The Vertex-Face Correspondence and the Elliptic $6j$-symbols

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Abstract

A new formula connecting the elliptic $6j$-symbols and the fusion of the vertex-face intertwining vectors is given. This is based on the identification of the $k$ fusion intertwining vectors with the change of base matrix elements from Sklyanin’s standard base to Rosengren’s natural base in the space of even theta functions of order $2k$. The new formula allows us to derive various properties of the elliptic $6j$-symbols, such as the addition formula, the biorthogonality property, the fusion formula and the Yang-Baxter relation. We also discuss a connection with the Sklyanin algebra based on the factorised formula for the $L$-operator.

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1 Introduction

The theory of elliptic hypergeometric series has been rapidly developing. In [1], Frenkel and Turaev took the initiative introducing a notion of modular hypergeometric functions $r+1\omega_r$ as an elliptic analogue of the very-well-poised balanced basic hypergeometric series $r+1\varphi_r$. This was based on an observation that the fusion of the face type Boltzmann weights studied by Date et.al.[2] can be regarded as an elliptic analogue of the $q$-6$j$ symbols.

The theory of biorthogonal rational functions led Spiridonov and Zhedanov to a generalization of the elliptic 6$j$-symbols[3]. Deeper consideration of the well-poisedness and very-well-poisedness conditions for general elliptic hypergeometric series resulted in a change of notations to $r+3V_{r+2}$ [4], which is now considered as a proper elliptic analogue of the very-well-poised basic hypergeometric series $r+1\varphi_r$. At the same time, they reached a relevant scheme for dealing with a family of biorthogonal rational functions as the generalized eigenvalue problem (GEVP) associated with two Jacobi matrices.

Recently Rosengren found a GEVP which is relevant specific to a generalisation of the elliptic 6$j$-symbols[5, 6]. It is deeply related with the representation of the Sklyanin algebra[7] on the space of theta functions. He found a natural basis of the space as a set of solutions of the GEVP and identified a change of base matrix elements between the two natural bases depending on a different parameters with a generalisation of the elliptic 6$j$-symbols. He then succeeded to derive an expression of the generalised “elliptic 6$j$-symbols” in terms of $12V_{11}$.

Such a connection with the Sklyanin algebra reminds us of the work by Takebe[8] on the diagonalisation of the higher spin eight-vertex model by using the algebraic Bethe ansatz. In fact, Rosengren’s natural basis turned out to be a fusion of the vertex-face intertwining vectors realised as the vectors in the space of theta functions. Such vectors are used to map (or gauge transform) a higher dimensional representation of the Sklyanin algebra associated with the fusion of the elliptic $R$-matrix to the Bethe ansatz operators of the fusion of the eight-vertex Solid-on-Solid (SOS) model. This makes the diagonalisation of the model simpler. The eigen states (the Bethe states) are then given by the fused intertwining vectors.

The purpose of this paper is to make a direct connection between the generalised
elliptic 6j-symbols and the fusion of the vertex-face intertwining vectors. In a context of
the solvable lattice models, it is standard to take intertwining vectors as the elements
of the vector space $V$, on which tensor product $V \otimes V$ the fusion of the $R$-matrix
acts. They map a fusion of the elliptic $R$-matrix to a fusion of the face weight of the
SOS model. By making a connection between Takebe’s realisation of the intertwining
vectors and the standard one, we find an exact relation between Rosengren’s change
of base matrix elements and the fusion of the standard intertwining vectors. For this
purpose, we fully use the dual of the intertwining vectors and their fusions. In the
process, we also give a formula connecting the elliptic binomial coefficients and the
intertwining vectors.

This paper is organised as follows. In the next section, we make a quick review of
the fusion of the eight-vertex model and the eight-vertex SOS model as well as their
vertex-face correspondence relationship. These follow the results in [2, 9]. In Sec.3, we
make a connection between the standard intertwining vectors and Rosengren’s vectors
of natural basis. This leads us to a new formula for the generalised elliptic 6j-symbols.
We then derive various properties of the elliptic 6j-symbols. We also make a brief
comment on a connection with the Sklyanin algebra. Since we use a slightly different
notation from [5, 6], in the Appendix, we give a derivation of the elliptic binomial
theorem, the elliptic Jackson’s summation formula and the expression of the elliptic
6j-symbols in terms of $12V_{11}$ following the idea of Rosengren. The other idea of proof
can be found in [10, 11].

1.1 Notations

Let $p = e^{-\frac{\pi K'}{K}}$, $q = -e^{-\frac{\pi \lambda}{2K}}$ and $\zeta = e^{-\frac{\pi \lambda u}{2K}}$. We introduce $x$, $\tau$ and $r$ by $x = -q$, $\tau = \frac{2K'}{K'}$ and $r = \frac{K'}{K}$. Then $p = e^{2\pi i \tau} = x^{2r}$. The parameter $r$ plays a role of restriction height
in the restricted SOS models. Through this paper, we assume $\text{Im} \tau > 0$. Let $\tilde{p} = e^{2\pi i r}$. We use the theta functions

\begin{align*}
\vartheta_1(u|\tau) &= 2\tilde{p}^{1/8}(\tilde{p}; \tilde{p})_\infty \sin \pi u \prod_{n=1}^{\infty} (1 - 2\tilde{p}^n \cos 2\pi u + \tilde{p}^{2n}), \\
\vartheta_0(u|\tau) &= -ie^{\pi i (u+\tau/4)} \vartheta_1 \left(u + \frac{\tau}{2}\right), \\
\vartheta_2(u|\tau) &= \vartheta_1 \left(u + \frac{1}{2}\right). 
\end{align*}
\[ \vartheta_3(u|\tau) = e^{\pi i (u + \tau/3)} \vartheta_1 \left( u + \frac{\tau + 1}{2} | \tau \right). \]

We also use the symbol \([u]\) defined by

\[ [u] = x^{\frac{u^2}{2}} u^2 e^{2x(u^2)} = C \vartheta_1 \left( \frac{u}{\tau} | \tau \right), \quad C = x^{-\frac{3}{4}} e^{-\frac{\pi i}{4} \frac{\tau}{4}}. \]

The elliptic shifted factorials are defined by

\[ [u]_n = \prod_{j=0}^{n-1} [u + j] \]

with the convention

\[ [u_1, u_2, \ldots, u_k]_n = \prod_{i=1}^{k} [u_i]_n. \]

## 2 Fusion and the Vertex-Face Correspondence

According to [2, 9], we here make a brief review of the fusion of the Boltzmann weight associated with the eight-vertex model and the eight-vertex solid-on-solid (SOS) model as well as their vertex-face correspondence relationship.

### 2.1 Fusion of Baxter’s elliptic \(R\)-matrix

The eight-vertex model is a two-dimensional square lattice model. For each vertex, we associate a dynamical variable (a spin) \(\varepsilon_j \in \{+, -\}\) with each edge \(j\) (Figure 1).

\[ R(u - v)^{\varepsilon_1 \varepsilon_2}_{\varepsilon'_1 \varepsilon'_2} = v^{\varepsilon'_2} \varepsilon_2 \]

\[ u^{\varepsilon'_1} \varepsilon_1 \]

Figure 1: The vertex model weight

Let \( V = \mathbb{C}v_{\varepsilon_1} \oplus \mathbb{C}v_{\varepsilon_2} \). We regard \( R(u) \in \text{End}(V \otimes V) \) by

\[ R(u)v_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum_{\varepsilon'_1, \varepsilon'_2} R(u)^{\varepsilon_1 \varepsilon_2}_{\varepsilon'_1 \varepsilon'_2} v_{\varepsilon'_1} \otimes v_{\varepsilon'_2}. \]

We have only the eight possible configurations, \( R_{++}, R_{+-}, R_{-+}, R_{--}, R_{++}, R_{+-}, R_{-+}, R_{--} \). To each of them, we assign the corresponding matrix element of Baxter’s elliptic
$R$-matrix as the Boltzmann weight\[12].

$R(u) = R_0(u) \begin{pmatrix} a(u) & b(u) & d(u) \\ b(u) & c(u) & d(u) \\ c(u) & b(u) & a(u) \end{pmatrix}, \quad (2.1)$

where

$$R_0(u) = z^{-\frac{r-1}{2r}} \frac{p \zeta^2 x^4; p}{x^4; p} R_0(u) = \frac{a(u)(1)}{a(u)(0)} \frac{b(u)(1)}{b(u)(0)} \frac{c(u)(1)}{b(u)(0)} \frac{d(u)(1)}{b(u)(0)}$$

with $z = \zeta^2 = x^{2a}$. Note that the $R$-matrix $R(u)$ is the two dimensional representation of the universal $R$-matrix of the vertex type elliptic quantum group $A_{q,p}$ [13]. The $R$-matrix (2.1) satisfies the Yang-Baxter equation (YBE), the unitarity relation, the crossing symmetry relation and the initial condition given as follows.

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v), \quad (2.2)$$

$$R(u)PR(u)P = \text{id}, \quad (2.3)$$

$$R(-u - 1) = (\sigma^y \otimes 1)^{-1} (PR(u)P)^t \sigma^y \otimes 1, \quad (2.4)$$

$$R(0) = P, \quad \lim_{u \to -1} R(u) = P - \text{id}. \quad (2.5)$$

Here $^t$ denotes the transposition with respect to the first vector space in the tensor product and $P(v_{x_1} \otimes v_{x_2}) = v_{x_2} \otimes v_{x_1}$.

Fusion of $R(u)$ was considered systematically in [2] (see also [14] for the $2 \times 2$ fusion case). Let $V_j, V_j$ be copies of $V$. Let us define the operator $\Pi_{1\ldots k}$ by

$$\Pi_{1\ldots k} = \frac{1}{k!} (P_{1k} + \cdots + P_{k-1k} + I) \cdots (P_{13} + P_{23} + I)(P_{12} + I).$$

This yields the projection on the space $V^{(k)}$ of the symmetric tensors in $V^{\otimes k}$. We define

$$R_{1\ldots k}^{(k,1)}(u) = \Pi_{1\ldots k} R_{1j}(u + k - 1) \cdots R_{k-1j}(u + 1) R_{kj}(u) \in \text{End}(V^{(k)} \otimes V_j).$$
The $R$-matrix $R_{1\cdots k,j}^{(k,1)}(u)$ then satisfies
\[ R_{1\cdots k,j}^{(k,1)}(u)\Pi_{1\cdots k} = R_{1\cdots k,j}^{(k,1)}(u). \]

By fusing $R_{1\cdots k,j}^{(k,1)}(u)$, $l$ times, we then define the $k \times l$ fusion $R$-matrix as follows.
\[ R^{(k,l)}(u) = \Pi_{1\cdots k} R_{1\cdots k,j}^{(k,1)}(u) \prod_{i=1}^{k} R^{(k,1)}(u-l+1). \quad (2.6) \]

This is an operator in $\text{End}(V^{(k)} \otimes V^{(l)})$. It satisfies
\[ R^{(k,l)}(u) = R^{(k,l)}(u)\Pi_{1\cdots k} = R^{(k,l)}(u)\Pi_{1\cdots i}. \quad (2.7) \]

Using the YBE (2.2) repeatedly and (2.7), we verify that $R^{(k,l)}(u)$ satisfies the YBE on $V^{(k)} \otimes V^{(l)} \otimes V^{(m)}$.
\[ R^{(k,l)}(u-v)R^{(k,m)}(u)R^{(l,m)}(v) = R^{(l,m)}(v)R^{(k,m)}(u)R^{(k,l)}(u-v). \]

### 2.2 Fusion of the eight-vertex SOS model

The eight-vertex SOS model is also a two-dimensional square lattice model. The dynamical variables $a_j$ are called local heights. They take values in $\mathbb{Z}$. For each face, we associate a local height $a_j$ with each vertex $j$ (Figure 2).

![Figure 2: The SOS model face weight](image)

We allow only the configurations satisfying the so-called the admissibility condition $|a_j - a_k| = 1$ for any two adjacent local heights $a_j$ and $a_k$. Then we have only the six possible configurations for each face. We assign the following Boltzmann weight (or face weight) $W \left( \begin{array}{c} a_1 & a_2 \\ a_4 & a_3 \end{array} \right) u$ to each configuration.

\[ W \left( \begin{array}{c} a_1 & a_2 \\ a_4 & a_3 \end{array} \right) u = R_0(u), \]
\[ W \left( \begin{array}{cc} a & a \pm 1 \\ a \pm 1 & a \end{array} \right) = R_0(u) \left[ \frac{a \mp u}{a[1 + u]} \right]. \tag{2.8} \]
\[ W \left( \begin{array}{cc} a & a \pm 1 \\ a \pm 1 & a \end{array} \right) = R_0(u) \left[ \frac{a \pm 1}{a[1 + u]} \right]. \]

Note that these weights can be understood as the matrix elements of the dynamical \( R \)-matrix \( R(u, a) \) obtained as the two dimensional representation of the universal dynamical \( R \)-matrix of the face type elliptic quantum group \( B_{q, \lambda}(\hat{sl}_2) \) [13, 16]. \( B_{q, \lambda}(\hat{sl}_2) \) is a central extension of Felder’s elliptic quantum group[17].

The face weights enjoy the following face-type YBE, unitarity, and crossing symmetry relations.

\[ \sum_g W \left( \begin{array}{cc} a & b \\ f & g \end{array} \right) W \left( \begin{array}{cc} f & g \\ e & d \end{array} \right) W \left( \begin{array}{cc} b & c \\ g & d \end{array} \right) = \sum_g W \left( \begin{array}{cc} a & g \\ f & e \end{array} \right) W \left( \begin{array}{cc} a & b \\ g & c \end{array} \right) W \left( \begin{array}{cc} g & c \\ e & d \end{array} \right), \tag{2.9} \]
\[ \sum_e W \left( \begin{array}{cc} a & b \\ e & c \end{array} \right) W \left( \begin{array}{cc} a & e \\ d & c \end{array} \right) W \left( \begin{array}{cc} b & c \\ e & d \end{array} \right) = 1, \tag{2.10} \]
\[ W \left( \begin{array}{cc} a & b \\ d & c \end{array} \right) = (-) \frac{a + d + b - c}{2} \frac{b}{a} W \left( \begin{array}{cc} d & a \\ c & b \end{array} \right) - u - 1. \tag{2.11} \]

In addition, the following initial conditions are the key for the fusion of the SOS weights.

\[ W \left( \begin{array}{cc} a & b \\ d & c \end{array} \right) |u = 0 \right) = \delta_{b,d}, \]
\[ W \left( \begin{array}{cc} a & b \\ d & c \end{array} \right) |u = -1 \right) = 0 \quad \text{if } |a - c| = 2, \]
\[ W \left( \begin{array}{cc} a & a \pm 1 \\ a \pm 1 & a \end{array} \right) |u = -1 \right) = -W \left( \begin{array}{cc} a & a \pm 1 \\ a \pm 1 & a \end{array} \right) |u = -1 \right). \]

The \( k \times l \) fusion of the face weight is obtained following the two steps[2]. Firstly we define
\[ W^{(k,1)} \left( \begin{array}{cc} a & b \\ d & c \end{array} \right) |u \right). \]
Then the RHS is independent of the choice of $a_1, .., a_{k-1}$ provided $|a-a_1| = |a_1-a_2| = \cdots = |a_{k-1}-b| = 1$. Secondly we define

$$
W^{(k,l)} \left( \begin{array}{ccc} a & b & u \\ d & c & l \end{array} \right)
= \sum_{a_1,..,a_{l-1}} W^{(k,1)} \left( \begin{array}{ccc} a & b & u-l+1 \\ a_1 & b_1 & u \end{array} \right) W^{(k,1)} \left( \begin{array}{ccc} a & b & u-l+2 \\ a_1 & b_1 & u \end{array} \right) \cdots W^{(k,1)} \left( \begin{array}{ccc} a_{l-1} & b_{l-1} & u \\ d & c & l \end{array} \right).
$$

(2.12)

Then the RHS is independent of the choice of $b_1, .., b_{l-1}$ provided $|b-b_1| = |b_1-b_2| = \cdots = |b_{l-1}-c| = 1$. In $W^{(k,l)}$, the admissible condition for the dynamical variables is extended to $a-b \in \{-k, -k+2, .., k\}$ for any two horizontally adjacent local heights $a,b$, while $a-d \in \{-l, -l+2, .., l\}$ for any two vertically adjacent local heights $a,d$.

The $k \times l$ fusion face weight $W^{(k,l)}$ satisfies the face type YBE.

$$
\sum_g W^{(k,l)} \left( \begin{array}{ccc} a & b & u \\ f & g & l \end{array} \right) W^{(k,m)} \left( \begin{array}{ccc} f & g & u+v \\ e & d & v \end{array} \right) W^{(m,l)} \left( \begin{array}{ccc} b & c & u-v \\ e & d & l \end{array} \right)
= \sum_g W^{(m,l)} \left( \begin{array}{ccc} a & g & u-v \\ f & e & l \end{array} \right) W^{(k,m)} \left( \begin{array}{ccc} a & b & v \\ g & c & u \end{array} \right) W^{(k,l)} \left( \begin{array}{ccc} g & c & u \\ e & d & l \end{array} \right).
$$

(2.13)

### 2.3 The vertex-face correspondence

The vertex-face correspondence is a relationship between Baxter’s $R$-matrix and the SOS face weight $W \left( \begin{array}{ccc} a_1 & a_2 & u \\ a_4 & a_3 \end{array} \right)$. Let us consider the following vectors in $V$

$$
\psi(u)^a_b = \psi_+(u)^a_b v_+ + \psi_-(u)^a_b v_-, \\
\psi_+(u)^a_b = \phi_0 \left( \frac{(a-b)u+a}{2r} \frac{\tau}{2} \right), \quad \psi_-(u)^a_b = \phi_3 \left( \frac{(a-b)u+a}{2r} \frac{\tau}{2} \right)
$$

(2.14)

with $|a-b| = 1$. Baxter showed that the following identity holds[12].

$$
\sum_{\varepsilon_1,\varepsilon_2} R(u-v)^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1 \varepsilon_2} \psi_{\varepsilon_1}(u)^a_b \psi_{\varepsilon_2}(v)^b_c = \sum_{\varepsilon \in Z} \psi_{\varepsilon}(v)^b_c \psi_{\varepsilon}(u)^a_b W \left( \begin{array}{ccc} a & b & u-v \\ b' & c \end{array} \right).
$$

(2.15)
We call $\psi(u)^a_b$ the (vertex-face) intertwining vectors.

$$\psi_\varepsilon(u)^a_b = \begin{array}{c} a \\ u \\ b \end{array} \quad \psi_\varepsilon^*(u)^b_a = \begin{array}{c} u \\ a \\ b \end{array}$$

(a) \hspace{2cm} (b)

Figure 3: (a) The intertwining vector; (b) the dual intertwining vector

In addition, applying the crossing symmetry properties of $R$ (2.4) and $W$ (2.11) to (2.15), we obtain the following relation.

$$\sum_{\varepsilon_1,\varepsilon_2} R(u - v)^{\varepsilon_1\varepsilon_2}_{\varepsilon_1'\varepsilon_2'} \psi_\varepsilon^*(u)^a_b \psi_\varepsilon^*(v)^b_c = \sum_{\varepsilon \in \mathbb{Z}} \psi_\varepsilon^*(v)^b_c \psi_\varepsilon^*(u)^a_b W\left(\begin{array}{c} c \\ b \\ a \end{array} | u - v \right),$$

(2.16)

where we set

$$\psi_\varepsilon^*(u)^a_b = -\frac{a - b}{2|b||u|} C^2 \psi_{-\varepsilon}(u - 1)^a_b$$

(2.17)

with $|a - b| = 1$. Defining $\psi^*(u)^a_b \in V^*$ by

$$\psi^*(u)^a_b\ v_\varepsilon = \psi_\varepsilon^*(u)^a_b, \quad v_\varepsilon \in V,$$

we call $\psi^*(u)^a_b$ the dual intertwining vectors. In fact, by a direct calculation, one can verify that the following inversion relations hold.

$$\sum_{\varepsilon = \pm} \psi_\varepsilon^*(u)^a_b \psi_\varepsilon(u)^b_a = \delta_{a,c},$$

(2.18)

$$\sum_{a = b \pm 1} \psi_\varepsilon^*(u)^a_b \psi_\varepsilon(u)^b_a = \delta_{\varepsilon',\varepsilon}.$$
Now let us consider the fusion of the vertex-face relationships (2.15), (2.16). We define the $k$ fusion of the intertwining vectors by\[2\]

$$\psi^{(k)}(u)_b^a = \Pi_{1\ldots k} \psi(u + k - 1)_c^a \otimes \psi(u + k - 2)_c^b \otimes \cdots \otimes \psi(u)_b^{ck-1}. \quad (2.20)$$

Here the RHS is independent of the choice of $c_1, \ldots, c_{k-1}$ provided $|a - c_1| = |c_1 - c_2| = \cdots = |c_{k-1} - b| = 1$. The local heights $a$ and $b$ now satisfy the extended admissible condition $a - b \in \{-k, -k+2, \ldots, k\}$. For $k > 1$, the basis $\{v^{(k)}_{\mu}\}_{\mu = -k,-k+2,\ldots,k}$ of $V^{(k)}$ is given by a fusion of the basis vectors $v_{\varepsilon_i} \ (\varepsilon_i = \pm)$ of $V$, $k$ times.

$$v^{(k)}_{\mu} = \Pi_{1\ldots k} v_{\varepsilon_1} \otimes v_{\varepsilon_2} \otimes \cdots \otimes v_{\varepsilon_k} = \frac{1}{k!} \sum_{\sigma \in S_k} v_{\varepsilon_{\sigma(1)}} \otimes v_{\varepsilon_{\sigma(2)}} \otimes \cdots \otimes v_{\varepsilon_{\sigma(k)}}, \quad (2.21)$$

where $S_k$ being the symmetric group and we set $\mu = \sum_{j=1}^k \varepsilon_j$. Substituting (2.14) to (2.20), we obtain

\[
\psi^{(k)}(u)_b^a = \sum_{\mu \in \{-k,-k+2,\ldots,k\}} v^{(k)}_{\mu} \psi^{(k)}_{\mu}(u)_b^a,
\]

\[
\psi^{(k)}_{\mu}(u)_b^a = \sum_{\varepsilon_1,\ldots,\varepsilon_k = \pm, \mu = \sum_{j=1}^k \varepsilon_j} v^{(k)}_{\varepsilon_1(u + k - 1)}^a v^{(k)}_{\varepsilon_2(u + k - 2)}^c \cdots v^{(k)}_{\varepsilon_k(u)}^b c_{k-1}. \quad (2.22)
\]

From (2.15), (2.6), and (2.12), it follows that the fused intertwining vectors satisfy the $k \times l$ fusion vertex-face correspondence relations with respect to $R^{(k,l)}$ and $W^{(k,l)}$.

\[
\sum_{\mu_1,\mu_2} R^{(k,l)}(u - v)_{\mu_1,\mu_2}^a v^{(k)}_{\mu_1}(u)_b^a v^{(l)}_{\mu_2}(v)_c^b = \sum_{\nu \in \mathbb{Z}} v^{(k)}_{\nu^a} v^{(l)}_{\nu^b} W^{(k,l)}(\nu^a \nu^b) u^c^{(k,l)} \psi^{(k)}_{\mu}(u)_c^b \psi^{(l)}_{\nu}(u)_b^c (u - v).
\]

(2.23)

Similarly, the dual intertwining vectors can be fused $k$ times in the following way\[9\].

$$\psi^{*(k)}(u)_b^a = \sum_{c_1,\ldots,c_{k-1}} \psi^{*(u + k - 1)}_a^c \otimes \psi^{*(u + k - 2)}_c^b \otimes \cdots \otimes \psi^{*(u)}_b^{ck-1} \quad (2.24)$$

with the property

$$\Pi_{1\ldots k} \psi^{*(k)}(u)_b^a = \psi^{*(k)}(u)_b^a \Pi_{1\ldots k}. \quad (2.24)$$
Written out in component form, the last relation indicates that the RHS of
\[
\psi^*_{\mu}(u)_a^b = \sum_{c_1, \ldots, c_{k-1}} \psi^*_{\epsilon_1}(u + k - 1)^c_1 \psi^*_{\epsilon_2}(u + k - 2)^c_2 \cdots \psi^*_{\epsilon_k}(u)_{c_k-1}^b
\]
is independent of the choice of \(\epsilon_1, \ldots, \epsilon_k\) provided \(\mu = \epsilon_1 + \cdots + \epsilon_k\). As above, it follows immediately from (2.16), (2.6) and (2.12), that we have
\[
\sum_{\mu_1, \mu_2} R^{(k,l)}_{\mu_1 \mu_2}(u - v)_{\mu_1 \mu_2} \psi^*_{\epsilon_1}(u)^{a}_{\mu_1} \psi^*_{\epsilon_2}(v)^{b}_{\mu_2} = \sum_{b' \in \mathbb{Z}} \psi^*_{\epsilon_1}(u)^{a}_{\mu_1} \psi^*_{\epsilon_2}(v)^{b'}_{\mu_1} W^{(k,l)}_{\mu_2} \begin{pmatrix} c & b' \\ b & a \end{pmatrix} (u - v).
\]
(2.25)

Finally, using (2.18) and (2.19), it is easy to verify the following inversion relations.
\[
\sum_{\mu \in \{-k, -k+2, \ldots, k\}} \psi^*_{\mu}(u)^{a}_{\mu} \psi^*_{\mu}(u)^{b}_{\mu} = \delta_{a, c}, \tag{2.26}
\]
\[
\sum_{a \in \{b - k, b - k + 2, \ldots, b + k\}} \psi^*_{\mu}(u)^{a}_{\mu} \psi^*_{\mu}(u)^{b}_{\mu} = \delta_{\mu', \mu}. \tag{2.27}
\]

3 The Elliptic 6\(j\)-symbols

3.1 The natural basis

Let \(\Theta_k\) be the space of even theta functions of order 2\(k\) with quasi-period (1, \(\tau\)) and zero characteristics.
\[
\Theta_k = \left\{ f(z) : \text{entire} \mid f(z + 1) = f(z), f(z + \tau) = e^{-2\pi ik(2z+\tau)} f(z), f(-z) = f(z) \right\}.
\]
This space forms a \(k + 1\) dimensional vector space.

Let us consider the vectors of \(\Theta_k\) given by
\[
e^k_n(z; \alpha, \beta) = [\alpha \pm rz]_n[\beta \pm rz]_{k-n}.
\]
(3.1)

Through this section, we use the abbreviation
\[
[u \pm z] = [u + z][u - z],
\]
\[
\vartheta_{\alpha}(u \pm z | \tau) = \vartheta_{\alpha}(u + z | \tau) \vartheta_{\alpha}(u - z | \tau),
\]
\[ \alpha = 0, 1, 2, 3. \] The vectors \( e^k_n(z; \alpha, \beta) \) \((n = 0, 1, \ldots, k)\) are linearly independent, if \( \alpha, \beta \) satisfy the following conditions[5].

\[
\frac{\alpha - \beta + j}{r} \notin \mathbb{Z} + r\mathbb{Z}, \ j = 1 - k, 2 - k, \ldots, k - 1,
\]

\[
\frac{\alpha + \beta + j}{r} \notin \mathbb{Z} + r\mathbb{Z}, \ j = 0, 1, \ldots, k - 1.
\]

Then a system of vectors \( \{e^k_n(z; \alpha, \beta)\}_{n=0}^k \) forms a basis of \( \Theta_k \), and is called the natural basis.

**Theorem 3.1 (Elliptic binomial theorem)[5]**

For generic parameters \( \alpha, \beta, \gamma \), we have

\[
[\alpha \pm rz]_k = \sum_{n=0}^{k} C^k_n(\alpha, \beta, \gamma)[\beta \pm rz]_n[\gamma \pm rz]_{k-n} \tag{3.2}
\]

with the coefficients

\[
C^k_n(\alpha, \beta, \gamma) = \frac{[1]_k}{[1]_{k-n}^n} \frac{[\alpha - \gamma, \alpha + \gamma + k - n]_n[\alpha - \beta, \alpha + \beta + n]_{k-n}}{[\beta - \gamma + n - k]_n[\gamma - \beta - n]_{k-n}[\beta + \gamma]_k}. \tag{3.3}
\]

We give a proof in the Appendix.

Rosengren showed that the change of base coefficients \( R^m_n(\alpha, \beta, \gamma, \delta; k; q, p) \) in

\[
e^k_n(z; \alpha, \beta) = \sum_{m=0}^{k} R^m_n(\alpha, \beta, \gamma, \delta; k; q, p)e^m_k(z; \gamma, \delta) \tag{3.4}
\]

can be regarded as a generalisation of the elliptic 6\( j \)-symbols. Furthermore he found an expression of the coefficients \( R^m_n(\alpha, \beta, \gamma, \delta; k; q, p) \) in terms of the elliptic analogue of the very-well-poised balanced basic hypergeometric series, \( 12V_{11} \).

**Theorem 3.2[5]**

\[
R^m_n(\alpha, \beta, \gamma, \delta; k; q, p) = \frac{[1]_k}{[1]_{k-m}^m} \frac{[\beta - \delta, \beta + \delta - 1 + k]_n[\alpha - \gamma, \alpha + \gamma + n]_n[\beta - \gamma, \beta + \gamma]_{k-n}[\beta - \gamma]_{k-m}}{[\gamma - \delta + m - k, \beta + \gamma]_m[\delta - \gamma - m]_{k-m}[\delta + \gamma, \beta - \gamma]_k}
\]

\[
\times_{12}V_{11}(\gamma - \beta - k; -n, -m, \alpha - \beta + n - k, \gamma - \delta + m - k, \gamma + \delta, \alpha - \beta + 1 - k; \gamma - \beta + 1). \tag{3.5}
\]

Here \( s+1V_s \) is defined by[4, 11]

\[
s+1V_s(u_0; u_1, \ldots, u_{s-4}) = \sum_{j=0}^{\infty} \frac{[u_0 + 2j]}{[u_0]} \prod_{i=0}^{s-4} \frac{[u_i]_j}{[u_0 + 1 - u_i]_j}
\]
with the balancing condition

\[ \sum_{i=1}^{s-4} u_i = \frac{s-7}{2} + \frac{s-5}{2} u_0. \]

A proof of the Theorem is given in the Appendix.

### 3.2 Relation with the intertwining vectors

In order to make a connection between \( R_m^o(\alpha, \beta, \gamma, \delta; k; q, p) \) and the vertex-face intertwining vectors \( \psi^{(k)}(u)_b^a \) and their duals \( \psi^{*(k)}(u)_b^a \), let us consider the standard basis of \( \Theta_k \) introduced by Sklyanin\[7\]. For \( k = 1 \), the following two vectors form a basis of \( \Theta_1 \).

\[
\begin{align*}
v_+(z) &= \vartheta_3(2z|2\tau) - \vartheta_2(2z|2\tau), \\
v_-(z) &= \vartheta_3(2z|2\tau) + \vartheta_2(2z|2\tau).
\end{align*}
\]

For \( k > 1 \), we obtain the basis \( \{v^{(k)}(z)\}_{\mu=-k,-k+2,..,k} \) of \( \Theta_k \) by fusing the basis vectors \( v_\varepsilon(z) (\varepsilon = \pm) \) of \( \Theta_1 \).

\[
v^{(k)}(z) = \Pi_{1,2,..,k} v_{\varepsilon_1}(z) \otimes v_{\varepsilon_2}(z) \otimes \cdots \otimes v_{\varepsilon_k}(z) \]

\[
= \frac{1}{k!} \sum_{\sigma \in S_k} v_{\varepsilon_{\sigma(1)}}(z) v_{\varepsilon_{\sigma(2)}}(z) \cdots v_{\varepsilon_{\sigma(k)}}(z),
\]

with \( \mu = \sum_{j=1}^{k} \varepsilon_j \). Now let us consider the intertwining vectors in the standard basis. We set

\[
\psi(u; z)_b^a = \sum_{\varepsilon = \pm} v_\varepsilon(z) \psi_\varepsilon(u)_b^a.
\]

Then from (2.14) and (3.6), standard addition formulae of the theta functions yield the following formula.

\[
\psi(u; z)_b^a = \vartheta_3 \left( z \pm \frac{(a-b)u + a}{2\tau} \right).
\]

Now we consider a fusion of the intertwining vectors \( \psi(u; z)_b^a \) by applying the same procedure as before (2.20). We set

\[
\psi^{(k)}(u; z)_b^a = \Pi_{1,2,..,k} \psi(u + k - 1; z)_{c_1}^a \otimes \psi(u + k - 2; z)_{c_2}^a \otimes \cdots \otimes \psi(u; z)_{c_k}^a.
\]

Remember that \( \psi^{(k)}(u)_b^a \) is independent of the choice of \( c_1,..,c_{k-1} \) and \( a, b \) satisfy the admissible condition \( a - b \in \{-k,-k+2,..,k\} \). Let us set \( a - b = k - 2n \ (n = \ldots) \)
0, 1, 2, ..., k). We evaluate $\psi^{(k)}(u; z)_b^a$ in the two ways. Firstly from (3.8) and (3.7), we obtain

$$\psi^{(k)}(u; z)_b^a = \sum_{\mu \in \{-k, -k+2, \ldots, k\}} v^{(k)}_\mu(z) \psi^{(k)}(u)_b^a,$$  \hspace{1cm} (3.11)

where $v^{(k)}_\mu(u)_b^a$ is given in (2.22). Secondly, from (3.9) and a choice $c_{j+1} = c_j + 1, c_0 = a$ for $j = 0, 1, 2, ..., n$ and $c_{j+1} = c_j - 1, c_k = b$ for $j = n, n+1, ..., k - 1$, we obtain[8]

$$\psi^{(k)}(u; z)_b^a = \prod_{j=0}^{n-1} \vartheta_3 \left( \frac{-u + a - k + 1 + 2j}{2r} \pm z \right) \prod_{j=0}^{k-n-1} \vartheta_3 \left( \frac{u + b + 2j + 1}{2r} \pm z \right) \times \prod_{j=0}^{k-n-1} \vartheta_1 \left( \frac{-u + a - k + 1 + 2j}{2r} \pm \left( z + \frac{\tau + 1}{2} \right) \right).$$  \hspace{1cm} (3.12)

Comparing this with the expression (3.1), we obtain the identification

$$\psi^{(k)}(u; z)_b^a = (-)^k e^{-\pi i k(\frac{\tau}{2} + 2 z)} C^{-k} e_n^k \left( z + \frac{\tau + 1}{2}; \frac{u + a - k + 1}{2}, \frac{-u - a - k + 1}{2} \right),$$  \hspace{1cm} (3.13)

where $C$ is a constant given in §1.1.

The formulae (3.11) and (3.13) indicate that the components $v^{(k)}_\mu(u)_b^a$ of the vertex-face intertwining vector play the role of the change of base matrix elements from $\{v^{(k)}_\mu(z)\}$ to $\{e_n^k(z; \frac{u + a - k + 1}{2}, \frac{-u - a - k + 1}{2})\}$ in $\Theta_k$. This role is similar to the one of the generalized group elements (= Babelon’s vertex-IRF transformations[18] ) in the theory of $q$-$6j$ symbols á la Rosengren[19]. ( See also §2.3 and §3 in [5].)

Remark 1. Using $a = b + k - 2n$ in the first factors of the first line in (3.12), we have

$$\psi^{(k)}(u; z)_b^a = (-)^k e^{-\pi i k(\frac{\tau}{2} + 2 z)} \prod_{j=0}^{n-1} \vartheta_1 \left( \frac{u - b + 2j + 1}{2r} \pm \left( z + \frac{\tau + 1}{2} \right) \right) \times \prod_{j=0}^{k-n-1} \vartheta_1 \left( \frac{u + b + 1 + 2j}{2r} \pm \left( z + \frac{\tau + 1}{2} \right) \right).$$

We hence obtain a second expression

$$\psi^{(k)}(u; z)_b^a = (-)^k e^{-\pi i k(\frac{\tau}{2} + 2 z)} C^{-k} e_n^k \left( z + \frac{\tau + 1}{2}; \frac{u - b + 1}{2}, \frac{u + b + 1}{2} \right).$$
Now let us apply the formula (3.13) and (3.11) to (3.2). In the LHS, we apply (3.13) with $n = k$, whereas in the RHS, we regard $\beta = \frac{-u-b-k+1}{2}$, $\gamma = \frac{-u-b-k+1}{2}$ and apply (3.13) and (3.11) with the replacement $a \to b$, $b \to c$. Then we get

$$\psi^{(k)}_{\mu}(u)^{a}_{a+k} = \sum_{n=0}^{k} C_{n}^{k}(\alpha, \beta, \gamma) \psi^{(k)}_{\mu}(u)^{b}_{c}.$$ 

Using the inversion relation (2.26), we obtain the following.

**Theorem 3.3**

$$C_{n}^{k}(\alpha, \beta, \gamma) = \sum_{\mu \in \{-k, -k+2, \ldots, k\}} \psi^{*}_{\mu}(u)^{c}_{b} \psi^{(k)}_{\mu}(u)^{a}_{a+k},$$

for $b - c = k - 2n$ and $\alpha = \frac{-u+a-k+1}{2}$, $\beta = \frac{-u+b-k+1}{2}$, $\gamma = \frac{-u-b-k+1}{2}$.

Similarly, substituting (3.13) into (3.4) and using (3.11), we obtain

$$\psi^{(k)}_{\mu}(u)^{a}_{b} = \sum_{m=0}^{k} R_{m}^{n}(\alpha, \beta, \gamma, \delta; k; q, p) \psi^{(k)}_{\mu}(u)^{c}_{d}, \quad (3.14)$$

for $a - b = k - 2n$, $c - d = k - 2m$ and $\alpha = \frac{-u+a-k+1}{2}$, $\beta = \frac{-u-a-k+1}{2}$, $\gamma = \frac{-u+c-k+1}{2}$, $\delta = \frac{-u-c-k+1}{2}$. Then from the inversion relation (2.26), we obtain the following formula for $R_{m}^{n}$.

**Theorem 3.4**

$$R_{m}^{n}(\alpha, \beta, \gamma, \delta; k; q, p) = \sum_{\mu \in \{-k, -k+2, \ldots, k\}} \psi^{*}_{\mu}(u)^{d}_{c} \psi^{(k)}_{\mu}(u)^{a}_{b}. \quad (3.15)$$

Remark 2. This formula should be compared with the scalar product expression for $R_{m}^{n}$ derived by Rosengren ((11.2) in [6]), where the scalar product is defined by Sklyanin’s invariant metric on $\Theta_{k}$. For this purpose, we need to study a scalar product formulae for the standard basis $\psi^{(k)}_{\mu}(z)$. This is an open problem.

The expression appeared in the RHS of (3.15) is nothing but a matrix $L^{(k)}$ introduced by Lashkevich and Pugai[15] for $k = 1$ and extended to higher $k$ by Kojima, Weston and the present author [9]. Namely,

$$L^{(k)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} u = \sum_{\mu \in \{-k, -k+2, \ldots, k\}} \psi^{*}_{\mu}(u)^{d}_{c} \psi^{(k)}_{\mu}(u)^{a}_{b} = R_{m}^{n}(\alpha, \beta, \gamma, \delta; k; q, p). \quad (3.16)$$
Combining (3.16) and (3.5), we obtain a full expression of $L^{(k)} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) u$ for arbitrary $k \in \mathbb{Z}_{>0}$.

**Corollary 3.5** For $a - b = k - 2n, c - d = k - 2m$,

$$L^{(k)} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left[1\right]_{k} \left[ -\frac{a + c}{2}, -u - \frac{a + c}{2} \right]_{m} \left[ -\frac{a + c}{2}, -u + \frac{a + c}{2} - k + 1 \right]_{n} \left[ -\frac{a + c}{2}, -u - \frac{a - c}{2} - k + 1 \right]_{k - n}$$

$$\times \sum_{j=0}^{\min(n, m)} \left[ \frac{a + c - k}{2} - k + 2j \right] \left[ \frac{a + c - k}{2} - k, -n, -m, a + n - k \right]_{j} \times \left[ 1, \frac{a + c}{2} + 1 + n - k, \frac{a + c}{2} + 1 + m - k, -\frac{a - c}{2} + 1 - n \right]_{j}$$

$$\times \left[ \frac{c + m - k, -u - k + 1, u + k - 1, \frac{a + c}{2} + 1}{} \right]_{j} \times \left[ \frac{c + m - k, -u - k + 1, u + k - 1, \frac{a + c}{2} + 1}{} \right]_{j}$$

In [9], some of the $L^{(k)}$-matrix elements are calculated by fusion. They agree with this formula.

**Remark 3.** The $L^{(k)}$-matrix plays an important role in the calculation of correlation functions for the fusion eight-vertex models by using the vertex-face correspondence. Especially, in the limit lattice size going to infinity, a semi-infinite product of $L^{(k)}$ gives rise to a non-trivial operator called the tail operator acting on the space of states of the corner transfer matrices. Roughly speaking, the tail operator fills a gap between the spaces of states of the fusion eight-vertex and the eight-vertex SOS models. In addition, the $L^{(k)}$-matrix itself is used to write down a commutation relation between the lattice vertex operators and the tail operator. Based on this, the tail operator is identified with a certain power of the half-current of the elliptic algebra $U_{q,p}(\hat{sl}_2)$ [15, 9], the algebra of the Drinfeld currents of the face type elliptic quantum group $B_{q,\lambda}(\hat{sl}_2)$ [16, 20]. Such consideration allows us to formulate the lattice models by using the representation theory of $U_{q,p}(\hat{sl}_2)$ [9].

Now using the formula (3.15), we can derive some properties of the elliptic 6j-symbols $R_{m}^{n}(\alpha, \beta, \gamma, \delta; k; q, p)$. First of all, the inversion relation (2.27) imply the addition formula of $R_{m}^{n}$.

**Proposition 3.6**

$$R_{n}^{m}(\alpha, \beta, \gamma, \delta; k; q, p) = \sum_{l=0}^{k} R_{l}^{n}(\alpha, \beta, \gamma, \delta; k; q, p) R_{l}^{m}(\alpha, \beta, \gamma, \delta; k; q, p).$$ (3.17)
Secondly, the two inversion relations (2.26) and (2.27) imply the biorthogonality property of $R_m^m$ which is equivalent to the biorthogonality condition derived first in [3].

**Proposition 3.7**

$$\sum_{m=0}^{k} R_m^m(\alpha, \beta, \gamma, \delta; k; q, p) R_m^m(\gamma, \delta, \alpha, \beta; k; q, p) = \delta_{n,l}. \quad (3.18)$$

Thirdly, from the fusion formulae for $\psi(u)^g_a$ and $\psi(u)^a_b$, we can verify the following fusion formula (combinatorial formula [5]) for $R_m^m$.

**Proposition 3.8**

$$R_m^m(\alpha, \beta, \gamma, \delta; k; q, p) = \sum_{0 \leq m_j \leq 1} \sum_{\sum_{j=1}^{k} m_j = m} R_{m_1}^{m_1}(\alpha_1, \gamma_1; 1; q, p) R_{m_2}^{m_2}(\alpha_2, \gamma_2; 1; q, p) \cdots R_{m_k}^{m_k}(\alpha_{k-1}, \gamma_{k-1}; 1; q, p), \quad (3.19)$$

where $n = \sum_{j=1}^{k} n_j \ (0 \leq n_j \leq 1)$.

**proof.** Substituting (2.20), (2.24) to (3.15), the formula follows from, for example, the identification $\alpha = \frac{-u+a-k+1}{2}, \alpha_j = \frac{-u+a_j-1-k+j}{2} \ (j = 1, 2, \ldots, k-1), \beta = \frac{-u-b}{2}, \gamma = \frac{-u+c-k+1}{2}, \gamma_j = \frac{-u+c_j-1-k+j}{2} \ (j = 1, 2, \ldots, k-1), \delta = \frac{-u-c_k}{2}$ for $a_{j-1} - a_j = 1 - 2n_j, c_{j-1} - c_j = 1 - 2m_j \ (j = 1, 2, \ldots, k)$ with $a_0 = a, a_k = b, c_0 = c, c_k = d$. $\square$

Finally, using the two types of the vertex-face correspondence relation (2.23), (2.25) and the inversion relations (2.26) and (2.27), we obtain the Yang-Baxter relation for $R_m^m$. (Figure 5)

**Theorem 3.9**

$$\sum_d W^{(k,l)} \left( \begin{array}{ccc} a & b & u-v \\ d & c & \end{array} \right) L^{(k)} \left( \begin{array}{ccc} d & c & u \\ f & e & \end{array} \right) L^{(l)} \left( \begin{array}{ccc} a & d & v \\ g & f & \end{array} \right) = \sum_d L^{(k)} \left( \begin{array}{ccc} a & b & u \\ g & d & \end{array} \right) L^{(l)} \left( \begin{array}{ccc} b & c & v \\ d & e & \end{array} \right) W^{(k,l)} \left( \begin{array}{ccc} g & d & u-v \\ f & e & \end{array} \right). \quad (3.20)$$

This formula is not obvious at all from either (3.4) or (3.5).
3.3 Connection with the Sklyanin algebra

Using the intertwining vectors and their duals, we can also construct a realisation of the Sklyanin algebra[21, 22, 23]. Let us define

$$\mathcal{L}^{(k)}(u)_a^b = \psi^{(k)}(u)_a^b \psi^{(k)}(u)_a^b \in \text{End} V^{(k)}$$

with $a - b \in \{-k, -k + 2, \ldots, k\}$. Then using (2.23), (2.25) and (2.26), we have

$$\sum_b \tilde{R}^{(k,l)}(u - v)\mathcal{L}^{(k)}(u)_a^b \otimes \mathcal{L}^{(l)}(v)_b^c = \sum_b \mathcal{L}^{(l)}(v)_a^b \otimes \mathcal{L}^{(k)}(u)_b^c \tilde{R}^{(k,l)}(u - v).$$

Here $\tilde{R}^{(k,l)}(u) = P^{(k)} R^{(k,l)}(u)$ with $P^{(k)}$ being the permutation operator $P^{(k)} u \otimes v = v \otimes u$, $u, v \in V^{(k)}$. Let $\mathcal{W}$ be a certain vector space with a basis $\{w_a\}_{a \in \mathbb{Z}}$. The matrix $\mathcal{L}^{(k)}$ can be regarded as a linear operator on $V^{(k)} \otimes \mathcal{W}$ by

$$\mathcal{L}^{(k)}(u)v^{(k)}_\mu \otimes w_b = \sum_{\mu, a} \mathcal{L}^{(k)}(u)_a^b v^{(k)}_\mu \otimes w_a.$$ 

One possible interpretation of the vector space $\mathcal{W}$ is given in the second paper in [22] as the space of theta functions spanned by the characters of the integrable representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ with a fixed level.

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A Proofs of Theorem 3.1 and 3.2

We here follows the idea of [5].

A.1 Proof of Theorem 3.1

We prove by an induction on \( k \). Let us first split the factor \([\alpha + k \pm rz]\) into two parts.

\[
[\alpha + k \pm rz] = A_n[\beta + n \pm rz] + B_n[\gamma + k - n \pm rz]. \tag{A.1}
\]

From the theta function identity

\[
[x + y][x - y][u + v][u - v] = [u + x][u - x][v + y][v - y] - [u + y][u - y][v + x][v - x]
\]

we find

\[
A_n = \frac{[\alpha + \gamma + k - n][\gamma - \alpha - n]}{[\gamma + \beta + k][\gamma - \beta + k - 2n]},
\]

\[
B_n = -\frac{[\beta + \alpha + k + n][\beta - \alpha + n - k]}{[\gamma + \beta + k][\gamma - \beta + k - 2n]}.
\]

Then substituting (A.1) into

\[
[\alpha \pm rz]_{k+1} = [\alpha \pm rz]_k[\alpha + k \pm rz][\gamma \pm rz]_{k-n+1}
\]

and using (3.2), one obtains

\[
[\alpha \pm rz]_{k+1} = \sum_{n=0}^{k} A_n C^k_n(\alpha, \beta, \gamma)[\beta \pm rz]_{n+1}[\gamma \pm rz]_{k-n}
\]

\[
+ \sum_{n=0}^{k} B_n C^k_n(\alpha, \beta, \gamma)[\beta \pm rz]_n[\gamma \pm rz]_{k-n+1}.
\]

This yields the following recursion relation

\[
C^{k+1}_n(\alpha, \beta, \gamma) = A_{n-1}C^k_{n-1}(\alpha, \beta, \gamma) + B_nC^k_n(\alpha, \beta, \gamma).
\]
Under the boundary conditions

\[ C_0^0 = 1, \quad C_{-1}^k = C_{k+1}^k = 0, \]

we find that \( C_n^k(\alpha, \beta, \gamma) \) in (3.3) solves the recursion relation. \( \square \)

### A.2 Elliptic Jackson’s summation formula

**Theorem A.1**

\[
10V_9(\beta - \gamma - k; -k, \alpha - \gamma, -\alpha - \gamma + 1 - k, \beta + rz, \beta - rz) = \frac{[\gamma - \beta, \gamma + \beta, \alpha + rz, \alpha - rz]_k}{[\alpha - \beta, \alpha + \beta, \gamma + rz, \gamma - rz]_k}
\]

**Proof.** Substitute \( C_n^k(\alpha, \beta, \gamma) \) in (3.3) into (3.2) and compare the RHS to the definition of \( 10V_9 \), we obtain the desired formula. \( \square \)

### A.3 Proof of Theorem 3.2

The equation (3.4) is equivalent to

\[
[\alpha \pm rz]_n[\beta \pm rz]_{k-n} = \sum_{m=0}^{k} R_m^m(\alpha, \beta, \gamma, \delta; k; q, p)[\gamma \pm rz]_m[\delta \pm rz]_{k-m}
\]

Substituting (3.2) to the LHS, we have

\[
[\alpha \pm rz]_n[\beta \pm rz]_{k-n} = \sum_{j=0}^{n} C_j^k(\alpha, \gamma, \beta')[\gamma \pm rz]_j[\beta' \pm rz]_{n-j}[\beta \pm rz]_{k-n}.
\]

Setting \( \beta' = \beta + k - n \) and using the formula

\[
[\beta \pm rz]_{l+m} = [\beta \pm rz]_l[\beta + l \pm rz]_m,
\]

we have

\[
[\alpha \pm rz]_n[\beta \pm rz]_{k-n} = \sum_{j=0}^{n} C_j^n(\alpha, \gamma, \beta + k - n)[\gamma \pm rz]_j[\beta \pm rz]_{k-j}.
\]

Similarly, for \( [\beta \pm rz]_{k-j} \) we have

\[
[\beta \pm rz]_{k-j} = \sum_{l=0}^{k-j} C_l^{k-j}(\beta, \delta', \delta)[\delta' \pm rz]_l[\delta \pm rz]_{k-j-l}.
\]
Then setting $\delta' = \gamma + j$, we have

$$[\alpha \pm rz]_{n}[\beta \pm rz]_{k-n} = \sum_{j=0}^{n} \sum_{l=0}^{k-j} C_{n}^{j}(\alpha, \gamma, \beta + k - n)C_{l}^{k-j}(\beta, \gamma + j, \delta)[\gamma \pm rz]_{j+l}[\delta \pm rz]_{k-j-l}.$$  

Comparing this with (3.4), we obtain

$$R_{m}^{n}(\alpha, \beta, \gamma, \delta; k; q, p) = \sum_{j=0}^{\min(n,m)} C_{j}^{n}(\alpha, \gamma, \beta + k - n)C_{m-j}^{k-j}(\beta, \gamma + j, \delta).$$

Substituting (3.3) into this, we obtain the formula (3.5). 

□

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