On the connection between orthant probabilities and the first passage time problem

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Abstract

This article describes a new Monte Carlo method for the evaluation of the orthant probabilities by sampling first passage times of a non-singular Gaussian discrete time-series across an absorbing boundary. This procedure makes use of a simulation of several time-series sample paths, aiming to record their first crossing instants. Thus, the computation of the orthant probabilities is traced back to the accurate simulation of a non-singular Gaussian discrete-time series. Moreover, if the simulation is also efficient, this method is shown to be more speedy than the others proposed in the literature. As example, we make use of the Davies-Harte algorithm in the evaluation of the orthant probabilities associated to the ARFIMA(0,d,0) model. Test results are presented that compare this method with currently available software.

Keywords: First passage time, orthant probabilities, simulation of Gaussian discrete-time series, Davies-Harte algorithm.

1 Introduction

For many computational problems in statistics and especially in economics (cf. for example Ku and Seneta, 1994), we have to compute the so-called orthant probabilities (Tong, 1990) of the form

\[ P \left( \bigcap_{i=1}^{k} \{ X_i < S_i \} \right) = \int_{O_k} \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x \right\} dx, \]

where the random vector \( X \equiv (X_1, X_2, \ldots, X_k) \) follows a multivariate normal distribution with zero mean and unity variance, \( \Sigma \) is a \( k \times k \) symmetric positive definite correlation matrix and

\[ O_k = \{ x \equiv (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_i < S_i, S_i \in \mathbb{R}, i = 1, 2, ..., k \} \]
are the so-called orthant regions. Since analytical closed forms of the multidimensional integral (1) are known only in a few special cases, their computation leads almost always to numerical methods.

Within numerical methods, the evaluation of (1) is full of history. The expansion in tetrachoric series was the first attempt (cf. for a review Gupta, 1963), a method quickly abandoned due to the slow convergence. A completely different approach is the Plackett’s formula (Plackett, 1954) consisting of an integral dimension-reduction formula based on ad hoc conditioning. This formula is used in the the routine G01HBF of the NAG Fortran software library (see http://www.nag.co.uk/) that evaluates (1) up to \( k = 10 \). Different numerical algorithms can be simply obtained via standard numerical integrations, such as the classical Monte-Carlo method or some adaptive quadrature formulae. For \( k = 2 \) and \( k = 3 \) Genz (2001, see also references therein) sets up a competitive algorithm based on Gaussian quadratures with adaptive integration. If \( k > 3 \), the infinite integration limits in (1) need to be carefully handled, either by using some type of transformation into a finite region (see Genz, 1992 and 1993 and references therein) or by using selected cutoff values. Genz has shown that the quadrature formulae take less time and are more accurate than the Monte Carlo algorithm for small \( k \), whereas the Monte Carlo method appears more efficient and has comparable accuracy with \( k \) greater than 6 or 7. In the last few years, a completely different approach has spread, based on probability simulation. In a review paper, Hajivassiliou et al. (1996) analyze the properties of several simulators and find that the one they propose with the label GHK (Geweke-Hajivassiliou-Keane) works better than all other methods by keeping a good balance between accuracy and computational costs. Here the idea consists of estimating (1) by means of recursive conditioned probabilities involving samples of truncated standard normal random variables (r.v.’s). When \( \Sigma \) has a tridiagonal form or sparse structures, there are different numerical methods available based on special decompositions of \( \Sigma \) (for similar matters see Tong, 1990). When the orthant regions have bounds like \( S_i = 0 \) for any \( i \), the interest has been focused on expressing (1) in terms of the correlation coefficients. Some solutions are available in closed forms for \( k = 2 \) and \( k = 3 \) while for \( k \geq 4 \) integral representations may be obtained. For example, Sun (1988) gives analytical decompositions of (1) into a combination of several low order integrals. A \( k \)-dimensional recursion formula is found in Zhongren and Kedem (1999) by using a polar coordinates trasformation.

In short, with up to \( k = 10 \), there are different numerical methods available by which orthant probabilities can be robustly and reliably computed.
at low to moderate accuracy levels. High accuracy or high dimension problems can require long computation times and it is still not clear what is the best method for this type of problem.

The aim of this paper is to explore the connection between the orthant probabilities (1) and the first passage time (FPT) problem for a non-singular Gaussian discrete time-series. This connection suggests a new way to evaluate the orthant probabilities by numerical methods. The keystones is choosing among some simulation procedures devoted to the construction of time-series paths. The mathematical kernel of the FPT problem for a non-singular Gaussian time-series is recalled in section 2 (for a more detailed analysis see Di Nardo, 2002). The Genz algorithm and the GHK simulator are summarized in section 3 together with some classical simulation procedures of the time-series. As a working example, in the last section the orthant probabilities are computed in connection with the ARFIMA(0, d, 0) model by means of the FPT approach. Comparisons are also given with the results reached via the Genz algorithm and the GHK simulator.

2 The mathematical kernel

Let us define the FPT r.v. of a discrete time-series $X_t$ through an absorbing time dependent boundary $S_t$ as

$$T = \min\{t : X_t \geq S_t\}. \quad (2)$$

In the following, suppose $P(X_0 = x_0) = 1$ with $x_0 < S_0$ and in order to simplify set $x_0 = 0$.

First we observe that

$$P(T = k) = P(X_1 < S_1, X_2 < S_2, \ldots, X_{k-1} < S_{k-1}, X_k \geq S_k). \quad (3)$$

When $X_t$ is a Gaussian time-series with zero mean, unity variance and symmetric positive definite correlation matrix $\Sigma$, it is

$$P(T = k) = \int_{D_k} \frac{1}{(2\pi)^{k/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x \right\} dx$$

where

$$D_k = \{x \equiv (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_i < S_i, i = 1, 2, \ldots, k - 1, x_k \geq S_k\}.$$
Setting $\mathbf{S} \equiv (S_1, S_2, \ldots, S_k)$ and $P_k(\mathbf{S}, \Sigma) = P\left(\cap_{i=1}^{k} \{X_i < S_i\}\right)$, equation (3) could be rewritten as

\[
P(T = k) = \begin{cases} 
1 - P_1(\mathbf{S}, \Sigma), & \text{if } k = 1, \\
P_{k-1}(\mathbf{S}, \Sigma) - P_k(\mathbf{S}, \Sigma), & \text{if } k > 1
\end{cases}
\]

by which it results

\[
P_k(\mathbf{S}, \Sigma) = 1 - P(T \leq k). \tag{4}
\]

The above equation provides the link between the orthant probabilities and the FPT distribution of a non-singular Gaussian time-series. This suggests the use of the machinery of the FPT problems for a speedy computing of (1).

The next result allows us to estimate the FPT distribution function via simulation of the time-series $X_t$.

**Theorem 2.1** For an upper-bounded boundary $S_t$, the FPT r.v. (2) is fair, i.e. $P(T < \infty) = 1$.

**Proof.** Denote by $\rho_{ij}$ the correlation coefficients of $X_i$ and $X_j$ for $i, j = 1, 2, \ldots, k$. Set $S_{\text{max}} = \max_{t \geq 0} S_t < \infty$ and $\rho_{\text{max}} = \max_{i \neq j} \rho_{ij}$. By using the Slepian inequality (Tong, 1990),

i) if $\rho_{\text{max}} > 0$ it is

\[
P_k(\mathbf{S}, \Sigma) \leq P_k(\mathbf{S}, \Sigma_{\rho_{\text{max}}}) \leq P_k(\mathbf{S}_{\text{max}}, \Sigma_{\rho_{\text{max}}}) \tag{5}
\]

where $\Sigma_{\rho_{\text{max}}}$ is the matrix with 1 on the principal diagonal and $\rho_{\text{max}}$ elsewhere and $\mathbf{S}_{\text{max}}$ is a vector whose $k$ components are equal to $S_{\text{max}}$. The random vector with normal distribution function $N(0, \Sigma_{\rho_{\text{max}}})$ has exchangeable r.v.’s and so (Tong, 1990)

\[
P_k(\mathbf{S}_{\text{max}}, \Sigma_{\rho_{\text{max}}}) = \int_{-\infty}^{\infty} \Phi^k \left( \frac{S_{\text{max}} + \sqrt{\rho_{\text{max}}}z}{\sqrt{1 - \rho_{\text{max}}}} \right) \phi(z)dz, \quad S \in \mathbb{R}
\]

where

\[
\phi(z) = \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{-z^2}{2} \right\}, \quad \Phi(x) = \int_{-\infty}^{x} \phi(z)dz, \quad x \in \mathbb{R}; \tag{6}
\]

ii) if $\rho_{\text{max}} \leq 0$ it is

\[
P_k(\mathbf{S}, \Sigma) \leq P_k(\mathbf{S}, I) \leq P_k(\mathbf{S}_{\text{max}}, I) = \Phi^k(S_{\text{max}}) \tag{7}
\]

where $I$ is the identity matrix and $\Phi(x)$ is given in (6).
From (5) and (7), it is
\[ \lim_{k \to \infty} P_k(S, \Sigma) = 0 \]  (8)
and the result follows taking the limit in (4) for \( k \) going to infinity.  

3 Numerical evaluations of orthant probabilities

Let us recall that the analytical results on FPT problems are mostly centered on stochastic processes of diffusion type, where the Markov property plays a leading role in handling the related transition probability density function (pdf), see Ricciardi et al. (1999). The FPT distribution also has an explicit analytical expression for the class of stochastic processes named the Levy type anomalous diffusion, in which the mean square displacement of the diffusive variable \( X_t \) scales with time as \( t^\gamma \) with \( 0 < \gamma < 2 \) (see Balakrishnan 1985, Molchan 1999, Rangarajan and Ding 2000). Nevertheless, apart from few special cases no closed forms of the FPT distribution are available in the literature. Thus numerical algorithms or simulation procedures have been resorted to in order to get more information on FPT features. In particular, simulation procedures are the tools mainly used, as they are especially suitable to being implemented on parallel computers (see Di Nardo et al., 2000).

A typical simulation procedure lies in sampling \( N_S \) values of the FPT r.v. \( T \), by a suitable construction of as many time-discrete paths, and then to record the instants when such realizations first cross the boundary.

Note that \( k \) consecutive observations of a standardized Gaussian time-series can be generated via the innovations algorithm from \( k \) i.i.d. standard normal r.v.’s with \( O(k^3) \) operations (Brockwell and Davies, 1991). However, if we add few hypotheses on the correlation function, faster methods are available. For example the Durbin-Levinson algorithm (Brockwell and Davies, 1991) is an efficient procedure with \( O(k^2) \) operations if \( X_t \) is stationary and \( \rho_t \) vanishing when \( t \) goes to infinity. If in addition the discrete Fourier transform of the circulant embedding vector
\[ \{\rho_0, \rho_1, \ldots, \rho_{k-1}, \rho_k, \rho_{k-1}, \ldots, \rho_1\} \]  (9)
is a positive real sequence, a still faster method is the Davies-Harte algorithm (Davies et al., 1987) with computational cost \( O(k \log k) \) (for a review on the methods generating realizations of a Gaussian stationary process see Percival, 1992).
Therefore in order to evaluate the orthant probabilities (1), we suggest
the following Monte Carlo method: choose a faster method simulating the
paths of $X_t$, record their first crossing instants through the boundary $S$ and
then estimate (1) through (4).

In the next section, a FORTRAN 77 implementation of this procedure
is compared with the Genz algorithm and the GHK simulator that appear
to be the more widespread methods computing orthant probabilities.

The Genz method (Genz, 1993) transforms the original integral (1) into
an integral over unit hypercube

$$P_k(S, \Sigma) = e_1 \int_0^1 e_2 \int_0^1 \cdots \int_0^1 e_k \int_0^1 d\mathbf{w},$$

(10)

where $d\mathbf{w} \equiv (dw_1, \ldots, dw_k)$,

$$e_i = \begin{cases}
\Phi \left( \frac{S_1}{c_{i1}} \right), & \text{for } i = 1 \\
\Phi \left( \frac{S_i - \sum_{j=1}^{i-1} c_{ij}\Phi^{-1}(w_j e_j)}{c_{ii}} \right), & \text{for } i = 2, \ldots, k,
\end{cases}$$

c_{ij} are elements of the lower triangular Cholesky decomposition of $\Sigma$, $\Phi(x)$
is given in (6) and $\Phi^{-1}(x)$ is its inverse. The idea of the Genz method is to
apply a standard Monte Carlo method to (10) and evaluate \{e_i\} by sampling
pseudo-random numbers over $(0, 1)$. Let us observe that the Cholesky decom-
position of $\Sigma$ needs $O(k^3)$ computations for a fixed $k$. The evaluation of the
integrand function in (10) takes $O(k^2)$ computations for each iteration and
such an evaluation is repeated until the estimated error is less than a prefixed
tolerance or the iterations number reaches a superior limit. The method
has been implemented in FORTRAN 77 and in GAUSS, and the routines
are available at http://www.sci.wsu.edu/math/faculty/genz/homepage up
to $k = 100$.

The GHK simulator (Hajivassiliou et al., 1996) estimates (1) by means
of

$$\hat{P}_k(S, \Sigma) = \frac{1}{N} \sum_{n=1}^N P(A_1)P(A_2|y_{1,n}) \cdots P(A_k|y_{1,n}; y_{2,n}; \ldots; y_{k-1,n})$$

(11)

where \{y_{i,n}\} are drawn sequentially from independent standard normal dis-
tributions truncated to the events $A_i$

$$A_i = \left\{ -\infty < Y_i < \frac{S_i - \sum_{j=1}^{i-1} c_{ji}Y_j}{c_{ii}} \right\}$$
with \( c_{ij} \) elements of the lower triangular Cholesky decomposition of \( \Sigma \) and \( \{Y_i\}_{i \in \mathbb{N}} \) a sequence of i.i.d.r.v.’s having standard normal distribution. Again the Cholesky decomposition of \( \Sigma \) takes \( O(k^3) \) computations for a fixed \( k \), whereas the evaluation of (11) is of order \( O(N \times k^2) \) for \( N \) sampling. The method has been implemented in FORTRAN 77 and in GAUSS, and the routines are available at http://econ.lse.ac.uk/staff/vassilis/ up to \( k = 40 \).

4 A working example

The Davies-Harte algorithm generates a path with \( k + 1 \) steps of a stationary Gaussian time-series having zero mean and autocovariances \( \rho_0, \ldots, \rho_k \), such that the finite Fourier transform of (9) is non-negative. Craigmiled (2003) has shown that the Davies-Harte algorithm is the most efficient way to simulate an ARFIMA\((0, d, 0)\) process (Hosking, 1981). Let us recall that the standardized ARFIMA\((0, d, 0)\) time-series is Gaussian, stationary with zero mean and correlation function

\[
\rho_k = \frac{d(d+1) \cdots (d+k-1)}{(1-d)(2-d) \cdots (k-d)}, \quad k = 1, 2, \ldots
\]  

(12)

if \( \rho_0 = 1 \). To simulate a realization \( X_1, X_2, \ldots, X_k \) with the Davies-Harte algorithm, the steps are:

i) for \( n = 0, 1, 2, \ldots, 2k - 3 \) evaluate

\[
g_n = \sum_{j=0}^{k-2} \gamma(j) \exp \left( \frac{2\pi i j n}{2k-2} \right) + \sum_{j=k-1}^{2k-3} \gamma(2k-2-j) \exp \left( \frac{2\pi i j n}{2k-2} \right); \quad (13)
\]

ii) for \( j = 0, 1, \ldots, k - 1 \) evaluate

\[
X_{j+1} = \frac{1}{2\sqrt{k-1}} \sum_{n=0}^{2k-3} Z_n \sqrt{g_n} \exp \left( \frac{2\pi i j n}{2k-2} \right)
\]

(14)

where \( Z_0, Z_{k-1} \) are real normal r.v.’s with zero mean and variance 2, \( \{Z_n\}_{n=1}^{k-2} \) is a sequence of independent complex normal r.v.’s with independent real and imaginary parts, each of variance 1, and \( Z_n = Z_{2k-2-n} \) for \( n = k, \ldots, 2k - 3 \).

Indeed, inverting equation (13) it is

\[
\rho_j = \frac{1}{2k-2} \sum_{n=0}^{2k-3} g_n \exp \left( \frac{2\pi i j n}{2k-2} \right), \quad j = 0, 1, \ldots, k - 1
\]
so that
\[
\text{Cov}(X_{p+1} X_{q+1}) = E (X_{p+1} X_{q+1})
\]
\[
= \frac{1}{4(k-1)} \sum_{n=0}^{2k-3} \sum_{l=0}^{2k-3} E(Z_n Z_l) \sqrt{g_n g_l} \exp \left( \frac{2\pi i [pn - lq]}{2k - 2} \right)
\]
\[
= \frac{1}{4(k-1)} \sum_{n=0}^{2k-3} 2\sqrt{g_k^2} \exp \left( \frac{2\pi i (p - q)k}{2k - 2} \right) = \rho_{p-q}.
\]

Then from (4), an estimation of (1) is
\[
\tilde{P}_k(S, \Sigma) = 1 - \frac{\text{number of first crossings before or equal } k}{N_S} \quad (15)
\]
where \(N_S\) is the number of simulated paths.

Observe that the step \(i)\) is evaluated just once for any fixed \(k\) with computational cost \(O(k \log k)\), whereas the step \(ii)\) is repeated \(N_S\) times each with \(O(k \log k)\) evaluations. To speed up the simulation with the fast Fourier transform algorithm, round \(k\) up to the nearest power of two and truncate the simulated series at the end.

In the following, we take down some results in the evaluation of the orthant probabilities with the Genz algorithm, the GHK simulator and the method suggested here. Specifically, Table 1 refers to the case of constant boundary \(S(t) = 1, d = 0.2\) and \(k \geq 20\) and Table 2 refers to the case of linear boundary \(S(t) = 2 - 0.01t, d = 0.3\) and \(k \geq 20\). In both cases, \(\tilde{P}_k\) stands for numerical evaluations of (1).

With regard to the Genz method, the maximum number \(N_{\text{max}}\) of integrand function evaluations has been allowed to \(k \times 10^3\), as suggested by the same author. With this choice, the computed value \(\tilde{P}_k\) has reached the absolute accuracy \(10^{-4}\) required in input, except in some cases marked with star in the tables. To the 99% confidence level, the estimated absolute error has been near \(10^{-4}\) for all \(k \geq 20\), except the cases marked with star in the tables where it has been of order \(10^{-3}\). In Tables 1 ÷ 2, \(CT_G\) represents an evaluation in seconds of the time employed by the method in computing \(\tilde{P}_k\) with such parameters.

With regard to the GHK method, the number \(N\) of allowed simulations has been set equal to \(N_S\) as indicated in Tables 1 ÷ 2. In order to generate the required \(N \times k\) pseudo-random uniform numbers over \((0, 1)\), it has been employed the routine G05CAF of the NAG software library. In Tables 1 ÷ 2,
$CT_G$ represents an evaluation in seconds of the time employed by the method in computing $\tilde{P}_k$ with such parameters.

The method here proposed has been implemented in a FORTRAN 77 program on a server ALPHA with operating system OSF1 5.0 and by using the NAG software library:

i) the routine G05FDF generates a vector of normal pseudo-random numbers by means of the Box and Muller method (see for example Rubinstein, 1981);

ii) the routine C06HBF computes the discrete Fourier cosine transform of the sequence (13);

iii) the routine C06EBF evaluates the discrete Fourier transform of the Hermitian sequence in (14) with the fast Fourier transform (see Brigham, 1974).

The time window size has been set equal to $2^5$ for $k = 20 ÷ 30$ and equal to $2^6$ for $k = 31 ÷ 40$. In Tables 1 ÷ 2, $N_S$ is the number of simulated sample paths whereas $CT_{FPT}$ represents an evaluation in seconds of the time employed by the routine in computing $\tilde{P}_k$ with such parameters.

As it would be theoretically expected, the GHK simulator and the Genz method cost more in computational time than the simulation procedure with the Davies-Harte algorithm. Especially in the case of non-constant boundary, the Genz method is time-consuming if one would reach the required accuracy. Furthermore, the proposed Monte Carlo method is particularly suited to be implemented on a parallel computer in order to further cut the computational time and to improve the approximation; this because the sample paths of the simulated time-series could be easily generated independently of one another.

References

[1] Balakrishnan V. (1985) Anomalous diffusion in one dimension. Physica A 132, 569–580.

[2] Brigham E.O. (1974) The Fast Fourier Transform. Prentice-Hall.

[3] Brockwell P.J., Davis R.A. (1991) Time Series: Theory and Methods. Springer-Verlag, New York, 2nd. ed.
[4] Craigmile P.F. (2003) Simulating a Class of Stationary Gaussian Processes Using the Davies-Harte Algorithm, with Application to Long Memory Processes. *Jour. Time Series Anal.* 24, 505–511.

[5] Davies R.B., Harte D.S. (1987) Test for Hurst effect. *Biometrika* 74, 95–101.

[6] Di Nardo E. (2002) On first-passage problem for a non-singular Gaussian discrete-time series. *Quad. Stat.* 4, 51–70.

[7] Di Nardo E., Nobile A.G., Pirozzi E., Ricciardi L.M., Rinaldi S. (2000) Simulation of Gaussian processes and first passage time densities evaluation. *Lecture Notes in Computer Science* 1798, 319-333.

[8] Genz A. (1992) Numerical Computation of Multivariate Normal Probabilities. *J. Comp. Graph Stat.* 1, 141–149.

[9] Genz A. (1993) Comparison of Methods for the Computation of Multivariate Normal Probabilities. *Computing Science and Statistics* 25, 400–405.

[10] Genz A. (2001) Numerical Computation of Rectangular Bivariate and Trivariate Normal and t Probabilities. Preprint.

[11] Gupta S.S. (1963) Probability Integrals of Multivariate Normal and Multivariate t. *Ann. Math. Statist.* 34, 792–828.

[12] Hajivassiliou V., McFadden D., Ruud P. (1996) Simulation of Multivariate Normal Rectangle Probabilities and their Derivatives. Theoretical and Computational results. *J. Econom.* 72 (1-2), 85–134.

[13] Hosking J.R.M. (1981) Fractional Differencing. *Biometrika* 68, 165–176.

[14] Ku S., Seneta E. (1994) The number of peaks in a stationary sample and orthant probabilities. *Jour. Time Series Anal.* 15, 385–403.

[15] Molchan G.M. (1999) Maximum of a fractional Brownian motion: Probabilities of small values. *Commun. Math. Phys.* 205, 97–111.

[16] Percival D.B. (1992) Simulating Gaussian Random Processes with Specified Spectra. *Computing Science and Statistics* 24, 534–538.

[17] Plackett R.L. (1954) A Reduction Formula for Normal Multivariate Integrals. *Biometrika* 41, 351–360.
[18] Rangarajan G., Ding M. (2000) Anomalous diffusion and the first passage time problem. Physical Review E 62, 120–133.

[19] Rangarajan G., Ding M. (2000) First passage time distribution for anomalous diffusion. Physics Letters A 273, 322–330.

[20] Ricciardi L.M., Di Crescenzo A., Giorno V., Nobile A.G. (1999) An Outline of Theoretical and Algorithmic Approaches to First Passage Time Problems with Applications to Biological Modeling. Math. Japonica 50, 247–322.

[21] Rubinstein R.Y. (1981) Simulation and the Monte Carlo Method. Wiley, New York.

[22] Sun H.J. (1988) A General Reduction Method for $n$-Variate Normal Orthant Probability. Comm. Stat. Theory and Methods 17, 3913–3921.

[23] Tong Y.L. (1990) The Multivariate Normal Distribution. Springer-Verlag, New-York.

[24] Zhongren N., Kedem B. (1999) On Normal Orthant Probabilities. Chinese J. Appl. Probab. Statist. 15, 262–275.
Table 1: Evaluations of the orthant probabilities $\tilde{P}_k$ for $S(t) = 1$ and $d = 0.2$. $CT_G$, $CT_{GHK}$ and $CT_{FPT}$ represent an evaluation in seconds of the time employed in the computation of $\tilde{P}_k$ respectively for the Genz method, the GHK method and the FPT method. $N_S$ is the number of the simulated sample paths in the FPT method.

| $k$ | $\tilde{P}_k$ | $CT_G$ | $\tilde{P}_k$ | $CT_{GHK}$ | $\tilde{P}_k$ | $CT_{FPT}$ | $N_S$ |
|-----|---------------|--------|---------------|------------|---------------|------------|-------|
| 20  | 0.0924        | 2.91   | 0.0927        | 0.49       | 0.0925        | 0.38       | 2000  |
| 21  | 0.0835        | 7.19   | 0.0835        | 0.54       | 0.0838        | 0.41       | 2100  |
| 22  | 0.0756        | 5.11   | 0.0757        | 0.58       | 0.0752        | 0.41       | 2100  |
| 23  | 0.0684        | 8.07   | 0.0687        | 0.64       | 0.0686        | 0.42       | 2200  |
| 24  | 0.0620        | 3.69   | 0.0622        | 0.68       | 0.0618        | 0.42       | 2200  |
| 25  | 0.0563        | 8.98   | 0.0572        | 0.80       | 0.0558        | 0.46       | 2400  |
| 26  | 0.0511        | 9.45   | 0.0512        | 0.93       | 0.0513        | 0.50       | 2650  |
| 27  | 0.0463        | 4.28   | 0.0465        | 1.03       | 0.0460        | 0.50       | 2650  |
| 28  | 0.0422        | 6.81   | 0.0422        | 1.05       | 0.0423        | 0.50       | 2650  |
| 29  | 0.0383        | 7.18   | 0.0387        | 1.15       | 0.0385        | 0.52       | 2750  |
| 30  | 0.0349        | 4.93   | 0.0348        | 1.25       | 0.0335        | 0.53       | 2800  |
| 31  | 0.0317        | 3.33   | 0.0320        | 2.31       | 0.0318        | 1.43       | 3900  |
| 32  | 0.0289        | 5.37   | 0.0291        | 2.45       | 0.0288        | 1.47       | 3950  |
| 33  | 0.0264        | 5.57   | 0.0266        | 2.58       | 0.0262        | 1.48       | 4000  |
| 34  | 0.0240        | 8.95   | 0.0243        | 2.76       | 0.0232        | 1.48       | 4000  |
| 35  | 0.0220        | 6.03   | 0.0222        | 2.88       | 0.0215        | 1.48       | 4000  |
| 36  | 0.0199        | 1.07   | 0.0202        | 3.01       | 0.0198        | 1.48       | 4000  |
| 37  | 0.0182        | 6.64   | 0.0185        | 3.17       | 0.0182        | 2.30       | 6300  |
| 38  | 0.0167        | 4.37   | 0.0167        | 5.39       | 0.0165        | 2.30       | 6300  |
| 39  | 0.0153        | 6.95   | 0.0154        | 6.80       | 0.0155        | 2.34       | 6400  |
| 40  | 0.0140        | 3.09   | 0.0140        | 6.18       | 0.0137        | 2.38       | 6500  |
Table 2: As in Table 1 for $S(t) = 2 - 0.01 t$ and $d = 0.3$.

| $k$ | $P_k$ | $CT_G$ | $P_k$ | $CT_{GHK}$ | $P_k$ | $CT_{FPT}$ | $N_S$ |
|-----|-------|--------|-------|-----------|-------|-----------|------|
| 20  | 0.6661| 25.00  | 0.6683| 0.74      | 0.6661| 0.62      | 3100 |
| 21  | 0.6520*| 26.52  | 0.6523| 0.93      | 0.6518| 0.71      | 3500 |
| 22  | 0.6381| 12.43  | 0.6397| 1.04      | 0.6381| 0.75      | 3700 |
| 23  | 0.6243| 45.02  | 0.6247| 1.25      | 0.6240| 0.84      | 4200 |
| 24  | 0.6107| 9.16   | 0.6101| 1.41      | 0.6106| 0.90      | 4500 |
| 25  | 0.5972| 33.16  | 0.5966| 1.52      | 0.5973| 0.90      | 4500 |
| 26  | 0.5838| 53.18  | 0.5847| 1.67      | 0.5832| 0.93      | 4650 |
| 27  | 0.5708| 36.85  | 0.5711| 1.80      | 0.5706| 0.94      | 4700 |
| 28  | 0.5578| 25.49  | 0.5584| 1.88      | 0.5578| 0.94      | 4700 |
| 29  | 0.5450| 26.58  | 0.5456| 2.00      | 0.5451| 0.94      | 4700 |
| 30  | 0.5323| 31.18  | 0.5331| 2.28      | 0.5324| 0.99      | 4850 |
| 31  | 0.5199| 19.32  | 0.5207| 2.96      | 0.5196| 1.91      | 4950 |
| 32  | 0.5075| 22.96  | 0.5077| 3.17      | 0.5080| 1.96      | 5100 |
| 33  | 0.4952| 31.59  | 0.4967| 3.47      | 0.4954| 2.03      | 5200 |
| 34  | 0.4833| 32.85  | 0.4833| 3.66      | 0.4847| 2.03      | 5250 |
| 35  | 0.4714| 34.15  | 0.4725| 4.04      | 0.4716| 2.12      | 5500 |
| 36  | 0.4596| 35.32  | 0.4596| 4.44      | 0.4596| 2.23      | 5800 |
| 37  | 0.4482| 24.34  | 0.4482| 4.70      | 0.4476| 2.29      | 5900 |
| 38  | 0.4368| 38.15  | 0.4362| 5.29      | 0.4357| 2.37      | 6100 |
| 39  | 0.4256| 26.13  | 0.4249| 5.73      | 0.4256| 2.50      | 6400 |
| 40  | 0.4146| 40.59  | 0.4133| 6.04      | 0.4146| 2.52      | 6500 |