Flux saturation number of superconducting rings

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The distributions of electrical current and magnetic field in a thin-film superconductor ring is calculated by solving the London equation. The maximum amount of flux trapped by the hole, the fluxoid saturation number, is obtained by limiting the current density by the depairing current. The results are compare it with similar results derived for the bulk case of a long hollow cylinder [Nordborg & Vinokur, Phys. Rev. B 62.12408 (2000)]. In the limit of small holes our result reduces to the Pearl solution for an isolated vortex in a thin film. For large hole radius, the ratio between saturation numbers in bulk and film superconductors is proportional to the square root of the hole size.

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Use of thin film superconductors integrated in nanodevices requires precise knowledge of the film behavior in the presence of both external and self induced magnetic fields. Recent experiments have shown that properly designed arrays of dots and antidots can serve as effective traps for magnetic flux. It has also been shown that patterned superconducting films allow for the motion of magnetic vortices to be guided over the film area, opening up for a new field often called fluxonics.

The models for flux trapping used in past years did not take into account the precise geometry of the patterned film samples. Those conventional models were based on approximations applicable either to a single vortex in an infinite film, equivalent to having a hole of zero radius, i.e., a Pearl vortex, or to an infinitely long hollow cylinder. Both geometries are far from realistic for thin film devices. However several recent works considers dynamics of vortices, current and field distributions in thin-film superconductor with finite hole size have been published. Yet, the flux saturation number, defined as the maximum number of flux quanta which can be trapped inside the hole, have so far not been calculated. Since the saturation number is a key concept for the vortex-antidot interaction and the overall effect of film patterning, we dedicate the present work to this calculation.

In this paper, we examine the problem of trapped flux in a thin-film superconducting ring by solving the London equations. The ring geometry allows modelling of a realistic situation with an isolated anti-dot in a thin-film sample. We consider the case of zero external magnetic field, and show that the saturation number for a thin-film ring can differ significantly from that of a hollow cylinder with the same radius. It is shown how the difference depends on the ratio between the hole radius and the thickness of the ring.

Consider a ring where the outer radius, \( r_2 \), is much larger than the inner radius, \( r_1 \), see Fig. 1. It is assumed that the film thickness, \( d \), is negligible compared to both the hole radius and the effective penetration length \( \lambda_{eff} = \lambda^2/d \), where \( \lambda \) is the London penetration depth of the superconductor. We want to determine the amount of flux trapped in the ring in a remanent state with flux present due to some magnetic prehistory. It will be assumed that flux pinning elsewhere is absent.

In the superconductor \( r_1 < r < r_2 \), and \( -d/2 < z < d/2 \), the distributions of current density \( \mathbf{j} \) and induction \( \mathbf{B} \) are given by the London equation, which in terms of the vector potential \( \mathbf{A} \), where \( \text{curl} \mathbf{A} = \mathbf{B} \), reads

\[
\text{curl}(\lambda^2 \mu_0 \mathbf{j} + \mathbf{A}) = 0. \tag{1}
\]

Due to the symmetry, the current density and vector potential have in cylindrical coordinates only azimuthal components, \( \mathbf{j} = (0,j,0) \) and \( \mathbf{A} = (0,A,0) \), respectively. From the Eq. (1) it then follows that

\[
\mu_0 \lambda^2 j(r) + A(r) = \frac{\Phi_f}{2\pi r}, \tag{2}
\]

where flux quantization implies that the constant \( \Phi_f \), the London fluxoid, is a free parameter restricted to an integer number of the flux quantum, \( \Phi_f = n\Phi_0 \), where \( \Phi_0 = h/2e \). The second term on the left-hand-side represents the flux through the area of radius \( r \);
\[ \int_0^r B(r') 2\pi r' \, dr' = A(r) \pi r, \] where \( B \) is the induction in the film plane, and is directed perpendicular to the plane.

In the absence of an applied field, the vector potential is only due to the current in the ring, and can be expressed as

\[ A = \frac{\mu_0}{4\pi} \int \frac{j}{|r - r'|} \, d^3r'. \] (3)

For the film geometry one can neglected variations of the current across the the sample thickness and average Eq. (3) over \( z \). Introducing the sheet current \( J(r) = J_{1/2} j(r) \, dz \), together with the dimensionless variables \( J'(r) = (\mu_0 \lambda_{\text{eff}})^2/\Phi_f J(r) \) and \( \tilde{r} = r/\lambda_{\text{eff}} \) the Eq. (2) becomes,

\[ J'(\tilde{r}) + \frac{1}{4\pi} \int_{r_1}^{r_2} \int_0^{2\pi} \frac{J'(\tilde{r}) \cos \theta}{\sqrt{\left(\tilde{r}/\tilde{r}\right)^2 + 1 - 2(\tilde{r}/\tilde{r}) \cos \theta}} \, d\theta d\tilde{r}' = 1/2\pi \tilde{r}. \] (4)

This Fredholm integral equation of the second kind was solved numerically by converting it into a set of linear equations corresponding to discrete values of the radial coordinate,

\[ J_i + \frac{1}{4\pi} \sum_{j} Q_{ij} \tilde{r}_j = \frac{1}{2\pi \tilde{r}_i}, \] (5)

\[ Q_{ij} = \int_0^{2\pi} \frac{\cos \theta}{\sqrt{(\tilde{r}_i/\tilde{r}_j)^2 + 1 - 2(\tilde{r}_i/\tilde{r}_j) \cos \theta}} \, d\theta. \] (6)

Results of such calculations are presented in Fig. 2. The upper panel shows the field and current distribution for the case where \( r_1 = \lambda_{\text{eff}} \), \( r_2 = 20\lambda_{\text{eff}} \). The field is mostly concentrated in the hole, where it has a pronounced peak at \( r_1 \), while a small peak from the return field is seen near \( r_2 \). The shielding current flows predominantly in the vicinity of the hole, reaching a maximum at the edge.

Plotted in the lower panel is the normalized current distribution for holes of different sizes ranging from \( r_1/\lambda_{\text{eff}} = 1 \) to 100, with all rings having \( r_2/\lambda_{\text{eff}} = 1000 \). For comparison, the figure includes also the current distribution around a single Pearl vortex in an infinite film,

\[ J_{\text{Pearl}}(\tilde{r}) = \left[ S_1(\tilde{r}/2) - K_1(\tilde{r}/2) - 2/\pi \right]/8. \] (7)

Here \( S_1 \) is the first order Struve function and \( K_1 \) the first order modified Bessel function of the second kind. For the small hole case, \( r_1 = \lambda_{\text{eff}} \), our numerical result is very close to the Pearl solution over the whole ring area except near the outer edge, where the sheet current has an upturn. The curves for the two larger hole radii also follow Eq. (7) for intermediate \( r \), but are seen to have an additional upturn at the inner edge. The current enhancement near \( r_1 \) is found to increase with the hole size, whereas near the outer edge the behavior is marginally influenced by the hole. Similar results for superconducting rings were reported previously by Brandt and Clem.\textsuperscript{16}

Our main interest lies in finding the fluxoid saturation value as function of the size of the hole in a thin film. It follows from Eqs. (7) and (8) that the magnitude of the sheet current depends linearly on \( \Phi_f \). Thus, the maximum \( \Phi_f \) corresponds to having a current at the inner edge, \( J(r_1) \), equal to the maximum current supported by the superconductor. We take this maximum value to be the depairing current,\textsuperscript{17}

\[ j_{dp} = \frac{\Phi_0}{3\sqrt{3}\pi \mu_0 \lambda_{\text{eff}} \xi}. \] (8)

Thus, the fluxoid saturation number, \( n_{\text{sat}} \), satisfies,

\[ \frac{n_{\text{sat}} \Phi_0}{j_{dp} d} = \frac{\Phi_f}{J(r_1)}, \] (9)
which gives
\[ n_{\text{sat}} = \frac{\lambda_{\text{eff}}}{3\sqrt{3}\xi J(\tilde{r}_1)}. \] (10)

Shown in Fig. 4 is the fluxoid saturation number as a function of the hole size. The following material parameters were here used, \( \xi = 3 \text{ nm}, \lambda = 150 \text{ nm}, d = 100 \text{ nm}, \) corresponding to YBa\(_2\)Cu\(_3\)O\(_x\) at low temperatures. The stepwise increase of \( n_{\text{sat}} \) versus \( r_1 \) seen in the main plot is essentially linear, but with an additional weak upward curvature. The inset shows that this behavior continues also for larger holes, \( r_1 > \lambda \).

For small holes we found that the current near the inner edge behaves essentially according to Eq. (7), which in this limit is well approximated by \( \tilde{J}(\tilde{r}) = 1/2\tilde{r} \). Thus, for small holes the saturation number is given by,
\[ n_{\text{sat}} = \frac{2r_1}{3\sqrt{3}\xi}, \quad r_1 \ll \lambda_{\text{eff}}. \] (11)

For larger holes where \( r_1 \gg \lambda_{\text{eff}} \), the current upturn near the edge prevents us from using the Pearl solution directly. Instead, we make a conjecture that the sheet current distribution is described by
\[ \tilde{J}(\tilde{r}) = \frac{1}{\pi\tilde{r}\sqrt{\tilde{r}^2 - r_1^2}}. \] (12)

Firstly, this analytical form fits excellently when compared to the current enhancement calculated numerically. Secondly, this form has the same asymptotic behavior as the Pearl solution, \( \tilde{J}(\tilde{r}) = 1/\pi\tilde{r}^2 \) for \( \tilde{r} \gg 1 \), consistent with the close agreement seen in Fig. 2 over a wide intermediate range of \( r \). Finally, the Eq. (12) has the diverging factor \( 1/\sqrt{\tilde{r} - r_1} \), commonly present in the edge behavior of thin superconductors in the Meissner state. As usual, the divergence is cut off at a distance \( \lambda_{\text{eff}} \) from the edge, giving for the ring geometry a maximum sheet current of \( \tilde{J}(\tilde{r}_1) = 1/(\sqrt{2\pi r_1}\tilde{r}_1^{3/2}) \). It then follows from Eq. (10) that for large holes one has,
\[ n_{\text{sat}} = \frac{\sqrt{2}r_1^{3/2}}{3\sqrt{3}\xi\lambda_{\text{eff}}^{1/2}}, \quad r_1 \gg \lambda_{\text{eff}}. \] (13)

It is of interest to compare these results with the fluxoid saturation number associated with a hole in a bulk superconductor. The case of a circular hole through an infinite superconductor was discussed in Ref. 11 using that for this geometry with \( r_1 \) being the hole radius, the induction is given by
\[ B(r) = n\Phi_0CK_0(r_1/\lambda), \quad r \geq r_1, \] (14)
where \( 1/C = \pi r_1\lambda/[2K_1(r_1/\lambda) + (r_1/\lambda)K_0(r_1/\lambda)] \), and \( K_0, K_1 \) are the zeroth-order and first-order modified Bessel functions. The current density around the hole here obtained simply by \( \mu_0\tilde{J}(r) = B'(r) \), and with

\[ \frac{n_{\text{bulk}}}{n_{\text{sat}}} = \sqrt{\frac{r_1}{2d}}. \] (17)

For typical patterned films with hole size \( r_1 = 500 - 600 \text{ nm} \), this ratio is close to 2.

Note that when the current density approaches the depairing value, the superconducting order parameter will be reduced. In particular, this will occur most easily in the vicinity of the hole. This means that one should
consider the saturation numbers derived in this work as upper limiting values for the number of flux quanta possibly trapped by the superconductor.

In summary, we have derived the fluxoid saturation number for thin superconducting films containing a circular hole of finite radius. For small holes, \( r_1 \ll \lambda_{\text{eff}} \), the result is approximately equal to the flux saturation number for bulk superconductors with a cylindric cavity. For large holes, \( r_1 \gg \lambda_{\text{eff}} \), the saturation number in thin films is less than in bulks by a factor \( \sqrt{r_1/2d} \). The present results, obtained by assuming the sample size to be much larger than the hole, should apply to superconducting films patterned with a single antidot. Even more, knowing the fluxoid saturation number, one may predict the interaction between vortices and antidot arrays in various experimental configurations.

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