QUANTUM THEORY OF SCALAR FIELD IN ISOTROPIC WORLD

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Fock representations are constructed for a free scalar field in the closed and quasi-Euclidean isotropic cosmological models. Invariance of their cyclic vector (vacuum) under isometries and the correspondence principle single out a class of unitarily equivalent representations.

1. Introduction

There is a certain hope that an analysis of applicability of the basic notions and methods of quantum field theory (QFT) in a space-time manifold which differs from the Minkowski world can shed light upon some problems of this theory as a whole. On the other hand, the idea has been repeatedly expressed [1–3] that the notion of a free particle, which is essentially non-pseudo-Euclidean, and, as long as QFT is adequate to the dynamics of elementary processes, it is natural to seek manifestations of a fundamental relationship (if any) between cosmology and the micro-world in natural to seek manifestations of a fundamental relationship (if any) between cosmology and the micro-world in the QFT structure.

It is from these positions that, as the first step, a study of quantum theory of a free scalar field in the de Sitter world was undertaken [4, 5]. It was shown there, in particular, that the notion of a free particle, which is commonly related to irreducible representations of the Poincaré group, can be related to the de Sitter group in a similar manner only if one invokes some particular formulation of the correspondence principle. In the present paper we make an attempt to extend the methods and results of Ref. [5] to the general case of isotropic models of the Universe. We consider in detail closed models [the group of isometries $O(4)$] and at the end give the modifications of the main results for quasi-Euclidean models [the group of isometries $O(3) \times T_3$].

A closed isotropic Universe $F_4$ may be geometrically reproduced as the rotation hypersurface

\[ x^a x^a = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \rho^2 = f^2 \left( \frac{x^0}{\rho} \right), \]

\[ f > 0, \quad |f'| < 1 \quad (1.1) \]

in the five-dimensional pseudo-Euclidean space-time $E_5$ with the metric

\[ ds^2 = (dx^0)^2 - dx^a dx^a. \]

It is convenient to choose the parameter $\rho$ in such a way that $f(0) = 1$.

In the coordinates $y^\mu$ ($y^0 \equiv \eta$) determined by the relations

\[ x^0 = \rho \int_0^\eta \sqrt{b^2(\eta') + \dot{b}(\eta')} \, d\eta', \]
\[ x^a = \rho b(\eta) n^a(y^1, y^2, y^3); \quad n^n n^a = 1, \]

where $b(\eta) = f(x^0/\rho)$ and $\dot{b}(\eta) = db/d\eta$, the metric of $F_4$ has the form

\[ ds^2 = g_{\mu\nu} dy^\mu dy^\nu = \rho^2 b^2(\eta) (dy^2 - h_{ij} dy^i dy^j) \quad (1.2) \]

where $h_{ij} = \partial_i n^a \partial_j n^a$, $\partial_i = \partial/\partial y^i$, Greek indices take the values from 0 to 3, Latin ones $i, j, k$ from 1 to 3 and $a, b$ from 1 to 4. If the metric is specified in the form (1.2), then one can obtain Eq. (1.1) by putting

\[ \eta = \int_0^{x^0/\rho} \sqrt{1 - |f'(\alpha)|^2} \frac{d\alpha}{f(\alpha)}. \]
De Sitter space corresponds to $b(\eta) = 1/\cos \eta$; for the Friedmann models with $p = \varepsilon/3$ and $p = 0$ ($\varepsilon =$ energy density and $p =$ pressure) one has $b(\eta) = \cos \eta$ and $b(\eta) = \cos^2(\eta/2)$, respectively [6].

2. The field equation

As follows from [7] and [5], the scalar field equation in a Riemannian space has the form

$$\Box + m^2 + R/6)\phi = 0 \quad (c = \hbar = 1),$$

(2.1)

where $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, $R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}$ is the Ricci tensor, and the sign of the curvature tensor is chosen so that

$$\nabla_\mu \nabla_\nu A_\sigma - \nabla_\nu \nabla_\mu A_\sigma = R^{\alpha}_{\sigma\mu\nu} A_\alpha$$

for any vector $A_\alpha$; $\Box = \nabla^\alpha \nabla_\alpha$ is the d’Alembert operator, and $\nabla_\alpha$ is a covariant derivative.

Eq. (2.1) is obtained by variation with respect to $A$ of the following action integral:

$$A = \int \sqrt{-g} d^4y$$

$$= \frac{1}{2} \int \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \left( m^2 + \frac{R}{6} \right) \phi^2 \right] \sqrt{-g} d^4y,$$

(2.2)

where $d^4y = dy^0 dy^1 dy^2 dy^3$. Varying the integral (2.2) with respect to $g^{\mu\nu}$, we obtain the (metric) energymomentum tensor (EMT)

$$T_{\mu\nu} = T_{\mu\nu}^{\text{can}} - \frac{1}{6} (R_{\mu\nu} + \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \phi^2,$$

(2.3)

where $T_{\mu\nu}^{\text{can}}$ is the canonical EMT,

$$T_{\mu\nu}^{\text{can}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} L.$$ The tensor (2.3) possesses the properties

$$T_{\mu\nu} = T_{\nu\mu}, \quad T^\alpha_\alpha = m^2 \phi^2, \quad \nabla^\sigma T_{\sigma\mu} = 0.$$

Let us pass to quantum field theory by specifying commutation relations on a certain spacelike hypersurface $\Sigma$ in $F_0$:

$$[\phi(M), \phi(M')] = 0, \quad [\partial_\mu \phi(M), \partial_\nu \phi(M')] = 0,$$

$$\int_{\Sigma} [\phi(M), \partial_\mu \phi(M')] f(M') d^4 \sigma^\mu(M') = i f(M'),$$

(2.4)

where $M, M' \in \Sigma$, $f(M)$ is an arbitrary function and $d\sigma^\mu(M)$ is the vector area element on $\Sigma$. In the Heisenberg picture (which is fixed by the choice of $\Sigma$), the operator $\phi$ satisfies Eq. (2.1) and the initial data (2.4). Let us choose as $\Sigma$ the surface $\eta = 0$, then $d\sigma^\mu = \delta^{0\mu} \rho^2 \sqrt{3} d^3 y$.

In $F_0$, due to (1.2), Eq. (2.1) has the form

$$\left[ \frac{1}{b^2} \frac{\partial^2}{\partial \eta^2} + \frac{b^2}{b} - \Delta + m^2 \rho^2 + \frac{b^2}{b} \right] \phi = 0,$$

where $\Delta = h^{-1/2} \partial_i \left( h^{1/2} h^{ik} \partial_k \right)$ is the Laplace operator on the sphere $n^a n^a = 1$. Separating the variables, we obtain $\phi$ as an expansion in harmonic polynomials on the sphere $n^a n^a = 1$:

$$\phi(\eta, y) = \frac{1}{b(\eta)} \sum_{s=0}^{\infty} \sum_{\sigma=1}^{\infty} u_{s\sigma}(\eta) P^{s\sigma}(y),$$

(2.5)

$$[\Delta + s(s+2)] P^{s\sigma}(y) = 0,$$

$$P^{s\sigma}(y) = \frac{\sqrt{2s(s+1)}}{\pi^{1/2}} P^{s\sigma}_{a_1 \ldots a_s} n^{a_1} \ldots n^{a_s},$$

(2.6)

where the tensors $P^{s\sigma}_{a_1 \ldots a_s} n^{a_1} \ldots n^{a_s}$, which are symmetric over all $a_i$ and have a zero trace over a pair of indices, are orthonormalized:

$$P^{s\sigma}_{a_1 \ldots a_s} P^{s\sigma'}_{a_1 \ldots a_s} = \delta^{s\sigma\sigma'}, \quad \sigma, \sigma' = 1, \ldots, (s+1)^2.$$

Here and henceforth $y = (y^1, y^2, y^3)$. The operator $u_{s\sigma}(\eta)$ evidently obeys the equation

$$\ddot{u} + [(s+1)^2 + m^2 \rho^2 b^2(\eta)] u = 0.$$ (2.7)

Let us introduce its two linearly independent solutions $u^{\pm}$, specifying them by the initial conditions

$$u^+_s(0) = \frac{1}{\sqrt{s_0}}, \quad \ddot{u}^+_s(0) = \pm \sqrt{s_0},$$

$$s_0 = \sqrt{(s+1)^2 + m^2 \rho^2}.$$ (2.8)

The Cauchy problem (2.7), (2.8) is equivalent to the Volterra integral equation

$$w^\pm(s_0, \eta) = \frac{1}{\sqrt{s_0}} e^{\pm is_0 \eta}$$

(2.10)

are solutions of the equation

$$\ddot{w} + s_0^2 w = 0$$

(2.11)

under the initial conditions (2.8); the function

$$g(\eta, \eta') = -\text{sign}(\eta - \eta')$$

$$\times \begin{vmatrix} w_1(\eta') & w_2(\eta') \\ \dot{w}_1(\eta') & \dot{w}_2(\eta') \end{vmatrix}^{-1} \begin{vmatrix} w_1(\eta) & w_2(\eta) \\ \dot{w}_1(\eta) & \dot{w}_2(\eta) \end{vmatrix}$$

$$= \frac{\text{sign}(\eta - \eta')}{2s_0} \sin[s_0(\eta - \eta')]$$

(2.12)

is a fundamental solution of Eq. (2.11); $w_1$ and $w_2$ are its any two linearly independent solutions. Evidently,

$$\left( u^+_s(\eta) \right)^* = u^-_s(\eta),$$

$$W(u^+_s, u^-_s) = u^+_s \dot{u}^-_s - u^-_s \dot{u}^+_s = -2i.$$

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It follows from Eq. (2.9) that the function $u^{±}_s(η)$ may be approximated with any precision by members of the uniformly convergent sequence of iterations:

\[ u^{±}_0(η) = w^{±}(s, η), \]

\[ u^{±}_n(η) = w^{±}(s, η), \]

\[ u^{±}_{n+1}(η) = w^{±}(s, η) \]

\[ + 2m^2ρ^2 \int_0^η [1 - b^2(η)]g(η, η')u^{±}_n(η') dη'. \]  

(2.12)

This method is convenient for finding asymptotic expressions for $s \gg mρ$. It is, in particular, easy to verify that

\[ u^+_s u^-_s = \frac{1}{s+1} \left[ 1 - \frac{m^2ρ^2b^2}{2(s+1)^2} + o \left( \frac{m^2ρ^2}{s^2} \right) \right]. \]  

(2.13)

It is clear that for $n = 0$

\[ u^+_s(η) = w^+_s(s+1, η) = \frac{1}{\sqrt{s+1}} e^{i\pm i(s+1)η}. \]

The operator $u_{σσ}(η)$ in (2.5) may be presented in the form

\[ u_{σσ}(η) = \frac{\sqrt{so}}{2} q_{σσ}(u^+_s + u^-_s) + \frac{i}{2\sqrt{so}} p_{σσ}(u^-_s - u^+_s), \]  

(2.14)

where $q_{σσ} = u_{σσ}(0)$ and $p_{σσ} = \dot{u}_{σσ}(0)$. Owing to (2.5) and to the completeness of the system of harmonic polynomials on the sphere $n^σ n^σ = 1$, one has

\[ u_{σσ}(η) = \rho(η) \int \phi(η, y) P^{σσ}(y) dσ(y), \]

\[ \dot{u}_{σσ}(η) = \rho(η) \int \left[ \partial_0 φ(η, y) + \frac{\dot{b}}{b} φ(η, y) \right] P^{σσ}(y) dσ. \]

It follows from (2.4) that

\[ [q_{σσ}, q_{σ's'}] = 0, \quad [p_{σσ}, p_{σ's'}] = 0, \]

\[ [q_{σσ}, p_{σ's'}] = i δ_{σσ'} δ_{σ's'}. \]  

(2.15)

Using these relations, the expressions (2.5) and (2.14) and the harmonic polynomials addition theorem [8], one can present the commutator of field operators $φ(M)$ taken at two arbitrary points $M_1$ and $M_2$ in $F_4$ as the following series:

\[ D(η_1, y_1; η_2, y_2) = i[φ(M_1), φ(M_2)] \]

\[ = \frac{i}{2b(η_1)b(η_2)} \sum_{s=0}^{∞} \frac{s+1}{2σ^2} C^*_s \left( n^σ(y_1)n^σ(y_2) \right) D_s(η_1, η_2), \]

where $C^*_s$ is a Gegenbauer polynomial and

\[ D_s(η_1, η_2) = \frac{1}{i} \left[ u^-_s(η_1)u^+_s(η_2) - u^+_s(η_1)u^-_s(η_2) \right]. \]

It is well known that knowledge of the commutation function $D$ makes it possible to solve the Cauchy problem for Eq. (2.1) with initial data on an arbitrary surface $Σ$. It is of interest to note that, according to [5], the function $D$ in de Sitter space in case $m = 0$ is concentrated on the light cone. On the other hand, any $F_4$ is conformal to de Sitter space, and, since conformal mappings preserve the light cone, one can evidently assert that $D$ for $m = 0$ is also concentrated on the light cone in any $F_4$. This property, which is natural for a field with zero rest mass, will only take place if the field equation is chosen in the form (2.1).

3. Fock representations

The next step in quantization is to construct a representation of the commutation relations. To this end, we introduce the operators [9]

\[ z^±_{σσ} = \frac{i}{\sqrt{2}} \left( p_{σσ} - \sum_{t, τ} T_{σσ, tτ} q_{tτ} \right), \quad z^±_{σσ} = (s^σ)^*, \]  

(3.1)

where $T = S + iQ$ is an arbitrary symmetric matrix $(T_{σσ, tτ} = T_{tτ, σσ})$ with a positive-definite imaginary part $Q$. From (2.15) it follows

\[ [z^+_σ, z^-_τ] = 0, \quad [z^-_σ, z^+_τ] = Q_{σσ, tτ}. \]

Reversing the equality (3.1), we have

\[ q_{σσ} = \frac{1}{\sqrt{2}} (z^-_{σσ} + z^+_σ), \]

\[ p_{σσ} = \frac{1}{\sqrt{2}} (T^*_{σσ, tτ} z^-_{tτ} + T_{σσ, tτ} z^+_t), \]  

(3.2)

where

\[ z^±_{σσ} = \sum_{tτ} Q_{σσ, tτ} z^±_{tτ}. \]

The state vector $|0\rangle$, defined by the relations

\[ z^-_{σσ}|0\rangle = 0, \quad \langle 0|0\rangle = 1, \]

will be called a quasi-vacuum. For each $T$, the states

\[ |s_1σ_1 ... s_Nσ_N\rangle = z^+_{s_1σ_1} ... z^+_{s_Nσ_N}|0\rangle \]

($N$-quasiparticulate states) form a basis of a certain Fock representation. Further, we will seek among these representations the one in which $|s_1σ_1 ... s_Nσ_N\rangle$ might be regarded as a state with a certain number $N$ of particles.

4. Conserved quantities and invariant quasi-vacuum states

It is, above all, evident that the quasi-vacuum of such a representation should be invariant under the group of isometries of the space-time $F_4$. It is, as is easily seen, the group $O(4)$ with the six generators

\[ Z = i \zeta^α_α ∂_α, \quad \zeta^α_α = δ^α_β (n^β\partial_α n^α - n^α\partial_β n^β) h^{ij}, \]  

(3.3)
corresponding to rotations in the \((ab)\) planes of the embedding space \(E_5\). Each Killing vector \(\zeta^\alpha\) determines a conserved (i.e., independent of the choice of the space-like hypersurface \(\Sigma\)) quantity

\[
M_{(ab)} = \int_\Sigma (\zeta^\alpha T_{\alpha\beta} d\sigma^\beta).
\]

Calculating these integrals according to [5], we obtain

\[
M_{(ab)} = \sum_{s,\sigma,\tau} (s+1) g_{s+1,\sigma} \mathcal{P}_{(ab)}^{\sigma\tau} p_{s+1,\tau},
\]

where \(\mathcal{P}_{(ab)}^{\sigma\tau}\) are antisymmetric matrices with the elements

\[
\mathcal{P}_{(ab)}^{\sigma\tau} = P_{\alpha_1...\alpha_s b_1...b_s} - P_{\alpha_1...\alpha_s a_1...a_s} P_{a_1...a_s b_1...b_s}.
\]

The requirement of invariance of a quasi-vacuum under isometries, expressed by the condition

\[
M_{(ab)} |0\rangle = \mu_{(ab)} |0\rangle,
\]

where the \(\mu\)'s are \(c\)-numbers, leads to the equalities

\[
T_{s,\sigma,\tau} = \delta_{s\sigma} \delta_{s\tau} T_s, \quad \mu_{(ab)} = 0,
\]

with \(T_s = S_s + iQ_s\), \(Q_s > 0\) being arbitrary numbers.

Let us introduce new parameters \(\lambda_s\) instead of \(T_s\):

\[
T_s = i s_0 \frac{1 - \lambda_s}{1 + \lambda_s}, \quad |\lambda_s| < 1,
\]

and form the new operators

\[
c^-_{s\sigma} = \frac{1 + \lambda_s}{\sqrt{1 - |\lambda_s|^2}} c_{s\sigma}^- \quad c^+_{s\sigma} = (c^-_{s\sigma})^*,
\]

obeying the canonical computation relations

\[
[c^+_{s\sigma}, c^-_{t\tau}] = 0, \quad [c^-_{s\sigma}, c^+_{t\tau}] = \delta_{s\sigma} \delta_{t\tau}.
\]

It is natural to call them quasi-particle creation and annihilation operators. The expressions (3.2) are considerably simplified:

\[
q_{s\sigma} = \frac{1}{\sqrt{2 s_0 (1 + |\lambda_s|^2)}} [(1 - \lambda_s) c^-_{s\sigma} + (1 + \lambda_s) c^+_{s\sigma}],
\]

\[
p_{s\sigma} = -i \sqrt{\frac{s_0}{2}} \frac{1}{\sqrt{1 - |\lambda_s|^2}} [(1 - \lambda_s) c^-_{s\sigma} - (1 + \lambda_s) c^+_{s\sigma}],
\]

and we finally obtain for \(M_{(ab)}\):

\[
M_{(ab)} = \sum_{s,\sigma,\tau} s c^+_{s\sigma} \mathcal{P}_{(ab)}^{\sigma\tau} c^-_{s\sigma}.
\]

Thus the requirement that the quasi-vacuum should be invariant under isometries selects a class of Fock representations, each of them being specified by a certain choice of the sequence of parameters \(\{\lambda_s\}\).

Furthermore, \(F_4\) admits conformal transformations, among which it is sufficient for us to consider the one-parameter subgroup determined by the generator \(Z_0 = \zeta^\alpha \partial_{\alpha}\), where the vector \(\zeta^\alpha = 0\) satisfies the generalized Killing equation

\[
\nabla_\mu \zeta^\nu + n_\nu \zeta_\mu = 2 b^2 \delta_{\mu\nu}.
\]

Since \(T_s = 0\) for the massless scalar field, \(m = 0\), the integral

\[
M_{(0)} = \int_\Sigma (\zeta^\alpha T_{\alpha\beta} d\sigma^\beta)
\]

is conserved. The quasi-vacuum is invariant under these transformations,

\[
M_{(0)} |0\rangle = \mu_{(0)} |0\rangle, \quad (4.1)
\]

only if \(\lambda_s = 0\). Thus for \(m = 0\) we obtain a single Fock representation with an invariant quasi-vacuum, which therefore may be called the vacuum. In this case,

\[
M_{(0)} = \frac{1}{2} \sum_{s,\sigma} (s + 1) (c^+_{s\sigma} c^-_{s\sigma} + c^+_{\sigma,s} c^-_{\sigma,s}).
\]

Lastly, in the general case, the relations (2.5), (2.14) and (4.1) lead to the following expression for the field operator:

\[
\phi(\eta,y) = \frac{1}{\sqrt{2b(y)}} \sum_{s=0}^{\infty} \sum_{\sigma=1} P_{s\sigma}(y) \times \{u_s(\eta)c^+_{s\sigma} + u^*_s(\eta)c^-_{s\sigma}\}, \quad (4.2)
\]

where

\[
u_s(\eta) = \frac{u^+_s + \lambda_s u^-_s}{\sqrt{1 - |\lambda_s|^2}}.
\]

5. The correspondence principle: the quasi-classical limit

Now we will consider the restrictions on the sequence \(\{\lambda_s\}\) following from the correspondence principle in the following formulation. If the amplitudes

\[
\psi_{s\sigma}(\eta,y) = \langle \phi(\eta,y)c^+_{s\sigma}|0\rangle = \frac{u^*_s(\eta) P_{s\sigma}(y)}{\sqrt{2b(y)}} \quad (5.1)
\]

are relativistic wave functions of a particle, they must be quasi-classical at large \(s\). Of course, this is only a necessary condition following from the interpretation of the quantum number \(s^2\) as the spatial momentum squared [5]. It means that at large \(s\) there must be such functions among \(\psi_s = \sqrt{\alpha} e^{iS}\) that \(\alpha = |\psi_s|^2\) and \(S = \arg \psi_s\) satisfy the Hamilton-Jacobi equation

\[
\delta^\alpha \partial_{\alpha}\delta_{\alpha} S = m^2
\]
and the continuity equation
\[ g^{\mu \nu} \nabla_\mu (\alpha \partial_\nu S) = 0 \]
(see more details in [5]). Separating the variables,
\[ \alpha = A(\eta)C(y), \quad S = T(\eta) + U(y), \]
we obtain the following four equations for the functions \( A, C, T, U \):
\[ \dot{T}^2 = m^2 \rho^2 b^2 + \kappa^2; \quad \left( \frac{\dot{A}}{A} + 2 \frac{\dot{b}}{b} \right) \dot{T} + \ddot{T} = \xi; \quad (5.2) \]
\[ h^{ij} \partial_i U \partial_j U = \kappa^2, \quad \frac{1}{C} h^{ij} \partial_i C \partial_j C + \Delta U = \xi, \quad (5.3) \]
where \( \kappa \) and \( \xi \) are real separation constants.

Let us first consider Eqs. (5.2). Since for the function (5.1)
\[ \dot{T} = -\frac{1}{|u_s|^2}, \quad A = \frac{|u_s|^2}{b^2 \rho^2}, \]
the second equation in (5.2) is satisfied if one puts \( \xi = 0 \). Let us write down the first equation using the asymptotic expression (2.13):
\[ \frac{(1 - |\lambda_s|^2) (s + 1)}{1 + |\lambda_s|^2 + 2 \left( 1 + \frac{m^2 \rho^2 b^2}{2(s+1)^2} + o\left( \frac{1}{s^2} \right) \right) \Re \left[ \lambda_s e^{-2i\lambda_s \eta (1 + o(1))} \right] }^2 = \kappa^2 + (m \rho b)^2. \quad (5.4) \]
Neglecting all terms of orders smaller than \( (s + 1)^2 \), we obtain
\[ \kappa^2 = (s + 1)^2 [1 + o(1)], \quad \lambda_s \to 0 \quad as \quad s \to \infty. \]
Therefore the l.h.s. of (5.4) may be expanded in powers of \( \lambda_s \):
\[ (s + 1)^2 \left( 1 + \frac{m^2 \rho^2 b^2}{2(s+1)^2} + o\left( \frac{1}{s^2} \right) \right) - 4 \left( 1 + \frac{m^2 \rho^2 b^2}{2(s+1)^2} + o\left( \frac{1}{s^2} \right) \right) \Re \left[ \lambda_s e^{-2i\lambda_s \eta (1 + o(1))} \right] + o(\lambda) \]
\[ = (s + 1)^2 [1 + o(1)] + m^2 \rho^2 b^2. \]
This equation is satisfied only if \( \kappa^2 = (s + 1)^2 + o(1) \) and
\[ \lim_{s \to \infty} (s + 1)^2 \lambda_s = 0. \quad (5.5) \]
One can also easily verify that there exist harmonic polynomials which exactly satisfy Eqs. (5.3) with \( \kappa^2 = (s + 1)^2 \) and \( \xi = 0 \).

Thus the correspondence principle formulated here selects among the Fock representations with \( O(4) \)-invariant quasi-vacua such representations that the sequences \( \{ \lambda_s \} \) characterizing them converge to zero faster than \( \{(s + 1)^{-2}\} \).

Let us note that these selected representations are unitarily equivalent to each other. To prove that, it is sufficient to establish their unitary equivalence to the representation where all \( \lambda_s = 0 \). In quite a similar manner to the corresponding calculation in [5], it can be shown that this takes place if the product
\[ \prod_{s=0}^\infty (1 - |\lambda_s|)^{-(s+1)^2/2} \]
converges. For its convergence, it is sufficient that
\[ |\lambda_s| \leq \text{const} \cdot s^{-(3/2+\epsilon)} \]
where \( \epsilon > 0 \) is an arbitrary number, and the set of sequences \( \{ \lambda_s \} \), selected by the correspondence principle, manifestly satisfies this requirement.
6. The limit of Minkowski space

Evidently, we should also require that as \( \rho \to \infty \), when the space-time curvature vanishes, the expression (4.3) for the field operator should turn into the usual decomposition into positive- and negative-frequency exponentials.

To consider this limiting transition, we introduce the coordinates

\[
t = \rho \eta, \quad y^1 = \frac{r}{\rho}, \quad y^2 = \theta, \quad y^3 = \chi,
\]

so that

\[
n^1 = \sin \frac{r}{\rho} \sin \theta \cos \chi, \quad n^3 = \sin \frac{r}{\rho} \cos \theta, \\
n^2 = \sin \frac{r}{\rho} \sin \theta \sin \chi, \quad n^4 = \cos \frac{r}{\rho}
\]

(6.1)

Then

\[
ds^2 = b^2 \left( \frac{t}{\rho} \right) \left[ dt^2 - dr^2 - \rho^2 \sin^2 \frac{r}{\rho} (d\theta^2 + \sin^2 \theta d\chi^2) \right],
\]

which turns into the Minkowski metric in spherical coordinates in the limit \( \rho \to \infty \):

\[
ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\chi^2).
\]

The expansion (4.3) can be represented in these coordinates in the form

\[
\phi(\eta, y) = \frac{1}{\sqrt{2b(\eta)\rho}} \sum_{s=0}^{\infty} \sum_{l=0}^{s} \sum_{n=-l}^{l} \left\{ Y_{sl}(y) u_s(\eta) c^+_{sln} + Y^*_{sl}(y) u^*_s(\eta) c^-_{sln} \right\},
\]

where

\[
Y_{sl}(y) = \frac{\sqrt{\Gamma(s + l + 3/2)(s + 1)}}{\Gamma(s - l + 1/2) \sin(r/\rho)} \times P^{(l+1/2)}_s (\cos(r/\rho)) Y_l(\theta, \chi),
\]

\(P^\nu_\mu\) are Legendre functions and \(Y_l\) are normalized spherical functions.

If we put

\[
s/\rho = k_s, \quad 1/\rho = \Delta k_s, \quad \sqrt{\rho} c^\pm_{sln} = \phi^\pm_{slm}(k),
\]

then

\[
\phi = \frac{1}{\sqrt{2}} \sum_{l, n} \sum_{k_s} \Delta k_s \sqrt{s/\rho} \left( \frac{\sin r/\rho}{s + 1} \right)^{-1/2} \times P^{-(l+1/2)}_s \left( \cos \frac{kr}{s + 1} \right) \sqrt{k_s} \times \left\{ Y_l(\theta, \chi) u_s \left( \frac{t}{\rho} \right) \phi^+_l(k_s) + Y^*_l(\theta, \chi) u^*_s \left( \frac{t}{\rho} \right) \phi^-_l(k_s) \right\}.
\]

Due to (2.10) and (2.12), the condition \( b(0) = 1 \) and the well-known equality

\[
\lim_{\nu \to \infty} \nu^\mu P^\mu_\nu = J_\mu(x),
\]

where \( J_\mu \) is a Bessel function, we obtain, as \( \rho \to \infty \) and \( k_s \to k > 0 \),

\[
\phi(t, r, \theta, \chi) = \frac{1}{\sqrt{2}} \sum_{l=0}^{\infty} \sum_{n=-l}^{l} \int dk \left\{ e^{ik^0 t} V_l(k, y) \phi^+_l(k) + e^{-ik^0 t} V^*_l(k, y) \phi^-_l(k) \right\},
\]

\[
k^0 = \sqrt{k^2 + m^2}, \quad V_l(k, y) = \int_{\rho \chi}^\infty J_{l+1/2}(kr) Y_l(\theta, \chi),
\]

if \( \lambda_s \to 0 \) as \( m \rho \to \infty \).

A priori, we may not consider \( \lambda_s \) to be independent of \( m \) and \( \rho \), but dimensional considerations imply that if such a dependence does exist, it should have the form \( \lambda_s = \lambda_s(m \rho) \).

The operators \( \phi^+_l(k) \) evidently obey the usual commutation rules for creation and annihilation operators

\[
\left[ \phi^+_l(k), \phi^+_m(k') \right] = 0, \quad \left[ \phi^+_l(k), \phi^-_{m'}(k') \right] = \delta_{lm} \delta_{nm} \delta(k - k').
\]

7. Quasi-Euclidean isotropic model

Let us now briefly discuss a particular case of open models, the quasi-Euclidean isotropic Universe. Its metric can be represented in the form

\[
ds^2 = a^2(\eta) [d\theta^2 - d\chi^2] - (d\xi^2 + d\eta^2 - d\chi^2)^2
\]

and is invariant with respect to the group \( O(3) \times T_3 \).

Solutions of the Klein-Gordon equation have the form

\[
\phi(\eta, \xi) = \frac{(2\pi)^{-3/2}}{a} \int u(\eta, \tilde{k}) e^{ik^0 t} e^{ik^1 \xi} \tilde{k} \tilde{k} = (k_1, k_2, k_3); \quad \Phi^= = (\xi^1, \xi^2, \xi^3) \quad k\xi = k_i \xi^i.
\]

The quantum numbers \( k_i \) are evidently eigenvalues of the generators \( Z_i \) of the subgroup \( T_3 \). Changes in the formulae of Sec. 2 are obvious, and we will not concentrate on them.

Instead of (3.1), we now have

\[
z^- (k) = \frac{1}{\sqrt{2}} \left\{ p(k) - \int d\tilde{k}' \langle T(k, \tilde{k}')q(\tilde{k}') \rangle \right\}, \quad z^+ (k) = (z^- (k))^*, \quad T(k, k') = T(k', k),
\]

\[
\int \text{Im} T(k, k') f(k) f(\tilde{k}) d\tilde{k} > 0
\]

for any function \( f(\tilde{k}) \neq 0 \).

The condition of quasi-vacuum invariance with respect to \( T_3 \) gives

\[
T(k, \tilde{k}) = T(k') \delta(k - \tilde{k}'),
\]
and with respect to $O(3)$:

$$T(k) = T(k), \quad k = \sqrt{k^2}.$$  

Quasiclassical solutions at large $k$ exist if

$$k^2 \lambda(k) = k^2 \left(\frac{i \sqrt{k^2 + m^2} - T(k)}{i \sqrt{k^2 + m^2} + T(k)}\right) \to 0 \quad (7.1)$$

as $k \to \infty$.

On the other hand, a representation of the commutation relations with a given function $T(k)$ is equivalent to the representation with $\lambda(k) = 0$ if $\int |\lambda(k)|^2 k^2 \, dk < \infty$, which is true if (7.1) is valid.

8. Conclusion

We conclude that, unlike QFT in de Sitter space, in which the condition of quasi-vacuum invariance with respect to isometries and the correspondence principle single out the unique Fock representation $\{\lambda_s = 0\}$, in our more general and consequently less symmetric case, it is apparently impossible to advance further than singling out a class of representations of the commutation relations which are equivalent to the one with all $\lambda_s = 0$ or $\lambda(k) = 0$. However, the representations, satisfying all the requirements formulated in Sec. 5 and 6, do not entirely fill this class. An interpretation of this fact probably requires an additional study.

A few words on some special cases of the closed model. The case of de Sitter space ($b = 1/\cos \eta$) has been, as we already mentioned, considered in detail in Ref. [5]. The case of Einstein’s static model ($b \equiv 1$) is the simplest. In this model, evidently,

$$u_s^{\pm}(\eta) = \frac{1}{\sqrt{s}} e^{\pm is\eta}.$$  

The quantity $M_{(0)}$ is conserved for any rest mass $m$ and may be interpreted as the energy:

$$M_{(0)} = \frac{1}{2} \sum_{s,\sigma} \sqrt{m^2 + \frac{(s+1)^2}{\rho^2}} (c^{+}_{s\sigma} c^{-}_{s\sigma} + c^{+}_{s\sigma} c^{-}_{s\sigma}).$$

The requirement (4.2) immediately leads to the condition $\lambda_s = 0$.

Furthermore, Eq. (2.7) has solutions in terms of cylindrical functions for the metric obtained by Staniukovich [10] ($b = e^\eta$, $a = \text{const} > 0$) (in a theory of gravity with a time-variable “constant” $\kappa$). The general solution (2.7) then has the form

$$u_s(\eta) = Z_{s+1/2}(\frac{m\rho}{a} e^{\eta})$$

where $Z_\nu(x)$ is any solution to the Bessel equation. The functions $u^{\pm}$ are then determined by the initial conditions (2.8).

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References

[1] P. Roman, Nuovo Cim. 37, 396 (1965).
[2] W. Thirring, “Special Problems in High Energy Physics”, Wien — NY, 1967.
[3] J-P. Vigier, in: Proc. Int. Workshop on Non-Local Quantum Field Theory, Preprint P2-3590, Dubna, 1968.
[4] E.A. Tagirov, E.D. Fediun’kin and N.A. Chernikov, Preprint JINR P2-3392, Dubna, 1967.
[5] N.A. Chernikov and E.A. Tagirov, Ann. Inst. Henri Poincaré 9, 109 (1968); Preprint JINR P2-3777, Dubna, 1967.
[6] L.D. Landau and E.M. Lifshitz, “Field Theory”, Fizmatgiz, Moscow, 1960.
[7] R. Penrose, in: “Gravitation and Topology”, Moscow, 1965.
[8] G. Bateman and A. Erdelyi, “Higher Transcedental Functions”, v.2, Nauka, M., 1966.
[9] N.A. Chernikov, in: Proc. Int. Workshop on Non-Local Quantum Field Theory, Preprint P2-3590, Dubna, 1968.
[10] K.P. Staniukovich, “Gravitational Field and Elementary Particles”, Nauka, M., 1965.

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COMMENT

The paper above is an English translation of an article written on the basis of Dr. K. Bronnikov’s graduation thesis at Moscow State University; the work was carried out with my assistance. The paper was published in 1968 as Joint Institute for Nuclear Research (Dubna) preprint P2–4151 in Russian. For this reason it is not so internationally known as, e.g., Ref. [5], but it had certainly influence in the former USSR on studies of quantum fields and particles interacting with gravitation, with further applications to cosmology and astrophysics, which began in the second half of the 1960s. Its results were essentially used in the first papers in this area by Ya.B. Zel’dovich, A.A. Starobinsky, A.A. Grib and others. In particular, it is known to me from a private communication by Prof. S.S. Gershtein that Ya.B. Zel’dovich estimated it very highly. Probably, the paper can not only be of historical interest, but also be useful for some present-day readers.

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14 September 2004