On the group velocity of oscillating neutrino states and the equal velocities assumption

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Abstract

In a two flavour world, the usual equal $p$, equal $E$, or equal $v$ assumptions used to derive the neutrino oscillation length can be replaced by a specific definition of the velocity of the oscillating state which, unlike those assumptions, is compatible with exact production kinematics. This definition is further vindicated by the analysis of $\pi_{\mu2}$ decay at rest.

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1 Introduction

Contradictory arguments are still exchanged in the discussion of neutrino oscillations in vacuum not to mention more complicated cases. People keep arguing about the superiority of their favourite assumption (equal $p$'s, $E$'s, or $v$'s \[3\]) in deriving an oscillation length. Some of the champions of the last approximation even claim that the usual formula must be corrected by a factor of two \[4\] while defenders of the standard result have spent some time showing that the equal $v$'s is kinematically untenable \[1,2\].

Although this last assumption is more at variance with exact production kinematics than the other two, treating it with care helps to uncover a weaker hypothesis which is both sufficient and compatible with kinematics, at least in a two flavour - two mass neutrino world.

Using the naive plane-wave model in this simple case, we shall show that:

1. The equal $v$ assumption does yield the same oscillation length as the other two provided velocity is treated consistently, i.e. the same value is used in the momentum-energy and in the position-time relations.

2. To derive this oscillation length, the only necessary assumption which is compatible with exact kinematics, and is, in some way, contained in the three usual approximations is a specific definition of the group velocity of the would-be wave packet.

3. This value of the group velocity must be postulated, but is vindicated by the equality of the derived oscillation length with that found by more elaborate methods; moreover, it seems difficult to avoid in the case of $\pi \to \mu \nu$ at rest.

2 Standard and 'equal velocities' assumptions

Being confined to a two mass world, we shall use simplified notations from the start and represent the neutrino born at space-time point $(0,0)$ in some charged current reaction involving charged lepton $l = e$ or $\mu$ by:

$$|0,0> = |\nu_l> = \cos \theta|1> + \sin \theta|2>$$  (1)

where $|h> = h = 1,2$ are mass eigenstates with eigenvalues $m_h$ and definite energy and momentum $E_h$ and $p_h$.

Assuming only the existence of a momentum operator commuting with the
Hamiltonian, \( H \) can be propagated by use of the space-time translation operator \( U = e^{-i(Ht - \vec{p} \cdot \vec{r})} \) which yields:

\[
U|0, 0 >= |x, t >= \cos \theta e^{-i(E_1 t - p_1 x)}|1 > + \sin \theta e^{-i(E_2 t - p_2 x)}|2 >
\]  

(2)

We have taken the \( x \) axis along the direction of motion. The values of \( E_h \) and \( p_h \) depend on the production kinematics; in the case of a \( \pi^0 \to 2 \) decay for example, they are fixed by the masses in the \( \pi \) rest frame and from there in the lab, once the \( \pi \)'s decay angle and velocity are given. Projection of \( |x, t > \) onto \( |\nu l' > = -\sin \theta |1 > + \cos \theta |2 > \) gives then

\[
A_{l \to l'}(x, t) = \sin \theta \cos \theta (e^{-i(E_2 t - p_2 x)} - e^{-i(E_1 t - p_1 x)})
\]  

(3)

for the amplitude to detect a neutrino of flavor \( l' \neq l \) at point \( x \) and time \( t \) relative to the production point at \( (0, 0) \), if that neutrino undergoes a C.C. interaction. The object of interest is the phase difference \( \delta \phi = \delta E t - \delta px \) (with \( \delta E = E_1 - E_2 \) and \( \delta p = p_1 - p_2 \)) which appears in the oscillation probability obtained by squaring (3):

\[
P_{l \to l'}(x, t) = \sin^2(2\theta) \sin^2(\frac{\delta \phi}{2})
\]  

(4)

To evaluate \( \delta \phi \), people postulate equal momenta, or equal energies, (see e.g. \[6\]) or even equal velocities (e.g. \[4\]) of the two definite mass components. We first show that all three assumptions yield the same expression without further approximations:

### 2.1 Equal energies

It is assumed here that \( E_1 = E_2 \), hence \( \delta \phi = -\delta px \). However the relation:

\[
\delta p^2 = \delta E^2 - \delta m^2
\]  

(5)

yields in this case:

\[
\delta p = -\frac{\delta m^2}{p_1 + p_2} = -\frac{\delta m^2}{2\bar{p}} \quad \text{and} \quad \delta \phi = \frac{\delta m^2}{2\bar{p}} x
\]

with an obvious definition for \( \bar{p} \).

Requiring then \( \delta \phi = 2\pi \) we find the well known result:

\[
L_{osc} = \frac{4\pi \bar{p}}{|\delta m^2|}
\]  

(6)
2.2 Equal momenta

Here, $\delta \phi = \delta E t$ and (11) yields $\delta E = \frac{\delta m^2}{E_1 + E_2}$. If $v$ is the velocity of the center of the would-be wave packet, one finds:

$$\delta \phi = \delta E t = \frac{\delta m^2}{E_1 + E_2} \frac{t}{2p} x$$

where the last equality holds exactly provided $v = \frac{2p}{E_1 + E_2}$. With this definition for $v$, the same oscillation length as in (2.1) is found without the traditional ultra-relativistic ingredients used to transform (11) by setting $t \approx x$ for the neutrino trajectory and $E_1 + E_2 \approx 2p$.

2.3 Equal velocities

In this case, $\delta \phi$ does not reduce to a single term and it is very important not to "approximate" $t$ by $x$, for in so doing one would arrive at:

$$\delta \phi = (\delta E - \delta p)x = \delta m e^{-\eta}x$$

upon introducing $\eta = \tanh^{-1}(v)$

The oscillating pattern would be described by:

$$\sin^2\left(\frac{\delta m}{2} e^{-\eta}x\right) \quad \text{and the oscillation length:} \quad L'_{\text{osc}} = \frac{2\pi e^\eta}{|\delta m|}$$

However, since

$$e^\eta = \frac{E_1 + p_1}{m_1} = \frac{E_2 + p_2}{m_2} \approx \frac{2(E + p)}{m_1 + m_2}$$

this yields finally

$$L'_{\text{osc.}} = \frac{4(E + p)\pi}{|\delta m^2|}$$

viz. twice the usual value in the relativistic regime where $E \approx p$; this is the origin of the claims of some authors.

However, if the hypothesis of equal velocities has any meaning, then the center of the wave packet moves with that velocity, not with velocity 1. Therefore, defining its position by $x = vt$ yields:

$$\delta \phi = \delta E(t - vx) = \delta m \gamma (1/v - v)x = \frac{\delta m}{v\gamma x}$$

Now

$$\frac{1}{v\gamma} = \frac{m_1}{p_1} = \frac{m_2}{p_2} = \frac{m_1 + m_2}{2\bar{p}}$$

$^1\gamma = 1/\sqrt{1 - v^2}$ as usual
Hence \( \delta \phi = \frac{\delta m^2}{2p} x \) and the formula found in (2.1) results.

Clearly, replacing \( \frac{1}{v} - v \) by \( 1 - v \) (equivalent to \( t \to x \) in [1]) cannot be harmless; in the situation of 2.2, \( v \) is only an overall factor, and replacing it by 1 would induce a relative error on the phase difference which goes to 0 with \( 1 - v \). Not so in the present case where the relative error is \( 1/(1 + v) \) - hence the factor 2 found above. Stated differently, the same value for \( v \) must be used in configuration and in momentum space.

One sees that in [6] for example, the trap is escaped because of the further equal \( E \)'s or equal \( p \)'s assumption beyond the first \( t \approx x \) approximation. Note that even if the 'equal \( v \)'s ' assumption is contradictory with production kinematics, as shown in [1] and [2], it must embody some approximate truth because, for oscillations to be observed, it is important that the two component waves do not separate too early, and therefore, that they do not have too different velocities. \( E_1 \approx E_2 \) and, say, \( m_1 \ll m_2 \) as assumed in [1] is possible only in ultra relativistic situations; but then very small velocity differences make huge differences in the \( \gamma \) factor, so \( \frac{\delta v}{v} < 1 \) is possible, though exact equality is excluded; this last impossibility makes the loss of coherence and the disappearance of oscillations inescapable after a certain distance.

3 A 'kinematically correct' assumption

3.1 Requirements

A connection between \( x \) and \( t \) is necessary to make predictions about observations which will register a position w.r.t. the creation point, but never measure the elapsed time. This connection should arise from a full quantum treatment using wave packets or field theory; it would then only be necessary to propagate the system in time, with the space evolution being taken care of by the wave equation.

In the simple plane wave approach, this relation must be imposed because plane waves are of infinite extent and with two different (mass) states involved one has apparently no mean of defining a 'center' either by averaging or by stationary phase arguments.

The three assumptions detailed above are just different ways of doing this. No further ingredient is needed in the equal \( v \) case, since there is an obvious definition for the group velocity. A very likely definition of this velocity has also been used in the equal \( p \) approximation, thereby avoiding the need of the 'ultra relativistic' \( v \approx 1 \) and \( E_1 + E_2 \approx 2p \). Finally, the equal \( E \) approximation circumvents the problem by yielding a stationary oscillation pattern from the beginning.  

\[ \text{This makes it the preferred hypothesis of the authors of [1, 2] because it avoids ambigu-} \]
It is clear now that if we endow the center of the would-be wave packet with the average momentum and energy of its components: $\bar{p} = \frac{p_1 + p_2}{2}$, $\bar{E} = \frac{E_1 + E_2}{2}$, and/or postulate that it has group velocity $v_g = \frac{\bar{p}}{\bar{E}}$, we find back the same expression for the phase difference in all three cases; it appears that this postulate embodies all that is needed to yield the usual $L_{osc}$:

$$v = \frac{p_1 + p_2}{E_1 + E_2} \Rightarrow \delta \phi = \delta px - \delta Et = (\delta p - \delta E \frac{E_1 + E_2}{p_1 + p_2}) x = \frac{(\delta p^2 - \delta E^2) x}{p_1 + p_2} = -\frac{\delta m^2 x}{2\bar{p}}$$

Indeed, this simple hypothesis is a/ implied by the (2.3) assumption, b/ identical with the definition of the group velocity that we have used in the (2.2) case and c/ not required in the (2.1) derivation, but not incompatible either.

### 3.2 The case of $\pi \rightarrow \mu \nu$ at rest

We show here that our postulate is actually necessary in the kinematically most simple case of $\pi \rightarrow \mu \nu$ at rest.

From $E_j = \frac{m_\pi^2 + m_\nu^2 - m_\mu^2}{2m_\pi}$ and $E_\mu + E_\nu = m_\pi$ one sees that:

$$m_1 < m_2 \Rightarrow E_1 < E_2 \Rightarrow E_{\mu,1} > E_{\mu,2} \Rightarrow p_{\mu,1} > p_{\mu,2}$$

and by momentum conservation $p_1 > p_2$ which, using again $E_1 < E_2$ shows that $v_1 > v_2$; as expected, all three assumptions are inconsistent with kinematics and the third is the worst since $v_1/v_2 = \frac{p_1/p_2}{E_1/E_2}$

Now, for oscillations to be observable, one must have $|p_1 - p_2| < \Delta p$ where $\Delta p$ is the quantal momentum uncertainty due to the localization of the source (the $\pi$ in this case), so that one cannot tell which neutrino mass eigenstate has been produced. Consequently, the phase relation is preserved between the components of the muon state which can be described by a linear superposition of amplitudes. In the plane-wave approximation, this wave function must read something like:

$$\psi = \cos \theta e^{-i(E_{\mu,1}t - p_1 x)} + \sin \theta e^{-i(E_{\mu,2}t - p_2 x)}$$

where $p_1$ and $p_2$ are the same as for the two neutrino mass states recoiling against $\mu$.\footnote{This shows that $\delta E = \frac{\delta m^2}{2m_\pi}$ contrary to the approximation of [9].}
Therefore $|\psi|^2 = 1 + \sin 2\theta \cos(\delta E_\mu t - \delta p x)$ and (assuming $\theta \leq \pi/2$) maximizing the probability gives $x = \frac{\delta E_\mu}{\delta p} t = v_\mu t$ for the center of the (very elementary) wave packet, with $\frac{\delta E_\mu}{\delta p}$ suggestive of a discrete version of the usual $v_g = \frac{\partial E}{\partial p}$.

However in the present case, $\frac{\delta E_\mu}{\delta p} = \frac{p_1 + p_2}{E_{\mu,1} + E_{\mu,2}} = \frac{\bar{p}}{E_{\mu}}$.

But this makes the above postulate for the neutrino velocity more likely, because to find this last value for $v_\mu$, we have to endow the muon state with the mean momentum $\bar{p}$ and energy $\bar{E}_\mu$ of its two components, and likewise for the neutrino in order to enforce energy and momentum conservation in the decay; in turn, this implies our neutrino velocity. Of course, other ways of averaging could do the job as far as conservation laws are concerned; but it is hard to see how $\theta$-dependent averages of $E$ and $p$ could yield back the same ($\theta$-independent) expression for the velocity of the muon obtained by the requirement of maximum probability.

Certainly, assuming $\Delta p > |p_1 - p_2|$ and using a superposition of two plane waves with $p = p_1$ and $p = p_2$ for the muon is not perfectly consistent, but not worse either than what is done on the neutrino side. The reason for which we had to use the muon argument is evidently that the ill-defined oscillating state (2) has a constant modulus. It is not possible to make any direct statement about its location in space-time without first defining its flavour content (as is done to calculate the oscillation length).

4 Restrictions and conclusion

The shortcomings of the plane wave description of neutrino oscillations were discussed long ago [10]; the necessity of using wave packets (see e.g. [13]) or field theory ([14], [15]) and of incorporating the neutrino production and/or detection processes in the description of the phenomenon ([10], [16]) has been the subject of many works and still feeds a continuing debate. However, the results of these more elaborate treatments always imply, up to damping factors, the oscillation pattern described by (6) in the relativistic limit, whenever oscillations are not washed out by resolution or loss of coherence.

The above analysis explains the robustness of this classical formula by exposing the only ingredient necessary to retrieve it; this turns out to be more general than what is used in standard derivations and does not contradict production kinematics. As a by-product, it also shows that the variable to be used in the oscillation length formula is the momentum rather than the energy, should the distinction apply. However this would happen in the non-relativistic regime where, due to phase-space limitations and admixture of ‘wrong’ helicity states, an oscillation probability disconnected from the rest of the process has a very limited meaning.

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4 Which is but a simplified and discretized version of standard stationary phase arguments used to define the center of a wave packet

5 Of course we discount incorrect treatments of the equal velocity assumption
Obviously, the restriction to two flavours is a weakness of the analysis presented here; it must be remarked, however, that the oscillation length has a clear physical interpretation only in this case. The corresponding parameters become unavoidably entangled with mixing matrix elements when dealing with more than two flavours. Nevertheless, it is our hope that the simplicity of the arguments and results presented here may shed some light on the general case.

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