Macroscopic realism of quantum work fluctuations

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We study the fluctuations of the work performed on a driven quantum system, defined as the difference between subsequent measurements of energy eigenvalues. These work fluctuations are governed by statistical theorems with similar expressions in classical and quantum physics. In this article we show that we can distinguish quantum and classical work fluctuations, as the latter can be described by a macrorealistic theory and hence obey Leggett-Garg inequalities. We show that these inequalities are violated by quantum processes in a driven two-level system and in a harmonic oscillator subject to a squeezing transformation.

I. INTRODUCTION

The thermodynamics of quantum systems has become a rapidly expanding field of research in the recent years [1, 2]. One of its main lines of research has been triggered by the discovery of fluctuation relations, especially the classical work fluctuation theorems [3, 4] and the derivation of their quantum-mechanical counterparts [5]. One prominent example of such a fluctuation theorem is the Jarzynski relation [6],

\[ \langle e^{-\beta w} \rangle_w = e^{-\beta \Delta F}, \]  
(1)

which relates the average work \( w \) performed during a non-equilibrium transformation with the free energy difference \( \Delta F \) between two thermal states at inverse temperature \( \beta = 1/kT \). It has been shown that, if the work performed on a quantum system is defined in a suitable way, the Jarzynski equality Eq. (1) holds for classical as well as for quantum systems [6]. While there has been some debate what “suitable” means in this context [3, 7, 8], a widely accepted definition of quantum work is given by the difference in the outcome of projective measurements of the Hamiltonian operator at different times. With this definition the quantum work becomes, in general, a fluctuating quantity, similar to the fluctuating work in classical non-equilibrium thermodynamics. While the (non-equilibrium) work is characterized by a (classical) probability density in both cases, the origin of the randomness is quite different: In classical physics energies have definite values, and work fluctuations stem from the random exchange of energy with the particles in the surrounding heat bath, while in quantum mechanics, they originate from quantum uncertainty and the resulting fundamental randomness of the outcome of measurement, i.e., Born’s rule and the projection postulate. In this work, we address the question: “How quantum is quantum work?”, by investigating to what extent the different origin of quantum and classical work fluctuation have measurable consequences.

Our approach to the problem will be to consider processes where the energy of the system is measured multiple times, see Fig.1, and where we can hence study temporal correlations in the work fluctuations. If the work behaves classically, it should be describable as a macroscopic, realistic variable, which is measurable in a non-invasive manner, and hence its correlations should obey the Leggett-Garg inequalities [9, 10]. Leggett-Garg inequalities have been used to analyze quantum effects in thermodynamics processes and heat engines [11]. While they only apply for correlation functions of dichotomic variables in their original form, entropic Leggett-Garg inequalities have been derived for correlations of more general variables.

The article is structured as follows: In Sec. II we introduce the definition of quantum work and its probability distribution. In Sec. III we recall the dichotomic and entropic Leggett-Garg inequalities and we discuss their application to the work done on a quantum system. In Sec. IV and Sec. V we investigate whether the inequalities are obeyed or violated for a driven two-level system and a squeezed harmonic oscillator, respectively. In Sec. VI we discuss the consequences and possible applications of our results.

II. QUANTUM WORK AND ITS PROBABILITY DISTRIBUTION

Consider an isolated quantum system with a time-dependent Hamiltonian \( H_t = H(\lambda_t) \), where \( \lambda_t \) is a varying control parameter. The state of the system obeys the Liouville-von Neumann equation \( i\hbar \dot{\rho}_t = [H_t, \rho_t] \), and, in general, work will be performed on it, i.e. energy will be injected into (or removed from) the system. In order to determine the work performed during a given process, (quantum as well as classically), one measures the energy of the system before and after the process. While in classical systems such measurements are unproblematic, the measurement on a quantum system will in general have random outcomes and it will change the state of the system. If one wants to measure the work performed on the system between the times \( t_0 \) and \( t_1 \) one has to probe the system energy at the beginning and the end of the driving. This will yield one of the eigenvalue \( E_k^{t_0} \) of \( H_{t_0} \) with a probability \( p_k = \text{tr}[\Pi_k^{t_0} \rho_0] \), and, subsequently, an eigenvalue \( E_k^{t_1} \) of \( H_{t_1} \), with a probability \( p_k = \text{tr}[\Pi_k^{t_1} \rho_1] \), where we have introduced the projection
operators on the energy eigenstates, \( \Pi_{k_0}^2 = |k_\alpha\rangle \langle k_\alpha| \) of \( H_t \) at \( t = t_\alpha, \alpha = 0, 1 \). The state at \( t_1 \) is \( \rho_1 = U_{1,0} \rho \ U_{1,0}^\dagger \), where \( U_{1,0} = T \exp[-i \int_{t_0}^{t_1} H_t dt] \) is the time evolution operator from time \( t_0 \) to \( t_1 \) (\( T \) denotes the time ordering operator) and \( \rho = \Pi_{k_0}^0 \) is the state conditioned on the outcome of the first measurement at \( t = t_0 \). Subsequent time evolution of the quantum state and measurements are described by the same formalism, cf., Fig. 1.

One obtains the work by merely subtracting the measured energies,

\[
W_{k_1,k_0} = E_{k_1}^1 - E_{k_0}^0,
\]

and since it depends on the random outcome of projective energy measurements it is an inherently fluctuating quantity. Its probability distribution is given by

\[
p(w) = \sum_{k_0,k_1} \delta(w - W_{k_1,k_0}) p_{k_1,k_0},
\]

where, according to the above arguments,

\[
p_{k_1,k_0} = \text{tr}[\Pi_{k_1}^1 U_{1,0} \Pi_{k_0}^0 \rho_0 \Pi_{k_0}^1 U_{1,0}^\dagger].
\]

is the joint probability distribution for measuring \( E_{k_0}^0 \) at \( t_0 \) and \( E_{k_1}^1 \) at \( t_1 \).

Assuming that the system is initially in a thermal state \( \rho_0 = \exp(-\beta H_0)/Z_0 \) with \( Z_0 = \text{tr}[\exp(-\beta H_0)] \), it is rather straightforward to use the characteristic function \( \chi(p) \) for the work probability distribution Eq. 3 and verify that the quantum Jarzynski equality Eq. (1) holds, where \( \langle \ldots \rangle_w = \int \cdots p(w) dw \) and \( \Delta F = F_1 - F_0 \) with the free energy \( F_t = -(1/\beta) \ln(Z_t) \).

Instead of measuring the energy only in the beginning and the end of a process one might perform several energy measurements at different times, see Fig. 1. Probing the system energy at three instants of time, \( t_0, t_1, t_2 \) leads to the joint probability distribution

\[
p_{k_2,k_1,k_0} = \text{tr}[\Pi_{k_2}^2 U_{2,1} \Pi_{k_1}^1 U_{1,0} \Pi_{k_0}^0 \rho_0 \Pi_{k_0}^1 U_{1,0}^\dagger \Pi_{k_1}^0 U_{2,1}^\dagger].
\]

By summing over indices one obtains, e.g., \( p_{k_1,k_0} \) and \( p_{k_2,k_1} \) from \( \rho_0 \). Note that, since in general \( \Pi_{k_0}^0, U_{i,j} \neq 0 \), the distribution arising from summing over the middle index \( k_1 \) is not equivalent to the distribution \( p_{k_2,k_1} \) without measurement at \( t_1 \).

From Eq. 5 we define the joint probability

\[
p(w_1, w_2) = \sum_{k_0,k_1,k_2} \delta(w_1 - W_{k_0,k_1}) \delta(w_2 - W_{k_1,k_2})
\]

\[
\times p_{k_2,k_1,k_0}
\]

to perform the work \( w_1 \) between \( t_0 \) and \( t_1 \), and the work \( w_2 \) between \( t_1 \) and \( t_2 \). Note that its marginal distribution \( p(w_1) = \int dw_2 p(w_1, w_2) \) is equivalent to Eq. 3 and hence fulfills the fluctuation relations, while the marginal \( p(w_2) = \int dw_1 p(w_1, w_2) \) will in general not fulfill such relations because the system is not in an equilibrium state at \( t_1 \).

Using Eq. 3 the probability for the total work \( w_{\text{tot}} = w_1 + w_2 \) yields

\[
p(w_{\text{tot}}) = \int dw_1 dw_2 \delta(w_{\text{tot}} - [w_1 + w_2]) p(w_1, w_2)
\]

\[
= \sum_{k_0,k_1,k_2} \delta(w_{\text{tot}} - W_{k_2,k_1,k_0}) p_{k_2,k_1,k_0},
\]

i.e., it depends, as one would expect, only on the difference between the first and the last energy measurements. While the intermediate measurements will, in general, influence the final energy measurement \( \text{[12]} \), one can prove that the Jarzynski relation \( \text{[1]} \) still holds for the total work \( \text{[12]} \).

The measurement back action and, in particular, the destruction of quantum mechanical coherence by the middle measurement (the first measurement acts on a thermal state with already vanishing coherences), presents a fundamental difference between the definition of work in quantum and classical contexts. One may, indeed, include non-invasiveness as a desired property of the definition of work, but that turns out to be incompatible with its relationship with the average energy of the system \( \text{[14]} \).

In this article, we shall instead retain the generally accepted definition of work, and address its invasiveness in a more quantitative manner. To this end, we shall appeal to the Leggett-Garg inequalities \( \text{[9]} \), which precisely concern correlations between measurements performed at different times on a quantum system. We note that Leggett-Garg inequalities have also been applied to characterize the quantumness of a quantum heat engine through the correlation between the working system observables at different times \( \text{[11]} \).
III. LEGGET-GARG INEQUALITIES FOR WORK MEASUREMENTS

Assuming macroscopic realism and noninvasive measurability of a dichotomic variable that is measured at different times, \( t_i \) with output values \( Q_i = \pm 1 \), Leggett and Garg derived the inequality \([9,10]\)

\[
C_{21} + C_{32} - C_{31} \leq 1.
\]

(8)

for the two-time correlation functions \( C_{ij} = \langle Q_i Q_j \rangle \). While Eq. (8) is obeyed for classical dynamics, measurements on a quantum systems may violate the Leggett-Garg inequality \([15,10]\). This violation is readily understood as a consequence of the measurement back action, which is absent in the classical case. In the present work, we shall use the Leggett-Garg inequality to assess how the definition of quantum work as the result of projective energy measurements necessarily implies a quantitative difference between the fluctuations of quantum and classical work. Note that Eq. (9) is not associated with the absolute magnitude of energy and work measurements, but only the statistical correlations of the variables \( Q_i = \pm 1 \), which we can associate with the projective measurements on two different eigenstates.

For systems with more eigenstates, the measurement outcome \( Q_i \) at time \( t_i \) may attain more than two values \( \{ q_i \} \), and an alternative, entropic Leggett-Garg inequality has been derived for the correlations between such multi-valued measurement outcomes \([17]\).

\[
H(Q_2|Q_0) \leq H(Q_2|Q_1) + H(Q_1|Q_0).
\]

(9)

Here, \( H(Q_j|Q_i) = - \sum_{q_i,q_j} p(q_j|q_i) \log p(q_j|q_i) \) is the (classical) conditional entropy, where \( p(q_j|q_i) \) is the conditional probability for outcome \( q_j \), given the earlier outcome \( q_i \) (occurring with probability \( p(q_i) \)).

We shall now apply the Leggett-Garg and entropic Leggett-Garg inequalities to the correlations Eq. (4) of energy measurements. These correlations reflect how the definition of work is affected by measurement back-action effects and, hence, to what extent the underlying thermodynamic transformation can be modelled as a classical process.

The mean conditional entropy for energy measurements at two instants of time is given by

\[
H(E^j|E^i) = - \sum_{k_j,k_i} p_{k_j,k_i} \log p_{k_j|k_i},
\]

(10)

where the conditional probability, \( p_{k_j|k_i} \), is the quantum mechanical transition probability \( |k_j \rangle \rightarrow |k_i \rangle \), between the eigenstates of the Hamiltonian at times \( t_i \) and \( t_j \), as governed by the unitary time evolution operator \( U_{j,i} \), defined above.

For a slowly varying Hamiltonian, the system will adiabatically follow the time dependent eigenstate in which it is prepared by the first measurement, and hence \( p_{k_j|k_0} = \delta_{k_j,k_0} \) (note the eigenenergies may generally differ and a

definite, nonvanishing amount of work is hence done on the system). Also, during the subsequent evolution and measurement, the system follows the same \( (k_i) \) eigenstate, and due to the definite outcomes, \( H(E^j|E^i) = 0 \) and the entropic Leggett-Garg equality Eq. (9) for energy measurements is trivially fulfilled. In the following sections, we shall hence consider systems that do not evolve adiabatically.

Noting that the probability distribution for the work done on the system is governed by the joint probability of the two pertaining energy measurements, \( w_{k_j,k_i} = E_{k_j}^j - E_{k_i}^i, \ p(w_{k_j,k_i}) = p(k_j,k_i) = p(k_j|k_i)p(k_i) \), we shall introduce the corresponding entropy \( H(w_{j|i}) = H(E^j,E^i) = - \sum_{k_i,k_j} p_{k_j,k_i} \log p_{k_j,k_i} \). Using the identity between conditional and joint entropies \([15]\), \( H(E^j|E^i) = H(E^j,E^i) - H(E^i) \), where \( H(E^i) = - \sum_{k_i} p_{k_i} \log p_{k_i} \) is the entropy of the distribution \( p_{k_i} \), we can rewrite the entropic Leggett-Garg inequality \([9]\), as a relation for the work distributions

\[
H(w_{21}) \leq H(w_{21}) + H(w_{10}) - H(E^i).
\]

(11)

Here, \( H(w_{j|i}) = - \sum_{p(w_{j,i}) \neq 0} p(w_{j|i}) \log p(w_{j|i}) \) is the entropy of the work distribution, which is discrete if the corresponding energy spectrum is discrete. Note, that Eq. (11) does not only depend on the entropy of work distributions but also on the entropy of the middle energy measurement \( H(E^i) \). Disregarding this term leads to an inequality that is more easily fulfilled, and which reflects that in a classical system the entropy of the work distribution, without the middle measurement at \( t_j \) is always smaller than with this measurement taking place, because classical measurements do not decrease the information \([17]\). Note that it is easier to observe a violation of the Leggett-Garg inequality if the entropy of the middle energy measurement \( H(E^i) \) is retained in Eq. (11).

As a side remark, we note that to go from Eq. (10) to Eq. (11) we assume that the the work distribution has no “degeneracies”, i.e. there are no \( k_j, k_i \) and \( k_j', k_i' \) with \( W_{k_j,k_i} = W_{k_j',k_i'} \). While such degeneracies may be easy to avoid, they can also be accounted for by using the grouping formula for the Shannon entropy \([18]\) where joint probabilities with \( W_{k_j,k_i} = W_{k_j',k_i'} \) are grouped together. If \( p = \{ p_1, \ldots, p_m \} \) is a probability distribution and \( q = \{ q_1, \ldots, q_n \} \) with \( q_j = \sum_{i \in I_j} p_i \) is another distribution formed by “grouping” the probabilities \( p_i \) which correspond to a subset of events with indices \( I_j \) the Shannon entropy yields

\[
H(q) = H(p) - \sum_j q_j H \left( \{ \frac{p_i}{q_j} | i \in I_j \} \right)
\]

(12)

\[= H(p) - H(q).\]

Hence, the grouping reduces the Shannon entropy by an amount given by weighted entropies of subset probabilities.
IV. VIOLATION OF THE CONVENTIONAL AND THE ENTROPIC LEGGETT-GARG INEQUALITIES FOR A TWO-LEVEL SYSTEM

We recall that the work is defined through projective measurements in the eigenstate basis of the time dependent Hamiltonian. One should hence determine the time evolution operator by solving the Schrödinger equation, and subsequently express it as a transformation between the eigenstates at different times. Since both the time evolution of the state of the system and the time dependence of the eigenstates are governed by unitary operations, the evolution with respect to the adiabatic basis is also unitary, and can, e.g., for the evolution between the first and the middle measurement, be expressed as

$$U_{1,0} = \begin{pmatrix} e^{\frac{i}{2}(\alpha - \beta)} \cos \frac{\theta}{2} & e^{\frac{i}{2}(\alpha - \beta)} \sin \frac{\theta}{2} \\ -e^{\frac{i}{2}(\alpha - \beta)} \sin \frac{\theta}{2} & e^{\frac{i}{2}(\alpha - \beta)} \cos \frac{\theta}{2} \end{pmatrix},$$  \hspace{1cm} (13)

with angles $\alpha$, $\beta$ and $\theta$. Note that this matrix pertains to the time dependent adiabatic basis, and when the mixing angle $\theta$ is small, it represents the case of adiabatic evolution (slowly varying Hamiltonian). In this limit, the system adiabatically follows the energy eigenstates of the system, the measurements are fully correlated and the entropies vanish. Rather than specifying a time dependent Hamiltonian, it is convenient and it represents no loss of generality, to represent the dynamics by the transformation matrix with respect to the time-dependent energy eigenstates $U_{1,0}$ and the similarly defined $U_{2,1}$. The matrix elements of these matrices then directly yield the joint probabilities in Eq. (4).

$$p_{k_1,k_0} = \text{tr}[\Pi_{k_0} U_{1,0} \Pi_{k_0}^0 \rho_0 \Pi_{k_0}^0 U_{1,0}^\dagger],$$ \hspace{1cm} (14)

For simplicity of analysis, we set $\alpha = \beta = 0$, so that $U_{1,0}$ becomes a real rotation matrix. The angle $\theta$ in Eq. (13) controls how much population is transfered between the eigenstates at different times, and, for simplicity, we shall assume that $U_{2,1} = U_{1,0}$.

Mutatis mutandis, we can obtain the joint probabilities $p_{k_2,k_0}$ and $p_{k_2,k_1}$, so that we can study the violation Eq. (8), where the dichotomic variable takes the values $\pm 1$ in the ground and excited state, and of Eq. (11).

In order to quantify the violation of the equations Eq. (8) and, respectively, Eq. (11) we define the Leggett-Garg parameters

$$K^\text{cor}_3 = \frac{1}{4}(1 - C_{01} - C_{02} + C_{12})$$ \hspace{1cm} (15)

and,

$$K^\text{cn}_3 = \frac{1}{2}(H(w_{21}) + H(w_{10}) - H(w_{20}) - H(E_1)),$$ \hspace{1cm} (16)

where negative values of the parameters are a signature of non-classical behavior. In Fig. 2 we plot $K^\text{cor}_3$ and $K^\text{cn}_3$ as functions of the angle $\theta$. Moreover, we plot $K^\text{cor}_3$ which follows from $K^\text{cor}_3$ by redefining $Q_1 \rightarrow -Q_1$, [10].

As initial state we chose a thermal state with $\beta = 1/\Delta E$, where $\Delta E$ is the energy splitting of the ground and excited state. Actually, for a two-level system the violation of the Leggett-Garg inequality does not depend on the initial state and temperature, as the matrix $U_{1,0}$ yields the same probability to obtain the same and the opposite eigenstates in the middle measurement for both initial outcomes.

Fig. 2 shows that for small finite $\theta$, all curves are in the grey area where the Leggett Garg inequality is violated. They, however, differ considerably. While either $K^\text{cor}_3$ or $K^\text{cn}_3$ violate the conditions for macroscopic realism for all angles except for multiples of $\pi/2$, non-classical correlations between measurements are not always revealed by the entropic Leggett-Garg inequality.

V. VIOLATION OF THE ENTROPIC LEGGETT-GARG INEQUALITY FOR A SQUEEZED HARMONIC OSCILLATOR

In this section, we study the entropic Leggett-Garg inequality for a harmonic oscillator. The quantum harmonic oscillator is in many aspects well described by classical physics, e.g., the evolution of the continuous position and momentum operators solve the same coupled linear equations as the classical coordinates. The work done on a harmonically trapped particles has been studied quite extensively in both the classical and the quantum case [19,22].

The harmonic oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega\left(a^+a + \frac{1}{2}\right),$$ \hspace{1cm} (17)

with $a = \sqrt{\hbar\omega/2}\left(x + (i/\omega)p\right)$ and $[a,a^+] = 1$. Driving the system with an arbitrary time-dependent potential strength $\omega_1$ will maintain the quadratic form of the
Hamiltonian, and hence result in linear coupled equations for the position and momentum operators or, equivalently, for the raising and lowering operators. Their time dependence in the Heisenberg picture can hence be represented by a Bogoliubov (squeezing) transformation [23, 24],

\[ a(\tau) = U_r^\dagger a U_r = \mu(\tau)a + \nu(\tau) a^\dagger, \]

where \( \mu(\tau) \) and \( \nu(\tau) \) are complex functions depending on details of the driving and \( U_r \) is the corresponding time evolution operator.

As in the case of the two-level system, we shall express the evolution with respect to the operators defining the energy measurements, \( H_t = \hbar\omega_0(\hat{a}^\dagger \hat{a} + \frac{1}{2}) \). These are the adiabatically evolved operators, and they are also given by a Bogoliubov transformation. Hence, without loss of generality, we can also represent the transformation of the quantum state, expressed in terms of the raising and lowering operators pertaining to the time dependent Hamiltonian as a Bogoliubov, or squeezing, transform,

\[ \hat{U}_{10}^\dagger a_{t_0} \hat{U}_{10} = \cosh r a_{t_1} + \sinh r e^{-i\phi} a_{t_1}^\dagger. \]

For simplicity, we omit the phase \( \phi \), and using Eq. (19) as the propagator between two energy measurements, the joint probability distribution (14) yields

\[ p_{k_1,k_0} = \text{tr}\left[ \Pi_{k_1} U_{10} \Pi_{k_0} ^\dagger \rho_0 \Pi_{k_0} U_{10}^\dagger \right] \]

\[ = G_{k_1,k_0}^{*}(r)\rho_{k_0,k_0}, \]

with \( \rho_{nn} = \langle n | \rho_0 | n \rangle \) and \( G_{mn}(r) = \langle m | U_r | n \rangle \). The transition matrix elements for squeezed number states \( G_{mn}(r) \) are provided in the Appendix A following analytical results in [25, 26]. With the above preparation, we are ready to address the entropic Leggett-Garg inequality [11], where the time evolution between \( t_0 \) and \( t_1 \) and between \( t_1 \) and \( t_2 \) are both governed by (19), but possibly with two different arguments \( r_{1} \) and \( r_{2} \).

Plotting the entropic Leggett-Garg parameter \( K_3^{en} \) as a function of \( r_{1} \) and \( r_{2} \) for a thermal initial state with \( \beta = 0.1(\hbar\omega_0)^{-1} \) in Fig. 3 reveals that also for the harmonic oscillator, Eq. (11) can be violated and the quantum work obeys non-classical statistics. Interestingly, even a vanishing small amount of squeezing is enough to violate the Leggett-Garg inequality. Increasing the squeezing strength increases the violation until a maximal violation is obtained at \( r_1 = r_2 \approx 0.02 \). Further increase of the squeezing parameter leads to a less pronounced violation and finally \( K_3^{en} \) turns positive indicating that the works statistics can no longer be distinguished from that of a classical process. This can be understood as a consequence of the fact that the squeezing of thermal states and number states generally broadens the number distribution, turning sub-Poissonian into super-Poissonian statistics [27]. Note also that for strong squeezing there is an asymmetry between \( r_1 \) (the squeezing between \( t_0 \) and \( t_1 \)) and \( r_2 \) (the squeezing between \( t_1 \) and \( t_2 \)). In this regime, too much squeezing in the second interval prevents the violation of the Leggett-Garg inequality.

It is an interesting question how the violation depends on the temperature of the thermal initial state. We recall that for the two-level system, the outcome correlations are independent of the outcome of the first measurement, and hence of the initial state. This is different for the oscillator, since the probability for measuring high energy outcomes in the first measurement depends on the initial temperature and do affect the subsequent correlations. In Fig. 4 we plot the \( K_3^{en} \) for different values of inverse temperatures \( \beta \) as a function of the squeezing parameter \( r \), where we set \( r_1 = r_2 \). Interestingly, the violation for larger \( \beta \), i.e. lower temperature, is less pronounced than for small \( \beta \). Hence, the quantum work statistics appears
more non-classical for higher initial temperatures. This can also be seen in the upper plot of Fig. 5, where we plot the minimal value of $K_{3n}$ as a function of $\beta$. This can be understood by the fact that for higher temperatures the system is more likely to start in a higher number state after the first measurement which is more strongly affected by the squeezing.

Fig. 4 also reveals that for increasing $\beta$ the values for the squeezing parameter $r_1 = r_2$, where the maximal violation occurs increases. This is studied more generally in Fig. 5, where we plot the value of $r$ leading to maximal Leggett-Garg violation as function of $\beta$. We observe a non-monotonous dependency with a maximum around $\beta = 1$ and approach towards a constant level for large $\beta$.

VI. CONCLUSION

We have shown that quantum work, defined according to Eq. (2) may show statistical correlations that cannot be described by classical macro-realism. This follows from the violation of Leggett-Garg and entropic Leggett-Garg inequalities for work measurements. In a driven two-level system as well as in a harmonic oscillator subject to squeezing, these inequalities are violated for certain driving parameter and initial temperatures. When both can be evaluated, the entropic and the normal Leggett-Garg inequalities do not necessarily identify the same correlations as non-classical, as their violation is only a sufficient but not a necessary criterion to abandon macro-realism. This points to the interest in developing tighter bounds to rule out the violation of macroscopic realism over broader parameter ranges.

The study of temporal correlations and their implication for both foundational and practical questions resembles the situation in the 1950’s, where a multitude of optical phenomena could be described by stochastically fluctuating classical fields, but where the Hanbury-Brown and Twiss measurements of (classical) intensity correlations spurred discussions about the general validity of classical modelling. This led to the insight that temporal fluctuations in intensity measurements can, indeed, exclude classical descriptions of the light field, and it stimulated the emergence of quantum optics as a research field. Non-classical properties of light are, e.g., witnessed by temporal noise correlations that violate Cauchy-Schwarz inequalities, anti-bunching, and higher-order interference effects, which have in several cases turned out to be useful properties, e.g., for precision sensing.

In this spirit, our work is an attempt to quantify temporal quantum correlations involved in thermodynamic processes and might be relevant for the evaluation and design of work extraction protocols and (measurement based) quantum thermal machines, where it was shown recently that the efficiency of cyclic processes may non-trivially involve correlations between subsequent cycles.

Appendix A: Analytical formula for $G_{mn}(z)$

In this appendix, we show the explicit expression for the matrix element $G_{mn}(z)$ derived in [25–27]. It yields

$$G_{mn}(z) = \begin{cases} (-1)^{m/2} \frac{\sqrt{m!}}{\cosh r} \sum_{i=1}^{N} \frac{(-4)^i (\sinh z)^{i+1}}{(2m+2i)!(1/2m-i)!} \frac{(2 \cosh z)^{-(m+n)/2}}{(2m+2i)!(1/2m-i)!} & \text{for } m, n \text{ even} \\ (-1)^{(m-1)/2} \frac{\sqrt{m!}}{\cosh r} \sum_{i=1}^{N} \frac{(-4)^i (\sinh z)^{i+1}}{(2m+2i)!(1/2m-i)!} \frac{(2 \cosh z)^{-(m+n)/2}}{(2m+2i)!(1/2m-i)!} & \text{for } m, n \text{ odd} \\ 0 & \text{else} \end{cases}$$  \hspace{1cm} (A1)$$

where the summation ends at $N = \min\{m/2, n/2\}$ for $n, m$ even and $N = \min\{(m-1)/2, (n-1)/2\}$ for $n, m$ odd, respectively.

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