GENERALIZED CAUCHY PROBLEMS FOR SPECIAL CONVOLUTIONARY DERIVATIVES

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ABSTRACT. In this paper, we deduce a local existence and uniqueness result for abstract Cauchy problems with a non-local convolutionary derivative induced by any driftless special Bernstein function with regular variation at infinity. To address the problem, we first show some regularity results for the potential measure in such case and then provide to show the result via Picard iterations. Finally, we prove a generalized version of Gronwall inequality for such non-local derivatives and we deduce some continuity properties of the solutions with respect to the initial datum or some parameters in a locally compact metric space.

Keywords: potential measure, inverse subordinator, special Bernstein function, generalized Gronwall inequality

CONTENTS

1. Introduction 2
2. Preliminaries and notation 3
   2.1. Bernstein functions and Lévy measures 3
   2.2. Subordinators and Potential measures 4
3. Hölder-regularity and power control of potential measures 5
4. Generalized Caputo derivatives and generalized fractional Ordinary Differential Equations 7
   4.1. Definition of the Picard operator 8
   4.2. Contraction property with respect to the Bielecki norm 10
   4.3. The affine autonomous case 11
5. Relaxation equations and Φ-exponential functions 13
6. The generalized Gronwall inequality 17
   6.1. The auxiliary operator $B$ 18
6.2. Proof of Theorem 6.1 20
7. Continuity with respect to parameters 21
   7.1. Continuous dependence on the initial datum 22
   7.2. Continuous dependence on a parameter 23
   7.3. A bound on the distance of the solutions 24
References 25
1. Introduction

The link between fractional calculus and inverse stable subordinators plays a fundamental role in the study of solutions of fractional (in time) differential equations (see [28] and references therein). Fractional calculus has been used to describe a large number of physical phenomena, for which such link can be exploited (see, for instance, [27]). In particular, the study of time-fractional parabolic equations and time-fractional differential-difference equations through the use of such probabilistic link lead to the characterization of the solutions of such equations with suitable initial data as probability density functions for some stochastic processes with randomly varying time (see [31]) and then spectral properties of such fundamental solutions have been used to reconstruct some sort of Fourier-like series representation in terms of orthogonal polynomials and Mittag-Leffler functions ([24, 23, 25, 7]). Let us also stress out that there are different standard methods to approach fractional differential equations, which rely on Laplace transform methods (see for instance [32, 18]), on the integral formulation of the problem (and then on solving non-linear Volterra integral equations [20]) or also on Picard iteration methods and fixed point theorems ([37]).

On the other hand, exploiting the link between general (eventually killed) subordinators and Bernstein functions (see [11, 33]) lead to a generalization of fractional calculus in order to consider eventual time-changes with respect to any inverse subordinator. A first generalization, which concerned complete Bernstein functions, and their representation with respect to Stieltjes measures, was considered in [21]. On the other hand, another generalization, obtained by means of semigroup theory, that worked with the whole class of Bernstein functions, since it makes use of the tail of the Lévy measure, has been achieved in [34]. Let us remark that these generalizations were not just technical improvements, since both other technical but more subtle problems ([9, 8, 16]), both other interesting models ([15, 10]), in particular a particular attention is given to the case in which one deals with special Bernstein functions, for whose the inverse of the generalized derivative operator can be explicitly expressed in terms of the density of the potential measure of a subordinator.

In this paper we focus on (eventually non-linear and non-autonomous) generalized fractional differential equations in the specific case of special Bernstein functions that are regularly varying at infinity. Indeed, by using the known control properties of the potential measure with respect to the Laplace exponent of the subordinator (see [11]), regular variation becomes an extremely powerful tool to exploit some regularity of both the potential measure and the potential density. Concerning regular variation and its link to probability models, we refer to [13] and [30]. In this case we are first able to prove a local existence and uniqueness theorem for generalized fractional Cauchy problems, by using a Picard iteration method similar to the one exploited in [37]. On the other hand, by also using the known properties of the Laplace transform of the moments of an inverse subordinator [35], we are also able to exploit a series representation of the solution of a general relaxation equation (i.e. an eigenvalue problem for the generalized fractional derivative). Let us remark that in case of Complete Bernstein Functions, these equations are already extensively studied in [21] and asymptotic properties of such solutions have been given in [22].

Another indispensable tool to study continuity properties of solutions of Cauchy problem with respect to parameters is Gronwall inequality [4]. Generalization of
the Gronwall inequality to the fractional case have been achieved for the Caputo derivative in [36] and then for the Caputo-Katugampola derivative in [1]. Finally, in [2], an extension of such inequality has been achieved for Caputo derivative with respect to other functions. We refer to references in [2] for the extension of the Gronwall inequality to Hadamard derivative and Ψ-Hilfer operators. Here, by using the approach exploited in [36], we obtain a generalized Gronwall inequality for special generalized Caputo derivative in the regularly varying case and we also exploit such inequality in terms of solutions of the relaxation equation. Thus we are also able to give some continuity results for solutions of the aforementioned Cauchy problems.

The paper is structured as follows:

- In Section 2 we exploit the notation and some known results on Bernstein functions and potential measures of subordinators;
- In Section 3 we investigate some regularity properties of the potential measure of a driftless special subordinator when its Laplace exponent is regularly varying at infinity;
- In Section 4 we give a local existence and uniqueness result for abstract Cauchy problems with non-local convolutionary time derivative induced by the tail of the Lévy measure of a special driftless Bernstein function that is regularly varying at infinity: such result relies on a Picard iteration technique together with a fixed point theorem;
- In Section 5 we exploit some properties of the solution of the relaxation equation (i.e. the eigenvalue problem for the aforementioned derivative): by using Laplace transform techniques we also exploit a series representation of such solutions in terms of sequential convolution products of the potential density;
- In Section 6 we show a generalization of the Gronwall inequality in the case of the aforementioned non-local derivative;
- Finally, in Section 7 we exploit some continuity results for solutions of non-local (in time) Cauchy problems: in this section we refer to the real case instead of a generic Banach space to lighten the notation, while the same proof can be actually used for generic abstract Cauchy problems.

2. Preliminaries and notation

2.1. Bernstein functions and Lévy measures. Let us give in this section some preliminary definitions.

**Definition 2.1.** We say a function \( \Phi \in C^\infty(0, +\infty) \) is a Bernstein function (see [33]) if and only if, \( \Phi(\lambda) \geq 0 \) and, for any \( n \in \mathbb{N} = \{1, 2, \ldots\} \), it holds

\[
(-1)^n \frac{d^n \Phi}{d\lambda^n} \leq 0.
\]

The convex cone (see [33, Corollary 3.8]) of Bernstein functions will be denoted as \( BF \).
A measure \( \mu \) on \( \mathbb{R}^d \setminus \{0\} \) is said to be a Lévy measure (see [11, Section 0.2]) if

\[
\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |x|^2) \mu(dx) < +\infty.
\]
We denote by $\mathcal{BLM}$ the subset of Lévy measures on $\mathbb{R}\setminus\{0\}$ such that $\mu(a, b) = 0$ for any $(a, b) \subseteq (-\infty, 0)$ and
$$\int_0^{+\infty} (1 \land x) \mu(dx) < +\infty.$$  
In particular we can consider these measure directly as measures on $\mathbb{R}^+$. 

For any Bernstein function, the following theorem, known as Lévy-Kintchine representation theorem, holds (see [33, Theorem 3.2])  

**Theorem 2.1.** A function $\Phi : (0, +\infty) \to \mathbb{R}$ belongs to $\mathcal{BF}$ if and only if there exist two constants $a, b \geq 0$ and a measure $\nu \in \mathcal{BLM}$ such that  
$$\Phi(\lambda) = a + b\lambda + \int_0^{+\infty} (1 - e^{-\lambda x}) \nu(dx).$$  
The measure $\nu$ is called the Lévy measure of $\Phi$ and the map $\Phi \in \mathcal{BF} \mapsto (a, b, \nu) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{BLM}$ is bijective. 

Let us give the following additional definitions. 

**Definition 2.2.** We say that a Bernstein function $\Phi \in \mathcal{BF}$ is driftless if $b = 0$. Moreover, we will suppose also $a = 0$ and $\nu(0, +\infty) = +\infty$. Given a Bernstein function $\Phi \in \mathcal{BF}$ we call conjugate of $\Phi$ the function 
$$\Phi^*(\lambda) = \frac{\lambda}{\Phi(\lambda)}.$$  
We say $\Phi$ is a special Bernstein function if $\Phi^*$ is a Bernstein function and the cone (see [33, Proposition 11.20]) of special Bernstein function will be denoted by $\mathcal{SBF}$. 

Moreover, we will work with regularly varying functions. In particular a function $f : [0, +\infty) \to \mathbb{R}$ is said to be regularly varying at infinity of order $\gamma$ if for any $c > 0$ it holds 
$$\lim_{t \to +\infty} \frac{f(ct)}{f(t)} = c^\gamma.$$  
When $\gamma = 0$, $f$ is said to be slowly varying at infinity. Moreover, we say that $f$ is slowly varying at $0^+$ if $t \to f(1/t)$ is slowly varying at infinity. Concerning regularly varying functions, we will mainly refer to [13, 30]. 

### 2.2. Subordinators and Potential measures. 

**Definition 2.3.** We call subordinator (see [11, Chapter III]) any increasing Lévy process $\sigma(t)$. 

Concerning the link between subordinators and Bernstein functions, we have the following Theorem (see [33, Theorem 5.1])  

**Theorem 2.2.** For any Bernstein function $\Phi$ there exists a unique subordinator $\sigma(t)$ such that $\mathbb{E}[e^{-\lambda \sigma(t)}] = e^{-i \Phi(\lambda)}$. 

For this reason, the function $\Phi$ is called Laplace exponent of $\sigma$. 


Definition 2.4. We will say that $\sigma$ is a driftless subordinator if and only if the Lévy triple of its Laplace exponent $\Phi$ is $(0,0,\nu)$ with $\nu(0,+\infty) = +\infty$. Let us recall that driftless subordinators are almost surely strictly increasing and $\sigma(t)$ is an absolutely continuous random variable for each $t > 0$. We say that $\sigma$ is a special subordinator if and only if its Laplace exponent $\Phi \in SBF$.

Now let us define another important class of processes that will play a main role in this paper.

Definition 2.5. Let $\sigma$ be a driftless subordinator. For any $y \geq 0$ let us define $L(t) = \inf\{\sigma(y) : y > t\}$. We will call $L(t)$ inverse subordinator and we will call potential measure of $\sigma$ the function $U(t) = \mathbb{E}[L(t)]$.

Remark 2.3. As we can see in [11], $U(t)$ is actually the distribution of the potential measure of $\sigma$. We will call it directly potential measure to simplify the notation.

Finally, let us recall the following Theorem concerning special subordinators (see [33, 11.3]).

Theorem 2.4. Let $\sigma$ be a special subordinator whose Laplace exponent $\Phi \in SBF$ admits Lévy triplet $(a,b,\nu)$. Then there exists a non-negative and non-increasing function $u(t)$ such that $\int_{0}^{t} u(t)dt < +\infty$ and

$U(dt) = c\delta_0(dt) + u(t)dt$ where $\delta_0(dt)$ is Dirac’s $\delta$ measure centered in 0 and

\[
\begin{cases}
0 & b > 0 \\
\frac{1}{\alpha + \nu(0, +\infty)} & b = 0.
\end{cases}
\]

In particular for a driftless special subordinator $\sigma$ with Lévy triple $(0,0,\nu)$ such that $\nu(0, +\infty) = +\infty$, $c = 0$ and $U$ is an absolutely continuous function with derivative (almost everywhere) given by $u$.

3. Hölder-regularity and power control of potential measures

Proposition 3.1. Let us suppose $\Phi$ is regularly varying at infinity with order $\gamma \in (0,1)$. Then the potential measure $U(t)$ is in $C^{\gamma-\varepsilon}_{loc}(\mathbb{R}^+)$ for any $\varepsilon \in (0,\gamma)$. Moreover, if $\lim \inf_{\lambda \to 0^+} \Phi(\lambda) = \alpha > 0$ (eventually $\alpha = +\infty$), then $U(t) \in C^{\gamma-\varepsilon}(\mathbb{R}^+)$ for any $\varepsilon \in (0,\gamma)$.

Proof. Let us recall (see [11]) that for any $t, s > 0$

$U(t+s) \leq U(t) + U(s)$. Moreover, being $L(t)$ increasing, so is $U(t)$. Consider $t > s$. We have

$\frac{U(t) - U(s)}{|t-s|^{\gamma-\varepsilon}} \leq \frac{U(t-s)}{|t-s|^{\gamma-\varepsilon}}$.

Now let us recall, by [11] Proposition III.1], that there exists a constant $C > 0$ such that

$U(t-s) \leq C \frac{1}{\Phi\left(\frac{t-s}{t-s}\right)}$.
and then we have
\[ \frac{U(t) - U(s)}{|t-s|^{\gamma - \varepsilon}} \leq \frac{C}{\Phi \left( \frac{1}{|t-s|} \right) |t-s|^{\gamma - \varepsilon}}. \]

Let us consider an interval \([a, b] \subset \mathbb{R}_+\) such that \(a \leq t < b\). Then \(|t-s| \leq |b - a| =: K\). Let us recall (see [17]) that there exists a slowly varying function \(\ell\) such that
\[ \Phi \left( \frac{1}{t-s} \right) |t-s|^{\gamma} = \ell \left( \frac{1}{t-s} \right). \]

Moreover we have (see [17])
\[ \lim_{|t-s| \to 0} |t-s|^{-\varepsilon} \ell \left( \frac{1}{t-s} \right) = +\infty \]
thus there exists a \(\delta > 0\) such that for any \(t, s \in [a, b]\) with \(|t-s| < \delta\) it holds
\[ |t-s|^{-\varepsilon} \ell \left( \frac{1}{t-s} \right) > 1. \]

Now let us distinguish two cases. If \(|t-s| \in (0, \delta)\) we have
\[ \frac{U(t) - U(s)}{|t-s|^{\gamma - \varepsilon}} \leq \frac{C}{\Phi \left( \frac{1}{|t-s|} \right) |t-s|^{\gamma - \varepsilon}} = \frac{C}{\ell \left( \frac{1}{|t-s|} \right) |t-s|^{\gamma - \varepsilon}} < C. \]

If \(|t-s| \in [\delta, K]\), then \(\frac{1}{t-s} \in \left[ \frac{1}{K}, \frac{1}{\delta} \right]\) and then, being \(\Phi\) a Bernstein function and thus a continuous function,
\[ \Phi \left( \frac{1}{t-s} \right) \geq \min_{\lambda \in \left[ \frac{1}{K}, \frac{1}{\delta} \right]} \Phi(\lambda) =: m \]
and then
\[ \frac{U(t) - U(s)}{|t-s|^{\gamma - \varepsilon}} \leq \frac{C}{\Phi \left( \frac{1}{|t-s|} \right) |t-s|^{\gamma - \varepsilon}} \leq \frac{C}{m \delta^{\gamma - \varepsilon}}. \]

Moreover, if \(\liminf_{\lambda \to 0} \Phi(\lambda) = \alpha > 0\), then let us distinguish two cases. If \(\alpha = +\infty\), let us consider a constant \(K(+\infty) > 0\) such that \(\Phi(\lambda) > 1 =: M(+\infty)\) for any \(\lambda > K(+\infty)\). If \(\alpha \in \mathbb{R}_+\), then let us consider a constant \(K(\alpha) > 0\) such that \(\Phi(\lambda) > \frac{\alpha}{2} =: M(\alpha)\) for any \(\lambda > K(\alpha)\). In any case, we know that if \(|t-s| \leq K(\alpha)\), then they are contained in a compact set of diameter \(K(\alpha)\) and there exists a constant \(\overline{C}\) such that
\[ U(t) - U(s) \leq \overline{C}|t-s|^{\gamma - \varepsilon}. \]
If \(|t-s| > K(\alpha)\), then we have
\[ \frac{U(t) - U(s)}{|t-s|^{\gamma - \varepsilon}} \leq \frac{C}{\Phi \left( \frac{1}{|t-s|} \right) |t-s|^{\gamma - \varepsilon}} \leq \frac{C}{M(\alpha) K(\alpha)^{\gamma - \varepsilon}} \]
concluding the proof. \(\square\)

**Corollary 3.2.** Let us suppose \(\Phi\) is regularly varying at infinity with order \(\gamma \in (0, 1)\). Then for any \(T > 0\) and any \(\varepsilon \in (0, \gamma)\) there exists a constant \(C(\varepsilon, T)\) such that
\[ U(t) \leq C(\varepsilon, T) t^{\gamma - \varepsilon}, \quad \forall t \in [0, T]. \]

**Proof.** This is direct consequence of the fact that \(U \in C^\gamma_{loc}(\mathbb{R}_+)\) together with the fact that \(U(0) = 0\). \(\square\)
Let us consider in particular a driftless special subordinator \( \sigma \), i.e. such that its Laplace exponent \( \Phi \) is a special Bernstein function. In general, since the potential measure \( U(t) = \mathbb{E}[L(t)] \), where \( L(t) \) is the inverse subordinator, is an increasing function, it is also of locally bounded variation, hence it admits a distributional derivative that is actually a Radon measure (see [3] for information on functions of bounded variation). However, for driftless special subordinators, it has been shown (see [33, Theorem 11.3]) that they are absolutely continuous functions. Let us denote by \( u(t) \) their derivative (almost everywhere) and let us recall (see [33]) that \( u \) is a non-increasing function.

In the following we will need some power control also on \( u(t) \) in the driftless special case. To do this, we just need the following easy Lemma.

**Lemma 3.3.** Let \( \Phi \) be a driftless special Bernstein function that is regularly varying at infinity with order \( \gamma \in (0, 1) \), of the form \( \Phi(\lambda) = \lambda^\gamma \ell(\lambda) \) where \( \ell(\lambda) \) is a slowly varying function. Then there exists a function \( \tilde{\ell}(t) \) slowly varying at 0 such that

\[
\frac{1}{\Gamma(\gamma)} t^{\gamma-1} \tilde{\ell}(t)
\]

as \( t \to 0^+ \).

*Proof.* Since \( \Phi \) is a regularly varying function we have, by a direct application of Karamata’s Tauberian theorem (see [11, Section 3.1]),

\[
U(t) \sim \frac{1}{\Gamma(1+\gamma)\Phi(1/t)} = \frac{t^\gamma}{\gamma \Gamma(\gamma) \ell(1/t)}.
\]

Let us observe that \( \tilde{\ell}(t) = \frac{t}{\ell(1/t)} \) is slowly varying at 0. Moreover, since \( \Phi \) is a special Bernstein function, \( U(t) \) admits a density \( u(t) \) that is non-increasing. Thus we obtain Equation (3.1) by Monotone Density Theorem (see [11, Chapter 0] or [30, Theorem 1.2.9]).

**4. Generalized Caputo derivatives and generalized fractional ordinary differential equations**

Let us now denote by \( \nu(s) = \nu(s, +\infty) \) the tail of the Lévy measure of a driftless subordinator \( \sigma \). Let us then introduce the following operator.

**Definition 4.1.** For any regular enough (for instance absolutely continuous) function \( f : \mathbb{R}^+ \to +\infty \) we define the generalized Caputo derivative of \( f \) induced by \( \sigma \) (with Laplace exponent \( \Phi \in BF \)) as

\[
\partial_t^\Phi f(t) = \frac{d}{dt} \int_0^t \nu(t-s)(f(s) - f(0))ds.
\]

These operators were firstly introduced in the context of semi-Markov processes by Toaldo in [34] and they actually generalize the classical Caputo fractional derivative, which falls in the case \( \Phi(\lambda) = \lambda^\gamma \).

For the particular case of driftless special subordinators, we can also introduce the special generalized fractional integral.

**Definition 4.2.** We call **special generalized fractional integral** induced by \( \Phi \in SBF \) the operator

\[
\mathcal{I}_t^\Phi f(t) = \int_0^t u(t-\tau)f(\tau)d\tau
\]
where \( u \) is the potential density of the driftless special subordinator \( \sigma \) with Laplace exponent \( \Phi \).

It has been shown in [29] that for regular enough functions \( f \) (as before, absolutely continuous is enough)

\[
\mathcal{I}_\Phi^t \partial_t^\Phi f(t) = \partial_t^\Phi \mathcal{I}_\Phi^t f(t) = f(t) - f(0)
\]

(4.1)

thus we can see the operator \( \mathcal{I}_\Phi^t \) as the inverse of the generalized Caputo derivative \( \partial_t^\Phi \) in case of a driftless special Bernstein function \( \Phi \). In the case \( \Phi(\lambda) = \lambda^\gamma \) we obtain the classical Riemann-Liouville fractional integral.

We are interested in some existence and uniqueness result for Cauchy problems of the form

\[
\left\{ \begin{array}{l}
\partial_t^\Phi f(t) = F(t, f(t)) \quad t \in [0, T] \\
f(0) = f_0
\end{array} \right. 
\]

in the case \( \Phi \in SBF \). Before proving an existence and uniqueness result, let us first show the following Lemma.

**Lemma 4.1.** The function \( f : [0, T] \to \mathbb{R} \) is a solution of the following generalized fractional Cauchy problem

\[
\left\{ \begin{array}{l}
\partial_t^\Phi f(t) = F(t, f(t)) \quad t \in [0, T] \\
f(0) = f_0
\end{array} \right. 
\]

(4.2)

if and only if it is a solution of the integral equation

\[
f(t) = f_0 + \mathcal{I}_\Phi^t F(t, f(t)), \quad t \in [0, T].
\]

(4.3)

**Proof.** If we suppose that \( f \) is solution of (4.2) then, applying \( \mathcal{I}_\Phi^t \) to both sides of the first equation and substituting \( f(0) = f_0 \) we get, by using also relation (4.1), Equation (4.3).

Vice versa, if we suppose \( f \) is solution of (4.3), we have just to apply the operator \( \partial_t^\Phi \) on both sides of the equation, recalling that \( \partial_t^\Phi f(0) = 0 \), to achieve the first equation of (4.2), while the initial condition follows from the fact that \( \mathcal{I}_\Phi^0 f(t) = 0 \). \( \square \)

Thus, to prove existence and uniqueness of the solution of the Cauchy problem (4.2), we just need to prove existence and uniqueness of the solution of the integral equation (4.3).

In the following we will consider the abstract Cauchy problem

\[
\left\{ \begin{array}{l}
\partial_t^\Phi f(t) = F(t, f(t)) \quad \text{a.e.} \, t \in [0, T] \\
f(0) = f_0
\end{array} \right. 
\]

(4.4)

where \( f : [0, T] \to X \), with \( (X, |\cdot|) \) any Banach space, and \( F : [0, T] \times X \to X \).

In particular we will prove the following local existence, uniqueness and regularity Theorem.

**Theorem 4.2.** Let \( \Phi \in BF \) and \( F : J \times X \to X \). Suppose the following conditions hold:

A1 \( \Phi \) is a driftless special Bernstein function;
A2 \( \Phi \) is regularly varying at infinity with order \( \gamma \in (0, 1) \);
A3 For any ball \( B_R \) in \( X \) there exists a constant \( C_R > 0 \) such that \( |F(t, x)| \leq C_R \) for almost any \( t \in J \) and any \( x \in B_R \);
A4 For any ball \( B_R \) in \( X \) there exists a constant \( L_R > 0 \) such that \( |F(t, x) - F(t, z)| \leq L_R |x - z| \) for almost any \( t \in J \) and any \( x, z \in B_R \).
Then for any initial datum \( f_0 \in B_R \) there exists \( T_1 > 0 \) such that equation (1.4) admits a unique solution \( f \in C^{\gamma - \varepsilon}(J_1, B_R) \) where \( J_1 = [0, T_1] \) and \( \varepsilon \in (0, \gamma) \).

The proof of the Theorem will be articulated in the following two Subsections:

- We first need to introduce what will be our Picard iterative operator, taking in consideration the structure of equation (4.3);
- Then we just need to show that the operator is actually a contraction when we choose the right norm on \( C^{\gamma - \varepsilon}(J_1, B_R) \).

### 4.1. Definition of the Picard operator.

Denote \( J = [0, T] \) and, for any \( \tau > 0 \), let us define the Bielecki norm on \( C(J, X) \) as

\[
|f|_\tau = \max_{t \in J} |f(t)|e^{-\tau t}.
\]

Now let us define the Picard operator

\[
A : (C(J, B_R), \| \cdot \|_\tau) \rightarrow (C(J, X), \| \cdot \|_\tau)
\]

for some \( R > 0 \), as

\[
Af(t) = f_0 + \mathcal{\mathcal{T}}^\mathcal{\gamma} \mathcal{T}^\mathcal{\varepsilon} F(t, f(t)),
\]

observing that any fixed point of \( A \) is solution of (4.3). Let us show the following lemma.

**Lemma 4.3.** The operator \( A \) is well defined and its codomain can be restricted to \( C^{\gamma - \varepsilon}(J, X) \) for any \( \varepsilon \in (0, \gamma) \).

**Proof.** Fix \( \delta > 0 \), \( \varepsilon \in (0, \gamma) \) and \( f \in C(J, B_R) \) and observe that

(4.5)

\[
|Af(t + \delta) - Af(t)| = \left| \int_0^t u(t - s)F(s, f(s))ds - \int_0^{t+\delta} u(t + \delta - s)F(s, f(s))ds \right|
\]

\[
\leq \int_0^t |u(t - s) - u(t + \delta - s)||F(s, f(s))|ds + \int_t^{t+\delta} u(t + \delta - s)||F(s, f(s))|ds
\]

\[
= I_1(t) + I_2(t).
\]

Let us first consider \( I_2(t) \). We have, by hypothesis \( A4 \),

\[
I_2(t) \leq CR \int_t^{t+\delta} u(t + \delta - s)ds = CRU(\delta).
\]

Now let us recall that for any \( t \in [0, T] \)

\[
U(t) \leq C(\varepsilon, T)t^{\gamma - \varepsilon}
\]

and \( U \) is locally \((\gamma - \varepsilon)\)-Hölder-continuous. Thus we have

\[
I_2(t) \leq C(R, \varepsilon, T)\delta^{\gamma - \varepsilon}.
\]

Now let us work with \( I_1 \). Arguing as before we have

\[
I_1(t) \leq CR \int_0^t (u(t - s) - u(t + \delta - s))ds = CR(U(t + \delta) - U(t) + U(\delta)).
\]

By using also the fact that \( U \) is locally Hölder-continuous we have

\[
I_1(t) \leq C(R, \varepsilon, T)\delta^{\gamma - \varepsilon}.
\]

We finally have

\[
|Af(t + \delta) - Af(t)| \leq C(R, \varepsilon, T)\delta^{\gamma - \varepsilon},
\]
and the proof is concluded. \[
\]

Now we need to restrict more the codomain of the operator. To do this, let us show the following result.

**Lemma 4.4.** There exists $T_1(R) \leq T$ such that for any $f \in C(J, B_R)$ it holds $Af_{|J_1} \in C(J_1, B_R)$.

**Proof.** Let us observe that

$$|Af(t)| \leq \int_0^t u(t-s)F(s, f(s))|ds \leq C_R U(t).$$

Now let us observe that $L(t)$ is left-continuous hence also $U(t)$ is left-continuous. Moreover, $U(0) = 0$, hence there exists $T_1 > 0$ such that $C_R U(T_1) < R$. Being $U$ non-decreasing, we have

$$\|Af(t)\|_{C(J_1, X)} \leq C_R U(T_1) < R,$$

where for any function $g \in C(J_1, X)$, $\|g\|_{C(J_1, X)} = \max_{t \in J_1} |g(t)|$.

Thus now we can restrict both domain and codomain of $A$. Thus from now on we will consider

$$A : (C^\gamma - \varepsilon(J_1, B_R), \|\cdot\|_\tau) \rightarrow (C^\gamma - \varepsilon(J_1, B_R), \|\cdot\|_\tau).$$

4.2. **Contraction property with respect to the Bielecki norm.** Now that we have shown that we choose domain and codomain of $A$ to be equal, we have to show that $A$ is a contraction. However, we cannot do this with respect to the Chebyshev norm $\|\cdot\|_{C(J_1, X)}$, but we will use the Bielecki norm. Let us first recall that this norm is equivalent to Chebyshev norm. Indeed we have

$$|f(t)e^{-\tau T_1} - f(t)|e^{-\tau t} \leq |f(t)|$$

that, taking the maximum for $t \in J_1$, gives the equivalence.

Thus we can show the following Proposition:

**Proposition 4.5.** There exists $\tau$ such that $A$ is a contraction on $(C^\gamma - \varepsilon(J_1, B_R), \|\cdot\|_\tau)$.

**Proof.** Let us consider $f, g \in C^\gamma - \varepsilon(J_1, B_R)$. We have

$$|Af(t) - Ag(t)| \leq \int_0^t u(t-s)F(s, f(s)) - F(s, g(s))|ds$$

$$\leq L_R \int_0^t u(t-s)|f(s) - g(s)|ds$$

$$= L_R \int_0^t u(t-s)|f(s) - g(s)|e^{-\tau s}e^{\tau s}ds$$

$$\leq L_R \|f - g\|_\tau \int_0^t u(t-s)e^{\tau s}ds.$$

Now let us observe that $u(t)$ is regularly varying at 0 with order $\gamma - 1$, hence there exists a slowly varying function $\ell$ such that $u(t) = t^{\gamma-1}\ell(t)$. Moreover, consider $\varepsilon_1 \in (0, \gamma)$ and a constant $C > 0$ such that $\ell(t) \leq Ct^{-\varepsilon_1}$. Thus we have $u(t) \leq Ct^{\gamma-\varepsilon_1-1}$ and we have

$$|Af(t) - Ag(t)| \leq CL_R \|f - g\|_\tau \int_0^t (t-s)^{\gamma-\varepsilon_1-1}e^{\tau s}ds.$$
Observe that \( \frac{1}{1 + \varepsilon_1 - \gamma} > 1 \), thus we can consider \( p \in \left(1, \frac{1}{1 + \varepsilon_1 - \gamma}\right) \) and use Hölder inequality to have
\[
|Af(t) - Ag(t)| \leq CL_R \|f - g\|_\tau \left( \int_0^t (t - s)^{p(\gamma - \varepsilon_1 - 1)} ds \right)^{\frac{1}{p'}} \left( \int_0^t e^{p' \tau s} ds \right)^{\frac{1}{p'}}
= L_R \|f - g\|_\tau \frac{1}{(p(\gamma - \varepsilon_1 - 1) + 1)^{\frac{1}{p'}}} T_1^{\frac{p(\gamma - \varepsilon_1 - 1) + 1}{p}} \left( \frac{1}{p' \tau} \right)^{\frac{1}{p'}} e^{\tau t}.
\]
Thus we achieve
\[
e^{-\tau t}|Af(t) - Ag(t)| \leq CL_R \|f - g\|_\tau \frac{1}{(p(\gamma - \varepsilon_1 - 1) + 1)^{\frac{1}{p'}}} T_1^{\frac{p(\gamma - \varepsilon_1 - 1) + 1}{p}} \left( \frac{1}{p' \tau} \right)^{\frac{1}{p'}}
\]
and then
\[
\|Af - Ag\|_\tau \leq CL_R \|f - g\|_\tau \frac{1}{(p(\gamma - \varepsilon_1 - 1) + 1)^{\frac{1}{p'}}} T_1^{\frac{p(\gamma - \varepsilon_1 - 1) + 1}{p}} \left( \frac{1}{p' \tau} \right)^{\frac{1}{p'}}.
\]
Since
\[
\lim_{\tau \to +\infty} CL_R \frac{1}{(p(\gamma - \varepsilon_1 - 1) + 1)^{\frac{1}{p'}}} T_1^{\frac{p(\gamma - \varepsilon_1 - 1) + 1}{p}} \left( \frac{1}{p' \tau} \right)^{\frac{1}{p'}} = 0
\]
there exists \( \tau > 0 \) such that
\[
CL_R \frac{1}{(p(\gamma - \varepsilon_1 - 1) + 1)^{\frac{1}{p'}}} T_1^{\frac{p(\gamma - \varepsilon_1 - 1) + 1}{p}} \left( \frac{1}{p' \tau} \right)^{\frac{1}{p'}} < 1
\]
concluding the proof. \( \square \)

Now we can conclude the proof of the Theorem. Indeed, we have that \( A \) is a contraction over \((C^{\gamma - \varepsilon}(J_1, B_R), \|\cdot\|_\tau)\) when \( \tau \) is big enough, hence, by contraction theorem (see [19]), we know there exists a unique fixed point \( f \in C^{\gamma - \varepsilon}(J_1, B_R) \) of \( A \). In particular \( f(t) = Af(t) \) is actually equation (4.33).

4.3. The affine autonomous case. A particularly easy case is given by the affine one. Indeed, in the affine case, we can show the following Corollary.

**Corollary 4.6.** Let \( F : X \to X \) be a bounded linear operator and let \( \xi \in X \). Suppose \( \Phi \in SBF \) is a driftless Bernstein function regularly varying at infinity with order \( \gamma \in (0, 1) \). Then there exists \( T \) depending only on \( \|F\|_{L(X,X)} := \sup_{|x|=1} |F(x)| \), \( |\xi| \) and \( |f_0| \) such that the problem
\[
\begin{align*}
\frac{\partial}{\partial t} f(t) &= \xi + Ff(t) \quad t \in [0, T] \\
f(0) &= f_0.
\end{align*}
\]
admits a unique continuous solution.

**Proof.** Let us fix \( R = |f_0| + 1 \) and consider the Picard operator \( A \) defined as in the previous Theorem. Starting from (4.35), this time let us observe that
\[
|\xi + Ff(t)| \leq |\xi| + \|F\|_{L(X,X)} |f(t)| \leq |\xi| + \|F\|_{L(X,X)} R
\]
and then
\[
I_2(t) \leq (|\xi| + \|F\|_{L(X,X)} R)U(\delta).
\]
Moreover, we have
\[ I_1(t) \leq (|\xi| + \|F\|_{L(X,X)}) R(U(t + \delta) - U(t) + U(\delta)) \]
thus obtaining
\[ |Af(t + \delta) - Af(t)| \leq (|\xi| + \|F\|_{L(X,X)}) R(U(t + \delta) - U(t) + 2U(\delta)). \]
Taking the limit as \( \delta \to 0 \), recalling that \( U \) is a continuous function such that \( U(0) = 0 \), we have that \( Af \in C(J, X) \) for any \( J = [0, T] \). Moreover, we have
\[ \|Af(t)\|_{C(J,X)} \leq (|\xi| + \|F\|_{L(X,X)}) R U(T) \]
thus, by using continuity, we only have to choose \( T \) small enough to have
\[ U(T) \leq \frac{R}{(|\xi| + \|F\|_{L(X,X)}) R}. \]

Let us define the left-continuous inverse of \( U \):
\[ U^{-}(u) = \min \{ x : F(x) \geq u \}, \quad \forall u > 0 \]
and set \( T = U^{-}(\frac{R}{2(|\xi| + \|F\|_{L(X,X)}) R}) \). Then, since \( U \) is continuous, we have
\[ U(T) = U \left( U^{-} \left( \frac{R}{2(|\xi| + \|F\|_{L(X,X)}) R} \right) \right) = \frac{R}{2(|\xi| + \|F\|_{L(X,X)}) R} < \frac{R}{(|\xi| + \|F\|_{L(X,X)}) R} \]
and then we have a suitable choice for \( T \) which depends only on \( R \) (and thus on \( |f_0| \), \( |\xi| \) and \( \|F\|_{L(X,X)} \) in such a way that \( |Af(t)| < R \) for any \( t \in [0, T] \). Thus we have that \( A : (C(J, B_R), \|\cdot\|_\gamma) \to (C(J, B_R), \|\cdot\|_\gamma) \) is well-defined.

Now observe that \( u \) is regularly varying at 0 with order \( \gamma - 1 \), then there exists a constant \( C(T) \) such that \( u(t) \leq t^{\frac{\gamma}{4}} \) for any \( t \in [0, T] \). Consider \( p = \frac{4-\gamma}{2(2-\gamma)} \) and observe that, for any \( f, g \in C(J, B_R) \),
\[ |Af(t) - Ag(t)| \leq C(T) \|F\|_{L(X,X)} \|f - g\|_\tau \int_0^t (t - s)^{\frac{2(2-\gamma)}{4}} e^{\tau s} ds \]
\[ \leq C(T) \|F\|_{L(X,X)} \|f - g\|_\tau \left( \frac{4}{8 - \gamma} \right)^{\frac{2(2-\gamma)}{4}} T^{\frac{2(2-\gamma)(8-\gamma)}{4(4-\gamma)}} \left( \frac{\gamma}{(4 - \gamma)\tau} \right)^{\frac{4-\gamma}{4}} e^{\tau t} \]
and then
\[ \|Af - Ag\|_\tau \leq C(T) \|F\|_{L(X,X)} \|f - g\|_\tau \left( \frac{4}{8 - \gamma} \right)^{\frac{2(2-\gamma)}{4}} T^{\frac{2(2-\gamma)(8-\gamma)}{4(4-\gamma)}} \left( \frac{\gamma}{(4 - \gamma)\tau} \right)^{\frac{4-\gamma}{4}} \]
and then one can chose \( \tau \) as
\[ \tau = \left( 2C(T) \|F\|_{L(X,X)} \left( \frac{4}{8 - \gamma} \right)^{\frac{2(2-\gamma)}{4}} T^{\frac{2(2-\gamma)(8-\gamma)}{4(4-\gamma)}} \left( \frac{\gamma}{4 - \gamma} \right)^{\frac{4-\gamma}{4}} \right)^{\frac{1}{4-\gamma}} \]
to obtain
\[ \|Af - Ag\|_\tau \leq \frac{1}{2} \|f - g\|_\tau \]
and conclude the proof.
Since we have shown that if we consider an affine autonomous problem like in Equation (4.6) then $T$ can be chosen to depend only on $|\xi|, \|F\|_{L(X,X)}$ and $|f_0|$, one can argue as in [3] Corollary 2, i.e. with a gluing argument, to show the following global uniqueness result.

**Corollary 4.7.** Under the hypotheses of Corollary 4.6, if there exists a solution $f \in C([0, +\infty), B_R)$, then this solution is unique.

**Remark 4.8.** Let us observe that if $\Phi \in SBF$ is a driftless Bernstein function which is also regularly varying at infinity of order $\gamma \in (0, 1)$, then we have for any $\varepsilon \in (0, \gamma)$ it holds $0 \leq u(t-s) \leq C_T(t-s)^{\gamma-\varepsilon-1}$. Then, in particular, we can consider any $\alpha \in (1 + \varepsilon - \gamma, 1)$ to have that $(t-s)^{\alpha}u(t-s)$ is integrable. In particular this means that the affine autonomous case can be treated as a Fredholm equation of the second kind and thus the solution can be expressed by Neumann series (see [14]).

### 5. Relaxation equations and $\Phi$-exponential functions

Now let us introduce another class of functions.

**Definition 5.1.** Given $\Phi \in BF$ a driftless Bernstein function, we will call $\Phi$-exponential function of rate $\lambda$ the function

$$e_{\Phi}(t; \lambda) = E_{\gamma}(\lambda t^\gamma), \quad t \geq 0$$

where $L$ is the inverse of the subordinator $\sigma$ whose Laplace exponent is $\Phi$.

Let us observe (see [12]) that if $\Phi = \lambda^\gamma$ for $\gamma \in (0, 1)$, then $e_{\Phi}(t; \lambda) = E_{\gamma}(\lambda t^\gamma)$, where

$$E_{\gamma}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k\gamma + 1)}, \quad t \in \mathbb{R}$$

is the one-parameter Mittag-Leffler function (see [20]).

Let us fix $\Phi \in SBF$ driftless Bernstein function and let us recall that, since we have asked for $a = 0$ and $\nu(0, +\infty) = +\infty$, then it can be shown that $L(t)$ is an absolutely continuous random variable for $t > 0$. Let us denote its density by $f_L(s; t)$. From now on, let us also denote by $L_{t\rightarrow \lambda}$ the Laplace transform operator, i.e., for any function $f : [0, +\infty) \rightarrow \mathbb{R}$ of exponential order

$$L_{t\rightarrow \lambda}[f(t)](\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t) dt$$

for any $\lambda \in \mathbb{C}$ for which such integral exists. It can be shown that there exists a real value $\omega = \text{abs}(f)$, called abscissa of convergence, such that if $\lambda > \omega$ then $L_{t\rightarrow \lambda}[f(t)](\lambda)$ exists, while if $\lambda < \omega$ $L_{t\rightarrow \lambda}[f(t)](\lambda)$ is not defined (see [5]). For this reason, we will just consider the case $\lambda \in \mathbb{R}$.

It is known that (see [26 Equation 3.13]) for any $\lambda > 0$

$$L_{t\rightarrow \lambda}[f_L(s; t)] = \frac{\Phi(\lambda)}{\lambda} e^{-s\Phi(\lambda)}.$$  \(5.1\)

On the other hand, it is not difficult to check (by definition) that

$$L_{t\rightarrow \lambda}[\Phi(t)](\lambda) = \frac{\Phi(\lambda)}{\lambda}.$$
Then, for any absolutely continuous function $f : [0, +\infty) \to \mathbb{R}$, we have

\[(5.2) \quad \mathcal{L}_{t \to \lambda}[\partial^\Phi_t f(t)](\lambda) = \Phi(\lambda) \mathcal{L}_{t \to \lambda}[f(t)](\lambda) - \frac{\Phi(\lambda)}{\lambda} f(0).\]

Now we can exploit the role of the $\Phi$-exponential function, by considering the eigenvalue problem for the operator $\partial^\Phi_t$ with a different approach.

**Proposition 5.1.** Let $\Phi \in \mathcal{B}\mathcal{F}$ be a driftless Bernstein function. Then, for any $\lambda \in \mathbb{R}$, the function $f(t) = f_0 e_\Phi(t; \lambda)$ is the unique solution of

\[(5.3) \quad \begin{cases} \partial^\Phi_t f(t) = \lambda f(t) & t > 0 \\ f(0) = f_0. \end{cases}\]

**Proof.** Let us first show that $f(t) = f_0 e_\Phi(t; \lambda)$ is a solution of the Cauchy Problem \[(5.3).\] For $\lambda = 0$ it is obvious, thus let us consider $\lambda \neq 0$. First of all, let us notice that $f(0) = f_0$. Let us then observe, by using equation \[5.1\]

\[\mathcal{L}_{t \to z}[f(t)](z) = f_0 \mathcal{L}_{t \to z} \left[ \int_0^\infty e^{\lambda s} f_L(s; t) ds \right](z) = f_0 \frac{\Phi(z)}{z} \int_0^\infty e^{\lambda s} e^{-\Phi(z)} ds = f_0 \frac{\Phi(z)}{z(\Phi(z) - \lambda)} \]

for any $z$ such that $\Phi(z) > \lambda$. Now, let us denote $\tilde{f}(z) = \mathcal{L}_{t \to z}[f(t)](z)$ and multiply everything by $\Phi(z) - \lambda$ to achieve

\[\tilde{f}(z)(\Phi(z) - \lambda) = f_0 \frac{\Phi(z)}{z}\]

and then

\[\frac{\Phi(z)}{z} (z \tilde{f}(z) - f_0) = \lambda \tilde{f}(z).\]

Dividing again everything by $z$ we achieve

\[(5.4) \quad \frac{\Phi(z)}{z} \left( \tilde{f}(z) - \frac{f_0}{z} \right) = \frac{\lambda}{z} \tilde{f}(z).\]

Now let us observe that $f(t)$ is in $L^\infty_{loc}(\mathbb{R}^+)$, hence it is also in $L^1_{loc}(\mathbb{R}^+)$. Thus the function $F(t) = \int_0^t f(s) ds$ is well defined and absolutely continuous. Moreover, let us recall that any Bernstein function $\Phi$ is increasing and continuous, hence we have that $\text{abs}(f(t)) = \min\{z > 0 : \Phi(z) = \lambda\}$. Let us set $z_0 = \text{abs}(f(t))$. Since $f \geq 0$, if $f \in L^1(\mathbb{R}^+)$, then $F(t)$ is bounded and thus abs$(F) = 0$. Let us then suppose that $f \notin L^1(\mathbb{R}^+)$. Then $\lim_{t \to +\infty} F(t) = +\infty$. Thus, by [3 Theorem 1.4.3] we know that for any $\varepsilon > 0$ there exists a constant $K_\varepsilon > 0$ such that $F(t) \leq K_\varepsilon e^{(z_0 + \varepsilon)t}$ for any $t > 0$. Thus in particular

\[\int_0^{+\infty} e^{-\lambda t} F(t) dt \leq K_\varepsilon \int_0^{+\infty} e^{-\lambda t} e^{(z_0 + \varepsilon)t} dt.\]

We can conclude that $F(t)$ is Laplace transformable with $\text{abs}(F(t)) \leq z_0$. In any case, we can consider the inverse Laplace transform of Equation \[(5.4)\] to achieve

\[\int_0^t \Phi(t - s)(f(t) - f(0)) = \lambda \int_0^t f(s) ds\]
and, being the right-hand side absolutely continuous, we can differentiate both sides to obtain the desired result.

Concerning uniqueness, let us suppose $g$ is solution of \(5.3\). Then, integrating both sides of the equation, we have

$$\int_0^t \nu(t - s)(g(t) - g(0)) = \lambda \int_0^t g(s) ds.$$  

Since, being $g$ a solution of the equation, it must belong to $L^1_{\text{loc}}$, it is Laplace transformable and, arguing as before, so it is its antiderivative $G(t) = \int_0^t g(s) ds$. We can then apply Laplace transform on both sides to achieve, after some algebraic manipulation and using the fact that $g(0) = f_0$,

$$L_{t \to z} [g(t)](z) = \frac{f_0 \Phi(z)}{z(\Phi(z) - \lambda)} = f(z)$$

for any $z > \max\{z_0, \text{abs}(g(t))\}$. Thus, by injectivity of the Laplace transform, we obtain that $g(t) = f(t)$ almost everywhere, completing the proof. \(\square\)

**Remark 5.2.** For $\lambda < 0$, we could obtain existence and uniqueness of a uniformly bounded solution of \(5.3\) also by Corollary 4.7 whenever $\Phi \in SBF$ and it is regularly varying. However, the Laplace transform approach is much more efficient when arguing for some simple linear abstract Cauchy problems.

Let us come back to the case in which $\Phi \in SBF$ is a driftless special subordinator regularly varying at $\infty$ with order $\gamma \in (0, 1)$. Concerning the $\Phi$-exponential functions, it will be useful to deduce some series representation.

To do that, let us define the following sequence of functions which are defined by recurrence:

\[
\begin{cases}
    u_0^\gamma(t) = 1; \\
u_1^\gamma(t) = U(t); \\
u_{k+1}^\gamma(t) = \int_0^t u(t - s)u_k^\gamma(s) ds \quad k \geq 1
  \end{cases}
\]

Let us first show the following Lemma

**Lemma 5.3.** Let $\Phi \in SBF$ be a driftless special subordinator regularly varying at $\infty$ with order $\gamma \in (0, 1)$. Then for any $\lambda > 0$ the series

$$\sum_{k=1}^{+\infty} k\lambda^k u_k^\gamma(t)$$

is totally convergent for $t \in [0, T]$ for any $T > 0$.

**Proof.** To do this let us fix $T > 0$ and $\varepsilon \in (0, \gamma)$ and let us observe by Corollary 3.2 that there exists $C_1 > 0$ such that $U(t) \leq C_1 t^{\gamma - \varepsilon}$ for any $t \in [0, T]$. Now, since $u(t)$ is regularly varying at $0$ with order $\gamma - 1$ we know there exists a constant $C_2$ such that $u(t) \leq C_2 t^{\gamma - 1 - \varepsilon}$ for any $t \in [0, T]$. So let us observe that

$$u_2^\gamma(t) \leq C_1 C_2 \int_0^t (t - s)^{\gamma - 1 - \varepsilon} s^{\gamma - \varepsilon} ds$$

Let us then set $w = \frac{t}{\beta}$ and $\beta = \gamma - \varepsilon$. We obtain

$$u_2^\gamma(t) \leq C_1 C_2 \beta^{2\beta} \int_0^1 (1 - w)^{\beta - 1} w^{\beta - 1} dw = C_1 C_2 \beta^{2\beta} \frac{\Gamma(\beta)\Gamma(\beta + 1)}{\Gamma(2\beta + 1)} = C_1 C_2 \frac{\beta}{\Gamma(2\beta + 1)} \Gamma(\beta + \beta^2).$$
Now we want to show in general that
\[ u_k^*(t) \leq C_1C_k^{k-1} \frac{\beta}{\Gamma(k+1)} (\Gamma(\beta)t^\beta)^k. \]

Let us argue by induction. We have already shown this for \( k = 1 \) and \( k = 2 \). Let us suppose it holds true for some \( k \geq 2 \). Then we have
\[ u_{k+1}^*(t) \leq C_1C_k^{k-1} \frac{\beta \Gamma(\beta)^k}{\Gamma(k+1)} \int_0^t (t-s)^{\beta-1} s^k ds. \]

Setting \( w = \frac{t}{s} \) we achieve
\[ u_{k+1}^*(t) \leq C_1C_k^{k-1} \frac{\beta \Gamma(\beta)^k}{\Gamma(k+1)} t^{(k+1)\beta}. \]

proving the claim. Hence we have
\[ \sum_{k=1}^{+\infty} k\lambda^k u_k^*(t) \leq \frac{C_1\beta}{C_2} \sum_{k=1}^{+\infty} k\lambda \frac{\Gamma(\beta)}{\Gamma(k+1)}^k \frac{k}{(k+1)^\beta} t^{(k+1)\beta}. \]

Now we only have to determine the radius of convergence of the power series \( g(y) = \sum_{k=1}^{+\infty} \frac{k\lambda^k}{\Gamma(k+1)} \) which is obviously \( \rho = +\infty \). Thus, being the power series on the right-hand side totally convergent as \( t \in [0, T] \) for any \( T > 0 \), so it is the series on the left-hand side. \( \square \)

Then we are ready to show the following Theorem.

**Theorem 5.4.** Let \( \Phi \in SBF \) be a driftless special subordinator regularly varying at \( \infty \) with order \( \gamma \in (0, 1) \). Then for any \( \lambda \in \mathbb{R} \) it holds
\[ \epsilon_\Phi(t; \lambda) = 1 + \sum_{k=1}^{+\infty} k\lambda^k u_k^*(t). \]

**Proof.** For \( \lambda = 0 \) it is obvious, so let us consider \( \lambda \neq 0 \). Let us define \( U_k(t) = \mathbb{E}[(L(t))^k] \) for \( k \geq 0 \). Let us work with \( \lambda > 0 \). Then we have, by monotone convergence theorem,
\[ \epsilon_\Phi(t; \lambda) = \sum_{k=0}^{+\infty} \frac{(\lambda^k U_k(t))}{k!}. \]

Now let us recall from [35, Equation 7] that
\[ \mathcal{L}_{t \to z}[U_k(t)](z) = k \frac{k!}{z^{\Phi^k(z)}}, \quad k \geq 1. \]

By a simple application of monotone convergence theorem, we have
\[ \mathcal{L}_{t \to z}[\epsilon_\Phi(t; \lambda)](z) = \frac{1}{z} + \sum_{k=1}^{+\infty} \frac{k\lambda^k}{z^{\Phi^k(z)}}. \]

Now let us observe that \( \mathcal{L}_{t \to z}[u_0^*(t)] = \frac{1}{2} \). Moreover, we have \( \mathcal{L}_{t \to z}[u_1^*(t)] = \mathcal{L}_{t \to z}[U(t)] = \frac{1}{2\Phi^1(z)} \).

Now let us show that
\[ \mathcal{L}_{t \to z}[u_k^*(t)] = \frac{1}{2\Phi^k(z)} \]

for any $k \geq 1$. This formula holds for $k = 1$, hence let us show it by induction. Suppose Formula (5.5) holds for $k \geq 1$. Then let us observe that $u(t)$ is Laplace transformable and $L_{t \to z}[u(t)](z) = z L_{t \to z}[U(t)](z) = \frac{1}{\Phi(z)}$. Hence we have:

$$L_{t \to z}[u^*_k(t)](z) = L_{t \to z}[u^*_k(t)](z) L_{t \to z}[u(t)](z) = \frac{1}{z\Phi^{k+1}(z)},$$

hence showing the claim.

Now that we have this, let us observe that in Lemma (5.3) we have shown that for any $\lambda > 0$ the series

$$\sum_{k=1}^{+\infty} k\lambda^k u_k^*(t)$$

absolutely converges for any $t \geq 0$. Again, by a simple application of the monotone convergence theorem we have

$$L_{t \to z} \left[ 1 + \sum_{k=1}^{+\infty} k\lambda^k u_k^*(t) \right] = \frac{1}{z} + \sum_{k=1}^{+\infty} \frac{k\lambda^k}{z\Phi^k(z)} = L_{t \to z} [e\Phi(t; \lambda)](z).$$

Finally, by injectivity of the Laplace transform, we get

$$e\Phi(t; \lambda) = 1 + \sum_{k=1}^{+\infty} k\lambda^k u_k^*(t).$$

Concerning $\lambda < 0$, let us observe that

$$\sum_{k=0}^{+\infty} \left| \frac{(\lambda^k U_k(t))}{k!} \right| = \sum_{k=0}^{+\infty} \frac{|\lambda|^k U_k(t)}{k!} = 1 + \sum_{k=1}^{+\infty} |\lambda|^k u_k^*(t)$$

hence absolute convergence of the series is ensured. Thus the proof follows analogously, by using dominated convergence theorem in place of the monotone convergence one. □

This series representation will come handy when proving continuous dependence on initial datum.

6. The generalized Gronwall inequality

Now we aim to show a Gronwall-like inequality for the special generalized fractional integral, following the lines of [36, Theorem 1]. In particular we will prove the following Theorem.

**Theorem 6.1.** Let $x, a, g \in L^1([0, T])$. Suppose that $a$ is non-negative and $g$ is non-decreasing and non-negative. Let $\Phi \in \mathcal{SBF}$ be a driftless Bernstein function that is regularly varying at infinity with order $\gamma \in (0, 1)$. Moreover, suppose that

$$x(t) \leq a(t) + g(t) \int_0^t x(s) ds.$$  

Then:

- It holds
  $$x(t) \leq \sum_{k=0}^{+\infty} B^k a(t)$$

  where $B^0$ is the identity operator and $B$ is defined as
  $$B a(t) = g(t) \int_0^t u(t-s)a(s)ds.$$
Lemma 6.3. We have indeed the following Lemma.

\[ x(t) \leq a(t) + C_2 \Gamma(\beta)g(t) \int_0^t E_{\beta, \beta}(C_2 \Gamma(\beta)g(T)(t-s))(t-s)^{\beta-1}a(s)ds, \]

where \( E_{\beta_1, \beta_2}(t) \) is the two-parameters Mittag-Leffler function (see [20]) defined as

\[ E_{\beta_1, \beta_2}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k\beta_1 + \beta_2)}; \]

- If \( a \) is non-decreasing then

\[ x(t) \leq a(t) \epsilon_{\delta}(t, g(T)). \]

The proof of this Theorem will be given in Subsection 6.2. Indeed we first need some technical Lemmas concerning the operator \( B \).

6.1. The auxiliary operator \( B \). For any function \( f \in L^1([0, T]) \) let us define the following operator

\[ Bf(t) = g(t) \int_0^t u(t-s)f(s)ds. \]

In this subsection we will prove some properties of the operator \( B \) and of its powers. First of all, let us show the following easy Lemma

**Lemma 6.2.** Let \( f_1, f_2 \in L^1([0, T]) \) with \( f_1(t) \leq f_2(t) \) almost everywhere in \([0, T]\). Then \( Bf_1(t) \leq Bf_2(t) \) for any \( t \in [0, T] \).

**Proof.** This property follows from the fact that \( g \) and \( u \) are non-negative functions. \( \Box \)

Now we want to prove some control on the function \( B^k f(t) \) for \( f \in L^1([0, T]) \). We have indeed the following Lemma.

**Lemma 6.3.** Fix \( \epsilon \in (0, \gamma) \) and define \( \beta = \gamma - \epsilon \). Then there exists a constant \( C_2 \) such that for any \( k \geq 1 \) and any non-negative \( f \in L^1([0, T]) \) it holds

\[ B^k f(t) \leq \frac{(C_2 \Gamma(\beta)g(t))^k}{\Gamma(k\beta)} \int_0^t (t-s)^{k\beta-1}f(s)ds. \]

**Proof.** Since we know that \( u(t) \) is a regularly varying function at 0 of order \( \gamma - 1 \), we know there exists \( C_2 > 0 \) such that \( u(t) \leq C_2 t^{\gamma-\epsilon-1} = C_2 t^{\beta-1} \). Then, for \( k = 1 \) we have

\[ Bf(t) = g(t) \int_0^t u(t-s)f(s)ds \leq C_2 g(t) \int_0^t (t-s)^{\beta-1}f(s)ds. \]

Now let us suppose Equation (6.3) holds for some \( k \geq 1 \). Then we have, by Lemma 6.2

\[ B^{k+1} f(t) = B(B^k f(t))(t) \leq g(t) \int_0^t u(t-s) \frac{(C_2 \Gamma(\beta)g(s))^k}{\Gamma(k\beta)} \int_0^s (s-\tau)^{k\beta-1}f(\tau)d\tau ds. \]
Now, since $g$ is increasing and $u(t-s) \leq C_2(t-s)^{\beta-1}$, we have, by also using the substitution $s = t - (t - s)w$,

\[
B^{k+1}f(t) \leq \left(\frac{C_2\Gamma(\beta)g(t)}{\Gamma(k\beta)\Gamma(\beta)}\right)^{k+1} \int_0^t (t-s)^{\beta-1} \int_0^s (s-\tau)^{k\beta-1} f(\tau)d\tau ds
\]

\[
= \left(\frac{C_2\Gamma(\beta)g(t)}{\Gamma(k\beta)\Gamma(\beta)}\right)^{k+1} \int_0^t f(\tau) \int_0^t (t-s)^{\beta-1}(s-\tau)^{k\beta-1} ds d\tau
\]

\[
= \left(\frac{C_2\Gamma(\beta)g(t)}{\Gamma(k\beta)\Gamma(\beta)}\right)^{k+1} \int_0^t f(\tau)(t-\tau)^{(k+1)\beta-1} \int_0^1 (1-w)^{\beta-1} w^{k\beta-1} dw d\tau
\]

\[
= \left(\frac{C_2\Gamma(\beta)g(t)}{\Gamma((k+1)\beta)}\right)^{k+1} \int_0^t f(\tau)(t-\tau)^{(k+1)\beta-1} d\tau,
\]

concluding the proof. \(\square\)

We have a different situation if $f(t)$ is a constant function, since we can control $B^k f$ by using the functions $u^*_k(t)$.

**Lemma 6.4.** For any $k \geq 1$ it holds

\[
(6.4) \quad B^k f(t) \leq (g(t))^k u^*_k(t)
\]

**Proof.** Let us first observe that $B^1 f(t) = g(t) V(t)$. Let us suppose Equation (6.4) holds for some $k \geq 1$. Thus, by Lemma 6.2 and the fact that $g(t)$ is non-decreasing, we have

\[
B^{k+1} f(t) = B(B^k f(t)) \leq B((g(t))^k u^*_k(t))(t) =
\]

\[
= g(t) \int_0^t u(t-s)(g(s))^k u^*_k(s) ds \leq (g(t))^{k+1} u^*_{k+1}(t),
\]

concluding the proof. \(\square\)

The power estimate given in Lemma 6.3 can be used first to prove the following result.

**Lemma 6.5.** Let $f \in L^1([0,T])$. Then:

- $\sum_{k=1}^{+\infty} B^k f(t)$ absolutely converges for any $t \in [0,T]$;
- $\lim_{k \to +\infty} B^k f(t) = 0$ uniformly in $[0,T]$;
- If $f \in L^\infty([0,T])$, then $\sum_{k=1}^{+\infty} B^k f(t)$ totally converges in $[0,T]$.

**Proof.** Let us observe that $|B^k f(t)| \leq B|f|(t)$ by definition of $B$. Moreover, by Lemma 6.2 we also have $-B|f|(t) \leq B f(t) \leq B|f|(t)$, hence, without loss of generality, we can consider $f \geq 0$. By Lemma 6.3 we have, for some $\varepsilon \in (0,\gamma)$, $C_2 > 0$ and $\beta = \gamma - \varepsilon$, choosing $k_0 \in \mathbb{N}$ such that $k_0\beta > 1$, we achieve

\[
\sum_{k=k_0}^{+\infty} B^k f(t) \leq \sum_{k=k_0}^{+\infty} \frac{(C_2\Gamma(\beta)g(t))^k}{\Gamma(k\beta)} \int_0^t (t-s)^{k\beta-1} f(s) ds
\]

\[
\leq \sum_{k=k_0}^{+\infty} \frac{(C_2\Gamma(\beta)g(t))^k}{\Gamma(k\beta)} T^{k\beta-1} \int_0^t f(s) ds
\]

\[
\leq \|f\|_{L^1([0,T])} \sum_{k=k_0}^{+\infty} \frac{(C_2\Gamma(\beta)g(T))^k}{\Gamma(k\beta)} T^{k\beta-1},
\]
where last inequality follows from the fact that \( f \in L^1([0, T]) \) and \( g \) is increasing. In particular we have that
\[
\sum_{k=k_0}^{\infty} \frac{(C_2 \Gamma(\beta)g(T))^k}{\Gamma(k\beta)} T^{k\beta-1} = (C_2 \Gamma(\beta)g(T))^{k_0} T^{k_0\beta-1} \sum_{k=0}^{\infty} \frac{(C_2 \Gamma(\beta)g(T)T^\beta)^k}{\Gamma(k\beta + k_0\beta)}
\]
\[
= (C_2 \Gamma(\beta)g(T))^{k_0} T^{k_0\beta-1} E_{\beta,k_0}(C_2 \Gamma(\beta)g(T)T^\beta).
\]
This is enough to prove that \( \sum_{k=0}^{\infty} B^k f(t) \) absolutely converges for any \( t \in [0, T] \).
Moreover, we have also that the series \( \sum_{k=k_0}^{\infty} B^k f(t) \) totally converges in \([0, T]\): this implies that it holds \( \lim_{k \to +\infty} B^k f(t) = 0 \) uniformly in \([0, T]\).

However, if \( f \in L^\infty([0, T]) \), we have for \( k < k_0 \)
\[
B^k f(t) \leq \frac{(C_2 \Gamma(\beta)g(T)T^\beta)^k}{\Gamma(k\beta + 1)} \|f\|_{L^\infty} \leq \frac{(C_2 \Gamma(\beta)g(T)T^\beta)^k}{\Gamma(k\beta + 1)} \|f\|_{L^\infty}
\]
obtaining the total convergence of the whole series. \( \square \)

Finally, let us show a last confrontation Lemma.

**Lemma 6.6.** For any non-negative increasing function \( f_1 \) and any non-negative function \( f_2 \) in \( L^1([0, T]) \) it holds
\[
B^k(f_1 f_2)(t) \leq f_1(t)B^k f_2(t)
\]
for any \( t \in [0, T] \).

**Proof.** Let us first observe that, being \( f_1 \) increasing
\[
B(f_1 f_2)(t) = g(t) \int_0^t u(t-s)f_1(s)f_2(s)ds \leq g(t)f_1(t) \int_0^t u(t-s)f_2(s)ds = f_1(t)B(f_2)(t).
\]
Now let us suppose Equation (6.5) holds for some \( k \geq 1 \). Then we have, by also using (6.6) and Lemma 6.2
\[
B^{k+1}(f_1 f_2)(t) = B(B^k(f_1 f_2))(t) \leq B(f_1 B^k f_2)(t) \leq f_1(t)B^{k+1} f_2(t),
\]
concluding the proof. \( \square \)

**6.2. Proof of Theorem 6.1**

**Proof.** Starting from the relation
\[
x(t) \leq a(t) + g(t) \mathcal{T}_t^x x(t)
\]
let us rewrite it as
\[
x(t) \leq a(t) + B x(t).
\]
Now, applying on both sides the operator \( B \) and recalling Lemma 6.2 we have
\[
B x(t) \leq B a(t) + B^2 x(t).
\]
Plugging last inequality in Equation (6.7) we have
\[
x(t) \leq a(t) + B a(t) + B^2 x(t).
\]
Now we want to show that
\[
x(t) \leq a(t) + \sum_{k=1}^{n-1} B^k a(t) + B^n x(t).
\]

Let us observe that (6.9) already holds for \( n = 2 \), since it coincides with Equation (6.8). Now let us suppose that Equation (6.9) holds for some \( n \geq 2 \). Then we can apply the operator \( B^n \) on both sides of Equation (6.7) to achieve
\[
B^n x(t) \leq B^n a(t) + B^{n+1} x(t),
\]
thus, plugging this relation in Equation (6.9), we have
\[
x(t) \leq a(t) + \sum_{k=1}^{n} B^k a(t) + B^{n+1} x(t).
\]
proving the claim.

Now let us observe that \( a \in L^1([0, T]) \) hence by Lemma 6.5 we know that \( \sum_{k=1}^{+\infty} B^k a(t) < +\infty \). Moreover, also \( x \in L^1([0, T]) \), hence, still by Lemma 6.5, \( \lim_{k \to +\infty} B^k x(t) = 0 \).

Thus, taking the limit as \( n \to +\infty \) in Equation (6.9) we obtain
\[
(6.10) \quad x(t) \leq a(t) + \sum_{k=1}^{+\infty} B^k a(t),
\]
proving the first part of the Theorem.

Concerning the second part, we have, by Lemma 6.3 and monotone convergence theorem, for any \( \varepsilon \in (0, \gamma) \), setting \( \beta = \gamma - \varepsilon \),
\[
\sum_{k=1}^{+\infty} B^k a(t) \leq \int_0^t \sum_{k=1}^{+\infty} \frac{(C_2 \Gamma(\beta) g(t))^k}{\Gamma(k\beta)} (t-s)^{k\beta-1} a(s) ds
\]
\[
= \int_0^t \sum_{k=0}^{+\infty} \frac{(C_2 \Gamma(\beta) g(t))^{k+1}}{\Gamma(k\beta + \beta)} (t-s)^{k\beta+\beta-1} a(s) ds
\]
\[
\leq C_2 \Gamma(\beta) g(t) \int_0^t E_{\beta,\beta}(C_2 \Gamma(\beta) g(T)(t-s))(t-s)^{\beta-1} a(s) ds.
\]
Then, plugging last inequality in (6.10) we obtain (6.2).

Concerning the third part of the Theorem, let us observe by Lemma 6.6 that
\[
B^k a(t) \leq a(t) B^k 1(t) \leq a(t)(g(T))^k u^*_k(t) \leq a(t)(g(T))^k u^*_k(t).
\]
Thus we get
\[
\sum_{k=0}^{+\infty} B^k a(t) \leq a(t) \left[ 1 + \sum_{k=1}^{+\infty} (g(T))^k u^*_k(t) \right]
\]
\[
\leq a(t) \left[ 1 + \sum_{k=1}^{+\infty} k(g(T))^k u^*_k(t) \right] = a(t) e_{\Phi}(t, (g(T))^k),
\]
where last equality follows from Theorem 5.4: this inequality completes the proof. \( \square \)

7. Continuity with respect to parameters

Since we now have a Gronwall-type inequality (that is to say Theorem 6.1), we can exploit some continuity properties of the solutions with respect to some parameters of the problem.
7.1. Continuous dependence on the initial datum. To work with continuous dependence on the initial datum, let us define, just for this subsection, the function

$$\Psi : f_0 \in \mathbb{R} \to (t \in [0, T(f_0)]) \to \Psi(t; f_0) \in \mathbb{R}$$

where, for fixed $f_0 \in \mathbb{R}$, $t \in [0, T] \mapsto \Psi(\cdot; f_0)$ is solution of the Cauchy problem

$$\begin{cases}
\partial_t^\alpha \Psi(t; f_0) = F(t, \Psi(t; f_0)) & t \in (0, T(f_0)) \\
\Psi(0; f_0) = f_0
\end{cases}$$

(7.1)

where $F$ and $\Phi$ satisfy the hypotheses of Theorem [4.2] and $T(f_0) = T_1$ as defined in the aforementioned Theorem. In particular we are stressing out the dependence of $T_1$ with respect to the initial datum $f_0$. Indeed, in Lemma [4.4] we have actually shown that $T_1$ depends on $R > 0$ such that $|f_0| < R$, thus we can just fix $R = |f_0| + 1$ to have that $T_1$ depends actually on $f_0$. Moreover, since $U$ is continuous, we are asking for

$$T_1 = U^\leftarrow \left(\frac{C_R}{R}\right).$$

Now let us consider $\tilde{f}_0 \in (f_0 - \delta, f_0 + \delta)$. If $\delta < 1$ we have that $|\tilde{f}_0| \leq |f_0| + \delta \leq R$. By using these observations, it is easy to prove the following Lemma

**Lemma 7.1.** Let $F$ and $\Phi$ satisfy the hypotheses of Theorem [4.2] and fix $R = |f_0| + 1$. Define $T_1 = U^\leftarrow \left(\frac{C_R}{R}\right)$. Then, for any $\delta \in (0, 1)$ and any $f_0 \in (f_0 - \delta, f_0 + \delta)$, the Cauchy problem

$$\begin{cases}
\partial_t^\alpha f(t) = F(t, f(t)) & t \in (0, T_1) \\
f(t) = \tilde{f}_0
\end{cases}$$

admits a unique solution in $\bigcap_{\varepsilon \in (0, \gamma)} C^\gamma - \varepsilon((0, T_1)).$

Thus we can use this Lemma to fix the time interval $J_1 = (0, T_1]$ as common interval for the existence of the solution for any initial datum sufficiently near $f_0$. We have then the following result.

**Proposition 7.2.** Fix $f_0 \in \mathbb{R}$ and $\delta \in (0, 1)$. The function $\tilde{f}_0 \in (f_0 - \delta, f_0 + \delta) \mapsto \Psi(\cdot; f_0) \in C^0((0, T(f_0)))$ is continuous in $f_0$.

Hence, in particular, since $f_0 \in \mathbb{R}$ is arbitrary, $\Psi(\cdot; f_0)$ is continuous with respect to $f_0 \in \mathbb{R}$.

**Proof.** Let us define the function $h(t; \tilde{f}_0) = |\Psi(t; \tilde{f}_0) - \Psi(t; f_0)|$. Since $\Psi(t; \tilde{f}_0)$ is solution of (7.1) (also by Lemma 7.1 which guarantees existence in $(0, T(f_0)]$ without changing the horizon $T$), then it is also solution of (4.3). We have

$$|\Psi(t; \tilde{f}_0) - \Psi(t; f_0)| \leq |\tilde{f}_0 - f_0| + \int_0^t u(t-s) \left| F(s, \Psi(s; \tilde{f}_0)) - F(s, \Psi(s; f_0)) \right| ds.$$

Moreover, we have, by Theorem 4.2 that $|\Psi(s; \tilde{f}_0)| \leq R$ and $|\Psi(s; f_0)| \leq R$, hence

$$|F(s, \Psi(s; \tilde{f}_0)) - F(s, \Psi(s; f_0))| \leq L_R h(s; \tilde{f}_0).$$

We obtain from Equation (7.2)

$$h(t; \tilde{f}_0) \leq |\tilde{f}_0 - f_0| + L_R \int_0^t u(t-s)h(s)ds,$$

hence, by Theorem 6.1 and the fact that $\varepsilon_R(t; \lambda)$ is increasing when $\lambda > 0$

$$h(t; \tilde{f}_0) \leq |\tilde{f}_0 - f_0| \varepsilon_R(t; L_R) \leq |\tilde{f}_0 - f_0| \varepsilon_R(T(f_0); L_R).$$
This completes the proof. □

7.2. Continuous dependence on a parameter. Now let us consider a function 
\( F : [0, T] \times \mathbb{R} \times U \to \mathbb{R} \) where \((V,d)\) is a locally compact metric space. Consider also \( \Phi \in \mathcal{BF} \). Let us suppose that \( \Phi \) and, for any \( v \in (V,d), \ (t,x) \in [0,T] \times \mathbb{R} \mapsto F(t,x,v) \) satisfy the hypotheses of Theorem 7.2. Moreover, suppose that \( v \in V \mapsto F(t,x,v) \) is locally Lipschitz uniformly with respect to \((t,x) \in [0,T] \times [-R,R] \) for any fixed \( R > 0 \).

Define then the function \( \Psi : v \in (V,d) \mapsto (t \in [0,T(v)]) \mapsto \Psi(t;v) \in \mathbb{R} \) where for fixed \( v \in (V,d) \), the function \( t \mapsto \Psi(t;v) \) is solution of

\[
\begin{align*}
\partial_t^\Phi \Psi(t;v) &= F(t, \Psi(t;v);v) & t \in (0,T(v)) \\
\Psi(0;v) &= f_0.
\end{align*}
\]

(7.3)

As before, we need, for a \( v_0 \in V \) fixed and a ball \( B_r(v_0) \) in \( V \), to fix a constant \( T \) (eventually depending only on \( v_0 \)) such that for every \( v \in B_r(v_0) \) the solution exists and is unique up to \( T \). To do this, let us fix \( R = |f_0| + 1 \) and \( x \in [-R,R] \).

Then we have

\[
|F(t,x,v)| \leq |F(t,x,v_0)| + |F(t,x,v) - F(t,x,v_0)| \leq C_R(v_0) + |F(t,x,v) - F(t,x,v_0)|.
\]

Now let us recall that \((V,d)\) is locally compact hence \( v_0 \) admits a neighbourhood \( V_0 \) such that \( V_0 \) is compact. Let us then consider a radius \( r > 0 \) so small that \( B_r(v_0) \subseteq V_0 \) and consider \( v \in B_r(v_0) \). Since \( F \) is locally Lipschitz, then there exists a constant \( L_r(R) \) such that

\[
|F(t,x,v)| \leq |F(t,x,v_0)| + |F(t,x,v) - F(t,x,v_0)| \leq C_R(v_0) + L_r d(v,v_0) \leq C_R(v_0) + r L_r(R).
\]

Hence we can set \( \bar{C}_R = C_R(v_0) + r L_r(R) \) to obtain

\[
|F(t,x,v)| \leq \bar{C}_R
\]

for any \( t \in [0,T] \), \( x \in [-R,R] \) and \( v \in B_r(v_0) \). Thus, arguing as we did for Lemma 7.1, we have the following Lemma.

**Lemma 7.3.** Let \((V,d)\) be a locally compact metric space, \( F : [0,T] \times \mathbb{R} \times V \to \mathbb{R} \) and \( \Phi \in \mathcal{BF} \). Suppose that \( \Phi \) and, for any fixed \( v \in V \), \( (t,x) \in [0,T] \times \mathbb{R} \mapsto F(t,x,v) \) satisfy the hypotheses of Theorem 7.2. Moreover, suppose that \( v \in V \mapsto F(t,x,v) \) is locally Lipschitz uniformly with respect to \((t,x) \in [0,T] \times [-R,R] \) for any fixed \( R > 0 \). Fix \( R = |f_0| + 1 \). Then there exists a constant \( r > 0 \) such that there exists a neighbourhood \( V_0 \) of \( v_0 \) with compact closure, \( B_r(v_0) \subseteq V_0 \) and defining \( L_r(R) \) such that

\[
|F(t,x,v) - F(t,x,v_0)| \leq L_r(R) d(v,v_0)
\]

for any \( t \in [0,T] \), \( x \in [-R,R] \) and \( v \in B_r(v_0) \), and \( T_1 = U^{\psi}\left(\frac{C_R + r L_r(R)}{R^{-\xi}}\right) \), then, for any \( v \in B_r(v_0) \), the Cauchy problem

\[
\begin{align*}
\partial_t^\Phi f(t) &= F(t, f(t); v) & t \in (0,T_1) \\
f(t) &= f_0
\end{align*}
\]

admits a unique solution in \( \bigcap_{\epsilon \in (0,\gamma)} C^{\gamma-\delta}((0,T_1)) \).

Now that we have a way to fix a common time interval, let us show the following continuity result.
Proposition 7.4. Let \((V, d)\) be a locally compact metric space, \(F : [0, T] \times \mathbb{R} \times V \to \mathbb{R}\) and \(\Phi \in \mathcal{BF}_d\). Suppose that \(\Phi\) and, for any fixed \(v \in V\), \((t, x) \in [0, T] \times \mathbb{R} \mapsto F(t, x; v) \in \mathbb{R}\) satisfy the hypotheses of Theorem 4.2. Moreover, suppose that \(v \mapsto F(t, x; v)\) is locally Lipschitz uniformly with respect to \((t, x) \in [0, T] \times [-R, R]\) for any fixed \(R > 0\). Then the function \(v \mapsto \Psi(\cdot; v)\) is continuous with respect to \(v \in V\).

Proof. Let us fix \(v_0 \in V\) and \(R = |f_0| + 1\). Then, by Lemma 7.3 there exists \(r > 0\) such that for any \(v \in B_r(v_0)\) the function \(\Psi(t; v)\) is well defined up to \(T_1 = U\left(\frac{C_0 + rL_r(R)}{R}\right)\) where \(L_r(R) > 0\) is such that \(|F(t, x; v) - F(t, x; v_0)| \leq L_r(R)d(v, v_0)\) for any \(v \in B_r(v_0), x \in [-R, R]\) and \(t \in [0, T]\).

Let us define the function

\[
h(t; v) = |\Psi(t; v) - \Psi(t; v_0)|.
\]

Since for any fixed \(v \in V\), \(\Psi(t; v)\) is solution of (7.3) up to the time horizon \(T_1\), we have

\[
h(t; v) \leq \int_0^t u(t-s)|F(s, \Psi(s; v); v) - F(s, \Psi(s; v_0); v_0)|ds.
\]

Now let us observe that, since \(\Psi(s; v)\) and \(\Psi(s; v_0)\) belong to \([-R, R]\) for any \(s \in [0, T]\) and \(v \in B_r(v_0)\),

\[
|F(s, \Psi(s; v); v) - F(s, \Psi(s; v_0); v_0)| \leq |F(s, \Psi(s; v); v) - F(s, \Psi(s; v_0); v_0)|
+ |F(s, \Psi(s; v_0); v_0) - F(s, \Psi(s; v_0); v_0)|
\leq L_r(R)d(v, v_0) + L_R h(s; v)
\]

Thus in particular we have

\[
h(t; v) \leq L_r(R)d(v, v_0)U(t) + L_R \int_0^t u(t-s)h(s; v)ds.
\]

Recalling that \(U(t)\) is a non-negative and non-decreasing function, we finally achieve, by Theorem 6.1,

\[
h(t; v) \leq L_r(R)U(t)d(v, v_0) \epsilon_\Phi(t; L_R)
\leq L_r(R)U(T)d(v, v_0) \epsilon_\Phi(T; L_R),
\]

concluding the proof. \(\square\)

7.3. A bound on the distance of the solutions. Let us now consider two solutions \(f_1, f_2\) of the Cauchy problems

\[
\begin{cases}
\theta_t^\Phi f_i(t) = F_i(t, f_i(t)) & t \in (0, T] \\
f_i(0) = f_i^0
\end{cases}
\]

where \(\Phi\) and \(F_i\) satisfy the hypotheses of Theorem 4.2. We want to show that if we can bound the distance between \(F_1\) and \(F_2\), then we can control also the distance between the solutions in terms of \(|f_0^1 - f_0^2|\). We have indeed the following proposition.

Proposition 7.5. Let \(F_i\) (\(i = 1, 2\)) and \(\Phi\) satisfy the hypotheses of Theorem 4.2 and let \(f_i\) be the solutions of the Cauchy problems (7.4). Moreover, suppose that for any \(R > 0\) there exists \(M_R > 0\) such that

\[
|F_1(t, x) - F_2(t, y)| \leq M_R \forall t \in [0, T], \forall x, y \in [-R, R].
\]
Let $T_0 > 0$ such that $f_i$ is well-defined in $[0, T_0)$ and set $T_* = \min\{T_1, T_2\}$. Then we have

$$\|f_1 - f_2\|_{C^0([0, T_*])} \leq (|f_0^1 - f_0^2| + M_R U(T)) \epsilon_\Psi(T, L_R).$$

**Proof.** Fix $R = \max\{|f_0^1|, |f_0^2|\} + 1$. Let us define $h(t) = |f_1(t) - f_2(t)|$ that is well-defined in $[0, T]$ where $T_* = \min\{T_1, T_2\}$. Since $f_i$ is solution of (7.4) we have

$$h(t) \leq |f_0^1 - f_0^2| + \int_0^t u(t-s)|F_1(s, f_1(s)) - F_2(s, f_2(s))|ds.$$

Now let us observe that

$$|F_1(s, f_1(s)) - F_2(s, f_2(s))| \leq |F_1(s, f_1(s)) - F_1(s, f_2(s))|$$

$$+ |F_1(s, f_2(s)) - F_2(s, f_2(s))|$$

$$\leq L_R h(s) + M_R$$

and then

$$h(t) \leq |f_0^1 - f_0^2| + M_R U(t) + L_R \int_0^t u(t-s)h(s)ds.$$

Hence, by Theorem 6.1 we have

$$h(t) \leq (|f_0^1 - f_0^2| + M_R U(t)) \epsilon_\Psi(t; L_R)$$

$$\leq (|f_0^1 - f_0^2| + M_R U(T)) \epsilon_\Psi(T; L_R).$$

Taking the supremum for $t \in [0, T_*]$ we conclude the proof. \qed

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