A Variation on Heawood List-Coloring for Graphs on Surfaces

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Abstract

We prove a variation on Heawood-list-coloring for graphs on surfaces, modeled on Thomassen's planar 5-list-coloring theorem. For $\epsilon > 0$ define the Heawood number to be $H(\epsilon) = \lfloor (7 + \sqrt{24 \epsilon + 1})/2 \rfloor$. We prove that, except for $\epsilon = 3$, every graph embedded on a surface of Euler genus $\epsilon > 0$ with a distinguished face $F$ can be list-colored when the vertices of $F$ have $(H(\epsilon) - 2)$-lists and all other vertices have $H(\epsilon)$-lists unless the induced subgraph on the vertices of $F$ contains $K_{H(\epsilon)-1}$.

Keywords: list-coloring, graph embeddings on surfaces, Heawood number

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1 Introduction

Thomassen's celebrated 5-list-coloring theorem for planar graphs, with its proof from "The Book" [1], establishes a stronger result [14]. It proves that if $G$ is a graph embedded in the plane, if $F$ is a face of the embedding, and if the vertices of $F$ have 3-lists and the remaining vertices have 5-lists, then $G$ is list-colorable. We investigate the extent to which an analogous theorem holds for graphs embedded on nonplanar surfaces. The same result does not hold with 3-lists on one face $F$ and 5-lists on the others, for example, when the induced subgraph on the vertices of $F$ contains a complete graph on four or more vertices. We also show that this result does not hold for locally planar graphs, graphs embedded with all noncontractible cycles suitably long, which are 5-list-colorable [3].

On the other hand every graph on a surface of Euler genus $\epsilon > 0$ can be $H(\epsilon)$-list-colored (see [7] for history), as well as $H(\epsilon)$-colored [6], where the
Heawood number is given by $H(\epsilon) = \left\lfloor \frac{7 + \sqrt{24 \epsilon + 1}}{2} \right\rfloor$. We prove the following analogue of Thomassen’s theorem.

**Theorem 1.1** Let $\epsilon \geq 1$, $\epsilon \neq 3$. Let $G$ be embedded on a surface of Euler genus $\epsilon$ with a distinguished face $F$. Let the vertices of $F$ have $(H(\epsilon) - 2)$-lists and those of $V(G) \setminus V(F)$ have $H(\epsilon)$-lists. Then $G$ is list-colorable unless $G$ contains $K_{H(\epsilon)-1}$ with all its vertices lying on $F$.

In [2] we proved a similar list-coloring result when there are $(H(\epsilon) - 1)$-lists on suitably far apart faces and $H(\epsilon)$-lists elsewhere.

**Proposition 1.2** For infinitely many values of $\epsilon \geq 1$, there is a surface of Euler genus $\epsilon$ and a graph $G$ such that $G$ embeds on that surface with all vertices on one face $F$, $G$ does not contain $K_{H(\epsilon)-2}$, and $G$ is not $(H(\epsilon) - 3)$-list-colorable.

Thus in this context the bound of $H(\epsilon) - 2$ for lists on the vertices of $F$ in Thm. 1.1 is best possible.

**2 Chromatic and topological background**

Recall that a graph can be $k$-colored if one of $k$ colors can be assigned to each vertex so that adjacent vertices receive different colors. Given a graph with a list of colors $L(v)$ for each vertex $v$, the graph can be $L$-list-colored if an element from $L(v)$ can be selected for each $v$ that gives a coloring of $G$. A graph is $k$-list colorable if whenever $|L(v)| \geq k$ for each $v$, the graph can be $L$-list-colored. All chromatic and topological definitions and basic results can be found in [7, 9].

Our results use the next two important theorems on list-coloring; if $v$ is a vertex of $G$, let $\deg(v)$ denote the degree of $v$ in $G$.

**Theorem 2.1** [15, 4] a. If $\Delta$ is the maximum degree of a graph $G$, then $G$ can be $\Delta$-list-colored except possibly when $G$ is an odd cycle or a complete graph.

b. If $G$ is a graph with lists $L$ such that $|L(v)| = \deg(v)$ for each vertex $v$, then $G$ can be $L$-list-colored except possibly when $G$ is (only) 1-connected and each block (i.e., each maximal 2-connected subgraph) is an odd cycle or a complete graph.

The use of "possibly" in the theorem means that there are lists which will prevent list-colorability; these preventing lists are characterized [15, 4], but are not needed explicitly here.

Let $L$ give lists for each vertex of a graph $G$. Then $G$ is $L$-critical if $G$ is
not $L$-list-colorable, but every proper subgraph of $G$ is $L$-list-colorable. When
$L(v) = \{1, 2, \ldots, k - 1\}$ for each vertex, $G$ is said to be $k$-critical.

**Theorem 2.2** [8]. Let $G$ be a graph with $n$ vertices and $e$ edges that does not
contain $K_4$, and let $L$ be lists for $G$ with $|L(v)| = k - 1$ for every vertex $v$
where $k \geq 4$. If $G$ is $L$-critical, then $2e \geq (k - 1)n + k - 3$.

Let $S_\epsilon$ denote a surface of Euler genus $\epsilon$. We use the following facts for a
multigraph $G$ embedded on $S_\epsilon$, which are derived from Euler's formula:

$$n - e + f \geq 2 - \epsilon$$

and

$$e \leq 3n + 3(\epsilon - 2),$$

when $G$ has $n$ vertices, $e$ edges, $f$ faces and, for the second inequality, when
each face has at least three boundary edges.

From the Heawood chromatic bound $H(\epsilon)$, it follows that the largest
complete graph that could possibly embed on $S_\epsilon$ is $K_{H(\epsilon)}$. In addition, the least
Euler genus $\epsilon$ for which $K_n$ embeds on $S_\epsilon$ is given by the inverse function

$$\epsilon = \frac{(n - 3)(n - 4)}{6}$$

(except that $K_7$ only embeds on the torus and not on the Klein
bottle, the two surfaces of Euler genus 2). From this we see that, given $\epsilon > 0$,
$K_{H(\epsilon)}$ is the largest complete graph that embeds on $S_\epsilon$ for

$$\left\lceil \frac{(H(\epsilon)-3)(H(\epsilon)-4)}{6} \right\rceil \leq \epsilon \leq \left\lfloor \frac{(H(\epsilon)-2)(H(\epsilon)-3)}{6} \right\rfloor - 1.$$

For ease of future computation, we spell out these bounds explicitly.

**Lemma 2.3** [2]. For $\epsilon > 0$, set $i = \left\lfloor \frac{H(\epsilon)-3}{3} \right\rfloor$ so that $H(\epsilon) = 3i + 3$, $3i + 4$ or
$3i + 5$ with $i \geq 1$. Then $K_{H(\epsilon)}$ is the largest complete graph that embeds on $S_\epsilon$ for
the following values of $\epsilon$:

a) If $H(\epsilon) = 3i + 3$, then

$$\frac{3i^2 - i}{2} \leq \epsilon \leq \frac{3i^2 + i - 2}{2}.$$  

b) If $H(\epsilon) = 3i + 4$, then

$$\frac{3i^2 + i}{2} \leq \epsilon \leq \frac{3i^2 + 3i + 1}{2}.$$  

c) If $H(\epsilon) = 3i + 5$, then

$$\frac{3i^2 + 3i + 2}{2} \leq \epsilon \leq \frac{3i^2 + 5i + 1}{2}.$$  

The one exception is the Klein bottle, with $\epsilon = 2$, $H(2) = 7$, $i = 1$, on which $K_6$
is the largest embedding complete graph.

By Lemma 2.3 when $H(\epsilon) \equiv 0, 2 \pmod{3}$ there are $i$ values of $\epsilon$ for which $K_{H(\epsilon)}$
is the largest complete graph embedding on $S_\epsilon$, and when $H(\epsilon) \equiv 1 \pmod{3}$,
there are $i + 1$.

We call the cases when $H(\epsilon) = 3i + 4, i \geq 1$, and $\epsilon = \frac{3i^2 + 3i}{2},$ the Special
Cases for surfaces and their largest embedding complete graph since only in
these cases it is possible numerically, from Euler's formula, to embed \( K_{H(e)+1} - E \), the complete graph on \( H(e) + 1 \) vertices minus one edge \( E \). When \( i = 1 \), Ringel proved [10] that \( K_8 - E \) does not embed on \( S_3 \), but when \( i > 1 \) and \( H(e) \equiv 1, 4, 10 \) (mod 12) he proved that \( K_{H(e)+1} - E \) does embed in these Special Cases. He left unresolved the other cases when \( H(e) \equiv 7 \) (mod 12) [11, p. 88].

3 Main result

When \( G \) is embedded on \( S_\varepsilon \), \( \varepsilon \geq 0 \), with a distinguished face \( F \) and contains \( K_{H(e)-1} \) with all its vertices lying on \( F \) and each having an \( (H(e) - 2) \)-list, then clearly \( G \) cannot be list-colored, and we say \( G \) contains an \( F \)-bad \( K_{H(e)-1} \). In the following \( G_F \) denotes the induced subgraph on \( V(F) \). For graphs \( G_1 \) and \( G_2 \), \( G_1 + G_2 \) denotes the join of \( G_1 \) with \( G_2 \).

Lemma 3.1. For all \( \varepsilon > 0 \), Theorem 1.1 holds for all graphs with \( n \) vertices, \( n \geq H(e) \), that embed on \( S_\varepsilon \).

Proof. We assume \( G \) does not contain an \( F \)-bad \( K_{H(e)-1} \). If \( n \geq H(e) - 2 \), then the graph can be \( (H(e) - 2) \)-list-colored. When \( n = H(e) - 1 \), if \( G = K_{H(e)-1} \), then by assumption it contains a vertex with an \( H(e) \)-list, all others with at least an \( (H(e) - 2) \)-list, and so can be list-colored. Otherwise \( G \) is a proper subgraph of \( K_{H(e)-1} \) with \( \Delta \leq H(e) - 2 \) and so can be list-colored by Thm. 2.1.

If \( n = H(e) \), suppose at most \( H(e) - 1 \) vertices lie on \( F \). We can list-color \( G_F \) since its maximum degree is at most \( H(e) - 2 \) and \( G_F \) does not contain \( K_{H(e)-1} \). In addition, for \( G, \Delta \geq H(e) - 1 \), the vertices of \( V(G) \setminus V(F) \) each have an \( H(e) \)-list, and thus the list-coloring of \( G_F \) extends to \( G \). Otherwise all \( H(e) \) vertices lie on \( F \). We know that for each vertex \( x \) on \( F \), \( G \setminus \{x\} \) is not the complete graph \( K_{H(e)-1} \) so that there are two vertices \( y, z \), distinct from \( x \), that are not adjacent. Applying this same fact to \( G \setminus \{y\} \) shows that \( G \) is a subgraph of \( K_{H(e)} \) minus at least two edges. Suppose two missing edges share a common vertex \( x \). Then \( G \setminus \{x\} \) can be list-colored since it is not \( K_{H(e)-1} \). In \( G \), \( x \) is adjacent to at most \( H(e) - 3 \) vertices and so the list-coloring extends to \( x \). Thus we assume \( G \) is a subgraph of \( K_{H(e)} \setminus \{ab, cd\} \), defined to be \( K_{H(e)} \) minus two independent edges, \( ab \) and \( cd \) with \( a, b, c, \) and \( d \) distinct vertices. \( K_{H(e)} \setminus \{ab, cd\} \) is the join of the 4-cycle \( (a, c, b, d) \) with \( K_{H(e)-4} \). \( K_{H(e)-4} \) can be \( (H(e) - 2) \)-list-colored. For \( x = a, b, c, d \), reset \( L(x) \) to be its original list minus the colors on \( K_{H(e)-4} \) so that each \( L(x) \) becomes a list of size at least 2.
Then the 4-cycle \((a, c, b, d)\) can always be list-colored by Thm. 2.1, and \(K_{H(e)} \setminus \{ab, cd\}\) is list-colorable, as is every possible subgraph. □

**Lemma 3.2.** If \(G\) embeds on \(S_\infty\), \(\epsilon > 0\), with vertices of degree \(H(\epsilon) - 2\) and \(H(\epsilon)\) and these have, respectively, lists of size \(H(\epsilon) - 2\) and \(H(\epsilon)\), if all of the former vertices lie on one face \(F\), and if the induced subgraph on \(V(F)\) does not contain \(K_{H(\epsilon)-1}\), then \(G\) can be list-colored.

**Proof.** If \(G = K_{H(\epsilon)-1}\), by assumption at least one vertex does not lie on \(F\) so that \(G\) can be list-colored. Otherwise by Thm. 2.1 \(G\) can be list-colored unless it is (only) 1-connected, contains at least two blocks, and possibly has coloring-preventing lists. We consider the block-cutvertex tree of \(G\), \(BCT(G)\), which consists of a vertex for each cutvertex and a vertex for each block with a cutvertex adjacent to a block-vertex if and only if the cutvertex lies on the block. Let \(P^*\) be a maximal path in \(BCT(G)\) and let \(v^*, w^*\) be the two endpoints of \(P^*\). Then \(v^*\) and \(w^*\) correspond to two blocks of \(G\), say \(G_{v^*}\) and \(G_{w^*}\). \(G_{v^*}\) has one vertex that is a cutvertex of \(G\), say \(v'\), and \(\deg(v') > \deg(v)\) for every \(v \neq v'\) in \(G_{v^*}\). Thus \(\deg(v') = H(\epsilon)\) and \(\deg(v) = H(\epsilon) - 2\) for every \(v \neq v'\) in \(G_{v^*}\). Thus \(G_{v^*}\) and similarly \(G_{w^*}\) are the complete graphs \(K_{H(\epsilon)-1}\). (Since \(H(\epsilon) \geq 6\), these blocks cannot be odd cycles.) Every vertex of \(G_{v^*}\), except for \(v'\), must lie on \(F\), and similarly every vertex of \(G_{w^*}\), except for \(w'\), its cutvertex, must lie on \(F\), and \(v' \neq w'\) (since \(2(H(\epsilon) - 2) > H(\epsilon) = \deg(v') = \deg(w')\)). Thus \(G\) contains two disjoint copies of \(K_{H(\epsilon)-1}\), and in fact it must contain at least three disjoint copies of \(K_{H(\epsilon)-1}\). Vertex \(v'\) is incident with two additional edges besides the \(H(\epsilon) - 2\) of \(G_{v^*}\). These two edges must lie on an odd cycle, for if instead each was an incident \(K_2\), then there would be a longer path than \(P^*\) in \(BCT(G)\). Thus the vertex \(z^*\) of \(BCT(G)\) corresponding to the odd cycle has degree at least three in \(BCT(G)\), and tracing a longest path from \(z^*\), that does not lie on \(P^*\), yields another leaf of \(BCT(G)\) that represents another (and disjoint) copy of \(K_{H(\epsilon)-1}\), say \(G_{z^*}\), with cutvertex \(x^*\) and with all other vertices of degree \(H(\epsilon) - 2\) and lying on \(F\).

Then \(G\) has at least \(3(H(\epsilon) - 1)\) vertices and at least \(3(H(\epsilon) - 1)(H(\epsilon) - 2)/2 + 3\) edges, and since \(\epsilon \leq 3n + 3(\epsilon - 2)\), we have \(3(H(\epsilon) - 1)(H(\epsilon) - 2)/2 + 3 \leq 9(H(\epsilon) - 1) + 3(\epsilon - 2)\), which is false for \(\epsilon \geq 5\) with \(H(\epsilon) \geq 9\).

Recall that the Euler genus of a connected graph equals the sum of the genera of its blocks [13]. Suppose \(\epsilon = 1\) with \(H(\epsilon) = 6\). Then a graph that
contains three copies of $K_5$ has Euler genus at least 3 so that three copies of $K_5$
cannot embed on the projective plane. For $\epsilon = 2$ with $H(\epsilon) = 7$, a graph
that contains three copies of $K_6$ has Euler genus at least 3 and so does not embed on
the torus or Klein bottle. When $\epsilon = 4$ with $H(\epsilon) = 8$, a graph that contains three
copies of $K_7$ has Euler genus at least 6 and so does not embed on the double
torus or the sphere with four crosscaps. When $\epsilon = 3$, three copies of $K_6$ can
embed on $S_k$, but by Euler's formula they cannot embed, each with 5 of their 6
vertices on the same face $F$. □

**Proposition 3.3.** Theorem 1.1 holds for all $\epsilon > 0$ provided $\epsilon \neq \frac{3^i + 3i}{2}$ for some
$i \geq 1$.

**Proof.** The proof is by induction on $n$, the number of vertices of $G$, and holds
for $n \leq H(\epsilon)$ by Lemma 3.1. We suppose $n \geq H(\epsilon) + 1$. Denote the number of
edges of $G$ by $e$ and the number of faces of the embedding by $f$.

Suppose $G$ contains a vertex $v$ of degree at most $H(\epsilon) - 3$. If $v$ does not
lie on $F$, then $G \setminus \{v\}$ does not contain an $F$-bad $K_{H(\epsilon)-1}$, by induction $G \setminus \{v\}$ can
be list-colored, and the coloring extends to $v$. If $v$ does lie on $F$, then suppose
the face $F$ becomes the face $F'$ in $G \setminus \{v\}$. If the vertices of $F'$ are a subset of the
vertices of $F$, then $G \setminus \{v\}$ does not contain an $F'$-bad $K_{H(\epsilon)-1}$, and the result
follows again by induction. Otherwise the face $F'$ is extended by at least one
vertex $v'$, and $v'$ has an $H(\epsilon)$-list. If the induced subgraph on $V(F')$ contains
$K_{H(\epsilon)-1}$, then that complete graph must contain $v'$ and so does not contain an $F'$-bad
$K_{H(\epsilon)-1}$. Thus we may assume $G \setminus \{v\}$ does not contain an $F'$-bad $K_{H(\epsilon)-1}$ in all
cases, it can be list-colored by induction, and the coloring extends to $v$. We conclude
that all vertices have degree at least $H(\epsilon) - 2$.

Suppose $G$ contains a vertex $v$ of degree less than $H(\epsilon)$ that does not lie
on $F$. Then $G \setminus \{v\}$ cannot contain an $F$-bad $K_{H(\epsilon)-1}$, by induction it can be list-
colored, and that coloring extends to $v$. Thus we may assume that all vertices of
degree $H(\epsilon) - 2$ and $H(\epsilon) - 1$ lie on $F$.

In addition $F$ cannot contain a vertex of degree $H(\epsilon) - 1$ or greater, for
suppose it did. Let $d_i$ denote the number of vertices of $G$ of degree $i$, let $d_{ai}$ denote
the number of vertices of $G$ of degree at least $i$, and let $d^F_{\geq H(\epsilon)}$ denote the
number of vertices of $F$ of degree at least $H(\epsilon)$.

Summing the face sizes we have
\[ 2e \geq |V(F)| + 3(f - 1) = d_{H(\epsilon)-2} + d_{H(\epsilon)-1} + d^F_{\geq H(\epsilon)} + 3(f - 1). \]
Then by Euler's formula $e \leq 3 n + 3 (\epsilon - 1) - d_{H(e)-2} - d_{H(e)-1} - d^{E}_{zH(e)}$.

Summing the vertex degrees we have
\[
2 e \geq (H(e) - 2) d_{H(e)-2} + (H(e) - 1) d_{H(e)-1} + H(e) (n - d_{H(e)-2} - d_{H(e)-1})
\]
\[
= H(e) n - 2 d_{H(e)-2} - d_{H(e)-1}.
\]

Combining the previous two inequalities gives
\[
(*) \quad d_{H(e)-1} + 2 d^{E}_{zH(e)} \leq 6 (\epsilon - 1) - (H(e) - 6) n \\
\leq 6 (\epsilon - 1) - (H(e) - 6) (H(e) + 1) \leq 0.
\]

The bound of 0 follows by evaluating, for $i \geq 1$, $H(e) = 3 i + 3$ and $\epsilon \leq \frac{3 i^2 + i - 2}{2}$, $H(e) = 3 i + 4$ and $\epsilon \leq \frac{3 i^2 + 3 i - 2}{2}$, and $H(e) = 3 i + 5$ and $\epsilon \leq \frac{3 i^2 + 5 i}{2}$. In the Special Cases with $H(e) = 3 i + 4$ and $\epsilon = \frac{3 i^2 + 3 i}{2}$, the upper bound of $(*)$ is 4. Since we are not in a Special Case, we conclude that $d_{H(e)-1} = d^{E}_{zH(e)} = 0$.

Thus $G$ has vertices of degree $H(e) - 2$ on $F$ and others of degree at least $H(e)$ off $F$. With this information we repeat the calculations above to get
\[
2 e \geq d_{H(e)-2} + 3 (f - 1),
\]
\[
2 e \geq (H(e) - 2) d_{H(e)-2} + H(e) d_{H(e)} + (H(e) + 1) (n - d_{H(e)-2} - d_{H(e)}) =
\]
\[
(H(e) + 1) n - 3 d_{H(e)-2} - d_{H(e)}.
\]

Combining the two previous inequalities gives
\[
(**) \quad (H(e) - 5) n \leq 6 (\epsilon - 1) + d_{H(e)-2} + d_{H(e)}.
\]

When $\epsilon = 1$ and $H(e) = 6$, we have $n \leq d_4 + d_5$, and since $n = d_4 + d_6 + d_{27}$ we have $n = d_4 + d_6$. By Lemma 3.2 the graphs on the projective plane can be list-colored. When $\epsilon = 2$ and $H(e) = 7$, we have $2 n \leq 6 + d_5 + d_7$, and since $n = d_5 + d_7 + 2 d_{28}$, we have $n \leq d_5 + d_7 + 2 d_{28} \leq 6$, a contradiction. (When $\epsilon = 3$, we are in an excluded case.) When $\epsilon = 4$ and $H(e) = 8$, similarly we get $3 n \leq 18 + d_6 + d_8$, so that $2 n \leq 2 d_6 + 2 d_8 + 2 d_{9} \leq 18$ and $n \leq 9 = H(e) + 1$. Thus $n = 9$, $n = d_6 + d_8$, and so the graphs on the surfaces of Euler genus 4 can be list-colored by Lemma 3.2.

When $\epsilon \geq 5$ and $H(e) \geq 9$, we have from $(**)$
\[
(H(e) - 5) n = (H(e) - 5) (d_{H(e)-2} + d_{H(e)} + d_{2H(e)+1})
\leq 6 (\epsilon - 1) + d_{H(e)-2} + d_{H(e)}
\]
so that $(H(e) - 6) (d_{H(e)-2} + d_{H(e)}) + (H(e) - 5) d_{2H(e)+1} \leq 6 (\epsilon - 1)$. If
\[ n = d_{H(e)-2} + d_{H(e)}, \] then by Lemma 3.2, these graphs can be list-colored.

Otherwise \( d_{2, H(e)+1} > 0. \) Then

\[
(***) \quad (H(e) - 6) n = (H(e) - 6) (d_{H(e)-2} + d_{H(e)} + d_{2, H(e)+1})
\]

\[ < (H(e) - 6) (d_{H(e)-2} + d_{H(e)}) + (H(e) - 5) d_{2, H(e)+1} \leq 6 (\epsilon - 1). \]

Thus \( H(e) + 1 \leq n < 6 (\epsilon - 1) / (H(e) - 6) \) and

\[ 0 < 6 (\epsilon - 1) - (H(e) + 1) (H(e) - 6) \leq 0 \]

by (\( \ast \)), giving a contradiction. \( \square \)

**Corollary 3.4.** In each Special Case, we may assume that all vertices have degree at least \( H(e) - 2 \), all vertices not on \( F \) have degree at least \( H(e) \), and

\[
d_{H(e)-1} + 2 d^F_{2, H(e)} \leq 4 \]

where \( d^F_{2, H(e)} \) is the number of vertices on \( F \) of degree at least \( H(e) \).

Note that the proof of Prop. 3.3 holds for all \( \varepsilon > 0 \) up to line (\( \ast \)) where the special results of Cor. 3.4 hold for the Special Cases.

**4 The Special Cases**

**Lemma 4.1.** If Theorem 1.1 holds also for \( n = H(e) + 1 \), then it holds for all graphs with \( n \) vertices that embed on \( S_n \), except possibly when \( \varepsilon = 3 \).

**Proof.** We know Thm. 1.1 holds for the non-Special Cases by Prop. 3.3, and we assume we are in a Special Case with \( H(\epsilon) = 3 i + 4, \epsilon = \frac{3i + 3}{2}, i > 1 \). If the theorem holds also for \( n = H(e) + 1 \), then that case joins the base cases of Lemma 3.1, and we may assume \( n \geq H(e) + 2 \) for the induction step. From the proof of Prop. 3.3 up to (\( \ast \)) we have for \( \varepsilon > 0 \)

\[
(****) \quad d_{H(e)-1} + 2 d^F_{2, H(e)} \leq 6 (\epsilon - 1) - (H(e) - 6) n
\]

\[ \leq 6 (\epsilon - 1) - (H(e) - 6) (H(e) + 2) \leq 0 \]

for \( i \geq 2 \) in the Special Cases. Thus again we conclude that \( d_{H(e)-1} = d^F_{2, H(e)} = 0. \)

Then the proof proceeds exactly as in the proof of Prop. 3.3. From (***)

we obtain \( (H(e) - 6) n < 6 (\epsilon - 1) \) so that \( H(e) + 2 \leq n < 6 (\epsilon - 1) / (H(e) - 6) \) and

\[ 0 < 6 (\epsilon - 1) - (H(e) + 2) (H(e) - 6) \leq 0 \] by (****), giving a contradiction, except when \( \varepsilon = 3 \). \( \square \)
Lemma 4.2. Let $G$ have $k$ vertices each with at least an $i$-list for some $i$, $1 \leq i < k$. If $G$ has at most $i$ vertices of degree $i$ or greater, then it can be list-colored.

Proof. Label the vertices in nonincreasing order by degree: $v_1, ..., v_i, v_{i+1}, ..., v_k$. The first at most $i$ vertices can be list-colored sequentially since when each is colored, it has at most $i-1$ colored neighbors. After that each remaining vertex has degree at most $i-1$ and with a $i$-list can always be list-colored. □

In the Special Cases, whether or not $K_{H(e)+1} - E$ embeds on $S_e$, we know that a graph on this surface with $n = H(e) + 1$ vertices is a subgraph of $K_{H(e)+1} - E$.

Proposition 4.3. Theorem 1.1 holds for all graphs with $n = H(e) + 1$ vertices in all Special Cases when $\varepsilon \geq 9$ and $H(e) \geq 10$.

Proof. For $\varepsilon \geq 9$ with $H(e) \geq 10$, we have $H(e) = 3i + 4$ and $\varepsilon = \frac{3i + 3}{2}$ for $i \geq 2$. Let $G$ be embedded on $S_e$, have $n = H(e) + 1$ vertices, and a face $F$ with $|V(F)| = j \leq H(e) + 1$. Then $G$ is a subgraph of $K_{H(e)+1} - E$, and $G_F$, the induced subgraph on $V(F)$, does not contain $K_{H(e)-1}$. By Cor. 3.4 all vertices of $F$ have degree at least $H(e) - 2$ and all of $V(G) \setminus V(F)$ have degree exactly $H(e)$, the maximum possible degree of $G$. In addition the vertices of $F$ satisfy $0 \leq d_{H(e)-1} + 2d_{H(e)} \leq 4$. Since $n = H(e) + 1$, when $d_{H(e)} > 0$, $d_{H(e)}^F$ counts the number of vertices of degree $H(e)$ that lie on $F$.

Suppose $d_{H(e)-1} + 2d_{H(e)}^F = 0$ so that all vertices of $V(F)$ have degree $H(e) - 2$. Then by Lemma 3.2, $G$ can be list-colored. Thus we suppose $0 < d_{H(e)-1} + 2d_{H(e)}^F \leq 4$.

Since the vertices of $V(G) \setminus V(F)$ have degree $H(e)$, we have that $G_F$ is a subgraph of $K_j - E$ and $G$ is a subgraph of the join $G_F + K_{H(e)+1-j}$. We label the vertices of $G$ $v_1, ..., v_j, ..., v_{H(e)+1}$ with the first $j$ lying on $F$ and the remaining outside; we know that some pair of vertices of $F$ is not adjacent. Our goal is to $(H(e) - 2)$-list-color $G_F$ in a way that extends to all of $G$.

Suppose $v_1$ and $v_2$ are not adjacent and that $c$ is a list-coloring of $G_F$. If $c(v_1) = c(v_2)$, then $c$ will extend to all of $G$ by coloring the vertices $v_{j+1}, ..., v_{H(e)}, v_{H(e)+1}$ sequentially. Similarly if either $c(v_1)$ or $c(v_2)$ does not lie in $L(v_k)$ for some $k$, $j + 1 \leq k \leq H(e) + 1$, then we can relabel $v_k$ to become
Suppose \( j \leq H(e) - 2 \). If \( L(v_1) \cap L(v_2) \neq \emptyset \), then we choose a common color for \( v_1 \) and \( v_2 \) and this extends to \( G_F \) with maximum degree at most \( H(e) - 3 \). Otherwise, without loss of generality, \( L(v_1) = \{1, 2, ..., H(e) - 2\} \) and \( L(v_2) = \{H(e) - 1, 2, H(e) - 4\} \). Since \( 2H(e) - 4 > H(e) \), there is a color for \( v_1 \) or for \( v_2 \) that does not lie in \( L(v_{H(e)+1}) \), and so \( G_F \) can be so list-colored with that coloring extending to \( G \).

We conclude that \( j = H(e) - 1, H(e), \) or \( H(e) + 1 \), giving us three cases, each divided into subcases according as \( d_{H(e)-1}^F + 2 d_{2H(e)}^F = 1, 2, 3 \) or 4. Since the proof techniques for the subcases of the first two cases are the same, we pick illustrative subcases. The third case is proved using Thm. 2.2.

Case I. \( j = H(e) - 1 \). Thus there are two adjacent vertices lying off \( F \), both of degree \( H(e) \); each vertex of \( F \) is adjacent to both of these and has degree \( H(e) - 2, H(e) - 1, \) or \( H(e) \) in \( G \).

Subcase A. \( d_{2H(e)}^F = 0 \) and \( d_{H(e)-1}^F = 1 \) or 3, or \( d_{2H(e)}^F = 1 \) and \( d_{H(e)-1}^F = 1 \). Suppose the first case holds with \( d_{2H(e)}^F = 0 \) and \( d_{H(e)-1}^F = 1 \). Then \( G_F \) is a subgraph of \( K_{H(e)-1} - E \) with \( H(e) - 2 \) vertices of degree \( H(e) - 4 \) (in \( G_F \), though degree \( H(e) - 2 \) in \( G \)) and one of degree \( H(e) - 3 \). Consider the complement \( G_F^c \), which has \( H(e) - 2 \) vertices of degree 2 and one of degree 1: there is no such graph. There is the same parity contradiction for the other two pairs of values.

In the following subcases we will apply Lemma 4.2 to a subgraph with \( k = H(e) - 3 \geq 7 \) vertices, each of which has a list size at least \( i = k - 1 \) or \( k - 2 \).

Subcase B. \( d_{2H(e)}^F = 0 \) and \( d_{H(e)-1}^F = 2 \). As argued in Subcase A, \( G_F^c \) has \( H(e) - 3 \) vertices of degree 2 and two of degree 1. Thus \( G_F^c \) consists of one path and (possibly) some cycles. Let \( v_1, v_2 \) be the first two vertices of the path. Since \( H(e) - 2 \leq |L(v_1) \cup L(v_2)| \leq 2H(e) - 4 \), either they can be colored in \( G_F^c \) with a common color or one can be colored with a color not in \( L(v_{H(e)+1}) \). Consider the remaining \( H(e) - 3 \) vertices of \( G_F^c - \{v_1, v_2\} \) which we must color. Depending on whether the path had two or more than two vertices, respectively, \( G_F^c - \{v_1, v_2\} \) consists of either cycles or cycles plus a path, resp. For each vertex in \( G_F^c - \{v_1, v_2\} \) we remove the one or two colors on \( v_1 \) and \( v_2 \) from each of their lists (when present), leaving lists of size at least \( H(e) - 4 \). Either
Subcase C. $d_{H(e)}^F = 0$ and $d_{H(e)-1} = 4$. In this case $G_F^c$ has $H(e) - 5$ vertices of degree 2 and four of degree 1, and the proof follows that of Subcase B.

Subcase D. $d_{H(e)}^F = 1$ or 2, respectively, and $d_{H(e)-1} = 0$. Then $G_F^c$ has $H(e) - 2$ (resp., $H(e) - 3$) vertices of degree 2 and one (resp., two) of degree 0. Thus $G_F^c$ consists of at least one cycle; let $v_1, v_2$ be two successive vertices of one such cycle $C^*$. We color as in Subcase B, and consider $G_F - \{v_1, v_2\}$ with the one or two colors of $v_1, v_2$ removed from the lists of these vertices, leaving lists of size at least $H(e) - 4$. Then at least $H(e) - 6$ (resp., $H(e) - 7$) vertices of $G_F - \{v_1, v_2\}$ have degree at most $H(e) - 6$, at most two vertices (resp., three vertices) have degree $H(e) - 4$ (there might be 2, resp., 3, when $C^*$ is a 3-cycle), and at most two vertices (when $C^*$ is larger than a 3-cycle) have degree at least $H(e) - 5$. By Lemma 4.2 with $k = H(e) - 3$ and $i = H(e) - 4$, the coloring of $v_1, v_2$ extends to $G_F$ and so to all of $G$.

Subcase E. $d_{H(e)}^F = 1$ and $d_{H(e)-1} = 2$. Then $G_F^c$ has $H(e) - 4$ vertices of degree 2, two of degree 1, one of degree 0, and consists of a path and (possibly) some cycles. The proof follows those of Subcases B and D.

Case II. $j = H(e)$. This case and its argument is parallel to that of Case I. There is only one vertex lying off $F$, it has degree $H(e)$, and it is adjacent to every vertex of $F$. The vertices of $F$ have degree $H(e) - 2$, $H(e) - 1$, or $H(e)$ in $G$. When $d_{H(e)}^F = 0$ and $d_{H(e)-1} = 1$ or 3, or $d_{H(e)}^F = 1$ and $d_{H(e)-1} = 1$, the same parity contradiction as in Subcase A occurs. Then in the remaining subcases we can apply Lemma 4.2 to a subgraph with $k = H(e) - 2 \geq 8$ vertices, each of which has a list of size at least $k - 1$ or $k - 2$.

When $d_{H(e)}^F = 0$ and $d_{H(e)-1} = 2$, the proof proceeds as in Subcase B; the only difference is that either $H(e) - 2$ or $H(e) - 4$ vertices in $G_F - \{v_1, v_2\}$ have degree at most $H(e) - 5$ and at most two have degree at least $H(e) - 4$. By Lemma 4.2 with $k = H(e) - 2$ and $i = H(e) - 4$, the coloring of $v_1$ and $v_2$ extends to $G_F$ and then to all of $G$. When $d_{H(e)}^F = 0$ and $d_{H(e)-1} = 4$, the proof technique is the same.

When $d_{H(e)}^F = 1$ or 2, respectively, and $d_{H(e)-1} = 0$, the proof is as in Subcase D. Specifically, at least $H(e) - 5$ (resp., $H(e) - 6$) vertices have degree
at most $H(e) - 5$ in $G_F - \{v_1, v_2\}$, at most two vertices (resp., three vertices) have degree $H(e) - 3$, and at most two vertices (two distinct neighbors of $v_1, v_2$ on a cycle) have degree at least $H(e) - 4$. By Lemma 4.2 with $k = H(e) - 2$ and $i = H(e) - 3$, the coloring of $v_1, v_2$ extends to $G_F$ and so to all of $G$. When $d_{H(e)}^e = 1$ and $d_{H(e)-1}^e = 2$, the proof follows that of previous cases, as in Case I.

Case III. $j = H(e) + 1$. All vertices of $G$ lie on $F$ and we must show that $G$ can be $(H(e) - 2)$-list-colored. If all vertices have degree $H(e) - 2$, then it can be list-colored by Lemma 3.2. Otherwise if $G$ cannot be list-colored, it contains $G'$, an $(H(e) - 2)$-list-critical subgraph, and we apply Thm. 2.2. $G'$ must have at least $H(e)$ vertices since $G$ does not contain $K_{H(e)-1}$, the only list-critical graph on $H(e) - 1$ or fewer vertices.

First suppose that $G'$ has $n' = H(e)$ vertices, and we apply Thm. 2.2 with $k = H(e) - 1$. Then $2e' \geq (H(e) - 2)H(e) + H(e) - 4$. The graph $G$ with $H(e) + 1$ vertices has, in addition, a vertex of degree at least $H(e) - 2$. Also $H(e) - 2$ additional diagonals can be added to the face $F$, triangulating that region. Thus for $G$ plus diagonals with $e'$ edges

$$2e' \geq H^2(e) - H(e) - 4 + 2(H(e) - 2) + 2(H(e) - 2) = H^2(e) + 2H(e) - 12.$$ We also have $2e' \leq 6n + 6(e - 2) = 6(H(e) + 1) + 6(e - 2)$. Thus $H^2(e) - 3H(e) - 18 \leq 6(e - 2)$. With $H(e) = 3i + 4$ and $e = \frac{3i^2 + 3i}{2}$, we reach a contradiction.

Similarly suppose that $G'$ has $n' = H(e) + 1$ vertices and we apply Thm. 2.2 as above. Then $2e' \geq (H(e) - 2)(H(e) + 1) + H(e) - 4$. $H(e) - 2$ additional diagonals can be added to $F$ so that $G$ plus diagonals with $e'$ edges satisfies $2e' \geq H^2(e) + 2H(e) - 10$ and again with $2e' \leq 6(H(e) + 1) + 6(e - 2)$ we reach a contradiction since $i \geq 2$. □

Then by Prop. 4.3, Lemma 4.1, and Prop. 3.3, we have Thm. 1.1.

5 Examples

The only $k$-critical graph on $k + 2$ vertices is the join $K_{k-1} + C_5$ [5]. As shown in [12] this graph with $k = H(e) - 1$ can sometimes embed on $S_e$.

Proof of Prop. 1.2. Suppose $K_{H(e)+1} - E$ embeds on $S_e$ so that we are in a Special Case with $\epsilon = \frac{3\epsilon^2 + 3\epsilon}{2}$ and $H(e) = 3i + 4$ (see the comments following
Lemma 2.3. By [10, 11], \(i > 1\), and such embeddings are possible when \(H(e) \equiv 1, 4, 10 \pmod{12}\) and perhaps (or perhaps not) when \(H(e) \equiv 7 \pmod{12}\). As described in [12], we can then obtain an embedding of \(K_{H(e) - 4} + C_5\) as follows. Suppose \(E = v_1 v_2\). Choose three additional vertices \(v_3, v_4, v_5\), and delete the edges \(v_2 v_3, v_3 v_4, v_4 v_5, v_5 v_1\). The remaining graph is \(K_{H(e) - 4} + C_5\). By Euler's formula this graph has too many edges to be embeddable with all vertices on one face (in which case then \(H(e) - 2\) diagonals could be added). Removing one vertex of \(K_{H(e) - 4}\) leaves \(K_{H(e) - 5} + C_5\) with all vertices on one face. This is an \((H(e) - 2)\)-critical graph which does not contain \(K_{H(e) - 2}\) and cannot be \((H(e) - 3)\)-list-colored. □

We ask if there are graphs as in Prop. 1.2 on all surfaces. The next example shows that for all surfaces, 3-lists on one face can prevent list-coloring even without the presence of complete graphs on four or more vertices.

Let \(G\) be a 2-connected outerplanar near-triangulation; that is, \(G\) is a triangulated polygon which consists of a cycle on all vertices plus diagonals that triangulate the interior of the cycle. These graphs are uniquely 3-colorable. Let the vertices on the exterior cycle be labelled successively \(v_1, v_2, \ldots, v_n\). Then for an orientable surface, select two edges on the exterior cycle, say \(v_1 v_2\) and \(v_{i+1} v_i\), with either \(v_1\) and \(v_{i+1}\) or \(v_2\) and \(v_i\) in different color classes of a 3-coloring. After identifying the edge \(v_1 v_2\) with \(v_{i+1} v_i\), this graph can be embedded on an orientable surfaces with all vertices lying on one face, and the graph cannot be 3-colored or 3-list-colored. For a nonorientable surface the edge \(v_1 v_2\) can be identified with a twist with \(v_i v_{i+1}\), the resulting graph can be embedded on a nonorientable surface and is not 3-list-colorable when either \(v_1\) and \(v_i\) or \(v_2\) and \(v_{i+1}\) lie in different color classes of a 3-coloring of the triangulated polygon.

If \(n\) is large and the two identified edges are distant in \(G\), then the resulting graph on a surface is locally planar and contains no induced \(K_4\).

This example also shows that Thm. 1.1 is best possible for the projective plane in terms of the list-size for vertices on the distinguished face \(F\). We ask if Thm. 1.1 is similarly best possible for small \(e\) such as \(e = 2\) and 4. We ask if there is a 4-list, 6-list theorem for the Klein bottle, and if there is a 5-list, 7-list theorem for \(e = 3\).

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