Tensor network formulation of the massless Schwinger model

Nouman Butt,1,∗ Simon Catterall,1 Yannick Meurice,2 and Judah Unmuth-Yockey1,†

1Department of Physics, Syracuse University, Syracuse NY 13244
2Department of Physics and Astronomy, The University of Iowa, Iowa City, IA 52242, USA

We construct a tensor network representation of the partition function for the massless Schwinger model on a two dimensional lattice using staggered fermions. The tensor network representation allows us to include a topological term. Using a particular implementation of the tensor renormalization group (HOTRG) we calculate the phase diagram of the theory. For a range of values of the coupling to the topological term \( \theta \) and the gauge coupling \( \beta \) we compare with results from hybrid Monte Carlo when possible and find good agreement.

I. INTRODUCTION

In recent years there has been a surge of interest in applying tensor network methods to calculate the properties of lattice spin and gauge models [1–9]. In low dimensions these formulations can avoid the usual sign problems associated with negative or complex probability weights that plague Monte Carlo approaches, and can yield very efficient computational algorithms [10–13]. For compact fields the general strategy has been to employ character expansions for all Boltzmann factors occurring in the partition function and subsequently to integrate out the original fields, yielding an equivalent formulation in terms of integer—or half-integer—valued fields. Typically local tensors can be built from these discrete variables and the partition function recast as the full contraction of all tensor indices.

However, writing local tensors for models with relativistic lattice fermions is more complicated [14–17]. One reason is tied to the Grassmann nature of the fermions which can induce additional, non-local sign problems which may be hard to generate from local tensor contractions. However, Gattringer et. al. have shown in Ref. [18] that a suitable dual formulation can be derived in the case of the massless Schwinger model which is free of these sign problems. Using this dual representation they have formulated a general Monte Carlo algorithm that can be used to simulate the model even in the presence of non-zero chemical potential and topological terms [19].

Other directions into the investigation of the Schwinger model have appeared in recent years as well. One approach has been the use of other numerical renormalization group methods like the density matrix renormalization group (DMRG) with matrix product states or matrix product operators (MPS or MPO). The massive Schwinger model with staggered fermions was investigated in Ref. [20] using the DMRG. In Ref. [21] the mass spectrum of the Schwinger model was calculated at zero and finite mass, and in Ref. [22] the authors studied the Schwinger model at finite temperature using the DMRG with MPO. The effect of truncation on the number of representations retained in the electric field basis for the Schwinger model was investigated in Ref. [23]. In Ref. [24] the confinement properties of the Schwinger model in the presence of a topological term were studied, and in Ref. [25] the authors considered the effects of a topological term on the vacuum structure of the model, again using the DMRG.

Similarly, Ref. [26] looked at a \( \mathbb{Z}_n \) formulation of the Schwinger model using the DMRG. They found that at large \( n \), one recovers similar results to the original continuous \( U(1) \) symmetry in the Schwinger model. Out of equilibrium properties were looked at in Ref. [27] for that same model.

On top of that, proposals and investigations into the potential for quantum simulations and computations of the Schwinger model were done in Refs. [28–30]. In Ref. [28] the lattice Schwinger model was considered for quantum simulation using cold atoms in an optical lattice. In Ref. [29] the authors considered general \( U(1) \) lattice gauge theories and they integrate out the gauge degrees of freedom, being left with a model of strictly matter, interacting non-locally. This model would be implemented using trapped ions. In Ref. [30], the authors considered the joint computation of the lattice Schwinger model using classical and quantum computers.

In this paper we show that the dual world-line formulation from Ref. [18] can be replicated by contraction of a suitable tensor network. It should be noted that a tensor formulation of the model allows for the definition of a transfer matrix, quantum Hamiltonian, and local Hilbert space. Rather than following a Monte Carlo strategy we instead use and follow the philosophy of the tensor renormalization group to coarse grain this tensor network. From this we calculate the partition function and free energy. We show that the results agree well with both Ref. [19] and conventional hybrid Monte Carlo simulations where the latter can be performed.

We start by reviewing the construction of the dual representation and show how the resulting dimer/loop representation can be obtained by the contraction of a suitable tensor network and derive the form of the fundamental tensor that is needed. We then describe the results of a coarsening of this tensor network using the HOTRG algorithm, calculate the free energy and its derivatives and

− nbutt@syr.edu
− jfunmuthyockey@gmail.com
compare the results to Monte Carlo simulations. We then
go on to add a topological term to the action with
coupling \( \theta \). We conclude with a summary of the advantages
and disadvantages of the method in this context.

II. MASSLESS SCHWINGER MODEL AND ITS
DUAL REPRESENTATION

We begin with the one-flavor staggered action for the
massless Schwinger model on a \( N_s \times N_x \) lattice with action

\[
S = S_F + S_g
\]

with

\[
S_F = \frac{1}{2} \sum_{x, \mu} \eta_\mu(x) \bar{\psi}(x) U_\mu(x) \psi(x + \mu) - \bar{\psi}(x + \mu) U_\mu^\dagger(x) \psi(x)
\]

and

\[
n = -\beta \sum_x \text{Re} [U_F(x)],
\]

where the Abelian gauge field \( U_\mu(x) = e^{i A_\mu(x)} \) lives on
the link between lattice sites \( x \) and \( x + \mu \) and the fermions
\( \psi(x) \) and \( \bar{\psi}(x) \) live at the sites. \( U_F \) is the usual Wilson
plaquette operator \( U_F(x) = \sum_{\nu < \nu'} U_\nu(x) U_{\nu'}(x + \mu) U^{\dagger}_{\nu'}(x + \nu) U^{\dagger}_{\nu}(x) \). The partition function for this model is then
given by

\[
Z = \int D[U] D[\bar{\psi}] D[\psi] e^{-S} = \int D[U] e^{\beta \sum_x \text{Re} [U_F(x)]} Z_F(U)
\]

with \( \int D[U] = \prod_x \int_{-\pi}^{\pi} dA_\mu(x)/2\pi \), \( \int D[\bar{\psi}] D[\psi] = \prod_x \int d\bar{\psi}(x) d\psi(x) \), and \( Z_F \) represents the part of the
partition function that depends on the fermion fields.

Following Ref. [18], and using the same notation for clarity, we first integrate out the fermions and generate an effective action depending only on the gauge fields. As a first step we redefine the link variables such that the staggered fermion phases \( \eta_\mu(x) \) can be absorbed into modified link variables \( U_\mu(x) \rightarrow \eta_\mu(x) U_\mu(x) \). Under this transformation the gauge action picks up an overall negative
sign but the measure is invariant. The Boltzmann factor associated with each bilinear fermion term can be written as the product of forward and backward hopping terms yielding a partition function

\[
Z_F = \int D[U] D[\bar{\psi}] D[\psi] \times \prod_x \sum_{k=0}^1 \left( \frac{1}{2} \bar{\psi}(x) U_\mu(x) \psi(x + \mu) \right)^k \times \sum_{k=0}^1 \left( \frac{1}{2} \bar{\psi}(x + \mu) U^{\dagger}_{\mu}(x) \psi(x) \right)^k.
\]

Notice that higher order terms in the expansion of the Boltzmann factors vanish because of the Grassmann nature of the fermions. There are several ways to generate a non-zero contribution to \( Z_F \). In each case, the Grassmann integration at each site must be saturated. To saturate the Grassmann integrations, exactly one forward and one backward hopping term must be associated with each site. This gives rise to a simple collection of possibilities. On the one hand, there may be a single forward and backward hopping term along the same link. This obviously saturates the integration, and is referred to as a dimer. On the other hand, there may be a forward and backward hop on two different links at a site. This indicates the passage of fermionic current through the site, and again saturates the integration measure there. Furthermore because of gauge invariance any non-dimer contribution to \( Z_F \) must correspond to a closed loop. Fig. 1 shows the allowed site contributions. A bold link indicates the presence of a \( \frac{1}{2} U \) or a \( -\frac{1}{2} U^\dagger \) factor along that link. Notice that the links are oriented corresponding to the presence of an arrow on each bold link whose direction is conserved through a site.

For a loop \( \ell \) with length \( L(\ell) \) one finds a contribution with absolute value

\[
\left( \frac{1}{2} \right)^{L(\ell)} \prod_{x, \mu \in \ell} (U_\mu)^{k_\mu(x)}
\]

where \( k_\mu(x) = \pm 1 \) distinguishes between \( U_\mu(x) \) and \( U^{\dagger}_\mu(x) \). In addition each loop carries a certain \( Z_2 \) phase which depends on the length of the loop and its winding along the temporal direction given by

\[
-(-1)^{\frac{1}{2} L(\ell)} (-1)^{W(\ell)}.
\]

Here, the overall negative sign is the usual one for closed fermion loops while the second factor keeps track of the number of forward hops which is exactly half the total length of the loop for a closed loop. Finally the factor \( (-1)^{W(\ell)} \) of the loop will be determined by the number of windings of the loop along the temporal direction assuming anti-periodic boundary conditions for the fermions. Using dimers and loops as basic constituents for non-zero contributions to the fermionic partition function we can write

\[
Z_F = \left( \frac{1}{2} \right)^V \sum_{l, \mu} (-1)^{N_\mu + \frac{1}{2} \sum_{l} L(\ell) + \sum_{l} W(\ell)} \times \prod_{\ell} \prod_{x, \mu \in \ell} (U_\mu)^{k_\mu(x)}(x).
\]

To proceed further we will need to construct this loop representation from the contraction of more basic objects located at sites and we take up this task in the next section.
FIG. 1. Sixteen non-zero possibilities for $\psi, \bar{\psi}$ integration at a site. These 16 possibilities end up being exactly the nonzero elements of the fermion tensor.

A. Tensor Formulation of the Fermionic Partition Function

We need to construct a local tensor which under contraction along lattice links yields $Z_F$. Let us ignore the overall sign for now and just deal with the magnitude. We allow two types of indices per link to capture separately the incoming and outgoing fermion lines making the fermion site tensor a rank eight object. Since each site is either the endpoint of a dimer, or has fermionic current incoming and outgoing from it is then modeled by the tensor structure (we leave off the gauge link factors for now)

$$T_{k_1\bar{k}_1k_2\bar{k}_2k_3\bar{k}_3k_4\bar{k}_4} = \begin{cases} 1 & \text{if any two } k_i \text{ and } \bar{k}_i \text{ are one and others are zero.} \\ 0 & \text{otherwise} \end{cases}$$

(9)

where each $(k_i, \bar{k}_i) = 0, 1$. A graphical representation of this tensor is shown in Fig. 2(a). By repeatedly contracting this site tensor with copies of itself over the lattice it can be seen that we generate the full set of closed loops and dimers for the model at zero gauge coupling excluding the overall factor of minus one for each closed fermion loop. The absolute value of the partition function at zero gauge coupling is then,

$$Z_F^{\beta=\infty} = \sum_{\{k, \bar{k}\}} \prod_x T_{k_1\bar{k}_1k_2\bar{k}_2k_3\bar{k}_3k_4\bar{k}_4}.$$ 

(10)

Here, $\{k, \bar{k}\}$ denote the set of $k, \bar{k}$ values for the entire lattice. Said another way, the 16 possible vertex configurations for fermion hopping in Fig. 1 are captured as nonzero tensor elements in the $T$ tensor.

B. Integrating out the gauge fields

The fermion partition function in the previous section does not include any contribution or interaction with the gauge fields. To proceed further we will employ a character expansion of the Boltzmann factors associated with the gauge action. This will ensure that each plaquette in the lattice will carry an integer variable. Integration of the link gauge field in the background of a particular set of fermion loops restricts the plaquette variables to change by plus or minus one on crossing any fermion line.

In this section, we will describe this in detail and, along with the tensor from the previous section, construct a tensor network that when fully contracted reproduces the full partition function for the massless Schwinger model.

To integrate the gauge links we first start by performing a character expansion on the Boltzmann factor corresponding to the pure gauge plaquette action

$$e^{-\beta \cos [A_\mu(x)+A_\nu(x+\mu)-A_\mu(x+\nu)-A_\nu(x)]} = \sum_{m=-\infty}^{m=\infty} I_m(-\beta)e^{im[A_\mu(x)+A_\nu(x+\mu)-A_\mu(x+\nu)-A_\nu(x)].}$$

(11)
Each plaquette \( p \) is now labeled by an integer \( m_p \). Note that \( I_m(\beta) = (-1)^m I_m(\beta) \). Furthermore, each link \( \ell \) is shared by two plaquettes \( p \) and \( p' \) each of which supplies a factor of \( e^{im_p A_i} \) and \( e^{-im_{p'} A_i} \). In addition, the link carries a factor of \( e^{ik_{\ell A_i}} \) or \( e^{-ik_{\ell A_i}} \) coming from \( Z_F \). Thus, in total, links carry two \( m \) indices inherited from their neighboring plaquettes together with a \( k \) and a \( \bar{k} \) index associated with the fermionic hopping terms. The integral over the field then gives

\[
\int_{-\pi}^{\pi} \frac{dA_i}{2\pi} e^{i(m_p-m_{p'}+k_\ell-k_{\bar{\ell}})A_i} = \delta_{m_p-m_{p'},} k_\ell-k_{\bar{\ell}},0 \tag{12}
\]

This allows us to write the partition function as

\[
Z = \sum_{\{m_p\}} \sum_{\{k_\ell,\bar{k}_{\ell}\}} \prod_p \delta_{m_p-m_{p'},} k_\ell-k_{\bar{\ell}},0 \prod_p I_{m_p}(\beta) \times \prod \mathcal{T}_{k_1,k_2,k_3,k_4,k_5} \times (-1)^{N_F+N_p+\frac{1}{2} \sum_i L_i} \tag{13}
\]

where \( \{m_p\} \) denotes the set of plaquette integers over the entire lattice, \( \{k_\ell,\bar{k}_{\ell}\} \) represent \( k \) indices over the links, and \( N_F = \sum p m_p \). At this point we have included all the minus signs for completeness. For periodic boundary conditions, the sum of winding numbers must always be zero, since one is restricted to the total charge-0 sector of the theory. Note that for this situation the overall \( \pm 1 \) factor is always positive \[18\].

Now, associated with each link are \( m \) fields and \( k \) fields, and a constraint between them. Associated with each plaquette is a single \( m \) field. This lets us define a link tensor, and a plaquette tensor. Link tensors have indices connecting to fermion tensors (the \( T \) tensors) living on each site, and gauge-field indices connecting to plaquette tensors (on each plaquette). We define this link tensor, \( A \), as,

\[
A_{m,m',m_2,m_3,m_4} = \delta_{m,-m_2+k_\ell-k_{\bar{\ell}},0} \delta_{m_2,-m_3+k_\ell-k_{\bar{\ell}},0} \delta_{m_3,-m_4+k_\ell-k_{\bar{\ell}},0} \delta_{m_4,-m+k_\ell-k_{\bar{\ell}},0} \tag{14}
\]

Fermion-like indices on link tensors are purely diagonal as seen from the definition involving the \( \delta \) function constraints on links. A diagram showing the relative position of the fermion and plaquette indices is shown in Fig. 2 (b). Since there is only a single \( m \) associated with each plaquette, a tensor definition must only depend on that single \( m \). A plaquette tensor, \( B \), can be defined as,

\[
B_{m,m_2,m_3,m_4} = \begin{cases} I_m(\beta) & \text{if } m_1 = m_2 = m_3 = m_4 \\ 0 & \text{otherwise} \end{cases} \tag{15}
\]

A graphical representation for the \( B \) tensor associated with plaquettes is shown in Fig. 2 (c).

These definitions of the \( A \) and \( B \) tensors allow us to write the partition function as follows,

\[
Z = \sum_{\{m_p\}} \sum_{\{k,\bar{k}\}} \left( \prod_p B_{m,m,m_2,m_3,m_4} \right) \left( \prod_{\ell} A_{m,m,k_\ell-k_{\bar{\ell}},k_\ell-k_{\bar{\ell}},0} \right) \times \left( \prod \mathcal{T}_{k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8} \right) \tag{16}
\]

This contraction over three unique tensor types can be represented as the tensor network shown in Fig. 3. Since the fermionic \( k \) indices always come in \( k, \bar{k} \) pairs, we can form a product state of those two indices to reduce the complexity of the notation,

\[
T \rightarrow T' = T_{k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8} = T_{K_1,K_2,K_3,K_4} \tag{17}
\]
A → A' = A_{m_1 m_2} (k_a \otimes k_b) (k_5 \otimes k_6) = A_{m_1 m_2} k_a k_b \quad (18)

The new enlarged K indices take values from 0 to 3, enumerating the four possible states each link can have: unoccupied, incoming, outgoing, and dimer. The A tensors are still diagonal in the new K indices.

III. TRANSFER MATRIX

Using the tensors defined in the previous sections, one can build a transfer matrix for this model. The transfer matrix can be defined as the product of two types of matrices. In this section, we first define and construct these two different matrices. Then, by combining these two matrices in the appropriate way we can define a transfer matrix. The partition function is the trace of the power of this final matrix.

The first type of matrix we define is the B matrix. It is made by contracting alternating B and A tensors along a time-slice.

\[ \mathcal{B}(m_1 \otimes \ldots \otimes m_N \otimes K_1 \otimes \ldots \otimes K_N)(m'_1 \otimes \ldots \otimes m'_N \otimes K'_1 \otimes \ldots \otimes K'_N) = \]

\[ B_{mm' m_1 m'_1} A_{m'_1 m''_1} K_{1} K'_1 B_{m''_1 m'''_1} m_2 m'_2 \times \]

\[ A_{m''_1 m'''_1} K_{2} K'_2 \ldots B_{m^{(N-1)}_1 m_N m'_N} \] \quad (19)

where a sum over repeated indices is implied. Diagrammatically \( \mathcal{B} \) is represented as Fig. 4. An important feature of this matrix is that it is diagonal, due to the diagonal nature of the B tensors, and the K indices in the A tensors. This means incoming states through this matrix do not change into other states.

In analogy with the construction of \( \mathcal{B} \) we define the \( \mathcal{A} \) matrix as the alternating contraction of \( T \) and \( A \) tensors along a time-slice,

\[ \mathcal{A}(m_1 \otimes \ldots \otimes m_N \otimes K_1 \otimes \ldots \otimes K_N)(m'_1 \otimes \ldots \otimes m'_N \otimes K'_1 \otimes \ldots \otimes K'_N) = \]

\[ A_{m_1 m'_1} K_{1} K'_1 A_{m_2 m'_2} K_{2} K'_2 \ldots A_{m_N m'_N} K_{N} K'_N \] \quad (20)

with a diagrammatic representation given by Fig. 5. This matrix has off-diagonal elements, and is responsible for the changing of states between time-slices. This matrix moves fermionic current across space, and through time, with the appropriate shift in the electric field to balance.

Using the definitions above we can recast the partition function into an alternating product of \( \mathcal{B} \) and \( \mathcal{A} \) matrices. This alternating product can be broken up, and recast as the \( N^\text{th} \) power of a single matrix,

\[ T_{\alpha \beta} = \sqrt{\mathcal{B}_{\alpha \delta} A_{\delta \gamma} \mathcal{B}_{\gamma \beta}} \] \quad (21)

where the square root is well-defined since \( \mathcal{B} \) is diagonal in all of its indices (and its matrix elements are positive). The indices in Eq. (21) are collective indices as defined before in the definitions of the \( \mathcal{B} \) and \( \mathcal{A} \) tensors. Now we can write the partition function as follows,

\[ Z = \text{Tr} [ T^N ] . \] \quad (22)

IV. FUNDAMENTAL TENSOR FOR TRG

A. Asymmetric tensor

In order to have efficient numerical calculations using the TRG, the tensor network structure should be translationally invariant. This means that for whatever fundamental tensor one uses, it must contract naturally with itself. That is, the top indices of the fundamental tensor should be compatible for contraction with the bottom indices, and the indices on the left side of the tensor
FIG. 5. Construction of matrix $\mathcal{A}$. In principle the construction continues to the left and right, alternating contraction between $\mathcal{A}$ and $T$ tensors. This matrix is responsible for moving fermionic current around in space and time, and adjusting the gradient of the electric field to compensate.

should be compatible for contraction with the indices on the right.

For this goal, we define a tensor, $\mathcal{M}$, using a single elementary plaquette tensor (the $B$), two link tensors (the $\mathcal{A}$s), and a single fermion $T$ tensor. This is shown diagrammatically in Fig. 6. As can be seen from the figure, there are two different types of indices associated with each direction in the tensor. Each direction has one $m$ index, and one $K$ index. However, repeated contraction of this tensor with itself in the appropriate pattern reproduces the partition function. This is the only fundamental tensor necessary to do that. The tensor is then explicitly given as,

$$
\mathcal{M}_{m_1 m_2 m_3 m_4 K_1 K_2 K_3 K_4} = \sum_{m_1', m_2', K_1, K_2} B_{m_1 m_1' m_2 m_2'} \times A_{m_2' m_3 K_1} T_{K_1 K_2 K_3} A_{m_4 m_4' K_2 K_4}'.
$$

(23)

Here the $K$ indices always have dimension four, however the $m$ indices run over all integers. The $m$ indices are constrained by the $K$ indices though. Looking at a single direction, the total size of the state-space associated with two of the indices is $D_{\text{bond}} = N_{\text{gauge}} \times 4$, where $N_{\text{gauge}}$ is the number of states allowed for the $B$ tensor index in practice.

**B. Symmetric tensor**

It’s possible to form a completely symmetric tensor in both space and time, as opposed to the asymmetric tensor constructed above. This tensor formulation relies on “dressing” the link fermion tensors in their surrounding gauge field configurations. This is possible because of how the $B$ tensor is completely diagonal in its four indices.

To construct the symmetric tensor, the first step is to separate the $B$ tensor into eight smaller pieces, four of which are associated with the adjacent link tensors, and the other four are associated with the four adjacent site tensors,

$$
B_{m_1 m_2 m_3 m_4} = \sum_{\alpha, \beta, \gamma, \sigma} b_{m_1 \sigma \alpha} b_{m_2 \beta \gamma} b_{m_3 \beta \gamma} b_{m_4 \gamma \sigma} = \sum_{\alpha, \beta, \gamma, \sigma, \rho, \lambda, \chi, \psi} b_{m_1 \psi \alpha} \delta_{\alpha \beta} b_{m_2 \beta \gamma} \delta_{\gamma \sigma} b_{m_3 \sigma \rho} \delta_{\rho \lambda} b_{m_4 \lambda \chi} \delta_{\chi \psi}.
$$

(24)

The $b$ tensors are also diagonal, and the $\delta$ matrices are simply Kronecker deltas. This decomposition can be seen graphically in Fig. 7. In principle, each of the above sums runs over all the integers; however, in practice one is forced to restrict the sum.

The $b$ tensors are contracted with adjacent $A$ tensors, and the Kronecker deltas are moved to the surrounding
the diagonal single, symmetric, translation invariant tensor, we split a contraction of the \( \tilde{T} \) tensors remaining. The partition function is simply \( B \) and can be seen in Fig. 9. At this point, there are no sets of indices (the \( K \) matrices) associated with that plaquette. The new \( \tilde{A} \) tensor has the form,

\[
\tilde{A}_{\{m_1 K m_2\}(m_1' K' m_2')} = \sum_{\alpha, \beta} b_{\alpha \beta m_1 m_1'} A_{\alpha \beta K K'} b_{\beta \beta m_2 m_2'}. \tag{25}
\]

This \( \tilde{A} \) matrix is diagonal, since it is diagonal in all three sets of indices (the \( K \)'s, and the \( m \)s) due to the aforementioned diagonal nature of the \( B \) tensor and the already diagonal nature of the \( K \) indices in the \( A \) tensor. This tensor can be seen in Fig. 8.

For the site tensor (\( T \) tensor), we now “wrap” it in Kronecker deltas which enforce that all four site tensors around a plaquette have the same \( m \)-plaquette number associated with that plaquette. The new \( \tilde{T} \) tensor has the form,

\[
\tilde{T}_{\{m_1 K_1 m_8\}(m_1 K_2 m_5)}(m_2 K_3 m_2)(m_3 K_4 m_7) = T_{\{m_1 K_1 m_8\}(m_1 K_2 m_5)} \delta_{m_3 m_2} \delta_{m_3 m_4} \delta_{m_5 m_6} \delta_{m_7 m_8}, \tag{26}
\]

and can be seen in Fig. 9. At this point, there are no \( B \) tensors remaining. The partition function is simply a contraction of the \( \tilde{A} \) and \( \tilde{T} \) tensors. To construct a single, symmetric, translation invariant tensor, we split the diagonal \( \tilde{A} \) into two halves using the singular value decomposition,

\[
\tilde{A}_{IJ} = \sum_{\alpha, \beta} U_{I \alpha} \lambda_{\alpha \beta} U_{J \beta}^\dagger = \sum_{\alpha, \beta, \gamma} (U_{I \alpha} \sqrt{\lambda_{\alpha \beta}}) (\sqrt{\lambda_{\beta \gamma}} U_{J \gamma}) = \sum_{\alpha} L_{I \alpha} L^\dagger_{J \alpha}. \tag{27}
\]

Furthermore, there are singular values with value zero, and they can be removed to decrease the size of the state space. This is equivalent to taking the square-root of the \( A \) matrix and removing the zero columns (rows). With the \( L \) matrices we can now form a symmetric tensor, by contracting four of these matrices with a \( \tilde{T} \),

\[
S_{ijkl}(\beta) = \sum_{\alpha, \beta, \gamma, \delta} \tilde{T}_{\alpha \beta \gamma \delta} L_{\alpha I} L_{\beta J} L_{\gamma K} L_{\delta L}. \tag{28}
\]

This tensor is symmetric in space and time, and since the \( L \) matrices are diagonal, its nonzero tensor elements are constrained by the fermion tensor, \( T \). This final \( S \) tensor satisfies the same constraint as the original fermion \( T \) tensor, however with tensor elements with values other than 1, instead given by linear combinations of modified Bessel functions which are functions of the gauge coupling.

V. NUMERICAL SIMULATION: HOTRG AND HMC

We implemented the HOTRG algorithm to evaluate \( \ln Z \) using the tensor defined in Eq. (23) as a translation invariant tensor for coarse-graining. We measured the average plaquette,

\[
\langle U_p \rangle = \frac{1}{N_s N_t} \frac{\partial \ln Z}{\partial \beta}, \tag{29}
\]

as a function of the gauge coupling and compared it to numerical data from Ref. [19]. In this case our computation using HOTRG completely agrees with the worm algorithm generated data. Moreover we can add a \( \theta \) term to the original action which results in new couplings, expressed as linear combinations of the gauge coupling and theta parameter, \( \eta = \frac{\beta}{2} - \frac{\theta}{4\pi} \) and \( \bar{\eta} = \frac{\beta}{2} + \frac{\theta}{4\pi} \). For the tensor construction here we only need to redefine the plaquette tensor, \( B \), with \( I_m(\beta) \) replaced by \( I_m(2\sqrt{\eta \bar{\eta}}) (\eta / \bar{\eta})^{m/2} \).

To ensure the formulation is valid, we measured a couple of observables, including the average plaquette \( \langle U_p \rangle \), and the topological charge, \( \langle Q \rangle \) as a function of the \( \theta \) parameter. The topological charge is defined as,

\[
\langle Q \rangle = \frac{1}{N_s N_t} \frac{\partial \ln Z}{\partial \theta}. \tag{30}
\]

The results of the calculation of the average plaquette as a function of \( \beta \) for different system sizes can be seen.
FIG. 9. The modified fermion tensor. The corners of the decomposed $B$ tensor are moved to the $T$ tensor at each site. These corners are Kronecker deltas, and enforce that each site around a plaquette has the same plaquette quantum number.

FIG. 10. Average Plaquette vs. $\beta$ for lattice sizes with $N_s = N_t = 4, 8, 16$ and compared with data from Ref. [19]. For this data $N_{\text{gauge}} = 3$ is sufficient to achieve similar accuracy to MC data.

FIG. 11. Average Plaquette vs. $\theta$ for a lattice with $N_s = N_t = 4$. Here $N_{\text{gauge}} = 5$ is necessary to achieve similar accuracy to the MC data.

FIG. 12. The topological charge as a function of $\theta$. Here we compare with Ref. [19]. We find a slightly larger range of plaquette quantum numbers are necessary—in contrast to the average plaquette—to achieve consistent results. In this case, the plaquette numbers had to be allowed to run from $m = -2$ to 2.

VI. CONCLUSIONS

In this paper we have constructed a tensor network formulation of the massless lattice Schwinger model with staggered fermions. We have considered both the usual action and one in which a topological term is added. The addition of the latter term induces a sign problem and...
renders the model intractable for a conventional hybrid Monte Carlo simulation.

Using the HOTRG algorithm we have computed the free energy and its derivatives and compared the results, where possible, with both hybrid Monte Carlo simulations and simulations based on a dual representation based on fermion loops. Where comparison is possible the agreement is good with the tensor network calculations being superior computationally to Monte Carlo. That said, we have experienced difficulties measuring observables for large values of the topological coupling \( \theta \). Typically the signal for an operator like the plaquette becomes very noisy after several iterations of the blocking scheme.

Additionally, arguments used for the positivity of terms in the sum of the partition function assume a complete lattice with boundary conditions and lattice size already achieved \[18\]. In contrast, the HOTRG does not know before-hand what the final size of the lattice will be, or what the boundary conditions will be at that size. This in turn gives the algorithm more freedom to choose which states are relevant during truncation, even though those very states may be projected out in the final step of blocking; making them useless.

A tensor construction scheme which uses an environment tensor might achieve better results at larger volumes, since, the forward-backward iteration from a complete lattice should retroactively adjust the intermediate states kept during truncation at smaller volumes.

Of course in the continuum limit the partition function should be independent of \( \theta \) and the difficulties are likely related at least in part to this fact — as the chiral symmetry of the lattice action is restored the system will develop chiral zero modes which will suppress the contribution of any topological field configurations to the partition function.

The \( \theta \) dependence is restored in the presence of a fermion mass. However in that case there are non-trivial \(-1\) factors which appear in the dual representation of the partition function. Part of the phase depends on the number of closed fermion loops appearing in any particular dual configuration. It is extremely hard to see how this phase can be reconstructed from the contraction of local tensors and we have not been able to generalize the tensor network described here to the case of non-zero masses. This should sound a cautionary note to the idea that tensor network formulations of lattice field theories are free of sign problems. In the case of fermion theories this may not be generically the case.

ACKNOWLEDGMENTS

The authors would like to thank the members of the QuLat collaboration for stimulating discussions. SC, YM, and JUY were supported by the U.S. Department of Energy (DOE) under Award Number de-sc0019139.

[1] Yuzhi Liu, Y. Meurice, M. P. Qin, J. Unmuth-Yockey, T. Xiang, Z. Y. Xie, J. F. Yu, and Haiyuan Zou. Exact blocking formulas for spin and gauge models. Phys. Rev. D, 88:056005, Sep 2013.

[2] Daisuke Kadoh, Yoshinobu Kuramashi, Yosihumi Nakamura, Ryo Sakai, Shinji Takeda, and Yusuke Yoshimura. Tensor network analysis of critical coupling in two dimensional \( \phi^4 \) theory. Journal of High Energy Physics, 2019(5):184, May 2019.

[3] Alexei Bazavov, Simon Catterall, Raghu G. Jha, and Judah Unmuth-Yockey. Tensor renormalization group study of the non-abelian higgs model in two dimensions. Phys. Rev. D, 99:114507, Jun 2019.

[4] Judah F. Unmuth-Yockey. Gauge-invariant rotor hamiltonian from dual variables of 3d \( u(1) \) gauge theory. Phys. Rev. D, 99:074502, Apr 2019.

[5] J. Unmuth-Yockey, Jin Zhang, A. Bazavov, Y. Meurice, and S.-W. Tsai. Universal features of the abelian polyakov loop in \( 1 + 1 \) dimensions. Phys. Rev. D, 98:094511, Nov 2018.

[6] Jing Chen, Hai-Jun Liao, Hai-Dong Xie, Xing-Jie Han, Rui-Zhen Huang, Song Chen, Zhong-Chao Wei, Zhi-Yuan Xie, and Tao Xiang. Phase transition of the four-dimensional ising model with higher-order tensor renormalization group. Phys. Rev. D, 100:054510, Sep 2019.

[7] Shinichiroy Akiyama, Yoshinobu Kuramashi, Takumi Yamashita, and Yusuke Yoshimura. Phase transition of four-dimensional ising model with higher-order tensor renormalization group. Phys. Rev. D, 100:054510, Sep 2019.

[8] Shun Wang, Zhi-Yuan Xie, Jing Chen, Bruce Normand, and Tao Xiang. Phase transitions of ferromagnetic potts models on the simple cubic lattice. Chinese Physics Letters, 31(7):070503, Jul 2014.

[9] J. F. Yu, Z. Y. Xie, Y. Meurice, Yuzhi Liu, A. Denbyecker, Haiyuan Zou, M. P. Qin, J. Chen, and T. Xiang. Tensor renormalization group study of classical \( xy \) model on the square lattice. Phys. Rev. E, 89:013308, Jan 2014.

[10] Alan Denbyecker, Yuzhi Liu, Y. Meurice, M. P. Qin, T. Xiang, Z. Y. Xie, J. F. Yu, and Haiyuan Zou. Controlling sign problems in spin models using tensor renormalization. Phys. Rev. D, 89:016008, Jan 2014.

[11] G. Evenbly and G. Vidal. Tensor network renormalization. Phys. Rev. Lett., 115:180405, Oct 2015.

[12] Michael Levin and Cody P. Nave. Tensor renormalization group approach to two-dimensional classical lattice models. Phys. Rev. Lett., 99:120601, Sep 2007.

[13] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang. Coarse-graining renormalization by higher-order singular value decomposition. Phys. Rev. B, 86:045139, Jul 2012.

[14] Ryo Sakai, Shinji Takeda, and Yusuke Yoshimura. Higher-order tensor renormalization group for relativistic fermion systems. Progress of Theoretical and Experimental Physics, 2017(6), 06 2017. 063B07.

[15] Yuya Shimizu and Yoshinobu Kuramashi. Grassmann
tensor renormalization group approach to one-flavor lattice schwinger model. \textit{Phys. Rev. D}, 90:014508, Jul 2014.

[16] Shinji Takeda and Yusuke Yoshimura. Grassmann tensor renormalization group for the one-flavor lattice Gross-Neveu model with finite chemical potential. \textit{Progress of Theoretical and Experimental Physics}, 2015(4), 04 2015.

[17] Daisuke Kadoh, Yoshinobu Kuramashi, Yoshifumi Nakamura, Ryo Sakai, Shinji Takeda, and Yusuke Yoshimura. Tensor network formulation for two-dimensional lattice $\mathcal{N} = 1$ wess-zumino model. \textit{Journal of High Energy Physics}, 2018(3):141, Mar 2018.

[18] Christof Gattringer, Thomas Klover, and Vasily Sazonov. Solving the sign problems of the massless lattice schwinger model with a dual formulation. \textit{Nuclear Physics B}, 897:732 – 748, 2015.

[19] Daisuke Kadoh, Christof Gattringer, Alexander Lehmann, and Christoph Weis. Simulation strategies for the massless lattice schwinger model in the dual formulation. \textit{Nuclear Physics B}, 924:63 – 85, 2017.

[20] T. M. R. Byrnes, P. Srijanesh, R. J. Bursill, and C. J. Hamer. Density matrix renormalization group approach to the massive schwinger model. \textit{Phys. Rev. D}, 66:013002, Jul 2002.

[21] M.C. Bañuls, K. Cichy, J.I. Cirac, and K. Jansen. The mass spectrum of the schwinger model with matrix product states. \textit{Journal of High Energy Physics}, 2013(11):158, Nov 2013.

[22] Boye Buyens, Frank Verstraete, and Karel Van Acoleyen. Hamiltonian simulation of the schwinger model at finite temperature. \textit{Phys. Rev. D}, 94:085018, Oct 2016.

[23] Boye Buyens, Simone Montangero, Jutho Haegeman, Frank Verstraete, and Karel Van Acoleyen. Finite-representation approximation of lattice gauge theories at the continuum limit with tensor networks. \textit{Phys. Rev. D}, 95:094509, May 2017.

[24] Joao C. Pinto Barros, Marcello Dalmonste, and Andrea Trombettoni. String tension and robustness of confinement properties in the schwinger-thirring model. \textit{Phys. Rev. D}, 100:036009, Aug 2019.

[25] Lena Funcke, Karl Jansen, and Steffan Kühn. Topological vacuum structure of the schwinger model with matrix product states. \textit{arXiv:1908.00551 [hep-lat]}, 2019.

[26] Elisa Ercolessi, Paolo Facchi, Giuseppe Magnifico, Saverio Pascazio, and Francesco V. Pepe. Phase transitions in $Z_n$ gauge models: Towards quantum simulations of the schwinger-weyl qed. \textit{Phys. Rev. D}, 98:074503, Oct 2018.

[27] Giuseppe Magnifico, Marcello Dalmonste, Paolo Facchi, Saverio Pascazio, Francesco V. Pepe, and Elisa Ercolessi. Real time dynamics and confinement in the $Z_n$ schwinger-weyl lattice model for 1+1 qed. \textit{arXiv:1909.04521 [quant-ph]}, 2019.

[28] E. Rico, T. Pichler, M. Dalmonste, P. Zoller, and S. Montangero. Tensor networks for lattice gauge theories and atomic quantum simulation. \textit{Phys. Rev. Lett.}, 112:201601, May 2014.

[29] Christine Muschik, Markus Heyl, Esteban Martinez, Thomas Monz, Philipp Schindler, Berit Vogell, Marcello Dalmonste, Philipp Hauke, Rainer Blatt, and Peter Zoller. U(1) wilson lattice gauge theories in digital quantum simulators. \textit{New Journal of Physics}, 19(10):103020, oct 2017.

[30] N. Klco, E. F. Dumitrescu, A. J. McCaskey, T. D. Morris, R. C. Pooser, M. Sanz, E. Solano, P. Lougovski, and M. J. Savage. Quantum-classical computation of schwinger model dynamics using quantum computers. \textit{Phys. Rev. A}, 98:032331, Sep 2018.