Almost every graph is divergent under the biclique operator

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ABSTRACT

A biclique of a graph $G$ is a maximal induced complete bipartite subgraph of $G$. The biclique graph of $G$ denoted by $KB(G)$, is the intersection graph of all the bicliques of $G$. The biclique graph can be thought as an operator between graphs. The iterated biclique graph of $G$ denoted by $KB^k(G)$, is the graph obtained by applying the biclique operator $k$ successive times to $G$. The associated problem is deciding whether an input graph converges, diverges or is periodic under the biclique operator when $k$ grows to infinity. All possible behaviors were characterized recently and an $O(n^4)$ algorithm for deciding the behavior of any graph under the biclique operator was also given. In this work we prove new structural results of biclique graphs. In particular, we prove that every false-twin-free graph with at least 13 vertices is divergent. These results lead to a linear time algorithm to solve the same problem.

Keywords: Bicliques; Biclique graphs; False-twin-free graphs; Iterated graph operators; Graph dynamics

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1 Introduction

Intersection graphs of certain special subgraphs of a general graph have been studied extensively. For example, line graphs (intersection graphs of the edges of a graph), interval graphs (intersection of intervals of the real line), clique graphs (intersection of cliques of a graph), etc [4, 5, 11, 14, 15, 28, 30].

The *clique graph* of $G$ denoted by $K(G)$, is the intersection graph of the family of all maximal cliques of $G$. Clique graphs were introduced by Hamelink in [20] and characterized by Roberts and Spencer in [35]. The computational complexity of the recognition problem of clique graphs had been open for more than 40 years. In [1] they proved that clique graph recognition problem is NP-complete.

The clique graph can be thought as an operator between graphs. The *iterated clique graph* $K^k(G)$ is the graph obtained by applying the clique operator $k$ successive times ($K^0(G) = G$). Then $K$ is called *clique operator* and it was introduced by Hedetniemi and Slater in [21]. Much work has been done on the scope of the clique operator looking at the different possible behaviors. The associated problem is deciding whether an input graph converges, diverges or is periodic under the clique operator when $k$ grows to infinity. In general it is not clear that the problem is decidable. However, partial characterizations have been given for convergent, divergent and periodic graphs restricted to some classes of graphs. Some of these lead to polynomial time recognition algorithms. For the clique-Helly graph class, graphs which converge to the trivial graph have been characterized in [3]. Cographs, $P_4$-tidy graphs, and circular-arc graphs are examples of classes where the different behaviors are characterized [8, 23]. Divergent graphs were also considered. For example in [32], families of divergent graphs are shown. Periodic graphs were studied in [11, 27]. In particular it is proved that for every integer $i$, there exist periodic graphs with period $i$ and also convergent graphs which converge in $i$ steps. More results about iterated clique graph can be found in [12, 13, 24, 25, 26, 33].

A biclique is a maximal bipartite complete induced subgraph. Bicliques have applications in various fields, for example biology: protein-protein interaction networks [6], social networks: web community discovery [22], genetics [2], medicine [31], information theory [19], etc. More applications (including
some of these) can be found in [29].

The biclique graph of a graph $G$ denoted by $KB(G)$, is the intersection graph of the family of all maximal bicliques of $G$. It was defined and characterized in [17]. However no polynomial time algorithm is known for recognizing biclique graphs. As for clique graphs, the biclique graph construction can be viewed as an operator $KB$ between graphs.

The iterated biclique graph $KB^k(G)$ is the graph obtained by applying to $G$ the biclique operator $KB$ $k$ times iteratively. It was introduced in [10] and all possible behaviors were characterized. It was proven that a graph $G$ is either divergent or convergent but it is never periodic (with period bigger than 1). In addition, general characterizations for convergent and divergent graphs are given. These results are based on the fact that if a graph $G$ contains a clique of size at least 5, then $KB(G)$ or $KB^2(G)$ contains a clique of larger size. Therefore, in that case $G$ diverges. Similarly if $G$ contains the gem or the rocket graphs as an induced subgraph, then $KB(G)$ contains a clique of size 5, and again $G$ diverges. Otherwise it is shown that after removing false-twin vertices of $KB(G)$, the resulting graph is a clique on at most 4 vertices, in which case $G$ converges. Moreover, it was proved that if a graph $G$ converges, it converges to the graphs $K_1$ or $K_3$, and it does so in at most 3 steps. These characterizations led to an $O(n^4)$ time algorithm for recognizing convergent or divergent graphs under the biclique operator.

In this work we show new results that lead to a linear time algorithm to solve the same problem. We study conditions for a graph to contain a $K_5$, a $C_5$, a butterfly, a gem or a rocket (see Figure 1) as induced subgraphs so we can decide divergence (since $K_5 \subseteq KB(C_5), KB(butterfly), KB(gem), KB(rocket)$). First we prove that if $G$ has at least 7 bicliques then it diverges. Then, we show that every false-twin-free graph with at least 13 vertices has at least 7 bicliques, and therefore diverges. Since adding false-twins to a graph does not change its $KB$ behavior, then the linear algorithm is based on the deletion of false-twin vertices of the graph and looking at the size of the remaining graph.

It is worth to mention that these results are indeed very different from the ones known for the clique operator, for which it is still an open problem to know the computational complexity of deciding the behavior of a graph under $K$. 



This work is organized as follows. In Section 2 the notation is given. Section 3 contains some preliminary results that we will use later. In Section 4 we prove that any graph with at least 7 bicliques diverges, and that every graph with at least 13 vertices with no false-twins vertices contains at least 7 bicliques. This leads to a linear time algorithm to decide convergence or divergence under the biclique operator.

2 Notation and terminology

Along the paper we restrict to undirected simple graphs. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $n = |V(G)|$ and $m = |E(G)|$. A subgraph $G'$ of $G$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq (V' \times V') \cap E$. When $E' = (V' \times V') \cap E$ say that $G' = (V', E')$ is an induced subgraph of $G$. A graph $G = (V, E)$ is bipartite when $V = U \cup W$, $U \cap W = \emptyset$ and $E \subseteq U \times W$. Say that $G$ is a complete graph when every possible edge belongs to $E$. A complete graph of $n$ vertices is denoted $K_n$. A clique of $G$ is a maximal complete induced subgraph while a biclique is a maximal bipartite complete induced subgraph of $G$. The open neighborhood of a vertex $v \in V(G)$ denoted $N(v)$, is the set of vertices adjacent to $v$ while the closed neighborhood of $v$ denoted by $N[v]$, is $N(v) \cup \{v\}$. Two vertices $u, v$ are false-twins if $N(u) = N(v)$. A vertex $v \in V(G)$ is universal if it is adjacent to all of the other vertices in $V(G)$. A path (cycle) of $k$ vertices, denoted by $P_k$ ($C_k$), is a set of vertices $v_1v_2...v_k \in V(G)$ such that $v_i \neq v_j$ for all $1 \leq i \neq j \leq k$ and $v_i$ is adjacent to $v_{i+1}$ for all $1 \leq i \leq k - 1$ (and $v_1$ is adjacent to $v_k$). A graph is connected if there exists a path between each pair of vertices. We assume that all the graphs of this paper are connected.

A rocket is a complete graph with 4 vertices and a vertex adjacent to two of them. A butterfly is the graph obtained by joining two copies of the $K_3$. 

![Graphs K_5, C_5, butterfly, gem and rocket, respectively.](image-url)
with a common vertex.

Given a family of sets $\mathcal{H}$, the \textit{intersection graph} of $\mathcal{H}$ is a graph that has the members of $\mathcal{H}$ as vertices and there is an edge between two sets $E, F \in \mathcal{H}$ when $E$ and $F$ have non-empty intersection.

A graph $G$ is an \textit{intersection graph} if there exists a family of sets $\mathcal{H}$ such that $G$ is the intersection graph of $\mathcal{H}$. We remark that any graph is an intersection graph \cite{36}.

Let $F$ be any graph operator. Given a graph $G$, the iterated graph under the operator $F^k$ is defined iteratively as follows: $F^0(G) = G$ and for $k \geq 1$, $F^k(G) = F^{k-1}(F(G))$. We say that a graph $G$ diverges under the operator $F$ whenever $\lim_{k \to \infty} |V(F^k(G))| = \infty$. We say that a graph $G$ converges under the operator $F$ whenever $\lim_{k \to \infty} F^k(G) = F^m(G)$ for some $m$. We say that a graph $G$ is periodic under the operator $F$ whenever $F^k(G) = F^{k+s}(G)$ for some $k, s, s \geq 2$.

The \textit{iterated biclique graph} $KB^k(G)$ is the graph obtained by applying iteratively the biclique operator $KB$ $k$ times to $G$.

In the paper we will use the terms convergent or divergent meaning convergent or divergent under the biclique operator $KB$.

By convention we arbitrarily say that the trivial graph $K_1$ is convergent under the biclique operator (observe that this remark is needed since the graph $K_1$ does not contain bicliques).

\section{Preliminary results}

We start with this easy observation.

\textbf{Observation 3.1} \textit{(\cite{16})}. \textit{If $G$ is an induced subgraph of $H$, then $KB(G)$ is a subgraph (not necessarily induced) of $KB(H)$}.

The following proposition is central in the characterization of convergent and divergent graphs under the biclique operator. Basically, it shows that if a graph contains a big complete subgraph, it is going to grow in one or two steps of $KB$. 

\begin{proof}
\end{proof}
Proposition 3.2 ([16]). Let $G$ be a graph that contains $K_n$ as a subgraph, for some $n \geq 4$. Then, $K_{2n-4} \subseteq KB(G)$ or $K_{(n-2)(n-3)} \subseteq KB^2(G)$.

Next theorem characterize the behavior of a graph under the biclique operator.

**Theorem 3.3 ([16]).** If $KB(G)$ contains either $K_5$ or the gem or the rocket as an induced subgraph, then $G$ is divergent. Otherwise, $G$ converges to $K_1$ or $K_3$ in at most 3 steps.

Notice that differently than the clique operator, a graph is never periodic under the biclique operator (with period bigger than 1). We remark the importance of the graph $K_5$ to decide the behavior of a graph under the biclique operator since we have that $KB(gem) = K_5$ and $K_5 \subseteq KB(rocket)$.

Observe that as proved in [16], the biclique graph does not change by the deletion or addition false-twin vertices since each pair of false-twins belongs to exactly the same set of bicliques. That is, for any graph $G$, $KB(G) = KB(G - \{v\})$ for any false-twin vertex $v$. It follows that the behavior of a graph under $KB$ does not change either. Therefore we focus our study on false-twins-free graphs. For that we need the following definition used in [16].

Consider all maximal sets of false-twin vertices $Z_1, ... Z_k$ and let $\{z_1, z_2, ..., z_k\}$ be the set of representative vertices such that $z_i \in Z_i$. The graph obtained by the deletion of all vertices of $Z_i - \{z_i\}$ for $i = 1, ..., k$, is denoted $Tw(G)$.

Observe that $Tw(G)$ has no false-twin vertices.

Using $Tw(G)$, as a corollary of Theorem 3.3, the next useful result was obtained.

**Corollary 3.4 ([16]).** A graph $G$ is convergent if and only if $Tw(KB(G))$ has at most four vertices. Moreover, $Tw(KB(G)) = K_n$ for $n = 1, ..., 4$.

Note that if some vertex lies in five bicliques, then $KB(G)$ contains a $K_5$ and then $G$ diverges. Therefore, Corollary 3.4 gives a polynomial time algorithm to test convergence of $G$: If some vertex lies in five bicliques answer that $G$ is divergent. Else, the computation of $KB(G)$ and $Tw(KB(G))$ is polynomial (we remark however that the number of bicliques of a graph can be exponential [34]). If $Tw(KB(G))$ has at most four vertices, answer that $G$ is convergent, otherwise answer that $G$ is divergent.
Constructing $KB(G)$ takes $O(n^4)$ time since for the case that it is done, each vertex belongs to at most 4 bicliques and then the input graph has at most $2n$ bicliques. Generating each biclique is $O(n^3)$ \([9, 10]\). Building $Tw(KB(G))$ can be done in $O(n+m)$ time using the modular decomposition \([18]\). Therefore the algorithm runs in $O(n^4)$ time.

4 Linear time algorithm

In this section we give a linear time algorithm for deciding whether a given graph is divergent or convergent under the biclique operator.

Motivated by Theorem 3.3 and Corollary 3.4, we study the structure of biclique graphs with false-twin vertices in order to find properties of graphs that contains $K_5$ so we can guarantee the divergence of the graph.

The following two lemmas answer that question.

**Lemma 4.1.** Let $G = KB(H)$ for some graph $H$. Let $b_1, b_2$ be false-twin vertices of $G$ and $B_1, B_2$ their associated bicliques in $H$. Suppose that there are no edges between vertices of $B_1$ and vertices of $B_2$. Then there exists a vertex $v \in H$ such that $v$ is adjacent to every vertex of $B_1$ and $B_2$. Furthermore, $G$ contains a $K_5$ as induced subgraph.

**Proof.** Let $b_1, b_2$ be false-twin vertices of $G$ and $B_1, B_2$ their associated bicliques in $H$, such that there are no edges between vertices of $B_1$ and vertices of $B_2$. Since $G$ is connected, take the shortest path from some vertex of $B_1$ to $B_2$. Let $w$ be the first vertex in the path such that $w \notin B_1$. Clearly, $w \notin B_2$. Let $v \in B_1$ be a vertex adjacent to $w$.

First, suppose that there exists a vertex $x \in B_1$ such that $x$ is not adjacent to $w$. Consider the following alternatives:

**Case 1:** $xv \in E(H)$. Then $\{x, v, w\}$ is contained in some biclique $B$, $B \neq B_1$ and $B \neq B_2$, such that it does not intersect $B_2$ since there is no edge between $B_1$ and $B_2$. This is a contradiction since $b_1$ and $b_2$ are false-twin vertices. It follows that every vertex in $B_1$ not adjacent to $w$ is not adjacent to $v$.

**Case 2:** $xv \notin E(H)$. Then there exists a vertex $y \in B_1$ adjacent to $v$ and $x$. By **Case 1**, $y$ must be adjacent to $w$. This is the same situation
as previous case but considering $y$ instead of $v$ and the biclique containing \{x, y, w\} instead of \{x, v, w\}. A contradiction.

We conclude that for all $x \in B_1$, $x$ is adjacent to $w$.

Now, the edge $vw$ is contained in a biclique $B$ that must intersect $B_2$. Since there are no edges between $B_1$ and $B_2$ there exists a vertex $z \in B_2$ such that $z$ is adjacent to $w$. The same argument used for $v \in B_1$ and $w$ also holds for $z \in B_2$ and $w$. That is, for all $z \in B_2$, $z$ is adjacent to $w$.

Finally, let $v, v'$ be adjacent vertices in $B_1$ and let $z, z'$ be adjacent vertices in $B_2$. Since $v, v', z, z'$ are adjacent to $w$, then \{v, w, z\}, \{v', w, z\}, \{v, w, z'\} and \{v', w, z'\} are contained in four different bicliques $B_3, B_4, B_5$ and $B_6$ such that $B_i \neq B_j$, for $1 \leq i \neq j \leq 6$. As $B_i \cap B_j \neq \emptyset$, for $2 \leq i \neq j \leq 6$ (Fig. 2), $K_5$ is an induced subgraph of $G$.

**Lemma 4.2.** Let $G = KB(H)$ for some graph $H$. Let $b_1, b_2, b_3$ be false-twin vertices of $G$ and let $B_1, B_2, B_3$ be their associated bicliques in $H$. Suppose that for any pair of bicliques $B_i, B_j$, $1 \leq i \neq j \leq 3$, there is an edge between some vertex of $B_i$ and some vertex of $B_j$. Then, $K_5$ is an induced subgraph of $G$.

**Proof.** Let $b_1, b_2, b_3$ be the false-twin vertices of $G$ and $B_1, B_2, B_3$ their associated bicliques in $H$ such that for any pair of bicliques $B_i, B_j$, $1 \leq i \neq j \leq 3$, there is an edge between some vertex of $B_i$ and some vertex of $B_j$. We will show that $H$ contains either a *butterfly*, a *gem*, a *rocket* or a $C_5$, or four...
mutually intersecting bicliques also intersecting with $B_1$, $B_2$ and $B_3$. In any case we obtain a $K_5$ in $G$. We have the following cases:

**Case 1:** There is a $K_3$ with one vertex in each biclique. Let $a \in B_1$, $b \in B_2$, $c \in B_3$ be the $K_3$. Now $ab$, $ac$ and $bc$ are contained in 3 different bicliques of $H$. It is easy to see that none of $B_1$, $B_2$ or $B_3$ are bicliques isomorphic to $K_2$, otherwise they would not intersect the biclique containing the opposite edge of the $K_3$ (e.g. $B_1$ with $bc$).

**Case 1.1:** One of the bicliques, say $B_1$, is isomorphic to $K_{1,r}$ where the vertex $a$ is in the partition of size one. As the biclique containing $bc$ must intersect $B_1$, there exists a vertex $d \in B_1$ adjacent to $b$ and not adjacent to $c$. Now, as $c \notin B_1$, there exists a vertex $e \in B_1$, such that $c$ is adjacent to $e$. Therefore $\{a, b, c, d, e\}$ induces a gem or a rocket depending on the edge $eb$. See Figure 3.

![Figure 3: Case 1.1](image)

**Case 1.2:** None of the bicliques $B_1$, $B_2$ and $B_3$ are isomorphic to $K_{1,r}$ where the vertex of the $K_3$ is in the partition of size one. As the biclique containing $bc$ has to intersect $B_1$, call $e \in B_1$ a vertex in that intersection and w.l.g. assume $e$ adjacent to $c$ and not to $b$.

**Case 1.2.1:** Suppose $e$ is adjacent to $a$. Now, as $B_1$ is not isomorphic to $K_{1,r}$, we have the following cases.

If there exists a vertex $g \in B_1$ adjacent to $e$ and not adjacent to $b$. Depending
on the edge $gc$, \{a, b, c, e, g\} induces a gem or \{a, b, e\}, \{b, c, e\}, \{a, c\} and \{g, e, c\} are contained in four mutually intersecting bicliques. See Figure 4.

Figure 4: Case 1.2.1 with $g$ adjacent to $e$ and not to $b$

Otherwise, assuming that every $g \in B_1$ adjacent to $e$ is adjacent to $b$, and considering that $b \notin B_1$, there exists $f \in B_1$ adjacent to $a$ and $b$. In this case \{a, b, c, e, f\} induces a gem or a rocket depending on the edge $fc$. See Figure 5.

Figure 5: Case 1.2.1 with $f$ adjacent to $a$ and $b$

Case 1.2.2: There exists $e \in B_1$ not adjacent to $a$ and $b$, and adjacent to $c$. Let $h \in B_1$ be a vertex adjacent to $e$ and $a$. Clearly, if $h$ is adjacent to
c, it must be adjacent to b, otherwise we would be in the case above. So, if h is adjacent to both, \{a, b, c, e, h\} induces a rocket. Therefore, we can assume that for every \( h \in B_1 \) adjacent to e and a, h is not adjacent to b and c. Moreover, this must be also true for every vertex in \( B_2 \) adjacent to b and every vertex in \( B_3 \) adjacent to c, that is, every vertex in \( B_2 \) adjacent to b is not adjacent to a and c, and every vertex in \( B_3 \) adjacent to c is not adjacent to a and b. Suppose that there exists \( k \in B_2 \) adjacent to b and not adjacent to h, then \{a, b, h\}, \{a, b, k\}, \{b, c\} and \{a, c\} are contained in four mutually intersecting bicliques. Then, we can assume \( k \) is adjacent to h. Indeed, assume that every vertex in \( B_1 \) adjacent to a is adjacent to every vertex in \( B_2 \) adjacent to b and to every vertex in \( B_3 \) adjacent to c. Also every vertex in \( B_2 \) adjacent to b is adjacent to every vertex in \( B_3 \) adjacent to c. Otherwise, we would obtain four mutually intersecting bicliques. Let \( j \in B_3 \) adjacent to c. Observe that if \( e \) is adjacent to \( k \) then \( e \) is also adjacent to \( j \), otherwise we are in case 1.2.1 considering the \( K_3 = \{h, k, j\} \). Then, depending on the edge \( ek \), \{e, h, k, j, c\} induces a rocket, or \{a, b, k, h\}, \{a, c, j, h\}, \{b, c, k, j\} and \{e, c, b\} are contained in four mutually intersecting bicliques. See Figure 6.

\[
\begin{align*}
\text{Figure 6: Case 1.2.2}
\end{align*}
\]

We covered all the cases when a \( K_3 \) is in \( H \).

Case 2: There is an induced \( C_4 = \{a, b, c, d\} \) in \( H \) such that \( a, b \in B_1 \), \( c \in B_2 \) and \( d \in B_3 \), that is, \( ab, bc, cd, ad \in E(H) \). Now as \( c \notin B_1 \), there exists either \( e \in B_1 \) adjacent to \( b \) and \( c \), or \( h \in B_1 \) adjacent to \( a \) and not adjacent
to c. We have the following cases:

Case 2.1: e is adjacent to b and c (the case where e is adjacent to a and d is analogous). Observe that e is not adjacent to d as we would obtain a triangle with one vertex in each biclique (case 1). Let \( k \in B_3 \) be a vertex adjacent to d. If \( k \) is adjacent to c then \( \{b, e, c, d, k\} \) induces a butterfly (otherwise case 1). Then assume every vertex \( k \in B_3 \) adjacent to d is not adjacent to c. Furthermore, if any vertex \( j \in B_2 \) adjacent to c, is also adjacent to d, then \( \{e, b, c, d, j\} \) induces a butterfly, a gem or a rocket depending on the edges \( ej, bj \). Therefore we can assume that every vertex \( j \in B_2 \) adjacent to c is not adjacent to d. See Figure 7.

**Figure 7: Case 2.1**

Case 2.1.1: There is some \( k \) not adjacent to b. Now as c \( \notin B_3 \), there exists \( \ell \in B_3 \) adjacent to k and not adjacent to c. If \( \ell \) is adjacent to b then \( \{\ell, b, c, d, k\} \) induces a \( C_5 \). We can assume \( \ell \) is not adjacent to b.

If \( k \) is adjacent to a then \( \{a, b, c, d\}, \{a, b, k\}, \{c, d, k\} \) and one of \( \{a, k, \ell\} \) or \( \{a, d, \ell\} \) depending on the edge \( al \), are contained in four different mutually intersecting bicliques. So we can assume \( k \) is not adjacent to a.

As a \( \notin B_2 \), either a is not adjacent to some vertex of \( B_2 \) that is adjacent to c, or a forms a triangle with two vertices of \( B_2 \).
Suppose first that $a$ is not adjacent to $j \in B_2$ such that $j$ is adjacent do $c$. Note that \{a, b, c, d\}, \{a, c, d, k\} and \{c, d, e\} are contained in three different mutually intersecting bicliques. See Figure 8.

![Diagram](image)

**Figure 8:** Case 2.1.1 with $a$ not adjacent to $j$

If $j$ is not adjacent to $b$ then \{b, c, j\} is contained in the fourth biclique (and we got four different mutually intersecting bicliques). So suppose $j$ is adjacent to $b$. If $j$ is not adjacent to $e$, the fourth biclique contains \{a, b, e, j\}. Finally, if $j$ is adjacent to $e$ then \{c, d, j\} is contained in the fourth biclique.

Suppose next that $a$ forms a triangle with two vertices of $B_2$. That is, there are two adjacent vertices $j, p \in B_2$ such that $j$ is adjacent to $c$ and $a$, and $p$ is adjacent to $a$ (see Figure 9). If $p$ is adjacent to $b$, then depending on the edge $ep$, \{a, b, c, e, p\} induces a *butterfly* or a *gem*. Assume therefore that $p$ is not adjacent to $b$. Now suppose that $j$ is not adjacent to $b$. Then, \{a, b, c, d\}, \{a, c, d, k\} and depending on the edge $dp$, either \{c, d, e\} and \{a, d, p\}, or \{c, d, j, p\} and \{a, b, p\} are contained in four different mutually intersecting bicliques. Finally, if $j$ is adjacent to $b$, depending on the edge $ej$, \{a, b, c, e, j\} induces a *gem* or a *rocket*.
Figure 9: Case 2.1.1 a form a triangle with 2 vertices of $B_2$

**Case 2.1.2:** Every vertex $k \in B_3$ adjacent to $d$ is adjacent to $b$. Now as $b \notin B_3$, there exists $m \in B_3$ adjacent to $k$ and $b$. Note that $m$ is not adjacent to $c$, otherwise case 1. Then $\{b, e, c, k, m\}$ induces a *butterfly, gem* or *rocket* depending on the edges $ek$ and $em$. See Figure [10]

**Case 2.2:** $h$ is adjacent to $a$ and not adjacent to $c$. By symmetry there exists $g \in B_1$ adjacent to $b$ and not adjacent to $d$. Assume that $g$ is not adjacent to $c$ and $h$ is not adjacent to $d$ (otherwise case 2.1).

Suppose first that there exists $k \in B_3$ adjacent to $d$ and $c$. Observe that $k$ is not adjacent to $b$ (case 1) and $k$ is not adjacent $a$ and $h$ at the same time (case 2.1 considering the $C_4 = \{b, c, k, a\}$). Depending on the edge $ak$, one of $\{b, c, k\}$ or $\{a, h, k\}$ along with $\{a, b, c, g\}$, $\{a, b, c, d\}$, $\{a, b, d, h\}$ are contained in four different mutually intersecting bicliques.

Suppose therefore that every $k \in B_3$ adjacent to $d$ is not adjacent to $c$. If $k$ is not adjacent to $b$ or $k$ is adjacent to $a$, then $\{a, b, c, g\}$, $\{a, b, c, d\}$, $\{a, b, d, h\}$ and $\{c, d, k\}$ are contained in four different mutually intersecting bicliques. Therefore $k$ is adjacent to $b$ and not adjacent to $a$. Since $c \notin B_3$, there exists $\ell \in B_3$ adjacent to $k$ and not adjacent to $c$. As $\ell$ is not adjacent to $b$ (case 2.1 considering the $C_4 = \{b, c, d, k\}$), then $\{a, b, c, g\}$, $\{a, b, c, d\}$, $\{a, b, d, h\}$
and \( \{b, d, k, \ell\} \) are contained in four different mutually intersecting bicliques. See Figure 11.

We covered all the cases when a \( C_4 \) is in \( H \) with all of the vertices in the bicliques \( B_1, B_2 \) and \( B_3 \).

**Case 3:** There is an induced \( C_k \), \( 5 \leq k \leq 9 \) in \( H \) with at least one vertex from each biclique \( B_1, B_2 \) and \( B_3 \). For the case \( k = 5 \) there is nothing to do. Finally, for \( 6 \leq k \leq 9 \), it is easy to see that, as each biclique containing two consecutive edges of the \( C_k \) has to intersect \( B_1, B_2 \) and \( B_3 \), then we would obtain a smaller cycle and therefore this case cannot occur.

Since we covered all cases the proof is done.

Next, we present the main theorem of this section. This theorem shows that almost every graph is divergent under the biclique operator. We remark that the linear time algorithm for recognizing convergent or divergent graphs given later in this section is based on this theorem.

**Theorem 4.3.** Let \( G \) be a graph. If \( G \) has at least 7 bicliques, then \( G \) diverges.
under the biclique operator.

Proof. By way of contradiction, suppose that $G$ has at least 7 bicliques and $G$ converges under the biclique operator. By Corollary 3.4, $Tw(KB(G)) = K_n$ for $n = 1, ..., 4$. Consider the following cases.

Case $n = 1$. Then $KB(G) = K_1$ is a contradiction since $G$ has at least 7 bicliques.

Case $n = 2$. Then $KB(G) = K_2$ or $KB(G)$ is bipartite with more than two vertices. In the first case $G$ has only 2 bicliques and therefore a contradiction. If $KB(G)$ is bipartite with more than two vertices $KB(G)$ is not a biclique graph [17] and that leads to a contradiction.

Case $n = 3$. Since $G$ has at least 7 bicliques it follows that in $KB(G)$ there exists a set of false-twin vertices of size at least three. Consider the bicliques $B_1, B_2, B_3$ of $G$ associated to the three false-twin vertices. If there is a pair of bicliques $B_i, B_j$ such that there is no edge between any vertex of $B_i$ and any vertex of $B_j$, by Lemma 4.1 it follows that $K_5$ is an induced subgraph of $KB(G)$. Otherwise, for every two pair of bicliques $B_i, B_j$ there is an edge between some vertex of $B_i$ and some vertex of $B_j$ and by Lemma 4.2 $KB(G)$
contains $K_5$ as an induced subgraph. In any case, by Theorem 3.3 $G$ diverges under the biclique operator, a contradiction.

**Case** $n = 4$. There are two alternatives. Suppose that $KB(G)$ has a set of false-twin vertices of size at least three. Then following the proof of the case $n = 3$ we arrive to a contradiction. Otherwise, there are only two possible graphs isomorphic to $KB(G)$ ($KB(G)$ has 7 or 8 vertices and it has no set of three false-twin vertices, see Fig. 12). By inspection, using the characterization given in [17] we prove that these two graphs are not biclique graphs. We conclude that this case can not occur.

![Figure 12: Unique two possible graphs for case $n = 4$.](image)

Since we covered all cases, $G$ diverges under the biclique operator and the proof is finished.

The next step is to study graphs without false-twin vertices with at least 7 bicliques. This will complete the idea of the linear time algorithm for recognizing divergent and convergent graphs under the biclique operator.

**Theorem 4.4.** Let $G$ be a false-twin-free graph. If $G$ has at least 13 vertices then $G$ has at least 7 bicliques.

**Proof.** We prove the result by induction on $n$. For $n = 13$, by inspection of all graphs without false-twin vertices the result holds. Suppose now that $n \geq 14$. By a Theorem in [7] there is a vertex $v$ such that $G - \{v\}$ has no false-twin vertices. Consider the graph $G' = G - \{v\}$. If $G'$ is connected, since it has at least 13 vertices, by inductive hypothesis it has at least 7 bicliques. Now as $G'$ is an induced subgraph of $G$ we conclude that $G$ also has at least
7 bicliques. Suppose now that $G'$ is not connected. Let $G_1, G_2, \ldots, G_s$ be the connected components of $G'$ on $n_1, n_2, \ldots, n_s$ vertices respectively. Since $G$ has no false-twin vertices, it can be at most one $G_i$ such that $n_i = 1$. If there is no component with at least 13 vertices, then by inductive hypothesis this component has at least 7 bicliques and so does $G$. Therefore every component has at most 12 vertices. Now, by inspection we can verify that every component $G_i$ (but maybe one with just 1 vertex) has at least $\left\lceil \frac{n_i}{2} \right\rceil$ bicliques. Also, since $G'$ is disconnected, $v$ along with at least one vertex of each of the $s$ components is a biclique in $G$ isomorphic to $K_{1,s}$ that is lost in $G'$. Summing up and assuming the worst case, that is, there exists one $n_i = 1$ (suppose $i = s$) we obtain that the number of bicliques of $G$ is at least

$$\left( \sum_{i=1}^{s-1} \left\lceil \frac{n_i}{2} \right\rceil \right) + 1 \geq \left\lceil \frac{11}{2} \right\rceil + 1 = 7$$

as we wanted to prove. Now the proof is complete. \qed

Theorem 4.4 implies that the number of convergent graphs without false-twin vertices is finite since convergent graphs without false-twin vertices have at most 12 vertices. This fact leads to the following linear time algorithm.

**Algorithm:** Given a graph $G$, build $H = Tw(G)$. If $H$ has at least 13 vertices, answer “$G$ diverges” and STOP. Otherwise, build $Tw(KB(H))$. If $Tw(KB(H))$ has at most 4 vertices answer “$G$ converges” and STOP. Otherwise, answer “$G$ diverges” and STOP.

The algorithm has $O(n + m)$ time complexity. For this observe that $H$ can be built in $O(n + m)$ time using the modular decomposition [18]. Finally, if $H$ has at most 12 vertices any further operation takes $O(1)$ time complexity.

## 5 Conclusions

In [16] it is given an $O(n^4)$ time algorithm to recognize convergent and divergent graphs under the biclique operator. In this paper we prove that graphs without false-twin vertices with at least 13 vertices diverge. This shows that “almost every” graph is divergent and as a direct consequence, we obtain a
linear time algorithm for recognizing the behavior of a graph under the biclique operator. We remark that in contrast as the iterated clique operator, no polynomial time algorithm is known for recognizing any of its possible behaviors.

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