Gaussian unitary ensemble with jump discontinuities and the coupled Painlevé II and IV systems

Xiao-Bo Wu\textsuperscript{1} and Shuai-Xia Xu\textsuperscript{2,*}

\textsuperscript{1} School of Mathematics and Computer Science, Shangrao Normal University, Shangrao 334001, People’s Republic of China
\textsuperscript{2} Institut Franco-Chinois de l’Energie Nucléaire, Sun Yat-sen University, Guangzhou 510275, People’s Republic of China

E-mail: wuxiaobo201207@163.com and xushx3@mail.sysu.edu.cn

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Abstract
We study the orthogonal polynomials and the Hankel determinants associated with the Gaussian weight with two jump discontinuities. When the degree $n$ is finite, the orthogonal polynomials and the Hankel determinants are shown to be connected with the coupled Painlevé IV system. Using this connection, we obtain a sequence of special function solutions to the coupled Painlevé IV system. In the double scaling limit as the jump discontinuities tend to the edge of the spectrum and the degree $n$ grows to infinity, we establish the asymptotic expansions for the Hankel determinants and the orthogonal polynomials, which are expressed in terms of solutions of the coupled Painlevé II system. As applications, we re-derive the recently found Tracy–Widom type expressions for the gap probability of there being no eigenvalues in a finite interval near the extreme eigenvalue of large Hermitian matrix from the Gaussian unitary ensemble (GUE) and the limiting conditional distribution of the largest eigenvalue in the GUE by considering a thinned process.

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1. Introduction and statement of results

Consider the Gaussian unitary ensemble (GUE), where the joint probability density function of the ordered eigenvalues is given by

\[ \rho_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{i=1}^{n} e^{-\lambda_i^2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^2; \]  

(1.1)

see [19, 27]. Here \( Z_n \), known as the partition function, is a normalization constant. It is well known that the density function (1.1) can be expressed in the determinantal form

\[ \rho_n(\lambda_1, \ldots, \lambda_n) = \det [K_n(\lambda_i, \lambda_j)]_{1 \leq i, j \leq n}, \]  

(1.2)

where

\[ K_n(x, y) = e^{-\frac{1}{2}(x^2 + y^2)} \sum_{k=0}^{n-1} H_k(x)H_k(y). \]  

(1.3)

The polynomial \( H_k(x) \) therein is the normalized \( k \)th degree Hermite polynomial, orthonormal with respect to the Gaussian weight \( e^{-x^2} \) on the real axis.

Introduce the Hankel determinant

\[ D_n(s_1, s_2; \omega_1, \omega_2) = \det \left( \int_R x^{j+k} w(x; s_1, s_2; \omega_1, \omega_2) dx \right)_{j,k=0}^{n-1} \]

\[ = \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{n} w(x_j; s_1, s_2; \omega_1, \omega_2) \]

\[ \times \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 dx_1 \cdots dx_n, \]  

(1.4)

where

\[ w(x; s_1, s_2; \omega_1, \omega_2) = e^{-x^2} \begin{cases} 1 & x < s_1, \\ \omega_1 & s_1 < x < s_2, \\ \omega_2 & x > s_2, \end{cases} \]  

(1.5)

with the constants \( \omega_k \geq 0, k = 1, 2. \) If \( \omega_1 = \omega_2 = 1 \), the Hankel determinant \( D_n(s_1, s_2; 1, 1) \) is corresponding to the pure Gaussian weight \( e^{-x^2} \) and can be evaluated explicitly

\[ D_n^{\text{GUE}} = D_n(s_1, s_2; 1, 1) = (2\pi)^{n/2} e^{-\gamma_2/2} \prod_{k=1}^{n-1} k!; \]  

(1.6)

see [27, equation (4.1.5)]. There exist a system of monic orthogonal polynomials \( \pi_n(x) = \pi_n(x; s_1, s_2) = x^n + \cdots, n \geq 0, \) orthogonal with respect to the weight function \( w(x) = w(x; s_1, s_2; \omega_1, \omega_2) \)

\[ \int_R \pi_m(x) \pi_n(x) w(x) dx = \gamma_2^{-2} \delta_{m,n}, \quad m, n \in \mathbb{N}. \]  

(1.7)
The orthogonal polynomials satisfy the three term recurrence relation
\[ z\pi_n(z) = \pi_{n+1}(z) + \alpha_n\pi_n(z) + \beta_n^2\pi_{n-1}(z), \] (1.8)
where \( \alpha_n = \alpha_n(s_1, s_2) \) and \( \beta_n = \beta_n(s_1, s_2) \) are the recurrence coefficients. The polynomials \( \gamma_n\pi_n(x) \) are the normalized orthogonal polynomials and the leading coefficients \( \gamma_n = \gamma_n(s_1, s_2) \) are connected to the Hankel determinant \( D_n(s_1, s_2) \) by
\[ D_n(s_1, s_2) = \prod_{j=0}^{n-1} \gamma_j^2(s_1, s_2). \] (1.9)

The Hankel determinants generated by the Gaussian weight (1.5) with discontinuities appear naturally in random matrix theory. Consider the gap probability that there is no eigenvalue in the finite interval \((s_1, s_2)\) for the GUE matrices. On account of (1.1) and (1.4), the gap probability can be expressed as a ratio of the Hankel determinants
\[ \text{Pro}(\lambda_j \notin (s_1, s_2) : j = 1, \ldots, n) = \frac{D_n(s_1, s_2; 0, 1)}{D_n^{\text{GUE}}}, \] (1.10)
where \( \lambda_1 < \ldots < \lambda_n \) are the eigenvalues of a matrix in GUE and \( D_n(s_1, s_2; \omega_1, \omega_2) \) is defined in (1.4). For the gap probability on the infinite interval \((s, +\infty)\), we have the distribution of the largest eigenvalue
\[ \text{Pro}(\lambda_n < s) = \frac{D_n(s, s; 0, 0)}{D_n^{\text{GUE}}}. \] (1.11)

When the size \( n \) is finite, the gap probability, or equivalently the Hankel determinant \( D_n(s, s; 0, 0) \), can be evaluated in terms of solutions to the Painlevé IV equation [20, 31]. It is also shown in [20] that these solutions can be expressed in terms of classical special functions. In the large \( n \) limit, the distribution of the largest eigenvalue converges to the celebrated Tracy–Widom distribution
\[ \lim_{n \to +\infty} \text{Pro}(\lambda_n < \sqrt{2n + \frac{s}{\sqrt{2n}}}) = \exp \left( - \int_s^{+\infty} (x - s)q_{\text{HM}}(x)dx \right), \] (1.12)
where \( q_{\text{HM}}(s) \) is the Hastings–Mcleod solution of the second Painlevé equation \( q''(x) - 2q^3(x) - xq(x) = 0 \) with the asymptotic behavior \( q_{\text{HM}}(x) \sim Ai(x) \) as \( x \to +\infty \); see [30]. In [1], the asymptotics of the Hankel determinants and the orthogonal polynomials associated with the Gaussian weight with a single discontinuity near the soft edge are considered. As an application, they establish the deformed Tracy–Widom distribution, which is expressed in terms of the Ablowitz–Segur solutions to the second Painlevé equation characterized by the asymptotic behavior \( q_{\text{AS}}(x) \sim kAi(x) \) as \( x \to +\infty \), \( 0 < k < 1 \). The Hankel determinants and orthogonal polynomials associated with the Gaussian weight with a single jump discontinuity have also been considered in [21, 28, 32, 34].

The Hankel determinants with more discontinuities also arise naturally in the studies of the gap probability in random matrix theory. As an example, we proceed to consider the thinned process in GUE by removing each eigenvalue \( \lambda_1 < \ldots < \lambda_\nu \) of the GUE independently with probability \( p \in (0, 1) \). The thinned process is first introduced by Bohigas and Pato in [2, 3] with motivations arising from nuclear physics. It is observed in [3] that the remaining and removed eigenvalues can be interpreted as observed and unobserved particles, respectively. If we know the information that the largest observed particle \( \lambda_{\nu}^\text{max} \) is less than \( \gamma \), then the conditional
distribution of the largest eigenvalue $\lambda_n$ of the original GUE can be expressed by the ratio of Hankel determinants

$$\text{Pro}(\lambda_n < x | \lambda_{\text{max}}^T < y) = \frac{D_n(y, x; p, 0)}{D_n(y, x; p, p)}, \quad x > y,$$

(1.13)

and

$$\text{Pro}(\lambda_n < x | \lambda_{\text{max}}^T < y) = \frac{D_n(x, y; 0, 0)}{D_n(x, y; p, p)}, \quad x < y,$$

(1.14)

with $D_n(s_1, s_2; \omega_1, \omega_2)$ defined in (1.4); see [1, 6, 8, 9]. It is noted that other thinned random matrices in the situation of the circular ensemble have been considered in [6] and also in [4] with applications in the studies of Riemann zeros.

Recently, the limits of (1.10) and (1.13) are studied in [7, 9] by considering the Fredholm determinants of the Airy kernel with several discontinuities. More generally, the limits of the gap probabilities on any finite union of intervals near the extreme eigenvalues are considered in [9]. In [33], the second author of the present paper and Dai derive the asymptotics of (1.13) via the Fredholm determinants of the Painlevé XXXIV kernel which is a generalization of the Airy kernel. In both [9, 33], Tracy–Widom type expressions for the limiting distributions are established by using solutions to the coupled Painlevé II system. In [7, 33], the asymptotic expansions of the Tracy–Widom type distributions are also studied with the multiplicative constant determined by Riemann’s zeta-function.

The jump discontinuities in (1.5) are known as the Fisher–Hartwig singularities of jump type in the literature. Recently, the studies of the asymptotics of the Hankel determinants with Fisher–Hartwig singularities have attracted great interest. In [5], the Hankel determinants generated by a general weight function on the real axis with a one-cut regular potential and several Fisher–Hartwig singularities are studied. When the singularities are separated from each other and bounded away from the soft edges, the asymptotic expansions for large Hankel determinants are established including the nontrivial constant terms. This extends the result in [21] for the Hankel determinants generated by the Gaussian weight with a single discontinuity. The applications in the statistical properties of the characteristic polynomials and the gap probability of certain thinned process in random matrices are also considered therein. In the analogues of the Toeplitz determinants generated by a general weight function on the unit circle with several Fisher–Hartwig singularities, the asymptotic expansions have been worked out earlier in [13]. When the Fisher–Hartwig singularities merge together or approach the soft edges with certain speed, some critical transitions happen. The transition asymptotics of the Hankel determinants when two Fisher–Hartwig singularities are merging together in the bulk have been derived in [10] and expressed in terms of the Painlevé V transcendent. The asymptotic expansions for large Hankel determinants with a Fisher–Hartwig singularity approaching the soft edge have also been studied in [1, 32, 34], where the Painlevé II transcendent are involved. In [8], a transition is observed in the asymptotics of the Hankel determinants with a Fisher–Hartwig singularity in the bulk when the amplitude of jump varies with the size $n$. We also refer to [11] for a recent study of the Hankel determinants with Fisher–Hartwig singularities of various types of transitions and the applications in the rigidity of the eigenvalues of large random Hermitian matrix.

The present work is devoted to the studies of the Hankel determinants and the orthogonal polynomials associated with the Gaussian weight with two jump discontinuities both as the degree $n$ is finite and as $n$ tends to infinity. When the degree $n$ is finite, we show that the Hankel determinants and the orthogonal polynomials are described by the coupled Painlevé IV system. Using these relations and the fact that the moments of the weight function (1.5) can be written
in terms of a combination of the error functions, we obtain a sequence of special function solutions to the coupled Painlevé IV system. The generalization to the Gaussian weight with more than two jump discontinuities are also considered. Particularly, we show that the Hankel determinants generated by the Gaussian weight with more than two discontinuities are related to a coupled Painlevé IV system of higher dimension. As the jump discontinuities tend to the largest eigenvalue of GUE and the degree \( n \) grows to infinity, we establish asymptotic expansions for the Hankel determinants and the orthogonal polynomials. The asymptotics are expressed in terms of solutions to the coupled Painlevé II system. As applications, our results reproduce the asymptotic expansions of the gap probability in a finite interval near the largest eigenvalue of GUE and the conditional distribution of the largest eigenvalue of GUE as defined in (1.10) and (1.13), respectively, which are obtained previously in [9, 33].

### 1.1. Statement of results

#### 1.1.1. The coupled Painlevé IV system.

We introduce the Hamiltonian

\[
H_{IV}(a_1, a_2, b_1, b_2; x; s) = -2(a_1b_1 + a_2b_2 + x)(a_1 + a_2) + 2(a_1b_1(x - s) + a_2b_2(x + s) + nx) - (a_1b_1^2 + a_2b_2^2).
\]  

The Hamiltonian is associated with a degenerate Garnier system in two variables in the studies of the classification of four-dimensional Painlevé-type equations by Kawakami, Nakamura and Sakai [26, equations (3.11)–(3.13)]; see remark 6 below for this relation. The coupled Painlevé IV system can be written as the following Hamiltonian system

\[
\begin{aligned}
\frac{da_1}{dx} &= \frac{\partial H_{IV}}{\partial b_1}(a_1, a_2, b_1, b_2; x, s) = -2a_1(a_1 + a_2 + b_1 - x + s), \\
\frac{da_2}{dx} &= \frac{\partial H_{IV}}{\partial b_2}(a_1, a_2, b_1, b_2; x, s) = -2a_2(a_1 + a_2 + b_2 - x - s), \\
\frac{db_1}{dx} &= -\frac{\partial H_{IV}}{\partial a_1}(a_1, a_2, b_1, b_2; x, s) = b_1^2 + 2b_1(a_1 + a_2 - x + s) + 2(a_1b_1 + a_2b_2 + n), \\
\frac{db_2}{dx} &= -\frac{\partial H_{IV}}{\partial a_2}(a_1, a_2, b_1, b_2; x, s) = b_2^2 + 2b_2(a_1 + a_2 - x - s) + 2(a_1b_1 + a_2b_2 + n).
\end{aligned}
\]  

Eliminating \( b_1 \) and \( b_2 \) from the system, we find that \( a_1 \) and \( a_2 \) solve a couple of second order nonlinear differential equations

\[
\begin{aligned}
&\frac{d^2 a_1}{dx^2} - \frac{1}{2a_1}\left(\frac{da_1}{dx}\right)^2 - 6a_1(a_1 + a_2)^2 + 8a_1(a_1 + a_2)x - 8a_1^2s + 2(2n - 1)a_1 - 2a_1(x - s)^2 = 0, \\
&\frac{d^2 a_2}{dx^2} - \frac{1}{2a_2}\left(\frac{da_2}{dx}\right)^2 - 6a_2(a_1 + a_2)^2 + 8a_2(a_1 + a_2)x + 8a_2^2s + 2(2n - 1)a_2 - 2a_2(x + s)^2 = 0.
\end{aligned}
\]  

If \( a_2 = 0 \), we recover from the above equations the classical Painlevé IV equation with special values of the parameters (see [17] and [29, equation (32.2.4)])

\[
\begin{aligned}
&y''_{IV} = \frac{1}{2y_{IV}}y'^2_{IV} + 3\frac{v^3_{IV}}{2} + 4x_1y^2_{IV} + 2(x^2 + 1 - 2n)y_{IV}, \\
&y_{IV}(x) = -2a_1(x + s; s).
\end{aligned}
\]
1.1.2. Orthogonal polynomials of finite degree: the coupled Painlevé IV system. Our first result shows that, when the degree $n$ is finite, several quantities of the orthogonal polynomials associated with the weight function (1.5) can be expressed in terms of the coupled Painlevé IV system. These quantities include the Hankel determinants, the recurrence coefficients, leading coefficients and the values of the orthogonal polynomials at the jump discontinuities of (1.5).

We are interested in the Gaussian weight with two jump discontinuities, thus without loss of generality we assume that the parameters in (1.5) satisfy

\[ s_1 < s_2, \quad \omega_1 > 0, \quad \omega_2 > 0, \quad \omega_1 \neq \omega_2, \quad \omega_1 \neq 1. \]  

(1.19)

**Theorem 1.** Let $s_k$ and $\omega_k, k = 1, 2$ be given by (1.19) and $D_n(s_1, s_2) = D_n(s_1, s_2, \omega_1, \omega_2)$ be the Hankel determinant defined in (1.4), we denote

\[ F(s_1, s_2) = \frac{\partial}{\partial s_1} \ln D_n(s_1, s_2) + \frac{\partial}{\partial s_2} \ln D_n(s_1, s_2), \]  

(1.20)

and

\[ x = \frac{s_1 + s_2}{2}, \quad s = \frac{s_2 - s_1}{2}. \]  

(1.21)

Then $F(s_1, s_2)$ is related to the Hamiltonian for the coupled Painlevé IV system by

\[ F(s_1, s_2) = H_{IV}(x; s) - 2nx. \]  

(1.22)

Moreover, let $\alpha_n(s_1, s_2), \beta_n(s_1, s_2)$ be the recurrence coefficients defined in (1.8), $\gamma_n(s_1, s_2)$ be the leading coefficient of the orthonormal polynomial defined in (1.7) and $\pi_n(x) = \pi_n(x; s_1, s_2)$ be the monic orthogonal polynomial defined in (1.7), we have for $n \geq 1$

\[ \alpha_n(s_1, s_2) = \frac{a_1(x; s)b_1^2(x; s) + a_2(x; s)b_2^2(x; s)}{2(a_1(x; s)b_1(x; s) + a_2(x; s)b_2(x; s) + n)}, \]  

(1.23)

\[ \beta_n^2(s_1, s_2) = \frac{1}{2}(a_1(x; s)b_1(x; s) + a_2(x; s)b_2(x; s) + n), \]  

(1.24)

\[ \gamma_{n-1}^2 = \frac{1}{4\pi i}e^{r^2}y(x; s) \neq 0, \]  

(1.25)

\[ \frac{d}{dx} \ln \gamma_{n-1}(s_1, s_2) = a_1(x; s) + a_2(x; s), \]  

(1.26)

\[ \pi_n(s_1) = \frac{2\pi i}{\omega_1 - 1}e^{-2nx + i}a_1(x; s)b_1(x; s)^2 \]  

\[ \frac{y(x; s)}{y(x; s)}, \]  

(1.27)

\[ \pi_n(s_2) = \frac{2\pi i}{\omega_2 - \omega_1}e^{2nx + i}a_2(x; s)b_2(x; s)^2 \]  

\[ \frac{y(x; s)}{y(x; s)}, \]  

(1.28)

\[ \gamma_{n-1}^2 \pi_{n-1}(s_1) = \frac{2}{\omega_1 - 1}e^{-x^2}a_1(x; s), \]  

(1.29)

\[ \gamma_{n-1}^2 \pi_{n-1}(s_2) = -\frac{2}{\omega_2 - \omega_1}e^{x^2}s^2a_2(x; s), \]  

(1.30)

where $a_k(x; s)$ and $b_k(x; s), k = 1, 2$ satisfy the coupled Painlevé IV system (1.16) and $y(x; s)$ is connected with $a_k(x; s), k = 1, 2, \frac{dy}{dx} = 2(a_1 + a_2 - x)y$. 

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Remark 1. It is noted that the moments of (1.5) are linear in $\omega_k$ and hence the orthogonal polynomials $\pi_n(x)$ and $\pi_{n-1}(x)$ are entire in $\omega_k$, $k = 1, 2$. Therefore, it is seen from (1.30) that

$$a_2(x; s) = \frac{\omega_2 - \omega_1}{2} \gamma_{n-1}^{2} \pi_{n-1}(s_2^2) e^{-\frac{s_2^2}{2}} \to 0, \quad \omega_1 \to \omega_2,$$

and from (1.25) and (1.28) that

$$a_2(x; s) b_2(x; s)^2 = 2(\omega_2 - \omega_1) \gamma_{n-1}^{2} \pi_n(s_2^2) e^{-\frac{s_2^2}{2}} \to 0, \quad \omega_1 \to \omega_2.$$  

Similarly, it follows from (1.25), (1.27) and (1.29) that

$$a_1(x; s) = \frac{\omega_1 - 1}{2} \gamma_{n-1}^{2} \pi_{n-1}(s_1^2) e^{-\frac{s_1^2}{2}} \to 0, \quad \omega_1 \to 1,$$

and

$$a_1(x; s) b_1(x; s)^2 = 2(\omega_1 - 1) \gamma_{n-1}^{2} \pi_n(s_1^2) e^{-\frac{s_1^2}{2}} \to 0, \quad \omega_1 \to 1.$$  

Thus, using (1.31), (1.33) and recalling (1.17) and (1.18), the functions $y_{IV}(x) = -2a_1(x + s; s)$ and $y_{IV}(x) = -2a_2(x - s; s)$ solve the classical Painlevé IV equation as $\omega_1 \to \omega_2$ and $\omega_1 \to 1$, respectively. This allows us to take the limit $\omega_1 \to \omega_2$ or $\omega_1 \to 1$ in theorem 1, which then implies that the Hankel determinants and the orthogonal polynomials associated with the weight function (1.5) with a single jump discontinuity are related to the solutions of the classical Painlevé IV equation.

It is observed that the moments of the weight function (1.5) can be written in terms of a combination of the error functions, or equivalently in terms of the parabolic cylinder functions with special parameters. This fact, together with theorem 1, allows us to obtain a sequence of special function solutions to the couple of nonlinear differential equation (1.17) and express the Hamiltonian associated with the special function solutions in terms of the Wronskian determinant of a combination of the error functions.

Corollary 1. We denote

$$\varphi(s_1, s_2) = \frac{\sqrt{\pi}}{2} e^{\frac{(s_1 + s_2)^2}{4}} \left( \text{erfc}(-s_1) + \omega_1 \text{erfc}(s_1) + (\omega_2 - \omega_1)\text{erfc}(s_2) \right)$$

with the error function given by

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} \, dt,$$

and the Wronskian determinant involving the function $\varphi$

$$\tau_n(s_1, s_2) = \det \left( \delta^{j+k} \varphi(s_1, s_2) \right)_{j,k=0}^{n-1}, \quad \delta := \frac{\partial}{\partial s_1} + \frac{\partial}{\partial s_2}, \quad n \geq 1,$$

with $\tau_0(s_1, s_2) = 1$. Then the Hankel determinant (1.4) and the Hamiltonian can be written in terms of the Wronskian determinant

$$D_n(s_1, s_2) = 2^{n(n-1)} e^{\frac{(s_1 + s_2)^2}{4}} \tau_n(s_1, s_2),$$

$$H_{IV} \left( \frac{s_1 + s_2}{2} : \frac{s_2 - s_1}{2} \right) = \delta \ln \tau_n(s_1, s_2),$$

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where \( n \geq 1 \). Moreover, we obtain the following special function solutions to the couple of nonlinear differential equation (1.17) with \( n \geq 1 
abla_1\) and \( n \geq 1 \nabla_2\).

\[
\begin{align*}
\nabla_1 & \left( \frac{s_1 + s_2}{2}; \frac{s_2 - s_1}{2} \right) = \frac{1}{2} \frac{\partial}{\partial s_1} \ln \frac{\tau_{n-1}(s_1, s_2)}{\tau_1(s_1, s_2)} + \frac{s_1 + s_2}{4}, \\
\nabla_2 & \left( \frac{s_1 + s_2}{2}; \frac{s_2 - s_1}{2} \right) = \frac{1}{2} \frac{\partial}{\partial s_2} \ln \frac{\tau_{n-1}(s_1, s_2)}{\tau_1(s_1, s_2)} + \frac{s_1 + s_2}{4}.
\end{align*}
\]

1.1.3. The coupled Painlevé II system. To state our main results on the asymptotics of the orthogonal polynomials, we introduce the following coupled Painlevé II system in dimension four:

\[
\begin{align*}
\frac{d v_1}{dx} - \frac{\partial H_{II}}{\partial v_1} &= 2(v_1 + v_2 + \frac{x}{2}) - w_1^2, \\
\frac{d v_2}{dx} - \frac{\partial H_{II}}{\partial v_2} &= 2(v_1 + v_2 + \frac{x + s}{2}) - w_2^2, \\
\end{align*}
\]

where \( v_k = v_k(x; s), w_k = w_k(x; s), k = 1, 2 \) and the Hamiltonian \( H_{II} = H_{II}(v_1, v_2, w_1, w_2; x; s) \) is given by

\[
H_{II}(v_1, v_2, w_1, w_2; x; s) = -(v_1 + v_2)^2 - (v_1 + v_2)x + v_1 w_1^2 + v_2 w_2^2 - sv_2.
\]

The coupled Painlevé II system appears in both of the degeneration schemes of the Garnier system in two variables [24, equations (3.5)–(3.7)] and the Sasano system [25, equations (3.22) and (3.23)] by Kawakami.

Eliminating \( w_1 \) and \( w_2 \) from the Hamiltonian system (1.42) gives us the following nonlinear equations for \( v_1 \) and \( v_2 \):

\[
\begin{align*}
v_{1xx} - \frac{v_{1x}^2}{2v_1} - 4v_1(v_1 + v_2 + \frac{x}{2}) &= 0, \\
v_{2xx} - \frac{v_{2x}^2}{2v_2} - 4v_2(v_1 + v_2 + \frac{x + s}{2}) &= 0.
\end{align*}
\]

Let \( v_k(x; s) = u_k(x; s)^2 = u_k(x)^2, k = 1, 2 \), the above equations are further simplified to

\[
\begin{align*}
u_{1xx} - xu_1 - 2u_1(u_1^2 + u_2^2) &= 0, \\
u_{2xx} - (x + s)u_2 - 2u_2(u_1^2 + u_2^2) &= 0.
\end{align*}
\]

If \( v_2(x) = u_2(x)^2 = 0 \), then (1.45) is reduced to the classical second Painlevé equation

\[
q'' - 2q^3 - qx = 0.
\]

The functions \( v_k(x; s) \) and \( u_k(x; s), k = 1, 2 \), are also connected with \( H_{II}(x; s) \) by

\[
\frac{d}{dx} H_{II}(x; s) = -(v_1(x; s) + v_2(x; s)) = -(u_1(x; s)^2 + u_2(x; s)^2),
\]
which can be obtained by taking derivative on both side of (1.43); see also [33, equation (7.37)].

The existence of solutions to the coupled Painlevé II system are established in [9]; see also [33].

**Proposition 1.** (Claeys and Doeraene [9]). For the parameters $\omega_k, k = 1, 2$ as given in (1.19) and $s > 0$, there exist real-valued and pole-free solutions $v_k(x; s)$ (or $u_k(x; s)$), $k = 1, 2$, to the coupled nonlinear differential equation (1.44) (or (1.45)) subject to the boundary conditions as $x \to +\infty$

\[
v_1(x; s) = u_1(x; s)^2 \sim (1 - \omega_1)\text{Ai}(x)^2,
\]

\[
v_2(x; s) = u_2(x; s)^2 \sim (\omega_1 - \omega_2)\text{Ai}(x + s)^2,
\]

(1.48)

where $\text{Ai}$ is the standard Airy function.

**1.1.4. Asymptotics of the Hankel determinants and applications in random matrices.** Our second result gives the asymptotics of the Hankel determinants expressed in terms of the solutions to the coupled Painlevé II system.

**Theorem 2.** Let $s_1$ and $\omega_k$, $k = 1, 2$ be given by (1.19) and $s_k$ be related to $t_k$, $k = 1, 2$ by

\[
s_1 = \sqrt{2n + \frac{t_1}{2n^{1/6}}}, \quad s_2 = \sqrt{2n + \frac{t_2}{2n^{1/6}}},
\]

with $t_1 < t_2$, then we have the asymptotics of the Hankel determinant $D_n(s_1, s_2) = D_n(s_1, s_2; \omega_1, \omega_2)$ defined in (1.4) as $n \to \infty$

\[
D_n(s_1, s_2) = D_{n}^{\text{GUE}} \exp \left( - \int_{t_1}^{+\infty} (\tau - t_1) \left( u_1(\tau; t_2 - t_1) \right)^2 + u_2(\tau; t_2 - t_1)^2 \right) d\tau \left( 1 + O(n^{-1/6}) \right),
\]

(1.49)

where $D_{n}^{\text{GUE}}$ is the Hankel determinant associated with the Gaussian weight with expression given in (1.6), $u_k(x; s)$, $k = 1, 2$, are solutions to (1.45) subject to the boundary conditions (1.48) and the error bound is uniform for $t_1, t_2$ in any compact subset of $\mathbb{R}$.

**Remark 2.** When $s \to 0$, it is shown in [9, equation (1.28)] that

\[
u_1(x; s)^2 + u_2(x; s)^2 = q^2(x; \omega_2) + O(s),
\]

where $q(x; \omega_2)$ is the Ablowitz–Segur solution to the second Painlevé equation (1.46) with the boundary condition as $x \to +\infty$

\[
q(x; \omega_2) \sim \sqrt{1 - \omega_2} \text{Ai}(x).
\]

(1.50)

Therefore, as $t_2 - t_1 \to 0$, the formula (1.49) is reduced to

\[
D_n(s_1) = D_{n}^{\text{GUE}} \exp \left( - \int_{t_1}^{+\infty} (\tau - t_1)q^2(\tau; \omega_2)d\tau \right) \left( 1 + O(n^{-1/6}) \right),
\]

(1.51)

where $D_{n}(s_1)$ is the Hankel determinant associated with the weight function (1.5) with one jump discontinuity by taking $s_1 = s_2 = \sqrt{2n + \frac{t_1}{2n^{1/6}}}$. The expansion agrees with the result from [1] where the case with a single jump discontinuity is considered.
As an application of the asymptotics of the Hankel determinants, we derive the gap probability of there being no eigenvalues in a finite interval near the extreme eigenvalues of large GUE by using (1.10) and (1.49). The result except the error term is first obtained by Claeys and Doeraene in [9].

**Corollary 2.** (Claeys and Doeraene [9]). Let $s_1$ and $s_2$ be as in theorem 2, we have the asymptotic approximation of the gap probability of finding no eigenvalues of GUE in the finite interval $(s_1, s_2)$

\[
\text{Pro}(\lambda_j \notin (s_1, s_2) : j = 1 \ldots n) = \exp \left(-\int_{t_1}^{+\infty} (\tau - t_1)(u_1(\tau; t_2 - t_1)^2 + u_2(\tau; t_2 - t_1)^2) d\tau \right)
\times \left(1 + O(n^{-1/6})\right),
\]

where $u_k(x; s)$, $k = 1, 2$, are solutions to (1.45) subject to the boundary conditions (1.48) with the parameters $\omega_1 = 0$, $\omega_2 = 1$ and the error bound is uniform for $t_1$ and $t_2$ in any compact subset of $\mathbb{R}$.

In the second application, we derive from (1.49) and (1.51) the large $n$ limit of the distribution (1.13) in the thinning and conditioning GUE. This reproduces a recent result from [9, 33].

**Corollary 3.** (Claeys and Doeraene [9], Xu and Dai [33]). Let $s_1$ and $s_2$ be as in theorem 2, we have the asymptotics of the conditional gap probability

\[
\text{Pro}(\lambda_n < s_2 | \lambda_{\max}^X < s_1) = \exp \left(-\int_{t_1}^{+\infty} (\tau - t_1)(u_1(\tau; t_2 - t_1)^2 + u_2(\tau; t_2 - t_1)^2 + q^2(\tau; p)) d\tau \right)
\times \left(1 + O(n^{-1/6})\right),
\]

where $u_k(x; s)$, $k = 1, 2$, are solutions to (1.45) subject to the boundary conditions (1.48) with the parameters $\omega_1 = \rho \in (0, 1)$, $\omega_2 = 0$, $q(x; p)$ is the Ablowitz–Segur solution to the second Painlevé equation (1.46) with the asymptotics (1.50) and the error bound is uniform for $t_1$ and $t_2$ in any compact subset of $\mathbb{R}$.

1.11.5. Asymptotics of the coupled Painlevé IV system. Next, we show that the scaling limit of the coupled Painlevé IV system leads to the coupled Painlevé II system.

**Theorem 3.** Let $s_k$, $\omega_k$, $k = 1, 2$ be as in theorem 2 and $x = (s_1 + s_2)/2$, $s = (s_2 - s_1)/2$, then we have the asymptotics of the coupled Painlevé IV system as $n \to \infty$

\[
a_1(x; s) = -\frac{1}{\sqrt{2n^{1/6}}} \left( v_1(t_1; t_2 - t_1) + \frac{v_{14}(t_1; t_2 - t_1)}{2n^{1/3}} + O(n^{-2/3}) \right),
\]

\[
a_2(x; s) = -\frac{1}{\sqrt{2n^{1/6}}} \left( v_2(t_1; t_2 - t_1) + \frac{v_{24}(t_1; t_2 - t_1)}{2n^{1/3}} + O(n^{-2/3}) \right),
\]

\[
b_1(x; s) = \sqrt{2n} \left( 1 - \frac{v_{14}(t_1; t_2 - t_1)}{2v_1(t_1; t_2 - t_1)n^{1/3}} + O(n^{-2/3}) \right),
\]

\[
b_2(x; s) = \sqrt{2n} \left( 1 - \frac{v_{24}(t_1; t_2 - t_1)}{2v_2(t_1; t_2 - t_1)n^{1/3}} + O(n^{-2/3}) \right).
\]
where \( v_1(x; s) \) and \( v_2(x; s) \) are solutions to the coupled Painlevé II system (1.44) subject to the boundary conditions (1.48), \( H_0(x; s) \) is the Hamiltonian associated to these solutions and the subscript \( s \) in \( v_k(x; s) \) denotes the derivative of \( v_k(x; s) \) with respect to \( x \) for \( k = 1, 2 \).

### 1.1.6. Asymptotics of the orthogonal polynomials

Applying theorems 1 and 3, we obtain the asymptotics of the recurrence coefficients, leading coefficients of the orthogonal polynomials and the values of the orthogonal polynomials at \( s_1 \) and \( s_2 \).

**Theorem 4.** Let \( s_k, \omega_k, k = 1, 2 \) be as in theorem 2, we have the asymptotics of the recurrence coefficients and leading coefficients of the orthogonal polynomials as \( n \to \infty \)

\[
\alpha_n = -\frac{1}{\sqrt{2}} \left( v_1(t_1; t_2 - t_1) + v_2(t_1; t_2 - t_1) \right) n^{-1/6} + O(n^{-1/2}),
\]

\[
\beta_n = \frac{1}{\sqrt{2}} n^{1/2} - 2^{-3/2} \left( v_1(t_1; t_2 - t_1) + v_2(t_1; t_2 - t_1) \right) n^{-1/6} + O(n^{-1/2}),
\]

\[
\gamma_{n-1} = 2^{\frac{3}{2}} \frac{1}{\sqrt{2}} \left( \frac{1}{\omega_1} \right)^{1/2} \left( \frac{ne}{2} \right)^{n/2} e^{n^{1/3}} u_1(t_1; t_2 - t_1)(1 + O(n^{-1/3})),
\]

Moreover, we derive the asymptotics of the values of the orthogonal polynomials at \( s_k \) as \( n \to \infty \)

\[
\pi_n(s_1) = \left( \frac{2\pi}{1 - \omega_1} \right)^{1/2} \left( \frac{ne}{2} \right)^{n/2} e^{n^{1/3}} u_1(t_1; t_2 - t_1)(1 + O(n^{-1/3})),
\]

\[
\pi_n(s_2) = \left( \frac{2\pi}{\omega_1 - \omega_2} \right)^{1/2} \left( \frac{ne}{2} \right)^{n/2} e^{n^{1/3}} u_2(t_1; t_2 - t_1)(1 + O(n^{-1/3})).
\]

Here \( v_3(x; s) \) and \( q_k(x; s), k = 1, 2 \) are solutions to (1.44) and (1.45) subject to the boundary conditions (1.48). \( H_0(x; s) \) is the Hamiltonian corresponding to these solutions.

**Remark 3.** When \( s_1 = s_2 \), the weight function (1.5) is reduced to the Gaussian weight with a single jump discontinuity. Similar asymptotic expansions for the recurrence coefficients and the orthogonal polynomials in the case of a single jump discontinuity have been proved in [1]. For example, we have the asymptotic expansion of the recurrence coefficients [1, theorem 5]

\[
\alpha_n = \frac{1}{\sqrt{2}} g^2(x; \omega_2)n^{-1/6} + O(n^{-1/2}),
\]

where \( q(x; \omega_2) \) is the Ablowitz–Segur solution to the second Painlevé equation (1.46) with the boundary condition (1.50). As mentioned in remark 2, it is shown in [9, equation (1.28)] that

\[
v_1(t_1; t_1 - t_2) - v_2(t_1; t_1 - t_2) = u_1^2(t_1; t_1 - t_2) - u_2^2(t_1; t_1 - t_2) \rightarrow q^2(x; \omega_2), \quad \text{as} \quad t_1 \to t_2,
\]

Moreover, we derive the asymptotics of the values of the orthogonal polynomials at \( s_k \) as \( n \to \infty \)

\[
\pi_n(s_1) = \left( \frac{2\pi}{1 - \omega_1} \right)^{1/2} \left( \frac{ne}{2} \right)^{n/2} e^{n^{1/3}} u_1(t_1; t_2 - t_1)(1 + O(n^{-1/3})),
\]

\[
\pi_n(s_2) = \left( \frac{2\pi}{\omega_1 - \omega_2} \right)^{1/2} \left( \frac{ne}{2} \right)^{n/2} e^{n^{1/3}} u_2(t_1; t_2 - t_1)(1 + O(n^{-1/3})).
\]
where \( q(x; \omega_2) \) is the Ablowitz–Segur solution to the second Painlevé equation (1.46) with the boundary condition (1.50). Formally, this allows us to recover (1.64) by taking limit \( t_1 \to t_2 \) in (1.59).

### 1.2. Organization of the rest of this paper

The rest of the paper is organized as follows. In section 2, we consider the Riemann–Hilbert (RH) problem for the orthogonal polynomials associated with the Gaussian weight with two jump discontinuities (1.5). We show that the RH problem is equivalent to the one for the coupled Painlevé IV system. The properties of the Painlevé IV system are studied, including the Lax pair and the Hamiltonian formulation. We then prove theorem 1 which relates the Hankel determinants and the orthogonal polynomials to the coupled Painlevé IV system. Applying theorem 1 and writing the moments of (1.5) in terms of the error functions, we prove corollary 1. At the end of this section, we show that the Hankel determinants generated by the Gaussian weight with more than two discontinuities can be related to a coupled Painlevé IV system of higher dimension. In section 3, we study the asymptotics of the orthogonal polynomials orthogonal with respect to (1.5) by performing Deift–Zhou nonlinear steepest descent analysis of the RH problem for the orthogonal polynomials. Finally, the proofs of theorems 2–4 are given in section 4.

### 2. Orthogonal polynomials and the coupled Painlevé IV system

In this section, we will relate the Hankel determinants and the orthogonal polynomials associated with the weight function (1.5) to the coupled Painlevé IV system. The connections are collected in theorem 1. The derivations are based on the RH problem representation of the orthogonal polynomials. Then we prove corollary 1 by using theorem 1. Finally, the generalization to Hankel determinants generated by the Gaussian weight with more than two discontinuities are studied, which are shown to be connected with a coupled Painlevé IV system of higher dimension.

#### 2.1. Riemann–Hilbert problem for the orthogonal polynomials

In this subsection, we first consider the RH problem for the orthogonal polynomials with respect to (1.5), which was introduced by Fokas et al [18]. We then derive several identities relating the logarithmic derivative of the Hankel determinants to the solution of the RH problem. At the end of the subsection, we transform the RH problem to a model RH problem with constant jumps.

##### 2.1.1. Riemann–Hilbert problem for \( Y \)

(a) \( Y(z; s_1, s_2) \) (\( Y(z) \) for short) is analytic in \( \mathbb{C} \setminus \mathbb{R} \);

(b) \( Y(z) \) satisfies the jump condition

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R},
\]

where \( w(x) = w(x; s_1, s_2; \omega_1, \omega_2) \) is defined in (1.5);

(c) The behavior of \( Y(z) \) at infinity is

\[
Y(z) = \left( I + \frac{Y_1}{z} + O \left( \frac{1}{z^2} \right) \right) \begin{pmatrix} z^0 & 0 \\ 0 & z^{-\pi} \end{pmatrix}, \quad z \to \infty;
\]  

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(d) \( Y(z) = \begin{pmatrix} O(1) & O(\ln |z-s_k|) \\ O(1) & O(\ln |z-s_k|) \end{pmatrix} \) as \( z \to s_k \) for \( k = 1, 2 \).

For \( \omega_k \geq 0, k = 1, 2 \), it follows from the Sokhotski–Plemelj formula and Liouville’s theorem that the unique solution of the RH problem for \( Y \) is given by

\[
Y(z) = \begin{pmatrix} \pi_n(z) & 1/2\pi i \int_{\mathbb{R}} \frac{\pi_n(x)w(x)}{x-z} \, dx \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & -\gamma_{n-1}^2 \int_{\mathbb{R}} \frac{\pi_{n-1}(x)w(x)}{x-z} \, dx \end{pmatrix},
\]

(2.2)

where \( \pi_n(z) \) and \( \gamma_{n-1} \) are defined in (1.7); see [18].

We establish two differential identities expressing the logarithmic derivative of the Hankel determinant \( D_n \) in terms of the solution \( Y \).

**Proposition 2.** Let \( s_k \) and \( \omega_k, k = 1, 2 \) be given by (1.19) and \( F(s_1, s_2) \) be the logarithmic derivative of the Hankel determinant as defined in (1.20), we have the following relations

\[
F(s_1, s_2) = \frac{1-\omega_1}{2\pi i} e^{-z^2} (Y^{-1}Y')_{11}(s_1) + \frac{\omega_1-\omega_2}{2\pi i} e^{-z^2} (Y^{-1}Y')_{11}(s_2),
\]

and

\[
F(s_1, s_2) = 2 \lim_{z \to \infty} z(Y(z)z^{-\omega_1} - I)_{11},
\]

(2.3)

(2.4)

where \( Y \) is defined in (2.2).

**Proof.** According to (1.20), it follows by taking logarithmic derivative on both sides of the equation (1.9) that

\[
F(s_1, s_2) = -2 \sum_{j=0}^{n-1} \gamma_j^{-1} \left( \frac{\partial \gamma_j}{\partial s_1} + \frac{\partial \gamma_j}{\partial s_2} \right)
\]

(2.5)

\[
= \sum_{j=0}^{n-1} \left( (1-\omega_1) e^{-z^2} \gamma_j^2 \pi_j(s_1)^2 + (\omega_1-\omega_2) e^{-z^2} \gamma_j^2 \pi_j(s_2)^2 \right).
\]

(2.6)

Applying the Christoffel–Darboux identity, we obtain

\[
F(s_1, s_2) = (1-\omega_1) e^{-z^2} \gamma_{n-1}^2 (\pi_n'(s_1) \pi_{n-1}(s_1) - \pi_n(s_1) \pi_n'(s_1))
\]

\[
+ (\omega_1-\omega_2) e^{-z^2} \gamma_{n-1}^2 (\pi_n'(s_2) \pi_{n-1}(s_2) - \pi_n(s_2) \pi_n'(s_2)).
\]

(2.7)

Then, the differential identity (2.3) follows from the definition of \( Y \) and (2.7).

To prove (2.4), we use a change of variable in (1.7) and obtain

\[
\gamma_j^{-2} = \gamma_j(s_1, s_2)^{-2} = \int_{\mathbb{R}} \pi_j(x)^2 w(x) \, dx
\]

\[
= \int_{\mathbb{R}} \pi_j(x+s_k)^2 w(x+s_k) \, dx, \quad k = 1, 2.
\]

(2.8)

Taking derivative with respect to \( s_k \) on both sides of (2.8) for \( k = 1, 2 \) and using the orthogonality and the definition of the weight function (1.5), we have

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\[-2\gamma_j^{-1} \frac{\partial}{\partial s_1} \gamma_j = -2\gamma_j^2 \int_{\mathbb{R}} x \pi_j(x)^2 w(x)dx - (\omega_1 - \omega_2)e^{-i\gamma_j^2} \pi_j(s_2)^2, \quad (2.9)\]

and
\[-2\gamma_j^{-1} \frac{\partial}{\partial s_2} \gamma_j = -2\gamma_j^2 \int_{\mathbb{R}} x \pi_j(x)^2 w(x)dx - (1 - \omega_1)e^{-i\gamma_j^2} \pi_j(s_1)^2. \quad (2.10)\]

Combining the formulas with (2.5) and (2.6) and using the Christoffel–Darboux formula once again, we obtain
\[F(s_1, s_2) = -2\gamma_n^2 \int_{\mathbb{R}} x \left( \frac{d}{dx} \pi_n(x) \pi_{n-1}(x) - \pi_n(x) \frac{d}{dx} \pi_{n-1}(x) \right) w(x)dx. \quad (2.11)\]

From
\[\pi_n(x) = x^n + p_n x^{n-1} + \cdots,\]
we have the decomposition
\[x \frac{d}{dx} \pi_n(x) = n \pi_n(x) - p_n \pi_{n-1}(x) + \cdots.\]

Substituting this into (2.11) and using the orthogonality, we obtain (2.4). This completes the proof of proposition 2.

We define
\[\Phi(z; x, s) = \sigma_1 e^{x^2/2} Y(z + x)e^{-\frac{1}{2}(z+x)^2}\sigma_1, \quad (2.12)\]

where the variables $x, s$ are related to $s_1$ and $s_2$ by (1.21) and the Pauli matrix $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\Phi(z) = \Phi(z; x, s)$ satisfies the following RH problem.

### 2.1.2. Riemann–Hilbert problem for $\Phi$.

(a) $\Phi(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(b) $\Phi(z)$ satisfies the jump condition
\[\Phi_+ (z) = \Phi_- (z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z < -s,\]
\[\Phi_+ (z) = \Phi_- (z) \begin{pmatrix} 1 & 0 \\ \omega_1 & 1 \end{pmatrix}, \quad -s < z < s,\]
\[\Phi_+ (z) = \Phi_- (z) \begin{pmatrix} 1 & 0 \\ \omega_2 & 1 \end{pmatrix}, \quad z > s.\]

(c) The behavior of $\Phi(z)$ at infinity is
\[\Phi(z) = \Phi^{(\infty)}(z)e^{\frac{1}{2}(z^2 + x^2)\gamma_j^2 - i\pi_j} \quad \text{with} \quad \Phi^{(\infty)}(z) = I + \sum_{k=1}^{\infty} \frac{\Phi_k(x, s)}{z^k}. \quad (2.13)\]
The behavior of $\Phi(z)$ near $-s$ is
\[
\Phi(z) = \Phi^{(-)}(z) \left( I + \frac{1 - \omega_1}{2\pi i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ln(z + s) \right) E^{(-s)},
\]
(2.14)
where $\arg(z + s) \in (-\pi, \pi)$. Here, $\Phi^{(-)}(z)$ is analytic near $z = -s$ and has the following expansion
\[
\Phi^{(-)}(z) = P_0(x, s)(I + P_1(x, s)(z + s) + O((z + s)^2)).
\]
(2.15)
The piecewise constant matrix $E^{(-s)}$ is given by
\[
E^{(-s)} = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{Im } z > 0, \\
\begin{pmatrix} 1 & -\omega_1 \\ -\omega_1 & 1 \end{pmatrix}, & \text{Im } z < 0.
\end{cases}
\]

The behavior of $\Phi(z)$ near $s$ is
\[
\Phi(z) = \Phi^{(+)}(z) \left( I + \frac{\omega_1 - \omega_2}{2\pi i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ln(z - s) \right) E^{(s)},
\]
(2.16)
where $\arg(z - s) \in (-\pi, \pi)$. Here, $\Phi^{(+)}(z)$ is analytic near $z = s$ and has the following expansion
\[
\Phi^{(+)}(z) = Q_0(x, s)(I + Q_1(x, s)(z - s) + O((z - s)^2)).
\]
(2.17)
The piecewise constant matrix $E^{(s)}$ is defined by
\[
E^{(s)} = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{Im } z > 0, \\
\begin{pmatrix} 1 & 0 \\ -\omega_2 & 1 \end{pmatrix}, & \text{Im } z < 0.
\end{cases}
\]

**Remark 4.** The unique existence of solution to the RH problem for $Y$ and the invertible transformation (2.12) imply the existence of unique solution of the RH problem for $\Phi$ for $\omega_k \geq 0$, $k = 1, 2$, $x = \frac{4t + a_0}{2}$ $\in \mathbb{R}$ and $s = \frac{2i}{\alpha} \in \mathbb{R} \setminus \{0\}$. Therefore, the coefficients $\Phi_1(x, s)$, $\Phi_2(x, s)$, $P_0(x, s)$, $P_1(x, s)$, $Q_0(x, s)$ and $Q_1(x, s)$ in the expansions of $\Phi(z; x, s)$ are pole free for $x \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$.

**Remark 5.** If $\omega_1 = \omega_2$, the logarithmic singularity in (2.16) disappears. The RH problem for $\Phi$ is reduced to that for the Painlevé IV equation with special choices of the Stokes multipliers; see [17, proposition 5.4].

### 2.2. Lax pair and the coupled Painlevé IV system

In this section, we show that the solution $\Phi(z; x, s)$ of the RH problem satisfies a system of differential equations in $z$ and $x$ when the parameter $s$ is fixed. The compatibility condition
\( \Phi_{z}(z; x, s) = \Phi_{s}(z; x, s) \) gives us the coupled Painlevé IV system. The Hamiltonian for the system is also derived.

**Proposition 3.** We have the following Lax pair

\[
\Phi_{z}(z; x, s) = A(z; x, s)\Phi(z; x, s), \quad \Phi_{s}(z; x, s) = B(z; x, s)\Phi(z; x, s),
\]

where

\[
A(z; x, s) = (z + x)\sigma_{3} + A_{\infty}(x, s) + \frac{A_{1}(x, s)}{z + s} + \frac{A_{2}(x, s)}{z - s},
\]

\[
B(z; x, s) = z\sigma_{3} + A_{\infty}(x, s),
\]

with the coefficients given below

\[
A_{\infty}(x, s) = \begin{pmatrix} 0 & y(x; s) \\ -2(a_{1}(x; s)b_{1}(x; s) + a_{2}(x; s)b_{2}(x; s) + n) / y(x; s) & 0 \end{pmatrix},
\]

\[
A_{k}(x, s) = \begin{pmatrix} a_{k}(x; s)b_{k}(x; s) & a_{k}(x; s)y(x; s) \\ -a_{k}(x; s)b_{k}(x; s)/y(x; s) & -a_{k}(x; s)b_{k}(x; s) \end{pmatrix}, \quad k = 1, 2.
\]

The compatibility condition of the Lax pair gives us the coupled Painlevé IV system

\[
\begin{aligned}
\frac{dy}{dx} &= 2(a_{1} + a_{2} - x)y, \\
\frac{d\alpha_{1}}{dx} &= -2a_{1}(a_{1} + a_{2} + b_{1} - x + s), \\
\frac{d\alpha_{2}}{dx} &= -2a_{2}(a_{1} + a_{2} + b_{2} - x - s), \\
\frac{db_{1}}{dx} &= b_{1}^{2} + 2b_{1}(a_{1} + a_{2} - x + s) + 2(a_{1}b_{1} + a_{2}b_{2} + n), \\
\frac{db_{2}}{dx} &= b_{2}^{2} + 2b_{2}(a_{1} + a_{2} - x - s) + 2(a_{1}b_{1} + a_{2}b_{2} + n).
\end{aligned}
\]

Eliminating \( b_{1} \) and \( b_{2} \) from the system, it is seen that \( a_{1} \) and \( a_{2} \) satisfy the following nonlinear differential equations

\[
\begin{aligned}
\frac{d^{2}a_{1}}{dx^{2}} - \left( \frac{d\alpha_{1}}{dx} \right)^{2} &= 6a_{1}(a_{1} + a_{2})^{2} + 8a_{1}(a_{1} + a_{2})x - 8a_{1}^{2} + 2(2n - 1)a_{1} - 2a_{1}(x - s)^{2} = 0, \\
\frac{d^{2}a_{2}}{dx^{2}} - \left( \frac{d\alpha_{2}}{dx} \right)^{2} &= 6a_{2}(a_{1} + a_{2})^{2} + 8a_{2}(a_{1} + a_{2})x + 8a_{2}^{2} + 2(2n - 1)a_{2} - 2a_{2}(x + s)^{2} = 0.
\end{aligned}
\]

**Proof.** Since the jump matrices of the RH problem for \( \Phi(z; x, s) \) are independent of the variables \( z \) and \( x \), we have that \( \Phi_{z}(z; x, s), \Phi_{s}(z; x, s) \) and \( \Phi(z; x, s) \) satisfy the same jump condition. Thus, the coefficient \( A(z; x, s) \) in the differential equations is meromorphic for \( z \) in the complex plane with only possible isolated singularities at \( z = 0, \pm s \) and the coefficient \( B(z; x, s) \) is analytic for \( z \) in the complex plane. Then, it follows from the local behavior of \( \Phi(z; x, s) \) as \( z \to \infty, z \to \pm s \) that the coefficients \( A(z; x, s) \) and \( B(z; x, s) \) are rational functions in \( z \) with the form given in (2.19) and (2.20). Using the fact that \( \det \Phi = 1 \), we have \( \text{tr}A = \text{tr}B = 0 \) and thus all the coefficients \( A_{k}, k = 0, 1, 2 \) in (2.19) are trace-zero. Using the master equation in (2.18) and the local behavior \( \Phi(z) \) at \( z = \pm s \), we have

\[
\det A_{k} = 0, \quad k = 1, 2.
\]
We denote $(A_k)_{11} = a_k b_k$ for $k = 1, 2$.

Substituting the behavior of $\Phi$ at infinity into the master equation of the Lax pair (2.18), we find after comparing the coefficients of $z^0$ and $z^{-1}$ on both sides of the equation that

$$A_\infty = [\Phi_1, \sigma_3],$$

(2.25)

and

$$A_1 + A_2 = -n \sigma_3 + [\Phi_2 + x \Phi_1, \sigma_3] + [\sigma_3, \Phi_1],$$

(2.26)

where $\Phi_k$ is the coefficient of $z^{-k}$ in the large $z$ asymptotic expansion of $\Phi(z)$. In view of (2.25), we get

$$A_\infty = \begin{pmatrix} 0 & -2(\Phi_1)_{12} \\ 2(\Phi_1)_{21} & 0 \end{pmatrix}.$$

(2.27)

From the diagonal entries of the equation (2.26), we find the relation

$$2(\Phi_1)_{12}(\Phi_1)_{21} = n + (A_1 + A_2)_{11} = a_1 b_1 + a_2 b_2 + n.$$

(2.28)

We define

$$y = -2(\Phi_1)_{12},$$

(2.29)

then the above relations imply that

$$(A_\infty)_{12} = y, \quad (A_\infty)_{21} = -\frac{2}{y}(a_1 b_1 + a_2 b_2 + n).$$

(2.30)

We define $(A_k)_{12} = a_k y$ for $k = 1, 2$. Then, the other entries of $A_k$ can be expressed in terms of $a_k$ and $b_k$ for $k = 1, 2$, as given in (2.22).

Similarly, the coefficient $B(z) = B(z; x, s)$ can be determined by using the behavior of $\Phi$ at infinity

$$B(z) = \Phi_1(z)\Phi(z)^{-1} = z \sigma_3 + \begin{pmatrix} 0 & -2(\Phi_1)_{12} \\ 2(\Phi_1)_{21} & 0 \end{pmatrix} = z \sigma_3 + A_\infty.$$

(2.31)

The compatibility condition $\Phi_{zx} = \Phi_{xz}$ gives us the zero-curve equation

$$A_x = B_2 + [A, B] = 0.$$  

(2.32)

Substituting (2.19) and (2.20) into the above equation, the compatibility condition is equivalent to the system

\[
\begin{align*}
\frac{dA_\infty}{dx} &= x[A_\infty, \sigma_3] - [A_1, \sigma_3] - [A_2, \sigma_3], \\
\frac{dA_1}{dx} &= -[A_1, A_\infty] + s[A_1, \sigma_3], \\
\frac{dA_2}{dx} &= -[A_2, A_\infty] - s[A_2, \sigma_3],
\end{align*}
\]

(2.33)

which is known as the degenerate Schlesinger equations.

We then obtain the system of differential equations (2.23). Eliminating $b_1$ and $b_2$ from the system, we obtain the differential equations for $a_1$ and $a_2$ as given in (2.24). This completes the proof of proposition 3. ☐
Proposition 4. The Hamiltonian for the coupled Painlevé IV system is
\[
H_{\text{IV}}(a_1, a_2, b_1, b_2; x, s) = -2(a_1b_1 + a_2b_2 + n)(a_1 + a_2) + 2(a_1b_1(x - s) + a_2b_2(x + s) + nx) - (a_1b_1^2 + a_2b_2^2).
\] (2.34)

And the coupled Painlevé IV system (2.23) can be written as the Hamiltonian system:

\[
\begin{align*}
\frac{da_1}{dx} &= H_{\text{IV},a_1}(a_1, a_2, b_1, b_2; x, s), \\
\frac{da_2}{dx} &= H_{\text{IV},a_2}(a_1, a_2, b_1, b_2; x, s), \\
\frac{db_1}{dx} &= -H_{\text{IV},b_1}(a_1, a_2, b_1, b_2; x, s), \\
\frac{db_2}{dx} &= -H_{\text{IV},b_2}(a_1, a_2, b_1, b_2; x, s),
\end{align*}
\] (2.35)

where \(H_{\text{IV},a}\) denotes the partial derivative of \(H_{\text{IV}}\) with respect to \(a\).

Proof. With the local singularities \(z = \pm s\) fixed in the system of differential equations with rational coefficients (2.18), the Hamiltonian associated with the isomonodromy deformation of the system with respect to \(x\) is given by

\[
H_{\text{IV}}(x; s) = -\text{Res}_{s=\infty} \text{tr} \left( \phi^{(\infty)}(z)^{-1} \left[ \frac{d}{dz} \phi^{(\infty)}(z) \right] \frac{d}{dx} \Theta(z; x) \sigma_3 \right) = -2(\Phi_1)_{11}.
\] (2.36)

where \(\Theta(z; x) = (\frac{1}{4} z^2 + xz)\sigma_3\), \(\phi^{(\infty)}(z)\) and \(\Phi_1 = \Phi_1(x, s)\) are defined in (2.13); see for instance [23, equation (1.11)] and [22, equation (3)]. Using (2.3), (2.4) and (2.12), we have

\[
H_{\text{IV}}(x; s) = F(s_1, s_2) + 2nx
= 2nx + \frac{1 - \omega_1}{2\pi i} (\Phi^{-1}\Phi_2)_{12}(-s) + \frac{\omega_1 - \omega_2}{2\pi i} (\Phi^{-1}\Phi_2)_{12}(s).
\] (2.37)

From (2.14) and (2.16), we get

\[
H_{\text{IV}}(x; s) = 2nx + \frac{1 - \omega_1}{2\pi i} (P_1)_{12} + \frac{\omega_1 - \omega_2}{2\pi i} (Q_1)_{12},
\] (2.38)

where \(P_1 = P_1(x, s)\) and \(Q_1 = Q_1(x, s)\) are defined in (2.15) and (2.17), respectively. Substituting the expansions (2.14) and (2.16) into the master equation of (2.18), we obtain

\[
\frac{1 - \omega_1}{2\pi i} P_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} P_0^{-1} = A_1,
\] (2.39)

\[
P_1 + \frac{1 - \omega_1}{2\pi i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [(x - s)\sigma_3 + A_{\infty} - \frac{A_1}{2s}] P_0 = 0,
\] (2.40)

\[
\frac{\omega_1 - \omega_2}{2\pi i} Q_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Q_0^{-1} = A_2,
\] (2.41)

\[
Q_1 + \frac{\omega_1 - \omega_2}{2\pi i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [(x + s)\sigma_3 + A_{\infty} + \frac{A_1}{2s}] Q_0 = 0.
\] (2.42)
Now
\[
P_0^{-1} = \left( \begin{array}{cc}
0 & 0 \\
1 & 0 \\
\end{array} \right)
\]
\[
= \left( \begin{array}{cc}
(P_0)_{12} & -(P_0)_{12}^2 \\
(P_0)_{22} & (P_0)_{22}^2 \\
\end{array} \right).
\]
Then, a substitution of the above equation into (2.39) gives
\[
\frac{1 - \omega_1}{2\pi i}(P_0)_{12} = \frac{1 - \omega_1}{2\pi i}(P_0)_{22}
\]
\[
= \frac{1 - \omega_1}{2\pi i}(P_0)_{12} - a_i y, \quad \frac{1 - \omega_1}{2\pi i}(P_0)_{12}(P_0)_{22} = a_i b_1. \quad (2.43)
\]
Let
\[
\mathcal{A} = (x-s)\sigma_3 + A_\infty - \frac{A_2}{2s},
\]
we obtain from (2.40) that
\[
\frac{1 - \omega_1}{2\pi i}(P_1)_{12} = \frac{1 - \omega_1}{2\pi i}(2(P_0)_{12}(P_0)_{22}A_{11} - (P_0)_{12}^2A_{21} + (P_0)_{22}^2A_{12})
\]
\[
= 2a_i b_1 A_{11} + a_1 y A_{21} - \frac{a_i b_1^2}{y} A_{12}
\]
\[
= 2a_i b_1(x-s) - 2a_i (a_1 b_1 + a_2 b_2 + n) - a_i b_1^2 + \frac{1}{2s} a_i a_2 (b_1 - b_2)^2. \quad (2.44)
\]
Similarly, we get after some straightforward calculations
\[
\frac{\omega_1 - \omega_2}{2\pi i}(Q_0)_{22} = \frac{-a_2 b_1^2}{y}, \quad \frac{\omega_1 - \omega_2}{2\pi i}(Q_0)_{12} = -a_2 y,
\]
\[
\frac{\omega_1 - \omega_2}{2\pi i}(Q_0)_{12}(Q_0)_{22} = a_2 b_2, \quad (2.45)
\]
and
\[
\frac{\omega_1 - \omega_2}{2\pi i}(Q_1)_{12} = 2a_2 b_2(x+s) - 2a_2 (a_1 b_1 + a_2 b_2 + n) - a_2 b_2^2 - \frac{1}{2s} a_i a_2 (b_1 - b_2)^2. \quad (2.46)
\]
Then, the expression of the Hamiltonian (2.34) follows directly by substituting (2.44) and (2.46) into (2.38). In view of the Hamiltonian (2.34), it is seen that the coupled Painlevé IV system (2.23) is equivalent to the Hamiltonian system (2.35). This completes the proof of proposition 4. □

**Remark 6.** We note that the Lax pair (2.18) is equivalent to that appeared in the studies of the degeneration scheme of the Garnier system with five regular singular points in [26] by Kawakami et al. Let \( Y_{KNS}(\zeta; t_1, t_2; \theta^a, \theta^b, \theta^c) \) be the system used in [26, equation (3.11)], we have
\[
\Phi(\zeta; x, s) = e^{\frac{1}{4}(x-s)^2 - (x-s)(\zeta-x)} e^{i(x-s)^2} \sigma_1 Y_{KNS}\left(-\sqrt{2}(\zeta-s); \sqrt{2}(x-s), \sqrt{2}(x+s); 0, 0, -n\right) \sigma_1,
\]
where the Pauli matrix \( \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\
1 & 0 \end{array} \right) \). The functions \( u, p_1, p_2, q_1, q_2 \) appeared in the coefficients of the Lax pair [26, equation (3.11)] are related to \( y, a_1, a_2, b_1, b_2 \) in (2.21) and (2.22) by
\[
y(x, s)e^{-2xs-s^2} = \frac{\sqrt{q}}{u(t_1, t_2)}, \quad a_2(x,s) = \frac{1}{2\sqrt{2}} p_1(t_1, t_2), \quad b_4(x, s) = -\sqrt{2} q_1(t_1, t_2), \quad (2.48)
\]
where \( t_1 = \sqrt{2}(x - s) \), \( t_2 = \sqrt{2}(x + s) \) and \( k = 1, 2 \). Moreover, the Hamiltonian (2.34) can be expressed in terms of the Hamiltonians \( H_{\text{Gar}, 1}^{1+1} (t_1, t_2) \) and \( H_{\text{Gar}, 2}^{1+1} (t_1, t_2) \) given in [26, equations (3.12) and (3.13)]

\[
H_{\text{IV}}(x; s) = \sqrt{2} (H_{\text{Gar}, 1}^{1+1} (t_1, t_2) + H_{\text{Gar}, 2}^{1+1} (t_1, t_2)) + 2ns, \tag{2.49}
\]

with \( t_1 = \sqrt{2}(x - s) \), \( t_2 = \sqrt{2}(x + s) \).

### 2.3. Proof of theorem 1

The relation (1.22) follows from (2.37). Let \( Y_1 \) and \( Y_2 \) be the coefficients of \( 1/z \) and \( 1/z^2 \) in the expansion of \( Y \) near infinity (2.1), we have the following relations for the recurrence coefficients \( \alpha_n = \alpha_n(s_1, s_2), \beta_{n-1} = \beta_{n-1}(s_1, s_2) \) and the leading coefficient \( \gamma_n = \gamma_n(s_1, s_2) \) of the monic orthogonal polynomial of degree \( n - 1 \):

\[
\alpha_n = (Y_1)_{11} + \frac{(Y_2)_{12}}{(Y_1)_{12}}, \quad \beta_n^2 = (Y_1)_{12}(Y_1)_{21} \quad \text{and} \quad \gamma_n^2 = -\frac{1}{2\pi i} (Y_1)_{21}; \tag{2.50}
\]

see [15]. In view of (2.12), it is then seen that

\[
(Y_1)_{11} = -\Phi_1 \text{ or } nx, \quad (Y_1)_{12} = e^{-z^2} (\Phi_1)_{21}, \quad (Y_1)_{21} = e^{z^2} (\Phi_1)_{12}, \tag{2.51}
\]

and

\[
(Y_2)_{12} = e^{-z^2} ((n + 1)x(\Phi_1)_{21} + (\Phi_2)_{21}). \tag{2.52}
\]

where \( \Phi_1 \) and \( \Phi_2 \) are defined in (2.13). From the relation (2.26), we have

\[
x(\Phi_1)_{21} + (\Phi_2)_{21} = \frac{1}{2} (A_1 + A_2)_{21} + (\Phi_1)_{11}(\Phi_1)_{21}. \tag{2.53}
\]

Substituting (2.51) into (2.50) and recalling (2.28)-(2.30), we obtain that

\[
\beta_n^2 = (\Phi_1)_{12}(\Phi_1)_{21} = \frac{1}{2} (a_1(x; s)b_1(x; s) + a_2(x; s)b_2(x; s) + n), \tag{2.54}
\]

and

\[
\gamma_n^2 = -\frac{1}{2\pi i} e^{z^2} (\Phi_1)_{12} = \frac{1}{4\pi i} e^{z^2} y(x; s) \neq 0. \tag{2.55}
\]

On account of (2.23), we have

\[
\frac{d}{dx} \ln \gamma_n = a_1(x; s) + a_2(x; s). \tag{2.56}
\]

Inserting (2.51)-(2.53) into (2.50) yields

\[
\alpha_n = -(\Phi_1)_{11} + \frac{(\Phi_2)_{21} + x(\Phi_1)_{21}}{(\Phi_1)_{21}}
\]

\[
= \frac{1}{2} \frac{(A_1 + A_2)_{21}}{(\Phi_1)_{21}},
\]

\[
= \frac{a_1(x; s)b_1(x; s) + a_2(x; s)b_2(x; s)}{2(a_1(x; s)b_1(x; s) + a_2(x; s)b_2(x; s) + n)}. \tag{2.57}
\]
In summary, we obtain (1.23)–(1.26) by collecting (2.54)–(2.57).

From (2.2), (2.12), (2.14) and (2.16), it is seen that

\[
\pi_n(s_1) = (Y)_{11}(s_1) = e^{\frac{1}{2}i^2-xi}(\Phi_{22})(s_1) = e^{\frac{1}{2}i^2-xi}(P_0)_{22},
\]

(2.58)

\[
\pi_n(s_2) = (Y)_{11}(s_2) = e^{\frac{1}{2}i^2+xi}(\Phi_{22})(s_2) = e^{\frac{1}{2}i^2+xi}(Q_0)_{22}.
\]

(2.59)

Therefore, we obtain (1.27) and (1.28) after replacing the expressions of \((P_0)_{22}\) and \((Q_0)_{22}\) by (2.43) and (2.45). Similarly, we derive (1.29) and (1.30) by considering \((Y)_{21}(s_k), k = 1, 2\).

This completes the proof of theorem 1.

2.4. Proof of corollary 1

As (1.21), we denote

\[
x = \frac{s_1 + s_2}{2}, \quad s = \frac{s_2 - s_1}{2}.
\]

(2.60)

From the multiple integral representation of the Hankel determinant in (1.4), we see that

\[
D_n(s_1, s_2) = \det \left( \int_{\mathbb{R}} t^{ij} w(t + x) dt \right)_{jk=0}^{n-1}.
\]

(2.61)

By (1.5), the moments of \(w(t + x)\) can be written as

\[
m_k(x, s) = \int_{\mathbb{R}} t^k w(t + x) dt
\]

\[
= e^{-x^2} \left( \int_{-\infty}^{-x} \int_{-\infty}^{t} e^{-t^2-2ut} \, dt \, dt + \omega_1 \int_{-\infty}^{+\infty} \int_{-\infty}^{t} e^{-t^2-2ut} \, dt \, dt \right.
\]

\[
+ (\omega_2 - \omega_1) \int_{-\infty}^{+\infty} \int_{-\infty}^{t} e^{-t^2-2ut} \, dt \, dt \bigg),
\]

(2.62)

where \(k \geq 0\). Taking derivative on both side of the above equation yields

\[
\frac{d}{dx} (e^{x^2} m_k(x, s)) = -2 e^{x^2} m_{k+1}(x, s).
\]

(2.63)

Thus, we have for \(k \geq 0\)

\[
\frac{d^k}{dx^k} (e^{x^2} m_0(x, s)) = (-2)^k e^{x^2} m_k(x, s).
\]

(2.64)

With \(\varphi\) given by (1.35), we get

\[
m_0(x, s) = \int_{\mathbb{R}} w(t + x) dt = \int_{\mathbb{R}} w(t) dt = e^{-\frac{\varphi(s_1 + s_2)^2}{4}} \varphi(s_1, s_2).
\]

(2.65)

Then, we obtain from (2.60), (2.64) and (2.65) for \(k \geq 0\)

\[
m_k(x, s) = (-2)^k e^{\frac{\varphi(s_1 + s_2)^2}{4}} \delta^k \varphi(s_1, s_2),
\]

(2.66)
where $\delta = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Substituting (2.66) into (2.61), we obtain (1.38). The relation (1.39) then follows from (1.20), (1.22) and (1.38). To derive (1.40), we take derivative on (2.8) and obtain

$$
\frac{\partial}{\partial s_1} \gamma_{n-1} = \frac{\partial}{\partial s_1} \int_{\mathbb{R}} \pi_{n-1}(t)^2 w(t) dt = (1 - \omega_1) \pi_{n-1}(s_1) e^{-s_1^2}.
$$

(2.67)

This, together with (1.29), gives

$$
a_1(x, s) = \frac{\partial}{\partial s_1} \ln \gamma_{n-1} = \frac{1}{2} \frac{\partial}{\partial s_1} \ln \frac{D_{n-1}}{D_n}.
$$

(2.68)

We then obtain (1.40) by substituting (1.38) into (2.68). The equation (1.41) can be derived in a similar way. This completes the proof of corollary 1.

2.5. Generalization to the Gaussian weight with more than two discontinuities

In this section, we consider the generalization to the Gaussian weight with more than two discontinuities

$$
u(t; \tilde{s}, \tilde{\omega}) = e^{i\omega t} \begin{cases} 
1, & t \leq s_1, \\
\omega_k, & s_k < t < s_{k+1}, 1 \leq k \leq m - 1, \\
\omega_m, & t \geq s_m,
\end{cases}
$$

(2.69)

where

$$
\tilde{s} = (s_1, \ldots, s_m), \quad \tilde{\omega} = (\omega_1, \ldots, \omega_m),
$$

(2.70)

$s_1, \ldots, s_m, \omega_k \geq 0, 1 \leq k \leq m$ and $\omega_1 \neq 1, \omega_k \neq \omega_{k+1}$ for $1 \leq k \leq m - 1, m \geq 2$. There exist a unique sequence of monic polynomials $\{\pi_n(z; \tilde{s}, \tilde{\omega})\}_{n=0}^{\infty}$ orthogonal with respect to the weight function (2.69). Similar to (2.2), we define

$$
Y(z; \tilde{s}, \tilde{\omega}) = \begin{pmatrix} 
\pi_n(z; \tilde{s}, \tilde{\omega}) \\
-2\pi i n_{n-1} \pi_{n-1}(z; \tilde{s}, \tilde{\omega})
\end{pmatrix} = \begin{pmatrix} 
1 \\
-2\pi i n_{n-1}
\end{pmatrix} \frac{\pi_n(x; \tilde{s}, \tilde{\omega}) u(x; \tilde{s}, \tilde{\omega})}{x - z} \cdot \frac{1}{\pi_{n-1}(x; \tilde{s}, \tilde{\omega}) u(x; \tilde{s}, \tilde{\omega})} dx.
$$

(2.71)

Then, $Y(z; \tilde{s}, \tilde{\omega})$ satisfies a similar RH problem as $Y$ given in section 2.1. Let $\Phi(z; x, \tilde{c}, \tilde{\omega})$ be given by

$$
\Phi(z; x, \tilde{c}, \tilde{\omega}) = \sigma_1 e^{\frac{2}{\delta} \sigma_3} Y(z + x; \tilde{s}, \tilde{\omega}) e^{-\frac{2}{\delta} (z + x)^2 \sigma_3} \sigma_1,
$$

(2.72)

where $x = s_1$ and $\tilde{c} = (0, \ldots, s_m - s_1)$. Then $\Phi(z; x, \tilde{c}, \tilde{\omega})$ satisfies the following RH problem.

2.5.1. Riemann–Hilbert problem for $\Phi(z; x, \tilde{c}, \tilde{\omega})$.

(a) $\Phi(z; x, \tilde{c}, \tilde{\omega})$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.
(b) $\Phi(z, x, \vec{c}, \vec{\omega})$ satisfies the jump condition

$$
\Phi_+ (z, x, \vec{c}, \vec{\omega}) = \Phi_- (z, x, \vec{c}, \vec{\omega}) \begin{cases}
(1 & 0), \\
(1 & 1), & z < c_1, \\
(1 & 0), & c_{k-1} < z < c_k, 1 < k \leq m, \\
(1 & 0), & z > c_m,
\end{cases}
$$

where $\vec{c} = (c_1, \ldots, c_m)$.

(c) The behavior of $\Phi(z, x, \vec{c}, \vec{\omega})$ at infinity is

$$
\Phi(z, x, \vec{c}, \vec{\omega}) = \left(I + \frac{\Phi_1}{z} + \frac{\Phi_2}{z^2} + O\left(\frac{1}{z^3}\right)\right) e^{i \frac{1}{4} z^2 + x(\omega_1 z - \nu_3)}; \quad (2.73)
$$

(d) The behavior of $\Phi(z, x, \vec{c}, \vec{\omega})$ near $c_k$ is

$$
\Phi(z, x, \vec{c}, \vec{\omega}) = \Phi^{(c_k)}(z, x, \vec{c}, \vec{\omega}) \left(I + \frac{\omega_{k-1} - \omega_k}{2\pi i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ln(z - c_k) \right) E^{(k)}, \quad (2.74)
$$

where $\arg(z - c_k) \in (-\pi, \pi)$, $1 \leq k \leq m$. Here, $\Phi^{(c_k)}(z)$ is analytic near $z = c_k$ and has the following expansion

$$
\Phi^{(c_k)}(z) = Q_{c_k}(x)(I + Q_{c_k}(x)(z - c_k) + O((z - c_k)^2)), \quad (2.75)
$$

with $1 \leq k \leq m$. The piecewise constant matrix $E^{(k)}$, $1 \leq k \leq m$ is defined by

$$
E^{(k)} = \begin{cases}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{Im } z > 0, \\
\begin{pmatrix} 1 & 0 \\ -\omega_k & 1 \end{pmatrix}, & \text{Im } z < 0.
\end{cases}
$$

**Proposition 5.** Given $c_1, \ldots, c_m$, we have the following Lax pair for $\Phi(z, x) = \Phi(z, x, \vec{c})$

$$
\Phi_+(z, x) = A(z, x)\Phi(z, x), \quad \Phi_+(z, x) = B(z, x)\Phi(z, x), \quad (2.76)
$$

where

$$
A(z, x) = (x + z)\sigma_3 + A_{\infty}(x) + \sum_{k=1}^m \frac{A_k(x)}{z - c_k}, \quad (2.77)
$$

$$
B(z, x) = x\sigma_3 + A_{\infty}(x), \quad (2.78)
$$

with the coefficients given below

$$
A_{\infty}(x) = \begin{pmatrix} 0 & y(x) \\ -2 \sum_{k=1}^m a_k(x)b_k(x) + n & y(x) \end{pmatrix}, \quad (2.79)
$$
\[ A_k(x) = \begin{pmatrix} a_k(x) b_k(x) & a_k(x) y(x) \\ -a_k(x) b_k^2(x)/y(x) & -a_k(x) b_k(x) \end{pmatrix}, \quad 1 \leq k \leq m. \tag{2.80} \]

The compatibility condition of the Lax pair gives us the coupled Painlevé IV system

\[
\begin{align*}
\frac{dy}{dx} &= 2 \left( \sum_{j=1}^{m} a_j - x \right) y, \\
\frac{da_k}{dx} &= -2a_k \left( \sum_{j=1}^{m} a_j + b_k - x - c_k \right), \\
\frac{db_k}{dx} &= b_k^2 + 2b_k \left( \sum_{j=1}^{m} a_j - x - c_k \right) + 2 \left( \sum_{j=1}^{m} a_j b_j + n \right). \tag{2.81}
\end{align*}
\]

Let \( H_{IV}(x; \vec{c}) = H_{IV}(a_1, \ldots, a_m, b_1, \ldots, b_m; x, \vec{c}) \) be given by

\[
H_{IV}(x; \vec{c}) = 2nx + 2 \sum_{k=1}^{m} a_k b_k(x + c_k) - 2 \left( \sum_{k=1}^{m} a_k b_k + n \right) \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} a_k b_k^2, \tag{2.82}
\]

then the coupled Painlevé IV system (2.81) are equivalent to the following Hamilton equations

\[
\frac{da_k}{dx} = \frac{\partial H_{IV}}{\partial b_k}, \quad \frac{db_k}{dx} = -\frac{\partial H_{IV}}{\partial a_k}, \quad 1 \leq k \leq m. \tag{2.83}
\]

Moreover, eliminating \( b_1, \ldots, b_m \) from the system, then \( a_1, \ldots, a_m \) satisfy the following nonlinear differential equations with \( 1 \leq k \leq m \)

\[
\frac{d^2 a_k}{dx^2} - \frac{1}{2a_k} \left( \frac{da_k}{dx} \right)^2 - 4a_k \sum_{j=1}^{m} a_j \left( \sum_{i=1}^{m} a_i - x - c_j \right) \\
- 2a_k \left( \sum_{j=1}^{m} a_j - x - c_k \right)^2 + 2(2n - 1)a_k = 0. \tag{2.84}
\]

**Proof.** Since the jump matrices of the RH problem for \( \Phi(z; x) \) are independent of the variables \( z \) and \( x \), we see that the coefficient \( A(z) \) is rational in \( z \) with possible isolated singularity at \( z = c_k, 1 \leq k \leq m \). Using the fact that \( \det \Phi = 1 \), we have \( \text{tr} A = 0 \) and thus all the coefficients \( A_k, 1 \leq k \leq m \) in (2.77) are trace-zero. From (2.76) and the local behavior \( \Phi(z) \) at \( z = c_k \), we have

\[
\det A_k = 0, \quad 1 \leq k \leq m.
\]

Thus, the coefficients \( A_k, 1 \leq k \leq m \) can be parameterized as (2.80) with a free parameter \( y \) to be defined later.

Substituting the behavior of \( \Phi \) at infinity into (2.76) and comparing the coefficients of \( z^0 \) and \( z^{-1} \) on both sides of the equation, we obtain

\[
A_\infty = [a_1, \sigma_1], \tag{2.85}
\]

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and
\[ \sum_{k=1}^{m} A_k = -n\sigma_3 + [\Phi_2 + x\Phi_1, \sigma_3] + [\sigma_3, \Phi_1]\Phi_1. \]  
(2.86)

From (2.85), we get
\[ A_{\infty} = \begin{pmatrix} 0 & -2(\Phi_1)_{12} \\ 2(\Phi_1)_{21} & 0 \end{pmatrix}. \]  
(2.87)

Comparing the diagonal entries of both side of the equation (2.86), we see
\[ 2(\Phi_1)_{12}(\Phi_1)_{21} = \sum_{k=1}^{m} a_k b_k + n. \]  
(2.88)

Let
\[ y = -2(\Phi_1)_{12}, \]  
(2.89)

then we arrive at (2.79) by using (2.87) and (2.88). Similarly, the coefficient \( B(z) = B(z; x) \) can be determined by using the behavior of \( \Phi \) at infinity
\[ B(z) = z\sigma_3 + \begin{pmatrix} 0 & -2(\Phi_1)_{12} \\ 2(\Phi_1)_{21} & 0 \end{pmatrix} = z\sigma_3 + A_{\infty}. \]  
(2.90)

Substituting (2.77) and (2.78) into the zero-curve equation (2.32), we obtain the following degenerate Schlesinger equations
\[
\begin{cases}
\frac{dA_k}{dx} = -[A_k, A_{\infty}] - c_k[A_k, \sigma_3], \\
\frac{dA_{\infty}}{dx} = x[A_{\infty}, \sigma_3] - \sum_{k=1}^{m} [A_k, \sigma_3].
\end{cases}
\]  
(2.91)

We then obtain the system of differential equations (2.81). It is straightforward to check that the Hamilton equations (2.83) are indeed the equations (2.81). Finally, eliminating \( b_1, \ldots, b_m \) from (2.81) gives us (2.84). This completes the proof of proposition 5.

It is direct to see that the Lax pair for \( \Phi(z; x, \vec{c}, \vec{\omega}) \), the coupled Painlevé IV system, the Hamiltonian formulation and the system of nonlinear differential equations for \( a_1, \ldots, a_m \), given in proposition 5 are consistent with those obtained in proposition 3, when the number of regular singularities \( m = 2 \) in (2.76).

Moreover, similar to theorem 1, we show that the logarithmic derivative of the Hankel determinant generated by the Gaussian weight with more than two discontinuities can be expressed in terms of the Hamiltonian of \( \Phi(z; x, \vec{c}, \vec{\omega}) \).

**Proposition 6.** Let \( D_n(s_1, s_2, \ldots, s_m) \) be the Hankel determinant generated by (2.69) and denote
\[ F(s_1, s_2, \ldots, s_m) = \sum_{k=1}^{m} \frac{\partial}{\partial s_k} \ln D_n(s_1, s_2, \ldots, s_m). \]  
(2.92)
then we have
\[ F(s_1, s_2, \ldots, s_m) = H_{IV}(s_1; \vec{c}) - 2ns_1, \tag{2.93} \]
where \(H_{IV}(x; \vec{c})\) is given in (2.82) with \(\vec{c} = (0, s_2 - s_1, \ldots, s_m - s_1)\).

**Proof.** As (2.3), we have the differential identity
\[ F(s_1, s_2, \ldots, s_m) = \frac{1}{2\pi i} \sum_{k=1}^m (\omega_{k-1} - \omega_k) e^{-\frac{i}{2} \omega_k(Y^{-1} Y')_2}(s_k), \tag{2.94} \]
where \(\omega_0 = 1\). This, together with (2.72), implies
\[ F(s_1, s_2, \ldots, s_m) = \sum_{k=1}^m \omega_{k-1} \omega_k (\Phi^{-1} \Phi_z)_{12}(c_k), \tag{2.95} \]
where \(c_1 = 0\) and \(c_k = s_k - s_1, 2 \leq k \leq m\). Substituting (2.74) into (2.76), we obtain
\[ \frac{\omega_{k-1} - \omega_k}{2\pi i} (\Phi^{-1} \Phi_z)_{12}(c_k) = 2a_k b_k(x + c_k) - 2ak \left( \sum_{j=1}^m a_j b_j + n \right) - a_k b_k^2 - \sum_{j=1, j \neq k}^m \frac{a_k a_j (b_k - b_j)^2}{c_k - c_j}. \tag{2.96} \]
Note that
\[ \sum_{k=1}^m \sum_{j=1, j \neq k}^m a_k a_j (b_k - b_j)^2 \frac{1}{c_k - c_j} = 0. \tag{2.97} \]
We obtain (2.93) by substituting (2.96) into (2.95). This completes the proof. \(\square\)

### 3. Nonlinear steepest descent analysis of the Riemann–Hilbert problem for \(Y\)

In this section, we take \(s_1 = \sqrt{2n} + \frac{1}{\sqrt{2n}}\) and \(s_2 = \sqrt{2n} + \frac{\sqrt{2n}}{6}\) in the weight function (1.5). Then, we perform Deift–Zhou nonlinear steepest descent analysis [14–16] for the RH problem for \(Y(z; s_1, s_2)\) as \(n \to \infty\). The analysis will allow us to find the asymptotics of the Hankel determinants and the orthogonal polynomials associated with (1.5). The analysis of the RH problem with one jump singularity near the soft edge \(\sqrt{2n}\) in the weight function (1.5) is considered in [1, 34]. The RH method has also been applied in the studies of the asymptotics of the Hankel determinants generated by Gaussian type weight functions with Fisher–Hartwig singularities on the real axis; see for instance [5, 8, 10].

#### 3.1. The first transformation: \(Y \to T\)

The first transformation is defined by
\[ T(z) = (2n)^{-\frac{1}{4}n\sigma_3} e^{-\frac{1}{4}i\sigma_3} Y(\sqrt{2n} z) e^{\left(\frac{i}{2} - g(z)\right)\sigma_3}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.1} \]
where the constant \(l = -1 - 2\ln 2\). The \(g\)-function therein is defined by
\[ g(z) = \frac{2}{\pi} \int_{-1}^1 \ln(z - x) \sqrt{1 - x^2} \, dx, \tag{3.2} \]
where the logarithm takes the principal branch $\text{arg}(z - x) \in (-\pi, \pi)$. We then introduce the $\phi$-function

$$\phi(z) = z\sqrt{z^2 - 1} - \ln \left( z + \sqrt{z^2 - 1} \right),$$

(3.3)

where the principal branches are chosen. The $\phi$-function and $g$-function are related by

$$2 \left[ g(z) + \phi(z) \right] - 2z^2 - l = 0, \quad z \in \mathbb{C} \setminus (-\infty, 1].$$

(3.4)

As a consequence, $T$ is normalized at infinity

$$T(z) = I + O(1/z),$$

and satisfies the same behavior as $Y(\sqrt{2nz})$ near $z = \lambda_k, k = 1, 2$:

$$T(z) = \begin{pmatrix} O(1) & O(\ln|z - \lambda_k|) \\ O(1) & O(\ln|z - \lambda_k|) \end{pmatrix},$$

while $T$ satisfies the following jump condition

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & \theta(x)e^{-2\phi(x)} \\ 0 & 1 \end{pmatrix}, \quad x \in (1, +\infty),$$

$$T_+(x) = T_-(x) \begin{pmatrix} e^{2\phi_+(x)} & \theta(x) \\ 0 & e^{2\phi_-(x)} \end{pmatrix}, \quad x \in (-1, 1),$$

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & e^{-2\phi_+(x)} \\ 0 & 1 \end{pmatrix}, \quad x \in (-\infty, -1),$$

(3.5)

where $\theta(x) = \begin{cases} 1 & x < \lambda_1, \\ \omega_1 & \lambda_1 < x < \lambda_2, \text{ with } \lambda_1 = 1 + \frac{\beta}{2n} \text{ and } \lambda_2 = 1 + \frac{\beta}{2n}, \\ \omega_2 & x > \lambda_2, \end{cases}$

3.2. The second transformation: $T \rightarrow S$

In the second transformation, we define

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lens}, \\ T(z) \begin{pmatrix} 1 & 0 \\ -e^{2\phi(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the upper lens}, \\ T(z) \begin{pmatrix} 1 & 0 \\ e^{2\phi(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the lower lens}, \end{cases}$$

(3.6)
Figure 1. The jump contours and regions for the RH problem for $S$ when $\lambda_1 > 1$.

where the regions are illustrated in figure 1. Then $S$ satisfies the jump condition

$$S_+(z) = S_-(z)J_\gamma(z).$$

(3.7)

For $\lambda_1 > 1$, we have

$$J_\gamma(z) =
\begin{cases}
\begin{pmatrix} 1 & \omega_2 e^{-2n_\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (\lambda_2, +\infty), \\
\begin{pmatrix} 1 & \omega_1 e^{-2n_\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (\lambda_1, \lambda_2), \\
\begin{pmatrix} 0 & e^{-2n_\phi(z)} \\ -e^{2n_\phi(z)} & 0 \end{pmatrix}, & z \in (1, \lambda_1), \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-1, 1), \\
\begin{pmatrix} 1 & 0 \\ e^{2n_\phi(z)} & 1 \end{pmatrix}, & z \text{ on lens,} \\
\begin{pmatrix} 1 & 0 \\ -2n_\phi(z) & 1 \end{pmatrix}, & z \in (-\infty, -1),
\end{cases}
$$

(3.8)

where the contours are indicated in figure 1.

For $\lambda_1 < 1 < \lambda_2$, we have

$$J_\gamma(z) =
\begin{cases}
\begin{pmatrix} 1 & \omega_2 e^{-2n_\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (\lambda_2, +\infty), \\
\begin{pmatrix} 1 & \omega_1 e^{-2n_\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (1, \lambda_2), \\
\begin{pmatrix} e^{2n_\phi(z)} & \omega_1 \\ 0 & e^{2n_\phi(z)} \end{pmatrix}, & z \in (\lambda_1, 1), \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-1, \lambda_1), \\
\begin{pmatrix} 1 & 0 \\ e^{2n_\phi(z)} & 1 \end{pmatrix}, & z \text{ on lens,} \\
\begin{pmatrix} 1 & e^{-2n_\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (-\infty, -1).
\end{cases}
$$
For $\lambda_1 < \lambda_2 < 1$, we have

$$J_S(z) = \begin{cases} 
(1 - 2\omega_2 e^{-2n\phi(z)}) & z \in (1, +\infty), \\
(1 - 2\omega_2 e^{-2n\phi(z)}) & z \in (\lambda_2, 1), \\
(0 - 2\omega_2 e^{-2n\phi(z)}) & z \in (\lambda_1, \lambda_2), \\
(0 - 1) & z \in (-1, \lambda_1), \\
(-1 - 1) & z \in (\lambda_1, 1), \\
(1 - 1) & z \in (1, +\infty), \\
(1 - 1) & z \in (\lambda_2, 1), \\
(0 - 1) & z \in (\lambda_1, \lambda_2), \\
(0 - 1) & z \in (-\infty, -1).
\end{cases}$$

We also mention that $S(z)$ satisfies the same behavior as $T(z)$ near $z = \lambda_k$, $k = 1, 2$ for $z$ outside the lens-shaped region indicated in figure 1:

$$S(z) = \begin{pmatrix} O(1) & O(\ln |z - \lambda_1|) \\
O(1) & O(\ln |z - \lambda_1|) \end{pmatrix}. \quad (3.9)$$

This, together with the relation (3.6), determines the behavior of $S(z)$ for $z$ inside the lens-shaped region and near $\lambda_k, k = 1, 2$.

3.3. Global parametrix

The global parametrix solves the following approximating RH problem, with the jump along $(-1, \lambda_1)$:

(a) $N(z)$ is analytic in $\mathbb{C} \setminus [-1, \lambda_1]$;

(b) $N_+(x) = N_-(x) \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}, \quad x \in (-1, \lambda_1); \quad (3.10)$

(c) $N(z) = I + O(z^{-1}), \quad z \to \infty. \quad (3.11)$

The solution of the RH problem is constructed explicitly (see [32]):

$$N(z) = \begin{pmatrix} \eta(z) + \eta^{-1}(z) & \eta(z) - \eta^{-1}(z) \\
\eta(z) - \eta^{-1}(z) & \eta(z) + \eta^{-1}(z) \end{pmatrix}, \quad \eta(z) = \left(\frac{z - \lambda_1}{z + 1}\right)^{1/4}, \quad (3.12)$$
Figure 2. The jump contours and regions for the RH problem for $\Psi$ for $s > 0$.

where the branch is chosen such that $\eta(z)$ is analytic in $\mathbb{C} \setminus [-1, \lambda_1]$, and $\eta(z) \sim 1$ as $z \to \infty$.

3.4. Local parametrix near $z = 1$

The jump matrices for $S(z)$ are not close to the identity matrix near the node points $z = \pm 1$. Thus, local parametrices have to be constructed in the neighborhoods of $z = \pm 1$. Near $z = -1$, the parametrix $P^{(-)}(z)$ can be constructed in terms of the Airy function \cite{12,15}. We proceed to find a local parametrix $P^{(1)}(z)$ in $U(1, r)$, which is an open disc centered at $z = 1$ with radius $r > 0$. The parametrix solves the following RH problem:

3.4.1. Riemann–Hilbert problem for $P^{(1)}$.

(a) $P^{(1)}(z)$ is analytic in $U(1, r) \setminus \Sigma_3$;
(b) On $\Sigma_3 \cap U(1, r)$, $P^{(1)}(z)$ satisfies the same jump condition as $S(z)$,
\begin{equation}
P^{(1)}_+(z) = P^{(1)}_-(z) J(z), \quad z \in \Sigma_3 \cap U(1, r);
\end{equation}
(c) $P^{(1)}(z)$ satisfies the following matching condition on $\partial U(1, r)$:
\begin{equation}
P^{(1)}(z) N^{-1}(z) = I + O \left( n^{-1/3} \right);
\end{equation}
(d) $P^{(1)}(z)$ satisfies the same behavior as $S(z)$ as $z \to \lambda_k$ for $k = 1, 2$; see (3.9).

To construct the local parametrix, we introduce the following model RH problem, which shares the same jump condition as $P^{(1)}(z) e^{-\pi n(1+i)}$.

3.4.2. The Riemann–Hilbert problem for $\Psi$.

(a) $\Psi(\zeta; x, s)$ (or $\Psi(\zeta)$ for short) is analytic in $C \setminus \bigcup_{j=0}^4 \Sigma_j$, where the jump contours are indicated in figure 2.
(b) $\Psi(\zeta)$ satisfies the jump condition for $s > 0$
\[ \Psi_+(\zeta) = \Psi_-(\zeta) \begin{cases} 
\begin{pmatrix} 1 & \omega_2 \\
0 & 1 \end{pmatrix}, & \zeta \in (s, +\infty), \\
\begin{pmatrix} 1 & \omega_1 \\
0 & 1 \end{pmatrix}, & \zeta \in (0, s), \\
\begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix}, & \zeta \in \Sigma_2, \\
\begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}, & \zeta \in \Sigma_3, \\
\begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix}, & \zeta \in \Sigma_4. 
\end{cases} \] (3.15)

(c) As \( \zeta \to \infty \),
\[
\Psi(\zeta) = \left( \begin{array}{c}
1 \\
ir(x, s) \\
0 \\
1
\end{array} \right) \left[ I + \frac{\Psi_1(x, s)}{\zeta} + O\left(\zeta^{-2}\right) \right] \times \zeta^{-\frac{1}{2}} I + i\sigma_1 \frac{I + \omega}{\sqrt{2}} e^{-\left(\frac{\omega}{2}\zeta^2 + i\omega^2 \zeta \sigma_3 \right)},
\] (3.16)

where \( r(x, s) = i(\Psi_1(x, s))_{12} \).

(d) As \( \zeta \to 0 \),
\[
\Psi(\zeta) = \Psi^{(0)}(\zeta) \left[ I + \frac{1 - \omega_1}{2\pi i} \begin{pmatrix} 0 & 1 \\
0 & 0 \end{pmatrix} \ln \zeta \right] E,
\] (3.17)

where \( \Psi^{(0)}(\zeta) \) is analytic at \( \zeta = 0 \) with the expansion
\[
\Psi^{(0)}(\zeta) = \hat{P}_0(x, s)(I + \hat{P}_1(x, s)\zeta + O(\zeta^2)),
\] (3.18)

and the piecewise constant matrix
\[
E = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}, & \zeta \in I, \\
\begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix}, & \zeta \in II, \\
\begin{pmatrix} 1 - \omega_1 & -\omega_1 \\
1 & 1 \end{pmatrix}, & \zeta \in III, \\
\begin{pmatrix} 1 & -\omega_1 \\
0 & 1 \end{pmatrix}, & \zeta \in IV.
\end{cases}
\]

(e) As \( \zeta \to s \),
\[
\Psi(\zeta) = \Psi^{(1)}(\zeta) \left( I + \frac{\omega_1 - \omega_2}{2\pi i} \begin{pmatrix} 0 & 1 \\
0 & 0 \end{pmatrix} \ln(\zeta - s) \right) \hat{E},
\] (3.19)

where \( \Psi^{(1)}(\zeta) \) is analytic at \( \zeta = s \) with the following expansion
\[
\Psi^{(1)}(\zeta) = \hat{Q}_0(x, s)(I + \hat{Q}_1(x, s)(\zeta - s) + O((\zeta - s)^2)).
\] (3.20)
Here, the piecewise constant matrix
\[
\hat{E} = \begin{cases} 
  \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{Im } \zeta > 0, \\
  \begin{pmatrix} 1 & -\omega_2 \\ 0 & 1 \end{pmatrix}, & \text{Im } \zeta < 0.
\end{cases}
\]

The RH problem for \( \Psi \) appears recently in the studies of the Fredholm determinants of Painlevé II kernel and Painlevé XXXIV kernel in [33] by the second author of the present work and Dai. It also arises in the studies of the determinants of the Airy kernel with several discontinuities in [9] by Claeys and Doeraene, when the number of discontinuities therein equals to two. The existence of solution to the RH problem for \( \Psi \) is proved in [9, 33]. It is also shown therein that \( \Psi(\zeta; x, s) \) satisfies the following Lax pair
\[
\Psi_x(\zeta; x, s) = -i \left( \zeta + x + v_1 + v_2 + \frac{v_1 x}{2 \zeta} + \frac{v_2 x}{4 \zeta^2 (\zeta - s)} \right) \Psi(\zeta; x, s),
\]
\[
\Psi_s(\zeta; x, s) = \begin{pmatrix} 0 & i \\ -i \zeta - 2i(v_1 + v_2 + \frac{x}{2}) & 0 \end{pmatrix} \Psi(x; s).
\]

The compatibility condition of the Lax pair is described by the coupled Painlevé II system (1.42). Moreover, the Hamiltonian (1.43) is related to the coefficient of \( 1/\zeta \) in the large-\( \zeta \) expansion of \( \Psi(\zeta) \) in (3.16) by
\[
H_{01}(x; s) = \frac{x^2}{4} + r(x; s);
\]
see [33, equation (4.21)].

Using (3.3), we define the conformal mapping
\[
f(z) = \left( \frac{3}{2} \phi(z) \right)^{2/3} = 2(z - 1) + \frac{1}{5}(z - 1)^2 + O((z - 1)^3),
\]
from a neighborhood of \( z = 1 \) to that of the origin. Then the local parametrix \( P^{11}(z) \) can be constructed for \( z \in U(1, r) \) as follows
\[
P^{11}(z) = E(z) \Psi \left( n^{2/3}(f(z) - f(\lambda_1)), n^{2/3}(f(\lambda_2) - f(\lambda_1)), n^{2/3}(f(\lambda_3) - f(\lambda_1)) \right) e^{n(z^{2/3})},
\]
and the pre-factor
\[
E(z) = N(z) \left( 1 - i\sigma_1 \right)^{\frac{1}{2}} \left( n^{2/3}(f(z) - f(\lambda_1)) \right)^{\sigma_1/4}
\times \begin{pmatrix} 1 \\ iH_{11}(n^{2/3}(f(\lambda_1)), n^{2/3}(f(\lambda_2) - f(\lambda_1))) \end{pmatrix}.
\]

where \( H_{11}(x; t) \) is the Hamiltonian given in (1.43) and related to the coefficient of \( 1/\zeta \) in the large-\( \zeta \) expansion of \( \Psi(\zeta) \) by (3.23).
Proposition 7. For

\[ \lambda_k = \frac{s_k}{\sqrt{2n}} = 1 + \frac{t_k}{2n^{2/3}} \]

with bounded real parameters \( t_k, k = 1, 2 \), such that \( t_1 < t_2 \), the local parametrix defined in (3.25) and (3.26) solves the RH problem for \( P^{(1)} \). Moreover, we have the expansion as \( n \to \infty \) for \( z \in \partial U(1, r) \):

\[ P^{(1)}(z)N(z)^{-1} = I + \frac{\Delta(z)}{n^{1/3}} + O(n^{-2/3}), \]

where

\[ \Delta(z) = \frac{H(z)(n^{2/3}f(\lambda_1); n^{2/3}(f(\lambda_2) - f(\lambda_1)))}{2(f(z) - f(\lambda_1))^{1/2}}N(z)(\sigma_3 - i\sigma_1)N^{-1}(z) = O(1), \]

and \( H_\tau(x; i) \) is the Hamiltonian defined in (1.43).

**Proof.** Taking the principal branch for the fractional power, it follows from (3.24) that

\[ (f(x) - f(\lambda_1))^\frac{1}{2} = (f(x) - f(\lambda_1))^\frac{1}{2} e^{\frac{\pi}{2}i}, \quad x < \lambda_1. \]

This, together with the expression of \( N(z) \) in (3.12), implies that \( E(z) \) is analytic for \( z \) in \( U(1, r) \). Recalling the properties of \( \Psi(z) \) in (3.15)–(3.19), it is then seen that the jump condition and the local behaviors near \( \lambda_k, k = 1, 2 \), in the RH problem for \( P^{(1)} \), are fulfilled.

We then proceed to check the matching condition (3.14). Substituting the large-\( \zeta \) behavior of \( \Psi(z) \) (3.16) into (3.25) leads us to the expansion for \( z \) on \( \partial U(1, r) \) as \( n \to \infty \):

\[ P^{(1)}(z)N(z)^{-1} = N(z) \left[ \frac{1}{\sqrt{n}}(I - i\sigma_1) \left( \frac{n^{2/3}(f(z) - f(\lambda_1))}{\sqrt{n}^2 f(\lambda_1)^2} \right) \right] \]

\[ \times \left( I + \frac{\Psi_1(n^{2/3}f(\lambda_1), n^{2/3}(f(\lambda_2) - f(\lambda_1)))}{n^{2/3}(f(z) - f(\lambda_1))} + O(n^{-2/3}) \right) \]

\[ \left[ n^{2/3}(f(z) - f(\lambda_1)) \right]^{-\frac{\sigma_1}{2}} \frac{1}{\sqrt{2}}(I + i\sigma_1)e^{\rho(z; \lambda_1)i\sigma_1}N(z)^{-1}, \]

where

\[ \rho(z; \lambda_1) = \frac{2}{3} f(z)^{3/2} - \frac{2}{3} (f(z) - f(\lambda_1))^{3/2} - f(\lambda_1)(f(z) - f(\lambda_1))^{1/2} \]

\[ = \frac{1}{(f(z) - f(\lambda_1))^{1/2}} \left( \frac{2}{3} f(z)^2 \left( 1 - \frac{f(\lambda_1)}{f(z)} \right) \right)^{1/2} \]

\[ - \frac{2}{3} f(z)^2 \left( 1 - \frac{f(\lambda_1)}{f(z)} \right)^2 - f(z)f(\lambda_1) \left( 1 - \frac{f(\lambda_1)}{f(z)} \right) \]

\[ = \frac{f(\lambda_1)^2}{4(f(z) - f(\lambda_1))^{1/2}} + O(f(\lambda_1)). \]

Here \( f(\lambda_1) \sim 2(\lambda_1 - 1) as \lambda_1 \to 1 \). Inserting the definition of \( \Psi_1 \) in (3.16) and (3.30) into (3.29), we obtain (3.27). For \( z \in \partial U(1, r) \), the denominator in (3.28), namely \( f(z) - f(\lambda_1) \), is
bounded away from zero. It follows from proposition 1 and (1.47), that \( H(x; s) \) is analytic for real variables \( x \) and \( s \) and thus bounded. Therefore, the factor \( \Delta \) defined in (3.28) is bounded for \( z \in \partial U(1, r) \). Thus, we obtain the matching condition (3.14) and complete the proof of proposition 7.

In the above proposition, we have derived the matching condition (3.14) for bounded real parameters \( t_k, k = 1, 2 \). By using the asymptotic analysis of \( \Psi(z; x, s) \) as \( x \to +\infty \) performed in [9, 33], the matching condition (3.14) can be established when the parameters \( t_1 \) and \( t_2 \) are allowed to grow with \( n \) at a certain speed.

**Proposition 8.** Let

\[
\lambda_k = \frac{s_k}{\sqrt{2n}} = 1 + \frac{t_k}{2n^{2/3}}
\]

with

\[-c_1 \leq t_1 < t_2 \leq c_2 n^{1/3}, \quad t_2 - t_1 \leq c_3, \]

(3.31)

for any given positive constants \( c_k, k = 1, 2, 3 \), we have the matching condition (3.14).

**Proof.** Comparing (3.16) and [9, equation (43)], it is seen that the model RH problem \( \Psi(\zeta; x, s) \) is related to the one in [9] by

\[
\Psi(\zeta; x, s) = i \begin{pmatrix} 1 & 0 \\ ir(x, s) & 1 \end{pmatrix} \sigma_1 \Psi_{CD}(\zeta; x, \bar{y}),
\]

(3.32)

where \( \Psi_{CD}(\zeta; x, \bar{y}) \) denotes the one used in [9] with the local singularities \( y = (0, s) \) and \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). The asymptotic analysis of \( \Psi_{CD}(\zeta; x, \bar{y}) \) as \( x \to +\infty \) has been performed in [9].

Tracing back a series of invertible transformations \( \Psi_{CD} \to A \to B \to C \to D \) in [9, equations (78), (79), (82) and (91)], we have

\[
\Psi(\zeta; x, s) = i \begin{pmatrix} 1 & 0 \\ ir(x, s) & 1 \end{pmatrix} \sigma_1 x^{3/2} \left( \frac{i}{4} \right)^{1/2} D \left( \frac{\zeta}{x} \right) \left( \frac{\zeta}{x} + 1 \right)^{\sigma_3/4} \times \frac{(I - i \sigma_1)}{\sqrt{2}} e^{-\frac{i}{2}(\zeta + s)^{3/2} \sigma_3},
\]

(3.33)

for large positive \( x \) and \( |\zeta/x| > 1 + \delta, \delta > 0 \). Here \( D(z) \) is the solution to an RH problem with jump \( J_D(z) = I + O(1/x) \), uniformly for \( z \) on the jump contours and the behavior \( D(z) = I + O(1/z) \) near infinity. The solution admits the following expansion

\[
D(z) = I + \frac{D_1}{z} + O\left( \frac{1}{z} \right), \quad z \to \infty,
\]

(3.34)

where

\[
D_1 = O(1/x), \quad x \to +\infty.
\]

(3.35)

As \( x \to +\infty \) and \( \zeta \to +\infty \), the formula (3.33) can be further simplified to

\[
\Psi(\zeta; x, s) = i \begin{pmatrix} 1 & 0 \\ ir(x, s) + \frac{i}{4} x^2 & 1 \end{pmatrix} \sigma_1 x^{3/2} \sigma_3 \left( I + O\left( \frac{1}{\sqrt{x}} \right) \right) \left( I + O\left( \frac{x}{\zeta} \right) \right) \times \frac{1}{\sqrt{2}} (I - i \sigma_1) e^{-\frac{i}{2}(\zeta + s)^{3/2} \sigma_3},
\]

(3.36)
Now, using the expansion (3.36) we prove the matching condition (3.14) for \( t_k, \ k = 1, 2 \) given in (3.31) and \( t_k \geq d, \ d > 0 \) is big enough. To this end, it is seen after recalling (3.24) that
\[
n^{2/3} f(\lambda_1) \sim t_1, \quad \frac{n^{2/3} (f(z) - f(\lambda_1))}{n^{2/3} f(\lambda_1)} \sim n^{2/3} t_1^{-1} f(z),
\]
for \( z \in \partial U(1, r), \ n \) large and \( \lambda_1 \) given in (3.31). Thus, using (3.36) in the large \( n \) expansion of (3.25), we obtain after some straightforward calculations
\[
P^{(1)}(z) N^{-1}(z) = N(z) \frac{I + i\sigma_1}{\sqrt{2}} \left[ I + O(1/\sqrt{n^{2/3} t_1}) + O\left(\frac{t_1}{n^{2/3}}\right)\right]
\times \frac{I - i\sigma_1}{\sqrt{2}} N^{-1}(z), \tag{3.37}
\]
for \( z \in \partial U(1, r) \) and \( n \) large. This leads us to the matching condition (3.14) for \( t_k, \ k = 1, 2 \) given in (3.31) and \( t_k \geq d, \ d > 0 \) is big enough.

While, for \(-c_1 < t_k \leq d \) the matching condition (3.14) has been verified in proposition 7. Thus, we complete the proof of proposition 8.

**Remark 7.** We note that the nonlinear steepest descent analysis of the RH problem for \( Y(z; x_1, x_2) \) performed in this section can be generalized to the RH problem for \( Y(z; \vec{s}) \) associated with the weight function with more than two discontinuities (2.69) near the soft edge. In the later case, a model RH problem associated with the coupled Painlevé II system of higher dimension used in [9] will be needed in the construction of the local parametrix near the soft edge.

### 3.5. The final transformation: \( S \to R \)

The final transformation is defined by
\[
R(z) = \begin{cases} 
S(z) N^{-1}(z), & z \in \mathbb{C} \setminus \{U(-1, r) \cup U(1, r) \cup \Sigma_S\}, \\
S(z) \left( P^{(1)}(z) \right)^{-1}, & z \in U(-1, r) \setminus \Sigma_S, \\
S(z) \left( P^{(1)}(z) \right)^{-1}, & z \in U(1, r) \setminus \Sigma_S.
\end{cases} \tag{3.38}
\]

From the matching condition (3.14), we have
\[
\|J_R(z) - I\|_{L^2; L^\infty(\Sigma_R)} = O(n^{-1/3}), \tag{3.39}
\]
where the error bound is uniform for the parameters \( t_1 \) and \( t_2 \) specified by (3.31) according to proposition 8. Thus, by a standard argument as given in [12, 14, 15], we have the estimate
\[
R(z) = I + O(n^{-1/3}), \tag{3.40}
\]
where the error bound is uniform for \( z \) in whole complex plane and uniform for the parameters \( t_1 \) and \( t_2 \) specified by (3.31).

### 4. Proofs of theorems 2–4

In this section, we will prove the main results on the asymptotics of the Hankel determinants, the orthogonal polynomials, the recurrence coefficients and the leading coefficients. Moreover, we will derive the asymptotics of the coupled Painlevé IV system.
4.1. Proof of theorem 2: asymptotic of the Hankel determinants

From (1.20), we see that
\[
\ln D_n(s_1, s_2) - \ln D_n(s_1 + c_0, s_2 + c_0) = - \int_{t_1}^{t_1 + c_0} F(x, x + s_2 - s_1) dx,
\]
where \(c_0 > 0\), \(s_1 = \sqrt{2n + \frac{1}{2n^{1/6}}}\), \(s_2 = \sqrt{2n + \frac{3}{2n^{1/6}}}\) for \(t_1 < t_2\) in any compact subset of \(\mathbb{R}\) as given in theorem 2. We will prove theorem 2 in the following way. In lemma 1, we will derive the asymptotics of \(F(x, x + s_2 - s_1)\) uniformly for \(x \in [\sqrt{2n - \frac{1}{2n^{1/6}}}, \sqrt{2n + c_0 + \frac{3}{2n^{1/6}}}]\), for any given constants \(c_0 > 0\), \(k = 0, 1, 2\), by using the differential identity (2.3) and the non-linear steepest descent analysis of \(Y\) carried out in section 3. It is worth to mention that the variable \(n^{1/6}(x - \sqrt{2n})\) is allowed to grow as \(n^{1/6}\), while the \(n^{1/6}\) scaling of the distance between the variables \(x\) and \(x + s_2 - s_1\) of the function \(F(x, x + s_2 - s_1)\) is given by \(n^{1/6}(s_1 - s_2)\) and remains bounded. In lemma 2, we will estimate the integration constant \(\ln D_n(s_1, c_0, s_2 + c_0)\) with \(c_0 > 0\) by using the Hankel determinants generated by the Gaussian weight. Applying lemmas 1 and 2, we then prove theorem 2.

**Lemma 1.** Let
\[
s_k = \sqrt{2n + \frac{t_k}{2n^{1/6}}}, \quad k = 1, 2,
\]
and \(F(s_1, s_2)\) be the logarithmic derivative of the Hankel determinant defined in (1.20), we have
\[
F(s_1, s_2) = \sqrt{2n}n^{1/6}H_{\mathbb{2}}(t_1; t_2 - t_1) + O(n^{-1/6}),
\]
where \(H_{\mathbb{2}}(x; s)\) is the Hamiltonian for the coupled Painlevé II system as defined in (1.43). The error bound is uniform for \(-c_1 \leq t_1 < t_2 \leq c_2n^{1/6}\) and \(t_2 - t_1 \leq c_3\) for any given positive constants \(c_k\), \(k = 1, 2, 3\); see also (3.1).

**Proof.** Tracing back the series of invertible transformations (3.1), (3.6) and (3.38)
\[
Y \rightarrow T \rightarrow S \rightarrow R,
\]
we have
\[
Y_+(\sqrt{2n}z) = (2n)^{1/4}e^{\frac{1}{4}n^{1/6}R(z)E(z)\Psi_+}
\times \left( n^{1/3}(f(z) - f(\lambda_1)); x_n, s_n \right) e^{\frac{n^{2}z}{\sigma}}, \quad \lambda_1 < z < 1 + r,
\]
where \(E(z)\) as defined in (3.26) is analytic for \(|z - 1| < r\). With the parameters specified by (3.31), we have
\[
x_n = n^{2/3}f(\lambda_1) = t_1 + O(n^{-1/3}), \quad s_n = n^{2/3}(f(\lambda_2) - f(\lambda_1)) = t_2 - t_1 + O(n^{-1/3}).
\]
(4.3)

Thus, substituting (4.2) and the estimate (3.40) into the differential identity (2.3), we obtain
\[
F(s_1, s_2) = \frac{1 - \omega_1}{\sqrt{2\pi i}}n^{1/6}(\Psi^{-1}\Psi_+)^{(2)}(0) + \frac{\omega_1 - \omega_2}{\sqrt{2\pi i}}n^{1/6}(\Psi^{-1}\Psi_+)^{(2)}(s_n) + O(n^{-1/6}),
\]
(4.4)
where the error bound is uniform for \( s_1 \) and \( s_2 \) specified by (3.31). Using the expansions of \( \Psi(z) \) near \( z = 0 \) and \( z = s \) in (3.17) and (3.19), we have

\[
F(s_1, s_2) = \frac{1 - \omega_1}{\sqrt{2\pi i}} h^{1/6}(\hat{P}_{121}(x_n, s_n)) + \frac{\omega_1 - \omega_2}{\sqrt{2\pi i}} h^{1/6}(\hat{Q}_1)_{21}(x_n, s_n) + O(n^{-1/6}),
\]

(4.5)

where \( \hat{P}_1 \) and \( \hat{Q}_1 \) are defined in (3.18) and (3.20), respectively.

Next, we express \( \hat{P}_1 \) and \( \hat{Q}_1 \) in terms of the coupled Painlevé II system. Applying the differential equation (3.21), we obtain

\[
\frac{1 - \omega_1}{2\pi i} \hat{P}_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{P}_0^{-1} = \begin{pmatrix} \frac{\nu_{1k}}{2} & -i\nu_1 \\ -i\nu_1 & -\frac{\nu_{1k}}{2} \end{pmatrix},
\]

(4.6)

\[
\hat{P}_1 + \frac{1 - \omega_1}{2\pi i} \left[ \hat{P}_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \hat{P}_0^{-1} \begin{pmatrix} \frac{\nu_{2k}}{2s} & i + \frac{\nu_2}{s} \\ -i \left( x + v_1 + v_2 - \frac{\nu_{2k}}{4v_2} \right) & \frac{\nu_{2k}}{2s} \end{pmatrix} \hat{P}_0.
\]

(4.7)

Now

\[
\hat{P}_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{P}_0^{-1} = \begin{pmatrix} \hat{P}_0(\hat{P}_0)_{11}(\hat{P}_0)_{21} & (\hat{P}_0^2)_{21} \\ -(\hat{P}_0^2)_{21} & (\hat{P}_0^2)_{11} \end{pmatrix}.
\]

Then, this, together with (4.6), leads us to

\[
\frac{1 - \omega_1}{2\pi i} (\hat{P}_0)_{11} = -i\nu_1, \quad \frac{1 - \omega_1}{2\pi i} (\hat{P}_0)_{21} = i\nu_1, \quad \frac{1 - \omega_1}{2\pi i} (\hat{P}_0)_{11} (\hat{P}_0)_{21} = -\frac{\nu_{1k}}{2}.
\]

(4.8)

From the (21) entry of the matrix equation (4.7), it is seen that

\[
(\hat{P}_1)_{21} = -i \left( x + v_1 + v_2 - \frac{\nu_{2k}}{4v_2} \right) (\hat{P}_0)_{11} + \frac{\nu_{2k}}{s} (\hat{P}_0)_{11} (\hat{P}_0)_{21} + i \left( 1 + \frac{\nu_2}{s} \right) (\hat{P}_0^2)_{21}.
\]

(4.9)

Substituting (4.8) into (4.9), we obtain

\[
\frac{1 - \omega_1}{\pi i} (\hat{P}_1)_{21} = -2v_1 \left( x + v_1 + v_2 - \frac{\nu_{2k}}{4v_2} \right) - \frac{\nu_{1k}\nu_{2k}}{s} + \left( 1 + \frac{\nu_2}{s} \right) \frac{\nu_{1k}}{2v_1}.
\]

(4.10)

Similarly, we have that

\[
\frac{\omega_1 - \omega_2}{2\pi i} (\hat{Q}_0)_{11} = -i\nu_2, \quad \frac{\omega_1 - \omega_2}{2\pi i} (\hat{Q}_0)_{21} = i\frac{\nu_{2k}}{4v_2}.
\]

(4.11)
Therefore, we obtain from (1.42), (4.3) (4.5), (4.10) and (4.12) that

\[ F(s_1, s_2) = \frac{1}{\sqrt{2}}(1 + O(e^{-cn^{1/4}})). \]

Next we derive the asymptotic expansion for the Hankel determinant \( D_n \) defined by (1.4) when the jump discontinuities of the weight function (1.5) are large enough.

**Lemma 2.** For \( s_2 > s_1 \geq \sqrt{2n} + c_9 \) and any given positive constant \( c_9 \), we have the asymptotic approximation for the Hankel determinant \( D_n(s_1, s_2) = D_n(s_1, s_2; \omega_1, \omega_2) \) defined by (1.4)

\[ D_n(s_1, s_2) = D_n^{\text{GUE}} \left( 1 + O(e^{-cn^{1/4}}) \right), \]

where \( c \) is some positive constant and \( D_n^{\text{GUE}} \), given explicitly in (1.6), is the Hankel determinant associated with the pure Gaussian weight.

**Proof.** For \( s_2 > s_1 \geq \sqrt{2n} + c_9 \), we have \( \lambda_2 > \lambda_1 \geq 1 + \frac{c_9}{\sqrt{2n}} \geq 1 \), where \( \lambda_k = \frac{\sqrt{2n}}{\sqrt{2n}} \), \( k = 1, 2 \). On account of (3.3), we have for \( \lambda > 1 \)

\[ \phi(\lambda) = 2 \int_1^{\lambda} \sqrt{x^2 - 1} \, dx. \]

Therefore, \( \phi(\lambda) \) is strictly increasing for \( \lambda > 1 \) and there exists some constant \( c > 0 \) such that

\[ n\phi(\lambda) > n\phi(\lambda_1) \geq n\phi(1 + \frac{c_9}{\sqrt{2n}}) > cn^{1/4} \]

for \( \lambda > \lambda_1 \geq 1 + \frac{c_9}{\sqrt{2n}} \). Thus, the jump matrices \( J_S(z) \) defined in (3.8) tend to the identity matrix exponentially fast for \( z \in (\lambda_1, \lambda_2) \cup (\lambda_2, +\infty) \). Therefore, we have

\[ S(z) = (I + \epsilon_a(z))S_0(z), \]

where \( S_0(z) \) is solution to the RH problem for \( S \) when the parameters \( \omega_1 = \omega_2 = 1 \) and \( \lambda_1 = 1 \) therein and \( z \) is bounded away from \( \lambda_k, k = 1, 2 \). Here the error term

\[ |\epsilon_a(z)| < ce^{-n\phi(\lambda_1)} \]

for some constant \( c > 0 \). Tracing back the sequence of transformations \( Y \rightarrow T \rightarrow S \), given in (3.1) and (3.6), we have

\[ Y(\sqrt{2n}z) = (2n)^{1/2}e^{\frac{i\pi}{4}n}e^{\frac{i\pi}{4}n} (I + \epsilon_a(z))S_0(z)e^{n\phi(z)n^{1/4}}z^{n\phi(z)}, \]
where $e^{ng(3n^3)} = (I + O \left( \frac{1}{n} \right)) e^{3n^3}$. In view of (2.2) and the differential identity (2.4), we obtain

$$F(s_1, s_2) = 2p_n + \delta_n = \delta_n, \quad |\delta_n| < ce^{-n(\lambda_1)}, \quad (4.20)$$

where $\lambda_1 = \frac{1}{2n},$ and $c$ is some positive constant. Here $p_n$ is the sub-leading coefficient of the monic Hermite polynomial of degree $n$ and thus $p_n = 0.$ We have the estimate (4.20) for any $s_1$ and $s_2$ in $[\sqrt{2n} + c_0, +\infty)$ such that $s_2 > s_1$. Therefore, given $s_2 > s_1 \geq \sqrt{2n} + c_0$ and $\tau \geq 0$, we have

$$|F(s_1 + \sqrt{2n\tau}, s_2 + \sqrt{2n\tau})| < ce^{-n(\lambda_1 + \tau)}, \quad (4.21)$$

for some constant $c > 0$. Recalling (1.20), we have

$$F(s_1 + s, s_2 + s) = \frac{d}{ds} \ln D_n(s_1 + s, s_2 + s), \quad (4.22)$$

for $s \in \mathbb{R}.$ We integrate with respect to $s$ on both sides of the above equation and obtain

$$\ln D_n(s_1, s_2) - \ln D_n(s_1 + \sqrt{2nL}, s_2 + \sqrt{2nL}) = -\int_0^{\sqrt{2nL}} F(s_1 + s, s_2 + s)ds$$

$$= -\sqrt{2n} \int_0^{L} F(s_1 + \sqrt{2n\tau}, s_2 + \sqrt{2n\tau})d\tau. \quad (4.23)$$

It is seen from (4.21) and (4.17) that

$$\left| \int_0^{L} F(s_1 + \sqrt{2n\tau}, s_2 + \sqrt{2n\tau})d\tau \right| < c \int_0^{+\infty} e^{-n(\lambda_1 + \tau)}d\tau < c_1 e^{-c_2 n^{1/4}}, \quad (4.24)$$

for some positive constants $c_k, k = 1, 2.$ As $L \to +\infty$, $\ln D_n(s_1 + \sqrt{2nL}, s_2 + \sqrt{2nL})$ tends to the Hankel determinant associated with the pure Gaussian weight $D_n^{\text{GUE}}$, given explicitly in (1.6). Thus, let $L \to +\infty$ in (4.23) and applying (4.24), we get (4.16) and complete the proof of lemma 2.

**Proof of Theorem 2.** Now, we are ready to prove theorem 2. Let

$$s_1 = \sqrt{2n} + \frac{t_1}{\sqrt{2n^{1/6}}}, \quad s_2 = \sqrt{2n} + \frac{t_2}{\sqrt{2n^{1/6}}}$$

with $t_k, k = 1, 2$ in any compact subset of $\mathbb{R}$ such that $t_1 < t_2.$ Similarly to (4.23), we integrate (4.22) with respect to $s$ to obtain for given $c_0 > 0$

$$\ln D_n(s_1, s_2) - \ln D_n(s_1 + c_0, s_2 + c_0) = -\int_0^{c_0} F(s_1 + s, s_2 + s)ds. \quad (4.25)$$

In view of (4.16), we have

$$D_n(s_1 + c_0, s_2 + c_0) = D_n^{\text{GUE}} \left( 1 + O \left( e^{-c_1 n^{1/4}} \right) \right), \quad (4.26)$$
Recalling (1.47), we obtain from an integration by parts that
\[
\int_0^c F(s_1 + s, s_2 + s)ds = \sqrt{2n}^{1/6} \int_0^c H_{11} \left( \sqrt{2n}^{1/6} (s_1 + s - \sqrt{2n}); t_2 - t_1 \right) ds + O(n^{-1/6}). \tag{4.27}
\]

By a change of variable \( \tau = \sqrt{2n}^{1/6}(s_1 + s - \sqrt{2n}) \), we have
\[
\sqrt{2n}^{1/6} \int_0^c H_{11}(\sqrt{2n}^{1/6}(s_1 + s - \sqrt{2n}); t_2 - t_1)ds = \int_{t_1}^{t_0} H_{11}(\tau; t_2 - t_1)d\tau,
\tag{4.28}
\]
where
\[
t_0 = t_1 + \sqrt{2c_0}n^{1/6}. \tag{4.29}
\]
Recalling (1.47), we obtain from an integration by parts that
\[
\int_{t_1}^{t_0} H_{11}(\tau; t_2 - t_1)d\tau = \int_{t_1}^{t_0} (\tau - t_1) \left( u_1(\tau; t_2 - t_1)^2 + u_2(\tau; t_2 - t_1)^2 \right) d\tau
\tag{4.30}
\]
\[
\quad + (t_0 - t_1)H_{11}(t_0; t_2 - t_1),
\]
where \( u_1(\tau) \) and \( u_2(\tau) \) are solutions to the coupled nonlinear differential equations (1.45) subject to the boundary conditions (1.48) as \( x \to +\infty \). On account of (1.48) and (4.29), we have
\[
\int_{t_1}^{t_0} (\tau - t_1) \left( u_1(\tau; t_2 - t_1)^2 + u_2(\tau; t_2 - t_1)^2 \right) d\tau = O\left( e^{-cn^{1/4}} \right), \tag{4.31}
\]
for some constant \( c > 0 \). We see from (1.47), (1.48) and (4.29) that \( H_{11}(t_0; t_2 - t_1) \) is also exponentially small. Thus, it follows from (4.27)–(4.31) that
\[
\int_0^c F(s_1 + s, s_2 + s)ds = \int_{t_1}^{t_0} (\tau - t_1) \left( u_1(\tau; t_2 - t_1)^2 + u_2(\tau; t_2 - t_1)^2 \right) d\tau
\tag{4.32}
\]
\[
\quad + O(n^{-1/6}).
\]

Inserting (4.26) and (4.32) into (4.25), we obtain (1.49). This completes the proof of theorem 2.

### 4.2. Proof of theorem 3: asymptotics of the coupled Painlevé IV

From this section, the parameters in (1.5) are defined by \( s_1 = \sqrt{2n} \lambda_1 = \sqrt{2n} + \sqrt{\frac{1}{5}} \) and \( s_2 = \sqrt{2n} \lambda_2 = \sqrt{2n} + \sqrt{\frac{1}{3}} \), with \( t_1 \) and \( t_2 \) in any compact subset of \( \mathbb{R} \). Tracing back the sequence of transformations \( Y \to T \to S \to R \), given in (3.1), (3.6) and (3.38), we have the expression for large \( z \):
\[
Y(\sqrt{2n}z) = (2n)^{-\frac{1}{12}} e^{\frac{i}{6} \delta_2} R(z) N(z) e^{\frac{i}{3} \delta_3} e^{\frac{i}{12} \delta_3}. \tag{4.33}
\]
where \( l = -1 - 2 \ln 2 \). From the definition of \( g(z) \) in (3.2), it is seen that

\[
e^{ag(z)\sqrt 3} z^{-n\sqrt 3} = I + O\left(\frac{1}{z^2}\right), \quad z \to \infty.
\]

By the expression of \( N(z) \) in (3.12), we have the expansion

\[
N(z) = I + \frac{N_1}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty,
\]

(4.34)

where \( N_1 = -\frac{1}{4}(1 + \lambda_1) \sigma_2 \) and the Pauli matrix \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). It follows from the expansion of the jump for \( R(z) \) in (3.27) and (3.28) that

\[
R(z) = I + \frac{R_1(z)}{n^{1/3}} + O\left(\frac{1}{n^{2/3}}\right), \quad n \to \infty.
\]

(4.35)

Here, \( R_1 \) satisfies the jump relation

\[
R_+^{(1)}(z) - R_-^{(1)}(z) = \Delta(z),
\]

(4.36)

where \( \Delta(z) \) is given in (3.28) and \( R_1(z) = O(1/z) \) for \( z \) large. Applying Cauchy’s theorem, it is seen that

\[
R_1^{(1)}(z) = \begin{cases} 
\sqrt{1 + \lambda_1} H_0(n^{2/3} f(\lambda_1); n^{2/3} f(\lambda_2) - f(\lambda_1)) (\sigma_3 - i \sigma_1) - \Delta(z), & z \in U(1, r), \\
\sqrt{1 + \lambda_1} H_0(n^{2/3} f(\lambda_1); n^{2/3} f(\lambda_2) - f(\lambda_1)) (\sigma_3 - i \sigma_1), & z \notin U(1, r).
\end{cases}
\]

(4.37)

Thus, we get from the expression the following expansion as \( z \to \infty \)

\[
R(z) = I + \frac{R_1}{z} + O\left(\frac{1}{z^2}\right),
\]

(4.38)

and

\[
R_1 = \frac{\sqrt{1 + \lambda_1}}{2\sqrt{f(\lambda_1)n^{1/3}}} H_0(n^{2/3} f(\lambda_1); n^{2/3} f(\lambda_2) - f(\lambda_1)) (\sigma_3 - i \sigma_1)
\]

\[
= \frac{1}{2n^{1/3}} H_0(t_1; t_2 - t_1)(\sigma_3 - i \sigma_1) + O(n^{-2/3}),
\]

(4.39)

where use is also made of (3.24). In view of (3.12), (3.24) and (3.28), we have the following expansion as \( z \to \lambda_1 \)

\[
R_1^{(1)}(z) = -\frac{1}{10} H_0(t_1; t_2 - t_1)(\sigma_3 - i \sigma_1) + O(n^{-2/3}) + O(z - \lambda_1).
\]

(4.40)

Substituting (4.34), (4.38) and (4.39) into (4.33) yields
According to (4.44) and (4.48) and in view of the fact that \(Y(z)\) has at most logarithm singularity at \(s_1\), we have

\[
a_1(x; s) = \frac{1}{\gamma(x; s)} e^{-x^2} \lim_{z \to s} (z + s)(Y(z + x)Y^{-1}(z + x))_{21},
\]

\[
= \frac{1}{\gamma(x; s)} e^{-x^2} \lim_{z \to s_1} (z - s_1)(Y(z)Y^{-1}(z))_{21},
\]

\[
= \frac{1}{\gamma(x; s)} e^{-x^2} \lim_{z \to s_1} \sqrt{2n(z - s_1)}Y(Y^{-1}(\sqrt{2n}z))_{21}.
\]

Inserting (4.2) into (4.49), it is seen that

\[
a_1(x; s) = \frac{(2n)^{n-a}}{\gamma(x; s)} e^{-x^2 - af} \left( R(\lambda_1)E(\lambda_1) \left( \lim_{\zeta \to 0} \zeta \Psi(\zeta)\Psi^{-1}(\zeta) \right) E^{-1}(\lambda_1)R^{-1}(\lambda_1) \right)_{21}.
\]
where use is made of the fact that $E(z)$ and $R(z)$ are analytic at $z = \lambda_1$. It follows from the behavior of $\Psi(z)$ near $z = \lambda_1$ as given in (3.17) that

$$
\lim_{\zeta \to 0} \zeta \Psi_{\zeta}(\zeta) \Psi^{-1}(\zeta) = \frac{1 - \omega_1}{2\pi i} \hat{P}_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{P}_0^{-1}.
$$

(4.51)

From the expression (3.26), we get for $k = 1, 2$,

$$
E(\lambda_k) = \frac{1}{\sqrt{2}} (I - i\sigma_1) n^{\nu_1/6} 2^{\nu_1/2} \left( -iH_0(t_1; t_2 - t_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (I + O(n^{-2/3})) \right).
$$

(4.52)

Thus, we obtain from (3.40), (4.50)–(4.52) that

$$
a_1(x; s) = \frac{(1 - \omega_1)(2n)^{-\nu_1}}{2\pi i y(x; s)} e^{-x^2 - \mu} \left( \hat{P}_0 \right)^{\nu_1/3} + i(\hat{P}_0)^{11/3} (\hat{P}_0)_{21} + H_0(t_1; t_2 - t_1)(\hat{P}_0)^{0/3} + O(n^{-1/3}) \right).
$$

Using (4.8) and (4.43), we obtain the asymptotic approximation of $a_1(x; s)$ as stated in (1.54).

Similarly, from (3.19), (3.25) and (4.45), we have

$$
a_2(x; s) = \frac{(2n)^{-\nu_1}}{y(x; s)} e^{-x^2 - \mu} \left( R(\lambda_2)E(\lambda_2) \frac{\omega_1 - \omega_2}{2\pi i} Q_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \times (\hat{P}_0)^{-1} E^{-1}(\lambda_2) R^{-1}(\lambda_2)_{21} \right).
$$

(4.53)

Substituting (3.40), (4.11), (4.43) and (4.52) into (4.53), we obtain the asymptotic expansion of $a_2(x; s)$ as given in (1.55). In view of (4.46) and (4.47), we obtain the asymptotics of $b_1(x; s)$ and $b_2(x; s)$ by considering the (2, 2) entry of the matrices in (4.50) and (4.53):

$$
b_1(x; s) = \frac{(1 - \omega_1)}{2\pi i a_1(x; s)} \left( -i(\hat{P}_0)^{2/3} + O(n^{-1/3}) \right)
$$

$$
= \sqrt{2n} \left( 1 - \frac{\nu_1 t_1; t_2 - t_1}{2t_1(t_1; t_2 - t_1)} + O(n^{-2/3}) \right), \quad n \to \infty,
$$

(4.54)

and

$$
b_2(x; s) = \frac{(\omega_1 - \omega_2)}{2\pi i a_2(x; s)} \left( -i(\hat{P}_0)^{2/3} + O(n^{-1/3}) \right)
$$

$$
= \sqrt{2n} \left( 1 - \frac{\nu_2 t_1; t_2 - t_1}{2t_2(t_1; t_2 - t_1)} + O(n^{-2/3}) \right), \quad n \to \infty,
$$

(4.55)

where $x = \frac{t_1 + t_2}{2} = \sqrt{n} + \frac{t_1 + t_2}{2\sqrt{3}n^{1/6}}$ and $s = \frac{t_1 - t_2}{2} = \frac{t_1 - t_2}{2\sqrt{3}n^{1/6}}$. This completes the proof of theorem 3.

4.3. Proof of theorem 4: asymptotics of the orthogonal polynomials

From (1.54)–(1.57), we have

$$
a_1(x; s)b_1(x; s)^2 = -\sqrt{2n}^{5/6} \left( \nu_1(t_1; t_2 - t_1) - \frac{1}{2} \nu_1 t_1; t_2 - t_1 n^{-1/3} + O(n^{-2/3}) \right),
$$

(4.56)
\[ a_2(x; s)b_2(x; s)^2 = -\sqrt{2} n^{5/6} \left( v_2(t_1; t_2 - t_1) \frac{1}{2} v_{2t}(t_1; t_2 - t_1) n^{-1/3} + O(n^{-2/3}) \right), \]

(4.57)

\[ a_1(x; s)b_1(x; s) + a_2(x; s)b_2(x; s) = -n^{1/3} \left( (v_1(t_1; t_2 - t_1) + v_2(t_1; t_2 - t_1)) \right. \]
\[ \left. + O(n^{-2/3}) \right), \]

(4.58)

Substituting (4.56)–(4.58) into (1.23), (1.24), (1.27) and (1.28), we obtain the asymptotics of the recurrence coefficients and \( \pi_n(s_k) \), \( k = 1, 2 \), as given in (1.59), (1.60), (1.62) and (1.63), respectively. In view of (2.50), (4.34), (4.39) and (4.42), we derive the asymptotic expansion for the leading coefficient \( \gamma_{n-1} \) of the orthonormal polynomial of degree \( n - 1 \):

\[
\gamma_{n-1} = \left( -\frac{1}{2\pi i} (Y_1)_{21} \right)^{1/2}
\]
\[
= \left( \frac{\sqrt{2} n^{5/6} e^{-\frac{m}{n}} (R_1 + N_1)_{21}}{-2\pi i} \right)^{1/2}
\]
\[
= 2^{\frac{1}{2}} n^{\frac{3}{4}} e^{\frac{\pi}{2}} \left( 1 + \frac{H_{11}(t_1; t_2 - t_1)}{2n^{1/3}} \right) + O(n^{-2/3}) \quad n \to \infty. \quad (4.59)
\]

Thus, we complete the proof of theorem 4.

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ORCID iDs

Xiao-Bo Wu https://orcid.org/0000-0002-2179-7130
Shuai-Xia Xu https://orcid.org/0000-0003-4524-0596

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