Precessional motion of a vortex in a finite-temperature Bose-Einstein condensate

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We study the precessing motion of a vortex in a Bose-Einstein condensate of atomic gases. In addition to the former zero-temperature studies, finite temperature systems are treated within the Popov and semiclassical approximations. Precessing vortices are discussed utilizing the rotating frame of reference. The relationship between the sign of the lowest excitation energy and the direction of precession is discussed in detail.

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I. INTRODUCTION

The Gross-Pitaevskii (GP) approximation is often used to analyze the Bose-Einstein condensates (BEC) of atomic gases. The GP approximation only treats the condensate fraction, leaving out the normal component. Therefore, the GP equation treats systems at zero temperature. The Bogoliubov equations describe the excitations supported by the condensate in the zero-temperature limit. The Bogoliubov excitation spectra agree with experimental results, such as the collective oscillation modes (including the Tkachenko waves) of a vortex lattice and those on Bragg spectroscopy of elongated condensates. Since the physical systems exist at finite temperatures, attempts to incorporate effects of temperature into the various approximations have been made. Among these, the Hartree-Fock-Bogoliubov-Popov (Popov) approximation (Sec. V. F in Ref. 7) is one of the most common finite-temperature approximations.

The Popov approximation treats the density of the normal gas component as a mean field. The condensate is described with the GP equation, extended to include the mean field potential. The excitation spectra and the wavefunctions of the normal gas are given by eigenequations. Because of computational complexity, we use the Popov equations only for excitations below a certain cutoff energy \( E_{\text{cut}} \). Therefore, a precessing vortex state as a deviation from the axisymmetric configuration. In this work we aim to treat the precessing vortex in a 2D geometry directly within the Popov approximation in order that the relation between the precession and the excitations are directly taken into account from the outset.

II. APPROXIMATIONS

The condensate is treated with a nonlinear Schrödinger equation (NLSE) within the Popov approximation. The thermal atoms are described using the Popov equations which are eigenequations. Because of computational complexity, we use the Popov equations only for excitations below a certain cutoff energy \( E_{\text{cut}} \). The excitation energies \( \varepsilon > E_{\text{cut}} \) are taken into account within the semiclassical approximation. The density of the condensate is \( n_0(\mathbf{r}) \equiv |\phi(\mathbf{r})|^2 \), where \( \phi(\mathbf{r}) \) is the condensate wavefunction. The normal particle density coming from the excitations below \( E_{\text{cut}} \) is \( n_1(n_2) \). Therefore, the particle number density is \( n(\mathbf{r}) = n_0(\mathbf{r}) + n_1(\mathbf{r}) + n_2(\mathbf{r}) \). The condensate is described with the NLSE

\[
-\nabla^2 \phi + V - \mu + g(n_0 + 2n_1 + 2n_2) - \mathbf{w}_{\text{rot}} \cdot \mathbf{r} \times \mathbf{p} \phi = 0 \quad (1)
\]

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where \( C = \hbar^2/(2m) \) and \( g = 4\pi\hbar^2a/m \). The mass of a Na atom \( m = 38.17 \times 10^{-27} \) kg, and the scattering length \( a = 2.75 \) nm for Na atoms are employed. We use the cutoff energy \( E_{\text{cut}} = 10\hbar\omega_{\text{rot}} \). Angular velocity of rotation is \( \omega_{\text{rot}} \) and the rotation axis is parallel with the \( z \) axis (\( \omega_{\text{rot}} = e_z\omega_{\text{rot}} \)). Excitations below the cutoff \( E_{\text{cut}} \) are eigenstates of the Popov equations
\[
\begin{align*}
-C\nabla^2 + V - \mu + 2g(n_0 + n_1 + n_2) \\
-\omega_{\text{rot}} \cdot r \times p)u_q - g\phi^2v_q &= \varepsilon_qu_q, \\
-C\nabla^2 + V - \mu + 2g(n_0 + n_1 + n_2) \\
+\omega_{\text{rot}} \cdot r \times p)v_q + g\phi^2u_q &= \varepsilon_qv_q.
\end{align*}
\]
These reduce to the Bogoliubov equations if we neglect \( n_1 \) and \( n_2 \). The wavefunctions \( u_q \) and \( v_q \) obey the normalization condition
\[
\int \left( |u_q|^2 - |v_q|^2 \right) dr = 1.
\]
The density \( n_1 \) is a weighted sum of the wavefunctions \( u \) and \( v \):
\[
n_1(r) = \left\{ \sum_q \left( |u_q|^2 + |v_q|^2 \right) f(\varepsilon_q) + |v_q|^2 \right\}
\]
\[
f(\varepsilon) = \frac{1}{\varepsilon_b/\hbar^2 - 1}.
\]
The higher-energy range \( \varepsilon > E_{\text{cut}} \) is described within the semiclassical approximation which neglects the derivatives of the amplitudes of the wavefunctions \( u \) and \( v \) and the second derivatives of their phases. We also neglect the phase of the condensate wavefunction \( \phi \) here. Then the Popov equations reduce into algebraic form (Ref. \[14\], Eqs. (5)). The expression for \( n_2 \), in analogy with Eq. (4), is:
\[
n_2(r) = \int \frac{dp}{2\pi} \left( \frac{\varepsilon_{\text{HF}}}{\varepsilon} f\left( \frac{\varepsilon}{2} \right) - \frac{1}{2} \right) \Theta(\varepsilon - E_{\text{cut}})
\]
where the Hartree-Fock (HF) energy
\[
\varepsilon_{\text{HF}}(r, p) = \frac{p^2}{2m} + V - \mu + 2g(n_0 + n_1 + n_2)
\]
and energies
\[
\varepsilon(r, p) = \sqrt{\varepsilon_{\text{HF}}^2(r, p) - g^2n_0},
\]
\[
\varepsilon(r, p) = \varepsilon(r, p) - \omega_{\text{rot}} \cdot r \times p
\]
are functions of \( r \) and \( p \). The noncondensate densities \( n_1 \) and \( n_2 \) are determined from Eqs. (4) and (6). They are treated as mean field potentials throughout the above equations. Thus the numerical procedure needs to be selfconsistent such that it is repeated until the solution reaches convergence in which all the equations are simultaneously satisfied.

The angular momenta of the condensate \( \phi \) and of the wavefunctions \( u, v \) are
\[
A(\phi) = e_z \int \phi^*(r \times p)\phi^2 dr, \quad (10)
\]
\[
A(u_q) = e_z \int u_q^*(r \times p)u_q^2 dr, \quad (11)
\]
\[
A(v_q) = e_z \int v_q^*(r \times p)v_q^2 dr \quad (12)
\]
where \( U_q \equiv \int |u_q|^2 dr \) and \( V_q \equiv \int |v_q|^2 dr \). Within the Bogoliubov theory \((T = 0)\), the excitation energies depend linearly on the angular velocity \( \omega_{\text{rot}} \) as follows:
\[
\varepsilon = \varepsilon_{\text{lab}} - \hbar \omega_{\text{rot}} q_0, \quad (13)
\]
\[
q_0 \equiv \text{Re} \left\{ \left[ A(u_q) - A(\phi) \right] U_q + \left( A(v_q) + A(\phi) \right) V_q \right\}
\]
\[
/(U_q + V_q). \quad (14)
\]
The value \( q_0 \) is useful for characterizing an excitation also for finite-temperature systems. But as for the \( \omega \) dependence of \( \varepsilon \), there are deviations from Eq. (15) due to changes of density in the normal component \( n_2(r) + n_2(r) \) for finite-temperature systems. Figure indicates these deviations. Changes in the excitation energies modify the normal-component density which in turn affects the whole system, including the \( \varepsilon \)’s themselves. Therefore, the spectra for each value of \( \omega_{\text{rot}} \) needs to be calculated individually.

III. OFF-CENTERED VORTEX

The sign of the excitation energy of the core-localized excitation and the direction of the precessional motion are related at zero temperature. The predicted precession frequency fits well with results of the experiments. A finite-temperature extension of the Bogoliubov equation, the Popov approximation, shows that the sign of the core-localized excitation becomes positive if the direction of the precessing motion and the sign of the excitation energy correspond to each other, the precessing motion of the vortex must also be inverted as the lowest excitation energy raises from negative to positive values.

We extend the 2D treatment of the Bogoliubov equations to finite-temperature Popov equations. It makes possible to treat the slightly off-centered vortices directly within the Popov approximation.

Assume a BEC system has a vortex line perpendicular to the \( z \) axis. Particles are confined with a harmonic trap along the \( x \) and \( y \) axes
\[
V(x, y) = \frac{m \omega_{\text{trap}}^2}{2} (x^2 + y^2)
\]
with \( \omega_{\text{trap}} = 2\pi \times 200 \). The system has finite \( z \) thickness and it is uniform along the \( z \) axis. Periodic boundary conditions along \( z \) are used. Hence this system is
not a two-dimensional BEC, but rather a three-dimensional BEC having restricted geometry. We treat a system having $10^3$ atoms within a $z$ thickness of $10 \mu m$. The Thomas-Fermi (TF) radius $R_{\text{TF}} = 6.793 \mu m$ is used as the scale of length.

Equations (1) - (9) are repeatedly solved until convergence into a self-consistent solution. While the vortexfree ($A(\phi) = 0$) and centered-vortex ($A(\phi) = 1$) configurations are most likely, there also exists a solution with an off-centered vortex ($0 < A(\phi) < 1$) in a narrow window of $\omega$. Figures 2(a-c) represent such a typical system. Here the angular momentum is $A(\phi) = 0.863$. The particle number of the condensate is $48\%$ of the total particle number. The noncondensate density has a characteristic peak at the core of the vortex, like those in the axisymmetric studies.

The displacement of the vortex core $\Delta r$ is unrestricted in the numerical processes, unlike for the axisymmetric situations ($\Delta r = 0$). Therefore, the displacement $\Delta r$ depends on temperature $T$ and the rotation frequency $\omega_{\text{rot}}$ as presented in Fig. 3(a).

Figure 3(a) plots rotation frequencies at which the system is static. Let us denote the displacement and the rotation frequency at the static point as $\Delta r'$ and $\omega'_{\text{rot}}$. When $\Delta r < \Delta r'$, $\omega_{\text{rot}} = \omega'_{\text{rot}}$, and $T = 0$, the vortex has an instability and it tends to move inward. When $\Delta r > \Delta r'$, the direction is outward. As $\omega_{\text{rot}}$ increases in Fig. 3(a), the system has a wider range of displacements $\Delta r$ having an inward instability. This instability brings the vortex to the center of system and makes the vortex state more sustainable. At the finite temperature $T = 0.1 T_c$, the range of $\Delta r$ having an inward instability is almost the same. But the positive value of the lowest excitation energy $\epsilon'$ in Fig. 3(b) requires that the displacement $\Delta r$ is stable and the vortex tends to remain at $\Delta r = \Delta r'$.

The lowest excitation energies $\epsilon'$ in Fig. 3(b) differ significantly between the $T = 0$ system and those of the $T = 0.1 T_c$ system. It is small and negative ($0 > \epsilon' > -0.004 \hbar \omega_{\text{rot}}$) for the $T = 0$ system, while it is positive in the $T = 0.1 T_c$ system. The corresponding rotation

![FIG. 1: Dependence of $\epsilon'$ (lowest $\epsilon$) on $\omega_{\text{rot}}$ in an axisymmetric system ($\Delta r = 0$). The solid lines indicate the $\epsilon'$ at $T = 0$ (Bogoliubov approximation) and $T/T_c = 0.1, 0.3$, and 0.5 (Popov approximation). The dotted lines represent Eq. (13). The angular momentum $q_\theta = -1$. The dotted line and the solid line overlap at $T = 0$. The two lines are separated for $T > 0$, which means that “equality” between the rotation velocity $\omega_{\text{rot}}$ and the excitation energy in Eq. (13) is lost at finite temperatures.](image1)

![FIG. 2: Density profiles of a system at the temperature $T = 0.5 T_c$, and angular momentum of the condensate $A(\phi) = 0.863$ in rotating coordinates with $\omega_{\text{rot}} = 0.300 \omega_{\text{tr}}$. (a) Particle density $n_0(x, y)$ of the condensate, (b) Particle density $n_1(x, y) + n_2(x, y)$ of the noncondensate, (c) Excitation spectra $\epsilon_q$ vs. $q_\theta$. The solid line shows $1 - \omega_{\text{rot}} q_\theta / \omega_{\text{tr}}$. Its slope indicates the rotation velocity $\omega_{\text{rot}}$. The line touches the dipole modes at $q_\theta = \pm 1$.](image2)
frequencies $\omega_{\text{rot}}$ in Fig. 3(a) differ only a little. This indicates that the direction of the precessional motion which is represented by the sign of $\omega_{\text{rot}}$ and the sign of the lowest core excitation energy are not related in the $T > 0$ systems. This point is discussed further in the next section.

Figures 4(a-b) compare the spectral densities

$$g_1(j) = \int \frac{d\varepsilon}{\mathcal{E}} \sum_{q,j \leq \frac{\mathcal{E}}{\mathcal{E}_{\text{rot}}} < (j+1)} (|u_q|^2 + |v_q|^2) f(\varepsilon_q) + |v_q|^2,$$

(16)

$$g_2(\varepsilon'') = \frac{d^3 p d\varepsilon}{\mathcal{E} \varepsilon} \left( f(\varepsilon) + \frac{1}{2} \right) \frac{1}{2} \delta(\varepsilon'' - \varepsilon),$$

(17)

and the angular momenta of the noncondensate

$$L_1(j) = e_z \cdot \frac{1}{\hbar} \sum_{q,j \leq \frac{\mathcal{E}}{\mathcal{E}_{\text{rot}}} < (j+1)} \int \frac{d\varepsilon}{\mathcal{E}} [u_q^* (r \times p) u_q + v_q^* (r \times p) v_q'] f(\varepsilon_q) + v_q (r \times p) v_q'] f(\varepsilon_q),$$

(18)

$$L_2(\varepsilon'') = e_z \cdot \frac{1}{\hbar} \int \frac{d^3 p d\varepsilon}{\mathcal{E} \varepsilon} (r \times p) \{ f(\varepsilon(r, p)) - \frac{1}{2} \left( \frac{\varepsilon_{\text{HF}}(r, p)}{\varepsilon(r, p)} - 1 \right) \} \times \delta(\varepsilon'' - \varepsilon),$$

(19)

to verify the mutual consistency of the Popov approximation and that of the semiclassical approximation. Above, $g_1$ and $L_1$ are obtained within the Popov approximation, while $g_2$ and $L_2$ are obtained within the semiclassical approximation. Results of these two approximations are consistent with each other for $\varepsilon > 5\hbar\omega_{\text{rot}}$.

The core-localized mode is mainly affected by the particle densities inside the core, while $g_1$ and $g_2$ account for the densities over the whole area of the system. Since our main interest is the core-localized mode, we employ calculations with reduced accuracy for $n_2$ and $g_2$ for larger $x, y$ (around $|x| > 1.2 R_{\text{TF}}$ or $|y| > 1.2 R_{\text{TF}}$). This affects the plot for $g_1$ in Fig. 3(a), but it has little effect on $\varepsilon'$ as shown in the dependence of $\varepsilon'$ on the cutoff energy, $E_{\text{cut}}$.

IV. THE SIGN AND DIRECTION OF VORTEX PRECESSION

The angular velocity $\omega_{\text{prec}}$ of the precessional motion of a vortex is related with that of the rotating frame $\omega_{\text{rot}}$ through

$$\omega_{\text{prec}} + \omega_{\text{rot}} = \text{const.}$$

(20)

within the GP equations. The core-localized excitation has the lowest excitation energy $\varepsilon'$ within the range of $\omega_{\text{rot}}$ we treat. Using variational Lagrangean analysis, it can be shown that

$$\varepsilon' = 0 \text{ at } \omega_{\text{prec}} = 0$$

(21)

for a small displacement $\Delta r$. This relation remains valid for $0 < \Delta r < 0.5 R_{\text{TF}}$ within the accuracy $0 > \varepsilon' > -0.004 \hbar \omega_{\text{rot}}$ in the present system, see Fig. 3(b).

Equations (18), (20), and (21) lead to the relation

$$\varepsilon' = \hbar q_0 \omega_{\text{prec}}.$$  

(22)

Therefore, the direction of the precessional motion and the sign of the core-localized excitation correspond to each other at $T = 0$.

If the coordinate transformation in Eq. (20) were not valid, the relation Eq. (22) between the excitation energy $\varepsilon'$ and the angular velocity $\omega_{\text{prec}}$ of the precessional motion would not hold. The next section describes how this occurs at finite temperatures.

An easier way to disprove Eq. (22) is as follows. Figure 3(b) displays $\varepsilon'$ for $\omega_{\text{prec}} = 0$. It shows that Eq. (21) is no longer satisfied for $T > 0$. Therefore, it becomes impossible to satisfy Eq. (22). The sign of the lowest excitation and the direction of the precessional motion are
The velocities of the noncondensate and normal component below the cutoff are negligible \((n_0 + n_1 \ll n)\). This indicates adiabaticity between the rotating trap and the normal gas. Within the Popov approximation, the angular velocity of the precessional motion of a vortex is restricted to that of a normal gas and a confining trap.

We have considered static systems with \(\omega_{\text{prec}} = 0\) in a rotating frame with the angular velocity \(\omega_{\text{rot}}\). These two \(\omega\)'s may be transformed between each other using the simple relationship between the stationary and rotating frames of reference in Eq. (20), valid at zero temperature. But taking into account the normal component, this relation is only valid when the normal component is rotating at the angular velocity \(\omega_{\text{rot}}\). Varying \(\omega_{\text{rot}}\) will change the density profiles through Eqs. (13) and (14). Then the coordinate transformation Eq. (20) does not hold. Adiabaticity required within the Popov framework restricts the recognizing of \(\omega_{\text{rot}}\) as the angular velocity of precession \(\omega_{\text{prec}}\). This is another reason why Eq. (22) does not hold at finite temperatures.

VI. DISCUSSION

It is confirmed that the sign of the lowest excitation energy \(\varepsilon'\) is, in general, unrelated with the direction of the precessional motion of a vortex within the Popov approximation. Within the Bogolubov approximation, the excitation energy \(\varepsilon'\) of core-localized mode and the precession frequency \(\omega_{\text{prec}}\) are proportional to each other as shown in Eq. (22). The derivation of Eq. (22) shows that the angular velocity (and the direction) \(\omega_{\text{prec}}\) of the precessional motion arises from the coordinate transformation Eq. (20), which does not have any explicit relation with the core-localized excitation. Therefore, the core-localized excitation is responsible for the inward/outward motions of vortex and not explicitly related to the precessional motion.

We think that this nature of the core-localized excitation does not change even at a finite \(T\). However, a
FIG. 5: (a) Velocities of all the atoms (solid line) and the normal component (dashed line) along the x axis (y = 0). The velocity follows that of the rotating frame (dotted line) outside the condensate (|x| > RF). Temperature T = 0.5Tc and angular momentum A(s) = 0.863. (b) Density profiles along the x axis for comparison with (a). The plots of n0 (dashed line) and n1 + n2 (dotted line) are equivalent to those on the x axes in Figs. 2a-b.

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