Gröbner-Shirshov bases for semirings∗

L. A. Bokut†
School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
Sobolev Institute of Mathematics, Russian Academy of Sciences
Siberian Branch, Novosibirsk 630090, Russia
Email: bokut@math.nsc.ru

Yuqun Chen‡ and Qiuhui Mo
School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
Email: yqchen@scnu.edu.cn
scnuhuashimomo@126.com

In Memorial of Jean-Louis Loday

Abstract: In the paper, we establish Gröbner-Shirshov bases for semirings and commutative semirings. As applications, we obtain Gröbner-Shirshov bases and A. Blass’s (1995) and M. Fiore -T. Leinster’s (2004) normal forms of the semirings \( \mathbb{N}[x]/(x = 1 + x + x^2) \) and \( \mathbb{N}[x]/(x = 1 + x^2) \) with one generator \( x \) and one defining relation, correspondingly.

Key words: Gröbner-Shirshov basis, semiring, congruence, normal form.

AMS 2010 Subject Classification: 16Y60, 16S15, 13P10

1 Introduction

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [37, 39], free Lie algebras [38, 39] and implicitly free associative algebras [38, 39] (see also [2, 5]), by H. Hironaka [30] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [20] for ideals of the polynomial algebras.

∗Supported by the NNSF of China (No. 11171118), the Research Fund for the Doctoral Program of Higher Education of China (No. 20114407110007) and the NSF of Guangdong Province (No. S201101003374).
†Supported by RFBR 09-01-00157, LSS-3669.2010.1 and SB RAS Integration grant No. 2009.97 (Russia) and Federal Target Grant Scientific and educational personnel of innovation Russia for 2009-2013 (government contract No. 02.740.11.5191).
‡Corresponding author.
Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra, see, for example, the books [11, 19, 21, 22, 27, 28], the papers [2, 14, 5], and the surveys [6, 14, 16, 17, 18].

Up to now, different versions of Composition-Diamond lemma are known for the following classes of algebras apart those mentioned above: (color) Lie super-algebras [32, 33, 34], Lie $p$-algebras [33], associative conformal algebras [15], modules [26, 31] (see also [24]), right-symmetric algebras [11], dialgebras [9], associative algebras with multiple operators [13], Rota-Baxter algebras [10], and so on.

It is well known Shirshov’s result [36, 39] that every finitely or countably generated Lie algebra over a field $k$ can be embedded into a two-generated Lie algebra over $k$. Actually, from the technical point of view, it was a beginning of the Gröbner-Shirshov bases theory for Lie algebras (and associative algebras as well). Another proof of the result using explicitly Gröbner-Shirshov bases theory is refereed to L.A. Bokut, Yuqun Chen and Qiuhui Mo [12].

A.A. Mikhalev and A.A. Zolotykh [35] prove the Composition-Diamond lemma for a tensor product of a free algebra and a polynomial algebra, i.e. they establish Gröbner-Shirshov bases theory for associative algebras over a commutative algebra. L.A. Bokut, Yuqun Chen and Yongshan Chen [7] prove the Composition-Diamond lemma for a tensor product of two free algebras. Yuqun Chen, Jing Li and Mingjun Zeng [25] prove the Composition-Diamond lemma for a tensor product of a non-associative algebra and a polynomial algebra.

L.A. Bokut, Yuqun Chen and Yongshan Chen [8] establish the Composition-Diamond lemma for Lie algebras over a polynomial algebra, i.e. for “double free” Lie algebras. It provides a Gröbner-Shirshov bases theory for Lie algebras over a commutative algebra. Yuqun Chen and Yongshan Chen [23] establish the Composition-Diamond lemma for matabelian Lie algebras.

In this paper, we establish Gröbner-Shirshov bases for semirings and commutative semirings. We show that for a given monomial ordering on the free (commutative) semiring, each ideal of the free (commutative) semiring algebra has the unique reduced Gröbner-Shirshov basis.

In 2004, M. Fiore and T. Leinster [29] find a strongly normalizing reduction system and a normal form of the semiring $\mathbb{N}[x]/(x = 1 + x + x^2)$ where $\mathbb{N}$ is the set of natural numbers which is regarded as a semiring and $(x = 1 + x + x^2)$ is the congruence on the semiring $\mathbb{N}[x]$ generated by $x = 1 + x + x^2$ $(x = 1 + x^2)$. In 1995, A. Blass [3] finds a normal form of the semiring $\mathbb{N}[x]/(x = 1 + x^2)$. Now, we use the Composition-Diamond lemma for commutative semirings to find Gröbner-Shirshov bases and Fiore-Leinster’s and Blass’s normal forms for the above semirings respectively. Also we show that each congruence of the semiring $\mathbb{N}$ is generated by one element and that the commutative semiring $\mathbb{N}[x]$ is not Noetherian.

## 2 Free semiring

Let $A$ be an $\Omega$-system, $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ where $\Omega_n$ is the set of $n$-ary operations, for example, ary$(\delta) = n$ if $\delta \in \Omega_n$. We will call $A$ an $\Omega$-groupoid and each $\omega$ would be called a product.
Let $k$ be a field and $kA$ the groupoid algebra over $k$, i.e. the $k$-linear space with a $k$-basis $A$ and linear multiple products $\omega \in \Omega$ that are extended by linearity from $A$ to $kA$. Such $kA$ is called an $\Omega$-algebra. If $\rho$ is a congruence relation on $A$ generated by pairs $(a_i, b_i)$, $i \in I$, then, as $\Omega$-algebras,

$$
(kA/\rho) \cong kA/Id(a_i - b_i, i \in I),
$$

where $Id(S)$ is the $\Omega$-ideal of $kA$ generated by $S$. It means that any “monomial” linear basis of $kA/Id(a_i - b_i, i \in I)$, i.e. a basis that consists of elements of $A$, is a set of normal forms for $A$.

Now let $(A, \circ, \cdot, \theta, 1)$ be a semiring, i.e. $(A, \circ, \theta)$ is a commutative monoid, $(A, \cdot, 1)$ is a monoid, $\theta \cdot a = a \cdot \theta = \theta$ for any $a \in A$, and $\cdot$ is distributive relative to $\circ$ from left and right.

Some people call “rig” instead of “semiring”.

The class of semirings is a variety. So a free semiring $kRig$ is a monoid, $\theta$ for any $u$, $w$ is called an $\Omega$-ideal of $kA$. It means that any “monomial” linear basis of $kA/Id(a_i - b_i, i \in I)$, i.e. a basis that consists of elements of $A$, is a set of normal forms for $A$.

For any $u, v, \omega \in \Omega$ we denote $\frac{\omega}{\theta}$, where $\omega = \theta = \theta$ for any $a \in A$, and $\cdot$ is distributive relative to $\circ$ from left and right.

Let $k$ be a field. We call the groupoid algebra $kRig(X)$ a semiring algebra. Any element in $kRig(X)$ is called a monomial and any element in $kRig(X)$ is called a polynomial.

For any $u_1 \circ u_2 \circ \cdots \circ u_n \in Rig(X)$, $u_i \in X^*$, we denote $|u|_o = n$.

For any $u \circ u \circ u \circ \cdots \circ u \in Rig(X)$, $u \in X^*$, we will denote it by $u^n$.

For any $u = u_1 \circ u_2 \circ \cdots \circ u_m \circ u_{m+1} \circ \cdots \circ u_n$, $v = v_1 \circ v_2 \circ \cdots \circ v_m \circ v_{m+1} \circ \cdots \circ v_t$, where $w_i, u_i, v_j \in X^*$, such that

$$
u_i \neq v_j \text{ for any } i = m + 1, \ldots, n, j = m + 1, \ldots, t,$$

we denote

$$
lcm_o(u, v) = w_1 \circ w_2 \circ \cdots \circ w_m \circ u_{m+1} \circ \cdots \circ u_n \circ v_{m+1} \circ \cdots \circ v_t
$$

the least common multiple of $u$ and $v$ in $Rig(X)$ with respect to $\circ$.

Throughout this paper, we denote $\mathbb{N}$ the set of natural numbers, $Rig(X|S)$ the semiring with generators $X$ and relations $S$.

We want to create Gröbner-Shirshov bases theory for $Rig(X)$. As for semigroups or groups, it is enough to create Gröbner-Shirshov bases theory for the algebra $kRig(X)$. Let us remind that $\theta$ and $0$ are different elements of $kRig(X)$ since $\theta \in Rig(X)$ and $0 \notin Rig(X)$. 


3 Composition-Diamond lemma for semirings

Let $* \notin X$. By a $*$-monomial we mean a monomial in $\text{Rig}(X \cup *)$ with only one occurrence of $*$. Let $u$ be a $*$-monomial and $s \in k\text{Rig}(X)$. Then we call
\[ u|_s = u|_{s \rightarrow s} \]
an $s$-monomial. For example, if
\[ u = x_1 x_2 \circ x_2 \star x_3 \in \text{Rig}(X \cup *) \]
then
\[ u|_s = u|_{s \rightarrow s} = x_1 x_2 \circ x_2 s x_3. \]

Let $>$ be any monomial ordering on $\text{Rig}(X)$, i.e. $>$ is a well ordering such that for any $v, w \in \text{Rig}(X)$ and $u$ a $*$-monomial,
\[ w > v \Rightarrow u|_w > u|_v. \]

For every polynomial $f \in k\text{Rig}(X)$, $f$ has the leading monomial $\bar{f}$. If the coefficient of $\bar{f}$ is 1, then we call $f$ to be monic.

For any set $S \subseteq k\text{Rig}(X)$, we say $S$ monic if any $s \in S$ is monic.

**Definition 3.1** Let $> be a monomial ordering on $\text{Rig}(X)$. Let $f, g$ be two monic polynomials in $k\text{Rig}(X)$.

(I) If there exist $a, b \in X^*$, such that $|\text{lcm}_o(\bar{f} a, b\bar{g})|_o < |\bar{f} a|_o + |a\bar{g} b|_o$ then we call $(f, g)_w = f a \circ u - b g \circ v$ the intersection composition of $f$ and $g$ with respect to $w$ where $w = \text{lcm}_o(\bar{f} a, b\bar{g}) = \bar{f} a \circ u = b\bar{g} \circ v$.

(II) If there exist $a, b \in X^*$, such that $|\text{lcm}_o(\bar{f}, a\bar{g} b)|_o < |\bar{f}|_o + |a\bar{g} b|_o$ then we call $(f, g)_w = f \circ u - a\bar{g} b \circ v$ the inclusion composition of $f$ and $g$ with respect to $w$ where $w = \text{lcm}_o(\bar{f}, a\bar{g} b) = \bar{f} \circ u = a\bar{g} b \circ v$.

In the above definition, $w$ is called the ambiguity of the composition. Clearly,
\[ (f, g)_w \in \text{Id}(f, g) \quad \text{and} \quad (f, g)_w < w, \]
where $\text{Id}(f, g)$ is the ideal of $k\text{Rig}(X)$ generated by $f, g$.

**Remark:** We regard $k\text{Rig}(X)$ as an $\Omega$-algebra. In the Definition 3.1 the ideal $\text{Id}(f, g)$ means $\Omega$-ideal. In this paper, the ideal of $k\text{Rig}(X)$ will be $\Omega$-ideal.

Let $f, g$ be polynomials and $g$ monic with $\bar{f} = a\bar{g} b \circ u$ for some $a, b \in X^*, u \in \text{Rig}(X)$. Then the transformation
\[ f \rightarrow f - \alpha a\bar{g} b \circ u \]
is called the elimination of the leading term (ELT) of $f$ by $g$, where $\alpha$ is the coefficient of the leading term of $f$.  

4
Definition 3.2 Suppose that $w$ is a monomial, $S$ a set of monic polynomials in $kRig(X)$ and $h$ a polynomial. Then $h$ is trivial modulo $(S; w)$, denoted by $h \equiv 0 \mod(S, w)$, if $h = \sum \alpha_i a_i s_i b_i \circ u_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $u_i \in Rig(X)$, $s_i \in S$ and $a_i s_i b_i \circ u_i < w$.

The set $S$ is called a Gröbner-Shirshov basis in $kRig(X)$ if any composition in $S$ is trivial modulo $S$ and corresponding to $w$.

A set $S$ is called a minimal Gröbner-Shirshov basis in $kRig(X)$ if $S$ is a Gröbner-Shirshov basis in $kRig(X)$ and for any $f, g \in S$ with $f \neq g$, $\beta a b \in X^*$, $u \in Rig(X)$, s.t., $f = \alpha a \circ b \circ \beta a b \circ u$.

Denote
$$\text{Irr}(S) = \{w \in Rig(X) \mid w \neq \alpha a \circ b \circ u \text{ for any } a, b \in X^*, u \in Rig(X), s \in S\}.$$ 

A Gröbner-Shirshov basis $S$ in $kRig(X)$ is reduced if for any $s \in S$, $\text{supp}(s) \subseteq \text{Irr}(S - \{s\})$, where $\text{supp}(s) = \{u_1, u_2, \ldots, u_n\}$ if $s = \sum_{i=1}^{n} \alpha_i u_i$, $0 \neq \alpha_i \in k$, $u_i \in Rig(X)$.

If the set $S$ is a Gröbner-Shirshov basis in $kRig(X)$, then we call also $S$ is a Gröbner-Shirshov basis for the ideal $Id(S)$ or the algebra $kRig(X)|S| := kRig(X)/Id(S)$.

Let $I$ be an ideal of $kRig(X)$. Then there exists uniquely the reduced Gröbner-Shirshov basis $S$ for $I$, see Theorem 3.5.

If a subset $S$ of $kRig(X)$ is not a Gröbner-Shirshov basis for $Id(S)$ then one can add to $S$ a nontrivial composition $(f, g)_w$ of $f, g \in S$ and continue this process repeatedly (actually using the transfinite induction) in order to obtain a set $S^\text{comp}$ of generators of $Id(S)$ such that $S^\text{comp}$ is a Gröbner-Shirshov basis in $kRig(X)$. Such a process is called Shirshov algorithm.

Suppose that $S = \{u_i - v_i \mid i \in I\}$ where for any $i$, $u_i, v_i \in Rig(X)$. In this case we call $u_i - v_i$ a semiring relation. It is clear that if $S$ is a set of semiring relations then so is $S^\text{comp}$. In order to find a normal form of the semiring $Rig(X|S) = Rig(X)/\rho(S)$, where $\rho(S)$ is the congruence of $Rig(X)$ generated by the set $\{(u_i, v_i)|i \in I\}$, it is enough to find a monomial $k$-basis of the semiring algebra $kRig(X|S)$.

Lemma 3.3 Let $S$ be a Gröbner-Shirshov basis in $kRig(X)$ and $s_1, s_2 \in S$. If $w = \alpha a s_1 b \circ u = c s_2 d \circ v$ for some $a, b, c, d \in X^*$, $u, v \in Rig(X)$, then
$$\alpha a s_1 b \circ u \equiv c s_2 d \circ v \mod(S, w).$$

Proof: There are two cases to consider.

(1) $\text{lcm}_v(\alpha a s_1 b, c s_2 d) = \alpha a s_1 b \circ c s_2 d$ which means there exists $u_1 \in Rig(X)$ such that $u = u_1 \circ c s_2 d$, $v = \alpha a s_1 b \circ u_1$.

Then
$$\alpha a s_1 b \circ u - c s_2 d \circ v$$
$$= \alpha a s_1 b \circ u_1 \circ c s_2 d - c s_2 d \circ \alpha a s_1 b \circ u_1$$
$$= \alpha a s_1 b \circ c s_2 d \circ u_1 - \alpha a s_1 b \circ c s_2 d \circ u_1 + \alpha a s_1 b \circ c s_2 d \circ u_1 - \alpha s_1 b \circ c s_2 d \circ u_1$$
$$= -\alpha a s_1 b \circ c(s_2 - s_1) d \circ u_1 + a(s_1 - s_1) b \circ c s_2 d \circ u_1.$$
Since $s_2 - s_2 < s_2$ and $s_1 - s_1 < s_1$, we have
\[ as_1b \circ c(s_2 - s_2)d \circ u_1 < as_1b \circ cs_2d \circ u_1 = w \]
and
\[ a(s_1 - s_1)b \circ cs_2d \circ u_1 < as_1b \circ cs_2d \circ u_1 = w. \]

It follows that
\[ as_1b \circ u \equiv cs_2d \circ v \mod (S, w). \]

(II) $|lcm_o(a\overline{s_1}b, cs_2d)|_o < |a\overline{s_1}b|_o + |cs_2d|_o$, i.e. $\overline{s_1} = u_1 \circ u_2 \circ \cdots \circ u_m \circ u_{m+1} \circ \cdots \circ u_n$, $\overline{s_2} = v_1 \circ v_2 \circ \cdots \circ v_m \circ v_{m+1} \circ \cdots \circ v_t$ such that
\[ au_1b = cv_1d, \ au_2b = cv_2d, \ldots, au_mb = cv_md \]
and
\[ au_i \neq cv_j \text{ for any } i = m + 1, \ldots, n, \ j = m + 1, \ldots, t, \]
where $u_i, v_j \in X^*$. In this case, there exists $u' \in \text{Rig}(X)$ such that
\[ u = cv_{m+1}d \circ cv_{m+2}d \circ \cdots \circ cv_td \circ u', \ v = au_{m+1}b \circ au_{m+2}b \circ \cdots \circ au_nb \circ u'. \]

There are four subcases to consider.

1) $u_1b_1 = c_1v_1$ for some $b_1, c_1 \in X^*$. In this subcase, $c = ac_1, b = b_1d$. Then
\[
\begin{align*}
as_1b \circ u - cs_2d \circ v & = as_1b_1d \circ ac_1v_{m+1}d \circ ac_1v_{m+2}d \circ \cdots \circ ac_1vTd \circ u' \\
& \quad - ac_1s_2d \circ au_{m+1}b_1d \circ au_{m+2}b_1d \circ \cdots \circ au_nb_1d \circ u' \\
& = a(s_1b_1 \circ c_1v_{m+1} \circ c_1v_{m+2} \circ \cdots \circ c_1v_t - c_1s_2 \circ u_{m+1}b_1 \circ u_{m+2}b_1 \circ \cdots \circ u_nb_1)d \circ u' \\
& = a((s_1, s_2)_{w'})d \circ u'
\end{align*}
\]
where $w' = s_1b_1 \circ c_1v_{m+1} \circ c_1v_{m+2} \circ \cdots \circ c_1v_t = c_1s_2 \circ u_{m+1}b_1 \circ u_{m+2}b_1 \circ \cdots \circ u_nb_1$.

Since $S$ is a Gröbner-Shirshov basis in $k\text{Rig}(X)$, we have
\[ (s_1, s_2)_{w'} \equiv 0 \mod (S, w'). \]

Then
\[
\begin{align*}
as_1b \circ u - cs_2d \circ v & = a((s_1, s_2)_{w'})d \circ u' \\
& \equiv 0 \mod (S, aw'd \circ u') \\
& \equiv 0 \mod (S, w).
\end{align*}
\]

2) $a_1u_1 = v_1d_1$ for some $a_1, d_1 \in X^*$. This subcase is similar to subcase 1). We omit the proof.
3) \( u_1 = c_1v_1d_1 \) for some \( c_1, d_1 \in X^* \). Then \( c = ac_1, d = d_1b \) and

\[
as_1b \circ u - cs_2d \circ v = as_1b \circ ac_1v_{m+1}d_1b \circ ac_1v_{m+2}d_1b \circ \cdots \circ ac_1v_d1b \circ u' - ac_1s_2d_1b \circ au_{m+1}b \circ au_{m+2}b \circ \cdots \circ au_nb \circ u' = a((s_1, s_2)w) \circ b \circ u'
\]

where \( w' = \overline{s_1} \circ c_1v_{m+1}d_1 \circ c_1v_{m+2}d_1 \circ \cdots \circ c_1v_d1d_1 - c_1s_2d_1 \circ u_{m+1} \circ u_{m+2} \circ \cdots \circ u_n \) \( b \circ u' \)

Since \( S \) is a Gröbner-Shirshov basis in \( kRig\langle X \rangle \), we have

\[
(s_1, s_2)w' \equiv 0 \mod(S, w').
\]

Similar to the subcase 1), we have

\[
as_1b \circ u - cs_2d \circ v \equiv 0 \mod(S, w').
\]

4) \( a_1u_1b_1 = v_1 \) for some \( a_1, b_1 \in X^* \). This subcase is similar to the subcase 3). We omit the proof.

The lemma is proved. \( \blacksquare \)

**Theorem 3.4** (Composition-Diamond lemma for semirings) Let \( S \) be a set of monic polynomials in \( kRig\langle X \rangle \), \( > \) a monomial ordering on \( Rig\langle X \rangle \) and \( Id(S) \) the \( \Omega \)-ideal of \( kRig\langle X \rangle \) generated by \( S \). Then the following statements are equivalent.

1. \( S \) is a Gröbner-Shirshov basis in \( kRig\langle X \rangle \).
2. \( f \in Id(S) \Rightarrow \bar{f} = a\overline{s}b \circ u \) for some \( a, b \in X^*, u \in Rig\langle X \rangle \) and \( s \in S \).
3. \( f \in Id(S) \Rightarrow f = \alpha_1a_1s_1b_1 \circ u_1 + \alpha_2a_2s_2b_2 \circ u_2 + \ldots + \alpha_n a_nb_n \circ u_n \) where \( a_1s_1b_1 \circ u_1 > a_2s_2b_2 \circ u_2 > \ldots > a_ns_nb_n \circ u_n \), \( 0 \neq \alpha_i \in k, a_i, b_i \in X^*, u_i \in Rig\langle X \rangle, s_i \in S \).

\( \textbf{Proof:} \ (1) \Rightarrow (2) \) Let \( 0 \neq f \in Id(S) \). Then

\[
f = \sum_{i=1}^{n} \alpha_i a_i s_i b_i \circ u_i
\]

where each \( \alpha_i \in k, a_i, b_i \in X^*, u_i \in Rig\langle X \rangle, s_i \in S \).

Let \( w_i = a_is_ib_i \circ u_i \) and we arrange this leading terms in non-increasing ordering by

\[
w_1 = w_2 = \ldots = w_m > w_{m+1} \geq \ldots \geq w_n.
\]

Now we prove the result by induction on \( m \).
If $m = 1$, then $\bar{f} = a_1\overline{sb}_1 \circ u_1$.

Now we assume that $m \geq 2$. Then

$$a_1\overline{sb}_1 \circ u_1 = w_1 = w_2 = a_2\overline{sb}_2 \circ u_2.$$ 

Since $S$ is a Gröbner-Shirshov basis in $k\text{Rig}(X)$, by Lemma 3.3, we have

$$a_2s_2b_2 \circ u_2 - a_1s_1b_1 \circ u_1 = \sum \beta_jc_jd_j \circ v_j$$

where each $\beta_j \in k, c_j, d_j \in X^*, v_j \in \text{Rig}(X)$, $s_j \in S$, and $c_j\overline{s}_j \circ v_j < w_1$. Therefore, since

$$\alpha_1a_1s_1b_1 \circ u_1 + \alpha_2a_2s_2b_2 \circ u_2 = (\alpha_1 + \alpha_2)a_1s_1b_1 \circ u_1 + \alpha_2(a_2s_2b_2 \circ u_2 - a_1s_1b_1 \circ u_1),$$

we have

$$f = (\alpha_1 + \alpha_2)a_1s_1b_1 \circ u_1 + \alpha_2\sum \beta_jc_jd_j \circ v_j + \sum_{i=3}^n \alpha_i a_is_ib_i \circ u_i.$$ 

If either $m > 2$ or $\alpha_1 + \alpha_2 \neq 0$, then the result follows from the induction on $m$. If $m = 2$ and $\alpha_1 + \alpha_2 = 0$, then the result follows from the induction on $w_1$.

$(2) \iff (2')$ is clear.

$(2) \implies (3)$ For any $f \in k\text{Rig}(X)$, by the ELTs, we can obtain that $f + Id(S)$ can be expressed as a linear combination of elements of $\text{Irr}(S)$. Now suppose $\alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_nu_n = 0$ in $k\text{Rig}(X|S)$ with $u_i \in \text{Irr}(S)$, $u_1 > u_2 > \ldots > u_n$ and each $\alpha_i \neq 0$. Then, in $k\text{Rig}(X)$,

$$g = \alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_nu_n \in Id(S).$$ 

By $(2)$, we have $u_1 = \overline{g} \notin \text{Irr}(S)$, a contradiction. So $\text{Irr}(S)$ is $k$-linearly independent. This shows that $\text{Irr}(S)$ is a $k$-basis of $k\text{Rig}(X|S)$.

$(3) \implies (2)$ Let $0 \neq f \in Id(S)$. Suppose that $\bar{f} \in \text{Irr}(S)$. Then

$$f + Id(S) = \alpha(\bar{f} + Id(S)) + \sum \alpha_i(u_i + Id(S)),$$

where $\alpha, \alpha_i \in k, u_i \in \text{Irr}(S)$ and $\bar{f} > u_i$. Therefore, $f + Id(S) \neq 0$, a contradiction. So $f = a\overline{sb} \circ u$ for some $a, b \in X^*, u \in \text{Rig}(X), s \in S$.

$(2) \implies (1)$ By the definition of the composition, we have $(f,g)_w \in Id(S)$. If $(f,g)_w \neq 0$, then by $(2)$, $(\overline{f}, \overline{g})_w = a_1\overline{sb}_1 \circ u_1$ for some $a_1, b_1 \in X^*, u_1 \in \text{Rig}(X), s_1 \in S$. Let

$$h = (f,g)_w - \alpha_1a_1s_1b_1 \circ u_1,$$

where $\alpha_1$ is the coefficient of $(f,g)_w$. Then $\overline{h} < (f,g)_w$ and $h \in Id(S)$. By induction, we can get the result. \hfill \blacksquare

Suppose that $>$ is a monomial ordering on $\text{Rig}(X)$ and $I$ an ideal of $k\text{Rig}(X)$. Then there exists a Gröbner-Shirshov basis $S \subset k\text{Rig}(X)$ for the ideal $I = Id(S)$, for example, we may take $S = I$. By Theorem 3.4, we may assume that the leading terms of the elements of $S$ are different with each other. For any $g \in S$, denote

$$\Delta_g = \{ f \in S | f \neq g \text{ and } \bar{f} = a\overline{gb} \circ u \text{ for some } a, b \in X^*, u \in \text{Rig}(X) \}$$

8
and $S_1 = S - \cup_{g \in S} \Delta_g$.

For any $f \in Id(S)$ we show that there exists an $s_1 \in S_1$ such that $\overline{f} = a\overline{s_1}b \circ u$ for some $a, b \in X^*$, $u \in Rig(X)$.

In fact, by Theorem 3.4, $\overline{f} = a'\overline{h}b' \circ u'$ for some $a', b' \in X^*$, $u' \in Rig(X)$ and $h \in S$. Suppose that $h \in S - S_1$. Then we have $h \in \cup_{g \in S} \Delta_g$, say, $h \in \Delta_g$, i.e. $h \neq g$ and $\overline{h} = a\overline{g}b \circ u$ for some $a, b \in X^*$, $u \in Rig(X)$. We claim that $\overline{h} > \overline{g}$. Otherwise, $\overline{h} < \overline{g}$. It follows that $\overline{h} = a\overline{g}b \circ u > ahb \circ u$ and so we have an infinite descending chain

$$\overline{h} > ahb \circ u > a^2\overline{h}b^2 \circ aub \circ u > a^3\overline{h}b^3 \circ a^2ub^2 \circ aub \circ u > \ldots$$

which contradicts that $>$ is well ordered.

Suppose that $g \not\in S_1$. Then by the above proof, there exists a $g_1 \in S$ such that $g \in \Delta_{g_1}$ and $\overline{g} > \overline{g_1}$. Since $>$ is well ordered, there must exist an $s_1 \in S_1$ such that $\overline{f} = a_1\overline{s_1}b_1 \circ u_1$ for some $a_1, b_1 \in X^*$, $u_1 \in Rig(X)$.

Let $f_1 = f - a_1s_1b_1 \circ u_1$, where $a_1$ is the coefficient of the leading term of $f$. Then $f_1 \in Id(S)$ and $\overline{f} > \overline{f_1}$.

By induction on $\overline{f}$, we know that $f \in Id(S_1)$ and hence $I = Id(S_1)$. Moreover, by Theorem 3.4, $S_1$ is clearly a minimal Gröbner-Shirshov basis for the ideal $Id(S)$.

Assume that $S$ is a minimal Gröbner-Shirshov basis for the ideal $I$.

For any $s \in S$, we have $s = s' + s''$, where $supp(s') \subseteq Irr(S - \{s\})$, $s'' \in Id(S - \{s\})$. Since $S$ is a minimal Gröbner-Shirshov basis, we have $\overline{s} = \overline{s'}$ for any $s \in S$.

Then $S_2 = \{s'|s \in S\}$ is the reduced Gröbner-Shirshov basis for the ideal $I$. In fact, it is clear that $S_2 \subseteq Id(S) = I$. For any $f \in Id(S)$, by Theorem 3.4, $\overline{f} = a_1\overline{s_1}b_1 \circ u_1 = a_1s_1b_1 \circ u_1$ for some $a_1, b_1 \in X^*$, $u_1 \in Rig(X)$.

Suppose that $S$, $R$ are two reduced Gröbner-Shirshov bases for the ideal $I$. For any $s \in S$, by Theorem 3.4,

$$\overline{s} = a\overline{r}b \circ u, \overline{r} = c\overline{s_1}d \circ v$$

for some $a, b, c, d \in X^*$, $u, v \in Rig(X)$ and hence $\overline{s} = a\overline{s_1}db \circ avb \circ u$. Since $\overline{s} \in supp(s) \subseteq Irr(S - \{s\})$, we have $s = s_1$. It follows that $a = b = c = d = 1$ and $u = v = \emptyset$ and so $\overline{s} = \overline{r}$.

If $s \neq r$ then $0 \neq s - r \in I = Id(S) = Id(R)$. By Theorem 3.4, $\overline{s-r} = a_1\overline{r_1}b_1 \circ u_1 = c_1\overline{s_1}d_1 \circ v_1$ for some $a_1, b_1, c_1, d_1 \in X^*$, $u_1, v_1 \in Rig(X)$ with $\overline{r_1}, \overline{s_1} < \overline{s} = \overline{r}$. This means that $s_2 \in S - \{s\}$ and $r_1 \in R - \{r\}$. Noting that $\overline{s-r} \in supp(s) \cup supp(r)$, we have either $\overline{s-r} \in supp(s)$ or $\overline{s-r} \in supp(r)$. If $\overline{s-r} \in supp(s)$ then $\overline{s-r} \in Irr(S - \{s\})$ which contradicts $\overline{s-r} = c_1\overline{s_1}d_1 \circ v_1$; if $\overline{s-r} \in supp(r)$ then $\overline{s-r} \in Irr(R - \{r\})$ which contradicts $\overline{s-r} = a_1\overline{r_1}b_1 \circ u_1$. This shows that $s = r$ and then $S \subseteq R$. Similarly, $R \subseteq S$.

Therefore, we have proved the following theorem.

**Theorem 3.5** Let $I$ be an ideal of $kRig(X)$ and $> a$ monomial ordering on $Rig(X)$. Then there exists uniquely the reduced Gröbner-Shirshov basis $S$ for $I$.

### 4 Composition-Diamond lemma for commutative semirings

In this section, we will give Gröbner-Shirshov bases theory for commutative semirings which is almost the same as the case of semirings. The compositions of commutative
semiring are simpler.

A semiring \((A, o, \cdot, \theta, 1)\) is commutative if \((A, \cdot, 1)\) is a commutative monoid. The class of commutative semirings is a variety. A free commutative semiring \(\text{Rig}[X]\) generated by a set \(X\) is defined as usual. Let \([(X), \cdot, 1]\) be the free commutative monoid generated by \(X\). If one fixes some linear ordering \(<\) on the set \([X]\), then any element of \(\text{Rig}[X]\) has an unique form \(\theta\), or \(w = u_1 \circ u_2 \circ \cdots \circ u_n\), where \(u_i \in [X]\), \(u_1 \leq u_2 \leq \ldots \leq u_n\), \(n \geq 1\).

For any \(u, v \in [X]\), we denote \(\text{lcm}(u, v)\) the least common multiple of \(u\) and \(v\) in \([X]\). Then there exist uniquely \(a, b \in [X]\) such that \(\text{lcm}(u, v) = au = bv\).

**Definition 4.1** Let \(<\) be a monomial ordering on \(\text{Rig}[X]\). Let \(f, g\) be two monic polynomials in \(\text{Rig}[X]\) and \(\overline{f} = u_1 \circ u_2 \circ \cdots \circ u_n\), \(\overline{g} = v_1 \circ v_2 \circ \cdots \circ v_m\) where each \(u_i, v_j \in [X]\). For any pair \((a, b) \in \{(a_{ij}, b_{ij}) \mid 1 \leq i \leq n, \ 1 \leq j \leq m\}\) where \(a_{ij}, b_{ij} \in [X]\) such that \(\text{lcm}(u_i, v_j) = a_{ij}u_i = b_{ij}v_j\), we call \((f, g)w = a_f \circ u - b_g \circ v\) the composition of \(f\) and \(g\) with respect to \(w\) where \(w = \text{lcm}_a(a_{ij}, b_{ij}) = a_{ij} \circ u = b_{ij} \circ v\).

**Definition 4.2** Suppose that \(w\) is a monomial in \(\text{Rig}[X]\), \(S\) a set of monic polynomials in \(\text{Rig}[X]\) and \(h\) a polynomial. Then \(h\) is trivial modulo \((S, w)\), denoted by \(h \equiv 0 \text{ mod } (S, w)\), if \(h = \sum \alpha_i a_i s_i \circ u_i\), where each \(\alpha_i \in k\), \(a_i \in [X]\), \(s_i \in \text{Rig}[X]\), \(s_i \in S\) and \(a_i \overline{s_i} \circ u_i < w\).

The set \(S\) is called a Gröbner-Shirshov basis in \(\text{Rig}[X]\) if any composition in \(S\) is trivial modulo \(S\) and corresponding to \(w\).

**Remark** For any given monic polynomials \(f, g \in \text{Rig}[X]\), there are finitely many compositions \((f, g)w\). Therefore we may use computer to realize Shirshov’s algorithm to find a Gröbner-Shirshov basis \(S^\text{comp}\) for a finite set \(S\) in \(\text{Rig}[X]\). However, the reduced Gröbner-Shirshov basis of \(\text{Id}(S)\) is generally infinite even if both \(S\) and \(X\) are finite, see Example 5.9.

The following theorems can be similarly proved to Theorems 3.4 and 3.5 respectively. We omit the detail.

**Theorem 4.3** (Composition-Diamond lemma for commutative semirings) Let \(S\) be a set of monic polynomials in \(\text{Rig}[X]\) and \(>\) a monomial ordering on \(\text{Rig}[X]\). Then the following statements are equivalent.

1. \(S\) is a Gröbner-Shirshov basis in \(\text{Rig}[X]\).
2. \(f \in \text{Id}(S) \Rightarrow \overline{f} = a\overline{s} \circ u\) for some \(a \in [X], u \in \text{Rig}[X]\) and \(s \in S\).
3. \(f \in \text{Id}(S) \Rightarrow f = a_1 a_1 s_1 \circ u_1 + a_2 a_2 s_2 \circ u_2 + \ldots + a_n a_n s_n \circ u_n\), where \(a_1 \overline{s_1} \circ u_1 > a_2 \overline{s_2} \circ u_2 > \ldots > a_n \overline{s_n} \circ u_n\), \(a_i \in k\), \(a_i \in [X]\), \(u_i \in \text{Rig}[X]\), \(s_i \in S\).
4. \(\text{Irr}(S) = \{w \in \text{Rig}[X] \mid w \neq a\overline{s} \circ u\) for any \(a \in [X], u \in \text{Rig}[X], s \in S\} is a \(k\)-basis of \(\text{Rig}[X]/S = \text{Rig}[X]/\text{Id}(S)\).

**Theorem 4.4** Let \(I\) be an ideal of \(\text{Rig}[X]\) and \(>\) a monomial ordering on \(\text{Rig}[X]\). Then there exists uniquely the reduced Gröbner-Shirshov basis \(S\) for \(I\).
5 Applications

In 2004, M. Fiore and T. Leinster [29] find a strongly normalizing reduction system and a normal form of the semiring $\mathbb{N}[x]/(x = 1 + x + x^2)$. Actually $\mathbb{N}[x]/(x = 1 + x + x^2) = \text{Rig}[x]/(x = 1 \circ x \circ x^2)$. Now, we use the Composition-Diamond lemma for commutative semirings, i.e. Theorem 4.3 to find a Gröbner-Shirshov basis and a normal form of this semiring.

We define a monomial ordering on $\text{Rig}[x]$ first. We order $[x]$ by degree ordering $\prec$: $x^n \prec x^m \iff n < m$.

For any $u \in \text{Rig}[x]$, $u$ can be uniquely expressed as $u = u_1 \circ u_2 \circ \cdots \circ u_n$, where $u_1, u_2, \ldots, u_n \in [x]$, and $u_1 \preceq u_2 \preceq \cdots \preceq u_n$. Denote $\text{wt}(u) = (u_n, u_{n-1}, \ldots, u_1)$.

We order $\text{Rig}[x]$ as follows: for any $u, v \in \text{Rig}[x]$, if one of the sequences is not a prefix of other, then $u < v \iff \text{wt}(u) < \text{wt}(v)$ lexicographically; if the sequence of $u$ is a prefix of the sequence of $v$, then $u < v$.

Then, it is clear that $<$ on $\text{Rig}[x]$ is a monomial ordering.

**Theorem 5.1** Let the ordering on $\text{Rig}[x]$ be as above. Then $k\text{Rig}[x|x = 1 \circ x \circ x^2] = k\text{Rig}[x|S]$ and $S$ is a Gröbner-Shirshov basis in $k\text{Rig}[x]$, where $S$ consists of the following relations

1. $x^4 = 1 \circ 1 \circ x^2$,
2. $x \circ x^3 = 1 \circ x^2$,
3. $1 \circ x^2 \circ x^n = x^n \ (1 \leq n \leq 3)$.

**Proof:** We denote $i \land j$ the composition of the type $i$ and type $j$.

Let us check all the possible compositions.

For $1 \land 1$, there is no composition.

For $1 \land 2$, the ambiguities $w$ of all possible compositions are: 1) $x^4 \circ x^6$ 2) $x^2 \circ x^4$

For $1 \land 3$, the ambiguities $w$ of all possible compositions are:

3) $x^4 \circ x^6 \circ x^{n+4}$ 4) $x^2 \circ x^4 \circ x^{n+2}$ 5) $x^{4-n} \circ x^{6-n} \circ x^4$

where $1 \leq n \leq 3$.

For $2 \land 2$, the ambiguity $w$ of all possible composition is: 6) $x \circ x^3 \circ x^5$

For $2 \land 3$, the ambiguities $w$ of all possible compositions are:

7) $x \circ x^3 \circ x^5 \circ x^{n+3}$ 8) $x \circ x^3 \circ x^{5-n} \circ x^{3-n}$ 9) $x^4 \circ x^2 \circ 1 \circ x^n$

10) $x^{n+2} \circ x^n \circ 1 \circ x^2$ 11) $x \circ x^3 \circ x^{n+1}$

where $1 \leq n \leq 3$. 

11
For $3 \land 3$, the ambiguities $w$ of all possible compositions are:

12) $1 \circ x^2 \circ x^n \circ x^{n+2} \circ x^{n+m}$  
13) $x \circ x^3 \circ x^2 \circ 1 \circ x^m$  
14) $1 \circ x^2 \circ x^p \circ x^{p-2} \circ x^{p+m-2}$  
15) $1 \circ x^2 \circ x^t \circ x^{2+t-n} \circ x^{t-n}$  
16) $1 \circ x^n \circ x^2 \circ x^4 \circ x^{m+2}$  
17) $1 \circ x \circ x^2 \circ x^3$

where $1 \leq n, m, t \leq 3$, $2 \leq p \leq 3$ and $t \geq n$.

We have to check that all these compositions are trivial $\text{mod}(S, w)$. Here, for example, we just check 1), 11), 12) and 17). Others are similarly proved.

For 17), let $f = 1 \circ x \circ x^2 - x$, $g = 1 \circ x \circ x^2 - x$. Then $w = 1 \circ x \circ x^2 \circ x^3$ and

$$(f, g)_w = (1 \circ x \circ x^2 - x) \circ x^3 - (1 \circ x \circ x^2 - x) \circ 1$$

$$= 1 \circ x^2 - x \circ x^3$$

$$\equiv 0.$$ 

From this it follows that we have the relation 2.

For 11), there are three cases to consider.

Case 1. $n = 1$, $f = 1 \circ x \circ x^2 - x$, $g = x \circ x^3 - 1 \circ x^2$. Then $w = x \circ x^3 \circ x^2$ and

$$(f, g)_w = (1 \circ x \circ x^2 - x)x - (x \circ x^3 - 1 \circ x^2) \circ x^2$$

$$= 1 \circ x^2 \circ x^2 - x^2$$

$$\equiv 0.$$ 

Case 2. $n = 2$, $f = 1 \circ x^2 \circ x^2 - x^2$, $g = x \circ x^3 - 1 \circ x^2$. Then $w = x \circ x^3 \circ x^3$ and

$$(f, g)_w = (1 \circ x^2 \circ x^2 - x^2)x - (x \circ x^3 - 1 \circ x^2) \circ x^3$$

$$= 1 \circ x^2 \circ x^3 - x^3$$

$$\equiv 0.$$ 

By Case 1 and Case 2 we have the relations 3.

Case 3. $n = 3$, $f = 1 \circ x^2 \circ x^3 - x^3$, $g = x \circ x^3 - 1 \circ x^2$. Then $w = x \circ x^3 \circ x^4$ and

$$(f, g)_w = (1 \circ x^2 \circ x^3 - x^3)x - (x \circ x^3 - 1 \circ x^2) \circ x^4$$

$$= 1 \circ x^2 \circ x^4 - x^4$$

$$\equiv 1 \circ x \circ x^3 - x^4$$

$$\equiv 1 \circ 1 \circ x^2 - x^4$$

$$\equiv 0.$$ 

By Case 3 we have the relation 1.

For 12), let $f = 1 \circ x^2 \circ x^n - x^n$, $g = 1 \circ x^2 \circ x^m - x^m$. Then $w = 1 \circ x^2 \circ x^n \circ x^{n+2} \circ x^{n+m}$ and

$$(f, g)_w = (1 \circ x^2 \circ x^n - x^n) \circ x^{n+2} \circ x^{n+m} - (1 \circ x^2 \circ x^m - x^m) \circ x^n \circ 1 \circ x^2$$

$$= x^{n+m} \circ 1 \circ x^2 - x^n \circ x^{n+2} \circ x^{n+m}$$

$$\equiv x^{n+m} \circ 1 \circ x^2 - 1 \circ x^2 \circ x^{n+m}$$

$$\equiv 0.$$ 

12
For 1), let \( f = x \circ x^3 - 1 \circ x^2, \ g = x^4 - 1 \circ 1 \circ x^2 \). Then \( w = x^4 \circ x^6 \) and

\[
\begin{align*}
(f, g)_w &= (x \circ x^3 - 1 \circ x^2)x^3 - (x^4 - 1 \circ 1 \circ x^2) \circ x^6 \\
&= 1 \circ 1 \circ x^2 \circ x^6 - x^3 \circ x^5 \\
&= 1 \circ 1 \circ x^2 \circ x^2 \circ x^4 - 1 \circ x^2 \\
&= 1 \circ x^2 \circ x^2 \circ x^4 - 1 \circ x^2 \\
&= x^2 \circ x^4 - 1 \circ x^2 \\
&= 1 \circ x^2 - 1 \circ x^2 \\
&= 0.
\end{align*}
\]

So \( S \) is a Gröbner-Shirshov basis in \( kRig[x] \).

The above proof implies that \( kRig[x| x = 1 \circ x \circ x^2| = kRig[x, S] \). \( \blacksquare \)

By Theorems 4.3 and 5.1 we have the following corollary.

**Corollary 5.2** (29) A normal form of the semiring \( Rig[x| x = 1 \circ x \circ x^2| \) is the set

\[
\{1 \circ (m+1) \circ x^2, \ 1^m \circ x^m, \ 1^m \circ (x^3)^m, \ \langle x \rangle^m \circ (x^2)^m, \ (x^2)^m \circ (x^3)^m|m, n \geq 0\}.
\]

In 1995, A. Blass [3] finds a normal form of the semiring \( \mathbb{N}[x]/(x = 1 + x^2) \). Clearly, \( \mathbb{N}[x]/(x = 1 + x^2) = Rig[x| x = 1 \circ x^2| \). We use Theorem 4.3 to find a Gröbner-Shirshov basis and a normal form of this semiring which is different from [3].

**Theorem 5.3** Let the ordering on \( Rig[x] \) be as in Theorem 5.1. Then \( kRig[x| x = 1 \circ x^2| = kRig[x, S] \) and \( S \) is a Gröbner-Shirshov basis in \( kRig[x] \), where \( S \) consists of the following relations

1. \( 1 \circ x^2 = x \),
2. \( x \circ x^4 = 1 \circ x^3 \),
3. \( x^5 = 1 \circ x^4 \),
4. \( 1 \circ x^3 \circ x^n = x^n \ (3 \leq n \leq 4) \).

**Proof:** Let us check all the possible compositions.

For 1 \( \land 1 \), the ambiguity \( w \) of all possible composition is: 1) \( 1 \circ x^2 \circ x^4 \)

For 1 \( \land 2 \), the ambiguities \( w \) of all possible compositions are:

\[
2) \ 1 \circ x^2 \circ x^5 \ 3) \ x^2 \circ x^4 \circ x \ 4) \ x^3 \circ x \circ x^4 \ 5) \ x^6 \circ x^4 \circ x
\]

For 1 \( \land 3 \), the ambiguities \( w \) of all possible compositions are: 6) \( x^3 \circ x^5 \ 7) \ x^7 \circ x^5 \)

For 1 \( \land 4 \), the ambiguities \( w \) of all possible compositions are:

\[
8) \ 1 \circ x^2 \circ x^5 \circ x^{n+2} \ 9) \ x \circ x^3 \circ 1 \circ x^n \ 10) \ x^{n-2} \circ x^n \circ 1 \circ x^3 \\
11) \ x^2 \circ 1 \circ x^3 \circ x^n \ 12) \ x^5 \circ x^3 \circ 1 \circ x^n \ 13) \ x^{n+2} \circ x^n \circ 1 \circ x^3
\]
where $3 \leq n \leq 4$.

For $2 \land 2$, the ambiguity $w$ of all possible composition is: 14) $x \circ x^4 \circ x^7$

For $2 \land 3$, the ambiguities $w$ of all possible compositions are: 15) $x^2 \circ x^5$  16) $x^8 \circ x^5$

For $2 \land 4$, the ambiguities $w$ of all possible compositions are:

\begin{align*}
17) & \ x \circ x^4 \circ x^7 \circ x^{n+4} & 18) & \ x \circ x^4 \circ x^{4-n} \circ x^{7-n} & 19) & \ x^6 \circ x^3 \circ 1 \circ x^n \\
20) & \ x^{n+3} \circ x^n \circ 1 \circ x^3 & 21) & \ x \circ x^4 \circ x^{n+1}
\end{align*}

where $3 \leq n \leq 4$.

For $3 \land 3$, there is no composition.

For $3 \land 4$, the ambiguities $w$ of all possible compositions are:

\begin{align*}
22) & \ x^8 \circ x^{n+5} \circ x^5 & 23) & \ x^2 \circ x^{n+2} \circ x^5 & 24) & \ x^{5-n} \circ x^{8-n} \circ x^5
\end{align*}

where $3 \leq n \leq 4$.

For $4 \land 4$, the ambiguities $w$ of all possible compositions are:

\begin{align*}
25) & \ x^{n+m} \circ x^{m+3} \circ x^m \circ x^3 \circ 1 & 26) & \ x^{m-3} \circ x^{m+n-3} \circ x^m \circ x^3 \circ 1 & 27) & \ x \circ x^4 \circ x^4 \circ x^3 \circ 1 \\
28) & \ x^6 \circ x^{n+3} \circ x^3 \circ 1 \circ x^m & 29) & \ x^4 \circ 1 \circ x^3 \circ x^3
\end{align*}

where $3 \leq n, m \leq 4$.

We have to check that all these compositions are trivial $\text{mod}(S, w)$. Here, for example, we just check 1), 4), 16), 21), 23), 25) and 27). Others are similarly proved.

For 1), let $f = 1 \circ x^2 - x$, $g = 1 \circ x^2 - x$. Then $w = 1 \circ x^2 \circ x^4$, and

\[(f, g)_w = (1 \circ x^2 - x) \circ x^4 - (1 \circ x^2 - x)x^2 \circ 1 = 1 \circ x^3 - x \circ x^4 \equiv 0.
\]

It follows that we have the relation 2.

For 4), let $f = 1 \circ x^2 - x$, $g = x \circ x^4 - 1 \circ x^3$. Then $w = x^3 \circ x \circ x^4$, and

\[(f, g)_w = (1 \circ x^2 - x)x \circ x^4 - (x \circ x^4 - 1 \circ x^3) \circ x^3 = 1 \circ x^3 \circ x^3 - x^2 \circ x^4 \equiv 1 \circ x^3 \circ x^3 - x^3 \equiv 0.
\]

Then we have the relation 4 for $n = 3$.

For 21), there are two cases to consider.

Case 1. $n = 3$, let $f = x \circ x^4 - 1 \circ x^3$, $g = 1 \circ x^3 \circ x^3 - x^3$. Then $w = x \circ x^4 \circ x^4$, and

\[(f, g)_w = (x \circ x^4 - 1 \circ x^3) \circ x^4 - (1 \circ x^3 \circ x^3 - x^3)x = x^4 - 1 \circ x^3 \circ x^4 \equiv 0.
\]

Then we have the relation 4 for $n = 4$. 14
Case 2. $n = 4$, let $f = x \circ x^4 - 1 \circ x^3$, $g = 1 \circ x^3 \circ x^{-4}$. Then $w = x \circ x^4 \circ x^5$, and

$$ (f, g)_w = (x \circ x^4 - 1 \circ x^3) \circ x^4 - (1 \circ x^3 \circ x^4 - x^4)x $$

$$ = x^5 - 1 \circ x^3 \circ x^5 $$

$$ \equiv x^5 - 1 \circ x^4 $$

$$ \equiv 0. $$

Then we have the relation 3.

For 16), let $f = x \circ x^4 - 1 \circ x^3$, $g = x^5 - 1 \circ x^4$. Then $w = x^8 \circ x^5$, and

$$ (f, g)_w = (x \circ x^4 - 1 \circ x^3) x^4 - (x^5 - 1 \circ x^4) \circ x^8 $$

$$ = 1 \circ x^4 \circ x^8 - x^4 \circ x^7 $$

$$ \equiv 1 \circ x^4 \circ x^3 \circ x^7 - x^4 \circ x^7 $$

$$ \equiv x^4 \circ x^7 - x^4 \circ x^7 $$

$$ \equiv 0. $$

For 23), let $f = x^5 - 1 \circ x^4$, $g = 1 \circ x^3 \circ x^n - x^n$. Then $x^2 \circ x^{n+2} \circ x^5$, and

$$ (f, g)_w = (x^5 - 1 \circ x^4) \circ x^2 \circ x^{n+2} - (1 \circ x^3 \circ x^n - x^n) x^2 $$

$$ = x^{n+2} - 1 \circ x^4 \circ x^2 \circ x^{n+2} $$

$$ \equiv x^{n+2} - 1 \circ x^3 \circ x^{n+2} $$

There are two cases to consider.

Case 1. $n = 3$. We have

$$ (f, g)_w \equiv x^{n+2} - 1 \circ x^3 \circ x^{n+2} $$

$$ \equiv x^5 - 1 \circ x^3 \circ x^5 $$

$$ \equiv 1 \circ x^4 - 1 \circ x^3 \circ 1 \circ x^4 $$

$$ \equiv 1 \circ x^4 - 1 \circ x^4 $$

$$ \equiv 0. $$

Case 2. $n = 4$. We have

$$ (f, g)_w \equiv x^{n+2} - 1 \circ x^3 \circ x^{n+2} $$

$$ \equiv x^6 - 1 \circ x^3 \circ x^6 $$

$$ \equiv x \circ x^5 - 1 \circ x^3 \circ x \circ x^5 $$

$$ \equiv x \circ x^5 - 1 \circ x^2 \circ x^5 $$

$$ \equiv x \circ x^5 - x \circ x^5 $$

$$ \equiv 0. $$

For 25), let $f = 1 \circ x^3 \circ x^n - x^n$, $g = 1 \circ x^3 \circ x^m - x^m$. Then $w = x^{n+m} \circ x^{m+3} \circ x^m \circ x^3 \circ 1$, and

$$ (f, g)_w = (1 \circ x^3 \circ x^n - x^n)x^m \circ 1 \circ x^3 - (1 \circ x^3 \circ x^m - x^m) \circ x^{n+m} \circ x^{3+m} $$

$$ = x^m \circ x^{n+m} \circ x^{3+m} - x^{n+m} \circ 1 \circ x^3 $$

$$ \equiv x^m \circ x^m - 1 \circ x^3 \circ x^{n+m} $$

$$ \equiv x^{n+m} - 1 \circ x^3 \circ x^{n+m}. $$
There are three cases to consider.

Case 1. \( n = m = 3 \). We have

\[
(f, g)_w \equiv x^m x^n - 1 \circ x^3 \circ x^{n+m} \\
\equiv x^6 - 1 \circ x^3 \circ x^6 \\
\equiv x \circ x^5 - 1 \circ x^3 \circ x \circ x^5 \\
\equiv x \circ x^5 - 1 \circ x^2 \circ x^5 \\
\equiv x \circ x^5 - x \circ x^5 \\
\equiv 0.
\]

Case 2. \( n = m = 4 \). We have

\[
(f, g)_w \equiv x^m x^n - 1 \circ x^3 \circ x^{n+m} \\
\equiv x^8 - 1 \circ x^3 \circ x^8 \\
\equiv x^3 \circ x^7 - 1 \circ x^3 \circ x^3 \circ x^7 \\
\equiv x^3 \circ x^7 - x^3 \circ x^7 \\
\equiv 0.
\]

Case 3. \( n = 3, m = 4 \), or \( m = 3, n = 4 \). We have

\[
(f, g)_w \equiv x^m x^n - 1 \circ x^3 \circ x^{n+m} \\
\equiv x^7 - 1 \circ x^3 \circ x^7 \\
\equiv x^2 \circ x^6 - 1 \circ x^3 \circ x^2 \circ x^6 \\
\equiv x^2 \circ x^6 - x^3 \circ x \circ x^6 \\
\equiv x^2 \circ x^6 - x^2 \circ x^6 \\
\equiv 0.
\]

For 27), let \( f = 1 \circ x^3 \circ x^3 - x^3, \) \( g = 1 \circ x^3 \circ x^4 - x^4 \). Then \( w = x \circ x^4 \circ x^4 \circ x^3 \circ 1 \), and

\[
(f, g)_w = (1 \circ x^3 \circ x^3 - x^3) x \circ 1 \circ x^3 - (1 \circ x^3 \circ x^4 - x^4) \circ x \circ x^4 \\
= x^4 \circ x \circ x^4 - x^4 \circ 1 \circ x^3 \\
\equiv 1 \circ x^3 \circ x^4 - x^4 \circ 1 \circ x^3 \\
\equiv 0.
\]

Therefore \( S \) is a Gröbner-Shirshov basis in \( kRig[x] \).

The above proof implies that \( kRig[x|x = 1 \circ x^2] = kRig[x|S] \).

We complete the proof. \( \blacksquare \)

By Theorems 4.3 and 5.3 we have the following corollary.
Corollary 5.4 A normal form of the semiring Rig\([x | x = 1 \circ x^2]\) is the set
\[
\{(1^{on} \circ x^{om})x^t, 1^{on} \circ x^3, 1^{on} \circ (x^4)^{om} \mid n, m \geq 0, \ 0 \leq t \leq 3\}.
\]

In order to compare another normal form of the semiring Rig\([x | x = 1 \circ x^2]\)
\[
\{1^{on} \circ x^2 \circ x^4, 1^{on} \circ (x^2)^{om}, (x^2)^{om} \circ (x^4)^{ot}, 1^{on} \circ (x^4)^{ot} \mid n, m, t \geq 0\}
\]
given by A. Blass \[3\], we need the following lemma.

Lemma 5.5 Suppose that \(\Gamma, \Sigma\) are two subsets of the semiring Rig\((X)\) and \(\rho\) is a congruence on Rig\((X)\). Suppose that \(\Gamma\) is a normal form of Rig\((X|\rho)\). If \(f : \Gamma \rightarrow \Sigma\) is a bijective mapping such that for any \(u \in \Gamma\), \(f(u)\rho = up\), then \(\Sigma\) is also a normal form of Rig\((X|\rho)\).

Proof: For any \(u \in \text{Rig}(X)\), since \(\Gamma\) is a normal form of the semiring Rig\((X|\rho)\), there is uniquely \(v \in \Gamma\), such that \(up = vp\). Hence \(up = f(v)\rho\), where \(f(v) \in \Sigma\).

For any two different \(u, v \in \Sigma\), if \(up = vp\), then \(f^{-1}(u)\rho = f^{-1}(v)\rho\) and hence \(f^{-1}(u) \neq f^{-1}(v)\), a contradiction. This shows that \(\Sigma\) is a normal form of the semiring Rig\((X|\rho)\). \(\blacksquare\)

Corollary 5.6 (\[3\]) A normal form of the semiring Rig\([x | x = 1 \circ x^2]\) is the set
\[
\{1^{on} \circ x^2 \circ x^4, 1^{on} \circ (x^2)^{om}, (x^2)^{om} \circ (x^4)^{ot}, 1^{on} \circ (x^4)^{ot} \mid n, m, t \geq 0\}.
\]

Proof: We denote
\[
\Gamma = \{(1^{on} \circ x^{om})x^t, 1^{on} \circ x^3, 1^{on} \circ (x^4)^{om} \mid n, m \geq 0, \ 0 \leq t \leq 3\},
\]
\[
\Sigma = \{1^{on} \circ x^2 \circ x^4, 1^{on} \circ (x^2)^{om}, (x^2)^{om} \circ (x^4)^{ot}, 1^{on} \circ (x^4)^{ot} \mid n, m, t \geq 0\}
\]
and \(\rho\) the congruence on Rig\([x]\) generated by \(\{x = 1 \circ x^2\}\).

Define
\[
f : \quad \Gamma \rightarrow \Sigma,
\]
\[
1^{on} \circ x^{om} \mapsto 1^{o(n+m)} \circ (x^2)^{om},
\]
\[
(1^{on} \circ x^{om})x \mapsto 1^{on} \circ (x^2)^{o(n+m)},
\]
\[
(1^{on} \circ x^{om})x^2 \mapsto (x^2)^{o(n+m)} \circ (x^4)^{om},
\]
\[
(1^{on} \circ x^{om})x^3 \mapsto (x^2)^{on} \circ (x^4)^{o(n+m)},
\]
\[
1^{on} \circ x^3 \mapsto 1^{on} \circ x^2 \circ x^4,
\]
\[
1^{on} \circ (x^4)^{om} \mapsto 1^{on} \circ (x^4)^{om}.
\]

Then \(f\) is a bijective mapping and for any \(u \in \Gamma\), \(f(u)\rho = up\) since \(f(u)\) is obtained by \(u\) replacing \(x, x^3\) for \(1 \circ x^2, x^2 \circ x^4\) respectively.

Now the result follows from Corollary 5.4 and Lemma 5.5. \(\blacksquare\)
Let $X$ be a well ordered set, $Z$ the integer ring and $Z(X)$ the semigroup ring over $Z$. It is easy to see that $(Z(X), \circ, \cdot)$ is a semiring with the operations $f \circ g := f + g, f \cdot g := fg$, where $f, g$ are polynomials in $Z(X)$. Now, we represent the semiring $Z(X)$ by generators and defining relations.

Let $X^{-1} = \{x^{-1} | x \in X\}$. We define a monomial ordering on $\text{Rig}(X \cup X^{-1} \cup 1^{-1})$ first.

For any $x, y \in X$, we define $x^{-1} > x > y^{-1} > y$ if $x > y$ and $x > 1^{-1} > 1$. Then we define the inverse deg-lex ordering $\leq$ on $\{X \cup X^{-1} \cup 1^{-1}\}^*$. We order $\text{Rig}(X \cup X^{-1} \cup 1^{-1})$ as follows: for any $u, v \in \text{Rig}(X \cup X^{-1} \cup 1^{-1})$, $u > v \iff \text{wt}(u) > \text{wt}(v)$ lexicographically.

Then, it is clear that $>$ on $\text{Rig}(X \cup X^{-1} \cup 1^{-1})$ is a monomial ordering.

**Theorem 5.7** Let the ordering be as above. Then $Z(X) \cong \text{Rig}(X \cup X^{-1} \cup 1^{-1})|S)$ as semirings and a Gröbner-Shirshov basis $S$ in $k\text{Rig}(X \cup X^{-1} \cup 1^{-1})$ consists of the following relations:

1. $x \circ x^{-1} = \theta$,
2. $1 \circ 1^{-1} = \theta$,
3. $x^{-1}y^{-1} = xy$,
4. $xy^{-1} = x^{-1}y$,
5. $x^i1^{-1} = x^{-i}$,
6. $1^{-1}x^i = x^{-i}$,

where $x, y \in X$, $\epsilon = \pm 1$. As a result, a normal form of the semiring $\text{Rig}(X \cup X^{-1} \cup 1^{-1})|S)$ is the set

$$\text{Irr}(S) \ = \ \{x_{i_1}^{e_1}x_{i_2}^{e_2} \cdots x_{i_m}^{e_m} \circ x_{21}^{e_2}x_{22}^{e_2} \cdots x_{2n_2}^{e_2} \circ \cdots \circ x_{m1}^{e_m}x_{m2}^{e_m} \cdots x_{mn_m}^{e_m} \mid x_{ij} \in X, m \geq 0, e_i = \pm 1, n_i \geq 0, i = 1, \ldots, m\},$$

where $x_{i_1}^{e_1}x_{i_2}^{e_2} \cdots x_{i_m}^{e_m} = 1^{e_1}$ if $n_i = 0$.

**Proof:** It is easy to see that

$$\sigma : Z(X) \rightarrow \text{Rig}(X \cup X^{-1} \cup 1^{-1})|S), \ (\epsilon x_{i_1}^{e_1}x_{i_2}^{e_2} \cdots x_{i_t}^{e_t}) \mapsto x_{i_1}^{e_1}x_{i_2}^{e_2} \cdots x_{i_t}^{e_t}, \ 0 \mapsto \theta$$

is a semiring isomorphism, where $\epsilon = \pm 1$. Since $\text{Irr}(S) = \sigma(Z(X))$, $\text{Irr}(S)$ is a $k$-basis of $k\text{Rig}(X \cup X^{-1} \cup 1^{-1})|S)$. Therefore, by using Theorem 5.4, $S$ is a Gröbner-Shirshov basis in $k\text{Rig}(X \cup X^{-1} \cup 1^{-1})$.

18
Let \((\mathbb{N}, \circ, \cdot)\) be the natural numbers semiring, where for any \(n, m \in \mathbb{N}\), \(n \circ m := n + m\), \(n \cdot m := n \times m\). Then \((\mathbb{N}, \circ, \cdot) = \text{Rig}[x \mid x = 1]\). For any congruence \(\rho\) on \(\mathbb{N}\), we have \(\mathbb{N}/\rho = \text{Rig}[x \mid x = 1, \rho]\). Let the ordering on \(\text{Rig}[x]\) be defined as in Theorem 5.1. By Shirshov algorithm, we are able to find a Gröbner-Shirshov basis \(\{x = 1, \rho\}\) for the set \(\{x = 1, \rho\}\). Suppose \(\{x = 1\} \cup S = \{x = 1, \rho\}\). Then by Theorem 4.4, we may assume that \(\{x = 1\} \cup S\) is the reduced Gröbner-Shirshov basis. Since \(\{x = 1\} \cup S\) is minimal, each element in \(S\) has the form \(1 \circ n = 1 \circ m\), \(n > m \in \mathbb{N}\) and \(S\) contains only one element, say, \(1 \circ m = 1 \circ n\), \(n > m \in \mathbb{N}\). It follows that the congruence \(\rho\) on \(\mathbb{N}\) is generated by one element \((n, m)\).

Thus, we have the following corollary.

**Corollary 5.8** Each congruence on the semiring \(\mathbb{N}\) is generated by one element. In particular, \(\mathbb{N}\) is Noetherian.

For a commutative algebra \(k[X|S]\) with \(|X| < \infty\), it is well known that a reduced Gröbner-Shirshov basis of \(k[X|S]\) must be finite. It is also well known that if the ring \(R\) is Noetherian then so is the polynomial ring \(R[X]\) if \(|X| < \infty\). However, it is not the case for the semiring \(\mathbb{N}[x]\).

**Example 5.9** Considering the semiring \(\mathbb{N}[x]/(x + 1 = x) = \text{Rig}[x \mid x \circ 1 = x]\), it is easy to have that \(k\text{Rig}[x \mid x \circ 1 = x] = k\text{Rig}[x|S]\) where \(S = \{x^n \circ 1 = x^n \mid n \geq 1\}\) is the reduced Gröbner-Shirshov basis in \(k\text{Rig}[x]\) with the ordering in Theorem 5.1.

Now, we construct an ascending chain of ideals in \(k\text{Rig}[x]\) as follows.

\[
I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots
\]

where \(I_n = \text{Id}(x \circ 1, x^2 \circ 1, \ldots, x^n \circ 1)\).

For any \(n \geq 1\), \(x^{n+1} \circ 1 \notin I_n\). Otherwise, there exist \(n \geq i \geq 1, a, b \in [x], u \in \text{Rig}[x]\) such that \(x^{n+1} \circ 1 = a(x^i \circ 1) b \circ u\). This is a contradiction because \(S\) is a minimal Gröbner-Shirshov basis in \(k\text{Rig}[x]\). Hence

\[
I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots
\]

Let us define congruence relation \(\rho_n\) on \(\mathbb{N}[x]\) generated by the set

\[
\{(x^i \circ 1, x \circ 1), \ 2 \leq i \leq n\}.
\]

Since \((x^{n+1} \circ 1, x \circ 1) \notin \rho_n\), we have an infinite ascending chain of congruences

\[
\rho_1 \subsetneq \rho_2 \subsetneq \ldots \subsetneq \rho_n \subsetneq \ldots
\]

Thus, we have the following corollary.

**Corollary 5.10** \(\mathbb{N}[x]\) is not Noetherian.

**Acknowledgement**: We are grateful to Marcelo Fiore who took our attention to his and T. Leinster’s paper [29].
References

[1] William W. Adams and Philippe Loustaunau, An introduction to Gröbner bases, Graduate Studies in Mathematics, Vol. 3, American Mathematical Society (AMS), 1994.

[2] G.M. Bergman, The diamond lemma for ring theory, *Adv. Math.*, **29**, 178-218 (1978).

[3] A. Blass, Seven trees in one, *Journal of Pure and Applied Algebra*, **103**, 1-21 (1995).

[4] L.A. Bokut, Insolvability of the word problem for Lie algebras, and subalgebras of finitely presented Lie algebras, *Izvestija AN USSR (mathem.)*, **36**(6), 1173-1219 (1972).

[5] L.A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, **15**, 117-142 (1976).

[6] L.A. Bokut and Yuqun Chen, Gröbner-Shirshov bases: Some new results, Proceedings of the Second International Congress in Algebra and Combinatorics, World Scientific, 35-56 (2008).

[7] L.A. Bokut, Yuqun Chen and Yongshan Chen, Composition-Diamond lemma for tensor product of free algebras, *Journal of Algebra*, **323**, 2520-2537 (2010).

[8] L.A. Bokut, Yuqun Chen and Yongshan Chen, Groebner-Shirshov bases for Lie algebras over a commutative algebra, *Journal of Algebra*, **337**, 82-102 (2011).

[9] L.A. Bokut, Yuqun Chen and Cihua Liu, Gröbner-Shirshov bases for dialgebras, *International Journal of Algebra and Computation*, **20**(3), 391-415 (2010).

[10] L.A. Bokut, Yuqun Chen and Xueming Deng, Gröbner-Shirshov bases for Rota-Baxter algebras, *Siberian Math. J.*, **51**(6), 978-988 (2010).

[11] L.A. Bokut, Yuqun Chen and Yu Li, Gröbner-Shirshov bases for Vinberg-Koszul-Gerstenhaber right-symmetric algebras, *Fundamental and Applied Mathematics*, **14**(8), 55-67 (2008) (in Russian). *J. Math. Sci.*, **166**, 603-612 (2010).

[12] L.A. Bokut, Yuqun Chen and Qiuhui Mo, Gröbner-Shirshov bases and embeddings of algebras, *International Journal of Algebra and Computation*, **20**, 875-900 (2010).

[13] L.A. Bokut, Yuqun Chen and Jianjun Qiu, Gröbner-Shirshov bases for associative algebras with multiple operators and free Rota-Baxter algebras, *Journal of Pure and Applied Algebra*, **214**, 89-100 (2010).

[14] L.A. Bokut, Yuqun Chen and K.P. Shum, Some new results on Groebner-Shirshov bases, in: Proceedings of International Conference on Algebra 2010, Advances in Algebraic Structures, 2012, pp.53-102.

[15] L.A. Bokut, Y. Fong and W.-F. Ke, Composition-Diamond lemma for associative conformal algebras, *Journal of Algebra*, **272**, 739-774 (2004).
[16] L. A. Bokut, Y. Fong, W.-F. Ke and P. S. Kolesnikov, Gröbner and Gröbner-Shirshov bases in algebra and conformal algebras, *Fundamental and Applied Mathematics*, 6(3), 669-706 (2000).

[17] L. A. Bokut and P. S. Kolesnikov, Gröbner-Shirshov bases: from their incipiency to the present, *J. Math. Sci.*, 116(1), 2894-2916 (2003).

[18] L. A. Bokut and P. S. Kolesnikov, Gröbner-Shirshov bases, conformal algebras and pseudo-algebras, *J. Math. Sci.*, 131(5), 5962-6003 (2005).

[19] L. A. Bokut and G. Kukin, Algorithmic and Combinatorial algebra, Kluwer Academic Publ., Dordrecht, 1994.

[20] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations, *Aequationes Math.*, 4, 374-383 (1970).

[21] B. Buchberger, G. E. Collins, R. Loos and R. Albrecht, Computer algebra, symbolic and algebraic computation, Computing Supplementum, Vol.4, New York: Springer-Verlag, 1982.

[22] B. Buchberger and Franz Winkler, Gröbner bases and applications, London Mathematical Society Lecture Note Series, Vol.251, Cambridge: Cambridge University Press, 1998.

[23] Yongshan Chen and Yuqun Chen, Groebner-Shirshov bases for matabelian Lie algebras, *Journal of Algebra*, 358, 143-161 (2012).

[24] Yuqun Chen, Yongshan Chen and Chanyan Zhong, Composition-Diamond lemma for modules, *Czechoslovak Math. J.*, 60(135), 59-76 (2010).

[25] Yuqun Chen, Jing Li and Mingjun Zeng, Composition-Diamond lemma for non-associative algebras over a polynomial algebra, *Southeast Asian Bull. Math.*, 34, 629-638 (2010).

[26] E. S. Chibrikov, On free Lie conformal algebras, *Vestnik Novosibirsk State University*, 4(1), 65-83 (2004).

[27] David A. Cox, John Little and Donal O’Shea, Ideals, varieties and algorithms: An introduction to computational algebraic geometry and commutative algebra, Undergraduate Texts in Mathematics, New York: Springer-Verlag, 1992.

[28] David Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Math., Vol.150, Berlin and New York: Springer-Verlag, 1995.

[29] M. Fiore and T. Leinster, An objective representation of the Gaussian integers, *Journal of Symbolic Computation*, 37, 707-716 (2004).

[30] H. Hironaka, Resolution of singularities of an algebraic variety over a field if characteristic zero, I, II, *Ann. of Math.*, 79, 109-203, 205-326 (1964).

[31] S.-J. Kang and K.-H. Lee, Gröbner-Shirshov bases for irreducible $sl_{n+1}$-modules, *Journal of Algebra*, 232, 1-20 (2000).
[32] A.A. Mikhalev, The junction lemma and the equality problem for color Lie superalgebras, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, 5, 88-91 (1989). English translation: *Moscow Univ. Math. Bull.*, 44, 87-90 (1989).

[33] A.A. Mikhalev, The composition lemma for color Lie superalgebras and for Lie p-superalgebras, *Contemp. Math.*, 131(2), 91-104 (1992).

[34] A.A. Mikhalev, Shirshov’s composition techniques in Lie superalgebra (non-commutative Gröbner bases). *Trudy. Sem. Petrovsk.*, 18, 277-289 (1995). English translation: *J. Math. Sci.*, 80, 2153-2160 (1996).

[35] A.A. Mikhalev and A.A. Zolotykh, Standard Gröbner-Shirshov bases of free algebras over rings, I. Free associative algebras, *International Journal of Algebra and Computation*, 8(6), 689-726 (1998).

[36] A.I. Shirshov, On free Lie rings, *Mat. Sb.*, 45, 113-122 (1958) (in Russian).

[37] A.I. Shirshov, Some algorithmic problem for ε-algebras, *Sibirsk. Mat. Z.*, 3, 132-137 (1962).

[38] A.I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, 3(2), 292-296 (1962) (in Russian). English translation: *SIGSAM Bull.*, 33(2), 3-6 (1999).

[39] Selected works of A.I. Shirshov, Eds L.A. Bokut, V. Latyshev, I. Shestakov, E. Zelmanov, Trs M. Bremner, M. Kochetov, Birkhäuser, Basel, Boston, Berlin, 2009.