Abstract. In this study, we examine the asymptotic behavior of solutions to nonlinear Schrödinger equations with time-dependent harmonic oscillators and prove the time-decay property of solutions in the case of a long range power type nonlinearity.

1. Introduction

Throughout this paper, we will consider nonlinear Schrödinger equations with time-dependent harmonic potentials;

$$\left\{ \begin{array}{l}
 i\partial_t u(t, x) - \left( -\Delta/2 + \sigma(t)|x|^2/2 \right) u(t, x) = \nu F_L(u(t, x))u(t, x) + \mu F_S(u(t, x))u(t, x), \\
 u(0, x) = u_0(x),
\end{array} \right.$$  

where \((t, x) \in \mathbb{R} \times \mathbb{R}^n, n \in \{1, 2, 3\}, \nu, \mu \in \mathbb{R}, F_L : \mathbb{C} \rightarrow \mathbb{R} \text{ and } F_S : \mathbb{C} \rightarrow \mathbb{R} \text{ are}\) nonlinear terms that are defined later. We assume the following assumption on the coefficient of harmonic oscillator \(\sigma(t)\);

**Assumption 1.1.** Suppose \(\sigma : \mathbb{R} \rightarrow \mathbb{R}\) and \(\sigma \in L^\infty(\mathbb{R})\), we define \(y_1(t)\) and \(y_2(t)\) as linearly independent solutions to

$$y_j''(t) + \sigma(t)y_j(t) = 0.$$  

Then \(y_1(t)\) and \(y_2(t)\) satisfy the following conditions

$$|y_2(t)| \geq |y_1(t)| \text{ as } |t| \gg 1, \text{ and } \lim_{|t| \rightarrow \infty} |y_2(t)| = \infty.$$  

Moreover, \(y_1(t), y_2(t), y_1'(t), \text{ and } y_2'(t)\) are continuous functions.

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Remark 1.2. If \( \sigma(t) = 0 \), then we have \( y_1(t) = \text{const} \) and \( y_2(t) = t \) and if \( \sigma(t) = -1 \), then we have \( y_1(t) = \sinh t \) and \( y_2(t) = \cosh t \). In the case where \( \sigma(t) \) decays as \( t^2 \sigma(t) \to k \) with \( 0 \leq k < 1/4 \), we obtain a linearly independent solution to \((2)\) such that for \( \lambda = (1 - \sqrt{1 - 4k})/2 \in [0, 1/2) \) and for some constants \( c_{1, \pm} \neq 0 \) and \( c_{2, \pm} \neq 0 \),

\[
\lim_{t \to \pm \infty} \frac{y_1(t)}{|t|^{\lambda}} = c_{1, \pm}, \quad \lim_{t \to \pm \infty} \frac{y_2(t)}{|t|^{1-\lambda}} = c_{2, \pm}
\]

hold. Models of \( \sigma(t) \) for \( k = 0 \) (i.e., \( \lambda = 0 \)) can be observed in, e.g., Naito [20] and Willett [26] and the models of \( \sigma(t) \) for \( \lambda \neq 0 \) can be observed in, e.g., Geluk–Marić–Tomić [8] (simplified models can be observed in Kawamoto [14] and Kawamoto–Yoneyama [17]).

In addition, we introduce some of the fundamental solutions to Hill’s equation, which include

\[
\zeta_j''(t) + \sigma(t) \zeta_j(t) = 0, \quad \begin{cases} 
\zeta_1(0) = 1, & \zeta_2(0) = 0, \\
\zeta_1'(0) = 0, & \zeta_2'(0) = 1.
\end{cases}
\]

Here, let \( v \) be a free solution associated with \((1)\), that is, \( v \) is a solution to

\[
\begin{align*}
\begin{cases}
i \partial_t v(t, x) + (\Delta/2 - \sigma(t)|x|^2/2) v(t, x) = 0, \\
v(0, x) = v_0 \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),
\end{cases}
\end{align*}
\]

and \( U_0(t, s) \) be a propagator of \( H_0(t) := -\Delta/2 + \sigma(t)|x|^2/2 \), that is, a family of unitary operators whose elements satisfy

\[
i \partial_t U_0(t, s) = H_0(t) U_0(t, s), \quad i \partial_s U_0(t, s) = -U_0(t, s) H_0(s), \quad U_0(s, s) = \text{Id}_{L^2(\mathbb{R}^n)}.
\]

Then, \( v(t, \cdot) = U_0(t, 0) v_0 \) holds. Based on the results obtained in [17] (see Kawamoto [16] and Korotyaev [18]), we derive the following dispersive estimate

\[
\|v(t, \cdot)\|_\infty = \|U_0(t, 0) v_0\|_\infty \leq C |\zeta_2(t)|^{-n/2} \|v_0\|_1,
\]

where \( \| \cdot \|_q, 1 \leq q \leq \infty \) denotes \( \| \cdot \|_{L^q(\mathbb{R}^n)} \). We now introduce the motivation for this paper. Let \( \phi(t, x) \) be a solution to \((1)\) with \( \sigma(t) \equiv 0, \mu = 0, \) and \( F_L(u(t, x)) = |u(t, x)|^{\rho_0-1} \) for some \( \rho_0 > 1 \), and suppose \( \phi(0, x) \) is included in some suitable function space. According to studies by Strauss [24], Barab [2], Tsutsumi–Yajima [25], and so on, there exists \( \phi_\pm \in L^2(\mathbb{R}^n) \) such that

\[
\lim_{t \to \pm \infty} \phi(t, \cdot) - e^{it\Delta/2} \phi_\pm = 0
\]

holds for \( 1 + 2/n < \rho_0 \) and fails for \( 1 < \rho_0 \leq 1 + 2/n \). Therefore, in this sense, when \( \rho_0 > 1 + 2/n \) the nonlinearity is termed short-range and when \( \rho_0 \leq 1 + 2/n \) the nonlinearity is termed long-range. Therefore, the power \( \rho_0 = 1 + 2/n \) denotes
a threshold. For the case $\rho_0 = 1 + 2/n$, the long-range scattering theory has been considered in some papers, e.g., Hayashi–Ozawa [13], Ozawa [21], Hayashi–Naumkin [12] among others. On the other hand, when we focus on the case where $t^2\sigma(t) \rightarrow k \in (0, 1/4)$ and $\zeta_2(t) = \mathcal{O}(|t|^{1-\lambda})$ for $|t| \gg 1$, it is expected that weak dispersive estimates

$$\|v(t, \cdot)\|_\infty = \|U_0(t, 0)v_0\|_\infty \leq C|\zeta_2(t)|^{-n/2} \|v_0\|_1 \leq C|t|^{-n(1-\lambda)/2} \|v_0\|_1,$$

will alter the threshold of nonlinearity from $1 + 2/n$ to $1 + 2/n(1-\lambda)$. Therefore, in this paper, we consider the manner in which $\sigma(t)$ affects the thresholds of nonlinearity.

As an introduction to the main Theorems, we state the following conditions on the $\sigma(t)$:

**Assumption 1.3.** For some $a_{1, \pm} \in \mathbb{R}$ and $a_{2, \pm} \neq 0$,

$$\lim_{t \rightarrow \pm \infty} \frac{\zeta_1(t)}{|y_2(t)|} = a_{1, \pm}, \quad \lim_{t \rightarrow \pm \infty} \frac{\zeta_2(t)}{|y_2(t)|} = a_{2, \pm}$$

hold. Moreover there exist $r_0 > 0$ and $c > 0$ such that for all $t \in (-\infty, -r_0] \cup [r_0, \infty)$, $|\zeta_2(t)| > c$ holds.

If we consider the asymptotic behavior for the short-range case ($\nu = 0, \mu \neq 0$) only, every arguments in Sect. 4 work well under the Assumption 1.3. However, when we consider long-range nonlinearities ($\nu \neq 0$), the decay condition $|\zeta_1(t)|/|\zeta_2(t)| \rightarrow 0$ as $|t| \rightarrow \infty$ acts very important role, this term appears in MDFM-decomposition (23). Indeed, by using such decay condition fully, the long-range scattering theory can be established ([12,13,21] among others). To consider the case of $\nu \neq 0$ and to imitate approaches of previous works, we additionally assume the following conditions on $y_1$ and $y_2$.

**Assumption 1.4.** For some $b_{1, \pm} \neq 0$, $b_{2, \pm} \neq 0$ and $\delta_0 > 0$

$$\lim_{t \rightarrow \pm \infty} \frac{\zeta_1(t)}{|y_1(t)|} = b_{1, \pm}, \quad \lim_{t \rightarrow \pm \infty} \frac{\zeta_2(t)}{|y_2(t)|} = b_{2, \pm}, \quad \frac{|y_1(t)|}{|y_2(t)|} \leq C|t|^{-\delta_0}$$

holds. Moreover, there exist $r_0 > 0$ and $c > 0$ such that for all $t \in (-\infty, -r_0] \cup [r_0, \infty)$, $|\zeta_2(t)| > c$ holds.

In this paper, we examine the asymptotic behavior of the solution to (1) using the approach of Hayashi–Naumkin [12]. To imitate this approach, we set for some $\gamma > 0$,

$$H^{\gamma, 0} := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \|u\|_{\gamma, 0} := \left( \int (1 + |\xi|^2)^\gamma |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2} < \infty \right\}$$

and

$$H^{0, \gamma} := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \|u\|_{0, \gamma} := \left( \int (1 + |x|^2)^\gamma |u(x)|^2 \, dx \right)^{1/2} < \infty \right\}$$

where $\hat{\cdot}$ indicates a Fourier transform. Finally, we present the following assumptions on nonlinearities:
Assumption 1.5. Consider Assumptions 1.1 and 1.3 or Assumptions 1.1 and 1.4. Let \( \tilde{F}_L, \tilde{F}_S: \mathbb{R} \to \mathbb{R} \) satisfy the following

\[
\tilde{F}_S(\phi(t)) \leq C|\phi(t)|^{\rho_S}, \quad \tilde{F}_L(\phi(t)) \leq C|\phi(t)|^{\rho_L},
\]

where \( \rho_S > 0 \) and \( \rho_L > 0 \) satisfy

\[
\sup_{|t| \geq r_0} |\xi_2(t)|^{-n\rho_S/2}t^{1+\delta_1} \leq C, \quad \sup_{|t| \geq r_0} |\xi_2(t)|^{-n\rho_L/2}t^1 \leq C,
\]

for some \( \delta_1 > 0 \). Assume that for some \( \gamma > n/2 \) and \( \phi(t) = \phi(t, \cdot), \psi(t) = \psi(t, \cdot) \) there exists constant \( C > 0 \) such that

\[
F(\phi(t)) = F(|\phi(t)|) \quad (6)
\]
\[
\|F(\phi(t))\|_\infty \leq C\tilde{F}(\|\phi(t)\|_\infty), \quad (7)
\]
\[
\|F(\phi(t))\|_{\gamma,0} \leq C\tilde{F}(\|\phi(t)\|_\infty) \|\phi(t)\|_{\gamma,0}, \quad (8)
\]
\[
\|F(\phi(t))\|_{0,\gamma} \leq C\tilde{F}(\|\phi(t)\|_\infty) \|\phi(t)\|_{0,\gamma}, \quad (9)
\]
\[
\|F(\phi(t))\|_\infty \leq C\tilde{F}(\|\phi(t)\|_\infty), \quad (10)
\]
\[
\|F(\phi(t))\|_{\gamma,0} \leq C\tilde{F}(\|\phi(t)\|_\infty), \quad (11)
\]

hold, where \( k = \infty \) or \( 2 \) and \( (F, \tilde{F}) = (F_L, \tilde{F}_L) \) or \( (F_S, \tilde{F}_S) \).

We denote \( F_S \) and \( F_L \) as short-range nonlinearity and long-range nonlinearity, respectively.

Remark 1.6. We consider the case in which \( \sigma(t) \) decays in \( t \). The power type nonlinearities \( F_S(u(t)) = |u(t)|^{2/n(1-\lambda)} + \delta_2 \) and \( F_L(u(t)) = |u(t)|^{2/n(1-\lambda)} \) with some \( \delta_2 > 0 \) that satisfies the Assumption 1.5 (see, Lemma 2.1. and 2.3. in [12]) can be considered examples. Because \( \xi_j(t) \) can be written as

\[
\xi_j(t) = c_{j,1}y_1(t) + c_{j,2}y_2(t) \quad (12)
\]

with constants \( c_{j,1}, c_{j,2} \in \mathbb{R}, (c_{j,1}, c_{j,2}) \neq (0, 0) \), Assumption 1.3 is the equivalent of assuming that \( c_{2,2} \neq 0 \) and Assumption 1.1. On the other hand, Assumption 1.4 is the equivalent of assuming that \( c_{1,2} = 0, c_{2,2} \neq 0 \) and Assumption 1.1. In the case of short-range nonlinearity, it is enough to assume Assumptions 1.1 and 1.3 to obtain a dispersive estimate. On the other hand, the Assumption 1.4 is needed to derive the dispersive estimates of long-range nonlinearity. The \( \sigma(t) \), which satisfies Assumptions 1.1 and 1.3, can be constructed with little difficulty; however, constructing the \( \sigma(t) \) that satisfies 1.1 and 1.4 can be very difficult. When \( \sigma(t) \) is non-continuous, we can construct \( \sigma(t) \) as presented in [17]. We summarize several models that satisfy our assumptions;
| $\sigma(t)$ | $kt^{-2}$ | $t^2\sigma(t) \to 0$ or $0$ | $-1$ |
|------------|------------|------------------------|----------|
| $\rho_S$   | $\rho_S > 2/(n(1 - \lambda))$ | $\rho_S > 2/n$ | $\rho_S > \max(n/2 - 1, 0)$, |
| $\rho_L$   | $\rho_L = 2/(n(1 - \lambda))$ | $\rho_L = 2/n$ | $\times$ |

where $k \in [0, 1/4]$ and $\lambda = (1 - \sqrt{1 - 4k})/2$.

Remark 1.7. If $f(u(t)) = F_S(u(t))$ or $F_L(u(t))$ satisfies $f(0) = f'(0) = 0$ and for all $z_1, z_2 \in \mathbb{C}$,

$$|f'(z_1) - f'(z_2)| \leq C \left\{ \begin{array}{ll} |z_1 - z_2| \cdot \max(|z_1|^{p-2}, |z_2|^{p-2}) & \text{if } p \geq 2, \\
|z_1 - z_2|^{p-1} & \text{if } p \leq 2,
\end{array} \right.$$  

we obtain (8) and (9) with $\tilde{f}(u(t)) = |u(t)|^{p-1}$, where $f' = \partial f / \partial z$ and $\partial f / \partial \bar{z}$, $f = \tilde{F}_S$ or $\tilde{F}_L$ according to the result of Lemma 3.4. in Ginibre–Ozawa–Velo [9]. Therefore, as for the power type nonlinearities such as $f(u(t)) = |u(t)|^{p-1}$ with some $p > \max(1, n/2)$, the Assumption 1.5 is then guaranteed with $n/2 < \gamma < p$. The log-like nonlinearity such as $f(u(t)) = \tilde{f}(u(t)) = (\log(1 + |u(t)|))^{-(p-1)}$ with $p \geq 1 + 2/n$ does not satisfy the assumptions presented in 1.5, and therefore, we must relax the Assumption 1.5 to include log-like nonlinearities such as nonlinearity suits when $\sigma(t) = -1$.

Consequently, we obtain the following dispersive estimates and asymptotics of the solution:

Theorem 1.8. Let $u_0 \in H^{\gamma, 0} \cap H^{0, \gamma}$ with $\gamma > n/2$, $\|u_0\|_{\gamma, 0} + \|u_0\|_{0, \gamma} = \varepsilon' \leq \varepsilon$, where $\varepsilon > 0$ is sufficiently small. If $\nu = 0$, then under Assumptions 1.1, 1.3, and 1.5, there exists a unique global solution to (1) such that $u \in C(\mathbb{R}; H^{\gamma, 0} \cap H^{0, \gamma})$ and

$$\|u(t)\|_\infty \leq C\varepsilon'(1 + |\zeta_2(t)|)^{-n/2} \quad (13)$$

hold, and if $\nu \neq 0$, then under Assumptions 1.1, 1.4, and 1.5, there exists a unique global solution to (1) such that $u \in C(\mathbb{R}; H^{\gamma, 0} \cap H^{0, \gamma})$ and (13) hold.

Theorem 1.9. Let $u_0 \in H^{\gamma, 0} \cap H^{0, \gamma}$ with $\gamma > n/2$, $\|u_0\|_{\gamma, 0} + \|u_0\|_{0, \gamma} = \varepsilon' \leq \varepsilon$, where $\varepsilon > 0$ is sufficiently small, and $u \in C(\mathbb{R}; H^{\gamma, 0} \cap H^{0, \gamma})$ is a global solution to (1), which is presented in Theorem 1.8. Moreover, consider the Assumptions 1.1, 1.4, and 1.5. Then there exist $W \in L^\infty \cap L^2$ and $\tilde{C}_1(\varepsilon', \nu) \geq 0$ and $\tilde{C}_2(\varepsilon', \mu) \geq 0$ with $\tilde{C}_1(\varepsilon', \nu) = 0$ iff $\nu = 0$ and $\tilde{C}_2(\varepsilon', \mu) = 0$ iff $\mu = 0$,

$$\left\| \mathcal{F}(U_0(0, t)u(t, \cdot))(t) \exp \left\{ i \int_{r_0}^t F_L(|\zeta_2(\tau)|^{-n/2}) \mathcal{F}(U_0(0, \tau)u(\tau, \cdot))d\tau \right\} - W \right\|_k$$

$$\leq C\varepsilon'^t - \delta_0 + \tilde{C}_1(\varepsilon', \nu) + C\varepsilon'^t - \delta_1 + \tilde{C}_2(\varepsilon', \mu) \quad (14)$$

holds for $t \geq r_0$, where $k = 2$ or $\infty$, $\tilde{C}_1(\varepsilon', \nu) < \alpha < \min(\gamma/2 - n/4, 1)$ and $\delta_0$ and $\delta_1$ are equivalent to those in Assumptions 1.4 and 1.5, respectively.
**Remark 1.10.** In the case where \( t^2 \sigma(t) \to 0 \), one has \( \delta_0 = 1 \) and find that (14) corresponds to the (1.4) in [12] with \( n/2 < \gamma < 1 + 2/n \) and \( F_L(u) = |u|^{2/n} \). Since the assumption \( t^2 \sigma(t) \to 0 \) includes \( \sigma(t) \equiv 0 \), our result may be a natural extension of result of [12]. On one hand, for \( \sigma(t) = kt^{-2}, k \in [0, 1/4] \) with some additional assumptions (see, [17]), one has \( |\xi_2(t)/\xi_1(t)|/|t|^{1-2\lambda} \to \text{const} \neq 0 \) as \( |t| \to \infty \) with \( \lambda = (1 - \sqrt{1 - 4k})/2 \). Then,

\[
\|u(t)\|_\infty \leq C \epsilon' (1 + |t|)^{-n(1-\lambda)/2}
\]

and

\[
\| \mathcal{F}(U_0(0, t)u(t, \cdot)) (t) \exp \left\{ iv \int_{r_0}^r F_L(|\xi_2(\tau)|^{-n/2} \mathcal{F}(U_0(0, \tau)u(\tau, \cdot)) d\tau \right\} - W \|_k \leq C \epsilon' t^{-(1-2\lambda)\alpha + \tilde{C}_1(\epsilon', \nu)} + C \epsilon' t^{-\delta_1 + \tilde{C}_2(\epsilon', \mu)}
\]

hold for \( F_L(u) = |u|^{2/(n(1-\lambda))} \), \( n/2 < \gamma < 1 + 2/(n(1 - \lambda)) \) and \( \alpha < \min(1/2 + 1/(n(1 - \lambda)) - n/4, 1) \). On the other hand, for \( \sigma(t) = -1, \xi_1(t) = \cosh t \) and \( \xi_2(t) = \sinh t \) hold, and then letting \( \nu = 0, F_3(u) = |u|^\rho \) with \( \rho > \max(0, n/2 - 1) \) and \( n/2 < \gamma < 1 + \rho \), we find

\[
\|u(t)\|_\infty \leq C \epsilon' (1 + |\sinh t|)^{-n/2}.
\]

In the case where \( \sigma(t) \) decays in \( t \), the second Theorem implies the threshold of short-range and long-range nonlinearities to be \( 2/n(1 - \lambda) \). For \( \sigma(t) = -1 \), if we put \( \nu = 0 \) and \( F_3(u(t)) = |u(t)|^{\delta_2} \) with some \( \delta_2 > 0 \), then we can obtain the Theorems with \( \xi_2(t) = \sinh(t) \). However, the short-range nonlinearity will ideally be \( (\log(1 + |u|^{-1}))^{-2/n - \delta_3} \) with some \( \delta_3 > 0 \) because \( F_3(|\xi_2(t)|^{-1} \leq c|t|^{-n/2 + \delta_1} \) holds for such nonlinearity. To justify this argument, we must prove (8) in the Assumption 1.5 for this nonlinearity.

In the above Theorems 1.8 and 1.9, we reveal the asymptotic behavior of the solutions to (1) for both cases of long-range and short-range nonlinearities with generalized conditions of the coefficients of harmonic potential. Such results, particularly in the case where \( \sigma(t) \) decays in \( t \), are yet to be observed, and we are of the impression that this result is not mathematically and physically interesting. Similarly, Carles [4] and Carles–Silva [5] considered nonlinear Schrödinger equations (NLS) with harmonic potential and time-dependent potentials, in these papers the global existence of solutions to (1) was proven under the more generalized potential including our models, and besides that they found the asymptotics of solutions of (1) for the case where \( \sigma(t) \) satisfying \( t^2|\sigma(t)| < 1/4 \) as \( t \to \infty \). Moreover for the case where \( \sigma(t) = -1, [3] \) also considered the global well-posedness of the solution, scattering theory, and so on. On the other hand, the results of [3–5] have not dealt with \( L^\infty \) asymptotics, \( L^2 \) asymptotics for the case where \( 0 < \lim t^2|\sigma(t)| \leq 1/4 \) and \( H^{\gamma,0} \cap H^{0,\gamma} \)-wellposedness with \( \gamma > n/2 \) ( \( H^{1,0} \cap H^{0,1} \)-wellposed can be found in [5] even for the case \( 0 < \lim t^2|\sigma(t)| \leq 1/4 \). We believe that this result is new and mathematically interesting. As a time-independent
case, Hani–Thomann [11] proved the asymptotics in the case when $\sigma(t) \equiv \sigma \neq 0$.

Our approach is applicable to different types of nonlinearities (see, e.g., Dodson [7], Masaki–Miyazaki–Uriya [19], Shimomura [23] and so on). Moreover, it has been determined that the approach used in [12] works well for the study on the lifespan of solutions to NLS (see, e.g., Sagawa–Sunagawa [22]), and hence our approach maybe applicable to such studies.

In Sect. 3, we found the $H^{\gamma,0} \cap H^{0,\gamma}$-wellposedness (local-in-time) of solution to (1). The key approaches have been established by [4,5], in particular the pseudo-energy $\alpha(t)$ which acts very important roles were found by §5 of [4]. In Sect. 2, we summarize these and introduce some decomposition theorem of propagators. By combing their approaches and choosing suitable decompositions of propagators, we are finally able to find the $H^{\gamma,0} \cap H^{0,\gamma}$-wellposedness, which is indispensable in order to imitate the approach of [12].

2. Preliminaries and auxiliary results

In this section, we shall introduce some important lemma that appears in the proof. In this section, we always assume Assumptions 1.1, 1.3, and 1.5. Throughout this paper, we use a constant $C > 0$, which is always positive and is independent of any other parameters under consideration. We may use the notations

$$u(t, \cdot) = u(t), \text{ (resp. } v(t, \cdot) = v(t), \text{ } w(t, \cdot) = w(t), \text{ and so on)}$$

and

$$G(u(t)) = vF_L(u(t))u(t) + \mu F_S(u(t))u(t).$$

We use the notation $p := -i\nabla, x^2 = |x|^2$ and $p^2 = |p|^2 = -\Delta$. For $t$-depend functions $A_1(t)$ and $A_2(t)$, the operator $|A_1(t)p + A_2(t)x|$ is defined as

$$|A_1(t)p + A_2(t)x| = \left|e^{-iA_2(t)x^2/(2A_1(t))}(A_1(t)p)e^{iA_2(t)x^2/(2A_1(t))}\right|$$

$$= e^{-iA_2(t)x^2/(2A_1(t))}|A_1(t)p|e^{iA_2(t)x^2/(2A_1(t))}.$$

To simplify the proof, we state the following lemma;

**Lemma 2.1.** Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$ and $\gamma > 0$. Then for all $\phi \in L^2(\mathbb{R}^n)$, there exists $C > 0$ such that

$$\left\|\left(a_1 p + a_2 x^2 + a_3 p^2 + a_4 x^2\right)^\gamma (p^2 + x^2 + 1)^{-\gamma} \phi\right\|_2 \leq C \|\phi\|_2$$

and

$$\left\|(p^2 + x^2 + 1)^{\gamma}(\langle p \rangle)^{2\gamma} + \langle x \rangle^{2\gamma})^{-1}\phi\right\|_2 \leq C \|\phi\|_2$$

hold.
This Lemma can be easily proved using the positiveness of harmonic oscillator $p^2 + x^2 \geq 1$ and Calderón–Vaillancourt Theorem. By noting $p = -i\nabla_x$ and $x = i\nabla_p$, for constants $A_3$ and $A_4$,

$$
\begin{align*}
\left( \frac{x}{p} \right) e^{-iA_3x^2} &= e^{-iA_3x^2} \left( \frac{x}{p - 2A_3x} \right), \\
\left( \frac{x}{p} \right) e^{-iA_4p^2} &= e^{-iA_4p^2} \left( \frac{x + 2A_4p}{p} \right).
\end{align*}
$$

(15)

Moreover, we denote $(x \cdot p + p \cdot x)/2$ by $A$. In the case of constant $A_5$, it satisfies

$$
\left( \frac{x}{p} \right) e^{-iA_5A} = e^{-iA_5A} \left( \frac{e^{A_5x}}{e^{-A_5p}} \right),
$$

(16)

and the proof can be found in, e.g., §2 of [17] and §2.3. of [14]. In addition, we occasionally use the notation $\langle \cdot \rangle = (1 + \cdot^2)^{1/2}$. 

2.1. Auxiliary results

For the sake of using the approach presented in [12], we must consider the Lemma 2.3 in [12] and the operator $J$ in the case when harmonic potential exists. To obtain these, we employ the formula in [18];

**Lemma 2.2.** The propagator $U_0(t, 0)$ can be decomposed into the following form

$$
U_0(t, 0) = e^{i\zeta_1(t)x^2/(2\zeta_1(t))} e^{i(\log |\zeta_1(t)|)A} e^{-i\zeta_2(t)p^2/(2\zeta_1(t))} S^v(t),
$$

(17)

where $(Sf)(x) = e^{-in\pi/2} f(-x)$ for $f \in L^2(\mathbb{R}^n)$ and $v(t)$ is the number of zeros in the elements of $\{ \tau \in [0, t] \mid \zeta_1(\tau) = 0 \}$ for $t \geq 0$ or $\{ \tau \in [t, 0] \mid \zeta_1(\tau) = 0 \}$ for $t \leq 0$. (see [17, 18]).

Based on the operator calculation (see, e.g., below (2.2) of [17]), we obtain

$$
\left( e^{-i(\log |\zeta_1(t)|)A} S^v \right)(x) = |\zeta_1(t)|^{-n/2} v(|\zeta_1(t)|^{-1} x).
$$

Using this lemma, the following proposition holds;

**Proposition 2.3.** Define $f(u(t))$ and $\tilde{f}(u(t))$ as either $F_S(u(t))$ and $\tilde{F}_S(u(t))$ or $F_L(u(t))$ and $\tilde{F}_L(u(t))$, respectively. Let $\gamma > 0$, then

$$
\|U_0(0, t)| f(u(t)) u(t)\|_{0, \gamma} \leq C \tilde{f}(\|u(t)\|_{\infty}) \|U_0(0, t) u(t)\|_{0, \gamma}
$$

(18)

holds for all $t \in \{ t \in \mathbb{R} \mid \zeta_1(t) \neq 0 \}$.

**Proof.** By (17), we have

$$
U_0(0, t) = \left( S^v(t) \right)^{-1} e^{i\zeta_2(t)p^2/(2\zeta_1(t))} e^{i(\log |\zeta_1(t)|)A} e^{-i\zeta_1(t)p^2/(2\zeta_1(t))}
$$

$$
= e^{i\zeta_2(t)p^2/(2\zeta_1(t))} e^{i(\log |\zeta_1(t)|)A} e^{-i\zeta_1(t)p^2/(2\zeta_1(t))} \left( S^v(t) \right)^{-1}.
$$
where $S$ commutes with $p^2$, $A$, and $x^2$. We define $v(t, x) = e^{-i\xi_1(t)x^2/(2\xi_1(t))}(S^u(t))^{-1}u(t, x)$ and $w(t, x) := v(t, |\xi_1(t)|x)$. Then using Lemma 2.3 of [12] and Assumption 1.5,

$$\|x\|^2 U_0(0, t)f(u(t))u(t)\|_2 = \|\xi_1(t)\|^2/2 \left\| x\|^2 e^{i\xi_2(t)p^2/(2\xi_1(t))}f(w(t))w(t)\right\|_2 \leq C\|\xi_1(t)\|^2/2 \left\| f(w(t))\right\|_\infty \left\| x\|^2 e^{i\xi_2(t)p^2/(2\xi_1(t))}w(t)\right\|_2 \leq C\|\xi_1(t)\|^2/2 \left\| f(w(t))\right\|_\infty \left\| x\|^2 e^{i\xi_2(t)p^2/(2\xi_1(t))}w(t)\right\|_2.$$ By using $\|\xi_1(t)\|^2/2 w(t) = e^{i(\log |\xi_1(t)|)A} e^{-i\xi_1(t)x^2/(2\xi_1(t))}(S^u(t))^{-1}u(t)$ and $\|w(t)\|_\infty = \|u(t)\|_\infty$, we have (18).

2.2. Pseudo-energy

Using Proposition 2.3, we obtain the estimate for $\|\cdot\|_{0, \gamma}$, which played a very important role in [12]. However, it is proven only on the region $t \in \{ t \in \mathbb{R} \mid \xi_1(t) \neq 0 \}$ and hence we now prove the same statement of Proposition 2.3 on $\mathbb{R} \setminus \{ t \in \mathbb{R} \mid \xi_1(t) \neq 0 \}$. We then introduce $\alpha(t)$ here and notice that we can observe that this $\alpha(t)$ exhibits this good relation, see also §5 of [4];

**Lemma 2.4.** Define the operator $\alpha(t)$ as

$$\alpha(t) := (\xi_2(t)p - \xi_2'(t)x)^2.$$ Then for all $(t, s) \in \mathbb{R}^2$ and $\gamma > 0$,

$$\langle \alpha(t)\rangle^\gamma U_0(t, s) = U_0(t, s) \langle \alpha(s)\rangle^\gamma$$

holds, where $\langle \cdot \rangle := (1 + \cdot^2)^{1/2}$. In particular, by remarking $\xi_2'(0) = 1$ and $\xi_2(0) = 0$,

$$\langle x\rangle^\gamma U_0(0, t) = \left(\xi_2(0)p - \xi_2'(0)x\right)^\gamma U_0(0, t) = U_0(0, t) \langle \alpha(t)\rangle^\gamma.$$ 

**Proof.** We first state that $S$ commutes with $p^2$, $A$, $x^2$, and $\alpha(t)$. We then have

$$\alpha(t)U_0(t, s) = (S^u(t))(S^v(t))^{-1}\alpha(t)\tilde{U}_0(0, t)\tilde{U}_0(0, s),$$

where $\tilde{U}_0(t, s) = (S^v(s))(S^v(t))^{-1}U_0(t, s)$. Now we prove $\alpha(t)\tilde{U}_0(t, 0) = \tilde{U}_0(t, 0)x^2$. Indeed, by noting (15), (16), and that

$$\xi_1(t)\xi_2'(t) - \xi_1'(t)\xi_2(t) = 1,$$

we have

$$\alpha(t)\tilde{U}_0(0, t) = e^{i\xi_1(t)x^2/(2\xi_1(t))}\left(\xi_2(t)p - \frac{1}{\xi_1(t)}x\right)^2 e^{-iA\log |\xi_1(t)|} e^{-i\xi_2(t)p^2/(2\xi_1(t))}$$

$$= e^{i\xi_1(t)x^2/(2\xi_1(t))} e^{-iA\log |\xi_1(t)|} \left(\frac{\xi_2(t)}{\xi_1(t)}p - x\right)^2 e^{-i\xi_2(t)p^2/(2\xi_1(t))}$$

$$= \tilde{U}_0(0, t)(-x)^2.$$
Next, we prove that \( x^2 \tilde{U}_0(0,s) = \tilde{U}_0(0,s) \alpha(s) \). Indeed,

\[
x^2 \tilde{U}_0(0,s) = e^{i \xi_2(s) p^2/(2 \xi_1(s))} \left( x - \frac{\xi_2(s)}{\xi_1(s)} p \right)^2 \left( \frac{e^i \log \xi_1(s) A}{e^{-i \xi_1(s) x^2/(2 \xi_1(s))}} \right)^2 e^{i \log \xi_1(s) A} \left( x - \frac{\xi_2(s)}{\xi_1(s)} p \right)^2 e^{-i \xi_1(s) x^2/(2 \xi_1(s))} = \tilde{U}_0(0,s) \alpha(s),
\]

which proves

\[
\alpha(t) \tilde{U}_0(0,t) = \alpha(t) \tilde{U}_0(0,0) \tilde{U}_0(0,s) = \tilde{U}_0(t,s) \alpha(s)
\]

By using

\[
\langle \alpha(t) \rangle^\gamma \tilde{U}_0(t,s) = \left( 1 + \tilde{U}_0(t,s) \alpha(s)^2 \tilde{U}_0(s,t) \right)^{\gamma/2} \tilde{U}_0(t,s) = \tilde{U}_0(t,s) \langle \alpha(s) \rangle^\gamma,
\]

we have Lemma.

\[\square\]

Remark 2.5. For to simplify the proof, we use the Korotyaev’s decomposition formula in the proof of this lemma. However, this lemma can be proven without using Korotyaev’s decomposition formula but with using commutator calculation.

In this paper, we present \( \alpha(t) \) pseudo-energy in \( t \). We then obtain the following important estimate;

**Proposition 2.6.** We define \( f(u(t)) \) and \( \tilde{f}(u(t)) \) as either \( \tilde{F}_S(u(t)) \) and \( \tilde{F}_S(u(t)) \) or \( \tilde{F}_L(u(t)) \) and \( \tilde{F}_L(u(t)) \), respectively. Let \( \gamma > 0 \), then for all \( t \in [t \in \mathbb{R} | \xi_2(t) \neq 0] \),

\[
\left\| \langle \alpha(t) \rangle^{\gamma/2} f(u(t))u(t) \right\|_2 \leq C \tilde{f}(\|u(t)\|_\infty) \left\| \langle \alpha(t) \rangle^{\gamma/2} u(t) \right\|_2 \tag{19}
\]

and

\[
\| U_0(0,t) f(u(t))u(t) \|_{0,\gamma} \leq C \tilde{f}(\|u(t)\|_\infty) \| U_0(0,t)u(t) \|_{0,\gamma} \tag{20}
\]

hold.

**Proof.** Because \( \langle x \rangle^2 U_0(0,t) = U_0(0,t) \langle \alpha(t) \rangle \) and \( U_0(0,t) \) is the unitary operator on \( L^2(\mathbb{R}^n) \), (21) is proven as a sub-consequent of (19), and therefore, we only prove (19). Using

\[
|\alpha(t)|^{\gamma/2} = |\xi_2(t)|^\gamma e^{i \xi_2(t) x^2/(2 \xi_2(t))} |p|^{\gamma} e^{-i \xi_2(t) x^2/(2 \xi_2(t))}
\]

and

\[
e^{-i \xi_2(t) x^2/(2 \xi_2(t))} f(u(t))u(t) = f \left( e^{-i \xi_2(t) x^2/(2 \xi_2(t))} u(t) \right) e^{-i \xi_2(t) x^2/(2 \xi_2(t))} u(t),
\]

we have

\[
\left\| \langle \alpha(t) \rangle^{\gamma/2} f(u(t))u(t) \right\|_2 \leq C \tilde{f}(\|u(t)\|_\infty) \left\| \langle \alpha(t) \rangle^{\gamma/2} u(t) \right\|_2.
\]
for $l(t) = e^{-i\xi_2(t)\frac{t^2}{2}} u(t)$ we have
$$\|a(t)|^{\gamma/2} f(u(t))u(t)\|_2 = |\xi_2(t)|^{\gamma} \|p|^{\gamma} f(l(t))l(t)\|_2.$$ Then by Assumption 1.5, we have
$$\|a(t)|^{\gamma/2} f(u(t))u(t)\|_2 \leq C|\xi_2(t)|^{\gamma} f_\gamma(\|l(t)\|_\infty) \|p|^{\gamma} l(t)\|_2$$
$$= C f_\gamma(\|u(t)\|_\infty) \|a(t)|^{\gamma/2} u(t)\|_2.$$ \hfill \Box

**Proposition 2.7.** We define $f(u(t))$ and $\tilde{f}(u(t))$ as either $F_S(u(t))$ and $\tilde{F}_S(u(t))$ or $F_L(u(t))$ and $\tilde{F}_L(u(t))$, respectively. Let $\gamma > 0$. Then for all $t \in \mathbb{R}$,
$$\|U_0(0, t) f(u(t))u(t)\|_{0, \gamma} \leq C \tilde{f}(\|u(t)\|_\infty) \|U_0(0, t)u(t)\|_{0, \gamma} \tag{21}$$
holds.

**Proof.** It is enough to prove that $\{t \in \mathbb{R} \mid \xi_1(t) = 0\} \cap \{t \in \mathbb{R} \mid \xi_2(t) = 0\} = \emptyset$ but this clearly holds from $\xi_1(t)\xi_2'(t) - \xi_1'(t)\xi_2(t) = 1$ for all $t \in \mathbb{R}$ holds. \hfill \Box

2.3. MDFM decomposition

In this subsection, we introduce the so called MDFM decomposition of $U_0(t, 0)$. A non-singular type decomposition (MDMDFM decomposition) was presented in [18], and this decomposition was modified by Adachi–Kawamoto [1], and Kawamoto [16]. However, to imitate the approach of [12], we must obtain the MDFM decomposition. To consider this issue, the following lemma, which was obtained by Kawamoto [15], is very useful;

**Lemma 2.8.** Let $a(t)$, $b(t)$ and $c(t)$ be
$$a(t) = \frac{1 - \xi_1(t)}{2\xi_2(t)}, \quad b(t) = \frac{\xi_2(t)}{2}, \quad c(t) = \frac{1 - \xi_1(t)}{2\xi_2(t)}.$$ Then the following decomposition of the propagator $U_0(t, 0)$ holds;
$$U_0(t, 0) = e^{-ia(t)x^2} e^{-ib(t)p^2} e^{-ic(t)x^2}. \tag{22}$$

**Remark 2.9.** In [15], only the case where $\sigma(t) \geq 0$ was considered. However, every argument that proves (22) works even if $\sigma(t)$ is negative.

Using this lemma, we obtain the MDFM-decomposition;

**Theorem 2.10.** For $\phi \in \mathcal{S}(\mathbb{R}^n)$, let us define
$$(\mathcal{M}(\tau) \phi)(x) = e^{ix^2/(2\tau)} \phi(x), \quad (D(\tau) \phi)(x) = \frac{1}{(i\tau)^{n/2}} \phi(x/\tau).$$
Then the following MDFM decomposition holds;
$$U_0(t, 0) = \mathcal{M}\left(\frac{\xi_2(t)}{\xi_2'(t)}\right) D(\xi_2(t)) \mathcal{F} \mathcal{M}\left(\frac{\xi_2(t)}{\xi_1(t)}\right) \tag{23}$$
Proof. For $\phi \in \mathcal{S}(\mathbb{R}^n)$,

\[
(U_0(t, 0)\phi)(x) = e^{-ia(t)x^2/(4\pi ib(t)))^{n/2}} \int e^{ix \cdot y/(4\pi ib(t)))} e^{-ic(t)y^2} \phi(y)dy
\]

\[
= e^{-i(a(t)-1/(4b(t)))x^2/2} D(2b(t)) \mathcal{F}[\psi](x),
\]

where $\psi(y) = e^{i(1/(4b(t)) - c(t))y^2} \phi(y)$. Together with

\[
a(t) - \frac{1}{4b(t)} = -\frac{\zeta_2'(t)}{2\zeta_2(t)}, \quad \frac{1}{4b(t)} - c(t) = \frac{\zeta_1(t)}{2\zeta_2(t)},
\]

we can obtain (23). \hfill \Box

Using this lemma, we can prove the $\| \cdot \|_\infty$ decay estimate;

**Lemma 2.11.** Let $u(t, x)$ be a smooth function and $|t| \geq r_0$. Then under Assumptions 1.1 and 1.3,

\[
\|u(t)\|_\infty \leq C |\zeta_2(t)|^{-n/2} \| \mathcal{F} U_0(0, t)u(t)\|_\infty + C |\zeta_2(t)|^{-n/2} \left| \frac{\zeta_1(t)}{\zeta_2(t)} \right|^\alpha \|U_0(0, t)u(t)\|_{0, \gamma}
\]

(24)

holds for $|t| \geq r_0$, where $\alpha \in (0, 1)$ and $\gamma > n/2 + 2\alpha$.

**Proof.** The proof can be obtained by imitating the approach used in Lemma 2.2. of [12]. The identity $u(t) = U_0(t, t)u(t)$ and (22) yields

\[
u(t) = e^{-ia(t)x^2/(4\pi ib(t)))^{n/2}} \int e^{ix \cdot y/(4\pi ib(t)))} e^{-ic(t)y^2} (U_0(0, t)u(t))(y)dy
\]

\[
= e^{i(\zeta_2'(t)/(2\zeta_2(t)))x^2/(4\pi ib(t)))^{n/2}} \int e^{-ix \cdot y/(2\pi b(t)))} \left\{ e^{i(\zeta_1(t)/(2\zeta_2(t)))y^2} - 1 + 1 \right\} (U_0(0, t)u(t))(y)dy
\]

\[
= e^{i(\zeta_2'(t)/(2\zeta_2(t)))x^2/(2\pi b(t)))^{n/2}} \mathcal{F}[U_0(0, t)u(t)](t, x/\zeta_2(t)) + R(t, x)
\]

with

\[
R(t, x) = e^{i(\zeta_2'(t)/(2\zeta_2(t)))x^2/(4\pi ib(t)))^{n/2}} \int e^{-ix \cdot y/(2\pi b(t)))} \left\{ e^{i(\zeta_1(t)/(2\zeta_2(t)))y^2} - 1 \right\} (U_0(0, t)u(t))(y)dy.
\]

Here, for any $0 < \alpha < 1$,

\[
|e^{i(\zeta_1(t)/(2\zeta_2(t)))y^2} - 1| \leq 2 \left| \sin \left( \frac{\zeta_1(t)}{2\zeta_2(t)} \right) \right| \leq C \left| \frac{\zeta_1(t)}{\zeta_2(t)} \right|^{\alpha} |y|^{2\alpha}
\]

holds. Therefore, we get

\[
|e^{i(\zeta_1(t)/(2\zeta_2(t)))y^2} - 1| \leq C \left| \frac{\zeta_1(t)}{\zeta_2(t)} \right|^{\alpha} |y|^{2\alpha}.
\]
Hence, $R(t, x)$ can be estimated as
\[
\| R(t) \|_{\infty} \leq C |\xi_2(t)|^{-n/2} \left| \frac{\xi_1(t)}{\xi_2(t)} \right|^\alpha \| \gamma_2 u_0(0, t) u(t, x) \|_1 \\
\leq C |\xi_2(t)|^{-n/2} \left| \frac{\xi_1(t)}{\xi_2(t)} \right|^\alpha \| u_0(0, t) u(t) \|_{0, \gamma}
\]
for $|t| \geq r_0$ and $\gamma > n/2 + 2\alpha$.

2.4. MDMDFM decomposition

In this subsection, we present the MDMDFM decomposition, which was proposed in [18] and modified in [1] (see §7) and [16] (see, Lemma 2.2. of [16]). This decomposition is used to prove local well-posedness. For the proof of local well-posedness, we employ the Sobolev inequality, that is,
\[
\| u(t) \|_{\infty} \leq C \| u(t) \|_{\gamma, 0}.
\]
However, it is difficult to obtain an estimate for $\| \cdot \|_{H^\gamma, \alpha}$, which also played a very important role in [12]. The reason for this difficulty is because if we calculate $pU_0(t, 0)$, it satisfies
\[
pU_0(t, 0) = U_0(t, 0) \left( \frac{1 + \frac{\xi_1'}{\xi_1} p + \frac{\xi_1'}{\xi_1} x}{\xi_1} \right) = U_0(t, 0) \left( \xi_2' p + \xi_2' x \right)
\]
Therefore, the term associated with $x$ appears again. This problem arises because of the non-commutativity of $p$ and the propagator. Hence, it difficult to include the norm $\| \cdot \|_{\gamma, 0}$ in the function space on which the principle of contraction mapping can be applied (see, (33)). A simple idea for overcoming this difficulty is that
\[
\| u(t) \|_{\infty} = \| e^{-i\xi_2(t)x^2/(2\xi_2(t))} u(t) \|_{\infty} \leq C \| e^{-i\xi_2(t)x^2/(2\xi_2(t))} u(t) \|_{\gamma, 0}
\]
\[
= |\xi_2(t)|^{-\gamma/2} \left\| \alpha(t) \right\|_{\gamma/2}^\gamma u(t) \right\|_2.
\]
(25)
The same function space in (33) of Sect. 3 works well. However, $\xi_2(t)$ includes some zero points on $t \in [-r_0, r_0]$ and therefore (25) fails on these points. In particular, we never remove the condition $\xi_2(0) = 0$, and therefore it is difficult to prove well-posedness near $t = 0$. We believe that this difficulty occurs because the decomposition (17) and (22) exhibit singularities in $\xi_2(t) = 0$. On the other hand, the decomposition of $U_0(t, 0)$ without singularities was also proven in [18] and [1], and therefore we use such a decomposition to prove local well-posedness.

We define $a_1(t)$ and $a_2(t)$ as
\[
a_1(t) = \frac{(y_1(t)y_2(t) - y_1(t)y_2'(t))}{(y_1(t)^2 + y_2(t)^2)}, \quad a_1(0) = 1,
\]
(26)
\[
a_2(t) = \frac{-(y_1(t)y_1(t) + y_2(t)y_2'(t))}{(y_1(t)^2 + y_2(t)^2)}, \quad a_2(0) = 0,
\]
(27)
where $y_1(t)$ and $y_2(t)$ are the solutions to (2). We then have the following lemma;
Lemma 2.12. Let \(a_1(t)\) and \(a_2(t)\) be defined as in (26) and (27), respectively. The propagator for \(U_0(t, 0)\) then exhibits the following factorization;

\[
U_0(t, 0) = M \left( \frac{-1}{a_2(t)} \right) (i)^{n/2} \mathcal{D} \left( \frac{1}{\sqrt{a_1(t)}} \right) e^{-i \int_0^t a_1(\tau) d\tau (p^2 + x^2)/2}.
\]

A merit of this decomposition is that each component of the decomposition exhibits no singular point for any \(t \in [-T, T]\), \(T \geq r_0\);

Lemma 2.13. Let \(T \geq r_0\) be a large constant. Then for all \(t \in [-T, T]\), there exists a constant \(C_T > 0\) such that

\[
|a_1(t)| \leq C_T, \quad |a_1(t)|^{-1} \leq C_T \quad |a_2(t)| \leq C_T.
\]

The proof can be observed in Lemma 2.2. of [16] and therefore we omit the proof.

Using Lemmas 2.12 and 2.13, we prove the local well-posedness.

3. Proof of local well-posedness

In this section, we shall prove the local well-posedness of the solution to (1) in the \(L^\infty\) sense. In this section, we consider the Assumptions 1.1, 1.3, and 1.5. Let \(T \geq r_0\) be a large number and prove the local well-posedness on \(t \in [-T, T]\). For simplicity, we only consider the case where \(t \in [0, T]\) because the case where \(t \in [-T, 0]\) can be proven in the same way. \(\tilde{C}_T\) is a finite positive constant and depends only on \(T\). To prove this, we introduce the \(L_T\) norm and function space \(L_{T, M}\) as follows

\[
\|\phi\|_{L_T} = \sup_{t \in [0, T]} \|(p^2 + x^2 + 1)^{\nu/2} \phi(t)\|_2 + \sup_{t \in [0, T]} \|\phi(t)\|_\infty,
\]

and for some \(M > 0\),

\[
L_{T, M} := \left\{ \phi \in C([0, T]; \mathcal{S}') : \|\phi\|_{L_T} \leq M \right\}.
\]

Theorem 3.1. Let \(T\) be an enough large so that \(T \geq r_0\) and fixed. Assume \(\|u_0\|_{\mathcal{F}, 0} + \|u_0\|_{0, \nu} = \varepsilon' \leq \varepsilon\) for some sufficiently small \(\varepsilon > 0\), where \(\gamma > 0\). Then, there exists an \(\varepsilon = \varepsilon(T)\), a finite positive constant \(C(T) > 0\), which depends only on \(T\), and a unique solution to (1) such that

\[
\|u\|_{L_T} \leq C(T)\varepsilon. \tag{28}
\]

Moreover for all \(0 < \varepsilon' < \varepsilon\),

\[
\|u\|_{L_T} \leq C(T)\varepsilon'. \tag{29}
\]
Proof. First, using the commutator calculation, we have
\[
D \left( \frac{1}{\sqrt{a_1(t)}} \right)^{-1} x D \left( \frac{1}{\sqrt{a_1(t)}} \right) \frac{x}{\sqrt{a_1(t)}},
\]
and therefore, for \( \phi \in \mathcal{H}(\mathbb{R}^n) \) and \( t, s \in [0, T] \), we obtain
\[
\| (p^2 + x^2 + 1)^{\gamma/2} U_0(t, s) \phi \|_2
= \left\| \left( p^2 + x^2 + 1 \right)^{\gamma/2} \mathcal{M} \left( \frac{-1}{a_2(t)} \right) \left( \frac{1}{\sqrt{a_1(t)}} \right)^{1/2} D \left( \frac{1}{\sqrt{a_1(t)}} \right)^{-1} \left( -i \right)^{n/2} \mathcal{M} \left( \frac{1}{a_2(s)} \right) \phi \right\|_2
= \left\| \left( p^2 - a_2(t)x^2 \right)^{1/2} \mathcal{M} \left( \frac{-1}{a_2(t)} \right) \left( \frac{1}{\sqrt{a_1(t)}} \right)^{1/2} D \left( \frac{1}{\sqrt{a_1(t)}} \right)^{-1} \left( -i \right)^{n/2} \mathcal{M} \left( \frac{1}{a_2(s)} \right) \phi \right\|_2
= \left\| \left( \sqrt{a_1(t)} p - \frac{a_2(t)}{\sqrt{a_1(t)}} x \right)^2 + \frac{x^2}{a_1(t)} + 1 \right\|^{\gamma/2} \left( p^2 + x^2 + 1 \right)^{-\gamma/2} \left( p^2 + x^2 + 1 \right)^{\gamma/2} \mathcal{M} \left( \frac{1}{a_2(s)} \right) \phi \right\|_2
= \left\| \right\|^{\gamma/2} \left( p^2 + x^2 + 1 \right)^{\gamma/2} \mathcal{M} \left( \frac{1}{a_2(s)} \right) \phi \right\|_2.
\]
By Lemmas 2.1 and Lemma 2.13, the last term of the above equation is smaller than
\[
\tilde{C}_T \left\| \right\|^{\gamma/2} \left( p^2 + x^2 + 1 \right)^{\gamma/2} \mathcal{M} \left( \frac{1}{a_2(s)} \right) \phi \right\|_2.
\]
where \( p^2 + x^2 + 1 \) commutes with \( e^{-i \int_0^t a_1(\tau) d\tau (p^2 + x^2)^{1/2}} D \left( \frac{1}{\sqrt{a_1(t)}} \right)^{-1} \left( -i \right)^{n/2} \mathcal{M} \left( \frac{1}{a_2(s)} \right) \phi \right\)_2, and \( e^{-i \int_0^t a_1(\tau) d\tau (p^2 + x^2)^{1/2}} \) is unitary on \( L^2(\mathbb{R}^n) \). A similar calculation as above yields the following
\[
\| (p^2 + x^2 + 1)^{\gamma/2} U_0(t, s) \phi \|_2 \leq \tilde{C}_T \left\| \right\|^{\gamma/2} \left( p^2 + x^2 + 1 \right)^{\gamma/2} \phi \right\|_2.
\]
Now we prove this theorem by employing the contraction mapping principle. Let \( \Xi \) as
\[
(\Xi(u))(t) := U_0(t, 0)u_0 - i \int_0^t U_0(t, s)(\nu F_L(u(s))u(s) + \mu F_S(u(s))u(s))ds
\]
Then it is enough to show that
\[
(\Xi(u))(t) = U_0(t, 0)u_0
\]
(I). \( \Xi \) maps \( L_{T,M} \) to \( L_{T,M} \).

(II). \( \Xi \) becomes contraction map, i.e., \( \Xi \) satisfies

\[
\exists \alpha \in [0, 1) \text{ s.t. } \forall u, v \in L_{T,M}; \quad \| \Xi(u) - \Xi(v) \|_{L_T} \leq \alpha \| u - v \|_{L_T}.
\]

(I). First, we assume that \( u \in L_{T,M} \), i.e., for all \( t \in [0, T] \)

\[
\| (p^2 + x^2 + 1)^{\gamma/2} u(t) \|_2 \leq M, \quad \| u(t) \|_\infty \leq M.
\]

and estimate

\[
\| (p^2 + x^2 + 1)^{\gamma/2} \Xi[u](t) \|_2.
\]

By the definition of \( \Xi \) we estimate

\[
\| (p^2 + x^2 + 1)^{\gamma/2} \Xi[u](t) \|_2 \\
\leq \| (p^2 + x^2 + 1)^{\gamma/2} U_0(t, 0) u_0 \|_2 + \int_0^t \| (p^2 + x^2 + 1)^{\gamma/2} U_0(t, s) G(u(s)) \|_2 ds \\
\leq \bar{C}_T \| (p^2 + x^2 + 1)^{\gamma/2} u_0 \|_2 + \bar{C}_T \int_0^t \| (p^2 + x^2 + 1)^{\gamma/2} G(u(s)) \|_2 ds,
\]

where \( G(u(s)) = v F_L(u(s)) u(s) + \mu F_S(u(s)) u(s) \). By Assumption 1.5 and Lemma 2.1, we obtain

\[
\| (p^2 + x^2 + 1)^{\gamma/2} F_L(s) u(s) \|_2 \\
\leq C(\| (p^2 + x^2 + 1)^{\gamma/2} F_L(u(s)) u(s) \|_2 + \| x \|^{\gamma/2} F_L(u(s)) u(s) \|_2) \\
\leq C(\bar{F}_L(\| u(s) \|_\infty) \| (p^2 + x^2 + 1)^{\gamma/2} u(s) \|_2 + \bar{F}_L(\| u(s) \|_\infty) \| x \|^{\gamma/2} u(s) \|_2) \\
\leq C \bar{F}_L(\| u(s) \|_\infty) \| (p^2 + x^2 + 1)^{\gamma/2} u(s) \|_2 \\
\leq CM^{\beta_L} \| (p^2 + x^2 + 1)^{\gamma/2} u(s) \|_2,
\]

and therefore, for \( C_M = |v| M^{\beta_L} + |\mu| M^{\beta_S} \)

\[
\| (p^2 + x^2 + 1)^{\gamma/2} \Xi[u](t) \|_2 \\
\leq \bar{C}_T \| (p^2 + x^2 + 1)^{\gamma/2} u_0 \|_2 \\
+ \bar{C}_T \int_0^t \left( |v| \bar{F}_L(\| u(s) \|_\infty) + |\mu| \bar{F}_S(\| u(s) \|_\infty) \right) \| (p^2 + x^2 + 1)^{\gamma/2} u(s) \|_2 ds \\
\leq \bar{C}_T \| (p^2 + x^2 + 1)^{\gamma/2} u_0 \|_2 + C_M \bar{C}_T \int_0^t \| (p^2 + x^2 + 1)^{\gamma/2} u(s) \|_2 ds \\
\leq (\bar{C}_T \varepsilon + C_M \bar{C}_T TM) \quad (31)
\]

holds. Next estimate

\[
\| \Xi(u)(t) \|_\infty.
\]
Using Sobolev’s inequality, for all $\psi \in \mathcal{D}((p^2 + x^2 + 1)^{\gamma})$, we obtain,
\[
\|\psi\|_\infty \leq C\|p^\gamma \psi\|_2 \leq C\|(p^2 + x^2 + 1)^{\gamma/2}\psi\|_2.
\] (32)

Hence, by the same calculations in (31) we have
\[
\|\Xi(u)\|_{LT} \leq \left(\tilde{C}_T \varepsilon + C_M \tilde{C}_T TM\right).
\]

Consequently, we get
\[
\|\Xi(u)\|_{LT} \leq \tilde{C}_T \varepsilon + C_M TM.
\]

Here $C_M TM$ is smaller than $C(\|\mu\| M^{1+\rho S} + |\nu| M^{1+\rho L})T$ and hence by putting $M = C(T)\varepsilon$ for some $C(T) > 0$, we have
\[
\|\Xi(u)\|_{LT} \leq \tilde{C}_T \varepsilon + C\tilde{C}_T \left(C(T)^{1+\rho L} + C(T)^{1+\rho S}\right)\varepsilon \left(\|\nu\|^{\rho L} + |\mu|^{\rho S}\right) T.
\]

Let $C(T)$ be $C(T) = 4\tilde{C}_T$ and $T > 0$ be so that
\[
T(C\tilde{C}_T(C(T)^{\rho L} + C(T)^{\rho S})) < \frac{1}{4(\|\nu\|^{\rho L} + |\mu|^{\rho S})}.
\]

Then for any fixed large $T > r_0 > 0$, there exists $\varepsilon = \varepsilon(T)$ such that the above inequality is fulfilled by taking $\varepsilon > 0$ sufficiently small compared to $C(T)$ and some constants. For such $C(T)$ and $T$, we notice
\[
\|\Xi(u)\|_{LT} \leq \frac{M}{4} + \frac{M}{4} = \frac{M}{2}.
\]

Then means $\Xi : L_{T,M} \to L_{T,M}$.

(II). By using (11) and the same argument as in Propositions 2.3 and Proposition 2.6, we get
\[
\|p^2 + x^2 + 1\|_{LT} (G(u(s)) - G(v(s))) \|_2 \\
\leq C (\|\nu\| F_L(u(s))\|_\infty + |\nu| F_L(v(s))\|_\infty + |\mu| \|F_S(u(s))\|_\infty + |\mu| \|F_S(v(s))\|_\infty) \\
\times \|(p^2 + x^2 + 1)(u(s) - v(s))\|_2.
\]

By using this inequality and the same argument as in the proof for (I), we have
\[
\|(p^2 + x^2 + 1)(\Xi(u)(t) - \Xi(v)(t))\|_2 \leq C_M \tilde{C}_T T \sup_{t \in [0,T]} \|(p^2 + x^2 + 1)(u(t) - v(t))\|_2
\]
and
\[
\|\Xi(u)(t) - \Xi(v)(t)\|_\infty \leq C_M \tilde{C}_T T \sup_{t \in [0,T]} \|(p^2 + x^2 + 1)(u(t) - v(t))\|_2.
\]
Hence, we get
\[ ||| \Xi(u) - \Xi(v) |||_{L_T} \leq C(|v|M^{\rho_L} + |\mu|M^{\rho_S}) \tilde{C}_T ||u - v||_{L_T}. \]

By taking \( M = C(T)\varepsilon' \) and supposing
\[ T(C\tilde{C}_T(C(T)^{\rho_L} + C(T)^{\rho_S})) \leq \frac{1}{2(|v|(|\varepsilon|)^{\rho_L} + |\mu|(|\varepsilon|)^{\rho_S})}, \]
we get
\[ ||| \Xi(u) - \Xi(v) |||_{L_T} \leq \frac{1}{2} ||u - v||_{L_T}. \]

By using the contraction mapping theorem (see, e.g., Cazenave [6]), we have (28). Moreover by taking \( M = C(T)\varepsilon' \) and by using \( \varepsilon' > (\varepsilon')^{-1} \) we also have (29). These complete the proof. \( \square \)

4. Proof of Theorems 1.8 and 1.9

In this section, we prove Theorem 1.8 with \(|t| \geq \rho_0\). Then, using Theorem 3.1, we can derive Theorem 1.8. Theorem 1.9 can be obtained as a sub-consequence of Theorem 1.8. Similarly in Sect. 3, we only consider the case where \( t \geq \rho_0 \). In this section, we always consider Assumptions 1.1, 1.3, and 1.5 and if necessary, we assume Assumption 1.4 additionally. Let \( T > \rho_0 \) be the same one given in Sect. 3. We define the function space \( X_T \) as follows
\[ X_T := \left\{ \phi \in C([\rho_0, T]; \mathcal{S}') : \|\phi\|_{X_T} \right\}, \]
where \( C_1(\varepsilon', \nu), C_2(\varepsilon', \mu) > 0 \) are sufficiently small constants that are later included.

We set \( u(t, x) \) is the solution to (1). Then according to Theorem 3.1, we have
\[ \|u(\rho_0)\|_{Y,0} \leq C(\rho_0)\varepsilon', \quad \|u(\rho_0)\|_{0,Y} \leq C(\rho_0)\varepsilon' \]
under the assumption \( \|u_0\|_{Y,0} + \|u_0\|_{0,Y} = \varepsilon' \leq \varepsilon \). Because \( \rho_0 \) is a given constant, we assume that \( \varepsilon' > 0 \) is sufficiently small compared to \( \tilde{C}_{\rho_0} \), i.e., \( 0 < C(\rho_0)\varepsilon' \ll 1 \). From this, we rewrite \( C(\rho_0) \) as \( C \) and in the following, we assume
\[ \|u(\rho_0)\|_{Y,0} \leq C\varepsilon', \quad \|u(\rho_0)\|_{0,Y} \leq C\varepsilon'. \]

Then by the same calculations in Sect. 3,
\[ \|U_0(0, \rho_0)u(\rho_0)\|_{Y,0} \leq C\varepsilon', \quad \|U_0(0, \rho_0)u(\rho_0)\|_{0,Y} \leq C\varepsilon'. \quad (34) \]

Furthermore we set the following lemma to prove the Theorem 1.8.
Lemma 4.1. Assume $\|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma} = \epsilon' \leq \epsilon$ for some sufficiently small $\epsilon > 0$, where $\gamma > n/2$. Then there exists a finite interval $[r_0, T]$ and a unique solution to (1) such that

$$\|u\|_{X_T} \leq T_{\epsilon'},$$

where a constant $T_{\epsilon'} > 0$ satisfies that for given $T$, $T_{\epsilon'} \to 0$ as $\epsilon' \to 0$.

Proof. Since $\|u\|_{L_T} \leq C_T \epsilon'$ holds, we have

$$(1 + t)C_1(\epsilon', \nu) + e^{C_2(\epsilon', \nu)} \leq C \left\| (p^2 + x^2 + 1)U_0(0, t)u(t) \right\|_{2} \leq C_T \left\| (p^2 + x^2 + 1)u(t) \right\|_{2} \leq C_T \epsilon'$$

and

$$(1 + |\xi_2(t)|)^{n/2} \|u(t)\|_{\infty} \leq C_T \left\| (p^2 + x^2 + 1)u(t) \right\|_{2} \leq C_T \epsilon'.$$

Theorem 4.2. Let $u$ be the local solutions to (1) stated in Lemma 4.1. Then for any $t \in [-T, -r_0] \cup [r_0, T]$,

$$(1 + |\xi_2(t)|)^{n/2} \|u(t)\|_{\infty} \leq C \epsilon'$$

where a constant $C > 0$ does not depend on $T$.

Proof. First, for $0 \leq s \leq r_0$, we use the obtained result in Theorem 3.1

$$(1 + s) \tilde{f}(\|u(s)\|_{\infty}) \leq C(\epsilon')^{\rho_S} \text{ or } C(\epsilon')^{\rho_L}$$

and for $s \geq r_0$, obtained result in Lemma 4.1

$$\|u(s)\|_{\infty} \leq T_{\epsilon'}(1 + |\xi_2(s)|)^{-n/2} \leq T_{\epsilon'}|\xi_2(s)|^{-n/2}.$$

Then, we estimate

$$\sup_{t \in [r_0, T]} \left( (1 + t)C_1(\epsilon', \nu) + e^{C_2(\epsilon', \nu)} \right)^{-1} \|U_0(0, t)u(t)\|_{0, \gamma} \leq C \epsilon',$$

where $C > 0$ is independent of $T$. By using Duhamel’s formula, we have

$$\|U_0(0, t)u(t)\|_{0, \gamma} \leq \|u_0\|_{0, \gamma} + \int_0^t \left\| (x)^{\gamma} U_0(0, s)G(u(s)) \right\|_{2} ds.$$

According to Proposition 2.3, we obtain

$$\|U_0(0, t)u(t)\|_{0, \gamma} \leq \|u_0\|_{0, \gamma} + \int_0^t \left( |v|\tilde{F}_L(\|u(s)\|_{\infty}) + |\mu|\tilde{F}_S(\|u(s)\|_{\infty}) \right) \|U_0(0, s)u(s)\|_{0, \gamma} ds \leq \|u_0\|_{0, \gamma} + C \int_0^t \left( |v|(T_{\epsilon'})^{\rho_L}(1 + s)^{-1} + |\mu|(T_{\epsilon'})^{\rho_S}(1 + s)^{-1-\delta_1} \right) \|U_0(0, s)u(s)\|_{0, \gamma} ds.$$
Define $C|v|(T_{v})^{PL} =: C_{1}(\varepsilon', v)$ and $C|\mu|(T_{v})^{PS} =: C_{2}(\varepsilon', \mu)$. Then using the Gronwall inequality, we obtain

$$\left\|U_{0}(0, t)u(t)\right\|_{0, \gamma} \leq C\left((1 + t)^{C_{1}(\varepsilon', v)} + e^{C_{2}(\varepsilon', \mu)}\right)\left\|u_{0}\right\|_{0, \gamma}. \tag{35}$$

Next we estimate $\|u(t)\|_{\infty}$. From Lemma 2.11, it holds that for $t \geq r_{0}$

$$\|u(t)\|_{\infty} \leq C\varepsilon'(1 + |\xi_{2}(t)|)^{-n/2}\left|\frac{\xi_{1}(t)}{\xi_{2}(t)}\right|^{\alpha}\left((1 + t)^{C_{1}(\varepsilon', v)} + e^{C_{2}(\varepsilon', \mu)}\right)\|u_{0}\|_{0, \gamma}$$

$$+ (1 + |\xi_{2}(t)|)^{-n/2}\|\mathcal{F}U_{0}(0, t)u(t)\|_{\infty}. \tag{36}$$

and therefore we shall estimate the term $(1 + |\xi_{2}(t)|)^{-n/2}\|\mathcal{F}U_{0}(0, t)u(t)\|_{\infty}$ in the following manner;

Through simple calculations, it holds that

$$i\partial_{t} \left(U_{0}(0, t)u(t)\right) = U_{0}(0, t)G(u(t)) = U_{0}(0, t)\left(vF_{L}(u(t))u(t) + \mu F_{S}(u(t))u(t)\right). \tag{37}$$

The term $U_{0}(0, t)F_{L}(u(t))u(t)$ is estimated as

$$U_{0}(0, t)F_{L}(u(t))u(t)$$

$$= \mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{1}(t)}\right)\mathcal{F}^{-1}\mathcal{D}(\xi_{2}(t))^{-1}\mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{1}(t)}\right)F_{L}(u(t))u(t)$$

$$= \mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{1}(t)}\right)\mathcal{F}^{-1}\mathcal{D}(\xi_{2}(t))^{-1}F_{L}\mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{2}(t)}\right)\mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{1}(t)}\right)u(t)$$

$$= \mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{1}(t)}\right)\mathcal{F}^{-1}F_{L}\left(|\xi_{2}(t)|^{-n/2}\mathcal{D}(\xi_{2}(t))^{-1}\mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{2}(t)}\right)u(t)\right)$$

$$\times \mathcal{D}(\xi_{2}(t))^{-1}\mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{2}(t)}\right)u(t). \tag{38}$$

For simplicity, we denote

$$\mathcal{M}_{1} = \mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{1}(t)}\right), \quad \mathcal{D} = \mathcal{D}(\xi_{2}(t)), \quad \mathcal{M}_{2} = \mathcal{M}\left(-\frac{\xi_{2}(t)}{\xi_{2}(t)}\right).$$

According to $\mathcal{F}\mathcal{M}_{1}^{-1}U_{0}(0, t) = \mathcal{D}^{-1}\mathcal{M}_{2}$, the above equation is equivalent to

$$\mathcal{M}_{1}\mathcal{F}^{-1}F_{L}\left(|\xi_{2}(t)|^{-n/2}\mathcal{D}\mathcal{M}_{1}^{-1}U_{0}(0, t)u(t)\right)\mathcal{F}\mathcal{M}_{1}^{-1}U_{0}(0, t)u(t).$$

Here, we denote $v(t) := U_{0}(0, t)u(t)$. The above term is then equivalent to

$$\left\{(\mathcal{M}_{1} - 1)\mathcal{F}^{-1}F_{L}\left(|\xi_{2}(t)|^{-n/2}\mathcal{M}_{1}^{-1}v(t)\right)\mathcal{M}_{1}^{-1}v(t)$$

$$+ \mathcal{F}^{-1}\left(F_{L}\left(|\xi_{2}(t)|^{-n/2}\mathcal{M}_{1}^{-1}v(t)\right)\mathcal{M}_{1}^{-1}v(t) - F_{L}\left(|\xi_{2}(t)|^{-n/2}\hat{v}(t)\right)\hat{v}(t)\right)\right\}$$

$$+ \mathcal{F}^{-1}F_{L}\left(|\xi_{2}(t)|^{-n/2}\hat{v}(t)\right)\hat{v}(t). \tag{39}$$
By performing a Fourier transform on both sides of (37), we obtain
\[ i \partial_t \hat{v}(t) = v \mathcal{F} U_0(0, t) F_L(u(t)) u(t) + \mu \mathcal{F} U_0(0, t) F_S(u(t)) u(t). \]

By using
\[ U_0(0, t) F_L(u(t)) u(t) = (38) = (39), \]
we obtain
\[ i \partial_t \hat{v}(t) = v F_L \left( |\xi_2(t)|^{-n/2} \hat{v}(t) \right) \hat{v}(t) \]
\[ + v (I_1(t) + I_2(t)) \]
\[ + \mu \hat{Q}(t), \]

where
\[ I_1(t) := \mathcal{F} (\mathcal{M}_1 - 1) \mathcal{F}^{-1} F_L \left( |\xi_2(t)|^{-n/2} \mathcal{M}^{-1}_1 v(t) \right) \mathcal{M}^{-1}_1 v(t), \]
\[ I_2(t) := F_L \left( |\xi_2(t)|^{-n/2} \mathcal{M}^{-1}_1 v(t) \right) \mathcal{M}^{-1}_1 v(t) - F_L \left( |\xi_2(t)|^{-n/2} \hat{v}(t) \right) \hat{v}(t) \]

and
\[ Q(t) := U_0(0, t) F_S(u(t)) u(t). \]

Let us define
\[ \hat{w}(t) := B(t) \hat{v}(t), \quad B(t) := \exp \left( i v \int_{r_0}^t \left( F_L \left( |\xi_2(\tau)|^{-n/2} \hat{v}(\tau) \right) \right) d\tau \right). \]

Then \( \hat{w} \) satisfies
\[ i \partial_t \hat{w}(t) = v B(t) (I_1(t) + I_2(t)) + \mu B(t) \hat{Q}(t). \]

Integrating both sides in \( t \) from \( r_0 \) to \( t \),
\[ \hat{w}(t) - \hat{w}(r_0) = -i \int_{r_0}^t B(\tau) \left( v (I_1(\tau) + I_2(\tau)) + \mu \hat{Q}(\tau) \right) d\tau, \]

and therefore
\[ \| \mathcal{F} U_0(0, t) u(t) \|_\infty \]
\[ = \| \hat{w}(t) \|_\infty \]
\[ \leq C \epsilon + C \int_{r_0}^t \left\{ |v| \| I_1(\tau) \|_\infty + \| I_2(\tau) \|_\infty + |\mu| \| \hat{Q}(\tau) \|_\infty \right\} d\tau \quad (40) \]
holds, when \( \gamma' > n/2 \),
\[ \| \hat{w}(r_0) \|_\infty = \| \mathcal{F} U_0(0, r_0) u(r_0) \|_\infty \leq C \| U_0(0, r_0) u(r_0) \|_1 \leq C \| U_0(0, r_0) u(r_0) \|_{0, \gamma'} \leq C \epsilon' \]
We now estimate each term in integration:

\[
\|I_1(t)\|_\infty = \left\| \mathcal{F} (M_1) \mathcal{F}^{-1} F_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) \hat{M}_0^{-1} v \right\|_\infty \\
\leq C \left\| (M_1) \mathcal{F}^{-1} F_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) \hat{M}_0^{-1} v \right\|_1 \\
\leq C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| x^{2\alpha} \mathcal{F}^{-1} F_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) \hat{M}_0^{-1} v \right\|_1,
\]

where \(0 < \alpha < 1\). Here by using Schwarz’s inequality and Lemma 2.11, for \(\gamma > n/2 + 2\alpha\) and \(\gamma' > n/2\), the above inequality is smaller than

\[
C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| \mathcal{F}^{-1} F_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) \hat{M}_0^{-1} v \right\|_{0,\gamma} \\
\leq C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| \xi_2(t) \right\|^{n/2} \left\| F_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right\|_{\gamma,0} \\
\leq C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| \xi_2(t) \right\|^{n/2} \tilde{F}_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) \left\| |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right\|_{\gamma,0} \\
\leq C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| \xi_2(t) \right\|^{-n\rho_L/2} \left\| \hat{M}_0^{-1} v \right\|^{\rho_L} \left\| \hat{M}_0^{-1} v \right\|_{0,\gamma} \\
\leq C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| \xi_2(t) \right\|^{-n\rho_L/2} \left\| v \right\|^{\rho_L} \left\| v \right\|_{0,\gamma},
\]

where we use (35), and above inequality gives

\[
\|I_1(t)\|_\infty \leq C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| \xi_2(t) \right\|^{-n\rho_L/2} \left\| v \right\|^{\rho_L} \left\| v \right\|_{0,\gamma}.
\]  

On the other hand,

\[
\|I_2(t)\|_\infty = \left\| F_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) \hat{M}_0^{-1} v - F_L \left( |\xi_2(t)|^{-n/2} \hat{v} \right) \hat{v} \right\|_\infty \\
\leq C \left( \tilde{F}_L \left( |\xi_2(t)|^{-n/2} \hat{M}_0^{-1} v \right) \left\| \hat{M}_0^{-1} v \right\|_\infty \right) \left\| \hat{M}_0^{-1} v - \hat{v} \right\|_\infty \\
\leq C \left\| \xi_2(t) \right\|^{-n\rho_L/2} \left\| v \right\|^{\rho_L} \left\| \left( \hat{M}_0^{-1} - 1 \right) v \right\|_1 \\
\leq C \left\| \frac{\xi_1(t)}{\xi_2(t)} \right\| \left\| \xi_2(t) \right\|^{-n\rho_L/2} \left\| v \right\|^{\rho_L} \left\| v \right\|_{0,\gamma},
\]  

(42)
and

\[ \| \hat{Q}(t) \|_{\infty} = \| \mathcal{F}U_{0}(0, t)F_{S}(u) \|_{\infty} \]
\[ \leq C \| \mathcal{F}M_{1}\mathcal{F}^{-1}D^{-1}M_{2}F_{S}(u) \|_{\infty} \]
\[ \leq C \| \mathcal{F}^{-1}F_{S}\left( |\xi_{2}(t)|^{-n/2}M_{1}^{-1}v \right)M_{1}^{-1}v \|_{\infty} \]
\[ \leq C|\xi_{2}(t)|^{-n \rho_{S}/2} \| M_{1}^{-1}v \|_{\rho_{S}} \| M_{1}^{-1}v \|_{0, \gamma, 0} \]
\[ \leq C|\xi_{2}(t)|^{-n \rho_{S}/2} \| v \|_{0, \gamma, 0}^{\rho_{S}} \| v \|_{0, \gamma, 0} \] (43)

Using (40) – (43) and together with

\[ \| v(t) \|_{0, \gamma, 0} \leq C \varepsilon^{\prime} \left( (1 + t)C_{1}(\varepsilon^{\prime}, v) + e^{C_{2}(\varepsilon^{\prime}, \mu)} \right) \leq C \varepsilon^{\prime} (1 + t)C_{1}(\varepsilon^{\prime}, v) \]

for \( t \geq r_{0} \), we obtain

\[ \| \mathcal{F}U_{0}(0, t)u(t) \|_{\infty} \]
\[ \leq C \varepsilon^{\prime} + \int_{r_{0}}^{t} \left\{ |\varepsilon^{\prime}| \left| \xi_{1}(\tau) \right|^{\alpha} \left| \xi_{2}(\tau) \right|^{-n \rho_{L}/2} \| v \|_{0, \gamma, 0}^{\rho_{L}} \| v \|_{0, \gamma, 0} + |\mu||\xi_{2}(\tau)|^{-n \rho_{S}/2} \| v \|_{0, \gamma, 0}^{\rho_{S}} \| v \|_{0, \gamma, 0}^{\rho_{S}} \right\} d\tau \]
\[ \leq C \varepsilon^{\prime} + C \varepsilon^{\prime} \int_{r_{0}}^{t} \left\{ (\varepsilon^{\prime})^{\rho_{L}}|v| \left| \xi_{1}(\tau) \right|^{\alpha} \left| \xi_{2}(\tau) \right|^{-n \rho_{L}/2} |\tau|^{C_{1}(\varepsilon^{\prime}, v)(1+\rho_{L})} \right. \]
\[ \left. + (\varepsilon^{\prime})^{\rho_{S}}|\mu||\xi_{2}(\tau)|^{-n \rho_{S}/2} |\tau|^{C_{1}(\varepsilon^{\prime}, v)(1+\rho_{S})} \right\} d\tau \]
\[ \leq C \varepsilon^{\prime} + C \varepsilon^{\prime} ((\varepsilon^{\prime})^{\rho_{L}} + (\varepsilon^{\prime})^{\rho_{S}}) \]
\[ \times \int_{r_{0}}^{t} \left\{ |v| \left| \xi_{1}(\tau) \right|^{\alpha} |\tau|^{-1+C_{1}(\varepsilon^{\prime}, v)(1+\rho_{L})} + |\mu||\tau|^{-1-C_{1}(\varepsilon^{\prime}, v)(1+\rho_{L})} \right\} d\tau. \] (44)

Here we note \( C_{1}(\varepsilon^{\prime}, 0) = 0 \), \( C_{1}(\varepsilon^{\prime}, v) \) and \( C_{2}(\varepsilon^{\prime}, \mu) \to 0 \) as \( \varepsilon^{\prime} \to 0 \), respectively, and then based on Assumption 1.3 with \( v = 0 \) and \( \delta_{1} > C_{2}(\varepsilon^{\prime}, v)(1 + \rho_{S}) \), we obtain the above term, which is smaller than \( C \varepsilon^{\prime} \), and based on Assumption 1.4 with \( v \neq 0 \), \( \delta_{1} > C_{2}(\varepsilon^{\prime}, \mu)(1 + \rho_{S}) \) and \( \alpha \delta_{0} > C_{1}(\varepsilon^{\prime}, v)(1 + \rho_{L}) \), we obtain the above inequality, which is smaller than

\[ C \varepsilon^{\prime} + C \varepsilon^{\prime} ((\varepsilon^{\prime})^{\rho_{L}} + (\varepsilon^{\prime})^{\rho_{S}}) \int_{r_{0}}^{t} |\tau|^{-1-C_{1}(\varepsilon^{\prime}, v)(1+\rho_{L})} + |\tau|^{-1-C_{2}(\varepsilon^{\prime}, v)(1+\rho_{S})} d\tau \]

and therefore, we finally obtain

\[ \| \mathcal{F}U_{0}(0, t)u(t) \|_{\infty} \leq C \varepsilon^{\prime}. \] (45)

Combining (35), (36), and (45), we finally derive Theorem 4.2. \( \square \)
4.1. Proof of Theorems 1.8 and 1.9

We now prove Theorem 1.8. From Theorems 3.1 and 4.2, we have

\[ \|u\|_{X_T} \leq C \left( \|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma} \right) = C \varepsilon' \]

for all \( t \in [-T, T] \). Because the constant \( C \) does not depend on \( T \), we apply the continuation argument and obtain Theorem 1.8. Moreover, Theorem 1.9 can be proven by imitating the proof of Theorem 1.2. in [12]. By Lemma 4.1 and the same calculation in (30) and (31), we get

\[
\left\| (p^2 + x^2 + 1)^{\gamma/2} u(t) \right\|_2 \leq q(t) \left\| (p^2 + x^2 + 1)^{\gamma/2} u_0 \right\|_2 + q(t) C(\varepsilon') \int_0^t \left\| (p^2 + x^2 + 1)^{\gamma/2} u(s) \right\|_2 ds,
\]

where \( q(t) \) satisfies for \( t \geq r_0 \mid q(t) \mid \leq \mid \xi_2(t) \mid^{4\gamma} \). Since \( \xi_2(t) \) is continuous we have \( u(t) \in C(\mathbb{R}; H^{\gamma,0} \cap H^{0,\gamma}) \) by using Gronwall inequality.

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