Non-conforming Crouzeix-Raviart element approximation for Stekloff eigenvalues in inverse scattering

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Abstract
In this paper, we use the non-conforming Crouzeix-Raviart element method to solve a Stekloff eigenvalue problem arising in inverse scattering. The weak formulation corresponding to this problem is non-self-adjoint and indefinite, and its Crouzeix-Raviart element discretization does not meet the condition of the Strang lemma. We use the standard duality technique to prove an extension of the Strang lemma. And we prove the convergence and error estimate of discrete eigenvalues and eigenfunctions using the spectral perturbation theory for compact operators. Finally, we present some numerical examples not only on uniform meshes but also on adaptive refined meshes to show that the Crouzeix-Raviart method is efficient for computing real and complex eigenvalues as expected.

Keywords Stekloff eigenvalue · Non-conforming Crouzeix-Raviart element · Strang lemma · Error estimates

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1 Introduction

Steklov eigenvalue problems have important physical background and many applications. For instance, they appear in the analysis of stability of mechanical oscillators immersed in a viscous fluid (see [29] and the references therein), in the study of surface waves [10], in the study of the vibration modes of a structure in contact with an incompressible fluid [11], and in the analysis of the antiplane shearing on a system of collinear faults under slip-dependent friction law [19]. Hence, the finite element methods for solving these problems have attracted more and more scholars’ attention. Until now, systematical and profound studies on the finite element approximation mainly focus on Steklov eigenvalue problems which satisfy $H^1$-elliptic condition (see, e.g., [3–6, 11, 13, 15, 22, 36, 43, 44, 48, 54, 56] and the references therein).

Recently Cakoni et al. [21] study a new Stekloff eigenvalue problem arising from the inverse scattering theory:

$$\Delta u + k^2 n(x) u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \gamma} = -\lambda u \text{ on } \partial \Omega, \quad (1.1)$$

where $\Omega$ is a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$), $\frac{\partial u}{\partial \gamma}$ is the outward normal derivative, $k$ is the wave number, and $n(x) = n_1(x) + i n_2(x)$ is the index of refraction that is a bounded complex valued function with $n_1(x) > 0$ and $n_2(x) \geq 0$.

Note that the weak formulation of (1.1) (see (2.1)) does not satisfy $H^1$-elliptic condition. Cakoni et al. [21] analyze the mathematics properties of (1.1) and use conforming finite element methods to solve it. Liu et al. [45] then study error estimates of conforming finite element eigenvalues for (1.1).

The non-conforming Crouzeix-Raviart element (the C-R element) was first introduced by Crouzeix and Raviart in [30] in 1973 to solve the stationary Stokes equation. It has also been used to solve linear elasticity equations (see [16, 35]), the Laplace equation/eigenvalues (see [7, 14, 17, 23–25, 33]), Darcy’s equation [2], Steklov eigenvalue (see [3, 13, 43, 48, 56]), Maxwell eigenvalue (see [18]), Stokes eigenvalue (see [40, 55]), and so on.

In this paper, we will study the C-R element approximation for the problem (1.1). As we know, the convergence and error estimates of the non-conforming finite element method for an eigenvalue problem are based on the convergence and error estimates of the non-conforming finite element method for the corresponding source problem. Hence, in Section 3, we first extend the Strang lemma to derive the error estimates for the source problem, then in Section 4, we use the spectral approximation theory to complete the error estimate for the eigenvalue problem. The features of our work are as follows:

1. The Strang lemma (see [51]) is a fundamental analysis tool. However, the sesquilinear form of the problem here is non-self-adjoint and indefinite, and the C-R element discretization does not meet the condition of the Strang lemma. To overcome this difficulty, referring to §5.7 in [17], we use the standard duality technique to prove an extension of the Strang lemma (see Theorem 2). Based on the theorem, we prove the convergence and error estimates of the C-R method for the corresponding source problem. The current paper, to our knowledge, is a
rare investigation of applying and extending the Strang lemma to elliptic boundary value problem that the corresponding sesquilinear form is non-selfadjoint and indefinite.

2. Cakoni et al. [21] write (1.1) as an equivalent eigenvalue problem of the Neumann-to-Dirichlet operator $T$. In this paper, we write the C-R element approximation of (1.1) as an equivalent eigenvalue problem of the discrete operator $T_h$, and prove $T_h$ converges to $T$ in the sense of norm in $L^2(\partial\Omega)$. Thus, using Babuška-Osborn spectral approximation theory [8], we prove first the error estimate of the C-R finite element eigenfunctions in the norm in $L^2(\partial\Omega)$ and the error estimate of C-R finite element eigenvalues, then we use the fundamental relationship (4.16) about eigenvalues and eigenfunctions to get the error estimate of the C-R finite element eigenfunctions in the norm $\| \cdot \|_h$ for the problem (1.1).

This method to analyze the error estimation of non-conforming elements for eigenvalue problems was proposed by Boffi et al. 10 years ago for the Laplace eigenvalue problem, for example, see Section 11 in [14] and Section 3 in [57].

3. We implement some numerical experiments not only on uniform meshes but also on adaptive refined meshes. It can be seen that the C-R method is efficient for computing real and complex eigenvalues as expected. In addition, we discover, when the index of refraction $n(x)$ is real and $\Omega$ is the L-shaped domain or the square with a slit, the C-R element eigenvalues approximate the exact ones from above, and numerical results in [21, 45] show conforming finite element eigenvalues approximate the exact ones from below; thus, we get the upper and lower bounds of eigenvalues.

It should be pointed out that the theoretical analysis and conclusions in this paper are also valid for the extension Crouzeix-Raviart element [39].

In this paper, regarding the basic theory of finite element methods, we refer to [8, 17, 26, 28, 47, 50].

Throughout this paper, the letter $C$ (with or without subscripts) denotes a positive constant independent of $h$, which may not be the same constant in different places. For simplicity, we use the symbol $a \lesssim b$ to mean that $a \leq C b$.

2 Preliminary

In this paper, we assume $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a polygonal ($d = 2$) or polyhedral ($d = 3$) domain, and assume $n \in L^\infty(\Omega)$. Let $H^\rho(\Omega)$ denote the Sobolev space on $\Omega$ with the norm $\| \cdot \|_{\rho, \Omega}$ (denoted by $\| \cdot \|_{\rho}$ for simplicity) and the seminorm $| \cdot |_{\rho, \Omega}$ (denoted by $| \cdot |_{\rho}$ for simplicity) and $H^0(\Omega) = L^2(\Omega)$, and let $H^\rho(\partial\Omega)$ denote the Sobolev space on $\partial\Omega$ with the norm $\| \cdot \|_{\rho, \partial\Omega}$ and the seminorm $| \cdot |_{\rho, \partial\Omega}$. Denote

$$a(u, v) = (\nabla u, \nabla v) - (k^2 nu, v), \quad (u, v) = \int_\Omega u \bar{v} dx, \quad \langle u, v \rangle = \int_{\partial\Omega} u \bar{v} ds.$$  

Cakoni et al. [21] give the weak form of (1.1): Find $\lambda \in \mathbb{C}, u \in H^1(\Omega) \setminus \{0\}$, such that

$$a(u, v) = -\lambda \langle u, v \rangle, \quad \forall v \in H^1(\Omega). \quad (2.1)$$
The source problem associated with (2.1) is as follows: Find \( \varphi \in H^1(\Omega) \) such that
\[
a(\varphi, v) = \langle f, v \rangle, \quad \forall v \in H^1(\Omega). \tag{2.2}
\]
The Neumann eigenvalue problem associated with \( n(x) \) is to find \( k^2 \in \mathbb{C} \) and a nontrivial \( u \) such that
\[
\Delta u + k^2 n(x) u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \gamma} = 0 \text{ on } \partial \Omega. \tag{2.3}
\]
In this paper, we always assume \( k^2 \) is not an interior Neumann eigenvalue of (2.3). Under this assumption, according to [21], the Neumann-to-Dirichlet map \( T : L^2(\partial \Omega) \rightarrow L^2(\partial \Omega) \) can be defined as follows. Let \( f \in L^2(\partial \Omega) \), define
\[
a(Af, v) = \langle f, v \rangle, \quad \forall v \in H^1(\Omega), \tag{2.4}
\]
and \( Tf = (Af)' \), where ‘ denotes the restriction to \( \partial \Omega \). Then (2.1) can be stated as the operator form:
\[
Tu = \mu u. \tag{2.5}
\]
(2.1) and (2.5) are equivalent, namely, if \((\mu, u) \in \mathbb{C} \times L^2(\partial \Omega)\) is an eigenpair of (2.5), then \((\lambda, Au)\) is an eigenpair of (2.1), \( \lambda = -\mu^{-1} \); conversely, if \((\lambda, u)\) is an eigenpair of (2.1), then \((\mu, u')\) is an eigenpair of (2.5), \( \mu = -\lambda^{-1} \).

From [21], we know \( T : L^2(\partial \Omega) \rightarrow L^2(\partial \Omega) \) is compact. If \( n(x) \) is real, then \( T \) is also self-adjoint.

Consider the dual problem of (2.1): Find \( \lambda^* \in \mathbb{C}, u^* \in H^1(\Omega) \setminus \{0\} \) such that
\[
a(v, u^*) = -\lambda^* \langle v, u^* \rangle, \quad \forall v \in H^1(\Omega). \tag{2.6}
\]
The source problem associated with (2.6) is as follows: Find \( \varphi^* \in H^1(\Omega) \) such that
\[
a(v, \varphi^*) = \langle v, g \rangle, \quad \forall v \in H^1(\Omega). \tag{2.7}
\]
Define the corresponding Neumann-to-Dirichlet operator \( T^* : L^2(\partial \Omega) \rightarrow L^2(\partial \Omega) \) by
\[
a(v, A^* g) = \langle v, g \rangle, \quad \forall v \in H^1(\Omega), \tag{2.8}
\]
and \( T^* g = (A^* g)' \). Then (2.6) has the equivalent operator form:
\[
T^* u^* = -\lambda^{*-1} u^*. \tag{2.9}
\]
It can be proved that \( T^* \) is the adjoint operator of \( T \) in the sense of inner product \( \langle \cdot, \cdot \rangle \). In fact, from (2.4) and (2.8) we have
\[
\langle Tf, g \rangle = a(Af, A^* g) = \langle f, A^* g \rangle = \langle f, T^* g \rangle, \quad \forall f, g \in L^2(\partial \Omega).
\]
Since \( T^* \) is the adjoint operator of \( T \), the primal and dual eigenvalues are connected via \( \lambda = \overline{\lambda^*} \).

Let \( \pi_h = \{\kappa\} \) be a regular \( d \)-simplex partition of \( \Omega \) (see [28], pp. 131). We denote \( h = \max_{\kappa \in \pi_h} h_\kappa \) where \( h_\kappa \) is the diameter of element \( \kappa \). Let \( \mathcal{E}_h \) denote the set of all \((d-1)\)-faces of elements \( \kappa \in \pi_h \). We split this set as \( \mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b \), where \( \mathcal{E}_h^i \) and \( \mathcal{E}_h^b \) are the sets of inner and boundary edges, respectively. Let \( S^h \) be the C-R element space defined on \( \pi_h \):
\[ S^h = \{ v \in L^2(\Omega) : v \mid_\kappa \in P_1(\kappa), \text{ } v \text{ is continuous at the barycenters of the } (d-1)\text{-faces of element } \kappa, \forall \kappa \in \pi_h \}. \]

The C-R element approximation of (2.1) is: Find \( \lambda_h \in \mathbb{C}, u_h \in S^h \setminus \{0\} \), such that

\[ a_h(u_h, v) = -\lambda_h \langle u_h, v \rangle, \quad \forall v \in S^h, \tag{2.10} \]

where \( a_h(u_h, v) = \sum_{\kappa \in \pi_h} \int_{\kappa} (\nabla u_h \cdot \nabla \bar{v} - k^2 n(x) u_h \bar{v}) \, dx \).

Define \( \| v \|_h = \left( \sum_{\kappa \in \pi_h} \| v \|_{1, \kappa}^2 \right)^{\frac{1}{2}} \), \( \| v \|_{1, \kappa} = \int_{\kappa} \sum_{i=1}^{d} |\frac{\partial v}{\partial x_i}|^2 + |v|^2 \, dx \). Evidently, \( \| \cdot \|_h \) is the norm on \( S^h \) and it is easy to know that \( a_h(\cdot, \cdot) \) is not uniformly \( S^h \)-elliptic.

The C-R element approximation of (2.2) is: Find \( \varphi_h \in S^h \), such that

\[ a_h(\varphi_h, v) = \langle f, v \rangle, \quad \forall v \in S^h. \tag{2.11} \]

Since \( k^2 \) is not an interior Neumann eigenvalue of (2.3), from the spectral approximation theory [26], we know that when \( h \) is properly small, \( k^2 \) is also not a C-R element eigenvalue for (2.3). So the discrete source problem (2.11) is uniquely solvable. Thus, we can define the discrete operator \( A_h : L^2(\partial \Omega) \to S^h \), satisfying

\[ a_h(A_h f, v) = \langle f, v \rangle, \quad \forall v \in S^h. \tag{2.12} \]

Let us denote by \( \delta S^h \) the function space defined on \( \partial \Omega \), which contains restriction of functions in \( S^h \) to \( \partial \Omega \). Define the discrete operator \( T_h : L^2(\partial \Omega) \to \delta S^h \subset L^2(\partial \Omega) \), satisfying \( T_h f = (A_h f)' \). Then (2.10) has the equivalent operator form:

\[ T_h u_h = \mu_h u_h, \tag{2.13} \]

namely, if \( (\mu_h, u_h) \in \mathbb{C} \times L^2(\partial \Omega) \) is an eigenpair of (2.13), then \( (\lambda_h, A_h u_h) \) is an eigenpair of (2.10), \( \lambda_h = -\mu_h^{-1} \); conversely, if \( (\lambda_h, u_h) \) is an eigenpair of (2.10), then \( (\mu_h, u'_h) \) is an eigenpair of (2.13), \( \mu_h = -\lambda_h^{-1} \).

The non-conforming finite element approximation of (2.6) is given by: Find \( \lambda^*_h \in \mathbb{C}, u^*_h \in S^h \setminus \{0\} \) such that

\[ a_h(v, u^*_h) = -\overline{\lambda^*_h} \langle v, u^*_h \rangle, \quad \forall v \in S^h. \tag{2.14} \]

The C-R element approximation of (2.7) is: Find \( \varphi^*_h \in S^h \), such that

\[ a_h(v, \varphi^*_h) = \langle v, g \rangle, \quad \forall v \in S^h. \tag{2.15} \]

Define the discrete operator \( A^*_h : L^2(\partial \Omega) \to S^h \) satisfying

\[ a_h(v, A^*_h g) = \langle v, g \rangle, \quad \forall v \in S^h, \tag{2.16} \]

and denote \( T^*_h g = (A^*_h g)' \), then (2.14) has the following equivalent operator form

\[ T^*_h u^*_h = -\overline{\lambda^*_h}^{-1} u^*_h. \tag{2.17} \]

It can be proved that \( T^*_h \) is the adjoint operator of \( T_h \) in the sense of inner product \( \langle \cdot, \cdot \rangle \). Hence, the primal and dual eigenvalues are connected via \( \lambda_h = \overline{\lambda^*_h} \).

We need the following regularity estimates which play an important role in our theoretical analysis. Note that for \( v \in H^\frac{1}{2}(\partial \Omega) \), \( \langle f, v \rangle \) has a continuous extension, still denoted by \( \langle f, v \rangle \), to \( f \in H^{-\frac{1}{2}}(\partial \Omega) \).
Lemma 1 For any \( f \in H^{-\frac{1}{2}}(\partial\Omega) \), let \( \langle f, v \rangle \) be the dual product on \( H^{-\frac{1}{2}}(\partial\Omega) \times H^\frac{1}{2}(\partial\Omega) \) in (2.2), then there exists a unique solution \( \varphi \in H^1(\Omega) \) to (2.2) such that
\[
\|\varphi\|_1 \lesssim \|f\|_{-\frac{1}{2},\partial\Omega}.
\] (2.18)

Proof Denote
\[
b(u, v) = \int_\Omega \nabla u \cdot \nabla v + n(x)u\bar{v}dx.
\]
Referring to the proof of (14.11) in \([28]\), we can prove by contradiction that there exists a positive constant \( C_0 \) independent of \( v \) such that for all \( v \in H^1(\Omega) \),
\[
\|v\|_1 \leq C_0 \sqrt{\text{Re} b(v, v)} \equiv C_0 \left( \int_\Omega |\nabla v|^2 + n_1(x)|v|^2dx \right)^{\frac{1}{2}}. \tag{2.19}
\]
From (2.19), it is easy to verify that \( \sqrt{\text{Re} b(v, v)} \) is a norm on \( H^1(\Omega) \) that is equivalent to the norm \( \|\cdot\|_1 \) and \( b(\cdot, \cdot) \) can be used as an inner product on \( H^1(\Omega) \).
(2.3) can be rewritten as: Find \( \tilde{\lambda} \in \mathbb{C}, u \in H^1(\Omega) \setminus \{0\} \) such that
\[
b(u, v) = \tilde{\lambda}(nu, v), \quad \forall v \in H^1(\Omega), \tag{2.20}
\]
Since \( k^2 \) is not an interior Neumann eigenvalue of (2.3), \( k^2 + 1 \) is not an eigenvalue of (2.20). Define the map \( B : H^1(\Omega) \rightarrow H^1(\Omega) \) by
\[
b(Bg, v) = (ng, v), \quad \forall v \in H^1(\Omega). \tag{2.21}
\]
Then \( B \) is compact, \( \frac{1}{k^2 + 1} \) is not an eigenvalue of \( B \). So by Fredholm alternative, we deduce \( (B - \frac{1}{k^2 + 1}I)^{-1} : H^1(\Omega) \rightarrow H^1(\Omega) \) is bounded.
From (2.21), we obtain
\[
a(\varphi, v) = b(\varphi, v) - (k^2 + 1)(n\varphi, v) = b(\varphi, v) - (k^2 + 1)b(B\varphi, v) = b((I - (k^2 + 1)B)\varphi, v). \tag{2.22}
\]
By Riesz representation theorem, there exists \( \hat{f} \in H^1(\Omega) \) such that
\[
\langle f, v \rangle = b(\hat{f}, v), \quad \forall v \in H^1(\Omega). \tag{2.23}
\]
From (2.22) and (2.23), (2.2) can be expressed as the following operator form:
\[
\left( \frac{1}{k^2 + 1}I - B \right) \varphi = \frac{1}{k^2 + 1} \hat{f}. \tag{2.24}
\]
Hence, (2.24), i.e., (2.2) admits a unique solution:
\[
\varphi = \left( \frac{1}{k^2 + 1}I - B \right)^{-1} \frac{1}{k^2 + 1} \hat{f}.
\]
From (2.23) we get \( \|\hat{f}\|_1 \lesssim \|f\|_{-\frac{1}{2},\partial\Omega} \), and thus (2.18) holds. \( \square \)

Lemma 2 Assume that \( \Omega \subset \mathbb{R}^2 \) is a polygonal with the largest interior angle \( \omega \), and \( \varphi \) is the solution of (2.2).
i) If $f \in L^2(\partial\Omega)$, then $\varphi \in H^{1+\frac{r}{2}}(\Omega)$ and
$$
\|\varphi\|_{1+\frac{r}{2}} \leq C_\Omega \|f\|_{0,\partial\Omega}.
$$

(2.25)

ii) If $f \in H^{\frac{1}{2}}(\partial\Omega)$, then $\varphi \in H^{1+r}(\Omega)$ satisfying
$$
\|\varphi\|_{1+r} \leq C_\Omega \|f\|_{\frac{1}{2},\partial\Omega}.
$$

(2.26)

Here $r = 1$ when $\omega < \pi$, and $r < \frac{\pi}{\omega}$ when $\omega > \pi$, and $C_\Omega$ is a priori constant dependent on $\Omega$ and wave number $k$ but independent of the right-hand side $f$ of the equation.

**Proof** Consider the auxiliary boundary value problem:

$$
\Delta \varphi_1 + \varphi_1 = 0, \quad \frac{\partial \varphi_1}{\partial \gamma} = f,
$$

(2.27)

$$
\Delta \varphi_2 + \varphi_2 = -k^2n(x)(\varphi_1 + \varphi_2) + \varphi_1 + \varphi_2, \quad \frac{\partial \varphi_2}{\partial \gamma} = 0.
$$

(2.28)

Let $\varphi_1$ and $\varphi_2$ be the solution of (2.28) and (2.28), respectively, then it is easy to see that $\varphi = \varphi_1 + \varphi_2$. Since $\Omega \subset \mathbb{R}^2$, from classical regularity results (see [32], or Proposition 4.1 in [3] and Proposition 4.4 in [11]), we have

$$
\|\varphi_1\|_{1+(\frac{1}{2}+s)r} \lesssim \|f\|_{s,\partial\Omega}, \quad s = 0, \frac{1}{2},
$$

and from classical regularity result for the Laplace problem with homogeneous Neumann boundary condition, we have

$$
\|\varphi_2\|_{1+r} \lesssim \| -k^2n(x)(\varphi_1 + \varphi_2) + \varphi_1 + \varphi_2\|_{0,\Omega}.
$$

Thus, we get

$$
\|\varphi\|_{1+(\frac{1}{2}+s)r} \lesssim \|\varphi_1\|_{1+(\frac{1}{2}+s)r} + \|\varphi_2\|_{1+(\frac{1}{2}+s)r} \lesssim \|f\|_{s,\partial\Omega} + \|\varphi\|_0, \quad s = 0, \frac{1}{2}.
$$

Substituting (2.18) into the above inequality, we get (2.25) and (2.26).

**Remark 1** (Regularity in $\mathbb{R}^3$). When $\Omega \subset \mathbb{R}^3$ is a polyhedral domain, the regularity of the solution of Neumann problem (2.28) has been discussed by many scholars. Referring Theorem 4 in [49], Remark 2.1 in [36], [41], and [32], and using the argument of Lemma 2 in this paper, we think the following regularity hypothesis $R(\Omega)$ is reasonable:

**Hypothesis $R(\Omega)$** Let $\varphi$ be the solution of (2.2) with $f \in L^2(\partial\Omega)$. When $\Omega \subset \mathbb{R}^3$, we have $\varphi \in H^{1+r_3}(\Omega)$ for all $r_3 \in (0, \frac{1}{2})$ and

$$
\|\varphi\|_{1+r_3} \leq C_\Omega \|f\|_{0,\partial\Omega}.
$$

(2.29)

It is easy to know that Lemmas 1–2 and Remark 1 are also valid for the dual problem (2.7).
The consistency term and the extension of the Strang lemma

Define $S^h + H^1(\Omega) = \{ w_h + w : w_h \in S^h, w \in H^1(\Omega) \}$.

Let $\varphi$ and $\varphi^*$ be the solution of (2.2) and (2.7), respectively. Define the consistency terms: For any $v \in S^h + H^1(\Omega)$,
\begin{align*}
D_h(\varphi, v) &= a_h(\varphi, v) - \langle f, v \rangle, \\
D^*_h(v, \varphi^*) &= a_h(v, \varphi^*) - \langle v, g \rangle.
\end{align*}

In order to analyze error estimates of the consistency terms, we need the following trace inequalities.

Lemma 3 For any $\kappa \in \pi_h$, the following trace inequalities hold:
\[
\| w \|_{0, \partial \kappa} \lesssim h^{-\frac{1}{2}} \| w \|_{0, \kappa} + h^\frac{1}{2} | w |_{1, \kappa}, \\
\| \nabla w \|_{0, \partial \kappa} \lesssim h^{-\frac{1}{2}} \| w \|_{0, \kappa} + h^{-\frac{1}{2}} | w |_{1, \kappa} + h^{-\frac{s}{2}} | w |_{1+s, \kappa} \left( \frac{1}{2} \leq s \leq 1 \right). \tag{3.3}
\]

Proof The conclusion is followed by using the trace theorem on the reference element and the scaling argument (see, e.g., Lemma 2.2 in [56]).

The following Green’s formula (see (2.7) in [20], (3.11) in [12] and Corollary 2.2 in [37]) will play a crucial role in our analysis:
\[
\int_{\partial \kappa} (\nabla w \cdot \gamma) v ds = \int_{\kappa} \Delta w v dx + \int_{\kappa} \nabla w \cdot \nabla v dx \ \forall \kappa \in \pi_h, \tag{3.4}
\]
where $w \in H^{1+\epsilon}(\kappa)$ with $\Delta w \in L^2(\kappa)$ and $v \in H^{1-\epsilon}(\kappa)$ with $0 \leq \epsilon < \frac{1}{2}$.

Now we will explain that the solution $\varphi$ of (2.2) satisfies
\[
\Delta \varphi \in L^2(\Omega). \tag{3.5}
\]
By Theorem 2.4.2.7 in [38], we can deduce that if $D \subset \Omega$ and $\text{dist}(\partial D, \partial \Omega) > 0$, then $\varphi \in W^{2,1}(D)$. Thus, by Green’s formula, from (2.2), we get
\[
- \int_{\Omega} \Delta \varphi \bar{v} dx - k^2 \int_{\Omega} n \varphi \bar{v} dx = 0, \ \forall \bar{v} \in C_0^\infty(\Omega),
\]
thus, we get $\Delta \varphi = -k^2 n \varphi$ a.e. $\Omega$, which indicates $\Delta \varphi \in L^2(\Omega)$.

Lemma 4 Suppose that $\varphi \in H^{1+t}(\Omega)(0 < t < \frac{1}{2})$ is the solution of (2.2) and Hypothesis $R(\Omega)$ holds, then
\[
\| \nabla \varphi \cdot \gamma \|_{L_{-\frac{1}{2}, \ell}^{1+t}} \lesssim \| \varphi \|_{1+t, \kappa} \ \forall \kappa \in \pi_h, \ \ell \in \partial \kappa. \tag{3.6}
\]

Proof Inequality (3.6) is contained in the proof of Corollary 3.3 on page 1384 of [12] or Lemma 2.1 in [20]. Green’s formula (3.4) and the inverse trace theorem (see page 387 in [42], or page 1767 in [20]) are used in the proof.
Based on the standard argument (see, e.g., [3, 43, 56]), the following consistency error estimates will be proved.

**Theorem 1** Let $\varphi$ and $\varphi^*$ be the solution of (2.2) and (2.7), respectively, and suppose that $\varphi, \varphi^* \in H^{1+t}(\Omega)$ with $t \in [s, 1]$ and **Hypothesis** $R(\Omega)$ holds, then

$$
|D_h(\varphi, v)| \lesssim h^t \|\varphi\|_{1+t} \|v\|_h, \quad \forall v \in S^h + H^1(\Omega),
$$

(3.7)

$$
|D_h^*(v, \varphi^*)| \lesssim h^t \|\varphi^*\|_{1+t} \|v\|_h, \quad \forall v \in S^h + H^1(\Omega),
$$

(3.8)

where $s = \frac{r}{2}$ when $\Omega \subset \mathbb{R}^2$, $s = r_3$ when $\Omega \subset \mathbb{R}^3$.

**Proof** Let $[[\cdot]]$ denote the jump across an inner face $\ell \in E_{i}^h$. By (3.4) we deduce

$$
D_h(\varphi, v) = a_h(\varphi, v) - \langle f, v \rangle = \sum_{\kappa \in \pi_h} \int_{\kappa} (-\Delta \varphi - k^2 n(x) \varphi) \tilde{v} dx + \sum_{\kappa \in \pi_h} \int_{\partial \kappa} \frac{\partial \varphi}{\partial \gamma} \tilde{v} ds = \sum_{\ell \in E^h} \int_{\ell} \frac{\partial \varphi}{\partial \gamma} [[\tilde{v}]] ds.
$$

(3.9)

Let $\ell$ be a $(d-1)$-face of $\kappa$, define

$$
P_\ell f = \frac{1}{|\ell|} \int_{\ell} f ds, \quad P_k f = \frac{1}{|\kappa|} \int_{\kappa} f dx.
$$

For $\ell \in E^i_h$, suppose that $\kappa_1, \kappa_2 \in \pi_h$ such that $\kappa_1 \cap \kappa_2 = \ell$. Since $[[\tilde{v}]]$ is a linear function vanishing at the barycenters of $\ell$, we have

$$
\int_{\ell} \frac{\partial \varphi}{\partial \gamma} [[\tilde{v}]] ds = \int_{\ell} \left( \frac{\partial \varphi}{\partial \gamma} - P_\ell \left( \frac{\partial \varphi}{\partial \gamma} \right) \right) [[\tilde{v}]] ds
$$

$$
= \int_{\ell} \left( \frac{\partial \varphi}{\partial \gamma} - P_\ell \left( \frac{\partial \varphi}{\partial \gamma} \right) \right) ( [[\tilde{v}]] - P_\ell [[\tilde{v}]] ) ds
$$

$$
= \int_{\ell} \frac{\partial \varphi}{\partial \gamma} ( [[\tilde{v}]] - P_\ell [[\tilde{v}]] ) ds.
$$

(3.10)

Then, when $t \in [\frac{1}{2}, 1]$, using Schwarz inequality, we deduce

$$
\left| \int_{\ell} \frac{\partial \varphi}{\partial \gamma} [[\tilde{v}]] ds \right| \leq \sum_{i=1,2} \| \nabla \varphi \cdot \gamma - P_\ell (\nabla \varphi \cdot \gamma) \|_{0,\ell} \| v \|_{\kappa_i} - P_\ell (v|_{\kappa_i}) \|_{0,\ell}
$$

$$
\leq \sum_{i=1,2} \| \nabla (\varphi - \varphi_I) \cdot \gamma \|_{0,\ell} \| v \|_{\kappa_i} - P_{k_i} (v|_{\kappa_i}) \|_{0,\ell},
$$

(3.11)

where $\varphi_I \in S^h$ be the C-R element interpolation function of $\varphi$. By Lemma 3 and the standard error estimates for $L^2$-projection, we deduce

$$
\| \nabla (\varphi - \varphi_I) \cdot \gamma \|_{0,\ell} \lesssim h^{t-\frac{1}{2}} \| \varphi \|_{1+t, \kappa_I}, \quad \| v \|_{\kappa_i} - P_{k_i} (v|_{\kappa_i}) \|_{0,\ell} \lesssim h^{\frac{1}{2}} \| v \|_{1, \kappa_i}.
$$

Substituting the above two estimates into (3.11), we obtain

$$
\left| \int_{\ell} \frac{\partial \varphi}{\partial \gamma} [[\tilde{v}]] ds \right| \lesssim \sum_{i=1,2} h^t \| \varphi \|_{1+t, \kappa_i} \| v \|_{1, \kappa_i},
$$

(3.12)
and substituting (3.12) into (3.9), we conclude that (3.7) holds.

When \( t < \frac{1}{2} \), from (3.10), we deduce that

\[
\left| \int_{\ell} \frac{\partial \varphi}{\partial \gamma} \frac{[[v]]}{[v]} ds \right| \leq \| \nabla \varphi \cdot \gamma \|_{t-\frac{1}{2}, \ell} \| [[v]] - P_{\ell}[[v]] \|_{\frac{1}{2}-t, \ell}.
\]

By using inverse estimate, Lemma 3, and the error estimate of \( L^2 \)-projection, we derive

\[
\| [[v]] - P_{\ell}[[v]] \|_{\frac{1}{2}-t, \ell} \lesssim h_{\ell}^{-\frac{1}{2}} \| [[v]] - P_{\ell}[[v]] \|_{0, \ell} \lesssim \sum_{i=1,2} h_{\ell_i}^i \| v \|_{1, \kappa_i}.
\]

Substituting the above estimate and (3.6) into (3.13), we obtain

\[
\left| \int_{\ell} \frac{\partial \varphi}{\partial \gamma} \frac{[[v]]}{[v]} ds \right| \lesssim \sum_{i=1,2} \| \varphi \|_{1+t, \kappa_i} h_{\ell_i}^i \| v \|_{1, \kappa_i},
\]

plugging the above inequality into (3.9) we also get (3.7).

Using the same argument as above, we can prove (3.8).

The C-R element approximation (2.11) of (2.2) does not satisfy the condition of the Strang lemma, that is \( a_h(\cdot, \cdot) \) is not uniformly \( S^h \)-elliptic. To overcome this difficulty, inspired by the work in §5.7 in [17], next we use the standard duality technique to prove an extension version of the well-known Strang lemma.

First, we will use the standard duality argument to prove that \( \| \varphi - \varphi_h \|_0 \) is a quantity of higher order than \( \| \varphi - \varphi_h \|_h \).

Introduce the auxiliary problem: Find \( \psi \in H^1(\Omega) \), such that

\[
a(v, \psi) = (v, g), \quad \forall \ v \in H^1(\Omega). \tag{3.14}
\]

Let \( \psi \) be the solution of (3.14). Then from elliptic regularity estimates for homogeneous Neumann boundary value problem, we know that there exists \( r_N > 0 \), such that

\[
\| \psi \|_{1+r_N} \lesssim \| g \|_0. \tag{3.15}
\]

Let \( \hat{E}_h(v, \psi) = a_h(v, \psi) - (v, g) \), then

\[
\left| \hat{E}_h(v, \psi) \right| \lesssim h^{r_N} \| \psi \|_{1+r_N} \| v \|_h, \quad \forall v \in S^h + H^1(\Omega). \tag{3.16}
\]

In fact, by (3.4) and \( \frac{\partial \psi}{\partial \gamma} |_{\partial \Omega} = 0 \), we deduce

\[
\hat{E}_h(v, \psi) = a_h(v, \psi) - (v, g) = \sum_{\kappa \in \pi_h} \int_{\kappa} v(-\Delta \psi - k^2 n(x) \psi) dx + \sum_{\kappa \in \pi_h} \int_{\partial \kappa} v \frac{\partial \psi}{\partial \gamma} ds - \int_{\Omega} v g dx \sum_{\ell \in \mathcal{E}_h^I} \int_{\ell} \frac{\partial \psi}{\partial \gamma} ds + \int_{\partial \Omega} v \frac{\partial \psi}{\partial \gamma} ds.
\]

Then, using the same argument as the part of Theorem 1 after (3.9), we can get (3.16).
Lemma 5 Let \( \varphi \) and \( \varphi_h \) be the solution of (2.2) and (2.11), respectively, and let \( \varphi^* \) and \( \varphi_h^* \) be the solution of (2.7) and (2.15), respectively, then

\[
\|\varphi - \varphi_h\|_0 \lesssim h^{r_N} \|\varphi - \varphi_h\|_h, \tag{3.17}
\]

\[
\|\varphi^* - \varphi_h^*\|_0 \lesssim h^{r_N} \|\varphi^* - \varphi_h^*\|_h. \tag{3.18}
\]

Proof Let \( I_h^C \) be the conforming linear interpolation operator on \( \pi_h \), then \( I_h^C \psi \in H^1(\Omega) \cap S^h \). According to (2.2) and (2.11), we have \( a_h(\varphi - \varphi_h, I_h^C \psi) = 0 \); thus, from (3.16), the interpolation error estimate and (3.15), we deduce

\[
\langle \varphi - \varphi_h, g \rangle \leq \langle \varphi - \varphi_h, \psi \rangle + a_h(\varphi - \varphi_h, \psi - I_h^C \psi) \leq h^{r_N} \|\psi\|_{1 + r_N} \|\varphi - \varphi_h\|_h \leq h^{r_N} \|g\|_0 \|\varphi - \varphi_h\|_h.
\]

By Riesz representation theorem, we obtain the desired result (3.17). Using the same argument as (3.17), we can prove (3.18).

Now we are ready to prove the following extension of the Strang lemma.

Theorem 2 Let \( \varphi \) and \( \varphi_h \) be the solution of (2.2) and (2.11), respectively, then

\[
\inf_{v \in S^h} \|\varphi - v\|_h + \sup_{v \in S^h \setminus \{0\}} \frac{|D_h(\varphi, v)|}{\|v\|_h} \lesssim \|\varphi - \varphi_h\|_h \tag{3.19}
\]

Let \( \varphi^* \) and \( \varphi_h^* \) be the solution of (2.7) and (2.15), respectively, then

\[
\inf_{v \in S^h} \|\varphi^* - v\|_h + \sup_{v \in S^h \setminus \{0\}} \frac{|D_h^*(\varphi, v^*)|}{\|v\|_h} \lesssim \|\varphi^* - \varphi_h^*\|_h \tag{3.20}
\]

Proof Denote

\[
A_h(u, v) = a_h(u, v) + K(u, v), \quad \forall u, v \in S^h + H^1(\Omega),
\]

where \( K > \|k^2 n\|_{0, \infty} \). Then we know that \( A_h \) satisfies the uniform \( S^h \)-ellipticity:

\[
|A_h(v, v)| \geq \min\{1, K - \|k^2 n\|_{0, \infty}\} \|v\|_h^2, \quad \forall v \in S^h.
\]

Thus, for any \( v \in S^h \),

\[
\|\varphi_h - v\|_h^2 \lesssim \|A_h(\varphi_h - v, \varphi_h - v)\|_h^2
\]

\[
= C \left| a_h(\varphi - v, \varphi_h - v) + \langle f, \varphi_h - v \rangle - a_h(\varphi, \varphi_h - v) + K \|\varphi_h - v\|_0^2 \right|.
\]
When $\|\varphi_h - v\|_h \neq 0$, dividing both sides of the above by $\|\varphi_h - v\|_h$, we obtain

$$
\|\varphi_h - v\|_h \lesssim \|\varphi - v\|_h + \frac{|a_h(\varphi, \varphi_h - v) - \langle f, \varphi_h - v \rangle|}{\|\varphi_h - v\|_h} + K \|\varphi_h - v\|_0
$$

From the triangular inequality and (3.17), we get

$$
\|\varphi - \varphi_h\|_h \leq \|\varphi - v\|_h + \|v - \varphi_h\|_h
$$

which together with $\|\varphi - \varphi_h\|_h \geq \inf_{v \in S^h} \|\varphi - v\|_h$, we obtain the first inequality in (3.19).

Similarly we can prove (3.20). The proof is completed.

Now we can state the error estimates of C-R element approximation for (2.2) and (2.7).

**Theorem 3** Under the conditions of Theorems 1 and 2, we have

$$
\|\varphi - \varphi_h\|_h \leq C h^t \|\varphi\|_{1+t}, \quad (3.21)
$$

$$
\|\varphi - \varphi_h\|_{0, \partial \Omega} \leq C h^s \|\varphi - \varphi_h\|_h \quad (3.22)
$$

$$
\|\varphi^* - \varphi^*_h\|_h \leq C h^t \|\varphi^*\|_{1+t}, \quad (3.23)
$$

$$
\|\varphi^* - \varphi^*_h\|_{0, \partial \Omega} \leq C h^s \|\varphi^* - \varphi^*_h\|_h. \quad (3.24)
$$

**Proof** From Theorem 2, the interpolation error estimate, and Theorem 1, we can obtain (3.21) and (3.23).

According to (2.2) and (2.11), we have $a_h(\varphi - \varphi_h, I^C_h \varphi^*) = 0$; thus, from (3.7), the interpolation error estimate and the regularity estimates (2.25) and (2.29), we deduce

$$
\langle \varphi - \varphi_h, g \rangle = \langle \varphi - \varphi_h, g \rangle - a_h(\varphi - \varphi_h, \varphi^*) + a_h(\varphi - \varphi_h, \varphi^*)
$$

$$
= -D_h^s(\varphi - \varphi_h, \varphi^*) + a_h(\varphi - \varphi_h, \varphi^* - I^C_h \varphi^*)
$$

$$
\lesssim h^s \|\varphi^*\|_{1+s} \|\varphi - \varphi_h\|_h \lesssim h^s \|g\|_{0, \partial \Omega} \|\varphi - \varphi_h\|_h.
$$

By Riesz representation theorem, we obtain the desired result (3.22).

Using the same argument as (3.22), we can prove (3.24).
Remark 2 Consider the Neumann boundary problem: find $\varphi \in H^1(\Omega)$ such that

$$a(\varphi, v) = \langle f, v \rangle + (\zeta, v), \quad \forall v \in H^1(\Omega).$$

(3.25)

Let $\varphi$ and $\varphi_h$ be the exact solution and the C-R element solution of (3.25), respectively,

$$D_h(\varphi, v) = a_h(\varphi, v) - \langle f, v \rangle - (\zeta, v),$$

and let $\varphi^*$ and $\varphi_h^*$ be the exact solution and the C-R element solution of the dual problem of (3.25), respectively. Then the analysis and conclusions in this section are also valid for (3.25).

4 Error estimates of discrete Stekloff eigenvalues

In this paper, we suppose that $\{\lambda_j\}$ and $\{\lambda_{j,h}\}$ are enumerations of the eigenvalues of (2.1) and (2.10) respectively, and let $\lambda_m$ be the $m$th eigenvalue with the algebraic multiplicity $q$ and the ascent $\alpha$, $\lambda_m = \lambda_{m+1} = \cdots, \lambda_{m+q-1}$. When $\|T_h - T\|_{0,\partial\Omega} \to 0$, the eigenvalues $\lambda_m, \cdots, \lambda_{m+q-1}$ of (2.10) will converge to $\lambda$ (see Lemma 5 on page 1091 of [34]). Let $M(\lambda)$ be the space of generalized eigenvectors associated with $\lambda$ and $T$, let $M_h(\lambda_{j,h})$ be the space of generalized eigenvectors associated with $\lambda_{j,h}$ and $T_h$, and let $M_h(\lambda) = \sum_{j=m}^{m+q-1} M_h(\lambda_{j,h})$. In view of the dual problem (2.6) and (2.14), the definitions of $M(\lambda^*)$, $M_h(\lambda_{j,h}^*)$, and $M_h(\lambda^*)$ are analogous to $M(\lambda)$, $M_h(\lambda_{j,h})$, and $M_h(\lambda)$.

Given two closed subspaces $V$ and $U$, denote

$$\delta(V, U) = \sup_{u \in V} \inf_{v \in U} \|u - v\|_{0,\partial\Omega}, \quad \hat{\delta}(V, U) = \max\{\delta(V, U), \delta(U, V)\}.$$

And denote $\hat{\lambda}_h = \frac{1}{q} \sum_{j=m}^{m+q-1} \lambda_{j,h}$. Thanks to [8], we get the following Theorem 4.

Theorem 4 Suppose $M(\lambda), M(\lambda^*) \subset H^{1+t}(\Omega)$ (t $\in$ [r, 1] for $\Omega \subset \mathbb{R}^2$, and t $\in$ [r, 1] for $\Omega \subset \mathbb{R}^3$), and Hypothesis R($\Omega$) holds. Then

$$\hat{\delta}(M(\lambda), M_h(\lambda)) \lesssim h^{s+t},$$

(4.1)

$$|\hat{\lambda}_h - \lambda| \lesssim h^{2t},$$

(4.2)

$$|\lambda - \lambda_{j,h}| \lesssim h^{2t}, \quad j = m, m + 1, \cdots, m + q - 1;$$

(4.3)

suppose $u_h$ is an eigenfunction corresponding to $\lambda_{j,h}$ ($j = m, m + 1, \cdots, m + q - 1$), $\|u_h\|_{0,\partial\Omega} = 1$, then there exists an eigenfunction $u$ corresponding to $\lambda$, such that

$$\|u_h - u\|_{0,\partial\Omega} \lesssim h^{(s+t)\frac{1}{2}},$$

(4.4)

$$\|u_h - u\|_h \lesssim h^t + h^{(s+t)\frac{1}{2}};$$

(4.5)

where $s = \frac{r}{2}$ when $\Omega \subset \mathbb{R}^2$, $s = r_3$ when $\Omega \subset \mathbb{R}^3$. 
Proof. Note that \( \| T f - T_h f \|_{0, \partial \Omega} = \| Af - A_h f \|_{0, \partial \Omega} = \| \varphi - \varphi_h \|_{0, \partial \Omega} \), from (3.22) with \( t = s \), we deduce
\[
\| T - T_h \|_{0, \partial \Omega} = \sup_{f \in L^2(\partial \Omega), \| f \|_{0, \partial \Omega} = 1} \| T f - T_h f \|_{0, \partial \Omega}
\lesssim \sup_{f \in L^2(\partial \Omega), \| f \|_{0, \partial \Omega} = 1} h^{2s} \| Af \|_{1+s} \lesssim h^{2s} \| f \|_{0, \partial \Omega}
\lesssim h^{2s} \to 0 \quad (h \to 0).
\]
(4.6)

Thus, from Theorem 7.1, Theorem 7.2 (inequality (7.12)), Theorem 7.3, and Theorem 7.4 in [8], we get
\[
\hat{\delta}(M(\lambda), M_h(\lambda)) \lesssim \| (T - T_h) |_{M(\lambda)} \|_{0, \partial \Omega},
\]
\[
| \lambda - \hat{\lambda}_h | \lesssim \sum_{i, j = m}^{m+q-1} | \langle (T - T_h) \varphi_i, \varphi_j^* \rangle | + \| (T - T_h) |_{M(\lambda)} \|_{0, \partial \Omega} \| (T^* - T_h^*) |_{M(\lambda^*)} \|_{0, \partial \Omega},
\]
\[
| \lambda - \hat{\lambda}_h | \lesssim \left\{ \sum_{i, j = m}^{m+q-1} | \langle (T - T_h) \varphi_i, \varphi_j^* \rangle | \right\}^{1/\alpha} + \| (T - T_h) |_{M(\lambda)} \|_{0, \partial \Omega} \| (T^* - T_h^*) |_{M(\lambda^*)} \|_{0, \partial \Omega},
\]
(4.7)
\[
\| u_h - u \|_{0, \partial \Omega} \leq C \| (T_h - T) \|_{M(\lambda)} \|_{0, \partial \Omega},
\]
(4.8)
\[
\| u_h - u \|_{0, \partial \Omega} \leq C \| (T_h - T) \|_{M(\lambda)} \|_{0, \partial \Omega} \|_{0, \partial \Omega},
\]
(4.9)

where \( \varphi_m, \ldots, \varphi_{m+q-1} \) are any basis for \( M(\lambda) \) and \( \varphi_m^*, \ldots, \varphi_{m+q-1}^* \) are the dual basis in \( M(\lambda^*) \).

From (3.22) with \( t \in [r, 1] \), we obtain
\[
\| (T - T_h) |_{M(\lambda)} \|_{0, \partial \Omega} = \sup_{f \in M(\lambda), \| f \|_{0, \partial \Omega} = 1} \| T f - T_h f \|_{0, \partial \Omega}
\lesssim h^{s+t} \sup_{f \in M(\lambda), \| f \|_{0, \partial \Omega} = 1} \| Af \|_{1+t}.
\]
(4.10)

Similarly, we have
\[
\| (T^* - T_h^*) |_{M(\lambda^*)} \|_{0, \partial \Omega} \lesssim h^{s+t} \sup_{f \in M(\lambda^*), \| f \|_{0, \partial \Omega} = 1} \| A^* f \|_{1+t}.
\]
(4.11)

Substituting (4.11) into (4.7) and (4.10), we get (4.1) and (4.4), respectively.

The remainder is to prove (4.2), (4.3) and (4.5). An easy calculation shows that
\[
\langle (T - T_h) \varphi_i, \varphi_j^* \rangle = \langle T \varphi_i, \varphi_j^* \rangle - \langle T_h \varphi_i, \varphi_j^* \rangle
\]
\[
= a_h(A \varphi_i, A^* \varphi_j^*) - a_h(A_h \varphi_i, A_h^* \varphi_j^*)
\]
\[
= a_h(A \varphi_i - A_h \varphi_i, A^* \varphi_j^*) + a_h(A_h \varphi_i, A^* \varphi_j^* - A_h^* \varphi_j^*)
\]
\[
= a_h(A \varphi_i - A_h \varphi_i, A^* \varphi_j^*) + a_h(A \varphi_i, A^* \varphi_j^* - A_h \varphi_j)
\]
\[
- a_h(A \varphi_i - A_h \varphi_i, A^* \varphi_j^* - A_h^* \varphi_j^*).
\]
(4.12)
By (3.1) and (3.2) with \( f = \varphi_i, \varphi = A\varphi_i, g = \varphi_i^*, \) and \( \varphi^* = A^*\varphi^*_j, \) we obtain

\[
a_h(A\varphi_i - A_h\varphi_i, A^*\varphi^*_j) = D_h^*(A\varphi_i - A_h\varphi_i, A^*\varphi^*_j) + \langle T\varphi_i - T_h\varphi_i, \varphi^*_j \rangle,
\]

\[
a_h(A\varphi_i, A^*\varphi^*_j - A_h\varphi^*_j) = D_h(A\varphi_i, A^*\varphi^*_j - A_h\varphi^*_j) + \langle \varphi_i, T^*\varphi^*_j - T_h^*\varphi^*_j \rangle.
\]

Substituting the above two relations into (4.13), we get

\[
\langle (T - T_h)\varphi_i, \varphi^*_j \rangle = -D_h^*(A\varphi_i - A_h\varphi_i, A^*\varphi^*_j) - D_h(A\varphi_i, A^*\varphi^*_j - A_h\varphi^*_j) + a_h(A\varphi_i - A_h\varphi_i, A^*\varphi^*_j - A_h\varphi^*_j),
\]

which together with (3.7), (3.8), (3.21), and (3.23) yields

\[
\left\| \langle (T - T_h)\varphi_i, \varphi^*_j \rangle \right\| \lesssim h^2 t.
\]

Substituting (4.15), (4.11), and (4.12) into (4.8) and (4.9), we get (4.2) and (4.3), respectively.

From (2.1) and (2.4), we get

\[
a(u, v) = a(A(\lambda u), v), \quad \forall v \in H^1(\Omega),
\]

noting that \( k^2 \) is not an eigenvalue of (2.3), we have \( u = -\lambda Au. \) Similarly, using (2.10) and (2.12), we can get \( u_h = -\lambda_h A_h u_h. \) Thus, we have the identity

\[
u_h - u = (A - A_h)(\lambda_h u_h - \lambda u) - A(\lambda_h u_h - \lambda u) + \lambda Au - \lambda A_h u.
\]

From the triangular inequality, (3.21), (2.18), (2.25), (2.29), (4.3), and (4.4), we deduce

\[
\|u_h - u\| \leq \|A - A_h\|(\lambda u - \lambda_h u_h)\|_h + \|\lambda_h u_h - \lambda u\|_h + \|\lambda Au - \lambda A_h u\|_h
\]

\[
\lesssim h^s\|A(\lambda u - \lambda_h u_h)\|_{1+s} + \|\lambda_h u_h - \lambda u\|_{0,\partial\Omega} + h' \lesssim h^{(t+s)\frac{1}{2}} + h',
\]

i.e., (4.5) holds. The proof is completed.

\( \square \)

5 Numerical experiments

Consider the problem (1.1) on the test domain \( \Omega \subset \mathbb{R}^2, \) where \( \Omega = (-\sqrt{2}, \sqrt{2})^2 \) is a square, or \( \Omega = (-1, 1)^2 \setminus ([0, 1] \times (-1, 0]) \) is an L-shaped domain with the largest inner angle \( \omega = \frac{3}{2}\pi, \) or \( \Omega = \left(-\sqrt{2}, \sqrt{2}\right)^2 \setminus \{0 \leq x \leq \sqrt{2}, y = 0\} \) is a square with a slit which the largest inner angle \( \omega = 2\pi, \) or \( \Omega \) is a unit disk, and \( k = 1, n(x) = 4 \) or \( n(x) = 4 + 4i. \)

We use Matlab 2012a to solve (1.1) on a Lenovo ideaPad PC with 1.8GHZ CPU and 8GB RAM. Our program is compiled under the package of iFEM [27].

Referring to [21, 45], when \( n = 4, \) we sort eigenvalues in descending order, and when \( n = 4 + 4i, \) we arrange complex eigenvalues by their imaginary parts from large to small.

For the unit disk, the exact Stekloff eigenvalues are given in [21, 45], and when \( n = 4, \) the largest six eigenvalues are \( \lambda_1 = 5.151841, \lambda_{2,3} = 0.223578, \lambda_{4,5} = \)
Table 1 The eigenvalues on the square: $n = 4$

| dof   | $\lambda_{1,h}$ | $\lambda_{2,h}$ | $\lambda_{3,h}$ | $\lambda_{4,h}$ | $\lambda_{5,h}$ | $\lambda_{6,h}$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 3136  | 2.2018805       | -0.2116751      | -0.2116708      | -0.9069429      | -2.759883       | -2.752381       |
| 12416 | 2.2023533       | -0.2121076      | -0.2121070      | -0.9077740      | -2.7664177      | -2.7646187      |
| 49408 | 2.2024690       | -0.2122160      | -0.2122159      | -0.9079851      | -2.7683097      | -2.7678463      |
| 197120| 2.2024977       | -0.2122431      | -0.2122431      | -0.9080383      | -2.7687870      | -2.7686695      |

$-1.269100, \quad \lambda_6 = -2.472703$, and when $n = 4 + 4i$, the four complex eigenvalues with the largest imaginary parts are $\lambda_1 = -0.320506 + 3.121689i, \lambda_{2,3} = -0.136861 + 1.396737i, \lambda_4 = -1.353076 + 0.791723i$. For the L-shaped and the slit domain, the reference eigenvalues of the exact eigenvalues are listed in Tables 11 and 12.

5.1 Numerical experiments on uniform meshes

We adopt a uniform mesh $\pi_h$ for each domain. The numerical results are listed in Tables 1, 2, 3, 4, 5, 6, 7, and 8. The error curves of the C-R eigenvalues are showed in Figs. 1, 2, 3, and 4.

From Lemma 2, the regularity results, we know that for the square domain $2r = 2$, for the L-shaped domain $2r \approx \frac{4}{3}$, for the unit square with a slit $2r \approx 1$. From Fig. 1, we can see that the convergence order of $\lambda_{1,h}, \lambda_{2,h}, \ldots, \lambda_{6,h}$ is approximately equal to 2 on the square domain; from Fig. 2, we can see that the convergence order of $\lambda_{2,h}$ is approximately equal to $\frac{4}{3} \approx 1.333333$ on the L-shaped domain, and the eigenfunction corresponding to $\lambda_2$ has lower smoothness than others; from Fig. 3, we can see that the convergence order of $\lambda_{2,h}$ is approximately equal to 1 on the slit domain, and the eigenfunction corresponding to $\lambda_2$ is also less smoother than others, which are coincide with the theoretical results. Although there is an effect of reginal approximation for the computation on the disk, namely, replacing the disk $\Omega$ with a similar polygonal $\Omega^h$, from Tables 4 and 8 and Fig. 4, we see that C-R element eigenvalues can approximate the exact ones.

For the square and the L-shaped domain, we also compare the numerical results in Tables 1, 2, 5, and 6 with Tables 5.2, 5.3, 5.5, and 5.6 in [45] and find that, with the increase of dof (or the decrease of mesh size $h$), the eigenvalues obtained by C-R element and the conforming element are getting closer.

Table 2 The eigenvalues on the L-shaped domain: $n = 4$

| dof   | $\lambda_{1,h}$ | $\lambda_{2,h}$ | $\lambda_{3,h}$ | $\lambda_{4,h}$ | $\lambda_{5,h}$ | $\lambda_{6,h}$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 9344  | 2.5335485       | 0.8592520       | 0.1246281       | -1.0845725      | -1.0901869      | -1.4147102      |
| 37120 | 2.5333019       | 0.8583814       | 0.1245509       | -1.0851154      | -1.0909141      | -1.4163502      |
| 147968| 2.5332364       | 0.8580275       | 0.1245311       | -1.0852527      | -1.0911151      | -1.4167642      |
| 590848| 2.5332194       | 0.8578847       | 0.1245261       | -1.0852873      | -1.0911726      | -1.4168682      |
### Table 3 The eigenvalues on the square with a slit: $n = 4$

| dof  | $\lambda_{1,h}$ | $\lambda_{2,h}$ | $\lambda_{3,h}$ | $\lambda_{4,h}$ | $\lambda_{5,h}$ | $\lambda_{6,h}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 12448 | 1.4848728 | 0.4698829 | -0.1840366 | -0.6893862 | -1.897837 | -1.9264514 |
| 49472 | 1.4847611 | 0.4658257 | -0.1841411 | -0.6900139 | -1.8995947 | -1.9278655 |
| 197248 | 1.4847266 | 0.4637839 | -0.1841672 | -0.6900592 | -1.8998016 | -1.9283610 |
| 787712 | 1.4847163 | 0.4627589 | -0.1841737 | -0.6900708 | -1.8998539 | -1.9285538 |

### Table 4 The eigenvalues on the unit disk: $n = 4$

| dof  | $\lambda_{1,h}$ | $\lambda_{2,h}$ | $\lambda_{3,h}$ | $\lambda_{4,h}$ | $\lambda_{5,h}$ | $\lambda_{6,h}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 128628 | 5.1514757 | 0.2235716 | 0.2235710 | -1.2689792 | -1.2689809 | -2.4724133 |
| 201444 | 5.1516049 | 0.2235738 | 0.2235738 | -1.2690218 | -1.2690228 | -2.4725174 |
| 359676 | 5.1517065 | 0.2235759 | 0.2235759 | -1.2690553 | -1.2690557 | -2.4725980 |
| 809421 | 5.1517811 | 0.2235773 | 0.2235773 | -1.2690803 | -1.2690804 | -2.4726559 |

### Table 5 The eigenvalues on the square: $n = 4 + 4i$

| dof  | $\lambda_{1,h}$ | $\lambda_{2,h}$ | $\lambda_{3,h}$ | $\lambda_{4,h}$ | $\lambda_{5,h}$ | $\lambda_{6,h}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 3136 | 0.687353 | -0.342514 | -0.342525 | -0.948908 | -2.779702 | -2.786716 |
|      | +2.494448i | +0.85089i | +0.850899i | +0.539844i | +0.53745i | +0.539647i |
| 12416 | 0.686749 | -0.342915 | -0.342916 | -0.949807 | -2.792169 | -2.794033 |
|      | +2.495075i | +0.850782i | +0.850784i | +0.540029i | +0.539739i | +0.540444i |
| 49408 | 0.686601 | -0.343014 | -0.343014 | -0.950034 | -2.7975417 | -2.7995897 |
|      | +2.495238i | +0.850755i | +0.850756i | +0.540079i | +0.540498i | +0.540656i |
| 197120 | 0.686564 | -0.343038 | -0.343038 | -0.950091 | -2.796245 | -2.796367 |
|      | +2.495280i | +0.850749i | +0.850749i | +0.540092i | +0.540671i | +0.540711i |

### Table 6 The eigenvalues on the L-shaped domain: $n = 4 + 4i$

| dof  | $\lambda_{1,h}$ | $\lambda_{2,h}$ | $\lambda_{3,h}$ | $\lambda_{4,h}$ | $\lambda_{5,h}$ | $\lambda_{6,h}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 9344 | 0.513857 | 0.398298 | -0.076964 | -1.438567 | -1.654555 | -2.513849 |
|      | +2.881404i | +1.459758i | +1.042587i | +0.803689i | +0.766423i | +0.570528i |
| 37120 | 0.514176 | 0.397512 | -0.077125 | -1.440022 | -1.66531 | -2.516699 |
|      | +2.882086i | +1.459328i | +1.042656i | +0.804437i | +0.766548i | +0.571289i |
| 147968 | 0.514259 | 0.397218 | -0.077165 | -1.440388 | -1.657092 | -2.517426 |
|      | +2.882263i | +1.459129i | +1.042672i | +0.80463i | +0.766548i | +0.571486i |
| 590848 | 0.514280 | 0.397106 | -0.077175 | -1.440479 | -1.657258 | -2.517610 |
|      | +2.882308i | +1.459043i | +1.042677i | +0.804678i | +0.766534i | +0.571536i |
Table 7  The eigenvalues on the square with a slit: \( n = 4 + 4i \)

| dof    | \( \lambda_{1,h} \) | \( \lambda_{2,h} \) | \( \lambda_{3,h} \) | \( \lambda_{4,h} \) | \( \lambda_{5,h} \) | \( \lambda_{6,h} \) |
|--------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 12448  | 0.918974             | 0.299813             | −0.262446            | −0.741837            | −2.615356            | −2.840331            |
|        | +1.770802i           | +1.003519i           | +0.757437i           | +0.608741i           | +0.561764i           | +0.493956i           |
| 49472  | 0.919206             | 0.296211             | −0.262573            | −0.742028            | −2.618344            | −2.845935            |
|        | +1.770795i           | +1.001745i           | +0.757447i           | +0.608765i           | +0.562409i           | +0.493673i           |
| 197248 | 0.919276             | 0.294417             | −0.262604            | −0.742076            | −2.619113            | −2.847993            |
|        | +1.770791i           | +1.000826i           | +0.757449i           | +0.608772i           | +0.562579i           | +0.493444i           |
| 787712 | 0.919297             | 0.293522             | −0.262612            | −0.742088            | −2.619310            | −2.848830            |
|        | +1.770789i           | +1.000356i           | +0.75745i            | +0.608774i           | +0.562623i           | +0.493306i           |

Table 8  The eigenvalues on the unit disk: \( n = 4 + 4i \)

| dof    | \( \lambda_{1,h} \) | \( \lambda_{2,h} \) | \( \lambda_{3,h} \) | \( \lambda_{4,h} \) |
|--------|----------------------|----------------------|----------------------|----------------------|
| 128628 | −0.320420            | −0.136864            | −0.136865            | −1.352964            |
|        | +3.124755i           | +1.39673i            | +1.39673i            | +0.79174i            |
| 201444 | −0.320451            | −0.136863            | −0.136863            | −1.353004            |
|        | +3.124732i           | +1.396733i           | +1.396733i           | +0.791734i           |
| 359676 | −0.320475            | −0.136862            | −0.136862            | −1.353035            |
|        | +3.124714i           | +1.396735i           | +1.396735i           | +0.79173i            |
| 809421 | −0.320492            | −0.136862            | −0.136862            | −1.353058            |
|        | +3.124700i           | +1.396736i           | +1.396736i           | +0.791726i           |

Fig. 1  The error curves of the first six eigenvalues on the square (left: \( n = 4 \), right: \( n = 4 + 4i \))
Fig. 2 The error curves of the first six eigenvalues on the L-shaped domain (left: $n = 4$, right: $n = 4 + 4i$)

5.2 Numerical experiments on adaptive meshes

In practical finite element computations, it is desirable to carry out the computations in an adaptive fashion (see, e.g., [1, 9, 50, 52, 53] and references cited therein). For the C-R element approximation of Steklov eigenvalue problem, the a posteriori error estimates have been developed by [48]. Referring to [48], in this subsection, we give the a posteriori error estimators by formal deduction and implement adaptive computation for (1.1).

Let $\ell \in E_h^i$ shared by elements $\kappa_1$ and $\kappa_2$, i.e., $\ell = \partial \kappa_1 \cap \partial \kappa_2$. We choose a unit normal vector $\gamma_\ell$, pointing outwards $\kappa_2$, and set the jumps of the normal derivatives of $v_h$ across $\ell$ as:

$$[[\nabla v_h]]_Y = \nabla v_h|_{\kappa_2} \cdot \gamma_\ell - \nabla v_h|_{\kappa_1} \cdot \gamma_\ell.$$ 

Fig. 3 The error curves of the first six eigenvalues on the square with a slit (left: $n = 4$, right: $n = 4 + 4i$)
Denote $\gamma_\ell = (\gamma_{\ell 1}, \gamma_{\ell 2})$, then the tangent $t_\ell = (-\gamma_{\ell 2}, \gamma_{\ell 1})$ on $\ell$, and we write the jumps of the tangential derivatives of $v_h$ across $\ell$ as:

$$[[\nabla v_h]]_\ell = \nabla v_h|_{\kappa_2} \cdot t_\ell - \nabla v_h|_{\kappa_1} \cdot t_\ell.$$ 

Notice that these values are independent of the chosen direction of the normal vector $\gamma_\ell$.

Now we define the a posteriori error indicators $\eta_\kappa(u_h)$ on $\kappa$ and $\eta(u_h)$ on $\Omega$ for the primal eigenfunction $u_h$:

For each $\ell \in \mathcal{E}$, let

$$J_{\ell, t}(u_h) = \begin{cases} [[\nabla u_h]]_\ell, & \text{if } \ell \in \mathcal{E}^i, \\ 0, & \text{if } \ell \in \mathcal{E}^b, \end{cases} 
J_{\ell, \gamma}(u_h) = \begin{cases} [[\nabla u_h]]_\ell, & \text{if } \ell \in \mathcal{E}^i, \\ 2(\nabla u_h \cdot \gamma_\ell - \lambda_h u_h)|_\ell, & \text{if } \ell \in \mathcal{E}^b, \end{cases}$$

and let

$$\eta_\kappa(u_h)^2 = |\kappa| k^2 n u_h \|_{0, \kappa}^2 + \frac{1}{2} \sum_{\ell \in \partial \kappa} |\ell| \| J_{\ell, \gamma}(u_h) \|^2_{0, \ell} + \frac{1}{2} \sum_{\ell \in \partial \kappa} |\ell| \| J_{\ell, t}(u_h) \|^2_{0, \ell},$$

$$\eta(u_h)^2 = \sum_{\kappa \in \pi_h} \eta_\kappa(u_h)^2.$$ 

Similarly, we define the a posteriori error indicators $\eta_\kappa(u_h^*)$ on $\kappa$ and $\eta(u_h^*)$ on $\Omega$ for the dual eigenfunction $u_h^*$.

We use $\sum_{\kappa \in \pi_h} (\eta_\kappa^2(u_h) + \eta_\kappa^2(u_h^*))$ as the a posteriori error indicator of $\lambda_h$. Using the indicator and consulting the existing standard adaptive algorithms (see, e.g., [27, 31, 46]), we solve (1.1). From Figs. 2 and 3, we find that the eigenfunction associated with $\lambda_2$ is singular, so in our numerical experiments, we compute the approximation of the second eigenvalue $\lambda_2$, and the numerical results on the L-shaped domain and the slit domain are listed in Table 9 and Table 10, respectively.

We show the curves of error and the a posteriori error estimators obtained by adaptive computing for the eigenvalue $\lambda_{2, h}$ in Figs. 5 and 6. It can be seen from them that the error curves and the error estimators’ curves are both basically parallel to the
Table 9 The second eigenvalues on adaptive meshes on the L-shaped domain

| l  | dof   | $\lambda_{2,h}(n = 4)$ | l  | dof   | $\lambda_{2,h}(n = 4 + 4i)$ |
|----|-------|------------------------|----|-------|-----------------------------|
| 1  | 9344  | 0.859246               | 1  | 9344  | 0.398302+1.459749i          |
| 2  | 10022 | 0.858839               | 2  | 9494  | 0.398303+1.459755i          |
| 25 | 202370| 0.857844               | 37 | 216512| 0.397113+1.459061i          |
| 26 | 225490| 0.857841               | 38 | 242902| 0.397112+1.459059i          |
| 27 | 249481| 0.857838               | 39 | 286483| 0.397108+1.459048i          |
| 28 | 276807| 0.857832               | 40 | 310425| 0.397092+1.459023i          |
| 29 | 331662| 0.857821               | 41 | 340309| 0.397092+1.459023i          |
| 30 | 387329| 0.857817               | 42 | 391833| 0.397084+1.459016i          |

Table 10 The second eigenvalues on adaptive meshes on the square with a slit

| l  | dof   | $\lambda_{2,h}(n = 4)$ | l  | dof   | $\lambda_{2,h}(n = 4 + 4i)$ |
|----|-------|------------------------|----|-------|-----------------------------|
| 1  | 12448 | 0.469884               | 1  | 12448 | 0.299812+1.003523i          |
| 2  | 12472 | 0.467948               | 2  | 12607 | 0.298475+1.002875i          |
| 25 | 170854| 0.461819               | 60 | 241012| 0.292782+0.999958i          |
| 26 | 192640| 0.461803               | 61 | 250930| 0.292782+0.999958i          |
| 27 | 222566| 0.461788               | 62 | 260992| 0.292761+0.999943i          |
| 28 | 261309| 0.461786               | 63 | 295455| 0.292741+0.999931i          |
| 29 | 298511| 0.461783               | 64 | 311350| 0.292738+0.999931i          |
| 30 | 335598| 0.461779               | 65 | 338930| 0.292725+0.999926i          |

Fig. 5 The error curves of the second eigenvalues on the L-shaped domain (left: $n = 4$, right: $n = 4 + 4i$)
line with slope $-1$, which indicate that the a posteriori error estimators of numerical eigenvalues are reliable and efficient and $\lambda_{2,h}$ achieves the convergence rate $O(h^2)$.

From tables and figures, we also see that under the same dof, the accuracy of approximate eigenvalues computed on adaptive meshes is far higher than that of approximate eigenvalues computed on uniform meshes.

**Remark 3** (The lower/upper bound of the Stekloff eigenvalues).

We find in Tables 1, 2, 3, 4, 9, and 10 that when the index of refraction $n(x)$ is real, all series of eigenvalues computed by C-R element show the tendency to decrease as the increase of dof except $\lambda_1$ on the square and $\lambda_1 \sim \lambda_3$ on the unit disk. Note that the numerical results in [21, 45] indicate that the conforming finite element eigenvalues approximate the exact ones from below when $n(x)$ is real. So we also use the P1 conforming element to compute, and obtain reference values of the exact eigenvalues by averaging the P1 conforming eigenvalues $\lambda_{j,h}^C$ and the C-R element eigenvalues $\lambda_{j,h}$. We list them in Tables 11 and 12. The property of monotone convergence of the conforming finite element eigenvalues is easy to prove. However, the property of monotone convergence of the C-R nonconforming finite element eigenvalues is a

### Table 11: The reference eigenvalues $\lambda_j(L)$ on the L-shaped domain and $\lambda_j(Slit)$ on the square with a slit: $n = 4$

| j | $\lambda_{j,h}$ | $\lambda_{j,h}^C$ | $\lambda_j(L)$ | $\lambda_{j,h}$ | $\lambda_{j,h}^C$ | $\lambda_j(Slit)$ |
|---|-------|-------|----------|-------|-------|---------|
| 1 | 2.533219 | 2.533209 | 2.533214 | 1.484716 | 1.484710 | 1.484713 |
| 2 | 0.8578847 | 0.8577495 | 0.8578171 | 0.4627589 | 0.4612150 | 0.4619870 |
| 3 | 0.1245261 | 0.1245229 | 0.1245245 | $-0.1841737$ | $-0.1841765$ | $-0.1841751$ |
| 4 | $-1.085287$ | $-1.085303$ | $-1.085295$ | $-0.6900708$ | $-0.6900769$ | $-0.6900738$ |
| 5 | $-1.091173$ | $-1.091207$ | $-1.091190$ | $-1.899854$ | $-1.899878$ | $-1.899866$ |
| 6 | $-1.416868$ | $-1.416912$ | $-1.416890$ | $-1.928554$ | $-1.928784$ | $-1.928669$ |
Table 12 The reference eigenvalues $\lambda_j(L)$ on the L-shaped domain and $\lambda_j(\text{Slit})$ on the square with a slit: $n = 4 + 4i$

| $j$ | $\lambda_{j,h}$ | $\lambda_{j,h}^C$ | $\lambda_j(L)$ | $\lambda_{j,h}$ | $\lambda_{j,h}^C$ | $\lambda_j(\text{Slit})$ |
|-----|-----------------|-------------------|----------------|-----------------|-------------------|----------------------------|
| 1   | 0.5142799       | 0.5143106         | 0.5142952      | 0.9192965       | 0.9193164         | 0.9193065                   |
|     | $+2.882308i$  | $+2.882334i$      | $+2.882321i$   | $+1.770789i$    | $+1.770782i$      | $+1.770786i$               |
| 2   | 0.3971057       | 0.3969716         | 0.3970387      | 0.2935223       | 0.2917372         | 0.2926298                   |
|     | $+1.459043i$  | $+1.458911i$      | $+1.458977i$   | $+1.000356i$    | $+0.9993946i$     | $+0.9998754i$              |
| 3   | $-0.0771754$   | $-0.0771792$      | $-0.0771773$   | $-0.2626120$    | $-0.2626151$      | $-0.2626135$               |
|     | $+1.042677i$  | $+1.042673i$      | $+1.042675i$   | $+0.7574501i$   | $+0.7574481i$     | $+0.7574491i$              |
| 4   | $-1.440479$    | $-1.440535$       | $-1.440507$    | $-0.7420884i$   | $-0.7420981i$     | $-0.7420933$               |
|     | $+0.8046784i$ | $+0.8047093i$     | $+0.8046939i$  | $+0.6087744i$   | $+0.6087755i$     | $+0.6087749i$              |
| 5   | $-1.657258$    | $-1.657409$       | $-1.657333$    | $-2.619310$     | $-2.619442$       | $-2.619376$                |
|     | $+0.7665341i$ | $+0.7664933i$     | $+0.7665137i$  | $+0.5626232i$   | $+0.5626581i$     | $+0.5626407i$              |
| 6   | $-2.517610$    | $-2.517765$       | $-2.517687$    | $-2.848830$     | $-2.850239$       | $-2.849534$                |
|     | $+0.5715365i$ | $+0.5715597i$     | $+0.5715481i$  | $+0.4933060i$   | $+0.4929997i$     | $+0.4931528i$              |

non-trivial result. In 2014, Carstensen and Gedicke [25] prove rigorously the monotonicity for the classical Laplace eigenvalue problem. For the Stekloff eigenvalue problem considered in this paper, it is meaningful to study the monotone convergence of C-R element eigenvalues.

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