Compatibility of any pair of 2-outcome measurements characterizes the Choquet simplex

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$1_K \equiv 1$ is the unit constant functional. A measurement belonging to $A(K)^m$ is called $m$-outcome. Two measurements $(f_k)_{k=1}^m$ and $(g_j)_{j=1}^n$ on $A(K)$ are called compatible (or jointly measurable) if there exists a family $(h_{k,j})_{1\leq k \leq m, 1\leq j \leq n}$ in $A(K)$ such that

$$h_{k,j} \geq 0, \quad f_k = \sum_{j'=1}^n h_{k,j'}, \quad g_j = \sum_{k'=1}^m h_{k',j} \quad (1 \leq k \leq m, 1 \leq j \leq n).$$

In [5] Plávala showed that for finite-dimensional $K$, $K$ is a simplex if and only if any pair of 2-outcome measurements are compatible. In the physical context, a compact convex set $K$ corresponds to a state space of a general physical system and Plávala’s result indicates that incompatibility of measurements [3] characterizes non-classicality of a physical system. The proof in [5] depends on the notion of the maximal face and is not straightforwardly applicable to infinite-dimensional compact sets. The purpose of this paper is to generalize this result to an arbitrary compact convex set $K$ (Theorems 1 and 2).

The first main result of this paper is the following theorem.

**Theorem 1** Let $K$ be a compact convex subset of a locally convex Hausdorff space. Then $K$ is a Choquet simplex if and only if any pair $(f_k)_{k=1}^m$ and $(g_j)_{j=1}^n$ of 2-outcome measurements on $A(K)$ are compatible.

Here a compact convex set $K$ is called a Choquet simplex if the Banach dual $A(K)^\ast$ equipped with the positive cone $A(K)^\ast_+ = \{ f \in A(K) \mid f(x) \geq 0 \ (\forall x \in K) \}$ is a lattice.

A Banach space $E$ is said to have a Banach predual $E_\ast$ if $E_\ast$ is Banach space such that its Banach dual $(E_\ast)^\ast$ is isometrically isomorphic to $E$. We can and do take such $E_\ast$, if exists, as a closed linear subspace of the Banach dual $E^\ast$. The second main result of this paper is then the following theorem.

**Theorem 2** Let $K$ be a compact convex subset of a locally convex Hausdorff space. Suppose that $A(K)$ has a Banach predual. Then the following conditions are equivalent.

(i) $K$ is a Bauer simplex.

(ii) $K$ is a Choquet simplex.

(iii) Any pair $(f_k)_{k=1}^m$ and $(g_j)_{j=1}^n$ of 2-outcome measurements on $A(K)$ are compatible.

Here $K$ is a Bauer simplex if the set $\partial K$ of extremal points of $K$ is compact and any point in $K$ is the barycenter of a unique boundary measure.

The above theorems are rephrased in terms of order-unit Banach space by using the following one-to-one correspondence ([1], Sections II.1 and II.2). A pair $(E, e)$ of Archimedean ordered linear space $E$ and an order unit $e \in E$ is called an order-unit Banach space if the order-unit norm $\| \cdot \|$ on $E$ induced by $e$ is complete. We write by $E_\ast_+$ the positive cone of $E$. For each order-unit
Banach space \((E,e)\) its dual \(E^*\) is a base-normed ordered linear space with the positive cone \(E^*_+ := \{ \phi \in E^* \mid \phi(a) \geq 0 \ (a \in E_+) \}\) and the base \(S(E) := \{ \phi \in E^*_+ \mid \|\phi\| = \phi(e) = 1 \}\). Each element of \(S(E)\) is called a state on \(E\). \(S(E)\) is a weakly* compact convex subset of \(E^*\) and the order-unit Banach space \((A(S(E)), 1_{S(E)})\) is shown to be order and isometrically isomorphic to \((E,e)\).

Conversely for any compact convex subset \(K\) of a locally convex Hausdorff space, \((A(K), 1_K)\) is an order-unit Banach space and \(K\) is continuously affine isomorphic to the set \(S(A(K))\) of states on \(A(K)\).

We also define the \((m\text{-outcome})\) measurement on an order-unit Banach space \((E,e)\) as a finite-sequence (belonging to \(E^m\)) of positive elements of \(E\) normalized to \(e\). The compatibility of measurements is defined in the same way as that of measurements on \(A(K)\). By this correspondence we can readily see that Theorems 1 and 2 are respectively equivalent to

**Theorem 3** Let \((E,e)\) be an order-unit Banach space. Then \(E^*\) is a lattice if and only if any pair of 2-outcome measurements \((a_j)_{j=1}^2\) and \((b_k)_{k=1}^2\) on \((E,e)\) are compatible.

**Theorem 4** Let \((E,e)\) be an order-unit Banach space. Suppose that \(E\) has a Banach predual \(E_*\). Then the following conditions are equivalent.

(i) \(S(E)\) is a Bauer simplex.

(ii) \(E^*\) is a lattice.

(iii) Any pair of 2-outcome measurements \((a_j)_{j=1}^2\) and \((b_k)_{k=1}^2\) on \((E,e)\) are compatible.

In the rest of this paper, after introducing some preliminaries in Section 2, we prove in Section 3 the main results Theorems 3 and 4.

**2 Preliminary**

This section reviews necessary results of order-unit Banach spaces with Banach preduals and those of simplexes. The reader is referred to [1] and [2,4] for the complete proofs of the facts given in this section.

Let \((E,e)\) be an order-unit Banach space with a Banach predual \(E_*\). Then the predual \(E_*\) endowed with the predual positive cone

\[ E_*^+ := \{ \psi \in E_* \mid \psi(a) \geq 0 \ (\forall a \in E_+) \} \]

has the base

\[ S_* := \{ \psi \in E_*^+ \mid \psi(e) = 1 \} = E_* \cap S(E) \]

and the base norm on \(E_*\) induced by \(S_*\) coincides with the predual norm

\[ \|\psi\| = \sup_{a \in E : \|a\| \leq 1} |\psi(a)| \ (\psi \in E_*), \]
which is the norm on $E^*$ restricted to $E_*$. Moreover the positive cone $E_+^*$ is weakly* closed (i.e. $\sigma(E, E_*)$-closed). Therefore by the bipolar theorem $E_+^*$ is the dual cone of $E_{++}$, i.e.

$$E_+^* = \{ a \in E \mid \psi(a) \geq 0 ( \forall \psi \in E_{++}) \} .$$

By the Banach-Alaoglu theorem, the unit ball of $E$ is weakly* compact and hence so is any interval

$$[a, b] := \{ x \in E \mid a \leq x \leq b \} \quad (a, b \in E).$$

A net $(x_i)_{i \in I}$ in $E$ is called bounded if $x_i \leq a \ (\forall i \in I)$ for some $a \in E$ and increasing if $i \leq i'$ implies $x_i \leq x_{i'} \ (i, i' \in I)$. We denote the supremum of the subset $\{ x_i \mid i \in I \}$ of $E$ by $\sup_{i \in I} x_i$ if it exists.

The following lemma will be used in the proof of Theorem \[4\].

**Lemma 1** Let $(E, e)$ be an order-unit Banach space with a Banach predual $E_*$. Then any bounded increasing net $(x_i)_{i \in I}$ in $E$ is weakly* convergent to the supremum $\sup_{i \in I} x_i$.

**Proof** Take an element $a \in E$ such that $x_i \leq a \ (i \in I)$. For a fixed $i_0 \in I$ we have $x_{i_0} \in [x_{i_0}, a]$ eventually and by the compactness of $[x_{i_0}, a]$ there exists a subnet $(x_{i(j)})_{j \in J}$ weakly* converging to some $x \in [x_{i_0}, a]$. Since $x_{i_0} \leq x_{i(j)}$ eventually for each $i \in I$, the weak* closedness of the positive cone $E_+^*$ implies $x_i \leq x$ for all $i \in I$. Moreover if $x_i \leq b \in E \ (i \in I)$, then again by the weak* closedness of $E_+^*$ we have $x \leq b$. Therefore $x = \sup_{i \in I} x_i$. Take arbitrary $\psi \in E_{++}$. Then $(\psi(x_i))_{i \in I}$ is an increasing net in $\mathbb{R}$ and

$$\psi(x) \geq \sup_{i \in I} \psi(x_i) \geq \sup_{j \in J} \psi(x_{i(j)}) = \psi(x),$$

which implies $\psi(x_i) \uparrow \psi(x)$. Since $E_*$ is the linear span of $E_{++}$, this implies $x_i \xrightarrow{\text{weakly*}} x$. \[ \square \]

Let $(E, e)$ be an order-unit Banach space. We define the linear order $\leq$ on the double dual space $E^{**}$ corresponding to the double dual cone

$$E^{**} := \{ x'' \in E^{**} \mid \langle x'', \psi \rangle \geq 0 \ (\forall \psi \in E_+^*) \},$$

where we introduced the notation $\langle x'', \psi \rangle := x''(\psi) \ (x'' \in E^{**}, \psi \in E^*)$. As usual we regard $E$ as a norm-closed subspace of $E^{**}$. Then by the bipolar theorem we have $E^{**} \cap E = E_+$, which implies that the order on $E^{**}$ restricted to $E$ coincides with the original order on $E$. Moreover $(E^{**}, e)$ is an order-unit Banach space with the Banach predual $E^*$ and $E_+^{**}$ is the weak* closure of $E_+$ by the bipolar theorem. Let us introduce the sets

$$\mathcal{E}(E) := \{ x \in E \mid 0 \leq x \leq e \},$$

$$\mathcal{E}(E^{**}) := \{ x'' \in E^{**} \mid 0 \leq x'' \leq e \} .$$

Each element of $\mathcal{E}(E)$ or $\mathcal{E}(E^{**})$ is called an *effect*. 
Lemma 2 Let \((E, e)\) be an order-unit Banach space. Then \(\mathcal{E}(E)\) is weakly* dense in \(\mathcal{E}(E^{**})\).

Proof Suppose that \(\mathcal{E}(E)\) is not weakly* dense in \(\mathcal{E}(E^{**})\). Then by the Hahn-Banach separation theorem, there exist \(\psi \in E^{*}\) and \(x'' \in \mathcal{E}(E^{**})\) such that

\[
\sup_{x \in \mathcal{E}(E)} \langle x, \psi \rangle < \langle x'', \psi \rangle.
\]

Since the unit ball of \(E\) coincides with

\[
\{-e + 2x \mid x \in \mathcal{E}(E)\},
\]

we have

\[
\|\psi\| = \sup_{x \in \mathcal{E}(E)} \langle -e + 2x, \psi \rangle < \langle -e + 2x'', \psi \rangle \leq \| -e + 2x''\|\|\psi\| \leq \|\psi\|,
\]

which is a contradiction. \(\square\)

The following well-known characterizations of the Choquet and Bauer simplices are also necessary.

Lemma 3 ([1], II.3.1 and II.3.11) Let \((E, e)\) be an order-unit Banach space. Then \(E^{*}\) is a lattice if and only if \(E\) has the Riesz decomposition property, i.e.

\[
0 \leq u \leq v_1 + v_2, \quad 0 \leq v_1, v_2
\]

\[
\implies \exists u_1, u_2 \in E: 0 \leq u_j \leq v_j (j = 1, 2), \quad u = u_1 + u_2
\]

holds for any \(u, v_1, v_2 \in E\).

Lemma 4 ([1],) Let \((E, e)\) be an order-unit Banach space. Then \(S(E)\) is a Bauer simplex if and only if \(E\) is a lattice.

3 Proofs of Theorems 3 and 4

In this section, we first prove Theorem 3 and then Theorem 4 by reducing to Theorem 3.

Let \((E, e)\) be an order-unit Banach space. For each effect \(a \in \mathcal{E}(E)\) there corresponds a 2-outcome measurement \((a, e - a)\) on \((E, e)\). Two effects \(a, b \in E\) are said to be compatible if the corresponding 2-outcome measurements \((a, e - a)\) and \((b, e - b)\) are compatible. The following lemma is immediate from the definition of the compatibility (cf. [5], Proposition 7).

Lemma 5 Let \((E, e)\) be an order-unit Banach space. Then effects \(a, b \in \mathcal{E}(E)\) are compatible if and only if there exists \(c \in E\) such that

\[
0, a + b - e \leq c \leq a, b.
\]

We also introduce the following notion of orthogonality.
Definition 1 Positive elements $a$ and $b$ in an ordered linear space $E$ is said to be orthogonal if

$$0 \leq c \leq a, b \implies c = 0$$

for any $c \in E$.

Then we have

Lemma 6 Let $(E,e)$ be an order-unit Banach space satisfying the condition (iii) of Theorem 4 and let $a, b \in E$ be orthogonal positive elements. Suppose that $a \neq 0$. Then for any $\psi \in S(E)$ with $\psi(a) = \|a\|$, it holds that $\psi(b) = 0$.

Proof The statement is obvious when $b = 0$ and we assume $b \neq 0$. Then since the effects $\|a\|^{-1}a$ and $\|b\|^{-1}b$ are compatible, by Lemma 5 there exists $c \in E$ such that

$$0, \|a\|^{-1}a + \|b\|^{-1}b - e \leq c \leq \|a\|^{-1}a, \|b\|^{-1}b.$$

This implies

$$0 \leq \min(\|a\|, \|b\|)c \leq a, b$$

and hence the orthogonality of $a$ and $b$ implies $c = 0$. Therefore

$$\|a\|^{-1}a + \|b\|^{-1}b - e \leq 0.$$

Thus for any state $\psi \in S(E)$ with $\psi(a) = \|a\|$ we have

$$0 \leq \|b\|^{-1}\psi(b) \leq \psi(e - \|a\|^{-1}a) = 0,$$

which implies $\psi(b) = 0$. $\square$

Proof of Theorem 4

(i) $\implies$ (ii) is obvious.

(ii) $\implies$ (i). This can be proved in the same way as in Proposition II.3.2 of [1] by noting that the set

$$\{ w \in E \mid x, y \leq w \leq z \} = [x, z] \cap [y, z]$$

for any $x, y, z \in E$ is weakly$^*$ compact.

(ii) $\implies$ (iii). Assume (ii) and take arbitrary 2-outcome measurements $(a_1, a_2) = (a_1, e - a_1)$ and $(b_1, b_2) = (b_1, e - b_1)$ on $(E,e)$. Then by $a_1 \leq e = b_1 + b_2$ and the Riesz decomposition property of $E$, there exist elements $c_{1,1}, c_{1,2} \in E$ such that

$$0 \leq c_{1,j} \leq b_j (j = 1, 2), \quad a_1 = c_{1,1} + c_{1,2}.$$

If we put $c_{2,j} := b_j - c_{1,j} \geq 0$, we have

$$b_j = c_{1,j} + c_{2,j} (j = 1, 2),$$

$$a_2 = e - a_1 = b_1 + b_2 - c_{1,1} - c_{1,2} = c_{2,1} + c_{2,2}.$$

Thus $(a_1, a_2)$ and $(b_1, b_2)$ are compatible.
Compatibility of any measurements characterizes the Choquet simplex

(3) $\implies$ (1). We assume (3) and prove the Riesz decomposition property of $E$. Take elements $u, v_1, v_2 \in E$ satisfying $$0 \leq u \leq v_1 + v_2, \quad 0 \leq v_1, v_2.$$ Let

$$A := \{ (x_1, x_2) \in E \times E \mid 0 \leq x_j \leq v_j (j = 1, 2), x_1 + x_2 \leq u \},$$

which is non-empty by $(0, 0) \in A$, and define a partial order $\leq$ on $A$ by

$$(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \leq y_2 \quad ((x_1, x_2), (y_1, y_2) \in A).$$

Let $D$ be a directed subset of $A$ and for each $\delta \in D$ we write $\delta = (x^\delta_1, x^\delta_2)$. Then for each $j = 1, 2$ the net $(x^\delta_j)_{\delta \in D}$ in $E$ is increasing and upper bounded by $v_j$. Thus by Lemma 1

$$x^\delta_j \xrightarrow{\text{weakly }^*} \sup_{\delta \in D} x^\delta_j =: x_j \in E.$$

By the weak* closedness of $E_+$, we have $(x_1, x_2) \in A$ and hence $(x_1, x_2)$ is a supremum of $D$ in $A$. Therefore by Zorn’s lemma there exists an element $(u_1, u_2) \in A$ maximal with respect to the order $\leq$ on $A$.

We next prove that $u - u_1 - u_2 \geq 0$ and $v_1 - u_1 \geq 0$ are orthogonal. Take $c \in E$ such that

$$0 \leq c \leq u - u_1 - u_2, v_1 - u_1.$$

Then this implies $(u_1, u_2) \leq (u_1 + c, u_2) \in A$ and hence by the maximality of $(u_1, u_2)$ we have $c = 0$. Therefore $u - u_1 - u_2$ and $v_1 - u_1$ are orthogonal. Similar proof also applies to the orthogonality of $u - u_1 - u_2$ and $v_2 - u_2$.

Now assume $u - u_1 - u_2 \neq 0$. We then take a state $\psi \in S(E)$ such that

$$\psi(u - u_1 - u_2) = \| u - u_1 - u_2 \| > 0.$$  

Then by Lemma 2 we have $\psi(v_1 - u_1) = \psi(v_2 - u_2) = 0$ since $u - u_1 - u_2$ and $v_j - u_j$ are orthogonal $(j = 1, 2)$. Hence, by noting $u \leq v_1 + v_2$, we have

$$0 < \psi(u - u_1 - u_2) = \psi(u_1 - v_1 - v_2) \leq 0,$$

which is a contradiction. Therefore $u = u_1 + u_2$, which proves the Riesz decomposition property of $E$.

Proof of Theorem 3. The proof of (3) $\implies$ (1) in Theorem 1 also applies to the “only if” part of the statement. To establish the “if” part, we assume that any pair of 2-outcome measurements on $(E, e)$ is compatible and show that $E^*$ is a lattice.

We first prove that any 2-outcome measurements on $(E^{**}, e)$ is compatible. Since each 2-outcome measurements corresponds to an effect by

$$\mathcal{E}(E^{**}) \ni x'' \mapsto (x'', e - x''),$$
we have only to prove that any effects $a'', b'' \in \mathcal{E}(E^{**})$ are compatible. By Lemma 2 there exist nets $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ in $\mathcal{E}(E)$ such that $a_i \xrightarrow{\text{weakly}^*} a''$ and $b_i \xrightarrow{\text{weakly}^*} b''$. Since the effects $a_i$ and $b_i$ are compatible for each $i \in I$, there exists $c_i \in E$ such that

$$0, a_i + b_i - e \leq c_i \leq a_i, b_i.$$ 

Since $c_i$ is bounded, the Banach-Alaoglu theorem implies that there exists a subnet of $(c_i)_{i \in I}$ weakly* converging to some $c'' \in E^{**}$. By the weak* closedness of the positive cone $E^{**}_+$, we have

$$0, a'' + b'' - e \leq c'' \leq a'', b'' ,$$

which implies the compatibility of the effects $a''$ and $b''$.

Now by Theorem 4, $E^{**}$ is a lattice and hence by Corollary 1 of [2] $E^*$ is a lattice, which completes the proof. □

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