DIRAC OSCILLATOR VIA R-DEFORMED HEISENBERG ALGEBRA

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Abstract

The complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg algebra is investigated.

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1 Introduction

The relativistic Dirac oscillator proposed by Moshinsky-Szczepaniac [1] is a spin $\frac{1}{2}$ object with the Hamiltonian which in the non-relativistic limit leads to that of a 3-dimensional isotropic oscillator shifted by a constant term plus a $\vec{L} \cdot \vec{S}$ coupling term for both signs of energy. There they construct a Dirac Hamiltonian, linear in the momentum $\vec{p}$ and position $\vec{r}$, whose square leads to the ordinary harmonic oscillator in the non-relativistic limit. The Dirac oscillator have been investigated in several context [2].

The R-deformed Heisenberg algebra or Wigner-Heisenberg algebraic technique [3] was recently super-realized for the SUSY isotonic oscillator [4, 5]. The R-Heisenberg algebra has also been investigated for the three-dimensional non-canonical oscillator to generate a representation of the orthosympletic Lie superalgebra $osp(3/2)$ [6].

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The R-Heisenberg algebra has been found relevant in the context of integrable models \cite{7}, and the Calogero interaction \cite{8, 9}. Recently it has been employed for bosonization of supersymmetry in quantum mechanics \cite{10}, and the discrete space structure for the 3D Wigner quantum oscillator has been investigated \cite{12}. In this work, we obtain the complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg (RDH) algebra.

\section{3D Wigner Oscillator}

In this Section, we provide a three dimension presentation of the Wigner system with its bosonic sector to be the 3D isotropic oscillator (assumed to be of spin-\(\frac{1}{2}\), to aid factorization).

The R-deformed Heisenberg (or Wigner-Heisenberg) algebra is given by following (anti-)commutation relations \((A,B)_+ \equiv AB + BA\) and \((A,B)_- \equiv AB - BA\):

\begin{equation}
H = \frac{1}{2}[a^-, a^+]_+, \quad [H, a^+]_- = \pm a^+, \quad [a^-, a^+]_- = 1 + cR, \quad [R, a^+]_+ = 0, \quad R^2 = 1, \quad (1)
\end{equation}

where \(c\) is a real constant associated to the Wigner parameter \cite{4}. Note that when \(c = 0\) we have the standard Heisenberg algebra.

It is straightforward, following the analogy with the Ref. \cite{4}, to define the superrealizations for the ladder operators \(a^\mp (\vec{\sigma} \cdot \vec{L} + 1)\) for \(H_W \equiv H(\vec{\sigma} \cdot \vec{L} + 1)\) taking the explicitly forms

\begin{equation}
a^\mp = a^\mp (\vec{\sigma} \cdot \vec{L} + 1) = \frac{1}{\sqrt{2}} \left\{ \mp \Sigma_1 \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \pm \frac{1}{r} (\vec{\sigma} \cdot \vec{L} + 1) \Sigma_1 \Sigma_3 - \Sigma_1 r \right\} \quad (2)
\end{equation}

which satisfy together with \(H_W \equiv H(\vec{\sigma} \cdot \vec{L} + 1)\) all the algebraic relations of the RDH algebra with the constant \(\frac{c}{2}\) replaced by \((\vec{\sigma} \cdot \vec{L} + 1)\) and \(R = \Sigma_3\). Note that \((\vec{\sigma} \cdot \vec{L} + 1)\) commutes with all the basic elements \((a^\mp\) and \(H_W)\) of the RDH algebra.

It may be observed that the RDH algebra that gets defined here is in fact three dimensional (one dimension for \(r\) and two for \((\vec{\sigma} \cdot \vec{L} + 1)\)) and is identically satisfied on any arbitrary three dimensional wave function.

On the eigenspaces of the operator \((\vec{\sigma} \cdot \vec{L} + 1)\), the 3D Wigner algebra gets reduced to a 1D from with \((\vec{\sigma} \cdot \vec{L} + 1)\) replaced by its eigenvalue \(\mp (\ell + 1), \ell = 0, 1, 2, \cdots\), where \(\ell\) is the orbital angular momentum quantum number. The eigenfuncitons of \((\vec{\sigma} \cdot \vec{L} + 1)\) for the eigenvalues \((\ell + 1)\) and \(-(\ell + 1)\) are respectivaly given by the well known spin-spherical harmonic \(y_{\pm}\).

Now, considering simultaneous eigenfuncitons of the mutually commuting \(H_W\) and \((\vec{\sigma} \cdot \vec{L} + 1)\) by
\[
\psi_{W,+} = \left( \frac{\tilde{R}_{1,+}(r)}{\tilde{R}_{2,+}(r)} \right) y_+ , \quad (\tilde{\sigma} \cdot \tilde{L} + 1) \psi_{W,+} = (\ell + 1) \psi_{W,+}, \tag{3}
\]
\[
\psi_{W,-} = \left( \frac{\tilde{R}_{1,-}(r)}{\tilde{R}_{2,-}(r)} \right) y_- , \quad (\tilde{\sigma} \cdot \tilde{L} + 1) \psi_{W,-} = -(\ell + 1) \psi_{W,-}, \tag{4}
\]

(where the use of the subscript \(+\)(\(-\)) indicates association with \([y_+(y_-)]\), we observe that
the positive semi-definite form of \(H_W\) the ladder relations and the form of \(H_W\) dictat that
the ground state energy \(E_w^{(0)}(\tilde{\sigma} \cdot \tilde{L} + 1) \geq 0\), where \(E_W(\tilde{\sigma} \cdot \tilde{L} + 1)\) indicates a function of
\(\tilde{\sigma} \cdot \tilde{L} + 1\), is determined by the annihilation condition which reads as two cases.

\section{The Dirac Oscillator Model}

Adding an "anomalous momentum" in the form of a (nonlocal) linear and hermitian inter-
action, \(\tilde{\alpha}.\tilde{\pi} \equiv -iM\omega\beta\tilde{\alpha}.\tilde{r} = (\tilde{\alpha}.\tilde{r})^1\), in the (noncovariant) Dirac free particle equation with
mass \(M\) and spin-\(\frac{1}{2}\), in the natural sistem of units,

\[
i\frac{\partial \psi}{\partial t} = (\tilde{\alpha}.\tilde{p} + M\beta)\psi , \tag{5}
\]

one obtains the equation for the Dirac oscillator \([1]\):

\[
i\frac{\partial \psi}{\partial t} = \{\tilde{\alpha}.(\tilde{p} + \tilde{\pi}) + M\beta\}\psi , \tag{6}
\]

where \(M\) and \(\omega\) are, respectively, the mass of the particle and the frequency of the oscillator,
and the matrices \((\tilde{\alpha}, \beta)\) satisfy the following properties:

\[
[\alpha_i, \beta_j] = 0, \quad [\alpha_i, \alpha_j] = 2\delta_{ij}1, \quad \beta^2 = 1 = \alpha_i^2 , \quad (i, j = 1, 2, 3). \tag{7}
\]

Writing the Dirac spinor in terms of the upper and lower components, respectively, \(\psi_1\) and
\(\psi_2\), \(\Psi(\tilde{r}, t) = \exp(-iEt) \begin{bmatrix} \psi_1(\tilde{r}) \\ \psi_2(\tilde{r}) \end{bmatrix}\) the standard representation of the matrices \(\tilde{\alpha}\) and \(\beta\).

\section{The Dirac oscillator via RDH algebra}

In this section, we implement a new realization of the Dirac oscillator in terms of elements
of the R-deformed Heisenberg algebra. To solve the equation Dirac, following the usual
procedure, we consider the second order differential equation,

\[
\tilde{H}_D\psi(\tilde{r}) = E\psi(\tilde{r}), \tag{8}
\]
where $\tilde{H}_D$ is a second order Hamiltonian, $\tilde{H}_D = H_D^2 + M^2 1$, $\tilde{E} = \frac{E^2 - M^2}{2M}$. In the spherical polar coordinate system, we obtain the non-relativistic form of the Hamiltonian $U$ [11], for an isotropic 3D SUSY harmonic oscillator with spin-$\frac{1}{2}$.

We consider a unitary operator in terms of the radial projection of the spin,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_r \end{bmatrix} = U^{-1} = U^\dagger,$$

(9)

to obtain the following relation between the transformed Dirac Hamiltonian, $\tilde{H}_D$, the 3D Wigner Hamiltonian, $H_W$, and the SUSY Hamiltonian, $H_{SUSY}$ [11]:

$$H_{SUSY} = U \tilde{H}_D U^\dagger = H_W - \frac{1}{2} \{ 1 + 2(\vec{\sigma} \cdot \vec{L} + 1) \Sigma_3 \} \omega \Sigma_3.$$  

(10)

4.1 The energy spectrum of the Dirac oscillator

The energy spectra of the operators $\tilde{H}_D$ and $H_{SUSY}$ are identical, since these operators are related by a unitary transformation. However, the relation between the principal quantum number $N$ and the angular momentum ($\ell$) is different, in each case. Obviously, the energy spectrum associated with the two types of eigenspaces belonging to the eigenvalues $\pm (\ell + 1)$:

Case(i) $\vec{\sigma} \cdot \vec{L} + 1 \rightarrow \ell + 1 = j + \frac{1}{2}, \quad j = \ell + \frac{1}{2}$

$$\tilde{E}_{N\ell} = \frac{E^2 - M^2}{2M} = \begin{cases} 2m\omega = \tilde{E}^{+}_{N(\ell+1)}, \\ 2(m+1)\omega = \tilde{E}^{-}_{N\ell}, \end{cases}$$

(11)

where $m = 0, 1, 2, \ldots$.

Case(ii) $\vec{\sigma} \cdot \vec{L} - 1 \rightarrow - (\ell + 1) = -(j + \frac{1}{2}), \quad j = (\ell + 1) - \frac{1}{2}$

$$\tilde{E}_{N\ell} = \frac{E^2 - M^2}{2M} = \begin{cases} (N + j + 3/2)\omega = \tilde{E}^{+}_{N\ell}, \quad N = j - 1/2, j + 3/2, j + 7/2, \ldots, \\ (N + j + 5/2)\omega = \tilde{E}^{-}_{N(\ell+1)}, \quad N = j + 1/2, j + 5/2, \ldots. \end{cases}$$

5 Conclusion

In this work we investigate the Dirac oscillator with the help of techniques of super-realization of the R-deformed Heisenberg algebra.

The Dirac oscillator with different interactions has been treated by Castaños et al. and by Dixit et al. [2]. These works motivate the construction of a new linear Hamiltonian in terms of the momentum, position and mass coordinates, through a set of seven mutually anticommuting 8x8-matrices yielding a representation of the Clifford algebra $\mathbb{C}l_7$. The seven elements of the Clifford algebra $\mathbb{C}l_7$ generate the three linear momentum components, the three position coordinates components and the mass, and their squares are the 8x8-identity matrix $I_{8\times8}$. Results of our analysis on Dirac oscillator via the Clifford algebra $\mathbb{C}l_7$ are in preparation.
In a forthcoming paper we show that the Dirac oscillator equation can be resolved algebraically without having to transform it into a second order differential equation. Therefore, the important connection for the Dirac 3D-isotropic oscillator with the linear ladder operators of the R-deformed Heisenberg algebra, satisfying the concomitant general oscillator quantum rule of Wigner, have explicited in this work.

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