Exotic solutions in string theory

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Summary. — Solutions of classical string theory, correspondent to the world sheets, mapped in Minkowsky space with a fold, are considered. Typical processes for them are creation of strings from vacuum, their recombination and annihilation. These solutions violate positiveness of square of mass and Regge condition. In quantum string theory these solutions correspond to physical states $|DDF\rangle + |sp\rangle$ with non-zero spurious component.

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Introduction

In this work we want to draw theoretists' attention to the fact that classical mechanics of Nambu-Goto string in its covariant Hamiltonian formulation contains solutions with quite exotic properties. Particularly, it contains solutions with negative square of mass.

Example 1. Phase space of open string is described by infinite set of canonical oscillator variables $a_n^\mu$, restricted by reality condition $(a_n^\mu)^* = a_{-n}^\mu$ and Virasoro constraints $L^n = \sum_k a_k^\mu a_{-k}^\mu = 0$. Let consider the following set of oscillators:

$$
a^0_\mu = (1, \alpha, 0), \quad a^{\pm 1}_\mu = (\alpha, 1/2, \pm 1/2i), \quad a^{\pm 2}_\mu = (0, \alpha/2, \pm \alpha/2i), \quad \text{others } a^n_\mu = 0.
$$

Here we consider a theory in 3-dimensional Minkowsky space, $a^n_\mu = (a^0_0, a^0_1, a^0_2)$; $\alpha$ is real parameter.

Let's show, that given set satisfies Virasoro constraints. At $|n| > 4$ $L^n = 0$, because in each term of the sum $\sum a_k^\mu a_{-k}^\mu$ one of oscillator variables vanishes. Due to a property $L^{-n} = (L^n)^*$, the check is needed only for conditions $L^n = 0$ at $0 \leq n \leq 4$. These conditions have a form:

$$(a^2_\mu)^2 = 0, \quad a_\mu^1 a_\mu^2 = 0, \quad 2a_\mu^0 a_\mu^2 + (a_\mu^1)^2 = 0, \quad a_{-1}^\mu a_\mu^2 + a_\mu^0 a_\mu^1 = 0, \quad (a_\mu^0)^2 + 2a_{-1}^\mu a_\mu^1 + 2a_{-2}^\mu a_\mu^1 = 0.$$
their validity can be easily proven by direct substitution of $a^m_\mu$.

Total momentum of the string is defined by an expression: $P_\mu = \sqrt{\pi} a^0_\mu$. Thus, square of string’s mass $P^2 = \pi(1 - a^2)$ is positive at $|a| < 1$ and negative at $|a| > 1$.

Let’s consider a function $a_\mu(\sigma) = 1/\sqrt{\pi} \sum_n a^m_\mu e^{i a \sigma}$. In this example $a_0(\sigma) = (1 + 2a \cos \sigma)/\sqrt{\pi}$, at $|a| > 1/2$ this function is not everywhere positive. Solutions, for which $a_0(\sigma)$ has variable sign, will be further called exotic. In this paper we will consider the properties of such solutions. In Section 1 we describe the geometrical method for reconstruction of string dynamics and show several examples of exotic dynamics. In Section 2 we consider the properties of exotic solutions in Lagrangian theory. In Section 3 the appearance of exotic solutions in quantum string theory is discussed.

1. – Exotic solutions in Hamiltonian theory

1.1. Geometrical reconstruction of exotic solutions.

Let’s introduce a function, related with string’s coordinates and momenta by expressions [1]

1. $Q_\mu(\sigma) = x_\mu(\sigma) + \int_0^\sigma d\tilde{\sigma} p_\mu(\tilde{\sigma})$,
2. $x_\mu(\sigma) = (Q_\mu(\sigma) + Q_\mu(-\sigma))/2, \quad p_\mu(\sigma) = (Q'_\mu(\sigma) + Q'_\mu(-\sigma))/2$

(x, p are even functions of $\sigma$). In terms of oscillator variables, introduced earlier:

3. $Q_\mu(\sigma) = X_\mu + \frac{P_\mu}{\sqrt{\pi}} \sigma + \frac{1}{\sqrt{\pi}} \sum_{n \neq 0} \frac{a^m_n}{in} e^{i n \alpha}, \quad Q'_\mu(\sigma) = a_\mu(\sigma)$.

The curve, defined by the function $Q_\mu(\sigma)$ (further called supporting curve) has the following properties:

1. the curve is light-like: $Q_\mu^2(\sigma) = 0$, this property is equivalent to Virasoro constraints on oscillator variables;
2. the curve is periodic: $Q_\mu(\sigma + 2\pi) - Q_\mu(\sigma) = Const = 2P_\mu \quad (P_\mu$ is total momentum of the string);
3. the curve coincides with the world line of one string end: $x_\mu(0, \tau) = Q_\mu(\tau)$; the world line of another end is the same curve, shifted onto the semi-period: $x_\mu(\tau, \tau) = Q_\mu(\pi + \tau)$;
4. the whole world sheet is reconstructed by this curve as follows: $x_\mu(\sigma, \tau) = (Q_\mu(\sigma_1) + Q_\mu(\sigma_2))/2, \quad \sigma_{1,2} = \tau \pm \sigma$, see fig.1.

These properties can be easily proven from the definition of $Q_\mu(\sigma)$ and known mechanics in oscillator variables, see Appendix 1.

Further consideration will be restricted to the supporting curves, whose time component $Q_0(\sigma)$ is non-monotonomous function, see fig.2. Such curves can be explicitly constructed, specifying tangent vector in the form $Q'_\mu(\sigma) = a_0(\sigma)(1, \tilde{n}(\sigma))$, where $\tilde{n}(\sigma)^2 = 1, \tilde{n}(\sigma)$ is $2\pi$-periodic function, and $a_0(\sigma)$ is $2\pi$-periodic function of variable sign(1). Such

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(1) Substitution $\tilde{n}(\sigma) = (\cos \sigma, \sin \sigma)$ and $a_0(\sigma) = (1 + 2a \cos \sigma)/\sqrt{\pi}$ corresponds to oscillator variables from Example 1.
supporting curves necessarily have singularities (cusps)\(^{(2)}\).

For the curves with \(a_0(\sigma) > 0\) (correspondent solutions are further called *normal*) vector \(P_\mu\) is time-like or light-like: \(P^2 \geq 0\), because \(P_\mu = \int_0^{2\pi} d\sigma a_\mu(\sigma)/2\) is represented as a sum of light-like vectors, directed into the future. For non-monotonous supporting curves \(P^2\) might take the both signs. At first, let’s study the case of time-like \(P_\mu\). In this case we can consider the evolution in center-of-mass frame (CMF), where \(P_\mu\) is directed along the time axis.

**Example 2.** Let each period of supporting curve contains a single non-monotonous interval, whose sizes are small comparing with the period. E.g. \(Q'_\mu(\sigma) = a_0(\sigma)(1, \cos \sigma, \sin \sigma)\) with \(a_0(\sigma) = 1+1.4 \cos 2\sigma - 0.6 \sin 3\sigma\). For this choice \(P_\mu = (\sigma, 0, 0)\) is directed along the time axis. The graph of function \(a_0(\sigma)\) and correspondent supporting curve are shown on fig.3.

Simplified model of the world sheet, stretched onto such supporting curve, is shown on fig.4. Equal-time slice of the world sheet at \(t < t_A\) is a single connected curve (“permanent component”), consisting of two segments \(L\) and \(a\). At \(t = t_A\) an additional open string \(bc\) appears. At \(t = t_B\) a process of recombination occurs, the strings exchange by their segments: \((La) + (bc) \rightarrow (Lb) + (ac)\). Short string \(ac\) disappears at \(t = t_B\). Further evolution (till the end of a period) contains only permanent component \(Lb\).

Realistic image of the world sheet (fig.5) contains more details. Particularly, one can find here singular lines \((fRAd\) and \(gBRe\)). Slices of the world sheet have cusps along these lines. These two lines are congruent and homotetic to the supporting curve \((cABh)\) with the coefficient 1/2. They are created by cusp points \(A, B\) on the supporting curve\(^{(3)}\). General structure of equal-time slices is the same: creation of new open string in point \(A\), its recombination with the permanent component in point \(R\), and annihilation of new string in point \(B\). Equal-time slices of this surface are shown on fig.6.

\(^{(2)}\) When \(Q'_\mu(\sigma)\) changes its sign, vector \(Q'_\mu(\sigma)\), lying on the light cone, passes through the origin: \(Q'_\mu(\sigma^*) = 0\). In this point the supporting curve has a cusp.

\(^{(3)}\) Cusp on supporting curve in point \(Q(\sigma^*)\) creates a singularity on the world sheet, located along the line \((Q(\sigma^*) + Q(\sigma))/2\), i.e. the supporting curve, contracted twice to the point \(Q(\sigma^*)\).
Fig. 7 shows equal-time slices $x_0(\sigma_1, \sigma_2) = t$ on the plane of parameters $(\sigma_1, \sigma_2)$. Band $\sigma_1 \leq \sigma_2 \leq \sigma_1 + 2\pi$ on this plane uniquely represents the world sheet. Slice for the moment $t : t_A < t < t_R$ is shown on this figure in bold (two disconnected parts). In further evolution small part is expanded and recombinates with long part in point $R$. After recombination new small part is shrunk in point $B$, and new long part continues to move in right-upward direction.

Straight lines, passing through points $\sigma_i = \sigma_{A,B} + 2\pi n$, $i = 1, 2$, $n \in \mathbb{Z}$, correspond to cusp lines on the world sheet. On these lines $Q'_0(\sigma)$ changes its sign. Vertical grey bands show regions with $Q'_0(\sigma_1) < 0$, horizontal grey bands correspond to $Q'_0(\sigma_2) < 0$.

Example 3. $a_0(\sigma) = 1 + 4\cos 2\sigma + 2\cos 3\sigma - 2\sin 3\sigma$, see fig. 8. This supporting curve contains two non-monotonous intervals in each period. Their interference leads to more complicated processes, particularly, appearance of closed strings in the evolution, see fig. 9.
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Fig. 6. String dynamics for Example 2. Bold lines are the strings, thin line is the supporting curve (projected to space component of CMF). The supporting curve has two segments: \( a_0 > 0 \) and \( a_0 < 0 \), directed in Minkowsky space forward and backward in time respectively. There is a long string, which is permanently present in the system. The common features of its evolution are shown on the upper frame sequence. Time is measured in periods. Central sequence shows in details the time interval, when the end of the string passes the part with \( a_0 < 0 \). Additional short string appears near \( t = 0.171 \). At \( t = 0.218 \) the recombination (\( * \)) occurs: short string is attached to the long string and a part of long string is detached. New short string disappears at \( t = 0.265 \). Lower frames show the details of recombination process: \((Lab) + (cde) \rightarrow (cab) + (Lde)\), segments \( L, c \) are exchanged.

Fig. 7. Equal-time slices on parameters plane. Right part shows details of recombination process. Arrows on strings (solid lines) show their orientation. Arrows on dashed lines show the direction of evolution.
Fig. 8. Function $a_0(\sigma)$ and supporting curve for Example 3. Two cusps of supporting curve $A_1$ and $A_2$ create in the vicinity of their middle point $B = (A_1 + A_2)/2$ a surface, shown on the right part of the figure. Here the lines $aBc$ and $dBe$ are homotetic with the coefficient $1/2$ to the parts of supporting curves near cusps $A_1$ and $A_2$ respectively. When the curve $aBc$ moves parallelly to itself, keeping the vertex on the curve $dBe$, it spans the world sheet. The part of the surface $(aBd + cBd)$ is spanned by the curve $aBc$ in motion along segment $Bd$, the part $(aBe + cBe)$ is spanned by $aBc$ in motion along $Be$. Equal time slice of the obtained surface contains closed string $aecd$.

Fig. 9. String dynamics for Example 3. On the first frame the following 4 disconnected elements can be found: Z-shaped long open string in the center, two shorter open strings in the corners of supporting curve and closed string $B$, looking like a butterfly. In the process of further evolution the long string recombinates with shorter strings, adopting the shape of reverted Z (frame 5). On frame 3 the closed string $B$ disappears, and new closed string $B'$ appears. On frame 6 it enters into recombination with the long string and is included in it. There are no closed strings on frame 7, here the long string in the center is connected. On the frame 8 the string $B$ is detached from the long string. The frame 9 coincides with the frame 1, and the evolution is repeated again.
Example 4. \( a_0(\sigma) = \cos\sigma \), see fig.10. For this supporting curve the period is space-like: \( P_\mu = (0, \pi/2, 0) \), \( P^2 < 0 \). Correspondent world sheet is stretched not in time direction, but in the space one. There is no center-of-mass frame for this system. In such cases one can use simultaneous reference frame, where the period has purely space direction (like in given example). In this frame the world sheet has finite size in temporal direction, the evolution is restricted in time, see fig.11. In other reference frames the world sheet is sloped to time axis, the formation of strings, appearing in equal-time slice, has finite spatial sizes. It moves with the mean velocity, equal to the slope \( |\vec{P}|/P_0 \), which is greater than 1, the light velocity in our units. However, we will show in the next Section, that all parts of this system do not exceed the light velocity. This motion is shown on fig.12. It becomes clear from this figure, that there is no direct hyper-light transfer of the system, this transfer occurs through a sequence of creation, annihilation and recombination processes. Not initial string but its exact copy approaches the finite point.

**Fig.10.** Supporting curve for Example 4. \( A_n = A_0 + Pn \) — creation points, \( B_n = B_0 + Pn \) — annihilation points, \( R_n = (A_0 + B_0)/2 + Pn \) — recombination points.

**Fig.11.** String dynamics for Example 4, observed in simultaneous reference frame. An infinite number of strings, periodically located in one space direction, simultaneously appear on the first frame. On the frame 4 the strings recombine. On the last frame they simultaneously disappear. Further observation will show the empty space.

**Fig.12.** The same evolution, observed in other reference frame. Here we see the moving string formation, which periodically repeats its form. On the first frame strings \( a_1 \) and \( b_1 \) are presented. On the third frame string \( a_1 \) disappears and string \( b_2 \) appears. The string \( b_1 \) deforms to the string \( a_2 \). The recombination occurs on the frame 5. The last frame is exact copy of the first one, shifted by the distance \( \Delta x \). This distance is greater than \( ct \), the path of light during the evolution.

1.2. Energetic flows on the world sheet.

Let’s consider a connected component in equal-time slice of the world sheet. Total
momentum of this component is given by the formula, derived in Appendix 2:

\[
\Delta P_\mu = \frac{1}{2}(-Q_\mu(\sigma_1) + Q_\mu(\sigma_2)) \bigg|_{(\sigma_1, \sigma_2)}^{(\sigma', \sigma')},
\]

where \((\sigma_1, \sigma_2)^{i:j}\) are respectively initial and final points of the component on the parameters plane. For appearing/disappearing component (e.g. short component, shown in bold on fig.7) we immediately have

\[
\Delta P_\mu = \frac{1}{2}(-Q_\mu(\sigma_1) + Q_\mu(\sigma_2)) \bigg|_{(\sigma_1, \sigma_2)}^{(\sigma', \sigma')} = 0
\]

(both ends of the string lie on the same edge of the world sheet).

For the permanent component, which starts and ends on different edges, we have

\[
\frac{1}{2}(-Q_\mu(\sigma_1) + Q_\mu(\sigma_2)) \bigg|_{(\sigma_1, \sigma_2)}^{(\sigma', \sigma'+2\pi)} = \frac{1}{2}(-Q_\mu(\sigma') + Q_\mu(\sigma'+2\pi)) = P_\mu,
\]

the result is total momentum of the string.

Angular moment of the component is given by the formula

\[
\Delta M_{\mu\nu} = \frac{1}{4}Q[^{\mu}Q^{\nu}]_{(\sigma_1, \sigma_2)}^{(\sigma', \sigma')} + \frac{1}{4} \left( -\int_{\sigma_1}^{\sigma'} + \int_{\sigma'}^{\sigma_2} \right) Q[^{\mu}dQ^{\nu}].
\]

Here \(Q^\mu = Q^\mu(\sigma_i)\) and square brackets denote antisymmetrization of indices: \(A[^{\mu}B^{\nu}] = A^\mu B^\nu - A^\nu B^\mu\). \(\Delta M_{\mu\nu}\) is defined by two intervals of supporting curve: \(\sigma \in [\sigma_1, \sigma_1'] \cup [\sigma_2, \sigma_2']\).

It is not changed in reparametrizations of the supporting curve, conserving boundary points of these intervals. Generally, it depends on position of boundary points.

For appearing/disappearing components we have

\[
\Delta M_{\mu\nu} = \frac{1}{4}Q[^{\mu}Q^{\nu}]_{(\sigma', \sigma')} + \frac{1}{4} \left( -\int_{\sigma'}^{\sigma'} + \int_{\sigma'}^{\sigma'} \right) Q[^{\mu}dQ^{\nu}] = 0,
\]

and for permanent component:

\[
M_{\mu\nu} = \frac{1}{4}Q[^{\mu}Q^{\nu}]_{(\sigma, \sigma)}^{(\sigma', \sigma'+2\pi)} + \frac{1}{4} \left( -\int_{\sigma}^{\sigma'} + \int_{\sigma'}^{\sigma'+2\pi} \right) Q[^{\mu}dQ^{\nu}] =
\]

\[
= \frac{1}{2}Q[^{\mu}(\sigma')P^{\nu}] + \frac{1}{4} \int_{\sigma'}^{\sigma'+2\pi} Q[^{\mu}dQ^{\nu}], \quad \frac{dM_{\mu\nu}}{d\sigma'} = \frac{1}{2}(Q[^{\mu}(\sigma')P^{\nu}] + P[^{\mu}Q^{\nu}(\sigma')) = 0.
\]

Thus, total momentum and angular moment for appearing/disappearing components vanish, and creation/annihilation processes do not violate the conservation laws. For permanent component total momentum and angular moment are presented as complete parametric invariants of the supporting curve, independent on the position of ends \((\sigma, \sigma')\), thus, they are conserved in evolution.
Vanishing of total energy for non-permanent component implies, that density of energy is not everywhere positive on the string. In Appendix 2 we derive a formula for linear density of energy \( dP_0/dl \), \( dl = |d\vec{x}| \) – element of length on equal-time slice:

\[
\frac{dP_0}{dl} = \text{sign}(Q'_1 Q'_{20}) \sqrt{\frac{2Q'_{10}Q'_{20}}{(Q'_1 Q'_2)}}.
\]

**Remark:** argument of this square root is not negative, because it can be presented in the form \( 2(n_1 n_2)^{-1} \), where \( n_{i\mu} = Q'_{i\mu}/Q'_{00} \) are light-like vectors, directed into the future.

For monotonous supporting curves \( (Q'_0 > 0) \) \( dP_0/dl \) is always positive. For non-monotonous supporting curves the density of energy on the string changes its sign in passage through the cusps (where the sign of \( (Q'_{10} Q'_{20}) \) is changed). Parts of the world sheet, shown on fig.7 in white and double grey, have positive density of energy; parts, shown in grey, correspond to negative density. Signs of density of energy are also shown on the world sheet fig.5. In recombination process, shown on fig.6, the density of energy is positive on the segments \( L, b, c, e \), and negative on the segments \( a, d \).

**1.3. Gauges.** Virasoro constraints generate reparametrizations of supporting curve (see Appendix 1). Gauges to Virasoro constraints select particular parametrization on this curve. For example, the curve can be parameterized by cone time: \( \sigma \sim (kQ) \) with \( k^2 = 0 \), e.g. \( k_{\mu} = (1, 1, 0, 0,...) \). In this parametrization \( (\text{light cone gauge}) \) Virasoro constraints can be explicitly resolved:

\[
Q_{\pm} = (Q_0 \pm Q_1)/\sqrt{2}, \quad \vec{Q}_{\pm} = (Q_2, ..., Q_{d-1}), \quad Q'_- = P_-/\pi > 0, \quad Q'_+ = \frac{\pi}{\sqrt{2}} \vec{Q}'_\pm \geq 0.
\]

Another possibility is to parameterize the curve by the time in CMF (for \( P^2 > 0 \)): \( \sigma \sim (PQ) \). Such gauge was originally considered by Rohrlich [2]. In this case \( Q'^0 = |\vec{Q}'_0| = \sqrt{P^2}/\pi > 0 \) \( (Q^0 \) is a component of \( Q_{\mu} \) along \( P_{\mu} \), \( \vec{Q}' \) is a component, orthogonal to \( P_{\mu} \). The both parametrizations can be introduced only on monotonous supporting curves, because we require that the component of \( Q_{\mu} \) along gauge axis \( (k_{\mu} \text{ or } P_{\mu}) \) should be monotonous in \( \sigma \). As a result, exotic solutions are absent in light cone and Rohrlich’s gauges. Non-monotonous supporting curves can be considered in parametrization of another type, see Appendix 5.

**1.4. Regge condition.** For normal solutions the spatial components of angular moment \( M_{ij} \) in CMF satisfy the following inequality: \( |M_{ij}| \leq P^2/2\pi \). It can be proven by purely geometrical methods, see Appendix 3. Alternatively, we can write the constraint \( L_0 = 0 \leftrightarrow P^2/2\pi = -\sum_{n>0} a_n^* a_n \) in terms of variables \( \alpha^n = (a_n^* + ia_n)/\sqrt{2} \):

\[
\frac{P^2}{2\pi} = \sum_{n \neq 0} \alpha^n \alpha^n + \sum_{n>0} a_n^* a_n - \sum_{n>0} a_{n+j}^* a_{n-j}^* , \quad M_{ij} = \sum_{n \neq 0} \frac{1}{n} \alpha^n \alpha^n.
\]

For normal solutions we can fix Rohrlich’s gauge: \( a_0^0 = 0 \) at \( n \neq 0 \), and for parametrically invariant variables \( (M_{ij}, P^2/2\pi) \) we will have: \( |M_{ij}| = |\sum \frac{1}{n} \alpha^n \alpha^n| \leq \sum \alpha^n \alpha^n \leq \sum \alpha^n \alpha^n + \sum a_i^* a_i = P^2/2\pi \).
For exotic solutions the Rohrlich’s gauge cannot be fixed, and negative contribution of \(a_0^0\)-oscillators in (7) can violate Regge-condition (and the condition \(P^2 \geq 0\)). An explicit example:

**Example 5:** \(a_\mu(\sigma) = \alpha \cos 2\sigma (1, \cos \sigma, \sin \sigma)\). In this case \(P_\mu = 0\) (supporting curve is closed in Minkowsky space, correspondent world sheet has finite sizes both in space and time directions). \(M_{12} = \frac{1}{2} \int Q_1 dQ_2 = -\pi \alpha^2/12 \neq 0\) (area, restricted by the curve on (12)-plane, does not vanish).

### 1.5. DDF variables.

In quantum theory the operators, introduced by Del Giudice, Di Vecchia and Fubini [3], play an important role. Here we will consider their classical analogs:

\[
\vec{A}_n = \int_{0}^{2\pi} d\sigma \vec{Q}'_\perp \exp \pi Q^2 / P^-.
\]

(8)

For normal solutions we can introduce the light cone gauge \(\pi Q^- / P^- = \sigma\), in this case \(\vec{A}_n\) coincide with transverse oscillator variables \(\vec{A}_n = \int_{0}^{2\pi} d\sigma \vec{Q}'_\perp e^{i\sigma}\) (formula (8) gives parametrically-invariant expression for them). For normal solutions the set of variables \(\{\vec{A}_n\}\) defines the supporting curve uniquely (up to translations and reparametrizations).

For exotic solutions we cannot introduce the light cone gauge, but variables \(\vec{A}_n\) are still defined, both for time-like and space-like \(P_\mu\). In this case the supporting curve is not uniquely defined by \(\{\vec{A}_n\}\), see Appendix 4. A deformation of supporting curve is possible, preserving \(\{\vec{A}_n\}\).

![Fig.13. The set of DDF variables defines unique supporting curve in sector of normal solutions, and a family of supporting curves in exotic sector.](image)

**Remark.**

In theories with gauge symmetries the actual phase space is formed by a complete set of gauge-invariant variables. Other variables can be ignored, because their variation transforms the system along gauge equivalent states, indistinguishable physically. All states of the system, correspondent to the same values of gauge invariants, can be identified\(\textsuperscript{4}\).

In string theory variables \(\{\vec{A}_n\}\) together with total momentum and mean coordinate can be taken as such complete set only for normal solutions. In this case other dynamical variables influence only parametrization of supporting curve and do not influence physical observables.

\(\textsuperscript{4}\) For details of this procedure, known as gaugeless reduction, see [4].
For exotic solutions it is not so. A family of curves $Q_\mu(\sigma)$ exist, different by their shape, which correspond to the same set $\{\vec{A}_n\}$, see fig.13. Identification of all states with fixed $\{\vec{A}_n\}$ will lead to mixing of exotic and normal states, i.e. identification of physically distinct states. Later we will show, that such identification occurs in standard covariant quantization of string theory.

2. – Exotic solutions in Lagrangian theory

Action of the string is defined as

$$A = \int L \, d\sigma_1 \, d\sigma_2, \quad L = \sqrt{ (\partial_1 x \, \partial_2 x)^2 - (\partial_1 x)^2 (\partial_2 x)^2 }, \quad \partial_\sigma = \partial/\partial\sigma_\alpha.$$ 

The world sheets, considered in the previous Section, have the following properties:

1. They belong to the class “open string”: infinite band, mapped to Minkowsky space. The only difference from the usually considered world sheets is that now this mapping is not trivial (has a fold, see figs.4,5).

2. The argument of the square root in $L$ for these surfaces is not negative: $(Q'_1 Q'_2)^2/16 \geq 0$. Such surfaces are usually called time-like: in each regular point of the surface one of two tangent vectors $Q'_1 \pm Q'_2$ is time-like, another one is space-like. This also implies, that velocity of points on the string does not exceed the light velocity\(^{(5)}\).

3. The argument of the square root in $L$ vanishes on the cusp lines (figs.5,7): $(Q'_1 Q'_2)^2/16 = 0$. There are two possibilities for a choice of sign of the square root in passage through the cusp line:

   a) Non-negative value is chosen for the square root: $L = |Q'_1 Q'_2|/4$. In this case we can show (see Appendix 2), that Lagrange-Euler equations on each of two patches of the world sheet, separated by the cusp line, are satisfied, but the boundary conditions on the cusp line are not: momentum, flowing from one patch, is not equal to the momentum, entering into another patch. In this case the considered world sheet is not extremal surface for action $A$.

   b) The square root changes sign in passage through the cusp line: $L = (Q'_1 Q'_2)/4$, $\text{sign}(Q'_1 Q'_2) = \text{sign}(Q'_{10} Q'_{20})$ (parts, marked by $+$ and $-$ on fig.5, contribute to the action with opposite signs). The string, whose Lagrangian changes sign in passage through singular points (“polarized strings”) were initially considered in work [5]. In this case the Lagrange-Euler equations and boundary conditions are satisfied, and the extremum of action is reached on considered surface.

Thus, exotic solutions of covariant Hamiltonian theory correspond to solutions of Lagrangian theory of type $b$. In the case $a$ such solutions should be excluded.

\(^{(5)}\) Considering equal-time slices of the world sheet in the vicinity of regular point, the action can be rewritten in the form $\int \sqrt{1 - \vec{v}_\perp^2} \, dtdl$, where $dl$ is element of length on equal-time slice and $\vec{v}_\perp$ is a component of velocity, orthogonal to a string: only this component has physical (parametrically invariant) sense. For time-like surfaces $|\vec{v}_\perp| \leq 1$. 
Remark. String theory is usually considered in conformal coordinates \((\tau, \sigma)\), related with isotropic ones \((\sigma_1, \sigma_2)\) by expression \(\sigma_{1,2} = \tau \pm \sigma\). In these coordinates \(\dot{x} = \partial x/\partial \sigma = (Q_1 + Q_2)/2\), \(\dot{x}' = \partial x/\partial \sigma' = (Q_1' - Q_2)/2\). One can consider a family of straight lines \(\tau = (\sigma_1 + \sigma_2)/2 = \text{Const}\) on parameters plane fig.7 (instead of equal-time slices) and obtain correspondent foliation of the world sheet by strings in Minkowsky space (in this case the strings are connected curves). Namely this foliation occurs in action \(A\), reformulated in terms of coordinates \((\tau, \sigma)\). For normal solutions (monotonous supporting curves) in each regular point of the world sheet the vector \(x\) is space-like, \(\dot{x} > 0\), \(\dot{x}_0 > 0\). For exotic solutions it is not so: in white regions of fig.7 \(x'^2 < 0\), \(\dot{x}' > 0\), \(\dot{x}_0 > 0\); in grey regions \(x'^2 > 0\), \(\dot{x}' < 0\); in double grey regions \(x'^2 < 0\), \(\dot{x}' > 0\), but \(\dot{x}_0 < 0\). Usually such solutions are rejected from consideration on early stage of theory construction, by explicit requirement \(x'^2 < 0\), \(\dot{x}' > 0\), \(\dot{x}_0 > 0\), see e.g. [6]. This requirement, however, remains unused in further development of the theory. To exclude the exotic solutions from Hamiltonian theory, one should explicitly impose the requirement \(a_0(\sigma) > 0\) \forall \sigma\). In the space of oscillator variables \((a_0^+)^n\) this requirement defines a region (linearly connected and compact for each fixed \(a_0^+ = P_0/\sqrt{\pi}\)). Boundary of this region is composed of patches, whose explicit expression can be found in exclusion of variable \(z = e^{i\sigma}\) from the system \(a_0(z) = 0\). For finite number of excited oscillators this expression can be written as polynomial of oscillator variables [7], with the degree, increasing together with the number of excited oscillators. Needless to say, that determination of this region is a hard problem. Until it will not be solved, exotic solutions exist in Hamiltonian theory, and should also be present on a quantum level.

3. – Exotic solutions in quantum theory

Standard covariant quantization of string theory works in 26-dimensional Minkowsky space. Physical space of states is defined as \(L^n|\text{phys}\rangle = 0\), \(n \geq 0\) (only a half of constraints is imposed). In this theory the property \(a_0(\sigma) > 0\) cannot be directly tested, because operator \(a_0(\sigma)\) does not commute with \(L^n\) and, consequently, does not take definite values on the physical vectors. Only indirect methods can be used for study of exotic sector in quantum theory.

Square of string’s mass is defined by the expression \(P^2/2\pi = \sum a_i^+ a_i^n - \sum a_0^+ a_0^n\). In classical string theory the excitation of \(a_0^n\)-oscillators gives negative contribution to \(P^2/2\pi\) and can lead to arbitrarily great negative square of mass.

In standard covariant quantum string theory the vacuum is annihilated by operators \(a_0^n|0\rangle = 0\), \(n > 0\). Time component of oscillator variables has a commutator \([a_0^n, a_0^{n+}] = -n\), opposite to the commutator of space components: \([a_i^n, a_i^{n+}] = n\). Operators \(a_0^{n+}\) create from vacuum the states \((a_0^{n+})^k|0\rangle\), for which correspondent occupation number operator \(a_0^n\)\(a_0^n\) takes negative eigenvalues, e.g. \((a_0^{1+})^0|0\rangle = -a_0^{1+}|0\rangle\) (for odd \(k\) the states \((a_0^{n+})^k|0\rangle\) have also negative norm). As a result, the excitation of \(a_0^n\)-oscillators in quantum theory gives positive contribution to \(P^2/2\pi\), and square of mass is not negative(6). Analogously, considering the expression (7), we can prove that Regge condition in quantum theory is not violated.

Such exclusion of tachionic and non-Regge regions, present in exotic sector on classical

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(6) Exception is a single tachionic state, which appears after the redefinition \(P^2/2\pi \rightarrow P^2/2\pi - 1\) (introduction of intercept).
level, has absolutely different mechanism, than the requirement \( a_0(\sigma) > 0 \). Actually, this is not the exclusion of these states, but another definition of their square of mass. This redefinition maps the regions in the phase space, violating positiveness of \( P^2 \) and Regge condition, to the regions of spectrum, where these conditions are not violated.

Now lets consider DDF states \( \prod_{n<0,i}(A_{ni})^{k_{ni}}|0\rangle \), operators \( A_{ni} \) are defined by expression (8) (no special ordering is required). On these states \( a_n|DDF\rangle = 0 \), \( n > 0 \), classically this corresponds to \( a_-(\sigma) = \text{Const} \), i.e. light cone parametrization. Due to the theorem [8], any physical state can be presented in a form \( |\text{phys}\rangle = |DDF\rangle + |sp\rangle \), where \( |DDF\rangle \) belongs to the space, covered by DDF-states, and \( |sp\rangle \) is a spurious state, which has zero norm and is orthogonal to all states in the physical space. First of all, note that exotic solutions necessarily have non-constant \( a_-(\sigma) \). Consequently, they cannot correspond to pure DDF-states, but should have non-zero spurious component. Then, actual space of states is obtained in factorization of the physical space by spurious states: \( |DDF\rangle + |sp\rangle \sim |DDF\rangle \). This factorization is equivalent to one described in subsection 1.5, which leads to identification of normal and exotic solutions.

Remarks.

1. Such situation occurs in standard representation of quantum string theory. In work [9] alternative representation was considered. It was shown, that in pseudo-Euclidean space, containing equal number of spatial and temporal directions (e.g. in \( d = 3 + 3 \)), quantum string theory can be represented in positively defined extended space of states, where correspondent Virasoro algebra has zero central charge. In this theory the square of string’s mass has the expression, analogous to (7), but all occupation number operators in this theory take positive eigenvalues. As a result, tachionic regions of classical theory are preserved in quantization.

2. Light cone and Rohrlich’s gauges exclude exotic solutions on the classical level. Correspondent quantum theory also does not contain the exotic solutions. It is well known for standard light cone quantization, applicable in 26 dimensions. In works [10-12] special subsets in the phase space of 4-dimensional open string theory were considered, for which the quantization in Rohrlich’s gauge and in (slightly modified) light cone gauge has no anomalies. The obtained spectra also do not contain the exotic sector.

Quantum string theory in the gauge, described in Appendix 5, is anomaly-free as well. It contains exotic solutions classically and preserves them on the quantum level.

Conclusion

We considered a class of solutions in Nambu-Goto theory of open string, which correspond to the world sheets, mapped in Minkowsky space with a fold.

1. Such solutions exist in covariant Hamiltonian formulation of the theory. Typical processes for these solutions are creation of strings from vacuum, their recombination and annihilation. These solutions can violate positiveness of square of mass and Regge condition \( |M_{ij}| \leq P^2/2\pi \). Such solutions are present, until a special requirement \( a_0(\sigma) > 0 \) will be explicitly imposed.

In non-covariant formulations the exotic solutions are absent for commonly used gauges (Rohrlich’s and light cone) and present for a gauge of special type, described here in Appendix 5.
2. In Lagrangian formulation a behavior of the square root in Lagrangian near its branching point plays defining role. If positive sign of the square root is chosen, the exotic solutions are absent (and should be excluded from correspondent Hamiltonian theory). If Lagrangian changes its sign in passage through the branching point, the exotic solutions are present.

3. In covariant quantum theory imposition of the requirement $a_0(\sigma) > 0$ is a subtle task. Until this requirement is not imposed, the exotic solutions are present.

Standard covariant quantization of string theory operates in the space of states with indefinite metric. This leads to redefinition of string’s square of mass, which maps the regions of classical theory with $P^2 < 0$ and $|M_{ij}| > P^2/2\pi$ to the regions of quantum spectra with $P^2 > 0$ and $|M_{ij}| < P^2/2\pi$. After that redefinition, the exotic solutions correspond to physical states $|DDF\rangle + |sp\rangle$ with non-zero spurious component. Factorization of the physical space by spurious states leads to mixing of exotic and normal solutions.

An example of covariant quantization of string theory in positively defined extended space of states is known, which contains exotic solutions in spectrum.

In non-covariant quantization the gauges, excluding exotic solutions classically, do not have them on the quantum level. The gauge, considered in Appendix 5, contains the exotic solutions classically and preserves them on the quantum level.

Finally, we want to describe the following problem, which we foresee in further development of string theory. One day somebody can find quantum representation of string theory in $d = 3 + 1$, which will be free of anomalies and will operate in positively defined extended space of states. This theory, however, necessarily should contain exotic states. Their exclusion is possible only in specific gauges, which do not contain exotic solutions classically. This requirement narrows the class of admissible gauges.

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The images, presented in this work, are constructed by computer program [13], which can create static 3D models of the world sheets and present string dynamics as a film. This program is developed in the frames of a project “Visualization of complex physical phenomena and mathematical objects in virtual environment”, supported by INTAS 96-0778 grant.

Appendix 1. Geometrical reconstruction of the world sheets

Properties 1,2 of supporting curve follow from its definition. Property 3 follows from 4 and 2. Let’s prove the property 4:

$$x_\mu(\sigma, \tau) = \frac{(Q_\mu(\sigma_1) + Q_\mu(\sigma_2))/2, \quad \sigma_{1,2} = \tau \pm \sigma.}$$

This formula was obtained in [1] by direct solution of Hamiltonian equations in $(x, p)$-representation. Here we will reproduce the proof of this formula in oscillator representation.

Coordinates and momenta of the string are defined by expressions(7)

$$x_\mu(\sigma) = X_\mu + \frac{i}{\sqrt{\pi}} \sum_{n \neq 0} a_n^{\mu} \cos n\sigma, \quad p_\mu(\sigma) = \frac{1}{\sqrt{\pi}} \sum_{n} a_n^{\mu*} \cos n\sigma, \quad$$

(7) See e.g. [6]. Difference of notations: $a_n^{\mu}$ in our work corresponds to $i\sqrt{\pi}a_n^{\mu*}$ in [6] ($n > 0$).
so that formulae (1)-(3) are valid. Poisson brackets for canonical variables:
\[
\{a^{\mu}_n, a^{\nu}_k\} = i n g_{\mu \nu} \delta^{k-n}, \quad \{X_\mu, P_\nu\} = g_{\mu \nu}.
\]
Hamiltonian of the system is an arbitrary linear combination of Virasoro constraints: 
\[
H = \sum c^{k} L^k \ (c^{0} = c^{-k}).
\]
Coefficients \(c^k\) influence only parametrization of the world sheet. The choice \(H = L^0\) corresponds to conformal parametrization (where \((x' \pm \dot{x})^2 = 0\)). This Hamiltonian generates phase rotations \(a^{\sigma}_\mu(\tau) = a^{\sigma}_\mu(0)e^{i\mu \tau}\) and shifts \(X_\mu(\tau) = X_\mu(0) + (P_\mu/\pi) \tau\).

Using (3), we see that the evolution of function \(Q_\mu(\sigma)\) is the shift of its argument: \(Q_\mu(\sigma, \tau) = Q_\mu(\sigma + \tau, 0)\). Then, using (2), we have the following evolution for coordinates and momenta:
\[
x_\mu(\sigma, \tau) = (Q_\mu(\tau + \sigma, 0) + Q_\mu(\tau - \sigma, 0))/2, \quad p_\mu(\sigma, \tau) = (Q'_\mu(\tau + \sigma, 0) + Q'_\mu(\tau - \sigma, 0))/2.
\]
Introducing isotropic coordinates \(\sigma_{1,2} = \tau \pm \sigma\), obtain formula (9).

**Remark:** Poisson brackets for \(Q_\mu(\sigma)\): \(\{Q_\mu(\sigma), Q_\nu(\sigma)\} = -2g_{\mu \nu} \partial(\sigma - \bar{\sigma})\), where \(\partial(\sigma) = [\sigma/2\pi] + \frac{1}{2}\), \([x]\) is integer part of \(x\), a derivative \(\partial(\sigma)' = \Delta(\sigma)\) is periodical delta-function. As a result, we have \(\{Q_\mu(\sigma), Q^2(\sigma)/4\} = \Delta(\sigma - \bar{\sigma})Q'_\mu(\sigma)\), and for \(H = \int d\sigma F(\sigma)Q^2(\sigma)/4\) : \(Q_\mu(\sigma) = \{Q_\mu(\sigma), H\} = F(\sigma)Q'_\mu(\sigma)\), linear combinations of constraints generate shifts of points in tangent direction to the supporting curve, or equivalently – reparametrizations of this curve.

Reduced phase space of string, obtained in factorization of the phase space by the action of gauge group, is actually a set of all possible supporting curves, which are considered as geometric images, without respect to their parametrization (two different parametrizations of the curve correspond to the same point of the reduced phase space). All physical observables in string theory are parametric invariants of supporting curve. The world sheet is also reconstructed by the supporting curve in parametrically invariant way, see fig.1.

**Appendix 2. Densities of momentum and angular moment**

Action of the string is equal to an area of the world sheet
\[
A = \int L d\sigma_1 d\sigma_2, \quad L = \sqrt{(\partial_1 x \partial_2 x)^2 - (\partial_1 x)^2(\partial_2 x)^2} = \sqrt{-\frac{1}{4} L^{\mu \nu} L^{\alpha \beta} \epsilon_{\alpha \beta \mu \nu}}, \quad L^{\mu \nu} = \epsilon_{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x^\nu,
\]
where \(\epsilon_{12} = -\epsilon_{21} = 1, \ \epsilon_{11} = \epsilon_{22} = 0\). \(L^{\mu \nu} d\sigma_1 d\sigma_2\) is tensor element of area.

Condition for extremum of action has a form:
\[
(10) \quad \partial_\alpha p^\alpha = 0, \quad p^\alpha = \frac{\delta A}{\delta \partial_\alpha x^\alpha} = -\frac{L^{\mu \nu}}{L} \epsilon_{\alpha \beta} \partial_\beta x^\nu,
\]
\(p^\alpha\) is momentum density, flowing through a section \(\sigma_\alpha = \text{Const}\). Integral form of this condition:
\[
\int_C p^\mu_\alpha \epsilon_{\alpha \beta} d\sigma_\beta = 0 \quad \text{has a sense of momentum conservation law: total momentum, flowing through a closed contour} \ C \ \text{on the world sheet, vanishes (here} \ d\sigma_\alpha = (d\sigma_1, d\sigma_2) \ \text{is tangent element to the curve} \ C, \ \epsilon_{\alpha \beta} d\sigma_\beta = (d\sigma_2, -d\sigma_1) \ \text{is the normal element, see fig.14).}
\]

**Fig.14.** Tangent and normal elements to the curve on parameters plane.

**Remarks.**

1. (Conventions). Usually the action of the string is multiplied by a factor \((2\pi\alpha')^{-1}\), \(\alpha'\) is a dimensional constant, defining the tension of the string. In our work we choose the system of units \(2\pi\alpha' = 1\).
Also the sign minus is usually introduced in the action of the string, by analogy with the mechanics of point particle, where $A = -ms$, $s$ is interval of the world line. This sign is actually needed only at the choice of metric with signature $(- + +)$. For this choice it provides positivity of energy (time component of covariant vector $p_\mu = g_{\mu\nu}\partial L/\partial \dot{x}_\nu$), see table below ($\vec{v} = \frac{d\vec{x}}{dx}$, $\gamma = (1-v^2)^{-1/2}$):

| signature | $L$ | $p^\mu = \partial L/\partial \dot{x}_\mu$ | $p_\mu = g_{\mu\nu}p^\nu$ |
|-----------|-----|---------------------------------|------------------|
| $- + + \cdots$ | $-m\sqrt{-\vec{v}^2}$ | $m\gamma(-1, \vec{v})$ | $m\gamma(1, \vec{v})$ |
| $+ - - \cdots$ | $m\sqrt{\vec{v}^2}$ | $m\gamma(1, -\vec{v})$ | $m\gamma(1, \vec{v})$ |

For signature $(+ - -)$, adopted in our work, positive Lagrangian corresponds to positive energy. Analogously, the choice of sign plus in string’s action corresponds to positive density of energy for normal solutions (see below).

2. In the points of the world sheet, where $L = 0$, $p^\mu$ has a singularity. Such cases are often rejected from consideration (see e.g. [6]). This rejection has no ground. It is shown in [14], that points with $L = 0$ for the theory in 3- and 4-dimensional Minkowsky space are topologically stable. They do not disappear in small deformations of the world sheet. Such singular cases occupy volumes in the phase space (are not rare). For the world sheets, considering in our work, $L = 0$ on cusp lines (where $\partial_1 x$ or $\partial_2 x$ vanishes). Thus, such singular cases should be considered in details.

Let’s subdivide the world sheet into patches with $L \neq 0$, separated by lines with $L = 0$ (isolated points with $L = 0$ are also rejected from the patches). Variation of action in inner regions of $L \neq 0$ patches gives equations (10) for these regions. Due to these equations, variation of action is reduced to boundary terms: $\delta A = \sum_\delta \int_{\partial_\delta} \delta x_\mu (dP^\mu_+ - dP^-_\mu)$, where $dP_\mu = p^\nu_\mu \epsilon_{\alpha\beta} d\sigma_\beta$ is a differential flow of momentum through the element $d\sigma_\beta$, see fig.15. This expression is well defined, only if $dP_\mu$ has finite limits on the boundary, correspondingly $dP^\mu_+$ and $dP^-_\mu$ for each of two patches, adjacent to $d\sigma_\beta$.

Annulation $\delta A = 0$ for any $\delta x$ implies that $dP^\mu_+ = dP^-_\mu$, limits of differential flows should coincide on the boundary. In other words: momentum, flowing from one patch, should be equal to momentum, entering into another patch, so that momentum conservation is satisfied globally on the world sheet(8). (For isolated singular point the momentum, flowing through a closed contour around this point should vanish. Analogously, boundary condition on a free edge of the world sheet has a form $dP_\mu = 0$ – flow of momentum through the free edge should vanish.)

![Fig.15. Flows of momentum through the line $L = 0$.](image)

Let’s substitute the representation of the world sheet $x(\sigma_1, \sigma_2) = (Q(\sigma_1) + Q(\sigma_2))/2$ into (10):

$$\partial_{1,2}x = Q_{1,2}/2, \quad L_{\mu\nu} = Q_{1,\mu}Q_{2,\nu}/4, \quad -\frac{1}{2}L_{\mu\nu}L_{\mu\nu} = (Q_1Q_2)^2/16.$$

(8) In this work we actually consider only such world sheets, for which the nominator in $p^\mu_\mu$ vanishes simultaneously with the denominator, and resolution of this 0/0 ambiguity gives smooth ($C^\infty$) function $p^\mu_\mu(\sigma_1, \sigma_2)$. For such world sheets the limits $dP^\pm_\mu$ automatically coincide, and the equations $\partial_\alpha p^\mu_\alpha = 0$ have sense everywhere on the world sheet.
EXOTIC SOLUTIONS IN STRING THEORY

\[ \mathcal{L} = (Q'_1Q'_2)/4, \quad p^i_1,2 = Q'_i(\sigma_{2,1})/2, \]

consequently, the equations (10) are valid: \( \partial_1 p_1 = \partial_2 p_2 = 0 \). Boundary conditions on a free edge are also satisfied (e.g. for the edge \( \sigma_1 = \sigma_2 \) we have \( p_1 = p_2, \; d\sigma_1 = d\sigma_2 \Rightarrow dP = p_1 d\sigma_2 - p_2 d\sigma_1 = 0 \)). Thus, the world sheets of the form \( x = (Q_1 + Q_2)/2 \) satisfy Lagrange-Euler equations and boundary conditions, also in the case, if the supporting curve \( Q(\sigma) \) is not monotous.

Remark: this result is valid in the case of Lagrangian theory of type \( b \), see Section 2. For the theory of type \( a \) we obtain \( \mathcal{L} = |Q'_1Q'_2|/4, \; p^i_1,2 = Q'_i(\sigma_{2,1})/2 \cdot \text{sign}(Q'_1Q'_2) \), function \( p^0_i \) is discontinuous on cusp lines fig.7. In this case the flows \( dP^\mu_i \) do not coincide on the cusp lines (e.g. for \( \sigma_1 = \sigma_{A\beta} + 2\pi n \) we have \( dP^+ \cdot dP^- = \pm Q_2 d\sigma_2 \neq 0 \)). Further consideration is concerned to the theory of type \( b \).

The momentum, flowing through the element \((d\sigma_1, d\sigma_2)\), is given by an expression

\[ dP^\mu = p^\mu_0 \epsilon_{\alpha\beta} d\sigma_\beta = (\mp Q'_i d\sigma_1 + Q'_2 d\sigma_2)^\mu / 2. \]  

Calculating \( \int dP^\mu \) along the curve, connecting points \((\sigma_1, \sigma_2)^i\) and \((\sigma_1, \sigma_2)^f\), obtain formula (4).

Density of angular moment:

\[ dM_{\mu\nu} = x_{[\mu} dP_{\nu]} = \frac{1}{2} (Q_1 + Q_2)_{[\mu} d(-Q_1 + Q_2)_{\nu]}. \]

Integrating this density along the curve, have

\[ \Delta M_{\mu\nu} = \frac{1}{4} Q'_1 Q'_2 \mid^{(\sigma_1, \sigma_2)^f}_{(\sigma_1, \sigma_2)^i} + \frac{1}{4} \int^{(\sigma_1, \sigma_2)^f}_{(\sigma_1, \sigma_2)^i} (-Q_{1}^{\mu} Q_{1}^{\nu} + Q_{2}^{\mu} Q_{2}^{\nu}). \]

This formula can be identically rewritten to the form (5).

Let’s derive a formula (6) for linear density of energy on the string. Let’s consider equal-time slice of the world sheet:

\[ dx_0(\sigma_1, \sigma_2) = (Q'_1 d\sigma_1 + Q'_2 d\sigma_2)/2 = 0 \Rightarrow d\sigma_1 = \frac{Q'_1}{Q'_2} d\sigma_2. \]

From (11): \( dP_0 = (-Q'_1 d\sigma_1 + Q'_2 d\sigma_2)/2 = Q'_2 d\sigma_2 \). Element of length on the string:

\[ dx(\sigma_1, \sigma_2) = (Q'_1 d\sigma_1 + Q'_2 d\sigma_2)/2 = \left( -Q'_1 Q'_2 \right) d\sigma_2, \; dl = |dl| = \left| \frac{1}{2} \right| \left( \sqrt{Q'_1 Q'_2} \right) d\sigma_2, \]

or after elementary transformations: \( dl = \sqrt{(Q'_1 Q'_2)/2Q'_2} \cdot |d\sigma_2| \) (argument of the square root is not negative).

The linear density of energy:

\[ \frac{dP_0}{dl} = \sqrt{\frac{2Q'_1}{Q'_2}} \cdot \frac{d\sigma_2}{|d\sigma_2|}. \]

To find the sign of \( d\sigma_2 \), let’s consider normal element \( \epsilon_{\alpha\delta} d\sigma_\beta = (d\sigma_2, -d\sigma_1) \), see fig.14. Expression (11) actually represents the flow of momentum in the direction of normal element. In our case the normal element should correspond to the Lorentz vector \( (Q'_1 d\sigma_2 - Q'_2 d\sigma_1) \), \textit{directed into the future}. In this case we calculate the flow of momentum through the equal-time slice from past to the future. This flow is usually identified with total momentum of the string.

Thus, \( Q_1 d\sigma_2 - Q_2 d\sigma_1 > 0 \Rightarrow (Q'_1 + Q'_2)^2/Q'_1 > 0 \), \textit{sign} \( d\sigma_2 = \text{sign} Q'_1 \). From (12) we also have \textit{sign} \( d\sigma_1 = - \text{sign} Q'_2 \). These relations define \textit{global orientation} for each connective component in equal time slice, shown by arrows on solid lines on fig.7. Arrows on dashed lines define the direction into the future. These two kinds of arrows are everywhere related by the rule, depicted on fig.14.

Finally, from (13) we have (6).
Appendix 3. Geometrical interpretation of Regge condition

In center-of-mass frame $P_\mu = (\sqrt{P^2}, 0)$ we have $M_{ij} = \frac{1}{2} \oint Q_i dQ_j = \frac{1}{2} S_{ij}$, $S_{ij}$ is oriented area, restricted by a projection of supporting curve onto a plane $(ij)$. For this area the following inequalities are valid: $|S_{ij}| \leq L^2_{ij}/4 \pi \leq L^2/4 \pi = P^2/\pi$, where $L_{ij}$ and $L$ are total lengths of supporting curve in projection to $(ij)$-plane and to space component of CMF respectively.

**Remarks.**

1. Here the first inequality follows from well-known geometrical fact: among all planar curves the ratio (area / square of length) reaches maximum on a circle. The last identity is due to $2P_0 = \int_0^{2\pi} d\sigma a_0 = \int_0^{2\pi} d\sigma |\vec{Q}| = L$.

2. Equality $|M_{ij}| = P^2/2\pi$ is reached on the supporting curve, whose projection to space component of CMF is a circle (lying in $(ij)$-plane). Correspondent string is a straight line, rotating at constant angular velocity in CMF [10].

3. For exotic solutions this argumentation is not valid. In this case $M_{ij} = S_{ij}/2$ as earlier, but $2P_0 = L_+ - L_-$, where $L_+$ is total length of parts of supporting curve with $a_0(\sigma) > 0$; $L_-$ is length of parts with $a_0(\sigma) < 0$. Generally $2P_0 \leq L = L_+ + L_-$.  

Appendix 4. DDF variables and light cone gauge

1. Normal solutions. For given set $\{\vec{A}_n\}$ the function $\vec{Q}_+ (\sigma)$ can be reconstructed: $\vec{Q}_+ (\sigma) = \frac{1}{2\pi} \sum_n \vec{A}_n e^{-in\sigma}$, then using the relation $Q_+ = \frac{\pi}{2P_-} \vec{Q}_+^2$ and one integration, we can find the supporting curve $Q_+ (\sigma)$.

2. Exotic solutions. In this case we cannot reconstruct the function $\vec{Q}_+ (\sigma)$:

$$\frac{1}{2\pi} \sum_n \vec{A}_n e^{-in\sigma} = \int_0^{2\pi} d\sigma \vec{Q}_+ (\sigma) \Delta (\pi Q_- (\sigma)/P_- - \tilde{\sigma}) = \frac{P_-}{\pi} \sum_i \vec{T}_+ (\sigma_i),$$

here $\Delta(\sigma) = \frac{1}{2\pi} \sum_n e^{in\sigma}$ is $2\pi$-periodical delta-function; $\sigma_i$ are solutions of the equation $\pi Q_- (\sigma)/P_- = \tilde{\sigma}$; $\vec{T}_+ = \vec{Q}_+ / |\vec{Q}_+|$ is transverse component of tangent vector $\vec{T}_+ = Q_\mu / |Q_\mu|$, see fig.16. For non-monotonous intervals of supporting curve we can reconstruct not vectors $\vec{T}_+$ themselves (they are sufficient to obtain the supporting curve), but their sum in a slice $Q_- = Const$. Obviously, we can deform the supporting curve in such a way, that this sum will not be changed. Particularly, we can add to $\vec{T}_+ (\sigma)$ near point $\sigma_i$ any function of the form $\delta \vec{T}_+ (\sigma) = \vec{f}(\pi Q_- (\sigma)/P_-)$, where $\vec{f} \in C^\infty$ and has finite support $[ab]$, and subtract this function in the vicinity of point $\sigma_i$. Thus, for exotic solutions variables $\{\vec{A}_n\}$ do not define the shape of supporting curve uniquely.

![Fig.16. DDF variables are defined by sum of vectors $T$ in three points $\sigma_{1,2,3}$.](image-url)
Appendix 5. Tangent angle gauge

Let’s transfer the mechanics to CMF: introduce orthonormal basis of vectors \( N^\alpha_\mu \) : \( N^\alpha_\mu N^\beta_\mu = g^{\alpha \beta} \), with \( N^\alpha_\mu = P_\mu / \sqrt{P^2} \), and decompose the supporting curve by this basis: \( Q_\mu(\sigma) = N^\alpha_\mu Q^\alpha(\sigma), \ a_\mu(\sigma) = N^\alpha_\mu a^\alpha(\sigma) \), etc. For the components \( a^\alpha(\sigma) \) we write the following parametrization:

\[
(14) \quad a^\alpha(\sigma) = a^0(\sigma) (1, \cos(\sigma + \theta), \sin(\sigma + \theta)), \quad a^0(\sigma) = \sum_n \alpha_n e^{i\sigma \alpha_n}, \alpha_n = \alpha_{-n}.
\]

From the condition \( \int_0^{2\pi} d\sigma a^\alpha(\sigma) = 2(\sqrt{P^2}, 0) \) we have \( a^0 = \sqrt{P^2}/\pi, \ a_1 = a_{-1} = 0 \).

Substituting this parametrization into general symplectic form, defining the string dynamics\(^9\), obtain:

\[
\Omega = dP_\mu \wedge dZ_\mu - dS \wedge d\theta - i\pi \sum_{n \geq 2}^{1} \frac{1}{n(n^2 - 1)} d\alpha_n \wedge d\alpha_n^*, \quad Z_\mu = X_\mu + S \Gamma_\mu, \\
X_\mu = Q_\mu(0) + \int_0^{2\pi} d\sigma a^0 q^\mu + \frac{P_\mu}{2 \pi \sqrt{P^2}} \int_0^{2\pi} d\sigma q^0, \quad q^\alpha(\sigma) = \int_0^\sigma d\sigma a^\alpha(\sigma), \quad \Gamma_\mu = N^1_\nu \partial N^2_\nu / \partial P_\mu.
\]

- Here \( Z_\mu \) is canonical mean coordinate, conjugated to \( P_\mu \) (compare with [10]). From these expressions one can obtain \( Q_\mu(\sigma) \) as a function of \( Z_\mu \) and oscillator variables, and then reconstruct the supporting curve as \( Q_\mu(\sigma) = Q_\mu(0) + q_\mu(\sigma) \).
- \( S = \frac{1}{2} \int Q^0 dQ^2 \) is string’s angular moment in CMF (spin). Here we consider \( d = (2 + 1) \)-dimensional Minkowsky space, where the spin is a scalar variable, which can take both signs.

The obtained symplectic form corresponds to the following Poisson brackets:

\[
\{ Z_\mu, P_\nu \} = g_{\mu \nu}, \quad \{ S, \theta \} = 1, \quad \{ \alpha_n, \alpha_m^* \} = -i\pi n(n^2 - 1)\delta_{nm}, \quad n, m \geq 2.
\]

It’s convenient to introduce new oscillator variables:

\[
a_n = \pi^{-1/2} (n(n^2 - 1))^{-1/2} \alpha_n, \quad \{ a_n, a_m^* \} = -i\delta_{nm}.
\]

Substituting (14) into the definition of spin, we obtain a constraint (mass shell condition):

\[
(15) \quad \Phi = \frac{P^2}{2\pi} - S = -\sum_{n \geq 2} n a_n^* a_n = 0.
\]

This constraint should be used as Hamiltonian of the system. One can easily prove, that it generates the reparametrization of supporting curve \( Q_\mu(\sigma) \to Q_\mu(\sigma + \tau) \).

Generators of Lorentz group are defined by expression, analogous to written in [10]:

\[
(16) \quad M_{\mu \nu} = X_{[\mu} P_{\nu]} + S N^{1}_{[\mu} N^{2}_{\nu]}.
\]

They generate Lorentz transformation of a coordinate frame \( (N^0_\mu, N^k_\mu e^1_i) \), \( e_1 = (\cos \theta, \sin \theta) \), \( e_2 = (-\sin \theta, \cos \theta) \), by which the supporting curve is decomposed with scalar (\( \theta \)-independent) coefficients.

Remarks.

1. Parameter \( \varphi = \sigma + \theta \) is equal to polar angle of vector \( \vec{Q}(\sigma) \), tangent to the supporting curve on CMF plane, see fig.17. (That’s why we call this parametrization as tangent angle gauge, TAG). Function \( a^0(\sigma) \) is equal to the radius of curvature \( a^0 = dQ^0 / d\sigma = dL / d\varphi = R \).

\(^9\) For the details of this technique see [10, 12].
Correspondent spin-mass spectrum is shown on fig.18b. It consists of linear Regge trajectories and is equal to multiplicity of eigenstates of operator \( L(\sigma) \). TAG also includes the supporting curves, for which the function \( a^0(\sigma) \) is not positive (those, for which \( P^2 > 0 \) and polar angle of vector \( d\vec{Q}/dQ^0 \) is monotonic in \( \sigma \), see fig.17). Thus, TAG includes the solutions from normal sector, for which the string has no cusps, and a definite admixture of solutions from exotic sector.

3. Region of variation of variables \( (P^2/2\pi, S) \) and typical examples of supporting curves are shown on fig.18a. Normal solutions occupy on this plane the sector \( 0 \leq S \leq P^2/2\pi \). For exotic solutions \( S \leq P^2/2\pi \), here \( S \) is not bounded from below (see (15)), exotic solutions violate the Regge condition.

Quantization of this mechanics is straightforward. Canonical operators:

\[
[Z_\mu, P_\nu] = ig_{\mu\nu} \quad [S, \theta] = i, \quad [a_n, a^*_m] = \delta_{nm}, \quad n, m \geq 2.
\]

Oscillator part of the mechanics can be realized in a positively defined Fock space: \( a_k|0\rangle = 0, \quad |\{n_k\}\rangle = \prod_{k \geq 2} \sqrt{n_k} (a^+_k)^{n_k} |0\rangle \), with the occupation numbers \( n_k = a^+_k a_k = 0, 1, 2, \ldots \). Complete space of states can be defined as a direct product of this Fock space onto the space of functions \( \Psi(P_\mu, \theta), 2\pi\text{-periodic for } \theta \), with definition of operators \( Z_\mu = i\partial/\partial P_\mu \), \( S = i\partial/\partial \theta \) (eigenvalues \( S \in \mathbb{Z} \)). Physical subspace is selected by the constraint \( (P^2/2\pi - S - \sum kn_k)_{\text{phys}} = 0 \).

Correspondent spin-mass spectrum is shown on fig.18b. It consists of linear Regge trajectories \( P^2/2\pi - S = T, \quad T = \sum_{k \geq 2} kn_k = 0, 2, 3, \ldots \). The multiplicity of states is constant on the trajectory and is equal to multiplicity of eigenstates of operator \( T \), e.g. \( T = 0 : |0\rangle, \quad T = 2 : |1\rangle, \quad T = 3 : |1\rangle, \quad T = 4 : |2\rangle, |1\rangle \) (2 states), etc.

**Remark:** in our approach \( Z_\mu, P_\mu \) and \( S \) are taken as independent canonical variables, for which...
the commutators are postulated directly from Poisson brackets. It was shown in [10], that in this case the Lorentz generators, defined by expression (16), have no anomaly in commutators.

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