Irregular Identification of Structural Models with Nonparametric Unobserved Heterogeneity

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Abstract

One of the most important empirical findings in microeconometrics is the pervasiveness of heterogeneity in economic behaviour (cf. Heckman 2001). This paper shows that cumulative distribution functions and quantiles of the nonparametric unobserved heterogeneity have an infinite efficiency bound in many structural economic models of interest. The paper presents a relatively simple check of this fact. The usefulness of the theory is demonstrated with several relevant examples in economics, including, among others, the proportion of individuals with severe long term unemployment duration, the average marginal effect and the proportion of individuals with a positive marginal effect in a correlated random coefficient model with heterogenous first-stage effects, and the distribution and quantiles of random coefficients in linear, binary and the Mixed Logit models. Monte Carlo simulations illustrate the finite sample implications of our findings for the distribution and quantiles of the random coefficients in the Mixed Logit model.

Keywords: Irregular Identification; Semiparametric Models; Nonparametric Unobserved Heterogeneity.

JEL classification: C14; C31; C33; C35

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1 Introduction

A tenet in empirical microeconometrics research is the pervasiveness of heterogeneity in behaviour of otherwise observationally equivalent individuals (cf. Heckman 2001). This paper shows that, for a large class of structural economic models, regular identification of functionals of nonparametric unobserved heterogeneity (UH), that is, identification of these functionals with a finite efficiency bound, implies certain necessary smoothness conditions on the functional, leading to a practically simple check for regularity (or lack thereof). In particular, this paper uses these implications to show that cumulative distribution functions (CDFs) and quantiles of UH often have infinite efficiency bounds in many empirically relevant economic models with nonparametric UH. These results have important practical implications, as these parameters are relevant for policy analysis, and they explain why any inferences on such parameters are expected to be unstable in empirical work. In particular, if a parameter is irregularly identified, then no regular estimator with a parametric rate of convergence exists (see Chamberlain 1986).

These observations are applicable to a wide class of models with nonparametric UH. We consider first continuous mixtures, which have been commonly employed as a modeling device to account for UH in a variety of economic settings ranging from labour to industrial organization; see Compiani and Kitamura (2016) for a recent review. The canonical example is a tightly specified structural parametric model that is made flexible by allowing all (or a subset) of parameters to be individual specific, thereby accounting for UH. We show that if the mapping from the individual specific parameters to the conditional likelihood is smooth, then there will be many functionals of UH that will not be regularly identified. Heuristically, smoothness of the conditional likelihood translates into a multicollinearity problem, as we further explain below. There are important economic applications that fall under this setting, see, e.g., Heckman and Singer (1984a, 1984b) for the study of unemployment duration. We demonstrate the usefulness of these results in the context of duration data by establishing an infinite efficiency bound for the distribution and quantiles of UH in the structural model of unemployment duration with two spells and nonparametric UH recently proposed by Alvarez, Borovicková and Shimer (2016).

The results are then extended to several classes of Random Coefficients (RC) models. These models have a long history in economics; see, e.g., Masten (2017) for a review of the literature. Applying our results to these models is technically more involved because these models have discontinuous conditional likelihoods given UH. We consider first RC models where UH is independent of regressors and establish an infinite efficiency bound for the distribution and quantiles of UH in binary and linear RC models. Establishing the zero information in the linear RC model is particularly challenging because the discontinuity in the conditional likelihood leads to potential discontinuities in the scores of the model. Given these results, we extend them to a triangular RC model with a continuous endogenous variable, where we show irregular identification of the average marginal effect (AME) and the proportion of individuals with a positive
marginal effect. The irregularity of the AME is driven by a positive mass of individuals with small first-stage effects. The irregular identification of the CDF and quantiles of the distribution of random or correlated effects holds more generally.

The models treated up to this point are indexed by the distribution of UH, and only by that distribution. However, a simple and powerful observation of this paper is that our analysis can be trivially extended to more complex semiparametric models indexed by UH and additional (possibly infinite-dimensional) parameters. We illustrate this point with several examples, including semiparametric mixture models where some parameters are fixed and others are random. A leading example is the popular RC Logit or Mixed Logit model, which is one of the most commonly used models in applied choice analysis. This model was introduced by Boyd and Mellman (1980) and Cardell and Dunbar (1980) and it is widely used in environmental economics, industrial economics, marketing, public economics, transportation economics and other fields. Applying our results to this model we obtain an infinite efficiency bound for CDFs and quantiles of the RC. The Mixed Logit example nicely illustrates the most appealing feature of our method of proof, which is its simplicity. Two lines of proof and a simple application of dominated convergence suffice. This should be contrasted with direct efficiency bounds calculations, which are particularly challenging for this model (or for any of the models we consider for that matter). These results have practical implications for proposed estimators of the Mixed Logit model. We report Monte Carlo simulations supporting our theoretical findings for “fixed grid” estimators of the distribution and quantiles in the Mixed Logit model (cf. Bajari, Fox and Ryan 2007 and Fox, Kim and Yang 2016). Further illustrations demonstrating the utility of our results in semiparametric settings are gathered in an Appendix and include examples on mixed proportional duration models and measurement error models with two measurements identified by means of Kotlarski’s lemma.

The parameters (functionals) we consider are of interest in their own. For example, labour economists are interested in the proportion of individuals at risk of severe long term unemployment, and more generally, social scientists are interested in evaluating the effects of treatments and policy interventions (e.g. average marginal effects and average signs). The functionals that we entertain, such as CDFs and quantiles of UH, are also used as inputs in subsequent counterfactual exercises. Our research limits the kind of inferences that are attainable with these parameters in models where UH is nonparametric.

What can be done to obtain regular identification of CDFs and quantiles of UH in these models? We show in several examples that functional form assumptions that restrict the conditional likelihood of observables given heterogeneity do not generally help for the purpose of achieving regularity of quantiles and CDFs if UH is still nonparametric. Thus, our results show that restricting UH is somewhat necessary to attain finite efficiency bounds for the distribution and quantiles of UH in many of the aforementioned models. Commonly used strategies in practice,
such as the use of parametric distributions for UH or considering discrete heterogeneity, indeed restore the regular identification of functionals of UH but can be deemed too strong. We find necessary conditions of regular identification under semiparametric restrictions on UH, although we recognize that giving general primitive assumptions for these conditions seems difficult. Our recommendation for inference on CDFs and quantiles of UH is to use flexible semiparametric specifications such as sieve methods; see, e.g., Shen (1997), Chen (2007), Bajari, Fox and Ryan (2007), Hu and Schennach (2008), Bester and Hansen (2007), Chen and Liao (2014), Fox, Kim and Yang (2016) and references therein, coupled with regularization (penalization) to reduce the high variance of estimates of functionals of UH when the conditional likelihood is a very smooth function of UH, as illustrated in this paper with the Mixed Logit model.

The rest of the paper is organized as follows. After a literature review, Section 3 sets notation and considers the class of continuous mixtures, where the method is most transparent. This section illustrates the theoretical results in the structural model of Alvarez, Borovicková and Shimer (2016). Section 4 extends the analysis to several classes of RC models. Section 5 extends further the analysis to semiparametric models, illustrating the theory with the Mixed Logit model. Section 6 discusses different strategies, some of them considered in the literature, to regularize the estimation of CDFs and quantiles of UH. Section 7 reports the results of some Monte Carlo simulations for the CDF and quantiles of the distribution of UH in the Mixed Logit model. Section 8 concludes. An Appendix contains proofs of the main results, further results on nonlinear RC models, examples and simulations.

2 Literature Review

Our paper relates to a number of studies providing sufficient conditions for nonparametric identification for the distribution of UH in the aforementioned models. See, among many others, Elbers and Ridder (1982), Heckman and Singer (1984a, 1984b) and Alvarez, Borovicková and Shimer (2016) for structural models of unemployment duration, Beran and Hall (1992), Beran, Feuerverger and Hall (1996), and Hoderlein, Klemela and Mammen (2010) for linear RC, Ichimura and Thompson (1998), Gautier and Kitamura (2013) and Hoderlein and Sherman (2015) for binary RC, Briesch, Chintagunta and Matzkin (2010) and Fox, Kim, Ryan and Bajari (2012) for RC multinomial choice models, Hoderlein, Holzmann and Meister (2017) for triangular RC models, Masten (2017) for simultaneous RC models, and Lewbel and Pendakur (2017) for nonlinear RC models. For a review of nonparametric identification results see Matzkin (2007, 2013) and Lewbel (2019). What differentiates our paper from these and other related studies is our focus on establishing whether identification is regular or not.

Establishing an infinite efficiency bound for functionals of UH in these models is a priori a rather challenging task. The main reason is that characterizing the so-called tangent space of
the model and projections onto it is generally quite complicated in the models we study here, and it may explain the relative lack of theoretical work on semiparametric efficiency bounds in RC and related models. See Newey (1990) for a review of semiparametric efficiency bounds and some of the related concepts. Our method of proof avoids the complications in directly computing the tangent space, projections and the Fisher information, which is the standard approach in the literature for obtaining efficiency bounds (see, e.g., Chamberlain 1986, Khan and Tamer 2010). Our indirect method of proof is relatively much simpler. The basic tool is a dominated convergence theorem, with regularity conditions that are easy to check in many models (although not in all models). The main building block is a fundamental result by van der Vaart (1991), who found a necessary condition for regular estimation of a parameter. The main observation of our paper consists in systematically exploiting the implications that van der Vaart’s (1991) necessary condition has on the smoothness of certain influence functions. van der Vaart (1991), Groeneboom and Wellner (1992) and Bickel, Klassen, Ritov and Wellner (1998) have also used the necessary condition of van der Vaart (1991) to show that CDFs are irregularly identified in some specific univariate exponential and uniform mixture models. Relative to this work, our contribution is to derive sufficient conditions for a general method of proof, thereby extending the scope of applications to models of economic interest. In particular, we allow for multidimensional UH, semiparametric models and non-smooth conditional likelihoods such as those that arise with RC models.

Although not the focus of this paper, a large class of models for which our results are applicable are panel data models with fixed effects. Within this setting, Chamberlain (1992) established regular identification of the AME in a linear RC panel data model, while Arellano and Bonhomme (2012) showed the identification of the full distribution of UH in a model with limited serial dependence in errors. Graham and Powell (2012) pointed out the irregular identification of the AME when regressors exhibit little variation across periods, while Bonhomme (2011) derived conditions for regular and irregular identification of moments of UH in nonlinear panel data. Our research is highly complementary to these papers, as we consider different models and our approach for proving irregular identification is different and exploits the smoothness implications of regular identification.

We illustrate the theoretical results with some Monte Carlo simulations implementing the “fixed grid” nonparametric CDF estimator of Bajari, Fox and Ryan (2007) and Fox, Kim, Ryan and Bajari (2011), and further investigated in Fox, Kim and Yang (2016). We contribute to the literature on the Mixed Logit model by proving the infinite efficiency bound for the CDF and quantiles of the nonparametric distribution of RC. We report further finite sample evidence on the performance of their computationally attractive “fix grid” estimator for CDFs and quantiles, as well as some regularized variants, complementing recent work in econometrics by Horowitz and Nesheim (2019) and Heiss, Hetzenecker and Osterhaus (2019).
3 Basic Setting and Results

Let \( \{(Z_i, \alpha_i)\}_{i=1}^n \) denote an independent and identically distributed (iid) sample with the same distribution as \((Z, \alpha)\). The observed data is \(Z_1, ..., Z_n\), while \(\alpha_i\) denotes the \(i\)-th individual’s UH. Assume each observation \(Z_i\) has a probability law \(\mathbb{P}\) and a density with respect to (wrt) a \(\sigma\)-finite measure \(\mu\) given by

\[
f_{\eta_0}(z) = \int_{\mathcal{A}} f_{z/\alpha}(z) d\eta_0(\alpha),
\]

where \(f_{z/\alpha}(z)\) denotes the known conditional density of \(Z\) given \(\alpha\), and \(\eta_0\) is the unknown distribution of \(\alpha\) with support on \(\mathcal{A} \subseteq \mathbb{R}^{d_\alpha}\) (the results can potentially be extended to abstract heterogeneity spaces, but for simplicity of exposition we focus on the Euclidean case). The assumption of known conditional density \(f_{z/\alpha}(z)\) is relaxed in Section 5.

Suppose we are interested in estimating a moment of UH, \(\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)]\), for a measurable function \(r(\cdot) \in L_2(\eta_0)\), where, henceforth, \(\mathbb{E}_{\eta_0}\) denotes the expectation under the distribution \(\eta_0\) and \(L_p(\nu)\) denotes the space of (equivalence classes of) real-valued measurable functions \(h\) such that \(\int |h|^p d\nu < \infty\), for a generic measure \(\nu\). Henceforth, we drop the sets of integration in integrals and the qualification \(\nu\)—almost surely for simplicity of notation. So, for example, a function in \(L_2(\nu)\) is discontinuous when there is no continuous function in its equivalence class. Also, we drop the reference to the measure \(\nu\) in \(L_2(\nu)\) when \(\nu = \mathbb{P}\), and write simply \(L_2\). We will be concerned with regular identification of \(\phi(\eta_0)\), i.e. identification of \(\phi(\eta_0)\) with a finite efficiency bound, when UH is nonparametric as formally defined below.

The basic message of this paper is based on two observations. First, from a general result in van der Vaart (1991), we prove that a necessary condition for regular identification of \(\phi(\eta_0)\) when UH is nonparametric is the existence of a measurable function \(s(Z)\) with zero mean and finite variance such that

\[
r(\alpha) - \phi(\eta_0) = \int s(z) f_{z/\alpha}(z) d\mu(z).
\]

Second, if the mapping \(\alpha \rightarrow f_{z/\alpha}\) is continuous (smooth), then under mild regularity conditions, (2) implies that \(r(\cdot)\) must be also continuous (smooth). The bulk of this paper is a formalization of the second observation and its application to some economic models of interest.

The precise sense of UH being nonparametric is the usual one, formalized as follows. Let \(H\) denote a class of distributions on \(\mathcal{A}\), and assume \(\eta_0 \in H\). Let \(\eta_t \in H\) be a parametric submodel indexed by \(t \in [0, \varepsilon)\), for some \(\varepsilon > 0\), such that for a \(b \in L_2(\eta_0)\) the classical mean square differentiability condition holds,

\[
\int \left[ \frac{d\eta_t^{1/2} - d\eta_0^{1/2}}{t} - \frac{1}{2} b d\eta_0^{1/2} \right]^2 \rightarrow 0 \text{ as } t \downarrow 0.
\]
Then, a formal definition of nonparametric UH is given as follows. Denote by $T(\eta_0)$ the linear span of the $b's$ in (3) and let $L^2_2(\nu)$ denote the subspace of functions in $L_2(\nu)$ with zero $\nu$–mean.

**Definition 3.1** UH is nonparametric if $T(\eta_0)$ is dense in $L^2_2(\eta_0)$.

Henceforth, we assume, unless otherwise stated, that UH is nonparametric. The first result in this section, which follows from an application of van der Vaart (1991), shows that, in the presence of nonparametric UH in model (1), regular identification of $E_{\eta_0}[r(\alpha)]$ requires necessarily that (2) holds.

**Lemma 3.1** If UH is nonparametric, then (2) is necessary for regular identification of $\phi(\eta_0)$.

We note that Severini and Tripathi (2006, 2012) and Bonhomme (2011) have found related results in the context of nonparametric instrumental variables and nonlinear panel data models, respectively. Also, Escanciano (2020) has shown that (2) is also sufficient for semiparametric identification of $\phi(\eta_0)$ in model (1). Note that we are not assuming here that $\eta_0$ or $s$ in (2) are identified. This generality is important because these functions may not be identified in many structural economics models under weak assumptions, which does not prevent us from identifying and estimating certain functionals of them (cf. Hurwicz 1950).  

We now proceed with the main insight of this paper, which is that if the mapping $\alpha \to f_{z/\alpha}$ is continuous (smooth), then, under regularity conditions, $r(\cdot)$ must be also continuous (smooth). This simple observation follows by dominated convergence, and it implies non-regularity of CDFs, signs, quantiles, and other functionals of UH in “smooth models” satisfying the following assumption. Let $N$ denote an open subset of $A \subset \mathbb{R}^{d_\alpha}$.

**Assumption 1** (i) $\alpha \to f_{z/\alpha}(z)$ is continuous on $N$ a.e.-$\mu$; (ii) for all $\alpha \in N$ there exists a neighborhood of $\alpha$, say $\Gamma_0 \subset N$, such that for all $s$ satisfying (2),

$$\int |s(z)| \sup_{\alpha \in \Gamma_0} f_{z/\alpha}(z) d\mu(z) < \infty.$$  \hspace{1cm} (4)

Assumption 1(i) is easy to check. Assumption 1(ii) is a dominance condition. The main complication in checking Assumption 1(ii) is that $s$ belongs to $L_2(\mathbb{P})$ but not necessarily to $L_1(\mu)$ or $L_2(\mu)$. We verify these conditions in a number of examples below.

**Lemma 3.2** Let the conditional density $f_{z/\alpha}(z)$ satisfy Assumption 1. Then, $r(\alpha)$ in (2) is continuous in $\alpha$ on $N$.

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1Of course, if $\eta_0$ is identified, so is $\phi(\eta_0)$ (since $r$ is known). Identification of $\phi(\eta_0)$ follows from (2) because we can find an identified function $\tilde{s}(Z)$, depending only on $f_{z/\alpha}$ and $r$, such that $r(\alpha) = E[\tilde{s}(Z) | \alpha]$ holds, and thus by iterated expectations $\phi(\eta_0) = E_{\eta_0}[r(\alpha)] = E_{\eta_0}[E[\tilde{s}(Z) | \alpha]] = E[\tilde{s}(Z)]$. 

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The following corollary is a direct consequence of the previous two lemmas.

**Corollary 3.1** Let Assumption 1 hold. The CDF \( \phi(\eta_0) = \mathbb{E}_{\eta_0}[1(\alpha \leq \alpha_r)] \), for \( \alpha_r \in N \), is not regularly identified.

Quantiles of UH are nonlinear functionals, and are not covered by the previous results. To extend the theory to a more general setting including nonlinear functionals we need to introduce some notation. A functional \( \phi(\eta_0) : H \to \mathbb{R} \) is said to be differentiable if there exists an \( r_\phi \in L^0(\eta_0) \) such that for all paths satisfying (3), it holds

\[
\lim_{t \to 0} \frac{\phi(\eta_0) - \phi(\eta_0)}{t} = \mathbb{E}_{\eta_0}[r_\phi(\alpha)b(\alpha)].
\]

Under nonparametric UH such \( r_\phi \) is unique, as in Newey (1994). This function \( r_\phi \) plays the role of the preceding moment function \( r \).

To illustrate with an example, consider the scalar UH case and assume \( \eta_0 \) is absolute continuous with a strictly positive Lebesgue density in a neighborhood of \( \phi(\eta_0) \), where \( \phi(\eta_0) \) is such that

\[
\int_{-\infty}^{\phi(\eta_0)} d\eta_0(\alpha) = \tau, \quad \tau \in (0,1).
\]

That is, \( \phi(\eta_0) \) is the \( \tau \)-quantile of \( \eta_0 \). It is well-known, see, e.g., Lemma 21.3 in van der Vaart (1998), that the quantile functional is differentiable under the conditions above with influence function

\[
r_\phi(\alpha) = -\frac{1(\alpha < \phi(\eta_0)) - \tau}{\hat{\eta}_0(\phi(\eta_0))},
\]

where \( \hat{\eta}_0 \) is the density pertaining to \( \eta_0 \). From our results, the discontinuity of the influence function \( r_\phi(\cdot) \) implies irregular identification. Next result, formalizes this finding.

**Corollary 3.2** Let Assumption 1 hold. Assume \( \eta_0 \) is absolute continuous with a strictly positive Lebesgue density in a neighborhood of \( \phi(\eta_0) \) satisfying (5). If \( \phi(\eta_0) \in N \), then the \( \tau \)-quantile of the nonparametric UH distribution is not regularly identified.

**Remark 3.1** Henceforth, whenever we discuss identification of quantiles, we implicitly assume that the components of UH have densities that satisfy the conditions in Corollary 3.2. This example illustrates how our results are applicable to nonlinear differentiable functionals.

We discuss now the complications of the more standard approach of computing the Fisher Information or the efficiency bound. Define the so-called tangent space of scores \( S := \{ s \in L^0_2 : s(z) = \mathbb{E}[b(\alpha) | Z] \text{ for some } b \in T(\eta_0) \} \). Then, a standard result in linear inverse problems is that all solutions \( s \) of equation (2) have the same orthogonal projection onto the closure of \( S \) (see Engl, Hanke and Nuebauer, 1996). Denote by \( s^* \) such orthogonal projection, the so-called
efficient score. The efficiency bound is given by the variance of \( s^*(Z) \) (see e.g. Newey 1990, van der Vaart 1998, Bickel et al. 1998, and Escanciano 2020). Thus, an alternative to our approach is to compute \( s^*(Z) \) and checking that it has infinite variance. However, computing \( s^*(Z) \) can be cumbersome, particularly because characterizing the mean squared closure of \( S \) can be a rather difficult task in the models we analyze here. In fact, to the best of our knowledge, the analytical expression for \( s^* \) remains unknown for the functionals and models we study. In passing, we note that these arguments show that it suffices to check the dominance condition (4) for \( s \) in the closure of \( S \). This additional information will turn out to be quite useful in some of our applications, such as the linear RC model.

3.1 An Application To A Structural Model of Unemployment

We illustrate the applicability of the previous results in the context of a structural model of unemployment with nonparametric UH. Nonparametric heterogeneity has played a critical role in rationalizing unemployment duration ever since the seminal contributions by Elbers and Ridder (1982) and Heckman and Singer (1984a, 1984b). Recent work by Alvarez et al. (2016) is motivated from this perspective. These authors have shown nonparametric identification of the distribution of UH in their nonparametric structural model for unemployment with two spells. Specifically, Alvarez, Borovicková and Shimer (2016) propose a structural model for transitions in and out of employment that implies a duration of unemployment given by the first passage time of a Brownian motion with drift, a random variable with an inverse Gaussian distribution. The parameters of the inverse Gaussian distribution are allowed to vary in arbitrary ways to account for UH in workers. These authors investigate nonparametric identification of the distribution of UH, \( \eta_0 \), when two unemployment spells \( Z_i = (t_{i1}, t_{i2}) \) are observed on the set \( \mathcal{T}^2, \mathcal{T} \subseteq [0, \infty) \). The reduced form parameters \( \alpha = (\alpha_1, \alpha_2)' \in \mathbb{R} \times [0, \infty) \) are functions of structural parameters. The distribution of \( Z_i \) is absolutely continuous with Lebesgue density \( f_{\eta_0}(t_1, t_2) \) given, up to a normalizing constant, by

\[
 f_{\eta_0}(t_1, t_2) = \int_{\mathbb{R} \times [0, \infty)} \frac{\alpha_2^2}{t_1^{3/2} t_2^{3/2}} e^{-\frac{(\alpha_1 t_1 - \alpha_2)^2}{2 t_1^2} - \frac{(\alpha_1 t_2 - \alpha_2)^2}{2 t_2^2}} d\eta_0(\alpha_1, \alpha_2). \tag{6}
\]

Alvarez, Borovicková and Shimer (2016) show that \( \eta_0 \) is nonparametrically identified up to the sign of \( \alpha_1 \), but they do not investigate if specific functionals of this distribution are regularly or irregularly identified, which is the focus of study here. Specifically, we show that the CDF of \( \eta_0 \) at a point, and other functionals of \( \eta_0 \) with discontinuous influence functions, such as quantiles, have infinite efficiency bounds. These functionals are important parameters. For example, \( \phi(\eta_0) = \mathbb{E}_{\eta_0} [1(\alpha_1 \leq \alpha_{10}) 1(\alpha_2 \leq \alpha_{20})] \), for a fixed \( \alpha_{10} < 0 < \alpha_{20} \) and large absolute values of \( \alpha_{10} \) and \( \alpha_{20} \), quantifies the proportion of individuals at risk of severe long term unemployment (an individual with parameters \( \alpha_1 \) and \( \alpha_2 \), \( \alpha_1 \leq \alpha_{10} \) and \( \alpha_2 \leq \alpha_{20} \), has a probability larger or
equal than $1 - \exp(2\alpha_{10}\alpha_{20})$ of remaining unemployed forever). We apply our previous results to this example for a generic moment $\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha_1, \alpha_2)]$, under the following mild condition.

**Assumption 2** (i) Let the set $T \subseteq [0, \infty)$ be a convex set with a non-empty interior; (ii) the moment function $r$ is locally bounded.

**Proposition 3.1** Under Assumption 2, if $\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha_1, \alpha_2)]$ is regularly identified, then

$$r(\cdot) \in \{b(\alpha_1, \alpha_2) \in L^0_2(\eta_0) : b(\alpha_1, \alpha_2) = C_1 + C_2\alpha_2^2 e^{2\alpha_1\alpha_2} h(\alpha_1^2, \alpha_2^2)\},$$

for constants $C_1$ and $C_2$ and a continuous function $h(u, v)$ defined on $(0, \infty)^2$ that, if $T$ is bounded, is an infinite number of times differentiable at $u \in (0, \infty)$, for all $v \in (0, \infty)$.

For the purpose of proving an infinite efficiency bound for CDFs and quantiles only the continuity part of Proposition 3.1 suffices. Thus, an implication of Proposition 3.1 is that the CDF of UH at the fixed point $(\alpha_{10}, \alpha_{20})$, i.e. $\phi(\eta_0) = \mathbb{E}[1(\alpha_1 \leq \alpha_{10})1(\alpha_2 \leq \alpha_{20})]$, is not regularly identified because $r_{\phi}(\alpha_1, \alpha_2) = 1(\alpha_1 \leq \alpha_{10})1(\alpha_2 \leq \alpha_{20})$ is not continuous when $(\alpha_{10}, \alpha_{20})$ is in the interior of the support of $\eta_0$.

**Corollary 3.3** Under Assumption 2(i), the CDFs and quantiles of UH in the model (6) are not regularly identified.

## 4 Random Coefficient Models

Random coefficient models have long been used in economics to model nonparametric UH. There is by now an extensive literature on nonparametric identification of UH in these models, see, e.g., Masten (2017) and references therein. In this paper we focus on establishing irregular identification of CDFs and quantiles of the distributions of RC. To the best of our knowledge, this is the first paper to do so in this generality.

A general class of random coefficient models, including nonlinear models, is given by

$$Y_i = m(X_i, \alpha_i),$$

where $Z_i = (Y_i, X_i)$ are observed, but $\alpha_i$ is unobserved and independent of $X_i$ with support $\mathcal{A}$. Assume $m : \mathcal{X} \times \mathcal{A} \to \mathbb{R}^r$ is a measurable map, where $\mathcal{X}$ is the support of $X$. The functional form of $m$ is known, and the nonparametric part is given by the distribution of $\alpha_i$. The assumptions of known $m$ and the independence of $\alpha_i$ and $X_i$ are relaxed below. The density of the data is

$$f_m(y, x) = \int_{\mathcal{A}} 1(y = m(x, \alpha)) d\eta_0(\alpha),$$
where $1(A)$ denotes the indicator function of the event $A$. In this setting, the dominating measure $\mu$ is defined on $\mathcal{Z} = \mathcal{Y} \times \mathcal{X}$ as $\mu(B_1 \times B_2) = \nu_{\mathcal{Y}}(B_1) \nu_{\mathcal{X}}(B_2)$, where $B_1$ and $B_2$ are Borel sets of $\mathcal{Y}$ and $\mathcal{X}$, respectively, $\nu_{\mathcal{Y}}$ is either the counting measure for discrete outcomes or the Lebesgue measure $\lambda(\cdot)$ for continuous outcomes, and $\nu_{\mathcal{X}}(\cdot)$ is the probability measure for $X$. The main challenge we face with RC models is that $f_{z/\alpha}(z) = 1 (y = m(x, \alpha))$ is not continuous, and thus the previous results need to be generalized. The generalization is non-trivial, particularly for continuous outcomes, and in some cases it requires delicate technical work. We consider first the binary choice RC model. Section 10.1 in the Appendix contains some generic results for nonlinear RC, as well as discussion on some RC models for which our conclusions do not hold.

### 4.1 Binary Choice Random Coefficient

The binary choice random coefficient model is given by

$$Y_i = 1 \left( X_i' \alpha_i \geq 0 \right),$$

where we observe $Z_i = (Y_i, X_i)$ but $\alpha_i$ is unobservable. The random vector $\alpha_i$ is independent of $X_i$, normalized to $|\alpha_i| = 1$ and satisfies $\mathbb{P}(\alpha_i = 0) = 0$. As in the existing literature, we assume $\eta_0$ is absolutely continuous wrt the uniform spherical measure $\sigma(\cdot)$ in $S_{d_{\alpha} - 1}$, where $S_{d_{\alpha} - 1} = \{ b \in \mathbb{R}^{d_{\alpha}} : |b| = 1 \}$ denotes the unit sphere in $\mathbb{R}^{d_{\alpha}}$. The density of the data for a positive outcome (i.e. the choice probability function) is given by

$$f_{\eta_0}(x) = \int_{S_{d_{\alpha} - 1}} 1 (x's \geq 0) d\eta_0(s). \quad (8)$$

Ichimura and Thompson (1998) and Gautier and Kitamura (2013) found sufficient conditions for nonparametric identification of $\eta_0$, but they did not investigate whether identification was regular or irregular, which is the focus here.

By (8) and Lemma 3.1 a necessary condition for regular identification of $\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)]$ under nonparametric UH is

$$r(\alpha) - \phi(\eta_0) = \int 1 (x'\alpha \geq 0) s(1, x) dv_X(x), \quad (9)$$

for some $s \in L^0_{d_{\alpha} - 1}$. The following result provides necessary conditions for regular identification.

Write $\alpha = (\alpha_1, \alpha_2)'$. **Proposition 4.1** If the distribution of $X/|X|$ is absolutely continuous, then $r(\cdot)$ in (9) must be uniformly continuous on $S_{d_{\alpha} - 1}$. If $X = (1, X')$ and $\alpha_2'X$ is absolutely continuous, then $r(\alpha_1, \alpha_2)$ is an absolutely continuous function of $\alpha_1$.

An implication of this proposition is that functionals such as the CDF and quantiles of random coefficients are not regularly identified in the binary RC model. To the best of our knowledge, this result is new in the literature.
Corollary 4.1 Under the conditions of Proposition 4.1, the CDFs and quantiles of UH in the binary RC model are not regularly identified.

4.2 Linear Random Coefficient

The linear RC model has a long history in econometrics, see, e.g., Hildreth and Huock (1968) and Swamy (1970). This model is given by

$$Y_i = X_i' \alpha_i,$$

where we observe a $d_z$-dimensional vector $Z_i = (Y_i, X_i)$, but $\alpha_i$ is unobservable and independent of $X_i$. The dimension of $X_i$ and $\alpha_i$ is $d_\alpha$, so $d_z = d_\alpha + 1$. Like in Hoderlein, Klemelä and Mammen (2010), we normalize $X_i$ so that $|X_i| = 1$.

The density of the data is

$$f_{\eta_0}(z) = \int_{\mathbb{R}^{d_\alpha}} 1(y = x'\alpha) \, d\eta_0(\alpha). \quad (10)$$

Nonparametric identification and estimation of $\eta_0$ has been studied by Beran and Hall (1992), Beran, Feuerverger and Hall (1996), and Hoderlein, Klemelä and Mammen (2010), among others. These authors exploit the relation between (10) and the Radon transform. In this paper we study necessary conditions for regular identification of $\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)]$, for a measurable function $r(\cdot)$ with $\mathbb{E}_{\eta_0}[r^2(\alpha)] < \infty$, and regular identification of quantiles of the components of $\alpha$.

By Lemma 3.1 a necessary condition for regular identification of $\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)]$ under nonparametric UH is

$$r(\alpha) - \phi(\eta_0) = \int s(x'\alpha, x) \, dv_X(x), \quad (11)$$

for some $s \in L^0_2$. Under suitable conditions scores in the tangent space $S = \{ s \in L^0_2 : s(z) = \mathbb{E}[b(\alpha)|Z] \text{ for some } b \in T(\eta_0) \}$ are continuous, but providing conditions under which elements of the closure of $S$ are continuous is much harder. In fact, without additional restrictions elements in the closure of $S$ can be potentially very discontinuous. We shall provide regularity conditions below that guarantee that any element of the closure of $S$ can be written as

$$s(z) = \frac{g(z)}{f_{\eta_0}(z)},$$

where $g(z)$ has an squared integrable weak derivative with respect to the first argument $y$. As we show below, this last condition will be instrumental for checking the sufficient conditions for the dominated convergence theorem in Lemma 3.2.

Let $\eta_{0,\alpha}$ denote the Lebesgue density of $x'\alpha$ when $\alpha$ has distribution $\eta_0$. The set $\eta_0 T(\eta_0)$ is defined as $\eta_0 T(\eta_0) := \{ \eta_0 b : b \in T(\eta_0) \}$, while the definition of a Sobolev space $H^{\rho_0}(A)$ is provided after (24) in the Appendix.
Assumption 3 For $d_\alpha > 1$ and $N$ as in Assumption 1: (i) the distribution $\eta_0$ is bounded, has bounded support, with a corresponding density $\eta_{0,z}$ that is continuous and satisfies $\inf_{z \in N} \eta_{0,z}(x^t \alpha) \geq 1/l(x)$ for a positive measurable function $l(\cdot)$ such that $\mathbb{E}_X [l^2(X)] < \infty$; (ii) $X$ is absolutely continuous with a bounded density $f_X(\cdot)$; (iii) $\eta_0 T(\eta_0) \subseteq H^\infty(A)$, where $\rho_0 + (d_\alpha - 1)/2 > 2$; (iv) $r$ belongs to the closure of $T(\eta_0)$.

The bounded support of Assumption 3(i) is often considered in the literature, see, e.g., Hoderlein, Klemelä and Mammen (2010). If the infinite efficiency bound holds in a model with bounded support of $\alpha$ it also holds in the more general model where the support is unrestricted. A sufficient condition for the continuity of $\eta_{0,z}$ is that the Fourier transform of the density of $\eta_0$ is integrable, which was also assumed in Hoderlein, Klemelä and Mammen (2010). Assumptions 3(i-ii) establish a link between the tails of $\eta_0$ and $f_X(\cdot)$. Assumption 3(iii) imposes a mild smoothness condition on the tangent space of $UH$. This assumption and Assumption 3(iv) allow but do not require nonparametric $UH$.

Proposition 4.2 Under Assumption 3 and if $r$ satisfies (11), then it must be continuous on $N$.

Corollary 4.2 Under the conditions of Proposition 4.2, the CDFs and quantiles of $UH$ are not regularly identified in the linear RC model.

4.3 Correlated Random Coefficients: AME

The independence assumption between regressors and $UH$ rules out important models and parameters in economics, such as the Average Marginal Effect (AME) $\phi(\eta_0) = \mathbb{E}_{\eta_0} [\gamma_i]$ and the Proportion of individuals with a Positive AME (PPAME), $\phi(\eta_0) = \mathbb{E}_{\eta_0} [1 (\gamma_i > 0)]$, where $\gamma_i$ is the coefficient of an endogenous continuous variable in a RC triangular system. We extend our previous results to these cases. We will show that under nonparametric $UH$ these important parameters are not regularly identified. These results appear to be new in the literature under this generality. For simplicity, we focus on a triangular model, but the same arguments are applicable to a wide class of random coefficient models, including simultaneous equation models, nonlinear models with endogeneity, or variations of these models that include covariates, multiple endogenous variables, and mixed random and non-random coefficients.

Consider the triangular model:

$$
Y_1 = \gamma Y_2 + U_1, \quad Y_2 = \delta X + U_2,
$$

where $\gamma, U_1, \delta$ and $U_2$ are RC, and we observe $Z = (Y_1, Y_2, X)'$. The variable $Y_2$ is a continuous treatment variable, possibly endogenous, in the sense that $U_1$ and $U_2$ are correlated, and $X$ is an instrument, independent of all the random coefficients. Suppose, the researcher is interested
in the AME $\phi(\eta_0) = \mathbb{E}_{\eta_0}[\gamma]$ or the PPAME $\phi(\eta_0) = \mathbb{E}_{\eta_0}[1(\gamma > 0)]$. We will provide conditions under which both parameters have an infinite efficiency bound. To see this, we obtain the reduced forms

$$Y_1 = \gamma \delta X + \gamma U_2 + U_1 \equiv \pi_1 X + \pi_0,$$
$$Y_2 = \delta X + U_2,$$

which, with some abuse of notation, are jointly written as $Y = \alpha_0 + \alpha_1 X$, where $Y = (Y_1, Y_2)'$, $\alpha = (\alpha_0, \alpha_1)$, $\alpha_0 = (\pi_0, U_2)'$ and $\alpha_1 = (\pi_1, \delta)'$. Proposition 4.2 can then be applied to the reduced form. Because the corresponding influence functions for the AME and PPAME are $r_{AME}(\alpha) = \pi_1/\delta$ and $r_{PPAME}(\alpha) = 1(\pi_1 > 0)1(\delta > 0) + 1(\pi_1 < 0)1(\delta < 0)$, respectively, and they are discontinuous functions of $\alpha_1 = (\pi_1, \delta)'$, non-regularity follows from Proposition 4.2. Consider the following assumption. Let $N$ be an open set in the interior of $\mathcal{A}$, the support of the reduced form random coefficient $\alpha$.

**Assumption 4** (i) Assumption 3 holds with the reduced form $Y = \alpha_0 + \alpha_1 X$; (ii) $X$ independent of the random coefficients $(\gamma, U_1, \delta, U_2)$; (iii) $(p_0, u_2, 0, d_0) \in N$ for some $(p_0, u_2, d_0)$; (iv) $(p_0, u_2, p_1, 0) \in N$ for some $(p_0, u_2, p_1)$.

**Proposition 4.3** Suppose (12) and Assumption 4(i-ii) holds. If in addition Assumption 4(iii) or Assumption 4(iv) holds, then the PPAME is not regularly identified. If Assumption 4(iv) holds and $\mathbb{E}[\gamma^2] < \infty$, then the AME is not regularly identified.

Proposition 4.3 proves non-regularity for the AME and the PPAME. The condition $\mathbb{E}[\gamma^2] < \infty$ ensures that the AME is a continuous functional in $L_2(\eta_0)$. If $f_{\delta^2}$ denotes the (Lebesgue) density of $\delta^2$ and $h(u) = \mathbb{E}[\pi_1^2 | \delta^2 = u] f_{\delta^2}(u)$, then a sufficient condition for $\mathbb{E}[\gamma^2] < \infty$ is $\lim_{u \to 0^+} h(u)/u^\rho < \infty$ for some $\rho > 0$ and $\mathbb{E}[\pi_1^2] < \infty$; see Khuri and Casella (2002, pg. 45).

Intuitively, non-regularity of the AME comes from the presence of a set of individuals with near-zero first-stage effects (Assumption 4(iv)), although $\mathbb{P}(\delta = 0) = 0$. When the instrument satisfies a monotonicity restriction, in the sense that $\mathbb{P}(\delta > 0) = 1$ or $\mathbb{P}(\delta < 0) = 1$, then regular identification of the AME might be possible. Indeed, Heckman and Vytlacil (1998) and Wooldridge (1997, 2003, 2008) show that with homogenous first-stage effects regular estimation by IV methods holds. Masten (2017, Proposition 4) gives conditions for nonparametric identification of the distribution of $\gamma$, but he did not discuss efficiency bounds for the AME or the PPAME under his conditions. Khan and Tamer (2010) and Graham and Powell (2012) show irregularity of the AME in different models where $\mathbb{E}[\gamma^2] = \infty$. We show irregularity of the AME in a setting where $\mathbb{E}[\gamma^2] < \infty$. See also Florens et al. (2008), Masten and Torgovitsky (2016), and the extensive literature following the seminal contributions by Imbens and Angrist (1994) and Heckman and Vytlacil (2005) for identification results on conditional and weighted AME or their discrete versions.
The PPAME is non-regular under more general conditions than the AME, because it has a discontinuous influence function under more general conditions than that of the AME. Heckman, Smith and Clements (1997) provide bounds for the analog to PPAME in the binary treatment case, and identification when gains are not anticipated at the time of the program. The irregularity of the PPAME also follows from a more general principle that we describe in the next section: if irregularity holds in a model with exogenous effects, it also holds in the model with endogenous effects.

5 Extension to Semiparametric Models

This section extends our results to semiparametric models. The main point is as follows, if a functional is non-regularly identified in a model, it will be non-regularly identified in a larger model that nests the original model as a special case. Information can only decrease (or remain the same) when we know less. This basic observation has important implications, and it widens substantially the applicability of our results as illustrated with the Mixed Logit model here and with further examples in the Appendix.

5.1 The Mixed Logit Model

Consider first a conditional semiparametric mixture model with density

\[ f_{\lambda_0,\theta_0}(y, x) = \int f_{y/x,\alpha}(y; \theta_0) d\eta_0(\alpha), \]

where \( \theta_0 \) is an additional unknown parameter, finite or infinite-dimensional. The basic idea here is that irregularity of \( \phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)] \) in the model where \( \theta_0 \) is known implies irregularity in the model where \( \theta_0 \) is unknown.

We illustrate our point with the random coefficients Logit model, also known as the Mixed Logit—one of the most commonly used models in applied choice analysis. Fox, Kim, Ryan and Bajari (2012) have recently shown nonparametric identification for the semiparametric Mixed Logit model. Here, we show that the identification of the CDF and quantiles of the distribution of RC is necessarily irregular when UH is nonparametric. The CDF and quantiles of this distribution are important parameters in applications of discrete choice.

The data \( Z_i = (Y_i, X_i) \) is a random sample from the density (wrt \( \mu \) below),

\[ f_{\lambda_0}(y, x) = \int f_{y/x,\alpha}(y; \theta_0) d\eta_0(\alpha), \]

where \( \lambda_0 = (\theta_0, \eta_0) \in \Theta \times H, \theta_0 = (\theta_{01}, ..., \theta_{0J})' \),

\[ f_{y/x,\alpha}(y; \theta_0) = \frac{\exp(\theta_{0y} + x'\alpha)}{1 + \sum_{j=1}^{J} \exp(\theta_{0j} + x'\alpha)}, \]
for $x = (x_0, x_1, ..., x_J) \in \mathcal{X}$ and $y \in \mathcal{Y} = \{0, 1, ..., J\}$. The consumer can choose between $j = 1, ..., J, J < \infty$, mutually exclusive inside goods and one outside good ($y = 0$). The utility for the inside good is normalized so that $\theta_{00} = 0$ and $x_0 = 0$. The random coefficients $\alpha$ are independent of the regressors $X$, and have a distribution $\eta_0$. The main result below also applies to the correlated random coefficient case. In fact, non-regular identification for CDFs and quantiles is proved even when $\theta_0$ is known. This will imply non-regularity when $\theta_0$ is unknown and/or when random coefficients are dependent of the characteristics.

The measure $\mu$ is defined on $\mathcal{Z} = \mathcal{Y} \times \mathcal{X}$ as $\mu(B_1 \times B_2) = \tau(B_1) \nu_X(B_2)$, where $B_1 \subset \mathcal{Y}$, $B_2$ is a Borel set of $\mathcal{X}$, $\tau(\cdot)$ is the counting measure and $\nu_X(\cdot)$ is the probability measure for $X$.

The vector $\alpha$ and covariates $x_y$ are $K$-dimensional. The parameter space $\Theta$ is an open set of $\mathbb{R}^J$. The set $H$ consists of measurable functions $\eta : \mathbb{R}^K \to \mathbb{R}$ whose support $A$ has a non-empty interior and $\int_A d\eta(\alpha) = 1$.

Applying the necessary condition for regular identification to a continuous linear functional $\phi(\eta) \in \mathbb{R}$ with influence function $r_\phi$ in the model where $\theta_0$ is known, it must be true that for some $s \in L_2$,

$$r_\phi(\alpha) - \phi(\eta_0) = \int f_{y/x,\alpha}(y; \theta_0) s(y, x) d\mu(y, x). \quad (13)$$

It is straightforward to show that the right hand side in (13) is continuous in $\alpha$ in the interior of its support. In fact, more is true in general: it is an analytic function of $\alpha$ (a function that is infinitely differentiable with a convergent power series expansion). But continuity suffices for proving the non-regularity of CDFs and quantiles of $\eta_0$. This follows without computing least favorable distributions and efficiency bounds, simply by dominated convergence. We gather the proof here to illustrate the simplicity of our method of proof.

**Proposition 5.1** $r_\phi$ in (13) is continuous in the interior of $A$.

**Proof of Proposition 5.1:** Write

$$\int f_{y/x,\alpha}(y, x) d\mu(y, x) = \sum_{j=0}^J \int f_{y/x,\alpha}(j; \theta_0) s(j, x) v_X(dx).$$

Each of the summands in the last expression is continuous in $\alpha$ in the interior of its support, by continuity and boundedness of $f_{y/x,\alpha}(j; \theta_0)$ and the dominated convergence theorem. ■

Proposition 5.1 implies that identification of the CDF and quantiles of the distribution of $\eta_0$ under the conditions specified in Fox et al. (2012) must be irregular. Bajari, Fox and Ryan (2007) propose a simple estimator of the CDF of $\eta_0$, and Fox, Kim and Yang (2016) show its consistency (in the weak topology) and obtain its rates of convergence. Proposition 5.1 implies that the estimator in Fox et al. (2016), or any other estimator for that matter, cannot achieve regular parametric rates of convergence. The lack of regularity is not evident from the rates
established in Fox et al. (2016). Let $F_0$ be the CDF pertaining to $\eta_0$ and $\hat{F}_\eta$ the “fixed grid” estimator of Bajari et al. (2007), Fox et al. (2011) and Fox et al. (2016) based on $D$ grid points ($D \equiv D(n)$, where $n$ is the sample size). The order of the bias established in Fox et al. (2016) is $D^{-s/K}$ where $s$ is the smoothness of the mapping $\alpha \to f_{y/x,\alpha}$ (here $s = \infty$). This suggests that parametric rates might be attainable, but our results show that this is not possible (at least in a local uniform sense). The order of the variance for $\hat{F}_\eta$ is inversely related to the minimum eigenvalue of the $D \times D$ matrix $\Psi_D$ with $(d_1, d_2)$ element, $1 \leq d_1, d_2 \leq D$, given by

$$E[g'(X, \alpha_{d_1})g(X, \alpha_{d_2})],$$

where $g(x, \alpha_d) = (f_{y/x,\alpha_d}(0; \theta), ..., f_{y/x,\alpha_d}(J; \theta))'$ are conditional choice probabilities when UH is evaluated at the $d - th$ grid point $\alpha_d$, $d = 1, ..., D$. This minimum eigenvalue quantifies the level of multicollinearity in the least squares regression of Fox et al. (2016), and we conjecture that given the high smoothness of the mapping $\alpha \to f_{y/x,\alpha}$ this term will go to zero exponentially fast, so it will be the main determinant in the (slow) rate of convergence of $\hat{F}_\eta$. A detailed theoretical analysis of this issue is beyond the scope of this paper, but see the discussion in the next section and the Monte Carlo simulations below, which support these claims.

6 Regularization

The previous examples show that regular identification of CDFs and quantiles of UH in the models considered may require restricting the nature of heterogeneity. In this section we investigate how common approaches considered in the literature address the lack of regularity of these functionals. Additionally, we provide a necessary condition for CDFs and quantiles to be regularly identified when UH is semiparametric and a discussion on how smoothness of $\alpha \to f_{z/\alpha}$ translates into a multicollinearity problem for sieve and related estimators.

Our first observation is derived from the main idea in the previous section: functional form assumptions that restrict the conditional likelihood may not help with the irregular identification of CDFs and quantiles if still the mapping $\alpha \to f_{z/\alpha}$ is smooth, while UH is nonparametric. For example, knowing the finite dimensional parameters of a semiparametric mixture, knowing the functional forms of the idiosyncratic error terms in Kotlarski’s lemma, or knowing the functional form of the baseline hazard in the mixed proportional hazard model do not help in restoring regular identification of CDFs and quantiles of UH when UH is nonparametric.

We discuss how restrictions on UH translate into regularity of functionals of UH. Denote by $\overline{T(\eta_0)}$ the mean squared closure of $T(\eta_0)$ in $L_2(\eta_0)$. That UH is not nonparametric formally means that $\overline{T(\eta_0)}$ is a strict subset of $L_2^0(\eta_0)$. The extension of the necessary condition for regular identification of $\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)]$, for a measurable function $r(\cdot)$ with $\mathbb{E}_{\eta_0}[r^2(\alpha)] < \infty$, is given in the following lemma. Let $\Pi_V$ denote the orthogonal projection operator onto $V$, where $V$
denotes the closure of $V$ in the norm topology.

**Lemma 6.1** The necessary condition for regular identification of $\phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)]$ is

$$\Pi_{\bar{T}(\eta_0)}r(\alpha) = \Pi_{\bar{T}(\eta_0)}\mathbb{E}[s(Z)|\alpha], \text{ for some } s \in L^0_2.$$  

(15)

The mismatch in smoothness between $r(\alpha)$ and $\mathbb{E}[s(Z)|\alpha]$, which was the source of irregularity in the examples studied, may now be restored by the projection onto $\bar{T}(\eta_0)$. We briefly discuss how different restrictions on UH translate into regularity of CDFs and quantiles in view of this general characterization.

A popular approach in practice is to consider a parametric distribution for the UH. A leading example of parametric model is a finite mixture with known and finite support points. Parametric heterogeneity leads to a finite dimensional tangent space $T(\eta_0)$, which is then closed $\bar{T}(\eta_0) = T(\eta_0)$, and which is generated by the scores of the specified distribution. Denote by $l_l$ the score of UH, i.e. $T(\eta_0) = span(l_l)$, assume $\mathbb{E}_{\eta_0}[l_\eta(\alpha)l'_\eta(\alpha)]$ is non-singular, and define the projected score $s_0(Z) = \mathbb{E}[l_\eta(\alpha)|Z]$. Then, simple algebra shows that a solution to (15) in $s$ is given by $s_r$ defined by

$$s_r(Z) = \lambda_r s_0(Z),$$

where $\lambda_r$ is a solution to

$$\mathbb{E}[s_0(Z)s'_0(Z)] \lambda_r = \mathbb{E}[r(\alpha)l'_\eta(\alpha)].$$

(16)

If the Fisher information for $\eta_0$ is positive, which means $\mathbb{E}[s_0(Z)s'_0(Z)]$ is non-singular, then there is a unique solution $\lambda_r$ of (16), and $\phi(\eta_0)$ is regularly identified. More generally, $\phi(\eta_0)$ may be regularly identified even when $\eta_0$ is not, and this corresponds to the system in (16) having some solution in $\lambda_r$. The drawback of the parametric approach is the high misspecification risk, which can be quantified by the dimension and form of the model’s tangent space. If the dimension of $T(\eta_0)$ is $D$, then the tangent space of the model is at most $D$–dimensional and given by $S := \{ s \in L^0_2 : s(z) = \lambda l_\eta s_0(z) \text{ for some } \lambda \in \mathbb{R}^D \}$. Estimators for functionals of UH will be in general inconsistent when the model is misspecified.

As usual, a semiparametric approach is more robust to misspecification. In Lemma 6.1 we have derived the necessary condition for regular identification of moments when UH is semiparametric, so $\bar{T}(\eta_0)$ is a strict subset of $L^0_2(\eta_0)$ of infinite dimension. Examples of semiparametric models include finite mixtures with unknown support points and sieve methods with incomplete sieve basis. Existing rate results for finite mixtures with unknown support points suggest irregularity of the CDFs in general (see, e.g., Chen 1995 and Heinrich and Kahn 2018), although we are not aware of any paper investigating semiparametric efficiency bounds for finite mixtures with unknown support points. We recognize that, although the sufficient condition for semiparametric restrictions in Lemma 6.1 is general, it may be hard to find primitive conditions for
it, as computing the closure of $T(\eta_0)$ and the projections onto it may not be straightforward in applications.

As a practical approach, we recommend a sieve method where the span of $\{l_\eta(\alpha)\}$ increases with the sample size, i.e. $D \rightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality normalize $l_\eta$ so that $E_{\eta_0} [l_\eta(\alpha)l_\eta'(\alpha)]$ is the identity matrix. A key quantity for sieve estimation is the minimum eigenvalue of the Fisher information matrix $E_s [s_0(Z)s_0'(Z)]$, denoted by $\xi_{\min} \equiv \xi_{\min}(D)$; see Fox, Kim and Yang (2016) and (16). We provide a useful bound for $\xi_{\min}$. To that end, we assume the score operator $Ab = E[b(\alpha)|Z]$ from $L^2(\eta_0)$ to $L^2$ is compact. A well known sufficient condition for this is

$$\int \frac{f^2_{z/\alpha}(z)}{f_{\eta_0}(z)} d\eta_0(\alpha)d\mu(z) < \infty. \tag{17}$$

Under this condition, $A$ has a sequence of singular values $\{\mu_d\}_{d=1}^\infty$ (see Engl, Hanke and Neubauer, 1996). Then, the following bound follows essentially from Blundell, Chen and Kristensen (2007, Lemma 1).

**Lemma 6.2** If (17) holds, then $\xi_{\min}(D) \leq \mu_D^2$.

Since $\mu_D \rightarrow 0$ as $D \rightarrow \infty$, Lemma 6.2 implies that also $\xi_{\min}(D) \rightarrow 0$. This is the multicollinearity problem mentioned above. Furthermore, the score operator $A$ is an integral operator with kernel $K(z,\alpha) = f_{z/\alpha}(z)/f_{\eta_0}(z)$, and it is well known that the smoother the mapping $\alpha \rightarrow K(z,\alpha)$, the faster the singular values $\mu_D$ go to zero. In particular, for analytical kernels the singular values decay exponentially fast to zero (Hille and Tamarkin 1931). The minimum eigenvalue $\xi_{\min}(D)$ is also closely related to the sieve measure of ill-posedness $\tau_D$ proposed in econometrics (see Chen 2007 and Blundell, Chen and Kristensen 2007) through the relation

$$\tau_D^2 = \frac{1}{\xi_{\min}(D)}.$$  

Prior to this paper, Blundell, Chen and Kristensen (2007, Lemma 1) obtained the bound $\tau_D \geq 1/\mu_D$ in a nonparametric IV setting. Thus, the modest contribution here is the interpretation in terms of the minimum eigenvalue of the Fisher information matrix. For applications of sieve estimators along this line and the important role of $\tau_D$ (or $\xi_{\min}(D)$) see, e.g., Chen (2007), Bajari, Fox and Ryan (2007), Hu and Schennach (2008), Bester and Hansen (2007), Chen and Liao (2014), Fox, Kim and Yang (2016) and references therein. Next section investigates the finite sample performance of the sieve “fixed grid” method of Fox, Kim and Yang (2016) and a regularized version to reduce the variance of estimates of the CDFs and quantiles of UH.

7 Monte Carlo

This section illustrates some of the theoretical ideas in a Monte Carlo study on the Mixed Logit model. Specifically, we consider the “fixed grid” nonparametric estimator of Bajari et al. (2007)
and Fox et al. (2016), and evaluate the performance of this estimator for estimating the CDF and quantiles of UH.\footnote{We thank Jeremy Fox for sharing the Matlab code to implement their estimator.} We also provide a variant of this estimator that performs a Singular Value Decomposition (SVD) of the resulting design matrix to reduce the variance of the estimator. To introduce the estimator, consider a discrete approximation of the distribution of UH of the form

\[ \eta_0(\alpha) \approx \sum_{d=1}^{D} \theta_d \delta_{\alpha_d}(\alpha), \]  

(18)

where \( \theta_d \) are probabilities, adding up to one, over a finite support \( \{\alpha_d\}_{d=1}^{D} \) of size \( D \) in \( \mathcal{A} \). In Fox et al. (2016) \( D \), and thus the discrete support, is allowed to increase with the sample size \( n \). Define \( Y_{i,j} \) as the binary choice equals 1 whenever individual \( i \)'s choice is \( j \), and zero otherwise. Define the regression error term \( \varepsilon_{i,j} = Y_{i,j} - f_{\eta_0}(j, X_i) \). The least squares estimator uses the regression equation

\[ Y_{i,j} = \int f_{y/X_i,\alpha}(j) d\eta_0(\alpha) + \varepsilon_{i,j}, \]

with the approximation in (18) to obtain the approximated linear regression model

\[ Y_{i,j} \approx \sum_{d=1}^{D} \theta_d f_{y/X_i,\alpha_d}(j) + \varepsilon_{i,j}. \]

Fox et al. (2016) proposes running a regression of \( Y_{i,j} \) on the regressors \( Z_{i,j} := f_{y/X_i,\alpha_d}(j) \) subject to the constrains on the probabilities \( \theta_d \), i.e.

\[ \hat{\theta} = \arg \min_{\theta \in \Delta_d} \frac{1}{nJ} \sum_{i=1}^{n} \sum_{j=0}^{J} \left( Y_{i,j} - \sum_{d=1}^{D} \theta_d Z_{i,j}^d \right)^2, \]  

(19)

where \( \theta = (\theta_1, ..., \theta_D)' \in \Delta_d = \{(p_1, ..., p_D) : 0 \leq p_d \leq 1 \text{ and } \sum_{d=1}^{D} p_d = 1\} \). The least squares problem in (19) is convex and can be efficiently solved by standard routines (such as lsqnonlin in Matlab). The estimator of the CDF of \( \eta_0 \) at \( \alpha_0 \) is then given by

\[ \hat{F}_\eta(\alpha_0) = \sum_{d=1}^{D} \hat{\theta}_d \mathbb{1}(\alpha_d \leq \alpha_0), \]  

(20)

and from the CDF we define the quantile estimators as usual.

For simplicity of computation, in the Monte Carlo we apply this estimator to the Mixed Logit model without fixed parameters, so

\[ f_{y/x,\alpha}(y) = \frac{\exp \left( x'_y \alpha \right)}{1 + \sum_{j=1}^{J} \exp \left( x'_j \alpha \right)}, \]
for $x = (x_0, x_1, ..., x_J) \in \mathcal{X}$ and $y \in \mathcal{Y} = \{0, 1, ..., J\}$. Smoothness of mapping $\alpha \rightarrow f_{y|x,\alpha}$ translates into high correlation of the regressors $Z_{i,j}^d$ when $D$ is large (for $d$’s corresponding to nearby $\alpha_d$’s), suggesting that methods that account for multicollinearity may reduce the variances of the resulting estimators. We suggest using the SVD of the design $nJ \times D$ matrix $Z = (Z_{i,j}^d)$, by adding the linear constrain $V_{p-D}\theta = 0$ to (19), where $V_{p-D} = (v_{p-D}, v_{p-D+1}, ..., v_D)$ denotes the last $p - D$ left singular vectors of $Z$ (where as usual, they are ordered according to the singular values from largest to smallest). This is the classical Principal Component Regression adapted to the constrained case where $\theta$’s are probabilities. The resulting estimator is

$$\tilde{\theta} = \arg \min_{\theta \in \Delta_d, V_{p-D}\theta = 0} \frac{1}{nJ} \sum_{i=1}^n \sum_{j=0}^J \left( Y_{i,j} - \sum_{d=1}^D \theta_d Z_{i,j}^d \right)^2,$$

which solves a convex problem and can be equally computed by routines such as lsqin in Matlab. Let $\tilde{F}_\eta(\alpha_0) = \sum_{d=1}^D \tilde{\theta}_d 1(\alpha_d \leq \alpha_0)$ denote the corresponding CDF estimator. We compare below the performance of the resulting CDFs and quantile estimators based on $\hat{\theta}$ and $\tilde{\theta}$, respectively.

The Monte Carlo setting we consider is taken from a recent study by Heiss, Hetzenecker and Osterhaus (2019). The data generating process we consider is as follows. The number of products (not including outside good) is $J = 3$. The number of product characteristics is $K = 2$. The characteristics are generated as independent uniforms on $[0, 1]$. The random coefficient distribution is a mixture of two bivariate normal distributions with probability weights $(1/2, 1/2)$, means $(-2.2, -2.2)$ and $(1.3, 1.3)$ and equal variances $\Sigma_1 = \Sigma_2 = \Sigma$ given by

$$\Sigma = \begin{bmatrix} 0.8 & 0.15 \\ 0.15 & 0.8 \end{bmatrix}.$$ 

To generate the grid $\{\alpha_d\}_{d=1}^D$ we use a Halton sequence with points spread on $[-5, 5] \times [-5, 5]$. The fixed grid covers the support of the true distribution with probability close to one. We consider different values for the number of points in the grid $D \in \{25, 100, 500\}$ and sample sizes $n \in \{100, 500, 1000\}$. For computing $\hat{\theta}$ we set the number of components $p$ to 5 throughout (we have investigated with values of $p$ between 3 and 10 and obtain qualitatively similar results). We set $p$ deterministically in simulations to save time, but in practice we recommend cross-validation to select $p$. The number of Monte Carlo simulations is $M = 500$. To evaluate the performance of CDFs’ estimators we compute the integrated absolute bias

$$\text{Bias}(\tilde{F}) = \frac{1}{ML} \sum_{m=1}^M \sum_{l=1}^L \left| \tilde{F}_{\eta,m}(\alpha_l) - F_0(\alpha_l) \right|,$$

where $\{\alpha_l\}_{l=1}^L$ is an additional equally spaced grid over $[-5, 5] \times [-5, 5]$ with $L = 121$, $\tilde{F}_{\eta,m}$ is the fixed grid CDF estimator (cf. 20) for the $m$ – $th$ Monte Carlo simulation, and $F_0$ denotes the true CDF pertaining to $\eta_0$. 

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We also report the Root integrated Mean Squared Error defined as

\[
RMSE(\hat{F}) = \sqrt{\frac{1}{ML} \sum_{m=1}^{M} \sum_{l=1}^{L} \left( \hat{F}_{n,m}(\alpha_l) - F_0(\alpha_l) \right)^2}.
\]

The quantities \( Bias(\tilde{F}) \) and \( RMSE(\tilde{F}) \) are analogously defined.

Table 1 reports the bias and root mean squared errors for the CDFs estimators \( \hat{F} \) and \( \tilde{F} \). The first observation is that the bias is small even for small sample sizes such as \( n = 100 \), and it does not depend much on \( D \), which is consistent with our discussion in Section 5.1. The regularization causes \( \tilde{F} \) to have a slightly larger bias than \( \hat{F} \) in some cases, although the difference is not substantial, and for small samples the bias of \( \tilde{F} \) is even smaller. On the other hand, the variance of \( \hat{F} \) is systematically larger than that of \( \tilde{F} \), particularly for moderate and large values of \( D \), consistent with our claims that the level of multicollinearity increases dramatically with the number of points \( D \).

| \( n \) | \( D \) | Bias(\( \hat{F} \)) | Bias(\( \tilde{F} \)) | RMSE(\( \hat{F} \)) | RMSE(\( \tilde{F} \)) |
|------|------|----------------|----------------|----------------|----------------|
| 100  | 25   | 0.0781         | 0.0729         | 0.1791         | 0.1059         |
| 500  | 25   | 0.0663         | 0.0713         | 0.1380         | 0.0933         |
| 1000 | 25   | 0.0605         | 0.0708         | 0.1231         | 0.0904         |
| 100  | 100  | 0.0799         | 0.0682         | 0.1896         | 0.0999         |
| 500  | 100  | 0.0606         | 0.0639         | 0.1428         | 0.0855         |
| 1000 | 100  | 0.0511         | 0.0630         | 0.1284         | 0.0831         |
| 100  | 500  | 0.0784         | 0.0651         | 0.1906         | 0.0982         |
| 500  | 500  | 0.0541         | 0.0602         | 0.1452         | 0.0835         |
| 1000 | 500  | 0.0440         | 0.0592         | 0.1303         | 0.0805         |

\( M = 500 \) simulations.

Table 2 reports the RMSE for the medians of the marginal distributions of UH (denoted by RMSEQ1 and RMSEQ2 for \( \hat{F} \) and RMSEQ1-PCR and RMSEQ2-PCR for \( \tilde{F} \), respectively). Results for other quantile levels are reported in the Appendix. We do not report the bias separately to save space, but we note that the bias for quantiles is much larger than the bias for CDFs. We observe substantial gains in terms of RMSE of the regularization by SVD, with the benefits increasing with the number of grid points. Importantly, in both cases, CDFs and quantiles, the reported results are consistent with much slower rates of convergence than parametric, lending support on the infinite efficiency bounds established in this paper.
| $n$  | $D$ | RMSEQ1 | RMSEQ1-PCR | RMSEQ2 | RMSEQ2-PCR |
|------|-----|--------|------------|--------|------------|
| 100  | 25  | 1.6624 | 0.8061     | 1.4621 | 0.7085     |
| 500  | 25  | 0.8492 | 0.5232     | 0.8713 | 0.4155     |
| 1000 | 25  | 0.8008 | 0.4923     | 0.7386 | 0.3254     |
| 100  | 100 | 1.6084 | 0.6315     | 1.8392 | 0.6514     |
| 500  | 100 | 0.9411 | 0.4995     | 0.9409 | 0.2970     |
| 1000 | 100 | 0.8947 | 0.1847     | 0.9786 | 0.1832     |
| 100  | 500 | 1.6373 | 0.6360     | 1.6270 | 0.5974     |
| 500  | 500 | 1.0599 | 0.2710     | 0.9917 | 0.2639     |
| 1000 | 500 | 0.9374 | 0.1879     | 0.9669 | 0.1766     |

$M = 500$ simulations.

8 Conclusions

We have established irregular identification of CDFs and quantiles (or more generally, functionals with discontinuous influence functions) of nonparametric UH in some structural economic models. Example applications include the structural model of unemployment with two spells in Alvarez et al. (2015), the binary and linear RC models (possibly with correlated effects), the AME in a triangular model with near zero first-stage effects, and the distribution and quantiles of UH in the Mixed Logit model. These are only some applications, but the results are applicable more widely. Further examples in the Appendix include mixed proportional duration models, and measurement error models with two measurements identified by means of Kotlarski’s lemma. Furthermore, as we discuss in the Appendix, we expect our approach to be applicable to many situations where the so-called Information Operator (see e.g. Begun, Hall, Huang and Wellner (1983)) is a smoothing operator.

The most appealing feature of our method of proof is its simplicity, relative to alternative approaches that directly compute efficiency bounds, which are particularly difficult to compute in the models we have studied. Instead, we exploit some necessary smoothness conditions that the influence function of a regularly identified functional must satisfy. The Mixed Logit example is illustrative of the easiness in the application of our method of proof. In contrast, directly computing the Fisher information and the efficiency bound in this model is rather challenging (and were unknown prior to this paper). The practical implications of the irregularity of CDFs and quantiles have been investigated in a Monte Carlo study. We have found substantial benefits from regularizing the fixed grid estimator of Bajari et al. (2007), Fox et al. (2011) and Fox et al. (2016), without sacrificing much of its appealing computational simplicity. Future research on the theoretical properties of regularized estimators is guaranteed.
9 Appendix A: Proofs of Main Results

Proof of Lemma 3.1: First, the functional \( \eta_0 \rightarrow \phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)] \) is differentiable with influence function

\[
\chi(\alpha) = \Pi_{T(\eta_0)} r(\alpha),
\]

where \( \Pi_{\overline{T}} \) denotes the orthogonal projection operator onto the closure of \( V, \overline{V} \). To see this, note that by linearity of \( \eta_0 \rightarrow \phi(\eta_0) \), for all \( b \in T(\eta_0) \),

\[
\lim_{t \to 0} \frac{\phi(\eta_t) - \phi(\eta_0)}{t} = \mathbb{E}_{\eta_0}[r(\alpha)b(\alpha)] = \mathbb{E}_{\eta_0}[\Pi_{T(\eta_0)} r(\alpha)b(\alpha)].
\]

Since UH is nonparametric \( \Pi_{T(\eta_0)} r(\alpha) = r(\alpha) - \phi(\eta_0) \). On the other hand, by Lemma 25.34 in van der Vaart (1998) the adjoint of the score operator is given by

\[
A^* s = \mathbb{E} [s(Z)|\alpha] - \mathbb{E} [s(Z)].
\]

The lemma then follows from Theorem 3.1 and Theorem 4.1 in van der Vaart (1991), which establish that a necessary condition for positive Fisher information for \( \phi(\eta_0) \) is

\[
r(\alpha) - \phi(\eta_0) = \mathbb{E} [s(Z)|\alpha],
\]

since \( \mathbb{E} [s(Z)] = 0 \). ■

Proof of Lemma 3.2: Let \( \alpha_n, \alpha \in \mathbb{N} \) such that \( \alpha_n \to \alpha \), and define \( h_n(z) = s(z)f_{z/\alpha_n}(z) \). Note (i) implies \( h_n(z) \to h(z) := s(z)f_{z/\alpha}(z) \) a.e-\( \mu \). Also, by the dominance condition, for a sufficiently large \( n \),

\[
\int |h_n(z)| \, d\mu(z) < \infty.
\]

We conclude by dominated convergence that

\[
\int s(z)f_{z/\alpha_n}(z) \, d\mu(z) \to \int s(z)f_{z/\alpha}(z) \, d\mu(z).
\]

■

Proof of Corollary 3.1: By Lemma 3.2 if the influence function of the functional is discontinuous then the functional is not regularly identified. Since the indicator is not continuous, this proves the lemma. ■

Proof of Corollary 3.2: Lemma 21.3 in van der Vaart (1998) shows the pathwise differentiability of the quantile functional with an influence function

\[
r_{\phi}(\alpha) = \frac{-\{1(\alpha \leq \phi(\eta_0)) - \tau\}}{\hat{\eta}_0(\phi(\eta_0))}.
\]
That is, under the regularity conditions of the corollary, the quantile functional \( \eta_0 \rightarrow \phi(\eta_0) \) satisfies, for all \( b \in T(\eta_0) \),

\[
\lim_{t \to 0} \frac{\phi(\eta_t) - \phi(\eta_0)}{t} = \mathbb{E}_{\eta_0}[r_\phi(\alpha)b(\alpha)].
\]

From Van der Vaart (1991) it follows that a necessary condition for the quantile functional to be differentiable is

\[
r_\phi(\alpha) - \phi(\eta_0) = \int s(z)f_{z/\alpha}(z)d\mu(z).
\]

By Lemma 3.2 if the influence function of the functional is discontinuous then the functional is not regularly identified. Since the influence function of the quantile is not continuous, this proves the lemma. ■

**Proof of Proposition 3.1:** By substitution of \( f_{z/\alpha}(t_1, t_2) \) we obtain

\[
\mathbb{E}[s(Z) | \alpha] = \int_{T^2} s(t_1, t_2)f_{z/\alpha}(t_1, t_2)dt_1dt_2 = C\beta^2 e^{2\alpha\beta} h(\alpha_1^2, \alpha_2^2),
\]

where

\[
h(u, v) = \int_{T^2} s(t_1, t_2)\frac{1}{t_1^{3/2}t_2^{3/2}} s(u, v; t_1)s(u, v; t_2)dt_1dt_2
\]

and

\[
s(u, v; t) = \exp\left(-\frac{ut}{2} - \frac{v}{2t}\right), \quad t \in T, \quad (u, v) \in (0, \infty).
\]

We check that the conditions for an application of the Leibniz’s rule hold. These conditions are

1. The partial derivative \( \partial^m s(u, v; t_1)s(u, v; t_2)/\partial^m u \) exists and is a continuous function on an open neighborhood \( B \) of \( (u, v) \), for a.s. \( (t_1, t_2) \in T^2 \).

2. There is a positive function \( h_m(t_1, t_2) \) such that

\[
\sup_{(u, v) \in B} \left| \frac{\partial^m s(u, v; t_1)s(u, v; t_2)}{\partial^m u} \right| \leq h_m(t_1, t_2) \tag{21}
\]

and

\[
\int_{T^2} s(t_1, t_2)\frac{1}{t_1^{3/2}t_2^{3/2}} h_m(t_1, t_2)dt_1dt_2 < \infty. \tag{22}
\]

Simple differentiation and induction show that for any integer \( m \geq 0 \)

\[
\frac{\partial^m s(u, v; t_1)s(u, v; t_2)}{\partial^m u} = 2^{-m}(-1)^m(t_1 + t_2)^m s(u, v; t_1)s(u, v; t_2).
\]
Therefore, by monotonicity we can find \( u^* \) and \( v^* \) such that (21) holds with
\[
h_m(t_1, t_2) = 2^{-m}(t_1 + t_2)^m s(u^*, v^*; t_1) s(u^*, v^*; t_2).
\]
Furthermore, by \( \mathbb{E} [s(Z) | \alpha] < \infty \) for all \( \alpha \) in a local neighborhood (by local boundedness of \( r \)), and the boundedness of \( \mathcal{T} \), condition (22) holds. The continuity of \( h(u, v) \) is a special case of the previous arguments with \( m = 0 \) (note the term \( (t_1 + t_2)^m \) is one and the boundedness of \( \mathcal{T} \) is not needed in this case).

**Proof of Proposition 4.1:** Define
\[
b(\alpha) = \mathbb{E} [s(Y_i = 1, X_i) | \alpha_i = \alpha] = \int 1 (x' \alpha \geq 0) s(1, x) dv_X(x).
\]
We prove that \( b \) is continuous and by compactness of the sphere is therefore uniformly continuous. Since the halfspaces \( 1 (x' \alpha \geq 0) \) and \( 1 (x' \alpha_0 \geq 0) \) intersect in sets having surface measure of order \( |\alpha - \alpha_0| \), it follows from the absolutely continuity of the angular component of \( X \) that
\[
|b(\alpha) - b(\alpha_0)| = O (|\alpha - \alpha_0|).
\]
When \( x = (1, \tilde{x}) \), then
\[
b(\alpha) = \int 1 (\tilde{x}' \alpha_2 \geq -\alpha_1) s(1, 1, \tilde{x}) dv_X(\tilde{x}),
\]
\[
= \int 1 (u \geq -\alpha_1) s_{\alpha_2}(u) f_{\alpha_2}(u) du,
\]
where \( s_{\alpha_2}(u) = \mathbb{E} \left[ s(Y_i = 1, 1, \tilde{X}_i) | \alpha_2 \tilde{X}_i = u \right] \) and \( f_{\alpha_2} \) denotes the density of \( \alpha_2 \tilde{X}_i \). The absolute continuity in \( \alpha_1 \) follows from the integrability of \( s_{\alpha_2}(u) f_{\alpha_2}(u) \) and Royden (1968, Chapter 5).

**Proof of Corollary 4.1:** The proof follows as in Corollaries 3.1 and 3.2.

For a function \( a \in L_1(\lambda) \cap L_2(\lambda) \), define the Fourier transform \( \hat{a}(t) = \int e^{it'\alpha} a(\alpha) d\alpha \), where \( i = \sqrt{-1} \). Use the notation
\[
\tilde{g}(p, x) = \int e^{ipy} g(y, x) dy,
\]
for the Fourier transform with respect to just the first argument (for \( g(\cdot, x) \in L_1(\lambda) \cap L_2(\lambda) \)). Define the norms
\[
|g|_{1, \rho}^2 = \int_{S^{d-1}} \int_{\mathbb{R}} |\tilde{g}(p, x)|^2 (1 + |p|^2)\rho dp dx \tag{23}
\]
and
\[
|g|_\rho^2 = \int |\hat{g}(t)|^2 (1 + |t|^2)\rho dt. \tag{24}
\]
The Sobolev space $H^\rho(A)$ is defined as the set of measurable functions $g$ such that $|g|_\rho < \infty$.

**Proof of Proposition 4.2:** Define the score operator $A : T(\eta_0) \to L_2$

$$Ab(z) = \frac{Rb\eta_0(z)}{f_{\eta_0}(z)}1(f_{\eta_0}(z) > 0),$$

where $R$ denotes the Radon transform

$$Ra(y, x) = \int a(\alpha) 1(y = x\alpha)d\alpha.$$  

Define $g(z) = s(z)f_{\eta_0}(z)$ and $a(\alpha) = b(\eta_0(\alpha))$. Since $f_{\eta_0}(z)$ and $\eta_0$ are bounded, it follows that $g$ and $a$ are in $L_1(\lambda) \cap L_2(\lambda)$. From the definition of $Ra(y, x)$

$$\sup_{y,x} |Ra(y, x)| \leq \int |a(\alpha)|d\alpha < \infty, \quad (25)$$

and since the supports of $\alpha$ and $X$ are bounded, the support of $Y$ is also bounded and $Ra \in L_2(\lambda)$, so we can view $R : L_2(\lambda) \to L_2(\lambda)$.

First, we show that if $s$ belongs to the closure of the range of $A$, then $g(z) = s(z)f_{\eta_0}(z)$ belongs to the closure of the range of $R$. Indeed, if $s_n$ is a sequence in the range of $A$ converging to $s$ in $L_2$, then $g_n = s_nf_{\eta_0}(z) \equiv Ra_n$ and clearly

$$\int |g_n(z) - g(z)|^2 dz \leq \int |s_n(z) - s(z)|^2 f_{\eta_0}(z)dz \to 0.$$

Next, we shall show that any function $g$ in the closure of the range of $R$ will have an squared integrable weak derivative with respect to the first argument (in $y$). By Theorem 2.4.1 in Ramm and Katsevich (1996) and Assumption 3(iii) it follows that $|g|_{1,\rho} < \infty$ for $\rho = \rho_0 + (d_\alpha - 1)/2$. While by well known results in Fourier analysis, with $\partial_y g$ denoting the weak derivative with respect to $y$

$$\int_{S^{d_\alpha - 1}} \left| \tilde{\partial_y} g(p, x) \right|^2 dpdx \leq \int_{S^{d_\alpha - 1}} \int |p|^2 |\tilde{g}(p, x)|^2 dpdx$$

$$\leq \int_{S^{d_\alpha - 1}} \int |\tilde{g}(p, x)|^2 (1 + |p|^2)\rho dpdx$$

$$< \infty,$$

and similarly, by Cauchy-Schwarz

$$\int_{S^{d_\alpha - 1}} \int \left| \tilde{\partial_y} g(p, x) \right| dpdx \leq \int_{S^{d_\alpha - 1}} \int (1 + |p|^2)^{1/2} |\tilde{g}(p, x)| dpdx$$

$$\leq C \left( \int_{S^{d_\alpha - 1}} \int (1 + |p|^2)^{1-\rho} dpdx \right)^{1/2}$$

$$< \infty,$$ because $\rho > 2.$
Thus \( \partial_y g(p, x) \in L_1(\lambda) \cap L_2(\lambda) \) and by Plancherell’s theorem \( \partial_y g(\cdot) \in L_2(\lambda) \), as we claimed.

Define \( \varphi(\cdot) = \partial_y g(\cdot) \in L_2(\lambda) \). We proceed to verify the conditions of the dominated convergence theorem, see Lemma 3.2. First, we show that \( g(y, x) \) is continuous in \( y \). Indeed, by the bounded support assumption
\[
g(y, x) = \int_{-\infty}^{y} \varphi(u, x) \, dx
\]
is absolutely continuous in \( y \) (see Royden 1968, Chapter 5).

Next, by independence of \( \alpha_i \) and \( X_i \),
\[
P[Y_i \leq y \mid X_i = x] = P[x' \alpha_i \leq y],
\]
and taking derivatives we conclude \( f_{\eta_0}(z) = \eta_{0,x}(y) \). Thus, \( f_{\eta_0}(z) \) is also continuous in \( y \) by Assumption 3(i). Moreover,
\[
\inf_{\alpha \in N} \eta_{0,x}(x' \alpha) \geq 1/l(x) > 0,
\]
which yields the continuity of \( \alpha \to s(x' \alpha, x) \) in \( N \). Furthermore, by Cauchy-Schwarz and
\[
\int \sup_{\alpha \in \Gamma_0} |s(x' \alpha, x)| f_X(x) \, dx = \int \sup_{\alpha \in \Gamma_0} |g(x' \alpha, x)| \sup_{\alpha \in \Gamma_0} \left| \frac{f_X(x)}{f_{\eta_0}(x' \alpha, x)} \right| \, dx
\]
\[
\leq \left( \int |\varphi(u, x)|^2 \, du \, dx \right)^{1/2} \left( \int \sup_{\alpha \in \Gamma_0} \left| \frac{f_X(x)}{f_{\eta_0}(x' \alpha, x)} \right|^2 \, dx \right)^{1/2}
\]
\[
\leq C \int l^2(x) f_X(x) \, dx
\]
\[
\leq C.
\]
Thus, by dominated convergence \( r \) must be continuous in \( N \). \( \blacksquare \)

**Proof of Corollary 4.2:** The proof follows as in Corollaries 3.1 and 3.2. \( \blacksquare \)

**Proof of Proposition 4.3:** A necessary condition for a reduced form functional \( \phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)] \) to be regularly identified is
\[
r(\alpha) - \phi(\eta_0) = \int s(\alpha_0 + \alpha_1 x, x) \, dv_X(x), \quad \alpha = (\alpha_0', \alpha_1') = (\pi_0, U_2, \pi_1, \delta)'.
\]
Thus, by Proposition 4.2 \( r(\alpha) \) must be continuous in \( N \). However, the influence function for the PPAME
\[
r_{PPAME}(\alpha) = 1(\pi_1 > 0)1(\delta > 0) + 1(\pi_1 < 0)1(\delta < 0)
\]
is discontinuous at the points \((p_0, u_2, 0, d_0)\) or \((p_0, u_2, p_1, 0)\). Conclude that the PPAME is not regularly identified. As for AME, by \( \mathbb{E} [\gamma^2] < \infty \) this functional is differentiable in the sense of van der Vaart (1991) with an influence function \( r_{AME}(\beta) = \pi_1/\delta \). Since there is no continuous
function that is \( \eta_0 \)-a.s equal to \( r_{AME}(\beta) = \pi_1/\delta \) when \((p_0, u_2, p_1, 0)\) is a point in the interior of the support, we conclude that the AME is not regularly identified.

**Proof of Lemma 6.1:** By Lemma 25.34 in van der Vaart (1998) the so-called score operator is given by

\[
Ab(z) = \mathbb{E}[b(\alpha)|Z], \ b \in T(\eta_0)
\]

Thus, by the law of iterated expectations

\[
\mathbb{E}[Ab(Z)s(Z)] = \mathbb{E}[b(\alpha)s(Z)]
= \mathbb{E}[b(\alpha)\mathbb{E}[s(Z)|\alpha]]
= \mathbb{E}\left[b(\alpha)\Pi_{T(\eta_0)}\mathbb{E}[s(Z)|\alpha]\right].
\]

In Lemma 3.2 we have shown that the functional \( \eta_0 \to \phi(\eta_0) = \mathbb{E}_{\eta_0}[r(\alpha)] \) is differentiable with influence function

\[
\chi(\alpha) = \Pi_{T(\eta_0)}T(\alpha).
\]

The lemma then follows from Theorem 3.1 in van der Vaart (1991).

**Proof of Lemma 6.2:** The sieve measure of ill-posedness (cf. Blundell, Chen and Kristensen 2007) is

\[
\tau_D = \sup_{b \in T(\eta_0), b \neq 0} \frac{\|b\|}{\|Ab\|}
\]

Since \( T(\eta_0) = \text{span}(l_\eta) \) and \( \mathbb{E}_{\eta_0}[l_\eta(\alpha)l'_\eta(\alpha)] \) is the identity then \( b = \lambda l_\eta \) and \( \|b\|^2 = \lambda^2 \), while \( \|Ab\|^2 = \lambda^2 \mathbb{E}\left[s_0(Z)s'_0(Z)\right] \lambda \). Thus,

\[
\tau_D^2 = \sup_{\lambda \in \mathbb{R}, \lambda \neq 0} \frac{|\lambda|^2}{\lambda^2 \mathbb{E}\left[s_0(Z)s'_0(Z)\right] \lambda} = \frac{1}{\xi_{\text{min}}(D)}.
\]

The bound then follows from Lemma 1 in Blundell, Chen and Kristensen (2007).

## 10 Appendix B: Further Results

### 10.1 Nonlinear RC

In this section we describe a generic approach that can be used for generic nonlinear RC models with continuous outcomes. We also illustrate how certain invertible RC models are ruled out by
our conditions. For the generic RC model in (7), the regularity condition reads

$$r(\alpha) - \phi(\eta_0) = \mathbb{E}[s(m(X_1, \alpha), X_1)].$$

(26)

Again, the main difficulty in proving that the right hand side of (26) is continuous is that the score function \(s(\cdot)\) is only known to be in \(L_2\) (thus, \(s\) is potentially very discontinuous). To overcome this difficulty, we resort to Fourier analysis and use the so-called Parseval’s identity (see Rudin 1987, pg. 187). To describe the method, assume \(X\) is absolutely continuous with density \(f_X(x)\), and define

$$g(z) = s(z)f_{\eta_0}(z) \quad \text{and} \quad w(z, \alpha) = \frac{1}{f_{\eta_0}(z)} \frac{(y = m(x, \alpha))}{f_X(x)} 1(f_{\eta_0}(z) > 0).$$

Note that \(g \in L_1(\lambda)\), and since \(f_{\eta_0}\) is bounded, also \(g \in L_2(\lambda)\). Let \(\eta_{m,x}\) denote the density of \(m(x, \alpha)\) when \(\alpha\) has density \(\eta_0\). Under our conditions below, \(w(\cdot, \alpha) \in L_1(\lambda) \cap L_2(\lambda)\), and by Parseval’s identity, if \(r\) satisfies (26) then

$$r(\alpha) - \phi(\eta_0) = \int \hat{g}(t) \bar{w}(t, \alpha) dt,$$

(27)

where, for a generic function \(h \in L_1(\lambda)\), \(\hat{h}(t) = (2\pi)^{-d/2} \int e^{-it\cdot z} h(z) dz\) denotes the Fourier transform, with \(i = \sqrt{-1}\), \(\overline{v}\) denotes the complex conjugate of \(v\) and

$$\bar{w}(t, \alpha) = (2\pi)^{-d/2} \int \frac{f_X(x)}{\eta_{m,x}(x^* \alpha)} e^{i(t_1 m(x, \alpha) + t_2 x)} dx.$$

This integral representation is now amenable to our Lemma 3.2 under the following assumption.

**Assumption 5** (i) The vector \(X\) is absolutely continuous with a bounded density \(f_X(\cdot)\); (ii) the density \(\eta_{m,x}\) is continuous and satisfies \(\inf_{\alpha \in \mathcal{N}} \eta_{m,x}(m(x, \alpha)) > 1/l(x)\) for an a.s. positive measurable function \(l(\cdot)\) such that \(\mathbb{E}_X[l^2(X)] < \infty\); (iii) the function \(\alpha \rightarrow m(x, \alpha)\) is continuous a.s. in \(x\); (iv) for all \(\hat{g}\) satisfying (27),

$$\int |\hat{g}(t)| \sup_{\alpha \in \mathcal{F}_0} |\bar{w}(t, \alpha)| dt < \infty.$$  

(28)

**Proposition 10.1** Under Assumption 5 and if \(r\) satisfies (11), then \(r(\cdot)\) must be continuous on \(N\).

**Proof of Proposition 10.1**: First, we need to check that \(g\) and \(w(z, \alpha)\) are in \(L_1(\lambda) \cap L_2(\lambda)\), so we can apply Parseval’s identity. From \(s \in L_2\) and the definition of \(g(z) = s(z)f_{\eta_0}(z)\), it is clear that \(g \in L_1(\lambda)\). Next, note

$$f_{\eta_0}(z) \leq \int_{\mathbb{R}^d} d\eta_0(\alpha) = 1.$$

30
Thus, $g$ also belongs to $L_2(\lambda)$. Furthermore, by independence of $\alpha_i$ and $X_i$,

$$\mathbb{P} [Y_i \leq y | X_i = x] = \mathbb{P} [m(x, \alpha_i) \leq y],$$

and taking derivatives we conclude $f_{\eta_0}(z) = \eta_{m,x}(y)$. Then, for $p = 1$ or $2$,

$$\int |w(z, \alpha)|^p \, dz = \int \left| \frac{f_X(x)}{\eta_{m,x}(x', \alpha)} \right|^p \, dx \leq \int l^p(x) |f_X(x)|^p \, dx \leq C \int l^p(x) f_X(x) \, dx < \infty,$$

because $f_X$ is bounded. Then, we can apply Parseval’s identity and obtain

$$r(\alpha) - \phi(\eta_0) = \int \hat{g}(t) \overline{w(t, \alpha)} \, dt.$$

We now proceed to verify the conditions of Lemma 3.2 with $\hat{g}(\cdot)$ playing the role of $s$ and $\overline{w(t, \alpha)}$ that of the conditional density. Note

$$\overline{w(t, \alpha)} = (2\pi)^{-d_x/2} \int \frac{f_X(x)}{\eta_{m,x}(m(x, \alpha))} e^{i(t_1 m(x, \alpha) + t_2 x)} \, dx.$$

Under the conditions of the proposition the function $\alpha \rightarrow \overline{w(t, \alpha)}$ is continuous on $N$ since $\eta_{m,x}(\cdot)$ and $m(x, \cdot)$ are continuous and $\eta_{m,x}(m(x, \alpha))$ is bounded away from zero on $N$. Furthermore, the dominance condition holds from (28). Conclude applying one more time dominated convergence under the dominance condition Assumption 5(iii). ■

Among the conditions of Assumption 5, the most important one is (28). We will see that in a class of invertible models this condition fails to be satisfied. Consider the canonical monotonic nonseparable model

$$Y_i = m(X_i, \alpha_i)$$

with a scalar $\alpha_i$ and where $\alpha \rightarrow m(x, \alpha)$ is strictly increasing with inverse $m^{-1}(y, x)$. Then, if we define $s(Y_i, X_i) = 1(m^{-1}(Y_i, X_i) \leq 0)$, then the regularity condition of Lemma 3.1 is satisfied with $r(\alpha) = 1(\alpha \leq 0)$, proving that the necessary condition for regular identification of the CDF at 0 (or at any other point in fact) holds. In invertible models like this, regularity of CDFs and quantiles is satisfied even in cases where $m$ is not known, but identified. Our results do not apply to invertible models where heterogeneity can be recovered as an identified function of observables.

To give a specific example, consider the model $Y_i = X_i + \alpha_i$, where $s(Y_i, X_i) = 1(Y_i \leq X_i)$ solves (2) with $r(\alpha) = 1(\alpha \leq 0)$, which is discontinuous at 0. This is of course an unrealistic
model, but the idea is simply to illustrate which of our assumptions is key for the results to hold. In this example, Assumption 5(i-ii) is satisfied under mild conditions, since \( \eta_{m,x}(m(x, \alpha)) = \eta_0(\alpha) \), but the integrability condition \((28)\) fails, since for \( s(Y_i, X_i) = 1(Y_i \leq X_i) \)

\[
\int |\hat{g}(t)| \sup_{\alpha \in \Gamma_0} |\hat{w}(t, \alpha)| \, dt = \inf_{\alpha \in \Gamma_0} \eta_0(\alpha) \int |\hat{g}(t_1)| \, dt_1 = \infty,
\]

where \( \hat{g}(t_1) = \int 1(\alpha \leq 0) \eta_0(\alpha) e^{it_1 \alpha} d\alpha \). Note that the discontinuity implies the lack of integrability.

### 10.2 Identification under Kotlarski’s Assumptions

There is a growing literature in econometrics identifying the distribution of latent variables by means of Kotlarski’s Lemma (see Prakasa Rao (1983) for a description of the method). In this setting we observe \( Z = (Y_1, Y_2) \) satisfying

\[
Y_1 = \alpha_1 + \alpha_2 \\
Y_2 = \alpha_1 + \alpha_3,
\]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3)' \) is a vector of UH with independent components, and (with some abuse of notation) Lebesgue densities \( \eta_{0j} \), for \( j = 1, 2, 3 \). The density of the data is given by

\[
f_{\eta_0}(y_1, y_2) = \int 1(y_1 = \alpha_1 + \alpha_2)1(y_2 = \alpha_1 + \alpha_3) \eta_{01}(\alpha_1) \eta_{02}(\alpha_2) \eta_{03}(\alpha_3) d\alpha_1 d\alpha_2 d\alpha_3 = \int \eta_{02}(y_1 - \alpha_1) \eta_{03}(y_2 - \alpha_1) \eta_{01}(\alpha_1) d\alpha_1.
\]

Consider a parametric submodel where \( \eta_{02} \) and \( \eta_{03} \) are known and continuous. The model reduces then to our original setting where \( f_{z/\alpha}(z) = \eta_{02}(y_1 - \alpha) \eta_{03}(y_2 - \alpha) \) is known and continuous in \( \alpha \). If the dominance condition of Lemma 3.2 is satisfied, then the CDF and quantiles of \( \eta_{01} \) will be irregularly identified.

### 10.3 Mixed Proportional Hazard Models

The Mixed Proportional Hazard Model leads to a conditional density for duration \( Y \) given a vector of covariates \( X \) given by

\[
f_{\eta_0}(y, x) = \int \phi(x) \psi(y) \alpha e^{-\phi(x)\Psi(y)\alpha} d\eta_0(\alpha),
\]

where \( \phi(x) \) is a transformation of covariates, \( \Psi(y) \) is the baseline cumulative hazard, with derivative \( \psi \), and \( \alpha \) denotes UH. In submodel where \( \phi(x) \) and \( \psi(y) \) are known, the model fits
our original formulation with \( f_{z/\alpha}(z) = \phi(x)\psi(y)\alpha e^{-\phi(x)\Psi(y)\alpha} \) known and continuous as a function of \( \alpha \). Indeed, Horowitz (1999) has established very slow rates of convergence (logarithmic) for the CDF of \( \alpha \), consistent with the irregular identification.

10.4 Anatomy of the general problem

The necessary condition for regular estimation in van der Vaart (1991) is quite general, and in its abstract form reads as

\[
\tilde{\psi} \in R(A^*),
\]

where \( \tilde{\psi} \) is the so-called gradient, which for our original moment functional is \( \tilde{\psi}(\alpha) = r(\alpha) - \phi(\eta_0) \), and \( A^* \) is the adjoint of the so-called score operator \( A \). In many semiparametric models, \( A^* \) is a smoothing integral operator, in the sense that

\[
A^*s = \int s(z)k(z,\alpha)d\mu(z)
\]

is an operator from \( L_2 \) to \( L_2(\eta_0) \) with a kernel function \( k \) such that \( \alpha \to k(z,\alpha) \) is smooth, at least for some submodel. We expect our results to be potentially applicable in this general setting.

10.5 Further Simulation Results

We report here further results for estimation of quantiles in the Mixed Logit Model. The setting is that of the Monte Carlo section, the only different being that other quantile levels \( \tau \) different from the median (\( \tau = 0.5 \)) are considered. Table 3 report the RMSE. We observe that, as expected, the RMSE at more extreme quantiles are larger than those for the median. Again, the gains from regularization are substantial, particularly for large values of \( R \).
| \( \tau \) | \( n \) | \( D \) | \( \text{RMSEQ1} \) | \( \text{RMSEQ1-PCR} \) | \( \text{RMSEQ2} \) | \( \text{RMSEQ2-PCR} \) |
|---|---|---|---|---|---|---|
| 0.25 | 100 | 25 | 1.6739 | 0.9308 | 1.7238 | 0.7270 |
| 0.25 | 500 | 25 | 1.3247 | 0.8674 | 1.3270 | 0.6305 |
| 0.25 | 1000 | 25 | 1.0596 | 0.8075 | 1.1569 | 0.6250 |
| 0.25 | 100 | 100 | 1.8369 | 0.6098 | 1.8217 | 0.6608 |
| 0.25 | 500 | 100 | 1.4038 | 0.4929 | 1.3763 | 0.5504 |
| 0.25 | 1000 | 100 | 1.3153 | 0.4832 | 1.2041 | 0.5036 |
| 0.25 | 100 | 500 | 1.8075 | 0.5893 | 1.8696 | 0.6275 |
| 0.25 | 500 | 500 | 1.4529 | 0.4520 | 1.4698 | 0.4894 |
| 0.25 | 1000 | 500 | 1.2954 | 0.4444 | 1.2481 | 0.4500 |
| 0.75 | 100 | 25 | 1.5719 | 0.9803 | 1.6573 | 0.9928 |
| 0.75 | 500 | 25 | 1.1938 | 0.8077 | 1.3158 | 0.7953 |
| 0.75 | 1000 | 25 | 0.8941 | 0.7045 | 1.1489 | 0.7525 |
| 0.75 | 100 | 100 | 1.8192 | 0.8616 | 1.7989 | 0.8099 |
| 0.75 | 500 | 100 | 1.2178 | 0.6029 | 1.1809 | 0.5936 |
| 0.75 | 1000 | 100 | 0.8947 | 0.5495 | 0.9476 | 0.5358 |
| 0.75 | 100 | 500 | 1.9017 | 0.8107 | 1.9402 | 0.8329 |
| 0.75 | 500 | 500 | 1.2381 | 0.5666 | 1.2324 | 0.5467 |
| 0.75 | 1000 | 500 | 0.9606 | 0.4885 | 0.9533 | 0.5102 |
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