Effects of dark sectors’ mutual interaction on the growth of structures

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Abstract

We present a general formalism to study the growth of dark matter perturbations when dark energy perturbations and interactions between dark sectors are present. We show that the dynamical stability on the growth of structure depends on the form of coupling between dark sectors. By taking the appropriate coupling which enables the stable growth of structure, we find that the effect of the interaction between dark sectors overwhelms that of dark energy perturbation on the growth of dark matter perturbation. Due to the influence of the interaction, the growth index can differ from the value without interaction by an amount up to the observational sensibility, which provides an opportunity to probe the interaction between dark sectors through future observations on the growth of structure.

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I. INTRODUCTION

There has been growing observational evidence indicating that our universe has entered the epoch of accelerated expansion [1, 2]. Within the framework of Einstein gravity, this acceleration can be attributed to the so-called dark energy (DE) with negative pressure, which dominates the content of the universe at present. The leading interpretation of such DE is a cosmological constant with equation of state (EoS) $w = -1$ [3]. Though the cosmological constant is consistent with the observation data, at the fundamental level it fails to be convinced: the vacuum energy density falls far below the value predicted by any sensible quantum field theory by many orders of magnitude, and it unavoidably leads to the coincidence problem, i.e., “why are the vacuum and matter energy densities of precisely the same order today?” More sophisticated models have been proposed to replace the cosmological constant by a dynamical dark energy in the conjectures relating the DE either to a scalar field called quintessence with $w > -1$, or to an exotic field called phantom with $w < -1$. It would be fair to say that there is no clear winner in sight to explain the nature of DE at the moment. On the other hand, the remarkable discovery of the cosmic acceleration might also be doubted as the result of a modification of standard Einstein gravity at large distances. This happens in $f(R)$ theories [4] and in braneworld models [5]. How to distinguish such modified gravity theory from that of DE is an imperative task which will bring about not only a breakthrough in cosmology, but also in the field of high energy physics.

The DE is usually not supposed to clump on the scales of the largest cosmic structures, and the most powerful way where its nature can be unveiled is to investigate the expansion history of universe [6]. However, the current observations of cosmic expansion history cannot break the degeneracies among different approaches trying to explain the cosmic acceleration. Considering that the rate of expansion as a function of cosmic time can in turn affect the process of structure formation, an alternative channel to study the clustering properties of cosmic structure has been used to detect the possible effects of DE. It is expected that the growth history of dark matter (DM) perturbations can provide a complementary probe to distinguish the modified gravity from DE [7]. However, most of these studies have neglected the effect of DE perturbations. The DE perturbations do not affect the background evolution, but they are crucial in determining the DE clustering properties [9], which consequently have effects on the evolution of DM perturbations. Recently it has been found that for small sound speed and DE EoS prominently away from −1, the presence of DE perturbations may leave a significant imprint on the growth function of DM perturbations [8].

In the minimal picture, DM does not feel any significant interactions from DE. Although this picture is consistent with current observations, considering that DE accounts for a large fraction of the universe, it is natural, in the framework of field theory, to consider its interaction with the remaining fields in the universe. The possibility that
DE and DM can interact has been widely discussed in [10]-[30]. It has been shown that the coupling between DE and DM can provide a mechanism to alleviate the coincidence problem [10, 11, 13, 14]. In addition, it has been argued that an appropriate interaction between DE and DM can influence the perturbation dynamics and affect the lowest multipoles of the CMB angular power spectrum [15, 16]. Furthermore it was suggested that the dynamical equilibrium of collapsed structures such as clusters would be modified due to the coupling between DE and DM [19, 24]. In the presence of such coupling, there has been some concerns about the stability of the perturbations [21]. However, it was proved in [22] that the stability of the curvature perturbation depends on the forms of coupling between dark sectors. Since observational signatures on the dark sectors’ mutual interaction have been found in the probes of the cosmic expansion history [20, 25, 26], it is interesting to ask whether the interaction can have an effect on the growth of structure, whether it can provide a more consistent check for the coupling between dark sectors.

In this paper we are going to study the effect of the interaction between DE and DM on the growth function of DM perturbations. Recently there is an attempt on this study[27]. Here we will incorporate the DE perturbations in our study. We will restrict our investigation to constant sound speed as well as constant EoS of DE. From our formalism we will see that the growth of DM perturbations gets more modification due to the dark sectors’ interaction than that of the DE perturbations. This provides an opportunity to probe the interaction between DE and DM through the growth history of DM.

The organization of the paper is as follows. In the following section we will provide our analytical framework for the perturbation equations at the linear level. In Sec.III, we will present our numerical results and discuss the growth function of DM. We will give our conclusions and discussions in the last section.

II. ANALYTICAL FORMALISM

In this section we will derive the second order differential equations for the perturbations of DM and DE with couplings between them in a spatially flat Friedmann-Robertson-Walker (FRW) background. The perturbed space-time at first order reads,

$$ds^2 = a^2[-(1 + 2\psi)dt^2 + 2\partial_i B dx^i + (1 + 2\phi)\delta_{ij} dx^i dx^j + D_{ij} Edx^i dx^j],$$  \tag{1}

where $\psi, B, \phi, E$ is scalar metric perturbations, $a$ is the cosmic scale factor and $D_{ij} = (\partial_i \partial_j - \frac{1}{3}\delta_{ij} \nabla^2)$.

We work with general stress-energy tensor

$$T^{\mu\nu} = \rho U^\mu U^\nu + p(g^{\mu\nu} + U^\mu U^\nu),$$  \tag{2}

$$3$$
for a two-component system consisting of DE and DM. Each energy-momentum tensor satisfies the conservation law

\[ \nabla_\mu T^\mu_\nu (\lambda) = Q^\nu_\lambda \]  

where \( Q^\nu_\lambda \) denotes the interaction between different components and \( \lambda \) denotes either the DM or the DE sector. The perturbed energy-momentum tensor reads,

\[ \delta \nabla_\mu T^\mu_\nu (\lambda) = \frac{1}{a^2} \left\{ -2[p'_\lambda + 3H(p_\lambda + \rho_\lambda)]\psi + \delta p'_\lambda + (p_\lambda + \rho_\lambda)\theta_\lambda + 3H(\delta p_\lambda + \delta \rho_\lambda) + 3(p_\lambda + \rho_\lambda)\phi' \right\} = \delta Q^\nu_\lambda \]

\[ \partial_i \delta \nabla_\mu T^\mu_i (\lambda) = \frac{1}{a^2} \left\{ [p'_\lambda + H(p_\lambda + \rho_\lambda)]\nabla^2 B + [(p'_\lambda + \rho'_\lambda) + 4H(p_\lambda + \rho_\lambda)]\theta_\lambda \right. \\
+ (p_\lambda + \rho_\lambda)\nabla^2 B' + \nabla^2 \delta p_\lambda + (p_\lambda + \rho_\lambda)\theta'_\lambda + (p_\lambda + \rho_\lambda)\nabla^2 \psi \right\} = \partial_i \delta Q^i_\lambda \]

where \( \theta = \nabla^2 v \), \( v \) is the potential of three velocity and the prime denotes the derivative with respect to the conformal time \( \tau \). The perturbed Einstein equations yield, in the linear order

\[ \nabla^2 \phi + 3H(\nabla \psi - \phi') + H \nabla^2 B - \frac{1}{6} [\nabla^2]^2 E = -4\pi G a^2 \delta \rho \]

\[ \nabla^2 \psi - \nabla^2 \phi' + 2H^2 \nabla^2 B - \frac{a''}{a} \nabla^2 B + \frac{1}{6} [\nabla^2]^2 E' = -4\pi G a^2 (p + \rho) \theta \]

\[ -\partial^i \partial_\nu \psi - \partial^i \partial_\nu \phi + \frac{1}{2} \partial^i \partial_\nu E' + H \partial^i \partial_\nu E' + \frac{1}{6} \partial^i \partial_\nu [\nabla^2 E - 2H \partial^i \partial_\nu B - \partial^i \partial_\nu B'] = 8\pi G a^2 \Pi^i_j \]

where \( \delta \rho \) is the total energy perturbation, \( \delta \rho = \sum_\lambda \delta \rho_\lambda \) and \( (p + \rho) \theta = \sum_\lambda (p_\lambda + \rho_\lambda) \theta_\lambda \).

Considering an infinitesimal transformation on the coordinates[31],

\[ \tilde{x}^\mu = x^\mu + \delta x^\mu \]

\[ \delta x^0 = \xi^0(x^\mu) \]

\[ \delta x^i = \partial^i \beta(x^\mu) + v^i(x^\mu)(\partial_i v^i = 0) \]

the perturbed quantities behave as,

\[ \tilde{\psi} = \psi - \xi^0' - \frac{a'}{a} \xi^0 \]

\[ \tilde{B} = B + \xi^0 - \beta' \]

\[ \tilde{\phi} = \phi - \frac{1}{3} \nabla^2 \beta - \frac{a'}{a} \xi^0 \]

\[ \tilde{E} = E - 2\beta \]

\[ \tilde{v} = v + \beta' \]

\[ \tilde{\theta} = \theta + \nabla^2 \beta'. \]
Inserting eq(7) into the medium part of eq(4), we obtain the behavior of \( \delta Q^\mu \) on the right hand side of eq(4)

\[
\delta \tilde{Q}^0 = \delta Q^0 - Q^0 \xi^0 + Q^0 \xi^\nu Q^\nu
\]

\[
\delta \tilde{Q}_p = \delta Q_p + Q^0 \beta',
\]

(8)

where

\[
\delta Q^i = \partial^i \delta Q_p + \delta Q^i_t (\partial_t \delta Q^t = 0).
\]

and \( \delta Q_p \) is the potential of three vector \( \delta Q^i \). This is consistent with the results got from Lie derivatives,

\[
L_{\delta x} Q^\nu = \delta x^\sigma Q^\nu - Q^\sigma \delta x^\nu
\]

\[
\delta \tilde{Q}^\nu = \delta Q^\nu - L_{\delta x} Q^\nu,
\]

(9)

which shows that \( Q^\nu \) is covariant.

We expand the metric perturbations in Fourier space by using scalar harmonics[31],

\[
\tilde{\psi}^Y(s) = (\psi - \xi^0 - a' \xi^0) Y(s)
\]

\[
\tilde{B}^Y_i(s) = (B - k \xi^0 - \beta') Y_i(s)
\]

\[
\tilde{\phi}^Y(s) = (\phi - \frac{1}{3} k \beta - \frac{a'}{a} \xi^0) Y(s)
\]

\[
\tilde{E}^Y_{ij}(s) = (E + 2k \beta) Y_{ij}(s)
\]

\[
\tilde{\theta}^Y(s) = (\theta + k \beta') Y^Y(s)
\]

(10)

and the perturbed conservation equations eq(4) read

\[
\delta \nu' + 3H \frac{\delta p}{\rho} - w \delta \lambda = -(1 + w)k v - 3(1 + w)\phi' + (2\psi - \delta \lambda) \frac{a^2 Q^0}{\rho} + \frac{a^2 \delta Q^0}{\rho}
\]

\[
(v + B)' + H(1 - 3w)(v + B) = k \frac{a^2 Q^0}{\rho} - \frac{w'}{1 + w} (v + B) + \frac{a^2 Q^0}{\rho} v - \frac{a^2 Q^0}{1 + w} B + \frac{a^2 \delta Q^0}{(1 + w)\rho}
\]

(11)

The interaction \( Q^\nu,_{\lambda} \) can be decomposed into two parts,

\[
Q^\nu,_{\lambda} = Q_{\lambda} U^\nu,_{\lambda} + F^\nu_{\lambda}
\]

(12)

where \( U^\nu,_{(\lambda)} \) is the total four-velocity as defined in[31][21], \( F^\nu_{\lambda} = h^\nu_{\mu} Q^\mu,_{\lambda} \) is a spacial vector and vanishes in back ground \( F^\nu_{\lambda} = 0[22] \), \( h^\nu_{\mu} = g^\nu_{\mu} + U^\nu,_{(total)} U^\mu,_{(total)} \) is the projection operator. The perturbed quantities read

\[
\delta Q^0_\lambda = \delta Q_{\lambda} U^0,_{(total)} + Q_{\lambda} \delta U^0,_{(total)}
\]

\[
\delta Q_{p\lambda} = (Q_{\lambda} v,_{(total)} + f_{\lambda}) U^0,_{(total)}
\]

(13)
and Eq(11) can go back to Eq(20) and Eq(21) in [21], if we neglect the anisotropic stress \( \pi_\lambda \).

Constructing the gauge invariant quantities[31],

\[
\Psi = \psi - \frac{1}{k} \mathcal{H}(B + \frac{E'}{2k}) - \frac{1}{k}(B' + \frac{E''}{2k}) \\
\Phi = \phi + \frac{1}{6} E - \frac{1}{k} \mathcal{H}(B + \frac{E''}{2k}) \\
\delta \rho^I_\lambda = \delta \rho_\lambda - \rho_\lambda \left( \frac{v_\lambda + B}{k} \right) \\
\delta p^I_\lambda = \delta p_\lambda - \rho_\lambda \left( \frac{v_\lambda + B}{k} \right) \\
\Delta_\lambda = \delta_\lambda - \rho_\lambda \left( \frac{v_\lambda + B}{k} \right) \\
V_\lambda = v_\lambda - \frac{E}{2k} \\
\delta Q^I_\lambda = \delta Q_\lambda - \frac{Q^I_\lambda}{\mathcal{H}} \left( \phi + \frac{E}{6} \right) + Q^0_\lambda \left( \frac{1}{\mathcal{H}} (\phi + \frac{E}{6}) \right)' \\
\delta Q^I_{p\lambda} = \delta Q_{p\lambda} - Q^0_\lambda \frac{E'}{2k},
\]

the perturbed Einstein equations eq(5) become

\[
\Phi = 4\pi G a^2 \sum_\lambda \left( \Delta_\lambda + \frac{\alpha^2 Q^0_\lambda V_\lambda}{\rho_\lambda} \right) \rho_\lambda \\
k(\mathcal{H}\Psi - \Phi') = 4\pi G a^2 \sum_\lambda (\rho_\lambda + \rho_\lambda) V_\lambda \\
\Psi = -\Phi,
\]

where we have assumed that the pressure perturbation of DE is isotropic \( \Pi^i_\lambda = 0 \).

Using the gauge invariant quantities eq(14), we can obtain the linear perturbation equations for DM from eq(11),

\[
\begin{align*}
\Delta'_m + \left[ \frac{\rho'_m V'_m}{\rho_m k} \right] &= -kV_m - 3\Phi' + 2a^2 Q^0_m \Phi - \Delta_m \frac{a^2 Q^0_m V'_m}{\rho_m} - \Delta_m \frac{a^2 Q^0_m}{\rho_m k} \rho_m + \frac{a^2 \delta Q^0_m}{\rho_m} \\
&+ \frac{a^2 Q^0_m}{\rho_m} \Phi - \frac{a^2 Q^0_m}{\rho_m} \left( \frac{\Phi}{\mathcal{H}} \right)' \\
V'_m &= -\mathcal{H}V_m + k\Psi - \frac{a^2 Q^0_m V'_m}{\rho_m} + \frac{a^2 \delta Q^I_m}{\rho_m}
\end{align*}
\]

Considering the pressure perturbation of DE [21, 22]

\[
\frac{\delta p_d}{\rho_d} = C_e^2 \delta_d - (C_e^2 - C_a^2) \frac{\rho'_d v_d + B}{\rho_d k},
\]

where \( C_e^2 = \frac{\delta p_d}{\rho_d} \bigg|_{\text{rf}} \) is the sound speed in DE rest frame , \( C_a^2 = \frac{\rho'_v}{\rho_d} \) is the adiabatic sound speed , and further noting
Similarly for the DE perturbation we have

\[
\Delta'_d + \left[ \frac{\rho'_d V_d}{\rho_d k} \right]' + 3\mathcal{H}C^2_e (\Delta_d + \frac{\rho'_d V_d}{\rho_d k}) - 3\mathcal{H}(C^2_e - C^2_m)\frac{\rho'_d V_d}{\rho_d k} - 3\mathcal{H}(\Delta_d + \frac{\rho'_d V_d}{\rho_d k})\Phi = -k(1+w)V_d -3(1+w)\Phi + 2\Psi \frac{a^2Q^0}{\rho_d} \frac{\rho'_d V_d}{\rho_d k} \frac{\rho'_d}{\rho_d} + \frac{a^2\delta Q^0_d}{\rho_d} \frac{\rho'_d}{\rho_d} \Phi \frac{2\Psi}{\rho_d} \frac{a^2\delta Q^0_d}{\rho_d} \frac{\rho'_d}{\rho_d} \Phi - \frac{a^2Q^0}{\rho_d} \frac{\rho'_d}{\rho_d} \Phi
\]

\[
(19)
\]

Inserting eq. (17) into eq. (16) to eliminate \( V_m \), in the subhorizon approximation \( k \gg aH \), we obtain the second order equation for the DM perturbation

\[
\Delta''_m = -\mathcal{H} \left[ \frac{2a^2Q^0}{\rho_m} \right] \Delta'_m + ( - \Delta_m \frac{a^2Q^0}{\rho_m} + \frac{a^2\delta Q^0_d}{\rho_m}) \mathcal{H} \Delta_m = \frac{a^2Q^0}{\rho_m} + \left[ \frac{a^2Q^0}{\rho_m} \right] \mathcal{H} \Delta_m = \frac{a^2Q^0}{\rho_m} + \left[ \frac{a^2Q^0}{\rho_m} \right] \Delta_m
\]

\[
(20)
\]

Similarly for the DE perturbation we have

\[
\Delta''_d = -3\mathcal{H}C^2_e \Delta_d - \frac{a^2Q^0}{\rho_d} \frac{\rho'_d}{\rho_d} \Delta_d + \left\{ \mathcal{H}(1-3w) - \frac{w}{1+w} \frac{\rho'_d}{\rho_d} \right\} \mathcal{H} \Delta_d = \frac{3\mathcal{H}C^2_e - 6wH + \frac{a^2Q^0}{\rho_d} \frac{\rho'_d}{\rho_d} \Delta_d - k(\frac{a^2Q^0}{\rho_d}) \frac{\rho'_d}{\rho_d} \Delta_d = \frac{3\mathcal{H}C^2_e - 6wH + \frac{a^2Q^0}{\rho_d} \frac{\rho'_d}{\rho_d} \Delta_d - k(\frac{a^2Q^0}{\rho_d}) \frac{\rho'_d}{\rho_d} \Delta_d
\]

\[
+3(\mathcal{H} + (1+w)\Psi + \frac{a^2\delta Q^0_d}{\rho_m} \left[ \frac{2a^2Q^0}{\rho_m} \right] \mathcal{H} \Delta_d - \frac{w}{1+w} \frac{\rho'_d}{\rho_d} \frac{a^2Q^0}{\rho_m} \right) \Delta_d = \frac{\Delta_d - k(\frac{a^2Q^0_d}{\rho_d}) \frac{\rho'_d}{\rho_d} \Delta_d - \frac{3\mathcal{H}C^2_e - 6wH + \frac{a^2Q^0}{\rho_d} \frac{\rho'_d}{\rho_d} \Delta_d - k(\frac{a^2Q^0}{\rho_d}) \frac{\rho'_d}{\rho_d} \Delta_d
\]

\[
+3 \left[ \mathcal{H} + \frac{a^2\delta Q^0_d}{\rho_d} \frac{\rho'_d}{\rho_d} \right] \Delta_d = \frac{\Delta_d - k(\frac{a^2Q^0}{\rho_d}) \frac{\rho'_d}{\rho_d} \Delta_d - \frac{3\mathcal{H}C^2_e - 6wH + \frac{a^2Q^0}{\rho_d} \frac{\rho'_d}{\rho_d} \Delta_d - k(\frac{a^2Q^0}{\rho_d}) \frac{\rho'_d}{\rho_d} \Delta_d
\]

Changing the conformal time into the cosmic proper time, we can rewrite the above second order equations as

\[
\tilde{\Delta}_d + 2 \left( \mathcal{H} + \frac{a^2Q^0}{\rho_m} \right) \tilde{\Delta}_d + \left[ \frac{a^2Q^0}{\rho_m} \mathcal{H} + \left( \frac{a^2Q^0}{\rho_m} \right)^2 + \frac{1}{a} \left( \frac{a^2Q^0}{\rho_m} \right)^2 \frac{\rho'_d}{\rho_d} \right] \tilde{\Delta}_d = \mathcal{H} \Delta_d = \mathcal{H} \Delta_d
\]

\[
(21)
\]

\[
\tilde{\Delta}_d + \left\{ \mathcal{H}(1-3w) - \frac{w}{1+w} \frac{\rho'_d}{\rho_d} \right\} \mathcal{H} \Delta_d = \left\{ \mathcal{H}(1-3w) - \frac{w}{1+w} \frac{\rho'_d}{\rho_d} \right\} \mathcal{H} \Delta_d
\]

\[
+3 \left[ \tilde{\mathcal{H}} + \mathcal{H} \frac{a^2\delta Q^0_d}{\rho_m} \frac{\rho'_d}{\rho_d} \right] \Delta_d = \frac{\tilde{\mathcal{H}} + \mathcal{H} \frac{a^2\delta Q^0_d}{\rho_m} \frac{\rho'_d}{\rho_d} \Delta_d}{\mathcal{H} \Delta_d}
\]

where the dot denotes the derivative with respect to the proper time \( t \). For the convenience in the following discussions, we can further rewrite the second order perturbation equations for DE and DM into dimensionless form

\[
\frac{d^2\ln \Delta_m}{d\ln a}^2 = \frac{1}{2} - \frac{3}{2} (1 - \Omega_m) \right\} \frac{d\ln \Delta_m}{d\ln a} + \frac{1}{2} \mathcal{H} \Delta_m - \frac{a^2Q^0}{\rho_m} e^{-\ln \Delta_m} - \frac{2a^2Q^0}{\rho_m} \frac{d\ln \Delta_m}{d\ln a} + \frac{1}{a} \frac{d}{d\ln a} \left( \frac{a^2Q^0}{\rho_m} \right) e^{-\ln \Delta_m}
\]

\[
+ \frac{3}{2} \left| \Omega_m \Delta_m + (1 - \Omega_m) \right| \Delta_d e^{-\ln \Delta_m}
\]

\[
(24)
\]
\[
\frac{d^2 \ln \Delta_d}{d \ln a^2} + \left( \frac{d \ln \Delta_d}{d \ln a} \right)^2 + \left[ \frac{1}{2} - \frac{3}{2} w (1 - \Omega_m) \right] \frac{d \ln \Delta_d}{d \ln a} + \left[ 3 C_e^2 - 6 w + \frac{2 a Q_0^d}{H \rho_d} \right] \frac{d \ln \Delta_d}{d \ln a} \\
= -3 \left( \frac{1}{H} \frac{d H}{d \ln a} + 1 \right) C_e^2 - \frac{1}{a H} \frac{d}{d \ln a} \left( \frac{a^2 Q_0^d}{H \rho_d} \right) + \left[ 1 - 3w - \frac{w}{1 + w \rho_d} \frac{1}{d \rho_d} + \frac{a Q_0^d}{H \rho_d} \right] \times \left\{ -3 C_e^2 + 3 w - \frac{a Q_0^d}{H \rho_d} \right\} \\
- \frac{k \delta Q_{\rho d}^f}{\rho_d H^2} e^{-\ln \Delta_d} - \frac{k^2 C_e^2}{a^2 H^2} + (1 + w) \frac{3}{2} \left( \Omega_m \Delta_m + (1 - \Omega_m) \Delta_d \right) e^{-\ln \Delta_d} + 3 \left[ \frac{d}{d \ln a} + \frac{w}{1 + w \rho_d} \frac{1}{d \rho_d} \right] \\
+ \frac{a \delta Q_{\rho d}^f}{H \rho_d} \left[ 1 - 3w - \frac{w}{1 + w \rho_d} \frac{1}{d \rho_d} \right] + \frac{a Q_0^d}{H \rho_d} \left( \frac{a^2 \delta Q_{\rho d}^f}{\rho_d} \right) e^{-\ln \Delta_d}. \tag{25}
\]

In the subhorizon approximation, from the perturbed Einstein equations eq(15) we can get the “Poisson equation”

\[
-k^2 = \frac{3}{2} H^2 \left( \Omega_m \Delta_m + (1 - \Omega_m) \Delta_d \right) \tag{26}
\]

where we have used Friedmann equation in the derivation. This equation can be used to build the bridge between the matter perturbations to the metric perturbations.

Since the nature of DE and DM remains unknown, it will not be possible to derive the precise form of the interaction between them from the first principle. One has to assume a specific coupling from the outset [11, 29, 30] or determine it from phenomenological requirements [12, 25]. From eq(8) and eq(9), we know that \( Q^\nu \) is a covariant vector, which does not need to depend on the four velocity. For the generality, we can assume the phenomenological description of the interaction between dark sectors in the comoving frame as

\[
Q_m^\nu = \left[ \frac{3 H}{a^2} (\delta_1 \rho_m + \delta_2 \rho_d), 0, 0, 0 \right]^T \\
Q_d^\nu = \left[ \frac{3 H}{a^2} (\delta_1 \rho_m + \delta_2 \rho_d), 0, 0, 0 \right]^T, \tag{27}
\]

where \( \delta_1, \delta_2 \) are small positive dimensionless constants and superindex \( T \) is the transpose of the vector. Choosing positive sign in the interaction, one can ensure the direction of energy transfer from DE to DM, which is required to alleviate the coincidence problem [17, 23] and avoid some unphysical problems such as negative DE density etc [21, 25]. In the subhorizon approximation \( k >> aH \),

\[
\delta Q_m^{\nu} \approx \frac{3 H}{a^2} (\delta_1 \delta_\rho_m + \delta_2 \delta_\rho_d) \\
\delta Q_d^{\nu} \approx \frac{3 H}{a^2} (\delta_1 \delta_\rho_m + \delta_2 \delta_\rho_d) \\
\delta^\nu_m \approx \delta \rho_m \\
\delta^\nu_d \approx \delta \rho_d. \tag{28}
\]

Further noting that the gauge-invariant momentum transfer \( \delta Q_{p\lambda}^I \) refers to the intrinsic momentum transfer between dark sectors and as explained in [31], such intrinsic momentum transfer is due to the collision of particles from
different fluids. Such collision can produce acoustics in DM fluid as well as pressure which may resist the squeeze of
the attraction of gravity and hinder the growth of gravity fluctuations during tightly coupled photon baryon period,
we therefore set \( \delta Q_{\rho m} \approx \delta Q_{\rho d} \approx 0 \)[22]. This is a choice of interaction and the result should not heavily depend on such
setting.

Employing the above interaction form, we can finally arrive at perturbation equations for dark sectors with constant
EoS of DE

\[
\frac{d^2 \ln \Delta_m}{d \ln a^2} = - \left( \frac{d \ln \Delta_m}{d \ln a} \right)^2 - \left[ \frac{1}{2} - \frac{3}{2} w(1 - \Omega_m) \right] \frac{d \ln \Delta_m}{d \ln a} - \left( 3 \delta_1 + \frac{6 \delta_2}{r} \right) \frac{d \ln \Delta_m}{d \ln a} + \frac{3 \delta_2}{r} \frac{d \ln \Delta_d}{d \ln a} \exp(\ln \frac{\Delta_d}{\Delta_m}) \\
+ \frac{3[\exp(\ln \frac{\Delta_d}{\Delta_m}) - 1]}{r} \left( \delta_2 + 3 \delta_1 \delta_2 + 3 \frac{\delta_2}{r} (1 + \frac{H'}{H} + 1) - \frac{\delta_2 r'}{r} \right) + \frac{3}{2} \left[ \Omega_m + (1 - \Omega_m) \exp(\ln \frac{\Delta_d}{\Delta_m}) \right]
\]

(29)

\[
\frac{d^2 \ln \Delta_d}{d \ln a^2} = - \left( \frac{d \ln \Delta_d}{d \ln a} \right)^2 - \left[ \frac{1}{2} - \frac{3}{2} w(1 - \Omega_m) \right] \frac{d \ln \Delta_d}{d \ln a} + \frac{3}{2} \left[ \Omega_m \exp(\ln \frac{\Delta_m}{\Delta_d}) + (1 - \Omega_m) \right] - \frac{k^2 C_e^2}{a^2 H^2} \\
+ \left[ 3 \delta_2 + 6 \delta_1 r + 6 w - 3 C_a^2 + 3 (C_e^2 - C_a^2) \delta_1 r + \frac{\delta_2}{1 + w} + \frac{C_e^2 C_p^4}{1 + w \rho_d} \right] \frac{d \ln \Delta_d}{d \ln a} \\
- 3 \delta_1 r \exp(\ln \frac{\Delta_m}{\Delta_d}) \frac{d \ln \Delta_m}{d \ln a} + \left[ \frac{H'}{H} + 1 \right] (w - C_e^2) + 3 \delta_1 \left( \frac{H'}{H} + 1 \right) (r + r') - \delta_1 \left[ \frac{H'}{H} + 1 \right] (r + r') \exp(\ln \frac{\Delta_m}{\Delta_d}) \\
+ 3 \left[ w - C_e^2 + \delta_1 r (1 - \exp(\ln \frac{\Delta_m}{\Delta_d})) \right] \left[ (1 - 3w) - \frac{3 C_e^2 - C_a^2}{1 + w} (1 + w + \delta_1 r + \delta_2) - 3 (\delta_1 r + \delta_2) - \frac{C_e^2 C_p^4}{1 + w \rho_d} \right],
\]

(30)

where \( r = \rho_m / \rho_d \) and the prime denotes \( d/d \ln a \).

III. NUMERICAL RESULTS

In this section we present numerical results of solving the above perturbation equations. We concentrate on the
behavior of the evolution of the DM perturbation.

It was argued that the presence of the interaction between DE and DM may give rise to dynamical instabilities in
the growth of structure [32]. In our general form of the phenomenological interaction, when we choose the coupling
between dark sectors in proportion to the DE energy density by setting \( \delta_2 = 0 \) while keeping \( \delta_1 \) nonzero, we do find
the consistent fast growth of the fluctuations of DM, as seen in Fig. 1. However, when we choose the dark sectors’
mutual interaction in proportion to the energy density of DE by taking \( \delta_1 = 0 \) and \( \delta_2 \neq 0 \), we have the stable DM
perturbation. This result tells us that the stability in the growth of structure depends on the type of coupling between
dark sectors, which is consistent with the findings in the curvature perturbation in [22].

In the following we focus on the interaction proportional to the energy density of DE to keep the stability of the
matter density perturbation. Initial conditions are given at the redshift \( z = 3200 \) where approximately is the time
of matter-radiation equality, and we take the adiabatic initial conditions and assume zero initial time derivatives of
DM and DE perturbations. In our analysis we do not allow $w < -1$. As to $k$, we choose its value above $0.01h\text{Mpc}^{-1}$ so that there is large scale structure data on the matter power spectrum [3, 8]. For the sound speed of DE, $C_e^2$, we restrict it to be positive and smaller than unity. In Fig 2, we show the evolution of perturbations of DM and DE for different values of parameters $C_e^2, w, k$ and the strength of the coupling $\delta_2$ between dark sectors.

Without the interaction between dark sectors, we observed that there is the sensible influence of the DE perturbation on the evolution of DM perturbation when the sound speed is tiny enough and the EoS of DE substantially deviates from $-1$. In Fig 2 a, we can see this qualitative behavior that for smaller $C_e^2$, the DE perturbation grows. If we take $w$ further away from $-1$, it will grow more and will influence the DM perturbation. In Fig 2 b, it shows that for fixed $C_e^2$, DE perturbation grows when $w$ deviates from $-1$. This is consistent with the result in [8]. However, when $C_e^2$ is not so tiny and $w$ close to $-1$, in the subhorizon approximation, we observed that the influence of the DE perturbation is suppressed. For bigger $k$, DE influence is even smaller, see Fig 2 c. This result will not change when the interaction presents. The influence of the interaction between dark sectors can start to appear in the very recent epoch. In Fig 2 d, it shows there in the red circle the small deviation caused by the coupling between dark sectors in the matter density perturbation at recent time.

To see more clearly of the influence of different parameters on the growth history of the DM perturbation, we
introduce the growth index $\gamma$ with the definition [8]

$$\gamma_m = (\ln \Omega_m)^{-1} \ln \left( \frac{a}{\Delta_m} \frac{d\Delta_m}{da} \right).$$  \hspace{1cm} (31)

The growth index is generically not constant which was first emphasized and investigated in terms of cosmological parameters in [34]. The growth index has been argued as a useful way in principle to distinguish the modified gravity models from DE models [7, 35]. In Fig 3, we show that the influence of the DE perturbation on the growth index for the lack of interaction between dark sectors. The red lines in the figure mark the result without DE perturbation. In Fig 3a, we see that for fixed DE EoS $w$, when $C_e^2$ decreases, the growth index with the DE perturbation deviates
Figure 3: These figures illustrate the behaviors of dark fluctuations in different cases when varying the effective sound speed, dark energy EoS, wave number, and coupling. The solid lines are for the DM perturbation while the dotted lines are for the DE perturbation.

more from the result without DE perturbation. This deviation will be even more prominent when DE EoS is further away from $-1$ as shown in Fig. 3 b. We can see clearly that the difference between the growth index with and without DE perturbation can be as big as 0.03. Our numerical result further supports what found in [8]. In Fig 3 c, d, we observed that in the subhorizon approximation the value of $k$ does not influence as much as the parameters $w, C_e^2$ on the result of the growth index.

In Fig 4, we present our numerical results when we incorporate the interaction between DE and DM. The solid lines are for the results with DE perturbation, while the dotted lines are for the results without DE perturbation.
Figure 4: The growth index behavior when the interaction between DE and DM presents. Solid lines are for the result with DE perturbation, while dotted lines are for the result without DE perturbation.

It is clearly shown that the growth index got more influenced from the interaction between dark sectors than the DE perturbation. If we take the best fitting value of $\delta_2$ from observations of the expansion history of the universe, $\delta_2 \sim 10^{-2}$ [25, 26], its influence on the growth index will even overwhelm that of the DE perturbation. Although the enhancement of the growth index due to the DE perturbation and the interaction shown in Fig. 3, 4 is clear, the available accuracy from the observations such as DUNE[28] etc. is calculated for the ΛCDM model and may not be true for our interacting cosmology, since the current values of $\Omega_{c0}, \delta_{c0}$ may not typically be equal in the two cases as argued in [33]. However, this phenomenon is interesting, as it opens the possibility that the future measurement of the growth factor may be helpful to reveal the presence of the interaction between DE and DM.

**IV. CONCLUSIONS AND DISCUSSIONS**

The cosmological observations have provided a firm evidence for significant physics beyond standard models. It is clear now that the formation of structure in the universe demands DM and the accelerated expansion of our universe requires some kind of DE or a significant infrared modification on Einstein gravity. DE and DM are two major components of the cosmic energy budget. It is reasonable to explore possible interactions between them in the framework of field theory.

In this paper we have concentrated on the time evolution of the DM perturbation. We have derived general equations to describe the perturbations of DM and DE which incorporates the interactions between them. It was argued that the interaction between dark sectors might give rise to dynamical instabilities on the growth of structure.
However, we observed that this instability depends on the form of the coupling. For example, if we choose the interaction in proportion to the energy density of DE, we did observe the stable growth of structure. This result is consistent with what is found in the curvature perturbation [22].

Besides the interaction between dark sectors, we have also discussed the effects of the nonvanishing DE perturbation on the evolution of DM perturbation. Usually in the discussion of the growth of structure, the DE perturbation was neglected. For the minimal picture without the coupling between dark sectors, the influence of the DE perturbation on the growth function of DM perturbation has been examined in [8]. In our general formalism, by taking the appropriate coupling between dark sectors which enables the stable growth of structure, we have found that the effect of the interaction between dark sectors overwhelms that of the DE perturbation on the growth function of DM perturbation. When the DE EoS $w$ is in the vicinity of $-1$ somewhere around the best fitted value at the moment, the DE perturbation is suppressed, however, when the interaction presents, the growth index can differ from the value without interaction by a big amount up to the observational sensibility. This provides an interesting way to probe the interaction between dark sectors through the observations on the growth of structure in large scale [25, 26].

It would be of great interest to confront our theoretical work to observations to constrain the interaction between dark sectors. However, the data available on the growth of structure are still poor and there is still a long way to go before we can talk about precision cosmology in this respect. We will leave our work in this direction in the future. On the other hand, it is also very interesting to include the interaction between dark sectors to modify the code in studying the N-body cosmological simulations on the structure formation. Work in this direction is in progress.

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