures on \( \mathbb{C} \) (considered as a tangent space to itself).
The argument of the complex dilatation. The angle $\alpha$ is defined only up to addition of $\pm \pi$. Replacing it by $2\alpha$ is a way to kill this indeterminacy. We will see that the expression $2\alpha$ also appears naturally.

But $\mu_f$ determines the direction of the image ellipse only together with $T_f$. In fact, the preimages of circles with the center $0$ under $T_f$ are also ellipses and $\alpha$ is the direction of the minor axes of these ellipses. Therefore, $\mu_f$ determines the direction of the preimage, rather than image, ellipses. This crucial feature is rarely mentioned.

An approach to the geometric interpretation of the complex dilatation. By the definition $\mu_f$ at a point depend only on the tangent map $T_f$ at this point. In fact, one can define the complex dilatation $\mu_T$ of any orientation-preserving real linear map $T: \mathbb{C} \to \mathbb{C}$ in such a way that $\mu_f = \mu_{T_f}$ for any differentiable map $f$ as above. Namely, let

$$
\begin{pmatrix}
u_1 & \nu_2 \\
v_1 & \nu_2
\end{pmatrix}
$$

be the matrix of $T$ with respect to the standard basis of $\mathbb{C} = \mathbb{R}^2$. Let

$$
T_1 = u_1 + i\nu_1, \quad T_2 = u_2 + i\nu_2, \quad \text{and}
$$

$$
T_z = (T_1 - iT_2)/2, \quad T_{\overline{z}} = (T_1 + iT_2)/2,
$$

and define the complex dilatation of $T$ as $\mu_T = T_{\overline{z}}/T_z$. Clearly, $\mu_f = \mu_{T_f}$ for any differentiable map $f$ as above, and Alhfors’s interpretation of $\mu_f$ applies mutatis mutandis to $\mu_T$. Therefore, the real linear maps $\mathbb{C} \to \mathbb{C}$ are at heart of the matter. Our approach is based on the following observations.

- A natural measure of distortion of the conformal structure of $\mathbb{C}$ by a real linear map $T: \mathbb{C} \to \mathbb{C}$ is the pull-back by $T$ of the standard conformal structure on $\mathbb{C}$.

- The set $U(\mathbb{C})$ of all conformal structures on $\mathbb{C}$ carries a canonical structure of a model of the hyperbolic plane. In fact, $U(\mathbb{C})$ can be naturally identified with the open unit disc $U$ together with its structure of the Klein model of the hyperbolic plane.

- The standard isomorphism of the Klein model with the Poincaré unit disc model transforms this measure of distortion into the classical complex dilatation.

The angle $2\alpha$ naturally appears already in the Klein model, and the replacement of $D_f$ by $d_f$ results from the transition from the Klein to the Poincaré model.

Complex dilatation and Arnold’s approach to altitudes in hyperbolic geometry. A conformal structure on $\mathbb{C}$ is defined as a (positive or negative) definite quadratic form
on $\mathbb{C}$ considered up to a non-zero real factor. So, one of the key elements of our approach is the canonical structure of the Klein model of the hyperbolic plane on the space of definite quadratic forms on $\mathbb{C}$ considered up to a non-zero real factor. This is also a key element of Arnold’s approach to the triangle altitudes theorem in Lobachevsky (hyperbolic) geometry [Ar]. The author has to admit that he missed this fact even after writing an exposition of Arnold’s ideas [I2] and completing the first version of the present paper. (In [I2] the Poisson bracket of quadratic forms on $\mathbb{C} = \mathbb{R}^2$ was replaced by the commutator of traceless $2 \times 2$ real matrices.)

**The action of real linear maps.** Let $u : \mathbb{C} \to \mathbb{C}$ is a real linear orientation-preserving isomorphism. Then there is a unique map $u^* : \mathcal{U} \to \mathcal{U}$ such that $\mu_{Tu} = u^*(\mu_T)$ for any orientation-preserving real linear map $T : \mathbb{C} \to \mathbb{C}$. Moreover, $u^* : \mathcal{U} \to \mathcal{U}$ is complex-analytic. Let us quote J.H. Hubbard [H], p. 162:

> The fact that $u^*$ is analytic has far-reaching consequences. When you dig down to where the complex analytic structure of Teichmüller space comes down from (a highly nontrivial result, with a rich and contentious history, involving Ahlfors, Rauch, Grothendieck, Bers, and many others), you will find that this is the foundation of it all (see ...). Thus do little acorns into mighty oak trees grow.

Uniqueness of $u^*$ is straightforward. The complex-analyticity can be proved by a direct calculation. See [H], Proposition 4.8.10. The approach of this paper offers a conceptual explanation of the complex-analyticity. Taking the pull-backs of conformal structures by $u$ leads a map $\mathcal{U}(\mathbb{C}) \to \mathcal{U}(\mathbb{C})$. By the very definition of the Klein model of the hyperbolic plane, this induced map is its automorphism (i.e. preserves lines, angles, distances, etc.). Passing to the Poincaré model turns this map into an automorphism $u^* : \mathcal{U} \to \mathcal{U}$ of the latter. The property $\mu_{Tu} = u^*(\mu_T)$ is immediate, and hence this recovers the classical map $u^*$. On the other hand, it is well known that the automorphisms of the Poincaré model are complex-analytic. See Section 4 for the details.

**The paper.** The main part of the paper, namely Sections 1 – 4, is devoted to a theory of the complex dilatation of real linear maps $\mathbb{C} \to \mathbb{C}$ based on the above observations. The main result is Theorem 3.2. In Section 5 this theory is used to deal with real linear maps $V \to W$ from one complex vector space of dimension 1 to another. The complex dilatation of such a map is not a number, but a tensor. The author resisted the temptation to present this theory in such generality from the very beginning.

The present paper grow out of the observation that the replacement of $D_f$ by $d_f$ corresponds to the transition from the Klein to the Poincaré model of the hyperbolic plane. An preliminary exposition [I1] of this idea was written back in 1995 in response to a question of J.D. McCarthy about the geometric meaning of the complex dilatation. I am grateful to him for asking this question, his interest, and many conversations.
1. Quadratic forms and conformal structures on $\mathbb{C}$

Conformal structures. Recall that a quadratic form on a real vector space $V$ is a function $q : V \to \mathbb{R}$ such that $q(v) = B(v, v)$ for some symmetric bilinear form $B : V \times V \to \mathbb{R}$ and all $v \in V$. The bilinear form $B$ can be reconstructed from the quadratic form $q$ as its polarization $B_q(v, w) = (q(v + w) - q(v) - q(w))/2$. A quadratic form $q$ is positive definite if $q(v) > 0$ for all $v \neq 0$, and negative definite if $q(v) < 0$ for all $v \neq 0$. It is definite if it is either positive or negative definite. A quadratic form on a complex vector space $V$ is a quadratic form on $V$ considered as a real vector space.

A conformal structure on a real or complex vector space $V$ is a definite quadratic form on $V$ considered up to multiplication by a non-zero real number, or, equivalently, as a positive definite quadratic form considered up to multiplication by a positive real number. The conformal structure determined by a quadratic form $q$ is called the conformal class of $q$ and denoted by $[q]$. The set $\mathcal{U}(V)$ of all conformal structures on $V$ is a subspace of the projective space $\mathbb{P}Q(V)$ associated with the vector space $Q(V)$ of quadratic forms on $V$. Namely, $\mathcal{U}(V)$ is the image in $\mathbb{P}Q(V)$ of the set of all definite forms.

We are interested only in the case when $V$ is a complex vector space of dimension 1. It turns out that in this case $\mathcal{U}(V)$ has a canonical structure of a hyperbolic plane, similar to the Klein model of the hyperbolic geometry, and for $V = \mathbb{C}$ the space $\mathcal{U}(V) = \mathcal{U}(\mathbb{C})$ can be identified with the unit disc $\mathbb{U} = \{ u + iv \mid u^2 + v^2 < 1 \}$ in the complex plane, which is nothing else but the set of points of the Klein model. It is also the set of points of the Poincaré model of hyperbolic geometry, and passing from the Klein model to the Poincaré model is a key step in our approach to the complex dilatation.

The space $Q(\mathbb{C})$ of quadratic forms on $\mathbb{C}$. For most of the paper we will deal with the case $V = \mathbb{C}$. Our first goal is to identify $\mathcal{U}(\mathbb{C})$ with $\mathbb{U}$.

Let us denote by $X, Y : \mathbb{C} \to \mathbb{R}$ the maps $X(z) = \text{Re} \, z$ and $Y(z) = \text{Im} \, z$. As is well known, $q$ is a quadratic form on $\mathbb{C}$ if and only if there exist $a, b, c \in \mathbb{R}$ such that $q(x + iy) = ax^2 + 2bxy + cy^2$ for every $x, y \in \mathbb{R}$ or, what is the same, if

$$ q = aX^2 + 2bXY + cY^2. $$

Clearly, $(X^2, 2XY, Y^2)$ is basis of $Q(\mathbb{C})$, and the coefficients $a, b, c$ are the coordinates of $q$ with respect to this basis. It is well known and easy to check that $q = aX^2 + 2bXY + cY^2$ is a definite quadratic form if and only if its determinant

$$ D(q) = D(a, b, c) = ac - b^2 $$

is positive, i.e. $D(q) = ac - b^2 > 0$. In other terms, $[q] \in \mathcal{U}(\mathbb{C})$ if and only if
D(q) > 0. Let us stress the crucial fact is that the determinant D is itself a quadratic form on Q(C). This phenomenon is specific for dimension 2.

Another convenient basis of Q(C) consists of quadratic forms

\[ n = X^2 + Y^2, \quad r = X^2 - Y^2, \quad i = 2XY. \]

The corresponding coordinates (t, r, s) are related to (a, b, c) by the equations

\[ s = b, \quad t = (a + c)/2, \quad r = (a - c)/2. \]

In these coordinates D has the diagonal form \( D(t, r, s) = t^2 - r^2 - s^2 \). The basis \((n, r, i)\) has the advantage of being closely related with the structure of C. In fact,

\[ n(z) = |z|^2 = z\overline{z}, \quad r(z) = \text{Re} z^2, \quad i(z) = \text{Im} z^2 \]

for all \( z \in \mathbb{C} \). In particular, \([n]\) is the standard conformal structure on \( \mathbb{C} \).

A direct sum decomposition of Q(C). Now we are almost ready to identify \( U(C) \) with \( \mathbb{R} \). Let \( C \) be a copy of \( \mathbb{C} \), which we will treat as a complex vector space. Let us identify \( Q(C) \) with \( \mathbb{R} \oplus \mathbb{C} \) by the map \((t, r, s) \mapsto (t, r + is)\). Let us set \( z = r + is \) and consider \((t, z)\) as “mixed” coordinates on \( Q(C) = \mathbb{R} \oplus \mathbb{C} \). In the coordinates \((t, z)\) the determinant \( D \) takes the form \( D(t, z) = t^2 - |z|^2 \). Hence the space \( U(C) \) of conformal structures on \( \mathbb{C} \) is defined by the homogeneous inequality \( t^2 - |z|^2 > 0 \), or, equivalently, by the homogeneous inequality \( |z/t|^2 < 1 \).

The plane \( \mathbb{C} = \mathbb{R} \oplus \mathbb{C} \) in \( Q(C) = \mathbb{R} \oplus \mathbb{C} \) corresponds to a projective line in the projective plane \( \mathbb{P}Q(C) \). The complement \( A(C) \) to this projective line is an affine plane. Clearly, \( U(C) \subset A(C) \). We will use the map \((t, z) \mapsto z/t \in \mathbb{C} \) to identify \( A(C) \) with \( \mathbb{C} \). This identification takes \([n]\) to \( \mathbb{R} \) and \( U(C) \) to the unit disc \( \mathbb{U} \) in \( \mathbb{C} \).

This is the promised identification of \( U(C) \) with \( \mathbb{U} \).

Using the copy \( C \) of \( \mathbb{C} \) allows us to keep the distinction between the quadratic forms on \( \mathbb{C} \) represented by elements of \( \mathbb{C} \) and complex numbers, which are elements of \( \mathbb{C} \). By this reason the space \( U(C) \) is identified with the unit disc \( \mathbb{U} \) in \( \mathbb{C} \), not in \( \mathbb{C} \).

Diagonalization. For \( a, c \in \mathbb{R} \) let \( q_{a,c} \in Q(C) \) be the quadratic form \( aX^2 + cY^2 \). Obviously, \( q_{a,c} \) is a definite form if and only if \( a, c \) are of the same sign. As is well know, every quadratic form on \( \mathbb{C} = \mathbb{R}^2 \) can be turned into the form \( q_{a,c} \) for some \( a, c \) by a rotation. Let us state this diagonalization theorem more precisely.

To begin with, let \( V, W \) be real vector spaces and \( L: V \rightarrow W \) be a real linear map. For \( p \in Q(W) \) let the pull-back \( L^*p \in Q(V) \) be the composition \( L^*p = p \circ L \). If \( U \) is another vector space and \( K: U \rightarrow V \) is a linear map, then \( (L \circ K)^* = K^* \circ L^* \).
Let $V$ be a complex vector space. For $\tau \in \mathbb{C}$ the multiplication map $m_\tau : V \rightarrow V$ is defined by $m_\tau(v) = \tau v$. Taking pull-backs by $m_\tau$ leads to the induced map $m_\tau^* : Q(V) \rightarrow Q(V)$. If $\tau \neq 0$, then $m_\tau^*$ induces a map $\mathbb{P}m_\tau^* : \mathbb{P}Q(V) \rightarrow \mathbb{P}Q(V)$.

Finally, let $r_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the counter-clockwise rotation by an angle $\theta \in \mathbb{R}$ around the origin $(0, 0) \in \mathbb{R}^2$. As is well known, the standard identification $\mathbb{R}^2 \rightarrow \mathbb{C}$ turns $r_0$ into the multiplication map $m_\tau : \mathbb{C} \rightarrow \mathbb{C}$, where $\tau = e^{i\theta} = \cos \theta + i \sin \theta$. Now we are ready to state our form of the diagonalization theorem.

1.1. Theorem. For every quadratic form $q$ on $\mathbb{C}$ there exist $a, c, \theta \in \mathbb{R}$ such that

$$q = m_\tau^* q_{a,c},$$

where $\tau = e^{i\theta}$. Moreover, one can assume that $|a| \geq |c|$.

Proof. Let $q \in Q(\mathbb{C}) = Q(\mathbb{R}^2)$. By the spectral theorem for symmetric operators in $\mathbb{R}^2$ there is a basis $\{v, w\}$ of $\mathbb{R}^2$ orthonormal with respect to $n(x, y) = x^2 + y^2$ and orthogonal with respect to $q$. After interchanging $v$ and $w$, if necessary, we may assume that $\langle v, w \rangle$ has the same orientation as the standard basis. Then there is a rotation $r_0$ taking $(v, w)$ to the standard basis of $\mathbb{R}^2$. If $r_0$ is such a rotation, then $q = r_0^* q_{a,c}$ for some $a, c \in \mathbb{R}$. Since $r_{\pi/2}^* q_{a,c} = q_{c,a}$ and $r_0^* \circ r_{\pi/2}^* = r_{0 + \pi/2}^*$, one can assume that $|a| \geq |c|$. ■

The conformal class of a diagonalized form. Let $a, c > 0$ and $\tau \in \mathbb{C}$, $\tau \neq 0$, and let us consider the quadratic form $q = m_\tau^* q_{a,c}$. Our next goal is to identify the conformal class $[q]$ of $q$ as an element of $\mathbb{U}$. Let us consider first the case $q = q_{a,c}$. The $(t, \tau, s)$-coordinates of $q_{a,c}$ are $((a + c)/2, (a - c)/2, 0)$ and hence

$$q_{a,c} = ((a + c)/2, (a - c)/2) \in \mathbb{R} \oplus \mathbb{C}$$

after the identification $Q(\mathbb{C}) = \mathbb{R} \oplus \mathbb{C}$. It follows that

$$[q_{a,c}] = \frac{a - c}{a + c} \in \mathbb{C}$$

after the identification $\mathbb{A}(\mathbb{C}) = \mathbb{C}$. In order to deal with the general case $q = m_\tau^* q_{a,c}$ we need to study the action of the pull-backs $m_\tau^*$ on the space of quadratic forms $Q(\mathbb{C})$ and of the induced maps $\mathbb{P}m_\tau^*$ on projective plane $\mathbb{P}Q(\mathbb{C})$.

1.2. Lemma. The map $m_\tau^* : Q(\mathbb{C}) \rightarrow Q(\mathbb{C})$ respects the direct sum decomposition $Q(\mathbb{C}) = \mathbb{R} \oplus \mathbb{C}$. It acts on the summand $\mathbb{R}$ as the multiplication by $|\tau|^2$ and on the summand $\mathbb{C}$ as the multiplication map $m_{\rho}$, where $\rho = \overline{\tau}^2$. 

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**Proof.** The proof is an almost straightforward calculation. Obviously, if \( z \in \mathbb{C} \), then
\[
\mathsf{m}_\tau^* n(z) = |\tau|^2 |z|^2, \quad \mathsf{m}_\tau^* r(z) = \text{Re} \tau^2 z^2, \quad \text{and} \quad \mathsf{m}_\tau^* i(z) = \text{Im} \tau^2 z^2.
\]
If we present \( \tau^2 \) in the form \( \tau^2 = \alpha + i \beta \) with \( \alpha, \beta \in \mathbb{R} \), then
\[
\text{Re} \tau^2 z^2 = \alpha \text{Re} z^2 - \beta \text{Im} z^2 \quad \text{and} \quad \text{Im} \tau^2 z^2 = \beta \text{Re} z^2 + \alpha \text{Im} z^2
\]
for all \( z \in \mathbb{C} \). It follows that \( \mathsf{m}_\tau^* n = |\tau|^2 n \) and
\[
(\mathsf{m}_\tau^* r = \alpha r - \beta i, \quad \mathsf{m}_\tau^* i = \beta r + \alpha i,
\]
and hence \( \mathsf{m}_\tau^* \) leaves both summands \( \mathbb{R}, \mathbb{C} \) invariant, and acts on \( \mathbb{R} \) as the multiplication by \( |\tau|^2 \). Under the identification \( \mathbb{Q}(\mathbb{C}) = \mathbb{R} \oplus \mathbb{C} \) the forms \( r, i \) correspond to 1, \( i \in \mathbb{C} \) respectively, and the formulas (3) take the form
\[
(\mathsf{m}_\tau^* (1) = \alpha - \beta i, \quad \mathsf{m}_\tau^* (i) = \beta + \alpha i.
\]
Therefore, \( \mathsf{m}_\tau^* \) acts on \( \mathbb{C} \) as the multiplication by \( \alpha - i \beta = \overline{\tau}^2 \).

**1.3. Theorem.** Suppose that \( \tau \in \mathbb{C} \) and \( \tau \neq 0 \). Then the map \( \mathbb{P}\mathsf{m}_\tau^* \) leaves both \( \mathcal{A}(\mathbb{C}) \) and \( \mathcal{U}(\mathbb{C}) \) invariant. In particular, \( \mathbb{P}\mathsf{m}_\tau^* \) induces a map
\[
\mathsf{m}_\tau^{**} : \mathcal{A}(\mathbb{C}) \longrightarrow \mathcal{A}(\mathbb{C}).
\]
The identification \( \mathcal{A}(\mathbb{C}) = \mathbb{C} \) turns \( \mathsf{m}_\tau^{**} \) into \( \mathsf{m}_\sigma : \mathbb{C} \longrightarrow \mathbb{C} \), where \( \sigma = \overline{\tau}/\tau \).

**Proof.** Lemma 1.2 implies that \( \mathbb{P}\mathsf{m}_\tau^* \) leaves the projective line corresponding to the plane \( \mathbb{C} \) in \( \mathbb{Q}(\mathbb{C}) = \mathbb{R} \oplus \mathbb{C} \) invariant and hence leaves \( \mathcal{A}(\mathbb{C}) \) invariant. Lemma 1.2 also implies that if \( \mathsf{m}_\tau^* (t, z) = (t', z') \), then \( |t'| = |\tau|^2 |t'| \) and \( |z'| = |\tau|^2 |z'| \).

It follows that \( \mathsf{m}_\tau^* \) leaves invariant the cone in \( \mathbb{Q}(\mathbb{C}) \) defined by the homogeneous inequality \( t^2 - |z|^2 > 0 \). Hence the induced map \( \mathbb{P}\mathsf{m}_\tau^* \) leaves \( \mathcal{U}(\mathbb{C}) \) invariant.

Let \( (t, z) \in \mathbb{R} \oplus \mathbb{C} \) be a representative of \( \mu \in \mathcal{A}(\mathbb{C}) \). Then \( \mu = z/t \) under the identification \( \mathcal{A}(\mathbb{C}) = \mathbb{C} \). Lemma 1.2 implies that
\[
\mathsf{m}_\tau^*(t, z) = (|\tau|^2 t, \overline{\tau}^2 z)
\]
and hence that \( (|\tau|^2 t, \overline{\tau}^2 z) \) is a representative of \( \mathsf{m}_\tau^{**}(\mu) \). Hence
\[
\mathsf{m}_\tau^{**}(\mu) = \overline{\tau}^2 z/|\tau|^2 t = \overline{\tau}^2/|\tau|^2 (z/t).
\]
under the identification \( A(C) = C \). It follows that this identification turns \( m^{**}_\tau \) into the multiplication map \( m_\sigma : C \to C \), where \( \sigma = \overline{\tau^2}/|\tau|^2 = \overline{\tau^2}/\tau \overline{\tau} = \overline{\tau}/\tau \). ■

1.4. Corollary. If \( q \in Q(C) \) is a definite form and \( \tau \in C, \tau \neq 0 \), then

\[
[m_\tau^* q] = (\overline{\tau}/\tau)[q] \in C.
\]

1.5. Corollary. There is a unique conformal structure on \( C \) fixed by all maps \( \mathbb{P}m_\tau^* \) with \( \tau \neq 0 \). It is equal to the conformal class \([n]\) of \( n\).

Proof. Theorem 1.3 implies that a conformal structure on \( C \) is fixed by all maps \( \mathbb{P}m_\tau^* \) with \( \tau \neq 0 \) if and only if as a point of \( \mathbb{U} \) it is fixed by the multiplication by all \( \sigma \in C \) of the form \( \sigma = \overline{\tau}/\tau \) with \( \tau \neq 0 \). But \( \sigma \) has such form if and only if \( |\sigma| = 1 \), and there is exactly one point of \( \mathbb{U} \) fixed by the multiplication by all such \( \sigma \), namely, the point 0. Since 0 corresponds to \([n]\), this proves the corollary. ■

1.6. Theorem. Let \( a, c \in \mathbb{R} \) and \( \tau \in C \). Suppose that \( a, c \) are positive and \( \tau \neq 0 \). Let \( q = m_\tau^* q_{a,c} \). Then under the identification \( A(C) = C \) the conformal class

\[
[q] = (\overline{\tau}/\tau) \frac{a-c}{a+c} \in C.
\]

Proof. As we already saw, \([q_{a,c}] = \frac{a-c}{a+c} \in C\). By the definition of \( m^{**}_\tau \) we have

\[
[m_\tau^* q_{a,c}] = m^{**}_\tau[q_{a,c}] = m^{**}_\tau\left(\frac{a-c}{a+c}\right).
\]

It remains to apply Theorem 1.3. ■

2. The Poincaré invariant of a quadratic form

From the Klein model to the Poincaré model. The open unit disc \( \mathbb{U} \subset C \) is the set of points of the Klein model of hyperbolic geometry. It is also the set of points of the Poincaré model of hyperbolic geometry. There is a canonical isomorphism between these two structures on \( \mathbb{U} \). In order to describe it, let us use the upper hemisphere

\[
S_+^2 = \{ (u + iv, w) \mid u^2 + v^2 + w^2 = 1, \ w > 0 \} \subset C \times \mathbb{R}
\]
as an intermediary between the Klein and the Poincaré models.
Let \( \pi \) be the orthogonal projection of the upper hemisphere \( S^2_+ \) to the equatorial disc \( U = \{ (u + iv) \in \mathbb{C} \mid u^2 + v^2 < 1 \} \). In other terms, \( \pi(u + iv, w) = u + iv \in \mathbb{C} \). Let \( S \) be the stereographic projection of the upper hemisphere \( S^2_+ \) to the equatorial disc from the point \( s = (0 + i0, -1) \). As is well known, the map \( \Omega = S \circ \pi^{-1} \) transforms the Klein structure of the hyperbolic plane on \( U \) into the Poincaré one.

The Poincaré invariant of quadratic forms on \( \mathbb{C} \). For every definite quadratic form \( q \) on \( \mathbb{C} \) we define its Poincaré invariant \( P(q) \) as the image \( P(q) = \Omega([q]) \in U \) of its conformal class \([q]\) under the map \( \Omega \). Our next goal is to compute \( P(q) \) for \( q = m^*_x q_{a,c} \). To begin with, observe that

\[
(5) \quad P(m^*_x q) = (\tau/\tau) P(q)
\]

for every \( q \in Q(\mathbb{C}) \) and \( \tau \in \mathbb{C} \), \( \tau \neq 0 \). Indeed, the map \( \Omega \) obviously commutes with the multiplication maps \( m_\sigma \) for \( |\sigma| = 1 \), and hence (5) follows from Corollary 1.4. We need also an explicit description of the map \( \Omega \).

2.1. Lemma. Suppose that \( \gamma \in \mathbb{R} \), \( \sigma \in \mathbb{C} \) and \( \gamma > 0 \), \( |\sigma| = 1 \).

If \( p = \frac{1 - \gamma^2}{1 + \gamma^2} \sigma \), then \( \Omega(p) = \frac{1 - \gamma}{1 + \gamma} \sigma \).

**Proof.** The counter-clockwise rotation of \( \mathbb{C} \times \mathbb{R} \) along the vertical axis \( o \times \mathbb{R} \) by the angle \( - \arg \sigma \) reduces the proof to the case \( \sigma = 1 \). In this case everything happens in the vertical coordinate plane \( v = 0 \), and we can ignore the \( v \)-coordinate and write

\[
p = \left( \frac{1 - \gamma^2}{1 + \gamma^2}, 0 \right).
\]

By using the well known identity

\[
\left( \frac{1 - \gamma^2}{1 + \gamma^2} \right)^2 + \left( \frac{2\gamma}{1 + \gamma^2} \right)^2 = 1
\]

and the assumption \( \gamma > 0 \), we see that

\[
\pi^{-1}(p) = \left( \frac{1 - \gamma^2}{1 + \gamma^2}, \frac{2\gamma}{1 + \gamma^2} \right).
\]

We need to prove that the stereographic projection \( S \) takes this point to

\[
\left( \frac{1 - \gamma}{1 + \gamma}, 0 \right).
\]
By the definition of $S$ this means that the points 

$$(0, -1), \quad \left(\frac{1 - \gamma}{1 + \gamma}, 0\right), \quad \text{and} \quad \left(\frac{1 - \gamma^2}{1 + \gamma^2}, \frac{2\gamma}{1 + \gamma^2}\right)$$

are contained in the same line. This happens if and only if the vectors 

$$\left(\frac{1 - \gamma}{1 + \gamma}, 0\right) - (0, -1) = \left(\frac{1 - \gamma}{1 + \gamma}, 1\right) \quad \text{and} \quad \left(\frac{1 - \gamma^2}{1 + \gamma^2}, \frac{2\gamma}{1 + \gamma^2}\right) - (0, -1) = \left(\frac{1 - \gamma^2}{1 + \gamma^2}, \frac{(1 + \gamma)^2}{1 + \gamma^2}\right)$$

are proportional, i.e. if and only if 

$$\frac{1 - \gamma^2}{1 + \gamma^2} / \frac{1 - \gamma}{1 + \gamma} = \frac{(1 + \gamma)^2}{1 + \gamma^2}.$$ 

A straightforward verification of this identity completes the proof. ■

2.2. **Theorem.** If the form $q = m^* q_{a,c}$ is definite, then $c/a$ is positive and 

$$P(q) = \frac{1 - \gamma}{1 + \gamma} (\tau/\tau),$$

where $\gamma$ is the positive square root $\sqrt{c/a}$.

**Proof.** If $m^* q_{a,c}$ is definite, then $q_{a,c}$ is also definite, and hence the real numbers $a, c$ are of the same sign. Therefore $c/a$ is positive. By Theorem 1.6

$$[q] = \frac{a - c}{a + c} (\tau/\tau) = \frac{1 - c/a}{1 + c/a} (\tau/\tau) = \frac{1 - \gamma^2}{1 + \gamma^2} (\tau/\tau).$$

An application of Lemma 2.1 completes the proof. ■

**Remark.** Lemma 2.1 leads to a simple formula for the map $\Omega^{-1}$. Given $\mu \in \mathbb{U}$, let $\sigma = \mu/|\mu|$ and $\gamma = (1 - |\mu|)/(1 + |\mu|)$. Then $|\sigma| = 1$, $\gamma > 0$,

$$|\mu| = \frac{1 - \gamma}{1 + \gamma}, \quad \mu = \frac{1 - \gamma}{1 + \gamma} \sigma, \quad \frac{1 - \gamma^2}{1 + \gamma^2} = \frac{2|\mu|}{1 + |\mu|^2}, \quad \text{and hence}$$

$$\Omega^{-1}(\mu) = \frac{1 - \gamma^2}{1 + \gamma^2} \sigma = \frac{2|\mu|}{1 + |\mu|^2} \sigma = \frac{2\mu}{1 + |\mu|^2}.$$
3. The dilatation of real linear maps $C \to C$

The Poincaré dilatation of a real linear map $C \to C$. For an invertible real linear map $T : C \to C$ the pull-back $T^*[n] \in U$ is a natural measure of distortion of the standard conformal structure $[n]$ by $T$. Let us define Poincaré conformal dilatation $\pi_T$ as

$$\pi_T = \Omega (T^*[n]).$$

In order to relate the Poincaré conformal dilatation $\pi_T$ with the classical complex dilatation $\mu_T$ of $T$, let us consider the image $E$ of the unit circle in $C$ under $T$. As is well known, $E$ is an ellipse with the center at the origin.

3.1. Theorem. (i) Let $D$ be the ratio of the major axis of $E$ to the minor axis. Then

$$|\pi_T| = \frac{D - 1}{D + 1}.$$

(ii) If $\alpha$ is an angle between $R \subset C$ and a direction of the maximal distortion of $T$, then

$$\arg \pi_T = 2\alpha.$$

Proof. Let $q = T^*n$. Then $\pi_T = P(q)$ and $q(z) = n(T(z)) = \|T(z)\|^2$ for every $z \in C$. By Theorem 1.1, $q = m^*_\tau q_{a,c}$ for some $a, c \in R$ and $\tau \in C$ such that $|\tau| = 1$ and $|a| \geq |c|$. Since $q$ is positive definite together with $n$, we may assume that $a \geq c > 0$. Let $\gamma = \sqrt{c/a}$. For every $z \in C$

$$\|T(z)\|^2 = q(z) = m^*_\tau q_{a,c}(z) = q_{a,c}(m_\tau(z)) = q_{a,c}(\tau z).$$

At the same time $\|z\|^2 = \|\tau z\|^2$ because $|\tau| = 1$. It follows that the square of the distortion of $T$ in the direction of a non-zero vector $z \in C = R^2$ is equal to

$$\|T(z)\|^2/\|z\|^2 = \|q_{a,c}(\tau z)\|^2/\|z\|^2 = \|q_{a,c}(\tau z)\|^2/\|\tau z\|^2.$$

Since $a \geq c$, the ratio $\|q_{a,c}(w)\|^2/\|w\|^2$ achieves its maximum $a$ when $w \in R$, and its minimum $c$ when $w \in iR$. Hence $\|T(z)\|^2/\|z\|$ achieves its maximum $\sqrt{a}$ when $\tau z \in R$, and its minimum $\sqrt{c}$ when $\tau z \in iR$.

The length of the major axis of the ellipsis $E$ is equal to the maximal distortion of $T$, i.e. to $\sqrt{a}$, and the length of its minor axis is equal to the minimal distortion, i.e. to $\sqrt{c}$. It follows that the ratio $D$ of the axes of $E$ is equal to $\gamma^{-1} = \sqrt{a/c}$. Since $|\tau/\tau| = 1$, this fact together with Theorem 2.2 implies the part (i) of the theorem.
The distortion of $T$ in the direction of vector $z$ is maximal when $w = \tau z \in \mathbb{R} \subset \mathbb{C}$, or, what is the same, when $\arg(\tau z) = 0$ or $\pi$. Equivalently, the distortion is maximal when $\arg z = -\arg \tau$ or $-\arg \tau + \pi$. Therefore $\alpha = -\arg \tau$ or $-\arg \tau + \pi$ and hence $2\alpha = -2\arg \tau$. On the other hand, $2\arg \tau = -\arg(\tau/\tau)$ and Theorem 2.2 implies that $\arg(\tau/\tau) = \arg \pi T$. Hence $2\alpha = \arg \pi T$. This proves the part (ii).

3.2. Main Theorem. The Poincaré conformal dilatation of a real linear orientation-preserving automorphism $T: \mathbb{C} \rightarrow \mathbb{C}$ is equal to its classical complex dilatation.

Proof. By Theorem 3.1, the Poincaré conformal dilatation satisfies properties (i) and (ii) from the Introduction (or, rather, their analogues for real linear maps) characterizing the classical complex dilatation. ■

4. The action of real linear maps $\mathbb{C} \rightarrow \mathbb{C}$

Automorphisms induced by a real linear invertible maps $\mathbb{C} \rightarrow \mathbb{C}$. Let $u: \mathbb{C} \rightarrow \mathbb{C}$ is a real linear invertible map. Then the pull-back map $u^*: \mathbb{Q}(\mathbb{C}) \rightarrow \mathbb{Q}(\mathbb{C})$ is an automorphism of the real vector space $\mathbb{Q}(\mathbb{C})$ and hence induces an automorphism $\mathbb{P}u^*: \mathbb{P}\mathbb{Q}(\mathbb{C}) \rightarrow \mathbb{P}\mathbb{Q}(\mathbb{C})$ of the projective plane $\mathbb{P}\mathbb{Q}(\mathbb{C})$. Clearly, a quadratic form $q \in \mathbb{Q}(\mathbb{C})$ is definite if and only if the pull-back $u^*q$ is definite, and hence $\mathbb{P}u^*$ leaves the space $\mathbb{U}(\mathbb{C})$ of conformal structures on $\mathbb{C}$ invariant. After the identification of $\mathbb{Q}(\mathbb{C})$ with $\mathbb{R} \oplus \mathbb{C}$ and $\mathbb{A}(\mathbb{C})$ with $\mathbb{C}$ the space $\mathbb{U}(\mathbb{C})$ turns into the open unit disc $\mathbb{U}$ in the complex plane $\mathbb{C}$, and the automorphism $\mathbb{P}u^*$ turns into an automorphism of the projective plane $\mathbb{P}(\mathbb{R} \oplus \mathbb{C})$ leaving $\mathbb{U}$ invariant.

The structure of the Klein model of the hyperbolic plane on $\mathbb{U}$ is defined in terms of $\mathbb{U}$ and the geometry of the projective plane $\mathbb{P}(\mathbb{R} \oplus \mathbb{C})$. Therefore this structure is invariant under any automorphism of this projective plane leaving $\mathbb{U}$ invariant. In particular, it is invariant under $\mathbb{P}u^*$. In other terms, $\mathbb{P}u^*$ is an automorphism of the Klein model. Passing to the Poincaré unit disc model of the hyperbolic plane turns $\mathbb{P}u^*$ into an automorphism of the Poincaré model. In more details, using the map $\Omega$ to pass to the Poincaré model turns $\mathbb{P}u^*$ into the map (denoted by $u^*$ again)

$$u^* = \Omega \circ \mathbb{P}u^* \circ \Omega^{-1}: \mathbb{U} \rightarrow \mathbb{U}$$

and this map is an automorphism of the Poincaré unit disc model. It is easy to see that if $u$ is orientation preserving, then $u^*: \mathbb{U} \rightarrow \mathbb{U}$ is also orientation-preserving.
4.1. Lemma. If \( T: \mathbb{C} \rightarrow \mathbb{C} \) is a real linear invertible map, then \( \pi_{T \circ u} = u^*(\pi_T) \).

Proof. Obviously, \((T \circ u)^* n = (u^* \circ T^*)(n) = u^*(T^* n)\) and hence
\[
[(T \circ u)^* n] = P u^* (T^* n).
\]

By applying \( \Omega \) to the both part of this equality we see that
\[
\pi_{T \circ u} = \Omega \left( [(T \circ u)^* n] \right) = \Omega \circ P u^* \circ \Omega^{-1}(\Omega[T^* n]) = u^*(\pi_T). \]

4.2. Theorem. If \( u \) is orientation-preserving, then \( u^* \) is complex analytic and
\[
\mu_{T \circ u} = u^*(\mu_T)
\]
for every real linear orientation-preserving invertible map \( T: \mathbb{C} \rightarrow \mathbb{C} \).

Proof. If \( u \) is orientation-preserving, then \( u^* \) is also orientation-preserving. Since every orientation-preserving automorphism of the Poincaré model is complex analytic, it follows that \( u^* \) is complex analytic. Moreover, if \( u \) is orientation-preserving, then \( T \circ u \) is orientation-preserving together with \( T \). Therefore, Theorem 3.2 implies that \( \mu_{T \circ u} = \pi_{T \circ u} \) and \( \mu_T = \pi_T \). It remains to apply Lemma 4.1. ■

5. Complex vector spaces of dimension 1

**Quadratic forms associated with non-zero vectors.** The goal of this section is to extend the notion of the Poincaré conformal dilatation from \( \mathbb{C} \) to complex vector spaces \( V \) of dimension 1. Let us start with elementary observations about such vector spaces. A basis of \( V \) over \( \mathbb{C} \) is just a non-zero vector \( v \in V \). Hence, for any non-zero \( v \in V \) there is a unique isomorphism of complex vector spaces \( f_v: V \rightarrow \mathbb{C} \) taking \( v \) to \( 1 \in \mathbb{C} \). For a non-zero vectors \( v \in V \) we will denote by \( q_v \) the pull-back quadratic form \( f_v^* n \). If \( V = \mathbb{C} \) and \( v = 1 \), then \( f_1 = f_v = \text{id}_\mathbb{C} \) and hence \( q_1 = n \).

5.1. Lemma. Let \( u, v \) be two non-zero vectors in \( V \). Let \( a \) be the unique complex number such that \( v = au \). Then \( f_u(v) = a \) and \( f_u = af_v = m_a \circ f_v \).

Proof. Obviously, \( f_u(v) = f_u(au) = af_u(u) = a1 = a \) and \( af_v(v) = a1 = a \). Hence \( f_u \) and \( af_v \) agree on the basis of \( V \) formed by the vector \( v \). It follows that \( f_u = af_v \). Since \( af_v = m_a \circ f_v \), this completes the proof. ■
5.2. Lemma. Let \( u, v \) be two non-zero vectors in \( V \), and let \( a \) be the unique complex number such that \( v = au \). Then \( q_v = |a|^{-2} q_u \).

**Proof.** Let us consider the case \( V = C \), \( u = 1 \) first. In this case \( v = a \) and we need to prove that \( q_a = |a|^{-2} n \). Let \( b = a^{-1} \). Then \( m_b(a) = 1 \), and hence \( f_a = m_b \). It follows that \( q_a = m_b^* n \). By Section 1 \( m_b^* n = |b|^2 n \), and hence

\[
q_a = m_b^* n = |b|^2 n = |a|^{-2} n.
\]

This proves the lemma in the case \( V = C \), \( u = 1 \).

In the general case \( f_u(v) = a \) by Lemma 5.1 and \( f_a(a) = 1 \) by the definition of \( f_a \). It follows that \( f_a \circ f_u(v) = f_a(f_u(v)) = f_a(a) = 1 \) and hence \( f_v = f_a \circ f_u \) and

\[
q_v = f_v^* n = (f_a \circ f_u)^* n = f_u^* (f_a^* n) = f_u^* q_a.
\]

By the already proved special case, \( q_a = |a|^{-2} q, = |a|^{-2} n \). It follows that

\[
q_v = f_u^* q_a = f_u^* (|a|^{-2} n) = |a|^{-2} f_u^* n = |a|^{-2} q_u. \quad \blacksquare
\]

5.3. Corollary. The conformal class of \( q_v \) does not depend on the choice of non-zero \( v \in V \).

**Proof.** Indeed, if \( u, v \) are non-zero vectors in \( V \), then \( v = au \) for some non-zero \( a \in C \) and hence \( q_v = |a|^{-2} q_u \). Since \( |a|^{-2} \) is a positive real number, the quadratic forms \( q_w \) and \( q_v \) have the same conformal class. \( \blacksquare \)

The Poincaré conformal dilatation of real linear maps. Let \( V, W \) be complex vector spaces of dimension 1. Let \( T : V \to W \) be a real-linear map. Let us choose non-zero vectors \( v \in V, w \in W \). If we identify both \( V \) and \( W \) with \( C \) by the maps \( f_v : V \to C \) and \( f_w : W \to C \) respectively, the map \( T \) will turn into a real linear map \( C \to C \), allowing to speak about its Poincaré complex dilatation as defined in Section 3. Using these identification is equivalent to replacing \( T \) by the map \( f_w \circ T \circ f_v^{-1} \) considered as a map \( C \to C \). Naively, one may try to define the Poincaré conformal dilatation of \( T \) as the Poincaré complex dilatation \( \pi_T(v, w) \) of \( f_w \circ T \circ f_v^{-1} \). As we will see in a moment, \( \pi_T(v, w) \) does not depends on \( w \), but does depends on \( v \).

5.4. Lemma. \( \pi_T(v, w) \) does not depend on \( w \) and hence can be denoted by \( \pi_T(v) \).

**Proof.** It is sufficient to prove that the conformal class of the form \( (f_w \circ T \circ f_v^{-1})^* n \) does not depend on \( w \). Since \( (f_w \circ T \circ f_v^{-1})^* n = (T \circ f_v^{-1})^* (f_w^* n) = (T \circ f_v^{-1})^* q_w \), Corollary 5.3 implies that this is indeed the case. \( \blacksquare \)
5.5. Lemma. Let \( u, v \) be two non-zero vectors in \( V \), and let \( a \) be the unique complex number such that \( v = au \). Then \( \pi_T(v) = (\overline{a}) \pi_T(u) \).

Proof. Let \( w \in W \), \( w \neq 0 \). By the definition, \( \pi_T(v) \) and \( \pi_T(u) \) are equal to the Poincaré complex dilatations of the maps \( f_w \circ T \circ f_v^{-1} \) and \( f_w \circ T \circ f_u^{-1} \) respectively. Lemma 5.1 implies that \( f_u = m_a \circ f_v \) and hence \( f_v^{-1} = f_u^{-1} \circ m_a \) and
\[
    f_w \circ T \circ f_v^{-1} = f_w \circ T \circ f_u^{-1} \circ m_a = (f_w \circ T \circ f_u^{-1}) \circ m_a.
\]
It follows that
\[
(6) \quad (f_w \circ T \circ f_u^{-1})^* n = ((f_w \circ T \circ f_u^{-1}) \circ m_a)^* n = m_a^* \left( (f_w \circ T \circ f_u^{-1})^* n \right).
\]
Let \( q = (f_w \circ T \circ f_u^{-1})^* n \). Then (6) implies that \( (f_w \circ T \circ f_u^{-1})^* n = m_a^* q \), and hence
\[
\pi_T(v) = P(m_a^* q) = (\overline{a}/a) P(q) = (\overline{a}/a) \pi_T(u),
\]
where the second equality follows from (5). \( \blacksquare \)

The dual and the conjugate vector spaces. Lemma 5.5 means that the Poincaré complex dilatation \( \pi_T(v) \) is a quantity depending on the choice of a basis of \( V \). Moreover, this quantity depends on the choice in a way making it a tensor in a classical sense. Now we will explain what this means in the modern language.

Let \( V \) be a complex vector space. Recall that its dual vector space \( V^* \) has as vectors the complex linear maps \( V \rightarrow \mathbb{C} \). If \( a \in \mathbb{C} \) and \( f \in V^* \), then \( af \) is defined by \( af(v) = a(f(v)) = f(av) \). The conjugate vector space \( \overline{V} \) of \( V \) is equal to \( V \) as a real vector space, but the multiplication by \( a \in \mathbb{C} \) in \( \overline{V} \) is defined as the multiplication by \( \overline{a} \) in \( V \). We will denote this multiplication by \( v \mapsto a \cdot v \), so that \( a \cdot v = \overline{a} v \).

Suppose now that the complex dimension of \( V \) is 1. Every non-zero \( v \in V \) forms a basis of both \( V \) and \( \overline{V} \). The dual basis of \( \overline{V}^* \) consists of the map \( \nu^*: \overline{V} \rightarrow \mathbb{C} \) defined by \( \nu^*(w) = \overline{f_v(w)} \). Indeed, \( \nu^*(v) = f_v(v) = 1 \), and
\[
\nu^*(a \cdot w) = \overline{f_v(a \cdot w)} = \overline{f_v(\overline{a}w)} = \overline{a} \overline{f_v(w)} = a \overline{f_v(w)} = a \nu^*(w),
\]
i.e. \( \nu^* \) is complex-linear. The tensor product \( \overline{V}^* \otimes V \) is also of complex dimension 1. Any non-zero vector \( v \in V \) leads to a basis of \( \overline{V}^* \otimes V \) consisting of the vector \( \nu^* \otimes v \).

5.6. Theorem. Let \( V, W \) be complex vector spaces of dimension 1. Let \( T: V \rightarrow W \) be a real linear map. Then \( \pi_T(v) (\nu^* \otimes v) \) does not depend on the choice of a non-zero \( v \in V \).
Suppose that \( v, u \in V \) are non-zero. Then \( v = au \) for some \( a \in \mathbb{C}^* \). By Lemma 5.1 \( f_u = a f_v \). It follows that \( u^\bullet = \overline{a} v^\bullet \). Therefore \( (\overline{a})^{-1} u^\bullet = v^\bullet \) and

\[
v^\bullet \otimes v = ( (\overline{a})^{-1} u^\bullet ) \otimes au = (a/\overline{a}) (u^\bullet \otimes u)
\]

By Lemma 5.5 \( \pi_T(v) = (\overline{a}/a) \pi_T(u) \). It follows that

\[
\pi_T(v) (v^\bullet \otimes v) = (\overline{a}/a) \pi_T(u) (v^\bullet \otimes v) = (a/\overline{a}) \pi_T(u) (u^\bullet \otimes u) = \pi_T(u) (u^\bullet \otimes u).
\]

Therefore, \( \pi_T(v) (v^\bullet \otimes v) \) indeed does not depend on the choice of \( v \). ■

**Poincaré conformal dilatation as a tensor.** In view of Theorem 5.6 we can define the Poincaré conformal dilatation \( \pi_T \) of \( T \) by the formula \( \pi_T(v) (v^\bullet \otimes v) \in V^* \otimes V \) for any non-zero \( v \in V \). Suppose that \( V = W = \mathbb{C} \). In this case the identification of \( \mathbb{C}^* \otimes \mathbb{C} \) with \( \mathbb{C} \) by the isomorphism taking \( 1^\bullet \otimes 1 \) to \( 1 \) turns \( \pi_T \in \mathbb{C}^* \otimes \mathbb{C} \) into the Poincaré conformal dilations \( \pi_T \) from Section 3, as one easily checks. This easily implies that the Poincaré conformal dilation agrees with the classical one in general.

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