A FULLY POLYNOMIAL TIME APPROXIMATION SCHEME FOR A NP-HARD PROBLEM

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Abstract. We present a novel feasibility criteria for the intersection of convex sets given by inequalities. This criteria allows us to easily assert the feasibility by analyzing the unconstrained minimum of a certain convex function, that we form with the given sets. Next an algorithm is presented which extends the idea to a particular non-convex case: assert the inclusion of the intersection of a set of balls with equal radii in another ball with a different radius. Given a certain condition on the radii is met, our method can decide if the inclusion happens or not. The condition on the radii can be seen as a generalization of linear programming. Next we apply the results to approximate the solution of a NP-hard problem.

Key words. feasibility criteria, convex optimization, non-convex optimization, quadratic programming, N-P hard

AMS subject classifications. to be filled in xx

1. Introduction. some intro ...

1.1. Definitions and convergence results. In this paper we need to be able to minimize non-smooth convex functions. For this we begin with the definition of the so called "sub-gradient" of a convex function:

Definition 1.1 (subgradient). Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a convex function. We call the **sub-differential** of \( f \) in the point \( X \) the set

\[
\partial f(X) = \left\{ d_f \in \mathbb{R}^n \mid f(Y) \geq f(X) + d_f^T \cdot (Y - X) \quad \forall Y \in \mathbb{R}^n \right\}
\]

and we call \( d_f \in \partial f(X) \) a **sub-gradient** of \( f \) at the point \( X \).

Next we present from the literature, [1], the "sub-gradient descend" algorithm. Let \( X_0 \in \mathbb{R}^n \) and

\[
X_k = X_{k-1} - \alpha_k \cdot f_d(X_{k-1})
\]

where \( f_d(X) \in \partial f(X) \) is a sub-gradient and \( \alpha_k > 0 \) is a step size either constant, either \( \alpha_k \to 0 \) with \( \sum_{k=1}^{\infty} \alpha_k = \infty \).

The following result is known about the convergence of the sub-gradient descent algorithm, [1]:

Theorem 1.2. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a convex function with bounded sub-gradients, i.e. \( \exists L > 0 \) such that for all \( X \in \mathbb{R}^n \), \( \forall f_d \in \partial f(X) \) one has \( \|f_d\| < L \). If \( \|X_0 - X^*\| \leq T \) and the step size \( \alpha_k = \frac{T}{\sqrt{k\|f_d(X_k)\|}} \) then

\[
f(X_k) - f(X^*) \leq \frac{L \cdot T}{\sqrt{k}}
\]

Proof. See [1]
1.2. Convex domains of interest. Let \( X \in \mathbb{R}^{n \times 1}, n, m \in \mathbb{N}_+ \) and let \( g_k : \mathbb{R}^{n \times 1} \to \mathbb{R} \) be convex functions for \( k \in \{1, \ldots, m\} \). Then we define the convex sets:

\[
S_k = \left\{ X \in \mathbb{R}^{n \times 1} \left| g_k(X) \leq 0 \right. \right\} \tag{1.4}
\]

and we are interested if the set

\[
S = \bigcap_{k=1}^m S_k \tag{1.5}
\]

is empty or not. For this we define the following function:

\[
G(X) = \sum_{k=1}^m g_k^+(X) \tag{1.6}
\]

where

\[
g_k^+(X) = \begin{cases} 
g_k(X) & g_k(X) \geq 0 \\ 
0 & g_k(X) \leq 0 
\end{cases} \tag{1.7}
\]

2. Main results. We begin with a simple but important lemma concerning the feasibility of the (1.5).

2.1. Convex feasibility. Let us give the following lemma:

**Lemma 2.1.** Let

\[
X^* = \arg\min_{X \in \mathbb{R}^n} G(X) \tag{2.1}
\]

then the following are equivalent:

1. The set \( S \) is not empty, i.e \( \exists X_0 \in \mathbb{R}^n \) such that

   \[
g_k(X_0) \leq 0 \quad \forall k \in \{1, \ldots, m\} \]

2. \( g_k(X^*) \leq 0 \quad \forall k \in \{1, \ldots, m\} \)

**Proof.** For 2 \( \Rightarrow \) 1, indeed if \( g_k(X^*) \leq 0 \) for all \( k \in \{1, \ldots, m\} \) then \( X^* \in S \neq \emptyset \). For 1 \( \Rightarrow \) 2 let \( X_0 \) such that \( g_k(X_0) \leq 0 \) for all \( k \in \{1, \ldots, m\} \) and assume that \( \exists k \) such that \( g_k(X^*) > 0 \). Then

\[
0 = G(X_0) < G(X^*) \tag{2.2}
\]

which is a contradiction with the fact that \( X^* \) is the minimum of \( G \). \( \square \)

**Remark 2.2.** The feasibility of the intersection of \( m \) convex sets (given by function inequalities) can be asserted by examining the minimum of a non-smooth convex function.

**Remark 2.3 (Complexity analysis for linear feasibility).** We consider the feasibility problem

\[
A \cdot X + B \leq 0 \tag{2.3}
\]
where \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times 1} \). We assume that \( A = \begin{bmatrix} A_1^T \\ \vdots \\ A_m^T \end{bmatrix} \) and \( B = [b_1 \ldots b_m]^T \) with \( \|A_k^T\| \leq 1 \). We therefore obtain \( g_k(X) = A_k^T \cdot X + b_k \). Let \( X \in \mathbb{R}^n \) then

\[
g(x) = \partial g(X) = \partial \left( \sum_{k=1}^{m} g_k^+(X) \right) = \sum_{k=1}^{m} \partial g_k^+(X)
\]

Please note that \( \partial g_k^+(X) = \{0, A_k\} \) therefore for \( \|G_d\| \) with \( G_d \in \partial G(x) \) we obtain

\[
\|G_d\| \leq \sum_{k=1}^{m} \|A_k\| \leq m
\]

It follows from Theorem 1.2 that minimizing \( G \) using sub-gradient descent we obtain after \( p \) steps

\[
G(X_p) - G(X^*) \leq \frac{m \cdot T}{\sqrt{p}}
\]

Therefore if one wants \( G(X_p) - G(X^*) < \epsilon \) follows

\[
p \geq \frac{m^2 \cdot T^2}{\epsilon^2}
\]

In addition to this, each step consists of evaluating \( G \) which the sum of \( m \) functions \( g_k^+ \). Evaluating \( g_k^+ \) takes \( n \) flops, therefore evaluating \( G \) takes \( m \cdot n \) flops. Overall, one needs \( \mathcal{O} \left( \frac{m^3 \cdot n \cdot T^2}{\epsilon^2} \right) \) flops to assert with \( \epsilon \) precision the feasibility of the linear program. This complexity, although polynomial (for a known bounded \( T \)), is quite disappointing from practical point of view. Even worse, in general, an upper bound on \( T \), as a function of \( n, m, A, B \), is exponential, see [3]. However we will use a similar technique in a following subsection to prove the existence of fully polynomial time approximation schemes for various problems.

2.2. An \( NP \)-hard example. In [2] is given an example of a quadratic maximization problem, which is known to be \( NP \)-hard. That is

\[
\max \sum_{k=1}^{n} x_i \cdot (x_i - 1) + \sum_{k=1}^{n} x_i \cdot s_i = f(X)
\]

s.t \( \sum_{k=1}^{n} x_i \cdot s_i \leq M \)

\[
0 \leq x_i \leq 1
\]

where \( s_i \in \mathbb{Z} \). The authors of the mentioned paper argue that because for any feasible point one has \( x_i \cdot (x_i - 1) \leq 0 \) follows that \( f(X) \leq M \). Obtaining the maximum of \( f(X^*) = M \) is equivalent to finding a subset of integers \( s_k \) which add up to \( M \).

Please note that \( f(X) = \sum_{k=1}^{n} x_i^2 + \sum_{k=1}^{n} (s_i - 1) \cdot x_i = X^T \cdot X + S^T \cdot X \), where \( \hat{S} = [s_1 - 1 \ldots s_n - 1] \). Let \( S = [s_1 \ldots s_n] \) and it is easy to see that

\[
f(X) = X^T \cdot X + 2 \cdot \frac{\hat{S}^T}{2} \cdot X + 1 \cdot \frac{1}{4} \cdot S^T \cdot \hat{S} - \frac{1}{4} \cdot \hat{S}^T \cdot \hat{S}
\]

\[
= \left( X + \frac{1}{2} \hat{S} \right)^T \cdot \left( X + \frac{1}{2} \hat{S} \right) - \frac{1}{4} \cdot \hat{S}^T \cdot \hat{S}
\]
It follows that the optimizer of (2.8) is also the optimizer of the following problem

$$\max \quad \left\| X - \frac{-1}{2} \cdot \hat{S} \right\|^2$$

s.t \quad \begin{cases} S^T \cdot X \leq M \\ 0 \leq x_i \leq 1 \end{cases}

(2.10)

However, in the following we treat another, similar problem, then we return to the above at the end.

2.3. Test for the inclusion of an intersection of balls in a ball. We want to solve the following non-convex optimization problem:

$$\max \quad (X - C)^T \cdot (X - C)$$

s.t \quad (X - C_k)^T \cdot (X - C_k) \leq R^2 \quad \forall k \in \{1, \ldots, m\}

(2.11)

that is, we want to find the point an intersection of spheres which is the furthest to $C \in \mathbb{R}^n$.

Let us define the following sets:

$$B_0 = \left\{ X \in \mathbb{R}^n \mid (X - C)^T \cdot (X - C) \leq r^2 \right\}$$

$$B_k = \left\{ X \in \mathbb{R}^n \mid (X - C_k)^T \cdot (X - C_k) \leq R^2 \right\}$$

(2.12)

$$C_1 = \bigcap_{k=1}^m B_k \quad C_2 = B_0$$

for some $R, r > 0$. In order to solve the problem (2.11) we propose to develop a test which can assert if $C_1 \subseteq C_2$

**Lemma 2.4.** Let $f(X)$ be a convex function on $\mathbb{R}^n$ and $0 < \mu < 1$. Then

$$g(X) = \begin{cases} f(X) & f(X) \geq 0 \\ \mu \cdot f(X) & f(X) < 0 \end{cases}$$

(2.13)

is also convex on $\mathbb{R}^n$

**Proof.** One can observe that $g(X) = \mu \cdot f(X) + (1 - \mu) \cdot f^+(X)$ where $\mu, (1 - \mu) > 0$ and $f, f^+$ are convex functions, hence $g$ is convex because it is a positive combination of convex functions. \qed

**Remark 2.5.** For the above function it is also true that $g(X) - \mu \cdot f(X)$ is convex. Indeed

$$h(X) = g(X) - \mu \cdot f(X) = \begin{cases} (1 - \mu) \cdot f(X) & f(X) \geq 0 \\ 0 & f(X) \leq 0 \end{cases}$$

(2.14)

for which one can use the above proof to show that $h$ is convex since $1 - \mu > 0$ and therefore $h = (1 - \mu) \cdot f^+$. 

Let us define the functions, with $\eta, \mu > 0$:

$$
\begin{align*}
    f_k(X) &= \frac{1}{r^2} \cdot (X - C_k)^T \cdot (X - C_k) - 1 \\
    f(X) &= \frac{1}{r^2} \cdot (X - C)^T \cdot (X - C) - 1 \\
    g_k(X) &= \begin{cases} 
        \eta \cdot f_k(X) & f_k(X) \geq 0 \\
        \mu \cdot f_k(X) & f_k(X) < 0 
    \end{cases} \\
    h(x) &= \begin{cases} 
        -f(X) & f(X) \leq 0 \\
        0 & f(X) \geq 0 
    \end{cases}
\end{align*}
$$

(2.15)

with $\mu < \frac{1}{\eta}$ then using Lemma 2.4 we obtain that $g_k$ are convex functions. Moreover using Remark 2.5 one can show that $g_k - \mu \cdot f_k$ is convex. Next let

$$
G(X) = \eta \cdot \sum_{k=1}^{m} g_k(X) + \mu \cdot h(X)
$$

(2.16)

for some $\eta > 0$ to be defined later. We also define:

$$
X^* = \arg\min_{X \in \mathbb{R}^n} G(X)
$$

(2.17)

**Definition 2.6.** We say that a point $X \in \mathbb{R}^n$ is $\epsilon > 0$ deep inside the set $C_1$ if $f_k(X) \leq -\epsilon$ for all $k$.

Concerning the above defined function $G$, we have the following lemma

**Lemma 2.7.** For a given fixed $\epsilon > 0$, if

$$
\mu < \frac{\epsilon}{2 + \eta \cdot m \cdot (1 - \epsilon)} \quad \eta \geq \frac{R^2}{r^2} \cdot \frac{1}{m}
$$

(2.18)

we obtain that $G$ is convex on $\mathbb{R}^n$ and $f_k(X^*) < \epsilon$ for all $k \in \{1, \ldots, m\}$

**Proof.** Let us assume that $\exists k$ such that $f_k(X^*) \geq \epsilon$. Then

$$
G(X^*) > \epsilon - m \cdot \eta \cdot (1 - \epsilon)
$$

while for all $X_0$ deep inside $C_1$ (i.e. $f_k(X) \leq -\epsilon$) one has

$$
G(X_0) \leq -\eta \cdot m \cdot (1 - \epsilon)
$$

(2.19)

(2.20)

For the given choice of $\mu$ one obtains $G(X_0) < G(X^*)$. Indeed

$$
G(X_0) < G(X^*) \iff -\eta \cdot m \cdot \epsilon + \mu \leq \epsilon - m \cdot \eta \cdot (1 - \epsilon)
$$

(2.21)

Next, for convexity, let us consider

$$
G = \eta \cdot \sum_{k=1}^{m} (g_k - \mu \cdot f_k) + \mu \cdot \left( \eta \cdot \sum_{k=1}^{m} f_k(X) + h(X) \right)
$$

(2.22)
Since $g_k - \mu \cdot f_k$ is convex, it is sufficient to prove the convexity of

\begin{equation}
\hat{G}(X) = \eta \cdot \sum_{k=1}^{m} f_k(X) + h(X) = \eta \cdot \sum_{k=1}^{m} f_k - f + h
\end{equation}

Using Remark 2.5 it is easy to see that

\begin{equation}
f + h = \begin{cases} f & f(X) \geq 0 \\ 0 & f(X) \leq 0 \end{cases}
\end{equation}

is convex. Next $\eta \cdot \sum_{k=1}^{m} f_k - f$ is a twice differentiable function hence for proving its convexity we prove that its hessian is positively defined. The hessian can simply be computed to be

\begin{equation}
\left( \eta \cdot \frac{2}{R^2} \cdot m - \frac{2}{r^2} \right) \cdot I_n \succeq 0
\end{equation}

which happens indeed for the given choice of $\eta$.

Remark 2.8. By making the function convex with the above choice of $\eta$ we assure the fact that the global minimum of $G$ can be easily found. Next, we want to use the found minimum to assert the inclusion of $C_1$ in $C_2$. For this we have to impose some extra constraints on the parameter $\eta$

Lemma 2.9. Compute $X^\star$ with $\eta < \frac{\mu}{2} \cdot \frac{1}{m}$. Then if $f_k(X^\star) < -\epsilon$ for all $k$ and $f(X^\star) \leq -\epsilon$ follows that $\forall X$ such that $f_k(X) \leq -\epsilon$ for all $k$ and $f(X) \geq \epsilon$ (i.e we say with $\epsilon$ precision that $\forall X \in C_1 \setminus C_2$)

Proof. Indeed, if $\exists X_0$ with $f_k(X_0) \leq -\epsilon$ for all $k$ and $f(X_0) \geq \epsilon$ follows that

\begin{equation}
G(X_0) \leq -\mu \cdot \eta \cdot m \cdot \epsilon
\end{equation}

while

\begin{equation}
G(X^\star) \geq \mu \cdot \epsilon - \eta \cdot \mu \cdot m
\end{equation}

For the given choice of $\eta$ one obtains $G(X_0) < G(X^\star)$

\begin{equation}
G(X_0) < G(X^\star) \iff -\mu \cdot \eta \cdot m \cdot \epsilon < \mu \cdot \epsilon - \mu \cdot \eta \cdot m \iff \eta < \frac{\epsilon}{1-\epsilon} \cdot \frac{1}{m}
\end{equation}

Remark 2.10. It is interesting to see in what conditions exists $\eta$ such both the above lemmas are fulfilled. This means that the global minimum can be easily found (due to the result given by Lemma 2.7) and it confers information about the sets $C_1$ and $C_2$, see Lemma 2.9. This can be accomplished if:

\begin{equation}
\frac{R^2}{m} \cdot \frac{1}{r^2} \leq \frac{\epsilon}{1-\epsilon} \cdot \frac{1}{m} \iff R \leq \sqrt{\frac{\epsilon}{1-\epsilon} \cdot r}
\end{equation}

where $0 \leq \epsilon < 1$ is the desired precision.

Remark 2.11. The condition (2.29) can be seen as a generalization to maximizing a linear objective function, since a linear objective can be approximated by a ball with an infinite radius, therefore such a restriction on the quotient of the radii seems natural.
Before moving forward to our final example, let us have a synthesis of the presented results.

**Theorem 2.12.** Let \( \eta, \mu, r, R \) chosen as shown above, and let \( X^* \) be the solution to (2.17). Then we have the following alternatives with \( \epsilon \) accuracy, for a given \( \epsilon > 0 \):

1. if \( f_k(X^*) \leq -\epsilon \ \forall k \) and \( f(X^*) \leq -\epsilon \) then \( C_1 \subseteq C_2 \).
2. if \( f_k(X^*) \leq -\epsilon \ \forall k \) and \( f(X^*) > -\epsilon \) then \( \exists \ X \in C_1 \setminus C_2 \).
3. if \( \exists k \ - \epsilon < f_k(X^*) \) and \( f(X^*) \leq -\epsilon \) then \( \exists X \in C_1 \setminus C_2 \).
4. if \( \exists k \ - \epsilon < f_k(X^*) \) and \( f(X^*) \geq -\epsilon \) then \( \exists X \in C_1 \setminus C_2 \).

**Proof.** Let’s say that one obtains \( X^* \), the solution to (2.17), then:

1. if

\[
(2.30) \quad f_k(X^*) \leq -\epsilon \ \forall k \quad f(X^*) \leq -\epsilon
\]

we use Lemma 2.9 to conclude that for all \( X \) satisfying \( f_k(X) \leq -\epsilon \) for all \( k \), one has \( f(X) \leq \epsilon \) hence we say that \( C_1 \subseteq C_2 \) with \( \epsilon \) precision.

2. if

\[
(2.31) \quad f_k(X^*) \leq -\epsilon \ \forall k \quad f(X^*) > -\epsilon
\]

we say with \( \epsilon \) precision that \( \exists X \in C_1 \setminus C_2 \) and this point is \( X^* \). We actually found a point not \( \epsilon \) deep in \( C_2 \) but \( \epsilon \) deep in \( C_1 \).

3. Please note that due to the choice of \( \mu, f_k(X^*) \leq \epsilon \) for all \( k \). But if

\[
(2.32) \quad \exists k \ - \epsilon < f_k(X^*) \leq \epsilon \quad f(X^*) \leq -\epsilon
\]

let us assume that \( \exists X_0 \) such that \( f_k(X_0) \leq -\epsilon \) for all \( k \) and \( f(X_0) \geq \epsilon \). Then due to the choice of \( \eta \) (as in Lemma 2.9) one has

\[
(2.33) \quad G(X_0) < -\eta \cdot \mu \cdot \epsilon \cdot m < \mu \cdot \epsilon - \eta \cdot \mu \cdot m
\]

but

\[
(2.34) \quad G(X^*) \geq \mu \cdot \epsilon - \eta \cdot \mu \cdot m
\]

since at least for one \( k \in \{1, \ldots, m\} \) \( f_k(X^*) > -\epsilon \). This involves \( G(X_0) < G(X^*) \) which is a contradiction, hence we conclude with \( \epsilon \) accuracy that \( \exists X \in C_1 \setminus C_2 \).

4. Finally, if

\[
(2.35) \quad \exists k \ - \epsilon < f_k(X^*) \leq \epsilon \quad f(X^*) \geq -\epsilon
\]

then we conclude again that \( \exists X \in C_1 \setminus C_2 \). The point \( X^* \) is close to the frontier of \( C_1 \) and not \( \epsilon \) deep inside \( C_2 \).

**2.4. Approximation of the maximum distance to a point over a polytope.** We want to close this section with an approximate solution to (2.10). For this we first want to approximate the polytope with an intersection of balls. Therefore we replace the constraints as follows:

\[
(2.36) \quad 0 \leq x_k \iff X \in B(C_{k+}, R) \quad x_k \leq 1 \iff X \in B(C_{k-}, R)
\]

Let \( P = [1 \ldots 1]^T \in \mathbb{R}^n \). Then let \( P_{k+} = \frac{1}{2} \cdot P - \frac{1}{2} \cdot e_k, P_{k-} = \frac{1}{2} \cdot P + \frac{1}{2} \cdot e_k \) and \( C_{k+} = P_{k+} + R \cdot e_k \) and \( C_{k-} = P_{k-} - R \cdot e_k \). Therefore the balls are tangent to the
planes forming the hypercube in the middle of the faces of the hypercube. In order
to choose \( R \) we first evaluate the distance from \( P_{k\pm} \) to one of the vertexes \( V \) of the
facet of the hypercube and call it \( d^2 = \sum_{k=1}^{n-1} \frac{1}{4^k} = \frac{n-1}{4^2} \), therefore the distance from
\( C_{k\pm} \) to \( V \) is
\[
(2.37) \quad \frac{R^2}{R^2} = R^2 + \frac{n-1}{4} \iff \frac{R^2}{R^2} = 1 + \frac{n-1}{4} \leq 1 + \epsilon \iff R \geq \sqrt{\frac{n-1}{4\cdot \epsilon}}.
\]
For the other constraint \( S^T \cdot X \leq M \), let \( Q \) such that \( S^T \cdot Q - M = 0 \) and \( Q \) is strictly
inside the hypercube. Then consider the ball \( B(C_S, R) \) where \( C_S = Q - R \cdot \frac{S}{\|S\|} \). The
ball is tangent to the hyperplane in the point \( Q \). For a vertex \( V \) of the polytope, on
the hyperplane defined by \( S \), one has \( \|V - Q\|^2 \leq \sum_{k=1}^{n} 1 = n \), therefore the distance
from the center of the ball \( C_S \) to such a vertex is
\[
(2.38) \quad \hat{R}^2 \leq R^2 + n \iff \frac{\hat{R}^2}{R^2} \leq 1 + \frac{n}{4} \leq 1 + \epsilon \iff R \geq \sqrt{n \epsilon}
\]
hence we solve the approximate problem:
\[
\max \quad (X - \frac{1}{2} \cdot \hat{S})^T \cdot (X - \frac{1}{2} \cdot \hat{S})
\]
\[
\text{s.t.} \quad \begin{cases}
(X - C_{k+})^T \cdot (X - C_{k+}) \leq R^2 & \forall k \\
(X - C_{k-})^T \cdot (X - C_{k-}) \leq R^2 & \forall k \\
(X - C_S)^T \cdot (X - C_S) \leq R^2
\end{cases}
\]
(2.39)

Next, in order to use the above presented theory we need
\[
(2.40) \quad r \geq R \cdot \frac{1}{\sqrt{q(\epsilon)}} \geq \sqrt{n \epsilon} \cdot \frac{1}{\sqrt{q(\epsilon)}}
\]
which can happen if the distance from a point inside the intersection of balls to \( \frac{1}{4} \cdot \hat{S} \)
is greater then \( \sqrt{\frac{n \epsilon}{q(\epsilon)}} \). It is sufficient to test this for the point \( 0 \). This means that we need
\[
(2.41) \quad \|\hat{S}\| \geq 2 \cdot \sqrt{n \epsilon} \cdot \frac{1}{\sqrt{q(\epsilon)}}
\]
where \( S = [s_1 \ldots s_n] \) and \( \hat{S}^T = [s_1 - 1 \ldots s_n - 1] \). Having the original goal
in mind, that is finding whether there is a subset of integers \( s_k \) which add up to \( M \),
we modify the problem like so: we increase the number of integers with one additional
integer, \( s_{n+1} > \sum_{k=1}^{n} s_k^2 + \text{abs}(M) \) such that it cannot be used with other already
preset (initially given) integers to obtain the sum \( M \). With the newly added integer,
we want to assure
\[
(2.42) \quad \|\hat{S}_{\text{new}}\|^2 = s_{n+1}^2 + \|\hat{S}\|^2 \geq 4 \cdot \frac{n+1}{\epsilon} \cdot \frac{1}{q(\epsilon)^2}
\]

hence we have the new problem given by
\[
s_{n+1}^2 \geq \max \left\{ \left( \sum_{k=1}^{n} s_k^2 + \text{abs}(M) \right)^2 \cdot 4 \cdot \frac{n+1}{\epsilon} \cdot \frac{1}{q(\epsilon)^2} - \|S\|^2 \right\}
\]
\[
S_{\text{new}} = [S^T \ s_{n+1}]^T \quad \hat{S}_{\text{new}} = S_{\text{new}} - [1 \ldots 1]^T
\]
\[
(2.43) \quad R_{\text{new}} \geq \sqrt{\frac{n+1}{\epsilon}}
\]
and we solve

\[
\begin{aligned}
\max & \quad \|X + \frac{1}{2} \cdot \hat{S}\|^2 \\
\text{s.t} & \quad \left\{ \begin{array}{l}
X \in B(C_{k \pm}, R) \\
X \in B(C_S, R)
\end{array} \right. \quad \forall k \in \{1, \ldots, n, n + 1\}
\end{aligned}
\]

(2.44)

with the updated \(C_{k \pm}\) and \(C_S\) obtained as shown above but with \(S_{new} \in \mathbb{Z}^{n+1}\) this time.

### 2.5. Complexity analysis.

In order to assert the inclusion of an intersection of balls in another given ball with the framework presented above, one just has to obtain the global minimum of the function \(G\) given by (2.16). We showed that with proper choices of the parameters involved, namely \(\eta, \mu, G\) is convex but non-smooth.

We can use the result given Theorem 1.2 to obtain the minimum of \(G\). For a given initial point \(X_0\) let \(T > 0\) be a given such that \(\|X_0 - X^*\| \leq T\). As we have showed above, for the chosen parameters, the optimum \(X^*\) is close to the unit cube, therefore if \(X_0 = 0\) then \(T \leq 2 \cdot \sqrt{n}\). We can now obtain an upper bound on the subgradients of \(f_k\) and call it \(L_k\). Indeed \(\frac{\partial f_k}{\partial X} = 2R^2 \cdot (X - C_k)^T\) hence let

\[
L_k = \max \|X - C_k\|^2 \quad \text{s.t} \quad \|X - X_0\|^2 \leq T^2
\]

(2.45)

then follows

\[
\left\| \frac{\partial G}{\partial X} \right\| \leq \frac{2 \cdot m}{R^2} \cdot \max\{L_1, \ldots, L_m\} + \frac{2}{\eta^2} \cdot L_0 = L \in \mathcal{O}(m)
\]

(2.46)

Hence using sub-gradient descent, after \(p\) steps one obtains:

\[
G(X_p) - G(X^*) \leq \frac{T \cdot L}{\sqrt{p}}
\]

(2.47)

For obtaining and \(\epsilon\) accuracy, for the the position of \(X^*\), one has to make \(p\) steps where

\[
p \geq \frac{L^2 \cdot T^2}{\epsilon^2} \in \mathcal{O}\left(\frac{m^2 \cdot T^2}{\epsilon^2}\right)
\]

(2.48)

Each step requires the evaluation of the subgradient of \(G\) which costs \(m \cdot n\) flops, one obtains the final complexity for finding \(X^*\) to \(\epsilon\) accuracy to be

\[
\mathcal{O}\left(\frac{m^3 \cdot n \cdot \sqrt{n^2}}{\epsilon^2}\right)
\]

(2.49)

Once \(X^*\) is found, one can use Theorem 2.12 to assert to \(\epsilon\) precision the inclusion of the intersection of spheres in another sphere (with properly chosen radii).

### 3. Conclusion and future work.

As future work, besides a more detailed proof of the convexity of the function presented in subsection 2.5, one should study the numerical stability of the presented algorithm and eventually test it on some benchmark problems.

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