Lane-Emden equations perturbed by nonhomogeneous potential in the super critical case

Abstract: Our purpose of this paper is to study positive solutions of Lane-Emden equation

$$-\Delta u = Vu^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}$$

perturbed by a non-homogeneous potential $V$ when $p \in [p_c, \frac{N+2}{N-2})$, where $p_c$ is the Joseph-Lundgren exponent. When $p \in (\frac{N}{N-2}, p_c)$, the fast decaying solution could be approached by super and sub solutions, which are constructed by the stability of the $k$-fast decaying solution $w_k$ of $-\Delta u = u^p$ in $\mathbb{R}^N \setminus \{0\}$ by authors in [9]. While the fast decaying solution $w_k$ is unstable for $p \in (p_c, \frac{N+2}{N-2})$, so these fast decaying solutions seem not able to disturbed like (0.1) by non-homogeneous potential $V$. A surprising observation that there exists a bounded sub solution of (0.1) from the extremal solution of $-\Delta u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$ and then a sequence of fast decaying solutions and slow decaying solutions could be derived under appropriated restrictions for $V$.

Keywords: Lane-Emden Equation; Potential; Decaying Solution; Singularity

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1 Introduction

Our concern in this paper is to consider fast decaying solutions of weighted Lane-Emden equation in punctured domain

$$\begin{cases} 
-\Delta u = Vu^p & \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\
\quad u > 0 & \text{in} \quad \mathbb{R}^N \setminus \{0\},
\end{cases}$$

where $p > 1$, $N \geq 3$ and the potential $V$ is a locally Hölder continuous function in $\mathbb{R}^N \setminus \{0\}$.

During the last years there has been a renewed and increasing interest in the study of the semilinear elliptic equations with potentials, motivated by great applications in mathematical fields and physical fields, e.g. the well known scalar curvature equation in the study of Riemannian geometry, the scalar field equation for standing wave of nonlinear Schrödinger and Klein-Görden equations, the Matukuma equation, see a survey [17, 21] and more references on decaying solutions at infinity see [1, 5–7, 10, 14]. For Lane-Emden equation (1.1) involving nonhomogeneous potential $V(x) = |x|^\alpha (1 + |x|)^{\beta - \alpha_0}$, the authors in [3, 4] showed the nonexistence provided $\beta > -2$ and $p \leq \frac{N+\beta}{N-2}$, also see [1, Theorem 3.1]. In [8], the infinitely many positive solutions of problem (1.1) are constructed for $p \in (\frac{N+\beta}{N-2}, \frac{N+\alpha_0}{N-2}) \cap (0, +\infty)$ with $\alpha_0 \in (-N, +\infty)$ and $\beta \in (-\infty, \alpha_0)$, by dealing...
with the distributional solutions of
\[-\Delta u = Vu^p + k\delta_0 \quad \text{in} \quad \mathbb{R}^N,\]
where \( k > 0 \), \( \delta_0 \) is Dirac mass at the origin and \( p = \frac{N + \alpha_0}{N-2} \) is the critical exponent named Serrin exponent, the value for problem (1.2) with recoverable isolated singularities. Compared to the case \( V \equiv 1 \), problem (1.1) would have totally different isolated singular solution structure for the supercritical case \( p \geq \frac{N+\alpha_0}{N-2} \), due to the behavior of potential at infinity.

When \( V = 1 \), equation (1.1) is well known as Lane-Emden-Fowler equation
\[-\Delta u = u^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},\]
which has been extensively studied in the last decades. The authors in [4] showed the nonexistence of positive solutions of problem (1.3) for \( p \leq \frac{N}{N-2} \); and when \( p > \frac{N}{N-2} \), problem (1.3) always has a singular solution \( w_p(x) = c_p|x|^{-\frac{p}{p-1}} \) with
\[c_p = \left(\frac{2}{p-1}(N - 2 - \frac{2}{p-1})\right)^{\frac{1}{p-1}}.\]

When \( p \in (\frac{N}{N-2}, \frac{N+2}{N-2}) \), a branch of fast decaying solutions of (1.3) can be derived by phase analysis as following:
(i) A sequence \( k \)-fast decaying solutions \( w_k \) with \( k > 0 \) such that
\[\lim_{|x| \to 0^+} w_{p,k}(x)|x|^{-\frac{p}{p-1}} = c_p \quad \text{and} \quad \lim_{|x| \to \infty} w_{p,k}(x)|x|^{N-2} = k.\]
(ii) \( \lim_{k \to \infty} w_{p,k}(x) = w_p(x) \) for \( x \in \mathbb{R}^N \setminus \{0\}. \)

Here and in the sequel, a function \( u \in C^2(\mathbb{R}^N \setminus \{0\}) \) is called a \( v \)-fast decaying if \( u \) has the asymptotic behavior at infinity \( \lim_{|x| \to \infty} u(x)|x|^{N-2} = \nu \) for \( \nu > 0 \). It is worth noting that there is a critical exponent, named by Joseph-Ludgren exponent:
\[p_c = 1 + \frac{4}{N - 4 + 2\sqrt{N-1}} \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right),\]
such that the fast decaying solutions \( w_{p,k} \) is stable for \( p \in (\frac{N}{N-2}, p_c) \), semistable for \( p = p_c \) and unstable for \( p \in (p_c, \frac{N+2}{N-2}) \). Thanks to the stability of \( \{w_k\}_k \) for \( p \in (\frac{N}{N-2}, p_c) \), solutions with multiple singular points are derived for Lane-Emden equation in a bounded smooth domain, see the references [7, 18, 19, 23]. Moreover, for the supercritical case that \( p \geq \frac{N+2}{N-2} \), problem (1.3) has been studied in [11, 12, 18] and the references therein.

In particular, the authors in [12, 13] constructed infinitely many solutions of (1.3) with \( p > \frac{N+2}{N-2} \) in an exterior domain by analyzing the related linearization problem at \( w_p \).

Involving a nonhomogeneous potential \( V \), we can't transform (1.1) into ODE to obtain the symmetric solutions by using the phase analysis, nor the variational method fails to apply due to the singularity at the origin. Thanks to the stability of \( k \)-fast fast decaying solution \( w_k \) of (1.3) for \( p \in (\frac{N}{N-2}, p_c) \), the Schauder fixed point theorem could be applied to obtain fast decaying solution of (1.1) by constructing a solution \( v_k \) of the problem
\[-\Delta v = V(w_k + \nu)^p - w_k^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}\]
for \( k > 0 \) sufficiently small. And then a \( \tilde{v}_k \)-fast decaying solution \( u_{\tilde{v}_k} := v_k + w_k \) of (1.1) is derived, see the reference [9]. Precisely,
\begin{itemize}
  \item[(a)] let \( p \in (\frac{N}{N-2}, p_c) \), potential \( V \) is a Hölder function verifying that

\[|V(x) - 1| \leq c_0|x|^{\tau_0} \quad \text{for} \quad x \in B_1(0),\]

  \end{itemize}

  \text{global control,}

\[0 \leq V(x) \leq c_\infty(1 + |x|)^\beta \quad \text{for} \quad |x| > 0,\]
where $c_0 > 0$, $c_\infty \geq 1$, $\beta < (N-2)p - N$ and

$$\tau_0 > \tau^*_p := \frac{2}{p-1} \left( \frac{2 - 2}{N - 2} \right) - \sqrt{\left( \frac{2}{p-1} - \frac{2}{N - 2} \right)^2 - 2 \left( 2 - \frac{2}{p-1} \right)}.$$

Then there exists $v_0 > 0$ such that for any $v \in (0, v_0)$, problem (1.1) has a $v$-fast decaying solution $u_v$, which has singularity at the origin as $\lim_{|x| \to 0} u_v(x)|x|^{\frac{\mu}{2}} = c_p$.

(b) When $V$ is radially symmetric and decreasing with respect to $|x|$, then there exists $\nu_0 = +\infty$ and $u_\infty = \lim_{v \to +\infty} u_v$, and $u_\infty$ is a solution of (1.1) verifying (1.7) and $\frac{1}{\gamma} |x|^a \leq V(|x|) \leq \gamma |x|^a$ for $|x| > 1$.

Then $v_0 = +\infty$ and $u_\infty = \lim_{v \to +\infty} u_v$, and $u_\infty$ is a solution of (1.1) verifying (1.7) and $\frac{1}{\gamma} |x|^a \leq u_\infty(x)|x|^{\frac{\mu}{2}} \leq c$ for any $|x| \geq 1$ and some $c > 1$.

While the fast decaying solution $w_k$ of (1.3) is unstable for $p \in (p_c, \frac{N+2}{N-2})$, so it seems not able to disturb (1.3) at the fast decaying solution $w_k$ by nonhomogeneous potential $V$. This is our motivation to show the existence of fast decaying solutions of problem (1.1) for $p \in \left[ p_c, \frac{N+2}{N-2} \right]$ and our results on fast decaying solutions state as follows.

**Theorem 1.1.** Assume that $p_c$ is given by (1.5), $p \in \left[ p_c, \frac{N+2}{N-2} \right]$, the potential $V \in C^1(\mathbb{R}^N \setminus \{0\})$ satisfies that

$$\left( 1 + \rho_0 |x|^2 \right)^{\frac{1}{2}(p(N-2) - N)} \leq V(x) \leq 1, \text{ for } x \in \mathbb{R}^N \setminus \{0\},$$

where $\rho_0 = \frac{1}{N(N-2)}$.

Then there exists $\nu^* > 0$ such that for $v \in (\nu^*, +\infty)$, problem (1.1) has a $v$-fast decaying solution $u_v$, which has singularity at the origin as

$$\lim_{|x| \to 0} u_v(x)|x|^{\frac{\mu}{2}} = c_p.$$

where $c_p$ is given in (1.4).

Unlike the case of $p \in (\frac{N}{N-2}, p_c)$, the Schauder fixed point theorem fails to build the lower bound for problem (1.1), due to lack the stability of fast decaying solution $w_k$ of (1.3) for $p \in \left[ p_c, \frac{N+2}{N-2} \right]$. Note that

$$U_\mu(x) = (N(N-2))^{\frac{N-2}{4}} \left( \frac{\mu}{1 + \mu^2 |x|^2} \right)^{\frac{N-4}{2}},$$

with $\mu > 0$ is the extremal solution of $-\Delta U_\mu = U_\mu^{\frac{N+2}{N-2} - p}$ in $\mathbb{R}^N$, which is possible to provide a sub fast decaying solution for (1.1) if $U_\mu^{\frac{N+2}{N-2} - p} \leq 1$ and $U_\mu < w_\rho$, the slow decaying solution of (1.3). Based on this observation, we construct pairs sub-super solution $(U_\mu^*, w_k)$ with $\mu^* = (N(N-2))^{-\frac{1}{2}}$ and $k > k^*$ for some $k^* > 0$ and iterating procedure with initial data $w_k$ could be applied to approach a sequence of fast decaying.

Our final interest is to study the limit of $\{u_v\}_v$ as $v \to +\infty$ and the result states as follows.

**Theorem 1.2.** Assume that $p_c$ is given by (1.5), $p \in \left[ p_c, \frac{N+2}{N-2} \right]$, $V \in C^1(\mathbb{R}^N \setminus \{0\})$ satisfies (1.6) and for some $\gamma > 1$, $a \in ((N-2)p - N - 2, 0)$

$$\frac{1}{\gamma} |x|^a \leq V(|x|) \leq \gamma |x|^a \text{ for } |x| > 1.$$

Let $u_v$ be a $v$-fast decaying solution of problem (1.1) with $v \in (\nu^*, +\infty)$ derived by Theorem 1.1. Then the limit of $\{u_v\}_v$ as $v \to +\infty$ exists, denoting $u_\infty = \lim_{v \to +\infty} u_v$, and $u_\infty$ is a solution of (1.1) verifying (1.7) and

$$\frac{1}{c} \leq u_\infty(x)|x|^{\frac{\mu}{2}} \leq c \text{ for } |x| \geq 1,$$

where $c > 1$. 

The rest of this paper is organized as follows. In Section 2, we show qualitative properties of the solutions to elliptic problem with homogeneous potential and some basic estimates. Section 3 is devoted to build fast decaying solutions of (1.1) by iteration method. Section 4 is devoted to the slow decaying solution as the limit of fast decaying solutions.

2 Preliminary

2.1 Singularity at the origin

Since the lower bound $U_\mu'$ does not blow up at the origin, so we have to provide the classification of singularity at the origin of positive solution of (1.1).

**Theorem 2.1.** Assume that $p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$, the potential $V \in C^1(\mathbb{R}^N \setminus \{0\})$ satisfies (1.6).

Let $u$ be a positive solution of (1.1), then $u$ is removable at the origin or

$$\lim_{|x| \to 0^+} u(x)|x|^\frac{2}{p-1} = c_p,$$

(2.1)

where $c_p > 0$ is defined in (1.4).

**Proof.** In order to apply [16, Theorem 3.3], we need to check that $V$ verifies the conditions:

$$\frac{1}{c} |x|^p \leq V(x) \leq c|x|^p \quad \text{in } B_1(0) \setminus \{0\}$$

and

$$|\nabla \log V(x)| \leq \frac{c}{|x|}$$

for some $c \geq 1$.

Let $V(0) = 1$ and from (1.6), there exists $c > 0$ such that

$$|V(x) - 1| \leq c|x|, \quad \forall x \in B_{\frac{1}{2}}(0)$$

and so we have that $|\nabla V(0)| \leq c$.

Now we apply [16, Theorem 3.3] to obtain that $u$ is removable at the origin or there exists $c > 1$ such that

$$\frac{1}{c} |x|^\frac{2}{p-1} \leq u(x) \leq c|x|^\frac{2}{p-1}, \quad \forall x \in B_1(0) \setminus \{0\}. \quad (2.2)$$

Finally, we improve the singularity when $u$ is not removable. Let

$$t = -\ln |x|, \quad \omega = \frac{x}{|x|} \in S^{N-1}$$

and

$$v(t, \omega) = |x|^\frac{2}{p-1} u(r, \omega).$$

Because of (1.1), $v \in C^2(\mathbb{R} \times S^{N-1})$ is bounded and we have that

$$v_{tt} + C_0 v_t + \Delta_0 v = -c_p^{-1} v + V(r, \omega) v^p = 0 \quad \text{in } \mathbb{R} \times S^{N-1},$$

where $C_0 = \frac{N-2}{p-1} (\frac{N+2}{N-2} - p)$. Thanks to (2.2), we conclude that for each sequence $\{t_k\}, t_k \to -\infty$ as $k \to +\infty$, there exists a subsequence still denoting $\{t_k\}$ such that

$$\Delta_0 v_{\infty} - c_p^{-1} v_{\infty} + v_{\infty}^p = 0 \quad \text{in } S^{N-1}. \quad (2.3)$$

It is shown in Appendix B in [16] that (2.3) has only solutions $v_{\infty} \equiv 0$ or $v_{\infty} \equiv c_p$. Note that the limit set of a $C^2$ function is connected, then we have that

$$v(t, \omega) \to 0 \quad \text{or } v(t, \omega) \to c_p \quad \text{as } t \to -\infty$$

Therefore, we obtain (2.1).
2.2 Basic estimate

In this subsection, some estimates are introduced, which play important roles in our construction of fast-decaying solutions for problem (1.1). Denote
\[ I(x) = c_N |x|^{2-N}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \]
which is the fundamental solution of \(-\Delta I = \delta_0\) in \(\mathbb{R}^N\) and \(c_N > 0\) is a normalized constant.

Lemma 2.2. Suppose that \( f \in L^1(\mathbb{R}^N) \) is a nonnegative function satisfying \( |f(x)| \leq c|x|^{-r} \) for \( |x| > r \), with \( r > N \) and some \( r > 0, c > 0 \). Then
\[
\lim_{|x| \to \infty} (\Gamma * f)(x)|x|^{N-2} = c_N \int_{\mathbb{R}^N} f(x) dx. \tag{2.4}
\]

Proof. By the decay condition of \( f \), we have that for any \( \epsilon > 0 \), there exists \( R > r_0 \) such that for \( R \) large,
\[
\int_{B_R(0)} f(x) dx \geq (1 - \epsilon)\|f\|_{L^1(\mathbb{R}^N)}.
\]
For \( |x| \geq 8R \gg 1 \), there holds \( (1 - \epsilon)|x|^{2-N} \leq |x - y|^{2-N} \leq (1 + \epsilon)|x|^{2-N} \) for \( y \in B_R(0) \) and
\[
(\Gamma * f)(x) = c_N \int_{B_R(0)} \frac{f(y)}{|x - y|^{N-2}} dy + c_N \int_{\mathbb{R}^N \setminus B_R(0)} \frac{f(y)}{|x - y|^{N-2}} dy,
\]
which yields that for \( |x| \) large,
\[
(1 - \epsilon)\|f\|_{L^1(\mathbb{R}^N)} \leq |x|^{-r} \int_{B_R(0)} f(y) dy \leq (1 + \epsilon)\|f\|_{L^1(\mathbb{R}^N)}
\]
and
\[
\int_{\mathbb{R}^N \setminus B_R(0)} \frac{f(y)}{|x - y|^{N-2}} dy \leq c \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|y|^{-r}}{|x - y|^{N-2}} dy \leq c \int_{B_R(0)} \frac{|y|^{-r}}{|x - y|^{N-2}} dy + c \int_{B_R(0)} \frac{|y|^{-r}}{|x - y|^{N-2}} dy \leq R^{2-N} \int_{\mathbb{R}^N \setminus B_R(0)} |y|^{-r} dy + (|x| - R)^{-r} \int_{B_R(0)} |x - y|^{2-N} dy \leq cR^{2-r} + c(|x| - R)^{-r} R^2.
\]
Passing to the limit as \( \epsilon \to 0 \) and letting \( R = |x|/8 \to +\infty \), we see that \( |x| \to +\infty \) and then (2.4) holds. \( \square \)

We remark that \( k \)-fast decaying solution \( w_k \) of (1.3) verifies the integral equation \( w_k = \Gamma * (w_k^p) \) and
\[
c_N \int_{\mathbb{R}^N} w_k(x)^p dx = k
\]
by applying Lemma 2.2 directly.

For \( \mu > 0 \), denote
\[
U_\mu(x) = \left( N(N-2) \right)^{\frac{N-2}{2}} \left( \frac{\mu}{1 + \mu^2 |x|^2} \right)^{\frac{N-2}{2}}, \quad \forall x \in \mathbb{R}^N,
\]
which verifies the equation \(-\Delta U_\mu = \frac{2}{\mu^2} U_\mu^{\frac{N-2}{2}} \) in \(\mathbb{R}^N\). We observe that
\[
\lim_{|x| \to +\infty} U_\mu(x)|x|^{N-2} = \left( N(N-2) \right)^{\frac{N-2}{2}} \frac{1}{\mu^2} U_\mu^{\frac{N-2}{2}}.
\]
Recall that

$$\mu^* = \left( N(N - 2) \right)^{-\frac{1}{2}},$$  \hspace{1cm} (2.6)

and for \( p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right) \), we denote

$$V_p = U_{\mu^*}^{\frac{N}{2}-p} \text{ in } \mathbb{R}^N,$$

which is radially symmetric and decreasing with respect to \( r = |x| \), then

$$V_p(x) \leq 1 = V_p(0), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$  \hspace{1cm} (2.7)

**Lemma 2.3.** Let \( \mu^* = \left( N(N - 2) \right)^{-\frac{1}{2}} \), \( w_k \) be the k-fast decaying solution of (1.3) with \( k > 0 \) and \( U_{\mu^*} \) is defined in (2.5). Then there exists \( k^* > 0 \) such that for any \( k \geq k^* \)

$$w_k \geq U_{\mu^*} \text{ in } \mathbb{R}^N \setminus \{0\}. \hspace{1cm} (2.8)$$

**Proof.** 

**Step 1:** we show \( U_\mu < w_p \) in \( \mathbb{R}^N \setminus \{0\} \) for \( \mu = \mu^* \). This is equivalent to that

$$\mu < (c_p/c_N)^{\frac{2}{p-2}} \left( |x|^{-a} + \mu^2 |x|^{-a} \right) \quad \text{for any } x \in \mathbb{R}^N \setminus \{0\},$$

where \( a = \frac{4}{(p-1)(N-2)} \in (1, 2) \) for \( p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right) \). Let \( r = |x| \) and for \( \mu > 0 \)

$$f_\mu(r) = (c_p/c_N)^{\frac{2}{p-2}} \left( r^{-a} + \mu^2 r^{-a} \right) - \mu \quad \text{for } r > 0,$$

which achieve the minimum at point \( r_0 = \sqrt{\frac{\mu}{\mu^2}} \mu^{-1} \) and we want to show

$$f_\mu^{*}(r_0) = (c_p/c_N)^{\frac{2}{p-2}} \left( r_0^{-a} + \mu^* r_0^{-a} - \mu^* \right) > 0.$$

In fact, \( f_\mu^{*}(r_0) > 0 \) is equivalent to

$$c_p^{\frac{2}{p-2}} \left( \left( \frac{a}{2-a} \right)^{\frac{p-2}{2}} + \left( \frac{2-a}{a} \right)^{\frac{p-2}{2}} \right) > \left( N(N - 2) \right)^{\frac{p-2}{2}},$$

which holds by showing

$$c_p^{\frac{2}{p-2}} \geq \left( N(N - 2) \right)^{\frac{p-2}{p-1}}.$$

This could be written as

$$\frac{2}{p-1} \left( N - 2 - \frac{2}{p-1} \right) \geq N(N - 2),$$

which is true for \( p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right) \) by direct computation.

**Step 2:** we show \( U_{\mu^*} < w_p \) in \( \mathbb{R}^N \setminus \{0\} \) for \( \mu = \mu^* \). Fix \( R > 1 \) large enough such that

$$U_{\mu^*}(x) \leq 2 \left( N(N - 2) \right)^{\frac{p-2}{p-1}} |x|^{2-N} \quad \text{for any } |x| > R.$$  \hspace{1cm} (2.9)

There exists \( \varepsilon_0 > 0 \) such that

$$U_{\mu^*}(x) \leq w_p(x) - \varepsilon_0 \quad \text{for any } 0 < |x| \leq R.$$  \hspace{1cm} (2.10)

Therefore, there exists \( k^* > 2 \left( N(N - 2) \right)^{\frac{p-2}{p-1}} \) such that

$$w_k(x) \geq w_p(x) - \varepsilon_0 \quad \text{for any } 0 < |x| \leq R$$

and

$$w_k(x) \geq 2 \left( N(N - 2) \right)^{\frac{p-2}{p-1}} |x|^{2-N} \quad \text{for any } |x| > R,$$

which conclude our results. \( \Box \)
3 Fast decaying solutions

Proof of Theorem 1.1. Our proof divides into five steps.

Step 1. Existence by iteration method. We initiate from \( v_0 := w_k \), denote by \( v_n \) iteratively the unique solution of

\[
    v_n = \Gamma \ast (Vv_{n-1}^p) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

that is,

\[
    \begin{cases}
        -\Delta v_n = Vv_{n-1}^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\
        \lim_{|x| \to 0} v_n(x)|x|^{N-2} = 0.
    \end{cases}
\]

As \( v_0 = \Gamma \ast (v_0^p) \) in \( \mathbb{R}^N \setminus \{0\} \) and \( V \leq 1 \), we have that

\[
    v_1 \leq v_0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Inductively, we see that for any \( n \), we obtain that

\[
    v_n \leq v_0 = \Gamma \ast (v_0^p) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Now we show that \( U_{\mu^*} \) is a lower bound for \( \{v_n\}_n \) for \( k \in [k^*, +\infty) \). From (2.7) and the assumption that

\[
    V_p \leq V \leq 1,
\]

we obtain that \( U_{\mu^*} \) is a sub solution of (1.1), i.e.

\[
    -\Delta U_{\mu^*} = V_p U_{\mu^*}^p \leq V U_{\mu^*}^p \leq V w_k^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

i.e. \( U_{\mu^*} \geq \Gamma \ast (V U_{\mu^*}^p) \), then

\[
    v_1 \geq U_{\mu^*} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Inductively, we see that for any \( n \in \mathbb{N} \), we have that

\[
    v_n \geq U_{\mu^*} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

so \( \{v_n\}_n \) has a lower barrier \( U_{\mu^*} \). Therefore, the sequence \( \{v_n\}_n \) converges. Denote \( u_{v_k} = \lim_{n \to \infty} v_n \), then for any compact set \( K \) in \( \mathbb{R}^N \setminus \{0\} \), \( u_{v_k} \) verifies the equation

\[
    -\Delta u = V u^p \quad \text{in} \quad K,
\]

and then \( u_{v_k} \) is a classical solution of (1.1) verifying

\[
    U_{\mu^*} \leq u_{v_k} \leq w_k \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Here we let

\[
    v_k(V) = c_N \int_{\mathbb{R}^N} V u_{v_k}^p \, dx,
\]

we also replace \( v_k \) by \( v_k(V) \) if it is not confusing. Then

\[
    k \leq v_k \leq \bar{v}_k \leq k + c_{\delta_k} k^p \quad \text{and} \quad \lim_{|x| \to +\infty} u_{v_k}(x)|x|^{N-2} = v_k
\]

hold by Lemma 2.2. Here and in what follows, we always denote \( u_{v_k} \) as the \( v_k \)-fast decaying solution of (1.1) derived by the sequence \( v_k \) defined in (3.1) with initial value \( w_k \).

Step 2: the mapping \( k \mapsto v_k \) is increasing and \( V \mapsto v_k(V) \) is increasing in the sense that \( v_k(V_1) \geq v_k(V_2) \) if \( V_1 \geq V_2 \). For \( k^* \leq k_1 < k_2 \) by the increasing monotonicity of \( k \mapsto w_k \), we have that \( w_{k_1} < w_{k_2} \). Let \( \{v_{n,k_i}\} \) be sequence of (3.1) with the initial data \( v_{i,0} = w_{k_i} \), here \( i = 1, 2 \).

Let

\[
    v_{i,n} = \lim_{|x| \to +\infty} v_{i,n}(x)|x|^{N-2}, \quad i = 1, 2, \quad n = 1, 2, 3, \ldots
\]
We see that
\[ v_{1,1} = c_N \int_{\mathbb{R}^N} V w_{k_1}^p \, dx \geq c_N \int_{\mathbb{R}^N} V w_{k_2}^p \, dx = v_{2,1} \]
and
\[ v_{1,1} \geq v_{2,1} \quad \text{in} \quad \mathbb{R}^N \backslash \{0\}. \]
Inductively, we have that for any \( n \in \mathbb{N} \),
\[ v_{1,n} \geq v_{2,n}, \]
which implies that the limit \( u_{v_{k_1}} \) of \( \{v_{n,k_1}\} \) and the limit \( u_{v_{k_2}} \) of \( \{v_{n,k_2}\} \) as \( n \to +\infty \) verifies that
\[ \lim_{|x| \to +\infty} u_{v_{k_2}}(x)|x|^{N-2} \geq \lim_{|x| \to +\infty} u_{v_{k_1}}(x)|x|^{N-2}, \]
that is,
\[ v_{k_2} \geq v_{k_1}. \]
As a conclusion, for any \( k \in [k^*, +\infty) \), there exists a \( v^* := v_{k^*} > 0 \) such that problem (1.1) has a solution \( u_{v_k} \) such that
\[ \lim_{|x| \to +\infty} u_{v_k}(x)|x|^{N-2} = v_k. \]
We conclude \( u_{v_{k_1}} \geq u_{v_{k_2}} \) in \( \mathbb{R}^N \backslash \{0\} \) for \( k^* \leq k_2 \leq k_1 < +\infty \) by \( v_{1,n} \geq v_{2,n} \) in \( \mathbb{R}^N \backslash \{0\} \) for any \( n \in \mathbb{N} \). That means that the mapping \( k \mapsto v_k \) is increasing.

Similarly, we can obtain that the mapping: \( V \to v_k(V) \) is increasing.

**Step 3: we prove that the mapping** \( k \in (k^*, +\infty) \mapsto v_k \) **is continuous.** For \( 0 < k_1 < k_2 \), we have that \( w_{k_1} < w_{k_2} \).

Let \( \{v_{n,k}\} \) be sequence of (3.1) with the initial data \( v_{0,i} = w_{k_i}, \) here \( i = 1, 2, \) Let
\[ v_{n,i} = \lim_{|x| \to +\infty} v_{n,k_i}(x)|x|^{N-2}, \quad i = 1, 2, \quad n = 1, 2, 3, \ldots. \]
We see that
\[ v_{1,1} = c_N \int_{\mathbb{R}^N} V w_{k_1}^p \, dx < c_N \int_{\mathbb{R}^N} V w_{k_2}^p \, dx = v_{1,2} \]
and
\[ 0 < v_{1,2} - v_{1,1} = c_N \int_{\mathbb{R}^N} V (w_{k_2}^p - w_{k_1}^p) \, dx \leq c_N \int_{\mathbb{R}^N} (w_{k_2}^p - w_{k_1}^p) \, dx = k_2 - k_1. \]
Inductively, we have that for any \( n \in \mathbb{N} \),
\[ 0 \leq v_{n,2} - v_{n,1} \leq k_2 - k_1, \]
which implies that the limit \( u_{v_{k_1}} \) of \( \{v_{n,k_1}\} \) and the limit \( u_{v_{k_2}} \) of \( \{v_{n,k_2}\} \) as \( n \to +\infty \) verifies that
\[ 0 \leq \lim_{|x| \to +\infty} u_{v_{k_2}}(x)|x|^{N-2} - \lim_{|x| \to +\infty} u_{v_{k_1}}(x)|x|^{N-2} \leq k_2 - k_1, \]
that is,
\[ 0 \leq v_{k_2} - v_{k_1} \leq k_2 - k_1. \]
As a conclusion, \( k \mapsto v_k \) is increasing and continuous.

Let
\[ v_\infty = \lim_{k \to +\infty} v_k, \]
then we have that for any \( k \in [k^*, +\infty) \), there exists \( \nu \in \nu^* , v_\infty \) (\( \nu = v^* \) if \( v_\infty = v^* \)) such that problem (1.1) has a solution \( u_{v_k} \) such that
\[ \lim_{|x| \to +\infty} u_{v_k}(x)|x|^{N-2} = v_k. \]
Step 4, we prove that \( v_\infty = +\infty \). By the increasing monotonicity of the mapping: \( V \to v_k(V) \), we only have to prove that \( v_k(V_p) \to +\infty \) as \( k \to +\infty \) for \( \mu > \mu^* \).

By contradiction, we may assume that

\[
v_\infty(V_p) < +\infty,
\]

where we recall \( V_p = \frac{\|u\|^p_{L^p}}{\mu^2} \) in \( \mathbb{R}^N \). Now fix \( \bar{\nu} \in [\nu^*, v_\infty(V_p)] \), then there exist \( \alpha_1 = p(N - 2) - N - 2 \in (-2, 0) \) and \( l_1 > 1 \) such that

\[
\bar{\nu}_l^N V^\frac{1}{p+1} > v_\infty(V_p)
\]

and denote

\[
\psi_1(x) = l_1^\frac{2-\alpha_1}{p+1} u_0(l_1^{\frac{1}{p}} x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.
\]

Let \( \bar{k} \) be the number such that \( v_k = \bar{\nu} \). By direct computation, we have that

\[
\psi_1(x) \leq l_1^\frac{2-\alpha_1}{p+1} w_k(l_1^{\frac{1}{p}} x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}
\]

and

\[
-\Delta \psi_1 = V_{l_1} \psi_1^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

where \( V_{l_1}(x) := l_1^{N+1} V_p(l_1^{-1} x) \leq V_p(x) \) by the decreasing monotonicity of \( V_p \).

Note that \( w_{l_1^\alpha N, l_1^\alpha} (x) = l_1^\frac{2-\alpha_1}{p+1} w_k(l_1^{-1} x) \) and then we may initiate the iteration (3.1) with \( v_0 = w_{l_1^\alpha N, l_1^\alpha} \) and \( \psi_1 \) is a lower bound, so we have a solution \( u_{I_{l_1^\alpha N, l_1^\alpha}} \) of (1.1) such that

\[
\psi_2 \leq u_{I_{l_1^\alpha N, l_1^\alpha}} \leq w_{I_{l_1^\alpha N, l_1^\alpha}},
\]

it yields

\[
v_{I_{l_1^\alpha N, l_1^\alpha}} > \bar{\nu}_l^N V^\frac{1}{p+1} > v_\infty(V_p),
\]

which contradicts (3.2). Thus, we have that \( v_\infty(V_p) = +\infty \).

Finally, we prove (1.7). From Theorem 2.1, we only have to rule out the case that \( u_{\nu_1} \) is removable at the origin. If \( u_{\nu_1} \) is removable at the origin, then \( u_{\nu_1} \) is a bounded classical sub solution of

\[
-\Delta u = u^p \quad \text{in} \quad \mathbb{R}^N.
\]

Since \( u_{\nu_1} \leq w_k \) and \( u_{\nu_1} \) is bounded at the origin, there exists \( |x_1| > 0 \) small enough such that

\[
w_k(x) \leq w_p(x + x_1), \quad \forall x \in \mathbb{R}^N \setminus B_1(0)
\]

and

\[
u_{\nu_1}(x) \leq w_p(x + x_1), \quad \forall x \in B_1(0).
\]

Now consider the sequence \( \nu_n = I_{n^*} (\nu_{\nu_1}) \) with the initial data \( v_0 = u_{\nu_1} \), and this sequence is increasing and controlled by function \( w_k \) and \( w_p(x_1 + \cdot) \), the limit of \( \{\nu_n\} \) will be a bounded positive classical solution of

\[
-\Delta u = u^p \quad \text{in} \quad \mathbb{R}^N.
\]

This contradicts [16, Theorem 1.1], which says the nonexistence of bounded positive solution of (3.3). We complete the proof. \( \square \)
4 Limit of fast decaying solutions

Theorem 4.1. Assume that \( p_c \) is given by (1.5), \( p \in \left[ p_c, \frac{N+2}{N-2} \right) \), \( V \in C^1(\mathbb{R}^N \setminus \{0\}) \) is radially symmetric, decreasing with respect to \( |x| \), satisfies (1.6) and (1.8).

Let \( u_\nu \) be a \( \nu \)-fast decaying solution of problem (1.1) with \( \nu \in (\nu^*, +\infty) \) derived by Theorem 1.1. Then the limit of \( \{u_\nu\}_\nu \) as \( \nu \to +\infty \) exists, denoting \( u_\infty = \lim_{\nu \to +\infty} u_\nu \), and \( u_\infty \) is a solution of (1.1) verifying (1.7) and (1.9).

Proof: Recall the mapping \( \nu \in (\nu^*, +\infty) \to u_\nu \) is increasing, where \( u_\nu \) is a \( \nu \)-fast decaying solution of problem (1.1), so our aim is to show the existence of sequence \( \{u_\nu\}_\nu \) as \( \nu \to +\infty \). To this end, we show a priori estimates for \( \{u_\nu\}_\nu \).

Step 1. Radial symmetry. We recall that the solution \( u_\nu \) is approaching by sequence \( v_n = \Gamma^*(V_{n-1})^{1/nu} \) with initial data \( v_0 = v_k \). Note that \( v_k \) and \( V \) are radially symmetric and decreasing in \( r = |x| \), so is \( \{v_n\} \) for any \( n \in \mathbb{N} \). Therefore, \( u_\nu \) is radially symmetric and decreasing in \( r = |x| \).

Step 2. Uniform estimates. It is standard to show that \( u_\nu \) is a very weak solution of (1.1) in the distributional sense that \( u_\nu \in L^1_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N, Vdx) \) satisfies the identity

\[
\int_{\mathbb{R}^N} u_\nu(-\Delta)\xi \, dx = \int_{\mathbb{R}^N} Vu_\nu^p \xi \, dx, \quad \forall \xi \in C^\infty_c(\mathbb{R}^N). \tag{4.1}
\]

We claim that there exists \( c > 0 \) independent of \( \nu \) such that

\[
\|u_\nu\|_{L^1_{\text{loc}}(\mathbb{R}^N)} \leq c \quad \text{and} \quad \|u_\nu\|_{L^p_{\text{loc}}(\mathbb{R}^N, Vdx)} \leq c.
\]

Indeed, we recall that

\[
U_1(x) = \frac{c_N}{(1 + |x|^2)^{\frac{N+2}{2}}}, \quad \forall x \in \mathbb{R}^N,
\]

then

\[
-\Delta U_1 = U_1^{\frac{N+2}{2}} \quad \text{in} \quad \mathbb{R}^N,
\]

where \( c_N = (N(N-2))^\frac{N+2}{2} \).

For \( \varepsilon \in (0, \frac{1}{2}) \), we denote

\[
\eta_\varepsilon(x) = \eta_0(\varepsilon |x|), \quad \forall x \in \mathbb{R}^N,
\tag{4.2}
\]

where \( \eta_0 : [0, +\infty) \to [0, 1] \) is a smooth increasing function such that

\[
\eta_0(t) = 0, \quad \forall \ t \geq 2 \quad \text{and} \quad \eta_0(t) = 1, \quad \forall \ t \in [0, 1].
\]

Take \( U_1\eta_\varepsilon^2 \) as a test function of (4.1), then by H"{o}lder inequality, we have that

\[
\int_{\mathbb{R}^N} Vu_\nu^p U_1\eta_\varepsilon^2 \, dx = \int_{\mathbb{R}^N} u_\nu(-\Delta) (U_1\eta_\varepsilon^2) \, dx
\]

\[
= \int_{\mathbb{R}^N} u_\nu(U_1^{\frac{N+2}{2}} - \eta_\varepsilon^2 + 4\eta_\varepsilon \nabla U_1 \cdot \nabla \eta_\varepsilon + U_1(-\Delta)(\eta_\varepsilon^2)) \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^N} Vu_\nu^p U_1\eta_\varepsilon^2 \, dx \right)^\frac{1}{p} \left( \int_{\mathbb{R}^N} V^{1-p} U_1^{1-p} \eta_\varepsilon^{2p} \, dx \right)^\frac{1}{2p}
\]

\[
+ \varepsilon \|\eta_0\|_{C^1(\mathbb{R})} \int_{B_{\frac{1}{2}}(0) \setminus B_{\frac{1}{2}}(0)} V^{1-p} U_1^{1-p} \eta_\varepsilon^{2p} \nabla |U_1|^{p-1} \, dx
\]
Indeed, from \[22, \text{Theorem 2.1}\], there exists for some $c_0 > 0$ depends on $\varepsilon$ and $V$, but it is independent of $v$. Then we have that
\[
c_N(1 + \frac{1}{\varepsilon^2}) \frac{c_0}{\epsilon} \left( \int_{B_1(0)} V u^p \right) \leq \int_{B_{\frac{1}{2}}(0)} V u^p \leq \int_{B_{\frac{1}{2}}(0)} V u^p \leq \int_{\mathbb{R}^N} V u^p \leq c_0^{\frac{p}{p-1}},
\]
that is,
\[
\|u_v\|_{L^p_{\text{loc}}(\mathbb{R}^N, Vdx)} \leq c
\]
for some $c > 0$ independent of $v$.

Furthermore,
\[
\int_{\mathbb{R}^N} u_v U_1 x^2 \leq c_0 \left( \int_{\mathbb{R}^N} V u^p \right)^{\frac{1}{p}} \leq c_0^{\frac{p}{p-1}}
\]
and
\[
\|u_v\|_{L^1_{\text{loc}}(\mathbb{R}^N)} \leq c.
\]

**Step 3. the limit of $\{u_v\}_v$**. From Theorem 1.1, the mapping $v \in [v^*, \infty) \rightarrow u_v$ is increasing and uniformly bounded in $L^1_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N, Vdx)$, so there exists $u_\infty \in L^1_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N, Vdx)$ such that
\[
u \to u_\infty \quad \text{as} \quad v \to +\infty \quad \text{a.e. in } \Omega \text{ and in } L^1_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N, Vdx).
\]

It is known in [20] that $u_v$ is also a weak solution of (1.1), i.e.
\[
\int_{\mathbb{R}^N} u_v(-\Delta) \xi dx = \int_{\mathbb{R}^N} V u^p \xi dx, \quad \forall \xi \in C^\infty_c(\mathbb{R}^N). \tag{4.3}
\]
Passing to the limit of (4.3), we obtain that $u_\infty$ is a weak solution of (1.1) in the sense of (4.1).

Note that $u_v$ is radially symmetric and decreasing with respect to $|x|$, so is $u_\infty$. Then we have that $u_\infty \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ and then $V u^p_\infty$ is in $L^{\infty}_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$. By standard regularity results, we have that $u$ is a classical solution of (1.1).

Since $u_v$ verifies (1.7) at the origin for any $v > 0$ and $u_\infty$ is the limit of an increasing sequence $\{u_v\}_v$, then we have that
\[
\liminf_{|x| \to 0} u_\infty(x)|x|^{\frac{2}{p-1}} \geq c_p. \tag{4.4}
\]
Next we claim
\[
\limsup_{|x| \to 0} u_\infty(x)|x|^{\frac{2}{p-1}} \leq c_p, \tag{4.5}
\]
Indeed, from [22, Theorem 2.1], there exists $c > 0$ such that
\[
u_\infty(x) \leq c|x|^{-\frac{2}{p-1}}, \quad \forall x \in B_1(0) \setminus \{0\},
\]
and by [2, Theorem B], then we have that (4.5) and (1.7) hold true for $u_\infty$.

Finally, we claim (1.9). As in [15], we denote
\[
\nu(x) = u_\infty\left(\frac{x}{|x|^2}\right) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.
\]
By direct computation, we have that
\[
\nabla u(x) = \nabla u_\infty \left( \frac{x}{|x|^2} \right) \frac{1}{|x|^2} - 2 \left( \nabla u_\infty \left( \frac{x}{|x|^2} \right) \cdot x \right) \frac{x}{|x|^2}
\]
and
\[
\Delta u(x) = \frac{1}{|x|^4} \Delta u_\infty \left( \frac{x}{|x|^2} \right) + \frac{2(2-N)}{|x|^4} \left( \nabla u_\infty \left( \frac{x}{|x|^2} \right) \cdot x \right).
\]
Let \( u^r(x) = |x|^{2-N} v(x) \), then for \( x \in \mathbb{R}^N \setminus \{0\} \), we obtain that
\[
-\Delta u^r(x) = -\Delta v(x) |x|^{2-N} - 2 \nabla v(x) \cdot (\nabla |x|^{2-N}) = |x|^{-2-N}(-\Delta)u \left( \frac{x}{|x|^2} \right) = V^r(x) u^r(x)^p,
\]
where
\[
V^r(x) = |x|^{-2-N+p(N-2)} V_p \left( \frac{x}{|x|^2} \right).
\]
Direct computation implies that
\[
-\Delta u^r(x) = V^r(x) u^r(x)^p, \quad \forall x \in \mathbb{R}^N \setminus \{0\},
\]
where
\[
V^r(x) \sim |x|^\alpha \text{ as } |x| \to 0 \quad \text{with } \alpha = -\alpha_1 - 4 + (p-1)(N-2)
\]
and \( \alpha_1 = (N-2)p - N - 2 \in (-2, 0) \).

Now claim that \( p \in \left( \frac{N+\alpha}{N-2}, \frac{N+2+\alpha}{N-2} \right) \). Note that \( p > \frac{N}{N-2} \) follows by the fact \( p > \frac{N}{N-2} \) and \( p < \frac{N+2+\alpha}{N-2} \) is equivalent to \((N-2)p - N - 2 < -4 + (p-1)(N-2) - \alpha_1 \), which is true thanks to \( \alpha \leq 0 \). Then by [22, Theorem 2.1] and [16, Theorem 3.3], we have that
\[
\frac{1}{c} |x|^{-\frac{2N+\alpha}{N-2}} \leq u^r(x) \leq c |x|^{-\frac{2N+\alpha}{N-2}}, \quad \forall x \in B_1(0) \setminus \{0\},
\]
where \( c > 1 \) and
\[
\frac{2+\alpha}{p-1} = -(N-2) + \frac{2 + \alpha_1}{p-1}.
\]
By Kelvin transformation, we turn back that
\[
\frac{1}{c} |x|^{-\frac{2N+\alpha}{N-2}} \leq u_\infty(x) \leq c |x|^{-\frac{2N+\alpha}{N-2}}, \quad \forall x \in \mathbb{R}^N \setminus B_1(0)
\]
for some \( c > 1 \).

**Proof of Theorem 1.2.** In Theorem 4.1, we have proved the limit \( \{u_r\} \), as \( r \to +\infty \) when \( V \) is radially symmetric and decreasing with respect to \( |x| \). Now we would prove the limit when \( \{u_r\} \), has no symmetric property.

From assumptions (1.6) and (1.8), there exists a radially symmetric potential \( V_r \geq V \) in \( \mathbb{R}^N \) and there exists \( l_0 > 1 \) such that
\[
V(x) \geq V_r(l_0 x) := V_{r,l_0}(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.
\]
From the proof of Theorem 1.1, there exist a mapping \( k \to u_{v_k(V)} \), by the monotonicity, we have that
\[
l_{l_0}^{\frac{2N+2-N}{p}} u_{v_k(V_{r,l_0})} = u_{v_k(V_{r,l_0})} \leq u_{v_k(V)} \leq u_{v_k(V)} \quad \text{in } \mathbb{R}^N \setminus \{0\},
\]
where
\[
l_{l_0}^{\frac{2N+2-N}{p}} v_k(V_{r,l_0}) = v_k(V_{r,l_0}) \leq v_k(V) \leq v_k(V_{r,l_0}).
\]
Therefore, we obtain that the mapping \( v \in (0, \infty) \to u_v \) is increasing and uniformly bounded in \( L^1_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N, Vdx) \). The left proof is the same as the one of Theorem 4.1.

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