Feynman’s Propagator Applied to Network Models of Localization

H. Mathur

Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106-7079

(January 8, 2022)

Network models of dirty electronic systems are mapped onto an interacting field theory of lower dimensionality by interpreting one space dimension as time. This is accomplished via Feynman’s interpretation of anti-particles as particles moving backwards in time. The method developed maps calculation of the moments of the Landauer conductance onto calculation of correlation functions of an interacting field theory of bosons and fermions. The resulting field theories are supersymmetric and closely related to the supersymmetric spin-chain representations of network models recently discussed by various authors. As an application of the method, the two-edge Chalker-Coddington model is shown to be Anderson localized, and a delocalization transition in a related two-edge network model (recently discussed by Balents and Fisher) is studied by calculation of the average Landauer conductance.

PACS:

I. INTRODUCTION

Dirty electronic systems exhibit a variety of interesting phases and transitions in their transport properties at low temperature. For example consider two dimensional electrons moving in a random potential and a strong perpendicular field, as in a quantum Hall experiment. Generically the wavefunctions are localized, with tails that decay over a length scale called the localization length, which leads to insulating behaviour at low temperature. However if the parameters (e.g., the magnetic field or electron density) are tuned to special isolated values the localization length diverges. Such delocalization transitions were postulated shortly after the discovery of the quantum Hall effect and invoked in order to explain it [1]. Since then impressive strides have been made in experimental characterization of these transitions [3]; but a satisfactory theoretical understanding has not yet emerged [3].

An important theoretical advance was made by Chalker and Coddington who introduced a simplified network model of the quantum Hall transition [4]. Numerical studies show this model produces the same universal behaviour at the delocalization transition as more literal (and more complicated) models of the quantum Hall system [5]. Because the network model is relatively simple and is based on a clear physical picture of the transition, it seems a promising starting point for a controlled approximate analysis of the transition.

More recently it has been observed that the random bond Ising model is closely related to a variation on the network model [5]. Progress in analysis of network models is therefore desirable from this point of view also.

Conductance is a sensitive probe of delocalization. The purpose of this paper is to introduce a technique suitable for calculation of the conductance of network models. Following Landauer we imagine electrons are injected from the source into the sample where they undergo multiple scattering [6]. Eventually an electron may either get scattered forward into the drain or it may get scattered backward into the source (see fig 1). We wish to calculate the probability of forward scattering.

It is often fruitful in statistical mechanics to map a problem onto a quantum field theory of lower dimensionality by reinterpreting one space dimension as time. In applying that strategy here it becomes necessary to take into account the fact that electrons will then appear to move both forwards and backwards in “time”.

Figure 1. Landauer’s picture: The electron undergoes multiple scattering in the sample and is eventually scattered forward into the drain. In the field theory representation this corresponds to a process involving pair creation and annihilation.
A similar situation is encountered in quantum electrodynamics: in a famous paper, *Theory of Positrons*, Feynman showed that it is possible to regard positrons as electrons moving backwards in time. Following Feynman it is possible here to interpret the x-axis in fig 1 as time provided we regard an electron moving to the left as a sort of anti-particle of an electron moving to the right. For example the process in which a right moving electron is scattered to the left could be regarded as a process in which the particle meets it’s anti-particle leading to their mutual annihilation. From this perspective, the simple process depicted in fig 1 where an electron zig-zags through the sample could be regarded as a process in which a particle-antiparticle pair is created (at the earlier kink in the trajectory) and annihilated (at the later kink).

Based on this interpretation, it is possible to map a two-dimensional network model onto a one-dimensional field theory of particles and anti-particles. The statistics of the particles may be taken to be either Bose or Fermi. For technical reasons it is most convenient to introduce both species. Thus calculation of the conductance is mapped onto calculation of the correlation functions of an interacting field theory of bosons and fermions.

For clarity we illustrate the method on two-edge network models which are essentially one-dimensional. They map onto quantum mechanics problems of interacting bosons and fermions—a zero-dimensional field theory. Generalization of the mapping to a two-dimensional network model is straightforward. Indeed the method should be much more broadly applicable to dirty electronic systems. Since an important obstacle to non-perturbative analysis of random systems has been the lack of suitable representations of the problem, it is hoped this method may prove useful. The present method is closely related to supersymmetric spin-chain representations of network models that have recently been discussed by several authors.

As an application of the method we analyse two different network models. The first is essentially a one-dimensional Chalker-Coddington model which exhibits Anderson localization: for a large enough sample the zero temperature conductance is found to decay exponentially with sample size.

The second model was recently discussed in context of quantum transport by Balents and M.P.A. Fisher. It is related to a model of glass first studied by Dyson and to a special version of the random bond Ising model introduced by McCoy and Wu. An enlightening discussion of these connections, with references, is given in the paper of Balents and M.P.A. Fisher. This model is known to have a critical point at a special value of its parameters which is of considerable interest as a simple example of a random quantum critical point. Much is known about this model and its relations, particularly through the application of real space renormalization group methods by D.S. Fisher. In their recent paper Balents and M.P.A. Fisher have applied supersymmetry methods to calculate the exact two-parameter scaling function of the Green’s function of this model. Here we shall study it’s delocalization transition by calculating the conductance. A summary of our results for this model is given in section IVC.

## II. FIELD THEORY FORMULATION

In this section a simple two-edge network model is introduced and it’s Landauer conductance defined. The main result is an exact formal expression for the Landauer conductance, eq (42-44), within a field theory formulation of the problem.

### A. Landauer Conductance

The models considered in this paper consist of two counter-propagating “edge states” coupled by tunnelling. Along each one-dimensional edge electrons can propagate in only one direction (see fig 2). The electron wave function has two components: $\psi_+(x)$, the amplitude to be on the right moving edge at position $x$, and $\psi_-(x)$, the amplitude to be on the left moving edge at position $x$. The time independent Schrödinger equation governing this model is

$$
\left( -i \frac{\partial}{\partial x} m(x) \right) \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = E \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.
$$

The tunnelling amplitude $m(x)$ is some given function. Eventually we will take tunnelling to be a random process and will be interested in performing averages over an ensemble of different realizations of $m(x)$ with statistics to be described below. We are interested in solutions at some fixed energy $E$ (the fermi energy).

For later reference, it is useful to rewrite eq (1) in the form

$$
- i \frac{\partial}{\partial x} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} E & -m(x) \\ m^*(x) & -E \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}
$$

(2)
Figure 2 (a) A two edge network model. (b) Schematic picture of the random function \( m(x) \).

An obvious generalization of this model is to consider \( 2N \) counter-propagating edge states. With suitable statistical assumptions about the tunnelling these generalizations lead to the Chalker-Coddington model of the quantum Hall transition [4] or the random bond Ising model [5]. The model studied here can be considered a special anisotropic case in which tunnelling between alternate pairs of edges has been turned off.

Following Landauer’s method [6], the model is separated into the sample (region between \( x_i \) and \( x_f \)) and the probes (regions to the left of \( x_i \) and to the right of \( x_f \)); see fig 2. The left probe is called the source; the right probe, the drain. In contrast to the sample the probes are assumed to be disorder free. \( m(x) \) is therefore random only for \( x_i < x < x_f \); it is constant in the probes. For simplicity we shall assume that the edges are uncoupled in the probes (that is, \( m(x) = 0 \))\(^1\). This is shown schematically in fig 2 (b).

The wave eq (2) is now trivially soluble in the probes. We find

\[
\begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix} = \begin{pmatrix}
\alpha \\ 
\beta
\end{pmatrix} \begin{pmatrix}
\exp iEx \\
\exp -iEx
\end{pmatrix}
\begin{pmatrix}
\gamma \\
\delta
\end{pmatrix}
\begin{pmatrix}
dr
\end{pmatrix}
\tag{3}
\]

For a given realization of \( m(x) \) the connection between \( (\gamma, \delta) \) and \( (\alpha, \beta) \) can be found by integrating eq (2) across the sample

\[
\begin{pmatrix}
\gamma \\
\delta
\end{pmatrix} = T \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}.
\tag{4}
\]

\( T \) is a \( 2 \times 2 \) matrix called the transfer matrix of the sample. The transfer matrix is the focus of numerical studies of network models [3].

Here we are interested in solving eq (2) subject to scattering boundary conditions in the probes so that

\[
\alpha \rightarrow 1, \ \beta \rightarrow r; \ \gamma \rightarrow t, \ \delta \rightarrow 0.
\tag{5}
\]

According to Landauer the zero-temperature conductance of the sample (at Fermi energy \( E \)) is given by

\[
g = \frac{e^2}{h} |t|^2.
\tag{6}
\]

Returning to eq (2) it is useful to regard \( x \rightarrow \text{time} \) and to interpret it as the time-dependent Schrödinger equation of a two-level system. An awkward feature of such a reinterpretation is that the \( 2 \times 2 \) “Hamiltonian” matrix in eq (2) is not Hermitean\(^2\). Sadly this feature will persist through much of our analysis and appears to be more generally pervasive [8,12,13].

We can now imagine calculating the retarded propagator for eq (2). Given the wave function (values of \( \psi_+ \) and \( \psi_- \)) at a particular \( x \)-slice, the retarded propagator gives the wave function for “future” values of \( x \). For example, if the wave function is known at \( x_i \), the retarded propagator gives the wave function inside the sample (and beyond, in the drain). This is in very much the same spirit as the transfer matrix (indeed the retarded propagator is the transfer matrix for a special value of its arguments). It is not helpful for calculating the scattering amplitudes.

\(^1\)It is not essential for our approach to assume \( m(x) = 0 \) in the probes. However, it simplifies the analysis. It would be expected that the transport properties of the sample should not be modified by the nature of the probes. This can be checked explicitly under some circumstances.

\(^2\)Perhaps this is a good place to emphasize that in this paper although the “Hamiltonian” is frequently non-Hermitean, it is always a well defined operator in a well defined Hilbert space. There are no difficulties of interpretation as our “Hamiltonian” is merely used as a calculational device. The true Schrödinger equation of the model, eq (1), is governed by a Hermitean Hamiltonian.
While solving eq (2) subject to scattering boundary conditions we do not know the complete wave function for any \( x \)-slice. Instead we know the positive frequency component \( \psi_+ = 1 \) at the earlier slice \( x_i \) and we know the negative frequency component \( \psi_- = 0 \) at a later slice \( x_f \). We would like to reconstruct the wave function between the slices using this information; or at least we would like to know \( \psi_+(x \rightarrow x_f) = t \).

In *Theory of Positrons* Feynman showed precisely this would be achieved by a modification of the retarded propagator, now known as the Feynman propagator \([\mathbb{6}]\). Feynman derived an integral equation obeyed by his propagator and a perturbation series for it; this was the main focus of his paper. Peripherally, in two Appendixes, he showed the same results could be deduced from a second quantized theory of particles and anti-particles. This development will now be used to provide a field theory representation of the two edge network model (eq 2) \(^3\).

**B. Deduction From Second Quantization**

The purpose of this section is to derive a second quantized Hamiltonian from which one can calculate the Feynman propagator and scattering amplitude \( t \) of eq (2).

1. Fermion Representation

Return to the interpretation of eq (2) as a time dependent two level system. Introduce \( a^\dagger_+ \) which creates the right moving state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \); \( a^- \) which creates the left moving state \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \); and \( a_+ \) and \( a_- \), the corresponding destruction operators. The \( a \)'s obey fermi (anti-)commutation rules. Also introduce the “time”-dependent Hamiltonian

\[
H(x) = (a^\dagger_+ a^-_+ - a^\dagger_- a^+_-) \begin{pmatrix} E & -m(x) \\ m^*(x) & -E \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.
\]

(7)

This Hamiltonian is not useful for our purpose. It can be shown to generate the retarded propagator.

Instead we must now introduce \( e^R \) fermions, related to the \( a \) fermions via the particle-hole transformation

\[
\begin{align*}
  c^R &= a^\dagger_+; &  c^- &= a_-; \\
  c^R &= a_+; &  c^- &= a^-.
\end{align*}
\]

(8)

The superscript \( R \) is superfluous for the moment; its function will become apparent later. In terms of the \( c^R \) fermions the Hamiltonian is

\[
H_F^R(x) = E(c^R c^R + c^- c^-) - m(x)c^R c^R + m^*(x)c^- c^-.
\]

(9)

The scattering amplitude \( t \) can be computed straightforwardly using this Hamiltonian; see eq (19) below. In the remainder of this subsection that result will be derived.

Note that \( H_F \) is non-Hermitian—this is traceable to the non-Hermiticity of the “Hamiltonian” of eq (2). The reader troubled by this non-Hermiticity should again read footnote 2. Note also that \( H_F \) does not conserve the total number of \( c \)-fermions. Instead we may regard the \( c_- \) fermion as the “anti-particle” of the \( c_+ \). What is conserved then is the total “charge”—the difference in the number of particles and anti-particles, given by

\[
Q_F = c^R c^R - c^- c^-.
\]

(10)

The S-matrix corresponding to \( H_F^R \) is the solution to

\[
- i \frac{\partial}{\partial x} S_F^R(x, x_i) = H_F^R(x) S_F^R(x, x_i)
\]

subject to the initial condition \( S_F^R(x_i, x_i) = 1 \).

Turning to the derivation of eq (19), we first obtain a useful formula, following Feynman. Let \( (e_+(x), e_-(x)) \) be a solution to the wave eq (2). Construct the operator \( \tilde{F} \equiv e_+(x) c^R_+ + e_-(x) c^R_- \). \( \tilde{F} \) is transformed by the \( S \)-matrix as follows:

\[
S_F^R(x, x_i) \tilde{F} \tilde{S}_F^{-R-1}(x, x_i) = e_+(x) c^R_+ + e_-(x) c^R_-.
\]

(12)

This is Feynman’s formula. To prove it, regard eq (12) as an ansatz; in other words assume that \( SFS^{-1} \) is of this form with \( e_+(x) \) and \( e_-(x) \) some suitable functions. Upon differentiation with respect to \( x \), it will be seen that the ansatz is consistent provided \( (e_+, e_-) \) obey the wave eq (2).

Carrying out this plan, we note

\[
- i \frac{\partial}{\partial x} S_F^{-R-1}(x, x_i) = - S_F^{-R-1}(x, x_i) H_F^R(x)
\]

(13)

obtained by differentiating \( S_F^{-R-1} S_F^R = 1 \) and using the evolution eq (11) for the \( S \)-matrix. Eq (11), (12) and (13) together yield
\[-i \frac{\partial}{\partial x} \left( S_F R S_F^{-1} \right) = [H_F, S_F R S_F^{-1}] \]
\[= e_+(x)[H_F, c_R^+] + e_-(x)[H_F, c^R].\]
\[(14)\]

The relevant commutators are
\[[H_F, c_R^+] = Ec_R^+ + mc^R\]
\[[H_F, c^R] = -Ec^R - mc_R^+\]
\[(15)\]

Hence the derivative of the left side of the ansatz eq (12) is
\[-i \frac{\partial}{\partial x} \left( S_F R S_F^{-1} \right) = c_R^+ [Ec_+(x) - m(x)e_-(x)]
+ e_R^+ [m^*(x)e_+(x) - Ee_-(x)].\]
\[(16)\]

On comparing with the corresponding \( x \)-derivative of the right hand side of the ansatz it is seen that indeed \((e_+, e_-)\) must obey the wave eq (2); this completes the proof of Feynman’s formula eq (12).

Next let \((e_+, e_-)\) be the special solution that obeys scattering boundary conditions:
\[e_+(x_i) = 1; \quad e_-(x_i) = v;\]
\[e_+(x_f) = t; \quad e_-(x_f) = 0.\]
\[(17)\]

This leads to the scattering formula
\[S_F^R(x_f, x_i)(c^R_R + R c_F^R) S_F^{-1}(x_f, x_i) = t c^R_R \]
\[\Rightarrow S_F^R(x_f, x_i) c^R_R = t c^R_R S_F^R(x_f, x_i) - R S_F^R(x_f, x_i) c^R_R,\]
\[(18)\]

a useful special case of Feynman’s formula, eq (12).

Finally analyse the matrix element \( <0|c^R_R S_F^R(x_f, x_i) c^R_R|0> \) here \(|0>\) is the vacuum for the \( c^R \)-fermions. Use of the scattering formula, eq (18), reveals
\[t = \frac{<0|c^R_R S_F^R(x_f, x_i) c^R_R|0>}{<0|S_F^R(x_f, x_i)|0>}.\]
\[(19)\]

Eq (19) shows that the scattering amplitude \( t \) can be computed by studying the evolution of the single particle state \( c^R_R|0> \) under the Hamiltonian of eq (9). Note that the vacuum amplitude in the denominator of eq (19) is not trivial. In Feynman’s words, “It differs from unity because, for example, a pair could be created which eventually annihilates itself”.

Eq (19) is the main result of this subsection. By itself, eq (19) is not especially helpful. Our eventual purpose is to compute averages over the random tunnelling process \( m(x) \). Since \( m(x) \) appears in both numerator and denominator of eq (19), this form is not particularly well adapted for averaging.

2. Boson Representation

Tracing through the derivation of eq (19) it is seen that the Fermi statistics of the \( c^R \) particles plays no crucial role; only the commutators of eq (15) are essential. Hence we may replace the \( c^R \) fermions with \( b^R \) bosons governed by the Hamiltonian
\[H_B^R(x) = E(b^R b^R_R + b^R_R b^R_R) + m(x)b^R b^R_R + m^*(x)b^R_R b^R_R.\]
\[(20)\]

Apart from the replacement \( c^R \rightarrow b^R \), this differs from eq (9) in the sign of the pair creation term (the term proportional to \( b^R b^R_R \)). This sign change ensures commutation relations of the desired form similar to eq (15):
\[[H_B^R, b^R_R] = E b^R_R + m^*(x)b^R_R,
[H_B^R, b^R_R] = -E b^R_R - m(x)b^R_R.\]
\[(21)\]

Note that due to the sign change in the pair creation term the boson Hamiltonian, eq (20), is actually Hermitian. Again the \( b^R \) boson may be regarded as the anti-particle of the \( b^R \). The Hamiltonian conserves the difference in the number of particles and anti-particles, that is, the charge
\[Q_B = b^R_R b^R_R - b^R_R b^R_R.\]
\[(22)\]

The entire analysis of the preceding subsection can now be carried over essentially unchanged. The S-matrix corresponding to \( H_B^R \) is the solution to
\[-i \frac{\partial}{\partial x} S_B^R(x, x_i) = H_B^R(x) S_B^R(x, x_i) \]
\[(23)\]
subject to the initial condition \( S_B^R(x_i, x_i) = 1 \). Feynman’s formula, eq (12), and the scattering formula, eq (18), apply upon making the replacement \( c^R \rightarrow b^R \). Using the scattering formula we can deduce
\[t = \frac{<0|b^R R S_B^R(x_f, x_i) b^R_R|0>}{<0|S_B^R(x_f, x_i)|0>},\]
\[(24)\]

—the bosonic analogue of eq (19).

C. Analysis of the Vacuum Problem

Remarkably, the vacuum amplitudes for the bosons and fermions cancel; that is,
\[<0|S_B^R(x_f, x_i)|0> <0|S_F^R(x_f, x_i)|0> = 1.\]
\[(25)\]

We have previously derived a fermionic and bosonic expression for the scattering amplitude, neither well suited for performing an average over the random tunnelling process, \( m(x) \). Eq (25) will enable us to weld these expressions into a form suitable for averaging. We return to this point in section E below. Here we focus on proving eq (25).
To calculate the vacuum amplitude, following Feynman, let us analyze a series of problems that interpolate smoothly between a soluble limit and the problem we want to solve. Introduce a truncated problem for which \( m(x) \) is left unchanged for \( x_0 < x < x_f \) and is set equal to zero for \( x_i < x < x_0 \). By varying \( x_0 \) we obtain the desired series of problems. Evidently \( x_0 = x_i \) is the case we wish to solve. On the other hand, if \( x_0 = x_f \), there is no tunnelling and the problem is trivially soluble.

Denote the scattering coefficients for the truncated problem \( r(x_0) \) and \( t(x_0) \). For the soluble case, \( r(x_f) = 0 \) and \( t(x_f) = 1 \).

The fermion S-matrix for the truncated problem will be written as \( S^{RF}_{x_0}(x, x) \). It obeys

\[
-i \frac{\partial}{\partial x} S^{RF}_{x_0}(x, x) = H^R_F(x) S^{RF}_{x_0}(x, x) \tag{26}
\]

with \( S^{RF}_{x_0}(x_0, x_0) = 1 \). The truncated boson S-matrix is similarly defined.

The vacuum amplitude for the truncated fermion problem will be denoted \( C^R_F(x_0) \). Thus

\[
C^R_F(x_0) = <0 | S^{RF}_{x_0}(x_f, x_0) | 0> \tag{27}
\]

Analogous expressions can be written for the bosonic case. When \( x_0 = x_f \) the vacuum amplitude is unity for both bosons and fermions.

The strategy now is to study the variation of \( C(x_0) \) with \( x_0 \) by writing a differential equation for it. A formal solution of the differential equation can be obtained which will suffice to prove the cancellation of the vacuum amplitude.

For this purpose it is useful to write the solution to eq (26) as a formal series

\[
S^{RF}_{x_0}(x_f, x_0) = 1 + i \int_{x_0}^{x_f} dx_1 H^R_F(x_1) + i^2 \int_{x_0}^{x_f} dx_1 \int_{x_0}^{x_1} dx_2 H^R_F(x_1) H^R_F(x_2) + \ldots \tag{28}
\]

By differentiation of this series it follows

\[
-i \frac{\partial}{\partial x_0} S^{RF}_{x_0}(x_f, x_0) = - S^{RF}_{x_0}(x_f, x_0) H^R_F(x_0) \tag{29}
\]

Taking the vacuum expectation of eq (29) obtain

\[
-i \frac{\partial}{\partial x_0} C^R_{x_0} = - <0 | S^{RF}_{x_0}(x_f, x_0) H^R_F(x_0) | 0>\;
H^R_F(x_0) = - m(x_0) c^R_+ c^R_- + \text{others} \tag{30}
\]

In eq (30) we have explicitly written only that term of \( H^R_F \) (eq 9) which does not annihilate the vacuum.

According to the scattering formula, eq (18),

\[
S^{RF}_{x_0}(x_f, x_0) c^R_+ = t(x_0) c^R_+ S^{RF}_{x_0}(x_f, x_0) - r(x_0) S^{RF}_{x_0}(x_f, x_0) c^R_- . \tag{31}
\]

Applying this to the matrix element in eq (30) yields the differential equation obeyed by \( C^R_F(x_0) \):

\[
-i \frac{\partial}{\partial x_0} C^R_F(x_0) = - m(x_0) r(x_0) C^R_F(x_0) . \tag{32}
\]

We seek a solution subject to \( C^R_F(x_f) = 1 \).

Define \( L(x_0) \) as the solution of

\[
-i \frac{\partial}{\partial x_0} L(x_0) = - m(x_0) r(x_0) \tag{33}
\]

with the initial condition \( L(x_f) = 0 \). Given \( m(x) \) one can imagine solving the truncated problem for various values of \( x_0 \) and, in principle, computing \( L(x_0) \). The solution of eq (32) is then

\[
C^R_F(x_0) = \exp L(x_0) . \tag{34}
\]

Note that this solution obeys the differential equation (32) and the initial condition \( C^R_F(x_f) = 1 \).

The same analysis leads to the boson vacuum amplitude

\[
C^R_B(x_0) = \exp -L(x_0) . \tag{35}
\]

The sign change is traceable to the sign difference in the pair creation terms of the Fermi and Bose Hamiltonians. Hence \( C^R_F(x_0) C^R_B(x_0) = 1 \) for all \( x_0 \) and in particular for \( x_0 = x_i \), which establishes the desired result, eq (25).

It is interesting that Feynman appears to have devoted some effort to interpreting Dirac particles as bosons even though it violated the spin-statistics theorem. In particular, the result of eq (25), which is of no obvious value in quantum electrodynamics, is derived in the Theory of Positrons.

D. Conjugate Amplitude

The Landauer conductance is given by \( |t|^2 \). Thus expressions for \( t^* \) analogous to eq (19) and (24) will be needed. To obtain them consider the equation conjugate to eq (2):

\[
-i \frac{\partial}{\partial x} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} -E & m(x) \\ -m^* (x) & E \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} . \tag{36}
\]

Eq (36) is constructed to have the property that if \( (\epsilon_+, \epsilon_-) \) is a solution to eq (2), then the complex conjugate \( (\epsilon_+, \epsilon_-) \) is a solution to eq (36). Hence if eq (36) is solved subject to scattering boundary conditions, the scattering amplitude will be \( t^* \).

On the other hand, direct comparison reveals that eq (2) is transformed into eq (36) by the replacements \( E \to -E, m \to -m^*, m^* \to -m \). Hence introduce \( c^A \) fermions governed by the Hamiltonian
\[ H^A_F(x) = -E(c^A_l c^A_+ + c^A_+ c^A_-) + m^*(x)c^A_+ c^A_- - m(x)c^A_+ c^A_. \] (37)

This Hamiltonian is obtained from eq (9) by making the replacements indicated above. By the reasoning that lead from eq (9) to (19), \( H^A_F \) generates the conjugate amplitude via

\[
t^* = \begin{cases} <0|c^A_+ S^A_B(x_f, x_i)c^A_+|0> \\ <0|S^A_B(x_f, x_i)|0> \end{cases} \]

where \( S^A_B(x_f, x_i) \) is the S-matrix corresponding to \( H^A_F \).

Similarly, one can write

\[
t^* = \begin{cases} <0|b^A_+ S^A_B(x_f, x_i)b^A_+|0> \\ <0|S^A_B(x_f, x_i)|0> \end{cases} \]

where \( S^A_B(x_f, x_i) \) is the S-matrix of the boson Hamiltonian

\[ H^A_B = -E(b^A_+ b^A_+ + b^A_- b^A_-) - m^*(x)b^A_+ b^A_- - m(x)b^A_+ b^A_. \]

(40)

Finally note that the vacuum amplitudes for the A bosons and fermions cancel, as they do for their R counterparts:

\[
<0|S^A_B(x_f, x_i)|0> <0|S^A_B(x_f, x_i)|0> = 1.
\]

(41)

E. Supersymmetry

The results of the previous subsections can now be assembled into an expression for the Landauer conductance suitable for averaging over the random tunnelling process \( m(x) \). Simultaneously introduce \( c^R \) and \( c^A \) fermions and \( b^R \) and \( b^A \) bosons governed by the total Hamiltonian

\[ H_{\text{SUSY}}(x) = H^R_F + H^A_F + H^R_B + H^A_B = E \hat{M} + m(x) \hat{A} + m^*(x) \hat{B} \]

(42)

Here \( \hat{M} \equiv (c^R_+ c^R_+ + c^R_- c^R_- - R \rightarrow A) + (b^R_+ b^R_+ + b^R_- b^R_- - R \rightarrow A) \) and \( \hat{A} \equiv -c^R_+ c^R_- - c^A_+ c^A_- + b^R_+ b^R_- - b^A_+ b^A_- \) and \( \hat{B} \equiv c^R_+ c^R_- + c^A_+ c^A_- + b^R_+ b^R_- - b^A_+ b^A_. \)

As usual the corresponding S-matrix obeys

\[-i \frac{\partial}{\partial x} S_{\text{SUSY}}(x, x_i) = H_{\text{SUSY}}(x) S_{\text{SUSY}}(x, x_i) \]

(43)

and the initial condition \( S_{\text{SUSY}}(x_i, x_i) = 1 \).

The Landauer conductance can be calculated using

\[ |t|^2 = <0|c^A_+ S^R_F S^A_B S^R_B S^A_+|0> <0|c^R_+ S^R_F S^R_B S^A_B S^A_+|0>. \]

(44)

Using eq (19), (38), (25) and (41), this product is easily seen to be \(|t|^2 \).

Eq (44) is our principal tool to analyse the network model. It shows that the conductance can be calculated by following the evolution of a two fermion state under the Hamiltonian \( H_{\text{SUSY}} \). The principal advantage of this expression is due to the absence of a denominator which makes it well suited for averaging over the random tunnelling process \( m(x) \). The averaging will be discussed further in sections III and IV assuming different distributions for \( m(x) \). The reader has perhaps noticed that one could develop various other expressions for the Landauer conductance that involve evolving, instead of a two fermion state, a two boson state or a one boson, one fermion state; but eq (44) is the form we shall use here.

Finally let us briefly examine some symmetries of \( H_{\text{SUSY}} \). Evidently, it conserves the R fermion charge; that is,

\[ [Q^R_F, H_{\text{SUSY}}] = 0 \]

(45)

where \( Q^R_F \equiv c^R_+ c^R_+ - c^R_- c^R_- \). Similarly it also conserves the R boson charge as well as the corresponding quantity for the A boson and fermion. \( H_{\text{SUSY}} \) also possesses a supersymmetry analogous to that displayed by the models studied in refs [13]. Specifically the Hamiltonian, eq (42), commutes with the supercharges \( Q^R_{F B}, Q^A_{F B}, Q^R_{B F} \) and \( Q^A_{B F} \) which are given by

\[
Q^R_{F B} \equiv c^R_+ b^R_+ + c^R_- b^R_-,
Q^R_{B F} \equiv c^R_+ b^R_+ - c^R_- b^R_-;
\]

(46)

\( Q^R_{F B} \) and \( Q^A_{B F} \) are obtained by replacing \( R \rightarrow A \) in eq (46).

It will be seen below that after disorder averaging we must analyse an effective interacting Hamiltonian instead of the non-interacting (but random) Hamiltonian of eq (42). The effective interacting Hamiltonian will also possess the symmetries discussed above.

III. TWO EDGE CHALKER-CODDINGTON MODEL

A. Disorder Average

First let us assume that the real and imaginary parts of \( m(x) \) are independent gaussian white noise processes with zero mean. Thus \( \langle m(x) \rangle_{\text{ens}} = 0 \). Here \( \langle \ldots \rangle_{\text{ens}} \) denotes an average over the ensemble of \( m(x) \) realizations. The variance is given by

\[
7
\]
\[ m^*(x), m(x') \rvert_{\text{ens}} = D\delta(x - x') \]
\[ m(x), m(x') \rvert_{\text{ens}} = 0. \]  
(47)

Essentially the same statistics result if it is assumed that \( m(x) \) is a rapidly fluctuating phase factor. These assumptions are similar to those made in the Chalker-Coddington model. For simplicity we suppose \( E = 0 \). It is shown in Appendix A that this entails no loss of generality or modification of behavior; but the averaging is simpler to describe.

To calculate the average Landauer conductance, we must average the \( S \)-matrix \( S_{\text{SUSY}}(x_f, x_i) \) over the ensemble described above. This is accomplished by expanding the \( S \)-matrix in a formal series (eq 28) and averaging term by term. The result is
\[
\langle S_{\text{SUSY}}(x_f, x_i) \rangle_{\text{ens}} = \exp\left[-(D/2)H_{CC}(x_f - x_i)\right] \]  
(48)

where
\[
H_{CC} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}. \]  
(49)

\( \hat{A} \) and \( \hat{B} \) are defined below eq (42). The ensemble averaged conductance is then given by
\[
[g]_{\text{ens}} = \frac{e^2}{h} < 0|c^\dagger_R c^\dagger_A \exp\left(-\frac{D}{2}H_{CC}(x_f - x_i)\right) c_R^\dagger c_A^\dagger|0 >. \]  
(50)

Thus in order to calculate the average conductance we need to study the evolution of a two fermion state under the effective Hamiltonian \( H_{CC} \). In contrast to \( H_{\text{SUSY}} \), \( H_{CC} \) is not random; but the price paid is that it is interacting.

B. Anderson Localization

The evaluation of eq (50) is simplified by the observation that all of the fermionic bilinears that appear in \( H_{CC} \) (namely \( c^R_R, c^R_A, c^A_R, c^A_A \) and \( c^A_R c^A_A \)) annihilate the two fermion state. Hence the expression for the average conductance, eq (50), simplifies to
\[
[g]_{\text{ens}} = \frac{e^2}{h} < 0|B \exp\left(-\frac{D}{2}H_{\text{boson}}^\text{boson}(x_f - x_i)\right)|0 > \]  
(51)

Here \( H_{\text{boson}}^\text{boson} \) is the purely bosonic part of \( H_{CC} \) (written explicitly below) and \( |0 > \) is the boson vacuum. Physically then we need only calculate the boson vacuum amplitude.

Now let us analyse the boson Hamiltonian
\[
H_{\text{boson}}^\text{boson} = (b^R_R b^R_R - b^A_A b^A_A)(b^R_R b^A_R - b^A_A b^R_A) \\
+ (b^R_R b^R_R - b^A_A b^A_A)(b^R_R b^A_R - b^A_A b^R_A). \]  
(52)

In contrast to the full Hamiltonian \( H_{CC} \), the bosonic part is Hermitian. Note that it is also positive definite since it is of the form \( \hat{D}^\dagger \hat{D} + \hat{D} \hat{D}^\dagger \) where \( \hat{D} \equiv (b^R_R b^R_R - b^A_A b^A_A) \). After some manipulation we obtain the more revealing form
\[
H_{CC}^\text{boson} = h_n + h_+ + h_-; \]  
with
\[
h_n \equiv 2n^R_R n^A_A + n^R_R + n^A_A + 2; \]
\[
h_+ \equiv -2b^R_R b^A_R b^A_A; \]
\[
h_- \equiv -2b^R_R b^A_R b^A_A. \]  
(53)

Here \( n^R_R = b^R_R b^R_R \) etc. Although lengthy, eq (53) has a simple content. Consider the bosonic state \( |n > B \), \( n = 0, 1, 2, \ldots |n > B \) is defined as a normalized state that contains \( n \) bosons of each kind (R+, R-, A+ and A-). \( n = 0 \) corresponds to the boson vacuum. Inspection of eq (53) shows that these states are closed under the action of \( H_{CC}^\text{boson} \). Hence we need only consider the block of \( H_{CC}^\text{boson} \) that lies within the invariant subspace spanned by these states.

Our plan therefore is to find the eigenstates of \( H_{CC}^\text{boson} \) that lie within the subspace spanned by \( |n > B \). Expansion of the boson vacuum in terms of these eigenstates will then allow straightforward evaluation of the boson vacuum amplitude and the average Landauer conductance, eq (51). Note that the Hermiticity of \( H_{CC}^\text{boson} \) ensures that its eigenstates form a complete set and it is therefore appropriate to use them as a basis.

The coupled boson Hamiltonian, \( H_{CC}^\text{boson} \), is solved in section C below. It is found to have a continuum of eigenstates denoted \( |k > \) with eigenvalue \( (1 + k^2)/2 \). Here \( k \in [0, \infty] \). The eigenstates are orthogonal and are normalized so that \( <k|0 > = 1 \) and
\[
<p|k >= \frac{2}{\pi} \cosh^2 \frac{\pi k}{2} \frac{1}{k \sinh \frac{\pi k}{2}} \delta(k - p). \]  
(54)

It follows from eq (54) and the presumed completeness of the eigenfunctions of \( H_{CC}^\text{boson} \)
\[
\frac{\pi}{2} \int_0^\infty dk \frac{k \sinh \frac{\pi k}{2}}{\cosh^2 \frac{\pi k}{2}} |k < k| = I. \]  
(55)

Here \( I \) denotes the identity matrix in the subspace spanned by \( |n > B \).

Inserting the resolution of the identity eq (55) into the conductance formula, eq (51), yields
\[
[g]_{\text{ens}} = \frac{\pi e^2}{2h} \exp\left(-\frac{D}{4}(x_f - x_i)\right) \times \int_0^\infty dk \frac{k \sinh \frac{\pi k}{2}}{\cosh^2 \frac{\pi k}{2}} \exp\left(-\frac{Dk^2}{4}(x_f - x_i)\right). \]  
(56)

In the limit of large sample size we find
\[
[g]_{\text{ens}} \approx \frac{e^2}{h} \left(\frac{\pi}{D(x_f - x_i)}\right)^\frac{1}{2} \exp\left(-\frac{D}{4}(x_f - x_i)\right). \]  
(57)
Thus the model exhibits Anderson localization: the conductance decays exponentially for sufficiently large sample size as generally expected of a dirty one dimensional quantum wire. Eq (57) agrees with the result obtained by ref [8].

C. Solution of Coupled Boson Hamiltonian

We wish to solve the Schrödinger equation

$$H_{CC}^{boson} |\psi\rangle = \lambda |\psi\rangle$$

(58)

within the subspace spanned by $|n\rangle_B$. Expand the eigenstate as

$$|\psi\rangle = a_0|0\rangle_B + a_1|1\rangle_B + \ldots + a_n|n\rangle_B + \ldots$$

(59)

The effect of $H_{CC}^{boson}$, eq (53), on the states $|n\rangle_B$ is easily computed:

$$h_n|n\rangle_B = (4n^2 + 4n + 2)|n\rangle_B;$$
$$h_1|n\rangle_B = -2(n + 1)^2|n+1\rangle_B;$$
$$h_2|n\rangle_B = -2n^2|n-1\rangle_B.$$  

(60)

Eq (59) and (60) together allow us to write the Schrödinger equation as a three term recurrence relation

$$(4n^2 + 4n + 2)a_n - 2n^2a_{n-1} - 2(n+1)^2a_{n+1} = \lambda a_n$$

(61)

subject to $a_{-1} = 0$. Our goal now is to solve eq (61) for different $\lambda$ and then ortho-normalize the solutions. The process of orthonormalization will weed out the disallowed values of $\lambda$. Note that from the Hermiticity and positive definiteness of $H_{CC}^{boson}$ mentioned following eq (52), we are already assured that the allowed $\lambda$ must be positive and real.

To solve the recurrence relation we introduce the generating function

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots$$

(62)

From the recurrence relation, eq (61), we can easily construct the differential equation obeyed by $f$:

$$x(x-1)^2 \frac{d^2 f}{dx^2} + (3x - 1)(x - 1) \frac{df}{dx} + \left(x - 1 + \frac{\lambda}{2}\right)f = 0.$$  

(63)

This equation has three regular singular points: at $x = 0, 1$ and $\infty$; it is therefore a Riemann P-equation. One solution is analytic at $x = 0$ as can be verified by directly substituting the series, eq (62), in the differential eq (63). This is the solution we seek; it generates the solution to eq (61).

The solution to a Riemann P-equation can always be expressed in terms of Hypergeometric functions. Making the standard transformations (see, e.g., ref [17], chapter 5) we find the analytic solution is given by

$$f(x) = (1-x)^\mu F(\mu + 1, \mu + 1; x);$$

$$where \quad \mu = \frac{1}{2} + \frac{\sqrt{1-2\lambda}}{2}.$$  

(64)

Note that for $\lambda > 1/2$, $\mu$ becomes complex. Eq (64) therefore points to some change in behaviour at $\lambda = 1/2$. Below we will find that in fact the allowed eigenvalues are $\lambda > 1/2$.

The coefficients $a_n$ can be extracted from $f(x)$ via the contour integral

$$a_n = \oint_C \frac{dx}{2\pi i} \frac{1}{x^{n+1}} f(x).$$

(65)

Here $C$ is a contour that encircles the origin but not the branch point $x = 1$. This contour integral cannot be expressed in elementary form in general but the large $n$ behaviour of $a_n$ can be obtained by using an integral representation for the hypergeometric function (see Appendix B).

Let us focus on the solutions to eq (61) with $\lambda > 1/2$. Write $\lambda = 1/2 + k^2/2$ where $k \in [0, \infty]$. The large $n$ asymptotic behaviour is

$$a_n \approx \frac{2}{\pi} \cosh \frac{\pi k}{2} \frac{1}{(2k \sinh \frac{\pi k}{2})^{1/2}} \frac{1}{\sqrt{n}} \cos \left(\frac{k}{2} \ln n + \phi(k)\right).$$

(66)

Here the phase $\phi(k) = \arg[\Gamma(ik)\Gamma(1/2 - ik/2)/\Gamma^2(1/2 + ik/2)]$. Thus the solutions for $\lambda > 1/2$ decay slowly (as $1/\sqrt{n}$) and exhibit weak logarithmic oscillations.

Fortunately, it turns out that this asymptotic behaviour is sufficient to orthonormalize the eigenstates. Let $b_n$ be a solution to eq (61) with eigenvalue $\rho$:

$$(4n^2 + 4n + 2)b_n - 2n^2b_{n-1} - 2(n+1)^2b_{n+1} = \rho b_n.$$  

(67)

This result is not surprising when one considers the analogous result of Strum-Liouville theory where it is known that the overlap integral of distinct eigenfunctions can be expressed as a surface term. See, for example, ref [17], p 719-720.
Thus the asymptotic behaviour of \( a_n \) is sufficient to evaluate the orthonormalization sum \( \lim_{N \to \infty} \sum_{n=0}^{N} a_n b_n \). Use of eq (66), (68) and the delta function representation\(^5\)

\[
\lim_{L \to \infty} \frac{1}{k} \sin k L = \pi \delta(k)
\]

yields eq (54).

We have so far focussed on solutions with \( \lambda > 1/2 \). Using similar arguments one can show that solutions with \( 0 < \lambda < 1/2 \) do not decay fast enough for large \( n \) to be ortho-normalizable even as continuum eigenfunctions. They are analogous to the exponential negative energy solutions of the free particle Schrödinger equation in elementary quantum mechanics. In the same spirit, the solutions for \( \lambda > 1/2 \) are analogous to the positive energy continuum of plane wave solutions.

Having shown that there are no eigenstates with \( \lambda < 1/2 \), we have justified the step from eq (54) to eq (55). Alternatively one might try to prove the completeness relation, eq (55), directly. Forming the matrix element of eq (55) between the states \(| n > B \) and \(| m > B \), one should check whether

\[
\frac{\pi}{2} \int_{0}^{\infty} dk \sinh \frac{\pi k}{2} \cos \frac{\pi k}{2} a_m(k) a_n(k) = \delta_{nm}.
\]

We have verified this analytically (and more extensively numerically) for a few small values of \( n \) and \( m \) for which the \( a \)'s can be computed directly from eq (61). It may be possible to construct the general proof using eq (64-65), but this passes beyond conventional standards of rigour and good taste in theoretical physics.

### IV. DELOCALIZATION TRANSITION IN DYSON GLASS

Now let us analyse the case in which \( m(x) \) is purely imaginary. We rewrite eq (2) as

\[
-\frac{\partial}{\partial x} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} E & -im \\ im & -E \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.
\]

Here \( m(x) \) is a real white noise process with mean \( m_0 \). We have recycled the symbol \( m \) to conserve the finite resources of the alphabet. As noted in the introduction, this model is related to a model of glass first analysed by Dyson [14] and to a special version of the random bond Ising Model introduced by McCoy and Wu [13]. It has been extensively studied previously in its various incarnations [14,13] and it is known to have a critical point at \( m_0 = 0 \) and \( E = 0 \). Here we shall use the Landauer conductance to investigate this critical point. For simplicity, the discussion is limited to two circumstances. First we set \( E = 0 \) and study the behaviour of the conductance as \( m_0 \to 0 \). Next we set \( m_0 = 0 \) and tune the Fermi energy, \( E \).

#### A. Elementary Solution for \( E = 0 \)

The techniques developed in this paper can be used to calculate the disorder averaged Landauer conductance of this model; but for the special case \( E = 0 \) a much more complete solution can be obtained by elementary means. The content of this solution is instructive and it is therefore described.

When \( E = 0 \) the transfer matrix of eq (71) has a particularly simple explicit form for arbitrary \( m(x) \). Define

\[
M = \int_{x_i}^{x_f} dx m(x).
\]

Then the transfer matrix

\[
T = \begin{pmatrix} \cosh M & \sinh M \\ \sinh M & \cosh M \end{pmatrix}.
\]

Use of eq (4), (5) and (73) immediately yields

\[
t = \tanh M
\]

and

\[
g = \frac{e^2}{\hbar} \tanh^2 M.
\]

Since \( m \) is a Gaussian white noise process \( M \) is a Gaussian random variable. If we suppose

\[
[m]_{ens} = m_0 \quad \text{and} \quad [\delta m(x) \delta m(x')]_{ens} = D \delta(x - x'),
\]

where \( \delta m(x) \equiv m(x) - m_0 \), it is easy to calculate

\[
[M]_{ens} = m_0 (x_f - x_i) \quad \text{and} \quad [(\delta M)^2]_{ens} = D (x_f - x_i).
\]

From eq (77) the full distribution of \( M \) can be reconstructed:
\[ P(M) = \frac{1}{\sqrt{2\pi D(x_f-x_i)}} \exp \left( -\frac{[M-m_0(x_f-x_i)]^2}{2D(x_f-x_i)} \right). \] (78)

It is now altogether straightforward to obtain statistical information about the Landauer conductance from eq (75) and (78). In particular it can be shown: (i) At the critical point the average conductance decays as a power of the system size \((x_f-x_i)^{-1/2}\). (ii) Away from the critical point \((m_0 \neq 0)\) the average conductance decays exponentially (Anderson localization). The localization length diverges as \(m_0^{-\nu}\) where the exponent \(\nu = 2\). (iii) The conductance has a very broad distribution; the average is dominated by large rare fluctuations. (iv) Away from the critical point the typical conductance also decays exponentially but with a distinct localization length that diverges with exponent \(\nu = 1\) as the critical point is approached.

In summary although eq (71) is essentially trivial to solve for \(E = 0\), the content of the solution is not trivial. It illustrates many general features of random critical systems.

**B. Tuning Fermi Energy**

1. **Disorder Averaging**

The localization length is expected to diverge as \((\ln E)^2\) as \(E \to 0\). To probe this singularity we shall make one simplification. The frequency \(E\) will be considered to be complex, and we shall approach the singularity at the origin along the imaginary axis. Thus we shall replace \(E \to i\omega\) in eq (71) and study \([g]_{\text{ens}}(\omega)\).

Note that for imaginary frequency eq (71) is real and therefore the scattering amplitude \(t\) is also real. In general \(t\) is not a suitable probe of delocalized behaviour as it decays exponentially (due to phase fluctuations) even when \(|t|^2\) does not. In this case however we may look for the transition by calculating the average of \(t\) which is not only real, but also positive (and therefore free of sign fluctuations\(^6\)). This constitutes a simplification because to calculate \(t\) the \(A\) fermions and bosons are not needed.

Even if one insists on calculating \(|t|^2\) the \(A\) fermions and bosons are unneeded. Since \(t\) is real \(|t|^2 = t^2\) and we may write

\[ |t|^2 = <0|e^{RS_B^R}(x_f,x_i)c^{R\dagger}_+|0> <0|b^{R\dagger}_+S_B^R(x_f,x_i)b^{R}\rangle |0>. \] (79)

Having noted these possibilities, we now disregard them. Instead we use eq (44) to calculate \(|t|^2\). Hence we need to average the S-matrix \(S_{\text{SU SY}}(x_f,x_i)\) over the white noise ensemble of real \(m(x)\) with \([m(x)]_{\text{ens}} = 0\) and \([m(x),m(x')]_{\text{ens}} = D\delta(x-x')\). The result is

\[ [S_{\text{SU SY}}(x_f,x_i)]_{\text{ens}} = \exp \left( -\frac{D}{2}H_D(x_f-x_i) \right) \] (80)

where

\[ H_D \equiv \omega\hat{N} + \frac{D}{2}\hat{K}^2. \] (81)

\(\hat{N}\) counts the total number of particles regardless of species and \(K \equiv i\hbar\{c^{R\dagger}_+c^{R}_+,e^{R\dagger}_+e^{R}_+,c^{A\dagger}_+c^{A}_+\} + c^{\dagger}_+c^{\dagger}_+ - b^{R\dagger}_+b^{R}_+ + b^{R\dagger}_-b^{R}_- - b^{A\dagger}_+b^{A}_- + b^{A\dagger}_-b^{A}_+\}.\) The ensemble averaged conductance is then given by

\[ [g]_{\text{ens}} = \frac{e^2}{h} <0|c^A_+c^R_+\exp \left( -\frac{D}{2}H_D(x_f-x_i) \right) c^{R\dagger}_+c^{A\dagger}_+|0>. \] (82)

2. **Critical Behaviour**

Once again all the fermionic bilinears in \(\hat{K}^2\) annihilate \(c^{R\dagger}_+c^{A\dagger}_+|0>\); hence the conductance is determined by the boson vacuum amplitude

\[ [g]_{\text{ens}} = \frac{e^2}{h} \exp[-2\omega(x_f-x_i)] \times <0|_{\text{B}} \exp \left( -\frac{D}{2}H_D^{\text{boson}}(x_f-x_i) \right) |0>_B. \] (83)

As before \(H_D^{\text{boson}}\) is hermitean and positive definite. It has an invariant subspace spanned by the states \(|n>_B, n = 0, 1, 2, \ldots\) \(|n>_B\) is here defined as the state with 2 \(n\) bosons of each kind. Hence we may focus on the block of \(H_D^{\text{boson}}\) that lies within the subspace spanned by \(|n>_B\).

---

\(^6\)Proof: Consider a series of truncated problems as in section II C. First note that for \(x_0 = x_f, t(x_0) = 1\). Then observe that \(t(x_0)\) can never vanish; if it did, the wave function vanishes in the drain, and by integrating eq (71) backwards, it vanishes in the sample and source also. Since it cannot pass through zero, by continuity, \(t(x_0)\) must remain positive for all \(x_0\) including \(x_f\).
Our plan is to find eigenstates of this block and to expand the boson vacuum in terms of these eigenstates. In contrast to the two edge Chalker-Coddington case the eigenstates are discrete and can be labelled by an integer \( l = 1, 2, 3, \ldots \). The eigenvalue problem is solved approximately in the relevant small \( \omega \) limit in the next subsection. It will be found that the eigenvalues are given by

\[
\lambda_l = \frac{2\pi^2 l^2}{(\ln \frac{\omega}{D})^2}
\]

(84)

and the overlap with the vacuum is

\[
| l|0 \rangle_B = \frac{1}{|2 \ln \frac{\omega}{D}|} \sinh \pi k_l
\]

(85)

where \( k_l \equiv l\pi/\ln(\omega/D) \). Putting together these results

\[
[g]_{\text{ens}} \approx e^2/h \left( \frac{1}{2 \ln \frac{\omega}{D}} \right) \exp \left( -2\pi^2 D \left( \frac{1}{\ln \frac{\omega}{D}} \right)^2 (x_f - x_i) \right).
\]

(86)

for large samples. Eq (86) shows that the sample is Anderson localized away from the critical point. The localization length diverges as \((\ln \omega)^2\) as \( \omega \to 0 \), in agreement with previous work (Balents and Fisher and refs therein).

3. Solution of Boson Problem

This calculation is very similar to the corresponding calculation by Balents and Fisher of the average Green’s function. For this reason, after pointing out the special feature of the conductance calculation (discussion leading to eq (89)) we present the remaining steps in outline. The reader interested in more details should consult the appropriate sections of Balents and Fisher.

We wish to solve

\[
H_D^{\text{boson}} |\psi\rangle = \lambda |\psi\rangle
\]

(87)

within the subspace spanned by the states \( |n \rangle_B \) defined in the previous subsection. To this end expand \( |\psi\rangle \) as

\[
|\psi\rangle = a_0 |0 \rangle_B + a_1 |1 \rangle_B + \ldots + a_n |n \rangle_B + \ldots
\]

(88)

A difficulty arises if we assume the states are normalized so that \( <n|b|m>_B = \delta_{mn} \). In this case the effect of \( H_D^{\text{boson}} \) on \( |n \rangle_B \) is to yield a linear combination of \( |n+1 \rangle_B, |n \rangle_B \) and \( |n-1 \rangle_B \) with coefficients that involve products such as \( \sqrt{n(n+1)} \). Such non-integer coefficients would then also appear when eq (87) is written as a recurrence relation analogous to eq (61) and would defeat the generating function technique that was used to solve eq (61).

However if we work with an unusual normalization such that \( <n|b|m>_B = (n+1)\delta_{mn} \) simple coefficients result. Adopting this convention we find

\[
\left( \frac{4\omega}{D} m + 4m^2 + m + 1 \right) a_m
\]

\[
-m(2m-1)a_{m-1} - (m+1)(2m+3)a_{m+1} = \lambda a_m.
\]

(89)

The corresponding generating function obeys a differential equation with three singular points, two regular and one irregular. Thus the solutions are related not to the common functions of mathematical physics but to the more obscure Mathieu or Spheroidal functions. Rather than pursue this direction we closely follow Balents and Fisher to obtain an approximate solution to the Schrödinger eq (89), valid in the interesting limit of small \( \omega/D \).

First we solve eq (89) for \( \omega = 0 \) using the generating function method. This solution should be accurate for \( n \ll D/\omega \). Setting \( a_0 = 1 \) we find

\[
a_n \approx \frac{1}{2n} \left( \sinh \frac{\pi k}{\pi k} \right)^{1/2} \cos(k \ln n - \phi(k))
\]

(90)

for \( 1 \ll n \ll D/\omega \). Here we have introduced \( k \) via \( \lambda = 2k^2 \) and \( \phi(k) \equiv \arg \Gamma(1 - ik) + 2k \ln 2 \). Next we approximate the large \( n \) behaviour by taking the continuum limit of eq (89):

\[
n^2 \frac{d^2}{dn^2} a + 3n \frac{d}{dn} a + \left( \frac{\lambda}{2} + 1 - \frac{\omega}{D} \right) a = 0.
\]

(91)

Eq (91) is soluble by Laplace transforms; but an even quicker solution may be effected by introducing \( f(n) \equiv na(n) \) which obeys the equation

\[
n^2 \frac{d^2}{dn^2} f + n \frac{d}{dn} f + \left( \frac{\lambda}{2} - \frac{\omega}{D} \right) f = 0.
\]

(92)

Precisely this equation was analysed by Balents and Fisher in Appendix D of their paper. Borrowing their results we can find the values of \( \lambda \) for which eq (91) has a solution that decays as \( n \to \infty \) and smoothly matches eq (90) for \( 1 \ll n \ll D/\omega \). This yields the quantized energy levels of eq (84).

The eigenstates may be approximately normalized by taking the continuum solution to apply everywhere. The normalization sum \( \sum_{n=0}^{\infty} a_n^2/(n+1) \) may then be replaced by an integral. The relevant integral is evaluated asymptotically by Balents and Fisher (Appendix D). Transcription of their result yields eq (85).
C. Summary

For reference we summarize the results of this section:

At the critical point ($m_0 = 0, \omega = 0$) the average conductance decays as the inverse square root of the sample size. Off the critical point the average conductance decays exponentially on a length scale called the localization length. For $\omega = 0$ we find the localization length of the average conductance diverges as $m_0^{-\nu}$ as $m_0 \to 0$, with exponent $\nu = 2$; for $m_0 = 0$, we find it diverges as $(\ln \omega)^2$. These results for the average conductance localization length are consistent with previous calculations of other non-local correlation functions for these models [10, 13].

For $\omega = 0$, the model is completely soluble by elementary methods. In this simple case it shows many of the features generally expected of random critical systems [13]. In particular the Landauer conductance is very broadly distributed and the average is dominated by large, rare fluctuations. Away from the critical point the typical conductance also decays exponentially but with a localization length different from the average conductance. The localization length for the typical conductance diverges with exponent 1 as $m_0 \to 0$.

ACKNOWLEDGMENTS

It is a pleasure to thank Leon Balents, Matthew Fisher, Steve Girvin, Phil Taylor and Tanmay Vachaspati for encouragement, illuminating discussions and telling me about their work prior to publication. The author is supported by an Alfred P. Sloan Research Fellowship.

APPENDIX A:

The purpose of this Appendix is to show that the results of section III are not modified when $E \neq 0$. This can be done in various ways. The method followed here is chosen because it also illustrates how to average the S-matrix when the random Hamiltonian, $H_{SU3}$, is composed of two non-commuting pieces.

Factorise the S-matrix as

$$S_{SU3}(x, x_i) = \exp[i E \hat{M}(x - x_i)]U(x, x_i). \quad (A1)$$

Then eq (42-43) imply that $U$ obeys

$$-i \frac{\partial}{\partial x} U(x, x_i) = [m(x) \hat{A}(x) + m^*(x) \hat{B}] U(x, x_i) \quad (A2)$$

and the initial condition $U(x_i, x_i) = 1$. Here

$$\hat{A}(x) = \exp[-i E \hat{M}(x - x_i)] \hat{A} \exp[i E \hat{M}(x - x_i)],$$

$$\hat{B}(x) = \exp[-i E \hat{M}(x - x_i)] \hat{B} \exp[i E \hat{M}(x - x_i)]. \quad (A3)$$

In this interaction representation it is easy to show

$$\hat{A}(x) = \exp[-i 2 E(x - x_i)] \hat{A},$$

$$\hat{B}(x) = \exp[i 2 E(x - x_i)] \hat{B}. \quad (A4)$$

The disorder average of $U$ can be computed as in section III.

$$[U(x, x_i)]_{\text{ens}} = \exp \left( -\frac{D}{2} (x - x_i)(\hat{A}\hat{B} + \hat{B}\hat{A}) \right). \quad (A5)$$

Recall $\hat{A}\hat{B} + \hat{B}\hat{A} \equiv H_{CC}$ (eq 49). Hence

$$[S_{SU3}(x, x_i)]_{\text{ens}} = \exp[i E \hat{M}(x - x_i)]$$

$$\times \exp \left( -\frac{D}{2} H_{CC}(x - x_i) \right). \quad (A6)$$

Finally note

$$\hat{M} e^{R t} |c_i^A| 0 > = 0. \quad (A7)$$

Eq (44), (A6) and (A7) together show that

$$[g]_{\text{ens}} = \frac{e^2}{\hbar} < 0|\hat{c}_i^A \hat{c}_i^R | \exp \left( -\frac{D}{2} H_{CC}(x_f - x_i) \right) e^{R t} |c_i^A| 0 > \quad (A8)$$

—indepedent of $E$ as claimed in the paper.

APPENDIX B:

The contour integral in eq (65) is evaluated here asymptotically for large $n$. $f(x)$ is defined in eq (64). As in the paper, we focus on $\lambda > 1/2$ and write $\lambda = (1 + k^2)/2$. Thus $\mu = -1/2 + i k/2$.

Use the integral representation of the hypergeometric function

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_1^\infty dt (t - x)^{a - c} (t - 1)^{c - b - 1}$$

$$\times \frac{\Gamma(c - a - b - 1)}{(c - b)(c - a - b)} \int_1^\infty dt t^a (t - 1)^{c - b - 1} f_n(t) \quad (B1)$$

valid provided $\text{Re} > \text{Re} > 0$ (ref [7], chapter 5).

Substitute eq (B1) in eq (65) with $a \to a + 1$, $b \to a + 1$ and $c \to 1$. Exchange the order of the $t$ and $x$ integrals to obtain

$$a_n = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_1^\infty dt t^a (t - 1)^{c - b - 1} f_n(t) \quad (B2)$$

where

$$f_n(t) = \int_0^C \frac{dx}{2\pi i x^{n+1}} (t - x)^{-1/2 - ik/2} (1 - x)^{-1/2 + ik/2}. \quad (B3)$$

Here $C$ is a contour that encircles the origin but not the branch point at $x = 1$. 
Note that the integrand in eq (B3) has branch points at \( x = 1 \) and \( x = t \) where \( t \) is some point on the positive real axis to the right of 1 (and to be eventually integrated over the range from 1 to \( \infty \)). To be consistent with the conventions of the integral representation, eq (B1), we draw branch cuts along the positive real axis from 1 to \( +\infty \) and \( t \) to \( +\infty \). Also the phase of \((t-x)\) and \((1-x)\) must both be taken to be zero when \( x \) lies on the real axis to the left of 1.

The contour \( C \) is now deformed to pass above and below the branch cut. It is closed by a small circle around \( x = 1 \) and by a big circle at \( \infty \). To be consistent with the geometric series; however we need to expand about one

\[
\text{Substitute eq (B9) in eq (B7). This leads to the result of eq (66).}
\]

\[
F(a, b, c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, a+b-c+1; 1-x) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} \times F(c-a, c-b, c-a-b+1; 1-x). \quad (B8)
\]

\[
\text{We obtain}
\]

\[
F\left(\frac{1}{2} + ik, \frac{1}{2}, 1; e^{-s}\right) \approx \frac{\Gamma(-ik)}{\Gamma^2(1/2 - ik/2)} + \frac{\Gamma(ik)}{\Gamma^2(1/2 + ik/2)} s^{-ik}. \quad (B9)
\]

[1] The Quantum Hall Effect, 2nd ed, edited by R.E. Prange and S.M. Girvin (Springer-Verlag, New York 1990).

[2] S.L. Sondhi, S.M. Girvin, J.P. Carini and D. Shahar, Rev Mod Phys 69, 315 (1997).

[3] B. Huckestein, Rev Mod Phys 67, 357 (1995).

[4] J.T. Chalker and P.D. Coddington, J Phys C 21, 2665 (1988).

[5] S. Cho and M.P.A. Fisher, cond-mat 9607173.

[6] For a review, see A. Szafer and A.D. Stone, IBM J Res Dev 32, 384 (1988); for a careful derivation from linear response, H.U. Baranger and A.D. Stone, Phys Rev B 40, 8169 (1989).

[7] R.P. Feynman, Phys Rev 76, 749 (1949).

[8] M.R. Zirnbauer, Annalen der Physik 3, 513 (1994).

[9] L. Balents, M.P.A. Fisher, and M. Zirnbauer, Nucl Phys B 483, 681 (1997).

[10] I.A. Gruzberg, N. Read and S. Sachdev, Phys Rev B 55, 10593 (1997).

[11] I. Gruzberg, N. Read and S. Sachdev, cond-mat 9704032.

[12] J. Kondov and J.B. Marston, cond-mat 9612224.

[13] L. Balents and M.P.A. Fisher, cond-mat 9706009.

[14] F.J. Dyson, Phys Rev 92, 1331 (1953).

[15] B.M. McCoy and T.T. Wu, Phys Rev 176, 631 (1968).

[16] D.S. Fisher, Phys Rev B50, 3799 (1994) and Phys Rev B51, 6411 (1995).

[17] P.M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York 1953).