On the mass relation of a meson nonet

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Abstract

It is pointed out that the omission of the effects of the transition between quarkonia or the assumption that the transition between quarkonia is flavor-independent would result in the inconsistent results for the pseudoscalar meson nonet. It is emphasized that the mass relation of the non-ideal mixing meson nonets should incorporate the effects of the flavor-dependent transition between quarkonia. The new mass relations of a meson nonet are presented.

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I. Introduction

In the framework of the quark model, a $qar{q}$ meson nonet contains two isoscalar states $\eta_8$ and $\eta_1$. In general, these two isoscalar states can mix, which results in two physical states $\eta$ and $\eta'$.

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix}
= U
\begin{pmatrix}
\eta_8 \\
\eta_1
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\eta_8 \\
\eta_1
\end{pmatrix}.
\]

(1)

In the $\eta_8 = (u\bar{d} + d\bar{s} - 2s\bar{s})/\sqrt{6}$ and $\eta_1 = (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$ basis, the mass-squared matrix $M^2$ describing the mixing of $\eta_8 - \eta_1$ can be written as \[\text{(2)}\]

\[
M^2 = \begin{pmatrix}
M^2_8 & M^2_{8\bar{s}} \\
M^2_{8\bar{s}} & M^2_1
\end{pmatrix},
\]

which satisfies

\[
UM^2U^{-1} = \begin{pmatrix}
m^2_8 & 0 \\
0 & m^2_{\eta'}
\end{pmatrix},
\]

(3)

From Eqs. (2) and (3), one can obtain

\[
M^2_8 + M^2_1 = m^2_8 + m^2_{\eta'}.
\]

(4)

Some authors believe that $M^2_8 = m^2_8$ and $M^2_1 = m^2_1$ [2, 3], where $m_8$ and $m_1$ are the masses of the bare (before mixed) states $\eta_8$ and $\eta_1$, respectively. Then based on the original Gell-Mann-Okubo mass formula [4]

\[
m^2_8 = \frac{1}{3}(4m^2_K - m^2_{\pi^0}), \quad m^2_1 = \frac{1}{3}(2m^2_K + m^2_{\pi^0}),
\]

(5)

it is argued [2] that the mass spectrum of a meson nonet is linear:

\[
M^2_8 + M^2_1 = m^2_8 + m^2_{\eta'} = 2m^2_K.
\]

(6)

However, for the pseudoscalar nonet, $m^2_8 + m^2_{\eta'} = 1.217$ (GeV)$^2$, but $2m^2_K = 0.492$ (GeV)$^2$, therefore, it is believed [3] that the original Gell-Mann-Okubo mass formula is invalid for the pseudoscalar nonet. Based on Regge phenomenology, the new version of Gell-Mann-Okubo mass formula is given as [3]

\[
2M^2_8 - M^2_N = 4m^2_K - 3m^2_{\pi^0},
\]

(7)
where $M_N^2 = \cos^2 \alpha m_N^2 + \sin^2 \alpha m_N'^2$, $M_S^2 = \sin^2 \alpha m_N^2 + \cos^2 \alpha m_N'^2$ and $\sin \alpha = \frac{\sqrt{2} \cos \theta + \sin \theta}{\sqrt{3}}$.

Here, we want to emphasize that $M_S^2$ and $M_N^2$, the diagonal elements of the mass matrix $M^2$ describing the mixing of $\eta_8 - \eta_1$, are rather different from $m_S^2$ and $m_N^2$, the masses-squared of the bare octet $\eta_8$ and singlet $\eta_1$, respectively, which are employed in the original Gell-Mann-Okubo mass formula. The left-hand side and the right-hand side of Eq. (6) are not balance does not result from that the original Gell-Mann-Okubo mass formula is invalid for the pseudoscalar nonet, but from that the effects of the transition amplitudes $A_{m_{88}}$ and $A_{m_{11}}$ are not considered, where $A_{m_{88}}$ and $A_{m_{11}}$ denote the transition amplitudes of $\eta_8 \leftrightarrow \eta_8$ and $\eta_1 \leftrightarrow \eta_1$, respectively. Also, the so-called new version Gell-Mann-Okubo mass formula, Eq. (7), is in fact derived in the presence of the flavor-independent transition between quarkonia. The assumption that the transition between quarkonia is flavor-independent would give the inconsistent results for the pseudoscalar meson nonet[5]. The main purpose of this work is to emphasize that the mass relation of the non-ideal mixing nonets should incorporate the effects of the flavor-dependent transition between quarkonia.

II. Effects of the transition amplitudes of $q\overline{q} \leftrightarrow q'\overline{q}'$

In the $N = (u\overline{u} + d\overline{d})/\sqrt{2}$ and $S = s\overline{s}$ basis, the general form of the mass-squared matrix describing the mixing of the physical states $\eta$ and $\eta'$ can be written as[1]

$$M'^2 = \begin{pmatrix} M_N^2 & A_{NS} \\ A_{NS} & M_S^2 \end{pmatrix},$$

(8)

with

$$M_N^2 = m_N^2 + A_{NN}, \quad M_S^2 = m_S^2 + A_{SS},$$

(9)

where $m_N$ and $m_S$ are the masses of the bare states $N$ and $S$, respectively; $A_{NN}$, $A_{SS}$ and $A_{SN}$ are the transition amplitudes of $N \leftrightarrow N$, $S \leftrightarrow S$ and $N \leftrightarrow S$, respectively. If the transition between quarkonia is flavor-dependent[3, 4, 8], $A_{NN}$, $A_{SS}$ and $A_{SN}$ usually are parameterized as[3, 4, 10]

$$A_{SS} = A, \quad A_{NN} = r^2 A, \quad A_{SN} = r A.$$  

(10)
\( r^2 = 2 \), i.e., \( A_{SS} = A \), \( A_{NN} = 2A \) and \( A_{SN} = \sqrt{2}A \) means that the transition between quarkonia is flavor-independent \([11, 12]\). Owing to

\[
(N, S) = (\eta_8, \eta_1) R = (\eta_8, \eta_1) \begin{pmatrix} \sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \sqrt{\frac{3}{2}} \end{pmatrix},
\]

the mass-squared matrices \( M^2 \) and \( M'^2 \) can be connected by

\[
M^2 = RM'^2 R^{-1}.
\]

\( N \) is the orthogonal partner of \( \pi^0 \), the isovector state of a meson nonet, and one can expect that \( N \) degenerates with \( \pi^0 \) in effective quark masses, therefore we can assume \( m^2_N = m^2_{\pi^0} \), which is a widely adopted assumption \([10, 13, 14]\). According to the other form of the original Gell-Mann-Okubo mass formula \([4]\) \( m^2_N + m^2_S = 2m^2_K \), one can get

\[
M^2_8 = \frac{1}{3}(4m^2_K - m^2_{\pi^0}) + \frac{1}{3}Ar^2 - \frac{2\sqrt{2}}{3}Ar + \frac{2}{3}A, \tag{13}
\]

\[
M^2_{18} = -\frac{2\sqrt{2}}{3}(m^2_K - m^2_{\pi^0}) + \frac{\sqrt{2}}{3}Ar^2 - \frac{1}{3}Ar - \frac{\sqrt{2}}{3}A, \tag{14}
\]

\[
M^2_1 = \frac{1}{3}(2m^2_K + m^2_{\pi^0}) + \frac{2}{3}Ar^2 + \frac{2\sqrt{2}}{3}Ar + \frac{1}{3}A, \tag{15}
\]

where

\[
A = \frac{(m^2_{\eta'} - 2m^2_K + m^2_{\pi^0})(m^2_{\eta} - 2m^2_K + m^2_{\pi^0})}{2(m^2_{\pi^0} - m^2_K)}, \tag{16}
\]

\[
r^2 = \frac{(m^2_{\eta} - m^2_{\pi^0})(m^2_{\eta'} - m^2_{\eta})}{(m^2_{\pi'} - 2m^2_K + m^2_{\pi^0})(m^2_{\eta} - 2m^2_K + m^2_{\pi^0})}. \tag{17}
\]

Comparing Eq. \((3)\) with Eqs. \((13)\)~\((15)\), one can have

\[
M^2_8 = m^2_8 + Am_{88}, \tag{18}
\]

\[
M^2_1 = m^2_1 + Am_{11}, \tag{19}
\]

\[
Am_{88} = \frac{1}{3}Ar^2 - \frac{2\sqrt{2}}{3}Ar + \frac{2}{3}A, \tag{20}
\]

\[
Am_{11} = \frac{2}{3}Ar^2 + \frac{2\sqrt{2}}{3}Ar + \frac{1}{3}A, \tag{21}
\]

which shows that \( M^2_8 \) and \( M^2_1 \) are rather different from \( m^2_8 \) and \( m^2_1 \).

Based on the above relations, the mass relation of a meson nonet can be read as

\[
M^2_8 + M^2_1 = m^2_{\eta} + m^2_{\eta'} = 2m^2_K + A(r^2 + 1). \tag{22}
\]
If $A$ is set to be zero, Eq. (22) can be reduced to Eq. (1). Therefore, it is in the absence of the effects of the transition amplitudes $A_{m88}$ and $A_{m11}$ that the argument $m_\eta^2 + m_{\eta'}^2 = 2m_K^2$ is given.

Both Eq. (6) and Eq. (22) are deduced from the original Gell-Mann-Okubo mass formula, the only difference between Eqs. (6) and (22) is that Eq. (22) incorporates the effects of the transition amplitudes $A_{m88}$ and $A_{m11}$. However, for the pseudoscalar nonet Eq. (6) is obviously invalid while Eq. (22) formula does hold with a high accuracy (both sides of Eq. (22) are equal to 1.217 (GeV)$^2$), which indicates that Eq. (6) is invalid for the pseudoscalar nonet does not result from that the original Gell-Mann-Okubo mass formula is invalid for the pseudoscalar nonet, but from the omission of the effects of the transition amplitudes $A_{m88}$ and $A_{m11}$.

**III. Inconsistency from the flavor-independent transition amplitudes of $q\bar{q} \leftrightarrow q'\bar{q}'$**

Now we turn to discuss why the so-called new version of Gell-Mann-Okubo mass formula Eq. (7) is in fact derived from the assumption that the transition between quarkonia is flavor-independent.

From Eqs. (3), (8) and (12), we have

$$M_N^2 + M_S^2 = m_\eta^2 + m_{\eta'}^2,$$

then from Eqs. (7) and (23), the following equations can be given

$$M_N^2 = \frac{2m_\eta^2 + 2m_{\eta'}^2 + 3m_{a^0}^2 - 4m_K^2}{3}, \quad M_S^2 = \frac{4m_K^2 - 3m_{a^0}^2 + m_\eta^2 + m_{\eta'}^2}{3}.$$  

(24)

With $m_N^2 + m_S^2 = 2m_K^2$ and $m_N^2 = m_{a^0}^2$, from Eqs. (7) and (24), $A_{NN}$ and $A_{SS}$ can be read as

$$A_{NN} = M_N^2 - m_N^2 = \frac{2m_\eta^2 + 2m_{\eta'}^2 - 4m_K^2}{3},$$

$$A_{SS} = M_S^2 - m_S^2 = \frac{m_\eta^2 + m_{\eta'}^2 - 2m_K^2}{3}.$$  

(25)

(26)

So $A_{NN}$ and $A_{SS}$ satisfy

$$A_{NN} = 2A_{SS},$$

(27)
which implies that the transition between quarkonia is flavor-independent, and one can deduce that

\[ A_{SN} = \sqrt{2} A_{SS}. \]  

(28)

We note that with the assumption \( m_{N}^2 = m_{\pi^0}^2 \), the original Gell-Mann-Okubo mass formula \( m_{S}^2 + m_{N}^2 = 2m_{K}^2 \) can be re-expressed by

\[ 2m_{S}^2 - m_{N}^2 = 4m_{K}^2 - 3m_{\pi^0}^2. \]  

(29)

From Eqs. (7), (27) and (29), we can have

\[ 2M_{S}^2 - M_{N}^2 \equiv 2m_{S}^2 - m_{N}^2 = 4m_{K}^2 - 3m_{\pi^0}^2, \]  

(30)

which shows that the implicit assumption \( A_{NN} = 2A_{SS} \) exists in Eq. (7).

In fact, based on \( M_{N}^2 = \cos^2 \alpha m_{\eta}^2 + \sin^2 \alpha m_{\eta'}^2, M_{S}^2 = \sin^2 \alpha m_{\eta}^2 + \cos^2 \alpha m_{\eta'}^2 \) and \( \sin \alpha = \frac{\sqrt{2} \cos \theta + \sin \theta}{\sqrt{3}} \), the following relation can be given

\[ 2M_{S}^2 - M_{N}^2 \equiv \cos^2 \theta m_{\eta}^2 + \sin^2 \theta m_{\eta'}^2 + \sqrt{2} \sin 2\theta (m_{\eta}^2 - m_{\eta'}^2). \]  

(31)

From Eqs. (1), (2) and (3), one can have

\[ \cos^2 \theta = \frac{m_{\eta'}^2 - M_{S}^2}{m_{\eta'}^2 - m_{\eta}^2}, \quad \sin 2\theta = \frac{2M_{18}^2}{m_{\eta'}^2 - m_{\eta}^2}. \]  

(32)

According to Eqs. (13), (14), (31) and (32), one can have

\[ 2M_{S}^2 - M_{N}^2 \equiv \cos^2 \theta m_{\eta}^2 + \sin^2 \theta m_{\eta'}^2 + \sqrt{2} \sin 2\theta (m_{\eta}^2 - m_{\eta'}^2) = 4m_{K}^2 - 3m_{\pi^0}^2 - A(r^2 - 2). \]  

(33)

Obviously, if \( r^2 \) is set to be 2, Eq. (33) can be reduced to Eq. (7). Therefore, it is in the presence of the assumption that the transition between quarkonia is flavor-independent, that Eq. (7) is derived.

Under the assumption that the transition between quarkonia is flavor-independent, i.e., from Eqs. (27) and (28), the mass matrix Eq. (8) can be reduced to

\[ M'_{n^2} = \begin{pmatrix} m_{N}^2 + 2A & \sqrt{2}A \\ \sqrt{2}A & m_{S}^2 + A \end{pmatrix}, \]  

(34)
where \( A \equiv A_{SS} \). Diagonalizing the above mass matrix, one can have

\[
m^2_N + m^2_S + 3A = m^2_\eta + m^2_{\eta'},
\]

(35)

\[
(m^2_N + 2A)(m^2_S + A) - 2A^2 = m^2_\eta m^2_{\eta'},
\]

(36)

By eliminating \( A \) from the above two relations, one can get Schwinger’s original nonet mass formula\(^{[15]}\)

\[
(4m^2_K - 3m^2_\eta - m^2_{\pi^0})(3m^2_\eta' + m^2_{\pi^0} - 4m^2_K) = 8(m^2_K - m^2_{\pi^0})^2,
\]

(37)

here we still use \( m^2_N + m^2_S = 2m^2_K \) and the assumption \( m^2_N = m^2_{\pi^0} \).

For the pseudoscalar nonet, the left-hand side of Eq. (37) is 0.1178 (GeV)\(^4\), while the right-hand side of Eq. (37) is 0.4140 (GeV)\(^4\), which clearly shows that for the pseudoscalar meson nonet, the assumption that the transition between quarkonia is flavor-independent can result in the inconsistent results.

However, if we assume that the transition between quarkonia is flavor-dependent, i.e., we use the mass matrix

\[
\begin{pmatrix}
  m^2_N + r^2 A & rA \\ rA & m^2_S + A
\end{pmatrix}
\]

to describing the mixing of \( \eta \) and \( \eta' \), in the presence of \( m^2_N + m^2_S = 2m^2_K \) and \( m^2_N = m^2_{\pi^0} \), the new version of Schwinger’s nonet mass formula can be read as

\[
[2r^2m^2_K - (1+r^2)m^2_\eta - (r^2 - 1)m^2_{\pi^0}][(1+r^2)m^2_\eta' + (r^2 - 1)m^2_{\pi^0} - 2r^2m^2_K] = 4r^2(m^2_K - m^2_{\pi^0})^2.
\]

(38)

For the pseudoscalar nonet, both sides of Eq. (38) are equal to 0.6788 (GeV)\(^4\).

The original Gell-Mann-Okubo mass formula and the assumption \( m^2_N = m^2_{\pi^0} \) are incorporated in both Eq. (37) and Eq. (38), however, for the pseudoscalar nonet, Eq. (37) is obviously invalid, while Eq. (38) holds with a high accuracy, which indicates that for the pseudoscalar meson nonet, the assumption that the transition between quarkonia is flavor-independent can result in the inconsistent results. The correct mass relation of the pseudoscalar meson nonet should include the effects of the flavor-dependent transition between quarkonia.

### IV. Concluding remarks

Generally speaking, the mixing of the isoscalar states of a meson nonet results from some
extra interactions which not only cause the breaking of SU(3) flavor symmetry but also lead to the transition between quarkonia. As pointed out by Ref.\[16\], in the absence of the transition between quarkonia, the breaking of SU(3) flavor symmetry only results in the ideal mixing of the isoscalar states of a meson nonet. That is to say, for the ideal mixing meson nonet, the effects of the transition between quarkonia can be ignored, however, for the non-ideal mixing meson nonet, the effects of the transition between quarkonia should be rather important. Therefore, for the almost ideal mixing meson nonets, the effects of the flavor-independent transition between quarkonia and those of the flavor-dependent transition between quarkonia can be ignored. In fact, the results of Refs.\[2, 3, 5, 18\] all indicate that for the almost ideal mixing meson nonets such as the vector, tensor and axial vector meson nonets, both Eq. (3) based on the omission of the transition amplitudes and Eq. (37) based on the assumption that the transition is flavor-independent can hold with a higher accuracy. However, for the pseudoscalar nonet which is the non-ideal mixing meson nonet, both Eq. (3) and Eq. (37) are obviously invalid while both Eq. (22) and Eq. (38) which all incorporate the effects of the flavor-dependent transition between quarkonia hold with rather high accuracy, which shows that the correct mass relation of the non-ideal mixing meson nonets should include the effects of the flavor-dependent transition. Therefore, the mass relations such as Eqs. (22), (33) and (38) which include the effects of the flavor-dependent transition between quarkonia hold not only for the non-ideal mixing meson nonets but also for the almost ideal mixing meson nonets.

In conclusion, we point out that the invalidity of Eq. (3) for the pseudoscalar meson nonet arises from the omission of the effects of the transition between quarkonia, and that the so-called new version of Gell-Mann-Okubo mass formula in fact is based on the assumption that the transition between quarkonia is flavor-independent which can result in the inconsistent results for the pseudoscalar meson nonet. We emphasize that the correct mass relation of the non-ideal mixing meson nonets should include the effects of the flavor-dependent transition between quarkonia. The new mass relations such as Eqs. (22), (33) and (38) which include the effects of the flavor-dependent transition between quarkonia hold for all the meson nonets.
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