Generalized Monge gauge

S. Habib Mazharimousavi,∗ S. Danial Forghani,† and S. Niloufar Abtahi‡

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin 10, Turkey.

(Dated: February 13, 2017)

Monge gauge in differential geometry is generalized. The original Monge gauge is based on a surface defined as a height function $h(x, y)$ above a flat reference plane. The total curvature and the Gaussian curvature are found in terms of the height function. Getting benefits from our mathematical knowledge of general relativity, we shall extend the Monge gauge toward more complicated surfaces. Here in this study we consider the height function above a curved surface namely a sphere of radius $R$. The proposed height function is a function of $\theta$ and $\varphi$ on a closed interval. We find the first, second fundamental forms and the total and Gaussian curvatures in terms of the new height function. Some specific limits are discussed and two illustrative examples are given.

PACS numbers: 02.40.-k; 02.40.Hw; 02.40.Sf

Keywords: Monge Gauge; Spherically Symmetry; Membrane;

I. INTRODUCTION

Applications of pure mathematics in applied science are not rare. For decades differential geometry has been known as a useful tool to study lipid membranes [1]. Tu and Ou-Yang [2] have published a review work on the recent theoretical advances in elasticity of membranes which covers most of the recent and initial papers in this context. Another review paper on the same issue has been published by Deserno in [3]. In this work, by using differential geometry, fluid lipid membrane as a 2-dimensional surface has been investigated. In [4] the differential geometry of proteins and its applications have been presented. Application of helicoidal membrane in connecting the stacked endoplasmic reticulum sheets inside the cell has been worked out by Terasaki et al in [5] and highlighted by Marshall in [6]. To the first order approximation, biological membranes are made of a bilayer of phospholipid molecules embedded in water [3, 7, 8]. The physics of cell membranes including physics behind the self-assembly, molecular structures and electrical properties are given in [9].

Far from the biological (cell-) membranes, in the Einstein theory of gravity there are objects which are called timelike thin-shells. These are basically membranes in spacetime. As the theory of general relativity itself, the mathematical structures of such objects are very rich. Although, initially the spacetime has been assigned to be 3+1-dimensional but the lower dimensional spacetime i.e., 2+1 is also very well known model. The analogy between the 1+1-dimensional thin-shells in 2+1-dimensional spacetime and the 2-dimensional surfaces in 3-dimensional space is the point that we shall get into in our formalism. More precisely, in this paper, we shall try to borrow concepts / mathematical rules of general relativity mainly in connections to the concept of thin-shells to develop the basic tools used in the study of (cell-) membranes. One of the most used concept in the early development of the biological membranes is the so called Monge Gauge (MG). The MG [3] is a parametrization / mapping of a 2-dimensional surface which is defined by a height function $h(x, y)$ over a flat plane as a function of orthonormal coordinate on the plane $x$ and $y$ into a 3-dimensional flat space of coordinates $x, y$ and $z$. This is usually shown as

$$h: \left\{ \mathbb{R}^2 \ni (x, y) \rightarrow h(x, y) \right\}.$$  

(1)

Using the standard definition of the total and the Gaussian curvature one finds [3]

$$\kappa = \frac{h_{xx}(1 + h_y^2) + h_{yy}(1 + h_x^2) - 2h_{xy}h_xh_y}{2\left(1 + h_x^2 + h_y^2\right)^2}.$$  

(2)

∗ Electronic address: habib.mazhari@emu.edu.tr
† Electronic address: danial.forghani@emu.edu.tr
‡ Electronic address: sayedeh.abtahi@emu.edu.tr
\[ \kappa_G = \frac{h_{,xx}h_{,yy} - h_{,xy}^2}{(1 + h_{,xx}^2 + h_{,yy}^2)^2} \]  

(3)

respectively. Introducing \( \nabla = \left( \frac{\partial}{\partial y} \right) \) the above usually are compacted as

\[ 2\kappa = \nabla \cdot \left( \frac{\nabla h}{\sqrt{1 + (\nabla h)^2}} \right) \]  

(4)

and

\[ \kappa_G = \frac{\det (\partial^2 h)}{(1 + (\nabla h)^2)^2} \]  

(5)

in which the Hessian \( \partial^2 h \) is given by

\[ \partial^2 h = \begin{pmatrix} h_{,xx} & h_{,xy} \\ h_{,yx} & h_{,yy} \end{pmatrix} \]  

(6)

For instance, in the famous paper of Seifert and Langer \[10\], the height function was set to \( h(x, y) = h \exp [iqx] + \text{c.c.} \) in which \( h \) and \( q \) are two constants. For more recent work one may look at the work of Bingham, Smye and Olmsted \[11\].

As of \[10\] and \[11\] the other studies where the MG has been used, the original unperturbed surface is locally flat. However, in the case of a blood cell we may not be able to consider its membrane flat. Therefore and fluctuation from its original rest shape needs to be analyzed exactly without using MG. In this study we show that in such cases one may use a proper MG which of course is not the one introduced above. Spherically symmetry is the most common symmetry which occurs in nature and hence in this study we concentrate on spherically symmetric MG.

II. THE SPHERICAL MG

Let’s start with a spherical shell of radius \( R \) with some fluctuation on its surface given by the spherical height function \( h(\theta, \varphi) \) in which \( \theta \) and \( \varphi \) are the polar and azimuthal angle. Unlike the Cartesian height function, the non fluctuated sphere is given by \( h(\theta, \varphi) = R \). Our aim is to find the first and second fundamental forms as well as the extrinsic curvature tensor and the scalar curvature of the hypersurface \( \Sigma \) in terms of \( h(\theta, \varphi) \) only. We start with the definition of the normal vector on the surface \( \Sigma \) given by

\[ n_\gamma = \frac{1}{\sqrt{\Delta}} \frac{dF}{dx^\gamma} \bigg|_\Sigma \]  

(7)

in which \( F := r - h(\theta, \varphi) = 0 \) is the definition of the surface \( \Sigma \) in three dimensional flat spherical symmetric bulk space \( M \) with line element

\[ ds_M^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \]  

(8)

Herein \( \Delta \) is defined as

\[ \Delta = g^{\alpha\beta} \frac{dF}{dx^\alpha} \frac{dF}{dx^\beta} \bigg|_\Sigma \]  

(9)

in which \( g^{\alpha\beta} \) is the metric tensor of the bulk and \( n_\gamma n^\gamma = 1 \). Let’s note that the Greek letters \( \alpha, \beta, \ldots = 1, 2, 3 \) are used for the bulk space while the Latin letters \( i, j, \ldots = 2, 3 \) shall be used for the hypersurface. Using the definition (1) we find

\[ n_r = \frac{1}{\sqrt{\Delta}} \]  

(10)
\[ n_\theta = -\frac{1}{\sqrt{\Delta}} h,\theta \]  \hspace{1cm} (11)

and

\[ n_\varphi = -\frac{1}{\sqrt{\Delta}} h,\varphi \]  \hspace{1cm} (12)

with

\[ \Delta = 1 + \frac{h,\theta}{h^2} + \frac{h,\varphi}{h^2 \sin^2 \theta}. \]  \hspace{1cm} (13)

The induced metric on the hypersurface \( \Sigma \) is given by

\[ g_{ij} = \partial x^\alpha / \partial \xi^i \partial x^\beta / \partial \xi^j g_{\alpha\beta} \]  \hspace{1cm} (14)

in which the line element on the hypersurface is written as

\[ ds^2_\Sigma = g_{ij} d\xi^i d\xi^j. \]  \hspace{1cm} (15)

Explicitly one finds

\[ ds^2_\Sigma = (h^2 + h,\theta^2) d\theta^2 + (h^2 \sin^2 \theta + h,\varphi^2) d\varphi^2 + 2h,\theta h,\varphi d\theta d\varphi \]  \hspace{1cm} (16)

or simply

\[ g_{ij} = \begin{bmatrix} h^2 + h,\theta^2 & h,\theta h,\varphi \\ h,\theta h,\varphi & h^2 \sin^2 \theta + h,\varphi^2 \end{bmatrix} \]  \hspace{1cm} (17)

with its inverse

\[ g^{ij} = \frac{1}{h^4 \sin^2 \theta \Delta} \begin{bmatrix} h^2 \sin^2 \theta + h,\varphi^2 & -h,\theta h,\varphi \\ -h,\theta h,\varphi & h^2 + h,\theta^2 \end{bmatrix}. \]  \hspace{1cm} (18)

Having the induced metric and the normal vector on the hypersurface \( \Sigma \), one can use the definition of extrinsic curvature tensor of the surface \( \Sigma \)

\[ K_{ij} = -n_\gamma \left( \frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right) \bigg|_\Sigma \]  \hspace{1cm} (19)

to find the nonzero components of the extrinsic curvature or second fundamental form. We note that \( \Gamma^\gamma_{\alpha\beta} \) are the Christoffel symbols of the second kind of the bulk space with the only nonzero components given by

\[ \Gamma^\theta_{\varphi\varphi} = \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \Gamma^\theta_{\theta\theta} = \frac{1}{r} \]  \hspace{1cm} (20)

\[ \Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\varphi\varphi} = -r \sin^2 \theta, \]  \hspace{1cm} (21)

and

\[ \Gamma^\varphi_{\varphi\theta} = \Gamma^\theta_{\varphi\varphi} = \frac{\cos \theta}{\sin \theta}, \quad \Gamma^\theta_{\varphi\varphi} = -\sin \theta \cos \theta. \]  \hspace{1cm} (22)

The components of the extrinsic curvature tensor are found to be

\[ K_{\theta\theta} = \frac{h^2 + 2h,\theta - hh,\theta}{h\sqrt{\Delta}} \]  \hspace{1cm} (23)

\[ K_{\varphi\varphi} = \frac{h^2 \sin^2 \theta - hh,\theta \sin \theta \cos \theta + 2h,\varphi^2 - hh,\varphi}{h\sqrt{\Delta}} \]  \hspace{1cm} (24)
in which

\[ K_{\varphi \theta} = K_{\theta \varphi} = \frac{2h,\theta h,\varphi + hh,\varphi \cos \theta - hh,\theta \varphi}{h \sqrt{\Delta}}. \]  

Next, we find \( K_i^j \) in order to find the total and the Gaussian curvature. Using the induced metric one finds

\[ K_\theta^\theta = \frac{(h^2 - hh,\theta \theta + 2h,\theta^2) \sin^3 \theta + (hh,\varphi \varphi + hh,\varphi h,\varphi \theta - h^2,\theta \varphi,\theta \varphi) \sin \theta - h,\theta h^2 \cos \theta}{h^4 \sin^3 \theta \Delta^{3/2}}, \]  

\[ K_\varphi^\varphi = \frac{(h^2 + h,\theta^2) (h \sin^2 \theta - h,\theta \sin \theta \cos \theta - h,\varphi \varphi) \sin \theta + (2hh,\varphi \varphi + hh,\varphi h,\varphi \theta) \sin \theta - h,\theta h^2 \cos \theta}{h^4 \sin^3 \theta \Delta^{3/2}}, \]  

\[ K_\theta^\varphi = \frac{(h,\theta h,\varphi - hh,\varphi \varphi) \sin^3 \theta + h,\varphi (h^2 + h,\varphi^2) \sin^2 \theta \cos \theta + (h,\theta h,\varphi \varphi - h,\varphi h,\varphi \theta) h,\varphi \sin \theta + h^3,\varphi \cos \theta}{h^4 \sin^3 \theta \Delta^{3/2}} \]  

and

\[ K_\varphi^\theta = \frac{(h^2 + h,\theta^2) (-h,\varphi \sin \theta + h,\varphi \cos \theta) + \sin \theta (h,\theta h,\varphi (h + h,\varphi \varphi))}{h^4 \sin^3 \theta \Delta^{3/2}}. \]  

Finally we find the mean curvature \( \kappa = \frac{1}{2} tr (K_i^j) \) and the Gaussian curvature \( \kappa_G = \det (K_i^j) \), given by

\[ \kappa = \frac{\alpha_1 \sin^3 \theta + \alpha_2 \sin^2 \theta \cos \theta + \alpha_3 \sin \theta + \alpha_4 \cos \theta}{2h^4 \Delta^2 \sin^3 \theta} \]  

in which

\[ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 3hh,\varphi \varphi + 2h^3 - h^2,\varphi \varphi \\ -h,\theta (h^2 + h,\theta^2) \\ h,\varphi (3h - h,\theta \theta) + 2h,\theta h,\varphi h,\varphi \theta - h,\varphi \varphi (h^2 + h,\theta^2) \\ -2h,\theta h^2,\varphi \end{pmatrix}, \]  

and

\[ \kappa_G = \frac{(h^2,\varphi + (h^2 + h,\theta^2) \sin^2 \theta) (\beta_1 \sin^4 \theta + \beta_2 \sin^3 \theta \cos \theta + \beta_3 \sin^2 \theta + \beta_4 \sin \theta \cos \theta + \beta_5 \cos^2 \theta)}{h^7 \Delta^3 \sin^6 \theta} \]  

in which

\[ \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} = \begin{pmatrix} h (h^2 + 2h,\theta - hh,\theta \theta) \\ -h,\theta (h^2 + 2h,\theta - hh,\theta \theta) \\ h,\varphi h^2 + (h,\theta h,\varphi \varphi - h^2,\varphi \varphi + 2h,\theta h,\varphi) h + 4h,\theta h,\varphi h,\varphi \theta - 2h,\varphi h,\varphi \varphi - 2h^2,\varphi h,\theta \theta \\ 2h,\varphi (-2h,\varphi h,\varphi + h,\varphi h, h) \\ -hh^2,\varphi \end{pmatrix} \]  

respectively.

### III. SPECIAL CASES

Our first special case is the perfect sphere with \( h(\theta, \varphi) = R \). This in turn implies \( h,\theta = h,\varphi = 0 \) and therefore

\[ K_i^j = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix} \]
and therefore $\kappa = \frac{1}{R}$ and $\kappa_G = \frac{1}{R^2}$.

In the second special case we set $h(\theta, \varphi) = h(\varphi)$ and consequently $h, \theta = 0$. These result in

$$g_{ij} = \begin{bmatrix} h^2 & 0 \\ 0 & h^2 \sin^2 \theta + h^2, \varphi \end{bmatrix}$$

(35)

and

$$K_i^j = \begin{bmatrix} \frac{h^2 \sin^2 \theta + h^2}{h^2 \sin^2 \Delta \Delta} & \frac{h, \varphi h^2 \sin^2 \theta \cos \theta + h, \varphi h \cos \theta}{h^2 \sin^2 \theta - h, \varphi + 2h, \varphi} \\ \frac{h^2 \sin^2 \theta \Delta \Delta}{h^2 \sin^2 \theta \Delta \Delta} & \frac{h^2 \sin^2 \theta - h, \varphi + 2h, \varphi}{h^2 \sin^2 \theta \Delta \Delta} \end{bmatrix}$$

(36)

in which $\Delta = 1 + \frac{h, \varphi}{h^2 \sin^2 \theta}$. Following $K_i^j$ one finds

$$\kappa = \frac{2h^2 \sin^2 \theta + 3h^2, \varphi - hh, \varphi}{2h^2 \Delta \Delta \sin^2 \theta}$$

(37)

and

$$\kappa_G = \frac{h^2 \sin^4 \theta + \left(2h^2, \varphi - hh, \varphi \right) \sin^2 \theta - h^2, \varphi \cos^2 \theta}{(h^2 \sin^2 \theta + h^2, \varphi)^2}.$$  

(38)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sphere.png}
\caption{A sphere with a bump in left and a miss/reverse bump on right. The exact value of the parameters in Eq. (44) are as follow: $R = 1$, $\alpha = 40$ and $\varepsilon_0 = \pm 0.2$. The positive and negative $\varepsilon_0$ give the left and the right respectively.}
\end{figure}

The next special case is given by $h(\theta, \varphi) = h(\theta)$ with $h, \varphi = 0$. The induced metric tensor becomes

$$g_{ij} = \begin{bmatrix} h^2 + h^2, \theta & 0 \\ 0 & h^2 \sin^2 \theta \end{bmatrix}$$

(39)

with the second fundamental form

$$K_i^j = \begin{bmatrix} \frac{h^2 - hh, \theta + 2h^2, \theta}{h^2 \Delta \Delta} & 0 \\ 0 & \frac{\left(h^2 + h^2, \theta \right) \left(h \sin \theta - h, \theta \cos \theta \right)}{h^4 \sin \theta \Delta \Delta} \end{bmatrix}.$$  

(40)

Finally we find

$$\kappa = \frac{\left(3hh, \theta + 2h^3 - h^2 hh, \theta \right) \sin \theta - h^3, \theta \cos \theta - h^2 h, \theta \cos \theta}{2h \left(h^2 + h^2, \theta \right)^2 \sin \theta}$$

(41)

and

$$\kappa_G = \frac{\left(h^2 - hh, \theta + 2h^2, \theta \right) \left(h \sin \theta - h, \theta \cos \theta \right)}{h \left(h^2 + h^2, \theta \right)^2 \sin \theta}.$$  

(42)
with

$$\Delta = 1 + \frac{h^2_0}{h^2}.$$  \hfill (43)

A. Example 1: A spheres with a bump

In Fig. 1 we plot a sphere with a bump, whose equation is given by

$$h(\theta) = R \left(1 + \varepsilon_0 \exp(-\alpha \theta^2)\right)$$  \hfill (44)

in which $R$ is the radius of the background sphere and $\varepsilon_0$ and $\alpha$ are constants. Depending on the value of $\varepsilon_0$ and $\alpha$ the size of the bump changes and even get a reversed shape with negative $\varepsilon_0$ (see Fig. 1 right). Using (41) and (42) we find

$$\kappa = \left(\frac{\lambda_1 \varepsilon_0^2 e^{-2\alpha \theta^2} + \lambda_2 \varepsilon_0 e^{-\alpha \theta^2} + (1 + \Delta) \sin \theta}{2R \left(1 + \varepsilon_0 e^{-\alpha \theta^2}\right)^2 \Delta^{3/2} \sin \theta}\right)$$  \hfill (45)

$$\kappa_G = \frac{\left(1 + \varepsilon_0 e^{-\alpha \theta^2} (-4\alpha^2 \theta^2 + 2\alpha + 2) + \varepsilon_0^2 e^{-2\alpha \theta^2} \left(4\alpha^2 \theta^2 + 2\alpha + 1\right)\right) \left(\varepsilon_0 e^{-\alpha \theta^2} (2\alpha \theta \cos \theta + \sin \theta) + \sin \theta\right)}{R^2 \sin \theta \left(1 + \varepsilon_0 e^{-\alpha \theta^2}\right) \left(1 + 2\varepsilon_0 e^{-\alpha \theta^2} + \varepsilon_0^2 e^{-2\alpha \theta^2} \left(4\alpha^2 \theta^2 + 1\right)\right)^2}$$  \hfill (46)

in which

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} (2\alpha + 4\alpha^2 \theta^2 + \Delta + 1) \sin \theta + 2\Delta \alpha \theta \cos \theta \\ 2\Delta \alpha \theta \cos \theta + (2 - 4\alpha^2 \theta^2 + 2\alpha + 2\Delta) \sin \theta \end{pmatrix}$$

and

$$\Delta = 1 + \frac{4\varepsilon_0^2 \alpha^2 \theta^2 e^{-2\alpha \theta^2}}{\left(1 + \varepsilon_0 e^{-\alpha \theta^2}\right)^2}. \hfill (47)$$
At $\theta = 0$ one finds $\Delta = 1$ and consequently

$$\kappa = \frac{2\alpha \varepsilon_0 + \varepsilon_0 + 1}{R(1 + \varepsilon_0)^2}$$

(48)

and

$$\kappa_G = \frac{(2\varepsilon_0\alpha + \varepsilon_0 + 1)^2}{R^2(1 + \varepsilon_0)^4} = \kappa^2.$$  

(49)

### B. Example 2: A sphere with a delta type inflation

In our second example we consider

$$h = R \left(1 + \frac{\zeta}{((\theta - \theta_0)^2 + \zeta^2)\pi}\right)$$

(50)

in which $\theta_0$, $\zeta$, and $R$ are constants. This model provides a delta-type inflation at $\theta = \theta_0$ on a sphere of radius $R$ such that a smaller $\zeta$ produces more localized and sharper inflation. In Fig. 2 we plot $h$ for $R = 1$, $\theta_0 = \frac{\pi}{2}$ and $\zeta = 0.1$. The sharp symmetric deformation at the equator of the sphere is displayed in this figure.

FIG. 3: $\kappa$ (red/above) and $\kappa_G$ (blue/below) in terms of $\theta$ for the second example i.e., Eq. (50). The specific values of parameters are as given in Fig. 2.

Using the general equations we find the total curvature given by

$$\kappa = \frac{-x\zeta \left(\Lambda^2\pi^2H^4 + 4\zeta^2x^2\right)\cot \theta + \pi H^2 \Lambda \left(\Lambda^2\pi^2H^4 - \pi \left(3x^2 - \zeta^2\right)\zeta H\Lambda + 6\zeta^2x^2\right)}{R\Lambda \left(\Lambda^2\pi^2H^4 + 4\zeta^2x^2\right)^2}$$

(51)
in which \(x = \theta - \theta_0\), \(H = (\theta - \theta_0)^2 + \zeta^2\) and \(\Lambda = 1 + \frac{\zeta^2}{\pi^2}\). In addition, the Gaussian curvature is also obtained as
\[
\kappa_G = \frac{(-6\Lambda \zeta \pi H x^2 + 2\Lambda \zeta^3 \pi H + \Lambda^2 \pi^2 H^4 + 8\zeta^2 x^2) \left(\Lambda \pi H^2 - 2\zeta x \cot \theta\right) \pi H^2}{R^2 \Lambda \left(\Lambda^2 \pi^2 H^4 + 4\zeta^2 x^2\right)}.
\]
(52)

The limit of \(\kappa\) and \(\kappa_G\) when \(\theta \to \theta_0\) are found to be
\[
\lim_{\theta \to \theta_0} \kappa = 1 + \frac{\zeta^2 + \zeta^3}{R \zeta (1 + \zeta)^2},
\]
(53)
\[
\lim_{\theta \to \theta_0} \kappa_G = \frac{2 + \zeta^2 + \zeta^3}{R^2 (1 + \zeta)^3}
\]
(54)

while their limits when \(\zeta \to 0\) becomes \(\frac{1}{R}\) and \(\frac{1}{R^2}\), respectively, for all \(\theta\) except for \(\theta = \theta_0\). At \(\theta = \theta_0\) we find \(\lim_{\zeta \to 0} \kappa = \infty\) and \(\lim_{\zeta \to 0} \kappa_G = \frac{2}{R^2}\). In Fig. 3 we plot \(\kappa\) and \(\kappa_G\) in terms of \(\theta\) for the specific choice of the parameters presented in Fig. 2.

IV. CONCLUSION

Finding the effect of a small shape fluctuation on the free energy of a membrane helps to predict the possible changes on the shape of such objects due to diverse kind of external perturbations. For instance when one taps on a soap bobble, it causes a change in its shape. We believe that the change of the shape of the soap bobble follows the minimum change of its free energy. Therefore knowing how to find the geometric properties of such structure, makes our job simpler. Some of the 2-dimensional surfaces can be approximated as a local flat surface and any fluctuation is considered as the MG. But in more general cases the unperturbed surface may not fit on a flat surface neither locally nor globally. In such cases a different version of the MG may be more helpful. In this work we have introduced the spherical MG in its most general form. With some specific cases and examples we have shown the application of this formalism. Our next step will be to find the MG for toroidal and cylindrical surfaces.

[1] P. Canham, J. Theor. Biol, 26, 61 (1970);
W. Helfrich, Z Naturforsch C, 28, 693 (1973);
R. Lipowsky, Nature, 349, 475 (1991);
U. Seifert, Adv. Phys., 46, 13 (1997);
Z. C. Ou-Yang, J. X. Liu and Y. Z. Xie, Geometric methods in the elastic theory of membranes in liquid crystal phases World Scientific, Singapore (1999);
I. M. Mladenov, P. A. Djondjorov, M. T. Hadzhilazova and V. M. Vassilev, Commun Theor Phys, 59, 213 (2013);
J. T. Jenkins, J. Math. Biol., 4, 149 (1977);
O. Zhongcan and W. Helfrich, Phys. Rev. Lett., 59, 2486 (1987);
O. Zhongcan and W. Helfrich, Phys. Rev. A, 39, 5280 (1989);
Z. C. Ou-Yang, Phys. Rev. A, 41, 4517 (1990);
U. Seifert, K. Berndl and R. Lipowsky, Phys. Rev. A, 44, 1182 (1991);
R. Podgornik, S. Svetina and B. Zekš, Phys. Rev. E, 51, 544 (1995);
D. H. Boal, M. Rao, Phys. Rev. A, 46, 3037 (1992);
R. Capovilla and J. Guven, J. Phys. A Math. Gen., 35, 6233 (2002);
R. Capovilla, J. Guven, J. A. Santiago, Phys. Rev. E, 66, 021607 (2002);
Z. C. Tu and Z. C. Ou-Yang, Phys. Rev. E, 68, 061915 (2003);
Z.C. Tu, J. Chem. Phys., 132, 084111 (2010);
Z.C. Tu, Chin. Phys. B, 22, 028701 (2013).
[2] Z. C. Tu and Z. C. Ou-Yang, Advances in Colloid and Interface Science, 208, 66 (2014).
[3] M. Deserno, Chem. and Phys. of Lipids, 185, 11 (2015).
[4] A. Goriely, A. Hausrath and S. Neukirch, Biophys. Rev. Lett., 3, 77 (2008).
[5] M. Terasaki, et al. Cell 154, 285 (2013).
[6] W. F. Marshall, Cell 154, 265 (2013).
[7] R. Lipowsky and E. Sackmann, (1995) Handbook In Biological Physics vols 1,2 (Amsterdam: Elsevier).
[8] D. Boal, (2002) Mechanics of the Cell (Cambridge: Cambridge U. Press).
[9] H. G. L. Coster, J. Biological Phys. 29, 363 (2003).
[10] U. Seifert and S. A. Langer, Europhys. Lett., 23, 71 (1993).
[11] R. J. Bingham, S. W. Smye and P. D. Olmsted, Europhys. Lett., 111, 18004 (2015).