A universal boundary value problem for partial differential equations

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Abstract

A new boundary value problem for partial differential equations is discussed. We consider an arbitrary solution of an elliptic or parabolic equation in a given domain and no boundary conditions are assumed. We study which restrictions the boundary values of the solution and its normal derivatives must satisfy. Linear integral equations for the boundary values of the solution and its normal derivatives are obtained, which we call the universal boundary value equations. A universal boundary value problem is defined as a partial differential equation together with the boundary data which specify the values of the solution on the boundary and its normal derivatives and satisfy to the universal boundary value equations.

For the equations of mathematical physics such as Laplace’s and the heat equation the solution of the universal boundary value problem is presented. Applications to cosmology and quantum mechanics are mentioned.
1 Introduction

An ordinary boundary value problem is defined as a partial differential equation in a domain together with the boundary conditions. A solution to the boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

The form of the boundary conditions depends on the type of the partial differential equation (elliptic, parabolic or hyperbolic), see for example [1]. Examples of boundary value problems for Laplace’s equation are the Dirichlet problem, the Neumann problem and the third boundary value problem.

In this note a new boundary value problem is discussed. We consider an arbitrary solution of an elliptic or parabolic equation in a given domain and no boundary conditions are assumed. We study which restrictions the boundary values of the solution and its normal derivatives must satisfy. Linear integral equations for the boundary values of the solution and its normal derivatives are obtained which we shall call the universal boundary value equations.

Our aim is to obtain a relation for the boundary values of a function and its derivatives if the function is a solution of the differential equation in the given domain.

If \( u \) is a solution of Laplace’s equation \( \Delta u = 0 \) in a domain \( G \) with the boundary \( S \) let \( \gamma_0 u = u|_S \) be its trace on the boundary and \( \gamma_1 u = \partial u/\partial \nu|_S \) the trace of the normal derivative. Then the universal boundary value equations have the form

\[
A \gamma_0 u + B \gamma_1 u = 0, \quad \int_S \gamma_1 u ds = 0,
\]

where \( A \) and \( B \) are integral operators on \( S \) (see Eqs (2) and (3) below). Conversely, if \( u_0 \) and \( u_1 \) are two functions on \( S \) which satisfy the universal boundary value equations \( Au_0 + Bu_1 = 0, \int_S u_1 ds = 0 \) then there exists a solution \( u \) of Laplace’s equation \( \Delta u = 0 \) such that \( u|_S = u_0, \partial u/\partial \nu|_S = u_1 \).

A very simple example is given by the 1-dimensional "Laplace’s equation" \( u''(x) = 0, \quad a < x < b \). In this case the universal boundary value equations are the following two relations: \( u(a) - u(b) = u'(a)(a - b), \quad u'(a) = u'(b) \) among four numbers (boundary values) \( u(a), u(b), u'(a), u'(b) \) (compare Eqs (2) and (3) below).

For the equations of mathematical physics such as Laplace’s and the heat equation the solution of the universal boundary value problem is presented. Applications to cosmology and quantum mechanics are mentioned.
2 Universal boundary value problem for Laplace’s equation

Let us consider Laplace’s equation

\[ \Delta u = 0, \quad x \in G \subset \mathbb{R}^3, \tag{1} \]

in the bounded domain \( G \) of the Euclidean space \( \mathbb{R}^3 \) with the boundary \( S = \partial G \in C^3 \) which satisfies the additional condition of strong convexity type (an exact formulation of this condition is given below).

To study the boundary value problems for Laplace’s equation (1) we introduce the functional space \( C^1_{\text{norm}}(\bar{G}) \) which is the space of continuously differential functions on domain \( G \) with the "proper normal derivative" \( \frac{\partial u}{\partial \nu} \) on the boundary \( \partial G \) (see [1]). Here \( \nu \) is the continuous vector field of the unit exterior normal vectors on the surface \( S = \partial G \).

**Theorem 1.** Let the function \( u \in C^2(G) \cap C^1_{\text{norm}}(\bar{G}) \) satisfies the equation (1) and the surface \( S \in C^3 \) satisfies the strong convexity condition (see below). Then the boundary values \( u_0 = u|_S \) and \( u_1 = \frac{\partial u}{\partial \nu}|_S \) satisfy the equations (universal boundary value equations)

\[ u_0(x) + \frac{1}{2\pi} \int_S \frac{\cos \phi_{xy}}{|x - y|^2} u_0(y) ds_y - \frac{1}{2\pi} \int_S \frac{1}{|x - y|} u_1(y) ds_y = 0, \quad x \in S; \tag{2} \]

\[ \int_S u_1 ds = 0. \tag{3} \]

Conversely, let be given the functions \( u_0 \in C(S) \) and \( u_1 \in C(S) \) which satisfy the universal boundary value equations (2), (3). Then there exists the unique function \( u \in C^2(G) \cap C^1_{\text{norm}}(\bar{G}) \) such that

\[ \Delta u = 0, \quad x \in G \]

\[ u|_S = u_0, \quad \frac{\partial u}{\partial \nu}|_S = u_1. \]

Here

\[ \cos \phi_{x,y} = \frac{\partial}{\partial y} \frac{1}{|x - y|^2}. \]

The additional condition of strong convexity on the surface \( S \in C^3 \):

There exists a constant \( c_0 > 0 \) such that

\[ |(\nu(u, v), d^2 r(u, v))| \geq c_0((du)^2 + (dv)^2) \]

for any region

\[ \{r = r(u, v), \quad (u, v) \in D\} = S_1 \subset S. \]
The equations (2), (3) are the universal boundary value equations for Laplace’s equation.

The second part of the theorem is the statement of correct solvability of the universal boundary value problem to the Laplace equation in the space $C^2(G) \cap C^{1,\text{norm}}(\bar{G})$.

**Sketch of the proof. Derivation of universal boundary equations.**

Assume that the function $u$ satisfies the conditions:

$$u \in C^2(G) \cap C^1(\bar{G}), \Delta u = 0, x \in G$$

Then according to the Green formula

$$u(x) + \frac{1}{4\pi} \int_S \frac{\cos \phi_{xy}}{|x-y|^2} u_0(y) dS_y - \frac{1}{4\pi} \int_S \frac{1}{|x-y|} u_1(y) dS_y = 0, x \in G.$$ 

By passage to the limit as $x \to S$, $x \in G$ in the Green formula and using the jump formula for the double layer potential

$$\frac{1}{4\pi} \int_S \frac{\cos \phi_{xy}}{|x-y|^2} u_0(y) dS_y \to \frac{1}{4\pi} \int_S \frac{\cos \phi_{xy}}{|x-y|^2} u_0(y) dS_y - \frac{1}{2} u_0(x)$$

we obtain the equality (2):

$$u(x) + \frac{1}{2\pi} \int_S \frac{\cos \phi_{xy}}{|x-y|^2} u_0(y) dS_y - \frac{1}{2\pi} \int_S \frac{1}{|x-y|} u_1(y) dS_y = 0, x \in S.$$ 

The equality (3) is the consequence of the Ostrogradskii-Gauss theorem.

Now let the function $u$ satisfies the conditions of the theorem:

$$u \in C^2(G) \cap C^{1,\text{norm}}(\bar{G}) \text{ and } \Delta u = 0, x \in G.$$ 

Let $S_h$ is a surface which is parallel to the surface $S$, i.e. $S_h$ is the result of displacement of any point of the surface $S$ in the direction of interior normal on the distance $h > 0$. Then $S_h$ is $C^2$ surface for any sufficiently small $h$ according to the assumption of the theorem. Let $G_h$ is the subdomain of domain $G$ with the boundary $S_h$. Hence the function $u|_{G_h} \in C^2(G_h) \cap C^1(\bar{G}_h)$ and therefore the functions $u|_{S_h}$ and $\frac{\partial u}{\partial n}|_{S_h}$ satisfy the equations (2), (3) on the surface $S_h$.

The assumptions of the theorem is sufficient to prove the equality (2), (3) for the functions $u|_S$ and $\frac{\partial u}{\partial n}|_S$ by passage to the limit as $h \to +0$.

**Solution of the universal boundary value problem.**

Let the functions $u_0, u_1 \in C(S)$ satisfy the universal boundary value equations (2) and (3).

Let us consider the following Neumann problem:

$$\Delta v = 0, \frac{\partial v}{\partial n}|_S = u_1$$

which has the unique solution $v$ according to the condition (3) and the Lyapunov-Steklov theorem. Let us set $\hat{u}_0 = v|_S$. Then according to the first statement of the theorem the
functions \( \hat{u}_0 \) and \( u_1 \) satisfy the equation (2). The functions \( u_0 \) and \( u_1 \) also satisfy this equation according to the assumption of the theorem.

Hence the following equality holds for the function \( w = \hat{u}_0 - u_0 \):

\[
2\pi w(x) - \int_S w(y) \frac{\cos \phi_{xy}}{|x - y|^2} dS_y = 0, \ x \in S;
\]

then according to the potential theory ([1])

\[
w = C_0 = \text{const}.
\]

Therefore the function \( u = v - C_0 \) is the solution of universal boundary value problem:

\[
\Delta u = 0, \ u|_S = u_0, \ \frac{\partial u}{\partial \nu}|_S = u_1.
\]

**Remark.** The universal boundary value equations for Poisson’s equation

\[
\Delta u = f, \ x \in G
\]

have the form

\[
2\pi u_0(x) + \int_S \frac{\cos \phi_{xy}}{|x - y|^2} u_0(y) - \frac{u_1(y)}{|x - y|} dS_y - \int_G \frac{f(y)}{|x - y|} dy = 0, \ x \in S, \quad (4)
\]

\[
\int_S u_1 ds = \int_G f(y) dy.
\]

### 3 Universal boundary value problem for the heat equation

In this section we obtain the universal boundary value equations for the heat equation.

**Theorem 2.** Let \( G = R_+ \times R_+ \) be the quarter-plane \( \{(t, x) : t, x > 0\} \); let the function \( u = u(t, x) \in C(\bar{R}_+, C^1(\bar{R}_+) \cap C^2(R_+)) \cap C^1(R_+, C(R_+)) \) is bounded on the domain \( G \) and satisfies the heat equation:

\[
u''_t - u''_{xx} = 0, \ (t, x) \in G.x
\]

(5)

Then the function \( u \) has the boundary values

\[
v(x) = u(+0, x), \ x > 0,
\]

\[
\phi(t) = u(t, +0), \ t > 0,
\]

\[
\psi(t) = u'_x(t, +0), \ t > 0,
\]
the boundary values are continuous functions $v, \phi, \psi \in C(\mathbb{R}_+)$ and they satisfy the relation

$$
\phi(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{\xi^2}{4t}} v(\xi) d\xi - \frac{1}{\sqrt{\pi t}} \int_0^t \frac{1}{\sqrt{\tau}} \psi(t - \tau) d\tau
$$

(6)

The relation (6) is the universal boundary value equation for the heat equation on the quarter-plane $G$.

4 On general boundary value problems

Here we discuss an interpretation of the universal boundary value equations by using a functional analysis framework.

A rather general boundary value problem for elliptic equations is discussed by Vishik [2] (see also [3]) by investigating of extensions of the minimal differential operator or restriction of the maximal differential operator in a Hilbert space. Different boundary conditions were related in [2] with different extensions of the minimal differential operator (self-adjoint extensions, solvable extensions or extensions with another properties), but the universal boundary value equations were not mentioned.

Let $L$ be a linear differential operator of the second order with the domain $D(L) = C^2(G) \cap C^1(\bar{G})$. Let the operators $\gamma_0, \gamma_1$ are the trace maps defined on the domain $D(\gamma_0) = D(\gamma_1) = D(L)$ by $\gamma_0 u = u|_S$, $\gamma_1 u = \frac{\partial u}{\partial \nu}|_S$.

Let us introduce an operator $T : D(L) \to C(S) \oplus C(S) \oplus C(G) \equiv Y$ (where $Y$ is the direct sum of Banach spaces) on the domain $D(T) = D(L)$ by the formula $Tu = (\gamma_0 u, \gamma_1 u, Lu)$.

Instead of the spaces of continuous functions one can use the Sobolev spaces.

The aim of the investigation in [2] was to obtain a description of linear subspaces $\mathcal{P}$ in the space of boundary values such that the restriction $L_\mathcal{P}$ of maximal operator $L_{\text{max}}$ (without any conditions to boundary values) on the subspace $\{u \in D(L_{\text{max}}) : (\gamma_0 u, \gamma_1 u) \in \mathcal{P}\}$ has the bounded inverse operator or compact inverse operator. The general boundary value problem according to [2] is the elliptic partial differential equation together with the equation on boundary values which specify the subspace $\mathcal{P}$.

The problem of finding the solvable restrictions of the maximal elliptic operator $L_{\text{max}}$ is reduced to the problem of describing the set of operators $A, B$ in the space $C(S)$ such that the intersection of three subspaces in the space $Y$: $\text{Im}(T)$, $\{(v_0, v_1, f) \in Y : Av_0 + Bv_1 = \theta\}$ and $\{(v_0, v_1, f) \in Y : f = \hat{f}\}$ consists of the unique point for arbitrary $\hat{f} \in C(G)$. The third component of the space $Y$ has the special role in the approach of [2]. The aim of our investigation is to present the image $\text{Im}(T)$ by the linear equation on the space $Y$ (the equation (7) below) such that all of three components of the image of operator $T$ are having the same rights in this equation.
Our investigation of the universal boundary value problem for Laplace’s equation (1) and for the heat equation (4) in principle can be extended to the universal boundary value problem for more general linear differential equations. One can proceed as follows. The universal boundary equation for the differential equation
\[ Lu = f \]
can be defined as an equation in the space \( Y \):
\[ \{ (v_0, v_1, f) \in Y : Av_0 + Bv_1 + Cf = \theta \}, \tag{7} \]
where linear operators \( A, B, C \) are defined on the spaces \( C(S), C(S), C(G) \) respectively and take values in some Banach space \( Z \). Of course, the main task is to find an explicit form of the operators \( A, B, C \).

For example the equation (7) for Poisson equation \( \Delta u = f, \ x \in G \) is the generalization of equations (2), (3) in the form of equation (7):
\[
2\pi u_0(x) + \int_S \frac{\cos \phi_{xy}}{|x - y|^2} u_0(y) - \frac{u_1(y)}{|x - y|} ds_y - \int_G \frac{f(y)}{|x - y|} dy = 0, \ x \in S.
\]
\[
\int_S u_1 ds = \int_G f(y) dy.
\]

The universal boundary value problem for the homogeneous equation is the problem of description of the kernel of the maximal operator \( L \) in terms of boundary values of the elements of this kernel,
\[ u \in \text{Ker}(L) \iff \{ (v_0, v_1) \in C(S) \times C(S) : Av_0 + Bv_1 = 0 \}. \tag{8} \]
Eq (8) is an abstract form of the universal boundary value equations (2), (3) for Laplace equation (1).

In [4] the problem of describing relations between boundary values is discussed and by using the Fourier transform an infinite number of equations for some functionals from the boundary functions is obtained.

## 5 Discussions and Conclusions

In this note Laplace’s and the heat equations are considered and the restrictions on the boundary values of the solutions and its normal derivatives are studied. The linear integral equations for the boundary values of the solution and its normal derivatives are obtained which are called the universal boundary value equations.
The universal boundary value problem appeared in the cosmological considerations [5]. One had to define an operator \( e^\tau \frac{d^2}{dt^2} v(t) = u(\tau, t) \) without specifying boundary conditions on the quarter-plane \( \tau, t > 0 \). To this end one of the authors (I.V.) obtained the universal boundary value equations for the heat and Laplace’s equations. Concerning applications of the operator \( e^\tau \frac{d^2}{dt^2} v(t) \) on the half-plane to nonlinear equations see [6].

Restrictions on the Cauchy data are mentioned in the studying the wave equation on non-globally-hyperbolic manifold [7]. In this work the description of the kernel (8) for operator of Cauchy problem for wave equation on non-globally-hyperbolic manifold is obtained.

There could be important applications of the universal boundary value problem in quantum mechanics when the Schrodinger equation with the degenerated Hamiltonian is considered [8, 9]. In this paper it was obtained a description of the kernel (8) for operator of the Cauchy problem for Schrodinger equation with degenerated Hamiltonian (see theorem 2.2 in [9]) and solution of an optimal problem (see theorem 11.1 in [9]) on the set (7) for this equation.

It would be interesting to obtain generalizations of the universal boundary value equations considered in this paper to other partial differential equations.

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