MULTIPLICATIVE STRUCTURES ON THE TWISTED K-THEORY OVER FINITE GROUPS

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Abstract. Let $K$ be a finite group and let $G$ be a finite group acting over $K$ by automorphisms. In this paper we study two different but intimate related subjects: on the one side we classify all possible multiplicative and associative structures which one can endow the twisted $G$-equivariant K-theory over $K$, and on the other, we classify all possible monodical structures which one endow the category of twisted and $G$-equivariant bundles over $K$. We achieve this classification by encoding the relevant information in the cochains of a sub double complex of the double bar resolution associated to the semi-direct product $K times G$; we use known calculations of the cohomology of $K$, $G$ and $K times G$ to produce concrete examples of our classification.

In the case on which $K = G$ and $G$ acts by conjugation, the multiplication map $G times G 	o G$ is a homomorphism of groups and we define a shuffle homomorphism which realizes this map at the homological level. We show that the categorical information that defines the Twisted Drinfeld Double can be realized as the dual of the shuffle homomorphism applied to any 3-cocycle of $G$. We use the pullback of the multiplication map in cohomology to classify the possible ring structures that Grothendieck ring of representations of the Twisted Drinfeld Double may have, and we include concrete examples of this procedure.

Introduction

The purpose of this work is to investigate a relationship existing among certain tensor categories attached to a semi-direct product of groups (they include the Twisted Drinfeld Double of a discrete group), their fusion algebras and the cohomology groups of the semi-direct product. These tensor categories, as well as some of our results, are closely related to abelian extensions of Hopf algebras and the cohomological description of $\text{Opext}(\mathbb{C}^G, \mathbb{C}^F)$, given by Kac in [16], cf. [19].

The abelian extension theory of Hopf algebras was generalized to coquasi-Hopf algebras by Masuoka in [20]. Some of our results and constructions can be framed in the abelian extension theory of coquasi-Hopf algebras in the particular case where the matched pair of groups is a semi-direct product. However, our approach to these tensor categories does not follow Masuoka’s point of view, instead we use the concept of pseudomonoids in a suitable 2-monoidal 2-category associated to a group.

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There are some reasons why we prefer to use the pseudomonoid approach: First, it exhibits more clearly the relationship between the cohomology of semi-direct products and the multiplicative structures of the $G$-equivariant twisted $K$-theory over a finite group $K$. Second, some terms of a spectral sequence associated to the double complex have a direct interpretation in terms of obstructions and classification of the possible pseudomonoid structures. Third, some of our constructions and results make sense in categories different from the category of sets; in particular, if we change to the Cartesian categories of locally compact topological spaces and change discrete group cohomology by the Borel-Moore cohomology \[21\], we have a much general theory where the categories of coquasi-Hopf algebras may not capture all the desired information.

There are two main reasons for our interest in the fusion algebra of the tensor categories defined in this paper. On the one side, semisimple tensor categories can be encoded in a combinatorial structure divided in two parts: the fusion algebra (or the Gorthendick ring of the tensor category) and a non-abelian cohomological information provided by the $F$-matrices of the $6j$-symbols, \[24\].

And on the other, these fusion algebras are generalizations of $^wK([G/G])$, the $w$-twisted stringy $K$-theory of the groupoid $[G/G]$ (see \[25\]).

In the case that the semi-direct product is finite, the associated tensor categories are fusion categories, and they belong to a bigger family of fusion categories denoted group-theoretical fusion categories for which many interesting results have been established cf. \[12, 22\]. Since an explicit description of the fusion rules of the tensor categories studied in this paper, via induction and restriction of projective representations of certain subgroups of $G$ already appear in \[26\], Theorem 4.8], our approach focuses in determining the number of fusion category and fusion algebra structures associated to a fix semi-direct product. We accomplish this task in several steps. First, we show that the information encoding a pseudomonoid with strict unit in the 2-category of $G$-sets with twists over the group $K$ is equivalent to a 3-cocycle in $\mathbb{Z}^3(\text{Tot}^*(A^*\ast(K \times G, \mathbb{T})))$, where the double complex $A^*\ast(K \times G, \mathbb{T})$ is the sub double complex without the 0-th row of the double bar resolution $C^*\ast(K \times G, \mathbb{T})$ and whose total cohomology calculates the cohomology of $K \times G$. Second, we show that the information encoded in a pseudomonoid with strict unit in the 2-category of $G$-sets with twists $K$, is precisely the precise information required to endow the category $\text{Bun}_G(K)$ of projective $G$-equivariant complex vector bundles with a monodical structure; hence the isomorphism classes of bundles Groth($\text{Bun}_G(K)$) becomes a fusion algebra, and this fusion algebra structure could be alternatively understood as a twisted $G$-equivariant K-theory ring over $K$. Third, we define the twisted $G$-equivariant K-theory over the group $K$ and we show the conditions under which this group could be endowed with a multiplicative structure making it a ring; we define the group of multiplicative structures by $MS_G(K)$ and we show that this group could be calculated by the use of a spectral sequence associated to the complex $\text{Tot}^*(A^*\ast(K \times G, \mathbb{T}))$. We study the canonical homomorphism $H^3(\text{Tot}^*(A^*\ast(K \times G, \mathbb{T}))) \xrightarrow{\phi} MS_G(K)$ and we give an explicit description of its kernel and its cokernel; a multiplicative structure on $MS_G(K)$ not appearing in the image of $\phi$ endows the twisted $G$-equivariant K-theory of $K$ with a ring structure, which is not the fusion algebra of the tensor categories $\text{Bun}_G(K)$, or in other words an algebra structure, which is not possible to categorify.
Of particular interest is the case on which $G = K$ and $G$ acts on itself by conjugation. In this case, the cohomological information which was used in \cite{9} to define the Twisted Drinfeld Double $D^n(G)$ for $w \in Z^3(G, \mathbb{T})$ defines an element in $Z^3(\text{Tot}^*(A^{**}(G \times G, \mathbb{T})))$. The formulae defining this 3-cocycle were reminiscent of the formulae appearing in \cite{11} Theorem 5.2] on the proof of the Eilenberg-Zilber theorem, and we conjectured that there had to exist a way to define for any cocycle in $Z^n(G, \mathbb{T})$ a $n$-cocycle in $Z^n(\text{Tot}^*(A^{**}(G \times G, \mathbb{T})))$ having similar properties as the ones defined for $n = 3$. We show in this paper that indeed this is the case and its proof is based on two facts: first that the multiplication map $\mu : G \times G \rightarrow G$, $\mu(k, g) = kg$ is a homomorphism of groups, and second, on a construction of an explicit Shuffle homomorphism at the chain level, whose dual $\tau^\vee : C^*(G, \mathbb{T}) \rightarrow \text{Tot}^*(C^{**}(G \times G, \mathbb{T}))$ applied to $w$ recovers the cocycle defined in \cite{9}, and moreover that in cohomology equals the pullback of $\mu$, e.i. $\mu^* = \tau^* : H^*(G, \mathbb{T}) \rightarrow H^*(G \times G, \mathbb{T})$. Since the group $G \times G$ is isomorphic to the product $G \times G$ we get that the map $H^3(\text{Tot}^*(A^{**}(K \times G, \mathbb{T}))) \rightarrow \text{MS}_G(K)$ is surjective, and since we know that the cohomology class of the 3-cocycle that is defined in \cite{9} could be recovered from the cohomology class of $\mu^*w$, we give a simple procedure to determine the fusion algebras of $\text{Rep}(D^n(G))$ which are isomorphic to the $G$-equivariant K-theory ring $\mathbb{K}U_G(K)$; at the end of this work this procedure exemplified in some interesting cases.

This paper is organized as follows. In Section 1 we provide background material on the semi-direct products and the double bar complex associated to its cohomology. In Section 2 the Shuffle homomorphism of a trivializable semi-direct product is defined and some of its properties are shown. In Section 3 the 2-monoidal 2-category of twisted $G$-sets with strict unit and the 2-category of pseudomonoids in this 2-category are defined and described using the complex $\text{Tot}^*(A^{**}(K \times G, \mathbb{T}))$. In Section 4 the tensor category of equivariant vector bundles over a group, tensor functors and monoidal natural isomorphism associated to the 2-category of pseudomonoids in the 2-category of twisted $G$-sets with strict unit are defined. In Section 5 the obstruction to the existence of multiplicative structures over the twisted $G$-equivariant K-theory over a group $K$ is described using the spectral sequence associated to the filtration $F^r := A^{**r}$ of the double complex $A^{**}$. In Section 6 several concrete examples are completely calculated. We finish with an appendix in Section 7 on which we give the explicit relation of our tensor categories and coquasi-bialgebras.

1. Preliminaries

1.1. Semi-direct products. Let $K$ be a discrete group endowed with an action of the discrete group $G$ defined through a homomorphism $\rho : G \rightarrow \text{Aut}(K)$; for simplicity, for $g \in G$ and $k \in K$ denote the action by $g(k) := \rho(g)(k)$. Denote by $K \rtimes G$ the group defined by the semi-direct product of $G$ with $K$; as a set $K \rtimes G := K \times G$ and the product structure is defined by $(a, g)(b, h) := (a \rho(g)(b), gh)$.

The group $K \rtimes G$ fits in the short exact sequence

$$1 \rightarrow K \rightarrow K \rtimes G \rightarrow G \rightarrow 1$$

and we say that $K \rtimes G$ is isomorphic to another split extension

$$1 \rightarrow K \rightarrow E \rightarrow G \rightarrow 1$$
whenever there is an isomorphism $\psi : K \rtimes G \cong E$ such that $\pi_2 = p \circ \psi$. In what follows we will outline the conditions under which the semi-direct product $K \rtimes G$ is isomorphic to the direct product $K \times G$ as split extensions.

For this recall that $\text{Inn}(K)$ is the group of inner automorphisms of the group $K$, i.e. the automorphisms of $K$ induced by conjugation, and that it fits in the short exact sequences

$$1 \rightarrow Z(K) \rightarrow K \xrightarrow{\tau} \text{Inn}(K) \rightarrow 1$$

$$1 \rightarrow \text{Inn}(K) \rightarrow \text{Aut}(K) \rightarrow \text{Out}(K) \rightarrow 1$$

where $Z(K)$ denotes the center of $K$ and $\text{Out}(K)$ denotes the group of outer automorphisms of $K$.

**Proposition 1.1.** The semi-direct product $K \rtimes G$ is isomorphic to the product $K \times G$ if and only if there is a homomorphism $\sigma : G \rightarrow K$ such that the following diagram commutes

$$\begin{array}{ccc}
G & \xrightarrow{\sigma} & K \\
\downarrow{\rho} & & \downarrow{\tau} \\
\text{Inn}(K) & & \\
\end{array}$$

where the action of $G$ on $K$ is given by the inner automorphisms defined by $\rho$.

**Proof.** Suppose that there is a homomorphism $\sigma : G \rightarrow K$ with the desired properties. Define the isomorphism of sets

$$\psi : K \rtimes G \rightarrow K \times G, \quad (a,g) \mapsto (a\sigma(g),g);$$

note that on the one hand

$$\psi((a,g) \cdot (b,h)) = \psi(ab\rho(g)(b),gh)$$

$$= \psi(a\sigma(g)b\sigma(g)^{-1},gh)$$

$$= (a\sigma(g)b\sigma(h),gh)$$

and on the other hand

$$\psi(a,g)\psi(b,h) = (a\sigma(g),g)(b\sigma(h),h)$$

$$= (a\sigma(g)b\sigma(h),gh);$$

therefore the map $\psi$ is the desired isomorphism of groups.

For the converse let us suppose that $\psi : K \rtimes G \rightarrow K \times G$ is an isomorphism of split extensions. In particular we have that

$$\pi_2(\psi(a,g)) = g$$

since the projection on the second coordinate on both groups produce the anchor map to $G$. Define the map

$$\phi : G \rightarrow K, \quad g \mapsto \pi_1(\psi(1,g))$$

and hence we have that $\psi(1,g) = (\phi(g),g)$. Since $\psi$ is an isomorphism we have that

$$(\phi(gh),gh) = \psi(1,gh) = \psi(1,\psi(1,h)) = (\phi(g),g)(\phi(h),h) = (\phi(g)\phi(h),gh)$$

and therefore we obtain that the map $\phi$ is moreover a homomorphism of groups.
Denote by $\alpha : K \to K$ the isomorphism of groups induced by $\psi$ when $\alpha$ is defined by the equation
$$\psi(a, 1) = (\alpha(a), 1).$$
Applying $\psi$ to both sides of the equation
$$(1, g) \cdot (a, 1) = (g(a), g)$$
we obtain on the one side
$$\psi(1, g) \psi(a, 1) = (\phi(g), g)(\alpha(a), 1) = (\phi(g)\alpha(a), g)$$
and on the other
$$\psi(g(a), g) = \psi(g(a), 1) \cdot (1, g) = (\alpha(g(a)), 1)(\phi(g), g) = (\alpha(g(a))\phi(g), g),$$
implying that
$$\phi(g)\alpha(a)\phi(g)^{-1} = \alpha(g(a))$$
and applying $\alpha^{-1}$ we get
$$\alpha^{-1}(\phi(g))\alpha\alpha^{-1}(\phi(g))^{-1} = g(a)$$
which implies the desired equation
$$\tau(\alpha^{-1}(\phi(g))) = \rho(g).$$
Hence the homomorphism
$$\sigma := \alpha^{-1} \circ \phi : G \to K$$
fits into the diagram
$$\begin{array}{ccc}
G & \xrightarrow{\sigma} & K \\
\downarrow{\rho} & & \downarrow{\tau} \\
\text{Inn}(K). & & \\
\end{array}$$

**Corollary 1.2.** The semi-direct product $K \rtimes \text{Inn}(K)$ is isomorphic as split extensions to the direct product $K \times \text{Inn}(K)$ if and only if the natural short exact sequence
$$1 \to Z(K) \to K \to \text{Inn}(K) \to 1$$
 splits.

**Corollary 1.3.** If the semi-direct product $K \rtimes G$ is isomorphic to $K \times G$ then the action of $G$ on $K$ is given by inner automorphisms of $K$, and there is a homomorphism $\sigma : G \to K$ that realizes such inner automorphisms.

**Example 1.4.** Let $K = G$ and consider the conjugation action of $K$ on itself. In this case $\rho = \tau$ and we can take $\sigma = \text{Id}_K$. Therefore the map
$$\mu : K \times K \to K, \quad (a, g) \mapsto ag$$
is a homomorphism of groups and
$$\mu \times \pi_2 : K \times K \to K \times K, \quad (a, g) \mapsto (ag, g)$$
is an isomorphism.
Remark 1.5. Note that whenever we have a homomorphism \( \sigma : G \to K \) such that \( \rho = \tau \circ \sigma \) then the map

\[
\mu : K \rtimes G \to K, \quad (k,g) \mapsto k\sigma(g)
\]

becomes a group homomorphism. Moreover, the map

\[
K \rtimes G \to K \rtimes K, \quad (k,g) \mapsto (k,\sigma(g))
\]

is a homomorphism of groups.

1.2. Bar resolution. Let us find an explicit model for the homology of the group \( K \rtimes G \). For this, let us first setup the notation for the explicit model for the bar resolution that we will use.

Take \( H \) a discrete group and define the complex \( C_\ast(EH, \mathbb{Z}) \) with

\[
C_n(EH, \mathbb{Z}) := \mathbb{Z}H_\otimes n+1
\]

and with differential \( \partial_H : C_n(EH, \mathbb{Z}) \to C_{n-1}(EH, \mathbb{Z}) \) defined by the equation on generators

\[
\partial_H(h_1, h_2, ..., h_{n+1}) = (h_2, h_3, ..., h_{n+1}) + \sum_{i=1}^{n} (-1)^i (h_1, ..., h_i h_{i+1}, ..., h_{n+1}).
\]

The complex \((C_\ast(EH, \mathbb{Z}), \partial_H)\) becomes a complex in the category of \( ZH \)-modules if we endow each \( C_n(EH, \mathbb{Z}) \) with the left \( ZH \)-module structure defined by the equation

\[
h \cdot (h_1, ..., h_{n+1}) := (h_1, ..., h_{n+1} h^{-1}).
\]

The augmentation map

\[
\epsilon : C_0(EH, \mathbb{Z}) \to \mathbb{Z}, \quad \epsilon(h) = 1
\]

is a map of \( ZH \)-modules and the complex \( C_\ast(EH, \mathbb{Z}) \) becomes a \( ZH \)-free resolution of the trivial \( ZH \)-module \( \mathbb{Z} \),

\[
\epsilon : C_\ast(EH, \mathbb{Z}) \to \mathbb{Z}.
\]

The elements

\[
(h_1, h_2, ..., h_n, 1)
\]

generate the \( ZH \)-module \( C_n(EH, \mathbb{Z}) \) and therefore we could write them using the “bar notation”

\[
[h_1|h_2|...|h_n] := (h_1, h_2, ..., h_n, 1);
\]

the differential \( \partial_H \) in this base becomes

\[
\partial_H [h_1|h_2|...|h_n] = [h_2|h_3|...|h_n] + \sum_{i=1}^{n} (-1)^i [h_1|...|h_i h_{i+1}|...|h_n] \\
+ (-1)^{n+1} h_n^{-1} \cdot [h_1|...|h_{n-1}].
\]

For a left \( ZH \)-module \( W \), the homology groups of \( H \) with coefficients in \( W \) are defined as

\[
H_\ast(H, W) := H_\ast(C_\ast(EH, \mathbb{Z}) \otimes_{ZH} W)
\]

and the cohomology groups of \( H \) with coefficients in \( W \) as

\[
H^\ast(H, W) := H^\ast(\text{Hom}_{ZH}(C_\ast(EH, \mathbb{Z}), W)).
\]

Since we have a canonical isomorphism of \( Z \)-modules

\[
\text{Hom}_{ZH}(C_n(EH, \mathbb{Z}), W) \cong \text{Maps}(H^n, W),
\]
the cohomological differential $\delta_H$ in terms of the bar notation becomes

$$(\delta_H f)[h_1|h_2|...|h_n] = f[h_2|h_3|...|h_n] + \sum_{i=1}^{n} (-1)^i f[h_1|...|h_i|h_{i+1}|...|h_n] + (-1)^{n+1} h_{n-1}^{-1} \cdot f[h_1|...|h_{n-1}]$$

1.3. Cohomology of $K \times G$. Consider the double complex

$$C_*(EG, \mathbb{Z}) \otimes \mathbb{Z} C_*(EK, \mathbb{Z})$$

with differentials $\partial_G \otimes 1$ and $1 \otimes \partial_K$. Denote by $(g_1, ..., g_{p+1}|k_1, ..., k_{q+1})$ a generator in $C_p(EG, \mathbb{Z}) \otimes \mathbb{Z} C_q(EK, \mathbb{Z})$ and define the action of $(k, g) \in K \times G$ by the equation

$$(k, g) \cdot (g_1, ..., g_{p+1}|k_1, ..., k_{q+1}) := (g_1, ..., g_{p+1}g^{-1}|g(k_1), ..., g(k_{q+1})k^{-1})$$

A straightforward computation shows that indeed it is an action and therefore $C_p(EG, \mathbb{Z}) \otimes \mathbb{Z} C_q(EK, \mathbb{Z})$ becomes a free $\mathbb{Z}(K \times G)$-module. Since the differentials $\partial_G \otimes 1$ and $1 \otimes \partial_K$ are also maps of $\mathbb{Z}(K \times G)$-modules, we could take the total complex

$$\text{Tot}_r(C_*(EG, \mathbb{Z}) \otimes \mathbb{Z} C_*(EK, \mathbb{Z}))$$

whose degree $n$-component is

$$\text{Tot}_n(C_*(EG, \mathbb{Z}) \otimes \mathbb{Z} C_*(EK, \mathbb{Z})) := \bigoplus_{p+q=n} C_p(EG, \mathbb{Z}) \otimes \mathbb{Z} C_q(EK, \mathbb{Z})$$

and whose differential is

$$\partial_G \otimes 1 \oplus (-1)^p 1 \otimes \partial_K$$

thus obtaining a free $\mathbb{Z}(K \times G)$ complex. Since the homology of the total complex of this double complex is just $\mathbb{Z}$ in degree 0,

$$H_n(\text{Tot}_*(C_*(EG, \mathbb{Z}) \otimes \mathbb{Z} C_*(EK, \mathbb{Z})), \partial_G \otimes 1 \oplus (-1)^p 1 \otimes \partial_K) = \mathbb{Z},$$

we have that

$$\text{Tot}_*(C_*(EG, \mathbb{Z}) \otimes \mathbb{Z} C_*(EK, \mathbb{Z})) \overset{\sim}{\to} \mathbb{Z}$$

is a free $\mathbb{Z}(K \times G)$ resolution of $\mathbb{Z}$.

Making use of the bar notation we take the elements

$$[g_1|...|g_p||k_1|...|k_q] := (g_1, ..., g_p, 1|k_1, ..., k_q, 1)$$

as a set of generators of $C_p(EG, \mathbb{Z}) \otimes \mathbb{Z} C_q(EK, \mathbb{Z})$ as a $\mathbb{Z}(K \times G)$-module; in this base we have the equality

$$(g_1, ..., g_p, g||k_1, ..., k_q, k) = (k^{-1}, g^{-1}) \cdot [g_1|...|g_p||g(k_1)|...|g(k_q)].$$

This choice of base provides us with an isomorphism of $\mathbb{Z}$-modules

$$\text{Hom}_{\mathbb{Z}(K \times G)}(C_p(EG, \mathbb{Z}) \otimes \mathbb{Z} C_q(EK, \mathbb{Z}), \mathbb{Z}) \cong \text{Maps}(G^p \times K^q, \mathbb{Z})$$

that allows us to transport the dual of the differentials $\partial_G \otimes 1$ and $1 \otimes \partial_K$ to $\text{Maps}(G^p \times K^q, \mathbb{Z})$; the induced differentials will be denoted by $\delta_G$ and $\delta_K$. Therefore we obtain:

**Definition 1.6.** Let $\overline{C}^{p,q}(K \times G, \mathbb{Z})$, $p, q \geq 0$, be the double complex

$$\overline{C}^{p,q}(K \times G, \mathbb{Z}) := \text{Maps}(G^p \times K^q, \mathbb{Z})$$
with differentials $\delta_G : C^{p,q} \rightarrow C^{p+1,q}$ and $\delta_K : C^{p,q} \rightarrow C^{p,q+1}$ defined by the equations

$$(\delta_G F)[g_1, \ldots, g_{p+1}, |k_1, \ldots, k_q] = F[g_1, \ldots, g_{p+1}, |k_1, \ldots, k_q]$$

$$+ \sum_{i=1}^{p} (-1)^i F[g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}, |k_1, \ldots, k_q]$$

$$+ (-1)^{p+1} F[g_1, \ldots, g_p, |g_{p+1}(k_1), \ldots, g_{p+1}(k_q)]$$

$$(\delta_K F)[g_1, \ldots, g_p, |k_1, \ldots, k_{q+1}] = F[g_1, \ldots, g_p, |k_2, \ldots, k_{q+1}]$$

$$+ \sum_{j=1}^{q} (-1)^j F[g_1, \ldots, g_p, |k_1, \ldots, k_j k_{j+1}, \ldots, k_{q+1}]$$

$$+ (-1)^{q+1} F[g_1, \ldots, g_p, |k_1, \ldots, k_q],$$

where by convention $G^0 \times K^0$ is the set with one point.

Hence we have

**Lemma 1.7.** The cohomology of the total complex of the double complex $\overline{C}^{*,*}(K \rtimes G, \mathbb{Z}), \delta_G, \delta_K$ is the cohomology of the group $K \rtimes G$, i.e.

$$H^*(\overline{C}^{*,*}(K \rtimes G, \mathbb{Z})), \delta_G \oplus (-1)^p \delta_K \cong H^*(K \rtimes G, \mathbb{Z}).$$

We can take a smaller double complex, more suited for our work, which is called the normalized double complex. Let us define it

**Definition 1.8.** The normalized double complex of $\overline{C}^{*,*}(K \rtimes G, \mathbb{Z}), \delta_G, \delta_K$ is the double complex

$$C^{p,q}(K \rtimes G, \mathbb{Z}) \subset \overline{C}^{p,q}(K \rtimes G, \mathbb{Z})$$

consisting of maps $F : G^p \times K^q \rightarrow \mathbb{Z}$ such that $F[g_1, \ldots, g_p, |k_1, \ldots, k_q] = 1$ whenever $g_i = 1$ or $k_j = 1$. The differentials on $C^{p,q}$ are also $\delta_G$ and $\delta_K$. We setup $C^{0,0} = \mathbb{Z}$.

It is known in homological algebra that the normalized complex of the bar resolution is quasi-isomorphic to the bar resolution (see page 215 in [13]). Therefore we have

**Lemma 1.9.** The induced map on total complexes

$$\text{Tot}^*(C^{*,*}(K \rtimes G, \mathbb{Z})) \rightarrow \text{Tot}^*(\overline{C}^{*,*}(K \rtimes G, \mathbb{Z}))$$

is a quasi-isomorphisms. Then

$$H^*(\text{Tot}^*(C^{*,*}(K \rtimes G, \mathbb{Z})), \delta_G \oplus (-1)^p \delta_K) \cong H^*(K \rtimes G, \mathbb{Z}).$$

We can generalize our definition of the double complex to other coefficients. Denote by $T$ the group $S^1$ and consider the exact sequence of coefficients

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow T \rightarrow 0.$$

We will denote the complex $C^{p,q}(K \rtimes G, T)$ as the complex generated by maps $F : G^p \times K^q \rightarrow T$ on which $F$ is 1 if one of the entries is the identity, and the differentials $\delta_G$ and $\delta_K$ are the same ones as Definition 1.8 but changing the sums by multiplications. Note also that we set up $C^{0,0}(K \rtimes G, T) := T$.

Then

$$H^*(\text{Tot}^*(C^{*,*}(K \rtimes G, T)), \delta_G \oplus \delta_K^{(-1)^p}) \cong H^*(K \rtimes G, T).$$
1.3.1. **Decomposition of the cohomology of** $K \times G$. Note that the 0-th row of the double complex is isomorphic to the normalized bar cochain complex of $G$

\[(C^\ast,0(K \times G, \mathbb{Z}), \delta_G) \cong (C^\ast(G, \mathbb{Z}), \delta_G),\]

and the action of the differential $\delta_K$ on this row is trivial. Therefore, if we define

**Definition 1.10.** Let

\[A^{\ast,\ast}(K \times G, \mathbb{Z}) := C^{\ast,\ast}0(K \times G, \mathbb{Z})\]

be the sub double complex of $C^{\ast,\ast}(K \times G, \mathbb{Z})$ with trivial 0-th row, and differentials $\delta_G$ and $\delta_K$.

Then we obtain

**Lemma 1.11.** There is a canonical isomorphism of double complexes

\[(C^{\ast,\ast}(K \times G, \mathbb{Z}), \delta_G \oplus (-1)^p \delta_K) \cong (A^{\ast,\ast}(K \times G, \mathbb{Z}), \delta_G \oplus (-1)^p \delta_K) \oplus (C^\ast(G, \mathbb{Z}), \delta_G),\]

which induces a canonical isomorphism in cohomology

\[H^\ast(K \times G, \mathbb{Z}) \cong H^\ast(Tot^\ast(A^{\ast,\ast}(K \times G, \mathbb{Z}))) \oplus H^\ast(G, \mathbb{Z}),\]

and the respective one with coefficients in $\mathbb{T}$,

\[H^\ast(K \times G, \mathbb{T}) \cong H^\ast(Tot^\ast(A^{\ast,\ast}(K \times G, \mathbb{T}))) \oplus H^\ast(G, \mathbb{T})\]

where $A^{\ast,\ast}(K \times G, \mathbb{T}) = C^{\ast,\ast}0(K \times G, \mathbb{T})$.

2. **The case of the trivializable semi-direct product $K \times G$**

Whenever the semi-direct product $K \times G$ is isomorphic to $K \times G$ we know by Proposition 1.1 and Remark 1.5 that there exists a homomorphism $\sigma : G \rightarrow K$ such that $K \times G \rightarrow K \times K$, $(k, g) \mapsto (k, \sigma(g))$ is a homomorphism of groups. In this case we have the homomorphisms

\[K \times G \rightarrow K \times K \xrightarrow{\mu} K, \quad (k, g) \mapsto (k, \sigma(g)) \mapsto k\sigma(g),\]

which induce the following homomorphism at the level of their homologies

\[H_\ast(K \times G, \mathbb{Z}) \rightarrow H_\ast(K \times K, \mathbb{Z}) \xrightarrow{\mu} H_\ast(K, \mathbb{Z}).\]

Since the interesting information lies on the homomorphism $\mu_\ast$ we will investigate its properties.

2.1. **The Shuffle homomorphism**. In what follows we will describe how to obtain the map $\mu_\ast$ at the chain level using the explicit models for $H_\ast(K \times K, \mathbb{Z})$ and $H_\ast(K, \mathbb{Z})$ defined previously. Its definition needs some preparation.

Take a base element $[g_1|...|g_p|k_1|...|k_q]$ in $C_p(EK, \mathbb{Z}) \otimes \mathbb{Z} C_q(EK, \mathbb{Z})$ and think of it as a way to represent $\Delta^p \times \Delta^q$ as the product of the simplices $[g_1|...|g_p] \times [k_1|...|k_q]$ in $BK$. For $\lambda \in Shuff(p, q)$ a $(p, q)$-shuffle, i.e. an element in the symmetric group $\mathfrak{S}_{p+q}$ such that $\lambda(i) < \lambda(j)$ whenever $1 \leq i < j \leq p$ or $p + 1 \leq i < j \leq p + q$, we can define an element

\[\lambda[g_1|...|g_p|k_1|...|k_q] \in C_p+q(EK, \mathbb{Z})\]

such that $\lambda[g_1|...|g_p|k_1|...|k_q] := [s_1|...|s_{p+q}]$ with

\[s_\lambda(i) = \begin{cases} g_i & \text{if } i \leq p \\ (g_{p+\lambda(i)} - i + 1 \cdots g_{p-1} g_{p+\lambda(i)-1} \cdots g_{p-1} g_p)^{-1} & \text{if } i > p. \end{cases}\]
From the Eilenberg-Zilber theorem it follows that the set
\[ \{ \lambda[g_1|...|g_p|k_1|...|k_q] : \lambda \in \text{Shuff}(p,q) \} \]
is a subdivision in simplices of dimension \( p + q \) of the product of the simplices \([g_1|...|g_p] \times [k_1|...|k_q] \).

An equivalent way to see the elements \( \lambda[g_1|...|g_p|k_1|...|k_q] \) is the following: a \((p, q)\)-shuffle can also be understood as a way to parameterize a lattice path of minimum distance from the point \((0,0)\) to the point \((p,q)\); one moves one unit to the right in the steps \(\lambda(1),...,\lambda(p)\) and one unit up in the steps \(\lambda(p+1),...,\lambda(p+q)\).

We label the horizontal and vertical edges in the rectangular lattice defined by the points \((0,0), (p,0), (p,q), (0,q)\) by the following rule:

- The horizontal path between \((i-1,j)\) and \((i,j)\) is labeled with \(g_i\) independent of \(j\).
- The vertical path between \((p,j-1)\) and \((p,j)\) is labeled with \(k_j\) and all the other vertical paths are labeled in such a way that the squares become commutative squares (when thinking of the labels as maps). This implies that the vertical edge from \((i,j-1)\) to \((i,j)\) is labeled with
\[ g_{i+1} \cdots g_{j-1} g_j k_j (g_{i+1} \cdots g_{j-1} g_j)^{-1}. \]

Since \(\lambda\) parameterizes a path in the lattice, the element \(\lambda[g_1|...|g_p|k_1|...|k_q]\) encodes the labels that the path \(\lambda\) follow in order to go from \((0,0)\) to \((p,q)\).

For example, consider the \((3,2)\)-shuffle
\[ \lambda := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix} \]
then the element \(\lambda[g_1|g_2|g_3|k_1|k_2]\) can be seen as the path

\[
\begin{array}{ccc}
(2,2) & \xrightarrow{g_3} & (3,2) \\
\downarrow{g_3 k_2 g_3^{-1}} & & \downarrow{k_2} \\
(1,1) & \xrightarrow{g_2} & (2,1) & \xrightarrow{g_3} & (3,1) \\
\downarrow{g_2 g_3 k_3 (g_2 g_3)^{-1}} & & \downarrow{k_1} \\
(0,0) & \xrightarrow{g_1} & (1,0) & \xrightarrow{g_2} & (2,0) & \xrightarrow{g_3} & (3,0) \\
\end{array}
\]
and therefore
\[ \lambda[g_1|g_2|g_1|k_1|k_2] = [g_1|g_2 g_3 k_1 g_3^{-1} g_2^{-1}] g_2 [g_3 k_2 g_3^{-1}] g_3. \]

We can now define the map of complexes which in homology realizes \(\mu_\ast\):

**Definition 2.1.** Let
\[ \tau : C_p(EK, \mathbb{Z}) \otimes_{\mathbb{Z}} C_q(EK, \mathbb{Z}) \to C_{p+q}(EK, \mathbb{Z}) \]
be the graded homomorphism defined on generators by the equation
\[ \tau(g_1, ... g_p, g | k_1, ..., k_q, k) := \sum_{\lambda \in \text{Shuff}(p,q)} (g_k)^{-1} \cdot (−1)^{|\lambda|} \lambda[g_1|...|g_p|g k_1 g^{-1}|...|g k_q g^{-1}] \]
where \(|\lambda|\) denotes the sign of the permutation \(\lambda\). This is usually called the **Shuffle homomorphism**.
We claim that the Shuffle homomorphism $\tau$ is an admissible product in the sense described in [11, IV-55, p. 118]

**Theorem 2.2.** The graded homomorphism

$$\tau: \text{Tot}_*(C_*(EK, \mathbb{Z}) \otimes_\mathbb{Z} C_*(EK, \mathbb{Z})) \to C_*(EK, \mathbb{Z})$$

is a chain map that satisfies the equations

$$\tau((a^{-1}, h^{-1}) \cdot (g_1, ... g_p, g||k_1, ..., k_q, k)) = (ha)^{-1} \cdot \tau(g_1, ... g_p, g||k_1, ..., k_q, k)$$

$$(\epsilon \otimes \epsilon)(k, g) = \epsilon(kg).$$

Hence it induces a graded homomorphism in homology

$$\tau_*: H_*(K \times K, \mathbb{Z}) \to H_*(K, \mathbb{Z})$$

which is equal to the one induced by the pushforward $\mu_*$.

**Proof.** Let us start by proving the equations. For the first one, we have on the one side

$$\tau((a^{-1}, h^{-1}) \cdot (g_1, ... g_p, g||k_1, ..., k_q, k))$$

$$= \tau(g_1, ... g_p, gh||h^{-1}k_1h, ..., h^{-1}k_qh, h^{-1}kh)$$

$$= \sum_{\lambda \in \text{Shuff}(p, q)} (gkh)^{-1} \cdot (-1)^{|\lambda|} \lambda[g_1, ..., g_p, gk_1g^{-1}, ..., gk_qg^{-1}]$$

and on the other

$$(ha)^{-1} \cdot \tau(g_1, ... g_p, g||k_1, ..., k_q, k)$$

$$= (ha)^{-1} \sum_{\lambda \in \text{Shuff}(p, q)} (gk)^{-1} \cdot (-1)^{|\lambda|} \lambda[g_1, ..., g_p, gk_1g^{-1}, ..., gk_qg^{-1}]$$

$$= \sum_{\lambda \in \text{Shuff}(p, q)} (gkh)^{-1} \cdot (-1)^{|\lambda|} \lambda[g_1, ..., g_p, gk_1g^{-1}, ..., gk_qg^{-1}];$$

this in particular implies that $\tau$ is a homomorphism which preserves the respective module structures, namely that

$$\tau((a, h)(g_1, ... g_p, g||k_1, ..., k_q, k)) = \mu(a, h)\tau(g_1, ... g_p, g||k_1, ..., k_q, k).$$

For the second, simply note that

$$(\epsilon \otimes \epsilon)(k, g) = 1 \otimes 1 = 1 = \epsilon(kg)$$

which implies that $\epsilon \circ \mu(k, g) = (\epsilon \otimes \epsilon)(k, g)$.

The proof of the fact that $\tau$ is a chain map, namely that

$$\partial_K \circ \tau = \tau \circ (\partial_K \otimes 1 + (-1)^p 1 \otimes \partial_K),$$

is essentially included in the proof of [11, Theorem 5.2]: the decomposition in simplices of dimension $p+q$ of the product of the simplices $[g_1, ..., g_p] \times [k_1, ..., k_q]$ in $BK$ is done by choosing appropriately $p + q$ edges with the use of the $(p, q)$-shuffles as follows

$$\tau[g_1, ..., g_p||k_1, ..., k_q] = \sum_{\lambda \in \text{Shuff}(p, q)} (-1)^{|\lambda|} \lambda[g_1, ..., g_p, k_1, ..., k_q].$$

With this decomposition of $[g_1, ..., g_p] \times [k_1, ..., k_q]$, its boundary can be calculated as $(\partial_K \times 1 + (-1)^p 1 \otimes \partial_K)[g_1, ..., g_p] \times [k_1, ..., k_q]$ or alternatively as $\partial_K(\tau[g_1, ..., g_p||k_1, ..., k_q])$; then it follows that $\tau$ is a chain map.
Now, since \( \tau \) is a chain map which preserves the module structures, then it induces a chain map at the level of the coinvariants

\[
\text{Tot}_s(C_*(EK, \mathbb{Z}) \otimes_{\mathbb{Z}} C_*(EK, \mathbb{Z})) \otimes_{\mathbb{Z}[K \ltimes K]} \mathbb{Z} \to C_*(EK, \mathbb{Z}) \otimes_{K} \mathbb{Z}
\]

which defines a homomorphism

\[
\tau_* : H_*(K \ltimes K, \mathbb{Z}) \to H_*(K, \mathbb{Z}).
\]

This homomorphism \( \tau_* \) must be equal to the pushforward \( \mu_* \) since \( \tau \) preserves the module structures defined by \( \mu \). \( \square \)

If we have a homomorphism of groups \( \sigma : G \to K \) for which the map \( \psi : K \times G \to K \times G, (k, g) \mapsto (k, \sigma(g)) \) induces an isomorphism of semi-direct products, then we have that the map

\[
\tau \circ (\sigma \otimes 1) : C_p(EK, \mathbb{Z}) \otimes_{\mathbb{Z}} C_q(EK, \mathbb{Z}) \to C_{p+q}(EK, \mathbb{Z})
\]

\[
[g_1|...|g_p||k_1|...|k_q] \mapsto \tau[\sigma(g_1)|...|\sigma(g_p)||k_1|...|k_q]
\]

induces a chain map

\[
\tau \circ (\sigma \otimes 1) : \text{Tot}_s(C_*(EG, \mathbb{Z}) \otimes_{\mathbb{Z}} C_*(EK, \mathbb{Z})) \to C_*(EK, \mathbb{Z})
\]

preserving their respective module structures, and hence inducing a homomorphism

\[
(\tau \circ (\sigma \otimes 1))_* : H_*(K \ltimes G, \mathbb{Z}) \to H_*(K, \mathbb{Z})
\]

which is equal to the pushforward map of the composition \( \mu \circ \psi \); i.e.

\[
(\mu \circ \psi)_* = (\tau \circ (\sigma \otimes 1))_* : H_*(K \ltimes G, \mathbb{Z}) \to H_*(K, \mathbb{Z}).
\]

2.2. The dual of the Shuffle homomorphism. Dualizing the map \( \tau \), we obtain a homomorphism

\[
\tau^\vee : \overline{C}^*(K, \mathbb{Z}) \to \bigoplus_{p+q=n} \overline{C}^{p,q}(K \ltimes K, \mathbb{Z})
\]

\[
(\tau^\vee F)[s_1|...|s_n] \mapsto \bigoplus_{p+q=n} F(\tau[s_1|...|s_p||s_{p+1}|...|s_{p+q}])
\]

which induces a cochain map

\[
\tau^\vee : (\overline{C}^* (K, \mathbb{Z}), \delta_K) \to (\text{Tot}^*(\overline{C}^{*,*}(K \ltimes K), \mathbb{Z}), \delta_G \oplus (-1)^p \delta_K);
\]

here we have kept the notation of the differentials as \( \delta_G \) and \( \delta_K \) in order to avoid confusions. A straightforward calculation shows that the cochain map \( \tau^\vee \) preserves normalized cochains

\[
\tau^\vee : (C^* (K, \mathbb{Z}), \delta_K) \to (\text{Tot}^*(C^{*,*}(K \ltimes K, \mathbb{Z}), \delta_G \oplus (-1)^p \delta_K)
\]

and therefore it induces a homomorphism at the level of cohomologies

\[
\tau^* : H^*(K; \mathbb{Z}) \to H^*(K \ltimes K, \mathbb{Z})
\]

which by Theorem 2.2 is equal to the pullback of the homomorphism of groups \( \mu \). If we bundle up our previous discussion we have
Theorem 2.3. The homomorphism in cohomology
\[ \tau^*: H^*(K, \mathbb{Z}) \to H^*(K \times K, \mathbb{Z}) \]
defined by the cochain map
\[ \tau^\vee: C^n(K, \mathbb{Z}) \to \bigoplus_{p+q=n} C^{p,q}(K \times K, \mathbb{Z}) \]

\[ (\tau^\vee F)[s_1|...|s_n] \mapsto \bigoplus_{p+q=n} F(\tau[s_1|...|s_p][s_{p+1}|...|s_{p+q}]) \]
is equal to
\[ \mu^*: H^*(K, \mathbb{Z}) \to H^*(K \times K, \mathbb{Z}) \]
which is the pullback of the group homomorphism \( \mu: K \times K \to K, (k, g) \mapsto kg \).

2.3. Further properties of the Shuffle homomorphism. Consider the homomorphisms
\[ \iota_K: K \to K \times K, \quad \iota_G: K \to K \times K \]
\[ x \mapsto (x, 1_K) \quad g \mapsto (1_K, g) \]

and note that they fit into the commutative diagram

\[ \begin{array}{ccc}
K & \xrightarrow{\iota_K} & K \\
\downarrow{\mu} & & \downarrow{\mu} \\
K & \xrightarrow{\iota_G} & K
\end{array} \]

which induces the commutative diagram

\[ \begin{array}{ccc}
H^*(K, \mathbb{Z}) & \xrightarrow{\iota_K^*} & H^*(K \times K, \mathbb{Z}) \\
\downarrow{\mu^*} & & \downarrow{\mu^*} \\
H^*(K, \mathbb{Z}) & \xrightarrow{\iota_G^*} & H^*(K, \mathbb{Z}).
\end{array} \]

From Lemma 1.11 we know that there is a canonical isomorphism
\[ H^P(K \times K, \mathbb{Z}) \cong H^P(A^{+*}(K \times K, \mathbb{Z})) \oplus H^P(K, \mathbb{Z}) \]
and the homomorphism \( \iota_G^* \) is precisely the projection on the second component of the direct sum \( H^P(A^{+*}(K \times K, \mathbb{Z})) \oplus H^P(K, \mathbb{Z}) \).

Moreover, by defining the chain map
\[ \tau_1^\vee: C^n(K, \mathbb{Z}) \to \bigoplus_{p+q=n, \ q \geq 0} A^{p,q}(K \times K, \mathbb{Z}) \]

\[ (\tau_1^\vee F)[s_1|...|s_n] \mapsto \bigoplus_{p+q=n, \ q \geq 0} F(\tau[s_1|...|s_p][s_{p+1}|...|s_{p+q}]) \]

we have that the chain map
\[ \tau^\vee: C^*(K, \mathbb{Z}) \to \text{Tot}^*(A^{+*,0}(K \times K, \mathbb{Z})) \oplus C^{+,0}(K \times K, \mathbb{Z}) \]
is isomorphic to the chain map
\[ \tau_1^\vee + 1: C^*(K, \mathbb{Z}) \to \text{Tot}^*(A^{+,0}(K \times K, \mathbb{Z})) \oplus C^{+,0}(K \times K, \mathbb{Z}) \]


and therefore we have that at the cohomological level we obtain the commutative
diagram

\[
\begin{array}{ccc}
H^p(K, \mathbb{Z}) & \xrightarrow{\tau^*_1} & H^p(\text{Tot}^*(A^{*,*}(K \ltimes K, \mathbb{Z}))) \\
\downarrow & & \downarrow \\
H^p(K, \mathbb{Z}) & \xrightarrow{\iota^*_K|_A} & H^p(\text{Tot}^*(B^{*,*}(K \ltimes G, \mathbb{Z})))
\end{array}
\]

for all \( p > 0 \), where at the cochain level \( \iota^*_K|_A \) is simply the projection map on the 0-th column

\[ \iota^*_K|_A : \text{Tot}^*(A^{*,*}(K \ltimes K, \mathbb{Z})) \to A^{*,0}(K \ltimes K, \mathbb{Z}) \cong C^{*,0}(K, \mathbb{Z}). \]

Therefore, if

**Definition 2.4.** Let

\[ B^{*,*}(K \ltimes G, \mathbb{Z}) := C^{*,>0,>0}(K \ltimes G, \mathbb{Z}) \]

be sub double complex of \( C^{*,*}(K \ltimes K, \mathbb{Z}) \) with trivial 0-th row and trivial 0-th column, and differentials \( \delta_G \) and \( \delta_K \).

We obtain

**Lemma 2.5.** The homomorphism

\[ H^p(K, \mathbb{Z}) \oplus H^p(\text{Tot}^*(B^{*,*}(K \ltimes K, \mathbb{Z}))) \to H^p(\text{Tot}^*(A^{*,*}(K \ltimes K, \mathbb{Z}))) \]

\[ x \oplus y \mapsto \tau^*_1 x + y \]

is an isomorphism for all \( p > 0 \).

**Proof.** The short exact sequence of complexes

\[ 0 \to \text{Tot}^*(B^{*,*}(K \ltimes K, \mathbb{Z})) \to \text{Tot}^*(A^{*,*}(K \ltimes K, \mathbb{Z})) \to C^{*,0}(K, \mathbb{Z}) \to 0 \]

induces a long exact sequence in cohomology groups

\[ \to H^p(\text{Tot}^*(B^{*,*}(K \ltimes K, \mathbb{Z}))) \to H^p(\text{Tot}^*(A^{*,*}(K \ltimes K, \mathbb{Z}))) \xrightarrow{\iota^*_K|_A} H^p(K, \mathbb{Z}) \to \]

which splits by diagram (2.3).

Therefore we can conclude that there is a canonical splitting of the cohomology of \( K \ltimes K \) in terms of the cohomology of \( K \) and of the cohomology of the double complex \( B^{*,*}(K \ltimes K, \mathbb{Z}) \).

**Proposition 2.6.** The homomorphism

\[ H^p(K, \mathbb{Z}) \oplus H^p(\text{Tot}^*(B^{*,*}(K \ltimes K, \mathbb{Z}))) \oplus H^p(K, \mathbb{Z}) \xrightarrow{\cong} H^p(K \ltimes K, \mathbb{Z}) \]

\[ x \oplus y \oplus z \mapsto \tau^*_1 x + y + \pi^*_2 z \]

is an isomorphism for all \( p > 0 \). Here \( \pi_2 : K \ltimes K \to K, (a, g) \mapsto g \) denotes the homomorphism induced by the projection on the second coordinate satisfying \( \pi_2 \circ \iota_G = 1 \).
3. Categorical definitions

3.1. 2-category of discrete $G$-sets with twist. We shall fix a discrete group $G$. We define the 2-category of discrete $G$-sets with twist as follows:

1. Objects will be called discrete $G$-sets with twist, and they are pairs $(X, \alpha)$, where $X$ is a discrete left $G$-set and $\alpha \in Z^2_G(X, \mathbb{T})$ is a normalized 2-cocycle in the $G$-equivariant complex of $X$, i.e. a map

$$\alpha : G \times G \times X \to \mathbb{T},$$

such that

$$\alpha[\tau\rho][x] \alpha[\sigma\tau][\rho][x]^{-1} \alpha[\sigma][\tau][\rho][x]^{-1} = 1$$

for all $\sigma, \tau, \rho \in G, x \in X$.

Note that the previous equation is equivalent to the equation $\delta_G \alpha = 1$, when we see $\alpha$ as element in $C^{2,1}(X \times G, \mathbb{T})$.

2. Let $(X, \alpha_X), (Y, \alpha_Y)$ be discrete $G$-sets with twist. A 1-cell from $(X, \alpha_X)$ to $(Y, \alpha_Y)$, also called a $G$-equivariant map, is a pair $(L, \beta)$,

$$(X, \alpha_X) \xrightarrow{(L, \beta)} (Y, \alpha_Y)$$

where

- $L : X \to Y$ is a morphism of $G$-sets,
- $\beta \in C^1_G(X, \mathbb{T})$ is a normalized cochain such that $\delta_G(\beta) = L^*(\alpha_Y)(\alpha_X)^{-1}$, i.e., a map

$$\beta : G \times X \to \mathbb{T}$$

such that

$$\beta[\tau][x] \beta[\sigma][\tau][x]^{-1} \beta[\sigma][\tau] = \alpha_Y[\sigma][\tau][L(x)] \alpha_X[\sigma][\tau][x]^{-1},$$

for all $\sigma, \tau \in G, x \in X$.

3. Given two 1-cells $(L, \beta), (L, \beta') : (X, \alpha_X) \to (Y, \alpha_Y)$, a 2-cell $\theta : (L, \beta) \Rightarrow (L, \beta')$

$$\begin{array}{ccc}
(X, \alpha_X) & \xrightarrow{\theta} & (Y, \alpha_Y) \\
\downarrow \downarrow & \Downarrow \Downarrow & \downarrow \downarrow \\
(L, \beta) & \Rightarrow & (L, \beta')
\end{array}$$

is 0-cochain $\theta \in C_0^0(X, \mathbb{T})$ such that $\delta_G(\theta) = \beta' \beta^{-1}$, i.e., a map $\theta : X \to \mathbb{T}$, such that

$$\theta[x][\tau][\sigma][x]^{-1} = \beta'[\sigma][x][\beta][\sigma][x]^{-1}$$

for all $\sigma \in G, x \in X$.

Let us define the composition on 1-cells and 2-cells in the following way. Let

$$(F, \beta_F) : (X, \alpha_X) \to (Y, \alpha_Y)$$

and

$$(G, \beta_G) : (Y, \alpha_Y) \to (Z, \alpha_Z)$$

two 1-cells, define their composition as

$$(G, \beta_G) \circ (F, \beta_F) = (G \circ F, F^*(\beta_G)\beta_F) : (X, \alpha_X) \to (Z, \alpha_Z),$$
and if $\theta : (L, \beta) \Rightarrow (L, \beta')$ and $\theta' : (L, \beta') \Rightarrow (L, \beta'')$ are 2-cells, their composition is the product of the maps, namely

$$\theta' \circ \theta =: \theta' \theta : (L, \beta) \Rightarrow (L, \beta'')$$

A straightforward calculation implies that

**Lemma 3.1.** The composition of 1-cells and 2-cells satisfies the axioms of a 2-category.

### 3.2. Pseudomonoids

A strict 2-monoidal 2-category is a triple $(\mathcal{B}, \boxtimes, I)$ where $\mathcal{B}$ is a 2-category, $\boxtimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is a 2-functor and $I$ is an object in $\mathcal{B}$, such that $\boxtimes \circ (\boxtimes \times \text{Id}_B) = \boxtimes \circ (\text{Id}_B \times \boxtimes)$ and $I \boxtimes X = X \boxtimes I = X$ for every object in $\mathcal{B}$ (for more details see [6]).

**Definition 3.2.** Given a strict 2-monoidal 2-category $(\mathcal{B}, \boxtimes, I)$, a **pseudomonoid** in $\mathcal{B}$ consists of:

- an object $C \in \mathcal{B}$,

  together with:

- a **multiplication** 1-cell $m : C \boxtimes C \to C$,
- an **identity-assigning** 1-cell $I : I \to C$,

  together with the following 2-isomorphisms:

- the **associator**:

  $$(3.1)$$

  $\begin{array}{cccc}
    C \boxtimes C \boxtimes C & \xrightarrow{\text{id}_{\boxtimes} m} & C \boxtimes C \\
    m \circ \text{id}_{\boxtimes} & \Downarrow & \Downarrow \\
    C \boxtimes C & \xrightarrow{m} & C \boxtimes C
  \end{array}$$
• the left and right unit laws:

\[
\begin{array}{ccc}
\text{I} \boxtimes \text{C} & \xrightarrow{\boxtimes \text{id}} & \text{C} \boxtimes \text{C} \\
\downarrow{\ell} & = & \downarrow{\imath} \\
& \text{C} & \xleftarrow{\text{id} \boxtimes \ell} \text{C} \boxtimes \text{I}
\end{array}
\]

(3.2)

such that the following diagrams commute:

• the pentagon identity for the associator:

\[
\begin{array}{ccc}
m \circ (m \boxtimes \text{id}) \circ (m \boxtimes \text{id}) & \xrightarrow{a \circ (m \boxtimes \text{id})} & m \circ (m \boxtimes \text{id}) \circ (m \boxtimes \text{id}) \\
\downarrow{m \circ (m \boxtimes \text{id})} & = & \downarrow{m \circ (m \boxtimes \text{id})} \\
& m & m \circ (m \boxtimes \text{id})
\end{array}
\]

(3.3)

where we use the equalities

\[
(m \boxtimes \text{id}) \circ (m \boxtimes \text{id}) = m \boxtimes m
\]

in order to compose the upper 1-cells \(a \circ (m \boxtimes \text{id})\) and \(a \circ (\text{id} \boxtimes \text{id} \boxtimes m)\),
equalities which follow from the fact that \(\boxtimes\) is a 2-functor.

• the triangle identity for the left and right unit laws:

\[
\begin{array}{ccc}
m \circ (m \boxtimes \text{id}) \circ (m \boxtimes \text{id}) & \xrightarrow{a \circ (m \boxtimes \text{id})} & m \circ (m \boxtimes \text{id}) \circ (m \boxtimes \text{id}) \\
\downarrow{m \circ (m \boxtimes \text{id})} & = & \downarrow{m \circ (m \boxtimes \text{id})} \\
& m & m \circ (m \boxtimes \text{id})
\end{array}
\]

(3.4)

where we use the fact that \(\text{C} \boxtimes \text{I} = \text{C} = \text{I} \boxtimes \text{C}\) to make sense of the diagonal arrows.

**Remark 3.3.** The 2-category of \(G\)-sets with twist has a 2-monoidal structure,
where the product of two objects \((X, \alpha_X), (Y, \alpha_Y)\) is given by

\[
(X, \alpha_X) \boxtimes (Y, \alpha_Y) = (X \times Y, \alpha_X \boxtimes \alpha_Y),
\]

where \(X \times Y\) is a \(G\)-set with the diagonal \(G\)-action and \(\alpha_X \boxtimes \alpha_Y := \pi^1_2(\alpha_X) \pi^2_1(\alpha_Y)\).
In an analogous way we construct the product \(\boxtimes\) for 1-cells and 2-cells. A unit object
is any fixed \(G\)-set with one element and the constant function 1 for 2-cocycle.

3.3. **Pseudomonoids in the 2-category of \(G\)-sets with twist.** We are interested in studying pseudomonoids in the 2-category of \(G\)-sets with twists, but in order to get normalized cocycles we are forced to consider only pseudomonoids
where the identity-assigning 1-cell is strict in the sense that the cochain (the second component of the 1-cell) is trivial, and furthermore, that the unit constraints
in diagram (3.2) are identities, namely that the diagram (3.2) commutes. We will
call these pseudomonoids with strict unit.
Proposition 3.4. A pseudomonoid with strict unit in the 2-category of $G$-sets with twists is equivalent to the following data:

- A monoid $(K, m, 1)$, where $K$ is a $G$-set, $m$ is $G$-equivariant and $1 \in K$ is a $G$-invariant element.
- $\alpha \in C^{2,1}(K \times G, \mathbb{T})$, $\beta \in C^{1,2}(K \times G, \mathbb{T})$, $\theta \in C^{0,3}(K \times G, \mathbb{T})$ such that $\alpha \oplus \beta \oplus \theta$ is a three cocycle in $\operatorname{Tot}(A^{*,*}(K \times G, \mathbb{T})), \delta_G \oplus \delta^{-(1)^P}_K$ with $A^{*,*}(K \times G, \mathbb{T})$ the double complex introduced in Definition 1.10.

Proof. A pseudomonoid in the 2-category of $G$-sets is:

i) An object $C = (K, \alpha)$ where $K$ is a $G$-set and $\alpha : G \times G \times K \to \mathbb{T}$ such that $\alpha$ is normalized in the components of the group $G$ and that $\delta_G \alpha = 1$.

ii) A multiplication 1-cell $m = (m, \beta) : (K \times K, \alpha \boxtimes \alpha) \to (K, \alpha)$ such that $m : K \times K \to K$ is a $G$-equivariant map, and a map $\beta : G \times K \times K \to \mathbb{T}$ satisfying the equation $\delta_G \beta = m^* \alpha \cdot (\alpha \boxtimes \alpha)^{-1}$.

iii) An identity-assigning 1-cell $I = (1_K, \gamma) : (\{\ast\}, 1) \to (K, \alpha)$ where $1_K : \{\ast\} \to K$ is a map choosing a $G$-invariant element $1_K := 1_K(\ast)$ in $K$, and $\gamma : G \times \{\ast\} \to \mathbb{T}$ is the constant map 1 because we are only considering pseudomonoids with strict unit. The cochain condition on $\gamma$ reads $\delta_G \gamma[g_1 | g_2 | \ast] = \alpha[g_1 | g_2 | 1_K]$ and since $\gamma$ is the constant function it follows that $\alpha[g_1 | g_2 | 1_K] = 1$, namely that $\alpha$ is normalized in the $K$ variable.

iv) The left hand side of diagram (3.2) translates to the diagram

\[
\begin{array}{ccc}
\{\ast\} \times K, 1 \boxtimes \alpha & \xrightarrow{(1_K \times \text{id}, \gamma \boxtimes 1)} & (K \times K, \alpha \boxtimes \alpha) \\
\downarrow{\pi_2, 1} & & \downarrow{(m, \beta)} \\
(K, \alpha) & & (K, \alpha)
\end{array}
\]

where $\pi_2$ is the projection on the second component. The composition of the 1 at the level of the $G$-sets implies the equation $m(1_K, h) = h$ for any element $h \in K$. Now, the composition of the 1-cells at the level of the cochains implies the equation $\gamma[g | \ast] \cdot \beta[g | 1_K | h] = 1$ with $g \in G$ and $h \in K$. Since the cochain $\gamma$ is equal to the constant function 1, we have that $\beta[g | 1_K | h] = 1$. Applying the same arguments as
above we conclude that $\beta[g||h_11_K] = 1$ and therefore the left and right unit
laws imply that $\beta$ is normalized on the $K$ components, and moreover that
$1_K$ is a unit for the multiplication map $m$.

v) Diagram (3.1) translates to the diagram

$$(K \times K, \alpha \boxtimes \alpha)$$

\[ (m \times \text{id}, \beta \boxtimes 1) \quad (\text{id} \times m, 1 \boxtimes \beta) \]

whose commutativity at the level of $G$-sets implies that the multiplication
$m : K \times K \to K$ is associative, and at the level of cochains the diagram
implies the equation

$$\delta_G \theta [g]\alpha_1 h_2 h_3 = \beta [g]h_2 h_3 \beta [g]|h_1| h_2 |h_3|^{-1} \beta [g]|h_1| h_2 h_3 |h_1|^{-1}.$$  

vi) Diagram (3.3) translates into the equation

$$\delta_G \theta [k_1 k_2 k_3 k_4] = \theta [k_1 k_2 |k_3 k_4] \theta [k_1 |k_2 k_3 |k_4] \theta [k_2 |k_3 |k_4]$$

for all elements $k_1, k_2, k_3, k_4$ in $K$.

vii) Diagram (3.4) translates into the equality

$$\theta [1_K |k_2] = 1$$

for all $k_1, k_2 \in K$, because the unit constraints are trivial.

From ii) we have that the multiplication map $m : K \times K \to K$ is a $G$-equivariant
map and v) tells us that the multiplication $m$ is associative. From iii) we know
that $K$ is provided with a $G$ invariant element $1_K$ and iv) tells us that this element
is a left and right unit for the multiplication $m$. We have then that $(K, m, 1_K)$ is
a monoid endowed with a $G$-action compatible with $m$ and $1_K$.

Since $K$ is a $G$-equivariant monoid with unit, we can use the notation of Definition 1.6
to see that equation (3.5) can be written as

$$\delta_G (\beta) \delta_K (\alpha) = 1,$$

and equation (3.6) becomes

$$\delta_G (\theta) \delta_K (\beta)^{-1} = 1$$

and equation (3.7) becomes

$$\delta_K (\theta) = 1.$$
are normalized in all variables since their normalization on coordinates of $G$ follow from the definition of the 2-category of $G$-sets with twists.

Summarizing we have that $\alpha \in C_2^2(K \rtimes G, T), \beta \in C_1^1(K \rtimes G, T)$ and $\theta \in C_0^3(K \rtimes G, T)$ such that $\alpha \oplus \beta \oplus \theta$ is a three cocycle in $\left(\text{Tot}(A^*\cdot^*(K \rtimes G, T)), \delta_G \oplus \delta_K^{-1}\right)$ because we have that

$$(\delta_G \oplus \delta_K^{-1})(\alpha \oplus \beta \oplus \theta) = \delta_G(\alpha) \oplus \delta_K(\alpha) \delta_G(\beta) \oplus \delta_K(\beta)^{-1} \delta_G(\theta) \oplus \delta_K(\theta)$$

or in a diagram

$$\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\delta_K & \delta_G & \delta_K & \delta_G \\
\beta & 1 & \delta_K & 1 \\
\theta & 1 & \delta_K & 1 \\
\end{array}$$

From the construction above, it is easy to see that if we are given the $G$ equivariant monoid $(K, m, 1_K)$ with unit, plus the cocycle $\alpha \oplus \beta \oplus \theta$ then one can construct in a unique way a pseudomonoid with strict unit in the 2-category of $G$-sets with twists. This finishes the proof. □

Remark 3.5. For a fixed $G$ equivariant monoid $(K, m, 1_K)$, the possible pseudomonoid structures with strict unit in the 2-category of $G$-sets associated over $(K, m, 1_K)$ are classified by elements in $Z^3(\text{Tot}(A^{\cdot,\cdot}(K \rtimes G, T)))$, namely, 3-cocycles in the total complex of $A^{\cdot,\cdot}(K \rtimes G, T)$.

Definition 3.6. Given pseudomonoids $(C, m, I, a)$ and $(C', m', I', a')$ in a 2-category $\mathcal{B}$, a morphism $\mathcal{F} : (C, m, I, a) \rightarrow (C', m', I', a')$ consists of:

- 1-cell $\mathcal{F} : C \rightarrow C'$

equipped with:

- a 2-isomorphism

\[\begin{array}{ccc}
C \otimes C & \overset{F_2}{\longrightarrow} & C' \otimes C' \\
\downarrow F \quad & & \downarrow m' \\
C' & \overset{F_2}{\longrightarrow} & C
\end{array}\]
a 2-isomorphism

\[
\begin{array}{c}
\text{I} \\
\downarrow \scriptstyle F_0 \\
C \\
\downarrow \scriptstyle F \\
C'
\end{array}
\]

such that diagrams expressing the following laws commute:

- compatibility of \(F_2\) with the associator:

\[
\begin{array}{c}
m' \circ (m' \boxtimes \text{id}) \circ (F \boxtimes F \boxtimes F) \\
\downarrow \scriptstyle a' \circ (F \boxtimes F \boxtimes F) \\
m' \circ (F \boxtimes F) \circ (\text{id} \boxtimes m') \\
\downarrow \scriptstyle F \circ m \circ (\text{id} \boxtimes m)
\end{array}
\]

- compatibility of \(F_0\) with the left unit law:

\[
\begin{array}{c}
m' \circ (\text{id} \boxtimes m') \circ (F \boxtimes F) \\
\downarrow \scriptstyle m' \circ (F \boxtimes F) \circ (\text{id} \boxtimes m') \\
m' \circ (F \boxtimes F) \circ (\text{id} \boxtimes \text{id}) \\
\downarrow \scriptstyle m' \circ (F \boxtimes F) \circ (\text{id} \boxtimes \text{id})
\end{array}
\]

- compatibility of \(F_0\) with the right unit law:

\[
\begin{array}{c}
m' \circ (F \boxtimes \text{I'}) \\
\downarrow \scriptstyle m' \circ (F \boxtimes \text{I'}) \\
m' \circ (F \boxtimes \text{I'}) \\
\downarrow \scriptstyle m' \circ (F \boxtimes \text{I'})
\end{array}
\]

Definition 3.7. Given two pseudomonoids with strict unit \(\mathcal{K} = (K, m, 1, \alpha, \beta, \theta)\) and \(\mathcal{K}' = (K', m', 1', \alpha', \beta', \theta')\) in the 2-category of \(G\)-sets with twists, a morphism \(\mathcal{F} : \mathcal{K} \to \mathcal{K}'\) is a morphism of pseudomonoids (as in Definition 3.6) such that the 2-isomorphism \(F_0\) of diagram (3.9) is an identity.

Proposition 3.8. A morphism of pseudomonoids with strict unit in the 2-category of \(G\)-sets with twists \(\mathcal{F} : \mathcal{K} \to \mathcal{K}'\) consists of the triple \(\mathcal{F} = (F, \chi, \kappa)\) with

\[
F : K \to K'
\]

a \(G\)-equivariant morphism of monoids, and cochains \(\chi \in C^{1,1}(K \rtimes G, \mathbb{T})\) and \(\kappa \in C^{0,2}(K \rtimes G, \mathbb{T})\) such that

\[
(\delta_G + \delta^{(-1)^n}_\Delta)(\chi \oplus \kappa^{-1}) = F^* \alpha' / \alpha \oplus F^* \beta' / \beta \oplus F^* \theta' / \theta.
\]

Proof. Following Definition 3.6, a morphism \(\mathcal{F} : \mathcal{K} \to \mathcal{K}'\) consists of:

i) A 1-cell \((F, \chi) : (K, \alpha) \to (K', \alpha')\), i.e. a \(G\)-equivariant map \(F : K \to K'\) and a normalized cochain \(\chi \in C_G^1(K, \mathbb{T})\) such that

\[
\delta_G \chi = F^* \alpha' / \alpha.
\]

(3.10)
ii) A 2-cell \( \kappa \in C_0^0(K \times K, T) \)

\[
\begin{array}{c}
(K \times K, \alpha \boxtimes \alpha) \xrightarrow{(m, \beta)} (K, \alpha) \\
(F \times F, \chi) \downarrow \downarrow (F, \chi) \\
(K', \alpha' \boxtimes \alpha') \xrightarrow{(m', \beta')} (K', \alpha')
\end{array}
\]

such that

\[
\delta_G \kappa = (\beta \cdot m^* \chi) / (\chi \boxtimes \chi \cdot F^* \beta').
\]

Note that at the level of the \( G \) sets the diagram is commutative, therefore \( F : K \to K' \) preserves the multiplication; and since

\[
\delta_K \chi = \chi \boxtimes \chi \cdot (m^* \chi)^{-1},
\]

we can rewrite the equation above as:

\[
(3.11) \quad \delta_G \kappa \cdot \delta_K \chi = \beta / F^* \beta'.
\]

iii) The commutativity of the diagram (3.9)

\[
\begin{array}{c}
(\{\ast\}, 1) \\
\downarrow \downarrow \\
(K, \alpha) \xrightarrow{(F, \chi)} (K', \alpha')
\end{array}
\]

implies that the map \( F : K \to K' \) preserves the unit and that \( \chi(g, 1) = 1 \) for any \( g \in G \), namely that \( \chi \) is normalized in the \( K \) variable and therefore we could say that \( \chi \in C^{1,1}(K \rtimes G, T) \).

iv) The commutativity of \( \kappa \) with the associator. This implies that \( F \) preserves the associativity of the multiplications \( m \) and \( m' \), and that the following equation is satisfied

\[
\theta'[F(a_1)|F(a_2)|F(a_3)] \kappa[a_2|a_3] \kappa[a_1|a_2a_3] = \kappa[a_1|a_2] \kappa[a_1a_2|a_3] \theta[a_1|a_2|a_3]
\]

for all \( a_1, a_2, a_3 \in K \). Note that this last equation can be written as

\[
(3.12) \quad \delta_K \kappa = \theta / F^* \theta'.
\]

v) Compatibility with the left and right units, but as the 2-cells \( F_0, l, r, l' \) and \( r' \) are identities, then this implies that \( \kappa[1|a] = 1 = \kappa[a|1] \) and therefore we have that \( \kappa \) is normalized in the \( K \) variables and we can assume that \( \kappa \in C^{0,2}(K \rtimes G, T) \).

From the previous arguments it follows that \( F : K \to K' \) is a \( G \)-equivariant morphism of monoids, and moreover, calculating the differential, we have that

\[
(\delta_G \oplus \delta_K^{(-1)})^p(\chi \oplus \kappa^{-1}) = \delta_G(\chi) \oplus \delta_K(\chi)^{-1} \delta_G(\kappa)^{-1} \oplus \delta_K(\kappa)^{-1} = F^* \alpha'/\alpha + F^* \beta'/\beta + F^* \theta'/\theta,
\]

where the second equality follows from equations (3.10), (3.11) and (3.12). In a diagram
This finishes the proof.

**Definition 3.9.** Given morphisms $F = (F, F_0, F_2)$ and $G = (G, G_0, G_2)$ from $(C, m, I, a)$ to $(C', m', I', a')$ pseudomonoids in $B$, a 2-morphism $s : F \to G$ is a 2-cell $s : F \to G$ in $B$ such that the following diagrams commute:

- compatibility with $F_2$ and $G_2$:

\[
\begin{array}{ccc}
\delta G & \cdot \ m' & \cdot (\delta G) \\
\downarrow F_2 & \downarrow & \downarrow G_2 \\
F \circ m & \cdot s \circ m & \cdot G \circ m
\end{array}
\]

(3.13)

- compatibility with $F_0$ and $G_0$:

\[
\begin{array}{ccc}
F \circ I & \cdot F_0 & \cdot G_0 \\
\downarrow s \circ I & & \downarrow G \circ I \\
\end{array}
\]

(3.14)

**Proposition 3.10.** Given morphisms of pseudomonoids with strict unit in the 2-category of $G$-sets with twists $F = (F, \chi, \kappa)$ and $F' = (F', \chi', \kappa')$, with $F, F' : K \to K'$, a 2-morphism $\gamma : F \to F'$ is a cochain $\gamma : C^{0,1}(K \times G, T)$ such that

\[
(\delta_G \oplus \delta_K^{(-1)p})(\gamma) = (\chi'/\chi, \kappa/\kappa').
\]

**Proof.** Let $\gamma : (F, \chi) \to (F', \chi')$ be the 2-cell defined by the 2-morphism, then we have that

\[
\delta_G(\gamma) = \chi'/\chi.
\]

Now, by diagram (3.13) we have that

\[
\delta_K(\gamma) = \kappa/\kappa',
\]

and by diagram (3.14) we have that $\gamma$ is a normalized cochain. □

**Remark 3.11.** Propositions 3.4, 3.8 and 3.10 imply that the relevant information encoded in cochains for the 2-category of pseudomonoids with strict unit of $G$-sets with twists, is given by the cochains of the total complex

\[
\left(\text{Tot}(A^{*,*}(K \times G, T)), \delta_G \oplus \delta_K^{(-1)p}\right)
\]
For a fixed \( G \)-equivariant monoid \((K, m, 1_K)\), Proposition 3.3 tells us that the 3-cocycles \( Z^3(\text{Tot}^*(A^{*}*(K \rtimes G, T))) \) are in one to one correspondence with the set of possible pseudomonoid structures with strict unit in the 2-category of \( G \)-sets with twist over \( K \). If we only consider invertible morphisms of pseudomonoids as defined in Proposition 3.8, we may define a groupoid which encodes the equivalence classes of pseudomonoid structures over \( K \). Let us be more explicit.

### 3.4. Equivalence classes of pseudomonoid structures over a fixed monoid.

**Definition 3.12.** Fix a \( G \)-equivariant monoid \((K, m, 1_K)\). Define the groupoid \( \text{Psdmn}^G(K) \) whose set of objects is \( Z^3(\text{Tot}^*(A^{*}*(K \rtimes G, T))) \) and whose morphisms are invertible morphisms of pseudomonoids as defined in Proposition 3.8. The groupoid \( \text{Psdmn}^G(K) \) encodes the information of all pseudomonoid structures over \( K \) and its coarse moduli space \( |\text{Psdmn}^G(K)| \), i.e. the set of equivalence classes defined by the morphisms, is the set of equivalence classes of pseudomonoid structures on \( K \).

Note that a morphism in \( \text{Psdmn}^G(K) \) consists of a triple \((F, \chi, \kappa)\) where \( F : K \to K \) must be a \( G \)-equivariant automorphism. If we denote by 
\[
\text{Aut}_G(K) := \{ f \in \text{Aut}(K) : f(gk) = gf(k) \text{ for all } g \in G \}
\]
then we have that the group \( \text{Aut}_G(K) \) is isomorphic to the subgroup of \( \text{Aut}(K \rtimes G) \) which leaves the \( G \) fixed; for a \( G \)-equivariant automorphism \( f \in \text{Aut}_G(K) \) we can associate the automorphism \( \tilde{f} \in \text{Aut}(K \rtimes G) \) by the equation
\[
\tilde{f}(k, g) := (f(k), g).
\]
Since the automorphism \( \tilde{f} \) leaves \( G \) fixed, then the groups of automorphism act on the double complex \( A^{*}*(K \rtimes G, \mathbb{Z}) \); we claim

**Lemma 3.13.** The set of equivalence classes of pseudomonoid structures on \( K \) is isomorphic can be described by the quotient
\[
|\text{Psdmn}^G(K)| \cong H^3(\text{Tot}^*(A^{*}*(K \rtimes G, T)))/\text{Aut}_G(K).
\]

**Proof.** We first perform the quotient with the morphisms \((F, \chi, \kappa)\) where \( F \) is the identity on \( K \); this quotient is precisely \( H^3(\text{Tot}^*(A^{*}*(K \rtimes G, T))) \). Then we see that elements of \( H^3(\text{Tot}^*(A^{*}*(K \rtimes G, T))) \) lying on the same orbit of the action of \( \text{Aut}_G(K) \) define equivalent pseudomonoid structures. \( \square \)

In particular we may conclude that if \( H^3(\text{Tot}^*(A^{*}*(K \rtimes G, T))) = 0 \), then all pseudomonoid structures with strict unit in the 2-category of \( G \)-sets with twist over \( K \) are isomorphic to the trivial one. Since we are interested in finding pseudomonoid structures with strict unit in the 2-category of \( G \)-sets with twist over \( K \) non isomorphic to the trivial one, we will calculate the group \( H^3(\text{Tot}^*(A^{*}*(K \rtimes G, T))) \), the group \( \text{Aut}_G(K) \) and the set \(|\text{Psdmn}^G(K)|\) for some particular examples.

The main tool we will use in order to calculate the group \( H^3(\text{Tot}^*(A^{*}*(K \rtimes G, T))) \) will be the Lyndon-Hochschild-Serre spectral sequence. This spectral sequence can be obtained if the complex \( \text{Tot}^*(A^{*}*(K \rtimes G, T)) \) is filtered by the complexes
\[
F^n := \text{Tot}^*(A^{*}_{\geq n}*(K \rtimes G, T))
\]
thus defining a spectral sequence whose second page becomes
\[
E_2^{p,q} = H^p(G, H^q(K, \mathbb{T}))
\]
The dihedral group $D_n$ acts on $H^q(K, \mathbb{T})$ through the induced action of $G$ on $K$; note in particular that

$$E_2^{p,q} = H^q(K, \mathbb{T})^G \text{ and } E_2^{p,0} = H^p(G, \mathbb{T}).$$

On the other hand we have that

$$\text{Aut}_G(K) \cong C_{\text{Aut}(K)}(\rho(G))$$

where $\rho : G \to \text{Aut}(K)$ determines the action of $G$, and

$$C_{\text{Aut}(K)}(\rho(G)) := \{ f \in \text{Aut}(K) : [f, \rho(g)] = 1 \text{ for all } g \in G \}$$

is the centralizer of $\rho(G)$ on $\text{Aut}(K)$, consisting of the automorphisms of $K$ which commute with the $G$-action. In particular when $G = \text{Aut}(K)$ we have that $\text{Aut}_G(K) = Z(\text{Aut}(K))$. Let us see some examples:

3.4.1. $K = \mathbb{Z}/p$ for prime $p > 2$, and $G = \text{Aut}(\mathbb{Z}/p) = \mathbb{Z}/(p - 1)$. Here we have that $\text{Aut}_G(K) = Z(\text{Aut}(K)) \cong \mathbb{Z}/(p - 1)$ and that

$$H^n(G, H^3(K, \mathbb{T})) = H^3(K, \mathbb{T})^G \cong (\mathbb{Z}/p)^{\mathbb{Z}/(p-1)} = 0$$

$$H^1(G, H^2(K, \mathbb{T})) = \text{Hom}(\mathbb{Z}/(p - 1), 0) = 0$$

$$H^2(G, H^1(K, \mathbb{T})) = H^2(\mathbb{Z}/(p - 1), \mathbb{Z}/p) = 0$$

where the last equality follows from the fact that $H^n(G, M)$ is annihilated by $|G|$ for all $n > 0$ \[III.10.2\]. In this case $H^3(\text{Tot}^*(A^{**}(K \times G, \mathbb{T}))) = 0$ and therefore all pseudomonoid structures on $K = \mathbb{Z}/p$ are equivalent to the trivial one. This example could be generalized as follows:

3.4.2. Groups with order relatively prime. Let us recall two facts. First, if $G$ is a finite group of order $m$, $r$ is a positive integer with $(m, r) = 1$ and $A^r = 0$, then $H^n(K, A) = 0$ for all $n$ and all subgroups $K$ of $G$, see \[17\, Proposition 1.3.1]. And second, if $e$ is the exponent of $H^2(G, \mathbb{T})$ then $e^2$ divides the order of $G$, see \[17\, Theorem 2.1.5].

Then for the case on which $|G|$ is relatively prime to $|K|$ we obtain that

$$H^1(G, H^2(K, \mathbb{T})) = 0$$

and therefore we have that $H^3(\text{Tot}^*(A^{**}(K \times G, \mathbb{T}))) \cong H^3(K, \mathbb{T})^G$ and

$$|\text{Psdmn}^G(K)| \cong H^3(K, \mathbb{T})^G/\text{Aut}_G(K).$$

3.4.3. The dihedral group $D_n$ as a semi-direct product. The dihedral group is isomorphic to $\mathbb{Z}/n \rtimes \mathbb{Z}/2$ when $\mathbb{Z}/2$ acts on $\mathbb{Z}/n$ by multiplication of $-1$. Since the induced action of $\mathbb{Z}/2$ on the cohomology ring $H^*(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}[x]/\langle nx \rangle$ maps $x \mapsto -x$, we have that $x^2 \mapsto x^2$ and therefore $H^4(\mathbb{Z}/n, \mathbb{Z})^{\mathbb{Z}/2} = \mathbb{Z}/n$ and

$$H^2(\mathbb{Z}/2, H^2(\mathbb{Z}/n, \mathbb{Z})) = H^2(\mathbb{Z}/n, \mathbb{Z})^{\mathbb{Z}/2} = \begin{cases} \mathbb{Z}/2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Since

$$H^4(D_n, \mathbb{Z}) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/n & \text{if } n \text{ is even} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/n & \text{if } n \text{ is odd} \end{cases}$$

we know that

$$H^3(\text{Tot}^*(A^{**}(\mathbb{Z}/n \rtimes \mathbb{Z}/2, \mathbb{T}))) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/n & \text{if } n \text{ is even} \\ \mathbb{Z}/n & \text{if } n \text{ is odd} \end{cases}$$
and therefore

\(|Psdmn^{Z/2}(Z/n)| = \begin{cases} \frac{(Z/2 \oplus Z/n)}{\text{Aut}(Z/n)} & \text{if } n \text{ is even} \\ (Z/n)/\text{Aut}(Z/n) & \text{if } n \text{ is odd} \end{cases}\)

because in this case \(\text{Aut}_{Z/2}(Z/n) = \text{Aut}(Z/n)\).

In particular, when \(n = 4\) we have that \(\text{Aut}(Z/4) = Z/2\). and therefore the action of \(\text{Aut}_{Z/2}(Z/4)\) on \(H^3(\text{Tot}^{\ast}(A^{\ast \ast}(Z/4 \times Z/2, T)))\) is trivial. Hence

\(|Psdmn^{Z/2}(Z/4)| = Z/2 \oplus Z/4|.

3.5. The case of the group acting on itself by conjugation. Perhaps the most known pseudomonoid with strict unit in the 2-category of \(G\)-sets with twist was introduced by Dijkgraaf, Pasquier and Roche in [9, Section 3.2] while defining the quasi Hopf algebra \(D^w(G)\) with for \(w \in Z^3(G; T)\). In the equations (3.2.5) and (3.2.6) of [9] they defined a 3-cocycle \(\alpha_w \oplus \beta_w \oplus \theta_w \in Z^3(\text{Tot}(A^{\ast \ast}(G \times G, T)))\) by the equations

\[
\alpha_w[g|h][x] := \frac{w[g|h|x]w[ghxh^{-1}g^{-1}|g|h]}{w[g|h][h]} \quad \beta_w[g|x|y] := \frac{w[g|x|y]w[gxg^{-1}|gyg^{-1}|g]}{w[gxg^{-1}|g|y]} \quad \theta_w[x|y|z] := w[x|y|z],
\]

where \(\alpha_w\) was used to define the algebra law, \(\beta_w\) to define the coalgebra law, and \(\theta_w\) encoded the fact that the coproduct is quasicoassociative (and other coherence).

This quasi Hopf algebra \(D^w(G)\) is known as the Twisted Drinfeld Double of \(G\) twisted by \(w\) (cf. [25]). Firstly we claim the following.

Lemma 3.14. The 3-cocycle \(\alpha_w \oplus \beta_w \oplus \theta_w\) equals \(\tau^w_1 w\), the image of \(w\) under the restricted shuffle homomorphism \(\tau^w_1 w\) defined in (2.2). (\(\blacksquare\))

Therefore by Lemma 1.11, Theorem 2.2 and Lemma 2.5 we get

Proposition 3.15. The cohomology class

\([\alpha_w \oplus \beta_w \oplus \theta_w] \oplus [w] \in H^3(\text{Tot}^{\ast}(A^{\ast \ast}(G \times G, T))) \oplus H^3(G, T)\)

\(^1\)In order to get exactly the same formulæ it is necessary to change \(G\) by \(G^{\text{op}}\).
is equal to $\mu^*[w]$ where $\mu : G \times G \to G, (a, g) \mapsto ag$ is the multiplication map. Moreover we obtain the isomorphism

$$H^3(G, \mathbb{T}) \oplus H^3(\text{Tot}^*(B^{*\ast}(G \rtimes G, \mathbb{T}))) \to H^3(\text{Tot}^*(A^{*\ast}(G \rtimes G, \mathbb{T})))$$

$[u] \oplus [x] \mapsto [\alpha u \oplus \beta w \oplus \theta w] + [x].$

Now, since $G \rtimes G \cong G \times G$, we have that the Lyndon-Hochschild-Serre spectral sequence collapses at the second page. And since the action of $G$ on $G$ is given by conjugation, then the action of $G$ on $H^3(G, \mathbb{T})$ is trivial. Hence we have that $H^3(\text{Tot}(B^{*\ast}(G \rtimes G, \mathbb{T})))$ sits in the middle of the short exact sequence

$$0 \to H^3(G, \text{Hom}(G, \mathbb{T})) \to H^3(\text{Tot}(B^{*\ast}(G \rtimes G, \mathbb{T}))) \to \text{Hom}(G, H^2(G, \mathbb{T})) \to 0.$$

Moreover, in the present situation we have $\rho(G) = \text{Inn}(G)$, and therefore

$$\text{Aut}_G(G) = C_{\text{Aut}(G)}(\text{Inn}(G)),$$

namely the group of automorphisms of $G$ which commute with all inner automorphisms.

With the previous calculations at hand we can calculate the group $H^3(\text{Tot}(B^{*\ast}(G \rtimes G, \mathbb{T})))$ and $\text{Aut}_G(G)$ in some particular examples:

3.5.1. $G$ simple and non abelian: When $G$ is simple and nonabelian, its abelianization $G/[G, G]$ is trivial. Therefore $\text{Hom}(G, \mathbb{T})$ and $\text{Hom}(G, H^2(G, \mathbb{T}))$ are trivial and hence

$$H^3(\text{Tot}(B^{*\ast}(G \rtimes G, \mathbb{T}))) = 0.$$

So we have that

$$|\text{Psddm}^G(G)| \cong H^3(G, \mathbb{T})/\text{Aut}_G(G).$$

When $G$ is the alternating group $A_n$ we have that $\text{Aut}_n(A_n) = C_{\mathfrak{S}_n}(A_n) = 1$ and therefore

$$|\text{Psddm}^{A_n}(A_n)| \cong H^3(A_n, \mathbb{T}), \text{ for } n \neq 6 \text{ and } n > 4.$$

In particular when $n = 5$ we have that $H^3(A_5, \mathbb{T}) = \mathbb{Z}/120$ and hence

$$|\text{Psddm}^{A_5}(A_5)| \cong \mathbb{Z}/120.$$

3.5.2. Binary icosahedral group. The binary icosahedral group $\overline{A}_5$ is a subgroup of $SU(2)$ that can be obtained as the pullback of the diagram

$$\begin{array}{ccc}
\overline{A}_5 & \to & SU(2) \\
\downarrow & & \downarrow \\
A_5 & \to & SO(3)
\end{array}$$

where $A_5$ embeds in $SO(3)$ as the group of isometries of an icosahedron. This group satisfies $H_1(A_5, \mathbb{Z}) = 0$, $H_2(A_5, \mathbb{Z}) = 0$ and $H_3(A_5, \mathbb{Z}) = \mathbb{Z}/120$ (see [11, Page 279]), and therefore $H^1(\overline{A}_5, \mathbb{Z}) = H^2(\overline{A}_5, \mathbb{Z}) = H^3(\overline{A}_5, \mathbb{Z}) = 0$ and $H^4(\overline{A}_5, \mathbb{Z}) = \mathbb{Z}/120$. Hence $H^1(\overline{A}_5, \mathbb{T}) = H^2(\overline{A}_5, \mathbb{T}) = 0$, $H^3(\overline{A}_5, \mathbb{T}) = \mathbb{Z}/120$ and $H^4(\text{Tot}(B^{*\ast}(\overline{A}_5 \rtimes \overline{A}_5; \mathbb{T}))) = 0$.

In this case $\text{Inn}(\overline{A}_5) \cong A_5$ and $\text{Aut}(\overline{A}_5) \cong \mathfrak{S}_5$, therefore $\text{Aut}_{\overline{A}_5}(G) = C_{\mathfrak{S}_5}(A_5) = 1$ and

$$|\text{Psddm}^{A_5}(A_5)| \cong \mathbb{Z}/120.$$

A similar argument applies to any superperfect group since by definition they are the ones such that $H_1(G, \mathbb{Z}) = 0$ and $H_2(G, \mathbb{Z}) = 0$. 

3.5.3. The dihedral group $G = D_n$ with $n$ odd. In this case
\[ \text{Hom}(D_n, T) = \mathbb{Z}/2, \quad H^2(D_n; \mathbb{Z}/2) = \mathbb{Z}/2 \quad \text{and} \quad H^2(D_n; T) = 0, \]
hence
\[ H^3(\text{Tot}^* (B^{*,*}(D_n \times D_n; T))) = H^2(D_n, \text{Hom}(D_n, T)) = \mathbb{Z}/2. \]
Since $H^3(D_n; T) = \mathbb{Z}/2 \oplus \mathbb{Z}/n$ we have that the isomorphism classes of pseudomonoid structures coming from the Twisted Drinfeld Double construction are $\mathbb{Z}/2 \oplus \mathbb{Z}/n$ and that there is an independent pseudomonoid structure which comes from $H^2(D_n; \text{Hom}(D_n; T)) = \mathbb{Z}/2$. In this case we have that $\text{Aut}_{D_n}(D_n) = Z(D_n) = 1$ and therefore
\[ |\text{Psdmn}^{D_n}(D_n)| \cong (\mathbb{Z}/n \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2). \]

3.5.4. The symmetric group $G = S_n$ for $n \geq 4$. From [1, VI-5] we know that
\[ \text{Hom}(S_n, T) = \mathbb{Z}/2, \quad H^2(S_n; \mathbb{Z}/2) = \mathbb{Z}/2 \]
\[ H^3(S_n; T) = \mathbb{Z}/2 \quad \text{and} \quad H^3(S_n; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \]

hence we get the exact sequence
\[ 0 \to \mathbb{Z}/2 \to H^3(\text{Tot}^* (B^{*,*}(S_n \times S_n; T))) \to \mathbb{Z}/2 \to 0. \]
In particular we could say that the nontrivial element in $H^2(S_n, \text{Hom}(S_n, T)) = \mathbb{Z}/2$ induces a pseudomonoid structure on $S_n$ which is not isomorphic to any structure coming from the construction of the Twisted Drinfeld Double. This follows from the fact that for $n \neq 2$ and $n \neq 6$, $\text{Aut}(S_n) = \text{Inn}(S_n) = S_n$ and therefore \( \text{Aut}_{S_n}(S_n) = Z(S_n) = 1 \) is the trivial group. Whenever $n = 6$ we know that $\text{Out}(S_6) = \mathbb{Z}/2$; nevertheless $\text{Aut}_{S_n}(S_6) = 1$. Therefore we have that
\[ |\text{Psdmn}^{S_n}(S_n)| = H^3(S_n, T) \oplus H^3(\text{Tot}^* (B^{*,*}(S_n \times S_n; T))) \]
\[ H^3(S_n, T) = \begin{cases} \mathbb{Z}/12 \oplus \mathbb{Z}/2 & \text{if } n = 4, 5 \\ \mathbb{Z}/12 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \geq 6 \end{cases} \]

3.5.5. G cyclic group. When $G = \mathbb{Z}/n$ we get that
\[ H^3(\text{Tot}^* (B^{*,*}(\mathbb{Z}/n \times \mathbb{Z}/n; T))) = H^2(\mathbb{Z}/n, \text{Hom}(\mathbb{Z}/n, T)) = \mathbb{Z}/n \]
and $H^3(\mathbb{Z}/n, T) = \mathbb{Z}/n$. The action of $\text{Aut}_{\mathbb{Z}/n}(\mathbb{Z}/n) = \mathbb{Z}/n^\times$, which is the multiplicative group of units in $\mathbb{Z}/n$, on $H^1(\mathbb{Z}/n, T) = \mathbb{Z}/n$ is given by multiplication and while on $H^3(\mathbb{Z}/n, T) = \mathbb{Z}/n$ is given by the square of the multiplication; hence we get that
\[ |\text{Psdmn}^{\mathbb{Z}/n}(\mathbb{Z}/n)| \cong (\mathbb{Z}/n \oplus \mathbb{Z}/n) / \mathbb{Z}/n^\times \]
where the action is given by
\[ \mathbb{Z}/n^\times \times (\mathbb{Z}/n \oplus \mathbb{Z}/n) \to (\mathbb{Z}/n \oplus \mathbb{Z}/n), \quad (a, (x, y)) \mapsto (ax, a^2y). \]

For example when $n = 4$ we have that
\[ |\text{Psdmn}^{\mathbb{Z}/4}(\mathbb{Z}/4)| \cong (\mathbb{Z}/4) / (\mathbb{Z}/4^\times) \times \mathbb{Z}/4. \]

\(^2\text{See Group Cohomology of Symmetric Groups in http://groupprops.subwiki.org}\)
4. The monoidal category of equivariant vector bundles on a pseudomonoid

Let $G$ be a group and $K = (K, m, 1, \alpha, \beta, \theta)$ a pseudomonoid with strict unit in the 2-category of $G$-sets with twists.

We define the category $Bun_G(K)$ of $G$-equivariant finite dimensional bundles over $K$ as follows:

An object is a $K$-graded finite dimensional Hilbert space $H = \bigoplus_{k \in K} H_k$ and a twisted $G$-action
\[
\triangleright : G \to U(H)
\]
such that
\begin{itemize}
  \item $\sigma \triangleright H_k = H_{\sigma k}$
  \item $\sigma \triangleright (\tau \triangleright h_k) = \alpha[\sigma|\tau|][k](\sigma \triangleright h_k)$
  \item $\mathbf{e} \triangleright h = h$
\end{itemize}
for all $\sigma, \tau \in G, k \in K, h_k \in H_k$. Morphisms in the category are linear maps that preserve the grading and the twisted action, i.e., a linear map $f : H \to H'$ is a morphism if
\begin{itemize}
  \item $f(H_k) \subset H'_k$,
  \item $f(\sigma \triangleright h) = \sigma \triangleright f(h)$
\end{itemize}
for all $\sigma \in G, k \in K$ and $h \in H$.

We define a monoidal structure on $Bun_G(K)$ as follows:

**Proposition 4.1.** Let $H$ and $H'$ be objects in $Bun_G(K)$, then the tensor product of Hilbert spaces $H \otimes H'$ is a $G$-equivariant $K$-bundle with $K$-grading $(H \otimes H')_k = \bigoplus_{x,y \in K : xy = k} H_x \otimes H'_y$ and twisted $G$-action
\[
\sigma \triangleright (h_x \otimes h'_y) := \beta[\sigma||x||y](\sigma \triangleright h_x \otimes \sigma \triangleright h'_y),
\]
for all $k \in K, \sigma \in G, h_x \in H_x$ and $h'_y \in H'_y$.

**Proof.** We just need to see that
\[
\sigma \triangleright (\tau \triangleright (h_x \otimes h'_y)) = \alpha[\sigma||\tau||xy](\sigma \triangleright h_x \otimes \sigma \triangleright h'_y)
\]
for all $\sigma, \tau \in G, x, y \in K$. In fact we have
\[
\sigma \triangleright (\tau \triangleright (h_x \otimes h'_y))
\begin{align*}
&= \beta[\sigma||x||y]\left(\sigma \triangleright (\tau \triangleright h_x \otimes \tau \triangleright h'_y)\right) \\
&= \beta[\sigma||x||y]\beta[\sigma||\tau x||\tau y]\left(\sigma \triangleright (\tau \triangleright h_x) \otimes \sigma \triangleright (\tau \triangleright h'_y)\right) \\
&= \beta[\sigma||x||y]\beta[\sigma||\tau x||\tau y]\alpha[\sigma||\tau||] \alpha[\sigma||y||] \left((\sigma \triangleright h_x) \otimes (\sigma \triangleright h'_y)\right) \\
&= \frac{\beta[\sigma||x||y]\beta[\sigma||\tau x||\tau y]\alpha[\sigma||\tau||] \alpha[\sigma||y||]}{\beta[\sigma||x||y]} \left((\sigma \triangleright h_x) \otimes (\sigma \triangleright h'_y)\right) \\
&= \alpha[\sigma||\tau||xy]\left((\sigma \triangleright h_x) \otimes (\sigma \triangleright h'_y)\right),
\end{align*}
\]
where the last equality follows from the equation $\delta_G(\beta)\delta_K(\alpha) = 1$ obtained in (3.5) in the proof of Proposition 3.4.

$\square$
Now, for $\mathcal{H}$, $\mathcal{H}'$ and $\mathcal{H}''$ objects in $Bun_G(K)$ the associativity constraint
\[
\Theta : (\mathcal{H} \otimes \mathcal{H}') \otimes \mathcal{H}'' \to \mathcal{H} \otimes (\mathcal{H}' \otimes \mathcal{H}''),
\]
for the monoidal structure $\otimes$ is defined by
\[
\Theta((h_x \otimes h'_y) \otimes h''_z) = \theta[x|y|z]^{-1}h_x \otimes (h'_y \otimes h''_z)
\]
for all $x, y, z \in K$, $h_x \in \mathcal{H}_x$, $h'_y \in \mathcal{H}'_y$ and $h''_z \in \mathcal{H}''_z$.

In order to see that $\Theta$ is natural isomorphism in $Bun_G(K)$ we need to see that it preserves the grading, which follows from the definition, and that $\Theta$ is compatible with the $G$-action. The compatibility follows from the equalities

\[
\Theta(\sigma \triangleright [(h_x \otimes h'_y) \otimes h''_z]) = \beta[\sigma|x|y|z]\theta[\sigma|x|y]\Theta((\sigma \triangleright h_x \otimes \sigma \triangleright h'_y) \otimes \sigma \triangleright h''_z)
\]
\[
= \beta[\sigma|x|y|z]\beta[\sigma|x|y] \frac{\theta[\sigma \cdot x|\sigma \cdot y|\sigma \cdot z]}{\theta[x|y|z]} \sigma \triangleright h_x \otimes (\sigma \triangleright h'_y \otimes \sigma \triangleright h''_z)
\]
\[
= \beta[\sigma|x|y|z] \frac{\theta[x|y|z]}{\theta[x|y|z]} \sigma \triangleright h_x \otimes (\sigma \triangleright h'_y \otimes \sigma \triangleright h''_z)
\]
\[
= \sigma \triangleright \theta[x|y|z]^{-1}h_x \otimes (h'_y \otimes h''_z)
\]
\[
= \sigma \triangleright \Theta((h_x \otimes h'_y) \otimes h''_z),
\]

where the equality between the third and the fourth line follows from the equation $\delta_1(\theta)\delta_2(\beta)^{-1} = 1$ obtained in (3.6) in the proof of Proposition 3.4.

The pentagonal axiom follows directly from the 3-cocycle condition of $\theta$.

Finally we define the unit object $\underline{C}$ as the one dimensional Hilbert space $\mathbb{C}$ graded only at the unit element $e \in K$, endowed with trivial $G$-action. From the normalization of all cochains, it follows that $\underline{C}$ is unit object in $Bun_G(K)$ with respect to the tensor product. All in all, we have just proved that

**Proposition 4.2.** For $K = (K, m, 1, \alpha, \beta, \theta)$ a pseudomonoid with strict unit in the 2-category of $G$-sets with twists, the triple $(Bun_G(K), \otimes, \underline{C})$, endowed with the tensor product $\otimes$ and the unit element $\underline{C}$ is a monoidal category (or tensor category).

**Remark 4.3.** Let $G$ be a group acting by the right over an abelian group $H$. Then $G$ also acts by the left over $\hat{H}$ the abelian group of all characters of $H$. In the case that the 3-cocycle $\alpha \oplus \beta \oplus \theta \in Z^3(Tot(A^* \circ (\hat{H} \times G, T)))$ is trivial, then the tensor category $Bun_G(H)$ is just $\text{Rep}(\hat{H} \times G)$ the tensor category of finite dimensional unitary representation of the semi-direct product $H \rtimes G$, where $G$ acts by the left on $H$ as $g \cdot a := ag^{-1}$, for all $g \in G, a \in H$.

**Remark 4.4.** Consider the case of a group $G$ acting on itself by conjugation and the pseudomonoid $K^w_G = (G, m, 1, \alpha_w, \beta_w, \theta_w)$ defined in section 3.5. In the case that $G$ is finite, the tensor category $Bun_G(K^w_G)$ is exactly the category of representations of $\text{Rep}(D^w(G))$ of the Twisted Drinfeld Double $D^w(G)$. Note that the quasi-Hopf algebra $D^w(G)$ is defined only for $G$ finite, but $Bun_G(K^w_G)$ is defined for an arbitrary discrete group. So, the tensor category $Bun_G(K^w_G)$ is a generalization of the twisted Drinfeld Double of a finite group.
The Twisted Drinfeld Double $D^w(G)$ is important at least for the following two reasons. First, $D^w(G)$ is a quasi-triangular quasi-Hopf algebra [9] and then $\text{Rep}(D^w(G))$ is a braided tensor category and it defines representations of the braid group and knots and links invariants [24, Chapter 1]. Second, $\text{Rep}(D^w(G))$ is a Modular Tensor Category (MTC), so it defines a 3D-TQFT using Reshetikhin-Turaev construction, see [3, Chapter 4] and [24, Chapter 4].

For any group, the tensor category $\text{Bun}_G(\mathcal{K}_G^{\text{w}})$ is a braided tensor category, so it defines representations of the braid group and knots and links invariants [24, Chapter 1]. Second, $\text{Rep}(D^w(G))$ is not a MTC if $G$ is infinite. In fact, the category of finite dimensional unitary representation of $G$ is a full tensor subcategory of $\text{Bun}_G(\mathcal{K}_G^{\text{w}})$ and it has infinite many isomorphism classes of irreducible representations, so in particular $\text{Bun}_G(\mathcal{K}_G^{\text{w}})$ would also have infinitely many simple objects.

4.1. Morphism of pseudomonoids, monoidal functors and natural isomorphisms. A morphism in the 2-category of pseudomonoids in the 2-category $G$-sets with twists induce a monoidal functor between the associated monoidal categories.

**Proposition 4.5.** Let $\mathcal{F} = (F, \chi, \kappa) : \mathcal{K} \to \mathcal{K}'$ be a morphism of pseudomonoids. Then $\mathcal{F}$ induces a monoidal functor from the monoidal categories $(\text{Bun}_G(\mathcal{K}), \otimes, \Box)$ and $(\text{Bun}_G(\mathcal{K}'), \otimes', \Box')$.

**Proof.** Let $\mathcal{F} = (F, \chi, \kappa) : \mathcal{K} \to \mathcal{K}'$ be a morphism of pseudomonoids as defined in Proposition 3.8. We define a functor $F : \text{Bun}_G(\mathcal{K}) \to \text{Bun}_G(\mathcal{K}')$ in the following way: for $\mathcal{H}$ an object in $\text{Bun}_G(\mathcal{K})$, the $K'$-graded Hilbert space $F(\mathcal{H})$ is the direct sum

$$F(\mathcal{H})_y = \bigoplus_{x \in K : F(x) = y} \mathcal{H}_x.$$ 

The twisted $G$-action on $F(\mathcal{H})$ is defined as follows: take $h'_y \in F(\mathcal{H})_y$ defined by the element $h'_y = h_x$ for some vector $h_x \in \mathcal{H}_x$ with $F(x) = y$. Define the twisted $G$-action $\triangleright'$ on $h'_y$ by

$$\sigma \triangleright' h'_y := \chi[\sigma|x](\sigma \triangleright h_x),$$

and note that

$$\sigma \triangleright' (\tau \triangleright' h'_y) = \sigma \triangleright' (\kappa[\tau|x]|x \triangleright h_x)$$

$$= \chi[\tau || x] \kappa[\sigma|\tau x] \sigma \triangleright (\tau \triangleright h_x)$$

$$= \chi[\tau || x] \kappa[\sigma|\tau x] \alpha[\sigma|\tau || x] \sigma \triangleright h_x$$

$$= \alpha'[\sigma|\tau || y] \chi[\sigma \tau || x] \sigma \triangleright h_x$$

$$= \alpha'[\sigma|\tau || y] \sigma \triangleright' h'_y$$

where the equality between the third and the fourth lines follows from the equation $\delta_G \chi = F^* \alpha'/\alpha$ obtained in (3.10) in the proof of Proposition 3.8. Therefore the Hilbert space $F(\mathcal{H})$ is an object in $\text{Bun}_G(\mathcal{K}')$.

Let us check that the functor $F$ is monoidal: take $\mathcal{H}, \mathcal{H}'$ two objects in $\text{Bun}_G(\mathcal{K})$ and consider the map

$$R : F(\mathcal{H}) \otimes' F(\mathcal{H}') \to F(\mathcal{H} \otimes \mathcal{H}')$$

$$h_{x_1} \otimes' h_{x_2} \mapsto \kappa(x_1, x_2)^{-1} h_{x_1} \otimes h_{x_2}$$

where $h_{x_1} \in \mathcal{H}_{x_1}$ and $h_{x_2} \in \mathcal{H}'_{x_2}$ but we see them both as elements in $F(\mathcal{H})_{F(x_1)}$ and $F(\mathcal{H}')_{F(x_2)}$ respectively, and the element $h_{x_1} \otimes h_{x_2}$ we see it as an element in
\[
\mathbf{F}(\mathcal{H} \otimes \mathcal{H}')_{F(x_1 \otimes x_2)}. \text{ Let us show that the map } R \text{ is a morphism in } \text{Bun}_G(\mathcal{K}'), \text{ i.e. that } R(\sigma \triangleright' (h_{x_1} \otimes' h_{x_2})) = \sigma \triangleright' R(h_{x_1} \otimes' h_{x_2}). \text{ For he left hand side we have:}
\]

\[
R(\sigma \triangleright' (h_{x_1} \otimes' h_{x_2})) = \beta'[\sigma]|F(x_1)|F(x_2)| x_1 \chi[\sigma|x_2]R(\sigma \triangleright h_{x_1} \otimes' \sigma \triangleright h_{x_2}) = \beta'[\sigma]|F(x_1)|F(x_2)| x_1 \chi[\sigma|x_2] \kappa[\sigma|x_2]^{-1}(\sigma \triangleright h_{x_1} \otimes \sigma \triangleright h_{x_2}),
\]

and for the right hand side we have:

\[
\sigma \triangleright' R(h_{x_1} \otimes' h_{x_2}) = \sigma \triangleright' \kappa[x_1|x_2]^{-1}(h_{x_1} \otimes h_{x_2}) = \chi[\sigma|x_2] \kappa[x_1|x_2]^{-1}(\sigma \triangleright h_{x_1} \otimes h_{x_2});
\]

and note that the two expressions coincide due to the equation \(\delta_G \kappa \cdot \delta_K \chi = \beta/F'\beta'\) that appeared in (3.11) under the proof of Proposition 3.8.

The hexagonal axiom of the monoidal functor follows directly from the equation \(\delta_K \kappa = \theta/F'\theta'\) and we will not reproduce its proof. \(\square\)

**Proposition 4.6.** Given morphisms of pseudomonoids with strict unit in the 2-category of G-sets with twists \(F = (F, \chi, \kappa)\) and \(F' = (F', \chi', \kappa')\), with \(F, F' : \mathcal{K} \to \mathcal{K}'\), a 2-morphism \(\gamma : F \Rightarrow F'\), induces a monoidal natural isomorphism between the monoidal functors \(F\) and \(F'\).

**Proof.** Using the notation defined above, we define the transformation between \(F\) and \(F'\) as follows:

\[
\begin{align*}
\mathbf{F}(\mathcal{H}) & \to \mathbf{F}'(\mathcal{H}) \\
h_x \mapsto \gamma(x)^{-1} h_x.
\end{align*}
\]

Equations \(\delta_1 \gamma = \chi'/\chi\) and \(\delta_2 \gamma = \kappa/\kappa'\) shown in the proof of Proposition 3.10 imply that the transformation is natural and monoidal, respectively. \(\square\)

### 4.2. Automorphisms of pseudomonoids and their action on the monoidal category of equivariant vector bundles.

Let us fix \(\mathcal{K} = (K, m, 1, \alpha, \beta, \theta)\) a pseudomonoid with strict unit in the 2-category of G-sets with twists and let \((\text{Bun}_G(\mathcal{K}), \otimes, \underline{\underline{\circ}})\) be the monoidal category of G-equivariant bundles over \(K\).

Take \(F = (F, \chi, \kappa) : \mathcal{K} \to \mathcal{K}\) an invertible morphism of the pseudomonoid \(\mathcal{K}\) and note that Proposition 4.5 tells us that the induced monoidal functor

\[
\mathbf{F} : (\text{Bun}_G(\mathcal{K}), \otimes, \underline{\underline{\circ}}) \to (\text{Bun}_G(\mathcal{K}), \otimes, \underline{\underline{\circ}})
\]

becomes an automorphism of the monoidal category \((\text{Bun}_G(\mathcal{K}), \otimes, \underline{\underline{\circ}})\).

If we denote by

\[
\text{Aut}_{\text{Psmnd}}(\mathcal{K})
\]

the 2-group of automorphisms of the pseudomonoid \(\mathcal{K}\), whose morphisms are invertible morphisms \(F : \mathcal{K} \to \mathcal{K}\) and whose 2-morphisms are the natural transformations between functors \(\gamma : F \Rightarrow F'\), and

\[
\text{Aut}_\otimes(\text{Bun}_G(\mathcal{K}))
\]

the 2-group of automorphisms of the monoidal category \((\text{Bun}_G(\mathcal{K}), \otimes, \underline{\underline{\circ}})\), whose morphisms are invertible monoidal functors and whose 2-morphisms are monoidal
natural transformations, then we have that Propositions 4.5 and 10 imply that there is a 2-functor
\[
\text{Aut}_{\text{Psmnd}}(\mathcal{K}) \to \text{Aut}_{\otimes}(\text{Bun}_G(\mathcal{K}))
\]
\[
\gamma : F \mapsto F' \mapsto \gamma : F \mapsto F'
\]
from the 2-group of automorphisms of the pseudomonoid \( \mathcal{K} \) to the 2-group of automorphisms of \( \text{Bun}_G(\mathcal{K}) \).

To understand the previous action in more detail, let us start by studying the category \( \text{Aut}_{\text{Psmnd}}(\mathcal{K}) \).

An automorphism \( F = (F, \chi, \kappa) : \mathcal{K} \to \mathcal{K} \) consists of a \( G \)-equivariant automorphism \( F \in \text{Aut}_G(\mathcal{K}) \), together with a degree 2 cochain \( \chi \oplus \kappa^{-1} \) in \( \text{Tot}^*(A^*\otimes(K \rtimes G, T)) \) such that
\[
(\delta_G \oplus \delta_K^{-1})(\chi \oplus \kappa^{-1}) = F^*\alpha/\alpha \oplus F^*\beta/\beta \oplus F^*\theta/\theta.
\]

The automorphism \( F \) lies on the image of the forgetful functor
\[
\text{Aut}_{\text{Psmnd}}(\mathcal{K}) \to \text{Aut}_G(\mathcal{K})
\]
\[
F = (F, \chi, \kappa) \mapsto F;
\]
if and only if the cohomology classes \([\alpha \oplus \beta \oplus \theta]\) and \( F^*[\alpha \oplus \beta \oplus \theta] \) are equal as cohomology classes in \( H^2(\text{Tot}^*(A^* \otimes(K \rtimes G, T))) \).

If we define
\[
\text{Aut}_G(\mathcal{K};[\alpha \oplus \beta \oplus \theta]) := \{ F \in \text{Aut}_G(\mathcal{K}) | F^*[\alpha \oplus \beta \oplus \theta] = [\alpha \oplus \beta \oplus \theta]\}
\]
we have that the 2-group of automorphisms of \( \mathcal{K} \) sits in the exact sequence
\[
0 \to H^2(\text{Tot}^*(A^* \otimes(K \rtimes G, T))) \to \text{Aut}_{\text{Psmnd}}(\mathcal{K}) \to \text{Aut}_G(\mathcal{K};[\alpha \oplus \beta \oplus \theta]) \to 0
\]
where \( H^2(\text{Tot}^*(A^* \otimes(K \rtimes G, T))) \) denotes the degree 2-cocycles and \( \text{Tot}^1(A^* \otimes(K \rtimes G, T)) \) parameterizes the 2-morphisms between the morphisms of \( Z^2(\text{Tot}^*(A^* \otimes(K \rtimes G, T))) \).

If we take equivalence classes of automorphisms in \( \text{Aut}_{\text{Psmnd}}(\mathcal{K}) \) defined by the 2-morphisms, we obtain a group which is usually denoted by
\[
\pi_1(\text{Aut}_{\text{Psmnd}}(\mathcal{K}));
\]
this group sits in the middle of the short exact sequence
\[
0 \to H^2(\text{Tot}^*(A^* \otimes(K \rtimes G, T))) \to \pi_1(\text{Aut}_{\text{Psmnd}}(\mathcal{K})) \to \text{Aut}_G(\mathcal{K};[\alpha \oplus \beta \oplus \theta]) \to 0;
\]
and by the Lyndon-Hochschild-Serre spectral sequence we know that there is an exact sequence
\[
0 \to H^1(G, \text{Hom}(K, T)) \to H^2(\text{Tot}^*(A^* \otimes(K \rtimes G, T))) \to H^2(K, T)^G \xrightarrow{d_2} H^2(G, \text{Hom}(K, T)),
\]
where \( d_2 \) is the differential of the second page.
Furthermore, if we take the group of 2-morphisms of the identity morphism in $\text{Aut}_{\text{Psmnd}}(K)$, we obtain a group which is usually denoted by

$$\pi_2(\text{Aut}_{\text{Psmnd}}(K))$$

and is equal to $H^1(\text{Tot}(A^{*\ast}(K \times G, T))) = \text{Hom}(K, T)^G$.

### 4.3. The Grothendieck ring associated to the monoidal category

Consider the isomorphism classes of objects in the monoidal category $\text{Bun}_G(K)$. Since the objects could be understood as finite dimensional vector spaces which are $K$-graded endowed with a projective $G$-action, we can add them up and moreover we can multiply them by using the tensor product of the monoidal category. What we obtain is a semi-ring which we can make into a ring by applying the standard Grothendieck construction argument of K-theory. Denoting by $\text{Groth}(\text{Bun}_G(K))$ the Grothendieck ring constructed from the monoidal category $\text{Bun}_G(K)$, we have a functor

$$\text{Psmnd}_G \to \text{Rings}$$

$$K \mapsto \text{Groth}(\text{Bun}_G(K))$$

from the 2-category of pseudomonoids with strict unit in the 2-category $G$-sets with twists, to the category of rings.

The ring $\text{Groth}(\text{Bun}_G(K))$ can also be understood as the $\alpha$-twisted $G$-equivariant K-theory of the monoid $K$ where the multiplication is induced by the pushforward $m_*$ of the multiplication $m : K \times K \to K$. This twisted K-theory ring was the main motivation of this work and is the subject of the next section.

In the case on which $G = K$ and $G$ acts on $G$ by the left adjoint action we have seen that any 3-cocycle $w \in Z^3(G; T)$ induces a 3-cocycle $\alpha_w \oplus \beta_w \oplus \theta_w \in Z^3(\text{Tot}^*(A^{*\ast}(G \times G, T)))$ that makes $K := (G, m, 1, \alpha_w, \beta_w, \theta_w)$ into a pseudomonoid with strict unit in the 2-category $G$-sets with twists. In this case the Grothendieck ring $\text{Groth}(\text{Bun}_G(K))$ is isomorphic to the Grothendieck ring of representations

$$\text{Groth}(\text{Rep}(D^w(G)))$$

of the Twisted Drinfeld Double $D^w(G)$ of the group $G$ (see [25, Section 3]), which is also isomorphic to the $w$-twisted stringy K-theory

$$wK_{\text{st}}([G/G])$$

of the groupoid $[G/G]$ [25, Prop. 18] c.f. [2, 3, 14].

#### 4.3.1. Automorphisms

Since the 2-functor

$$\text{Aut}_{\text{Psmnd}}(K) \to \text{Aut}_{\otimes}(\text{Bun}_G(K))$$

induces a homomorphism

$$\pi_1(\text{Aut}_{\text{Psmnd}}(K)) \to \pi_1(\text{Aut}_{\otimes}(\text{Bun}_G(K))),$$

we have that there is a homomorphism of groups

$$\pi_1(\text{Aut}_{\text{Psmnd}}(K)) \to \text{Aut}(\text{Groth}(\text{Bun}_G(K)))$$

which composed with the inclusion

$$H^2(\text{Tot}(A^{*\ast}(K \times G, T))) \to \pi_1(\text{Aut}_{\text{Psmnd}}(K))$$
defines a homomorphism

\[ H^2(Tot(A^{*G}(K \times G, T))) \rightarrow \text{Aut}(\text{Groth}(\text{Bun}_G(K))). \]

The previous morphism will be of interest when we compare it with the group of automorphisms of the twisted equivariant K-theory ring in section 5.7.1.

5. The fusion product and the twisted G-equivariant K-theory ring

Whenever \( X \) is a finite \( G \)-CW complex with \( G \) a finite group, the elements in \( H^3_G(X; \mathbb{Z}) \) classify the isomorphism classes of projective unitary stable and equivariant bundles over \( X \), and these bundles provide the required information to define equivariant Fredholm bundles over \( X \); the homotopy groups of the space of section of a such bundle is one way to define the twisted \( G \)-equivariant K-theory groups of \( X \) (see [4]). The homotopy classes of automorphisms of a projective unitary stable and equivariant bundle over \( X \) are in one to one correspondence with \( H^2_G(X; \mathbb{Z}) \) and this group acts on the twisted \( G \)-equivariant K-theory groups.

Whenever the space \( X \) is a discrete \( G \)-set, there is an equivalent but easier way to define the twisted \( G \)-equivariant K-theory groups of \( X \). Let us review it.

5.1. Twisted \( G \)-equivariant K-theory. Take a normalized 2-cocycle \( \alpha : G \times G \times X \rightarrow \mathbb{T} \) and define an \( \alpha \)-twisted \( G \)-vector bundle over \( X \) as a finite dimensional \( X \)-graded complex vector space \( E \), which can alternatively seen as a finite dimensional complex vector bundle \( p : E \rightarrow X \) with finite support, endowed with a \( G \) action such that \( p \) is \( G \) equivariant, the action of \( G \) on the fibers is complex linear, and such that the composition of the action on \( E \) satisfies the equation

\[ g \cdot (h \cdot z) = \alpha(g, h||p(z))(gh \cdot z) \]

for all \( z \) in \( E \). Two \( \alpha \)-twisted \( G \)-vector bundles over \( X \) are isomorphic if there exists a \( G \) equivariant map \( E \rightarrow E' \) of complex vector bundles which is an isomorphism of vector bundles.

Definition 5.1. The \( \alpha \)-twisted \( G \)-equivariant K-theory of \( X \) is the Grothendieck group

\[ KU_G(X; \alpha) \]

associated to the semi-group of isomorphism classes of \( \alpha \)-twisted \( G \)-vector bundles over \( X \) endowed with the direct sum operation.

If we have a \( G \)-equivariant map \( F : Y \rightarrow X \) then the pullback of bundles induces a group homomorphism

\[ F^* : KU_G(X; \alpha) \rightarrow KU_G(Y; F^* \alpha). \]

5.1.1. For a normalized cochain \( \chi \in C^1_G(X; \mathbb{T}) \) with \( \delta_G \chi = \alpha' / \alpha \) then there is an induced isomorphism of groups

\[ \chi : KU_G(X; \alpha) \xrightarrow{\sim} KU_G(X; \alpha') \]

where \( \chi(E) := E \) and the \( G \)-action \( ' \) on \( z \in \chi(E) \) is given by the equation:

\[ h' \cdot z := \chi[h||p(z)](h \cdot z). \]

Since cohomologous twistings induce isomorphic twisted K-theory groups, we have that \( H^2_G(X; \mathbb{T}) \) classifies the isomorphism classes of twistings for the \( G \)-equivariant K-theory of \( X \). And since the isomorphisms \( \chi \) and \( \chi \cdot (\delta_G \gamma) \) are equal, we have
that the group \( H^1_G(X; T) \) acts on the \( \alpha \)-twisted \( G \)-equivariant K-theory group \( KU_G(X; \alpha) \) by automorphisms.

5.2. **Pushforward.** For a \( G \)-equivariant map \( F : Y \to X \) and \( \alpha \in Z^2_G(X; T) \) there is a pushforward map

\[
F_* : KU_G(Y; F^*\alpha) \to KU_G(X; \alpha)
\]

defined at the level of vector bundles as follows

\[
(F_* E)_x := \bigoplus_{\{y \in Y | F(y) = x\}} E_y
\]

where the \( G \)-action on \( F_* E \) is the one induced by the \( G \)-action on \( Y \) and the \( G \)-action on \( E \).

5.3. **External product.** If we consider two \( G \)-sets with twist \((X, \alpha_X)\) and \((Y, \alpha_Y)\), the external product is the homomorphism

\[
KU_G(X; \alpha_X) \times KU_G(Y; \alpha_Y) \cong KU_G(X \times Y; \pi_1^*\alpha_X \cdot \pi_2^*\alpha_Y)
\]

where \((E \boxtimes F)_{(x,y)} := E_x \otimes F_y\) and \(\pi_1\) and \(\pi_2\) denote the projections of \(X \times Y\) on the first and the second coordinate respectively.

5.4. **Multiplicative structures on Twisted Equivariant K-theory.** In the particular case on which the \( G \)-set \(X\) is endowed with the additional structure of a \( G \)-equivariant multiplication map \(m : X \times X \to X\)

and moreover that the cohomology class \([\alpha]\) of the twisting is multiplicative i.e. \(\pi_1^*[\alpha] \cdot \pi_2^*[\alpha] = m^*[\alpha]\), then the \( \alpha \)-twisted \( G \)-equivariant K-theory group can be endowed with a product structure. This construction could be done for \( G \)-equivariant H-spaces, but for clarity we will restrict ourselves to the case on which the \( G \)-set is a \( G \)-equivariant monoid with unit.

Let \(K\) be a \( G \)-equivariant discrete monoid with unit and denote by \(m : K \times K \to K\) the multiplication of the monoid. Take a twist \(\alpha \in Z^2_G(K; T)\) that is multiplicative, i.e. that there exist a cochain \(\beta \in C^1_G(K \times K; T)\) such that

\[
\delta_G \beta = \frac{m^*\alpha}{\pi_1^*\alpha \cdot \pi_2^*\alpha}
\]

or equivalently \(\delta_G \beta \cdot \delta_K \alpha = 1\), then we can compose the following morphisms

\[
KU_G(K; \alpha) \times KU_G(K; \alpha) \xrightarrow{\boxtimes} KU_G(K \times K; \pi_1^*\alpha \cdot \pi_2^*\alpha) \xrightarrow{\beta} KU_G(K \times K; m^*\alpha) \xrightarrow{m} KU_G(K; \alpha)
\]

thus producing a product structure

\[
\ast : KU_G(K; \alpha) \times KU_G(K; \alpha) \to KU_G(K; \alpha)
\]

\[
(E, F) \mapsto m_* (\beta(\boxtimes(E, F))).
\]

It is a simple calculation to see that the product \(\ast\) previously defined is associative whenever the cohomology class \([\beta] \in H^1_G(K \times K; T)\) satisfies the equation

\[
\delta_K [\beta] = 1
\]
as a cohomology class in $H^3_G(K \times K \times K; \mathbb{T})$. We therefore have that if there exists a cochain $\theta \in C^3_G(K \times K \times K; \mathbb{T})$ such that

$$\delta_G \theta = \delta_K \beta$$

then the product $\star$ previously defined endows the group $\mathbb{K}U_G(K; \alpha)$ with a ring structure. Summarizing:

**Proposition 5.2.** Consider $\alpha \in Z^2_G(K; \mathbb{T})$ and $\beta \in C^{1,2}(K \times G; \mathbb{T})$ satisfying the equations

$$\delta_G \beta \cdot \delta_K \alpha = 1, \quad \delta_2 [\beta] = 1.$$  
Then the group $\mathbb{K}U_G(K; \alpha)$ endowed with the product structure $\star \beta$ becomes a ring. Let us denote this ring by

$$\mathbb{K}U_G(K; \alpha, \beta) := (\mathbb{K}U_G(K; \alpha), \star \beta)$$
and let us call the pair $(\alpha, \beta)$ a multiplicative structure for $K$.

5.4.1. Many of the features of the twisted $G$-equivariant K-theory rings are better understood if we work with the notation introduced in section 1.

Recall that the double complex $A^{*,*} := A^{*,*}(K \times G; \mathbb{T})$ is the subcomplex of $C^{*,*}(K \times G; \mathbb{T})$ disregarding the 0-th row. Consider the subcomplex $A^{*,*,>3}$ of $A^{*,*}$ defined as subcomplex of $C^{*,*}(K \times G; \mathbb{T})$ where we disregard the first four rows.

The double complex $A^{*,*/A^{*,*,>3}}$ consists of the second, third and fourth rows of the double complex $C^{*,*}(K \times G; \mathbb{T})$, and we have that if $\mathbb{K}U_G(K; \alpha)$ can be made into a ring is because there exists $\beta$ and $\theta$ such that the cochain $\alpha \oplus \beta \oplus \theta$ becomes a 3-cocycle in the complex $\text{Tot}^*(A^{*,*/A^{*,*,>3}})$ and the 3-cocycle can be seen in a diagram as follows

\[
\begin{array}{cccc}
4 & & & \\
3 & \theta & 1 & \\
2 & \beta & \delta_G & 1 \\
1 & \alpha & \delta_G & 1
\end{array}
\]

\[
\begin{array}{c}
\delta_G \\
(\delta_K)^{-1}
\end{array}
\]

**Proposition 5.3.** If the cocycle $\alpha \in Z^2_G(K; \mathbb{T})$ can be lifted to a 3-cocycle $\alpha \oplus \beta \oplus \theta$ in the complex $\text{Tot}^*(A^{*,*/A^{*,*,>3}})$ then the the group $\mathbb{K}U_G(K; \alpha)$ can be endowed with the ring structure $\mathbb{K}U_G(K; \alpha, \beta)$.

5.4.2. Let us see another way to understand the conditions under which the twist $\alpha$ can define a multiplicative structure on $\mathbb{K}U_G(K; \alpha)$. Consider the filtration of the double complex $A^{*,*}$ given by the subcomplexes $F_r : = A^{*,*}\geq r$. The spectral sequence that the filtration defines abuts to the cohomology of the total complex of $A^{*,*}$

$$E_{\infty}^{*,*} \Rightarrow H^*(\text{Tot}(A^{*,*}))$$

and has for first page

$$E_1^{p,q} = H_G^p(K^q; \mathbb{T})$$
with differential
\[ d_1 : E^{p,q}_1 \to E^{p,q+1}_1, \quad d_1[x] := [(\delta_K)^{(-1)} x]. \]

If we have a twist \( \alpha \), its cohomology class \([\alpha]\) is an element in \( E^{2,1}_1 \). The element \( d_1[\alpha] \) is the first obstruction to lift \( \alpha \) to a 3-cocycle in \( \text{Tot}^* (A^* \cdot) \), that is, \( d_1[\alpha] = 1 \) if and only if there exists \( \beta \in C^{1,2}(K \times G; T) \) such that \( \delta_G \beta \cdot \delta_K \alpha = 1 \). Note furthermore that
\[
\alpha \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
Lemma 5.5. Consider \((\alpha, \beta)\) and \((\alpha', \beta')\) multiplicative structures on \(K\). If \(\alpha \oplus \beta\) and \(\alpha' \oplus \beta'\) are cohomologous as 3-cocycles in \(\text{Tot}^*(A^{*,*}/A^{*,*}>2)\), then the rings \(\mathbb{K}U_G(K; \alpha, \beta)\) and \(\mathbb{K}U_G(K; \alpha', \beta')\) are isomorphic.

Proof. If \(\alpha \oplus \beta\) and \(\alpha' \oplus \beta'\) are cohomologous as 3-cocycles in \(\text{Tot}^*(A^{*,*}/A^{*,*}>2)\), then there exists a 2-cochain \(\chi \oplus \kappa^{-1}\) with \(\chi \in A^{1,1}\) and \(\kappa \in A^{0,2}\) such that
\[
\delta_G \oplus \delta_K^{-1}(\chi \oplus \kappa^{-1}) = \alpha' / \alpha \oplus \beta' / \beta,
\]
namely that \(\delta_G \chi = \alpha' / \alpha\) and \((\delta_K \chi)^{-1} (\delta_G \kappa)^{-1} = \beta' / \beta\), or diagramatically
\[
\begin{array}{c|c|c|c}
3 & \delta_G & \beta' / \beta \downarrow \delta_K^{-1} & \chi \downarrow \delta_G \alpha' / \alpha \\
2 & \kappa^{-1} & \beta / \beta & 1 \mid \beta / \beta \\
1 & 0 & 1 & 2 \\
0 & & & \\
\end{array}
\]

The isomorphism \(\chi : \mathbb{K}U_G(K; \alpha, \beta) \xrightarrow{\sim} \mathbb{K}U_G(K; \alpha', \beta')\) induces an isomorphism of rings
\[
\tilde{\chi} : \mathbb{K}U_G(K; \alpha, \beta) \xrightarrow{\sim} \mathbb{K}U_G(K; \alpha', \beta'')
\]
where \(\beta'' := \beta(\delta_K \chi)^{-1}\). Since \((\delta_K \chi)^{-1} (\delta_G \kappa)^{-1} = \beta' / \beta\), we have that \(\beta'' = \beta' (\delta_G \kappa)\), therefore the isomorphism \(\tilde{\beta''}\) and \(\tilde{\beta'}\) are equal and we obtain that \(\mathbb{K}U_G(K; \alpha', \beta'') \cong \mathbb{K}U_G(K; \alpha', \beta')\). \(\square\)

The short exact sequence of complexes
\[
0 \rightarrow A^{*,*}>2/A^{*,*}>3 \rightarrow A^{*,*}/A^{*,*}>2 > A^{*,*}/A^{*,*}>2 \rightarrow 0
\]
induces a long exact sequence in cohomology groups
\[
\rightarrow H^3(\text{Tot}^*(A^{*,*}/A^{*,*}>3)) \rightarrow H^3(\text{Tot}^*(A^{*,*}/A^{*,*}>2)) \rightarrow H^4(\text{Tot}^*(A^{*,*}>2/A^{*,*}>3)) \rightarrow
\]
\[
[\alpha \oplus \beta \oplus \theta] \rightarrow [\alpha \oplus \beta] \rightarrow [(\delta_G \beta)^{-1}]
\]
and we see that Lemma 5.3 and Proposition 5.4 imply that the subgroup
\[
\text{MS}_G(K) := \{[\alpha \oplus \beta] \in H^3(\text{Tot}^*(A^{*,*}/A^{*,*}>2)) | [\delta_K \beta] = 1\}
\]
is precisely the group of equivalence classes of multiplicative structures associated to the twisted \(G\)-equivariant K-theory of the monoid \(K\). We define

Definition 5.6. The group
\[
\text{MS}_G(K) := \{[\alpha \oplus \beta] \in H^3(\text{Tot}(A^{*,*}/A^{*,*}>2)) | [\delta_G \beta] = 1\}
\]
will be called the group of multiplicative structures for the \(G\)-equivariant K-theory of the monoid \(K\).

And therefore we have that

Proposition 5.7. The elements of the group \(\text{MS}_G(K)\) are in one to one correspondence with the set of isomorphism classes of ring structures (in the sense of Lemma 5.3) on the twisted \(G\)-equivariant K-theory to the monoid \(K\).

In particular we have that there are at most \(\#(\text{MS}_G(K))\) of different multiplicative structures in the twisted \(G\)-equivariant K-theory groups of \(K\).
5.6. Automorphisms of the twisted equivariant K-theory ring. From section 5.1.1 we know that if \( \chi \in C^1_G(K; \mathbb{T}) \) satisfies \( \delta_K \chi = 1 \) then the map \( \chi \) induces an isomorphism of groups

\[
\chi : \mathbb{K}U_G(K; \alpha) \xrightarrow{\cong} \mathbb{K}U_G(K; \alpha).
\]

Whenever \((\alpha, \beta)\) is a multiplicative structure, it follows that the map \( \chi \) induces an isomorphism of rings whenever the homomorphism \( \delta_K \chi \) is the identity map, namely that \( [\delta_K \chi] = 1 \) as a cohomology class in \( H^1_G(K \times K; \mathbb{T}) \). If we define the group of multiplicative elements by

\[
H^1_G(K; \mathbb{T})_{\text{mult}} := \{ [\chi] \in H^1_G(K; \mathbb{T}) | \delta_K [\chi] = \pi^*_1 [\chi] \cdot \pi^*_2 [\chi] \cdot m^* [\chi]^{-1} = 1 \}
\]

we have then that the group of automorphisms of the twisted \( G \)-equivariant K-theory ring is equal to the multiplicative elements in \( H^1_G(K; \mathbb{T}) \), i.e.

\[
\text{Aut}(\mathbb{K}U_G(K; \alpha, \beta)) = H^1_G(K; \mathbb{T})_{\text{mult}}
\]

Using the spectral sequence of section 5.4.2 we see that the multiplicative terms appear in the second page of the spectral sequence

\[
H^1_G(K; \mathbb{T})_{\text{mult}} = E^{1,1}_2
\]

since \( d_1 [\chi] = [\delta_K \chi]^{-1} \) is the obstruction of being multiplicative. If furthermore a multiplicative element satisfies \( d_2 [\chi] = 1 \), then we have that \( [\chi] \) can be lifted to an element \( [\chi \oplus k] \) in \( H^2(\text{Tot}(A^* \cdot *)) \). This means that we have the exact sequence

\[
0 \to H^2(C^*(K; \mathbb{T})^G) \to H^2(\text{Tot}(A^* \cdot *)) \to H^1_G(K; \mathbb{T})_{\text{mult}} \xrightarrow{d_2^1} H^3(C^*(K; \mathbb{T})^G)
\]

where the \( G \)-invariant cochains come from the first page of the spectral sequence, i.e. \( E^{0,1}_1 = C^*(K; \mathbb{T})^G \), and its cohomology appears in the second page, i.e. \( E^{0,2}_2 = H^0(C^*(K; \mathbb{T})^G, \delta_K) \).

5.7. Relation between the Grothendieck ring associated to a monoidal category and the twisted equivariant K-theory ring. Let us consider a \( G \)-equivariant monoid with unit \( K \). We have seen that to a pseudomoid with strict unit in the 2-category of \( G \)-sets with twist \( K = (K, m, 1, \alpha, \beta, \theta) \) over \( K \) we can associate the ring \( \text{Groth}(\text{Bun}_G(K)) \) of isomorphism classes of objects in the monoidal category \( \text{Bun}_G(K) \). At the same time, since \( \alpha \oplus \beta \oplus \theta \in Z^3(\text{Tot}^*(A^* \cdot *(K \times G, \mathbb{T}))) \), then \((\alpha, \beta, \theta)\) is a multiplicative structure and we get that

\[
\text{Groth}(\text{Bun}_G(K)) \cong \mathbb{K}U_G(K; \alpha, \beta)
\]
as rings. We have therefore a canonical map

\[
(5.2) \quad H^3(\text{Tot}(A^* \cdot *)) \to \text{MS}_G(K) \quad [\alpha \oplus \beta \oplus \theta] \mapsto [\alpha \oplus \beta]
\]

from the isomorphism classes of pseudomonoid structures with strict unit of \( G \)-sets with twist over \( K \), to the group of multiplicative structures of the \( G \)-equivariant twisted K-theory groups of \( K \). Let us understand this map in more detail.
Consider the projection homomorphism between the exact sequences of complexes

\[
\begin{array}{ccccccc}
0 & \rightarrow & A^{*,>2} & \rightarrow & \pi & \rightarrow & A^{*,>2} / A^{*,>2} \\
\rightarrow & \rightarrow & \phi & \rightarrow & 0
\end{array}
\]

and the homomorphism between the long exact sequences induced

\[
\begin{array}{ccccccc}
H^3(\text{Tot}(A^{*,*})) & \rightarrow & H^3(\text{Tot}(A^{*,>2})) & \rightarrow & H^4(\text{Tot}(A^{*,>2})) & \rightarrow & 0 \\
\phi & \rightarrow & \nu & \rightarrow & p
\end{array}
\]

\[
\begin{array}{ccccccc}
H^3(\text{Tot}(A^{*,>2})) & \rightarrow & H^3(\text{Tot}(A^{*,>3})) & \rightarrow & H^4(\text{Tot}(A^{*,>3})) & \rightarrow & 0 \\
\delta & \rightarrow & \varphi & \rightarrow & p
\end{array}
\]

where \(\nu\) and \(\varphi\) are the connection homomorphisms with \(\nu[\alpha \oplus \beta] = \delta_2[\beta]\).

By definition

\[
\text{MS}_G(K) = \ker(\varphi)
\]

and furthermore note that the map \(\pi\) is injective since \(H^3(\text{Tot}(A^{*,>3})) = 0\). Therefore the canonical map defined in \[(5.20)\] is precisely the map

\[
H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} \text{MS}_G(K).
\]

In complete generality it is difficult to give an explicit description of how the kernel and the cokernel of the map \(H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} \text{MS}_G(K)\) looks like, but for calculations we can give the following description:

**Theorem 5.8.** The kernel of the homomorphism \(H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} \text{MS}_G(K)\) is isomorphic to

\[
\ker \left( H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} \text{MS}_G(K) \right) \cong \iota \left( H^3(C^*(K, \mathbb{T})^G) \right)
\]

and the cokernel is isomorphic to

\[
\text{coker} \left( H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} \text{MS}_G(K) \right) \cong \ker \left( H^4(C^*(K, \mathbb{T})^G) \rightarrow H^4(\text{Tot}(A^{*,*})) \right).
\]

**Proof.** Let us start with the kernel of the map \(\phi\). From the long exact sequences of cohomologies defined above we get that

\[
\ker \left( H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} \text{MS}_G(K) \right) \cong \iota \left( H^3(\text{Tot}(A^{*,>2})) \right)
\]

where the group \(H^3(\text{Tot}(A^{*,>2}))\) consists of elements in \(C^{0,3}\) that are closed under the differentials \(\delta_G\) and \(\delta_K\) and therefore

\[
H^3(\text{Tot}(A^{*,>2})) = Z^3(K, \mathbb{T})^G.
\]

Now, since the elements \(\delta_K(C^2(K, \mathbb{T})^G)\) are all zero in \(H^3(\text{Tot}(A^{*,*}))\) we have that

\[
\iota \left( Z^3(K, \mathbb{T})^G \right) = \iota \left( H^3(C^*(K, \mathbb{T})^G) \right)
\]

and therefore

\[
\ker \left( H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} \text{MS}_G(K) \right) \cong \iota \left( H^3(C^*(K, \mathbb{T})^G) \right).
\]
For the cokernel we have that the long exact sequence in cohomologies defined above implies that
\[ \text{coker } \left( H^3(\text{Tot}(A^*)) \overset{\phi}{\to} MS_G(K) \right) \cong H^4(\text{Tot}(A^{*,>2})) \cap \ker(p) \cap \ker(\iota). \]

Now, the projection map \( p : H^4(\text{Tot}(A^{*,>2})) \to H^4(\text{Tot}(A^{*,>2}/A^{*,>3})) = H^4_G(K^3, T) \)
has for kernel the elements in the fourth cohomology of the \( G \)-invariant \( K \)-chains
\[ \ker(p) = H^4(C^*(K, T)^G), \]
and this follows from the spectral sequence that converges \( H^*(\text{Tot}(A^{*,>2})) \) associated to the filtration \( \text{Tot}(A^{*,>q,*}) \). Since the natural homomorphism
\[ H^4(C^*(K, T)^G) \to H^4(\text{Tot}(A^{*,*})) \]
coincides with the map \( \iota \), we have the desired isomorphism
\[ \text{coker } \left( H^3(\text{Tot}(A^*)) \overset{\phi}{\to} MS_G(K) \right) \cong \ker \left( H^4(C^*(K, T)^G) \to H^4(\text{Tot}(A^{*,*})) \right). \]

\[ \square \]

For calculation purposes let us understand the kernel and the cokernel of the map \( \phi \) from the point of view of the spectral sequence of section 5.4.2.

**Proposition 5.9.** Consider the filtration of the complex \( \text{Tot}(A^{*,*}) \) defined by the subcomplexes \( \text{Tot}(A^{*,>q,*}) \) and consider the spectral sequence that it defines which converges to \( H^*(\text{Tot}(A^{*,*})) \). Then we have the isomorphisms
\[ \ker \left( H^3(\text{Tot}(A^{*,*})) \overset{\phi}{\to} MS_G(K) \right) \cong E^{0,3}_1, \]
\[ \text{coker } \left( H^3(\text{Tot}(A^{*,*})) \overset{\phi}{\to} MS_G(K) \right) \cong d_2(E^{2,1}_2) + d_2(E^{1,2}_2). \]

In the particular case on which the spectral sequence collapses at the second page we conclude that
\[ 0 \to H^3(C^*(K, T)^G) \to H^3(\text{Tot}(A^{*,*})) \overset{\phi}{\to} MS_G(K) \to 0. \]

**Proof.** The first page of the spectral sequence associated to the filtration \( \text{Tot}(A^{*,>q,*}) \) is given by \( E^{0,q}_1 = C^q(K, T)^G \) and \( E^{p,q}_1 = H^p_G(K^q, T) \) whenever \( q > 0 \), and therefore one has that the second page is given by
\[ E^{0,q}_2 = H^q(C^*(K, T)^G) \]
and for \( p > 0 \) and \( q > 0 \)
\[ E^{p,q}_2 = \frac{\ker \left( H^p_G(K^q, T) \overset{\delta}{\to} H^p_G(K^{q+1}, T) \right)}{\text{im} \left( H^p_G(K^{q-1}, T) \overset{\delta}{\to} H^p_G(K^q, T) \right)}. \]

The map
\[ H^4(C^*(K, T)^G) \to H^4(\text{Tot}(A^{*,*})) \]
coincides with the standard map \( E^{2,4}_2 \to H^4(\text{Tot}(A^{*,*})) \) and its kernel consists of the images of \( E^{3,2}_2 \) and \( E^{2,1}_3 \) under the differentials \( d_2 \) and \( d_3 \) respectively, since we know that the map
\[ E^{0,4}_4 \to H^4(\text{Tot}(A^{*,*})). \]
is injective. Therefore we have that
\[ d_2(E_{2}^{1,2}) \subset \ker \left( H^4(C^*(K, T)^G) \to H^4(Tot(A^{*}*)) \right) \]
and the above inclusion is an equality whenever \( d_3(E_{3}^{2,1}) = 0 \). In the case that
\( d_3(E_{3}^{2,1}) \neq 0 \) we could abuse of the notation and say that
\[ d_3(E_{3}^{2,1}) + d_2(E_{2}^{1,2}) = \ker \left( H^4(C^*(K, T)^G) \to H^4(Tot(A^{*}*)) \right) . \]

A similar argument could be used to calculate \( \iota \left( H^3(C^*(K, T)^G) \right) \). Since \( E_2^{0,3} = H^3(C^*(K, T)^G) \) we have that its image \( \iota(E_2^{0,3}) \subset H^3(Tot(A^{*}*)) \) is equal to the image of the canonical map
\[ E_2^{0,3} \to H^3(Tot(A^{*}*)). \]
Since the image is isomorphic to the group to which it converges, in this case \( E_4^{0,4} \), then we can conclude that \( \iota \left( H^3(C^*(K, T)^G) \right) \cong E_4^{0,3} \) and therefore
\[ \ker \left( H^3(Tot(A^{*}*)) \to MS_G(K) \right) \cong E_4^{0,3}. \]

Finally, whenever the spectral sequence collapses at the second page we have that \( d_2 = 0 = d_3 \) and therefore \( \phi \) is surjective. Since we have in this case we have that \( E_4^{0,3} = E_2^{0,3} = H^3(C^*(K, T)^G) \), the proposition follows. \( \square \)

From Theorem 5.3.8 we can deduce two things.
- If \( K = (K, m, 1, \alpha, \beta, \theta) \) is a pseudomonoid with strict unit in the 2-category of \( G \)-sets with twists such that \( [\alpha \oplus \beta \oplus \theta] \) lies in the image of \( \iota \), then the Grothendieck ring \( \text{Groth}(Bun_G(K)) \) is isomorphic to the untwisted ring \( \text{KU}_G(K) \).
- Multiplicative structures \((\alpha', \beta') \) in \( MS_G(K) \) such that \( \alpha' \oplus \beta' \oplus \theta' \) belongs to \( Z^3(Tot(A^{*}*/A^{*}+3)) \) and \( \delta_K \theta' \neq 0 \), define ring structures \( \text{KU}_G(K; \alpha', \beta') \) which cannot be obtained as the Grothendieck ring \( \text{Groth}(Bun_G(K)) \) for any pseudomonoid \( K \) with strict unit in the 2-category of \( G \)-sets with twists.

5.7.1. Relation between the automorphisms. We have seen that the isomorphism classes of automorphisms of \( K \) that leave the monoid \( K \) fixed is isomorphic to the group \( H^2(Tot(A^{*}*(K \rtimes G, T))) \). Since the automorphism group of \( K_\text{add}(K; \alpha, \beta) \) is \( H^2_G(K; T)_{\text{mult}} \), we have that there is an induced map
\[ H^2(Tot(A^{*}*)) \to H^2_G(K; T)_{\text{mult}} \]
which matches the homomorphism of that appears in the exact sequence 5.41
\[ 0 \to H^2(C^*(K; T)^G) \to H^2(Tot(A^{*}*)) \to H^2_G(K; T)_{\text{mult}} \to d_2 H^3(C^*(K; T)^G). \]

Note that in the case that the spectral sequence collapses at the second page we get the short exact sequence
\[ 0 \to H^2(C^*(K; T)^G) \to H^2(Tot(A^{*}*)) \to H^1_G(K; T)_{\text{mult}} \to 0. \]

A more elaborate analysis of the homomorphisms
\[ H^3(Tot(A^{*}*)) \to MS_G(K) \text{ and } H^2(Tot(A^{*}*)) \to H^2_G(K; T)_{\text{mult}} \]
will depend on the choice of the group \( G \) and of the \( G \)-equivariant monoid \( K \). In the next chapter we will calculate explicitly the previous homomorphisms for several
examples, and from them we will deduce interesting information with regard to the twisted equivariant K-theory rings.

6. Examples

The main objective of this section is to use Proposition 5.9 to calculate the kernel and cokernel of the homomorphism

$$H^3(\text{Tot}(A^*, *)) \xrightarrow{\phi} MS_G(K)$$

for different choices of $G$ and $K$, in order to show the different twisted $G$-equivariant K-theory rings over $K$ that can appear.

6.1. Trivial action of $G$ on $K$. In this case we have that the spectral sequence defined in Proposition 5.9 collapses at the second page and moreover we have that $C^*(K, T) = C^*(K, T)^G$. Therefore we obtain the short exact sequence

$$0 \to H^3(K, T) \to H^3(\text{Tot}(A^*, *)) \xrightarrow{\phi} MS_G(K) \to 0,$$

thus implying that

$$MS_G(K) \cong H^3(\text{Tot}^*(B^{*,*}(K \times G, T))),$$

and moreover that all multiplicative structures for the $G$-equivariant K-theory of $K$ can be obtained from the ring structures defined by the Grothendieck rings of the monoidal categories $Bun_G(K)$. We also obtain the short exact sequence

$$0 \to H^2(K, T) \to H^2(\text{Tot}(A^*, *)) \to \text{Hom}(K, \text{Hom}(G, T)) \to 0$$

where in this case $H^3_\text{mult}(K, T)^G = \text{Hom}(K, \text{Hom}(G, T))$.

Furthermore, if $[\theta] \in H^3(K, T)$ is non trivial then we can define a non-trivial pseudomoinoid with strict unit in the 2-category of $G$-sets with twist $K = (K, m, 1, 0, 0, \theta)$ with $[0 \oplus 0 \oplus \theta]$ non-zero in $H^3(\text{Tot}(A^*, *))$, such that

$$\text{Groth}(Bun_G(K)) \cong R(G) \otimes_{\mathbb{Z}} \mathbb{Z}[K]$$

where $R(G)$ is the Grothendieck ring of finite dimensional complex representations of $G$ and $\mathbb{Z}[K]$ is the group ring of $K$, since we know that $R(G) \otimes_{\mathbb{Z}} \mathbb{Z}[K]$ is isomorphic to the non-twisted ring structure on $\mathbb{KU}_G(K)$.

6.1.1. $G = \mathbb{Z}/n$ and $K = \mathbb{Z}/m$. In this case we have that $H^3(K, T) = \mathbb{Z}/m$ and

$$H^3(\text{Tot}(A^*, *)) = \mathbb{Z}/m \oplus \mathbb{Z}/(n, m)$$

where $(n, m)$ is the maximum common denominator of the pair $n, m$. Therefore $MS_G(K) = \mathbb{Z}/(n, m)$ and we have that all non trivial multiplicative structures come from the group

$$\mathbb{Z}/(n, m) \cong H^3(\text{Tot}(B^{*,*})) \subset H^3(\text{Tot}(A^*, *))$$.
6.2. Adjoint action of $G$ on itself. From Lemma [2.3] we know that in this case we have the split short exact sequence

$$0 \longrightarrow H^\ast(Tot(B^{\ast\ast})) \longrightarrow H^\ast(Tot(A^{\ast\ast})) \longrightarrow H^\ast(G, T) \longrightarrow 0$$

induced by the short exact sequence of complexes

$$0 \rightarrow \text{Tot}(B^{\ast\ast}) \subset \text{Tot}(A^{\ast\ast}) \rightarrow C^{\ast>0}(G, T) \rightarrow 0.$$

Since we have the inclusion $H^\ast(\text{Tot}(B^{\ast\ast})) \subset H^\ast(\text{Tot}(A^{\ast\ast}))$, we can deduce that the groups $E^\ast_{2,0} = H^3(G, T) \hspace{1mm} G$ of the 0-th column of the second page of the spectral sequence converging to $H^\ast(\text{Tot}(A^{\ast\ast}))$ defined in Proposition [5.9] are unaffected by the differentials $d_i$ for $i > 1$; this follows from the injectivity between the spectral sequences associated to the filtrations $B^{\ast\ast\ast}$ and $A^{\ast\ast\ast}$ of $\text{Tot}(B^{\ast\ast})$ and $\text{Tot}(A^{\ast\ast})$ respectively.

Therefore we have that $E^\ast_{2,0} = E^\ast_{2,0} = H^3(G, T) \hspace{1mm} G$ and $d_2(E^1_{2,0}) = 0 = d_3(E^2_{2,1})$, and by Proposition [5.9] we get the short exact sequence

$$(6.1) \quad 0 \rightarrow H^3(C^\ast(G, T) \hspace{1mm} G) \rightarrow H^3(\text{Tot}(A^{\ast\ast})) \overset{\phi}{\rightarrow} MS\hspace{1mm}C(G) \rightarrow 0.$$

Moreover, the cokernel of the inclusion

$$E^\ast_{\infty,q} \rightarrow H^q(\text{Tot}(A^{\ast\ast}))$$

should match the cohomology group $H^q(\text{Tot}(B^{\ast\ast}))$ since this piece is built from the groups $E^\ast_{r,s}$ with $r, s > 1$ and $r + s = p$, therefore we have the canonical isomorphism

$$H^q(C^\ast(G, T) \hspace{1mm} G) \oplus H^q(\text{Tot}(B^{\ast\ast})) \overset{\sim}{\rightarrow} H^q(\text{Tot}(A^{\ast\ast})), \ x \oplus y \mapsto x + y$$

which in particular implies that

$$H^q(\text{Tot}(A^{\ast\ast})) \overset{\sim}{\rightarrow} H^q(G, T).$$

Then we can conclude that the composition of the maps

$$H^3(\text{Tot}(B^{\ast\ast})) \subset H^3(\text{Tot}(A^{\ast\ast})) \overset{\phi}{\rightarrow} MS\hspace{1mm}C(G)$$

is an isomorphism, and therefore we obtain the canonical isomorphism

$$(6.2) \quad H^3(\text{Tot}(B^{\ast\ast})) \overset{\sim}{\rightarrow} MS\hspace{1mm}C(G), \quad [\alpha \oplus \beta] \mapsto [\alpha \oplus \beta].$$

Also we obtain the short exact sequence

$$0 \rightarrow H^2(C^\ast(G, T) \hspace{1mm} G) \rightarrow H^2(\text{Tot}(A^{\ast\ast})) \rightarrow H^2_G(G, T)_{\text{mult}} \rightarrow 0,$$

with $H^2_G(G, T)_{\text{mult}} \cong H^2(\text{Tot}(B^{\ast\ast})) \cong \text{Hom}(\text{Hom}(G, T)).$

Now we will study the multiplicative structures that define the pseudomonoids constructed via the formalism introduced in [3], and whose properties were outlined in section [5.3]. For this purpose we need to calculate the composition of the maps

$$H^3(G, T) \overset{\tau_G}{\rightarrow} H^3(\text{Tot}(A^{\ast\ast})) \overset{\phi}{\rightarrow} MS\hspace{1mm}C(G)$$

where $\tau_G$ is the induced map in cohomology which was defined at the chain level in (2.2). This calculation will be carried out using the ring structure of the ring $H^\ast(G, Z)$ together with the pullback map $\mu^* : H^\ast(G, Z) \rightarrow H^\ast(G \rtimes G, Z)$ induced by the multiplication $\mu : G \rtimes G \rightarrow G$ as we have in Theorem [2.3]. Since we have the isomorphism

$$H^3(\text{Tot}(A^{\ast\ast})) \oplus H^3(G, T) \overset{\cong}{\rightarrow} H^3(G \rtimes G, T), \quad x \oplus y \mapsto x + \pi_2^* y$$
Doubles is equivalent to the map Grothendieck rings of representations Groth(Rep(D)
Since in this case H and therefore we obtain that the map 
Since
\[ \phi \circ \tau^*_1 : H^3(G, \mathbb{T}) \rightarrow MS_G(G) \]
is equivalent to the composition of homomorphisms
\[ H^3(G, \mathbb{T}) \xrightarrow{\mu} H^3(G \rtimes G, \mathbb{T})/(H^3(C^*(K; \mathbb{T})^G) \oplus H^3(G, \mathbb{T})) \xrightarrow{\tau} MS_G(G). \]
Therefore we simply need to find the restriction to H^3(Tot(B^{*+}))* of the image of \( \mu^* \) in H^3(G \rtimes G, \mathbb{T}),
\[ H^3(G, \mathbb{T}) \xrightarrow{\mu^*} H^3(G \rtimes G, \mathbb{T}) \xrightarrow{\tau^*_1} H^3(Tot(B^{*+}))* \cong MS_G(G). \]
Let us see some examples.

6.2.1. Cyclic groups. Consider G = K = \( \mathbb{Z}/n \) and note that as rings \( H^*/(G; \mathbb{Z}) = \mathbb{Z}[x]/(nx) \) where \( |x| = 2 \). By the Kunneth theorem we have that \( H^2(G \times G; \mathbb{Z}) \cong H^2(G; \mathbb{Z}) \otimes H^0(G; \mathbb{Z}) \oplus H^0(G; \mathbb{Z}) \otimes H^2(G; \mathbb{Z}) \)
and moreover we have that
\[ \mu^* x = x \otimes 1 + 1 \otimes x. \]
Since \( x^2 \) is the generator of H^4(G; \mathbb{Z}) we obtain
\[ \mu^* x^2 = x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2 \]
where
\[ H^4(Tot(B^{*+})*(G \times G, \mathbb{Z})) = \langle x \otimes x \rangle \cong \mathbb{Z}/n \]
and therefore we obtain that the map
\[ \phi \circ \tau^*_1 : H^3(G, \mathbb{T}) \rightarrow MS_G(G) \]
is equivalent to the map
\[ H^4(G; \mathbb{Z}) \rightarrow H^4(Tot(B^{*+})*(G \times G, \mathbb{Z})), \ x^2 \rightarrow 2x \otimes x. \]
Since in this case H^3(G; \mathbb{T}) \cong MS_G(G) \cong \mathbb{Z}/n we have that \( \phi \circ \tau^*_1 : \mathbb{Z}/n \rightarrow \mathbb{Z}/n. \)
Therefore, when n is odd, the map \( \phi \circ \tau^*_1 \) is an isomorphism and therefore the Grothendieck rings of representations Groth(Rep(D^{w}(G))) of the Twisted Drinfeld Doubles D^{w}(G) for w \in H^3(G, \mathbb{T}) are all non-isomorphic.

Meanwhile when n is even, the cocycle \( w \in Z^3(G, \mathbb{T}) \) whose cohomology class is \( \frac{\beta}{2} \in \mathbb{Z}/n \) has for Grothendieck ring of representations Groth(Rep(D^{w}(G))) a ring isomorphic to Groth(Rep(D(G))), which is by definition the ring \( KU_G(G) = R(G) \otimes ZG \). Moreover, the multiplicative structures defined by odd numbers in \( \mathbb{Z}/n \cong MS_G(G) \) define Grothendieck rings of representations associated to the respective pseudomonoids which cannot be recovered via the Grothendieck ring of representations associated to the Twisted Drinfeld Double construction.

Note furthermore that in this case the automorphisms groups are isomorphic
\[ H^2(Tot(A^{*+})*(G \times G, \mathbb{T}))) \cong H^1_G(G, \mathbb{T})_{\text{mult}} \cong \mathbb{Z}/n. \]
6.2.2. Quaternionic group. Consider $G = K = Q_8$ the quaternionic group and recall that $Q_8 \subset SU(2)$ and that it sits in the short exact sequence

$$0 \to \mathbb{Z}/2 \to Q_8 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0.$$  

From the Serre vibration

$$SU(2)/Q_8 \to ESU(2)/Q_8 \to BSU(2)$$

one can deduce that the integral cohomology ring of $Q_8$ is

$$H^*(Q_8, \mathbb{Z}) = \mathbb{Z}[x, y, e]/(x^2, y^2, xy, 2x, 2y, xe, ye, 8e)$$

with $|x| = |y| = 2$ and $|e| = 4$.

Since the cohomology of $Q_8$ is all of even degree, by the Kunneth isomorphism we have that

$$H^4(Q_8 \times Q_8, \mathbb{Z}) \cong \bigoplus_{j=0}^4 H^j(Q_8, \mathbb{Z}) \otimes H^{4-j}(Q_8, \mathbb{Z})$$

and since $Q_8 \times Q_8 \cong Q_8 \times Q_8$ we have that

$$H^*(Q_8 \times Q_8, \mathbb{Z}) \cong H^*(Q_8 \times Q_8, \mathbb{Z}).$$

In this case we have that

$$MS_{Q_8}(Q_8) \cong H^4(Tot(B^*\times^*(Q_8 \times Q_8, \mathbb{Z}))) = \langle x \otimes x, x \otimes y, y \otimes x, y \otimes y \rangle \cong (\mathbb{Z}/2)^{\oplus 4}$$

and since

$$\mu^* e = e \otimes 1 + 1 \otimes e$$

we can deduce that the map

$$\phi \circ \tau_1^* : H^3(Q_8, \mathbb{T}) \cong MS_{Q_8}(Q_8)$$

is the trivial map. A nice consequence of the triviality of the map $\phi \circ \tau_1^*$ is that for all $w \in Z^3(Q_8, \mathbb{T})$, the Grothendieck ring of representations of the $w$-Twisted Drinfeld Double is isomorphic as rings to $\mathbb{K}U_{Q_8}(Q_8)$, which is the Grothendieck ring of representations of the untwisted Drinfeld double $D(Q_8)$.

Note also that the automorphisms groups are isomorphic

$$H^2(Tot(A^*\times^*(Q_8 \times Q_8, \mathbb{T}))) \cong H^1_{Q_8}(Q_8, \mathbb{T})_{\text{mult}} \cong (\mathbb{Z}/2)^{\oplus 4}.$$  

6.2.3. $G$ simple and non-abelian. In section 3.5.1 we showed that $H^3(Tot(B^*\times^* (G \rtimes G, \mathbb{T}))) = 0$ and therefore we have that $MS_G(G) = 0$. Hence all Grothendieck rings of representations for all pseudomonoids are isomorphic to the ring $\mathbb{K}U_G(G)$.

6.2.4. $G$ binary icosahedral. The same result applies to the binary icosahedral group since we showed in section 3.5.2 that $H^3(Tot(B^*\times^*(G \rtimes G, \mathbb{T}))) = 0$ and hence $MS_G(G) = 0$.

6.3. $\mathbb{Z}/n$ acted by $\mathbb{Z}/2$. Consider the action of $G = \mathbb{Z}/2$ on $K = \mathbb{Z}/n$ given by multiplication of $-1$. The group $\mathbb{Z}/n \rtimes \mathbb{Z}/2$ is isomorphic to the dihedral group $D_n$ of rigid symmetries of the regular polygon of $n$ sides.
6.3.1. \( n \) odd. Let us suppose that \( n \) is odd and recall that in this case \( H^1(D_n, \mathbb{T}) = \mathbb{Z}/2, H^2(D_n, \mathbb{T}) = 0 \) and \( H^3(D_n, \mathbb{T}) = \mathbb{Z}/2 \oplus \mathbb{Z}/n. \) Since \( H^3(\mathbb{Z}/2, \mathbb{T}) = \mathbb{Z}/2 \) we can conclude that \( H^3(\text{Tot}(A^{*\ast})) = \mathbb{Z}/n. \) Now, applying the spectral sequence defined in Proposition 5.9 we have that \( E_1^{1,q} = H^n_1(K^q, \mathbb{T}) = \mathbb{Z}/2, \) since the only fixed point of the \( \mathbb{Z}/2 \) action is the \( p \)-tuple of zeroes, and therefore one obtains that \( E_2^{1,q} = 0 \) for \( q > 0 \) since a simple calculation shows that the maps \( d_1 : E_1^{1,2i-1} \to E_1^{1,2i} \) are all isomorphisms. Moreover, since \( E_2^{2,q} = H^n_2(K^q, \mathbb{T}) = 0 \) because \( H^2(\mathbb{Z}/2, \mathbb{T}) = 0 \) we have that \( E_3^{2,1} = 0 \) and we can conclude that

\[
\text{coker} \left( H^3(\text{Tot}(A^{*\ast})) \xrightarrow{\phi} MS_G(K) \right) = 0;
\]

hence we have that the map \( H^3(\text{Tot}(A^{*\ast})) \xrightarrow{\phi} MS_G(K) \) is surjective.

It remains now to calculate \( \ker \left( H^3(\text{Tot}(A^{*\ast})) \xrightarrow{\phi} MS_G(K) \right) \). Applying the same argument as before we have that \( E_2^{0,3} = H^3(C^*(K, \mathbb{T})^G) \) and we already know that \( E_2^{1,2} = 0 = E_2^{1,1}; \) therefore \( \iota(H^3(C^*(K, \mathbb{T})^G)) \) coincides with the image of the canonical map \( E_2^{0,3} \to H^3(\text{Tot}(A^{*\ast})) \) and this map must be surjective. Therefore we have that

\[
\ker \left( H^3(\text{Tot}(A^{*\ast})) \xrightarrow{\phi} MS_G(K) \right) = \mathbb{Z}/n
\]

and we can conclude that

\[
MS_G(K) = 0;
\]

i.e. all multiplicative structures on the \( \mathbb{Z}/2 \)-equivariant K-theory of \( \mathbb{Z}/n \) are trivial and all the Grothendieck rings Groth(\text{Bun}_{\mathbb{Z}/2}(K)) are isomorphic to the ring \( \mathbb{K}_{\mathbb{Z}/2}(\mathbb{Z}/n) \) for any \( K = (\mathbb{Z}/n, m, 1, \alpha, \beta, \theta). \) By Remark 4.3 \( \mathbb{KU}_{\mathbb{Z}/2}(\mathbb{Z}/n) \) is just the ring of isomorphism classes of representations of the dihedral group \( D_n. \)

In this case the automorphisms groups are both trivial

\[
H^2(\text{Tot}(A^{*\ast}(\mathbb{Z}/n \times \mathbb{Z}/2, \mathbb{T}))) = 0 = H^1_{\text{mult}}(\mathbb{Z}/n, \mathbb{T})
\]

6.3.2. \( n \) even. Let us now suppose that \( n \) is even; in this case \( H^1(D_n, \mathbb{T}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2, H^2(D_n, \mathbb{T}) = \mathbb{Z}/2 \) and \( H^3(D_n, \mathbb{T}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/n. \) Since \( H^3(\mathbb{Z}/2, \mathbb{T}) = \mathbb{Z}/2 \) we have that \( H^3(\text{Tot}(A^{*\ast})) = \mathbb{Z}/2 \oplus \mathbb{Z}/n. \) Now, we also have that

\[
E_1^{1,q} = H^n_1(K^q, \mathbb{T}) \cong \text{Maps}(\mathbb{Z}/2^q, \mathbb{Z}/2)
\]

since the fixed points of the \( \mathbb{Z}/2 \) action on \( K^q \) consists of \( q \)-tuples of points with either 0 or \( \frac{n}{2} \) for entries. It is a simple calculation to show that the differential \( d_1 : E_1^{1,i} \to E_1^{1,i+1} \) is precisely the differential of the cohomology of the group \( \mathbb{Z}/2 \) with coefficients in \( \mathbb{Z}/2 \) and therefore we get that

\[
E_2^{1,q} \cong H^q(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2.
\]

The groups \( E_2^{1,q} \) are trivial because \( H^2(\mathbb{Z}/2, \mathbb{T}) = 0. \)

Let us now calculate the group \( E_2^{0,1} = H^1(C^*(K, \mathbb{T})^G). \) This group consists of the maps \( f : \mathbb{Z}/n \to \mathbb{T} \) such that \( f \) is invariant under the \( G \)-action, namely \( f(x) = f(-x), \) and that \( \delta_1 f = 0, \) which means that \( f \) is a homomorphism. The only \( G \)-invariant homomorphisms are the ones that take values in the subgroup \( \mathbb{Z}/2 \subset \mathbb{T} \) and therefore we have that

\[
E_2^{0,1} = H^1(C^*(K, \mathbb{T})^G) = \mathbb{Z}/2.
\]
The information we have obtained so far on the cohomology groups of the second page of the spectral sequence is the following

| 3 | ? |
|--|---|
| 2 | ? \(\mathbb{Z}/2\) |
| 1 | \(\mathbb{Z}/2\) \(\mathbb{Z}/2\) 0 |
| 0 | \(\mathbb{T}\) \(\mathbb{Z}/2\) 0 \(\mathbb{Z}/2\) |
| 0 | 1 | 2 | 3 |

and since we know that \(H^1(D_n, \mathbb{T}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2\) and \(H^2(D_n, \mathbb{T}) = \mathbb{Z}/2\) we can deduce that \(E_\infty^{0,2} = E_2^{0,2} = 0\) and therefore \(E_2^{0,3} = E_4^{0,3} = \mathbb{Z}/n\). Hence we have that

\[
H^3(C^*(G, \mathbb{T})^G) = \mathbb{Z}/n, \quad MS_G(K) = \mathbb{Z}/2 \quad \text{and}
\]

\[
H^3(\text{Tot}(A^{*,*})) \xrightarrow{\phi} MS_G(K), \quad \mathbb{Z}/n \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2
\]

is the canonical projection map.

In this case the automorphisms groups are isomorphic

\[
H^2(\text{Tot}(A^{*,*}(\mathbb{Z}/n \times \mathbb{Z}/2, \mathbb{T})) \cong H^1_{\mathbb{Z}/2}(\mathbb{Z}/n, \mathbb{T})_{\text{mult}} \cong \mathbb{Z}/2.
\]

7. Appendix: Relation with (coquasi) bialgebras

The classical Tannakian duality permits to reconstruct a compact group from its finite dimensional representations and a generalization of this statement was done for quantum groups (Hopf algebras); see [15] for a survey on classical and quantum Tannakian duality. The Tannakian principle works in the following way. If \(C\) is a \(\mathbb{C}\)-linear abelian tensor category with a faithful, exact and strong tensor functor \(C \rightarrow \text{Vec}_C\) (called fiber functors), then there exists a unique bialgebra \(H\) that solves certain universal problem and such that the category of corepresentations of \(H\) is canonically equivalent to \(C\) as a tensor category, see [15, Theorem 3 and Proposition 4].

The tensor category \(\text{Bun}_G(K)\) is endowed with a natural forgetful functor \(U : \text{Bun}_G(K) \rightarrow \text{Vec}_C\), that is exact and faithful. However, unless \(\theta \in Z^3(K, \mathbb{T})\) has trivial cohomology, the functor \(U\) is not monoidal. The functor \(U\) is only quasi-monoidal (a quasi-fiber functor) and a reconstruction of a Hopf algebra in general is not possible. A generalized version of Tannakian duality using quasi-fiber functors was developed by Shahn Majid in [18], using a generalization of the notion of Hopf algebra, called coquasi-Hopf algebra, previously defined (in a dual version) by Drinfeld [10].

In what follows we will show how \(\text{Bun}_G(K)\) can be understood as the tensor category of corepresentations associated to an explicit coquasi-bialgebra, and for this purpose we will show that the input necessary for defining such coquasi-Hopf algebra is equivalent to the information encoded in a pseudomonoid with strict unit in the 2-category of \(G\)-sets with twists.

7.1. Coquasi-bialgebras.
7.1.1. Coalgebras and comodules. Let $k$ be a field. A coalgebra over $k$ is a vector space over $k$ together with two linear maps $\Delta : C \to C \otimes C$, $\varepsilon : C \to k$ (called comultiplication and counit respectively) such that $(C \otimes \Delta)\Delta = (\Delta \otimes C)\Delta$ and $(C \otimes \varepsilon)\Delta = (\varepsilon \otimes C)\Delta = C$. We shall use the Sweedler’s notation omitting the sum symbol, that is $\Delta(c) = c_1 \otimes c_2$ if $c \in C$.

If $C$ is a coalgebra, a right $C$-comodule is a vector $k$-space $M$ with a linear map $\rho : M \to M \otimes C$ such that $(\rho \otimes C)\rho = (M \otimes \Delta)\rho$ and $(M \otimes \varepsilon) = M$. Again for the comodule structure we shall use sweedler’s notation omitting the sum symbol, i.e., $\rho(m) = m_0 \otimes m_1$, $m_0 \in M, m_1 \in C$. If $M, N$ are $C$-comodules, a comodule map is a linear map $f : M \to N$ such that $\rho_N f = (f \otimes C)\rho_M$. We shall denote by $\mathcal{M}^C$ the category of right $C$-comodules.

If $C, C'$ are coalgebras $C \otimes C'$ is a coalgebra with comultiplication $\Delta(c \otimes c') = (c_1 \otimes c'_1) \otimes (c_2 \otimes c'_2)$ and counit $\varepsilon(c \otimes c') = \varepsilon(c)\varepsilon(c')$.

For a coalgebra $C$ the space $C^*$ is an associative algebra with the convolution product $f * g(c) = f(c_1)g(c_2)$ and unit $\varepsilon$.

7.1.2. Coquasi-bialgebras. Coquasi-bialgebras and coquasi-Hopf algebras are generalizations of the notion of bialgebras and Hopf algebras and they were defined (in a dual version) by Drinfeld in [10]. We shall recall the definition: a coquasi-bialgebra is a five-tuple $(H, \Delta, m, 1_H, \phi)$ where $H$ is a coassociative coalgebra with counit, $m : H \otimes H \to H, h \otimes g \to hg$ is a coalgebra map, $1_H$ is a grouplike element (i.e. $\Delta(1_H) = 1_H \otimes 1_H$) which is a unit for $m$, and $\phi \in (H \otimes H \otimes H)^*$ is a convolution invertible map (called the coassociator), satisfying the identities

\begin{align*}
(1.1) \quad \phi(g \otimes 1_H \otimes h) &= \varepsilon(g)\varepsilon(h), \\
(1.2) \quad m(m \otimes id_H) * \phi &= \phi * m(id_H \otimes m), \\
(1.3) \quad \phi(d_1 f_1 g_1 h_1) \phi(d_2 f_2 g_2 h_2) &= \phi(d_1 \otimes f_1 \otimes g_1) \phi(d_2 \otimes f_2 \otimes g_2) \phi(f_1 \otimes g_3 \otimes h_2)
\end{align*}

for all $f, g, h \in H$.

Remark 7.1. The notion of quasi-bialgebra is closely related with the notion of pseudomonoid, in fact a quasi-bialgebra is a pseudomonoid in CoAlg the 2-category of coalgebras, see [23, Chapter 15].

The category of right $H$-comodules $\mathcal{M}^H$ is monoidal, where the tensor product is over the base field and the comodule structure of the tensor product is induced by the multiplication. The associator is given by

$$\Phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

$$\Phi_{U,V,W}((u \otimes v) \otimes w) = \phi^{-1}(u_1, v_1, w_1)u_0 \otimes (v_0 \otimes w_0)$$

for $u \in U, v \in V, w \in W$ and $U, V, W \in \mathcal{M}^H$.

Example 7.2. Let $K$ be a discrete group and $\theta \in Z^3(K, T)$ a normalized 3-cocycle. The group algebra $\mathbb{C}K$ is a quasi-bialgebra with comultiplication $\Delta(u_\sigma) = u_\sigma \otimes u_\sigma$, counit $\varepsilon(u_\sigma) = 1$ and coassociator

$$\phi_\theta(u_\sigma, u_\tau, u_\rho) = \theta(\sigma, \tau, \rho),$$

for all $\sigma, \tau, \rho \in K$.

The coquasi-bialgebra $(\mathbb{C}K, \Delta, \varepsilon, \phi_\theta)$ is a coquasi-Hopf algebra, with $S(u_\sigma) = u_{\sigma^{-1}}$, $\alpha \equiv 1$ and $\beta(u_\sigma) = \theta(\sigma, \sigma^{-1}, \sigma)$ for all $\sigma \in G$. 
7.2. Coquasi-bialgebras associated to pseudomonoids. Let $G$ be a finite group and $K = (K, m, 1, \alpha, \beta, \theta)$ a pseudomonoid with strict unit in the 2-category of $G$-sets with twists; let us denote $C^G := \text{Maps}(G, C)$, let $\delta_\sigma \in C^G$ be the function that assigns 1 to $\sigma$ and 0 otherwise, and let $\delta_{\sigma, \tau}$ be the Dirac’s delta associated to the pair $\sigma, \tau \in G$, namely $\delta_{\sigma, \tau}$ is 1 whenever $\sigma = \tau$ and 0 otherwise.

**Theorem 7.3.** The vector space $C^G \# K := C^G \otimes K$ with basis $\{\delta_\sigma \# x | \sigma \in G, x \in K\}$ is a coquasi-bialgebra with product

$$ (\delta_\sigma \# x)(\delta_\tau \# y) = \delta_{\sigma, \tau} \beta[\sigma|x|y] \delta_{\sigma} \# xy, $$

coproduct,

$$ \Delta(\delta_\sigma \# x) = \sum_{a, b \in G; ab = \sigma} \alpha[a\mid b\mid x] \delta_a \# bx \otimes \delta_b \# x, $$

associator

$$ \phi(\delta_\sigma \# x, \delta_\tau \# y, \delta_\rho \# z) = \delta_{\sigma, \tau} \delta_{\tau, \rho} \theta|x\mid y\mid z|, $$

counit $\varepsilon(\delta_\sigma \# x) = \delta_{\sigma, e}$, and unit $1\# e$ for all $\sigma, \tau, \rho \in G$, $x, y, z \in K$.

**Proof.** Let us check that $\Delta$ is coassociative,

$$ (\Delta \otimes \text{id})\Delta(\delta_\sigma \# x) = (\text{id} \otimes \Delta)(\sum_{ab = \sigma} \alpha[a\mid b\mid x] \delta_a \# bx \otimes \delta_b \# x) $$

$$ = \sum_{abc=\sigma} \alpha[a\mid b\mid c\mid x] \alpha[a\mid b\mid c\mid x] \delta_a \# bx \otimes \delta_b \# cx \otimes \delta_c \# x $$

and

$$ (\text{id} \otimes \Delta)\Delta(\delta_\sigma \# x) = (\text{id} \otimes \Delta)(\sum_{ab = \sigma} \alpha[a\mid b\mid x] \delta_a \# bx \otimes \delta_b \# x) $$

$$ = \sum_{abc=\sigma} \alpha[a\mid b\mid c\mid x] \alpha[a\mid b\mid c\mid x] \delta_a \# bx \otimes \delta_b \# cx \otimes \delta_c \# x $$

Since $\delta_G(\alpha) = 1$, $\Delta$ is coassociative. It is immediate to see that $1\# e$ is a grouplike element and the unit for the product.

Now we shall see that the product is a coalgebra map:

$$ \Delta((\delta_\sigma \# x)(\delta_\tau \# y)) = \Delta(\beta[\sigma|x|y] \delta_{\sigma} \# xy) $$

$$ = \sum_{ab=\sigma} \beta[ab\mid x|y] \alpha[a\mid b\mid xy] \delta_a \# b(xy) \otimes \delta_b \# xy $$

and

$$ (m \otimes m)\Delta(\delta_\sigma \# x \otimes \delta_\tau \# y) = m \otimes m(\sum_{a, b, r, s \in G; ab = rs = \sigma} \alpha[a\mid b\mid x] \alpha[r\mid s\mid y] \delta_a \# bx \otimes \delta_r \# s \otimes \delta_b \# x \otimes \delta_s \# y) $$

$$ = \sum_{ab=\sigma} \alpha[a\mid b\mid x] \alpha[a\mid b\mid y] \beta[a\mid bx\mid by] \beta[b\mid x\mid y] \delta_a \# (bx) \otimes \delta_b \# xy. $$

Since $\delta_G \beta = m^* \alpha \cdot (\alpha \otimes \alpha)^{-1} = (\delta_K \alpha)^{-1}$ we have

$$ \beta[ab\mid x|y] \alpha[a\mid b\mid xy] = \alpha[a\mid b\mid x] \alpha[a\mid b\mid y] \beta[a\mid bx\mid by] \beta[b\mid x\mid y] $$

for all $a, b \in G$, $x, y \in K$, so the multiplication is a coalgebra map.
Let us check axiom (7.2)

\[ m(m \circ \text{id}) \ast \phi(\delta_\sigma \# x, \delta_\tau \# y, \delta_\rho \# z) = \delta_{\sigma, \tau, \rho} \theta[\sigma | x | y] \beta[\sigma | x | y | z] \theta[\sigma | y | z] \delta_\sigma \# xyz \]

and

\[ \phi \ast (m \circ \text{id}) (\delta_\sigma \# x, \delta_\tau \# y, \delta_\rho \# z) = \delta_{\sigma, \tau, \rho} \theta[\sigma | x | \sigma y] \beta[\sigma | y | z] \beta[\sigma | x | y | z] \delta_\sigma \# xyz. \]

Since \( \delta_G(\theta) \delta_K(\beta)^{-1} = 1 \) we have

\[ \beta[\sigma | x | y] \beta[\sigma | x | y | z] \theta[\sigma | y | z] = \theta[\sigma | x | y | z] \beta[\sigma | y | z] \beta[\sigma | x | y | z] \]

for all \( \sigma \in G, x, y, z \in G \), then \( m(m \circ \text{id}) \ast \phi = \phi \ast (m \circ \text{id}) \).

Finally in order to prove (2.1) and (2.2), note that the projection \( \pi : C^G \# K \to CK, \delta_\sigma \# x \mapsto \delta_\sigma.x_\# x \) is a coalgebra map and \( \phi = \phi_0 \circ \pi \) (see Example 7.2), then since \( \phi_0 \in (CK \otimes CK \otimes CK)^* \) is a coassociator for \( CK \), \( \phi \) is a coassociator for \( C^G \# K \).

**Remark 7.4.** If \( \theta \) is trivial, \( C^G \# K \) is a Hopf algebra.

**Theorem 7.5.** Let \( G \) be a finite group and \( K = (K, m, 1, \alpha, \beta, \theta) \) a pseudomonoid with strict unit in the 2-category of \( G \)-sets with twists. The tensor category of right \( C^G \# CK \)-comodules is tensor isomorphic to the monoidal category \( \text{Bun}_G(K) \) of equivariant vector bundles on \( K \).

**Proof.** Let \( V = \bigoplus_{x \in K} V_x \) be an object in \( \text{Bun}_G(K) \), we define a right \( C^G \# CK \)-comodule structure over \( V \) by

\[ \rho(v_x) = \sum_{\sigma \in G} \sigma \triangleright v_x \otimes \delta_\sigma \# x, \]

for all \( v_x \in V_x, x \in K \).

The coassociativity of \( \rho \) follows from the equations

\[(\rho \circ \text{id}) \rho(v_x) = \sum_{\tau, \sigma \in G} \rho(\tau \triangleright v_x) \otimes \delta_\tau \# x \]

\[ = \sum_{\sigma, \tau \in G} \sigma \triangleright (\tau \triangleright v_x) \otimes \delta_\sigma \# \tau x \otimes \delta_\tau \# x, \]

\[(\text{id} \circ \Delta) \rho(v_x) = \sum_{\tau \in G} \tau \triangleright v_x \otimes \Delta(\delta_\tau \# x) \]

\[ = \sum_{\sigma, \tau, \rho \in G} \alpha[a | b | c] \tau \triangleright v_x \otimes \delta_\alpha \# bx \otimes \delta_\beta \# c \]

\[ = \sum_{\sigma, \tau, \rho \in G} \alpha[\sigma | \tau | x] (\sigma \triangleright v_x) \otimes \delta_\sigma \# \tau x \otimes \delta_\tau \# x. \]

and the fact that \( \sigma \triangleright (\tau \triangleright v_k) = \alpha[\sigma | \tau | k] (\sigma \triangleright v_k) \) for all \( \sigma, \tau \in G, k \in K, v_k \in V_k \),

\[(\text{id} \circ \Delta) \rho = (\rho \circ \text{id}) \rho. \]

The counity axiom follows from

\[(\text{id} \circ \varepsilon) \rho(v_x) = \sum_{\sigma \in G} \sigma \triangleright v \otimes \varepsilon(\delta_\sigma \# x) \]

\[ = \varepsilon \triangleright v_x = v_x \]

\]
Now, for \( f : V \to W \) a morphism in \( \text{Bun}_G(\mathcal{K}) \), we shall see that \( f \) is also a morphism of comodules, and this follows from

\[
(f \otimes \text{id}) \rho_V(v_x) = \sum_{\sigma \in G} f(\sigma \triangleright v) \otimes \delta_\sigma \# x
\]

\[
= \sum_{\sigma \in G} \sigma \triangleright f(v_x) \otimes \delta_\sigma \# x
\]

\[
= \rho_W(f(v_x)),
\]

where last equality follows because \( f(v_x) \in W_x \).

Conversely, if \((V, \rho)\) is a right \( \mathbb{C}_G \# \mathbb{C}K \)-comodule, using the coalgebra epimorphism \( \pi : \mathbb{C}_G \# \mathbb{C}K \to \mathbb{C}K, \delta_\sigma \# x \mapsto \delta_\sigma, v_x \), \( V \) is a \( \mathbb{C}K \)-comodule by \((\text{id} \otimes \pi) \rho\). Recall that \( \mathbb{C}K \)-comodules are just \( K \)-graded vector spaces, then

\[
V = \bigoplus_{x \in K} V_x, \quad \text{where } V_x = \{ v \in V : (\text{id} \otimes \pi) \rho = v \otimes x \}.
\]

Since \( \{ \delta_\sigma \# x \mid \sigma \in G, x \in K \} \) is a basis of \( \mathbb{C}_G \# \mathbb{C}K \) and by definition \( \rho(V_z) \subset V \otimes \mathbb{C}_G \# x \) for all \( x \in G \), the comodule map defines and is defined by unique linear maps \( \triangleright : \mathbb{C}_G \otimes V_x \to V \) \((x \in G)\) such that

\[
\rho(v_x) = \sum_{\sigma \in G} \sigma \triangleright v_x \# x,
\]

for all \( v_x \in V_x \).

First we shall see that \( \tau \triangleright V_x \subseteq V_{\tau x} \) for all \( \tau \in G, x \in X \):

\[
\sum_{\tau \in G} \rho(\tau \triangleright v_x) \otimes \delta_\tau \# x = (\text{id} \otimes \Delta) \rho(v_x)
\]

\[
= (\text{id} \otimes \Delta) \rho(v_x)
\]

\[
= \sum_{\sigma, \tau \in G} \alpha[\sigma \mid \tau \mid x](\sigma \tau) \triangleright v_x \otimes \delta_\sigma \# \tau x \otimes \delta_\tau \# x,
\]

so

\[
\rho(\tau \triangleright v_x) = \sum_{\sigma \in G} \alpha[\sigma \mid \tau \mid x](\sigma \tau) \triangleright v_x \otimes \delta_\sigma \# \tau x,
\]

then

\[
(\text{id} \otimes \pi) \rho(\tau \triangleright v_x) = \sum_{\sigma \in G} \alpha[\sigma \mid \tau \mid x](\sigma \tau) \triangleright v_x \otimes \tau(\delta_\sigma \# \tau x)
\]

\[
= \alpha[\sigma] \mid x \tau \triangleright v_x \otimes \tau x
\]

\[
= \tau \triangleright v_x \otimes \tau x,
\]

and therefore \( \tau \triangleright v_x \in V_{\tau x} \). Moreover we have

\[
\sum_{\sigma, \tau \in G} \sigma \triangleright (\tau \triangleright v_x) \otimes \delta_\sigma \# \tau x \otimes \delta_\tau \# x = (\rho \otimes \text{id}) \rho(v_x)
\]

\[
= (\text{id} \otimes \Delta) \rho(v_x)
\]

\[
= \sum_{\sigma, \tau \in G} \alpha[\sigma \mid \tau \mid x](\sigma \tau) \triangleright v_x \otimes \delta_\sigma \# \tau x \otimes \delta_\tau \# x,
\]

so

\[
\sigma \triangleright (\tau \triangleright v_x) = \alpha[\sigma \mid \tau \mid x](\sigma \tau) \triangleright v_x
\]
for all $v_x \in V_x$, $x \in K$, $\sigma, \tau \in G$.

The counit axiom
\[
v_x = (\text{id} \otimes \varepsilon) \rho_v(v_x)
\]
\[
= \sum_{\sigma \in G} \sigma \triangleright v \otimes \varepsilon(\delta_\sigma \# x)
\]
\[
= e \triangleright v_x,
\]
for all $v_x \in V_x$.

Now, let $(V, \rho_V), (W, \rho_W)$ be right $\mathbb{C}^G \otimes \mathbb{C}K$-comodules. Let us see that every morphism of comodules $f : V \rightarrow W$ defines a morphism between the equivariant twisted vector bundles associated.

If $v_x \in V_x$,
\[
(id \otimes \pi) \rho_W(f(v_x)) = (id \otimes \pi)(f \otimes \text{id}) \rho_V(v_x)
\]
\[
= (f \otimes \text{id})(id \otimes \pi) \rho_V(v_x)
\]
\[
= (f \otimes \text{id})(v_x \otimes x)
\]
\[
= f(v_x) \otimes x,
\]
then $f(v_x) \in W_x$. Since $f(v_x) \in W_x$,
\[
\sum_{\sigma \in G} \sigma \triangleright f(v_x) \otimes \delta_\sigma \# x = \rho_W(f(v_x))
\]
\[
= (f \otimes \text{id}) \rho_V(x)
\]
\[
= \sum_{\sigma \in G} f(\sigma \triangleright v_x) \otimes \delta_\sigma \# x,
\]
then $\sigma \triangleright f(v_x) = f(\sigma \triangleright v_x)$ for all $x \in X$, $v_x \in V_x, \sigma \in G$.

Finally we shall prove that the functors previously defined are monoidal. Let $V,W \in \mathcal{MC}^G \otimes \mathbb{C}K$, then the grading of the twisted vector bundles associated to $V \otimes W$ is given by $(V \otimes W)_k = \bigoplus_{x,y \in K : xy = k} V_x \otimes W_y$, in fact, if $v_x \in V_x, w_y \in W_y$, then
\[
(id \otimes \pi) \rho_{V \otimes W}(v_x \otimes w_y) = \text{id} \otimes \pi \left( \sum_{\sigma, \tau \in G} \beta[\sigma|x|y] \sigma \triangleright v_x \otimes \tau \triangleright w_y \otimes \delta_\sigma \delta_\tau \# xy \right)
\]
\[
= \sum_{\sigma \in G} \beta[\sigma|x|y] \sigma \triangleright v_x \otimes \sigma \triangleright w_y \otimes \pi(\delta_\sigma \# xy)
\]
\[
= v_x \otimes w_y \otimes xy,
\]
then $W_x \otimes V_y \subset (V \otimes W)_{xy}$. The comodules $V$ and $W$ are finite dimensional, so $(V \otimes W)_k = \bigoplus_{x,y \in K : xy = k} V_x \otimes W_y$ and since
\[
\sum_{\sigma \in G} \sigma \triangleright (v_x \otimes w_y) \otimes \delta_\sigma \# xy = \rho_{V \otimes W}(v_x \otimes w_y)
\]
\[
= \sum_{\sigma \in G} \beta[\sigma|x|y] \sigma \triangleright v_x \otimes \sigma \triangleright w_y \otimes \delta_\sigma \# xy
\]
the twisted action on $V \otimes W$ is given by $\sigma \triangleright (v_x \otimes w_y) = \beta[|x|y] \sigma \triangleright v_x \otimes \sigma \triangleright w_y$.

We shall check that the associator induced by the coquasi-biagebra structure agrees with the associator of $\text{Bun}_G(K)$. 

Let $U, V, W$ be right comodules. If $u_x \in U_x, v_y \in V_y$ and $w_z \in W_z$, then

$$
\Phi_{U, V, W}(u_x \otimes v_y) \otimes w_z = \phi^{-1}((u_x)_1, (v_y)_1, (w_z)_1)(u_x)_0 \otimes ((v_y)_0 \otimes (w_z)_0)
$$

$$
= \sum_{\sigma, \tau, \rho \in G} \phi^{-1}(\delta_{\sigma \# x, \delta_{\tau \# y}, \delta_{\rho \# z})\sigma \triangleright u_x \otimes \tau \triangleright v_y \otimes \rho \triangleright w_z
$$

$$
= \theta|x|y|z|^{-1}u_x \otimes v_y \otimes w_z.
$$

\begin{remark}
Whenever $G$ acts by conjugation on $K = G$ and we consider a 3-cocycle $w \in Z^3(G, \mathbb{T})$, the coquasi-bialgebra $\mathbb{C}^G \# \mathbb{C}G$ defined by the pseudomonoid $(G, m, 1, \alpha_w, \beta_w, \theta_w)$ as in section 3.5, is the dual of the Twisted Drinfeld Double of the finite group $G$ defined in [4, Section 3.2].
\end{remark}

We finish with a corollary of the results of this appendix and the ones of section 6.2. A quasi-isomorphism between coquasi-Hopf algebras $(H, \psi)$ and $(H', \psi')$ is a pair $(F, \theta)$ consisting of a coalgebra isomorphism $F : H \to H'$ and a convolution invertible map $\theta \in (H \otimes H')^*$ such that $\theta(1 \otimes h) = \theta(h \otimes 1) = \varepsilon(h)$ and

$$
\theta(g_1 \otimes h_1)F(g_2h_2) = F(g_1)F(h_2)\theta(g_2 \otimes h_2),
$$

for all $g, h \in H$. A quasi-isomorphism is called an isomorphism of quasi-Hopf algebras if additionally

$$
\theta(f_1 \otimes g_2)\theta(f_2g_2 \otimes h_1)\psi(f_3 \otimes g_3 \otimes h_3) = \psi'(F(f_1) \otimes F(g_1) \otimes F(h_1))\theta(g_2 \otimes h_2)\theta(f_2 \otimes g_3 \otimes h_3).
$$

Coquasi-isomorphisms and isomorphisms of coquasi-Hopf algebras have categorical interpretations. Isomorphisms correspond to tensor equivalences and quasi-isomorphism with quasi-monoidal equivalences (or non-coherent monoidal equivalence). In the case of cosemisimple coquasi-Hopf algebras, it was proved in [8, Prop. 4], that two coquasi-Hopf algebras are coquasi-isomorphic if and only if they have equivalent Grothendieck rings.

In general, the Twisted Drinfeld Double $D^w(G)$ of a finite group $G$, is not isomorphic to a Hopf algebra, or equivalently, the tensor category of representations of $D^w(G)$ is not equivalent to the category of representations of any Hopf algebra. However, by the isomorphism outlined in (6.2), if $G$ is finite and acts over itself by conjugation, the Grothendieck ring of $\text{Bun}_G(K)$ for any 3-cocycle in $Z^3(\text{Tot}^*(A^{*\infty}(G \times G, T)))$, is always equivalent to the Grothendieck ring of $\text{Bun}_G(K')$, where the 3-cocycle associated lives in $Z^3(\text{Tot}^*(B^{*\infty}(G \times G, T)))$. By Theorem 7.3 and Theorem 7.5, $\text{Bun}_G(K')$ is the category of representation of a Hopf algebra, so in particular we can conclude

\begin{corollary}
The Twisted Drinfeld Double of a finite group is always quasi-isomorphic to a Hopf algebra.
\end{corollary}

\begin{references}
[1] A. Adem and R. J. Milgram. Cohomology of finite groups, volume 309 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1994.

[2] A. Adem, Y. Ruan, and B. Zhang. A stringy product on twisted orbifold K-theory. Morfismos, 11(2), 2007.
\end{references}
[3] B. Bakalov and A. Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[4] N. Bárdenas, J. Espinoza, M. Joachim, and B. Uribe. Classification of twists in equivariant K-theory for proper and discrete actions. *http://arxiv.org/abs/1202.1880*, 2012.

[5] E. Becerra and B. Uribe. Stringy product on twisted orbifold K-theory for abelian quotients. *Trans. Amer. Math. Soc.*, 361(11):5781–5803, 2009.

[6] F. Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[7] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[8] A. A. Davydov. Finite groups with the same character tables, Drinfel’d algebras and Galois algebras. In *Algebra (Moscow, 1998)*, pages 99–111. de Gruyter, Berlin, 2000.

[9] R. Dijkgraaf, V. Pasquier, and P. Roche. Quasi Hopf algebras, group cohomology and orbifold models. *Nuclear Phys. B Proc. Suppl.*, 18B:60–72 (1991), 1990. Recent advances in field theory (Annecy-le-Vieux, 1990).

[10] V. G. Drinfel’d. Quasi-Hopf algebras. *Algebra i Analiz*, 1(6):114–148, 1989.

[11] S. Eilenberg and S. Mac Lane. On the groups of $H(\Pi, n)$. I. *Ann. of Math. (2)*, 58:55–106, 1953.

[12] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.

[13] P. J. Hilton and U. Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[14] J. Hu and B.-L. Wang. Delocalized chern character for stringy orbifold K-theory. *http://arxiv.org/abs/1110.0953*, 2011.

[15] A. Joyal and R. Street. An introduction to Tannaka duality and quantum groups. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*., pages 413–492. Springer, Berlin, 1991.

[16] G. I. Kac. Group extensions which are ring groups. *Mat. Sb. (N.S.)*, 76(118):473–496, 1968.

[17] G. Karpilovsky. *The Schur multiplier*, volume 2 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1987.

[18] S. Majid. Tannaka-Kreın theorem for quasi-Hopf algebras and other results. In *Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990)*, volume 134 of *Contemp. Math.*., pages 219–232. Amer. Math. Soc., Providence, RI, 1992.

[19] A. Masuoka. Hopf algebra extensions and cohomology. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*., pages 167–209. Cambridge Univ. Press, Cambridge, 2002.

[20] A. Masuoka. Cohomology and coquasi-bialgebra extensions associated to a matched pair of bialgebras. *Adv. Math.*, 173(2):262–315, 2003.

[21] C. C. Moore. Group extensions and cohomology for locally compact groups. *Trans. Amer. Math. Soc.*, 221(1):1–33, 1976.
[22] V. Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8(2):177–206, 2003.

[23] R. Street. *Quantum groups*, volume 19 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2007. A path to current algebra.

[24] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, revised edition, 2010.

[25] S. Willerton. The twisted Drinfeld double of a finite group via gerbes and finite groupoids. *Algebr. Geom. Topol.*, 8(3):1419–1457, 2008.

[26] S. J. Witherspoon. Products in Hochschild cohomology and Grothendieck rings of group crossed products. *Adv. Math.*, 185(1):136–158, 2004.

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