Conservation and invariance properties of submarkovian semigroups

A.F.M. ter Elst\textsuperscript{1} and Derek W. Robinson\textsuperscript{2}

Abstract

Let $\mathcal{E}$ be a Dirichlet form on $L^2(X)$ and $\Omega$ an open subset of $X$. Then one can define Dirichlet forms $\mathcal{E}_D$, or $\mathcal{E}_N$, corresponding to $\mathcal{E}$ but with Dirichlet, or Neumann, boundary conditions imposed on the boundary $\partial \Omega$ of $\Omega$. If $S$, $S^D$ and $S^N$ are the associated submarkovian semigroups we prove, under general assumptions of regularity and locality, that $S_t \varphi = S^D_t \varphi$ for all $\varphi \in L^2(\Omega)$ and $t > 0$ if and only if the capacity $\text{cap}_\Omega(\partial \Omega)$ of $\partial \Omega$ relative to $\Omega$ is zero. Moreover, if $S$ is conservative, i.e. stochastically complete, then $\text{cap}_\Omega(\partial \Omega) = 0$ if and only if $S^D$ is conservative on $L^2(\Omega)$. Under slightly more stringent assumptions we also prove that the vanishing of the relative capacity is equivalent to $S^D_t \varphi = S^N_t \varphi$ for all $\varphi \in L^2(\Omega)$ and $t > 0$.
1 Introduction

In two earlier papers [RoS] [ElR] the relationships between the invariance of a set $\Omega$ under the action of a submarkovian semigroup $S$ and capacity conditions on the boundary $\partial\Omega$ of the set were explored. In the current paper we demonstrate that these features are connected to the conservative property for the semigroup $S^D$ obtained by imposing Dirichlet boundary conditions on $\partial\Omega$. Under quite general conditions the latter property is equivalent to the capacity of $\partial\Omega$ relative to $\Omega$ being zero. Alternatively these conditions are equivalent to the equality $S^t_\Omega \varphi = S^N_\Omega \varphi$ for all $\varphi \in L^2(\Omega)$ and $t > 0$, where $S^N$ is the semigroup obtain by imposing Neumann boundary conditions on $\partial\Omega$. This latter result is related to the work of Arendt and Warma [ArW1] [ArW2] on boundary conditions on the Laplacian on arbitrary domains and a number of our arguments are similar.

The analysis of [RoS] was for a semigroup $S$ on $L^2(\mathbb{R}^d)$ generated by a second-order, divergence-form, elliptic operator $H$ with $W^{1,\infty}$-coefficients and an open set $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. Then it was established that $S_t L^2(\Omega) \subseteq L^2(\Omega)$ for all $t > 0$ if and only if the capacity $\text{cap}(\partial\Omega)$ of $\partial\Omega$ measured with respect to the form $h$ associated with $H$ is zero. It was also remarked that this equivalence fails if the coefficients of $H$ are not Lipschitz continuous. The problem is that the degeneracy of the coefficients can differ depending whether one approaches the boundary $\partial\Omega$ from $\Omega$ or from $\Omega^c$. The situation was clarified in [ElR] by the demonstration that invariance could be completely characterized by a condition on the capacities relative to $\Omega$ and $\Omega^c$ with no regularity required of the coefficients or the boundary $\partial\Omega$. In addition the set $\Omega$ is allowed to be measurable. The results of [ElR] were derived in the general framework of local Dirichlet forms and the current discussion will also be carried out in this framework.

We assume throughout that $X$ is a locally compact $\sigma$-compact metric space equipped with a positive Radon measure $\mu$ such that $\text{supp }\mu = X$. Let $\mathcal{E}$ be a Dirichlet form on $X$. The Dirichlet form is called \textit{regular} if $D(\mathcal{E}) \cap C_c(X)$ is dense both in $D(\mathcal{E})$, with the graph norm, and in $C_0(X)$, with the supremum norm. Throughout this paper we assume that $D(\mathcal{E}) \cap C_c(X)$ is dense in $C_0(X)$. Moreover, we also require throughout that $\mathcal{E}$ is \textit{local} in the sense that $\mathcal{E}(\psi, \varphi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ with $\varphi \psi = 0$. This notion appears slightly stronger than locality as defined in [FOT] but if $\mathcal{E}$ is regular then it is equivalent by a result of Schmuland [Sch]. Let $\Omega$ be an open subset of $X$. We associate with the form $\mathcal{E}$ a second form $\mathcal{E}_D$ which corresponds abstractly to $\mathcal{E}$ with Dirichlet boundary conditions imposed on $\partial\Omega$. The latter form is defined by first setting

$$D_\Omega = D_\Omega(\mathcal{E}) = \{ \varphi \in D(\mathcal{E}) : \text{supp }\varphi \text{ is a compact subset of } \Omega \} .$$

Since $D(\mathcal{E}) \cap C_c(X)$ is dense in $C_0(X)$ it follows that $D_\Omega$ is dense in $L^2(\Omega)$. Then we define $D_\Omega$ as the closure of $D_\Omega$ with respect to the graph norm on $D(\mathcal{E})$. Since $1_\Omega \varphi = \varphi$ for all $\varphi \in D_\Omega$ and the multiplication operator $\varphi \mapsto 1_\Omega \varphi$ is continuous on $L^2(X)$, it follows that $D_\Omega \subseteq L^2(\Omega)$. Here and in the sequel we identify $L^2(\Omega)$ in a natural way with the subspace $\{ 1_\Omega \varphi : \varphi \in L^2(X) \}$. Finally $\mathcal{E}_D(= \mathcal{E}_D) = D(\mathcal{E}_D) = \{ \varphi \in D(\mathcal{E}) : \varphi = 0 \text{ on } \partial\Omega \}$ with domain $D(\mathcal{E}_D) \subseteq L^2(\Omega)$ by $\mathcal{E}_D = \mathcal{E} |_{D_\Omega}$. Subsequently, in Section 3, we introduce a second form $\mathcal{E}_N$ which corresponds to the introduction of Neumann boundary conditions on $\partial\Omega$. But the definition of $\mathcal{E}_N$ is more complicated and its analysis requires stronger assumptions. Therefore we first concentrate on the relatively simple form $\mathcal{E}_D$.

It follows straightforwardly that $\mathcal{E}_D$ is a Dirichlet form on $L^2(\Omega)$ and $\mathcal{E}(\mathcal{E}_D) \cap C_c(\Omega)$ is dense in $C_0(\Omega)$. Let $H_D(= H_{\Omega,D})$ and $S^D(= S^{0,D})$ denote the operator and semigroup on
\( L_2(\Omega) \) associated with \( \mathcal{E}_D \). Since \( S \) and \( S^D \) are submarkovian semigroups they extend to all the \( L_p \)-spaces including \( L_\infty(X) \) and \( L_\infty(\Omega) \).

Next we define the capacity and relative capacity of a set with respect to the form \( \mathcal{E} \). If \( \Omega \) is a subset of \( X \) and \( A \subseteq \overline{\Omega} \) then the relative capacity introduced earlier by Arendt and Warma \cite{ArW1} \cite{ArW2}.

Theorem 1.1 Adopt the foregoing definitions and assumptions. Let \( \Omega \) a subset of \( X \) and \( A \subseteq \overline{\Omega} \) then the relative capacity \( \text{cap}_\Omega(A) \in [0, \infty] \) is defined by

\[
\text{cap}_\Omega(A) = \text{cap}_{\Omega,\mathcal{E}}(A) = \inf \{ \| \varphi \|^2_{D(\mathcal{E})} : \varphi \in D(\mathcal{E}) \text{ and there exists an open } V \subset X \text{ such that } A \subseteq V \text{ and } \varphi \geq 1 \text{ a.e. on } V \cap \Omega \}.
\]

If \( \Omega = X \) then \( \text{cap}(A) = \text{cap}_\mathcal{E}(A) = \text{cap}_{X,\mathcal{E}}(A) \) is the capacity of the set \( A \). This version of relative capacity is the one used in \cite{ElR}, but it is probably different from the definition of relative capacity introduced earlier by Arendt and Warma \cite{ArW1} \cite{ArW2}.

If \( \mathcal{E} \) is regular and \( \Omega \) is measurable then it follows from \cite{ElR}, Theorem 1.1, that \( S \) leaves \( L_2(\Omega) \) invariant if and only if there exist \( A_1, A_2 \subseteq \partial \Omega \) such that \( A_1 \cup A_2 = \partial \Omega \) and \( \text{cap}_\Omega(A_1) = 0 = \text{cap}_{R,\mathcal{E}}(A_2) \). In particular, if \( \text{cap}_\Omega(\partial \Omega) = 0 \) then \( L_2(\Omega) \) is \( S \)-invariant.

Our main result gives a criterion for the validity of the converse of the latter statement.

**Theorem 1.1** Adopt the foregoing definitions and assumptions. Let \( \Omega \) be an open subset of \( X \). Consider the following conditions.

I. \( S^D \) is conservative, i.e. \( S^D_t 1_\Omega = 1_\Omega \) for all \( t > 0 \).

II. \( S_t \varphi = S^D_t \varphi \) for all \( \varphi \in L_2(\Omega) \) and \( t > 0 \).

III. \( \text{cap}_\Omega(\partial \Omega) = 0. \)

Then \( \mathbb{I} \implies \mathbb{II} \implies \mathbb{III} \). In particular Conditions \( \mathbb{II} \) and \( \mathbb{III} \) imply that \( L_2(\Omega) \) is \( S \)-invariant.

Moreover, if \( S \) is conservative then \( \mathbb{III} \implies \mathbb{II} \). Finally, if \( \mathcal{E} \) is regular, then \( \mathbb{III} \implies \mathbb{II} \).

The theorem applies directly if \( \mathcal{E} \) is the form of a second-order, divergence-form, elliptic operator with real measurable coefficients on \( L_2(R^d) \). Then \( \mathcal{E} \) is regular, local and the corresponding semigroup \( S \) is conservative. We will discuss this example more fully in Section 4. The equivalence \( \mathbb{II} \iff \mathbb{III} \) generalizes a result of Arendt and Warma for the Laplacian (see \cite{ArW2}, Proposition 2.5).

One can draw a stronger conclusion if the capacity \( \text{cap}(\partial \Omega) = 0 \) and \( \mathcal{E} \) is regular, since this immediately implies that \( \text{cap}_{\Omega}(\partial \Omega) = 0 = \text{cap}_{\mathcal{E}}(\partial \Omega) \). There is, however, a converse to this statement if \( |\partial \Omega| = 0 \). Then the conditions \( \text{cap}_{\Omega}(\partial \Omega) = 0 = \text{cap}_{\mathcal{E}}(\partial \Omega) \) imply that \( \text{cap}(\partial \Omega) = 0 \) by \cite{ElR}, Lemma 2.9. (The condition \(|\partial \Omega| = 0 \) is essential since \( \text{cap}(\partial \Omega) \geq |\partial \Omega| \).)

Therefore if \( |\partial \Omega| = 0 \) then \( \text{cap}(\partial \Omega) = 0 \) is equivalent to both \( S^D \) and \( S^{\mathcal{E}} \) being conservative or to the conditions \( S_t \varphi = S^D_t \varphi \) and \( S_t \psi = S^{\mathcal{E}}_t \psi \) for all \( \varphi \in L_2(\Omega), \psi \in L_2(\overline{\Omega}) \) and \( t > 0 \).

In Section 3 we will give a further characterization of the condition \( \text{cap}_\Omega(\partial \Omega) = 0 \) in terms of Neumann boundary conditions.

## 2 Dirichlet boundary conditions

In this section we prove Theorem \( \mathbb{III} \). The proof depends on a couple of standard results which we use throughout this paper.

First, the \( S \)-invariance of \( L_2(\Omega) \) is equivalent to the condition \( 1_\Omega \varphi \in D(\mathcal{E}) \) for all \( \varphi \in D(\mathcal{E}) \) or for all \( \varphi \) in a core of \( \mathcal{E} \). These criteria are a corollary of a general result.
of Ouhabaz [Ouh], Theorem 2.2, for local accretive forms (see also [FOT], Theorem 1.6.1, and [EiR], Proposition 2.1).

Secondly, we need an order relation between the semigroups $S$ and $S^D$. Note that each bounded operator $A$ on $L_2(\Omega)$ can be extended to a bounded operator on $L_2(X)$, still denoted by $A$, via $\varphi \mapsto A(1_\Omega \varphi) \in L_2(\Omega) \subset L_2(X)$ for all $\varphi \in L_2(X)$. In particular $S^D_t$ extends to a bounded operator on $L_2(X)$. Note that $\lim_{t \to 0} S^D_t \varphi = 1_\Omega \varphi$ for all $\varphi \in L_2(X)$.

**Proposition 2.1** If $\Omega$ is open and $\varphi \in L_2(X)_+$ then $0 \leq S^D_t \varphi \leq S_t \varphi$ for all $t > 0$.

The proposition follows from an adaptation of the reasoning of [Are], Section 4.2. Alternatively it can be deduced from [Ouh], Theorem 2.24. The proof relies on the following extension of Lemma 4.2.3 of [Are].

**Lemma 2.2** Let $\varphi \in D(\mathcal{E}_D)$ and $\psi \in D(\mathcal{E}_+)$ satisfy

$$\langle \chi, \varphi \rangle + \mathcal{E}_D(\chi, \varphi) \leq \langle \chi, \psi \rangle + \mathcal{E}(\chi, \psi)$$

(1)

for all $\chi \in D(\mathcal{E}_D)_+$. Then $\varphi \leq \psi$.

**Proof** There exist $\varphi_1, \varphi_2, \ldots \in D_\Omega$ such that $\lim \|\varphi_n - \varphi\|_{D(\mathcal{E})} = 0$. Then, however, $\text{supp}(\varphi_n - \psi)_+ \subseteq \text{supp} \varphi_n \subset \Omega$ since $\psi \geq 0$. So $(\varphi_n - \psi)_+ \in D_\Omega$ for all $n \in \mathbb{N}$. Moreover, $\lim (\varphi_n - \psi)_+ = (\varphi - \psi)_+$ in $D(\mathcal{E})$. Hence $(\varphi - \psi)_+ \in D(\mathcal{E}_D)$.

Secondly, set $\chi = (\varphi - \psi)_+$ in (1). Then one deduces that

$$\|(\varphi - \psi)_+\|^2 = (\varphi - \psi)_+, \varphi - \psi) \leq -\mathcal{E}(\varphi - \psi)_+, \varphi - \psi) = -\mathcal{E}(\varphi - \psi)_+ \leq 0,$$

where we used locality of $\mathcal{E}$ in the last equality. Hence $(\varphi - \psi)_+ = 0$ or, equivalently, $\varphi \leq \psi$. \hfill $\square$

**Proof of Proposition 2.1** Let $\tau \in L_2(\Omega)_+$. Set $\varphi = (I + H_D)^{-1}\tau$ and $\psi = (I + H)^{-1}\tau$. Then $\varphi \in D(H_D) \subseteq D(\mathcal{E}_D)$ and $\psi \in D(H) \subseteq D(\mathcal{E})$. Moreover, $\psi \geq 0$ because $\tau \geq 0$ and $S$ is submarkovian. Now

$$\langle \chi, \varphi \rangle + \mathcal{E}_D(\chi, \varphi) = \langle \chi, (I + H_D)\varphi \rangle = \langle \chi, \tau \rangle = \langle \chi, \psi \rangle + \mathcal{E}(\chi, \psi)$$

for all $\chi \in D(\mathcal{E}_D)$. Therefore $(I + H_D)^{-1}\tau \leq (I + H)^{-1}\tau$ by Lemma 2.2. Similarly, $(I + \lambda H_D)^{-1}\tau \leq (I + \lambda H)^{-1}\tau$ for all $\lambda > 0$ and $\tau \in L_2(\Omega)_+$. Then $S^D_t\tau \leq S_t\tau$ for all $t > 0$ since $S_t\tau = \lim_{n \to \infty}(I + n^{-1}tH)^{-n}\tau$ with a similar expression for $S^D_t$. Finally, since $S^D_t\tau = 0$ for all $\tau \in L_2(\Omega)$ the proposition follows. \hfill $\square$

**Corollary 2.3** If $\Omega_1 \subseteq \Omega_2$ are open then $0 \leq S^\Omega_1,D \varphi \leq S^\Omega_2,D \varphi$ for all $\varphi \in L_2(X)_+$ and $t > 0$.

**Proof** This follows from Proposition 2.1 with $X$ replaced by $\Omega_2$, $\mathcal{E}$ replaced by $\mathcal{E}_\Omega_2$ and $S$ by $S^\Omega_2,D$. \hfill $\square$

Now we turn to the proof of Theorem 1.1.

**Proof of Theorem 1.1** (1) $\Rightarrow$ (2). Let $\varphi \in L_1(\Omega) \cap L_2(\Omega)_+$ and $t > 0$. Then $S^D_t \varphi \leq S_t \varphi$ by Proposition 2.1. Therefore using Condition 2 and the positivity and contractivity of $S$ one has

$$\|\varphi\|_1 = (1_\Omega, \varphi) = (S^D_t 1_\Omega, \varphi) = (1_\Omega, S^D_t \varphi) \leq (1_\Omega, S_t \varphi) \leq (1, S_t \varphi) = \|S_t \varphi\|_1 \leq \|\varphi\|_1.$$
Hence all three inequalities are in fact equalities. Since the second inequality in (2) is an equality it follows that \( \langle 1_{\Omega^c}, S_t \varphi \rangle = 0 \). Therefore \( 1_{\Omega^c} S_t \varphi = 0 \) and \( S_t \varphi \in L_2(\Omega) \). Since the first inequality in (2) is an equality one deduces from the order relation \( S_t^D \varphi \leq S_t \varphi \) of Proposition 2.1 that \( S_t^D \varphi = S_t \varphi \). But this immediately implies that \( S_t^D \psi = S_t \psi \) for all \( t > 0 \) and \( \psi \in L_2(\Omega) \). Thus Condition \( \text{II} \) is established.

\( \text{II} \Rightarrow \text{III} \). If \( \varphi \in L_2(\Omega) \) then \( S_t \varphi = S_t^D \varphi \in L_2(\Omega) \). So \( L_2(\Omega) \) is \( S \)-invariant. Next, let \( K \subset X \) compact. Since \( X \) is locally compact and \( D(\mathcal{E}) \cap C_c(X) \) is dense in \( C_0(X) \) there exist an open set \( V \) and a \( \varphi \in D(\mathcal{E}) \cap C_c(X) \) such that \( \varphi \geq 1_V \geq 1_K \) pointwise. Then \( 1_\Omega \varphi \in D(\mathcal{E}) \cap L_2(\Omega) \), by \( S \)-invariance of \( L_2(\Omega) \), and

\[
\lim_{t \to 0} (1_\Omega \varphi, (I - S_t^D)1_\Omega \varphi) = \lim_{t \to 0} (1_\Omega \varphi, (I - S_t)1_\Omega \varphi) = \mathcal{E}(1_\Omega \varphi)
\]

exists. Therefore \( 1_\Omega \varphi \in D(\mathcal{E}_D) \). By definition of \( \mathcal{E}_D \) there exist \( \psi_1, \psi_2, \ldots \in D_\Omega \) such that \( \lim_{n \to \infty} \psi_n = 1_\Omega \varphi \) in \( D(\mathcal{E}) \). Then \( 1_\Omega \varphi - \psi_n \in D(\mathcal{E}), K \cap \partial \Omega \subset V \setminus \text{supp} \psi_n \), the set \( V \setminus \text{supp} \psi_n \) is open and \( 1_\Omega \varphi - \psi_n \geq 1 \) a.e. on \( (V \setminus \text{supp} \psi_n) \cap \Omega \) for all \( n \in \mathbb{N} \). Therefore \( \text{cap}_\Omega(K \cap \partial \Omega) \leq \|1_\Omega \varphi - \psi_n\|_{\mathcal{E}(\mathcal{E})} \) for all \( n \in \mathbb{N} \) and \( \text{cap}_\Omega(K \cap \partial \Omega) = 0 \). Since \( X \) is \( \sigma \)-compact one deduces that \( \text{cap}_\Omega(\partial \Omega) = 0 \).

\( \text{III} \Rightarrow \text{II} \). Suppose that \( S \) is conservative. If \( \varphi \in L_1(\Omega) \cap L_2(\Omega) \) then

\[
(\varphi, S_t^D \underline{1}_\Omega \varphi) = (S_t^D \varphi, \underline{1}_\Omega) = (S_t \varphi, \underline{1}_\Omega) = (\varphi, \underline{1}_\Omega S_t \underline{1}_\Omega) = (\varphi, \underline{1}_\Omega)
\]

for all \( t > 0 \). Therefore \( S_t^D \underline{1}_\Omega = \underline{1}_\Omega \) for all \( t > 0 \).

\( \text{II} \Rightarrow \text{III} \). Finally, suppose that \( \mathcal{E} \) is regular. We shall prove that if \( \varphi \in D(\mathcal{E}) \) then \( 1_\Omega \varphi \in D(\mathcal{E}_D) \). We argue as in the proof of [EIR], Theorem 2.4.

Since \( \text{cap}_\Omega(\partial \Omega) = 0 \) for all \( n \in \mathbb{N} \) there exist \( \psi_n \in D(\mathcal{E}) \) and an open \( V_n \subset X \) such that \( \partial \Omega \subset V_n, \psi_n \geq 1 \) almost everywhere on \( V_n \cap \Omega \) and \( \|\psi_n\|_{D(\mathcal{E})} \leq 1/n \). Without loss of generality we may assume that \( 0 \leq \psi_n \leq 1 \). Let \( \varphi \in D(\mathcal{E}) \cap C_c(X) \). Let \( n \in \mathbb{N} \). Define \( \varphi_n = (\varphi - \varphi \psi_n) \underline{1}_\Omega \in L_2(\Omega) \). Then \( \text{supp} \varphi_n \) is compact and

\[
\text{supp} \varphi_n \subset \overline{\Omega} \cap V_n^c \subset \overline{\Omega} \cap V_n^c \subset \Omega.
\]

Hence there exists a \( \chi \in D(\mathcal{E}) \cap C_c(\Omega) \) such that \( \chi|_{\text{supp} \varphi_n} = 1 \). Then \( \varphi_n = (\varphi - \varphi \psi_n) \chi \in D(\mathcal{E}) \). So \( \varphi_n \in D_\Omega \subset D(\mathcal{E}_D) \). It follows from locality that

\[
\mathcal{E}(\varphi_n) \leq \mathcal{E}(\varphi_n) + \mathcal{E}((\varphi - \varphi \psi_n) \underline{1}_\Omega^c)
\]

\[
= \mathcal{E}(\varphi - \varphi \psi_n) \leq 2 \mathcal{E}(\varphi) + 4 \mathcal{E}(\varphi) \|\psi_n\|_\infty^2 + 4 \mathcal{E}(\psi_n) \|\varphi\|_\infty^2 \leq 6 \mathcal{E}(\varphi) + 4 \|\varphi\|_\infty^2
\]

for all \( n \in \mathbb{N} \). So the sequence \( \varphi_1, \varphi_2, \ldots \) has a weakly convergent subsequence \( \varphi_{n_1}, \varphi_{n_2}, \ldots \) in the Hilbert space \( D(\mathcal{E}_D) \). Clearly \( \lim_{n \to \infty} \varphi_n = 1_\Omega \varphi \) in \( L_2(\Omega) \). So \( 1_\Omega \varphi \in D(\mathcal{E}_D) \) for all \( \varphi \in D(\mathcal{E}) \cap C_c(X) \).

Since \( D(\mathcal{E}) \cap C_c(X) \) is dense in \( D(\mathcal{E}) \) by regularity it follows that \( L_2(\Omega) \) is \( S \)-invariant by [EIR], Proposition 2.1. Moreover, by density, \( D(\mathcal{E}) \cap L_2(\Omega) \subset D(\mathcal{E}_D) \). Since the converse inclusion is obvious it follows that \( D(\mathcal{E}) \cap L_2(\Omega) = D(\mathcal{E}_D) \). Hence \( S_t \varphi = S_t^D \varphi \) for all \( \varphi \in L_2(\Omega) \) and \( t > 0 \). This completes the proof of Theorem \( \text{II} \). \( \square \)

### 3 Neumann boundary conditions

The form corresponding to \( \mathcal{E} \) with Neumann boundary conditions on \( \partial \Omega \) is defined in terms of the truncations of \( \mathcal{E} \). If \( \chi \in D(\mathcal{E}) \cap L_\infty(X)_+ \) then the truncated form \( \mathcal{E}_\chi \) is given by
\[ D(\mathcal{E}_x) = D(\mathcal{E}) \cap L_\infty(X) \] and
\[ \mathcal{E}_x(\varphi) = \mathcal{E}(\chi \varphi, \varphi) - 2^{-1} \mathcal{E}(\chi, \varphi^2) \]
for all \( \varphi \in D(\mathcal{E}) \cap L_\infty(X) \). It has three basic properties:
\[ 0 \leq \mathcal{E}_x(\varphi) \leq \| \varphi \|_\infty \mathcal{E}(\varphi), \quad (3) \]
\[ \mathcal{E}_x(0 \vee \varphi \wedge 1) \leq \mathcal{E}_x(\varphi), \quad (4) \]
and
\[ \text{if } 0 \leq \chi_1 \leq \chi_2 \text{ then } 0 \leq \mathcal{E}_{\chi_1}(\varphi) \leq \mathcal{E}_{\chi_2}(\varphi) \quad (5) \]
where all three properties are valid for all \( \varphi \in D(\mathcal{E}) \cap L_\infty(X) \). These properties are established in [BoH], Proposition 1.4.1.1.

It follows from (3) that \( \mathcal{E}_x \) can be extended to \( D(\mathcal{E}) \) by continuity. The extension, which we continue to denote by \( \mathcal{E}_x \), still satisfies the Markovian property (4) and the monotonicity property (5).

Next for each open subset \( \Omega \) of \( X \) define the convex subset \( \mathcal{C}_\Omega \) of \( D(\mathcal{E}) \) by
\[ \mathcal{C}_\Omega = \{ \chi \in D(\mathcal{E}) \cap L_\infty(X), 0 \leq \chi \leq 1_\Omega \}. \]
It follows that \( \mathcal{C}_\Omega \) is a directed set with respect to the natural order. In particular if \( \chi_1, \chi_2 \in \mathcal{C}_\Omega \) then \( \chi_{12} = \chi_1 + \chi_2 - \chi_1 \chi_2 \in \mathcal{C}_\Omega \). Moreover, \( \chi_{12} - \chi_1 = \chi_2(1_\Omega - \chi_1) \geq 0 \) and \( \chi_{12} - \chi_2 = \chi_1(1_\Omega - \chi_2) \geq 0 \). Therefore it follows from (5) that \( \chi \mapsto \mathcal{E}_x \) is a monotonically increasing net of quadratic forms with the common domain \( D(\mathcal{E}) \). Then one can define a form \( \mathcal{E}_N (= \mathcal{E}_\Omega_N) \) by \( D(\mathcal{E}_N) = D(\mathcal{E}) \) and
\[ \mathcal{E}_N(\varphi) = \lim_{\chi \in \mathcal{C}_\Omega} \mathcal{E}_x(\varphi) = \sup\{ \mathcal{E}_x(\varphi) : \chi \in \mathcal{C}_\Omega \}. \]
Since \( \mathcal{E}_N \) is defined as a limit of quadratic forms it is automatically a quadratic form on \( L_2(X) \) and it follows from (3) and (4) that \( \mathcal{E}_N \) satisfies the continuity property
\[ 0 \leq \mathcal{E}_N(\varphi) \leq \mathcal{E}(\varphi) \quad (6) \]
and the Markovian property
\[ \mathcal{E}_N(0 \vee \varphi \wedge 1) \leq \mathcal{E}_N(\varphi) \quad (7) \]
for all \( \varphi \in D(\mathcal{E}) \). We emphasize that \( \mathcal{E}_N \) is a form on \( L_2(X) \).

The definition of \( \mathcal{E}_N \) is motivated by the theory of second-order elliptic operators. Let \( X = \mathbb{R} \) and define \( \mathcal{E} \) by \( D(\mathcal{E}) = W^{1,2}(\mathbb{R}) \) and \( \mathcal{E}(\varphi) = \int_{\mathbb{R}} |\varphi'|^2 \). Then \( \mathcal{E}_x(\varphi) = \int_{\Omega} \chi |\varphi'|^2 \) and \( \mathcal{E}_N(\varphi) = \int_{\Omega} |\varphi'|^2 \).

Our aim is to compare the forms \( \mathcal{E}_D \) and \( \mathcal{E}_N \) on \( L_2(\Omega) \) but in general \( \mathcal{E}_N \) is not closed nor even closable. In fact it is closed under quite general assumptions (see Proposition 3.6 below) but in any case one can introduce the relaxation \( \mathcal{E}_N \) of \( \mathcal{E}_N \).

The relaxation \( \hat{t} \) of a quadratic form \( t \) is variously called the lower semi-continuous regularization (see [EkT], page 10) or the relaxed form (see [Dal], page 28). It is the closure of the largest closable form which is less than or equal to \( t \) (see [Sim] Theorem 2.2). In particular, if \( t \) is closable then \( \hat{t} \) is the closure.

The relaxation \( \mathcal{E}_N \) of \( \mathcal{E}_N \) is automatically a Dirichlet form; it is positive, closed and satisfies (7). Moreover, it satisfies \( \mathcal{E}_N(\varphi) \leq \mathcal{E}(\varphi) \) for all \( \varphi \in D(\mathcal{E}) \) by (6). Let \( H_N (= H_{\Omega,N}) \) and \( \mathcal{S}_N (= \mathcal{S}_{\Omega,N}) \) denote the operator and submarkovian semigroup on \( L_2(X) \) associated with \( \mathcal{E}_N \).  

5
Remark 3.1 If \( \varphi \in D(\mathcal{E}) \cap C_c(\Omega^c) \) then \( \mathcal{E}_\chi(\varphi) = 0 \) for all \( \chi \in \mathcal{C}_\Omega \) by locality. Therefore \( \mathcal{E}_N(\varphi) = 0 \) and \( \hat{\mathcal{E}}_N(\varphi) = 0 \). Since \( \varphi \in D(\mathcal{E}) \cap C_c(\Omega^c) \) is dense in \( C_c(\Omega^c) \) and \( C_c(\Omega^c) \) is dense in \( L_2(\Omega^c) \) one deduces that \( \hat{\mathcal{E}}_N(\varphi) = 0 \) for all \( \varphi \in L_2(\Omega^c) \). Hence if \( \varphi \in D(\hat{\mathcal{E}}_N) \) then \( \mathbb{1}_\Omega \varphi \in D(\hat{\mathcal{E}}_N) \) and \( \hat{\mathcal{E}}_N(\varphi) = \hat{\mathcal{E}}_N(\mathbb{1}_\Omega \varphi) \). In particular, the space \( L_2(\Omega^c) \) is invariant under \( S^N \).

**Proposition 3.2** Let \( \Omega \subseteq \mathbb{R}^d \) be open. If \( S_t^D \varphi = S_t^N \varphi \) for all \( \varphi \in L_2(\Omega) \) and \( t > 0 \) then \( L_2(\Omega) \) is \( S^N \)-invariant and \( \text{cap}_\partial(\partial \Omega) = 0 \).

**Proof** Since \( S^D \) leaves \( L_2(\Omega) \) invariant the \( S^N \)-invariance follows immediately. But the latter property implies that if \( \varphi \in D(\hat{\mathcal{E}}_N) \) then \( \mathbb{1}_\Omega \varphi \in D(\hat{\mathcal{E}}_N) \). Next let \( \varphi \in D(\mathcal{E}) \). Then \( \varphi \in D(\hat{\mathcal{E}}_N) \) and

\[
\lim_{t \downarrow 0} t^{-1}(\mathbb{1}_\Omega \varphi, (I - S_t^D)\mathbb{1}_\Omega \varphi) = \lim_{t \downarrow 0} t^{-1}(\mathbb{1}_\Omega \varphi, (I - S_t^N)\mathbb{1}_\Omega \varphi) = \hat{\mathcal{E}}_N(\mathbb{1}_\Omega \varphi) < \infty.
\]

So \( \mathbb{1}_\Omega \varphi \in D(\mathcal{E}_D) \). The rest of the proof is then a repetition of the argument that \( \text{II} \to \text{III} \) in Theorem 3.1.

Under more stringent assumptions (see Theorem 3.7) we will prove that Proposition 3.2 has a converse. One key condition is strong locality.

We define \( \mathcal{E} \) to be **strongly local** if \( \mathcal{E}(\varphi, \psi) = 0 \) for all \( \varphi, \psi \in D(\mathcal{E}) \) and \( a \in \mathbb{R} \) such that \( (\varphi + a\mathbb{1})\psi = 0 \). This condition corresponds to locality in the sense of [BoH].

Strong locality gives a couple of useful implications.

**Lemma 3.3** Suppose \( \mathcal{E} \) is strongly local and regular. Then

\[
\mathcal{E}_N(\varphi) = \sup\{\mathcal{E}_\chi(\varphi) : \chi \in D(\mathcal{E}) \cap C_c(\Omega), 0 \leq \chi \leq 1\}
\]

for all \( \varphi \in D(\mathcal{E}) \).

**Proof** First notice that there are \( \chi_1, \chi_2, \ldots \in D(\mathcal{E}) \cap C_c(\Omega)_+ \) such that \( \chi_n \uparrow \mathbb{1}_\Omega \). Then \( \mathcal{E}_N(\varphi) = \lim_{n \to \infty} \mathcal{E}_{\chi_n}(\varphi) \) for all \( \varphi \in D(\mathcal{E}) \) (see the discussion on page 82 in [ERS], which requires \( \mathcal{E} \) to be regular and strongly local).

Thus if \( \mathcal{E} \) is regular and strongly local then one can replace the set \( C_\Omega \) by the set \( \{\chi \in D(\mathcal{E}) \cap C_c(\Omega) : 0 \leq \chi \leq 1\} \) in the definition of \( \mathcal{E}_N \).

Next we establish that if \( \mathcal{E} \) is strongly local then there is an order relation between \( S^D \) and \( S^N \).

**Proposition 3.4** Let \( \Omega \subseteq \mathbb{R}^d \) be open. If \( \mathcal{E} \) is strongly local then \( \mathcal{E}_D \subseteq \mathcal{E}_N \). Moreover, \( 0 \leq S_t^D \varphi \leq S_t^N \varphi \) for all \( \varphi \in L_2(X)_+ \) and \( t > 0 \).

**Proof** Clearly \( \mathcal{E}_N(\varphi) = \mathcal{E}(\varphi) \) for all \( \varphi \in D_\Omega \). But \( D_\Omega \) is dense in \( D(\mathcal{E}_D) \). Hence it follows from (II) that \( \mathcal{E}_N(\varphi) = \mathcal{E}(\varphi) = \mathcal{E}_D(\varphi) \) for all \( \varphi \in D(\mathcal{E}_D) \). Therefore \( \mathcal{E}_D \subseteq \mathcal{E}_N \).

Since \( L_2(\Omega) \) is \( S^D \)-invariant and \( S_t^D \varphi = 0 \) for all \( \varphi \in L_2(\Omega^c) \), by definition, it suffices to prove the order property of the semigroups for all \( \varphi \in L_2(\Omega)^c \).

Let \( \varepsilon > 0 \) and define the form \( \mathcal{E}_{N\varepsilon} \) by \( \mathcal{E}_{N\varepsilon} = \mathcal{E}_N + \varepsilon \mathcal{E} \). Then \( \mathcal{E}_{N\varepsilon} \) is a Dirichlet form and \( D(\mathcal{E}_{N\varepsilon}) \cap C_c(X) \) is dense in \( C_0(X) \). Moreover, \( D_\Omega(\mathcal{E}_{N\varepsilon}) = D_\Omega(\mathcal{E}_N) \cap (\mathcal{E}_{N\varepsilon})_D = (1 + \varepsilon)D \). Therefore we can apply Proposition 2.1 to deduce that \( 0 \leq S_t^{D+\varepsilon} \varphi \leq S_t^{N\varepsilon} \varphi \) for all \( t > 0 \) and \( \varphi \in L_2(\Omega)_c \) where \( S^{N\varepsilon} \) is the semigroup associated with the Dirichlet form \( \mathcal{E}_{N\varepsilon} \). Since \( \lim_{t \to 0} S_t^{N\varepsilon} = S_t^N \) strongly for all \( t > 0 \) by [Kat], Theorem VIII.3.11, the proposition is established.
Remark 3.5 The semigroup domination property of Proposition 3.4 can be characterized in terms of the forms $E_D$ and $E_N$ by a general result of Ouhabaz (see [Ouh], Theorem 2.24). In particular it follows that $D(E_D)$ is an ideal of $D(E_N)$, i.e. if $0 \leq \varphi \leq \psi$ with $\varphi \in D(E_N)$ and $\psi \in D(E_D)$ then $\varphi \in D(E_D)$.

Under the additional assumption that $E$ is regular one can deduce that $S$-invariance of $L_2(\Omega)$ suffices for equality of $S$ and $S^N$ in restriction to $L_2(\Omega)$.

Proposition 3.6 Let $\Omega \subset \mathbb{R}^d$ be open. Assume $E$ is regular and strongly local and that $L_2(\Omega)$ is $S$-invariant. Then

$$E_N(\varphi) = E(\mathbb{1}_\Omega \varphi)$$

for all $\varphi \in D(E)$. Therefore $E_N$ is closed. Moreover, $S^N$ leaves $L_2(\Omega)$ invariant and $S_t^N \varphi = S_t \varphi$ for all $\varphi \in L_2(\Omega)$ and $t > 0$.

Proof Fix $\varphi \in D(E) \cap C_c(X)$. Then $\mathbb{1}_\Omega \varphi \in D(E)$ since $L_2(\Omega)$ is $S$-invariant. Moreover, $\chi = \mathbb{1}_\Omega \chi$ for all $\chi \in C_\Omega$. Therefore

$$E_N(\varphi) = \lim_{\chi \in C_\Omega} \left( E(\varphi, \chi \varphi) - 2^{-1} E(\chi, \varphi^2) \right)$$

$$= \lim_{\chi \in C_\Omega} \left( E(\varphi, \chi \mathbb{1}_\Omega \varphi) - 2^{-1} E(\mathbb{1}_\Omega \chi, \varphi^2) \right)$$

$$= \lim_{\chi \in C_\Omega} \left( E(\mathbb{1}_\Omega \varphi, \chi \mathbb{1}_\Omega \varphi) - 2^{-1} E(\chi, (\mathbb{1}_\Omega \varphi)^2) \right) = E_N(\mathbb{1}_\Omega \varphi)$$

where we have used locality. Thus $E_N(\varphi) = E_N(\mathbb{1}_\Omega \varphi)$. Next choose $\psi \in D(E) \cap C_c(X)$ with $\psi \geq 1_K$ where $K = \text{supp} \varphi$. Then, replacing $\psi$ by $0 \lor \psi \land 1$ if necessary, one can assume $0 \leq \psi \leq 1$ and $\psi = 1$ on $K$. Set $\chi = \mathbb{1}_\Omega \psi$. Then $\chi \in C_\Omega$ and $\chi = 1$ on $K \cap \Omega$. Therefore $\chi \mathbb{1}_\Omega \varphi = \mathbb{1}_\Omega \varphi$ and

$$E_N(\mathbb{1}_\Omega \varphi) = E(\mathbb{1}_\Omega \varphi, \mathbb{1}_\Omega \varphi) - 2^{-1} E(\chi, (\mathbb{1}_\Omega \varphi)^2) = E(\mathbb{1}_\Omega \varphi)$$

by strong locality. Hence $E_N(\mathbb{1}_\Omega \varphi) = E(\mathbb{1}_\Omega \varphi)$. But in the previous paragraph we established that $E_N(\varphi) = E_N(\mathbb{1}_\Omega \varphi)$. Therefore $E_N(\varphi) = E(\mathbb{1}_\Omega \varphi)$ for all $\varphi \in D(E) \cap C_c(X)$. This equality then extends to all $\varphi \in D(E)$ by regularity of $E$. The remaining statements of the Proposition 3.6 are straightforward. □

We now prove a kind of converse of Proposition 3.2.

Theorem 3.7 Assume $E$ is regular and strongly local. The following conditions are equivalent.

I. $\text{cap}_\Omega(\partial \Omega) = 0$.

II. $S_t^D \varphi = S_t^N \varphi$ for all $\varphi \in L_2(\Omega)$ and all $t > 0$.

Proof The implication II $\Rightarrow$ I is established by Proposition 3.2 without the regularity and strong locality.

I $\Rightarrow$ II. Suppose $\text{cap}_\Omega(\partial \Omega) = 0$. Then the implication III $\Rightarrow$ II of Theorem 1.1 gives $S_t^D \varphi = S_t \varphi$ for all $\varphi \in L_2(\Omega)$ and $t > 0$. But $\text{cap}_\Omega(\partial \Omega) = 0$ also implies that $L_2(\Omega)$ is
$S$-invariant. Therefore Proposition 3.6 gives $S_t \varphi = S_t^N \varphi$ for all $\varphi \in L_2(\Omega)$ and $t > 0$. Hence by combination of these conclusions one obtains Statement III of the theorem. □

It follows from Theorems 1.1 and 3.7 that the relative capacity condition $\text{cap}_\Omega(\partial \Omega) = 0$ is equivalent to $S^D_t \varphi = S_t \varphi$ for all $\varphi \in L_2(\Omega)$ and $t > 0$ or to $S^D_t \varphi = S^N_t \varphi$ for all $\varphi \in L_2(\Omega)$ and $t > 0$. It is not equivalent, however, to $S_t^N \varphi = S_t \varphi$ for all $\varphi \in L_2(\Omega)$ and $t > 0$. A counterexample can be given as follows. Define the form $\mathcal{S}$ associated with $h$.

The foregoing results can be applied to degenerate elliptic operators on $\mathbb{R}^d$.

Let $(c_{kl})$ be a symmetric $d \times d$-matrix with coefficients $c_{kl} \in L_\infty(\mathbb{R}^d)$ such that $C(x) = (c_{kl}(x))$ is positive-definite for almost all $x \in \mathbb{R}^d$. Define the positive quadratic form $h$ by $D(h) = W^{1,2}(\mathbb{R}^d)$ and

$$h(\varphi) = \sum_{k,l=1}^d (\partial_k \varphi, c_{kl} \partial_l \varphi).$$

We call $h$ the degenerate elliptic form with coefficients $(c_{kl})$. Further let $\hat{h}$ denote the relaxation of $h$. It is established in [ERSZ1], Theorem 1.1, that $\hat{h}$ is a regular, strongly local, Dirichlet form. (The relaxation is referred to as the viscosity form in [ERSZ1] and the definition of locality used in this reference corresponds to strong locality as defined in Section 3.) Moreover, the submarkovian semigroup $\mathcal{S}$ associated with $\hat{h}$ is conservative by Theorem 3.7 of [ERSZ2]. Therefore all the statements of Theorems 1.1 and 3.7 are equivalent for $\hat{h}$ and the corresponding elliptic operator $\hat{H}$ and submarkovian semigroup $\mathcal{S}$.

The form $(\hat{h})_D$ corresponding to $\hat{h}$ and the open subset $\Omega \subset \mathbb{R}^d$ is the $D(\hat{h})$-closure of the restriction of $\hat{h}$ to $C^\infty_0(\Omega)$. Therefore, if $c_{kl} = \delta_{kl}$, i.e. if $h = \hat{h}$ is the form of the Laplacian, then $h_D$ corresponds to the usual Laplacian with Dirichlet boundary conditions. It is not the case, however, that $h_N$ always corresponds to the Laplacian with Neumann boundary conditions. The form $l_\Omega$ of the Laplacian with Neumann boundary conditions is usually defined with the domain $D(l_\Omega) = W^{1,2}(\Omega)$. Note that $D(l_N) \subset L_2(\Omega)$. But, by Remark 3.1 one can always write $D(h_N) = D_{\Omega,N} \oplus L_2(\Omega^c)$ with $D_{\Omega,N}$ a subspace of $L_2(\Omega)$. If $|\partial \Omega| > 0$ then clearly $D_{\Omega,N} \neq W^{1,2}(\Omega) = D(l_N)$ and $D_{\Omega,N}$ contains elements which are not in $D(l_N)$. But $D_{\Omega,N}$ can be a strict subset of $W^{1,2}(\Omega)$ even if $|\partial \Omega| = 0$. If, for example, $d = 1$ and $\Omega = (-1,0) \cup (0,1)$ then $D_{\Omega,N} = W^{1,2}(-1,1) \ominus W^{1,2}(-1,0) \oplus W^{1,2}(0,1) = W^{1,2}(\Omega)$.

Theorem 4.4 gives, in principle, a practical way of concluding that the semigroup $S^D$ corresponding to Dirichlet boundary conditions is conservative on $L_\infty(\Omega)$. It suffices to verify that $\text{cap}_{\Omega_n}(\partial \Omega) = 0$. But calculating the relative capacity is not straightforward. The next proposition gives sufficient and practical conditions to make the verification.

**Proposition 4.1** Let $\Omega \subset \mathbb{R}^d$ be open. Let $h_1$ and $h_2$ be degenerate elliptic forms with coefficients $(c_{kl}^{(1)})$ and $(c_{kl}^{(2)})$. Suppose there exists an $a \in \mathbb{R}$ such that $C^{(1)}(x) \leq a C^{(2)}(x)$ for almost every $x \in \Omega$. Moreover, suppose that $\text{cap}_{\Omega,h_2}(\partial \Omega) = 0$. Then the semigroup $S^{(1)}_D$ associated with $(\hat{h}_1)_D$ is conservative.
The proof relies on the fact that the relaxation depends locally on the coefficients of the form.

**Lemma 4.2** Let $h_1$ and $h_2$ be degenerate elliptic forms with coefficients $(c_{kl}^{(1)})$ and $(c_{kl}^{(2)})$. Let $U \subset \mathbb{R}^d$ be an open set and suppose that $c_{kl}^{(1)}|_U = c_{kl}^{(2)}|_U$ for all $k, l \in \{1, \ldots, d\}$. Let $\varphi \in L_2(\mathbb{R}^d)$ and suppose that $\text{supp} \varphi \subset U$. Then $\varphi \in D(h_1)$ if and only if $\varphi \in D(h_2)$ and in this case $\hat{h}_1(\varphi) = \hat{h}_2(\varphi)$.

**Proof** Without loss of generality we may assume that $C^{(2)}(x) = 0$ for all $x \in U^c$. Vogt [Vog] proved that there exists a measurable function $p: \mathbb{R}^d \to \mathbb{R}^{d \times d}$, with values in the orthogonal projections, such that the degenerate elliptic form $k_1$ with coefficients $x \mapsto (C^{(1)}(x))^{1/2} p(x) (C^{(1)}(x))^{1/2}$ is closable and $\hat{h}_1 = \overline{k_1}$. Following the constructive proof in [Vog] it follows that the degenerate elliptic form $k_2$ with coefficients $x \mapsto (C^{(1)}(x))^{1/2} \mathbb{1}_U(x) p(x) (C^{(1)}(x))^{1/2}$ is closable and $\hat{h}_2 = \overline{k_2}$. Then the rest of the proof of the lemma is clear.

**Proof of Proposition 4.1** For all $k, l \in \{1, \ldots, d\}$ define $c_{kl}^{(3)}: \mathbb{R}^d \to \mathbb{R}$ by $c_{kl}^{(3)} = \mathbb{1}_\Omega c_{kl}^{(1)}$. Then $C^{(3)}(x) \leq a C^{(2)}(x)$ for almost every $x \in \mathbb{R}^d$. Hence $\text{cap}_{\Omega, \hat{h}_3} (\partial \Omega) = 0$, where $h_3$ is the degenerate elliptic forms with coefficients $(c_{kl}^{(3)})$. Therefore the semigroup $S^{(3)}_D$ associated with $(\hat{h}_3)_D$ is conservative. But $(\hat{h}_3)_D = (\hat{h}_1)_D$ by Lemma 4.2 Hence the semigroup $S^{(1)}_D$ associated with $(\hat{h}_1)_D$ is conservative.

The assumptions of Proposition 4.1 are satisfied in many cases, see [ElR] Section 3, or under the more stringent condition $\text{cap}_{\Omega} (\partial \Omega) = 0$ see [RoS].

**Acknowledgement**

The authors are indebted to Adam Sikora for explaining that Dirichlet boundary conditions could be used to characterize subspaces invariant under the semigroup generated by a degenerate elliptic operator and for sketching a proof based on finite propagation speed methods.

Part of the work was carried out whilst the first author was visiting the Australian National University with partial support from the Centre for Mathematics and its Applications and part of the work was carried out whilst the second author was visiting the University of Auckland with financial support from the Faculty of Science.

**References**

[Are] ARENDT, W., Heat kernels, 2006. Internet Seminar, [http://tulka.mathematik.uni-ulm.de/2005/lectures/internetseminar.pdf](http://tulka.mathematik.uni-ulm.de/2005/lectures/internetseminar.pdf)

[ArW1] ARENDT, W. and WARMA, M., Dirichlet and Neumann boundary conditions: What is in between? *J. Evol. Équ.* 3 (2003), 119–135.

[ArW2] ——, The Laplacian with Robin boundary conditions on arbitrary domains. *Potential Anal.* 19 (2003), 341–363.
Bouleau, N. and Hirsch, F., *Dirichlet forms and analysis on Wiener space*, vol. 14 of de Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 1991.

Dal Maso, G., *An introduction to Γ-convergence*, vol. 8 of Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston Inc., Boston, MA, 1993.

Ekeland, I. and Temam, R., *Convex analysis and variational problems*. North-Holland Publishing Co., Amsterdam, 1976.

Elst, A.F.M. ter and Robinson, D.W., Invariant subspaces of submarkovian semigroups. *J. Evol. Equ.* 8 (2008), 661–671.

Elst, A.F.M. ter, Robinson, D.W. and Sikora, A., Small time asymptotics of diffusion processes. *J. Evol. Equ.* 7 (2007), 79–112.

Elst, A.F.M. ter, Robinson, D.W., Sikora, A. and Zhu, Y., Dirichlet forms and degenerate elliptic operators. In Koelink, E., Neerven, J. van, Pagter, B. de and Sweers, G., eds., *Partial Differential Equations and Functional Analysis*, vol. 168 of Operator Theory: Advances and Applications. Birkhäuser, 2006, 73–95. Philippe Clement Festschrift.

Elst, A.F.M. ter, Robinson, D.W., Sikora, A. and Zhu, Y., Second-order operators with degenerate coefficients. *Proc. London Math. Soc.* 95 (2007), 299–328.

Fukushima, M., Oshima, Y. and Takeda, M., *Dirichlet forms and symmetric Markov processes*, vol. 19 of de Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 1994.

Kato, T., *Perturbation theory for linear operators*. Second edition, Grundlehren der mathematischen Wissenschaften 132. Springer-Verlag, Berlin etc., 1980.

Ouhabaz, E.-M., *Analysis of heat equations on domains*, vol. 31 of London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2005.

Robinson, D.W. and Sikora, A., Degenerate elliptic operators: capacity, flux and separation. *J. Ramanujan Math. Soc.* 22 (2007), 385–408.

Schmuland, B., On the local property for positivity preserving coercive forms. In MA, Z.M. and Röckner, M., eds., *Dirichlet forms and stochastic processes*. Walter de Gruyter & Co., Berlin, 1995, 345–354. Papers from the International Conference held in Beijing, October 25–31, 1993, and the School on Dirichlet Forms, held in Beijing, October 18–24, 1993.

Simon, B., A canonical decomposition for quadratic forms with applications to monotone convergence theorems. *J. Funct. Anal.* 28 (1978), 377–385.

Vogt, H., The regular part of symmetric forms associated with second order elliptic differential expressions, 2008.