Onset of “hard” turbulence in Benard Convection and small-scale universality.

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Anomalous scaling of small-scale fluctuations of velocity and its derivatives, a feature of “strong” (“hard”) turbulence, is directly related to extreme rare events in turbulent flows and other stochastic processes. The direct transition from the low Reynolds number “weak” Gaussian turbulence to fully developed “strong” turbulence at a critical Reynolds number \(Re_{cr} \approx 8.91\) was recently theoretically predicted and tested in high resolution numerical simulations of V. Yakhot & D. A. Donzis, Phys. Rev. Lett. 110, 044501 (2017) & PhysicalD, 382-385, 12 (2018) on an example of a flow excited by a Gaussian random force. In this paper we study the onset of “hard” turbulence in Benard (RB) convection, where, depending on the Rayleigh number, turbulence is produced by both instabilities of the bulk flow and of the wall boundary layers. The developed theory predicts non-monotonic behavior of the low-Reynolds number moments of velocity derivatives \(M_2n(Re) \propto Re^{2n}\), observed in the direct numerical simulations of Schumacher et.al (Phys.Rev.E, 98,033120 (2018)), The calculated magnitudes of anomalous exponents \(\rho_{2n}\) in flows stirred by forces obeying Gaussian or exponential statistics are slightly different, which may indicate existence of universality classes defined by production mechanisms.

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I. INTRODUCTION

Transition from laminar to turbulent flow was discovered and analyzed by Osborne Reynolds in 1883, who reported emergence of “sinuous” motions out of a direct and steady water flow in a pipe. Moreover, Reynolds quantified the phenomenon in terms of dimensionless parameter \(Re = UL/ν\) later called Reynolds number. Here \(U\) and \(L\) are mean velocity across the pipe of radius \(L\). In this work, Reynolds introduced a critical parameter \(Re = Re_{cr}\), so that at \(Re \leq Re_{cr}\) the flow was laminar, with steady parabolic velocity profile \(U(r)\). He noticed the appearance of irregular or random fluctuations \(v(x,t)\) when \(Re > Re_{cr}\). With increase of \(Re > Re_{cr}\), the amplitude and degree of randomness increased which made analysis of the flow very hard. Interestingly, Reynolds was the first to suggest description of this flow using statistical methods. To this day, the question of the structure and statistics of velocity fluctuations \(v(x,t)\) as a function of \(Re - Re_{cr} \rightarrow \infty\) remains open.

Depending on geometry and physical mechanisms, various laminar flows become unstable at widely different Reynolds numbers \(Re = UL/ν\), where \(V\) and \(L\) are characteristic velocity and length scale of a flow. One can introduce dimensionless critical number \(Re_{cr}\) marking first instability of a laminar flow pattern. As \(Re - Re_{cr} \rightarrow 0\), low - intensity velocity fluctuations are described as, usually Gaussian, random field, which can loosely be called “weak or soft turbulence”. Some qualitative ideas can be obtained from the Landau theory considering a stationary flow \(v_0(x)\) with a small time-dependent perturbation \(v_1(x,t) = A(t)f(x) \propto f(x)e^{iωt}e^{-κxt}\) where \(ωi \gg |γ|\). In the vicinity of transition point one can write

\[
\frac{d|A|^2}{dt} = 2(Re - Re_{cr})|A|^2 - γ|A|^4 \tag{1}
\]

when \(Re - Re_{cr} > 0\), the amplitude saturates at \(|A|_{max} \propto \sqrt{(Re - Re_{cr})/γ}\). Extrapolating this into interval \(Re >> Re_{cr}\), we obtain \(|A|_{max} \propto \sqrt{Re}\). This result can numerically be accurate when \(Re - Re_{cr} > Re_{cr}\), though small enough for the \(O(A^0)\) non-linear contributions to (1) be neglected. The important feature of (1) is that no randomness is present in Landau theory which assumes that equation (1) is the result of an averaging over high-frequency modes \(ωi \gg γ \equiv O(Re - Re_{cr}) \rightarrow 0\), where \(γ\) is a characteristic frequency (inverse period) of the first time-dependent mode \(v_1(x,t)\) generated by instability. Landau assumed that with further increase of the Reynolds number, the field \(v_1\) becomes unstable i.e. its perturbation \(v_2(x,t)\) grows into a periodic flow with frequency \(γ_2 \approx 2γ\) and so on. While this theory is physically appealing, its main drawback the fact that the Reynolds number of “second” instability generating small-scale fluctuations is unknown and it is not clear how one can calculate it when \(|A|\) is not small. The attempts to treat (1) as a first two terms of the Taylor expansion by adding a few high-order powers in \(A\) led to unsurmountable complications [2].

The passage to strong turbulence involves a few steps: a. laminar or regular low - Reynolds number field \(U(x,t)\) which is a solution to the Navier-Stokes equations b. theoretically or experimentally understanding its stability; c. study of fluctuations and their interactions with each other and with a mean flow. Each step of this program is extremely involved and difficult due to in general complex geometry and lack of a small parameter. Not surprisingly, the strong turbulence problem is a subject of more than a century of experimental and theoretical efforts. In this paper we are interested in a completely different kind of transition to turbulence not involving laminar velocity field \(v_0(x)\).

A. Direct Transition.

To study dynamics of velocity fluctuations it is useful to define the Reynolds number \(Re = \nu_{rms}L/ν = \sqrt{\nu_{rms}L/ν}\) based on fluctuating velocity \(v\) for which \(v = 0\). Below, interested in fluctuations only, \(Re\) denotes the Reynolds...
number based exclusively on a fluctuating velocity field $v$. To avoid difficulties related to geometry, boundary conditions and instabilities of a laminar flow, the dynamics of a flow governed by the Navier-Stokes equations in an infinite fluid stirred by a Gaussian random forcing on a scale $r \approx L = O(1)$ has been studied in Refs. [3]-[4] (the density is taken $\rho = 1$ without loss of generality):

$$\partial_t v + v \cdot \nabla v = -\nabla p + v \nabla^2 v + f$$

(2)

$$\nabla \cdot v = 0.$$  

A random Gaussian noise $f$ is defined by the correlation function $[1]$:

$$f_i(k, \omega) f_j(k', \omega') = (2\pi)^{d+1} D_0(k) \delta(k + k')$$

(3)

where the four-vector $k = (k, \omega)$ and projection operator is: $P_i(k) = \delta_i - \frac{\partial}{\partial k_i}$. It is clear from (2)-(3) that in the limit $D_0 \to 0$ the nonlinearity is small and $v(k) \approx G^0 f = O(\sqrt{D_0})$, where the “bare” Green function is $G^0 = 1/(i\omega + \nu k^2)$. In this limit the velocity field is Gaussian with the derivative moments $M_{2n} = \left( \partial_x v_x \right)^{2n} / \left( \partial_x v_x \right)^{2n} \approx (2n - 1)!!$.  

All renormalized perturbation expansions, applied to the problem (2)-(3), failed to derive anomalous scaling in the strongly non-linear regime and, instead, one can seek a non-perturbative solution satisfying two asymptotic constraints: in the “weak turbulence” range $Re \ll Re^{tr}_{2n-1}$, the moments $M_{2n} = \Gamma(Re, n) \approx A_n Re^{2n}$ obeying, in general, multi-scaling laws with not yet known amplitudes $A_n$ and exponents $\rho_{2n}$. The two limiting curves cross at the $n$-dependent transitional Reynolds numbers $Re^{tr}_{2n}$, investigated in detail in Refs.[3]-[4].  

Thus, in the low-Reynolds number interval $Re < Re^{tr}_{2n}$:

$$M_{2n} = (2n - 1)!! \approx A_{2n}(Re^{tr}_{2n})^{\rho_{2n}}$$

(4)

On Fig.1, these ideas have been confirmed by direct numerical simulations (DNS) of the the moments of derivatives $M_{2n}$ vs Reynolds number based on the Taylor scale $R_\lambda = \sqrt{5 \nu^2 v_{rms}^2}$. We can see horizontal lines corresponding to the Re-independent normalized Gaussian moments $M_{2n} = (2n - 1)!!$ for $2 \leq n \leq 6$.  

To find transitional Reynolds numbers $Re^{tr}_{2n}$ a new concept has been introduced in Refs.[3]-[4]: define $v_{2n} = L^2 (\partial_x v_x)^{2n+2} \ll A_{2n} Re^{\alpha_{2n}}$ and $\tilde{R}_{\lambda,n}^{tr} = \sqrt{5 \nu v_{2n}} \approx 8.91$ derived in Refs.[5]-[8]. To calculate large scale transitional Reynolds number

$$R_{\lambda,2}^{tr} \equiv \tilde{R}_{\lambda,2}^{tr} = \sqrt{\frac{5 \nu v_{2n}}{2}}$$

(5)

where $K = v_{2n}^2 / 2$, we use the theoretical result $\nu_T = 0.0845 K^2 / E$ practically identical to the one used in engineering turbulence modeling for the last fifty years [9], which gives immediately $R_{\lambda,2}^{tr} \approx 8.88$. The somewhat “unexpected” consequence of this result, is clearly seen on Fig.1, where the onsets of anomalous scaling for different moments $M_n$ are observed at very different $Re^{tr}_{2n}$ but at a single $n$-independent $\tilde{R}_{\lambda,n}^{tr} \approx 9.0$. If amplitudes $A_{2n}$ are not an exponentially increasing function of the moment order $n$, then $A_{2n}$ is a weakly dependent function of $n$. Therefore, we conclude that $A_{2n} \approx \text{const}$, which can be found from the low-order moments. One can easily express $Re^{tr}_{2n}$ in terms of $\tilde{R}_{\lambda,n}^{tr}$ [3] - [4] and close the equation (4) for $\rho_{2n}$. The results are presented in Table 1 and compared with the data on the middle panel of Fig.1.

B. Direct transition: numerical procedure for high-Reynolds number limit.

In addition to the “classic” problem of anomalous exponents $d_n$ and $\rho_n$, the study of Refs.[3]-[4] opened up a new question of possible universality of transitional Reynolds number $R_{\lambda,n}^{tr} = \sqrt{5 \nu v_{2n}} \approx 8.91$ derived in the Renormalization Group analysis of turbulence in the limit $r \to L$ [5]-[9]. The possible universality of this result may have important consequences for numerical simulations demonstrated in Fig.1 where the analytic theory is compared to the low Reynolds number DNS on the two left panels. In the Gaussian forcing case [3]-[4]:

$$Re^{tr}_n = C(\tilde{R}_{\lambda,n}^{tr})^{\nu_{2n}} > \frac{\nu_{2n}}{\nu_{2n}}$$

(6)

and at transition points

$$e_n = \left( \frac{E}{\tilde{E}} \right)^n (2n - 1)!! \approx C^{\nu_{2n}} (\tilde{R}_{\lambda,n}^{tr})^{\nu_{2n}}$$

where $\tilde{R}_{\lambda,n}^{tr} \approx 8.91$ independent on $n$. One can easily derive a simple estimate $Re \approx 1.5 R_{\lambda,2}^{tr}$ giving $Re^{tr}_{2n} \approx 100 - 200$ resulting in $C \approx 100 - 200$. This closes the equation for exponents $d_n$ and $\rho_{2n}$: if, as in the problem (2) - (3), $\tilde{E} \approx D_0 = O(1)$, then $\rho_{2n} = d_n + n$. The details are presented in [4]. The possible universality of transitional $R_{\lambda,n}^{tr}$ enables high-Reynolds number computations of flows based on the low-Reynolds number data obtained either theoretically or numerically. The matching procedure is qualitatively demonstrated on the rightmost panel of Fig.1 on an example of the moment $e_2(y)$ where $y \propto Re_{2} - Re^{tr}_{2}$. It consists of three main steps: a. calculate or compute the moments of derivatives in the linear low-Reynolds number limit $Re \lesssim R_{\lambda,n}^{tr} \approx 120$ or $\tilde{R}_{\lambda,n} \lesssim 8.91$ b. This allows evaluation of the exponents $d_n$ and $\rho_n$. c. Extrapolation of an assumed high-Reynolds number solution $e_n = C Re^{d_n}$ back to the transition point, $Re \to Re^{tr}_n$. Plot the resulting dependence in the entire range $Re \gtrsim Re^{tr}_n$. Below we generalize this scheme to a much more complex situation.

II. COMPLEX TRANSITION. BOUNDARY LAYER EFFECTS.

The simplified problem of Refs. [3]-[4], described above, dealt with an artificial situation of direct transition between a well-defined Gaussian state of a fluid and the non-linearity-dominated strong turbulence. In other words,
Re	\rightarrow\infty. In general, the problem is very hard, for it involves the first instability leading to rolls, generation of the low-Re “weak turbulence” which is a precursor to the strong “hard” turbulence we are interested in this paper. The number of both experimental and theoretical publications dealing with RB convection published in the last few decades is enormous and it is impossible even briefly review them. Majority of he work in the field dealt with the large-scale global properties of the phenomenon leading to predictions of the heat transfer as a function of various large-scale parameters. Here we are interested in the small-scale velocity and velocity derivatives fluctuations, which is a relatively new and interesting topic. This problem has been addressed in the DNS published in a recent paper [10] on the RB convection with Prandtl and Reynolds numbers varying in the range 0.005 \leq Pr \leq 100 and 1 \leq Re \leq 1000. At large Re \approx 100 – 2000 the scaling exponents of the first two moments of kinetic energy dissipation rate were similar to those observed in Ref.[4], indicating possible universality. On the other hand, the low-Re behavior of a flow, reflecting some structural transitions, was much more complex and appearance of anomalous scaling at Re \approx 100 was definitely not from a Gaussian state of Refs.[3]-[4]. Unlike in the direct transition of Refs.[3]-[4], where the moments of derivatives M_{2n}(Re) were monotonic functions of Reynolds number, in Benard convection this was not so: the curves had a well - pronounced minima in the low-Re interval [10]. While this paper shed light on many important phenomena related to the Prandtl number dependence of the heat transfer, the details of statistics of the dissipation rate fluctuations, including the non-monotonic behavior of the moments, remained somewhat unresolved, mainly due to large difference between thermal and viscous boundary layers substantially complicating the situation. Below, we consider a greatly simplified problem of convection in a gap \( H \) between two infinite plates.

A. Phenomenology.

In this paper we are interested in the small-scale behavior of a flow between two infinite plates separated by the gap \( H \). The low plate at \( z = -H/2 \) is heated by an electric current \( I \). Due to the energy conservation, the heat flux averaged over horizontal planes \( z = z_0 \) is \( J(z) \)
and we keep the top and bottom plates under constant temperature difference $\Delta$.

We consider the coupled three-dimensional equations of motion for velocity and temperature fluctuations $v_i$ and $T$, respectively:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j^2} + \alpha g T \delta_{i3}, \quad (7)$$

$$\frac{\partial T}{\partial t} + v_j \frac{\partial T}{\partial x_j} = \frac{\partial^2 T}{\partial x_j^2} = \kappa \frac{\partial^2 \Theta}{\partial x_j^2} = \nu \frac{\partial^2 v_i}{\partial x_j^2}, \quad (8)$$

Here the horizontally averaged temperature $\Theta = \Theta(z)$ and $\partial v_i = 0$.

According to (7), to understand small-scale features of a flow, we have to investigate temperature fluctuations acting as a forcing term in the Navier-Stokes equation (7). Below, we use the theory of probability density (PDF) of temperature fluctuations in RB convection developed in the nineties [11]-[12]. In the low-Rayleigh number linear and weakly non-linear regimes, the following results, relevant for this study, have been established [13]-[17].

1. At $Ra < 1708$ the heat transfer is governed by conduction with heat flux $J = \kappa \frac{\partial T}{\partial z} = \nu \frac{\partial^2 v_i}{\partial x_j^2} = \text{const}$ and $v = 0$;
2. First, at $Ra \approx Ra_{cr} \approx 1700$, instability of a linear temperature profile with $v = 0$, typical of conduction, leads to the formation of a “quasi-steady” large-scale flow pattern called rolls.
3. Then, in the interval $6 \times 10^{-6} - 10^0 - 10^7$, weak fluctuations around this ordered flow field lead to the low- amplitude, $O(Ra - Ra_{cr}) \ll Ra_{cr}$, velocity and temperature fluctuations. In this range, according to Krishnamurti [14] and Busse [15], convection consists of ordered rolls with embedded small-scale fluctuations they call “convection elements”. It is important that, while rolls are characterized by the length-scale $r \approx H$, the small-scale elements “live” on the scale $r \ll H$, independent on $H$. Quoting Busse [15], “At moderate Prandtl numbers, turbulent convection at Rayleigh numbers of the order of $10^1 - 10^2$ exhibits the typical structure of relatively steady large-scale cells in which highly fluctuating (both in space and in time) small-scale convection elements are imbedded”. Similar results have been reported in a detailed study of Castaing et al. [13], showing a few peaks on the heat transfer curves before the rise of “hard” turbulence at $Ra \approx 4 \times 10^7$. In a relatively recent paper, P. Tong et al. [17] reported two competing mechanisms of heat transfer coming from the structures in the bulk of a cell and from viscous sublayers prior the onset of “hard” turbulence. The most important lesson from the existing experimental for what follows is emergence of a few qualitatively different contributions to the heat transfer in a soft turbulence range of $Ra$ variation.

To solve (7)-(8), following [10]-[12], the domain of variation of both velocity and temperature fields, can be subdivided in two parts. a. The wall region with thin ($\eta << H$) velocity and temperature boundary layers (BL). Due to the no-slip boundary conditions $v_{top} = v_{bot} = 0$, strong wall shear leads to the boundary layer instability manifested in discrete bursts in the directions of the bulk. (The associated with it temperature fluctuations are often called “plumes”. Similar mechanism of turbulence production in channel flows, responsible for the low Reynolds number Blasius scaling of the friction coefficient, was recently discussed in [16]. This leads to generation of velocity/temperature fluctuations in the bulk. b. Thus, we will study convection outside the boundary layers, using the phenomenology of the BL physics as an approximate boundary condition for equations defined in the bulk. In this domain turbulence can be assumed isotropic and homogeneous.

4. In the high Reynolds number limit $Ra \gg Ra_{cr}$, the flow becomes strongly non-linear and the notion of well-separated plumes invalid: due to strong interaction they lose their individuality in the bulk of the cell. This limit is characterized by strong small-scale intermittency and anomalous scaling.

### III. PROBABILITY DENSITY $P(X)$. LOW “REYNOLDS NUMBER”.

In this Section we use the modified theory of probability density developed for the problems of passive scalar and RB convection in Ref.[10]-[12]. Since in the field of the large-scale rolls, $\frac{\partial u}{\partial x} \approx \nu \frac{\partial^2 v_i}{\partial x_j^2}$, we define the “low Reynolds number regime” by the range where non-linearity in (7) can be neglected,

$$V \frac{\partial v_i}{\partial x} - \nu \frac{\partial^2 v_i}{\partial x^2} = \alpha g T \quad \text{(9)}$$

plus no-slip boundary conditions on solid walls. With $E_3 = \nu \frac{\partial u}{\partial x}^2$:

$$\frac{\langle \nabla T \nabla \rangle^2}{\langle \nabla T \rangle^2} = \frac{v_3 T}{v_3 T}$$

From the heat equation we have:

$$\frac{\langle \nabla T \rangle^2}{\langle \nabla T \rangle} = \frac{v_3 T}{v_3 T}$$

Defining $\bar{E}_3 = \nu \frac{\partial u}{\partial x}^2$, gives:

$$\bar{E} = 2 \nu \frac{\partial v_i}{\partial z} \frac{\partial^2 v_i}{\partial x^2} + \nu \frac{\partial^2 v_i}{\partial x^2} \langle \nabla \rangle^2 = \nu \langle \nabla \rangle^2 T^2$$

Based on the theory [11]-[12] supported by experimental data [13]-[15], [17], we conclude that there exist two mechanisms of dissipation of kinetic energy $E_1 \propto T^2$ and $E_2 \propto b |T|$, with continuous $O(T^2)$ contribution coming from the “convection elements” and the $O(T)$ one from the discrete plumes arising from the BL instability at $Ra \geq 1700 \nu |b| / H$ with the typical rising velocity $V$. Obtaining this estimate we rely on the concept “marginally stable” boundary layer introduced by Malkus [18] and discussed Castaing et al. [13].

We are interested in probability density $P(X)$ in the limit of small $X = T/T_{rms}$. This range includes heat conduction regime, formation of weakly fluctuating rolls and plumes coming boundary layers. As in Refs. [10]-[12], it is assumed that in the central part of the cell the fluid is...
well mixed and turbulence there can be assumed homogeneous and isotropic. Following [11], [12] multiplying (8) by $T^{2n-1}$ gives:

$$-(2n-1)T^{2n-2} (\nabla T)^2 = T^{2n-1} v_3 \partial \Theta \partial z$$

With $X^2 = T^2/T^3$, $Y^2 = (\nabla T)^2/(\nabla T)^2$ and $W = v_3 T/\bar{v}_3$. These equations can be rewritten:

$$(2n-1) X^{2n-2} Y^2 = X^{2n-2} W$$

and introducing conditional means gives [10]-[12]:

$$(2n-1) \int X^{2n-2} r_1(X) P(X) dX = \int X^{2n-2} r_3(X) P(X) dX$$

where

$$r_1(X) = \frac{\int Y^2(x) \delta(X(x) - X) dx}{\int \delta(X(x) - X) dx}$$

and

$$r_3(X) = \frac{X}{v_3 X} \int \frac{v_3(x) \delta(X(x) - X) dx}{\int \delta(X(x) - X) dx}$$

$r_1(X)$ and $r_3(X)$ are conditional expectation values of temperature dissipation and production rates for fixed magnitude of dimensional temperature $X$. After simple manipulations one obtains a formal expression for probability density $P(X)$ [10]-[12]:

$$P(X) = \frac{C}{r_1(X)} \exp \left[ - \int_0^X \frac{r_3(u) du}{u r_1(u)} \right]$$

or

$$P(X) = \frac{C}{r_1(X)} \exp \left[ - \int_0^X \frac{u v_3(u) du}{u r_1(u)} \right]$$

(10)

We can evaluate this expression in the limit $X \to 0$. First, according to [11]-[12], positive definite conditional dissipation rate

$$r_1(X) \approx \alpha + \beta X^2 = \alpha (1 + \frac{\beta}{\alpha} X^2)$$

Since positive temperature fluctuations (blobs of hotter fluid) are carried by positive velocity fluctuations $v_3$, we conclude that $v_3(T) \approx -v_3(-T)$.

As $Ra - Ra_{cr} \to 0$, the fluctuations of the large-scale rolls, called “convection elements” are very weak, lacking any typical velocity scale. Therefore, in this limit by the symmetry: $v_3(X) \propto X$. At large Rayleigh number the instability of viscous sublayers leads to plumes emitted with a typical velocity $V = y V_p$ where we introduce an artificial Reynolds number $y \propto Re - Re(p)$ with $Re(p)$ denoting the Reynolds number of first instability of boundary layer (manifested in peaks in a heat flux curve) resulting in weak discrete bursts. This means that in this theory $y \geq 0$ and the conditionally averaged velocity can be written as:

$$\frac{v_3(X)}{v_{rms}} \approx \gamma X + 2y V_p/v_{rms} \approx \gamma X + 2\kappa y$$

where $V_p/v_{rms} = O(1)$. We can see that when $y = 0$, the resulting Gaussian flow is dominated by the weak small-scale elements. Substituting all this into (10) gives:

$$P(X, y) = \frac{C(y)}{(1 + \frac{2}{\alpha} X^2)} \exp \left[ - \int_0^X \frac{\gamma u + 2\kappa y du}{\alpha (1 + \frac{2}{\alpha} u^2)} \right]$$

and the probability density of temperature fluctuations in the central part of convection cell with $\alpha = \kappa = 1$ is:

$$P(X, y) = \frac{C(y)}{(1 + \frac{2}{\gamma} X^2)^{1 + \frac{2}{\gamma}}} \exp(-2y \arctan(\sqrt{\beta} X))$$

with $C(y) = 1/2 \int_0^\infty \Pi(X, y) dX$ and $\beta \approx 1.4$ estimated in [12]. As $y \to 0$, this expression gives Gaussian with the half-width $\delta \approx \sqrt{\gamma / \beta}$. It has been found in Ref.[12] that although the derivation is, strictly speaking, valid for $\frac{2}{\gamma} X^2 \to 0$, the result agrees very well with numerical simulations in a much broader interval. An interesting feature of this expression is the dependence of the PDF on Reynolds number $y$. This is the consequence of a qualitative transition happening in the flow $y > 0$. The behavior of the PDF as a function of “Reynolds number” $y$ is shown on Fig.1.

A. Moments of dissipation rate. Low-Re regime.

Based on the above derivation: the conditional mean of kinetic energy dissipation rate is approximated by the expression:

$$\bar{\varepsilon} \approx y X + X^2$$

(12)

and thus, the normalized moments of the dissipation rate are calculated readily

$$\epsilon_n(y) = \frac{\int_0^\infty (y X + X^2)^n P(X, y) dX}{\left( \int_0^\infty (y X + X^2)^2 P(X, y) dX \right)^{n/2}}$$

This expression is valid when Reynolds number is so small that the non-linearity in (7) can be neglected, but large enough to allow for the relatively weak boundary layer instability leading to isolated discrete plumes. This mechanism is similar to the one considered in [16] leading to an intermediate Blasius scaling in a channel flow.

In the limit $y \to 0$, in the interval $y X < X^2$, the probability density $P(X)$ is close to the Gaussian with the first few low-order moments $\epsilon_n \propto X^{2n} \approx (2n - 1)!!.$

It is interesting that the expression (12) with $y \propto Re - Re^{tr}$, reflects two competing mechanisms experimentally
observed by Tong et al. [17]. Indeed, when $y \to 0$, the scale-lacking-excitations dominate while as $y$ grows, the plumes moving with velocity $V = yV_p$ take over. One can also see, as the “Reynolds number” $y$ grows, due to appearance of discrete plumes $yX > X^2$, the PDF (11) varies to close- to -exponential which is an immediate approximation breaks down.

function of Reynolds number. It is promising that the low-Re range behavior of a flow can be addressed numerically using direct numerical simulations. In a general case of the Reynolds number dependent moments, the exponents $d_n(y)$ are found from the equation:

$$e_n(y) = \frac{(\mathcal{E}/\overline{\mathcal{E}})^n}{C_{d_n}(y)} = C_{d_n}(y) \left( \frac{\overline{\mathcal{E}}_{\Lambda, n}}{\overline{\mathcal{E}}_{\Lambda, 0}} \right)^{\frac{n d_n(y)}{2}} \tag{13}$$

where $e_n(y)$ are found from Fig.3. The result is.

$$d_n(y) = \frac{1}{2} [n(2.19 \ln C + \frac{3}{2}) - \frac{\ln e_{2n}}{\ln C}] + \sqrt{\frac{1}{4} [n(2.19 \ln C + \frac{3}{2}) - \frac{\ln e_{2n}}{\ln C}]^2 + \frac{3}{2} n \frac{\ln e_{2n}}{\ln C}} \tag{14}$$

IV. SUMMARY AND DISCUSSION. UNIVERSALITY.

The direct transition from Gaussian case with $M_{2n} = (2n - 1)!$ was investigated both theoretically and numerically in Ref 3-4. The results, based on transitional Reynolds number $R_{tr}$, are presented on Fig.1. It is clear from Fig.2 that in the case of RB convection the...
how universal this number is?

The universality of the Reynolds number based on “turbulent” viscosity $R_{\lambda,T} \approx 10.0$, derived from dynamic Renormalization Group [5]-[8], widely used in engineering [9], is known for many years. In fact, it is the basis of the so-called $K – E$ modeling (see Section I) and Ref.[9]. Numerical and experimental data on flows past the cylinder, decaying turbulence and even flow past various industrial applications like cars, gave for the Reynolds number based on “turbulent viscosity” $R_{\lambda}^{T} \approx 9.0 – 11.0$. In Ref.[5] the transition to anomalous scaling $R_{\lambda}^{T} \approx 9.0$ has been first reported in the DNS of the Navier- Stokes equations on a periodic domain driven by a force $f \approx \alpha v$ defined at the large scales with $\lambda T = k \approx 1 – 2$, completely different from the one discussed in Refs.[3]-[4]. Possible universality of this number may be not too startling. Indeed, while in open, far from equilibrium, system the “bare” $Re_{cr}$ of the first instability of a laminar pattern may vary in a broad interval, the “dressed” one, characterizing transition from “normal” to anomalous dimensions ( intermittency) can be fixed at $R_{\lambda}^{tr} \approx 9.$.

To study the role of the forcing statistics we, assuming for the sake of the argument, universality of transitional constants $R_{\lambda,2}^{T} \approx 9.0$, and $C \approx 90 – 100$, evaluated the exponents $d_n$ from the expression (14). The ratio of exponents $\rho_{2n} = d_n + n$ in the flows driven by gaussian and exponential forces, respectively, is plotted on Fig.5, for $y = 4.5$, in the huge, not experimentally realizable interval $2 \leq n \leq 1000$. One can see the ratio varying in the range $0.925 \leq \frac{\rho_2}{\rho_1} \leq 0.955$, which, though quite close to unity, may indicate existence of universality classes reflecting mechanisms driving turbulence flow.

To conclude the paper we would like to pose a question which can readily be resolved in future numerical and physical experiments: how general is the passage to turbulence, described in this paper, in a typical wall flow where the randomness-generating bulk and wall-layer instabilities often coexist ? Given the results of Ref.[16], this generality may not be impossible. The role of weak-to-strong turbulence transition in chemical kinetics, combustion and mixing in high-Reynolds number fluids may be of importance in various, at present not explained, processes.

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