Simple Estimation of Frequency Response of Two-layer Pressure-sensitive-paint Model

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1. Introduction

Pressure-sensitive paint (PSP), which is a molecular sensor utilizing the luminescent light modulated by oxygen concentration, is a powerful experimental fluid measurement tool to obtain surface pressure distributions. Thus far, many researchers have developed and modified this experimental measurement technique. The reader should refer to Liu and Sullivan1) for the details of this technique. Although this technique has been used to measure the steady-pressure distribution during the early stage of application, it is recently being adopted and applied to unsteady flows. When the PSP technique is applied to unsteady flow fields, the frequency response of PSP is important.

The frequency response of PSP1,2) is dominated by the following three factors:

- frequency response of oxygen diffusion in the PSP layers,
- frequency response modulation based on lifetime of the PSP layers, and
- integration of total luminescent light radiated from PSP layers and evaluation of frequency response of the entire PSP system.

With regard to the first factor, two analytical formula have been developed for uniform paint in which the diffusivity coefficient does not change: the harmonic response model and the eigen function model. The harmonic response model assumes the Fourier expansion of both spatially homogeneous directions and temporal direction. On the other hand, the eigen function model allows us to use arbitrary temporal and spatial distribution while needing to calculate it with a sufficient number of eigen modes to reduce the truncation error of the model.2) With regard to the second factor, though time lag based on PSP lifetime was numerically calculated by Pandey and Gregory,2) it can be simply approximated by analytical formula of the first-order lag system to estimate the frequency response.2) Different from the previous two factors, integration of total luminescent light cannot be calculated using the analytical expression and requires numerical calculation. This kind of frequency response prediction is employed for various PSPs.

Recently, fast-response sprayable PSP has been developed. One of the fast-response PSPs is polymer/ceramic (PC)-PSP.4,5) The structure of PC-PSP can be generally explained by the two-layer model in which the top layer is assumed to have more ceramic particles and the bottom layer is assumed to have more polymers, whereas the bottom layer is not the white primary layer but a part of the PC-PSP. In reality, the ratio of ceramic particles and polymer does not change discontinuously but continuously. However, the change in the layer characteristics is considered to be affected by whether or not the particle ratio is over a certain threshold (so called “critical pigment volume concentration”), and therefore, the resulting change in a diffusivity to be sharp to some extent. Additionally, the previous study showed that this assumption of sudden change in layers is reasonable.3) Recently, Pandey and Gregory3) constructed a numerical model to predict the frequency response of the two-layer PSP model. In their study, numerical differentiation and integration are used for all three factors discussed above to predict the frequency response. Use of numerical simulation is computationally expensive and cumbersome for checking the validity of the simulation including grid convergence. Therefore, it is preferred to predict using an analytically expressed formula and to minimize the use of numerical differentiation and integration.3) In addition to that, the analytical formulation, which gives us information of the parameter sensitivity and asymptotic behaviors of the model when the parameters substantially changes, is useful to consider the parameter effects.

In this short paper, we derive the analytical formulation/ approximation of harmonic response of first and second factors of the oxygen diffusion and the luminescence quenching, respectively in a two-layer model to predict the frequency response. Using those formula, the numerical integration to account for the total luminescent light is only required and computational costs are significantly suppressed without checking the grid convergence for the first and second factors of the model. Results using the present formula are compared with the numerical results of a previous study and the validity is demonstrated.

2. Present Formula

Figure 1 shows the schematic of a two-layer model. The
prediction formula is briefly derived as follows. 

2.1. Oxygen diffusion

First, the oxygen concentration $C(z,t)$ is solved for the two-layer model. The wall parallel directions are assumed to be uniform and the resulting diffusion equation for the vertical direction is

$$\frac{\partial C(z,t)}{\partial t} = D_1 \frac{\partial^2 C(z,t)}{\partial z^2},$$

(1)

where the subscript $i$ is 1 or 2, $D_1$ and $D_2$ are diffusion coefficients for the top and bottom layers, respectively, and $z$ is the distance from the top surface of the paint. This equation is linear and the solution can be expressed by the superposition of harmonic response. Therefore, the following boundary conditions are considered for harmonic response:

$$C(0,t) = C_{\text{ref}} + C_{\omega_0}e^{i\omega t},$$

(2)

$$C(l_1 + 0,t) = C(l_1 - 0,t),$$

(3)

$$D_1 \frac{\partial C}{\partial z}(l_1 - 0,t) = D_2 \frac{\partial C}{\partial z}(l_1 + 0,t),$$

(4)

$$D_2 \frac{\partial C}{\partial z}(l_2,t) = 0,$$

(5)

where $z = l_1$ is the interface of the top and bottom layers and $z = l_2$ is the bottom of the paint. Here, $\omega$ is the angular frequency of the change in the oxygen concentration outside and $t$ is the time. $C_{\text{ref}}$ is the reference (averaged) value of the oxygen concentrations. Hereafter, the quantity with the subscript $\omega$ corresponds to the complex value which indicates the amplitude and the phase. The equation can be solved using the following formulation for the top layer:

$$C(z,t) = C_{\text{ref}} + C_{\omega(z)}e^{i\omega t},$$

(6)

whereas

$$E = \sqrt{D_1} \left( \exp \left( \frac{2\omega}{D_1} (l_1 + i) \right) + \exp \left( \frac{2\omega}{D_1} (l_1 + i) \right) \right) \times \left( \exp \left( \frac{2\omega}{D_2} (l_1 + i) \right) + \exp \left( \frac{2\omega}{D_2} (l_1 + i) \right) \right)$$

(7)

$$G = \sqrt{D_1} \left( \exp \left( \frac{2\omega}{D_1} (l_1 + i) \right) + 1 \right) \times \left( \exp \left( \frac{2\omega}{D_2} (l_1 + i) \right) + \exp \left( \frac{2\omega}{D_2} (l_1 + i) \right) \right),$$

(8)

$$H = \sqrt{D_2} \left( \exp \left( \frac{2\omega}{D_2} (l_1 + i) \right) - 1 \right) \times \left( \exp \left( \frac{2\omega}{D_2} (l_1 + i) \right) + \exp \left( \frac{2\omega}{D_2} (l_1 + i) \right) \right).$$

(9)

Although we can derive the solution for the bottom layer, it is not shown in this short paper because the luminescence from the bottom layer is ignored, as it was in the previous study. It should be noted that the solution derived for the two-layer model derived here is new. This equation is valid for finite amplitude of oscillation in oxygen fluctuation.

2.2. Luminescence lifetime

The equation of the number of luminophores in excited states can be reduced to an equation of luminescent intensity per length $I(z,t)$ as follows:

$$\tau_{\text{ref}} \frac{d}{dt} \left( \frac{I(z,t)}{I_{\text{ref}}} \right) + \left( A(z) + B(z) \frac{C(z)}{C_{\text{ref}}} \right) \left( \frac{I(z,t)}{I_{\text{ref}}} \right) = 1,$$

(10)

where $\tau_{\text{ref}}$ is the representative lifetime of the luminescence and $I_{\text{ref}}$ is the averaged luminescence intensity per unit length, which is constant. Here, $A(z)$ and $B(z)$ are Stern-Volmer coefficients. The equation above is nonlinear and difficult to be analyzed straightforwardly. Therefore, the harmonic response of the small-amplitude intensity is assumed to be $I(z,t) = I_{\text{ref}} + I_{\omega(z)}e^{i\omega t}$, and that of the small-amplitude concentration is assumed to be $C(z) = C_{\text{ref}} + C_{\omega(z)}e^{i\omega t}$ under the following boundary condition for the top of the paint

$$C(0,t) = C_{\text{ref}} + C_{\omega_0}e^{i\omega t},$$

(11)

whereas the superscript prime represents quantity in terms of small amplitude fluctuation, the product of which is ignored. It should be considered that $C_{\omega_0}(z)$ and $C_{\omega_1}(z)$ are written using the same expression as follows, but the amplitude of the latter is limited to be small. Actually, the small-amplitude concentration becomes

$$C(z,t) = C_{\text{ref}} + C_{\omega(z)}e^{i\omega t},$$

(12)

$$C_{\omega(z)} = C_{\omega_0} \exp \left( - \frac{2\omega}{D_1} \frac{1}{2} \right) \times \left( E - \frac{F}{G - H} \right) \quad z < l_1,$$

(13)

Then, we get
where the relation \( A(z) + B(z) = 1 \) is utilized and the second-order perturbation term of \( B(T_{a}(z)/T_{ref}) (C_{a}(z)/C_{ref}) \) is approximately neglected to derive the equation under the assumption of small perturbation. This approximation is the same as stated by Kameda.\(^2\)

2.3. Luminescence integration

Next, the luminescence from each position inside the paint is integrated while considering optical depth. Similar to the previous study, only the harmonic response in the top layer is integrated.

\[
T_{a} = \int_{0}^{l_{1}} I_{a}(z)e^{-\epsilon z}dz
\]

\[
= \int_{0}^{l_{1}} I_{ref}\left( \frac{B(z) C_{a}(z)}{1 + i\omega\tau_{ref}} \right) e^{-\epsilon z}dz,
\]

where \( T_{a} \) is the harmonic response of the integration of the luminescence light intensity throughout the paint and \( \epsilon \) is optical depth. Here, we define the ideal response of PSP luminescence considering low-frequency approximation: \( \omega \rightarrow 0 \) and sufficiently fast diffusion of oxygen \( C_{a}(z) = C_{a0} \):

\[
T_{a,\text{ideal}} = \int_{0}^{l_{1}} I_{a,\text{ideal}}(z)e^{-\epsilon z}dz
\]

\[
= \int_{0}^{l_{1}} I_{ref}\left( \frac{B(z) C_{a0}}{C_{ref}} \right) e^{-\epsilon z}dz.
\]

If \( B(z) \) is constant, we get the frequency response of the pressure measurement \( P' \) compared with the low-frequency approximation \( P'_{\text{ideal}} \) as follows:

\[
\frac{P'_{m}}{P'_{m,\text{ideal}}} = \frac{1}{B} T_{a,\text{ideal}}
\]

\[
= \int_{0}^{l_{1}} \left( \frac{C_{a}(z)}{1 + i\omega\tau_{ref}} \right) e^{-\epsilon z}dz
\]

\[
= \left( \frac{C_{a0}}{C_{ref}} \right) \int_{0}^{l_{1}} e^{-\epsilon z}dz
\]

\[
= \left( \frac{C_{a0}}{C_{ref}} \right) \frac{1}{1 + i\omega\tau_{ref}} \int_{0}^{l_{1}} e^{-\epsilon z}dz.
\]

This equation includes the integration that cannot be solved as a simple analytical form and requires numerical integration. To estimate frequency response, we simply conduct numerical integration, Eq. (19), of the present formula. The temporal integration required in the previous study is not required.

Finally, gain and phase-lag are calculated as follows:

\[
\text{Gain} = 20\log_{10} \left( \frac{P'_{m}}{P'_{m,\text{ideal}}} \right), \quad \text{Phase} = \text{arg} \left( \frac{P'_{m}}{P'_{m,\text{ideal}}} \right).
\]

3. Validation

The present formula is validated applying a test problem conducted by Pandey and Gregory.\(^3\) The model parameter of single- and two-layer models are shown in Table 1, where two different types of PSP are considered: typical and rough. Here, those two models (i.e., single- and two-layer models) for each of the two PSPs, a total of four models, are computed using the experimental data employed by Pandey and Gregory.\(^3\) The parameters used were exactly the same as those in the previous study,\(^3\) and were chosen to obtain the best-fit results when compared with the experiments in the previous study.

The present formula can be applied to the single-layer model proposed by Kameda\(^2\) with setting \( l_{1} = l_{2} \) to confirm the validity of the formulation. For the calculation, 100 points inside the top layer are used. Figure 2 shows the results obtained by the present formula compared with the numerical simulation conducted by Pandey and Gregory.\(^3\) The present results for all four cases agree fairly well with those of the previous models based on the numerical simulation. The experimental data is expressed well by the two-layer model of both present and previous computations. This result shows that the present formulation is appropriately derived and approximated.

4. Computational Time

Finally, the computational times of the previous numerical model and the present model are compared. The previous numerical model was implemented using second-order finite differencing and second-order Runge-Kutta schemes. The computations of typical PSP responses (Table 1) for 1,000 Hz oscillation were carried out. Similar to the previous study, 800 points were used for all PSPs, and thus 60 points were inside the top layer. A total of five cycles were computed for the previous model, with at least one cycle being executed and su

| Model parameter set | Lifetime \( \mu s \) | Top-layer thickness \( \mu m \) | Bottom-layer thickness \( \mu m \) | Top-layer diffusivity \( \text{m}^{2}/\text{s} \) | Bottom-layer diffusivity \( \text{m}^{2}/\text{s} \) | Hiding factor \( \epsilon \) \( \text{1/m} \) |
|---------------------|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Typical single (Kameda’s parameter\(^2\)) | 7.04 | 3 | 0 | 1.96 \( \times \) 10\(^{-7} \) | N/A | 0 |
| Typical double (Pandey and Gregory’s parameter\(^3\)) | 7.04 | 3 | 37 | 9.78 \( \times \) 10\(^{-8} \) | 9.78 \( \times \) 10\(^{-11} \) | 1.67 \( \times \) 10\(^{6} \) |
| Rough single (Kameda’s parameter\(^2\)) | 6.75 | 24 | 0 | 1.74 \( \times \) 10\(^{-3} \) | N/A | 0 |
| Rough double (Pandey and Gregory’s parameter\(^3\)) | 6.75 | 24 | 80 | 7.51 \( \times \) 10\(^{-6} \) | 9.78 \( \times \) 10\(^{-11} \) | 3.13 \( \times \) 10\(^{6} \) |

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computed after the solution reached a quasi-steady condition. Because of the diffusion number restriction, temporal step size was limited to 0.01 $\mu$s and the duration of the five cycles at 5 ms corresponds to 500,000 temporal steps. In this case, the previous model costs approximately 11.06 s with the Intel Xeon E5620 2.4 GHz processor and the Intel Fortran compiler without optimization options. This computational time is an average of 10 runs. On the other hand, the cost of the present model when computing 60 points inside the top layer is only 2.224 ms using the same processor, compiler and options, and this computational time is average over 1,000 runs. Although the time for the previous model can be reduced applying an implicit time integration technique, the present model is approximately 5,000 times faster than previous numerical model based on this comparative study. The results are summarized in Table 2.

5. Conclusions

In this short paper, we derived a simple formula of frequency response for a two-layer PSP model which requires only one numerical integration. This procedure is computationally cheaper than the previous procedures. The validation study shows that the present formula gives almost the same results as the previous numerical model.

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References

1) Liu, T. and Sullivan, J. P.: *Pressure- and Temperature-Sensitive Paints*, John Wiley & Sons, New Jersey, 2005.
2) Kameda, M.: Effect of Luminescence Lifetime on the Frequency Response of Fast-response Pressure-sensitive Paints, *Trans. Jpn. Soc. Mech. Eng. Ser. B*, 78 (2012), pp. 1942–1950.
3) Pandey, A. and Gregory, J. W.: Frequency-response Characteristics of Polymer/ceramic Pressure-sensitive Paint, *AIAA J.*, 54 (2016), pp. 174–185.
4) Sugimoto, T., Sugioka, Y., Numata, D., Nagai, H., and Asai, K.: Characterization of Frequency Response of Pressure-sensitive Paints, *AIAA J.*, 55 (2017), pp. 1460–1464.
5) Sugioka, Y., Arakida, K., Kasai, M., Nonomura, T., Asai, K., Egami, T., and Nakakita, K.: Evaluation of the Characteristics and Coating Film Structure of Polymer/ceramic Pressure-sensitive Paint, *Sensors*, 18 (2018), No. 4041.