THE $\frac{m}{n}$ PIERI RULE

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Abstract. The Pieri rule is an important theorem which explains how the operators $e_k$ of multiplication by elementary symmetric functions act in the basis of Schur functions $s_\lambda$. In this paper, for any $m/n \in \mathbb{Q}$ we study the relationship between the "rational" version of the operators:

$$e_k^{m/n} : \Lambda \to \Lambda$$

given by the elliptic Hall algebra, and the "rational" version $s_\lambda^{m/n}$ of the basis given by the Maulik-Okounkov stable basis construction. The answer is inspired by geometry, but relevant to combinatorial and representation-theoretic considerations.

1. Introduction

In this paper, we study a certain algebra $\mathcal{A}$ known by many names in the theory of quantum groups: the elliptic Hall algebra, the shuffle algebra, the doubly deformed $\mathcal{W}_{1+\infty}$ algebra, the Ding-Iohara algebra, and quantum toroidal $\mathfrak{gl}_1$. For each rational number $\frac{m}{n} \in \mathbb{Q} \cup \{\infty\}$, the algebra $\mathcal{A}$ contains a commutative subalgebra isomorphic to a polynomial ring in countably many variables:

$$\mathbb{Q}(q, t)[e_1^{m/n}, e_2^{m/n}, ...] \cong \mathcal{A}_{m/n} \subset \mathcal{A}$$

where $q, t$ are parameters. The algebra $\mathcal{A}$ acts on the ring of symmetric functions:

$$\mathcal{A} \curvearrowright \Lambda = \mathbb{Q}(q, t)[x_1, x_2, ...]^{\text{Sym}}$$

where $e_k^0 \in \mathcal{A}_0$ acts on $\Lambda$ by the operator of multiplication by the $k$–th elementary symmetric function, and $e_k^\infty \in \mathcal{A}_\infty$ acts on $\Lambda$ by the $k$–th Macdonald $q$–difference operator. The full algebra $\mathcal{A}$ can be thought of as interpolating between these two extremes, and we will spell out the interactions between generators for different $\frac{m}{n}$.

The same principle applies for bases of the representation $\Lambda$. At $\frac{m}{n} = \frac{0}{1} = 0$, we consider the basis of Schur functions, in which the operators of multiplication by symmetric functions are described quite nicely by the Pieri rule. At $\frac{m}{n} = \frac{1}{0} = \infty$, we consider the basis of Macdonald polynomials in which the $q$–difference operators are diagonal, and hence quite presentable. We will seek to interpolate between these two bases, i.e. to define a basis:

$$\{s_\lambda^{m/n}\}_{\lambda \text{ partition}}$$

of $\Lambda$, for any $\frac{m}{n} \in \mathbb{Q}$. There are a number of properties one wants from such a basis, but the one we will mostly be concerned with is that the generators:

$$e_k^{m/n} \in \mathcal{A}_{m/n} \subset \mathcal{A}$$
act "nicely" in it. Such a choice of (1.1) is given by the Maulik-Okounkov stable basis ([8]), which we recall in Section 4. We will prove the following result:

**Theorem 1.2. (The \( m/n \) Pieri rule):** For any coprime \((m, n) \in \mathbb{Z} \times \mathbb{N}\) and any positive integer \(k\), we have:

\[
e_k^{m/n} \cdot s_{\mu}^{m/n} = \sum s_{\lambda}^{m/n} (-1)^{ht} \prod_{i=1}^{k} \prod_{j=1}^{n} \chi_j(R_i) \lfloor \frac{mj}{n} \rfloor - \lfloor \frac{mj}{n} \rfloor (1.3)
\]

where the sum goes over all vertical \(k\)-strips of \(n\)-ribbons of shape \(\lambda \setminus \mu\). We write \(ht\) for the height \(^1\) of such a \(k\)-strip, and:

\[
\chi_j(R_i) = q^{x} t^{-y}
\]

where \((x, y)\) are the coordinates of the \(j\)-th box in the \(i\)-th ribbon, counted in northwest-southeast direction. These notions will be explained in more detail in Subsection 2.5. ²

Though we will not prove it in the interest of clarity, the above theorem also holds in case \(m\) and \(n\) are not coprime. For example, when \(m = 0\) and \(n\) is an arbitrary positive integer, the operators \(e_0^{0/n}\) give rise to multiplication by the image of the \(k\)-th elementary symmetric function under the plethysm \(x_i \rightarrow x_i^n\). Since the basis \(s_0^\lambda\) consists of the usual Schur functions, formula (1.3) specializes to the Lascoux-Leclerc-Thibon ribbon tableau formula (12) of [6]. If we further also specialize \(n = 1\), we obtain the usual Pieri rule for multiplication by elementary symmetric functions.

One application of Theorem 1.2 is that it allows us to define an action of \(U_{\gamma}(\hat{\mathfrak{gl}}_n)\) on \(\Lambda\), which extends the construction in Theorems 5.1 and 5.4 of [6] to general \(m\). Another application of Theorem 1.2 is the particular case \(k = 1\) and \(\mu = \emptyset\), when formula (1.3) becomes:

\[
e_1^{m/n} \cdot 1 = \sum_{i=1}^{n} s_{(i, 1^{n-i})}^{m/n} \cdot q^{\sum_{j=1}^{i-1} \lfloor \frac{mj}{n} \rfloor} (-t)^{\sum_{j=1}^{n} \lfloor \frac{mj}{n} \rfloor} (1.3)
\]

The above equality gives a new interpretation of the "symmetric function" side of the rational shuffle conjecture, and it is expected to equal the "combinatorial side" provided by the Hikita polynomial. See [5] for a review of the rational shuffle conjecture, and also for connections with knot theory and representation theory. According to a general framework in representation theory, the above equality reflects a certain resolution of the unique finite-dimensional irreducible module of the rational Cherednik algebra (with quantization parameter \(c = \frac{m}{n}\)) by standard modules corresponding to hook diagrams. This should be a bigraded version of the BGG-Koszul resolution of [2].

The structure of this paper is the following. In Section 2 we recall certain basic definitions concerning symmetric functions, partitions and Young diagrams. In

\(^1\)This notion is called spin in [6].

\(^2\)By replacing the word "vertical" with the word "horizontal", we obtain formulas for operators of multiplication by complete symmetric functions.
Section 3 we recall the definition of the algebra $A$, its alternative presentation as a shuffle algebra, and the way the shuffle algebra helps us understand the action of $A$ on the ring of symmetric functions. In Section 4, we recall the definition of the basis $s^{m/n}$ from [8] and prove Theorem 1.2. In Section 5, we show how to extend formulas (1.3) to an action of $U_v(\hat{sl}_n)$ on the vector space of symmetric polynomials, which depends on the additional parameter $m$.

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2. Definitions and notations: symmetric functions and partitions

2.1. Much of the present paper is concerned with the ring of symmetric functions in infinitely many variables $x_1, x_2, ...$, over the field $\mathbb{K} = \mathbb{Q}(q, t)$:

$$\Lambda = \mathbb{K}[x_1, x_2, ...]^{\text{Sym}}$$

There are a number of bases of this vector space, perhaps the most basic one consisting of monomial symmetric functions:

$$m_\lambda = \text{Sym} \left[ x_1^{\lambda_1} x_2^{\lambda_2} ... \right]$$

where $\lambda$ goes over all partitions. Particular instances of these are the power-sum functions:

$$p_k = m_{(k)} = x_1^k + x_2^k + ...$$

as well as the elementary and complete symmetric functions:

$$e_k = m_{(1,1,...,1)} = \sum_{i_1 < ... < i_k} x_{i_1} ... x_{i_k}, \quad h_k = \sum_{\lambda \vdash k} m_\lambda = \sum_{i_1,...,i_k} x_{i_1} ... x_{i_k}$$

The ring $\Lambda$ is generated by each of these particular symmetric functions:

$$\Lambda = \mathbb{K}[p_1, p_2, ...] = \mathbb{K}[e_1, e_2, ...] = \mathbb{K}[h_1, h_2, ...]$$

and as a vector space, it is spanned by:

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} ..., \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} ..., \quad h_\lambda = h_{\lambda_1} h_{\lambda_2} ...$$

as $\lambda = (\lambda_1 \geq \lambda_2 \geq ...)$ goes over all partitions of natural numbers.

2.2. We will consider two inner products on $\Lambda$, and note that all of them will by graded, symmetric and respect the bialgebra (product and coproduct) structure of $\Lambda$. We will not go into what this means, but observe that such an inner product is uniquely determined by the pairing of $p_k$ with itself:

$$\langle p_k, p_k \rangle = k$$

By Gram-Schimdt, there is a unique orthogonal basis $\{s_\lambda\}$ of $\Lambda$ such that:

$$\langle s_\lambda, s_\mu \rangle = 0$$
if $\lambda \neq \mu$, and:

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} m_\mu c_\lambda^\mu,$$

where the dominance ordering on partitions is:

$$\mu \leq \lambda \quad \text{if} \quad \mu_1 + \ldots + \mu_i \leq \lambda_1 + \ldots + \lambda_i \quad \forall i \quad (2.1)$$

and $|\mu| = |\lambda|$. The symmetric polynomials $s_\lambda$ are called **Schur functions**, and they play a very important role in the representation theory as the characters of irreducible representations of the special linear group. Note that we have:

$$e_k = s_{(1,\ldots,1)} \quad h_k = s_{(k)} \quad p_k = \sum_{i=0}^{k-1} (-1)^i s_{(k-i,1,\ldots,1)}$$

2.3. There is a 1-1 correspondence between partitions and Young diagrams, the latter being simply stacks of $1 \times 1$ boxes placed in the corner of the first quadrant. For example, the following Young diagram:

![Young diagram](image1)

represents the partition $(4, 3, 1)$, because it has 4 boxes on the first row, 3 boxes on the second row, and 1 box on the third row. The monomials displayed in Figure 1 are called the **weights** of the boxes they are in, and are defined by the formula:

$$\chi_{\Box} = q^x t^{-y} \quad (2.2)$$

where $(x, y)$ are the coordinates of the southwest corner of the box. We call the integer:

$$o_{\Box} = x - y \quad (2.3)$$

the **content** of the box, and note that the content is constant across diagonals (in this paper, the word "diagonal" will only refer to those in southwest-northeast direction). Finally, every box in a Young diagram comes with numbers denoted by:

$$a(\Box), \ l(\Box)$$

known as the **arm** and **leg** lengths, respectively. These numbers count the distance between the given box and the right and top borders of the partition, respectively. For example, the box of weight $t^{-1}$ in Figure 1 has arm length equal to 2 and leg length equal to 1. Moreover, a Young diagram has **inner and outer corners**: the example in Figure 1 has 4 inner corners (of weights $t^{-3}, qt^{-2}, q^3 t^{-1}, q^4$) and 3
outer corners (of weights $qt^{-3}, q^3t^{-2}, q^4t^{-1}$).

2.4. In this paper, we will also consider another inner product, generated by:

$$\langle p_k, p_k \rangle_{q,t} = k \cdot \frac{1 - q^k}{1 - t^k} \quad (2.4)$$

This is known as Macdonald inner product. By the same Gram-Schmidt principle, there is a unique orthogonal basis $\{P_\lambda\}$ of $\Lambda$ such that:

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0$$

if $\lambda \neq \mu$, and:

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} m_\mu c_\mu^{\lambda}$$

The symmetric functions $P_\lambda$ are called Macdonald polynomials. We will also consider a certain renormalization of these polynomials:

$$M_\lambda = \frac{P_\lambda}{\prod_{\square \in \lambda} (t - l(\square) - q^{a(\square)} + 1)} \quad (2.5)$$

Let us also recall the operator $\nabla : \Lambda \rightarrow \Lambda$ of Bergeron-Garsia which is diagonal in the basis of Macdonald polynomials:

$$\nabla \cdot M_\lambda = M_\lambda \prod_{\square \in \lambda} \chi(\square) \quad (2.6)$$

This operator is usually defined to be diagonal in modified Macdonald polynomials, so there will be an implicit plethysm $X \rightarrow X/(1 - t)$ connecting our notations with the more common ones in the literature.

2.5. Given two partitions, we will write $\mu \prec \lambda$ if the Young diagram of $\mu$ is completely contained in that of $\lambda$. This is equivalent with requiring that $\mu_i \leq \lambda_i$ for all $i$ and it is different from the dominance ordering (2.1). If we are in this situation, we call:

$$\lambda \setminus \mu$$

a skew diagram, meaning a subset of boxes in the first quadrant obtained by removing a Young diagram from a larger one. If $\lambda \setminus \mu$ is connected and contains no $2 \times 2$ square, then we call it a ribbon (or border strip). The quantity:

$$ht(B) = \max_{\square \in B} l(\square) - l(\square)$$

is called the height of a ribbon $B$. We may use the term $n$--ribbon if $|\lambda \setminus \mu| = n$. The boxes of an $n$--ribbon are indexed $\square_1, \ldots, \square_n$ going from northwest to southeast, and note that their contents are consecutive integers. Given any two disjoint $n$--ribbons, we say that one is next to the other if their first common edge (equivalently their last common edge) is vertical. A vertical $k$--strip of $n$--ribbons $\{B_1, \ldots, B_k\}$ is a collection of disjoint $n$--ribbons such that no two are next to each other. The height of such a $k$--strip is defined as:

$$ht = \sum_{i=1}^{k} ht(B_i)$$
The following Lemma is an easy combinatorial exercise:

**Lemma 2.7.** Any skew Young diagram $\lambda \setminus \mu$ of size $kn$ can be covered by at most one vertical $k$-strip of $n$-ribbons. Hence the matrix coefficients of (1.3) are either zeroes or monomials $\pm q^x t^y$.

**Proof** We will prove the statement by induction on $k$, where the case $k = 1$ is obvious. Let us assume a certain skew diagram $\lambda \setminus \mu$ can be covered by a vertical $k$-strip of $n$-ribbons, and show that the covering is unique. Note that there is a unique candidate for the $n$-ribbon $B_{\text{out}}$ which contains the northwest-most square of $\lambda \setminus \mu$: indeed, this ribbon must start from this square and trace the external boundary of $\lambda \setminus \mu$. We call $B_{\text{out}}$ the outer ribbon, and note that if it fails to end on a right vertical boundary of $\lambda \setminus \mu$, then we violate the condition that the skew diagram can be covered by a vertical strip of $n$-ribbons. Therefore, removing $B_{\text{out}}$ leaves us with yet another skew Young diagram which can be covered by a $k - 1$ strip of $n$-ribbons, so we can repeat the argument. Since at each step, the outer ribbon that we remove is unique, we conclude that the initial covering is unique. 

2.6. Let us consider a certain game. Start with any skew diagram $\lambda \setminus \mu$ and bubble down its rows according to the following procedure:

1. start with the topmost row of length $l$, and slide it diagonally (in the southwest direction) on top of the next row of length $l'$
2. on the second row, we will now have two overlapping strips of lengths:
   \[
   \max(l, l') + a \quad \text{and} \quad \min(l, l') - a \quad \text{for some } a > 0
   \]
3. take the longest of the two strips and slide it diagonally on top of the next row down, and repeat the procedure
4. when we obtain a row of length $n$, we remove it and go back to step (1)

If we can remove all the boxes of $\lambda \setminus \mu$ by applying the above sequence of moves, without ever obtaining a horizontal strip of more than $n$ boxes, we call the skew diagram $\lambda \setminus \mu$ a winner.

**Lemma 2.8.** A skew Young diagram is a winner if and only if it can be covered by a vertical strip of $n$-ribbons.

**Proof** Assume $D$ can be covered by a vertical strip of $n$-ribbons. Let $B_{\text{out}}$ be the outer ribbon of this covering (see the proof of Lemma 2.7) and suppose it has height $h$. Then as we bubble down the first row according to the above procedure, after $h - 1$ steps the blocks of the original ribbon $B_{\text{out}}$ will form a horizontal strip of length $n$. We remove this strip, and then repeat the game for the remaining skew diagram, which can be covered by a vertical strip of one less $n$-ribbons.
3. The elliptic Hall and shuffle algebras

3.1. In this section, we will present a certain algebra $\mathcal{A}$ known as the spherical elliptic Hall algebra. The following is a renormalized formulation of the presentation in [1], where this algebra was first defined. Set:

$$\alpha_k = (1 - t^k)(1 - q^k t^{-k}) \in K$$

Let $\mathbb{Z}_2^2$ denote the right half plane lattice, which we think of including the positive vertical half-line. Define the algebra $\mathcal{A}$ to be generated by elements $p_v$ for any $v \in \mathbb{Z}_2^2$ modulo the following relations:

$$[p_{kv}, p_{lv}] = 0 \ (3.1)$$

for any $k, l > 0$ and any $v \in \mathbb{Z}_2^2$, while:

$$[p_v, p_{v'}] = \theta_{v+v'} \ (3.2)$$

for any clockwise oriented lattice triangle $\{0, v, v + v'\} \subset \mathbb{Z}_2^2$ with no lattice points inside and on the first two edges, where:

$$\theta_v(z) = \sum_{k \geq 0} \theta_{kv} z^k := \frac{1 - q^{-1}}{(q - t)(1 - t^{-1})} \cdot \exp \left( \sum_{k \geq 1} \alpha_k p_{kv} z^k \right) \quad (3.3)$$

for any $v = (n, m) \in \mathbb{Z}_2^2$ such that $\gcd(m, n) = 1$.

3.2. Note that the condition (3.1) ensures that:

$$\mathbb{K}[p_1^{m/n}, p_2^{m/n}, ...] =: \mathcal{A}_{m/n} \hookrightarrow \mathcal{A}$$

are all commutative subalgebras, where we denote $p_{k}^{m/n} = p_{kn, km}$. Our basic module for $\mathcal{A}$ will be the ring of symmetric functions:

$$\Lambda := \mathbb{K}[x_1, x_2, ...]^\text{Sym}$$

where the algebra $\mathcal{A}$ acts by:

$$p_{k,0} = \text{multiplication by } p_k \quad (3.3)$$

$$p_{k, mk} = \nabla^{m} p_{k,0} \nabla^{-m}$$

$$p_{0, k}(M_{\lambda}) = M_{\lambda} \sum_{i \geq 0} q^{\lambda_i - i} t^{-i} \quad (3.4)$$

Because of the defining relations (3.1) - (3.2), this is enough to define the action of the whole algebra $\mathcal{A}$, although one needs to check that the defining relations are met (see [10] for a survey). Hence the operators:

$$p_k^{m/n} = p_{kn, km} : \Lambda \rightarrow \Lambda$$

thus defined interpolate between the operators (3.3) of multiplication by $p_k$ and the Macdonald eigenoperators (3.4). To obtain an understanding of how these operators explicitly act on symmetric functions for general $m$ and $n$, we turn to an incarnation of $\mathcal{A}$ known as the shuffle algebra.
3.3. Consider an infinite set of variables $z_1, z_2, \ldots$, and take the $\mathbb{K}$–vector space:

$$V = \bigoplus_{N \geq 0} \mathbb{K}(z_1, \ldots, z_N)^{\text{Sym}}$$

(3.5)

We can endow it with a $\mathbb{K}$–algebra structure by the so-called shuffle product:

$$R_1(z_1, \ldots, z_N) \ast R_2(z_1, \ldots, z_{N'}) =$$

$$= \text{Sym} \left[ R_1(z_1, \ldots, z_N)R_2(z_{N+1}, \ldots, z_{N+N'}) \prod_{i=1}^{N} \prod_{j=N+1}^{N+N'} \omega \left( \frac{z_i}{z_j} \right) \right]$$

(3.6)

where:

$$\omega(x) = \frac{(1 - xq)(t - x)}{(1 - x)(t - xq)}$$

(3.7)

and Sym denotes the symmetrization operator:

$$\text{Sym} \left( R(z_1, \ldots, z_N) \right) = \sum_{\sigma \in S(N)} R(z_{\sigma(1)}, \ldots, z_{\sigma(N)})$$

3.4. The shuffle algebra is defined as the subalgebra $S \subset V$ consisting of rational functions of the form:

$$R(z_1, \ldots, z_N) = \frac{r(z_1, \ldots, z_N)}{\prod_{1 \leq i < j \leq N} (tz_i - qz_j)}$$

(3.8)

where $r$ is a symmetric Laurent polynomial that satisfies the following wheel conditions (introduced in [3]):

$$r(z_1, \ldots, z_N) = 0 \quad \text{whenever} \quad \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{pmatrix} = \begin{pmatrix} q & 1 & t \\ t & q & t \end{pmatrix} \text{ or } \begin{pmatrix} t & q & t \\ 1 & q & t \end{pmatrix}$$

(3.9)

These conditions impose quite significant restrictions on the set of elements of $S$, as was studied in [3] and [11]. In particular, they ensure that for any skew partition $\lambda \backslash \mu$ of size $N$, we may plug in the set of weights of its boxes:

$$\{ \chi_{\square} \}_{\square \in \lambda \backslash \mu}$$

into the Laurent polynomial $r$. This will make the polynomial vanish because of the wheel conditions, but the order of vanishing precisely cancels out the order of vanishing of the denominator of (3.8). This implies that the quantity:

$$R(\lambda \backslash \mu) := R(\chi_{\square})_{\square \in \lambda \backslash \mu} \in \mathbb{K}$$

(3.10)

is a well-defined constant for any shuffle element $R \in S$.

3.5. The connection between the above and the elliptic Hall algebra is given by:

**Theorem 3.11.** (see [12], [11]) If we let $A^+ \subset A$ be the subalgebra generated by $p_{n,m}$ with $n > 0$, \footnote{So $A^+$ is obtained from $A$ by removing the generators $p_{0,k}$} then we have an isomorphism of algebras:

$$A^+ \rightarrow S, \quad p_{1,m} \mapsto z_1^m$$
By the above, $S$ is a model for the elliptic Hall algebra (except for the generators $p^\infty_k$), whose elements are certain rational functions. One of the reasons why this is relevant is that these rational functions are kernels that describe the action of $A^+$ in the representation $\Lambda$ of Subsection 3.2. Explicitly, it was shown in [9] that an element $R(z_1, \ldots, z_N) \in S \cong A^+$ acts in the basis of modified Macdonald polynomials by the formula:

$$R \cdot M_{\mu} = \sum_{\mu \prec \lambda} M_\lambda \cdot R(\lambda \setminus \mu) \prod_{\Box \in \lambda \setminus \mu} \left( t - q \chi_{\Box} \right) \prod_{\square \in \mu} \omega \left( \frac{\chi_{\Box}}{\chi_{\square}} \right)$$

(3.12)

where the sum goes over all skew diagrams $\lambda \setminus \mu$ of size $N$. In particular, setting $N = 1$ and $R(z_1) = 1$ gives us the first $q,t$--Pieri rule for Macdonald polynomials.

3.6. So to find out how the generator $p^{m/n}_k \in A$ acts on $\Lambda$ in the basis of modified Macdonald polynomials, we need to find out which element of the shuffle algebra it corresponds to, and then evaluate that shuffle element at the set of weights of various skew diagrams. It was shown in [11] that the isomorphism of Theorem 3.11 sends:

$$A \ni p^{m/n}_k \mapsto P^{m/n}_k \in S$$

(3.13)

where the rational function $P^{m/n}_k(z_1, \ldots, z_N)$ with $N = kn$ is given by:

$$P^{m/n}_k = \frac{(1-t)^N}{(1-q)^N} \frac{1}{\text{Sym}} \left[ \prod_{i=1}^{N} \frac{z_{r^{m,n}(i)}}{i} \sum_{j=0}^{k-1} \frac{t^j z_{n+1} \cdots z_{n+j}}{q^{z_{n+1}} \cdots q^{z_{n+j}}} \prod_{1 \leq i < j \leq N} \omega \left( \frac{z_i}{z_j} \right) \right]$$

where:

$$r^{m,n}(i) = \left[ \frac{mi}{n} \right] - \left[ \frac{(m+1) i}{n} \right]$$

Plugging this into (3.12) gives us explicit "shuffle formulas" for the generators of the elliptic Hall algebra $A^+$, and the way they act in the basis of Macdonald polynomials. In particular, plugging in $m = 0$ and $n = 1$ gives us formulas for the operators of multiplication by $p_k$ in the basis of Macdonald polynomials. Such formulas may be considered to be $q,t$--Pieri rules for Macdonald polynomials, and the coefficients boil down to sums over standard tableaux. However, instead of working with the above explicit presentation of $P^{m/n}_k$, we prefer a certain implicit characterization which was worked out in [11]:

Lemma 3.14. The rational function $P^{m/n}_k$ is the unique symmetric rational function in $N = kn$ variables and of homogenous degree $M = km$ such that:

- $P^{m/n}_k$ satisfies the wheel conditions (3.9)

- the degree of $P^{m/n}_k$ in any number of $i \in (0, N)$ of its variables is $< \frac{mi}{n}$
we have the normalization:

\[ P_{k}^{m/n}(1, q, \ldots, q^{N-1}) = \frac{(1 - q) \cdots (1 - q^{N})}{(t - q) \cdots (t - q^{N})} \cdot q^{\frac{M_{N-M+N-k}}{2}} \]

### 3.7

Recall that one of the fundamental features of the algebra \( \mathcal{A} \) is that, for any rational number \( m/n \), we have a commutative subalgebra which is isomorphic to a ring of polynomials in countably many variables:

\[ \Lambda \cong \mathcal{A}_{m/n}, \quad p_{k} \rightarrow p_{k}^{m/n} \]

Our \( \frac{m}{n} \) Pieri rules are concerned with the generators \( e_{k}^{m/n} \) that correspond to elementary symmetric functions under the above map, and the way they act on symmetric functions \( \Lambda \). In virtue of formula (3.12), to compute these operators explicitly in the basis of Macdonald polynomials, we need to obtain formulas for the corresponding shuffle elements. Along the same lines as (3.13), one shows that:

**Proposition 3.15.** Under the isomorphism \( \mathcal{A}^{\pm} \cong \mathcal{S} \) of Theorem 3.11, we have \( e_{k}^{m/n} \rightarrow E_{k}^{m/n} \), where:

\[
E_{k}^{m/n} = \frac{(1 - t)^{N}(1 - q)^{N}}{(t - q)^{N}} \text{Sym} \left[ \frac{\prod_{i=1}^{N} \frac{z_{m,n}^{r_{m,n}(i)}}{[k]_{t}!(1 - \frac{z_{mn}}{q^{2}})}}{\prod_{i=1}^{k-1} \frac{z_{m,n}^{r_{m,n}(i)}}{(1 - \frac{z_{mn}}{q^{2}})}} \prod_{1 \leq i < j \leq N} \omega \left( \frac{z_{i}}{z_{j}} \right) \right]
\]

where we write \([x]_{t} = t^{1-x} - t\) and \([x]_{1} = [1][-1][x]_{1}\).

However, we will not use the above explicit formula for \( E_{k}^{m/n} \), and so do not give a complete proof of the above Proposition. We leave it to the interested reader: it is not easy, but straightforward with the machinery developed in [11]. As in the previous paragraph, we will use the following implicit characterization à la [11]:

**Lemma 3.16.** The rational function \( E_{k}^{m/n} \) of Proposition 3.15 is the unique symmetric rational function in \( N = kn \) variables and of homogenous degree \( M = km \) such that:

- \( E_{k}^{m/n} \) satisfies the wheel conditions (3.9)

- the degree of \( E_{k}^{m/n} \) in any number \( i \in (0, N) \) of its variables is \( \leq \frac{m_{i}}{n} \)

- for any composition \( k = k_{1} + k_{2} \), we have:

\[
\lim_{\xi \rightarrow \infty} \frac{E_{k}^{m/n}(\xi z_{1}, \ldots, \xi z_{k_{1}n}, w_{1}, \ldots, w_{k_{2}n})}{\xi^{k_{1}m}} = E_{k_{1}}^{m/n}(z_{1}, \ldots, z_{k_{1}n}) E_{k_{2}}^{m/n}(w_{1}, \ldots, w_{k_{2}n})
\]

- we have the normalization:

\[ E_{k}^{m/n}(1, q, \ldots, q^{N-1}) = \delta_{k}^{1} \frac{(1 - q) \cdots (1 - q^{N})}{(t - q) \cdots (t - q^{N})} \cdot q^{\frac{M_{N-M+N-k}}{2}} \] (3.17)
Proof Proving that the explicit formula in Proposition 3.15 satisfies the above four bullets is a straightforward application of the machinery of [11], so we leave it to the interested reader. Note the fourth bullet, which claims that $E_k^{m/n}$ evaluated at $z_i = q^{n-i}$ vanishes unless $k = 1$. This clearly holds for the rational function in Proposition 3.15, because the only term which can survives evaluation is the identity permutation in the Sym, and this vanishes for $k > 1$ because of the term:

$$
\left(1 - \frac{z_n}{q^{z_{n+1}}} \right)
$$

that appears in the numerator. The difficult part is proving uniqueness, so let us consider a collection of shuffle elements $R_k$, $k \in \mathbb{N}$, satisfying the conditions in the four bullets ($m$ and $n$ will be fixed, and so we suppress them from the notation). We will obtain inductive formulas for the shuffle elements $R_k$, thus proving that they are unique. For any composition $\rho = (\rho_1, \rho_2, ..., \rho_l)$ of $N = kn$, let us write:

$$
R_k^\rho(y_1, ..., y_l) = R_k(y_1, ..., y_1q^{\rho_1-1}, ..., y_l, ..., y_lq^{\rho_l-1}) =
$$

$$
\frac{r_k(y_1, ..., y_1q^{\rho_1-1}, ..., y_l, ..., y_lq^{\rho_l-1})}{\prod_{1 \leq i < j \leq l} \prod_{a=1}^{\rho_i} \prod_{b=1}^{\rho_j} (y_jq^{b} - y_iq^{a-1}t)(y_jq^{b} - y_iq^{a+1}t^{-1})}
$$

Note that $R_k^{(1,...,1)}$ is $R$ itself, while (3.17) gives us:

$$
R_k^{(1)}(y_1) = y_1^m \left(1 - q \right)(1 - q^n) \frac{q^{\frac{m-n+1}{2}}}{(t - q)(t - q^n)}
$$

For $k > 1$

Our approach will be to obtain formulas for $R_k^\rho$ inductively, first in increasing order of $k$ and then in decreasing lexicographic order of $\rho$. By (3.12), $r_k$ of (3.18) is a Laurent polynomial that satisfies the wheel conditions (3.9). These conditions imply that $r_k$ vanishes whenever:

$$
y_iq^{a+1} = y_jq^{b}t \quad \text{for} \quad 1 < a \leq \rho_i, \quad 1 \leq b \leq \rho_j
$$

$$
y_iq^{a}t = y_jq^{b+1} \quad \text{for} \quad 1 \leq a \leq \rho_i, \quad 1 \leq b \leq \rho_j
$$

where we assume that $\rho_i \geq \rho_j$. If the inequality is the other way, then we simply change the roles of $i$ and $j$. The above zeroes are counted with the correct multiplicities, so the above fraction may be simplified to:

$$
R_k^\rho(y_1, ..., y_l) = \frac{r_k^\rho(y_1, ..., y_l)}{\prod_{1 \leq i < j \leq l} \prod_{a=1}^{\rho_i} \prod_{b=1}^{\rho_j} (y_jq^{b} - y_iq^{a-1}t)(y_jq^{b}t - y_iq^{a+1})}
$$

where $r_k^\rho(y_1, ..., y_l)$ is some Laurent polynomial in $l$ variables. The problem with the above formula is that the denominator goes over all pairs $i < j$, but tacitly makes the assumption that $\rho_i \geq \rho_j$. We do not want to make this assumption, so we could either use a much more complicated way to label the indices in the above formula, or find a better way to write it. We choose the latter approach. For any choice of:

$$
y_i = q^{x_i}t^{-y_i} \quad \text{and} \quad y_j = q^{x_j}t^{-y_j}
$$

the specializations (3.18) correspond to the weights of two horizontal strips of lengths $\rho_i$ and $\rho_j$ which start at the boxes of coordinates $(x_i, y_i)$ and $(x_j, y_j)$. Let us consider all ways to translate one horizontal strip over the other such that
they partially overlap, by which we mean that the resulting set of boxes can be divided into two horizontal strips of lengths:

$$\max(\rho_i, \rho_j) + b \quad \text{and} \quad \min(\rho_i, \rho_j) - b,$$

(3.20)

for some $b > 0$. Figure 2 below shows a certain example of partial overlapping:

\[
\begin{array}{cccc}
  y_i & & & \\
  & y_j & & \\
  & & y_j/y & \\
  & & & y_j/v
\end{array}
\]

Figure 2

In the above, we have $\rho_i = 4$ and $\rho_j = 6$. In the partial overlap on the left we have $b = 1$, whereas in the one on the right we have $b = 3$. We also allow the smallest of the resulting strips to have length 0. There are precisely $2\min(\rho_i, \rho_j)$ such translations, and they correspond to the terms which appear in the denominator of formula (3.19). So we can rewrite this formula as:

$$R_k^\rho(y_1, ..., y_l) = \prod_{i<j} \prod_{b \in S_{ij}^\pm} (y_j q^b - y_i t^\pm 1)$$

(3.21)

where the set $S_{ij}^+$ (respectively $S_{ij}^-$) consists of all integers $b$ which make the specialization $y_i = y_j q^b$ correspond to a translation that makes the $j$-th horizontal strip partially overlap the $i$-th horizontal strip, with the former (respectively latter) being to the left of the other. The two instances of partial overlap in Figure 3.7 correspond to $y_i = y_j/q$ for the example on the left, and $y_i = y_j q^5$ for the example on the right. Note that the cardinality of each of the sets $S_{ij}^\pm$ is equal to $\min(\rho_i, \rho_j)$. The advantage of (3.21) is that it makes no assumptions about the relative sizes of $\rho_i$ and $\rho_j$, because they are encoded in the definition of the sets $S_{ij}^\pm$.

We will now obtain inductive formulas for the Laurent polynomial $r_k^\rho$, first in increasing order of $k$ and then in decreasing lexicographic order of $\rho$, which will prove that this Laurent polynomial is unique. Because of the degree restrictions in the second bullet, we have:

$$\text{total deg } r_k^\rho = m + 2 \sum_{i<j} \min(\rho_i, \rho_j)$$

(3.22)

and:

$$\frac{m \rho_i}{n} \leq \deg_{y_i} r_k^\rho \leq \frac{m \rho_i}{n} + 2 \sum_{i \neq j} \min(\rho_i, \rho_j)$$

in each variable $y_i$. The space of Laurent polynomials satisfying the above degree conditions is quite large, but it is constrained by the following recurrence relations:

$$R_k^\rho(y_1, ..., y_l)|_{y_i = y_j q^b} = R_k^{\rho(i+j)}(y_1, ..., y_l)|_{y_i = y_j q^b}$$

(3.23)
for any $i < j$ and any $b \in S_{ij}^\pm$, where $\rho(i \not\leftrightarrow j)$ denotes the composition obtained from $\rho$ by replacing $\rho_i$ and $\rho_j$ by the two numbers in (3.20). Note this new composition is strictly greater than $\rho$ in lexicographic order. Moreover, the specializations in (3.23) precisely correspond to the translations of horizontal strips which go into the definition of the sets $S_{ij}^\pm$ (see Figure 2 for a visual description).

If we regard (3.23) as a Laurent polynomial in the single variable $y_l$, note that we have $2 \sum_{i < j} \min(\rho_i, \rho_j)$ conditions that involve this variable. The degree constraints (3.22) imply that $r_\rho$ only has terms $y_l^d$ for:

$$\frac{m\rho_i}{n} \leq d \leq \frac{m\rho_i}{n} + 2 \sum_{i \neq j} \min(\rho_i, \rho_j)$$

Note that there are as many such $d$ as there are conditions (3.23), unless $n|\rho_l$, when we have one extra $d$. However, in this latter case, the third bullet in the statement of the Lemma tells us that the least order term in $y_l$ is nothing but a product of $r_{\rho_l}$ and $r_{k-\rho_l}$ (up to linear factors from the denominator of (3.18)). Thus, we can apply Lagrange polynomial interpolation:

$$r_\rho^R(y_1, \ldots, y_l) = \text{least order term} \cdot \prod_{i < l} \prod_{a \in S^\pm_{il}} \left(\frac{y_l q^a}{y_i} - 1\right) +$$

$$+ \sum \sum_{i < l} \sum_{a \in S^\pm_{il}} \left[\left(\frac{y_l q^a}{y_i}\right)^{\left[\frac{m\rho_i}{n}\right]+1} r_\rho^{\rho(i \not\leftrightarrow j)}(y_1, \ldots, y_l)|_{y_l q^a = y_i} \prod_{j < l} \prod_{b \in S^\pm_{jl}} \frac{y_l q^b - y_j}{y_l q^b - y_j}\right]$$

where $\prod^*$ means that we exclude the linear factor which vanishes in the denominator (and also exclude the corresponding factor in the numerator). The first term is only thought to exist if $n|\rho_l$. To use the above recurrence, we need to translate this information in terms of the rational functions $R^R_k$:

$$R^R_k(y_1, \ldots, y_l) = \delta_{\rho,\rho_l} \gamma \cdot R^R_{k-1}(y_1, \ldots, y_{l-1}) + \sum_{a \in S^\pm_{il}} \gamma_{i,a}^\pm \cdot R^R_k(\rho(i \not\leftrightarrow l))(y_1, \ldots, y_l)|_{y_l q^a = y_i}$$

(3.24)

where $\rho' = (\rho_1, \ldots, \rho_{l-1})$ and:

$$\gamma = y_l^\frac{m\rho_i}{n} (1 - q)(1 - q^a) q^{\frac{m\rho_i - n + n - 1}{2}} \prod_{a \in S^\pm_{il}} \frac{y_l q^a - y_i}{y_l q^a t^{\pm 1} - y_i}$$

$$\gamma_{i,a}^\pm = \left(\frac{y_l q^a}{y_i}\right)^{\left[\frac{m\rho_i}{n}\right]+1} \prod_{b \in S^\pm_{il}} \frac{y_l q^b - y_j}{y_l q^b - y_j} \prod_{b \in S^\pm_{il}} \frac{y_l q^b - y_j t^{\pm 1}}{y_l q^b - y_j t^{\pm 1}}$$

(3.25)

The first term in (3.24) arises because of the third and fourth bullets in the statement of the Lemma. Recall that the $*$ above the product signs mean that we discard the factor which vanishes from the denominator, and also the corresponding factor from the numerator. The above formula concludes the proof, because it shows how the rational function $R_k = R_k^{(1,\ldots,1)}$ can be reconstructed inductively from the functions $R_{k'}$ for $k' < k$. \qed
4. Stable bases

4.1. We will now describe a construction of Maulik and Okounkov ([8]), known as the stable basis. This construction is geometric and very general, but we will simply spell out the definition in our particular setup. For any \( \frac{m}{n} \in \mathbb{Q} \), Maulik-Okounkov prove the existence of a unique integral basis \( \{ s^{m/n}_\lambda \} \) partition of \( \Lambda \):

\[
 s^{m/n}_\lambda = \sum_{\mu \subseteq \lambda} M_\mu \cdot c^\mu_\lambda(q,t)
\]

where \( M_\mu \) are the renormalized Macdonald polynomials (2.5), such that:

\[
 c^\mu_\lambda(q,t) = \prod_{\Box \in \lambda} \left( t^{-l(\Box)} - q^{a(\Box) + 1} \right)
\]

and \( c^\mu_\lambda \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \) are such that for all \( \mu < \lambda \) in the dominance ordering:

\[
 \deg_\prec c^\mu_\lambda < \frac{m}{n}(o_\mu - o_\lambda) + \max_\mu \\
 \deg_\succ c^\mu_\lambda \geq \frac{m}{n}(o_\mu - o_\lambda) + \min_\mu
\]

where we define the upper degree of a Laurent polynomial \( c(q,t) \) as:

\[
 \deg_\prec c(q,t) = \text{order of } c(az, bz) \text{ as } z \to \infty
\]

and the lower degree as:

\[
 \deg_\succ c(q,t) = \text{order of } c(az, bz) \text{ as } z \to 0
\]

and set:

\[
 a_\lambda = \sum_{\Box \in \lambda} a(\Box) \\
 \text{min}_\lambda = -\sum_{\Box \in \lambda} l(\Box) \\
 \text{max}_\lambda = |\lambda| + \sum_{\Box \in \lambda} a(\Box)
\]

From now on, the phrase "term of highest/lowest degree" of a Laurent polynomial \( c \) will refer to the sum of monomials in \( q, t \) for which the order in (4.4)/(4.5) is attained. For example, we have:

\[
 \text{h.d. } (q - t + t^{-1}q^{-1}) = q - t, \\
 \text{l.d. } (q - t + t^{-1}q^{-1}) = t^{-1}q^{-1}
\]

4.2. Let us explain conditions (4.2) and (4.3): they require that the Newton polygon of \( c^\mu_\lambda(q,t) \) lies in a certain strip, and the only case when the rightmost side of the strip is allowed to be attained is when \( \mu = \lambda \). The notations \( \prec \) and \( \succ \) for these orders refer to the preferred directions along Young diagrams which are specified by conditions (4.4) and (4.5). Although we will neither need nor prove this statement, it is well-known from the geometry of the Hilbert scheme that \( s^0_\lambda \) given by the above definition coincides with the usual Schur function \( s_\lambda \). Moreover, note that it is immediate from the definition that:

\[
 s^{r+1}_\lambda = \frac{\nabla s^r_\lambda}{\prod_{\Box \in \lambda} \lambda(\Box)}
\]

for any rational number \( r \). Finally, when \( r = \infty \), note that conditions (4.2) and (4.3) boil down to requiring that \( c^\mu_\lambda = 0 \) for \( \mu < \lambda \). Therefore, \( s^\infty_\lambda = P_\lambda \) are the usual Macdonald polynomials. However, these are not integral, and so

---

4 A symmetric function is integral if it expands in terms of Schur functions with coefficients in \( \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \). A basis is integral if it consists entirely of integral functions.

5 For example, the function \( c(q,t) = q - t + t^{-1}q^{-1} \) has \( \deg_\prec c = 1 \) and \( \deg_\succ c = -2 \).
4.3. Our main Theorem 1.2 involves showing how the operator $e^{m/n}_k$ acts on the stable basis. To obtain the required formula, we need to compute bounds on the upper and lower degrees of the matrix coefficients of this operator in the basis $M_\lambda$. By formula (3.12), we need to bound the upper and lower degrees of the expressions:

$$E_k^{m/n}(\lambda\setminus\mu)$$

as $\lambda\setminus\mu$ goes over all skew diagrams of $N = kn$ boxes, where $E_k^{m/n}$ is the shuffle element of Proposition 3.15 and Lemma 3.16. For any diagram $\lambda\setminus\mu$, let us write:

$\#_{\lambda\setminus\mu}$ = the number of pairs of boxes in $\lambda\setminus\mu$ of the same content

Then we will prove the following:

**Proposition 4.6.** For any skew diagram $\lambda\setminus\mu$ of $N = kn$ boxes, we have:

$$\deg_{\downarrow} E_k^{m/n}(\lambda\setminus\mu) \leq \frac{m}{n}(o_\lambda - o_\mu) + \frac{k(n-1)}{2} + \#_{\lambda\setminus\mu}$$ (4.7)

$$\deg_{\uparrow} E_k^{m/n}(\lambda\setminus\mu) \geq \frac{m}{n}(o_\lambda - o_\mu) - \frac{k(n+1)}{2}$$

In the first relation, we have equality if and only if $\lambda\setminus\mu$ is a vertical $k$–strip of $n$–ribbons $B_1, \ldots, B_k$ (see Subsection 2.5 for the definition), in which case the term of highest degree is:

$$\text{h.d. } E_k^{m/n}(\lambda\setminus\mu) = \left(\frac{q}{t}\right)^{\#_{\lambda\setminus\mu}} t^{\#_{\lambda\setminus\mu}}$$ (4.8)

where $\#_{\lambda\setminus\mu}$ denotes the number of boxes $\blacksquare \in \lambda\setminus\mu$ which are precisely $p$ diagonal steps away from an inner corner or vertical inner boundary of $\lambda\setminus\mu$, counted with multiplicity $p+1$, and $\chi_{i,j}$ denotes the content of the $j$–th box in the ribbon $B_i$.

**Proof** Let us prove relations (4.7) and (4.8) on $\deg_{\downarrow}$, and leave the analogous relation on $\deg_{\uparrow}$ as an exercise. We write $R_k = E_k^{m/n}(z_1, \ldots, z_N)$ with $N = kn$ and use the notations in the proof of Lemma 3.16. In particular, we will prove a slightly more general statement.

**Claim 4.9.** Let us consider a union $D = \{D_1, \ldots, D_l\}$ of horizontal strips $D_i$ lying in the first quadrant. Then we have:

$$\deg_{\downarrow} R_k(D) \leq \frac{m}{n} o_D + \#_D + \frac{k(n-1)}{2}$$ (4.10)

by which we mean that we apply the rational function $R_k$ to the set of weights of the boxes in $D$. We write $o_D = \sum \square \in D o_\square$ and $\#_D$ stands for the number of pairs of boxes of the same content in $D$. 

---

$^6$That is, the sum of monomials which produces the top degree term in (4.4)
Let \( \rho = (\rho_1, \ldots, \rho_t) \) denote the composition of \( N = kn \) determined by the lengths of our horizontal strips. We will prove this claim by induction in increasing order of \( k \), and then in decreasing lexicographic order of \( \rho \). The base case is dealt with by (3.17). The quantity whose lower degree we need to bound is:

\[
R_k(D) = R'_k(y_1, \ldots, y_t)
\]

where we set \( y_i \) to be the weight of the leftmost box of the \( i \)-th strip, and we denote \( o_i \) the content of the box of weight \( y_i \). The above quantity is a certain evaluation of the LHS of (3.24), so we may replace it by the corresponding evaluation of the RHS of (3.24). We may apply the induction hypothesis to the terms that appear in the RHS:

\[
deg_{\gamma} R'_{k-1}(y_1, \ldots, y_{t-1}) \leq \frac{m}{n} \cdot o_{D'} + \frac{(k-1)(n-1)}{2}
\]

\[
deg_{\gamma} R_k^{\langle i, o \rangle}(y_1, \ldots, y_t) |_{y_i \to y_i} \leq \frac{m}{n} \cdot o_{D'_{i,a}} + \frac{k(n-1)}{2}
\]

where \( \#' \) and \( \#_{i,a}^{\pm} \) denote the number of pairs of boxes of the same content among the arguments of each \( R \) in the LHS, computed with respect to the boxes in the sets \( D' \) and \( D'_{i,a} \). These latter sets of boxes are what is obtained from the set \( D \) by removing the \( l \)-th strip, respectively by performing the substitution \( y_i q^a = y_i \) (which amounts to translating the entire \( l \)-th horizontal strip). As for the coefficients \( \gamma \) and \( \gamma_{i,a}^{\pm} \), these are just products of linear factors, and we claim that:

\[
deg_{\gamma} \gamma = \#D - \#' + \frac{m}{n}(o_D - o_{D'}) + \frac{n-1}{2}
\]

\[
deg_{\gamma} \gamma_{i,a}^{\pm} \leq \#D - \#_{i,a}^{\pm} + \frac{m}{n}(o_D - o_{D'_{i,a}})
\]

Once we prove the above claims, we will have proved the induction step that establishes Claim 4.9. So let us show how to prove the more difficult equality (4.12) and leave the first as an analogous exercise. According to the formula at the end of the proof of Lemma 3.16, the coefficient \( \gamma_{i,a}^{\pm} \) consists of three parts:

\[
deg_{\gamma} \left( \frac{y_i q^a}{y_i} \right)^{\lceil \frac{m y_i}{n} \rceil} = (o_i + a - o_i) \left( \lceil \frac{m y_i}{n} \rceil + 1 \right)
\]

\[
deg_{\gamma} \prod_{b \in S_{jl}^l} \frac{y_i q^b - y_j}{y_i q^b - a - y_j} = -\max(o_i + a - o_i, 0) + \sum_{b \in S_{jl}^l} \max(o_i + b, o_j) - \max(o_i + b - a, o_j)
\]

\[
deg_{\gamma} \sum_{b \in S_{jl}^l} \frac{y_i q^b - a - y_j}{y_i q^b - y_j t^{b+1}} = \sum_{b \in S_{jl}^l} \max(o_i + b - a, o_j + 1) - \max(o_i + b, o_j + 1)
\]

The first equation is clear, so let’s focus on the last two. The first terms in the RHS (of each of the last two lines) comes from the * in the products: there is a factor missing. Adding the three equalities together gives us:

\[
deg_{\gamma} \gamma_{i,a}^{\pm} = (o_i + a - o_i) \left( \lceil \frac{m y_i}{n} \rceil + 1 \right) - \max(o_i + a - o_i, 0) + \sum_{b \in S_{jl}^l} \max(o_i + b - a, o_j + 1)
\]
It is straightforward to show that the first term on the last line equals simply unwinding the definition of \( o \), where in (4.10) can only be attained if we can perform certain moves of the following kinds:

1. remove an entire horizontal strip of length \( n \)
2. slide a horizontal strip diagonally until it partially overlaps with another horizontal strip

and manage to eliminate the entire set of boxes \( D \) without ever obtaining a horizontal strip of length \( > n \). We are free to choose which strips to remove/slide at each step, and when \( D = \lambda \setminus \mu \) is a skew Young diagram there is a preferred choice which will make all but one of the coefficients \( \gamma_{i,a} \) vanish.

The thing to do is to let \( D_l \) be the topmost row of \( \lambda \setminus \mu \). If we slide it diagonally onto a row \( D_l \) that is more than one step down, then the corresponding coefficient \( \gamma_{i,a} \) vanishes because of the numerator of (4.13) and the fact that no translation in \( S_{jl}^+ \) can be a diagonal slide, for all rows \( j \neq l \). So the only slide that produces a non-zero coefficient is precisely one row down, thus producing two overlapping strips. Take the longest of these strips: again, if we slide it more than one row down, the corresponding coefficient \( \gamma_{i,a} \) vanishes because of the numerator of (4.13).

Repeating this argument, we see that the only way to obtain a non-zero coefficient is by sliding one row at a time and then removing a row whenever it has length \( n \). This is precisely the game in Subsection 2.6, and we proved in Lemma 2.8 that the game ends with all boxes removed, thus producing equality in (4.10), only if \( \lambda \setminus \mu \) can be covered by a vertical \( k \)-strip of \( n \)-ribbons.

In this case, we need only evaluate the term of highest degree in \( R(\lambda \setminus \mu) \), and we will do so by using the same formula (3.24). Letting \( D = \lambda \setminus \mu \) and writing \( \rho_1, \ldots, \rho_l \) for the lengths of the rows of \( D \) ordered bottom to top, we have:

\[
\text{h.d. } R_k(D) = \left( \text{h.d. } R_k(D') \right) \cdot \left( \text{h.d. } \gamma_{i,a}^{+} \right) = \left( \text{h.d. } R_k(D') \right) \cdot \left( \frac{\mu}{\rho} \right)^{\frac{n^2}{n}}.
\]

\[
\prod_{\beta \neq \alpha} \frac{\nu^{\beta} - \nu^{\alpha}}{\nu^{\beta} - \nu^{\alpha}} \prod_{\beta \neq \alpha} \frac{\nu^{\beta} - \nu^{\alpha}}{\nu^{\beta} - \nu^{\alpha}} \prod_{\beta \neq \alpha} \frac{\nu^{\beta} - \nu^{\alpha}}{\nu^{\beta} - \nu^{\alpha}}
\]

where \( D' \) is the multiset of boxes obtained by sliding the first row of \( D \) diagonally down onto its second row. We continue sliding the top row as many times as the
height of the outer ribbon $B_{\text{out}}$ of $\lambda \setminus \mu$. As explained in Subsection 2.6, at the end of this procedure we will have straightened the outer ribbon into a horizontal strip of length $n$. When we remove this horizontal strip, we have:

$$\text{h.d. } R_k(D) = (\text{h.d. } R_k(D \setminus B_{\text{out}})) \cdot (\text{h.d. } \gamma) \left( \frac{q}{t} \right)^* .$$

$$\cdot \left( \frac{q}{t} \right)^{b_{\text{out}}} \prod_{\square \text{ start of some row}} \frac{\chi_{\square}(D \setminus B_{\text{out}})}{t-1} - 1 \prod_{\square \text{ start of some row}} \frac{\chi_{\square}(D \setminus B_{\text{out}})}{t-1}$$

where $b_{\text{out}}$ denotes the number of boxes $\square \in B_{\text{out}}$ which are precisely $p$ steps diagonally from an inner corner or vertical inner boundary of $\lambda \setminus \mu$, counted with multiplicity $p + 1$. The exponent $*$ of $q/t$ on the first line precisely amounts to the ratio between the quantity:

$$\prod_{i=1}^{n} \chi_i(B_{\text{out}}) \left| \frac{m_i}{n} \right| - \left| \frac{m(i-1)}{n} \right|$$

before and after bubbling down the outer ribbon. We will now use the definition of $\gamma$ in (3.25), and also cancel like factors in the second line above:

$$\text{h.d. } R_k(D) = (\text{h.d. } R_k(D \setminus B_{\text{out}})) \prod_{\square \text{ on the same diagonal as } \square \in D \setminus B_{\text{out}}} \left( \frac{q}{t} \right)^{b_{\text{out}}} \prod_{\square \text{ inner corner of } \lambda \setminus \mu} \left( \frac{q}{t} \cdot \frac{\chi_{\square}(D \setminus B_{\text{out}})}{t-1} - 1 \right) \prod_{\square \text{ outer corner of } \lambda \setminus \mu} \left( \frac{q}{t} \cdot \frac{\chi_{\square}(D \setminus B_{\text{out}})}{t-1} - 1 \right)$$

Iterating the above for the skew diagram $D \setminus B_{\text{out}}$, which is covered by a vertical strip of one less $n-$ribbons than $D$, we obtain precisely (4.8).

\[\square\]

4.4. We will now use the previous result to obtain Theorem 1.2.

**Proof of Theorem 1.2:** Let us consider the class:

$$\sigma = e_{k}^{m/n} \cdot s_{\mu}^{m/n} = e_{k}^{m/n} \cdot \sum_{\nu \leq \mu} M_{\nu} c_{\nu}(q, t)$$

for some positive integer $k$ and some partition $\mu$. We may compute the RHS using formula (3.12):

$$\sigma = \sum_{\lambda} M_{\lambda} \cdot \sum_{\nu \leq \mu} c_{\nu}(q, t) E_{k}^{m/n}(\lambda \setminus \nu) \prod_{\square \in \lambda \setminus \nu} \left( t - q \cdot \omega(\lambda \setminus \nu) \right) \prod_{\square \in \nu} \omega(\lambda \setminus \nu) =: \sum_{\lambda} M_{\lambda} \cdot d^{\lambda}$$

We want to prove that $\sigma$ equals the RHS of (1.3). By the uniqueness of the stable basis satisfying the properties of Subsection 4.1, it is enough to prove the following assertions about the coefficients $d^{\lambda}$ of the above expression:

$$\text{deg}_< d^{\lambda} \leq \frac{m}{n}(o_{\lambda} - o_{\mu}) + \max_{\lambda} + \frac{k(n-1)}{2}$$

and:

$$\text{deg}_< d^{\lambda} \geq \frac{m}{n}(o_{\lambda} - o_{\mu}) + \min_{\lambda} + \frac{k(n-1)}{2}$$
and that the first inequality becomes an equality precisely when $\lambda \setminus \mu$ can be covered by a $k$–strip of $n$–ribbons $B_1, \ldots, B_k$, with the term of highest degree equal to the monomial:

$$h.d. \ d^\lambda = (-1)^{|\lambda\setminus\mu|} q^{\max_{\lambda\setminus\mu}} (-1)^{ht} \prod_{i=1}^{k} \prod_{j=1}^{n} \chi_j(R_i) \left[ \frac{m}{n} \right] - \left[ \frac{m(n-1)}{n} \right] \quad (4.16)$$

with the notations of Theorem 1.2. We will prove inequality (4.14) and relation (4.16), and leave (4.15) as an analogous exercise. We have:

$$\deg_{\lambda\setminus\mu} c^\mu(q, t) \leq \frac{m}{n}(a_\nu - a_\mu) + \max_\nu$$

by (4.5), with equality if and only if $\mu = \nu$, while Proposition 4.6 gives us:

$$\deg_{\lambda\setminus\mu} E_k^{m/n}(\lambda\setminus\nu) \leq \frac{m}{n}(a_\lambda - a_\nu) + \#_{\lambda\setminus\nu} + \frac{k(n-1)}{2}$$

It is straightforward to compute the upper and lower limits of a product of linear factors, so note that we have:

$$\deg_{\lambda\setminus\mu} \prod_{\square \in \lambda\setminus\mu} \left[ (t-q\chi_\square) \prod_{\square \in \nu} \omega \left( \frac{\chi_\square}{\chi_\boxdot} \right) \right] = \sum_{\square \in \lambda\setminus\mu} [1 + \max(0, a_\square) + \# \text{ of } \square \in \nu \text{ on the same diagonal as } \square] = \max_{\lambda\setminus\mu} - \max_{\lambda\setminus\nu} - \#_{\lambda\setminus\nu}$$

Adding the above three inequalities gives us precisely the desired (4.15). As for (4.16), we need to compare the highest order terms. These only occur when $\nu = \mu$. By (4.5), we have:

$$h.d. \ c^\mu_{\mu}(q, t) = (-1)^{|\mu|} q^{\max_{\mu}} \quad (4.17)$$

while Proposition 4.6 gives us a formula for h.d. $E_k^{m/n}(\lambda\setminus\mu)$. Finally, it is straightforward to compute the lowest order term of a product of linear factors:

$$\prod_{\square \in \lambda\setminus\mu} \left[ (t-q\chi_\square) \prod_{\square \in \mu} \omega \left( \frac{\chi_\square}{\chi_\boxdot} \right) \right] = (-1)^{kn} \prod_{\square \in \lambda\setminus\mu} \left[ (t-q\chi_\square) \prod_{\square \in \mu} \omega \left( \frac{\chi_\square}{\chi_\boxdot} \right) \right] = (-t)^{kn}$$

where the sets $D_1, D_2, D_3, D_4$ consist of those boxes $\square = (x, y) \in \lambda\setminus\mu$ that are on the same diagonal as a box $\boxdot = (x_0, y_0)$ which is an inner corner, outer corner, vertical inside edge or horizontal inside edge of $\lambda\setminus\mu$. We force a factor out of the above formula to obtain:

$$= (-t)^{kn} \prod_{\square \in \lambda\setminus\mu} \left[ (t-q) \prod_{\square \in \mu} \omega \left( \frac{\chi_\square}{\chi_\boxdot} \right) \right] \prod_{\square \in D_1 \cup D_2} q^{\tau_0} \prod_{\square \in D_2 \cup D_3} q^{t+1} y_0 - y - 1 \prod_{\square \in D_3 \cup D_4} (-1)$$

and

$$= (-t)^{kn} \prod_{\square \in \lambda\setminus\mu} \left[ (t-q) \prod_{\square \in \mu} \omega \left( \frac{\chi_\square}{\chi_\boxdot} \right) \right] \prod_{\square \in D_1 \cup D_2} q^{\tau_0} \prod_{\square \in D_2 \cup D_3} q^{t+1} y_0 - y - 1 \prod_{\square \in D_3 \cup D_4} (-1)$$
where recall that \( b_{\lambda \setminus \mu} \) counts the number of \( \square \) which are \( p \) steps in the northeast direction from an inner corner or vertical inside edge, counted with multiplicity \( p + 1 \). Here we have used the simple observation that:

\[
\sum_{\square_1 = (x,y) \in D_1 \cup D_3} (x - x_0 + 1) = \sum_{\square_2 = (x,y) \in D_1 \cup D_3} (y - y_0 + 1) = b_{\lambda \setminus \mu}
\]

But now observe that:

\[
\sum_{\square_1 = (x,y) \in \lambda \setminus \mu} (x + 1) = \max_\lambda - \max_\mu, \quad \sum_{\square_2 = (x,y) \in \lambda \setminus \mu} (y - y_0 - 1) = -\#_{\lambda \setminus \mu} - kn
\]

so when \( \lambda \setminus \mu \) is a vertical \( k \)-strip of \( n \)-ribbons \( B_1, ..., B_k \), the highest degree term in the above formula amounts to:

\[
(-1)^{kn} \left( \frac{t}{q} \right)^{\#_{\lambda \setminus \mu}} \prod_{\text{inner corner of } \mu} b_{\lambda \setminus o} b_{\lambda \setminus o} \left( \frac{q}{t} \right) - 1 \prod_{\text{outer corner of } \mu} b_{\lambda \setminus o} b_{\lambda \setminus o} \left( \frac{q}{t} \right) - 1 \left( -1 \right)^{ht} q^{\max_\lambda - \max_\mu} t^{-\#_{\lambda \setminus \mu}}
\]

Multiplying the above factor with (4.8) and (4.17) gives us the desired (4.16).

\[\square\]

5. The \( U_v(\hat{\mathfrak{g}}_{\mathfrak{l}}) \)-module structure

5.1. The algebra \( \mathcal{A} \) of Subsection 3.1 is the positive half of its Drinfeld double:

\[DA = \langle p_v, v \in \mathbb{Z}^2 \setminus (0,0) \rangle\]

The algebra \( DA \) has twice as many generators as \( \mathcal{A} \), and relations (3.1) and (3.2) need to be slightly amended:

\[
[p_{kn,km}, p_{ln,lm}] = k \delta^0_{k+l} \frac{1 - q^{-k}}{1 - t^k} \left( \frac{q}{t} \right)^{\frac{kn}{k}} - \left( \frac{q}{t} \right)^{\frac{kn}{k}}
\]

for any \( k, l > 0 \) and any coprime \( m, n \), and:

\[
[p_v, p_{v'}] = \left( \frac{t}{q} \right)^{\alpha(v,v')} \theta_{v + v'}
\]

in the notation of Subsection 3.1. In the above, \( \alpha(v, v') \) denotes a certain integer that needs to be added to the formula when the lattice points \( v \) and \( v' \) lie on opposite sides of the vertical axis. We will not review the exact definition here, as we will not need it, but the interested reader may find it in [10]. It is also shown in \textit{loc cit} that the whole double algebra \( DA \) acts on \( \Lambda \), where:

\[
p_{-n,m} = -p^\dagger_{n,m} \cdot \left( \frac{t}{q} \right)^{\frac{2}{q}}
\]

for all \( n > 0 \). The adjoint is taken with respect to the inner product (2.4).
5.2. Let us consider the quantized infinite dimensional Heisenberg algebra:

$$U_v(\hat{\mathfrak{gl}}_1) = \langle \ldots, \pi_{-2}, \pi_{-1}, \pi_1, \pi_2, \ldots \rangle / [\pi_k, \pi_l] = k\delta_{k+l} v^{kn-k_l}$$

(5.3)

Then relation (5.1) says that for any fixed slope \(m/n\), the algebra \(DA\) contains a copy of \(U_v(\hat{\mathfrak{gl}}_1)\) via the embedding:

$$U_v(\hat{\mathfrak{gl}}_1) \hookrightarrow DA \quad \pi_k \rightarrow p_k^{m/n}$$

(5.4)

where we set \(v = \sqrt{q/t}\) and:

$$p_k^{m/n} = p_{kn,km} \quad p_{-k}^{m/n} = p_{-kn,-km} \cdot \left( -v^{-k} \frac{1 - q^k}{1 - q^{-k}} \right)$$

for any \(k > 0\). Instead of working with the generators \(\pi_k\) of \(U_v(\hat{\mathfrak{gl}}_1)\), which are like power-sum functions, we can introduce generators \(\varepsilon_k\) which behave like elementary symmetric functions. By this we mean that:

$$\varepsilon_{\pm 1} = \pi_{\pm 1}, \quad \varepsilon_{\pm 2} = \frac{\pi_{\pm 2}^2 - \pi_{\pm 2}}{2}, \quad \varepsilon_{\pm 3} = \frac{\pi_{\pm 3}^3 - 3\pi_{\pm 3}\pi_{\pm 1} + 2\pi_{\pm 2}}{6} \quad \text{etc}$$

Let us denote the images of these generators under the map (5.4) by \(e_k^{m/n} \in DA\).

When \(k > 0\), the formulas by which these act on \(\Lambda\) were given in Theorem 1.2. We will now give the analogue of this result for \(k < 0\).

**Theorem 5.5.** For any \((m, n) \in \mathbb{Z} \times \mathbb{N}\) and any positive integer \(k\), we have:

$$e_{-k}^{m/n} \cdot s_{\lambda}^{m/n} = \frac{(-1)^{kn}}{(qt)^{kn-n_k}} \cdot \sum_{\mu} s_{\mu}^{m/n} (-1)^{\text{width}} \prod_{i=1}^{k} \prod_{j=1}^{n} \chi_j(R_i) \left| \frac{m_j}{n} \right| - \left| \frac{m_j - 1}{n} \right|$$

(5.6)

where the sum goes over all horizontal \(k\)-strips of \(n\)-ribbons of shape \(\lambda \setminus \mu\), and the remaining notations are as in Theorem 1.2.

The proof of the above Theorem goes through almost word by word as that of Theorem 1.2, and so we will not repeat it. The starting point is the same formula (3.12), which tells us how the operators \(e_k^{m/n}\) act in the basis of renormalized Macdonald polynomials. By (5.2), the operator \(e_{-k}^{m/n}\) is the adjoint of \(e_k^{-m/n}\) up to an overall constant, and so its matrix coefficients in the basis of Macdonald polynomials are obtained from (3.12) by switching \(M_{\mu}\) and \(M_{\lambda}\) (up to a product of linear factors). This accounts for two differences between the present formula (5.6) and (1.3):

- we have \(-m\) in the exponents of Theorem 5.5, as opposed from the exponent \(m\) that appeared in Theorem 1.2
- we consider horizontal strips of ribbons (and their width) instead of vertical strips of ribbons (and their height). The reason for this is the sign in (5.2), which tells us that we need to flip the sign of power sum functions as we go from positive to negative. This has the effect of replacing elementary symmetric by complete symmetric functions.
5.3. For us, the complementary side of the quantum Heisenberg algebra $U_v(\hat{gl}_n)$ is the Kac-Moody algebra $U_\infty(\hat{sl}_n)$. The latter is generated by $x_i^-, \kappa_i, x_i^+$ for $i \in \{0, \ldots, n-1\}$ under the following relations:

$$\kappa_i x_i^+ \kappa_i^{-1} = v^{i(\delta_i^+ - 2\delta_i + \delta_{i+1})} x_i^+$$

$$[x_i^+, x_j^-] = \delta_i^i \cdot \kappa_i^{-1} \frac{\kappa_i - \kappa_i^{-1}}{v - v^{-1}}$$

for any $i, j \in \{0, \ldots, n-1\}$, and:

$$x_i^\sigma x_{i \pm 1}^\sigma - (v + v^{-1}) x_i^\sigma x_{i \pm 1}^\sigma + x_{i \pm 1}^\sigma x_i^\sigma = 0$$

$$[x_i^\sigma, x_j^\sigma] = 0$$

for any $\sigma \in \{+, -\}$ and any $j \neq i \pm 1$. We then construct:

$$U_v(\hat{gl}_n) = U_v(\hat{gl}_1) \otimes U_v(\hat{sl}_n)$$

where the two tensor factors are thought to commute. The authors of [6] constructed an action of $U_v(\hat{gl}_n)$ on any vector space $V$ with a basis $|\lambda\rangle$ indexed by partitions. We will now introducing the extra number $m$ into their construction.

5.4. To do so, let us define the crust of a partition $\lambda$ as the infinite ribbon starting at $(0, \infty)$, sitting upon the Young diagram of $\lambda$ and ending at $(\infty, 0)$. Starting at a given inner (resp. outer) corner $\blacksquare$ of $\lambda$, we follow the crust of $\lambda$ (resp. $\lambda \setminus \blacksquare$) toward the southeast. If the $k$-th step of this walk is down, as opposed to right, we call it a downstep. Similarly, we can start at $\blacksquare$ and follow the crust toward the northwest. If the $k$-th step of this walk is left, as opposed to up, we call it a leftstep. Note that there are finitely many downsteps and leftsteps.

Given coprime natural numbers $m$ and $n$, we call a function $\alpha : \mathbb{Z} \to \mathbb{Z}$ admissible if it is $n$-periodic and satisfies:

$$\alpha(j) + \alpha(1-j) = \delta_{j \equiv 0} - \delta_{j \equiv 1} + 2 \left\lfloor \frac{mj}{n} \right\rfloor - 2 \left\lfloor \frac{m(j-1)}{n} \right\rfloor - \frac{2m}{n} \tag{5.8}$$

for all $j \in \mathbb{Z}$.

**Theorem 5.9.** Consider any coprime natural numbers $m, n$, and any admissible function $\alpha$. For any vector space $V$ with a basis $|\lambda\rangle$ indexed by partitions, there is an action of $U_v(\hat{gl}_n)$ on $V$ via:

$$\kappa_i |\lambda\rangle = |\lambda\rangle \cdot \alpha(\# \text{ outer corners of content } \equiv i) - (\# \text{ inner corners of content } \equiv i)$$

\[
x_i^+ |\lambda\rangle = \sum \text{ inner corner } \lambda^\blacksquare \cdot \chi^\blacksquare \prod \text{ downstep } j \text{ steps } v^{\delta_n i - \delta_{n} i - 1 + \alpha(j)} \prod \text{ leftstep } j \text{ steps } v^{-\alpha(j)}
\]

\[
x_i^- |\lambda\rangle = \sum \text{ outer corner } \lambda^\blacksquare \cdot \chi^\blacksquare \prod \text{ downstep } j \text{ steps } v^{-\alpha(j)} \prod \text{ leftstep } j \text{ steps } v^{\delta_n i + \delta_{n} i - 1 - \alpha(j)}
\]
Meanwhile, for all $k > 0$ we set:

$$
\varepsilon_k |\mu\rangle = \sum |\lambda\rangle (-1)^{ht} \prod_{i=1}^{k} \prod_{j=1}^{n} \chi_j (R_i) \left\lfloor \frac{m j}{n} \right\rfloor - \left\lfloor \frac{m(j-1)}{n} \right\rfloor
$$

(5.10)

where the sum goes over all vertical $k -$ strips of $n -$ ribbons of shape $\lambda \setminus \mu$, and:

$$
\varepsilon_{-k} |\lambda\rangle = \frac{(-1)^{k n}}{(q t)^{\frac{kn}{2}}} \sum |\mu\rangle (-1)^{\text{width}} \prod_{i=1}^{k} \prod_{j=1}^{n} \chi_j (R_i) \left\lfloor \frac{m j}{n} \right\rfloor - \left\lfloor \frac{m(j-1)}{n} \right\rfloor
$$

(5.11)

where the sum goes over all horizontal $k -$ strips of $n -$ ribbons of shape $\lambda \setminus \mu$.

Remark 5.12. A particularly natural admissible function $\alpha$ is:

$$
\alpha(j) = \left\lfloor \frac{m j}{n} \right\rfloor + \left\lfloor \frac{m j}{n} \right\rfloor - \frac{2mj}{n}
$$

However, we do not know why this choice (or any other one) would be preferable from the point of view of the representation theory of $U_q(\mathfrak{gl}_m)$. We also do not know what role the number $m$ or the parameters $q$ and $t$ play in the representation theory, where the only natural parameter is $v = \sqrt{\frac{q}{t}}$. We believe that these structures come from the representation theory of the quantum toroidal (double affine) algebra of $\mathfrak{gl}_{1,n}$, though we do not know how to make this precise.

Remark 5.13. The action constructed in [6] corresponds to $m = 0$ and $\alpha(j) = 0$ for all $j$. Note that this case does not exactly fit into the framework of the above theorem, because $\gcd(0, n) \neq 1$. In fact, to account for this issue, the formulas of loc. cit and ours differ by a certain twist by $v$ of the formulas defining $\varepsilon_{\pm k}$. For simplicity, we stick to the choice of coprime $(m, n)$ so as to not overburden the notation.

**Proof** We will first prove the relations between the $x_i^+, \kappa_i, x_i^+$, which do not require the function $\alpha$ to be admissible in the sense of (5.8). This choice will only come into play when we check commutation with $\varepsilon_k$. Let us start by showing that:

$$
x_i^+ x_j^+ |\lambda\rangle = x_j^+ x_i^+ |\lambda\rangle
$$

(5.14)

for any Young diagram $\lambda$ and any $j \notin \{i - 1, i, i + 1\}$. Both sides of the equation add a box $\square$ of content $\equiv i$ and a box $\blacksquare$ of content $\equiv j$, and we need to check that the coefficients are the same. Suppose without loss of generality that $\square$ is situated southeast of $\blacksquare$, i.e. $o_\square > o_{\blacksquare}$. If we let:

$$
\gamma_1 = \langle \lambda + \square | x_j^+ | \lambda \rangle \cdot \langle \lambda + \blacksquare | x_j^+ | \lambda \rangle
$$

then the coefficients of the two sides of (5.14) differ from $\gamma_1$ simply by certain powers of $v$. Explicitly,

$$
(\lambda + \square + \blacksquare | x_i^+ x_j^+ | \lambda) = \gamma_1 \cdot v^{\beta(\square) - \beta(\blacksquare)} = (\lambda + \square + \blacksquare | x_j^+ x_i^+ | \lambda)
$$

where we write $\beta(j) = \alpha(j + 1) - \alpha(j)$. Hence, equality (5.14) holds. The case of $-$ is dealt with analogously. Let us now show that:

$$
x_i^+ x_j^+ x_{i \pm 1}^+ |\lambda\rangle - (v + v^{-1}) x_j^+ x_{i \pm 1}^+ x_i^+ |\lambda\rangle + x_{i \pm 1}^+ x_i^+ x_j^+ |\lambda\rangle = 0
$$

(5.15)
All terms in the above add two boxes \( \Box_1, \Box_2 \) of content \( \equiv i \) and a box \( \blacksquare \) of content \( \equiv i \pm 1 \) to \( \lambda \). Let us assume that they are ordered \( \blacksquare, \Box_1, \Box_2 \) from northwest to southeast (the other two cases are dealt with analogously) and compute the coefficients of the three operators in (5.15). If we let:

\[
\gamma_2 = \langle \lambda + \Box_1 | x_i^+ | \lambda \rangle \cdot \langle \lambda + \Box_2 | x_i^+ | \lambda \rangle \cdot \langle \lambda + \blacksquare | x_{i+1}^+ | \lambda \rangle
\]

then the coefficients of the two sides of (5.15) differ from \( \gamma_2 \) by certain explicit powers of \( v \):

\[
\begin{align*}
(\lambda + \Box_1 + \Box_2 + \blacksquare | x_i^+ x_i^+ x_{i+1}^+ | \lambda) &= \gamma_2 \cdot v^{\beta(o\Box_1 - o\blacksquare) + \beta(o\Box_2 - o\blacksquare) + \beta(o\Box_2 - o\Box_1)} \\
(\lambda + \Box_1 + \Box_2 + \blacksquare | x_i^+ x_i^+ x_{i+1}^+ | \lambda) &= \gamma_2 \cdot v^{\beta(o\Box_1 - o\blacksquare) + \beta(o\Box_2 - o\blacksquare) + \beta(o\Box_2 - o\Box_1) + 1} \\
(\lambda + \Box_1 + \Box_2 + \blacksquare | x_i^+ x_i^+ x_{i+1}^+ | \lambda) &= \gamma_2 \cdot v^{\beta(o\Box_1 - o\blacksquare) + \beta(o\Box_2 - o\blacksquare) + \beta(o\Box_2 - o\Box_1) + 2}
\end{align*}
\]

where recall that \( \beta(j) = \alpha(j + 1) - \alpha(j) \). The 1’s and 2’s which appear in the exponents come from the Kronecker deltas in the exponents of the formula for the operator \( x_i^+ \). Adding the above three quantities together proves (5.15). The case of \( – \) is dealt with analogously. The next thing to prove is that:

\[
\langle x_i^+ x_j^- | \lambda \rangle = \langle x_j^- x_i^+ | \lambda \rangle
\]  

(5.16)

for any Young diagram \( \lambda \) and any \( j \neq i \). Both sides of the above equation add a box \( \Box \) of content \( \equiv i \) and remove a box \( \blacksquare \) of content \( \equiv j \), so we need to check that the coefficients are the same. Suppose that the box \( \Box \) is situated southeast of \( \blacksquare \), i.e. \( o\Box \geq o\blacksquare \) (the other case is analogous). If we let:

\[
\gamma_3 = \langle \lambda + \Box | x_i^+ | \lambda \rangle \cdot \langle \lambda - \blacksquare | x_j^- | \lambda \rangle
\]

then the coefficients of the two sides of (5.16) differ from \( \gamma_3 \) simply by certain powers of \( v \). Explicitly,

\[
(\lambda + \Box - \blacksquare | x_i^+ x_j^- | \lambda) = \gamma_3 \cdot v^{\beta(o\Box - o\blacksquare)} = (\lambda + \Box - \blacksquare | x_j^- x_i^+ | \lambda)
\]

which proves (5.16). However, when \( i = j \) there is one case when the above argument breaks down: when \( \blacksquare = \Box \). Thus, the terms which survive in:

\[
\langle x_i^+ x_j^- | \lambda \rangle - \langle x_j^- x_i^+ | \lambda \rangle
\]

(5.17)

are those in which we add an inner corner of \( \lambda \) and then we remove it, or when we remove an outer corner of \( \lambda \) and then we add it back. It is easy to see what the contribution is in these cases, and so the commutator (5.17) equals:

- **outer corner downstep j steps** leftstep j steps
  - \( \sum_{\text{of content } \equiv i} \prod \gamma^{\delta_{n_{ij}} - \delta_{n_{ij} - 1}} \prod v^{-\delta_{n_{ij}} + \delta_{n_{ij} - 1}} \)
  - \( \sum_{\text{of content } \equiv i} \prod \gamma^{\delta_{n_{ij}} - \delta_{n_{ij} + 1}} \prod v^{-\delta_{n_{ij}} + \delta_{n_{ij} + 1}} \)

In each of the two sums above, the \( j \)-th step southeast of \( \blacksquare \) gives a term of \( v^0 \) unless:

- \( j \) is a downstep and \( j - 1 \) is a rightstep (in which case the \( j \)-step starts from an outer corner)
- \( j \) is a rightstep and \( j - 1 \) is a downstep (in which case the \( j \)-step starts from an inner corner)
The same analysis pertains to the the \(j\)-step northwest of \(\boxdot\). We conclude that the commutator (5.17) equals:

\[
\begin{align*}
|\lambda\rangle & \sum_{\text{outer corner of content} \equiv i} v^{\text{(# o.c. \equiv i SE of } \boxdot \text{) - (# i.c. \equiv i SE of } \boxdot \text{) - (# o.c. \equiv i NW of } \boxdot \text{) + (# i.c. \equiv i NW of } \boxdot \text{)}} |\lambda\rangle \\
-|\lambda\rangle & \sum_{\text{inner corner of content} \equiv i} v^{\text{(# o.c. \equiv i SE of } \boxdot \text{) - (# i.c. \equiv i SE of } \boxdot \text{) - (# o.c. \equiv i NW of } \boxdot \text{) + (# i.c. \equiv i NW of } \boxdot \text{)}}
\end{align*}
\]

where we use the abbreviations "o.c." for outer corner and "i.c." for inner corner, \(\equiv i\) stands for the content of the corner in question being congruent to \(i\) modulo \(n\), and SE/NW for southeast/northwest. As we go through the corners of content \(\equiv i\) of \(\lambda\) in order from the northwest to the southeast, the above sum takes the form:

\[
|\lambda\rangle \cdot (\pm v^{a_1} \pm ... \pm v^{a_k})
\]

(5.18)

where:

\[
a_1 = (\# \text{ outer corners } \equiv i) - (\# \text{ inner corners } \equiv i) - 1
\]

we have \(a_j - a_{j-1} = \{2, 0, 0, -2\}\) depending on whether the pair \((j-1\text{ th corner } \equiv i, \text{ j th corner } \equiv i)\) is (inner, inner), (inner, outer), (outer, inner) or (outer, outer),

\[
a_k = - (\# \text{ outer corners } \equiv i) + (\# \text{ inner corners } \equiv i) + 1
\]

and the \(j\)-th sign is + or − depending on whether the \(j\)-th corner of content \(\equiv i\) is outer or inner. Because of this, we conclude that (5.18) equals:

\[
|\lambda\rangle \cdot \frac{v^{a_1+1} - v^{-a_k-1}}{v-v^{-1}} = \kappa_i |\lambda\rangle - \frac{1}{v-v^{-1}} \kappa_i^{-1} |\lambda\rangle
\]

Proving the commutation relations between \(\kappa_i\) and \(x_j^\pm\) is trivial, since the coefficients of the latter operators are of no consequence, and all that matters is that they add/remove a box of content \(\equiv j\). With this, we have shown that \(x_i^-, \kappa_i, x_i^+\) satisfy the relations of \(U_v(\mathfrak{sl}_n)\). Note once again that this does not make any assumptions on the function \(\alpha : \mathbb{Z} \rightarrow \mathbb{Z}\).

Let us now show that the generators \(x_i^-, \kappa_i, x_i^+\) commute with the \(\varepsilon_{\pm k}\) of (5.10)-(5.11). This is clear for \(\kappa_i\), since the operators \(\varepsilon_{\pm k}\) add entire \(n\)-ribbons to a Young diagram, and adding a whole ribbon is easily seen to preserve the difference between the number of outer corners and inner corners of any given content modulo \(i\). So let us prove that \(x_i^+\) and \(x_i^-\) commute with \(\varepsilon_k\): we will show the case when the sign is + and \(k > 0\) (the other cases are analogous). To simplify the explanation, we will only study the case \(k = 1\) when we only add a single \(n\)-ribbon, though it should be clear from the proof that the argument works when one adds an arbitrary set of \(n\)-ribbons (not necessarily in a vertical strip). Thus, the commutation relation we need to prove is:

\[
x_i^+ e_1 |\mu\rangle = e_1 x_i^+ |\mu\rangle
\]

(5.19)

The two sides of the above equality go over all ways to add an inner corner of content \(\equiv i\) and an \(n\)-ribbon, albeit in different orders. Diagramatically:
\[
\text{LHS: } \mu \xrightarrow{\square} \mu + \square \xrightarrow{\text{ribbon}} R \lambda \quad \text{RHS: } \mu \xrightarrow{\text{ribbon}} R' \lambda - \square \xrightarrow{\square} \lambda
\]

(5.20)

We must first construct a bijection between the two ways to add ribbons/boxes above, and then to prove that the corresponding coefficients in the two sides of (5.19) match. In the LHS, the box $\square$ and the ribbon $R$ may be in the following relative positions:

1. They might not touch; in this case we set $\square = \square$

2. $\square$ might be completely covered by $R$; in this case we let $\square$ be the box one step northeast of $\square$

3. $\square$ might be directly west of the first box of $R$; in this case we let $\square$ be the last box of $R$

4. $\square$ might be directly south of the last box of $R$; in this case we let $\square$ be the first box of $R$

We set $R' = R + \square - \square$, and note that this is also a ribbon (thus yielding the desired bijection between the two sides of (5.20)) unless:

\[\overline{R} = R + \square\]

is an $n + 1$ ribbon. But in this case, we can break up the $n + 1$ ribbon $\overline{R}$ into an $n$-ribbon and an extra box in two ways (the extra box being one of the two endpoints of $\overline{R}$) and the contribution of these two ways to the LHS of (5.19) will cancel each other out. The reason for this is the presence of $(-1)^{ht}$ in the formula for $e_1$, since chopping off an endpoint from $\overline{R}$ in either of two ways leaves us with $n$-ribbons with height of different parities.

Therefore, let us prove that the two sides of (5.19) have equal coefficients for each of the four items above. In case (1), let us assume that the box $\square = \square$ lies to the southeast of the ribbon $R = R'$, and let:

\[\gamma_4 = \langle \mu + R + \square | x_i^+ e_1 | \mu \rangle\]

Then the coefficients of the two sides of (5.19) differ by an explicit power of $v$:

\[\langle \mu + R + \square | x_i^+ e_1 | \mu \rangle = \gamma_4 \cdot v^{\alpha(j+n)-\alpha(j)}\]

where $j$ is the number of steps between the ribbon $R$ and the box $\square$. The above equals $\gamma_4$ because $\alpha$ is assumed to be $n$-periodic. In case (2), let us write:

\[\gamma_5 = \langle \mu + R + \square | e_1 x_i^+ | \mu \rangle\]

Then we have:

\[\langle \mu + R + \square | x_i^+ e_1 | \mu \rangle = \gamma_5 \cdot v^{\alpha(n-j+1)+\alpha(j)} \left( \frac{q}{t} \right)^{-\left[ \frac{n + j}{t} \right] + \left[ \frac{m(j+1)}{t} \right] + \frac{m}{t}}\]

where $\square$ is the $j$-th box of $R$, or equivalently, $\square$ is the $j$-th box of $R'$. The above equals $\gamma_5$ because of assumption (5.8) and the fact that $v = \sqrt{\frac{t}{T}}$. Note that
\[ j \notin \{1, n\}, \text{ since this would contradict the assumption that } R \text{ completely covers } \square. \]

In case (3), let us write:

\[ \gamma_6 = (\mu + R + \square | e_1 x_i^+ | \mu) \]

We have \( R + \square = R' + \bullet = \overline{R} \), where the boxes \( \square \) and \( \bullet \) lie at the west and east ends of the \( n + 1 \) ribbon \( \overline{R} \). The corresponding coefficient equals:

\[ \langle \mu + R' + \bullet | x_i^+ e_1 | \mu \rangle = \gamma_6 \cdot \prod_{\text{right step of } \overline{R}} q^{j \cdot \left\lfloor \frac{m_j n}{n} \right\rfloor} \prod_{\text{down step of } \overline{R}} t^{-\left\lfloor \frac{m_j n}{n} \right\rfloor + \left\lfloor \frac{m_j(n+1)}{n} \right\rfloor} \]

where in the above, we call \( j \in \{1, \ldots, n\} \) a down step of \( \overline{R} \) if \( \chi_{j+1}(\overline{R}) = t \chi_j(\overline{R}) \), and a right step of \( \overline{R} \) if \( \chi_{j+1}(\overline{R}) = q \chi_j(\overline{R}) \). The above coefficient is equal to \( \gamma_6 \) because of assumption (5.8) and the fact that \( v = \sqrt{2} \). Case (4) is dealt with analogously to case (3), so we will leave it as an exercise to the interested reader.

The hardest part of this proof is showing that the operators \( \varepsilon_{\pm k} \) satisfy the relations in the quantum Heisenberg algebra \( U_q(\hat{gl}_1) \). However, this holds in virtue of Theorems 1.2 and 5.5. Indeed, by (5.4), the abstract relations satisfied by the \( \varepsilon_{\pm k} \) in the quantum Heisenberg algebra are satisfied by the concrete operators:

\[ \varepsilon_{\pm k}^{m/n} \colon \Lambda \longrightarrow \Lambda \text{ in the basis } s_{\lambda}^{m/n} \]

We do not know of a direct proof of the fact that the operators (5.10)-(5.11) respect the relations in the quantum Heisenberg algebra, other than appealing to (the rather involved) Theorems 1.2 and 5.5.

\[ \blacksquare \]

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