Polar topologies on sequence spaces in non-archimedean analysis

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Abstract

The purpose of the present paper is to develop a theory of a duality in sequence spaces over a non-archimedean vector space. We introduce polar topologies in such spaces, and we give basic results characterizing compact, $C$-compact, complete and $AK$-complete subsets related to these topologies.

Key words : Locally $K$-convex topologies, non archimedean sequence spaces, Schauder basis, separated duality.

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1. Introduction

The duality $\langle \lambda, \lambda^\alpha \rangle$, where $\lambda$ is a scalar sequence space, was studied by Köthe and Toeplitz [7] and it has been reformulated by Köthe [6] using the theory of locally convex spaces. After, the duality $D_{\lambda,\lambda^\beta}E$ has been studied by Chillingworth [2], Matthews [8], T. Komura and Y. Komura [4]. In this work, we are interested to a duality in non-archimedean sequence spaces. We consider a separated duality $\langle X, Y \rangle$ of vector spaces over a non-archimedean valued field $K$ (n.a); in [1] Ameziane and Babahmed gave a fundamental properties of this duality. Afterwards we take $E(X)$ and $E(Y)$ two vector-valued sequence spaces over $X$ and $Y$ respectively such that $E(Y) \subset E(X)^\beta$ that are endwed with the separated duality $\langle E(X), E(Y) \rangle$ by the canonic bilinear form (p.108). We introduce the notion of polar topoogies over $E(X)$; and by the linear maps $\pi_j^X$ and $\delta_j^X$ which we define in this paper; we study the polar topologies compatible with the duality $\langle E(X), E(Y) \rangle$ using the basic duality $\langle X, Y \rangle$. Finally we characterize $C$- compact, $AK$-complete and complete subsets of $E(X)$ relatively at these topologies. This study was useful in the study that we made in [3].

Throughout this paper, $K$ is a non-archimedean (n.a) non trivially valued complete field with valuation $|\cdot|$. $X$ and $Y$ are two n.a topological vector spaces over $K$ (or $K$ vector spaces) that are in separated duality $\langle X, Y \rangle$. The duality theory for locally $K$-convex spaces can be found more exten-sively in [1], [9] , [11] and [12].

2. Preliminary

A nonempty subset $A$ of a $K$-vector space $X$ is called $K$-convex if $\lambda x + \mu y + \gamma z \in A$ whenever $x, y, z \in A$, $\lambda, \mu, \gamma \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$, $|\gamma| \leq 1$ and $\lambda + \mu + \gamma = 1$. $A$ is said to be absolutely $K$-convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$. For a nonempty set $A \subset X$ its $K$-convex hull $c(A)$ and absolutely $K$-convex hull $c_0(A)$ are respectively the smallest $K$-convex and absolutely $K$-convex set that contains $A$. If $A$ is a finite set $\{x_1, \ldots, x_n\}$ we sometimes write $c_0(x_1, \ldots, x_n)$ instead of $c_0(A)$.

An absolutely $K$-convex subset of a locally $K$-convex space $X$ is called $K$- closed if for every $x \in X$ the set $\{ |\lambda| : \lambda \in K, \lambda x \in A \}$ is closed in $|K|$. If the valuation on $K$ is discrete every absolutely $K$-convex set $A$ is $K$-closed. If $K$ has a dense valuation an absolutely $K$-convex set $A$ is
$K$-closed if and only if from $x \in E$, $\lambda x \in A$ for all $\lambda \in K, |\lambda| < 1$ it follows
that $x \in A$. Intersections of $K$-closed sets are $K$-closed. For an absolutely $K$-convex set $A$ the $K$-closed hull of $A$ is the smallest subset of $X$ that is $K$-closed and contains $A$, it is denoted by $K_c(A)$. If $K$ is discrete we have $K_c(A) = A$ and if $K$ is dense, $K_c(A) = \cap \{\lambda A : \lambda \in K \text{ and } |\lambda| > 1\}$ ([1] p. 220).

A topological vector space $X$ over $K$ is called locally $K$-convex if $X$ has a base of zero consisting of locally $K$-convex sets.

Let $(X, \tau)$ a locally $K$-convex space, $\tau$ is define by a family of n.a. semi-norms $\tau$- continuous over $X$, and if $K$ is discrete, we can suppose that $N_0 = \{p(x)/x \in X\} \subset |K|$ for every $p \in \mathcal{P}$ ([9]); where $(\mathcal{P})$ is a family of n.a semi-norms which define the topology $\tau$.

If $p$ is a (n.a) semi-norm over $X$, $B_p(0, 1)$ is the set $\{x \in X : p(x) \leq 1\}$.

A sequence $(e_i)_i$ is a Schauder basis for $X$ if every $x \in X$ can be written uniquely as $x = \sum_{i=1}^{\infty} \lambda_i x_i$ where the coefficient functionals $f_j : x \mapsto \lambda_j$ are continuous.

Let $X$ a $K$-vector space and $M$ a subset of $X$, a $K$-convex filter over $M$, is a filter $F$ over $M$ having a basis $B$ consisting of $K$-convex subsets of $M$; this basis is called $K$-convex basis of $K$-convex filter $F$.

The order of all filters on $M$ induces an order on all $K$-convex filters on $M$. A maximal element of the ordered set of $K$-convex filter on $M$ is called maximal $K$-convex filter of $M$.

Let $(x_i)_{i \in I}$ a net on $M$; for all $i \in I$, put $F_i = \{x_j/j \geq i\}$. $(F_i)_{i \in I}$ is a filter over $M$ called filter associated to a net $(x_i)_{i \in I}$. Conversely, if $F = (F_i)_{i \in I}$ is a filter over $M$, for all $i \in I$ let $x_i \in F_i$; over $I$ we define the following order: $i \leq j \Leftrightarrow F_j \subset F_i$. $(x_i)_{i \in I}$ is a net in $M$ called a net associated to a filter $F$.

**Proposition 1.** Let $X$ a locally $K$-convex space, $M$ a subset of $X$ and $F = (F_i)_{i \in I}$ a maximal $K$-convex filter over $M$.

1. $F$ converges or not having any clusterpoint.
2. Let $(x_i)_{i \in I}$ a net associated to a $F$; if $(x_i)_{i \in I}$ converges to $x_0$, $F$ converges to $x_0$.

**Proof.** 1. Let $x_0$ a cluster point of $F$ and $(U_j)_{j \in J}$ a $K$-convex neighbourhood base of $x_0$, $F' = \{F_i \cap U_j/i \in I \text{ and } j \in J\}$ is a $K$-convex filter which converges to $x_0$ and it is coarsest than $F$, then $F = F'$.

2. $x_0$ is a clusterpoint of $(x_i)_{i \in I}$, then it is a clusterpoint of $F$, and so $F$ converges to $x_0$. □
Proposition 2. Let $X, Y$ two $K$–vector spaces, $f : X \rightarrow Y$ a linear map and $\mathcal{F} = (F_i)_{i \in I}$ a maximal $K$–convex filter over $X$ that having $\mathcal{B}$ us a $K$–convex basis; $f(\mathcal{B})$ is a $K$–convex basis of a maximal $K$–convex filter over $Y$.

A subset $A$ of a locally $K$–convex space $X$ is compactoid if for each neighbourhood $U$ of zero there exist $x_1, \ldots, x_n \in X$ such that $A \subset U + c_0 (x_1, \ldots, x_n)$. An absolutely $K$–convex subset $A$ of $X$ is said to be $C$–compact if every convex filter on $A$ has a clusterpoint on $A$. $K$ is $C$–compact if and only if $K$ is spherically complete.

Proposition 3. Let $M$ be a subset of $X$. The following are equivalent:

(i). $M$ is $C$–compact;

(ii). Every maximal $K$–convex filter over $M$ converges;

(iii). Any family of closed and $K$–convex subsets of $M$ whose intersection is empty contains a finite subfamily whose intersection is empty.

Let $\mathcal{B}$ a basis of a filter $\mathcal{F}$ on a subset $M$ of $X$; the smallest $K$–convex filter containing $\mathcal{B}$, is called $K$–convex filter generated by $\mathcal{B}$ and is denoted by $\mathcal{F}_0(\mathcal{B})$. We show that $\mathcal{F}_0(\mathcal{B}) = \{ F \subset M/\text{there exists } B \in \mathcal{B} : c(B) \subset F \}$, and $c(\mathcal{B})$ is $K$–convex basis of $\mathcal{F}_0(\mathcal{B})$, that is to say $\mathcal{F}_0(\mathcal{B}) = \mathcal{F}(c(\mathcal{B}))$.

If $(x_i)_{i \in I}$ is a net in $X$; $(x_i)_{i \in I}$ converges to $x_0$ if and only if the filter $K$–convex associated with $(x_i)_{i \in I}$ converges to $x_0$.

Proposition 4. Let $X, Y$ two $K$–vector spaces, $f : X \rightarrow Y$ a linear map, $M$ a subset of $X$ and $\mathcal{B}$ a base of filter on $M$. Then $f(\mathcal{B})$ is a base of filter on $f(M)$, and we have $\mathcal{F}_c(f(\mathcal{B})) = f(\mathcal{F}_c(\mathcal{B}))$.

$((\omega(X), \tau_\omega(X)))$ = the linear space of all sequences in $X$ endowed with the product topology $\tau_\omega(X)$ which is generated by the family of n.a semi-norms $(p_n)_{n \in \mathbb{N}, p \in (P)}$, $p_n(\varphi) = p(x_n)$ for all $\varphi = (x_n)_{n \in \omega(X)}$ and all $p \in (P)$, if $X$ is a locally $K$-convex space and $(P)$ is a family of n.a semi-norms which define his topology; this space is noted $\omega(K)$ (or $\omega$, for short) in case when $X = K$. A sequence space over $X$ is a subspace of $\omega(X)$.

We define the following sequence spaces over $X$

$$c_0(X) = \{(x_k)_{k \in \omega(X)} : (x_k)_{k} \text{ converges to zero}\}$$

$$c(X) = \{(x_k)_{k \in \omega(X)} : (x_k)_{k} \text{ converges in } X\},$$

$$\varphi(X) = \{(x_k)_{k \in \omega(X)} : \text{there exists } k_0 \in \mathbb{N} : x_k = 0 \text{ for all } k \geq k_0\},$$

$$m(X) = \{(x_k)_{k \in \omega(X)} : (x_k)_{k} \text{ is bounded in } X\}.$$
Over \( m(X) \) we define the sequence of n.a semi-norms \( (\mathfrak{p})_{p \in (\mathcal{P})} \) by:
\[
\mathfrak{p}(\mathfrak{r}) = \sup_{k} p(x_k) \text{ for all } \mathfrak{r} = (x_k)_k \in m(X).
\]
Let \( \tau_{\infty}(X) \) be the topology on \( m(X) \) defined with the sequence of n.a semi-norms \( (\mathfrak{p})_{p \in (\mathcal{P})} \).

3. Polar topologies

Let \( X \) and \( Y \) two \( K \)-vector spaces placed in separating duality \( \langle X,Y \rangle \). If \( A \) is a subset of \( X \), we denote by \( A^0 = \{ y \in Y / |\langle x, y \rangle| \leq 1 \text{ for all } x \in A \} \) the polar of \( A \) and \( A^{\infty} = \{ x \in X / |\langle x, y \rangle| \leq 1 \text{ for all } y \in A^0 \} \) the bipolar of \( A \).

\( A^0 \) is absolutely \( K \)-convex and \( \sigma(Y, X) \)-bounded.

For each absolutely \( K \)-convex subset \( A \) of \( Y \), \( K_c (\mathcal{A}^0(Y,X)) = A^\infty([1], \text{ corollary 4.3, p. 233}) \). A subset \( A \) of \( Y \) is said to be \( X \)-closed if for every \( y \in Y \setminus A \), there exits \( x \in X \) such that \( |\langle x, y \rangle| > 1 \) and \( |\langle x, A \rangle| \leq 1 \). Intersections of \( X \)-closed sets are \( X \)-closed. For a subset \( A \) of \( Y \) the \( X \)-closed hull \( X_c(A) \) of \( A \) is the smallest \( X \)-closed subset of \( Y \) that contains \( A \). For each subset \( A \) of \( Y \), \( X_c(A) = A^\infty([1], \text{ proposition 2.5, p. 224}) \). Using these two results and by [1], theorem 4.2, p. 233 we have: for all absolutely \( K \)-convex subset \( A \) of \( Y \), \( A \) is \( X \)-closed, if and only if, \( A \) is \( K \)-closed and \( \sigma(Y, X) \)-closed.

Let \( \mathcal{A} \) be a family of \( \sigma(Y, X) \)-bounded subsets of \( Y \) such that

(a) \( \mathcal{A} \) is directed by inclusion,

(b) \( Y = \bigcup_{A \in \mathcal{A}} A \),

(c) there exists \( \lambda_0 \in K, |\lambda_0| > 1 \) such that \( \lambda_0 A \in \mathcal{A} \), for all \( A \in \mathcal{A} \).

A topology \( \tau \) on \( X \) is called polar topology of \( \mathcal{A} \)-convergence, if \( \tau \) has a fundamental system of zero-neighbourhood \( (F.S.N) \) consisting of \( \{ A^0/A \in \mathcal{A} \} \).

A vector topology \( \tau \) on \( X \) is called polar topology if there exists a family \( \mathcal{A} \) of \( \sigma(Y, X) \)-bounded subsets of \( Y \) which has the properties (a), (b) and (c), such that \( \tau \) is a polar topology of \( \mathcal{A} \)-convergence. It is defined by the family of n.a. semi-norms \( (P_A)_{A \in \mathcal{A}} \), where \( P_A(x) = \sup \{|\langle x, y \rangle|/y \in A\} \).

If \( \mathcal{A} \) is the family of all subsets of \( Y \) that are:

1. Absolutely \( K \)-convex, weakly bounded and weakly \( C \)-compacts, we have the \( C \)-compact topology \( \tau_c(X,Y) = \tau_c \),

2. Absolutely convex and \( \sigma(Y, X) \)-compact, we have the Mackey topology \( \tau_m(X,Y) = \tau_m \),
3. $\sigma(Y,X)$—bounded and $X$—closed, we have the $X$—closed topology $\tau_e(X,Y) = \tau_e$.

4. $\sigma(Y,X)$—bounded, we have the strong topology $\tau_b(X,Y)$.

A locally $K$—convex topology $\tau$ on $X$ is called compatible with the duality $\langle X,Y \rangle$ or $(X,Y)$—compatible if $Y$ is isomorphic to the topological dual of $X$ provided with the topology $\tau$. The weak topology $\sigma(X,Y)$ is the coarsest topology among all topologies $(X,Y)$—compatible, and the upper bound topology of all topologies $(X,Y)$—compatible topology is the finest among all the topologies $(X,Y)$—compatible.

We say that $X$ is semi-reflexive if $X$ is isomorphic to the strong topological dual of $Y$ and if $\tau$ is a locally $K$—convex topology on $X$ we say that $X$ is $\tau$—reflexive if $X$ is semi-reflexive and $\tau = \tau_b(X,X')$.

For further information about polar topology of $\mathcal{A}$—convergence and general properties of locally $K$—convex spaces we refer to [1], [11] and [12].

If $A \subset \omega(X)$, the $\beta$—dual of $A$ is the subspace of $\omega(Y)$ which is defined by $A^\beta = \{(y_n)_n \in \omega(Y): \text{lim}_n \langle x_n, y_n \rangle = 0 \text{ for all } (x_n)_n \in A\}$. $A$ is called perfect if $A^{\beta\beta} = A$. If $A$ is perfect then $\varphi(X) \subset A$. For all $A \subset \omega(X)$, $A^\beta$ is perfect. We define $B^\beta$ if $B \subset \omega(Y)$ on the same way.

A subset $D$ of $\omega(X)$ is said to be solid if for every $\varpi = (x_k)_k \in D$ and $\alpha = (\alpha_k)_k \in \omega$ such that $|\alpha_k| \leq 1$ for all $k$, we have $\alpha\varpi = (\alpha_kx_k)_k \in D$. The solid hull $S(D)$ of $D$ is the smallest solid set of sequence containing $D$.

A topology on $E(X)$, with respect the duality $\langle E(X), E(X)^\beta \rangle$, will be called solid if the elements of the determining family of weakly bounded subsets of $E(X)^\beta$ are solids sets.

Let $E(X)$ and $E(Y)$ be two sequence spaces on $X$ and $Y$ respectively such that $E(Y) \subset E(X)^\beta$, we define on the pair $(E(X), E(Y))$ the following duality $\langle (x_n)_n, (y_n)_n \rangle = \sum_{n=1}^{\infty} \langle x_n, y_n \rangle$ for all $(x_n)_n \in E(X)$ and all $(y_n)_n \in E(Y)$.

If $\varphi(X) \subset E(X)$ and $\varphi(Y) \subset E(Y)$, the duality $\langle E(X), E(Y) \rangle$ is separate.

In the sequel $\langle E(X), E(Y) \rangle$ denotes a duality of this type.

$S(E(Y)) \subset [S(E(X))]^\beta$ and $\langle S(E(X)), S(F(Y)) \rangle$ is a separating duality extending the separating duality $\langle E(X), F(Y) \rangle$, therefore, we can assume that $E(X)$ and $F(Y)$ are solid.

For all $j \geq 1$, we consider the following linear mappings:
\[ \pi^X_j : E(X) \rightarrow X \quad \delta^X_j : X \rightarrow E(X) \]
\[ (x_n) \mapsto x_j \quad a \mapsto \delta_j(a) \]

where \( \delta_j(a) \) is the sequence with \( a \) in the \( j \)-th place and 0’s elsewhere.

We define also \( \pi^Y_j \) and \( \delta^Y_j \).

Let \( x = (x_k) \in \omega(X) \), for all \( n \geq 1 \) \( x^{[n]} = \sum_{j=1}^{n} \delta_j(x_j) \) is called the \( n \)-th section of \( x \).

We have: \( \pi^X_j \circ \delta^X_j = id_X \), \( \pi^Y_j \circ \delta^Y_j = id_Y \), \( (\pi^X_j)^* / Y = \delta^Y_j \) and \( (\delta^X_j)^* / F(Y) = \pi^Y_j \) where \( u^* \) is the algebraic adjoint of the linear map \( u \).

**Proposition 5.** Let \( A \) be a subset of \( E(X) \) if \( A \) is solid, \( A^o \) is solid and we have: \( A^o = [A \cap \varphi(X)]^o \).

**Definition 1.** Let \( A \) a subset of \( \omega(X) \).

1. is said that \( A \) is \( \delta^X \)-saturated if for all \( (x_n) \in A, \delta^X_j(x_j) \in A \).
2. It is said that \( A \) is \( \delta^X \)-saturated if \( A \) is \( \delta^X_j \)-saturated for all \( j \geq 1 \).
3. It is said that \( A \) is \( \pi^X \)-saturated if: \( x_j \in \pi^X_j(A) \) for all \( j \geq 1 \) \( \Rightarrow (x_n) \in A \).

If \( A \) is solid, \( A \) is \( \delta^X \)-saturated.

\( \varphi(X) \) is \( \delta^X \)-saturated and not \( \pi^X \)-saturated.

If \( p \) is a n.a. semi-norm on \( X \), \( \{(x_n) \in \omega(X) / \sup_n p(x_n) \leq 1 \} \) is \( \pi^X \)-saturated.

The following results are demonstrated in a direct:

**Proposition 6.** Let \( A \) be a subset of \( E(X) \).

1. If \( A \) is \( \pi^X \)-saturated, \( S(A) \) is \( \pi^X \)-saturated.
2. If \( A \) is \( \delta^X \)-saturated, \( S(A) \) and \( c_0(A) \) are \( \delta^X \)-saturated, and \( A^o \) is \( \delta^Y \)-saturated and \( \pi^Y \)-saturated.
3. \( \left[ \pi^X_j(A) \right]^o \subset \pi^Y_j(A^o) \) for all \( j \geq 1 \).
4. If \( A \) is \( \delta^X \)-saturated, \( \left[ \pi^X_j(A) \right]^o = \pi^Y_j(A^o) \).
5. If \( A \) is \( \delta^X \)-saturated,

\[ A^o = \pi^X_j \left[ \pi^Y_j(A^o) \right] = \left\{ (y_k) \in F(Y) / \sup_k |\langle x_k, y_k \rangle| \leq 1 \text{ for all } (x_k) \in A \right\} \].

6. \( S(A)^o \subset S(A^o) \); and if \( A \) is \( \delta^X \)-saturated, \( A^o = S(A)^o = S(A^o) \).
7. If $A$ is $\delta^X$-saturated and $F(Y)$-closed, $\pi_j^X(A)$ is $Y$-closed for all $j \geq 1$.
8. If $A$ is $\pi^X$-saturated and $\pi_j^X(A)$ is $Y$-closed for all $j \geq 1$, $A$ is $F(Y)$-closed.

Corollary 1. Let $A$ be a subset of $E(X)$ $\delta^X$-saturated and $\pi^X$-saturated.
For $A$ is $F(Y)$-closed, it is necessary and enough that $\pi_j^X(A)$ be $Y$-closed for all $j \geq 1$.

Proposition 7. Let $A$ be an absolutely $K$-convex subset of $E(X)$.
1. If $A$ is $K$-closed and $\delta_j^X$-saturated, $\pi_j^X(A)$ is $K$-closed.
2. If $A$ is $\pi^X$-saturated and $\pi_j^X(A)$ is $K$-closed for all $j \geq 1$, $A$ is $K$-closed.

Proposition 8. Let $\tau$ be a topology on $E(X)$ and $\tau_j$ the topology image reciprocal of $\tau$ by the linear map $\delta_j^X$ on $X$. If $\tau$ admits as S.F.N of $0 \{A^o/A \in A\}$, then $\left\{\left[\pi_j^Y(A)\right]^o / A \in A\right\}$ is a F.S.N. of $0$ for $\tau_j$.

Proof. ([1], proposition 2.9).

Proposition 9. For all $j \geq 1$, $\pi_j^X$ is $(\sigma(E(X), F(Y)), \sigma(X, Y))$-continuous and $\delta_j^X$ is $(\sigma(X, Y), \sigma(E(X), F(Y)))$-continuous.

Proof. $(\pi_j^X)^*(Y) \subset F(Y)$ and $(\delta_j^X)^*(F(Y)) \subset Y$, and the result follows from ([9], p. 128).

Proposition 10. 1. $\left[\pi_j^X(A)\right]^o = (\delta_j^Y)^{-1}(A^o)$ for all $A \in E(X)$.
2. $\left[\delta_j^X(B)\right]^o = (\pi_j^Y)^{-1}(B^o)$ for all $B \subset X$.
3. $\pi_j^X(A) \subset B \Rightarrow \delta_j^Y(B^o) \subset A^o$ for all $A \in E(X)$ and for all $B \subset X$.
4. $\delta_j^X(B) \subset A \Rightarrow \pi_j^Y(A^o) \subset B^o$ for all $A \in E(X)$ and for all $B \subset X$.
5. $(\pi_j^X)^{-1}(D^o) = \left[\delta_j^Y(D)\right]^o$ for all $D \subset Y$.
6. $(\delta_j^X)^{-1}(C^o) = \left[\pi_j^Y(C)\right]^o$ for all $C \subset F(Y)$.
7. $(\pi_j^X)^*(D) \subset C \Rightarrow \pi_j^X(C^o) \subset D^o$ for all $D \subset Y$ and for all $C \in E(Y)$.
8. $(\delta_j^X)^*(C) \subset D \Rightarrow \delta_j^X(D^o) \subset C^o$ for all $D \subset Y$ and for all $C \subset E(Y)$.

Proof. ([1], proposition 2.8).
A polar topology of $\mathcal{A}$–convergence on $E(X)$ is said solid, if all $A \in \mathcal{A}$ is solid. Thus, any polar, solid topology admits a F.S.N from 0 consisting of solid subsets.

If $\tau$ is the polar topology of $\mathcal{A}$–convergence on $E(X)$ such that $A$ is $\delta^Y$–saturated for all $A \in \mathcal{A}$, $\tau$ coincides with the polar topology of $S(\mathcal{A})$–convergence (proposition 6), and then $\tau$ is a polar and solid topology.

**Proposition 11.** Let $\tau$ be a polar topology of $\mathcal{A}$–convergence over $E(X)$ and $\tau_j$ the topology image reciprocal of $\tau$ by the linear map $\delta^X_j$ on $X$.

1. $\tau_j$ is the polar topology of $\pi^Y_j(A)$–convergence.
2. $\pi^X_j$ is $(\tau, \tau_j)$–continuous if and only if $\delta^Y_j \circ \pi^Y_j(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$.

**Proof.** ([1], proposition 3.8).

**Proposition 12.** If $\tau$ is the weak topology (resp. Mackey, resp. $C$–compact, resp. $E(X)$–closed; resp. strong) of $E(X)$ for all $j \geq 1$, $\tau_j$ is the weak topology (resp. Mackey, resp. $C$–compact, resp. $X$–closed; resp. strong) on $X$.

**Proof.** ([1], proposition 3.9).

**Proposition 13.** Let $\tau$ a polar topology of $\mathcal{A}$–convergence on $E(X)$, for all $j \geq 1$, we have:

1. $\delta^X_j$ is $(\tau_j, \tau)$–continuous;
2. If $\tau$ is solid, $\pi^X_j$ is $(\tau, \tau_j)$–continuous;
3. If $\pi^X_j$ is $(\tau, \tau_j)$–continuous, $\delta^X_j$ is $(\tau_j, \tau)$–closed.

**Proof.**
1. $\tau_j$ is a polar topology of $\pi^Y_j(\mathcal{A})$–convergence, and we have:
   
   $\delta^X_j \left( \left[ \pi^Y_j(A) \right]^\circ \right) \subset A^\circ$ for all $A \in \mathcal{A}$.

   2. If $\tau$ is solid, we have:
   
   $\pi^X_j(A^\circ) \subset \left[ \pi^Y_j(A) \right]^\circ$ for all $A \in \mathcal{A}$.

   3. Let $M$ a closed in $(X, \tau_j)$, there exists $A \in \mathcal{A}$ such that $\left[ \pi^Y_j(A) \right]^\circ \subset M^\circ$, therefore $A^\circ \subset \delta^X_j(M^\circ) = \left[ \delta^X_j(M) \right]^\circ$.

Let $\tau$ be a locally $K$–convex topology on $E(X)$ such that $E(X)$ be $\tau$–pol; if $\tau$ is $(E(X) F(Y))$–compatible, $\tau$ is a polar topology of $\mathcal{A}$–convergence, where $\mathcal{A}$ is constituted of $\sigma(F(Y), E(X))$–bounded and
$E(X)$—closed subsets of $F(Y)$, ([1], theorem 4.3). For all $j \geq 1$, $\tau_j$ is the polar topology of $\pi_j^Y(A)$—convergence on $X$ and $X$ is $\tau_j$—polar if all $A \in \mathcal{A}$ is $\delta^Y$—saturated, $\pi_j^X(A)$ is $\sigma(Y,X)$—bounded and $X$—closed (Proposition 6), and then $\tau_j$ is $(X,Y)$—compatible.

If $K$ is spherically complete, we have the following theorem:

**Theorem 1.** Suppose that $K$ be spherically complete, and let $\tau$ a locally $K$—convex topology on $E(X)$; if $\tau$ is $(E(X),F(Y))$—compatible, $\tau_j$ is $(X,Y)$—compatible, for all $j \geq 1$.

**Proof.** $\tau$ is a polar topology of $\mathcal{A}$ convergence, where $\mathcal{A}$ consists of absolutely $K$ convex, $\sigma(E(Y),E(X))$—bounded and $\sigma(E(Y),E(X))$—C—compact subsets of $F(Y)$ ([1], theorem 4.4). For all $j \geq 1$, $\pi_j^Y$ is $(\sigma(F(Y),E(X)), \sigma(Y,X))$—continuous, then $\pi_j^Y(A)$ is absolutely $K$—convex, $\sigma(Y,X)$—bounded and $\sigma(Y,X)$—C—compact for all $A \in \mathcal{A}$ and then $\tau_j$ is $(X,Y)$—compatible. $\blacksquare$

**Theorem 2.** Let $\tau$ a solid and polar topology on $E(X)$; if $E(X)$ is $\tau$—barreled, $X$ is $\tau_j$—barreled for all $j \geq 1$.

**Proof.** Let $B$ a $\tau_j$—barrel in $X$; $\delta_j^X$ is $(\tau_j, \tau)$—closed, then $\delta_j^X(B)$ is a $\tau$—barrel into $E(X)$ and then $(\delta_j^X)^{-1}(\delta_j^X(B))$ is a neighborhood of 0 in $(X, \tau_j)$ then $B$ is a neighborhood of 0 for $\tau_j$. $\blacksquare$

**Remark 1.** Instead of assuming that $\tau$ is solid, we can assume only that $\pi_j^X$ be $(\tau, \tau_j)$—continuous for all $j \geq 1$.

A subset $A$ of $E(X)$ said to be $\delta^X$—stable if for all $x = (x_k) \in E(X)$ such that there exists $j \geq 1$ satisfying $\delta_j^X(x_j) \in A$, then $x \in A$.

Let $A \subset E(X)$ such that $A \cap \{\delta_j^X(a)/a \in X$ and $j \geq 1\} = \phi$, $A$ is $\delta^X$ stable.

**Definition 2.** Let $\tau$ a vector topology on $E(X)$; we say that $E(X)$ is $\delta^X\tau$—barreled if every $\tau$—barrel $\delta^X$—stable, is a neighborhood of 0.

If $E(X)$ is $\tau$—barreled, it is $\delta^X\tau$—barreled.

**Theorem 3.** Let $\tau$ a polar and solid topology on $E(X)$; if there exists $j \geq 1$ such that $X$ is $\tau_j$—barreled, $E(X)$ is $\delta^X\tau$—barreled.
Proof. Let $B$ a $\tau$-barrel $\delta^X$-stable in $E(X)$; $\delta_j^X$ is $(\tau_j, \tau)$-continuous, so $(\delta_j^X)^{-1}(B)$ is a $\tau_j$-barrel, and then $(\delta^X)^{-1}(B)$ is a neighborhood of 0 in $(X, \tau_j)$ and hence $(\pi_j^X)^{-1}[(\delta_j^X)^{-1}(B)]$ is a neighborhood of 0 in $(E(X), \tau)$. $B$ is $\delta^X$-stable, then $(\pi_j^X)^{-1}[(\delta_j^X)^{-1}(B)] \subset B$ and then $B$ is a neighborhood of 0 in $(E(X), \tau)$.

Theorem 4. Suppose that $X$ and $Y$ are semi-reflexive, and let $\tau$ a topology on $E(X)$ which is $(E(X), F(Y))$-compatible. If $E(X)$ is $\tau$-reflexive, $X$ is $\tau_j$-reflexive for every $j \geq 1$.

Proof. $\tau = \tau_b(E(X), E(X)'') = \tau_b(E(X), F(Y))$; so for all $j \geq 1 \tau_j = \tau_b(X, Y)$ (Proposition 12). $Y$ is semi-reflexive, then $\tau_j$ is $(X, Y)$-compatible ([1], proposition 5.9) and then $\tau_j = \tau_b(X, (X, \tau_j)'').$

Corollary 2. If $K$ is spherically complete and $\tau$ is a topology on $E(X)$ which is $(E(X), F(Y))$-compatible and solid such that $E(X)$ is $\tau$-barreled, then $X$ is $\tau_j$ reflexive for any $j \geq 1$.

Proof. For all $j \geq 1$, $\tau_j$ is $(X, Y)$-compatible (theorem 1) and $X$ is $\tau_j$-barreled for all $j \geq 1$, then $X$ is $\tau_j$-reflexive ([1], theorem 5.2).

4. Compactness and $C$-compactness

Let $\tau$ a polar topology on $E(X)$ such that $\pi_j^X$ be $(\tau, \tau_j)$-continuous for all $j \geq 1$. If $M$ is a compact subset of $(E(X), \tau)$; $\pi_j^X(M)$ is a compact subset of $(X, \tau_j)$ for all $j \geq 1$.

In order to study the converse, we introduce the notion of $TK$-convergent net.

Definition 3. A net $(x_i)_{i \in I}$ in $E(X)$ is called $TK$-convergent if for all $j \geq 1$, $(x_i^j)_{i \in I}$ is convergent in $(X, \tau_j)$.

Theorem 5. Let $M$ a subset of $E(X)$; $M$ is relatively compact in $(E(X), \tau)$ if and only if:

(i.) $\pi_j^X(M)$ is relatively compact in $(X, \tau_j)$ for all $j \geq 1$;

(ii.) All $TK$-convergent net in $M$ converges in $(E(X), \tau)$. 
Proof. N.C.] \( \pi_j^X \) is \((\tau, \tau)\)–continuous for all \( j \geq 1 \), then \( \pi_j^X(M) \) is relatively compact in \((X, \tau_1)\). Let \((x^i_j)_{i \in I}\) a \(TK\)–convergent net in \(M\). For all \( j \geq 1 \) let \( x_j \in X \) such that \((x^i_j)_{i \in I}\) converges to \( x_j \) in \((X, \tau_j)\). \((x^i_j)_{i \in I}\) has a cluster point \( z = (z_n) \) in \((E(X), \tau)\). For all \( j \geq 1 \), \( z_j \) is a cluster point of \((x^i_j)_{i \in I}\) in \((X, \tau_j)\); then \( z_j = x_j \). \((x_n)\) is the unique cluster point of \((x^i)_{i \in I}\), therefore \((x^i)_{i \in I}\) converges to \((x_n)\) in \((E(X), \tau)\).

S.C.] Let \((x^i)_{i \in I}\) a net in \(M\), and let \(A\) the family of \(\sigma(F(Y), E(X))\)–bounded subset of \(F(Y)\) which defines the topology \(\tau\). For any \( j \geq 1 \), \(\tau_j\) is the polar topology of \(\pi_j^Y(A)\)–convergence on \(X\).

Let \( x_1 \) a cluster point of \((x^i_1)_{i \in I}\) in \((X, \tau_1)\). For all \( A \in A \) and for all \( i \in I \), there exists \( i_A > i \) such that \( x^i_1 \in \left[ \pi_j^X(A) \right]^0 \). Consider the sub family \( (i_A)_{A \in A} \) of \( I \), it is ordered by: \( i_A \leq i_B \iff A \subset B \) for all \( A, B \in A \). \((i_A)_{A \in A}\) is a filter on the right family. Let \( A_0 \in A; i_A \geq i_{A_0} \Rightarrow A_0 \subset A \Rightarrow \left[ \pi_j^X(A) \right]^0 \subset \left[ \pi_j^X(A_0) \right]^0 \Rightarrow x^i_1 \subset \left[ \pi_j^X(A_0) \right]^0 \). Therefore \((x^i_1)_{A \in A}\) converges to \( x_1 \) in \((X, \tau_1)\).

Let \( x_2 \) a cluster point of \((x^i_2)_{A \in A}\) in \((X, \tau_2)\). for all \( A \in A \), there exists \( l_1(i_A) > i_A \) such that \( x^i_2 \in \left[ \pi_j^X(A) \right]^0 \).

Let \( A_0 \in A; i_A \geq i_{A_0} \Rightarrow A \supset A_0 \Rightarrow \left[ \pi_j^X(A) \right]^0 \supset x^i_2 \supset \left[ \pi_j^X(A_0) \right]^0 \). Therefore \((x^i_2)_{A \in A}\) converges to \( x_2 \) in \((X, \tau_2)\).

Let \( x_3 \) a cluster point of \((x^i_3)_{A \in A}\) in \((X, \tau_3)\). For all \( A \in A \), there exists \( l_2(i_A) \rightarrow l_1(i_A) \) such that \( x^i_3 \in \left[ \pi_j^X(A) \right]^0 \). \((x^i_3)_{A \in A}\) converges to \( x_3 \) in \((X, \tau_3)\).

Inductively, for all \( j \geq 3 \) and for all \( A \in A \), there exists \( l_j(i_{A-1}) \rightarrow l_1(i_A) \) such that \((x^i_{j+1})_{A \in A}\) converges to \( x_{j+1} \) in \((X, \tau_{j+1})\).

Put \( y = (x^{j+1}, x^{l_1(i_A)}, x^{l_2(i_{A-1})}, ..., x^{l_k(i_{A-2})}) \) \(A \in A\).

For all \( j \geq 1 \), \((x^i_j)_{A \in A}\) converges to \( x_j \) in \((X, \tau_j)\); therefore \( y \) is \(TK\)–convergent, and hence it converges to \( x \) in \((E(X), \tau)\). Hence \( x \) is a cluster point of \((x^i)_{i \in I}\), and then \( M \) is relatively compact. ■

Corollary 3. Let \( M \) a subset of \(E(X)\), \( M \) is compact in \((E(X), \tau)\) if and only if:

(i) \( \pi_j^X(M) \) is compact in \((X, \tau_j)\) for all \( j \geq 1 \),

(ii) Any \(TK\)–convergent net in \( M \) converges to an element of \( M \) in \((E(X), \tau)\).

To give version of theorem 5 using the filters, we need introduce the
following definition:

**Definition 4.** Let $M$ a subset of $E(X)$ and $F$ a filter on $M$; we say that $F$ is $TK-$ convergent if for all $j \geq 1$ the filter generated by $\pi_j^X(F)$ converges in $(X, \tau_j)$.

Every convergent filter is $TK-$convergent, and if $F$ is a $TK-$convergent filter and $F'$ is a filter finer than $F$, $F'$ is $TK-$convergent.

**Proposition 14.** Let $M$ a subset of $E(X)$.

1. If $F = (F_i)_{i \in I}$ is a $TK-$convergent filter on $M$, any net associated to $F$ is $TK-$convergent.
2. If $(x^i)_{i \in I}$ is a $TK-$convergent net, the $K-$convex filter associated to $(x^i)_{i \in I}$ is $TK-$convergent.

**Theorem 6.** Let $M$ a subset of $E(X)$; $M$ is compact in $(E(X), \tau)$ if and only if:

(i.) $\pi_j^X(M)$ is compact in $(X, \tau_j)$ for all $j \geq 1$;
(ii.) Any $TK-$convergent filter on $M$ converges to an element of $M$.

**Proof.** N.C.] Let $F$ a $TK-$convergent filter on $M$. For any $j \geq 1$ let $x_j \in X$ such that $\pi_j^X(F)$ converges to $x_j$ in $(X, \tau_j)$. $F$ has at least one cluster point $z = (z_n)$ in $M$. For all $j \geq 1$, $z_j$ is a cluster point of $\pi_j^X(F)$, therefore $z_j = x_j$; then $(x_n)$ is the unique cluster point of $F$ in $M$, so $F$ converges to $(x_n)$ in $(M, \tau)$.

S.C.] Let $F$ a maximal filter on $M$; for all $j \geq 1$ $\pi_j^X(F)$ is a maximal filter on $\pi_j^X(M)$, therefore it converges to $x_j$ in $(X, \tau_j)$, and then $F$ is $TK-$ convergent, therefore it converges to an element of $M$. 

**Definition 5.** Let $M$ a subset of $E(X)$, we say that $M$ is an $AK-$complete subset of $(E(X), \tau)$ if every $x = (x_n)$ element of $E(X)$ such that $(x^{[n]})$ is a Cauchy sequence in $(M, \tau)$; $x \in M$ and $(x^{[n]})$ converges to $x$ in $(E(X), \tau)$.

We say that $M$ is relatively $AK-$complete if its closure $\overline{M}$ in $(E(X), \tau)$ is $AK-$ complete.

If $M$ is complete, it is $AK-$complete.

Any closed subset of a set $AK-$complete is $AK-$complete.

In the following result, we characterize the subsets solid and relatively compact of $(E(X), \tau)$. 

Theorem 7. Let $M$ a solid subset of $E(X)$, $M$ is relatively compact in $(E(X), \tau)$ if and only if:
(i.) $\pi_j^X(M)$ is relatively compact in $(X, \tau_j)$ for all $j \geq 1$,
(ii.) $x^{[i]} \xrightarrow{i \to \infty} x$ uniformly on $M$ in $(E(X), \tau)$,
(iii.) $M$ is relatively $AK$–complete in $(E(X), \tau)$.

Proof. N.C.] If $M$ is relatively compact, $M$ is relatively complete, and then it is relatively $AK$–complete.

Suppose we did not (ii.) there exists $A \in \mathcal{A}$ a sequence $(i^i)_i$ in $M$ and a strictly increasing sequence of integers $(j_i)_i$ such that $i^i x^{[j_i]} - i^i x \notin A^\circ$ for all $i \geq 1$. The sequence $(i^i x^{[j_i]} - i^i x)_i$ is $TK$–convergent to $0$, so it converges to $0$ in $(E(X), \tau)$ which is absurd.

S.C.] Let $(\alpha^i x)_{\alpha \in D}$ a net in $M$ such that for all $j \geq 1 (\alpha^i x)_\alpha \in D$ converges to $x_j$ in $(X, \tau_j)$. Let $A \in \mathcal{A}$ for all $i \geq 1 \alpha^i x^{[i]} - x^{[i]} = \sum_{n=1}^i \delta_n^X(\alpha^i x_n - x_n) \in A^\circ$
for $\alpha$ sufficiently large. So for all $i \geq 1 \alpha^i x^{[i]} - \alpha^i x \rightarrow x^{[i]}$ in $(E(X), \tau)$ in particular $x^{[i]} \in \mathcal{M}$ for all $i \geq 1$. Using this convergence and (ii), we can choose $\alpha$ as $x^{[i]} - x^{[j]} = (x^{[i]} - \alpha^i x^{[i]}) + (\alpha^i x^{[j]} - \alpha^i x) + (\alpha^i x - \alpha^i x^{[j]}) + (\alpha^i x^{[j]} - x^{[j]}) \in A^\circ$ for $i, j$ sufficiently great. Therefore $(x^{[i]})$ is a Cauchy net in $\mathcal{M}$ and then $x^{[i]} \rightarrow_{i \to \infty} x$ in $(E(X), \tau)$. From this convergence and (ii), we can choose $i$ such that $\alpha^i x - x = (\alpha^i x - \alpha^i x^{[i]}) + (\alpha^i x^{[i]} - x^{[i]}) + (x^{[i]} - x) \in A^\circ$ for $\alpha$ large enough, so $(\alpha^i x)_{\alpha \in D}$ converges to $x$ in $(E(X), \tau)$ and hence $M$ is relatively compact (theorem 5).

Corollary 4. Let $M$ a solid subset of $E(X)$; $M$ is compact in $(E(X), \tau)$ if and only if:
(i.) $\pi_j^X(M)$ is compact in $(X, \tau_j)$ for all $j \geq 1$,
(ii.) $x^{[i]} \xrightarrow{i \to \infty} x$ uniformly on $M$ in $(E(X), \tau)$
(iii.) $M$ is $AK$–complete in $(E(X), \tau)$.

Corollary 5. The envelope solid of a relatively compact subset of $(E(X), \tau)$ is not necessarily relatively compact.

Proof. Let $x = (x_n) \in E(X)$ such that $(x^{[i]})_i$ does not converge to $x$ in $(E(X), \tau)$ so $(z^{[i]})_i$ does not converge to $z$ uniformly on $S(x)$ and then $S(x)$ is not relatively compact.

Proposition 15. 1. Let $(x^i)_{i \in I}$ a net in $E(X)$; if $\mathcal{F}$ is a $K$–convex filter associated with $(x^i)_{i \in I}$, $\pi_j^X(\mathcal{F})$ is a $K$–convex filter associated with a net $(x^j)_{i \in I}$ for all $j \geq 1$. 
2. Let $\mathcal{F}$ a $K$–convex filter on $E(X)$; if $(x^i)_{i \in I}$ is a net associated to $\mathcal{F}$, $(x^i_j)_{i \in I}$ is a net associated to $\pi_j^X(\mathcal{F})$ for all $j \geq 1$.

**Theorem 8.** Let $M$ a $K$–convex subset of $E(X)$; $M$ is $C$–compact in $(E(X), \tau)$ if and only if:

(i) $\pi_j^X(M)$ is $C$–compact in $(X, \tau_j)$ for all $j \geq 1$,

(ii) Any $K$–convex and $TK$–convergent filter on $M$ admits a cluster point in $M$.

**Proof.** N.C.] Obvious.

S.C.] Let $\mathcal{F}$ a maximum $K$–convex filter of $M$. For any $j \geq 1$, $\pi_j^X(\mathcal{F})$ is a maximum $K$–convex filter of $\pi_j^X(M)$ (proposition 2), so $\pi_j^X(\mathcal{F})$ converges to $x_j$ in $(X, \tau_j)$. $\mathcal{F}$ is then $TK$–convergent, so it admits a cluster point in $M$, and hence $\mathcal{F}$ converges in $(E(X), \tau)$ (Proposition 1). $\blacksquare$

**Proposition 16.** Let $M$ a $K$–convex subset of $E(X)$; if $M$ is $C$–compact, any $K$–convex and $TK$–convergent filter on $M$ has a unique cluster point in $M$.

**Proof.** Let $\mathcal{F}$ a $K$–convex and $TK$–convergent filter on $M$. For all $j \geq 1$ let $x_j \in X$ such that $\pi_j^X(\mathcal{F})$ converges to $x_j$ in $(X, \tau_j)$. $\mathcal{F}$ admits at least one cluster point $(z_n)$ in $M$. For all $j \geq 1$, $z_j$ is a cluster point of $\pi_j^X(\mathcal{F})$ in $(X, \tau_j)$, and then $x_j = z_j$. So $(x_j)$ is the only cluster point of $\mathcal{F}$ in $M$. $\blacksquare$

5. $AK$–completion and completion

Let $M$ a subset of $E(X)$ and $\tau$ a topology on $E(X)$, we put:

$$S_M = \left\{x \in M | x^{[n]} \underset{n \to \infty}{\longrightarrow} x \ in \ (E(X), \tau) \right\}.$$ 

If $M$ is a subspace of $E(X)$, we say that $M$ is an $AK$–space if $S_M = M$.

**Proposition 17.** Let $\tau$ a polar topology of $A$ convergence on $E(X)$; $(E(X), \tau)$ is $AK$–complete.

**Proof.** Let $x = (x_n) \in E(X)$ such that $(x^{[n]})$ is a Cauchy sequence in $(E(X), \tau)$. For all $A \in A$ there exists $n_0 \geq 1$ such that $x^{[n]} - x^{[m]} \in A^c$ for all $n \geq m \geq n_0$, and then $x^{[n]} - x \in A^c$ for all $n \geq n_0$, then $x^{[n]} \underset{n \to \infty}{\longrightarrow} x$ in $(E(X), \tau)$. $\blacksquare$
Corollary 6. Let $M$ a subset of $E(X)$. $M$ is $AK$–complete if and only if $M$ contains every element $x$ of $E(X)$ such that $(x^{[i]})$ is the Cauchy sequence in $M$.

Corollary 7. Let $\tau'$ a locally $K$–convex topology on $E(X)$ coarser than $\tau$; any $AK$–complete subset of $(E(X), \tau')$ is complete in $(E(X), \tau)$.

Proof. Let $M$ an $AK$–complete subset of $(E(X), \tau')$, and either $x \in E(X)$ such that $(x^{[i]})$ is a Cauchy sequence in $(M, \tau)$, $(x^{[i]})$ is a Cauchy sequence in $(M, \tau')$, so $x \in M$ and hence $M$ is $AK$–complete in $(E(X), \tau')$ (Corollary 6).

For all $x = (x_n) \in E(X)$, we put $\psi_x : E(Y) \rightarrow c_0(K)$

$$\psi_x : \begin{array}{l}
y_n \rightarrow (x_n, y_n)_n \\
\end{array}$$

$\psi_x$ is a linear map.

Lemma 1. For any $x \in E(X)$, $\psi_x$ is $(\sigma(E(Y), E(X)), \sigma(c_0(K), m(K)))$–continuous.

Proof. $c_0(K)^\beta = m(K)$ and $(c_0(K), m(K))$ is a separating duality. Let $(\alpha_n) \in m(K)$; $E(X)$ is solid, then $(\alpha_n x_n) \in E(X)$, and we have $\psi_x((\alpha_n x_n)^\circ) \subset \{\alpha_n\}^\circ$.

Proposition 18. $(E(X), \sigma(E(X), E(Y)))$ is an $AK$–space.

Proof. Let $x = (x_n) \in E(X)$. For all $y = (y_n) \in E(Y)$, $(x_n, y_n) \in c_0(K)$; there exists $i_0 \geq 1$ such that $\sup_{n \geq i_0} |(x_n, y_n)| \leq 1$, then $x^{[i]} - x \in \{y\}^\circ$ for all $i \geq i_0$, and then $x^{[i]} \xrightarrow{i \to \infty} x$ in $(E(X), \sigma(E(X), E(Y)))$.

Proposition 19. Suppose that $K$ be local, and let $\tau$ a $(E(X), F(Y))$–compatible topology on $E(X)$; if $\tau$ is solid, $(E(X), \tau)$ is an $AK$–space.

Proof. Let $A$ a family of $(\sigma(F(Y), E(X))$–compacts and absolutely $K$–convex subsets of $F(Y)$ such that $\tau$ be a polar topology of $A$–convergence ([1], theorem 4.5.) Let $x = (x_n) \in E(X)$; for all $A \in A$, $\psi_x(A)$ is solid and $\sigma(c_0(K), m(K))$–compact in $c_0(K)$. Then $z^{[i]} \xrightarrow{i \to \infty} z$ uniformly on $z \in \psi_x(A)$ in $(c_0(K), \sigma(c_0(K), m(K)))$ (theorem 7); there exists $i_0 \geq 1$ such that $\left| z^{[i]} - z, e \right| \leq 1$ for all $i \geq i_0$ and for all $z \in \psi_x(A)$, then $x^{[i]} - x \in A^\circ$ for all $i \geq i_0$, and so $x^{[i]} \xrightarrow{i \to \infty} x$ in $(E(X), \tau)$.

We have the following result which is a kind of reciprocal of theorem 1:
Theorem 9. Suppose that $K$ be local, and let $\tau$ a polar and solid topology on $E(X)$ for separating duality $\langle E(X), E(X)^\beta \rangle$. If $\tau_j$ is $(X, Y)$–compatible for all $j \geq 1$, $\tau$ is $(E(X), E(X)^\beta)$–compatible.

Proof. $E(X)^\beta = (E(X), \sigma(E(X), E(X)^\beta)) \subset (E(X), \tau')$. Let $f \in (E(X), \tau')$ and $x = (x_n) \in E(X)$. $(E(X), \tau)$ is an $AK$–space (proposition 19), therefore $x^{[i]} \xrightarrow{i \to \infty} x$ in $(E(X), \tau)$, and then $f(x) = \lim f(x^{[i]}) = \sum_j f\delta_j^X(x_j)$. For all $j \geq 1$, $f\delta_j^X \in (X, \tau_j)' = Y$; therefore $f(x) = \sum_j \langle x_j, y_j \rangle$, with $y_j = f\delta_j^X$ for all $j \geq 1$. Hence $(y_j) \in E(X)^\beta$, and so $(E(X), \tau)' \subset E(X)^\beta$. ■

Let $\mathcal{C}$ a family of subsets of $F(Y)$ such that:
1. $\mathcal{C}$ is the right filtering for inclusion;
2. There exist $\lambda_0 \in K$, $|\lambda_0| > 1$ such that $\lambda_0 A \in \mathcal{C}$ for all $A \in \mathcal{C}$;
3. $\lambda_j^Y(A)$ is $\sigma(Y, X)$–bounded for all $j \geq 1$ and for all $A \in \mathcal{C}$
4. The subspace of $E(Y)$ generated by $\cup \{ A/A \in \mathcal{C} \}$ contains $\varphi(Y)$.

We put: \[
\begin{aligned}
\mathcal{C}(X) &= \left\{ (x_n) \in \omega(X)/ \sup_{(y_n) \in A} \sum_n \langle x_n, y_n \rangle < \infty \text{ for all } A \in \mathcal{C} \right\} \\
\mathcal{C}(Y) &= \text{subspace generated by } \cup \{ A/A \in \mathcal{C} \}. 
\end{aligned}
\]

If $\mathcal{C}$ is the family of all finite subsets of $F(Y)$, $\mathcal{C}(X) = F(Y)^\beta$.

$\varphi(X) \subset \mathcal{C}(X)$ and $\langle \mathcal{C}(X), \mathcal{C}(Y) \rangle$ is a separating duality defined by the bilinear form:
$$\langle (x_n), (y_n) \rangle = \sum_n \langle x_n, y_n \rangle \text{ for all } (x_n) \in \mathcal{C}(X) \text{ and for all } (y_n) \in \mathcal{C}(Y).$$

If $\tau$ is the polar topology of $\mathcal{A}$–convergence of $E(X)$, $(\mathcal{A}(X), \tau_\mathcal{A})$ is defined, where $\tau_\mathcal{A}$ is the polar topology defined on $\mathcal{A}$ by the family $\mathcal{A}$, and we have:
1. $E(X) \subset \mathcal{A}(X) \subset F(Y)^\beta$;
2. $\tau_\mathcal{A}/E(X) = \tau$.

Proposition 20. Let $\tau$ a polar topology of $\mathcal{A}$–convergence on $E(X)$.

1. $S_{\mathcal{A}(X), \tau_\mathcal{A}} \subset E(X)$,
2. $(\mathcal{A}(X), \tau_\mathcal{A})$ is $AK$–complete.

Proof. 1. Let $x = (x_n) \in S_{\mathcal{A}(X), \tau_\mathcal{A}}$; $x^{[i]} \xrightarrow{i \to \infty} x$ $(\tau_\mathcal{A})$, therefore $(x^{[i]})$ is Cauchy sequence in $(E(X), \tau)$ $(\tau = \tau_\mathcal{A}/E(X))$, and then $x \in E(X)$ (proposition 17).
2. Let \((x^{[i]})\) a Cauchy sequence in \((\mathcal{A}(X), \tau_{\mathcal{A}})\); for all \(A \in \mathcal{A}\), there exists \(i_0 \geq 1\) such that for all \(i, j \geq i_0\) \(\sup_i \left\{ \sum_{n=i+1}^{j} \langle x_n, y_n \rangle \right\} / (y_n) \in A \leq 1\).

We have on the one hand, sup \(\left\{ \sum_{n=i_0}^{n} \langle x_n, y_n \rangle \right\} / (y_n) \in A \leq 1\), therefore \(\sup_i \left\{ \sum_{n=i+1}^{\infty} \langle x_n, y_n \rangle \right\} / (y_n) \in A \leq 1\), therefore \(\sup_i \left\{ \langle x^{[i]} - x, (y_n) \rangle \right\} / (y_n) \in A \leq 1\), and then \(x^{[i]} \xrightarrow{i=\infty} x (\tau_{\mathcal{A}})\).

**Theorem 10.** Let \(\tau\) a solid and polar topology of \(\mathcal{A}\)–convergence on \(E(X)\). For \(E(X)\) is a closed subspace of \((\mathcal{A}(X), \tau_{\mathcal{A}})\) it is necessary and sufficient that any Cauchy net \(TK\)–convergent of \(E(X)\) converges in \((E(X), \tau)\).

**Proof.** N.C.] \(A\) is solid for all \(A \in \mathcal{A}\), therefore \(A^\circ = [A \cap \varphi(X)]^\circ\).

Let \((x^i)_{i \in I}\) a Cauchy and \(TK\)–convergent net in \((E(X), \tau)\). For all \(j \geq 1\), let \(x_j \in X\) such that \((x^j)_{i \in I}\) converges in \((X, \tau_j)\) to \(x_j, \tau_j\) is the polar topology of \(\pi_j^Y(\mathcal{A})\)–convergence on \(X\). Let \(A \in \mathcal{A}\), there exists \(k_0 \in I\) such that for all \(r, s \geq k_0\) \(\sum_{j=1}^{\infty} \langle x^i_j - x^i_j, y_j \rangle \leq 1\) for all \(N \geq 1\) and for all \(y \in A\). There exists \(k_j \in I\) such that for all \(r \geq k_j\) \(\left\| \langle x^i_j - x, y_j \rangle \right\| \leq 1\) for all \((y_n) \in A\). Let \(r_0 = \max\{k_0, k_1, \ldots, k_N\}\) for all \(r \geq r_0\) we have:

\[
\sum_{j=1}^{N} \left\| x^i_j - x, y_j \right\| \leq \max_{1 \leq j \leq N} \left\| \langle x^i_j - x, y_j \rangle \right\| \leq 1\] for all \((y_n) \in A\).

\[
\sum_{j=1}^{N} \left\| x^i_j - x, y_j \right\| \leq 1\] for all \((y_n) \in A\) and for all \(s \geq r_0\); therefore \(x^s - x \in [A \cap \varphi(X)]^\circ\) for all \(s \geq r_0\). Furthermore, \(x = x^s - (x^s - x) \in \mathcal{A}(X)\). Therefore \((x^i)_{i \in I}\) converges to \(x\) in \((\mathcal{A}(X), \tau_{\mathcal{A}})\), and then \(x \in E(X)\) and \((x^i)_{i \in I}\) converges to \(x\) in \((E(X), \tau)\).

S.C.] Let \((x^i)_{i \in I}\) a net in \(E(X)\) which converges to \(x\) in \((\mathcal{A}(X), \tau_{\mathcal{A}})\). \((x^i)_{i \in I}\) is a Cauchy and \(TK\)–convergent net in \((E(X), \tau)\) \((\tau = \tau_{\mathcal{A}/E(X)}\)\), therefore \((x^i)_{i \in I}\) converges to \(x\) in \((E(X), \tau)\).
Lemma 2. Let $L$ and $M$ two $K$-vector spaces, $\tau$ a topology on $L$, $L \xrightarrow{\pi} M \xrightarrow{\delta} L$ two linear maps such as $\pi \circ \delta = \text{id}_M$, and $\tau_\delta$ the inverse image topology of $\tau$ by $\delta$ on $M$. The application $\psi : (M, \tau_\delta) \rightarrow (\delta(M), \tau)$, $x \mapsto \delta(x)$, is an homeomorphism.

Proof. If $\mathcal{U}$ is a F.S.N of 0 for $\tau$; a F.S.N of 0 for $\tau_\delta$ is $\delta^{-1}(\mathcal{U}) = \{\delta^{-1}(U) / U \in \mathcal{U}\}$, and we have: $\psi^{-1}(U \cap \delta(M)) = \delta^{-1}(U)$ for all $U \in \mathcal{U}$. □

Theorem 11. Let $\tau$ a polar and solid topology of $\mathcal{A}$-convergence on $E(X)$; $(E(X), \tau)$ is complete if and only if:
(i.) $(X, \tau_j)$ is complete for all $j \geq 1$;
(ii.) $E(X)$ is a closed subspace of $(\mathcal{A}(X), \tau_A)$.

Proof. N.C.] $\delta_j^X$ is $(\tau, \tau_j)$-closed for all $j \geq 1$ (proposition 13), therefore $\delta_j^X(X)$ is a closed subspace of $(E(X), \tau)$, hence $(\delta_j^X(X), \tau)$ is complete. Now $(\delta_j^X(X), \tau) \simeq (X, \tau_j)$ (lemma 2), therefore $(X, \tau_j)$ is complete. Furthermore $E(X)$ is a closed subspace of $(\mathcal{A}(X), \tau_A)$ (theorem 10).
S.C.] Let $(x^i)_{i \in I}$ a Cauchy net in $(E(X), \tau)$. For $j \geq 1$, $(x^i_j)_{i \in I}$ is Cauchy in $(X, \tau_j)$ so it converges, and then $(x^i)_{i \in I}$ is $TK$-convergent in $(E(X), \tau)$ so it converges in $(E(X), \tau)$, (theorem 10). □

Remark 2. We can replace (ii) of theorem 11 by:
(ii) Any Cauchy $TK$-convergent net in $(E(X), \tau)$ converges in $(E(X), \tau)$.

Corollary 8. Let $\tau$ a polar and solid topology of $\mathcal{A}$-convergence on $E(X)$. If $E(X)$ is a closed subspace of $(\mathcal{A}(X), \tau_A)$: $(E(X), \tau)$ is sequentially complete if and only if $(X, \tau_j)$ is sequentially complete for all $j \geq 1$.

Lemma 3. Let $\tau$ a vector topology on $E(X)$; if $\tau$ is solid, $S_{E(X)}$ is the closure of $\varphi(X)$ in $(E(X), \tau)$.

Proof. $S_{E(X)} \subset \varphi(X)$. Let $x = (x_n) \in \varphi(X)$ and $U$ a solid neighborhood of 0, it is $z = (z_n) \in \varphi(X)$ as $x - z \in U$. Since $U$ is solid $x[i] - x \in U$ for $i$ large enough, then $x[i] \xrightarrow{i \rightarrow \infty} x$ in $(E(X), \tau)$ and hence $x \in S_{E(X)}$. □

Proposition 21. Let $\tau$ a solid and polar topology of $\mathcal{A}$-convergence on $E(X)$; if $(X, \tau_j)$ is complete for all $j \geq 1$, $(S_{E(X)}, \tau)$ is complete.
Proof. $S_{E(X)} = \varphi(X)$ (lemma 3), therefore $(S_{E(X)}, \tau)$ is a closed subspace of $(A(X), \tau_A)$, and then $(S_{E(X)}, \tau)$ is complete.

Application: Let $(X, \|\cdot\|)$ a n.a Banach space, we consider $m(X)$ endowed with the n.a. norm $\|\cdot\|_\infty$. We have $c_0(X) = S_m(X)$, and $\|\cdot\|_\infty$ defines a polar and solid topology on $m(X)$, therefore $(c_0(X), \|\cdot\|_\infty)$ is complete.

Theorem 12. Let $\tau$ a solid and polar topology of $A$–convergence on $E(X)$; if $E(X)$ is an $AK$–space, $(E(X), \tau)$ is complete if and only if $(X, \tau_j)$ is complete for all $j \geq 1$.

Proof. N.C.] Obvious.

S.C.] $E(X)$ is an $AK$–space, therefore $E(X) = S_{(E(X), \tau)}$. Now $S_{(A(X), \tau_A)} \subset E(X)$ (proposition 20) and $S_{(E(X), \tau)} \subset S_{(A(X), \tau_A)}$, therefore $E(X) = S_{(E(X), \tau)} = S_{(A(X), \tau_A)}$, and then $E(X)$ is a closed subspace of $(A(X), \tau_A)$.

Hence $(E(X), \tau)$ is complete (theorem 11).

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