Exact quantisation of $\text{U}(1)^3$ quantum gravity via exponentiation of the hypersurface deformation algebroid

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Abstract

The $\text{U}(1)^3$ model for 3+1 Euclidian signature general relativity (GR) is an interacting, generally covariant field theory with two physical polarisations that shares many features of Lorentzian GR. In particular, it displays a non-trivial realisation of the hypersurface deformation algebroid with non-trivial, i.e. phase space dependent structure functions rather than structure constants. In this paper we show that the model admits an exact quantisation. The quantisation rests on the observation that for this model and in the chosen representation of the canonical commutation relations the density unity hypersurface algebra can be exponentiated on non-degenerate states. These are states that represent a non-degenerate quantum metric and from a classical perspective are the relevant states on which the hypersurface algebra is representable. The representation of the algebra is exact, with no ambiguities involved and anomaly free. The quantum constraints can be exactly solved using groupoid averaging and the solutions admit a Hilbert space structure that agrees with the quantisation of a recently found reduced phase space formulation. Using the also recently found covariant action for that model, we start a path integral or spin foam formulation which, due to the Abelian character of the gauge group, is much simpler than for Lorentzian signature GR and provides an ideal testing ground.
for general spin foam models. The solution of $U(1)^3$ quantum gravity communicated in this paper motivates an entirely new approach to the implementation of the Hamiltonian constraint in quantum gravity.

Keywords: hypersurface deformation algebroid, quantum gravity, Bergmann Komar group

1. Introduction

The initial value formulation of general relativity (GR) [1] is the starting point for both numerical GR [2] producing black hole merger templates of ever increasing accuracy [3] and canonical quantisation [4]. A central ingredient in this approach to the dynamics of both classical and quantum gravity are the initial value constraints, known as the spatial diffeomorphism and Hamiltonian constraints. With respect to an ADM foliation [5] of spacetime they play a dual role first as temporal-spatial and temporal-temporal projections of the Einstein equations and second as generator gauge transformations as well as dynamical equations for the two physical polarisations of the spacetime metric.

These constraints form a closed algebroid [6] under Poisson brackets known as the hypersurface deformation algebroid (HDA) [7]. The term algebroid rather than algebra emphasises the fact that in contrast to a Poisson Lie algebra the constraints do not close with structure constants but with structure functions that have a non-trivial dependence on the phase space point that one is considering. Thus, multiple Poisson brackets produce always a linear combination of constraints, however, the coefficients of these linear combinations are functions on the phase space which become more and more complicated the more Poisson brackets one computes. This is in contrast to 1-parameter groups of spacetime diffeomorphisms whose generating vector fields do form a true Lie algebra and not just an algebroid. Indeed it is known that the 1-1 correspondence between the canonical constraint algebroid and the spacetime diffeomorphism algebra holds only ‘on shell’ i.e. when the Einstein equations are satisfied.

Due to the structure functions, a formal exponentiation of the generators of the algebroid does not yield an object that one could call a (Lie) group. However, any algebroid can be turned into a Lie algebra by computing all multiple Poisson brackets between the algebroid generators and adding the right hand sides so obtained to the set of generators of a true Lie algebra (one should restrict to a linearly independent set). This will in general be an infinite dimensional Lie algebra, even if the number of algebroid generators is finite. That Lie algebra can now be exponentiated in the usual way and yields a Lie group. In what follows we will call that object a groupoid from the point of view of the underlying algebroid or a group from the point of view of the Lie algebra obtained from the algebroid as above. See [8] and references therein for details. We also understand that the Bergman-Komar group [6] is mathematically defined in this way.

The fact that the constraint algebra of GR is an algebroid and not an algebra is one of the many reasons why it continues to be so difficult to construct a theory of quantum gravity. In recent decades some progress has been made using the connection formulation [9] and gave rise to a quantisation programme coined loop quantum gravity (LQG) [10]. The name arose because of the similarity of LQG to lattice gauge theory [11] in which gauge covariant Wilson loop functionals play a fundamental role. While it has been possible to define the quantum constraint on a common, dense, invariant domain [12] of a rigorously defined Hilbert space representation of the canonical commutation and * relations [13] and while the corresponding quantum algebroid indeed closes and in that sense is mathematically consistent, it closes with the wrong quantum structure functions [14], thereby exhibiting a physical anomaly. To
improve this, there have been at least four suggestions: In the master constraint [15] approach structure functions are omitted altogether by a classical equivalent reformulation of the constraints in terms of a single constraint. In the reduced phase space approach [16] one solves the constraints classically and thus avoids the issue of quantum structure functions. In [17] one uses classically equivalent constraints which are linear combinations of the original ones with phase space dependent coefficients (‘electric shifts’ and non-standard density weight for the Hamiltonian constraint) and tries to define those on a space of distributions over a dense subspace of the Hilbert space rather than that dense subspace itself. Finally in [18] one uses renormalisation techniques to define a renormalisation group flow of structure operators whose fixed point should be the physically correct ones.

These four programmes have been tested in various model situations, see e.g. [17, 19–21] and references therein. Here we wish to focus on two models, parametrised field theory (PFT) in two spacetime dimensions [22] and the U(1)$^3$ model for 3+1 Euclidian GR [23]. Both models are much simpler than GR but still share with GR the fact that they exhibit a non-trivial hypersurface deformation algebra. The constraints of PFT are yet much simpler than those of the U(1)$^3$ theory in that in their density weight two version they form a true (centrally extended) algebra rather than an algebroid while this is no longer the case for the U(1)$^3$ model which is therefore a much better testing ground.

The application of [12] to PFT can be found in [24], the application of [17] to PFT in [25] and the application of [18] to PFT in [26]. The application of [16] to U(1)$^3$ can be found in [27], the application of [17] to U(1)$^3$ in [28]. The work [26] shows that in PFT it is possible to find the correct anomaly free fixed point algebra using renormalisation and the work [29] that one can modify the constraints of PFT to have non-trivial structure functions with density weight unity constraints which still form a closed quantum algebroid on a space of distributions which are non-degenerate in the sense that the quantum metric is diagonal with non-degenerate eigenvalues. The work [27] shows that the non standard density weight is motivated by a space of distributions representing degenerate quantum metrics. Note that the work [28] is purely classical.

In this paper we show that the U(1)$^3$ model for 3+1 Euclidian GR, or U(1)$^3$ quantum gravity for short, can be solved exactly following analogous steps as in LQG. What makes this possible is the fact that, while the constraints of the U(1)$^3$ model display exactly the same algebraic structure as those for 3+1 Euclidian gravity, the latter is for the non-Abelian gauge group SU(2) rather than the Abelian U(1)$^3$. This has the consequence that all constraints are at most linear in the connection rather than quadratic. While this does not turn U(1)$^3$ into a free or topological theory, it is in fact highly interacting and displays two propagating polarisations, the linearity in the connection makes it possible to exponentiate the HDA.

In LQG one already exponentiated the spatial diffeomorphism constraints. This is relatively straightforward as those constraints form a closed and true sub Lie algebra of the hypersurface deformation algebra. In LQG the spatial diffeomorphisms act by unitary operators but 1-parameter subgroups do not act weakly continuously. Similarly in this paper we show that the exponentiated Hamiltonian constraints act by unitary operators, but 1-parameter ‘subgroupoids’ do not act weakly continuously. Just as in the classical theory, while the composition of exponentiated quantum Hamiltonian constraint actions can be computed in a form as closed as in the classical theory commutators thereof cannot be written as an action of an exponentiated spatial diffeomorphism due to the structure operators involved. Yet, the resulting expression is precisely the action of the corresponding classically exponentiated constraints and thus is represented without anomaly, directly on Hilbert space, without considering any dual spaces and without any ambiguities. It is on dual spaces that one can compute infinitesimal actions
and these can also been shown to be anomaly free, giving a representation of the HDA, without any ambiguities.

The Hilbert space representation considered in this paper for $U(1)^3$ theory is similar to but different from the LQG representation. It is based on generalised holonomies of the connection. The generalisation consists in modifying all three ingredients of an LQG spin network function: Graphs, spins and intertwiners are replaced by divergence free smearing functions. This can considered as ‘thickening’ the graph edges, to allow real valued rather than half integral spin quantum numbers on the edges and to take care of the Abelian nature of the gauge group by replacing invariant intertwiners at vertices by the divergence free condition.

This more general state space is precisely what allows the exponentiation of the Hamiltonian constraint. Just as for the spatial diffeomorphism constraint, that action makes use of the classical exponentiation, i.e. the Hamiltonian flow of the corresponding Hamiltonian vector field and that flow preserves a suitable space of non-degenerate smearing functions. That flow can be worked out as usual by Taylor expansion and as expected acts highly non-linearly on the space of smearing functions. This is in sharp contrast to the exponentiated spatial diffeomorphism constraint which has a linear action. In fact, for the $U(1)^3$ model the exponentiated Hamiltonian constraint flow can be defined for any density weight of the Hamiltonian constraint including density weight one. Then the action is not polynomial in the smearing function, not even when truncating the Taylor series to finite order. However, no matter how non-polynomial that action is, it maps the space of divergence free, density weight unity vector fields onto themselves. To the best of knowledge of the author, such a non-polynomial representation of the HDA has not been discussed previously in the literature and it is interesting to see how the Hamiltonian constraint in fact naturally generates it without any guess work about loop attachments ever necessary.

The lessons to be learnt from the present paper for actual GR, in the opinion of the author, are as follows:

1. The present exposition once again stresses the importance of the quantum non-degeneracy condition for a faithful representation of the HDA [29].
2. The model shows that there is no obstacle in using the natural density unity form of the Hamiltonian constraint. As shown in [29] density unity is enforced as soon as one considers as a cosmological constant or additional matter terms in GR.
3. The model shows that one can obtain anomaly free closure of the constraint algebra directly on the kinematical Hilbert space without invoking dual spaces, in particular the closure is off-shell.

The architecture of this paper is as follows:

In section 2 we briefly outline the classical description of the $U(1)^3$ model both in its covariant and canonical formulation [27].

Section 3 is the key section of the present paper. We show that any classical constraint linear in momentum $p$ admits a unitary representation in a Hilbert space representation of the CCR and the $*$ relations based on a cyclic vacuum $\Omega$ for the Weyl operators depending only on $p$ which is annihilated by the conjugate variable $q$. This holds no matter how non-linearly the dependence of the constraint on $q$ maybe. In particular, this allows to extend $U(1)^3$ QG by a quantum cosmological constant and the spatial Ricci scalar that allows to treat Lorentzian signature as well. The resulting action of the constraints is free of any (ordering) ambiguities.
In section 4 we define such a representation of the CCR and ∗ relations for the $U(1)^3$ model and apply the theorem to the spatial diffeomorphism and Hamiltonian constraint. We compute explicitly the first few terms of the Taylor expansion mentioned above and demonstrate unitarity, and anomaly freeness of the exponentiated constraints.

In section 5 we compute the dual action of the constraints on suitable distributions and verify anomaly freeness of the algebroid.

In section 6 we solve the quantum constraints by groupoid averaging and demonstrate that we arrive precisely at a reduced phase space quantisation of the reduced phase space description of [27]. In particular we can construct the unitary 1-parameter group generated by the physical Hamiltonian. The resulting theory is a kind of self-interacting, non-polynomial quantum electrodynamics with two propagating polarisations.

In section 7 we construct new non-relational, weak Dirac observables which are not related to any gauge fixing condition, both classically and quantum mechanically. We establish that they weakly commute with all quantum constraints.

In section 8 we develop a path integral formulation of the reduced description of the model that by construction is equivalent to the canonical operator theory. An interesting aspect is that instead of the usual heuristic undefined Lebesgue measure expressions, the systematic derivation yields automatically Bohr and discrete measures instead.

In section 9 we unfold the reduced phase space path integral and arrive at a covariant formulation of the rigging map (‘projector’ into the constraint kernel) which depends on the exponent of the classical action. This is interesting because the rigging map is difficult to construct when the constraints do not form a Lie algebra [30]. We transform that path integral over $U(1)^3$ connections and tetrads into a path integrals over connections and a $B$ field ($BF$ formulation) which is subject to $U(1)^3$ simplicity constraints. This reformulation is the starting point for a systematic spin foam treatment which should be much simpler in this Abelian setting. The fact that this model receives a substantial amount of guidance from the canonical treatment layed out in this paper may help to deepen the bridge between canonical and covariant LQG. The details of the spin foam formulation of $U(1)^3$ can be worked out using the results of the present paper and are reserved for future research.

In section 10 we summarise and conclude. In particular we compare the action of the Hamiltonian constraint in the usual LQG representation with that of the present paper. This is possible because one can mollify the form factors of the usual formulation to arrive at form factors of the present formulation. Among other things, a main difference between the two actions is that the lapse function in the present setting becomes part of the form factor in a non-linear fashion, i.e. it is not simply a coefficient in an expansion of spin (or charge) network functions.

In appendix we show that the present work extends to Lorentzian signature and a cosmological constant.

2. Classical $U(1)^3$ theory

Our exposition will be minimal. The details can be found in [23, 27, 28].

A possible action has recently been found in [27]

$$\int_M d^4X \det \left( \left\{ e^K_{\rho} \right\} \right) F_{\mu\nu}^I e^I_\mu e^J_\nu$$  \hspace{1cm} (2.1)
with spacetime tensor indices \( \mu, \nu, \rho = 0, 1, 2, 3 \) and frame indices \( I, J, K = 0, 1, 2, 3 \). The field \( e^I_a \) is a co-tetrad with inverse \( e^I_b \). The field \( F_{\mu \nu} \) is not the curvature of the Palatini formulation of Lorentzian vacuum GR but rather is constrained by

\[
F_{\mu \nu}^k = \frac{1}{2} \delta^k_{\mu} \epsilon_{\nu m k} F_{\mu \nu}^l =: F_{\mu \nu}^k = \frac{1}{2} \delta_{\mu}^k \epsilon_{\nu m k} \]

(2.2)

where \( A^I_{\mu} \) is a spacetime \((1,1)\) connection with \( j, k, \ell = 1, 2, 3 \). If we would add to \( F_{\mu \nu}^k \) the quadratic term \( \epsilon^{\mu \nu} \delta_{\kappa \ell \mu} \partial_{\mu} A^\kappa_a \partial_{\ell} \) we would obtain precisely the self-dual formulation of Euclidian signature GR \cite{27} and one would pass from the Abelian group \((1,1)\) to the non-Abelian group \( SU(2) \). The action (2.1) could be generalised by an Immirzi parameter \cite{27,31} but for the purpose of this paper (2.1) will be sufficient.

A careful Dirac constraint analysis of (2.1) reveals \cite{27} that the 3+1 split \( M \cong \mathbb{R} \times \sigma \) of the action, after getting rid of second class constraints, exhibits the following canonical data:

There are conjugate pairs \((A^I_{\mu}, E^a_I)\) with \( a, b, c = 1, 2, 3 \) and constraints

\[
G_j = \partial_a E^a_j, \\
D_a = F_{ab} E^b_j - A^I_{\mu} G_j, \\
C = F_{ab} \epsilon^{a b c} E^c_j \delta_{k m} \delta_{n \mu n} | \det(E) |^{-1/2}
\]

(2.3)

which are produced in the Dirac analysis directly in the natural density unity form. The constraints generate \((1,1)\) gauge transformations, spatial diffeomorphisms and Hamiltonian transformations respectively. Their smeared versions

\[
G [s] := \int_\sigma d^3 x s^j G_j, \quad D [u] := \int_\sigma d^3 x u^a D_a, \quad C [M] := \int_\sigma d^3 x M C,
\]

(2.4)

yields the Poisson algebra

\[
\{ G[s], G[t] \} = 0, \quad \{ D[u], G[s] \} = -G[u[s]], \quad \{ C[M], G[s] \} = 0, \quad \{ D[u], D[v] \} = -D[u,v], \quad \{ D[u], C[M] \} = -C[u,M] \\
\{ C[M], C[N] \} = -D[q^{-1}][M dN - N dM]
\]

(2.5)

which is computed using the fundamental Poisson brackets

\[
\{ E^a_I (x), A^b_{\mu} (y) \} = \delta^a_b \delta^I_\mu \delta (x, y)
\]

(2.6)

and where \( s^I, M \) are scalars, \( u \) is a vector field, \( u[s], u[M] \) is the action of a vector field on scalars and \( [u,v] \) the commutator vector field. Finally and most importantly for the present paper the metric field

\[
q_{ab} := e^I_a e^I_b \delta_{\mu \nu}, \quad E^3_I := \sqrt{\det(q)} q^{ab} e^I_b
\]

(2.7)

is a spatial metric derived from the field \( e^I_a \) which appears in the action (2.1), i.e. \( e^I_a \) for \( \mu = a, l = j \).

The four last relations in (2.5) show that \( D[u], C[M] \) represent the \( HDA \) \( \mathfrak{h} \) \cite{7}. The algebra does not close with the structure constants (i.e. smearing functions are independent of \( A, E \)) but with structure functions encoded in the inverse \( (q^{-1})^{ab} := q^{ab}, q^{ac} q_{cb} = \delta^a_b \) of the metric. The whole chain of relations (2.7) and the appearance of \( q^{-1} \) make it transparent that the very formulation of \( \mathfrak{h} \) assumes that the metric non-degenerate, i.e. nowhere singular. This is of course a necessary condition of classical GR as otherwise signature, curvature etc would not be well defined on the whole spacetime manifold. The classical hypersurface deformation algebra therefore reminds us of this basic but important non-degeneracy condition. Also, when going through the details of the classical calculation that leads to (2.5), one makes use of that.
non-degeneracy in every single step of the calculation. One may argue that for the density weight two version of the Hamiltonian constraint the classical non-degeneracy condition is not necessary. However, as shown in [29] only density weight unity is possible in Lorentzian vacuum quantum GR or even Euclidean quantum GR with a cosmological constant. As shown in [12], density weight unity is also consistent with any known matter coupling.

Now in LQG [10] this non-degeneracy condition is dealt with as follows: The state space of LQG is the closed span of (so called spin network) functions that exhibit excitations of the quantum metric only on the edges of finite graphs, therefore a typical LQG state is actually degenerate almost everywhere. Nevertheless, operators corresponding to $q_{ab}$ that appear in (2.5) must be densely defined on such states. This is indeed possible [12] exploiting singular properties of the LQG volume operator [32]. Obviously that operator must then vanish where the metric is not excited, i.e. the inverse quantum metric vanishes at zero quantum metric while the inverse classical metric diverges at zero classical metric. While this comes out somewhat naturally from the LQG framework, it has the following far reaching consequence: The last Poisson bracket relation in (2.5) relies on the fact that the inverse classical metric is nowhere vanishing. Would it be non-vanishing almost everywhere then the right hand side of that last relation would actually vanish because the Riemann/Lebesgue integral does not ‘see’ this set. This is precisely what one observes when one computes the dual action of the operators $C[M]$ on suitable spaces of distributions [33]. While this behaviour obviously depends on the selected space of distributions and while the commutator on the LQG Hilbert space itself does not vanish [12] it is clear that a proper representation of $\hbar$ in the quantum theory must be subject to some kind of quantum non-degeneracy in order to circumvent these difficulties if one does not want to change the density weight of the Hamiltonian constraint [17] which is forbidden by cosmological constant, Lorentzian rather than Euclidian signature vacuum terms and matter [29]. On the other hand, it is precisely due to the fact that the inverse volume operator of LQG vanishes except at vertices of a graph which ensures that the Hamiltonian constraint $C$ is densely defined. Thus we see that the domains of $\hbar$ and $C$ in LQG presently exclude each other. The key to reconcile them is to directly work with the exponentiated version of $C$ which frees us from defining $C$ separately and allows us to exclusively work with $\hbar$. This what is achieved in the $U(1)^3$ model as we show in the next section.

3. Quantum U(1)$^3$ theory

We will use some of the properties of LQG but modify them in some important details. We first define a Hilbert space representation of the CCR and $^\ast$ relations (we work in units with $\hbar = 1$ and drop an analog of Newton’s constant)

$$\begin{align*}
\left[ E^i_j (x), E^k_l (y) \right] &= \left[ A^i_j (x), A^k_l (y) \right] = 0, \\
\left[ E^i_j (x), A^k_l (y) \right] &= \delta^i_k \delta^j_l \delta (x,y), \\
\left[ E^i_j (x) \right]^\ast - E^i_j (x) &= \left[ A^i_j (x) \right]^\ast - A^i_j (x) = 0.
\end{align*}$$

(3.1)

Abusing the notation we will denote the representing operators and abstract algebra elements (3.1) by the same symbol.

The representation is based on a cyclic vector $\Omega$ which is a vacuum for the electric field $E^i_j (x)$, that is

$$E^i_j (x) \Omega = 0.$$

(3.2)
We will excite the vacuum by Weyl elements

\[ w[F] := \exp \left( -i \int_\sigma d^3 x \ F^a_j(x) \ A^j_a(x) \right). \]  
(3.3)

As a consequence of (3.1)–(3.3) (more precisely the corresponding Weyl relations induced by them) the excited states \( w[F] \Omega \) are simultaneous eigenstates of the operator valued distributions \( E^a_j(x) \)

\[ E^a_j(x) \ w[F] \Omega = F^a_j(x) \ w[F] \Omega \]  
(3.4)

and we necessarily have [29]

\[ \langle \Omega, \ w[F] \Omega \rangle = \delta_{F,0} \]  
(3.5)

where \( \delta_{F,0} \) is indeed non-vanishing (namely unity) if and only if \( F^a_j(x) \equiv 0 \) thereby displaying discontinuity of the excited states and non-separability of this Hilbert space of Narnhofer-Thirring type [33]. Together with the Weyl relations \( w[F] w[F'] = w[F' - F] \) the Hilbert space is the closure of the span of these excited states equipped with the inner product induced from (3.5). We see that in this representation it is quite easy to find everywhere non-degenerate states because by the spectral theorem

\[ q_{ab}(x) \ w[F] \Omega = q_{ab}^F(x) \ w[F] \Omega, \quad q_{ab}^F = \frac{1}{2|\det(F)|} \epsilon_{acdef} F^e_c F^d_k F^f_m \delta_{jm} \delta_{kn} \]  
(3.6)

thus as expected the eigenvalue \( q_{ab}^F \) is everywhere well defined if \( \det(F) \) is nowhere vanishing.

At this point one might wonder why one does not proceed the same way in LQG and chooses to work with the much more complicated space of SNWSs. There are actually two reasons. One is gauge invariance, the other is the fact that the constraints of LQG depend quadratically on \( A \) and not only linearly. Concerning gauge invariance, in order to solve the non-Abelian Gauss constraint one uses non-Abelian holonomies. It would be very difficult to solve the Gauss constraint using the Weyl elements (3.3). Concerning the quadratic dependence of the constraints on the connection we note that the minimal smearing dimension of a field is dictated by the dynamics [29]. For the Hamiltonian constraint of \( U(1)^3 \) theory, if we smear \( A \) in 3d as in (3.3) then the electric field dependence is diagonal and the action of \( C \) is roughly by multiplying a Weyl element by a functional linear in \( A \) smeared in 3d. It thus has the same form as the exponent of \( w[F] \) and thus has a chance to be written as (limits) of multiplication operators acting on the \( w[F] \Omega \). If we did the same smearing for the Hamiltonian constraint of Euclidian GR we would again get a diagonal electric field dependence but now the resulting function is not of the form of a linear functional of \( A \) smeared in 3d but rather a quadratic expression in \( A \) smeared in 3d. Thus to find a quadratic functional of \( A \) smeared in many dimensions as the exponent of \( w[F] \) we must lower the smearing dimension of \( w[F] \) to \( 1 \leq k \leq 2 \) which means that \( E \) now no longer multiplies by a function but by a \( \delta \) distribution in \( 3 - k \) dimensions. In order that the cosmological term also be well defined with a density weight \( \delta \) Hamiltonian constraint the unique choice is \( w = k = 1 \) [29]. The choice \( k = 1 \) is the natural choice taken in LQG from the perspective of gauge covariance as the holonomy is anyway along a 1-dimensional curve.

However, in \( U(1)^3 \) we have the luxury to use smearing dimension 3 for \( w[F] \) and thus also \( C \) can be considered with any density weight, including unity. The density weight \( \delta \) valued Hamiltonian constraint is defined by

\[ C_\delta = C |\det(E)|^{(d-1)/2} = \frac{\epsilon_{abcd} E^b_i E^c_j E^d_k}{|\det(E)|^{|2-\delta|/2}} \]  
(3.7)
where $B^a = \epsilon^{abc} F_{bc}/2$ is the magnetic field of $A$.

We start by quantising the Gauss constraint which is in fact diagonal on the $w[F]$ and imposes the condition that the smearing functions be divergence free

$$\partial_a F^a_j = 0.$$  \hspace{1cm} (3.8)

That is all there is to do to solve the Gauss constraint, no complicated closure constraints as in LQG have to be imposed.

The state of affairs is not as simple with respect to the spatial diffeomorphism and Hamiltonian constraints because in contrast to the Gauss constraint, which is independent of $A$, they depend linearly on $A$. However, the connection smeared against a function

$$A[F] = < F, A > = \int_\sigma d^3 x A^a_f F^a_j$$

is not a well defined operator in the Hilbert space, just $w[F] = e^{-iA[F]}$ is. We could now proceed as in LQG or loop quantum cosmology [34, 35] and replace this by say the regulated expression $A_{\epsilon}[F] = \sin(\epsilon A[F]) / \epsilon = i(w[F_{\epsilon}] - w[-F_{\epsilon}]) / (2\epsilon)$. $F_{\epsilon} = \epsilon F$ but this introduces ambiguities as we could use other approximants which are subleading as $\epsilon \to 0$ and anyway we cannot take the limit $\epsilon \to 0$.

Given the fact that only the exponential of $A[F]$ is well defined it is a natural question to ask whether at least the exponentials of $D[a]$, $C[G]$ are well defined. The idea is to regulate the $A$ dependence in those constraints in terms of Weyl elements of the form $w[F_{\epsilon}]$ and then to exponentiate those regulated constraints. This has the chance to bring the whole $A$ dependence of the constraints, which is not defined in its linear version, into the exponent where it can itself become a (operation on $a$) Weyl element which would be well defined. Now while the Weyl elements $w[F_{\epsilon}]$ are not continuous as operators with respect to $\epsilon$, the functions $F_{\epsilon}$ are continuous in a suitable topology (e.g. as Schwartz functions or simply pointwise in the sense of smooth functions). Then taking the limit $\epsilon \to 0$ on the set of those functions serves as a well motivated definition of the exponentiated and regulator free operator. We will first investigate this for general constraints linear in the momenta and then turn to the concrete $U(1)^3$ model constraints.

**Theorem 3.1.** Consider a phase space with configuration variables $E^I$ where $I$ is from any index set and conjugate momenta $A_I$ and let $C_M = A_I G_M^I(E)$ be phase space functions linear in $A$ and of arbitrary dependence in $E$ in precisely this ordering and labelled by and depending linearly on $M$. Consider a Hilbert space representation of the CCR and $*$ relations with cyclic vacuum $E^I \Omega = 0$, dense set of vectors of the form $w[F] \Omega$, $w[F] = e^{-iA[F]}$, $A[F] = A_I F^I$ and inner product $< \Omega, w[F] \Omega > = \delta_{F,0}$. Suppose that $G_M^I(E) \Omega = 0$. Let

$$C_{M,\epsilon} := \sum_I \frac{i}{2\epsilon} \{ w[\epsilon h_I] - w[-\epsilon h_I] \} \quad G_M^I(E); \quad (h_I)^I := \delta_I^J.$$  \hspace{1cm} (3.10)

Then

$$e^{-iC_{M,\epsilon}} w[F] \Omega = w[F_{M,\epsilon}(F)] \Omega$$

$$\text{where}$$

$$\lim_{\epsilon \to 0} F_{M,\epsilon}(F) = \left[ e^{X_M K} \right](F, 0).$$

(3.12)

Here $X_M$ is the classical Hamiltonian vector field of $C_M$ and $K^I(E,A) := E^I$ the I-the coordinate function.
We have by the spectral theorem and due to $G_M^j \Omega = 0$

$$C_{M,\epsilon} w[F] \Omega = G_M^j [F] \frac{i}{2\epsilon} [w[F + \epsilon h]] - w[F - \epsilon h] \Omega. \tag{3.13}$$

We introduce multiplication and discrete derivative operations

$$\left[ G_M^j w \right] (F) := G_M^j (F) w[F], \quad (\Delta_{\epsilon,l} w) [F] := \frac{1}{2\epsilon} [w[F + \epsilon h] - w[F - \epsilon h]] \tag{3.14}$$

on the space of operator valued functionals $F \mapsto f[F]$. Then

$$C_{M,\epsilon} w[F] \Omega = i \left( \sum_l G_M^l \Delta_{\epsilon,l} w \right) [F] \Omega = i \left( X_{M,\epsilon} w \right) [F] \Omega. \tag{3.15}$$

We now show by induction that

$$C_{M,\epsilon}^{n+1} w[F] \Omega = i^n \left( X_{M,\epsilon}^{n} w \right) [F]. \tag{3.16}$$

To see this we write $\left( X_{M,\epsilon}^{n} w \right) [F]$ explicitly

$$\left( X_{M,\epsilon}^{n} w \right) [F] = \frac{1}{(2\epsilon)^n} \sum_{\sigma_1, \ldots, \sigma_n, l_1, \ldots, l_n} G_M^{l_1} (F) G_M^{l_2} (F + \epsilon \sigma_1 h_{l_1}) \ldots$$

$$\times G_M^{l_n} (F + \epsilon (\sigma_1 h_{l_1} + \ldots + \sigma_{n-1} h_{l_{n-1}})) \ w[F + \epsilon (\sigma_1 h_{l_1} + \ldots + \sigma_n h_{l_n})] \tag{3.17}$$

with $\sigma_k = \pm 1, k = 1, \ldots, n$, which one also can see by induction. Using again that $G_M^j (E)$ is diagonal on $w[F] \Omega$ with eigenvalue $G_M^j (F)$ and the induction assumption we get

$$C_{M,\epsilon}^{n+1} w[F] \Omega = C_{M,\epsilon}^{n} C_{M,\epsilon} w[F] \Omega = i^n C_{M,\epsilon} \left( X_{M,\epsilon}^{n} w \right) [F]$$

$$= \frac{i^{n+1}}{(2\epsilon)^n} \sum_{\sigma_1, \ldots, \sigma_n, l_1, \ldots, l_n} G_M^{l_1} (F) G_M^{l_2} (F + \epsilon \sigma_1 h_{l_1}) \ldots$$

$$\times G_M^{l_n} (F + \epsilon (\sigma_1 h_{l_1} + \ldots + \sigma_{n-1} h_{l_{n-1}})) \ C_{M,\epsilon} w[F + \epsilon (\sigma_1 h_{l_1} + \ldots + \sigma_n h_{l_n})]$$

$$= \frac{i^{n+1}}{(2\epsilon)^n} \sum_{\sigma_1, \ldots, \sigma_n, l_1, \ldots, l_n} G_M^{l_1} (F) G_M^{l_2} (F + \epsilon \sigma_1 h_{l_1}) \ldots$$

$$\times G_M^{l_n} (F + \epsilon (\sigma_1 h_{l_1} + \ldots + \sigma_{n+1} h_{l_{n+1}})) \ w[F + \epsilon (\sigma_1 h_{l_1} + \ldots + \sigma_n h_{l_n})]$$

$$= i^{n+1} \left( X_{M,\epsilon}^{n+1} w \right) [F] \Omega. \tag{3.18}$$

It follows using formal Taylor expansion

$$e^{-i C_{M,\epsilon} w} [F] \Omega = \left( e^{X_{M,\epsilon} w} \right) [F] \Omega. \tag{3.19}$$

The operator $w[F]$ on the Hilbert space of square integrable functions with respect to the Bohr measure acts by multiplication

$$(w[F] \psi) (A) = e^{-i A F} \psi (A) =: w_A [F] \psi (A). \tag{3.20}$$

Therefore (3.19) when evaluated at $A$ may also be written

$$\left\{ e^{-i C_{M,\epsilon} w} [F] \right\} (A) = \left( e^{X_{M,\epsilon} w} \right) [F] \Omega (A) \tag{3.21}$$
as \( X_{M^i} \) does not act on \( A \). We introduce the coordinate function on the classical phase space \( K^i(E,A) = E^i \) whence \( w_A[F] = \exp(-iA_iK^i(F,0)) \). We formally extend \( \Omega \) to be the constant function of \( F \) i.e. \( \Omega(A,F) = \Omega(A) \) so that

\[
\{ e^{-iC_{M^i}} w[F] \Omega \} (A) = \{ e^{X_{M^i} \cdot w[A]} \Omega \} (A,F) = \{ e^{X_{M^i \cdot w_A} e^{-X_{M^i \cdot}} \Omega} \} (A,F)
\]

where \( X_{M^i}, \Omega(A,F) = 0 \) was used, i.e. that \( \Delta_{\mu,i} \) annihilates constant functions. Now \( e^{X_{M^i} \cdot w_A} e^{-X_{M^i \cdot}} = \exp \left( -i A_i e^{X_{M^i} \cdot} K^i(\cdot,0) e^{-X_{M^i \cdot}} \right) = \exp \left( -i A_i e^{X_{M^i \cdot}} K^i(\cdot,0) e^{-X_{M^i \cdot}} \right) . \)

Finally

\[
\left[ e^{X_{M^i \cdot}} K^i(\cdot,0) e^{-X_{M^i \cdot}} \right] (F) = \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left[ X_{M^i \cdot}, K^i(\cdot,0) \right] \right](F) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ X_{M^i \cdot}, K^i \right](F) \]

which converges pointwise in phase space to

\[
\left[ e^{X_{M^i \cdot}} K^i \right] (F,0)
\]

i.e. the Hamiltonian flow of \( C_M \).

Theorem 3.1 motivates the following definition.

**Definition 3.1.** The exponentiated constraints are densely defined by

\[
U(M) w[F] \Omega := e^{-iC_M \cdot w[F]} \Omega := w \left[ \{ e^{X_{M^i \cdot}} K^i \} (F) \right] \Omega.
\]

We note that the linearity of \( C_M \) in \( A \) is essential: It means that the Hamiltonian flow \( e^{X_{M^i \cdot}} \) preserves the polarisation of the phase space, i.e. it maps functions of \( E \) again to functions of \( E \) only. This is no longer true for say a quadratic dependence on \( A \) which is why what follows only applies to the \( U(1)^3 \) truncation of Euclidian GR. On the other hand, the density \( \delta \) cosmological term \( V_M = \Lambda \int d^3x | \det(E) |^{1/2} \) just depends on \( E \) and thus trivially preserves the \( E \) polarisation. We could therefore simply add it to \( C_M \) and keep the definition (3.26). However, then the cosmological constant contribution completely drops out from \( e^{X_{M^i \cdot}} \) thus this cannot be the correct generalisation of theorem 3.1 when an additional potential term is present.

**Theorem 3.2.** Keep all assumptions as in theorem 3.1 except that the constraints are generalised by a potential term \( C_M = A_i G_M^i(E) + V_M(E) \) such that also \( V_M(E) \Omega = 0 \). Then

\[
e^{-iC_M \cdot w[F]} \Omega = e^{-i\alpha_M \cdot \omega[F]} \\
\]

where \( X_M \) is the Hamiltonian vector field of \( A_i G_M^i \) and the phase \( \alpha_M := [\alpha_M^i]_{i=1} \) is given by

\[
\alpha_M^i (F) = \int_0^t dt \left[ V_M (e^{X_{M^i \cdot}} K) \right] (F).
\]

**Proof.** We follow exactly the same steps as in the proof of theorem 3.1. Then we find

\[
e^{-iC_M \cdot w[F]} \Omega = \left( e^{X_{M^i \cdot} - V_M \omega[K]} \right)_{K=F} = \left( e^{X_{M^i \cdot} - V_M \omega[K]} e^{-(X_{M^i \cdot} - V_M \omega[K])} \right)_{K=F} \\
\times \left[ e^{X_{M^i \cdot} - V_M \omega} \Omega \right]_{K=F} = \left[ e^{X_{M^i \cdot} K} \right] (F) \left[ e^{X_{M^i \cdot} - V_M \omega[K]} \right]_{K=F}.
\]

The difference to the previous situation is that still \( X_M \Omega = 0 \) but \( (V_M \Omega)(F) = V_M(F) \Omega \neq 0 \) because \( V_M \) is a multiplication operator of the space of \( F \) dependent functions. We define the
second factor by the Trotter product (note that we consider the continuous space of smearing functions itself, not the discontinuous space of functions of connections labelled by them)
\[
\left[ e^{(X_0 - \iota V_M) \Omega} \right] (K)_{K=F} := \lim_{N \to \infty} \left[ e^{\frac{N}{N} X_0} e^{-\iota \frac{N}{N} V_M} \Omega \right] (K)_{K=F}.
\] (3.30)

Using \( e^{-\frac{N}{N} X_0} \Omega = \Omega \) and
\[
e^{\frac{N}{N} X_0} e^{-\iota \frac{N}{N} V_M} e^{\frac{N}{N} X_0} = e^{-\iota \frac{N}{N} V_M} e^{\frac{\iota N}{N} X_0 K}
\] (3.31)

we can compute (3.30) exactly
\[
\left[ e^{(X_0 - \iota V_M) \Omega} \right] (K)_{K=F} = \lim_{N \to \infty} \left[ e^{-\iota \sum_{k=1}^{N} V_M \left( e^{\frac{k}{N} X_0 K} \right)} \right] (F) \Omega
= e^{-\iota \left[ \int_0^1 \mathrm{d}t \left( e^{\iota t \Omega} \right) \right]} (F) \Omega.
\] (3.32)

Given the fact that the action of the quantum constraints is dictated by their classical Hamiltonian flow on ‘polarised’ functions (in the sense of geometric quantisation [36]) we obtain the following expected result.

**Theorem 3.3.** The exponentiated constraints have the following properties:

1. unitarity \( U(M)^\dagger = U(-M) = U(M)^{-1} \)
2. weak discontinuity
3. anomaly freeness (in the sense defined below, see definition 3.2)

**Proof.**

1. By the assumed linearity in \( M \) we have \( X_M = -X_{-M} \) and by elementary properties of Hamiltonian vector fields \( e^{-X_M} e^{X_M} = \text{id} \) is the identity canonical transformation. Next
\[
U(-M) U(M) w[F] \Omega = e^{-\iota \alpha(M)} U(-M) w \left[ \left( e^{X_M} K \right) (F) \right] \Omega
= e^{-\iota \left[ \alpha(M) + \alpha(-M) \right]} \left( e^{\iota X_M (K) (F)} \right) w \left[ e^{X_M} e^{X_{-M}} F \right] \Omega = w[F] \Omega
\] (3.33)
as, using again linearity in \( M \) i.e. \( V_{-M} = -V_M \)
\[
\alpha_{-M}(e^{X_M K})(F) = \int_0^1 \mathrm{d}t V_{-M}(e^{X_M e^{-\iota X_M} K})(F)
= - \int_0^1 \mathrm{d}t V_M(e^{-\iota X_M} K)(F) = -\alpha_M(F).
\] (3.34)

Thus \( U(M)^{-1} = U(-M) \) and \( U(M) \) has an inverse on the dense span of the \( w[F] \Omega \). Then
\[
ge w[F] \Omega, U(M) w[F'] \Omega \ge \equiv e^{-\iota \alpha(M)} \delta_{F, \left[ e^{\iota X_M K}(F') \right]} e^{-\iota \alpha(M)} \delta_{e^{-\iota X_M K}(F'), F'}
= e^{-\iota \alpha(M)} \delta_{e^{-\iota X_M K}(F'), F'} = e^{\iota \alpha(M)} \delta_{[e^{-\iota X_M K}(F), F'], F'}
= < U(-M) w[F] \Omega, w[F'] \Omega >
\] (3.35)
One parameter unitary subgroups are of the form

\[ (3.38) \]

\[ (3.37) \]

\[ (3.39) \]

where \( \kappa_{\alpha, \beta}^\gamma \) are structure functions on the phase space, i.e. they may depend non-trivially on \( E \). Note that they cannot depend on \( A \) because

\[
\{ C_M, C_N \} = 2 (A_1 G^{\alpha}_{[M]}) G^{\alpha}_{[N]} + \{ V_{[M,A_1]} G_{[\alpha]} \} \tag{3.37}
\]

contains terms at most linear in \( A \) which themselves must combine to the constraint operators. Recall the identity

\[
e^X e^{-X} = 1 + st \left[ M, N \right] + O \left( s^3, s^3 t, s^3 t^2 \right) = 1 + st \left( M - V_{st} C_{-V_{st}} \right) + O \left( s^3, s^3 t, s^3 t^2, t^3 \right). \tag{3.38}
\]

We need the composition law of the \( U(M) \)

\[
U(M) U(N) w[F] \Omega = e^{-i \alpha_F} U(M) w \left[ (e^{X_K} K)(F) \right] \Omega
\]

\[
e^{-i (\alpha_F + \alpha_K) + \alpha_K (e^{X_K} K)(F)} \Omega.
\]

Making use of the automorphism property of the Hamiltonian flow for a general function \( H \) on the phase space and with the coordinate function \( P_I(E, A) = A_I \)

\[
\left( e^{X_K} H \right) (E, A) = \left[ H \left( e^{X_K} K, e^{X_K} P \right) \right] (A, E)
\]

and applied to \( H = e^{X_K} K \)

\[
U(M) U(N) w[F] \Omega = e^{-i (\alpha_F + \alpha_K) + \alpha_K (e^{X_K} K)(F)} \Omega.
\]

Iterating with \( U_1 = U(M) \), \( X_j = X_{M_j} \), \( \alpha_j = \alpha_{M_j} \), \( j = 1, \ldots, N \)

\[
U_1 \ldots U_N w[F] \Omega = e^{-i (\alpha_F + \alpha_K) + \alpha_K (e^{X_K} K)(F)} \ldots + \alpha_K (e^{X_K} e^{X_K} (e^{X_K} K)(F)) \Omega
\]

\[
\times W \left[ \left( e^{X_K} e^{X_K} K \right)(F) \right] \Omega.
\]

It follows that

\[
U(sM) U(sN) U(-sM) U(-sN) w[F] \Omega
\]

\[
e^{-i (\alpha_F + \alpha_K) + \alpha_K (e^{X_K} K)(F)} + \alpha_K (e^{X_K} e^{X_K} (e^{X_K} K)(F)) \Omega
\]

\[
\times W \left[ \left( e^{X_K} e^{X_K} e^{X_K} K \right)(F) \right] \Omega
\]

\[
e^{-i (\alpha_F + \alpha_K) + \alpha_K (e^{X_K} K)(F)} + \alpha_K (e^{X_K} e^{X_K} (e^{X_K} K)(F)) \Omega
\]

\[
\times W \left[ F + st \left[ X_K, X_{M_K} \right] K(F) + O \left( s^3, s^3 t, s^3 t^2, t^3 \right) \right] \Omega.
\]

We also expand the phase in (3.43) keeping terms up to second order and note that the flows only need to be expanded to linear order as the integrals are already of first order.
\[
\int_0^t \left[ e^{-ix_N} e^{-itx_N} e^{ix_M} e^{-itx_M} - e^{-itx_N} e^{-itx_M} \right] V_N + \int_0^t \left[ e^{ix_N} e^{-itx_N} e^{ix_M} e^{-itx_M} - e^{-itx_N} e^{-itx_M} \right] V_M \\
= \text{st} \left( x_M V_N - x_N V_M \right) + O \left( s^3, s^3 t, s^3 t^2, t^3 \right). 
\]

(3.44)

We now write (3.36) in the explicit form

\[
C_M = M^\alpha \left[ A_I G_\alpha^J \left( E \right) + V_\alpha \left( E \right) \right],
\]

\[
\{ C_M, C_N \} = M^\alpha N^\beta \left[ A_I G_\beta \left( E \right) + V_\beta \left( E \right) \right] \kappa_{\alpha\beta} \gamma \left( E \right)
\]

\[
=: A_I H_{M,N}^I \left( E \right) + L_{M,N} \left( E \right) = \{ C_M - V_M, C_N - V_N \} + \{ C_M - V_M, V_N \}
\]

\[- \{ C_N - V_N, V_M \} 
\]

(3.45)

which is again at most linear in \( A_I \) and where we have ordered all dependence on \( E \) to the right. Thus replacing \( G_{M}^I \), \( V_M \) by \( H_M \), \( L_{M,N} \) and following the same steps as for \( C_M \) we define

\[
U([M,N]) \ w[F] \Omega := \exp \left( i \left\{ C_M, C_N \right\} \right) \ w[F] \Omega = e^{i\alpha_{M,N}(F)} \ w \left[ \left( e^{-X_{M,N}} K \right) \left( F \right) \right] \Omega
\]

(3.46)

where \( X_{M,N} \) is the Hamiltonian vector field of \( A_I H_{M,N}^I \) and \( \alpha_{M,N}(F) \) is the phase

\[
\int_0^1 \left[ e^{iX_{M,N}} L_{M,N} \right]. 
\]

(3.47)

Since

\[
X_{sM,N} = \text{st} \ X_{M,N} = \text{st} \left[ X_{C_M - V_M, C_N - V_N} \right] = \text{st} \left[ X_{M,N} \right]
\]

(3.48)

and

\[
\alpha_{M,N,s}(F) = \alpha_{M,s}^\nu \left( F \right) = \text{st} \left[ L_{M,N} \left( F \right) \right] + O \left( s^3, s^3 t, s^3 t^2, t^3 \right)
\]

\[
= \text{st} \left( x_M V_N - x_N V_M \right) \left( F \right) + O \left( s^3, s^3 t, s^3 t^2, t^3 \right). 
\]

(3.49)

Now due to our choice of representation \( E^I = i\partial/\partial A_I \) we have \([A_I, E^J] = -i\delta^I_J = -i[A_I, E^I] \) and thus expect \([C_M, C_N] = -i(C_M, C_N) = i(C_M, C_N) \) to leading order in the quantum theory i.e. to leading order

\[
U(M) U(N) U(-M) U(-N) = 1 - [C_M, C_N] = 1 + i \{ C_M, C_N \} = U([M,N]).
\]

(3.50)

These relations establish that the composition law (3.41) has resulted in the leading order exponentiated substitute (3.50) for the infinitesimal version (3.36), i.e. that the representation of the \( U(M) \) is free of anomalies in the sense of the subsequent definition 3.2.

\[\square\]

The precise statement of anomaly freeness for the exponentiated versions is given in the following definition.

**Definition 3.2.** Suppose that operators \( U(M) = \exp(-iC_M) \), \( U([M,N]) = U(i\{C_M, C_N\}) \) are defined on the common, dense, invariant domain given by the linear span of the \( w[F]\Omega \) and suppose that

\[
U(sM) U(tN) U(-sM) U(-tN) \ w[F] \Omega = e^{-i\alpha_{s,t}} \ w[F_{s,t}] \Omega, \ U([sM, tN]) \ w[F] \Omega
\]

\[
= e^{-i\alpha_{s,t}} \ w[F_{s,t}] \Omega.
\]

(3.51)
Then the $U(M)$ are said to be represented free of anomalies on the Hilbert space $\mathcal{H}$ with dense span of the $w[F]\Omega$ iff $U(0) = U_{[[0,0]]} = 1_{\mathcal{H}}$ and

$$\left[ \frac{d}{ds} \frac{d}{dt} \left\{ F_{\alpha \beta} - F'_{\alpha \beta} \right\} \right]_{s,t=0} = 0 = \left[ \frac{d}{ds} \frac{d}{dt} \{ \alpha_{s,t} - \alpha'_{s,t} \} \right]_{s,t=0}. \quad (3.52)$$

This definition is general enough to encompass the situation that the 1-parameter groups $s \mapsto U(sM)$ are not weakly continuous; thus, while we cannot take the derivatives or even limits of the $U(sM)$ we can take derivatives or limits of the $F_{\alpha \beta}$. Note also that we insist on independent quantisations of $U(M) = \exp(-iC_M)$, $U([M,N]) = \exp(-i\{C_M,C_N\})$ as otherwise we can trivially obtain anomaly freeness by declaring $U([M,N]) = U(M)U(N)U(-M)U(-N)$.

An expected but unusual property of the operators $U(M)$ is that their products $U(M_1)U(M_2)$ in general cannot be written in the form $U(M_3)$, not even when $M_1, M_2$ are close to zero and thus $U(M_1), U(M_2)$ are ‘close’ to $\text{id}_M$. This is precisely due to the fact that the $[X_{M_1}, X_{M_2}]$ is not a linear combination with structure constants of the $X_M$, but with structure functions, i.e. that we have a Lie algebroid rather than a Lie algebra structure.

We now provide the details about the concrete situation in the $U(1)^3$ model. We treat only the case of vanishing potential. The inclusion of a non zero cosmological constant or a Lorentzian signature term is straightforward given the general theory above up to one non-trivial observation: It is known that the constraint algebra of vacuum GR with cosmological constant closes. But that statement uses the full quadratic dependence of the constraints on the connection $A$. In the appendix we show that the statement persists if in addition to dropping the quadratic term in $A$ we also drop the quadratic term in the spin connection involved in the Ricci scalar from the Hamiltonian constraint. This is consistent because both quadratic terms are responsible for turning the Abelian $U(1)^3$ into the non-Abelian $SU(2)$.

First we integrate the smeared constraints by parts and write them in the form

$$D[u] = \langle A, G_u \rangle, \quad C_\delta [M] = \langle A, G_{\delta,M} \rangle,$$

$$(G_u)^{(a)}_\delta(x) = -2[\partial_\delta (E^{(a)}_j u^b)](x), \quad (G_{\delta,M})^{(a)}_\delta(x) = -2[\partial_\delta (M^{b \delta} E^{(a)}_j E^{b}_j) \det(E)](x) \quad (3.53)$$

thereby displaying the functions $G^{(a)}_\delta$ of theorem 3.1 where the index $I = (a, j, x)$ is compound and summation over $I$ means summing over $a, j$ and integrating over $x$. Correspondingly, the $h_I$ of the theorem are given by

$$[h^{(a)}_\delta(x, \delta)]_\sigma^{(b)}(y) = \delta^{(b)}_\delta \delta^{(a)}_\sigma \delta(x, y) \quad (3.54)$$

where $\kappa \mapsto \delta_\kappa$ is a mollified $\delta$ distribution, i.e. a 1-parameter family of smooth functions converging to the $\delta$ distribution on Schwartz space over $\sigma$. With their help we define the analog of the discrete derivative $\Delta_{\alpha \beta}$ on functionals of the functions $F^{(a)}_j(x)$ of the theorem by

$$\left( \Delta_{\sigma,\kappa, a, j, x} W[F] \right)(x) := \frac{1}{2\epsilon} \left\{ W[F + \epsilon h^{(a)}_\delta(x, \kappa)] - W[F - \epsilon h^{(a)}_\delta(x, \kappa)] \right\}. \quad (3.55)$$

With all the $E^{(a)}_j(x)$ ordered to the outmost right, the assumption $G_\delta \Omega = M_\delta \Omega = 0$ is met if we set for $\delta < 2$

$$\left( \det(E) |^{(\delta-2)/2} E^{(a)}_j E^{(a)}_k \right)(x) \Omega := \lim_{s \to 0^+} \left( s + \det(E) |^{(\delta-4)/2} \right)^{-1} E^{(a)}_j E^{(a)}_k \Omega = 0. \quad (3.56)$$

Then the theorem applies, with the understanding that the limit $\epsilon \to 0$ at the level of the functions $F$ is accompanied by the limit $\kappa \to 0$. 

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As a result we obtain the hypersurface deformation groupoid (HDG) with explicit action on the common dense and invariant domain $D$ given by the span of the $w[F]\Omega$

$$U(u) := e^{-iD[u]}, \quad U(M) := e^{-iC[M]},$$

$$U(u) \ w[F] \ \Omega = w \left[ \left( e^{X_u} K \right)(F) \right] \ \Omega$$

$$U(M) \ w[F] \ \Omega = w \left[ \left( e^{X_M} K \right)(F) \right] \ \Omega$$

(3.57)

with $X_u, X_M$ the Hamiltonian vector field of $D[u], C[M]$ respectively.

The action of $U(u)$ in (3.57) is in fact the same as in LQG which works even in the non-Abelian setting because $D[u]$ is linear in $A$. As $D[u]$ is also linear in $E$, the flow of $D[u]$ preserves the linearity in $F$. For this reason, $e^{\delta D[u]K}$ can be worked out in closed form

$$\left[ \left( e^{X_u} K \right)(F) \right]_{ij}^a(x) = \left[ e^{L_u} F \right]_{ij}^a(x), \quad \left[ L_u F \right]_{ij}^a(x) = \left[ \left( u^b F \right)_{ij}^a - u^b_{ij} F^b \right](x)$$

(3.58)

where $L_u$ is the Lie derivative acting on vector field densities of weight one. One can check the implication

$$\partial_a F_{ij}^a = 0 \Rightarrow \partial_a \left( L_u F_{ij}^a \right) = 0$$

(3.59)

which means that solutions $w[F]\Omega$ of the Gauss constraint are mapped to solutions of the Gauss constraint. Furthermore the flow preserves the space of vector field densities.

As far as $C_\delta(M)$ is concerned, the fact that $D(u)$ generates spatial diffeomorphisms and the phase space dependent integrand $C_\delta(x)$; $C[M] = \int d^3 x M(x)$ $C_\delta(x)$ is a scalar density of weight $\delta$ implies

$$\{D(u), C_\delta(M)\} = -C_\delta(L_u M), \quad L_u M = u^a M_{,a} - (\delta - 1) u^a_M M \quad (3.60)$$

i.e. $M$ acquires the geometrical interpretation of a scalar density of weight $-(\delta - 1)$. Accordingly the net weight of $M_\delta(E) := M |\det(E)|^{(\delta - 2)/2}$ is always $-1$. It follows

$$\{ C_\delta(M), E^a_j(x) \} = 2 \left( \epsilon_{ijk} E^b_k E^c_i M_\delta(E) \right)_{,b}(x) \quad (3.61)$$

whence

$$\left[ (X_M K)^a_j(F) \right](x) = 2 \left[ \epsilon_{ijk} F^b_k F^c_i M_\delta(F) \right]_{,b}(x).$$

(3.62)

Obviously (3.62) is no longer linear in $F$, polynomial only for $\delta = 2$. However, for any $\delta$ (3.62) is divergence free because $\epsilon_{ijk} F^b_k F^c_i$ is antisymmetric in $a, b$. Thus the flow $e^{X_M \delta}$ preserves the space of solutions to the Gauss constraint.

To illustrate the degree of complexity of the flow, let us work out the first few orders for the simplest (polynomial) case $\delta = 2$. We set

$$[B_{M}(F,G)]_{ij}^a := 2\epsilon_{ijk} [M \left( F^b_k G^c_i \right)_{,b} \Rightarrow B_{M}(F,G) = B_{M}(G,F),$$

$$B_{M}(F,F) = (X_M K)(F), \quad \partial_a[B_{M}(F,G)]_{ij}^a = 0$$

(3.63)

which maps a pair of triples of divergence free vector densities to another triple of divergence free vector densities. It is also symmetric in $F, G$.

As an aside, note that

$$[B_{M}(F,G)]_{ij}^a = -\epsilon^{abc} \partial_b \left( \omega_M \right)^c_i, \quad \left( \omega_M \right)_i^c = M_{\epsilon de} \epsilon^{dei}. \quad (3.64)$$

Thus if $M$ transforms as a density of weight $-1$ and $F, G$ as vector densities of weight $+1$ then $\omega_M$ is a 1-form.
We have
\[
X_M K = B_M (K, K)
\]
\[
X_M^2 K = B_M (X_M K, K) + B_M (K, X_M K) = 2 B_M (K, B_M (K, K))
\]
\[
X_M^3 K = 2 B_M (X_M K, B_M (K, K)) + 2 B_M (K, X_M B_M (K, K))
= 2 B_M (B_M (K, K), B_M (K, K)) + 4 B_M (K, B_M (K, B_M (K, K))).
\] (3.65)

In general, \(X_M^N\) is a nested polynomial of order \(n + 1\) in \(K\) involving \(n\) bilinear forms \(B_M\) in all possible ways. It may well be possible to find the recursion relation for the numerical coefficients among these possible terms but we will not need them for what follows.

We end this section with the remark that for non-integer density weight or density weight smaller than two, the space \(\mathcal{F}\) to which the \(F\) belong is restricted to non-degenerate elements, that is \(\det(F) \neq 0\) anywhere, the quantum trace of the classical condition that the classical metric be non-degenerate. This will be further analysed in the next section.

### 4. Dual representation

The unitary operators \(U[u], U[M]\) are defined densely on the span \(\mathcal{D}\) of the \(w[F]\Omega\) (finite linear combinations of those), with \(F\) non-degenerate, i.e. in \(\mathcal{H}\) and not some space of distributions. They do not act weakly continuously there, thus their algebra can only be compared to the exponentiated classical hypersurface deformation algebra and this is what we did in the previous section, thereby establishing anomaly freeness on \(\mathcal{H}\) in the sense defined. One may obtain an infinitesimal action on a certain space \(L \subset \mathcal{D}^*\) of distributions on \(\mathcal{D}\) where \(\mathcal{D}^*\) is the algebraic dual of \(\mathcal{D}\) i.e. all linear functionals without continuity conditions. A general element \(l \in L\) maybe written
\[
l = \sum_F l[F] < w[F]\Omega, \ldots >_{\mathcal{H}}
\] (4.1)

and for any operator \(A\) with dense and invariant domain \(\mathcal{D}\) we define its dual \(A'\) on \(L\) by
\[
[A'] [w[F]\Omega] := l(Aw[F]\Omega) = \sum_{F'} l[F'] < w[F']\Omega, Aw[F]\Omega >.
\] (4.2)

While the sum in (4.1) is over uncountably many \(F\), the condition that \(Aw[F]\Omega \in \mathcal{D}\) makes sure that (4.2) is finite. We may thus construct \(U'[u], U'[M], U'[M,N]\)
\[
[U'[u] [w[F]\Omega] = l[(e^{\lambda u} K) (F)], [U'[M] [w[F]\Omega] = l[(e^{\lambda u} K) (F)],
[U'[[M,N] [w[F]\Omega] = l[(e^{X_{MN}} K) (F)]
\] (4.3)

where \(X_u, X_M, X_{M,N}\) are the Hamiltonian vector fields of \(D[u], C[M], \{C[M], C[N]\}\) respectively.

For given \(u,M\) we define the one parameter groups \(s \mapsto U'[su], s \mapsto U'[sM]\) to be continuous in the \(L, \mathcal{D}\) topology if \(l[U[su]\psi], l[U[sM]\psi]\) is continuous in \(s\) for all \(l \in L, \psi \in \mathcal{D}\). By (4.3) this is equivalent to the requirement that the coefficient functions \(l\) are continuous on the chosen space \(\mathcal{F}\) that we sum over. Consider \(\mathcal{F}\) to be some standard space e.g. divergence free Schwartz functions so that the flows \(e^{\lambda X_u}, e^{\lambda X_M}\) which formally involve spatial derivatives of arbitrarily high orders are well defined. This is however not sufficient: Unless the density weight of \(C_{\delta}\) is an integer larger than or equal to two, the functions \(F\) must be everywhere...
regular, i.e. \( \det(F) \neq 0 \) everywhere because the flow of the Hamiltonian constraint involves arbitrarily large negative powers of \( |\det(F)| \). This is the condition of quantum non-degeneracy [29].

We now compute the infinitesimal generators

\[
\begin{align*}
(-iD'[u] l)[F] &:= \frac{d}{ds} l((e^{Xs}K)[F] = (X_u)F = \{ D\{u\}, l(K) \}_{K=F} \\
(-iC'_\delta[M] l)[F] &:= \frac{d}{ds} l((e^{Xs}K)[F] = (X_M l)[F] = \{ C_\delta[M], l(K) \}_{K=F}
\end{align*}
\]

which are nothing but classical Poisson brackets with functions of \( E \) only. Therefore the algebra of the \( D'[u], D'[M] \) is precisely an anti-representation of the HDA \( \mathfrak{h} \) since

\[
i \left( \{ C_\delta(M), C_\delta(N) \}^\prime l \right)[F] = \left( \frac{d}{ds} \frac{d}{dt} (U'(\{sM,sN\})l)[F] \right)_{s=t=0} = \{ \{ C_\delta(M), C_\delta(N) \}, l(K) \}_{K=F}
\]

and closes without anomalies irrespective of the density weight of the Hamiltonian constraint provided the functionals \( l \) are restricted to smooth regular functions representing non-degenerate quantum metrics.

5. Solutions to the constraints by groupoid averaging

A general strategy to solve quantum constraints is to use ‘group averaging’ [30], that is, to construct a so-called anti-linear rigging map \( \eta : \mathcal{D} \rightarrow \mathcal{S} \subset \mathcal{D}^\ast \) that maps the common, dense invariant domain \( \mathcal{D} \) of the constraints to a subspace \( \mathcal{S} \) of algebraic distributions on \( \mathcal{D} \) which is in the kernel of the dual to all constraints. That is

\[
(\eta \psi)[C_\alpha \psi'] = 0
\]

for all \( \psi, \psi' \in \mathcal{D} \) and \( \alpha \) is some index that labels a complete set of constraints. In case that the constraints \( C_\alpha \) are the self-adjoint generators of a Lie algebra \( \mathfrak{h} \) we may pass to the corresponding unitary Lie group \( \mathfrak{g} \) generated by composition of the \( g = U[M] = \exp(-i \sum C_\alpha M^\alpha) \), \( M^\alpha \in \mathbb{R} \) and if \( \mathfrak{g} \) admits a left invariant Haar measure \( \mu \) then we may set

\[
\eta \psi := \int d\mu(g) < g \psi, \psi >_{\mathcal{H}}
\]

which satisfies \( \eta \psi' = (\eta \psi)[\psi'] \) by unitarity \( g^{-1} = g^{-1} \) which may be considered as the exponentiated version of (5.1)). Furthermore,

\[
< \eta \psi, \eta \psi' >_{\mathcal{H}} := (\eta \psi')[\psi]
\]

defines an inner product on those solutions.

It appears that we have good chances to apply this formalism given the theory of section 3 which provides us with unitarities \( U[u], U[M] \) labelling the constraints. Unfortunately, the \( U[u], U[M] \) do not generate a group but a groupoid. While for a group we have at least formally a composition law \( U[M_1] U[M_2] = U[M_3] \) where \( M_3 \) is in general an infinite Baker–Campbell–Hausdorff series in \( M_1, M_2 \) involving the structure constants of \( \mathfrak{l} \), for a groupoid such a relation does not hold, ‘words’ formed by taking products of the ‘alphabet letters’ \( U[M] \) are in general
independent of each other. Thus, there can be no group structure, no Haar measure and no rigging map as above.

Let \( \mathcal{A} \) be the set of alphabet letters consisting of all \( U[u], U[M] \) and let \( \mathcal{W}_N \) be the set of words \( w \) with \( N \) letters of the form \( w = a_1 \ldots a_N \) where \( w \) is not reducible to a word with fewer letters, by using the fact that the alphabet consists both \( a, a^{-1} \). We may try to form a discrete sum

\[
\eta \psi = \psi + \sum_{N=1}^{\infty} \sum_{w \in \mathcal{W}_N} \omega(w) \langle w \psi, \psi \rangle
\]

(5.4)

where we have included a ‘weight’ function \( \omega \). Then one may ask that

\[
(\eta \psi) [\alpha \psi'] = (\eta \psi) [\psi'] \quad \forall \ a \in \mathcal{A}.
\]

(5.5)

Since \( w \in \mathcal{W}_N \) contains words \( a w', w' \in \mathcal{W}_{N-1} \) we see that \( a^i w = a^{-1} w = w' \) can both increase and decrease word length by one unit so (5.5) does not lead to an immediate contradiction. However, even in case that \( \mathcal{A} \) has finitely many unitary letters \( a_1, \ldots, a_k \) which have the same finite order \( a_i^k = 1, i = 1, \ldots, k \) but otherwise are free (no other relations) there are relations for an infinite number of words to check. We therefore consider this approach to the groupoid situation as impractical.

The idea well known in the literature [37] is to pass to equivalent constraints that do form an algebra. Given first class constraints \( C_\alpha \) on a phase space with coordinates \( \{y_\alpha, x^\alpha, p_\mu, q^\mu\} \) one may solve the constraints for the \( y_\alpha \) and rewrite them in the form

\[
\tilde{C}_\alpha = y_\alpha + h_\alpha (x, p, q).
\]

(5.6)

These constraints are strictly Abelian \( \{\tilde{C}_\alpha, \tilde{C}_\beta\} = 0 \) and therefore can be subjected to group averaging. Among the caveats to this strategy is the fact that the function \( h_\alpha \) in general has several branches unless the constraints \( C_\mu \) involve the \( y_\alpha \) only linearly. This caveat is actually absent for the \( U(1)^3 \) as in fact all momenta appear at most linearly in the constraints.

In the \( U(1)^3 \) situation we may write the constraints just in terms of the curvatures \( F_{j}^{ab} \) modulo a term proportional to the Gauss constraint which annihilates the states \( w |F\rangle \Omega \) as \( F \) is divergence-free. Thus \( F_{ab} \) only depends on the transversal parts \( A_{a \perp} \) while the longitudinal parts \( A_{a l} = A_{a 0} - A_{a 1} \) drop both from the states and the constraints. It is thus natural to select the four momenta \( A_{a \perp}, j = 1, 2 \) as the \( y_\alpha \) and the two momenta \( A_{a l}^\perp \) as the \( p_\mu \) with corresponding conjugate configuration coordinates \( E_{j}^{\parallel}, E_{j}^{\perp} \) as the \( x^\alpha, q^\mu \) respectively. Precisely this description of the reduced phase space has been given in [27] of which we provide some details in the next section.

Thus instead of the \( U(M) = \exp(-i C_M) \) we consider the \( \hat{U}(M) = \exp(-i \tilde{C}_M) \), \( \tilde{C}_M = \sum_\alpha M^\alpha C_\alpha \) which inherit the action from section 3

\[
\hat{U}(M) w |F\rangle \Omega = w \left[ (\hat{E}^{K}) |F\rangle \right] \Omega
\]

(5.7)

where \( \hat{X}_M \) is the Hamiltonian vector field of \( \tilde{C}_M \) because the \( \tilde{C}_M \) is still linear in the momenta.

Still we cannot just integrate over the \( M^\alpha \in \mathbb{R} \) because the \( \hat{U}(M) \) are weakly discontinuous unitarities which is why integrals with respect to \( M \) of \( \langle \psi, U(M) \psi \rangle \) would simply vanish as the matrix element is supported on Haar measure zero sets. We are thus forced to consider instead the discrete (i.e. summation) measure to construct the rigging map

\[
\eta \psi = \sum_M \langle \hat{U}(M) \psi, \psi \rangle_H
\]

(5.8)
similar to the LQG approach to averaging the spatial diffeomorphism group [38]. We have

\[
(\eta \psi) \left[ \hat{U}(M') \psi' \right] = \sum_M < \hat{U}(M - M') \psi, >_\mathcal{M} = (\eta \psi)[\psi']
\]  

(5.9)

where unitarity and Abelian nature of the \( \hat{U}(M) \) was used.

For \( \psi = w[F] \Omega, w[F] = \exp(-i < A, F >) \) this can be further detailed as follows: Using \( \partial_\alpha F^\alpha_j = 0 \) we may split \( F^\alpha_j = F^\alpha_j^{\perp} \) into the components \( R^\alpha = F^\alpha_j^{\perp}(x), \alpha = 1, 2 \) and \( \eta^\alpha = F^\alpha_j^{\perp} \) and we split the coordinate functions \( [K(E, A)]^\eta_j(x) = F^\alpha_j(x) \) into the corresponding parts \( T^\alpha, Q^\alpha \) where it is understood that summation over \( \alpha, \mu \) includes an integral over \( x \). In this notation \( w[F] = w[(R, S)] = \exp(-i < y, R > - i < p, S >) \). Then

\[
\hat{U}(M) w[F] \Omega = \hat{U}(M) w[(R, S)] \Omega = w \left[ (e^{\tilde{X}_M} F) \right] \Omega = w \left[ (e^{\tilde{X}_M} T)(R, S), (e^{\tilde{X}_M} Q)(R, S) \right] \Omega = w \left[ (R + M, e^{\tilde{X}_M} Q)(R, S) \right] \Omega
\]  

(5.10)

where \( (\tilde{X}_M)T(R, S) = (M + T)(R, S) = M + R \) was used (\( M \) is the constant function) and \( \tilde{X}_M \) is the Hamiltonian vector field of \( \tilde{C}_M \). Thus for a general function

\[
\psi = \sum_F \psi(F) w[F] \Omega = \sum_{R, S} F \psi(R, S) w[(R, S)] \Omega
\]  

(5.11)

with \( \psi(F) \neq 0 \) for at most countably many \( F \) we have

\[
\sum_M \hat{U}(M) \psi = \sum_{R, S, M} \psi(R, S) w \left[ (R + M, e^{\tilde{X}_M} Q)(R, S) \right] \Omega
\]  

\[
= \sum_{R, S, M} \psi(R, S) w \left[ \hat{M}, (e^{\tilde{X}_M} Q)(R, S) \right] \Omega
\]  

\[
= \sum_{R, S, M} \psi(R, S) w \left[ \hat{M}, (e^{\tilde{X}_M} Q)(R, S) \right] \Omega
\]  

(5.12)

where we have introduced a new summation variable \( \hat{M} = M + R \) in the second step and in the third we used that the \( \tilde{X}_M \) are Abelian.

Next we have by unitarity and due to the Abelian property

\[
< \sum_M \hat{U}(M) \psi, < \sum_{M'} \hat{U}(M') \psi' >_\mathcal{M} = \sum_{M, M'} < \hat{U}(M - M') \psi, \psi' >
\]  

\[
= \left[ \sum_{M'} 1 \right] \left[ \sum_M < \hat{U}(M) \psi, \psi' > \right]
\]  

\[
= \text{Vol}(\mathcal{M}) \cdot < \eta \psi, \eta \psi' >_{\eta}
\]  

(5.13)

where \( \mathcal{M} \) is the space of \( M \) that we sum over which is the same as the space \( \mathcal{R} \) of \( R \). Accordingly we have for the rigging inner product
\[ \text{Vol}(\mathcal{M}) < \eta\psi, \eta\psi' > \eta \]
\[ = \sum_{R,S,M,R',S',M'} \psi^*(R,S) \psi'(R',S') \delta_{MM'} \delta \left( e^{x_\mu} e^{-x_\mu} Q(R,S), (e^{x_\mu} e^{-x_\mu} Q)(R',S') \right) \]
\[ = \text{Vol}(\mathcal{M}) \sum_{R,S,R',S'} \psi^*(R,S) \psi'(R',S') \delta_{O_\mu(R,S),O_\mu(R',S')} \]
\[ = \text{Vol}(\mathcal{M}) \sum_O \left[ \sum_R \psi(R, \delta_{O_Q^{-1}(R),O}) \right]^* \left[ \sum_{R'} \psi'(R', \delta_{O_Q^{-1}(R'),O}) \right] \quad (5.14) \]

where we used that \( e^{x_\mu} \) is invertible in the second step and rewrote the arguments in the remaining Kronecker, in the third we introduced the notation

\[ O_H := \left[ e^{x_\mu} H \right]_{z=-T} \quad (5.15) \]

which is the projection of the phase space function \( H \) to the relational gauge invariant observable corresponding to the gauge \( T = 0 \) [37] and in the last we solved the Kronecker using \( S = \delta_{O_Q^{-1}(R),O} \) which is the inversion of \( O_Q(R,S) = 0 \) at fixed \( O \).

It follows that the physical Hilbert space obtained by the rigging method can be identified with the Hilbert space with dense span given by the \( \sum_O \psi(O) w[(0,O)]\Omega \) via the unitary map \( V : \eta\psi \mapsto \hat{\psi} \) with

\[ \hat{\psi}(O) = \sum_R \psi(R, \delta_{O_Q^{-1}(R),O}) . \quad (5.16) \]

As expected, the physical states depend only on (relational) Dirac observables \( O \) corresponding to the configuration variables \( q^\mu \) not subject to the gauge fixing \( x^\alpha = 0 \) that defines these relational observables. As is well known [37], the phase space defined by the relational observables corresponding to \( q^\mu, p_\mu \) and the gauge fixing condition \( x^\alpha = 0 \) is completely equivalent to reduced phase space obtained by solving the constraints for \( y_\alpha \) in the gauge \( x^\alpha = 0 \) and keeping \( q^\mu, p_\mu \) as ‘true degrees of freedom’. Whenever that gauge fixing is not complete but leaves a 1-parameter family of residual gauge transformations, one may use the generator of those residual transformations as physical or reduced Hamiltonian. We thus turn to the reduced phase space description in the next section and compute and quantise the corresponding physical Hamiltonian.

6. Reduced phase space, physical Hamiltonian and quantisation

The classical part of this section is a slightly generalised version of a part of [27]. We thus will briefly and refer the reader to [27] for details.

The classical constraints can be written in density \( \delta \) form

\[ D^i_j = F^k_{ab} E^a_k E^b_j | \det(E)|^{(\delta-2)/2}, \quad C^a = F^a_{ij} \epsilon^{ijk} E^k_i | \det(E)|^{(\delta-2)/2} \quad (6.1) \]

with \( F^a_{ij} = 2 \partial_{[a} A^i_{b]} \), where \( G_j = \partial_{a} E^a_j = 0 \) was used so that \( E^a_j = E^a_j^\perp \) is already transversal.

The gauge condition we wish to impose is on \( E^a_2 \), \( \alpha = 1, 2 \) and correspondingly we want to solve (6.1) for \( A^a_2 \). We can use the Gauss constraint to install the three Coulomb gauges \( A^a_2 = A^a_2 - A^a_2 = 0, j = 1, 2, 3 \) as it was done in [27]. Note that while \( \partial_a E^a_3 = 0 \) defining the solutions
$E_{a_1}^\perp$ does not require a background metric, the definition of $A_{a_2}^\perp$ does require a background metric. We will pick one and proceed as in [27].

In this paper we slightly deviate from [27] and just impose two Coulomb gauges $A_\alpha^a = 0$, $\alpha = 1,2$ and also only solve two Gauss constraints $G_\alpha = \partial_\alpha E_\alpha^a = 0$. Thus we keep as true degrees of freedom the three canonical pairs $A_\alpha^a$, $E_\alpha^a$ and keep the Gauss constraint $G_3 = \partial_\alpha E_3^a$ still in place, to be dealt with later. We will see that after having reduced the six constraints $G_\alpha, D_\alpha, C$ we obtain a theory with a reduced Hamiltonian $H$ and a constraint $G := G_3$ under which $H$ is invariant and $H$ will be linear in $F_{ab}$ and non-linear in $E^3$ [27]. The resulting theory is thus a special type of non-linear, self-interacting, local electrodynamics.

Proceeding to the details, in this paper we consider just the case $\sigma = \mathbb{R}^3$. More general manifolds can be treated with adapted methods (matching boundary/decay conditions of the fields to the corresponding differentiable structure). We pick a global Cartesian coordinate system $x^a$, $a = 1,2,3$ on $\sigma$. The gauge condition on the solution $E_\alpha^a$ of $\partial_\alpha E_\alpha^a = 0$ that we pick is (remember $\alpha = 1,2$)

$$E_\alpha^a = \delta_\alpha^a.$$  \hspace{1cm} (6.2)

To distinguish the coordinate directions from the frame directions, we write $x, y, z$ for $a = 1,2,3$. Then (6.2) is a compact notation for $E_1^1 = E_2^2 = 1$, $E_3^3 = E_1^3 = E_2^1 = E_2^3 = 0$. Obviously the Gauss constraints $\partial_a E_\alpha^a = 0$ are identically satisfied. The reason why we do not impose $E_\alpha^a = 0$ is that we want to keep the model as close as possible to GR and thus insist on non-degenerate metrics, thus $\det (E_\alpha^a) = E_3^3 \neq 0$ is still possible.

We need to show that the six conditions (6.2) can be always installed no matter from which configuration of the $E_\alpha^a$ we start from and that the six constraints $G_\alpha, D_\alpha, C$ can always be solved for. To do this we rewrite (6.1) in terms of the density $-1$ inverse $E_a^\alpha$ and the density $+1$ magnetic field $B_j^a$

$$\det (E) E_a^\alpha = \frac{1}{2} \epsilon_{abc} \epsilon^{jkl} E_k^b E_l^c, \quad 2 B_j^a = \epsilon^{abc} F_{bc}$$  \hspace{1cm} (6.3)

from which

$$D_j^0 = \epsilon_{abc} B_k^a E^j_c, \quad C^0 = B_j^a E_a^j$$  \hspace{1cm} (6.4)

which have density weight zero. As $E_a^\alpha$ is non-degenerate we can decompose $B_j^1 = u^1 E_j^1$, $B_j^2 = v^2 E_j^2$ and find

$$D_j^0 = B_j^1 E_a^1 + B_j^2 E_a^2 = v_3 - B_j^3 E_a^3$$

$$D_j^0 = B_j^1 E_a^1 + B_j^3 E_a^3 = B_j^3 E_a^3 - u^3$$

$$D_j^0 = B_j^2 E_a^2 + B_j^3 E_a^3 = v_3 - v^3$$

$$C^0 = B_j^1 E_a^1 + B_j^2 E_a^2 + B_j^3 E_a^3 = u^1 + v^1 + B_j^3 E_a^3$$  \hspace{1cm} (6.5)

which can be solved algebraically for $u^3, v^1, v^2, v^3$ thus

$$B_j^1 = u^1 E_j^1 + u^3 E_j^3 + [B_j^1 E_b^1] E_j^3$$

$$B_j^2 = u^2 E_j^1 + (B_j^2 E_b^1 - u^1) E_j^2 + [B_j^2 E_b^1] E_j^3.$$  \hspace{1cm} (6.6)

The coefficients $u^1, u^2$ are constrained by the Bianchi identities $\partial_a B_\alpha^a = 0$

$$\delta^{\alpha\beta} E_\beta^a u_{\alpha,a} = - \left[ (B_j^1 E_j^1) E_a^3 \right]_a =: -t$$

$$\epsilon^{\alpha\beta} E_\beta^a u_{\alpha,a} = - \left[ (B_j^1 E_j^1) E_a^3 + (B_j^2 E_j^2) E_a^3 \right]_a =: -r$$  \hspace{1cm} (6.7)
where \( \partial_{\alpha}E_{\alpha}^a = 0 \) was used and \( \epsilon^{\alpha\beta} \) is the skew symbol in 2 dimensions. We introduce with \( I, J, K, \ldots \in \{ x, y \} \) the 2-dimensional divergence and curl

\[
d := \delta^{\alpha\beta} E_{\alpha}^I u_{\beta,J}, \quad c := \epsilon^{\alpha\beta} E_{\alpha}^I u_{\beta,J},
\]

(6.8)

then

\[
\delta^{\alpha\beta} E_{\alpha}^u u_{\beta,z} = -(t + d), \quad \epsilon^{\alpha\beta} E_{\alpha}^u u_{\beta,z} = -(r + c).
\]

(6.9)

The two equations (6.9) provide a quasi-linear (even linear) first order PDE system in two functions \( u_1, u_2 \). By the Cauchy–Kowalewskaja (CK) theorem [39] maximal analytic and unique solutions of (6.9) exist for real analytic ‘initial data’ \( u_0^I \) on a surface transversal to the \( z \) coordinate lines (e.g. the surface \( z = 0 \), real analytic inhomogeneities \( t, d \) and real analytic \( E_{\alpha}^u \) provided that the matrix

\[
\begin{pmatrix}
E_1^u & E_2^u \\
-E_2^u & E_1^u
\end{pmatrix}
\]

(6.10)

is non-degenerate i.e. \( \delta^{\alpha\beta} E_{\alpha}^u E_{\beta}^u > 0 \) (non-characteristic condition). In that case we can solve (6.9) for \( u_{\alpha,z} \) and then can compute the Taylor expansion of \( u_{\alpha} \) off \( z = 0 \) by CK iteration of the PDE system. We can argue the same way by solving instead for the \( x \) and \( y \) derivatives. It is not possible that \( \delta^{\alpha\beta} E_{\alpha}^x E_{\beta}^x = 0 \) for more than one of \( a = x, y, z \) because otherwise this would imply that say \( E_{\alpha}^x = 0; \quad I = x, y, z; \quad \alpha = 1, 2 \) and the metric would be degenerate. Thus w.l.o.g. we may pick the \( z \) direction as long as the non-characteristic condition above holds.

We may solve (6.9) also in case that \( E_1^u = E_2^u = 0 \) as long as the metric is non degenerate. For this requires that \( \det(\{ E_{\alpha}^I \}) \neq 0 \) and \( \partial_\alpha E_{\alpha}^I = \partial t E_{\alpha}^I = 0 \) ensures that there exist functions \( e_\alpha \) with \( E_{\alpha}^I = e_\alpha E_{\alpha}^I \) by simple connectedness of \( \sigma = \mathbb{R}^3 \). It follows that \( x, y, z \mapsto (\hat{x}, \hat{y}, \hat{z}) = (\epsilon_1(x, y, z), \epsilon_2(x, y, z), z) \) is a diffeomorphism. Furthermore, as (6.9) vanishes identically we have with \( u_{\alpha}(x, y, z) =: u(\epsilon)(x, y, z), \epsilon_2(x, y, z), z) \)

\[
d = \delta^{\alpha\beta} \epsilon^{IJ} e_{\alpha,J} u_{\beta,I} = \det (E_{\alpha}^I) \epsilon^{\alpha\beta} \partial_{\alpha} \hat{u}_{\beta} = -t
\]

\[
c = \epsilon^{\alpha\beta} \epsilon^{IJ} e_{\alpha,J} u_{\beta,I} = \det (E_{\alpha}^I) \delta^{\alpha\beta} \partial_{\alpha} \hat{u}_{\beta} = -r.
\]

(6.11)

Switching to those coordinates and denoting by \( \hat{i}, \hat{r} \) the transformed functions \( t/\det (E_{\alpha}^I), r/\det (E_{\alpha}^I) \) the solution is obtained as

\[
\hat{u}_{\alpha} = \Delta_2^{-1} \left[ \epsilon^{\beta\alpha} \hat{t}_{\beta} + \hat{r}_{\alpha} \right] + \epsilon^{\alpha\beta} \hat{h}_{\beta}
\]

(6.12)

where \( \Delta_2 = \delta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} \) and \( \hat{h}_{\alpha} \) is a homogeneous solution of (6.11)

\[
\hat{h}_{1,1} - \hat{h}_{2,2} = \hat{h}_{1,2} + \hat{h}_{2,1} = 0
\]

(6.13)

i.e. a solution of the Cauchy–Riemann (CR) equations in \( \hat{x}, \hat{y} \) which means that \( \hat{h}_1 + i\hat{h}_2 \) is a holomorphic function in \( \hat{x} + i\hat{y} \).

This shows that unique (up to the holomorphic function freedom at characteristic surfaces) and maximal analytic solutions to the constraints \( D_I = 0, C = 0 \) in terms of \( B_{\alpha}^I \) always exist if the \( B_{\alpha}^I, E_{\alpha}^I \) are real analytic which can in principle be computed to arbitrary precision using Taylor expansion. Solutions may also exist outside the analytic category but this will not be of relevance for what follows.

To see whether the gauge \( E_{\alpha}^u = \delta_{\alpha}^u \) can be installed we consider a general gauge transformation of \( E_{\alpha}^u \)
\[
\delta E^a_\alpha(x) = \left\{ \int d^3y \left[ \xi^l E^l_\alpha + N \xi^l \right](y), E^a_\alpha(x) \right\} \\
= \left\{ \int d^3y B^b \left[ \xi_{\beta} E^b_{\alpha} + N E^b_{\alpha} \right](y), E^a_\alpha(x) \right\} \\
= \epsilon_{abc} \partial_b \left[ \epsilon_{\alpha b} E^b_{\xi^l} + N\delta^l_{\alpha} E^l_{\alpha}(x) \right] (6.14)
\]

which shifts \(E^a_\alpha\) by a divergence free vector density thus preserving the Gauss constraint \(\partial_a E^a_\alpha = 0\). If we work with density weight \(\delta\) constraints then the relation between \(\xi^l, N\) in (6.14) and the density weight zero shift and lapse functions \(n^a, n\) is

\[
n^a \left| \det (E) \right|^{(\delta - 1)/2} = \xi^l E^l_\alpha \left| \det (E) \right|^{-1}, \quad n \left| \det (E) \right|^{(\delta - 2)/2} = \frac{N}{2} \left| \det (E) \right|^{-1}. (6.15)
\]

To install \(E^a_\alpha = \delta^a_\alpha\) we must have \(\delta E^a_\alpha = E^a_\alpha - \delta^a_\alpha = \Delta^a_\alpha\) with \(\partial_a \Delta^a_\alpha = 0\). Since \(\mathbb{R}^3\) is simply connected we find \(w_{\alpha a}\) with \(\Delta^a_\alpha = \epsilon_{abc} \partial_b w_{ca}\). We could impose w.l.g. \(\delta^{ab} \partial_a w_{ba} = 0\) but this will not be needed. It follows that (6.14) is solved by

\[
\epsilon_{abc} E^{b}_{\xi^l} + N\delta^l_{\alpha} E^l_{\alpha} = w_{\alpha a} + \partial_a g_{\alpha} (6.16)
\]

where \(g_{\alpha}\) are free functions. These are six equations for six free functions \(\xi^l, N, g_{\alpha}\) and we can solve them as follows:

Contracting (6.16) by \(E^i_{\beta}\) we obtain the equivalent system

\[
\epsilon_{\alpha \beta} \xi^i \equiv Z^i_{\alpha} (6.17)
\]

We first solve algebraically for \(\xi^l, N\). For \(j = 3\) and \(j = \beta\)

\[
-\epsilon_{\alpha \beta} \xi^\beta = Z^\beta_{\alpha}
\]

which yields

\[
\xi^\alpha = \epsilon^{\alpha \beta} Z^\beta_{\alpha}, \quad \xi_3 = -\frac{1}{2} \epsilon^{\alpha \beta} Z_{\alpha \beta}, \quad N = -\frac{1}{2} \delta^{\alpha \beta} Z_{\alpha \beta}, (6.19)
\]

and that \(Z_{\alpha \beta}\) does not have a trace free symmetric part

\[
Z_{(\alpha \beta)} = \frac{1}{2} \delta_{\alpha \beta} \delta^{\gamma \delta} Z_{\gamma \delta} = 0. (6.20)
\]

The system (6.20) only involves \(g_{\alpha}\). Its independent ingredients are only two equations

\[
Z_{12} + Z_{21} = 0, \quad Z_{11} - Z_{22} = 0 (6.21)
\]

which maybe written as

\[
E^1_{\alpha} [w_{a1} + \partial_a g_{1}] = 0, \quad E^1_{\alpha} [w_{a1} + \partial_a g_{1}] = 0, \quad E^2_{\alpha} [w_{a2} + \partial_a g_{2}] = 0. (6.22)
\]

With the relabelling

\[
u_1 := g_2, \quad w_2 := g_1, \quad t := E^1_{\alpha} w_{a2} + E^2_{\alpha} w_{a1} r := E^1_{\alpha} w_{a1} - E^2_{\alpha} w_{a2} (6.23)
\]

the system (6.22) turns into exactly the system (6.7) which therefore has a unique maximal analytic solution \(g_{\alpha}\) given suitable initial conditions. Having obtained such a solution \(g_{\alpha}\) the gauge is installed by using that \(g_{\alpha}\) in (6.19).

Once the gauge is installed, we ask what residual gauge transformations leave \(E^a_\alpha = \delta^a_\alpha\) invariant. Then we must solve (6.22) with \(w_{\alpha a} = 0\) and \(E^a_\alpha = \delta^a_\alpha\)

\[
g_{2, x} + g_{1, y} = 0, \quad g_{1, x} - g_{2, y} = 0 (6.24)
\]
i.e. \( g = g_1 + ig_2 \) is a holomorphic function of \( x + iy \) and its \( z \) dependence is not constrained. Then (6.19) becomes with \( Z^\alpha_\alpha = E^\alpha_\alpha g_{\alpha\alpha}, \ Z^\alpha_\beta = \partial_\alpha g_\beta \)

\[
\xi^\alpha = \epsilon^{\alpha\beta} E^\beta_3 \partial_\alpha g_\beta, \ \xi_3 = -\frac{1}{2} \epsilon^{\alpha\beta} \partial_\alpha g_\beta, \ N = -\frac{1}{2} \delta^{\alpha\beta} \partial_\alpha g_\beta. \quad (6.25)
\]

The relation between \( \xi^\beta, N \) and the shift and lapse fields \( n^\mu, n \) with density weight zero is given in (6.15) for the density \( \delta \) form of the constraints. In the given gauge \( \text{det}(E) = E^3_3 \) and \( |\text{det}(E)|^{(1+\delta)/2} n^\delta = E^\delta_3 E^\delta_3 + \delta_m^\delta \xi^\alpha, \ |\text{det}(E)|^{\delta/2} n = \frac{N}{\alpha} \). If we want to keep the relation with GR intact then we want that \( n^\delta \) decays at infinity while \( n \) approaches a constant \( c \) as to obtain the Minkowski metric asymptotically with Minkowski/Euclidian time coordinate \( \chi^0 = ct \) where \( c > 0 \) is some parameter which has the value of the speed of light when \( t \) is the usual asymptotic time but we allow variable \( c \) corresponding to 1-parameter reparametrisation freedom \( t \rightarrow t' = ct/c'. \)

Furthermore

\[
3^a - \delta^{ab} = \delta^\delta a^b b^a a^a b^b - \delta^{ab} = E^a_3 E^a_3 + \delta_1^\delta b^b + \delta_2^\delta b^b - \delta^{ab} = E^a_3 E^a_3 - \delta_3^\delta b^b \quad (6.26)
\]

also decays, i.e. \( E^3_3 \) and \( E^\delta_3 = 1 \) decay. Thus \( \xi^\delta \) must decay while \( N \) approaches \( 2c. \) Since \(-2\xi_3 = g_{2x} - g_{1y} = g_{2x} - -2g_{1y}, \) and \( \xi_3 = \epsilon^{\alpha\beta} E^\beta_3 g_{\alpha\beta}, \) the decay of \( \xi_3 \) can be ensured by demanding \( g_{\alpha\beta} = 0, \) \( g_{2x} = g_{1y} = 0 \) which means that \( g_{\alpha\beta} \) just depends on \( x^\delta \) and holomorphicity \( g_{1x} = g_{2y} = -c \) is a constant. Then \( N = 2c \) and \( n \) approaches \( c \) asymptotically indeed. Moreover, \( \xi_3 = 0, \xi^\alpha = -c\epsilon^{\alpha\beta} \partial_\beta E^\delta_3. \) As shown in [27] the analysis can be completed by providing suitable decay behaviour of all fields in order to make all constraints functionally differentiable and finite.

Thus, as intended we obtain a 1-parameter freedom of residual gauge transformations which give rise to a reduced Hamiltonian \( H. \) This is defined as follows. Let \( G \) be any functional of the true degrees of freedom \( E^a_3, A^a_\mu \) and let \( n, n^\delta \) be the 1-parameter family of lapse and shift functions just found. Let \( E^a_\alpha = \partial^\alpha_\alpha \) be the constrained/gauge fixed pure gauge configuration degrees of freedom and \( A^\alpha_\mu \) be the constrained/gauge fixed pure gauge momentum degrees of freedom whose existence we have demonstrated above but whose explicit expression will not be needed in what follows. Then

\[
\{ H^\delta, G \} = \{ \bar{D}^\delta (\bar{n}) + C^\delta (n), G \} \quad (6.27)
\]

where \( D^\delta, C^\delta \) are taken in the density weight \( \delta \) form. The Hamiltonian determined by (6.27) has been computed in [27] for \( \delta = 1 \) using an abstract argument valid for general theories at most linear in the momenta. To make exposition self-contained, we provide here an alternative argument which uses the concrete structure of our constraints. We expand

\[
C^\delta_\mu (n^\mu) := \bar{D}^\delta (\bar{n}) + C^\delta (n) = \int \text{d}^3 x \ n^\mu [B^\mu_\alpha h^{\delta,\alpha}_\mu + h^\delta_\mu] \quad (6.28)
\]

where \( n^0 = n \) and \( h^{\delta,\alpha}_\mu \) depends only on \( E^\delta_\alpha \) while \( h^\delta_\mu \) depends linearly on \( E^\delta_3 \) and on \( E^\delta_\alpha. \) We denote by \( B^\mu_\alpha \) the solution of (6.28) at \( E^\delta_\alpha = E^\delta_3 \) and denote the values of \( h^{\delta,\alpha}_\mu, h^\delta_\mu \) by \( h^{\delta,\alpha}_\mu, h^\delta_\mu \) respectively. Moreover, the solution of the gauge stability

\[
\{ C^\delta_\mu (n^\mu), E^\delta_\alpha \} = 0 = \epsilon^{\alpha\beta} \partial_\beta [n^\mu h^{\delta,\alpha}_\mu] \quad \Rightarrow \quad n^\mu h^{\delta,\alpha}_\mu = \partial_\alpha g^{\delta,\alpha} \quad (6.29)
\]

at \( E^\delta_\alpha = E^\delta_3 \) is denoted by \( \bar{n}^\mu, \bar{g}^{\delta,\alpha}. \) Thus we have

\[
B^\mu_\alpha h^{\delta,\alpha}_\mu + \bar{h}_\mu = 0, \ \bar{n}^\mu \bar{h}^{\delta,\alpha}_\mu = \partial_\alpha \bar{g}^{\delta,\alpha}. \quad (6.30)
\]
Thus
\[ \{ H^\delta, G \} = \int d^3 x \, \bar{\eta}^\mu \left[ \mathcal{B}^a_\mu \left\{ \delta^\alpha_{\mu \alpha}, G \right\}_{\mathcal{E}_a^{\alpha} = \mathcal{E}_a^{\alpha}} + \{ \delta^\mu_{\mu}, G \} \right] \]
\[ = \int d^3 x \, \bar{\eta}^\mu \left[ \mathcal{B}^a_\mu \left\{ \delta^\alpha_{\mu \alpha}, G \right\} + \{ \delta^\mu_{\mu}, G \} \right] \]
\[ = \int d^3 x \left[ \mathcal{B}^a_\mu \left( \{ \bar{\eta}^\mu \delta^\alpha_{\mu \alpha}, G \} - \delta^\alpha_{\mu \alpha} \{ \bar{\eta}^\mu, G \} \right) + \bar{\eta}^\mu \{ \delta^\delta_{\mu}, G \} \right] \]
\[ = \int d^3 x \left[ \mathcal{B}^a_\mu \partial_\alpha \{ g^\delta_{\alpha}, G \} + \delta^\delta_{\mu} \{ \bar{\eta}^\mu, G \} \right] + \bar{\eta}^\mu \{ \delta^\delta_{\mu}, G \} \]

where in the first step we used that the Poisson bracket must involve the constraint and that \( \{ B^a_\alpha, G \} = 0 \), in the second we used that while \( h^\alpha_{\mu \beta}, \delta^\mu_{\mu} \) depend on \( E_a^\alpha \) before fixing to \( E_a^\alpha \), the Poisson bracket with \( G \) only cares about the \( E_a^3 \) dependence, in the third we used the Leibniz rule, in the fourth we used both identities (6.30), in the fifth we integrated by parts (the decay behaviour of the fields excludes a boundary term [27]), used the Bianchi identity on the magnetic field and applied the Leibniz rule again. It follows that

\[ H^\delta = \int d^3 x \, \bar{\eta}^\mu \, \delta^\mu_{\mu} \]
\[ = \int d^3 x \left[ \bar{\xi}_a e^{b\beta} B^a_{\beta} E_a^\beta + \bar{N} B^a_3 E_a^3 \right] \]
\[ = \int d^3 x \left[ \bar{\xi}_a e^{a\beta} B^a_{\beta} E_a^\beta + \bar{N} B^a_3 E_a^3 \right] \]
\[ = \int d^3 x \left[ \bar{\xi}_a e^{b\beta} B^a_{\beta} E_a^\beta + \bar{N} B^a_3 E_a^3 \right] \]
\[ = c \int d^3 x \left[ - \delta^a_{\mu} E_a^3 B^a_{\mu} E_a^\mu + 2 B^a_3 E_a^3 \right] \]
\[ = c \int d^3 x \left[ \delta^a_{\mu} E_a^\mu \right]^{-1} \left[ - \delta^a_{\mu} E_a^3 B^a_3 \left( \delta^a_{\mu} E_a^\mu - \delta^3_{\mu} E_a^3 \right) + 2 B^a_3 \right] \]
\[ = c \int d^3 x \left[ \delta^a_{\mu} E_a^\mu \right]^{-1} \left[ - \delta^a_{\mu} E_a^3 B^a_3 + B^a_3 \left( 2 + \delta^a_{\mu} E_a^\mu \right) \right] \]
\[ =: < R, f(E) > \] (6.32)

where we used
\[ |\det (\bar{E})| E_a^3 = \delta^a_{\mu}, \quad |\det (\bar{E})| E_a^\alpha = \left[ - \delta^a_{\mu} E_a^\mu + \delta^a_{\mu} E_a^3 \right] \] (6.33)

and the notation
\[ f_a (E) = \frac{1}{E_a^3} \left[ \delta^a_{\mu} \left( 2 + \delta^a_{\mu} E_a^3 \right) - \delta^a_{\mu} \delta^a_{\mu} E_a^3 \right] . \] (6.34)

Note that \( H = H^\delta \) is independent of the choice of \( \delta \). This is true in general: The reduced Hamiltonian does not care about the density weight which was used for the constraint because one can redefine lapse and shift functions to unity density weight and finds unique values for those when gauge fixing.
It follows that the reduced theory is a kind of non-linear electrodynamics with gauge invariant electric and magnetic fields \( E^a := E^a_\lambda, B^a := B^a_\lambda = e^{abc} \partial_b A_c, A_a = A^a_\mu \) which is subject to the Gauss constraint \( G = \partial_a E^a = G_3 \). As the Hamiltonian is gauge invariant and linear in momentum \( A_a \) and since the configuration observables \( O \) that were extracted via the rigging map of the previous section are to be identified with the variables \( E^a_3 \), in order to match the representation of the reduced phase space quantised theory with the Dirac quantised theory we pick the following physical Hilbert space:

The vacuum is annihilated by the electric field \( E^a \Omega = 0 \) and excited states are given by \( \langle F, A \rangle = \int d^3 x F^a A_a \) where

\[
\langle F, A \rangle = \int d^3 x F^a A_a. \tag{6.35}
\]

The Gauss constraint is solved by asking that \( F \) is divergence free and the non-degeneracy is met by asking that \( F^a \) is nowhere vanishing. Since the Hamiltonian is linear in momentum \( A_a \) its quantisation in this representation is straightforward by the general results of section 3.

Again \( H \) itself is not defined but its unitary, weakly discontinuous one parameter group is

\[
U(t) \ w[F, \Omega] := \exp(-itH) \ w[F, \Omega] = w \left( (e^{X_H} K) (F) \right) \Omega \tag{6.36}
\]

with the Hamiltonian vector field \( X_H \) on the phase space with canonical pair \( (A_a, E^a) \) and \( K^a(A_a, E) = E^a \) is the coordinate function on the phase space which does not depend on \( A \) and thus we write \( K(F) = K(., F) = F^a. \) We have

\[
(X_H K)^a \left( F \right) = e^{abc} \partial_b f_c (F) \tag{6.37}
\]

which shows that the Hamiltonian flow of \( H \) preserves the divergence free vector field densities. Let \( \Lambda_1 \) be the space of 1-forms and \( V^1 \) the space of divergence free vector (pseudo) densities then we have maps

\[
f: V^1 \rightarrow \Lambda_1; \ F \mapsto f(\Lambda_1; \ F) = \Lambda_1; \ F \mapsto f(F), \tag{6.38}
\]

where \( f \) is given by (6.34). Then \( X_H = \text{rot} \circ f \) and the first few terms of the Taylor expansion of \( (e^{X_H} K) \) are

\[
X_H K = \text{rot} [f(K)], \ X_H^2 K = \text{rot} [f(\text{rot} [f(K)])], \ X_H^3 K = \text{rot} [f(\text{rot} [f(\text{rot} [f(K)])])]. \tag{6.39}
\]

The relation (6.36) states that, similar to coherent states for the harmonic oscillator, the quantum evolution of the states \( \langle F, \Omega \rangle \) with \( F \) the eigenvalue of the electric field, stays \textit{exactly on the classical trajectory}. The mathematical reason for this is the same as for coherent states, namely that the Hamiltonian flow preserves the polarisation: For coherent states, holomorphic functions of \( z = q - ip \) are preserved while here functions of \( E \) are preserved.

Furthermore, (6.36) states that we have access to the \textit{non-perturbative scattering matrix} in this interacting quantum field theory. Its matrix elements are exactly computable

\[
\langle \ w[F], \Omega, U(t) \ w[F'], \Omega \rangle = \delta_{F, (e^{X_H} K)(F')}. \tag{6.40}
\]

To interpret this model physically, note that in the given gauge the inverse density weight two metric is explicitly given by

\[
Q^{ab} = \begin{pmatrix}
1 + [E^1]^2 & E^1 E^2 & E^1 E^3 \\
E^1 E^2 & 1 + [E^2]^2 & E^2 E^3 \\
E^1 E^3 & E^2 E^3 & [E^3]^2
\end{pmatrix}. \tag{6.41}
\]
We write $E^I = h^I$, $I = 1, 2$ and $E^3 = 1 + h^3$ with

$$h^2(x, y, z) = - \int_{-\infty}^{\infty} ds \left( \partial_3 h^3 \right)(x, y, s)$$

(6.42)

assuming that $h^t$ vanish at infinity so that $h^t, h^3$ are two independent polarisations of the metric which could be called ‘gravitons’. To first order in $h^t$ we have $Q^{ab} = \delta^{ab} + 2h^a(\delta^{(a} \delta^{b)}) + 2h^b \delta^{(a} \delta^{b)}$.

Thus, we may interpret the matrix elements of the scattering matrix as graviton propagation or metric perturbation amplitudes. The fact that the matrix elements only take values 0, 1 between the Weyl states $w[F] \Omega$ looks strange at first but the fact that the $w[F] \Omega$ are similar to semiclassical coherent states in a Fock space makes this look less strange: Consider the coherent states $\psi(\Omega) = w[F] \Omega$ which due to the infinite number of degrees of freedom involved is very sharply peaked at $Z = e^{i\hbar \omega} Z$. Thus the modulus of the coherent state scattering amplitude is

$$| \langle \psi_Z, U(t) \psi_Z \rangle |^2 = e^{-\int d^4x \left[ Z^{\mu} - e^{i\omega \cdot \cdot \cdot} Z^{\mu} \right] \left[ Z^{\mu} - e^{i\omega \cdot \cdot \cdot} Z^{\mu} \right]}$$

(6.44)

which due to the infinite number of degrees of freedom involved is very sharply peaked at $Z = e^{i\hbar \omega} Z$ and thus is almost a Kronecker $\delta$. That it is not exactly a Kronecker $\delta$ is because in the Fock representation the Hamiltonian unitaries are weakly continuous.

The analogs of graviton states in our representation are therefore not the $w[F] \Omega$. To obtain the analog of graviton states we note that next to $\Omega$ there are many other ground states of the Hamiltonian namely $w[F] \Omega$ with $F = \text{const}$. This is because the Hamiltonian flow on $E^a$ is a first order equation $E^a = \text{rot}(|f(E)|^a$ and while $f_e(E)$ depends algebraically on $E$, the rotation operation maps a constant one form to zero. Thus, given constant initial data, these stay invariant in time as well. Thus we may pick as the Minkowski state the state

$$\Omega_M := w[F] \Omega, \quad F^a = \delta^a_i$$

(6.45)

which indeed yields $Q^{ab} = \delta^{ab}$ (note that $F^a_M$ is divergence free). From the point of view of $\Omega$, $\Omega_M$ is a highly excited ground state. We consider now Fock states as excitations of $\Omega_M$. Given a wave vector $0 \neq \vec{k} \in \mathbb{R}^3$ pick two transversal real valued vectors $\vec{e}_\lambda(\vec{k})$, $\lambda = 1, 2$ such that with $\vec{e}_3(\vec{k}) = \vec{k}/||\vec{k}||$ these three vectors constitute a right oriented ONB of $\mathbb{R}^3$ with Euclidian metric. Consider the even/odd functions under reflection at the origin

$$F^a_{\tilde{\vec{k}}, \lambda, +} (x) = e^\lambda (\vec{k}) \cos \left( \vec{k} \cdot x \right), \quad F^a_{\tilde{\vec{k}}, \lambda, -} (x) = e^\lambda (\vec{k}) \sin \left( \vec{k} \cdot x \right)$$

(6.46)

and for ‘occupation numbers’ $n \in \mathbb{N}$ the ‘multi-particle state’

$$\Psi_{\{k, \lambda, n\}} = \sum_{i=1}^{N} n_i F_{\tilde{\vec{k}}, \lambda, n_i} \Omega_M$$

(6.47)

with pairwise different $\tilde{\vec{k}}_1, \ldots, \tilde{\vec{k}}_N$. Then
where $S_N$ is the symmetric group in $N$ elements. We have exploited that the Kronecker is non-vanishing iff the number of labels are the same up to a permutation of the arguments. Thus (6.48) looks like a Fock space inner product except that plane wave smeared creation operators have been replaced by Weyl elements which are Kronecker normalised rather than $\delta-$distribution normalised. If one replaces the plane wave smearing by wave packet smearing, the analogy gets even closer.

The scattering matrix elements between the ‘Fock states’ (6.47) is now rather non-trivial because the form of the Fock states is not at all preserved in time and reflects both the interaction and our non-perturbative treatment thereof. This may also be seen as follows. The analog of the Wightman N-point functions are the Heisenberg correlators

$$W_{F_1,\ldots,F_N}(t_1,\ldots,t_N) := \langle \Omega, U(t_1) \, w [F_1] \, U(t_1)^{-1} \ldots U(t_N) \, w [F_N] \, U(t_N)^{-N} \, \Omega \rangle$$

$$= \langle \Omega, \, w \left[ \sum_{l=1}^{N} (e^{i \pi F_l} K) (F_l) \right] \, \Omega \rangle$$

$$= \delta_{\sum_{i=1}^{N} (e^{i \pi F_i} K)(F_i),0}$$  (6.49)

which cannot be written as a Gaussian in the $F_i$ and thus does not correspond to a (quasi-)free state, thereby demonstrating that this is an interacting QFT in $3+1$ dimensions. Using the theorems of section 3 we note that we can actually quantise all interacting QFT in any dimension as long as in the Hamiltonian formulation it is at most linear in momentum. We will come back to this in section 10.

The amplitudes (6.49) motivate a path integral formulation that we will address in section 8. Before we communicate another interesting observation.

### 7. Non-relational weak quantum Dirac observables

In [40] for the SU(2) theory (Euclidian GR) the observation was made that the seven constraints $G_j = D_a = C = 0$ can be solved by making the Ansatz (which is w.l.g. for non-degenerate metrics) $B^\beta_i = \lambda_\beta B_i$ and $D_a = 0$ requires $\lambda_\beta = 0$, $C = 0$ requires $\delta^{\beta\lambda} \lambda_\lambda = 0$ and the Bianchi identity $D_a B^\beta_i = 0$ requires $E^\alpha_i D_a \lambda_\beta = 0$ when $G_j = D_a E^\alpha_i = 0$ holds where $D_a$ is the covariant derivative of $A$. For the SU(2) theory it is very difficult to solve the condition $E^\alpha_i D_a \lambda_\beta = 0$.

However, we may apply the same Ansatz in the U(1) theory and find that $B^\beta_i = \lambda_\beta E^\alpha_i$ solves all constraints when $\lambda_\lambda$ is a constant, symmetric, trace free matrix.

Now consider the question to construct (weak) Dirac observables $O$ that depend only on $E$. Thus $\{D_u, O\} = \{C[N], O\} = 0$ for all $u, N$ at least on the constraint surface $G_j = D_a = C = 0$ of the $A, E$ phase space. Note that $O$ is trivially invariant under Gauss gauge transformations. This leads to the conditions

$$F^\beta_{ab} E^\alpha_{ij} = \epsilon_{\beta[\alpha} F^\beta_{ab} E^\alpha_{ij} = 0, \quad F^\beta_{ab} = 2 \partial_{a\alpha} \frac{\delta O}{\delta E^\alpha_i}.$$  (7.1)
The relation to the above Ansatz becomes now clear because the requirement
\[\epsilon^{abc} \frac{\partial}{\partial x^a} \kappa_{bc}(x,y) = \lambda_{ab} E_k^a \]  
(7.2)
with \(\lambda_{ab}\) constant, symmetric, tracefree solves (7.1) and the Bianchi identity when the Gauss constraint holds. The question is whether (7.2) can be solved for \(O\). The surprising answer is that it can for any such matrix \(\lambda\) which enables us construct an infinite number of non-configuration, non-relational Dirac observables.

We claim that on the constraint surface defined by the Gauss constraint
\[O_\lambda [E] := \frac{1}{2} \lambda^k \int_\sigma d^3 x \int d^3 y E_k^a(x) \kappa_{ab}(x,y) E^b_k(y)\]  
(7.3)
solves (7.2) with the symmetric Green function \(\kappa_{ab}(x,y) = \kappa_{ba}(y,x)\)
\[\kappa_{ab}(x,y) := -\epsilon_{abc} \delta^{cd} \frac{\partial}{\partial x^d} \kappa_{bc}(x,y), \kappa_{\Delta}(x,y) = -\frac{1}{4\pi ||x-y||},\]  
(7.4)
Obviously \(\kappa_{\Delta}\) is the Green function of the Laplace operator in flat Euclidian space. The astonishing fact is that \(O_\lambda\) is (weakly) spatially diffeomorphism invariant even though \(\kappa_{ab}\) heavily depends on the Euclidian background metric \(\delta_{ab}\). A similar counter-intuitive effect is observed for knot invariants [41] which are also constructed using the background dependent Green function (7.4).

We have
\[\epsilon^{abc} \frac{\partial}{\partial x^a} \kappa_{bc}(x,y) = -[\delta_{da} \delta_{eb} - \delta_{de} \delta_{ad}] \delta_{ef} \frac{\partial^2}{\partial x^e \partial x^f} \kappa_{\Delta}(x,y)\]  
\[= - \left[ \delta_{ab} \frac{\partial^2}{\partial x^b \partial x^e} - \delta_{ab} \Delta, \kappa_{\Delta}(x,y) = \frac{1}{\delta^2} \delta_{e}(x,y) - \delta_{ab} \left( \frac{\partial^2}{\partial x^b \partial x^e} \Delta^{-1} \right) (x,y) = P_{e\perp}_{ab}(x,y)\]  
(7.5)
\[\delta \delta_{ab} \kappa_{ab}(x,y) = \kappa_{ab}(x,y) = \kappa_{ab}(y,x)\]  
(7.6)
Thus by construction
\[\frac{\delta}{\delta E_k^a(x)} O_\lambda = \lambda^k \left[ \kappa_{ab} \cdot E_k^b \right](x).\]  
(7.7)
where \(G_k = \partial_\lambda E_k^a\) is the Gauss constraint. It follows that \(G_k\) is a weak Dirac observable, more precisely it is a strict Dirac observable on the partial constraint surface defined by the Gauss constraint only. Note that (7.6) of course trivially obeys the Bianchi identity also away from the Gauss constraint surface.

Since the constraint equation (7.1) are linear in \(O\) we can construct an infinite number of weak Dirac observables as follows: Let \(F: \mathbb{R}^N \rightarrow \mathbb{R}\) be any \(C^1(\mathbb{R}^N)\) function, let \(\lambda_i, i = 1,\ldots,N\) be any symmetric, tracefree, constant matrices then
\[F[E] := F(O_{\lambda_1}[E], \ldots, O_{\lambda_N}[E])\]  
(7.8)
is a strong Dirac observable on the Gauss constraint surface. Note that \( F \) need not be a polynomial. What is really astonishing is that it is generally believed that a spatially diffeomorphism invariant and Gauss invariant function of \( E^*_J \) should be built from curvature invariants of the metric \( q_{ab} \) where \( Q^{ab} = \delta^{ij} E^*_J E^*_k =: \det (q) q^{ab} \), e.g. as a function of the infinite tower of integrals of the form

\[
O_N := \int \mathrm{d}^3 x \sqrt{\operatorname{det} (q)} \operatorname{Tr} \left( R (q) \right)^N , \quad R_{ab} e^{cd} = R_{abef} q^{ef} q^{cd}
\]  

(7.9)

which are highly non-polynomial and rather complicated functions of \( E^*_J \).

However, (7.8) is not of this form and even more, it is also invariant under the Hamiltonian constraint. An obvious reason for this should be that (7.9) is exactly invariant under spatial diffeomorphisms while (7.8) is only when the Gauss constraint holds. Of course the space of functions (7.8) is much smaller than the space of functions of the (7.9) as the space of symmetric and trace-free matrices is just five dimensional while (7.9) involves infinitely many algebraically independent elements (although there are Gauss–Bonnet type relations among them, i.e. certain linear combinations yield topological invariants).

On the Hilbert space spanned by the \( w[F] \Omega \) constructed in section 3 the \( O_\lambda \) are diagonal with eigenvalue \(-O_\lambda[F] \). It follows

\[
\left[ U[u]^{-1} O_\lambda U[u] - O_\lambda \right] w[F] \Omega = - \left[ O_\lambda \left[ (\epsilon^{xy} K) (F) \right] - O_\lambda[F] \right] w[F] \Omega \\
= - \left[ (\epsilon^{xy} - 1) O_\lambda \circ K \right] w[F] \Omega = 0 \\
\left[ U[M]^{-1} O_\lambda U[u] - O_\lambda \right] w[F] \Omega = - \left[ O_\lambda \left[ (\epsilon^{xy} K) (F) \right] - O_\lambda[F] \right] w[F] \Omega \\
= - \left[ (\epsilon^{xy} - 1) O_\lambda \circ K \right] w[F] \Omega = 0
\]

(7.10)

because we consider divergence free \( F \). More in detail, by construction \( X_a O_\lambda \), \( X_M O_\lambda \) are both proportional to \( G_i \) and \( X_a G_i \propto G_i \), \( X_M G_i = 0 \). Thus \( [\epsilon^{xy} - 1] O_\lambda \), \( [\epsilon^{xy} - 1] O_\lambda \circ G_i \) which is then evaluated at \( F \) but \( G_i[F] \equiv 0 \). Thus on the span of the solutions of the Gauss constraint, the operator \( O_\lambda \) and more generally (7.8) strongly commutes with the constraints and thus preserves the physical Hilbert space constructed in section 5.

8. Path integral formulation of the reduced theory

We are interested in writing the summation (rather than integral) kernel of the propagator \( U(t) \) as a 'path integral'. Usually at this point one performs a Wick rotation in time \( t \) to obtain a contraction operator \( e^{-itH} \) rather than a unitary one \( e^{-itH} \) which has better chances to result in a rigorously defined measure on the space of field histories. However this is only true if \( H \) is bounded from below which for our \( H \) is far from being the case. Thus Wick rotation appears of little use and we thus stick to physical rather than imaginary time in what follows. For this reason, our considerations will be largely heuristic.

Let \( \mathcal{F} \) be the set of all divergence free vector field densities. When we write \( \sum_{\mathcal{F}} \) we mean \( \sum_{F \in \mathcal{F}} \) in what follows. Then the \( w[F] \Omega, F \in \mathcal{F} \) form an orthonormal basis of the non-separable Hilbert space \( \mathcal{H} \) and we may, given \( t \) invoke a partition of \([0,t] \) into segments \([kt/N, (k + 1)t/N] \), \( k = 0, \ldots, N - 1 \) of length \( t/N \) and \( N - 1 \) resolutions of unity.
\[<w[F_N]\Omega, U(t) w[F_0] \Omega> = \sum_{F_1,\ldots,F_{N-1}} \prod_{k=0}^{N-1} <w[F_{k+1}]\Omega, U\left(\frac{t}{N}\right) w[F_k] \Omega>\]

\[= \sum_{F_1,\ldots,F_{N-1}} \prod_{k=0}^{N-1} \delta_{F_{k+1}, (\epsilon \hat{x}^a K)(F_k)} \]  \hspace{1cm} (8.1)

Note that \(\delta_{F,F'}\) really means

\[\delta_{F,F'} = \prod_{x \in \mathbb{R}^3; \alpha = 1,2,3} \delta_{F_{x\alpha}, F'_{x\alpha}} \]  \hspace{1cm} (8.2)

where the product over all \(\alpha = 1,2,3\) is redundant because \(F, F'\) are divergence free but since these are Kronceker functions rather than \(\delta\) distributions, the redundant factors of unity do not cause any singularity.

The Bohr measure on the Bohr compactification \(\mathbb{R}_B\) of the real line has the property

\[\mu_B(T_k) = \delta_{k,0}; \quad T_k : \mathbb{R}_B \mapsto \mathbb{C}, \quad y \mapsto e^{iky}. \]  \hspace{1cm} (8.3)

Consider a partition \(P,\) of \(\mathbb{R}^3\) into cells \(\square\) of coordinate volume \(\epsilon^3\) and centre \(p_{\square}\) and denote by \(\lim\) the limit as the partition reaches the continuum. Then

\[\delta_{F,F'} = \lim \frac{1}{\epsilon} \prod_{\square \in P, \alpha} \delta_{F_{\epsilon x_{\alpha}, F'_{\epsilon x_{\alpha}}}} = \lim \frac{1}{\epsilon} \prod_{\square \in P, \alpha} \mu_B(T_{\epsilon x_{\alpha}}) \]  \hspace{1cm} (8.4)

where we introduced the product Bohr measure on products of the Bohr line and \(C\) denotes the space of ‘Bohr connections’. Thus (8.1) becomes

\[<w[F_N]\Omega, U(t) w[F_0] \Omega> = \sum_{F_1,\ldots,F_{N-1}} \int \prod_{l=0}^{N-1} d\mu_B\left(\epsilon^3 C^l\right) \times \exp \left(i \sum_{l=0}^{N-1} \int d^3 x C^l_0(x) \left[F^a_{l+1} - \left(e^{\hat{\mathcal{H}} K} K^a\right)(F_l)\right](x)\right). \]

(8.5)

Setting \(F^a_{\frac{t}{N}, x} := F^a_{\frac{t}{N}}(x), \quad \hat{F}^a_{\frac{t}{N}, x} := F^a_{\frac{t}{N}+1}(x) - F^a_{\frac{t}{N}}(x), \quad C^a_{\frac{t}{N}, x} := C^a_0(x)\) we have due to linearity of \(X_H\) in \(A\)

\[\int d^3 x C^a_0(x) [F^a_{\frac{t+1}{N}} - (e^{\hat{\mathcal{H}} K} K^a)(F_{\frac{t}{N}})](x)\]

\[= \frac{i}{N} \int d^3 x [C^a_0 \hat{F}^a - H(C,F)] \left(\frac{H}{N}\right) x + O((t/N)^2). \]  \hspace{1cm} (8.6)

To free ourselves from the restriction \(\partial_t F^a = 0\) when summing over \(F\) we introduce a Kronecker \(\prod_k \delta_{C^k}\) and remove by Bohr integration with respect to a zero component \(C_0\) of the connection. Using these manipulations we find
\[ < w[F_N] \Omega, U(t) w[F_0] \Omega > = \sum_{F_1, \ldots, F_{N-1}} \int_{t=0}^{N-1} \prod_{l=0}^{N-1} d\mu_B(C_l) \times \exp \left( i \frac{1}{N} \sum_{l=0}^{N-1} \int d^3x \left[ C_0 \hat{F}^{\alpha} - C_0 \partial_a F^a + H(C,F) \right] \left( \frac{lt}{N} \right)^{\epsilon} \right). \] (8.7)

Formally taking the limit \( N \to \infty \) keeping final \( F_N = F_f = F(i) \) and initial \( F_0 = F_i = F(0) \) fixed and integrating by parts we find

\[ < w[F_i] \Omega, U(i) w[F_i] \Omega > = \int [d\mu_D(F)] \int [d\mu_B(C)] \times \exp \left( -i \int_0^t d^3x \left\{ [\partial_0 C_0 - \partial_a C_0] F^a + H[C,F] \right\} \right) \] (8.8)

where \( \int [d\mu_D(F)] := \sum_{[F]} \) is the discrete (counting) measure. The exponent depends on \( F^a \) which interpret as the electric field \( E^a \), the magnetic field or spatial spatial curvature \( B^a = \epsilon^{abc} \partial_0 C_a \) of \( C \) and the temporal spatial curvature \( \partial_b C_{ab} - \partial_a C_0 \).

Performing the integral over \( C_a \) yields back \( \partial_a F^a = 0 \) and the classical equations of motion \( \hat{F}^a = \delta H/\delta C_a \) while performing the integral over \( E \), if it could be done explicitly, would yield a pure connection formulation [27].

9. Spin foam model

As is well known [37, 42] the heuristic relation between the rigging map \( \eta \) and the path integral over the full unconstrained phase space is as follows:

Let \( q^\alpha, p_a \) be canonical coordinates on the full phase space, let \( F_\alpha \) be all first class constraints, let \( S_0 \) be all second class constraints and let \( G^\alpha \) be a complete system of gauge fixing conditions for the \( F_\alpha \). Split \( (q^\alpha, x^I, Q^A) \), \( (p_a, \pi_\alpha, \gamma, P_A) \) where \( Q^A, P_A \) are referred to as the true degrees of freedom. Then

\[ < \eta \psi, \eta \psi' > := \lim_{T \to \infty} \int [dq \ dp] \psi^*(\phi_{-T, T} Q_T) \psi' (\phi_{-T, T} Q_{-T}) \delta[F] \delta[S] \delta[G] |\det[F,G]| |\det[S]| |\det[S]|^{1/2} e^{\int_{-T}^{T} \sum_{\alpha} \epsilon^\alpha \eta_\alpha} \] (9.1)

where \( \Omega_0 \) is some cyclic reference vector. The notation is that square brackets denote the product over all \( t \in [-T, T] \) of instantaneous quantities which are denoted by round brackets, e.g. \( \delta[F] = \prod_t \delta(F_t) \) and \( \delta(F_t) = \prod_{(\alpha)} \delta(F_{\alpha}(t)) \). The notation \( \psi, \psi' \to \Omega_0 \) in the denominator means that the denominator equals the numerator except for the replacement \( \psi, \psi' \to \Omega_0 \).

In [27] the first and second class classification of all constraints following from the covariant action (2.1) has been derived so that we can identify the above structures for the U(1)** model.

1. The unconstrained phase space has configuration variables \( (q^\alpha) = (A^\mu_{(1)}, \hat{e}_{(1)}^\alpha, \hat{\epsilon}_{(0)}^\alpha) \) and momentum variables \( (p_a) = (M^\mu_{(2)}, H, P_I) \) with spacetime indices \( \mu, \nu, \rho, \ldots = t, 1, 2, 3 \) and \( j = 1, 2, 3 \). The relation between \( \hat{e}^\alpha \) and \( e^\mu \) is that \( \hat{e}^\alpha = |\det(e^\mu_{(1)})|^{1/2} e^\mu \) is a half density.
2. Let
\[ \sigma^{\mu \nu} = 2 \varepsilon^{[\mu} \varepsilon_{\nu]} + \varepsilon^{\beta \delta} \varepsilon_{[\beta} \varepsilon_{\delta]} . \] (9.2)

Primary constraints are
\[ T^{\mu}_j = M^{\mu}_j - \sigma^{\mu}_{,\mu}, \quad M^{\mu}_j, \quad \tilde{P}^{\mu}_j \]
with \( a, b, c, \ldots = 1, 2, 3 \) spatial indices.

3. The first nine primary constraints are the triad conditions. Then \( M^{\mu}_j \) and seven of the sixteen \( \tilde{P}^{\mu}_j, \tilde{P}^{\mu}_j \) are stabilised by the secondary constraints
\[ G_j = \partial_0 \sigma^{\mu a}_j, \quad D_a = F^{\mu b}_{ab} \varepsilon^b_{,0} + \varepsilon_{ab} F^{\mu k}_{ab} \varepsilon^k_{,j}, \quad T_j = \varepsilon^j_0 \]
with \( F^{\mu b}_{ab} = 2 \partial_0 A^{b}_{ab} \). Note that \( T_j = 0 \) is the ‘time gauge’ which in the U(1)\(^3\) theory is a necessary constraint and not a convenient gauge fixing condition.

4. The Lagrange multipliers \( v'_a, v'_j, \tilde{v}'_a, \tilde{v}'_i \) of the primary constraints get fixed to
\[ v'_a = [\varepsilon^0_0]^{-1} \left[ F^{\mu b}_{ab} \varepsilon^b_{,0} + \varepsilon_{ab} F^{\mu k}_{ab} \varepsilon^k_{,j} \right], \quad v'_j = 0, \quad \tilde{v}'_a = -[\varepsilon^0_0]^{-1} \left[ \partial_0 \sigma^{ab} + \varepsilon_{ab} \varepsilon^0_0 \right] \]
(9.5)
to stabilise the remaining nine of the \( \tilde{P}^{\mu}_j, \tilde{P}^{\mu}_j \), the time gauge \( T_j \) and the triad conditions \( T^{\mu}_a \), while \( v'_j, \tilde{v}'_i \) remain free. No tertiary constraints arise.

5. The 14 first class constraints are given by
\[ \tilde{P}^{\mu}_j, \quad M^{\mu}_j, \quad \tilde{G}_j = \partial_0 M^{\mu}_j, \]
\[ \tilde{D}_a = F^{\mu b}_{ab} \tilde{A}^b_j + \sum_{L=0}^{3} \left\{ 2 \left[ (\tilde{t}^{\mu}_{,b} \varepsilon^b_{,0})_{,b} - (\tilde{t}^{\mu}_{,b} \varepsilon^b_{,0})_{,b} - (\tilde{t}^{\mu}_{,0} \varepsilon^b_{,0})_{,a} \right] + \frac{1}{2} \left[ (\tilde{t}^{\mu}_{,0} \varepsilon^b_{,0} - \tilde{t}^{\mu}_{,0} \varepsilon^b_{,0}) \right] \right\} \]
\[ \tilde{C} = v'_j M^{\mu}_j + \tilde{v}'_a M^{\mu}_j + \sum_{L} \tilde{v}'^L \tilde{t}^{\mu}_{,L} - \tilde{A}^b_j \partial_0 \sigma^{ab}_{j} - \frac{1}{2} \tilde{F}^{\mu b,ab}_{ab} \]
(9.6)
where in the last equation the above fixed values for 21 of 28 Lagrange multipliers have to be assumed. The 12 second class pairs are
\[ \left\{ \tilde{t}^{\mu}_L (x), T^a_k (y) \right\} = \delta^a_b \delta (x, y), \quad \left\{ T^a_j (x), \tilde{t}^{\mu}_k (y) \right\} = \delta^a_b \delta^b_j \varepsilon^0_0 (x) \delta (x, y) . \]
(9.7)

The unconstrained phase space thus has 2(12 + 16) = 56 degrees of freedom. The 14 first class constraints and 24 second class constraints remove 14 + 14 + 24 = 52 degrees of freedom leaving 4 physical degrees of freedom, i.e. 2 canonical pairs. We thus identify the canonical structure as follows:

1. First class set
\[ \{ F^a \} = \left\{ \tilde{P}^{\mu}_j, \quad M^{\mu}_j, \quad \tilde{G}_j, \quad \tilde{D}_a, \quad \tilde{C} \right\} \]
(9.8)
where the latter 7 constraints are solved for \( M^{\mu}_j, A^j_a \).

2. Corresponding gauge fixing conditions
\[ \{ G^a \} = \left\{ \varepsilon^{\mu 0}_0, \quad A^j_a, \quad A^j_a \right\} . \]
(9.9)
3. Second class set
\[ \{S_t\} = \left\{ \left(T, \tilde{t}^i, \tilde{e}^i, T^a\right) \right\} \]  
(9.10)

which are solved for \( \tilde{e}_t, \tilde{t}^i, \tilde{e}^i, \).

4. The split of canonical pairs is thus
\[ \{(\sigma^\alpha, \pi_\alpha)\} = \left\{ \left(\tilde{\sigma}^\mu_0, \tilde{P}^\mu_0\right), \left(A^\mu_i, M^\mu_i\right), \left(A^\mu_{\perp}, M^\mu_{\perp}\right) \right\} \]
\[ \{(x^i, y_j)\} = \left\{ \left(\tilde{e}^i_t, \tilde{\hat{t}}^i\right), \left(\tilde{e}^i, \tilde{\hat{e}}^i\right) \right\} \]
\[ \{Q^A, P_A\} = \left\{ \left(A^A_{\perp}, M^A_{\perp}\right) \right\} \].

(9.11)

The aim is now to rewrite (9.1) in a covariant form, i.e. we aim at keeping only the variables \( A^\mu_i, \tilde{e}^\mu_0, \tilde{e}^\mu_0 \) that appear in the classical action (2.1). Thus we want to get rid of \( M^\mu_i, P^\mu_i, P^\mu_\mu \).

First we note that
\[ \det[[S, S]] = \left[ (\tilde{e}^\mu_0)^9 \right], \det[[F, G]] = \det[[F_r, G_r]], \det[[F_p, G_p]] = 1 \]  
(9.12)

where \( F_r, G_r \) denote the list of secondary first class constraints and gauge fixing conditions only, i.e. \( \{F_r\} = \{\tilde{G}_r, \tilde{D}_r, \tilde{C}\} \) and \( \{G_r\} = \{A^\mu_{\perp}, M^\mu_{\perp}\} \) while \( F_p, G_p \) denote the list of primary first class constraints and gauge fixing conditions only, i.e. \( \{F_p\} = \{\tilde{P}^\mu_0, \tilde{M}^\mu_0\} \) and \( \{G_p\} = \{\tilde{e}^\mu_0, A^\mu_i\} \).

The Liouville measure \( dF_p \ dG_p \) is invariant under canonical transformations (at each time) and the symplectic potential \( F_p, dG_p \) changes by an exact differential, hence its time integral is invariant for canonical transformations that decay at early and late times. Therefore, by the Fadeev–Popov method we can drop \( \delta[G_p] \mid \det[[F_p, G_p]] \) from the numerator and denominator path integral and we can then carry out the integral \( \int dF_p \delta[F_p] = 1 \). Thus altogether we can drop \( dF_p \delta[G_p] \delta[F_p] \mid \det[[F_p, G_p]] \).

Next, \( Q_{\tilde{g}, T} \) is invariant under asymptotically trivial gauge transformations generated by \( F_r \) so we can identify \( Q \) with the \( F_r \) gauge invariant projection or relational observable \( Q := Q^G, F_r \) corresponding to the gauge fixing \( G_r \). Therefore
\[ B_T(Q) := \psi^* ((O_Q)_T), B_T^p(Q) := \Omega^0_T ((O_Q)_T) \Omega^0_T ((O_Q)_{-T}) \]
are gauge invariant. By the same Fadeev–Popov argument as above, we can thus drop \( \delta[G_r] \mid \det[[F_r, G_r]] \) from numerator and denominator path integral.

Furthermore, we can carry out the intervals over \( \tilde{P}^\mu_0, \tilde{P}^\mu_0, M^\mu_0 \) enforcing \( \tilde{\mu}_0 = 0, \ M^\mu_0 = 0, \ M^\mu_0 = \sigma^\mu_0 \). This simplifies (9.1) to
\[ \lim_{T \to \infty} \frac{\int_T^T dF_r^\mu d\sigma^\mu_0 d\tilde{\hat{e}}^i d\tilde{\hat{t}}^i \delta[F_r]^T_{T=0}}{B_T(Q) \to B_T^p(Q)} \]
(9.14)

We note that the secondary first class constraints at \( T^\mu = \tilde{P}^\mu_0 = M^\mu_0 = 0 \), simplify to
\[ \tilde{G}_r = \partial^\mu \sigma^\mu_0, \tilde{D}_r = F^\mu a \sigma^a_0 - A^\mu_0 \tilde{G}_r, \tilde{C} = -A^\mu_0 \tilde{G}_r - \frac{1}{2} F^\mu a \sigma^a \]
(9.15)

We bring the constraints \( F_r \) into the exponents using corresponding Lagrange multipliers \( \lambda^i, \lambda^a, \lambda \) which turns (9.1) into the form
\[ \lim_{T \to \infty} \frac{\int dF_r^\mu d\tilde{\hat{e}}^i d\tilde{\hat{t}}^i [\tilde{\psi}^i)]^T \delta[\tilde{\psi}^i]}{B_T(Q) \to B_T^p(Q)} \]
\[ \left( \int_T^T d\tau \left[ \frac{\sigma^\mu_0}{2} A^\mu_0 + [\lambda^i, \lambda^a] A^\mu_0 + [\lambda^a, \lambda^i] A^\mu_0 + \frac{1}{2} \sigma^a \right] \lambda^a \right) \]} B_T(Q) \]
(9.16)
We shift $A'_f + \lambda' = \lambda A'_f \rightarrow \tilde{A}'_f$ and drop the tilde again. Then all dependence of the integrand on $\lambda'$ has disappeared, yielding an infinite constant which drops out of the fraction leaving us with

$$\lim_{T \to \infty} \int \frac{dA'_f}{dA'_f} \frac{d\nu}{d\nu} d\lambda \mid \hat{e}'_j \rangle \mid \hat{e}'_j \rangle \exp \left( i \int_{-T}^{T} dt \int d^3x \mid \sigma''_{ij} \mid \hat{e}'_j \rangle \right) B_r (Q) \frac{B'_r (Q) \to B'_f (Q)}{B_r (Q) \to B'_f (Q)} (9.17)$$

where we integrated by pars in the exponent. Since the support of the integrand is at $\hat{e}'_j = 0$ this reduces $\sigma''_{ij}$ to

$$\sigma''_{ij} = \hat{e}'_j \hat{e}'_j, \quad \sigma''_{ab} = 2 \hat{e}'_a \hat{e}'_b + \epsilon_{\hat{e}'_a} \epsilon_{\hat{e}'_b} \hat{e}'_i. \quad (9.18)$$

We perform a field redefinition as follows

$$\lambda = \lambda', \quad \lambda^a = \lambda'^a$$

$$\hat{e}'_j = \hat{f}'_j, \quad \hat{e}'_j = \hat{f}'_j / \sqrt{\lambda}, \quad \hat{f}'_0 = \sqrt{\lambda} f'_0, \quad \hat{e}'_0 = \left[ \hat{f}'_0 - \sqrt{\lambda} \lambda'^a \sqrt{f'_0} \right] / \lambda'$$

(9.19)

whose pointwise Jacobian is easily computed to be $(\lambda')^{-1}$. Note also $| \hat{e}'_0 | / | \hat{e}'_0 | = | \hat{e}'_0 | / \lambda | \hat{e}'_0 |$. After this transformation, as one can check, the exponent no longer depends on $\lambda', \lambda'^a$, all dependence on those is a fixed function multiplying the integrand which can thus be integrated out and yield cancelling factors. Relabelling $\hat{f}$ by $\hat{e}$ again we find that (9.1) becomes

$$\lim_{T \to \infty} \int \frac{dA'_f}{dA'_f} \frac{d\nu}{d\nu} d\lambda \mid \hat{e}'_j \rangle \mid \hat{e}'_j \rangle \exp \left( i \int_{-T}^{T} dt \int d^3x F^{KL}_{\mu \nu} \hat{e}'_k \hat{e}'_L \right) B_r (Q) \frac{B'_r (Q) \to B'_f (Q)}{B_r (Q) \to B'_f (Q)} (9.20)$$

which is almost what one would expect (i.e. the integrand is the exponent of the classical action) except for the measure factor $| \hat{e}'_0 | / | \hat{e}'_0 |$ and the fact that the integral is supported on the time gauge $\hat{e}'_0 = 0$.

Expression (9.20) has an exponent linear in the connection and thus integrating over it yields an integral over $\hat{e}'_L$ supported at the solution of its classical field equation. However, to test techniques of spin foam models [43] in this Abelian U(1) gauge group theory one rather aims at a BF formulation of (9.20) rather than a tetrad formulation. This will bring out the significance of the time gauge: The imposition of the tetrad time gauge translates into a covariant simplicity constraint on the B field. To see this we note that

$$F^{KL}_{\mu \nu} \hat{e}'_k \hat{e}'_L = F'_{\mu \nu} \left[ 2 \hat{e}'_0 \hat{e}'_j + \epsilon_{\hat{e}'_a} \epsilon_{\hat{e}'_b} \hat{e}'_i \right] =: F'_{\mu \nu} \hat{e}'_j (9.21)$$

i.e. the B field of the BF formulation of the U(1) model is constrained to be of the form

$$B'_{ij} = \sigma'_{ij}. \quad (9.22)$$

To avoid confusion, note that this $B$ field has nothing to do with the magnetic field of $A$. The left hand side has 18 degrees of freedom, the right hand side only 13 with the time gauge imposed, thus there must be 5 simplicity constraints on $B$ that ensure that it has the form (9.22). To discover those, we try to solve (9.22) for $\hat{e}'_L$ while $\hat{e}'_0 = 0$. We have

$$B'_{ij} = B'_{ij} = \hat{e}'_0 \hat{e}'_j, \quad B'_{ij} = 2 \hat{e}'_0 \hat{e}'_j + \epsilon_{\hat{e}'_a} \epsilon_{\hat{e}'_b} \hat{e}'_i. \quad (9.23)$$

Eliminating $\hat{e}'_0^2$ from the second equation via the first, dualising $B'_{ij} := \epsilon_{abc} B'_c$, and using the formal inverse $B'_B B'_B = \delta'_B$, the second equation in (9.23) can be written as

$$\left( \hat{e}'_0 \right)^2 B'_a = \hat{e}'_0 \epsilon_{abc} \epsilon_{d'j} B'_d + \text{det} (B) B'_a \quad (9.24)$$
and contracting with $B^a_k$ finally yields

$$ (\tilde{e}'_0)^2 B^a_k B^a_k = \det(B) \left[ \delta_{jk} + \tilde{e}'_0 \epsilon_{jk} \tilde{e}'_0 B^a_k \right]. \quad (9.25) $$

These are 9 equations for 4 unknowns $\tilde{e}'_0$ and they can be solved if and only if the matrix $M_k := \tilde{B}_k^a B^a_k$ has no symmetric trace free piece (besides $B^a_k$ being non-degenerate, which is classically granted if $\tilde{e}'_0$ is a half densitised tetrad given the first relation in (9.23)). If that is the case we find

$$ \tilde{e}'_0 = \frac{3 \det(B)}{\delta_{jk} M_k}, \quad \tilde{e}'_0 = \frac{\tilde{e}'_0 \epsilon_{jk} M_k B^a_k}{\det(B)}. \quad (9.26) $$

Surprisingly the condition that the tracefree symmetric part of $M_k$ vanishes can be stated covariantly. To see this we compute the covariant density

$$ S_{jk}(B) := \epsilon_{\mu\nu\rho\sigma} B^j_\mu B^k_\nu B^a_{\rho\sigma} = 4 \epsilon_{abc} B^a_\nu B^b_\nu B^c_\nu = 8 \epsilon_{abc} M_{jk} \quad (9.27) $$

which directly yields the symmetric piece of $M$. Thus the five $U(1)^3$ simplicity constraints read

$$ S(B) - \frac{1}{3} \operatorname{Tr}(S(B)) = 0. \quad (9.28) $$

Conversely, inserting (9.22) with time gauge installed into $S(B)$ yields that

$$ S(B) = 6 \tilde{e}'_0 \det \left\{ \tilde{e}'_0 \right\} 13 \quad (9.29) $$

has only a trace part.

To rewrite the rigging map or path integral (9.20) from the tetrad into the BF formulation we thus proceed as follows: We define the map from the 18 dimensional space of tensors $B^{j\mu}_j$ to the 5 dimensional space of trace free symmetric matrices

$$ B \mapsto S_T(B) := S(B) - 13 \operatorname{Tr}(S(B)) / 3 \quad (9.30) $$

and we use the first equation in (9.23) and (9.26) to define a map from tensors $B^{j\mu}_j$ to a 13 dimensional space of tetrads in time gauge

$$ B \mapsto \tilde{E}(B); \quad E^\rho_k = 0. \quad (9.31) $$

Note that (9.31) just uses the trace and antisymmetric part of $B^a_k \tilde{B}_{ak}$ while $S_T(B)$ is the trace free symmetric part.

Therefore (9.30) and (9.31) defines an invertible map $B \mapsto (S_T(B), \tilde{E}(B))$ whose inverse we write as $(M, \tilde{e}) \mapsto b(M, \tilde{e})$. Then we determine a measure factor $\rho(B)$ such that

$$ \int d^{18}B \rho(B) \delta^5(S_T(B)) f(B) $$

$$ = \int d^{13} \tilde{e} \tilde{d}^3M \left| \det(\partial(b(M, \tilde{e}) / \partial(M, \tilde{e})) \right| \delta^5(M) \rho(b(M, \tilde{e})) f(b(M, \tilde{e})) $$

$$ = \int d^{13} \tilde{e} \left| \det(\partial(b(M, \tilde{e}) / \partial(M, \tilde{e})) |_{M=0} \rho(b(0, \tilde{e})) f(b(0, \tilde{e})) $$

$$ = \int d^{13} \tilde{e} |\tilde{e}'_0|^{9/2} f(b(0, \tilde{e})) \quad (9.32) $$

where the identity $S_T(b(M, \tilde{e}) = M$ was used, thus

$$ \rho(b(0, \tilde{e})) = \frac{|\tilde{e}'_0|^{9/2}}{\left| \det(\partial(b(M, \tilde{e}) / \partial(M, \tilde{e})) |_{M=0} \right|} = J(\tilde{e}). \quad (9.33) $$
Thus for instance
\[ \rho(B) := J(\hat{E}(B)) \] (9.34)

where the identity \( \hat{E}(b(M, \hat{e})) = \hat{e} \) was used. The details are reserved for future investigations.

The investigations of section 8 suggest that the naive Lebesgue measures \( dA, d\hat{e}, dB \) should rather be replaced by the Bohr \( d\mu_B(A) \) and discrete \( d\mu_D(\hat{e}) d\mu_D(B) \) measures respectively.

Thus we have brought the \( U(1)^3 \) model into the starting point for the usual spin foam treatment using the simplicity constraint (9.28). It is possible that the simpler Abelian context and the fact that the canonical treatment could be carried out in all details can help to deepen the relation between canonical and covariant LQG.

10. Summary and outlook

As we have seen, \( U(1)^3 \) quantum gravity is a quantum integrable model for vacuum quantum gravity in either signature with a possible cosmological constant. That quantum \( U(1)^3 \) theory can be developed to such an extent as has been layed out in this paper may seem astonishing at first. In retrospect, however, there is a single and simple mathematical reason for this: The constraints or physical Hamiltonian of the theory are at most linear in one of the variables of a canonical pair. This makes the system behave almost classically in the polarisation (in the sense of geometric quantisation [36]) adapted to this partly linear structure.

More realistic theories such as GR in either signature are typically at least quadratic in both variables of a canonical pair and therefore this remarkable simplicity of the quantum solution of \( U(1)^3 \) theory cannot be expected for such theories. Yet, the following lessons most likely can be transferred to those more realistic theories:

1. **Exponentiated vs. infinitesimal action**
   It is the action of the exponentiated Hamiltonian constraint, and not its generator, that can be implemented without anomalies on the kinematical Hilbert space. In LQG, that was already known for the much simpler spatial diffeomorphism constraint. For the Hamiltonian constraint the situation was less clear because while for the spatial diffeomorphism constraint there is one ‘missing smearing dimension’, for the Hamiltonian constraint the number of smearing dimensions is precisely correct in its density unity version. The more fundamental reason for failure of existence of the generator of the anomaly free action of the Hamiltonian constraint is weak discontinuity.

2. **‘Off-shell’ kinematical vs. dual action**
   The action of the Hamiltonian constraint can be defined directly on the kinematical Hilbert space, it does not require any dual states (distributions) or non standard operator topology relying on spatial diffeomorphism invariance. The algebroid closes off-shell in that sense.

3. **Quantum non-degeneracy**
   The natural and, for non-integer density weight and/or density weight less than two necessary, common, dense and invariant domain on which the quantum HDA acts consists of states which are quantum non-degenerate, i.e. the volume operator of any open region has non-zero volume expectation (even eigen-)value. This is not only semiclassically expected as the very definition of the HDA and its derivation (the quite non-trivial Poisson bracket calculation makes crucial use of classical non-degeneracy at every single step of the computation) critically rely on npn-degeneracy, in retrospect, this also explains why in LQG
an anomaly free quantum constraint algebroid is so difficult to obtain: The spin network states (SNWSs) which one uses as common, dense and invariant domain fail to be quantum non-degenerate. Therefore the quantum analogs of substantial parts of the classical Poisson bracket calculation in terms of commutators which depend on integrals (understood as infinite Riemann sums) and integrations by parts are impossible to reproduce on SNWS which can at most depict finitely many terms of the actually necessary infinite number of terms in the Riemann sums.

4. Density weight
In the $\text{U}(1)^3$ model one can work with any density weight on the non-degenerate domain, thereby showing that the natural density weight unity is not ruled out by the quantum theory. In more realistic theories with at least quadratic dependence in all canonical variables, density weight unity is in fact the only choice [29].

5. Detailed action of the Hamiltonian constraint
The Hilbert space representation of LQG and the one used here for the $\text{U}(1)^3$ model are in some sense very similar as they are both of the discontinuous Narnhofer-Thirring type. In both representations one can solve the $\text{SU}(2)$ or $\text{U}(1)^3$ Gauss constraint by considering appropriate subspaces defined by a restriction on the smearing functions of the connection (‘form factors’).

However, in LQG the connection is smeared along 1d paths while here we use a 3d smearing. This is possible because the constraints are linear in the connection, the quadratic dependence on the connection in the non-Abelian case enforces to smear in 1d only [29]. Therefore in LQG the Hamiltonian constraint acts by attaching loops to a graph in the vicinity of a vertex while here the Hamiltonian acts in a sense everywhere as the state is non-degenerate. Their action is therefore difficult to compare. To examine the closer relation between the two, we consider the $\text{U}(1)^3$ model but exactly quantised as in LQG [44], i.e. in terms of charge networks along 1d graphs rather than 3d smearings. Such a charge network smears the connection with a form factor

$$\left(F\text{_{CNW}}\right)^a_{\alpha}(x) = \sum_e n^e_\alpha \int_e dy^a \, \delta(x,y)$$

(10.1)

excited on a graph $\gamma$ with edges $e$, charges $n^e_\alpha \in \mathbb{Z}$ which solve the Gauss constraint $\sum_{b(e)=v} \bar{n}^e_\alpha = \sum_{f(e)=v} \bar{n}^e_\alpha$ where $b(e), f(e)$ denote the beginning/final point of an oriented edge $e$ and $v$ is a vertex of $\gamma$.

To obtain a closely related form factor in the present treatment of the theory we simply mollify the $\delta$ distribution (see also [45] for a similar procedure in linearized gravity)

$$\left(F_\epsilon\right)^a_{\alpha}(x) = \sum_e n^e_\alpha \int_e dy^a \, \delta_\epsilon(x,y)$$

(10.2)

where $\epsilon \to \delta_\epsilon$ is a sequence of positive, smooth functions of rapid decrease that converge to the $\delta$ distribution in the topology of the tempered distributions on $\mathbb{R}^3$, e.g. a Gaussian. Then it is easy to check that

$$\partial_\alpha \left(F_\epsilon\right)^a_{\alpha} = - \sum_v \delta_\epsilon(v) \left[ \sum_{f(e)=v} n^e_\alpha - \sum_{b(e)=v} n^e_\alpha \right] = 0$$

(10.3)

is indeed divergence free and
The whole action of the constraint consists in a change of the form factor, that is, it does not map a state \(w[F]\Omega\) into a linear combination of several such states but simply changes the graph everywhere along the graph, not only in the vicinity of a vertex, although its action in the vicinity of the vertex is strongest. As more and more actions are included, the derivatives will also eventually move the whole graph.

The lapse function has become part of the new form factor. This is similar to the finite action of the spatial diffeomorphisms which map graphs to diffeomorphic images and in this sense they respectively their generating vector fields also become part of the form factor.

The whole action of the constraint consists in a change of the form factor, that is, it does not map a state \(w[F]\Omega\) into a linear combination of several such states but simply a single state \(w[F]\Omega\).

By contrast, in the current implementation of the Hamiltonian constraint in LQG or \(U(1)^3\) theory exactly treated as in LQG we have:

i. The constraint acts only in the vicinity of vertices and does change the graph there.

ii. The charges are only changed in the vicinity of a vertex and more and more actions only change the graph ever more closely to a vertex.

iii. Rather the charges have changed in a very non-linear and even non-algebraic way as there are derivatives involved and they change everywhere along the graph, not only in the vicinity of a vertex, although its action in the vicinity of the vertex is strongest. As more and more actions are included, the derivatives will also eventually move the whole graph.

iv. The lapse function is not mapped a state \(w[F]\Omega\) into a linear combination of several such states but simply a single state \(w[F]\Omega\).

The determinant factor is determined by (10.4). Formula (10.7) has the following features:

i. For \(\delta > 0\) (10.7) is still concentrated on the support of \(F\), i.e. ‘the graph is not changed, no loop is created’.

ii. Rather the charges have changed in a very non-linear and even non-algebraic way as there are derivatives involved and they change everywhere along the graph, not only in the vicinity of a vertex, although its action in the vicinity of the vertex is strongest. As more and more actions are included, the derivatives will also eventually move the whole graph.

iii. The lapse function has become part of the new form factor. This is similar to the finite action of the spatial diffeomorphisms which map graphs to diffeomorphic images and in this sense they respectively their generating vector fields also become part of the form factor.

iv. The whole action of the constraint consists in a change of the form factor, that is, it does not map a state \(w[F]\Omega\) into a linear combination of several such states but simply a single state \(w[F]\Omega\).
iii. The lapse function evaluated at vertices is a coefficient in an expansion of CNW functions and does not get part of the form factor.

iv. The image of the constraint, even at a single vertex, is a non-trivial expansion of CNW functions with coefficients that depend on inverse volume operator factors.

Accordingly, the current paper suggests that the quantisation of the Hamiltonian constraint in LQG be changed following the above rules. Basically, one should try to exponentiate the Hamiltonian constraint. This is a non-trivial technical challenge because of the non-Abelian gauge group and because the limit $\epsilon \to 0$ of (10.7) (returning to the sharp rather than mollified graph) is singular. However, for theories with quadratic dependence on connections, smearing just in 1d cannot be avoided. This points again to the emphasis on the non-degenerate sector of the theory [29].

6. Interacting QFT

As a by-product of the present work we have shown:

For any spacetime dimension $1+D$ a field theory on $\mathbb{R} \times \sigma$ ($\sigma$ any D-manifold) whose classical Hamiltonian $H$ is at most linear in the momenta $\pi_I$ of the fields $\phi^I$ such that both $H, V$ vanish at $\phi^I = 0$ where $V$ is the potential of $H$, admits a quantisation in terms of its 1-parameter unitary group $t \mapsto U(t)$ in a representation of the Weyl algebra generated by $\phi^I, \pi_I$ of Narnhhofer Thirring type with vacuum $\phi^I \Omega = 0$ cyclic for the Weyl operators $w[F] = \exp(-i \pi [F]), \pi [F] := \int \partial^D x F^I \pi_I$ with suitable test functions $F^I$. It is given by

$\begin{equation}
U(t) w[F] \Omega = e^{-i \alpha t [F]} w[F] \Omega, \alpha_t [F] = \int_0^t ds V(F_s), F_s = (e^{X_0} K)(F)
\end{equation}$

where $X_0$ is the Hamiltonian vector field of $H - V$ and where $K'(\phi, \pi) = \phi'$ denotes the $I$th configuration coordinate function. The scalar product is specified by $\langle \Omega, w[F] \Omega \rangle \equiv \delta_{F,0}$. The dependence of $H, V$ on $\phi$ can be arbitrarily non-linear, even non-polynomial and can be background metric independent. The classical Legendre transform of $H$ with respect to $\pi$ is singular (just yielding the classical equation of motion for $\phi$), but with respect to $\phi$ yields an arbitrarily non-linear Lagrangian in terms of $\pi$ and its first time derivative $\dot{\pi}$. The resulting N-point functions of this QFT (i.e. vacuum expectation values of monomials) of the time translates $U(t) w[F] U(-t)$ are not Gaussians in the test functions and thus define an interacting QFT in that sense. Among the drawbacks of these QFT’s are: The Hilbert space underlying this representation of the Weyl algebra is non-separable, the unitary group is weakly discontinuous, the Hamiltonian is not bounded from below, the Lagrangean generically is not Poincaré invariant.

The results of the present paper suggest the following routes for the quantisation of the actual, untruncated theory:

A. Perturbation theory of quantum integrable model

Just as in usual QFT on the Minkowski space background one splits the Hamiltonian into a free (and thus integrable) part and an interaction term and performs perturbation theory, here we may use background independent perturbation theory by considering the terms of full GR containing the terms quadratic in the connection $\Lambda$ as perturbation and the $U(1)^3$ part as ‘free’ (rather: integrable) part. See [46] for more details where such a strategy is suggested for general theories which allow for ‘consistent deformations’, that is, one can continuously change the coupling constants of the theory without changing the number of physical (i.e. propagating) degrees of freedom.
B. Connection to twistor string theory
It has been pointed out in [47] that the $U(1)^3$ model has a possible connection to twistor string theory (TST). Basically $U(1)^3$ theory can be considered as weak coupling limit of self-dual gravity which has a twistor action formulation. The corresponding twistor solutions serve as the target space of TST. As TST is connected to $N = 4$ super Yang Mills theory (SYM) whose quantum theory is under good control, one may benefit from this correspondence and learn something about quantum gravity, first about the weak coupling limit, then, via A. about the strong coupling limit.

C. Renormalisation
Given the importance of quantum non-degeneracy for the $U(1)^3$ model HDA, one may use it as a guiding principle for LQG as well using renormalisation:
If one truncates LQG to a given finite graph, then the metric is everywhere non-degenerate when one probes the volume only in neighbourhoods of vertices and such neighbourhoods can be chosen to partition the spatial manifold $\sigma$ if it is compact (IR regularisation).
Considering the graph as a label for short distance resolution (UV regulator) one obtains an infinite family of theories labelled by all possible finite resolutions and each finite resolution theory is non-degenerate. Using renormalisation methods one can flow in a fixed point meaning that all finite resolution theories descend from a common continuum theory using coarse graining. That continuum theory is expected to be quantum non-degenerate and to allow for a proper representation of the HDA.

Data availability statement
No new data were created or analysed in this study.

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Appendix. Inclusion of cosmological constant and Lorentzian term
We include both terms within the Hamiltonian framework by truncating the Hamiltonian constraint of full Lorentzian GR with cosmological constant to the ‘Abelian’ terms. We expect that one can do this also within the Lagrangian framework by performing appropriate truncations of linear combinations of the Holst action [48] and a second order derivative (with respect to the tetrad, i.e. using its spin connection) action in the sense of [27]. Namely by 1. constraining both Lorentz connections to be SU(2) connections, e.g. $A^a_{\mu b} \propto \epsilon_{jkl} A_{\mu jl}$ and 2. dropping terms quadratic in both connections.

The Hamiltonian constraint of Lorentzian GR with cosmological constant in its density two form is given by
\[ C = \left[ F^L_{ab} + \beta R^L_{ab} \right] \epsilon_{jkl} E^a_j E^b_l + 3! \Lambda \det(E) \]  
(A.1)
where $F = 2dA + A \wedge A$ and $R = 2d\Gamma + \Gamma \wedge \Gamma$ and where $\Gamma$ is the spin connection of $E$. Here $\beta$ is related to the Immirzi parameter.
The terms $A \wedge A$ and $\Gamma \wedge \Gamma$ ensure that (A.1) is invariant under local $SU(2)$ transformations. We now simply drop both terms and keep the same label for the truncated constraint. Then (A.1) is still a density of weight two and it is invariant under $U(1)^3$ transformations $A \mapsto A - df$, $E \mapsto E$ because $F = 2dA$ is exact and $E$ is itself a Gauss invariant. Thus the truncated $C$ still closes with the full spatial diffeomorphism constraint and the truncated Gauss constraint $G_j = \partial_x E^i_j$. To see that it still obeys the HDA relations we smear it against test functions $M$ and set $C(M) = T(M) + V(M) + W(M)$ where

$$T(M) = \int d^3x A^i_j \partial_\mu G^{ab}_{Mj}, \quad V(M) = \int d^3x \Gamma^i_j \partial_\mu G^{ab}_{Mj},$$

$$W(M) = \int d^3x \Lambda E^i_j \epsilon_{abc} G^{ab}_{Mj},$$

(A.2)

with

$$G^{ab}_{Mj} := M^{ijkl} E^k_a E^l_b.$$ \hspace{1cm} (A.3)

We clearly have

$$\{C(M), C(N)\} = \{T(M), T(N)\} + \{T(M), V(N) + W(N)\} - \{T(N), V(M) + W(M)\}. \quad \text{(A.4)}$$

A straightforward calculation yields the result quoted in the main text

$$\{T(M), T(N)\} = - \int d^3x \omega^a Q^{ab} (F^b_\mu E^i_j) \quad \text{(A.5)}$$

where $\omega^a = MN^a - M^a N$, $Q^{ab} = E^j_\mu E^i_j$. Thus we already obtain the expected linear combination of Gauss and spatial diffeomorphism constraints. The calculation benefits from the observation that the Poisson bracket is proportional to

$$\int d^3x \int d^3y (M(x) N(y) - M(y) N(x))$$

(A.6)

before evaluating (derivatives of) the delta distribution $\delta(x, y)$. Therefore only terms that lead to derivatives of the delta distribution contribute, ultra-local terms vanish and in the corresponding integration by parts we only need to keep the terms that involve derivatives of (A.6) and lead to $\omega^a$.

To see that the remaining terms in (A.4) vanish we use the important observation [9] that the spin connection has a potential $\Gamma^i_j = \frac{\delta F^i_j}{\delta x^j(x)}$, $U = \int d^3x E^i_j \Gamma^i_j$ (plus potential boundary terms). It follows

$$\{T(M), V(N)\} - \{T(N), V(M)\} = \beta \int d^3x \int d^3y \left[ G^{ab}_{Mj,k}(x) G^{cd}_{Nk,l}(y) - M \leftrightarrow N \right]$$

$$\times \frac{\delta^2 U}{\delta E^i_j(x) \delta E^i_j(y)} = 0$$

(A.7)

since the second functional derivatives commute. Finally
\[
\{T(M), W(N)\} - \{T(N), W(M)\} = 3 \int d^3x \int d^3y \left[ G_{ab}^{ij} (x) N(y) - M \leftrightarrow N \right] \epsilon_{acd} \epsilon^{ijkl} \\
	imes (E^a_i E^b_j') (y) \delta (x, y)
\]
\[
= -3 \int d^3x \int d^3y \omega_b \epsilon_{jmn} E^b_m E^a_n \epsilon_{acd} \epsilon^{ijkl} E^i_k E^j_l
\]
\[
= 0 \quad \text{(A.8)}
\]

as claimed. It follows that the potential term \( V + W \) corresponding to Lorentzian signature and cosmological constant can be treated by the methods of section 3 as well.

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