Contextual Bandits with Random Projection

Xiaotian Yu

Department of Computer Science and Engineering
The Chinese University of Hong Kong
Shatin, N.T., Hong Kong

XTYU@CSE.CUHK.EDU.HK

Abstract

Contextual bandits with linear payoffs, which are also known as linear bandits, provide a powerful alternative for solving practical problems of sequential decisions, e.g., online advertisements. In the era of big data, contextual data usually tend to be high-dimensional, which leads to new challenges for traditional linear bandits mostly designed for the setting of low-dimensional contextual data. Due to the curse of dimensionality, there are two challenges in most of the current bandit algorithms: the first is high time-complexity; and the second is extreme large upper regret bounds with high-dimensional data. In this paper, in order to attack the above two challenges effectively, we develop an algorithm of Contextual Bandits via RAndom Projection (CBRAP) in the setting of linear payoffs, which works especially for high-dimensional contextual data. The proposed CBRAP algorithm is time-efficient and flexible, because it enables players to choose an arm in a low-dimensional space, and relaxes the sparsity assumption of constant number of non-zero components in previous work. Besides, we provide a linear upper regret bound for the proposed algorithm, which is associated with reduced dimensions.

1. Introduction

The Multi-Armed Bandit (MAB) problem was proposed and investigated by Robbins in 1952, which has attracted great interests from numerous researchers in operation research and computer science Robbins (1952); Auer et al. (2002); Bubeck and Cesa-Bianchi (2012). The fundamental issue in the MAB problem and its variants focuses on the exploration-exploitation trade-off, which refers to an algorithm trying to maximize cumulative rewards in sequential decisions but the algorithm has only limited knowledge about the mechanism of generating the rewards Auer (2002).

As a natural and important variant of the basic MAB problem, contextual bandits with linear payoffs, which are also known as linear bandits, are sequential decision-making problems with side information Wang et al. (2005); Dani et al. (2008); Abbasi-Yadkori et al. (2011); Chu et al. (2011). Specifically, given feature information of arm space for each of \( T \) rounds, a learner is required to choose one of \( K \) arms. Linear bandits contain a basic assumption of linearly mapping from the arm space to the reward space Filippi et al. (2010), which should be the most common case in reality.

Recently, contextual bandits with linear payoffs have been successfully applied into many practical applications. In Tang et al. (2013), the demonstration of advertisements was based on users’ input information on web pages. The authors formulated the problem of automatic layout selection in online advertisements as a contextual bandit problem. These personalized
advertisements are expected to improve click-through rates of web links. For these models of sequential decisions, recommendation algorithms always receive additional contextual information from users, which could be greatly useful for the online sequential decisions.

In the big data era, it is pretty common to encounter high-dimensional and/or sparse contextual information. In this case, traditional bandit algorithms, which are mostly designed in the setting of low-dimensional data, are facing new challenges in applications. Due to the curse of dimensionality, there are two challenges in most the of current bandit algorithms. The first is high time-complexity; and the second is extremely large upper regret bounds with high-dimensional data. Specifically, traditional contextual bandits (e.g., LinUCB in Chu et al. (2011)) contain inverse operations in the original contextual space, which will be time-consuming for computations. Besides, the regret bounds of linear bandits in Chu et al. (2011); Abbasi-Yadkori et al. (2011) are related to the original dimension of contextual data, which can lead to increasing regret bounds with the curse of dimensionality. This will be even worse when the original dimension of contextual data is larger than the total sequential rounds of playing bandits.

There have been some efforts on context bandits with linear payoffs in high-dimensional and/or sparse contextual data Deshpande and Montanari (2012); Carpentier et al. (2012). The corresponding bandit algorithms are named BallExp in Deshpande and Montanari (2012), and SLUCB in Carpentier et al. (2012).

However, in SLUCB, the authors assumed that the contextual data contain $S$ non-zero components, which may not be flexible in applications, especially for cases with high-dimensional dense data. In BallExp, the upper and lower regret bounds are relatively loose, and are still closely related to the original dimension of contextual data. Besides, both SLUCB and BallExp adopted the technique of ball exploration in the high-dimensional space, which will be time-consuming in applications. For rigorous analysis of time complexity for these two algorithms, interested readers can refer to Dani et al. (2008).

Random projection is a powerful and popular technique to deal with high-dimensional data Fern and Brodley (2003); Zhang et al. (2016), which maps high-dimensional data onto a low-dimensional space. Note that random projection does not contain the assumption of sparsity in high-dimensional data. The most common case in random projection is to construct a Gaussian random matrix, where each element is an i.i.d. sample following a standard normal distribution. It has been proved to preserve the Euclidean distance within an error ball Dasgupta and Gupta (1999). Besides, the error bounds for inner products in random projection have been investigated Kabán (2015), which will be an effective tool for analysis of upper regret bounds in contextual bandits.

In this paper, to tackle the aforementioned two challenges effectively, we propose an algorithm of Contextual Bandits via RAndom Projection (CBRAP) in the setting of linear payoffs, which works especially for high-dimensional data. Note that, for simplicity in the work, we assume that contextual bandits have linear payoff functions. But the framework of our bandit algorithm can be easily generalized to the case of relaxing the linear assumption. Specifically, our proposed algorithm adopts random projection to map the high-dimensional contextual information onto a low-dimension space, where we should design a random matrix. The proposed CBRAP algorithm is time-efficient and flexible, because it enables players to choose an arm in a low-dimensional space, and relaxes the sparsity assumption of constant number of non-zero components in previous work. Besides, we prove an upper regret bound
for the proposed algorithm, and show the bound to be better than the traditional ones with appropriate reduced dimensions. By comparing with three benchmark algorithms (i.e., LinUCB, BallExp and SLUCB), we demonstrate improved performance on cumulative payoffs of the CBRAP algorithm during its sequential decisions on both synthetic and real-world datasets, as well as its superior time-efficiency.

In summary, we make the following contributions.

- For contextual bandits in the setting of linear payoffs, we develop an efficient and practical algorithm named CBRAP by taking advantage of random projection.
- We derive an upper regret bound for the proposed CBRAP algorithm, which guarantee the worst case is associated with the reduced dimensions. Besides, our algorithm is more flexible and time-efficient than BallExp and SLUCB in high-dimensional settings.
- We evaluate the CBRAP algorithm via a series of experiments with synthetic and real-world datasets. Compared with the three benchmarks, we demonstrate the proposed algorithm’s improved performance of cumulative payoffs during sequential decisions, as well as its time-efficiency.

2. Preliminary and Related Work

In this section, we first introduce notions and the definition of sub-Gaussian of a random variable, which will be used in this paper. Then, we present the process of contextual bandits with linear payoffs, as well as the metric of bandits. Finally, we provide a brief survey on random projection.

2.1 Notations and Definition of Sub-Gaussian

The total sequential rounds of playing bandits is $T$. For each round $t \in [T]$ with $[T] = \{1, 2, \cdots, T\}$, a learner receives contextual information from the set of $X \in \mathbb{R}^n$, where $n$ can be an extremely large integer representing a high-dimensional space. In this work, high-dimensional contextual data precisely mean that $T \leq n$ or even $T \ll n$, which is the case mentioned in Carpentier et al. (2012). Let $K \in \mathbb{N}_+$ be the number of arms and $\pi_{t,y} \in [0,1]$ the reward of arm $y$ on round $t$ with $y \in [K]$ and $[K] = \{1, 2, \cdots, K\}$. We adopt $\| \cdot \|_2$ to denote the $\ell^2$ norm of a vector $x \in \mathbb{R}^n$, and $I_{m \times m}$ to denote the identity matrix with dimensions of $m \times m$. For a positive definite matrix $A \in \mathbb{R}^{m \times m}$, the weighted norm of vector $x$ is defined as $\|x\|_A = \sqrt{x^T A x}$. The inner product is represented as $\langle \cdot, \cdot \rangle$, and the weighted inner product is $x A^T y = \langle x, y \rangle_A$.

Mathematically, we give the following definition on the sub-Gaussian of a random variable.

**Definition 1** (Buldygin and Kozachenko (1980)). A random variable $\xi$ is sub-Gaussian if there exists an $R \geq 0$ such that

$$
E[\exp(\lambda \xi)] \leq \exp\left(\frac{\lambda^2 R^2}{2}\right),
$$

where $\lambda \in \mathbb{R}$, $E[\cdot]$ is the expectation of a random variable and $\exp(\cdot)$ denotes the exponential operation.
Given a set $\mathcal{F}$, $\xi$ is conditionally $R$-sub-Gaussian if there exists $R \geq 0$, we have $E[\exp(\lambda \xi) | \mathcal{F}] \leq \exp(\frac{\lambda^2 R^2}{2})$, $\forall \lambda \in \mathbb{R}$.

2.2 Contextual Bandits with Linear Payoffs

As shown in Auer (2002); Chu et al. (2011), an algorithm (denoted by $A$) for contextual bandits with linear payoffs usually contains the following three steps at round $t$:

1. contextual information $x_{t,y} \in \mathcal{X}$ for all $y \in [K]$ is revealed to the bandit algorithm $A$;
2. the bandit algorithm $A$ chooses an arm $a_t \in [K]$, which follows an underlying distribution $\Pi(x_{t,a}, \theta^*)$ with $\theta^* \in \mathbb{R}^n$ being the unknown true parameter vector; and
3. a stochastic payoff $\pi_{t,a_t} \in [0,1]$ is revealed to the bandit algorithm $A$.

In the above stochastic setting of step 3, for the chosen arm $a_t$ at round $t$, we usually assume that there is an underlying distribution $\Pi(x_{t,a_t}, \theta^*)$ with the first moment information being $\langle x_{t,a_t}, \theta^* \rangle$, so that $\pi_{t,a_t}$ is a sample from $\Pi(x_{t,a_t}, \theta^*)$. Thus, we have

$$\pi_{t,a_t} = x_{t,a_t}^T \theta^* + \eta_t,$$

where $\eta_t$ is a random noise satisfying the assumption of conditionally $R$-sub-Gaussian. That is, $\forall \lambda \in \mathbb{R}$, we have

$$E[\exp(\lambda \eta_t) | \mathcal{F}_t] \leq \exp\left(\frac{\lambda^2 R^2}{2}\right),$$

where $\mathcal{F}_t$ is the $\sigma$-algebra of $\sigma(\{x_{i,a_i}\}_{i \in [t]}, \{\eta_i\}_{i \in [t-1]})$ and $R \geq 0$. Eq. (3) implies that $E[\eta_t | \mathcal{F}_t] = 0$.

A popular measure in demonstrating the performance of an algorithm for solving MAB problems is regret, which is defined as the difference between the expected payoff of the optimal decision in hindsight and that of the algorithm. Mathematically, the regret of the algorithm $A$ is defined as

$$Regret(T) \triangleq E\left[\sum_{t=1}^T \max_{y \in [K]} x_{t,y}^T \theta^* - \sum_{t=1}^T \pi_{t,a_t}\right].$$

2.3 Random Projection

One common technique for dimensionality reduction is to perform linear random projection Baraniuk et al. (2010); Fodor (2002). In this paper, we consider projecting the contextual data of $\mathcal{X} \in \mathbb{R}^n$ onto a low-dimensional space of $\mathcal{Z} \in \mathbb{R}^m$. Without loss of generality, we denote the random projection matrix by $M \in \mathbb{R}^{m \times n}$. Then, we have

$$z = Mx,$$

where $z \in \mathcal{Z}$ and $x \in \mathcal{X}$.

In Blum (2006), $M$ is constructed as a random matrix where each element follows a normal distribution of $\mathcal{N} \sim (0, \hat{\sigma}^2)$. By setting $\hat{\sigma}^2 = 1/m$ in the next section, we name our algorithm of CBRAP with Standard Gaussian (SG) matrix (abbreviated as CBRAP.SG).

In Achlioptas (2003), the authors proposed new methods for constructing sparse random sign matrix for dimensionality reduction. In the ensuing section, we name our algorithm of CBRAP with Random Sign (RS) matrix (abbreviated as CBRAP.RS).
In addition to the above work, there have been other ways of constructing a matrix for random projection Li et al. (2006); Ailon and Chazelle (2006); Clarkson and Woodruff (2013); Lu et al. (2013). These investigations consider how to speed up the dimensionality reduction, or how to conduct the random projection with the assumption of low-rank matrix. In this paper, since our focus is to conduct the dimensionality reduction of contextual bandits in a general way, we only consider the construction of random matrix in Blum (2006); Achlioptas (2003).

2.4 Related Work

Contextual bandits are important variants of traditional MAB problems and match many real applications Langford and Zhang (2008); Li et al. (2010); Tang et al. (2013); Wang et al. (2005). Contextual bandits with linear payoffs have been intensively investigated in previous work Abbasi-Yadkori et al. (2012); Abe et al. (2003); Abe and Long (1999); Chu et al. (2011); Kaelbling (1994). As shown in Chu et al. (2011), the traditional upper regret bound for the LinUCB algorithm is

\[
\text{Regret}(T) \leq O(\sqrt{Tn \ln^3(KT \ln(T)/\delta)}), \tag{6}
\]

where \(\delta \in (0, 1)\) is a confidence parameter. We know that the dimension of contextual information of \(n\) in Eq. (6) will increase when the dimension of context space increases. Roughly, we have the upper regret bound as \(R(T) \leq O(T)\) when \(n = T\). This will be even worse when the dimension of context data becomes larger, especially for \(n \ll T\).

In Abbasi-Yadkori et al. (2012), Abbasi-Yadkori et al. studied a sparse variant of stochastic linear bandits. For high-dimensional bandits, Carpentier and Munos Carpentier et al. (2012) attacked high-dimensional stochastic linear bandits with the sparsity assumption of \(S\) non-zero component, where the algorithm is named SLUCB. The upper regret bound in Carpentier et al. (2012) is \(O(S\sqrt{T})\)). In real applications, the sparsity assumption may be unreasonable, especially for high-dimensional dense data.

In Deshpande and Montanari (2012), the authors proposed an algorithm named BallExp for high-dimensional linear bandits, where the regret bound is relatively loose, and is directly related to the dimension of data.

Recently, by adopting additional assumptions of margin and compatibility conditions in Bastani and Bavati (2015), the authors investigated high-dimensional covariates in online decision-making.

From prior work, it is urgent and important to develop a flexible and practical algorithm for contextual bandits with high-dimensional data, where we do not have additional assumptions (e.g., sparsity or the margin condition). This motivates our proposed CBRAP algorithm in the next section.

3. The CBRAP Algorithm

In this section, we firstly present the overview of CBRAP, and then provide theoretical analyses of a practical upper regret bound and time complexity for the algorithm.
Algorithm 1 CBRAP

1: input: \( m, T, \beta \in \mathbb{R}_+ \) and \( \alpha \in \mathbb{R}_+ \)
2: for \( p = 1, 2, \cdots, m \) do
3:   for \( q = 1, 2, \cdots, n \) do
4:     generate a random value \( b_{pq} \) based on SG or RS
5:     \( M(p, q) \leftarrow b_{pq} \)
6:   end for
7: end for
8: \( A_0 \leftarrow \mathbf{I}_{m \times m} \)
9: \( b_0 \leftarrow 0_m \)
10: for \( t = 1, 2, \cdots, T \) do
11:   observe context \( x_{t,y} \in \mathbb{R}^n \) for all \( y \in [K] \)
12:   \( z_{t,y} \leftarrow Mx_{t,y} \) for all \( y \in [K] \)
13:   if \( t == 1 \) then
14:     \( A_t \leftarrow A_{t-1} \)
15:     \( b_t \leftarrow b_{t-1} \)
16:   else
17:     \( A_t \leftarrow A_{t-1} + z_{t-1,a_{t-1}}z_{t-1,a_{t-1}}^T \)
18:     \( b_t \leftarrow b_{t-1} + \pi_{t-1,a_{t-1}}z_{t-1,a_{t-1}} \)
19: end if
20: \( \theta^*_z \leftarrow A_t^{-1}b_t \)
21: for \( y \in [K] \) do
22:   \( v_{t,y} \leftarrow \beta \|z_{t,y}\| \)
23:   \( \tilde{r}_{t,y} \leftarrow \langle \theta^*_z, z_{t,y} \rangle \)
24:   \( \text{ucb}_{t,y} \leftarrow \tilde{r}_{t,y} + v_{t,y} \)
25: end for
26: choose the arm \( a_t \leftarrow \arg \max_{y \in [K]} \text{ucb}_{t,y} \) (break ties arbitrarily)
27: observe the reward \( \pi_{t,a_t} \)
28: end for

3.1 Overview of CBRAP

Our proposed bandit algorithm is shown in Algorithm 1, which is named CBRAP. As depicted in Algorithm 1, the basic idea of CBRAP algorithm is to project the high-dimensional data onto a low-dimensional space, and maintains a confidence set of the unknown optimal parameter \( \theta^*_z \in \mathbb{R}^m \) in a low-dimensional space. \( \theta^*_z \) is corresponding to the original true parameter \( \theta^* \) in \( n \)-dimensional space.

Our main contribution in the CBRAP algorithm is two-fold. First, we construct a random matrix for contextual bandits from Step 2 to Step 7 in Algorithm 1. Note that the designed random matrix is flexible, and can be revised based on users’ needs. Here we just consider the random matrix in Blum (2006); Achlioptas (2003). Second, via the designed random matrix \( M \), we conduct dimensionality reduction for the contextual information in Algorithm 1.
4. Theoretical Analysis

For linear stochastic bandits, we consider the model as \( \pi_t = (x_t, \theta_*) + \eta_t \), where \( x_t \in \mathbb{R}^n \), \( \theta_* \in \mathbb{R}^n \) and \( \eta_t \) is conditionally R-sub-Gaussian with respect to \( \sigma \)-algebra as \( \mathcal{F}_t = \sigma(\{x_i\}_{i \in [t]}, \{\eta_i\}_{i \in [t-1]}) \).

We assume that there exists an approximated feature mapping \( z(x) \in \mathbb{R}^m \) for each arm \( x \in \mathbb{R}^n \) with \( 1 \leq m \leq n \) and a mapping \( \zeta(\theta_*) \in \mathbb{R}^l \) for the underlying parameter \( \theta_* \) such that for all \( x \in \bigcup_{t=1}^T \mathcal{D}_t \), \( |z(x) - x^T \theta_*| \leq \epsilon \), where \( \mathcal{D}_t \) is the decision set at time \( t \). Here the error \( \epsilon > 0 \) occurs due to dimension reduction.

Let \( \hat{\mu}_t \) be the \( \ell_2 \)-regularized least-squares estimate of \( \zeta(\theta_*) \) with regularization parameter \( \lambda > 0 \):

\[
\hat{\mu}_t = V_t^{-1}Z_t^T Y_t, \tag{7}
\]

where \( Z_t \in \mathbb{R}^{t \times m} \) is the matrix whose rows are \( z(x_1)^T, \ldots, z(x_t)^T \), \( Y_t = (\pi_1, \ldots, \pi_t) \) and \( V_t = Z_t^T Z_t + \lambda I_m \). Then we have the following lemma showing that with high probability, \( \zeta(\theta_*) \) lies in an ellipsoid centered at \( \hat{\mu}_t \).

**Lemma 1.** Assume that for all \( x \in \mathcal{D}_t \), \( \|z(x)\|_2 \leq L \), \( \|\zeta(\theta_*)\|_2 \leq S \) and \( |x^T \theta_*| \leq B \) and \( z(x)^T \zeta(\theta_*) \) is a \((\gamma, \epsilon)\)-approximation of \( x^T \theta_* \) on \( \mathcal{D} \times \theta_* \). Let \( \beta_t(\delta) = R \sqrt{m \log((1 + tL^2)/\delta)} + \sqrt{\lambda S + \epsilon \sqrt{t}} \). Then, for any \( \delta > 0 \), with probability at least \( 1 - (1 - \delta)^2 \), \( \zeta(\theta_*) \) lies in the set \( C_t \), where \( C_t \) is defined as follows:

\[
C_t = \{ \mu \in \mathbb{R}^m : \|\mu - \hat{\mu}_t\|_{V_t} \leq \beta_t(\delta(t)) \}. \tag{8}
\]

**Proof.** We have

\[
\|\hat{\mu}_t - \zeta(\theta_*)\|_{V_t} = \|V_t^{-\frac{1}{2}}V_t^{-1}Z_t^T Y_t - V_t^{-\frac{1}{2}}V_t^{-1}V_t \zeta(\theta_*)\|_2 \\
= \|Z_t^T (Y_t - Z_t \zeta(\theta_*)) - \lambda \zeta(\theta_*)\|_{V_t^{-1}} \\
\leq \|Z_t^T (Y_t - X_t^T \theta_*)\|_{V_t^{-1}} + \|\lambda \zeta(\theta_*)\|_{V_t^{-1}} \\
+ \|Z_t^T (X_t^T \theta_* - Z_t \zeta(\theta_*))\|_{V_t^{-1}}, \tag{9}
\]

where \( X_t = (x_1, \ldots, x_t) \).

The first term in Eq. (9) could be bounded by \( R \sqrt{m \log((1 + tL^2/\lambda m))} + 2 \log(1/\delta) \) with probability \( 1 - \delta \) via directly applying Theorem 2 in Abbasi-Yadkori et al. (2011). The second term can be easily bounded by \( \sqrt{\lambda S} \). For the third term, we can derive that eigenvalues of \( Z_t V_t^{-1} Z_t^T \) are smaller than 1 by conducting Singular-Value Decomposition (SVD) and then we have

\[
\|Z_t^T (X_t^T \theta_* - Z_t \zeta(\theta_*))\|_{V_t^{-1}} \\
\leq \|X_t^T \theta_* - Z_t \zeta(\theta_*)\|_2 \leq \epsilon \sqrt{t}, \tag{10}
\]

which completes proof by combining the bounds on the first two terms. \( \square \)
Based on Lemma 1, we now give a regret upper bound of CBRAP. In CBRAP, we construct a fixed random matrix $M \in \mathbb{R}^{m \times n}$, $z(x) = Mx$.

**Theorem 1.** Assume the same conditions as shown in Lemma 1. Without loss of generality, let $b_{m,t} = R\sqrt{\log \left((1 + tL^2/\lambda m)\right)} + 2\log(1/\delta) + \sqrt{S} + B$ and $a_{m,t} = \sqrt{2\log \left(1 + \frac{L^2}{\lambda m}\right)}$. For any $\delta > 0$, with probability at least $(1 - 2T\exp(-m\epsilon^2/8))(1 - \delta)$, the regret upper bound of CBRAP is

$$R_T \leq 2a_{m,T}b_{m,T}m\sqrt{T} + 2(a_{m,T}\sqrt{m} + 1)TLS\epsilon_1.$$  

**(Proof.** The proof follows Dani et al. (2008); Abbasi-Yadkori et al. (2011). Let $E$ denote the event that for all $x \in D$, $|z(x)\top \zeta(\theta_*) - x\top \theta_*| \leq \epsilon$. We know that $E$ holds with probability at least $1 - \gamma$. When $E$ holds and $\zeta(\theta_*)$ lies in the ellipsoid $C_t$, The instantaneous regret

$$r_t = x_{t,i}\top \theta_* - x_{t,i}\top \theta_*$$

$$\leq z(x_{t,i})\top \zeta(\theta_*) - z(x_{t,i})\top \zeta(\theta_*) + 2\epsilon$$

$$\leq z(x_{t,i})\top \mu_t - z(x_{t,i})\top \zeta(\theta_*) + 2\epsilon$$

$$\leq \|z(x_{t,i})\|_{V_{t-1}^{-1}} (\|\mu_t - \mu_t + \zeta(\theta_*) - \mu_t\|_{V_{t-1}}) + 2\epsilon$$

$$\leq 2\beta_b(t - 1)\|z(x_{t,i})\|_{V_{t-1}^{-1}} + 2\epsilon.$$  

(12)

Combined with the fact that $r_t \leq 2B$,

$$r_t \leq 2(\beta_b(t - 1) + B) \min\{\|z(x_{t,i})\|_{V_{t-1}^{-1}}, 1\} + 2\epsilon.$$  

(13)

Following Lemma 11 in Abbasi-Yadkori et al. (2011), we know that

$$\sum_{t=1}^{T} \min\{\|z(x_{t,i})\|_{V_{t-1}^{-1}}, 1\} \leq 2m \log \left(1 + \frac{TL^2}{\lambda m}\right)$$  

(14)

Therefore, the cumulative regret could be bounded by

$$R_T = \sum_{t=1}^{T} r_t$$

$$\leq \sum_{t=1}^{T} 2(\beta_b(t - 1) + B) \min\{\|z(x_{t,i})\|_{V_{t-1}^{-1}}, 1\} + 2T\epsilon$$

$$\leq 2(\beta_b(T) + B) \sqrt{T \sum_{t=1}^{T} \min\{\|z(x_{t,i})\|_{V_{t-1}^{-1}}, 1\}} + 2T\epsilon$$

$$\leq 2(b_{m,T}\sqrt{m} + \epsilon\sqrt{T})a_{m,T}\sqrt{mT} + 2T\epsilon.$$  

(15)

Based on Kabán (2015), for $x \in \mathbb{R}^n$, we have $\Pr\{|x\top \theta_* - x\top M\top \theta_*| > \epsilon_1\|x\|\|\theta_*\|\} < 2\exp\left(-\frac{m\epsilon^2}{8}\right)$. By taking $\zeta(\theta_*) = M\theta_*$, which is the operation in CBRAP, we directly derive the result.

When $m = d$, we can set $\zeta(\theta_*) = M^{-1}\theta_*$ and thus have $\epsilon_1 = 0$, which leads to recovering the regret of linear stochastic bandits, which is $O(\sqrt{T})$.  

$\square$
5. Discussions

This is an errata for the theoretical results in CBRAP: Contextual Bandits with RAndom Projection in AAAI 2017.

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