REGULARITY OF WEAK SOLUTION OF VARIATIONAL PROBLEMS
MODELING THE COSSEURAT MICROPOLAR ELASTICITY

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Abstract. In this paper, we consider weak solutions of the Euler-Lagrange equation to a variational energy functional modeling the geometrically nonlinear Cosserat micropolar elasticity of continua in dimension three, which is a system coupling between the Poisson equation and the equation of \( p \)-harmonic maps (\( 2 \leq p \leq 3 \)). We show that if a weak solution is stationary, then its singular set is discrete for \( 2 < p < 3 \) and has zero 1-dimension Hausdorff measure for \( p = 2 \). If, in addition, it is a stable-stationary weak solution, then it is regular everywhere when \( p \in [2, \frac{32}{15}] \).

1. Introduction

General continuum models involving independent rotations were introduced by the Eugene and Francois Cosserat brothers in 1909 [1], and were later rediscovered in 1960’s (see Eringen [2]). The micromorphic balance equations derived by Eringen [2] were formally justified by [3, 4] as a more realistic continuum model based on molecular dynamics and ensemble averaging. The major difficulty of mathematical treatment in the finite strain case comes from the geometrically exact formulation of the theory and the appearance of nonlinear manifolds that are necessary to describe the microstructure. Among many variants and vast body of results of Cosserat theory available in the literature, P. Neff [5, 6, 7] has made some systematical analysis of the Cosserat theory for micropolar elastic bodies by establishing the existence of minimizers in the framework of calculus of variations. Very recently, in an interesting article [8], Gaste l has shown a partial regularity theorem of minimizing weak solutions to a Cosserat energy functional for microplar elastic bodies.

The elastic body \( \Omega \subset \mathbb{R}^3 \) is assumed to be a bounded Lipschitz domain. The elastic body can be deformed by a translation mapping \( \phi : \Omega \to \mathbb{R}^3 \), and \( \phi(x) - x \) denotes the (small) dislocation for \( x \in \Omega \). Furthermore, the micropolar structure of the material associates each point \( x \in \Omega \) with an orthonormal frame that is free to rotate in \( \mathbb{R}^3 \) by an orthogonal matrix \( R(x) \in SO(3) \). Both translations and rotations induce material stresses that are given by \( R^t \nabla \phi - I_3 \) and \( R^t \nabla R \) respectively. The Cosserat energy functional stored in the elastic body \( \Omega \) consists of the contributions by both translations and rotations. For a pair of translation and rotation maps \( (\phi, R) : \Omega \to \mathbb{R}^3 \times SO(3) \), the contribution of rotational stresses to the Cosserat energy is given by

\[
\lambda \int_{\Omega} |R^t \nabla R|^p \, dx = \lambda \int_{\Omega} |\nabla R|^p \, dx
\]

for some \( \lambda > 0 \) and \( 2 \leq p \leq 3 \), while the contribution of translational stresses is given by

\[
\int_{\Omega} |P(R^t \nabla \phi - I_3)|^2 \, dx,
\]

1
where $\mathbb{P} : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}$ is the linear map defined by
\[
\mathbb{P}(A) = \sqrt{\mu_1}\text{devsym } A + \sqrt{\mu_3}\text{skew } A + \sqrt{\mu_2}(\text{tr} A)I_3, \quad A \in \mathbb{R}^{3 \times 3},
\]
and
\[
\text{devsym } A = \frac{1}{2}(A + A^t) - (\text{tr} A)I_3, \quad \text{skew } A = \frac{1}{2}(A - A^t),
\]
denotes the deviatoric symmetric part of $A$ and the skew-symmetric part of $A$ respectively. The constants $\mu_1, \mu_3$, and $\mu_2$ are assumed to be positive parameters in this paper.

The elastic body $\Omega$ may be subject to external forces, such as gravity or electromagnetic forces, that can be modeled by
\[
\int_{\Omega} \langle \phi - x, f \rangle \, dx + \int_{\Omega} \langle R, M \rangle \, dx,
\]
where $f : \Omega \to \mathbb{R}^3$ and $M : \Omega \to \mathbb{R}^{3 \times 3}$ are given functions. Collecting together all these terms, the Cosserat energy functional is given by
\[
\text{Cosser}(\phi, R) = \int_{\Omega} \left( |\mathbb{P}(R^t \nabla \phi - I_3)|^2 + |\nabla R|^p + \langle \phi - x, f \rangle + \langle R, M \rangle \right) \, dx. \tag{1.1}
\]
Recall that $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a minimizer of the Cosserat energy functional, if
\[
\text{Cosser}(\phi, R) \leq \text{Cosser}(\tilde{\phi}, \tilde{R}),
\]
holds for any $(\tilde{\phi}, \tilde{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$, with $(\tilde{\phi}, \tilde{R}) = (\phi, R)$ on $\partial \Omega$.

The existence of minimizers of $\text{Cosser}(\phi, R)$ in the Sobolev spaces, under the Dirichlet boundary condition, has been obtained by Neff [6]. By direct calculations, any minimizer $(\phi, R)$ of $\text{Cosser}(\phi, R)$ solves the Euler-Lagrange equation, called as the Cosserat equation:
\[
\begin{cases}
\text{div}(R\mathbb{P}\mathbb{P}(R^t \nabla \phi - I_3)) = \frac{1}{p}f,
\text{div}(|\nabla R|^{p-2} \nabla R) - \frac{2}{p} \nabla \phi (\mathbb{P}\mathbb{P}(R^t \nabla \phi - I_3))^t - \frac{1}{p}M \parallel T_RSO(3).
\end{cases} \tag{1.2}
\]
Here $T_RSO(3)$ denotes the tangent space of $SO(3)$, at $R \in SO(3)$, that is given by
\[
T_RSO(3) = \left\{ Q \in \mathbb{R}^{3 \times 3} \mid R^t Q + Q^t R = 0 \right\},
\]
and $\mathbb{P}^t : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}$ is the adjoint map of $\mathbb{P}$.

When $\mu_1 = \mu_2 = \mu_3 = 1$, we have that $\mathbb{P} = \mathbb{P}^t = \text{Id}$ is the identity map. Hence
\[
|\mathbb{P}(R^t \nabla \phi - I_3)|^2 = |\nabla \phi|^2 - 2\langle R, \nabla \phi \rangle + 3,
\]
and the Cosserat equation (1.2) reduces to the following simplified form:
\[
\begin{cases}
\Delta \phi = \text{div} R + \frac{1}{p}f,
\text{div}(|\nabla R|^{p-2} \nabla R) + \frac{2}{p} \nabla \phi - \frac{1}{p}M \parallel T_RSO(3).
\end{cases} \tag{1.3}
\]

We would like to remark that the system (1.2) and (1.3) are systems coupling between the Poisson equation for the macroscopic translational deformation variable $\phi : \Omega \to \mathbb{R}^3$ and the (nonlinear) $p$-harmonic map equation for the microscopic rotational deformation variable $R : \Omega \to SO(3)$.

By extending the techniques in the study of minimizing $p$-harmonic maps by Schoen-Uhlenbeck [13], Hardt-Lin [9], Fuchs [10], and especially Luckhaus [14], Gastel has recently shown in an interesting article [8] that any minimizer $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$
of the Cosserat energy functional $\text{Coss}(\phi, R)$ of the Cosserat functional (1.1) belongs to $C^{1,\alpha} \times C^\alpha$ in $\Omega$ away from a singular set $\Sigma$ of isolated points for all $2 \leq p < 3$. Moreover, $\Sigma$ is shown to be an empty set when $p \in [2, \frac{12}{5}]$ by extending stability inequality arguments by Schoen-Uhlenbeck [14], Xin-Yang [15], and Chang-Chen-Wei [16].

An interesting question to ask is whether the regularity result on minimizers of the Cosserat functional in [8] remains to hold for certain classes of weak solutions to the Cosserat equation (1.2). In this paper, we will answer this question affirmatively. To address it, we first need to introduce a few definitions.

For $1 \leq p < \infty$, recall the Sobolev space

$$W^{1,p}(\Omega, SO(3)) = \left\{ R \in W^{1,p}(\Omega, \mathbb{R}^{3 \times 3}) \mid R(x) \in SO(3), \text{ a.e. } x \in \Omega \right\}.$$

**Definition 1.1.** For $2 \leq p \leq 3$, given $f \in H^{-1}(\Omega, \mathbb{R}^3)$ and $M \in W^{-\frac{1}{p-1}}(\Omega, \mathbb{R}^{3 \times 3})$, a pair of maps $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a weak solution to the Cosserat equation (1.2), if it satisfies (1.2) in the sense of distributions, i.e.,

\[
\begin{align*}
\int_{\Omega} (\langle \nabla R \cdot \nabla \phi - I_3, \nabla \psi_1 \rangle + \frac{1}{2}(f, \psi_1)) \, dx &= 0, \\
\int_{\Omega} (\langle |\nabla R|^p \nabla R, \nabla \psi_2 \rangle + \frac{2}{p}(\nabla (R^t \nabla \phi - I_3), \nabla \psi_2) + \frac{1}{p}(M, \psi_2)) \, dx &= 0,
\end{align*}
\]

hold for any $\psi_1 \in H^1_0(\Omega, \mathbb{R}^3)$ and $\psi_2 \in W^{1,p}(\Omega, T_R SO(3)) \cap L^\infty(\Omega, \mathbb{R}^{3 \times 3}).$

It is readily seen that any minimizer $(\phi, R)$ of the Cosserat energy functional (1.1) is a weak solution of the Cosserat equation (1.2). A restricted class of weak solutions of (1.2) is the class of stationary weak solutions, which is defined as follows.

**Definition 1.2.** For $2 \leq p \leq 3$, $f \in H^{-1}(\Omega, \mathbb{R}^3)$, and $M \in W^{-\frac{1}{p-1}}(\Omega, \mathbb{R}^{3 \times 3})$, a weak solution $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ to the Cosserat equation (1.2) is called a stationary weak solution, if, in addition, $(\phi, R)$ is a critical point of the Cosserat energy functional (1.1) with respect to the domain variations, i.e.,

$$\left. \frac{d}{dt} \right|_{t=0} \text{Coss}(\phi_t, R_t) = 0,$$

where $(\phi_t(x), R_t(x)) = (\phi(x + tY(x)), R(x + tY(x)))$ for $x \in \Omega$, and $Y \in C^\infty_0(\Omega, \mathbb{R}^3)$.

It is easy to check that any minimizer $(\phi, R)$ of the Cosserat energy functional (1.1) is a stationary weak solution of the Cosserat equation (1.2). It can also be shown by a Pohozaev argument that any regular solution $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3 \times SO(3))$ of the Cosserat equation (1.2) is a stationary weak solution.

In section 2 below, we will show that when $\mu_1 = \mu_c = \mu_2 = 1$, any stationary weak solution $(\phi, R)$ of Cosserat equation (1.3) satisfies the following stationarity identity: for any $Y \in C^\infty_0(\Omega, \mathbb{R}^3)$, it holds that

\[
\begin{align*}
\int_{\Omega} (|\nabla \phi|^2 - 2(R, \nabla \phi) + |\nabla R|^p (-\text{div} Y) \, dx &+ \int_{\Omega} (\langle f, Y \cdot \nabla \phi \rangle + \langle M, Y \cdot \nabla R \rangle) \, dx \\
&+ \int_{\Omega} (2\nabla \phi \otimes \nabla \phi : \nabla Y - 2R_{ij} \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_j}{\partial x_k} + p|\nabla R|^p \nabla R \otimes \nabla Y \rangle \, dx = 0. \tag{1.5}
\end{align*}
\]

As a direct consequence of (1.5), we will establish an almost energy monotonicity inequality for stationary weak solutions to (1.3) when $\mu_1 = \mu_c = \mu_2 = 1$ holds. This, combined with
the symmetry of $SO(3)$, enables us to extend the compensated regularity technique by Hélein [19], Evans [20], and Toro-Wang [21] to show the following partial regularity.

**Theorem 1.3.** For $2 \leq p < 3$, $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, \mathbb{R}^{3 \times 3})$, if $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a stationary weak solution to the Cosserat equation (1.3), then there exist $\alpha \in (0,1)$ and a closed set $\Sigma \subset \Omega$, whose $(3-p)$-dimensional Hausdorff measure $H^{3-p}(\Sigma) = 0$, such that $(\phi, R) \in C^{1,\alpha}(\Omega \setminus \Sigma, \mathbb{R}^3) \times C^\alpha(\Omega \setminus \Sigma, SO(3))$. Furthermore, $\Sigma$ is a discrete set when $p \in (2,3)$.

We would like to point out that the discreteness of singular set $\Sigma$ for $2 < p < 3$ is a corollary of $H^1 \times W^{1,p}$-compactness property of weakly convergent stationary weak solutions of the Cosserat equation (1.3), which is a consequence of monotonicity inequality (2.3) and the Marstrand Theorem (see [23]).

To further improve the estimate of the singular set $\Sigma$ for a stationary weak solution $(\phi, R)$ of the Cosserat equation (1.2) both for $p = 2$ and $2 < p < 3$, we restrict our attention to a subclass of stationary weak solutions that are stable.

**Definition 1.4.** For $2 \leq p < 3$, $f \in H^{-1}(\Omega, \mathbb{R}^3)$, and $M \in W^{-1,p'}(\Omega, \mathbb{R}^{3 \times 3})$, a weak solution $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ to the Cosserat equation (1.2) is called a stable weak solution, if, in addition, the second order variation of the Cosserat energy functional at $(\phi, R)$ is nonnegative, i.e.,

$$\frac{d^2}{dt^2} \bigg|_{t=0} \text{Coss}(\phi_t, R_t) \geq 0,$$

where $(\phi_t, R_t) \in C^2((-\delta, \delta), H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3)))$ for some $\delta > 0$, satisfying $(\phi_0, R_0) = (\phi, R)$, is a variation of $(\phi, R)$ in the target space $\mathbb{R}^3 \times SO(3)$.

From the definition, any minimizer $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ of the Cosserat energy functional $\text{Coss}(\cdot, \cdot)$ is a stable weak solution of the Cosserat equation (1.2).

In section 3, we will establish in the stability Lemma 3.2 that any stable weak solutions $(\phi, R)$ of Cosserat equation satisfies the following stability inequality:

$$\int_\Omega ((p+1)|\nabla R|^{p-2}|
abla \psi|^2 - 2|\nabla R|^p|\psi|^2) \, dx \geq 0, \quad \forall \psi \in C_0^\infty(\Omega).$$

Utilizing the stability inequality (1.7), we can extend the ideas by Hong-Wang [17] and Lin-Wang [18] to establish a pre-compactness property of stable-stationary weak solutions of the Cosserat equation for $p = 2$, which can be employed to improve the estimate of singular set $\Sigma$. Moreover, by applying the non-existence theorem on stable $p$-harmonic maps from $S^2$ to $SO(3)$ for $p \in [2, \frac{24}{13}]$ that was established by Schoen-Uhlenbeck [14], Xin-Yang [15], and Chang-Chen-Wei [16], we prove a complete regularity result for stable stationary weak solutions to the Cosserat equation (1.3) when $p$ belongs to the range $[2, \frac{24}{13}]$. More precisely, we have

**Theorem 1.5.** For $p \in [2, \frac{24}{13}]$, $f \in L^\infty(\overline{\Omega}, \mathbb{R}^3)$, and $M \in L^\infty(\overline{\Omega}, \mathbb{R}^{3 \times 3})$, if $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a stable stationary weak solution to the Cosserat equation (1.3), then there exists $\alpha \in (0,1)$ such that $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \times C^\alpha(\Omega, SO(3))$.

Now we would like to mention a couple of questions.
Remark 1.6. 1). It remains to be an open question whether Theorem 1.3 remains to be true when $\frac{32}{15} < p < 3$. The main difficulty arises from that we can't rule out the existence of nontrivial stable p-harmonic maps from $S^2$ to SO(3) when $p$ lies in the interval $(\frac{32}{15}, 3)$.

2). It remains to be open whether Theorem 1.5 and Theorem 1.7 hold true when the positive constants $\mu_1, \mu_c, \mu_2$ are not necessarily equal. The main difficulty is that it is unknown whether an almost energy monotonicity inequality holds for stationary weak solutions $(\phi, R)$ of the Cosserat equation (1.2) when $p$ is not an identity map.

The paper is organized as follows. In section 2, we will derive both stationarity identity and almost energy monotonicity inequality for stationary weak solutions $(\phi, R)$ of the Cosserat equation (1.2). In section 3, we will rewrite the Cosserat equation (1.3) into a form in which the nonlinearity exhibits div-curl structure s. In section 4, we will prove an $\epsilon_0$-regularity theorem for stationary weak solutions $(\phi, R)$ of the Cosserat equation (1.3), and apply Marstrand’s theorem to obtain a refined estimate of the singular set when $2 < p < 3$. In section 5, we will derive the stability inequality for stable weak solutions and obtain the full regularity for stable stationary weak solutions $(\phi, R)$ of the Cosserat equation (1.3) when $p \in [2, \frac{32}{15}]$.

2. Stationarity identity and almost monotonicity inequality

This section is devoted to the derivation of stationarity identity and almost energy monotonicity inequality for stationary weak solutions to the Cosserat equation (1.3).

Lemma 2.1. For $2 \leq p < 3$, assume $\mu_1 = \mu_c = \mu_2 = 1$, $f \in L^2(\Omega, \mathbb{R}^3)$, and $M \in L^{p-1}(\Omega, SO(3))$. If $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a stationary weak solution of the Cosserat equation (1.2), then for any $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$ it holds that

\[
\int_\Omega \left( 2\nabla \phi \otimes \nabla \phi : \nabla Y + p|\nabla R|^p \nabla R \otimes \nabla R : \nabla Y - 2R^{ik} \frac{\partial \phi^k}{\partial x_j} \frac{\partial \phi^j}{\partial x_i} \right) dx \tag{2.1}
\]

\[
= \int_\Omega \left( (|\nabla \phi|^2 - 2\langle R, \nabla \phi \rangle + |\nabla R|^p) \text{div} Y - \langle Y(x) \cdot \nabla \phi, f \rangle - \langle Y(x) \cdot \nabla R, M \rangle \right) dx.
\]

Proof. For $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$, there is a sufficiently small $\delta > 0$ so that $\text{dist}(\text{supp}(Y), \partial \Omega) > \delta$. Define $(\phi_t, R_t)(x) = (\phi, R)(x + tY(x))$ for $x \in \Omega$ and $t \in (-\delta, \delta)$. Since $(\phi, R)$ is a stationary weak solution of (1.3), we have that

\[
0 = \frac{d}{dt} \bigg|_{t=0} \int_\Omega \left( (|\nabla \phi|^2 - 2\langle R, \nabla \phi \rangle + |\nabla R|^p) \text{div} Y + \langle \phi_t - x, f \rangle + \langle R_t, M \rangle \right) dx.
\]

Applying change of variables and direct calculations, it is not hard to see that

\[
\int_\Omega (|\nabla \phi|^2 - 2\langle R, \nabla \phi \rangle + |\nabla R|^p) (-\text{div} Y) dx + \int_\Omega (\langle Y \cdot \nabla \phi, f \rangle + \langle Y \cdot \nabla R, M \rangle) dx
\]

\[
+ \int_\Omega (2\nabla \phi \otimes \nabla \phi : \nabla Y - 2R^{ik} \frac{\partial \phi^k}{\partial x_j} \frac{\partial \phi^j}{\partial x_i} + p|\nabla R|^p \nabla R \otimes \nabla R : \nabla Y) dx = 0. \tag{2.2}
\]

This yields (2.1). \qed

By choosing suitable test variation fields $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$, we will obtain an almost energy monotonicity inequality for stationary weak solutions to the Cosserat equation (1.3).
Corollary 2.2. For $2 \leq p < 3$, assume $\mu_1 = \mu_2 = 1$, $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, SO(3))$. If $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a stationary weak solution of the Cosserat equation \([1,3]\), then for any $x \in \Omega$ and $0 < r_1 < r_2 < \text{dist}(x, \partial \Omega)$, it holds that

$$
\text{Coss}_x((\phi, R), r_1) + \int_{r_1}^{r_2} r^{p-3} \int_{\partial B_r} (p|\nabla R|^p - |\partial R|^{p-2} |\frac{\partial R}{\partial r}|^2 + |\frac{\partial \phi}{\partial r}|^2) dH^2 dr \\
\leq \text{Coss}_x((\phi, R), r_2),
$$

(2.3)

where $\text{Coss}_x((\phi, R), r)$ is the modified renormalized Cosserat energy defined by

$$
\text{Coss}_x((\phi, R), r) := e^{Cr^2} r^{p-3} \int_{B_r(x)} ((|\nabla R|^p + |\nabla \phi|^2) dx + Cr^3,
$$

(2.4)

where $C > 0$ depends on $p$, $\|f\|_{L^\infty(\Omega)}$, and $\|M\|_{L^\infty(\Omega)}$.

Proof. For simplicity, assume $x = 0 \in \Omega$ and $0 < r < \text{dist}(0, \partial \Omega)$ and write $B_r = B_r(0)$. Let $Y(x) = x \eta_\epsilon(|x|)$, where $\eta_\epsilon \in C_0^\infty(B_r)$ is chosen such that $\eta_\epsilon \to \chi_{B_r}$ as $\epsilon \to 0$. Plugging $Y$ into (2.1) and sending $\epsilon$ to 0, we obtain that

$$(p - 3) \int_{B_r} |\nabla R|^p dx + r \int_{\partial B_r} |\nabla R|^p dH^2 - \int_{B_r} |\nabla \phi|^2 dx + r \int_{\partial B_r} |\nabla \phi|^2 dH^2 = -4 \int_{B_r} \langle R, \nabla \phi \rangle dx + 2r \int_{\partial B_r} \langle R, \nabla \phi \rangle dH^2 - \int_{B_r} \langle (x \cdot \nabla \phi, f) + (x \cdot \nabla R, M) \rangle dx
$$

$$+ pr \int_{\partial B_r} |\nabla R|^{p-2} |\frac{\partial R}{\partial r}|^2 dH^2 + 2r \int_{\partial B_r} |\frac{\partial \phi}{\partial r}|^2 dH^2 - 2 \int_{\partial B_r} x^i R^j k \frac{\partial \phi^k}{\partial r} dH^2.\quad (2.5)$$

It is easy to estimate

$$\begin{align*}
2r \int_{\partial B_r} \langle R, \nabla \phi \rangle dx &\leq Cr^2 \int_{\partial B_r} |\nabla \phi|^2 dH^2 + Cr^2, \\
-4 \int_{B_r} \langle R, \nabla \phi \rangle dx &\leq Cr \int_{B_r} |\nabla \phi|^2 dx + Cr^2, \\
-2 \int_{\partial B_r} R : x \otimes \frac{\partial \phi}{\partial r} dH^2 &\leq r \int_{\partial B_r} |\frac{\partial \phi}{\partial r}|^2 dH^2 + Cr^3, \\
- \int_{B_r} \langle x \cdot \nabla \phi, f \rangle dx &\leq Cr \int_{B_r} |\nabla \phi|^2 dx + C\|f\|_{L^\infty(\Omega)}^2 r^4, \\
- \int_{B_r} \langle x \cdot \nabla R, M \rangle dx &\leq Cr \int_{B_r} |\nabla R|^p dx + C\|M\|_{L^\infty(\Omega)}^{p/2} r^4.
\end{align*}$$

Substituting these estimates into (2.5) yields

$$(p - 3) \int_{B_r} |\nabla R|^p dx + r \int_{\partial B_r} |\nabla R|^p dH^2 - \int_{B_r} |\nabla \phi|^2 dx + r \int_{\partial B_r} |\nabla \phi|^2 dH^2
$$

$$\geq pr \int_{\partial B_r} |\nabla R|^{p-2} |\frac{\partial R}{\partial r}|^2 dH^2 + r \int_{\partial B_r} |\frac{\partial \phi}{\partial r}|^2 dH^2 - Cr \int_{B_r} |\nabla R|^p dx - Cr \int_{B_r} |\nabla \phi|^2 dx - Cr^2 \int_{\partial B_r} |\nabla \phi|^2 dH^2 - C(1 + \|f\|_{L^\infty(\Omega)}^2 + \|M\|_{L^\infty(\Omega)}^{p/2}) r^2.\quad (2.6)$$
Hence we obtain for $0 < r \leq \min \{ 1, \text{dist}(0, \partial \Omega) \}$,
\[
\frac{d}{dr} \left\{ e^{Cr^2} r^{p-3} \int_{B_r} (|\nabla R|^p + |\nabla \phi|^2) \, dx \right\} \\
\geq e^{Cr^2} r^{p-3} \int_{B_r} (p|\nabla R|^{p-2} |\partial R| + 2|\partial \phi|^2) \, dx + (p - 2) e^{Cr^2} r^{p-4} \int_{B_r} |\nabla \phi|^2 \, dx \\
- C(1 + \|f\|_{L^\infty(\Omega)}^2 + \|M\|_{L^\infty(\Omega)}^2) r^2 \\
\geq r^{p-3} \int_{\partial B_r} (p|\nabla R|^{p-2} |\partial R| + 2|\partial \phi|^2) \, dH^2 \\
- C(1 + \|f\|_{L^\infty(\Omega)}^2 + \|M\|_{L^\infty(\Omega)}^2) r^2.
\] Integrating from $0 < r_1 \leq r_2 \leq \min \{ 1, \text{dist}(0, \partial \Omega) \}$, we obtain that the following monotonicity inequality:
\[
e^{Cr^2} r_2^{p-3} \int_{B_{r_2}} (|\nabla R|^p + |\nabla \phi|^2) \, dx + Cr_2^3 \\
\geq e^{Cr^2} r_1^{p-3} \int_{B_{r_1}} (|\nabla R|^p + |\nabla \phi|^2) \, dx + Cr_1^3 \\
+ \int_{r_1}^{r_2} r^{p-3} \int_{\partial B_r} (p|\nabla R|^{p-2} |\partial R| + 2|\partial \phi|^2) \, dH^2 \, dr,
\]
where $C > 0$ depends on $p$, $\|f\|_{L^\infty(\Omega)}$, and $\|M\|_{L^\infty(\Omega)}$. This completes the proof of (2.3).}

3. Div-curl structure of the Cosserat equation (1.3)

This section is devoted to rewriting of the Cosserat equation (1.3) into a form where the nonlinearity exhibits algebraic structures similar to that of $p$-harmonic maps into symmetric manifolds given by Hélein [19] and Toro-Wang [21].

Let $so(3)$ be the Lie algebra of $SO(3)$ or equivalently the tangent space of $SO(3)$ at $I_3$. Recall that a standard orthonormal base of $so(3)$ is given by
\[
\begin{align*}
\mathbf{a}_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \\
\mathbf{a}_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \\
\mathbf{a}_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]
For any $R \in SO(3)$,
\[
\{ \mathbf{V}_1(R) = \mathbf{a}_1 R, \mathbf{V}_2(R) = \mathbf{a}_2 R, \mathbf{V}_3(R) = \mathbf{a}_3 R \}
\]
forms an orthonormal base of $T_RSO(3)$, the tangent space of $SO(3)$ at $R$.

From (1.3) we have that for $i = 1, 2, 3$,
\[
\langle \text{div}(\nabla R^{p-2} \nabla R), \mathbf{V}_i(R) \rangle = -\frac{2}{p} \langle \nabla \phi, \mathbf{V}_i(R) \rangle + \frac{1}{p} \langle M, \mathbf{V}_i(R) \rangle.
\]

For $i = 1, 2, 3$, since $\mathbf{a}_i$ is skew-symmetric, we have that
\[
\langle |\nabla R|^{p-2} \nabla R, \nabla (\mathbf{V}_i(R)) \rangle = \langle |\nabla R|^{p-2} \nabla R, \mathbf{a}_i \nabla R \rangle = 0.
\]
Thus we can rewrite the Cosserat equation (1.3) as follows.

\[
\text{div}(|\nabla R|^{p-2}\nabla R) = \sum_{i=1}^{3} \text{div}\left(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle \mathbf{V}_i(R) \right)
\]

\[
= \sum_{i=1}^{3} \left[ \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle + \langle |\nabla R|^{p-2}\nabla R, \nabla(\mathbf{V}_i(R)) \rangle \right] \mathbf{V}_i(R)
\]

\[+ \sum_{i=1}^{3} \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle \nabla(\mathbf{V}_i(R)) \]

\[
= \sum_{i=1}^{3} \left( - \frac{2}{p} \langle \nabla \phi, \mathbf{V}_i(R) \rangle + \frac{1}{p} \langle M, \mathbf{V}_i(R) \rangle \right) \mathbf{V}_i(R) + \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle \nabla(\mathbf{V}_i(R)) \right].
\] (3.2)

From the above derivation, we see that for \( i = 1, 2, 3 \),

\[
\text{div}(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle) = \frac{2}{p} \langle \nabla \phi, \mathbf{V}_i(R) \rangle + \frac{1}{p} \langle M, \mathbf{V}_i(R) \rangle.
\] (3.3)

For \( i = 1, 2, 3 \), let \( Y_i : \Omega \to \mathbb{R} \) solve the auxiliary equation

\[
\Delta Y_i = \frac{2}{p} \langle \nabla \phi, \mathbf{V}_i(R) \rangle - \frac{1}{p} \langle M, \mathbf{V}_i(R) \rangle,
\] (3.4)

so that

\[
\text{div}(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle) + \nabla Y_i = 0.
\] (3.5)

Putting (3.2), (3.3), (3.4), (3.5) together, we obtain an equivalent form of (1.3)_2:

\[
\text{div}(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle + \nabla Y_i) \nabla(\mathbf{V}_i(R))
\]

\[
= \sum_{i=1}^{3} \left( \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle + \nabla Y_i \right) \nabla(\mathbf{V}_i(R))
\]

\[
- \sum_{i=1}^{3} \nabla Y_i \cdot \nabla(\mathbf{V}_i(R)) + \sum_{i=1}^{3} \left( - \frac{2}{p} \langle \nabla \phi, \mathbf{V}_i(R) \rangle + \frac{1}{p} \langle M, \mathbf{V}_i(R) \rangle \right) \mathbf{V}_i(R).
\] (3.6)

It is readily seen that as the leading order term of nonlinearity in the right hand side of the equation (3.6), \( \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle + \nabla Y_i \rangle \nabla(\mathbf{V}_i(R)) \) is the inner product of a divergence free vector field \( \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R) \rangle \) and a curl free vector field \( \nabla(\mathbf{V}_i(R)) \).

4. \( \epsilon_0 \)-regularity of stationary solutions of the Cosserat equation

In this section, we will establish an \( \epsilon_0 \)-regularity estimate and a partial regularity of stationary weak solutions of the Cosserat equation (1.3) and give a proof of Theorem 1.3. The key ingredient is the following energy decay lemma, under the smallness condition.

**Lemma 4.1.** For any \( 2 \leq p < 3 \), \( \mu_1 = \mu_c = \mu_2 = 1 \), \( f \in L^\infty(\Omega, \mathbb{R}^3) \) and \( M \in L^\infty(\Omega, SO(3)) \), there exist \( \epsilon_0 > 0 \) and \( \theta_0 \in (0, \frac{1}{2}) \) depending on \( p \), \( \|f\|_{L^\infty(\Omega)} \), and \( \|M\|_{L^\infty(\Omega)} \) such that if \( (\phi, R) \) is a stationary weak solution of the Cosserat equation (1.3), and satisfies, for \( x \in \Omega \) and \( 0 < r < \text{dist}(x, \partial \Omega) \),

\[
r^{p-3} \int_{B_r(x)} (|\nabla R|^p + |\nabla \phi|^2) \, dx \leq \epsilon_0^p,
\] (4.1)
Moreover, it holds that
\[
(\theta_0 r)^{p-3} \int_{B_{\theta_0 r}(x)} (|\nabla R|^p + |\nabla \phi|^2) \, dx
\leq \frac{1}{2} \max \left\{ r^{p-3} \int_{B_r(x)} (|\nabla R|^p + |\nabla \phi|^2) \, dx, \ r^p \right\}. \quad (4.2)
\]

**Proof.** We argue it by contradiction. Suppose that the conclusion were false. Then for any \( L > 0 \) with \( \|f\|_{L^\infty(\Omega)} + \|M\|_{L^\infty(\Omega)} \leq L \) and \( \theta \in (0, \frac{1}{2}) \), there exist \( \epsilon_k \to 0 \), \( x_k \in \Omega \), and \( r_k \to 0 \) such that
\[
r_k^{p-3} \int_{B_{r_k}(x_k)} (|\nabla R|^p + |\nabla \phi|^2) \, dx \leq \epsilon_k^p, \quad (4.3)
\]
but
\[
(\theta r_k)^{p-3} \int_{B_{\theta r_k}(x_k)} (|\nabla R|^p + |\nabla \phi|^2) \, dx > \frac{1}{2} \max \left\{ r_k^{p-3} \int_{B_k(x_k)} (|\nabla R|^p + |\nabla \phi|^2) \, dx, \ r_k^p \right\}. \quad (4.4)
\]

Define the rescaling maps
\[
\begin{align*}
R_k(x) &= R(x_k + r_k x), \\
\phi_k(x) &= r_k^{-2} \phi(x_k + r_k x), \\
f_k(x) &= r_k^{-2} f(x_k + r_k x), \\
M_k(x) &= r_k^p M(x_k + r_k x),
\end{align*}
\]
Then \((\phi_k, R_k)\) solves in \(B_1\)
\[
\begin{align*}
\Delta \phi_k &= r_k^2 \text{div}(R_k) + \frac{1}{2} f_k, \\
\text{div}(|\nabla R_k|^{p-2} \nabla R_k) &= \sum_{\alpha=1}^{3} \langle |\nabla R_k|^{p-2} \nabla R_k, V_\alpha(R_k) \rangle \nabla (V_\alpha(R_k)) \\
&\quad - \frac{2}{p} \sum_{\alpha=1}^{3} r_k^\frac{p}{2} \langle \nabla \phi_k, V_\alpha(R_k) \rangle - \langle M_k, V_\alpha(R_k) \rangle \nabla (V_\alpha(R_k)).
\end{align*}
\quad (4.5)
\]

Moreover, it holds that
\[
\int_{B_1} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx = r_k^{p-3} \int_{B_{r_k}(x_k)} (|\nabla R|^p + |\nabla \phi|^2) \, dx = \epsilon_k^p, \quad (4.6)
\]
and
\[
\theta r_k^{p-3} \int_{B_{\theta r_k}(x_k)} (|\nabla R|^p + |\nabla \phi|^2) \, dx > \frac{1}{2} \max \left\{ \int_{B_1} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx, \ r_k^p \right\}. \quad (4.7)
\]

Now we define the blow-up sequence:
\[
\begin{align*}
\hat{R}_k(x) &= \frac{R_k(x) - R_k}{\epsilon_k}, \\
\hat{\phi}_k(x) &= \frac{\phi_k(x) - \phi_k}{\epsilon_k^2}, \quad \forall x \in B_1,
\end{align*}
\]
where \( \overline{f} = \frac{1}{|B_1|} \int_{B_1} f \) denotes the average of \( f \) over \( B_1 \). Then \((\hat{\phi}_k, \hat{R}_k)\) solves, in \( B_1 \),

\[
\begin{cases}
\Delta \hat{\phi}_k = r_k^p \epsilon_k^{1-\frac{p}{2}} \text{div}(\hat{R}_k) + \frac{1}{3} \epsilon_k^{1-\frac{p}{2}} f_k, \\
\text{div}(|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k) = \epsilon_k \sum_{a=1}^{3} (|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k, V_{\alpha}(R_k)) \nabla (V_{\alpha}(\hat{R}_k)) \\
- \frac{1}{p} \sum_{a=1}^{3} [2r_k^p \epsilon_k^{1-\frac{p}{2}} (\nabla \hat{\phi}_k, V_{\alpha}(R_k)) - \epsilon_k^{1-p} (M_k, V_{\alpha}(R_k))] V_{\alpha}(R_k),
\end{cases}
\]

satisfies

\[
\int_{B_1} \hat{R}_k \, dx = 0, \quad \int_{B_1} \hat{\phi}_k \, dx = 0, \quad \int_{B_1} (|\nabla \hat{R}_k|^p + |\nabla \hat{\phi}_k|^2) \, dx = 1,
\]

and

\[
\theta^{p-3} \int_{B_0} (|\nabla \hat{R}_k|^p + |\nabla \hat{\phi}_k|^2) \, dx > \frac{1}{2} \max \left\{ 1, \frac{r_k^p}{\epsilon_k^p} \right\}.
\]

In particular, we have

\[
\frac{r_k^p}{\epsilon_k^p} \leq 2\theta^{p-3} \int_{B_0} (|\nabla \hat{R}_k|^p + |\nabla \hat{\phi}_k|^2) \, dx \leq 2\theta^{p-3}.
\]

This implies that

\[
r_k \leq C \epsilon_k.
\]

We may assume that there exist \( \phi_\infty \in H^1(B_1, \mathbb{R}^2) \), \( R_\infty \in W^{1,p}(B_1, SO(3)) \) such that, after passing to a subsequence,

\((\hat{\phi}_k, \hat{R}_k) \rightarrow (\phi_\infty, R_\infty) \) in \( H^1(B_1) \times W^{1,p}(B_1) \), \((\hat{\phi}_k, \hat{R}_k) \rightarrow (\phi_\infty, R_\infty) \) in \( L^2(B_1) \times L^p(B_1) \).

Then \((\phi_\infty, R_\infty)\) satisfies

\[
\begin{cases}
\phi_\infty = 0, \\
R_\infty = 0, \\
\int_{B_1} (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2) \, dx \leq 1.
\end{cases}
\]

Moreover, it follows from (4.11) that

\[
\|\epsilon_k^{-\frac{p}{2}} f_k\|_{L^\infty(B_1)} \leq C \epsilon_k^{-\frac{p}{2}} r_k^{\frac{p}{2}+2} \leq C r_k \rightarrow 0,
\]

\[
\|\epsilon_k^{1-p} M_k\|_{L^\infty(B_1)} \leq C \epsilon_k^{1-p} r_k^p \leq C \epsilon_k \rightarrow 0,
\]

and

\[
\|r_k^p \epsilon_k^{1-\frac{p}{2}} \text{div}(\hat{R}_k)\|_{L^p(B_1)} + \|r_k^p \epsilon_k^{1-p} \nabla \hat{\phi}_k\|_{L^2(B_1)} \leq C \epsilon_k^{1-\frac{p}{2}} r_k^p \leq C \epsilon_k \rightarrow 0.
\]

Hence, after sending \( k \rightarrow \infty \) in the equation (4.13), we conclude that \( \phi_\infty \) is a harmonic function and \( R_\infty \) is a \( p \)-harmonic function, i.e.,

\[
\begin{cases}
\Delta \phi_\infty = 0 \\
\text{div}(|\nabla \phi_\infty|^{p-2} \nabla \phi_\infty) = 0,
\end{cases} \quad \text{in } B_1.
\]
Lemma 4.2. The sequence \( \phi_k \) functions of bounded mean oscillations in \( \mathbb{R} \). For the convenience of readers, we sketch the proof here. Fix any point \( x_0 \in \mathbb{R}^3 \) and let \( \eta \) be a smooth cutoff function satisfying
\[ 0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } B_1, \quad \eta = 0 \text{ on } \mathbb{R}^3 \setminus B_2. \]
Then we have the following lemma, whose proof is based on the energy monotonicity inequality (2.3) and is similar to that by [20] and [21]. Denote by \( \text{BMO}(\mathbb{R}^3) \) the space of functions of bounded mean oscillations in \( \mathbb{R}^3 \).

**Lemma 4.2.** The sequence \( \{ \eta \hat{R}_k \}_{k \geq 1} \) is bounded in \( \text{BMO}(\mathbb{R}^3) \).

**Proof.** For the convenience of readers, we sketch the proof here. Fix any point \( x_0 \in \mathbb{R}^3 \) and 0 < \( r \leq \frac{1}{8} \), define \( y_k = x_k + r_k x_0 \in B_{r_k}(x_k) \). By the monotonicity inequality (2.3), we have
\[
\frac{1}{(rr_k)^3 - p} \int_{B_{rr_k}(y_k)} |\nabla R|^p \, dx \leq e^{C(rr_k)^2} \frac{1}{(rr_k)^3 - p} \int_{B_{rr_k}(y_k)} |\nabla R|^p \, dx
\]
Next we need to show that \( (\hat{\phi}_k, \hat{R}_k) \) converges strongly to \( (\phi_\infty, R_\infty) \) in \( H^1(B_1^\perp) \times W^{1,p}(B_1^\perp) \), which is based on the duality between the Hardy space and the BMO space. Let \( \eta : \mathbb{R}^3 \to \mathbb{R} \) be a smooth cutoff function satisfying
\[
0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } B_1, \quad \eta = 0 \text{ on } \mathbb{R}^3 \setminus B_2.
\]
Hence we have that for 0 < \( \theta < \frac{1}{2} \),
\[
\theta^{p-3} \int_{B_\theta} (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2) \, dx \leq C \theta^p (\|\nabla R_\infty\|_{L^\infty(B_2)} + \|\nabla \phi_\infty\|_{L^\infty(B_2)}^2)
\]
\[
\leq C \theta^p \int_{B_1} (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2) \, dx \leq C \theta^p.
\]
Applying the John-Nirenberg inequality yields that for any $1 \leq q < \infty$,
\[
\{\hat{R}_k\}_{k \geq 1} \text{ is bounded in } L^q(B_{\frac{3}{4}}).
\]
Since $\eta$ is smooth, it follows that for any $y \in B_r(x_0)$,
\[
|\langle \eta \hat{R}_k \rangle_{x_0,y} - \eta \langle \hat{R}_k \rangle_{x_0,y}| \leq C r^{-2} \int_{B_r(x_0)} |\hat{R}_k| \, dx.
\] (4.16)
Combining (4.15) with (4.16), it follows that for $x_0 \in B_{\frac{3}{4}}$,
\[
\begin{align*}
\frac{1}{r^3} \int_{B_r(x_0)} |\eta \hat{R}_k - (\eta \hat{R}_k)_{x_0,y}| \, dx \\
&\leq \frac{1}{r^3} \int_{B_r(x_0)} |\hat{R}_k - \eta \hat{R}_k_{x_0,y}| \, dx + \frac{1}{r^3} \int_{B_r(x_0)} |\eta \hat{R}_k_{x_0,y} - (\eta \hat{R}_k)_{x_0,y}| \, dx \\
&\leq \frac{1}{r^3} \int_{B_r(x_0)} |\hat{R}_k - (\hat{R}_k)_{x_0,y}| \, dx + C r^{-2} \int_{B_r(x_0)} |\hat{R}_k| \, dx \\
&\leq C + C \frac{r^2}{r^3} (\int_{B_{\frac{3}{4}}(x_0)} |\hat{R}_k|^3 \, dx) \frac{1}{r^2} \leq C.
\end{align*}
\]
Since $\eta = 0$ on $\mathbb{R}^3 \setminus B_{\frac{3}{4}}$, we have
\[
\sup_k \|\eta \hat{R}_k\|_{L^1(\mathbb{R}^3)} < \infty.
\]
Hence the above inequality remains to hold for $x_0 \in \mathbb{R}^3 \setminus B_{\frac{3}{4}}$ and $r > 0$. The proof is complete. \qed

**Lemma 4.3.** $\nabla \hat{R}_k$ converge strongly to $\nabla R_\infty$ in $L^p(B_{\frac{3}{4}})$, and $\nabla \hat{\phi}_k$ converge strongly to $\nabla \phi_\infty$ in $L^2(B_{\frac{3}{4}})$.

**Proof.** First notice that scalings of the equation (4.3) imply that for $i = 1, 2, 3$,
\[
\text{div} (|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k, \nabla \hat{\phi}_k) = -\frac{2}{p} \hat{R}_k^{\frac{p-2}{2}} (\nabla \hat{\phi}_k, \nabla \hat{\phi}_k) + \frac{1}{p} \hat{R}_k^{1-\frac{p}{2}} (M_k, \nabla \hat{\phi}_k) + \frac{1}{p} \hat{R}_k^{1-\frac{p}{2}} (M_k, M_k) (4.17)
\]
As in (4.4), let $Y_k^i : B_1 \to \mathbb{R}$ solve
\[
\begin{cases}
\Delta Y_k^i = \frac{2}{p} \hat{R}_k^{\frac{p-2}{2}} (\nabla \hat{\phi}_k, \nabla \hat{\phi}_k) - \frac{1}{p} \hat{R}_k^{1-\frac{p}{2}} (M_k, \nabla \hat{\phi}_k) \quad &\text{in } B_1, \\
Y_k^i = 0 \quad &\text{on } \partial B_1.
\end{cases}
\] (4.18)
It is easy to see that by $W^{2,2}$-theory, $Y_k^i$ satisfies
\[
\begin{align*}
\|\nabla Y_k^i\|_{L^2(B_1)} + \|\nabla^2 Y_k^i\|_{L^2(B_1)} &\leq C \hat{R}_k^{\frac{2}{p}} \|\nabla \hat{\phi}_k\|_{L^2(B_1)} + C \hat{R}_k^{1-\frac{p}{2}} \|M_k\|_{L^2(B_1)} \\
&\leq C \left( r_k^{\frac{2}{p}} \hat{R}_k^{\frac{2}{p}} + r_k^{1-\frac{p}{2}} \right) \leq C \hat{R}_k,
\end{align*}
\] (4.19)
where we have used (4.11) in the last step.
Adding the equations (4.17) and (4.18), we have that
\[
\text{div}(\langle |\nabla \widehat{R}_k|^p - \nabla \widehat{R}_k, \mathbf{V}_i(R_k) \rangle + \nabla Y^i_k) = 0 \quad \text{in} \quad B_1,
\]
(4.20)
and the blowup equation (4.18) becomes
\[
\begin{cases}
\text{div}(\langle |\nabla \widehat{R}_k|^p - \nabla \widehat{R}_k, \mathbf{V}_i(R_k) \rangle + \nabla Y^i_k) \\
= \epsilon_k \sum_{i=1}^{3} (\langle |\nabla \widehat{R}_k|^p - \nabla \widehat{R}_k, \mathbf{V}_i(R_k) \rangle + \nabla Y^i_k) \cdot \nabla (\mathbf{V}_i(\widehat{R}_k)) \\
- \frac{1}{\kappa} \sum_{i=1}^{3} \left[ 2 \eta \epsilon_k 1 - \frac{p}{\kappa} \langle \nabla \phi_k, \mathbf{V}_i(R_k) \rangle - \epsilon_k \langle \mathbf{M}_k, \mathbf{V}_i(R_k) \rangle \right] \mathbf{V}_i(R_k) \\
- \epsilon_k \sum_{i=1}^{3} \nabla Y^i_k \cdot \nabla (\mathbf{V}_i(\widehat{R}_k))
\end{cases}
\quad \text{in} \ B_1.
\]
(4.21)
Define
\[
H^i_k = \langle |\nabla \widehat{R}_k|^p - \nabla \widehat{R}_k, \mathbf{V}_i(R_k) \rangle + \nabla Y^i_k \cdot \nabla (\mathbf{V}_i(\widehat{R}_k)).
\]
Then it follows from (4.20) that \(H^i_k \in H^i_{\text{loc}}(B_1)\), the local Hardy space (see [19] and [20] for some basic properties of Hardy spaces). For any compact \(K \subset B_1\) and \(i = 1, 2, 3\), we can use \(\frac{3}{2} < \frac{p}{\kappa - 1} \leq 2\) and (4.19) to estimate
\[
\|H^i_k\|_{H^i(K)} \leq C \|\langle |\nabla \widehat{R}_k|^p - \nabla \widehat{R}_k, \mathbf{V}_i(R_k) \rangle + \nabla Y^i_k\|_{L^p(B_1)} \|\nabla (\mathbf{V}_i(\widehat{R}_k))\|_{L^p(B_1)}
\]
\[
\leq C \left[ \|\nabla \widehat{R}_k\|_{L^p(B_1)}^{p-1} + \|\nabla Y^i_k\|_{L^p(B_1)} \right] \|\nabla (\mathbf{V}_i(\widehat{R}_k))\|_{L^p(B_1)}
\]
\[
\leq C, \quad \forall k \geq 1.
\]
and
\[
\|H^i_k\|_{L^1(B_1)} \leq C \left[ \|\nabla \widehat{R}_k\|_{L^p(B_1)} + \|\nabla Y^i_k\|_{L^p(B_1)} \right] \|\nabla (\mathbf{V}_i(\widehat{R}_k))\|_{L^p(B_1)}
\]
\[
\leq C, \quad \forall k \geq 1.
\]
Assume \(\int_{\mathbb{R}^3} \eta \, dx \neq 0\). For \(i = 1, 2, 3\), set
\[
\mu^i_k = \frac{\int_{\mathbb{R}^3} H^i_k \eta \, dx}{\int_{\mathbb{R}^3} \eta \, dx}, \quad \forall k \geq 1.
\]
Then we have that
\[
\sup_{k \geq 1} \|\eta (H^i_k - \mu^i_k)\|_{H^i(\mathbb{R}^3)} \leq C \sup_{k \geq 1} \left( \|H^i_k\|_{H^i(\text{supp} \eta)} + \|H^i_k\|_{L^1(B_1)} \right) \leq C,
\]
(4.22)
and
\[
\|\mu^i_k\| \leq C \|H^i_k\|_{L^1(B_1)} \leq C.
\]
(4.23)
Observe that
\[
\text{div}(|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k - |\nabla R_\infty|^{p-2} \nabla R_\infty) \\
= \epsilon_k \sum_{i=1}^{3} H_k^i - \epsilon_k \sum_{i=1}^{3} \nabla Y_k^i \cdot \nabla (V_i(\hat{R}_k)) \\
- \frac{1}{p} \sum_{i=1}^{3} \left[2r_k \epsilon_k^{1-\frac{p}{2}} \langle \nabla \phi_k, V_i(\hat{R}_k) \rangle - \epsilon_k^{1-p} \langle M_k, V_i(\hat{R}_k) \rangle \right] V_i(\hat{R}_k).
\]

Multiplying this equation by \(\eta^2(\hat{R}_k - R_\infty)\) and integrating it over \(\mathbb{R}^3\), we obtain that
\[
\int_{B_1} \eta^2(|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k - |\nabla R_\infty|^{p-2} \nabla R_\infty) : \nabla (\hat{R}_k - R_\infty) \, dx \\
+ 2 \int_{B_1} \eta(|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k - |\nabla R_\infty|^{p-2} \nabla R_\infty) : \nabla \eta \otimes (\hat{R}_k - R_\infty) \, dx \\
= \epsilon_k \int_{B_1} \left[ \nabla Y_k^i \cdot \nabla (V_i(\hat{R}_k)) - H_k^i \right] \eta^2(\hat{R}_k - R_\infty) \, dx \\
+ \frac{1}{p} \sum_{i=1}^{3} \int_{B_1} \left[2r_k \epsilon_k^{1-\frac{p}{2}} \langle \nabla \phi_k, V_i(\hat{R}_k) \rangle - \epsilon_k^{1-p} \langle M_k, V_i(\hat{R}_k) \rangle \right] V_i(\hat{R}_k) \eta^2(\hat{R}_k - R_\infty) \, dx.
\]

It is not hard to see that
\[
\int_{B_1} \eta^2|\nabla \hat{R}_k - \nabla R_\infty|^p \, dx \\
\leq C \int_{B_1} \eta(|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k - |\nabla R_\infty|^{p-2} \nabla R_\infty)|\nabla \eta| |\hat{R}_k - R_\infty| \, dx \\
+ C\epsilon_k \int_{B_1} H_k^i \cdot \eta^2(\hat{R}_k - R_\infty) \, dx \\
+ C \epsilon_k \int_{B_1} \eta^2|\nabla Y_k^i| |\nabla \hat{R}_k| |\hat{R}_k - R_\infty| \, dx \\
+ C \int_{B_1} \left[2r_k \epsilon_k^{1-\frac{p}{2}} |\nabla \phi_k| + \epsilon_k^{1-p} |M_k| \right] \eta^2|\hat{R}_k - R_\infty| \, dx \\
= I_k + II_k + III_k + IV_k.
\]

Since
\[
|\nabla \hat{R}_k|^{p-2} \nabla \hat{R}_k \to |\nabla R_\infty|^{p-2} \nabla R_\infty \quad \text{in} \quad L^{\frac{p}{p-1}}(B_1), \quad \hat{R}_k \to R_\infty \quad \text{in} \quad L^p(B_1),
\]
we conclude that
\[
I_k \to 0.
\]

For \(III_k\), we have
\[
III_k \leq C\epsilon_k \|\nabla Y_k^i\|_{L^p(B_1)} \|\nabla \hat{R}_k\|_{L^2(B_1)} \|\hat{R}_k - R_\infty\|_{L^3(B_1)} \\
\leq C\epsilon_k \|\nabla Y_k^i\|_{H^1(B_1)} \|\nabla \hat{R}_k\|_{L^2(B_1)} \|\hat{R}_k - R_\infty\|_{L^3(B_1)} \\
\leq C\epsilon_k \to 0.
\]
We can apply (4.11) to estimate $IV_k$ by
\[
|IV_k| \leq C r_k^p \epsilon_k^{1-p} \|\nabla \phi_k\|_{L^2(B_1)} \|\widehat{R}_k - R_\infty\|_{L^2(B_1)} + C r_k^p \epsilon_k^{1-p} \|\widehat{R}_k - R_\infty\|_{L^1(B_1)} \\
\leq C r_k^p \epsilon_k^{1-p} \|\widehat{R}_k - R_\infty\|_{L^2(B_1)} + C r_k^p \epsilon_k^{1-p} \|\widehat{R}_k - R_\infty\|_{L^1(B_1)} \\
\leq C \epsilon_k \|\widehat{R}_k - R_\infty\|_{L^2(B_1)} \to 0.
\]

While the most difficult term $II_k$ can be estimated by employing the duality between $H^1(\mathbb{R}^3)$ and $\text{BMO}(\mathbb{R}^3)$ as follows.
\[
\int_{B_1} H_k^i \eta^2(\widehat{R}_k - R_\infty) \, dx \\
= \int_{B_1(0)} \eta(H_k^i - \mu_k^i) \eta(\widehat{R}_k - R_\infty) \, dx + \mu_k \int_{B_1} \eta^2(\widehat{R}_k - R_\infty) \, dx \\
= V_k + VI_k.
\]

It is easy to estimate
\[
|VI_k| \leq C |\mu_k^i| \int_{B_1} |\widehat{R}_k - R_\infty| \, dx \leq C \|H_k^i\|_{L^1(B_1)} \int_{B_1} |\widehat{R}_k - R_\infty| \, dx \to 0.
\]

We can apply Lemma 4.2 and (4.22) and (4.23) to estimate $V_k$ by
\[
|V_k| = |\int_{B_1} \eta(H_k^i - \mu_k^i) \eta(\widehat{R}_k - R_\infty) \, dx| \\
\leq C \|\eta(H_k^i - \mu_k^i)\|_{H^1(\mathbb{R}^3)} \|\eta(\widehat{R}_k - R_\infty)\|_{\text{BMO}(\mathbb{R}^3)} \leq C.
\]

Therefore we obtain that
\[
|II_k| \leq C \epsilon_k (|V_k| + |VI_k|) \leq C \epsilon_k \to 0.
\]

Putting all the estimates of $I_k, II_k, III_k, IV_k$ together, we arrive that
\[
\int_{B_1} |\nabla (\widehat{R}_k - R_\infty)|^p \, dx \to 0.
\]

Next, we are going to prove that
\[
\nabla \phi_k \rightarrow \nabla \phi_\infty \quad \text{in} \quad L^2(B_1/4).
\]

Since
\[
-\Delta \phi_k = r_k^p \epsilon_k^{1-p} \text{div}(\widehat{R}_k) + \frac{1}{2} \epsilon_k^{-p} f_k \quad \text{in} \quad B_1,
\]
and
\[
-\Delta \phi_\infty = 0 \quad \text{in} \quad B_1,
\]
multiplying both equations by $\eta^2(\widehat{\phi}_k - \phi_\infty)$, subtracting the resulting equations, and integrating over $\mathbb{R}^3$, we obtain that
\[
\int_{B_1} \eta^2 |\nabla (\widehat{\phi}_k - \phi_\infty)|^2 \, dx + 2 \int_{B_1} \eta |\nabla (\widehat{\phi}_k - \phi_\infty)| |\nabla \eta(\widehat{\phi}_k - \phi_\infty)| \, dx \\
= \int_{B_1} \left[ r_k^p \epsilon_k^{1-p} \text{div}(\widehat{R}_k) + \frac{1}{2} \epsilon_k^{-p} f_k \right] \eta^2(\widehat{\phi}_k - \phi_\infty) \, dx.
\]
Since \( \hat{\phi}_k \to \phi_\infty \) and \( \nabla \hat{\phi}_k \to \nabla \phi_\infty \) in \( L^2(B_{\frac{1}{4}}) \), we conclude that
\[
2 \int_{B_1} \eta \nabla (\hat{\phi}_k - \phi_\infty) \nabla \eta (\hat{\phi}_k - \phi_\infty) \, dx \to 0.
\]

Also, since
\[
\left\| r_k \frac{1 - p}{2} \text{div}(\hat{R}_k) + \frac{1}{2} \epsilon_k f_k \right\|_{L^2(B_1)} \leq C r_k \epsilon_k \frac{1 - p}{2} \| \nabla \hat{R}_k \|_{L^2(B_1)} + C \epsilon_k \frac{1 - p}{2} \leq C \epsilon_k \to 0,
\]
we conclude that
\[
\int_{B_{\frac{1}{4}}} \left[ r_k \frac{1 - p}{2} \text{div}(\hat{R}_k) + \frac{1}{2} \epsilon_k f_k \right] \eta (\hat{\phi}_k - \phi_\infty) \, dx \to 0.
\]

Thus we obtain that
\[
\int_{B_{\frac{1}{4}}} |\nabla (\hat{\phi}_k - \phi_\infty)|^2 \, dx \to 0.
\]

This completes the proof of Lemma \ref{lemma4.3}.

Next we apply Lemma \ref{lemma4.1} and the Marstrand Theorem to give a proof of Theorem \ref{thm1.3}.

**Proof of Theorem \ref{thm1.3}**. Define the singular set \( \Sigma \) by
\[
\Sigma = \left\{ x \in \Omega \mid \Theta^{3-p}((\phi, R), x) = \lim_{r \to 0} \text{Coss}_x((\phi, R), r) \geq \frac{1}{2} \kappa^p \right\}.
\]

Here \( \text{Coss}_x((\phi, R), r) \) denotes the modified renormalized Cosserat energy of \((\phi, R)\) in \( B_r(x) \) defined by \eqref{coss}, which is monotonically increasing with respect to \( r > 0 \) by Corollary \ref{cor2.2}. Hence the density function
\[
\Theta^{3-p}((\phi, R), x) = \lim_{r \to 0} \text{Coss}_x((\phi, R), r)
\]
exists for any \( x \in \Omega \) and is upper semicontinuous in \( \Omega \). From a simple covering argument (see \eqref{covering}), we know that
\[
H^{3-p}(\Sigma) = 0.
\]

For any \( x_0 \in \Omega \setminus \Sigma \), there exists \( r_0 > 0 \) such that \( B_{r_0}(x_0) \subset \Omega \), and
\[
\text{Coss}_{x_1}((\phi, R), \frac{r_0}{2}) = C r_0^2 \left( \frac{r_0}{2} \right)^{p-3} \int_{B_{r_0}(x_1)} (|\nabla R|^p + |\nabla \phi|^2) \, dx + C \left( \frac{r_0}{2} \right)^3 \leq \epsilon_0^p
\]
holds for all \( x_1 \in B_{\frac{r_0}{2}}(x_0) \).
Applying Lemma 4.1 repeatedly, we would obtain that there exists $\theta_0 \in (0, \frac{1}{2})$ such that

\[
(\theta_0^l r_0)^{p-3} \int_{B_{\theta_0^l r_0}(x_1)} (|\nabla R|^p + |\nabla \phi|^2) \, dx \\
\leq 2^{-l} \max \left\{ r_0^{p-3} \int_{B_{r_0}(x_0)} (|\nabla R|^p + |\nabla \phi|^2) \, dx, \frac{C r_0^p}{1 - 2 \theta_0^p} \right\} \quad (4.24)
\]

holds for all $x_1 \in B_{\frac{r_0}{2}}(x_0)$ and $l \geq 1$.

It follows from (4.24) that there exists $\alpha_0 \in (0, 1)$ such that

\[
r^{p-3} \int_{B_r(x_1)} (|\nabla R|^p + |\nabla \phi|^2) \, dx \\
\leq \left( \frac{r}{r_0} \right)^{p \alpha_0} \max \left\{ r_0^{p-3} \int_{B_{r_0}(x_0)} (|\nabla R|^p + |\nabla \phi|^2) \, dx, \frac{C r_0^p}{1 - 2 \theta_0^p} \right\} \\
\leq C(\epsilon_0) \left( \frac{r}{r_0} \right)^{p \alpha_0} \quad (4.25)
\]

holds for all $x_1 \in B_{\frac{r_0}{2}}(x_0)$ and $0 < r \leq \frac{r_0}{2}$. Thus, by Morrey’s decay Lemma [22], we conclude that $(\phi, R) \in C^{\alpha_0}(B_{\frac{r_0}{2}}(x_0))$. Since

\[\Delta \phi = \text{div}(R) + \frac{1}{2} f \quad \text{in} \quad B_{r_0}(x_0),\]

the higher order regularity theory of Poisson equation implies that $\phi \in C^{1,\alpha_0}(B_{\frac{r_0}{2}}(x_0))$. Since $x_0 \in \Omega \setminus \Sigma$ is arbitrary, we obtain that $(\phi, R) \in C^{1,\alpha_0}(\Omega \setminus \Sigma) \times C^{\alpha_0}(\Omega \setminus \Sigma)$.

Next we will employ the Marstrand Theorem [23] to show that the singular set $\Sigma$ is discrete for $2 < p < 3$. We argue it by contradiction. Suppose $\Sigma$ is not discrete. Then there exist a sequence of points $\{x_k\} \subset \Sigma$ and $x_0 \in \Sigma$ such that $x_k \to x_0$. Set $r_k = |x_k - x_0| \to 0$ and define

\[
(\phi_k, R_k, f_k, M_k)(x) = (\frac{x_k - x_0}{r_k}, f, \frac{r_k^{p+2}}{r_k} f, \frac{r_k^p M}{r_k^p})(x_0 + r_k x), \quad \forall x \in B_2.
\]

It is readily seen that $(\phi_k, R_k)$ is singular at 0 and $y_k = \frac{x_k - x_0}{r_k} \in \mathbb{S}^2$. Moreover, similar to (4.5), $(\phi_k, R_k)$ solves

\[
\begin{aligned}
\Delta \phi_k &= r_k^p \text{div}(R_k) + \frac{r_k}{2} f_k, \\
\text{div}(|\nabla R_k|^{p-2} \nabla R_k) &= \sum_{i=1}^{3} \langle |\nabla R_k|^{p-2} \nabla R_k, V_i(R_k) \rangle \nabla (V_i(R_k)) \quad \text{in} \quad B_2. \quad (4.26)
\end{aligned}
\]

\[
- \frac{1}{p} \sum_{i=1}^{3} \left[ 2r_k^p \langle \nabla \phi_k, V_i(R_k) \rangle - \langle M_k, V_i(R_k) \rangle \right] V_i(R_k)
\]
It follows from the monotonicity inequality \(2.3\) for \((\phi, R)\) and the scaling argument that \((\phi_k, R_k)\) also enjoys the following monotonicity inequality, i.e., for \(0 < r_1 < r_2 \leq 2\)

\[
e^{Cr_1^2r_1^{-p-3}}\int_{B_{r_1}} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx + Cr_1^3
\]

\[
+ \int_{r_1}^{r_2} r^{p-3} \int_{\partial B_r} (p|\nabla R_k|^{p-2} \left| \frac{\partial R_k}{\partial r} \right|^2 + \left| \frac{\partial \phi_k}{\partial r} \right|^2) \, dH^2 \, dr
\]

\[
\leq e^{Cr_2^2r_2^{-p-3}}\int_{B_{r_2}} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx + Cr_2^3. \tag{4.27}
\]

Moreover, for \(k > 1\) sufficiently large,

\[
\frac{1}{4} \epsilon_0^p \leq \int_{B_{2r}} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx = (2r_k)^{p-3} \int_{B_{2r_k}(x_0)} (|\nabla R|^p + |\nabla \phi|^2) \, dx \leq C. \tag{4.28}
\]

Hence

\[
\int_{B_2} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx \quad \text{is uniformly bounded above and below.}
\]

Then there exists \((\phi_\infty, R_\infty) \in H^1(B_2, \mathbb{R}^3) \times W^{1,p}(B_2, SO(3))\) such that, after passing to a subsequence,

\[
(\phi_k, R_k) \rightharpoonup (\phi_\infty, R_\infty) \quad \text{in} \quad H^1(B_2) \times W^{1,p}(B_2).
\]

It is not hard to see that by passing the limit \(k \to \infty\) in \((4.26)\), we see that \(\phi_\infty\) is a harmonic function in \(B_2\), and \(R_\infty\) is a \(p\)-harmonic map into \(SO(3)\) in \(B_2\). Moreover, it follows from the lower semicontinuity and the monotonicity inequality \((4.27)\) that for any \(0 < s \leq 2\), it holds

\[
\int_s^2 r^{p-3} \int_{\partial B_r} \left( p|\nabla R_\infty|^{p-2} \left| \frac{\partial R_\infty}{\partial r} \right|^2 + \left| \frac{\partial \phi_\infty}{\partial r} \right|^2 \right) \, dH^2 \, dr = 0,
\]

this follows from the fact that for any fixed \(0 < s \leq 2\),

\[
e^{Cs^2} s^{p-3} \int_{B_s} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx + Cs^3 \to \Theta^{3-p} \left( (\phi, R), x_0 \right), \quad \text{as} \quad k \to \infty.
\]

Therefore we must have that

\[
\left( \frac{\partial \phi_\infty}{\partial r}, \frac{\partial R_\infty}{\partial r} \right) = (0, 0),
\]

or equivalently \((\phi_\infty, R_\infty)\) is homogeneous of degree zero:

\[
(\phi_\infty(x), R_\infty(x)) = (\phi_\infty(x|x|), R_\infty(x|x|)), \quad x \in B_2. \tag{4.29}
\]

Since \(\phi_\infty\) is a smooth harmonic function with homogeneous degree zero, it follows that \(\phi_\infty\) is a constant.

Next we need to show\n
Claim 1.

\[
(\phi_k, R_k) \longrightarrow (\phi_\infty, R_\infty) \quad \text{in} \quad H^1(B_1) \times W^{1,p}(B_1).
\]
Assume the claim for the moment. Then it follows from (4.28) and $\phi_\infty = \text{constant}$ that $R_\infty : B_2 \to SO(3)$ is a nontrivial stationary $p$-harmonic map, which has at least two singular points 0 and $y_\infty \in S^2$ given by

$$y_\infty = \lim_{k \to \infty} \frac{x_k - x_0}{|x_k - x_0|}.$$ 

The singular set of $R_\infty$ contains the line segment $[0y_\infty]$ so that $H^1(\text{Sing}(R_\infty)) > 0$, which is impossible. Thus $\Sigma$ is a discrete set.

Finally, we would like to apply Marstrand theorem to prove Claim 1. To do it, we consider a sequence of Radon measures

$$\mu_k = (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx.$$ 

Since $\mu_k(B_2)$ is uniformly bounded, we may assume that there is a nonnegative Radon measure $\mu$ in $B_2$ such that after passing to a subsequence,

$$\mu_k \to \mu$$

as convergence of Radon measures. By Fatou’s lemma, we can decompose $\mu$ into

$$\mu = (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2) \, dx + \nu$$

for a nonnegative Radon measure $\nu$, called a defect measure. The monotonicity inequality (4.27) for $(\phi_k, R_k)$ implies that $\mu$ is a monotone measure in the following sense: for $x \in B_1$, $0 < r_1 < r_2 < \text{dist}(x, \partial B_2)$,

$$e^{Cr_1^p r_2^{p-3}} \mu(B_{r_1}(x)) + Cr_1^3 \leq e^{Cr_2^p r_2^{p-3}} \mu(B_{r_2}(x)) + Cr_2^3.$$ 

In particular, for any $x \in B_1$, the density function

$$\Theta^{3-p}(\mu, x) = \lim_{r \to 0} r^{p-3} \mu(B_r(x))$$

exists and is upper semicontinuous in $B_1$. Define the concentration set

$$S := \bigcap_{r>0} \left\{ x \in B_1 \mid \liminf_{k \to \infty} \int_{B_r(x)} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx \geq \frac{1}{2} r_0^p \right\}.$$ 

We claim that $S$ is a closed subset of $B_1$. In fact, let $\{x_k\}$ be a sequence of points in $S$ such that $x_k \to x_0 \in B_1$. If $x_0 \notin S$, then there exists $r_0 > 0$ and $\delta_0 > 0$ such that for $k > 1$ sufficiently large it holds that

$$r_0^{p-3} \int_{B_{r_0}(x_0)} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx \leq \frac{1}{2} r_0^p - \delta_0.$$
Taking $k$ large enough so that $|x_k - x_0| < \frac{r_0}{2}$ and applying the monotonicity inequality to each $(\phi_k, R_k)$, we have
\[
e^{C(\frac{r_0}{2})^2} \left( \frac{r_0}{2} \right)^{p-3} \int_{B_{\frac{r_0}{2}}(x_k)} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx + C \left( \frac{r_0}{2} \right)^3
\]
\[
\leq e^{C(r_0 - |x_k - x_0|)^2} \left( r_0 - |x_k - x_0| \right)^{p-3} \int_{B_{r_0 - |x_k - x_0|}(x_k)} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx
\]
\[
+ C \left( r_0 - |x_k - x_0| \right)^3
\]
\[
\leq e^{C(r_0 - |x_k - x_0|)^2} \left( \frac{r_0}{(r_0 - |x_k - x_0|)} \right)^{3-p} \int_{B_{r_0}(x_0)} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx
\]
\[
+ C \left( r_0 - |x_k - x_0| \right)^3
\]
\[
\leq e^{C\epsilon_0^2} \left( \frac{r_0}{(r_0 - |x_k - x_0|)} \right)^{3-p} \left( \frac{1}{2} \epsilon_0^p - \delta_0 \right) + C \left( r_0 - |x_k - x_0| \right)^3
\]
\[
\leq \frac{1}{2} \epsilon_0^p
\]
provided that $k$ large enough and $r_0$ is chosen sufficiently small. This contradicts to the fact $x_k \in S$. Hence $S$ is a closed subset.

Suppose $x_\ast \in B_1 \setminus S$. Then there exists $r_\ast > 0$ such that
\[
\liminf_{k \to \infty} r_\ast^{p-3} \int_{B_{r_\ast}(x_\ast)} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx < \frac{1}{2} \epsilon_0^p.
\]
Applying the $\epsilon_0$-regularity Theorem 1.3 we may conclude that after passing to another subsequence,
\[
R_k \to R_\infty \quad \text{in} \quad C^1_{\text{loc}} \cap W^{1,p}_{\text{loc}}(B_1 \setminus S),
\]
and
\[
\phi_k \to R_\infty \quad \text{in} \quad C^1_{\text{loc}} \cap H^1_{\text{loc}}(B_1 \setminus S).
\]
If we denote by $\text{Sing}(\phi_\infty, R_\infty)$ the set of discontinuity of $(\phi_\infty, R_\infty)$, and $\text{supp}(\nu)$ the support of the defect measure $\nu$. Then the above convergence implies that
\[
\text{Sing}(\phi_\infty, R_\infty) \cup \text{supp}(\nu) \subset S.
\]
On the other hand, if $\hat{x} \in S$, then after sending $k \to \infty$, we have that
\[
\frac{\mu(B_r(\hat{x}))}{r^{3-p}} \geq \frac{1}{2} \epsilon_0^p, \quad \forall r > 0.
\]
If $\hat{x} \notin \text{Sing}(\phi_\infty, R_\infty)$, then $(\phi_\infty, R_\infty)$ is regular near $\hat{x}$ and hence for $r$ sufficiently small,
\[
r^{p-3} \int_{B_r(\hat{x})} (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2) \, dx \leq \frac{1}{4} \epsilon_0^p,
\]
this implies that for small $r > 0$,
\[
\frac{\nu(B_r(\hat{x}))}{r^{3-p}} \geq \frac{1}{4} \epsilon_0^p,
\]
and hence $\hat{x} \in \text{supp}(\nu)$. Therefore, we conclude that
Lemma 4.4. \[
\text{Sing}(\phi_\infty, R_\infty) \bigcup \text{supp}(\nu) = S.
\]

Notice that if \(x \in S\), then
\[
\Theta^{3-p}(\mu, x) = \lim_{r \to 0} r^{3-p} \mu(B_r(x)) \geq \frac{1}{2} r_0^p.
\]

Moreover, for any compact subset \(K \subset B_1\), and any \(x \in S \cap K\),
\[
\frac{1}{2} r_0^p \leq \Theta^{3-p}(\mu, x) \leq r_K^{p-3} \mu(B_2) \leq r_K^{p-3} E_0,
\]
where \(r_K = \frac{1}{2} \text{dist}(K, \partial B_2) > 0\), and \(E_0 = \sup_k \int_{B_2} (|\nabla R_k|^p + |\nabla \phi_k|^2) \, dx\). Recall that by Federer-Ziemer theorem (see [22])
\[
\lim_{r \to 0} r^{p-3} \int_{B_r(x)} (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2) \, dy = 0
\]
holds for \(H^{3-p}\) a.e. \(x \in B_2\). Thus we obtain that

**Lemma 4.5.** For any compact \(K \subset B_1\), if \(x \in S \cap K\), then
\[
\frac{1}{2} r_0^p \leq \Theta^{3-p}(\mu, x) < C(K) < \infty.
\]

For \(H^{3-p}\) a.e. \(x \in S\),
\[
\Theta^{3-p}(\mu, x) = \Theta^{3-p}(\nu, x).
\]

It follows from Lemma 4.5 and standard covering arguments that for any compact set \(K \subset B_1\)
\[
\epsilon^p H^{3-p}(S \cap K) \leq \nu(S \cap K) \leq C H^{3-p}(S \cap K).
\]
Therefore
\[
\nu(S) = 0 \iff H^{3-p}(S) = 0.
\]

In particular, we have that

**Lemma 4.6.** \((\phi_k, R_k) \rightharpoonup (\phi_\infty, R_\infty)\) strongly in \(H^1(B_1) \times W^{1,p}(B_1)\) if and only if \(\nu(B_1) > 0\) if and only if \(H^{3-p}(S) > 0\).

Return to the proof of Claim 1. For \(2 < p < 3\), if
\[
(\phi_k, R_k) \rightharpoonup (\phi_\infty, R_\infty) \text{ in } H^1(B_1) \times W^{1,p}_{\text{loc}}(B_1),
\]
then by Lemma 4.6, we must have \(H^{3-p}(S) > 0\). Hence by Lemma 4.4, we have for \(H^{3-p}\) a.e. \(x \in S\),
\[
0 < \Theta^{3-p}(\nu, x) < \infty.
\]

Applying Marstrand Theorem to \(\nu\) and \(S\), we conclude that \(3 - p\) must be an integer, which is impossible. Hence Claim 1 is true. This completes the proof of Theorem 1.3.
5. Stable-stationary solutions of the Cosserat equation

This section is devoted to the proof of Theorem 1.5. More precisely, we will show that if \((\phi, R)\) is a stable stationary solution to the Cosserat equation (1.3). Then the singular set is empty for \(p\) belonging to the range \([2, \frac{32}{7}]\).

It is well-known that \(S^3\) is the universal cover of \(SO(3)\), and a locally isometric 2-to-1 covering map \(\pi : S^3 \to SO(3)\) is given by

\[
\pi(w, x, y, z) = \begin{pmatrix}
1 - 2y^2 - 2z^2 & 2xy - 2zw & 2xz + 2yw \\
2xy + 2zw & 1 - 2x^2 - 2z^2 & 2yz - 2xw \\
2xz - 2yw & 2yz + 2xw & 1 - 2x^2 - 2y^2
\end{pmatrix}, \quad \forall (w, x, y, z) \in S^3.
\]

In particular, the curvature operator of \(SO(3)\), \(R_{SO(3)}\), satisfies

\[
\langle R_{SO(3)}(v, w)v, w \rangle = |v|^2|w|^2 - \langle v, w \rangle^2, \quad v, w \in T_RSO(3).
\]

For \((\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1, p}(\Omega, SO(3))\), let

\[
(\phi_t, R_t) \in C^2((-\delta, \delta), H^1(\Omega, \mathbb{R}^3) \times W^{1, p}(\Omega, SO(3)))
\]

be a family of variations of \((\phi, R)\). Denote by

\[
\eta = \frac{d}{dt} |_{t=0} \phi_t, \quad \dot{\eta} = \frac{d^2}{dt^2} |_{t=0} \phi_t,
\]

and

\[
v = \frac{\partial R_t}{\partial t} |_{t=0}, \quad \dot{v} = \nabla_\phi \frac{\partial R_t}{\partial t} |_{t=0}.
\]

Applying the equation (1.3) for \((\phi, R)\) and direct calculations as in Smith [12], we obtain that

\[
\frac{d^2}{dt^2} |_{t=0} \text{Coss}(\phi_t, R_t)
= \frac{d^2}{dt^2} |_{t=0} \int_\Omega \left( |\nabla \phi_t|^2 - 2R_t \nabla \phi_t + |\nabla R_t|^p + (\phi_t - x) \cdot f + \langle R_t, M \rangle \right) dx
= \int_\Omega \left( 2|\nabla \eta|^2 - 4\langle v, \nabla \eta \rangle + p|\nabla R|^p - 2(\nabla v)^2 - \text{tr}(R_{SO(3)}(v, \nabla R) v, \nabla R) + p(2 - p)|\nabla R|^p \right) dx
\]

holds for any \(\eta \in H^1_0(\Omega, \mathbb{R}^3)\) and \(v \in H^1_0 \cap L^\infty(\Omega, T_RSO(3))\).

**Definition 5.1.** For \(2 \leq p < 3\), \(\mu_1 = \mu_2 = 1\), \(f \in L^\infty(\Omega, \mathbb{R}^3)\) and \(M \in L^\infty(\Omega, SO(3))\), a stationary weak solution \((\phi, R)\) of the Cosserat equation (1.3) is called a stable, stationary weak solution of the Cosserat equation (1.3) if, in addition,

\[
\frac{d^2}{dt^2} |_{t=0} \text{Coss}(\phi_t, R_t) \geq 0,
\]
or, equivalently,

\[
\int_{\Omega} \left( (2 |\nabla \eta|^2 - 4 \langle v, \nabla \eta \rangle + p |\nabla R|^{p-2} (|\nabla v|^2 - |\nabla R|^2 |v|^2) + p(p-2) |\nabla R|^{p-4} (\nabla R, \nabla v)^2 \right) dx \geq 0
\]  

(5.1)

holds for any \( \eta \in C^\infty_0(\Omega, \mathbb{R}^3) \) and \( v \in H^1_0(\Omega, T_RSO(3)) \).

**Lemma 5.2.** For \( 2 \leq p < 3, \mu_1 = \mu_c = \mu_2 = 1, f \in L^\infty(\Omega, \mathbb{R}^3) \) and \( M \in L^\infty(\Omega, SO(3)) \), if \( (\phi, R) \) is a stable, stationary weak solution of the Cosserat equation (1.3), then

\[
\int_{\Omega} \left( 6 |\nabla \omega|^2 - 4 \sum_{i=1}^{3} \psi \langle a_i R, \nabla \omega \otimes e^i \rangle + p(p+1) |\nabla R|^{p-2} |\nabla \psi|^2 - 2p |\nabla R|^p |\psi|^2 \right) dx \geq 0
\]  

(5.2)

holds for any \( \omega \in C^\infty_0(\Omega) \) and \( \psi \in C^\infty_0(\Omega) \). Here \( (e^1, e^2, e^3) \) is the standard base of \( \mathbb{R}^3 \). In particular,

\[
\int_{\Omega} \left( p+1 |\nabla R|^{p-2} |\nabla \psi|^2 - 2 |\nabla R|^p |\psi|^2 \right) dx \geq 0
\]  

(5.3)

holds for any \( \psi \in C^\infty_0(\Omega) \).

**Proof.** It is readily seen that (5.3) follows immediately from (5.2) by taking \( \omega = 0 \). Thus it suffices to show (5.2). For any \( \omega \in C^\infty_0(\Omega) \) and \( \psi \in C^\infty_0(\Omega) \), let \( \eta = \omega e^i \) and \( v = \psi a_i R \) and substitute them into (5.1) and then take summation over \( i = 1, 2, 3 \), we obtain that

\[
\int_{\Omega} \left( 2 \sum_{i=1}^{3} |\nabla (\omega e^i)|^2 - 4 \sum_{i=1}^{3} \psi \langle a_i R, \nabla \omega \otimes e^i \rangle + p(p-2) |\nabla R|^{p-4} \sum_{i=1}^{3} \langle \nabla R, \nabla (\psi a_i R) \rangle^2 \right.
\]

\[
+ p |\nabla R|^{p-2} \sum_{i=1}^{3} \left( |\nabla (\psi a_i R)|^2 - |\nabla R|^2 |\psi a_i R|^2 \right) \right) dx \geq 0.
\]  

(5.4)

Observe that

\[
\sum_{i=1}^{3} \langle \nabla R, \nabla (\psi a_i R) \rangle^2 = \sum_{i=1}^{3} \left[ \sum_{j=1}^{3} \nabla_j \psi \langle \nabla_j R, a_i R \rangle + \psi \langle \nabla R, a_i \nabla R \rangle \right]^2
\]

\[
= \sum_{i=1}^{3} \langle \nabla \psi \cdot \nabla R, a_i R \rangle^2 = |\nabla \psi \cdot \nabla R|^2 \leq |\nabla \psi|^2 |\nabla R|^2,
\]

\[
\sum_{i=1}^{3} |\nabla (\omega e^i)|^2 = 3 |\nabla \omega|^2, \quad \sum_{i=1}^{3} |\nabla R|^2 |\psi a_i R|^2 = 3 |\nabla R|^2 |\psi|^2,
\]

\[
\sum_{i=1}^{3} |\nabla (\omega e^i)|^2 - 4 \sum_{i=1}^{3} \psi \langle a_i R, \nabla \omega \otimes e^i \rangle + p(p-2) |\nabla R|^{p-4} \sum_{i=1}^{3} \langle \nabla R, \nabla (\psi a_i R) \rangle^2
\]

\[
+ p |\nabla R|^{p-2} \sum_{i=1}^{3} \left( |\nabla (\psi a_i R)|^2 - |\nabla R|^2 |\psi a_i R|^2 \right) \right) dx \geq 0.
\]  

(5.4)
and
\[
\sum_{i=1}^{3} |\nabla (\psi a_i R)|^2 = |\nabla \psi|^2 \sum_{i=1}^{3} |a_i R|^2 + 2\psi \nabla \psi \sum_{i=1}^{3} (a_i R, a_i \nabla R) + |\psi|^2 \sum_{i=1}^{3} (a_i \nabla R, a_i \nabla R)
\]
\[
= 3|\nabla \psi|^2 + \psi \nabla \psi \text{tr}(R^T a_i^T a_i \nabla R + \nabla R^T a_i^T a_i R) + |\psi|^2 \text{tr}(\nabla R^T \nabla R (\sum_{i=1}^{3} a_i^T a_i))
\]
\[
= 3|\nabla \psi|^2 + \psi \nabla \psi [\text{tr}(R^T \nabla R + \nabla R^T R) (a_i^T a_i)] + |\psi|^2 \text{tr}(\nabla R^T \nabla R (\sum_{i=1}^{3} a_i^T a_i))
\]
\[
= 3|\nabla \psi|^2 + |\nabla R|^2 |\psi|^2,
\]
where we have used
\[
a_1^T a_1 = \text{diag}(0, \frac{1}{2}, \frac{1}{2}), \quad a_2^T a_2 = \text{diag}(\frac{1}{2}, 0, \frac{1}{2}), \quad a_3^T a_3 = \text{diag}(\frac{1}{2}, \frac{1}{2}, 0),
\]
and
\[
(R, a_i \nabla R) = 0, \quad R^T \nabla R + \nabla R^T R = 0.
\]
Plugging these identities into (5.4), we obtain (5.2). \qed

Now we can extend the partial regularity theorem for stationary weak solutions of the Cosserat equation (1.3) obtained in the previous section to the class of stable weak solutions of the Cosserat equation (1.3). First, we consider Theorem 1.5 in the case that $p = 2$. Namely, we will show that

**Theorem 5.3.** For $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, SO(3))$, and $\mu_1 = \mu_c = \mu_2 = 1$, assume that $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, SO(3))$ is a stable, stationary weak solution of the Cosserat equation (1.3). Then $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \times C^{\alpha}(\Omega, SO(3))$ for some $\alpha \in (0, 1)$.

**Proof.** From the small energy regularity theorem obtained in the previous section, we know that there exists a closed singular set $\Sigma \subset \Omega$, with $H^1(\Sigma) = 0$, such that $(\phi, R) \in C^{1,\alpha}(\Omega \setminus \Sigma) \times C^{\alpha}(\Omega \setminus \Sigma)$ for some $0 < \alpha < 1$.

Now we want to show $\Sigma = \emptyset$. For, otherwise, there exists $x_0 \in \Sigma$ such that
\[
\Theta^1((\phi, R), x_0) \equiv \lim_{r \downarrow 0} r^{-1} \int_{B_r(x_0)} (|\nabla R|^2 + |\nabla \phi|^2) \, dx \geq \epsilon_0^2 > 0.
\]
For any sequence of radius $r_i \downarrow 0$, define the blow up sequence
\[
(\phi_i, R_i, f_i, M_i)(x) = (\phi, R, r_i^2 f, r_i^2 M)(x_0 + r_i x), \quad \forall x \in B_2.
\]
Then
\[
\lim_{i \to \infty} 2^{-1} \int_{B_2} (|\nabla R_i|^2 + |\nabla \phi_i|^2) \, dx = \Theta^1((\phi, R), x_0) \geq \epsilon_0^2.
\]
Thus there exists $(\phi_0, R_0) \in H^1(B_2, \mathbb{R}^3) \times H^1(B_2, SO(3))$ such that after passing to a subsequence,
\[
(\phi_i, R_i) \to (\phi_0, R_0) \text{ in } H^1(B_2, \mathbb{R}^3) \times H^1(B_2, SO(3)).
\]
Since \((\phi_i, R_i)\) satisfies
\[
\begin{aligned}
\Delta \phi_i &= r_i \text{div} R_i + \frac{1}{p} f_i \\
\Delta R_i + \frac{2}{p} r_i \nabla \phi_i - \frac{1}{p} M_i \perp T R_i \text{SO}(3),
\end{aligned}
\tag{5.5}
\]
it follows, after sending \(i \to \infty\), that on \(B_2\), \(\phi_0\) is a harmonic function and \(R_0\) is a harmonic map into \(\text{SO}(3)\). We now need

**Claim 2**: \((\phi_i, R_i) \to (\phi_0, R_0)\) in \(H^1(B_1, \mathbb{R}^3) \times H^1(B_1, \text{SO}(3))\).

We will apply the technique of potential theory by Hong-Wang [17] and Lin-Wang [18] to prove this claim. Let \(\nu \geq 0\) be a Radon measure in \(B_2\) such that
\[
\mu_i \equiv (|\nabla R_i|^2 + |\nabla \phi_i|^2) \, dx \rightharpoonup \mu \equiv (|\nabla R_0|^2 + |\nabla \phi_0|^2) \, dx + \nu
\]
as convergence of measures in \(B_2\). It suffices to show \(\nu \equiv 0\) in \(B_1\). Notice that \((\phi_i, R_i)\), solving (5.5), is indeed a stationary weak solution of the Euler-Lagrange equation of critical point of the Cosserart energy functional
\[
E_i(\hat{\phi}, \hat{R}) = \int_{B_2} (|\nabla \hat{R}|^2 + |\nabla \hat{\phi}|^2 - 2 r_i \langle \hat{R}, \nabla \hat{\phi} \rangle + \langle \hat{\phi} - x \rangle \cdot f_i + \langle \hat{R}, M_i \rangle) \, dx.
\]
In particular, the \(\epsilon_0\)-regularity theorem is applicable to \((\phi_i, R_i)\) and we conclude that if we define
\[
S = \bigcap_{r > 0} \left\{ y \in B_{\frac{3}{2}} \colon \lim_{i \to \infty} \min_{B_r(y)} (|\nabla R_i|^2 + |\nabla \phi_i|^2) \geq \epsilon_0^2 \right\}
\]
\[
= \left\{ y \in B_{\frac{3}{2}} \colon \Theta^1(\mu, y) = \lim_{r \to 0} r^{-1} \mu(B_r(y)) \geq \epsilon_0^2 \right\}.
\]
Then the following statements hold:
(i) \(S\) is closed with \(H^1(S) < \infty\), \(\text{supp}(\nu) \subset S\) and \(\Theta^1(\nu, y) = \Theta^1(\mu, y) \geq \epsilon_0^2\) for \(H^1\) a.e. \(y \in S\).
(ii) There exists \(\alpha \in (0, 1)\) such that
\[
(\phi_i, R_i) \to (\phi_0, R_0) \text{ in } (C^\alpha_{\text{loc}} \cap H^1_{\text{loc}})(B_{\frac{3}{2}} \setminus S).
\]
(iii)
\[
C_1(\epsilon_0) H^1(S) \leq \nu(B_{\frac{3}{2}}) \leq C_2(\epsilon_0) H^1(S).
\]
In particular, \(\nu \equiv 0\) if and only if \(H^1(S) = 0\). It follows from \(H^1(S) < +\infty\) that \(\text{Cap}_2(S) = 0\). Hence for any \(\delta > 0\), there exists \(\omega_\delta \in C^\infty_0(B_2)\) such that
\[
S \subset \text{int}(\{\omega_\delta = 1\}),
\]
and
\[
\int_{B_2} |\nabla \omega_\delta|^2 \, dx \leq \delta.
\tag{5.6}
\]
Hence for any \(a \in S\), there exists \(0 < r_a < \delta^2\) such that
\[
\omega_\delta \geq \frac{1}{2} \text{ on } B_{r_a}(a).
\]
From the compactness of $S$ and Vitali’s covering lemma, there exist $1 \leq l < \infty$ and \( \{a_m\}_{m=1}^l \subset S \) such that \( \{B_{r_{am}}(a_m)\}_{m=1}^l \) are mutually disjoint, and

\[
S \subset \bigcup_{m=1}^l B_{r_{am}}(a_m).
\]

From the definition of $S$, there exists a sufficiently large $i_l > 0$ such that

\[
\frac{\epsilon_0^2}{2} \leq \left( \frac{r_{am}}{5} \right)^{-1} \int_{B_{r_{am}}(a_m)} (|\nabla R_i|^2 + |\nabla \phi|^2) \, dx, \quad \forall i \geq i_l, \ m = 1, \cdots, l.
\] (5.7)

By the $W^{1,q}$-estimate on $\phi$, we know that

\[
\|\nabla \phi\|_{L^q(K)} \leq C(q, K)
\]
holds for any compact set $K \subset \Omega$ and $1 < q < \infty$. Hence for any $i \geq i_l$ and $m = 1, \cdots, l$, it follows from Hölder’s inequality that

\[
\left( \frac{r_{am}}{5} \right)^{-1} \int_{B_{r_{am}}(a_m)} |\nabla \phi_i|^2 \, dx \leq C \left( r_i r_{am} \right)^{-1} \int_{B_{r_i r_{am}}(a_0)} |\nabla \phi|^2 \, dx
\]

\[
\leq C(q) \left( r_i r_{am} \right)^{2-\frac{q}{q}} \leq C \delta^2 \leq \frac{1}{4} \epsilon_0^2,
\]
provided we choose $q = 12$ and $\delta \leq \left( \frac{\epsilon_0^2}{4C} \right)^{\frac{2}{3}}$ in the last step. Substituting this estimate into (5.7), we obtain that

\[
\frac{1}{4} \epsilon_0^2 \leq \left( \frac{r_{am}}{5} \right)^{-1} \int_{B_{r_{am}}(a_m)} |\nabla R_i|^2 \, dx, \quad \forall i \geq i_l, \ m = 1, \cdots, l.
\] (5.8)

Therefore for all $i \geq i_l$, we can bound

\[
H_{\delta^2}^1(S) \leq C \sum_{m=1}^l r_{am} = 5C \sum_{m=1}^l \left( \frac{r_{am}}{5} \right)
\]

\[
\leq \frac{20C}{\epsilon_0^2} \sum_{m=1}^l \int_{B_{r_{am}}(a_m)} |\nabla R_i|^2 \, dx
\]

\[
\leq \frac{80C}{\epsilon_0^2} \int_{\bigcup_{m=1}^l B_{r_{am}}(a_m)} |\nabla R_i|^2 \omega_5^2 \, dx
\]

\[
\leq \frac{80C}{\epsilon_0^2} \int_{B_2} |\nabla R_i|^2 \omega_5^2 \, dx.
\] (5.9)

It follows from the stability of $(\phi, R)$ and a scaling argument that $R_i$ satisfies the stability inequality (5.3) so that

\[
\int_{B_2} |\nabla R_i|^2 \omega_5^2 \, dx \leq \frac{3}{2} \int_{B_2} |\nabla \omega_5|^2 \, dx, \quad \forall i \geq i_l.
\] (5.10)

Plugging (5.10) into (5.9) and applying (5.6), we would obtain that

\[
H_{\delta^2}^1(S) \leq C(\epsilon_0) \delta.
\]
This, after sending $\delta \to 0$, would yield $H^1(S) = 0$ and hence Claim 2 is true.

It follows from the $H^1$-strong convergence of $(\phi_i, R_i)$ to $(\phi_0, R_0)$ and the energy monotonicity inequality (2.3), we conclude that

$$(\phi_0, R_0)(x) = (\phi_0, R_0)(\frac{x}{|x|}), \quad \forall x \in B_2,$$

is homogeneous of degree zero. Since $\phi_0$ is a harmonic function in $B_2$, it follows that $\phi_0$ is a constant. Thus

$$\int_{S^2} |\nabla_{S^2} R_0|^2 dH^2 = \Theta^1((\phi, R), x_0) \geq \epsilon_0^2,$$

and $R_0 \in C^\infty(S^2, SO(3))$ is a nontrivial harmonic map. Since $\Pi_1(S^3) = \{0\}$, it follows that there exists a nontrivial harmonic map $\hat{R}_0 \in C^\infty(S^2, S^3)$ such that $R_0 = \pi \circ \hat{R}_0$. Moreover, it follows from the stability inequality (5.11) that $\hat{R}_0$ is a stable harmonic map from $S^2$ to $S^3$, i.e.

$$\int_{S^2} (|\nabla_{S^2} \omega|^2 - |\nabla \hat{R}_0|^2 |\omega|^2) dH^2 \geq 0 \quad (5.11)$$

for any $\omega \in C^\infty(S^2, T_{\hat{R}_0}S^3)$. However it follows from Schoen-Uhlenbeck [14] that there is no nontrivial stable harmonic map from $S^2$ to $S^3$. We get a desired contradiction. Thus the singular set $\Sigma$ of $(\phi, R)$ is empty. \qed

Theorem 1.5 for the cases that $p > 2$ can be summarized into the following theorem.

**Theorem 5.4.** For $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, SO(3))$, and $\mu_1 = \mu_c = \mu_2 = 1$, if $p \in (2, \frac{32}{15})$ and $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a stable, stationary weak solution of the Cosserat equation (1.3), then there exists $\alpha \in (0, 1)$ such that $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \times C^\alpha(\Omega, SO(3))$.

**Proof.** It follows from $2 < p < 3$ and Theorem 1.3 that $\text{Sing}(\phi, R)$ is discrete. Suppose $\text{Sing}(\phi, R) \neq \emptyset$. Then there exist $x_0 \in \text{Sing}(\phi, R)$ and $r_0 > 0$ such that $\text{Sing}(\phi, R) \cap B_{r_0}(x_0) = \{x_0\}$. For $r_k \to 0$, define $(\phi_k, R_k)(x) = (\phi, R)(x_0 + r_k x)$ for $x \in B_2$. As in the proof of Theorem 1.3 we can apply the monotonicity inequality (2.3), Lemma 4.1 and Marstrand theorem to show that there exists a nontrivial $(\phi_0, R_0) \in H^1(B_1, \mathbb{R}^3) \times W^{1,p}(B_1, SO(3))$ such that, after passing to a subsequence, $(\phi_k, R_k) \to (\phi_0, R_0)$ strongly in $H^1(B_1, \mathbb{R}^3) \times W^{1,p}(B_1, SO(3))$. Hence $(\phi_0, R_0)$ is of homogeneous degree zero, $\phi_0$ is constant and $R_0 \in C^{1,\alpha}(B_1 \setminus \{0\}, SO(3))$ is a nontrivial, stable, stationary $p$-harmonic map. However, it follows from the stability Lemma 6.3 and Proposition 6.4 in Gastel [8] that for $p \in (2, \frac{32}{15})$, any stable stationary $p$-harmonic map $R(x) = R(\frac{x}{|x|}) \in C^{1,\alpha}(B_1 \setminus \{0\}, SO(3))$ must be constant. We get a desired contradiction. Hence $\text{Sing}(\phi, R) = \emptyset$ when $p \in (2, \frac{32}{15}]$. This completes the proof. \qed

Finally we would like to point out that Theorem 1.5 follows from Theorem 5.3 and Theorem 5.4.

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