THE SKEIN CATEGORY OF THE ANNULUS

K. AL QASIMI AND J.V. STOKMAN

Abstract. We construct the skein category $S$ of the annulus and show that it is equivalent to the affine Temperley-Lieb category of Graham and Lehrer. It leads to a skein theoretic description of the extended affine Temperley-Lieb algebras. We construct an endofunctor of $S$ that corresponds, on the level of tangle diagrams, to the insertion of an arc connecting the inner and outer boundary of the annulus. We use it to define and construct towers of extended affine Temperley-Lieb algebra representations. It allows us to construct a tower of representations acting on spaces of link patterns on the punctured disc which play an important role in the study of loop models. We finally describe this tower in terms of fused extended affine Temperley-Lieb algebra representations.

1. Introduction

In [16, 17] Kauffman constructed a knot invariant based on an elementary combinatorial rule for eliminating crossings in the associated knot diagram. In doing so he introduced the two skein relations, the Kauffman skein relation (3.1) and the loop removal relation (3.2). It has led to skein theory, the study of knots and links in 3-manifolds modulo the Kauffman skein relation and the loop removal relation, see, e.g., [25, 22, 20, 1].

The current paper deals with the 3-manifold $A \times [0, 1]$, with $A$ the annulus in the complex plane.

We introduce and study the skein category $S$ of the annulus $A$. It is the linear category with objects the nonnegative integers and morphisms $\text{Hom}_S(m, n)$ the linear skein of the annulus with $m$ marked points on the inner boundary and $n$ marked points on the outer boundary. In other words, $\text{Hom}_S(m, n)$ consists of the ambient isotopy classes of $(m, n)$-tangle diagrams on the annulus modulo the congruence relation generated by the Kauffman skein relation and the loop removal relation.

Following closely Przytycki [23, §3], we construct a relative version of the Kauffman bracket to prove that the skein category $S$ is equivalent to Graham and Lehrer’s [12] affine Temperley-Lieb category. As a consequence it follows that the endomorphism algebra $\text{End}_S(n) := \text{Hom}_S(n, n)$ is isomorphic to Green’s [14] $n$-affine diagram algebra, also known as the (extended) affine Temperley-Lieb algebra.

We will define an endofunctor $I$ of $S$ called the arc-insertion functor, which on the level of morphisms inserts a new arc connecting the inner and outer boundary of the annulus in a particular way while undercrossing all arcs it meets along the way. On the level of endomorphisms it provides a tower of algebras

$$\text{End}_S(0) \overset{I_0}{\to} \text{End}_S(1) \overset{I_1}{\to} \text{End}_S(2) \overset{I_2}{\to} \cdots$$
with connecting maps $\mathcal{Z}_n$ the algebra maps $\mathcal{Z}_{|\text{End}_S(n)} : \text{End}_S(n) \to \text{End}_S(n + 1)$. This tower was considered before in the context of knot theory \cite{3} and in the context of fusion of extended affine Temperley-Lieb algebra representations \cite{9} respectively. It differs from the arc-tower from e.g. \cite{9, 5}, which is defined with respect to the two-step algebra embedding $\text{End}_S(n) \to \text{End}_S(n + 2)$ that corresponds to the identification of an idempotent subalgebra of $\text{End}_S(n + 2)$ with $\text{End}_S(n)$.

We introduce and study towers $V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots$ of extended affine Temperley-Lieb algebra representations. These are chains of left $\text{End}_S(n)$-modules $V_n$ ($n \in \mathbb{Z}_{\geq 0}$) connected by morphisms $\phi_n : V_n \to \text{Res}^{\mathcal{Z}_n}_{n+1}(V_{n+1})$ of $\text{End}_S(n)$-modules, where $\text{Res}^{\mathcal{Z}_n}_{n+1}(V_{n+1})$ is the TL$_{n+1}$-module $V_{n+1}$ viewed as TL$_n$-module via the algebra map $\mathcal{Z}_n : \text{End}_S(n) \to \text{End}_S(n + 1)$.

Our motivation for studying such towers stems from integrable models in statistical physics with extended affine Temperley-Lieb algebra symmetry. Examples are inhomogeneous dense loop models and inhomogeneous XXZ spin-$\frac{1}{2}$ chains with quasi-periodic boundary conditions, see, e.g., \cite{15, 6, 11} and references therein. In this context the representation space $V_n$ of the tower represents the state space of the model at system size $n$ and the connecting maps relate the models of different system sizes.

We introduce a special tower of extended affine Temperley-Lieb algebra representations, which we will call the link pattern tower. It depends on a free parameter $v$. For even $n$ the representation space is spanned by ambient isotopy classes of $(0,n)$-tangle diagrams in $A$ without crossings and without loops, connecting $n$ marked points on the outer boundary of $A$. For odd $n$ the tangle diagrams include a defect line connecting the outer boundary to the inner boundary, and we add the rule that Dehn twists of the defect line may be removed by a weight factor $v$ (see (10.3)). The $\text{End}_S(n)$-action is described as follows. The skein class of an $(n,n)$-tangle diagram on the annulus acts on a diagram $D \in V_n$ by placing $D$ inside the $(n,n)$-tangle diagram, removing crossings and contractible loops by the skein relations, and removing noncontractible loops by a particular weight factor depending on $v$ (see (10.2)).

For $v = 1$ the representation spaces may be naturally identified with spaces of link patterns on the punctured disc $\mathbb{D}$ by shrinking the hole of the annulus to a point. The resulting representations arise in \cite{15, 6} as the state spaces of the inhomogeneous dense $O(\tau)$ loop models on the half-infinite cylinder.

For even $n$ the connecting maps $\phi_n : V_n \to V_{n+1}$ of the link pattern tower correspond, from the skein theoretic perspective, to the insertion of a defect line, undercrossing all arcs it meets along the way. These maps were considered before in \cite{4} in the study of the inhomogeneous dense loop model on the half-infinite cylinder. The connecting maps $\phi_n : V_n \to V_{n+1}$ for odd $n$ are more subtle. From a skein theoretic perspective they can be described as follows. The connecting map $\phi_n$ acts on a diagram by detaching the defect line from the inner boundary and reconnecting it to the outer boundary in two different ways, either encircling the hole of the annulus before reattaching it to the outer boundary, or not. These two contributions are given explicit weights depending on $v$ and on the Temperley-Lieb algebra parameter, see Theorem 10.3.
We show that the link pattern tower is nondegenerate for generic parameter values, in the sense that the induced morphisms \( \hat{\phi}_n : \text{Ind}^{\mathcal{I}_n}(V_n) \to V_{n+1} \) of \( \text{End}_S(n+1) \)-modules are surjective, where \( \text{Ind}^{\mathcal{I}_n}(V_n) \) is the \( \text{End}_S(n+1) \)-module obtained by inducing \( V_n \) along the algebra map \( \mathcal{I}_n : \text{End}_S(n) \to \text{End}_S(n+1) \). We relate the link pattern tower to the recently introduced fusion \([9]\) of extended affine Temperley-Lieb algebra modules. We construct for each \( n \in \mathbb{Z}_{\geq 0} \) a fused \( \text{End}_S(n+1) \)-module \( W_{n+1} \) and a morphism \( \psi_n : W_{n+1} \to V_{n+1} \) of \( \text{End}_S(n+1) \)-modules such that \( \hat{\phi}_n \) factorizes through \( \psi_n \). The \( \text{End}_S(n+1) \)-module \( W_{n+1} \) is obtained by fusing the \( \text{End}_S(n) \)-module \( V_n \) with an one-dimensional \( \text{End}_S(1) \)-module.

In our future work \([4]\) we use the link pattern tower to construct a tower of solutions to quantum Knizhnik-Zamolodchikov (qKZ) equations. The tower consists of \( V_n \)-valued solutions of qKZ equations (\( n \in \mathbb{Z}_{\geq 0} \)) which are compatible with respect to the connecting maps \( \phi_n \). At the stochastic/combinatorial value of the extended affine Temperley-Lieb parameter, the \( V_n \)-valued solution in the tower reduces to the ground state of the dense \( O(1) \) loop model on the half-infinite cylinder with perimeter \( n \). In that case the tower structure gives explicit recursion relations of the ground states with respect to the system size, leading to a refinement of the results in \([6]\).

The structure of the paper is as follows. In Sections 2 and 3 we define the category of tangle diagrams, \( \mathcal{T} \) and the skein category of the annulus, \( \mathcal{S} \), respectively. Some definitions might seem verbose given that pictures provide an intuitive understanding. However, the detail is required as there are subtle issues that need to be made precise. In Sections 4 and 5 we introduce the Temperley-Lieb category and show that it is equivalent to \( \mathcal{S} \). The Temperley-Lieb category was originally defined in terms of affine diagrams. We use an equivalent definition using tangle diagrams, which is more convenient for our purposes. In Section 6 we define the extended affine Temperley-Lieb algebra, \( \text{TL}_n \), algebraically and we show that it is isomorphic to \( \text{End}_S(n) \). This utilises the equivalence of categories established in Sections 4 and 5. Section 7 follows with us recalling the extended affine braid group, \( \text{B}_n \) and the extended affine Hecke algebra, \( \text{H}_n \). We recall how the algebra \( \text{TL}_n \) is a quotient of \( \text{H}_n \), and how \( \text{H}_n \) is a quotient of \( \mathbb{C}[\text{B}_n] \). In Section 8 we define the arc insertion functor \( \mathcal{I} : \mathcal{S} \to \mathcal{S} \) and we discuss how the resulting tower of extended affine Temperley-Lieb algebras lift to extended affine braid groups and extended affine Hecke algebras. In Section 9 we introduce the notion of towers of extended affine Temperley-Lieb algebra modules. In Section 10 we construct the link pattern tower. We show in Section 11 how the link pattern tower is related to fusion. Finally we discuss a \( B \)-type presentation of the extended affine Temperley-Lieb algebra in Section 12.

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2. The category of tangle diagrams

Consider the annulus

\[ A := \{ z \in \mathbb{C} \mid 1 \leq |z| \leq 2 \} \]
in the complex plane. We write $\partial A = C_i \cup C_o$ for the boundary of $D$, with $C_i := S^1$ the unit circle and $C_o = 2S^1$ (the indices “$i$” and “$o$” stand for inner and outer, respectively).

Set $\zeta_n := \exp(2\pi i/n)$ for $n \in \mathbb{Z}_{>0}$. We write $\mathcal{M}_i^{(n)}$ for the $n$ marked points $\{\zeta_n^{j-1}\}_{j=1}^n$ on $C_i$ and $\mathcal{M}_o^{(n)}$ for the $n$ marked points $\{2\zeta_n^{j-1}\}_{j=1}^n$ on $C_o$. Furthermore we set $\mathcal{M}_i^{(0)} := \emptyset =: \mathcal{M}_o^{(0)}$.

Let $m, n \in \mathbb{Z}_{\geq 0}$ with $m + n$ even. An $(m,n)$-tangle $T$ in $A \times \mathbb{R}$ is a disjoint union of smooth loops and arcs in $A \times \mathbb{R}$ satisfying:

a. The loops are in the interior of $A \times \mathbb{R}$.

b. The marked points $\{(p,0) \mid p \in \mathcal{M}_i^{(m)} \cup \mathcal{M}_o^{(n)}\}$ on the boundary of $A \times \mathbb{R}$ are the endpoints of the arcs.

c. The arcs intersect the boundary of $A \times \mathbb{R}$ transversally (and hence intersect the boundary only at their endpoints).

Let proj : $A \times \mathbb{R} \to A$ be the projection on the first component, proj$(z,h) := z$ (we think of $h$ as the height parameter). Consider the projection $D := \text{proj}(T)$ of an $(m,n)$-tangle $T$ in $A \times \mathbb{R}$ which is in general position with respect to proj. This means in particular that $\#(T \cap \text{proj}^{-1}(p)) \leq 2$ for all $p \in D$ and that the number of crossing points, i.e. the number of points $p \in D$ such that $\#(T \cap \text{proj}^{-1}(p)) = 2$, is finite. At a crossing point of $D$ we order the two arc segments of $D$ that transversally cross at $p$ by the relative heights of their preimages in $T$ near the fiber at $p$. Concretely, if $T \cap \text{proj}^{-1}(p) = \{(p,h_1),(p,h_2)\}$ with $h_1 > h_2$ and if $L_i$ is the arc segment of $T$ near $(p,h_i)$ for $i = 1,2$, then proj$(L_1)$ is said to overcross proj$(L_2)$ at $p$. Then $D$, together with the crossing data, is called an $(m,n)$-tangle diagram in $A$.

If we draw a picture of an $(m,n)$-tangle diagram then we label the inner points $\xi_i^{m-1}$ on the diagram by $i$ ($i = 1, \ldots, m$) and the outer points $2\zeta_n^{j-1}$ by $j$ ($j = 1, \ldots, n$). An example of a $(3,5)$-tangle diagram is given in figure 1.

![Figure 1. An example of a (3,5)-tangle diagram.](image)

We say that two $(m,n)$-tangle diagrams $D$ and $D'$ in $A$ are equivalent if there exists a smooth ambient isotopy $h : A \times [0,1] \to A$ fixing $\partial A$ pointwise, satisfying $h(D,1) = D'$ and respecting the crossing data. If $D$ is an $(m,n)$-tangle diagram we write $\overline{D} \in \text{Hom}_T(m,n)$ for its equivalence class.

**Definition 2.1.** The category $\mathcal{T}$ of tangle diagrams in the annulus is the category with objects $\mathbb{Z}_{\geq 0}$ and morphisms $\text{Hom}_T(m,n)$ the equivalence classes of $(m,n)$-tangle diagrams in $A$ if $m+n$ is even, respectively the empty set if $m+n$ is odd. The composition map

$$\text{Hom}_T(m,n) \times \text{Hom}_T(k,m) \to \text{Hom}_T(k,n), \quad (\overline{D}, \overline{D'}) \mapsto \overline{D \circ D'}$$
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is defined as follows: \( D \circ D' \) is the \((k,n)\)-tangle diagram obtained from \( D \) and \( D' \) by shrinking the radial size of \( D' \) by a factor two, inserting \( D' \) into \( D \), and restoring the thickness of the resulting annulus by a smooth rescaling map in the radial direction. The identity morphism \( \text{Id}_n \in \text{End}_T(n) \) is the equivalence class of the tangle diagram with straight line arcs from \( \xi_{j-1}^n \) to \( 2\xi_{j-1}^n \) for \( j = 1, \ldots, n \) and no loops.

An example of the composition of two tangle diagrams is given in (2.1).

\[
\begin{align*}
(2.1) & \quad 1 \circ 2 = 1 \circ 2 \\
2 & \quad 1 \quad 2 \\
3 & \quad 1 \quad 2 \\
4 & \quad 1 \quad 2
\end{align*}
\]

3. THE SKEIN CATEGORY OF THE ANNULUS

Write \( \mathbb{C}[\text{Hom}_T(m,n)] \) for the complex vector space with linear basis the equivalence classes of \((m,n)\)-tangle diagrams in \( A \). We take it to be \( \{0\} \) if \( m + n \) is odd. Extend the category \( T \) of tangle diagrams in \( A \) to a linear category by replacing the morphisms \( \text{Hom}_T(m,n) \) by \( \mathbb{C}[\text{Hom}_T(m,n)] \) and extending the composition maps complex bilinearly. Denote the resulting linear category by \( \text{Lin}(T) \). The skein category on the annulus is a quotient category of \( \text{Lin}(T) \). The quotient is defined in terms of the following \emph{skein relations}, due to Kauffman [16, 17].

\begin{lemma}
Let \( t^\frac{1}{4} \) be a nonzero complex number. The linear and transitive closure of the following local relations on the equivalence classes of tangle diagrams in \( A \) extend to a congruence relation on the linear category \( \text{Lin}(T) \):
\begin{enumerate}
\item The Kauffman skein relation \( (D, t^\frac{1}{4}D' + t^{-\frac{1}{4}}D'') \) with \( D, D', D'' \) three tangle diagrams that are identical except in a small open disc in \( A \) where they are as shown
\item The loop removal relation \( (D, -(t^\frac{1}{4} + t^{-\frac{1}{4}})D) \) with \( D, D' \) two tangle diagrams that are identical except in a small open disc in \( A \) where they are as shown
\end{enumerate}
\end{lemma}

\begin{definition}
The skein category \( S = S(t^\frac{1}{4}) \) of the annulus \( A \) is the quotient of the linear category \( \text{Lin}(T) \) by the congruence relation from Lemma 3.1.
\end{definition}

Thus \( S \) is a complex linear category with objects \( \mathbb{Z}_{\geq 0} \) and with morphisms \( \text{Hom}_S(m,n) \) being the equivalence classes in \( \mathbb{C}[\text{Hom}_T(m,n)] \) with respect to the congruence relation from Lemma 3.1. If \( m + n \) is odd then \( \text{Hom}_S(m,n) = \{0\} \). If \( D \) is a tangle diagram in
A then we will write $[D]$ for the corresponding element in $\text{Hom}_S(m,n)$. If no confusion can arise, then we simply write $D$ for the corresponding element $[D]$.

As is customary in skein theory, we write the Kauffman skein relation in $\text{Hom}_S(m,n)$ as

\[(3.1) \quad t^{\frac{1}{4}} \cdot D_1 + t^{-\frac{1}{4}} \cdot D_2 = D_3,\]

and the loop removal relation in the skein module $\text{Hom}_S(m,n)$ as

\[(3.2) \quad t^{\frac{1}{2}} + t^{-\frac{1}{2}} = -1,\]

with the disc showing the local neighbourhood in $A$ where the tangle diagrams differ. We will also write down identities in skein modules by depicting both sides of the equation as linear combinations of the tangle diagrams $D$ representing $[D]$.

**Remark 3.3.** The important observation, due to Kauffman [16, 17], is that $[D] \in \text{Hom}_S(m,n)$ is invariant under the Reidemeister moves $\Omega_1$, $\Omega_2$ and $\Omega_3$ (see Figure 2) and their mirror versions, applied to the $(m,n)$-tangle diagram $D$ in $A$. Hence $[D]$ represents an ambient isotopy class of a framed $(m,n)$-tangle in $A \times \mathbb{R}$, with endpoints of its arcs at $\{(\xi_m^{-1}, 0)\}_{i=1}^m \cup \{(2\xi_n^{-1}, 0)\}_{j=1}^n$ (suitable interpreted if $m$ and/or $n$ is zero).

\[\Omega_1' \quad \Omega_2 \quad \Omega_3\]

**Figure 2.** Reidemeister moves.

Note that the Reidemeister 1 moves are only satisfied up to a scalar multiple,

\[-t^{\frac{3}{4}} \quad -t^{-\frac{3}{4}}\]

4. **The affine Temperley-Lieb category**

The affine Temperley-Lieb category, introduced in [12], has affine diagrams as morphisms. These diagrams have also been introduced and defined in a slightly different way in [14]. Essentially they are ambient isotopy classes of nonintersecting arcs drawn on a rectangle that periodically repeats on an infinite horizontal strip. They can be visualised
as nonintersecting arcs on the surface of a cylinder, or equivalently as nonintersecting arcs on the annulus $A$. To avoid introducing more notation we give the definition of affine diagrams directly in terms of tangle diagrams in $A$.

**Definition 4.1** (Affine diagram). Let $m, n \in \mathbb{Z}_{\geq 0}$. An $(m,n)$-affine diagram is an $(m,n)$-tangle diagram $D$ in $A$ without crossings and without contractible loops in $A$. We write $D_{m,n}$ for the subclass of $\text{Hom}_T(m,n)$ consisting of the equivalence classes $\overline{D}$ of $(m,n)$-affine diagrams $D$ in $A$.

Note that the $D_{m,n} \subseteq \text{Hom}_T(m,n)$ ($m, n \in \mathbb{Z}_{\geq 0}$) do not define a subcategory of $T$ since the composition rule applied to affine diagrams may produce equivalence classes of tangle diagrams with contractible loops. This can be remedied as follows.

Let $\text{Lin}_c(T)$ be the linear quotient category of $\text{Lin}(T)$ obtained by the congruence relation that is induced from the linear and transitive closure of only the loop removal relation (3.2). The sublabel “c” stands for contractible, signifying that in $\text{Lin}_c(T)$ contractible loops are removed by a multiplicative factor. Note that $\text{Lin}_c(T)$ depends on $t^\frac{1}{2}$.

**Definition 4.2** ([12]). The affine Temperley-Lieb category $\mathcal{T}L = \mathcal{T}L(t^\frac{1}{2})$ is the linear subcategory of $\text{Lin}_c(T)$ with objects $\mathbb{Z}_{\geq 0}$ and morphisms $\text{Hom}_{\mathcal{T}L}(m,n)$ the subspace of $\mathbb{C}[\text{Lin}_c(T)]$ spanned by the equivalence classes of the elements from $D_{m,n}$.

If $D$ is an $(m,n)$-affine diagram then we write $\{D\}$ for the corresponding element in $\text{Hom}_{\mathcal{T}L}(m,n)$. If $D$ is an $(m,n)$-affine diagram and $D'$ is an $(k,m)$-affine diagram then

$$\{D\} \circ \{D'\} = (-t^\frac{1}{2} - t^{-\frac{1}{2}})^{l(D'')} \{D''_c\}$$

in $\text{Hom}_{\mathcal{T}L}(k,n)$, with $D''$ the $(k,n)$-tangle diagram in $A$ obtained by inserting $D'$ inside $D$ (in the same way as in Definition 2.1), with $l(D'')$ the number of loops in $D''$ contractible in $A$, and with $D''_c \in D_{k,n}$ the affine diagram obtained from $D''$ by removing the contractible loops.

Note that $\text{Hom}_{\mathcal{T}L}(m,n) = \{0\}$ if $m + n$ is odd, and that

$$\{\{D\} \mid D \text{ (m,n)-affine diagram}\}$$

is a linear basis of $\text{Hom}_{\mathcal{T}L}(m,n)$.

As stated before, in the literature (see, e.g., [12]) an $(m,n)$-affine diagram $D$ is sometimes drawn as a horizontal strip in $\mathbb{R}^2$ containing a periodically repeating rectangular diagram. This rectangle is obtained by cutting the annulus and “straightening” it such that the points on the top and bottom of the rectangle are numbered increasingly from left to right. It is convenient to make a specific choice of the cut depending on the number of marked points on the inner and outer boundary. For an $(m,n)$-affine diagram we will cut the annulus along the straight line segment from $\xi_{\text{max}(m,n)}$ to $2\xi_{\text{max}(m,n)}$. The resulting rectangle is referred to as the fundamental rectangle of $D$.

An example is depicted in figure 3. Note that the arc connecting point 1 and 2 on the top of the horizontal strip crosses the cut. Furthermore, the noncontractible loop encircling the hole of the annulus is the horizontal line in the interior of the fundamental rectangle.
5. THE EQUIVALENCE OF $\mathcal{S}$ AND $\mathcal{T}\mathcal{L}$

In this section we show that the linear categories $\mathcal{S}$ and $\mathcal{T}\mathcal{L}$ are equivalent. The subtle point is to show that the obvious linear functor from $\mathcal{T}\mathcal{L}$ to $\mathcal{S}$ is faithful. The proof for this uses a relative version of the Kauffman bracket for $(m,n)$-tangle diagrams in $\mathcal{A}$, compare with the proof of [23, Thm. 3.1].

**Theorem 5.1.** The linear categories $\mathcal{S}$ and $\mathcal{T}\mathcal{L}$ are equivalent.

**Proof.** Consider the essentially surjective linear functor $F : \mathcal{T}\mathcal{L} \to \mathcal{S}$ which is the identity on objects and maps $\{D\}$ to $[D]$ for an $(m,n)$-affine diagram $D$. It is clearly well defined since the loop removal relation holds in $\mathcal{S}$ as well as in $\mathcal{T}\mathcal{L}$.

Let $D$ be an $(m,n)$-tangle diagram in $\mathcal{A}$. The Kauffman skein relation and the loop removal relation allow to write $[D]$ as a linear combination of classes $[D'] \in \text{Hom}_\mathcal{S}(m,n)$ with the $D'$s being $(m,n)$-affine diagrams. It follows that the functor $F$ is full. It remains to show that $F$ is faithful.

Suppose that $m+n$ is even and let $D$ be an $(m,n)$-tangle diagram in $\mathcal{A}$ with $k$ crossing points. Let $\mathcal{S}_D$ be the set of cardinality $2^k$ containing the $(m,n)$-tangle diagrams $S$ without crossings that are obtained from $D$ by removing each crossing $\bigotimes$ in $D$ by either $\bigotimes$ or $\bigcirc$. For $S \in \mathcal{S}_D$ let $h_D(S)$ (respectively $v_D(S)$) be the number of crossing points at which $\bigotimes$ is replaced by $\bigotimes$ (respectively $\bigcirc$). Let $c_D(S)$ be the number of loops in $S$ that are contractible in $\mathcal{A}$, and write $\tilde{S}$ for the $(m,n)$-affine diagram obtained from $S$ by removing these contractible loops.

It is easy to see that there exists a well defined linear map $\hat{\psi} : \mathbb{C}[\text{Hom}_\mathcal{T}(m,n)] \to \text{Hom}_{\mathcal{T}\mathcal{L}}(m,n)$ satisfying

\begin{equation}
\hat{\psi}(D) := \sum_{S \in \mathcal{S}_D} (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{c_D(S)} t^{(h_D(S) - v_D(S))} / 4 [\tilde{S}]
\end{equation}

for all $(m,n)$-tangle diagrams $D$ in $\mathcal{A}$. Direct computations show that the map $\hat{\psi}$ respects the Kauffman skein relation (5.1) and the loop removal relation (5.2), so it gives rise to a linear map

$\psi : \text{Hom}_\mathcal{S}(m,n) \to \text{Hom}_{\mathcal{T}\mathcal{L}}(m,n)$

satisfying $\psi([D]) = \hat{\psi}(D)$ for $(m,n)$-tangle diagrams $D$ in $\mathcal{A}$.
By the Kaufmann skein relation (3.1) and the loop removal relation (3.2), the linear map $\psi$ is the inverse of the linear map $F : \text{Hom}_{T\mathcal{L}}(m,n) \to \text{Hom}_{S}(m,n)$. This shows that $F$ is faithful.

Remark 5.2. The special case $\text{End}_{S}(0) \simeq \text{End}_{T\mathcal{L}}(0)$ was established for general surfaces in [23, Lem. 3.3].

Definition 5.3. We call $\psi(D) = \hat{\psi}(D) \in \text{Hom}_{T\mathcal{L}}(m,n)$ (see (5.1)) the relative Kauffman bracket of the $(m,n)$-tangle diagram $D$ in $A$.

Remark 5.4. Note that for $(m,n) = (0,0)$, the relative Kauffman bracket $\psi([D])$ of a link diagram $D$ in $A$ lands in the algebra $\text{End}_{T\mathcal{L}}(0)$, which is isomorphic to the ring of polynomials in one variable (the variable corresponds to the equivalence class of a noncontractible loop in $A$, see the next section). Evaluating the resulting polynomial at $-t^\frac{1}{2} - t^{-\frac{1}{2}}$ can be thought of as closing the hole of the annulus and viewing the link diagram as element in the skein module of the disc (or equivalently, of the plane). As a result one obtains the usual Kauffman [16] bracket of $D$, viewed as a link diagram in the plane (see [18] and [20, §1.7]).

6. The extended affine Temperley-Lieb algebra

Write $\text{TL}_0 := \mathbb{C}[X]$ for the algebra of complex polynomials in one variable $X$, and write $\text{TL}_1 := \mathbb{C}[\rho, \rho^{-1}]$ for the algebra of complex Laurent polynomials in the variable $\rho$. Write $\text{TL}_2$ for the complex associative unital algebra with generators $e_1, e_2, \rho, \rho^{-1}$ and defining relations

$$
e_i^2 = (-t^\frac{1}{2} - t^{-\frac{1}{2}})e_i,$$
$$\rho e_i = e_{i+1}\rho,$$
$$\rho\rho^{-1} = 1 = \rho^{-1}\rho,$$
$$\rho^2 e_1 = e_1,$$

where the indices are taken modulo two. Finally, for $n \geq 3$ let $\text{TL}_n$ be the complex associative unital algebra with generators $e_1, e_2, \ldots, e_n, \rho, \rho^{-1}$ and defining relations

$$
e_i^2 = (-t^\frac{1}{2} - t^{-\frac{1}{2}})e_i,$$
$$e_i e_j = e_j e_i \quad \text{if} \quad i - j \neq \pm 1,$$
$$e_i e_{i+1} e_i = e_i,$$
$$\rho e_i = e_{i+1}\rho,$$
$$\rho\rho^{-1} = 1 = \rho^{-1}\rho,$$
$$\left(\rho e_1\right)^{n-1} = \rho^n(\rho e_1),$$

where the indices are taken modulo $n$. Note that $\text{TL}_n = \text{TL}_n(t^\frac{1}{2})$ for $n \geq 2$ depends on the nonzero complex parameter $t^\frac{1}{2}$, which we omit from the notations if no confusion can arise. Note that the last defining relation $\left(\rho e_1\right)^{n-1} = \rho^n(\rho e_1)$ can be replaced by

$$\rho^2 e_{n-1} = e_1 e_2 \cdots e_{n-1}.$$
Remark 6.1. The definition for $n = 2$ and $n \geq 3$ can be placed at the same footing by describing $\text{TL}_n$ in terms of the smaller set $e_1, e_2, \ldots, e_{n-1}, \rho, \rho^{-1}$ of algebraic generators. The defining relations then are

$$
e_i^2 = (-t_i^\frac{1}{2} - t_i^{-\frac{1}{2}})e_i, \quad 1 \leq i < n,
$$

$$e_ie_j = e_je_i \quad 1 \leq i, j < n \text{ and } i - j \neq \pm 1,
$$

$$e_ie_{i\pm 1}e_i = e_i, \quad 1 \leq i, i \pm 1 < n,
$$

$$\rho e_i = e_{i+1}\rho, \quad 1 \leq i < n - 1,
$$

$$\rho^2e_{n-1} = e_1\rho^2,
$$

$$\rho\rho^{-1} = 1 = \rho^{-1}\rho,
$$

$$\rho^2e_{n-1} = e_1e_2\cdots e_{n-1}.
$$

(6.2)

Definition 6.2 ([14]). $\text{TL}_n$ is called the $(n)$th extended affine Temperley-Lieb algebra.

Denote $\mathcal{T}\mathcal{L}_n := \text{End}_{\mathcal{T}\mathcal{L}}(n)$ for the algebra of endomorphisms of $n$ in the affine Temperley-Lieb category $\mathcal{T}\mathcal{L}$. The following theorem is due to Green [14] for $n \geq 3$.

Theorem 6.3. a. $\text{TL}_0 \simeq \mathcal{T}\mathcal{L}_0$ with the algebra isomorphism $\text{TL}_0 \to \mathcal{T}\mathcal{L}_0$ defined by

$$X \mapsto \begin{array}{c}
\end{array}
$$

b. $\text{TL}_1 \simeq \mathcal{T}\mathcal{L}_1$ with the algebra isomorphism $\text{TL}_1 \to \mathcal{T}\mathcal{L}_1$ defined by

$$\rho \mapsto \begin{array}{c}
\end{array}
$$

c. $\text{TL}_2 \simeq \mathcal{T}\mathcal{L}_2$ with the algebra isomorphism $\text{TL}_2 \to \mathcal{T}\mathcal{L}_2$ defined by

$$\rho \mapsto \begin{array}{c}
\end{array}, \quad e_1 \mapsto \begin{array}{c}
\end{array}, \quad e_2 \mapsto \begin{array}{c}
\end{array}
$$

(6.3)

d. If $n \geq 3$ then $\text{TL}_n \simeq \mathcal{T}\mathcal{L}_n$ with the algebra isomorphism $\text{TL}_n \to \mathcal{T}\mathcal{L}_n$ defined by

$$\rho \mapsto \begin{array}{c}
\end{array}, \quad e_i \mapsto \begin{array}{c}
\end{array}
$$

(6.4)

for $i = 1, \ldots, n$ (with the indices and the labels of the marked points taken modulo $n$).

Proof. a and b are well known (see, for instance, [20 §1.7] and [21 §4]), while d is due to Green [14 Prop. 2.3.7].
Proof of c: A direct check shows that there exists a unique unital algebra homomorphism \( \phi : \mathcal{T}L_2 \to \mathcal{T}L_2 \) satisfying (6.1).

Recall that the set \( \mathcal{D}_2 \) of \((2,2)\)-affine diagrams form a linear basis of \( \mathcal{T}L_2 \). The \((2,2)\)-affine diagrams can be described explicitly as follows.

The \((2,2)\)-affine diagram \( \phi(\rho^m) \) for \( m \in \mathbb{Z} \) is obtained from the identity element of \( \mathcal{T}L_2 \) by winding the outer boundary counterclockwise by an angle of \( m\pi \). It follows that the pairwise distinct \((2,2)\)-affine diagrams \( \phi(\rho^m) \) \((m \in \mathbb{Z})\) form the subset of \( \mathcal{D}_2 \) consisting of \((2,2)\)-affine diagrams whose arcs all connect the inner boundary with the outer boundary. The remaining \((2,2)\)-affine diagrams are the diagrams of the form

![Diagrams](image)

in which \( r \) nonintersecting, noncontractible loops are inserted for some \( r \in \mathbb{Z}_{>0} \). For \( r = 2k \) the resulting four types of \((2,2)\)-affine diagrams are

\[
\phi(e_2(e_1e_2)^k), \quad \phi(e_1(e_2e_1)^k), \quad \phi(\rho e_1(e_2e_1)^k), \quad \phi(\rho e_2(e_1e_2)^k).
\]

For \( r = 2k + 1 \) they are

\[
\phi(\rho(e_1e_2)^k), \quad \phi(\rho(e_2e_1)^k), \quad \phi((e_2e_1)^k), \quad \phi((e_1e_2)^k).
\]

Hence \( \phi \) maps the subset

\[
(6.5) \quad \{ \rho^m \}_{m \in \mathbb{Z}} \cup \{(e_2e_1)^k, \rho(e_2e_1)^k, e_1(e_2e_1)^k, \rho e_1(e_2e_1)^k \}_{k \in \mathbb{Z}_{>0}} \\
\cup \{(e_1e_2)^k, \rho(e_1e_2)^k, e_2(e_1e_2)^k, \rho e_2(e_1e_2)^k \}_{k \in \mathbb{Z}_{>0}}
\]

of \( \mathcal{T}L_2 \) bijectively onto the linear basis \( \mathcal{D}_2 \) of \( \mathcal{T}L_2 \). By the defining relations in \( \mathcal{T}L_2 \) we see that (6.5) spans \( \mathcal{T}L_2 \). We conclude that \( \phi \) is an isomorphism of algebras. \( \square \)

Remark 6.4. By Theorem 5.1 we now also have a skein-theoretic description \( \text{End}_S(n) \) of the \( n \)th extended affine Temperley-Lieb algebra, \n
\[ \mathcal{T}L_n \simeq \mathcal{T}L_n \simeq \text{End}_S(n). \]

The skein-theoretic description of the finite Temperley-Lieb algebra is described in [16].

7. THE EXTENDED AFFINE BRAID GROUP AND THE EXTENDED AFFINE HECKE ALGEBRA

In this section we take \( n \geq 3 \). The affine Temperley-Lieb algebra \( \mathcal{TL}_n \) of type \( \hat{A}_{n-1} \) is the subalgebra of \( \mathcal{T}L_n \) generated by \( e_1, e_2, \ldots, e_n \), see [7]. The defining relations of \( \mathcal{TL}_n \) are given by the first three lines in (6.1). Note that \( \mathbb{Z} \) acts on \( \mathcal{TL}_n \) by algebra automorphisms with \( m \in \mathbb{Z} \) acting by \( e_i \mapsto e_{i+m} \) (with the indices modulo \( n \)). Let \( \mathcal{TL}_n^\rho \) be the corresponding crossed product algebra \( \mathbb{Z} \ltimes \mathcal{TL}_n \). Note that \( \mathcal{TL}_n^\rho \) is isomorphic to the algebra generated by \( e_1, \ldots, e_n, \rho^\pm 1 \) with defining relations all but the last relation in (6.1). In this identification the element of \( \mathcal{TL}_n^\rho \) associated to the automorphism \( e_i \mapsto e_{i+1} \) corresponds to the generator \( \rho \). It follows that

\[ \mathcal{T}L_n \simeq \mathcal{TL}_n^\rho / \langle \rho^2 e_{n-1} - e_1 e_2 \cdots e_{n-1} \rangle \]
with \( \langle \rho^2 e_{n-1} - e_1 e_2 \cdots e_{n-1} \rangle \) the two-sided ideal generated by \( \rho^2 e_{n-1} - e_1 e_2 \cdots e_{n-1} \).

In \([7]\) the affine Temperley-Lieb algebra \( \overline{\text{TL}}_n \) is realized as a quotient of the affine Hecke algebra of type \( \hat{A}_{n-1} \). We recall this here, and give the extension of this result to \( \text{TL}_n \).

**Definition 7.1.** The extended affine Hecke algebra \( H_n \) of type \( \hat{A}_{n-1} \) is the unital complex associative algebra with generators \( T_1, T_2, \ldots, T_n, \rho, \rho^{-1} \) and defining relations

\[
\begin{align*}
(T_i - t^{-\frac{1}{2}})(T_i + t^{\frac{1}{2}}) &= 0, \\
T_i T_j &= T_j T_i \quad \text{if} \quad i - j \neq \pm 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\
\rho T_i &= T_{i+1} \rho, \\
\rho \rho^{-1} &= 1 = \rho^{-1} \rho,
\end{align*}
\]

(7.1)

where the indices are taken modulo \( n \).

Note that \( T_i \in H_n \) is invertible with inverse \( T_i^{-1} = T_i - t^{-\frac{1}{2}} + t^{\frac{1}{2}} \). The affine Hecke algebra of type \( \hat{A}_{n-1} \) is the subalgebra \( \mathcal{P}_n \) of \( H_n \) generated by \( T_1, T_2, \ldots, T_n \). The defining relations of \( \mathcal{P}_n \) are given by the first three lines in (7.1). The extended affine Hecke algebra \( H_n \) is isomorphic to the crossed product algebra \( \mathbb{Z} \ltimes \mathcal{P}_n \), where \( m \in \mathbb{Z} \) acts on \( \mathcal{P}_n \) by the algebra automorphism \( T_i \mapsto T_{i+m} \) (with the indices modulo \( n \)).

**Proposition 7.2.** There exists a unique surjective algebra map \( \psi_n : H_n \to \text{TL}_n \) satisfying \( \rho \mapsto \rho \) and \( T_i \mapsto e_i + t^{-\frac{1}{2}} \). The kernel of \( \psi_n \) is the two-sided ideal in \( H_n \) generated by the elements

\[
(7.2) \quad T_i T_{i+1} T_i - t^{-\frac{1}{2}} T_i T_{i+1} T_i - t^{\frac{1}{2}} T_{i+1} T_i T_i + t^{-1} T_i T_{i+1} + t^{-1} T_{i+1} T_i - t^{-\frac{3}{2}}, \quad i \in \mathbb{Z}/n\mathbb{Z}.
\]

**Proof.** Fan and Green \([7]\) showed that the kernel of the unique surjective algebra map \( \overline{\psi}_n : \overline{\mathcal{P}}_n \to \overline{\text{TL}}_n \) satisfying \( T_i \mapsto e_i + t^{-\frac{1}{2}} \) is generated by the elements \( (7.2) \) (see also \([13]\)). The proposition now follows since the \( \mathbb{Z} \)-actions on \( \overline{\mathcal{P}}_n \) and \( \overline{\text{TL}}_n \) are intertwined by \( \overline{\psi}_n \). \( \square \)

The extended affine braid group \( \mathcal{B}_n \) is the group generated by \( \sigma_1, \sigma_2, \ldots, \sigma_n, \tilde{\rho} \) with defining relations

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if} \quad i - j \neq \pm 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\tilde{\rho} \sigma_i &= \sigma_{i+1} \tilde{\rho},
\end{align*}
\]

(7.3)

where the indices are taken modulo \( n \), see e.g. \([13]\) for details. Recall that \( \mathcal{B}_n \) can be realized topologically in terms of \( n \) strands in \( \mathbb{C}^* \times [0, 1] \) starting at \( \{(2z_n^{\frac{1}{n}} - 1, 0)\}_{j=1}^n \) and ending at \( \{(2z_n^{\frac{1}{n}} - 1, 1)\}_{j=1}^n \), cf., e.g., \([13]\).
Given a braid in $B_n$, project it onto the cylinder $2S^1 \times [0, 1]$ and map $2S^1 \times [0, 1]$ homeomorphically onto $A$ by collapsing the wall of the cylinder inwards onto $A \times \{0\}$. This results in an $(n, n)$-tangle diagram in $A$, which we subsequently interpret as an element in the linear skein $\text{End}_S(n)$. This defines a surjective algebra map $\mu_n : \mathbb{C}[B_n] \to \text{End}_S(n)$ satisfying

$$\mu_n(\sigma_i) = e_i + t^{-\frac{1}{4}} e_i$$

Note that $\mu_n(\tilde{\rho}) = \rho$ and $\mu_n(\rho) = t^\frac{1}{4} T_i$, where the last equality follows from the Kauffman skein relation (3.1). Note also that $\mu_n(\sigma_i^{-1}) = t^{-\frac{1}{4}} e_i + t^\frac{1}{4}$.

**Remark 7.3.** Let $\nu_n : \mathbb{C}[B_n] \to H_n$ be the surjective algebra map satisfying $\nu_n(\tilde{\rho}) = \rho$ and $\nu_n(\sigma_i) = t^\frac{1}{4} T_i$, then we have the commuting diagram

$$
\begin{array}{ccc}
\mathbb{C}[B_n] & \xrightarrow{\nu_n} & H_n \\
\downarrow{\mu_n} & & \downarrow{\psi_n} \\
\text{End}_S(n) & & \\
\end{array}
$$

of algebra maps.

### 8. The Arc Insertion Functor

Let $D$ be an $(m, n)$-tangle diagram. We write $D^{ins}$ for the $(m + 1, n + 1)$-tangle diagram obtained from $D$ as follows. Add to $D$ two extra marked points $\xi_{m+1}^m$ and $2\xi_{n+1}^n$ and connect them with an arc in the fundamental rectangle of $D$ that undercrosses all arcs of $D$ it meets. Finally apply an ambient isotopy of $A$ that stabilizes the inner and outer boundary of $A$ and moves the $m + 1$ marked points on the inner boundary to
(i = 1, . . . , m + 1) and the n + 1 marked points on the outer boundary to 2τ_{n+1}^j (j = 1, . . . , n + 1). Note that D_{ins} only depends on D.

Example 8.1. For the (0, 2)-tangle diagram D_1 and the (1, 3)-tangle diagram D_2 given by

\[
D_1 := \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\]

we get

\[
D_1^{ins} = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\]

Proposition 8.2. There exists a unique linear functor \( \mathcal{I} : \mathcal{S} \to \mathcal{S} \) satisfying \( \mathcal{I}(n) = n + 1 \) (\( n \in \mathbb{Z}_{\geq 0} \)) and satisfying

(8.1) \[ \mathcal{I}([D]) := [D^{ins}] \]

for tangle diagrams D. We call \( \mathcal{I} \) the arc insertion functor.

Proof. Consider the linear map \( \tilde{\mathcal{I}} : \mathbb{C}[\text{Hom}_T(m, n)] \to \text{Hom}_S(m + 1, n + 1) \) defined by \( \tilde{\mathcal{I}}(D) := [D^{ins}] \) for \( D \in \text{Hom}_T(m, n) \). It is obvious that the expression \( \tilde{\mathcal{I}}(D) \) is unaltered when the Kauffman skein relation \( \text{(3.1)} \) is applied to one of the crossings of \( D \). It is also unaltered if the loop removal relation \( \text{(3.2)} \) is applied to \( D \), since the Reidemeister move \( \Omega_2 \) holds true in \( \text{Hom}_S(m + 1, n + 1) \). Hence \( \tilde{\mathcal{I}} \) descends to a well defined linear map \( \mathcal{I} : \text{Hom}_S(m, n) \to \text{Hom}_S(m + 1, n + 1) \) satisfying (8.1) for all \((m, n)\)-tangle diagrams \( D \).

Clearly \( \mathcal{I} \) maps the identity morphism to the identity morphism. So it remains to show that \( \mathcal{I} \) respects compositions of morphisms. But this follows from the rule that the inserted line does not cross the cut of the fundamental rectangle. \( \square \)

For \( n \in \mathbb{Z}_{\geq 0} \), the unit preserving algebra map \( \mathcal{I}_{|\mathcal{S}_n} : \text{End}_S(n) \to \text{End}_S(n + 1) \) can be interpreted as an algebra map \( \mathcal{I}_n : \text{T}L_n \to \text{T}L_{n+1} \) since \( \text{T}L_n \cong \text{End}_S(n) \) (see Theorem 6.3 and Remark 6.4). In the following proposition we explicitly compute \( \mathcal{I}_n \) on the generators of \( \text{T}L_n \).

Proposition 8.3. a. \( \mathcal{I}_0(X) = t^\frac{1}{4} \rho + t^{-\frac{3}{4}} \rho^{-1} \).

b. \( \mathcal{I}_1(\rho) = \rho(t^{\frac{1}{4}} e_1 + t^{-\frac{3}{4}}) \) and \( \mathcal{I}_1(\rho^{-1}) = (t^{\frac{1}{4}} e_1 + t^{-\frac{3}{4}}) \rho^{-1} \).

c. For \( n \geq 2 \) we have

\[
\begin{align*}
\mathcal{I}_n(e_i) &= e_i, \quad i = 1, \ldots, n - 1, \\
\mathcal{I}_n(e_n) &= (t^{\frac{1}{4}} e_n + t^{-\frac{3}{4}}) e_{n+1} (t^{-\frac{1}{4}} e_n + t^{\frac{1}{4}}), \\
\mathcal{I}_n(\rho) &= \rho(t^{-\frac{1}{4}} e_n + t^{\frac{3}{4}}), \\
\mathcal{I}_n(\rho^{-1}) &= (t^{\frac{1}{4}} e_n + t^{-\frac{3}{4}}) \rho^{-1}.
\end{align*}
\]
Proof. These are direct computations in the skein module.

Proof of a: We have $X = \bullet$ so

$$I_0(X) = t^\frac{1}{2} + t^{-\frac{1}{2}} = t^\frac{1}{2}\rho + t^{-\frac{1}{2}}\rho^{-1}$$

by the Kauffman skein relation (3.1).

Proof of b: We have $\rho = \bullet$ so

$$I_1(\rho) = t^\frac{1}{2} = \rho(t^\frac{1}{2}e_1 + t^\frac{1}{2})$$

by applying the Kauffman skein relation (3.1) to the crossing and rewriting the resulting expressions in terms of the generators of $\text{TL}_2$ (compare with the proof of Theorem 6.3). In a similar way one proves the explicit formula for $I_1(\rho^{-1}) \in \text{TL}_2$.

Proof of c: The formulas for $I_n(\rho^{\pm 1}) \in \text{TL}_{n+1}$ are obtained by a similar computation as in b.

For $1 \leq i < n$, applying the arc insertion functor to $e_i \in \text{TL}_n$ does not introduce crossings. The resulting $(n+1,n+1)$-affine diagram represents the generator $e_i$ in $\text{TL}_{n+1}$, so $I_n(e_i) = e_i$.

Note that applying the arc insertion functor to $e_n \in \text{TL}_n$ introduces two crossings. Resolving both crossings with the Kauffman skein relation (3.1) and expressing the resulting linear combination of four $(n+1,n+1)$-affine diagrams in terms of the generators of $\text{TL}_{n+1}$ yield the formula

$$I_n(e_n) = t^\frac{1}{2}e_ne_{n+1} + t^{-\frac{1}{2}}e_{n+1}e_n + e_{n+1} + e_n = (t^\frac{1}{2}e_n + t^{-\frac{1}{2}})e_{n+1}(t^{-\frac{1}{2}}e_n + t^\frac{1}{2}).$$

□

Let $n \geq 3$ for the remainder of this section (with appropriate adjustments the following remark can be extended to $n \geq 0$).

Let $I_n^B : \mathcal{B}_n \rightarrow \mathcal{B}_{n+1}$ be the group homomorphism that topologically is described by sticking in an additional braid between the $n$th and the first braid, with the new braid running “behind” all other braids (but not wrapping around the pole). For example,
It is the unique group homomorphism satisfying
\[
\mathcal{I}_n^{br}(\sigma_i) = \sigma_i, \quad i = 1, \ldots, n - 1,
\]
\[
\mathcal{I}_n^{br}(\sigma_n) = \sigma_n \sigma_{n+1} \sigma_n^{-1},
\]
\[
\mathcal{I}_n^{br}(\tilde{\rho}) = \tilde{\rho} \sigma_n^{-1}.
\]
Extending \(\mathcal{I}_n^{br}\) linearly to an algebra map \(\mathcal{I}_n^{br} : \mathbb{C}[B_n] \to \mathbb{C}[B_{n+1}]\), we have
\[
\mu_{n+1} \circ \mathcal{I}_n^{br} = \mathcal{I}|_{S_n} \circ \mu_n
\]
with \(\mu_n : \mathbb{C}[B_n] \to S_n\) the algebra map as defined in the previous section.

In addition, it is easy to show that there exists a unique unit preserving algebra map \(\mathcal{I}_n^{ha} : H_n \to H_{n+1}\) satisfying
\[
\mathcal{I}_n^{ha}(T_i) = T_i, \quad i = 1, \ldots, n - 1,
\]
\[
\mathcal{I}_n^{ha}(T_n) = T_n T_{n+1} T_n^{-1},
\]
\[
\mathcal{I}_n^{ha}(\rho) = t^{-\frac{1}{2}} \rho T_n^{-1},
\]
and
\[
\nu_{n+1} \circ \mathcal{I}_n^{br} = \mathcal{I}_n^{ha} \circ \nu_n
\]
with \(\nu_n : \mathbb{C}[B_n] \to H_n\) as defined in the previous section.

**Remark 8.4.** The maps \(\mathcal{I}_n^{br}\) and \(\mathcal{I}_n\) were constructed before in [3, 9].

9. **Towers of extended affine Temperley-Lieb algebra modules**

In [3] the sequence \(\{\mathcal{I}_n\}_{n \in \mathbb{Z}_{\geq 0}}\) of algebra maps \(\mathcal{I}_n : TL_n \to TL_{n+1}\) was used to study affine Markov traces. In [9] it was used to study fusion of affine Temperley-Lieb representations. In the next two sections we use the sequence \(\{\mathcal{I}_n\}_{n \in \mathbb{Z}_{\geq 0}}\) of algebra maps to introduce the notion of towers of extended affine Temperley-Lieb representations. We construct examples that are relevant for understanding the dependence of dense loop models and Heisenberg XXZ spin-\(\frac{1}{2}\) chains on their system size (cf. [15, 6, 4]).

We first introduce some notations. Let \(A\) be a \(\mathbb{C}\)-algebra. Write \(\mathcal{C}_A\) for the category of left \(A\)-modules. Write \(\text{Hom}_A(M, N)\) for the space of morphisms \(M \to N\) in \(\mathcal{C}_A\), which we will call intertwiners. Suppose that \(\eta : A \to B\) is a (unit preserving) morphism of \(\mathbb{C}\)-algebras. Write \(\text{Ind}^\eta : \mathcal{C}_A \to \mathcal{C}_B\) and \(\text{Res}^\eta : \mathcal{C}_B \to \mathcal{C}_A\) for the corresponding induction
and restriction functor. Concretely, if $M$ is a left $A$-module then
\[ \text{Ind}^\eta(M) := B \otimes_A M \]
with $B$ viewed as right $A$-module by $b \cdot a := b\eta(a)$ for $b \in B$ and $a \in A$. If $N$ is a left $B$-module then $\text{Res}^\eta(N)$ is the complex vector space $N$, viewed as $A$-module by $a \cdot n := \eta(a)n$ for $a \in A$ and $n \in N$. The restriction functor $\text{Res}^\eta$ is right adjoint to $\text{Ind}^\eta$. If $M$ is a left $A$-module and $N$ a left $B$-module, then the corresponding linear isomorphism
\[ \text{Hom}_A(M, \text{Res}^\eta(N)) \cong \text{Hom}_B(\text{Ind}^\eta(M), N) \]
is given by $\phi \mapsto \widehat{\phi}$ with $\widehat{\phi} \in \text{Hom}_B(\text{Ind}^\eta(M), N)$ defined by
\[ \widehat{\phi}(Z \otimes_A m) := Z\phi(m) \]
for $Z \in B$ and $m \in M$.

We use the shorthand notation $C_n$ for $C_{\text{TL}_n}$. For a left $\text{TL}_{n+1}$-module $V_{n+1}$ we use the shorthand notation $V^\ell_{n+1}$ for the left $\text{TL}_n$-module $\text{Res}^\eta_{n+1}(V_{n+1})$.

**Definition 9.1.** A tower
\[ V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots \]
of extended affine Temperley-Lieb algebra modules consists of a sequence $\{(V_n, \phi_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ with $V_n$ a left $\text{TL}_n$-module and $\phi_n \in \text{Hom}_{\text{TL}_n}(V_n, V^\ell_{n+1})$.

**Example 9.2.** The interpretation of the extended affine Temperley-Lieb algebras as the endomorphism spaces of the skein category $S$ immediately produces examples of towers of extended affine Temperley-Lieb algebra modules. Indeed, for any $m \in \mathbb{Z}_{\geq 0}$ we can construct the following natural tower $\{(V_n^{(m)}, \phi_n^{(m)})\}_{n \in \mathbb{Z}_{\geq 0}}$. Take
\[ V_n^{(m)} := \text{Hom}_S(m + n, n) \]
and view the space $V_n^{(m)}$ as a left module over $\text{TL}_n \simeq \text{End}_S(n)$ with representation map
\[ \pi_n^{(m)}(Y)Z := Y \circ Z \]
for $Y \in \text{End}_S(n)$ and $Z \in V_n^{(m)} = \text{Hom}_S(m + n, n)$. Take as intertwiners
\[ \phi_n^{(m)} := \mathcal{I}_{|\text{Hom}_S(m + n, n)} : V_n^{(m)} \rightarrow V_{n+1}^{(m)} \]
Note that there are other intertwiners one can take here, for instance $V_n^{(m)} \ni Z \mapsto \mathcal{I}(Z) \circ R$ with $R \in \text{End}_S(m + n)$. A refinement of this example will play an important role in the next section.

In the definition of towers $\{(V_n, \phi_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ of extended affine Temperley-Lieb algebra modules we do not require conditions on the intertwiners $\phi_n$, in particular allowing trivial intertwiners. A stronger version of towers avoiding trivial intertwiners is as follows.

**Definition 9.3.** We say that the tower $\{(V_n, \phi_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ of the extended affine Temperley-Lieb algebra modules is nondegenerate if $\widehat{\phi}_n : \text{Ind}^\eta_{\mathbb{Z}}(V_n) \rightarrow V_{n+1}$ is surjective for all $n \in \mathbb{Z}_{\geq 0}$. 


In particular, for a nondegenerate tower \( \{ (V_n, \phi_n) \}_{n \in \mathbb{Z}_{\geq 0}} \) of extended affine Temperley-Lieb algebra modules, the module \( V_{n+1} \) is a quotient module of \( \text{Ind}^{Z_n}(V_n) \),
\[
V_{n+1} \simeq \text{coim}(\hat{\phi}_n).
\]

We give an example of a nondegenerate tower of extended affine Temperley-Lieb algebra modules in the next section.

10. THE LINK PATTERN TOWER

Motivated by applications to integrable models in statistical physics [15, 6, 4], in particular to the dense loop model and the Heisenberg XXZ spin-\( 1/2 \) chain, we are going to construct a family of towers of extended affine Temperley-Lieb algebra representations acting on spaces of link patterns on the punctured disc. We again use the skein categorical context to build the tower.

We first build the extended affine Temperley-Lieb algebra modules. For \( n = 2k \) with \( k \in \mathbb{Z}_{\geq 0} \), note that \( \text{Hom}_{S}(0, 2k) \) is a \((\text{TL}_{2k}, \text{TL}_0)\)-bimodule. Define for \( u \in \mathbb{C} \),
\[
V_{2k}(u) := \text{Hom}_{S}(0, 2k) \otimes_{\text{TL}_0} \mathbb{C}(u),
\]
with \( \mathbb{C}(u) \) the one-dimensional representation of \( \text{TL}_0 = \mathbb{C}[X] \) defined by \( X \mapsto u \). We use the shorthand notation \( Y_u \) for \( Y \otimes_{\text{TL}_0} 1 \) with \( Y \in \text{Hom}_{S}(0, 2k) \).

For \( n = 2k + 1 \) with \( k \in \mathbb{Z}_{\geq 0} \) note that \( \text{Hom}_{S}(1, 2k + 1) \) is a \((\text{TL}_{2k+1}, \text{TL}_1)\)-bimodule and define for \( v \in \mathbb{C}^\ast \),
\[
V_{2k+1}(v) := \text{Hom}_{S}(1, 2k + 1) \otimes_{\text{TL}_1} \mathbb{C}(v),
\]
with \( \mathbb{C}(v) \) the one-dimensional representations of \( \text{TL}_1 = \mathbb{C}[\rho^{\pm 1}] \) defined by \( \rho \mapsto v \). We use the notation \( Z_v \) for \( Z \otimes_{\text{TL}_1} 1 \) with \( Z \in \text{Hom}_{S}(1, 2k + 1) \).

Remark 10.1. The left \( \text{TL}_{2k} \)-module \( V_{2k}(u) \) and the left \( \text{TL}_{2k+1} \)-module \( V_{2k+1}(v) \) are examples of the so-called standard \( \text{TL}_{N} \)-modules \( W_{j,z}[N] \) from [9, §4.2] (the extended affine Temperley-Lieb algebra \( \text{TL}_{N} \) is denoted by \( \text{TL}^a_{N} \) in [9]). Concretely, writing \( u = x + x^{-1} \) with \( x \in \mathbb{C}^\ast \), we have
\[
V_{2k}(u) = W_{0,x}[2k],
\]
\[
V_{2k+1}(v) = W_{1,\rho}[2k + 1].
\]

Next we study towers having \( V_{2k}(u) \) and \( V_{2k+1}(v) \) as building blocks. For this we need special elements in the skein modules \( \text{End}_S(0), \text{End}_S(1) \) and \( \text{Hom}_S(0, 2) \). Let \( \emptyset \in \text{End}_S(0) \) be the skein class of the empty tangle diagram in \( A \) and \( 1 \in \text{End}_S(1) \) the identity morphism. Then \( V_0(u) = \mathbb{C}\emptyset_u \) and \( V_1(v) = \mathbb{C}1_v \). For \( V_2(u) \), note that the skein module \( \text{Hom}_S(0, 2) \) is a free right \( \text{TL}_0 = \mathbb{C}[X] \)-module with \( \text{TL}_0 \)-basis \( \{ [c_+], [c_-] \} \), where
\[
c_+ = \begin{array}{c}
\includegraphics[scale=0.2]{c_plus}
\end{array},
\]
\[
c_- = \begin{array}{c}
\includegraphics[scale=0.2]{c_minus}
\end{array}.
\]
In particular, \( V_2(u) \) is two-dimensional with linear basis \( \{(c_+)_u, (c_-)_u\} \). Write \( U := t^\frac{1}{4}[c_+] + v[c_-] \in \text{Hom}_S(0, 2) \), so in pictures,

\[
U = t^\frac{1}{4} \begin{array}{cc}
1 & 2 \\
1 & \end{array} + v \begin{array}{cc}
2 & 1 \\
1 & \end{array}.
\]

**Lemma 10.2.** Let \( u \in \mathbb{C} \) and \( v \in \mathbb{C}^* \).

(i) Define the linear map \( \phi_0 : V_0(u) \to V_1(v) \) by \( \phi_0(\emptyset_u) := 1_v \). Then

\[
\text{Hom}_{\text{TL}_0}(V_0(u), V_1(v)^2) = \begin{cases} 
\mathbb{C} \phi_0 & \text{if } u = t^\frac{1}{2}v + t^{-\frac{1}{2}}v^{-1}, \\
\{0\} & \text{otherwise}.
\end{cases}
\]

(ii) Let \( u = t^\frac{1}{2}v + t^{-\frac{1}{2}}v^{-1} \). Define the linear map \( \phi_1 : V_1(v) \to V_2(u) \) by \( \phi_1(1_v) := U_u \). Then

\[
\text{Hom}_{\text{TL}_1}(V_1(v), V_2(u)^2) = \mathbb{C} \phi_1.
\]

**Proof.**

(i) Note that \( \phi_0 \in \text{Hom}_{\text{TL}_0}(V_0(u), V_1(v)^2) \) if and only if \( I_0(X)1_v = u1_v \). Proposition 8.3(a) gives \( I_0(X)1_v = (t^\frac{1}{2}v + t^{-\frac{1}{2}}v^{-1})1_v \), hence the result.

(ii) Take an arbitrary element \( Z_u \in V_2(u) \) with \( Z \in \text{Hom}_S(0, 2) \). The linear map \( \chi : V_1(v) \to V_2(u) \) defined by \( \chi(1_v) = Z_u \) is in \( \text{Hom}_{\text{TL}_1}(V_1(v), V_2(u)^2) \) if and only if

\[
I_1(\rho)Z_u = vZ_u
\]

in \( V_2(u) \). By Proposition 8.3(b) we have \( I_1(\rho) = \rho(t^{-\frac{1}{4}}e_1 + t^\frac{1}{4}) \). A direct computation in \( \text{Hom}_S(0, 2) \) shows that

\[
\rho(t^{-\frac{1}{4}}e_1 + t^\frac{1}{4}) \circ [c_+] = -t^{-\frac{3}{4}}[c_-], \\
\rho(t^{-\frac{1}{4}}e_1 + t^\frac{1}{4}) \circ [c_-] = t^\frac{1}{4}[c_+] + t^{-\frac{1}{4}}[c_-] \circ X,
\]

where we have used the loop removal relation \( \text{(8.2)} \) in the derivation of the first identity.

Writing \( m_{\alpha, \beta} := \alpha(c_+)_u + \beta(c_-)_u \in V_2(u) \) with \( \alpha, \beta \in \mathbb{C} \) we obtain from \( \text{(10.1)} \),

\[
\rho(t^{-\frac{1}{4}}e_1 + t^\frac{1}{4})m_{\alpha, \beta} = vm_{\alpha', \beta'}
\]

with

\[
\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = M \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad M := \begin{pmatrix} 0 & t^\frac{1}{4}v^{-1} \\ -t^{-\frac{3}{4}}v^{-1} & 1 + t^{-\frac{1}{2}}v^{-2} \end{pmatrix}.
\]

The result follows since \( M \) has eigenvalue one with eigenspace \( \mathbb{C} \begin{pmatrix} t^\frac{1}{4} \\ v \end{pmatrix} \). \( \square \)

The following theorem shows that \( \phi_0 \) and \( \phi_1 \) can be extended to a nondegenerate tower

\[
V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} \cdots
\]

of extended affine Temperley-Lieb representations when \( u = t^\frac{1}{4}v + t^{-\frac{1}{2}}v^{-1} \).
Note that the intertwiners $\phi_0$ and $\phi_1$ can alternatively be characterized by the formulas
\[
\phi_0(Y_v) = (I(Y))_v, \quad Y \in \text{End}_S(0),
\]
\[
\phi_1(Z_v) = (I(Z) \circ U)_v, \quad Z \in \text{End}_S(1)
\]
since $I(\emptyset)_v = 1_v$ and $(I(1) \circ U)_u = U_u$.

**Theorem 10.3.** Let $v \in \mathbb{C}^*$ and set $u := t^k v + t^{-k} v^{-1}$ and $k \in \mathbb{Z}_{\geq 0}$.

(i) There exist unique intertwiners $\phi_{2k} \in \text{Hom}_{TL_{2k}}(V_{2k}(u), V_{2k+1}(v)^2)$ and
\[
\phi_{2k+1} \in \text{Hom}_{TL_{2k+1}}(V_{2k+1}(v), V_{2k+2}(u)^2)
\]
satisfying
\[
\phi_{2k}(Y_v) := (I(Y))_v, \quad Y \in \text{Hom}_S(0, 2k),
\]
\[
\phi_{2k+1}(Z_v) := (I(Z) \circ U)_u, \quad Z \in \text{Hom}_S(1, 2k+1).
\]

(ii) The tower
\[
V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \ldots
\]

of extended affine Temperley-Lieb algebra representations is nondegenerate if $v \neq t^{\frac{1}{2}}$.

**Proof.** (i) If the maps $\phi_{2k}$ and $\phi_{2k+1}$ are well-defined, then they are obviously intertwiners. To prove that $\phi_{2k}$ and $\phi_{2k+1}$ are well-defined we have to show that $(I(Y) \circ I(X))_v = uI(Y)_v$ in $V_{2k+1}(v)$ for $Y \in \text{Hom}_S(0, 2k)$ and $(I(Z) \circ (I(\rho) \circ U))_u = v(I(Z) \circ U)_u$ in $V_{2k+2}(u)$ for $Z \in \text{Hom}_S(1, 2k+1)$. This is analogous to the proof of Lemma 10.2.

(ii) Denote by $C^{(2k)} \in \text{Hom}_S(0, 2k)$ and $C^{(2k+1)} \in \text{Hom}_S(1, 2k+1)$ the skein classes of the tangle diagrams

![Diagram](image)

respectively. We claim that $V_{2k}(u) = TL_{2k} \cdot (C^{(2k)})_u$ and $V_{2k+1}(v) = TL_{2k+1} \cdot (C^{(2k+1)})_v$.

Let $M_{2k}$ be the vector space spanned by the link patterns (noncrossing perfect matchings) on a strip of $\{1, \ldots, 2k\}$, viewed as left module over the subalgebra $TL_{2k}^{fin}$ of $TL_{2k}$ generated by $e_1, \ldots, e_{2k-1}$ (see, e.g., [10]). Let $L^{(2k)} \in M_{2k}$ be the link pattern connecting $j$ to $2k + 1 - j$ for $j = 1, \ldots, 2k$. By wrapping the strip on the annulus in such a way that $\{1, \ldots, 2k\}$ correspond to the marked points $2j_{2k} - 1$ $(j = 1, \ldots, 2k)$, we get an injective $TL_{2k}^{fin}$-module morphism $M_{2k} \hookrightarrow V_{2k}(u)$ mapping $L^{(2k)}$ to $(C^{(2k)})_u$. By inserting in addition an arc in $A$ connecting 1 to $2j_{2k+1}$ not intersecting the cut of the fundamental rectangle and not intersecting the other arcs, we get an injective $TL_{2k}^{fin}$-module morphism $M_{2k} \hookrightarrow V_{2k+1}(v)$. In this case, $L^{(2k)}$ is mapped to $(C^{(2k+1)})_v$.

With these observations and the fact that $\rho \in TL_n$ can be used to turn diagrams counterclockwise by an angle of $2\pi/n$, the claim is a consequence of $M_{2k} = TL_{2k}^{fin} \cdot L^{(2k)}$. 
This in turn is easy to establish using the alternative description of link patterns of \{1, \ldots, 2k\} in terms of Dyck paths (see, e.g., [11, §2.4]).

Now note that
$$\hat{\phi}_{2k}(1 \otimes_{\text{TL}_{2k}} (C(2k))^u) = (\mathcal{I}(C(2k)))^u = (C(2k+1))^u$$

hence \(\hat{\phi}_{2k} \in \text{Hom}_{\text{TL}_{2k+1}}(\text{Ind}_{2k}(V_{2k}(u)), V_{2k+1}(v))\) is surjective. By a direct computation we have
$$\hat{\phi}_{2k+1}(e_{k+1} \cdots e_{2k} e_{2k+1} \otimes_{\text{TL}_{2k+1}} (C(2k+1))^v) =$$
$$= (e_{k+1} \cdots e_{2k} e_{2k+1} \mathcal{I}(C(2k+1)))^u =$$
$$= (t^{\frac{1}{4}} v^2 - t^{\frac{3}{4}})(C(2k+2))^u,$$

hence \(\hat{\phi}_{2k+1} \in \text{Hom}_{\text{TL}_{2k+2}}(\text{Ind}_{2k+1}(V_{2k+1}(v)), V_{2k+2}(v))\) is surjective if \(v^2 \neq t^{\frac{1}{2}}\).

\(\square\)

**Remark 10.4.** The two skein classes \(\rho^{-k} \circ C(2k)\) and \(\rho^{-k} \circ C(2k+1)\) play an important role in determining the normalisation of the ground state of the dense loop model (see [11, 15]).

Fix \(v \in \mathbb{C}^*\) and set \(u = t^{\frac{1}{4}} v + t^{-\frac{3}{4}} v^{-1}\) for the remainder of this section. For \(n = 2k\) the representation space \(V_{2k}(u)\) consists of the equivalence classes of the skein module \(\text{Hom}_S(0, 2k)\) with respect to the congruence relation obtained as the linear and transitive closure of the noncontractible loop removal relation

\[
(10.2) \quad \begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{example1} & \quad \includegraphics[width=0.2\textwidth]{example2}
\end{array}
\]

\[
= (t^{\frac{1}{4}} v + t^{-\frac{3}{4}} v^{-1})
\]

For \(n = 2k+1\) odd, the representation space \(V_{2k+1}(v)\) consists of the equivalence classes of the skein module \(\text{Hom}_S(1, 2k+1)\) with respect to the congruence relation obtained as the linear and transitive closure of the following Dehn twist removal relation

\[
(10.3) \quad \begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{example3} & \quad \includegraphics[width=0.2\textwidth]{example4}
\end{array}
\]

\[
1 = v
\]

Let \(\tilde{C}_{2k}\) be a set of representatives of the ambient isotopy classes of \((0, 2k)\)-affine diagrams without loops. Let \(\tilde{C}_{2k+1}\) be a set of representatives of the ambient isotopy classes of the \((1, 2k + 1)\)-affine diagrams whose defect line lies within the fundamental rectangle of the diagram. By definition, the defect line of a \((1, 2k+1)\)-tangle diagram is the arc connecting the inner boundary of \(A\) with the outer boundary of \(A\). Then \(\tilde{B}_{2k} := \{([D])_u \mid D \in \tilde{C}_{2k}\}\) is a linear basis of \(V_{2k}(u)\) and \(\tilde{B}_{2k+1} := \{([D])_v \mid D \in \tilde{C}_{2k+1}\}\) is a linear basis of \(V_{2k+1}(v)\).

**Definition 10.5.** Let \(u = t^{\frac{1}{4}} v + t^{-\frac{3}{4}} v^{-1}\). We call the tower

\[
V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \ldots
\]
of extended affine Temperley-Lieb algebra representations the link pattern tower. We call \( v \in \mathbb{C}^* \) the twist weight and \( t^\frac{v}{4}v + t^{-\frac{v}{4}}v^{-1} \) the noncontractible loop weight of the link pattern tower.

Note that the intertwiners \( \phi_{2k} \) of the link pattern tower are simply given by the insertion of an arc in the underlying \((0, 2k)\)-tangle diagrams connecting the outer boundary with the inner boundary. This newly inserted arc is the defect line.

The intertwiners \( \phi_{2k+1} \) in the link pattern tower are more subtle. The intertwiner \( \phi_{2k+1} \) acts on the representative of a \((1, 2k + 1)\)-tangle diagram by detaching the defect line from the inner boundary and re-attaching it to the outer boundary in two different ways: either by enclosing the hole, or not. These two contributions get different weights \( t_{1/4} \) and \( v \), respectively. In Theorem 10.3 we have described the operation \( \phi_{2k+1} \) as the composition of arc insertion and composing with the linear combination \( U \in \text{Hom}_{S}(0, 2) \) of the two basic \((0, 2)\)-tangle diagrams \( c_+ \) and \( c_- \).

Example 10.6.

Let \( D := \{ z \in \mathbb{C} \mid |z| \leq 2 \} \) be the unit disc of radius two and \( D^* := D \setminus \{ 0 \} \). Let \( \mathcal{L}_{2k} \) be the set of link patterns in \( D^* \) connecting the \( 2k \) marked points \( \{ 2i\xi_{2k}^{-1} \}_{i=1}^{2k} \), i.e., it the set of perfect noncrossing matchings within \( D^* \) of the marked points \( \{ 2i\xi_{2k}^{-1} \}_{i=1}^{2k} \). For \( n = 2k + 1 \) odd, let \( \tilde{\mathcal{L}}_{2k+1} \) be the set of link patterns in \( D \) connecting the \( 2k + 2 \) marked points \( \{ 0 \} \cup \{ 2i\xi_{2k+1}^{-1} \}_{i=1}^{2k+1} \). In this context we call the line connecting to the puncture 0 the defect line. Since the defect line is now connected to 0 instead of the hole of the annulus we are loosing the information about how exactly the defect line is connected to 0. In other words, adding Dehn twists to the defect line does not change the link pattern. We are now going to realize the link pattern tower on the space of link patterns when the twist weight \( v \) is one.

For this consider the map \( A \rightarrow \mathbb{D} := \{ z \in \mathbb{C} \mid |z| \leq 2 \} \) given by \( re^{i\theta} \mapsto 2e^{\frac{2\pi}{4\xi_{2k}}}e^{i\theta} \). It is a radial map that fixes the outer boundary \( 2S^1 \) and maps the inner boundary \( S^1 \) of \( A \) onto \( 0 \in \mathbb{D} \). In this way the set \( \tilde{\mathcal{C}}_{n} \) of affine diagrams in \( A \) labelling a basis of the \( n \)th representation space in the link pattern tower is identified with \( \mathcal{L}_{n} \) for \( n \in \mathbb{Z}_{\geq 0} \). This results in a vector space identification of \( L_{n} \) with the complex vector space \( \mathbb{C}[\mathcal{L}_{n}] \) having \( \mathcal{L}_{n} \) as its linear basis. We now transport the TL\(_{2k}\)-module structure on \( V_{2k}(u) \) and the TL\(_{2k+1}\)-module structure on \( V_{2k+1}(v) \) to \( \mathbb{C}[\mathcal{L}_{2k}] \) and \( \mathbb{C}[\mathcal{L}_{2k+1}] \) respectively through these isomorphisms. It leads to an explicit realization of the link pattern tower with twist
weight one as a tower \( \{ (\phi_n, C[L_n]) \}_{n \in \mathbb{Z}_{\geq 0}} \) of extended affine Temperley-Lieb algebra representations acting on link patterns on \( \mathbb{D}^* \).

Note that the descriptions of the intertwiners \( \phi_{2k} \) and \( \phi_{2k+1} \) in terms of link patterns are as before: \( \phi_{2k} \) is the insertion of a defect line, and \( \phi_{2k+1} \) is detaching the defect line from the puncture 0 and re-attaching it to the outer boundary in two different ways. Note though that the crucial second description of \( \phi_{2k+1} \), in which a second defect line is added first and then the two defect lines are detached from the puncture 0 and connected to each other in two different ways, requires that one works on the annulus \( A \) instead of on the punctured disc \( \mathbb{D}^* \).

**Example 10.7.**

(1) Example of the action of \( e_2 \in TL_3 \) on \( C[L_3] \):

(2) Example of the action of \( e_2 \in TL_4 \) on \( C[L_4] \):

(3) Example of the intertwiner \( \phi_2 \) acting on \( C[L_2] \):

(4) Example of the intertwiner \( \phi_3 \) acting on \( C[L_3] \) (it corresponds to the second example from Example 10.6 with \( v = 1 \)):

**Remark 10.8.** The link pattern tower \( \{ (C[L_n], \phi_n) \}_{n \in \mathbb{Z}_{\geq 0}} \) with twist weight one plays an important role in the study of the dense loop model on the semi-infinite cylinder [15, 6, 4]. The representation space \( C[L_n] \) is the state space of the model of system size \( n \). In [6] the dense loop model of system size \( 2k + 1 \) is related to the dense loop model of system size \( 2k \) through the map \( \phi_{2k} \). The results in this paper allows one to also relate the dense loop model of system size \( 2k + 2 \) to the dense loop model of system size \( 2k + 1 \) through the (nontrivial) intertwiner \( \phi_{2k+1} \). We will return to this in [4], in which we
also derive recursion relations for associated ground states and for associated solutions of quantum Knizhnik-Zamolodchikov equations.

11. The link pattern tower and fusion

Let \( d_n \in \text{TL}_n \simeq \text{End}_S(n) \) be the skein class of the \((n,n)\)-tangle diagram

\[
\text{In terms of the algebraic generators of TL}_n, \text{ the element } d_{n+1} \text{ can be expressed as}
\]

\[
d_n = (t^{\frac{1}{2}} e_{n-1} + t^{-\frac{1}{2}}) \cdots (t^{\frac{1}{2}} e_1 + t^{-\frac{1}{2}}) \rho.
\]

Note that \( d_n \in \text{TL}_n \) is invertible and that

\[
(11.1) \quad d_{j+1} \mathcal{I}(Z) = \mathcal{I}(Z) d_{j+1} \quad \forall Z \in \text{Hom}_S(i, j).
\]

In particular, \( d_{n+1} \) lies in the centralizer of \( \mathcal{I}_n(\text{TL}_n) \) in \( \text{TL}_{n+1} \). Hence the algebra map

\[
\epsilon_n : \text{TL}_n \otimes \text{TL}_1 \rightarrow \text{TL}_{n+1}
\]

satisfying \( \epsilon_n(1 \otimes \rho) = d_{n+1} \) and \( \epsilon_n(Z \otimes 1) = \mathcal{I}_n(Z) \) for \( Z \in \text{TL}_n \). This is the special case \( \epsilon_{n,1} \) of the algebra maps \( \epsilon_{n,m} : \text{TL}_n \otimes \text{TL}_m \rightarrow \text{TL}_{n+m} \) \((n, m \in \mathbb{Z}_{\geq 0})\) defined in \([9] (3.14)\).

The algebra maps \( \epsilon_{n,m} \) were used in \([9]\) to define the fusion product of modules over extended affine Temperley-Lieb algebras as follows.

**Definition 11.1** ([9]). The fusion product of a left \( \text{TL}_n \)-module \( M_1 \) and a left \( \text{TL}_m \)-module \( M_2 \) is the left \( \text{TL}_{n+m} \)-module

\[
M_1 \times_f M_2 := \text{Ind}^{\epsilon_{n,m}}(M_1 \otimes M_2).
\]

Consider the link pattern tower

\[
V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \ldots
\]

where \( u = t^{\frac{1}{2}} v + t^{-\frac{1}{2}} v^{-1} \). Consider the canonical surjective intertwiners

\[
\pi_{2k} : \text{Ind}^{\epsilon_{2k}}(V_{2k}(u)) \rightarrow V_{2k}(u) \times_f V_1(v),
\]

\[
\pi_{2k+1} : \text{Ind}^{\epsilon_{2k+1}}(V_{2k+1}(v)) \rightarrow V_{2k+1}(v) \times_f V_1(v^{-1}),
\]

of \( \text{TL}_{2k+1} \)-modules and \( \text{TL}_{2k+2} \)-modules respectively, defined by

\[
\pi_{2k}(Y \otimes_{\text{TL}_{2k}} w_{2k}) := Y \otimes_{\text{TL}_{2k} \otimes \text{TL}_1} (w_{2k} \otimes 1_v),
\]

\[
\pi_{2k+1}(Z \otimes_{\text{TL}_{2k+1}} w_{2k+1}) := Z \otimes_{\text{TL}_{2k+1} \otimes \text{TL}_1} (w_{2k+1} \otimes 1_{v^{-1}}),
\]

for \( Y \in \text{TL}_{2k+1}, w_{2k} \in V_{2k}(u) \) and \( Z \in \text{TL}_{2k+2}, w_{2k+1} \in V_{2k+1}(v) \).

**Proposition 11.2.** Let \( u = t^{\frac{1}{2}} v + t^{-\frac{1}{2}} v^{-1} \). Let

\[
V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \ldots
\]
be the link pattern tower. Then the intertwiner \( \hat{\phi}_n \) factors through \( \pi_n \). In other words, there exist unique intertwiners

\[
\psi_{2k} : V_{2k}(u) \to V_{2k+1}(v), \\
\psi_{2k+1} : V_{2k+1}(v) \to V_{2k+2}(u)
\]

of \( \text{TL}_{2k+1} \)-modules and \( \text{TL}_{2k+2} \)-modules respectively, such that \( \psi_n \circ \pi_n = \hat{\phi}_n \) for all \( n \in \mathbb{Z}_{\geq 0} \).

**Proof.** It suffices to show that for all \( w_{2k} \in V_{2k}(u) \) and \( w_{2k+1} \in V_{2k+1}(v) \),

\[
d_{2k+1}(w_{2k}) = \phi_{2k}(w_{2k}), \\
d_{2k+2}(w_{2k}+1) = \phi_{2k+1}(w_{2k+1}).
\]

Write \( w_{2k} = Y_u \) with \( Y \in \text{Hom}_S(0,2k) \) and \( w_{2k+1} = Z_v \) with \( Z \in \text{Hom}_S(1,2k+1) \). Then

\[
d_{2k+1}(w_{2k}) = (d_{2k+1}I(Y))_v = (I(Y)\rho)_v = \phi_{2k}(w_{2k}),
\]

where we used (11.1) and the fact that \( d_1 = \rho \in \text{TL}_1 \) for the second equality. To prove the second equality of (11.2), first note that

\[
d_{2k+2}(w_{2k}+1) = (d_{2k+2}I(Z)U)_u = (I(Z)d_2U)_u
\]

by (11.1). Now \( d_2 = (t^2e_1 + t^{-2})\rho = I_1(\rho^{-1})\rho^2 \) in \( \text{TL}_2 \) by Proposition 8.3(b). Furthermore we have \( \rho^2U = U \) in \( \text{Hom}_S(0,2) \), so we conclude that

\[
d_{2k+2}(w_{2k}+1) = (I(Z)d_2(\rho^{-1})U)_u = \phi_{2k+2}(Z_v),
\]

where the last step follows from the proof of Lemma 10.22. \( \square \)

**Corollary 11.3.** Let \( u = t^{2}v + t^{-2}v^{-1} \) with \( v^2 \neq t^{4} \). Let

\[
V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \cdots
\]

be the link pattern tower.

Then the left \( \text{TL}_{2k+1} \)-module \( V_{2k+1}(v) \) is a quotient of \( V_{2k}(u) \times_f V_1(v) \) and the left \( \text{TL}_{2k+2} \)-module \( V_{2k+2}(u) \) is a quotient of \( V_{2k+1}(v) \times_f V_1(v^{-1}) \).

**Example 11.4.** Let \( u = t^{2}v + t^{-2}v^{-1} \) with \( v^2 \neq t^{4} \). Recall the identification of

\[
V_{2k}(u) = \mathcal{W}_{0,t^{4},v}[2k], \quad V_{2k+1}(v) = \mathcal{W}_{t^{4},v}[2k+1]
\]

with the standard modules from [9] (see Remark 10.11). Then [9] (4.26) shows that

\[
V_1(v) \times_f V_1(v^{-1}) \simeq V_2(u).
\]

12. **A \( B \)-type presentation of the extended affine Temperley-Lieb algebra**

Let \( B_n^B \) be the braid group of type \( B_n \), i.e. the group with generators \( \sigma_0^B, \ldots, \sigma_{n-1}^B \) and defining relations the braid relations associated to the type \( B \) Coxeter diagram

\[
\begin{array}{ccccccc}
\sigma_0^B & \sigma_1^B & \sigma_2^B & \ldots & \sigma_{n-2}^B & \sigma_{n-1}^B \\
\end{array}
\]
It is known that $B_n^B$ is isomorphic to the extended affine braid group $B_n$, with the isomorphism given by
\begin{align}
\sigma_0^B &\mapsto \rho_0 \sigma_{n-1} \cdots \sigma_1^{-1} \\
\sigma_i^B &\mapsto \sigma_i \quad (1 \leq i < n),
\end{align}
see [8, Rem. 1.1] and references within. We discuss now a similar $B$-type presentation of the extended affine Temperley-Lieb algebra $TL_n$.

Przytycki’s $B$-type affine Temperley-Lieb algebra $TL_n^B$ is defined as follows (see [23, Thm. 3.13]). For $n \geq 2$, $TL_n^B$ is the unital complex associative algebra with generators $\alpha, \tau, e_{1}, \cdots, e_{n-1}$ and defining relations
\begin{align}
e_i^2 &= \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)e_i, \\
e_i e_j &= e_j e_i \quad \text{if } |j - i| \geq 2, \\
e_i e_{i \pm 1} &= e_i, \\
\tau e_i &= e_i \tau \quad \text{if } i > 1, \\
e_1 \tau e_1 &= \alpha e_1 = e_1 \alpha, \\
\tau^2 &= -t^{\frac{1}{2}} \alpha \tau - t.
\end{align}

For $n = 0$ and $n = 1$ we set $TL^B_0 := \mathbb{C}[\alpha]$ and $TL^B_1 := \mathbb{C}[\tau, \tau^{-1}]$.

Note that $\tau$ is invertible with inverse $\tau^{-1} = -t^{-1} \tau - t^{\frac{1}{2}} \alpha$. Hence $\alpha = -t^{\frac{1}{2}} \tau - t^{\frac{1}{2}} \tau^{-1}$, and $\alpha$ is central. For $n = 1$ we define $\alpha$ by this formula.

Note that the assignments
\begin{align}
\sigma_0^B &\mapsto -t^{-\frac{3}{4}} \tau \\
\sigma_i^B &\mapsto t^{\frac{i}{2}} e_i + t^{-\frac{i}{4}} \quad (1 \leq i < n)
\end{align}

define a surjective algebra map $\mu^B_n : \mathbb{C}[B_n^B] \to TL^B_n$. In particular, $TL^B_n$ is isomorphic to a quotient of the group algebra $\mathbb{C}[B_n^B]$.

Recall the algebra map $\mu_n : \mathbb{C}[B_n] \to TL_n$ from Section 7. The following result is an algebraic reformulation of [23, Thm. 3.13(a)], see also Remark [12.2].

**Proposition 12.1.** There exists a unique isomorphism $TL^B_n \xrightarrow{\sim} TL_n$ of algebras such that the diagram
\[
\begin{array}{cccc}
\mathbb{C}[B_n^B] & \xrightarrow{\sim} & \mathbb{C}[B_n] \\
\mu^B_n \downarrow & & \downarrow \mu_n \\
TL^B_n & \xrightarrow{\sim} & TL_n
\end{array}
\]

of algebra maps is commutative, with the isomorphism $\mathbb{C}[B_n^B] \xrightarrow{\sim} \mathbb{C}[B_n]$ given by (12.1).
Proof. We have to show that there exists a well defined algebra map \( f_n : TL_n^B \to TL_n \) satisfying \( f_n(e_i) = e_i \) (1 \( \leq\) i < n) and
\[
f_n(\tau) = -t^4 \rho \mu_n(\sigma_n^{-1}) \cdots \mu_n(\sigma_1^{-1}),
\]
and that there exists a well defined algebra map \( g_n : TL_n \to TL_n^B \) satisfying \( g_n(e_i) = e_i \) (1 \( \leq\) i < n) and
\[
g_n(\rho) = -t^{-4} \tau \mu_n^B(\sigma_n^B) \cdots \mu_n^B(\sigma_1^B).
\]
We omit the proof as it is a straightforward check that all the algebra relations are respected by \( f_n \) and \( g_n \). For these checks it is convenient to use the presentation of \( TL_n \) in terms of the generators \( \rho^\pm, e_1, \ldots, e_{n-1} \) as given in Remark 6.1. \( \square \)

Remark 12.2. Combining Proposition [12.1] with Theorem [6.3] and Remark [6.4] yields an isomorphism \( TL_n^B \cong \text{End}_S(n) \) given by
\[
\tau \mapsto -t^4, \quad e_i \mapsto e_i \text{ for } 1 \leq i < n.
\]
for 1 \( \leq\) i < n. Note that under this isomorphism,
\[
\alpha \mapsto \text{Id}_n.
\]
This is the algebra isomorphism \( TL_n^B \cong \text{End}_S(n) \) from [23] Thm. 3.13(a)].

Using the B type presentation of \( TL_n \) the algebra maps \( \mathcal{I}_n : TL_n \to TL_{n+1} \) takes on a simple form.

Corollary 12.3. The algebra maps \( \mathcal{I}_n : TL_n \to TL_{n+1} \), viewed as algebra maps \( TL_n^B \to TL_{n+1}^B \) via the identification \( TL_n^B \cong TL_n \), satisfies \( \tau \mapsto \tau \) and \( e_i \mapsto e_i \) for 1 \( \leq\) i < n.

Using Corollary [12.3] in combination with [23] Thm. 3.13(b)] it follows that the algebra map \( \mathcal{I}_n : TL_n \to TL_{n+1} \) is injective.

References
[1] F. Bonahon, H. Wong, Representations of the Kauffman bracket skein algebra I: invariants and miraculous cancellations, Invent. Math. 204 (2016), 195–243.
[2] J. Belletête, A.M. Gainutdinov, J.L. Jacobsen, H. Saleur, R. Vasseur, On the correspondence between boundary and bulk lattice models and (logarithmic) conformal field theories, arXiv:1705.07769.
[3] S. Al Harbat, Markov trace on a tower of affine Temperley-Lieb algebras of type A, J. Knot Theory Ramifications 24 (2015), no. 9, 1550049, 28pp.
[4] K. Al Qasimi, B. Nienhuis, J.V. Stokman, Towers of qKZ equations and applications to loop- and vertex models, in preparation.

[5] A. Cox, P. Martin, A. Parker, C. Xi, Representation theory of towers of recollement: theory, notes and examples, Journal of Algebra 302 (2006), 340–360.

[6] P. Di Francesco, P. Zinn-Justin, J.-B. Zuber, Sum rules for the ground states of the O(1) loop model on a cylinder and the XXZ spin chain, J. Stat. Mech. (2006), P08011, 22pp.

[7] C.K. Fan, R.M. Green, On the affine Temperley-Lieb algebras, J. London Math. Soc. (2) 60 (1999), no. 2, 366–380.

[8] A. Gadbled, A.-L. Thiel and E. Wagner, Categorical action of the extended braid group of affine type A, Commun. Contemp. Math. 19 (2017), no. 3, 1650024, 39 pp. MR3631925

[9] A.M. Gainutdinov, H. Saleur, Fusion and braiding in finite and affine Temperley-Lieb categories, arXiv:1606.04530

[10] J. de Gier, Loops, matchings and alternating-sign matrices, Discrete Math. 298 (2005), 365–388.

[11] J. de Gier, P. Pyatov, Factorized solutions of Temperley-Lieb qKZ equations on a segment, Adv. Theor. Math. Phys. 14 (2010), 795–877.

[12] J.J. Graham, G.I. Lehrer, The representation theory of affine Temperley-Lieb algebras, Enseign. Math. (2) 44 (1998), no. 3-4, 173–218.

[13] J.J. Graham, G.I. Lehrer, Diagram algebras, Hecke algebras and decomposition numbers at roots of unity, Ann. Sci. École Norm. Sup. (4) 36 (2003), no. 4, 479–524.

[14] R.M. Green, On representations of affine Temperley-Lieb algebras, in: “Algebras and modules, II” (Geiranger, 1990), 245–261, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998.

[15] M. Kasatani, V. Pasquier, On polynomials interpolating between the stationary state of a $O(n)$ model and a Q.H.E. ground state, Comm. Math. Phys. 276 (2007), no. 2, 397–435.

[16] H.R. Morton, Invariants of links and 3-manifolds from skein theory and from quantum groups, in: “Topics in knot theory” (Erzurum, 1992), 107–155, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 399, Kluwer Acad. Publ., Dordrecht, 1993.

[17] H.R. Morton, Skein theory and the Murphy operators, in: “Knots 2000 Korea, Vol. 2” (Yongpyong), J. Knot Theory Ramifications 11 (2002), no. 4, 475–492.

[18] J.H. Przytycki, State models for knot polynomials, Topology 26 (1987), 395–407.

[19] J.H. Przytycki, Fundamentals of Kauffman bracket skein modules, Kobe J. Math. 16 (1999), 45–66.

[20] J.H. Przytycki, V. Turaev, The Conway and Kauffman modules of a solid torus, (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988), Issled. Topol. 6, 79–89, 190; English translation in J. Soviet Math. 52 (1988), no. 1, 2799–2805.

KdV Institute for Mathematics, University of Amsterdam, Science Park 105-107, 1098 XG Amsterdam, The Netherlands.

E-mail address: s.k.s.k.s.alqasemi@uva.nl

KdV Institute for Mathematics, University of Amsterdam, Science Park 105-107, 1098 XG Amsterdam, The Netherlands.

E-mail address: j.v.stokman@uva.nl