A Strategy for Maker in the Clique Game
which Helps to Tackle some Open Problems by Beck

Heidi Gebauer

Abstract

We study Maker/Breaker games on the edges of the complete graph, as introduced by Chvátal and Erdős. We show that in the \((m : b)\) clique game played on \(K_N\), the complete graph on \(N\) vertices, Maker can achieve a \(K_q\) for \(q = \left(\frac{m}{\log_2(b+1)} - o(1)\right) \cdot \log N\), which partially solves an open problem by Beck. Moreover, we show that in the \((1:1)\) clique game played on \(K_N\) for a sufficiently large \(N\), Maker can achieve a \(K_q\) in only \(O(2^{\frac{2q}{3}})\) moves, which improves the previous best bound and answers a question of Beck. Finally we consider the so called tournament game. A tournament is a directed graph where every pair of vertices is connected by a single directed edge. The tournament game is played on \(K_N\). At the beginning Breaker fixes an arbitrary tournament \(T_q\) on \(q\) vertices. Maker and Breaker then alternately take turns at claiming one unclaimed edge \(e\) and selecting one of the two possible orientations. Maker wins if his graph contains a copy of the goal tournament \(T_q\); otherwise Breaker wins. We show that Maker wins the tournament game on \(K_N\) with \(q = (1 - o(1)) \log_2 N\) which supports the random graph intuition: the threshold for \(q\) is asymptotically the same for the game played by two “clever” players and the game played by two “random” players.

This last result solves an open problem of Beck which he included in his list of the seven most humiliating open problems.

1 Introduction

In this paper we study games played by two opponents on edges of the complete graph \(K_N\) on \(N\) vertices. The two players alternately take turns at claiming some number of unclaimed edges until all edges are claimed. One of the players, called Maker, aims to create such a graph which possesses some fixed property \(P\). The other player, called Breaker, tries to prevent Maker from achieving his goal: Breaker wins if, after all \(\binom{n}{2}\) edges were claimed, Maker’s graph does not possess \(P\). A widely studied game of this kind is the \(q\)-clique game where \(P = K_q\), the clique on \(q\) vertices. An immediate question is how large \(q\) can be (in terms of \(N\)) such that Maker can achieve a \(K_q\) in the game on \(K_N\). Amazingly, for the ordinary \((1:1)\) game, i.e., the game where Maker and Breaker each take one edge

*Institute of Theoretical Computer Science, ETH Zurich, CH-8092 Switzerland. Email: gebauerh@inf.ethz.ch.
Theorem 1.1. (Beck, [1]) If \( q \leq \lfloor f(N) \rfloor \) then Maker has a winning strategy in the (1:1) \( q \)-clique game. If \( q \geq \lceil f(N) \rceil \) then Breaker has a winning strategy.

For the biased variant of the \( q \)-clique game, where Maker claims, say, \( m \) edges per move and Breaker claims, say, \( b \) edges per move, however, not so much is now. We denote this game by \((m : b)\) clique game and let \( f(N, m, b) \) denote the largest \( q \) such that Maker can achieve a \( K_q \) in the \((m : b)\) clique game on \( K_N \). Let 
\[
\begin{align*}
g(N, m, b) &= \begin{cases} 
\left\lfloor \frac{2}{\log(m+b) - \log m} \cdot \log N - 2 \log_c \log_c N + 2 \log_c e - 2 \log_c 2 - 1 + \frac{2 \log c}{\log c_0} + o(1) \right\rfloor, & \text{if } m > b \\
\left\lfloor \frac{2}{\log(m+b) - \log m} \cdot \log N - 2 \log_c \log_c N + 2 \log_c e - 2 \log_c 2 - 1 + o(1) \right\rfloor, & \text{if } m \leq b
\end{cases}
\end{align*}
\]
where \( c = \frac{m+b}{m} \) and \( c_0 = \frac{m}{m-b} \).

Beck poses the following open problem.

Open Problem 1.2. (Open Problem 30.2, [1]) Is it true that \( f(N, m, b) = g(N, m, b) \)?

We will show that this does not hold in full generality by proving the following.

Theorem 1.3. In the \((m : b)\) clique game played on \( K_N \) Maker has a strategy to achieve a \( K_q \) with 
\[
q = \left( \frac{m}{\log(b+1)} - o(1) \right) \cdot \log N
\]
If \( m \) and \( b \) are close to each other (say \( \frac{b}{2} \leq m \leq 2b \)) and large enough then by Theorem 1.3, 
\[
f(N, m, b) \geq \left( \frac{m}{\log(2m+1)} - o(1) \right) \log N \geq \left( \frac{2}{\log(\frac{m}{2})} + o(1) \right) \log N = g(N, m, b).
\]
In particular, \( f(N, m, m) \geq \left( \frac{m}{\log(m+1)} - o(1) \right) \log N > (2 + o(1)) \log N = g(N, m, m) \) for \( m \geq 6 \). This connects to the following open problem by Beck.

Open Problem 1.4. (Open Problem 31.1, [1])

(a) Is it true that in the \((2:2)\) clique game on \( K_N \) Maker has a strategy to achieve a \( K_q \) for 
\[
q = 2 \log N - 2 \log \log N + O(1) ?
\]

(b) Is it true that in the \((2:2)\) clique game on \( K_N \) Breaker can prevent Maker from achieving a \( K_q \) for 
\[
q = 2 \log N - 2 \log \log N + O(1) ?
\]

Open Problem 1.4 is still unsolved but Theorem 1.3 (for \( m = b \geq 6 \)) points out that it is not implausible that the answer to (b) is no.

Theorem 1.1, Open Problem 1.2, Theorem 1.3 and Open Problem 1.4 are about the issue of building a \( q \)-clique on a complete graph containing as few vertices as possible. Another issue is to build a clique fast.
Open Problem 1.5. (Open Problem 25.1, [1]) Playing the (1:1) clique game on the infinite complete graph $K_{\infty}$ (or at least a “very large” finite $K_N$), how long does it take for Maker to build a $K_q$?

Let $s(q)$ denote the minimum number of moves Maker needs to achieve a $K_q$. Theorem [1] directly implies the following.

Corollary 1.6. Maker can build a $K_q$ on the graph $K_N$ with $N \geq q \cdot 2^\frac{q}{2}(1 + o(1))$.

Hence $s(q) \leq \frac{1}{2}(\frac{N}{2}) \leq (1 + o(1))q^2 2^q$. The best known bound to our knowledge is $s(q) \leq 2^{q+2}$ which has been discovered by Beck [2] and, independently, by Pekeč and Tuza. From the other side, Breaker can prevent Maker from building a $K_q$ in $2^\frac{q}{2}$ moves [1], thus $s(q) \geq 2^\frac{q}{2}$.

Beck asks whether the bound $O(2^q)$ can be improved and, if yes, whether $s(q)$ is closer to the upper bound of (roughly) $2^q$ or to the lower bound of $2^\frac{q}{2}$. We will show that $s(q) \leq O\left(2^\frac{q^2}{2}\right)$, which means that $s(q)$ actually is closer to the lower bound.

Theorem 1.7. Let $q, r$ be integers such that $q$ is sufficiently large and let $N \geq q^5 \cdot 2^q \cdot r$. In the (1 : 1) game on $K_N$ Maker can in $q^5 \cdot 2^q \cdot r$ moves achieve that for some $\{v_1, \ldots, v_q\} \cup \{w_1, \ldots, w_r\} \subseteq V(G)$, (i) every edge $(v_i, u)$ with $i \in \{1, \ldots, q\}$ and $u \in \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_q\} \cup \{w_1, \ldots, w_r\}$ belongs to Maker’s graph, and (ii) for every pair $i, j \in \{1, \ldots, q\}$, the edge $(w_i, w_j)$ has not been claimed by Breaker.

We can now combine Theorem 1.7 and Theorem 1.1. First we apply Theorem 1.7 which allows Maker to obtain in his graph a $q$-clique $C$ and a vertex set $\{w_1, \ldots, w_r\}$ where every $w_i$ is connected to every vertex of $C$. Then we apply Theorem 1.1 which lets Maker build a $K_{2\log r - 2\log \log r - 2}$ on the subgraph induced by $\{w_1, \ldots, w_r\}$. Hence altogether Maker can achieve a $K_{q+2\log r - 3\log \log r}$ in $q^5 \cdot 2^q \cdot r + \binom{q}{2}$ moves. If we replace $q$ with $\frac{q}{3}$ and $r$ with $q^2 2^\frac{q}{2}$ we obtain the following.

Corollary 1.8. For a large enough $N$ Maker can build a $K_q$ in the game on $K_N$ in $2q^7 \cdot 2^\frac{q^2}{2}$ moves.

Another variation of the clique game is the so called tournament game. A tournament is a directed graph where every pair of vertices is connected by a single directed edge. The tournament game is played on $K_N$. At the beginning Breaker fixes an arbitrary tournament $T_q$ on $q$ vertices. Maker and Breaker then alternately take turns at claiming one unclaimed edge $e$ and selecting one of the two possible orientations. Maker wins if his graph contains a copy of the goal tournament $T_q$; otherwise Breaker wins. Beck [1] showed that Maker has a winning strategy for $q = (\frac{1}{2} - o(1)) \log N$. Actually, he even proved the stronger statement that Maker has a strategy to achieve that his graph contains a copy of all possible $T_q$. However, the random graph intuition (which says that the threshold for $q$ is asymptotically the same for the game played by two “clever” players and the game played by two “random” players) suggests that Maker already has a winning strategy if $q = (1 - o(1)) \log N$. Beck [1] included the following open problem in his list of the seven most humiliating open problems.

Open Problem 1.9. Is it true that Maker has a winning strategy for the tournament game with $q = (1 - o(1)) \log N$?
We prove that the answer to Open Problem 1.9 is yes.

**Theorem 1.10.** Maker has a winning strategy for the tournament game with \( q = (1 - o(1)) \log N \)

**Maker’s Strategy**  For proving the claimed results we will analyze adapted versions of the following, natural Maker’s Strategy for the ordinary (1:1) \( q \)-clique game: Maker first selects an arbitrary vertex \( v_1 \). In each of his next moves he claims an edge incident to \( v_1 \) until all edges incident to \( v_1 \) have been occupied. We refer to this sequence of moves as *processing* \( v_1 \). In this way Maker can connect (in his graph) at least \( N - 1 \) vertices to \( v_1 \). So his task is now reduced to achieving a \((q - 1)\)-clique in a graph with roughly \( N^2 \) vertices. It seems very plausible that by applying this strategy recursively Maker can, for \( q = \log N \), achieve a \( q \)-clique in \( 2N \) steps. However, there is one obstacle: While Maker claims edges incident to \( v_1 \) Breaker can already claim other edges in the graph, which might later bring Maker into troubles. To prevent this, Maker roughly proceeds as follows. After connecting \( v_1 \) to as many vertices as possible he deletes all vertices whose degree is above some threshold \( t \) and then continues in the resulting graph, where every vertex is connected to almost all (i.e., all but at most \( t \)) of the other vertices. By choosing \( t \) appropriately Maker can guarantee that both of these restrictions (i.e., fewer vertices and smaller degrees) do not have a significant influence. By a careful analysis we can show that Maker can achieve a \((\log N - o(1))\)-clique in \( N \) steps. We denote the above strategy by \( S \).

For the clique size our result is weaker than Beck’s result by a factor of 2. However, the described strategy turns out to be helpful for some variations of the clique game: For the biased clique game we consider the following adaptation of Maker’s strategy \( S \): At the beginning, instead of selecting one vertex \( v_1 \), he occupies an \( m \)-clique \( C \) in his graph. (In the more detailed analysis in Section 2 we will show how Maker can achieve this.) Let \( v_1, \ldots, v_m \) denote the vertices of \( C \). As long as there are vertices \( v \) for which \((v, v_1), (v, v_2), \ldots, (v, v_m)\) are all unclaimed, as his move, Maker fixes such a \( v \) and connects \( v \) to \( v_1, \ldots, v_m \). In this way Maker can achieve that in his graph roughly \( \frac{N - m}{(m+1)} \) vertices are adjacent to every \( v_i \in \{v_1, \ldots, v_m\} \).

A handwaving analysis (neglecting the fact that Breaker might have claimed edges which are non-incident to a clique edge) gives that Maker can achieve a \( K_q \) for \( q = \frac{m}{\log(m+1)} \log N \), which is actually roughly the same as we get in our careful analysis.

An adaption of the strategy \( S \) can also be used to prove Theorem 1.7. Basically the only additional requirement is that after processing \( q \) vertices the required set of \( r \) vertices is still present, which causes an additional factor of roughly \( r \).

Finally, for the tournament game Maker can adapt his strategy \( S \) as follows. Let \( T \) be the goal-tournament of Maker on the vertex set \( \{u_1, \ldots, u_q\} \). During the game Maker will maintain so called *candidate sets* \( V_1, \ldots, V_q \) such that every \( v_i \in V_i \) is still suitable for the part of vertex \( u_i \). In round \( i \) Maker basically selects a vertex \( v_i \) in \( V_i \) and proceeds in such a way that for every \( j > i \) he finally possesses \( \frac{|V_j|}{2} \) edges of the form \((v_i, v_j)\) where \( v_j \in V_j \) and the orientation of \((v_i, v_j)\) equals
In this way Maker reduced his task to occupying a fixed tournament on $q - 1$ vertices in the subgraph induced by those vertices in $V_{i+1} \cup V_{i+2} \cup \ldots \cup V_q$ which are in Maker’s graph adjacent to $v_i$. Note that in each such round the number of vertices in $V_j$ is roughly halved, which suggests that Maker has a winning strategy for $q = (1 - o(1)) \log(N)$.

**Notation** Let $G$ be a graph on $n$ vertices and let $v$ be a vertex in $V(G)$. By $d(v)$ we denote the ordinary degree of $v$ in $G$. The complementary degree $\bar{d}_G(v)$ of $v$ is the number of vertices different from $v$ in $G$ which are non-adjacent to $v$, i.e., $\bar{d}_G(v) = n - 1 - d(v)$. If there is no danger of confusion we sometimes just write $\bar{d}(v)$ for $\bar{d}_G(v)$.

If we consider the course of a game then $d_B(v)$ denotes the degree of $v$ in Breaker’s graph.

For a subset $S = \{v_1, \ldots, v_i\} \subseteq V(G)$, the *subgraph induced by $S$, $G[v_1,\ldots, v_i]$, denotes the graph obtained from deleting all vertices of $V(G) \setminus S$ in $G$.

## 2 The Biased Game

The following is a well known fact in graph theory.

**Observation 2.1.** Let $G$ be a graph on $n$ vertices with $\bar{d}(v) \leq d$ for every vertex $v \in V(G)$. Then $G$ contains a clique of size $\frac{n}{d+1}$.

This can be seen by considering the following greedy algorithm for building a clique: In every round select an arbitrary vertex, add it to the clique and delete all its neighbors. In this way, we deleted at most $d$ vertices per clique-vertex and thus get a clique of size at least $\frac{n}{d+1}$.

**Proposition 2.2.** For every $q, m, b$ there is an $n = n(q, m, b)$ such that in the $(m : b)$ clique game played on $K_n$ Maker has a strategy to achieve a $K_q$.

**Proof:** It suffices to consider the case where $m = 1$. We proceed by induction on $q$. Clearly, Maker can always achieve a $K_1$. Suppose now that $q > 1$. Let $n := [5b^2(b + 1)^2] \cdot n(q - 1, 1, b) + 1$ and let \{v_1, \ldots, v_n\} be the vertex set of a $K_n$. Maker uses the following strategy. Until all edges incident to $v_1$ have been occupied he claims in each of his moves one edge of the form $(v_i, v_1)$ for some $i$. In this way he can in total occupy at least $\frac{n-1}{b+1}$ edges incident to $v_1$. In the meantime Breaker has claimed at most $b \cdot (n - 1)$ edges. Maker iteratively removes every vertex $v \in \{v_2, \ldots, v_n\}$ with $d_B(v) \geq 2b(b + 1)$. Let $W$ denote the set of remaining vertices which are in Maker’s graph adjacent to $v_1$. By construction, $|W| \geq \frac{b-1}{b+1} - \frac{b(n-1)}{2(b+1)} = \frac{n-1}{2(b+1)}$ and $\bar{d}(v) \leq 2b(b + 1)$ for every vertex $v \in W$.

By Observation 2.1 $W$ contains a clique $K$ of size $n' := \frac{|W|}{2b(b+1)+1} \geq \frac{n-1}{2b(b+1) + 1}$. Note that no edge of $K$ has been claimed by either of the players. By our choice of $n$ we have $n' \geq n(q - 1, 1, b)$ and therefore Maker can achieve a $K_{q-1}$ on $K$, which together with $v_1$ forms a $K_q$. \[\square\]
Proof of Theorem 1.3. Choose \( C = C(m, b) \) in such a way that in the \((m : b)\) clique game played on \( K_C \) Maker has a strategy to achieve a \( K_m \). (Proposition 2.2 guarantees that such a \( C \) exists). Note that since we consider \( b \) and \( m \) as constants, \( C \) is also a constant. Throughout this section \( \text{game} \) means the \((m : b)\) clique game.

The next lemma shows how Maker can reduce his task to occupying a clique with \( m \) vertices less (than the original clique) in some appropriate subgraph.

**Lemma 2.3.** Let \( G \) be a graph on \( n \) vertices such that \( \bar{d}(v) \leq d \) for every \( v \in V(G) \) and \( n \geq C(d+1) \). Let \( q \geq 1 \) and let

\[
n' := \frac{n - C - md - 2b \cdot \binom{C}{2}}{b + 1} - \frac{bn}{q} = \frac{n}{(b + 1)} + \frac{b(b+1)^2}{q-b(b+1)} - \frac{C + md + 2b \cdot \binom{C}{2}}{b + 1}
\]

Maker can achieve that for some \( \{v_1, \ldots, v_m\} \cup \{w_1, \ldots, w_n\} \subseteq V(G) \), (i) every edge \((v, u)\) with \( i \in \{1, \ldots, m\} \) and \( u \in \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\} \cup \{w_1, \ldots, w_n\} \) belongs to Maker’s graph, (ii) \( \bar{d}(w_i) \leq d + q \) for every \( i \) with \( 1 \leq i \leq n' \), and (iii) the subgraph induced by \( \{w_1, \ldots, w_n\} \) contains no Breaker’s edge.

Before proving Lemma 2.3 we first show its consequences. For integers \( n, d \) let \( K(n, d) \) denote the class of graphs \( G \) on \( n \) vertices with \( \bar{d}(v) \leq d \) for every \( v \in V(G) \).

**Corollary 2.4.** Let \( d, n, q, s, \) be integers and let \( n' \) be defined as in Lemma 2.3. If for every \( G' \in K(n', d + q) \) Maker can obtain a \( K_s \) in the game on \( G' \) then he can achieve a \( K_{s+m} \) in the game on \( G \) for every \( G \in K(n, d) \).

**Proof of Lemma 2.3:** Maker proceeds as follows.

Round 1 Maker selects a set \( S \) of \( C \) vertices which form a clique in \( G \). (Such a set \( S \) exists due to Observation 2.1 and the assumption that \( n \geq C(d+1) \).) Then he occupies a clique \( K_m \) on \( S \) (this is possible by the definition of \( C \)). Let \( v_1, \ldots, v_m \) denote the clique-vertices. Note that in the meantime Breaker occupied at most \( b \binom{C}{2} \) edges.

Round 2 Let \( U := V(G) \setminus S \). Maker removes all vertices in \( U \) which are incident to at least one Breaker-edge. Let \( U' \) denote the resulting vertex set. Note that \( |U'| \geq |U| - 2b \binom{C}{2} = n - C - 2b \binom{C}{2} \). Note that the graph induced by \( \{v_1, \ldots, v_m\} \cup U' \) contains no Breaker-edge. However, it is possible that some vertices \( u \) in \( U' \) are not connected to all vertices in \( \{v_1, \ldots, v_m\} \). (Note that \( \bar{d}(u) \) can be larger than 0). Let \( U'' \) be the vertex set obtained by deleting all vertices in \( U' \) which are non-adjacent to at least one of the \( v_i \). Note that

\[
|U''| \geq |U'| - md \geq |U| - 2b \binom{C}{2} - md = n - C - 2b \binom{C}{2} - md
\]

and that every edge \((v_i, u)\) with \( 1 \leq i \leq m \) and \( u \in U'' \) is present.
Round 3 As long as there are vertices \( u \in U'' \) where \((u, v_i)\) is unclaimed for every \( i \in \{1, \ldots, m\} \), as his
move Maker selects such a \( u \) and occupies the edges \((u, v_1), (u, v_2), \ldots, (u, v_m)\). Note that he
can do at least
\[
  n_{\text{rem}} := \frac{|U''|}{b+1}
\]  
(such moves. Let \( u_1, \ldots, u_{n_{\text{rem}}} \) denote the corresponding vertices of \( U'' \). Note that Maker
possesses every edge \((u, v_i)\) with \( u \in \{u_1, \ldots, u_{n_{\text{rem}}}\} \) and \( 1 \leq i \leq m \).

Round 4 During Round 3 Breaker has claimed at most \( bn \) edges. Maker iteratively deletes every vertex
in \( \{u_1, \ldots, u_{n_{\text{rem}}}\} \) which has degree at least \( q \) in Breaker’s graph. In this way at most \( \frac{bn}{q} \)
vertices are deleted (otherwise we would have deleted more edges than Breaker occupied.)
Let \( \{w_1, \ldots, w_{n_{\text{rem}}-\frac{bn}{q}}\} \subseteq \{u_1, \ldots, u_{n_{\text{rem}}}\} \) denote the set of non-deleted vertices. Removing all
Breaker’s edges gives a subgraph with vertex set \( \{v_1, \ldots, v_m\} \cup \{w_1, \ldots, w_{n_{\text{rem}}-\frac{bn}{q}}\} \) such that
for every \( w \in \{w_1, \ldots, w_{n_{\text{rem}}-\frac{bn}{q}}\} \), \( \bar{d}(w) \leq d + q \), and Maker possesses every edge \((v_i, u)\) with
\( 1 \leq i \leq m \) and \( u \in \{v_1, \ldots, v_i-1, v_{i+1}, \ldots, v_m\} \cup \{w_1, \ldots, w_{n_{\text{rem}}-\frac{bn}{q}}\} \).

We have
\[
n_{\text{rem}} - \frac{bn}{q} = \frac{|U''|}{b+1} - \frac{bn}{q} \quad \text{(by (2))}
\geq \frac{n - C - 2b(C^2/2) - md}{b+1} - \frac{bn}{q} \quad \text{(by (1))}
= n'
\]

We can analyze Maker’s strategy by applying Corollary [2.4] repeatedly. Let
\[
q := \left( \frac{m}{\log(b+1)} \right) \cdot (\log N - 5\log\log N)
\]  
(3)
For simplicity we assume that \( q \) is divisible by \( m \). (For the case where \( q \) is not divisible by \( m \) we can
then follow similar lines.) Our goal is to show that in the game on \( K_N \) Maker can achieve a \( K_q \).

Let \( n \) be a large enough integer and let \( n' \) be defined as in Lemma [2.3] Then
\[
n' \geq \frac{n}{(b+1) + \frac{b(b+1)}{q-b(b+1)}} - (c_1 d + c_2)
\]  
(4)
for appropriate constants \( c_1, c_2 \geq 0 \).

Proposition 2.5. Let \( r := b + 1 + \frac{b(b+1)}{q-b(b+1)} \). Let \( i \in \{1, \ldots, \frac{q}{m}\} \). If
\[
\frac{n}{r^i} - i \cdot (c_1 q^2 + c_2) \geq C(q^2 + 1)
\]
then for every \( G \in K(n, (\frac{q}{m} - i) \cdot q) \), Maker can achieve a \( K_{i-m} \) in the game on \( G \).
Proof: We apply induction. For \( i = 1 \) the claim is clearly true. Indeed, if \( \frac{n}{r^i} - (c_1 q^2 + c_2) \geq C(q^2 + 1) \) then \( n \geq C(q^2 + 1) \). By assumption and Observation 2.3, \( G \) contains a clique of size at least \( \frac{n}{q^2 + 1} \geq C \). By our choice of \( C \) Maker can obtain the desired clique.

Assume now that \( i \geq 2 \). Let \( G \in K(n, (\frac{q}{m} - i) \cdot q) \) and let \( n' \) be defined as in Lemma 2.3. Suppose that

\[
\frac{n}{r^{i-1}} - i \cdot (c_1 q^2 + c_2) \geq C(q^2 + 1)
\]

(5)

By (4) (for \( d = (\frac{q}{m} - i) \cdot q \)) we obtain \( n \leq r \cdot (n' + c_1 \cdot (\frac{q}{m} - i) \cdot q + c_2) \leq r \cdot (n' + c_1 q^2 + c_2) \). Together with (5) this gives

\[
\frac{n' + c_1 q^2 + c_2}{r^{i-1}} - i \cdot (c_1 q^2 + c_2) \geq C(q^2 + 1)
\]

(6)

Thus

\[
\frac{n'}{r^{i-1}} - (i - 1) \cdot (c_1 q^2 + c_2) = \frac{n'}{r^{i-1}} + c_1 q^2 + c_2 - i \cdot (c_1 q^2 + c_2) \geq \frac{n' + c_1 q^2 + c_2}{r^{i-1}} - i \cdot (c_1 q^2 + c_2) \geq C(q^2 + 1)
\]

(7)

By induction Maker can achieve a \( K_{(i-1)m} \) in the game on \( G' \) for every \( G' \in K(n, (\frac{q}{m} - (i - 1)) \cdot q) \). Together with Corollary 2.4 this concludes our proof.

We now complete the proof of Theorem 1.3. Note that \( K_N \) is the unique element of \( K(N, 0) \). By Proposition 2.5 (for \( i = \frac{q}{m} \)) Maker can achieve a \( K_q \) in the game on \( K_N \) if

\[
\frac{N}{r^m} - \frac{q}{m} \cdot (c_1 q^2 + c_2) \geq C(q^2 + 1).
\]

Recall that \( r = b + 1 + \frac{b(b+1)}{q - b(b+1)} \). We have \( r \leq (b+1) \cdot (1 + \frac{b}{q - b(b+1)}) \leq (b+1) \cdot e^{\frac{b}{q - b(b+1)} - (b+1) \cdot e^{-b} \leq (b+1) \cdot e^{\frac{b}{q} \cdot e^{2b}} \). Hence \( r^m \leq (b+1)^m \cdot e^{2b} \). Thus

\[
\frac{N}{r^m} \geq \frac{N}{(b+1)^m \cdot e^{2b}}
\]

(7)

Hence

\[
\frac{N}{r^m} - \frac{q}{m} \cdot (c_1 q^2 + c_2) \geq \frac{N}{r^m} - q^4
\]

\[
\geq \frac{N}{(b+1)^m} \cdot e^{2b} - q^4 \quad \text{(by 7)}
\]

\[
\geq \frac{N}{(b+1)^m} \cdot e^{2b} - \left( \frac{m}{\log(b+1)} \right)^4 \log^4 N \quad \text{(by 3)}
\]

\[
\geq \frac{N}{2 \log N - 5 \log \log N} \cdot e^{2b} - \left( \frac{m}{\log(b+1)} \right)^4 \log^4 N
\]

\[
\geq \log^5 N \cdot e^{2b} - \left( \frac{m}{\log(b+1)} \right)^4 \log^4 N
\]

\[
\geq C \cdot \left( \frac{m}{\log(b+1)} \right)^2 \log^2 N + 1
\]

\[
\geq C(q^2 + 1)
\]

By Proposition 2.5 Maker can achieve the required clique.
3 Building a Clique Fast

Throughout this section by game we mean the ordinary (1:1) clique game. For integers $n, d$ let $K(n, d)$ denote the class of graphs $G$ on $n$ vertices with $d(v) \leq d$ for every $v \in V(G)$.

**Proof of Theorem 1.7:** Let $C(q, r)$ denote the constellation Maker is claimed to achieve in Theorem 1.7.

**Lemma 3.1.** Let $G \in K(n, d)$, let $q \geq 1$ and let $v_1 \in V(G)$. Maker can achieve in $\frac{n}{2}$ moves that for some $W \subseteq V(G) \setminus \{v_1\}$ with $|W| = \frac{n-1-d}{2} - \frac{n}{q}$, (i) $(v_1, w)$ belongs to Maker’s graph for every $w \in W$, (ii) $\bar{d}(w) \leq d + q$ for every $w \in W$, and (iii) the subgraph induced by $W$ contains no Breaker’s edge.

**Proof:** Maker proceeds as follows.

**Round 1** Maker removes all vertices in $V(G) \setminus \{v_1\}$ which are non-adjacent to $v_1$. Note that by assumption at most $d$ vertices are deleted. In his next $\frac{n-1-d}{2}$ moves Maker occupies an unclaimed edge incident to $v_1$. (Since there are at least $n - 1 - d$ vertices and in the first $\frac{n-1-d}{2}$ moves Breaker can collect at most $\frac{n}{2} - d$ edges, Maker can make these moves.) Let $V'$ denote the set of vertices which are in Maker’s graph adjacent to $v_1$. Note that $|V'| \geq \frac{n-1-d}{2}$.

**Round 2** During Round 1 Breaker has occupied at most $n$ edges. Let $W$ denote the vertex set resulting from deleting iteratively every vertex $v$ with Breaker’s degree at least $q$ from $V'$. Note that $|W| \geq |V'| - \frac{n}{q}$ (otherwise the number of Breaker’s edges which were deleted is larger than $n$). Finally we delete all Breaker’s edges in $W$, which increases $\bar{d}(w)$ by at most $q$ for every $w \in W$. Clearly, $v_1$ and $W$ fulfill (i), (ii) and (iii).

Note that Maker makes at most $\frac{n-1-d}{2} \leq \frac{n}{2}$ moves during Round 1 and no move during Round 2. So the required constellation can be obtained in $\frac{n}{2}$ moves.

We have

$$\frac{n-1-d}{2} - \frac{n}{q} = \frac{n}{2} - \frac{d + 1}{2} \geq \frac{n}{2} + \frac{d + 1}{2} = (d + 1)$$

(8)

The following is a consequence of Lemma 3.1 and (8).

**Corollary 3.2.** Let $d, i, n, q, r, s$ be integers. If for every $G' \in K\left(\frac{n}{2} + \frac{d}{q} - (d + 1), d + q\right)$ Maker can in $s$ moves obtain a $C(i - 1, r)$ on $G'$ then he can achieve a $C(i, r)$ on $G$ in $s + \frac{n}{2} \geq (d + 1)$ moves for every $G \in K(n, d)$.

We will analyze Maker’s strategy by applying Corollary 3.2 repeatedly. We fix a $q$ and let

$$N := q^5 \cdot 2^q \cdot r$$

(9)

Our goal is to show that in the game on $K_N$ Maker can achieve a $C(q, r)$.
Proposition 3.3. Let $i \in \{0, 1, \ldots, q\}$. If
\[
\frac{n}{\left(2 + \frac{4}{q^2}\right)^i} - iq^2 \geq r(q^2 + 1)
\]
then for every $G \in K(n, (q-i) \cdot q)$, Maker can achieve a $C(i, r)$ on $G$ in $n$ moves.

Proof: We proceed by induction. We first consider the case where $i = 0$. Let $G \in K(n, q^2)$ and suppose that $n \geq r(q^2 + 1)$. By Observation 2.1, $G$ contains a clique of size $\frac{n}{q^2 + 1}$ which by assumption is at least $r$. So there is a $C(0, r)$ in $G$.

Suppose now that $i \geq 1$ and assume that
\[
\frac{n}{\left(2 + \frac{4}{q^2}\right)^i} - iq^2 \geq r(q^2 + 1)
\]
(10)

Let
\[
n' := \frac{n}{2 + \frac{4}{q^2}} - ((q - i) \cdot q + 1) \geq \frac{n}{2 + \frac{4}{q^2}} - q^2
\]
(11)

Note that $n \leq (n' + q^2) \cdot \left(2 + \frac{4}{q^2}\right)$. By (10),
\[
r(q^2 + 1) \leq \frac{n' + q^2}{\left(2 + \frac{4}{q^2}\right)^{i-1}} - iq^2 \leq \frac{n'}{\left(2 + \frac{4}{q^2}\right)^{i-1}} + q^2 - iq^2 = \frac{n'}{\left(2 + \frac{4}{q^2}\right)^{i-1}} - (i-1)q^2
\]

By induction, for every $G \in K(n', (q-i+1) \cdot q)$ Maker can achieve a $C(i-1, r)$ on $G$ in $n'$ moves. By Corollary 3.2 for $d = (q-i) \cdot q$ and $s = n'$, Maker can achieve a $C(i, r)$ on $G$ in $\frac{n}{2} + n'$ moves, for every $G \in K(n, q - i)$. Since $n' \leq \frac{n}{2}$, the number of moves is at most $n$.

We now conclude the proof of Theorem 1.7. Note that $K_N$ is the unique element of $K(N, 0)$. It suffices to show that Maker can achieve a $C(q, r)$ on $K_N$ in $N$ moves. We have
\[
\left(2 + \frac{4}{q - 2}\right)^q = 2^q \cdot \left(1 + \frac{2}{q - 2}\right)^q \leq 2^q \cdot \left(1 + \frac{4}{q}\right)^q \leq 2^q \cdot e^4
\]
(12)

Hence by (9) and (12),
\[
\frac{N}{\left(2 + \frac{4}{q - 2}\right)^q} - q^3 \geq \frac{N}{(2^q \cdot e^4)} - q^3 \geq \frac{q^5 \cdot r}{e^4} - q^3 \geq r(q^2 + 1)
\]

Proposition 3.3 for $i := q$ and $n := N$ yields that on $K_N$ Maker can achieve a $C(q, r)$ in $N = q^5 \cdot 2^q \cdot r$ moves.
4 Building a Tournament

We need some notation first. We assume that Maker colors his edges red and Breaker colors his edges blue. We say that Maker wins $T_s$ on $G$ if for every tournament $T$ on $s$ vertices Maker has a strategy to achieve a red copy of $T$ in the (1:1) game on $G$.

Proof of Theorem 1.10.

The next lemma describes how Maker can reduce his task of occupying a fixed tournament $T$ to the task of occupying a given tournament with one vertex less. In addition to the clique game Maker will maintain so called candidate sets $V_1, \ldots, V_s$ in such a way that every vertex $v_i \in V_i$ is still suitable for the part of vertex $i$.

**Lemma 4.1.** Let $G$ be a graph such that $\bar{d}(v) \leq d$ for every $v \in V(G)$. Let $q, r$ be integers, let $T_r$ be a partition of $V(G)$ such that $|V_i| \geq n$ for every $i$ with $1 \leq i \leq r$. Maker can achieve that for some $v_1 \in V_1$ and for some subsets $V_2', V_3', \ldots, V_r'$ with $V_i' \subseteq V_i$ for $i$, $2 \leq i \leq r$, (i) for every $i$ with $2 \leq i \leq r$, $v_1$ is in Maker’s graph adjacent to at least $n - d - \frac{rn}{q}$ vertices in $V_i'$ in such a way that the orientation of $(v_1, v_i)$ equals the orientation of $(u_1, u_i)$, (ii) $\bar{d}(v) \leq d + q^2$, for every $v \in V_2' \cup V_3' \cup \ldots \cup V_r'$, and (iii) the subgraph induced by $V_2' \cup V_3' \cup \ldots \cup V_r'$ contains no Breaker’s edge.

**Proof:** We can assume wlog that $|V_i| = n$ for every $i$. Maker selects an arbitrary vertex $v_1 \in V_1$. Then he proceeds as follows.

**Round 1** Maker removes all vertices in $V(G) \setminus \{v_1\}$ which are non-adjacent to $v_1$. For every $i$, let $\tilde{V}_i$ denote the set of vertices in $V_i$ which were not deleted. Note that $|\tilde{V}_i| \geq |V_i| - d$.

**Round 2** Until all edges incident to $v_1$ have been claimed Maker applies the following strategy: If Breaker occupies an edge $(v_1, v_i)$ with $v_i \in \tilde{V}_i$ for some $i \in \{1, \ldots, r\}$ then Maker occupies – if possible – an edge $(v_1, u_i)$ with $w_i \in \tilde{V}_i$. If Maker cannot occupy such an edge (because all edges connecting $v_1$ with $\tilde{V}_i$ are already occupied) or if Breaker occupies an edge which is not incident to $v_1$ then Maker claims an arbitrary edge incident to $v_1$. In this way Maker can in $rn$ moves achieve to possess $|\tilde{V}_i| \geq \frac{|V_i| - d}{2} \geq \frac{n - d}{2}$ edges connecting $v_1$ with $\tilde{V}_i$ for every $i$ with $2 \leq i \leq r$. Hence for $i, i = 2, \ldots, r$ there is a subset $W_i \subseteq V_i$ with

$$|W_i| \geq \frac{n - d}{2}$$

such that Maker’s graph contains $(v_1, w)$ for every $w \in W_i$.

**Round 3** In Round 2 Breaker occupied at most $rn$ edges. In every $W_i$ Maker iteratively removes those vertices with Breaker’s degree at least $q^2$. Let $V'_i \subseteq W_i$ denote the vertices in $W_i$ which were not deleted. Note that $|V'_i| \geq |W_i| - \frac{rn}{q^2}$ (otherwise Maker deleted more edges than Breaker added).
Hence $|V'_i| \geq |W_i| - \frac{dn}{q^2} \geq \frac{d-n}{q^2}$ (the last inequality is by (13)). Deleting all Breaker’s edges increases $\bar{d}(v)$ by at most $q^2$ for every $v \in V'_1 \cup V'_2 \cup \ldots \cup V'_r$ and assures (iii).

For integers $d, n, r$ let $K(n, r, d)$ denote the class of graphs $G$ with a partition $V_1 \cup V_2 \cup \ldots \cup V_r$ of $V(G)$ such that $|V_i| \geq n$ for every $i \in \{1, \ldots, r\}$ and $\bar{d}(v) \leq d$ for every $v \in V(G)$. Lemma 4.1 directly implies the following.

**Corollary 4.2.** Let $d, n, r, s$ be integers with $r \geq 2$.

If for every $G' \in K\left(\frac{d-n}{q^2}, r - 1, d + q^2\right)$ Maker can win $T_s$ on $G'$ then he can win $T_{s+1}$ on $G$ for every $G \in K(n, r, d)$.

We can analyze Maker’s strategy by applying Corollary 4.2 repeatedly. Let

$$q := \log N - 6 \log \log N$$

(14)

Our goal is to show that Maker can win $T_q$ on $K_N$.

**Proposition 4.3.** Let $i \in \{1, \ldots, q\}$. If

$$\frac{n}{(2 + \frac{4}{q^2})^i} - i \cdot q^3 \geq 1$$

then for every $G \in K(n, i, (q-i)q^2)$ Maker wins $T_i$ on $G$.

**Proof:** We apply induction. For $i = 1$ the claim is clearly true. Indeed, if $\frac{n}{2+q^2} - q^3 \geq 1$ then $n \geq (q-1)q^2$ and therefore $K(n, i, (q-1) \cdot q^2)$ is well-defined. Since Maker wins $T_1$ on every graph we are done for the case where $i = 1$.

Suppose now that $i \geq 2$. Let $G \in K(n, i, (q-i)q^2)$ and assume that

$$\frac{n}{(2 + \frac{4}{q^2})^i} - i \cdot q^3 \geq 1$$

(15)

Let

$$n' := \frac{n - (q-i)q^2}{2} - \frac{i n}{q^2} \geq \frac{n - (q-i)q^2}{2} - \frac{n}{q} = \frac{n}{2 + \frac{1}{q^2}} - \frac{(q-1)q^2}{2}$$

Note that $n \leq \left(n' + \frac{(q-1)q^2}{2}\right)(2 + \frac{4}{q^2})$. Together with (15) this implies the following.

$$\frac{n' + \frac{(q-1)q^2}{2}}{(2 + \frac{4}{q^2})^{i+1}} - iq^3 \geq 1$$

(16)

We have

$$1 \leq \frac{n' + \frac{(q-1)q^2}{2}}{(2 + \frac{4}{q^2})^{i-1}} - iq^3 \leq \frac{n'}{(2 + \frac{4}{q^2})^{i-1}} + q^3 - iq^3 = \frac{n' - iq^3}{(2 + \frac{4}{q^2})^{i-1}} - (i - 1)q^3$$
By induction Maker can win $T_{i−1}$ on $G'$ for every $G' \in K(n', i − 1, (q − i + 1)q^2)$. Together with Corollary 4.2 this implies that Maker wins $T_i$ on $G$.

We now complete the proof of Theorem 1.10. We just need to show that Maker can win $T_q$ on all $G \in K(\frac{N}{q}, q, 0)$. (We assume here for simplicity that $N$ is divisible by $q$.) By Proposition 4.3 (for $i = q$) we just need to show that $\frac{N}{q} - q^4 \geq 1$.

Note that

$$\left(2 + \frac{4}{q - 2}\right)^q = 2^q \cdot \left(1 + \frac{2}{q - 2}\right)^q \leq 2^q \cdot e^{\frac{2q}{q - 2}} \leq 2^q \cdot e^4$$

(17)

By (14) and (17) we have

$$\frac{N}{q} - q^4 \geq \frac{N}{q} - q^4 \geq \frac{N}{q} - q^4 \geq \frac{N}{q} - q^4 \geq \frac{N}{q} - q^4 \geq \frac{\log^5 N}{e^4} - \log^4 N \geq 1$$

By Proposition 4.3 Maker can win $T_q$ on all $G \in K(\frac{N}{q}, q, 0)$. Therefore he can win $T_q$ on $K_N$.

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