Nondivergence Elliptic and Parabolic Problems with Irregular Obstacles

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Abstract. We prove the natural weighted Calderón and Zygmund estimates for solutions to elliptic and parabolic obstacle problems in nondivergence form with discontinuous coefficients and irregular obstacles. We also obtain Morrey regularity results for the Hessian of the solutions and Hölder continuity of the gradient of the solutions.

Contents

1. Introduction
2. Preliminaries
  2.1. Notations
  2.2. Basic assumptions
  2.3. Weighted Lebesgue and Sobolev spaces
3. Elliptic fully nonlinear obstacle problems
4. Elliptic linear obstacle problems
5. Morrey regularity results and Hölder continuity of the gradient
6. Parabolic obstacle problem
References

1. Introduction

We study in this paper the following elliptic obstacle problems:

\[
\begin{align*}
  a_{ij}(x)D_{ij}u & \leq f \quad \text{in } \Omega, \\
  (a_{ij}(x)D_{ij}u - f) (u - \psi) & = 0 \quad \text{in } \Omega, \\
  u & \geq \psi \quad \text{in } \Omega, \\
  u & = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(1.1)

and

\[
\begin{align*}
  F(x, D^2u) & \leq f \quad \text{in } \Omega, \\
  (F(x, D^2u) - f) (u - \psi) & = 0 \quad \text{in } \Omega, \\
  u & \geq \psi \quad \text{in } \Omega, \\
  u & = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.2)

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Here $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with its boundary $\partial \Omega \in C^{1,1}$. The coefficient matrix $(a_{ij}(x))$ and the fully nonlinear operator $F(x,M)$ are supposed to be uniformly elliptic, see Section 2. The nonhomogeneous term $f \in L^p_w(\Omega)$ is given, as is the obstacle function $\psi \in W^{2,p}_w(\Omega)$, $\psi \leq 0$ a.e. on $\partial \Omega$, where a weight $w$ in some Muckenhoupt class and the range of $p$ will be clarified later.

We also consider the following parabolic obstacle problem:

$$
\begin{cases}
  u_t - a_{ij}(x,t)D_{ij}u &\geq f \quad \text{in } \Omega_T, \\
  (u_t - a_{ij}(x,t)D_{ij}u - f)(u - \psi) & = 0 \quad \text{in } \Omega_T, \\
  u &\geq \psi \quad \text{in } \Omega_T, \\
  u & = 0 \quad \text{on } \partial_p \Omega_T,
\end{cases}
$$

(1.3)

where $\Omega_T := \Omega \times (0,T)$, $T > 0$, and $\partial_p \Omega_T := (\partial \Omega \times [0,T]) \cup (\Omega \times \{t = 0\})$ with $\partial \Omega \in C^{1,1}$. Here the coefficient matrix $(a_{ij}(x,t))$ is uniformly parabolic, see Section 2; the nonhomogeneous term $f$ is in $L^p_w(\Omega_T)$ with $p > 2$ and $w$ in a Muckenhoupt class, and the obstacle function is $\psi \in W^{2,1}L^p_w(\Omega_T)$ with $\psi \leq 0$ a.e. on $\partial_p \Omega_T$.

The main purpose of this study is to investigate existence, uniqueness and regularity properties of solutions to the obstacle problems (1.1), (1.2) and (1.3) in the framework of weighted Lebesgue spaces. The weighted Lebesgue spaces $L^p_w$ not only generalize the classical Lebesgue spaces $L^p$, but also are closely related to Morrey spaces $L^{p,\theta}$. In particular, knowing the fact that the Hardy-Littlewood maximal function of the characteristic function of a ball is a Muckenhoupt weight (see [13]), we are able to obtain an optimal Morrey regularity for the Hessian of the solutions to (1.1) and (1.2). This leads to a higher integrability result of the Hessian of the solutions and Hölder continuity of the gradient of the solutions.

In this paper we deal with discontinuous coefficients $a_{ij}$, irregular obstacle functions $\psi$ and discontinuous nonhomogeneous terms $f$ given in the weighted Lebesgue spaces. We notice that if $\partial \Omega$, $a_{ij}$, $f$ and $\psi$ are smooth enough, for instance, $\partial \Omega \in C^{2,\alpha}$, $a_{ij}, f \in C^\alpha(\bar{\Omega})$ for some $\alpha > 0$, and $\psi \in C^2(\bar{\Omega})$, then the obstacle problem (1.1) has a unique strong solution $u \in C^{1,1}(\bar{\Omega})$, see [16], and furthermore, the $C^{1,1}$ regularity of solutions for various types of obstacle problems has been extensively investigated under appropriate regularity assumptions on the boundary of domain, the obstacle, the nonhomogeneous term, see [15, 18, 21].

In the case of discontinuous coefficients and irregular nonhomogeneous terms, but without obstacles, the regularity results for elliptic and parabolic equations in nondivergence form have been obtained in [6, 7, 10, 11, 13, 27] for the elliptic case, and in [1, 5, 17, 26] for the parabolic case. In particular, weighted $W^{2,p}$ estimates were established in a series of papers [2, 3, 4]. Here we want to extend these results for nondivergence structure problems from the non-obstacle case to the obstacle case. More precisely, we shall establish the weighted $W^{2,p}$ estimates of solutions to the elliptic obstacle problems (1.1) and (1.2), and parabolic obstacle problem (1.3), by essentially proving that the Hessian of solutions is as regular as the nonhomogeneous terms and the Hessian of the associated obstacles.

Our approach is mainly based on a new general approximation argument in the literature. Unlike other approximation arguments which have in general penalty terms as in [10], we find a better approximation of Heaviside functions in order to use specially redesigned reference equations (3.4), (4.5) and (6.6). The choice of such an approximation method seems to be appropriate to our theory, as the problem under consideration is in the setting of Lebesgue spaces and one can easily control the $L^p$-norm of the nonhomogeneous term in a reference equation. Although this
approximation method does not involve penalty terms, we can utilize comparison principles to show that the solution is in the constraint set. Furthermore, this approach can be extended to the fully nonlinear obstacle problems.

This paper is organized as follows. In the next section we introduce some background and review weighted Lebesgue and Sobolev spaces. In Section 3 and 4 we establish the weighted $W^{2,p}$ estimates for the elliptic fully nonlinear obstacle problem (1.1) and elliptic linear obstacle problem (1.2), respectively. In Section 5 we present Morrey regularity results and obtain Hölder continuity of the gradient of the solutions for the elliptic obstacle problems. Finally, in the last section we prove the weighted $W^{2,p}$ estimates for parabolic linear obstacle problem (1.3).

2. Preliminaries

2.1. Notations. We start with some standard notations and terminologies.

(1) For $y \in \mathbb{R}^n$ and $r > 0$, $B_r(y) := \{x \in \mathbb{R}^n : |x - y| < r\}$ denotes the open ball in $\mathbb{R}^n$ with center $y$ and radius $r$.

(2) For $(y,s) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, $Q_r(y,s) := B_r(y) \times (s - r^2, s + r^2)$ denotes the parabolic cylinder with middle center $(y,s)$, radius $r$, height $r^2$.

(3) For a Lebesgue measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes the Lebesgue measure of $E$.

(4) For an integrable function $h : E \to \mathbb{R}$ with a bounded measurable set $E \subset \mathbb{R}^n$, we denote $\overline{h}_E$ the integral average of $h$ on $E$ by

$$\overline{h}_E := \int_E h(x) \, dx = \frac{1}{|E|} \int_E h(x) \, dx.$$ 

(5) $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the Euclidean inner product in $\mathbb{R}^n$.

(6) $S(n)$ denotes the set of real $n \times n$ symmetric matrices. For $M \in S(n)$, $\|M\|$ denotes the $(L^2, L^2)$-norm of $M$, that is, $\|M\| = \sup_{|x|=1} |Mx|$, and we write $M \geq 0$ to mean that $M$ is a non-negative definite symmetric matrix.

(7) The summation convention of repeated indices are used.

(8) For the sake of convenience, we employ the letter $c$ to denote any universal constants which can be explicitly computed in terms of known quantities, and so $c$ might vary from line to line.

2.2. Basic assumptions. For the problem (1.1), the coefficient matrix $A = (a_{ij}) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is assumed to be symmetric (that is, $a_{ij} \equiv a_{ji}$) and uniformly elliptic in the following sense:

**Definition 2.1.** We say that the coefficient matrix $A$ is uniformly elliptic if there exist positive constants $\lambda$ and $\Lambda$ such that

$$\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2$$

(2.1)

for almost every $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$.

For the problem (1.2), the fully nonlinear operator $F = F(x,M)$ is assumed to be uniformly elliptic in the following sense:

**Definition 2.2.** We say that the fully nonlinear operator $F$ is uniformly elliptic if there exist positive constants $\lambda$ and $\Lambda$ such that

$$\lambda \|N\| \leq F(x, M + N) - F(x, M) \leq \Lambda \|N\|$$

(2.2)

for almost every $x \in \Omega$ and all $M, N \in S(n)$ with $N \geq 0$. 
We also assume that $F(x,0) \equiv 0$, for simplicity, and that $F = F(x,M)$ is a convex function of $M \in S(n)$.

For the parabolic problem (1.3), the coefficient matrix $A = (a_{ij}) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is assumed to be symmetric and uniformly parabolic in the following sense:

**Definition 2.3.** We say that the coefficient matrix $A = A(x,t)$ is uniformly parabolic if there exist positive constants $\lambda$ and $\Lambda$ such that

$$\lambda |\xi|^2 \leq \langle A(x,t)\xi,\xi \rangle \leq \Lambda |\xi|^2 \quad (2.3)$$

for almost every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and all $\xi \in \mathbb{R}^n$.

### 2.3. Weighted Lebesgue and Sobolev spaces.

**Definition 2.4.** Let $1 < s < \infty$. We say that $w$ is a weight in Muckenhoupt class $A_s$, or an $A_s$ weight, if $w$ is a locally integrable nonnegative function on $\mathbb{R}^n$ with

$$[w]_s := \sup_B \left( \int_B w(x) \, dx \right) \left( \int_B w(x)^{-\frac{1}{s-1}} \, dx \right)^{s-1} < +\infty , \quad (2.4)$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. If $w$ is an $A_s$ weight, we write $w \in A_s$, and $[w]_s$ is called the $A_s$ constant of $w$.

We note that the Muckenhoupt classes $A_s$ are monotone in $s$, more precisely, $A_{s_1} \subset A_{s_2}$ for $1 < s_1 \leq s_2 < \infty$. The weighted Lebesgue space $L^p_w(\Omega)$, $1 < p < \infty$, $w \in A_s$ with $1 < s < \infty$, consists of all measurable functions $g$ on $\Omega$ such that

$$|| g ||_{L^p_w(\Omega)} := \left( \int_\Omega |g|^p w \, dx \right)^{\frac{1}{p}} < +\infty .$$

The weighted Sobolev space $W^{m,p}_w(\Omega)$, $m \in \mathbb{N}$, $1 < p < \infty$, $w \in A_s$ with $1 < s < \infty$, is defined by a class of functions $g \in L^p_w(\Omega)$ with weak derivatives $D^\alpha g \in L^p_w(\Omega)$ for all multiindex $\alpha$ with $|\alpha| \leq m$. The norm of $g$ in $W^{m,p}_w(\Omega)$ is defined by

$$|| g ||_{W^{m,p}_w(\Omega)} := \left( \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha g|^p w \, dx \right)^{\frac{1}{p}} .$$

For a detailed discussion of the weighted Lebesgue and Sobolev spaces, we refer the readers to [22, 24] and references therein. We will use the following embedding lemma later in the proof of Theorem 3.3, see [4, Remark 2.4].

**Lemma 2.5.** Let $n_0 < p < \infty$ for some $n_0 > 1$, and let $w \in A_{n_0}$. Suppose that $f \in L^p_{w_0}(\Omega)$. Then $f \in L^{p,n_0}_{\frac{n_0}{p}}(\Omega)$ for some small $\kappa = \kappa \left( n, \frac{1}{n_0}, [w]_{\frac{n_0}{p}} \right) > 0$ with the estimate

$$|| f ||_{L^{p,n_0}_{\frac{n_0}{p}}(\Omega)} \leq c || f ||_{L^p_{w_0}(\Omega)} , \quad (2.5)$$

for some positive constant $c = c(n, n_0, p, [w]_{\frac{n_0}{p}}, \text{diam}(\Omega))$. 


3. Elliptic fully nonlinear obstacle problems

In order to measure the oscillation of \( F = F(M,x) \) with respect to the variable \( x \), we define

\[
\beta_F(x,x_0) := \sup_{M \in S(n) \setminus \{0\}} \frac{|F(M) - F(x_0,M)|}{\|M\|},
\]

and set \( \beta(x,x_0) := \beta_F(x,x_0) \) for the sake of simplicity.

We first need the following weighted \( W^{2,p} \) estimate for convex fully nonlinear equations without obstacle. This can be found in [3].

**Lemma 3.1.** Let \( n_0 < p < \infty \), where \( n_0 := n - \nu_0 \) for some \( \nu_0 = \nu_0 \left( \frac{\lambda}{n} \right) > 0 \), and let \( w \in A_{n_0}^\nu \). Suppose that \( \partial\Omega \in C^{1,1} \) and \( f \in L^p_w(\Omega) \). Then there exists a small \( \delta = \delta(n,\lambda,\Lambda,p,w,\partial\Omega) > 0 \) such that if

\[
\sup_{x_0 \in \Omega, 0 < r \leq R_0} \left( \frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} \beta(x,x_0)^n \, dx \right)^{\frac{1}{n}} \leq \delta \tag{3.1}
\]

for some \( R_0 > 0 \), then the problem

\[
\begin{cases}
F(x,D^2u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{3.2}
\]

has a unique solution \( u \in W^{2,p}_w(\Omega) \) with the estimate

\[
\|u\|_{W^{2,p}_w(\Omega)} \leq c\|f\|_{L^p_w(\Omega)},
\tag{3.3}
\]

for some positive constant \( c = c(n,\lambda,\Lambda,p,w,\partial\Omega,\text{diam}(\Omega),R_0) \).

We will use the following comparison principle for fully nonlinear operators, see [8, Theorem 2.10].

**Lemma 3.2.** Suppose that \( U \) is a bounded domain and that \( f \in L^p(U) \), \( 1 < p < \infty \). Let \( u_1, u_2 \in C(\overline{U}) \) be supersolution and subsolution of the equation \( F(x,D^2u) = f \) in \( U \), respectively, with \( u_1 \geq u_2 \) in \( \partial U \). Then we have \( u_1 \geq u_2 \) in \( U \).

We now state the first main result in this paper, the weighted \( W^{2,p} \) estimate for the obstacle problem [12].

**Theorem 3.3** (Main Theorem 1). Let \( n_0 < p < \infty \), where \( n_0 := n - \nu_0 \) for some \( \nu_0 = \nu_0 \left( \frac{\Lambda}{n} \right) > 0 \), and let \( w \in A_{n_0}^\nu \). Suppose that \( \partial\Omega \in C^{1,1} \), \( f \in L^p_w(\Omega) \) and \( \psi \in W^{2,p}_w(\Omega) \). Then there exists a small \( \delta = \delta(n,\lambda,\Lambda,p,w,\partial\Omega) > 0 \) such that if

\[
\sup_{x_0 \in \Omega, 0 < r \leq R_0} \left( \frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} \beta(x,x_0)^n \, dx \right)^{\frac{1}{n}} \leq \delta \tag{3.4}
\]

for some \( R_0 > 0 \), then the fully nonlinear obstacle problem \([12]\) has a unique solution \( u \in W^{2,p}_w(\Omega) \) with the estimate

\[
\|u\|_{W^{2,p}_w(\Omega)} \leq c \left( \|f\|_{L^p_w(\Omega)} + \|\psi\|_{W^{2,p}_w(\Omega)} \right),
\tag{3.5}
\]

for some positive constant \( c = c(n,\lambda,\Lambda,p,w,\partial\Omega,\text{diam}(\Omega),R_0) \).

**Proof.** First, in order to approximate the Heaviside function, we choose a non-decreasing smooth function \( \Phi_\varepsilon \in C^\infty(\mathbb{R}) \), see for instance [23, 25], satisfying

\[
\Phi_\varepsilon(s) \equiv 0 \quad \text{if} \quad s \leq 0; \quad \Phi_\varepsilon(s) \equiv 1 \quad \text{if} \quad s \geq \varepsilon,
\]

where \( \varepsilon > 0 \) is a small positive number.

[1] Reference to the main result.

[2] Reference to the main result.

[3] Reference to the main result.

[4] Reference to the main result.

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[24] Reference to the main result.

[25] Reference to the main result.
and 
\[ 0 \leq \Phi_{\varepsilon}(s) \leq 1, \quad \forall s \in \mathbb{R}. \]
Let \( g(x) := f(x) - F(x, D^2 \psi(x)) \) for \( x \in \Omega \). Since \( f \in L^p_w(\Omega) \), \( \psi \in W^{2,p}_w(\Omega) \) and \( F \) is Lipschitz in \( M \), we have \( g \in L^p_w(\Omega) \) with the estimate
\[
\|g\|_{L^p_w(\Omega)} \leq c \left( \|f\|_{L^p_w(\Omega)} + \|F(\cdot, D^2 \psi)\|_{L^p_w(\Omega)} \right)
\leq c \left( \|f\|_{L^p_w(\Omega)} + \|\psi\|_{W^{2,p}_w(\Omega)} \right).
\]
We write \( g^+ = \max\{g, 0\} \) and \( g^- = \max\{-g, 0\} \), and consider the following problem:
\[
\begin{cases}
F(x, D^2 u_\varepsilon) = g^+ \Phi_{\varepsilon}(u_\varepsilon - \psi) + f - g^+ & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{3.6}
\]
We note that the above problem (3.6) has a unique solution. Indeed, according to Lemma 3.1, it follows that for each \( v_0 \in L^p_w(\Omega) \), there is \( v \in W^{2,p}_w(\Omega) \) satisfying
\[
\begin{cases}
F(x, D^2 v) = g^+ \Phi_{\varepsilon}(v_0 - \psi) + f - g^+ & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Since \( 0 \leq \Phi_{\varepsilon}(\cdot) \leq 1 \), we can deduce from Lemma 3.1 that
\[
\|v\|_{W^{2,p}_w(\Omega)} \leq R,
\]
where \( R \) is independent of \( v_0 \). Defining \( v = Sv_0 \), we see that \( S \) maps the \( R \)-ball in \( L^p_w(\Omega) \) into itself and \( S \) is compact, as \( W^{2,p}_w(\Omega) \) is a compact subset of \( L^p_w(\Omega) \) (see for instance [19]). By Schauder’s fixed point theorem, there is \( u_\varepsilon \) such that \( u_\varepsilon = Su_\varepsilon \), which is the solution to the problem (3.6).

Now, it follows from Lemma 3.3 that
\[
\|u_\varepsilon\|_{W^{2,p}_w(\Omega)} \leq c \left( \|g^+ \Phi_{\varepsilon}(u_\varepsilon - \psi)\|_{L^p_w(\Omega)} + \|f\|_{L^p_w(\Omega)} + \|g^+\|_{L^p_w(\Omega)} \right)
\leq c \left( \|g^+\|_{L^p_w(\Omega)} + \|f\|_{L^p_w(\Omega)} + \|g^+\|_{L^p_w(\Omega)} \right)
\leq c \left( \|f\|_{L^p_w(\Omega)} + \|g\|_{L^p_w(\Omega)} \right)
\leq c \left( \|f\|_{L^p_w(\Omega)} + \|\psi\|_{W^{2,p}_w(\Omega)} \right).
\]
Hence \( \{u_\varepsilon\} \) is uniformly bounded in \( W^{2,p}_w(\Omega) \). Using Lemma 2.5 and Sobolev embedding, we can find a subsequence \( \{u_{\varepsilon_k}\}_{k=1}^\infty \) with \( \varepsilon_k \searrow 0 \), and a function \( u \in W^{2,p}_w(\Omega) \cap C^\alpha(\overline{\Omega}) \) such that \( u_{\varepsilon_k} \) converges to \( u \) weakly in \( W^{2,p}_w(\Omega) \) and strongly in \( C^\alpha(\overline{\Omega}) \) for some \( \alpha > 0 \).

Next, we claim that \( u \) is a solution of the fully nonlinear obstacle problem (1.2). First, we see that \( u = 0 \) on the boundary \( \partial \Omega \) since \( u_{\varepsilon_k} \) uniformly converges to \( u \) and \( u_{\varepsilon_k} = 0 \) on \( \partial \Omega \) for every \( k \). Also, we have from (3.6) that
\[
F(x, D^2 u_{\varepsilon_k}) = g^+ \Phi_{\varepsilon_k}(u_{\varepsilon_k} - \psi) - g^+ + f \leq f \quad \text{in } \Omega,
\]
for every \( k \), and hence \( F(x, D^2 u) \leq f \) in \( \Omega \).

We now prove that \( u \geq \psi \) in \( \Omega \). For each \( m \in \mathbb{N} \), \( \Phi_{\varepsilon_k}(u_{\varepsilon_k} - \psi) \) converges to 0 on the set \( \{ u < \psi - \frac{1}{m} \} \), by the uniform convergence of the \( u_{\varepsilon_k} \). Therefore, \( F(x, D^2 u) = f - g^+ \) on the set \( \{ u < \psi - \frac{1}{m} \} \), for each \( m \in \mathbb{N} \). Since \( \{ u < \psi \} = \bigcup_{m=1}^\infty \{ u < \psi - \frac{1}{m} \} \), we have \( F(x, D^2 u) = f - g^+ \) on the set \( \{ u < \psi \} \). We note that \( u, \psi \in C(\Omega) \), since \( u, \psi \in W^{2,p}(\Omega) \) for some \( p > \frac{n}{n-1} \) by Lemma 2.5. Hence,
V := \{u < \psi\} is an open set. Now suppose that V \neq \emptyset. From the definition of g, we have

\[ F(x, D^2 \psi) = f - g \text{ in } V. \]

Also it is clear that

\[ F(x, D^2 u) = f - g^+ \leq f - g \text{ in } V, \]

and that

\[ u = \psi \text{ on } \partial V. \]

Then we obtain from Lemma 3.2 that u \geq \psi in V, which contradicts the definition of the set V. We thus conclude that V = \emptyset and u \geq \psi in \Omega.

Finally, we prove that F(x, D^2 u) = g on the set \{u > \psi\}. To do this, observe that for each m \in \mathbb{N}, \Phi_\varepsilon^k(u_\varepsilon^k - \psi) converges to 1 almost everywhere on the set \{u > \psi + \frac{1}{m}\}. Therefore, we obtain

\[ F(x, D^2 u) = g^+ + f - g^+ = f \]

on the set \{u > \psi\} = \bigcup_{m=1}^{\infty} \{u > \psi + \frac{1}{m}\}.

Consequently, u \in W^{2,p}_w(\Omega) is a solution to (1.2) with the estimate

\[ \|u\|_{W^{2,p}_w(\Omega)} \leq \liminf_{k \to \infty} \|u_\varepsilon^k\|_{W^{2,p}_w(\Omega)} \leq c \left( \|f\|_{L^p_w(\Omega)} + \|\psi\|_{W^{2,p}_w(\Omega)} \right). \]

4. Elliptic linear obstacle problems

We start with the small bounded mean oscillation (BMO) assumption on the coefficient matrix A for the linear obstacle problem (1.1).

Definition 4.1. We say that the coefficient matrix A is (δ, R)-vanishing if

\[ \sup_{0 < r \leq R} \sup_{y \in \mathbb{R}^n} \left( \frac{1}{B_r(y)} \int_{B_r(y)} |A(x) - \overline{A}_{B_r(y)}|^2 \, dx \right)^{\frac{1}{2}} \leq \delta, \]

where \( \overline{A}_{B_r(y)} = \frac{1}{B_r(y)} \int_{B_r(y)} A(x) \, dx \) is the integral average of A on the ball B_r(y).

We note that one can take R = 1 for simplicity, which is due to the scaling invariance property. On the other hand, \( \delta > 0 \) is invariant under such a scaling. The assumption (4.1) on the coefficient matrix is weaker than the vanishing mean oscillation (VMO) or continuity assumption on the coefficient matrix, see [4, 5] for more details.

We next introduce the weighted \( W^{2,p} \) estimates for linear elliptic equations without obstacle, see [3].
Lemma 4.2. Let $2 < p < \infty$ and let $w \in A_{\frac{1}{p}}$. Suppose that $\partial \Omega \in C^{1,1}$ and $f \in L^p_w(\Omega)$. There exists a small $\delta = \delta(\Lambda, p, n, w, \partial \Omega) > 0$ such that if $A$ is uniformly elliptic and $(\delta, R)$-vanishing, then the problem

$$
\begin{align*}
\begin{cases}
a_{ij}D_{ij}u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
$$

has a unique solution $u \in W^{2,p}_w(\Omega)$ with the estimate

$$
\|u\|_{W^{2,p}_w(\Omega)} \leq c \|f\|_{L^p_w(\Omega)},
$$

for some positive constant $c = c(n, \Lambda, \Lambda, p, w, \partial \Omega, \text{diam}(\Omega))$.

We also need the following maximum principle for linear equations, see for instance [8, Theorem 2.10] and [11].

Lemma 4.3. Suppose that $U$ is a bounded domain and that $A = (a_{ij})$ is uniformly elliptic and $(\delta, R)$-vanishing. If $u$ satisfies

$$
\begin{align*}
\begin{cases}
a_{ij}D_{ij}u &\leq 0 \quad \text{in } U, \\
u &\geq 0 \quad \text{on } \partial U,
\end{cases}
\end{align*}
$$

then $u \geq 0$ in $U$.

Now we state and prove the second main result in this paper, the global weighted $W^{2,p}$ estimate for the linear elliptic obstacle problem [11].

Theorem 4.4 (Main Theorem 2). Let $2 < p < \infty$ and let $w \in A_{\frac{1}{p}}$. Suppose that $\partial \Omega \in C^{1,1}$, $f \in L^p_w(\Omega)$ and $\psi \in W^{2,p}_w(\Omega)$. Then there exists a small $\delta = \delta(n, \Lambda, \Lambda, p, w, \partial \Omega) > 0$ such that if $A$ is uniformly elliptic and $(\delta, R)$-vanishing, then there is a unique solution $u \in W^{2,p}_w(\Omega)$ to the obstacle problem [11] with the estimate

$$
\|u\|_{W^{2,p}_w(\Omega)} \leq c \left( \|f\|_{L^p_w(\Omega)} + \|\psi\|_{W^{2,p}_w(\Omega)} \right),
$$

for some positive constant $c = c(n, \Lambda, \Lambda, p, w, \partial \Omega, \text{diam}(\Omega))$.

Proof. Since $\partial \Omega \in C^{1,1}$, there exists an extension $\overline{\psi}$ of $\psi$ to $\mathbb{R}^n$ with $\overline{\psi} = \psi$ a.e. in $\Omega$, and

$$
\|\overline{\psi}\|_{W^{2,p}_w(\mathbb{R}^n)} \leq c \|\psi\|_{W^{2,p}_w(\Omega)},
$$

for some constant $c$ depending only on $n, p, w, \partial \Omega$ and $\text{diam}(\Omega)$, see [12]. Let $g = f - a_{ij}D_{ij}\overline{\psi}$ in $\mathbb{R}^n$ (we extend $f$ to zero outside $\Omega$). Since $f \in L^p_w(\mathbb{R}^n)$ and $\overline{\psi} \in W^{2,p}_w(\mathbb{R}^n)$, we have $g \in L^p_w(\mathbb{R}^n)$ with the estimate

$$
\|g\|_{L^p_w(\mathbb{R}^n)} \leq c \left( \|f\|_{L^p_w(\mathbb{R}^n)} + \|\overline{\psi}\|_{W^{2,p}_w(\mathbb{R}^n)} \right)
\leq c \left( \|f\|_{L^p_w(\Omega)} + \|\psi\|_{W^{2,p}_w(\Omega)} \right).
$$

Now let $\varphi$ denote a standard mollifier with support in $B_1$, and set $\varphi_\varepsilon(x) := \varepsilon^{-n} \varphi(x/\varepsilon)$. We define the usual regularizations $a_{ij}^\varepsilon := a_{ij} * \varphi_\varepsilon$, $\overline{\psi}^\varepsilon := \overline{\psi} * \varphi_\varepsilon$, $f_\varepsilon := f * \varphi_\varepsilon$, and $g_\varepsilon := f_\varepsilon - a_{ij}^\varepsilon D_{ij}\overline{\psi}^\varepsilon$. We note that for each $\varepsilon > 0$, the matrix $(a_{ij}^\varepsilon) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is uniformly elliptic with the same ellipticity constants.
Furthermore, \( g_\varepsilon \to g \) almost everywhere, as \( \varepsilon \to 0 \), and
\[
\|g_\varepsilon\|_{L^p_\varepsilon(\mathbb{R}^n)} \leq c \left( \|f_\varepsilon\|_{L^p_\varepsilon(\mathbb{R}^n)} + \|\psi_\varepsilon\|_{W^{2,p}_\varepsilon(\mathbb{R}^n)} \right)
\]
\[
\leq c \left( \|f\|_{L^p(\mathbb{R}^n)} + \|\psi\|_{W^{2,p}(\mathbb{R}^n)} \right)
\]
\[
\leq c \left( \|f\|_{L^p_\varepsilon(\Omega)} + \|\psi\|_{W^{2,p}_\varepsilon(\Omega)} \right).
\]

Let \( \Phi_\varepsilon(s) \) be the function in the proof of Theorem 3.3. We then consider the problem:
\[
\begin{cases}
  a_{ij}^\varepsilon D_{ij}u_\varepsilon + g_\varepsilon^+ \Phi_\varepsilon(u_\varepsilon - \psi_\varepsilon) + f_\varepsilon \leq g_\varepsilon^+ \quad \text{in } \Omega, \\
  u_\varepsilon = 0 \quad \text{on } \partial\Omega.
\end{cases}
\]

(4.5)

According to Lemma 4.2 for each \( \nu_0 \in L^p_\varepsilon(\Omega) \) there is \( v \in W^{2,p}_w(\Omega) \) for which
\[
\begin{cases}
  a_{ij}^\varepsilon D_{ij}v = g_\varepsilon^+ \Phi_\varepsilon(v_0 - \psi_\varepsilon) + f_\varepsilon \quad \text{in } \Omega, \\
  v = 0 \quad \text{on } \partial\Omega.
\end{cases}
\]

By the fact that \( 0 \leq \Phi_\varepsilon(\cdot) \leq 1 \) and Lemma 4.2 we find that
\[
\|v\|_{W^{2,p}_w(\Omega)} \leq R,
\]
where \( R \) is independent of \( \nu_0 \). We set \( v = Sv_0 \). Then we see that \( S \) maps the \( R \)-ball in \( L^p_\varepsilon(\Omega) \) into itself and \( S \) is compact. It follows from Schauder’s fixed point theorem that there is the unique \( u_\varepsilon \) such that \( u_\varepsilon = Su_\varepsilon \), which is the solution to the problem (4.3). Lemma 4.2 now yields
\[
\|u_\varepsilon\|_{W^{2,p}_w(\Omega)} \leq c \left( \|g_\varepsilon^+ \Phi_\varepsilon(u_\varepsilon - \psi_\varepsilon)\|_{L^p_\varepsilon(\Omega)} + \|f_\varepsilon\|_{L^p_\varepsilon(\Omega)} + \|g_\varepsilon^+\|_{L^p_\varepsilon(\Omega)} \right)
\]
\[
\leq c \left( \|g_\varepsilon^+\|_{L^p_\varepsilon(\Omega)} + \|f_\varepsilon\|_{L^p_\varepsilon(\Omega)} + \|g_\varepsilon^+\|_{L^p_\varepsilon(\Omega)} \right)
\]
\[
\leq c \left( \|f\|_{L^p_\varepsilon(\mathbb{R}^n)} + \|g_\varepsilon\|_{L^p_\varepsilon(\mathbb{R}^n)} \right)
\]
\[
\leq c \left( \|f\|_{L^p(\mathbb{R}^n)} + \|\psi\|_{W^{2,p}(\mathbb{R}^n)} \right).
\]

Hence \( \{u_\varepsilon\} \) is uniformly bounded in \( W^{2,p}_w(\Omega) \cap W^{1,2}_0(\Omega) \). So we can find a subsequence \( \{u_{\varepsilon_k}\}_{k=1}^{\infty} \) with \( \varepsilon_k \to 0 \), and a function \( u \in W^{2,p}_w(\Omega) \cap W^{1,2}_0(\Omega) \) such that \( u_{\varepsilon_k} \) converges to \( u \) weakly in \( W^{2,p}_w(\Omega) \cap W^{1,2}_0(\Omega) \), and \( u_{\varepsilon_k} \) converges to \( u \) almost everywhere, as \( \varepsilon_k \to 0 \).

We next claim that \( u \) is a solution of the obstacle problem (1.1). Since \( u \in W^{1,2}_0(\Omega) \), \( u = 0 \) on \( \partial\Omega \). It also follows from (4.5) that
\[
a_{ij}^\varepsilon D_{ij}u_{\varepsilon_k} = g_\varepsilon^+ \Phi_{\varepsilon_k}(u_{\varepsilon_k} - \psi_{\varepsilon_k}) - g_\varepsilon^+ + f_{\varepsilon_k} \leq f_{\varepsilon_k} \quad \text{in } \Omega,
\]
for every \( k \). Passing to the limit \( k \to \infty \), we obtain that \( a_{ij}D_{ij}u \leq f \) a.e. in \( \Omega \).

We now show that \( u \geq \psi \) in \( \Omega \). To do this, fix \( k \in \mathbb{N} \), and the note that \( \Phi_{\varepsilon_k}(u_{\varepsilon_k} - \psi_{\varepsilon_k}) = 0 \) on the set \( V_k := \{u_{\varepsilon_k} < \psi_{\varepsilon_k}\} \). Hence, \( a_{ij}^\varepsilon D_{ij}u_{\varepsilon_k} = f_{\varepsilon_k} - g_{\varepsilon_k}^+ \) in \( V_k \). If \( V_k \neq \emptyset \), then it follows from the definition of \( g_{\varepsilon_k} \) that
\[
a_{ij}^\varepsilon D_{ij}u_{\varepsilon_k} = f_{\varepsilon_k} - g_{\varepsilon_k}^+ = f_{\varepsilon_k} - g_{\varepsilon_k} - g_{\varepsilon_k}^- = a_{ij}^\varepsilon D_{ij}\psi_{\varepsilon_k} - g_{\varepsilon_k}^- \leq a_{ij}^\varepsilon D_{ij}\psi_{\varepsilon_k} \quad \text{in } V_k.
\]
Since \( u_{e_k} = \overline{\psi}_{e_k} \) on \( \partial V_k \), we discover that
\[
\begin{aligned}
& a_{ij} D_{ij} (u_{e_k} - \overline{\psi}_{e_k}) \leq 0 \quad \text{in} \quad V_k, \\
& u_{e_k} - \overline{\psi}_{e_k} \geq 0 \quad \text{on} \quad \partial V_k.
\end{aligned}
\]  
(4.6)

Then in light of Lemma 4.3, \( u_{e_k} - \overline{\psi}_{e_k} \geq 0 \) in \( V_k \), which contradicts the definition of the set \( V_k \), and we conclude that \( V_k = \emptyset \) and \( u_{e_k} \geq \overline{\psi}_{e_k} \) in \( \Omega \). But then since \( k \in \mathbb{N} \) is arbitrary, passing to the limit \( k \to \infty \), we discover that \( u \geq \overline{\psi} \) a.e. in \( \Omega \). Therefore, \( u \geq \psi \) a.e. in \( \Omega \), as \( \overline{\psi} = \psi \) a.e. in \( \Omega \).

We next show that \( a_{ij} D_{ij} u = f \) on the set \( \{ u > \psi \} \). Observe that for each \( m \in \mathbb{N} \), \( \Phi_{e_k} (u_{e_k} - \overline{\psi}_{e_k}) \) converges to 1 almost everywhere on the set \( \{ u > \overline{\psi} + \frac{1}{m} \} \), to find
\[
a_{ij} D_{ij} u = g^+ + f - g^+ = f
\]
on the set \( \{ u > \psi \} = \bigcup_{m=1}^{\infty} \{ u > \overline{\psi} + \frac{1}{m} \} \). Consequently, \( u \in W^{2,p}_\infty (\Omega) \) is a solution to (1.2) with the estimate
\[
\| u \|_{W^{2,p}_\infty (\Omega)} \leq \lim \inf_{k \to \infty} \| u_{e_k} \|_{W^{2,p}_\infty (\Omega)} \leq c \left( \| f \|_{L^p(\Omega)} + \| \psi \|_{W^{2,p}_\infty (\Omega)} \right).
\]  
(4.7)

Now it remains to prove the uniqueness. Let \( u_1, u_2 \) be solutions to (1.1) and assume that the open set \( G := \{ u_2 > u_1 \} \) is nonempty. Since \( u_2 > u_1 \geq \psi \) in \( G \), we know that \( a_{ij} D_{ij} u_2 = f \) in \( G \). Therefore, we have
\[
a_{ij} D_{ij} (u_2 - u_1) = f - a_{ij} D_{ij} u_1 \geq 0 \quad \text{in} \quad G
\]
and
\[
u_2 = u_1 \quad \text{on} \quad \partial G.
\]
Then Lemma 4.3 implies that \( u_2 - u_1 \leq 0 \) in \( G \), which is a contradiction, and therefore \( u_1 = u_2 \). This finishes the proof. \( \square \)

5. Morrey regularity results and Hölder continuity of the gradient

The Morrey space \( L^{p,\theta}(\Omega) \) with \( p \in [1, \infty) \) and \( \theta \in [0, n] \) consists of all measurable functions \( g \in L^p(\Omega) \) for which the norm
\[
\| g \|_{L^{p,\theta}(\Omega)} := \left( \sup_{y \in \Omega, r > 0} \frac{1}{r^\theta} \int_{B_r(y) \cap \Omega} |g(x)|^p \, dx \right)^{\frac{1}{p}}
\]
is finite. The Sobolev-Morrey space \( W^{2,p,\theta}(\Omega) \) consists of all functions \( g \in W^{2,p}(\Omega) \) such that the second order derivatives belongs to the Morrey space \( L^{p,\theta}(\Omega) \). A natural norm of this space is defined by
\[
\| g \|_{W^{2,p,\theta}(\Omega)} := \| g \|_{L^p(\Omega)} + \| D^2 g \|_{L^{p,\theta}(\Omega)}.
\]
We note that for \( p \in [1, \infty) \), \( L^{p,0}(\Omega) \cong L^p(\Omega) \) and \( L^{p,n}(\Omega) \cong L^\infty(\Omega) \). Hence, we deal with only the case \( 0 < \theta < n \) in this section.

We now state and prove the main results in this section, the Morrey regularity results for the elliptic obstacle problems (1.2) and (1.1).

**Theorem 5.1.** Let \( 0 < \theta < \infty \), where \( n_0 := n - n_0 \) for some \( n_0 = n_0 (\frac{1}{n}, n) > 0 \), and let \( 0 < \theta < n \). Suppose that \( \partial \Omega \in C^{1,1} \), \( f \in L^{p,\theta}(\Omega) \) and \( \psi \in W^{2,p,\theta}(\Omega) \). Then there exists a small \( \delta = \delta(n, \lambda, \Lambda, p, \theta, \partial \Omega) > 0 \) such that if (3.4) is satisfied for
some $R_0 > 0$, then the fully nonlinear obstacle problem (1.1) has a unique solution $u \in W^{2,p,\theta}(\Omega)$ with the estimate

$$\|u\|_{W^{2,p,\theta}(\Omega)} \leq c \left( \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p,\theta}(\Omega)} \right),$$

for some positive constant $c = c(n, \lambda, p, \theta, \partial\Omega, \text{diam}(\Omega), R_0)$.

**Theorem 5.2.** Let $2 < p < \infty$ and let $0 < \theta < n$. Suppose that $\partial\Omega \in C^{1,1}$, $f \in L^p(\Omega)$ and $\psi \in W^{2,p,\theta}(\Omega)$. There exists a small $\delta = \delta(n, \lambda, p, \theta, \partial\Omega) > 0$ such that if $A$ is uniformly elliptic and $(\delta, R)$-vanishing, then the obstacle problem (1.1) has a unique solution $u \in W^{2,p,\theta}(\Omega)$ and we have the estimate

$$\|u\|_{W^{2,p,\theta}(\Omega)} \leq c \left( \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p,\theta}(\Omega)} \right),$$

for some positive constant $c = c(n, \lambda, p, \theta, \partial\Omega, \text{diam}(\Omega))$.

**Proof of Theorem 5.1 and 5.2.** Throughout the proof, we use the number $m_0$ to denote the number $n_0$ when proving Theorem 5.1 and the number 2 when proving Theorem 5.2 respectively.

We first recall that for a locally integrable function $h : \mathbb{R}^n \to \mathbb{R}$, the Hardy-Littlewood maximal function of $h$ is defined by

$$\mathcal{M}h(x) := \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |h(y)| dy,$$

for $x \in \mathbb{R}^n$. From [28, Proposition 2], we see that if $\sigma \in (0, 1)$, then

$$\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma \in A_1,$$

where $\chi_{B_r(x_0)}$ is the characteristic function of $B_r(x_0)$. Hence, it follows from the fact that $p > m_0$ and the monotonicity of the classes $A_\sigma$ that

$$\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma \in A_1 \subset A_{\frac{m_0}{n_0}}$$

with $\left[\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\right]_{\frac{m_0}{n_0}} \leq c(n, m_0, p, \sigma)$.

We now fix any $\sigma \in (\frac{m_0}{n}, 1)$. Then by Theorem 3.3 and 3.4, we have

$$\int_{B_r(x_0) \cap \Omega} |D^2u|^p \, dx = \int_{\Omega} |D^2u|^p \left(\mathcal{M}\chi_{B_r(x_0)}\right)^\sigma \, dx$$

$$\leq \int_{\Omega} |D^2u|^p \left(\mathcal{M}\chi_{B_r(x_0)}\right)^\sigma \, dx$$

$$\leq c \left( \int_{\Omega} |f|^p \left(\mathcal{M}\chi_{B_r(x_0)}\right)^\sigma \, dx + \int_{\Omega} |D^2\psi|^p \left(\mathcal{M}\chi_{B_r(x_0)}\right)^\sigma \, dx \right),$$

for some positive constant $c$ depending only on $n, \lambda, p, \theta, \partial\Omega$ and $\text{diam}(\Omega)$.

We next use the following set decomposition

$$\Omega = (B_{2r}(x_0) \cap \Omega) \cup \left( \bigcup_{k=1}^{\infty} (B_{2^{k+1}r}(x_0) \setminus B_{2^kr}(x_0)) \cap \Omega \right),$$
to find
\[
\int_{\Omega} |f|^p (M\chi_{B_r(x)})^\sigma \, dx = \int_{B_{2r}(x) \cap \Omega} |f|^p (M\chi_{B_r(x)})^\sigma \, dx \\
+ \sum_{k=1}^{\infty} \int_{(B_{2k+1,r}(x) \setminus B_{2k,r}(x)) \cap \Omega} |f|^p (M\chi_{B_r(x)})^\sigma \, dx.
\] (5.3)

Since \(M\chi_{B_r(x)} \leq 1\), we have the estimate
\[
\int_{B_{2r}(x) \cap \Omega} |f|^p (M\chi_{B_r(x)})^\sigma \, dx \leq \int_{B_{2r}(x) \cap \Omega} |f|^p \, dx \leq r^\theta \|f\|_{L^p,\sigma(\Omega)}^p.
\] (5.4)

We note that for each \(x \in B_{2k+1,r}(x_0) \setminus B_{2k,r}(x_0)\) and for each \(\rho > (2^{k+1} - 1)r\),
\[
0 < \int_{B_\rho(x)} \chi_{B_r(x_0)}(y) \, dy \leq \frac{|B_r(x_0)|}{|B_\rho(x)|} = \left(\frac{r}{\rho}\right)^n.
\]

Since \(2^{k+1} - 1 \geq 2^k - 1 \geq 2^{k-1}\), it follows that
\[
\int_{B_\rho(x)} \chi_{B_r(x_0)}(y) \, dy \leq \left(\frac{r}{2^{k-1}r}\right)^n = \frac{1}{2^n(n-1)},
\]
and hence
\[
(M\chi_{B_r(x_0)}(x))^\sigma = \left(\sup_{\rho > 0} \int_{B_\rho(x)} \chi_{B_r(x_0)}(y) \, dy\right)^\sigma \leq \frac{1}{2^n(n-1)}.
\]

Therefore, we deduce that for each \(k = 1, 2, \ldots\),
\[
\int_{(B_{2k+1,r}(x_0) \setminus B_{2k,r}(x_0)) \cap \Omega} |f|^p (M\chi_{B_r(x)})^\sigma \, dx \\
\leq \frac{1}{2^n(n-1)} \int_{(B_{2k+1,r}(x_0) \setminus B_{2k,r}(x_0)) \cap \Omega} |f|^p \, dx \\
\leq \frac{1}{2^n(n-1)} \int_{B_{2k+1,r}(x_0) \cap \Omega} |f|^p \, dx \\
\leq c(n) \left(\frac{2^{k+1}r}{2^n(n-1)}\right)^\theta \|f\|_{L^p,\sigma(\Omega)}^p = c(n) 2^{(\sigma n + \theta) - (\sigma n - \theta)k} r^\theta \|f\|_{L^p,\sigma(\Omega)}^p.
\] (5.5)

We combine (5.3) and (5.5) with (5.3) to derive
\[
\int_{\Omega} |f|^p (M\chi_{B_r(x)})^\sigma \, dx \leq c(n) r^\theta \left(1 + 2^{(\sigma n + \theta) - (\sigma n - \theta)k} \sum_{k=1}^{\infty} 2^{-(\sigma n - \theta)k}\right) \|f\|_{L^p,\sigma(\Omega)}^p \\
\leq cr^\theta \left(\sum_{k=0}^{\infty} 2^{-(\sigma n - \theta)k}\right) \|f\|_{L^p,\sigma(\Omega)}^p \\
\leq cr^\theta \|f\|_{L^p,\sigma(\Omega)}^p.
\]

Similarly, we find
\[
\int_{\Omega} |D^2\psi|^p (M\chi_{B_r(x)})^\sigma \, dx \leq cr^\theta \|D^2\psi\|_{L^p,\sigma(\Omega)}^p.
\]

Thus
\[
\int_{B_r(x_0) \cap \Omega} |D^2u|^p \, dx \leq cr^\theta \left(\|f\|_{L^p,\sigma(\Omega)}^p + \|D^2\psi\|_{L^p,\sigma(\Omega)}^p\right).
\]
Dividing the both sides by \( r^\theta \) and taking the supremum with respect to \( x_0 \in \Omega \) and \( r > 0 \), we conclude that \( D^2 u \in L^{p,\theta}(\Omega) \) with the desired estimates (5.1) and (5.2). This completes the proof. □

For the fully nonlinear obstacle problem (1.2), we have H"older continuity of the gradient of the solution when \( p > n \), by the Sobolev embedding theorem. However, for the linear obstacle problem (1.1), we cannot obtain such a result directly, when \( 2 < p \leq n \). Nevertheless, the Morrey regularity result (see Theorem 5.2) and the following Sobolev-Morrey embedding lemma allow to prove H"older continuity of the gradient of the solution for appropriate values of \( p \) and \( \theta \).

Lemma 5.3. [9, Lemma 3.III and Lemma 3.IV] Suppose that \( \Omega \) is a bounded domain with \( \partial \Omega \in C^{1,1} \). Let \( v \in W^{1,p,\theta}(\Omega) \). If \( p + \theta > n \), then \( v \in C^{\alpha,0}(\Omega) \) for \( \alpha = 1 - \frac{n-\theta}{p} \) and we have the estimate

\[
\|v\|_{C^{\alpha,0}(\Omega)} \leq c \|v\|_{W^{1,p,\theta}(\Omega)},
\]

where \( c \) is a positive constant depending only on \( n, p, \theta \) and \( \partial \Omega \).

We now state the H"older continuity result of the gradient of the solution to the linear obstacle problem (1.1).

Theorem 5.4. Under the assumptions of Theorem 5.2, let \( u \in W^{2,p,\theta}(\Omega) \) be the solution to the obstacle problem (1.1). If \( p + \theta > n \), then \( Du \in C^{\alpha,0}(\Omega) \) for \( \alpha = 1 - \frac{n-\theta}{p} \) and we have the estimate

\[
\|Du\|_{C^{\alpha,0}(\Omega)} \leq c \left( \|f\|_{L^{p,\theta}(\Omega)} + \|\psi\|_{W^{2,p,\theta}(\Omega)} \right),
\]

for some positive constant \( c = c(n, \Lambda, p, \theta, \partial \Omega, \text{diam}(\Omega)) \).

Proof. The proof follows directly from Theorem 5.2 and Lemma 5.3. □

6. Parabolic obstacle problem

In this section we consider the parabolic obstacle problem (1.3). As in the elliptic case, we first introduce the small BMO assumption on the coefficient matrix \( A(x,t) \).

Definition 6.1. We say that the coefficient matrix \( A = A(x,t) \) is \((\delta,R)\)-vanishing if

\[
\sup_{0 < r \leq R} \sup_{(y,s) \in \mathbb{R}^n \times \mathbb{R}} \left( \int_{Q_r(y,s)} |A(x,t) - \overline{A}_{Q_r(y,s)}|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \delta,
\]

where \( \overline{A}_{Q_r(y,s)} = \int_{Q_r(y,s)} A(x,t) \, dx \, dt \) is the integral average of \( A(x,t) \) on the parabolic cylinder \( Q_r(y,s) \).

We remark that one can take \( R = 1 \) as in the elliptic case, which is due to the scaling invariance property. On the other hand, \( \delta > 0 \) is invariant under such a scaling.

We now provide the definitions of the Muckenhoupt classes and weighted Sobolev space in the parabolic version. For a given \( 1 < s < \infty \), we say that \( w \) is a weight in Muckenhoupt class \( A_s \), or an \( A_s \) weight, if \( w \) is a locally integrable nonnegative function on \( \mathbb{R}^{n+1} \) with

\[
[w]_s := \sup_{Q} \left( \int_{Q} w(x,t) \, dx \, dt \right) \left( \int_{Q} w(x,t)^{-\frac{1}{s-1}} \, dx \, dt \right)^{s-1} < +\infty,
\]

for all \( Q \).
where the supremum is taken over all parabolic cylinders $Q \subset \mathbb{R}^{n+1}$. If $w$ is an $A_s$ weight, we write $w \in A_h$, and $[w]_h$ is called the $A_s$ constant of $w$.

The weighted Lebesgue space $L^p_w(\Omega_T)$, $1 < p < \infty$, $w \in A_h$ with $1 < s < \infty$, consists of all measurable functions $g = g(x,t)$ on $\Omega_T$ such that

$$
\|g\|_{L^p_w(\Omega_T)} := \left( \int_{\Omega_T} |g(x,t)|^p w(x,t) \, dx \, dt \right)^{\frac{1}{p}} < +\infty.
$$

The weighted Sobolev space $W^{2,1}L^p_w(\Omega_T)$, $1 < p < \infty$, $w \in A_h$ with $1 < s < \infty$, is defined by a class of functions $g = g(x,t) \in L^p_w(\Omega_T)$ with distributional derivatives $D^\alpha_t D^\beta_x g(x,t) \in L^p_w(\Omega_T)$ for $0 \leq 2r + |\alpha| \leq 2$. The norm of $g$ in $W^{2,1}L^p_w(\Omega_T)$ is defined by

$$
\|g\|_{W^{2,1}L^p_w(\Omega_T)} := \left( \sum_{\alpha = 0}^{2r + |\alpha|} \sum_{\beta = 0}^{2s + |\beta|} \int_{\Omega_T} |D^\alpha_t D^\beta_x g(x,t)|^p w(x,t) \, dx \, dt \right)^{\frac{1}{p}}.
$$

In addition, let $W^{2,1}_0L^p_w(\Omega_T)$ be the closure in the $W^{2,1}L^p_w(\Omega_T)$ norm of the space

$$
\mathcal{C} = \{ \phi \in C^\infty(\Omega_T) : \phi(x,t) = 0 \text{ for } (x,t) \in \partial_p \Omega \}.
$$

We will utilize the following weighted $W^{2,1}L^p$ estimate for linear parabolic equations without obstacle, see [2].

**Lemma 6.2.** Let $2 < p < \infty$ and let $w = w(x,t) \in A_h$. Suppose that $\partial \Omega \in C^{1,1}$ and $f \in L^p_w(\Omega_T)$. Then there exists a small $\delta = \delta(\Lambda, p, n, w, \partial \Omega, T) > 0$ such that if $A$ is uniformly parabolic and $(\delta, R)$-vanishing, then the following problem

$$
\begin{align*}
\begin{cases}
  u_t - a_{ij} D_{ij} u &= f & \text{in } \Omega_T, \\
  u &= 0 & \text{on } \partial_p \Omega_T,
\end{cases}
\end{align*}
$$

has a unique solution $u \in W^{2,1}L^p_w(\Omega_T)$ with the estimate

$$
\|u\|_{W^{2,1}L^p_w(\Omega_T)} \leq c \|f\|_{L^p_w(\Omega_T)} \tag{6.3}
$$

for some positive constant $c = c(n, \lambda, \Lambda, p, w, \partial \Omega, \text{diam}(\Omega), T)$.

For an open set $U \subset \mathbb{R}^{n+1}$, $C^{2,1}(U)$ ($C^{2,1}(\overline{U})$) is defined by a set of continuous functions in $U$ (in $\overline{U}$) having continuous derivatives $D_x u, D^2_x u, D_t u$ in $U$ (in $\overline{U}$). We also define the parabolic boundary $\partial_p U$ to be the set of all points $(x,t) \in \partial U$ such that for any $r > 0$, the parabolic cylinder $Q_r(x,t)$ contains points not in $U$. We remark that in the special case $U = \Omega_T = \Omega \times (0,T]$, the parabolic boundary $\partial_p U$ of $U$ coincides with $\partial_p \Omega_T = (\partial \Omega \times [0,T]) \cup (\Omega \times \{t = 0\})$.

The following maximum principle for linear parabolic equations can be found in [20] Lemma 2.1.

**Lemma 6.3.** Let $U \subset \Omega_T$ be a bounded domain. Suppose that $A = (a_{ij})$ is uniformly parabolic with $a_{ij} \in C(\mathbb{R}^{n+1})$. If $u \in C^{2,1}(\overline{U})$ satisfies

$$
\begin{align*}
\begin{cases}
  u_t - a_{ij} D_{ij} u &\geq 0 & \text{in } U, \\
  u &\geq 0 & \text{on } \partial_p U,
\end{cases}
\end{align*}
$$

then $u \geq 0$ in $U$.

Let us now state and prove the last main result in this paper regarding the parabolic obstacle problem (1.3).
Theorem 6.4 (Main Theorem 3). Let $2 < p < \infty$ and let $w = w(x, t) \in A_{\infty}$. Suppose that $\partial \Omega \in C^{1,1}$, $f \in L^p_w(\Omega_T)$ and $\psi \in W^{2,1}L^p_w(\Omega_T)$. There exists a small $\delta = \delta(n, \lambda, \Lambda, p, w, \partial \Omega, T) > 0$ such that if $A$ is uniformly parabolic and $(\delta, R)$-vanishing, then the obstacle problem $(L\phi)$ has a solution $u \in W^{2,1}L^p_w(\Omega_T)$ and we have the estimate

$$\|u\|_{W^{2,1}L^p_w(\Omega_T)} \leq c \left( \|f\|_{L^p_w(\Omega_T)} + \|\psi\|_{W^{2,1}L^p_w(\Omega_T)} \right),$$

for some positive constant $c = c(n, \lambda, \Lambda, p, w, \partial \Omega, \text{diam}(\Omega), T)$.

Proof of Theorem 6.4. We first note that since $\partial \Omega \in C^{1,1}$, there exists an extension $\tilde{\psi}$ of $\psi$ to $\mathbb{R}^{n+1}$ with $\tilde{\psi} = \psi$ a.e. in $\Omega_T$, and

$$\left\| \tilde{\psi} \right\|_{W^{2,1}L^p_w(\mathbb{R}^{n+1})} \leq c \left\| \psi \right\|_{W^{2,1}L^p_w(\Omega_T)},$$

for some constant $c = c(n, p, w, \partial \Omega, \text{diam}(\Omega), T)$, see [12]. Let $g = -f + \tilde{\psi}_t - a_{ij}D_{ij}\tilde{\psi}$ in $\mathbb{R}^{n+1}$ (we extend $f$ to zero outside $\Omega_T$). Define $f \in L^p_w(\mathbb{R}^{n+1})$ and $\tilde{\psi} \in W^{2,1}L^p_w(\mathbb{R}^{n+1})$. Then we see that $g \in L^p_w(\mathbb{R}^{n+1})$ with the estimate

$$\|g\|_{L^p_w(\mathbb{R}^{n+1})} \leq c \left( \|f\|_{L^p_w(\mathbb{R}^{n+1})} + \|\tilde{\psi}\|_{W^{2,1}L^p_w(\mathbb{R}^{n+1})} \right),$$

and that

$$\|g\|_{L^p_w(\mathbb{R}^{n+1})} \leq c \left( \|f\|_{L^p_w(\mathbb{R}^{n+1})} + \|\tilde{\psi}\|_{W^{2,1}L^p_w(\mathbb{R}^{n+1})} \right),$$

for some constant $c = c(n, p, w, \partial \Omega, \text{diam}(\Omega), T)$. We now let $\varphi$ denote a standard mollifier with support in $Q_1$, and define $\varphi_c(x, t) := \varepsilon^{-(n+1)}\varphi((x, t)/\varepsilon)$. We then consider the regularizations $a_{ij}^c := a_{ij}^* \varphi_c$, $f_c := f^* \varphi_c$, $\bar{\psi}_c := \varphi_c \varphi_c$, $(\bar{\psi}_c)_t := (\varphi_c)_t$, and $g_c := -f + (\psi_c)_t - a_{ij}^c D_{ij}\varphi_c$. We note that for each $\varepsilon > 0$, the matrix $(a_{ij}^c) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times n}$ is uniformly parabolic with the same constants $\lambda$ and $\Lambda$. Moreover, we see that $g_c \to g$ almost everywhere, as $\varepsilon \to 0$, and that

$$\|g_c\|_{L^p_w(\mathbb{R}^{n+1})} \leq c \left( \|f_c\|_{L^p_w(\mathbb{R}^{n+1})} + \|\bar{\psi}_c\|_{W^{2,1}L^p_w(\mathbb{R}^{n+1})} \right),$$

for some constant $c = c(n, p, w, \partial \Omega, \text{diam}(\Omega), T)$. We next let $\Phi_c(s) := s\Phi_c(s)$ be the function in the proof of Theorem 3.3 and define $\Psi_c(s) := s\Psi_c(s)$ for $s \in \mathbb{R}$. Then the function $\Psi_c \in C^\infty(\mathbb{R})$ is non-decreasing and satisfies

$$\Psi_c(s) \equiv 0 \quad \text{if} \quad s \leq 0; \quad \Psi_c(s) \equiv s \quad \text{if} \quad s \geq \varepsilon,$$

and

$$0 \leq \Psi_c(s) \leq s, \quad \forall s \geq 0.$$

Now let us look at the following problem:

$$\begin{align*}
(u_c)_t - a_{ij}^c D_{ij}u_c &= -\Psi_c(g_c) \Phi_c(u_c - \bar{\psi}_c) + \Psi_c(g_c) + f_c & \text{in} \Omega_T, \\
u_c &= 0 & \text{on} \partial_p \Omega_T.
\end{align*}$$

(6.6)

According to Lemma 6.2, we find that for each $v_0 \in L^p_w(\Omega_T)$, there exists a function $v \in W^{2,1}L^p_w(\Omega_T)$ such that

$$\begin{align*}
v_t - a_{ij} D_{ij}v &= -\Psi_c(g_c) \Phi_c(v_0 - \bar{\psi}_c) + \Psi_c(g_c) + f_c & \text{in} \Omega_T, \\
v &= 0 & \text{on} \partial_p \Omega_T.
\end{align*}$$

(6.6)
Recall that $0 \leq \Phi_\varepsilon(s) \leq 1$ and $0 \leq \Psi_\varepsilon(s) \leq |s|$ for all $s \in \mathbb{R}$, to find from Lemma 6.2 that
\[
\|v\|_{W^{2,1}L^p(\Omega_T)} \leq R,
\]
where $R$ is independent of $v_0$. Let us write $v = Sv_0$. Then observe that $S$ maps the $R$-ball in $L^p_u(\Omega_T)$ into itself and that $S$ is compact. Thus it follows from Schauder’s fixed point theorem that there is a unique $u_\varepsilon$ such that $u_\varepsilon = Su_\varepsilon$, which is the solution to the problem (6.6).

Lemma 6.2 now yields
\[
\begin{align*}
\|u_\varepsilon\|_{W^{2,1}L^p(\Omega_T)} &\leq c \left( \|\Psi_\varepsilon(g_\varepsilon) \Phi_\varepsilon(u_\varepsilon - \overline{\psi}_\varepsilon)\|_{L^p_u(\Omega_T)} + \|\Psi_\varepsilon(g_\varepsilon)\|_{L^p_u(\Omega_T)} + \|f_\varepsilon\|_{L^p_u(\Omega_T)} \right) \\
&\leq c \left( \|g_\varepsilon\|_{L^p_u(\Omega_T)} + \|g_\varepsilon\|_{L^p_u(\Omega_T)} + \|f_\varepsilon\|_{L^p_u(\Omega_T)} \right) \\
&\leq c \left( \|g_\varepsilon\|_{L^p_u(\mathbb{R}^{n+1})} + \|f\|_{L^p_u(\mathbb{R}^{n+1})} \right) \\
&\leq c \left( \|f\|_{L^p_u(\Omega_T)} + \|\psi\|_{W^{2,1}L^p(\Omega_T)} \right).
\end{align*}
\]
Hence $\{u_\varepsilon\}$ is uniformly bounded in $W^{2,1}L^p_u(\Omega_T)$. So we can find a subsequence $\{u_{\varepsilon_k}\}_{k=1}^\infty$ with $\varepsilon_k \searrow 0$, and a function $u \in W^{2,1}L^p_u(\Omega_T)$ such that $u_{\varepsilon_k}$ converges to $u$ weakly in $W^{2,1}L^p_u(\Omega_T)$, and $u_{\varepsilon_k}$ converges to $u$ almost everywhere. Since $f_\varepsilon, g_\varepsilon, \overline{\psi}_\varepsilon$ and $\Psi_\varepsilon$ are smooth functions, Lemma 6.2 implies that $u_\varepsilon \in W^{2,1}L^q(\Omega_T)$ for all $q \in (2, \infty)$, and so $u_\varepsilon \in C^\alpha(\Omega_T)$ for some $\alpha \in (0,1)$. Then, by Schauder’s theorem, $u_\varepsilon, D_\varepsilon u_\varepsilon, D^2_\varepsilon u_\varepsilon$ and $D_\varepsilon u_\varepsilon$ belong to $C^\alpha(\Omega_T)$, and so we conclude that $u_\varepsilon \in C^{2,1}(\Omega_T)$.

We next claim that $u$ is a solution of the obstacle problem (1.3). Observe that $u = 0$ on $\partial_\varepsilon \Omega_T$. We recall (6.6) to discover that
\[
(u_{\varepsilon_k})_t - a^\varepsilon_{ij}D_\varepsilon u_{\varepsilon_k} = \Psi_{\varepsilon_k}(g_{\varepsilon_k}) - (1 - \Phi_{\varepsilon_k}(u_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k})) + f_{\varepsilon_k} \geq f_{\varepsilon_k} \quad \text{in } \Omega.
\]
Passing to the limit $k \to \infty$, we find that $u_t - a^\varepsilon_{ij}D_\varepsilon u \geq f$ a.e. in $\Omega_T$.

We next want to show that $u \geq \psi$ in $\Omega_T$. To do this, fix $k \in \mathbb{N}$. We then observe that $\Phi_{\varepsilon_k}(u_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k}) = 0$ on $V_k := \{u_{\varepsilon_k} < \overline{\psi}_{\varepsilon_k}\}$, and so we discover that $(u_{\varepsilon_k})_t - a^\varepsilon_{ij}D_\varepsilon u_{\varepsilon_k} = \Psi_{\varepsilon_k}(g_{\varepsilon_k}) - f_{\varepsilon_k}$ in $V_k$. If $V_k \neq \emptyset$, then $u_{\varepsilon_k} \geq \overline{\psi}_{\varepsilon_k}$ in $\Omega_T$. On the other hand, if $V_k = \emptyset$, then it follows from the definition of $\Psi_{\varepsilon_k}$ and $g_{\varepsilon_k}$ that
\[
(u_{\varepsilon_k})_t - a^\varepsilon_{ij}D_\varepsilon u_{\varepsilon_k} = \Psi_{\varepsilon_k}(g_{\varepsilon_k}) + f_{\varepsilon_k} \geq g_{\varepsilon_k} + f_{\varepsilon_k} \geq \overline{\psi}_{\varepsilon_k} + f_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k} = \overline{\psi}_{\varepsilon_k} + f_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k} + \overline{\psi}_{\varepsilon_k} \geq \overline{\psi}_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k} - \varepsilon_k \in V_k.
\]
Define $\overline{u}_{\varepsilon_k}(x,t) := u_{\varepsilon_k}(x,t) + \varepsilon_k t$. It is a straightforward to check
\[
\left\{
\begin{array}{lr}
(\overline{u}_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k})_t - a^\varepsilon_{ij}D_\varepsilon(\overline{u}_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k}) \geq 0 & \quad \text{in } V_k, \\
\overline{u}_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k} \geq 0 & \quad \text{on } \partial_\varepsilon V_k,
\end{array}
\right.
\]
where we have used the fact that $u_{\varepsilon_k} = \overline{\psi}_{\varepsilon_k}$ on $\partial_\varepsilon V_k$. Then from Lemma 6.3 we have
\[
\overline{u}_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k} \geq 0 \quad \text{in } V_k,
\]
and thus

\[ u_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k} \geq -\varepsilon_k T \geq -\varepsilon_k T \quad \text{in} \quad V_k. \]

Recalling the definition of \( V_k \), we see that \( u_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k} \geq -\varepsilon_k T \) in \( \Omega_T \). Passing to the limit \( k \to \infty \), we discover that \( u - \overline{\psi} \geq 0 \) a.e. in \( \Omega_T \). Therefore, we conclude that \( u - \psi \geq 0 \) a.e. in \( \Omega_T \).

Finally, we show that \( u_t - a_{ij} D_{ij} u = f \) on the set \( \{ u > \psi \} \). To prove this, we find that for each \( m \in \mathbb{N} \), \( \Phi_{\varepsilon_k}(u_{\varepsilon_k} - \overline{\psi}_{\varepsilon_k}) \) converges to 1, and \( \Phi_{\varepsilon_k}(g_{\varepsilon_k}) \) converges to \( g^+ \) almost everywhere on the set \( \{ u > \psi + \frac{1}{m} \} \). Then

\[ u_t - a_{ij} D_{ij} u = -g^+ + g^+ + f = f \quad \text{on} \quad \{ u > \psi \} = \bigcup_{m=1}^{\infty} \{ u > \psi + \frac{1}{m} \}. \]

As a consequence, we conclude that the problem \([12]\) has a solution \( u \in W^{2,1} L^p_w(\Omega_T) \) and we have the desired estimate

\[ ||u||_{W^{2,1} L^p_w(\Omega_T)} \leq \liminf_{k \to \infty} ||u_{\varepsilon_k}||_{W^{2,1} L^p_w(\Omega_T)} \leq c \left( ||f||_{L^p_w(\Omega_T)} + ||\psi||_{W^{2,1} L^p_w(\Omega_T)} \right). \]

\[ \square \]

**Remark 6.5.** We remark that the uniqueness of a solution to the parabolic obstacle problem \([13]\) is not evident in general. However, when \( D_k a_{ij} \) exist and are bounded, one can obtain the uniqueness of a solution by coerciveness, see for instance \([10]\).

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