Existence and coexistence in first-passage percolation

Daniel Ahlberg∗

Abstract

We consider first-passage percolation with i.i.d. non-negative weights coming from some continuous distribution under a moment condition. We review recent results in the study of geodesics in first-passage percolation and study their implications for the multi-type Richardson model. In two dimensions this establishes a dual relation between the existence of infinite geodesics and coexistence among competing types. We explain the limitations of the current techniques by presenting a partial result in dimensions $d \geq 2$.

1 Introduction

In first-passage percolation the edges of the $\mathbb{Z}^d$ nearest neighbour lattice, for some $d \geq 2$, are equipped with non-negative i.i.d. random weights $\omega_e$, inducing a random metric $T$ on $\mathbb{Z}^2$ as follows: For $x, y \in \mathbb{Z}^d$, let

$$T(x, y) := \inf \left\{ \sum_{e \in \pi} \omega_e : \pi \text{ is a self-avoiding path from } x \text{ to } y \right\}.$$  \hspace{1cm} (1)

Since its introduction in the 1960s by Hammersley and Welsh [11], a vast body of literature has been generated seeking to understand the large scale behaviour of distances, balls and geodesics in this random metric space. The state of the art has been summarized in various volumes over the years, including [2, 14, 16, 24]. We will here address questions related to geodesics, and shall for this reason make the common assumption that the edge weights are sampled from a continuous distribution. Since many of the results we shall rely on require a moment condition for their conclusions to hold, we shall assume in what follows that $\mathbb{E}[Y^d] < \infty$, where $Y$ denotes the minimum weight among the $2d$ edges connected to the origin.

Later work of Richardson [23] and Cox and Durrett [6] extended the above radial convergence to simultaneous convergence in all directions. Their results show that the ball $\{ z \in \mathbb{Z}^d : T(0, z) \leq t \}$
in the metric $T$ once rescaled by $t$ approaches the unit ball in the norm $\mu$. The unit ball in $\mu$, henceforth denoted by $\text{Ball} := \{x \in \mathbb{R}^d : \mu(x) \leq 1\}$, is therefore commonly referred to as the \textit{asymptotic shape}, and known to be compact and convex with non-empty interior. However, little else is known regarding the properties of the shape in general, and this we shall see is a major obstacle for our understanding of many other features of the model.

Questions regarding geodesics were posed already in the work of Hammersley and Welsh. However, it took until the mid 1990s before Newman and co-authors \cite{18, 19, 21, 22} initiated a systematic study of the geometry of geodesics in first-passage percolation. Under the assumption of continuous weights there is almost surely a unique path attaining the minimum in (1); we shall denote this path $\text{geo}(x, y)$ and refer to it as the \textit{geodesic} between $x$ and $y$. The graph consisting of all edges on $\text{geo}(0, y)$ for some $y \in \mathbb{Z}^d$ is a tree spanning the lattice. Understanding the properties of this object, such as the number of topological ends, leads one to the study of \textit{infinite geodesics}, i.e. infinite paths $g = (v_1, v_2, \ldots)$ of which every finite segment is a geodesic. We shall write $\mathcal{T}_0$ for the collection of infinite geodesics starting at the origin. A simple compactness argument shows that the cardinality $|\mathcal{T}_0|$ of $\mathcal{T}_0$ is always at least one.\footnote{Consider the sequence of finite geodesics between the origin and $n\mathbf{e}_1$, where $\mathbf{e}_1$ denotes the first coordinate vector. Since the number of edges that connect to the origin is finite, one of them must be traversed for infinitely many $n$. Repeating the argument results in an infinite path which by construction is a geodesic.}

As a means to make rigorous progress on Newman’s conjecture, Haggström and Pemantle \cite{10} introduced a model for competing growth on $\mathbb{Z}^d$, for $d \geq 2$, known as the \textit{two-type Richardson model}. In this model, two sites $x$ and $y$ are initially coloured red and blue respectively. As time evolves an uncoloured site turns red at rate 1 times the number of red neighbours, and blue at rate $\lambda$ times the number of blue neighbours. A central question of interest is for which values of $\lambda$ there is positive probability for both colours to coexist, in the sense that they both are responsible for the colouring of infinitely many sites.

There is an intimate relation between the existence of infinite geodesics and coexistence in the Richardson model that we shall pay special interest in. In the case of equal strength competitors ($\lambda = 1$), one way to construct the two-type Richardson model is to equip the edges of the $\mathbb{Z}^d$ lattice with independent exponential weights, thus exhibiting a direct connection to first-passage percolation. The set of sites eventually coloured red in the two-type Richardson model is then equivalent to the set of sites closer to $x$ than $y$ in the first-passage metric. That is, an analogous way to phrase the question of coexistence is whether there are infinitely many points closer to $x$ than $y$ as well as infinitely many points closer to $y$ than $x$ in the first-passage metric. As before, a compactness argument will show that on the event of coexistence there are disjoint infinite geodesics $g$ and $g'$ that respectively originate from $x$ and $y$. Haggström and Pemantle \cite{10} showed that, for $d = 2$, coexistence of the two types occurs with positive probability, and deduced as a corollary that

$$\mathbb{P}(|\mathcal{T}_0| \geq 2) > 0.$$  

Their results were later extended to higher dimensions and more general edge weight distributions in parallel by Garet and Marchand \cite{9} and Hoffman \cite{12}. In a later paper Hoffman \cite{13} showed that in two dimensions, coexistence of four different types has positive probability, and
that $\mathbb{P}(|\mathcal{S}_0| \geq 4) > 0$. The best result currently known is a strengthening of Hoffman’s result due to Damron and Hanson \cite{Damron}, showing that

$$\mathbb{P}(|\mathcal{S}_0| \geq 4) = 1.$$ 

In this paper we shall take a closer look at the relation between existence of infinite geodesics and coexistence in competing first-passage percolation. We saw above that on the event of coexistence of various types, a compactness argument could give the existence of equally many infinite geodesics, and it is conceivable that it is possible to locally modify the edge weight in such a way that these geodesics are re-routed through the origin. Conversely, interpreting infinite geodesics as ‘highways to infinity’, along which the different types may escape their competitors, it seems that the existence of a given number of geodesics should accommodate an equal number of surviving types. These heuristic arguments suggest a duality between existence and coexistence, and it is this dual relation we shall make precise.

Given sites $x_1, x_2, \ldots, x_k$ in $\mathbb{Z}^d$, we let $\text{Coex}(x_1, x_2, \ldots, x_k)$ denote the event that for every $i = 1, 2, \ldots, k$ there are infinitely many sites $z \in \mathbb{Z}^d$ for which $j = i$ minimizes the distance $T(x_j, z)$. (The continuous weight distribution assures that there are almost surely no ties.) In two dimensions the duality between existence and coexistence that we prove takes the form:

$$\exists x_1, x_2, \ldots, x_k \text{ such that } \mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 0 \iff \mathbb{P}(|\mathcal{S}_0| \geq k) > 0.$$

Probability Turning the above heuristic into a proof is more demanding than it may seem. In order to derive this relation we will rely on the ergodic theory for infinite geodesics recently developed by Ahlberg and Hoffman \cite{Ahlberg}. The full force of this theory is currently restricted to two dimensions, which prevents us from obtaining an analogue to (2) in higher dimensions. In higher dimensions we deduce a partial result based on results of Damron and Hanson \cite{Damron} and Nakajima \cite{Nakajima}.

1.1 The dual relation

Before we state our results formally, we remind the reader that $Y$ denote the minimum weight among the $2d$ edges connected to the origin. We recall (from \cite{Hoffman}) that $\mathbb{E}[Y^2] < \infty$ is both necessary and sufficient in order for the shape theorem to hold in dimension $d \geq 2$.

**Theorem 1.** Consider first-passage percolation on the square lattice with continuous edge weights satisfying $\mathbb{E}[Y^2] < \infty$. For any $k \geq 1$, including $k = \infty$, and $\varepsilon > 0$ we have:

(i) If $\mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 0$ for some $x_1, x_2, \ldots, x_k$ in $\mathbb{Z}^2$, then $\mathbb{P}(|\mathcal{S}_0| \geq k) = 1$.

(ii) If $\mathbb{P}(|\mathcal{S}_0| \geq k) > 0$, then $\mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 1 - \varepsilon$ for some $x_1, x_2, \ldots, x_k$ in $\mathbb{Z}^2$.

In dimensions higher than two we establish parts of the above dual relation. We define the number of sides of the asymptotic shape as the number of tangent planes to $\partial$Ball. Hence, the number of sides is finite if and only if the shape is a (finite) convex polygon ($d = 2$), polyhedron ($d = 3$), or polytope ($d \geq 4$).

**Theorem 2.** Consider first-passage percolation on the $d$-dimensional cubic lattice, for $d \geq 2$, with continuous edge weights. For any $k \geq 1$, including $k = \infty$, and $\varepsilon > 0$ we have
(i) If $\mathbb{E}\left[\exp(\alpha \omega_i)\right] < \infty$ and $\mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 0$ for some $\alpha > 0$ and $x_1, x_2, \ldots, x_k$ in $\mathbb{Z}^d$, then $\mathbb{P}(|\mathcal{T}_0| \geq k) = 1$.

(ii) If $\mathbb{E}[Y^d] < \infty$ and the shape has at least $k$ sides, then $\mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 1 - \varepsilon$ for some $x_1, x_2, \ldots, x_k$ in $\mathbb{Z}^d$.

In Section 2 we shall review the recent development in the study of infinite geodesics that will be essential in for the deduction, in Section 3 of the announced dual result. Finally, in Section 4 we prove the partial result valid in higher dimensions.

1.2 A mention of our methods

There is one connection between existence and coexistence that is easy to prove, and which has been hinted at above. Namely, if Geos($x_1, x_2, \ldots, x_k$) denotes the event that there exist $k$ pairwise disjoint infinite geodesics, each originating from one of the points $x_1, x_2, \ldots, x_k$, then

$$\text{Coex}(x_1, x_2, \ldots, x_k) \subseteq \text{Geos}(x_1, x_2, \ldots, x_k).$$

(3)

To see this, let $V_i$ denote the set of sites closer to $x_i$ than to any other $x_j$, for $j \neq i$, in the first-passage metric. On the event Coex($x_1, x_2, \ldots, x_k$) each set $V_i$ is infinite, and for each $i$ a compactness argument gives the existence of an infinite path contained in $V_i$, which by construction is a geodesic. Since $V_1, V_2, \ldots, V_k$ are pairwise disjoint, due to uniqueness of geodesics, so are the resulting infinite geodesics.

Let $\mathcal{N}$ denote the maximal number of pairwise disjoint infinite geodesics. Since $\mathcal{N}$ is invariant with respect to translations (and measurable) it follows from the ergodic theorem that $\mathcal{N}$ is almost surely constant. Hence, positive probability for coexistence of $k$ types implies the almost sure existence of $k$ pairwise disjoint geodesics. That $|\mathcal{T}_0| \leq \mathcal{N}$ is trivial, given the tree structure of $\mathcal{T}_0$. The inequality is in fact an equality, which was established by different means in [1, 20]. Together with (3), this resolves the first part of Theorems 1 and 2.

Above it was suggested that infinite geodesics should, at least heuristically, be thought of as ‘highways to infinity’ along which the different types may escape the competition. The concept of Busemann functions, and their properties, will be central in order to make this heuristic precise. These functions have their origin in the work of Herbert Busemann [4] on metric spaces. In first-passage percolation Busemann functions first appeared in the work of Newman [21], and were further developed by Hoffman [12, 13]. They are at this point indispensable in the study of infinite geodesics in various models for spatial growth, and have been used to construct stationary measures and stationary solutions to differential equations by Cator and Pimentel [5] and Bakhtin, Cator and Khanin [3].

Finally, we remark that it is (in two dimensions) widely believed that the asymptotic shape is not a polygon, in which case it follows from [13] that both $\mathbb{P}(|\mathcal{T}_0| = \infty) = 1$ and that for every $k \geq 1$ there are $x_1, x_2, \ldots, x_k$ such that $\mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 0$. The latter was extended to infinite coexistence by Damron and Hochman [8]. Thus, proving that the shape is non-polygonal would make our main theorem obsolete. However, understanding the asymptotic shape is a notoriously hard problem, which is the reason an approach sidestepping Newman’s curvature assumption has been developed in the first place.
2 Geodesics and Busemann functions

In this section we review the recent developments in the study of infinite geodesics in first-passage percolation. We shall focus on the two-dimensional setting, and remark on higher dimensions only at the end. We make no claim in providing a complete account of previous work, and instead prefer to focus on the results that will be of the greatest significance for the purposes of this paper. A more complete description of these results, save those reported in the more recent studies [1, 20], can be found in [2].

2.1 Infinite geodesics according to Newman with collaborators

The study of geodesics in first-passage percolation was pioneered by Newman and co-authors [18, 19, 21, 22] in the mid 1990s. Their work gives rise to a precise conjectural picture of the structure of infinite geodesics. In order to describe these results we shall need some notation. First, we say that an infinite geodesic \( g = (v_1, v_2, \ldots) \) has asymptotic direction \( \theta \), in the unit circle \( S^1 := \{ x \in \mathbb{R}^2 : |x| = 1 \} \), if the limit \( \lim_{k \to \infty} v_k/|v_k| \) exists and equals \( \theta \). Second, two infinite geodesics \( g \) and \( g' \) are said to coalesce if their symmetrical difference \( g \Delta g' \) is finite. The conjectures originating from the work of Newman and co-authors can be summarized as, under mild conditions on the weight distribution, the following hold:

(a) with probability one, every infinite geodesic has an asymptotic direction;
(b) for every direction \( \theta \) there is an almost surely unique geodesic with direction \( \theta \);
(c) for every direction \( \theta \), with probability one, any two geodesics with direction \( \theta \) coalesce.

In particular, these statements would imply that \( |\mathcal{T}_0| = \infty \) almost surely. Another property that could be added to this list is that of asymptotically linear Busemann functions. However, we shall save this discussion for later.

Licea and Newman [21, 18] proved conditional versions of these statements under an additional curvature assumption of the asymptotic shape. While this assumption seems plausible for a large family of edge weight distributions, there is no known example for which it is known to hold. Rigorous proofs of the corresponding statements for a rotation invariant first-passage like model, where the asymptotic shape is known to be a Euclidean disc, has been obtained by Howard and Newman [15]. Since proving properties like strict convexity and differentiability of the boundary of the asymptotic shape in standard first-passage percolation appears to be a major challenge, later work has focused on obtaining results without assumptions on the shape.

2.2 Busemann functions

Busemann functions first appeared in the first-passage literature in the work of Newman [21], as a means of describing the microscopic structure of the boundary of a growing ball in the first passage metric. The usefulness of Busemann functions in order to describe properties of geodesics was explored in later work of Hoffman [12, 13].

Given an infinite geodesic \( g = (v_1, v_2, \ldots) \) in \( \mathcal{T}_0 \) the Busemann function \( B_g : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R} \) of \( g \) is defined as the limit

\[
B_g(x, y) := \lim_{k \to \infty} [T(x, v_k) - T(y, v_k)].
\]
As observed by Hoffman [12], with probability one the limit in (4) exists for every \( g \in T_0 \) and all \( x, y \in \mathbb{Z}^2 \), and satisfies the following properties:

- \( B_g(x, y) = B_g(x, z) + B_g(z, y) \) for all \( x, y, z \in \mathbb{Z}^2 \);
- \( |B_g(x, y)| \leq T(x, y) \);
- \( B_g(x, y) = T(x, y) \) for all \( x, y \in g \) such that \( x \in \text{geo}(0, y) \).

In [12] Hoffman used Busemann functions to establish that there are at least two disjoint infinite geodesics almost surely. In [13] he used Busemann functions to associate certain infinite geodesics with sides (tangent lines) of the asymptotic shape. The approach involving Busemann functions in order to study infinite geodesics was later developed further in work by Damron and Hanson [7] and Ahlberg and Hoffman [1]. Studying Busemann functions of geodesics, as opposed to the geodesics themselves, has allowed these authors to establish rigorous versions of Newman’s conjectures regarding the structure of geodesics. Describing parts of these results in detail will be essential in order to understand the duality between existence of geodesics and coexistence in competing first-passage percolation.

### 2.3 Linearity of Busemann functions

We shall call a linear functional \( \rho : \mathbb{R}^2 \to \mathbb{R} \) supporting if the line \( \{ x \in \mathbb{R}^2 : \rho(x) = 1 \} \) is a supporting line to \( \partial \text{Ball} \) through some point, and tangent if \( \{ x \in \mathbb{R}^2 : \rho(x) = 1 \} \) is the unique supporting line (i.e. the tangent line) through some point of \( \partial \text{Ball} \). Given a supporting functional \( \rho \) and a geodesic \( g \in T_0 \) we say that the Busemann function of \( g \) is asymptotically linear to \( \rho \) if

\[
\limsup_{|y| \to \infty} \frac{1}{|y|} |B_g(0, y) - \rho(y)| = 0.
\]

As an example of the usefulness of the asymptotic linearity we mention that (5), together with the third of the properties of Busemann functions exhibited by Hoffman, provides information on the direction of \( g = (v_1, v_2, \ldots) \) in the sense that the set of limit points of the sequence \( \langle v_k/|v_k| \rangle_{k \geq 1} \) is contained in the arc of \( \partial \text{Ball} \) corresponding to the set \( \{ x \in S^1 : \mu(x) = \rho(x) \} \).

Building on the work of Hoffman [13], Damron and Hanson [7] showed that for every tangent line of the asymptotic shape there exists a geodesic whose Busemann function is described by the corresponding linear functional. In a simplified form their result reads as follows:

**Theorem 3.** For every tangent functional \( \rho : \mathbb{R}^2 \to \mathbb{R} \) there exists, almost surely, a geodesic in \( T_0 \) whose Busemann function is asymptotically linear to \( \rho \).

While Damron and Hanson proved existence of geodesics with linear Busemann functions, later work of Ahlberg and Hoffman [11] showed that every geodesic has a linear Busemann function, and that the associated linear functionals are unique.

**Theorem 4.** With probability one, for every geodesic \( g \in T_0 \) there exists a supporting functional \( \rho : \mathbb{R}^2 \to \mathbb{R} \) such that the Busemann function of \( g \) is asymptotically linear to \( \rho \).
To address uniqueness, note that the set of supporting functionals is naturally parametrized by the direction of their gradients. Due to convexity of the shape, these functionals stand in 1-1 correspondence with the unit circle $S^1$. We shall from now on identify the set of supporting functionals with $S^1$. The next result, also from [1], addresses ergodic aspects and uniqueness.

**Theorem 5.** There exists a closed deterministic set $C \subseteq S^1$ such that, with probability one, the (random) set of supporting functionals $\rho$ for which there exists a geodesic in $T_0$ with Busemann function asymptotically linear to $\rho$ equals $C$. Moreover, for every $\rho \in C$ we have

$$\mathbb{P}(\exists \, \text{two geodesics in } T_0 \text{ with Busemann function linear to } \rho) = 0.$$

From Theorem 3 it follows that $C$ contains all tangent functionals, and that if Ball has at least $k$ sides (i.e. tangent lines), then $|T_0| \geq k$ almost surely. By Theorem 4 it follows that every geodesic has a linear Busemann function, and by Theorem 5 that if $T_0$ has size at least $k$, then there exist $k$ geodesics with distinct Busemann functions. All these observations will be essential in proving part (ii) of Theorem 1.

Due to the connection between asymptotic directions and linear Busemann functions mentioned above, Theorems 3-5 may be seem as rigorous, although somewhat weaker, versions of Newman’s conjectures (a)-(b). The same is true for conjecture (c), as we describe next.

### 2.4 Coalescence

An aspect of the above development that we have ignored so far is that of coalescence. For instance, Theorem 3 is a simplified version of a stronger statement proved by Damron and Hanson. They prove that for every tangent functional $\rho : \mathbb{R}^2 \to \mathbb{R}$ there exists, almost surely, a family of geodesics $\Gamma = \{ \gamma_z : z \in \mathbb{Z}^2 \}$, where $\gamma_z \in T_z$, such that any one geodesic in $\Gamma$ has Busemann function linear to $\rho$ and any two geodesics in $\Gamma$ coalesce. (The latter of course implies that the Busemann functions of all geodesics in $\Gamma$ coincide.) In a similar spirit, Ahlberg and Hoffman show the following.

**Theorem 6.** For every supporting functional $\rho \in C$, with probability one, any two geodesics $g \in T_y$ and $g' \in T_z$ with Busemann function asymptotically linear to $\rho$ coalesce.

Coalescence was irrelevant for the proof of Theorem 3 in [7], but instrumental for the deduction of Theorems 4 and 5 in [1]. In short, the importance of coalescence lies in the possibility to apply the ergodic theorem to asymptotic properties of shift invariant families of coalescing geodesics. We remark that although it has been suggested that coalescence should fail for large $d$, it seems plausible that results analogous to Theorems 3-5 should hold for all $d \geq 2$.

The results described above together address the cardinality of the set $T_0$. Recall that $N$ denotes the maximal number of pairwise disjoint infinite geodesics and is almost surely constant. As a corollary to Theorems 4 and 5 it follows that also $|T_0|$ is almost surely constant. Clearly $|T_0| \leq N$, and due to the coalescence property in Theorem 5 we obtain equality.

**Corollary 7.** With probability one $|T_0|$ is constant and equal to $N$.

The corollary was first established in [1] via the study of coalescence and Busemann functions. A more direct argument, assuming a stronger moment condition, was later given by Nakajima [20].
2.5 Geodesics in higher dimensions

Whether the description of geodesics detailed above remains correct also in higher dimensions is at this point unknown. What we do know is that the argument behind Theorem 3 can be extended to dimensions $d \geq 3$ under minor adjustments; see also [2]. However, the proofs of Theorems 4-6 exploit planarity in a much more fundamental way, and do not extend easily to higher dimensions. On the other hand, the argument of Nakajima [20] shows that Corollary 7 remains valid in all dimensions under the condition that $\mathbb{E}[\exp(\alpha \omega_x)] < \infty$ for some $\alpha > 0$. These properties will be sufficient in order to obtain the partial result stated in Theorem 2.

3 The dual relation in two dimensions

With the background outlined in the previous section we are now ready to prove Theorem 1.

3.1 Coexistence implies existence

The short proof of part (i) is an easy consequence of Corollary 7. Suppose that for some choice of $x_1, x_2, \ldots, x_k$ in $\mathbb{Z}^2$ we have $\mathbb{P}(\mathrm{Coex}(x_1, x_2, \ldots, x_k)) > 0$. By (3) we have $\mathbb{P}(N \geq k) > 0$, and since $N$ is almost surely constant it follows from Corollary 7 that

\[ \mathbb{P}(|\mathcal{I}_0| \geq k) = 1. \]

While the above argument is short, it hides much of the intuition for why the implication holds. We shall therefore give a second argument based on coalescence that may be more instructive, even if no more elementary. This argument will make explicit the heuristic that geodesics are ‘highways to infinity’ along which the different types will have to move in order to escape the competition.

Again by Corollary 7 either $\mathcal{I}_0$ is almost surely infinite, in which case there is nothing to prove, or $\mathbb{P}(|\mathcal{I}_0| = k) = 1$ for some integer $k \geq 1$. Suppose the latter. We shall argue that for any choice of $x_1, x_2, \ldots, x_{k+1}$ in $\mathbb{Z}^2$ we have $\mathbb{P}(\mathrm{Coex}(x_1, x_2, \ldots, x_{k+1})) = 0$.

We first note that for any finite number of competing types, for each infinite geodesic $g$ there may be only one of the competing types that reach infinitely many sites along $g$. Which of them it is can be read out from the Busemann function of $g$; it is the one whose starting position minimizes $B_g(0, x_i)$. (The minimizer is unique due to unique geodesics.) Since there from each point in $\mathbb{Z}^2$ are $k$ infinite geodesics, if there are $k+1$ competing types, then at least one of them will not reach infinitely many sites along any geodesic in $\mathcal{I}_0$. Suppose that the type left out starts at a site $x$. Since for each geodesic in $\mathcal{I}_x$ there is a geodesic in $\mathcal{I}_0$ with which it coalesces (as of Theorem 6), it follows that for each geodesics $g \in \mathcal{I}_x$ the type starting at $x$ will be closer to at most finitely many sites along $g$. Choose $n$ so that these sites are all within distance $n$ from $x$. Next note that for only finitely many sites in $\mathbb{Z}^2$ the geodesic to $x$ will diverge from all geodesics in $\mathcal{I}_x$ within distance $n$ from $x$. Consequently, all but finitely many sites in $\mathbb{Z}^2$ will lie closer to some other starting point, implying that the $k+1$ types do not coexist.
3.2 Existence implies coexistence

Central in the proof of part (ii) is the linearity of Busemann functions. According to Theorem 4 every geodesic has an asymptotically linear Busemann function, and by Theorem 5 there is a deterministic set $\mathcal{C}$ of linear functionals that are the ones that `describe' these Busemann functions. In addition, by Theorems 6 and 7 for each linear functional $\rho \in \mathcal{C}$ there is almost surely a unique geodesic $g \in \mathcal{G}$ whose Busemann function is asymptotically linear to $\rho$, and any geodesic $g'$ with Busemann function asymptotically linear to $\rho$ coalesces with $g$. In particular $|\mathcal{G}| = |\mathcal{C}|$ almost surely, and we shall below write $B_\rho$ for the Busemann function of the (almost surely unique) geodesic $g$ corresponding to $\rho$.

Now, let $k$ be an integer and suppose that $|\mathcal{G}| \geq k$ with positive probability. Then, indeed, $|\mathcal{G}| = |\mathcal{C}| \geq k$ almost surely. Fix $\varepsilon > 0$ and let $\rho_1, \rho_2, \ldots, \rho_k$ be distinct elements of $\mathcal{C}$. In order to show that $\mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 1 - \varepsilon$ for some choice of $x_1, x_2, \ldots, x_k$, we shall choose these points so that with probability $1 - \varepsilon$ we have $B_{\rho_i}(x_i, x_j) < 0$ for all $i = 1, 2, \ldots, k$ and $j \neq i$. On this event, for each $i$, the site $x_i$ is closer to all points along the geodesic corresponding to $\rho_i$ than any of the $x_j$ for $j \neq i$, implying that Coex($x_1, x_2, \ldots, x_k$) occurs.

Given $\rho \in \mathcal{C}$, $z \in \mathbb{Z}^2$, $\delta > 0$ and $M \geq 1$ we let $A_\rho(z, \delta, M)$ denote the event that

$$|B_\rho(z, y) - \rho(y - z)| < \delta|y - z| \quad \text{for all } |y - z| \geq M.$$ 

Due to linearity of geodesics (Theorems 4 and 5) there exists for every $\rho \in \mathcal{C}$ and $\delta, \gamma > 0$ an $M < \infty$ such that

$$\mathbb{P}(A_\rho(z, \delta, M)) > 1 - \gamma \quad \text{for every } z \in \mathbb{Z}^2. \quad (6)$$

We further introduce the following notation for plane regions related to $\rho$:

$$H_\rho(z, \delta) := \{y \in \mathbb{R}^2 : \rho(y - z) \leq -\delta|y - z|\};$$

$$C_\rho(z, \delta) := \{y \in \mathbb{R}^2 : |\rho(y - z)| \leq \delta|y - z|\}.$$ 

Note that on the event $A_\rho(z, \delta, M)$ we have for all $y \in H_\rho(z, \delta)$ such that $|y - z| \geq M$ that $B_\rho(z, y) < 0$. Hence, $H_\rho(z, \delta)$ corresponds to sites that are likely to be at a further distance to far out vertices along the geodesic corresponding to $\rho$ compared to $z$.

Given $\rho_1, \rho_2, \ldots, \rho_k$ we now choose $\delta > 0$ so that the cones $C_{\rho_i}(0, \delta)$, for $i = 1, 2, \ldots, k$, intersect only at the origin. Next, we choose $M$ large so that for all $i$

$$\mathbb{P}(A_{\rho_i}(z, \delta, M)) > 1 - \varepsilon/k.$$ 

Finally, due to the choice of $\delta$ we may choose $x_1, x_2, \ldots, x_k$ so that $|x_i - x_j| \geq M$ for all $i \neq j$ and such that for each $i$ the set $H_{\rho_i}(x_i, \delta)$ contains $x_j$ for all $j \neq i$. (For instance, position the sites on a circle of large radius, in positions roughly corresponding to the directions of $\rho_1, \rho_2, \ldots, \rho_k$.) Due to these choices we will on the event $\bigcap_{i=1,2,\ldots,k} A_{\rho_i}(x_i, \delta, M)$, which occurs with probability at least $1 - \varepsilon$, have for all $i = 1, 2, \ldots, k$ that $B_{\rho_i}(x_i, x_j) < 0$ for all $j \neq i$, as required.

It remains to show that if $|\mathcal{G}| = \infty$ with positive probability, then it is possible to find a sequence $(x_i)_{i \geq 1}$ for which Coex($x_1, x_2, \ldots$) occurs with probability close to one. If $|\mathcal{G}| = \infty$ with positive probability, then it does with probability one, and $|\mathcal{C}| = \infty$ almost surely. Let $(\rho_i)_{i \geq 1}$ be an increasing sequence in $\mathcal{C}$ (considered as a sequence in $[0, 2\pi]$). By symmetry we
may assume that each \( \rho_i \) corresponds to an angle in \((0, \pi/2)\). Fix \( \varepsilon > 0 \) and set \( \varepsilon_i = \varepsilon/2^i \). We choose \( \delta_1 \) so that \( C_{\rho_1}(0, \delta_1) \) intersect each of the lines \( C_{\rho_j}(0, 0) \), for \( j \geq 2 \), only at the origin, and \( M_1 \) so that \( \mathbb{P}(A_{\rho_1}(z, \delta_1, M_1)) > 1 - \varepsilon_1 \). Inductively we choose \( \delta_i \) so that \( C_{\rho_1}(0, \delta_i) \) intersects each cone \( C_{\rho_j}(0, \delta_j) \) for \( j < i \) and each line \( C_{\rho_j}(0, 0) \) for \( j > i \) only at the origin, and \( M_i \) so that \( \mathbb{P}(A_{\rho_1}(z, \delta_i, M_i)) > 1 - \varepsilon_i \).

For any sequence \((x_i)_{i \geq 1}\), we have \( \mathbb{P}(\bigcap_{i \geq 1} A_{\rho_1}(x_i, \delta_i, M_i)) > 1 - \varepsilon \). It remains only to verify that we may choose the sequence \((x_i)_{i \geq 1}\) so that for each \( i \geq 1 \) we have \( x_i - x_j \geq M_i \) and \( x_j \in H_{\rho_i}(x_i, \delta_i) \) for all \( j \neq i \). For \( i \geq 1 \) we take \( \nu_{i+1} \in \mathbb{Z}^2 \) such that \( |\nu_{i+1}| > \max\{M_1, M_2, \ldots, M_{i+1}\} \), \( \rho_{i+1}(\nu_{i+1}) > \delta_{i+1}|\nu_{i+1}| \) and \( \rho_j(\nu_{i+1}) < -\delta_j|\nu_{i+1}| \) for all \( j \leq i \). We note that this is possible since the sequence \( (\rho_i)_{i \geq 1}\) is increasing and the cone-shaped regions \( C_{\rho_i}(0, \delta_i) \) and \( C_{\rho_j}(0, \delta_j) \) for \( i \neq j \) intersect only at the origin. Finally, take \( x_1 = (0, 0) \), and for \( i \geq 1 \) set \( x_{i+1} = x_i + \nu_{i+1} \).

### 4 Partial duality in higher dimensions

The proof of Theorem 2 is similar to that of Theorem 1. So, instead of repeating all details we shall only outline the proof and indicate at what instances our current understanding of the higher dimensional case inhibits us from deriving the full duality. In the sequel we assume \( d \geq 2 \).

The proof of the first part of the theorem is completely analogous. Suppose that

\[
\mathbb{P}(\text{Coex}(x_1, x_2, \ldots, x_k)) > 0
\]

for some choice of \( x_1, x_2, \ldots, x_k \) in \( \mathbb{Z}^d \), possibly infinitely many. Then \( \mathcal{N} \geq k \) almost surely, and by (Nakajima's version, which requires an exponential moment assumption, of) Corollary 7 we have \( |\mathcal{Z}_0| \geq k \) almost surely.

For the second part of the argument we will need to modify slightly the approach from the two dimensional case. In the general case we do not know that every geodesic has an asymptotically linear Busemann function. However, from (the higher dimensional version of) Theorem 8 we know that if the shape has at least \( k \) sides (that is, tangent hyperplanes), then, almost surely, there are \( k \) geodesics in \( \mathcal{Z}_0 \) which all have asymptotically linear Busemann functions described by different linear functionals. Based on this we may repeat the proof of part (ii) of Theorem 1 to obtain coexistence of \( k \) types with probability arbitrarily close to one.

In the case the shape has infinitely many sides, then with probability one there are infinitely many geodesics in \( \mathcal{Z}_0 \) with asymptotically linear Busemann functions, all described by different linear functionals. Let \( (\rho_i)_{i \geq 1} \) be a sequence of such linear functionals. Denote by \( L_i \) the intersection of the hyperplane \( \{x \in \mathbb{R}^d : \rho_i(x) = 0\} \) and the \( x_1x_2 \)-plane, i.e., the plane spanned by the first two coordinate vectors. Each \( L_i \) has dimension zero, one or two, and by exploiting the symmetries of \( \mathbb{Z}^d \) we may assume that sequence \( (\rho_i)_{i \geq 1} \) is chosen so that they all have dimension one. Each \( L_i \) is then a line through the origin in the \( x_1x_2 \)-plane, and by restricting to a subsequence we may assume that the sequence \( (\nu_i)_{i \geq 1} \) of normal vectors of these lines is monotone (considered as elements in \([0, 2\pi)\)). We may now proceed and select a sequence of points \((x_i)_{i \geq 1}\) in the \( x_1x_2 \)-plane in an analogous manner as in the two-dimensional case, leading to coexistence of infinitely many types with probability arbitrarily close to one.
References

[1] D. Ahlberg and C. Hoffman. Random coalescing geodesics in first-passage percolation. Preprint, see arXiv:1609.02447.

[2] A. Auffinger, M. Damron, and J. Hanson. 50 years of first-passage percolation, volume 68 of University Lecture Series. American Mathematical Society, Providence, RI, 2017.

[3] Y. Bakhtin, E. Cator, and K. Khanin. Space-time stationary solutions for the Burgers equation. J. Amer. Math. Soc., 27(1):193–238, 2014.

[4] H. Busemann. The geometry of geodesics. Academic Press Inc., New York, N. Y., 1955.

[5] E. Cator and L. P. R. Pimentel. Busemann functions and equilibrium measures in last passage percolation models. Probab. Theory Related Fields, 154(1-2):89–125, 2012.

[6] J. T. Cox and R. Durrett. Some limit theorems for percolation processes with necessary and sufficient conditions. Ann. Probab., 9(4):583–603, 1981.

[7] M. Damron and J. Hanson. Busemann functions and infinite geodesics in two-dimensional first-passage percolation. Comm. Math. Phys., 325(3):917–963, 2014.

[8] M. Damron and M. Hochman. Examples of nonpolygonal limit shapes in i.i.d. first-passage percolation and infinite coexistence in spatial growth models. Ann. Appl. Probab., 23(3):1074–1085, 2013.

[9] O. Garet and R. Marchand. Coexistence in two-type first-passage percolation models. Ann. Appl. Probab., 15(1A):298–330, 2005.

[10] O. Häggström and R. Pemantle. First passage percolation and a model for competing spatial growth. J. Appl. Probab., 35(3):683–692, 1998.

[11] J. M. Hammersley and D. J. A. Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif, pages 61–110. Springer-Verlag, New York, 1965.

[12] C. Hoffman. Coexistence for Richardson type competing spatial growth models. Ann. Appl. Probab., 15(1B):739–747, 2005.

[13] C. Hoffman. Geodesics in first passage percolation. Ann. Appl. Probab., 18(5):1944–1969, 2008.

[14] C. D. Howard. Models of first-passage percolation. In Probability on discrete structures, volume 110 of Encyclopaedia Math. Sci., pages 125–173. Springer, Berlin, 2004.

[15] C. D. Howard and C. M. Newman. Geodesics and spanning trees for Euclidean first-passage percolation. Ann. Probab., 29(2):577–623, 2001.

[16] H. Kesten. Aspects of first passage percolation. In École d’été de probabilités de Saint-Flour, XIV—1984, volume 1180 of Lecture Notes in Math., pages 125–264. Springer, Berlin, 1986.
[17] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B*, 30:499–510, 1968.

[18] C. Licea and C. M. Newman. Geodesics in two-dimensional first-passage percolation. *Ann. Probab.*, 24(1):399–410, 1996.

[19] C. Licea, C. M. Newman, and M. S. T. Piza. Superdiffusivity in first-passage percolation. *Probab. Theory Related Fields*, 106(4):559–591, 1996.

[20] S. Nakajima. Ergodicity of the number of infinite geodesics originating from zero. Preprint, see arXiv:1807.05900.

[21] C. M. Newman. A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1017–1023, Basel, 1995. Birkhäuser.

[22] C. M. Newman and M. S. T. Piza. Divergence of shape fluctuations in two dimensions. *Ann. Probab.*, 23(3):977–1005, 1995.

[23] D. Richardson. Random growth in a tessellation. *Proc. Cambridge Philos. Soc.*, 74:515–528, 1973.

[24] R. T. Smythe and J. C. Wierman. *First-passage percolation on the square lattice*, volume 671 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978.