Practical techniques of QCD

vs

exact results of solvable models

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Abstract

We check quantitatively the validity of some popular phenomenological approaches of QCD in simple models. Dispersion sum rules are considered within the ladder approximation of a field-theoretic model with OPE given by ordinary loop diagrams which are computable for any number of loops. A correlator of two currents within the model complies with all requirements of standard QCD sum rules approach for fitting low-lying resonances such as fast convergence and good stability while the exact spectrum contains no resonance. Optimized PT as it is inspired by the principle of minimal sensitivity is analyzed within a quantum mechanical model and is shown to work well as compared to pure asymptotic expansion in the coupling constant or Padé approximation. Renormalon technique is tested within another quantum mechanical model and is found to fail to detect its low-energy structure.

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1 Introduction

With considerable and steady improvement of experimental data (as a recent review see, e.g., [1]) requirements to the accuracy of theoretical predictions are becoming stricter and one encounters an urgent necessity of quantitative check of phenomenological methods used to treat the data (e.g., [2]). In the electroweak sector of the Standard Model (SM) perturbation theory (PT) works well because of smallness of the coupling constant and there is no immediate need in its revision that was confirmed by the accurate estimate of the top quark mass based on electroweak radiative corrections [3]. For processes involving strong interactions PT calculations are now of the comparable precision with data that caused efforts to modify ordinary PT [2]. Also PT is known not to be complete due to existence of instantons and confinement. Several widely used approaches were suggested in QCD for describing the properties of hadrons and were successfully used for phenomenological predictions while their real precision has never been checked as well as their real connection with fundamental lagrangian and properties of QCD. At present it seems that this latter problem will be solved only within the lattice approximation that is going to attain a status of ultimate judge of validity of computational methods. Not everything is yet computable within lattice so some phenomenological methods have no quantitative tests. In these circumstances it is instructive to analyze some models.

In this paper we give a quantitative analysis of several methods used in QCD within exactly solvable models. They are dispersion sum rules, optimized PT, and partial resummation of PT for pinning down the non-PT effects (renormalons). The common feature of the above techniques is an attempt to obtain the non-PT information on the strong interactions studying asymptotic series in strong coupling constant (within ordinary PT) or large momentum asymptotic expansions (within operator product expansion (OPE)).

First we consider a two-point correlator in a field-theoretic model and restrict ourselves to a kind of massless $\phi^3$ ladder approximation. The spectral density of the correlator within the approximation is known explicitly and does not contain any resonances. Meanwhile making use of the standard sum rules technique with a simple ”resonance + continuum” model of the spectrum predicts parameters of the ”resonance” very accurately in a sense that all necessary criteria of stability are perfectly satisfied. Though the use of sum rules implies that hadron properties are mainly determined by several leading terms of asymptotic expansion of the correlator of relevant interpolating currents in deep Euclidean domain that cannot be guaranteed by itself. This problem was studied within several model of quantum mechanics or field theories in space-time dimension less than four and sum rules proved to be successful though it was stressed that there is no criterion for establishing the validity of the technique. In two-dimensional electrodynamics (the Schwinger model) [4] sum rules calculations were directly checked and stability of the result with respect to inclusion of
higher order corrections was proposed as an intrinsic criterion of sum rules applicability. In
the present paper we demonstrate within another exactly soluble model an opposite example
when truncation of the asymptotic expansion of a correlator leads to missing some main
properties of its spectral density while the formal stability requirement for corresponding
sum rules is completely satisfied. This situation reflects some general property of sum rules
and can be realized in QCD as well. This is also the first four-dimensional field-theoretical
example with known exact answer.

Second, within a quantum mechanical model we study the problem of resummation of
an asymptotic perturbation series via optimization of the perturbative expansion. In most
physically interesting models of quantum field theory the conventional perturbation theory
forms an asymptotic series in the coupling constant which can be used for calculation of
the Green’s functions only if the effective parameter of the expansion is small enough while
for practical purposes there exist powerful methods of refining the expansion. We also note
that there are two different approaches to such refinement. One is more mathematical and
consists in dealing directly with an asymptotic series applying methods of resummation like
Borel (with hypotheses of higher order behavior) or Padé technique. Alternative approach
relies on changing the splitting of the whole interaction into exact part and perturbation. It
is closer to Ritz variational technique. We think that such an approach is more physically
relevant.

In the last part of our paper we discuss nonperturbative (power-like) corrections that
for cases that have no simple formulation in terms of OPE are now mostly based on using
renormalons (for a concise up-to-date review see [5]). Because the expansion parameter
– a running coupling constant $\alpha_s$ – is sufficiently large for moderate energies, predictions
differ strongly depending on a way one chooses to handle a strong coupling constant in
infrared region. We mimic infrared renormalons of QCD by resumming a Born series for
s-wave scattering within an exactly soluble model of quantum mechanics. Numerical esti-
mates show that traditional technique (e.g. [6]) based on introduction of nonperturbative
power corrections fails to determine the low-energy mass scale of the model analogous to
a typical resonance mass in QCD. An alternative approach exploiting a modified running
coupling constant of the model and nonperturbative continuation of evolution equations into
an infrared region gives solid and accurate estimate of this scale.

2 Dispersion sum rules

The problem of checking this technique is not new and fairly well understood. It is known
that the solution to the sum rules problems is not unique. Several toy models have been
considered where dispersion sum rules work well. We here consider an example [7] focusing
mainly on the statement that the formal result is nice and satisfies all requirements of the
technique but the exact spectral density is different and there is no intrinsic criterion to tell us
what is going wrong. This is also a almost realistic model – four dimensional field-theoretical
model within commonly used approximation that reveals no explicitly pathological behavior.

2.1 The model

We consider a field-theoretical model in four dimensions in the approximation that is equiv-
alent to ladder massless $\phi^3$ one. In the $\phi^3$ model the expansion parameter is dimensionless
ratio of the dimensionful coupling constant and the energy of the process in question, so
the interaction vanishes at large energies. On the other hand at low energies the parameter
of the expansion becomes large and the model requires a nonperturbative treatment. Thus
it would be instructive to investigate the model along the line of ordinary QCD methods.
Not to be taken quite seriously it gives nevertheless a way to go beyond the perturbation
theory because here the explicit expressions for diagrams in any order of loop expansions are
known \[8, 9, 10\]. The model is attractive also because other attempts to break the bound of
perturbation theory tend to be in dimensions different from 4.

We modify the usual $\phi^3$ model slightly to make it more convenient for our purpose. The
Lagrangian of our model reads

$$
\mathcal{L} = \mathcal{L}_0 + e_1 \varphi^2 A + e_2 \phi^2 A + \ldots
$$

where $\mathcal{L}_0$ is a free kinetic term for all fields, ellipsis stands for other interactions that are
considered to be small and neglected, and we choose $e_2 = -e_1 = e$. All questions of stability
of the model (the existence of a stable ground state, for example) remain beyond the scope
of our toy consideration.

We study a correlator of two composite operators $j = \varphi\phi$ in the ladder approximation.
The correlator has the form \[9\]

$$
\Pi(q) = i \int \langle 0 | T j(x) j(0) | 0 \rangle e^{iqx} dx, \quad \Pi(Q^2) = \frac{1}{16\pi^2} \ln \left( \frac{\mu^2}{Q^2} \right) + \Delta \Pi(Q^2), \quad Q^2 = -q^2,
$$

where

$$
\Delta \Pi(Q^2) = \frac{1}{16\pi^2} \sum_{L=2}^{\infty} \left( -\frac{e^2}{16\pi^2 Q^2} \right)^{L-1} \binom{2L}{L} \zeta(2L-1), \quad \binom{2L}{L} = \frac{(2L)!}{L! L!},
$$

where $\zeta(z)$ is the Riemann’s $\zeta$-function. Since the coupling constant $e$ is dimensionful the
expansion \(8\) simulates power corrections or OPE of the ordinary QCD. Note that this
particular subset of diagrams does not lead to infrared problems within PT as in a general
superrenormalizable theory. Setting $e^2/16\pi^2 = 1$ we get

$$
p(Q^2) \equiv 16\pi^2 \Pi(Q^2) = \ln \left( \frac{\mu^2}{Q^2} \right) + \Delta p(Q^2),
$$
with
\[ \Delta p(Q^2) = \sum_{L=2}^{\infty} \left( -\frac{1}{Q^2} \right)^{L-1} \binom{2L}{L} \zeta(2L-1) \]  
(5)
in analogy with QCD where the scale is given and all condensates are expressed through \( \Lambda_{\text{QCD}} \).

What we see first is the alternating character of the series (5) in Euclidean direction \((Q^2 > 0)\) due to the special choice of interaction (1). This was actually the reason to create all decorations for the simple \( \phi^3 \) model in which every term has the same sign for Euclidean \( q \). Note that the series (5) is convergent for \( Q^2 > 4 \) as it is seen from the asymptotic behavior of the coefficients at large \( L \)
\[ \binom{2L}{L} \sim \frac{4^L}{\sqrt{\pi L}}. \]  
(6)
The sum (5) has the following closed form \([10]\]
\[ -\sum_{n=1}^{\infty} nQ^2 \left[ \left( 1 + \frac{4}{n^2Q^2} \right)^{-\frac{1}{2}} - 1 + \frac{2}{n^2Q^2} \right] \]
and contains no resonances.

The spectral density for the function \( \Delta p(Q^2) \) reads
\[ \Delta \rho(s) = -\sum_{n=1}^{\infty} \frac{ns\sqrt{s}}{\sqrt{\frac{4}{n^2} - s}} \theta \left( \frac{4}{n^2} - s \right) \theta(s). \]  
(7)
Unfortunately the whole spectral density of the correlator \( \rho(s) = 1 + \Delta \rho(s) \) is negative in some domains. In case \( e_1e_2 > 0 \) it would have a correct positive sign but the cut would be situated in the wrong place of the complex plane (negative semiaxis). So, any choice of the interaction sign leads to unphysical spectral density and the set of ladder diagrams is hardly representative for the exact correlator \( \langle 0|Tj(x)j(0)|0 \rangle \). The same situation is realized in QCD. If one takes seriously the leading order correlator for vector currents, for example, one finds that the spectral density contains an unphysical pole at \( Q^2 = \Lambda_{\text{QCD}}^2 \) due to running coupling constant and does not satisfy the spectrality condition being negative at \( s < \Lambda_{\text{QCD}}^2 \).

We omit these delicate points and proceed as in QCD. Namely, whether the expansion (5) can be described successfully with the simple formula
\[ \rho^{\text{test}}(s) = F\delta(s - m^2) + \theta(s - s_0) \]  
(8)
for the spectral density \( \rho(s) \).

An explicit expansion for the correlator reads
\[ \Delta p(Q^2) = -\frac{6\zeta(3)}{Q^2} + \frac{20\zeta(5)}{Q^4} - \frac{70\zeta(7)}{Q^6} + \frac{252\zeta(9)}{Q^8} + \ldots \]  
(9)
while the "test" form of the correlator is

\[ p^{\text{test}}(Q^2) = \ln \left( \frac{\mu^2}{Q^2 + s_0} \right) + \frac{F}{Q^2 + m^2}. \]  

(10)

We connect expressions (4) and (10) by means of sum rules. There are many essentially equivalent ways to extract physical parameters from OPE (4) and representation (10) however the most widely used are local duality or finite energy sum rules (FESR) [11] and Borel technique. We check them in sequence.

2.2 Finite energy and Borel sum rules

First we use FESR

\[ \int_0^{s_0} s^k \rho(s) ds = \int_0^{s_0} s^k \rho^{\text{test}}(s) ds \]  

(11)

where \( k = 0, 1, 2 \) because the test spectral density has three parameters to be determined. Eqs. (11) now become

\[ F = s_0 - 6\zeta(3), \quad Fm^2 = \frac{1}{2} s_0^2 - 20\zeta(5), \quad Fm^4 = \frac{1}{3} s_0^3 - 70\zeta(7) \]  

(12)

that leads to the equation for determination of the duality threshold \( s_0 \)

\[ \frac{1}{12} s_0^4 - 2\zeta(3) s_0^3 + 20\zeta(5) s_0^2 - 70\zeta(7) s_0 + 420\zeta(7) \zeta(3) - 400\zeta(5)^2 = 0. \]

It has solutions \( s_0 = 16.9 \) and \( s_0' = 2.24 \), and for corresponding parameters \( F = 9.67 \) and \( m^2 = 12.6 \) whilst \( F' = -4.97 \) and \( m'^2 = 3.67 \). We will study the first solution because the second one gives an unnatural relation between the "resonance mass" and the "duality interval" \( s_0' < m'^2 \).

Now we check the Borel sum rules approach [12]. The Borel transformation \( p(M^2) \) of the function \( p(Q^2) \) is

\[ p(M^2) = 1 - \frac{6\zeta(3)}{M^2} + \frac{20\zeta(5)}{M^4} - \frac{35\zeta(7)}{M^6} + \frac{42\zeta(9)}{M^8} + \ldots, \]  

(13)

the continuum contribution gives

\[ c(M^2) = \exp \left( -\frac{s_0}{M^2} \right), \]  

(14)

and the resonance contribution reads

\[ r(M^2) = \frac{F}{M^2} \exp \left( -\frac{m^2}{M^2} \right). \]  

(15)

We have plotted these functions in Fig. 1. As we see our "test" representation accurately simulates the asymptotic form of the correlator at \( M^2 > 8 \). Equating the functions \( p(M^2) \)
and $r(M^2) + c(M^2)$ we obtain the Borel sum rules to determine parameters of the resonance. The sum rules look like ordinary QCD sum rules. Fig. 2 shows the dependence of mass $m^2$ on the Borel variable $M^2$ with the parameters $s_o$ and $F$ given by FESR for the different numbers of power corrections included. For other close values of the parameters $s_o$ and $F$ the results are not very different from fig. 2. A stable region is reached for $8 < M^2 < 18$ where, on the one hand, higher order power corrections are small and, on the other hand, the continuum contribution eq. (14) does not prevail over the resonance one eq. (15). The best stability is obtained at the optimal values $s_o = 16.6, F = 9.39$ that shows a consistency of the Borel sum rules and FESR. Furthermore, the curves in fig. 2 show that inclusion of higher order power corrections does not destroy the Borel sum rules and even enlarges the region of stability.

Let us emphasize that the sum rules are perfectly saturated by the artificially introduced resonance and show very good stability though the exact spectral density does not contained any resonance singularities. The reason of this phenomenon is quite transparent: using sum rules we neglect the high order terms of large momentum expansion which do not affect the rough integral characteristics of the spectral density but are essentially responsible for its local behavior. The extreme sensitivity of the local form of the spectral density to the high orders of perturbative expansion can be easily demonstrated within considered model. For example, substituting $\zeta$-functions for $L > 2$ by units in the series (9) one neglects the terms with $n > 2$ in eq. (7) and gets the modified spectral density

$$\Delta \tilde{\rho}(s) = -s \sqrt{s} \theta(s) \left( \frac{\theta(4 - s)}{\sqrt{4 - s}} + \frac{\theta(1 - s)}{\sqrt{1 - s}} \right).$$

As we see the tiny variation of the coefficients of the asymptotic expansion leads to a drastic change of the spectral density at low scale $s < 4/9$. Indeed, the modified spectral density has two singular points at $s = 4$ and 1 while the original expression has an infinite number of singularities at $s_n = 4/n^2, n = 1, 2, \ldots$. At the same time this variation does not really affect the sum rules result that changes slightly (less than 10%): $\tilde{s}_0 = 17.9, \tilde{F} = 10.7, \tilde{m}^2 = 13.0$.

The lesson we draw is that our toy model containing no resonances can be well fitted by a standard "resonance + continuum" ansatz with good stability properties. While stability criteria look appropriate for determining parameters of resonance in established channels there is no intrinsic criterion of the reliability of QCD sum rules predictions for channels where there is no experimental data about the spectral density as, for instance, for gluonia. This makes sum rules predictions questionable.

Our model has recently been reanalyzed in [13] where some mathematical criteria for validity of the "resonance + continuum" ansatz have been used in addition to a simple stability requirement. It was shown that making use of Holder inequalities that are valid due to the positiveness of the spectral density puts some additional constraints on the structure
of the series in $1/Q^2$ and can help in ruling out some cases of spurious resonances. Note however that such a technique is not applicable in cases of non-diagonal correlators where the spectral density must not be positively defined.

## 3 Optimized perturbation theory

Some methods have been suggested to improve convergence of conventional perturbation theory. We would divide them into two groups depending on either one knows coefficients of expansion for $n$-term that is rare or one knows the structure of interaction and makes educated guess about the best zero order approximation. First is a pure mathematical problem (like a direct summation with Borel-like technique) and we don’t touch it concentrating on the second.

A broad class of methods is represented by general approach of $\delta$-expansion. The basic idea of the optimized $\delta$-expansion is to introduce the artificial parameter $\delta$ which interpolates between the original theory with Hamiltonian $H$, and another theory, with Hamiltonian $H_0(\lambda)$ ($\lambda$ is a set of auxiliary parameters not present in the original theory), which is soluble and reflects some main properties of the theory we are interested in. One defines a new Hamiltonian depending on $\delta$

$$H_\delta = H_0(\lambda) + \delta(H - H_0(\lambda))$$

and any desired quantity is evaluated as a perturbation series in $\delta$, which is set equal to unity at the end of the calculations. So this parameter $\delta$ simply marks relevant terms of the expansion developed. Convergence of the series is achieved by an optimization procedure e.g. by fixing the parameters $\lambda$ at every finite order of the expansion in $\delta$ according to principle of minimal sensitivity (PMS) at the point where the result is least sensitive to their variation, or the principle of fastest apparent convergence (FAC) at the point where the next term in the series vanishes, or some other criterion. Though the above procedure is not rigorous it gives good numerical results in most cases. The method has been mostly advanced in studying the anharmonic oscillator where the convergence has been rigorously established.

It seems instructive to consider the simplest model which, nevertheless, retains most relevant features of the real problem and, we hope, can help to gain some intuition to cure some difficulties of PT expansion in QCD.
3.1 The model

The problem we will study is the ground state in the spectrum of the stationary Schrödinger equation in three dimensions [17]

\[ H(\alpha)\psi(\mathbf{r}) \equiv (-\Delta + U(\alpha, r))\psi(\mathbf{r}) = E\psi(\mathbf{r}), \quad r = |\mathbf{r}|, \]  

(17)

where we set \(2m = \hbar = 1\) and

\[ U(\alpha, r) = \begin{cases} -\frac{\pi^2}{4} - \alpha + \alpha^2 r, & r < 1, \\ 0, & r > 1. \end{cases} \]  

(18)

The choice of the potential is transparent: at \(\alpha = 0\) the depth of the well is equal to the value at which the first s-wave bound state appears. The exact solution for the ground state energy \(E(\alpha)\) is given by the transcendental equation

\[ \sqrt{-E} = \alpha^{2/3} \frac{\text{Bi}(\xi_0)\text{Ai}'(\xi_1) - \text{Ai}(\xi_0)\text{Bi}'(\xi_1)}{\text{Bi}(\xi_0)\text{Ai}(\xi_1) - \text{Ai}(\xi_0)\text{Bi}(\xi_1)}, \]

\[ \xi_0 = -\alpha^{-4/3} \left( \frac{\pi^2}{4} + \alpha + E \right), \quad \xi_1 = -\alpha^{-4/3} \left( \frac{\pi^2}{4} + \alpha - \alpha^2 + E \right), \]  

(19)

where \(\text{Ai}(z)\) and \(\text{Bi}(z)\) are Airy functions [18]. Using the asymptotic expansion of Airy functions at large negative \(\xi_{0,1}\) (small \(\alpha\)) we obtain an asymptotic series for the ground state energy \(E(\alpha)\) is given by the transcendental equation

\[ \sqrt{-E} = \alpha^{2/3} \frac{\text{Bi}(\xi_0)\text{Ai}'(\xi_1) - \text{Ai}(\xi_0)\text{Bi}'(\xi_1)}{\text{Bi}(\xi_0)\text{Ai}(\xi_1) - \text{Ai}(\xi_0)\text{Bi}(\xi_1)}, \]

(20)

Numerically it reads

\[ \tilde{E}(\alpha) = -\alpha^2 \left( 1 - 1.7026\alpha + 1.2602\alpha^2 - 0.7864\alpha^3 + \ldots \right), \]  

(21)

and we find that the series converges fairly badly near the point \(\alpha \sim 1\). In fact, the series diverges for any positive \(\alpha\) because the coefficients of the expansion (20) in high orders grow factorially. That reflects the presence of a singularity of the function \(E(\alpha)\) at the origin in the complex \(\alpha\) plane. The form of the singularity can be found directly from eq. (19): \(E(\alpha)\) has a cut along the negative semiaxis and a branch point at \(\alpha = 0\). For sufficiently small \(|\alpha|\) it is an analytical function for \(|\arg(\alpha)| < \pi\); therefore the series (20) is Borel recoverable [19], i.e. we can restore complete information on the function \(E(\alpha)\) from its asymptotic expansion. The presence of the singularity in the Green’s function of eq. (17) reflects the fact that at \(\alpha = 0\) the spectrum of eq. (17) changes qualitatively and a discrete part of the spectrum appears.
Consider now the model (17) within the perturbation theory framework. At \( r < 1 \) the potential (18) consists of two parts: one is a constant and the other depends linearly on \( r \).

The Schrödinger equation with the constant part of the potential only has bound states for any positive \( \alpha \) with the ground state energy \( E_0(\alpha) \) determined by the equation

\[
\sqrt{-E_0} = \sqrt{\frac{\pi^2}{4} + \alpha + E_0} \cot \left( \sqrt{\frac{\pi^2}{4} + \alpha + E_0} \right).
\] (22)

If \( \alpha \) is small enough one can consider the constant part of the potential to be responsible for creation of the bound state while the linear term is a perturbation because it is suppressed by an extra power of \( \alpha \). Then one can search for the ground state energy of eq. (17) as a series in \( \alpha \) using perturbation theory. The ground state energy of the unperturbed Hamiltonian (the solution of eq. (22)) is an analytical function at the origin of the complex \( \alpha \) plane and can be expanded in a convergent series for any finite positive \( \alpha \)

\[
E_0(\alpha) = -\frac{\alpha^2}{4} \left( 1 - \left( \frac{1}{2} - \frac{2}{\pi^2} \right) \alpha + \left( \frac{5}{16} - \frac{11}{6\pi^2} - \frac{3}{\pi^4} \right) \alpha^2 - \left( \frac{7}{32} - \frac{13}{8\pi^2} - \frac{3}{2\pi^4} - \frac{6}{\pi^6} \right) \alpha^3 + \ldots \right).
\] (23)

However, after inclusion of perturbations the ground state energy of the whole Hamiltonian (the solution of eq. (19)) gains the singularity and the full series (20) becomes divergent.

### 3.2 Optimized perturbative expansion

Our purpose now is to develop the optimized perturbation theory (OPT) for our toy model. Following the general idea of \( \delta \)-expansion we have to choose the “unperturbed” Hamiltonian. The Hamiltonian

\[
H_0(\alpha') = -\Delta + U_0(\alpha', r),
\] (24)

where

\[
U_0(\alpha', r) = \begin{cases} 
-\frac{\pi^2}{4} - \alpha', & r < 1, \\
0, & r > 1,
\end{cases}
\] (25)

is the potential of the spherical well with adjustable depth seems to be the simplest and most appropriate choice. Here the set of parameters \( \lambda \) in eq. (16) reduces to the single parameter \( \alpha' \) characterizing the depth of the well and we consider \( 0 < \alpha' < 2\pi^2 \) so that the trial Hamiltonian (24) has only one bound state. Then the perturbation reads

\[
\delta (H(\alpha) - H_0(\alpha')) \equiv \delta V(\alpha, \alpha', r) = \begin{cases} 
\delta(\alpha' - \alpha + \alpha^2 r), & r < 1, \\
0, & r > 1.
\end{cases}
\] (26)
Expanding in $\delta$ and setting $\delta = 1$ we obtain the $n$-th order approximation for the ground state energy as a series

$$E_n(\alpha, \alpha') = E_0(\alpha') + E^{(1)}(\alpha, \alpha') + \ldots + E^{(n)}(\alpha, \alpha').$$

(27)

The zero order approximation $E_0(\alpha')$ (the ground state energy of the Hamiltonian (24)) is determined by transcendental equation (22) with parameter $\alpha'$ instead of $\alpha$. For corrections we have

$$E^{(1)}(\alpha, \alpha') = \int \! dr \, \psi_0^*(\alpha', r)V(\alpha, \alpha', r)\psi_0(\alpha', r),$$

$$E^{(2)}(\alpha, \alpha') = \int_0^\infty \! \frac{dE}{E_0(\alpha') - E} \int \! dr \, \psi_0^*(\alpha', r)V(\alpha, \alpha', r)\psi_0(\alpha', E, r)|^2,$$

(28)

$$\ldots,$$

where the quantity $\psi_0(\alpha', r)$ is the wave function of the ground state of the Hamiltonian (24) and $\psi_E(\alpha', E, r)$ is the $s$-wave function with energy $E$ which belongs to the continuous part of the spectrum of the Hamiltonian (24). These wave functions can be easily obtained but the explicit expressions are too large to be presented here.

For arbitrary $\alpha'$ explicit analytical expressions for the terms of the expansion (27) are absent and we have to analyze this expansion numerically. However, if $\alpha'$ is small enough we can expand the right hand side of eq. (27) in the series in $\alpha'$

$$E_0(\alpha') = -\frac{\alpha'^2}{4} \left( 1 - \left( \frac{1}{2} - \frac{3}{2} \pi^2 \right) \alpha' + \left( \frac{5}{16} - \frac{11}{6\pi^2} - \frac{3}{\pi^4} \right) \alpha'^2 + \ldots \right),$$

$$E^{(1)}(\alpha, \alpha') = \frac{(\alpha' - \alpha)\alpha'}{2} \left( 1 - \left( \frac{3}{4} - \frac{3}{\pi^2} \right) \alpha' + \ldots \right) +$$

$$+ \frac{\alpha'^2}{4} \left( 1 + \frac{4}{\pi^2} - \left( \frac{3}{4} - \frac{2}{\pi^2} + \frac{12}{\pi^4} \right) \alpha' + \ldots \right),$$

$$E^{(2)}(\alpha, \alpha') = -\left( \frac{(\alpha' - \alpha)}{2} \left( 1 + \ldots \right) + \alpha^2 \left( \frac{1}{4} + \frac{1}{\pi^2} \right) \left( 1 + \ldots \right) \right)^2.$$

(29)

The $\delta$-expansion (27) generates three types of perturbative series.

Setting $\alpha' = \alpha$ in eq. (27) without further expansion of $E_0(\alpha)$, $E^{(i)}(\alpha, \alpha)$ in $\alpha$ we reproduce the standard perturbation theory of quantum mechanics with the unperturbed Hamiltonian $H_0(\alpha)$ and the perturbation $V(\alpha, \alpha, r)$.

Expanding $E_0(\alpha)$, $E^{(i)}(\alpha, \alpha)$ in $\alpha$ we derive the asymptotic series (20). We should note that the standard analysis of the positronium spectrum consists exactly in this machinery, i.e. every term of perturbative expansion around the nonrelativistic Coulomb solution is expanded in the fine structure constant.

In our model if $\alpha' - \alpha = O(\alpha^2)$ the effective parameter of the expansion is proportional to

$$\frac{1}{E_0(\alpha')} \int \! dr \, \psi^*(\alpha', r)V(\alpha, \alpha', r)\psi(\alpha', r) \sim \alpha.$$

(30)
So $E_n(\alpha, \alpha')$ after expansion in $\alpha$ correctly reproduces the first $n$ terms of eq. (20). Thus if we are interested in the asymptotic expansion only the choice of the initial approximation does not play any role.

If, however, we intend to go beyond the asymptotic expansion in $\alpha$ we should choose the “unperturbed” Hamiltonian in order to provide the best convergence of the expansion. The most natural way to optimize the expansion (27) is to fix the parameter $\alpha'$ in $n$-th order according to PMS criterion at the value $\alpha_{n}^{PMS}$ which is defined by the equation

$$\left.\frac{\partial E_n}{\partial \alpha'}\right|_{\alpha' = \alpha_{n}^{PMS}} = 0,$$

or according to FAC criterion at the value $\alpha_{n}^{FAC}$ so that

$$E^{(n+1)}|_{\alpha' = \alpha_{n}^{FAC}} = 0.$$

To solve eqs. (31, 32) for arbitrary $\alpha$ one has to proceed numerically. However, for small $\alpha$ the roots of eqs. (31, 32) can be found as a series in $\alpha$. This allow us to observe some qualitative features of OPT. At $n = 1$ using eq. (29) we find

$$\alpha_{1}^{PMS} = \alpha \left(1 - \left(\frac{1}{2} + \frac{2}{\pi^2}\right)\alpha + \ldots\right),$$

$$\alpha_{1}^{FAC} = \alpha \left(1 - \left(\frac{1}{2} + \frac{2}{\pi^2}\right)\alpha + \ldots\right).$$

As $n \to \infty$ we have the formal series

$$\alpha_{\#}^{\infty} \sim \alpha(1 + a_1^{\#} \alpha + a_2^{\#} \alpha^2 + \ldots),$$

where # stands for PMS or FAC and the coefficients $a_n^{\#}$ are completely determined by eqs. (31, 32). Moreover, by construction

$$E_{\infty}(\alpha, \alpha_{\#}^{\infty}) = E_0(\alpha_{\#}^{\infty}(\alpha)).$$

Power counting arguments show that $\alpha_{n}^{PMS} - \alpha_{n}^{FAC} = O(\alpha^{n+2})$, i.e. $\alpha_{\infty}^{PMS}$ and $\alpha_{\infty}^{FAC}$ have identical asymptotic expansions. Thus in the limit $n \to \infty$ all corrections in the optimized expansion vanish in our model for both the PMS and FAC optimization prescriptions and we obtain

$$E_{\infty}(\alpha, \alpha_{\#}^{\infty}) = E_0(\alpha_{\#}^{\infty}(\alpha)).$$

Since the function $E_0(\alpha_{\#}^{\infty})$ can be expanded in a convergent series in $\alpha_{\#}^{\infty}$ (eq. (23)) the singularity of the function $E(\alpha)$ at $\alpha = 0$ is absorbed by the function $\alpha_{\#}^{\infty}(\alpha)$, i.e. the series (33) must be a divergent asymptotic expansion. However, we can search for the values $\alpha_{n}^{\#}$ which satisfy eqs. (31) (32) at every order numerically rather than as a series in $\alpha$.

Thus we have three kinds of perturbative expansion for the ground state energy: the asymptotic series, the standard perturbation theory and OPT. The asymptotic series seems to be the most primitive tool and we expect to obtain the best results using OPT. This assumption is completely confirmed by numerical analysis given in the next section.
3.3 Numerical evaluation

The results of the numerical analysis are given in Tables 1-3. In Table 1 we present the exact value \( E(\alpha) \) and the results of the asymptotic expansion \( \tilde{E}_n(\alpha) \) up to the \( n \)-th order \((n = 0, 1, 2, 3)\). In Table 2 we compare the exact value \( E(\alpha) \), the result of the optimized expansion \( E_1(\alpha, \alpha_{1}^{PMS}) \) and the results of the standard perturbation theory up to the \( n \)-th order \((n = 0, 1)\), i.e. the values of \( E_0(\alpha) \) and \( E_1(\alpha, \alpha) \) not expanded in \( \alpha \). The \([i, j]\) Padé approximants \( E_{i,j}(\alpha) \) with \( i + j \leq 3 \) are shown in Table 3.

We give numerical estimates for three different values of the parameter \( \alpha \) that represent typical cases:

- \( \alpha = 0.1 \), the asymptotic expansion is applicable and justified;
- \( \alpha = 0.5 \), the asymptotic expansion is still applicable but one has to deal with high order corrections to achieve a satisfactory accuracy. An improvement of the asymptotic perturbation theory is desirable;
- \( \alpha = 1.0 \), the asymptotic expansion in principle can provide only \( \sim 2\% \) accuracy after summation of \( \sim 13 \) terms. Then the terms start to grow. The asymptotic perturbation theory must be reformulated.

As we can see, a naive attempt to improve the convergence of the perturbative expansion by simply keeping the exact (not expanded) value of \( E_n(\alpha, \alpha) \) which sums up some next-to-leading corrections gives a good result but is essentially insufficient if \( \alpha \) is large enough. This shows that optimization is important for convergence. Though we cannot directly demonstrate that without optimization (for any fixed \( \alpha' \)) the series \((27)\) becomes divergent, as is the case for the anharmonic oscillator \([16]\), the numerical analysis clearly shows the advantage of the optimized expansion. Indeed, in all cases the best convergence is achieved within OPT. Even for \( \alpha = 1.0 \) taking only the first order correction we reach \( \sim 2\% \) accuracy. Using the PMS prescription we can also correctly estimate the error of the result. For example, for \( \alpha = 1.0 \) the naive estimate is \((E^{(1)}(1.0, \alpha^{PMS})/E_0(\alpha^{PMS}))^2 \sim 0.02 \) which coincides with the real uncertainty (see Table 3).

We should note that in our numerical analysis for all values of \( \alpha \) the asymptotic series \((20)\) is truncated far from the critical order (which depends on \( \alpha \)) where it begins to diverge. The bad convergence of the series reveals itself only in the fact that its terms decrease quite slowly. On the other hand even the third order correction is hardly available. So the accuracy of the perturbation theory is restricted rather by technical reasons than by the asymptotic character of the series. Thus the optimization is not only a resummation prescription that is useful in high orders of the asymptotic expansion but also gives an opportunity to improve
the accuracy of perturbation theory in low orders. This is an important benefit, especially for nontrivial systems where high order calculations are impossible.

A remark about the Padé approach is in order. As we can see, some Padé approximants are closer to the exact result than the plain asymptotic expansion. However, it is not possible to make a choice between various approximants until the exact result or the general structure of the series are known. Moreover in high orders where the asymptotic character of the series reveals itself the Padé theory becomes useless. The reason is that because of their specific structure Padé approximants can correctly reproduce only pole-like singularities while the function $E(\alpha)$ has a branch point at $\alpha = 0$. On the other hand OPT seems to be the most appropriate tool to deal with such a problem because it speeds up the convergence of the perturbation series choosing the most natural initial approximation for a specific model not by a mathematical trick. This is an automatic summation device that does not need input information on the form of the singularity of the bound state energy in the coupling constant but reproduces it via optimization. This feature can be observed in our toy model, where the forms of the singularity of the ground state energy at $\alpha = 0$ in the auxiliary and original theories are essentially different, but OPT reproduces the correct singular $\alpha$ dependence through the optimized value of auxiliary parameter $\alpha'$. This general property of optimized perturbation theory allows one to cope even with Borel nonsummable series \cite{16} while the class of problems where the Padé theory can be successfully applied is quite restricted.

Note that perturbation theory for Green’s functions is much more involved and one of the main problems is uniformity of convergence with respect to momentum. Next section is devoted partly to this problem.

4 Testing renormalon technique

In this section we check ideas of method based on notion of infrared renormalon within the quantum mechanical model. The technique we further refer to as a standard one presently consists in resumming bubble chains with principal value prescription for singularities in the Borel plane. Rich phenomenology can be built on such a base \cite{20} though the real sensitivity of the approach to the infrared physics is unclear as well as an unambiguous disentangle of perturbative and nonperturbative (condensate) contributions \cite{21}. Some other approaches use mostly the modified running of the coupling constant \cite{22, 23}, a specific recipe for scale setting \cite{24, 25, 26} or some modification of $\beta$ function to produce evolution of running coupling constant at small momenta \cite{27}. Initially there is no preference between these techniques because no exact results on the behavior of PT in large orders or in IR domain are known. Some general properties of the quantum field theory to be respected (like analyticity) give no much help to distinguish between possibilities. Yet in phenomenological applications,
the existing methods give different numerical results lying on the edge of experimental errors and the final selection of a working frame will eventually be based on how well the particular technique fits experimental data.

In this section we investigate two different approaches within a quantum mechanical model that reflects some general features of renormalons.

4.1 The model

We consider the problem of potential scattering with

\[ V(r) = V_0 \delta(r - r_0) \]  

and limit ourselves to s-wave amplitudes [28]. Such a potential can be considered as a kind of confining (not completely) one. We study a value of wave function at the origin (a free wave function is normalized to 1). The exact solution for scattering of the plain wave with momentum \( k \) reads

\[ \psi(k) \equiv \psi(k, r = 0) = \left( 1 + \frac{V_0}{k} e^{ikr_0} \sin(kr_0) \right)^{-1}. \]  

where we set \( 2m = \hbar = 1 \) as in the previous section. To study scattering of wave packages with distributed momentum we consider an integral of the form

\[ \Psi(\lambda) = \int_0^\infty \psi(k)W(k, \lambda) dk \]  

where \( W(k, \lambda) \) is a normalized weight function of a package depending on a set of parameters \( \lambda \), \( \int_0^\infty W(k, \lambda) dk = 1 \). It is more convenient to deal with a function \( F(\lambda) \)

\[ \Psi(\lambda) = 1 + F(\lambda) \]

so that \( F \) vanishes if the scattering potential is switched off. Because of oscillating factors in eq. (39) integrals (40) are not well suitable for the PT analysis (they are “Minkowskian” quantities). For \( m = r_0^{-1} > |V_0| \) there are no bound states in the potential (38) and we can carry out the Wick rotation because \( \psi(k) \) is analytic in upper semi-plane (\( \text{Im} \ k > 0 \)) that corresponds to the physical sheet in energy \( E \sim k^2 \). In “Euclidean” region the exact expression for \( \psi(k) \) becomes

\[ \psi(q) = \left( 1 + \frac{V_0}{2q}(1 - e^{-2q/m}) \right)^{-1} \]  

where \( k = iq, q > 0 \). The last formula can be obtained by PT from a Born series for the standard Lippmann-Schwinger equation of potential scattering since we deal with a finite range potential. Each term of the Born series for \( \psi(q) \) contains contributions of different
kinds: a power like or “perturbative” (modeling logarithmic terms of a QCD series) and exponentially suppressed or “nonperturbative” (modeling power corrections in QCD). We are trying to use an analogy with QCD as close as possible (even through terminology) though one must remember a toy character of the model. The parameter $V_0$ like $\Lambda_{QCD}$ determines the scale at which the perturbation theory series becomes poorly convergent while $m$ like $\rho$-meson mass determines the scale of nonperturbative effects. Note that we do not fix the sign of our parameter $V_0$ and will study both attractive and repulsive interaction.

At high energies ($q \gg m$) the expansion parameter is a running coupling constant $\alpha(q) = V_0/2q$ (trivial asymptotic freedom as in superrenormalizable theories)

$$\psi(q) = \psi^{as}(\alpha) + \psi^{np}(m, V_0, q)$$

where

$$\psi^{as}(\alpha) = \sum_{n=0}^{\infty} (-\alpha)^n$$

and $\psi^{np}(m, V_0, q)$ stands for exponentially suppressed “nonperturbative” terms

$$\psi^{np}(m, V_0, q) = \frac{V_0}{2q} e^{-2q/m} + \ldots$$

Within the present model we classify terms with respect to their behavior at large $q$: power like vanishing – PT, faster than any power – non-PT. With the standard renormalization group (RG) terminology we have

$$\beta(\alpha) = q \frac{\partial \alpha}{\partial q} = -\alpha.$$ (44)

Resummation in eq. (42) (in the spirit of RG) results in definition of a new (renormalization group improved) “running” coupling constant

$$\alpha_{as}(q) = \frac{V_0}{2q + V_0}$$ (45)

with a $\beta$ function

$$\beta^{as}(\alpha_{as}) = q \frac{\partial \alpha_{as}}{\partial q} = -\alpha_{as}(1 - \alpha_{as}).$$ (46)

The use of this expansion parameter allows us to improve the perturbation theory and to sum up all “perturbative” power terms of the series (42)

$$\psi^{as}(q) = 1 + \alpha_{as}(q).$$ (47)

Now we turn to consideration of the wave package of a specific form given by the following weight function

$$W(q, Q) = Q e^{-Q/q} \frac{e^{-Q/q}}{q^2}.$$

This weight function has a bump of the width $\sqrt{3} Q$ located at $q \sim Q/2$ so the above wave package can be considered as a “probe” of the scattering potential at the scale $\sim 2/Q$. 

4.2 Borel resummation and modified $\beta$ function

It is easy to see that our observable (40) suffers from the renormalon. Substituting $\psi(q)$ in eq. (40) by its asymptotic expansion (47) we obtain

$$F_{\text{as}}(\alpha) = \int_0^\infty \alpha_{\text{as}}(q)W(q,Q)dq.$$ (48)

The quantity $F_{\text{as}}(\alpha)$ has a typical structure of QCD observable containing renormalon, i.e. it is an integral of some weight function multiplied by a running coupling constant over the interval that includes strong coupling domain. Note that in our model the use of the running coupling constant (45) in the integrand accounts for all perturbative corrections. One can only dream about that in QCD where a trick based on a pure assumption motivated by the “naive nonabelianization” [29] is used.

After integration we get the series with factorially growing coefficients

$$F_{\text{as}}(\alpha) = \sum_{n=1}^\infty (-1)^n n!\alpha^n.$$ (49)

Properties of the series (49) depend crucially on the sign of $\alpha$ or $V_0$. Let us consider first repulsive potential $V_0 > 0$. Then the alternating sign series (49) is Borel summable (in QCD it corresponds to an ultraviolet renormalon). The Borel image

$$\tilde{F}_{\text{as}}(u) = -\frac{u}{1+u}$$ (50)

has a pole at $u = -1$ and is a regular function on the positive semiaxis. So the Borel summation leads to an unambiguous result ($\alpha = V_0/2Q$)

$$F_B(\alpha) = \frac{e^{-1/\alpha}}{\alpha}E_1 \left( \frac{1}{\alpha} \right) - 1$$ (51)

where $E_1(x)$ is the integral exponent [15]

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt.$$

One could naively hope that the Borel summation provides us with nonperturbative information on the function $F(Q)$ and on the magnitude of nonperturbative parameter $m$. However numerical analysis shows that $F_B(Q)$ does not approximate well the exact function $F(Q)$ for intermediate $Q \sim m$ (Fig. 3). Indeed, $\alpha_{\text{as}}(q)$ is the best expansion parameter (exponentially accurate) at large momenta but $\psi_{\text{as}}(q)$ does not approximate well the function $\psi(q)$ at small $q$ and eq. (51) tells us nothing about the parameter $m$ that measures exponentially suppressed “nonperturbative” contributions. Alternatively, one can compute the function $F(Q)$ within the modified perturbation theory with sufficient accuracy even at very small $Q$. For this purpose one has to choose a relevant expansion parameter. First we write

$$\alpha_{\text{as}}(q) = \alpha_{\mu}^{(1)}(q) \left( 1 - \frac{\mu - V_0}{V_0} \alpha_{\mu}^{(1)}(q) \right)^{-1} = \alpha_{\mu}^{(1)}(q) \sum_{n=0}^\infty \left( \frac{\mu - V_0}{V_0} \right)^n \alpha_{\mu}^{(1)}(q)^n,$$
\[ \alpha^{(1)}(q) = \frac{V_0}{2q + \mu} \]

where \( \mu \) is a parameter. Now we limit ourselves to only two terms of this expansion that is reasonable in PT region and define new expansion parameter

\[ \alpha^{(2)}(q) = \alpha^{(1)}(q) \left( 1 + \frac{\mu - V_0}{V_0} \alpha^{(1)}(q) \right) = \frac{V_0}{2q + \mu} \left( 1 + \frac{\mu - V_0}{2q + \mu} \right) \]  

(52)

with a \( \beta \) function

\[ \beta^{(2)}(\alpha) = -\alpha + O(\alpha^2) \]

that is a PT transformation. The perturbation theory series for \( \psi(q) \) in \( \alpha^{(2)} \) reads

\[ \psi^{as}(q) = 1 + \alpha^{(2)}(q) + O(\alpha(q)^2). \]  

(53)

Taking the first order term in eq. (53) and fixing the parameter \( \mu \) at some value we find the function \( F^{\mu}(Q) = \int_0^\infty \alpha^{(2)}(q) W(q, Q) dq \) to be very close to the exact function \( F(Q) \) up to the very small \( Q \) (Fig. 3). Note that at very large \( Q \) the function \( F_B(Q) \) becomes closer to exact result \( F(Q) \) than \( F^{\mu}(Q) \) because \( F_B(Q) \) and \( F(Q) \) have the same asymptotic expansion. On the other hand lifting this too strong condition we get the function \( F^{\mu}(Q) \) that approximates the exact function \( F(Q) \) uniformly for all finite \( Q \). Because at the optimal \( \mu \) the function \( \psi^{(2)}(q) \) is close to \( \psi(q) \) for all finite \( q \) this approximation is universal in a sense that it works well for various forms of scattering package. The optimal value \( \mu^{opt}(m, V_0) \) can be extracted from experiment (in our model the exact solution plays the role of experimental data). It turns out to be very sensitive to the variation of \( m \). So in this way we obtain the real information on the scale \( m \) at which nonperturbative effects become important.

The case of attractive potential \( V(r) = -V_0 \delta(r - r_0), V_0 > 0 \) at first sight seems to be completely different. The running coupling constant

\[ \alpha_{as}(q) = \frac{V_0}{2q - V_0} \]

with \( \beta \) function \( \beta(\alpha) = -\alpha(1 + \alpha) \) has a “Landau pole” at \( q = V_0/2 > 0 \). The series (49) now becomes

\[ F^{as}(\alpha) = \sum_{n=1}^{\infty} n!\alpha^n \]  

(54)

and it is not Borel summable: we encounter an “infrared renormalon”. Its Borel image has a pole on the positive semiaxis in full analogy with QCD (when the QCD \( \beta \) function is taken in one loop approximation). So Borel procedure leads to an ill-defined representation in case of attractive potential. Standard technique could consider it as a signal of the presence of nonperturbative contributions. One should stress however that exact function undergoes no qualitative change in the low energy domain (see also [30]). Following the line of QCD
renormalon technique we define the result of Borel summation by deforming the integration contour in the complex $u$ plane. The result obtained in this way depends on the specific form of an integration contour while an appropriate nonperturbative part (condensates or power corrections in QCD) must cancel this dependence. In our model we use the principal value (PV) prescription to define the sum of the series

$$F_B(\alpha) = \text{PV} \int_0^\infty e^{-u/\alpha} \frac{u}{1-u/\alpha} \frac{du}{\alpha} = e^{-1/\alpha} \frac{\alpha}{\alpha} \text{Ei} \left( \frac{1}{\alpha} \right) - 1$$

(55)

where $\text{Ei}(x)$ is an integral exponent $[18]$

$$\text{Ei}(x) = \text{PV} \int_{-\infty}^x \frac{e^t}{t} dt,$$

$\alpha = V_0/2Q > 0$. Within the renormalon picture one should search for the exact function $F(Q)$ in the form

$$F(Q) = F_B(\alpha) + Ce^{-1/\alpha} + \ldots$$

(56)

Here the first term has the same asymptotic expansion as the exact function $F$, a constant $C$ gives the leading exponentially suppressed correction and ellipsis stands for “higher twist” contributions. The power of the exponent in (54) is determined by the position of the pole in the Borel image. To find the value of $C$ one has to use purely nonperturbative method or extract it from experiment. Note that the form of first term, the value of the constant $C$, and high order corrections do depend on summation prescription while the whole sum does not by construction. The parameter $C(m, V_0)$ is quite sensitive to the variation of $m$ so it can be considered as a “probe” of nonperturbative effects what is the main issue of the renormalon calculations. However, we find (Fig. 4 that eq. (56) poorly approximates the exact function $F(Q)$ for $Q \sim m$. The reason is the same as in the case of Borel summable series. Namely, $\alpha_{as}(q)$ is a bad expansion parameter at small momenta. The only difference with previous case is that for Borel nonsummable series the running coupling $\alpha_{as}$ becomes singular at some point and in principle cannot be used for approximation of regular function $\psi(q)$. Stress once again however that it still accumulates all PT terms exactly as in the previous case. Moreover eq. (56) has no relation to the exact function $F(Q)$ because the leading non-PT term in the expansion of $F(Q)$ at large $Q$ has completely different $\alpha$ dependence

$$\sqrt{\frac{\alpha V_0}{m}} \exp \left( -2 \sqrt{\frac{V_0}{\alpha m}} \right).$$

As a consequence, the parameterization (56) is not universal i.e. one gets essentially different values of $C$ for different scattering packages.

Again, introducing a running coupling constant of the form (52) with an appropriate $\mu$ we obtain a uniform approximation of the function $F(Q)$ for all $Q$ (Fig. 4). In QCD this
prescription corresponds to the use of (probably mass dependent) RG equation for strong coupling constant with infrared regular solution [22, 27, 31].

Note that in this way as within renormalon picture we determine the scale where non-perturbative effects become crucial rather than find the exact form of $F(Q)$. However, the use of an appropriate expansion parameter allows us to achieve much higher accuracy at intermediate momenta in calculations of the integral observables like $F(Q)$.

5 Conclusion

While it is clear that sum rules should be applied with great care to cases where the physical spectrum is not known we found that there is no intrinsic criterion within the technique for detection that something goes wrong. All criteria of convergence, stability, existence of a mass gap are perfectly satisfied. Still this is not completely artificial construction but a model based on an approximation of quantum field theory.

The optimized perturbation theory shows that knowledge of the potential allows to reduce the necessary number of PT terms to a bare minimum still providing good accuracy. We would like to stress that in our model we used more than simple play with a finite piece of PT series, we chose the leading approximation differently. In QCD, however, the choice of zero order approximation is practically unique because there is no other solvable approximation but free relativistic field theory and no real optimization is therefore possible at present. In this respect a simple change of a renormalization prescription has no deep physical justification though can be quite successful phenomenologically in some particular cases.

The analysis of two possible ways of extracting information on low energy domain of a quantum mechanical model shows that approach based on optimization of the PT through introduction of a flexible expansion parameter using the freedom of choice of the scheme is more efficient for moderate energies than direct Borel resummation technique. In our model also the singularity of Borel image for attractive potential does not correspond to leading non-PT asymptotics of the exact function that is one of main reasons for using the renormalon technique in phenomenological applications of QCD. Though obtained in a toy model, these observations may serve as a ground for using a modified running of the coupling constant of QCD in the infrared domain for phenomenological applications instead (or in addition to) the standard renormalon technique.

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Tables

| α  | $E(\alpha)$ | $E_0(\alpha)$ | $E_1(\alpha)$ | $E_2(\alpha)$ | $E_3(\alpha)$ |
|----|-------------|---------------|---------------|---------------|---------------|
| 0.1 | 0.2101      | 0.2500        | 0.2074        | 0.2105        | 0.2104        |
| 0.5 | 2.426       | 6.250         | 0.9292        | 2.898         | 2.284         |
| 1.0 | 2.144       | 25.00         | -17.57        | 13.94         | -5.721        |

Table 1: The exact ground state energy $E(\alpha)$ and the result of the asymptotic expansion $\hat{E}_n(\alpha)$ up to the $n$-th order ($n = 0, 1, 2, 3$) (in units of $10^{-2}$).

| α  | $E(\alpha)$ | $E_1(\alpha, \alpha_{1}^{PMS})$ | $E_0(\alpha)$ | $E_1(\alpha, \alpha)$ |
|----|-------------|---------------------------------|---------------|------------------------|
| 0.1 | 0.2101      | 0.2104                          | 0.2430        | 0.2092                 |
| 0.5 | 2.426       | 2.424                           | 5.531         | 1.915                  |
| 1.0 | 2.144       | 2.116                           | 20.13         | -4.200                 |

Table 2: The exact ground state energy $E(\alpha)$, the result of the optimized expansion up to the first order $E_1(\alpha, \alpha_{1}^{PMS})$ and the results of the standard perturbation theory up to the $n$-th order $E_n(\alpha, \alpha)$ ($n = 0, 1$) (in units of $10^{-2}$).

| α  | $E(\alpha)$ | $E^{[1,1]}(\alpha)$ | $E^{[0,2]}(\alpha)$ | $E^{[2,1]}(\alpha)$ | $E^{[1,2]}(\alpha)$ | $E^{[0,3]}(\alpha)$ |
|----|-------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 0.1 | 0.2101      | 0.2104               | 0.1874               | 0.2104               | 0.2104               | 0.2104               |
| 0.5 | 2.426       | 2.366                | 2.340                | 2.430                | 2.424                | 2.562                |
| 1.0 | 2.144       | 0.5388               | 5.759                | 1.833                | 1.600                | 4.331                |

Table 3: The exact ground state energy $E(\alpha)$ and various $[i, j]$ Padé approximants $E^{[i,j]}(\alpha)$ with $i + j \leq 3$ (in units of $10^{-2}$).
**Figure Captions**

Fig. 1. The resonance contribution $r(M^2)$ to the Borel sum rules and the function $\hat{r}(M^2) = p(M^2) - c(M^2)$.

Fig. 2. The mass $m^2$ plotted as a function of Borel variable $M^2$ with two (a), three (b) and four (c) orders of power corrections included. The arrows mark stability interval of the Borel sum rules.

Fig. 3. Numerical results for repulsive potential $V_0 = 1$, $m = 3$.
Function $F^\mu(Q)/F(Q)$, the optimal value $\mu = 6$ (line a).
Function $F_B(Q)/F(Q)$ (line b).

Fig. 4. Numerical results for attractive potential $V_0 = 1$, $m = 3$.
Function $F^\mu(Q)/F(Q)$ for the optimal value $\mu = 3.5$ (line a).
Function $F_B(Q)/F(Q)$ (line b).
Function $(F_B(Q) + Ce^{-1/\alpha})/F(Q)$ for an optimal value $C = -0.06$ (line c).
Fig. 1
Fig. 2
Fig. 3
Fig. 4