The difference of growth rate distributions between sales and profits

Atushi Ishikawa\textsuperscript{1}, Shouji Fujimoto\textsuperscript{1} and Masashi Tomoyose\textsuperscript{2}

\textsuperscript{1}Kanazawa Gakuin University, 920-1392, Japan
\textsuperscript{2}University of the Ryukyus, 903-0213, Japan

E-mail: ishikawa@kanazawa-gu.ac.jp

Abstract. Using numerical simulations, the authors exhibit the difference between two types of the growth rate distributions, the one of which is observed in both positive and negative data such as profits, and the other of which is in non-negative data such as sales. In the simulation, firstly the Langevin equation generates both positive and negative variables, the growth rate distributions of which are linear functions of the logarithmic growth rate. By superposing the variables not to be negative, we find that the growth rate distributions of the non-negative variables have wider tails than line shape on a log-log scale. At the same time, two types of Non–Gibrat’s Laws in the middle scale range are also confirmed as observed in real economic data.

1. Introduction
In order to understand economic and social systems, it is important to study collective behavior of the elements. One of remarkable examples is Pareto’s Law \cite{1}:

\[ P(v) \propto v^{-(\mu+1)} \quad \text{for} \quad v > v_{th}, \tag{1} \]

which is observed in economic data such as assets, sales, profits, income of firms, the number of employees and personal income (denoted by \( v \)) over some size threshold \( v_{th} \). Here, \( P(v) \) is the probability density function (pdf) and the power \( \mu \) is called Pareto index. It is important to understand the mechanism generating Pareto’s Law because a small number of firms in the power–law range dominate a large percent of the overall sales, profits and so forth. Pareto’ Law has been well investigated by using various models (\cite{2}–\cite{5} for instance). Recently, by using no model, it has been shown that Pareto’s Law can be explained by laws observed in a massive amount of digitized economic data. Fujiwara et al. \cite{6, 7} point out that Pareto’s Law can be derived from the Law of Detailed Balance and from Gibrat’s Law \cite{8}.

Detailed Balance is time-reversal symmetry:

\[ P_{12}(v_1, v_2) = P_{12}(v_2, v_1) \tag{2} \]

as observed in the equilibrium system which corresponds to a stable economy. Here, firm sizes at the two successive points in time are denoted by \( v_1 \) and \( v_2 \), and \( P_{12}(v_1, v_2) \) is the joint pdf. The growth rate \( R \) is defined as the ratio \( R = v_2/v_1 \). Gibrat’s Law states that the conditional pdf of the growth rate \( Q(R|v_1) \) is independent of the initial value \( v_1 \) as follows:

\[ Q(R|v_1) = Q(R), \tag{3} \]
where \( Q(R|v_1) \) is defined as \( Q(R|v_1) = \frac{P_R(v_1,R)}{P(v_1)} \) by using the pdf \( P(v_1) \) and the joint pdf \( P_{1,R}(v_1,R) \).

In these analyses of economic data, two types of the growth rate distributions are observed. For instance in Ref. [9], it is reported that the shape of the growth rate distribution of profits is different from that of sales by employing Japanese firms data [10]. The growth rate distribution of profits can be approximated by linear functions of the logarithmic growth rate \( r = \log_{10} v_2/v_1 \) as follows:

\[
\log_{10} q(r) = c - t_+ r \quad \text{for } r > 0 ,
\]

\[
\log_{10} q(r) = c + t_- r \quad \text{for } r < 0 .
\]

Here, \( q(r) \) is the pdf of \( r \), and is related to that of \( R \) by \( \log_{10} q(r) = \log_{10} Q(R) + r + \log_{10}(\ln 10) \). The linear approximation is, however, not suitable for the growth rate distribution of sales because the distribution has a curvature and wider tails than line shape on a log–log scale.

This difference has been observed in other literature employing not only Japanese firms data but also European and North American firms data (see [11]–[14] for instance). This aspect has been also observed in other economic data. Data having the growth rate distribution with no wide tail are income of firms [12, 15, 9], and GDP [16] for instance. In contrast, data having the growth rate distribution with wide tails are personal income in Japan [6], and assets and sales in France and the number of employees in the UK [7] for instance.

From these observations, we have made a simple hypothesis that the difference between shapes of growth rate distributions comes from being subtracted or not [9]. On the one hand, for example, profits are calculated by a subtraction of total expenditure from total sales at a rough estimate. The values can be both positive and negative. On the other hand, sales are not calculated by any subtraction. The values cannot be negative. This hypothesis can explain above observations in the literatures. On one hand, profits, income of firms and GDP are calculated by a subtraction, therefore, the growth rate distributions have no wide tail. The growth rate distributions are approximated by linear functions of \( r \) on a log–log scale. On the other hand, assets, sales, the number of employees and personal income are not calculated by any subtraction, therefore, the growth rate distributions have wide tails. The linear approximation is not applicable for the growth rate distributions.

In order to verify this hypothesis, one of the authors (A. I) has studied temporal changes of high–sales in Japan and observed that the growth rate distributions of temporal sales changes have no wide tail as expected [9]. This confirms our hypothesis that the growth rate distributions of both positive and negative variables are approximated by linear functions of \( r \) and that those of non–negative variables have wider tails than line shape on a log–log scale.

This hypothesis is not restricted in the large scale range. It is well known that Pareto’s Law (1) breaks below the size threshold \( v_{th} \) [17, 18]. It is also important to study the distribution in the middle scale range because the majority of firms are middle-sized. In the middle size range, one of the authors (A. I) has showed that the log–normal distribution:

\[
P(v) = C \ v^{-(\mu+1)} e^{-\alpha \ln^2 \frac{v}{v_{th}}} \quad \text{for } v < v_{th}
\]

can also be deduced using Detailed Balance and First Non–Gibrat’s Law [19] along works in Ref. [6, 7]. Non–Gibrat’s Law, observed below the size threshold \( v_{th} \), describes the dependence of the growth rate distribution on the initial value. In Ref. [20], the following is proved. If the growth rate distribution is described by linear functions of \( r \) even in the middle scale range as

\[
\log_{10} q(r|v_1) = c(v_1) - t_+ (v_1) \ r \quad \text{for } r > 0 ,
\]

\[
\log_{10} q(r|v_1) = c(v_1) + t_- (v_1) \ r \quad \text{for } r < 0 ,
\]
under Detailed Balance, the dependence of the initial value is uniquely fixed as follows:

$$t_{\pm}(v_1) = t_{\pm}(v_{th}) \pm \alpha \ln \frac{v_1}{v_{th}}.$$  \hfill (9)

This is verified in empirical data [20]. The probability of positive growth increases and the probability of negative growth decreases as the initial value $v_1$ increases in the middle scale range. In this paper, we designate Eqs. (7)–(9) First Non–Gibrat’s Law.

For the growth rate distributions of assets or sales in the middle scale range, other type of Non–Gibrat Law is reported [21]–[23]. The probability of positive and negative growth decreases as the initial value increases. Let this law be designated by Second Non–Gibrat’s Law. Let us make another hypothesis that the difference between two shapes of the growth rate distributions is related to the difference between two Non-Gibrat’s Laws in the middle scale range.

In this study, we clarify these two hypotheses by using numerical simulations as follows:

(i) We generate both positive and negative variables, the growth rate distributions of which are linear functions of $r$. The variables follow Pareto’s Law in the large scale range, and follow the log–normal distribution in the middle scale range. In the middle scale range, the growth rate distributions follow First Non–Gibrat’s Law.

(ii) By superposing the variables not to be negative, we find that the growth rate distributions of the non–negative variables have wider tails than line shape on a log–log scale.

(iii) We also find that the growth rate distributions of the non–negative variables follow Second Non–Gibrat’s Law in the middle scale range.

2. Empirical Observations

In this section, we present the results of empirical analyses of our previous studies in order to set up the simulation model. We have employed the database “CD Eyes 50” published by TOKYO SHOKO RESEARCH, LTD [10]. In the database, top 500 thousand sales data of Japanese firms are available. This database is thought to be approximately exhaustive in the large scale range. The database also includes both positive and negative profits data. The number of firms with positive profits is approximately 300 thousand and that with negative profits is approximately 40 thousand. The remaining approximately 160 thousand data do not include profits data. We have excluded firms with negative profits, the number of which is significantly less than that with positive profits. This is due to the fact that the negative profits data gathered from high–sales data are exclusive. Despite the fact that we do not have a complete picture of the positive profits data for firms in the middle scale range, we have decided to employ this data to investigate the consistency of the laws with the data.

Hereafter, let “non–negative variables” and “both positive and negative variables” be denoted by $x$ and $v$, respectively. Figure 1 shows the scatter plots of profits and sales in the database. In each scatter plot, Detailed Balance (2) has been approximately confirmed by applying the one-dimensional (1–dim) Kolmogorov–Smirnov (K–S) test [9].

Firstly, let us analyze profits data. In order to identify Gibrat’s Law in the large scale range and Non–Gibrat’s Law in the middle scale range, we divide the range of the initial profits $v_1$ into logarithmically equal bins as $v_1 \in [10^{2+0.2(n-1)}, 10^{2+0.2n}]$ ($n = 1, 2, \cdots, 20$). The conditional pdfs of the profits growth rate are shown in Fig. 2. In this figure, we observe that the growth rate distributions can be approximated by linear functions of $r$ in the range for $n \geq 6$ ($v_1 \geq 10^3$). By applying the linear approximations (7) and (8) to the data in Fig. 2, the relation between $v_1$ and $t_{\pm}(v_1)$ is obtained (Fig. 3). On the one hand, Fig. 3 shows that $t_{\pm}(v_1)$ hardly responds to $v_1$ for $n = 16, \cdots, 20$ ($v_1 \geq 10^5$). This means that Gibrat’s Law (3) or (4)–(5) is valid in the large scale range. On the other hand, $t_{+}(v_1)$ linearly increases and $t_{-}(v_1)$ linearly decreases symmetrically with $\log_{10} v_1$ for $n = 6, \cdots, 10$ ($10^3 \leq v_1 < 10^5$). This is First Non-Gibrat’s Law (7)–(9), which
Figure 1. The scatter plots of profits and sales in the database. Here, \(v_1\) and \(v_2\) \((x_1\) and \(x_2\)) represent profits (sales) in 2004 and 2005, respectively. The number of profits and the number of sales are 177,492 and 406,385 sets, respectively. In each case, Detailed Balance (2) is approximately confirmed by applying the 1-dim K–S test.

Figure 2. Conditional pdfs \(q(r|v_1)\) of the log profits growth rate \(r = \log_{10} \frac{v_2}{v_1}\) from 2004 \((v_1)\) to 2005 \((v_2)\). The data points are classified into twenty bins of the initial profits with equal magnitude in logarithmic scale, \(v_1 \in [10^{2+0.2(n-1)}, 10^{2+0.2n}] \ (n = 1, 2, \ldots, 20)\) thousand yen.

can be derived analytically from the linear approximations (7) and (8) under Detailed Balance (2) [20]. In Fig. 3, the parameters are estimated as follows: \(\alpha \sim 0\) for \(v > v_{th}\) (the large scale range), \(\alpha \sim 0.14\) for \(v_{\min} < v < v_{th}\) (the middle scale range), \(t_+(v_{th}) \sim 2, t_-(v_{th}) \sim 1, v_{th} \sim 10^{2+0.2(16-1)} = 10^5\) thousand \((= 100\) million\) yen and \(v_{\min} \sim 10^{2+0.2(6-1)} = 10^3\) thousand \((= 1\) million\) yen. Rigorously, a constant parameter \(\alpha\) must not take different values. In the database, however, a large number of firms stay in the same range for two successive years. This parameterization is approximately valid for describing the pdf.

Figure 4 shows that the resultant pdf (6) behaves as Pareto’s Law (1) in the large scale range where Gibrat’s Law is hold \((\alpha \sim 0)\), and behaves as the log–normal distribution in the middle scale range where First Non–Gibrat’s Law is hold \((\alpha \sim 0.14)\).
Figure 3. The relation between the lower bound of each bin $v_1 \in [10^{2+0.2(n-1)}, 10^{2+0.2n})$ and $t_\pm(v_1)$. From the left, each point on the graph represents $n = 1, 2, \cdots, 20$. The values are measured by the least square method in the region $0 \leq |r| \leq 2$ in Fig. 2.

Figure 4. The resultant pdf of positive profits (6) behaves as Pareto’s Law (1) in the large scale range where Gibrat’s Law is hold ($\alpha \sim 0$), and behaves as the log–normal distribution in the middle scale range where First Non–Gibrat’s Law is hold ($\alpha \sim 0.14$).

Secondly, we analyze sales data. In the same manner as in profits data analysis, we divide the range of the initial sales $x_1$ into logarithmically equal bins as $x_1 \in [10^{6+0.2(n-1)}, 10^{6+0.2n}) (n = 1, 2, \cdots, 10)$. The conditional pdfs of the sales growth rate are shown in Fig. 5. Note that the database does not include sales data in the middle scale range. Therefore, only Gibrat’s Law (3) is observed. In Fig. 5, the growth rate distributions hardly change as the initial sales value $x_1$ increases. In this case, Pareto’s Law (1) is induced from Gibrat’s Law (3) and from Detailed Balance (2) [6, 7]. Figure 6 shows that the sales data follow Pareto’s Law (1) in the large scale range. We should not pay attention the break of Pareto’s Law because this comes from the exhaustiveness of sales in the database.

Figure 5. Conditional pdfs $q(r|x_1)$ of the log sales growth rate $r = \log_{10} x_2/x_1$ from 2004($x_1$) to 2005($x_2$). The data points are classified into ten bins of the initial sales with equal magnitude in logarithmic scale, $x_1 \in [10^{6+0.2(n-1)}, 10^{6+0.2n}) (n = 1, 2, \cdots, 10)$ thousand yen.

As stated in introduction, the difference of growth rate distributions between sales and profits exists. Most distinguishing feature is that the growth rate distributions of profits are approximated by linear functions of $r$ ($n \geq 6$ in Fig. 2), however, those of sales are not (Fig. 5). This feature is also observed in existing literatures employing various economic data in many
countries ([11]—[16] for instance). We have made a simple hypothesis that the difference between shapes of growth rate distributions comes from being subtracted or not [9].

In order to confirm this hypothesis, one of the authors (A. I) has studied temporal changes of high–sales in Japan [9]. Here, let high–sales at the three successive points in time be denoted by $x_0$, $x_1$, and $x_2$. We consider two temporal changes $x_1 - x_0 \equiv v_1$ and $x_2 - x_1 \equiv v_2$, which can be both positive and negative. The data are classified into the following four quadrants: $(v_1 > 0, v_2 > 0)$, $(v_1 < 0, v_2 > 0)$, $(v_1 < 0, v_2 < 0)$, and $(v_1 > 0, v_2 < 0)$. In each quadrant, distributions in the growth rate of temporal sales changes $R = |v_2/v_1|$ are shown in Fig. 7. In four cases, no wide tail is observed as expected. It is possible to compare growth rate distributions of profits with those of temporal sales changes only in first quadrant because exhaustive negative profits data are not available. These two growth rate distributions are same kind in the sense that they have no wide tail. This confirms our hypothesis that the growth rate distributions of both positive and negative variables are approximated by linear functions of $r$ and that those of non–negative variables have wide tails on a log–log scale. At the same time, Detailed Balance and First Non–Gibrat’s Law have been also approximately observed in four quadrants.

**Figure 6.** The sales data in the database follow Pareto’s Law in the large scale range. This database does not include middle-seized sales data. The break of Pareto’s Law comes from the exhaustiveness of sales in the database. We should not pay attention the break.

**Figure 7.** Conditional pdfs $q(r|v_1)$ in the log growth rate of the temporal sales change $r = \log_{10}|v_2/v_1|$ for cases $(v_1 < 0, v_2 > 0)$, $(v_1 > 0, v_2 > 0)$, $(v_1 < 0, v_2 < 0)$ and $(v_1 > 0, v_2 < 0)$. In each figure, data points are classified into four bins of the initial temporal sales change with equal magnitude in logarithmic scale, $|v_1| \in [10^{4.5+0.5(n-1)}, 10^{4.5+0.5n}]$ $(n = 1, 2, \cdots, 4)$ thousand yen. Here, $v_2 = x_2 - x_1$ is the change between 2004 ($x_1$) and 2005 ($x_2$), and $v_1 = x_1 - x_0$ is between 2003 ($x_0$) and 2004 ($x_1$).
Unfortunately, the database we have employed does not include sales data in the middle scale range. By following the reports [21]–[23], Second Non–Gibrat’s Law, that the probability of positive and negative growth decreases as the initial value increases, is observed in the middle scale range. First and Second Non–Gibrat’s Laws are depicted in Figs. 8 and 9, respectively.

Figure 8. First Non–Gibrat’s Law:
The horizontal axis is the logarithm of the growth rate \( r \) and the vertical axis is the logarithm of its pdf \( \log_{10} q(r|v_1) \). The dashed lines with arrow means the change of the growth rate distribution as the initial profits \( v_1 \) increases in the middle scale range. This corresponds to the case \( 6 \leq n \leq 10 \) in Fig. 2.

Figure 9. Second Non–Gibrat’s Law:
The horizontal axis is the logarithm of the growth rate \( r \) and the vertical axis is the logarithm of its pdf \( \log_{10} q(r|x_1) \). The dashed lines with arrow means the change of the growth rate distribution as the initial sales \( x_1 \) increases in the middle scale range.

3. TFI Simulation Model
In Ref. [24], based on the work by Takayasu et al. [5], the authors have proposed the simulation model which leads to Pareto’s Law in the large scale range where Gibrat’s Law is hold, and to the log-normal distribution in the middle scale range where First Non–Gibrat’s Law is hold. The model is given by the Langevin equation

\[
v(t + 1) = b(v(t), t)v(t) + f(t),
\]

where \( b(v(t), t) \) is a non–negative multiplicative stochastic noise and \( f(t) \) is a random additive noise.

In the simulation, we adopt the distribution of \( b(v(t), t) \) to be Eqs. (7)–(9) with \( \alpha = 0 \) for \( v > v_{\text{th}} \) (Gibrat’s Law), \( \alpha \neq 0 \) for \( v_{\text{int}} < v < v_{\text{th}} \) (First Non–Gibrat’s Law) and \( \alpha = 0 \) for \( v < v_{\text{int}} \) keeping \( t_\pm(x) \) continuously (Fig. 10).

Figure 10 corresponds to Fig. 3 observed in empirical profits data. As the middle scale range in which First Non–Gibrat’s Law is hold, we observe the range \( \max(v_{\text{min}}, v_{\text{int}}) < v < v_{\text{th}} \). Here, \( v_{\text{int}} \) is depicted in Fig. 10 and \( v_{\text{min}} \) is defined as the bottom bound of the range where Detailed Balance is observed. In this paper, we refer to the combination of Gibrat’s Law in the large scale range and Non–Gibrat’s Law in the middle scale range as Extended-Gibrat’s Law. We also adopt the distribution of \( f(t) \) to be Weibull distribution:

\[
\frac{x}{\lambda} \left( 1 - \left( \frac{x}{\lambda} \right)^k \right) \exp\left( -\left( \frac{x}{\lambda} \right)^k \right).
\]

The typical scatter plot of the simulation is shown in Fig. 11, where we take \( t_+ (v_{\text{th}}) = 2.5, t_- (v_{\text{th}}) = 1.5, v_{\text{th}} = 10^6, k = 0.5, \lambda = 30 \) and \( \alpha = 0.1 \). In order to claim that this model is consistent with the empirical observation, Detailed Balance and Extended–Gibrat’s law must be satisfied. From Eq. (10), Extended–Gibrat’s law (7)–(9) is hold in the range where a noise \( f(t) \) is negligible. Detailed Balance (2) is also confirmed. The validity is not rejected within the 5% significance level by applying the 1-dim K–S test in the range approximately over \( v_{\text{min}} = 10^4 \) (Fig. 12). It is worth noting that Detailed Balance is not imposed in this simulation a priori.

\[1\] The last parameterization is imposed to exclude immoderate slopes of growth rate distributions.
The resultant pdf of \( v \) is shown in Fig. 13, where the log-normal distribution in the middle scale range is observed in addition to Pareto’s Law in the large scale one. From these results, we recognize that this simulation model fits with the empirical observations.

Figure 10. Continuous functions \( t_\pm(v) \). The horizontal axis is logarithmic scale.

Figure 11. The scatter plot of data points under Extended–Gibrat’s Law. The number of data points is 500,000.

Figure 12. Each p value of the 1–dim K–S test for the scatter plot in Fig. 11. We compared the distribution sample for 
\[
P(v_1 \in [10^{3+0.2(n-1)}, 10^{3+0.2n}], v_2)
\]
with another sample for 
\[
P(v_1, v_2 \in [10^{3+0.2(n-1)}, 10^{3+0.2n}])
\]
\( n = 1, 2, \cdots, 20 \). Detailed Balance is verified in the range approximately over \( 10^4 \).

Figure 13. The resultant pdf \( P(v) \). In the large scale range \((v > 10^6)\), Pareto’s Law is observed. In the middle scale one \((10^4 < v < 10^6)\), the log–normal distribution is observed.

4. Simulation Model for Profits and Sales
The simulation in the previous section generates the stochastic variables \( v \), the growth rate distributions of which are adopted to be linear functions of \( r \) (7) and (8). The multiplicative noise \( b \) and the additive one \( f \) are both positive, therefore, the stochastic variables \( v \) are also positive. However, in the case that the growth rate distributions have no wide tail, the stochastic variables \( v \) should be both positive and negative in our hypothesis.

From the results of analyses in temporal changes of sales data [9], we extend the model to generate Detailed Balance and First Non-Gibrat’s Law in the second, third and fourth quadrants,
as observed in the first quadrant. This can be realized by the modified Langevin equation [25]:

$$v(t + 1) = a(t) b(v(t), t) v(t) + f(t),$$  \hspace{1cm} (11)

where $a(t) = 1$ with probability 0.5 and $a(t) = -1$ with probability 0.5. In this case, the distribution of $f$ is adopted to be Gaussian $N(0, \sigma^2)$. The typical scatter plot of the simulation is shown in Fig. 14, where we take $t_+(v_{th}) = 2.6$, $t_-(v_{th}) = 1.4$, $v_{th} = 10^6$, $\sigma = 20$ and $\alpha = 0.2$. In each quadrant, Detailed Balance is approximately confirmed by applying the 1–dim K–S test (Fig. 15). Even in modified Langevin Eq. (11), Extended–Gibrat’s Law (7)–(9) is hold in the range where a noise $f(t)$ is negligible.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{scatter_plot.png}
\caption{The scatter plots of data points in four quadrants. The total number of data points is 500,000.}
\end{figure}

Using these both positive and negative variables $v$, we obtain non–negative variables $x$ by assuming the change of $x$ to be $v$:

$$v(t) = x(t + 1) - x(t).$$ \hspace{1cm} (12)

In the next simulation, non–negative variables $x$ are updated iteratively with the restriction $x > 0$ as follows:

$$x(t + 1) = \begin{cases} 
  x(t) + v(t) & \text{if} \quad x(t + 1) > 0, \\
  x(t) & \text{if} \quad x(t + 1) < 0.
\end{cases}$$ \hspace{1cm} (13)

The resulting scatter plot of $x$ is shown in Fig. 16. Detailed Balance is approximately verified by applying the 1–dim K–S test (Fig. 17).
Figure 15. P values of the 1-dim K–S test for each scatter plot in Fig. 14. We compared the distribution sample for \( P(\mid v_1 \mid \in [10^{3+0.2(n-1)}, 10^{3+0.2n}], \mid v_2 \mid) \) with another sample for \( P(\mid v_1 \mid, \mid v_2 \mid \in [10^{3+0.2(n-1)}, 10^{3+0.2n}]) \) \((n = 1, 2, \cdots, 20)\). In each quadrant, Detailed Balance is approximately verified.

Figure 16. The scatter plot of non-negative data points \( x \), the number of which is 500,000.

Figure 17. P values of the 1-dim K–S test for the scatter plot in Fig. 16. We compared the distribution sample for \( P(x_1 \in [10^{4+0.2(n-1)}, 10^{4+0.2n}], x_2) \) with another sample for \( P(x_1, x_2 \in [10^{4+0.2(n-1)}, 10^{4+0.2n}]) \) \((n = 1, \cdots, 20)\). Detailed Balance is approximately confirmed.

The main result in this paper is the growth rate distributions of non-negative variables \( x \) shown in Fig. 18. The point is that the growth rate distributions have wide tails. In addition, Second Non–Gibrat’s Law is also observed in the middle scale range. The probability of positive and negative growth decreases as the bin size of the classification of \( x \) increases as expected.
The resultant cumulative pdf of $x$ is shown in Fig. 19.

**Figure 18.** Conditional pdfs $q(r|x_1)$ of the log growth rate $r = \log_{10} \frac{x_2}{x_1}$. The data points are classified into twenty bins of the initial variables with equal magnitude in logarithmic scale, $x_1 \in [10^{4+0.2(n-1)}, 10^{4+0.2n}]$ ($n = 1, 2, \cdots, 20$).

**Figure 19.** The resultant cumulative pdf $P(v)$.

5. Conclusion and future issues
In this paper, by using numerical simulations, we explain the difference between two types of the growth rate distributions. One is observed in both positive and negative data such as profits, the distribution of which can be approximated by linear functions on a log–log scale. The other is observed in non–negative data such as sales, the distribution of which has wide tails.

In the previous study [24], the simulation model generates only non–negative variables. At the same time, the growth rate distributions are adopted to be linear functions on a log–log scale. To resolve this discrepancy, in this study, the model has been modified to generate both positive and negative variables. The variables follow Detailed Balance and First Non–Gibrat’s Law in the second, third and fourth quadrants, as observed in the first quadrant of the original model [24]. This is consistent with the results of analyses in temporal changes of sales data [9].
Using both positive and negative variables \( v \), we have obtained non-negative variables \( x \) by assuming the change of \( x \) to be \( v \). In the joint pdf of \( x \), we have confirmed Detailed Balance. At the same time, we have found that the growth rate distributions have wide tails and Second Non-Gibrat’s Law is observed in the middle scale range. As a matter of fact, Second Non-Gibrat’s Law with respect to \( x \) is not related to First Non-Gibrat’s Law with respect to \( v \) because Second Non-Gibrat’s Law is also observed in the case that the growth rate distribution of \( v \) follows Gibrat’s Law only \( (\alpha = 0) \). Second Non-Gibrat’s Law is induced from the superposition of variables \( v \). This result may be interpreted as the relation between stock and flow.

In this paper, we assume that there are similar joint pdfs in four quadrants of \( v \). However, they are thought to be different from each other in an unstable economy \([26, 27]\). This can be controlled by the factor \( a \) in Eq. (11). As a next step, quantitative analyses of the simulation including the above situation are under investigation. The analytic derivation for the result in the simulation should be also examined. The reason will be clarified why the growth rate distributions of \( v \) are approximated by linear functions on a log-log scale. These researches might provide useful knowledge in the management of credit risk.

**Acknowledgments**

The authors are grateful to APFA7 & Tokyo Tech – Hitotsubashi Interdisciplinary Conference, where this work was completed and especially to Prof. H. Yoshikawa and Prof. T. Watanabe for useful comments. This work was supported in part by a Grant-in-Aid for Scientific Research (C) (No. 20510147) from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

[1] Pareto V 1897 *Cours d’Economique Politique* (London: Macmillan)
[2] Kesten H 1973 *Acta Math.* 131 207
[3] Levy M and Solomon S 1996 *Int. J. Mod. Phys.* C 7 595
[4] Sornette D and Cont R 1997 *J. Phys.* I 7 431
[5] Takayasu H, Sato A and Takayasu M 1997 *Phys. Rev. Lett.* B 79 966
[6] Fujiwara Y, Souma W, Aoyama H, Kaizoji T and Aoki M 2003 *Physica* A 321 598
[7] Fujiwara Y, Guilmi C D, Aoyama H, Gallegati M and Souma W 2004 *Physica* A 335 197
[8] Gibrat R 1932 *Les inegalites economiques* (Paris: Sirey)
[9] Ishikawa A 2009 Power-Law and Log-Normal Distributions in Temporal Changes of Firm-Size Variables *Economics* 3 2009-11
[10] TOKYO SHOKO RESEARCH, LTD., http://www.tsr-net.co.jp/
[11] Amaral L A N, Buldyrev S V, Havlin S, Leschhorn H, Maass P, Salinger M A, Stanley H E and Stanley M H R 1997 *J. Phys. I France* 7 621
[12] Okuyama K, Takayasu M and Takayasu H 1999 *Physica* A 269 125
[13] Matia K, Fu D, Buldyrev S V, Pammolli F, Riccaboni M and Stanley H E 2004 *Europhys. Lett.* 67 498
[14] Gabaix X 2005 *The Granular Origins of Aggregate Fluctuations* (MIT and NBER Working Paper)
[15] Ishikawa A 2006 *Physica* A 363 367
[16] Canning D, Amaral L A N, Lee Y, Meyer M and Stanley H E 1998 *Economics Lett.* 60 335
[17] Badger W W 1980 *Mathematical Models as a Tool for the Social Science* (New York: Gordon and Breach) p87
[18] Montroll E W and Shlesinger M F 1983 *J. Stat. Phys.* 32 209
[19] Ishikawa A 2006 *Physica* A 367 425
[20] Ishikawa A 2007 *Physica* A 383 79
[21] Aoyama H 2004 9th Ann. Workshop on Economic Heterogeneous Interacting Agents (Kyoto)
[22] Aoyama H, Fujiwara Y and Souma W 2004 *The Physical Society of Japan* 2004 Autumn Meeting (Aomori)
[23] Takayasu H 2009 APFA7th Tokyo Tech - Hitotsubashi Interdisciplinary Conf. (Tokyo)
[24] Tomoyose M, Fujimoto S and Ishikawa A 2009 Proc. of the YITP Workshop on Econophysics (Kyoto) *Prog. Theor. Phys. Suppl.* 179 114
[25] Mizuno T, Takayasu M and Takayasu H 2004 *Physica* A 332 403
[26] Ishikawa A 2006 *Physica* A 371 525
[27] Ishikawa A 2007 Proc. of the YITP Workshop on Econophysics (Kyoto) *Prog. Theor. Phys. Suppl.* 179 103