CHARACTERIZATION OF FINITELY GENERATED
INFINITELY ITERATED WREATH PRODUCTS

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Abstract. Given a sequence of \((G_i)_{i \in \mathbb{N}}\) of finite transitive groups of degree \(n_i\), let \(W_\infty\) be the inverse limit of the iterated permutational wreath products \(G_m \wr \cdots \wr G_2 \wr G_1\). We prove that \(W_\infty\) is (topologically) finitely generated if and only if \(\prod_{i=1}^{\infty} (G_i/G'_i)\) is finitely generated and the growth of the minimal number of generators of \(G_i\) is bounded by \(d \cdot n_1 \cdots n_{i-1}\) for a constant \(d\). Moreover we give a criterion to decide whether \(W_\infty\) is positively finitely generated.

1. Introduction

Let \((G_i)_{i \in \mathbb{N}}\) be a sequence of finite transitive permutation groups of degree \(n_i\) and let \(W_m = G_m \wr \cdots \wr G_2 \wr G_1\) be the iterated (permutational) wreath product of the first \(m\) groups. The infinitely iterated wreath product is the inverse limit

\[ W_\infty = \lim_{\leftarrow} W_m = \lim_{\leftarrow} (G_m \wr \cdots \wr G_2 \wr G_1). \]

In a recent paper Bondarenko \[2\] studies some sufficient conditions on the sequence \((G_i)_{i \in \mathbb{N}}\) to get that the profinite group \(W_\infty\) is (topologically) finitely generated: under the conditions that the minimal number of generators \(d(G_i)\) of \(G_i\) is bounded by a constant \(d\) and \(\prod_{i=1}^{\infty} (G_i/G'_i)\) is finitely generated, using techniques from branch groups, he produces a finitely generated dense subgroup of \(W_\infty\).

Since \(\prod_{i=1}^{\infty} (G_i/G'_i)\) is a homomorphic image of \(W_\infty\), the second condition is clearly also a necessary condition: if \(W_\infty\) is generated as a profinite group by \(d\) elements, then \(d(\prod_{i=1}^{\infty} (G_i/G'_i)) \leq d\).

Another necessary condition comes from the observation that if \(K\) is a finite permutation group of degree \(n\) and \(H\) is finite, then \(d(H) \leq n \cdot d(H \wr K)\) (see the remark at the beginning of section 5). Since \(W_i = G_i \wr W_{i-1}\) where \(W_{i-1}\) is a permutation group of degree \(n_1n_2\cdots n_{i-1}\), it follows that if \(W_\infty\) is finitely generated by \(d\) elements, then \(d(G_i) \leq d \cdot n_1n_2\cdots n_{i-1}\) for every \(i > 1\).

The main result of this paper is that these two necessary conditions are also sufficient.

Theorem 1. Let \((G_i)_{i \in \mathbb{N}}\) be a sequence of transitive permutation groups of degree \(n_i\). The inverse limit \(W_\infty\) of the iterated wreath products \(G_m \wr \cdots \wr G_2 \wr G_1\) is finitely generated if and only if

1. \(\prod_{i=1}^{\infty} (G_i/G'_i)\) is finitely generated,
2. there exists an integer \(d\) such that \(d(G_i) \leq d \cdot n_1 \cdots n_{i-1}\) for every \(i > 1\).
Actually, we prove that there exists an absolute constant $k_0$ such that
\[ d(W_\infty) \leq \max(d + 2, d(W_{i_0})) + d \left( \prod_{i=1}^{\infty} (G_i/G'_i) \right). \]

where $i_0$ is the first index such that $n_1 \cdots n_{i_0-1} \geq \log_{60} k_0$. Indeed $k_0$ is the smallest positive integer with the property: if a finite group $L$ has a unique minimal normal subgroup $N$ and $|N| \geq k_0$, then $P_L(d) \geq \frac{1}{2} P_{L/N}(d)$ for each $d \geq 2$, where $P_L(d)$ (resp. $P_{L/N}(d)$) denotes the probability of generating $L$ (resp. $L/N$) with $d$ elements. The existence of such a constant is ensured by the main theorem in [19]. On the other hand we conjecture that for every $d \geq 2$ and every monolithic group $L$ with socle $N$
\[ P_L(d) \geq \frac{53}{90} P_{L/N}(d) \]

(1.1)

(the equality holds if $L = \text{Alt}(6)$ and $d = 2$). If this were true, our result would become
\[ d(W_\infty) \leq \max(d + 2, d(W_1)) + d \left( \prod_{i=1}^{\infty} (G_i/G'_i) \right). \]

For example, the inequality (1.1) is satisfied if the socle of $N$ is a direct power of alternating or sporadic simple groups [25]: this implies that if every non-abelian composition factor in the $G_i$’s is alternating or sporadic, then $d(W_\infty) \leq \max(d + 2, d(W_1)) + d \left( \prod_{i=1}^{\infty} (G_i/G'_i) \right)$.

The proof of Theorem 1 relies on a generalization to the “non-soluble” case of some results in [15] and [16]. In that papers the author considered the generation of the wreath product $W = H \wr K$ of two finite permutation groups $H$ and $K$ and a formula was found for $d(W)$ in the case where $H$ is soluble. Later, in [4], the minimal number of generators of a group $G$ was connected to some special homomorphic images of $G$ whose behavior can be studied with the help of an equivalence relation among the chief factors of $G$ (see section 2 for more details). Using these new techniques, we are able to control the “non-abelian” part of the problem and to produce a formula for $d(W)$ whenever the degree of $K$ is large enough.

Infinitely iterated wreath products appear in literature with several motivations. For example they can be viewed as automorphism groups of suitably constructed rooted trees and play a relevant role in the study of self-similar groups (see e.g. [9], [10]). Moreover, they provide a useful tool to construct examples and counterexamples in the context of profinite groups (see e.g. [20], [24], [17]). Bhattacharjee [1] and Quick [21] [22] considered wreath products of non-abelian simple groups with transitive action and proved that their inverse limit is generated by 2 elements even with positive probability. Recall that a profinite group $G$ may be viewed as a probability space with respect to the normalized Haar measure and that $G$ is called positively finitely generated (PFG) if for some $k$ a random $k$-tuple generates $G$ with positive probability. From the papers of Bhattacharjee and Quick, it follows that an infinitely iterated wreath product of transitive groups $G_i$’s is PFG when every $G_i$ is a nonabelian simple group. However in [17] an example is given of an infinitely iterated wreath product of transitive groups that is 2-generated but non PFG.
In Proposition [15] with the help of a result by Jaikin-Zapirain and Pyber [11], we will obtain a criterion that makes it possible to decide whether \( W_\infty \) is PFG from information on the structure of the transitive groups \( G_i \)'s and their degree \( n_i \)'s.

2. Generating crown-based powers

Let \( L \) be a monolithic primitive group and let \( A \) be its unique minimal normal subgroup. For each positive integer \( k \), let \( L^k \) be the \( k \)-fold direct product of \( L \). The crown-based power of \( L \) of size \( k \) is the subgroup \( L_k \) of \( L^k \) defined by

\[
L_k = \{(l_1, \ldots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \mod L\}.
\]

Equivalently, \( L_k = A^k \text{Diag} L^k \).

Let, as usual, \( d(G) \) denote the minimal number of generators of a finite group \( G \). In [4] it is proved that for every finite group \( G \) there exists a monolithic group \( L \) and a homomorphic image \( L_k \) of \( G \) such that

1. \( d(L/\text{soc} L) < d(G) \)
2. \( d(L_k) = d(G) \).

An \( L_k \) with this property will be called a generating crown-based power for \( G \). In [4] it is explained how \( d(L_k) \) can be computed in terms of \( k \) and the structure of \( L \). A key ingredient when one wants to determine \( d(G) \) from the behavior of the crown-based power homomorphic images of \( G \) is to evaluate for each monolithic group \( L \) the maximal \( k \) such that \( L_k \) is a homomorphic image. This integer \( k \) comes from an equivalence relation among the chief factors of \( G \). More generally, following [12], we say that two irreducible \( G \)-groups \( A \) and \( B \) are \( G \)-equivalent and we put \( A \sim_G B \), if there is an isomorphism \( \Phi : A \rtimes G \to B \rtimes G \) such that the following diagram commutes:

\[
\begin{array}{cccccc}
1 & \longrightarrow & A & \longrightarrow & A \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & \Phi & \downarrow & \Phi & & & & \downarrow & \\
1 & \longrightarrow & B & \longrightarrow & B \rtimes G & \longrightarrow & G & \longrightarrow & 1
\end{array}
\]

Note that two \( G \)-isomorphic \( G \)-groups are \( G \)-equivalent. In the particular case where \( A \) and \( B \) are abelian the converse is true: if \( A \) and \( B \) are abelian and \( G \)-equivalent, then \( A \) and \( B \) are also \( G \)-isomorphic. It is proved that two chief factors \( A \) and \( B \) of \( G \) are \( G \)-equivalent if and only if either they are \( G \)-isomorphic between them or there exists a maximal subgroup \( M \) of \( G \) such that \( G/\text{Core}_G(M) \) has two minimal normal subgroups \( N_1 \) and \( N_2 \) \( G \)-isomorphic to \( A \) and \( B \) respectively. For example, the minimal normal subgroups of \( L_k \) are all \( L_k \)-equivalent.

Let \( A = X/Y \) be a chief factor of \( G \). A complement \( U \) to \( A \) in \( G \) is a subgroup \( U \) of \( G \) such that \( UX = G \) and \( U \cap X = Y \). We say that \( A = X/Y \) is Frattini if \( X/Y \) is contained in the Frattini subgroup of \( G/Y \); this is equivalent to say that \( A \) is abelian and there is no complement to \( A \) in \( G \). The number \( \delta_G(A) \) of non-Frattini chief factors \( G \)-equivalent to \( A \) in any chief series of \( G \) does not depend on the series. Now, we denote by \( L_A \) the monolithic primitive group associated to \( A \), that is

\[
L_A = \begin{cases} 
A \rtimes (G/C_G(A)) & \text{if } A \text{ is abelian}, \\
G/C_G(A) & \text{otherwise}.
\end{cases}
\]
If $A$ is a non-Frattini chief factor of $G$, then $L_A$ is a homomorphic image of $G$. More precisely, there exists a normal subgroup $N$ such that $G/N \cong L_A$ and $soc(G/N) \sim_G A$ (in the following we will sometimes identify $soc L_A$ with $A$ as $G$-groups). Consider now all the normal subgroups $N$ with the property that $G/N \cong L_A$ and $soc(G/N) \sim_G A$: the intersection $R_G(A)$ of all these subgroups has the property that $G/R_G(A)$ is isomorphic to the crown-based power $(L_A)_{\delta_G(A)}$ for short). The socle $I_G(A)/R_G(A)$ of $G/R_G(A)$ is called the $A$-crown of $G$ and it is a direct product of $\delta_G(A)$ minimal normal subgroups $G$-equivalent to $A$. Later we will use the facts that

$$I_G(A) = \{ g \in G \mid g \text{ induces an inner automorphism on } A \}$$

and $A \sim_G B$ implies $I_G(A) = I_G(B)$. In particular, if $A$ and $B$ are chief factors of $G$ and $A \sim_G B$, then $R_G(A) = R_G(B)$ and $L_A \cong L_B$.

Note that if $L_k$ is a homomorphic image of $G$ for some $k \geq 1$ then $L$ is associated to a non-Frattini chief factor $A$ of $G$ ($L \cong L_A$) and $k \leq \delta_G(A)$. If $L_{A,k}$ is a generating crown-based power then $L_{A,\delta_G(A)}$ has the same property: in this case, by abuse of notation, we will say that $A$ is a generating chief factor for $G$.

The minimal number of generators of a generating crown-based power can be computed when $A$ is abelian with the help of the following formula: for an irreducible $G$-module $M$, set

$$r_G(M) = \dim_{End_G(M)} M \quad s_G(M) = \dim_{End_G(M)} H^1(G,M)$$

and define

$$h_G(M) = \begin{cases} 
\delta_G(M) & \text{if } M \text{ is a trivial } G\text{-module}, \\
\frac{s_G(M)-1}{r_G(M)} + 2 & \text{otherwise}.
\end{cases}$$

Note that, as $G/R \cong L_{M,k}$ where $R = R_G(M)$ and $k = \delta_G(M)$, we have $\delta_G(M) = \delta_G/R(M) = \delta_{L_{M,k}}(M)$. Moreover, if $\delta_G(M) > 0$, then $R \leq C_G(M)$ and $\dim_{End_G(M)} H^1(G,M) = \delta_G(M) + \dim_{End_G(M)} H^1(G/C_G(M),M)$ (see e.g. [1.2] in [10]) and therefore $r_G(M) = r_{G/R}(M)$ and $s_G(M) = s_{G/R}(M)$. We conclude that if $\delta_G(M) > 0$, then

$$h_G(M) = h_{L_{M,\delta_G(M)}}(M). \quad (2.1)$$

From a result by Gaschütz [8 Satz 2], we have that either $h_G(M) = d(L_{M,\delta_G(M)})$ or $h_G(M) < d(L_{M,\delta_G(M)})$. Therefore we have the following:

**Proposition 2.** If there exists an abelian generating chief factor of $A$ of $G$, then

$$d(G) = h_G(A).$$

In our discussion we will employ different arguments according to the existence or not of an abelian generating chief factor. In the first case it is useful to notice that

**Proposition 3.** Let $d(I_G)$ be the minimal number of generators of the augmentation ideal of $\mathbb{Z}G$ as a $G$-module. If $G$ has an abelian generating chief factor $A$, then

$$d(G) = d(I_G) = h_G(A).$$

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Proof. By a result of Cossey, Gruenberg and Kovács [3, Theorem 3]
\[ d(I_G) = \max\{h_G(M) \mid M \text{ irreducible } G\text{-module}\}, \]
thus \( d(I_G) \geq h_G(A) = d(G) \). Since \( d(I_G) \leq d(G) \), we have an equality. \( \square \)

Theorem 1 will be derived by an extension to the non-abelian crowns of the following:

**Proposition 4** (Proposition 1 [10]). If \( H \) is a finite group and \( G \) is a transitive permutation group of degree \( n \), then
\[ d(I_{H\langle G \rangle}) = \max \left\{ d(I_{H/H'}), \left[ \frac{d(I_H) - 2}{n} \right] + 2 \right\}. \]

### 3. Crowns in wreath products

Let \( H \) be a finite group and \( K \) a transitive group of degree \( n \) and denote by \( W = H \wr K = H^n \rtimes K \) the (permutational) wreath product of \( H \) and \( K \), where \( K \) permutes the components of the base subgroup \( H^n = H_1 \times \cdots \times H_n \).

In this section we want to study the relation between the chief factors of \( H \) and the chief factors of \( W \). First note that if \( A \) is an \( H \)-group then \( A^n \) can be seen as a \( W \)-group where \( H^n \) acts componentwise and \( K \) permutes the components of the elements. When dealing with \( A^n \) as a \( W \)-group we will usually refer to this action. We say that an \( H \)-group \( A \) is irreducible if the only \( H \)-groups contained in \( A \) are \( A \) and \( \{1\} \); we say that an \( H \)-group is trivial if the action of \( H \) on \( A \) is the trivial one, that is \( H = C_H(A) \).

**Proposition 5.** Let \( A \) and \( B \) be irreducible \( H \)-groups.

1. If \( A \) is a non-trivial \( H \)-group, then \( A^n \) is an irreducible non-trivial \( W \)-group.
2. If \( A \sim_H B \) then \( A^n \sim_W B^n \).
3. If \( A \) and \( B \) are non-trivial \( H \) groups and \( A \sim_H B \), then \( A^n \sim_W B^n \).
4. If \( A \) is a non-central chief factor of \( H \) and \( L \) is the associated monolithic group, then \( A^n \) is a chief factor of \( W \) and the monolithic primitive group associated to \( A^n \) is isomorphic to \( L \setminus K \).

Proof. (1) Let \( N \neq 1 \) a \( W \)-group contained in \( A^n = A_1 \times \cdots \times A_n \) and let \( 1 \neq (x_1, \ldots, x_n) \in N \) be a non trivial element. As \( K \) is transitive on the components, we can assume \( x_1 \neq 1 \). Note that \( C_A(H) \) is a proper \( H \)-subgroup of \( A \), hence \( C_A(H) = 1 \) by irreducibility of \( A \). Thus \( [x_1, H] \neq 1 \) and in particular \([x_1, H]\) is a non-trivial \( H \)-subgroup of \( A \), hence \([x_1, H] = A \). Therefore \( [(x_1, \ldots, x_n), H] = [x_1, H] \times \{1\} \) and \( [x_1, H] \) is contained in \( N \) and, by the transitivity of the action of \( K \), we conclude that \( A^n \leq N \).

(2) Let \( A \sim_H B \): there exists an isomorphism \( \Phi : A \rtimes H \to B \rtimes H \) such that the following diagram commutes:
\[
\begin{array}{cccccc}
1 & \longrightarrow & A & \longrightarrow & A \rtimes H & \longrightarrow & H & \longrightarrow & 1 \\
\downarrow^{\phi} & & \downarrow^{\Phi} & & \downarrow & & \downarrow & \\
1 & \longrightarrow & B & \longrightarrow & B \rtimes H & \longrightarrow & H & \longrightarrow & 1 \\
\end{array}
\]

(3.1)
Now define $Ψ : A^n × W → B^n × W$ by the position
$$((a_1, \ldots, a_n)(h_1, \ldots, h_n)k)^ψ = (a_1^ψ, \ldots, a_n^ψ)(h_1^ψ, \ldots, h_n^ψ)k.$$  
Thus $Ψ$ is a well defined isomorphism for which the following diagram is commutative:

$$
\begin{array}{cccccc}
1 & → & A^n & → & A^n × W & → & W & → & 1 \\
\downarrow{ψ} & & \downarrow{ψ} & & \parallel & & \parallel & &  \\
1 & → & B^n & → & B^n × W & → & W & → & 1 \\
\end{array}
$$

(3.2)

where $ψ$ is the restriction to $A^n$ of $Ψ$, and therefore $A^n ∼_W B^n$.

Assume, by contradiction, that $A^n ∼_W B^n$. We first consider the case where $A$ and $B$ are abelian. Then the $W$-equivalence relation is simply the $W$-isomorphism relation and $A^n ∼_W B^n$ implies that there exists a $W$-isomorphism $ψ : A^n → B^n$. Note that $C_{A^n}(K) = \text{Diag}(A^n) ≅ A$ and similarly $C_{B^n}(K) = \text{Diag}(B^n) ≅ B$. Since $ψ$ is a $W$-isomorphism, the restriction of $ψ$ to $C_{A^n}(K)$ is a $W$-isomorphism between $C_{A^n}(K) = \text{Diag}(A^n)$ and $C_{B^n}(K) = \text{Diag}(B^n)$. This implies that there is an $H$-isomorphism between $A$ and $B$, and we conclude that $A ∼_H B$.

We now consider the case where $A$ and $B$ are non-abelian. Assume that the diagram (3.2) is commutative. First of all we note that the minimal normal subgroups of $A^n × H^n$ contained in $A^n$ are the subgroups $A_i$. Moreover the $A_i^ψ$ are minimal normal subgroups of $(A^n × H^n)^ψ = B^n × H^n$ contained in $(A^n)^ψ = B^n$. It follows that $A_i^ψ = B_j$ for some $j$. In particular $A ≅ B$ as groups.

If $A_i^ψ = B_1$, then consider that $[\prod_{i>1} A_i, H_1] = 1$ implies

$$[\prod_{i>1} A_i, H_1]^ψ = \left[ \prod_{i>1} A_i^ψ, H_1^ψ \right] = \left[ \prod_{i>1} B_i, H_1^ψ \right] = 1$$

thus $H_1^ψ ≤ C_{B^n × H^n}(\prod_{i>1} B_i)$. Moreover, $H_1^ψ ≤ B^n × H_1$ since the right part of the diagram (3.2) commutes, and therefore

$$H_1^ψ ≤ C_{B^n × H^n}(\prod_{i>1} B_i) \cap (B^n × H_1) ≤ B_1 × H_1.$$

It follows that the following diagram commutes

$$
\begin{array}{cccccc}
1 & → & A_1 & → & A_1 × H_1 & → & H_1 & → & 1 \\
\downarrow{ψ} & & \downarrow{ψ} & & \parallel & & \parallel & &  \\
1 & → & B_1 & → & B_1 × H_1 & → & H_1 & → & 1 \\
\end{array}
$$

and $A_1 ∼_H B_1$. Since the action of $H$ on $A$ and $B$ is equal to the action of $H_1$ on $A_1$ and $B_1$ respectively, $A ∼_H B$ and we are done.

We are left with the case $A_i^ψ ≠ B_1$; then there exists $j ≠ 1$ such that $A_j^ψ = B_1$. Note that we can not argue as above, since now $A_i^ψ × H_i^ψ$ is contained in $B_1 B_j × H_1$ but not in $B_j × H_1$ and hence we can not simply “restrict” the diagram (3.2) to one component.
Let \( A \) which send sketch a direct proof.

**Proof.** This is a consequence of the definition (see remark after Proposition 1.2) and Theorem 11.4.10 in [24].

Since the right part of the diagram commutes, for every \( h \in H_1 \) there exist unique elements \( b_i \in B \) such that \( h^g = (b_1, \ldots, b_n) \): we define the map \( \beta : H_1 \to B_1 \) by sending \( h \) to the element \( h^g = (b_1, 1, \ldots, 1) \). Then \([H_1, A_j] = 1 \) implies \([H_1, B_1] = 1 \) and hence \( h^g \in H \) commutes with every element of \( B_1 \). It follows that the map \( \Theta : A_j \times H_1 \to B_1 \times H_1 \) defined by \((a, h)^g = a h^g h \) is a well defined homomorphism for which the following diagram is commutative

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & A_j & \longrightarrow & A_j \times H_1 & \longrightarrow & H_1 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & B_1 & \longrightarrow & B_1 \times H_1 & \longrightarrow & H_1 & \longrightarrow & 1
\end{array}
\]

and hence \( A_j \sim_{H_1} B_1 \) (note that the action of \( H_1 \) on \( A_j \) is the trivial one and it is not equivalent to the action of \( H \) on \( A \)).

Now, by definition,

\[
I_{H_1}(A_j) = \{ x \in H_1 \mid x \text{ induces an inner automorphism on } A_j \} = H_1,
\]

hence \( A_j \sim_{H_1} B_1 \) implies \( I_{H_1}(B_1) = I_{H_1}(A_j) = H_1 \). Then \( I_{H}(B^n) = (I_{H}(B))^n = H^n \), and since \( B^n \sim_{H} A^n \), we get \( I_{H}(A^n) = I_{H}(B^n) = H^n \). Therefore \( I_{H}(A) = H = I_{H}(B) \). As we will see in the subsequent Lemma 6 from the facts that \( I_{H}(A) = H = I_{H}(B) \) and that \( A \cong B \) as groups, we get that \( A \) and \( B \) are \( H \)-equivalent to the same trivial \( H \)-group. By transitivity, it follows that \( A \sim_{H} B \) and this gives the desired contradiction.

(4) Let \( A \) be a chief factor of \( H \). Then \( L \cong H/C_H(A) \) if \( A \) is non-abelian, \( L \cong A \times H/C_H(A) \) otherwise. Note that \( C_W(A^n) \leq \cap_{i=1}^n C_W(A_i) \leq H^n \), as the action of \( K \) on the components is faithful. Hence \( C_W(A^n) = C_H(A^n) \). Then \( W/C_W(A^n) \cong (H/C_H(A))/K \) and the result follows.

\[ \square \]

**Lemma 6.** Let \( A \) be a \( G \)-group with trivial center. If \( I_G(A) = G \) then \( A \) is \( G \)-equivalent to the trivial \( G \)-group \( A' \), where \( A' = A \) as a group.

**Proof.** This is a consequence of the definition (see remark after Proposition 1.2 in [12]) and Theorem 11.4.10 in [24], but for the readers’ convenience, we will sketch a direct proof.

Since \( A \) has trivial center and \( I_G(A) = G \), there is a homomorphism \( f : G \to A \) which send \( g \in G \) to the element \( f(g) \) in \( A \) such that \( a f(g) = a^g \) for every \( a \in A \). Let \( A^* \) be the trivial \( G \)-group equal to \( A \) as a group. Now we define

\[
\Phi : A^* \times G \to A \times G
\]

\[
(a, g) \mapsto af(g)^{-1}g.
\]

Note that, by definition of \( f \), for every \( g \in G \) the element \( f(g)^{-1}g \) centralizes the elements of \( A \) in \( A \times G \). Thus \((a_1, g_1)(a_2, g_2)^\Phi = (a_1a_2, g_1g_2)^\Phi = a_1a_2f(g_1g_2)^{-1}g_1g_2 = a_1(a_2f(g_2)^{-1})(f(g_1)^{-1}g_1)g_2 = a_1(f(g_1)^{-1}g_1)(a_2f(g_2)^{-1})g_2 = (a_1, g_1)^\Phi(a_2, g_2)^\Phi \), since \( a_2f(g_2)^{-1} \in \hat{A} \). This shows that \( \Phi \) is a homomorphism.
Then the following diagram is commutative:

\[
\begin{array}{cccccc}
1 & \longrightarrow & A^* & \longrightarrow & A^* \times G & \longrightarrow & G & \longrightarrow & 1 \\
\| & & \downarrow\phi & & \| & & \| & & \| \\
1 & \longrightarrow & A & \longrightarrow & A \times G & \longrightarrow & G & \longrightarrow & 1 \\
\end{array}
\]

and we conclude that \( A \sim_G A^* \). \( \Box \).

From now on, \( B \) will denote the base subgroup \( H^n \) of \( W = H \wr K = B \times K \).
Let us fix a chief series of \( H \) passing through the derived subgroup \( H' \) of \( H \)
\[
1 = N_t \lhd N_{t-1} \lhd \cdots \lhd N_1 \lhd N_0 = H. \tag{3.3}
\]

Since every \( N_i^n \) is normal in \( W \), we can refine the series \( (N_i^n)_i \) to get a \( W \)-chief series of \( B \) passing through the derived subgroup \( B' \)
\[
1 = M_{s_i} \lhd \cdots \lhd M_{s_{i-1}} = N_{i-1}^n \lhd \cdots \lhd M_{s_1} = N_1^p = B' \lhd \cdots M_1 \lhd M_0 = B. \tag{3.4}
\]

For every prime \( p \), let \( d_p(H/H') \) be the minimal number of generators of the
Sylow \( p \)-subgroup of \( H/H' \). Note that \( d_p(H/H') = h_{H/H'}(A) \) where \( A \) is a central non-Frattini chief-factor of \( H/H' \) of order \( p \). Moreover, if \( A = X/Y \) is a central non-Frattini (i.e. complemented) chief-factor of \( H \), then \( X \) can not be contained in \( H' \); therefore
\[
d_p(H/H') = h_H(\mathbb{F}_p) = h_{H/H'}(\mathbb{F}_p) \tag{3.5}
\]
where \( A \sim_H \mathbb{F}_p \) and \( \mathbb{F}_p \) is the irreducible trivial \( \mathbb{F}_p H \)-module.

**Proposition 4.** Let \( M = M_i/M_{i+1} \) be a non-Frattini chief factor of the series \( \mathfrak{A} \).

(1) If \( M_i \leq B' \), then there exists a non-Frattini chief factor \( A = X/Y \) of the series \( \mathfrak{A} \) contained in \( H' \) such that \( M = X^n/Y^n \). Moreover \( M \) is not \( W \)-equivalent to any chief factor of \( W/B' \), \( \delta_W(M) = \delta(H)(A) \) and \( L_M \cong L_A \rtimes K \).

(2) If \( B' \leq M_{i+1} < M_i \leq B \), then \( \delta_W(M) \leq \delta_K(M) + d_p(H/H')_rK(M) \).

(3) If \( B \leq M_{i+1} \), and \( M \) is not equivalent to any \( W \)-chief factor of \( B/B' \),
then the action of \( W \) on \( M \) induces an action of \( K \) on \( M \), \( \delta_W(M) = \delta_K(M) \) and the primitive monolithic group associated to \( M \) is the same in the two actions.

**Proof.** (1) We first prove that the map \( A = X/Y \mapsto A^n = X^n/Y^n \) gives a bijection between the set of non-Frattini chief factors of the series \( \mathfrak{A} \) contained in \( H' \) and the set of non-Frattini chief factors of the series \( \mathfrak{A} \) contained in \( B' \).
Let \( A = X/Y \) be a non-Frattini chief factor of the series \( \mathfrak{A} \) contained in \( H' \). Note that the central complemented chief factors of \( \mathfrak{A} \) lie above \( H' \). Then \( A \) is not central and hence, by Proposition 5, we have that \( A^n \) is a non-central chief factor of the series \( \mathfrak{A} \) contained in \( B' \). Moreover, if \( U \) is a complement to \( A \) in \( H \), then \( U \rtimes K \) is a complement to \( A^n \) in \( W \). This implies that the map is well defined.
To prove that the map is bijective, it is sufficient to show that if \( A = N_i/N_{i+1} \) is a Frattini chief factor of \( H \), then every chief factor \( X/Y \) of the series \( \mathfrak{A} \) with \( N^a_{i+1} \leq Y < X \leq N^a_i \) is Frattini. We can assume
\( N_{i+1} = 1; \) thus \( A \leq \text{Frat} \, H \) and \( A^n \leq (\text{Frat} \, H)^n = \text{Frat} \, B \leq \text{Frat} \, W \) and we are done.

To prove that \( \delta_H(A) = \delta_W(A^n) \) it is sufficient to show that \( A^n \) can not be equivalent to any \( W \)-chief factor containing \( B' \), indeed, from Proposition \( \text{[5]} \), we already know there are \( \delta_H(A) \) chief factors of \( \text{[6,4]} \) \( W \)-equivalent to \( A^n \) inside \( B' \). Assume, by contradiction, that \( A^n \sim_W M = X/Y \) where \( B' \leq Y \leq X \leq W \). Then \( I_W(A^n) = I_W(M) \). But, on one hand, \( I_W(A^n) = (I_H(A))^n \leq B \), on the other hand \( I_W(M) = XCG(W)(X) \). This implies that \( X \leq B \) and \( I_W(M) = B \). In particular, as \( B' \leq Y \), \( M \) is centralized by \( B \). Therefore the two factors \( A^n \) and \( M \) are abelian, the equivalence relation reduces to a \( W \)-isomorphism and hence \( A^n \) is centralized by \( B \). It follows that \( A \) is a central factor of \( H \), but this is a contradiction, since complemented central chief factors of \( \text{[3,3]} \) lie above \( H' \).

Finally, by Proposition \( \text{[5]} \), we get that \( L_M = L_A \cap K \).

(2) Set \( \overline{H} = H/H' \) and note that \( B \leq C_W(M) \), hence the action of \( W \) on \( M \) induces an action of \( K \) on \( M \). We follow the arguments of Lemma 2.1 \( \text{[15]} \) and Lemma 4.1 in \( \text{[16]} \). Since we are dealing with non-Frattini factors, we can assume that the Frattini subgroup of \( \overline{H} \) is trivial. The Sylow \( p \)-subgroup \( \overline{H}_p \) of \( \overline{H} \) is a vector space of dimension \( d = d_p(\overline{H}) \) generated, let say, by the elements \( h_1, \ldots, h_d \). Then the Sylow \( p \)-subgroup \( \overline{H}^n_p \) of \( \overline{H}^n \) is generated, as an \( \mathbb{F}_pK \)-module, by the elements \( (h_i, 1, \ldots, 1) \). In particular \( \overline{H}^n \) is the direct sum of \( d \) cyclic \( \mathbb{F}_pK \)-modules, and the number of complemented \( \mathbb{F}_pK \)-modules \( K \)-equivalent to \( M \) in \( \overline{H}^n \) is at most \( d_p(\overline{H})r_K(M) \) where \( r_K(M) = \text{dim}_{\text{End}_K(M)}(M) \) (see Lemma 2.1 \( \text{[15]} \)). It follows that \( \delta_W(M) \leq \delta_K(M) + d_p(\overline{H})r_K(M) \).

(3) It is sufficient to note that \( B \leq C_W(M) \) and that, by the first part of the proposition, \( M \) can not be equivalent to any chief factor contained in \( B' \).

\[ \square \]

Now we consider non-trivial \( W \)-modules (abelian \( W \)-groups) and the values of the function \( h_W \) on them.

**Proposition 8.** Let \( p \) be a prime and \( M \) be a non-trivial irreducible \( \mathbb{F}_pW \)-module.

1. If \( M \) is \( W \)-equivalent to a non-Frattini \( W \)-chief factor contained in \( B' \), then there exists a non-trivial irreducible \( \mathbb{F}_pH \)-module \( U \) such that \( M \sim_W U^n \) and \( h_W(M) \leq \left[ \frac{h_U(U)}{n} \right] + 2 \).
2. If \( M \) is \( W \)-equivalent to a non-Frattini \( W \)-chief factor of \( B/B' \), then \( h_W(M) \leq h_K(M) + d_p(H/H') \).
3. If \( M \) is not \( W \)-equivalent to any non-Frattini \( W \)-chief factor of \( B \) but \( \delta_W(M) = \delta_K(M) \geq 1 \), then \( h_W(M) = h_K(M) \).
4. If \( \delta_W(M) = 0 \), then \( h_W(M) \leq 2 \).

**Proof.**

(1) The first part follows from Propositions \( \text{[7]} \) the bound of \( h_W(M) \) is proved in \( \text{[16]} \) step 2.5).

(2) Since \( M \) is \( W \)-equivalent to a chief factor of \( B/B' \), \( B \) centralizes \( M \) and hence \( r_W(M) = r_K(M) \). Let \( \overline{H} = H/H' \). By Proposition \( \text{[7]} \) \( \delta_W(M) \leq \)
\[\delta_K(M) + d_p(\mathcal{H})r_K(M).\] Moreover (see (1.2) in \[15\])

\[
s_W(M) = \delta_W(M) + \dim\text{End}_W(M) \cdot H^1(W/C_W(M), M)
\leq \delta_K(M) + \dim\text{End}_K(M) \cdot H^1(K/C_K(M), M)
= d_p(\mathcal{H})r_K(M) + s_K(M).
\]

Therefore,

\[
h_W(M) = \left[\frac{s_W(M) - 1}{r_W(M)}\right] + 2
\leq \left[\frac{d_p(\mathcal{H})r_K(M) + s_K(M) - 1}{r_K(M)}\right] + 2
\leq h_K(M) + d_p(\mathcal{H}).
\]

(3) Since \(\delta_W(M) = \delta_K(M) \geq 1\), we have that \(M\) is not equivalent to any chief factor contained in \(B\) and hence \(B\) is contained in \(R_W(A)\) where \(A\) is a chief factor \(W\)-equivalent to \(M\) (every minimal normal subgroup of \(W/R_W(A)\) is \(W\)-equivalent to \(A\)). By the same arguments used to prove equation (2.1) it follows that \(h_W(M) = h_{W/B}(M) = h_K(M)\).

(4) This is proved in Lemma 1.5 of \[14\].

\[\square\]

4. NUMBER OF GENERATORS OF WREATH PRODUCTS

Let \(L\) be a monolithic primitive group with socle \(N\). Let us denote by \(P_L(d)\) (resp. \(P_{L/N}(d)\)) the probability of generating \(L\) (resp. \(L/N\)) with \(d\) elements, and, for \(d \geq d(L)\), let

\[P_{L,N}(d) = P_L(d)/P_{L/N}(d).\]

When \(N\) is non-abelian, the formula given in \[4\] to evaluate \(d(L_i)\) is the following:

**Theorem 9.** \[4\] Theorem 2.7 Let \(L\) be a monolithic primitive group with non-abelian socle \(N\) and let \(d \geq d(L)\). Then \(d(L_i) \leq d\) if and only if

\[t \leq \frac{P_{L,N}(d) |N|^d}{|C_{\text{Aut } L}(L/N)|}.
\]

In Theorem 1.1 in \[19\] it is proved that if \(|N|\) is large enough and \(d \geq 2\) random elements generate \(L\) modulo \(N\), then these elements almost certainly generate \(L\) itself.

**Theorem 10.** \[19\] Theorem 1.1 There exists a positive integer \(k_0\) such that, if \(L\) is a monolithic primitive group with socle \(N\) and \(|N| \geq k_0\), then for every \(d \geq d(L)\) we have \(P_{L,N}(d) \geq 1/2\).

**Proposition 11.** Let \(L\) be a monolithic primitive group with a non-abelian socle \(N\), \(K\) a transitive group of degree \(n\) and \(L^* = L \wr K\). Assume that \(|N|^n \geq k_0\). For every positive integer \(t\) and every integer \(d \geq d(L^*/\text{soc } L^*) - 2\), if \(d(L_t) \leq d \cdot n\), then \(d(L_t^*) \leq d + 2\).

**Proof.** Since \(L_t\) can be generated by \(nd\) elements, by Theorem 9 we have that

\[t \leq \frac{P_{L,N}(nd) |N|^{nd}}{|C_{\text{Aut } L}(L/N)|}.
\]
As \( N \leq C_{\text{Aut}}L(L/N) \) and \( P_{L,N}(nd) \leq 1 \), we deduce \( t \leq |N|^{nd-1} \).

Now, again by Theorem \( \text{[10]} \) to prove that \( d(L^n) \leq d+2 \), it is sufficient to prove that

\[
t \leq \frac{P_{L^*,M}(d+2)|M|^{d+2}}{|C^*|}
\]

where \( M = \text{soc} \ L^* \) and \( C^* = C_{\text{Aut}}L^*(L^*/M) \). By assumption \( d+2 \geq \max(d(L^*/M), 2) = d(L^*) \), where the last equation follows from \( \text{[18]} \), and \( |M| = |N|^n \geq k_0 \). Thus we can apply Theorem \( \text{[10]} \) to get that \( P_{L^*,M}(d+2) \geq 1/2 \). Moreover, if \( N = S^n \), where \( S \) is a simple non-abelian group and \( a \) a positive integer, from the proof of Lemma 1 in \( \text{[5]} \), \( |C^*| \leq na|S|^{na-1} |\text{Aut} S| \leq na|S|^{na+1} \). It follows that

\[
\frac{P_{L^*,M}(d+2)|M|^{d+2}}{|C^*|} \geq \frac{1}{2} \cdot \frac{|M|^{d+2}}{na|S|^{na+1}}.
\]

Since \( t \leq |N|^{nd-1} \) and \( M = N^n \), it is sufficient to check that \( \frac{|N|^{(d+2)}}{2na|S|^{na+1}} \geq |N|^{nd-1} \), that is \( |N|^{2na+1} = |S|^{2na+a} \geq 2na|S|^{na+1} \), and this follows from the fact that \( |S| \geq 60 \). \( \Box \)

**Proposition 12.** Let \( K \) be a transitive permutation group of degree \( n \geq \log_{60} k_0 \), where \( k_0 \) is the constant defined in Theorem \( \text{[14]} \) Then

\[
d(H \downarrow K) \leq \max \left( d(H/H' \downarrow K), \left\lceil \frac{d(H)}{n} \right\rceil + 2 \right).
\]

**Proof.** Set \( \overline{H} = H/H' \). When \( W = H \downarrow K \) has an abelian generating chief factor, by Proposition \( \text{[8]} \) \( d(G) = d(I_G) \), and then the result follows from Proposition \( \text{[4]} \)

\[
d(W) = d(I_W) = \max \left( d\left( I_{\overline{H}/K} \right), \left\lceil \frac{d(H) - 2}{n} \right\rceil + 2 \right).
\]

Now we assume that every generating chief factor is non-abelian and we argue by induction on \( |H| \), the case \( |H| = 1 \) being obviously true. Let \( M \) be a non-abelian generating chief factor of the series \( \text{[5,4]} \). If \( M \) is not contained in \( B' \), then, by Proposition \( \text{[2]} \) \( M \) is a \( K \)-group such that \( \delta_W(M) = \delta_K(M) \) and the crown-based power \( L_{M,\delta_W(M)} \) is a homomorphic image of \( K \). Therefore

\[
d(W) = d(L_{M,\delta_W(M)}) \leq d(K) \leq d(\overline{H} \downarrow K)
\]

and the result follows.

We are left with the case where \( M \) is a non-abelian chief factor contained in \( B' \). From Proposition \( \text{[7]} \) we know that there exists a non-abelian chief factor \( N \) of the series \( \text{[5,3]} \) such that \( \delta_W(M) = \delta_H(N) \) and \( L_M \cong L_N \downarrow K \). Set \( L = L_N \), \( L^* = L \downarrow K \) and \( \delta = \delta_H(N) \).

Let \( d_0 = \max \left( d(\overline{H} \downarrow K), \left\lceil \frac{d(H)}{n} \right\rceil + 2 \right) \); we want to apply Proposition \( \text{[11]} \) to prove that \( d(W) = d(L_*^n) \leq d_0 \). As \( |L/N| < |H| \), by induction we get

\[
d(L/N \downarrow K) \leq \max \left( d(L/L' \downarrow K), \left\lceil \frac{d(L/N)}{n} \right\rceil + 2 \right).
\]
Since $L/L'$ is a homomorphic image of $H$ and $L^*/M = L/N \triangleleft K$, we deduce that$$d(L^*/M) = d(L/N \triangleleft K) \leq \max \left( d(H/K), \left\lfloor \frac{d(H)}{n} \right\rfloor + 2 \right) = d_0.$$

Moreover $d_0 \geq \left\lceil \frac{d(H)}{n} \right\rceil + 2$, that is $n(d_0 - 2) \geq d(H) \geq d(L_0)$. Also, the assumption $n \geq \log_{20} k_0$, gives $|N|^n \geq k_0$. Therefore all the hypothesis of Proposition 11 are satisfied (for $d = d_0 - 2$) and we conclude that $d(W) = d(L_0^*) \leq d_0$. \hfill \Box

The previous result reduces the problem of finding a bound to $d(W)$ to the case where $H$ is an abelian group. Let$$\rho_{K,H,p} = \max_M h_K(M) + d_p(H/H')$$where $M$ ranges over the set of non trivial irreducible $\mathbb{F}_pK$-modules, with $\rho_{K,H,p} = 0$ if every irreducible $\mathbb{F}_pK$-module is trivial.

Proposition 13. If $H$ is abelian, then $d(H \triangleleft K) \leq \max_{p|H}(d(H \times K), \rho_{K,H,p})$.

Proof. Let $W = H \triangleleft K$ and let $M$ be a generating chief factor for $W$.

If $M$ is non-abelian, then $M$ can not be $W$-equivalent to any chief factor of $B = H^n$, hence $R_W(M) \geq B$ and $L_{M,\delta_W(M)}$ is a homomorphic image of $K$. It follows that$$d(W) = d(L_{M,\delta_W(M)}) \leq d(K) \leq d(H \times K)$$and we are done.

Now, let us assume that $M$ is abelian. If $M$ is central, by equation 3.5 it follows that $h_W(M) = h_{W/W'}(M) \leq d(W/W') \leq d(H \times K)$ since $W/[B, K] \cong H \times K$. Thus $d(W) = h_W(M) \leq d(H \times K)$ and the result follows.

Then we are left with the case where $M$ is non-central. By Proposition 8 (both (2) and (3)), $h_W(M) \leq h_K(M) + d_p(H)$ and therefore $d(W) = h_W(M) \leq \rho_{K,H,p}$. This completes the proof. \hfill \Box

5. Iterated Wreath products

Note that if $K$ is a permutation group of degree $n$, then$$d(H) \leq n \cdot d(H \triangleleft K);$$indeed, given a set$$\{g_i = (h_{i,1}, \ldots, h_{i,n})k_i \mid h_{i,j} \in H, k_i \in K, \ i = 1, \ldots, d\}$$of generators for $H \triangleleft K$, then $H$ can be generated by the elements $\{h_{i,j} \mid j = 1, \ldots, n, i = 1, \ldots, d\}$. Moreover,$$d(H \triangleleft K) \geq d(H/H' \times K/K')$$since $H/H' \times K/K'$ is a homomorphic image of $H \triangleleft K$.

This shows the “only if” implication of Theorem 11. The other implication is proved in the following theorem.

Theorem 14. Let $(G_i)_{i \in \mathbb{N}}$ be a sequence of transitive permutation groups of degree $n_i$. Let $G_i' = G_i/G_i'$ and denote by $W_m = G_m \triangleleft \cdots \triangleleft G_1$ the iterated permutational wreath product of the first $m$ groups. Assume that there exists two integers $c$ and $d$ with

$(i) \ d(\prod_{i=1}^\infty G_i) = c$
(ii) \( d(G_i) \leq d \cdot n_1 \cdots n_{i-1} \) for every \( i > 1 \).
Then, for \( e = \max(d + 2, d(W_{i_0})) \), where \( i_0 \) is the first index such that the degree \( n_1 \cdots n_{i_0} \) of \( W_{i_0} \) is at least \( \log_{60}(k_0) \), we get the following:

1. If \( M \) is a non-trivial irreducible \( \mathbb{F}_p W_m \)-module, where \( m \geq i_0 \), then

   \[
   h_{W_m}(M) \leq e + d_p \left( \prod_{i=i_0}^{m} \overline{G}_i \right);
   \]

2. \( d(W_m) \leq e + d \left( \prod_{i=i_0}^{m} \overline{G}_i \right) \) for every \( m \geq i_0 \);
3. The inverse limit of the iterated wreath products \( W_m \) is finitely generated and
   \( d \left( \lim_{W_m} W_m \right) \leq e + c \).

Proof.  

1. We argue by induction on \( m \). The case \( m = i_0 \) is trivial since \( h_{W_{i_0}}(M) \leq d(W_{i_0}) \leq e \). So let \( m > i_0 \) and let \( M \) be a non-trivial irreducible \( \mathbb{F}_p W_m \)-module. By Proposition \( \text{8} \) applied to \( W_m = G_m \wr W_{m-1} \), where \( n = n_1 \cdots n_{m-1} \) is the degree of \( W_{m-1} \), we get that either \( h_{W_m}(M) \leq \left[ \frac{h_{G_m}(U) - 2}{n} \right] + 2 \) for an \( \mathbb{F}_p G_m \)-module \( U \) contained in \( G_m' \), or

   \[
   h_{W_m}(M) \leq h_{W_{m-1}}(M) + d_p(\overline{G}_m);
   \]

   thus

   \[
   h_{W_m}(M) \leq \max \left( \left[ \frac{h_{G_m}(U) - 2}{n} \right] + 2, h_{W_{m-1}}(M) + d_p(\overline{G}_m) \right).
   \]

   Since \( h_{G_m}(U) \leq d(G_m) \leq d(n) \) implies \( \left[ \frac{h_{G_m}(U) - 2}{n} \right] + 2 \leq d + 2 \), and, by

   inductive hypothesis \( h_{W_{m-1}}(M) \leq e + d_p \left( \prod_{i=i_0}^{m-1} \overline{G}_i \right) \), we get

   \[
   h_{W_m}(M) \leq \max \left( d + 2, e + d_p \left( \prod_{i=i_0}^{m-1} \overline{G}_i \right) + d_p(\overline{G}_m) \right) \leq e + d_p \left( \prod_{i=i_0}^{m} \overline{G}_i \right).
   \]

2. Again, we argue by induction on \( m \), the case \( m = i_0 \) being trivial.
   So let \( m > i_0 \) that is \( n = n_1 \cdots n_{m-1} > \log_{60}(k_0) \). Proposition \( \text{12} \)

   applied to \( W_m = G_m \wr W_{m-1} \), gives

   \[
   d(W_m) \leq \max \left( d(\overline{G}_m \wr W_{m-1}), \left[ \frac{d(G_m)}{n} \right] + 2 \right) \leq \max \left( d(\overline{G}_m \wr W_{m-1}), d + 2 \right).
   \]  

   (5.1)

   Then we apply Proposition \( \text{13} \) to have

   \[
   d(\overline{G}_m \wr W_{m-1}) \leq \max_{p|\overline{G}_m} \left( d(\overline{G}_m \times W_{m-1}), \rho_{W_{m-1}, G_m \wr p} \right)
   \]

   (5.2)

   where

   \[
   \rho_{W_{m-1}, G_m \wr p} = \max_M \left( h_{W_{m-1}}(M) \right) + d_p(\overline{G}_m)
   \]

   and \( M \) ranges over the set of non trivial irreducible \( \mathbb{F}_p W_{m-1} \)-modules, with \( \rho_{W_{m-1}, G_m \wr p} = 0 \) if every irreducible \( \mathbb{F}_p W_{m-1} \)-module is trivial. By part (1) of this theorem, \( h_{W_{m-1}}(M) \leq e + d_p \left( \prod_{i=i_0}^{m-1} \overline{G}_i \right) \), and hence

   \[
   \rho_{W_{m-1}, G_m \wr p} \leq e + d_p \left( \prod_{i=i_0}^{m-1} \overline{G}_i \right) + d_p(\overline{G}_m) = e + d_p \left( \prod_{i=i_0}^{m} \overline{G}_i \right).
   \]  

(5.3)
Moreover, note that a crown-based power homomorphic image of $G_m \times W_{m-1}$ is either a homomorphic image of $W_{m-1}$ or a homomorphic image of $\overline{G}_m \times \overline{W}_{m-1}$ (in the latter case it is associated to a central chief factor). This implies that

$$d \left( G_m \times W_{m-1} \right) \leq \max \left( d \left( G_m \times \overline{W}_{m-1} \right), d \left( W_{m-1} \right) \right)$$

$$\leq \max \left( d \left( \prod_{i=i_0}^m \overline{G}_i \right), d \left( W_{m-1} \right) \right).$$

By inductive hypothesis we get $d \left( W_{m-1} \right) \leq e + d \left( \prod_{i=i_0}^{m-1} G_i \right)$, and therefore

$$d \left( G_m \times W_{m-1} \right) \leq \max \left( d \left( \prod_{i=i_0}^m G_i \right) + e + d \left( \prod_{i=i_0}^{m-1} G_i \right) \right)$$

$$\leq e + d \left( \prod_{i=i_0}^m G_i \right). \quad (5.4)$$

From (5.2), (5.3) and (5.4), we obtain that

$$d \left( G_m \wr W_{m-1} \right) \leq \max \left( d \left( G_m \times W_{m-1} \right), e + d \left( \prod_{i=i_0}^m G_i \right) \right)$$

$$\leq e + d \left( \prod_{i=i_0}^m G_i \right).$$

Since $d + 2 \leq e$, from (5.1) we conclude that

$$d \left( W_m \right) \leq \max \left( d \left( G_m \wr W_{m-1} \right), d + 2 \right)$$

$$\leq e + d \left( \prod_{i=i_0}^m G_i \right).$$

(3) This follows directly from (2) and the assumption that $d \left( \prod_{i=1}^\infty G_i \right) = c$. Indeed $d \left( W_m \right) \leq e + d \left( \prod_{i=i_0}^m G_i \right) \leq e + c$ for every $m$, and the same bound applies to the generating number of their inverse limit.

6. PROBABILITY OF GENERATING AN ITERATED WREATH PRODUCT

Once we know that a profinite group $G$ is finitely generated, it is natural to ask about the probability to find a set of generators for the group. A profinite group $G$ is called Positively Finitely Generated (PFG) if there exists an integer $t \geq d(G)$ such that a randomly chosen $t$-tuple generates $G$ with positive probability.

Note that it is possible to extend the definitions of $G$-equivalence and crowns to profinite groups (see [7]). Moreover, if $G$ is finitely generated then $\delta_G(A)$ is finite for every finite irreducible $G$-group $A$ and in particular this holds for the chief factors of $G$ [7, Theorem 12]. Recently, Jaikin-Zapirain and Pyber gave a characterization of PFG-groups in terms of non-abelian crowns:
Theorem 15 (Jaikin-Zapirain, Pyber [11]). A finitely generated profinite group $G$ is PFG if and only if there exists a constant $c$ such that for every non-abelian chief factor $A$ of $G$, $\delta_G(A) \leq l(A)^c$ where $l(A)$ is the minimal degree of a faithful transitive representation of $A$.

This allows us to characterize PFG infinitely iterated permutational wreath products.

Proposition 16. Let $(G_i)_{i \in \mathbb{N}}$ be a sequence of transitive permutation groups of degree $n_i$. Assume that the inverse limit $W_\infty$ of the iterated permutational wreath products $W_m = G_m \wr \cdots \wr G_1$ is finitely generated. Then $W_\infty$ is PFG if and only if there exists a constant $c$ such that for every non-abelian chief factor $A$ of $G_i$ and for every $i > 1$, $\delta_{G_i}(A) \leq l(A)^{cn_i-n_{i-1}}$.

Proof. Let $M$ be a non-abelian chief factor of $W = W_\infty$ such that $\delta_W(M) > 0$. Since $\delta_W(M)$ does not depend on the chosen chief series and is finite (Theorems 11 and 12 in [7]), then $\delta_W(M) = \delta_{W_i}(M)$ for some $i$; let $i$ be the smallest integer with this property. Without loss of generality we can assume $i > 1$. Since $\delta_{W_{i-1}}(M) < \delta_{W_i}(M)$, $M$ is equivalent to a non-abelian chief factor of $B = G_i^n$, the base subgroup of $W_i = G_i \wr W_{i-1}$, where $n = n_1 \cdots n_{i-1}$ is the degree of $W_{i-1}$. In particular $M$ is equivalent to a non-abelian chief factor contained in $B^l$, and from Proposition 15 it follows that there exists a non-abelian chief factor $A$ of $G_i$ such that $M \sim_{W_i} A^n$ and $\delta_{W_i}(M) = \delta_{G_i}(A)$. Since $l(M) = l(A)^n$ (see Proposition 5.2.7 in [11] and the comments afterwards), the result follows from the characterization of PFG-groups given by Jaikin-Zapirain and Pyber (Proposition 15). □

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