Entire solutions of the generalized Hessian inequality

Xiang Li, Jing Hao, Jiguang Bao*

Abstract In this paper, we discuss the more general Hessian inequality \( \sigma_k^\lambda(D_i(A(|Du|)D_ju)) \geq f(u) \) including the Laplacian, p-Laplacian, mean curvature, Hessian, k-mean curvature operators, and provide a necessary and sufficient condition on the global solvability, which can be regarded as generalized Keller-Osserman conditions.

Keywords generalized Hessian inequality · existence · nonexistence · Keller-Osserman condition

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1 Introduction and the statement of results

In this paper, we discuss the solvability of the generalized Hessian inequality

\[
\sigma_k^\lambda(D_i(A(|Du|)D_ju)) \geq f(u) \quad \text{in } \mathbb{R}^n, \quad (1.1)
\]

where

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n, \quad k = 1, 2, \cdots, n
\]

is the k-th elementary symmetric function, \( \lambda(D_i(A(|Du|)D_ju)) \) denotes the eigenvalues of the symmetric matrix of \( (D_i(A(|Du|)D_ju)) \), and \( A, f \) are two given positive continuous functions on \((0, +\infty)\).

The generalized Hessian operator \( \sigma_k(\lambda(D_i(A(|Du|)D_ju))) \), introduced by many authors [1, 2, 3], is an important class of fully nonlinear operator. It is a generalization

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of some typical operators we shall be interested in as follows: the m-k-Hessian operator
for the case \(A(p) = p^{m-2}, m > 1\) is treated by Trudinger and Wang [4]; the k-mean
curvature operator for the case \(A(p) = (1 + p^2)^{-\frac{1}{2}}\) is treaded by Concus and Finn
[16] and Peletier and Serrin [17]; the generalized k-mean curvature operator for the
case \(A(p) = (1 + p^2)^{-\alpha}, \alpha < \frac{1}{2}\) and \(A(p) = p^{2m-2} (1 + p^{2m})^{-\frac{1}{2}}, m > 1\) is treated by
Tolksdorf [5], Usami [6] and Suzuki [7], respectively.

In particular, (1.1) is the k-Hessian innequality for the case \(A(p) = 1\). For \(k = 1\),
Wittich (n=2 [21]), Haviland (n=3 [22]), Walter (n \(\geq 2\) [23]) proved the Laplacian
equation
\[
\Delta u = f(u) \text{ in } \mathbb{R}^n
\]
has no solution if and only if
\[
\int_{-\infty}^{\infty} \left( \int_{s}^{\infty} f(t)dt \right)^{-\frac{1}{2}} ds < \infty.
\]
Here and after, we omit the lower limit to admit an arbitrary positive number. Keller
[13] and Osserman [14] showed that the Laplacian inequality
\[
\Delta u \geq f(u) \text{ in } \mathbb{R}^n
\]
has a positive solution \(u \in C^2(\mathbb{R}^n)\) if and only if \(f\) satisfies the Keller-Osserman
condition
\[
\int_{-\infty}^{\infty} \left( \int_{s}^{\infty} f(t)dt \right)^{-\frac{1}{2}} ds = \infty. \tag{1.2}
\]
The condition (1.2) is often used to study the boundary blow-up (explosive, large)
solutions (see [18, 19, 20]). Ji and Bao [11] extended the above results from \(k = 1\)
to \(1 \leq k \leq n\), which can be regards as the generalized Keller-Osserman condition.
Naito and Usami [8] extended the above results from \(A(p) = 1\) to the generalized
Hessian inequality (1.1) for \(k = 1\) and got similar results.

In this paper, we shall extend this result from \(k = 1\) to \(1 \leq k \leq n\) for the generar-
alized Hessian inequality (1.1) and develop existence and nonexistence conditions of
entire solutions for (1.1). To state our results, we define a generalized k-convex entire
solution of (1.1) to be a function \(u \in \Phi^k(\mathbb{R}^n)\) which satisfies (1.1) at each \(x \in \mathbb{R}^n\),
where
\[
\Phi^k(\mathbb{R}^n) = \left\{ u \in C^1(\mathbb{R}^n) : A(\|Du\|)Du \in C^1(\mathbb{R}^n), \lambda(D_i(A(\|Du\|)D_ju)) \in \Gamma_k \text{ in } \mathbb{R}^n \right\},
\]
and
\[
\Gamma_k := \{ \lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0, l = 1, 2, \cdots, k \}.
\]

In (1.1), we assume that the positive function \(A \in C^1(0, \infty)\) satisfies
\[
pA(p) \in C[0, \infty) \text{ is strictly monotone increasing in } (0, \infty), \tag{1.3}
\]
and the positive function $f \in C(0, \infty)$ satisfies

$$f \text{ is monotone non-decreasing in } (0, \infty).$$  \hfill (1.4)

First we consider the case

$$\lim_{p \to \infty} pA(p) < \infty.$$  \hfill (1.5)

**Theorem 1.1.** Assume that $A$ satisfies (1.3), (1.5) and $f$ satisfies (1.4), then the inequality (1.1) has no positive solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$.

**Remark 1.2.** The $k$-mean curvature inequality (1.1) for the case $A(p) = (1 + p^2)^{-\frac{1}{2}}$ satisfies the Theorem 1.1, and the corresponding results were obtained by Cheng and Yau [9] and Tkachev [10].

Next we consider the case

$$\lim_{p \to \infty} pA(p) = \infty.$$  \hfill (1.6)

We now define a continuous function $\Psi : [0, \infty) \to [0, \infty)$ by

$$\Psi(p) = p \left( pA(p) \right)^k - \int_0^p (tA(t))^k \, dt, \quad p \geq 0.$$  \hfill (1.7)

It follows from the condition (1.3) that the inverse function of $\Psi$ on $[0, \infty)$ exists, denoted by $\Psi^{-1}$. For example, if $A(p) = p^{m-2}$, $m > 1$, then

$$\Psi(p) = \frac{m-1}{m} p^m \quad \text{and} \quad \Psi^{-1}(p) = \left( \frac{m}{m-1} p \right)^{\frac{1}{m}}.$$

A necessary and sufficient result is as follows.

**Theorem 1.3.** Assume that $A$ satisfies (1.3), (1.6) and $f$ satisfies (1.4), then inequality (1.1) has a positive solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$ if and only if

$$\int_{-\infty}^\infty \left( \Psi^{-1} \left( \int_0^s f^k(t) \, dt \right) \right)^{-1} \, ds = \infty.$$ \hfill (1.8)

For $k = 1$, $A(p) = 1$, (1.8) is exactly the Keller-Osserman condition (1.2). So we can regard (1.8) as a generalized Keller-Osserman condition.

If we strengthen the case (1.6) to

$$0 < \liminf_{p \to \infty} \frac{A(p)}{p^{m-2}} \leq \limsup_{p \to \infty} \frac{A(p)}{p^{m-2}} < \infty \quad \text{for some} \quad m > 1.$$ \hfill (1.9)

As a consequence of Theorems 1.3, we obtain the following corollary.
Corollary 1.4. Assume that $A$ satisfies (1.3), (1.9) and $f$ satisfies (1.4), then inequality (1.1) has a positive solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$ if and only if
\[
\int_0^\infty \left( \int_s^{s+1} f^k(t) dt \right)^{-\frac{1}{k(m-1)+1}} ds = \infty. \tag{1.10}
\]

Remark 1.5. Corollary 1.4 holds for the cases $A(p) = 1$, $m = 2$ which was obtained by Ji and Bao [11]; $A(p) = p^{m-2}$, $m > 1$ which was obtained by Feng and Bao [12]; $A(p) = (1 + p^2)^{-\alpha}$, $m = 2 - 2\alpha > 1$ and $A(p) = p^{2m-2}(1 + p^{2m})^{-\frac{1}{2}}$, $m > 1$, which are first obtained by authors of this paper.

Remark 1.6. Under the assumption of Corollary 1.4, if $f(u) = u^\gamma$, $\gamma \geq 0$, then the inequality (1.1) has a positive solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$ if and only if $\gamma \leq m - 1$.

If we strengthen the requirement of $f$ from (1.4) to the positive function $f \in C(\mathbb{R})$ satisfying
\[
f \text{ is monotone non-decreasing in } \mathbb{R}, \tag{1.11}
\]
then we have similarly following corollary:

Corollary 1.7. Assume that $A$ satisfies (1.3) and $f$ satisfies (1.11). If (1.5) holds, then the inequality (1.1) has no solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$; if (1.6) holds, then the inequality (1.1) has a solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$ if and only if (1.8) holds, in particular, if (1.9) holds, then the inequality (1.1) has a solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$ if and only if (1.10) holds.

Remark 1.8. Under the assumption of Corollary 1.7, if $f(u) = e^u$, then the inequality (1.1) has no solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$.

In particular, we will get a better regularity of solutions later about that if $A \in C^1[0, \infty)$, $A(0) \neq 0$, then $u \in C^2(\mathbb{R}^n) \cap \Phi^k(\mathbb{R}^n)$; otherwise we consider the case
\[
0 < \liminf_{p \to 0} \frac{A(p)}{p^{l-2}} \leq \limsup_{p \to 0} \frac{A(p)}{p^{l-2}} < \infty \text{ for some } l > 2, \tag{1.12}
\]
then $u \in W^{2,q}_{\text{loc}}(\mathbb{R}^n)$, $1 < q < \frac{l-1}{l-2}$, by embedding theorem, we have $u \in C^{1,\alpha}(\mathbb{R}^n) \cap \Phi^k(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$.

The rest of our paper is organized as follows. In Section 2, we introduce some results on radial solutions and the local existence of Cauchy problem associated to (1.1) as preliminaries. In Section 3, we give the comparison principle and prove Theorems 1.1, 1.3 and Corollary 1.4, 1.7.
2 Preliminary results on radial solutions

We need some properties of radial functions in the proof of the main theorem. For \( R > 0 \), let \( B_R := \{ x \in \mathbb{R}^n : |x| < R \} \).

**Lemma 2.1.** For any positive number \( a \), assume \( \varphi(r) \in C[0, R) \cap C^1(0, R) \) is a positive solution of the Cauchy problem of the implicit equation

\[
\begin{cases}
A (|\varphi'(r)|) \varphi'(r) = \left( \frac{nr^{k-n}}{C_n^k} \int_0^r s^{n-1} f^k(\varphi(s)) \, ds \right)^{\frac{1}{k}} =: F(r, \varphi), \ r > 0, \\
\varphi(0) = a.
\end{cases}
\tag{2.1}
\]

Then \( \varphi'(0) = 0, \ \varphi'(r) > 0 \) in \( (0, R) \), and it satisfies \( \varphi(r) \in C^1[0, R) \cap C^2(0, R) \) with \( A (\varphi'(r)) \varphi'(r) \in C^1[0, R) \), and the ordinary differential equation

\[
C_n^{k-1} A (\varphi'(r)) \varphi'(r) = \left( \frac{r^{k-n}}{C_n^k} \int_0^r s^{n-1} f^k(\varphi(s)) \, ds \right)^{\frac{1}{k}} = f^k(\varphi(r)).
\tag{2.2}
\]

**Proof.** Denote

\[
h(r) := \int_0^r A (|\varphi'(s)|) \varphi'(s) \, ds,
\]

then it satisfies \( h(0) = 0 \) and

\[
h'(r) = A (|\varphi'(r)|) \varphi'(r) = \left( \frac{nr^{k-n}}{C_n^k} \int_0^r s^{n-1} f^k(\varphi(s)) \, ds \right)^{\frac{1}{k}} > 0, \ 0 < r < R.
\]

It is easy to see that \( h(r) \in C^2(0, R) \). By (1.3) and (2.1), \( \varphi'(r) > 0 \) in \( (0, R) \).

\[
\lim_{r \to 0} \frac{h(r) - h(0)}{r - 0} = \lim_{r \to 0} h'(\xi) = \lim_{\xi \to 0} \left( \frac{n\xi^{k-n}}{C_n^k} \int_0^\xi s^{n-1} f^k(\varphi(s)) \, ds \right)^{\frac{1}{k}} = 0,
\]

where \( \xi = \xi(r) \in (0, r) \). Hence, \( h'(0) = 0 \) and \( h(r) \in C^1[0, R) \), which implies that \( \varphi'(0) = 0 \) and \( \varphi(r) \in C^1[0, R) \). One can see that

\[
\lim_{r \to 0} \frac{h'(r) - h'(0)}{r - 0} = \lim_{r \to 0} \left( \frac{n \int_0^r s^{n-1} f^k(\varphi(s)) \, ds}{C_n^k r^n} \right)^{\frac{1}{k}} = \left( \frac{f^k(a)}{C_n^k} \right)^{\frac{1}{k}},
\tag{2.3}
\]

consequently, \( h(r) \in C^2[0, R) \), which implies that \( A (\varphi'(r)) \varphi'(r) \in C^1[0, R) \).
A direct calculation using (2.1) leads to
\[
    h''(r) = \frac{(h'(r))^{1-k}}{k} \left( \frac{n(k-n)r^{k-n-1}}{C_n^k} \int_0^r s^{n-1}f^k(\varphi(s))ds + \frac{nr^{k-1}}{C_n^k}f^k(\varphi(r)) \right)
\]
\[
    \geq \frac{1}{C_n^k} \left( \frac{h'(r)}{r} \right)^{1-k} f^k(\varphi(r)) > 0,
\]
then \( h'(r) \in C^1[0, R] \) is a strictly monotone increasing function of \( r \), and by (1.3), \( g(\varphi') := A(\varphi') \varphi' \in C^1(0, \varphi'(R)) \) is a strictly monotone increasing function of \( \varphi' \), then there exists inverse function \( \varphi'(r) = g^{-1}(h'(r)) \in C^1(0, R) \), which implies \( \varphi(r) \in C^2(0, R) \).

By (2.4) and (2.1), we have
\[
    h''(r) = \frac{k-n}{k} h'(r) + \frac{n}{kC_n^k} \left( \frac{h'(r)}{r} \right)^{1-k} f^k(\varphi(r)),
\]
it is easy to verify that \( \varphi(r) \) satisfies equation (2.2). \( \square \)

**Remark 2.2.** In particular, if \( A \in C^1[0, R], A(0) \neq 0 \), consider the function \( H(r, \varphi') = h'(r) - g(\varphi') = 0 \), then \( H_{\varphi'}(0, 0) = A(0) \neq 0 \), hence we know from the implicit function theorem that there exists \( \varphi'(r) \in C^1(0, R) \), then we can strengthen the regularity to \( \varphi(r) \in C^2(0, R) \). Otherwise we assume that (1.12) and by (2.3), we have
\[
    \left( \frac{f^k(a)}{C_n^k} \right)^{\frac{1}{k}} = \lim_{r \to 0} \frac{h'(r)}{r} = \lim_{r \to 0} \frac{(\varphi'(r))^{1-k}}{r} = \lim_{r \to 0} \frac{\varphi''(r)}{r^{\frac{1}{k} - \frac{1}{q}}},
\]
for \( 1 < q < \frac{1}{1-k} \), we can strengthen the regularity to \( \varphi(r) \in C^2(0, R) \cap W^{2,q}(0, R) \).

**Lemma 2.3.** For any positive number \( a \), assume \( \varphi(r) \in C[0, R] \cap C^1(0, R) \) is a positive solution of the Cauchy problem (2.1). Then, for \( u(x) = \varphi(r) \), where \( r = |x| < R \), we have that
\[
    \lambda(D_i(A(|Du|) D_j u))
    = \left( (A(\varphi'(r)) \varphi'(r))', \frac{A(\varphi'(r)) \varphi'(r)}{r}, \ldots, \frac{A(\varphi'(r)) \varphi'(r)}{r} \right), \quad r \in [0, R),
\]
and then \( u(x) \in C^2(B_R \setminus \{0\}) \cap \Phi^k(B_R) \) is a solution of
\[
    \sigma_k \lambda(D_i(A(|Du|) D_j u))) = \frac{C_n^k r^{1-n}}{n} \left( r^{n-k} (A(\varphi'(r)) \varphi'(r))^k \right)' = f^k(u).
\]

**Proof.** By Lemma 2.1, we can get \( \varphi(r) \in C^1[0, R] \cap C^2(0, R) \) with \( A(\varphi'(r)) \varphi'(r) \in C^1[0, R] \), and it satisfies \( \varphi'(0) = 0, \varphi'(r) > 0 \) in \((0, R)\). For \( u(x) = \varphi(r) \), where \( r = |x| \in (0, R) \), \( i, j = 1, \ldots, n \), we have
\[
    u_i(x) = \varphi'(r) \frac{x_i}{r},
\]
\[
    (2.8)$$
\[ |Du| = \left| \varphi'(r) \frac{x}{r} \right| = \varphi'(r), \quad (2.9) \]

\[ u_{ij}(x) = \left( \frac{\varphi''(r)}{r^2} \right) x_i x_j - \left( \frac{\varphi'(r)}{r^3} \right) x_i x_j + \left( \frac{\varphi'(r)}{r} \right) \delta_{ij}. \]

Then by \((2.8)\) and \((2.9)\), we have

\[
D_i (A (|Du|) D_j u) = D_i \left( A (\varphi'(r)) \varphi'(r) \frac{x_j}{r} \right) \\
= (A (\varphi'(r)) \varphi'(r)) \frac{x_i x_j}{r^2} + A (\varphi'(r)) \varphi'(r) \frac{\delta_{ij} r - x_j x_i}{r^2} \\
= \left( (A (\varphi'(r)) \varphi'(r))' - A (\varphi'(r)) \varphi'(r) \right) \frac{x_i x_j}{r^2} + A (\varphi'(r)) \varphi'(r) \frac{\delta_{ij}}{r}. \quad (2.10)
\]

By \((2.8)\) and \(\varphi'(0) = 0\), we have

\[
0 \leq \lim_{x \to 0} |u_i(x)| = \lim_{x \to 0} |\varphi'(r)||\frac{x_i}{r}| \leq \lim_{r \to 0} \varphi'(r) = 0,
\]

which means

\[
\lim_{x \to 0} u_i(x) = 0.
\]

Similarly, using \((2.10)\) we have

\[
\lim_{x \to 0} D_i (A (|Du|) D_j u) \\
= \lim_{x \to 0} \left( \left( (A (\varphi'(r)) \varphi'(r))' - A (\varphi'(r)) \varphi'(r) \right) \frac{x_i x_j}{r^2} + A (\varphi'(r)) \varphi'(r) \frac{\delta_{ij}}{r} \right) \\
= \left( (A (\varphi'(0)) \varphi'(0))' \right) \delta_{ij}.
\]

Define

\[
u_i(0) = 0, \quad D_i (A (|Du|) D_j u) (0) = (A (\varphi'(0)) \varphi'(0))' \delta_{ij},
\]

then \(u(x) \in C^1(B_R) \cap C^2(B_R \setminus \{0\})\), with \(A (|Du|) Du \in C^1(B_R)\).

For \(r \in [0, R]\), let

\[
a = \begin{cases} 
\frac{(A(\varphi'(r))\varphi'(r))'}{r^2} - \frac{A(\varphi'(r))\varphi(r)}{r^3}, & r \in (0, R), \\
0, & r = 0,
\end{cases}
\quad b = \begin{cases} 
\frac{A(\varphi'(r))\varphi'(r)}{r}, & r \in (0, R), \\
(A (\varphi'(0)) \varphi'(0))', & r = 0,
\end{cases}
\]

then the matrix

\[
D_i (A (|Du|) D_j u) = ax^T x + bI.
\]

By calculations of linear algebra, we know that the eigenvalues of a symmetric matrix such as \(D_i (A (|Du|) D_j u)\) is \((ar^2 + b, b, \ldots, b)\). Hence

\[
\lambda(D_i (A (|Du|) D_j u)) = \begin{cases} 
\left((A (\varphi'(r)) \varphi'(r))', \frac{A(\varphi'(r))\varphi(r)}{r}, \ldots, \frac{A(\varphi'(r))\varphi(r)}{r}\right), & r \in (0, R), \\
\left((A (\varphi'(0)) \varphi'(0))', \ldots, (A (\varphi'(0)) \varphi'(0))'\right), & r = 0.
\end{cases}
\]
Since
\[ \lim_{r \to 0} \frac{A(\varphi'(r)) \varphi'(r)}{r} = (A(\varphi'(0)) \varphi'(0))', \]
we can always think that (2.6) holds, and equation (2.7) can thus be obtained easily from the definition of \( \sigma_k \).

Since \( f \) and \( \varphi \) are both monotone non-decreasing, for \( r \in [0, R) \),
\[ f(\varphi(r)) \geq f(\varphi(0)) = f(a) > 0. \]
Then we know that \( \frac{A(\varphi'(r)) \varphi'(r)}{r} > 0 \) and
\[ \sigma_k (\lambda(D_i (A(\|Du\|) D_j u))) = C_{n-1}^{k-1} \left( \frac{A(\varphi'(r)) \varphi'(r)}{r} \right)^{k-1} \left( (A(\varphi'(r)) \varphi'(r))' + \frac{n-k}{k} A(\varphi'(r)) \varphi'(r) \right) \]
which means
\[ (A(\varphi'(r)) \varphi'(r))' + \frac{n-k}{k} A(\varphi'(r)) \varphi'(r) > 0, \]
and then for \( 1 \leq l \leq k \),
\[ \sigma_l (\lambda(D_i (A(\|Du\|) D_j u))) \]
\[ = C_{n-1}^{l-1} \left( \frac{A(\varphi'(r)) \varphi'(r)}{r} \right)^{l-1} \left( (A(\varphi'(r)) \varphi'(r))' + \frac{n-l}{l} A(\varphi'(r)) \varphi'(r) \right) \]
\[ \geq C_{n-1}^{l-1} \left( \frac{A(\varphi'(r)) \varphi'(r)}{r} \right)^{l-1} \left( (A(\varphi'(r)) \varphi'(r))' + \frac{n-k}{k} A(\varphi'(r)) \varphi'(r) \right) \]
\[ > 0. \]
This implies that \( \lambda(D_i (A(\|Du\|) D_j u)) \in \Gamma_k \) is valid in \( B_R \). \hfill \Box

Obviously for \( u(x) = \varphi(r) \), we can see that \( u(x) \in C^2(B_R \setminus \{0\}) \cap C^1(B_R) \), with \( A(|Du|) Du \in C^1(B_R) \) is a solution of (2.7) if and only if \( \varphi(r) \in C^1[0, R) \cap C^2(0, R) \) with \( A(\varphi'(r)) \varphi'(r) \in C^1[0, R) \) is a solution of (2.2).

Remark 2.4. In particular, if \( A \in C^1[0, \infty) \), \( A(0) \neq 0 \), by Remark 2.2, we have \( \varphi(r) \in C^2[0, R) \), and
\[ \lim_{x \to 0} u_{ij}(x) = \lim_{x \to 0} \left( \left( \varphi''(r) - \frac{\varphi'(r)}{r} \right) \frac{x_i x_j}{r^2} + \left( \frac{\varphi'(r)}{r} \right) \delta_{ij} \right) = \varphi''(0) \delta_{ij}. \]
Define \( u_{ij}(0) = \varphi''(0)\delta_{ij} \), then it is straightforward to show that \( u(x) \in C^2(\mathbb{R}^R) \). Otherwise we assume (1.12), then by (2.5) and \( |\frac{\varphi''(r)}{r}| \leq 1 \), we have

\[
\limsup_{r \to 0} \frac{|u_{ij}(x)|}{r^{\frac{l-2}{l-1}}} = \limsup_{r \to 0} \left| \left( \frac{\varphi''(r)}{r} - \frac{\varphi'(r)}{r} \right) \frac{x_ix_j}{r^2} + \frac{\varphi'(r)}{r} \delta_{ij} \right|
\leq \limsup_{r \to 0} \frac{|\varphi''(r)| + \left| \frac{\varphi'(r)}{r} \right| + \left| \frac{\varphi'(r)}{r} \right|}{r^{\frac{l-2}{l-1}}}
= 3 \left( \frac{f^k(a)}{C_n^k} \right)^{\frac{1}{k}}.
\]

For \( \frac{l-2}{l-1}q < 1 \), we have \( D^2u(x) \in L^{nq}(B_R) \). Then it is straightforward to see \( u(x) \in W^{2,nq}(B_R) \).

Next, we discuss the local existence of (2.1) near \( r = 0 \). The equipment we use is Euler’s break line, and the process is similar to the proof of the existence theorem of ordinary differential equations (see [15]).

**Lemma 2.5.** For any positive number \( a \), there exist a positive number \( R \) such that the Cauchy problem (2.1) has a solution in \([0,R]\).

**Proof.** By Lemma 2.1, we know that \( \varphi'(r) = g^{-1}(F(r, \varphi)) \in C[0,R] \cap C^1(0,R) \) is a strictly monotone increasing function of \( r \). We define a functional \( G(\cdot, \cdot) \) on \( \mathcal{R} := [0,R] \times \{ \varphi \in C[0,R] : a \leq \varphi < 2a \} \), as

\[
G(r, \varphi) := g^{-1}(F(r, \varphi)),
\]

where \( R \) is a small enough positive constant. Then (2.1) can be rewritten as

\[
\varphi'(r) = G(r, \varphi).
\]

It is easy to see \( G > 0 \) for \( r > 0 \).

We defined a Euler’s break line on \([0,R]\) as

\[
\begin{cases}
\psi(r) = a, & 0 \leq r \leq r_1, \\
\psi(r) = \psi(r_{i-1}) + G(r_{i-1}, \psi)(r - r_{i-1}), & r_{i-1} < r \leq r_i, \ i = 2,3, \ldots, m,
\end{cases}
\]

where \( 0 = r_0 < r_1 < \ldots < r_m = R \) and \( m \in \mathbb{N} \).

**Step 1.** We want to make sure that \( a \leq \psi(r) < 2a \) for all \( r \in [0,R] \), i.e. \((r, \psi) \in \mathcal{R} \). In fact, it is obvious \( \psi(r) \geq a \). Since

\[
G(r, \psi) \leq g^{-1}\left( \left( \frac{n^{k-n}}{C^k_n} \int_0^r s^{n-1}ds f^k(\psi(s)) \right)^{\frac{1}{k}} \right)
\leq g^{-1}\left( \left( \frac{1}{C^k_n} \right)^{\frac{1}{k}} R f(\psi(R)) \right) < \infty.
\]
Then for the break line \((r, \psi)\), we have

\[
a \leq \psi(r) \leq a + g^{-1}\left(\frac{1}{C_k} \frac{1}{n} R f(\psi(R))\right) R \leq a + g^{-1}\left(\frac{1}{C_k} R f(\psi(R))\right) R.
\]

Therefore, we can choose \(R\) sufficiently small to make sure that \(\psi(r) < 2a\).

**Step 2.** We will prove that Euler’s break line \(\psi\) is an \(\varepsilon\)-approximation solution of (2.1). To do this, we only need to prove that for any small \(\varepsilon > 0\), there are appropriate points \(\{r_i\}_{i=1, \ldots, m}\) to make the break line satisfy

\[
\left|\frac{d\psi(r)}{dr} - G(r, \psi)\right| < \varepsilon, \quad r \in [0, R].
\]

By (2.11), \(\psi(r)\) has continuous derivatives in \([0, R]\) expect for a few points. There are unilateral derivatives at these individual points. If the derivative doesn’t exist, we consider the right derivative.

As a matter of fact, by (2.12), it is easy to see that

\[
\lim_{r \to 0} G(r, \psi) = 0
\]

is valid uniformly for any \((r, \psi) \in \mathcal{R}\). Then for each \(\varepsilon > 0\), there exists \(\bar{r} \in (0, R)\) such that for \(0 \leq r < \bar{r}\), we have

\[
G(r, \psi) < \varepsilon.
\]

Assume \(r_1 = \bar{r}\), then

\[
\left|\frac{d\psi(r)}{dr} - G(r, \psi)\right| = |G(r, \psi)| < \varepsilon, \quad 0 < r < \bar{r}.
\]

For \(\bar{r} \leq r \leq R\), by the proof of Lemma 2.1, we know that \(g^{-1} \in C[0, F(R, \psi)] \cap C^1(0, F(R, \psi)]\), then \(g^{-1}\) is Lipschitz continuous on \([F(\bar{r}, \psi), F(R, \psi)]\). Let \(r_{i-1} < r \leq r_i\), we have

\[
\left|\frac{d\psi(r)}{dr} - G(r, \psi)\right| \leq C |F(r_{i-1}, \psi) - F(r, \psi)|
\]

\[
\leq C \left(\frac{n}{C_k^k} \right)^{\frac{1}{n}} \left(\int_0^r s^{n-1} f_k(\psi(s))ds - \int_0^{r_{i-1}} s^{n-1} f_k(\psi(s))ds\right)^{\frac{1}{n}}
\]

\[
\leq C \left(\frac{n}{C_k^k} \right)^{\frac{1}{n}} \left((r_{i-1} - r_{i-n}) \int_0^r s^{n-1} f_k(\psi(s))ds + r_{i-1} \int_{r_{i-1}}^r s^{n-1} f_k(\psi(s))ds\right)^{\frac{1}{n}}
\]

\[
\leq C \left(\frac{1}{C_k^k} \right)^{\frac{1}{n}} (r_{i-1} - r_{i-n}) R^n f_k^k(2a) + \bar{r}^{k-n} (r_{i-1} - r_{i-1}^n) f_k^k(2a)^{\frac{1}{n}}.
\]
Since function \( r^{k-n} \) and \( r^n \) are both Liptchitz continuous on \([\bar{r}, R]\), for the above \( \varepsilon \), there exists \( \delta(\varepsilon) > 0 \) satisfying

\[
\max_{2 \leq i \leq m} |r_{i-1} - r_i| < \delta(\varepsilon),
\]

and then we have

\[
\left| \frac{d\psi(r)}{dr} - G(r, \psi) \right| < C|r_{i-1} - r| < \varepsilon.
\]

Thus, Euler’s break line \( \psi \) is an \( \varepsilon \)-approximation solution of (2.1).

**Step 3.** The next step is to find a solution of (2.1) by the Euler break line we defined. Assume \( \{\varepsilon_j\}_{j=1}^\infty \) is a positive constant sequence converging to 0. For each \( \varepsilon_j \), there is an \( \varepsilon_j \)-approximation solution \( \psi_j \) on \([0, R]\), defined as above. By Step 1, it is easy to know that

\[
|\psi_j'(r') - \psi_j'(r'')| = G(r_{i-1}, \psi_j)|r' - r''| \leq M|r' - r''|,
\]

where \((r', \psi_j), (r'', \psi_j) \in \mathcal{R}\). That is to say, \( \{\psi_j\} \) is equicontinuous and uniformly bounded \((r'' = 0)\). Then by the Ascoli-Arzela Lemma, we can find a uniformly convergent subsequence, still denoted as \( \{\psi_j\} \), without loss of generality.

Assume \( \lim_{j \to \infty} \psi_j = \varphi \). Since \( \psi_j \in C[0, R] \), we know that \( \varphi \in C[0, R] \). By \( \psi_j(0) = a \), we have \( \varphi(0) = a \).

Since \( \psi_j \) is an \( \varepsilon_j \)-approximation solution, we have

\[
\frac{d\psi_j(r)}{dr} = G(r, \psi_j) + \Delta_j(r), \tag{2.14}
\]

where \( |\Delta_j(r)| < \varepsilon_j \), for \( r \in [0, R] \). Integrating (2.14) from 0 to \( r(\leq R) \), we have

\[
\psi_j(r) = a + \left( \int_0^r G(s, \psi_j) ds + \int_0^r \Delta_j(s) ds \right).
\]

Let \( j \to \infty \),

\[
\varphi(r) = a + \lim_{j \to \infty} \left( \int_0^r G(s, \psi_j) ds + \int_0^r \Delta_j(s) ds \right) \tag{2.15}
\]

\[
= a + \int_0^r G(s, \varphi) ds.
\]

Since \( \varphi \in C[0, R] \), by (2.15), \( \varphi \in C^1(0, R] \). Differentiating (2.15), we have \( \varphi'(r) = G(r, \varphi), \ r > 0 \). Hence, we can see that \( \varphi \) satisfies (2.1) in \([0, R]\).

In fact, a local solution also exists for any real number \( a \) if we do not consider only the positive ones. Once \( a \) is positive, it is easy to know the solution \( \varphi \) is positive, too.
3 Proof of the Main Results

We will prove the main results by the comparison lemma.

Lemma 3.1. Let \( \varphi(r) \in C^1[0, R] \cap C^2(0, R) \) with \( A(\varphi'(r)) \varphi'(r) \in C^1[0, R] \) satisfying (2.2), with \( \varphi'(0) = 0 \) and \( \varphi(r) \to \infty \) as \( r \to R \). Then, if \( u(x) \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n) \) is a positive solution of (1.1), we have \( u(x) \leq \varphi(|x|) \) at each point in \( B_R \).

Proof. Let \( v(x) = \varphi(|x|) \), and by Lemma 2.3, we know \( v(x) \in C^2(B_R \setminus \{0\}) \cap C^1(B_R) \) is a solution of (2.7).

Let \( L[w] = \sigma_k^\frac{d}{2} \left( \lambda(D_i(A(|D(u-a)|)|D_jw)) \right) - f(w) \). Suppose to the contrary that \( u > v \) somewhere, then there is some constant \( a > 0 \) such that \( u - a \) touches \( v \) from below, which means \( u - a - v \leq 0 \) in \( B_R \). Suppose \( u - a \) touches \( v \) at some interior point \( x_0 \in B_R \). Then there is \( R' \in (0, R) \) such that \( x_0 \in B_{R'} \). Since \( v(x) = \varphi(|x|) \to \infty \) as \( x \to \partial B_R \) and \( u \) is bounded in \( B_R \), we can assume \( \sup_{\partial B_{R'}}(u - a - v) < 0 \).

It follows from (1.4) that in \( B_{R'} \),

\[
L[u - a] = \sigma_k^\frac{d}{2} \left( \lambda(D_i(A(|D(u-a)|)|D_j(u-a)))) \right) - f(u - a) \\
= \left( \sigma_k^\frac{d}{2} \left( \lambda(D_i(A(|Du|)|D_jw)) \right) - f(u) \right) + (f(u) - f(u - a)) \\
\geq 0 = L[v].
\]

Now \( u - a \) is a subsolution and \( v \) is a solution (with respect to \( L \)). By the maximum principle,

\[
0 = \sup_{B_{R'}}(u - a - v) = \sup_{\partial B_{R'}}(u - a - v) < 0,
\]

which is impossible. ☐

Lemma 3.2. The inequality (1.1) has a positive solution \( u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n) \) if and only if the Cauchy problem (2.1) has a positive solution \( \varphi(r) \in C^2(0, \infty) \cap C^1[0, \infty) \) with \( A(\varphi'(r)) \varphi'(r) \in C^1[0, \infty) \) for some positive number \( a \).

Proof. First, the sufficient condition is obvious. If there exists such a solution \( \varphi(r) \) of (2.1) for \( R = +\infty \), let \( u(x) = \varphi(r) \), \( r = |x| \). By Lemma 2.3 and Lemma 2.1, \( \sigma_k^\frac{d}{2} \left( \lambda(D_i(A(|Du|)|D_ju)) \right) = f(u) \) and \( \lambda(D_i(A(|Du|)|D_ju)) \in \Gamma_k \) for \( x \in \mathbb{R}^n \). Thus \( u(x) \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n) \) is a required solution of (1.1).

Next, we will prove the necessary condition. On the contrary, suppose that no such function \( \varphi(r) \) exists globally. Then for any positive number \( a \), the Cauchy problem (2.1) has a positive solution \( \varphi(r) \) on some interval which cannot be a global solution. Hence, we assume \( 0, R \) is the maximal interval in which the solution exists. Since \( \varphi'(r) > 0 \) for \( r > 0 \), we know \( \varphi(r) \to \infty \) as \( r \to R \). Then by Lemma 2.1, \( \varphi(|x|) \) satisfies (2.2). By Lemma 3.1, any positive solution \( u(x) \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n) \) of (1.1) would satisfy \( u(x) \leq \varphi(|x|) \) for \( x \in B_R \). In particular we have \( u(0) \leq \varphi(0) = a \). However, since \( a \) is arbitrary, we take \( a = \frac{u(0)}{2} \) and obtain a contradiction, which means the necessary condition holds. ☐
Proof of Theorem 1.1. Suppose to the contrary that inequality (1.1) has a positive solution \( u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n) \). Then, by Lemma 3.2, the Cauchy problem (2.1) has a positive solution \( \varphi(r) \in C^2(0, \infty) \cap C^1[0, \infty) \) with \( A(\varphi'(r)) \varphi'(r) \in C^1[0, \infty) \) for some positive number \( a \). Since \( f \) and \( \varphi \) are both monotone non-decreasing, it follows from (2.1) that

\[
A(\varphi'(r)) \varphi'(r) = \left( \frac{n r^k - n}{C_n^k} \int_0^r s^{n-1} f^k(\varphi(s)) ds \right)^{\frac{1}{k}} \leq \left( \frac{1}{C_n^k} \right)^{\frac{1}{k}} r f(\varphi(r)), \ r > 0. \tag{3.1}
\]

Substituting (3.1) in (2.2), we obtain

\[
C_{n-1}^k (A(\varphi'(r)) \varphi'(r))' \left( \frac{A(\varphi'(r)) \varphi'(r)}{r} \right)^{k-1} \geq \frac{k}{n} f^k(\varphi(r)), \ r > 0,
\]

which comes to

\[
\left( (A(\varphi'(r)) \varphi'(r))^k \right)' \geq \frac{k}{C_n^k} r^{k-1} f^k(\varphi(r)), \ r > 0. \tag{3.2}
\]

We now integrate (3.2) over \([0, r]\) to obtain

\[
(A(\varphi'(r)) \varphi'(r))^k \geq \frac{k}{C_n^k} \int_0^r s^{k-1} f^k(\varphi(s)) ds \geq \frac{1}{C_n^k} r^k f^k(a), \ r > 0,
\]

which leads to

\[
A(\varphi'(r)) \varphi'(r) \geq \left( \frac{1}{C_n^k} \right)^{\frac{1}{k}} r f(a), \ r > 0. \tag{3.3}
\]

By (1.3), we see that

\[
\left( \frac{1}{C_n^k} \right)^{\frac{1}{k}} r f(a) \leq A(\varphi'(r)) \varphi'(r) \leq \lim_{p \to \infty} p A(p) < \infty, \ r > 0.
\]

Letting \( r \to \infty \) in the above, we have a contradiction. This completes the proof. \( \square \)

Next we consider some properties of the function (1.7). By (1.3), we know

\[
\Psi'(p) = p \left( (p A(p))^k \right)' > 0, \ p > 0,
\]

then \( \Psi \) is strictly monotone increasing on \((0, \infty)\) and \( \Psi(0) = 0 \). By

\[
\Psi(p) + \int_0^1 (t A(t))^k dt = p (p A(p))^k - \int_1^p (t A(t))^k dt > (p A(p))^k, \ p > 1,
\]

we have \( \lim_{p \to \infty} \Psi(p) = \infty \). Hence the inverse function of \( \Psi \) on \([0, \infty)\) exists, denoted by \( \Psi^{-1} \). Clearly \( \Psi^{-1} \) is a strictly monotone increasing function and satisfies \( \lim_{p \to \infty} \Psi^{-1}(p) = \infty \).
Lemma 3.3. Assume that $A$ satisfies (1.3), (1.6) and $f$ satisfies (1.4). If
\[ \int_{s}^{\infty} \left( \Psi^{-1} \left( \int_{0}^{s} f^k(t) dt \right) \right)^{-1} ds < \infty, \]  
then the inequality (1.1) has no positive solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$.

Proof. Suppose to the contrary that the inequality (1.1) has a positive solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$. Then, by Lemma 3.2, the Cauchy problem (2.1) has a positive solution $\varphi(r) \in C^2(0, \infty) \cap C^1[0, \infty)$ with $A(\varphi'(r)) \varphi'(r) \in C^1[0, \infty)$ for some positive number $a$. As above we obtain (3.2) and (3.3). Letting $r \to \infty$ in (3.3), by (1.3), we have $\lim_{r \to \infty} \varphi'(r) = \infty$. Therefore $\lim_{r \to \infty} \varphi(r) = \infty$. Multiplying (3.2) by $\varphi' > 0$, we have
\[ \Psi'(\varphi'(r)) = \varphi'(r) \left( (A(\varphi'(r)) \varphi'(r))^k \right)' \geq \frac{k}{C_n} f^k(\varphi(r)) \varphi'(r), \quad r > 1, \]
and then integrating on $[1, r]$, we obtain
\[ \Psi(\varphi'(r)) \geq \frac{k}{C_n} \int_{\varphi(1)}^{\varphi(r)} f^k(s) ds, \quad r > 1. \]

Hence
\[ \left( \Psi^{-1} \left( \frac{k}{C_n} \int_{\varphi(1)}^{\varphi(r)} f^k(s) ds \right) \right)^{-1} \varphi'(r) \geq 1, \quad r > 1. \]

Integrating on $[1, r]$, we have
\[ \int_{\varphi(1)}^{\varphi(r)} \left( \Psi^{-1} \left( \frac{k}{C_n} \int_{\varphi(1)}^{s} f^k(t) dt \right) \right)^{-1} ds \geq r - 1, \quad r > 1. \]  
(3.5)

Letting $r \to \infty$ in (3.5), we have
\[ \int_{\varphi(1)}^{\infty} \left( \Psi^{-1} \left( \frac{k}{C_n} \int_{\varphi(1)}^{s} f^k(t) dt \right) \right)^{-1} ds = \infty, \]
which contradicts (3.4). This completes the proof.

Lemma 3.4. Assume that $A$ satisfies (1.3), (1.6) and $f$ satisfies (1.4). If (1.8) holds, then the inequality (1.1) has a positive solution $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \Phi^k(\mathbb{R}^n)$.

Proof. By Lemma 3.2, we only need to prove that the Cauchy problem (2.1) has a positive solution $\varphi(r) \in C^2(0, \infty) \cap C^1[0, \infty)$ with $A(\varphi'(r)) \varphi'(r) \in C^1[0, \infty)$ for some positive number $a$. Suppose to the contrary that no such solution of (2.1) exists. As in the proof of Lemma 3.2, the problem (2.1) has such a positive solution $\varphi(r)$ valid...
on the maximal existence interval $[0, R]$. By Lemma 2.1, we know that $\varphi$ satisfies (2.2).

Next, we show that $\varphi(R) = \lim_{r \to R} \varphi(r) = \infty$, $r \in [0, R)$. Suppose to the contrary that $\varphi(R) < \infty$. Then, by (2.1), $\varphi'(R) < \infty$ exists. By the continuation theorem of the Cauchy problem (2.1), $\varphi$ can be extended as a solution to the right beyond $R$. This contradicts the choice of $R$. Hence we have $\varphi(R) = \infty$.

Since $\varphi'(r) > 0$ for $0 < r < R$, then by (2.2) and (1.3), we have

$$C_{n-1}^{k-1} (A(\varphi'(r)) \varphi'(r))' \left( \frac{A(\varphi'(r)) \varphi'(r)}{r} \right)^{k-1} \leq f_k(\varphi(r)),$$  

which comes to

$$\left( (A(\varphi'(r)) \varphi'(r))^k \right)' \leq \frac{n}{C_n^k} r^{k-1} f_k(\varphi(r)), \quad 0 < r < R.$$

Multiplying the above by $\varphi' > 0$, we have

$$\Psi'(\varphi'(r)) = \varphi'(r) \left( (A(\varphi'(r)) \varphi'(r))^k \right)' \leq \frac{n R^{k-1}}{C_n^k} f_k(\varphi(r)) \varphi'(r), \quad 0 < r < R,$$

and then integrating on $[0, r], \ r < R$, we obtain

$$\Psi(\varphi'(r)) \leq \frac{n R^{k-1}}{C_n^k} \int_a^r f_k(s) \, ds, \quad 0 < r < R.$$

Hence

$$\left( \Psi^{-1} \left( \frac{n R^{k-1}}{C_n^k} \int_a^r f_k(s) \, ds \right) \right)^{-1} \varphi'(r) \leq 1, \quad 0 < r < R.$$

Integrating on $[0, r], \ r < R$, we have

$$\int_a^r \left( \Psi^{-1} \left( \frac{n R^{k-1}}{C_n^k} \int_a^r f_k(s) \, ds \right) \right)^{-1} \, ds \leq r, \quad 0 < r < R.$$

Letting $r \to R$ in the above, we have

$$\int_a^\infty \left( \Psi^{-1} \left( \frac{n R^{k-1}}{C_n^k} \int_a^s f_k(t) \, dt \right) \right)^{-1} \, ds \leq R < \infty,$$

which contradicts (1.8). This completes the proof.

Combining Lemma 3.3 and Lemma 3.4, we proof Theorem 1.3.
Lemma 3.5. Assume that (1.9) holds. Then
\[
0 < \liminf_{p \to \infty} \frac{\Psi(p)}{p^{k(m-1)+1}} \leq \limsup_{p \to \infty} \frac{\Psi(p)}{p^{k(m-1)+1}} < \infty.
\]
Consequently, we have
\[
0 < \liminf_{p \to \infty} \frac{\Psi^{-1}(p)}{p^{k(m-1)+1}} \leq \limsup_{p \to \infty} \frac{\Psi^{-1}(p)}{p^{k(m-1)+1}} < \infty.
\]

Proof. By (1.7) and (1.9), we have
\[
\limsup_{p \to \infty} \frac{\Psi(p)}{p^{k(m-1)+1}} \leq \limsup_{p \to \infty} \frac{p(pA(p))^k}{p^{k(m-1)+1}} < \infty.
\]
Next we show that there exist positive constants \(P\) and \(C\) such that
\[
\Psi(p) \geq Cp^{k(m-1)+1}, \quad p \geq P,
\]
which implies that
\[
\liminf_{p \to \infty} \frac{\Psi(p)}{p^{k(m-1)+1}} > 0.
\]
The condition (1.9) implies the existence of positive constants \(P_1, C_1\) and \(C_2\) such that
\[
C_1p^{k(m-1)+1} \leq p(pA(p))^k \leq C_2p^{k(m-1)+1}, \quad p \geq P_1.
\]
By (3.7), choose \( \theta > 0 \) so small that \((C_2/C_1) \theta^{k(m-1)} < 1/2\), then we have
\[
\frac{(\theta p)((\theta p)A(\theta p))^k}{p(pA(p))^k} \leq \frac{C_2(\theta p)^{k(m-1)+1}}{C_1p^{k(m-1)+1}} = \frac{C_2\theta^{k(m-1)+1}}{C_1} < \frac{1}{2} \theta, \quad p \geq P,
\]
where \( P = P_1/\theta \). We observe that
\[
\int_0^p (tA(t))^k \, dt = \int_0^{\theta p} (tA(t))^k \, dt + \int_{\theta p}^p (tA(t))^k \, dt
\]
\[
\leq (\theta p) ((\theta p)A(\theta p))^k + (p-\theta p)(pA(p))^k
\]
\[
= p(pA(p))^k \left(1 - \theta + \frac{(\theta p)((\theta p)A(\theta p))^k}{p(pA(p))^k}\right), \quad p \geq P.
\]
From (3.9), (3.8), and (3.7) it follows that
\[
\Psi(p) = p(pA(p))^k \left(1 - \frac{\int_0^p (tA(t))^k \, dt}{p(pA(p))^k}\right)
\]
\[
\geq p(pA(p))^k \left(\theta - \frac{(\theta p)((\theta p)A(\theta p))^k}{p(pA(p))^k}\right)
\]
\[
> \frac{1}{2} \theta p(pA(p))^k
\]
\[
\geq \frac{1}{2} \theta C_1p^{k(m-1)+1}, \quad p \geq P,
\]
which gives (3.6). This completes the proof. □

By Theorem 1.3 and Lemma 3.5, we can get Corollary 1.4 immediately.

Proof of Corollary 1.7. The proof of Corollary 1.7 is similar to the above. Most of the properties we need are almost the same as we have discussed. Since $f$ is now a positive, continuous and monotone non-decreasing function defined on $\mathbb{R}$, we do not need $a$ to be positive in Lemma 2.1, Lemma 2.5 and Lemma 3.2 to get the similar conclusions. □

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