A NOTE ON TWO SPECIES COLLISIONAL PLASMA IN BOUNDED DOMAINS

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Abstract. We construct a unique global-in-time solution to the two species Vlasov-Poisson-Boltzmann system in convex domains with the diffuse boundary condition, which can be viewed as one of the ideal scattering boundary model. The construction follows a new $L^2$-$L^\infty$ framework in [4]. In our knowledge this result is the first construction of strong solutions for two species plasma models with self-consistent field in general bounded domains.

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1. Introduction

One of the fundamental models for dynamics of dilute charged particles (e.g., electrons and ions) is the Vlasov-Maxwell-Boltzmann (VMB) system, in which particles interact with themselves through collisions and with their self-consistent electromagnetic field:

$$\begin{align*}
\partial_t F_+ + v \cdot \nabla_x F_+ + \frac{e_+}{m_+} (E + \frac{v}{c} \times B) \cdot \nabla_v F_+ &= Q(F_+, F_+) + Q(F_+, F_-), \\
\partial_t F_- + v \cdot \nabla_x F_- - \frac{e_-}{m_-} (E + \frac{v}{c} \times B) \cdot \nabla_v F_- &= Q(F_-, F_+) + Q(F_-, F_-).
\end{align*}$$

(1.1)

Here $F_\pm(t,x,v) \geq 0$ are the density functions for the ions ($+$) and electrons ($-$) respectively, and $e_\pm, m_\pm$ the magnitude of their charges and masses, $c$ the speed of light. The self-consistent electromagnetic field $E(t,x), B(t,x)$ in (1.1) is coupled with $F(t,x,v)$ through the Maxwell system (see [15]). Previous studies for the VMB system, for example the existence of global in time classical solution, uniqueness, and asymptotic behavior without boundaries, can be found in [15], [6].

Now formally as the speed of light $c \to \infty$, one can derive the so-called two species Vlasov-Poisson-Boltzmann (VPB) system, where $B(t,x) = 0$. And the field $E$, that we are interested in, is associated with an electrostatic potential $\phi$ as

$$E(t,x) := -\nabla_x \phi(t,x),$$

(1.2)

where the potential is determined by the Poisson equation:

$$-\Delta_x \phi(t,x) = \int_{\mathbb{R}^3} (F_+ - F_-) dv := \rho.$$  

(1.3)

In this paper we consider the zero Neumann boundary condition for $\phi$:

$$\frac{\partial \phi}{\partial n} = 0 \text{ for } x \in \partial \Omega.$$  

(1.4)
It turns out that the presence of all the physical constants does not create essential mathematical difficulties. Therefore, for simplicity we normalize all constants in (1.1) to be one, and the VPB system takes the form:

$$\begin{align*}
\partial_t F_+ + v \cdot \nabla_x F_+ + E \cdot \nabla_x F_+ &= Q(F_+, F_+) + Q(F_+, F_-), \\
\partial_t F_- + v \cdot \nabla_x F_- - E \cdot \nabla_x F_- &= Q(F_-, F_+) + Q(F_-, F_-).
\end{align*}$$  
(1.5)

The collision operator between particles measures “the change rate” in binary hard sphere collisions and takes the form of

$$Q(F_1, F_2)(v) := Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2)$$

$$:= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} [(v-u) \cdot \omega][F_1(v') F_2(u') - F_1(v) F_2(u)] d\omega du,$$

where $u' = u - [(u-v) \cdot \omega] \omega$ and $v' = v + [(u-v) \cdot \omega] \omega$. The collision operator enjoys a collision invariance: for any measurable $G_1, G_2$,

$$\int_{\mathbb{R}^3} \left[ 1 - |v|^2 \right] Q(G_1, G_1) dv = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \int_{\mathbb{R}^3} Q(G_1, G_2) = 0.$$

It is well-known that a global Maxwellian $\mu$ satisfies $Q(\cdot, \cdot) = 0$ where

$$\mu(v) := \frac{1}{(2\pi)^{3/2}} \exp \left( -\frac{|v|^2}{2} \right).$$

Throughout this paper, let’s use the notation

$$\iota = + \text{ or } -,$$ and denote $-\iota = \begin{cases} - & \text{if } \iota = + \\ + & \text{if } \iota = - \end{cases}$.

Being an important equation in both theoretic and application aspects, the Boltzmann equation has drawn attentions and there have been a lot of research activities in analytic study of the equation. Notably the nonlinear energy method has led to solutions of many open problems including global strong solution of both the VMB system and the VPB system, when the initial data are close to the Maxwellian $\mu$. One thing to note is that these results deal with idealized periodic domains or whole space, in which the solutions can remain bounded in $H^k$ for large $k$.

In many important physical applications, e.g. semiconductor and tokamak, the charged dilute gas is confined within a container, and its interaction with the boundary often plays a crucial role both in physics and mathematics. So it’s natural to consider the equation in a bounded domain $\Omega$, and the interaction of the gas with the boundary is described by suitable boundary conditions [2] [24]. In this paper we consider one of the physical conditions, a so-called diffuse boundary condition:

$$F_\iota(t, x, v) = c_\mu \mu(v) \int_{n(x) \cdot u > 0} F_\iota(t, x, u)(n(x) \cdot u) du \text{ for } (x, v) \in \gamma_\iota.$$  
(1.10)

Here, $\gamma_- := \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}$ and $n(x)$ is the outward unit normal at a boundary point $x$. A number $c_\mu$ is chosen to be $\sqrt{2\pi}$ so that $c_\mu \int_{n(x) \cdot u > 0} \mu(u)(n(x) \cdot u) du = 1$. Due to this normalization the distribution of (1.10) enjoys a null flux condition at the boundary:

$$\int_{\mathbb{R}^3} F_\iota(t, x, v)(n(x) \cdot v) dv = 0 \text{ for } x \in \partial \Omega.  
(1.11)$$

One can view this boundary condition as one of the ideal scattering model.

However, in general, higher regularity may not be expected for solutions of the Boltzmann equation in physical bounded domains. Such a drastic difference of solutions with boundaries had been demonstrated as the formation and propagation of discontinuity in non-convex domains [23] [7], and a non-existence of some second order derivatives at the boundary in convex domains [10]. Evidently the nonlinear energy method is not generally available to the boundary problems. In order to overcome such critical difficulty, Guo developed a $L^2$-$L^\infty$ framework in [13] to study global solutions of the Boltzmann equation with various boundary conditions. The core of the method lays in a direct approach (without taking derivatives) to achieve a pointwise bound using trajectory of the transport operator, which leads substantial development in various directions including [8] [7] [16] [17]. There are also studies on different type of collisional plasma models such as a Fokker-Planck equation with some boundary conditions (for example, see [14] and reference therein).

The main goal of the paper is to study the 2 species VPB system coupled of (1.5) with (1.9) and (1.4), which describes the dynamics of electrons in the absence of a magnetic field. From (1.7) and (1.11), a smooth solution of VPB with the diffuse BC (1.10) preserves total mass:

$$\int_{\Omega \times \mathbb{R}^3} F_\iota(t, x, v) dv dx \equiv \int_{\Omega \times \mathbb{R}^3} F_\iota(0, x, v) dv dx \text{ for all } t \geq 0.$$  
(1.12)
We assume that initially \( F_0(x,v) \) satisfies
\[
\int_{\Omega \times \mathbb{R}^3} (F_+(0,x,v) - F_-(0,x,v))dv = 0. \tag{1.13}
\]
Then \( \int_{\Omega} \{ \int_{\mathbb{R}^3} (F_+(t,x,v) - F_-(t,x,v))dv \} dx = 0 \) for all \( t > 0 \). This zero-mean condition guarantees a solvability of the Poisson equation (1.3) with the Neumann boundary condition (1.4).

There are some previous studies for the one-species VPB system (which is obtained by letting \( F_- = 0 \)) with physical boundary conditions. For example, the time asymptotics of a solution to the VPB system is studied under some a priori assumption on the solutions. In [23] renormalized solutions (no uniqueness) were constructed for the VPB system with diffuse boundary condition. Recently in [4] the authors constructed a unique global strong solution to the VPB system with diffuse boundary condition. They also had a weighted \( W^{1,p} \), \( 3 < p < 6 \) estimate for the solution of such system. This regularity result was later improved in [3] where the author obtained a weighted \( W^{1,\infty} \) estimate for the solution under the appearance of an external field with a favorable sign condition \( E \cdot n > 0 \) on the boundary which will be explained later.

We consider a perturbation around \( \mu \):
\[
F_\epsilon = \mu + \sqrt{\mu} f_\epsilon. \tag{1.14}
\]

Then the corresponding problem is given by
\[
\partial_t f_+ + v \cdot \nabla v f_+ - \nabla \phi \cdot \nabla v f_+ + \frac{v}{2} \nabla \phi f_+ - \frac{2}{\sqrt{\mu}} Q(\sqrt{\mu} f_+, u) - \frac{1}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f_+) - \frac{1}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f_-) = \Gamma(f_+ - f_+) - v \cdot \nabla \phi \sqrt{\mu}, \tag{1.15}
\]
\[
\partial_t f_- + v \cdot \nabla v f_- + \nabla \phi \cdot \nabla v f_- - \frac{v}{2} \nabla \phi f_- - \frac{2}{\sqrt{\mu}} Q(\sqrt{\mu} f_-, u) - \frac{1}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f_-) - \frac{1}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f_+) = \Gamma(f_- - f_+) + v \cdot \nabla \phi \sqrt{\mu},
\]
\[
f_\epsilon(t,x,v) = \frac{2}{\sqrt{\mu}} \int_{\mathbb{R}^3} \sqrt{\mu} f_\epsilon(t,x,u)(n(x) \cdot u)du \text{ for } (x,v) \in \gamma_-.
\]

For \( g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \), \( h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \), let
\[
Lg := \frac{1}{\sqrt{\mu}} \left[ \frac{2}{\sqrt{\mu}} Q(\sqrt{\mu} g_1, \mu) + Q(\mu, \sqrt{\mu} (g_1 + g_2)) \right] := \nu(v)g - Kg. \tag{1.18}
\]

Here the collision frequency is defined as
\[
\nu(v) := \frac{2}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} g_1, \mu) := 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (v - u) \cdot \omega |\mu(u)du| \omega \sim \langle v \rangle.
\]

It is well-known that for hard-sphere case,
\[
\frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} g_1, \mu) = \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\mu, \sqrt{\mu} g_1) = \int_{\mathbb{R}^3} k_2(v, u)g_1(u)du,
\]
\[
\frac{1}{\sqrt{\mu}} Q_{\text{loss}}(\mu, \sqrt{\mu} g_1) = \int_{\mathbb{R}^3} k_1(v, u)g_1(u)du,
\]
with
\[
k_1(v, u) = \frac{\pi}{2} |v - u| \frac{1}{8\mu} e^{-\frac{|v|^2}{4\mu} + \frac{1}{8\mu} |v|^2},
\]
\[
k_2(v, u) = \frac{\pi}{2} |v - u| - \frac{1}{8\mu} e^{-\frac{|v|^2}{8\mu} + \frac{1}{8\mu} |v|^2}.
\]

Thus
\[
Kg := \left[ \begin{array}{c} 
\frac{2}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} g_1, \mu) + Q(\mu, \sqrt{\mu} (g_1 + g_2)) \\
\frac{2}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} g_2, \mu) + Q(\mu, \sqrt{\mu} (g_1 + g_2))
\end{array} \right] := \begin{bmatrix} k_2(v, u)g_1(u)du & k_1(v, u)g_1(u)du \\
k_2(v, u)g_2(u)du & k_1(v, u)g_2(u)du
\end{bmatrix}.
\]

The nonlinear operator is defined as
\[
\Gamma(g,h) := \Gamma(g,h) - \Gamma_{\text{loss}}(g,h) := \frac{1}{\sqrt{\mu}} \left[ Q_{\text{gain}}(\sqrt{\mu} g_1, \sqrt{\mu} (h_1 + h_2)) - Q_{\text{loss}}(\sqrt{\mu} g_1, \sqrt{\mu} (h_1 + h_2)) \right] - \left[ Q_{\text{gain}}(\sqrt{\mu} g_2, \sqrt{\mu} (h_1 + h_2)) - Q_{\text{loss}}(\sqrt{\mu} g_2, \sqrt{\mu} (h_1 + h_2)) \right]. \tag{1.22}
\]
Then for $f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix}$, (1.15) becomes

$$\partial_t f + v \cdot \nabla_x f - q \nabla \phi \cdot \nabla_v f + q^v \nabla \phi f + Lf = \Gamma(f, f) - q_1 v \cdot \nabla \phi \sqrt{p},$$

(1.23)

where $q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $q_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let’s clarify some notations. We denote

$$w_\phi(v) = e^{\|v\|^2}.$$  

(1.24)

The boundary of the phase space $\gamma := \{(x, v) \in \partial \Omega \times \mathbb{R}^3 \}$ can be decomposed as

$$\gamma_- = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}, \quad \text{(the incoming set)},$$

$$\gamma_+ = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}, \quad \text{(the outgoing set)},$$

$$\gamma_0 = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}, \quad \text{(the grazing set)}.$$  

(1.25)

For any function $z(x, v) : \Omega \times \mathbb{R}^3 \to \mathbb{R}$, denote

$$|z|_{2, +}^2 = \int_{\gamma_+} z^2 d\gamma, \quad |z|_{2, -}^2 = \int_{\gamma_-} z^2 d\gamma, \quad |z|_{2, 0}^2 = \int_{\partial \Omega \times \mathbb{R}^3} z^2 |n(x) \cdot v| dv dx.$$  

Now for any vector-valued function $f, g : \Omega \times \mathbb{R}^3 \to \mathbb{R}^2$, with $f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix}$ and $g = \begin{bmatrix} g_+ \\ g_- \end{bmatrix}$, let’s clarify the following notations:

$$|f| = |f_+| + |f_-|, \quad f \cdot g = f_+ g_+ + f_- g_-,$$

(1.26)

and $(f, g) = \int_{\Omega \times \mathbb{R}^3} f \cdot g dv dx = \int_{\Omega \times \mathbb{R}^3} (f_+ g_+ + f_- g_-) dv dx,$

$$f^p := \begin{bmatrix} f^p_+ \\ f^p_- \end{bmatrix}, \quad \int f = \int \begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \int f_+ + \int f_-,$$

$$\partial f = \begin{bmatrix} \partial f_+ \\ \partial f_- \end{bmatrix}, \quad |f|_{2, +}^p := \int_{\gamma_+} |f|^p d\gamma \sim \int_{\gamma_+} (|f_+|^p + |f_-|^p) d\gamma,$$

(1.27)

$$|f|_{2, -}^p := \int_{\gamma_-} |f|^p d\gamma \sim \int_{\gamma_-} (|f_+|^p + |f_-|^p) d\gamma, \quad \text{and} \quad |f|_{\gamma, p}^p := \int_{\partial \Omega \times \mathbb{R}^3} |f|^p |n(x) \cdot v| dv dx,$$

$$\|f(t)\|_p^p := \int_{\Omega \times \mathbb{R}^3} |f|^p dv dx \sim \int_{\Omega \times \mathbb{R}^3} (|f_+(t)|^p + |f_-(t)|^p) dv dx, \quad \|f(t)\|_\infty = \sup_{(x, v) \in \Omega \times \mathbb{R}^3} |f_+(t)| + |f_-(t)|.$$  

1.1. A New Distance Function. Throughout this paper we extend $\phi_f$ for a negative time. Let

$$\phi_f(s, x, v) := \phi_{f_0}(s, x, v) \quad \text{for} \quad -\infty < s < 0,$$

(1.28)

where $\phi_{f_0}(x, v)$ satisfies $-\Delta \phi_{f_0}(x, v) = \int_\mathbb{R}^3 (f_0^{+} - f_0^{-}) \sqrt{p} dv$.

The characteristics (trajectory) is determined by the Hamilton ODEs for $f_+$ and $f_-$ separately

$$\frac{d}{ds} \begin{bmatrix} X^f_+(s; t, x, v) \\ V^f_+(s; t, x, v) \end{bmatrix} = -\nabla_x \phi_f(s, X^f_+(s; t, x, v)) \quad \text{for} \quad -\infty < s, t < \infty,$$

(1.29)

with $(X^f(t; t, x, v), V^f(t; t, x, v)) = (x, v)$.

For $(t, x, v) \in \mathbb{R} \times \Omega \times \mathbb{R}^3$, we define the backward exit time $t^f_{\text{b}}(t, x, v)$ as

$$t^f_{\text{b}}(t, x, v) := \sup\{s \geq 0 : X^f_+(s; t, x, v) \in \Omega \quad \text{for all} \quad \tau \in (t - s, t)\}.$$  

Furthermore, we define $x^f_{\text{b}}(t, x, v) := X^f_+(t - t^f_{\text{b}}(t, x, v); t, x, v)$ and $v^f_{\text{b}}(t, x, v) := V^f_+(t - t^f_{\text{b}}(t, x, v); t, x, v)$.

**Definition 1** (Distance Function). For $\varepsilon > 0$, for $\tau = +$ or $-\tau$ as in (1.29), define

$$\alpha_{f, \varepsilon, \tau}(t, x, v) := \chi \left( \frac{t - t^f_{\text{b}}(t, x, v) + \varepsilon}{\varepsilon} \right) \left| n(x^f_{\text{b}}(t, x, v)) \cdot v^f_{\text{b}}(t, x, v) \right|$$

(1.30)

$$+ \left[ 1 - \chi \left( \frac{t - t^f_{\text{b}}(t, x, v) + \varepsilon}{\varepsilon} \right) \right].$$

Here we use a smooth function $\chi : \mathbb{R} \to [0, 1]$ satisfying

$$\chi(0) = 0, \quad \tau \leq 0, \quad \text{and} \quad \chi(0) = 1, \quad \tau \geq 1,$$

(1.31)

$$\frac{d}{d\tau} \chi(\tau) \in [0, 4] \quad \text{for all} \quad \tau \in \mathbb{R}.$$
Note that \( \alpha_{f,c}(0, x, v) \equiv \alpha_{f_0,c}(0, x, v) \) is determined by \( f_0 \) and its extension \( f_0^\perp \). For the sake of simplicity, we could drop the superscription \( \perp \) in \( X_l^i, V_l^i, t^i_{x_0}, x^i_{v_0}, v^i_{v_0} \), unless they could cause any confusion.

Also denote
\[
\alpha_{f,c}(t, x, v) := \begin{bmatrix} \alpha_{f,c}(t, x, v) & 0 \\ 0 & \alpha_{f,c}(t, x, v) \end{bmatrix},
\]
and let \(|\alpha_{f,c}(t, x, v)| := |\alpha_{f,c}(t, x, v)| + |\alpha_{f,c}(t, x, v)|.

One of the crucial properties of the new distance function in (1.30) is an invariance under the Vlasov operator:
\[
\partial_t + v \cdot \nabla_x - \xi \nabla_\xi \alpha f_{f,c}(t, x, v) = 0.
\]

This is due to the fact that the characteristics solves a deterministic system (1.28) (See the proof in the appendix). This crucial invariance property under the Vlasov operator is one of the key points in our approach.

It is important to note that a different version of the distance function which has been used in the author's previous paper \[3\] to establish the regularity of the one-species VPB system is not applicable here. In \[3\], the weight \( \tilde{\alpha} \) took the form
\[
\tilde{\alpha}(t, x, v) = \left[ |v \cdot \nabla \xi(x)|^2 + \xi(x)^2 - 2(v \cdot \nabla \xi(x) \cdot v)\xi(x) - 2(E(t, \mathcal{E}) \cdot \nabla \xi(t, \mathcal{E}))\xi(x) \right]^{1/2},
\]
for \( x \in \Omega \) close to boundary, where \( \mathcal{E} := \{ x \in \partial \Omega : d(x, x) = d(x, \partial \Omega) \} \) is uniquely defined. And \( \xi(x) \) was assumed to be a \( C^2 \) function \( \xi : \mathbb{R}^3 \to \mathbb{R} \) such that \( \Omega = \{ x \in \mathbb{R}^3 : \xi(x) < 0 \} \), \( \partial \Omega = \{ x \in \mathbb{R}^3 : \xi(x) = 0 \} \), and \( \nabla \xi(x) \neq 0 \) when \( |\xi(x)| \ll 1 \). And the domain was assumed to be strictly convex:
\[
\sum_{i,j} \partial_{ij} \xi(x) \xi \eta_j \geq C\xi |\xi|^2 \text{ for all } \xi \in \mathbb{R}^3 \text{ and for all } x \in \Omega = \Omega \cup \partial \Omega.
\]

One of the crucial property this \( \tilde{\alpha} \) enjoys is the velocity lemma:
\[
|\{ \partial_t + v \cdot \nabla_x + E \cdot \nabla_v \} \tilde{\alpha}(t, x, v)| \lesssim |v|\tilde{\alpha},
\]
when under the sign condition
\[
E \cdot n > \delta \quad \text{on } \partial \Omega,
\]
where \( n \) is the outward normal vector. This can be seen by direct computation:
\[
|\{ \partial_t + v \cdot \nabla_x + E \cdot \nabla_v \} \tilde{\alpha}^2(t, x, v)| \sim |v|\tilde{\alpha}^2 + C_{\xi}(E, \nabla_x E, \partial_t E)|\xi(x)|,
\]
for some bounded function \( C_{\xi} \). Now under (1.39), we get an extra stronger control for \( \xi(x) \) from \( \tilde{\alpha}^2 \), and therefore the second term on the right-hand side of (1.37) can be bounded by:
\[
C_{\xi}|\xi(x)| \lesssim \inf_{y \in \partial \Omega} E(t, y) \cdot \nabla \xi(x)|\xi(E(t, \mathcal{E}) \cdot \nabla \xi(t, \mathcal{E}))\xi(x) \leq C_{\xi} \delta \tilde{\alpha}\tilde{\alpha}(t, x, v).
\]
Thus combining (1.37) and (1.38) we obtain (1.39). This means \( \tilde{\alpha}(t, x, v) \) retains its full power under the transport operator, which is crucially used for establishing the theories in \[3\].

Thus it’s clear that without the last term in (1.34), i.e. in the case \( E \cdot \nabla \xi = 0 \) on \( \partial \Omega \), in order to have the \( \xi(x) \) control from the second term on the right hand side of (1.37), we can only obtain
\[
|\{ \partial_t + v \cdot \nabla_x + E \cdot \nabla_v \} \tilde{\alpha}^2(t, x, v)| \lesssim |v|\tilde{\alpha}(t, x, v).
\]
Therefore \( \tilde{\alpha}(t, x, v) \) suffers a loss of power under the transport operator, and would result it’s been inapplicable for the situation here.

Therefore the previous distance function \( \tilde{\alpha} \) would work only under a crucial favorable sign condition (1.36). But for the two species VPB system, it’s clear from the equation (1.5) that if one requires the sign condition for the field for \( F_+ \), i.e. \(-\nabla \phi \cdot n > 0 \), then inevitably one would have \(+\nabla \phi \cdot n < 0 \), so the field for \( F_- \) would fail to satisfy the sign condition. We note that the similar \( \tilde{\alpha} \) has also been used by \[3\], \[12\], \[16\] in the study of one-species problem of Vlasov equation.

Thus one of the major benefit for this new distance function \( \alpha \) is that it only requires the zero-Neumann boundary condition \( E \cdot n = 0 \) (see Lemma 1, Proposition 5), and therefore with \( \pm \nabla \phi \cdot n = 0 \) from (1.4), we can apply this distance function to the two species VPB system (1.3).

1.2. Main Theorem. The main goal of this paper is the construction of a unique global strong solution of the two species VPB system with the diffuse boundary condition when the domain is \( C^2 \) and convex. Moreover an asymptotic stability of the global Maxwellian \( \mu \) is studied.

Here a \( C^1 \) domain means that for any \( p \in \partial \Omega \), there exists sufficiently small \( \delta_1 > 0, \delta_2 > 0 \), and an one-to-one and onto \( C^1 \)-map
\[
\eta_p: \{ x_1 \in \mathbb{R}^2 : |x_1| < \delta_1 \} \to \partial \Omega \cap B(p, \delta_2),
\]
\[
x_1 = (x_{1,1}, x_{1,2}) \mapsto \eta_p(x_{1,1}, x_{1,2}).
\]
A **convex** domain means that there exists \( C_\Omega > 0 \) such that for all \( p \in \partial \Omega \) and \( \eta_p \) and for all \( x_{\parallel} \) in (1.40)

\[
\sum_{i,j=1}^{2} \zeta_i \zeta_j \partial_i \partial_j \eta_p (x_{\parallel}) \cdot n(x_{\parallel}) \leq -C_\Omega |\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^2 .
\] (1.41)

**Theorem 1.** Assume a bounded open \( C^2 \) domain \( \Omega \subset \mathbb{R}^3 \) is convex (1.44). Let \( 0 < \delta < \theta \ll 1 \). Assume the neutral condition (1.13) and the compatibility condition

\[
f_0,\varepsilon (x,v) = c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} f_0,\varepsilon (x,u) \sqrt{\mu(u)} \{ n(x) \cdot u \} du \quad \text{on } \gamma_- .
\] (1.42)

Then there exists a small constant \( 0 < \varepsilon_0 \ll 1 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \) if an initial datum \( F_0 = \mu + \sqrt{\mu} f_0 \geq 0 \) satisfies

\[
\| w_0 f_0 \|_{L^\infty (\Omega \times [0,1])} < \varepsilon ,
\] (1.43)

and, recall the matrix definition of \( \alpha \) in (1.32),

\[
\| w_0 \alpha \|_{L^\infty (\Omega \times [0,1])} < \varepsilon
\]

and, for some \( C > 0 \),

\[
\| w_0 \alpha \|_{L^p (\Omega \times [0,1])} \leq e^{C t} \quad \text{for all } t \geq 0 ,
\] (1.47)

and, for \( 0 < \delta = \delta (p, \beta) \ll 1 \),

\[
\| \nabla_v f(t) \|_{L^2 (\Omega) L^{\frac{1}{2} + \delta} (\mathbb{R}^3)} \leq 1 \quad \text{for all } t \geq 0 .
\] (1.48)

Furthermore, if \( (f, \phi_f) \) and \( (g, \phi_g) \) are both solutions to (1.12), (1.10), (1.17) then

\[
\| f(t) - g(t) \|_{L^{\frac{1}{1 + \delta} (\Omega \times [0,1])}} \leq \varepsilon \| f(0) - g(0) \|_{L^{\frac{1}{1 + \delta} (\Omega \times [0,1])}} \quad \text{for all } t \geq 0 .
\] (1.49)

The proof of Theorem 1 devotes a nontrivial extension of the argument of [4] now for the two species VPB system. One of the major difference here is the \( L^2 \) coercivity estimate.

Now we illustrate the main ideas in the proof of Theorem 1 which largely follows the framework in [4]. In the energy-type estimate of \( \nabla_v f \) in \( \alpha \)-weighted \( L^p \)-norm, the operator \( v \cdot \nabla_x \) causes a boundary term to be controlled:

\[
f_0 \int_{\partial \Omega} \int_{v_{\parallel} \leq 0} \| \alpha \|_{L^2 (\Omega)} \| \nabla_x f \|_{L^p (\mathbb{R}^3)} \| n \cdot v \| dv \, ds > 0 .
\]

It turns out this integrand is integrable if

\[
\beta > \frac{p - 2}{p} \quad \text{so that } \quad |n \cdot v|^{p - 2} \| n \|_{L^1 (\mathbb{R}^3)} ^{p - 1} \in L^1 (\mathbb{R}^3) .
\] (1.50)

On the other hand to control the terms in the bulk we need a **bound of** \( \phi_f(t) \) in \( C^2 \). A key observation is that

\[
\bigg\| \int_{\mathbb{R}^3} \nabla_x f \sqrt{\mu} dv \bigg\|_{L^2 (\Omega)} \leq \sup_{x_{\parallel} \in \mathbb{R}^3} \sum_{i=\pm} \left| \frac{\sqrt{\mu}}{\alpha_{f,\varepsilon,\delta} (\Omega)} \right| \| \nabla_x f \|_{L^{\frac{1}{p}} (\mathbb{R}^3)} ,
\] (1.51)

which leads \( C^2,0 \)-bound of \( \phi_f \) by the Morrey inequality for \( p > 3 \) as long as

\[
\alpha_{f,\varepsilon,\delta} \in L^1 (\mathbb{R}^3) \quad \text{for some } \beta > \frac{p - 2}{p - 1} .
\] (1.52)

The proof of (1.52) can be found in [4], where the authors employ a change of variables \( v \mapsto (x_{\parallel} f(t,x,v), t_{\parallel} f(t,x,v)) \), and carefully compute and bound the determinant of the Jacobian matrix to get

\[
\int_{|v| \leq 1} \alpha_{f,\varepsilon,\delta}^{-\beta} dv \leq \int_{\text{boundary}} \frac{|(x - x_{\parallel} f(t,x,v))^{1-\beta} dv |}{|x - x_{\parallel} f(t,x,v)|^{\beta} dv } + \text{good terms} < \infty ,
\] (1.53)

which turns to be bounded as long as \( \beta p^* < 1 \).

In order to run the \( L^2 - L^\infty \) bootstrap argument we need to prove the \( L^2 \)-coercivity property of the solution \( f \) (Proposition 8). This is one of the major difference from [4], as here for the two species VPB system, the null space of the linear operator \( L \) in (1.13) is a six-dimensional subspace of \( L^2 (\mathbb{R}^3;\mathbb{R}^2) \) spanned by orthonormal vectors

\[
\left\{ \begin{bmatrix} \sqrt{\mu} \\ 0 \\ \sqrt{\nu} \end{bmatrix} , \begin{bmatrix} \sqrt{\mu} \\ \sqrt{\nu} \sqrt{\mu} \\ \sqrt{\nu} \sqrt{\mu} \end{bmatrix} , \begin{bmatrix} \frac{\mu^2 - \lambda^2}{2\sqrt{\mu}} \sqrt{\mu} \\ \frac{\mu^2 - \lambda^2}{2\sqrt{\mu}} \sqrt{\mu} \end{bmatrix} \right\} , \quad i = 1, 2, 3 .
\] (1.54)
of the equation (1.23), we properly choose a set of test functions: expansion, and then use change of variables to get the $L_f$∥(see Lemma 1 from [15] for the proof). And the projection of $f$ onto the null space $N(L)$ can be denoted by

\[
P_f(t, x, v) := \left\{ a_+(t, x) \left[ \frac{\sqrt{v}}{\mu} \right] + a_-(t, x) \left[ \frac{0}{\mu} \right] + b(t, x) \cdot \nabla_x c(t, x) \cdot \frac{v}{\sqrt{2}} \left[ \frac{\sqrt{v}}{\mu} \right] + \frac{c(t, x)}{2\sqrt{2}} \left[ \frac{\sqrt{v}}{\mu} \right] \right\}.
\]

Using the standard $L^2$ energy estimate of the equation, it is well-known (See [15]) that $L$ is degenerate: $\langle L_f, f \rangle \gtrsim \|\mu^{1/2}(I - P_f)\|_{L^2_{(t,x,v)}}$. Thus it’s clear that in order to control the $L^2$ norm of $f(t)$, we need a way to bound the missing $\|P(t)\|_{L^2}$ term.

From there we adopt the ideas from [7] and apply it to our setting (two species system). By using weak form of the equation (1.24), we properly choose a set of test functions:

\[
\psi_a \equiv \left\{ (|v|^2 - \beta_a)\sqrt{\mu}v \cdot \nabla_x \varphi_a \right\},
\psi_{b_1}^i \equiv \left\{ (v_i^2 - \beta_b)\sqrt{\mu}v_j \cdot \nabla_x \varphi_{b_1}^i \right\}, \quad i, j = 1, 2, 3,
\psi_{b_2}^i \equiv \left\{ (|v|^2 - \beta_b)\sqrt{\mu}v_j \cdot \nabla_x \varphi_{b_2}^i \right\}, \quad i \neq j,
\psi_c \equiv \left\{ (|v|^2 - \beta_c)\sqrt{\mu}v \cdot \nabla_x \varphi_c \right\},
\]

where $\varphi_{a \pm}(t, x, v_0(t, x))$, $\varphi_b(t, x)$, and $\varphi_c(t, x)$ solve

\[
-\Delta \varphi_{a \pm} = a \pm(t, x), \quad \partial_t \varphi_{a \pm}|_{\partial \Omega} = 0,
\]

\[
-\Delta \varphi_b = b(t, x), \quad \varphi_b|_{\partial \Omega} = 0, \quad \text{and } -\Delta \varphi_c = c(t, x), \quad \varphi_c|_{\partial \Omega} = 0,
\]

and carefully choose $\beta_a = 10, \beta_b = 1, \beta_c = 5$ to satisfy (71). Integrating against these test functions $\int_0^T \langle \psi, \frac{\partial f(t)}{\partial t} \rangle dt$, we can nicely extract the $L^2$ norms of the $N(L)$ projections of $f$: $\|a_\pm(t)\|_{L^2_{(t,x,v)}}, \|b(t)\|_{L^2_{(t,x,v)}}, \|c(t)\|_{L^2_{(t,x,v)}}$ through the term $\langle v \cdot \nabla_x f, \psi \rangle$. And therefore we recover the bound for the missing $\|P(t)\|_{L^2}$ term from the $L^2$ energy estimate of $f$.

Finally we use $L^2-L^\infty$ bootstrap argument to derive an exponential decay of $f$ in $L^\infty$. The main idea here is to control $f_+$ and $f_-$ separately along their trajectories $(X_+(s), V_+(s))$ and $(X_-(s), V_-(s))$ by using the double Duhamel expansion, and then use change of variables to get the $L^2$ bound. But here as we are working with the two species system, it’s important to note that in the process of the double Duhamel expansion, a mix of trajectories would occur [5,23]. That is if we start with either $t = +$ or $-$, both the $f_+$ and $f_-$ terms would appear in the first Duhamel expansion of $f$. From there we perform the second Duhamel expansion by expanding $f_+$ along $(X_+(s), V_+(s))$, and expanding $f_-$ along $(X_-(s), V_-(s))$. And then we treat them using two different change of variables

\[
u \mapsto X_+(s'; s, X_+(s; t, x, v), u), \quad \nu \mapsto X_-(s'; s, X_-(s; t, x, v), u)
\]

accordingly to get the bound with $\|f_+\|_{L^2} + \|f_-\|_{L^2}$ in the bulk. But thanks to the $L^2$ coercivity (Proposition [8] which gives control to the whole $\|f\|_{L^2}$, we can take the sum $\sum_{i=\pm} \|f_i\|_{L^2}$ and close the estimates.

2. Preliminary

In this section, we give some basic estimates of initial-boundary problems of the transport equation in the presence of a time-dependent field $E(t, x)$, and $f$ here is assumed to be a scalar valued function $f(t, x, v) : (0, \infty) \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$ satisfies

\[
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \psi f = H,
\]

where $H = H(t, x, v)$ and $\psi = \psi(t, x, v) \geq 0$. We assume that $E$ is defined for all $t \in \mathbb{R}$. Throughout this section $(X(t; s, x, v), V(t; s, x, v))$ denotes the characteristic which is determined by (1.28) with replacing $-t\nabla_x \phi_f$ by $E$.

**Lemma 1.** Assume that $\Omega$ is convex [L.47]. Suppose that $\sup_t \|E(t)\|_{C^0_1} < \infty$ and

\[
n(x) \cdot E(t, x) = 0 \quad \text{for } x \in \partial \Omega \text{ and for all } t.
\]

Assume $(t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3$ and $t + 1 \geq h_0(t, x, v)$. If $x \in \partial \Omega$ then we further assume that $n(x) \cdot v > 0$. Then we have

\[
n(x_0(t, x, v)) \cdot v_0(t, x, v) < 0.
\]

**Proof.** The proof is the same as that of Lemma 1 in [3]. But since we are going to use some of the argument for later purpose, let’s present the proof here.
Step 1. Note that locally we can parametrize the trajectory (see Lemma 15 in \cite{15} or \cite{22} for details). We consider local parametrization (1.40). We drop the subscript \( p \) for the sake of simplicity. If \( X(s; t, x, v) \) is near the boundary then we can define \((X_n, \hat{X}_n)\) to satisfy

\[
X(s; t, x, v) = \eta(X_1(s; t, x, v)) + X_n(s; t, x, v)[-n(X_1(s; t, x, v))].
\]  

(2.4)

For the normal velocity we define

\[
V_n(s; t, x, v) := V(s; t, x, v) \cdot [-n(X_1(s; t, x, v))].
\]

(2.5)

We define \( V_\parallel \) tangential to the level set \((\eta(X_1) + X_n(-n(X_1)))\) for fixed \( X_n \). Note that

\[
\frac{\partial (\eta(x_i) + x_n(-n(x_i)))}{\partial x_{\parallel,i}} \perp n(x_i) \quad \text{for} \quad i = 1, 2.
\]

We define \((V_{\parallel,1}, V_{\parallel,2})\) as

\[
V_{\parallel,i} := \left( V - V_n[-n(X_1)] \right) \cdot \left( \frac{\partial \eta(x_i)}{\partial x_{\parallel,i}} + X_n[-\frac{\partial n(x_i)}{\partial x_{\parallel,i}}] \right).
\]

(2.6)

Therefore we obtain

\[
V(s; t, x, u) = V_n[-n(X_1)] + V_\parallel \cdot \nabla x_i \eta(X_1) - X_nV_\parallel \cdot \nabla x_i n(X_1).
\]

(2.7)

Directly we have

\[
\dot{X}(s; t, x, u) = \dot{X}_1 \cdot \nabla x_i \eta(X_1) + \dot{X}_n[-n(X_1)] - X_n \dot{X}_\parallel \cdot \nabla x_i n(X_1).
\]

Comparing coefficients of normal and tangential components, we obtain that

\[
\dot{X}_n(s; t, x, v) = V_n(s; t, x, v), \quad \dot{X}_\parallel(s; t, x, v) = V_\parallel(s; t, x, v).
\]

(2.8)

On the other hand, from (2.4),

\[
\dot{V}(s) = V_n[-n(X_1)] - V_n \nabla x_i n(X_1) \dot{X}_i + V_\parallel \cdot \nabla x_i \eta(X_1) \dot{X}_1 + V_\parallel \cdot \nabla x_i \eta(X_1) - X_n \nabla x_i n(X_1) \dot{X}_\parallel - X_n \nabla x_i n(X_1) \dot{V}_\parallel - X_n \nabla x_i n(X_1) \dot{V}_\parallel - X_n \nabla x_i n(X_1) \dot{V}_\parallel.
\]

(2.9)

From (2.10) \([-n(X_1)]\), and \( \dot{V} = E \), we obtain that

\[
\dot{V}_n(s) = \left[ V_\parallel(s) \cdot \nabla x_i \eta(X_1(s)) \cdot V_\parallel(s) \cdot n(X_1(s)) \right] + E(s, X(s)) \cdot [-n(X_1(s))] - X_n(s) V_\parallel(s) \cdot \nabla x_i n(X_1(s)) \cdot V_\parallel(s) \cdot n(X_1(s)).
\]

(2.10)

Step 2. We prove (2.3) by the contradiction argument. Assume we choose \((t, x, v)\) satisfying the assumptions of Lemma 1. Let us assume

\[
X_n(t - t_b; t, x, v) + V_n(t - t_b; t, x, v) = 0.
\]

(2.11)

First we choose \(0 < \varepsilon \ll 1\) such that \(X_n(s; t, x, v) \ll 1\) and

\[
V_n(t; t, x, v) \geq 0 \quad \text{for} \quad t \leq t_b(t; x, v) + \varepsilon.
\]

(2.12)

The sole case that we cannot choose such \(\varepsilon > 0\) if there exists \(0 < \delta \ll 1\) such that \(V_n(s; t, x, v) < 0\) for all \(s \in (t - t_b(t; x, v), t - t_b(t; x, v) + \delta)\). But from (2.8) for \(s \in (t - t_b(t; x, v), t - t_b(t; x, v) + \delta)\),

\[
0 \leq X_n(s; t, x, v) = X_n(t - t_b(t; x, v); t, x, v) + \int_{t-t_b(t; x, v)}^s \frac{V_n(\tau; t, x, v)}{V_\parallel(t; x, v)} d\tau < 0.
\]

Now with \(\varepsilon > 0\) in (2.12), temporarily we define \(t_* := t - t_b(t; x, v) + \varepsilon, x_* = X(t - t_b(t; x, v) + \varepsilon; t, x, v), \) and \(v_* = V(t - t_b(t; x, v) + \varepsilon; t, x, v).\) Then \((X_n(s; t, x, v), X_\parallel(s; t, x, v)) = (X_n(s; t_*, x_*, v_*), X_\parallel(s; t_*, x_*, v_*))\) and \((V_n(s; t, x, v), V_\parallel(s; t, x, v)) = (V_n(s; t_*, x_*, v_*), V_\parallel(s; t_*, x_*, v_*)).\)

Now we consider the RHS of (2.10). From (1.41), the first term \( [\dot{V}_\parallel(s) \cdot \nabla x_i \eta(X_1(s)) \cdot V_\parallel(s) \cdot n(X_1(s))] \leq 0.\) By an expansion and (2.2) we can bound the second term

\[
E(s, X(s)) \cdot n(X_1(s))
\]

\[
= E(s, X_n(s), X_\parallel(s)) \cdot n(X_1(s))
\]

\[
= E(s, 0, X_1(s)) \cdot n(X_1(s)) + \|E(s)\|_{C^2} O(|X_1(s)|)
\]

\[
= \|E(s)\|_{C^2} O(|X_n(s)|).
\]

(2.13)

From (2.2) and assumptions of Lemma 1

\[
|V_\parallel(s; t, x, v)| \leq |v| + t_b(t, x, v)||E||_\infty \leq |v| + (1 + t)||E||_\infty.
\]

Combining the above results with (2.10), we conclude that

\[
\dot{V}_n(t; x_*, v_*) \leq (|v| + (1 + t)||E||_\infty)^2 X_n(s; t_*, x_*, v_*),
\]
and hence from (2.8) for $t - t_b(t, x, v) \leq s \leq t_*$,
\[
\frac{d}{ds} [X_n(s; t_*, x, v_*) + V_n(s; t_*, x, v_*]) \leq \langle (\nabla f + (1 + t)|E|) \rangle^2 [X_n(s; t_*, x, v_*) + V_n(s; t_*, x, v_*)].
\] (2.14)

By the Gronwall inequality and (2.11), for $t - t_b(t, x, v) \leq s \leq t_*$,
\[
[X_n(s; t_*, x, v_*) + V_n(s; t_*, x, v_*)] \leq [X_n(t - t_b(t, x, u)) + V_n(t - t_b(t, x, u))] e^{C \epsilon^2 (1 + t)|E|_{\infty}}^2 \leq 0.
\]

From (2.12) we conclude that $X_n(s; t, x, v) \equiv 0$ and $V_n(s; t, x, v) \equiv 0$ for all $s \in [t - t_b(t, x, u), t - t_b(t, x, u) + \epsilon]$. We can continue this argument successively to deduce that $X_n(s; t, x, v) \equiv 0$ and $V_n(s; t, x, v) \equiv 0$ for all $s \in [t - t_b(t, x, v), t]$. Therefore $x_n = 0 = v_n$ which implies $x \in \partial \Omega$ and $n(x) \cdot v = 0$. This is a contradiction since we chose $n(x) \cdot v > 0$ if $x \in \partial \Omega$.

\section*{Lemma 2.}

Assume that, for $\Lambda_1 > 0$, $\delta_1 > 0$,
\[
\sup_{t \geq 0} \epsilon^{\Lambda_1 \dagger} \|E(t)\|_{\infty} \leq \delta_1 \ll 1.
\] (2.15)

We also assume $\psi(t, x, v) \leq C(t, v)$ for some $C > 0$. For $\epsilon$ satisfying
\[
\epsilon > \frac{2\Lambda_1}{\Lambda_1} > 0,
\] (2.16)

there exists a constant $C_{\delta_1, \Lambda_1, \Omega} > 0$ such that, for all $t \geq 0$,
\[
\int_0^t \int_{\gamma \gamma_L} |h| d\gamma ds \leq C_{\delta_1, \Lambda_1, \Omega} \left\{ \|h_0\|_1 + \int_0^t \|h(s)\|_1 + \|\nabla f + v \cdot \nabla v + E \cdot \nabla v + \psi h(s)\|_1 \right\} ds.
\] (2.17)

If $E \in L^\infty$ does not decay but
\[
\|E(t)\|_{\infty} \leq \delta,
\] (2.18)

then for $\epsilon > 0$,
\[
\int_0^t \int_{\gamma \gamma_L} |h| d\gamma ds \leq C_{\delta_1, \epsilon, \Omega} \left\{ \|h_0\|_1 + \int_0^t \|h(s)\|_1 + \|\nabla f + v \cdot \nabla v + E \cdot \nabla v + \psi h(s)\|_1 \right\} ds \right\} ds
\] (2.19),

where we have time-dependent constant $C_{\delta_1, \epsilon, \Omega} > 0$.

\section*{Proof.}

See the proof of Lemma 6 in [4].

\section*{Lemma 3 (Green’s identity).}

For $p \in [1, \infty)$, we assume $f \in L^p_{loc}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$ satisfies
\[
\partial f + v \cdot \nabla f + E \cdot \nabla v f \in L^p_{loc}(\mathbb{R}_+; L^p_{loc}(\Omega \times \mathbb{R}^3)), \quad f \in L^p_{loc}(\mathbb{R}_+; L^p_{\gamma_+}(\gamma_+)).
\]

Then $f \in C^p_{\gamma_+(\Omega), L^p(\Omega \times \mathbb{R}^3)}$ and $f \in L^p_{loc}(\mathbb{R}_+; L^p(\gamma_+))$.

Moreover
\[
\|f(T)\|^p_p + \int_0^T |f|^p_p = \|f(0)\|^p_p + \int_0^T |f|^p_p + p \int_0^T \int_{\Omega \times \mathbb{R}^3} |\partial f + v \cdot \nabla f + E \cdot \nabla v f| |f|^{p-2} f.
\] (2.20)

\section*{Proof.}

See the proof of Lemma 5 in [4].

\section*{Proposition 2.}

Assume the compatibility condition
\[
f_0(x, v) = g(0, x, v) \text{ for } (x, v) \in \gamma_-.\] (2.21)
Let $p \in [1, \infty)$ and $0 < \vartheta < 1/4$. Assume
\[ \nabla_x f_0, \nabla_v f_0 \in L^p(\Omega \times \mathbb{R}^2), \]
\[ \nabla_x v_0 \partial_t g, \nabla_x v_0 \nabla_v g, \nabla_v v_0 \partial_t g, \nabla_v v_0 \nabla_v g \in L^p([0, T] \times \gamma_-), \]
\[ \nabla_x H, \nabla_v H \in L^p([0, T] \times \Omega \times \mathbb{R}^3), \]
\[ e^{-\vartheta |x|^2} \nabla x \psi, e^{-\vartheta |x|^2} \nabla_v \psi \in L^p([0, T] \times \Omega \times \mathbb{R}^3), \]
\[ e^{\vartheta |x|^2} f_0 \in L^\infty(\Omega \times \mathbb{R}^3), e^{\vartheta |x|^2} g \in L^\infty([0, T] \times \gamma_-), \]
\[ e^{\vartheta |x|^2} H \in L^\infty([0, T] \times \Omega \times \mathbb{R}^3). \]

Then for any $T > 0$, there exists a unique solution $f$ to (3.1) such that $\nabla_{x,v} f \in C^0([0, T]; L^p(\Omega \times \mathbb{R}^3)) \cap L^1((0, T); L^p(\gamma))$.

**Proof.** See the proof of Proposition 2 in [4]. \hfill \qed

**Lemma 4.** Assume $E(t,x) \in C^1_x$ is given and (2.24) and
\[ \sup_{t \geq 0} e^{\Lambda_2 \alpha} \| \nabla_x E(t) \|_\infty \leq \delta_2 \ll 1, \] (2.23)
with $\Lambda_2 + \delta_2 + \varepsilon \leq 1$. Then there exists $C > 0$ such that
\[ |\nabla_v X(s; t, x,v)| \leq Ce^{C\delta_2(\Lambda_2)^{-2}} |t-s|, \text{ for all } \max(t - t_b(t, x, v), -\varepsilon) \leq s \leq t. \] (2.24)

**Proof.** See the proof of Lemma 9 in [4]. \hfill \qed

### 3. $L^\infty$ estimate

Let $\varepsilon = + -$ as in (1.9). We set $F^0(t,x,v) \equiv \mu$ and $\phi^0 \equiv 0$. We then apply proposition 2 for $\ell = 0, 1, 2...$ to get a sequence $F^\ell$ such that
\[ \partial_t F^\ell + v \cdot \nabla_x F^\ell + t \nabla \phi^\ell \cdot \nabla_x F^\ell = Q_{\text{gain}}(F^\ell, F^\ell, F^\ell) - Q_{\text{loss}}(F^\ell, F^\ell, F^\ell), \]
\[ -\Delta \phi^\ell = \int_{\mathbb{R}^3} F^\ell - F^\ell \, dv, \quad \int_{\Omega} \phi^\ell \, dx = 0, \quad \left. \frac{\partial \phi^\ell}{\partial n} \right|_{\partial \Omega} = 0, \] (3.1)
and, on $(x,v) \in \gamma_-,$
\[ F^\ell(t,x,v) = c_\mu \mu \int_{\Omega} F^\ell(t,x,u)\{n(x) \cdot u\} \, du, \] (3.2)
and $F^\ell(0,x,v) = F_0(x,v).$

Then $F^\ell$ solves
\[ \left[ \partial_t + v \cdot \nabla_x - q v \nabla \phi^\ell \cdot \nabla_x + v \cdot q \nabla \phi^\ell \right] F^\ell = K f^\ell - q_1 v \cdot \nabla \phi^\ell \cdot \nabla_x \sqrt{\mu} + \Gamma_{\text{gain}}(f^\ell, f^\ell) - \Gamma_{\text{loss}}(f^\ell, f^\ell), \]
\[ -\Delta \phi^\ell = \int_{\mathbb{R}^3} (f^\ell - f^\ell) \sqrt{\mu} \, dv, \quad \int_{\Omega} \phi^\ell \, dx = 0, \quad \left. \frac{\partial \phi^\ell}{\partial n} \right|_{\partial \Omega} = 0, \] (3.3)

Denote the characteristics $(X^\ell_i, V^\ell_i)$ which solves
\[ \frac{d}{ds} X^\ell_i(s; t, x,v) = V^\ell_i(s; t, x,v), \]
\[ \frac{d}{ds} V^\ell_i(s; t, x,v) = -t \nabla \phi^\ell(s, X^\ell_i(s; t, x,v)), \]
\[ t^\ell_1(t,x,v) := \sup \{ s < t : X^\ell_i(s; t, x,v) \in \partial \Omega \}, \]
\[ x^\ell_1(t,x,v) := X^\ell_i(t^\ell_1(t,x,v); t, x, v), \]
\[ t^\ell_{-1}(t,x,v,v_1) := \sup \{ s < t^\ell_1 : X^\ell_i(s; t^\ell_1(t,x,v), v_1) \in \partial \Omega \}, \]
\[ x^\ell_{-1}(t,x,v,v_1) := X^\ell_i(t^\ell_{-1}(t,x,v,v_1); t^\ell_1(t,x,v),v_1), \] (3.5)
and inductively
\[ t^\ell_{-k}(t, x, v_1, \cdots, v_{k-1}) := \sup \{ s < t^\ell_{-k}(t, x, v_1, \cdots, v_{k-1}) : X^\ell_i(s; t^\ell_{-k}(t, x, v_1, \cdots, v_{k-1}), v_{k-1}) \in \partial \Omega \}, \]
\[ x^\ell_{-k}(t, x, v_1, \cdots, v_{k-1}) := X^\ell_i(t^\ell_{-k}(t, x, v_1, \cdots, v_{k-1}); t^\ell_1(t,x,v), x^\ell_{-1}(t,x,v,v_1), v_{k-1}), \] (3.6)
Proposition 3. Assume that for sufficiently small $M > 0$, such that
\[
\|w_\phi f_0\|_\infty < \frac{M}{2},
\] then there exits $T^*(M) > 0$ such that
\[
\sup_{0 \leq t \leq T^*} \max_{\ell} \|w_\phi f^\ell(t)\|_\infty \leq M.
\]
Proof. We define
\[
h^\ell(t, x, v) := w_\phi(v) f^\ell(t, x, v).
\]
By an induction hypothesis we assume
\[
\sup_{0 \leq t \leq T^*} \|h^\ell(t)\|_\infty \leq M.
\]
Then $h^{\ell+1}$ solves
\[
[\partial_t + v \cdot \nabla_x - q \Delta_x \phi^\ell \cdot \nabla_v + \frac{q}{2} v \Delta \phi^\ell - \frac{q}{w_\phi} \nabla \phi^\ell \cdot \nabla_v w_\phi] h^{\ell+1} = K_{w_\phi} h^\ell - q_1 v \cdot \nabla \phi^\ell w_\phi \sqrt{\mu} + w_\phi \Gamma_{\text{gain}} \left( \frac{h^\ell}{w_\phi}, \frac{h^\ell}{w_\phi} \right) - w_\phi \Gamma_{\text{loss}} \left( \frac{h^{\ell+1}}{w_\phi}, \frac{h^\ell}{w_\phi} \right),
\]
where $K_{w_\phi} (\cdot) = w_\phi K(\frac{1}{w_\phi} \cdot )$. The boundary condition is
\[
h^\ell x_{\gamma^{-1}} = c_n w_\phi \mu \int \nabla \phi^\ell \cdot \sqrt{\mu} (n \cdot u) \, du.
\]
We define
\[
\nu^\ell(t, x, v) := \begin{bmatrix} \nu_+^\ell(t, x, v) \\ 0 \end{bmatrix} := \begin{bmatrix} \nu(v) + \frac{v}{2} \cdot \nabla \phi^\ell - \frac{\nabla \phi^\ell \cdot \nabla w_\phi}{w_\phi} \\ 0 \\ \nu(v) - \frac{v}{2} \cdot \nabla \phi^\ell + \frac{\nabla \phi^\ell \cdot \nabla w_\phi}{w_\phi} \end{bmatrix}
\]
From (3.10), for $M \ll 1$, $\|\nabla \phi^\ell\|_\infty \ll 1$ and hence
\[
\nu^\ell(t, x, v) \geq \frac{\nu_0}{2} (v).
\]
Let
\[
g^\ell := -q_1 v \cdot \nabla \phi^\ell \sqrt{\mu} + \Gamma_{\text{gain}} \left( \frac{h^\ell}{w_\phi}, \frac{h^\ell}{w_\phi} \right) := \begin{bmatrix} g_+^\ell \\ g_-^\ell \end{bmatrix}.
\]
Note that
\[
|w_\phi g^\ell| \lesssim \|h^\ell\|_\infty + (v)\|h^\ell\|_\infty^2,
\]
where we have used
\[
|w_\phi \Gamma \left( \frac{h}{w_\phi}, \frac{h}{w_\phi} \right)| \lesssim (v)\|h\|_\infty^2.
\]
Consider the trajectories of $h^{\ell+1}$ and $h^{\ell+1}$ separately from (3.11),
\[
\frac{d}{ds} \left\{ e^{-\int_s^t \nu^\ell(r, X^\ell(s), V^\ell(s), V^\ell(s), V^\ell(s)) \, dr} h^{\ell+1}(s, X^\ell(s), V^\ell(s), V^\ell(s)) \right\} = e^{-\int_s^t \nu^\ell(r, X^\ell(s), V^\ell(s), V^\ell(s)) \, dr} \left\{ K_{w_\phi} \delta^\ell(s, X^\ell(s), V^\ell(s)) + w_\phi g^\ell(s, X^\ell(s), V^\ell(s)) \right\}.
\]
From (3.18) and (3.12), we have
\[
h^{\ell+1}(t, x, v)
\]
We define
\[ \tilde{w}_\theta(v) \equiv \frac{1}{w_\theta(v) \sqrt{\mu(v)}}. \] (3.20)

From (3.12),

the last line of (3.19) = \(1_{t_{i,1} > 0} \int_{t_{i,1}}^{x_{i,1}} \frac{1}{w_\theta(V_i(t_{i,1}^v))} \int_{n(\theta^v)} h_i^x(t_{i,1}, x_{i,1}, v_1) \tilde{w}_\theta(v_1)c_\mu n(x_{i,1}) \cdot v_1 \, dv_1. \)

We define \( \mathcal{V}(x) = \{ v \in \mathbb{R}^3 : n(x) \cdot v > 0 \} \) with a probability measure \( d\sigma = d\sigma(x) \) on \( \mathcal{V}(x) \) which is given by
\[ d\sigma \equiv c_\mu \mu(v) \{ n(x) \cdot v \} \, dv. \] (3.21)

Let
\[ \mathcal{V}_{j,i} := \{ v_j \in \mathbb{R}^3 : n(x_{j,i}^{(j-1)}) \cdot v_j > 0 \}. \] (3.22)

Then inductively we obtain from (3.10), (3.18) and (3.12),
\[ |h_{\epsilon}^{k+1}(t, x, v)| \leq 1_{t_{i,1} > 0} \int_{t_{i,1}}^{x_{i,1}} e^{-f_{i,1}^v} \left| h_{\epsilon}^{k+1}(0, X_{\epsilon}^t(0), V_{\epsilon}^t(0)) \right| + \int_t^x e^{-f_{i,1}^v} |K_{w_{\epsilon}^t} h_{\epsilon}^t + w_{\epsilon}^t g_{\epsilon}^t(s, X_{\epsilon}^t(s, t, x, v), V_{\epsilon}^t(s, t, x, v))| \, ds \]
\[ + 1_{t_{i,1} > 0} \tilde{w}_\theta(V_i(t_{i,1}^v)) \int_{\mathcal{V}_{j,i}} |H|, \] (3.23)

where \(|H|\) is bounded by
\[ \sum_{i=1}^{k-1} 1_{\{t_{i,1}^{(j-1)} < 0 < t_{i,1}^{(j-1)} \}} |h_{\epsilon}^{(j-1)}(0, X_{\epsilon}^{(j-1)}(0; v_1), V_{\epsilon}^{(j-1)}(0; v_1))| \, d\Sigma_{j,i}(0) \] (3.24)
\[ + \sum_{i=1}^{k-1} \int_{\max(t_{i,1}^{(j-1)}, 0)}^{t_{i,1}^{(j-1)}} 1_{\{t_{i,1}^{(j-1)} < 0 < t_{i,1}^{(j-1)} \}} \]
\[ \times |K_{w_{\epsilon}^t} h_{\epsilon}^t + w_{\epsilon}^t g_{\epsilon}^t(s, X_{\epsilon}^{(j-1)}(s; v_1), V_{\epsilon}^{(j-1)}(s; v_1))| \, d\Sigma_{j,i}(s) \, ds \]
\[ + 1_{\{0 < t_{i,1}^{(j-1)} \}} |h_{\epsilon}^{(j-1)}(t_{i,1}, x_{i,1}, v_{i-1})| \, d\Sigma_{j,i}(t_{i,1}, x_{i,1}, v_{i-1}) \] (3.25)

where
\[ d\Sigma_{j,i}^{k-1}(s) = \{ \Pi_{j,i}^{k-1} d\sigma_{j,i} \} \times \{ e^{-f_{j,i}^\nu} \tilde{w}_\theta(v) \} d\Sigma_{i}(s) \times \Pi_{j,i}^{k-1} \{ e^{-f_{j,i}^\nu} \tilde{w}_\theta(v) \} d\Sigma_{j,i}(s), \] (3.26)

and
\[ X_{i}^{j-1}(s; v_1) = X_{i}^{j-1}(s; t_{i,1}^{(j-1)}, x_{i,1}^{(j-1)}, v_{i-1}), \]
\[ V_{i}^{j-1}(s; v_1) = V_{i}^{j-1}(s; t_{i,1}^{(j-1)}, x_{i,1}^{(j-1)}, v_{i-1}). \] (3.27)

\textbf{Step 2-2.} We claim that there exist \( T > 0 \) and \( k_0 > 0 \) such that for all \( k \geq k_0 \) and for all \( (t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3 \), we have
\[ \int_{\Pi_{j,i}^{k-1} \mathcal{V}_{j,i}} 1_{\{t_{i,1}^{(k-1)}(t, x, v, u, v_{i-1}) > 0 \}} d\Sigma_{j,i}^{k-1} \lesssim \Omega \left\{ \frac{1}{2} \right\}^{k/5}. \] (3.28)

The proof of the claim is a modification of a proof of Lemma 14 of [10].

For \( 0 < \delta \ll 1 \) we define
\[ \mathcal{V}_{j,i}^\delta := \{ v_j \in \mathcal{V}_{j,i} : |v_j \cdot n(x_{j,i}^{(j-1)})| > \delta, \ |v_j| \leq \delta^{-1} \}. \] (3.29)

Choose
\[ T = \frac{2}{\delta^{2/3}(1 + \| \nabla \phi \|_{\infty})^{2/3}}. \] (3.30)

We claim that
\[ |t_{j,i}^{j-1} - t_{j+1,i}^{j-1}| \gtrsim \delta^3, \text{ for } v_j \in \mathcal{V}_{j,i}^\delta, \ 0 \leq t \leq T, \ 0 \leq t_{j,i}^{j-1}. \] (3.31)
For $j \geq 1$, 
\[
\left| \int_{t_{j+1}^{(j-1)}}^{t_{j}^{(j-1)}} v_{j} \cdot (t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) ds \right|^2 \\
= \left| x_{j+1,a}^{(j-1)} - x_{j,a}^{(j-1)} \right|^2 \\
\geq \left| (x_{j+1,a}^{(j-1)} - n(x_{j,a}^{(j-1)})) \right| \\
= \left| \int_{t_{j+1}^{(j-1)}}^{t_{j}^{(j-1)}} v_{j} \cdot (t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) \cdot n(x_{j,a}^{(j-1)}) ds \right| \\
= \left| \int_{t_{j+1}^{(j-1)}}^{t_{j}^{(j-1)}} \left( v_j - \int_{t_{j,a}^{(j-1)}}^{s} \nabla \phi^{(j)}(\tau, X_{j,a}^{(j-1)}; t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) \right) \cdot n(x_{j,a}^{(j-1)}) ds \right| \\
\geq |v_j \cdot n(x_{j,a}^{(j-1)})| \left| \int_{t_{j,a}^{(j-1)}}^{t_{j+1,a}^{(j-1)}} \nabla \phi^{(j)}(\tau, X_{j,a}^{(j-1)}; t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) \right| \cdot n(x_{j,a}^{(j-1)}) ds | \\
\geq |v_j \cdot n(x_{j,a}^{(j-1)})| |t_{j,a}^{(j-1)} - t_{j+1,a}^{(j-1)}| - \left| \int_{t_{j,a}^{(j-1)}}^{t_{j+1,a}^{(j-1)}} \nabla \phi^{(j)}(\tau, X_{j,a}^{(j-1)}; t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) \right| \cdot n(x_{j,a}^{(j-1)}) ds |. 
\]

Here we have used the fact if $x, y \in \partial\Omega$ and $\partial\Omega$ is $C^2$ and $\Omega$ is bounded then $|x - y|^2 \gtrsim |(x - y) \cdot n(x)|$. Hence 
\[
|v_j \cdot n(x_{j,a}^{(j-1)})| \lesssim \frac{1}{|t_{j,a}^{(j-1)} - t_{j+1,a}^{(j-1)}|} \left| \int_{t_{j,a}^{(j-1)}}^{t_{j+1,a}^{(j-1)}} V_{j,a}^{(j-1)}(s; t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) ds \right|^2 \\
+ \frac{1}{|t_{j,a}^{(j-1)} - t_{j+1,a}^{(j-1)}|} \left| \int_{t_{j,a}^{(j-1)}}^{t_{j+1,a}^{(j-1)}} \nabla \phi^{(j)}(\tau, X_{j,a}^{(j-1)}; t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) \right| \cdot n(x_{j,a}^{(j-1)}) ds | \\
\lesssim |t_{j,a}^{(j-1)} - t_{j+1,a}^{(j-1)}| \left| |v_j|^2 + \left| t_{j,a}^{(j-1)} - t_{j+1,a}^{(j-1)} \right|^2 \nabla \phi^{(j)} \right|^2 \\
+ \frac{1}{2} \sup_{t_{j,a}^{(j-1)}} \left| \nabla \phi^{(j)}(\tau, X_{j,a}^{(j-1)}; t_{j,a}^{(j-1)}, x_{j,a}^{(j-1)}, v_j) \right| \cdot n(x_{j,a}^{(j-1)}) |. 
\]

(3.33)

For $v_j \in \mathcal{V}_j^\delta$, $0 \leq t \leq T$, and $t_{j,a}^{(j-1)} \geq 0$, 
\[
|v_j \cdot n(x_{j,a}^{(j-1)})| \lesssim \left| t_{j,a}^{(j-1)} - t_{j+1,a}^{(j-1)} \right| \left| |\delta - 2 + T^3 \nabla \phi^{(j)} |^2 + ||\nabla \phi^{(j)}|| \right|. 
\]

We choose $T$ as (3.31) then prove (3.32).

Therefore if $t_{k,a}^{(k-1)} \geq 0$ then there can be at most \( \left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1 \) numbers of $v_m \in \mathcal{V}_{m,a}^\delta$ for $1 \leq m \leq k - 1$.

Equivalently there are at least $k - 2 - \left\lfloor \frac{C_\delta}{\delta} \right\rfloor$ numbers of $v_i \in \mathcal{V}_{i,a} \setminus \mathcal{V}_{k,a}^\delta$ for $0 \leq i \leq m$.

Let us choose $k = N \times \left( \left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1 \right)$ and $N = \left( \left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1 \right) \gg C > 1$. Then we have
\[
\int_{\Pi_{i=1}^{k-1} \mathcal{V}_j} 1 \left( t_{k,a}^{(k-1)}(t, x, v, v^1, \ldots, v^{k-1}) > 0 \right) d^3 \omega_{k-1} \\
\leq \sum_{m=1}^{\left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1} \left\{ \text{there are exactly m of } v_i \in \mathcal{V}_{i,a} \right\} \prod_{j=1}^{k-1} C_{0j} \mu(v_j)^{1/4} d v_j \\
\leq \sum_{m=1}^{\left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1} \left\{ \text{there are exactly m of } v_i \in \mathcal{V}_{i,a} \right\} \prod_{j=1}^{k-1} C_{0j} \mu(v_j)^{1/4} d v_j \\
\leq \left( \left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1 \right) \left( k - 1 \right) \left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1 \left\{ \delta \right\}^{k-2-\left\lfloor \frac{C_\delta}{\delta} \right\rfloor} \left\{ \int_{\mathcal{V}_{i,a}} C_{0j} \mu(v_j)^{1/4} d v_j \right\}^{\left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1} \\
\leq \left\{ C N \right\}^{\left\lfloor \frac{\delta}{C_\delta} \right\rfloor} \left\{ \frac{k}{N} \right\}^2 \left\{ \frac{k}{N} \right\}^{2-k} \left\{ \frac{k}{N} \right\}^{2-k} \leq \left\{ \frac{1}{2} \right\}^k ,
\]

where we have chosen $k = N \times \left( \left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1 \right)$ and $N = \left( \left\lfloor \frac{C_\delta}{\delta} \right\rfloor + 1 \right) \gg C > 1$.

**Step 2-3.** We define a notation 
\[
k_{\rho}(v, u) := \frac{1}{|v - u|} \exp \left\{ -g|v - u|^2 - \frac{|v|^2 - |u|^2}{|v - u|^2} \right\} ,
\]

(3.34)
For $0 < \frac{q}{r} < \varrho$, if $0 < \varrho < \varrho - \frac{q}{r}$ then
\[ k_\varrho(v, u) \frac{e^{\varrho |v|^2}}{e^{\varrho |u|^2}} \lesssim k_\varrho(v, u). \] (3.35)

See the proof in the appendix.

Moreover, for $0 < \frac{q}{r} < \varrho$, (see the proof of Lemma 7 in [13])
\[ \int_{\mathbb{R}^3} k_\varrho(v, u) \frac{e^{\varrho |v|^2}}{e^{\varrho |u|^2}} \, du \lesssim (\varrho)^{-1}. \] (3.36)

Then from (3.16), (3.14), and (3.23)-(3.27), and (3.29), if we choose $\ell \geq k_0$ and $0 \leq t \leq T$ where $k_0$ and $T$ in (3.29), we have
\[ |h^{\ell+1}_t(t, x, v)| \leq C_{k_0} \|e^{-\frac{\varrho}{r} t} h_0\|_\infty \]
\[ + \int_0^t e^{-\frac{\varrho}{r} (t-s)} \int_{\mathbb{R}^3} k_\varrho(V^{\ell}_t(s; t, x, v), u) \|h^{\ell}_t(s, X^\ell_t(s; t, x, v), u)\|_{\mathbb{R}^3} \, du \, ds \]
\[ + C_k \sup_{l} \int_{\max(t_{1,0}^{\ell-1}, 0)}^{t_{1,0}^{\ell-1}} e^{-\frac{\varrho}{r} (t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} k_\varrho(V^{\ell-1}_t(s; v_1), u) \times \|h^{\ell-1}_t(s, X^{\ell-1}_t(s; v_1), u)\| \{n(x_1) \cdot v_1\} \sqrt{\mu(v_1)} \, dv_1 \, du \, ds \]
\[ + \int_{\max(t_{1,0}^{\ell-1}, 0)}^{t_{1,0}^{\ell-1}} \langle V^{\ell}_t(s; t, x, v) \rangle e^{-\frac{\varrho}{r} (t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{-\varrho |v|^2}}{\varrho |u|^2} \, du \, ds \]
\[ + C_k \sup_{l} \int_{\max(t_{1,0}^{\ell-1}, 0)}^{t_{1,0}^{\ell-1}} \|V^{\ell-1}_t(s; v_1)\| \times e^{-\frac{\varrho}{r} (t-s)} \|e^{\frac{\varrho}{r} (t-s)} h^{\ell-1}_t(s)\|_{\infty}^2 \, ds \]
\[ + \int_{\max(t_{1,0}^{\ell-1}, 0)}^{t_{1,0}^{\ell-1}} \|e^{-\frac{\varrho}{r} (t-s)} \nabla \varphi \ell(s)\|_{\infty} \, ds \]
\[ + C_k \sup_{l} \int_{\max(t_{1,0}^{\ell-1}, 0)}^{t_{1,0}^{\ell-1}} \|e^{-\frac{\varrho}{r} (t-s)} \nabla \varphi^{\ell-1}(s)\|_{\infty} \, ds \]
\[ + \left\{ \frac{1}{2} \right\}^{k/5} \|e^{-\frac{\varrho}{r} (t-\ell(k-1))} h(t_{k-1}(k-1))\|_{\infty}, \]
where we used the abbreviation of (3.25).

From $\int_0^t \langle V^{\ell-1}_t(s; v_1) \rangle e^{-\frac{\varrho}{r} (t-s)} \|V^{\ell-1}_t(s; v_1)\|_{\mathbb{R}^3} \, ds \lesssim 1$ and (3.36), we derive that
\[ \|h^{\ell+1}_t(t)\|_{\infty} \lesssim \|h^{\ell+1}_0(t)\|_{\infty} + \|h^{\ell+1}_t(t)\|_{\infty} \lesssim_k \|h(0)\|_{\infty} + o(1) \|h(t_{k-1}(k-1))\|_{\infty} + \max_{t \geq 0} \sup_{0 \leq s \leq t} \|h^{\ell-1}(s)\|_{\infty} + \max_{t \geq 0} \sup_{0 \leq s \leq t} \|h^{\ell}(s)\|_{\infty}^2. \] (3.38)

By taking supremum in $\ell$ and choosing $M \ll 1$ and $0 \leq t \leq T^* \leq T$ with $T^* \ll 1$, we conclude (3.3).

\[ \square \]

4. Weighted $W^{1,p}$ estimates

**Proposition 4.** The main goal of this section is to prove the following weighted $W^{1,p}$ estimate for the sequence $f^\ell$ in (3.3). Let us choose $0 < \varrho < \varrho - \frac{q}{r} \ll 1$ and
\[ \frac{p-2}{p} < \beta < \frac{2}{3}, \quad \text{for} \quad 3 < p < 6. \] (4.1)

Assume $f^\ell$ solves (3.3), and for some $T > 0$
\[ \sup_{t \geq 0} \sup_{0 \leq s \leq T} \|w_0 f^\ell(t)\|_{\infty} \ll 1, \] (4.2)
\[ \sup_{t \geq 0} \sup_{0 \leq s \leq T} e^{\Lambda t} \|\nabla \varphi f^\ell(t)\|_{\infty} < \delta_1, \] (4.3)
Proof. \(\alpha\) Then for \(\iota\) where we define, for \(0 < \epsilon < 1\),

\[
\mathcal{E}'(t) := \|w_0 f(t)\|_\infty + \|w_\alpha f(t)\|_p + \|w_\alpha \alpha_{\iota,\epsilon} \nabla \nabla f(t)\|_p + \|\nabla \nabla f(t)\|_{p,\iota} < \infty.
\]

where we define, for \(0 < \epsilon < 1\),

\[
\mathcal{E}^{\iota+1}(t) := \|w_0 f^{\iota+1}(t)\|_\infty + \|w_\alpha f^{\iota+1}(t)\|_p + \|w_\alpha \alpha_{\iota,\epsilon} \nabla \nabla f^{\iota+1}(t)\|_p + \|\nabla \nabla f^{\iota+1}(t)\|_{p,\iota}.
\]

To prove this, we need the following results:

**Proposition 5.** Assume \(\phi_f(t, x)\) obtained from (1.16) with \(\nabla \phi_f\) satisfies (2.15) and

\[
sup_{t \geq 0} e^{\lambda t} \|\nabla^2 \phi_f(t)\|_\infty \leq \delta_2 \ll 1.
\]

Then for \(\iota = + \) or \(-\) as in (1.19), for all \(0 < \sigma < 1\) and \(N > 1\) and for all \(s \geq 0\), \(x, \in \Omega\),

\[
\int_{|u| \leq N} \frac{du}{\alpha_{\iota,\epsilon}(s, u, x)^{p}} \lesssim_{\sigma, \Omega, \Lambda_1, \Lambda_2, N, \kappa} 1,
\]

and, for any \(0 < \kappa \leq 2\),

\[
\int_{|u| \geq N} e^{-C|u-u|^2} \frac{1}{\alpha_{\iota,\epsilon}(s, u, x)^{p}} du \lesssim_{\sigma, \Omega, \Lambda_1, \Lambda_2, N, \kappa} 1.
\]

**Proof.** It is important to note that from (2.15) we have \(n(x) \cdot \nabla \phi_f = 0\) for all \(x, \in \Omega\). Thus for both trajectories \((X_\alpha; s, t, x, v), V_\alpha(s, t, x, v),\) their corresponding fields \(\nabla \phi_f \cdot n(x) = 0\). Therefore we can apply Proposition 3 from [1] to \(\alpha_{\iota,\epsilon}^{+}\) and \(\alpha_{\iota,\epsilon}^{-}\) separately to conclude (1.18) and (1.19).

**Lemma 5.** For any \(0 < \delta < 1\), we claim that if \((f, \phi_f)\) solves (1.10) then

\[
\|\phi_f(t)\|_{C^{1,1}(\Omega)} \lesssim_{\delta, \Omega} \|w_\alpha f(t)\|_\infty \text{ for all } t \geq 0.
\]

**Proof.** We have, for any \(p > 1\),

\[
\left\| \int_{\mathbb{R}^3} (f_+ - f_-) \sqrt{\mu(v)} dv \right\|_{L^p(\Omega)} \leq \Omega^{1/p} \left( \int_{\mathbb{R}^3} w_\alpha(v)^{-1} \sqrt{\mu(v)} dv \right) \|w_\alpha f(t)\|_\infty.
\]

Then we apply the standard elliptic estimate to (1.10) and deduce that

\[
\|\phi_f(t)\|_{W^{2,p}(\Omega)} \lesssim \|w_\alpha f(t)\|_\infty.
\]

On the other hand, from the Morrey inequality, we have, for \(p > 3\) and \(\Omega \subset \mathbb{R}^3\),

\[
\|\phi_f(t)\|_{C^{1,1}(\Omega)} \lesssim_{\delta, \Omega} \|\phi_f(t)\|_{W^{2,p}(\Omega)}.
\]

Now we choose \(p = 3/\delta\) for \(0 < \delta < 1\). Then we can obtain (1.10).

To close the estimate, we use the following lemma crucially.

**Lemma 6.** Assume (2.1). If \(\phi_f\) solves (1.10) then

\[
\|\phi_f(t)\|_{C^{2,1-1/\bar{p}}(\Omega)} \lesssim_{p, \Omega} \left( \int_{\mathbb{R}^3} (f_+ - f_-) \sqrt{\mu} dv \right)_{C^{0,1-1/\bar{p}}(\Omega)} \text{ for } p > 3.
\]

**Proof.** Applying the Schauder estimate to (1.10), we deduce

\[
\|\phi_f(t)\|_{C^{2,1-1/\bar{p}}(\Omega)} \lesssim_{p, \Omega} \left( \int_{\mathbb{R}^3} (f_+ - f_-) \sqrt{\mu} dv \right)_{C^{0,1-1/\bar{p}}(\Omega)} \text{ for } p > 3.
\]

By the Morrey inequality, \(W^{1,p} \subset C^{0,1-1/\bar{p}}\) with \(p > 3\) for a domain \(\Omega \subset \mathbb{R}^3\) with a smooth boundary \(\partial \Omega\), we derive

\[
\left\| \int_{\mathbb{R}^3} (f_+ - f_-) \sqrt{\mu} dv \right\|_{C^{0,1-1/\bar{p}}(\Omega)} \lesssim \left( \int_{\mathbb{R}^3} \mu^{-1/2} dv \int_{\mathbb{R}^3} f(t)_{L^p(\Omega \times \mathbb{R}^3)} + \int_{\mathbb{R}^3} \nabla_x f(t) - f(t) \sqrt{\mu} dv \right)_{L^p(\Omega)}.
\]
By the Hölder inequality, for $t = +$ or $-$ as in (1.1),
\[
\left| \int_{\mathbb{R}^3} \nabla_x f_t(t, x, v) \sqrt{\mu(v)} dv \right| \\
\leq \left\| \frac{\sqrt{\mu(v)}}{\alpha_{f, t, c}(t, x, \cdot)} \right\|_{L_p^p(\mathbb{R}^3)} \left\| \alpha_{f, t, c}(t, x, \cdot)^{\beta} \nabla_x f_t(t, x, \cdot) \right\|_{L_p^q(\mathbb{R}^3)} \\
= \left( \int_{\mathbb{R}^3} \frac{\mu(v)}{\alpha_{f, t, c}(t, x, v)} dv \right)^{\frac{p-1}{p}} \left\| \alpha_{f, t, c}(t, x, \cdot)^{\beta} \nabla_x f_t(t, x, \cdot) \right\|_{L_p^q(\mathbb{R}^3)}.
\]
(4.14)

Note that $\frac{2}{p-1} < \frac{p}{2} < \frac{2}{p} < 1$ from (1.1). We apply Proposition 3 and conclude that (4.14) \( \lesssim 1 \). Taking $L^p(\Omega)$-norm on (4.14) and from (4.13), we conclude (4.11).

We need some basic estimates to prove Proposition 1. Recall the decomposition of $L$ in (1.18). From (1.19)
\[
|\nabla_v \nu(v)| \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\omega| |\mu(u)| dw du \lesssim 1.
\]
(4.15)

Recall the definition of $k_{\beta}(v, u)$ from (3.34). From (1.20) and a direction computation, for $0 < \beta < \frac{1}{2}$,
\[
|\partial_{v_i} k_1(v, u + v)| = C_{k_1} |v_i| e^{-\frac{|v_i^2 + (u + v)^2|}{4}} \lesssim k_{\beta}(v, u + v),
\]
(4.16)

and
\[
|\partial_{v_i} k_2(v, u + v)| = C_{k_2} |v_i| e^{-\frac{|v_i^2 + (u + v)^2|}{4}} \lesssim k_{\beta}(v, u + v).
\]
(4.17)

For $g_1, g_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$, let $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, we define
\[
K_{\beta} g(v) := \int_{\mathbb{R}^3} \nabla_v k_2(v, u) (3g_1(u) + g_2(u)) du - \int_{\mathbb{R}^3} \nabla_v k_1(v, u) (g_1(u) + g_2(u)) du.
\]
(4.18)

From (3.65), (4.17), and (4.17),
\[
|w_{\beta} K_{\beta} g(v)| \lesssim \sum_i \int_{\mathbb{R}^3} |k_i(v, u + v)| \frac{w_{\beta}(v)}{w_{\beta}(u + v)} (|w_{\beta} \nabla_v g_1(u + v)| + |w_{\beta} \nabla_v g_2(u + v)|) du
\]
\[
\lesssim \int_{\mathbb{R}^3} k_{\beta}(v, u) w_{\beta} |\nabla_v g(u)| du,
\]
\[
|w_{\beta} K_{\beta} g(v)| \lesssim \int_{\mathbb{R}^3} |\nabla_v k_2(v, u + v)| \frac{w_{\beta}(v)}{w_{\beta}(u + v)} (|w_{\beta} g_1(u + v)| + |w_{\beta} g_2(u + v)|) du
\]
\[
\lesssim \int_{\mathbb{R}^3} k_{\beta}(v, u) w_{\beta}(v) |w_{\beta} g(u)| du
\]
\[
\lesssim \|w_{\beta} g\|_{L_{\infty}}.
\]
(4.19)

For $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ and $h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$, the nonlinear Boltzmann operator $\Gamma(g, h)$ in (1.22) equals
\[
\Gamma(g, h) = \begin{bmatrix} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega|(h_1 + h_2)(v + u_\perp) g_1(v + u_\parallel) \sqrt{\mu(v + u)} d\omega dv du - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega|(h_1 + h_2)(v + u_\parallel) g_2(v + u_\parallel) \sqrt{\mu(v + u)} d\omega dv du \\
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega|(h_1 + h_2)(v + u_\perp) g_2(v + u_\parallel) \sqrt{\mu(v + u)} d\omega dv du - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega|(h_1 + h_2)(v + u_\parallel) g_1(v) \sqrt{\mu(v + u)} d\omega dv du \\
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega|(h_1 + h_2)(v + u_\parallel) g_2(v) \sqrt{\mu(v + u)} d\omega dv du - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega|(h_1 + h_2)(v + u_\parallel) g_1(v) \sqrt{\mu(v + u)} d\omega dv du \\
\end{bmatrix}.
\]
(4.20)
where \( u_1 = (u \cdot \omega) \omega \) and \( u_\perp = u - u_1 \). Following the derivation of (1.20) in Chapter 3 of [9], by exchanging the role of \( \sqrt{\mu} \) and \( w^{-1} \), we have

\[
|w_{\varnothing} \Gamma(g, h)| \lesssim \|w_{\varnothing} g\|_\infty \int_{\mathbb{R}^3} k_{\varnothing}(v, u) |w_{\varnothing} h(u)| \, du,
\]

\[
|w_{\varnothing} \Gamma(g, h)| \lesssim \|w_{\varnothing} h\|_\infty \left( \int_{\mathbb{R}^3} k_{\varnothing}(v, u) |w_{\varnothing} g(u)| \, du + \langle v \rangle |w_{\varnothing} g(v)| \right) .
\]

By direct computations

\[
\nabla_v \Gamma(g, h)(v) = \nabla_v \Gamma_{\text{gain}}(g, h) - \nabla_v \Gamma_{\text{loss}}(g, h)
\]

\[
= \Gamma_{\text{gain}}(\nabla_v g, h) + \Gamma_{\text{gain}}(g, \nabla_v h) - \Gamma_{\text{loss}}(\nabla_v g, h) - \Gamma_{\text{loss}}(g, \nabla_v h) + \Gamma_v(g, h).
\]

Here we have defined

\[
\Gamma_v(g, h)(v) := \Gamma_{v, \text{gain}} - \Gamma_{v, \text{loss}}
\]

\[
= \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} |u \cdot \omega|(h_1 + h_2)(v + u \perp) g_1(v + u_1) \nabla_v \sqrt{\mu(v + u)} \, d\omega du - \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} |u \cdot \omega|(h_1 + h_2)(v + u) g_1(v) \nabla_v \sqrt{\mu(v + u)} \, d\omega du \right) .
\]

Note that

\[
|w_{\varnothing} \Gamma_{\text{gain}}(\nabla_v g, h)| + |w_{\varnothing} \Gamma_{\text{gain}}(g, \nabla_v h)|
\]

\[
\lesssim \left( \|w_{\varnothing} g\|_\infty + \|w_{\varnothing} h\|_\infty \right) \left( \|w_{\varnothing} \Gamma_{\text{gain}}(\nabla_v g, w_{\varnothing}^{-1})| + |w_{\varnothing} \Gamma_{\text{gain}}(w_{\varnothing}^{-1}, \nabla_v h)| \right)
\]

\[
\lesssim \left( \|w_{\varnothing} g\|_\infty + \|w_{\varnothing} h\|_\infty \right) \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \left( (u - v) \cdot \omega \right) \frac{|w_{\varnothing}(v)|}{|w_{\varnothing}(u)|} \left( \frac{\nabla_v h(u')}{w_{\varnothing}(u')} + \frac{\nabla_v g(v')}{w_{\varnothing}(v')} \right) \, d\omega du.
\]

Then following the derivation of (1.20) in Chapter 3 of [9], by exchanging the role of \( \sqrt{\mu} \) and \( w_{\varnothing}^{-1} \), we can obtain a bound of

\[
|w_{\varnothing} \Gamma_{\text{gain}}(\nabla_v g, h)| + |w_{\varnothing} \Gamma_{\text{gain}}(g, \nabla_v h)| \lesssim \left( \|w_{\varnothing} g\|_\infty + \|w_{\varnothing} h\|_\infty \right) \int_{\mathbb{R}^3} k_{\varnothing}(v, u) \frac{|w_{\varnothing}(v)|}{w_{\varnothing}(u)} (|w_{\varnothing} \nabla_v g(u)| + |w_{\varnothing} \nabla_v h(u)|) \, du
\]

\[
\lesssim \left( \|w_{\varnothing} g\|_\infty + \|w_{\varnothing} h\|_\infty \right) \int_{\mathbb{R}^3} k_{\varnothing}(v, u) (|w_{\varnothing} \nabla_v g(u)| + |w_{\varnothing} \nabla_v h(u)|) \, du.
\]

Clearly

\[
|w_{\varnothing} \Gamma_{\text{loss}}(g, \nabla_v h)| \lesssim \|w_{\varnothing} g\|_\infty \int_{\mathbb{R}^3} \frac{|w_{\varnothing}(v)|}{w_{\varnothing}(u)} |w_{\varnothing} \nabla_v h(u)| \, d\omega du
\]

\[
\lesssim \|w_{\varnothing} g\|_\infty \int_{\mathbb{R}^3} k_{\varnothing}(v, u) |w_{\varnothing} \nabla_v h(u)| \, du,
\]

\[
|w_{\varnothing} \Gamma_{\text{loss}}(\nabla_v g, h)| \lesssim \langle v \rangle \|w_{\varnothing} h\|_\infty |w_{\varnothing} \nabla_v g(v)|.
\]

For \( \Gamma_{v, \text{loss}}(g, h) \) defined in (4.23),

\[
|w_{\varnothing} \Gamma_{v, \text{loss}}(g, h)|
\]

\[
\lesssim \frac{|w_{\varnothing}(v)|}{w_{\varnothing}(v')} |w_{\varnothing} g| \int_{\mathbb{R}^3 \times \mathbb{R}^2} |(u - v) \cdot \omega| \frac{1}{w_{\varnothing}(u)} |w_{\varnothing} h(u)| \nabla_v \sqrt{\mu(u)} \, d\omega du
\]

\[
\lesssim \langle v \rangle \|w_{\varnothing} g\| \|w_{\varnothing} h\|_\infty.
\]

For \( \Gamma_{v, \text{gain}}(g, h) \), following the derivation of (1.20) in Chapter 3 of [9], by exchanging the role of \( \sqrt{\mu} \) and \( w_{\varnothing}^{-1} \)

\[
|w_{\varnothing} \Gamma_{v, \text{gain}}(g, h)| \lesssim \|w_{\varnothing} h\|_\infty \int_{\mathbb{R}^3 \times \mathbb{R}^2} |(u - v) \cdot \omega| \frac{|w_{\varnothing}(v)|}{w_{\varnothing}(v')} \frac{|w_{\varnothing} g(v')|}{w_{\varnothing}(v')} \nabla_v \sqrt{\mu(u)} \, d\omega du
\]

\[
\lesssim \|w_{\varnothing} h\|_\infty \int_{\mathbb{R}^3} k_{\varnothing}(v, u) |w_{\varnothing} g(u)| \, du.
\]

The next result is about estimates of derivatives on the boundary. Assume \( (4.22) \) and \( (4.23) \). We claim that for \((x, v) \in \gamma_\gamma\),

\[
|\nabla_{x,v} f^{\ell+1}(t, x, v)| \lesssim \langle v \rangle \sqrt{\mu(v)} \left( 1 + \frac{1}{|n(x) \cdot v|} \right) \times (4.20).
\]

(4.28)
\[
\int_{\{x: u > 0\}} \left\{ \langle u \rangle |f^{t+1} + |f^{t}\rangle + \langle u \rangle \langle f^{t+1} + |f^{t}\rangle \rangle + \mu(u) \langle f^{t+1} + |f^{t}\rangle \rangle \right\} d\mu(u) \\
+ \langle u \rangle \langle f^{t+1} + |f^{t}\rangle \rangle + \mu(u) \langle f^{t+1} + |f^{t}\rangle \rangle \right\} \sqrt{\mu(u)} (n(x) \cdot u) du.
\]

From (3.3),
\[
\partial_t f^{t+1}(t, x, v) = \frac{-1}{n(x) \cdot v} \left( \partial_t f^{t+1} + \sum_{i=1}^{2n} (v \cdot \tau_i) \partial_{\tau_i} f^{t+1} - q \nabla_x \phi^t \cdot \nabla_x f^{t+1} + q \nabla_x \phi^t \cdot \nabla_x f^{t+1} \right.
\]
\[
+ \frac{1}{2} \nabla_x \phi^t f^{t+1} + \nu f^{t+1} - K f^t - \Gamma_{\text{gain}}(f^t, f^{t+1}) + \Gamma_{\text{loss}}(f^t, f^{t+1}) + q_1 v \cdot \nabla_x \phi^t \sqrt{\mu} \left\}
\]

Let \(\tau_1(x)\) and \(\tau_2(x)\) be unit tangential vectors to \(\partial\Omega\) satisfying \(\tau_1(x) \cdot n(x) = 0 = \tau_2(x) \cdot n(x)\) and \(\tau_1(x) \times \tau_2(x) = n(x)\). Define the orthonormal transformation from \([n, \tau_1, \tau_2]\) to the standard basis \(\{e_1, e_2, e_3\}\), i.e. \(T(x)n(x) = e_1, T(x)\tau_1(x) = e_2, T(x)\tau_2(x) = e_3\), and \(T^{-1} = T^\top\). Upon a change of variable: \(u' = T(x)u\), we have
\[
n(x) \cdot u = (n(x) \cdot T^\top(x)u') = (n(x)' T^\top(x)u') = [T(x)n(x)]' u' = e_1 \cdot u' = u_1',
\]
then the RHS of the diffuse BC (52) equals
\[
c_\mu \sqrt{\mu(v)} \int_{u_1' > 0} f^t(t, x, T^\top(x)u') \sqrt{\mu(u')} (u_1') du'.
\]

Then we can further take tangential derivatives \(\partial_{\tau_i}\) as, for \((x, v) \in \gamma_\tau\),
\[
\partial_{\tau_i} f^{t+1}(t, x, v) = \frac{c_\mu \sqrt{\mu(v)}}{n(x) \cdot v} \left( \partial_{\tau_i} f^{t+1}(t, x, u) \sqrt{\mu(u)} (n(x) \cdot u) du \right.
\]
\[
+ c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} \nabla_v f^t(t, x, u) \frac{\partial T^\top(x)}{\partial \tau_i} T^\top(x) u \sqrt{\mu(u)} (n(x) \cdot u) du.
\]

We can take velocity derivatives directly to (4.34) and obtain that for \((x, v) \in \gamma_\tau\),
\[
\nabla_v f^{t+1}(t, x, v) = c_\mu \nabla_v \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} f^t(t, x, u) \sqrt{\mu(u)} (n(x) \cdot u) du,
\]
\[
\partial_t f^{t+1}(t, x, v) = c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} \partial_t f^t(t, x, u) \sqrt{\mu(u)} (n(x) \cdot u) du.
\]

For the temporal derivative, we use (1.24) again to deduce that
\[
\partial_t f^{t+1}(t, x, v) = \frac{c_\mu \sqrt{\mu(v)}}{n(x) \cdot u} \left\{ \langle u \rangle \cdot \nabla_x \phi^t f^{t+1} + \langle u \rangle \nabla_x f^{t+1} + \langle u \rangle \nabla_x \phi^t f^{t-1} + \nu f^{t+1} - K f^t \right.
\]
\[
+ \Gamma_{\text{gain}}(f^t, f^{t+1}) - \Gamma_{\text{loss}}(f^t, f^{t+1}) - q_1 u \cdot \nabla_x \phi^t f^{t-1} \right\} \sqrt{\mu(u)} (n(x) \cdot u) du.
\]

From (4.30), (4.33), (4.20), and (4.21), we conclude (4.29).

**Proof of Proposition 4**

**Step 1.** Note that by our choice of \(f^1\), we have \(\partial_t f^1(t, x, v) \mid_{\gamma_\tau} = 0\). Therefore combining (4.31), (4.30), and (4.32) and the assumption that \(|\nabla_{\tau, v} f_0|_{l, p} < \infty\), we get (4.5) is valid for \(l = 1\).

Thus it suffices to prove the following induction statement: there exist \(T^{**} < 1\) (and \(T^{**} < T^*(M)\)) and \(C > 0\) such that
\[
\text{if } \max_{0 \leq m \leq \ell} \sup_{0 \leq t \leq T^{**}} \mathcal{E}^{m}(t) \leq C \left\{ ||w_0 f_0||_p + ||w_0 \alpha f_0||_p + ||w_0 \alpha \phi f_0||_p + ||\nabla_{\tau, v} f_0||_{l, p} \right\} < \infty,
\]
then \( \sup_{0 \leq t \leq T^{**}} \mathcal{E}^{\ell+1}(t) \leq C \left\{ ||w_0 f_0||_p + ||w_0 \alpha f_0||_p + ||w_0 \alpha \phi f_0||_p + ||\nabla_{\tau, v} f_0||_{l, p} \right\} \).

Define
\[
\nu_\phi(t, x, v) := \begin{bmatrix} \langle \nu \rangle + \frac{c_\mu}{\sqrt{2}} \cdot \nabla_x \phi \langle \phi \rangle + 0 \\ 0 \end{bmatrix} \cdot \nabla_x \phi \langle \phi \rangle.
\]

From the assumption (4.2), we have that \(\nu + \frac{c_\mu}{\sqrt{2}} \cdot \nabla_x \phi \langle \phi \rangle \geq \frac{\nu(v)}{\sqrt{2}}\), and \(\nu - \frac{c_\mu}{\sqrt{2}} \cdot \nabla_x \phi \langle \phi \rangle \geq \frac{\nu(v)}{\sqrt{2}}\).
From (1.16), (1.24), and (4.21), we can easily obtain that, for $0 < \varrho < 1$

$$
\|w_\varrho f^{t+1}(t)\|_p^p + \int_0^t \|\nu_\varrho^{1/p} w_\varrho f^{t+1}\|_p^p + \int_0^t \|w_\varrho f^{t+1}\|_{p,+}^p

\lesssim \|w_\varrho f(0)\|_p^p + (1 + \|w_\varrho f\|_\infty) \int_0^t \int_{\Omega \times \mathbb{R}^3} |w_\varrho f^{t+1}(v)|^{p-1} \int_{\mathbb{R}^3} k_\varrho(v, u) \frac{w_\varrho(v)}{w_\varrho(u)} |w_\varrho f(u)| \mathrm{d}u

+ \|w_\varrho f\|_\infty \int_0^t \int_{\Omega \times \mathbb{R}^3} \langle v \rangle |w_\varrho f^{t+1}|^p + o(1) \int_0^t \|w_\varrho f^{t+1}\|_p^p + \int_0^t \|\nabla \varphi\|_p^p + \int_0^t \|w_\varrho f^{t+1}\|_{p,+}^p.
$$

(4.36)

Note that by the Hölder inequality, (3.36), and (3.35),

$$
\int_{\mathbb{R}^3} |w_\varrho f^{t+1}(v)|^{p-1} \int_{\mathbb{R}^3} k_\varrho(v, u) |w_\varrho f^t(u)| \mathrm{d}u \lesssim \|w_\varrho f^{t+1}\|_{L^p_{\varrho \nu}} \left\| \left( \int_{\mathbb{R}^3} k_\varrho(v, u) |w_\varrho f^t(u)| \mathrm{d}u \right)^{1/q} \right\|_{L^p_{\varrho \nu}}^{1/p}.
$$

(4.37)

From a standard elliptic theorem and (4.10), we have

$$
\int_0^t \|\nabla \varphi\|_p^p \lesssim \int_0^t \|w_\varrho f\|_p^p.
$$

(4.38)

Now we focus on $\int_0^t |w_\varrho f^{t+1}|_{p,+}^p$ in (4.36). We plug in (3.2) and then decompose $\gamma_+ \cup \gamma_+ \mathbb{R}$ where $\varepsilon$ is small but satisfies (2.16). This leads

$$
\int_0^t \|w_\varrho f^{t+1}\|_{p,+}^p \lesssim \int_0^t \|w_\varrho f^t\|_{p,+}^p + (1 + \|w_\varrho f\|_\infty) \int_0^t \|w_\varrho f^{t-1}\|_p^p + \int_0^t \|w_\varrho f^{t+1}\|_p^p.
$$

(4.39)

Collecting terms from (4.36), (4.37), (4.38), and (4.39), we conclude that for $\sup_{t \geq 0} \|w_\varrho f^t\|_\infty \ll 1$,

$$
\|w_\varrho f^{t+1}(t)\|_p^p + \int_0^t \|\nu_\varrho^{1/p} w_\varrho f^{t+1}\|_p^p + \int_0^t \|w_\varrho f^{t+1}\|_{p,+}^p

\lesssim \|w_\varrho f(0)\|_p^p + (1 + \|w_\varrho f\|_\infty) \int_0^t \|w_\varrho f^{t-1}\|_p^p + \int_0^t \|w_\varrho f^{t+1}\|_{p,+}^p

\lesssim \|w_\varrho f(0)\|_p^p + (1 + \|w_\varrho f\|_\infty) \int_0^t \|w_\varrho f^{t-1}\|_p^p + \int_0^t \|w_\varrho f^{t+1}\|_{p,+}^p.
$$

(4.40)

Step 2. By taking derivatives $\partial \in \{\nabla x, \nabla v\}$ to (3.39),

$$
[\partial_t + v \cdot \nabla f - q \nabla \varphi f \cdot \nabla v + \nu_\varrho \partial f^{t+1}] = w_\varrho G^{t+1}.
$$

(4.41)

where

$$
G^{t+1} = - \partial v \cdot \nabla f^{t+1} + q \partial \nabla \varphi f \cdot \nabla f^{t+1} + \partial \Gamma_{\text{gain}}(f^t, f^t) - \partial \Gamma_{\text{loss}}(f^{t+1}, f^t) - \partial \left[ \nu(v) + \frac{p}{2} \cdot \nabla \varphi f(t, x) \right] f^{t+1} - \partial K f^t - q_1 \partial (v \cdot \nabla \varphi f \cdot \sqrt{p}).
$$

(4.42)
Here we have used
\[ \nu_{\phi, w} = \nu_{\phi, w}(t, x, v) := \begin{bmatrix} \nu(v) + \frac{v}{2} \cdot \nabla \phi^f(t, x) + \frac{\nabla_x \phi^f \cdot \nabla_v w_\phi}{w_\phi} & 0 \\ 0 & \nu(v) - \frac{v}{2} \cdot \nabla \phi^f(t, x) + \frac{\nabla_x \phi^f \cdot \nabla_v w_\phi}{w_\phi} \end{bmatrix}. \] (4.43)

Denote
\[ \nu_{\phi, w} := \nu(v) + \frac{v}{2} \cdot \nabla \phi^f(t, x) + \frac{\nabla_x \phi^f \cdot \nabla_v w_\phi}{w_\phi}, \nu_{\phi, w} := \nu(v) - \frac{v}{2} \cdot \nabla \phi^f(t, x) + \frac{\nabla_x \phi^f \cdot \nabla_v w_\phi}{w_\phi}. \]

From (4.33) and (4.41), for \( \ell = 0 \) and \( - \) we have
\[ \frac{1}{p} |w_\phi \alpha_{\ell, \ell} \partial f_{\ell+1}^{f+1}|p-1 |[\partial_t + v \cdot \nabla_x - \ell \nabla \phi^f \cdot \nabla_v + \nu_{\phi, w}]|w_\phi \alpha_{\ell, \ell} \partial f_{\ell+1}^{f+1}| = \alpha_{\ell, \ell} |w_\phi \partial f_{\ell+1}^{f+1}|p-1 |[\partial_t + v \cdot \nabla_x - \ell \nabla \phi^f \cdot \nabla_v + \nu_{\phi, w}]|w_\phi \partial f_{\ell+1}^{f+1}| = \alpha_{\ell, \ell} |\partial f_{\ell+1}^{f+1}|p-1 |\partial f_{\ell+1}^{f+1}|. \] (4.44)

From (4.15), (4.43), (4.18), and (4.23)
\[ |G| \lesssim |\nabla_\nu f^{f+1}| + |\nabla^2 \phi^f||\nabla_\nu f^{f+1}| + |\Gamma_{\text{gain}}(\partial f^f, \partial f^g)| + |\Gamma_{\text{gain}}(\partial f^f, \partial f^g)| + |\Gamma_{\text{loss}}(\partial f^{f+1}, \partial f^g)| + |\Gamma_{\text{loss}}(\partial f^{f+1}, \partial f^g)| + |K \partial f^g| + |f^{\ell+1}| + |\Gamma_{\text{gain}}(f^{\ell+1}, f^g)| + |\Gamma_{\text{gain}}(f^{\ell+1}, f^g)| + |K \partial f^g| + w_\phi (v)^{-1/2}(|\nabla \phi^f| + |\nabla^2 \phi^f|) (1 + ||w_\phi f^{f+1}||_\infty). \] (4.45)

Now we apply Lemma 3 to (4.44) to both \( f^{f+1} \) and \( f^{\ell+1} \) separately and add them together to obtain
\[ \|w_\phi \alpha_{\ell, \ell} \partial f^{f+1}(t)\|_p^p + \int_0^t \|w_\phi \alpha_{\ell, \ell} \partial f^{f+1}(t)\|_p^p dt + \int_0^t |w_\phi \alpha_{\ell, \ell} \partial f^{f+1}|p^p dt \leq \|w_\phi \alpha_{\ell, \ell} \partial f(0)\|_p^p + \int_0^t \|w_\phi \alpha_{\ell, \ell} \partial f^{f+1}(t)\|_p^p dt + \int_0^t \|w_\phi \alpha_{\ell, \ell} \partial f^{f+1}(t)\|_p^p dt. \] (4.46)

First we consider (4.46). Directly, the contribution of \(|\nabla_\nu f^{f+1}| + |\nabla^2 \phi^f||\nabla_\nu f^{f+1}| \) of (4.45) in (4.46) is bounded by
\[ 1 + \sup_{0 \leq s \leq t} \|w_\phi \phi^f(s)\|_\infty + \sup_{0 \leq s \leq t} \|w_\phi f^{f+1}(s)\|_\infty. \] (4.47)

From (4.19), (4.23), and (4.25), the contribution of \(|\Gamma_{\text{gain}}(f^f, \partial f^g)| + |\Gamma_{\text{gain}}(f^f, \partial f^g)| + |\Gamma_{\text{loss}}(f^{f+1}, \partial f^g)| + |\Gamma_{\text{loss}}(f^{f+1}, \partial f^g)| + |K \partial f^g| \) of (4.45) in (4.46) is bounded by
\[ (1 + \sup_{0 \leq s \leq t} \|w_\phi f^f(s)\|_\infty + \sup_{0 \leq s \leq t} \|w_\phi f^{f+1}(s)\|_\infty) \times \int_0^t \int_{\Omega \times \mathbb{R}^3} \phi_{\ell, \ell} \partial f^{f+1}(v)|^{p-1} \int_{\mathbb{R}^3} \phi_{\ell, \ell}(v)^\delta k_p(v, u) w_\phi (v) \partial f^f(u) du \, dx \, dz. \] (4.48)

The estimate of (4.48) is carried out in Step 3.

From (4.25), the contribution of \(|\Gamma_{\text{loss}}(\partial f^{f+1}, \partial f^g)| \) of (4.45) in (4.46) is bounded by
\[ \sup_{0 \leq s \leq t} \|w_\phi f^f(s)\|_\infty \int_0^t \|w_\phi \phi^f w_\phi \alpha_{\ell, \ell} \partial f^{f+1}\|_p^p. \] (4.49)

For the \(|\partial f^{f+1}| \) contribution of (4.45) in (4.46), we bound
\[ \int_0^t \int_{\Omega \times \mathbb{R}^3} p |w_\phi \alpha_{\ell, \ell} \partial f^{f+1}|^{p-1} |\partial f^{f+1}| \, dx \, dz \]
\[ \lesssim \int_0^t \int_{\Omega \times \mathbb{R}^3} |w_\phi \phi^f w_\phi \alpha_{\ell, \ell} \partial f^{f+1}|^{p-1} |w_\phi f^{f+1}| \delta k_p(s, x, v) \right|_{(p-1)/p} \, dx \, dz \]
\[ \lesssim o(1) \int_0^t \int_{\Omega \times \mathbb{R}^3} |w_\phi \phi^f w_\phi \alpha_{\ell, \ell} \partial f^{f+1}|^{p} + (1 + \delta_1/A_1) \int_0^t \int_{\Omega \times \mathbb{R}^3} |w_\phi f^{f+1}|^{p}. \] (4.50)
Here we have used the fact that, from (1.24) and (4.3)
\[ |\alpha_{f,\ell}(s, x, v)| \]
\[ \leq 2(1_{s+1\leq t_{b}(s, x, v)}v_{b}(s, x, v) + 1_{s+1 < t_{b}(s, x, v) + 1}) \]
\[ \leq 1 + |v| + \int_{-1}^{0} |\nabla \phi^\ell(\tau, X(\tau; s, x, v))|d\tau + \int_{0}^{\infty} |\nabla \phi^\ell(\tau, X(\tau; s, x, v))|d\tau \]
\[ \leq (1 + \|w_{\phi}f_{0}\|_{\infty} + \delta_{1}/\Lambda_{1})(v), \]
and from (4.41),
\[ |\alpha_{f,\ell}(s, x, v)|^{\beta} \leq (1 + \delta_{1}/\Lambda_{1}) \times \frac{(\beta)}{(\beta-1)/p} \leq (1 + \delta_{1}/\Lambda_{1}). \]
From (4.26), the contribution of \( |\Gamma_{\nu,\text{loss}}(f^{\ell+1}, f^{\eta})| \) of (4.45) in (4.46) \( \phi \) is bounded by
\[ \|w_{\phi}f^{\ell+1}\|_{\infty} \int_{0}^{t} \int_{\Omega_{R}^{3}} p|w_{\phi}\alpha_{f,\ell,\epsilon}^{\beta}\partial f^{\ell+1}(v)^{p-1}| |\alpha_{f,\ell,\epsilon}(v)|^{\beta}(v) w_{\phi}(v)w_{\phi}(v)^{-1}\|w_{\phi}f^{\ell}(s, x, \cdot)\|_{L_{p}(R^{3})} \]
\[ \leq \|w_{\phi}f^{\ell+1}\|_{\infty} \Big\{ \int_{0}^{t} \int_{\Omega_{R}^{3}} |\alpha_{f,\ell,\epsilon}^{\beta}|^{\beta} p + \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}f^{\ell}|^{p} \Big\}, \]
where we have used, from (3.35), (3.36),
\[ |\alpha_{f,\ell,\epsilon}(v)|^{\beta} \leq 1. \]
From (4.17) and (4.22), the contribution of \( |\Gamma_{\nu,\text{gain}}| \) and \( |K_{\nu}f| \) in (4.16) \( \phi \) is bounded by
\[ (1 + \sup_{0 \leq s \leq t} \|w_{\phi}f^{\ell}\|_{\infty}) \int_{0}^{t} \int_{\Omega_{R}^{3}} |\alpha_{f,\ell,\epsilon}^{\beta}|^{\beta} p + (1 + \sup_{0 \leq s \leq t} \|w_{\phi}f^{\ell}(s)\|_{\infty}) \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}f^{\ell}|^{p}, \]
where we have used, for \( 1/p + 1/p^{*} = 1 \) and \( 0 < \tilde{\phi} \ll \phi \), from (4.23), (4.26),
\[ \int_{\Omega_{R}^{3}} |\alpha_{f,\ell,\epsilon}(v)|^{\beta}(w_{\phi}\partial f^{\ell+1}(v))^{p-1}| \int_{\Omega_{R}^{3}} \kappa_{\phi}(v, u)w_{\phi}(u)|f^{\ell}(u)|dudv \]
\[ \leq (1 + \sup_{0 \leq s \leq t} \|w_{\phi}f^{\ell}\|_{\infty}) \int_{0}^{t} \int_{\Omega_{R}^{3}} |\alpha_{f,\ell,\epsilon}^{\beta}(v)|^{\beta} p + (1 + \sup_{0 \leq s \leq t} \|w_{\phi}f^{\ell}(s)\|_{\infty}) \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}f^{\ell}|^{p}, \]
Note that from the standard elliptic estimate and (1.10),
\[ \|\phi^{\ell}(t)\|_{W^{2,p}(\Omega)} \leq \left( \int_{\Omega_{R}^{3}} (\int_{\Omega_{R}^{3}} \phi^{\ell}-\phi^{\ell}(t, x, v)\sqrt{\mu(v)}dv)dv \right)^{1/p} \leq \|f^{\ell}(t)\|_{L_{p}(\Omega \times R^{3})}. \]
Then from (4.54) we bound the contribution of \( w_{\phi}(v)^{-1/2}(\nabla \phi^{\ell} + |\nabla^{2} \phi^{\ell}|)(1 + \|w_{\phi}f^{\ell+1}\|_{\infty}) \) of (4.45) in (4.44) by
\[ (1 + \|w_{\phi}f^{\ell+1}\|_{\infty}) \int_{0}^{t} \int_{\Omega_{R}^{3}} p|w_{\phi}\alpha_{f,\ell,\epsilon}^{\beta}\partial f^{\ell+1}(v)^{p-1}| |\alpha_{f,\ell,\epsilon}(v)|^{\beta}(v) w_{\phi}(v)w_{\phi}(v)^{-1}\|w_{\phi}f^{\ell}(s, x, \cdot)\|_{L_{p}(R^{3})} \]
\[ \leq (1 + \|w_{\phi}f^{\ell+1}\|_{\infty}) \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}\alpha_{f,\ell,\epsilon}^{\beta}\partial f^{\ell+1}(v)^{p-1}w_{\phi}^{-1/4}(\nabla \phi^{\ell} + |\nabla^{2} \phi^{\ell}|) \]
\[ \leq (1 + \|w_{\phi}f^{\ell+1}\|_{\infty}) \left\{ o(1) \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}\alpha_{f,\ell,\epsilon}^{\beta}\partial f^{\ell+1}(v)^{p} + \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}\alpha_{f,\ell-1,\epsilon}\partial f^{\ell}(v)|^{p} + \int_{0}^{t} \int_{\Omega_{R}^{3}} \|\phi^{\ell}\|_{W^{2,p}} \int_{\Omega_{R}^{3}} w_{\phi}^{-p/4} \right\} \]
\[ \leq o(1) \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}\alpha_{f,\ell,\epsilon}^{\beta}\partial f^{\ell+1}(v)^{p} + \int_{0}^{t} \int_{\Omega_{R}^{3}} |w_{\phi}\alpha_{f,\ell-1,\epsilon}\partial f^{\ell}(v)|^{p} + \int_{0}^{t} \int_{\Omega_{R}^{3}} \|f^{\ell}\|^{p}. \]
where we have used, from (4.51), \( |\alpha_{f,\ell,\epsilon}(v)|^{\beta} w_{\phi}(v)^{-1/2} \leq w_{\phi}(v)^{-1/4}. \)

**Step 3.** We focus on (4.45). With \( N > 0 \), we split the \( u \)-integration of (4.45) into the integrations over \( \{u \leq N\} \) and \( \{u \geq N\} \).
For \(|u| \geq N\) and \(0 < \tilde{p} \ll p\), by Hölder inequality with \( \frac{1}{p} + \frac{1}{p^*} = 1 \)

\[
\int_{|u| \geq N} |\alpha_{f,t,c}^\beta(v)| k_\tilde{q}(v,u) \partial f^t(u) | \leq |\alpha_{f,t,c}^\beta(v)| \left( \sum_{i=\pm} \int_{|u| \geq N} k_\tilde{q}(v,u) \frac{1}{\alpha_{f,t,c}^\beta(u)\partial f^t(u)} \right)^{1/p^*} \left( \int_{|u| \geq N} k_\tilde{q}(v,u) |\alpha_{f,t,c}^\beta\partial f^t(u)|^p \right)^{1/p} \quad (4.56)
\]

\[
\lesssim |\alpha_{f,t,c}^\beta(v)| \left( \int_{|u| \geq N} k_\tilde{q}(v,u) |\alpha_{f,t,c}^\beta\partial f^t(u)|^p \right)^{1/p} \quad ,
\]

where have used Proposition 5 with \( \beta q < \frac{p-1}{p} \) = 1 from (4.1).

Then the contribution of \(|u| \geq N\) in (4.48) is bounded by

\[
\int_{\Omega} \int_{R^3} |u|^{p/\tilde{p}} w_\tilde{q}^\beta (v)| w_\tilde{q} \partial f^t(v)|^p - 1 |\alpha_{f,t,c}^\beta(v)| \frac{1}{v^{(p-1)/p} \alpha_{f,t,c}^\beta(v)} | w_\tilde{q}(v,u)| \int_{|u| \geq N} k_\tilde{q}(v,u) \right)^{1/p} \quad \ldots (4.57)
\]

where we have used, from (4.51), \( \frac{|\alpha_{f,t,c}^\beta(v)|}{v^{(p-1)/p}} \lesssim 1 \) for \( \beta \) in (4.1), (3.35), and (3.36).

The contribution of \(|u| \leq N\) in (4.48) is bounded by, from the Hölder inequality,

\[
\int_{\Omega} \int_{R^3} |u|^{p/\tilde{p}} w_\tilde{q}^\beta (v)| w_\tilde{q} \partial f^t(v)|^p - 1 \left[ \int_{|u| \leq N} \sum_{i=\pm} k_\tilde{q}(v,u) \frac{|\alpha_{f,t,c}^\beta(v)|}{v^{(p-1)/p} \alpha_{f,t,c}^\beta(v)} | w_\tilde{q}(v,u)| \int_{|u| \geq N} k_\tilde{q}(v,u) \right)^{1/p} \quad \ldots (4.58)
\]

where we have used (3.35) and the fact \( \frac{|\alpha_{f,t,c}^\beta(v)|}{v^{(p-1)/p}} \lesssim 1 \) from (4.51) and (4.1).

By the Hölder inequality, we bound an underlined \( u \)-integration inside (4.58) as

\[
\| w_\tilde{q}^\beta \partial f^t(\cdot) \|_{L^p(R^3)} \times \left( \sum_{i=\pm} \int_{R^3} e^{-p/\tilde{p}|v-u|^2} \frac{1}{|v-u|^{p/\tilde{p}+1}} \alpha_{f,t,c}^\beta(u) | w_\tilde{q}(v,u)| \int_{|u| \leq N} \right)^{1/q} \quad , \quad (4.59)
\]

where \( 1/p + 1/p^* = 1 \).

It is important to note that for \( t = + \) or \( - \),

\[
\left( \int_{R^3} e^{-p/\tilde{p}|v-u|^2} \frac{1}{|v-u|^{p/\tilde{p}+1}} \alpha_{f,t,c}^\beta(u) | w_\tilde{q}(v,u)| \int_{|u| \leq N} \right)^{1/p^*} \leq \left( \frac{1}{|v|^p} \right)^{1/p^*} \left( \frac{1}{\alpha_{f,t,c}^\beta(\cdot)} \right)^{1/p^*} \quad . \quad (4.60)
\]

By the Hardy-Littlewood-Sobolev inequality with

\[
1 + \frac{1}{p/p^*} = \frac{1}{3/p^*} + \frac{1}{2 - p^*}
\]
we have

\[
\left\| \frac{1}{\alpha f_{t-1,e,x}(\cdot)^p} \right\|_{L^p(R^3)}^{1/p} = \left\| \frac{1}{\alpha f_{t-1,e,x}(\cdot)^p} \right\|_{L^{p/p}(\mathbb{R}^3)}^{1/p} \\
\lesssim \left( \int_{\mathbb{R}^3} \frac{1}{\alpha f_{t-1,e,x}(v)^{3\beta/2}} dv \right)^{2/3}.
\]

(4.61)

For \(3 < p < 6\), we have \(\frac{3}{2} \frac{p-2}{p} < 1\) and \(\frac{2}{3} < \frac{p-1}{p}\). Importantly from (4.1) we have \(\frac{3\beta}{2} < 1\). Now we apply (4.8) in Proposition 5 to conclude that

\[
\left( \int_{\mathbb{R}^3} \frac{1}{\alpha f_{t-1,e,x}(v)^{3\beta/2}} dv \right)^{2/3} \lesssim_{p,\beta,M,\Omega} 1.
\]

Finally from (4.53), (4.59), (4.60), (4.61), and (4.67) we bound

\[
(4.62) \lesssim o(1) \int_0^t \left\| \nu_{\beta}^{1/p} w_\beta \alpha_{f_{t-1,e}}^{\beta} \partial f^{\ell+1} \right\|^p_p + (1 + \sup_{0 \leq s \leq t} ||w_\beta f^{\ell}(s)||_{\infty} + \sup_{0 \leq s \leq t} \left| \partial f^{\ell+1}(s) \right|_{\infty}) \int_0^t \left\| w_\beta \alpha_{f_{t-1,e}}^{\beta} \partial f^{\ell} \right\|^p_p.
\]

(4.62)

Collecting terms from (4.47), (4.48), (4.49), (4.50), (4.52), (4.53), (4.55), (4.61), and (4.62) we have

\[
(4.63) \lesssim o(1) \int_0^t \left\| \nu_{\beta}^{1/p} w_\beta \alpha_{f_{t-1,e}}^{\beta} \partial f^{\ell+1} \right\|^p_p \\
+ (1 + \sup_{0 \leq s \leq t} \left| \nabla^2 \hat{v}(s) \right|_{\infty}) \int_0^t \left\| w_\beta \alpha_{f_{t-1,e}}^{\beta} \partial f^{\ell+1} \right\|^p_p \\
+ (1 + \sup_{0 \leq s \leq t} ||w_\beta f^{\ell}(s)||_{\infty} + \delta / \Lambda_1) \int_0^t \left\| \partial f^{\ell+1} \right\|^p_p \\
+ (1 + \sup_{0 \leq s \leq t} ||w_\beta f^{\ell}(s)||_{\infty} + \sup_{0 \leq s \leq t} ||w_\beta f^{\ell+1}(s)||_{\infty}) \int_0^t \left( \left\| \partial f^{\ell} \right\|_p + \left\| w_\beta \alpha_{f_{t-1,e}}^{\beta} \partial f^{\ell} \right\|_p \right).
\]

(4.63)

Step 4. We focus on (4.48) \( w_{\beta} \). From (4.25) and (4.29),

\[
\int_{n(x) v < 0} |n(x) \cdot v|^{\beta} |w_\beta \nabla x \cdot f^{\ell+1}(t, x, v)|^p |n(x) \cdot v| dv \\
\lesssim \int_{n(x) v < 0} \langle v \rangle^p \mu(v) \nabla x \cdot f^{\ell+1}(t, x, v) |n(x) \cdot v|^{(\beta+1)(\beta-1)+1} \times |(120)|^p dv.
\]

(4.64)

Note that for \(0 < \tilde{\beta} \ll 1\) we have \(\mu(v) \nabla x \cdot f^{\ell+1}(t, x, v) \lesssim e^{-C|v|^2}\) for some \(C > 0\) when \(|v| \gg 1\).

On the other hand, from (4.11), we have

\[
(\beta - 1)p + 1 > \frac{p - 2}{p} p - p + 1 = -1, \quad |n(x) \cdot v|^{(\beta-1)p+1} \in L_1^{w_{\beta}}(\mathbb{R}^3).
\]

(4.65)
Now we bound $|(1.29)|^p$. For the first line of (1.29), we split the $u$-integration into $\gamma_+^*(x) \cup \gamma_+(x) \setminus \gamma_+^*(x)$ where $\varepsilon$ is small but satisfies (2.10). By the Hölder inequality
\[
\left\{ \sum_{\varepsilon = \pm} \int_{n(x) \cdot u > 0} \left| w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^*(s, x, u) \{ w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t, \varepsilon, u}(u) \} - \beta \{ u \} \mu(u) \{ n(x) \cdot u \} du \right|^p \right\}^{1/p} \\
\lesssim \left\{ \int_{\gamma_+^*(x)} \left| w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^*(s, x, u) \right|^p \{ n(x) \cdot u \} du \right\}^{1/p} \\
\times \left\{ \sum_{\varepsilon = \pm} \int_{\gamma_+(x) \setminus \gamma_+^*(x)} \left| w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon, u}(u) \right|^{-\beta} \{ n(x) \cdot u \} \mu(u) du \right\}^{1/p} \\
+ \left\{ \int_{\gamma_+(x) \setminus \gamma_+^*(x)} \left| w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon, u}(s, x, u) \right|^{-\beta} \{ n(x) \cdot u \} \mu(u) \{ n(x) \cdot u \} du \right\}^{1/p} \\
\times \left\{ \sum_{\varepsilon = \pm} \int_{\gamma_+(x) \setminus \gamma_+^*(x)} \left| w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon, u}(s, x, u) \right|^{-\beta} \{ n(x) \cdot u \} \mu(u) \{ n(x) \cdot u \} du \right\}^{1/p}, \\
p^* := \frac{p}{p-1}.
\]  
(4.66)
Note that $\alpha_{f, t, \varepsilon}(s, x, u) \neq |n(x) \cdot u|$ for $(s, x, u) \in \gamma_+$ in general. From (1.1), $\beta p^* < 1$. From (4.8) and (1.9) with $v = 0$, we have $\alpha_{f, t, \varepsilon}(s, x, u) \cdot \{ n(x) \cdot u \} \leq \alpha_{f, t, \varepsilon}(s, x, u) \in L^1_{\text{loc}}(\{ u \in \mathbb{R}^3 \})$. Since $1_{\gamma_+^*(x)}(v) \downarrow 0$ almost everywhere in $\mathbb{R}^3$ as $\varepsilon \downarrow 0$, by the dominant convergence theorem, for (4.3), we choose $\varepsilon := \frac{2\mu_1}{\delta_1} \ll \Omega_1$
\[
(4.67) \lesssim o(1) \int_{\gamma_+^*(x)} |w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^*(s, x, u)|^p \{ n(x) \cdot u \} du \\
+ \int_{\gamma_+(x) \setminus \gamma_+^*(x)} |w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^*(s, x, u)|^p \mu(u)\{ n(x) \cdot u \} du.
\]
Now applying Lemma 2 and (4.44) to $f^t$ and $\tilde{f}^t$ separately and adding them together, the last term of (4.67) has a bound as
\[
(4.68) \lesssim \|w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^*(0)\|_{\mu/\delta_1}^{1/\delta_1} + \int_0^t \|w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^t\|_{\mu/\delta_1}^{1/\delta_1} du + \|w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^\varphi\|_{\mu/\delta_1}^{1/\delta_1},
\]
where, from (4.41) and (4.48),
\[
(4.69) \lesssim \int_0^t \int_{\Omega \times \mathbb{R}^3} \|\partial_t + v \cdot \nabla_{x, \varepsilon} - q \nabla_\varphi \alpha_{f, t - 1, \varepsilon, u}^{\beta - 1} \cdot \nabla_{x, \varepsilon} + \nu \partial_t, w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^*(s, x, u) \|_{\mu/\delta_1}^{1/\delta_1} \|f^t\|_{\mu/\delta_1}^{1/\delta_1} \|f^\varphi\|_{\mu/\delta_1}^{1/\delta_1},
\]
\[
(4.70) \lesssim \int_0^t \int_{\Omega \times \mathbb{R}^3} \|\alpha^{\beta}_{f, t - 1, \varepsilon} (\nabla_{x, \varepsilon} f^t)\|_{\mu/\delta_1}^{1/\delta_1} \|\alpha^{\beta}_{f, t - 1, \varepsilon} (\nabla_{x, \varepsilon} f^\varphi)\|_{\mu/\delta_1}^{1/\delta_1} \|f^t\|_{\mu/\delta_1}^{1/\delta_1} + \|\alpha^{\beta}_{f, t - 1, \varepsilon} (\nabla_{x, \varepsilon} f^t)\|_{\mu/\delta_1}^{1/\delta_1} \|f^\varphi\|_{\mu/\delta_1}^{1/\delta_1},
\]
Clearly (4.70) \lesssim (4.80) \|f^t\|_{\mu/\delta_1}. And, from (4.39),
\[
(4.71) \lesssim \int_0^t \|w^{(\varepsilon)}_{\mu} \alpha^{\beta}_{f, t - 1, \varepsilon} \nabla_{x, \varepsilon} f^t\|_{\mu/\delta_1}^{1/\delta_1}.
\]
Now we consider the third term of (1.29). From the trace theorem $W^{1,\varphi}(\Omega) \rightarrow W^{1,\varphi}(\partial \Omega)$ and (1.51)
\[
\|\nabla \phi\|_{L^p(\partial \Omega)} \lesssim \|\nabla \phi\|_{W^{1,\varphi}(\Omega)} \lesssim \|\nabla \phi\|_{L^p(\Omega \times \mathbb{R}^3)}.
\]
Then
\[
(4.72) \int_{\partial \Omega} \left\{ \int_{n(x) \cdot u > 0} \left( |(f^{t + 1}| + |f^t|) + \mu(u)\{ n(x) \cdot u \} \right) \sqrt{\mu(u)\{ n(x) \cdot u \}} du \right\}^p dS_x \\
\lesssim (1 + \|w_\varphi f^{t + 1}\|_p + \|w_\varphi f^t\|_p \left( \sum_{m=1}^{t+1} \|\nabla \phi\|_{L^p(\Omega \times \mathbb{R}^3)} \right) \\
\lesssim (1 + \|w_\varphi f^{t + 1}\|_p + \|w_\varphi f^t\|_p \left( \sum_{m=1}^{t+1} \|\nabla \phi\|_{L^p(\Omega \times \mathbb{R}^3)} \right) + \|w_\varphi f^{t + 1}\|_{L^p(\Omega \times \mathbb{R}^3)} + \|w_\varphi f^t\|_{L^p(\Omega \times \mathbb{R}^3)}).
\]
For the second term of (4.29), by the Hölder inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \) for \( 3 < p < 6 \),

\[
\left\{ \int_{n \cdot u > 0} \left( |u| (|f|^p + |f'|^p) \right) + (1 + \| w_0 f^\ell \|_\infty + \| w_0 f^{\ell-1} \|_\infty) \int_{\mathbb{R}^3} \frac{k_\phi(u, u')^{1/q}}{|n \cdot u'|^{1/p}} k_\phi(u, u')^{1/p} (|f|^p (u') + f^{\ell-1}(u')) |n \cdot u'|^{1/p} du' \right\}^p
\]

\[
\lesssim \left( \int_{n \cdot u > 0} (|f'|^p + |f|^p) \{n \cdot u\} \sqrt{\rho} du \right)^p
\]

\[
+ (1 + \| w_0 f^\ell \|_\infty + \| w_0 f^{\ell+1} \|_\infty) \left( \int_{\mathbb{R}^3} k_\phi(u, u') |n \cdot u'|^{-q/p} du' \right)^{p/q} \int_{\mathbb{R}^3} k_\phi(u, u') (|f|^p (u') + f^{\ell+1}(u')) |n \cdot u'| du' du'
\]

\[
\lesssim (1 + \| w_0 f^\ell \|_\infty + \| w_0 f^{\ell+1} \|_\infty) \int_{n \cdot u > 0} (|f'|^p + |f|^p) \{n \cdot u\} du
\]

\[
\lesssim (1 + \| w_0 f^\ell \|_\infty + \| w_0 f^{\ell+1} \|_\infty) (\| w_0 f^\ell \|_\infty + \| w_0 f^{\ell+1} \|_\infty).
\]

(4.74)

Collecting terms from (4.64), (4.67), (4.69), (4.73), and (4.74) we derive that

\[
\lesssim \| w_0 \alpha_\beta_{f,0} \nabla s, f(0) \mu(u) \|_p + o(1) \left( \int_0^t \| w_0 f^\ell(s) \|_\infty + \| w_0 f^{\ell+1}(s) \|_\infty \right)
\]

\[
+ \left( \| w_0 f^\ell(s) \|_\infty + \sup_{0 \leq s \leq t} \| w_0 f^\ell(s) \|_\infty \right) \int_0^t \sum_{m=\ell-1}^\ell \| w_0 f^m(s) \|_p
\]

\[
+ \left( \sup_{0 \leq s \leq t} \| w_0 f^m(s) \|_\infty \right) \int_0^t \sum_{m=\ell-1}^\ell \| w_0 f^m(s) \|_p
\]

\[
\quad \left( \| w_0 f^\ell \|_p + \| w_0 f^{\ell+1} \|_p \right).
\]

(4.75)

**Step 5.** From (4.40), (4.43), (4.63), (4.75) we have

\[
\sup_{0 \leq s \leq t} \mathcal{E}^{\ell+1}(s)
\]

\[
\leq C_0 \left( \| w_0 \alpha_{f_0,0} |\nabla, f_0| \|_p + t \left( 1 + \| w_0 f^{\ell-1} \|_\infty + \sum_{m=\ell-1}^{\ell+1} \| w_0 f^m \|_\infty + \| \nabla^2 \phi^{\ell-1} \|_\infty + \| \nabla^2 \phi^\ell \|_\infty \right) \right) \max_{0 \leq m \leq \ell+1} \sup_{0 \leq s \leq t} \mathcal{E}^m(s) + o(1) \max_{0 \leq m \leq \ell+1} \sup_{0 \leq s \leq t} \mathcal{E}^m(s).
\]

(4.76)

On the other hand, from Lemma 6

\[
\| \nabla^2 \phi(t) \|_\infty + \| \nabla^2 \phi^{\ell-1}(t) \|_\infty \lesssim \| \mathcal{E}^\ell(t) + \mathcal{E}^{\ell-1}(t) \|^{1/p}.
\]

(4.77)

Therefore from (4.76), (4.77), and the induction hypothesis in (4.34), we first choose a small \( o(1) \), then large \( C \gg C_0 \), and finally small \( 0 < \eta^* \ll 1 \) to conclude

\[
\sup_{0 \leq s \leq t} \mathcal{E}^{\ell+1}(s) \leq C \left( \frac{C}{10} \| w_0 \alpha_{f_0,0} |\nabla, f_0| \|_p + \frac{1}{10} \sup_{0 \leq s \leq t} \mathcal{E}^m(s) \right)
\]

\[

\leq C \left( \| w_0 f_0 \|_\infty + \| w_0 f_0 \|_p + \| w_0 \alpha_{f_0,0} |\nabla, f_0| \|_p \right).
\]

This proves (4.34).

\[
\square
\]

5. \( L^1_t L^{4+}_v \) Bound of \( \nabla_v f^\ell \)

**Proposition 6.** Assume the initial condition satisfies (3.7), (4.3), and

\[
\| w_0 \nabla_v f_0 \|_{L^\tau_v} < \infty.
\]

(5.1)
Then for $T^{**} \ll 1$, the sequence (3.3) satisfies
\[
\sup_{t} \sup_{0 \leq t \leq T^{**}} \| \nabla v f^{\ell}(t) \|_{L^1_{\mu}(0) L^1_{v}(R^3)} \lesssim 1 \text{ for all } t \geq 0.
\] (5.2)

Proof. Step 1. Note that from (3.3) and (4.32), we have
\[
\begin{align*}
[\partial_t + v \cdot \nabla_x - q \nabla_x \phi^x \cdot \nabla_v + \nu(v) + q \frac{v}{2} \cdot \nabla_v \phi^v] \partial_t e^{\ell+1} \\
= - \partial_t e^{\ell+1} - \frac{1}{2} \partial_t \phi^x e^{\ell+1} - \partial_v e^{\ell+1} + \partial_v (K f^{\ell}) + q_1 (\Gamma_{\text{gain}}(f^{\ell}, f^{\ell})) - q_1 (\Gamma_{\text{loss}}(f^{\ell+1}, f^{\ell})) - q_1 (\partial_e \phi^e \sqrt{\Gamma} - \frac{\nu^2}{2} \partial_e \phi^e \sqrt{\Gamma})
\end{align*}
\] (5.3)

with the boundary bound for $(x, v) \in \gamma_-$
\[
|\partial_v e^{\ell+1}| \lesssim |v| \sqrt{\Gamma} \int_{v>0} |f^{\ell+1}| \sqrt{\Gamma} (n \cdot u) du \text{ on } \gamma_-. \] (5.4)

From (4.15), (4.19), (4.24), (4.25), (4.26), and (4.27), we obtain the following bound along the characteristics for $f_+$ and $f_-$ separately. For $\ell = +$ or $-\ell$ as in (1.9),
\[
\begin{align*}
|\partial_e f_+^{\ell+1} (t, x, v)| \\
\leq 1 \chi_{(t,v) \in \Gamma_+} |\partial_e f_+^{\ell+1} (0, X^e_0 (0; t, x, v), V^e_0 (0; t, x, v))| \\
+ \int_{t_{\min}}^{t} |\partial_e f_+^{\ell+1} (s; t, x, v), V^e_0 (s; t, x, v))| ds \\
+ \int_{t_{\min}}^{t} (1 + \|w \phi^{\ell+1} \|_{\infty} + \|w \phi^{\ell+1} \|_{\infty}) \int_{R^3} k_0 (V^e_0 (s, u)) |\partial_e f_+^{\ell+1} (s, X^e_0 (s, u))| ds ds \\
+ \int_{t_{\min}}^{t} \|f_+^{\ell+1} (s) \|_{\infty} \|\nabla_x \phi^x (s, X^e_0 (s, x, v))| ds \\
\end{align*}
\] (5.5)

where $\delta_1$ is in (4.33). Here we used that from (4.29), on the RHS of (4.33), $|\Gamma_{\text{loss}}(\partial_e f_+^{\ell+1}, f^{\ell})| \lesssim (\ell) \|w_\phi f^{\ell} \|_{\infty} |\partial_e f_+^{\ell+1}| \leq \frac{\nu(v)}{2} |\partial_e f_+^{\ell+1}|$, and thus can be absorbed to the LHS.

Note that if $|v| > 2 \frac{\nu(v)}{\Lambda_1}$, then from (4.33) and (4.4), for $0 \leq s \leq t$,
\[
|V^e_0 (s; t, x, v))| \geq |v| - \int_{0}^{t} |\nabla_x \phi^x (\tau, t, x, v)| d\tau \\
\geq |v| - \delta_1/\Lambda_1 \\
\geq |v| / 2.
\] (5.10)

Therefore
\[
\sup_{s, t, x} \left\| \frac{1}{w_\phi (V^e_0 (s; t, x, v))} \right\|_{L^r_{e}} \lesssim 1 \text{ for any } 1 \leq r \leq \infty.
\] (5.11)

We derive
\[
\begin{align*}
\| &5.5\|_{L^2_{e} L^{1+\delta}_{\mu}} \\
\lesssim \left( \int_{\Omega} \left( \int_{R^3} |w_{\phi} \partial_e f_+^{\ell+1} (0, X^e_0 (0), V^e_0 (0))|^{3} \int_{R^3} \frac{dv}{|w_\phi (V^e_0 (0))|^{1+\delta} \|w_\phi V^e_0 (0)\|} \right) \right)^{1/3} \\
\lesssim \left( \int_{\Omega \times R^3} |w_\phi (V^e_0 (0; t, x, v))| \partial_e f_+^{\ell+1} (0, X^e_0 (0; t, x, v), V^e_0 (0; t, x, v))|^{3} dv dx \right)^{1/3} \\
\lesssim \|w_\phi \partial_e f_0 \|_{L^3_{e}},
\end{align*}
\] (5.12)

where we have used a change of variables $(x, v) \mapsto (X^e_0 (0; t, x, v), V^e_0 (0; t, x, v))$ and (5.11). Clearly
\[
\|5.5\|_{L^2_{e} L^{1+\delta}_{\mu}} \lesssim \sup_{0 \leq s \leq t} \|w_\phi f_+^{\ell+1} (s)\|_{\infty}.
\] (5.13)
From $W^{1,2}(\Omega) \subset L^6(\Omega) \subset L^2(\Omega)$ for a bounded $\Omega \subset \mathbb{R}^3$, and the change of variables $(x,v) \mapsto (X^f_{t}(s; t, x, v), V^f_{t}(s; t, x, v))$ for fixed $s \in (\max\{t-t_n, 0\}, t)$,

$$
\|5.9\|_{L^2_{L^4}}^{1+s} \lesssim \|w_\alpha f^{t+1}\|_{\infty} \|\int^t_0 \|\mu^{1/8} \nabla_x \phi^f(s, X^f_{t}(s; t, x, v))\|_{L^2_{\nu, v}} \|\mu^{1/8} \|_{L^2_{L^4}}^{3(1+s)} \lesssim \|w_\alpha f^{t+1}\|_{\infty} \|\int^t_0 \|\nabla_x \phi^f(s)\|_{L^2_{\nu, v}} \|\phi^f(s)\|_{L^2_{L^4}} \lesssim \|w_\alpha f^{t+1}\|_{\infty} \|\int^t_0 \|w_\alpha f^1\|_2.
$$

(5.14)

**Step 2.** We claim

$$
\|5.7\|_{L^2_{L^4}^{1+s}} \lesssim \int^t_0 \|w_\alpha \alpha_{t-1,e}^{\beta} \partial_s f^t(s)\|_{L^p_{\nu, v}}.
$$

(5.15)

Now we have for $3 < p < 6$, by the Hölder inequality $\frac{1}{t} = \frac{1}{p} + \frac{1}{p}$,

$$
\|\int^t_{\max\{t-t_n, 0\}} |\partial_s f^t(s, X^f_{t}(s; t, x, v), V^f_{t}(s; t, x, v))|ds\|_{L^1_{\nu, v}(\mathbb{R}^3)} \lesssim \|w_\alpha \alpha_{t-1,e}^{\beta} \partial_s f^t(s, X^f_{t}(s; t, x, v), V^f_{t}(s; t, x, v))|ds\|_{L^1_{\nu, v}(\mathbb{R}^3)}
$$

(5.16)

$$
\lesssim \|w_\alpha (v)^{-1} \alpha_{t-1,e}^{\beta} (t, x, v)^{\beta}\|_{L^p_{\nu, v}(\mathbb{R}^3)} \times \int^t_0 \|w_\alpha \alpha_{t-1,e}^{\beta} \partial_s f^t(s, X^f_{t}(s; t, x, v), V^f_{t}(s; t, x, v))|ds\|_{L^p_{\nu, v}(\mathbb{R}^3)}
$$

where we have used $\alpha_{t-1,e}^{\beta} (t, x, v) = \alpha_{t-1,e}^{\beta} (s, X^f_{t}(s; t, x, v), V^f_{t}(s; t, x, v))$ for $t-t_n(t, x, v) \leq s \leq t$ and the change of variables $(x,v) \mapsto (X^f_{t}(s; t, x, v), V^f_{t}(s; t, x, v))$ and the Minkowski inequality.

For $\beta$ in (1.44), we have $\beta \frac{p}{p-1} < 1$ since $\frac{1}{p} < \frac{1}{4p}$ for $3 < p$. Therefore, we can choose $0 < \delta \ll 1$ so that $\beta$ in (1.44) satisfies

$$
\beta \times \frac{p + p\delta}{p - 1 - \delta} < 1. \quad (5.17)
$$

We apply Proposition 5 to conclude that

$$
\sup_{t,s} \|w_\alpha (v)^{-1} \alpha_{t-1,e}^{\beta} (t, x, v)^{\beta}\|_{L^p_{\nu, v}(\mathbb{R}^3)} = \sup_{t,s} \int_{\mathbb{R}^3} e^{-\beta \frac{p + p\delta}{p - 1 - \delta} |v|^2} \frac{1}{\alpha_{t-1,e}^{\beta} (t, x, v)^{\beta}} dv \lesssim 1. \quad (5.18)
$$

Finally, from (5.16), (5.18), and (1.47), we conclude the claim (5.15).

**Step 3.** We consider (5.8). We split the $u$-integration of (5.8) into two parts with $N \gg 1$ as

$$
\int_{|u| \leq N} k_\alpha(V^f_{t}(s), u) \nabla_v f^t(s, X^f_{t}(s), u) du
$$

(5.19)

$$
+ \int_{|u| \geq N} k_\alpha(V^f_{t}(s), u) \nabla_v f^t(s, X^f_{t}(s), u) du.
$$

(5.20)

Firstly, we bound (5.19). From the change of variables $(x,v) \mapsto (X^f_{t}(s; t, x, v), V^f_{t}(s; t, x, v))$ for $t-t_n(s, t, x, v) \leq s \leq t$

$$
\left\|\int_{|u| \leq N} k_\alpha(V^f_{t}(s, t, x, v), u) \nabla_v f^t(s, X^f_{t}(s; t, x, v), u) du \right\|_{L^2_{L^4}} \lesssim \left\|\int_{|u| \leq N} k_\alpha(v, u) \nabla_v f^t(s, x, u) du \right\|_{L^2_{L^4}}.
$$

(5.21)
If \(|v| \geq 2N\) then \(|v - u|^2 \geq |v|^2\) and \(k_\varphi(v, u) \lesssim \frac{C |v|^2}{|v-u|^2}\) for \(|v| \geq 2N\) and \(|u| \leq N\). For \(0 < \delta \ll 1\) with \(\frac{\delta(1 + \delta)}{1 - \delta} > 3\),

\[
\begin{align*}
&\lesssim C_N \left\| \int_{|u| \leq N} k_\varphi(v, u) \|
abla_v f^\varphi(s, x, u) \| \, dz \right\| \frac{3(1 + \delta)}{L_v^{1-\delta} ((|u| \leq 2N))} \| L_v^2 \\
&+ \left\| e^{-C|v|^2} \right\| L_v^{1/2} \left\| \int_{|u| \leq N} \frac{1}{|v - u|} \|
abla_v f^\varphi(s, x, u) \| \, du \right\| \frac{3(1 + \delta)}{L_v^{1-\delta} ((|u| \geq 2N))} \| L_v^3 \right. \\
&\lesssim \left\| \frac{1}{|v - u|} \|
abla_v f^\varphi(s, x, \cdot) \| \frac{3(1 + \delta)}{L_v^{1-\delta}} \right\| L_v^2 .
\end{align*}
\]

Then by the Hardy-Littlewood-Sobolev inequality with \(1 + \frac{1}{1 - \delta} = \frac{3}{2} + \frac{1}{1 + \delta}\), we derive that

\[
\begin{align*}
&\leq C_{N, \delta} \left\| \nabla_v f^\varphi(s, x, v) \right\| L_v^{1+\delta} = \| \nabla_v f^\varphi(s) \| L_v^{1+\delta}.
\end{align*}
\]

Combining the last estimate with \(5.21\), \(5.22\), we prove that

\[
\| \nabla_v f^\varphi(s, x, v) \| L_v^{1+\delta} \lesssim \| \nabla_v f^\varphi(s) \| L_v^{1+\delta}.
\]

Now we consider \(5.20\). Choose \(0 < \delta' \ll 1\). We have

\[
\begin{align*}
\lesssim & \int_{|u| \geq N} \frac{1}{\omega_\varphi(V^\varphi_t(s, t, x, v))} \left\| \frac{\omega_\varphi(V^\varphi_t(s, t, x, v))}{\omega_\varphi(u)} \right\| \frac{k_\varphi(V^\varphi_t(s, t, x, v), u)}{\alpha_{f-1,\varphi}(s, X^\varphi_t(s, t, x, v), u) \varphi} \right| \frac{1}{\omega_\varphi(V^\varphi_t(s, t, x, v))} \left\| \frac{\omega_\varphi(u)}{\omega_\varphi(V^\varphi_t(s, t, x, v))} \right| \alpha_{f-1,\varphi}(s, X^\varphi_t(s, t, x, v), u) \varphi \nabla_v f^\varphi(s, X^\varphi_t(s, t, x, v), u) \right| \, du.
\end{align*}
\]
Finally using (1.44) in Proposition 5 with \( \frac{\beta}{p} < \beta p < 1 \) from (1.44) and applying the change of variables \((x, v) \mapsto (X^t(s; t, x, v), V^t(s; t, x, v))\), we derive that

\[
\left\| \left\| (5.20) \right\| L^p_{t,x} \right\| L^p_{v}\]
\[
\lesssim \left\| \left\| \frac{1}{w_p(v)^{\beta}} w_p(u)^{\alpha_{\ell-1,\epsilon}} (s, x, u, u)^{\beta} \nabla_v f^\ell(s, x, u) \right\| L^p_{v,\epsilon} \right\|
\]
\[
\lesssim \left\| \left\| \frac{1}{w_p(v)^{\beta}} \right\| L^p_{v} \right\| \left\| w_p(u)^{\alpha_{\ell-1,\epsilon}} (s, x, u, u)^{\beta} \nabla_v f^\ell(s, x, u) \right\| L^p_{v,\epsilon}
\]
\[
\lesssim \left\| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f^\ell(s) \right\| L^p_{v}.
\]

Combining (5.23) and (5.25), we conclude that

\[
\left\| (5.20) \right\| L^p_{t,x} L^{s+\delta} \lesssim \left\| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f^\ell(s) \right\| L^p_{v,\epsilon}.
\]

Finally from (5.23) and (5.26), and using the Minkowski inequality, we conclude that

\[
\left\| (5.28) \right\| L^p_{t,x} L^{s+\delta}
\]
\[
\lesssim (1 + \| w_p f^\ell \|_{\infty} + \| w_p f^{\ell+1} \|_{\infty}) \int_0^t \left( \| \nabla_v f^\ell(s) \|_{L^2_{t,x}} + \| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f^\ell(s) \|_{L^p_{v}} \right) ds.
\]

Collecting terms from (5.5) - (5.9), (5.12), (5.6), (5.14), (5.13), (5.27), we derive

\[
\sup_{0 \leq s \leq t} \| \nabla_v f^\ell(s) \|_{L^2_{t,x}}
\]
\[
\lesssim \sup_{0 \leq s \leq t} \| \nabla_v f^{\ell+1}(s) \|_{L^2_{t,x}} + \sup_{0 \leq s \leq t} \| \nabla_v f^{\ell+1}(s) \|_{L^2_{t,x}}
\]
\[
\lesssim \left\| w_p \nabla_v f(0) \right\|_{L^2_{t,x}} + \sup_{0 \leq s \leq t} \| w_p f^{\ell+1}(s) \|_{\infty} + \sup_{0 \leq s \leq t} \| w_p f^\ell(s) \|_{\infty}
\]
\[
\quad + t(1 + \sup_{0 \leq s \leq t} \| w_p f^{\ell+1}(s) \|_{\infty} + \sup_{0 \leq s \leq t} \| w_p f^\ell(s) \|_{\infty})(\sup_{0 \leq s \leq t} \| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f^\ell(s) \|_{L^p_{v}} + \| \nabla_v f^\ell(s) \|_{L^2_{t,x}}).
\]

Therefore from (5.28) and (4.35), we can choose \( T^{**} \ll 1 \) and conclude (5.24).

\[
\square
\]

6. LOCAL EXISTENCE

**Theorem 7.** Let \( 0 < \tilde{\theta} < \vartheta \ll 1 \). Assume that for sufficiently small \( M > 0 \), \( F_{0} = \mu + \sqrt{\mu} f_{0} \geq 0 \) satisfying (5.7), (1.5), (5.1), and the compatibility condition (1.2).

Then there exists \( T^{*} \) \((M) > 0 \) and a unique solution \( F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geq 0 \) to (1.15), (1.16), and (1.17) in \([0, T^{*}(M)) \times \Omega \times \mathbb{R}^3\) such that

\[
\sup_{0 \leq t \leq T^{*}} \| w_p f(t) \|_{\infty} \leq M.
\]

Moreover

\[
\sup_{0 \leq t \leq T^{*}} \| \nabla_v f(t) \|_{L^2_{t,x}} \ll 1 \quad \text{for } 0 < \delta \ll 1,
\]

and

\[
\sup_{0 \leq t \leq T^{*}} \left\{ \left\| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f(t) \right\|_{p,\epsilon} + \int_0^t \left\| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f(t) \right\|_{p,\epsilon} ds \right\} < \infty.
\]

Furthermore, \( \| w_p f(t) \|_{\infty}, \| \nabla_v f(t) \|_{L^2_{t,x}} \) and \( \| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f(t) \|_{p,\epsilon} + \int_0^t \left\| w_p(\alpha_{\ell-1,\epsilon}) \nabla_v f(t) \right\|_{p,\epsilon} ds \) are continuous in \( t \).

**Proof.** Step 1. We claim that for \( T^{**} \ll 1 \), the whole sequence (5.33) satisfies

\[
f^{\ell} \to f \text{ strongly in } L^{\infty}((0, T); L^1((\Omega \times \mathbb{R}^3)).
\]

Note that \( f^{\ell+1} - f^{\ell} \) satisfies \((f^{\ell+1} - f^{\ell})|_{t=0} = 0, \) so

\[
\partial_t(f^{\ell+1} - f^{\ell}) + v \cdot \nabla_x (f^{\ell+1} - f^{\ell}) - \frac{q}{2} v^2 \cdot \nabla_x \phi^{f^{\ell+1} - f^{\ell}} + \frac{q}{2} v^2 \cdot \nabla_x \phi^{f^{\ell+1} - f^{\ell}} + \frac{q}{2} v^2 \cdot \nabla_x \phi^{f^{\ell+1} - f^{\ell}} + v [f^{\ell+1} - f^{\ell}]
\]
\[
= \partial_t f^{\ell+1} + v \cdot \nabla_x f^{\ell+1} - \frac{q}{2} v^2 \cdot \nabla_x \phi^{f^{\ell+1} - f^{\ell}} + \frac{q}{2} v^2 \cdot \nabla_x \phi^{f^{\ell+1} - f^{\ell}} + \frac{q}{2} v^2 \cdot \nabla_x \phi^{f^{\ell+1} - f^{\ell}} + v [f^{\ell+1} - f^{\ell}]
\]
\[
+ \Gamma_{\text{gain}} (f^{\ell+1} - f^{\ell}) - \Gamma_{\text{loss}} (f^{\ell+1} - f^{\ell}) + \Gamma_{\text{gain}} (f^{\ell} - f^{\ell}) + \Gamma_{\text{loss}} (f^{\ell} - f^{\ell}).
\]
By Lemma 3 for $L^{1+\delta}$-space with $0 < \delta < 1$, we obtain

\[
\|f^{t+1} - f^t\|_{1+\delta} + \int_0^t \|\nu^{1+\delta}_\phi (f^{t+1} - f^t)\|_{1+\delta} + \int_0^t \|f^{t+1} - f^t\|_{1+\delta} + \int_0^t \|f^{t+1} - f^t\|_{1+\delta},
\]

(6.6)

where $\nu^{1+\delta}_\phi$ is defined as (4.35).

Now for $0 < \delta < 1$, by the Hölder inequality with $1 = \frac{1}{2+\delta} + \frac{1}{\delta}$ and the Sobolev embedding $W^{1,1+\delta}(\Omega) \subset L^{\frac{3(1+\delta)}{2-\delta}}(\Omega)$ when $\Omega \subset \mathbb{R}^3$,

\[
\int_0^t \int_{\Omega \times \mathbb{R}^3} |\nabla x \phi \cdot f_{t-1} \cdot \nabla_v f^{t-1} - f^{t}|^\delta d\tau dx
\]

\leq \int_0^t \int_{\Omega \times \mathbb{R}^3} \|\nabla u f^{t-1}\|_{L^{\frac{3(1+\delta)}{2-\delta}}} \|\nabla_v f^{t-1}\|_{L^{1+\delta}} \|f^{t+1} - f^t\|_{1+\delta} + \int_0^t \|f^{t+1} - f^t\|_{1+\delta},
\]

(6.7)

\[
\lesssim \sup_{0 \leq s \leq t} \||\nabla_v f^{t-1}(s)\|_{L^{1+\delta}} \times \int_0^t \|f^{t+1} - f^t\|_{1+\delta} ds.
\]

A simple modification of (4.37) and (4.38) as

\[
\int_0^t \int_{\Omega \times \mathbb{R}^3} \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} k_v(v, u) |f^t(u) - f^{t-1}(u)| |f^{t+1}(v) - f^t(v)| \delta
\]

\leq \int_0^t \int_{\Omega \times \mathbb{R}^3} \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} k_v(v, u) \|f^t(u) - f^{t-1}(u)| |k_v(v, u) \| f^{t+1}(v) - f^t(v)| \delta
\]

\leq \int_0^t \int_{\Omega \times \mathbb{R}^3} \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f^t(v) - f^{t-1}(v)| \| f^{t+1}(v) - f^t(v)| \delta
\]

\leq \int_0^t \int_{\Omega \times \mathbb{R}^3} \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f^t - f^{t-1}|_{1+\delta} + \int_0^t \|f^{t+1} - f^t\|_{1+\delta},
\]

(6.8)

Then following the proof of (4.39) and applying (6.7) to (6.8), we can obtain

\[
\int_0^t \|f^{t+1} - f^t\|_{1+\delta} + \int_0^t \|f^t - f^{t-1}\|_{1+\delta} + \int_0^t \|f^{t+1} - f^t\|_{1+\delta}
\]

\leq (1) \int_0^t \|f^t - f^{t-1}\|_{1+\delta} + \|f^t - f^{t-1}(0)\|_{1+\delta}
\]

+ \sup_{0 \leq s \leq t} \{1 + ||\nabla_v f^{t-2}(s)||_{L^{1+\delta}} + ||w_\partial f^{t}(s)||_{L^{1+\delta}} + \|w_\partial f^{t-1}(s)\|_{L^{1+\delta}} \}
\]

\[
\left(\int_0^t \|f^t - f^{t-1}\|_{1+\delta} + \int_0^t \|f^{t+1} - f^t\|_{1+\delta} \right).
\]

(6.9)

Using (3.8), (3.2), (6.6), (6.7), (6.8), (6.9) and $[f^{t+1} - f^t]_{t=0} = 0$ we get

\[
\sup_{0 \leq s \leq t} \|f^{t+1}(s) - f^t(s)\|_{1+\delta} + \int_0^t \|f^{t+1} - f^t\|_{1+\delta} \leq \|O(t) + o(1)\| \left(\sup_{0 \leq s \leq t} \|f^t - f^{t-1}\|_{1+\delta} + \int_0^t \|f^t - f^{t-1}\|_{1+\delta} + \sup_{0 \leq s \leq t} \|f^{t-1} - f^{t-2}\|_{1+\delta} \right).
\]

(6.10)

Thus adding (6.10) with the same estimate (6.10) $f^{t+2} - f_{t+1}$ we get

\[
\sup_{0 \leq s \leq t} \|f^{t+1}(s) - f^t(s)\|_{1+\delta} + \int_0^t |f^{t+1} - f^{t+1}|_{1+\delta} + \|f^{t+2}(s) - f^{t+1}(s)\|_{1+\delta} + \int_0^t \|f^{t+2} - f^{t+1}\|_{1+\delta}
\]

\leq \|O(t) + o(1)\| \left(\sup_{0 \leq s \leq t} \|f^t - f^{t-1}\|_{1+\delta} + \int_0^t \|f^t - f^{t-1}\|_{1+\delta} + \sup_{0 \leq s \leq t} \|f^{t-1} - f^{t-2}\|_{1+\delta} \right).
\]
Therefore, inductively we have
\[ \sup_{0 \leq s \leq t} \| f^{t+1}(s) - f^t(s) \|_{1+\frac{\delta}{2}} + \int_0^t \| f^{t+1} - f^t \|_{1+\frac{\delta}{2}} \leq [O(t) + o(1)]^m. \]

Hence we derive stability
\[ \sup_{0 \leq s \leq t} \| f^t(s) - f^m(s) \|_{1+\frac{\delta}{2}} \leq [O(t) + o(1)]^{\min\{m, \ell\}}, \]
and this concludes (6.11).

Step 2. We combine (6.8) and (6.12) to get unique weak-* convergence (up to subsequence if necessary), \( (w_\delta f^t, w_\delta f^{t+1}) \rightharpoonup (w_\delta f, w_\delta f) \) weakly-* in \( L^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^3; \mathbb{R}^2) \cap L^\infty(\mathbb{R} \times \gamma; \mathbb{R}^2) \). For \( \varphi = [\varphi^+, \varphi^-] \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^3; \mathbb{R}^2) \),
\[ \int_0^T \langle f^{t+1}, -\partial_t - v \cdot \nabla_x + \varphi \rangle + \langle q f^{t+1}, \nabla_x \varphi \rangle + \langle \Gamma_{\text{gain}}(f^t, f^t), \varphi \rangle - \langle \Gamma_{\text{loss}}(f^{t+1}, f^t), \varphi \rangle \]
\[ = \int_0^T \langle K f^t, \varphi \rangle - \langle q f^t, \nabla_x \varphi \rangle + \langle \Gamma_{\text{gain}}(f^t, f^t), \varphi \rangle - \langle \Gamma_{\text{loss}}(f^{t+1}, f^t), \varphi \rangle \]
\[ + \int_0^T \langle f^{t+1}, \varphi \rangle_{\Gamma_\delta} - \int_0^T \langle c_\mu \sqrt{\mu} \int_{n \cdot u \geq 0} f^t \mu(n \cdot u) du, \varphi \rangle_{\Gamma_\delta}. \]

Except the underbraced terms in (6.14) all terms converges to limits with \( f \) instead of \( f^{t+1} \) or \( f^t \).

We define, for \( (t, x, v) \in \mathbb{R} \times \Omega \times \mathbb{R} \) and for \( 0 < \delta \ll 1 \),
\[ f^\delta_\delta(t, x, v) := \kappa_\delta(x, v) f^t(t, x, v) \]
\[ := \chi \left( \frac{|n(x) \cdot v|}{\delta} - 1 \right) \left[ 1 - \chi(\delta|v|) \right] \chi \left( \frac{|v|}{\delta} - 1 \right) f^t(t, x, v). \]

Note that \( f^\delta(t, x, v) = 0 \) if either \( |n(x) \cdot v| \leq \delta \), \( |v| \geq \frac{\delta}{2} \), or \( |v| \leq \delta \).

From (6.12),
\[ \left| \int_0^T \langle \Gamma_{\text{loss}}(f, f), \varphi \rangle \right| \]
\[ \leq \left| \int_0^T \langle \int_{\mathbb{R}^3} |v - u| (f^\delta_\delta(t, x, u) - f^\delta_\delta(t, x, u) + f^\delta_\delta(t, x, u) - f^\delta_\delta(t, x, u)) \sqrt{\mu(u)} du, \varphi(t, x, v) \rangle \right| dt \]
\[ + \left| \int_0^T \langle \int_{\mathbb{R}^3} |v - u| (f^\delta(t, x, u) + f^\delta(t, x, u)) \sqrt{\mu(u)} du, f^{t+1}(v), \varphi(t, x, v) \rangle \right| dt. \]

The second term converges to zero from the weak-\( * \) convergence in \( L^\infty \) and (3.8). The first term is bounded by, from (3.8),
\[ \left[ \int_0^T \left( \int_{\mathbb{R}^3} \kappa_\delta(x, u) (f^\delta(t, x, u) - f(t, x, u)) (u) \sqrt{\mu(u)} du \right)^2 \right]^{1/2} \]
\[ \times \sup_{0 \leq t \leq T} \| w_\delta f^{t+1}(t) \|_{L^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^3)} + O(\delta). \]

On the other hand, from Lemma 9 we have an extension \( \tilde{f}^\delta(t, x, v) \) of \( \kappa_\delta(x, u) f^t(t, x, u) \). We apply the average lemma (see Theorem 7.2.1 in page 187 of [9], for example) to \( \tilde{f}^\delta(t, x, v) \). From (6.3) and (4.8)
\[ \sup_{\ell} \left\| \int_{\mathbb{R}^3} \tilde{f}^\delta(t, x, u) \sqrt{\mu(u)} du \right\|_{H^{3/2}_x(\mathbb{R} \times \mathbb{R}^3)} < \infty. \]

Then by \( H^{3/4} \subseteq L^2 \), up to subsequence, we conclude that
\[ \int_{\mathbb{R}^3} \kappa_\delta(x, u) f^t(t, x, u) (u) \sqrt{\mu(u)} du \rightarrow \int_{\mathbb{R}^3} \kappa_\delta(x, u) f(t, x, u) (u) \sqrt{\mu(u)} du \] strongly in \( L^2_{t,x} \).

So we conclude that (6.14) \( \rightarrow 0 \) as \( \ell \rightarrow \infty \).

For (6.12) let us use a test function \( \varphi_1(v) \varphi_2(t, x) \). From the density argument, it suffices to prove a limit by testing with \( \varphi(t, x, v) \).
We use a standard change of variables \((v, u) \mapsto (v', u')\) and \((v, u) \mapsto (u', v')\) (for example see page 10 of [1]) to get
\[
\int_0^T \mathcal{G}_{\text{gain}}(f^r, f^t - f) \varphi \, dt
= \int_0^T \mathcal{G}_{\text{gain}}(f^t - f, f) \varphi \, dt + \int_0^T \mathcal{G}_{\text{gain}}(f^t - f, f) \varphi \, dt
= \sum_{i=\pm} \int_0^T \int_{\Omega \times R^3} \left( \int_{R^3} \int_{S^2} f^r_i(t, x, u) - f_1(t, x, u) + f^t_i(t, x, u) - f_1(t, x, u) \sqrt{\mu(v')} |v - u| \varphi_{1,i}(v') \, du \right)
\times f_i^t(t, x, v)^2 \varphi_{2,i}(t, x) \, dx \, dv dt
\]
(6.16)
\[
+ \sum_{i=\pm} \int_0^T \int_{\Omega \times R^3} \left( \int_{R^3} \int_{S^2} f^r_i(t, x, u) - f_1(t, x, u) \sqrt{\mu(v')} |v - u| \varphi_{1,i}(v') \, du \right)
\times \left( f_1^t(t, x, v) + f_1^t(t, x, v) \right) \varphi_{2,i}(t, x) \, dx \, dv dt.
\]
(6.17)
For \(N \gg 1\) we decompose the integration of (6.16) and (6.17) using
\[
1 = \{1 - \chi([|u| - N])\} \{1 - \chi([|v| - N])\}
+ \chi([|u| - N]) + \chi([|v| - N]) - \chi([|u| - N]) \chi([|v| - N]).
\]
(6.18)
Note that \(\{1 - \chi([|u| - N])\} \{1 - \chi([|v| - N])\} \neq 0\) if \(|v| \leq N + 1\) and \(|u| \leq N + 1\), and if \(\chi([|u| - N]) + \chi([|v| - N]) - \chi([|u| - N]) \chi([|v| - N]) \neq 0\) then either \(|v| \geq N\) or \(|u| \geq N\). From (5.3), the second part of (6.16) and (6.17) from (6.18) are bounded by
\[
\int_0^T \int_{\Omega \times R^3} \int_{R^3} \int_{S^2} \left[ \cdots \right] \times \{1 - \chi([|u| - N])\} \{1 - \chi([|v| - N])\}
\leq \sup \|w_\varphi f\|_\infty \|w_\varphi f\|_\infty \times \left\{ e^{-\frac{3}{2} |v|^2} + e^{-\frac{3}{2} |u|^2} \right\} \{1_{|v| \geq N} + 1_{|u| \geq N}\}
\leq O(\frac{1}{N}).
\]
Now we only need to consider the parts with \(\{1 - \chi([|u| - N])\} \{1 - \chi([|v| - N])\}\). Then
\[
= \sum_{i=\pm} \int_0^T \int_{\Omega \times R^3} \int_{R^3} \left( f^r_i(t, x, u) - f_1(t, x, u) + f^t_i(t, x, u) - f_1(t, x, u) \sqrt{\mu(v')} |v - u| \varphi_{1,i}(v') \, du \right)
\times \{1 - \chi([|v| - N])\} \int_{R^3} \sqrt{\mu(v')} |v - u| \varphi_{1,i}(v') \, du \, dx \, dv dt
\]
(6.19)
Now, let us define
\[
\Phi_{v,i}(u) := \{1 - \chi([|u| - N])\} \int_{R^3} \sqrt{\mu(v')} |v - u| \varphi_{1,i}(v') \, du \, dx \, dv dt
\]
(6.20)
For \(0 < \delta \ll 1\) we have \(O(\frac{1}{N})\) number of \(v_i \in R^3\) such that \(\{v \in R^3 : |v| \leq N + 1\} \subset \bigcup_{i=1}^{O(\frac{1}{N})} B(v_i, \delta)\). Since \(\Phi_{v,i}(u)\) is smooth in \(u\) and compactly supported, for \(0 < \varepsilon \ll 1\) we can always choose \(\delta > 0\) such that
\[
\Phi_{v,i}(u) - \Phi_{v,i}(u) < \varepsilon \quad \text{if} \quad v \in B(v_i, \delta).
\]
(6.21)
Now we replace \(\Phi_{v,i}(u)\) in the second line of (6.19) by \(\Phi_{v,i}(u)\) whenever \(v \in B(v_i, \delta)\). Moreover we use \(\kappa_\delta\)-cut off in (6.19). If \(v\) is included in several balls then we choose the smallest \(i\). From (6.21) and (5.3) the difference of (6.19) and the one with \(\Phi_{v,i}(u)\) can be controlled and we conclude that
\[
\int_0^T \int_{\Omega \times R^3} \int_{R^3} \Phi_{v,i}(u) \int_{R^3} \kappa_\delta(x, u)(f^r_i(t, x, u) - f_1(t, x, u) + f^t_i(t, x, u) - f_1(t, x, u)) \Phi_{v,i}(u) \, du \times \{1 - \chi([|v| - N])\} \int_{R^3} \sqrt{\mu(v')} |v - u| \varphi_{1,i}(v') \, du \, dx \, dv dt
\]
(6.22)
From Lemma[1] and the average lemma
\[
\max_{1 \leq i \leq O(\frac{1}{N})} \sup \| \int_{R^3} \kappa_\delta(x, u) f^t(t, x, u) \Phi_{v,i}(u) \, du \|_{H^{1/4}_{\varepsilon}(R \times R^3)} < \infty.
\]
(6.23)
For $i = 1$ we extract a subsequence $\ell_1 \subset \mathcal{I}_1$ such that
\[
\int_{\mathbb{R}^3} \kappa_\delta(x,u)f^\ell(t,x,u)\Phi_{v_i}(u)du \rightarrow \int_{\mathbb{R}^3} \kappa_\delta(x,u)f(t,x,u)\Phi_{v_i}(u)du \text{ strongly in } L^2_{t,x}.
\] (6.24)
Successively we extract subsequences $\mathcal{I}_{O(\frac{\alpha_3}{\alpha_1})} \subset \cdots \subset \mathcal{I}_2 \subset \mathcal{I}_1$. Now we use the last subsequence $\ell \in \mathcal{I}_{O(\frac{\alpha_3}{\alpha_1})}$ and redefine $f^\ell$ with it. Clearly we have (6.22) for all $i$. Finally we bound the last term of (6.22) by
\[
C_{\varphi_{3,N}} \max_i \int_0^T \sum_{i=\pm} \left\| \kappa_\delta(x,u)(f^\ell(t,x,u) - f(t,x,u))\Phi_{v_i}(u)du \sup_\ell \|w_\varphi f^\ell\|_\infty \right\|_{L^2_{t,x}} \rightarrow 0 \quad \text{as} \quad \ell \rightarrow \infty.
\]
Together with (6.22) we prove (6.16) $\rightarrow 0$. Similarly we can prove (6.17) $\rightarrow 0$.

Now we consider (6.12). From
\[
-(\Delta \phi^\ell - \Delta \phi) = \int \kappa_\delta(f_+^\ell - f^\ell - f_-^\ell - f^-_\varphi)\sqrt{\nabla} + \int (1 - \kappa_\delta)(f_+^\ell - f^\ell - f_-^\ell - f^-_\varphi)\sqrt{\nabla},
\]
we have
\[
\|\nabla \phi^\ell - \nabla \phi\|_{L^2_{t,x}} \leq \left\| \int \kappa_\delta(f^\ell - f)\sqrt{\nabla} \right\|_{L^2_{t,x}} + O(\delta) \sup_\ell \|w_\varphi f^\ell\|_\infty.
\] (6.25)
Then following the previous argument, we prove $\nabla \phi^\ell \rightarrow \nabla \phi$ strongly in $L^2_{t,x}$ as $\ell \rightarrow \infty$. Combining with $w_\varphi f^\ell \rightharpoonup w_\varphi f$ in $L^\infty$, we prove $\int_0^T (6.32)$ converges to $\int_0^T \langle qf, \{\nabla \phi^\ell \cdot \nabla \varphi + \varphi \cdot \nabla \phi^\ell \} \rangle$. This proves the existence of a (weak) solution $f \in L\infty$.

**Step 7.** We claim (6.27). By the weak lower-semicontinuity of $L^p$ we know that (if necessary we further extract a subsequence out of the subsequence of Step 6)
\[
w_\varphi \alpha_{t,c,x}^\ell \nabla \phi^\ell \rightarrow F, \quad \sup_{0 \leq t \leq T^*} \|F(t)\|_p \leq \liminf_{0 \leq t \leq T^*} \sup_{0 \leq t \leq T^*} \|w_\varphi \alpha_{t,c,x}^\ell \nabla \phi^\ell(t)\|_p,
\]
and
\[
\int_0^{T^*} \|F\|_p \leq \liminf_{0 \leq t \leq T^*} \int_0^{T^*} \|w_\varphi \alpha_{t,c,x}^\ell \nabla \phi^\ell(t)\|_p.
\]
We need to prove that
\[
F = w_\varphi \gamma_{t,c,x}^\ell \nabla \phi^\ell \text{ almost everywhere except } \gamma_0.
\] (6.26)
We claim that, up to some subsequence, for any given smooth test function $\psi \in C^\infty_0(\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0; \mathbb{R}^3)$
\[
\lim_{\ell \rightarrow \infty} \int_0^T \int_{\Omega \times \mathbb{R}^3} w_\varphi \alpha_{t,c,x}^\ell \nabla \phi^\ell(t)\psi \, dx \, dv = \int_0^T \int_{\Omega \times \mathbb{R}^3} w_\varphi \gamma_{t,c,x} \nabla \phi \psi \, dx \, dv.
\] (6.27)
We note that we need to extract a single subsequence, let say $(\ell_\ell) \subset \{\ell\}$, satisfying (6.27) for all test functions in $C^\infty_0(\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0; \mathbb{R}^3)$. Of course the convergent rate needs not to be uniform and it could vary with test functions.

For each $N \in \mathbb{N}$ we define a set
\[
S_N := \{(x,v) \in \bar{\Omega} \times \mathbb{R}^3 : \text{dist}(x,\partial \Omega) \leq \frac{1}{N} \text{ and } |n(x) \cdot v| \leq \frac{1}{N} \} \cup \{|v| > N\}.
\] (6.28)
For a given test function we can always find $N \gg 1$ such that
\[
sup(\psi) \subset (S_N)^c := \bar{\Omega} \times \mathbb{R}^3 \backslash S_N.
\] (6.29)
We will exam (6.27) by the identity obtained from the integration by parts
\[
\int_0^T \langle w_\varphi \gamma_{t,c,x}^\ell \nabla \phi^\ell(t)\psi \rangle = \int_0^T (\alpha_{t,c,x}^\ell \nabla \phi^\ell(t)\psi) + \sum_{i=\pm} \int_0^T \nabla \alpha_{t,c,x}^\ell \nabla \phi^\ell(t)\psi + \int_0^T \langle \nabla \phi^\ell(t)\psi \rangle.
\] (6.30)
(6.31)
(6.32)
We finish this step by proving the convergence of (6.30) and (6.31). From (1.30) and (3.5), if $(x,v) \in (S_N)^c$ then
\[
\sup_{t \geq 0} |\alpha_{t,c,x}^\ell(t,x,v)| \lesssim |v| + (t + \epsilon)^\beta \sup_{\ell \geq 0} \|\nabla \phi^\ell\|_{L^\infty} \lesssim N^\beta + (T^* + \epsilon)^\beta \sup_{\ell \geq 0} \|w_\varphi f^\ell\|_{L^\infty} \leq C_N < +\infty.
\]
Hence we extract a subsequence (let say \{\ell_N\}) out of subsequence in Step 6 such that \(\alpha^{\beta}_{f,t,N,\ell_N} \xrightarrow{\gamma} A_\ell \in L^\infty\) weakly-* in \(L^\infty((0,T^*) \times (S_N)^c) \cap L^\infty((0,T^*) \times (\gamma \cap (S_N)^c))\). Note that \(\alpha^{\beta}_{f,t,N,\ell_N}\) satisfies \([\partial_t + v \cdot \nabla_x - \ell \nabla_y f^\ell \cdot \nabla_v] \alpha^{\beta}_{f,t,N,\ell_N} = 0\) and \(\alpha^{\beta}_{f,t,N,\ell_N} \xrightarrow{\gamma} \|u - v\|^2\). By passing a limit in the weak formulation we conclude that \([\partial_t + v \cdot \nabla_x - \ell \nabla_y f^\ell \cdot \nabla_v] A_\ell = 0\) and \(A_\ell \xrightarrow{\gamma} \|u - v\|^2\). By the uniqueness of the Vlasov equation (\(\nabla \phi_f \in W^{1,p}\) for any \(p < \infty\)) we derive \(A_\ell = \alpha^{\beta}_{f,t,\ell_N}\) almost everywhere and hence conclude that \(\alpha^{\beta}_{f,t,N,\ell_N} \xrightarrow{\gamma} \alpha^{\beta}_{f,t,\ell_N} \) weakly-* in \(L^\infty((0,T^*) \times (S_N)^c) \cap L^\infty((0,T^*) \times (\gamma \cap (S_N)^c))\). (6.33)

Now the convergence of (6.30) and (6.31) is a direct consequence of strong convergence of (6.34) and the weak-* convergence of (6.33).

**Step 8.** We devote the entire Step 8 to prove the convergence of (6.32).

**Step 8-a.** Let us choose \((x,v) \in (S_N)^c\). From (6.24)
\[
\text{If } t^{\ell}_{b,v} \geq t + \epsilon \text{ then } \alpha^{f,t,\ell}_N(t,x,v) = 1.
\]
From now we only consider that case
\[
t^{\ell}_{b,v}(t,x,v) \leq \epsilon + t.
\]
If \(|v| \geq 2(\epsilon + T^*) \sup \|\nabla \phi^\ell\|_\infty\) then
\[
|V^{\ell}_{b}(t,x,v)| \geq |v| - \int_s^\ell \|\nabla \phi^\ell(\tau)\|_\infty \text{d} \tau
\geq (\epsilon + T^*) \sup \|\nabla \phi^\ell\|_\infty \text{ for all } \ell \text{ and } s \in [-\epsilon, T^*].
\]
For this case we need a version of velocity lemma of \(\tilde{\alpha}\) in (3.31), which shows up in the author's previous paper [3], but this time with neutral boundary condition \(\pm \nabla \phi^\ell \cdot n = 0\) on \(\partial \Omega\). So \(\tilde{\alpha}\) now takes the form
\[
\tilde{\alpha}(t,x,v) := \sqrt{\xi(x)^2 + |\nabla \xi(x) \cdot u|^2 - 2(u \cdot \nabla^2 \xi(x) \cdot u)x}.\]
From a direct computation,
\[
[\partial_t + u \cdot \nabla_x - \ell \nabla_y f^\ell \cdot \nabla_u]\{\xi(x)^2 + |\nabla \xi(x) \cdot u|^2 - 2(u \cdot \nabla^2 \xi(x) \cdot u)x)\}
= 2(u \cdot \nabla \xi) + 2(u \cdot \nabla^2 \xi - u) - 2u \cdot (u \cdot \nabla^2 \xi - u)\xi
\geq |u \cdot \nabla \xi|^2 + |\xi|^2 + \{u + 1 |u|\}(-2(u \cdot \nabla^2 \xi(x) \cdot u)(x) + |\nabla \phi^\ell \cdot \nabla \xi| |\nabla \xi| |u|.
\]
From the Neumann BC \((n(x) \cdot E(t,x) = 0 \text{ on } \partial \Omega)\), we have
\[
|\nabla \phi^\ell(t,x) \cdot \nabla \xi(x)|
\leq |\nabla \phi^\ell(t,x) \cdot \nabla \xi(x)| + |\nabla \phi^\ell(t)||c_{1,0}\|\xi||c_{2,0}\| |x - x_*|
\leq \Omega \|\nabla \phi^\ell(t)||c_{1,0}\|\xi),
\]
where \(x_\ast \in \partial \Omega\) such that \(|x - x_*| = \inf_{y \in \partial \Omega} |x - y|\).

By controlling the last term of (6.38) by (6.30) and using (6.36), we conclude that
\[
\frac{d}{ds} \tilde{\alpha}(s,X^f(s,t,x,u),V^f(s,t,x,u))^2
\leq \Omega \left(1 + |V^{f\ell}(s,t,x,u)| + \frac{1}{|V^{f\ell}(s,t,x,u)|}\right) \tilde{\alpha}(s,X^f(s,t,x,u),V^f(s,t,x,u))^2
\leq \Omega \left(1 + |V^{f\ell}(s,t,x,u)|\right) \tilde{\alpha}(s,X^f(s,t,x,u),V^f(s,t,x,u))^2,
\]
so
\[
|\tilde{\alpha}(s,X^f(s,t,x,u),V^f(s,t,x,u))| \geq \frac{1}{C_\Omega} \tilde{\alpha}(t,x,v)e^{-C_\Omega |\epsilon + T^*| \sup \|\nabla \phi^\ell\|_\infty}
\geq e^{-\frac{C_\Omega}{\sup \|\nabla \phi^\ell\|_\infty}} \times \frac{1}{N} \text{ for all } \ell \text{ and } s \in [-\epsilon, T^*].
\]
Especially at \(s = t - t^{\ell}_{b,v}(t,x,v)\), from (6.37),
\[
|n(x^{\ell}_{b,v}) \cdot v^{\ell}_{b,v}| \geq e^{-\frac{C_\Omega}{\sup \|\nabla \phi^\ell\|_\infty}} \times \frac{1}{N} \text{ for all } \ell.
\]
Step 8-b. From now on we assume \( V_{n,n}^{f}(t-t_{b,n}^{f},t,t,x,v) \) and

\[
|v| \leq 2(\varepsilon + T^{**}) \sup_{t} \|\nabla \phi\|_{\infty},
\]

or, from \((1.28)\), \( |V_{b,n}^{f} (s;t,x,v)| \leq 3(\varepsilon + T^{**}) \sup_{t} \|\nabla \phi\|_{\infty} \) for \( s \in [-\varepsilon,T^{**}] \). \hfill (6.41)

Let \((X_{n,n}^{f}, X_{\|a\|,n}^{f}, V_{n,n}^{f}, V_{\|a\|,n}^{f})\) satisfy \((2.3), (2.6), \) and \((2.10)\) with \( E = -\iota \nabla \phi^{\circ} \).

Let us define

\[
\tau_{1} := \sup \{ \tau > 0 : V_{n,n}^{f}(s;t,x,v) \geq 0 \text{ for all } s \in [t - t_{b,n}^{f}(t,x,v), \tau] \}.
\]

Since \((X_{n,n}^{f}(s;t,x,v), V_{n,n}^{f}(s;t,x,v))\) is \( C^{1} \) (note that \( \nabla \phi^{\circ} \in C_{t,x}^{1} \)) in \( s \) we have \( V_{n,n}^{f}(\tau_{1};t,x,v) = 0 \).

We claim that, there exists some constant \( \delta_{**} = O_{\varepsilon,T^{**}, \sup_{t} \|\nabla \phi\|_{C_{1}^{1}}} \) in \((6.43)\) which does not depend on \( \ell \) such that

\[
If \quad 0 \leq V_{n,n}^{f}(t-t_{b,n}^{f}(t,x,v),t,t,x,v) \leq \delta_{**} \text{and } (6.41),
\]

then \( V_{n,n}^{f}(s;t,x,v) \leq e^{C_{t-(t-t_{b,n}^{f}(t,x,v))}\|\nabla \phi\|_{C_{1}^{1}}} V_{n,n}^{f}(t-t_{b,n}^{f}(t,x,v),t,t,x,v) \) for \( s \in [t - t_{b,n}^{f}, \tau_{1}] \). \hfill (6.43)

For the proof we regard the equations \((2.8), (2.6), \) and \((2.10)\) as the forward-in-time problem with an initial datum at \( s = t - t_{b,n}^{f}(t,x,v) \). Clearly we have \( X_{n,n}^{f}(t-t_{b,n}^{f}(t,x,v),t,x,v) = 0 \) and \( V_{n,n}^{f}(t-t_{b,n}^{f}(t,x,v),t,x,v) \geq 0 \) from Lemma \[1\]

Again from Lemma \[1\] if \( V_{n,n}^{f}(t-t_{b,n}^{f}(t,x,v),t,x,v) = 0 \) then \( X_{n,n}^{f}(s;t,x,v) = 0 \) for all \( s \geq t - t_{b,n}^{f}(t,x,v) \). From now on we assume \( V_{n,n}^{f}(t-t_{b,n}^{f}(t,x,v),t,x,v) > 0 \). From \((2.10)\), as long as \( t - t_{b,n}^{f}(t,x,v) \leq s \leq T^{**} \) and

\[
V_{n,n}^{f}(s;t,x,v) \geq 0 \text{ and } X_{n,n}^{f}(s;t,x,v) \leq \frac{1}{N} \ll 1,
\]

then we have

\[
\hat{V}_{n,n}^{f}(s) = \underbrace{[V_{n,n}^{f}(s) \cdot \nabla^{2} \eta(X_{\|a\|,n}^{f}(s)) \cdot \nabla V_{\|a\|,n}^{f}(s)] \cdot \nabla(X_{\|a\|,n}^{f}(s))}_{\leq \text{from (6.41)}} - \nabla \phi^{\circ}(s) \cdot |\nabla \phi^{\circ}(s)|
\]

\[
= O(1) \sup_{t} \|\nabla \phi\|_{C_{1}^{1}} \times X_{n,n}^{f}(s) \quad \text{from (6.43) and (2.13)}
\]

\[
- X_{n,n}^{f}(s)[V_{\|a\|,n}^{f}(s) \cdot \nabla^{2} \eta(X_{\|a\|,n}^{f}(s)) \cdot \nabla V_{\|a\|,n}^{f}(s)] \cdot \nabla(X_{\|a\|,n}^{f}(s)) \in C(1+\varepsilon+T^{**})^{2}(\sup_{t} \|\nabla \phi\|_{C_{1}^{1}} \sup_{t} \|\nabla \phi\|_{\infty}) \times X_{n,n}^{f}(s).
\]

Let us consider \((6.45)\) together with \( \hat{X}_{n,n}^{f}(s;t,x,v) = V_{d,n}^{f}(s;t,x,v) \). Then, as long as \( s \) satisfies \((6.44)\),

\[
V_{n,n}^{f}(s) = V_{n,n}^{f}(t-t_{b,n}^{f}) + \int_{t-t_{b,n}^{f}}^{s} \hat{V}_{n,n}^{f}(\tau) \d \tau
\]

\[
\leq V_{n,n}^{f}(t-t_{b,n}^{f}) + \int_{t-t_{b,n}^{f}}^{s} C(1+\varepsilon+T^{**})^{2}(\sup_{t} \|\nabla \phi\|_{C_{1}^{1}} \sup_{t} \|\nabla \phi\|_{\infty}) \times X_{n,n}^{f}(\tau) \d \tau
\]

\[
= V_{n,n}^{f}(t-t_{b,n}^{f}) + \int_{t-t_{b,n}^{f}}^{s} C(1+\varepsilon+T^{**})^{2}(\sup_{t} \|\nabla \phi\|_{C_{1}^{1}} \sup_{t} \|\nabla \phi\|_{\infty}) \int_{t-t_{b,n}^{f}}^{\tau} V_{n,n}^{f}(\tau') \d \tau' \d \tau.
\]

Following the same argument of the proof of Lemma \[3\] we derive that

\[
V_{n,n}^{f}(s) \leq V_{n,n}^{f}(t-t_{b,n}^{f}) + C(1+\varepsilon+T^{**})^{2}(\sup_{t} \|\nabla \phi\|_{C_{1}^{1}} \sup_{t} \|\nabla \phi\|_{\infty}) \int_{t-t_{b,n}^{f}}^{s} \max_{s \in [s - (t-t_{b,n}^{f}(t,x,v))]} V_{n,n}^{f}(\tau') \d \tau'.
\]

From the Gronwall’s inequality, we derive that, as long as \((6.44)\) holds,

\[
V_{n,n}^{f}(s;t,x,v) \leq V_{n,n}^{f}(t-t_{b,n}^{f}(t,x,v)) e^{C(1+\varepsilon+T^{**})^{2}(\sup_{t} \|\nabla \phi\|_{C_{1}^{1}} \sup_{t} \|\nabla \phi\|_{\infty}) \times s - (t-t_{b,n}^{f}(t,x,v))}.
\]
Now we verify the conditions of (6.41) for all \(-\varepsilon \leq t - t^f_{b,\iota}(t, x, v) \leq s \leq T^*\). Note that we are only interested in the case of \(V_{n,\iota}^f(t - t^f_{b,\iota}(t, x, v); t, x, v) < \delta_{**}\). From the argument of (6.34), ignoring negative curvature term,

\[
|X^f_{n,\iota}(s; t, x, v)| \leq (\varepsilon + T^*)|V^f_{n,\iota}(t^f_{b,\iota}; t, x, v)| + C[1 + (\varepsilon + T^*)^2] \sup_t \|\nabla \phi^f\|_\infty \sup_t \|\nabla \phi^f\| C \int_{t^f_{b,\iota}}^s \int_{t^f_{b,\iota}}^{t^f_{b,\iota}} |X^f_{n,\iota}(\tau; t, x, v)| d\tau ds 
\leq (\varepsilon + T^*)|V^f_{n,\iota}(t^f_{b,\iota}; t, x, v)| + C \int_{t^f_{b,\iota}}^s \int_{t^f_{b,\iota}}^{t^f_{b,\iota}} |\tau - (t - t^f_{b,\iota})| |X^f_{n,\iota}(\tau; t, x, v)| d\tau.
\]

Then by the Gronwall’s inequality we derive that, in case of (6.35),

\[
|X^f_{n,\iota}(s; t, x, v)| \leq C_{**} + T^* |V^f_{n,\iota}(t - t^f_{b,\iota}; t, x, v)| \quad \text{for all} \quad -\varepsilon \leq t - t^f_{b,\iota} \leq s \leq T^*.
\]

If we choose

\[
\delta_{**} = \frac{o(1)}{T^* + \varepsilon} \times \frac{1}{N},
\]

then (6.40) holds for \(-\varepsilon \leq t - t^f_{b,\iota}(t, x, v) \leq s \leq T^*\). Hence we complete the proof of (6.43).

**Step 8-c.** Suppose that (6.41) holds and \(0 \leq V_{n,\iota}^f(t - t^f_{b,\iota}(t, x, v); t, x, v) < \delta_{**}\) with \(\delta_{**}\) of (6.48). Recall the definition of \(\tau_1\) in (6.32). Inductively we define \(\tau_2 := \sup \{\tau \geq 0 : V_{n,\iota}^f(s; t, x, v) \leq 0 \text{ for all } s \in [\tau_1, \tau]\} \quad \text{and} \quad \tau_3, \tau_4, \ldots\)

Clearly such points can be countably many at most in an interval of \([t - t^f_{b,\iota}, t]\). Suppose \(\lim_{k \to \infty} \tau_k = t\). Then choose \(k_0 \geq 1\) such that \(|\tau_{k_0} - t| \ll |V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)|\). Then, for \(s \in [\tau_{k_0}, t]\), from (6.35) and (6.41),

\[
|V^f_{n,\iota}(t; t, x, v)| \lesssim |V^f_{n,\iota}(t - t^f_{b,\iota}; t, x, v)|.
\]

Now we assume that \(\tau_{k_0} < t \leq \tau_{k_0+1}\). From the definition of \(\tau_3\) in (6.32) we split the case in two.

**Case 1:** Suppose \(V^f_{n,\iota}(s; t, x, v) > 0\) for \(s \in (\tau_{k_0}, t)\).

From (6.35) and (6.41),

\[
V^f_{n,\iota}(t; t, x, v) \lesssim \int_{\tau_{k_0}}^T X^f_{n,\iota}(s) \lesssim |V^f_{n,\iota}(t - t^f_{b,\iota}; t, x, v)|.
\]

**Case 2:** Suppose \(V^f_{n,\iota}(s; t, x, v) < 0\) for \(s \in (\tau_{k_0}, t)\).

Suppose

\[
-V_{n,\iota}^f(t; t, x, v) = |V^f_{n,\iota}(t; t, x, v)| \geq |V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)| \quad \text{for any} \quad 0 < A < \frac{1}{2}.
\]

From (6.35), now taking account of the curvature term this time, we derive that

\[
-V_{n,\iota}^f(t; t, x, v) = \int_{\tau_{k_0}}^t (-1)[V^f_{n,\iota}(s; t, x, v)) \cdot \nabla^2 \eta(X^f_{n,\iota}(s)) \cdot V^f_{n,\iota}(s)] \cdot n(X^f_{n,\iota}(s)) ds + C |V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)|,
\]

where we have used (6.31) and (6.41). From (6.31), the above inequality implies that, for \(|V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)| \ll 1,

\[
|V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)|^2 \lesssim \int_{\tau_{k_0}}^t (-1)[V^f_{n,\iota}(s; t, x, v)) \cdot \nabla^2 \eta(X^f_{n,\iota}(s)) \cdot V^f_{n,\iota}(s)] \cdot n(X^f_{n,\iota}(s)) ds.
\]

Note that \(|\nabla^2 V_{n,\iota}^f(s)|\) and \(|\nabla^2 X^f_{n,\iota}(s)|\) are all bound from \(\nabla \phi^f \in C^1\), (6.31), and (6.37). Hence we have

\[
\frac{1}{2} |V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)|^2 \lesssim \int_{\tau_{k_0}}^t (-1)[V^f_{n,\iota}(s; t, x, v)) \cdot \nabla^2 \eta(X^f_{n,\iota}(s)) \cdot V^f_{n,\iota}(s)] \cdot n(X^f_{n,\iota}(s)) ds.
\]

On the other hand, if \(-|V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)|^2 \leq \tau_{k_0}\) then \(|t - \tau_{k_0}| \leq |V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)|^4\), which implies that, from (6.35), (6.31), and (6.41),

\[
|V_{n,\iota}^f(t; t, x, v)| \lesssim |V_{n,\iota}^f(t - t^f_{b,\iota}; t, x, v)|^4 \quad \text{for any} \quad 0 < A < \frac{1}{2}.
\]
Now we consider $X_n^{t_f}(t; t, x, v)$. From (6.35) and $X_n^{t_f}(s; t, x, v) = V_n^{t_f}(s; t, x, v)$ together with (6.47) and (6.51)

$$X_n^{t_f}(t; t, x, v) \leq |V_n^{t_f}(t - t_{b_n}; t, x, v)| + \int_{t_{b_n}}^{t} \int_{t_{b_n}}^{t} \left| \frac{V_n^{t_f}(s) \cdot \nabla^2 \eta(X_n^{t_f}(s)) \cdot V_n^{t_f}(s) \cdot n(X_n^{t_f}(s))}{|s|} \right| ds dr \leq 0$$

$$\leq |V_n^{t_f}(t - t_{b_n}; t, x, v)|$$

$$+ \left| V_n^{t_f}(t - t_{b_n}; t, x, v) \right|^2 \int_{t_{b_n}}^{t} \left[ |V_n^{t_f}(s) \cdot \nabla^2 \eta(X_n^{t_f}(s)) \cdot V_n^{t_f}(s) \cdot n(X_n^{t_f}(s))| ds \right]$$

$$\leq |V_n^{t_f}(t - t_{b_n}; t, x, v)| - |V_n^{t_f}(t - t_{b_n}; t, x, v)|^2 \text{ from (6.52)}$$

$$\leq |V_n^{t_f}(t - t_{b_n}; t, x, v)| - |V_n^{t_f}(t - t_{b_n}; t, x, v)|^2 \leq 0,$$

for $|V_n^{t_f}(t - t_{b_n}; t, x, v)| \ll 1$. Clearly this cannot happen since $x \in \Omega$ and $x_n \geq 0$. Therefore our assumption (6.51) was wrong and we conclude (6.53).

Step 8-d. From (6.34), (6.40), (6.51), and (6.53) in Step 8-a and Step 8-b, we conclude that the same estimate (6.55) for $|V_n^{t_f}(t - t_{b_n}; t, x, v)| \ll 1$ in the case of (6.35) and (6.41). Finally from (6.34), (6.40), (6.53), and (6.55) we conclude that

$$|V_n^{t_f}(t - t_{b_n}; t, x, v)| \geq \begin{cases} 1 & (t, x, v) \in [0, T^*] (S_N)^c. 
\end{cases}$$

From (2.36), (2.37), (2.40), and (2.41) in Lemma 2.4 in [21], and combing with (6.51) we have

$$\nabla_{x,v}n(x_{b_n}) \leq \frac{1}{|v_n(x_{b_n})|} \left| V_n^{t_f}(t - t_{b_n}; t, x, v) \right|,$$

$$\nabla_{x,v}(x_{b_n}) \leq \frac{1}{|v_n(x_{b_n})|} \left| V_n^{t_f}(t - t_{b_n}; t, x, v) \right|$$

$$\leq \frac{1}{|v_n(x_{b_n})|} \left| V_n^{t_f}(t - t_{b_n}; t, x, v) \right|$$

Therefore from above we have

$$|\nabla_{x,v} \alpha_{f,\epsilon}^{b}(t, x, v)| \leq \beta |\alpha_{f,\epsilon}^{b}(t, x, v)|^{\beta - 1} |\nabla_{x,v} n(x_{b_n}) + \nabla_{x,v} v_n(x_{b_n})| \leq \chi |V_n^{t_f}(t - t_{b_n}; t, x, v)|^{\beta - 1} |V_n^{t_f}(t - t_{b_n}; t, x, v)|^{-1}.$$

Combing (6.55) and (6.56) we achieve

$$\sup_{t \in \Omega, (x,v) \in (S_N)^c} |\nabla_{x,v} \alpha_{f,\epsilon}^{b}(t, x, v)| \leq \frac{1}{|V_n^{t_f}(t - t_{b_n}; t, x, v)|^{2 - \beta}} \leq \epsilon, N, T^*.$$"
Lemma 7. In order to prove the proposition we need the following:

and the projection of $\|h_1(t)\| \leq \|h_1(t')\| \leq 1$ for $|t - t'| \ll 1$, we have

Then by (6.20) we have $\|h(t)\| - \|h(t')\| < \frac{1}{2^k} + O_k(\|t - t'\|)$. For large $k$, choosing $|t - t'| \ll 1$, we can prove $\|h(t)\| - \|h(t')\| \ll 1$ as $|t - t'| \ll 1$. Hence $w_\alpha f(t)\|_\infty$ is continuous in $t$.

The continuity of $\|
abla_v f(t)\|_{L^2}\|_{L^2} + |w_\alpha^\beta \nabla_v f(t)|_p + f_0 |w_\alpha^\beta \nabla_v f(t)|_p$ is an easy consequence of (5.5)–(6.19), (4.1), (4.15), (4.6) as well.

7. $L^2$ Coercivity

Proposition 8. Suppose $(f, \phi)$ solves (1.19), (1.21), and (1.23). Then there is $0 < \lambda_2 \ll 1$ such that for $0 \leq s \leq t$,

The null space of linear operator $L$ is a six-dimensional subspace of $L^2_\nu(\mathbb{R}^3; \mathbb{R}^2)$ spanned by orthonormal vectors

and the projection of $f$ onto the null space $N(L)$ is denoted by

In order to prove the proposition we need the following:

Lemma 7. There exists a function $G(t)$ such that, for all $0 \leq s \leq t$, $G(s) \lesssim \|f(s)\|_2^2$ and

(7.4)
Proof of Proposition Step 1. Without loss of generality we prove the result with \( s = 0 \). We have an \( L^2 \)-estimate from \( f_0^t (2e^{\lambda x}f, (1.23)) \)

\[
\| e^{\lambda x}f(t) \|_2^2 - \| f(0) \|_2^2 + \int_0^t | e^{\lambda x}(1 - P_\gamma) f_{\gamma} |_2^2 + \\
+ \int_0^t \int_{\Omega \times \mathbb{R}^3} v \cdot \nabla \phi f e^{2\lambda x} (f_+^2 - f_-^2) + 2 \int_0^t e^{\lambda x} (f, Lf) = \\
2 \int_0^t e^{2\lambda x} (f, \Gamma(f, f)) - 2 \int_0^t e^{2\lambda x} \int_{\Omega} \nabla \phi f \cdot \int_{\mathbb{R}^3} v \sqrt{\mu} (f_+ - f_-) \\
+ 2 \lambda_x \int_0^t \| e^{\lambda x} f(\tau) \|_2^2.
\]

where

\[
P_\gamma f := [P_\gamma f_+, P_\gamma f_-] := \left[ c_\mu \sqrt{\mu(u)} f_{\mu(u)>0} f_+ + c_\mu \sqrt{\mu(u)} f_{\mu(u)<0} f_- \right] du.
\]

On the other hand multiplying \( \sqrt{\mu(v)} \phi f(t, x) \) with a test function \( \psi(t, x) \) to \((1.23)\) and applying the Green’s identity, (from the charge conservation) we obtain

\[
\int_{\Omega} \nabla \phi f(t, x) \cdot \int_{\mathbb{R}^3} v \sqrt{\mu} (f_+ - f_-) d\sigma dx \\
= - \int_{\Omega} \phi f(t, x) \left( \int_{\mathbb{R}^3} v \cdot \nabla_\mu \sqrt{\mu} (f_+ - f_-) d\sigma \right) dx + \int_{\partial \Omega \times \mathbb{R}^3} \phi f(t, x) (f_+ - f_-) \sqrt{\mu(n \cdot v)} d\sigma dS_a \\
= \int_{\Omega} \phi f(t, x) \partial_\tau \left( \int_{\mathbb{R}^3} (f_+ - f_-) \sqrt{\mu} dx \right) dx + \int_{\partial \Omega \times \mathbb{R}^3} \phi f(t, x) (f_+ - f_-) \sqrt{\mu(n \cdot v)} d\sigma dS_a.
\]

From \((1.11)\), the last boundary contribution equals zero. Now we use \((1.10)\) and deduce that

\[
\int_0^t e^{2\lambda x} \int_{\Omega} \phi f(t, x) \partial_\tau \left( \int_{\mathbb{R}^3} (f_+ - f_-)(\tau) \sqrt{\mu} d\sigma dx \right) d\tau dx \\
= - \int_0^t e^{2\lambda x} \int_{\Omega} \phi f(t, x) \partial_\tau \Delta_x \phi f(\tau, x) d\sigma dx d\tau \\
= \frac{1}{2} \int_0^t e^{2\lambda x} \int_{\Omega} \partial_\tau |\nabla \phi f(\tau, x)|^2 d\sigma dx d\tau \\
= \frac{1}{2} \left( \int_{\Omega} e^{2\lambda x} |\nabla \phi f(t, x)|^2 d\sigma dx \right) - \frac{1}{2} \left( \int_{\Omega} |\nabla \phi f(0, x)|^2 d\sigma dx \right) \\
- \lambda_x \int_0^t e^{2\lambda x} \int_{\Omega} |\nabla \phi f(\tau, x)|^2 d\sigma dx d\tau.
\]

Hence we derive

\[
\| e^{\lambda x} f(t) \|_2^2 - \| f(0) \|_2^2 + \int_0^t \int_{\Omega \times \mathbb{R}^3} e^{2\lambda x} v \cdot \nabla \phi f(f_+^2 - f_-^2) + \\
+ 2C \int_0^t \int_{\Omega \times \mathbb{R}^3} e^{2\lambda x} (I - P_\gamma) f |_0^t + \int_0^t e^{2\lambda x} (1 - P_\gamma) f_{\gamma} |_0^t + \\
\lesssim \| f(0) \|_2^2 + \| \nabla \phi f(0) \|_2^2 + \int_0^t e^{2\lambda x} \sqrt{\vartheta(\Gamma(f, f))} \\
+ \{ \lambda_x + o(1) \} \int_0^t e^{2\lambda x} f |_0^t \|_2^2 + \lambda_x \int_0^t e^{2\lambda x} \nabla \phi f |_2^2.
\]

Now we apply Lemma and add \( o(1) \times (1.4) \) to the above inequality and choose \( 0 < \lambda_x \ll 1 \) to conclude \((1.4)\) except the full boundary control.

Step 2. Note that from \((1.23)\), \( P_\gamma f_{\pm} = z_\pm(t, x) \sqrt{\mu(v)} \) for a suitable functions \( z_\pm(t, x) \) on the boundary. Then for \( \iota = + \) or \(-, \) for \( 0 < \varepsilon < 1 \)

\[
| P_\gamma f_\iota^2 |_{2,2} = \int_{\partial \Omega} |z_\iota(t, x)|^2 dx \times \int_{\mathbb{R}^3} \mu(v) |n(x) \cdot v| dv \\
\lesssim \int_{\partial \Omega} |z_\iota(t, x)|^2 dx \times \int_{\gamma_\iota(x)} \mu(v)^{3/2} |n(x) \cdot v| dv \\
= \left[ 1_{\gamma_+ \setminus \gamma_{\iota}(x)} \mu^{1/4} P_\gamma f_{\lambda} + 1_{\gamma_- \setminus \gamma_{\iota}(x)} \mu^{1/4} P_\gamma f_{\lambda} \right].
\]
Since $P_s f = f - (1 - P_s)f$ on $\gamma_+$ we have $|1_{\gamma_x \gamma_+} \mu^{1/4} P_s f|_2^2 \leq |1_{\gamma_x \gamma_+} \mu^{1/4} f|_2^2 + \| (1 - P_s)f \|_2^2$. Therefore

$$
\int_0^t |P_s f|_{2,2}^2 \leq \int_0^t |1_{\gamma_x \gamma_+} \mu^{1/4} f|_2^2 + \int_0^t \| (1 - P_s)f \|_2^2. 
$$

(7.6)

Note that

$$
\left| \left[ \partial_t v \cdot \nabla x - q \nabla \phi \cdot \nabla v \right] (\mu^{1/4} f) \right|
\lesssim \mu^{1/4} \left\{ \| v \| \nabla x \phi + | v | \| \nabla x \phi \| + | L f | + | \Gamma (f, f) | \right\}.
$$

By the trace theorem Lemma 2

$$
\int_0^t |1_{\gamma_x \gamma_+} \mu^{1/4} f|_2^2 d t
\lesssim \| f_0 \|_2^2 + (1 + \| w f \|_\infty) \int_0^t \| f \|_2^2 + \int_0^t \| \nabla \phi \|_2^2.
$$

(7.7)

Adding $o(1) \times (7.6)$ to the result of Step 1 and using (7.7) we conclude (7.1).

**Proof of Lemma 7.** From the Green’s identity, a solution $f$ of (1.23) satisfies

$$
\begin{align*}
\langle f (t), \psi (t) \rangle - \langle f (s), \psi (s) \rangle & - \int_s^t \langle f (t), \partial_t \psi \rangle + \int_s^t \int_f (\psi \cdot f) (v \cdot n (x)) \mu f \phi f d x d t \\
& - \int_s^t (P f, v \cdot \nabla x \psi) - \int_s^t \langle (I - P) f, v \cdot \nabla x \psi \rangle + \int_s^t (q \sqrt{\mu f}, \nabla x \psi f \cdot \nabla v \left( \frac{1}{\sqrt{\mu}} \right)) \\
& = \int_s^t \langle \psi, \{ -L (I - P) f + \Gamma (f, f) \} \rangle - \int_s^t \int_f \psi (v \cdot \nabla x \phi f \sqrt{\mu}) \mu f \phi f. 
\end{align*}
$$

(7.8)

We use a set of test functions:

$$
\psi_a \equiv \left\{ \begin{array}{ll} - (| v |^2 - \beta_a) \sqrt{\mu v} \cdot \nabla x \phi f \end{array} \right\},
\psi_{b_{1,1}} \equiv \left\{ \begin{array}{ll} (| v |^2 - \beta_b) \sqrt{\mu} \partial_j \phi f \end{array} \right\}, i, j = 1, 2, 3,
\psi_{b_{1,2}} \equiv \left\{ \begin{array}{ll} (| v |^2 - \beta_b) \sqrt{\mu} \partial_j \phi f \end{array} \right\}, i \neq j,
\psi_c \equiv \left\{ \begin{array}{ll} (| v |^2 - \beta_c) \sqrt{\mu v} \cdot \nabla x \phi f \end{array} \right\},
$$

(7.9)

where $\phi f (t, x), \phi_b (t, x), \phi_c (t, x)$ solve

$$
\begin{align*}
- \Delta \phi f &= a_{\pm} (t, x) \quad \partial_\alpha \phi f \big|_{\partial \Omega} = 0, \\
- \Delta \phi_b &= b_j (t, x) \quad \phi_b \big|_{\partial \Omega} = 0, \quad \Delta \phi f = c (t, x) \quad \phi_c \big|_{\partial \Omega} = 0,
\end{align*}
$$

(7.10)

and $\beta_a = 10$, $\beta_b = 1$, and $\beta_c = 5$ such that for all $i = 1, 2, 3$,

$$
\begin{align*}
\int_{\mathbb{R}^3} (| v |^2 - \beta_a) \left( \frac{| v |^2 - 3}{2 \sqrt{2}} \right) v_i \mu (v) d v &= 0, \\
\int_{\mathbb{R}^3} (v_i^2 - \beta_b) \mu (v) d v &= 0, \\
\int_{\mathbb{R}^3} (v_i^2 - \beta_c) v_i \mu (v) d v &= 0.
\end{align*}
$$

(7.11)

**Step 1.** Estimate of $\int_{\mathbb{R}^3} \phi f$. From (7.9) and (7.11), we have $\psi_{b_{1,1}}, \psi_{b_{2,2}}$, and $\psi_c$. For $\psi = \psi_a$, because from definition $\phi = \phi f - \phi f - \phi f - \phi f$ we have $\psi_{b_{1,1}}, \psi_{b_{2,2}}$.

where

$$
\begin{align*}
\mu f \phi f \big|_{\psi = \psi_a} &= \int_{\mathbb{R}^3} (| v |^2 - \beta_a) (v_i) \mu d v \\
& = \int_s^t \int_{\Omega_2} (\nabla \phi f \cdot \nabla \phi f) \cdot \nabla \phi f = C_1 \int_s^t \| \nabla \phi f \|_2^2, \\
\end{align*}
$$

(7.12)

$$
\begin{align*}
\mu f \phi f \big|_{\psi = \psi_a} &= \int_{\mathbb{R}^3} - (| v |^2 - \beta_a) (v_i) \mu d v \\
& = 5.
\end{align*}
$$

(7.13)
Now we look at (7.8)_{c}. For \( \psi = \psi_c \), from oddness in velocity integration and (7.11), (7.8)_{c} becomes
\[
\int_s^t \langle Pf, v \cdot \nabla_x \psi_c \rangle = -C_2 \int_s^t \|c(\tau)\|^2,
\]  
(7.14)
where \( C_2 = 2 \int_{R^3} (|v|^2 - \beta_c) v_i^2 \left( \frac{|v|^2}{2c^2} \right) \mu(v) dv = 20\pi^{3/2} \).

For \( \psi = \psi_\alpha \), from oddness in velocity integration and (7.11), (7.8)_{c} becomes
\[
\int_s^t \langle Pf, v \cdot \nabla_x \psi_\alpha \rangle = -C_1 \int_s^t \|a_+(\tau)\|^2 + \|a_-(\tau)\|^2,
\]  
(7.15)
where \( C_1 = 5 \) as in (7.13).

For fixed \( i, j \), we choose test function \( \psi = \psi_{i,j}^{1,1} \) in (7.8) where \( \beta_3 \) and \( \varphi_3 \) are defined in (7.11) and (7.10). From oddness in velocity integration and definition of \( \beta_3 \), (7.8)_{c} in (7.8) yields
\[
(7.8)_{c}|_{\psi_{i,j}^{1,1}} := \int_s^t \langle Pf, v \cdot \nabla \psi_{i,j}^{1,1} \rangle = -C_3 \int_s^t \int_{\Omega} b_i(\partial_{ij} \Delta^{-1} b_j),
\]  
(7.16)
where \( C_3 := 2 \int_{R^3} (|v|^2 - \beta_3) \frac{x_j^2}{\sqrt{2}} \mu dv = 4\sqrt{\pi} \).

Next, we try test function \( \psi_{i,j}^{1,2} \) with \( i \neq j \) to obtain
\[
(7.8)_{c}|_{\psi_{i,j}^{1,2}} := \int_s^t \langle Pf, \psi_{i,j}^{1,2} \rangle = 2 \int_s^t \int_{\Omega \times R^3} (b \cdot \frac{v}{\sqrt{2}}) \sqrt{\mu} \cdot \nabla \psi_{i,j}^{1,2} = -C_4 \int_s^t \int_{\Omega} (b_j(\partial_{ij} \Delta^{-1} b_j) + b_i(\partial_{ij} \Delta^{-1} b_i)),
\]  
(7.17)
by oddness in velocity integral where \( C_4 := 14\sqrt{\pi} \). Note that the RHS of (7.10) cancel out with the first term in the RHS of (7.17), therefore combining them we get
\[
\left( \sum_{i,j} - \frac{C_4}{C_3} \times (7.8)_{c}|_{\psi_{i,j}^{1,1}} \right) + \left( \sum_{i,j} (7.8)_{c}|_{\psi_{i,j}^{1,2}} \right) = -C_4 \sum_{i,j} \int_s^t \int_{\Omega} b_i(\partial_{ij} \Delta^{-1} b_i) = -C_4 \int_s^t \|b(\tau)\|^2.
\]
Estimate of (7.8)_{p}: From (7.9),
\[
(7.8)_{p} = \int_s^t \langle q \sqrt{\mu} f, \nabla_x \phi_f \cdot \nabla_x \left( \frac{1}{\sqrt{\mu}} \psi \right) \rangle, \quad \psi = \psi_{a_\pm, b, c},
\]
\[
\lesssim \int_s^t \|w \phi f\|_{\infty} \int_{\Omega} \nabla \phi \cdot \nabla \psi_{a_\pm, b, c} \lesssim \int_s^t \|w \phi f(\tau)\|_{\infty} \|Pf(\tau)\|^2,  
\]  
(7.18)
by elliptic estimate \( \|\nabla \varphi\|_2 \lesssim \|\varphi\|_{H^2} \lesssim \|Pf\|_2 \).

Step 2. Estimate of \( c \): For boundary integral (7.8)_{b}, we decompose \( f_\gamma = P_\gamma f + 1_{+}(1 - P_\gamma) f \). Then from (7.11) and trace theorem \( \|\nabla \varphi\|_2 \lesssim \|\varphi\|_{H^2} \lesssim \|c\|_2 \),
\[
\int_s^t \int_{\gamma} \psi_c \cdot f(v \cdot n(x)) = \int_s^t \int_{\gamma} \psi_c \cdot 1_{+}(1 - P_\gamma) f d\gamma
\]
\[
\lesssim \varepsilon \int_s^t \|c(\tau)\|_2^2 + C_\varepsilon \int_s^t \|f(1 - P_\gamma) f(\tau)\|_{L^2}^2, \quad \varepsilon \ll 1.
\]  
(7.19)
If we define
\[
Re := \int_s^t \langle \psi, \{L(I - P)f - \Gamma(f, f)\} \rangle + \int_s^t \langle (I - P)f, v \cdot \nabla_x \psi \rangle,
\]  
(7.20)
then from (7.14), (7.18), elliptic estimate and Young’s inequality we have
\[
Re|_{\psi_c} \lesssim \varepsilon \int_s^t \|c\|_2^2 + \int_s^t \|Pf(\tau)\|_2^2 + \int_s^t \|v^{-1/2} \Gamma(f, f)(\tau)\|_2^2,
\]  
(7.21)
We also use even/oddness in velocity integration, (7.11), and Young’s inequality to estimate,
\[
(7.8)_{p}|_{\psi_c} = \int_s^t \langle f, \partial_t \psi_c \rangle = \int_s^t \langle (I - P)f, \partial_t \psi_c \rangle
\]
\[
\lesssim \varepsilon \int_s^t \|\nabla \Delta^{-1} \partial_t c(\tau)\|_2^2 + \int_s^t \|(I - P)f(\tau)\|_2^2.
\]  
(7.22)
Now, we choose a new test function $\psi^c_\varepsilon := \left[ \frac{1}{\sqrt{\lambda_c}} \right] \varphi_c \phi_c(t, x)$ Note that $\partial_t \varphi_c$ solves $-\Delta \partial_t \varphi_c = \partial_c \psi_c(t, x)$ with $\partial_t \varphi_c(t, x)|_{\partial \Omega} = 0$. We taking difference quotient for $\partial_t f$ and it replace first three terms in the LHS of (7.8). With help of Poincaré inequality $\|\partial_t \varphi_c\|_2 \lesssim \|\nabla \partial_t \varphi_c\|_2$, we can also compute (7.8)$_{f_j}|_{\psi=\psi^c_\varepsilon} = 0,$ and

\begin{equation}
(7.8)_f|_{\psi=\psi^c_\varepsilon} = \int_s^t (q \sqrt{\mu f}, \left[ \frac{\nabla \phi_f \cdot \varphi_c}{\nabla \phi_f \cdot \varphi_c} \right] \partial_t \varphi_c) \cdot \varepsilon \int_s^t \frac{\|\nabla \varphi^c_\varepsilon\|^2}{\|\nabla \varphi^c_\varepsilon\|^2 + (\|a_+(\tau)\|^2 + \|a_-(\tau)\|^2)}.
\end{equation}

\begin{equation}
\int_s^t \langle \psi(t), \psi(t) \rangle \lesssim \int_s^t \|\nabla \partial_t \varphi_c\|^2_2 + \int_s^t \|b(\tau)\|^2_2 + \int_s^t \|f(\tau)\|^2_2.
\end{equation}

Since $\psi^c_\varepsilon$ vanishes when it acts with $L f$ and $I (f, f)$, and boundary integral (7.8)$_B$ vanishes by Dirichlet boundary condition of $\varphi_\varepsilon$, from (7.23), (7.24), and (7.25), we obtain

\begin{equation}
\int_s^t \langle \psi(t), \psi(t) \rangle \lesssim C(t) - C(s) + \int_s^t \|\nabla \partial_t \varphi_c\|^2_2 + \int_s^t \|f(\tau)\|^2_2 + \int_s^t \|P \varphi_c\|^2_2 + \int_s^t \|b(\tau)\|^2_2 + \int_s^t \|f(\tau)\|^2_2.
\end{equation}

Step 3. Estimate of $a$: From mass conservation $\int_s^t a_\varepsilon(t, x) dx = 0$, $\varphi_{a_\varepsilon}$ in (7.10) is well-defined. Moreover, we choose $\varphi_{a_\varepsilon}$ so that has mean zero, $\int_\Omega \varphi_{a_\varepsilon}(t, x) dx = 0$. Therefore, Poincaré inequality $\|\varphi_{a_\varepsilon}\|_2 \lesssim \|\nabla \varphi_{a_\varepsilon}\|_2$ holds and these are also true for $\partial_t \varphi_{a_\varepsilon}$ which solves same elliptic equation with Neumann boundary condition.

For boundary integral (7.8)$_B$, we decompose $f_s = P_s f + \chi_{\varepsilon}(1 - P_s) f$. From Neumann boundary condition $\partial_n \varphi_\varepsilon = 0$ and oddness in velocity integral, $\int_s^t \psi \cdot (P_s f + \chi_{\varepsilon}(1 - P_s) f) = 0$ and we obtain similar estimate as (7.19),

\begin{equation}
\int_s^t \langle \psi(t), \psi(t) \rangle \lesssim \int_s^t \|\nabla \partial_t \varphi_c\|^2_2 + \int_s^t \|\nabla \partial_t \varphi_c\|^2_2 + \int_s^t \|f(\tau)\|^2_2 + \int_s^t \|b(\tau)\|^2_2.
\end{equation}

for $\varepsilon \ll 1$ where $C(t) := \|f(t), \psi(t)\| \lesssim \|f(t)\|^2_2$. For $\varepsilon \ll 1$ where $C(t) := \|f(t), \psi(t)\| \lesssim \|f(t)\|^2_2$. For $\varepsilon \ll 1$ where $C(t) := \|f(t), \psi(t)\| \lesssim \|f(t)\|^2_2$. For $\varepsilon \ll 1$ where $C(t) := \|f(t), \psi(t)\| \lesssim \|f(t)\|^2_2$.
and from the null condition on boundary (1.11), we have \( \frac{7.53}{7.53} B \frac{7.53}{7.53} B |_{\psi = \psi^*_0} = 0 \). Moreover,

\[
\int_s^t \langle Pf, v \nabla_x \psi^*_0 \rangle + \int_s^t \langle I - Pf, v \nabla_x \psi^*_0 \rangle \lesssim \varepsilon \int_s^t (\|\nabla^{-1} \partial_t a_+ (\tau)\|_2^2 + \|\nabla^{-1} \partial_t a_- (\tau)\|_2^2) + \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|I - Pf(\tau)\|_2^2,
\]

and from (1.7)

\[
\int_s^t \langle \psi^*_0, \Gamma(f, f) \rangle = 0.
\]

Now taking difference quotient, we obtain from (7.29), (7.30), and (7.31), for almost t,

\[
\int_s^t \langle \partial_t f, \psi^*_0 \rangle = \int_s^t \int_\Omega \nabla_x \varphi_a \cdot \partial_t a_+ + \partial_t \varphi_a \cdot \partial_t a_- \, dx
\]

\[
= \int_s^t (\|\nabla^{-1} \partial_t a_+ (\tau)\|_2^2 + \|\nabla^{-1} \partial_t a_- (\tau)\|_2^2) \lesssim \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|I - Pf(\tau)\|_2^2 + \int_s^t \|\nu^{-1/2} \Gamma(f, f)\|_2^2.
\]

Finally we change c into a in (7.21) and combine with (7.53), (7.53), (7.53), (7.53), (7.53), (7.53), and (7.53) with \( \varepsilon \ll 1 \) to obtain

\[
\int_s^t \|a(\tau)\|_2^2 + \int_s^t \|\nabla \phi f(\tau)\|_2^2
\]

\[
\lesssim G_a(t) - G_a(s) + \int_s^t \|I - Pf(\tau)\|_2^2 + \int_s^t \|1 - P_\nu f(\tau)\|_2^2
\]

\[
+ \int_s^t \|\nu^{-1/2} \Gamma(f, f)(\tau)\|_2^2 + \int_s^t \|w_\delta f(\tau)\|_\infty \|Pf f(\tau)\|_2^2 + \int_s^t \|\delta f(\tau)\|_2^2,
\]

for \( \varepsilon \ll 1 \) where \( G_a(t) := \int_{t \times \mathbb{R}^3} f(t)(\tau) \lesssim \|f(t)\|_2^2 \).

**Step 4. Estimate of b**: For fixed \( i, j \), we choose test function \( \psi = \psi^{i,j} b \) in (7.49) where \( \beta_0 \) and \( \varphi_0 \) are defined in (7.11) and (7.10). For boundary integration, contribution of \( P_\nu f \) vanishes by oddness.

\[
\int_s^t \langle Pf, v \nabla_x \psi^{i,j} b \rangle + \int_s^t \langle I - Pf, v \nabla_x \psi^{i,j} b \rangle \lesssim \varepsilon \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|1 - P_\nu f(\tau)\|_2^2,
\]

and similar as (7.22) and (7.21), we use oddness and definition of \( \beta_0 \) to vanish contribution of \( a \) and \( b \). We obtain

\[
\int_s^t \langle Pf, v \nabla_x \psi^{i,j} b \rangle \lesssim \varepsilon \int_s^t \|\nabla^{-1} \partial_t b_j (\tau)\|_2^2 + \int_s^t \|\partial_t b_i (\tau)\|_2^2 + \int_s^t \|1 - P_\nu f(\tau)\|_2^2,
\]

\[
\int_s^t \langle Pf, v \nabla_x \psi^{i,j} b \rangle \lesssim \varepsilon \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|\nabla^{-1} \partial_t b_j (\tau)\|_2^2 + \int_s^t \|\partial_t b_i (\tau)\|_2^2 + \int_s^t \|1 - P_\nu f(\tau)\|_2^2.
\]

Next, we try test function \( \psi^{i,j} b \) with \( i \neq j \). We also have the following three estimates using oddness of velocity integral,

\[
\int_s^t \langle Pf, v \nabla_x \psi^{i,j} b \rangle \lesssim \varepsilon \int_s^t \|\nabla^{-1} \partial_t b_j (\tau)\|_2^2 + \int_s^t \|\partial_t b_i (\tau)\|_2^2 + \int_s^t \|1 - P_\nu f(\tau)\|_2^2,
\]

\[
\int_s^t \langle Pf, v \nabla_x \psi^{i,j} b \rangle \lesssim \varepsilon \int_s^t \|\nabla^{-1} \partial_t b_j (\tau)\|_2^2 + \int_s^t \|\partial_t b_i (\tau)\|_2^2 + \int_s^t \|1 - P_\nu f(\tau)\|_2^2.
\]

To obtain estimate for \( \|\nabla^{-1} \partial_t b_j\|_2 \), we use a test function \( \frac{\sqrt{\frac{\partial_t \varphi_0}}{\sqrt{\frac{\partial_t \varphi_0}}}}{\sqrt{\frac{\partial_t \varphi_0}}{\sqrt{\frac{\partial_t \varphi_0}}}} \frac{\sqrt{\frac{\partial_t \varphi_0}}{\sqrt{\frac{\partial_t \varphi_0}}}}{\sqrt{\frac{\partial_t \varphi_0}}{\sqrt{\frac{\partial_t \varphi_0}}}} \).

Note that \( \partial_t \varphi_0 \) solves\(- \Delta \partial_t \varphi_0 = \partial_t b_j (t, x) \) with \( \partial_t \varphi_0 (t, x)|_{\partial \Omega} = 0 \). We taking difference quotient for \( \partial_t f \) in (1.10) and with help of Poincaré inequality, we get

\[
\frac{7.53}{7.53} \psi \bigg|_{\psi = \psi^{i,j} b} = 0,
\]

\[
\frac{7.53}{7.53} f \bigg|_{\psi = \psi^{i,j} b} = \int_s^t \int_{t \times \mathbb{R}^3} \sqrt{\partial_t f} \cdot \left[ \frac{\partial_t \varphi_0 t \cdot \partial_t \varphi_0}{\partial_t \varphi_0 t \cdot \partial_t \varphi_0} \right]
\]

\[
\lesssim \int_s^t \|w_\delta f\|_\infty \|\nabla^{-1} \partial_t b_j (\tau)\|_2^2 + \|\partial_t b_i (\tau)\|_2^2 + \|a_+ (\tau)\|_2^2 + a_-(\tau)\|_2^2.
\]
Moreover,
\[
\int_s^t \langle P_f, v \nabla \psi_{b_j} \rangle + \int_s^t \langle (I - P)f, v \nabla \psi_{b_j} \rangle \lesssim \varepsilon \int_s^t \| \nabla \Delta^{-1} \partial b_j(\tau) \|_2^2 + \int_s^t \| (a_+ \tau) \|_2^2 + \| a_- \tau \|_2^2 + \| c(\tau) \|_2^2 + \int_s^t \| (I - P)f \|_2^2. \tag{7.39}
\]

Since \( \psi_{b_j} \) vanishes when it acts with \( Lf \) and \( \Gamma(f, f) \), and boundary integral \( \text{(7.38)} \) vanishes by Dirichlet boundary condition of \( \partial \psi_k \), from \( \text{(7.39)} \), \( \text{(7.39)} \), and \( \text{(7.40)} \), we obtain
\[
\int_s^t \int \partial \psi_{b_j}(\tau, x) \partial b_j(\tau, x) dx = \int_s^t \| \nabla \Delta^{-1} \partial b_j(\tau) \|_2^2 \\
\lesssim \varepsilon \int_s^t \| \nabla \Delta^{-1} \partial b_j(\tau) \|_2^2 + \int_s^t \| (a_+ \tau) \|_2^2 + \| a_- \tau \|_2^2 + \| c(\tau) \|_2^2 + \int_s^t \| (I - P)f \|_2^2. \tag{7.40}
\]

Now we combine \( \text{(7.39)} \), \( \text{(7.39)} \), \( \text{(7.39)} \), \( \text{(7.39)} \), \( \text{(7.39)} \), \( \text{(7.39)} \), \( \text{(7.39)} \), and \( \text{(7.39)} \) for all \( i, j \) with proper constant weights. In particular, we note that RHS of \( \text{(7.10)} \) is cancelled by the first term on the RHS of \( \text{(7.14)} \). Therefore,
\[
\int_s^t \| b(\tau) \|_2^2 = - \sum_{i, j} \int_s^t \int \partial_j \Delta^{-1} b_i \\
\lesssim G_b(t) - G_b(s) + \int_s^t \| (I - P)f(\tau) \|_2^2 + \int_s^t \| (1 - P)\tilde{f}(\tau) \|_2^2 \\
+ \int_s^t \| \nabla \Delta^{-1} \Gamma(f, f)(\tau) \|_2^2 + \int_s^t \| w\phi(\tau) \|_2^2 + \| \tilde{P}f(\tau) \|_2^2 \\
+ \int_s^t \| c(\tau) \|_2^2 + \varepsilon \int_s^t \| (a_+ \tau) \|_2^2 + \| a_- \tau \|_2^2, \quad G_b(t) \lesssim \| f(t) \|_2^2, \quad \varepsilon \ll 1. \tag{7.41}
\]

Finally we combine \( \text{(7.26)} \), \( \text{(7.33)} \), and \( \text{(7.41)} \) with \( \varepsilon \ll 1 \) to conclude \( \text{(7.4)} \).

\[\square\]

8. Global Existence and Exponential decay

The following time-dependent interpolation estimate is crucial in the proof of Theorem 1.

**Lemma 8.** Assume \( \Omega \subset \mathbb{R}^3 \) with a smooth boundary \( \partial \Omega \). For \( 0 < D_1 < 1, 0 < D_2 < 1, \) and \( \Lambda_0 > 0 \),
\[
\| \nabla^2 \phi(t) \|_{L^\infty(\Omega)} \lesssim_{D_1, D_2, \Omega} e^{D_1 \Lambda_0 t} \| \phi(t) \|_{C^{1, 1, 1 - D_1}(\Omega)} + e^{-D_2 \Lambda_0 t} \| \phi(t) \|_{C^{2, 2}(\Omega)} \text{ for all } t \geq 0 \tag{8.1}
\]

**Proof.** Let \( \Omega_1 \) be an open bounded subset of \( \mathbb{R}^3 \) containing the closure \( \overline{\Omega} \). Suppose \( \phi(t) \in C^{2, D_2}(\Omega_1) \). From a standard extension theorem (e.g. see Lemma 6.37 of [11] in page 136) there exists a function \( \tilde{\phi}(t) \in C^{2, D_2}(\Omega_1) \) and \( \tilde{\phi}(t) \equiv 0 \) in \( \mathbb{R}^3 \) such that \( \tilde{\phi}(t) \equiv \phi(t) \) in \( \Omega \) and
\[
\| \tilde{\phi}(t) \|_{C^{1, 1, 1 - D_1}(\Omega_1)} \leq C_{D_1, D_2, \Omega_1} \| \phi(t) \|_{C^{1, 1, 1 - D_1}(\Omega)} \text{ and } \| \tilde{\phi}(t) \|_{C^{2, 2}(\Omega_1)} \leq C_{D_1, D_2, \Omega_1} \| \phi(t) \|_{C^{2, 2}(\Omega_1)}, \tag{8.2}
\]
where \( C_{D_1, D_2, \Omega_1} \) does not depend on \( \phi(t) \) and \( t \).

Choose arbitrary points \( x, y \) in \( \mathbb{R}^3 \). For \( 0 \leq s \leq 1, (1 - s)x + sy \in \overline{\Omega} \). Note that
\[
\nabla \phi(t, x) = \nabla \phi(t, (1 - s)x + sy) - \nabla \phi(t, x) - (1 - s)x + sy \nabla \phi(t, x)|D_2 \times (1 - s)x + sy = x|D_2 \\
+ (\frac{y - x}{|y - x|} \cdot \nabla \phi(t, x) |y - x| \\
= O(|x - y|^{1 + D_2}) + |D_2 \nabla \phi(t, x) |C^{0, D_2} + (\frac{y - x}{|y - x|} \cdot \nabla \phi(t, x) |y - x|.
\]

Taking an integration on \( s \in [0, 1] \), we obtain that
\[
\int_0^1 \left| \frac{y - x}{|y - x|} \cdot \nabla \phi(t, x) \right|^2 ds \leq \frac{1}{|y - x|} \int_0^1 |(y - x) \cdot \nabla \phi(t, (1 - s)x + sy) ds + \frac{1}{1 + D_2} |x - y|^{D_2} [\nabla^2 \phi(t)]_{C^{0, D_2}}. \tag{8.3}
\]

On the other hand, from an expansion along \( s \),
\[
\nabla \phi(t, y) - \nabla \phi(t, x) = \int_0^1 [(y - x) \cdot \nabla \phi(t, (1 - s)x + sy) ds.
\]
We plug this identity into (8.3) and deduce that for $0 < D_1 < 1$
\[
\left| \left( \frac{x-y}{|x-y|} \cdot \nabla \right) \nabla \tilde{\phi}(t, x) \right| \\
\leq \frac{1}{1 + D_2} \left| \nabla \tilde{\phi}(t, x) - \nabla \tilde{\phi}(t, y) \right| |x-y| + \frac{1}{1 + D_2} |x-y| D_2 |\nabla^2 \tilde{\phi}(t)|_{C^{0,1-D_2}}.
\]
(8.4)

Now let us choose
\[|x-y| = e^{-\lambda_0 t}, \quad \tilde{\omega} := \frac{x-y}{|x-y|} \in S^2.\]

From (8.3)
\[| (\tilde{\omega} \cdot \nabla) \nabla \tilde{\phi}(t, x)| \leq e^{D_1 \lambda_0 t} |\nabla \tilde{\phi}(t)|_{C^{0,1-D_1}} + \frac{1}{1 + D_2} e^{-D_2 \lambda_0 t} |\nabla^2 \tilde{\phi}(t)|_{C^{0,2}}.
\]

Taking supremum in $x$ and $\tilde{\omega}$ to the above inequality and using $|\nabla^2 \tilde{\phi}(t)|_{C^{0,2}} \leq M$, we get
\[\|\nabla^2 \tilde{\phi}(t)\|_{L^\infty_t(\Omega)} \leq e^{D_1 \lambda_0 t} |\nabla \tilde{\phi}(t)|_{C^{0,1-D_1}} + e^{-D_2 \lambda_0 t} |\nabla^2 \tilde{\phi}(t)|_{C^{0,2}}.
\]

Finally from (8.2) and the above estimate we conclude (8.1). □

Now we are ready to prove the global-in-time result.

**Proof of Theorem 7** Step 1. For $0 < M \ll 1$ and $0 < \delta_* \ll 1$, we first assume that an initial datum satisfies
\[
\|w_0 f_0\|_\infty + \|w_0 f_0\|_p + \|w_0 \alpha^{\rho}_{f_0,\epsilon} \nabla x \cdot f_0\|_p \leq \delta_* M, \\
\|w_0 \alpha^{\rho}_{\epsilon} \nabla x \cdot f_0\|_{L^1(\Omega \times \mathbb{R}^3)} + \|\nabla^2 \phi_0(0)\|_{L^\infty_t(\Omega)} < \infty.
\]
(8.5)

We will choose $M, \delta_*$ later. For the sake of convenience we choose a large constant $L > \max (M, \|\nabla^2 \phi(0)\|_\infty)$. In order to use the continuation argument along the lines of the local existence theorem, Theorem 7 we set
\[T = \sup \left\{ t \geq 0 : \|e^{\lambda \alpha \phi}(t)\|_\infty + \|\phi(t)\|_p \leq M, \right. \]
\[\left. \quad \text{and} \quad \|w_0 \alpha^{\rho}_{\epsilon} \nabla x \cdot f(t)\|_p + \int_0^t |w_0 \alpha^{\rho}_{\epsilon} \nabla x \cdot f(t)|_p \, dt < \infty, \right. \]
\[\left. \quad \text{and} \quad \|\nabla f(t)\|_{L^2(\Omega)} + |\nabla^2 \phi(t)|_{L^\infty_t(\Omega)} \leq L. \right\}
\]
(8.6)

Here for fixed $\delta \ll 1$, we choose $\lambda_0$ such that
\[20\sqrt{CC_2 M} \leq \lambda_0 \leq \min \left( \frac{\lambda_2}{2}, \frac{\nu_0}{4} \right), \quad \text{for} \quad M \ll 1,
\]
(8.7)

where $\lambda_2$ is obtained in Proposition 8. Note that from (4.10) the condition (4.4) holds for $M \ll 1$.

Step 2. We claim that
\[\sup_{0 \leq t \leq T} e^{\lambda_0 \phi(t)} \|\nabla^2 \phi(t)\|_\infty \leq C_2 M, \quad \text{with} \quad C_2 := C_2 + (C_1 C_p)^{1/p} \delta_*.
\]
(8.8)

Here $C_\Omega$ appears in (1.10), and $C_1$ in (4.11), and $C_p$ in Proposition 4.

From (4.10) and (8.3), for $0 \leq t \leq T$, for all $D_1 > 0$
\[\|\phi(t)\|_{C^{1,1-D_1}(\Omega)} \leq C_\Omega \|w_0 f(t)\|_\infty \leq C_\Omega M e^{-\lambda_0 t}.
\]
(8.9)

On the other hand, from Proposition 3 replacing $f^t$ and $f^{t+1}$ by $f$ in (4.63), (4.75), and by Gronwall’s inequality and (8.5), we derive that for $0 \leq t \leq T$
\[\|f(t)\|_p + \|w_0 \alpha^{\rho}_{f,\epsilon} \nabla x \cdot f(t)\|_p + \int_0^t |w_0 \alpha^{\rho}_{f,\epsilon} \nabla x \cdot f(s)|_p \, ds < \infty.
\]
(8.10)

Now we use Lemma 6 from (4.11), for $p > 3$ and $0 \leq t \leq T$,
\[\|\phi(t)\|_{C^{2,1}(\Omega)} \leq (C_1 C_p)^{1/p} \delta_* e^{\lambda_0 \phi(t)} \leq (C_1 C_p)^{1/p} \delta_* M.
\]
(8.11)
Finally we use an interpolation between $C^{1,1-D_1}(\bar{\Omega})$ and $C^{2,1-\frac{2}{p}}(\bar{\Omega})$ and derive an estimate of $C^2(\bar{\Omega})$: Applying Lemma 8 and (8.1) with $D_2 = 1 - \frac{2}{p}$, from (8.10) and (8.9), we derive that for all $0 < D_1 < 1$, $3 < p < 6$, $\Lambda_0 > 0$, and $0 \leq t \leq T$,

$$
\|\nabla^2 \phi_f(t)\|_\infty \leq e^{-[\lambda_\infty - D_1\Lambda_0]} \Lambda_1 M + e^{-[(1 - \frac{2}{p})\Lambda_0 - \frac{1}{p}C_p(1 + L)]t} (C_1 C_p)^{1/p} \delta_s M. 
$$

Then we choose

$$
\Lambda_0 = \frac{\lambda_\infty}{2} + \frac{C_p}{p}(1 + L) \quad \text{and then} \quad D_1 = \frac{\lambda_\infty}{2\Lambda_0}.
$$

In conclusion we have, for all $0 \leq t \leq T$,

$$
\|\nabla^2 \phi_f(t)\|_\infty \leq e^{-\frac{\lambda_\infty}{2\Lambda_0}} [C\Lambda_1 + (C_1 C_p)^{1/p}\delta_s] M.
$$

As long as $M \ll L$ then $\|\nabla^2 \phi_f(t)\|_\infty \leq L$ for all $0 \leq t \leq T$ and hence the claim (8.8) holds.

**Step 3.** We claim that there exists $T_\infty \gg 1$ such that, for $N \in \mathbb{N}$, $t \in [NT_\infty, (N + 1)T_\infty]$, and $(N + 1)T_\infty \leq T$,

\[
\begin{align*}
&\|w_\phi f(t)\|_\infty \\
&\leq e^{-\frac{\lambda_\infty}{2}(t - NT_\infty)} \|w_\phi f(NT_\infty)\|_\infty + o(1) \sup_{NT_\infty \leq s \leq t} e^{-\frac{\lambda_\infty}{2}(t - s)} \|w_\phi f(s)\|_\infty \\
&+ C_{T_\infty} \int_{NT_\infty}^{t} e^{-\frac{\lambda_\infty}{2}(t - s)} \|f(s)\|_{L^2_{x,v}} ds \\
&+ C_{T_\infty} \int_{NT_\infty}^{t} e^{-\frac{\lambda_\infty}{2}(t - s)} \|\nabla \phi_f(s)\|_\infty ds.
\end{align*}
\]

For the sake of simplicity we present a proof of (8.14) for $N = 0$. The proof for $N > 0$ can be easily obtained by considering $f(NT_\infty)$ as an initial datum.

As (8.9) we define $h(t, x, v) := w_\phi f(t, x, v)$. Then $h$ solves (8.10) and (8.12) with exchanging all $(h^\ell, h^{\ell + 1}, \phi^\ell)$ to $(h, h, \phi_f)$. We define

$$
\nu_{\phi_f, w_\phi}(t, x, v) := \left[ \begin{array}{c} \nu_{\phi_f, w_\phi,+} \\ 0 \\ \nu_{\phi_f, w_\phi,-} \end{array} \right] = \left[ \begin{array}{c} \nu(v) + \frac{\kappa}{2} \cdot \nabla \phi_f + \frac{\nabla \phi_f \cdot \nabla v}{w_\phi} \\ 0 \\ \nu(v) - \frac{\kappa}{2} \cdot \nabla \phi_f + \frac{\nabla \phi_f \cdot \nabla v}{w_\phi} \end{array} \right] .
$$

From (8.6) and (4.10), for $0 \leq t \leq T$

$$
\nu_{\phi_f, w_\phi, \pm} \geq \left\{ \nu_0 - \frac{\|\nabla \phi_f\|_\infty}{2} - 2\theta \|\nabla \phi_f\|_\infty \right\} \{v\}
\geq \left\{ \nu_0 - \frac{1}{2} - 2\theta M \right\} \{v\}
\geq \nu_0 \frac{1}{2} \{v\} .
$$

Then $h$ solves (3.18) along the trajectory with deleting all superscriptions of $\ell$ and $\ell + 1$ and exchanging $\nu^\ell$ to $\nu_{\phi_f, w_\phi}$ and with new $g$

$$
g := -q_1 v \cdot \nabla \phi_f \sqrt{\mu} + \Gamma \left( \frac{h}{w_\phi}, \frac{h}{w_\phi} \right) .
$$

We define a stochastic cycles for $\ell = +$ or $-$ as in (13.11),

$$
(t_{\ell,0}(t, x, v_1, \cdots, v_{\ell-1}), x_{\ell,0}(t, x, v_1, \cdots, v_{\ell-1})) ,
$$

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by deleting all superscriptions in (3.3) and (3.6). Then by deleting all superscriptions of \(\ell\) and \(\ell + 1\) from (3.37), we obtain the bound for \(h\):

\[
|h_\ell(t, x, v)| \
\leq O(k)\left|e^{-\frac{2\pi}{\ell}t}h_0\right|_\infty + O(k) \sup_{0 \leq s \leq t} \|e^{-\frac{2\pi}{\ell}(t-s)}h(s)\|_\infty^2 + O(k) \int_0^t \|e^{-\frac{2\pi}{\ell}(t-s)}\nabla \phi_{\frac{h_0}{\ell}}(s)\|_\infty ds \]

\[+ \left\{ O(k) + \left\{ \frac{1}{2} \right\} \right\} \sup_{0 \leq s \leq t} \|e^{-\frac{2\pi}{\ell}(t-s)}h(s)\|_\infty
\]

\[+ \int_{\max\{t_{1,0}\}}^{t_{1,\ell}} \int_\mathbb{R}^3 k_\ell(V_\ell(s; t, x, v), u)|h(s, X_\ell(s; t, x, v), u)|du ds \]

\[+ O(k) \sup_{t} \int_{\max\{t_{1,1,0}\}}^{t_{1,\ell}} \int_\mathbb{R}^3 k_\ell(V_\ell(s; t, x, v), u)|h(s, X_\ell(s; t, x, 1), u)||n(x_1) \cdot v|\sqrt{\mu(v)} du ds.
\]

(8.18)

For any large \(m \gg 1\) we define

\[k_{\ell, m}(v, u) = 1_{|v-u| \geq \frac{1}{m}}|u| \leq m\]

such that \(\sup_{t} \int_\mathbb{R}^3 |k_{\ell, m}(v, u) - k_\ell(v, u)| du \leq \frac{1}{m}\), and \(\|k_{\ell, m}(v, u)\|_{\infty} \leq m\).

Furthermore we split the time interval as, for each \(\ell, l\)

\[
\left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i}\right\} = \left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i} - \delta \right\} \cup \left\{ t_{l,i} - \delta \leq s \leq t_{l,i}\right\}
\]

(8.20)

where we choose a small constant \(0 < \delta \ll k\) later in (8.20).

For (8.18), we have

\[k_{\ell, m}(v; u, l) = 1_{|v-u| \geq \frac{1}{m}}|u| \leq m, k_\ell(v, u),
\]

such that \(\sup_{t} \int_\mathbb{R}^3 |k_{\ell, m}(v, u) - k_\ell(v, u)| du \leq \frac{1}{m}\), and \(\|k_{\ell, m}(v, u)\|_{\infty} \leq m\).

Furthermore we split the time interval as, for each \(\ell, l\)

\[
\left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i}\right\} = \left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i} - \delta \right\} \cup \left\{ t_{l,i} - \delta \leq s \leq t_{l,i}\right\}
\]

(8.20)

where we choose a small constant \(0 < \delta \ll k\) later in (8.20).

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\]

such that \(\sup_{t} \int_\mathbb{R}^3 |k_{\ell, m}(v, u) - k_\ell(v, u)| du \leq \frac{1}{m}\), and \(\|k_{\ell, m}(v, u)\|_{\infty} \leq m\).

Furthermore we split the time interval as, for each \(\ell, l\)

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\left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i}\right\} = \left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i} - \delta \right\} \cup \left\{ t_{l,i} - \delta \leq s \leq t_{l,i}\right\}
\]

(8.20)

where we choose a small constant \(0 < \delta \ll k\) later in (8.20).

For (8.18), we have

\[k_{\ell, m}(v; u, l) = 1_{|v-u| \geq \frac{1}{m}}|u| \leq m, k_\ell(v, u),
\]

such that \(\sup_{t} \int_\mathbb{R}^3 |k_{\ell, m}(v, u) - k_\ell(v, u)| du \leq \frac{1}{m}\), and \(\|k_{\ell, m}(v, u)\|_{\infty} \leq m\).

Furthermore we split the time interval as, for each \(\ell, l\)

\[
\left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i}\right\} = \left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i} - \delta \right\} \cup \left\{ t_{l,i} - \delta \leq s \leq t_{l,i}\right\}
\]

(8.20)

where we choose a small constant \(0 < \delta \ll k\) later in (8.20).

For (8.18), we have

\[k_{\ell, m}(v; u, l) = 1_{|v-u| \geq \frac{1}{m}}|u| \leq m, k_\ell(v, u),
\]

such that \(\sup_{t} \int_\mathbb{R}^3 |k_{\ell, m}(v, u) - k_\ell(v, u)| du \leq \frac{1}{m}\), and \(\|k_{\ell, m}(v, u)\|_{\infty} \leq m\).

Furthermore we split the time interval as, for each \(\ell, l\)

\[
\left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i}\right\} = \left\{ \max\{t_{l+1,0}\} \leq s \leq t_{l,i} - \delta \right\} \cup \left\{ t_{l,i} - \delta \leq s \leq t_{l,i}\right\}
\]

(8.20)

where we choose a small constant \(0 < \delta \ll k\) later in (8.20).
Case 3: For $|v_1| \leq \frac{m}{2}$ and $|u| \leq m$, we split the time integration as $\int_{\tau_{j-1},\tau_j,0}^{\tau_{j-1},\tau_j} = \int_{\tau_{j-1},\tau_j,0}^{\tau_{j-1},\delta} + \int_{\tau_{j-1},\delta}^{\tau_{j-1},\tau_j}$ and use (8.20) to conclude that

$$\int_{\tau_{j-1},\delta}^{\tau_{j-1},\tau_j} e^{-\frac{\mu(t-s)}{2}} \eta \leq O(k) \sup_{t} \int_{\tau_{j-1},\delta}^{\tau_{j-1},\tau_j} e^{-\frac{\mu(t-s)}{2}} \eta \, ds + O(k) \int_{\tau_{j-1},\delta}^{\tau_{j-1},\tau_j} e^{-\frac{\mu(t-s)}{2}} \eta \, ds$$

Combining (8.18), (8.21), and (8.22) we get

$$|h(s, X(s; t, x, v), u)| = |h_+(s, X(s; t, x, v), u) + |h_-(s, X(s; t, x, v), u)|$$

Now for $|h(s, X(s; t, x, v), u)|$ we use similar bounds for $|h_+(s, X(s; t, x, v), u)|$ and $|h_-(s, X(s; t, x, v), u)|$ separately and add them together to get

$$|h(s, X(s; t, x, v), u)|$$

where

$$t'_{j,\tau_j} = t'_{j,\tau_j}(s, X(s; t, x, v), u, v_1', \ldots, v'_{j-1}),$$

$$t'_{j-1,\tau_j} = t'_{j-1,\tau_j}(s, X(s; t, x, v), u, v_1', \ldots, v'_{j-2}),$$

$$x'_{j,\tau_j} = x'_{j,\tau_j}(s, X(s; t, x, v), u, v_1', \ldots, v'_{j-1}),$$

$$x'_{j-1,\tau_j} = x'_{j-1,\tau_j}(s, X(s; t, x, v), u, v_1', \ldots, v'_{j-2}).$$
We can obtain the exactly same lower bound of \( \det (\partial X(t,v)) \)

\[
\leq O(m(k) (\| e^{-\frac{\lambda}{t_0} h_0 } \|_\infty + \sup_{0 \leq s \leq t} \| e^{-\frac{\lambda}{t_0} (t-s) } h(s) \|_\infty )
\]

\[
+ O(m(k) \int_0^t \| e^{-\frac{\lambda}{t_0} (t-s) } \nabla \phi \frac{h}{\psi}(s) \|_\infty ds
\]

\[
+ \left\{ O(m(k) \delta + O(k) \frac{1}{m} + O(m(1)) \left( \frac{k}{2} \right)^{k/5} \right\} \sup_{0 \leq s \leq t} \| e^{-\frac{\lambda}{t_0} (t-s) } h(s) \|_\infty
\]

\[
+ O(m^2) \int_0^t \int_{|u| \leq m} \int_{|s| \leq m} e^{-\frac{\lambda}{t_0} (t-s) } \int_{|u| \leq m} |h_+(s', X_+(s'; s, X_+(s; t, x, v), u), u')| du'ds'dds'
\]

\[
+ O(m^2) \int_0^t \int_{|u| \leq m} \int_{|s| \leq m} e^{-\frac{\lambda}{t_0} (t-s') } \int_{|u| \leq m} |h_-(s', X_-(s'; s, X_-(s; t, x, v), u), u')| du'ds'dds'
\]

\[
+ O(k) \sup_{l,s,t} \int_0^t \int_{|u| \leq m} \int_{|u| \leq m} |h(s, X_+(s; t, x, v), u)| du'ds'dds'
\]

\[
+ O(k) \sup_{l,s,t} \int_0^t \int_{|u| \leq m} \int_{|u| \leq m} \int_{|u| \leq m} |h_+(s', X_+(s'; t', x', v'), u')| du'ds'dds'
\]

\[
+ O(k) \sup_{l,s,t} \int_0^t \int_{|u| \leq m} \int_{|u| \leq m} \int_{|u| \leq m} |h_-(s', X_-(s'; t', x', v'), u')| du'ds'dds'.
\]

Choose \( T \gg 1 \) and \( k \gg 1 \) in (8.29) and (8.31). Then we choose

\[
m = k^2 \quad \text{and} \quad \delta = \frac{1}{m^3 k},
\]

so that \( O(m(k) \delta + O(k) \frac{1}{m} + O(m(1)) \left( \frac{k}{2} \right)^{k/5} \ll 1 \).

Note that

\[
\frac{\partial X_\pm(s; t, x, v)}{\partial v_i} = -(t_i - s) \text{Id}_{3 \times 3}
\]

\[
\therefore \int_t^\tau \int_t^\tau \left( \frac{\partial X_\pm(\tau'; t, x, v)}{\partial v_j} \right) \left( \nabla \phi \frac{h}{\psi}(\tau', X_\pm(\tau'; t, x, v)) \right) d\tau' d\tau,
\]

Now we use Lemma 4. Note that from (8.8), the condition (4.7) of Lemma 4 is satisfied with \( \Lambda_2 = \frac{\lambda}{\tau} \) and \( \delta_2 = C_2 M \). From Lemma 4 and (8.24) we have for \( \iota = + \) or \( - \),

\[
\left| \frac{\partial X_\iota(s; t_i, x_i, v_i)}{\partial v_j} \right| \leq C_\iota \frac{4C_2 M}{(\lambda_\infty)^2} \left| t_i - \tau' \right|.
\]

From (8.28) and (8.5), the second term of RHS in (8.27) is bounded by

\[
CC_2 M e^{\frac{4CC_2 M}{(\lambda_\infty)^2} (\lambda_\infty)^2} \int_s^t \int_t^\tau (t_i - \tau') e^{-\frac{\lambda}{t_0} \tau'} d\tau' d\tau
\]

\[
\leq 4CC_2 M e^{\frac{4CC_2 M}{(\lambda_\infty)^2} (\lambda_\infty)^2} |t_i - s|.
\]

From our choice of \( \lambda_\infty \) in (8.7), we have

\[
\frac{4CC_2 M}{(\lambda_\infty)^2} e^{\frac{4CC_2 M}{(\lambda_\infty)^2} (\lambda_\infty)^2} \leq \frac{1}{10}.
\]

Therefore from (8.27), for \( 0 \leq s \leq t_1 - \delta \),

\[
\det \left( \frac{\partial X_\iota(s; t_i, x_i, v_i)}{\partial v_j} \right) = \det \left( -(t_i - s) \text{Id}_{3 \times 3} + o(1) \right)
\]

\[
\geq |t_i - s|^3
\]

\[
\geq \delta.
\]

We can obtain the exactly same lower bound of \( \det \left( \frac{\partial X_\iota(s'; s, X_\iota(t, x, v), u)}{\partial v_j} \right) \), \( \det \left( \frac{\partial X_\iota(s'; s, X_\iota(t, x, v), u)}{\partial u} \right) \), and \( \det \left( \frac{\partial X_\iota(s'; s, X_\iota(t, x, v), u)}{\partial u} \right) \) for \( 0 \leq s' \leq s - \delta \) and \( 0 \leq s' \leq t'_i - \delta \).
Now we apply the change of variables
\[ v_t \rightarrow X_\nu(s; t, x_0, v_t), \]
\[ v'_t \rightarrow X_\nu(s'; t', x'_0, v'_t), \]
\[ u \rightarrow X_\nu(s'; s, X_\nu(s; t, x, v), u), \]
\[ u \rightarrow X_\nu(s'; s, X_\nu(s; t, x, v), u), \]
and conclude (8.14) from (8.25) and (8.26).

Applying (8.14) successively, we achieve that
\[
\|w_\nu f(t)\|_\infty \leq e^{\nu_0 T_\infty} + \frac{1}{1 - e^{-\nu_0 T_\infty}} \sup_{0 \leq s \leq t} e^{-\nu_0 (t-s)} \|w_\nu f(s)\|_\infty
\]
\[
+ C_{T_\infty} e^{\nu_0 T_\infty} \int_0^t e^{-\nu_0 (t-s)} \|f(s)\|_2 ds + C_{T_\infty} e^{\nu_0 T_\infty} \int_0^t e^{-\nu_0 (t-s)} \|\nabla \phi_f(s)\|_\infty ds,
\]
(8.31)

where we have used
\[
e^{\nu_0 T_\infty} \{1 + e^{-\nu_0 T_\infty} + \cdots + e^{-\nu_0 N T_\infty}\} = e^{\nu_0 T_\infty}.
\]

Step 4. From Proposition 8.1 and (1.14) we have
\[
\|e^{\lambda_2 f(t)}\|_2^2 + \|e^{\lambda_2 f(t)} \nabla \phi(t)\|_2^2
\]
\[
+ \int_0^t \|e^{\lambda_2 f(\tau)}\|_2^2 + \|e^{\lambda_2 f(\tau)} \nabla \phi_f(\tau)\|_2^2 d\tau + \int_0^t e^{\lambda_2 f(t)} dt + \int_0^t e^{\lambda_2 f(t)} dt + \int_0^t e^{\lambda_2 f(t)} dt
\]
\[
\lesssim \|f_0\|_2^2 + \|\nabla \phi_0\|_2^2.
\]
(8.32)

Hence
\[
(8.31)_{L^2} \lesssim e^{-\min(\lambda_2, \lambda_2)} \|f_0\|_2 + \|\nabla \phi_0\|_2
\]
\[
\lesssim e^{-\min(\lambda_2, \lambda_2)} \|f_0\|_2 + \|\nabla \phi_0\|_2.
\]
(8.33)

Now we consider (8.34). In order to close the estimate in (8.31) we need to improve the decay rate of \(\|\nabla \phi_f(s)\|_\infty\).

We claim that, for \(\theta_{2, r, p} > 0\) (which is specified in (8.38)),
\[
\|\nabla \phi_f(s)\|_\infty \lesssim e^{-(1 + \theta_{2, r, p}) \lambda_\infty s} \{\sup_{t \geq 0} \|e^{\lambda_2 f(t)}\|_2 + \|e^{\lambda_\infty f(t)}\|_\infty\}.
\]
(8.34)

By Morrey’s inequality for \(\Omega \subset \mathbb{R}^3\) and \(r > 3\)
\[
|\nabla \phi_f|_\infty \lesssim |\nabla \phi_f|_{C^{0,1-\epsilon}(\Omega)} \lesssim |\nabla \phi_f|_{W^{1, r}(\Omega)}.
\]
(8.35)

Then applying the standard elliptic estimate to (1.14), we get
\[
|\nabla \phi_f(t)|_{W^{1, 2}(\Omega)} \lesssim \left| \int_{\mathbb{R}^3} (f_+ - f_-) \sqrt{\mu} dv \right|_{L^2(\Omega)} \lesssim e^{-\lambda_2 t} \sup_{t \geq 0} \|e^{\lambda_2 f(t)}\|_2,
\]
\[
|\nabla \phi_f(t)|_{W^{1, p}(\Omega)} \lesssim \left| \int_{\mathbb{R}^3} (f_+ - f_-) \sqrt{\mu} dv \right|_{L^p(\Omega)} \lesssim e^{-\lambda_\infty t} \sup_{t \geq 0} \|e^{\lambda_\infty f(t)}\|_\infty.
\]
(8.36)

Now we use the standard interpolation: For \(p > r > 3\),
\[
|\nabla \phi_f|_{W^{1, r}(\Omega)} \lesssim |\nabla \phi_f(t)|_{W^{1, 2}(\Omega)} |\nabla \phi_f(t)|_{W^{1, p}(\Omega)}^{1-\theta_{2, r, p}}
\]
for
\[
\theta_{2, r, p} := \frac{1 - \frac{2}{r}}{2 - \frac{2}{r}} = \frac{2}{3} \frac{p - 3}{p - 2},
\]
(8.38)

Then we derive
\[
\sup_{t \geq 0} \|e^{\theta_{2, r, p} \lambda_2 + (1 - \theta_{2, r, p}) \lambda_\infty t} \nabla \phi_f(t)\|_\infty
\]
\[
\lesssim \left( \sup_{t \geq 0} \|e^{\lambda_2 f(t)}\|_2 \right)^{\theta_{2, r, p}} \left( \sup_{t \geq 0} \|e^{\lambda_\infty f(t)}\|_\infty \right)^{1-\theta_{2, r, p}}
\]
\[
\lesssim \sup_{t \geq 0} \|e^{\lambda_2 f(t)}\|_2 + (1) \sup_{t \geq 0} \|e^{\lambda_\infty f(t)}\|_\infty.
\]
(8.39)
From our choice (8.47) and \(0 < p - 3 \ll 1\),

\[
\theta_{2,r,p} \lambda_2 + (1 - \theta_{2,r,p}) \lambda_\infty \geq (1 + \theta_{2,r,p}) \lambda_\infty. \tag{8.40}
\]

From (8.39)

\[
\frac{\partial f}{\partial t} \lesssim \int_0^t e^{-\gamma(t-s)} e^{-(1+\theta_{2,r,p}) \lambda_\infty s} \|e^{\lambda_{x,s}} f(s)\|_2 ds
\]

\[+ o(1) \int_0^t e^{-\gamma(t-s)} e^{-(1+\theta_{2,r,p}) \lambda_\infty s} \|e^{\lambda_{x,s}} w_\theta f(s)\|_\infty ds \]

\[\lesssim e^{-\min(\frac{\gamma}{\lambda'}, \lambda_\infty)} t \|f_0\|_2 + \|\nabla f_0\|_2 \quad \text{from (8.32)}
\]

\[+ o(1) e^{-\min(\frac{\gamma}{\lambda'}, \lambda_\infty)} t \sup_{0 \leq s \leq t} \|e^{\lambda_{x,s}} w_\theta f(s)\|_\infty.
\]

Multiplying \(e^{\lambda_{x,t}}\) and taking \(\sup_{t \geq 0}\) to (8.31) with \(\lambda_\infty \leq \min\left(\frac{\gamma}{\lambda'}, \frac{\lambda_\infty}{2}\right)\), and from (8.33) and (8.41), we obtain that

\[
\sup_{t \geq 0} e^{\lambda_{x,t}} \|w_\theta f(t)\|_\infty \lesssim \|w_\theta f(0)\|_\infty + \|f_0\|_2 + \|\nabla f_0\|_2 + o(1) \sup_{0 \leq s \leq t} e^{\lambda_{x,s}} \|w_\theta f(s)\|_\infty. \tag{8.42}
\]

By absorbing the last (small) term, we conclude that

\[
\sup_{0 \leq s \leq T} e^{\lambda_{x,t}} \|w_\theta f(t)\|_\infty \leq C\delta, M. \tag{8.43}
\]

If we choose \(\delta \ll 1/C\) then by the local existence theorem (Theorem 7) and continuity of \(\|w_\theta f(t)\|_\infty\), \(\|w_\theta f(t)\|_p + \|w_\theta \alpha_\theta^p \nabla_x \cdot \nabla f(s)\|_p + \int_0^s \|w_\theta \alpha_\theta^p \nabla_x \cdot \nabla f(s)\|_p ds\), and \(\nabla f(s)\|_p L^2(\gamma^{\lambda_{x,s}}(s))\), we conclude that \(T = \infty\).

Then the estimates of (1.47) and (1.48) are direct consequence of Proposition 4, Lemma 6, and Proposition 6. And (1.49) can be derived from (6.5)–(6.11) by replacing \(f^t\), \(f^{t+1}\) with \(f, g\). □

**APPENDIX A. AUXILIARY RESULTS AND PROOFS**

**Proof of (1.39).** Let \(t = + \text{ or } -\), from (1.28), for \(t - t_{b,\epsilon}(x, t, x, v) < s \leq t\),

\[
x_{b,\epsilon}(s, X \pm (s; t, t, x, v), V \pm (s; t, t, x, v)) = x_{b,\epsilon}(t, t, x, v)
\]

\[
v_{b,\epsilon}(s, X \pm (s; t, t, x, v), V \pm (s; t, t, x, v)) = v_{b,\epsilon}(t, t, x, v).
\]

Therefore

\[
[\partial_t + v \cdot \nabla_x \pm \nabla_x \phi(f) \cdot \nabla_v] \alpha_{f,\epsilon, \pm}(t, t, x, v)
\]

\[
= \frac{d}{ds} \alpha_{f,\epsilon, \pm}(s, X \pm (s; t, t, x, v), V \pm (s; t, t, x, v)) |_{s = t}
\]

\[
= \frac{d}{ds} \alpha_{f,\epsilon, \pm}(t, t, x, v) = 0.
\]

From (1.20) and (1.29),

\[
t_{b,\epsilon}(x, X \pm (s; t, t, x, v), V \pm (s; t, t, x, v)) = t_{b,\epsilon}(t, x, v) - (t - s).
\]

Therefore

\[
[\partial_t + v \cdot \nabla_x \pm \nabla_x \phi(f) \cdot \nabla_v](t - t_{b,\epsilon}(t, t, x, v))
\]

\[
= \frac{d}{ds} [s - t_{b,\epsilon}(s, X \pm (s; t, t, x, v), V \pm (s; t, t, x, v))] |_{s = t}
\]

\[
= \frac{d}{ds} [t - t_{b,\epsilon}(t, t, x, v)] = 0.
\]

These prove (1.33). □

**Proof of (3.35).** The proof follows the argument of Lemma 7 in [13]. Note

\[
k(v, u) e^{g v} = \frac{1}{|v - u|} \exp \left\{ -g|v - u|^2 - g \frac{|v|^2 - |u|^2}{|v - u|^2} + \vartheta |v|^2 - \vartheta |u|^2 \right\}. 
\]

Let \(v - u = \eta\) and \(u = v - \eta\). Then the exponent equals

\[-g|\eta|^2 - g \frac{|\eta|^2 - 2v \cdot \eta}{|\eta|^2} - \vartheta (|v - \eta|^2 - |v|^2) - g |v - u|^2 - g \frac{|v - u|^2}{|v - u|^2} + \vartheta |v|^2 - \vartheta |u|^2\]

\[= -2g|\eta|^2 + 4g v \cdot \eta - 4g \frac{|v \cdot \eta|^2}{|\eta|^2} - \vartheta (|\eta|^2 - 2v \cdot \eta)
\]

\[= (-2g - \vartheta) |\eta|^2 + (4g + 2\vartheta)v \cdot \eta - 4g \frac{|v \cdot \eta|^2}{|\eta|^2}.
\]
If $0 < \vartheta < 4\varrho$ then the discriminant of the above quadratic form of $|\eta|$ and $\frac{\eta}{|\eta|}$ is

$$(4\varrho + 2\vartheta)^2 - 4(-2\varrho - \vartheta)(-4\varrho) = 4\vartheta^2 - 16\varrho\vartheta < 0.$$ 

Hence, the quadratic form is negative definite. We thus have, for $0 < \vartheta < \varrho - \frac{3}{2}$, the following perturbed quadratic form is still negative definite

$$-(\varrho - \vartheta)|\eta|^2 - (\varrho - \vartheta)\frac{|\eta|^2 - 2v \cdot \eta}{|\eta|^2} - \vartheta\{\eta^2 - 2v \cdot \eta\} \leq 0.$$ 

Therefore we conclude. 

Recall $\kappa_{s}(x, v)$ in (A.13). Let us denote $f_{s}(t, x, v) := \kappa_{s}(x, v)f(t, x, v)$. We assume that $f(s, x, v) = e^{\vartheta}f_{0}(x, v)$ for $s < 0$. Then $\|f_{s}\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})} \lesssim \|f\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})} + \|f_{0}\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})}$, $\|f_{s}\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})} \lesssim \|f_{1}\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})} + \|f_{0}\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})}$.

**Lemma 9.** Assume $\Omega$ is convex in $\{0, L\}$ and $\sup_{0 \leq t \leq T} \|E(t)\|_{L^{\infty}(\Omega)} < \infty$. Let $E(t, x) = 1_{\Omega}(x)E(t, x)$ for $x \in \mathbb{R}^{3}$. There exists $\bar{f}(t, x, v) \in L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})$, an extension of $f_{s}$, such that $\bar{f}_{|\Omega \times \mathbb{R}^{3}} \equiv f_{s}$ and $\bar{f}_{|\gamma} \equiv f_{s}_{|\gamma}$ and $\bar{f}_{|t = 0} \equiv f_{s}_{|t = 0}$. Moreover, in the sense of distributions on $\mathbb{R} \times \mathbb{R}^{3} \times \mathbb{R}^{3}$

$$[\partial_{t} + v \cdot \nabla_{x} + qE \cdot \nabla_{v}]\bar{f} = h,$$

where

$$h(t, x, v) := \kappa_{s}(x, v)1_{t \in [0, \infty)} \{\partial_{t} + v \cdot \nabla_{x} + qE \cdot \nabla_{v}\}f$$

$$+ \kappa_{s}(x, v)1_{t \in (-\infty, 0)} \{1 + v \cdot \nabla_{x} + qE \cdot \nabla_{v}\}f_{0} \kappa_{s}(x, v)$$

$$+ f(t, x, v)\{v \cdot \nabla_{x} + qE \cdot \nabla_{v}\}f_{0} \kappa_{s}(x, v),$$

where $\tilde{t}_{b}^{E}, \tilde{t}_{E}^{E}, \tilde{t}_{f}^{E}, x_{b}^{E}$ are defined in (A.5).

Moreover,

$$[\|h\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})} \lesssim \|\partial_{t} + v \cdot \nabla_{x} + qE \cdot \nabla_{v}\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})} + \|f\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})} + \|\tilde{t}_{b}^{E} \cdot \nabla_{x} + qE \cdot \nabla_{v}\|_{L^{2}(\mathbb{R}^{x} \times \mathbb{R}^{3})}].$$

**Proof.** In the sense of distributions

$$[\partial_{t}f_{s} + v \cdot \nabla_{x}f_{s} + qE \cdot \nabla_{v}f_{s} = h]$$

Clearly $\|v \cdot \nabla_{x} + qE \cdot \nabla_{v}\kappa_{s}(x, v) \lesssim 1$. For $x \in \mathbb{R}^{3} \setminus \Omega$ we define

$$t_{b}^{E}(x, v) := \{\sup \{s \geq 0 : x - sv \in \mathbb{R}^{3} \setminus \Omega \text{ for all } \tau \in (0, s)\} \}$$

$$t_{f}^{E}(x, v) := \{\sup \{s \geq 0 : x + sv \in \mathbb{R}^{3} \setminus \Omega \text{ for all } \tau \in (0, s)\} \},$$

and $x_{b}^{E}(x, v) = x - t_{b}^{E}(x, v)\mathbf{i}(x, v)$, $x_{f}^{E}(x, v) = x + t_{f}^{E}(x, v)\mathbf{i}(x, v)$.

We define, for $x \in \mathbb{R}^{3} \setminus \Omega$,

$$f_{E}(t, x, v) := 1_{x_{b}^{E} \mathbf{i}(x, v) \in \partial \Omega}f_{s}(t - t_{b}^{E}(x, v), x_{b}^{E}(x, v), v)$$

$$+ 1_{x_{f}^{E} \mathbf{i}(x, v) \in \partial \Omega}f_{s}(t + t_{f}^{E}(x, v), x_{f}^{E}(x, v), v).$$

Recall that, from (0.13), $f_{s} \equiv 0$ when $n(x) \cdot v = 0$, and hence $f_{E} \equiv 0$ for $n(x) \cdot v = 0$. Since $\Omega$ is convex if $v \neq 0$ then $x_{b}^{E}(x, v) \in \partial \Omega \cap \{x_{f}^{E}(x, v) \in \partial \Omega\} = \emptyset$. Note that

$$f_{E}(t, x, v) = f_{s}(t, x, v) = f_{s}(t, x, v) \text{ for } x \in \partial \Omega.$$ 

And since for any $s > 0$, 

$$(t + s - t_{b}^{E}(x + sv, v), x_{b}^{E}(x + sv, v), v) = (t - t_{b}^{E}(x, v), x_{b}^{E}(x, v), v),$$

$$(t + s + t_{f}^{E}(x + sv, v), x_{f}^{E}(x + sv, v), v) = (t - t_{f}^{E}(x, v), x_{f}^{E}(x, v), v),$$

so in the sense of distribution, in $\mathbb{R}^{3} \setminus \Omega$,

$$[\partial_{t}f_{E} + v \cdot \nabla_{x}f_{E} = 0].$$

We define

$$\tilde{f}(t, x, v) := 1_{\Omega}(x)f_{s}(t, x, v) + 1_{\mathbb{R}^{3} \setminus \Omega}(x)f_{E}(t, x, v).$$

From (A.4), (A.7), and (A.8) we prove (A.1). The estimates of (A.3) are direct consequence of Lemma 2. 

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