Existence, construction and extension of continuous solutions of an iterative equation with multiplication

Chaitanya Gopalakrishna¹, Murugan Veerapazham², Suyun Wang³ & Weinian Zhang⁴,*

¹Statistics and Mathematics Unit, Indian Statistical Institute, R.V. College Post, Bengaluru 560059, India;
²Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Mangalore 575025, India;
³School of Mathematics, Lanzhou City University, Lanzhou 730070, China;
⁴School of Mathematics, Sichuan University, Chengdu 610064, China

Email: cberbalaje@gmail.com, murugan@nitk.edu.in, wangsy1970@163.com, matzwn@163.com

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Abstract The iterative equation is an equality with an unknown function and its iterates, most of which found from references are a linear combination of those iterates. In this paper, we work on an iterative equation with multiplication of iterates of the unknown function. First, we use an exponential conjugation to reduce the equation on \( \mathbb{R}_+ \) to the form of the linear combination on \( \mathbb{R} \), but those known results on the linear combination were obtained on a compact interval or a neighborhood near a fixed point. We use the Banach contraction principle to give the existence, uniqueness and continuous dependence of continuous solutions on \( \mathbb{R}_+ \) that are Lipschitzian on their ranges, and construct its continuous solutions on \( \mathbb{R}_+ \) sewing piece by piece. We technically extend our results on \( \mathbb{R}_+ \) to \( \mathbb{R}_- \) and show that none of the pairs of solutions obtained on \( \mathbb{R}_+ \) and \( \mathbb{R}_- \) can be combined at the origin to get a continuous solution of the equation on the whole \( \mathbb{R} \), but can extend those given on \( \mathbb{R}_+ \) to obtain continuous solutions on the whole \( \mathbb{R} \).

Keywords functional equation, iteration, nonlinear combination, contraction principle

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1 Introduction

Consider a map \( f : E \to E \) on a nonempty set \( E \). Its \( n \)-th-order iterate, denoted by \( f^n \), is defined recursively by \( f^0 = \text{id} \), the identity map on \( E \), and \( f^{n+1} = f \circ f^n \). Being an important operation in the present era of informatics, iteration gets more and more attractive to researchers and attention (see, for example, the book [4] and the survey [1]) is paid to those functional equations involving iteration, called iterative equations. The general form of such equations can be presented as

\[
\Phi(x, f(x), f^2(x), \ldots, f^n(x)) = 0,
\]

(1.1)
where $\Phi$ is a given function and $f$ is unknown.

There are many published papers on (1.1) with specific $\Phi$’s. In the case where $\Phi(x_0,\ldots,x_n) = x_n - F(x_0)$, (1.1) is of the form $f^n = F$, which is the iterative root problem [3, 10]. If $\Phi(x_0, x_1) = G(x_0, x_1) - x_1$, then (1.1) is of the form $G(x, f(x)) = f(x)$, to which the problem of the invariant curve $y = f(x)$ for the planar mapping $(x, y) \to (y, G(x, y))$ can be reduced [4]. In the case where $\Phi(x_0,\ldots,x_n) := a_1x_1 + \cdots + a_nx_n - F(x_0)$, a linear combination, the equation (1.1) becomes the following form:

$$a_1f(x) + \cdots + a_nf^n(x) = F(x),$$

(1.2)
called the polynomial-like iterative equation, which was investigated in various aspects such as continuous solutions [2, 5, 9, 12], differentiable solutions [16], convex solutions and decreasing solutions [13] and equivariant solutions [18].

It is also interesting to discuss $\Phi$ of the nonlinear combination, but no results on the nonlinear combination are found except for [6, 8, 11] and [14]. Actually, in [6, 8, 11], the general $\Phi$’s are considered and $\Phi$ are not Lipschitzian, which have great difficulties in discussion.

In this paper, we discuss the iterative equation (1.1) with multiplication, i.e.,

$$\Phi(x_0, x_1,\ldots, x_n) = \prod_{k=1}^{n} x_{k}^{\alpha_k} - G(x_0),$$

which is not Lipschitzian either. In this case, the equation can be presented as

$$(g(x))^{\alpha_1}(g^2(x))^{\alpha_2}\cdots(g^n(x))^{\alpha_n} = G(x),$$

(1.3)
where $G$ is given and $g$ is unknown. In Section 2, we see that the problem of solving (1.3) on $\mathbb{R}_+ := (0, \infty)$ can be reduced with an exponential conjugation to solving the standard polynomial-like iterative equation (1.2) on the whole $\mathbb{R}$. Noting that those known results on (1.2) were given either on a compact interval or on a neighborhood near a fixed point, our first effort will be made on (1.2) on the whole $\mathbb{R}$. Then we approach solutions of (1.3) on $\mathbb{R}_+$ in a twofold process. First, using the Banach contraction principle, we give sufficient conditions for the existence and uniqueness of continuous solutions of (1.3) on $\mathbb{R}_+$ that are Lipschitzian on their ranges in Section 3. We also prove that the obtained solution depends on $G$ continuously. Then using the second method, in Section 4 we construct its continuous solutions sewing piece by piece as done in [12, 15]. In Section 5, we technically extend our results on $\mathbb{R}_+$ to $\mathbb{R}_- := (-\infty, 0)$, and show that none of the pairs of solutions obtained on $\mathbb{R}_+$ and $\mathbb{R}_-$ can be combined at the origin to get a continuous solution of (1.3) on the whole $\mathbb{R}$. On the other hand, we also give some results on continuous solutions of (1.3) on the whole $\mathbb{R}$ extending those given on $\mathbb{R}_+$. Finally, in Section 6, we illustrate our results with examples and make some remarks on problems open for further discussion.

2 Preliminaries

Let $C_b(\mathbb{R})$ (resp. $C_b(\mathbb{R}_+)$) consist of all the bounded continuous self-maps of $\mathbb{R}$ (resp. $\mathbb{R}_+$). Then $C_b(\mathbb{R})$ is a Banach space with the uniform norm $\| \cdot \|$, defined by $\| f \| := \text{sup}\{|f(x)| : x \in \mathbb{R}\}$. Consider $g$ on $\mathbb{R}_+$. We can use the exponential map $\psi(x) = e^x$ to conjugate $g$ to a map on the whole $\mathbb{R}$, i.e., let $f(x) := \log g(e^x)$, a map from $\mathbb{R}$ into $\mathbb{R}$ (one-to-one if $g$ is one-to-one), and reduce (1.3) to the polynomial-like equation

$$a_1f(x) + a_2f^2(x) + \cdots + a_nf^n(x) = F(x), \quad x \in \mathbb{R},$$

(2.1)
where $F(x) := \log G(e^x)$.

**Proposition 2.1.** The map $g$ is a solution (resp. unique solution) of (1.3) in $X \subseteq C_b(\mathbb{R}_+)$ if and only if $f(x) := \psi^{-1}(g(\psi(x)))$ is a solution (resp. unique solution) of (2.1) in $Y \subseteq C_b(\mathbb{R})$, where $\psi(x) = e^x$ and $Y = \{ \psi^{-1} \circ g \circ \psi : g \in X \}$. 

Proof. Let \( g \) be a solution of (1.3) in \( \mathcal{X} \). Since \( \psi \) is a homeomorphism of \( \mathbb{R} \) onto \( \mathbb{R}_+ \), clearly \( \mathcal{Y} \subseteq \mathcal{C}_b(\mathbb{R}) \) and \( f \in \mathcal{Y} \). Also, for each \( x \in \mathbb{R} \), we have
\[
\sum_{k=1}^{n} a_k f^k(x) = \sum_{k=1}^{n} a_k \log g^k(e^x) = \log \left( \prod_{k=1}^{n} (g^k(e^x))^{a_k} \right) \log G(e^x) = F(x),
\]
implying that \( f \) is a solution of (2.1) on \( \mathbb{R} \). The converse follows similarly. Now, in order to prove the uniqueness, assume that (1.3) has a unique solution in \( \mathcal{X} \) and suppose that \( f_1 \) and \( f_2 \) are any two solutions of (2.1) in \( \mathcal{Y} \). Then by the “if” part of what we have proved above, there exist the solutions \( g_1 \) and \( g_2 \) of (1.3) in \( \mathcal{X} \) such that \( f_1 = \psi^{-1} \circ g_1 \circ \psi \) and \( f_2 = \psi^{-1} \circ g_2 \circ \psi \). By our assumption, we have \( g_1 = g_2 \) and therefore \( f_1 = f_2 \). The proof of the converse is similar.

By Proposition 2.1, it suffices to prove the existence for (2.1) on the whole \( \mathbb{R} \) in order to prove the existence of the solution of (1.3) on \( \mathbb{R}_+ \).

Let \( I := [a, b] \) and \( J := [c, d] \) be compact intervals in \( \mathbb{R} \) and \( \mathbb{R}_+ \) respectively with nonempty interiors. Let \( \mathcal{C}(\mathbb{R}, I) \) (resp. \( \mathcal{C}(I, I) \)) be the set of all the continuous maps of \( \mathbb{R} \) (resp. \( I \)) into \( I \). Similarly, we define \( \mathcal{C}(\mathbb{R}_+, J) \) and \( \mathcal{C}(J, J) \). For any \( f \in \mathcal{C}(\mathbb{R}, I) \) or \( \mathcal{C}(I, I) \), let \( \|f\| = \sup \{|f(x)| : x \in I\} \), and for any \( g \) in \( \mathcal{C}(\mathbb{R}_+, J) \) or \( \mathcal{C}(J, J) \), let \( \|g\| = \sup \{|g(x)| : x \in J\} \). For any map \( f \), let \( \mathcal{R}(f) \) denote the range of \( f \). For \( M, \delta \geq 0 \), let
\[
\mathcal{F}_{I, \delta, M}(\mathbb{R}) := \{ f \in \mathcal{C}_b(\mathbb{R}) : \mathcal{R}(f) = I, f(a) = a, f(b) = b \text{ and } \delta(x - y) \leq f(x) - f(y) \leq M(x - y), \\
\forall x, y \in I \text{ with } x \geq y \},
\]
\[
\mathcal{G}_{J, \delta, M}(\mathbb{R}_+) := \{ g \in \mathcal{C}_b(\mathbb{R}_+) : \mathcal{R}(g) = J, g(c) = c, g(d) = d \text{ and } (x/y)^\delta \leq g(x)/g(y) \leq (x/y)^M, \\
\forall x, y \in J \text{ with } x \geq y \}.
\]
Then it can be observed that \( \mathcal{F}_{I, \delta, M}(\mathbb{R}) \subseteq \mathcal{F}_{I, \delta_1, M_1}(\mathbb{R}) \) and \( \mathcal{G}_{J, \delta, M}(\mathbb{R}_+) \subseteq \mathcal{G}_{J, \delta_1, M_1}(\mathbb{R}_+) \), whenever \( \delta \geq \delta_1 \geq 0 \) and \( M_1 \geq M \geq 0 \).

**Proposition 2.2.** Let \( M, \delta \geq 0 \). Then \( g \in \mathcal{G}_{J, \delta, M}(\mathbb{R}_+) \) if and only if \( \psi^{-1} \circ g \circ \psi \in \mathcal{F}_{I, \delta, M}(\mathbb{R}) \), where \( \psi(x) = e^x \) and \( I = \log(\bar{J}) := \{ \log x : x \in J \} \).

**Proof.** Let \( g \in \mathcal{G}_{J, \delta, M}(\mathbb{R}_+) \) and \( I := \log(\bar{J}) = [a, b] \). Then \( a = \log c \) and \( b = \log d \). Clearly, \( f := \psi^{-1} \circ g \circ \psi \in \mathcal{C}_b(\mathbb{R}) \). Also, we have \( f(a) = \log g(e^a) = \log g(c) = \log c = a \), and similarly, \( f(b) = b \). So \( I \subseteq \mathcal{R}(f) \). The reverse inclusion follows by definitions of \( f \) and \( I \), because \( \mathcal{R}(g) = J \). Therefore \( \mathcal{R}(f) = I \).

Next, let \( x, y \in I \) with \( x \geq y \). Then there exist \( u, v \in J \) with \( u \geq v \) such that \( x = \log u \) and \( y = \log v \). So from the assumption on \( g \), we have
\[
\left( \frac{u}{v} \right)^\delta \leq \frac{g(u)}{g(v)} \leq \left( \frac{u}{v} \right)^M,
\]
implying that
\[
\delta \log \left( \frac{e^u}{e^v} \right) \leq \log \left( \frac{g(e^u)}{g(e^v)} \right) \leq M \log \left( \frac{e^u}{e^v} \right),
\]
i.e., \( \delta(x - y) \leq f(x) - f(y) \leq M(x - y) \). Therefore, \( f \in \mathcal{F}_{I, \delta, M}(\mathbb{R}) \). The converse follows similarly.

**Proposition 2.3.** If \( M < 1 \) or \( \delta > 1 \), then \( \mathcal{G}_{J, \delta, M}(\mathbb{R}_+) = \emptyset \). If \( M = 1 \) or \( \delta = 1 \), then \( \mathcal{G}_{J, \delta, M}(\mathbb{R}_+) = \{ g \in \mathcal{C}_b(\mathbb{R}_+) : |g| = \text{id} \} \).

**Proof.** Let \( g \in \mathcal{G}_{J, \delta, M}(\mathbb{R}_+) \). Then by Proposition 2.2, \( f := \psi^{-1} \circ g \circ \psi \in \mathcal{F}_{I, \delta, M}(\mathbb{R}) \), where \( \psi(x) = e^x \) and \( I = \log(\bar{J}) \). So for any \( x, y \in I \) such that \( x \geq y \), we have
\[
\delta(x - y) \leq f(x) - f(y) \leq M(x - y).
\]
If \( M < 1 \), then by setting \( y = a \) in (2.2), we get \( f(x) < x \), \( \forall x \in I \) with \( x > a \). This is a contradiction to the fact that \( f(b) = b \), because \( b > a \). So \( \mathcal{F}_{I, \delta, M}(\mathbb{R}) = \emptyset \) and hence \( \mathcal{G}_{J, \delta, M}(\mathbb{R}_+) = \emptyset \), whenever \( M < 1 \). A similar argument holds when \( \delta > 1 \).
If $M = 1$, then from (2.2), we have
\[ f(x) - f(y) \leq x - y, \quad \forall x, y \in I \text{ with } x \geq y. \] (2.3)

Now for $x = b$, (2.3) implies that $f(y) \geq y, \forall y \in I$ with $y < b$. Moreover, setting $y = a$ in (2.3), we have $f(x) \leq x, \forall x \in I$ with $x > a$. Thus $f(x) = x, \forall x \in I$, and therefore $f|_I = \text{id}$. This implies that $g|_I = \text{id}$. The reverse inclusion is trivial. So $G_{I, \delta, M}(\mathbb{R}_+^+; \delta \text{ where } 1 \leq \delta \leq M)$ of (2.3) implies that $\mathcal{G}_{I, \delta, M}(\mathbb{R}_+^+; \delta \text{ where } 1 \leq \delta \leq M)$.

Lemma 2.5

Lemma 2.6

Proposition 2.4. The set $\mathcal{F}_{I, \delta, M}(\mathbb{R})$ is a complete metric space under the metric induced by $\| \cdot \|$. 

Proof. It can be easily seen that $\mathcal{F}_{I, \delta, M}(\mathbb{R})$ is a closed subset of $C_b(\mathbb{R})$. So, since $C_b(\mathbb{R})$ is complete with respect to the metric induced by $\| \cdot \|$, it follows that $\mathcal{F}_{I, \delta, M}(\mathbb{R})$ is also complete. 

In view of Proposition 2.3, we cannot seek the solutions of (1.3) without imposing the conditions on $M$ and $\delta$. So, henceforth we assume that $0 < \delta \leq 1 \leq M$. We need the following six technical lemmas, the last three of which look similar to some lemmas given in [7] but we have to rewrite their proofs carefully because of the following difference: it is assumed in [7] that $f \in C(I, I)$ and $f$ is a homeomorphism of $I$ onto itself, implying that $f^{-1}$ is well defined on the whole domain $f$ of $f$, but this paper deals with $f \in C(\mathbb{R}, I)$ satisfying that $f|_I$ is a homeomorphism of $I$ onto itself. So $f$ is not a homeomorphism on $\mathbb{R}$, i.e., $f^{-1}$ is not defined on the whole $\mathbb{R}$. In this case, we can consider only the inverse of $f|_I$. For a specific example, the conclusion $L_f \in \mathcal{F}_{I, K_0, K_1}(\mathbb{R})$, made in [7, Lemma 3.2], is not true here, simply because we have defined $L_f$ only on $I$. However, even if we define it on the whole of $\mathbb{R}$, it does not belong to $\mathcal{F}_{I, K_0, K_1}(\mathbb{R})$, because in that case $\mathcal{R}(L_f) \neq I$. So, in view of this, we include their proofs here in order to avoid confusion.

Lemma 2.5 (See [17]). Let $f, g \in C(I, I)$ satisfy $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$, with $M \geq 1$. Then
\[ \|f^k - g^k\|_I \leq \left( \sum_{j=0}^{k-1} M^j \right) \|f - g\|_I \quad \text{for } k = 1, 2, \ldots \] (2.4)

Lemma 2.6 (See [17]). Let $f \in C(I, I)$ satisfy $f(a) = a$, $f(b) = b$ and $\delta(x - y) \leq f(x) - f(y) \leq M(x - y)$ for all $x, y \in I$ with $x \geq y$, where $0 < \delta \leq 1 \leq M$. Then $f$ is a homeomorphism of $I$ onto itself and
\[ \frac{1}{M}(x - y) \leq f^{-1}(x) - f^{-1}(y) \leq \frac{1}{\delta}(x - y) \] (2.5)

for each $x, y \in I$ with $x \geq y$.

Lemma 2.7 (See [17]). Let $f, g : I \rightarrow I$ be homeomorphisms satisfy $\delta(x - y) \leq f(x) - f(y) \leq M(x - y)$ and $\delta(x - y) \leq g(x) - g(y) \leq M(x - y)$ for all $x, y \in I$ with $x \geq y$, where $0 < \delta \leq 1 \leq M$. Then
\[ \delta \|f^{-1} - g^{-1}\|_I \leq \|f - g\|_I \leq M\|f^{-1} - g^{-1}\|_I. \]

For $\alpha_k \geq 0$ (1 \leq k \leq n) with \( \sum_{k=1}^{n} \alpha_k = 1 \) and $f \in \mathcal{F}_{I, \delta, M}(\mathbb{R})$, define $L_f : I \rightarrow I$ by
\[ \alpha_k \in (\mathbb{R}, I), \quad \text{defining } L_f : I \rightarrow I \quad \text{by} \]
\[ L_f(x) = \alpha_1 x + \alpha_2 f(x) + \cdots + \alpha_n f^{n-1}(x), \quad x \in I. \]

Lemma 2.8. Let $f \in \mathcal{F}_{I, \delta, M}(\mathbb{R})$, where $0 < \delta \leq 1 \leq M$. Then $L_f(a) = a$, $L_f(b) = b$, $\mathcal{R}(L_f) = I$ and
\[ K_0(x - y) \leq L_f(x) - L_f(y) \leq K_1(x - y) \] (2.6)

for each $x, y \in I$ with $x \geq y$, where $K_0 := \sum_{k=1}^{n} \alpha_k \delta^{k-1}$ and $K_1 := \sum_{k=1}^{n} \alpha_k M^{k-1}$.

Proof. It can be easily seen that $L_f(a) = a$, $L_f(b) = b$ and $\mathcal{R}(L_f) = I$. Also, for any $x, y \in I$ with $x \geq y$, we have
\[ L_f(x) - L_f(y) = \sum_{k=1}^{n} \alpha_k f^{k-1}(x) - \sum_{k=1}^{n} \alpha_k f^{k-1}(y) = \sum_{k=1}^{n} \alpha_k (f^{k-1}(x) - f^{k-1}(y)) \]
This completes the proof.

Lemma 2.9. Let $0 < \delta \leq 1 \leq M$ and $f \in \mathcal{F}_{I, \delta, M}(\mathbb{R})$. Then
\[
\frac{1}{K_1}(x - y) \leq L_f^{-1}(x) - L_f^{-1}(y) \leq \frac{1}{K_0}(x - y)
\]
for each $x, y \in I$ with $x \geq y$, where $K_0$ and $K_1$ are as in Lemma 2.8.

Proof. We see that it follows from the proof of Lemma 2.6, by noting from Lemma 2.8 that $L_f(a) = a,$ $L_f(b) = b$, $\mathcal{R}(L_f) = I$ and $L_f$ satisfies (2.6) with $0 < K_0 \leq 1 \leq K_1$.

Lemma 2.10. Let $0 < \delta \leq 1 \leq M$ and $f_1, f_2 \in \mathcal{F}_{I, \delta, M}(\mathbb{R})$. Then
\[
\|L_{f_1} - L_{f_2}\|_I \leq K_2\|f_1 - f_2\|_I \quad \text{and} \quad \|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \leq \frac{K_2}{K_0}\|f_1 - f_2\|_I,
\]
where $K_0$ and $K_1$ are as in Lemma 2.8 and $K_2 := \sum_{k=2}^{n} \alpha_k(\sum_{j=0}^{k-2} M^j)$.

Proof. Let $f_1, f_2 \in \mathcal{F}_{I, \delta, M}(\mathbb{R})$. Then for each $x \in I$, we have
\[
\|L_{f_1} - L_{f_2}\|_I \leq K_2\|f_1 - f_2\|_I
\]
Moreover, since $L_{f_1}, L_{f_2} : I \to I$ are homeomorphisms satisfying (2.6) with $0 < K_0 \leq 1 \leq K_1$, by Lemma 2.7, we have
\[
\|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \leq \frac{1}{K_0}\|L_{f_1} - L_{f_2}\|_I.
\]
Therefore, (2.8) and (2.9) together imply that
\[
\|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \leq \frac{K_2}{K_0}\|f_1 - f_2\|_I.
\]
This completes the proof.
3 Unique Lipschitzian solution on $\mathbb{R}_+$

In this section, we apply Banach’s contraction theorem to find a unique continuous solution of (1.3) on $\mathbb{R}_+$ that is Lipschitzian on its range.

**Theorem 3.1.** Let $0 < \alpha_1 < 1$, $\alpha_k \geq 0$ for $2 \leq k \leq n$ such that $\sum_{k=1}^{n} \alpha_k = 1$ and $G \in \mathcal{G}_{J,K_1,K_0,M}(\mathbb{R}_+)$, where $J = [c,d]$, $c < d$ and $0 < \delta \leq 1 \leq M$. Let $K_0$, $K_1$ and $K_2$ be as defined in Lemmas 2.8 and 2.10, respectively. If $K_2 < K_0$, then (1.3) has a unique solution in $\mathcal{G}_{J,\delta,M}(\mathbb{R}_+)$.

**Proof.** Let $G \in \mathcal{G}_{J,K_1,K_0,M}(\mathbb{R}_+)$, $a := \log c$ and $b := \log d$. Then we obtain the interval $I = [a,b]$ with $a < b$, which satisfies $I = \log J$. By Proposition 2.2, we have $F := \psi^{-1} \circ G \circ \psi \in \mathcal{F}_{I,K_1,K_0,M}(\mathbb{R})$, where $\psi(x) = e^x$.

Define $T : \mathcal{F}_{I,\delta,M}(\mathbb{R}) \to \mathcal{C}_{\delta}(\mathbb{R})$ by $Tf(x) = L_{f_1}^{-1}(F(x))$, $x \in \mathbb{R}$. By the definitions of $F$ and $L_f$, we have $Tf(a) = a$ and $Tf(b) = b$. This implies that $I \subseteq \mathcal{R}(Tf)$. Also, since $L_f^{-1} : I \to I$, we have $\mathcal{R}(L_f^{-1}) \subseteq I$, and therefore $\mathcal{R}(Tf) \subseteq I$. So, $\mathcal{R}(Tf) = I$. Furthermore, for any $x, y \in I$ with $x \geq y$, as $F \in \mathcal{F}_{I,K_1,K_0,M}(\mathbb{R})$, we have

$$
Tf(x) - Tf(y) = L_{f_1}^{-1}(F(x)) - L_{f_1}^{-1}(F(y)) \leq \frac{1}{K_0}(F(x) - F(y)) \quad \text{(by using Lemma 2.9)}
$$

and

$$
Tf(x) - Tf(y) = L_{f_1}^{-1}(F(x)) - L_{f_1}^{-1}(F(y)) \geq \frac{1}{K_1}(F(x) - F(y)) \quad \text{(again by using Lemma 2.9)}
$$

Hence $Tf \in \mathcal{F}_{I,\delta,M}(\mathbb{R})$, which proves that $T$ is a self-map on $\mathcal{F}_{I,\delta,M}(\mathbb{R})$.

We now prove that $T$ is a contraction. For $f_1, f_2 \in \mathcal{F}_{I,\delta,M}(\mathbb{R})$ and $x \in \mathbb{R}$, we have

$$
|Tf_1(x) - Tf_2(x)| = |L_{f_1}^{-1}(F(x)) - L_{f_2}^{-1}(F(x))|
\leq \|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \quad \text{(since $F(x) \in I$)}
\leq \frac{K_2}{K_0} \|f_1 - f_2\|_I \quad \text{(by using Lemma 2.10)}
\leq \frac{K_2}{K_0} \|f_1 - f_2\|_I,
$$

(3.1)

implying that $\|Tf_1 - Tf_2\| \leq \frac{K_2}{K_0} \|f_1 - f_2\|$. Since $0 < K_2 < K_0$, it follows that $T$ is a contraction. By Proposition 2.4, $\mathcal{F}_{I,\delta,M}(\mathbb{R})$ is complete, and hence by Banach’s contraction principle, $T$ has a unique fixed point in $\mathcal{F}_{I,\delta,M}(\mathbb{R})$, i.e., there exists a unique $f \in \mathcal{F}_{I,\delta,M}(\mathbb{R})$ such that $L_f^{-1}(F(x)) = f(x), \forall x \in \mathbb{R}$, which proves that $f$ is the unique solution of (2.1) in $\mathcal{F}_{I,\delta,M}(\mathbb{R})$. This implies by Propositions 2.1 and 2.2 that $g := \psi \circ f \circ \psi^{-1}$ is the unique solution of (1.3) in $\mathcal{G}_{I,\delta,M}(\mathbb{R}_+)$. The proof is completed. \(\square\)

Although Lemmas 2.8, 2.9 and 2.10 are true for $\alpha_1 \in [0,1]$, in Theorem 3.1 we have assumed that $\alpha_1 \in (0,1)$ for the following reason: if $\alpha_1 = 1$, then $g = G$ is the unique solution of (1.3) so that the problem is trivial. On the other hand, if $\alpha_1 = 0$, then we have $K_0 = \sum_{k=2}^{n} \alpha_k \delta^{k-1}$. So the condition (in Theorem 3.1) $K_2 < K_0$ is not satisfied, because

$$
K_2 = \sum_{k=2}^{n} \alpha_k \left( \sum_{j=0}^{k-2} M^j \right) \geq \sum_{k=2}^{n} \alpha_k \geq \sum_{k=2}^{n} \alpha_k \delta^{k-1} = K_0.
$$

Thus this theorem is not true for $\alpha_1 = 0$. In particular, one cannot solve the iterative root problem $g^n = G$ on $\mathbb{R}_+$ using this theorem.

**Corollary 3.2.** In addition to the assumptions of Theorem 3.1, suppose that $G|_J = \text{id}$. If $K_2 < K_0$, then $G$ is the unique solution of (1.3) in $\mathcal{G}_{I,\delta,M}(\mathbb{R}_+)$. 

Proof. It follows from Theorem 3.1, because clearly $G$ is a solution of (1.3) in $G_{J,\delta,M}(\mathbb{R}_+)$. \qed

Under the assumptions of Theorem 3.1, we show that the solution obtained depends continuously on the function $G$. More precisely, we have the following theorem.

**Theorem 3.3.** In addition to the assumptions of Theorem 3.1, suppose that $G_1 \in G_{J,K_1,\delta,K_0,M}(\mathbb{R}_+)$ and $g_1 \in G_{J,\delta,M}(\mathbb{R}_+)$ satisfy that $\prod_{k=1}^{n}(g_1^k(x))^{\alpha_k} = G_1(x)$ for all $x \in \mathbb{R}_+$. Then

$$\|g - g_1\| \leq \frac{d}{c(K_0 - K_2)}\|G - G_1\|.$$  \hspace{1cm} (3.2)

*Proof.* Given $G, G_1, g$ and $g_1$ as above, let $F(x) = \log G(e^x)$, $F_1(x) = \log G_1(e^x)$, $f(x) = \log g(e^x)$ and $f_1(x) = \log g_1(e^x)$, $\forall x \in \mathbb{R}$. Since $G, G_1 \in G_{J,K_1,\delta,K_0,M}(\mathbb{R}_+)$, by Proposition 2.2, we have $F, F_1 \in F_{I,K_1,\delta,K_0,M}(\mathbb{R})$, where $I = [a, b]$ such that $a = \log c$ and $b = \log d$. Using a similar argument, we see that $f, f_1 \in F_{I,\delta,M}(\mathbb{R})$. Moreover, $f$ and $f_1$ satisfy (2.1) and the equation

$$\sum_{k=1}^{n} \alpha_k f_1^k(x) = F_1(x), \hspace{1cm} x \in \mathbb{R},$$

respectively, implying that $L_f^{-1}(F(x)) = f(x)$ and $L_{f_1}^{-1}(F_1(x)) = f_1(x)$, $\forall x \in \mathbb{R}$. Therefore, for each $x \in \mathbb{R}$,

$$|f(x) - f_1(x)| = |L_f^{-1}(F(x)) - L_{f_1}^{-1}(F_1(x))| \leq \|L_f^{-1}(F(x)) - L_{f_1}^{-1}(F_1(x))\| \leq \|L_f^{-1} - L_{f_1}^{-1}\| + \frac{1}{K_0}|F(x) - F_1(x)| \leq \frac{K_2}{K_0} \|f - f_1\| + \frac{1}{K_0}|F - F_1||,$n

and hence

$$\|f - f_1\| \leq \frac{K_2}{K_0} \|f - f_1\| + \frac{1}{K_0}|F - F_1||.$$  \hspace{1cm} (3.3)

Since $K_2 < K_0$, the above inequality shows

$$\|f - f_1\| \leq \frac{1}{K_0 - K_2} \|F - F_1\|.$$  \hspace{1cm} (3.4)

Since the map $x \mapsto e^x$ is continuously differentiable on $I$ with the bounded derivative, it is a Lipschitzian map on $I$. In fact, $|e^x - e^y| < \epsilon|\log x - \log y|$, $\forall x, y \in I$. So for each $x \in \mathbb{R}_+$, we have

$$|g(x) - g_1(x)| = |e^{f_1(\log x)} - e^{f_1(\log x)}| < \epsilon|f(\log x) - f_1(\log x)| \leq \epsilon\|f - f_1\|,$n

implying that

$$\|g - g_1\| \leq \epsilon\|f - f_1\| \leq \frac{d}{K_0 - K_2} \|F - F_1\|$$  \hspace{1cm} (3.4)

Since the map $x \mapsto \log x$ is continuously differentiable on $J$ with the bounded derivative, it is a Lipschitzian map on $J$. In fact, $|\log x - \log y| < \frac{1}{2}|x - y|$, $\forall x, y \in J$. Therefore, for each $x \in \mathbb{R}$, we have

$$|F(x) - F_1(x)| = |\log G(e^x) - \log G_1(e^x)| \leq \frac{1}{c}|G(e^x) - G_1(e^x)| \leq \frac{1}{c}\|G - G_1\|,$n

implying that

$$\|F - F_1\| \leq \frac{1}{c}\|G - G_1\|.$$  \hspace{1cm} (3.5)

Then (3.2) follows from (3.4) using (3.5). \qed
The assumptions that $0 < \alpha_1 < 1$ and $\sum_{k=1}^{n} \alpha_k = 1$, made in Theorem 3.1, are not strong. In fact, if $\alpha_1 > 1$ or $\sum_{k=1}^{n} \alpha_k > 1$, then we can divide all the exponents $\alpha_k$’s in (1.3) by $\sum_{k=1}^{n} \alpha_k$ to get the normalized equation, but the assumptions on $G$ have to be changed suitably.

4 Infinitely many continuous solutions on $\mathbb{R}_+$

The method used in Section 3 is an application of Banach’s fixed point theorem, which gives a unique Lipschitzian solution. If we are only concerned with the continuity of solutions and do not require them to be Lipschitzian, we can find more solutions. In this section, we use another method, sewing piece by piece, to find infinitely many continuous solutions of (1.3) on $\mathbb{R}_+$.

Consider (1.3) with real $\alpha_k$’s ($1 \leq k \leq n$), and without loss of generality assume that $\alpha_n \neq 0$. Then (1.3) and its modified equation (2.1) can be represented equivalently as

\[
g^n(x) = \prod_{k=1}^{n-1} (g^k(x))^{\lambda_k} G(x) \tag{4.1}
\]

and

\[
f^n(x) = \sum_{k=1}^{n-1} \lambda_k f^k(x) + F(x), \tag{4.2}
\]

respectively, where $\lambda_k$’s are real for $1 \leq k \leq n - 1$. Let $\lambda := \sum_{k=1}^{n-1} \lambda_k$. We discuss for $\lambda \geq 0$ and $\lambda < 0$ separately.

First, we consider the case $\lambda > 0$. In 2007, Xu and Zhang [12] proved the existence of continuous solutions of (4.2) on the compact interval $I$ with the assumption that $\lambda \in [0,1)$. In what follows, solving (4.2) with $\lambda \in [0,1)$ on the whole $\mathbb{R}$, we obtain the solutions of (4.1) on $\mathbb{R}_+$.

Let $\mathbb{I} := [a,b]$, where $[a,b]$ denotes an open interval $(a,b)$, a semi-closed interval $[a,b)$ or $(a,b]$ or a closed interval $[a,b]$ in $\mathbb{R}$, and one or both of the endpoints of $\mathbb{I}$ may be infinite. Let $\mathbb{J} = [c,d]$ be an interval in $\mathbb{R}_+$, where $c$ and $d$ may be 0 and $\infty$, respectively. For $\zeta \in \mathbb{J}$, the closure of $\mathbb{I}$, $\eta \in \mathbb{I}$ and $\lambda \in [0,1)$, let

\[
R_{\mathbb{I},\lambda}[\mathbb{R};\mathbb{I}] := \{ f \in C_b(\mathbb{R}) : f |_{\mathbb{I}} \text{ is strictly increasing and satisfies (A1) and (A2)} \},
\]

\[
S_{\eta,\lambda}[\mathbb{R}_+;\mathbb{J}] := \{ g \in C_b(\mathbb{R}_+) : g |_{\mathbb{J}} \text{ is strictly increasing and satisfies (B1) and (B2)} \},
\]

where

(A1) $(f(x) - (1 - \lambda)x)(\zeta - x) > 0$ for $x \neq \zeta$;

(A2) $(f(x) - (1 - \lambda)\xi)(\zeta - x) < 0$ for $x \neq \zeta$;

(B1) $(g(x) - x^{1-\lambda})(\eta - x) > 0$ for $x \neq \eta$;

(B2) $(g(x) - \eta^{1-\lambda})(\eta - x) < 0$ for $x \neq \eta$.

Proposition 4.1. Let $\lambda \in [0,1)$. Then a map $g \in S_{\eta,\lambda}[\mathbb{R}_+;\mathbb{J}]$ for $\eta \in \mathbb{J}$ if and only if $f = \psi^{-1} \circ g \circ \psi \in R_{\mathbb{I},\lambda}[\mathbb{R};\mathbb{I}]$, where $\psi(x) = e^{x}$, $\zeta = \log \eta$ and $\tilde{\zeta} = \log(\tilde{\eta})$.

Proof. Let $g \in S_{\eta,\lambda}[\mathbb{R}_+;\mathbb{J}]$, where $\eta \in \mathbb{J}$. Since $f |_{\mathbb{I}} = \psi^{-1} \circ g |_{\mathbb{J}} \circ \psi$ and $g |_{\mathbb{J}}$ is strictly increasing, clearly $f |_{\mathbb{I}}$ is also strictly increasing. In order to prove that $f$ satisfies the condition (A1), consider any $x \in \mathbb{I}$ such that $x \neq \zeta$. Let $y \in \mathbb{J}$ be such that $x = \log y$. Then either $g(y) < y^{1-\lambda}$ or $g(y) > y^{1-\lambda}$ according as either $y > \eta$ or $y < \eta$, respectively. This implies that either $f(x) < (1 - \lambda)x$ or $f(x) > (1 - \lambda)x$ according as either $x > \zeta$ or $x < \zeta$, respectively. In any case, we have $(f(x) - (1 - \lambda)x)(\zeta - x) > 0$. By a similar argument, using the condition (B2) for $g$, we can prove that $f$ satisfies the condition (A2). Hence, $f \in R_{\mathbb{I},\lambda}[\mathbb{R};\mathbb{I}]$. The converse follows similarly. \( \Box \)

Lemma 4.2 (See [12]). Let $\lambda \in [0,1)$ and $F \in R_{\alpha,\lambda}[\mathbb{R};\mathbb{I}]$. Then for arbitrary $x_0 \in (a,b)$, (4.2) has a solution in $R_{\alpha,0}[I_1;I_1]$, where $I_1 = [a,x_0]$. More concretely, for arbitrary $x_0 \in (a,b)$, there exists a strictly
decreasing sequence \((x_1, x_2, \ldots, x_{n-1})\) in \((a, x_0)\) such that the sequence \((x_m)\) defined recursively by

\[
x_{n+m} = \sum_{j=1}^{n-1} \lambda_j x_{j+m} + F(x_m) \quad \text{for } m \geq 0
\]

(4.3)
satisfies the conditions (i) \(x_{m+1} \in (a, x_m)\) for \(m \geq 1\), and (ii) \((a, x_0) = \bigcup_{m=1}^{\infty} [x_m, x_{m-1}]\) and

\[
f(x) := \begin{cases} a, & \text{if } x = a, \\ f_m(x), & \text{if } x \in [x_m, x_{m-1}], \end{cases} \quad m \geq 1
\]
is a solution of (4.2) in \(R_{a,0}[I_1; I_1]\), where \(f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j]\) is an arbitrary order-preserving homeomorphism for \(1 \leq j \leq n - 1\) and \(f_m : [x_m, x_{m-1}] \to [x_{m+1}, x_m]\) is the order-preserving homeomorphism defined recursively by

\[
f_m(x) = \lambda_{n-1} x + \lambda_{n-2} f_{m-1}(x) + \cdots + \lambda_1 f_{m-n+2} \circ f_{m-n+3} \circ \cdots \circ f_{m-1}(x) + F \circ f_{m-n+1} \circ f_{m-n+2} \circ \cdots \circ f_{m-1}(x), \quad x \in [x_m, x_{m-1}] \quad \text{for } m \geq n.
\]

**Lemma 4.3** (See [12]). Let \(\lambda \in [0, 1)\) and \(F \in R_{a,\lambda}[I; I]\). Then for arbitrary \(x_0 \in [a, b]\), (4.2) has a solution in \(R_{a,0}[I_2; I_2]\), where \(I_2 = [x_0, b]\). Moreover, for arbitrary \(x_0 \in [a, b]\), there exists a strictly increasing sequence \((x_1, x_2, \ldots, x_n)\) in \((x_0, b)\) such that the sequence \((x_m)\) defined recursively by (4.3) satisfies the conditions (i) \(x_{m+1} \in (x_m, b)\) for \(m \geq 1\), and (ii) \([x_0, b) = \bigcup_{m=1}^{\infty} [x_{m-1}, x_m]\) and

\[
f(x) := \begin{cases} f_m(x), & \text{if } x \in [x_{m-1}, x_m], \quad m \geq 1, \\ b, & \text{if } x = b
\]
is a solution of (4.2) in \(R_{a,0}[I_2; I_2]\), where \(f_j : [x_{j-1}, x_j] \to [x_{j+1}, x_j]\) is an arbitrary order-preserving homeomorphism for \(1 \leq j \leq n - 1\) and \(f_m : [x_m, x_{m-1}] \to [x_{m+1}, x_m]\) is the order-preserving homeomorphism defined recursively by

\[
f_m(x) = \lambda_{n-1} x + \lambda_{n-2} f_{m-1}(x) + \cdots + \lambda_1 f_{m-n+2} \circ f_{m-n+3} \circ \cdots \circ f_{m-1}(x) + F \circ f_{m-n+1} \circ f_{m-n+2} \circ \cdots \circ f_{m-1}(x), \quad x \in [x_{m-1}, x_m] \quad \text{for } m \geq n.
\]

**Theorem 4.4.** Let \(\lambda \in [0, 1)\) and \(G \in S_{a,\lambda}[R_{a,\lambda}; I]\) such that \(R(G) = R(G[I])\), where \(J = [c, d]\). Then (4.1) has the solutions in \(S_{a,\lambda}[R_{a,\lambda}; I]\). Moreover, each solution depends on \(\lambda \) arbitrarily chosen orientation-preserving homeomorphisms \(f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j], \quad j = 1, 2, \ldots, n - 1\), where \(x_0 = \log d\) and \(x_1, x_2, \ldots, x_n\) are given as in Lemma 4.2.

**Proof.** Given \(G \in S_{a,\lambda}[R_{a,\lambda}; I]\), by Proposition 4.1, we have \(F = \psi^{-1} \circ G \circ \psi \in R_{a,\lambda}[R; I]\), where \(\psi(x) = e^x\) and \(I = \log(J)\). Also, since \(R(G) = R(G[I])\), it follows that \(R(F) = R(F[I])\). So \(F_I := F[I] \in R_{a,\lambda}[I; I]\).

Therefore, by Lemma 4.2, (4.2) has a solution \(\phi_1 \in R_{a,0}[I; I]\). Let \(f\) be the extension of the map \(\phi_1\) to \(\mathbb{R}\) defined by

\[
f(x) = \phi_1 \circ F_{-1} \circ F(x), \quad x \in \mathbb{R}.
\]

(4.4)

We assert that \(f\) is a solution of (4.2) in \(R_{a,0}[R; I]\). Being a strictly increasing continuous map, \(F_I : I \to R(F_I)\) has the inverse \(F_{-1}\), which is also strictly increasing and continuous on \(R(F_I)\). Therefore, as \(R(F) = R(F[I])\), clearly \(f\) is a well-defined map on \(\mathbb{R}\). Also, \(f\) is continuous on \(\mathbb{R}\), being the composition of continuous maps \(\phi_1\), \(F_{-1}\) and \(F\). Furthermore, as \(\phi_1 \in R_{a,0}[I; I]\), it follows that \(f\) is strictly increasing on \(I\), and satisfies the conditions (A1) and (A2). Therefore, \(f \in R_{a,0}[R; I]\). Moreover, for each \(x \in \mathbb{R}\),

\[
f^n(x) - \sum_{k=1}^{n-1} \lambda_k f^k(x) = f^{n-1}(f(x)) - \sum_{k=1}^{n-1} \lambda_k f^{k-1}(f(x))
\]

\[= f^{n-1}[f(f(x)) - \sum_{k=1}^{n-1} \lambda_k f^{k-1}[f(f(x)) \text{ (since } f(x) \in R(F) = R(F[I]) \subseteq I)]
\]
Therefore, \( f \) is a solution of (4.2) in \( \mathbb{R}_{a,0}[\mathbb{R};\mathbb{I}] \). Hence by Propositions 2.1 and 4.1, \( g = \psi \circ f \circ \psi^{-1} \) is a solution of (4.1) in \( S_{c,0}[\mathbb{R}_{a,0};\mathbb{J}] \). Furthermore, by Lemma 4.2, \( \phi_1 \) and hence \( g \) depend on \( n - 1 \) arbitrarily chosen orientation-preserving homeomorphisms \( f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j], j = 1, 2, \ldots, n - 1, \) where \( x_0 = b \).

For the other class \( S_{d,0}[\mathbb{R}_{a,0};\mathbb{J}] \), we can similarly prove the following result using Lemma 4.3.

**Theorem 4.5.** Let \( \lambda \in [0, 1) \) and \( G \in S_{d,0}[\mathbb{R}_{a,0};\mathbb{J}] \) such that \( \mathcal{R}(G) = \mathcal{R}(G_{\mathbb{J}}) \), where \( \mathbb{J} = [c, d] \). Then (4.1) has the solutions in \( S_{c,0}[\mathbb{R}_{a,0};\mathbb{J}] \). Moreover, each solution depends on \( n - 1 \) arbitrarily chosen orientation-preserving homeomorphisms \( f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j], j = 1, 2, \ldots, n - 1, \) where \( x_0 = \log c \) and \( x_1, x_2, \ldots, x_n \) are given as in Lemma 4.3.

In the special case \( \lambda = 0 \), (4.1) reduces to the equation

\[
g^n(x) = G(x),
\]

i.e., the problem of iterative roots for a given function \( G \). We have the following results for the solutions of (4.5) on \( \mathbb{R}_+ \).

**Corollary 4.6.** Let \( G \) be a continuous function on \( \mathbb{R}_+ \) such that \( G \) is strictly increasing on \( J \), \( G(c) = c \), \( G(d) < d \), \( \mathcal{R}(G) = [c, G(d)] \) and \( G(x) < x \) for \( x \in (c, d) \), where \( J = [c, d] \). Then (4.5) has the solutions on \( \mathbb{R}_+ \). Moreover, each solution depends on \( n - 1 \) arbitrarily chosen orientation-preserving homeomorphisms \( f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j], j = 1, 2, \ldots, n - 1, \) where \( x_0 = \log d \) and \( x_1, x_2, \ldots, x_n \) are given as in Lemma 4.2.

**Proof.** It follows from Theorem 4.4, because \( G \in S_{c,0}[\mathbb{R}_{a,0};\mathbb{J}] \) with \( J = [c, d] \) such that \( \mathcal{R}(F) = \mathcal{R}(F_{\mathbb{J}}) \).

We have the following analogous result for the case \( G(x) > x \), whose proof is similar.

**Corollary 4.7.** Let \( G \) be a continuous function on \( \mathbb{R}_+ \) such that \( G \) is strictly increasing on \( \mathbb{J} \), \( G(c) > c \), \( G(d) = d \), \( \mathcal{R}(G) = [G(c), d] \) and \( G(x) > x \) for \( x \in (c, d) \), where \( J = [c, d] \). Then (4.5) has the solutions on \( \mathbb{R}_+ \). Moreover, each solution depends on \( n - 1 \) arbitrarily chosen orientation-preserving homeomorphisms \( f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j], j = 1, 2, \ldots, n - 1, \) where \( x_0 = \log c \) and \( x_1, x_2, \ldots, x_n \) are given as in Lemma 4.3.

Theorems 4.4 and 4.5 each give infinitely many solutions of (4.1) on \( \mathbb{R}_+ \) since infinitely many choices can be made for the initial functions \( f_1, f_2, \ldots, f_{n-1} \) in Lemmas 4.2 and 4.3. Similar conclusions hold for Corollaries 4.6 and 4.7.

Next, we consider the case \( \lambda \leq 0 \). In 2013, assuming that \( \lambda \leq 0 \), Zhang et al. [15] proved the existence of continuous solutions of (4.2) on the compact \( I \). In what follows, solving (4.2) with \( \lambda \leq 0 \) on the whole \( \mathbb{R} \), we obtain solutions of (4.1) on \( \mathbb{R}_+ \). For compact intervals \( I = [a, b] \) and \( J = [c, d] \) of \( \mathbb{R} \) and \( \mathbb{R}_+ \), respectively, and for \( \lambda \in \mathbb{R} \), let

\[
\begin{align*}
A_\lambda[\mathbb{R}; I] &:= \{f \in C_b(\mathbb{R}) : f|_I \text{ is strictly increasing, } f(a) = \lambda a \text{ and } f(b) = \lambda b\}, \\
B_\lambda[\mathbb{R}_+; J] &:= \{g \in C_b(\mathbb{R}_+) : g|_J \text{ is strictly increasing, } g(c) = e^\lambda \text{ and } g(d) = d^\lambda\}.
\end{align*}
\]
Proposition 4.8. Let $\lambda \in \mathbb{R}$. Then $g \in \mathcal{B}_\lambda[\mathbb{R}_+; J]$ if and only if $f = \psi^{-1} \circ g \circ \psi \in \mathcal{A}_\lambda[\mathbb{R}; J]$, where $\psi(x) = e^x$ and $I = \log(J)$.

Proof. Let $g \in \mathcal{B}_\lambda[\mathbb{R}_+; J]$, where $\lambda \in \mathbb{R}$. Since $f|_I = \psi^{-1} \circ (g|_J) \circ \psi$ and $g|_J$ is strictly increasing, clearly $f|_I$ is also strictly increasing. Also, $f(a) = \log g(e^a) = \log(g(e^a)) = \log(e^\lambda) = \lambda \log e = \lambda a$ and similarly $f(b) = \lambda b$. Hence $f \in \mathcal{A}_\lambda[\mathbb{R}; I]$. The converse follows similarly. \qed

Remark 4.10. The proof of the above lemma (see [15, pp. 82–89]) shows steps to obtain those solutions.

Step 1. For each $\zeta, \xi \in (a, b)$ and $\lambda \leq 0$, let
\[
\mathcal{A}_\lambda[I] := \{ f \in \mathcal{C}(I, \lambda I) : f \text{ is strictly increasing on } I, f(a) = \lambda a, f(b) = \lambda b, f(x) > \lambda x \text{ for } x \in (a, b) \text{ and } f \text{ is linear on } [\zeta, b] \},
\]
\[
\mathcal{B}_\lambda[I] := \{ f \in \mathcal{C}(I, \lambda I) : f \text{ is strictly increasing on } I, f(a) = \lambda a, f(b) = \lambda b, f(x) < \lambda x \text{ for } x \in (a, b) \text{ and } f \text{ is linear on } [a, \xi] \}.
\]

In this step, we construct the solutions of (4.2) for $F \in \mathcal{A}_{1-\lambda}[I] \cup \mathcal{B}_{1-\lambda}[I]$ (see [15, Theorem 1]). This enables us to construct a sequence $(F_m)$ in $\mathcal{A}_{1-\lambda}[I] \cup \mathcal{B}_{1-\lambda}[I]$, which converges to a given function $F$ of a more general form and find the corresponding solutions $f_m$ for $m = 1, 2, \ldots$

Step 2. Using the sequential compactness of $(f_m)$ and verifying that its limit $f$ is a solution of (4.2), we arrive at the existence of the solution of (4.2) for $F \in \mathcal{A}_{1-\lambda}[I] \cup \mathcal{B}_{1-\lambda}[I]$, where
\[
\mathcal{A}_\lambda[I] := \{ f \in \mathcal{C}(I, \lambda I) : f \text{ is strictly increasing on } I, f(a) = \lambda a, f(b) = \lambda b \text{ and } f(x) > \lambda x \text{ for } x \in (a, b) \},
\]
\[
\mathcal{B}_\lambda[I] := \{ f \in \mathcal{C}(I, \lambda I) : f \text{ is strictly increasing on } I, f(a) = \lambda a, f(b) = \lambda b \text{ and } f(x) < \lambda x \text{ for } x \in (a, b) \}
\]
for $\lambda \leq 0$ (see [15, Theorem 2]).

Step 3. Dropping the assumption that the location of $F$ is below or above the line $y = (1-\lambda)x$ made in $\mathcal{A}_{1-\lambda}[I]$ and $\mathcal{B}_{1-\lambda}[I]$, we obtain the solutions of (4.2) for $F \in \mathcal{A}_{1-\lambda}[I; J]$ (see [15, Corollary 1]). In fact, given any $F \in \mathcal{A}_{1-\lambda}[I; J]$, let $\Gamma := \{ x \in I : F(x) = (1-\lambda)x \}$. Then $I = \Gamma \cup (\cup I_j)$ and $I_j$’s are disjoint open intervals, denoted by $(a_j, b_j)$’s $(a_j, b_j) \in \Gamma$ such that $F(x) \neq (1-\lambda)x$ for $x \in (a_j, b_j)$. Then either $F_j \in \mathcal{B}_{1-\lambda}[I]$ or $F_j \in \mathcal{A}_{1-\lambda}[I]$, where $F_j := F|_{I_j}$ for $j = 1, 2, \ldots$ By Step 2, for each $j$, the equation
\[
f^n(x) = \sum_{k=1}^{n-1} \lambda_k f^k(x) + F_j(x)
\]
has a solution $f_j \in \mathcal{A}_1[I; I_j]$, which depends on the choice of a sequence $(F_{j,m})$ in $\mathcal{A}_{1-\lambda}[I] \cup \mathcal{B}_{1-\lambda}[I]$. Then it follows that the function $f \in \mathcal{A}_1[I; I]$ defined by
\[
f(x) = \begin{cases} f_j(x), & \text{if } x \in I_j, \\ x, & \text{if } x \in \Gamma \end{cases}
\]
is a solution of (4.2) on $I$.

Since infinitely many choices can be made for each of the sequences $(F_{j,m})$’s, Lemma 4.9 indeed gives infinitely many solutions of (4.2) for $F \in \mathcal{A}_{1-\lambda}[I; J]$.

Theorem 4.11. Let $\lambda \leq 0$ and $G \in \mathcal{B}_{1-\lambda}[\mathbb{R}_+; J]$ such that $\mathcal{R}(G) = J^{1-\lambda} := \{ x^{1-\lambda} : x \in J \}$, where $J = [c, d]$. Then (4.1) has infinitely many solutions in $\mathcal{B}_1[\mathbb{R}_+; J]$. Moreover, each solution depends on the suitably chosen sequences $(F_{j,m})$’s for $j = 1, 2, \ldots$ as indicated in the above Remark 4.10.
Proof. Given \( G \in \mathcal{B}_1[\mathbb{R}_+; J] \), by Proposition 4.8, we have \( F = \psi^{-1} \circ G \circ \psi \in \mathcal{A}_1 \mathcal{L}[\mathbb{R}; I] \), where \( \psi(x) = e^x \) and \( I = \log(J) \). Also, since \( \mathcal{R}(G) = J^1 \mathcal{L} \), we have \( \mathcal{R}(F) = (1 - \lambda)I \). So \( F_1 := F|_I \in \mathcal{A}_1[I; I] \), and therefore by Lemma 4.9, (4.2) has a solution \( \phi_1 \) in \( \mathcal{A}_1[I; I] \). Let \( f \) be the extension of \( \phi_1 \) to \( \mathbb{R} \) as defined in (4.4). We prove that \( f \) is a solution of (4.2) in \( \mathcal{A}_1[\mathbb{R}_+; I] \). Furthermore, as indicated in Remark 4.10, the construction of \( h \) and \( \psi \) extend those solutions from \( G \) and \( \mathcal{R}(G) \), clearly \( f \) is a well-defined map on \( \mathbb{R} \). Also, \( f \) is continuous on \( \mathbb{R} \), being the composition of continuous maps \( \phi_1, F_1^{-1} \) and \( F \). Furthermore, as \( \phi_1 \in \mathcal{A}_1[I; I] \), it follows that \( f|_I \) is strictly increasing, \( f(\alpha) = \lambda \alpha \) and \( f(b) = \lambda b \). Therefore, \( f \in \mathcal{A}_1[I; I] \). Moreover, by a similar argument as in the proof of Theorem 4.4, it can be shown that \( f \) is a solution of (4.2) in \( \mathcal{A}_1[\mathbb{R}; I] \). Hence by Propositions 2.1 and 4.8, \( g = \psi \circ f \circ \psi^{-1} \) is a solution of (4.1) in \( \mathcal{B}_1[\mathbb{R}_+; J] \). Furthermore, as indicated in Remark 4.10, the construction of \( \phi_1 \) and hence that of \( g \) depend on the choice of sequences \( (F_{j,m}) \)'s for \( j = 1, 2, \ldots \).}

In the special case \( \lambda = 0 \), we have the following result for solutions of the iterative root problem (4.5) on \( \mathbb{R}_+ \).

**Corollary 4.12.** Let \( G \) be a continuous function on \( \mathbb{R}_+ \) such that \( G \) is strictly increasing on \( J \), \( G(c) = c, G(d) = d \) and \( \mathcal{R}(G) = J \), where \( J = [c, d] \). Then (4.5) has infinitely many solutions on \( \mathbb{R}_+ \). Moreover, each solution depends on the suitably chosen sequences \( (F_{j,m}) \)'s for \( j = 1, 2, \ldots \) as indicated in Remark 4.10.

**Proof.** It follows from Theorem 4.11, because \( G \in \mathcal{B}_1[\mathbb{R}_+; J] \) such that \( \mathcal{R}(G) = J \).

5 Extension to the whole \( \mathbb{R} \)

In the above two sections, we find the solutions of (1.3) on \( \mathbb{R}_+ \). In this section, we make an effort to extend those solutions from \( \mathbb{R}_+ \) to \( \mathbb{R}_- \) and the whole \( \mathbb{R} \).

In order to extend the solutions from \( \mathbb{R}_+ \) to its closure, we require the continuity of \( g \) and \( G \) at 0, i.e., we require the necessary conditions \( \lim_{x \to -\infty} F(x) = \log(G(0)), \lim_{x \to -\infty} f(x) = \log(g(0)) \) for all \( 1 \leq k \leq n \) and

\[
\sum_{k=1}^{n} a_k \log g^k(0) = \log(G(0)).
\]

These conditions can indeed be satisfied if \( \sum_{k=1}^{n} a_k = 1 \). \( G|_{\mathcal{R}(G)} = \text{id} \), \( 0 \in \mathcal{R}(G) \) and \( G \equiv g \) on \([0, +\infty)\).

Considering \( g \) on \((-\infty, 0)\), we have the following proposition.

**Proposition 5.1.** Let \( \alpha_k \in \mathbb{Z} \) for \( 1 \leq k \leq n \) such that \( \sum_{k=1}^{n} \alpha_k \) is odd. Then the map \( g \) is a solution (resp. unique solution) of (1.3) in \( \mathcal{X} \subseteq C_0(\mathbb{R}_-) \) if and only if \( h(x) := \psi^{-1}(g(\psi(x))) \) is a solution (resp. unique solution) of the equation

\[
(h(x))^\alpha_1((h^2(x))^{\alpha_2} \cdots (h^n(x)))^{\alpha_n} = H(x)
\]

in \( \mathcal{Y} \subseteq C_0(\mathbb{R}_+) \), where \( \psi(x) = -x \), \( H(x) = \psi^{-1}(G(\psi(x))) \) and \( \mathcal{Y} = \{\psi^{-1} \circ g \circ \psi : g \in \mathcal{X}\} \).

**Proof.** Let \( g \) be a solution of (1.3) in \( \mathcal{X} \). Since \( \psi \) is a homeomorphism of \( \mathbb{R}_+ \) onto \( \mathbb{R}_- \), clearly \( \mathcal{Y} \subseteq C_0(\mathbb{R}_+) \) and \( h \in \mathcal{Y} \). Also, for each \( x \in \mathbb{R}_+ \) and \( k \in \{1, 2, \ldots, n\} \), we have \( H(x) = -G(-x) \) and \( h^k(x) = -g^k(-x) \). Therefore,

\[
\prod_{k=1}^{n} (h^k(x))^{\alpha_k} = \prod_{k=1}^{n} (-g^k(-x))^{\alpha_k} = (-1)^{\sum_{k=1}^{n} \alpha_k} \prod_{k=1}^{n} (g^k(-x))^{\alpha_k} = -\prod_{k=1}^{n} (g^k(-x))^{\alpha_k} = -G(-x) = H(x),
\]

since \( \sum_{k=1}^{n} \alpha_k \) is odd, implying that \( h \) is a solution of (5.1) on \( \mathbb{R}_+ \). The converse follows similarly. Furthermore, the proof of uniqueness is similar to that of Proposition 2.1. \( \square \)
Using Proposition 5.1, we can indeed extend our result in Theorem 3.1 on the solutions of (1.3) from \( \mathbb{R}^+ \) to \( \mathbb{R}^- \), whenever those \( \alpha_k \in \mathbb{Z} \), \( k = 1, \ldots, n \), satisfy that \( \sum_{k=1}^{n} \alpha_k \) is odd. Furthermore, if \( \lambda_k \in \mathbb{Z} \) for \( 1 \leq k \leq n - 1 \), then by comparing the coefficients of \( g^k \) (\( 1 \leq k \leq n \)), in (4.1) and (1.3), we have \( \alpha_k = -\lambda_k \) for \( 1 \leq k \leq n - 1 \) and \( \alpha_n = 1 \), and moreover, the assumption that \( \sum_{k=1}^{n} \alpha_k \) is odd, made in Proposition 5.1, demands that \( 1 - \sum_{k=1}^{n-1} \lambda_k \) is odd, i.e., \( \lambda \) is even. Therefore, using Proposition 5.1, we can indeed extend our results in Theorems 4.4 and 4.5 on the solutions of (1.3) on \( \mathbb{R}^+ \) to \( \mathbb{R}^- \), whenever \( \lambda_k \in \mathbb{Z} \) for all \( 1 \leq k \leq n - 1 \) such that \( \lambda \) is even.

The second paragraph of this section and Proposition 5.1 extend a solution from \( \mathbb{R}^+ \) to its closure and from \( \mathbb{R}^+ \) to \( \mathbb{R}^- \), respectively. Thus, we can apply Theorem 3.1 to give the existence of (Lipschitzian) continuous solutions on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \) for self-maps with the compact range. However, none of the pair \( g_+ \) and \( g_- \), the solutions obtained in the existence results on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \), respectively, can be combined at 0. Actually, there is not a continuous self-map \( G \) on \( \mathbb{R} \) satisfying our required conditions. More precisely, for solutions obtained in Theorem 3.1, let us start with a self-map \( G \) of \( \mathbb{R} \). Since we need solutions obtained by Theorem 3.1 and Proposition 5.1, the map \( G \) is required to satisfy

\[
G|_{\mathbb{R}^+} \in G_{J_1,K_1,\delta,K_0M}(\mathbb{R}^+) \quad \text{and} \quad G|_{\mathbb{R}^-} \in G_{J_2,K_1,\delta,K_0M}(\mathbb{R}^-),
\]

where \( M, \delta \) and \( K_1 \)'s are given as in Theorem 3.1, and \( J_i = [c_i,d_i], i = 1,2 \) are compact subintervals of \( \mathbb{R}^+ \) and \( \mathbb{R}^- \), respectively. The first inclusion in (5.2) implies \( G(x) \geq c_1 > 0, \forall x \in \mathbb{R}^+ \), and the second inclusion in (5.2) implies that \( G(x) \leq d_2 < 0, \forall x \in \mathbb{R}^- \). The two inequalities imply that \( \lim_{x \to -0} G(x) \leq d_2 < 0 < c_1 \leq \lim_{x \to 0+} G(x) \), whenever the limits exist. Thus, \( G \) cannot be continuous at 0, and therefore we cannot give a continuous solution of (1.3) on the whole \( \mathbb{R} \) for a continuous self-map \( G \) on the whole \( \mathbb{R} \) by Proposition 5.1, extending continuously solutions obtained by Theorem 3.1. For a similar reason to the above extension of solutions obtained by using the fixed point method, we cannot give a result on continuous solutions on the whole \( \mathbb{R} \) by combining the pair \( g_+ \) and \( g_- \), the solutions obtained using the construction method used in Section 4.

On the other hand, we can think of some easier extension results just using results on \( \mathbb{R}^+ \). Let \( J := [c,d] \) be a compact interval in \( \mathbb{R}^+ \), and

\[
G_{J,\delta,M}(\mathbb{R}) := \{ g \in C_b(\mathbb{R}) : R(g) = J, g(c) = c, g(d) = d \text{ and } (x/y)^M \leq g(x)/g(y) \leq (x/y)^M, \forall x, y \in J \text{ with } x \geq y \}
\]

for constants \( M > \delta \geq 0 \).

**Corollary 5.2.** Suppose that \( G \in G_{J,\delta,M}(\mathbb{R}) \), where \( K_1 \)'s are given as in Theorem 3.1, satisfies \( G(x) = z_0 \) for all \( x \leq 0 \), where \( z_0 \) is a fixed point of \( G \). Then under the assumptions of Theorem 3.1, (1.3) has a solution in \( G_{J,\delta,M}(\mathbb{R}) \).

**Proof.** Given \( G \) as above, let \( I := \log J, \zeta_0 := \log z_0, G_+ := G|_{\mathbb{R}^+} \text{ and } F_+ := \psi^{-1} \circ G_+ \circ \psi \), where \( \psi(x) = e^x \) for all \( x \in \mathbb{R} \). Then by Theorem 3.1, the equation \( (g(x))^{\alpha_1} (g^2(x))^{\alpha_2} \cdots (g^n(x))^{\alpha_n} = G_+(x) \) has a unique solution \( g_+ \in G_{J,\delta,M}(\mathbb{R}) \), which is given by \( g_+ = \psi \circ f_+ \circ \psi^{-1} \), where \( f_+ \) is the unique solution of the equation \( \alpha_1 f(x) + \alpha_2 f^2(x) + \cdots + \alpha_n f^n(x) = F_+(x) \) in \( F_{I,\delta,M}(\mathbb{R}) \). Also, since \( G_+(z_0) = z_0 \), we have \( F_+(\zeta_0) = \zeta_0 \). Therefore, as \( f_+ \) is strictly increasing on \( I \), we get \( f_+(\zeta_0) = \zeta_0 \), implying that \( g_+(z_0) = z_0 \), where \( g_+ := \psi \circ f_+ \circ \psi^{-1} \in G_{J,\delta,M}(\mathbb{R}) \). Define a map \( g : \mathbb{R} \to \mathbb{R} \) by

\[
g(x) = \begin{cases} 
  g_+(x), & \text{if } x \in \mathbb{R}^+, \\
  z_0, & \text{if } x \leq 0.
\end{cases}
\]

Since \( \lim_{x \to -\infty} F_+(x) = \zeta_0 \), by using the proof of Theorem 3.1, we have

\[
\lim_{x \to -\infty} f_+(x) = \lim_{x \to -\infty} L_{J_+}^{1/2}(F_+(x)) = \zeta_0,
\]

implying that \( \lim_{x \to -0} g_+(x) = z_0 \). Therefore, \( g \) is continuous on \( \mathbb{R} \), and hence \( g \in G_{J,\delta,M}(\mathbb{R}) \). Furthermore, since \( g(z_0) = z_0 \), it is easily seen that \( g \) is a solution of (1.3) on \( \mathbb{R} \).
The above corollary shows that we can only use our result Theorem 3.1 on $\mathbb{R}^+$ and assume that $G$ is a constant on $\mathbb{R}^-$ to give a continuous solution on the whole $\mathbb{R}$. Similar results for such kinds of maps $G$ can be proved by using Theorems 4.4, 4.5 and 4.11 as well.

6 Examples and remarks

Example 6.1. Consider the equation

$$(g(x))^\frac{3}{4}(g^2(x))^\frac{1}{4} = G(x),$$

(6.1)

where $G : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$G(x) = \begin{cases} 
1, & \text{if } x \in (0, 1], \\
e^{(1+\log x)\log \sqrt{x}}, & \text{if } x \in [1, e], \\
e, & \text{if } x \in [e, \infty).
\end{cases}$$

Let $f(x) := \log g(e^x)$ and $F(x) := \log G(e^x)$ for $x \in \mathbb{R}$. Then (6.1) reduces to the polynomial-like equation

$$\frac{3}{4}f(x) + \frac{1}{4}f^2(x) = F(x),$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is the map defined by

$$F(x) = \begin{cases} 
0, & \text{if } x \leq 0, \\
x^2 + x \frac{2}{2} & \text{if } x \in [0, 1], \\
1, & \text{if } x \geq 1.
\end{cases}$$

Note that $F \in \mathcal{F}_{1, \frac{3}{4}}(\mathbb{R})$, where $I = [0, 1]$. Let $\delta = \frac{2}{3}$ and $M = 2$. Then $K_1\delta = \frac{1}{2}$ and $K_0M = \frac{3}{2}$, and therefore, $F \in \mathcal{F}_{J, K_1\delta, K_0M}(\mathbb{R})$. This implies by Proposition 2.2 that $G \in \mathcal{G}_{J, K_1\delta, K_0M}(\mathbb{R}^+)$, where $J = [1, e]$. Also, $K_2 = \frac{1}{4} < \frac{11}{12} = K_0$. Thus, all the hypotheses of Theorem 3.1 are satisfied. Hence, (6.1) has a unique solution $g$ in $\mathcal{G}_{J, \frac{3}{2}, 2}(\mathbb{R})$.

Example 6.2. Consider the equation

$$(g^2(x))^3 = G(x),$$

(6.2)

where $G : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$G(x) = \begin{cases} 
1, & \text{if } x \in (0, 1], \\
\sqrt{x}, & \text{if } x \in [1, e], \\
e^{\pi x}, & \text{if } x \in [e, \infty).
\end{cases}$$

Let $f(x) := \log g(e^x)$ and $F(x) := \log G(e^x)$ for $x \in \mathbb{R}$. Then (6.2) reduces to $-2f(x) + 3f^2(x) = F(x)$, which is equivalent to

$$f^2(x) - \frac{2}{3}f(x) = \frac{1}{3}F(x),$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is the map defined by

$$F(x) = \begin{cases} 
0, & \text{if } x \leq 0, \\
x^3 & \text{if } x \in [0, 1], \\
\frac{1}{3x}, & \text{if } x \geq 1.
\end{cases}$$
Note that $H := \frac{1}{3}F \in R_{0, \frac{1}{2}}[R; I]$, where $I = [0, 1]$. Therefore by Proposition 4.1, it follows that the map $G_1$ defined by $G_1(x) = e^{H(x)}$ lies in $S_{1, \frac{1}{2}}[R_+; J]$, where $J = [1, e]$. Also, since $R(H) = [0, \frac{1}{6}] = R(H_{[1]})$, we have $R(G_1) = [1, \sqrt{e}] = R(G_1 | J)$. Therefore, by Theorem 4.4,
\[
\frac{(g^2(x))^3}{(g(x))^4} = G_1(x),
\]
and hence (6.2) has a solution $g$ on $R_+$.

**Example 6.3.** Consider the equation
\[
(g^2(x))^3(g(x))^6 = G(x),
\]
where $G : R_+ \to R_+$ is defined by
\[
G(x) = \begin{cases} 
1, & \text{if } x \in (0, 1], \\
x^3, & \text{if } x \in [1, 2], \\
\frac{7x + 2}{x}, & \text{if } x \in [2, \infty).
\end{cases}
\]
Then $G \in B_{\delta}[R_+; J]$, where $J = [1, 2]$. Also, $R(G) = J^3$. Hence by Theorem 4.11, (6.3) has a solution in $B_{\delta}[R_+; J]$.

We make the following observations regarding the two approaches (i.e., using the fixed point theorem and constructing solutions piece by piece) considered to solve (1.3). First, the solutions $g$ of (1.3) obtained in Theorems 4.4 and 4.5 have exactly one fixed point at an end-point of $R(g)$, whereas each solution $g$ obtained in Theorems 3.1 and 4.11 has fixed points at both end-points of $R(g)$. Second, as noted before, using Theorem 3.1, we cannot solve the iterative root problem (4.5). On the other hand, we can indeed obtain the solutions of (4.5) using Corollaries 4.6, 4.7 and 4.12.

Additionally, we note that in Section 4 we did not complete our discussion for all $\lambda \in R$, because we have assumed that $0 \leq \lambda < 1$ in Theorems 4.4 and 4.5. We remark that these theorems are not necessarily valid for $\lambda \geq 1$, and therefore our current approach cannot be used in this case to solve (1.3) on $R_+$. More precisely, if $\lambda \geq 1$, then the sets $S_{c, \lambda}[R_+; J]$ and $S_{d, \lambda}[R_+; J]$ are not necessarily nonempty. In fact, if $G \in S_{c, \lambda}[R_+; [1, 2]]$, then by using the conditions (B1) and (B2), we have $1 < G(2) < 1/4$, which leads to a contradiction. We arrive at a similar contradiction that $1 < G(1) < 1$ if $G \in S_{d, \lambda}[R_+; [1, 2]]$. So both the sets $S_{c, \lambda}[R_+; [1, 2]]$ and $S_{d, \lambda}[R_+; [1, 2]]$ are empty.

Furthermore, as observed in Section 5, we can extend our results on the solutions of (1.3) (resp. (4.1)) on $R_+$ to $R_-$, whenever those $\alpha_k \in Z$, $k = 1, \ldots, n$ satisfy that $\sum_{k=1}^{n} \alpha_k$ is odd (resp. whenever those $\lambda_k \in Z$ for $k = 1, \ldots, n - 1$ satisfy that $\lambda$ is even). On the other hand, if $\alpha_k \in R \setminus Z$ for some $1 \leq k \leq n$, then for any $G, g \in C_b(R_-)$, the map $x \mapsto \prod_{k=1}^{n} (g^k(x))^{\alpha_k}$ is a multi-valued complex map, whereas the map $x \mapsto G(x)$ is a single valued real map. So in order to obtain the equality $\prod_{k=1}^{n} (g^k(x))^{\alpha_k} = G(x)$, we have to choose branches of the complex logarithm suitably, which depends not only on $x$ but also on each term of the product $\prod_{k=1}^{n} (g^k(x))^{\alpha_k}$. Therefore, solving (1.3) on $R_-$ in this case is very difficult. For a similar reason, solving (4.1) on $R_-$ is difficult if $\lambda_k \in R \setminus Z$ for some $1 \leq k \leq n - 1$. In Corollary 5.2, we only give some easy extension results on solutions of (1.3) on the whole $R$. It is still open for general $G$.

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