Parking Cars after a Trailer

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Abstract

Recently, the authors extended the notion of parking functions to parking sequences, which include cars of different sizes, and proved a product formula for the number of such sequences. We here give a refinement of that result involving parking the cars after a trailer. The proof of the refinement uses a multi-parameter extension of the Abel–Rothe polynomial due to Strehl.

1 The result

Parking sequences were introduced in [3] as an extension of the classical notion of parking functions, where we now take into account parking cars of different sizes. This extension differs from other extensions of parking functions [1, 5, 6, 7, 11] since the parking sequences are not invariant under permuting the entries. The main result in [3] is that the number of parking sequences is given by the product

\[(y_1 + n) \cdot (y_1 + y_2 + n - 1) \cdot \cdots \cdot (y_1 + \cdots + y_{n-1} + 2),\]

(1.1)

where the \(i\)th car has length \(y_i\). Note that this reduces to the classical \((n + 1)^{n-1}\) result of Konheim and Weiss [4] when setting \(y_1 = y_2 = \cdots = y_n = 1\). The proof in [3] is an extension of the circular argument by Pollak; see [8].

We now introduce a refinement of the result by adding a trailer.

Definition 1.1. Let there be \(n\) cars \(C_1, \ldots, C_n\) of sizes \(y_1, \ldots, y_n\), where \(y_1, \ldots, y_n\) are positive integers. Assume there are \(z - 1 + \sum_{i=1}^n y_i\) spaces in a row, where the trailer occupies the \(z - 1\) first spaces. Furthermore, let car \(C_i\) have the preferred spot \(c_i\). Now let the cars in the order \(C_1\) through \(C_n\) park according to the following rule:

Starting at position \(c_i\), car \(C_i\) looks for the first empty spot \(j \geq c_i\). If the spaces \(j\) through \(j + y_i - 1\) are empty, then car \(C_i\) parks in these spots. If any of the spots \(j + y_i - 1\) is already occupied, then there will be a collision, and the result is not a parking sequence.

Iterate this rule for all the cars \(C_1, C_2, \ldots, C_n\). We call \((c_1, \ldots, c_n)\) a parking sequence for \(\vec{y} = (y_1, \ldots, y_n)\) if all \(n\) cars can park without any collisions and without leaving the \(z - 1 + \sum_{i=1}^n y_i\) parking spaces.
As an example, consider three cars of sizes $\vec{y} = (2, 2, 1)$, a trailer of size 3, that is $z = 4$, and the preferences $\vec{c} = (5, 6, 2)$. Then there are $2 + 2 + 1 = 5$ available parking spaces after the trailer, and the final configuration of the cars is

|   | T |   | C₅ |   | C₁ |   | C₂ |
|---|---|---|----|---|----|---|----|
| 1 | 2 | 3 | 4  | 5 | 6  | 7 | 8  |

All cars are able to park, so this yields a parking sequence.

We now have the main result. Observe that when setting $z = 1$, this expression reduces to equation (1.1).

**Theorem 1.2.** The number of parking sequences $f(\vec{y}; z)$ for car sizes $\vec{y} = (y₁, \ldots, yₙ)$ and a trailer of length $z - 1$ is given by the product

$$f(\vec{y}; z) = z \cdot (z + y₁ + n - 1) \cdot (z + y₁ + y₂ + n - 2) \cdots (z + y₁ + \cdots + y_{n-1} + 1).$$

2 The proof

The first part of our proof comes from the following identity. Let $\bigcup$ denote disjoint union of sets.

**Lemma 2.1.** The number of parking sequences for car sizes $(y₁, \ldots, yₙ, y_{n+1})$ and a trailer of length $z - 1$ satisfies the recurrence

$$f(\vec{y}, y_{n+1}; z) = \sum_{L \cup R = \{1, \ldots, n\}} \left( z + \sum_{l \in L} yₗ \right) \cdot f(\vec{y}_L; z) \cdot f(\vec{y}_R; 1),$$

where $\vec{y}_S = (y_{s₁}, \ldots, y_{sₖ})$ for $S = \{s₁ < s₂ < \cdots < sₖ\} \subseteq \{1, \ldots, n\}$.

**Proof.** Consider the situation required for the last car $C_{n+1}$ to park successfully:

- Car $C_{n+1}$ must see, to the left of its vacant spot, the trailer along with a subset of the cars labeled with indices $L$ occupying the first $z - 1 + \sum_{l \in L} yₗ$ spots. Hence, the restriction $\vec{c}_L$ of $\vec{c} = (c₁, c₂, \ldots, c_{n+1})$ to the indices in $L$ must be a parking sequence for $\vec{y}_L$ and trailer of length $z - 1$. This can be done in $f(\vec{y}_L; z)$ possible ways.

- Car $C_{n+1}$ must have a preference $c_{n+1}$ that lies in the range $[1, z + \sum_{l \in L} yₗ]$.

- Car $C_{n+1}$ must see, to the right of its vacant spot, the complementary subset of cars labeled with indices $R = \{1, 2, \ldots, n\} - L$ occupying the last $\sum_{r \in R} yₗ$ spots. These cars must have parked successfully with preferences $\vec{c}_R$ and no trailer, that is, $z = 1$. This is enumerated by $f(\vec{y}_R; 1)$.

Now summing over all decompositions $L \cup R = \{1, 2, \ldots, n\}$, the recursion follows. \qed
The next piece of the proof of Theorem 1.2 utilizes a multi-parameter convolution identity due to Strehl [10]. Let \( \mathbf{x} = (x_{i,j})_{1 \leq i < j} \) and \( \mathbf{y} = (y_j)_{1 \leq j} \) be two infinite sets of parameters. For a finite subset \( A \) of the positive integers, define the two sums
\[
x^A_{>a} = \sum_{j \in A, j > a} x_{a,j} \quad \text{and} \quad y^A_{\leq a} = \sum_{j \in A, j \leq a} y_j.
\]
Define the polynomials \( t_A(\mathbf{x}, \mathbf{y}; z) \) and \( s_A(\mathbf{x}, \mathbf{y}; z) \) by
\[
t_A(\mathbf{x}, \mathbf{y}; z) = z \cdot \prod_{a \in A - \max(A)} (z + y^A_{\leq a} + x^A_{>a}),
\]
\[
s_A(\mathbf{x}, \mathbf{y}; z) = \prod_{a \in A} (z + y^A_{\leq a} + x^A_{>a}).
\]
Note that, when \( A \) is the empty set, we set \( t_A(\mathbf{x}, \mathbf{y}; z) \) to be 1. We directly have that
\[
(z + y^A_{\leq \max(A)}) \cdot t_A(\mathbf{x}, \mathbf{y}; z) = z \cdot s_A(\mathbf{x}, \mathbf{y}; z). \tag{2.1}
\]

Now Theorem 1, equation (6) in [10] states:

**Theorem 2.2** (Strehl). The polynomials \( s_L(\mathbf{x}, \mathbf{y}; z) \) and \( t_R(\mathbf{x}, \mathbf{y}; w) \) satisfy the following convolution identity:
\[
s_A(\mathbf{x}, \mathbf{y}; z + w) = \sum_{L \cup R = A} s_L(\mathbf{x}, \mathbf{y}; z) \cdot t_R(\mathbf{x}, \mathbf{y}; w). \tag{2.2}
\]

Strehl first interprets \( s_A(\mathbf{x}, \mathbf{y}; z) \) and \( t_A(\mathbf{x}, \mathbf{y}; z) \) as sums of weights on functions, then translates these via a bijection to sums of weights on rooted, labeled trees where the \( x_{i,j}'s \) record ascents, and the \( y_j's \) record descents. The proof of (2.2) then follows from the structure inherent in splitting a tree into two. A similar result using the same bijection was discovered by Eğecioğlu and Remmel in [2].

**Proof of Theorem 1.2**. The proof follows from noticing that our proposed expression for \( f(\vec{y}; z) \) is Strehl’s polynomial \( t_{\{1,2,\ldots,n\}}(1, \mathbf{y}; z) \). By induction we obtain
\[
f(\vec{y}; y_{n+1}; z) = \sum_{L \cup R = \{1,2,\ldots,n\}} \left( z + \sum_{l \in L} y_l \right) \cdot f(\vec{y}_L; z) \cdot f(\vec{y}_R; 1)
\]
\[
= \sum_{L \cup R = \{1,2,\ldots,n\}} (z + y^L_{\leq \max(L)}) \cdot t_L(1, \mathbf{y}; z) \cdot t_R(1, \mathbf{y}; 1)
\]
\[
= \sum_{L \cup R = \{1,2,\ldots,n\}} z \cdot s_L(1, \mathbf{y}; z) \cdot t_R(1, \mathbf{y}; 1)
\]
\[
= s_{\{1,2,\ldots,n\}}(1, \mathbf{y}; z + 1)
\]
\[
= t_{\{1,2,\ldots,n+1\}}(1, \mathbf{y}; z),
\]
where we used the recursion in Lemma 2.1, equation (2.1) and Theorem 2.2. □
3 Concluding remarks

The polynomial \( t_A(x, y; z) \) satisfies the following convolution identity; see [10, Equation (7)],
\[
t_A(x, y; z + w) = \sum_{B \cup C = A} t_B(x, y; z) \cdot t_C(x, y; w).
\] (3.1)

Hence it is suggestive to think of this polynomial as of binomial type and the polynomial \( s_A(x, y; w) \) as an associated Sheffer sequence; see [9]. When setting all the parameters \( x \) to be constant and also the parameters \( y \) to be constant, we obtain the classical Abel–Rothe polynomials. Hence it is natural to ask if other sequences of binomial type and their associated Sheffer sequences have multi-parameter extensions. Since the Hopf algebra \( k[x] \) explains sequences of binomial type, one wonders if there is a Hopf algebra lurking in the background explaining equations (3.1) and (2.2).

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