Beyond Octonions

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Abstract

We investigate Clifford Algebras structure over non-ring division algebras. We show how projection over the real field produces the standard Atiyah-Bott-Shapiro classification.

Quaternions and octonions may be presented as a linear algebra over the field of real numbers \( \mathbb{R} \) with a general element of the form

\[
Y = y_0e_0 + y_ie_i, \quad y_0, y_i \in \mathbb{R}
\]  

where \( i = 1, 2, 3 \) for quaternions \( \mathbb{H} \) and \( i = 1..7 \) for octonions \( \mathbb{O} \). We always use Einstein’s summation convention. The \( e_i \) are imaginary units, for quaternions

\[
e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k, \quad (2)
\]

\[
e_i e_0 = e_0 e_i = e_i, \quad (3)
\]

\[
e_0 e_0 = e_0, \quad (4)
\]

where \( \delta_{ij} \) is the Kronecker delta and \( \epsilon_{ijk} \) is the three dimensional Levi–Cevita tensor, as \( e_0 = 1 \) when there is no confusion we omit it. Octonions have the same structure, only we must replace \( \epsilon_{ijk} \) by the octonionic structure

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constant $f_{ijk}$ which is completely antisymmetric and equal to one for any of the following three cycles

$$123, 145, 176, 246, 257, 347, 365.$$  

(5)

The important feature of real, complex, quaternions and octonions is the existence of an inverse for any non-zero element. For the generic quaternionic or octonionic element given in (1), we define the conjugate $Y^*$ as an involution $(Y^*)^* = Y$, such that

$$Y^* = y_0 e_0 - y_i e_i,$$

(6)

introducing the norm as $N(Y) \equiv \|Y\| = YY^* = Y^*Y$ then the inverse is

$$Y^{-1} = \frac{Y^*}{\|Y\|}.$$  

(7)

The Norm is nondegenerate and positively definite. We have the decomposition property

$$\|XY\| = \|X\| \|Y\|$$  

(8)

$N(xy)$ being nondegenerate and positive definite obeys the axioms of the scalar product.

Going to higher dimensions, we define “hexagonions” ($X$) by introducing a new element $e_8$ such that

$$X = Q_1 + Q_2 e_8$$

$$= x_0 e_0 + \ldots + x_{16} e_{16}, \quad x_\mu \in \mathbb{R}$$

(9)

and

$$e_i e_j = -\delta_{ij} + C_{ijk} e_k.$$  

(10)

Now, we have to find a suitable form of the completely antisymmetric tensor $C_{ijk}$. Recalling how the structure constant is written for octonions

$$Q = Q_1 + Q_2 e_4$$

$$= x_0 e_0 + \ldots + x_7 e_7,$$

(11)

where $Q$ are quaternions, we have already chosen the convention $e_1 e_2 = e_3$ which is extendable to (11). We set $e_1 e_4 = e_5$, $e_2 e_4 = e_6$ and $e_3 e_4 = e_7$, but we still lack the relationships between the remaining possible triplets, $\{e_1, e_6, e_7\}; \{e_2, e_5, e_7\}; \{e_3, e_5, e_6\}$ which can be fixed by using

$$e_1 e_6 = e_1 (e_2 e_4) = -(e_1 e_2) e_4 = -e_3 e_4 = -e_7,$$

$$e_2 e_5 = e_2 (e_1 e_4) = -(e_2 e_1) e_4 = +e_3 e_4 = +e_7,$$

$$e_3 e_5 = e_3 (e_1 e_4) = -(e_3 e_1) e_4 = -e_2 e_4 = -e_6.$$
These cycles define all the structure constants for octonions. Returning to $\mathbb{X}$, we have the seven octonionic conditions, and the decomposition (9). We set $e_1e_8 = e_9$, $e_2e_8 = e_A$, $e_3e_8 = e_B$, $e_4e_8 = e_C$, $e_5e_8 = e_D$, $e_6e_8 = e_E$, $e_7e_8 = e_F$ where $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$ and $F = 15$. The other elements of the multiplication table may be chosen in analogy with (11). Explicitly, the 35 hexagonionic triplets are

\[(123), \ (145), \ (246), \ (347), \ (176), \ (365), \ (189), \ (28A), \ (38B), \ (48C), \ (58D), \ (68E), \ (78F), \ (1BA), \ (1DC), \ (1EF), \ (29B), \ (2EC), \ (2FD), \ (3A9), \ (49D), \ (4AE), \ (4BF), \ (3FC), \ (3DE), \ (5C9), \ (5AF), \ (5EB), \ (6FD), \ (6CA), \ (6BD), \ (79E), \ (7DA), \ (7CB).\]

This can be extended for any generic higher dimensional $\mathbb{F}^n$.

It can be shown by using some combinatorics that the number of such triplets $N$ for a general $\mathbb{F}^n$ algebra is $(n > 1)$

\[N = \frac{(2^n - 1)!}{(2^n - 3)! \cdot 3!}, \quad (12)\]

giving

| $\mathbb{F}^n$ | $n$ | $\dim$ | $N$ |
|---------------|-----|--------|-----|
| $\mathbb{Q}$  | 2   | 4      | 1   |
| $\mathbb{O}$  | 3   | 8      | 7   |
| $\mathbb{X}$  | 4   | 16     | 35  |

and so on.

One may notice that for any non-ring division algebra ($\mathbb{F}$, $n > 3$), $N > \dim(\mathbb{F}^n)$ except when $\dim = \infty$, i.e. a functional Hilbert space with a Cliff(0, $\infty$) structure.

It is clear that for any ring or non-ring division algebras, $e_i, e_j \in \mathbb{F}^n$, we have

\[\{e_i, e_j\} = -2\delta_{ij}. \quad (13)\]

As we explained in [1] and [2], treating quaternions and octonions as elements of $R^4$ and $R^8$ respectively, we can find the full set of matrices $R(4)$ and $R(8)$ that corresponds to any elements $e_i$ explicitly

For quaternions $e_i \leftrightarrow (E_i)_{\alpha\beta} = \delta_{\alpha\beta} - \delta_{ij} \delta_{\alpha0} + e_{i\alpha\beta}$,

\[\{E_i, E_j\} = -2\delta_{ij} \quad i, j = 1..3, \quad \alpha, \beta = 1..4,\]

For octonions $e_i \leftrightarrow (E_i)_{\alpha\beta} = \delta_{\alpha\beta} - \delta_{ij} \delta_{\alpha0} + f_{i\alpha\beta}$,

\[\{E_i, E_j\} = -2\delta_{ij} \quad i, j = 1..7, \quad \alpha, \beta = 1..8\]
Following, the same translation idea projecting our algebra $\mathbb{X}$ over $\mathbb{R}^{16}$, any $E_i$ is given by a relation similar to that given in (14),

$$\left( E_i \right)_{\alpha\beta} = \delta_{i\alpha} \delta_{\beta 0} - \delta_{i\beta} \delta_{\alpha 0} + C_{i\alpha\beta}. \tag{15}$$

But contrary to quaternions and octonions, the Clifford algebra (over the real field $\mathbb{R}^{16}$) closes only for a subset of these $E_i$’s, namely

$$\{E_i, E_j\} = -2\delta_{ij} \quad \text{for} \quad i, j, k = 1 \ldots 8 \quad \text{not} \quad 1 \ldots 15. \tag{16}$$

Because we have lost the ring division structure. We can find easily that another ninth $E_i$ can be constructed, in agreement with the Clifford algebra classification [3]. There is no standard $16$ dimensional representation for $\text{Cliff}$ (15). Following this procedure, we can give a simple way to write real Clifford algebras over any arbitrary Euclidean dimensions.

Sometimes, a specific multiplication table may be favored. For example in soliton theory, the existence of a symplectic structure related to the bi-hamiltonian formulation of integrable models is welcome. It is known from the Darboux theorem, that locally a symplectic structure is given up to a minus sign by

$$J_{\text{dim} \times \text{dim}} = \begin{pmatrix} 0 & -1_{\text{dim}} \\ 1_{\text{dim}} & 0 \end{pmatrix}, \tag{17}$$

this fixes the following structure constants

$$C\left( \frac{\text{dim}}{2} \right)_{1(\frac{\text{dim}}{2} + 1)} = -1, \tag{18}$$

$$C\left( \frac{\text{dim}}{2} \right)_{2(\frac{\text{dim}}{2} + 2)} = -1, \tag{19}$$

$$\vdots$$

$$C\left( \frac{\text{dim}}{2} \right)_{(\frac{\text{dim}}{2} - 1)(\text{dim} - 1)} = -1, \tag{20}$$

which is the decomposition that we have chosen in (14) for octonions

$$C_{415} = C_{426} = C_{437} = -1. \tag{22}$$

\footnote{Look to [2] for a non standard representation.}
Generally our symplectic structure is
\[
(1|E(E_{\frac{dim}{2}}))_{\alpha\beta} = \delta_{0\alpha}\delta_{\beta}E_{\frac{dim}{2}} - \delta_{0\beta}\delta_{\alpha}E_{\frac{dim}{2}} - \epsilon_{\alpha\beta}E_{\frac{dim}{2}}.
\] (23)

Moreover some other choices may exhibit a relation with number theory and Galois fields \[4\]. It is highly non-trivial how Clifford algebraic language can be used to unify many distinct mathematical notions such as Grassmanian \[5\], complex, quaternionic and symplectic structures.

The main result of this section, the non-existence of standard associative 16 dimensional representation of \textit{Cliff} \((0, 15)\) is in agreement with the Atiyah–Bott–Shapiro classification of real Clifford algebras \[3\]. In this context, the importance of ring division algebras can also be deduced from the Bott periodicity \[6\].

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