A note on combining chaotic dynamical systems using the fuzzy logic XOR operator.

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Abstract—In this paper we explore whatever combining two chaotic dynamical systems using the fuzzy logic operator XOR can maintain or not the chaotic properties of the resulting dynamical system. This study is motivated by techniques used in applications to secure communications, images encryption and cryptography.

Key words : Chaos, fuzzy logic, ergodic theory, full branch.

I. INTRODUCTION

Chaotic dynamical systems are commonly used as models in a wide range of applications, cryptography, image encryption and retrieval and achievement of associative memory properties. Extreme sensitivity to initial condition is an interesting property of chaotic systems. This property makes chaotic systems a worthy choice for constructing cryptosystems or for image encryption.

Another idea is to mix two or more chaotic dynamical systems to gain more “unpredictably” or more “confusion” in order to enhance encryption process. This rise a natural question, whatever combining two chaotic dynamical systems permits to maintain chaotic property of the resulting one? In particular whatever combining two chaotic dynamical systems by fuzzy logic operators, mainly the operator xor, gives rise to a chaotic dynamical system.

In other words consider two chaotic dynamical systems $(I, F)$ and $(I, G)$ is the dynamical system $(I, F \ast G)$ still chaotic? Where $\ast$ stands for any fuzzy logic operator.

We studied some combination of known chaotic dynamical systems and checked whatever the combination is still chaotic or not. This gave us some preliminary remarks about how to combine chaotic dynamical systems in order to maintain the chaotic properties of the resulting dynamical system.

II. DYNAMICAL SYSTEMS

A. Topological dynamics

A dynamical system $(X, F)$ consists of a compact metric space $X$ and a continuous self-map $F$.

A point $x \in X$ is said to be an equicontinuity point, or to be Lyapunov stable, if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$ one has $d(F^n(y), F^n(x)) < \epsilon$ for any integer $n \geq 0$.

We say that $(X, F)$ is sensitive if for any $x \in X$ we have:

$\exists \epsilon > 0, \forall \delta > 0, \exists y \in B_{\delta}(x), \exists n \geq 0$ such that $d(F^n(y), F^n(x)) > \epsilon$.

A dynamical system $(X, T)$ is transitive if for any nonempty open sets $U, V \subset \mathbb{R}^2$ there exists $n > 0$ with $U \cap F^{-n}(V) \neq \emptyset$. This is equivalent to the existence of a point with a dense orbit.

A dynamical system is said topologically mixing if for any nonempty open sets $U, V \subset \mathbb{R}^2, U \cap F^{-n}(V) \neq \emptyset$ for all sufficiently large $n$.

Let $(X, F)$ be a dynamical system, endow the set $X$ with a sigma algebra $\mathcal{B}$. The function $F$ preserves some measure $\mu$ on the sigma algebra $\mathcal{B}$ if for every $B \in \mathcal{B}$ we have $\mu(F^{-1}(B)) = \mu(B)$. We say then that $(X, \mathcal{B}, F, \mu)$ is a measurable dynamical system.

The topological support of a measure is defined as the set of all points $x \in X$ for which every open neighborhood of $x$ has positive measure.

A dynamical system $(X, \mathcal{B}, F, \mu)$ is ergodic if every invariant subset of $X$ is either of measure 0 or of measure 1. Equivalently, if for any measurable $U, V \subset X$, there exists some $n \in \mathbb{N}$ such that $\mu(U \cap F^{-n}(V)) > 0$.

If a dynamical system is ergodic then it is transitive on the topological support of the measure.

For dynamical systems on the real line a common used measure is the Lebesgue measure, the topological support of the Lebesgue measure is $\mathbb{R}$.
B. Chaotic dynamical systems on the interval

Dynamical systems defined on the real line have a particular behavior, a rich literature is devoted for the subject, we will recall here some results about their properties. In matter of chaos The Devaney’s chaos is seen as a combination of unpredictably (sensitivity) and regular behaviors (periodic points), transitivity ensuring that the system is undecomposable.

**Definition 1:** A topological dynamical system \((X, f)\) is chaotic in the sense of Devaney if :

1. Is transitive.
2. The set of periodic points is dense in \(X\).
3. Is sensitive to initial conditions.

It is know that for every dynamical system the conditions 1 and 2 implies the condition 3 which lead to the so called Modified Devaney definition of chaos.

For interval maps transitivity is enough to imply the other two conditions.

**Proposition 2:** An interval map is chaotic in the sense of Devaney if and only if it is transitive.

For the proof of this result you can look at [13].

III. RESULTS

A. Preliminary remarks

One issue when using the fuzzy logic xor operator is preserving the invariance of the resulting dynamical system if we want to combine two dynamical systems they must be defined on the same interval but this is not enough to ensure invariance of the resulting combined dynamical system.

If the two dynamical systems are defined on the interval \([0, 1]\) it is easy to show that the result will be invariant on the interval \([0, 1]\). One solution to overcome the problem of the invariance is to rescale every dynamical system defined on a given interval to \([0, 1]\).

Below we give two examples the first one of an xor combination of two chaotic dynamical systems which is not chaotic and the second one where the result is a chaotic dynamical system.

**Example 3:** Let us consider the two dynamical systems \([(0, 1], f_r)\) and \([(0, 1], T)\) where \(f_r\) and \(T\) are the logistic map and the tent map respectively.

\[
f_r(x) = r x(1 - x), T(x) = \begin{cases} -2x + 1, & 0 \leq x \leq 0.5 \\ 2x - 1, & 0.5 \leq x \leq 1 \end{cases}
\]

These two dynamical systems are well known chaotic systems, let us consider their fuzzy xor combination \(H\) defined by

\[
H(x) = \max(f_r(x), T(x)) - \min(f_r(x), T(x))
\]

It possesses two fixed points, the fixed point 0 is instable while the fixed point 0.23 is asymptotically stable. The basin of attraction of the point 0 contains 4 isolated points while the basin of attraction of 0.23 contain the hole interval except the basin of attraction of 0.

**Example 4:** Consider the two following dynamical systems the two dynamical systems \([(0, 1], T)\) and \([(0, 1], ST)\) where \(T\) is the tent map and \(ST\) that is defined by

\[
ST(x) = \begin{cases} -2x + 1, & 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq \frac{3}{4} \end{cases}
\]

The graph of the function \(ST\) is given below, you can see it as an inverted Tent map graph. The map \(ST\) is chaotic as the point \(\frac{1}{5}\) has a dense orbit. The map \(S \text{ xor } ST\) is chaotic.

B. Numerical experiments summary

Along with the two examples shown before we have tested some other dynamical systems, part of the results is shown in the following table.

| xor         | Doubling map | Cubic map | Logistic map | Tent map | Inverted Tent map |
|-------------|-------------|-----------|--------------|----------|-------------------|
| Doubling map | Non chaotic | Non chaotic | Non chaotic | Non chaotic | Non chaotic       |
| Cubic map   | Non chaotic | Non chaotic | Non chaotic | Non chaotic | Non chaotic       |
| Logistic map| Non chaotic | Non chaotic | Non chaotic | Non chaotic | Chaotic           |
| Tent map    | Non chaotic | Non chaotic | Non chaotic | Non chaotic | Chaotic           |
| Inverted Tent map | Non chaotic | Non chaotic | Non chaotic | Non chaotic | Chaotic           |

The observation of the results of the table suggests that combining two dynamical systems using the xor operator leads to the resulting dynamical system to be chaotic if their graphs have some form of symmetry to the horizontal line \(y = \frac{1}{2}\).

C. Mirror effect and number of full branches

The aimes of this section is to come with some criterion choice. The optimal situation is that the two dynamical systems have to be symmetrical to the horizontal line \(y = \frac{1}{2}\) this is what we will call a mirror effect. In this situation we can show that the combination is a chaotic dynamical using tools from ergodic theory.

In practical this result could be a tool to choose the dynamical systems to combine, the closest to symmetry to \(y = \frac{1}{2}\) they are the best chances the combination to work we have.

**Definition 5:** Let \(I \subseteq \mathbb{R}\) be an interval. A map \(f : I \rightarrow I\) is a full branch map if there exists a finite or countable partition \(P\) of \(I\) into subintervals such that for each \(w \in P\) the map \(f\mid_{int(w)} : \text{int}(w) \rightarrow \text{int}(I)\) is a bijection.

A map \(f\) is a piecewise continuous (resp \(C^1, C^2, affine\)) full branch map if for each \(w \in P\) the map \(f\mid_{int(w)} : \text{int}(w) \rightarrow \text{int}(I)\) is a homeomorphism (resp \(C^1\) diffeomorphism, \(C^2\) diffeomorphism, \(affine\)).

**Definition 6:** A full branch map has bounded distortion if

\[
\sup_{n \in \{1, 2\}} \sup_{w^{(n)} \in P^{(n)}} \sup_{x, y \in w^{(n)}} \log \left| \frac{Df^{(n)}(x)}{Df^{(n)}(y)} \right| < \infty
\]

here \(n\) stands for the \(n\)th derivative.

If the function is piecewise affine then the distortion is 0.

**Example 7:** The tent map, the doubling map and the logistic map have two full branches with bounded distortion, the cubic map has three full branches with bounded distortion.

**Proposition 8:** Consider \([(0, 1], f)\) and \([(0, 1], g)\) two dynamical systems, suppose that the graphs of \(f\) and \(g\) are symmetrical according to the horizontal line \(y = \frac{1}{2}\) and that the number of full branches of \(f\) and \(g\) are equal to \(k\).

The dynamical system \([(0, 1], f \text{ xor } g)\) has \(2k\) full branches.

**Proof.** Suppose that we have a partition \(P\) of \([0, 1]\) into subintervals such that for each \(w \in P\) the map \(f\mid_{int(w)} :
int (w) → [0, 1] is a bijection.  

As g is a mirror of f we obtain by symmetry

\[
(f \ xor g)(x) = \begin{cases} 
2 |f(x) - 0.5| & \text{if } f(x) \leq 0.5 \\
2 |f(x) - 0.5| & \text{if } f(x) \geq 0.5 
\end{cases}
\]

As f is a bijection there is a partition \( w = w_1 \cup w_2 \) such that:

\[
(f \ xor g)(x) = \begin{cases} 
2 |f(x) - 0.5| & \text{if } x \in w_1 \\
2 |f(x) - 0.5| & \text{if } x \in w_2 
\end{cases}
\]

Thus \( f \ xor g \) has two full branches on \( w \).

**Proposition 9:** Consider \(([0, 1], f)\) and \(([0, 1], g)\) two dynamical systems, suppose that the graphs of \( f \) and \( g \) are symmetrical according to the horizontal line \( y = \frac{1}{2} \) and that \( f \) and \( g \) have the full branch property then \( f \ xor g \) is chaotic.

**Proof.** Suppose that \( f \) has \( k \) branches then \( f \ xor g \) has \( 2k \) branches and a relevant partition \( w \).

As the graph of \( g \) is symmetrical to the graph of \( f \) then they have the same distortion.

On each branch of the partition we have

\[
\sup_n \sup_{x \in w_i} \sup_{y \in w_j} \sup_{x \in w_i} \sup_{y \in w_j} \log |D(f \ xor g)^n(x)| / |D(f \ xor g)^n(y)| \leq \max \left( \sup_n \sup_{x \in w_i} \sup_{y \in w_j} \sup_{x \in w_i} \sup_{y \in w_j} \log |Df^n(x)| / |Df^n(y)|, \sup_n \sup_{x \in w_i} \sup_{y \in w_j} \sup_{x \in w_i} \sup_{y \in w_j} \log |Dg^n(x)| / |Dg^n(y)| \right)
\]

Hence \( f \ xor g \) is a full branch map with bounded distortion. Then the Lebesgue measure is ergodic [10].

As the Lebesgue measure is ergodic then \( f \ xor g \) is transitive on the topological support of the Lebesgue measure. Hence it is chaotic.

**IV. CONCLUSION**

We investigated an ideal situation to combine chaotic one dimensional maps using the fuzzy logic xor operator.

Using tools from ergodic theory we were able to establish a result using slightly strong condition than Devaney’s chaos, it is worth noting that this condition is satisfied by a majority of classical chaotic maps on the interval.

**REFERENCES**

[1] Baptista M. S. Cryptography with chaos / Baptista M. S. Physics Letters A. – 240(1-2). - 1998. – P. 50–54.
[2] R. Chemlal, I Djellit, Coding Information and Problems of storage in Dynamical Systems, FACTA UNIVERSITATIS, SER ELEC ENERG, Vol 17, December 2004, 355-363.
[3] Dmitriev A. S. "Storing and Recognizing Information with One-Dimensional Dynamic Systems" (In Russian), Radiotechnika i Elektrononika (1991), vol. 36, no1, pp 101-108.
[4] A.A. Dmitriev Design of Message-Carrying Chaotic Sequences 2002 Nonlinear Phenomena in Complex Systems.
[5] H. K. Kwan,THREE-LAYER BIDIRECTIONAL ASYMMETRICAL ASSOCIATIVE MEMORY, IEEE 2003.
[6] Kocarev L. Chaos-based cryptography: A brief overview, IEEE Circuits and Systems Magazine, 2001, N°1. P6–21.
[7] Mykola Kushmar, Yuriy Fedkovych, Petro Kroialo, Hryhorii Kosovan. Encryption of the Images on the Basis of Two Chaotic Systems with the Use of Fuzzy Logic, 2020 IEEE 15th International Conference on Advanced Trends in Radioelectronics, Telecommunications and Computer Engineering (TCSET).