Randomness criteria in binary visibility graph perspective

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April 14, 2010

Abstract

By means of a binary visibility graph, we present a novel method to study random binary sequences. The behavior of the some topological properties of the binary visibility graph, such as the degree distribution, the clustering coefficient, and the mean path length have been investigated. Several examples are then provided to show that the numerical simulations confirm the accuracy of the theorems for finite random binary sequences. Finally, in this paper we propose, for the first time, three topological properties of the binary visibility graph as a randomness criteria.

1 Introduction

The relationship between time series analysis and complex networks have emerged [1, 2]. Zhang et al. introduced a method of mapping between time series and complex networks, they found that, the dynamics of time series are encoded into the topology of the corresponding network [3, 4]. Lacasa et

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al. have proposed an alternative mapping between time series and complex networks based on the visibility graph algorithm, they are able to discriminate uncorrelated randomness from chaos series [5, 6].

Recently, complex network theory has stimulated explosive interests in the study of social, informational, technological and biological systems, resulting in a deeper understanding of complex systems [7, 8, 9, 10]. We apply visibility algorithm as a new method for random binary sequences analysis, which converts binary sequences into complex networks. Whereas the previous works [11, 3] were focused on the dynamics of a complex system is usually recorded in the form of time series, which can be studied through its visibility graph from a complex network perspective. The intent of this paper is to propose a new binary visibility graph (BVG) which stands as a subgraph of the visibility graph. The rest of the paper is organized as follows. In Sec.II we introduce the BVG algorithm. In Sec.III we derive exact results for topological properties of the BVG such as degree distribution, local clustering coefficient, long distance visibility. we propose, for the first time, three topological properties of the BVG as a randomness criteria. This section is followed by an outlook section.

2 Construction of BVG

We start with the description of the visibility graph. By considering an arbitrary sampled time series \( \{u_t : t = 1, 2, ..., N\} \). Each data point of the time series is encoded into a node of the visibility graph. Two arbitrary data points \( u_i \) and \( u_j \) in the time series have visibility, and consequently become two nodes in the associated graph, if any other data point \( u_k \) such that \( i < k < j \) fulfills.

\[
    u_k < u_i + (u_i - u_j) \frac{i - k}{j - i}.
\]

(2-1)

An example of a time series containing 20 data points and the associated visibility graph derived from the visibility algorithm is illustrated in (Fig.1). By definition, any visibility graph extracted from a time series is always connected since each node see, at least its nearest neighbors and the degree of any node \( u_t \) with \( 1 < t < N \) is more than 2. Furthermore, the constructed graph inherits several properties of the series in its structure. Therefore, periodic series convert into regular graphs, random series convert into irregular random graphs and fractal series do so into scale-free networks [5]. It
is also found that a visibility graph is invariant under affine transformation of the series data since the visibility criterion is invariant under rescaling of both horizontal and vertical axes, and under horizontal and vertical transformation [12].

The BVG is an algorithm that maps a binary sequence into a graph (as shown in Fig.2). Here, we briefly describe the binary visibility algorithm in the following way:

Let \( \{x_i\}_{i=1,...,N} \) be a binary sequence of \( N \) bits. The algorithm assigns each bit of the binary sequence to a node in the BVG the algorithm is abbreviated as BVA. Two nodes \( i \) and \( j \) in the BVG are connected if one can draw a visibility line in the binary sequence joining \( x_i \) and \( x_j \) that does not intersect any intermediate bits. \( x_i(x_j) \) can only be 0 and 1. Therefore, \( i \) and \( j \) are two connected nodes if the succeeding geometrical criterion is satisfied with the binary sequence:

\[
x_i + x_j > x_n \quad \text{that} \quad x_n = 0 \quad \text{for all} \quad n \quad \text{such that} \quad i < n < j.
\] (2-2)

It is important to note that, given a binary sequence, its BVG is a subgraph of its associated visibility graph. consequently, as in the former case, the BVG associated with a binary sequence is always connected and undirected, since, each node sees at least its first neighbors (left-hand and right-hand). In what follows we will show that the simplicity of the binary version of the algorithm allows analytical solvability and geometrically simpler, this new method can attest to distinguish between random and non-random binary sequences.

3 Topological properties of the BVG

In order to investigate some statistical characteristics of the binary sequences, the following assumptions are made with respect to random binary sequences to be tested:

**Uniformity:** The occurrence of zeros and ones are of equal probabilities, i.e. if a sequence is of length \( n \), the expected number of ones (or zeros) is \( n/2 \).

**Scalability:** Any subsequences should have the same statistic characters with the sequence they randomly extracted from, i.e. any test applicable to a sequence can also be applied to the
subsequences.

**Consistency:** The behavior of a generator must be consistent across starting values (seeds).

Under above framework, The National Institute of Standards and Technology (*NIST*) statistical tests suite (which can be freely down-loaded from website http://csrc.nist.gov/rng/) for random binary sequences offers a battery of sixteen statistical tests [13]. In the following three subsections we will present three intuitive interpretations of the topological properties of the BVG.

### 3.1 Degree distribution

Let us consider a bi-infinite binary sequence created from a binary valued random variable $X$ (with $x$ as its values) such that $x \in \{0, 1\}$. For simplicity, we will label a generic bit $x_0$ as the “seed” bit hereafter. In order to obtain the degree distribution $P(k)$ [14] of the associated graph, we are going to estimate the probability of an arbitrary bit having $x_0$ value which can be observe, $k$ other bits. If $k$ bits are observed by $x_0$, there will be encounter with two bounding bits with values on each side, one on the right-hand side of $x_0$ and the other on its (L.H.S). So that the $k - 2$ visible bits will be located in that window, i.e. they are zeros. This implies the minimum possible degree is $k = 2$.

As these “inner” bits should appear sorted by its position from seed(being on the left or right side if depending in the position of the seed), Hence we can say that there are exactly $k - 1$ different possible configurations $\{C_i\}_{i=0,\ldots,k-2}$, where the index $i$ determines the number of inner bits on the right-hand side of $x_0$ (see Fig.3). It should be mentioned that the case where $k = 4$ and $x_0 = 0$ is an exception, since the seed is always in between two inner bits. In this paper, for a more exacting analysis, we study the cases $x_0 = 0$ and $x_0 = 1$, separately.

We are calculated for the first example a set of possible configurations for a seed bit $x_0$ with $k = 4$ result denoted in Fig.3. As it is observe the sign of the subindex in $x_i$ depending bit is whether, it is located at the (L.H.S) or (R.H.S) of $x_0$. Therefore, the boundings bits subindex directly indicates
the amount of bits located in that side. As an example, in $x_0 = 1$, $C_0$ is the configuration where none of the $k - 2 = 2$ inner bits are located in the (L.H.S) of $x_0$, and hence the left bounding bits are labeled as $x_{-1}$ and the right bounding bits are labeled as $x_3$. For $x_0 = 0$, $C_0$ is the configuration where one of the $k - 2 = 2$ inner bits are located in the (L.H.S) of $x_0$, and therefore the left bounding bits are labeled as $x_{-2}$ and the right bounding bits are labeled as $x_{n+1}$. Note that $n$ hidden bits can be located in the (R.H.S) of the inner bit. In $x_0 = 1$, $C_1$ is the configuration for which inner bits are located in the (L.H.S) of $x_0$ and another inner bits are located in its (R.H.S). For $x_0 = 0$, $C_1$ is the configuration for which $n_1$ hidden bits are located in the (L.H.S) of $x_0$ and $n_2$ hidden bits are located in its (R.H.S). Finally, in $x_0 = 1$, $C_2$ is the configuration for which both inner bits are located in the (L.H.S) of the seed. For $x_0 = 0$, $C_2$ is the configuration where one of the $k - 2 = 2$ inner bits are located in the (R.H.S) of $x_0$, and therefore the right bounding bits are labeled as $x_2$ and the left bounding bits are labeled as $x_{-(n+1)}$. Notice that $n$ hidden bits can be located in the (R.H.S) of the inner bit (see Fig. 3).

Consequently, $C_i$ corresponds to the configuration for which $i$ inner bits are placed at the (R.H.S) of $x_0$, and $k - 2 - i$ inner bits are placed at its (L.H.S). Each of these possible configurations have an associated probability $p_i \equiv p(C_i)$ that will result in $P(k)$ such that

$$P(k) = \sum_{i=0}^{k-2} p_i.$$  \hspace{1cm} (3-3)

Now, the calculation of a general relation for $P(k)$ should be done in the following steps:

In the first step, we are going to perform to calculation of Eq.(3), for $k = 2$, i.e. the probability that the seed bits have two and only two visible bits. These obviously will be the bounding bits that we will label $x_{-1}$ and $x_1$ for (L.H.S) and (R.H.S) of the seed, respectively. For $k \geq 2$, by taking into account the total probability that $x_0$ sees is 1. Because of any bit in the introduced binary visibility algorithm (sec.2), sees at least its first neighbors. Now, let us look at the particular case for Eq.(3), taken at $k = 2$:

For $x_0 = 0$:

$$p(x_0 = 0) = Prob(x_1, x_{-1} = 1) = \frac{1}{8}$$
For $x_0 = 1$:

$$p(x_0 = 1) = \text{Prob}(x_1, x_{-1} = 1) = \frac{1}{8}$$

Then,

$$P(k = 2) = p(x_0 = 0) + p(x_0 = 1) = \frac{1}{4} \quad \text{(3-4)}$$

In this step, we are going to perform to calculation of Eq.(3), for $k = 3$, i.e., for the seed which has three and only three observable bits. In this process, we encounter with two different configurations: $C_0$, in which $x_0$ has two bounding visible bits ($x_{-1}$ and $x_2$, respectively) and a (R.H.S) inner bit ($x_1$, and the same for $C_1$ but with the inner bit being placed at the (L.H.S) of the seed; so

$$P(k = 3) = p(C_0) + p(C_1) \equiv p_0 + p_1$$

Note that at this point for $x_0 = 0$, an arbitrary number $n$ of hidden bits $b_1, b_2, \ldots, b_n$ can eventually be located between the inner and the bounding bits, and this fact needs to be taken into account in the probability calculation. The geometrical restrictions for the $b_j$ hidden bits are $b_j = 0 (j=1, \ldots, n)$ for $C_0$ and $d_j = 0 (j=1, \ldots, n')$ for $C_1$. Then,

$$p_0(x_0 = 0) = \text{Prob}[(x_{n+1}, x_{-1} = 1) \cap \{b_j = 0\}_{j=1,\ldots,n}],$$

$$p_1(x_0 = 0) = \text{Prob}[(x_{-(n'+1)}, x_1 = 1) \cap \{d_j = 0\}_{j=1,\ldots,n'}].$$

At this stage we have to consider all the hidden bits totally configurations ($C_0$ without hidden bits, $C_0$ with a single hidden bit, $C_0$ with two hidden bits, and so on, and the same for $C_1$). With a little calculation, one obtains

$$p_0(x_0 = 0) = \frac{2}{32}[1 + \sum_{n=2}^{\infty}(\prod_{j=2}^{n} p(n_j))] = \frac{3}{32} \quad \text{(3-5)}$$

where the first term in the square bracket in Eq.(5) corresponds to the contribution of a configuration with no hidden bits and the second sums over the contributions of $n$ hidden bits.

$$p_0(x_0 = 1) = \text{Prob}[(x_2, x_{-1} = 1) \cap (x_1 = 0)] = \frac{2}{32},$$

For a similar result $p_1$ can be find. As a consequence of this similarity the configurations are symmetrical for be $C_0$, $C_1$. Ultimately, one gets

$$P(k = 3) = 2(p_0(x_0 = 0) + p_0(x_0 = 1)) = \frac{10}{32} \quad \text{(3-6)}$$
To continue the evaluation, we need to calculate the contributions due to the Eq.(3), for \( k = 4 \), i.e. for the seed which has four and only four observable bits. For \( x_0 = 1(x_0 = 0) \), we encounter with three different configurations: \( C_0 \), in which \( x_0 \) has two bounding visible bits \( x_{-1}, x_3(x_{-2}, x_{n+1}) \) respectively and two (R.H.S) inner bits \( x_1, x_2(x_{-1}, x_1) \) and the same for \( C_1 \) but with the inner bits being place at the (L.H.S) of the seed; so

\[
P(k = 4) = p(C_0) + p(C_1) + p(C_2) \equiv p_0 + p_1 + p_2
\]

Note at this point that for \( x_0 = 0 \), an arbitrary number \( n \) of hidden bits \( b_1, b_2, \ldots, b_n \) can eventually be located between the inner and the bounding bits, and this fact needs to be taken into account in the probability calculation. The geometrical restrictions for the \( b_j(b_i) \) hidden bits are \( b_j = 0(j=1,\ldots,n2)[b_i = 0( i=-1,\ldots,-n1)] \) for \( C_0 \) and the same for \( C_1, C_2 \). Then,

\[
p_0(x_0 = 0) = \text{Prob}[(x_{n2+1}, x_{-(n1+1)} = 1) \cap (\{b_j = 0\}_{j=1,\ldots,n2}) \cap (\{b_i = 0\}_{i=-1,\ldots,-n1})],
\]

Now, we need to consider every possible hidden bits configuration (\( C_0 \) without hidden bits, \( C_0 \) with a single hidden bit, \( C_0 \) with two hidden bits, and so on, and the same for \( C_1, C_2 \)). With a little calculation, one obtains

\[
p_0(x_0 = 0) = \frac{1}{32}[1 + 2 \sum_{n=2}^{\infty} (\prod_{j=2}^{n} p(n_j))] = \frac{2}{32} \quad \text{(3-7)}
\]

where the first term in the square bracket in Eq.(7) corresponds to the contribution of a configuration with no hidden bits and the second sums over the contributions of \( n1 \) and \( n2 \) hidden bits.

\[
p_0(x_0 = 1) = \text{Prob}[(x_3, x_{-1} = 1) \cap (x_1 = 0) \cap (x_2 = 0)] = \frac{1}{32}.
\]

We obtain similar results for \( p_1(p_2) \) and consequently the configuration provided by \( C_1(C_2) \) is symmetrical to the one provided by \( C_0 \). Ultimately, one gets

\[
P(k = 4) = 3(p_0(x_0 = 0) + p_0(x_0 = 1)) = \frac{9}{32} \quad \text{(3-8)}
\]

Let us proceed by tackling the case \( P(k = 5) \), that is, the probability that the seed has five and only five visible bits. Four different configurations arise: \( C_0 \), in which \( x_0 \) has two bounding visible bits
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\(x_{-1}, x_4\) respectively and three right-hand side inner bits \(x_1, x_2, x_3\) and the same for \(C_1, C_2, C_3\) but with the inner bits being placed at the left-hand side of the seed; so

\[
P(k = 5) = p(C_0) + p(C_1) + p(C_2) + p(C_3) \equiv p_0 + p_1 + p_2 + p_3,
\]

Then,

\[
p_0(x_0 = 1) = \text{Prob}[(x_4, x_{-1} = 1) \cap (x_1 = 0) \cap (x_2 = 0) \cap (x_3 = 0)] = \frac{1}{64},
\]

We can find an identical result for \(p_1(p_2,p_3)\) and consequently the configuration provided by \(C_1(C_2,C_3)\) is similar to the one provided by \(C_0\). Ultimately, one gets

\[
P(k = 5) = 4p_0(x_0 = 1) = \frac{4}{64} \quad \text{(3-9)}
\]

The results of the present calculations are summarized:

\[
P(x_0 = 0) = \begin{cases} 
\frac{4}{32} & k = 2 \\
\frac{6}{32} & k = 3, 4 \\
0 & k \geq 5 
\end{cases}
\]

\[
P(x_0 = 1) = \frac{(k - 1)}{2^{k+1}} \quad k \geq 2
\]

Therefore, we can argue that, for \(k \geq 5\):

\[
P(k) = \frac{(k - 1)}{2^{k+1}} \quad \text{(3-10)}
\]

But, in general,

\[
P(k) = P(x_0 = 0) + P(x_0 = 1) \quad \text{(3-11)}
\]

We can achieve that, the degree distribution \(P(k)\) of the associated BVG has the semiexponential form.

The values of \(\chi^2\) goodness-of-fit test between the theoretical prediction degree distribution Eq. (11) and numerical results demonstrated the measure of uniformity. In order to confirm further the accuracy of our analytical results for the case of finite binary sequences, we have performed several numerical simulations. We have generated random binary sequences of \(10^6\) bits and their associated
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In Fig. 4 we have plotted the degree distribution of the resulting graphs (triangles correspond to a sequence extracted from a CCCBG tent map [15], while circles correspond to one extracted from a CCCBG logistic map [15, 16], respectively). The line is the best fit of the theoretical, showing a perfect agreement with the numerics.

3.2 Local clustering coefficient distribution

By means of geometrical arguments, we can obtain the local clustering coefficient $C$ [8, 7, 17, 18, 14] of a BVG associated with a binary sequence. For a reference node $i$, $C$ means the rate of nodes connected to $i$ that are connected between each other, where $C$ represents the clustering. In other words, we have to work out from a reference node $i$ how many nodes from those visible to $i$ have mutual visibility (triangles), normalized with the set of possible triangles ($\binom{k}{2}$). In a first step, if a generic node $i$ has degree $k = 2$, these nodes are straightforwardly two bounding bits, hence having mutual visibility. Hence, in this condition there exists one triangle and $C(k = 2) = 1$. Now if a generic node $i$ has degree $k = 3$ ($k = 4, 5$), one (two, three) of its neighbors will be an inner bit (two, three bits), which will only have visibility of one of the bounding bits (by construction). We achieve that in this condition we can only form three (five, five) triangles out of three (six, ten) possible ones, thereby:

\[
C(k) = \begin{cases} 
1 & k = 2, 3 \\
\frac{5}{6} & k = 4 \\
\frac{2k-5}{\binom{k}{2}} & k \geq 5
\end{cases}
\]  

(3-12)

This relation between $k$ and $C$ for $k \geq 5$ allows us to deduce the local clustering coefficient distribution $P(C)$ as follows:

\[
P(k) = \frac{k-1}{2^{k+1}} = P\left(\frac{f(C) - C + 4}{C^2 \frac{f(C) - C + 4}{2C} + 4}\right),
\]

Where $f(C) = (C^2 - 32C + 16)^{\frac{3}{2}}$. In general,

\[
P(C) = \begin{cases} 
\frac{f(C) - C + 4}{C^2 \frac{f(C) - C + 4}{2C} + 4} & 0 < C \leq \frac{5}{10} \\
\frac{9}{32} & C = \frac{5}{6} \\
\frac{18}{32} & C = 1
\end{cases}
\]  

(3-13)
To confirm the validity of this latter relation within finite binary sequences, in Fig. 5 we illustrate the clustering distribution of a BVG associated with a random binary sequence of $10^6$ bits (circles) obtained numerically. The line is the best fit of the theoretical and triangles corresponds to the theoretical prediction ($C = 1, \frac{5}{6}$), in excellent agreement with the numerics. The values of $\chi^2$ goodness-of-fit test between the theoretical prediction clustering distribution Eq. (13) and numerical results demonstrated the measure of consistency.

3.3 Long distance visibility, mean degree, mean path length

The mean path length scaling [14], can be derived as below, let us first estimate the probability $P(n)$ that two bits separated by $n$ intermediate bits be two connected nodes in the graph. By taking into account a binary sequence to construct associated BVG. An arbitrary $x_0 = 1$ from the mentioned sequence can be “observe” $x_n = 1$ (and therefore would be connected to node $x_n$ in the graph) if and only if $x_i = 0$ for all $x_i (i = 1, 2, ..., n - 1)$. Then $P(n)$ may be estimated as

$$P(n) = \text{prob}[(x_0, x_n = 1) \cap \{x_i = 0\}_{i=1}^{n-1}] = \frac{1}{2^n}$$

(3-14)

Now, we can derive the mean degree $< k >$ of the binary visibility graph as follows:

$$< k > = \sum kP(k) = 3.5,$$

(3-15)

which we can be obtained from $P(n)$ as

$$< k > = 3.5 \sum_{n=1}^{\infty} P(n) = 3.5.$$ 

(3-16)

At this point, in the Fig. 6 to illustrate the adjacency matrix [14] of the BVG associated with a random binary sequence of 500 bits (if nodes $i$ and $j$ are connected, then the entry $i$, $j$ are filled in black and otherwise they are filled, blank). Since every bit $x_i$ has visibility of its first neighbors $x_{i-1}, x_{i+1}$, every node $i$ will be connected by construction to nodes $i - 1$ and $i + 1$: the graph is thus connected. The Fig. 6 indicates that the graph is very to exact homogeneous structure, i.e. the adjacency matrix is exactly filled around the main diagonal. Moreover, the matrix evidences a superposed compact structure, noticeably the visibility probability $P(n) = \frac{1}{2^n}$ that introduces some
shortcuts in the BVG, much in the vein of the small-world model [12]. Here, the $P(n)$ denotes, the shortcuts probability. From the Statistical point of view, we can interpret the graphs structure as nearly homogeneous, where by increasing the size of graphs, the size of the local neighborhood do not change. Hence, we can approximate its mean path length $L(N)$ as

$$L(N) \approx \sum_{n=1}^{N-1} nP(n) = \sum_{n=1}^{N-1} \frac{n}{2^n} = 2(1 - \frac{N + 1}{2N})$$

(3-17)

It is observe that, the logarithmic scaling emerged , denoting that the BVG associated with a generic random sequence is small world [12], which may be observed in the Fig. 5. The numerical results of $L(N)$ (circles) of a BVG associated with several random binary sequences of increasing size $N = 2^7, 2^8, ..., 2^{19}$ in the Fig. 7, have been plotted. The line is the best fit of the theoretical. The values of $\chi^2$ goodness-of-fit test between the theoretical prediction mean path length Eq. (17) and numerical results demonstrated the measure of scalability.

4 Conclusion and outlook

In this article, we have investigated the binary visibility graph, constructed from the random binary sequences. The present study illustrates the uselessness of the previous works in the analysis of random binary sequences [19, 20, 6]. We have also evaluated exact results on several topological properties of the BVG associated with generic uncorrelated random binary sequences, and numerical simulations confirmed its reliability for finite sequences, and the results show the three topological properties of the binary visibility graph as a excellent randomness criteria.

Furthermore, we do hope that our obtained results through this paper will pave the way for further studies on nonlinear dynamical systems.

5 Acknowledgments

The authors would like to express their heartfelt gratitude to Mr. D. Manzoori, Mr. S. Behnia for the nice editing of their paper.
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