Quark Mass Matrix with a Structure of a
Rank One Matrix Plus a Unit Matrix

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Abstract

A quark mass matrix model $M_q = M_{e}^{1/2}O_q M_{e}^{1/2}$ is proposed where $M_{e}^{1/2} = \text{diag}(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ and $O_q$ is a unit matrix plus a rank one matrix. Up- and down-quark mass matrices $M_u$ and $M_d$ are described in terms of charged lepton masses and additional three parameters (one in $M_u$ and two in $M_d$). The model can predict reasonable quark mass ratios (not only $m_u/m_c$, $m_c/m_t$, $m_d/m_s$ and $m_s/m_b$, but also $m_u/m_d$) and Kobayashi-Maskawa matrix elements.

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Recent observation of top quark mass by the CDF Collaboration [1] has brought a realistic study of quark mass matrix model within our reach more and more. Now, of the ten independent observable quantities in the three-quark-family scheme, we have already possessed experimental knowledge of nine quantities, i.e. except for one parameter (a CP-violation phase parameter). One of the criteria of model-building is how we can describe those observable quantities with a few parameters as possible. From the phenomenological point of view, for example, the Rosner-Worah model [2] (and its democratic-type version [3]), which has six adjustable parameters, can provide satisfactory predictions for six quark masses and four independent observable quantities of Kobayashi-Maskawa (KM) [4] matrix.

Recently, one of the authors (Y.K.) has proposed a lepton and quark mass matrix model [5]:

$$M_f = m_f^0 \cdot G O_f G \ ,$$

$$G = \text{diag}(g_1, g_2, g_3) \ ,$$

$$O_f = 1 + 3 a_f X(\phi_f) \ ,$$

where $$f = \nu, e, u, \text{and } d$$ are indices for neutrinos, charged leptons, up- and down-quarks, respectively. In the charged lepton mass matrix $$M_e$$, the parameter $$a_e$$ is chosen as $$a_e = 0$$, so that the parameters $$m_0^2 g_i^2$$ are fixed by charged lepton masses $$m_i^e$$ ($$m_1^e = m_e, m_2^e = m_\mu, m_3^e = m_\tau$$) as $$m_0^2 g_i^2 = m_i^e$$. Since the phase parameters $$\phi_q$$ are fixed at $$\phi_u = 0$$ and $$\phi_d = \pi/2$$ and the parameters $$m_0^u$$ and $$m_0^d$$ are fixed as $$m_0^u = m_0^d$$, the model includes only two adjustable parameters ($$a_u$$ and $$a_d$$) and provides reasonable values of quark mass ratios (not absolute values) and KM matrix parameters.

In spite of such phenomenological success, the following points in Ref. [5] are still unsatisfactory to us: (i) $$X(\phi_d)$$ is not a rank one matrix so that we must propose a complicated mechanism to explain the origin of this term. (ii) There are no reasonable explanation for nonzero phase terms which exist only in (1, 2) and (2, 1) matrix elements of $$X(\phi_d)$$. This ansatz is contrary to the philosophy of "democratic". (iii) Even though we accept that $$O_d = 1 + 3 a_d X(\pi/2)$$ is simple, the
inverse matrix $O_f^{-1}$ is not simple and it is difficult to account for $M_f = m^f_0 G O_f G$ in a seesaw type model $M_f \simeq m M_F^{-1} m$ with $m \propto G$ and $M_F \propto O_f^{-1}$.

In this paper we search for a possible form of the matrices $O_f$ in (1) with the following conditions: (a) The matrix form of each term in $O_f$ is as simple as possible. (b) The number of hierarchically different terms is as few as possible. In a Higgs mechanism model, the latter condition means Higgs fields are as few as possible.

The simplest form of $O_f$ is a unit matrix, but it leads to $M_u = M_d = M_e$ and fails to give the good predictions for quark mass spectrum and KM matrix. The next simple form of $O_f$ is a unit matrix plus a rank one matrix, which agrees with the condition (b). Therefore it is meaningful to study quark masses and mixing in the case that $O_q(q = u, d)$ is given by a unit matrix plus a rank one matrix with a complex coefficient.

In the present paper, we propose the following quark mass matrix:

$$M_f = (m^f_0/m^e_0) M_e^{1/2} O_f M_e^{1/2},$$

(5)

$$M_e^{1/2} = \text{diag}(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}),$$

(6)

$$O_f = 1 + 3a_f e^{i\alpha_f} X,$$

(7)

where $X \equiv X(0)$ is a rank one matrix. The inverse matrices of $O_f$ are also simple

$$O_f^{-1} = 1 + 3b_f e^{i\beta_f} X,$$

(8)

where

$$b_f e^{i\beta_f} = -a_f e^{i\alpha_f}/(1 + 3a_f e^{i\alpha_f}).$$

(9)

In a seesaw type model with heavy fermions, the inverse matrices $O_f^{-1}$ become more fundamental quantity than $O_f$ and the parameters $b_f$ and $\beta_f$ are more important than $a_f$ and $\alpha_f$.

In the model of Ref. [5], there are only two adjustable parameters $(a_u, a_d)$ because of the ansatz $\phi_d = \pi/2$, but it is difficult to give a reasonable explanation for the mass matrices in model building. In the present model, three parameters $(a_u, a_d, \alpha_d)$ are necessary but a matrix form of $O_f^{-1}$ as well as $O_f$ are simple. This makes model building easy.

If one feel the “democratic” type matrix form $X$ in the present model (7) somewhat mysterious, one may alternatively consider a diagonal matrix form
diag(0, 0, 1) by taking a suitable transformation of family basis, because the unit matrix term 1 in (7) is unchanged under this transformation.

The reason that we consider a democratic matrix from in $O_f$ is motivated by only a phenomenological reason suggested in Ref. [6], i.e., by the fact that for up-quark mass matrix with $\alpha_u = 0$, we can obtain the successful mass relation [6]

$$m_u/m_c \simeq 3m_c/4m_\mu, \quad (10)$$

for a small value of $\varepsilon_u \equiv 1/a_u$. Substituting the quark mass values which is given in eq. (12), the left hand side of eq. (10) is $4.0 \times 10^{-3}$, while the right hand side of eq. (10) is $3.6 \times 10^{-3}$. Note that the ratio $m_u/m_c$ is insensitive to the parameter $a_u$. The parameter $\varepsilon_u \equiv 1/a_u$ is determined by the mass ratio

$$m_c/m_t \simeq 2(m_\mu/m_\tau)\varepsilon_u. \quad (11)$$

The quark mass values [7] at an electroweak symmetry breaking energy scale $\mu = \Lambda_W \equiv \langle \phi^0 \rangle = (\sqrt{2}G_F)^{-1/2}/\sqrt{2} = 174$ GeV are

$$m_u = 0.0024 \pm 0.0005 \text{ GeV}, \quad m_c = 0.605 \pm 0.009 \text{ GeV}, \quad m_t = 174 \pm 10^{+13}_{-12} \text{ GeV},$$

$$m_d = 0.0042 \pm 0.0005 \text{ GeV}, \quad m_s = 0.0851 \pm 0.014 \text{ GeV}, \quad m_b = 2.87 \pm 0.03 \text{ GeV}, \quad (12)$$

where $\langle \phi^0 \rangle$ is a vacuum expectation value of a Higgs scalar field $\phi^0$ in the standard model and we have used $\Lambda^{(4)}_{\overline{MS}} = 0.26$ GeV.

Differently from the model given in Ref. [5], down-quark mass matrix $M_d$ with $\alpha_d \neq 0$ in the present model is not Hermitian. We will demonstrate that the present model with the form (7) also can provide reasonable predictions of quark mass ratios and KM matrix by adjusting our parameters $a_u$, $a_d$ and $\alpha_d$ (i.e., $b_u$, $b_d$ and $\beta_d$).

In the present model, a case $a_d \approx -1/2$ can provide phenomenologically interesting predictions as seen below. For small values of $|\alpha_d|$ and $\varepsilon_d \equiv -(2 + a_d^{-1})$, we obtain the down-quark mass ratios

$$\frac{m_s}{m_b} \simeq \frac{1}{2\kappa} \left(1 - 48\sqrt{\frac{2m_c m_\mu}{3m_\tau^2}} \right), \quad (13)$$

$$\frac{m_d}{m_s} \simeq \frac{16}{\kappa^2 m_\tau^2} \left(1 + 96\sqrt{\frac{2m_c m_\mu}{3m_\tau^2}} \right), \quad (14)$$


where

\[ \kappa = \sqrt{\sin^2 \frac{\alpha_d}{2} + \left( \frac{\varepsilon_d}{4} \right)^2} . \]  

(15)

We also obtain

\[ \frac{m_d m_s}{m_b^2} \simeq 4 \frac{m_e m_\mu}{m_\tau^2} , \]  

(16)

as a relation which is insensitive to the small parameters \(|\alpha_d|\) and \(\varepsilon_d\). The left hand side of eq. (16) is \(4.3 \times 10^{-5}\), while the right hand side of eq. (16) is \(6.8 \times 10^{-5}\) with the quark mass values (12).

Furthermore, we can obtain ratios of up-quarks to down-quarks, for example,

\[ \frac{m_u}{m_d} \simeq 6\kappa \sim 12 \frac{m_s}{m_b} . \]  

(17)

Suitable choice of small values of \(\varepsilon_d\) and \(\alpha_d\) ensures \(m_u/m_d \sim O(1)\) in spite of \(m_t \gg m_b\). From (9), a small value \(|\varepsilon_u| = 1/|a_u| \simeq 0\) means \(b_u \simeq -1/3\), while a small value \(|\varepsilon_d| = |2 + a_d^{-1}| \simeq 0\) means \(b_d \simeq -1\). It is noted that, in spite of the large ratio of \(m_t/m_b\), the ratio of \(b_d/b_u\) is not so large, i.e., \(b_d/b_u \simeq 3\).

Then, let us discuss the KM matrix elements \(V_{ij}\). The KM matrix \(V\) is given by

\[ V = U^u_L P U^{d \dagger}_L , \]  

(18)

where \(U^u_L\) and \(U^d_L\) are defined by

\[ U^u_L M_u M^\dagger_u U^{u \dagger}_L = \text{diag}(m_u^2, m_c^2, m_t^2) , \quad U^d_L M_d M^\dagger_d U^{d \dagger}_L = \text{diag}(m_d^2, m_s^2, m_b^2) , \]  

(19)

respectively, and \(P\) is a phase matrix. Here, we have considered that the quark basis for the mass matrix (5) can, in general, deviate from the quark basis of weak interactions by some phase rotations. The simplest case \(P = \text{diag}(1, 1, 1)\) cannot provide reasonable predictions of \(|V_{ij}|\). When we take

\[ P = \text{diag}(1, 1, -1) , \]  

(20)

we can obtain reasonable predictions for both quark mass ratios and KM matrix elements, although it is an open question why such a phase inversion is caused on the third family quark. The predictions of \(|V_{ij}|\) are sensitive to every value of \(\varepsilon_u\), \(\varepsilon_d\) and \(\alpha_d\), so that it is not adequate to express \(|V_{ij}|\) as simple approximate relations such as those in (10)–(11) and (13)–(17). Therefore, we will show only
numerical results for $|V_{ij}|$. For example, by taking $a_u = 28.65$, $a_d = -0.4682$, $\alpha_u = 0$ and $\alpha_d = 7.96^\circ$ ($b_u = -0.3295$, $b_d = -1.072$, $\beta_u = 0$ and $\beta_d = 18.5^\circ$), which are chosen by fitting the quark mass ratios, we obtain the following predictions of quark masses, KM matrix elements $|V_{ij}|$ and the rephasing-invariant quantity $J$ [8]:

$$m_u = 0.00228 \text{ GeV}, \quad m_c = 0.591 \text{ GeV}, \quad m_t = 170 \text{ GeV},$$
$$m_d = 0.00429 \text{ GeV}, \quad m_s = 0.0875 \text{ GeV}, \quad m_b = 3.02 \text{ GeV},$$

$$|V_{us}| = 0.223, \quad |V_{cb}| = 0.0542, \quad |V_{ub}| = 0.00309,$$
$$|V_{td}| = 0.0146, \quad |V_{ub}/V_{cb}| = 0.0570, \quad J = 2.30 \times 10^{-5}.$$

The prediction $|V_{cb}| = 0.0542$ in (22) is somewhat large in comparison with the experimental value $|V_{cb}| = 0.040 \pm 0.005$ [9]. If we use $P = (1, 1, -e^{i\delta})$ with a small phase value $\delta$ instead of $P = (1, 1, -1)$, we can obtain more excellent predictions without changing predictions of quark masses in (21): for example, when we take $\delta = -4.4^\circ$, we obtain

$$|V_{us}| = 0.223, \quad |V_{cb}| = 0.0400, \quad |V_{ub}| = 0.00274,$$
$$|V_{td}| = 0.0111, \quad |V_{ub}/V_{cb}| = 0.0686, \quad J = 1.55 \times 10^{-5}.$$

In the numerical predictions of quark masses (21), we have used a common enhancement factor of quark masses to lepton masses, $m_q^0/m_e^0 = m_q^0/m_e^0 = 3$, in order to compare with quark mass values at $\mu = \Lambda_X$ (12). It is an open question why we can set the factor $m_q^0/m_e^0$ as just three. Although we are happy if we can explain such the factor $m_q^0/m_e^0 = 3$ by evolving quark and lepton masses from $\mu = \Lambda_X$ to $\mu = \Lambda_W$, unfortunately, it is not likely to derive such a large factor $\sim 3$ from the conventional renormalization calculation.

At present, we have no theory to determine the parameters $a_f$ and $\alpha_f$. For charged leptons, we must take $a_e = 0$. For quarks, we have chosen $a_q$ from the phenomenological parameter fitting. However, in the present stage, we do not provide any unified understanding for $a_f$ and $\alpha_f$, i.e., they are nothing more than phenomenological parameters.

In conclusion, quark mass ratios and KM matrix elements can be fitted only by three parameters $a_u$, $a_d$ and $\alpha_d$ ($b_u$, $b_d$ and $\beta_d$) fairly well. If we take a seesaw-type model, we must consider that the parameters $b_f$ and $\beta_f$ in $O_f^{1}$ are more
fundamental ones rather than \( a_f \) and \( \alpha_f \) in \( O_f \). Then it is worth while that we can obtain a large ratio of \( m_t/m_b \) together with a reasonable ratio \( m_u/m_d \) without taking so hierarchically different values of \( b_u \) and \( b_d \), i.e., with taking \( b_u \simeq -1/3 \) and \( b_d \simeq -1 \), in contrast to \( a_u \simeq 30 \) and \( a_d \simeq -1/2 \) in the case of \( GO_f G \) picture.

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