The random case of Conley’s theorem: III. 
Random semiflow case and 
Morse decomposition*

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Abstract

In the first part of this paper, we generalize the results of the author [25, 26] from the random flow case to the random semiflow case, i.e. we obtain Conley decomposition theorem for infinite dimensional random dynamical systems. In the second part, by introducing the backward orbit for random semiflow, we are able to decompose invariant random compact set (e.g. global random attractor) into random Morse sets and connecting orbits between them, which generalizes the Morse decomposition of invariant sets originated from Conley [9] to the random semiflow setting and gives the positive answer to an open problem put forward by Caraballo and Langa [6].

Key words. Random dynamical systems; stochastic partial differential equations; random semiflow; Conley decomposition theorem; Morse decomposition; random attractor

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1 Introduction

This paper is the third and also the final part of the series of papers [25, 26], which aim at studying, in random setting, Conley decomposition theorem and Morse decomposition theorem. Both of these two theorems are originated from Conley [9]. My other two related (joint) works are [27, 28].

Conley’s fundamental theorem of dynamical systems [9] can be stated as follows.

Theorem 1.1. (Conley’s fundamental theorem of dynamical systems). Any flow on a compact metric space decomposes the space into a chain recurrent part and a gradient-like part.

For the importance of the theorem, it was adapted for maps on compact spaces by Franks [17], was later established for maps on locally compact metric spaces by Hurley [20, 21], and was extended by Hurley [22] for semiflows and maps on arbitrary metric spaces. Recently, the author [25, 26] extended Conley decomposition theorem to random dynamical systems (RDS) on Polish spaces. The results of [25, 26] was written for finite dimensional (i.e. random flow) case, so a natural question is whether the theorem also holds for infinite

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dimensional RDS (i.e. random semiflow). This is just what we answer in the first part of the present paper. Through simple observation, we can characterize the random chain recurrent set in terms of random attractors similar to [25, 26]. But when we consider the complete Lyapunov function for random semiflow, the construction in [26] (see expression (4.1) in [26]) is not applicable. The reason is as follows: 1) random semiflow is not defined for \( t < 0 \). 2) The random attractor is only forward invariant. Even if we introduce backward orbits for random semiflow as we do in Section 4, we cannot still define \( \tau(\omega, x) = -\infty \) when \( x \in A(\omega) \) (see (4.1) in [26]) because we cannot conclude that there must be a backward orbit through \( x \) which lie on the attractor. 3) Since backward orbits are not necessarily unique, \( \tau(\omega, x) \) is not well defined. To bypass the obstacles mentioned above, we will construct a new complete Lyapunov function following Conley [9] as well as Arnold and Schmalfuss [3], which has weaker properties than the complete Lyapunov function in [26].

Morse decomposition theorem, originated from Conley [9], is very useful in studying the inner structure of invariant sets, e.g. global attractor (see [18, 35] for comprehensive study of it), which can be stated at the abstract level as follows.

**Theorem 1.2. (Morse decomposition theorem).** Any flow restricted to an invariant compact set decomposes the compact set into finite number of invariant compact subsets (i.e. Morse sets) and connecting orbits between them.

To be more specific, let \( \varphi \) be a flow and \( S \) be an invariant compact set of \( \varphi \). Assume that \((A_i, R_i), i = 1, \ldots, n \) are attractor-repeller pairs of \( \varphi \) with

\[
\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = S \quad \text{and} \quad S = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset.
\]

Then the family \( D = \{M_i\}_{i=1}^n \) of invariant compact subsets of \( S \), defined by

\[
M_i = A_i \cap R_{i-1}, \quad 1 \leq i \leq n,
\]

is called a Morse decomposition of \( S \), and each \( M_i \) is called Morse set. If \( D \) is a Morse decomposition, \( M_D \) is defined to be \( \bigcup_{i=1}^n M_i \), which completely describes the asymptotic behaviors of \( \varphi \). For detailed definitions and further properties, see [9].

For example, consider the differential equation \( \dot{x} = (1 - x^2)x \) with \( x \in \mathbb{R} \) and assume that \( \varphi \) is the flow generated by it, see Figure 1.1. It is clear that \( \mathcal{A} := [-1, 1] \) is the global attractor of \( \varphi \). By restricting \( \varphi \) to \( \mathcal{A} \), it is easy to see that all attractors in \( \mathcal{A} \) are \( \emptyset, \mathcal{A}, \{1\}, \{-1, 1\} \). If we set \( A_0 = \emptyset, A_1 = \{-1\}, A_2 = \{-1, 1\}, A_3 = \mathcal{A} \), then the corresponding repellers are \( R_0 = \mathcal{A}, R_1 = [0, 1], R_2 = \emptyset, R_3 = \emptyset \) and hence the corresponding Morse sets are \( M_1 = \{-1\}, M_2 = \{1\}, M_3 = \emptyset \). Therefore, \( D = \{M_1, M_2, M_3\} \) is a Morse decomposition of \( \mathcal{A} \).

Similar to the deterministic case, the global random attractor is a very crude object as it is the largest invariant random compact set, which means that it includes all the
smaller invariant random sets of the systems and it also contains all the unstable manifolds associated to all these invariant parts of the attractor. The dynamics on the global random attractor is generally so complicated that the structure of global random attractor is really important for us to understand the asymptotic behavior of the system. In fact, this theory, closely related to stochastic bifurcation theory, “is still in its infancy” (Arnold [1]). Some specific models have been studied in this aspect, see, for example, Arnold and Boxler [2], Baxendale [3], Keller and Ochs [24], Schenk-Hoppe [32] etc for the finite dimensional case. Caraballo et al [7] studied the first stochastic bifurcation problem in an infinite dimensional case, i.e. they showed that the global random attractor of Chafee-Infante equation with a multiplicative noise undergoes a stochastic pitchfork bifurcation when the coefficient of linear term passes through the first eigenvalue of the negative Laplacian from below. For general theoretical results, Ochs [29] firstly considered the Morse decomposition of weak random attractors; Crauel et al [12] and Liu et al [28] considered the Morse decomposition for random flow on compact metric space (it is clear that their results also hold when we consider Morse decomposition of global random attractors instead of the entire state space). But these results are written only in random flow case, i.e. finite dimensional case; “it is at all not clear if something similar (i.e. Morse decomposition for global random attractors) could be true for SPDEs” (Caraballo and Langa [6]). The obstacles to this open problem may contain the following: 1) random backward orbits (see Section 4 for the definition) are not introduced for random semiflows to our best knowledge. 2) Generically, there is no uniqueness for random backward orbits. 3) Even for a point (or random variable) in an invariant random set, there may be backward orbit which does not lie on the invariant set through the point (or random variable). In the second part of the paper, by introducing random backward orbits and careful treating the difference between random semiflow and random flow, we give the positive answer to this open problem, i.e. we obtain that Morse decomposition for global random attractors in infinite dimensional case also holds. In fact, we prove that the corresponding result holds for any invariant random compact set.

2 Preliminaries

Throughout the paper, we assume that $X$ is a Polish space, i.e. a separable complete metric space. In this section, we will give some preliminaries for later use. Firstly we give the definition of continuous random dynamical systems (cf Arnold [1]).

**Definition 2.1.** Let $X$ be a metric space with a metric $d_X$. A random dynamical system (RDS), shortly denoted by $\varphi$, consists of two ingredients:

(i) A model of the noise, namely a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(t, \omega) \mapsto \theta_t \omega$ is a measurable flow which leaves $\mathbb{P}$ invariant, i.e. $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

(ii) A model of the system perturbed by noise, namely a cocycle $\varphi$ over $\theta$, i.e. a measurable mapping $\varphi : \mathbb{R}^+ \times \Omega \times X \to X$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$, such that $\varphi(t, \omega, \cdot) = \varphi(t, \omega) : X \to X$ is continuous for arbitrary $t \geq 0$ and $\omega \in \Omega$, moreover

$$\varphi(0, \omega) = \text{id}_X, \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \quad \text{for all} \quad t, s \in \mathbb{R}^+, \omega \in \Omega. \quad (1)$$

In [25] [26], $\varphi$ is defined for every $t \in \mathbb{R}$ (i.e. random flow), which is usually true for the RDS generated by finite dimensional random and stochastic differential equations, i.e.
random and stochastic ODEs. Here, \( \varphi \) is only defined for \( t \geq 0 \) (i.e. random semiflow), which is usually generated by random and stochastic PDEs.

**Definition 2.2.** A random set \( D \) is said to be forward invariant under the RDS \( \varphi \) if 
\[
\varphi(t, \omega)D(\omega) \subset D(\theta t \omega) \text{ for all } t \geq 0 \text{ almost surely; It is said to be invariant if } \varphi(t, \omega)D(\omega) = D(\theta t \omega) \text{ for all } t \geq 0 \text{ almost surely.}
\]

**Definition 2.3.** Assume that \( D \) is a random set, then the omega-limit set of \( D \), \( \Omega_D \), is defined to be 
\[
\Omega_D(\omega) := \bigcap_{t \geq 0} \bigcup_{s \geq t} \varphi(s, \theta^{-s} \omega)D(\theta^{-s} \omega).
\]

**Definition 2.4.** For given two random sets \( D, A \), we say \( A \) (pull-back) attracts \( D \) if 
\[
\lim_{t \to \infty} d(\varphi(t, \theta^{-t} \omega)D(\theta^{-t} \omega) \cap A(\omega)) = 0
\]
holds almost surely, where \( d(A) \) stands for the Hausdorff semi-metric between two sets \( A, B \), i.e. 
\[
d(A) := \sup_{x \in A} \inf_{y \in B} d_X(x, y); \text{ and we say } A \text{ attracts } D \text{ in probability or weakly attracts } D \text{ if }
\mathbb{P} - \lim_{t \to \infty} d(\varphi(t, \omega)D(\omega) \cap A(\theta t \omega)) = 0.
\]

By the measure preserving of \( \theta_t \), it is clear that pull-back attraction implies weak attraction.

**Remark 2.1.** (i) Clearly \( x \in \Omega_D(\omega) \) if and only if there exist sequences \( t_n \to \infty \) and 
\( x_n \in D(\theta^{-t_n} \omega) \) such that \( \varphi(t_n, \theta^{-t_n} \omega)x_n \to x \) as \( n \to \infty \).

(ii) For any two given random sets \( D_1 \) and \( D_2 \), we have 
\[
\Omega_{D_1 \cup D_2}(\omega) = \Omega_{D_1}(\omega) \cup \Omega_{D_2}(\omega)
\]
almost surely. By the definition of omega-limit sets, we clearly have 
\[
\Omega_{D_1 \cup D_2}(\omega) \supset \Omega_{D_1}(\omega) \cup \Omega_{D_2}(\omega),
\]
so we only need to show the converse inclusion holds. To see this, for arbitrary \( x \in \Omega_{D_1 \cup D_2}(\omega) \), there exist sequences \( t_n \to \infty \) and \( x_n \in D_1(\theta^{-t_n} \omega) \cup D_2(\theta^{-t_n} \omega) \) such that \( \varphi(t_n, \theta^{-t_n} \omega)x_n \to x \) as \( n \to \infty \). Hence there exists a subsequence such that 
\[
x_{n_k} \in D_1(\theta^{-t_{n_k}} \omega) \text{ or } x_{n_k} \in D_2(\theta^{-t_{n_k}} \omega)
\]
holds for all \( k = 1, 2, \ldots \) and \( \varphi(t_{n_k}, \theta^{-t_{n_k}} \omega)x_{n_k} \to x, k \to \infty \). That is, \( x \in \Omega_{D_1}(\omega) \) or \( x \in \Omega_{D_2}(\omega) \). Therefore, 
\[
\Omega_{D_1 \cup D_2}(\omega) \supset \Omega_{D_1}(\omega) \cup \Omega_{D_2}(\omega)
\]
(iii) If a random closed set \( E \) pull-back attracts \( D \), then \( \Omega_D \subset E \) almost surely. Indeed, if this is false, i.e. the set \( \tilde{\Omega} := \{ \omega | \Omega_D(\omega) \not\subset E(\omega) \} \) has positive probability, for any \( \omega \in \tilde{\Omega} \), assuming that \( x \in \Omega_D(\omega) \setminus E(\omega) \), then there exist sequences \( t_n \to \infty \) and \( x_n \in D(\theta^{-t_n} \omega) \) such that \( \varphi(t_n, \theta^{-t_n} \omega)x_n \to x \) as \( n \to \infty \). Hence by the definition of Hausdorff semi-metric and the fact that \( E \) pull-back attracts \( D \) we have 
\[
0 < d(\{ x \} \setminus E(\omega)) = \lim_{n \to \infty} d(\varphi(t_n, \theta^{-t_n} \omega)x_n \setminus E(\omega)) \leq \lim_{n \to \infty} d(\varphi(t_n, \theta^{-t_n} \omega)D(\theta^{-t_n} \omega) \setminus E(\omega)) = 0,
\]
a contradiction.
The following proposition comes from [30], which gives a relation between an $\bar{F}^\nu$-measurable function and an $F$-measurable one.

**Proposition 2.1.** Assume that $\nu$ is a positive measure on the measurable space $(X,F)$. Denote $\bar{F}^\nu$ the completion of the $\sigma$-algebra $F$ with respect to the measure $\nu$. If $f$ is an $\bar{F}^\nu$-measurable function, then there exists an $F$-measurable function $g$ such that $f = g$ $\nu$-a.e.

### 3 Conley decomposition for random semiflow

**3.1 Characterization of random chain recurrent set by random attractors (an observation)**

In this subsection, through simple observation, we obtain that, for random semiflow, the characterization of random chain recurrent set by random attractors holds similar to that in random flow case presented in [25, 26].

**Definition 3.1.** ([26]) (i) Assume that $\epsilon > 0$ is a random variable. A random open set $U$ is called $\epsilon$-absorbing if there exists a random variable $T > 0$ such that $U$ contains the $\epsilon$-neighborhood of $U_T(\omega) := \bigcup_{t \geq T} \varphi(t, \theta_t \omega)U\theta_t(\omega)$, i.e.

$$B_\epsilon(U_T(\omega)) \subset U(\omega).$$

And we call a random open set $U$ absorbing if it is $\epsilon$-absorbing for some random variable $\epsilon > 0$.

(ii) An invariant random closed set $A$ is called an (local) attractor if there exists an absorbing neighborhood $U$ of $A$ such that $A(\omega) = \Omega_U(\omega)$. And we call

$$B(A, U)(\omega) := \{x \mid \varphi(t, \omega)x \in U(\theta_t) \text{ for some } t \geq 0\}$$

the basin of attraction of $A$ with respect to $U$.

**Definition 3.2.** Assume that $A$ is an attractor with a random absorbing neighborhood $U$ and the basin of attraction of $A$ with respect to $U$, $B(A, U)$. Then we call

$$R(\omega) := X \setminus B(A, U)(\omega)$$

the repeller corresponding to $A$ with respect to $U$, and call $(A, R)$ an attractor-repeller pair of $\varphi$ (with respect to $U$).

**Remark 3.1.** (i) Generally speaking, a random attractor is only forward invariant, not necessarily invariant. In particular, when $\overline{U}$ is a random compact set or $\varphi$ is a random “flow” instead of “semiflow”, the random attractor is invariant, see Lemma 3.2, Proposition 3.6 and Remark 3.7 in [14] for details.

(ii) The repeller corresponding to $A, R$, is also forward invariant. In fact, if there exists some $x_0 \in R(\omega)$ and $t_0 > 0$ such that $\varphi(t_0, \omega)x_0 \notin R(\theta_{t_0} \omega)$, i.e. $\varphi(t_0, \omega)x_0 \in B(A, U)(\theta_{t_0} \omega)$, then by the definition of $B(A, U)$ we have

$$\varphi(t_1, \theta_{t_0} \omega) \circ \varphi(t_0, \omega)x_0 \in U(\theta_{t_1} \circ \theta_{t_0} \omega)$$

for some $t_1 \geq 0$, that is

$$\varphi(t_0 + t_1, \omega)x_0 \in U(\theta_{t_0 + t_1} \omega).$$
Hence we have $x_0 \in B(A, U)(\omega)$, a contradiction.

(iii) It is easy to see that Lemma 5.2 in [26] also holds when $\varphi$ is a random semiflow, i.e. for any given random attractor $A$ there exists a forward invariant absorbing neighborhood $U$ of $A$ such that $\Omega_U(\omega) = A(\omega)$. In this case, it follows immediately that $B(A, U)$ is a forward invariant random open set by the the definition of $B(A, U)$ and the forward invariance of $U$.

Similar to the proofs of Lemmas 3.4, 3.6 and 3.7 in [25], we can obtain that Theorem 6 in [26] also holds for random semiflow. In fact, the proofs of these lemmas are the same for both random semiflow and random flow. Hence by (iii) of Remark 3.1 we have the following

**Theorem 3.1.** Assume that $X$ is a Polish space and $\varphi$ is a random semiflow on $X$. Assume that $U$ is a forward invariant random absorbing set, $A$ is the random attractor determined by $U$ and $R$ is the random repeller corresponding to $A$ with respect to $U$, then

$$\mathcal{C}R_{\varphi} = \bigcap (A \cup R) \text{ almost surely,}$$

(2)

where $\mathcal{C}R_{\varphi}$ denotes the random chain recurrent set of $\varphi$ (see [25] for the definition for random flow; this definition is also applicable to random semiflow), and the intersection is taken over all forward invariant random absorbing sets.

**Remark 3.2.** In above theorem, we characterize the random chain recurrent set by random attractors for random semiflow on $X$. It is easy to see that the corresponding result also holds when RDS $\varphi$ is restricted to an invariant random compact set.

### 3.2 Complete Lyapunov function for random semiflow

Firstly for arbitrary random attractor $A$, we will show that we can construct a Lyapunov function $\tilde{l}$ for $\varphi$ with respect to $A$, see Lemmas 3.1 and 3.2 for the construction.

**Lemma 3.1.** Assume that $\mu$ is a positive measure on the product measurable space $(\Omega \times X, \mathcal{F} \times \mathcal{B}(X))$, $A$ is a random attractor with a forward invariant random absorbing neighborhood $U$, and $R$ is the random repeller corresponding to $A$ with respect to $U$. Then there exists an $\mathcal{F} \times \mathcal{B}(X)$-measurable function $\tilde{l}(\omega, x)$ such that

$$\tilde{l}(\omega, x) = \begin{cases} 0, & x \in A(\omega), \\ 1, & x \in R(\omega), \\ 0 < \tilde{l}(\omega, x) \leq 1, & x \in X \setminus (A(\omega) \cup R(\omega)). \end{cases}$$

Moreover, $\tilde{l}(\omega, x)$ is non-increasing $\mu$-a.e. along the orbits of the skew-product semiflow $\Theta$ corresponding to $\varphi$, i.e. $\Theta_t(\omega, x) := (\theta_t \omega, \varphi(t, \omega) x)$.

**Proof.** Let

$$\phi(\omega, x) := \frac{\text{dist}_X(x, A(\omega))}{\text{dist}_X(x, A(\omega)) + \text{dist}_X(x, R(\omega))}.$$ 

It is clear that

$$\phi(\omega, x) = \begin{cases} 0, & x \in A(\omega), \\ 1, & x \in R(\omega). \end{cases}$$

Let

$$l(\omega, x) = \sup_{i \geq 0} \phi(\theta_i \omega, \varphi(t, \omega)x).$$
By the forward invariance of $A$ and $R$, we have

$$l(\omega, x) = \begin{cases} 
0, & x \in A(\omega), \\
1, & x \in R(\omega), \\
0 < l(\omega, x) \leq 1, & x \in X \setminus (A(\omega) \cup R(\omega)).
\end{cases} \tag{3}$$

By the definition of $l(\omega, x)$, it is clear that

$$l(\theta_t \omega, \varphi(t, \omega)x) \leq l(\omega, x), \ \forall t > 0,$$

i.e. $l(\omega, x)$ is non-increasing along the orbits of $\Theta$.

Next we prove the measurability of $l(\omega, x)$. For $\forall a \in \mathbb{R}^+$, we have

$$\{(\omega, x) \mid l(\omega, x) > a\} = \{(\omega, x) \mid \sup_{t \geq 0} \phi(\theta_t \omega, \varphi(t, \omega)x) > a\}$$

$$= \Pi_{\Omega \times X} \{(t, \omega, x) \mid \phi(\theta_t \omega, \varphi(t, \omega)x) > a, \ t \geq 0\},$$

where $\Pi_{\Omega \times X}$ stands for the canonical projection of $\mathbb{R}^+ \times \Omega \times X$ to $\Omega \times X$. By the measurability of the maps

$$\phi : (\omega, x) \mapsto \phi(\omega, x), \ \theta : (t, \omega) \mapsto \theta_t \omega, \ \varphi : (t, \omega, x) \mapsto \varphi(t, \omega)x,$$

we know that the map

$$(t, \omega, x) \mapsto \phi(\theta_t \omega, \varphi(t, \omega)x)$$

is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X)$-measurable. Hence we obtain that

$$\{(t, \omega, x) \mid \phi(\theta_t \omega, \varphi(t, \omega)x) > a, \ t \geq 0\} \in \mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X).$$

By the projection theorem (see Proposition 2.3 in [25], which is originated from [8]) we have

$$\{(\omega, x) \mid l(\omega, x) > a\} \in (\mathcal{F} \times \mathcal{B}(X))^u,$$

that is, $l(\omega, x)$ is $(\mathcal{F} \times \mathcal{B}(X))^u$-measurable, where the superscript $u$ denotes the universal $\sigma$-algebra. By Proposition 2.24, there exists an $\mathcal{F} \times \mathcal{B}(X)$-measurable function $\tilde{l}(\omega, x) = l(\omega, x)$ $\mu$-a.e. This completes the proof of the lemma. \hfill \square

**Remark 3.3.** The function $\tilde{l}(\omega, x)$ obtained in Lemma 3.1 has two shortcomings:

- $0 < \tilde{l}(\omega, x) \leq 1$ when $x \in X \setminus (A(\omega) \cup R(\omega))$, not the desired $0 < \tilde{l}(\omega, x) < 1$;
- $\tilde{l}(\omega, x)$ is non-increasing along the orbits of $\Theta$ when $x \in X \setminus (A(\omega) \cup R(\omega))$, but not necessarily strictly decreasing.

In fact, in the following lemma we will show that the two shortcomings mentioned in Remark 3.3 can be partially improved, that is, $0 < \tilde{l}(\omega, x) < 1$ and strict decreasing along the orbits of $\Theta$ when $x \in X \setminus (A(\omega) \cup R(\omega))$ can hold in weaker sense—in probability.

**Lemma 3.2.** Assume that $A$ is a random attractor, and $R$ is the corresponding random repeller of $A$ with respect to some random absorbing neighborhood $U$. Then there exists a Lyapunov function $\hat{l}(\omega, x)$ which has the same properties as $\tilde{l}(\omega, x)$ constructed in Lemma 3.1. Furthermore, for arbitrary $\epsilon > 0$ and arbitrary random variable $x \in X \setminus (A \cup R)$, there exists an $\Omega_{\epsilon} \subset \Omega$ with the measure $\mathbb{P}(\Omega_{\epsilon}) < \epsilon$ such that on $\Omega \setminus \Omega_{\epsilon}$, $\hat{l}(\omega, x)$ has the following two fine properties:

$$0 < \hat{l}(\omega, x(\omega)) < 1; \tag{4}$$

$$\hat{l}(\theta_t \omega, \varphi(t, \omega)x(\omega)) < \hat{l}(\omega, x(\omega)), \ \forall t > 0. \tag{5}$$
Proof. Let 

\[ \hat{l}(\omega, x) := \frac{1}{2} \left[ \tilde{l}(\omega, x) + \int_0^{+\infty} e^{-t} \tilde{l}(\theta_t \omega, \varphi(t, \omega)x)dt \right]. \]  

(6)

It is clear that \( \hat{l} \) takes value 0 on \( A \) and takes value 1 on \( R \), respectively, by the forward invariance of \( A \) and \( R \). By the monotonicity of \( \tilde{l} \) along the orbits of \( \varphi \), it follows that \( \hat{l} \) has the same monotonicity and that

\[ \int_0^{+\infty} e^{-t} \tilde{l}(\theta_t \omega, \varphi(t, \omega)x)dt \leq \int_0^{+\infty} e^{-t} dt \cdot \tilde{l}(\omega, x) = \tilde{l}(\omega, x). \]

Hence we have \( \frac{1}{2} \tilde{l}(\omega, x) \leq \hat{l}(\omega, x) \leq \tilde{l}(\omega, x) \), which implies that \( 0 < \hat{l}(\omega, x) \leq 1 \) when \( x \in X \setminus (A(\omega) \cup R(\omega)) \). That is, \( \hat{l}(\omega, x) \) has the same properties as \( \tilde{l}(\omega, x) \).

For arbitrary random variable \( x \in B(A, U) \), we have

\[ \mathbb{P} - \lim_{t \to -\infty} \hat{l}(\theta_t \omega, \varphi(t, \omega)x) = 0. \]  

(7)

In fact, by the definition of \( B(A, U) \), there exists a \( t = t(\omega) \) such that \( \varphi(t, \omega)x(\omega) \in U(\theta_t \omega) \). Then for \( \forall \epsilon > 0 \) we have

\[ \lim_{s \to -\infty} \mathbb{P}\{ \omega \mid \text{dist}_X(\varphi(s, \omega)x(\omega), A(\theta_s \omega)) > \epsilon \} = \lim_{s \to -\infty} \mathbb{P}\{ \omega \mid \text{dist}_X(\varphi(s + t, \omega)x(\omega), A(\theta_{s+t} \omega)) > \epsilon \} = \lim_{s \to -\infty} \mathbb{P}\{ \omega \mid \text{dist}_X(\varphi(s, \theta_t \omega) \circ \varphi(t, \omega)x(\omega), A(\theta_s \circ \theta_t \omega)) > \epsilon \} \leq \lim_{s \to -\infty} \mathbb{P}\{ \omega \mid d(\varphi(s, \theta_t \omega)U(\theta_t \omega)|A(\theta_s \circ \theta_t \omega)) > \epsilon \} = 0, \]

where the inequality holds by the definition of Hausdorff semi-metric and the last equality holds by the fact \( \Omega_U = A \) and the measure preserving of \( \theta_t \).

Now we will prove that (4) and (5) hold. The idea of the proof is borrowed from [3].

If (4) does not hold, i.e. there existing some \( \epsilon_0 > 0 \) (without loss of generality, suppose \( \epsilon_0 \leq 1 \)) and some random variable \( x_0 \in B(A, U) \setminus A \), for arbitrary \( \Omega_{\epsilon_0} \) with \( \mathbb{P}(\Omega_{\epsilon_0}) < \epsilon_0 \), there exists an \( \tilde{\Omega} \subset \Omega \setminus \Omega_{\epsilon_0} \) with \( \mathbb{P}(\tilde{\Omega}) > 0 \) such that on \( \tilde{\Omega} \)

\[ \hat{l}(\omega, x_0(\omega)) = 1. \]

Then by (6), we have

\[ \hat{l}(\theta_t \omega, \varphi(t, \omega)x_0(\omega)) = 1, \ \forall \omega \in \tilde{\Omega}, \ \text{Leb} - \text{almost all } t \in \mathbb{R}^+, \]

a contradiction to (7).

If (5) does not hold, similar to the proof of (4), there existing some \( \epsilon_0 > 0 \) and some random variable \( x_0 \in B(A) \setminus A \), for arbitrary \( \Omega_{\epsilon_0} \) with \( \mathbb{P}(\Omega_{\epsilon_0}) < \epsilon_0 \), there exists an \( \tilde{\Omega} \subset \Omega \setminus \Omega_{\epsilon_0} \) with \( \mathbb{P}(\tilde{\Omega}) > 0 \) such that for arbitrary \( \omega \in \tilde{\Omega} \), there exists \( t_0 = t_0(\omega) > 0 \) satisfying

\[ \hat{l}(\theta_{t_0} \omega, \varphi(t_0, \omega)x_0(\omega)) = \hat{l}(\omega, x_0(\omega)). \]

By the monotonicity of \( \hat{l} \) along the orbits of \( \Theta \), we have

\[ \hat{l}(\theta_s \omega, \varphi(s, \omega)x_0(\omega)) = \hat{l}(\omega, x_0(\omega)) > 0 \text{ for all } 0 \leq s \leq t_0, \]  

(9)
and
\[ \tilde{l}(\theta_{s+t_0}\omega, \varphi(s + t_0, \omega)x_0(\omega)) = \tilde{l}(\theta_s\omega, \varphi(s, \omega)x_0(\omega)) \] for Leb-almost all \( s \geq 0 \).

Hence
\[ \tilde{l}(\theta_{nt_0+s}\omega, \varphi(nt_0 + s, \omega)x_0(\omega)) = \tilde{l}(\theta_s\omega, \varphi(s, \omega)x_0(\omega)) \] (10)
for all \( n \in \mathbb{N} \) and for Leb-almost all \( s \geq 0 \). Therefore for each \( \omega \in \tilde{\Omega} \), \( \exists \tau = \tau(\omega) \geq 0 \) such that for which both (9) and (11) hold, i.e. we have
\[ \tilde{l}(\theta_{nt_0+\tau}\omega, \varphi(nt_0 + \tau, \omega)x_0(\omega)) = \tilde{l}(\omega, x_0(\omega)) > 0, \quad \forall n \in \mathbb{N}. \] (11)

Letting \( n \to \infty \) in (11), we obtain a contradiction to (7). This terminates the proof of the lemma. \( \square \)

By completely similar to [26] in random flow case, we can also construct complete Lyapunov function for random semiflow; furthermore, we can discuss chain transitive components completely similar to random flow case. In fact, it is clear that semiflow or flow is not relevant in these steps. Hence here we only state associated results and omit details of the proof.

**Theorem 3.2.** Assume that \( X \) is a Polish space and \( \varphi \) is a random semiflow on \( X \). Then there exists a complete Lyapunov function \( L : \Omega \times X \to \mathbb{R}^+ \) for \( \varphi \) with the following properties:

(i) \( L \) is an \( \mathcal{F} \times \mathcal{B}(X) \)-measurable function;
(ii) \( L(\theta_t\omega, \varphi(t, \omega)x) \leq L(\omega, x) \) for \( \forall t > 0 \) \( \mu \)-a.e., recalling that \( \mu \) is defined in Lemma 3.1;
(iii) \( L(\theta_t\omega, \varphi(t, \omega)x) = L(\omega, x) \) for \( \forall t > 0 \) when \( x \in \mathcal{C}\mathcal{R}_\varphi(\omega) \);
(iv) If the random variable \( x \) is completely random non-chain recurrent, i.e. \( x(\omega) \in X \setminus \mathcal{C}\mathcal{R}_\varphi(\omega) \) \( \mathbb{P} \)-a.s., then for arbitrary \( \epsilon > 0 \) there exists an \( \Omega_\epsilon \subset \Omega \) satisfying \( \mathbb{P}(\Omega_\epsilon) < \epsilon \) such that for arbitrary \( \omega \in \Omega \setminus \Omega_\epsilon \), the following holds:
\[ L(\theta_t\omega, \varphi(t, \omega)x(\omega)) < L(\omega, x(\omega)), \quad \forall t > 0. \]

(v) The range of \( L \) on \( \mathcal{C}\mathcal{R}_\varphi(\omega) \) is a compact nowhere dense subset of \([0, 1] \);
(vi) \( L \) separates different random chain transitive components of \( \varphi \);
(vii) If \( C \) and \( C' \) are distinct random chain transitive components of \( \varphi \) with the property that for arbitrary random variables \( \epsilon, T > 0 \) there is an \( \epsilon\)-\( T \)-chain from \( C \) to \( C' \) \( \mathbb{P} \)-a.s. then \( L(\Omega, C) > L(\Omega, C') \).

**Remark 3.4.** Among the properties of complete Lyapunov function for random semiflow, (ii) and (iv) are weaker than that of random flow case obtained in [26]. The reason is that the Lyapunov function for attractor-repeller pair constructed in Lemma 3.2 has weaker properties.

## 4 Morse decomposition of global random attractors

Global random attractors were introduced by Crauel and Flandoli [14], Schmalfuss [33], and were studied for many SDEs, see [13, 16, 32, 34], among others. First let us recall the definition of global random attractor.

**Definition 4.1.** ([14]) Assume that \( \varphi \) is a random semiflow on a Polish space \( X \), then a random compact set \( A \) is called a global random attractor for \( \varphi \) if
A is invariant, i.e.

$$\varphi(t, \omega)A(\omega) = A(\theta_t \omega), \forall t \geq 0$$ (12)

for almost all $\omega \in \Omega$;

A pull-back attracts every bounded deterministic set, i.e. for any bounded deterministic set $B \subset X$, we have

$$\lim_{t \to \infty} d(\varphi(t, \theta_{-t} \omega)B|A(\omega)) = 0$$ (13)

almost surely.

The global random attractor for RDS $\varphi$ is the minimal random compact set which attracts all the bounded deterministic sets and it is the largest random compact set which is invariant in the sense of (12), see [13] for details. The global random attractor is uniquely determined by attracting deterministic compact sets, see [10] for details.

We now introduce “backward orbit” for random semiflow:

- For fixed $\omega$ and $x$, a mapping $\sigma_r(\omega) : \mathbb{R}^- \to X$ is called a backward orbit of $\varphi$ through $x$ driven by $\omega$ if it satisfies the cocycle property:

$$\sigma_0(\omega) = x, \sigma_{t+s}(\omega) = \varphi(s, \theta_t \omega) \circ \sigma_t(\omega) \text{ for } \forall t \leq 0, s \geq 0, t + s \leq 0.$$

- Let $M$ denote the set of all $X$-valued random variables and $x \in M$. A mapping $\sigma : \mathbb{R}^- \to M$ is called a backward orbit of $\varphi$ through $x$ if for all $\omega \in \Omega$, the following cocycle property holds:

$$\sigma_0(\omega) = x(\omega), \sigma_{t+s}(\omega) = \varphi(s, \theta_t \omega) \circ \sigma_t(\omega) \text{ for } \forall t \leq 0, s \geq 0, t + s \leq 0.$$

Also we can introduce “entire orbit” for random semiflow:

- For fixed $\omega$ and $x$, a mapping $\sigma_r(\omega) : \mathbb{R} \to X$ is called an entire orbit of $\varphi$ through $x$ driven by $\omega$ if it satisfies the cocycle property:

$$\sigma_0(\omega) = x, \sigma_{t+s}(\omega) = \varphi(s, \theta_t \omega) \circ \sigma_t(\omega) \text{ for } \forall t \in \mathbb{R}, s \geq 0.$$

- Let $x \in M$. A mapping $\sigma : \mathbb{R} \to M$ is called an entire orbit of $\varphi$ through $x$ if for all $\omega \in \Omega$, the following cocycle property holds:

$$\sigma_0(\omega) = x(\omega), \sigma_{t+s}(\omega) = \varphi(s, \theta_t \omega) \circ \sigma_t(\omega) \text{ for } \forall t \in \mathbb{R}, s \geq 0.$$

By the cocycle property of $\sigma$, it is clear that for arbitrary $t \geq 0$ and $\omega \in \Omega$ we have

$$\sigma_t(\omega) = \varphi(t, \omega) \circ \sigma_0(\omega).$$

That is, when an entire orbit $\sigma$ of $\varphi$ is restricted to $\mathbb{R}^+$ (called forward orbit), then it coincides with the orbit of $\varphi$, which is the same as the deterministic case.

We can give an alternative definition of (forward, backward) invariant sets for random semiflow:

- A random set $D$ is called forward invariant if $D = D^+_{\varphi}$ almost surely, where

$$D^+_{\varphi}(\omega) := \{x|\varphi(t, \omega)x \in D(\theta_t \omega) \text{ for all } t \geq 0\};$$
A random set $D$ is called **backward invariant** if $D = D_{\varphi}^-$ almost surely, where
\[D_{\varphi}(\omega) := \{ x | \exists \text{ a backward orbit } \sigma \text{ in } D \text{ through } x, \text{ i.e. } \sigma_t(\omega) \in D(\theta_t \omega), \forall t \leq 0 \};\]

A random set $D$ is called **invariant** if $D = D_{\varphi}$ almost surely, where
\[D_{\varphi}(\omega) := \{ x | \exists \text{ an entire orbit } \sigma \text{ in } D \text{ through } x, \text{ i.e. } \sigma_t(\omega) \in D(\theta_t \omega), \forall t \in \mathbb{R} \} .\]

**Remark 4.1.** (i) Clearly a random set $D$ is invariant if and only if it is both forward invariant and backward invariant. It is easy to verify that forward invariant and invariant sets defined above coincide with that of Definition 2.2, which is convenient for us to choose appropriate definition in the sequel.
(ii) If $D_1$ and $D_2$ are forward invariant, then clearly $D_1 \cup D_2$ and $D_1 \cap D_2$ are forward invariant; If $D_1$ and $D_2$ are invariant, then clearly $D_1 \cup D_2$ is invariant, while $D_1 \cap D_2$ is not necessarily invariant (since $D_1 \cap D_2$ is not necessarily backward invariant), which differs from random flow case, see page 35 in [1].

**Lemma 4.1.** Assume that $D$ is a forward invariant random compact set, then for any point on $\Omega_D$ there exists a backward orbit lying on $\Omega_D$ through this point.

**Proof.** By the definition of omega-limit set we know that, for any given $x \in \Omega_D(\omega)$, there exist sequences $t_n \rightarrow +\infty$, $x_n \in D(\theta_{-t_n} \omega)$ such that $\varphi(t_n, \theta_{-t_n} \omega)x_n \rightarrow x$, $n \rightarrow \infty$. For arbitrary $k \in \mathbb{Z}^-$, there exists an $N_0$ such that the set $\{ \varphi(t_n + k, \theta_{-t_n} \omega)x_n | n \geq N_0 \}$ is pre-compact, i.e. the closure of $\{ \varphi(t_n + k, \theta_{-t_n} \omega)x_n | n \geq N_0 \}$ is compact. In fact, by the forward invariance of $D$, when $t_n + k \geq 0$, we have
\[\{ \varphi(t_n + k, \theta_{-t_n} \omega)x_n | n \geq N_0 \} \subset D(\theta_k \omega).\]

By taking a subsequence, $\lim_{n \rightarrow \infty} \varphi(t_n + k, \theta_{-t_n} \omega)x_n$ exists and denote by $\tilde{x}_k$ the limit. Since $D(\theta_k \omega)$ is compact, we have $\tilde{x}_k \in D(\theta_k \omega)$.

For $k \leq t \leq k + 1$, $k \in \mathbb{Z}^-$, let $\sigma_t(\omega) = \varphi(t - k, \theta_k \omega)\tilde{x}_k$. Clearly $\sigma$ is a backward orbit of $\varphi$ through $x$. Moreover the backward orbit obtained in this way lies on $\Omega_D$. In fact, by the definition of $\Omega_D$, it is clear that $\tilde{x}_k \in \Omega_D(\theta_k \omega)$. Hence by the invariance of $\Omega_D$, we have $\sigma_t(\omega) \in \Omega_D(\theta_t \omega)$, $t \leq 0$. □

**Corollary 4.1.** Assume that $D$ is an invariant random compact set, then for any point on $D$, there exists a backward orbit lying on $D$ through this point.

**Proof.** It follows immediately from Lemma 4.1. □

In contrast to Lemma 4.1, given a forward invariant random compact set $D$, a natural question is, for any random variable $x \in \Omega_D$, does there exist a backward orbit lying on $\Omega_D$ through $x$? The answer is yes, see the following lemma.

**Lemma 4.2.** Assume that $D$ is a forward invariant random compact set, then for any random variable on $\Omega_D$ there exists a backward orbit lying on $\Omega_D$ through this random variable.

**Proof.** By Lemma 4.1, we know that, for given $k \in \mathbb{Z}^-$ and for each $\omega$, there exists an $\tilde{x}_k(\omega) \in \Omega_D(\theta_k \omega)$ from which we obtain a backward orbit from time $k$ to time 0 (present time). Hence we need only to show that we can select appropriate $\tilde{x}_k$ such that the map $\omega \mapsto \tilde{x}_k(\omega)$ is measurable. In other words, we need to show $\sigma_k \in M$, which implies $\sigma_s \in M$, $\forall k \leq s \leq 0$. For arbitrary $t > 0$, denote $\varphi^{-1}(t, \omega)x$ the preimage of $x$ under
ϕ. Consider the skew-product semiflow Θ corresponding to ϕ (see Lemma 3.1) which is an $\mathcal{F} \times \mathcal{B}(X)$-measurable mapping from $\Omega \times X$ to itself for fixed $t \geq 0$. The preimage of $(\omega, x)$ under $\Theta_t$ is $\Theta_t^{-1}(\omega, x) := (\theta_t \omega, \varphi^{-1}(t, \omega)x)$. Since $\Omega_D$ is a random compact set, we have $\text{graph}(\Omega_D) := \{(\omega, x) | x \in \Omega_D(\omega)\} \in \mathcal{F} \times \mathcal{B}(X)$, see page 59 of [8] or Proposition 2.4 in [11]. Hence we have $\Theta_t^{-1}(\text{graph}(\Omega_D)) \in \mathcal{F} \times \mathcal{B}(X)$ by the measurability of $\Theta_t$. It is clear that $\text{graph}(\varphi^{-1}(t, \omega)\Omega_D(\theta_t \omega)) = \Theta_t^{-1}(\text{graph}(\Omega_D))$, where

$\Omega_t^D(\omega) := \varphi^{-1}(t, \omega)\Omega_D(\theta_t \omega)$.

Therefore $\Omega_t^D$ is an $\mathcal{F}^u$-measurable random compact set, see page 59 of [8] or Proposition 2.4 in [11] again. By Lemma 2.7 in [11] we know that we may assume that $\Omega_t^D$ is an $\mathcal{F}$-measurable random compact set. In particular, for given $k \in \mathbb{Z}^-$, $\Omega_t^D \cap \Omega_D$ is a nonempty random compact set by (v) of Proposition 2.1 in [25]. By the measurable selection theorem (see Proposition 2.2 in [25]), we can choose a random variable $\tilde{x}_k \in \Omega_{-k}^D \cap \Omega_D$. This completes the proof. □

So we have

**Corollary 4.2.** Assume that $D$ is an invariant random compact set, then for any random variable on $D$, there exists a backward orbit lying on $D$ through this random variable.

Throughout this section, we use $S$ to denote the invariant random compact set we will decompose, say, $S$ is a global random attractor. By Corollaries 1.1 and 1.2 for any point (random variable) on $S$, there exists backward orbit lying on $S$ through this point (random variable). Afterwards, when we say backward orbits, we refer those lying on $S$ unless otherwise stated (since there may be backward orbit not lying on $S$ but lying on the entire state space — $X$).

**Definition 4.2.** An invariant random compact set $A \subset S$ is called a (local) attractor if there exists a random open neighborhood $U$ of $A$ relative to $S$ such that $\Omega_t(\omega) = A(\omega)$. (By Lemma 3.1 in [20], without loss of generality, we can assume that $U$ is forward invariant.) The basin of attraction of $A$ is defined by

$B(A)(\omega) := \{x \in S(\omega) | \varphi(t, \omega)x \in U(\theta_t \omega) \text{ for some } t \geq 0\}$

and the repeller $R$ corresponding to $A$ is defined by

$R(\omega) = S(\omega) \setminus B(A)(\omega)$.

$(A, R)$ is called an attractor-repeller pair.

Note that since $S$ is a random compact set, by Lemma 3.2 in [25] (the proof of Lemma 3.2 is also applicable here), $B(A)$ is independent of the choice of $U$.

**Lemma 4.3.** Assume that $(A, R)$ is an attractor-repeller pair in $S$, then $A$, $B(A)$, and $R$ are invariant random sets.
Proof. (i) The invariance of $A$ follows immediately from its definition.
(ii) The forward invariance of $B(A)$ follows directly from the definition of $B(A)$ and the forward invariance of $U$. For arbitrary $x \in B(A)(\omega)$, if for any backward orbit $\sigma$ through $x$, there exists some $t_0 < 0$ such that $\sigma t_0 \in R(\theta t_0 \omega)$. By the definition of $R$, we know that any point in $R$ can not enter into $B(A)$ in positive time, so we obtain that $R$ is forward invariant. Therefore,

$$
\varphi(-t_0, \theta t_0 \omega) \sigma t_0 (\omega) = \sigma_0 (\omega) = x \in R(\omega),
$$
a contradiction. That is, $B(A)$ is backward invariant.
(iii) By (ii) we only need to show the backward invariance of $R$. For arbitrary $x \in R(\omega)$, if for any backward orbit $\sigma$ through $x$, there exists some $t_0 < 0$ such that $\sigma t_0 \in B(A)(\theta t_0 \omega)$. Then by the forward invariance of $B(A)$, we have

$$
\varphi(-t_0, \theta t_0 \omega) \sigma t_0 (\omega) = \sigma_0 (\omega) = x \in B(A)(\omega),
$$
a contradiction. Hence $R$ is backward invariant. \hfill \square

Definition 4.3. Assume that $x$ is a random variable in $S$, and $\sigma$ is an entire orbit through $x$. Then the omega-limit set $\Omega_\sigma$ and the alpha-limit set $\Omega^*_\sigma$ of $\sigma$ are defined to be

$$
\Omega_\sigma(\omega) := \bigcap_{T \geq 0} \bigcup_{t \geq T} \{ \sigma_t(\theta - t \omega) \},
$$

and

$$
\Omega^*_\sigma(\omega) := \bigcap_{T \geq 0} \bigcup_{t \geq T} \{ \sigma_{-t}(\theta t \omega) \},
$$

respectively.

It is clear that

$$
\Omega_\sigma(\omega) := \bigcap_{T \geq 0} \bigcup_{t \geq T} \{ \varphi(t, \theta - t \omega) x(\theta - t \omega) \},
$$
i.e. the omega-limit set of $\sigma$ only depends on the random variable $x$, so $\Omega_\sigma$ can also be denoted by $\Omega_x$; while the alpha-limit set depends on the entire orbit $\sigma$. Clearly a point $y \in \Omega_\sigma(\omega)$ (respectively $y \in \Omega^*_\sigma(\omega)$) if and only if there exist sequences $t_n \rightarrow +\infty$ (respectively $t_n \rightarrow -\infty$) and $y_n = \sigma t_n(\theta - t_n \omega)$ such that $y_n \rightarrow y$ as $n \rightarrow +\infty$.

Lemma 4.4. Assume that $x$ is a random variable in $S$, and $\sigma$ is an entire orbit through $x$. Then $\Omega_\sigma$ and $\Omega^*_\sigma$ are invariant random compact sets.

Proof. The random variable $x$ can be regarded as a random set consisting of just a single point, so $\Omega_\sigma$ is an invariant random compact set.

For arbitrary $y \in \Omega^*_\sigma(\omega)$, there exist sequences $t_n \rightarrow +\infty$ and $y_n = \sigma_{-t_n}(\theta t_n \omega)$ such that $y_n \rightarrow y$ as $n \rightarrow +\infty$. For $s > 0$, we have

$$
\varphi(s, \omega) y = \lim_{n \rightarrow +\infty} \varphi(s, \omega) \circ \sigma_{-t_n}(\theta t_n \omega)
= \lim_{n \rightarrow +\infty} \varphi(s, \omega) \circ \sigma_{-t_n}(\theta t_n - s \circ \theta s \omega)
= \lim_{n \rightarrow +\infty} \sigma_{-t_n}(\theta t_n - s \circ \theta s \omega)
= \lim_{n \rightarrow +\infty} \sigma_{-t_n}(\theta t_n \circ \theta s \omega) \ (\text{let } t_n - s = \tau_n)
$$
where the 1st equality holds by the continuity property of \( \varphi \) with respect to \( x \), the 3rd equality holds by the cocycle property of \( \sigma \), and the last inclusion relation holds by the definition of \( \Omega^*_\sigma \). This verifies the forward invariance of \( \Omega^*_\sigma \).

For arbitrary \( y \in \Omega^*_\sigma(\theta_s\omega) \) with \( s > 0 \), there exist sequences \( t_n \to +\infty \) and \( y_n = \sigma_{-t_n}(\theta_{t_n} \circ \theta_s\omega) \) such that \( y_n \to y \) as \( n \to +\infty \). Then we have

\[
y = \lim_{n \to +\infty} \sigma_{-t_n}(\theta_{t_n} \circ \theta_s\omega) = \lim_{n \to +\infty} \sigma_{-(\tau_n - s)}(\theta_{\tau_n - s} \circ \theta_s\omega) \quad \text{(let } t_n + s = \tau_n) \\
= \lim_{n \to +\infty} \varphi(s, \omega)\sigma_{-\tau_n}(\theta_{\tau_n - s} \circ \theta_s\omega) \\
= \lim_{n \to +\infty} \varphi(s, \omega)\sigma_{-\tau_n}(\theta_{\tau_n} \circ \theta_s\omega) \\
= \varphi(s, \omega) \lim_{n \to +\infty} \sigma_{-\tau_n}(\theta_{\tau_n} \circ \theta_s\omega)
\]

where the last two equalities hold by the pre-compactness of \( \{\sigma_{-\tau_n}(\theta_{\tau_n} \circ \theta_s\omega)|n \in \mathbb{N}\} \), and by taking a subsequence we assume that the subsequence converges to \( x \in \Omega^*_\sigma \). This verifies \( \Omega^*_\sigma(\theta_s\omega) \subset \varphi(s, \omega)\Omega^*_\sigma(\omega) \).

Therefore, we have showed that \( \varphi(s, \omega)\Omega^*_\sigma(\omega) = \Omega^*_\sigma(\theta_s\omega) \), hence completed the proof.

\( \square \)

**Lemma 4.5.** Assume that \( x \) is a random variable with \( \sigma \) being an entire orbit through \( x \), and \( A \) is a random attractor with \( R \) being the corresponding repeller. Then we have:

(i) if \( x \in R \) almost surely, then \( \Omega_{\sigma} \subset R \) and \( \Omega^*_\sigma \subset R \) almost surely;

(ii) if \( x \in B(A) \setminus A \) almost surely, then \( \Omega_{\sigma} \subset A \) and \( \Omega^*_\sigma \subset R \) almost surely;

(iii) if \( x \in A \) almost surely, then \( \Omega_{\sigma} \subset A \) almost surely; if \( \Omega^*_\sigma \subset A \) almost surely, then \( \sigma \) lies on \( A \) almost surely, i.e. for arbitrary \( t \in \mathbb{R} \), we have \( \sigma_t \subset A \) almost surely;

(iv) if \( x \in B(A) \) almost surely, then \( \Omega_{\sigma} \subset A \) almost surely; if \( x \in B(R) := S \setminus A \) almost surely, then \( \Omega^*_\sigma \subset R \) almost surely.

**Proof.** (i) By the forward invariance of \( R \), the former is obvious. By the forward invariance of \( B(A) \) we obtain that all backward orbits through \( x \) must lie on \( R \), so by the definition of \( \Omega^*_\sigma \) we have \( \Omega^*_\sigma \subset R \) almost surely.

(ii) The former follows directly from Lemma 4.3 in [28], (it is clear that Lemma 4.3 also holds for random semiflow). Assume that \( U \) is a forward invariant random open neighborhood of \( A \) relative to \( S \) such that \( \Omega_U = A \) and let \( V = S \setminus U \). For arbitrary random variable \( y \in V \), let \( \sigma^y \) be a backward orbit through \( y \). Then by the forward invariance of \( U \), \( \sigma^y \) lies on \( V \). Hence we have \( \Omega^*_\sigma^y \subset V \) almost surely. If \( \Omega^*_\sigma^y \not\subset R \) with positive probability, letting \( R_1 := R \cup \Omega^*_\sigma^y \), then \( R_1 \) is an invariant random compact set by Lemma 4.4 and (ii) of Remark 4.11. Then we can choose a random variable \( z \) such that \( z \in R_1 \) almost surely and \( z \in R_1 \setminus R \) with positive probability. On one hand, \( \Omega_z \subset R_1 \) almost surely by the invariance of \( R_1 \), which implies

\[
\mathbb{P} - \lim_{t \to \infty} d(\varphi(t, \omega)z(\omega)|R_1(\theta_t\omega)) = 0.
\]

On the other hand \( z \in B(A) \) with positive probability, which implies that

\[
\lim_{t \to \infty} d(\varphi(t, \cdot)z(\omega)|A(\theta_t \cdot)) = 0
\]
with positive probability, a contradiction to the fact that \( R_1 \cap A = \emptyset \) almost surely. Therefore, for arbitrary random variable \( y \in V \), we have \( \Omega_{\sigma^y} \subset R \) almost surely.

Let \( U_n := \varphi(n, \theta_{-n})U(\theta_{-n}) \), \( n \in \mathbb{N} \), then we have \( U_{n+1} \subset U_n \) by the forward invariance of \( U \). Moreover, each \( U_n \) is a forward invariant random open neighborhood of \( A \) relative to \( S \) and \( \Omega_{U_n} = A \). Clearly we have

\[
A(\omega) = \lim_{n \to \infty} U_n(\omega).
\]

Letting \( V_n = S \setminus U_n \), for arbitrary random variable \( x \in V_n \), we have \( \Omega_{\sigma^x} \subset R \) almost surely by the above argument. Since \( n \) is arbitrary, for arbitrary random variable in \( S \setminus A \) with a backward orbit \( \sigma \), we have \( \Omega_{\sigma} \subset R \) almost surely. This completes the proof of (ii).

(iii) The former is trivial. Since \( A \) is forward invariant, any backward orbit through a random variable in \( S \setminus A \) must lie on \( S \setminus A \). If there exists some \( t_0 \in \mathbb{R} \) such that \( \mathbb{P}\{\omega \sigma_{t_0}(\omega) \not\in A(\theta_{t_0})\} = \delta > 0 \), then for all \( s \leq t_0 \) we have \( \mathbb{P}\{\omega \sigma_s(\omega) \not\in A(\theta_{s})\} \geq \delta \), i.e. \( \mathbb{P}\{\omega \sigma_{s}(\omega) \subset S(\theta_{s})A(\theta_{s})\} \geq \delta \). Then by the proof of (ii) it follows that \( \Omega_{\sigma} \subset R \) with positive probability, a contradiction to the fact that \( \Omega_{\sigma} \subset A \) almost surely.

(iv) The former follows directly from (ii) and (iii), while the later has been proved during the proof of (ii). \( \square \)

**Definition 4.4.** Assume that \( (A_i, R_i) \) are attractor-repeller pairs of \( \varphi \) on the invariant random compact set \( S \) with

\[
\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = S \text{ and } S = R_0 \supseteq R_1 \supseteq \cdots \supseteq R_n = \emptyset.
\]

Then the family \( D = \{M_i\}_{i=1}^n \) of invariant random compact sets, defined by

\[
M_i = A_i \cap R_{i-1}, \quad 1 \leq i \leq n,
\]

is called a Morse decomposition of \( S \), and each \( M_i \) is called Morse set. If \( D \) is a Morse decomposition, \( M_D \) is defined to be \( \bigcup_{i=1}^n M_i \).

**Remark 4.2.** (i) For \( i \neq j \), say \( i < j \), \( M_i \cap M_j = A_i \cap R_{i-1} \cap A_j \cap R_{j-1} \subset A_i \cap R_{j-1} \subset A_i \cap R_i = \emptyset \).

(ii) Each Morse set \( M_i \) is invariant, which is trivial if \( \varphi \) is a random flow, but requires explanation in the case of random semiflow. Clearly each \( M_i \) is forward invariant. For any point \( x \in M_i(\omega) = A_i(\omega) \cap R_{i-1}(\omega) \), there exists a backward orbit \( \sigma \) in \( A_i \) by the backward invariance of \( A_i \). By the forward invariance of \( R_{i-1} = B(A_{i-1}) \), any backward orbit through a point in \( R_{i-1} \) must lie on \( R_{i-1} \). Hence we have \( \sigma \) lying on \( M_i \), i.e. \( M_i \) is backward invariant.

**Lemma 4.6.** Assume that \( A_1, A_2 \subset S \) are two random attractors with basins of attraction \( B(A_1), B(A_2) \), respectively. Assume that \( D \) is a random compact set satisfying \( D \subset B(A_1) \cup B(A_2) \) almost surely. Then \( A_1 \cup A_2 \) pull-back attracts \( D \).

**Proof.** Denote

\[
\Omega_1 = \{\omega | D(\omega) \subset B(A_1)(\omega)\}, \quad \Omega_2 = \{\omega | D(\omega) \subset B(A_2)(\omega) \setminus B(A_1)(\omega)\}.
\]

Choose a random compact set \( D_1 \subset B(A_1) \) almost surely satisfying \( D_1(\omega) = D(\omega) \) for \( \omega \in \Omega_1 \) and choose \( D_2 \subset B(A_2) \) almost surely satisfying \( D_2(\omega) = D(\omega) \) for \( \omega \in \Omega_2 \). Then
we have $D \subset D_1 \cup D_2$ almost surely. Therefore, by (ii) and (iii) of Remark 4.1 and Lemma 4.3 in [28], we obtain for $\mathbb{P}$-almost all $\omega$

$$\Omega_D(\omega) \subset \Omega_{D_1 \cup D_2}(\omega) = \Omega_{D_1}(\omega) \cup \Omega_{D_2}(\omega) \subset A_1(\omega) \cup A_2(\omega).$$

By the definition of omega-limit set, it is clear that $\Omega_D$ pull-back attracts $D$, so $A_1 \cup A_2$ pull-back attracts $D$. This completes the proof. \qed

**Remark 4.3.** It is obvious that the result of Lemma 4.6 holds for finite case, i.e. if the random compact set $D \subset \bigcup_{i=1}^n B(A_i)$ almost surely, then $\bigcup_{i=1}^n A_i$ pull-back attracts $D$.

**Theorem 4.1.** Assume that $D = \{M_i\}_{i=1}^n$ is a Morse decomposition of $S$, determined by attractor-repeller pairs $(A_i, R_i)$, $i = 1, \ldots, n$. Then $M_D$ determines the limiting behavior of $\varphi$ on $S$. Moreover, there are no “cycles” between the Morse sets. More precisely, we have:

(i) For any random variable $x$ in $S$, there exists an entire orbit $\sigma$ through $x$ such that $\Omega_\sigma \subset M_D$ and $\Omega_\sigma^* \subset M_D$ almost surely.

(ii) If $\sigma$ is an entire orbit through the random variable $x$ satisfying that $\Omega_\sigma \subset M_p$ almost surely and $\Omega_\sigma^* \subset M_q$ almost surely for some $1 \leq p, q \leq n$, then $p \leq q$; Moreover, $p = q$ if and only if $\sigma$ lies on $M_p$.

(iii) If $\sigma_1, \ldots, \sigma_l$ are $l$ entire orbits through the random variables $x_1, \ldots, x_l$ respectively such that for some $1 \leq j_0, \ldots, j_l \leq n$, $\Omega_{\sigma_k} \subset M_{j_k}$ and $\Omega_{\sigma_k}^* \subset M_{j_k}$ for $k = 1, \ldots, l$, then $j_0 \leq j_l$. Moreover, $j_0 < j_l$ if and only if $\sigma_k$ does not lie on $M_{j_k}$ with positive probability for some $k$, otherwise $j_0 = \cdots = j_l$.

**Proof.** (i) Since $\emptyset = R_0^c \subsetneq R_1^c \subsetneq \cdots \subsetneq R_n^c = S$, let $\tilde{R}_i = R_i^c \setminus R_{i-1}^c$. Then $S = \bigcup_{i=1}^n \tilde{R}_i$ almost surely and $\tilde{R}_i = B(A_i) \setminus B(A_{i-1})$. Hence for arbitrary random variable in $\tilde{R}_i$, it is attracted by $A_i$ but not by $A_{i-1}$. For arbitrary random variable $x$ in $S$, choose $n$ random variables $x_1, \ldots, x_n$ such that $x_i \in \tilde{R}_i$ almost surely and $x(\omega) = x_i(\omega)$ when $\omega \in \Omega_i$, where $\Omega_i := \{\omega | x(\omega) \in \tilde{R}_i(\omega)\}$, $i = 1, \ldots, n$. By the fact $x_i \in \tilde{R}_i = \tilde{R}_i^c \cap \tilde{R}_{i-1}$ we know that $x_i$ is attracted by $A_i \cap \tilde{R}_{i-1} = M_i$ almost surely. Then by Lemma 4.6 we obtain for $\mathbb{P}$-almost all $\omega$

$$\Omega_\sigma(\omega) = \Omega_x(\omega) \subset \bigcup_{i=1}^n \Omega_{x_i}(\omega) \subset \bigcup_{i=1}^n M_i(\omega) = M_D(\omega).$$

Since $S = A_0^c \supseteq A_1^c \supseteq \cdots \supseteq A_n^c = \emptyset$, let $\tilde{A}_i = A_{i-1}^c \setminus A_i^c = B(R_{i-1}) \cap A_i$, $i = 1, \ldots, n$. Then $S = \bigcup_{i=1}^n \tilde{A}_i$ almost surely. By (iv) of Lemma 4.5, for given random variable $x \in \tilde{A}_i$, we have $\Omega_\sigma^* \subset R_{i-1}$ almost surely for any backward orbit $\sigma$ through $x$. Since $x \in \tilde{A}_i \subset A_i$, by the invariance of $A_i$, there exists a backward orbit $\sigma$ through $x$ lying on $A_i$ (we cannot guarantee generally that any backward orbit through $x$ must lie on $A_i$). Hence for this $\sigma$ we have $\Omega_\sigma^* \subset A_i$ almost surely. Therefore, we have obtained that for any random variable $x \in \tilde{A}_i$, there exists a backward orbit $\sigma$ through it such that $\Omega_\sigma^* \subset A_i \cap R_{i-1} = M_i$ almost surely. For arbitrary random variable $y \in S$, choose $n$ random variables $y_i$, $i = 1, \ldots, n$ such that $y_i \in \tilde{A}_i$ almost surely.
and \( y(\omega) = y_i(\omega) \) when \( \omega \in \Omega_i \), where \( \Omega_i := \{ \omega \mid y(\omega) \in \tilde{A}_i(\omega) \} \). By above argument, for each \( i \), there exists a backward orbit \( \sigma_i \) through \( y_i \) such that \( \Omega^*_\sigma_i \subset M_i \) almost surely. “Attaching” the corresponding parts of these \( \sigma_i \)'s together when \( y \) lies on \( \tilde{A}_i \), we obtain a backward orbit \( \sigma \) through \( y \). By the choice of \( \sigma \), we have

\[
\Omega^*_\sigma \subset \bigcup_{i=1}^n \Omega^*_\sigma_i \subset \bigcup_{i=1}^n M_i = M_D
\]

almost surely as desired.

(ii) Since \( \Omega_\sigma \subset M_p = A_p \cap R_{p-1} \) almost surely, we have \( x \in A_{p-1}^c \) almost surely. By the fact that \( \Omega^*_\sigma \subset M_q = A_q \cap R_{q-1} \) almost surely, we have \( \sigma \) lying on \( A_q \) almost surely by (iii) of Lemma 4.5. In particular, \( \sigma_0 = x \in A_q \) almost surely. Hence we have \( x \in A_{p-1}^c \cap A_q \) almost surely. If \( q < p \), then \( A_q \subset A_{p-1} \), hence \( A_q \cap A_{p-1}^c = \emptyset \) almost surely, a contradiction.

If \( \sigma \) lies on \( M_p \), then we have \( \Omega_\sigma, \Omega^*_\sigma \subset M_p \) almost surely by the fact that \( M_p \) is an invariant random compact set. That is, we must have \( p = q \). Conversely, if \( p = q \), the fact \( \Omega_\sigma = \Omega_x \subset M_p = A_p \cap R_{p-1} \) implies \( x \in R_{p-1} \) almost surely. It follows that \( \sigma \) lies on \( R_{p-1} \) since any backward orbit through a random variable in \( R_{p-1} \) must lie on it. \( \Omega^*_\sigma \subset M_q = A_q \cap R_{q-1} \) implies that \( \sigma \) lies on \( A_q \) by (iii) of Lemma 4.5. So we have obtained that \( \sigma \) lies on \( A_q \cap R_{q-1} = M_p \) almost surely.

(iii) follows from (ii) immediately.

\[ \square \]

**Remark 4.4.** In (i) of Theorem 4.1, we obtain that, for given random variable \( x \), there exists an entire orbit through \( x \) satisfying \( \Omega_\sigma \subset M_D \) and \( \Omega^*_\sigma \subset M_D \) almost surely. While in the deterministic case, any entire orbit has this property, see [9] for the flow case and [31] for the semiflow case. But their methods are not applicable here. We are not sure whether the similar result holds for random semiflow.

In Theorem 4.1 we give the positive answer to an open problem put forward by Caraballo and Langa [6], which confirms our belief that we may understand theoretically the structure of global random attractor in a way as we do in the deterministic case. But, as mentioned in introduction, the theory of dynamics on the global random attractor is so immature that we know very few examples for which the structure of global random attractors is well understood. Hence to understand them well, more concrete examples are needed at the first step. The global random attractor of Chafee-Infante reaction-diffusion equation perturbed by Stratonovich multiplicative noise may be the best understood model in infinite dimensional case, so we use it to illustrate our results. For this model in deterministic case, it is well studied, see Hale [18] and Henry [19] for instance.

**Example 4.1.** Consider the Chafee-Infante reaction-diffusion equation perturbed by Stratonovich multiplicative noise

\[
du = (\Delta u + \beta u - u^3)dt + \delta u \circ dW, \quad x \in [0, \pi] \text{ with } u(t, 0) = u(t, \pi) = 0.
\]  

(14)

This equation is well studied in [7] and very recently in [5] and [36]. Here we take it as an example to illustrate our results, for detailed analysis see [7]. Denote

\[
D = [0, \pi], \quad H = L^2(D),
\]

\[
\mathcal{K}^+ = \{ u \in H \mid u(x) \geq 0 \text{ almost everywhere} \},
\]

\[
\mathcal{K}^- = \{ u \in H \mid u(x) \leq 0 \text{ almost everywhere} \}.
\]
Assume that $\varphi$ is the RDS generated by (14) on $H$, then $\varphi$ is order preserving on $H$, i.e. if $u_0 \geq v_0$ almost everywhere, then

$$\varphi(t, \omega)u_0 \geq \varphi(t, \omega)v_0.$$ 

In particular, since $\{0\}$ is a random fixed point of $\varphi$, $K^\pm$ are invariant cones of $\varphi$. We can show that in each of these two cones, there exists a random compact absorbing set. Hence it follows that $\varphi$ has nontrivial random attractors $A^+$, $A^-$, respectively, in each of these two cones. Furthermore, there exist positive and negative random fixed points $\pm a(\omega) \in A^\pm(\omega)$ such that

$$0 \leq u \leq a(\omega) \quad \text{for all } u \in A^+(\omega),$$

$$-a(\omega) \leq u \leq 0 \quad \text{for all } u \in A^-(\omega).$$

It has been shown in [7] that (14) has global random attractor $A$, and the lower bound on the dimension of $A$ has also been obtained, but the exact structure of $A$ has not been obtained in [7]. The authors conjecture in [7] that, for $\lambda_1 < \beta < \lambda_2$ ($\lambda_1$, $\lambda_2$ are the first two eigenvalues of the negative Laplacian), the global random attractor

$$A(\omega) = A^+(\omega) \cup A^-(\omega),$$

with $A^\pm$ consisting of a one-dimensional manifold joining the origin to $\pm a(\omega)$. Moreover, since orbits near 0 in $K^+$ move away from the origin (i.e. $\{0\}$ is unstable), it is also conjectured in [7] that $a(\omega)$ is attracting in $K^+$ (symmetrically, $-a(\omega)$ is attracting in $K^-$). Very recently, Wang and Duan [36] confirm these two conjectures. Indeed, they show that the semiflow on the global random attractor of (14) is topologically equivalent to a well studied one-dimensional stochastic ODE (see [2], [13] for the study of this stochastic ODE), see [36] for details. Now we can use our abstract results here. Choose $\pm b(\omega)$ such that $0 < b(\omega) < a(\omega)$ and $-a(\omega) < -b(\omega) < 0$, then by the order preserving property of $\varphi$ and the fact that $\pm a(\omega)$ are attracting in $K^\pm$, respectively, we have $[-a(\omega), -b(\omega))$ and $(b(\omega), a(\omega)]$ being forward invariant and $\Omega_{[-a,-b]}(\omega) = \{a(\omega)\}$, $\Omega_{[b,a]}(\omega) = \{a(\omega)\}$. Therefore, it is clear that $\emptyset$, $\{a(\omega)\}$, $\{-a(\omega)\}$ and $A$ are all local random attractors on the global random attractor $A$. If we set

$$A_0 = \emptyset, \ A_1(\omega) = \{a(\omega)\}, \ A_2(\omega) = \{a(\omega), -a(\omega)\}, \ A_3(\omega) = A(\omega),$$

then the corresponding repellers are

$$R_0(\omega) = A(\omega), \ R_1(\omega) = [-a(\omega), 0], \ R_2(\omega) = \{0\}, \ R_3 = \emptyset,$$

respectively. So $D = \{M_1, M_2, M_3\}$ with

$$M_1 = \{a(\omega)\}, \ M_2 = \{-a(\omega)\}, \ M_3 = \{0\}$$

is a Morse decomposition of $A$, just consisting of random fixed points of $\varphi$. If we reset $A_0(\omega) = \emptyset$, $A_2(\omega) = \{a(\omega), -a(\omega)\}$, $A_3(\omega) = A(\omega)$, then $M_1(\omega) = \{-a(\omega)\}, M_2(\omega) = \{a(\omega)\}, M_3(\omega) = \{0\}$ are the corresponding Morse sets. These two are the finest Morse decompositions of $A$, consisting of just random fixed points. The Morse decomposition can be coarsen: if we set $A_0 = \emptyset$, $A_1(\omega) = \{-a(\omega), a(\omega)\}$, $A_2 = A$, then $R_0 = A$, $R_1(\omega) = \{0\}, \ R_2 = \emptyset$. Hence the corresponding Morse sets are $M_1 = \{-a(\omega), a(\omega)\}, M_2 = \{0\}$. 

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By Remark 3.2 we know that the random chain recurrent set on $\mathcal{A}$
\[
\mathcal{CR}_\varphi(\omega) = \bigcap \{A(\omega) \cup R(\omega)\} = \{-a(\omega), 0, a(\omega)\}
\]
for $\mathbb{P}$-almost all $\omega$, where the intersection is taken over all local random attractors on $\mathcal{A}$, i.e. $\emptyset$, $\{a(\omega)\}$, $\{-a(\omega)\}$, $\{a(\omega), -a(\omega)\}$ and $\mathcal{A}$. It is clear that the random chain recurrent set on $\mathcal{A}$ equals the union of Morse sets in (15).

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