BERRY–ESSEEN BOUNDS FOR SELF-NORMALIZED SUMS OF LOCAL DEPENDENT RANDOM VARIABLES

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Abstract

In this paper, we prove a Berry–Esseen bound with optimal order for self-normalized sums of local dependent random variables under some mild dependence conditions. The proof is based on Stein’s method and a randomized concentration inequality. As applications, we obtain optimal Berry–Esseen bounds for $m$-dependent random variables and graph dependency.

1 INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of independent random variables, and let

$$S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2, \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{n-1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = S_n/n$. We say $S_n/V_n$ is a self-normalized sum. We note that $S_n/\hat{\sigma}_n$ is the well-known Student’s $t$-statistic, which is one of the most important tools in statistical testing when the standard deviation of the underlying distribution is unknown. Based on the fact that

$$P(S_n/\hat{\sigma}_n \geq x) = P(S_n/V_n \geq x[n/(n + x^2 - 1)]^{1/2}), \quad x \geq 0,$$

it is common to study the self-normalized sum $S_n/V_n$. One of the most important advantages of self-normalized sums is that the range of Gaussian approximation can be much wider than their corresponding non-self-normalized sums under the same polynomial moment conditions. Berry–Esseen bounds of Gaussian approximation for self-normalized sums of independent random variables have been well-studied in the literature. For example, Berry–Esseen theorem for Student’s $t$-statistics were proved in Bentkus and Götze (1996) and Bentkus, Bloznelis and Götze (1996), and an exponential nonuniform Berry–Esseen bound was obtained in Jing and Wang (1999). Differing from the method in Bentkus and Götze (1996) and Bentkus et al. (1996), Stein’s method can be used to prove Berry–Esseen bounds with explicit constants (c.f. Shao (2005)).

If $(X_1, X_2, \ldots)$ is a family of dependent random variables, asymptotic theories on self-normalized sums have also been studied in the literature. For example, in de la Peña, Klass and Lai (2004), the authors studied self-normalized processes, and in Bercu and Touati (2008),
self-normalized sums for martingales were considered. Recently, a Cramér-type moderate deviation under some weak dependence assumptions was proved in Chen, Shao, Wu and Xu (2016). For more details, we refer to Shao and Wang (2013) for a survey.

Local dependence is also a commonly used dependence structure in applications. A family of local dependent random variables means that certain subset of the random variables are independent of those outside their respective neighborhoods. There are several forms of local dependence assumptions in the literature. For example, decomposable random variables were considered in Barbour, Karoński and Ruciński (1989), where a $L_1$ bound was also obtained; dependency neighborhoods was introduced in Rinott and Rotar (1996) to study error bounds for multivariate normal approximation. Uniform and nonuniform Berry–Esseen bounds for a general local dependence structure were established in Chen and Shao (2004). Recently, Fang (2019) proved an error bound of Wasserstein-2 distance for a generalized local dependence structure.

In this paper, our main purpose is to prove Berry–Esseen bounds for self-normalized sums of local dependent random variables. In Theorem 2.1 (see Section 2), under conditions (LD1) and (LD2), we provide a Berry–Esseen bound for self-normalized sums of local dependent random variables. Compared to the results for non-self-normalized sums in Theorem 2.2 in Chen and Shao (2004), under the same conditions (LD1) and (LD2), we do not require the fourth moment assumption to obtain the optimal convergence rate, which in turn shows robustness of self-normalized statistics. As applications, we obtain Berry–Esseen bounds for studentized statistics for $m$-dependent random variables and graph dependency.

The proof of our main result is based on Stein’s method and concentration inequality approach. The technique of concentration inequality approach has been applied to obtain sharp Berry–Esseen bounds for univariate and multivariate normal approximations in the literature, and we refer to Chen (1998); Chen and Shao (2001, 2004, 2007); Shao (2005); Shao and Zhou (2016) and Shao and Zhang (2021). In this paper, we develop new randomized concentration inequalities for local dependent random variables under some mild dependence conditions. The ideas are based on Chen and Shao (2004) and Shao (2005).

The rest of this paper is organized as follows. We give our main results in Section 2. Applications are given in Section 3. Some useful preliminary lemmas and a new randomized concentration inequality are proved in Section 4. We give the proof of our main result in Section 5.

2 MAIN RESULTS

Let $\{X_i\}_{i \in J}$ be a field of real-valued random variables satisfying that $\mathbb{E}X_i = 0$ for all $i \in J$. We introduce the following local dependence conditions.

(LD1) For any $i \in J$, there exists $A_i \subset J$ such that $X_i$ is independent of $\{X_j : j \notin A_i\}$;

(LD2) For any $i \in J$, there exists $B_i \subset J$ such that $B_i \supset A_i$ and $\{X_j : j \in A_i\}$ is independent of $\{X_j : j \notin B_i\}$.

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These dependence assumptions have been commonly used in the literature. We remark that we do not assume any structures on the index set. Put

\[
S = \sum_{i \in \mathcal{J}} X_i, \quad V = \sqrt{\left( \sum_{i \in \mathcal{J}} (X_i Y_i - \bar{X} \bar{Y}) \right)_+}, \quad W = S/V,
\]

where \( Y_i = \sum_{j \in A_i} X_i \), \( \bar{X} = \frac{\sum_{i \in \mathcal{J}} X_i}{|\mathcal{J}|} \), \( \bar{Y} = \frac{\sum_{j \in J} Y_j/|J|}{|\mathcal{J}|} \) and \((x)_+ = \max(x, 0)\) is the positive part of \( x \). Let \( \sigma = \sqrt{\text{Var}(S)} \). Here and in the sequel, we denote by \(|A|\) the cardinality of \( A \) for any \( A \subset J \). We remark that if \( \mathbb{E} X_i \) is not necessarily 0, then one can simply replace \( X_i \) by \( X_i - \mathbb{E} X_i \) in (2.1).

Moreover, let \( \kappa \) be any number such that \( \kappa \geq \max_{i \in \mathcal{J}} \{|\{j : B_i \cap A_j \neq \emptyset\}|, |\{j : i \in B_j\}|\} \) and let

\[
\beta_0 = \sum_{i \in \mathcal{J}} \mathbb{P}(|X_i| > \sigma/\kappa),
\]

\[
\beta_2 = \frac{1}{\sigma^2} \sum_{i \in \mathcal{J}} \mathbb{E} \{|X_i|^2 \mathbb{I}(|X_i| > \sigma/\kappa)\},
\]

\[
\beta_3 = \frac{1}{\sigma^3} \sum_{i \in \mathcal{J}} \mathbb{E} \{|X_i|^3 \mathbb{I}(|X_i| \leq \sigma/\kappa)\},
\]

\[
\theta = \frac{1}{\sigma^2} \sum_{i \in \mathcal{J}} \sum_{j \in A_i} \mathbb{E} |X_i X_j| \mathbb{I}(|X_i| \leq \sigma/\kappa, |X_j| \leq \sigma/\kappa).
\]

We have the following theorem.

**Theorem 2.1.** Under (LD1) and (LD2). We have

\[
\sup_{z \in \mathbb{R}} |\mathbb{P}(W \leq z) - \Phi(z)| \leq C \{ (1 + \theta) \kappa^2 \beta_3 + \kappa \beta_2 + \beta_0 \} + C \kappa^{1/2} (\theta + 1) |J|^{-1/2},
\]

where \( C > 0 \) is an absolute constant and \( \Phi \) is the standard normal distribution function.

**Remark 2.2.** We first make some remarks on \( \beta_j \)'s. If \( \max_i |X_i| \leq \sigma/\tau \) almost surely, then \( \beta_0 = 0 \). Otherwise, by Chebyshev’s inequality, we have

\[
\beta_0 \leq \kappa^2 \beta_2.
\]

If \( \mathbb{E} |X_i|^3 < \infty \), then

\[
\kappa \beta_2 + \kappa^2 \beta_3 \leq \frac{\kappa^2}{\sigma^3} \sum_{i \in \mathcal{J}} \mathbb{E} |X_i|^3.
\]

Moreover, by Hölder’s inequality, we have

\[
\theta \leq \left( \mathbb{E} \left\{ \frac{1}{\sigma^2} \sum_{i \in \mathcal{J}} \sum_{j \in A_i} |X_i X_j| \mathbb{I}(|X_i| \leq \sigma/\kappa, |X_j| \leq \sigma/\kappa) \right\}^{3/2} \right)^{2/3}
\]

\[
\leq \kappa_1 |\mathcal{J}|^{1/3} |\beta_3|^{2/3} \leq \kappa |\mathcal{J}|^{1/3} |\beta_3|^{2/3},
\]

where \( \kappa_1 = \max_{i \in \mathcal{J}} |A_i| \). If \( \beta_3 \) is of order \( O(|\mathcal{J}|^{-1/2}) \), then \( \theta \) is of order \( O(1) \).
Remark 2.3. Chen and Shao (2004) proved a Berry–Esseen bound of the same order as in Theorem 2.1 for non-self-normalized sums. Specifically, assume further that for any \( i \in J \), there exists \( C_i \subseteq J \) such that \( C_i \supset B_i \) and \( \{ X_j : j \in B_i \} \) is independent of \( \{ X_j : j \notin C_i \} \), and it follows that (see Theorem 2.4 of Chen and Shao (2004))

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(S/\sigma \leq z) - \Phi(z) \right| \leq 75(\kappa')^{p-1} \sum_{i \in J} \mathbb{E}|X_i/\sigma|^p,
\]

where \( \kappa' \geq \max(|\{ j : B_j \cap C_i \neq \emptyset \}|, |\{ j : i \in C_j \}|) \). Under the same dependence condition as in Theorem 1.1, Chen and Shao (2004) proved that for \( 2 < p \leq 4 \),

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(S/\sigma \leq z) - \Phi(z) \right| \leq C(1 + \kappa) \sum_{i \in J} \left( \mathbb{E}|X_i/\sigma|^{3+\gamma} + \mathbb{E}|Y_i/\sigma|^{3+\gamma} \right)
\]

\[+ C\kappa^{1/2} \left( \sum_{i \in J} \left( \mathbb{E}|X_i/\sigma|^p + \mathbb{E}|Y_i/\sigma|^p \right) \right)^{1/2},\]

which is of the best possible order \( O(n^{-1/2}) \) when \( p = 4 \) and \( \kappa = O(1) \). For the self-normalized sum \( S/V \), the best possible order can be obtained under a third moment condition.

Remark 2.4. Specially, if \( \{ X_i, i \in J \} \) is a field of independent random variables, then it is not hard to see that \( \kappa = \theta = 1 \). Then, by (2.3), \( |J|^{-1/2} \leq \beta_3 \). Hence, the right hand side of (2.2) reduces to \( C(\beta_0 + \beta_2 + \beta_3) \), which is as same as the results in Bentkus et al. (1996).

3 APPLICATIONS

3.1 Self-normalized sums of \( m \)-dependent random variables

Let \( d \geq 1 \) and let \( \mathbb{Z}^d \) denote the \( d \)-dimensional space of positive integers. For any \( i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \), we define the distance by \( |i-j| := \max_{1 \leq k \leq d} |i_k-j_k| \), and for \( A, B \subseteq \mathbb{Z}^d \), we define the distance between \( A \) and \( B \) by \( \rho(A, B) := \inf\{|i-j| : i \in A, j \in B\} \). Let \( J \) be a subset of \( \mathbb{Z}^d \), and we say a field of random variables \( \{ X_i : i \in J \} \) is an \( m \)-dependent random field if \( \{ X_i, i \in A \} \) and \( \{ X_j, j \in B \} \) are independent whenever \( \rho(A, B) > m \) for any \( A, B \subseteq J \). Choose \( A_i = \{ j : |i-j| \leq m \} \), \( B_i = \{ j : |i-j| \leq 2m \} \). We have \( |\{ j : A_j \cap B_i \}| \leq (6m+1)^d \) for all \( i \). Applying Theorem 2.1, we have the following theorem.

Theorem 3.1. Let \( \{ X_i, i \in J \} \) be an \( m \)-dependent field with \( \mathbb{E}X_i = 0 \) and assume that \( \mathbb{E}|X_i|^3 < \infty \). Let \( Y_i = \sum_{j \in A_i} X_i \). Assume that \( \sigma^2 := \sum_{i \in J} \mathbb{E}\{ X_i Y_i \} > 0 \). Let \( W \) be as in (2.1). Then,

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(W \leq z) - \Phi(z) \right| \leq C(m + 1)^{3d} (1 + (m + 1)^d |J|^{1/3} \gamma^{2/3})(\gamma + |J|^{-1/2}),
\]

where \( \gamma = \sum_{i \in J} \mathbb{E}|X_i|^3 / \sigma^3 \).
3.2 Graph dependency

We now consider a field of random variables \( \{X_i : i \in \mathcal{V}\} \) indexed by the vertices of a graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \). We say \( \mathcal{G} \) is a dependency graph if for any pair of disjoint sets \( \Gamma_1 \) and \( \Gamma_2 \) in \( \mathcal{V} \) such that no edge has one endpoint in \( \Gamma_1 \) and the other in \( \Gamma_2 \), then the sets of random variables \( \{X_i : i \in \Gamma_1\} \) and \( \{X_j : j \in \Gamma_2\} \) are independent. Let \( A_i = \{j \in \mathcal{V} : \text{there is an edge connecting } i \text{ and } j\} \), \( A_{ij} = A_i \cup A_j \), \( B_i = \cup_{j \in A_i} A_j \). Noting that \( |\{j : A_j \cap B_i\}| \leq C d^3 \), and applying Theorem 2.1, we have the following theorem.

**Theorem 3.2.** Let \( \{X_i, i \in \mathcal{V}\} \) be a field of random variables indexed by the vertices of a dependency graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \). Assume that \( \mathbb{E}X_i = 0 \). Let \( S = \sum_{i \in \mathcal{V}} X_i, Y_i = \sum_{j \in A_i} X_j \) and let \( V = \sqrt{\left( \sum_{i \in \mathcal{V}} (X_i Y_i - \bar{X} \bar{Y}) \right)_+} \). Put \( W = S/V \) and \( n = |\mathcal{V}| \). Let \( d \) be the maximal degree of \( \mathcal{G} \). Then, we have

\[
\sup_{z \in \mathbb{R}} |\mathbb{P}(W \leq z) - \Phi(z)| \leq C d^9 \left(1 + d^{6} n^{1/3} \gamma^{2/3} / n^{1/2} + \gamma\right),
\]

where \( \gamma = \sum_{i \in \mathcal{V}} \mathbb{E}|X_i|^3 \), and \( \sigma^2 = \sum_{i \in \mathcal{V}} \mathbb{E}\{X_i Y_i\} \).

Graph dependency was firstly discussed in Baldi and Rinott (1989), where a Berry–Esseen bound for non-self-normalized version of \( W \) is also proved. However, the self-normalized Berry–Esseen bound is new.

### 4 SOME PRELIMINARY LEMMAS AND PROPOSITIONS

In this section, we first prove some preliminary lemmas, and then prove an important concentration inequality, which is of independent interest.

#### 4.1 Some Preliminary lemmas

Let

\[
\bar{X}_i := X_i I(|X_i| \leq \sigma / \kappa), \quad \bar{Y}_i := \sum_{j \in A_i} \bar{X}_j, \quad \bar{\sigma} = \left( \sum_{j \in \mathcal{J}} \mathbb{E}\{\bar{X}_j \bar{Y}_j\} \right)^{1/2},
\]

\[
\bar{S} := \sum_{i \in \mathcal{J}} \bar{X}_i, \quad \bar{V} := \psi\left( \sum_{i \in \mathcal{J}} \bar{X}_i \bar{Y}_i \right), \quad \bar{W} := \bar{S} / \bar{V},
\]

where

\[
\psi(x) = ((x \lor 0.25\sigma^2) \land 2\sigma^2)^{1/2}.
\]

The following lemma provides bounds for \( \bar{\sigma}^2 \) and some tail and moment inequalities for \( \{\bar{X}_i, i \in \mathcal{J}\} \).

**Lemma 4.1.** Assume that (LD1) holds. We have

\[
|\bar{\sigma}^2 - \sigma^2| \leq 3\kappa \sigma^2 \beta_2,
\]

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\[
\mathbb{E}\left( \sum_{i \in \mathcal{J}} (\bar{X}_i Y_i - \mathbb{E}\{\bar{X}_i Y_i\}) \right)^2 \leq \kappa^2 \sigma^4 \beta_3, \tag{4.4}
\]
\[
\mathbb{P}\left\{ \left| \sum_{i \in \mathcal{J}} (\bar{X}_i Y_i - \mathbb{E}\{\bar{X}_i Y_i\}) \right| \geq \frac{\sigma^2}{2} \right\} \leq 4\kappa^2 \beta_3. \tag{4.5}
\]

Specially, if \( \beta_2 \leq 1/(150\kappa) \), we have
\[
0.98\sigma^2 \leq \sigma^2 \leq 1.02\sigma^2. \tag{4.6}
\]

**Proof of Lemma 4.1.** We first prove (4.3). Note that for any \( i, j \in \mathcal{J} \),
\[
|\mathbb{E}\{X_i X_j\} - \mathbb{E}\{\bar{X}_i \bar{X}_j\}| \
\leq \mathbb{E}|(X_i - \bar{X}_i)(X_j - \bar{X}_j)| + \mathbb{E}|\bar{X}_i (X_j - \bar{X}_j)| + \mathbb{E}|(X_i - \bar{X}_i)(X_j - \bar{X}_j)| \
\leq \frac{\sigma}{\kappa} (\mathbb{E}|X_i| I(|X_i| > \sigma/\kappa) + \mathbb{E}|X_j| I(|X_j| > \sigma/\kappa)) \
+ \mathbb{E}|\bar{X}_i (X_j - \bar{X}_j)| \mathbb{E}|(X_i - \bar{X}_i)| \mathbb{E}|X_j| > \sigma/\kappa) \
\leq 1.5(\mathbb{E}|X_i|^2 I(|X_i| > \sigma/\kappa) + \mathbb{E}|X_j|^2 I(|X_j| > \sigma/\kappa)).
\]

Thus,
\[
|\sigma^2 - \sigma^2| = \left| \sum_{i \in \mathcal{J}} \sum_{j \in A_i} \mathbb{E}\{X_i X_j\} - \sum_{i \in \mathcal{J}} \sum_{j \in A_i} \mathbb{E}\{\bar{X}_i \bar{X}_j\} \right| \
\leq 1.5 \sum_{i \in \mathcal{J}} \sum_{j \in A_i} (\mathbb{E}|X_i|^2 I(|X_i| > \sigma/\kappa) + |X_j|^2 I(|X_j| > \sigma/\kappa)) \
+ 1.5 \sum_{j \in \mathcal{J}} \sum_{i \in A_i} \mathbb{E}|X_j|^2 I(|X_j| > \sigma/\kappa)) \
\leq 3\sigma^2 \kappa \beta_2.
\]

This proves (4.3). Specially, (4.6) follows directly from (4.3).

Now, by Cauchy’s inequality, we have with \( c = \kappa^2 \),
\[
\mathbb{E}\{\bar{X}_i^2 \bar{Y}_i^2\} \leq \frac{\kappa}{2} \mathbb{E}\bar{X}_i^4 + \frac{\kappa^2}{2c} \sum_{j \in A_i} \mathbb{E}\bar{X}_j^4 = \frac{\kappa^2}{2} \mathbb{E}\bar{X}_i^4 + \frac{\kappa}{2} \sum_{j \in A_i} \mathbb{E}\bar{X}_j^4 \
\leq \frac{\kappa^2 \sigma}{2\tau} \mathbb{E}\bar{X}_i^4 + \frac{\kappa}{2\tau} \sum_{j \in A_i} \mathbb{E}\bar{X}_j^3, \tag{4.7}
\]
where we used the fact that \( |\bar{X}_i| \leq \sigma/\kappa \) in the last line. Taking summation over \( i \in \mathcal{J} \) on both sides of (4.7) yields
\[
\sum_{i \in \mathcal{J}} \mathbb{E}\{\bar{X}_i^2 \bar{Y}_i^2\} \leq \kappa^4 \beta_3. \tag{4.8}
\]
By (4.8), we have the left hand side of (4.4) is
\[
\mathbb{E} \left( \sum_{i \in J} \sum_{j \in B_i} (\bar{X}_i \bar{Y}_i - \mathbb{E}\{\bar{X}_i \bar{Y}_i\})(\bar{X}_j \bar{Y}_j - \mathbb{E}\{\bar{X}_j \bar{Y}_j\}) \right)
\]
\[
= \sum_{i \in J} \sum_{j \in B_i} \left\{ \frac{1}{2} \mathbb{E}\{\bar{X}_i^2 \bar{Y}_i^2\} + \frac{1}{2} \mathbb{E}\{\bar{X}_j^2 \bar{Y}_j^2\} \right\}
\]
\[
= \frac{1}{2} \left( \sum_{i \in J} \sum_{j \in B_i} \mathbb{E}\{\bar{X}_i^2 \bar{Y}_i^2\} + \sum_{j \in J} \sum_{i \in B_i} \mathbb{E}\{\bar{X}_j^2 \bar{Y}_j^2\} \right)
\]
\[
\leq \kappa^2 \sigma^4 \beta_3.
\]
This proves (4.4). Moreover, (4.5) follow immediately from (4.4) and the Markov inequality.

The following lemma will be useful in the proof of our main result. Let \( \xi_i = \bar{X}_i / \bar{V}, \eta_i = \bar{Y}_i / \bar{V} \) and \( W^{(i)} = \bar{W} - \eta_i \).

**Lemma 4.2.** Assume that (LD1) holds and assume that \( \beta_2 \leq 1/(150 \kappa) \). Let \( f \) be any absolutely continuous function such that \( \|f\| \leq 1 \) and \( \|f'\| \leq 1 \). We have
\[
\sum_{i \in J} \|\mathbb{E}\{\xi_i f(W - \eta_i)\}\| \leq 62 \kappa^2 \beta_3 + 2 \tau \beta_2.
\] (4.9)

**Proof.** Let \( N(A_i) = \{j : A_i \cap A_j \neq \emptyset\} \), \( N(B_i) = \{j : B_i \cap A_j \neq \emptyset\} \), and
\[
\bar{V}^{(i)} = \psi \left( \sum_{k \in A_i^c \cap A_j^c} \bar{X}_k \bar{X}_i \right).
\]
The left hand side of (4.9) can be bounded by
\[
\sum_{i \in J} \|\mathbb{E}\{\xi_i f(W - \eta_i)\}\| \leq \sum_{i \in J} \|\mathbb{E}\left\{ \frac{\bar{X}_i}{\bar{V}^{(i)}} f \left( \frac{\bar{S} - \bar{Y}_i}{\bar{V}^{(i)}} \right) \right\} - \mathbb{E}\left\{ \frac{\bar{X}_i}{\bar{V}^{(i)}} f \left( \frac{\bar{S} - \bar{Y}_i}{\bar{V}^{(i)}} \right) \right\} \|
\]
\[
+ \sum_{i \in J} \|\mathbb{E}\left\{ \frac{\bar{X}_i}{\bar{V}^{(i)}} f \left( \frac{\bar{S} - \bar{Y}_i}{\bar{V}^{(i)}} \right) \right\} \|
\]
\[
:= T_1 + T_2.
\]
Recall that \( \bar{V} \geq \sigma/2 \) and \( \bar{V}^{(i)} \geq \sigma/2 \). Then, it follows that
\[
\left| \frac{1}{\bar{V}} - \frac{1}{\bar{V}^{(i)}} \right| \leq \frac{1}{\bar{V}^{(i)}(\bar{V} + \bar{V}^{(i)})} \left| \sum_{j \in A_i} \sum_{k \in A_j} \bar{X}_j \bar{X}_k + \sum_{j \in A_i^c} \sum_{k \in A_j} \bar{X}_j \bar{X}_k \right|
\]
\[
\leq \frac{4}{\sigma^3} \left( \sum_{j \in A_i} \sum_{k \in A_j} \left| \bar{X}_j \bar{X}_k \right| + \sum_{j \in A_i^c} \sum_{k \in A_j} \left| \bar{X}_j \bar{X}_k \right| \right),
\] (4.10)
By (4.10),

\[ T_1 \leq \sum_{i \in J} \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_i| \left| \frac{1}{V} - \frac{1}{V^{(i)}} \right| \right\} \]

\[ \leq \frac{4}{\sigma^3} \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_j} \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_i \bar{X}_j \bar{X}_k| \right\} \]

\[ + \frac{4}{\sigma^3} \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_j} \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_i \bar{X}_j \bar{X}_k| \right\} \]

\[ \leq \frac{8\kappa^2}{3\sigma^2} \sum_{i \in J} \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_i^3| \right\} \]

\[ + \frac{8\kappa}{3\sigma^2} \sum_{j \in J} \sum_{i,j \in A_i} \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_j^3| \right\} \]

\[ + \frac{8}{3\sigma^2} \sum_{k \in A_i} \sum_{j,k \in A_j} \sum_{i,j \in A_i} \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_k^3| \right\}. \]

Let \( \bar{Y}_{ij} = \sum_{k \in A_i \cup A_j} \bar{X}_k \). Then, by \( |\bar{Y}_{ij} - \bar{Y}_n| \leq \sum_{k \in A_j} |\bar{X}_k| \leq \sigma \), we have

\[ \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_i^3| \right\} = \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) \mathbb{E}\{ |\bar{X}_i|^3 \} \right\}, \]

\[ \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) |\bar{X}_j^3| \right\} \leq \mathbb{E}\left\{ \left( 1 + \frac{2|\bar{S} - \bar{Y}_n|}{\sigma} \right) \mathbb{E}\{ |\bar{X}_j|^3 \} \right\} + 2 \mathbb{E}|\bar{X}_j|^3. \]

As \( |\bar{Y}_n| \leq \sigma, |\bar{Y}_{ij}| \leq 2\sigma \) and \( \mathbb{E}|\bar{S}| \leq \bar{\sigma} \leq 1.02\sigma \), we have

\[ \mathbb{E}|\bar{S} - \bar{Y}_n| \leq 2.02\sigma, \quad \mathbb{E}|\bar{S} - \bar{Y}_{ij}| \leq 3.02\sigma. \]

Substituting (4.12) and (4.13) to (4.11) yields

\[ T_1 \leq 62\kappa^2 \beta_3. \]

Moreover, observe that

\[ \sum_{i \in J} |\mathbb{E}\bar{X}_i| \leq \sum_{i \in J} \mathbb{E}\{|\bar{X}_i| 1(|\bar{X}_i| > \sigma/\kappa)\} \]

\[ \leq \frac{\kappa}{\sigma} \sum_{i \in J} \mathbb{E}\{|\bar{X}_i|^2 1(|\bar{X}_i| \geq \sigma/\kappa)\} = \kappa\sigma\beta_2. \]

For \( T_2 \), as \( \bar{V}^{(i)} \geq \sigma/2 \) and \( \|f\| \leq 1 \), by (4.15), we have

\[ |T_2| \leq \frac{2}{\sigma} \sum_{i \in J} |\mathbb{E}\{ \bar{X}_i \}| \leq 2\kappa\beta_2. \]

Combining (4.14) and (4.16), we complete the proof. \( \square \)
The next lemma provides upper bounds for the fourth moments of $\sum_{i \in J} \bar{X}_i$ and $\sum_{i \in J} \bar{Y}_i$.

**Lemma 4.3.** Under (LD1) and (LD2), and assume that $\beta_2 \leq 1/(150\kappa)$. We have

$$
E\left( \sum_{i \in J} \bar{X}_i \right)^4 \leq 1161(\theta + 1)\sigma^4, \quad (4.17)
$$

$$
E\left( \sum_{i \in J} \bar{Y}_i \right)^4 \leq 1161\kappa^2(\theta + 1)\sigma^4. \quad (4.18)
$$

As a consequence,

$$
E\left\{ \left( \sum_{i \in J} \bar{X}_i \right)^2 \left( \sum_{j \in J} \bar{Y}_j \right)^2 \right\} \leq 1161\kappa(\theta + 1)\sigma^4. \quad (4.19)
$$

**Proof.** We first prove the first inequality. Recall that $EX_i = 0$ and thus

$$
|E\bar{S}| = \left| \sum_{i \in J} E\bar{X}_i \right| = \left| \sum_{i \in J} E\{X_i I(|X_i| > \sigma/\kappa)\} \right| \leq \beta_2 \kappa \sigma. \quad (4.20)
$$

Let $\bar{X}_{i,0} = \bar{X}_i - E\{\bar{X}_i\}$ and let $\bar{Y}_{i,0} = \sum_{j \in A_i} \bar{X}_{j,0}$. Under (LD1), we have

$$
E|\bar{S} - E\bar{S}|^4 = \sum_{i \in J} E\left\{ \bar{X}_{i,0} \left( \left( \sum_{i \in J} \bar{X}_{j,0} \right)^3 - \left( \sum_{i \in J} \bar{X}_{j,0} \right)^2 \right) \right\}
$$

$$
= 3 \sum_{j \in J} \sum_{j \in A_i} E\left\{ \bar{X}_{i,0} \bar{X}_{j,0} \left( \sum_{k \in J} \bar{X}_{k,0} \right)^2 \right\}
$$

$$
- 3 \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in J} E\{X_{i,0}X_{j,0}X_{k,0}X_{i,0}\}
$$

$$
+ \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in J} E\{X_{i,0}X_{j,0}X_{k,0}X_{i,0}\}
$$

$$
\leq T_1 + T_2,
$$

where

$$
T_1 = \frac{9}{2} \sum_{j \in J} E\left\{ |X_{i,0}Y_{i,0}| \left( \sum_{k \in J} \bar{X}_{k,0} \right)^2 \right\},
$$

$$
T_2 = \frac{5}{2} \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in J} E\{X_{i,0}X_{j,0}X_{k,0}X_{i,0}\}.
$$

By (LD2), we have

$$
|T_1| \leq 9 \sum_{i \in J} E\left\{ |X_{i,0}Y_{i,0}| \left( \sum_{k \in J} \bar{X}_{k,0} \right)^2 \right\} + 9 \sum_{i \in J} E\left\{ |X_{i,0}Y_{i,0}| \left( \sum_{k \in J} \bar{X}_{k,0} \right)^2 \right\}
$$

$$
:= T_{11} + T_{12}.
$$
For $T_{11}$, noting that $(\tilde{X}_{i,0}, \tilde{Y}_{i,0})$ is independent of $\{\tilde{Y}_{k,0} : j \in B_i^c\}$, then

$$T_{11} = 9 \sum_{i \in J} \mathbb{E}[\tilde{X}_{i,0}\tilde{Y}_{j,0}] \times \mathbb{E}\left\{ \left( \sum_{k \in B_i^c} \tilde{X}_{k,0} \right)^2 \right\}. \quad (4.21)$$

Note that $|\tilde{X}_{i,0}| \leq 2\sigma/\kappa$. By Lemma 4.1, the second expectation of (4.21) can be bounded by

$$\mathbb{E}\left\{ \left( \sum_{k \in B_i^c} \tilde{X}_{k,0} \right)^2 \right\} \leq 2 \mathbb{E}\left\{ \left( \sum_{k \in J} \tilde{X}_{k,0} \right)^2 \right\} + 2 \mathbb{E}\left\{ \left( \sum_{k \in B_i} \tilde{X}_{k,0} \right)^2 \right\} \leq 2 \mathbb{E}|\tilde{S}|^2 + 8\sigma^2 \leq 10.04\sigma^2. \quad (4.22)$$

Recalling that $\theta = \sum_{i \in J} \sum_{j \in A_i} \mathbb{E}[|\tilde{X}_{i,j}/\sigma|^2]$, and noting that $|\tilde{Y}_{i}| \leq 1$, we have the first expectation of (4.21) is bounded by

$$\sum_{i \in J} \mathbb{E}[|\tilde{X}_{i,0}\tilde{Y}_{j,0}|] \leq \sum_{i \in J} (\mathbb{E}[|\tilde{X}_{i,0}|] + 3\lambda \mathbb{E}[|\tilde{X}_{i,0}|]) \leq \theta \sigma^2 + 3\sigma^2 \kappa \beta_2 \leq (\theta + 0.02)\sigma^2, \quad (4.23)$$

where we used (4.15) and the assumption that $\beta_2 \leq 1/(150\kappa)$ in the last line. Substituting (4.22) and (4.23) to (4.21) gives

$$T_{11} \leq 99(\theta + 1)\sigma^4. \quad (4.24)$$

For $T_{12}$, as $\sum_{k \in B_i} |\tilde{X}_{k,0}| \leq 2\sigma$, we have $(\sum_{k \in B_i} |\tilde{X}_{k,0}|)^2 \leq 4\sigma^2$ almost surely, and thus

$$T_{12} \leq 36 \sum_{i \in J} \mathbb{E}[|\tilde{X}_{i,0}\tilde{Y}_{j,0}|] \leq 36(\theta + 1)\sigma^4. \quad (4.25)$$

By (4.23) and noting that $(\sum_{k \in A_i} \tilde{X}_{k,0})^2 \leq 4\sigma^2$ almost surely, we have

$$|T_2| \leq 10(\theta + 1)\sigma^4. \quad (4.26)$$

By (4.20) and (4.24)–(4.26), and recalling that $\beta_2 \leq 1/(150\kappa^2)$ and $\kappa \geq 1$, we have

$$\mathbb{E}[|\tilde{S}|^4] \leq 8(\mathbb{E}[|\tilde{S}|] + |\mathbb{E}[\tilde{S}]|^4) \leq 8(145(\theta + 1)\sigma^4 + \beta_2^4\sigma^4) \leq 1161(\theta + 1)\sigma^4. \quad (4.27)$$

This proves (4.17).

For the second inequality, observe that

$$\sum_{i \in J} \tilde{Y}_i = \sum_{i \in J} \sum_{j \in A_i} \tilde{X}_j = \sum_{j \in J} \lambda_j \tilde{X}_j,$$

where $\lambda_j := |\{i : j \in A_i\}|$. Then, we have $\lambda_j \leq \kappa$ for all $j \in J$. Using a similar argument, we have (4.18) holds. The inequality (4.19) follows from (4.17) and (4.18) by applying Cauchy’s inequality.
4.2 Concentration inequalities

In this subsection, we prove a concentration inequality under (LD1) and (LD2). We denote by \(C, C_1, C_2, \ldots\) absolute constants.

**Proposition 4.4.** Assume that (LD1) and (LD2) hold and assume that \(\beta_2 \leq 1/(150\kappa)\) and \(\beta_3 \leq 1/(150\kappa^2)\). Let \(\mathcal{F}_i = \sigma(X_j : j \in A_i)\). For any \(z \in \mathbb{R}\) and \(\mathcal{F}_i\)-measurable random variables \(a\) and \(b\) such that \(a \leq b\), we have

\[
\mathbb{P}^{\mathcal{F}_i}(z + \frac{a}{\bar{V}} \leq \bar{W} \leq z + \frac{b}{\bar{V}}) \\
\leq \frac{2(b-a)}{\sigma} + C_1\kappa^2\left(\beta_3 + \sum_{j \in N(B_i)} (\mathbb{E}^{\mathcal{F}_i} |X_j/\sigma|^3 + \mathbb{E}^{\mathcal{F}_i} |S_{X_j}^3/\sigma^4|)\right),
\]

where \(\mathbb{P}^{\mathcal{F}_i}\) and \(\mathbb{E}^{\mathcal{F}_i}\) denote the conditional probability and conditional expectation given \(\mathcal{F}_i\), respectively.

**Proof of Proposition 4.4.** We use the idea of Proposition 3.2 in Chen and Shao (2004) to prove this proposition. Let \(\alpha = 20\kappa^2\beta_3\). Without loss of generality, we assume that \(b - a \leq \sigma/2\), otherwise the result is trivial. As \(\kappa \geq 1\) and by assumption we have \(\kappa^2\beta_3 \leq 1/150\). Therefore, we have

\[b - a + \alpha \leq 0.8\sigma.\]  \hfill (4.28)

Let

\[
f_{a,b,v}(w) = \begin{cases} 
-\frac{b-a+\alpha}{2v} & \text{if } w \leq z + a/v - \alpha/v, \\
\frac{1}{2\alpha}(w - \frac{a+\alpha}{v})^2 - \frac{b-a+\alpha}{2v} & \text{if } z + \frac{a-\alpha}{v} < w \leq z + \frac{a}{v}, \\
\frac{1}{2\alpha}(w - \frac{b}{v})^2 + \frac{b-a+\alpha}{2v} & \text{if } z + \frac{b}{v} < w \leq z + \frac{b+\alpha}{v}, \\
-\frac{b-a+\alpha}{2v} & \text{if } w > z + (b+\alpha)/v.
\end{cases}
\]

Then, we have

\[
f_{a,b,v}'(w) = \begin{cases} 
1 & \text{if } z + a/v \leq w \leq z + b/v, \\
0 & \text{if } w < z + a/v - \alpha/v \text{ or } w > z + b/v + \alpha/v, \\
\text{linear} & \text{otherwise.}
\end{cases}
\]

Let \(f := f_{a,b,\bar{V}}\). Then, it follows from \(\bar{V} \geq \sigma/2\) that \(\|f\| \leq (b-a+\alpha)/(2\bar{V}) \leq (b-a+\alpha)/\sigma\).

Fix \(i\). Recall that \(N(A_i) = \{j : A_i \cap A_j \neq \emptyset\}\) and \(N(B_i) = \{j : B_i \cap A_j \neq \emptyset\}\).
Let $\sigma_i = (\mathbb{E}\{\sum_{j \in N(B_i)} \bar{X}_j\}^2)^{1/2}$. Then, recalling that $\beta_3 \leq 1/(150\kappa^2)$ and by Hölder's inequality, we have

$$\sigma_i \leq \left(\frac{N(B_i)^2}{\mathbb{E}|\bar{X}_j|^3}\right)^{1/3} \leq \left(\kappa^2\sigma^3\beta_3\right)^{1/3} \leq 0.2\sigma.$$  

On one hand, as $\|f\| \leq (b - a + \alpha)/\sigma$, $\bar{V} \geq \sigma/2$, and $\{X_j : j \notin N(B_i)\}$ is independent of $\mathcal{F}_i$, we have

$$\left|\mathbb{E}^{\mathcal{F}_i}\left\{\left(\sum_{j \notin N(B_i)} \xi_j\right)f(W)\right\}\right| \leq \frac{2(b - a + \alpha)}{\sigma^2} \mathbb{E}\left|\sum_{j \notin N(B_i)} \bar{X}_j\right| \leq \frac{2\sigma_i}{\sigma^2}(b - a + \alpha) \leq \frac{b - a + \alpha}{2\sigma}. \quad (4.31)$$

Let

$$\dot{M}_j(t) = \bar{X}_j(1 - \bar{Y}_j \leq t \leq 0) - 1(0 < t \leq -\bar{Y}_j), \quad \bar{M}_j(t) = \mathbb{E}\{\dot{M}_j(t)\},$$

$$\bar{M}(t) = \sum_{j \in N(B_i)} \dot{M}_j(t), \quad \bar{M}(t) = \mathbb{E}\{\bar{M}(t)\}.$$

For the lower bound of the left hand side of (4.31), observe that

$$\mathbb{E}^{\mathcal{F}_i}\left\{\left(\sum_{j \notin N(B_i)} \xi_j\right)f(\bar{W})\right\} = \sum_{j \notin N(B_i)} \mathbb{E}^{\mathcal{F}_i}\{\xi_j(f(\bar{W}) - f(\bar{W} - \eta_j))\} + \sum_{j \notin N(B_i)} \mathbb{E}^{\mathcal{F}_i}\{\xi_j f(\bar{W} - \eta_j)\} \quad (4.32)$$

where

$$Q_1 = \mathbb{E}^{\mathcal{F}_i}\left\{\frac{f'(\bar{W})}{\bar{V}^2} \int_{-\infty}^{\infty} \bar{M}(t)dt\right\},$$

$$Q_2 = \mathbb{E}^{\mathcal{F}_i}\left\{\frac{1}{\bar{V}^2} \int_{-\infty}^{\infty} \left(f'(\frac{\bar{S} + t}{\bar{V}}) - f'(\frac{\bar{S}}{\bar{V}})\right) \bar{M}(t)dt\right\},$$

$$Q_3 = \mathbb{E}^{\mathcal{F}_i}\left\{\frac{1}{\bar{V}^2} \int_{-\infty}^{\infty} f'(\frac{\bar{S} + t}{\bar{V}})(\dot{M}(t) - \bar{M}(t))dt\right\},$$

$$Q_4 = \sum_{j \notin N(B_i)} \mathbb{E}^{\mathcal{F}_i}\{\xi_j f(\bar{W} - \eta_j)\}.$$

We now bound $Q_1$ to $Q_4$ one by one.

Observe that

$$\int_{-\infty}^{\infty} \bar{M}(t)dt = \sum_{j \in N(B_i)} \mathbb{E}\{\bar{X}_j \bar{Y}_j\} = \sigma^2 - \sum_{j \in N(B_i)} \mathbb{E}\{\bar{X}_j \bar{Y}_j\}. \quad (4.33)$$

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Now, by Hölder’s inequality, the second term of the R.H.S. of (4.33) is bounded by

\[
\left| \sum_{j \in \mathcal{N}(B_i)} \mathbb{E}\{\bar{X}_j \bar{Y}_j\} \right| \leq \left( |\mathcal{N}(B_i)|^{1/2} \sum_{j \in \mathcal{N}(B_i)} \mathbb{E} |\bar{X}_j \bar{Y}_j|^{3/2} \right)^{2/3} \\
\leq \left( \kappa^{1/2} \sum_{j \in \mathcal{N}(B_i)} \left[ \frac{\kappa^{3/2}}{2} \mathbb{E} |\bar{X}_j|^3 + \frac{1}{2\kappa^{3/2}} \mathbb{E} |\bar{Y}_j|^3 \right] \right)^{2/3} \\
\leq (\kappa^2 \beta_3)^{2/3} \sigma^2 \leq (1/150)^{2/3} \sigma^2 \leq 0.1 \sigma^2.
\]

Thus, we have

\[
\left| \int_{-\infty}^{\infty} M(t) dt - \sigma^2 \right| \leq 0.1 \sigma^2.
\]

Note that \(\sigma^2/4 \leq \bar{V}^2 \leq 2\sigma^2\). For \(Q_1\), we have

\[
Q_1 \geq 0.9 \mathbb{E} F \left\{ \frac{\sigma^2}{V^2} \mathbb{I}(z + a/\bar{V} \leq \bar{W} \leq a + b/\bar{V}) \right\} \\
\geq 0.45 \mathbb{P} F_i \left( z + a/\bar{V} \leq \bar{W} \leq a + b/\bar{V} \right).
\]

For \(Q_3\),

\[
|Q_3| \leq Q_{31} + Q_{32},
\]

where

\[
Q_{31} = \mathbb{E} F_i \int_{|t| \leq \sigma} \frac{1}{V^2} \left| f' \left( \frac{\bar{S} + t}{V} \right) \right| |\tilde{M}(t) - M(t)| dt, \\
Q_{32} = \mathbb{E} F_i \int_{|t| > \sigma} \frac{1}{V^2} |\tilde{M}(t) - M(t)| dt.
\]

For \(Q_{31}\), noting that \(\|f'\| \leq 1\), we have

\[
\frac{1}{v} \int_{|t| \leq \sigma} \left| f' \left( \frac{s + t}{v} \right) \right|^2 dt \leq \int_{|t| \leq \sigma/v} f' \left( \frac{s}{v} + t \right) dt \leq b - a + \alpha,
\]

and thus, by Cauchy’s inequality,

\[
Q_{31} \leq \frac{1}{8} \mathbb{E} F_i \int_{|t| \leq \sigma} \frac{1}{V^2} \left| f' \left( \frac{\bar{S} + t}{V} \right) \right|^2 dt + 2 \mathbb{E} F_i \int_{|t| \leq \sigma} \frac{1}{V^2} |\tilde{M}(t) - M(t)|^2 dt \\
\leq \frac{b - a + \alpha}{4\sigma} + \frac{8}{\sigma^2} \int_{|t| \leq \sigma} \mathbb{E} |\tilde{M}(t) - M(t)|^2 dt,
\]

(4.35)
where we used the fact that \( \tilde{V} \geq \sigma/2 \) and \( \hat{M}(t) \) is independent of \( F_i \). For the second term of (4.35), letting \((X_j^*, \hat{Y}_j^*)\) be an independent copy of \((X_j, \hat{Y}_j)\), we have

\[
\frac{1}{\sigma^2} \int_{|t| \leq \sigma} \mathbb{E}[\hat{M}(t) - M(t)]^2 dt \\
= \frac{1}{\sigma^2} \sum_{j \in N(B_i)^c} \sum_{l \in N(B_j)^c \cap N(B_i)^c} \int_{|t| \leq \sigma} \text{Cov}\{\hat{M}_j(t), \hat{M}_l(t)\} dt \\
= \frac{1}{\sigma^2} \sum_{j \in N(B_i)^c} \sum_{l \in N(B_j)^c} \int_{|t| \leq \sigma} \mathbb{E}\{\hat{X}_j \hat{X}_l (|\hat{Y}_j| \wedge |\hat{Y}_l|) 1(\hat{Y}_j \hat{Y}_l \geq 0) \\
- \hat{X}_j \hat{X}_l^* (|\hat{Y}_j| \wedge |\hat{Y}_l^*|) 1(\hat{Y}_j \hat{Y}_l^* \geq 0)\} dt \\
\leq \frac{1}{\sigma^2} \sum_{j \in J} \sum_{l \in N(B_j)} \mathbb{E}\{|\hat{X}_j \hat{X}_l^* (|\hat{Y}_j| \wedge |\hat{Y}_l^*|) + |\hat{X}_j \hat{X}_l^*^* (|\hat{Y}_j| \wedge |\hat{Y}_l^*|)\} \leq 2\kappa^2 \beta_3.
\]

Substituting (4.36) to (4.35) yields

\[ Q_{31} \leq \frac{0.25(b - a + \alpha)}{\sigma} + 16\kappa^2 \beta_3. \tag{4.37} \]

For \( Q_{32} \), noting that

\[ \mathbb{E} \int_{-\infty}^{\infty} |t| |\hat{M}(t)| dt \leq \frac{1}{2} \sum_{i \in J} \mathbb{E}\{|\hat{X}_i \hat{Y}_i^2|\} \leq \frac{\kappa^2}{2} \sum_{i \in J} \mathbb{E}|\hat{X}_i|^3, \tag{4.38} \]

we have

\[ Q_{32} \leq \frac{4}{\sigma^2} \mathbb{E} \int_{|t| > \sigma} |t||\hat{M}(t) - M(t)| dt \leq \frac{8}{\sigma^2} \mathbb{E} \int_{-\infty}^{\infty} |t| |\hat{M}(t)| dt \leq 4\kappa^2 \beta_3. \tag{4.39} \]

Combining (4.37) and (4.39), we have

\[ |Q_3| \leq \frac{0.125(b - a + \alpha)}{\sigma} + 20\kappa^2 \beta_3. \tag{4.40} \]

Let \( f_{a,b,V(j)} \) be defined as in (4.30) by taking \( v = \hat{V}(j) \). For \( Q_4 \), we have

\[
Q_4 = \sum_{j \in N(B_i)^c} \mathbb{E} F_i \left\{ \bar{X}_j \left[ \frac{1}{V} f \left( \frac{\bar{S} - \bar{Y}_j}{V} \right) - \frac{1}{\hat{V}(j)} f \left( \frac{\bar{S} - \bar{Y}_j}{\hat{V}(j)} \right) \right] \right\} \\
+ \sum_{j \in N(B_i)^c} \mathbb{E} F_i \left\{ \bar{X}_j \left[ \frac{1}{\hat{V}(j)} f \left( \frac{\bar{S} - \bar{Y}_j}{\hat{V}(j)} \right) - \bar{X}_j \frac{1}{\hat{V}(j)} f_{a,b,\hat{V}(j)} \left( \frac{\bar{S} - \bar{Y}_j}{\hat{V}(j)} \right) \right] \right\} \\
+ \sum_{j \in N(B_i)^c} \mathbb{E} F_i \left\{ \bar{X}_j \frac{1}{\hat{V}(j)} f_{a,b,\hat{V}(j)} \left( \frac{\bar{S} - \bar{Y}_j}{\hat{V}(j)} \right) \right\} \\
:= Q_{41} + Q_{42} + Q_{43}.
\]
For any \( j \in N(B_i)^c \), by (4.10) and (4.11), we have

\[
Q_{41} \leq \sum_{j \in N(B_i)^c} \mathbb{E}^{F_i} \left[ \left\{ \frac{\bar{X}_j}{V} f \left( \frac{\bar{S} - \bar{Y}_j}{V} \right) \right\} - \left\{ \frac{\bar{X}_j}{V} f \left( \frac{\bar{S} - \bar{Y}_j}{V} \right) \right\} \right] \\
\leq \sum_{j \in N(B_i)^c} \mathbb{E}^{F_i} \left\{ \left( \|f\| + \frac{2|S - \bar{Y}_j|}{\sigma} \right) |X_j| \left\| \frac{1}{V} - \frac{1}{V(U)} \right\| \right\} \\
\leq \frac{4}{\sigma^4} \sum_{j \in N(B_i)^c} \mathbb{E}^{F_i} \left\{ (b - a + \alpha) + 2|S - \bar{Y}_j| |X_j| \sum_{l \in N(A_j)} |\bar{X}_l|Y_l \right\} \\
\leq \frac{4(b - a + \alpha)}{3\sigma^4} Q_{44} + \frac{8}{3\sigma^4} Q_{45},
\]

where

\[
Q_{44} = \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \sum_{k \in A_l} \mathbb{E}^{F_i} \left\{ (|\bar{X}_j|^3 + |\bar{X}_l|^3 + |\bar{X}_k|^3) \right\}, \\
Q_{45} = \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \sum_{k \in A_l} \mathbb{E}^{F_i} \left\{ (|\bar{S} - \bar{Y}_j|) (|\bar{X}_j|^3 + |\bar{X}_l|^3 + |\bar{X}_k|^3) \right\}.
\]

Noting that \( |N(B_i)| \leq \kappa, |N(A_j)| \leq \kappa, |\{ j : A_j \cap A_l \neq \emptyset \}| \leq \kappa \) and \( |A_l| \leq \kappa \), we have

\[
Q_{44} \leq \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \sum_{k \in A_l} \mathbb{E}^{F_i} |\bar{X}_j|^3 + \sum_{k \in N(B_i)^c} \sum_{l \in A_j} \sum_{j : A_j \cap A_l \neq \emptyset} \mathbb{E}^{F_i} |\bar{X}_k|^3 \\
+ \sum_{l \in N(B_i)^c} \sum_{j : A_j \cap A_l \neq \emptyset} \sum_{k \in A_l} \mathbb{E}^{F_i} |\bar{X}_l|^3 + \sum_{k \in N(B_i)^c} \sum_{l \in A_j} \sum_{j : A_j \cap A_l \neq \emptyset} \mathbb{E}^{F_i} |\bar{X}_k|^3 \\
\leq 3\kappa^2 \sigma^3 \beta_3 + 2\kappa^2 \sum_{j \in N(B_i)^c} \mathbb{E}^{F_i} |\bar{X}_j|^3,
\]

where we used the fact that \( \mathbb{E}^{F_i} |\bar{X}_j|^3 = \mathbb{E} |\bar{X}_j|^3 \) for \( j \in N(B_i)^c \).

Now, we bound \( Q_{45} \). For any \( l \), let \( \bar{S}_l = \bar{S} - \sum_{k \in N(B_i) \cup A_j \cup A_l} \bar{X}_k \), then \( |\bar{S} - \bar{Y}_j| - |\bar{S}_l| \leq 2\sigma \). Also, if \( l \in N(B_i)^c \), we have \( \bar{S}_l \) is independent of \( \bar{X}_l \) and \( (\bar{S}_l, \bar{X}_l) \) is independent of \( F_i \). Noting that \( \sigma^2 \leq 1.02\sigma^2 \) by (4.6) and that \( \kappa \geq 1 \), we have

\[
\mathbb{E} |\bar{S}_l| \leq \mathbb{E} |\bar{S}| + 3\kappa \sigma \leq \bar{\sigma} + 3\sigma \leq 4.02\sigma.
\]

Therefore, for \( l \in N(B_i)^c \),

\[
\mathbb{E}^{F_i} |(\bar{S} - \bar{Y}_j) \bar{X}_l|^3 \leq \mathbb{E} |\bar{S}_l \bar{X}_l|^3 + 2\sigma \mathbb{E} |\bar{X}_l|^3 \\
\leq \mathbb{E} |\bar{S}_l| \mathbb{E} |\bar{X}_l|^3 + 2\sigma \mathbb{E} |\bar{X}_l|^3 \leq 6.02\sigma \mathbb{E} |\bar{X}_l|^3.
\]

For \( l \in N(B_i) \), we have

\[
\mathbb{E}^{F_i} |(\bar{S} - \bar{Y}_j) \bar{X}_l|^3 \leq \sigma \mathbb{E}^{F_i} |\bar{X}_l|^3 + \mathbb{E}^{F_i} |\bar{S} \bar{X}_l|^3.
\]
Hence, by (4.46) and (4.47), we then obtain
\[
Q_{45} \leq \mathbb{E}^{F_i} \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \sum_{k \in A_l} |\bar{S} - \bar{Y}_j| |\bar{X}_j|^3 \\
+ \mathbb{E}^{F_i} \sum_{k \in N(B_i)^c} \sum_{l: k \in A_l, j: A_j \cap A_l \neq \emptyset} |\bar{S} - \bar{Y}_j| |\bar{X}_k|^3 \\
+ \mathbb{E}^{F_i} \sum_{l \in N(B_i)^c} \sum_{j: A_j \cap A_l \neq \emptyset} |\bar{S} - \bar{Y}_j| |\bar{X}_l|^3 \\
+ \mathbb{E}^{F_i} \sum_{l \in N(B_i)} \sum_{j: A_j \cap A_l \neq \emptyset} |\bar{S} - \bar{Y}_j| |\bar{X}_l|^3 \\
\leq C_2 \sigma \kappa^2 \left( \sum_{j \in J} \mathbb{E}|\bar{X}_j|^3 + \sum_{j \in N(B_i)} \mathbb{E}^{F_i} |\bar{X}_j|^3 \right) + C_3 \kappa^2 \sum_{j \in N(B_i)} \mathbb{E}^{F_i} |\bar{S} \bar{X}_j^3| .
\]

By (4.45) and (4.48), we have
\[
Q_{41} \leq C_4 \kappa^2 \left( \beta_3 + \sum_{j \in N(B_i)} \mathbb{E}^{F_i} |\bar{X}_j/\sigma|^3 + \sum_{j \in N(B_i)} \mathbb{E}^{F_i} |\bar{S} \bar{X}_j^3/\sigma^4| \right) .
\]

For $Q_{42}$, observing that
\[
|f(w) - f_{a,b,V^{(j)}}| \leq \left| \frac{1}{V} - \frac{1}{V^{(j)}} \right| ,
\]
and by (4.10) and (4.45), we obtain
\[
Q_{42} \leq \frac{2}{\sigma} \sum_{j \in N(B_i)^c} \mathbb{E} \left\{ |\bar{X}_j| \left| \frac{1}{V} - \frac{1}{V^{(j)}} \right| \right\} \\
\leq \frac{8}{\sigma^3} \sum_{j \in N(B_i)^c} \sum_{k \in N(A_j)} \mathbb{E}^{F_i} \left\{ |\bar{X}_j | \bar{X}_k \bar{Y}_k | \right\} \\
\leq \frac{8}{3 \sigma^3} \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \mathbb{E}^{F_i} \left\{ |\bar{X}_j|^3 + |\bar{X}_k|^3 + |\bar{X}_l|^3 \right\} \\
\leq 8 \kappa^2 \beta_3 + 8 \kappa^2 \sum_{j \in N(B_i)} \mathbb{E}^{F_i} |\bar{X}_j/\sigma|^3 .
\]

For $Q_{43}$, as $\beta_2 \leq 1/(150 \kappa) \leq 1/150$, $j \in N(B_i)^c$, $V^{(j)} \geq 0.5 \sigma$ and $\| f_{a,b,V^{(j)}} \| \leq (b - a + \alpha)/\sigma$, by (4.15), we have
\[
|Q_{43}| \leq \frac{2 \| f_{a,b,V^{(j)}} \|}{\sigma} \sum_{j \in N(B_i)^c} |\mathbb{E}^{F_i} \bar{X}_j | = \frac{2}{\sigma^2} (b - a + \alpha) \sum_{j \in N(B_i)^c} |\mathbb{E} \bar{X}_j | \\
\leq 2(b - a + \alpha) \beta_2/\sigma \leq 0.2(b - a + \alpha)/\sigma .
\]
By (4.49)-(4.51), we have
\[ |Q_4| \leq 0.2(b - a + \alpha)/\sigma \]
\[ + C_9\kappa^2 \left( \beta_3 + \sum_{j \in N(B_i)} \mathbb{E}^F_i |X_j/\sigma|^3 + \sum_{j \in N(B_i)} \mathbb{E}^F_i |S X_j^3/\sigma^4| \right). \]  \hfill (4.52)

To bound \( Q_2 \), we note that
\[
\left( f'(\frac{S+t}{V}) - f'(\frac{S}{V}) \right)
= \begin{cases} 
\frac{1}{\alpha} \int_0^t \{ z + a/\bar{V} - \alpha/\bar{V} \leq (S + s)/\bar{V} \leq z + a/\bar{V} \} ds & \text{if } t \geq 0, \\
-\frac{1}{\alpha} \int_0^t \{ z + b/\bar{V} - \alpha/\bar{V} \leq (S + s)/\bar{V} \leq z + a/\bar{V} \} ds & \text{if } t < 0.
\end{cases}
\]

Then, recalling that \( \alpha = 20\kappa^2\sigma \beta_3 \), we have
\[ |Q_2| \leq \frac{4}{\sigma^2} \int_{|t| \geq \sigma} M(t) dt + \frac{8}{\alpha \sigma^2} \int_0^\sigma \int_0^t L(\alpha) ds |M(t)| dt \]
\[ + \frac{8}{\alpha \sigma^2} \int_0^\sigma \int_t^\sigma L(\alpha) ds |M(t)| dt \]
\[ \leq \frac{8}{\sigma^3} (\sigma \alpha^{-1} L(\alpha) + 1) \int_{-\infty}^\infty |t M(t)| dt \]
\[ \leq 4\kappa^2(\sigma \alpha^{-1} L(\alpha) + 1) \beta_3 \]
\[ \leq 0.2 L(\alpha) + 4\kappa^2 \beta_3, \]  \hfill (4.53)

where
\[ L(\alpha) = \lim_{n \to \infty} \sup_{y, x \in \mathbb{Q}} \mathbb{P}^F_i (y + (x - 1/n)/\bar{V} \leq W \leq y + (x + 1/n)/\bar{V} + \alpha/\bar{V} + 1/n), \]

the notation \( \mathbb{Q} \) denotes the set of rational numbers, and we applied (4.38) and Jansen's inequality in the last line.

Combining (4.31), (4.32), (4.34), (4.40), (4.52) and (4.53), we have
\[ 0.45 \mathbb{P}^F_i (z + a/\bar{V} \leq W \leq z + b/\bar{V}) \]
\[ \leq 0.9 \frac{b - a + \alpha}{\sigma} + 0.2 L(\alpha) \]
\[ + C_6\kappa^2 \left( \beta_3 + \sum_{j \in N(B_i)} \mathbb{E}^F_i |X_j/\sigma|^3 + \sum_{j \in N(B_i)} \mathbb{E}^F_i |S X_j^3/\sigma^4| \right). \]  \hfill (4.54)

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Rearranging (4.54) yields
\[ P^F_i(z + a/V \leq \bar{W} \leq z + b/V) \leq \frac{2(b - a + \alpha)}{\sigma} + 0.5L(\alpha) + C_9\kappa^2(\beta_3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{X}_j/\sigma|^3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{S}\bar{X}_j^3/\sigma^4|). \]  
(4.55)

Letting \( a = x - 1/n \) and \( b = x + 1/n + \alpha \), and \( z = y \) in (4.54) and taking supremum over \( y, x \in \mathbb{Q} \) and letting \( n \to \infty \), we have
\[ L(\alpha) \leq 0.5L(\alpha) + C_9\kappa^2(\beta_3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{X}_j/\sigma|^3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{S}\bar{X}_j^3/\sigma^4|), \]
and hence,
\[ L(\alpha) \leq C_9\kappa^2(\beta_3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{X}_j/\sigma|^3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{S}\bar{X}_j^3/\sigma^4|). \]  
(4.56)

Substituting (4.56) to (4.55), and recalling that \( \alpha = 20\kappa^2\beta_3 \), we have
\[ P^F_i(z + a/V \leq \bar{W} \leq z + b/V) \leq \frac{2(b - a)}{\sigma} + C_{10}\kappa^2(\beta_3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{X}_j/\sigma|^3 + \sum_{j \in N(B_i)} E^{F_i}|\bar{S}\bar{X}_j^3/\sigma^4|). \]
This completes the proof. \( \square \)

5 PROOF OF MAIN RESULTS

In this subsection, let \( \bar{X}_i, \bar{Y}_i, \bar{S}, \bar{V} \) and \( \bar{W} \) be defined as in (4.1). We use a truncation argument to prove Theorem 2.1. Specifically, we first prove a Berry–Esseen bound for \( \bar{W} \), and then prove an error bound for \( \sup_{z \in \mathbb{R}} |P(W \leq z) - P(\bar{W} \leq z)| \). Again, we denote by \( C, C_1, C_2, \ldots \) absolute positive constants that may take different values in different places.

Now, we give the following proposition, which provides a Berry–Esseen bound for \( \bar{W} \), and the proof is based on Stein’s method and the concentration inequality approach.

**Proposition 5.1.** Under (LD1) and (LD2). We have
\[ \sup_{z \in \mathbb{R}} |P(\bar{W} \leq z) - \Phi(z)| \leq C((\theta + 1)\kappa^2\beta_3 + \kappa\beta_2). \]

**Proof.** Without loss of generality, noting that \( \kappa \geq 1 \), we assume that \( \beta_2 \leq 1/(150\kappa) \) and \( \beta_3 \leq 1/(150\kappa^2) \), otherwise the inequality is trivial. Let \( f_z \) be the solution to the Stein equation
\[ f'(w) - wf(w) = I(w \leq z) - P(Z \leq z). \]
In what follows, we simply write \( f := f_z \). It can be shown that for all \( w, w' \in \mathbb{R} \),

\[
0 \leq f(w) \leq 1, \quad |f'(w)| \leq 1, \quad |f'(w) - f(w')| \leq 1,
\]

and for all \( w \) and \( t \),

\[
|f'(w + t) - f'(w)| \leq (|w| + 1)|t| + 1(z - |t| \leq w \leq z + |t|).
\]

(5.1)

Note that by the Stein equation, we have

\[
\mathbb{P}(\bar{W} \leq z) - \mathbb{P}(Z \leq z) = \mathbb{E}\{f'(\bar{W}) - \bar{W}f(\bar{W})\}.
\]

Let \( \bar{X}_i \) and \( \bar{Y}_i \) be defined as in (4.1) with \( \tau = \kappa \), and recall that

\[
\xi_i = \frac{\bar{X}_i}{V}, \quad \eta_i = \frac{\bar{Y}_i}{V}, \quad \bar{W}^{(i)} = \bar{W} - \eta_i.
\]

(5.2)

Then, it follows that

\[
\mathbb{E}\{\bar{W}f(\bar{W})\} = \sum_{i \in J} \mathbb{E}\{\xi_i(f(\bar{W}) - f(\bar{W}^{(i)}))\} + \sum_{i \in J} \mathbb{E}\{\xi_if(\bar{W}^{(i)})\}
= \mathbb{E}\left\{\frac{1}{V^2} \int_{-\infty}^{\infty} f'\left(\frac{\bar{W} + u}{V}\right)K(u)du\right\} + \sum_{i \in J} \mathbb{E}\{\xi_if(\bar{W}^{(i)})\},
\]

where

\[
K(u) = \sum_{i \in J} \bar{X}_i \{ I(-\bar{Y}_i \leq u \leq 0) - I(0 < u \leq -\bar{Y}_i) \}.
\]

(5.3)

By (5.1) and (5.2), we then obtain

\[
\mathbb{P}(\bar{W} \leq z) - \mathbb{P}(Z \leq z)
= \mathbb{E}\left\{f'(\bar{W}) \left(1 - \frac{\sum_{i \in J} \bar{X}_i \bar{Y}_i}{V^2}\right)\right\}
- \mathbb{E}\left\{1 \int_{-\infty}^{\infty} \left(f'(\frac{S + t}{V}) - f'(\frac{\bar{S}}{V})\right)\tilde{M}(t)dt\right\} - \sum_{i \in J} \mathbb{E}\{\xi_if(\bar{W}^{(i)})\}
\leq R_1 + R_2 + R_3 + R_4,
\]

(5.4)

where

\[
R_1 = \mathbb{E}\left|1 - \frac{1}{V^2} \sum_{i \in J} \bar{X}_i \bar{Y}_i\right|,
\]

\[
R_2 = \frac{8}{\sigma^3} \mathbb{E}\left\{\left(|\bar{W}| + 1\right) \int_{-\infty}^{\infty} |t\tilde{M}(t)|dt\right\},
\]

\[
R_3 = \frac{4}{\sigma^2} \sum_{i \in J} \mathbb{E}\{\bar{X}_i \bar{Y}_i |I(z - |\bar{Y}_i|/V \leq \bar{W} \leq z + |\bar{Y}_i|/V)\}.
\]

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\[ R_4 = \left| \sum_{i \in J} \mathbb{E} \{ \xi_i f(\tilde{W}^{(i)}) \} \right|. \]

For \( R_1 \), by (4.4)–(4.6), we have

\[
R_1 \leq \mathbb{E} \left\{ \left| 1 - \frac{1}{2\sigma^2} \sum_{i \in J} \tilde{X}_i \tilde{Y}_i \right| \left| \left( \sum_{i \in J} X_i Y_i > 2\sigma^2 \right) \right| \right. \\
+ \mathbb{E} \left\{ \left| 1 - \frac{4}{\sigma^2} \sum_{i \in J} \tilde{X}_i \tilde{Y}_i \right| \left| \left( \sum_{i \in J} X_i Y_i < \sigma^2 / 4 \right) \right| \right. \\
\leq \left( 1 + \frac{4\sigma^2 / \sigma^2}{2} \right) \mathbb{P} \left( \sum_{i \in J} \tilde{X}_i \tilde{Y}_i - \bar{\sigma}^2 > \sigma^2 / 2 \right) \\
+ \frac{4}{\sigma^2} \mathbb{E} \left| \sum_{i \in J} \tilde{X}_i \tilde{Y}_i - \bar{\sigma}^2 \right| \left| \left( \sum_{i \in J} \tilde{X}_i \tilde{Y}_i - \bar{\sigma}^2 \right) > \sigma^2 / 2 \right| \\
\leq 100\kappa^2 \beta_3.
\] 

(5.5)

For \( R_2 \), we have

\[
R_2 \leq \frac{4}{\sigma^3} \sum_{i \in J} \mathbb{E} \left\{ |\tilde{W}| + 1 \right| |\tilde{X}_i \tilde{Y}_i^2| \right\} \leq \frac{4}{\sigma^3} \sum_{i \in J} \mathbb{E} \left\{ \left( \frac{2}{\sigma} |\bar{S}| + 1 \right) |\tilde{X}_i \tilde{Y}_i^2| \right\}.
\]

Let

\[
\bar{S}^{(B_i)} = \sum_{j \notin B_i} \tilde{X}_j.
\]

Then, \( \bar{S}^{(B_i)} \) is independent of \( (\tilde{X}_i, \tilde{Y}_i) \), and \( |\bar{S} - \bar{S}^{(B_i)}| \leq \sigma \). Moreover, by (4.6),

\[
\mathbb{E} |\bar{S}^{(B_i)}| \leq \mathbb{E} |\bar{S}| + \bar{\sigma} + \sigma \leq 2.02\sigma.
\]

Thus, we have

\[
\mathbb{E} |\bar{S} \tilde{X}_i \tilde{Y}_i^2| \leq \mathbb{E} |\bar{S}^{(B_i)}| \mathbb{E} |\tilde{X}_i \tilde{Y}_i^2| + \sigma \mathbb{E} |\tilde{X}_i \tilde{Y}_i^2| \leq 3.02\sigma \mathbb{E} |\tilde{X}_i \tilde{Y}_i^2|.
\]

Moreover, by Hölder’s inequality, for any \( c > 0 \),

\[
\sum_{i \in J} \mathbb{E} |\tilde{X}_i \tilde{Y}_i^2| \leq \sum_{i \in J} \mathbb{E} \left( \frac{c^3 |\tilde{X}_i|^3}{3} + \frac{2|\tilde{Y}_i|^3}{3^3/2} \right) \\
\leq \sum_{i \in J} \mathbb{E} \left( \frac{c^3 |\tilde{X}_i|^3}{3} + \kappa^2 \sum_{j \in A_i} \frac{2|\tilde{X}_j|^3}{3^3/2} \right) \\
\leq \frac{c^3 \sigma \beta_3}{3} + \frac{2\kappa^3 \sigma \beta_3}{3^3/2}.
\]
Choosing $c = \kappa^{2/3}$, we have
\[
\sum_{i \in J} \mathbb{E}|\bar{X}_i \bar{Y}_i^2| \leq \kappa^2 \sigma^3 \beta_3.
\]
Therefore,
\[
R_2 \leq 16(\kappa + 1) \sum_{i \in J} \mathbb{E}|\bar{X}_i \bar{Y}_i^2/\sigma^3| \leq 16\kappa^2 \beta_3. \tag{5.6}
\]
For $R_3$, by Proposition 4.4 and noting that $|\bar{X}_i| \leq \sigma/\kappa$, we have
\[
R_3 \leq C_1 \left( \sum_{i \in J} \mathbb{E}|\bar{X}_i \bar{Y}_i^2/\sigma^3| + \kappa^2 \beta_3 \sum_{i \in J} \mathbb{E}|\bar{X}_i \bar{Y}_i^2| \right)
+ \kappa^2 \sum_{i \in J} \sum_{j \in N(B_i)} \mathbb{E}|\bar{X}_i \bar{Y}_i \bar{X}_j / \sigma^3| + \kappa^2 \sum_{i \in J} \sum_{j \in N(B_i)} \mathbb{E}|\bar{S}_i \bar{Y}_i \bar{X}_j / \sigma^4| \leq C_2 \kappa^2 (\beta_3 + \kappa^2 \beta_3 + \beta_3 + \beta_3) \leq C_3 (\theta + 1) \kappa^2 \beta_3. \tag{5.7}
\]
For $R_4$, by Lemma 4.2, we have
\[
R_4 \leq C_4 (\kappa^2 \beta_3 + \kappa \beta_2). \tag{5.8}
\]
By (5.4)–(5.8), we have
\[
\mathbb{P}(\bar{W} \leq z) - \mathbb{P}(W \leq z) \leq C_5 (\theta + 1) \kappa^2 \beta_3 + C_6 \kappa \beta_2.
\]
The lower bound can be obtained similarly. This completes the proof. \qed

Now we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Assume without loss of generality that
\[
\beta_2 \leq \frac{1}{150 \kappa}, \quad \beta_3 \leq \frac{1}{150 \kappa^2}.
\]
Recall the function $\psi$ in (4.2) and $\bar{S}$ in (4.1), and define
\[
\tilde{V} = \psi \left( \sum_{i \in J} (\bar{X}_i \bar{Y}_i - \bar{X} \bar{Y}) \right), \quad \tilde{W} = \bar{S} / \tilde{V}. \tag{5.9}
\]
Then, we have
\[
\sup_{z \in \mathbb{R}} |\mathbb{P}(\bar{W} \leq z) - \mathbb{P}(W \leq z)| \leq \mathbb{P}(\max_{i \in J} |X_i| > \sigma/\kappa) + \mathbb{P}(\sum_{i \in J} X_i \bar{Y}_i \leq \sigma^2/4).
\]
Now,
\[
\mathbb{P}(\max_{i \in J} |X_i| > \sigma/\kappa) \leq \sum_{i \in J} \mathbb{P}(|X_i| > \sigma/\kappa) = \beta_0.
\]
Also, by (4.5), we have

$$\mathbb{P}\left\{ \sum_{i \in J} \bar{X}_i \bar{Y}_i \leq \frac{\sigma^2}{4} \right\} \leq \mathbb{P}\left\{ \left| \sum_{i \in J} (\bar{X}_i \bar{Y}_i - \mathbb{E}[\bar{X}_i \bar{Y}_i]) \right| \geq \frac{\sigma^2}{2} \right\} \leq 4\kappa^2 \beta_3.$$ 

Then,

$$\sup_{z \in \mathbb{R}} |\mathbb{P} (\tilde{W} \leq z) - \mathbb{P} (W \leq z)| \leq 4\kappa^2 \beta_3 + \beta_0. \quad (5.10)$$

For $\epsilon > 0$, let

$$h_{z,\epsilon}(w) = \begin{cases} 1 & \text{if } w \leq z, \\ 0 & \text{if } w > z + \epsilon, \\ \text{linear} & \text{otherwise}. \end{cases}$$

Then, it follows that $h'_{z,\epsilon}(w) = I(z \leq w \leq z + \epsilon) / \epsilon$. Recall that $\tilde{W}$ and $\bar{V}$ as in (4.1), and observe that

$$\mathbb{P}(\tilde{W} \leq z) - \mathbb{P}(\bar{W} \leq z) \leq \mathbb{E}\{h_{z,\epsilon}(\tilde{W}) - h_{z,\epsilon}(\bar{W})\} + \mathbb{P}(z \leq \tilde{W} \leq z + \epsilon)$$

$$\leq \mathbb{E}\{h_{z,\epsilon}(\tilde{W}) - h_{z,\epsilon}(\bar{W})\} + \left( \Phi(z + \epsilon) - \Phi(z) \right) + 2 \sup_{z \in \mathbb{R}} |\mathbb{P}(\tilde{W} \leq z) - \Phi(z)|. \quad (5.11)$$

Note that

$$\left| \frac{1}{V} - \frac{1}{\bar{V}} \right| \leq \frac{|\mathcal{J}| |\bar{X}| \bar{Y}|}{\bar{V} (\bar{V} + \bar{V})},$$

and thus

$$|\mathbb{E}\{h_{z,\epsilon}(\tilde{W}) - h_{z,\epsilon}(\bar{W})\}| \leq \|h'_{z,\epsilon}\| \mathbb{E}|\tilde{W} - \bar{W}|$$

$$\leq \frac{1}{\epsilon} \mathbb{E}\left\{ \frac{|\bar{S}^2 \bar{Y}|}{\bar{V} \bar{V} (\bar{V} + \bar{V})} \right\}$$

$$\leq \frac{4}{\epsilon \sigma^3} \mathbb{E}|\bar{S}^2 \bar{Y}|$$

$$= \frac{4}{\epsilon \sigma^3 |\mathcal{J}|} \mathbb{E}\left| \left( \sum_{i \in \mathcal{J}} \bar{X}_i \right) \left( \sum_{j \in \mathcal{J}} \bar{Y}_j \right) \right|. \quad (5.12)$$

By the Hölder inequality and Lemma 4.3, we have for any $c > 0$,

$$\mathbb{E}\left| \left( \sum_{i \in \mathcal{J}} \bar{X}_i \right) \left( \sum_{j \in \mathcal{J}} \bar{Y}_j \right) \right| \leq \frac{c}{2} \mathbb{E}|\bar{S}^2| + \frac{1}{2c} \mathbb{E}\left| \left( \sum_{i \in \mathcal{J}} \bar{X}_i \right) \left( \sum_{j \in \mathcal{J}} \bar{Y}_j \right) \right|^2$$

$$\leq \frac{c}{2} \sigma^2 + \frac{1}{2c} \left( 1161 \kappa^2 (\theta + 1) \sigma^4 \right).$$

Choosing $c = 35\kappa (\theta + 1) \sigma$, and noting that $\sigma^2 \leq 1.02 \sigma^2$ by Lemma 4.1, we have the expectation term of the right hand side of (5.12) is bounded by

$$\mathbb{E}\left| \left( \sum_{i \in \mathcal{J}} \bar{X}_i \right) \left( \sum_{j \in \mathcal{J}} \bar{Y}_j \right) \right| \leq 35\kappa (\theta + 1) \sigma^3. \quad (5.13)$$
Substituting (5.13) to (5.12) yields
\[ |E\{h_{z,\epsilon}(\tilde{W}) - h_{z,\epsilon}(\bar{W})\}| \leq 140\kappa(\theta + 1)/(\epsilon|\mathcal{J}|). \]
Moreover, for the second term of the right hand side of (5.11),
\[ \Phi(z + \epsilon) - \Phi(z) \leq 0.4\epsilon. \]
Choosing \( \epsilon = 19(\kappa^{1/2}(\theta + 1))/|\mathcal{J}|^{1/2} \), and by (5.11), we have
\[ \mathbb{P}(\tilde{W} \leq z) - \mathbb{P}(\bar{W} \leq z) \leq 16\kappa^{1/2}(\theta + 1)|\mathcal{J}|^{-1/2} + 2\sup_{z \in \mathbb{R}}|\mathbb{P}(\tilde{W} \leq z) - \Phi(z)|. \] (5.14)
The same lower bound also holds by the same argument. Then, we have
\[ \sup_{z \in \mathbb{R}}|\mathbb{P}(\tilde{W} \leq z) - \mathbb{P}(\bar{W} \leq z)| \leq 16\kappa^{1/2}(\theta + 1)|\mathcal{J}|^{-1/2} + 2\sup_{z \in \mathbb{R}}|\mathbb{P}(\tilde{W} \leq z) - \Phi(z)|. \] (5.15)
By (5.10) and (5.15), and applying Proposition 5.1, we completes the proof. □

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