A CHRONOLOGY OF CONTINUED SQUARE ROOTS
AND OTHER CONTINUED COMPOSITIONS,
THROUGH THE YEAR 2016

DIXON J. JONES

ABSTRACT. An infinite continued composition is an expression of the form
\[ \lim_{n \to \infty} t_0 \circ t_1 \circ t_2 \circ \cdots \circ t_n(c), \]
where the \( t_i \) are maps from a set \( D \) to itself, the initial value \( c \) is a point in \( D \), and the order of operations proceeds from right to left.

This document is an annotated bibliography, in chronological order through the year 2016, of selected continued compositions whose primary sources have typically been obscure. In particular, we include continued square roots:
\[ a_0 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{\cdots}}}, \]
as well as continued powers, continued cotangents, and \( f \)-expansions. However, we do not include continued fractions, continued exponentials, or forms such as infinite sums and products in which the \( t_i \) are linear functions, because the literature on these forms is extensive and well-summarized elsewhere.

1. Introduction

This is a bibliography, in chronological order through the year 2016, of selected continued function compositions or, more briefly, continued compositions\(^1\) expressions of the form
\[ \lim_{n \to \infty} t_0 \circ t_1 \circ t_2 \circ \cdots \circ t_n(c), \] (1)
where the \( t_i \) are maps from a set \( D \) to itself, the initial value \( c \) is a point in \( D \), and the order of operations proceeds from right to left. Under these conditions the \( n \)th approximant
\[ t_0 \circ t_1 \circ t_2 \circ \cdots \circ t_n(c) \] (2)
is well-defined for each \( n = 0, 1, 2, \ldots \), and the continued composition converges if the sequence of approximants
\[ t_0(c), \quad t_0 \circ t_1(c), \quad t_0 \circ t_1 \circ t_2(c), \] (3)

\( ^1 \) In earlier drafts of this document, posted on a personal web site beginning in 2008, the author used the term “chain composition,” which seemed truer to German Ketten-constructions like Kettenbrüche. However, “continued” has long-entrenched connotations in English, and will probably be recognized more widely than “chain.”
and so on, has a unique, finite limit. Following Schoenefuss 1992, in emulation of a common notation for continued fractions we write expression (1) more compactly as $K_{i=0}^\infty t_i(c)$.

Many mathematical objects can be thought of as continued compositions, but this chronology is limited to work in which one of the following appears. (We assume that $D$ is the complex plane here, but of course $D$ varies in the chronology from item to item.)

- The continued, infinite, iterated, or nested square root

\[ a_0 + \sqrt{a_1 + \sqrt{a_2 + \ldots}}, \quad (4) \]

generated by $t_i(z) = a_i + \sqrt{c}$ with $c = 0$.

- The continued $p$th power

\[ a_0 + (a_1 + (a_2 + (\ldots)^p)^p)^p, \quad (5) \]

using $t_i(z) = a_i + z^p$, $p \in \mathbb{R}$, and $c = 0$ for $p > 0$; for $p < 0$ we use $c = \infty$. In Doppler 1832 we find perhaps the earliest observation that this expression becomes a regular continued fraction for $p = -1$; Herschfeld 1935 notes this as well, along with the fact that infinite sums comprise the case $p = 1$. An example of a continued square ($p = 2$) is given without comment in Dixon 1878. The case

\[ a_0 + (a_1 + (a_2 + (\ldots)^p)^p)^p, \]

where the $p_i$ are in $(0, 1)$ and may not all be equal (a form which could also be called a continued $r$th root, where $r_i = \frac{1}{p_i}$), is mentioned in Herschfeld 1935, Andrushkiw 1985, and Mukherjee 2013.

- Lehmer’s continued cotangent, so-named because the continued composition using $t_i(x) = (a_i x + 1)/(x - a_i)$ with $c = 0$ simplifies to

\[ \cot(\arccot a_0 - \arccot a_1 + \arccot a_2 - \ldots). \]

See Lehmer 1938.

- The continued logarithm, a name which has been applied to several disparate forms. Here we refer primarily to continued compositions in which $t_i(z) = a_i + \log_b(z)$. Other kinds of “continued logarithms” use variants of $t_i(z) = t_i(z) = z/\ln(z)$, for instance in approximating the Lambert $W$ function.\(^2\) The “binary (base 2) continued logarithm,” initiated by Gosper\(^3\) is

\(^2\)c.f. Yanghua Wang, The Ricker wavelet and the Lambert $W$ function. *Geophysical Journal International* (2015) **200**, 111–115. doi: 10.1093/gji/ggu384.

\(^3\)B. Gosper, Continued fraction arithmetic. Perl Paraphernalia. [http://perl.plover.com/classes/cftalk/INFO/gosper.txt](http://perl.plover.com/classes/cftalk/INFO/gosper.txt) See also J. M. Borwein, K. G. Hare, and J. G. Lynch, Generalized continued logarithms and related continued fractions. [https://arxiv.org/abs/1606.06984](https://arxiv.org/abs/1606.06984). 22 June 2016
a continued fraction of the form
\[ 2^{a_0} + \frac{2^{a_0}}{2^{a_1} + \frac{2^{a_1}}{2^{a_2} + \cdots}}, \]
where one is interested in the integer sequence \( \{a_i\} \).

- General continued compositions of the form (1), including \( f \)-expansions (Bissinger 1944), infinite processes (Thron 1961), Kettenoperationen or continued operations (Laugwitz 1990), and others (c.f. Kakeya 1924, Paulsen 2013). Because there is an extensive literature on \( f \)-expansions, and most of it dates from after 1950, only a few citations are included here. The book Schweiger 2016 gives a comprehensive overview of \( f \)-expansions and related topics.

Among the continued compositions not covered in this bibliography are:

- The infinite sum \( \sum a_i \), where \( t_i(z) = a_i + z \) and \( c = 0 \), and other examples for which the \( t_i(z) \) are linear in \( z \), such as the Engel expansion\(^4\)
\[ \frac{1}{a_0} \left( 1 + \frac{1}{a_1} \left( 1 + \frac{1}{a_2} (1 + \ldots) \right) \right), \]
where \( t_i(z) = (1 + z)/a_i \) and \( c = 0 \); and the Pierce expansion\(^5\)
\[ a_0(1 - a_1(1 - a_2(1 - \ldots))), \]
using \( t_i(z) = a_i(1 - z) \) and \( c = 0 \).

The infinite product \( \prod a_i \), generated by \( t_i(z) = a_iz \) with \( c = 1 \), is another continued composition of linear functions not covered here. However, we do include infinite products of finite continued compositions, of which Viète’s formula for \( \frac{2}{\pi} \) is probably the most prominent example; see Viète 1593\(^6\)

- Most types of continued fractions, such as the regular continued fraction
\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}, \]

generated by \( t_i(z) = a_i + 1/z \) with \( c = \infty \). (It is more standard to use \( t_i(z) = 1/(a_i + z) \) with \( c = 0 \), and tack on or ignore the term \( a_0 \).)

\(^4\)F. Engel, Entwicklung der Zahlen nach Stammbrüchen, Verhandlungen der 52. Versammlung deutscher Philologen und Schulmaenner in Marburg, 1913, 190–191.

\(^5\)T. A. Pierce, On an algorithm and its use in approximating roots of algebraic equations, The American Mathematical Monthly 36, no. 10, 1929, 523–525.

\(^6\)The exception being made for Viète’s formula points up the fact that we are generally excluding sequences in linear terms of two or more indices, such as \( \sum_i \prod_j a_{i,j} \).
The continued, infinite, or iterated exponential, or tower

\[ a_0 e^{a_1 e^{a_2 e^{\cdots}}}, \]

where \( t_i(z) = a_i e^z \) and \( c = 1 \). On the positive reals one can unambiguously define \( t_i(x) = a_i^x \) to obtain

\[ a^a^a^\cdots. \]

The literature on these forms is too extensive to list here.\(^7\)

2. The issue of associativity: continued versus iterated compositions

The superficially similar iterated composition

\[
\lim_{n \to \infty} t_n \circ t_{n-1} \circ \cdots t_2 \circ t_1 \circ t_0(c),
\]

where the order of operations proceeds from right to left, is sometimes confused with \( \prod \), particularly in the case where the \( t_i \) are identical, that is, when \( t_i = t \) for all \( i = 0, 1, 2 \ldots \) In such a case, the approximants

\[
t(c), \quad t \circ t(c), \quad t \circ t \circ t(c),
\]

and so on, can be viewed either as continued or as iterated compositions.

Denoting an iterated composition’s \( n \)th iterate by

\[
u_n(c) = t_n \circ t_{n-1} \circ \cdots \circ t_2 \circ t_1 \circ t_0(c),
\]

one has the forward recurrence relation

\[
u_n(c) = t_n \circ \nu_{n-1}(c),
\]

but note that, in general, this does not generate a continued composition. Indeed, using the abbreviation

\[
\prod_{j=0}^{n-1} t_j (c) = t_j \circ \prod_{j=0}^{n-2} t_j (c),
\]

one has the backward recurrence relation

\[
\prod_{j=0}^{n} t_j (c) = t_j \circ \prod_{j=0}^{n-1} t_j (c).
\]

Up until 1911, most examples of continued nonlinear function compositions\(^8\) were actually iterated compositions of the form \( \prod \), constructed using the method.

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\(^7\)More than 6,000 unannotated references related to continued fractions (including many of the sources cited in this chronology) are given in Claude Brezinski, *A Bibliography on Continued Fractions, Padé Approximation, Sequence Transformation and Related Subjects*, Ciencias [Sciences], 3. Prensas Universitarias de Zaragoza, Zaragoza, 1991. 348 pp. MR1148890, ISBN 847733238X.

\(^8\)A bibliography of 125 references on continued exponentials may be found in R. Arthur Knoebel, Exponentials reiterated, *The American Mathematical Monthly* **88**, no. 4, 1981, 235–252. MR0610484.

Here we consider “continued linear function compositions” to include continued fractions.
of successive substitution. In this procedure, one attempts to solve an equation of the form \( x = t(x) \) by repeatedly substituting \( t(x) \) for the argument of \( t \):

\[
\begin{align*}
  x &= t(t(x)) \\
  x &= t(t(t(x))) \\
  x &= t(t(t(t(x)))) \\
  \text{and so on,}
\end{align*}
\]

which amounts to an algorithm for computing a fixed point of \( t \). The conditions on \( t \) under which this algorithm would converge began to be rigorously addressed in the second half of the 19th century; we cite only a few of the many papers on forward iteration from this period.

Ramanujan’s continued square root identities from 1911 appear to be the first published examples of continued compositions \(^{[1]}\) in which the \( t_i \) are nonlinear and not identical.

3. Terminology

To name the expression \( \sqrt{a_0} + \sqrt{a_1} + \sqrt{a_2} + \cdots \), the works cited and quoted here have employed nearly every possible concatenation of one or more words from the set \{continued, infinite, nested, iterated\} with a word from the set \{root, radical, power\}. However, the text not directly quoted from primary sources attempts to follow two guidelines in naming function compositions:

1. Adjectives should be specific about the associativity of the expression. Thus, as discussed in Section \(^{[2]}\) continued as a modifier of a named function composition indicates formation by a backward recurrence relation, while iterated indicates a forward recurrence relation. On the other hand, the adjectives nested and infinite are ambiguous with regard to associativity.

2. Nouns should refer to functions rather than symbols. Thus we prefer continued square root and continued \( r \)th root to the less specific continued radical.

4. About the format

Citations are listed 1) in chronological order by year of publication, 2) alphabetically by author last name within a given year, and, if the author(s) published more than one paper in a year, 3a) chronologically when the order of publication is evident, or 3b) alphabetically by journal title. Undated references are listed last. Where possible, Mathematical Reviews (MR) numbers and book ISBNs are provided.

5. About this version

New listings are marked with a dagger \(†\). Items that were not in the previous version are

- ARCHIMEDES ∼250 BCE (page 6)
- VAN CEULEN 1596 (page 8)
- SNELLIUS 1621 (page 8)
- EULER 1744 (page 9)
6. Caveat lector

This bibliography should be considered an inventory, not a detailed or comprehensive history. Given the ever-increasing store of digitized source material, it should also be considered a work in progress. It was compiled and annotated by an interested layman who is not an authority on any of the mathematical topics it mentions, and who does not claim proficiency in languages other than English. Corrections, and suggestions for additional sources from 2016 and earlier, are welcome.

7. The Chronology

1. †(∼250 BCE) Archimedes of Syracuse. *Kuklou metrēsis* [Measurement of a circle]. For an English translation of the original Greek, see T. L. Heath, *The Works of Archimedes, ed. in Modern Notation, with Introductory Chapters*, C. J. Clay and Sons, Cambridge University Press Warehouse, London, 1897. Heath’s translation was reprinted by Dover Publications, Inc., Mineola, New York, 2002, ISBN-13: 978-0486420844.

   Archimedes almost certainly did not manipulate continued square roots. Nonetheless, it is appropriate to cite this fragmentary manuscript as the inspiration for many of the continued square root expressions, listed below, which arise in the calculation of π. Archimedes’ Proposition 2 is the first known use of perimeters of regular polygons (specifically, 96-gons obtained iteratively by halving the sides of a hexagon), inscribed in and circumscribed around a circle, to compute lower and upper bounds, respectively, for π. Making rational approximations to irrational lengths at each step, Archimedes obtains the famous inequality

\[
\frac{223}{71} < \pi < \frac{22}{7}.
\]

Sometimes credited, sometimes not, the “Archimedean algorithm” of polygon construction will be employed in Vīète 1593, Van Ceulen 1596, Euler 1744, Ensheim 1799, Catalan 1842, and many later sources.
2. (1593) François Viète, *Variorum de Rebus Mathematicis Responsorum Liber VII*. Reprinted, with an English translation of the Latin, in *Pi: A Source Book, 3rd edition*, edited by Len Berggren, Jonathan Borwein, and Peter Borwein, Springer-Verlag, New York, 2004, 53–67 and 690–706. ISBN 9780387205717. Cited or referenced in Rudio 1891, Bopp 1913, Lebesgue 1937, Osler 1999, Servi 2003, Levin 2005 and 2006, Lim 2007a, Osler 2007, Moreno and García-Caballero 2013a and 2013b, García-Caballero, Moreno, and Prophet 2014b, Osler 2016, Osler, Jacob, and Nishimura 2016, and Weisstein n.d.

This work is generally considered the first to have expressed a number using nested square root expressions, namely

\[
\frac{2}{\pi} = \sqrt{\frac{2}{2} \cdot \sqrt{\frac{2}{2} + \sqrt{\frac{2}{2} + \frac{1}{2}} \cdot \sqrt{\frac{2}{2} + \frac{1}{2} \sqrt{\frac{2}{2} + \frac{1}{2}}} \cdots}},
\]

(8)
a brute which has defeated a few subsequent authors and their typesetters. Indeed, H. W. Turnbull gives a version whose associativity is not easily parsed (*The World of Mathematics*, edited by James R. Newman, Simon and Schuster, New York, 1956, p. 121), which in turn led G. B. Thomas, Jr., to present a misinterpreted and incorrect version (*Calculus and Analytic Geometry, Alternate Edition*, Addison-Wesley Publishing Company, Inc, Reading, Massachusetts, 1972, p. 834). Herschfeld 1935 cites J. W. L. Glaisher (*Messenger of Mathematics* 2 (new series), 1873, p. 124) for the slightly cleaner

\[
\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots.
\]

(9)

but Catalan 1842 derived the reciprocal of this form earlier (and apparently unaware of Viète).

The formula is obtained by a variation on the Archimedean method. Rather than approximating a circle’s perimeter between inscribed and circumscribed regular polygons, Viète compares the areas of inscribed regular \(2^n\)-gons and \(2^{n+1}\)-gons. A notably clear explanation of Viète’s method is given in the appendix of Moreno and García-Caballero 2013b.

It is worth noting that the pages reprinted in *Pi: A Source Book* are not from Viète’s original 16th century work, but from *Francisci Vietae, Opera Mathematica, in Unum Volumen Congesta*, a nicely typeset compilation of Viète’s works published in 1646 by Frans van Schooten Junior, a professor at the Engineering School, Duytsche Mathematique, loosely connected to Leiden University. Viète did not use the “modern” square root notation employed by Van Schooten. In the original *Variorum*, equation (8) above is something like a diagram and description, almost a flow chart, in Latin,

\[9\]

Lefebvre 1897, Cajori 1928, and others cite the 1634 *Les Oeuvres Mathématiques* of Simon Stevin, edited by Albert Girard, as one of the earliest sources of the square root symbol as it is currently used. Cajori furthermore credits Descartes’ 1637 *Géométrie* with introducing the *vinculum* (the trailing “over-bar”) in conjunction with the surd symbol \(\sqrt{\cdot}\) to create \(\sqrt{-}\).
occupying about three quarters of the page that lies between “pages” 30 and 31 (page numbers appear only on one of two facing pages).

3. †(1596) Ludolph van Ceulen, *Vanden circkel*, Jan Andriesz, Delft. Cited in Bosmans 1910.

In this book, his first, Van Ceulen computes π to 20 decimal places, using the Archimedean method of regular polygons inscribed in and circumscribed around a circle. Van Ceulen’s computations are based on his expansions of polygon side lengths as finite continued square roots of the constant term 2 (sometimes with other integers or fractions as the rightmost term). As noted in Cajori 1928, Van Ceulen employed the “cossic” square root notation, used as early as 1525 by Christoff Rudolff (in his *Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeincklich die Coss genent werden*): a dot is printed immediately to the right of the surd symbol to indicate that the radical continues to the right. Thus, in cossic notation,

\[ \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} \]

is equivalent to the modern

\[ \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}. \]

One could view Van Ceulen’s finite expansions, more than Viète’s infinite one, as perhaps the earliest precursors of the continued square root form.

When Van Ceulen died in 1610, his widow Adrienne Simons undertook to publish Latin versions of many of his mathematical works (originally written in Dutch) with the help of Willebrordus Snellius (whose un-Latinized name is variously given as Willebrord Snellius or Snell or Snel or Snel van Royen, and who is more famous for the property of wave refraction known as Snell’s Law). The *Vanden circkel* translation appears, along with three of Van Ceulen’s pamphlets, in Snellius’s *Lvdolphi ` a Cevlen De Circvlo & Adscriptis Liber*, Lvgd. Batav. Apud Iodocum ` a Colster, 1619. However, Bosmans 1910 comments that this translation has few pages in common with the original text.

4. †(1621) Willebrordus Snellius, *Cyclometricus: de circuli dimensione secundum Logistarum abacos, & ad Mechanicem accuratissima; atque omnium parabilissima*. Elzevir, Lugdunum Batavorum [Leiden]. Cited in Bosmans 1910.

According to Bosmans 1910, Snellius enthusiastically began translating Van Ceulen 1596 from Dutch into Latin, but became bored with Van Ceulen’s enormous calculations. This drove him to find a more efficient version of the Archimedean algorithm. The crux of Snellius’ improvement is the observation that the perimeter of an inscribed n-gon converges to π twice as fast as that of a circumscribed n-gon. Like Van Ceulen, Snellius uses the cossic notation for his finite continued square roots.
5. (1669) Isaac Newton, *De analysi per aequationes numero terminorum infinitas*. For an English translation from the original Latin, see *The Mathematical Papers of Isaac Newton, Volume 2, 1667-1670*, edited by D. T. Whiteside, Cambridge University Press, London, 1968, 221–227. Alluded to in DOPPLER 1832, BOUCHÉ 1862, ISENKRAHE 1888, HEYMANN 1894A and 1894B, ISENKRAHE 1897, and HEYMANN 1901. Cited in TOURNÉS 1996.

Newton’s iterative process for finding roots of equations (later known as Newton’s method, or the Newton-Raphson method), was first revealed here. In this chronology, only TOURNÉS 1996 cites this work directly. However, several authors take Newton’s method to be the inspiration for, or a special case of, the application of iterated and/or continued compositions in solving equations. TOURNÉS 1996 remarks, “Without knowing whether there was transmission or rediscovery, the [method of successive substitution] is found in the West in Viète (1600), Kepler (1618), Harriot (1631), Oughtred (1652) and others, before Newton would reveal it systematically in *De analysi per aequationes numero terminorum infinitas* from 1669, then in the *Methodus fluxionum serierum infinitarum* from 1671 and various subsequent writings… Newton, considerably further than his predecessors, gradually extended the technique of obtaining roots by successive extractions…”

6. †(1744) Leonhard Euler, *Variis modis circuli quadraturam numeris proxime expressendi*. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 9, for the year 1737, 222–238. Eneström number 074. Alluded to in LERESGUE 1937; cited in BOPP 1913, and in Bell’s 2010 translation of Euler 1783.

In the first of nineteen sections, Euler uses inscribed and circumscribed 96-gons in the unit circle to establish the following bounds on $\pi$:

$$\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} < \pi < \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}.$$  \hspace{0.5cm} (10)

In the last section he represents an arc length $A$ on the unit circle by the expansion

$$A = \frac{\sin A}{\cos \frac{A}{2} \cdot \cos \frac{A}{4} \cdot \cos \frac{A}{8} \cdot \cos \frac{A}{16} \cdots}.$$  \hspace{0.5cm} (11)

The special case $A = \frac{\pi}{2}$ will appear in Euler 1763 (cited in CATALAN 1842 and 1872, and in CANDIDO 1908), while a reprise of (11) in Euler 1783 will be cited by Rudio in 1890 (see RUDIO 1891). However, none of these later writers seem to have been aware of this early paper. A note at The Euler Archive, [http://eulerarchive.maa.org/](http://eulerarchive.maa.org/) states that the paper “was presented to the St. Petersburg Academy on February 20, 1738.”
7. (1763) Leonhard Euler, Annotationes in locum quendam Cartesii ad circuli quadraturam spectantem. *Novi Commentarii academiae scientiarum imperialis Petropolitanae, pro Annis MDCLX. et MDCLXI.* 8, 24–27 (summary) and 157–168. Eneström number 275. For an English translation of the original Latin, see “Annotations to a certain passage of Descartes for finding the quadrature of the circle,” translated by Jordan Bell, online at https://arxiv.org/abs/0705.3423v1 23 May 2007. Cited in Catalan 1842 and 1872, and Candido 1908.

This article contains the expansion

$$\frac{\pi}{2} = \lim_{n \to \infty} \sec \frac{\pi}{2^n} \sec \frac{\pi}{2} \cdots \sec \frac{\pi}{2^n}, \tag{12}$$

a special case of identity (11) above in Euler 1744. In 1842, Catalan will independently rediscover Viète’s continued square root formula for \( \frac{2}{\pi} \), and will cite this paper and equation (12) above in his proof.

Catalan 1872 and Candido 1908 claim that equation (12) dates from 1760 and/or 1761. These citations allude to the years covered by the journal volume, not its publication date. A note at The Euler Archive, http://eulerarchive.maa.org/ states: “According to C. G. J. Jacobi, a treatise with this title was read to the Berlin Academy on July 20, 1758. According to the records, it was presented to the St. Petersburg Academy on October 15, 1759.”

8. (1783) Leonhard Euler, Variae observationes circa angulos in progressione geometrica progredientes. *Opuscula Analytica, Vol. I*, Typis Academiae Imperialis Scientiarum, Petropoli [St. Petersburg], 356–351. Eneström number 561. For an English translation of the original Latin, see “Various observations on angles proceeding in geometric progression,” translated by Jordan Bell, online at https://arxiv.org/abs/1009.1439 8 September 2010. Cited in Rudio 1891.

Alluding indirectly to his paper from 1744, Euler here uses the identity \( \sin s = 2 \sin \frac{s}{2} \cos \frac{s}{2} \) to derive the expansion (11) above for the length of an arc on the unit circle (using \( s \) rather than \( A \) to represent arc length). Rudio 1891 will take this expansion as a starting point for developing Viète’s formula for \( \frac{2}{\pi} \). The Euler Archive, http://eulerarchive.maa.org/ states that this paper “was presented to the St. Petersburg Academy on November 15, 1773.”

9. †(1799)“Citoyen Ensheim” [Moses Ensheim], *Recherches sur les calculs différentiel et intégral*. Paris, de l’imprimerie d’Agasse, rue de Poitevins, no. 13, an VII. 28 pages. Cited in Bopp 1913.

On pages 17 and 18 of this rare pamphlet, Ensheim approximates \( \pi \) by the Archimedean method of inscribed and circumscribed regular polygons in the unit circle. He uses successive substitution to develop a nested expression for \( \cos \frac{m}{2^n} \), and independently derives the identity (11) above from Euler 1744. Ensheim then applies his general formulas to the sequence of
3 \cdot 2^k\text{-gons}, k = 1, 2, 3, \ldots, to generate the following bounds on \pi (of which Euler’s inequality (10) above is a special case):

\[3 \cdot 2^q(2 - (2 + (2 + (2 \cdots + (2 + \sqrt{3})^k \cdots) \cdots)^k)^k)^k < \pi < 3 \cdot 2^{q+1} \frac{(2 - (2 + (2 + (2 \cdots + (2 + \sqrt{3})^k \cdots) \cdots)^k)^k)^k}{(2 + (2 + (2 + (2 \cdots + (2 + \sqrt{3})^k \cdots) \cdots)^k)^k)^k} ,\]

where the left-hand expression and the numerator and denominator on the right each contain \(q\) square roots (including \(\sqrt{3}\)). In a footnote to his discussion of the sequence 

\((2 + \sqrt{3})^k, (2 + (2 + \sqrt{3})^k) \cdots,\)

Ensheim expresses amazement that his fellow mathematicians “hardly treat this species of infinite radical.”

10. (1809) Carl Friedrich Gauss, *Theoria motus corporum coelestium in sectionibus conicis solem ambientium*, Frid. Perthes & I. H. Besser, Hamburg. For an English translation of the original Latin, see *Theory of the motion of the heavenly bodies moving about the sun in conic sections: a translation of Gauss’s “Theoria motus.” With an appendix*, translated by Charles Henry Davis, Little, Brown, and Co., Boston, 1857. Cited in *Isenkrahe 1888 and 1897*.

Without dwelling on its origin in celestial mechanics, the “Kepler problem” is to solve for \(E\) in the equation

\[E = M + \epsilon \sin E ,\]

given the quantities \(M\) and \(\epsilon\). In article 11 of Section 1, Gauss proposes an iterative solution, but not, however, the obvious regimen of successive substitution, which would produce the expansion

\[E = M + \epsilon \sin(M + \epsilon \sin(M + \cdots + \epsilon \sin M) \cdots) .\]

Instead, he makes some simplifying assumptions which yield

\[E = M + \epsilon \sin \epsilon \pm \frac{\lambda}{\mu \mp \lambda}(M + \epsilon \sin \epsilon - \epsilon) , \quad (13)\]

where \(\epsilon\) is an approximate value or initial guess for \(E\). Gauss then posits (in Davis’s translation): “If the assumed value \(\epsilon\) differs too much from the truth . . . at least a much more suitable value will be found by this method, with which the same operation can be repeated, once, or several times if it should appear necessary. It is very apparent, that if the difference of the first value \(\epsilon\) from the truth is regarded as a quantity of the first order, the error of the new value would be referred to the second order, and if the operation were further repeated, it would be reduced to the fourth order, the eighth order, etc. Moreover, the less the eccentricity \([e]\), the more rapidly will the successive corrections converge.”
11. (1821) Henri Gerner Schmidten, Mémoire sur l’intégration des équations linéaires. *Annales de Mathématiques Pures et Appliquées* 11, 269–316. Cited in “M.R.S.” 1830 and Tournès 1996.

The author’s first name is given in other sources as Henrik, and his surname as von Schmidten and de Schmidten. He begins his paper by assuming that a solution \( y \) exists for the implicit differential equation

\[
\varphi \cdot y = f \cdot y ,
\]

where \( \varphi \cdot y \) is “a function which contains the differential coefficients or the highest order differences in the proposed equation,” and \( f \cdot y \) is “any other function of the independent variables of the differential or differential coefficients.” If \( X \) is a solution of the homogenous equation \( \varphi \cdot X = 0 \), equation (14) can be written as \( \varphi \cdot y = \varphi \cdot X + f \cdot y \); from this, Schmidten isolates \( y \) on the left by applying the inverse \( \frac{1}{\varphi} \); the result is

\[
y = X + \frac{1}{\varphi} f \cdot y .
\]

He continues, “By means of this implicit relation we shall easily find the explicit value of \( y \) by successive substitutions; this will be

\[
y = X + \frac{1}{\varphi} f (X + \frac{1}{\varphi} f (X + \frac{1}{\varphi} f (X + \ldots .
\]

Some difficulties are immediately apprehended: “It will be seen, however, that the value of \( y \) will in general remain very complicated, unless \( \varphi \cdot y \) and \( f \cdot y \) are linear with respect to \( y \), which embraces a very extended and very important class of equations: that of linear equations... In this case we have

\[
y = X + \frac{1}{\varphi} f X + \frac{1}{\varphi} f (\frac{1}{\varphi} f X) + \frac{1}{\varphi} f (\frac{1}{\varphi} f (\frac{1}{\varphi} f X)) + \ldots
\]

and I propose to state its principal consequences...”

Tournès 1996 observes, “[T]he beginning of the text contains an abstract exposition of the method of successive substitutions, of great formal beauty and surprisingly modern.”

12. (1826) “Herrn Prof. Schmidten” [Henri Gerner Schmidten], Versuch über die Integration der Differential-Gleichungen. *Journal für die reine und angewandte Mathematik* 1, 137–151.

In this “[e]xcerpt from an essay read by the author, in the Royal Danish Academy of Sciences, Copenhagen, in the Danish language,” Schmidten furthers his applications of successive substitution. Beginning with a differential equation of the form

\[
F \left( x, y, y', y'', \ldots, y^{(n)} \right) = 0 ,
\]

he shows how, under helpful conditions, one may iteratively rearrange and integrate to produce a reduced equation

\[
f_n (x, y) = c_n + \psi y ,
\]
where $c_n$ is a constant of integration. Schmidten then assumes that $f_n$ can be inverted (with inverse $P$), by which one obtains

$$y = P(x, c_n + \psi y).$$

This, in turn, is the setup he needs to generate a solution by successive substitution:

$$y = P(x, c_n + \psi(P(x, c_n + \psi(\cdots \psi y \cdots))).$$

From his algorithm arise some formidable expressions. One of his solutions is

$$y = \alpha \int dx \left( \gamma + \frac{\delta x}{\int dx \left( \gamma + \frac{\delta x}{\int dx (\cdots)} \right)} \right).$$

13. (1830) “M. R. S.”, Note sur quelques expressions algébriques peu connues. *Annales de Mathématiques Pures et Appliquées* 20, 352–366.

The author of this paper, hiding behind an acronym, further develops the key idea in Schmidten 1821 about successive substitution. After several specific examples, he proposes that the general equation $F(x) = 0$ can be made to take the form $x = A + f(x)$, and this in turn can be used with the method of successive substitution to obtain

$$x = A + f(A + f(A + f(A + \ldots))).$$

He goes on to use $x = Af(x)$ for

$$x = Af(Af(Af(Af(\ldots)))),$$

as well as $x = A^{f(x)}$ to generate

$$x = Af(A^{f(Af(Af(\cdots)))}),$$

and $x = A + f(B + \varphi(x))$ to get

$$x = A + f(B + \varphi(A + f(B + \varphi(A + \ldots)))).$$

However, Monsieur R. S. then attempts to reverse the process, using the divergent sum

$$x = 1 - 2 + 4 - 8 + 16 - 32 + 64 - \ldots$$

to deduce the “equation” $x = 1 - 2x$, from which he concludes that $x = 1/3$. His argument that

$$a - a + a - a + a - a + \ldots$$

sums to $1/2a$ was one of many in a debate, reaching back to the early 18th century, about how to deal with divergent infinite sums; see for instance the Wikipedia entry on Grandi’s series at [http://en.wikipedia.org/wiki/History_of_Grandi’s_series](http://en.wikipedia.org/wiki/History_of_Grandi’s_series).

(It seems possible that this paper is the work of Henri Gerner Schmidten. As shown by two of the three papers credited to him here, Schmidten...
appears diffident about revealing his full name. He also seems to be the most prominent proponent of successive substitution during this time period. The work of “M. R. S.” fits nicely into the sequence of papers by Schmidten in the decade preceding and including 1830.

14. (1830) “Mr. de Schmidten, prof. des mathém. à Copenhague” [Henri Gerner Schmidten], Sur un principe général dans la théorie des séries. Journal für die reine und angewandte Mathematik 5, 388–396.

Schmidten broadens his conception of infinite nested function compositions by defining

\[ F(x) = \varphi[A_1 + f_1(x)\varphi(A_2 + f_2(x)\varphi(A_3 + \cdots))] , \]

where \( \varphi \) is an invertible function with inverse \( \psi(x) \), the \( f_i \) are functions of \( x \), and \( A_1, A_2, A_3 \ldots \) are quantities which depend on \( F(x) \). He furthermore assumes that \( x_1, x_2, x_3, \ldots \) are values of \( x \) for which \( f_i(x_i) = 0 \) for \( i = 1, 2, 3, \ldots \), and that these values exist, although he admits that \( x_i \) may not uniquely solve \( f_i(x) = 0 \). Under these assumptions, he is able to unwrap the finite approximants of \( F \) and solve for the \( A_i \) in sequential order:

\[ A_1 = \psi(F(x_1)) , \quad A_2 = \psi\left(\frac{\psi(F(x_2)) - A_1}{f_1(x_2)}\right) , \]
\[ A_3 = \psi\left(\frac{\psi(F(x_3)) - A_1}{f_1(x_3)f_2(x_3)}\right) , \]

and so on. He gives two examples in which these calculations are simplified: first, setting \( \varphi(x) = \frac{1}{x} \), and second, using \( \varphi(x) = x \), which turns \( F \) into an infinite sum:

\[ F(x) = A_1 + A_2f_1(x) + A_3f_1(x)f_2(x) + \cdots . \]

Furthermore, if for each \( i = 1, 2, 3 \ldots \) one has \( f_i(x) = x \), then the Taylor series for \( F \) is generated. Schmidten also considers functions defined implicitly, for instance

\[ y = \varphi(x, y) , \]

which expands using successive substitution into

\[ y = \varphi[x, \varphi(x, \varphi(x, \ldots))] . \]

(Such an expansion underlies the continued cotangent developed in Lehmer 1938.) On page 391, there is a tantalizing allusion to the “sum” \( \frac{1}{2} = 1 - 1 + 1 - 1 + \cdots \) mentioned in “M.R.S” 1830 — perhaps another clue in support of the idea that Schmidten was “M.R.S.”

15. (1832) Christian Doppler, Über Kettenwurzeln und deren Konvergenz. Jahrbücher des kaiserlichen königlichen polytechnischen Institutes in Wien 17, 175–200. Cited in Jones 2015.

The author is well-known for the Doppler effect in physics; this paper predates his wave propagation work by about ten years.
CONTINUED SQUARE ROOTS AND CONTINUED COMPOSITIONS, THROUGH 2016

Doppler’s paper contains perhaps the earliest use of the German Kettenwurzel, which he defines more generally than do most subsequent authors as

\[ A \sqrt[\alpha + B \sqrt[\beta + C \sqrt[\gamma + D \sqrt[\delta + E \sqrt[\epsilon + \ldots]]]]]{\ldots} \] \hspace{1cm} (15)

However, he goes on to say that “the exponent values \(a, b, c, d, e,\ldots\) as well as the values and signs of the multipliers \(A, B, C, D, E,\ldots\) are assumed to be periodic in their infinite progression.” This assumption is crucial for applying the method of successive substitution to obtain convergence results. Doppler recognizes that for special values of the exponents \(a, b, c,\ldots\) (which are not assumed to be positive integers) one obtains the continued fraction

\[
\frac{A}{\alpha + \frac{B}{\beta + \frac{C}{\gamma + \frac{D}{\delta + \frac{E}{\epsilon + \ldots}}}}} \hspace{1cm} (16)
\]

as well as what are called continued reciprocal powers:

\[
\frac{A}{a} \left( \frac{\alpha + B}{\beta + \frac{C}{c}} \left( \frac{\gamma + D}{d} \left( \delta + \ldots \right) \right) \right) \hspace{1cm} (17)
\]

and continued reciprocal roots:

\[
\frac{A}{\alpha + \frac{B}{\beta + \frac{C}{\gamma + \frac{D}{\delta + \ldots}}}} \hspace{1cm} (18)
\]

The form (15) turns up again in GÜNTHER 1880, while (17) does not appear again in the papers listed here until the 1990s (LAUGWITZ 1990, SCHOENEFUSS 1992).

Thus far, Doppler’s paper is the earliest to give

\[
\frac{1 + \sqrt{1 + 4a}}{2}
\]
as the limit of the continued square root

$$\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}.$$ \hspace{1cm} (19)

16. (1834) C. F. Eichhorn, *Principien einer allgemeinen Functionenrechnung*, Helwingschen Hof-Buchhandlung, Hannover.

This unusual book begins by restricting its “general function calculations” to linear functions \(f\), for which \(f(ca + cb) = cf(a) + cf(b)\). The author employs modern notation like \(f^n\) for the composition of \(f\) with itself \(n\) times, including \(f^{-1}\) for the inverse of \(f\); on the other hand, he also turns \(y + a\theta(y) = \psi(x)\) into the dubious

\[y(1 + a\theta(\ )) = \psi(x), \quad \text{and consequently} \quad y = \frac{\psi(x)}{1 + a\theta(\ )} \]

(p. 81). Pages 81–98 and 138–139 develop some infinitely nested expressions based on successive substitution. A review of this book in *Repertorium der gesammten deutschen Literatur* 2, 1834, 536–537, complains, “[T]his work will be intelligible and useful only to those analysts who have studied Euler, Legendre, Lacroix, Burg, and others, written with greater clarity and elegance . . . the errors of printing and calculation take the reader’s attention much more than is already the case, due to [the general difficulty of the subject].”

17. (1837) A. Pioch, *Mémoire sur la résolution des équations, suivi de notes sur l’évaluation des fonctions symétriques et sur la détermination des tangentes et des plans tangents*, Leroux, Brussels. Cited in *Realis* 1877.

I have not seen this work. A footnote in *Realis* 1877, credited to “E. C.” (likely E. Catalan, the journal editor), states, “An instructor in the military academy of Brussels, named Pioch, who died very young, had considered equations of the form \(x = f(x)\). His 1837 *Dissertation on the resolution of equations* contains some remarkable ideas.”

18. (1842) E. Catalan, Note sur le rapport de la circonférence au diamètre. *Annales de Mathématiques* 1ère série. 1, 190–196. Cited in *Lefevere*, 1897, *Wiernsberger* 1904b and 1905, and *Bopp* 1913.

Catalan rediscovers Viète’s formula (although Viète’s name is not mentioned), expressed as the reciprocal of equation \([9]\) above. He also derives the formula

$$\pi = \lim_{n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}},$$ \hspace{1cm} (20)

where there are \(n\) 2s under the radical. This expansion will repeatedly be rediscovered, for instance in *Fanien* 1850, *Didion* 1872, *Pierce* 1891, *Candido* 1908, *Hauser* 2003, and *Chang and Chang* 2016.
19. (1850) A. Fanien, Sur le calcul de \( \pi \). *Nouvelles Annales de Mathématiques* 1re série. 9, 190–192.

This is another derivation, via polygons inscribed in the unit circle, of the formula for \( \pi \) given in *Catalan 1842*, equation (20) above. In a footnote, Fanien claims that (20) was known to Viète.

20. (1855) A. Amiot, *Éléments de Géométrie: rédigés d’après le nouveau programme de l’enseignement scientifique des Lycées*, Paris, Dezobry, E. Mag- deleine et Cie, Lib.-Éditeurs.

This geometry textbook was reprinted many times between 1855 and 1875; sometime after 1860 the firm of Ch. Delagrave et Cie, Libraires-Éditeurs became the publisher. Pages 119 and 120 of the first edition develop the formula (20) above from *Catalan 1842*. No references are cited.

21. (1862) A. Bouché, *Premier essai sur la théorie des radicaux continus, et sur ses applications à l’algèbre et au calcul infinitésimal*. *Mémoires de la Société Académique de Maine et Loire* 12, Mallet-Bachelier, Paris, 81–151. Cited in *Realis 1877*.

In this journal’s Volume 9 (1861), the minutes of a meeting held on 9 January 1861 state: “M. Bouche summarizes his theory of continued radicals; this work is referred to the editorial board.” This long paper is apparently the result. Using successive substitution, Bouché develops continued \( r \)th root solutions to certain polynomial and other functional equations. The first chapter introduces the article’s primary target, the trinomial equation \( y^m - y - P = 0 \), where \( m > 1 \) and \( P > 0 \); successive substitution is accomplished with \( y = \sqrt[m]{P + y} \). The example \( y^3 - y - 7 = 0 \) is computed in detail in Chapter 2. In Chapter 3 the author considers the consequences of substituting \( y = z/k + t \) in his trinomial \( m \)-th degree equation, although focusing primarily on the case \( m = 2 \). The case \( m = 3 \) consumes Chapter 4; the polynomial \( y^m - Ay^n - By^p - Cy^q - D = 0 \) is the subject of Chapter 5, along with some formal observations on properties of a continued radical solution. The author extends his methods to \( x^m - Ax^{m-1} - Bx^{m-2} - \ldots - Gx - H = 0 \) in Chapter 6, to \( F(x) = x^m + \sum_{i=1}^m P_i x^{m-i} = 0 \) in Chapter 7, and to other functional equations like \( y^p = e \) in Chapter 8.

*Realis 1877* remarks that “the theory of continued radicals is far from new, and Mr. A. Bouché is not the first to have studied it.”

22. (1862) Léon Sancery, De la méthode des substitutions successives pour le calcul des racines des équations. *Nouvelles annales de mathématiques, 2nd series* 1, 305–312 and 384–400. Cited in *Netto 1896*.

Sancery supposes that \( F(x) = 0 \) is an algebraic or transcendental equation which may be expressed in the form \( x = \varphi(x) \), and for which \( x = \alpha \) is a solution. By successive substitution from an initial value \( x_1 \), he forms the sequence \( x_n = \varphi(x_{n-1}) \), and asks what conditions must be met by
φ(x) so that the \( x_n \) converge to \( α \). In short order he deduces the condition \( 0 < φ'(x) < 1 \) if the sequence \( \{x_n\} \) is to converge monotonically, and
\[-1 < φ'(x) < 0 \] if the sequence is alternating. He applies his tests to the quadratic equation \( x^2 + px + q = 0 \), recast as
\[ x = -\frac{q}{p} - \frac{x^2}{p}, \]
and to the cubic \( x^3 - px + q = 0 \), where again the linear term is isolated on the left. The limitations of these examples lead him to consider the case where \( φ(x) \) is invertible, and ultimately to the case in which \( F(x) = 0 \) can only be expressed as \( ψ(x) = φ(x) \), where \( ψ(x) \) is not a linear function.

23. (1872) “M. le général Didion” [Isidore Didion], Expression du rapport de la circonférence au diamètre et nouvelle fonction. *Comptes rendus hebdomadaires des séances de l’Académie des Sciences* 74, 36–39. Cited in Catalan 1872 and Wiernsberger 1904b.

General Didion independently rediscovers formula (20) above, along with related expressions arising from regular polygons inscribed in the unit circle. He seems to sense that the associativity of continued square roots of constant terms is ambiguous, and instead of
\[
\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}
\]
(21)
he prefers to write
\[
\sqrt{2 - \left[ \left( \sqrt{4 - C^2} + 2 \right)^{\frac{1}{2}} + 2 \right]^{\frac{1}{2}} + 2} \left[ \left( \sqrt{4 - C^2} + 2 \right)^{\frac{1}{2}} + 2 \right]^{\frac{1}{2}} + 2}
\]
(22)
explaining that “replacing the \( \sqrt{\cdots} \) sign by elevation to the power \( \frac{1}{2} \), the expression (21) for the [polygon] side will become (22), in which the indication of successive operations follows the natural order of writing, from left to right, and approaches that of series and continued fractions.”

24. (1872) E. Catalan, Sur une Communication récente de M. le général Didion, concernant une expression du rapport de la circonférence au diamètre. *Comptes rendus hebdomadaires des séances de l’Académie des Sciences* 74, 177. Cited in Wiernsberger 1904b.

Catalan immediately responds to Didion 1872 with a curt letter stating, first, that Catalan himself had established the general’s continued square root formulas thirty years earlier (Catalan 1842), and, second, that “[t]he true author of these various formulas is, if I am not mistaken, Euler. As early as 1760, this great Geometer gave this curious relation in the *Nouveaux Commentaires de Pétersbourg*:

\[
\frac{\pi}{2} = \sec \frac{\pi}{4} \sec \frac{\pi}{8} \sec \frac{\pi}{16} \cdots
\]
It is easy to see that this does not differ, in substance, from the main formulas in question.” Neither Catalan nor Didion seems to have been aware of Viète’s work.

25. (1877) J. J. Åstrand, Neue, einfache Transformation und Auflösung der Gleichungen von der Form \(x^n - ax \pm b = 0\). *Astronomische Nachrichten* 89, 347–350. Cited in Günther 1880 and 1881.

In this short note, the author proposes to solve the equation in the title by substituting \(y = \sqrt[n]{a}^n\) for \(x\), which yields the transformed equation \(y^n - y \pm c = 0\), where \(c = b/(a^{1/n})\). Successive substitution produces the solution

\[-\sqrt[n]{a}^n \sqrt[n]{c} + \sqrt[n]{c} + \sqrt[n]{c} + \cdots.

Åstrand then performs the arithmetic to solve \(x^3 - 7x + 7 = 0\), which he cites as an equation from Lagrange’s *Traité de Résolution des Équations Numériques*, 1808.

26. (1877) S. Realis, Sur quelques questions proposées dans la nouvelle correspondance, Question 142. *Nouvelle Correspondance Mathématique*, Brussels, 193–194.

The question, attributed to “É. Lucas,” asks for an equation whose roots are

\[x = \pm \sqrt{2} \pm \sqrt{2} \pm \sqrt{2} \pm \cdots. \quad (23)\]

In reply, Realis offers \(x^2 - x - 2 = 0\), and mentions its relation to the more general

\[x = \sqrt[n]{p} + \sqrt[n]{p} + \sqrt[n]{p} + \cdots.

However, Realis seems not to have appreciated the subtlety of M. Lucas’s ± signs, which, if assumed to have some periodic pattern, would produce various trigonometric identities. (If “É. Lucas” is in fact Edouard Lucas, then this question may foreshadow some of the results in Lucas 1878 concerning the connection between expressions of the form [23] and trigonometric half-angle formulas.)

27. (1878) T. S. E. Dixon, Continued roots. *The Analyst* 5, no. 1, 20–21. Cited in Jones 2008 and 2015.

This note, barely more than a page long, contains the earliest appearance yet of the English-language terms “continued root” and “continued power”. The author observes that the limit of

\[\sqrt[n]{q + p} \sqrt[p]{q + p} \sqrt[p]{q + p} \sqrt[p]{q + p} \cdots\]

is a root of the equation \(x^n - px = q\); that

\[\sqrt[p]{q - p} \sqrt[p]{q - p} \sqrt[p]{q - p} \sqrt[p]{q - p} \cdots\]
“solves” the equation $x^m + px^n = q$; and that

$$\sqrt{q-p}\sqrt{q-p}\sqrt{q-p}\cdots \text{ and } \frac{q}{p} - \frac{1}{p} \left( \frac{q}{p} - \frac{1}{p} \left( \frac{q}{p} - \cdots \right) \right)$$

are both roots of $x^2 + px = q$. (The latter notation, placing the exponent to the left of the left parenthesis, was independently proposed more than a century later in Jones 1991.) We even find a continued reciprocal root solution,

$$x = \sqrt{q} + \frac{\sqrt{q} + \cdots}{p + \sqrt{q} + \cdots}$$

(24)

to the equation $x^{m+n} + px^n = q$, anticipating Günther 1880. The results are mainly formal, with no rigorous consideration given to convergence, nor are any limitations on the values of $p$ or $q$ mentioned. Most of Dixon’s observations are covered or anticipated in Doppler 1832.

28. (1878) Edouard Lucas, Théorie des fonctions numériques simplement périodiques. *American Journal of Mathematics* 1, 184–240. For an English translation, see *The Theory of Simply Periodic Numerical Functions*, translated by Sidney Kravitz, edited by Douglas Lind, Fibonacci Association, 1969. Cited in Günther 1881, and Vellucci and Bersani 2016c.

Section XV of this paper deals with the “relation of the functions $U_n$ and $V_n$ with continued radicals,” where

$$U_n = \frac{a^n - b^n}{a - b}, \quad V_n = a^n + b^n,$$

and $a$ and $b$ are the two roots of the equation $x^2 = Px - Q$ for $P$ and $Q$ relatively prime integers. By successive substitution one may write

$$a = \sqrt{-Q + P\sqrt{-Q + P\sqrt{-Q + \cdots}}}$$

(Lucas’s Equation 88). Lucas also builds iterated formulas for $V_n$, and notes their resemblance to expressions for $\cos \frac{\pi}{2^n}$, $\cos \frac{\pi}{3\cdot2^n}$, and $\cos \frac{\pi}{5\cdot2^n}$.

29. (1880) S. Günther, Eine didaktisch wichtige Auflösung der trinomischen Gleichungen, *Zeitschrift für mathematische und naturwissenschaftlichen Unterricht [Hoffmann Z.*] 11 68–72. Originally published with the same title in *Verhandlungen der vierunddreissigsten Versammlung deutscher Philologen und Schulmänner in Trier, vom 24. bis 27. September 1879*. B. G. Teubner, Leipzig, 187–190. Cited in Scharwen 1880, Günther 1881, Hoffmann 1881, Isenkrahe 1888, Heymann 1894a, Isenkrahe 1897, Goldziher 1911, and Jones 2015.

In his introduction, Günther recounts the efforts of Lambert, Malfatti, Gauss, and others to solve the trinomial equation $xp \pm ax = \beta$, and mentions Åstrand 1877 in this regard, but also credits Jacob Bernoulli with earlier
but unspecified ideas underlying Åstrand’s method (possibly concerning successive substitution). Günther himself proposes a continued reciprocal root solution, similar to expression (16) above in Doppler 1832. One of his goals is to solve the “pension problem”: finding the rate of interest $q$ in the equation

$$r \frac{q^n - 1}{q - 1} = aq^n,$$

where $a$ is the single premium, $r$ the pension rate, and $n$ the time. Günther rewrites this as $q^{n+1} - \frac{a+r}{a}q^n = -r$ and invokes successive substitution to get

$$q = \sqrt[n]{-r} - \frac{a+r}{a} + \sqrt[n]{-r} - \frac{a+r}{a} + \ldots$$

A review in Jahrbuch über die Fortschritte der Mathematik 12 1882, p. 76, states: “After historical notes about the solution of the trinomial equation, the author gives the solution of the equation in the following form

$$x^{m+n} + ax^m = b$$

in the following form

$$x = \sqrt[1/n]{b} + \frac{a + \sqrt[1/n]{b} + \frac{a + \sqrt[1/n]{b} + \frac{a + \sqrt[1/n]{b} + \ldots}{a + \sqrt[1/n]{b} + \frac{a + \sqrt[1/n]{b} + \ldots}}}}{a + \sqrt[1/n]{b} + \frac{a + \sqrt[1/n]{b} + \ldots}}.$$

He then discusses the forms this structure takes in the special cases where $m/n = 1$, $m = 1$, $n = 1$, and carefully puts an end to defects of this solution, namely the lack of a readily detectable convergence, among others.” However, it emerges immediately with Schaewen 1880, and subsequently in Hoffmann 1881, Netto 1887, and Isenkrahe 1888, that Günther’s solution has defects which are not resolved.

30. (1880) “Gymnasialleher v. Schaewen” [Paul von Schaewen], Zur Lösung trinomischer Gleichungen. Zeitschrift für mathematische und naturwissenschaftlichen Unterricht [Hoffmann Z.] 11, 264–267. Cited in Heymann 1894a and 1894b, and in Goldziher 1911.

In this note, a high school teacher in Saarbrücken takes issue with several claims made in Günther 1880. After offering corrections to a few typographical errors in Günther’s paper, Schaewen works several examples showing that Günther’s continued reciprocal root algorithm converges to the intended root of the initial trinomial equation only in special cases, and gives some empirical deductions based on these calculations. In a gracious postscript, Dr. Günther admits that “[Schaewen’s] remark that the [continued reciprocal roots] are not suitable for numerical computation appears to be generally true. For the time being, as long as the nature of these new forms is still little studied, we can only conclude that in some instances
we have found [his observations] confirmed, and that in others there was a fairly rapid convergence.”

31. (1881) Siegmund Günther, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen*, Verlag von Louis Nebert, Halle a/S. Cited in Heymann 1894b.

Pages 75–82 give an overview of, and some additional detail about, the functions $U_n$ and $V_n$ defined in Lucas 1878, which can be used to generate continued square root representations of trigonometric functions of special angles.

32. (1881) K. E. Hoffmann, Ueber die Auflösung der trinomischen Gleichungen durch kettenbruchähnliche Algorithmen, *Archiv der Mathematik*, 33–45. Cited in Isenkrahe 1888 and 1897, Goldziher 1911, Thron 1961, and Jones 2015.

Hoffmann addresses some of the difficulties arising in Günther 1880. He observes that the trinomial equation

$$f(x) = x^m + px^n + q = 0$$

can be rearranged, on the one hand, in the form

$$x^n(x^{m-n} + p) = -q$$

to obtain “Algorithm A”:

$$x_{i+1} = \sqrt[n]{\frac{-q}{p + x_i^{m-n}}}$$

as the basis for successive substitution, starting at $x_0 = 0$. On the other hand, the trinomial may also take the form

$$x^{m-n} = -p - \frac{q}{x^n}$$

which yields “Algorithm B”:

$$x_{i+1} = m - \sqrt[n]{-p - \frac{q}{x_i^n}},$$

with initial value $x_0 = \infty$. He shows that some of Günther’s calculations mistakenly assume that iteration of each form yields the same limiting value. One of Hoffmann’s principle results is the following: “All real roots of trinomial equations can be found by [Günther’s] continued fraction-like algorithms, as follows: Algorithm A supplies roots lying in the interval

$$-\sqrt{\frac{np}{m}} \text{ to } m - \sqrt{\frac{np}{m}},$$

while one finds the roots lying outside this interval using Algorithm B.”

33. (1887) Eugen Netto, Ueber einen Algorithmus zur Auflösung numerischer algebraischer Gleichungen, *Mathematische Annalen* 29 141–147. Cited in Isenkrahe 1888 and 1897, and Goldziher 1911.
The paper’s notation has been somewhat simplified in what follows. The author notes that iteration of
\[ x_{k+1} = \sqrt[n]{x_k + a} \] (26)
has been used “many times” to solve the trinomial equation \( x^n - x - a = 0 \), but no references are cited. He sets \( f(x) = ax^{n-1} + bx^{n-2} + \cdots + c \), and lists \( f \)'s real roots as \( r_1, r_2, \ldots, r_i, \ldots \) in descending order (assuming initially that none are duplicated). From an arbitrary initial value \( x_0 \), he inductively defines \( x_{k+1} = \sqrt[n]{f(x_k)} \). For an arbitrary root \( r_i \), he establishes a sequence of distances \( \delta_k = x_k - r_i \), and deduces that for \( \delta_k \) sufficiently small,
\[ \delta_{k+1} = \frac{f'(r_i)}{n r_i^{n-1}} \delta_k = q(r_i) \delta_k \]
“with arbitrary precision.” He identifies six cases, accordingly as \( q(r_i) \) is positive or negative and of magnitude greater than, equal to, or less than 1; he meticulously analyzes the sequence of iterates \( x_i \) in each case; and closes with a detailed example using the trinomial \( x^n - x - a = 0 \).

Perhaps the most interesting aspect of this paper is its glimpse into the intricate structure of periodic points arising out of algorithm (26) under certain conditions. In the long paragraph beginning at the top of page 144, Netto discovers the “remarkable circumstance” that “if \( r_{2i+1} \) is negative and \( f'(r_{2i+1}) \) is greater than \( nr_{2i+1}^n \), there is between \( r_{2i} \) and \( r_{2i+1} \), and also between \( r_{2i+2} \) and \( r_{2i+1} \), an odd number of values \( x_0 \) such that two values \( x_0, x_1 \) periodically repeat so that the algorithm does not converge. In the same way, it is also possible that \( x_\lambda = x_0 \) without the equality \( x_\mu = x_0 \) even for \( \mu < \lambda \). Indeed, there may be an infinite number of values \( x_0 \) with this property... If we let \( \lambda \) pass through the sequence of primes, the existence of infinitely many such \( x_0 \) follows.” (Italics are in the original.) In light of Sharkovski’s results from 1964, Netto’s observations, even if only for a polynomial example, seem worthy of a retrospective look.

34. (1888) C. Isenkrahe, Über die Anwendung iteriter Funktionen zur Darstellung der Wurzeln algebraischer und transcendenter Gleichungen, Mathematische Annalen 31 309–317. Cited in Isenkrahe 1897, Goldziher 1911, Thron 1961, and Jones 2015.

After reviewing the results of Hoffmann 1881 and Netto 1887 concerning the solution of polynomial equations of the form \( x = f(x) \) by iterated function algorithms, Isenkrahe establishes general principles governing the convergence of such algorithms. Some of his main results are:

(a) Iteration of \( f(x) \) can only converge to a fixed value \( \xi_a \) when \( |f'\xi_a| < 1 \).

(b) If the iteration of \( f(x) \) converges to \( \xi_a \), this convergence is always more rapid the smaller the modulus of \( f'(x) \), and the slower, the nearer this modulus is to 1.
(c) For real values, the iteration approaches the limit $\xi_a$ only from one side if $f'(\xi_a)$ is positive, while the value oscillates around $\xi_a$ if $f'(\xi_a)$ is negative.

(d) Iteration of the function $\frac{f(x) - xf'(x)}{1 - f'(x)}$ yields the complete real and complex roots of the equation $x = f(x)$, depending on the starting point.

(Though elegantly stated, observations (a)–(c) were known at least as far back as SANCERY 1862.) Isenkrahe points out that Newton’s method for approximating roots of equations is an application of (d); he shows that Gauss’s iterative solution to the Kepler problem (see GAUSS 1809) is an example of Newton’s method, justifies its convergence, and offers an elegant simplification of formula (13) above. Isenkrahe closes by considering the consequences of his analysis for complex roots. Near the beginning of the paper, and also at the end, he intimates that a more extensive work on this subject is forthcoming; see ISENKRAHE 1897.

35. (1890) Maurice Fouché, Remarques sur la méthode des périmètres pour calculer le nombre $\pi$. Bulletin de la Société Mathématique de France 18, 135–138.

Presents a relatively short derivation of the reciprocal of Viète’s formula (9) above. Curiously, the author calls this only “the known formula” for $\pi/2$; Viète’s name is not mentioned. Compare CATALAN 1842.

36. (1891) George Winslow Pierce, The Life-Romance of an Algebraist, J. G. Cupples, Boston, 1891, 18. Cited in HERSCHFELD 1935.

Mentions in passing the formula

$$\pi = \lim_{n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}.$$  \hspace{1cm} (27)

given in CATALAN 1842 and many other sources, and rigorously interpreted in HERSCHFELD 1935.

37. (1891) Ferdinand Rudio, Über die Konvergenz einer von Vieta herrührenden eigentümlichen Produktdenwicklung. Zeitschrift für Mathematik und Physik, historisch-literarische Abteilung 36, 139–140. Cited in BOPP 1913, SCHUSKE AND THRON 1961, SIZER 1986, and CASTELLANOS 1988.

CASTELLANOS 1988 states that the convergence of Viète’s expression (8) above was proved for the first time in this short note. In fact, the note is a letter to an editor who more or less goaded Rudio to provide such a proof. In 1890, in the journal Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich 35, Rudio published a long article on the history of the circle quadrature problem. Mentioning (the reciprocal of) Viète’s formula for $\frac{2}{\pi}$, he added a footnote quoting the infinite product of secants
from Euler 1783\[10\] The following year, in *Zeitschrift für Mathematik und Physik, historisch-literarische Abteilung* 36, editor Moritz Cantor reviewed Rudio’s article. Cantor pointed out that a rigorous investigation of convergence had not yet been done for Viète’s formula, and suggested that Rudio should have done so. Rudio’s page-and-a-half response to Cantor uses the cosine half-angle formula to replace the secants in Euler’s infinite product with finite continued square roots, and carefully establishes the convergence of the resulting expansion.

38. (1894a) W. Heymann, *Ueber die Auflösung der Gleichungen vom fünften Grade.* *Zeitschrift für Mathematik und Physik* 39, 162–182, 193–202, 257–272, 321–354. Cited in Heymann 1894b and Goldziher 1911; plagiarized in Rabinowitsch 1911.

This paper spans more than 90 journal pages in four installments; it possesses two main “parts” (which are not quite synchronized with the installments), and offers a potpourri of section styles. Part I gives an overview of approaches to the quintic equation, citing the work of Klein, Abel, Paul Gordan, and others; many special cases are examined in detail. Our interest here is the fourth installment: Section C (titled “Ueber Kettenfunctionen”) of Part II\[11\] The author defines a *Kettenfunction* by “a system of equations

\[
x = \varphi_1 x_1, \quad x_1 = \varphi_2 x_2, \quad x_2 = \varphi_3 x_3, \ldots, \quad x_{n-1} = \varphi_n x_n
\]
or by an expression

\[
x = \varphi_1 \varphi_2 \varphi_3 \cdots \varphi_n x_n
\]

where the \(\varphi\) are given functions. For

\[
\varphi_k = a_k + \frac{1}{x_k} \quad (k = 1, 2, \ldots, n)
\]
an ordinary continued fraction arises; for

\[
\varphi_k = \sqrt[\varphi]{a_k + x_k}
\]

one obtains a continued root

\[
x = \sqrt[\varphi]{a_1 + \sqrt[\varphi]{a_2 + \sqrt[\varphi]{a_3 + \cdots \sqrt[\varphi]{a_n + x_n}}}}
\]

and in the same way one can speak of a continued power \([Kettenpotenz]\), a continued logarithm \([Kettenlogarithmus]\), and so on.” Heymann immediately follows this bold definition by limiting the discussion to periodic continued compositions, which, as we have seen in Doppler 1832 (not cited in this paper), are handled using successive substitution and fixed-point iteration: “The special subject of our paper does not allow us to enter

\[10\] In the footnote on page 17 of his 1890 quadrature paper, Rudio expresses doubt that Euler was aware of Viète’s formula.

\[11\] The sprawling organization of this paper has led to erroneous citations. Some later authors have cited Section C as a stand-alone paper called “Ueber Kettenfunctionen.”
into general questions about continued compositions. On the contrary, we restrict ourselves to...continued compositions of period 2... Let

\[ y = \varphi(x) \quad \text{and} \quad y = \psi(x) \tag{28} \]

be any two functions, but assume that they can easily be inverted.” The author’s interest in graphical iteration is exquisitely demonstrated in 25 figures, all on a separate oversized sheet.

In a postscript, Heymann reports having recently found Günther 1880 and Schaewen 1880, and comments that his approach to continued compositions resolves Günther’s difficulties. The date “June 1893” at the end indicates that this paper precedes Heymann 1894b, which is dated “December 1893.”

39. (1894b) W. Heymann, Theorie der An- und Umläufe und Auflösung der Gleichungen vom vierten, fünften und sechsten Grade mittelst goniometrischer und hyperbolischer Functionen. Journal für die reine und angewandte Mathematik 113, 267–302. Cited in Heymann 1901.

The prolific Heymann continues his research, initiated in Heymann 1894a, into the solution of equations by iterated functions. In Section I, he begins to document the existence of cyclic orbits (“indifferente Umläufe”), lays the groundwork for function iteration in polar coordinates, and briefly considers how iteration would produce imaginary solutions. Section II is devoted to a detailed investigation of \( n \)th-degree trinomial equations, \( n \geq 1 \), which are reduced to four cases corresponding to the possible arrangements of + and − signs between the three terms. His iterative solutions thus involve the functions \( \varphi(x) = x^r \) and \( \psi(x) = \pm x \pm c \). He makes the calculations (independently duplicated a century later in Jones 1991 and Schoenefuss 1992) for the ranges of \( c \) yielding an attracting fixed point. In Section III, Heymann tackles polynomials of the fourth, fifth, and sixth degree, simplified using circular and hyperbolic trig substitutions; he works out many numerical examples. Finally, in Section IV he considers the question of reformulating the pair of functions \( \varphi(x) \) and \( \psi(x) \) so that their intersection is unchanged, while the rate of convergence of the iterative algorithm is increased.

40. (1896) Karl Bochow, Eine einheitliche Theorie der regelmäßigen Vielecke: 2. Teil., E. Baensch, Magdeburg. Cited in Wiernsberger 1904b and Bochow 1910.

I have not seen this work. In its bibliography, Wiernsberger 1904b quotes Bochow: “I propose to show a general principle which...furnishes a series of equations giving the side and diagonals of any regular polygon.”

41. (1897) C. Isenkrahe, Das Verfahren der Funktionswiederholung, seine Geometrische Veranschaulichung und Algebraische Anwendung, B. G. Teubner, Leipzig. Cited in Goldziher 1911.
At the beginning and close of Isenkrahe 1881, the author vowed to say more on the topic of functional iteration, and in this booklet of 114 pages he apparently fulfills that promise. It was published as a “scientific supplement” to the annual report of the Kaiser-Wilhelm-Gymnasium, the high school in Trier, Germany, where Isenkrahe was a teacher and, eventually, director. Drawing upon his earlier paper, and citing a textbook by Eugen Netto (Vorlesungen über Algebra, Volume 1, B. G. Teubner, Leipzig, 1896) several times, the book gives a concise development of iterated functions as solutions to equations of the form \( F(x) = 0 \). The geometrical interpretation of fixed-point iteration is emphasized in nearly 80 graphs.

An interesting aspect of Isenkrahe’s approach is his handling of the associativity of infinite function compositions. Perhaps recognizing the difference between iterated and continued compositions, he invents the terms Bruchkette and Wurzelkette as the iterated equivalents of Kettenbrüchen (continued fractions) and Kettenwurzeln (continued roots). However, the expression of the iterated/continued dichotomy in mathematical notation is not so clear. For certain values of \( a_0 \) and \( a_1 \), he develops what we might call an \textit{iterated square} solution to the quadratic equation \( a_0 + a_1 x + x^2 = 0 \):

\[
x = -\frac{a_0}{a_1} - \frac{1}{a_1} \left( -\frac{a_0}{a_1} - \frac{1}{a_1} \left( -\frac{a_0}{a_1} - \frac{1}{a_1} \left( -\frac{a_0}{a_1} \cdots \right)^2 \right)^2 \right)^2.
\]

(29)

However, perhaps concerned that this looks too unconventional, he assures us that “becomes somewhat clearer when the powers are recorded as radicals with a fractional index,” that is,

\[
x = \frac{1}{\sqrt{2}} \left( -\frac{a_0}{a_1} - \frac{1}{\sqrt{2}} \left( -\frac{a_0}{a_1} - \frac{1}{\sqrt{2}} \left( -\frac{a_0}{a_1} \cdots \right)^{\frac{1}{2^{\frac{1}{2}}}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} = \cdots,
\]

(30)

where \( \frac{1}{\sqrt{2}} \sqrt[2]{x} = x^2 \). The trick of writing \( \frac{m}{\sqrt{2}} \) for \( x^{n/m} \) is repeated later, when Isenkrahe reprises his 1888 analysis of Hoffmann 1881 and the trinomial \( x^m - px^n + q = 0 \).

42. (1897) B. Lefebvre, S. J., Cours Développé d’Algèbre Élémentaire précédé d’un Aperçu Historique sur les Origines des Mathématiques Élémentaires et suivi d’un Recueil d’Exercices et de Problèmes, Tome I, Calcul Algébrique, Namur Librairie Classique de Ad. Wesmael-Charlier, Éditeur, Belgium.

This school textbook includes some problems (numbers 13, 23, and 30 on pages 306, 308, and 309, respectively) derived from Catalan 1842, and an interesting example of a telescoping continued square root of 6 terms. Incidentally, a long footnote spanning pages 105 to 107 contains a surprisingly detailed history of square root notation.

43. (1899) Karl Bochow, Problem 1740. Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht 30, 24–28. Solution by the proposer, \textit{ibid.}, 31, 191–198. Cited in Bochow 1910.
The problem reads roughly as follows: If $a$ is a number between 0 and $\frac{1}{2}$, one can write
\[2 \sin a\pi = \sqrt{2 + \lambda_1 \sqrt{2 + \lambda_2 \sqrt{2 + \lambda_3 \sqrt{2 + \lambda_4 \sqrt{2 + \ldots}}}}} \quad \text{(31)}\]
and at the same time $a$ can be written as a series
\[a = \frac{1}{2^2} + \frac{\lambda_1}{2^3} + \frac{\lambda_1 \lambda_2}{2^4} + \frac{\lambda_1 \lambda_2 \lambda_3}{2^5} + \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{2^6} + \ldots \quad \text{(32)}\]
in which each $\lambda_i$ is either $+1$ or $-1$.

Similarly, if every $\lambda_i$ is equal to $\pm 1$,
\[a = \frac{1}{2^2} + \frac{\lambda_1}{2^3} + \frac{\lambda_2}{2^4} + \frac{\lambda_3}{2^5} + \ldots \]
and if, simultaneously,
\[2 \sin a\pi = \sqrt{2 + \lambda_1 \sqrt{2 + \lambda_2 \sqrt{2 + \lambda_3 \sqrt{2 + \ldots}}}} \]
then the double sine, the continued square root, the value of $a$, and its series expansion “determine each other unambiguously so that this interrelation contains the general solution of the problem: for a given angle to find its [sine] function, and for a given function value its corresponding angle.”

44. (1899) Kasimir Cwojdzinski, Kettenwurzeln. *Archiv der Mathematik und Physik* 17, 29–35.

The paper begins with a discussion remarkably similar to Dixon 1878, but less compressed, giving many of the same formal continued root constructions of solutions to trinomial equations in one variable. The author defines the continued $n$th root
\[\pm a \sqrt{\pm b \pm a} \sqrt{\pm b \pm a} \sqrt{\pm b \pm \ldots} ,\]
as the limit of the sequence of approximants using the forward recurrence relation
\[w_{n+1} = \pm a \sqrt{\pm b \pm w_n} .\]
He then discusses some particular cases. When $w_{n+1} = a \sqrt{b - w_n}$, he argues that the even and odd approximants respectively increase and decrease toward a limit, provided that $b > a \sqrt{b}$. The cases $w_{n+1} = a \sqrt{b + w_n}$ and $w_{n+1} = -a \sqrt{-b - w_n}$ are claimed respectively to increase and decrease towards their limits. However, although more is offered than in Dixon 1878 to justify the formal expansions, the arguments for convergence are not rigorous. The author gives a numerical evaluation of
\[\frac{\pi}{\psi} \sqrt{e - \frac{\pi}{\psi} e - \ldots} ,\]
where $\psi = \frac{1}{2}(\sqrt{5} - 1)$, but his list of calculated approximants is incorrect. The paper is plagued with nearly two dozen typographical errors in its seven pages.
Continuing his investigations of fixed-point iteration from 1894, Heymann considers the question of periodic points, especially points of periods 2 and 3. Many examples are exhibited, including the continued square root
\[ \sqrt{a + \sqrt{b + \sqrt{a + \sqrt{b + \cdots}}}}. \]

He also applies his graphical theory to continued fractions and geometric series.

Cited in Wiernsberger 1904b and Bochow 1904.

This paper and Bochow 1904 comprise Bochow’s lecture at the Eleventh Annual General Meeting of the Association for the Advancement of Teaching in Mathematics and Natural Sciences at Dusseldorf during Whitsun Week 1902. The topic is the calculation of side lengths of regular polygons, resulting in the continued square root expansions

\[
2 \cos \left( \frac{p}{2q} \pi \right) = \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \cdots}}} \tag{33}
\]
\[
2 \cos \left( \frac{p}{3 \cdot 2q} \pi \right) = \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \cdots}}} \sqrt{2 \pm \sqrt{3}} \tag{34}
\]

where \( p, q \), and the periodic sequence of the signs are given in a table. (Compare Lucas 1878 and Bochow 1899).

Bochow’s lecture was followed by a discussion involving three other teachers. Notes on this discussion, written up by J. E. Böttcher of Leipzig, offer some suggestions on how Bochow’s methods could be implemented in the classroom.

Cited in Wiernsberger 1904a and 1904b.

This page-and-a-half research report exhibits, with little motivation or explanation, the continued radical \( \sqrt{2 \pm \sqrt{2 \pm \cdots}} \) (having signs in periodic sequence) as it pertains to the lengths of sides of regular polygons in the unit circle. The material is expanded upon in Wiernsberger 1904b.

Cited in Wiernsberger 1904a and 1904b.
von Winkelfunktionen.  

Unterrichtsblätter für Mathematik und Naturwissenschaften **10**, no. 1, 12–16. Cited in Wiernsberger 1904b and Bochow 1910.

In this continuation of the lecture Bochow 1902, the author works many examples demonstrating continued square root expansions of trigonometric functions in a regular 18-gon.

49. (1904) W. Heymann, Über die Auflösung der Gleichungen durch Iteration auf geometrischer Grundlage.  

Jahresbericht der technischen Staatselehranstalten in Chemnitz für die Zeit von Ostern 1903 bis Ostern 1904, J. C. F. Pickenhahn & S., Chemnitz. Cited in Goldziher 1911.

I have not seen this work. Goldziher states that the method of iterated functions was “applied to the pension problem in the 1880s [in Günther 1880] and after investigations of the process by Netto, Isenkrahe, and Heymann . . . Heymann [in the work cited here] succeeded in giving the final formula for the practical case.”

50. (1904a) Paul Wiernsberger, Sur les expressions formées de radicaux superposés.  

Comptes rendus hebdomadaires des séances de l’Académie des Sciences. **138**, 1401–1403. Cited in Wiernsberger 1904b.

Revealing more of the work that will eventually comprise Wiernsberger 1904b, the author looks not only at

\[
\sqrt{2 + \epsilon_1 \sqrt{2 + \epsilon_3 \sqrt{2 + \ldots,}}}, \quad |\epsilon_i| = 1,
\]

but also discusses convergence criteria for the more general

\[
\sqrt[n]{a_1 + \epsilon_1 \sqrt[n]{a_2 + \ldots + \epsilon_{h-1} \sqrt[n]{a_{h-1} + \ldots}}}, \quad |\epsilon_h| = 1.
\]

(35)

(36)

Of particular note is this result: “If all \(\epsilon_h\) are positive, the necessary and sufficient condition for the expression (36) to be convergent is that the numbers \(a_h\) all satisfy the inequality

\[
a_h^n < A \quad (h = 1, 2, 3, \ldots),
\]

(37)

where \(A\) is a finite number.” (On page 1402, equation [37] above is misprinted as \(a^n < A\), with no subscript on the \(a\). Other misprints include a missing \(n\) in a nest of \(n\)th roots on page 1403.) As of this writing, Wiernsberger’s is the earliest statement of this important convergence condition, predating Pólya 1916, Vijayaraghavan in Hardy, et al. 1927, and Herschfeld 1935 (all of which more rigorously invoke the lim sup). Wiernsberger also attempts convergence criteria for the case when the \(\epsilon_h\) are not all positive, but with less definitive results. All the proofs are sketched.

51. (1904b) Paul Wiernsberger, Recherches diverses sur des polygones réguliers et les radicaux superposés, A. Rey, Lyons. Cited in Bochow 1910.
Wiernsberger’s doctoral thesis gives an overview of 19th-century geometry of regular polygons, leading to detailed calculations involving the continued square root
\[ \sqrt{2 \pm \sqrt{2 \pm \ldots \pm \sqrt{2}}}, \]
where the signs occur in periodic sequence. Among the trigonometric formulas eventually derived are
\[
\sin \frac{\pi}{2}x = \frac{1}{2} \sqrt{2 + \epsilon_1 \sqrt{2 + \cdots + \epsilon_h \sqrt{2 + \cdots}}} \tag{38}
\]
and
\[
\cos \frac{\pi}{2}x = \frac{1}{2} \sqrt{2 - \epsilon_1 \sqrt{2 + \cdots + \epsilon_h \sqrt{2 + \cdots}}}, \tag{39}
\]
where the \( \epsilon_i = \pm 1 \) form a sequence unique to \( x \) (compare Bochow 1899, which Wiernsberger cites).

In the book’s last chapter, Wiernsberger undertakes what is, as of this writing, the first rigorous discussion of convergence for the general continued \( n \)th root
\[ \sqrt[n]{a_1 + \sqrt[n]{a_2 + \cdots + \sqrt[n]{a_h + \cdots}}}, \]
where the \( a_k \) are positive, \( n \) is greater than or equal to 2, and the positive root is always taken. Here the typo in the convergence condition (37) above from Wiernsberger 1904A is corrected; the proof is essentially the same as that given over thirty years later in Herschfeld 1935. Wiernsberger independently rediscovers Doppler’s limit (19) above for a continued square root of constant terms \( a > 0 \), and investigates special cases in which one negative square root appears in the infinite nest; as an example, Wiernsberger uses equation (19) above to show that \( \sqrt{a - \sqrt{a + \sqrt{a + \cdots}}} \) converges for \( a > (5 + 2\sqrt{5})/4 \).

The bibliography contains 105 items in chronological order, most of which concern topics related to the properties of regular polygons inscribed in and/or circumscribed around a circle. The earliest source is Section VII of Gauss’s *Disquisitiones arithemeticae* (1801), while the latest is Wiernsberger 1904A. In connection with continued square roots, the author cites Catalan 1842, Didion 1872, and several articles and books by Karl Bochow; however, Viète 1593 is absent.

52. (1905) Karl Bochow, *Die Funktionen rationaler Winkel, besonders die numerische Berechnung der Winkelfunktionen ohne Benutzung der trigonometrischen Reihen und ohne Kenntnis der Zahl \( \pi \).* E. Baensch, Magdeburg.

I have not seen this work. This book and Bochow 1896 are apparently quite rare, and have not been digitized as of this writing.

53. (1905) Paul Wiernsberger, Sur les polygones réguliers et les radicaux carrés superposés. *Journal für die reine und angewandte Mathematik* **130**, 144–152. MR1580680.
This is an overview and extension of Wiernsberger 1904b. In addition to developing the formulas (38), (39), and similar expressions for \( \sin \pi x \) and \( \cos \pi x \), Wiernsberger derives an expression for \( \pi \) as the infinite product of continued square roots

\[
\pi = \frac{2^{k-1}}{a} \times C_{2k,a} \times \lim_{h=\infty} \left\{ \frac{2}{C_{2k+1,2k-a}} \cdot \frac{2}{C_{2k+2,2k+1-a}} \cdots \frac{2}{C_{2h,2h-1-a}} \right\},
\]

(40)

where \( C_{m,n} \) is a continued square root representation of \( 2 \sin \frac{n\pi}{m} \) and \( 4a \leq 2^{k+1} < 2^{h} \); for \( a = 1 \) this yields the Catalan 1842 reciprocal version of Viète’s formula (8) above.

54. (1907) C. Alasia, Question 878. *Supplemento al Periodico di Mathematica* 11, no. 1, 16. Solution by the proposer, *ibid.* 11, no. 3, 46–47. Cited in Cipolla 1908.

The problem reads roughly as follows: Prove that

\[
\sin(45^\circ \pm x) = \sqrt{\frac{1 \pm \sin 2x}{2}}
\]

and deduce the relation

\[
2 \cos \left(60^\circ \pm \frac{30}{2^n}\right) = \sqrt{2 - \sqrt{2 - \sqrt{2 - \cdots - \sqrt{2}}}},
\]

where 2 is repeated \( n \) times on the right, and where the + sign or - sign on the left must be taken accordingly as \( n \) is odd or even. Of the 14 solutions received, 3 were incomplete and the rest were deemed “too cumbersome.”

55. (1908) Michele Cipolla, Intorno ad un radicale continuo. *Periodico di Matematica per l’insegnamento Secondario*, Series 3, 5, 179–185. Cited in Herschfeld 1935, Jones 2008, Moreno and García-Caballero 2012, García-Caballero, Moreno, and Prophet 2014b, and Vellucci and Bersani 2016c.

Uses the cosine half-angle formula and induction to prove: If \( i_j \) is +1 or -1, and

\[
\epsilon_{j,n-1} = \frac{1 - i_j i_{j+1} \cdots i_{n-1}}{2}
\]

for \( j = 1, \ldots, n - 1 \), then

\[
\sqrt{2 + i_{n-1}} \sqrt{2 + i_{n-2}} \sqrt{2 + \cdots + i_1} = 2 \cos(1 + 2\epsilon_{1,n-1} + 2^2 \epsilon_{2,n-1} + \cdots + 2^{n-1} \epsilon_{n-1,n-1}) \frac{\pi}{2^n + 1};
\]

the left side is a “left continued radical” (radicale continuo a sinistra) in the limit as \( n \to \infty \). An equivalent formulation using \( i_{n-j} \) instead of \( i_j \) gives a “right continued radical” (radicale continuo a destra) in the limit:
Setting

\[ \lambda_j = \frac{1 - i_1 i_2 \ldots i_j}{2} \]

for \( j = 1, \ldots, n - 1 \), one has

\[ \sqrt{2 + i_1 \sqrt{2 + i_2 \sqrt{2 + \cdots + i_{n-1} \sqrt{2}}}} = 2 \cos(\frac{\pi}{2^n+1}). \quad (42) \]

The paper gives examples using various periodic sequences \( \{i_j\} \) in the finite and infinite cases, including

\[ \sqrt{2 - \sqrt{2 - \sqrt{2 - \cdots}}} = 1 = \cdots - \sqrt{2 - \sqrt{2 - \sqrt{2}}} \]

and

\[ \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \cdots}}} = \frac{\sqrt{5} - 1}{2}. \]

56. (1908) G. Candido, Sul numero \( \pi \). Supplemento al Periodico di Matematica 11, 113–115. Cited in Herschfeld 1935.

Uses trig identities to independently develop several formulas already in existence at the time, including the reciprocal of equation (9) above (Viète is not mentioned; Catalan 1842 has the same result), and equation (27). The paper concludes with the formula

\[ \frac{\pi}{2} = \lim_{n \to \infty} 2^{n-1} \frac{\sqrt{2 - \sqrt{2 + \cdots + \sqrt{2}}}}{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}, \]

where there are \( n - 1 \) terms each in the numerator and denominator; this was essentially derived in Didion 1872.

57. (1910) Karl Bochow, Kettenwurzeln und Winkelfunktionen. Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht 41, 161–186. Cited in Szegő 1930, Herschfeld 1935, and Kommerell 1951.

The author gives a lengthy overview of his work over two decades on continued square roots with terms \( \pm 2 \) arising from trigonometric identities; see Bochow 1896, 1899, 1902, 1904, and 1905. Wiernsberger 1904B is also cited as a principal reference. Much of the paper is devoted to computing examples of finite continued square roots having up to six terms.

58. †(1910) H. Bosmans, S. J., Un émule de Viète: Ludolphe van Ceulen. Analyse de son “Traité du cercle.” Annales de la société scientifique de Bruxelles, 34, second partie, 88–139.

In this valuable overview and explication of Van Ceulen 1596, Bosmans shows in modern language and notation the geometric and trigonometric
tools Van Ceulen employed to obtain his continued square root approximations to $\pi$.

59. (1911) Karl Goldziher, Beiträge zur Praxis der für die Berechnung des Rentenzinsfusses verwendbaren speziellen trinomischen Gleichung. *Zeitschrift für Mathematik und Physik* 59, 410–431.

This is a survey of practical methods for solving trinomial equations arising in the computation of pension interest rates. The paper’s second section draws on the work of Günther, Hoffmann, Netto, and Isenkrahe from the 1880s concerning fixed point iteration, by which one generates pension equation solutions expressible as continued $r$th roots.

60. †(1911) Izko-Ewna Rabinowitsch, Beiträge zur auflösung der algebraischen gleichungen 5. grades. Buchdruckerei K. J. Wyss, Bern. 32 pp.

This doctoral dissertation is noted here primarily for its blatant plagiarism, on page 27, of the discussion of “Kettenfunktion,” “Kettenwurzel,” “Kettenbruch,” and “Kettenlogarithmus” given on page 322 of HEYMAN 1894A. The nearly perfect copy is marred only by its failure to record the index $p$ in defining a $p$th root function. Nowhere does Rabinowitsch cite or credit Heymann.

61. (1911) Srinivasa Ramanujan, Question 289. *Journal of the Indian Mathematical Society* 3, 90. Solution by the proposer, ibid. 4, 1912, 226. The problem and its solution are reprinted in HARDY, ET AL. 1927, p. 323. Cited in HERSHEYEL 1935, the solution to HANISCH 1955, SCHUSKE AND THRON 1961, BORWEIN AND DE BARRA 1991, BERNDT, CHOI, AND KANG 1999, RAO AND VANDEN BERGHE 2005, Mukherjee 2013, Lynd 2014, and Weisstein n.d. The problem is to prove the formulas

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}$$

and

$$4 = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \cdots}}}.$$  

These identities are notable because they do not represent the iteration of a forward recurrence relation to find a fixed point (although, in his notebooks, Ramanujan derived the more general precursors to these formulas by successive substitution; see BERNDT 1989). On p. 348 of HARDY, ET AL. 1927, and in HERSHEYEL 1935, it is noted that Ramanujan’s solution is incomplete, and rigorous convergence arguments are supplied.

62. †(1913) K. Bopp, Eine Schrift von Ensheim “Recherches sur les calculs différentiel et intégral” mit einem sich darauf beziehenden, nicht in die “Oeuvres” übergegangenen Brief. *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch-naturwissenschaftliche Klasse, Abteilung A. Mathematisch-physikalische Wissenschaften*, essay 7, 49 pages.
April 10, 1913 was the one hundredth anniversary of the death of Joseph-Louis Lagrange. In connection with this centenary, Karl Bopp’s essay reveals the existence of the pamphlet Ensheim 1799 (considered rare even in 1913), in the back of which was printed a previously unknown letter by Lagrange, written to Ensheim, dated 23 Nov. 1799. Bopp explicates the content of Ensheim 1799, including Ensheim’s independent rediscovery of the method of inscribed and circumscribed polygons in a circle to generate continued square root expressions as upper and lower bounds for π.

Bopp cites Euler 1744 (presumably unknown to Ensheim) for an earlier example of this polygonal approximation, and also notes the exchange between M. Cantor and F. Rudio which resulted in Rudio 1891.

63. (1914) Srinivasa Ramanujan, Question 507. Journal of the Indian Mathematical Society 5, 240. Solution by the proposer, ibid. 6, 1914, 74–77. The problem and its solution are reprinted in Hardy, et al. 1927, 327–329. Cited in Berndt and Bhargava 1993, Berndt 1994, and Berndt, Choi, and Kang 1999.

The problem states: “Solve completely
\[ x^2 = y + a, \quad y^2 = z + a, \quad z^2 = x + a; \]
and hence show that
\[
(a) \quad \sqrt{8 - \sqrt{8 + \sqrt{8 - \ldots}}} = 1 + 2\sqrt{3}\sin 20^\circ \\
(b) \quad \sqrt{11 - 2\sqrt{11 + 2\sqrt{-\ldots}}} = 1 + 4\sin 10^\circ \\
(c) \quad \sqrt{23 - 2\sqrt{23 + 2\sqrt{23 - \ldots}}} = 1 + 4\sqrt{3}\sin 20^\circ. 
\]

64. (1915) Srinivasa Ramanujan, Question 722. Journal of the Indian Mathematical Society 8, 240. Solution in M. B. Rao, Cyclic equations, ibid. 16, 1925, 139–154. Solution by G. N. Watson, ibid. 18, 1929, 113–117. The problem is reprinted in Hardy, et al. 1927, 332. Cited in Berndt and Bhargava 1993, Berndt 1994, and Berndt, Choi, and Kang 1999.

The problem states: “Solve completely
\[ x^2 = a + y, \quad y^2 = a + z, \quad z^2 = a + u, \quad u^2 = a + x; \]
and deduce that, if
\[ x = \sqrt{5 + \sqrt{5 + \sqrt{5 + x}}}, \]
then
\[ x = \frac{1}{2} \{2 + \sqrt{5 + \sqrt{15 - 6\sqrt{5}}} \} . \]

 Apparently Ensheim had sent some of his work to Lagrange, and presumed to print Lagrange’s reply on the inside back cover of his pamphlet. Following perfunctory excuses about its tardiness, Lagrange’s short letter notes the similarity of Ensheim’s calculus methods to John Landen’s some thirty years earlier, and closes with a generally complimentary flourish.

Notable in this connection is that Bopp expands Ensheim’s inequalities to an excessive degree, creating some of the most elaborate continued square root expressions to be found in any of the sources reviewed here.
and that, if
\[ x = \sqrt[5]{5 + \sqrt{5 - \sqrt{5 + x}}}, \]
then
\[ x = \frac{1}{4} \left[ \sqrt{5 - 2 + \sqrt{13 - 4\sqrt{5}}} + \sqrt{50 + 12\sqrt{5} - 2\sqrt{65 - 20\sqrt{5}}} \right]. \]

65. (1916) J. J. Ginsburg, Problem 460. *The American Mathematical Monthly* **23**, no. 6, 209. Solution by Nathan Altshiller, *ibid.* **24**, no. 1, 1917, 32–33. Cited in Herschfeld 1935, Sizer 1986, Borwein and de Barra 1991, and Jones 2008.

The problem asks for the limit of the continued square root
\[ \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}. \]

The solution considers the more general form
\[ \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} \]
and gives the limit as
\[ \frac{1 + \sqrt{1 + 4a}}{2}, \]
a result shown previously in Doppler 1832.

66. (1916) G. Pólya, Problem proposal 501. *Archiv der Mathematik und Physik, Series 3* **24**, 84. Solution by G. Szegő, *ibid.* **25**, 1917, 88–89. Cited in Herschfeld 1935, Pólya and Szegő 1925, and Schuske and Thron 1961.

The problem is to prove that for positive real \( a_i \),

\[ \lim_{n \to \infty} \sqrt{a_1 + \sqrt{a_2 + \sqrt{\cdots + \sqrt{a_n}}}} \]
converges or diverges accordingly as

\[ \limsup_{n \to \infty} \frac{\log \log a_n}{n} \]
is less than or greater than \( \log 2 \). The problem reappears in Pólya and Szegő 1925.

67. (1918a) Salvatore Pincherle, Sulle catene di radicali quadratici. *Rendiconti delle sessioni della Reale Accademia delle Scienze dell’istituto di Bologna. Classe di scienze fisiche. Nuova Serie* **22** (1917–18), 35–55. Cited in Pincherle 1918b and 1918c, Scarpis 1920, and Pólya and Szegő 1925.

Initiating a somewhat confusing record of continued square root publications in one year, this paper is the first of two parts, both parts having
the same title, but the second (Pincherle 1918b) appearing in a different journal. The author considers the continued square root
\[
\sqrt{a \pm \sqrt{a \pm \sqrt{a \pm \ldots}}} = a,
\]
where the signs occur in a periodic order. He introduces definitions and notation, then investigates convergence in the cases \(a > 2\) and \(a = 2\).

68. (1918b) Salvatore Pincherle, Sulle catene di radicali quadratici. Atti della Rendiconti Accademia della Scienze di Torino. Classe di scienze fisiche, matematiche e naturale. 53 1917–1918, 745–763. Cited in Pincherle 1918c, Scarpis 1920, Pólya and Szegö 1925, and Kuba and Schoissengeier 2002.

Presented in Turin, this paper is the second part of Pincherle 1918a, which was read in Bologna. Here, Pincherle repeats most of his notation and definitions for a new audience, then wraps up his initial investigation into the continued square root (45) above by considering the cases in which \(0 \leq a < 2\).

69. (1918c) Salvatore Pincherle, Sulle radici reali delle equazioni iterate di una equazione quadratica. Atti dell’ Accademia Nazionale dei Lincei Rendiconti. Classe di scienze fisiche, matematiche e naturali, Roma. Series 5, 27, 2nd sem., 177–183. Cited in Scarpis 1920, Pólya and Szegö 1925, and Kuba and Schoissengeier 2002.

The author inductively defines \(\alpha_1(x) = x^2 - a\) and \(\alpha_n(x) = \alpha_{n-1}(x) - a\). He observes that, to each arrangement of the signs + or − in the terms of the continued square root (45) above, having \(n\) radicals, there corresponds a different solution of \(\alpha_n(x) = 0\). He then gives a criterion for these solutions to be real. See also Vellucci and Bersani 2016a.

70. (1920) Umberto Scarpis, Catene periodiche di radicali. Giornale di matematiche di Battaglini 58, 1–13.

Writing in a journal intended for university students, the author presents a relatively accessible account of some results in Pincherle 1918a–c concerning
\[
+\sqrt{a \pm \sqrt{a \pm \sqrt{a \pm \ldots}}},
\]
where the signs are taken to be in a periodic sequence.

71. (1924) S. Kakeya, On a generalized scale of notations, Japanese Journal of Mathematics 1 95–108. Cited in Thron 1961 and Schweiger 2016.

This paper anticipates the \(f\)-expansion of Bissinger 1944. The author writes, “We have usually two methods of denoting the [real] numbers, namely the method of decimal fractions and of continued fractions. These two methods can be included in the same consideration, and a generalized
method of notation can be obtained, to which the usual methods belong as
special cases.

"We consider a function \( f(x) \) of the following properties:

1. \( f(x) \) is continuous and monotonic (having no constant part) in the
   interval \( 0 \leq x \leq 1 \).

We distinguish the two cases when \( f(x) \) is increasing and decreasing by
the names case A and case B.

2. \( f(0) = \alpha, f(1) = \beta \), where \( \alpha \) and \( \beta \) are integers.

\( \alpha \) is less or greater than \( \beta \) according as the case is A or B.

"If \( f(x) \) is differentiable almost everywhere. If \( Df(x) \) denotes one of the
derivatives of \( f(x) \), then \( Df(x) \), if it is not zero, is positive or negative
according as the case is A or B.

3. The measure of the set of \( x \) in the interval \( 0 \leq x \leq 1 \), for which
   \( |Df(x)| \leq 1 \) is zero. Namely \( |Df(x)| > 1 \) almost everywhere.

"Then evidently we have the inverse function \( g(x) \) of \( f(x) \) ."

Ultimately, expansions of the form \( a_1 + g(a_2 + g(a_3 + \ldots + g(a_n) \ldots)) \) are
developed for arbitrary real numbers. Kakeya gives examples of functions \( g \)
which produce decimal, continued fraction, continued square, and continued
logarithm expansions. No references are cited.

72. \( ^{14} \) (1925) Ossian Poirier, Angles et sinus: théorie des radicaux carrés sur 2
superposés. J. Hermann, Paris.

In the “Avertissement” at the front of this eccentric book, the author
admits that he is “étranger aux mathématiques.” Over 146 pages, he slowly
and elaborately develops the trigonometric theory of continued square roots
of \( \pm 2 \), and invents yet another notation system intended to simplify the
typography. Poirier’s idea is to write

\[
\sqrt{a \pm \sqrt{b \pm \sqrt{c \pm \sqrt{d}}} \ldots}
\]

to represent

\[
\sqrt{a \pm \sqrt{b \pm \sqrt{c \pm \sqrt{d}}} \ldots}
\]

He seems to have worked in isolation; no sources are cited. The book is mys-
terious. Almost nothing can be discovered about its author. In the dozens
of databases consulted for this bibliography, the name “Ossian Poirier” ap-
pears independently from this book only a few times, in connection with
the military, law enforcement, or society page notices; it cannot even be de-
termined whether these few disparate references are about the same person,
much less whether he was an amateur mathematician.

73. (1925) G. Szegö, Aufgabe 18. Jahresbericht der Deutschen Mathematiker
Vereinigung 33, 69. Solution (received October 6, 1924) by N. Obreschkoff,
ibid., 117–118. Cited in Szegö 1930 and Herschfeld 1935.

\( ^{14} \) Wherever this problem is cited, the date 1924 is given to Volume 33. The problem itself was
received on April 30, 1924; however, the date on the journal’s title page for Volume 33 is 1925.
The problem is to prove that
\[ \epsilon_0 \sqrt{2 + \epsilon_1 \sqrt{2 + \epsilon_2 \sqrt{2 + \cdots}}} = 2 \sin \left( \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{\epsilon_0 \epsilon_1 \epsilon_2 \cdots \epsilon_n}{2^n} \right), \]
where the \( \epsilon_i \) assume one of the values \(-1, 0, 1\). (The formula was discovered at least 25 years earlier; see Bochow 1899, and also Szegö 1930.)

Problem 161 asks for justification for the equation
\[ \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} = 1 + \frac{1}{1 + \frac{1}{1 + \cdots}}. \]

Problem 162 restates Pólya 1916: for positive real \( a_i \),
\[ \lim_{n \to \infty} \sqrt{a_1 + \sqrt{a_2 + \sqrt{\cdots + \sqrt{a_n}}}} \]
converges or diverges accordingly as
\[ \limsup_{n \to \infty} \frac{\log \log a_n}{n} \]
is less than or greater than \( \log 2 \). (The 1972 version of this book in English mistakenly gives the value 2 instead of \( \log 2 \).)

Problem 163 asks for proof that
\[ \lim_{n \to \infty} \sqrt{a_1 + \sqrt{a_2 + \sqrt{\cdots + \sqrt{a_n}}}} \]
converges if the series \( \sum_{n=1}^{\infty} 2^{-n} a_n (a_1 a_2 \cdots a_n)^{-1/2} \) converges.

Problem 183 restates Szegö 1925: prove that
\[ \epsilon_0 \sqrt{2 + \epsilon_1 \sqrt{2 + \epsilon_2 \sqrt{2 + \cdots}}} = 2 \sin \left( \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{\epsilon_0 \epsilon_1 \epsilon_2 \cdots \epsilon_n}{2^n} \right), \]
where the \( \epsilon_i \) assume one of the values \(-1, 0, 1\) (see also Bochow 1899).

Problem 184 asks for proof that every \( x \in [-2, 2] \) can be written in the form
\[ x = \epsilon_0 \sqrt{2 + \epsilon_1 \sqrt{2 + \epsilon_2 \sqrt{2 + \cdots}}}, \]
where the \( \epsilon_i \) assume either of the values \(-1 \) or 1. Problem 185 asks to show that a real number \( x \) is of the form \( x = 2 \cos k\pi \), \( k \) rational, if and only if the sequence \( \epsilon_i \) is periodic after a certain term.
In a note on page 348 discussing Ramanujan 1911, T. Vijayaraghavan is credited with the following result: If

$$a_n \geq 0, \quad T_n = \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots + \sqrt{a_n}}}}, \quad (48)$$

then a necessary and sufficient condition for the existence of $\lim_{n \to \infty} T_n$ is that

$$\limsup_{n \to \infty} \log \frac{a_n}{2^n} < \infty. \quad (49)$$

As pointed out in Sondow and Hadjicostas 2007, this is essentially equivalent to [50] in Herschfeld 1935, although it is rather more reminiscent of Pólya 1916 in its formulation; indeed, Problem 162 in Pólya and Szegö 1925 is cited by the Collected Papers editors and/or T. Vijayaraghavan as “a less precise form of the convergence criterion.” The earliest statement of this convergence criterion is in Wiernsberger 1904a.

Berndt, Choi, and Kang 1999 further comment that the note cited here “was considerably amplified in a letter from Vijayaraghavan to B. M. Wilson on January 4, 1928.” That letter is reproduced in Berndt and Rankin 1995.
It is proposed that
\[ \sqrt{20 + \sqrt{20 + \sqrt{12 + \sqrt{12 + \cdots}}} + \sqrt{6 + \sqrt{6 + \cdots}}} = 3. \]

The “proof” by formal manipulation uses the formula
\[ n + 1 = \sqrt{n(n + 1) + \sqrt{n(n + 1) + \cdots}}. \]

79. (1935b) Alexander Y. Boldyreff, Problem 75. *National Mathematics Magazine* 9, no. 4, 118. Solution by Dewey C. Duncan, *ibid.* 9, no. 7, 1935, 208–209. Solution by Theodore Bennett, *ibid.* 9, no. 8, 247–248. Cited in Herschfeld 1935, the solution to Dence 1983, and Sizer 1986.

It is proposed that
\[ a = \sqrt{ab + (a^{n-1} - b) \sqrt{ab + (a^{n-1} - b) \sqrt{\cdots}}} \]

where \( a, b \) are positive real numbers and \( n \) is a positive integer. The first solution involves only formal manipulations; the second justifies the existence of the limit as well.

80. (1935) Vincent C. Harris, Problem 78. *National Mathematics Magazine* 9, no. 6, 180. Solution by Theodore Bennett, *ibid.* 9, no. 8, 1935, 251–252. Cited in Herschfeld 1935, Dence 1983 and Sizer 1986.

The value of
\[ \sqrt{12 + \sqrt{48 + \sqrt{768 + \sqrt{196608 + \cdots}}}} \]

is determined to be \( 1 + \sqrt{13} \), using a method similar to Altshiller’s general solution of Ginsburg 1916. Aaron Herschfeld’s critique of the formal manipulation of the infinite expressions in the above Problems 75 and 78 is printed on page 252.

81. (1935) Aaron Herschfeld, On infinite radicals. *The American Mathematical Monthly* 42, no. 7, 419–429. Cited in the solution to Ogilvy 1949, the solution to Hanisch 1955, Schuske and Thron 1961, Thron 1961, Schuske and Thron 1962, Ogilvy 1970, Rohde 1974, Jones 1991 and 1995, Berndt, Choi, and Kang 1999, Laugwitz and Schoenefuss 1999, Levin 2005, Rao and Vanden Berghe 2005, Humphries 2007, Sondow and Hadjicostas 2007, Zimmermann and Ho 2008a, Efthimiou 2012, Gluzman and Yukalov 2012, Mukherjee 2013, Senadheera 2013, Xi and Qi 2013, Clark and Richmond 2014, Lynd 2014, Jones 2015, Lesher and Bersani 2016, Vellucci and Bersani 2016c, and Weisstein n.d.

Herschfeld proves that the “right infinite radical”
\[ \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}} \]
converges if
\[ \limsup_{n \to \infty} a_n^{2^{-n}} < +\infty, \quad (50) \]
where the \( a_n \) are nonnegative reals; compare Wiernsberger 1904b, Pólya 1916, and Vijayaraghavan in Hardy, et al. 1927 for essentially the same result. The author calculates the approximate value
\[ \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots}} \approx 1.757933}; \]
and gives the formula
\[ x(2 + x) = x\sqrt{2^2 + x\sqrt{2^4 + x\sqrt{2^6 + \cdots}}}. \]
Herschfeld follows Cipolla 1908 in distinguishing between these “right” infinite radicals (which are continued compositions) and “left” infinite radicals, which are iterated compositions of the form
\[ \cdots \sqrt{a_3 + \sqrt{a_2 + \sqrt{a_1}}}. \]
Herschfeld shows that a left infinite radical converges if and only if the sequence \( a_n \) has a limit \( a \). The limit of the left infinite radical is then
\[ \frac{1 + \sqrt{1 + 4a}}{2} \]
when \( a > 0 \) (compare Doppler 1832; and Ginsburg 1916, which Herschfeld cites). Lastly, his Theorem III states without proof that for \( a_n \geq 0 \) and \( r_n \in (0, 1] \), suppose the series \( \sum_{n=1}^{\infty} r_1 r_2 \cdots r_n \) converges; then the right infinite radical
\[ (a_1 + (a_2 + (a_3 + \cdots)^{r_3})^{r_2})^{r_1} \]
converges if and only if
\[ \limsup_{n \to \infty} a_n^{r_1 r_2 \cdots r_n} < +\infty. \]

82. Henri Lebesgue, Sur certaines expressions irrationnelles illimitées. *Bulletin of the Calcutta Mathematical Society* 29, 17–28. Cited in Aoki and Kojima 2016.

Lebesgue claims the ideas for this paper occurred to him while lecturing at the Collège de France in 1930; however, his insights unwittingly duplicate those of Rudio, Bochow, Wiernsberger, Cipolla, Pólya and Szegő, Póier, and others in explicating the continued square roots of \( \pm 2 \) arising from trigonometric half-angle formulas. He attributes formula (11) to Euler in this regard, and realizes that Viète had obtained essentially the same result. Fortunately, he finds out and acknowledges that his work comprises

\[ ^{15} \text{Herschfeld calls this the Kasner number, in honor of Edward Kasner, who “for approximately twenty-five years... has periodically suggested to his classes at Columbia University the investigation of the problem of ‘infinite radicals.’” The web site mathworld.wolfram.com calls this the “nested radical constant.”} \]
independent rediscoveries of at least one author’s work (see the next item below).

83. (1938) Henri Lebesgue, Sur certaines expressions irrationnelles illimitées. *Bulletin of the Calcutta Mathematical Society* **30**, 9–10. Cited in Aoki and Kojima 2016.

In this courteous note, about a page long, Lebesgue reports having been informed by his colleague Hadamard that the principal theorem proved in Lebesgue 1937 was previously proved, using the same methods, in Wiernsberger 1904b. Lebesgue makes a complimentary remark about the extent of Wiernsberger’s thesis bibliography, and directs readers to Wiernsberger’s papers from 1903, 1904, and 1905.

84. (1938) D. H. Lehmer, A cotangent analogue of continued fractions. *Duke Mathematical Journal* **4**, 323–340. MR1546053. Cited in Leighton 1940, Bissinger 1944, Spira and Scheeline 1974, Shallit 1976, Schoenefuss 1992, Rivoal 2007, and Schweiger 2016.

The author proposes the “continued iteration” of a function $f(x, y)$:

$$f(x_1, f(x_2, f(x_3, \ldots))) . \tag{51}$$

The $x_i$ are not initially defined, but later are assumed to be integers. Examples of (51) are given for various $f$:

$$f(x, y) = \begin{cases} 
  x + y & \text{yields an infinite sum} \\
  xy & \text{an infinite product} \\
  x + \frac{1}{y} & \text{a regular continued fraction} \\
  x + \frac{y}{c} & \text{a power series of terms } \frac{1}{c^i} .
\end{cases}$$

If $c = 10$ in the last case, one obtains a decimal expansion. The author’s interest is in

$$f(x, y) = \frac{1 + xy}{y - x} ,$$

which can be expressed as $f(x, y) = \cot(\arccot x - \arccot y)$ by using the identity

$$\cot(\alpha - \beta) = \frac{1 + \cot \alpha \cot \beta}{\cot \alpha - \cot \beta} .$$

In the notation of this bibliography, (51) can be expressed as a continued composition (1) using $t_i(x) = (a_i x + 1)/(x - a_i) = \cot(\arccot a_i - \arccot x)$, which produces

$$\cot(\arccot a_0 - \arccot a_1 + \arccot a_2 - \ldots) .$$

Lehmer uses integer terms “in order to obtain sequences of rational approximations to a real number.” After defining conditions for a “regular” continued cotangent expansion, he proves that all infinite regular continued cotangent expansions converge, and, in a certain sense, do so more rapidly
than continued fractions. He introduces a constant $\xi \approx 0.59263$, for which the continued cotangent expansion converges least rapidly. Some properties of continued cotangents and continued fractions are analogous: for instance, $x$ is rational or irrational accordingly as its continued cotangent expansion is finite or infinite.

85. (1940) Walter Leighton, Proper continued fractions. *The American Mathematical Monthly* 47, no. 5, 274–280. MR0002567. Cited in Bissinger 1944, Rivoal 2007, and Schweiger 2016.

From the introduction: “This paper generalizes the so-called ‘regular’ continued fraction expansion of a real number. The treatment includes as a special case the ‘continued cotangent’ expansion of Lehmer.”

86. (1941) Roy MacKay, Problem E 474. *The American Mathematical Monthly* 48, no. 5, 337. Solution by Eduardo Gaspar, *ibid.* 49, no. 3, 1942, 197. Cited in the solution to Ogilvy 1949.

The problem states: “For $k > 1$, define
\[
\begin{align*}
a_1 &= \sqrt{k(k-1)} \\
an &= \sqrt{k(k-1) + an-1} \\
b_1 &= \sqrt{k} \\
b_n &= \sqrt{kb_n-1}.
\end{align*}
\]
Prove that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = k$.”

87. (1944) B. H. Bissinger, A generalization of continued fractions. *Bulletin of the American Mathematical Society* 50, 868–876. MR0011338. Cited in Bissinger and Herzog 1945, Herzog and Bissinger 1945, Everett 1946, Rényi 1957, Swartz and Wendroff 1960, Thron 1961, Waterman 1970, Schoenefuss 1992, Martin 2004, and Schweiger 2016.

Introduces the $f$-expansion (52) of a real number $x \in (0,1)$,
\[
x = f(a_1 + f(a_2 + f(\ldots))) ,
\]
where the $a_i$ are positive integers and $f$ is monotone, strictly decreasing, defined on $(1, \infty)$ with $f(1) = 1$ and $f(\infty) = 0$, and is such that the chord defined by any two points on the graph of $y = f(t)$ has a slope bounded away from $-1$ for $t > f(2) + 1$. See also Kakeya 1924.

88. (1945) F. Herzog and B. H. Bissinger, A generalization of Borel’s and F. Bernstein’s theorems on continued fractions. *Duke Mathematical Journal* 12, 325–334. MR0012346. Cited in Bissinger and Herzog 1945.

Regarding the $f$-expansion (52) above, the authors write, “Borel...and F. Bernstein...have shown in the case $f(t) = t^{-1}$ (simple continued fractions) that (a) the set of all $x, 0 < x < 1$, for which $a_n(x) \leq k_n$ for all $n$, where the $k_n$ are positive integers, is of measure zero if and only if $\sum 1/k_n$
diverges, (b) the set of all \( x, 0 < x < 1 \), for which \( a_n(x) > 1 \) for all \( n \) is of measure zero. F. Bernstein showed these statements to be true also when only a fixed subsequence of the \( a_n(x) \) is considered. Certain consequences of statements (a) and (b), concerning the boundedness of the sequence \( a_n(x) \) as well as the occurrence of finitely many ones in the sequence \( a_n(x) \), were derived in these two papers.

“This paper is concerned with the problem of generalizing Borel’s and Bernstein’s results to \([f]-expansions\). The first results in this direction were obtained by Bissinger... who proved these statements to be true when \( f \) belongs to a certain class of polygonal functions. The proofs of these statements are based on the fact that the \( a_n(x) \) are statistically independent for polygonal functions. Consequently these proofs cannot be used to re-establish Borel’s and Bernstein’s results since, as already pointed out by them, the \( a_n(x) \) are not statistically independent for \( f(t) = t^{-1} \).

“This paper gives a solution of the problem which remained — to characterize analytically a [class of functions] which does include \( f(t) = t^{-1} \) and for which the statements (a) and (b) hold.”

89. (1945) B. H. Bissinger and F. Herzog, An extension of some previous results on generalized continued fractions. *Duke Mathematical Journal* **12**, 655–662. MR0015534. Cited in Schoenefuss 1992.

This paper builds on Bissinger 1944 and Herzog and Bissinger 1945 largely by loosening the restrictions on functions used to create \( f \)-expansions.

90. (1946) C. J. Everett, Representations for real numbers. *Bulletin of the American Mathematical Society* **52**, 861–869. MR0018221. Cited in Rechard 1950, Rényi 1957, Swartz and Wendroff 1960, Thron 1961, Waterman 1970, and Schweiger 2016.

Considers Bissinger’s \( f \)-expansions using continuous strictly increasing functions \( f(t) \) on \([0, p] \) for which \( f(0) = 0, f(p) = 1 \). Where Bissinger generalizes continued fraction expansions of real numbers, Everett generalizes decimal expansions; see Kakeya 1924 for an earlier treatment of both topics.

91. (1949) C. S. Ogilvy, Problem E 874. *The American Mathematical Monthly* **56**, no. 6, 404. Solution by C. W. Trigg, *ibid.* **57**, no. 3, 1950, 186. Cited in the solution to Herschfeld 1957, and Jones 2008.

The problem essentially asks for proof that the iterated composition

\[ \cdots \sqrt{x + \sqrt{x + \sqrt{x}}} \]

has an integral limit if and only if \( x \) is of the form \( n(n - 1) \), in which case the limit is \( n \). The solution generalizes this to \( r \)th roots, and gives \( x = (n - 1)n(n + 1) \) for \( r = 3 \). An editor’s note points to MacKay 1941 as a similar problem.
92. (1950) O. W. Rechard, The representation of real numbers, *Proceedings of the American Mathematical Society* 1, 674–681. Cited in Swartz and Wendorff 1960 and Schweiger 2016.

From the introduction: “Let \( p \geq 2 \) be a positive integer and denote by \( E_p \) the class of all continuous, strictly increasing functions \( f(x) \) on the interval \( 0 \leq x \leq p \) with \( f(0) = 0 \) and \( f(p) = 1 \). In a generalization of the decimal representation, [Everett 1946] has associated with every real number \( 0 \leq \gamma < 1 \) a sequence of integers [by means of an algorithm involving] \( f(x) \) ... an arbitrary function in the class \( E_p \).

“Some functions (for example, \( f(x) = x/p \), which leads to the representation of a number as a decimal to the base \( p \)) when employed in [Everett’s] algorithm ... yield one-one correspondences between real numbers and sequences of integers mod \( p \). On the other hand, any function, for example, whose graph has more than one point in common with any of the straight line segments connecting the points \((j, 0) \) and \((j + 1, 1), j = 0, 1, \ldots, p - 1, \) will obviously lead to a correspondence which is many-one. We shall denote by \( E_p^{*} \) the subclass of \( E_p \) consisting of those functions which in [Everett’s algorithm] give[s] rise to one-one correspondences.

“The present paper contains very simple characterizations of those correspondences between real numbers and sequences of integers mod \( p \) which can be obtained by applying [Everett’s algorithm] with functions from the classes \( E_p^{*} \) and \( E_p - E_p^{*} \) respectively. By means of these characterizations it is possible to settle two of the problems raised by Everett and to give an answer (albeit not a completely satisfactory one) to a third, namely that of characterizing the class \( E_p^{*} \) itself.”

93. (1951) Karl Kommerell, Berechnung der trigonometrischen und zyklometrischen Funktionen durch Kettenwurzeln. *Mathematisch-physikalische Semesterberichte zur Pflege des Zusammehangs von Schule und Universität* 2, 126–134.

The paper begins: “In the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, Vol. 41, 1910, K. Bochow developed formulas for the calculation of trigonometric and inverse trigonometric functions ... Bochow has only infinite continued square roots; he is therefore not in a position to estimate the error which one incurs when an infinite continued square root is terminated at a certain point; furthermore, the description and the argumentation are quite cumbersome. Finally, he does not have the courage to carry out the calculation. [His] formulas ... can now be proved very easily and can be used to calculate the functions quickly and reliably and to estimate the degree of accuracy of the calculation.” The Catalan 1842 formula for \( \pi \) (equation 20 above) and Viète’s formula for \( 2/\pi \) (equation 8 above) are discussed.

(In a few internet links, this article is incorrectly attributed to “Karl Koenig,” perhaps due to inaccurate optical character recognition of scanned pages.)
94. (1955) Herman Hanisch, Problem 4620. *The American Mathematical Monthly* 62, no. 1, 45. Solution by N. J. Fine, *ibid.* 63, no. 3, 1956, 194–195.

This is identical to the first problem in Ramanujan 1911, as noted in an editorial comment to the solution.

95. (1956) G. N. Wollan and D. M. Mesner, On a function defined by means of an infinite radical. Paper given at the 33rd annual meeting of the Indiana Section of the Mathematical Association of America, May 5, 1956. Abstract printed in *The American Mathematical Monthly* 63, no. 8, 614. Cited in Wollan and Mesner 1957.

From the abstract: “If $0 < x \leq 1$, then $x$ has a non-terminating binary representation $x = a_1a_2 \cdots a_n \cdots$. Let $\alpha_n = (-1)^{a_n}$ and

$$f_n(x) = \sqrt[k + \alpha_1]{k + \alpha_2\sqrt[k + \cdots + \alpha_n\sqrt[k]}},$$

$n = 1, 2, 3, \ldots$, and let $I$ denote the interval $0 < x \leq 1$. When $k > 2 + \sqrt{2}$, then $\lim_{n \to \infty} f_n(x)$ exists and is real for every $x$ in $I$ and this defines a function $f(x) = \lim_{n \to \infty} f_n(x)$. This function is discontinuous at every point in $I$ having a terminating binary representation and is continuous elsewhere in $I$. The function is not monotone in any sub-interval on $I$ and yet has a derivative equal to zero at each point of a dense set of points of $I$ and has a left derivative equal to zero at every point of discontinuity.”

96. (1957) Aaron Herschfeld, Problem E 1258. *The American Mathematical Monthly* 64, no. 3, 197. Solution by D. A. Freedman, *ibid.* 64, no. 9, 1957, 673.

“Prove that a necessary and sufficient condition for the rationality of

$$R = \sqrt[3]{a} + \sqrt[3]{a} + \cdots,$$

where $a$ is a positive integer, is that $a = N(N + 1)(N + 2)$, the product of three consecutive integers. In that case find $R$. $R$ is shown to equal $N + 1$. This problem was essentially solved in Ogilvy 1949.

97. (1957) A. Rényi, Representations for real numbers and their ergodic properties. *Acta mathematica Academiae Scientiarum Hungaricae* 8, 477–493. MR0097374. Cited in Thron 1961, Waterman 1970 and 1975, and Schweiger 2016.

Puts the $f$-expansion results of Bissinger 1944 and Everett 1946, as well as known theorems about regular continued fraction and decimal expansions, in a general context. An example is given expanding a real number $x \in [0, 2^n - 1]$ as a continued $n$th root. The $\beta$-expansion of a real number is introduced. A summary of this work is given in Chapter 10 of Schweiger 2016.
98. (1957) G. N. Wollan and D. M. Mesner, Some additional remarks on a function defined by means of an infinite radical. Paper given at the 34th annual meeting of the Indiana Section of the Mathematical Association of America, May 11, 1957. Abstract printed in *The American Mathematical Monthly* 64, no. 8, 621. 

From the abstract: “This paper presents some additional properties of the function $f(x)$ defined on $0 < x \leq 1$ by the relation $f(x) = \lim_{n \to \infty} f_n(x)$ where

$$f_1(x) = \sqrt{k + \alpha_1 \sqrt{k}}, \quad f_2(x) = \sqrt{k + \alpha_1 \sqrt{k + \alpha_2 \sqrt{k}}}, \quad f_n(x) = \sqrt{k + \alpha_1 \sqrt{k + \cdots + \alpha_n \sqrt{k}}} \text{ with } n \text{ nested root signs},$$

$n = 1, 2, \ldots$ and $\alpha_n = (-1)^{a_n}$ where $a_n$ is the $n$th digit in the nonterminating binary representation of $x$. . . .
The authors show that when $k > 2 + \sqrt{2}$, although the function has a denumerably infinite set of discontinuities and is not monotone in any subinterval, it is of bounded variation; although it has a value at each point of the interval with $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$, yet the set of values of the function is of measure zero. Furthermore the derivative exists almost everywhere and whenever it exists its value is zero, but there is a nondenumerable set of points (of measure zero) at which the derivative does not exist.”

99. (1958) P. J. Myrberg, Iteration von Quadratwurzeloperationen. *Annales Academiae Scientiarum Fennicae. Series A. I, Mathematica* 259, 16 pp. MR0108662. Cited in Schuske and Thron 1962 and Jones 2008.

The paper addresses convergence questions for the expression

$$z = \pm \sqrt{p \pm \sqrt{p \pm \sqrt{p + z}}}$$

and derives formulas involving trigonometric functions of special angles. Compare Lucas 1878, Cipolla 1908, and the works of Bochow, Wiernberger, and Pincherle.

100. (1960) B. K. Swartz and B. Wendroff, Continued function expansions of real numbers. *Proceedings of the American Mathematical Society* 11, no. 4, 634–639.

From the introduction: “We present a theory of continued function expansions of numbers which contains the generalized continued fractions of [Bissinger 1944] and the generalized decimal representations of [Everett 1946] . . . We generalize [the latter] by admitting a wider class of functions than those of the form $f^{-1}(x - n)$. [Rechard 1950] gave a necessary and sufficient condition that the correspondence between numbers and sequences resulting from Everett’s algorithm be $1 - 1$. This condition appears in our theory as a simple functional relation similar to one considered [in Schreier and Ulam, Eine Bemerkung über die Gruppe der topologischen
101. (1961) Georgellen Schuske and W. J. Thron, Infinite radicals in the complex plane. *Proceedings of the American Mathematical Society* **12**, 527–532. MR0151586. Cited in Thron 1961, Schuske and Thron 1962, and Lorentzen 1995 and 1998.

The following theorem is proved about continued square roots with complex terms: Let $\theta$ be a fixed number, $0 < \theta < \pi$. Define $g(\theta)$ by

$$g(\theta) = \begin{cases} 
\pi - \theta/2 & \text{when } 0 < \theta \leq 2\pi/3, \\
2(\pi - \theta) & \text{when } 2\pi/3 \leq \theta < \pi.
\end{cases}$$

Let $\epsilon$ be an arbitrarily small fixed positive number, $0 < \epsilon < \min[\theta, g(\theta)]$. Let $a \in A_\epsilon$ if and only if

$$|a| > 0 \quad \text{and} \quad -\theta + \epsilon < \arg a < g(\theta) - \epsilon.$$

Then the sequence of continued square root approximants $K_{i=0}^n \sqrt{a_i}$ converges if

$$a_n \in A_\epsilon \quad \text{for every } n,$$

and

$$\limsup_{n \to \infty} |a_n|^{2^{-n}} < \infty.$$

The proof makes use of the bound (50) above, from Herschfeld 1935.

102. (1961) W. J. Thron, Convergence regions for continued fractions and other infinite processes. *The American Mathematical Monthly* **68**, 734–750. MR0133444. Cited in Lorentzen 1995 and 1998, and Jones 2015.

The author begins with an arbitrary complex valued generating function

$$f(a_n^{(1)}, \ldots, a_n^{(k)}, z)$$

of $k + 1$ complex variables and calls it $t_n(z)$. He then inductively defines an “infinite process” as a sequence $\{T_n(z)\}$ such that

$$T_1(z) = t_1(z); \quad T_n(z) = T_{n-1}(t_n(z)), \quad n > 1.$$

He writes, “Through suitable choice of $z = c$ [where $c$ is the initial value], many well-known infinite processes can be obtained. Some of these are . . . infinite series, infinite products, infinite radicals, infinite exponentials, [and] two different kinds of continued fractions. [W]e are concerned mainly with describing methods . . . for obtaining convergence-region criteria for certain of these infinite processes. The methods have been successfully applied to all except the first two types. . . . That no nontrivial convergence regions exist for infinite series and infinite products will become clear after we have defined what a convergence region is.”

103. (1962) Georgellen Schuske and W. J. Thron, On periodic infinite radicals. *Annales Academiae Scientiarum Fennicae. Series A. I, Mathematica* **307**, 8 pp. MR0137937.
This paper appears to have been the first to prove that
\[ \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} \]
converges to
\[ \frac{1 + \sqrt{1 + 4a}}{2} \]
when \( a \) is any complex number, and where \( \sqrt{1 + 4a} \) is taken in accordance with this definition: \( \sqrt{x} \) is the square root whose real part is positive or 0, and is defined to lie on the positive imaginary axis if \( x \) is negative.

An independent rediscovery of this result is SIZER AND WIREDU 1996. Compare 
DOPPLER 1832 and GINSBURG 1916 for the case where \( a \) is a positive real number.

104. (1962) D. O. Shklarsky, N. N. Chentzov, and I. M. Yaglom, The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics. W. H. Freeman, San Francisco, ISBN 9780716704126. Reprinted by Dover Publications, Inc., Mineola, New York, 1993, ISBN 9780486277097. Cited in EFTHIMIOU 2012 and 2013.

Problem 195 reads: “Let some of the numbers \( a_1, a_2, \ldots, a_n \) be +1 and the rest be –1. Prove that
\[ 2 \sin \left( a_1 + \frac{a_1a_2}{2} + \frac{a_1a_2a_3}{4} + \cdots + \frac{a_1a_2\cdots a_n}{2^{n-1}} \right) 45^\circ = a_1 \sqrt{2 + a_2 \sqrt{2 + a_3 \sqrt{2 + \cdots a_n \sqrt{2}}}}. \]

For example, let \( a_1 = a_2 = \cdots = a_n = 1 \):
\[ 2 \sin \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \right) 45^\circ = 2 \cos \frac{45^\circ}{2^{n-1}} \]
\[ = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}. \]

The problem, which is unattributed, is essentially the same as Problem 183 in PÓLYA AND SZEGŐ 1925; see equation (47) above. Compare also BOCHOW 1899 and CIPOLLA 1908.

105. (1967) J. H. McKay, The William Lowell Putnam Mathematical Competition. The American Mathematical Monthly 74, no. 7, 771–777. Cited in BOISSEIEN AND DE BARRA 1991.

Putnam Exam problem A-6 asks for “justification” of the identity
\[ 3 = \sqrt{1 + 2 \sqrt{1 + 3 \sqrt{1 + \cdots}}}. \]
(53)

The original source, RAMANUJAN 1911, is not cited in the exam or in the published solution.
106. (1970) C. S. Ogilvy, To what limits do complex iterated radicals converge? The American Mathematical Monthly 77, no. 4, 388–389. MR1535864. Cited in Gerber 1973, Rohde 1974, Walker 1983, Jones 1991, and Laugwitz and Schoenefuss 1999.

This short note, printed in the old “Research Problems” section of the Monthly, asked a number of questions concerning the limiting value of \( \sqrt{a + \sqrt{a + \cdots}} \) for complex and negative values of \( a \). Some answers to these questions had already been given in Schuske and Thron 1961, 1962.

107. (1970) Michael S. Waterman, Some ergodic properties of multi-dimensional \( f \)-expansions. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 16, 77–103. MR0282939. Cited in Schweiger 2016.

From the introduction: “This paper is concerned with probabilistic aspects of the expansion of points in \( n \)-dimensional Euclidean space. The expansions we consider need not converge although previous work has required convergence... In 1869 Jacobi presented an extension of the continued fraction to two dimensions. Perron extended Jacobi’s work to \( n \)-dimensions. In 1964 Schweiger began an examination of the measure theoretic properties of Jacobi’s algorithm. It was this work which motivated our paper. However Schweiger has... published some results which also concern general \( F \)-expansions for \( n \)-dimensions. The class of algorithms he considers does not include the Jacobi algorithm and is a natural generalization of Rényi. Our results generalize most of Schweiger’s work and have the Jacobi algorithm as an example. We also include a central limit theorem and a law of the iterated logarithm.”

108. (1971) D. H. Lehmer, On the compounding of certain means. Journal of Mathematical Analysis and Applications 36, 183–200.

From the introduction: “By the compound of two means \( M(u, v) \) and \( M'(u, v) \) is meant the function \( M \times M'(u, v) \) defined as follows. Let the two sequences \( \{a_n\} \) and \( \{b_n\} \) be constructed recursively by

\[
\begin{align*}
  a_0 &= u, & b_0 &= v \\
  a_{i+1} &= M(a_i, b_i), & b_{i+1} &= M'(a_i, b_i).
\end{align*}
\]

Then \( M \times M'(u, v) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n \), provided both limits exist and are equal.” Lehmer defines two classes of means, \( \mu \) and \( M \) (both generalized from \( A, G \), and \( H \), the arithmetic, geometric and harmonic means, respectively) and considers compounds of means from each class. He develops a Taylor expansion for the mean \( R(u, v) = M_1 \times M_2(u, v) \) (where both are in \( M \), and \( M_1 = A \)), and shows that the series coefficients \( g_n \) are even integers which “satisfy recurrences vaguely like [those for] the Bernoulli numbers.” Using polynomials \( P_n(x) \) defined recursively by \( P_0(x) = 1 \) and \( P_{n+1}(x) = P_n^2(x) + 2x^2 \), he gives an alternate formulation for the \( g_n \). For
calculating the radius of convergence of $R$, he defines an additional polynomial $Q_n(x) = x^{2^n-1} P_n(1/x)$, and employs the following lemma: “The $2^n$ roots of $Q_{n+1}(x)$ are the $2^n$ values of the [continued square root]

$$-2 \pm \sqrt{-2 \pm \sqrt{-2 \pm \ldots \pm \sqrt{-2}},$$

where $n$ square roots are given signs in all $2^n$ ways.”

109. (1973) Leon Gerber, Complex iterated radicals. Proceedings of the American Mathematical Society 41, no. 1, 205–210. MR0318721. Cited in Walker 1983.

This is the first of at least three papers addressing the problems posed in Ogilvy 1970, all apparently unaware of the joint papers of Schuske and Thron from 1961 and 1962. From the abstract: “We prove the convergence of the sequence $S$ defined by $z_{n+1} = (z_n - c)^{1/2}$, $c$ real, for any choice of $z_0$. Let $k = \frac{1}{2} - c^{1/2}$. If $c < 0$ or $c = \frac{1}{2}$, $S$ has only one fixed point $w = \frac{1}{2} + k$ and converges to $w$ for any $z_0$. If $0 \leq c < \frac{1}{4}$, $S$ has the fixed points $w_1 = \frac{1}{2} + k$ and $w_2 = \frac{1}{2} - k$, and for any $z_0 \neq w_2$, $S$ converges to $w_1$. If $c > \frac{1}{4}$, $S$ has the fixed points $w_1 = \frac{1}{2} + ik$ and $w_2 = \frac{1}{2} - ik$ and converges to $w_1$ if $\text{Re}(z_0) \geq 0$ and to $w_2$ otherwise. We show that convergence is strictly monotone when the neighborhood system is the pencil of coaxial circles with $w_1$ and $w_2$ as limiting points, and give rates of convergence.” Ogilvy 1970 is the only reference cited.

110. (1973) R. K. Guy and J. L. Selfridge, The nesting and roosting habits of the laddered parenthesis. The American Mathematical Monthly 80, no. 8, 868–876. MR0347625.

This paper considers combinatorial aspects of the associativity of the continued exponential

$$a^{\cdot \cdot \cdot a}$$

with $k$ a’s, which, as the authors say, “is ambiguous until the order of the $k - 1$ operations has been indicated, say by the insertion of $k - 2$ pairs of parentheses.” (Although continued exponentials have been excluded from this bibliography, this reference is included because the nature of the underlying function used in this paper seems of less consequence than the issues of associativity that accompany compositions of multiple functions.)

111. (1974) Hanns-Walter Rohde, Complex iterated radicals. The American Mathematical Monthly 81, no. 1, 14–21. MR0338333. Cited in Walker 1983.

This is the second paper, essentially contemporaneous with Gerber 1973, to address the questions in Ogilvy 1970. The author defines a sequence $\{f_n\}$ inductively by $f_0(z) = 0$ and $f_n(z) = (a_n(z) + f_{n-1}(z))^n$, $n = 1, 2, 3, \ldots$, where the $a_n(z)$ are functions holomorphic in a simply connected domain $D$ of the complex plane, and the $r_n$ are complex numbers.
(Note that these \( f_n \) are defined as iterated compositions; see Section 2 above). Convergence of the \( f_n \) is established “in a simple way by means of a classical theorem due to Vitali . . . and the Monodromy theorem.” The questions in Ogilvy 1970 are addressed by letting \( r_n = \frac{1}{2} \) and \( a_n(z) = z \) for \( n = 1, 2, 3, \ldots \) Herschfeld 1935 is cited, but not the papers of Schuske and Thron from the early 1960s.

112. (1974) Robert Spira and Alexander Scheeline, Table errata: “A cotangent analogue of continued fractions” (Duke Mathematical Journal 4 (1938), 323–340) by D. H. Lehmer. Mathematics of Computation 28, 677. MR0345897.

The authors write: “A new calculation of D. H. Lehmer’s constant \( \xi \) to 1092D has revealed that the 80D value recorded on p. 334 of the original paper [Duke Mathematical Journal 4 (1938), 323–340; Zbl 19, 9] is correct to only 71D, corresponding to the eighth convergent to the infinite simple continued fraction appearing in equation (36) on the same page.”

113. (1975) M. S. Waterman, \( F \)-expansions of rationals. Aequationes Mathematicae 13, 263–268. MR0396457

From the abstract: “The ergodic theorem has been used to deduce results about the \( F \)-expansions of almost all \( x \) in \((0,1)\). A simple lemma from measure theory yields some corresponding statements about the expansions of the rationals, a set of measure zero.”

114. (1976) Jeffrey Shallit, Predictable regular continued cotangent expansions. Journal of Research of the National Bureau of Standards, Section B 80B, no. 2, 285–290. MR0429723. Cited in Rivoal 2007.

From the abstract: “Expansions of the form

\[
x = \cot(\arccot n_0 - \arccot n_1 + \arccot n_2 - \cdots)
\]

are discussed. It is shown that if \( x \) is of the form \( \frac{1}{2}(c + \sqrt{c^2 + 4}) \), then the \( n_i \) are predictable by a simple recurrence. Continued fractions derived from the expansion of \( x \) are also given.”

115. (1980) Kenneth B. Stolarsky, Mapping properties, growth, and uniqueness of Vieta (infinite cosine) products. Pacific Journal of Mathematics 89, no. 1, 209–227. MR0596932. Cited in Levin 2005.

From the abstract: “The natural logarithm of \( z \) can be written as an infinite product involving iterated square roots of \( z \). A Vieta product is defined to be a more general infinite product involving \( z \) raised to arbitrary fractional powers. Restricted to the unit circle, Vieta products generalize infinite cosine products studied . . . in connection with [Pisot-Vijayaraghavan] numbers. Vieta products are shown to have conformal mapping, monotonicity, and growth properties very similar to those of the natural logarithm. By using certain properties of Eulerian polynomials, the exponents
of \( z \) in a Vieta product are shown to be unique in a strong sense.” (Pisot-Vijayaraghavan or PV-numbers are real algebraic integers greater than 1 whose Galois conjugates have absolute value less than 1.)

116. (1983) Thomas P. Dence, Problem 1174. *Mathematics Magazine* **56**, 178. Solution by the Chico Problem Group, *ibid.* **57**, 1984, 299–300. Cited in Sizer 1986, Borwein and de Barra 1991, Jones 2008, and Efthimiou 2012.

The problem asks for real numbers \( A \) and \( k \) such that the sequence

\[
\sqrt{k}, \sqrt{k - \sqrt{k}}, \sqrt{k - \sqrt{k + \sqrt{k}}}, \sqrt{k - \sqrt{k + \sqrt{k - \sqrt{k}}}}, \ldots
\]

converges to \( A \), and to write \( k \) explicitly in terms of \( A \). The solution, adapted by the problems section editor, is as follows: “If \( k = 0 \), the sequence converges to 0. Otherwise, for all \( k \geq k_0 \approx 1.7548777 \) (where \( k_0 \) is the unique positive solution to \( k^3 - 2k^2 + k - 1 = 0 \)), the sequence converges to \( A = \sqrt{k - \frac{3}{4} - \frac{1}{2}}, \) whence \( k = A^2 + A + 1 \). Moreover, for each \( A \geq A_0 = \sqrt{k_0 - \frac{3}{4} - \frac{1}{2}} \approx 0.5024359 \), as well as for \( A = 0 \), but for no other value of \( A \), there is a (unique) value of \( k \) for which the sequence has limit \( A \).” Much the same question is asked, and the value 1.7548777 independently derived, in Zimmerman and Ho 2008.

117. (1983) Peter L. Walker, Iterated complex radicals. *The Mathematical Gazette* **67**, 269–273. Cited in Jones 1991.

From the introduction: “The question of the convergence of the sequence defined recursively by

\[ z_{n+1} = \sqrt{c + z_n} \quad (n \geq 0) \]

for a given complex value of \( c \), and a fixed determination of the square root was posed [in Ogilvy 1970] and partial solutions have been given [in Rohde 1974 and Gerber 1973]. It turns out that if \( c \) and \( z_n \), \( (n \geq 1) \) are chosen to have non-negative imaginary parts, then the sequence converges for any initial \( z_0 \). This note gives two solutions to the problem: the first uses some complex analytic machinery (Schwarz’[s] Lemma and the Riemann Mapping Theorem) while the second is entirely elementary, using only the properties of bilinear mappings.”

118. (1985) Edward L. Allen, Continued radicals. *The Mathematical Gazette* **69**, 261–263. Cited in Rao and Vanden Berghe 2005 and Clark and Richmond 2014.

Shows that, for any integer \( r > 1 \),

\[ \sqrt{a + \sqrt{a + \sqrt{\cdots}}} \]

converges to the unique positive root of \( x^r - x - a = 0 \); mentions that this root is \( k \) if \( a = k(k - 1) \) for \( r = 2 \) (compare Ogilvy 1949); and proves that
the partial products of the reciprocal of Viète’s formula for $\frac{2}{\pi}$, equation (8) above, are increasing and bounded. No literature is cited.

119. (1985) R. L. Andrushkiw, On the convergence of continued radicals with applications to polynomial equations. *Journal of the Franklin Institute* 319, 391–396. MR0790965. Cited in Johnson and Richmond 2008, Efthimiou 2012, Clark and Richmond 2014, and Lynd 2014.

The paper begins with an independent derivation of the result in Allen 1985 for the convergence of $\sqrt{a + \sqrt{a + \sqrt{\ldots}}}$ in the special case $a = 1$. It then develops the following generalization of Pólya 1916: Given $a_n \geq 0$, integers $r_n > 1$, let

$$\alpha = \limsup_{a_n \in S} \left( \frac{\log \log a_n}{n} \right), \quad S = \{ a_n \mid a_n > 1, n \in \mathbb{Z}^+ \},$$

$$r = \liminf_{n \to \infty} r_n, \quad R = \limsup_{n \to \infty} r_n.$$

Then

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}$$

converges if $\alpha < \log r$; diverges if $\alpha > \log R$; may converge or diverge if $\alpha \in [\log r, \log R]$; and converges if $a_n \in [0, 1]$ for all $n \in \mathbb{Z}^+$. (Compare Herschfeld 1935, Theorem III.) The author gives an error estimate for the approximants, and shows how roots of some polynomial equations can be approximated by continued $r$th roots. The only reference given is Pólya and Szegő 1925.

120. (1986) Walter S. Sizer, Continued roots. *Mathematics Magazine* 59, 23–27. MR0828417. Cited in Laugwitz 1990, Borwein and de Barra 1991, Pickover and Lakhtakia 1991, Jones 1991, Schoenefuss 1992, Jones 1995, Sizer and Wiedu 1996, Laugwitz and Schoenefuss 1999, Johnson and Richmond 2008, Zimmermann and Ho 2008a, Lim 2010, Efthimiou 2012, Gluzman and Yukalov 2012, Mukherjee 2013, Clark and Richmond 2014, Lynd 2014, Lesher and Lynd 2016, Vellucci and Bersani 2016c, and Weisstein n.d.

Presents an independent rediscovery of the continued square root convergence condition given in Wiernsberger 1904a (equation (37) above), and proves that any nonnegative real number can be represented as a continued square root with terms $\{a_i\}_{i=0}^\infty$ where, for $i \geq 1$, $a_i$ is 0, 1, or 2.

121. *(1987) R. B. Paris, An asymptotic approximation connected with the golden number. *American Mathematical Monthly* 94, 272–278. MR883295. Cited in Finch 2003.

The author writes, “The subject of this note is the repeated square root sequence $\{u_n\}$ defined by

$$u_1 = 1, \quad u_n = (1 + u_{n-1})^{1/2}, \quad (n \geq 2) \quad (54)$$
the \( n \)th term of which can be represented in the form of \( n - 1 \) nested square roots as
\[
  u_n = \sqrt{1 + \cdots + \sqrt{1 + \sqrt{1 + 1}}}.
\]

[...] Here, we shall not be concerned with the actual limiting value of (54) but with the considerably more subtle problem of the determination of the manner in which \( u_n \) approaches the limit \( \theta \) \( [= (1 + \sqrt{5})/2] \).” It is shown that
\[
  \theta - u_n \sim \frac{2K}{(2\theta)^n}
\]
where \( K \) is a constant (now known as the Paris constant) with approximate value 1.098630. In section 3, the sequence (54) is generalized “by taking \( u_1 = x \), where \( x \) denotes a real variable, and defining an iterated function \( u_n(x) \) by
\[
  u_1 = x, \quad u_n(x) = \left( \lambda + \mu u_{n-1}(x) \right)^{1/2}, \quad (n \geq 2)
\]
where \( \lambda \) and \( \mu \) are real constants. . . . If \( \mu^2 + 4\lambda \geq 0 \), the limit of the sequence, when it exists, is real and independent of \( x \) and is given by
\[
  u_{\infty} = \frac{1}{2}\mu + \frac{1}{2}(\mu^2 + 4\lambda)^{1/2}.
\]
The paper concludes with two special cases, one of which has \( \lambda = 2\mu^2 \) and \( \mu > 0 \), leading to a hypergeometric function which can be expressed in terms of elementary functions. In the particular case \( \lambda = \mu = \frac{1}{2} \), an algebraic form of Euler’s product (11) is derived as a function of \( x \), which in turn yields Viète’s formula (8) when \( x = 0 \).

122. (1988) Dario Castellanos, The ubiquitous \( \pi \), part I. Mathematics Magazine 61, 67–98. MR943543. Cited in Berndt 1994.

Section 2 derives Viète’s formula for \( \frac{2}{\pi} \) (equation (8) above), starting with a finite product formula for \( \sin \theta/\theta \), which becomes Euler’s formula (12) above (given in Rudio 1891) as the number of terms increase without bound. Viète’s formula is the case \( \theta = \frac{\pi}{2} \).

123. (1989) Bruce C. Berndt, Ramanujan’s Notebooks, Part II, Springer-Verlag New York, 107–112. MR0970033. Cited in Berndt and Bhargava 1993, Lorentzen 1998, Berndt, Choi, and Kang 1999, Levin 2005, Rao and Vanden Berghe 2005, and Vellucci and Bersani 2016c.

Entries 3, 4, 5 (given in several parts), and 6 in Chapter 12 of Ramanujan’s second notebook are identities and formulas related to continued square roots. Entries 3 and 4 are general formal identities, generated by successive substitution, of which the two formulas in Ramanujan 1911 are examples.
Generalizing from the examples of continued fractions and continued square roots, the author defines a *Kettenoperation*, or continued operation, as the sequence \( \{P_n\} \) defined by

\[
P_1 = f(p_1), \quad P_2 = f(p_1 + f(p_2)), \quad P_3 = f(p_1 + f(p_2 + f(p_3))), \ldots
\]

where the \( p_i \) are positive real numbers and \( f(x) > 0 \) for \( x > 0 \). He then explores convergence criteria for \( \{P_n\} \) when \( f(x) = x^\alpha \) for various real \( \alpha \).

He redisCOVERS the convergence condition ((37) and (50) above) for continued square roots, but also offers this generalization: Let \( f \) be a continuous and monotone nondecreasing for \( x > 0 \), with \( 0 < \alpha < 1 \). Moreover, let \( f(x) \) be strictly monotone increasing for \( x \geq x' \). Then a necessary and sufficient condition for the convergence of \( K_n=0 f(p_n) \) is that \( \lim \sup n f(p_n) < \infty \).

The author gives some rough convergence conditions for the cases \( f(x) = x^{-\alpha} \), where \( 0 < \alpha < 1 \) and \( 1 < \alpha \), but notes that his approach does not include continued fractions, for which \( \alpha = 1 \). Finally, he gives examples in which decreasing \( f \) yield divergent sequences.

The article begins by citing the identities (43) and (44) above, from Ramanujan 1911, and the continued square root of constants (19) (but traced back only to the solution of Ginsburg 1916). The authors continue:

"In this note we consider a general class of nested radicals with arithmetic progressions which includes \([19, 43, \text{and } 44 \text{ above}]\) as special cases. We deal with \( p \)-th roots, though more general monotone functions could be considered... We look for a finite nonnegative solution to the equation

\[
f(k) = \sqrt[k]{g(k) + k^\alpha f(k + a)}, \tag{55}
\]

where \( k, \alpha, a \geq 0, \ p > 1, \) and \( g : [0, \infty] \to [0, \infty] \) with \( y = \inf g > 0 \). We solve \( \text{(55)} \) iteratively. Take \( f_0, 0 < f_0 < g^{1/p} \) and define \( f_{n+1} \) by

\[
f_{n+1}^p(k) = g(k) + k^\alpha f_n(k + a).
\]

Then \( f_0 \leq g^{1/p} \leq f_1 \). Hence, inductively, if \( f_{n+1} \geq f_n \),

\[
f_{n+2}^p(k) - f_{n+1}^p(k) = k^\alpha [f_{n+1}(k + a) - f_n(k + a)] \geq 0,
\]

and \( f_n \) increases to \( f^* \) which takes values in \( (0, \infty) \) and which solves \( \text{(55)} \)."

A “growth condition” is then used to prove that \( f^* \) has the desired form.
The authors give several examples, then extend their method to a functional equation in two variables, and finally consider solutions to (55) which fail the growth condition.

126. (1991) John Gill, Inner composition of analytic mappings on the unit disk. *International Journal of Mathematics and Mathematical Sciences* **14**, no. 2, 221–226. MR1096859. Cited in Lorentzen 1995.

What we are here calling *continued compositions*, the author of this paper calls *inner compositions*. He writes, “In the present paper the following question is posed, and, to some extent, answered: Suppose each member of the sequence \( \{f_n\} \) is analytic on \( \text{Int}(D) \) and continuous on \( D \) with \( \text{Int}D \supseteq f(D) \) (it is not assumed that \( f_n \to f \)). Under what conditions does \( F(z) = f_1 \circ \cdots \circ f_n(z) \to \lambda, \) a constant, for all \( z \in D, \) as \( n \to \infty? \) Thus we are considering ‘inner’ compositions of essentially random sequences of functions mapping the unit disk into itself. Although our approach focuses on mappings of \( D \) into \( D, \) more general results are possible... We shall present several theorems describing conditions on the \( f_n \)s that imply \( F(D) \to \lambda. \)”

127. (1991) Dixon J. Jones, Continued powers and roots. *The Fibonacci Quarterly* **9**, no. 1, 37–46. MR1089518. Cited in Jones 1995, Laugwitz and Schoenfuss 1999, Mukherjee 2013, and Jones 2015.

Gives some convergence conditions for the continued \( p \)th power (5) above, when \( p \) and the terms \( a_i \) are nonnegative. It is shown that the continued \( p \)th power with nonnegative constant terms \( a_i = a \geq 0 \) converges if and only if

\[
a \geq 0 \quad \text{for } 0 < p < 1 ; \\
a = 0 \quad \text{for } p = 1 ; \text{ and} \\
0 \leq a \leq R \quad \text{for } p > 1 ,
\]

where

\[
R = \sqrt[p-1]{\frac{(p - 1)^{p-1}}{p^p}}.
\]

(These were given independently, and nearly simultaneously, in Schoenfuss 1992; similar calculations were made in Heymann 1894B.) For \( 0 < p < 1 \) and arbitrary nonnegative terms, a minor extension of Herschfeld’s convergence condition (50) (actually a special case of Herschfeld’s Theorem III) for continued \( r \)th roots is cited and proved. For \( p > 1, \) the continued \( p \)th power is shown to converge if \( \lim \sup_{i \to \infty} a_i < R, \) and to diverge if \( p \geq 1 \) and \( \lim \inf_{i \to \infty} a_i > R. \) A necessary and sufficient condition for convergence of

\[
b + 2(a + 2(b + 2(a + \cdots )))
\]
is given. Finally it is proved that, for $p > 1$, a continued $p$th power of positive terms $a_i$ converges if
\[ \frac{a_{i+1}^p}{a_i} \leq \frac{(p-1)^{p-1}}{p^p} \]
for all sufficiently large values of $i$.

128. (1991) C. A. Pickover and A. Lakhtakia, Continued roots in the complex plane. *Journal of Recreational Mathematics* **23**, 198–202.

The authors write, “[T]he striking beauty and complexity of the patterns resulting from ... iterative calculations has only recently been explored in detail, due in part to advances in computer graphics... The simple iterated functions discussed in the literature naturally lead to curiosity about the behavior of more complicated functions. We have found particularly visually interesting stability plots for
\[ z \to \sin(z + f^n(z)) \]
where $z$ is a complex variable, and $f^n(z)$ is a continued radical, and $n$ is the number of terms used in the nesting... In particular, we focus on the function defined recursively by $f^{n+1}(z) = \sqrt{z + f^n(z)}$, $n > 1$, with the initiator $f^1(z) = \sqrt{z}$. Julia sets of these iterated functions are depicted.

129. (1992) Lutz W. Schoenefuss, *Nichtautonome Differenzengleichungen und Kettenoperationen*, Mitteilungen aus dem mathematischen Seminar der Universität Giessen, no. 207, 110 pp. MR1156689. Cited in Laugwitz and Schoenefuss 1999, and Jones 2015.

From the introduction: “An infinite nested expression of the form
\[ f(a_0 + f(a_1 + f(a_2 + \ldots))) \]
which we call a continued operation, denoted by the symbol $K_{n=0}^\infty f(a_n)$, and its sequence of partial operations $f(a_0)$, $f(a_0 + f(a_1))$, $f(a_0 + f(a_1 + f(a_2)))$, etc., can be viewed as a generalization of infinite sums and continued fractions... [In] the first chapter of this work... we study the convergence of continued operations with monotonic functions $f$, in particular the family of functions $f(x) = x^\alpha$, $\alpha \in \mathbb{R}$. Besides numerous convergence results we obtain a complete survey of the convergence of $K_{n=0}^\infty a^n$, $\alpha \in \mathbb{R}$, for constant terms $a > 0$... Here we also make clear the special rôle of infinite sums.

“With a trick, the convergence question can be answered. Denoting the ‘kth tail’ of a convergent continued operation by $x_k$ for $k \in \mathbb{N}$:
\[ x_k = K_{n=k}^\infty f(a_n) \]
the numbers $x_k$ satisfy the nonautonomous difference equation
\[ x_k = f(a_k + x_{k+1}), \quad k \in \mathbb{N}. \]
Depending upon conditions on $f$, the convergence of $K_{n=0}^\infty f(a_n)$ is equivalent to the existence of a solution — or of a unique solution — to this
In certain cases a special asymptotic property distinguishes the one solution \( (x_n) \) — for which \( x_0 = K_{n=0}^\infty f(a_n) \) — from the other possible solutions of that equation: its minimality.

Thus one is lead in a completely natural way to the study of the general nonautonomous difference equation of first order

\[
x_{n+1} = F(n, x_n), \quad n \in \mathbb{N}.
\]

This happens in the third chapter, the heart of this work.

In order to get on the track of the asymptotic behavior of solutions of the equation, we examine first the limiting set of such a solution. Now the special case of an autonomous equation

\[
x_{n+1} = F(x_n), \quad n \in \mathbb{N}
\]

admits of three invariance characteristics of limiting sets, which permit a more exact localization. We can transfer these in an appropriate way to the general equation. Here we find our way using the corresponding approach from the theory of ordinary differential equations.

The transfer makes it necessary to define limit functions for the function \( F : \mathbb{N} \times D \to \mathbb{R}, \quad D \subseteq \mathbb{R} \). The totality of these we call the limiting set \( \Lambda(F) \). It is true that the formulation of the invariance results must be restricted to limit equations formed by limit functions, but nevertheless a beautiful structure develops, analogous to the limiting set \( L(x_0) \) of the solution sequence \( (x_n) \) — the limiting set \( \Lambda(F) \) of the function \( F \). This becomes more transparent if we use various facts from the theory of dynamical systems and results of the Ukrainian mathematician Sarkovskii.

In the fourth chapter we translate results about continued operations into the language of chapter 3. We explain how one can interpret the formation of a continued operation as a generalization of a summation procedure for linear difference equations.

The theory of the third chapter presupposes to a large extent so-called compactness characteristics of the function \( F \) and the solution \( (x_n) \). This makes more difficult the parallel development of this theory with the difference equations of the continued operations, because the interesting cases of the continued operations are usually those which lead to unbounded solutions and non-compact functions. But the remarks of the fourth chapter point also to a possible way out, namely stability analyses of the solutions.

The appendix consists of two parts: A supremely strange continued operation — the infinite exponential — is presented; then two continued radicals of Ramanujan are examined more precisely.

130. (1993) Alexander Abian and Sergei Sverchkov, Solutions and representations by iterated radicals. *International Journal of Mathematical Education in Science and Technology* 24, no. 3, 449–455. MR1229874. Cited in Rao and Vanden Berghe 2005.

From the abstract: “We give solutions of polynomial equations as well as some remarkable representations of functions as infinite iterations of
Many of the results are independent rediscoveries; for instance, the authors give continued $r$th root representations of roots of trinomial equations, and exhibit (without attribution) the Ramanujan identity (43) above. The main theorems involve an infinitely differentiable function $f$ having nonnegative $m$th derivatives for every $m = 0, 1, 2, \ldots$, and a function $g_n$ defined for every $n = 0, 1, 2, \ldots$ by

$$g_0(x) = f(x) \quad \text{and} \quad g_n(x) = \begin{cases} \frac{g_{n-1}^2(x) - g_{n-1}^2(0)}{x} & \text{for } x \neq 0 \\ (g_{n-1}(0))' & \text{for } x = 0. \end{cases}$$

With $a_n = g_{n-1}^2(0)$, it is proved that

$$f(x) = \sqrt[3]{a_1 + x} \sqrt[4]{a_2 + x} \sqrt[5]{a_3 + x} \cdots \sqrt[n]{a_n + xg_n(x)}.$$  \hfill (56)

If it is further assumed that the $m$th derivatives of $f$ are nonnegative on $(-\frac{1}{2}, \frac{1}{2})$ and $1 \leq g_n'(x) \leq g_n(x)$ for $n = 0, 1, 2, \ldots$, then

$$f(x) = \lim_{n \to \infty} \sqrt[3]{a_1 + x} \sqrt[4]{a_2 + x} \sqrt[5]{a_3 + x} \cdots \sqrt[n]{a_n + x} = \sqrt[3]{a_1 + x} \sqrt[4]{a_2 + x} \sqrt[5]{a_3 + x} \cdots.$$ \hfill (57)

131. (1993) Bruce C. Berndt and S. Bhargava, Ramanujan — for lowbrows. The American Mathematical Monthly 100, no. 7, 644–656. MR1237220. Cited in Berndt 1994 and Berndt, Choi, and Kang 1999.

In Section 3, the authors discuss the problems proposed in Ramanujan 1914 and Ramanujan 1915, including entries in Ramanujan’s notebooks pertaining to these and similar problems.

132. (1994) Bruce C. Berndt, Ramanujan’s Notebooks, Part IV, Springer-Verlag New York, 10–20. MR1261634. Cited in Berndt and Bhargava 1993, Berndt, Choi, and Kang 1999, Xi and Qi 2013, and Weisstein n.d.

The publisher’s blurb about this book states: “This is the first of two volumes devoted to proving the results found in the unorganized portions of [Ramanujan’s] second notebook and in the third notebook.” Entries 4 and 5 involve continued square roots. Entry 4 is a portion of the problem proposal Ramanujan 1914, which is solved by finding six nontrivial zeroes of a polynomial of degree 8; pages 11–17 of this book are devoted to rigorously calculating these zeroes. Entry 5 exhibits continued square roots that converge to these zeroes. The following two pages make use of two computer algebra packages to obtain closed form expressions.
133. (1995) Bruce C. Berndt and Robert A. Rankin, *Ramanujan: Letters and Commentary*, American Mathematical Society, Providence; London Mathematical Society, London. ISBN 9780821802878. MR1353909. Cited in Berndt 1994 and Berndt, Choi, and Kang 1999.

This book reproduces the text of a letter, dated 4 January 1928, from T. Vijayaraghavan to B. M. Wilson (one of the editors of Hardy, et al. 1927), “[r]egarding the justification of the formal processes in Ramanujan’s solution” of Ramanujan 1911. Vijayaraghavan continues, “I now do not remember whether I justified Ramanujan’s solution or simply established the criterion of convergency” (49) above. He then notes that “to prove that [equation (43) above] has the value 3 some additional remarks are necessary ... the only proof I can think of is unconscionably long and tedious ...” The remainder of the letter sketches this proof. In a commentary following the letter, the editors clarify and correct a few of Vijayaraghavan’s steps, and give a thumbnail biography of the man.

134. (1995) Dixon J. Jones, Continued powers and a sufficient condition for their convergence. *Mathematics Magazine* 68, no. 5, 387–392. MR1365650. Cited in Zimmerman and Ho 2008a, Gluzman and Yukalov 2012, Mukherjee 2013, Xi and Qi 2013, Kefalas 2014, Lynd 2014, and Jones 2015.

It is proved that, for real $p > 1$, the continued $p$th power $K_{i=0}^{\infty} a_p^i$ converges if $(a_n/R)^{p^n}$ is bounded, where

$$R = \frac{p - 1}{p^{p^{-1}}} = \frac{p^{-1} \sqrt{(p - 1)^{p-1}}}{p^p}.$$

(See Jones 1991 for an earlier appearance of the second form of $R$ above.) An error in Example III of this paper, noted by J. Nichols-Barrer in a Letter to the Editor, 69, no. 3, 238, is corrected in a Letter to the Editor, 69, no. 4, 316.

135. (1995) Lisa Lorentzen, A convergence question inspired by Stieltjes and by value sets in continued fraction theory. In Proceedings of the International Conference on Orthogonality, Moment Problems and Continued Fractions (Delft, 1994). *Journal of Computational and Applied Mathematics* 65 nos. 1–3, 233–251. MR1379134. Cited in Lorentzen 1998.

From the abstract: “Let $V$ be a subset of the complex plane $\mathbb{C}$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of self-mappings of $V$; i.e., $f_n(V) \subseteq V$. The question is then: Under what conditions will the sequence

$$F_n(w) := f_1 \circ f_2 \circ \cdots \circ f_n(w), \quad n = 1, 2, 3 \ldots$$

of composite maps converge to a constant function in $V$? In this paper we give a survey of some of the answers and open problems connected with this question.”
136. (1996) Walter S. Sizer and E. K. Wiredu, Convergence of some continued radicals in the complex plane. *Bulletin of the Malaysian Mathematical Society* 19, no. 1, 1–7. MR1465806. Cited in Johnson and Richmond 2008 and Mukherjee 2013.

Independently rediscovers the result from Schuske and Thron 1962 that the continued square root \( \sqrt[n]{a} \) above converges for all complex constants \( a \), where the principal value of the square root function is used. The authors note that, using a different branch of the square root function, there are values of \( a \) for which \( \sqrt[n]{a} \) does not converge. The MathSciNet review MR1465806 mistakenly displays the general continued square root \( \sqrt[n]{\sqrt[n]{\cdots\sqrt[n]{a}}/\cdots} \) above, instead of \( \sqrt[n]{a} \).

137. (1996) Dominique Tournès, *L’intégration approchée des équations différentielles ordinaires (1671-1914)*, thèse de doctorat de l’université Paris 7–Denis Diderot, June 1996. Reprinted by Presses Universitaires du Septentrion, Villeneuve d’Ascq, 1997, ISBN 9782284002093.

Chapter 4 of this doctoral thesis traces the history of “the method of successive approximations,” from ancient Greek times through the early 20th century. Subsection 2.1.1. gives a valuable appraisal of Schmidten 1821 in its historical context. Tournès writes, “In a dissertation published in 1821, Henri Gerner Schmidten gives us a panorama of the method of successive approximations and its possible applications in the various branches of analysis... [The] Schmidten text... shows that at the beginning of the 19th century the empirical technique of successive substitutions had already assumed the aspect of a very general fixed point problem perfectly theorized in its formal aspect.”

138. (1998) Lisa Lorentzen, Continued compositions of self-mappings. *Proceedings of the Second Asian Mathematical Conference 1995 (Nakhon Ratchasima)*, 34–38, World Sci. Publ., River Edge, NJ. MR1660544.

From the abstract: “Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of self-mappings of a set \( V \); i.e., \( f_n(V) \subset V \). Under what conditions will the sequence \( \{F_n(w)\}_{n=1}^{\infty} \) given by \( F_n(w) := f_1 \circ f_2 \circ \cdots \circ f_n(w) \) converge to a constant function in \( V \)? Answers to this question have applications in dynamical systems, Schur analysis, continued fractions and other similar structures like towers of exponentials and infinite radicals.” The author mentions the Denjoy-Wolff Theorem; notes the \( f_i \) which produce continued square roots and continued exponentials; states other results that pertain when the \( f_i \) are linear fractional transformations; and closes with a convergence theorem from a paper on continued exponentials (I. N. Baker and P. J. Rippon, Towers of exponents and other composite maps, *Complex Variables, Theory and Application: An International Journal* 12, no.1–4, 1989, 181–200).

139. (1999) Mihály Bencze, About Ramanujan’s nested radicals. *Octogon Mathematics Magazine* 7, no. 2, 15–21. MR1730807.
The note begins by quoting equation (47) above, citing PÓLYA AND Szegő 1925; it then give proof sketches of several common continued square root expressions, including (8), (43), (44) above, and a version of (68) below. The only other citation is the name “Kreyszig,” mentioned in connection with iteration of a real-valued function \( g(x) \) and the condition that such iteration converges if \( |g'(x)| < 1 \); this may refer to a result in a text by Erwin Kreyszig, but the result was known long before Kreyszig’s birth in 1922.

140. (1999) Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I*, Springer-Verlag, New York, ISBN 9780387989310. MR1721985. Originally published by McGraw-Hill, New York, 1978. Cited in Luo, Qi, Barnett, and Dragomir 2003.

In Section 8.4, on continued fractions and Padé approximants, the authors include a brief subsection on continued function representations. Examples are given of continued exponentials, continued square roots, and continued logarithms. Problems 8.32 to 8.36 develop convergence properties of these examples, with particular emphasis on continued exponentials.

141. (1999) Bruce C. Berndt, Youn-Seo Choi, and Soon-Yi Kang, The problems submitted by Ramanujan to the Journal of the Indian Mathematical Society. In *Continued fractions: from analytic number theory to constructive approximation. A volume in honor of L. J. Lange. Papers from the International Conference held at the University of Missouri, Columbia, MO, May 20–23, 1998.* Edited by Bruce C. Berndt and Fritz Gesztesy. Contemporary Mathematics, 236. American Mathematical Society, Providence, RI, ISBN 9780821812006. MR1665358 (collection) and MR1665361 (this paper).

Gives a detailed overview of the 58 problems Ramanujan submitted to the *Journal of the Indian Mathematical Society* between 1911 and 1919, including Ramanujan 1911, 1914, 1915 listed above.

142. (1999) Detlef Laugwitz and Lutz Schonefuss, Convergence of continued operations. In *Iteration Theory (ECIT ’96)*, Proceedings of the European Conference on Iteration Theory, edited by L. Gardini, et al., *Grazer Mathematische Berichte* 339, 243–250. MR1748827. Cited in Jones 2015.

From the summary: “For a monotonic bijection of \( \mathbb{R}_+ \) and a sequence \( a_n > 0 \) we consider both the continued operation

\[
f(a_0 + f(a_1 + f(a_2 + \cdots)))
\]

and the difference equation \( x_n = f(a_n + x_{n+1}) \). In Section 2, the convergence of the continued operation is discussed for \( a_n = c \), and is completely settled for \( f(x) = x^r \), \( r \in \mathbb{R} \), by means of fixed point iterations. The general case is considered in Section 3 (increasing \( f \)) and Section 4 (decreasing \( f \)), and results on the convergence of the continued operation are obtained.
in terms of positive solutions of the difference equation. Some conclusions can be drawn on the existence and uniqueness of positive solutions of this nonautonomous equation.”

143. (1999) Thomas J. Osler, The union of Vieta’s and Wallis’s products for pi. *The American Mathematical Monthly* **106**, no. 8, 774–776. MR1718586. Cited in Levin 2005.

Wallis’s product is

\[
\frac{2}{\pi} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdots.
\]  

(58)

The author shows that this expansion and Viète’s formula for \(\frac{2}{\pi}\), equation (8) above, are special cases of the following product:

\[
\frac{2}{\pi} = \prod_{n=1}^{\infty} \left( \frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \cdots + \frac{1}{2} \sqrt{\frac{1}{2}}}} \right) \
\times \prod_{n=1}^{\infty} \frac{2^{p+1}n - 1}{2^{p+1}n} - \frac{2^{p+1}n + 1}{2^{p+1}n}.
\]  

(59)

144. (2000) Juan Carlos Cortés López and Juan Ángel Aledo Sánchez, Cálculo geométrico del límite de sucesiones trigonométricas. *SUMA* **34**, 53–58. Cited in Cortés López 2000.

From the abstract: “At present, the calculation of the limit of a sequence, both in high school and in the first university courses, has been carried out from an exclusively analytical-algebraic approach. In this article we propose a rich geometric approach to initiate this theme in the classroom, particularizing our proposal for the trigonometric sequences.” By “trigonometric sequences” the authors mean not only the continued square roots arising from trig functions, but also limits like \(\lim_{x \to 0} \frac{\sin x}{x}\), all of which receive geometric interpretations.

145. (2000) Juan Carlos Cortés López, Algunas representaciones radicales infinitas de los números naturales. *Boletín Sociedad “Puig Adam”* **54**, 29–38. MR1751288.

The author independently rediscovers the two continued square root expansions of Ramanujan 1911 and their underlying structure, then extends this to cube roots and \(i\)th roots, and ultimately shows how any positive integer may be represented. The only reference is to a 1999 preprint later published as Cortés López and Aledo Sánchez 2000.

146. (2002) G. Kuba and J. Schoissengeier, Dyadische Kettenwurzeln. *Wissenschaftliche Nachrichten. Herausgegeben vom Bundesministerium für Bildung, Wissenschaft und Kultur* **119**, 23–24.
Citing two of Pincherle’s papers from 1917–1918 and Pólya and Szegő 1925, the authors write, “The aim of this note is to prove that every real number \(z\) can be represented in the interval \([0, 2]\) by means of dyadic continued square roots.” They take an unusual, symbolic approach to these representations: “Let \(\Sigma = \{+,-\}^N\) be the set of all sequences whose \(n\)th term \(\sigma(n)\) is either a plus or a minus symbol. For any sequence \(\sigma \in \Sigma\) let \(W(\sigma)\) be given by

\[
W(\sigma) := \sqrt{2\sigma(1)}\sqrt{2\sigma(2)}\sqrt{2\sigma(3)}\sqrt{2\sigma(4)}\sqrt{2} \cdots ,
\]

where we first assume that the expression \(W(\sigma)\) represents a well-defined real number for each \(\sigma \in \Sigma\).”

147. (2002) L. D. Servi, Problem 10973. *The American Mathematical Monthly* **109** (9), 854. Solution by Richard Stong, *ibid.* **111**, no. 7, 627–628.

The problem states: “With \(R_k(n)\) defined as below, prove that \(\lim_{k \to \infty} R_k(2)/R_k(3) = \frac{3}{2}\).

\[
R_k(n) = \sqrt[2]{2} - \sqrt[2]{2 + \sqrt[n]{2 + \cdots + \sqrt[2]{n}}} .
\]

148. †(2003) Steven R. Finch, *Mathematical Constants*. Cambridge University Press, Cambridge, England, 7–9. ISBN 0-521-81805-2. MR2003519.

Section 1.2 mentions the continued square root having terms \(a_i = 1\); subsection 1.2.1 analyzes its convergence (as an iterated function composition, using methods given in PARIS 1987) to the golden ratio \(\frac{1}{2}(1 + \sqrt{5})\); the subsection also alludes to the continued square root having terms \(a_i = i\) (whose limit is called the Kasner number in HERSCHELF 1935). Subsection 1.2.2 analyzes the continued cube root of terms \(a_i = 1\) (whose limit is called the plastic constant in LIM 2010).

149. (2003) Clemens Hauser, Pi, e und Kettenwurzeln. *Der mathematische und naturwissenschaftliche Unterricht* **56**, no. 4, 201–203.

From the abstract: “If we approach the circumference of the unit circle by means of a regular polygon, we obtain attractive representations of \(\pi\) in the form of continued square roots. Only the mathematics of the intermediate level are needed for this. Further investigations of continued square roots can be envisaged in the advanced level. Here, a connection to Euler’s
number $e$ can be established.” The author derives the long-established

$$
\pi = \lim_{n \to \infty} 2^n \sqrt[n]{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}},
$$

found at least as far back as Catalan 1842. The number $e$ arises in connection with the expression

$$
\ln 2 = \lim_{n \to \infty} 2^n \sqrt[n]{\sqrt{2 + \cdots + \sqrt{2 + 2.5 - 2}}},
$$

which is obtained by iterating hyperbolic trigonometric functions.

150. (2003) Qiu-Ming Luo, Feng Qi, Neil S. Barnett, Sever S. Dragomir, Inequalities involving the sequence $\sqrt[n]{a + \sqrt[n]{a + \cdots + \sqrt[n]{a}}}$. Mathematical Inequalities and Applications 6, no. 3, 413–419. MR1992481. Cited in Xi and Qi 2013.

From the abstract: “The convergence of the sequence [with terms]

$$
\sqrt[n]{a + \sqrt[n]{a + \sqrt[n]{a + \cdots + \sqrt[n]{a}}}}
$$

is proved, and some inequalities involving this sequence are established for $a > 0$. As by-product[s], two identities involving irrational numbers are obtained. Two open problems are proposed.” The authors define

$$
S_{n,t}(a) = \sqrt[n]{a + \sqrt[n]{a + \sqrt[n]{a + \cdots + \sqrt[n]{a}}}}
$$

and

$$
f_{n,t}(a) = \frac{a - S_{n+1,t}}{a - S_{n,t}},
$$

and derive upper and lower bounds for $f_{3,t}(a)$ for ranges of positive $a$.

151. (2003) L. D. Servi, Nested square roots of 2. The American Mathematical Monthly 110, no. 4, 326–330. MR1984573. Cited in Nyblom 2005, Jones 2008, Zimmerman and Ho 2008a, Epthimiou 2012, Nyblom 2012, Moreno and García-Caballero 2012, 2013a, and 2013b, Senadhheera 2013, García-Caballero, Moreno, and Prophet 2014a, 2014b, and 2014c, Lynd 2014, and Lesher and Lynd 2016, and Vellucci and Bersani 2016c.

Another independent rediscovery of continued square roots derived from iterated trigonometric functions. Compare Wiernsberger 1904b, Ci-polla 1908, Pólya and Szegő 1925, Myrberg 1958, and others above.
152. (2004) Adriana Berechet, Solving a conjecture about certain $f$-expansions. *Proceedings of the Romanian Academy – Series A: Mathematics, Physics, Technical Sciences, Information Science* 5, no. 3, 231–235. MR2122307.

From the abstract: “The conjecture asserts that the equivalence of the label sequence of the regular continued fraction expansion to the sequence $(\xi_n)_{n \in \mathbb{N}^+}$ associated with it ... still holds for the label sequence of any $f$-expansion satisfying [two conditions]. We prove that [the first] condition and a strengthening of a Lipschitz condition ... are sufficient to ensure a necessary and sufficient condition under which the asserted equivalence holds. The proof involves processes on several probability spaces and some associated dynamical systems relating the $f$-expansion ... to [random variables] on the probability space used in the concluding theorem.”

153. (2004) Greg Martin, The unreasonable effectualness of continued function expansions. *Journal of the Australian Mathematical Society* 77, no. 3, 305–319. MR2099803. Cited in Jones 2015.

From the introduction: “[W]e focus on the $f$-expansions introduced [in Bissinger 1944]... The purpose of this paper is to demonstrate that the function $f$ can be chosen so that the expansions of prescribed real numbers can have essentially any desired behavior. The following results, listed in roughly increasing order of unlikeliness, are representative of what we can prove.

**Theorem 1.** For any two real numbers $x, y \in (0,1)$, there exists a function $f$ such that the $f$-expansion of $x$ is the same as the usual continued fraction expansion of $y$.

**Theorem 2.** There exists a function $f$ such that the $f$-expansion of any rational or quadratic irrational terminates.

**Theorem 3.** There exists a function $f$ such that the $f$-expansion of a real number $x$ is periodic if and only if $x$ is a cubic irrational number.

**Theorem 4.** There exists a function $f$ such that, simultaneously for every integer $d \geq 1$, a real number $x$ is algebraic of degree $d$ if and only if the $f$-expansion of $x$ terminates with the integer $d + 1$.

“[W]e should confess what the reader might already suspect, that the functions giving the nice behaviors of Theorems 1–4 are infeasible for actual computations. Indeed, the existence of such functions is essentially a consequence of the existence of continuous functions on the interval $(0,1)$ with certain properties.”

This confession notwithstanding, the paper concludes with some computations involving functions $f_\alpha$ defined by

$$a_0 + (a_1 + (a_2 + (a_3 + (a_4 + \cdots)^{-\alpha})^{-\alpha})^{-\alpha})^{-\alpha},$$

where $\alpha > 0$. (For $\alpha = \frac{1}{2}$, this has been called a continued reciprocal square root.) Questions and conjectures are posed about the termination of such expansions.
154. (2005) Aaron Levin, A new class of infinite products generalizing Viète’s product formula for \( \pi \). *The Ramanujan Journal* 10, no. 3, 305–324. MR2193382. Cited in Levin 2006, Moreno and García-Caballero 2013a and 2013b, Senadheera 2013, García-Caballero, Moreno, and Prophet 2014a, Nishimura 2015b, Nishimura 2016, and Osler 2016.

From the abstract: “We show how functions \( F(z) \) which satisfy an identity of the form \( F(\alpha z) = g(F(z)) \) for some complex number \( \alpha \) and some function \( g(z) \) give rise to infinite product formulas that generalize Viète’s product formula for \( \pi \). Specifically, using elliptic and trigonometric functions we derive closed form expressions for some of these infinite products. By evaluating the expressions at certain points we obtain formulas expressing infinite products involving nested radicals in terms of well-known constants. In particular, simple infinite products for \( \pi \) and the lemniscate constant are obtained.”

155. (2005) M. A. Nyblom, More nested square roots of 2. *The American Mathematical Monthly* 112, no. 9, 822–825. MR2179862. Cited in Zimmerman and Ho 2008a, Efthimiou 2012, Nyblom 2012, Moreno and García-Caballero 2012 and 2013b, Senadheera 2013, García-Caballero, Moreno, and Prophet 2014a and 2014c, Lynd 2014, and Vellucci and Bersani 2016c.

This note’s principle result is: If \( x \geq 2 \) and \( k \) is a positive integer, then

\[
R_k(x) = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + x}}} \\
= \left( \frac{x + \sqrt{x^2 - 4}}{2} \right)^{1/2^k} + \left( \frac{x + \sqrt{x^2 - 4}}{2} \right)^{-1/2^k},
\]

and

\[
\tilde{R}_k(x) = \sqrt{-2 + \sqrt{2 + \cdots + \sqrt{2 + x}}} \\
= \left( \frac{x + \sqrt{x^2 - 4}}{2} \right)^{1/2^k} - \left( \frac{x + \sqrt{x^2 - 4}}{2} \right)^{-1/2^k},
\]

where there are \( k \) square roots in each of \( R_k \) and \( \tilde{R}_k \).

156. (2005) K. Srinivasa Rao and G. Vanden Berghe, On an entry of Ramanujan in his notebooks: a nested square root expansion. *Journal of Computational and Applied Mathematics* 173, no. 2, 371–378. MR2102903.

From the abstract: “In this letter, the elementary result of Ramanujan for nested roots, also called continued or infinite radicals, for a given integer \( N \), expressed by him as a simple sum of three parts \( (N = x + n + a) \) [see equation (68) below] is shown to give rise to two distinguishably different expansion formulas. One of these is due to Ramanujan and surprisingly, it is this other formula, not given by Ramanujan, which is more rapidly convergent!”
The author begins by comparing the Viète formula for \(\frac{2}{\pi}\) (equation (8) above) with
\[
\frac{2}{L} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2} + \cdots}}}}}
\] (60)

where
\[
L = \frac{B\left(\frac{1}{4}, \frac{1}{4}\right)}{2\sqrt{2}} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}} = 2.6220575542 \ldots
\]
is the lemniscate constant, \(B(x, y)\) is the beta function, and \(\Gamma(z)\) is the gamma function. The author continues, “We will see that the area enclosed by the curve \(C_4\) defined by \(x^4 + y^4 = 1\) is \(L\sqrt{2}\), and this will allow us to give a geometric meaning to the product formula (60)… We will show that the similarity between equations (8) and (60) goes beyond mere typographical appearances. We will see that (60) is related to the curve \(C_4\) in much the same way that Viète’s product is related to the circle.”

From the introduction: “The difference of two squares, \(x^2 - y^2 = (x + y)(x - y)\) and its immediate generalisations \(x^3 - y^3 = (x - y)(x^2 + xy + y), \ldots\), are among the most elementary identities. It is the purpose of this note to show that these simple formulas are at the heart of a number of advanced products, some finite, and some infinite. The reader may find some of these surprising, especially those involving trigonometric functions.” Viète’s formula (8) above is one of the products derived.

Let \(\mathbb{R}[x]\) be the ring of polynomials in \(x\) with real coefficients. The author writes: “We consider infinite nested radicals in which the arguments are positive polynomial sequences. It is shown that the evaluation of such a nesting is always finite, and we prove necessary and sufficient conditions for the evaluation to be a finite polynomial… In Section 2, we characterize when an infinite nested radical involving polynomials from \(\mathbb{R}[x]\) has a simple closed form as another polynomial in \(\mathbb{R}[x]\).” In Section 3 it is proved that,
for all positive polynomial sequences $a_n, b_n$, the limit

$$\lim_{n \to \infty} \sqrt[n]{a_1 + b_1 \sqrt[n]{a_2 + \cdots + b_{n-1} \sqrt[n]{a_n + b_n}}}$$

exists and is finite.

160. (2007a) Teik-Cheng Lim, Significance of an infinite nested radical number and its application in van der Waals potential functions. *MATCH Communications in Mathematical and in Computer Chemistry* 57, no. 3, 549–556. MR2337287.

From the abstract: “This paper demonstrates the relations between a new mathematical constant and its relevance to the molecular potential energy function commonly adopted in computational chemistry softwares. This mathematical constant, $1.7767750401$ (correct up to 12 decimal places), which fulfills the following infinite nested radical equation

$$n = \sqrt[n]{1 + \sqrt[n]{1 + \sqrt[n]{1 + \cdots}}}$$

$$= \sqrt[n]{n + \frac{1}{n} \times \sqrt[n]{n + \frac{1}{n} \times \sqrt[n]{n + \frac{1}{n} \times \cdots}},}$$

is shown to be applicable as the indices of a generalized Lennard-Jones potential energy function. This new potential function demonstrates very good agreement with the various versions of specific Lennard-Jones potential energy functions and the Buckingham potential function converted from the Lennard-Jones (12-6) function when the indices are positive integer multiples of the mathematical constant or when the indices are raised to the first four positive integer powers.”

161. (2007b) Teik-Cheng Lim, Infinite nested radicals: a mathematical poem. *The American Mathematical Monthly* 114, no. 3, 182. Cited in Lim 2010.

The “poem” consists of 9 identities, in as many lines, of the form

$$\sqrt[n]{n \times ((n + 1)^n - 1) + \sqrt[n]{n \times ((n + 1)^n - 1) + \cdots}}$$

$$= \sqrt[n]{n + ((n + 1)^n - 1) \times \sqrt[n]{n + ((n + 1)^n - 1) \times \cdots}}$$

for $n = 2, \ldots, 10$.

162. (2007) M. A. Nyblom, On the evaluation of a definite integral involving nested square root functions. *Rocky Mountain Journal of Mathematics* 37, number 4, 1301–1304. MR2360300. Cited in Nyblom 2012.

From the introduction: “[W]e shall in this paper take advantage of a structural feature of a class of nested square root functions, in order that we may evaluate their corresponding definite integrals which on first acquaintance appear rather intractable. In particular the functions in question will
be composed of a finite product of reciprocals of the form
\[ R_N(x) = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + x}}} \]
in which \( R_N(x) \) consists of \( N \) nested square roots. By restricting the intervals of integration to finite subintervals of \([2, \infty)\), we shall see that a simple application of a hyperbolic function substitution together with some standard identities will result in closed-form expressions for these definite integrals.” The main theorem is that, for \( \xi \geq 2 \) and \( N \) a positive integer,
\[
\frac{1}{2^N} \int_2^\xi \frac{dx}{\sqrt{2 + x} \cdot \sqrt{2 + \sqrt{2 + x}} \cdots \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + x}}}} = \left( \frac{\xi + \sqrt{\xi^2 - 4}}{2} \right)^{1/2^N} + \left( \frac{\xi + \sqrt{\xi^2 - 4}}{2} \right)^{-1/2^N} - 2.
\]
where the last expression in the denominator of the integrand is \( R_N(x) \).

163. (2007) Thomas J. Osler, Vieta-like products of nested radicals with Fibonacci and Lucas numbers. The Fibonacci Quarterly 45, no. 3, 202–204. MR2437033. Cited in García-Caballero, Moreno, and Prophet 2014a and 2014c.

From the abstract: “We present two infinite products of nested radicals involving Fibonacci and Lucas numbers. These products resemble Vieta’s classical product of nested radicals for \( \frac{\pi}{4} \) [equation (9) above]. A modern derivation of Vieta’s product involves trigonometric functions, while our product involves similar manipulations involving hyperbolic functions.”

164. (2007) T. Rivoal, Propriétés diophantiennes du développement en cotangent continue de Lehmer. Monatshefte für Mathematik 150, no. 1, 49–71. MR2297253. Cited in Schweiger 2016.

From the English abstract: “This article deals with an algorithm [Lehmer 1938] which enables us to write any real positive number as the sum of an alternating series of cotangents of integers \( n_{\nu}, \nu \geq 0 \), in a unique way. We continue the work begun by Lehmer and continued by Shallit: amongst other things, we give explicitly the link between the rational approximations of a given real number coming from this algorithm and the usual convergents of the same real number and we produce a quasi-optimal bound for the growth of the sequence \( (n_{\nu})_{\nu \geq 0} \) associated to an algebraic number. We also determine the regular continued fractions of an exceptional class of continued cotangent developments, which enables us to produce optimal irrationality measures of these expansions.”

165. (2007) Jonathan Sondow and Petros Hadjicostas, The generalized-Euler-constant function \( \gamma(z) \) and a generalization of Somos’s quadratic recurrence constant. Journal of Mathematical Analysis and Applications 332, no. 1, 292–314. MR2319662.
In Section 3, the authors define the generalized Somos constant $\sigma_t$, $t > 1$:

$$\sigma_t = \sqrt{1/\sqrt{2/\sqrt{3/\cdots}}} = 1^{1/t}2^{1/t^2}3^{1/t^3} \cdots = \prod_{n=1}^{\infty} n^{1/t^n}.$$  

This is shown to be related to the Ramanujan 1911 continued square root above.

166. (2008) Jamie Johnson and Tom Richmond, Continued radicals. *The Ramanujan Journal* **15**, no. 2, 259–273. MR2377579. Cited in Efthimiou 2012, Clark and Richmond 2014, and Lynd 2014, and Vellucci and Bersani 2016c.

From the abstract: “We consider the set of real numbers $S(M)$ representable as a continued [square root] whose [non-negative] terms $a_1, a_2, \ldots$ are all from a finite set $M$. We give conditions on the set $M$ for $S(M)$ to be (a) an interval, and (b) homeomorphic to the Cantor set.” The authors also derive an upper bound on the derivative of a finite approximant to a continued $r_t$th root.

167. (2008) Dixon J. Jones, Letter to the Editor. *Mathematics Magazine* **81**, no. 3, 230.

Provides some additional references for Zimmerman and Ho 2008a, and points out that the continued square roots derived from trigonometric identities have been independently rediscovered many times over the past century.

168. (2008) Teik-Cheng Lim, Two infinite nested radical constants. *The Mathematical Gazette*, **92**, no. 523, 96–97.

Computes constants $m = 0.4758608124$ and $n = 2.398384383$ (correct to 10 significant figures) satisfying

$$m = \sqrt[\infty]{m \cdot \sqrt[\infty]{m \cdot \sqrt[\infty]{m + m \cdot \sqrt[\infty]{\cdots}}}}$$

and

$$n = \sqrt[\infty]{n \cdot \sqrt[\infty]{n \cdot \sqrt[\infty]{n + n \cdot \sqrt[\infty]{\cdots}}} = \sqrt[\infty]{n \cdot n \cdot \sqrt[\infty]{n \cdot n \cdot \sqrt[\infty]{\cdots}}}$$

169. (2008a) Seth Zimmerman and Chungwu Ho, On infinitely nested radicals. *Mathematics Magazine* **81**, no. 1, 3–15. MR2380054. Cited in Efthimiou 2012, Efthimiou 2013, Mukherjee 2013, Senadheera 2013, Garcia-Caballero, Moreno, and Prophet 2014b, Lynd 2014, Lesher and Lynd 2016, and Vellucci and Bersani 2016c.

The authors write: “Is it possible to write any arbitrary integer, rational number, or indeed $\pi$ or $e$ as the limit of some sequence of nested radicals? And if an integer $k$ is such a limit, how many different sequences of radicals will converge to $k$? Although there seems to be some revived interest in this topic... previous research has not considered these questions... In this
paper we will make a systematic study of nested radicals, answering many such questions and suggesting further lines of research for the interested reader.” Eight references are cited.

170. (2008b) Seth Zimmerman and Chungwu Ho, Erratum: On Infinitely Nested Radicals. *Mathematics Magazine* **81**, no. 3, 190. MR2422950.

The journal editors write: “Authors Chungwu Ho and Seth Zimmerman have written to point out an error on page 14 of their paper ‘On Infinitely Nested Radicals,’ this *Magazine*, Vol. 81, February 2008: The gaps mentioned for the set $S_2$ do not exist. Gaps exist only for sets $S_a$ with $a \geq 3$.”

171. (2010) Thomas Koshy, Generalized nested Pell radical sums. *Bulletin of the Calcutta Mathematical Society* **102**, no. 1, 37–42. MR2680839.

From the abstract: “This article presents an extended Pell family of polynomial functions $g_n(x)$ which includes the well known Fibonacci, Lucas, Pell, and Pell-Lucas polynomials $f_n(x)$, $\ell_n(x)$, $p_n(x)$, and $q_n(x)$, respectively. It investigates the convergence of the sequence $\{S_n(x)\}$ of nested radical sums [continued square root approximants], where

$$S_n(x) = \sqrt{g_1(x) + \sqrt{g_2(x) + \sqrt{g_3(x) + \cdots + \sqrt{g_n(x)}}}};$$

it shows that the sequence converges and $\lim_{n \to \infty} S_n(x) < \lambda(\gamma)$, where $\gamma = \gamma(x) = x + \sqrt{x^2 + 1}$, $\gamma = \gamma(x) = (b\gamma + a)/\sqrt{5}$, $a = a(x) = g_0(x)$, and $b = b(x) = g_1(x)$.”

172. (2010) Teik-Cheng Lim, Continued nested radical fractions. *Mathematical Spectrum* **42** (2009-2010), 59–63. Cited in Lim 2011.

From the abstract: “We define a continued nested radical fraction (CNRF) as a hybrid of a continued fraction and a nested radical:

$$\text{CNRF} = a_0 + \frac{b_1}{a_1 + \frac{b_2}{\sqrt{a_2 + \cdots}}},$$

The note presents CNRF representations of the golden ratio using Fibonacci numbers as terms; of the “silver ratio” $1 + \sqrt{2}$ using Pell numbers as terms; and of the “plastic constant” $1.32471957\ldots$ with all terms equal to 1. The manipulations are purely formal; no rigorous convergence criteria are mentioned. Tables of values are given to support the claim of the CNRF’s superior convergence versus continued fraction and continued radical approximations. (The author’s “CNRF” is a form of the continued reciprocal root discussed in Jones 2015; see also Gütther 1880, Laugwitz 1990, Schoenefuss 1992, and Laugwitz and Schoenefuss 1999.)
173. (2011) John Gill, A mathematical note: convergence of infinite compositions of complex functions. Online at www.researchgate.net, accessed 2017-07-17.

From the abstract: “Inner Composition of analytic functions \((f_1 \circ f_2 \circ \cdots \circ f_n)(z)\) and Outer Composition of analytic functions \((f_n \circ f_{n-1} \circ \cdots \circ f_1)(z)\) are variations on simple iteration, and their convergence behaviors as \(n\) becomes infinite may reflect that of simple iterations of contraction mappings \((\phi\) defined on a simply-connected domain \(S\) with \(\phi(S) \subset \Omega \subset S\), \(\Omega\) compact). Several theorems are combined to give a summary of work in this area. In addition, recent results by the author and others provide convergence information about such compositions that involve functions that are not contractive, and in some cases, neither analytic nor meromorphic.”

174. (2011) Teik-Cheng Lim, Alternate continued nested radical fractions. Mathematical Spectrum 43 (2010-2011), 55–59.

Following the conventions established in Lim 2010, the author presents expansions of the golden ratio, silver ratio, plastic constant, and other real numbers using continued nested radical fractions of constant or periodic terms. The manipulations are again formal, with no convergence conditions given.

175. (2011) Samuel Gómez Moreno, Proposed Mayhem Problem M487. Crux Mathematicorum with Mathematical Mayhem 37, 137.

The problem states: “Let \(m\) be a positive integer. Find all real solutions to the equation

\[ m + \sqrt{m + \sqrt{m + \cdots + \sqrt{m + \sqrt{x}}} = x,} \]

in which the integer \(m\) occurs \(n\) times.”

176. (2012) Costas J. Efthimiou, A class of periodic continued radicals. The American Mathematical Monthly 119, no. 1, 52–58. MR2877666. Cited in Efthimiou 2013, Moreno and García-Caballero 2013b, Mukherjee 2013, Senadheera 2013, Clark and Richmond 2014, Lynd 2014, and Vellucci and Bersani 2016c.

The author writes: “In this brief article we find the values for a class of periodic continued radicals of the form

\[ a_0 \sqrt{2 + a_1 \sqrt{2 + a_2 \sqrt{2 + a_3 \sqrt{2 + \cdots}}},} \]

where for some positive integer \(n\),

\[ a_{n+k} = a_k, k = 0, 1, 2, \ldots, \]

and

\[ a_k \in \{-1, +1\}, k = 0, 1, 2, \ldots, n - 1. \]
Chebyshev polynomials are invoked; compare the virtually simultaneous Moreno and García-Caballero 2012 below. For continued square roots of the form studied here, compare Bochow 1899, Cipolla 1908, Pólya and Szegő 1925, and Wollan and Mesner 1956, 1957.

From the introduction: “There exists a more general approach for extrapolating asymptotic series in powers of a small parameter, or a variable, to finite and even infinite values of such variables. This approach is based on the self-similar approximation theory... In the frame of this theory, we have developed the methods of extrapolating asymptotic series by using several types of self-similar approximants, such as optimized approximants, nested exponentials, nested roots, iterated roots, and factor approximants...”

“In the present Letter, we advance a novel type of self-similar approximants that may be called self-similar continued radical approximants, or, for short, self-similar continued radicals. In a particular case, these continued radicals reduce to continued fractions... and, respectively, to Padé approximants. But, generally, their form is different and not reducible [sic] to continued fractions. The self-similar continued radicals could be transformed into expressions of the type of the numerical nested radicals..., which, however, is not convenient for the extrapolation procedure applied to functions.

“In Section 2, we explain how the self-similar continued radicals arise in the process of the self-similar renormalization of asymptotic series and prove the convergence of these root approximants. In Section 3, we demonstrate, by several examples from condensed-matter physics, that the continued radicals [continued $r$th roots] can be employed as approximants extrapolating asymptotic series and providing good accuracy. Possible generalizations for the continued radical approximants are also mentioned.”

The authors investigate the continued $p$th power

$$\left(1 + A_1 x(1 + A_2 x(\cdots(1 + A_k x)^p \cdots))^p \right)^p,$$

noting that the case $p = -1$ yields continued fractions; ultimately the restriction $|p| < 1$ is imposed. The continued $p_i$th power

$$\left(1 + A_1 x(1 + A_2 x(\cdots(1 + A_k x)^{p_{k_i}} \cdots)^{p_{i-1}})^{p_i} \right)^{p_0}$$

is mentioned in passing.

From the abstract: “The purpose of this note is to report a curious relation between the Chebyshev polynomials of degree $2n$ in a complex...”
variable and the nested square roots of depth $n$ of the form

$$\pm \sqrt{2 \pm \sqrt{2 \pm \ldots \pm \sqrt{2 + 2\xi}}},$$

$\xi$ being a complex number. Our approach leads us to generalize and recover, in a unified manner, the closed-form expressions recently given [in Servi 2003], corresponding to the case $\xi \in [-1, 1]$, and [in Nyblom 2005], corresponding to $\xi \in [1, \infty]$.

179. (2012) M. A. Nyblom, Some closed-form evaluations of infinite products involving nested radicals. Rocky Mountain Journal of Mathematics 42, no. 2, 751–758. MR2915517.

From the abstract: “By applying double and triple angle identities for hyperbolic and trigonometric cosine functions, we obtain closed-form evaluations for two families of infinite products involving nested radicals. The first group of results represents a generalization of the classic Viète infinite product expansion for $2/\pi$, while the second comprises variations on Viète type infinite products and infinite products involving nested square roots of 2. In addition, specific examples of Viète type infinite product expansions are presented for such numbers as $\frac{3\sqrt{3}}{2\pi}$ and $\frac{3}{\pi}$.”

180. (2013) Costas J. Efthimiou, A class of continued radicals. The American Mathematical Monthly 120, no. 5, 459–461. MR3035446.

From the introduction: “In Efthimiou 2012 the author discussed the values for a class of periodic continued radicals of the form

$$a_0 \sqrt{2 + a_1 \sqrt{2 + a_2 \sqrt{2 + a_3 \sqrt{2 + \cdots}}}},$$

(61)

where for some positive integer $n$, $a_{n+k} = a_k$, $k = 0, 1, 2, \ldots$, $a_k \in \{-1, +1\}$, $k = 0, 1, \ldots, n - 1$. It was also shown that the radicals given by equation (61) have limits two times the fixed points of the Chebychev polynomials $T_2n(x)$, thus unveiling an interesting relation between these topics.

“In Zimmerman and Ho 2008, the authors defined the set $S_2$ of all continued radicals of the form (61) (with $a_0 = 1$), and they investigated some of their properties by assuming that the limit of the radicals exists. In particular, they showed that all elements of $S_2$ lie between 0 and 2, any two radicals cannot be equal to each other, and $S_2$ is uncountable.

“[My] previous note partially bridged this gap, but left unanswered the question, ‘what are the limits if the radicals are not periodic?’ I answer the question in this note.”

181. (2013a) Samuel G. Moreno and Esther M. García, New infinite products of cosines and Viète-like formulae. Mathematics Magazine 86, no. 1, 15–25. Cited in Moreno and Garcia-Caballero 2013b, and García-Caballero, Moreno, and Prophet 2013c.
The authors write: “In this note we evaluate infinite products similar to the product (8) above, in Viète 1593, with the novelty that some of the plus signs are replaced by minus signs. We call these products Viète-like formulae. To derive them, first we manipulate a simple trigonometric relation (other than the double angle formula) in order to obtain a family of infinite products of cosines. Next, with the aid of a formula of Servi 2003 (see also Moreno and García-Caballero 2012 for a generalization), we transform these infinite products of cosines into infinite products of nested square roots of 2. These Viète-like expressions turn out to represent numbers like $\pi$, $\sqrt{3}$, and $\sqrt{5}-\sqrt{5}$.”

182. (2013b) Samuel G. Moreno and Esther M. García-Caballero, On Viète-like formulas. Journal of Approximation Theory 174, 90–112. MR3090772.
Cited in Moreno and García-Caballero 2013a, García-Caballero, Moreno, and Prophet 2014c, and Vellucci and Bersani 2016c.

From the abstract: “The very first recorded use of an infinite product in mathematics is the so-called Viète’s formula, in which each of its factors contains nested square roots of 2 with plus signs inside [equation (8) above] … and it can be proved by iterating the double angle formula $\sin 2x = 2 \cos x \sin x$, thus obtaining the infinite product $\frac{2}{\pi} = \prod_{n=2}^{\infty} \cos(\pi/2^n)$.

“This paper focuses, first, on the wide variety of iterations that the identity $\cos x = 2 \cos((\pi + 2x)/4) \cos((\pi - 2x)/4)$ admits; next, on the infinite products of cosines derived from these iterations and finally, on how these infinite products of cosines give rise to striking formulas.”

The main theorem proved is the following: Let $N$ be a positive integer and $(b_1, b_2, \ldots, b_N)$ a sequence for which $b_i$ equals either 1 or $-1$ and at least one $b_i$ equals $-1$. Define

$$\alpha = \sum_{i=1}^{N} \left( \prod_{j=1}^{i-1} b_{N-j+1} \right) 2^{N-i}, \quad \text{where } \prod_{j=1}^{0} = 1,$$

$$\sigma = \prod_{j=1}^{N} b_{N-j+1},$$

$$\hat{b}_k = b_{1+(k-1) \mod N}, \quad k = 1, 2, \ldots.$$  

For each complex number $z$, define the sequence $\{\rho_i(z)\}$ by

$$\rho_0(z) = z, \quad \rho_i(z) = \sqrt{2 + \hat{b}_i \rho_{i-1}}, \quad i \geq 1.$$  

Then

$$\frac{\sqrt{1 - z^2}}{\cos \left( \frac{\alpha \sigma}{2 \sigma -1} \frac{z}{2} \right)} = \prod_{j=0}^{\infty} \left( \prod_{i=jN+1}^{(j+1)N} \rho_i(z) \right).$$

In an appendix, the authors give a lucid explanation of the method used by François Viète to obtain his original formula (8) above.
183. (2013) Soumendu Sundar Mukherjee, An approximation inequality for continued radicals and power forms. Online at http://arxiv.org/abs/1303.4251

From the abstract: “In this article we derive an approximation inequality for continued radicals, generalizing an inequality of Herschfeld for continued square roots to arbitrary radicals [i.e., continued $r_i$th roots, $r_i \in (1, \infty)$], which is useful in exploring convergence issues and obtaining convergence rates. In fact, we generalize this inequality further to encompass the more general continued power forms. We demonstrate the use of this inequality by obtaining estimates for the convergence rates of several continued radicals including the famous Ramanujan radical [equation (43) above].”

184. (2013) William Paulsen, Asymptotic analysis and perturbation theory. CRC Press, Hoboken, ISBN 9781466515116.

Following Problem 18 in Section 3.5 (pp. 153–154), the author introduces a “continued-function” representation of a function $f(x)$ as follows: Let $h(x)$ be a simple function such that $h(0) = 1$. Then inductively define

\[ f_0(x) = f(x), \]
\[ g_n(x) = \text{the leading term of } f_n(x), \]
\[ f_{n+1}(x) = h^{-1} \left( \frac{f_n(x)}{g_n(x)} \right). \]

Then

\[ f(x) \sim g_0(x) \cdot h(g_1(x)) \cdot h(g_2(x)) \cdot h(\cdots), \]

where the relation $\sim$ (read “is similar to”) means that the ratio of $f$ and its expansion approaches 1 as $n$ increases without bound. Problems 19–22 ask the reader to compute continued square root expansions (using $h(x) = \sqrt{1+x}$) and continued exponential expansions ($h(x) = e^x$) for functions similar to divergent power series.

185. (2013) Jayantha Senadheera, On the periodic continued radicals of 2 and generalization for Vieta product. Online at http://arxiv.org/abs/1304.5659

From the abstract: “In this paper we study periodic continued radicals of 2. We show that any periodic continued radicals of 2 converg[e] to $2 \sin(q\pi)$, for some rational number $q$ depend[ing] on the continued radical. Furthermore we show that if $r_n$ is a periodic nested radicals [sic] of 2, which has $n$ nested roots, then the limit points of the sequence $2^n(2 \sin(q\pi) - r_n)$ have the form $\alpha\pi$, where $\alpha$ is an algebraic number. This result give[s] a set of sub sequences converg[ing] to $\alpha\pi$, for each $\alpha$. Also we show that limit of these sub sequences can be represented as Vieta like nested radical products. Hence this result generalizes the Vieta product for $\pi$. Several interesting examples are illustrated.”
186. (2013) Gaurav Tiwari, Complete elementary analysis of nested radicals. Online at [gauravtiwari.org]. Accessed 2017-07-17.

A web site (dated October 8, 2013) by a freelance web designer and blogger to “collect and expand what Ramanujan did with Nested Radicals and summarize all important facts in one article.”

187. (2013) Bo-Yan Xi, Feng Qi, Convergence, monotonicity, and inequalities of sequences involving continued powers. *Analysis (Berlin*) **33**, no. 3, 235–242. MR3118425.

From the summary: “…[T]he convergence, monotonicity, and inequalities of sequences involving continued powers √\(a-√\(a-\cdots-\sqrt[a]{a}\) are investigated and established.” The authors consider the cases \(a \in (1, \infty)\) and \(t \in (1, \infty)\); \(a \in (0, 1)\) and \(t \in (0, 1)\); \(a \in (-\infty, -1)\) and \(t\) odd; and \(a \in (-1, 0)\) and \(t\) odd.

188. (2014) Nikos Bagis, Solution of polynomial equations with nested radicals. Online at [http://arxiv.org/abs/1406.1948](http://arxiv.org/abs/1406.1948)

From the abstract: “In this note we present solutions of arbitrary polynomial equations in nested periodic radicals.” To obtain expressions that resemble continued \(r\)th roots, the author uses the trick, employed in Isenkrahe 1897, of setting \(d\sqrt{x} := x^{1/d}\), where \(d\) is a non-zero rational number. Compare Andrushkiw 1985.

189. (2014) Tyler Clark and Tom Richmond, Cantor sets arising from continued radicals, *The Ramanujan Journal* **33**, no. 3, 315–327. MR3182536.

From the abstract: “If \(a_1, a_2, a_3, \ldots\) are nonnegative real numbers and \(f_j(x) = \sqrt[a_j]{x + a_j}\), then \(\lim_{n \to \infty} f_1 \circ f_2 \circ \cdots \circ f_n(0)\) is a continued radical [continued square root] with terms \(a_1, a_2, a_3, \ldots\). The set of real numbers representable as a continued radical whose terms \(a_i\) are all from a set \(S = \{a, b\}\) of two natural numbers is a Cantor set. We investigate the thickness, measure, and sums of such Cantor sets.”

190. (2014a) Esther M. García-Caballero, Samuel G. Moreno, and Michael P. Prophet, New Viète-like infinite products of nested radicals with Fibonacci and Lucas numbers. *The Fibonacci Quarterly* **52**, 27–31. MR3181093.

Cited in Moreno and García-Caballero 2013b, and García-Caballero, Moreno, and Prophet 2014c.

From the abstract: “In a 2007 contribution by Osler in this Quarterly, the so-named Viète-like products were introduced as two eye-catching formulas representing either the \(n\)th Fibonacci number in terms of a product of nested radicals with the \(n\)th Lucas number inside, or vice-versa. As [in] the original and famous Viète’s infinite product, Osler’s infinite products have plus signs inside the nested radicals. In this paper we explore infinite products of nested square roots with Fibonacci and Lucas numbers with
the novelty that inside the radical symbols there are minus signs instead of plus signs.”

191. (2014b) Esther M. García-Caballero, Samuel G. Moreno, and Michael P. Prophet, The Golden Ratio and Viète’s Formula. *Teaching Mathematics and Computer Science* 12, 43–54.

From the abstract: “Viète’s formula uses an infinite product to express π. In this paper we find a strikingly similar representation for the Golden Ratio.”

192. (2014c) Esther M. García-Caballero, Samuel G. Moreno, and Michael P. Prophet, A complete view of Viète-like infinite products with Fibonacci and Lucas numbers. *Applied Mathematics and Computation* 247, 703–711. MR3270876.

From the abstract: “The main goal of this paper is to link the \( n \)th Fibonacci and Lucas numbers through certain infinite products of nested radicals. This work relies on recent results on Viète-like infinite products [which] appeared in MORENO AND GARCÍA-CABALLERO 2013. We will analyze in detail one particular case of these formulas and we will show how our treatment covers and extends previous results in the literature.”

193. (2014) Kyriakos Kefalas, On smooth solutions of non linear dynamical systems, \( f_{n+1} = u(f_n) \), part I. *Physics International* 5, no. 1, 112–127.

From the abstract: “We consider the dynamical system, \( f_{n+1} = u(f_n) \), (1) (where usually \( n \) is time) defined by a continuous map \( u \). Our target is to find a flow of the system for each initial state \( f_0 \), i.e., we seek continuous solutions of (1), with the same smoothness degree as \( u \). We start with the introduction of continued forms which are a generalization of continued fractions. With the use of continued forms and a modulator function (i.e., weight function) \( m \), we construct a sequence of smooth functions, which come arbitrarily close to a smooth flow of (1). The limit of this sequence is a functional transform, \( K_m[u] \), of \( u \), with respect to \( m \). The functional transform is a solution of (1), in the sense that \( K_m[u](y+c) \), is a flow of (1) for each translation constant \( c \). Here we present the first part of our work where we consider a subclass of dissipative dynamical systems in the sense [sic] that they have wandering sets of positive measure. In particular we consider strictly increasing real univariate maps, \( u : D \to D, D = (a + \infty) \), where \( a \geq 0 \), or \( a = -\infty \), with the property \( u(x) - x \geq \epsilon > 0 \), which implies that \( u \) has no real fixed points. We briefly give some mathematical and physical applications and we discuss some open problems. We demonstrate the method on the simple non-linear dynamical system \( f_{n+1} = u(f_n) + 1 \).” Ten references are cited.

194. (2014) Chris D. Lynd, Using difference equations to generalize results for periodic nested radicals. *The American Mathematical Monthly* 121, no. 1,
From the abstract included with an excerpt of the article at [http://www.math.uri.edu/~chris/pubs/NR%20Excerpt.pdf](http://www.math.uri.edu/~chris/pubs/NR%20Excerpt.pdf) “We investigate sequences of nested radicals [meaning here continued $r$th roots] where the indices, the coefficients, and the radicands are periodic sequences of real numbers. We show that one can determine the end behavior of a periodic nested radical by analyzing the basin of attraction of each equilibrium point, and each period-2 point, of the corresponding difference equation. Using this method of analysis, we prove a few theorems about the end behavior of nested radicals of this form. These theorems extend previous results on this topic because they apply to large classes of nested radicals that contain arbitrary indices, negative radicands, and periodic parameters with arbitrary periods. In addition, we demonstrate how to construct a periodic nested radical, of a general form, that converges to a predetermined limit; and we demonstrate how to construct a nested radical that converges asymptotically to a periodic sequence.”

195. (2014) Thomas J. Osler and Sky Waterpeace, Vieta’s product for pi from the Archimedean algorithm. *The Mathematical Gazette* **98**, 429–431. Cited in Osler 2016.

From the abstract: “In this paper we show how to derive [Viète’s] famous product of nested radicals for π from the Archimedean iterative algorithm for π. Only simple algebraic manipulations are needed.”

196. (2015) Dixon J. Jones, Continued reciprocal roots. *The Ramanujan Journal* **38**, 435–454. MR3414500. Cited in Vellucci and Bersani 2016c.

Proves that the continued reciprocal $r$th root

$$a_0 + \frac{1}{r^{a_1 + \frac{1}{r^{a_2 + \frac{1}{r^\ddots}}}}}$$

(62)

diverges if, and only if,

$$\limsup_{i \to \infty} a_i^p < 1,$$

where $r = 1/p$, $0 < p < 1$, and $a_i > 0$, $i = 0, 1, 2, \ldots$.

197. †(2015) Jörg Neunhäuserer, Continued logarithm representation of real numbers, https://doi.org/10.13140/rg.2.1.1763.1441.

From the abstract: “We introduce the continued logarithm representation of real numbers and prove results on the occurrence and frequency of digits with respect to this representation.”
The paper’s main theorem is the following generalization of Viète’s formula above for $\frac{2}{\pi}$:

$$\frac{slx}{x} = \sqrt{\frac{1 + sl'x}{2}}, \sqrt{\frac{1 + \sqrt{\frac{2}{1 + sl'x}}}{2}}, \sqrt{\frac{1 + \sqrt{\frac{2}{1 + \sqrt{\frac{2}{1 + sl'x}}}}}{2}} \ldots$$

where $slx$ is the lemniscate sine.

From the abstract: “In this paper, we give a new approach to prove inequalities for the Schwab-Borchardt mean, the lemniscatic mean and the arithmetic geometric mean. Additionally, we apply these means to inequalities for trigonometric functions or the lemniscate functions by considering several functional inequalities. One of these applications includes infinite product formulas for the lemniscate function and the arithmetic geometric mean by considering several functional equations.”

From the introduction: “Let $c$ be a real number with $c \geq 2$ and $\epsilon_1, \epsilon_2, \ldots$ an infinite sequence consisting of $\pm 1$. In this paper we are concerned with nested square roots of the form

$$R_c(\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_m) = \epsilon_1 \sqrt{c + \epsilon_2 \sqrt{c + \epsilon_3 \sqrt{c + \cdots + \epsilon_m \sqrt{c}}}$$

and

$$R_c(\epsilon_1, \epsilon_2, \ldots) := \lim_{m \to \infty} R_c(\epsilon_1, \epsilon_2, \ldots, \epsilon_m).$$
In the case of $c = 2$, it is known that the nested root (63) can be expressed as:

$$R_c(\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_m) = 2 \cos \pi \left( \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_m}{2^m} + \frac{1}{2^{m+1}} \right),$$  \hspace{1cm} (65)

where

$$a_i = \frac{1 - \epsilon_1 \cdots \epsilon_i}{2} = \begin{cases} 0 & (\text{if } \epsilon_1 \cdots \epsilon_i = 1) \\ 1 & (\text{if } \epsilon_1 \cdots \epsilon_i = -1). \end{cases}$$

Taking $\lim_{m \to \infty}$ of (64), we obtain a simple formula for the infinite nested square root:

$$R_2(\epsilon_1, \epsilon_2, \epsilon_3, \ldots) = 2 \cos \alpha \pi,$$  \hspace{1cm} (66)

where $\alpha$ is a real number defined by the 2-adic expansion

$$\alpha = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots.$$ 

These formulas were proved by Wiernsberger . . . in 1905, and about thirty years later Lebesgue . . . independently found the same formulas.

“The purpose of this paper is to give a generalization of the formulas (65) and (66) to the case $c \geq 2$. To accomplish the task, we need a suitable function which will take the place of $\cos x$. In the proof of the formulas (65) and (66), the duplication formula

$$2 \cos 2x = (2 \cos x)^2 - 2$$

was crucial. It is therefore natural to seek for a function $f(x)$ satisfying the functional equation

$$f(sx) = f(x)^2 - c,$$  \hspace{1cm} (67)

where $s$ is a constant depending only on $c$. Such functional equations were studied by Poincaré, who showed that there exists an entire function $f(x)$ satisfying (67).”

The authors also use their method to give a generalization of Viète’s formula (8) for $\frac{\pi}{2}$.

It should be noted that (65) is to be found in Bochow 1899 (expressed using the sine rather than the cosine function), and foreshadowed in Lucas 1878. Among several minor bibliographical errors, the ones most in need of correction are the references to “M. P. Wiernsberger”, which should be to “P. Wiernsberger.”

202. (2016) Mu-Ling Chang and Chia-Chin (Cristi) Chang, Evaluation of pi by nested radicals. Mathematics Magazine 89, no. 5, 336–337. MR3593652.

The authors prove the formula for $\pi$ in equation (20) above, apparently unaware of Catalan 1842 and of the formula’s many subsequent appearances.

203. (2016) Devyn A. Lesher and Chris D. Lynd, Convergence results for the class of periodic left nested radicals. Mathematics Magazine 89, no. 5, 319–335. MR3593651.
The authors write: “There are two fundamental questions that are found throughout the research on nested radicals: (1) Given a particular form of nested radical, under what conditions does it converge? (2) Given a particular form of nested radical, which numbers can be expressed as the limit of a nested radical?" 

"[...]In this paper we address both research questions, as they pertain to periodic left nested radicals. We provide numerical examples to illustrate each result.

“We also provide a recipe for constructing nested radicals with a predetermined end-behavior. If you choose the form of the nested radical and a limit, the recipe shows how to construct the unique nested radical of the chosen form that converges to the chosen limit. If you choose a periodic limiting sequence, the recipe shows how to construct a nested radical that asymptotically converges to the chosen periodic sequence.”

204. (2016) Ryo Nishimura, A generalization of Viète’s infinite product and new mean iterations. *The Australian Journal of Mathematical Analysis and Applications* **13**, no. 1, Art. 20, 9 pp. MR3590882.

From the abstract: “In this paper, we generalize Viète’s infinite product formula by use of Chebyshev polynomials. Furthermore, the infinite product formula for the lemniscate sine is also generalized. Finally, we obtain new mean iterations by use of these infinite product formulas.”

205. (2016) Thomas Osler, Iterations for the lemniscate constant resembling the Archimedean algorithm for $\pi$. *The American Mathematical Monthly* **123**, no. 1, 90–93. MR3453543.

From the abstract: “We give an iterative algorithm that converges to the lemniscate constant $L$. This algorithm resembles the famous Archimedean algorithm for $\pi$. The derivation is based on the recently discovered product of nested radicals for $\frac{2}{L}$ by Aaron Levin. Levin’s product closely resembles Vieta’s historic product for $\frac{2}{\pi}$."

206. (2016) Thomas J. Osler, Walter Jacob, and Ryo Nishimura, An infinite product of nested radicals for log $x$ from the Archimedean algorithm. *The Mathematical Gazette* **100**, no. 548, 274–278. MR3520821.

The authors show that log $x$ may be written as

$$
\frac{x - 1}{\sqrt{\frac{1}{2} + \frac{1}{2} \left( \frac{x}{2 \sqrt{2}} \right)} \sqrt{\frac{1}{2} + \frac{1}{2} \left( \frac{x}{2 \sqrt{2}} \right)} \sqrt{\frac{1}{2} + \frac{1}{2} \left( \frac{x}{2 \sqrt{2}} \right)} \cdots}
$$

based on a similar, more general expansion generated by the Archimedean algorithm.

\[16\] By “nested radicals” the authors mean both continued and iterated $r$th roots. — DJJ
207. (2016) Thomas J. Osler and Jesse M. Kosior, A sequence of good approximations for the period of a pendulum with large initial amplitude. *The Mathematical Scientist* **41**, no. 1, 40–44. MR3561649.

From the abstract: “We present three elementary approximate formulas for the period of a pendulum which starts at rest from a large angle of displacement. The first of these formulas is known, but the other two may be new. These three formulas result from taking the first three partial products of a new infinite product of nested radicals for the complete elliptic integral of the first kind that gives the exact period. Thus, more elementary approximations can be obtained from this exact product, but they become increasingly complex. Therefore, we stopped at three. We give a detailed table clearly displaying the accuracy of the approximations over the full range of possible initial angles of displacement. This infinite product of nested radicals is a special case of a new infinite product for the arithmetic-geometric mean that has appeared recently.”

208. (2016) Fritz Schweiger, *Continued Fractions and their Generalizations: A Short History of* $f$-**expansions**, Docent Press, Boston, ISBN 9781942795933.

From the introduction: “This book is about the history of $f$-expansions, their theory, their application, and their connection to other parts of mathematics... As a kind of background theory we occasionally use the language of fibred systems.” $g$-adic expansions of real numbers, where $f(x) = \frac{2}{7}$, and continued fraction expansions, where $f(x) = \frac{1}{x}$, are given as “prominent examples” of $f$-expansions; the latter are reviewed in Chapter 2, the former in Chapter 3. Infinite products are briefly covered in Chapter 4.

Chapter 5 looks at Kakeya 1924, which attempted to unify $g$-adic expansions and continued fractions under a common general form — the precursor by twenty years of Bissinger’s $f$-expansions. (Schweiger amusingly laments subsequent authors’ completely coincidental permutations of some of Kakeya’s terminology and notation.) Continued cotangents and Lehmer 1938 are the subject of Chapter 6. Chapter 7 discusses Bissinger 1944 and Everett 1946, which introduce $f$-expansions based on, respectively, increasing and decreasing functions $f$; the chapter’s closing comments concern Rechards 1950.

The Borel-Bernstein Theorem is taken up in Chapter 8, while ergodic properties of $f$-expansions are discussed in Chapter 9. Chapter 10 reviews the work of Rényi and others in extending Bissinger’s definitions, including the distinction between those $f$-expansions with independent versus dependent digits. Chapter 11 considers Gauss’s investigations into functional equations involving tails of continued fraction representations, and Kuzmin’s general result that proves one of Gauss’s conjectures. This leads to Chapter 12, Lévy’s alternate proof of the Gauss conjecture based on “the dual algorithm or the natural extension of continued fractions.”

The work of Gel’fond, Cigler, and Parry in 1959 and 1960 to extend Rényi’s $\beta$-expansions is dealt with in Chapter 13. Chapter 14 begins by
recalling a mostly-forgotten paper by T. E. McKinney from 1907 on \( \lambda \)-continued fractions, whose underlying map \( f_\lambda : [\lambda - 1, \lambda] \to [\lambda - 1, \lambda] \) is defined by

\[
f_\lambda(x) = \frac{\epsilon}{x} - \left\lfloor \frac{\epsilon}{x} + 1 - \lambda \right\rfloor
\]

where \( x \neq 0 \) and \( \epsilon = \text{sign} \ x \). After pointing to a special case of this by A. Hurwitz from 1888, Schweiger reviews the investigations of Nakada, Ito, and Tanaka into continued fractions of this type. Chapter 15, on discontinuous groups, looks at D. Rosen’s extension of continued fractions, which derives from linear fractional transformations in the group generated by \( S(z) = z + \lambda \) and \( T(z) = -\frac{1}{z} \).

Ergodic theory, and its connection to \( f \)-expansions, is alluded throughout the book, but becomes the focus of Chapters 16, 17, and 18, with invariant measure given particular attention in the latter two. C. Shannon’s concept of entropy as a measure of information is viewed in Chapter 19 as “a kind of average for the cylinders in an \( f \)-expansion (more general [sic] of a fibred system).” This leads naturally into a discussion, in Chapter 20, of Hausdorff dimension and P. Billingsley’s generalization, “which is adapted to problems connected with \( f \)-expansions.”

Multidimensional generalizations of continued fractions and \( f \)-expansions are taken up in Chapter 21. Here Schweiger is keen to point out a “serious flaw” in one of his own papers, and its subsequent detection by some later authors but perpetuation by others. The book’s final chapter is a three-page synopsis of aspects of chaos theory.

A bibliography of over 200 sources is provided. There is no index.

209. (2016a) Pierluigi Vellucci and Alberto Maria Bersani, The class of Lucas-Lehmer polynomials. *Rendiconti di Matematica e delle sue Applicazioni. Serie VII*, **37**, 43–62. Online at [https://arxiv:1060.01989](https://arxiv:1060.01989), accessed 2017-07-16. MR3622304. Cited in *Vellucci and Bersani 2016b* and *2016c*.

From the abstract: “In this paper we introduce a new sequence of polynomials, which follow the same recursive rule of the well-known Lucas-Lehmer integer sequence. We show the most important properties of this sequence, relating them to the Chebyshev polynomials of the first and second kind.” The polynomials are defined recursively by \( L_0(x) = x \) and \( L_n(x) = L_{n-1}^2(x) - 2 \). Zeroes of \( L_n \) turn out to be finite continued square roots with terms \( \pm 2 \).

210. (2016b) Pierluigi Vellucci and Alberto Maria Bersani, Ordering of nested square roots of 2 according to Gray code. Online at [https://arxiv.org/abs/1060.00222](https://arxiv.org/abs/1060.00222), accessed 2017-07-06. Cited in *Vellucci and Bersani 2016c*.

From the abstract: “In this paper we discuss some relations between zeros of Lucas-Lehmer polynomials and Gray code. We study nested square roots of 2 applying a ‘binary code’ that associates bits 0 and 1 to \( \oplus \) and \( \ominus \) signs in the nested form. This gives the possibility to obtain an ordering for
the zeros of Lucas-Lehmer polynomials, which assume the form of nested square roots of 2.”

211. (2016c) Pierluigi Vellucci and Alberto Maria Bersani, New formulas for \( \pi \) involving infinite nested square roots and Gray code. Online at [https://arxiv.org/abs/1606.09597v2](https://arxiv.org/abs/1606.09597v2) accessed 2017-07-06.

From the abstract: “In previous papers we introduced a class of polynomials which follow the same recursive formula as the Lucas-Lehmer numbers, studying the distribution of their zeros and remarking that this distribution follows a sequence related to the binary Gray code. It allowed us to give an order for all the zeros of every polynomial \( L_n \). In this paper, the zeros, expressed in terms of nested radicals, are used to obtain two new formulas for \( \pi \): the first can be seen as a generalization of the known formula

\[
\pi = \lim_{n \to \infty} 2^{n+1} \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}
\]

[where there are \((n+1)\) 2s under the outer radical], related to the smallest positive zero of \( L_n \); the second is an exact formula for \( \pi \) achieved thanks to some identities valid for \( L_n \).”

212. (Undated) Michael McGuffin and Brian Wong, The Museum of Nested Radicals, Online at [www.dgp.toronto.edu/~mjmcguff/](http://www.dgp.toronto.edu/~mjmcguff/) accessed 2018-06-15. Cited in Humphries 2007, Zimmerman and Ho 2008a, and Weisstein n.d.

This web site offers the formula

\[
x = \sqrt[n]{(1 - q)x^n + qx^{n-1} \sqrt[(n-1)]{(1 - q)x^n + qx^{n-1} \sqrt[n-2]{\cdots}}},
\]

with a number of special cases for various substitutions and values of \( q \) and \( n \). It also attributes to Ramanujan the formula

\[
x + n + a = \sqrt{ax + (n + a)^2 + \sqrt[3]{\cdots}}
\]

(68)

(where each square root subsumes its successors), again with several special cases listed, including the first problem from Ramanujan 1911. A few words of justification for these expressions are given; no sources are listed.

213. (Undated) Eric W. Weisstein, Nested Radical. From [MathWorld—A Wolfram Web Resource](http://mathworld.wolfram.com). Online at [mathworld.wolfram.com](http://mathworld.wolfram.com) accessed 2018-06-15.

Gives an overview of the results of Herschfeld, Ramanujan, and Viète, along with continued square root expressions for sines and cosines of special angles, the golden ratio, the silver ratio, and the plastic constant.
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5112 Fairchild Ave, Fairbanks, Alaska 99709-4523 USA

E-mail address: djones@alaska.edu