Monopoles, Dyons and Theta Term in Dirac-Born-Infeld Theory

N. Grandi∗, R.L. Pakman†, F.A. Schaposnik‡ and G. Silva†
Departamento de Física, Universidad Nacional de La Plata
C.C. 67, 1900 La Plata, Argentina

Abstract

We present dyon solutions to an $SU(2)$ Dirac-Born-Infeld (DBI) gauge theory coupled to a Higgs triplet. We consider different non-Abelian extensions of the DBI action and study the resulting solutions numerically, comparing them with the standard Julia-Zee dyons. We discuss the existence of a critical value of $\beta$, the Born-Infeld absolute field parameter, below which the solution ceases to exist. We also analyse the effect of modifying the DBI action so as to include the analogous of the $\theta$ term, showing that Witten formula for the dyon charge also holds in DBI theories.

*Becario CICBA
†Becario CONICET
‡Investigador CICBA
I. INTRODUCTION

The Dirac-Born-Infeld (DBI) action describes the low energy dynamics of D-branes [1]. In this respect, classical solutions to the DBI equations of motion have recently received much attention and several bion, soliton and instanton configurations have been already found [2]- [11].

Concerning monopole solutions to DBI theory, different possibilities have been discussed, either by considering extensions of the DBI action that admit BPS equations [3] (which are then necessarily the same as in the Yang-Mills-Higgs system [12]) or by coupling the DBI action to the usual symmetry breaking Higgs Lagrangian [3]- [10]. In this last case, ‘t Hooft-Polyakov-like monopole solutions arise provided $\beta$, the Born-Infeld “absolute field” parameter, is bigger than a critical value $\beta_c$ [7], [9].

We extend in the present work the analysis of purely magnetic solutions of ref. [9] by constructing electrically charged monopole solutions to an $SU(2)$ DBI theory coupled to a Higgs field in the adjoint, with a potential that breaks the symmetry down to $U(1)$. Since the DBI action contains a $(F_{\mu\nu}\tilde{F}^{\mu\nu})^2$ term which vanishes unless electric and magnetic fields are both present, dyons test more in detail DBI nonlinearities than purely magnetic monopoles. Moreover, one can discuss in this case some issues concerning Witten effect [13] and duality [14] in DBI models.

The paper is organized as follows: we start in Section II by discussing two alternative ways in which an Abelian DBI action can be extended to the case of an $SU(2)$ gauge theory depending on the trace operation used to define a scalar action from non-Abelian fields [15]. Then, we consider the addition of a Higgs action with a symmetry breaking potential and discuss the resulting equations of motion for both trace operations. In Section III we discuss dyon configurations by considering the usual Julia-Zee Ansatz [16]- [18]. We find the solutions numerically and discuss their main properties. The effect of adding a $\theta$-term is studied in section IV where the rules of charge quantization are discussed in detail. Finally, we present in section V a summary of our results and conclusions.

II. THE ACTION

For a non-Abelian gauge group, there are alternative definitions of the Dirac-Born-Infeld action [15], [20]- [25]. Basically, they differ in the way a scalar action is constructed using different trace operations over the group indices. As shown by Tseytlin [13], there is one which is singled out by the fact that it leads to an action which can be connected to the tree level string effective action for branes. The corresponding Lagrangian is

$$L_{DBI}^{Str} \equiv \beta^2 \text{Str} \left( 1 - \sqrt{-\det(g_{\mu\nu} + \frac{1}{\beta} F_{\mu\nu})} \right)$$

(1)

Here Str is a symmetric trace operation defined by the formula

$$\text{Str}(t_1, t_2, \ldots, t_N) \equiv \frac{1}{N!} \sum_{\pi} \text{tr}(t_{\pi(1)} t_{\pi(2)} \ldots t_{\pi(N)})$$

(2)

with $t^a$ the generators of the gauge group which we shall take for simplicity in the fundamental representation of $SU(2)$, normalized so that
Remarkably, definition (1) is equivalent to the familiar Born-Infeld form for the Lagrangian
\[ \mathcal{L}^{\text{Str}}_{\text{DBI}} = \beta^2 \text{Str} \left( 1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2} \right). \] (4)

In contrast, if one were to use the “tr” trace operation in the definition of the DBI action as a determinant like in (1), (of course one has to supplement the definition with some ordering rule for multiplying determinant elements) its explicit computation would not lead to the analogous of (4). One can instead directly define an alternative DBI Lagrangian as
\[ \mathcal{L}^{\text{tr}}_{\text{DBI}} = \beta^2 \text{tr} \left( 1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2} \right). \] (5)

We shall then consider both possibilities, taking as DBI non-Abelian Lagrangian (4) and (5).

Apart from this alternative related to the way the trace operation is defined, one has to decide how the Higgs field dynamics is introduced. One possibility is to construct DBI monopoles by demanding that the usual Yang-Mills-Higgs BPS relations also hold in the DBI case [4]. This amounts to define a Higgs field Lagrangian in a Born-Infeld-like way (i.e., with the scalar kinetic energy and potential terms also under a square root) in such a way that the model have a supersymmetric extension [4], [23]-[25]. One can then prove that the BPS relations coincide with those arising in the Yang-Mills-Higgs case [12], so that the resulting DBI monopole solutions are identical to the well-honored Prasad-Sommerfield exact solutions and have no specific features resulting from the DBI dynamics. Instead, we shall consider here, as already done in [4] for purely magnetic solutions, the usual $SU(2)$ Higgs field Lagrangian and a symmetry breaking potential not necessarily in the BPS limit.

We then propose the following Lagrangian for the Higgs field:
\[ \mathcal{L}_{\text{Higgs}} = \frac{1}{2} D^\mu \phi \cdot e A_\mu \phi - V[\phi] \] (6)
with the scalar triplet written in the form
\[ \phi = \phi^a t^a = \tilde{\phi} \cdot \tilde{t}, \] (7)
the symmetry breaking potential given by
\[ V[\phi] = \frac{\lambda}{4} \left( \tilde{\phi} \cdot \phi - \frac{\mu^2}{\lambda} \right)^2 \] (8)
and the covariant derivative defined as
\[ D_\mu \phi = \partial_\mu \phi + e A_\mu \phi. \] (9)

Concerning the field strength $F_{\mu\nu} = \tilde{F}_{\mu\nu} = e A_\mu \cdot \tilde{A}_\nu$, it is defined as
\[ \tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e A_\mu \wedge A_\nu \] (10)
In Born-Infeld theories, it is also convenient to define the canonically conjugated tensor,
\[ \tilde{G}_{\mu\nu} = -2 \frac{\partial \mathcal{L}_{\text{DBI}}}{\partial \tilde{F}^{\mu\nu}} \] (11)
(i) The equations of motion for $L_{DBI-Higgs}^{tr}$

When the usual trace operation "tr" is used, the DBI-Higgs Lagrangian reads

\[ L_{DBI-Higgs}^{tr} = \beta^2 \text{tr} \left( 1 - \sqrt{1 + \frac{1}{2} \frac{g_{\mu\nu} F_{\mu\nu}}{\beta^2} - \frac{1}{16 \beta^4} (F_{\mu\nu} \tilde{F}_{\mu\nu})^2} \right) + \frac{1}{2} D_{\mu} \tilde{\phi} \cdot D_{\nu} \tilde{\phi} - V[\phi]. \]  

(12)

The equations of motion take the form

\[ D_{\mu} \left( \frac{\tilde{F}_{\mu\nu} - \frac{1}{8 \beta^2} \left( \tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma} \right) \tilde{F}_{\mu\nu}}{\sqrt{1 + \frac{1}{2 \beta^2} F_{\mu\nu} F_{\mu\nu} - \frac{1}{64 \beta^4} (F_{\mu\nu} \tilde{F}_{\mu\nu})^2}} \right) = -e \tilde{\phi} \wedge D_{\nu} \tilde{\phi}, \]  

(13)

\[ D_{\mu} D_{\mu} \tilde{\phi} = \mu^2 \tilde{\phi} - \lambda \phi^2 \tilde{\phi}. \]  

(14)

(ii) The equations of motion for $L_{DBI-Higgs}^{Str}$

When the symmetric trace operation is used, the DBI-Higgs Lagrangian is defined as

\[ L_{DBI-Higgs}^{Str} = \beta^2 \text{Str} \left( 1 - \sqrt{1 + \frac{1}{2} \frac{g_{\mu\nu} F_{\mu\nu}}{\beta^2} - \frac{1}{16 \beta^4} (F_{\mu\nu} \tilde{F}_{\mu\nu})^2} \right) + \frac{1}{2} D_{\mu} \tilde{\phi} D_{\mu} \tilde{\phi} - V[\phi]. \]  

(15)

Then, the equations of motion read

\[ D_{ab\mu} \text{Str} \left( \frac{F_{\mu\nu} - \frac{1}{4 \beta^2} \left( F_{\rho\sigma} \tilde{F}^{\rho\sigma} \right) F_{\mu\nu}}{\sqrt{1 + \frac{1}{2 \beta^2} F_{\mu\nu} F_{\mu\nu} - \frac{1}{64 \beta^4} (F_{\mu\nu} \tilde{F}_{\mu\nu})^2}} \right) = -e (\phi \wedge D_{\nu} \phi)^a, \]  

(16)

\[ D_{\mu} D_{\mu} \tilde{\phi} = \mu^2 \tilde{\phi} - \lambda \phi^2 \tilde{\phi}. \]  

(17)

Note that in order to perform the trace operation in the resulting equations of motion, one has to expand the square root and then proceed to the explicit evaluation of traces. While the "normal" trace operation tr allows to reaccomodate the expansion as the square root appearing in (13), this is not the case for the symmetric trace. Then, one is left with equations of motion that correspond to a $1/\beta^2$ expansion. We just quote here the Lagrangian (15) and equations of motion for the gauge field (16) expanded to second order in $1/\beta^2$

\[ \mathcal{L}^{(2)} = -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{192 \beta^2} \left( (\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu})^2 + (\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu})^2 + 2(\tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma})(\tilde{F}^{\mu\nu} \tilde{F}^{\rho\sigma}) \right) + 2(\tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma})(\tilde{F}^{\mu\nu} \tilde{F}^{\rho\sigma}) \]  

(18)

\[ D_{\mu} \tilde{G}_{\mu\nu} = -e \tilde{\phi} \wedge D_{\nu} \tilde{\phi} \]  

(19)

where

\[ \tilde{G}_{\mu\nu} = \tilde{F}_{\mu\nu} - \frac{1}{24 \beta^2} \left( (\tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma}) \tilde{F}_{\mu\nu} + (\tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma}) \tilde{F}_{\mu\nu} + 2(\tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma}) \tilde{F}_{\mu\nu} + 2(\tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma}) \tilde{F}_{\mu\nu} \right) \]  

(20)
III. DYON SOLUTIONS

We consider the usual spherically symmetric Ansatz [16]-[18],
\begin{align}
\vec{A}_i(r) &= \frac{K(r)-1}{e} \vec{\Omega} \wedge \partial_i \vec{\Omega}, \\
\vec{A}_0(r) &= \frac{J(r)}{er} \vec{\Omega}, \\
\vec{\phi}(r) &= \frac{H(r)}{er} \vec{\Omega},
\end{align}
(21)
(22)
(23)

with
\begin{align}
\vec{\Omega} = \vec{\Omega}(\theta, \phi) = \frac{\vec{r}}{r}.
\end{align}
(24)

The appropriate boundary conditions for $K$, $J$ and $H$, \begin{align}
\lim_{r \to \infty} K(r) = 0, \quad \lim_{r \to \infty} \frac{1}{r} J(r) = M + \frac{b}{r}, \quad \lim_{r \to \infty} \frac{1}{r} H(r) = \frac{\mu e}{\sqrt{\lambda}}.
\end{align}
(25)

Here $M$ is a parameter with the dimensions of a mass which has to satisfy $M < e \mu / \sqrt{\lambda}$ to have an appropriate asymptotic behavior for $K(r)$ [18]. Concerning $b$, it determines, as we shall see, the electric charge. Concerning the conditions at the origin, we take
\begin{align}
K(0) = 1, \quad J(0) = 0, \quad H(0) = 0.
\end{align}
(26)

The electromagnetic $U(1)$ field strength $\mathcal{F}_{\mu\nu}$ is defined as usual [16] in the form
\begin{align}
\mathcal{F}_{\mu\nu} = \frac{\vec{\phi}}{|\vec{\phi}|} \cdot \left( \vec{F}_{\mu\nu} - \frac{1}{e|\vec{\phi}|^2} (D_\mu \vec{\phi} \wedge D_\nu \vec{\phi}) \right).
\end{align}
(27)

From (27) we define the $U(1)$ magnetic induction and electric field in the form
\begin{align}
B^i &= -\frac{1}{2} e^{ijk} \mathcal{F}_{jk}, \quad E^i = \mathcal{F}^{i0}.
\end{align}
(28)

Using Ansatz (21)-(24) one easily finds that
\begin{align}
B^i &= \frac{1}{er^2} x^i, \quad E^i = -\left( \frac{J(r)}{er} \right)' \frac{x^i}{r},
\end{align}
(29)

so that the magnetic flux is
\begin{align}
M = \int_{S^2} dS_i B^i = \frac{4\pi}{e},
\end{align}
(30)

It corresponds to that of a unit magnetic monopole located at the origin. Concerning the electric charge, it is defined as
\begin{align}
Q = \int_{S^2} dS_i E^i = \frac{4\pi b}{e},
\end{align}
(31)
In the case of a Born-Infeld theory, it is necessary to also define the electromagnetic projection of $G^a_{\mu\nu}$ which we shall call $\mathcal{G}_{\mu\nu}$. Following the same steps as those leading to $F_{\mu\nu}$ (see (27)), we start from

$$G^a_{\mu\nu} = \text{Tr} \left( \frac{1}{R} \left( F_{\mu\nu} - \frac{1}{2\beta^2} (\bar{F} \bar{F}) t^a \right) \right) \equiv \text{Tr} \left( O^a_{\mu\nu} F_{\rho\sigma} t^a \right) \tag{32}$$

Here Tr indicates any one of the two possible trace choices referred above and $R$ is defined as

$$R = \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu} F^{\mu\nu})^2} \tag{33}$$

Now, inspired in (27) we consider the shift

$$\bar{F}_{\mu\nu} \rightarrow \bar{F}_{\mu\nu} - \frac{1}{e|\phi|^2} D_\mu \bar{\phi} \wedge D_\nu \bar{\phi} \tag{34}$$

in (32) and then project the result on the $\bar{\phi} = \bar{\phi}/|\phi|$ direction. The answer is

$$\mathcal{G}_{\mu\nu} = \text{Tr} \left( O^a_{\mu\nu} \left( \bar{F}_{\rho\sigma} - \frac{1}{e|\phi|^2} D_\rho \bar{\phi} \wedge D_\sigma \bar{\phi} \right) \cdot \bar{\phi}/|\phi| \right) \tag{35}$$

From $\mathcal{G}_{\mu\nu}$ we now define the magnetic field $H_i$ and the electric induction $D_i$,

$$H^i = -\frac{1}{2} \varepsilon^{ijk} G_{jk}, \quad D^i = \mathcal{G}^{i0} \tag{36}$$

(i) The solution for $\mathcal{L}_{DBI-Higgs}^{\text{tr}}$

Inserting Ansatz (21)-(23) into the eqs. of motion (13)-(14) one gets

$$\rho^2 K'' = K(K^2 - J^2 + RH^2 - 1) + \frac{R}{2} \left( K' - \frac{1}{2\beta^2 \rho^3} P' J K \right) + \frac{\rho}{2\beta^2} J K \left( \frac{P'}{\rho^2} \right)'$$

$$\rho^2 H'' = 2H K^2 + \lambda H (H^2 - \rho^2)$$

$$\rho^2 J'' = 2JK^2 + \frac{R}{2} \left( \rho J' - J + \frac{1}{2\beta^2 \rho^2} P' (K^2 - 1) \right) - \frac{\rho}{2\beta^2} \left( \frac{P'}{\rho^2} \right)' (K^2 - 1) \tag{37}$$

Here we have

$$R^2 = 1 + \frac{1}{2\beta^2 \rho^4} ((K^2 - 1)^2 + 2\rho^2 K'^2 - 2J^2 K^2 - (J - \rho J')^2) - \frac{1}{4\beta^4 \rho^4} P'^2$$

$$P = \frac{J}{\rho} (K^2 - 1) \tag{38}$$

and we have used dimensionless variables and parameters defined as
\[ \rho = \frac{e^\mu r}{\sqrt{\lambda}}, \quad \hat{\lambda} = \lambda/e^2, \quad \hat{\beta} = \frac{\beta \lambda}{e \mu^2}, \quad \hat{M} = \frac{\sqrt{\lambda}M}{\mu} \]  \hspace{1cm} (39)

Concerning the energy density one has

\[ E = \int d^3x \Theta^{00} = \frac{4\pi \mu}{e \sqrt{\lambda}} \int d\rho \rho^2 \left( \frac{1}{\rho^4 R} \left( 2J^2 K^2 + (\rho J')^2 + \frac{1}{2\hat{\beta}^2} P^2 \right) + \right. \\
\left. 2\hat{\beta}^2 (R - 1) + \frac{\hat{\lambda}}{4} \left( \frac{H^2}{\rho^2} - 1 \right)^2 + \frac{1}{2\rho^4} \left( (H - \rho H')^2 + 2H^2 K^2 \right) \right) \]  \hspace{1cm} (40)

To obtain a detailed profile of the dyon solutions we have solved numerically the differential equations (37) employing a relaxation method for boundary value problems [26]. Such a method determines the solution by starting with an initial guess and improving it iteratively. The natural initial guess was the exact Prasad-Sommerfield solution [27] which corresponds to \( \hat{\lambda} = 0 \) and \( \hat{\beta} \to \infty \).

For \( \hat{\beta} \gtrsim 10 \), the solutions to eqs. (37) do not differ appreciably from the Julia-Zee dyon solution [18]. As \( \hat{\beta} \) decreases, the solution changes slowly: the dyon radius decreases and the (radial) electric and magnetic fields, \( \vec{E} \) and \( \vec{H} \) respectively, concentrate at the origin. Some of the solution profiles are depicted in figures (1)-(2). It should be noted that in the limit \( \hat{M} \to 0 \) we recover the DBI pure monopole solutions [3].

As it happens for other soliton-like solutions in DBI-Higgs theories [7], [9], there is a critical value of \( \hat{\beta} \) which we call \( \hat{\beta}_c \) such that for \( \hat{\beta} \leq \hat{\beta}_c \) the dyon solution ceases to exist. This can be clearly seen in figure 3, where the energy is plotted as a function of \( \hat{\beta} \). As \( \hat{\beta} \) approaches \( \hat{\beta}_c \) the derivative of the energy with respect to \( \hat{\beta} \) diverges. We have found that \( \hat{\beta}_c \sim 0.55 \) and it does not depend on \( \hat{\lambda} \) nor on \( \hat{M} \). This yields a critical dyon radius which is 0.85 of the standard Yang-Mills dyon radius (which can be recovered in the \( \beta \to \infty \) limit) for \( \hat{M} = \hat{\lambda} = 0.5 \).

The existence of \( \hat{\beta}_c \) is not a byproduct of our numerical method but a genuine effect. As we have thoroughly discussed in [9], the origin of this phenomenon could be traced back to the existence, in DBI theories, of a second dimensionful parameter \( \beta \) (\( [\beta] = \mu^2 \)) which enters together with \( \mu \) in the minimization of the energy. Indeed, by using approximate solutions, we have seen in [9] for pure monopoles (and the same analysis could be done for dyons) that there exist a region in the parameter space defined by the dimensionless combinations \( \hat{\lambda} \) and \( \hat{\beta} \), for which the energy has no minima. This region precisely corresponds to small values of \( \hat{\beta} \). Of course for the Yang-Mills-Higgs system, where the second dimensionless parameter is absent, solutions exist in the whole \( \hat{\lambda} \) range.

One can rephrase the analysis above by noting that when \( \hat{\beta} \) decreases, the dyon radius also decreases, as can be seen in figures [1,4]. The existence of a critical \( \hat{\beta} = \hat{\beta}_c \) then corresponds to the existence of a minimal radius below which the dyon (or monopole) cannot exist. This is reminiscent of an analogous phenomenon that takes place for self-gravitating monopoles [28]- [32]: they show an instability for sufficiently strong gravitational coupling, manifesting itself as an extremal blackhole in its exterior region and a more involved solution inside. In other words, non-linearities introduced by the DBI action have a similar effect as that produced when the coupling to gravity becomes relevant.
(ii) The solution for \( \mathcal{L}_{DBI-Higgs}^{Str} \)

As stated above, in order to handle the symmetric trace Lagrangian, one has to keep a finite number of terms in the expansion of the DBI square root. Then, to order \( 1/\beta^2 \), one can insert in the eqs. of motion (17)-(19) the Ansatz (16)-(18). We shall not display the resulting equations but briefly discuss their numerical solution.

For \( \hat{\beta} \geq 2 \) the solutions differ less than 1% from those arising when the usual trace ("tr") operation is considered. The profile of the solutions are indistinguishable from the solid line curves of figures [4-6]. As \( \hat{\beta} \) decreases, the dyon radius decreases with the same rate as in the usual trace case. This signals the existence of a \( \hat{\beta}_c \) also in the symmetric trace case. However, since the equations of motion are valid to order \( 1/\hat{\beta}^2 \), our analysis cannot be reliable for small \( \hat{\beta} \) and the region where one expects to find \( \hat{\beta}_c \) lies outside the validity range of our approximation.

IV. A \( \theta \) TERM

When the Yang-Mills-Higgs Lagrangian includes a CP violating \( \theta \)-term, a remarkable effect takes place: a dyon solution with quantum electric charge \( q = n_e e \) and magnetic charge \( g = 4\pi/e \) shifts its electric charge according to the relation (13)

\[
q = n_e e + \frac{e\theta}{2\pi}
\]

Here \( e \) is the unit electric charge, \( n_e \) an integer and a unit magnetic charge has been considered. Relation (13) was originally obtained considering an \( SU(2) \) gauge theory spontaneously broken to \( U(1) \) by the vacuum expectation value of a Higgs triplet, using semiclassical arguments and also by canonical methods. In this last approach, one defines the operator \( N \) that generates gauge transformations around the \( U(1) \) (electromagnetic) surviving symmetry and then imposes as an operator statement

\[
\exp(2\pi i N) = I
\]

Now, when the Lagrangian includes a CP violating term of the form

\[
\Delta L = \theta \frac{e^2}{32\pi^2} \text{tr} \tilde{F}_{\mu\nu} F^{\mu\nu}
\]

one can see that the condition (12) implies formula (13). For a general dyon solution, one gets

\[
q = n_e e + \frac{e\theta}{2\pi} \tilde{n}_m
\]

with the magnetic charge \( g \) expressed as a multiple of the unit 't Hooft-Polyakov charge,

\[
g = \frac{4\pi}{e} \tilde{n}_m
\]

In this section we analyze whether a similar phenomenon can take place in the non-Abelian Dirac-Born-Infeld theory we have described above.
Let us first recall that in Yang-Mills theory, a $\theta$-term can be generated by the action of an $SO(2)$ rotation followed by a scaling of the field strength \[\text{(19)}\]. Indeed, if one considers
\[
F_{\mu\nu} \rightarrow \frac{1}{\sqrt{\cos 2\alpha}} \left( \cos \alpha F_{\mu\nu} - \sin \alpha \tilde{F}_{\mu\nu} \right)
\]
then, a $\theta$ term of the form \[\text{(13)}\] can be generated from a $(-1/4)\text{tr}F_{\mu\nu}F^{\mu\nu}$ term, with $\alpha$ related to $\theta$ through the formula
\[
\tan(2\alpha) = \frac{e^{2\theta}}{8\pi^2}
\]
We shall now see that the same transformation changes the Dirac-Born-Infeld Lagrangian in such a way that when one computes the electric charge of dyons, Witten effect takes place exactly as in Yang-Mills theory.

We start then by analysing the effect of transformations \[\text{(16)}\] in the DBI action. For definiteness, we shall consider the case in which the DBI action is defined using the symmetric trace. Then, performing the change \[\text{(16)}\] in \[\text{(1)}\], one gets
\[
L_\theta^{DBI} = \beta^2 \text{Str} \left( 1 - \sqrt{-\det \left( g_{\mu\nu} + \frac{1}{\beta \sqrt{\cos 2\alpha}} \left( \cos \alpha F_{\mu\nu} - \sin \alpha \tilde{F}_{\mu\nu} \right) \right) } \right)
\]
which can be also written in the form
\[
L_\theta = \beta^2 \text{Str} \left( 1 - \sqrt{1 + \frac{1}{2\beta^2} \left( F_{\mu\nu}F^{\mu\nu} - \frac{e^{2\theta}}{8\pi^2} \tilde{F}_{\mu\nu}F^{\mu\nu} \right) - \frac{1}{16\beta^4} \left( \tilde{F}_{\mu\nu}F^{\mu\nu} + \frac{e^{2\theta}}{8\pi^2} F_{\mu\nu}F^{\mu\nu} \right)^2 } \right)
\]
In contrast with the case of YM theory, where the addition of a $\theta$-term does not change the eqs. of motion ($\tilde{F}F$ is a surface term), here, rotation \[\text{(16)}\] leads to eqs. of motion that differ from the $\theta = 0$ ones. Then, one has to see whether dyon solutions still exist for $\theta \neq 0$. Now, studying the modified system of eqs. of motion to the order worked out in the $\theta = 0$ case (i.e., up to the order $1/\beta^2$) one can easily see that its solutions coincide (if one rescales $\beta$ conveniently) with the $\theta = 0$ ones.

Having found that dyon solutions exist when a $\theta$ term is present, we are now ready to explicitly write the operator $N$ that generates the transformations associated with the surviving symmetry. One has to consider $U(1)$ transformations along the Higgs field direction $\phi$,
\[
\delta_{U(1)} \phi = 0 \\
\delta_{U(1)} A_\mu = \frac{1}{e} D_\mu (e \tilde{\phi})
\]
Then, the corresponding conserved current picks a contribution solely from the DBI Lagrangian,
\[ J_{\mu}^{U(1)} = \frac{\partial L_{DBI}^0}{\partial (\partial^\mu A^a)} \delta_{U(1)} A^{a\mu} \]  

(51) 

and the conserved charge \( N \) then takes the form

\[ N = \int d^3 x J_0^{U(1)} = \frac{1}{e} \int d^3 x \partial_i \left( G_{0i}^a \phi^a \right) - \frac{1}{e} \int d^3 x \partial_i \left( D^i G_{0i} \right)^a \]  

(52) 

where

\[ G_{\mu\nu}^a = \text{Str} \left( t^a \frac{1}{R_\theta} \left( F_{\mu\nu} - \frac{e^2 \theta}{8\pi^2} \tilde{F}_{\mu\nu} - \frac{1}{4\beta^2} (F_{\rho\sigma} \tilde{F}^{\rho\sigma} + \frac{e^2 \theta}{8\pi^2} F_{\rho\sigma} F^{\rho\sigma}) (\tilde{F}_{\mu\nu} + \frac{e^2 \theta}{8\pi^2} F_{\mu\nu}) \right) \right) \]  

(53) 

with

\[ R_\theta = \sqrt{1 + \frac{1}{2\beta^2} \left( F_{\mu\nu} F^{\mu\nu} - \frac{e^2 \theta}{8\pi^2} \tilde{F}_{\mu\nu} F^{\mu\nu} \right) - \frac{1}{16\beta^4} \left( \tilde{F}_{\mu\nu} F^{\mu\nu} + \frac{e^2 \theta}{8\pi^2} F_{\mu\nu} F^{\mu\nu} \right)^2} \]  

(54) 

It is important to note that

\[ \lim_{r \to \infty} G_{\mu\nu} = F_{\mu\nu} - \frac{e^2 \theta}{8\pi^2} \tilde{F}_{\mu\nu} \]  

(55) 

Now, use of the equations of motion makes the second term in the r.h.s. of (52) vanish. Then,

\[ eN = \int_{S^2_\infty} dS^i G_{0i}^a \tilde{\phi}^a = \int_{S^2_\infty} dS^i \left( F_{0i}^a - \frac{e^2 \theta}{8\pi^2} \tilde{F}_{0i}^a \right) \tilde{\phi}^a \]  

(56) 

or, using the magnetic charge \( M \) and electric charge \( Q \) defined by eqs. (30) and (31),

\[ eN = Q - \frac{\theta e^2}{8\pi^2} M \]  

(57) 

Condition (12) implies that \( N \) has to have integer eigenvalues \( n_e \). Calling \( q \) and \( g \) the eigenvalues of the electric and magnetic charge operators \( Q \) and \( M \) respectively, we then have

\[ q = en_e + \frac{\theta e^2}{8\pi^2} g \]  

(58) 

Now, we have seen that the DBI theory with a \( \theta \) term admits monopole solutions with unit magnetic charge \( g = (4\pi/e) \) so that formula (53) coincides with (14) obtained for the Georgi-Glashow model if one considers a solution with \( n_m \) units of magnetic charge. We then conclude that for the DBI model with \( \theta \) term the basic formulae (14) and (15) hold. One can then introduce the complex parameter \( \tau \) and, from the resulting discrete two-dimensional lattice, infer the existence of a discrete \( SL(2, \mathbb{Z}) \) symmetry. Of course to thoroughly study electric-magnetic duality, one should at this point consider the supersymmetric extension of DBI models but this goes beyond the scope of the present investigation.
V. SUMMARY AND CONCLUSIONS

We have seen that spontaneously broken Dirac-Born-Infeld $SU(2)$ gauge theory admits dyon solutions which, in the range $\hat{\beta} > \hat{\beta}_c$ behave as Julia-Zee dyons in Yang-Mills theory. As the absolute field parameter $\hat{\beta}$ decreases, the radius of the dyon also decreases so that the magnetic and electric field become more and more concentrated. For $\hat{\beta} \leq \hat{\beta}_c$ we have seen that the dyon solution ceases to exist, much in the way self-gravitating monopole and dyon solutions become unstable when coupling to gravity is sufficiently strong: in both cases there is a minimum radius below which the solution collapses.

Once the existence of monopole and dyon solutions is proven, it is natural to consider whether the analogue of Witten effect takes place in theories in which the gauge field dynamics is dictated by a DBI action. To study this issue, one has to include a theta term which, in the present case, arises naturally after an $SO(2)$ rotation in $F_{\mu \nu}$ is performed. Remarkably, although this shift greatly complicates the DBI dynamics, one can prove, using the Noether method, that the dyon electric charge is shifted exactly in the same way as in the Yang-Mills case. This makes natural to study the issue of duality in the supersymmetric extension of DBI theory [23], [12]. We hope to discuss this problem in a future work.

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REFERENCES

[1] See J. Polchinski, *Tasi Lectures on D-Branes*, hep-th/9611050 and references therein.
[2] G. Gibbons, Nucl. Phys. B 514 (1998) 603.
[3] C.G. Callan and J.M. Maldacena, Nucl. Phys. B 513 (1998) 198.
[4] J.P. Gauntlett, J. Gomis and P.K. Townsend, JHEP 01 (1998) 003.
[5] K. Shiraishi and S. Irenzaki, Int. Jour. of Mod. Phys. A6 (1991) 2635.
[6] A. Nakamura and K. Shiraishi, Hadronic Jour. 14 (1991) 369.
[7] E. Moreno, C. Núñez and F.A. Schaposnik, Phys. Rev. D58 (1998) 025015.
[8] K.G. Savvidy, hep-th/9810163.
[9] N. Grandi, E. Moreno and F.A. Schaposnik, Phys. Rev. D59 (1999) 125014.
[10] P.K. Tripathy, hep-th/9904186.
[11] J.H. Park, hep-th/9902081.
[12] H. Christiansen, C. Núñez and F.A. Schaposnik, Phys.Lett. B441 (1998) 185.
[13] E. Witten, Phys. Lett. B86 (1979) 283.
[14] See for example L. Alvarez-Gaumé and F. Zamora, in Trends in Theoretical Physics, eds. H.Falomir et al, AIP, New York, 1998, p.1.
[15] A.A. Tseytlin, Nucl. Phys. B501 (1997) 41.
[16] G. ‘t Hooft, Nucl. Phys. B 79 (1974) 276.
[17] A.M. Polyakov, JETP Lett. 20 (1974) 194.
[18] B. Julia and A. Zee, Phys. Rev. D11 (1975) 2227.
[19] G. Gibbons and D.A. Rasheed, Nucl.Phys.B454 (1995) 185.
[20] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, Nucl. Phys. B280 (1987) 599.
[21] A. Hashimoto and W. Taylor IV, Nucl. Phys. 503 (1997) 193.
[22] A. Hashimoto, Phys. Rev. D57 (1998) 6420.
[23] D. Brecher and M.J. Perry, Nucl. Phys. B527 (1998) 121.
[24] D. Brecher, Phys. Lett. B442 (1998) 117.
[25] S. Gonorazky, F.A. Schaposnik and G. Silva, Phys. Lett.B449 (1999) 187.
[26] W.H. Press, S.A. Teukolsky and W.T. Vetterling, *Numerical Recipes: The art of Scientific Computing*, Cambridge University Press, (1992).
[27] M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett 35 (1975) 760.
[28] K. Lee, V.P. Nair and E.J. Weinberg, Phys. Rev. D45 (1992) 2751.
[29] M.E. Ortiz Phys. Rev. D45 (1992) R2586.
[30] P. Breitenlohner, P. Forgács and D. Maison, Nucl. Phys. B383 (1992) 357; *ibid* B442 (1995) 126.
[31] D. Maison, to appear in the Proceedings of the Pacific Conference on Gravitation and Cosmology, Seoul, 1996, gr-qc/9605053.
[32] A. Lue and E.J. Weinberg, hep-th/9905223.
FIG. 1. Plot of $K(r)$, $J(r)/r$ and the Higgs field $H(r)/r$ (in dimensionless variables) for the dyon solution with $\hat{\lambda} = 0.5$ and $\hat{M} = 0.5$. The solid line corresponds to the solution with $\hat{\beta} = 10$ and the dashed line corresponds to the solution with $\hat{\beta} = 0.6$. 
FIG. 2.  Plot of $K(r)$, $J(r)/r$ and the Higgs field $H(r)/r$ (in dimensionless variables) for the dyon solution with $\hat{\lambda} = 0.5$ and $\hat{M} = 0.8$. The solid line corresponds to the solution with $\hat{\beta} = 10$ and the dashed line corresponds to the solution with $\hat{\beta} = 0.6$. 
FIG. 3. Energy of the dyon configuration as a function of $\hat{\beta}$ for $\hat{M} = 0.8$ and $\hat{\lambda} = 0.5$ (Similar curves are obtained for other values of $\hat{M}$ and $\hat{\lambda}$).