ON SPIKE SOLUTIONS FOR A SINGULARLY PERTURBED PROBLEM IN A COMPACT RIEMANNIAN MANIFOLD

SHENGBING DENG
School of Mathematics and Statistics
Southwest University, Chongqing 400715, China

ZIED KHEMIRI
University of Tunis El Manar Département de Mathématiques
Faculté des Sciences de Tunis, Campus Universitaire 2092 Tunis El Manar, Tunisia

FETHI MAHMoudi∗
Centro de Modelamiento Matemático, Universidad de Chile
Beauchef 851, Edificio Norte–Piso 7, Santiago de Chile

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Abstract. Let \((M, g)\) be a smooth compact riemannian manifold of dimension \(N \geq 2\) with constant scalar curvature. We are concerned with the following elliptic problem
\[-\varepsilon^2 \Delta_g u + u - u^{p-1} = 0, \quad u > 0, \quad \text{in } M.\]
where \(\Delta_g\) is the Laplace-Beltrami operator on \(M\), \(p > 2\) if \(N = 2\) and \(2 < p < \frac{2N}{N-2}\) if \(N \geq 3\), \(\varepsilon\) is a small real parameter. We prove that there exist a function \(\Xi\) such that if \(\xi_0\) is a stable critical point of \(\Xi(\xi)\) there exists \(\varepsilon_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0)\), problem (1) has a solution \(u_\varepsilon\) which concentrates near \(\xi_0\) as \(\varepsilon\) tends to zero. This result generalizes previous works which handle the case where the scalar curvature function of \((M, g)\) has non-degenerate critical points.

1. Introduction. We consider the following problem
\[-\varepsilon^2 \Delta_g u + u - u^{p-1} = 0 \quad \text{in } M\] (1)
where \((M, g)\) is a smooth compact Riemannian manifold without boundary of dimension \(N \geq 2\), \(\varepsilon > 0\) is a small parameter and \(p > 2\) if \(N = 2\), \(2 < p < 2^* = \frac{2N}{N-2}\) if \(N \geq 3\).

The energy functional \(J_\varepsilon\) associated to (1) is defined by
\[J_\varepsilon[u] = \int_M \left( \frac{\varepsilon^2}{2} |\nabla_g u|^2 + \frac{1}{2} u^2 - \frac{1}{p} u^p \right) d\mu_g \quad \text{for } u \in H^1(M).\]

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∗ Corresponding author.
There is an extensive literature regarding this problem in bounded domains of \( \mathbb{R}^n \) (with Neumann boundary condition). Solutions concentrating at points or in general positive dimensional manifolds has been found for \( 2 < p \leq \frac{2(n-k)}{n-k-2} \) for \( n-k \geq 3 \) and \( p > 2 \) for \( n-k = 2 \) (where \( k \) is dimension of the concentration manifold). In fact, replacing problem (1) with the following problem

\[
\begin{align*}
-\varepsilon^2 \Delta u + u - u^{p-1} &= 0, & \text{in } \Omega; \\
\frac{\partial u}{\partial \nu} &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

which arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation or of parabolic equations in chemotaxis, population dynamics and phase transitions. Many works has been devoted to study spike and bubble solutions. We refer the reader to the pioneering paper by Byeon and Park [2], see also Lin and al. [14, 19, 20], who established the existence of least-energy solutions to (2) and show that for \( \varepsilon \) small enough the least energy solution has a boundary spike, which approaches the maximum point of the mean curvature \( H \) of \( \partial \Omega \), as \( \varepsilon \) goes to zero. Later, in [4, 23] it has been proven that for any stable critical point of \( H \) one can construct single boundary spike layer, while in [9, 13, 25] the authors construct multiple boundary spike layer solutions at multiple stable critical points of \( H \). Multiple peak solutions has been also constructed, see [10] who proved that for any integer \( k \) there exist a boundary \( k \)-peak solutions, whose peaks collapse to a local minimum point of \( H \). We also mention the papers [2, 3, 7, 8, 11, 24] where interior spike layer solutions are constructed.

As a summary, we observe that since the equation is autonomous is the geometry of the domain who plays a crucial role in the location of point (or submanifold) concentration. This can be seen also once expanding the energy of a single boundary spike solution \( u_\varepsilon \), see Ni and Takagi [2, 14, 19]. They proved that the following asymptotic expansion holds

\[
J_\varepsilon[u_\varepsilon] = \varepsilon^N \left[ \frac{1}{2} I[w] - c_1 \varepsilon H(P_\varepsilon) + o(\varepsilon) \right],
\]

where \( c_1 > 0 \) is a generic constant, \( P_\varepsilon \) is the unique local maximum point of \( u_\varepsilon \) and \( H(P_\varepsilon) \) is the boundary mean curvature function at \( P_\varepsilon \in \partial \Omega \) and \( w \) is the unique ground state solution

\[
\begin{align*}
\Delta w - w + w^{p-1} &= 0, & w > 0, & \text{in } \mathbb{R}^N; \\
w(0) &= \max_{y \in \mathbb{R}^N} w(y), & \lim_{|y| \to +\infty} w(y) &= 0,
\end{align*}
\]

and \( I[w] \) is the ground-state energy

\[
I[w] = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 dy + \frac{1}{2} \int_{\mathbb{R}^N} w^2 dy - \frac{1}{p} \int_{\mathbb{R}^N} w^p dy.
\]

In the Riemannian setting, it turns out that is the scalar curvature function which is relevant for point concentration in \( M \) for problem (1) or some of its variants. For example, if one consider the following asymptotically critical elliptic problem

\[-\Delta_g u + h(x)u = u^{2^* - 1 - \varepsilon}, \quad u > 0 \quad \text{in } M,\]

where \( \Delta_g \) stands for the Laplace-Beltrami operator on \( M \), \( h(x) \) is a \( C^1 \) function on \( M \), \( 2^* = \frac{2N}{N-2} \) denotes the Sobolev critical exponent, \( \varepsilon \) is a small real parameter. Micheletti, Pistoia and Vétois [17] proved the existence of blowing-up solutions for this equation in any compact manifold with dimension \( N \geq 6 \). In particular they
prove that in the slightly subcritical case \( p_\varepsilon = \frac{N+2}{N-2} - \varepsilon \) blowing-up solutions at some point \( \xi_0 \) exist only for large potentials with respect to the potential of the Yamabe equation: \( h(\xi_0) > \frac{N-2}{4(N-1)} \text{Scal}_g(\xi_0) \), while in the slightly super-critical case \( p_\varepsilon = \frac{N+2}{N-2} + \varepsilon \) blowing-up solutions exist only for small potentials with respect to the potential of the Yamabe equation: \( h(\xi_0) < \frac{N-2}{4(N-1)} \text{Scal}_g(\xi_0) \). Using a Lyapunov-Schmidt reduction procedure S. Deng [5] proved that this problem admits a \( m \)-peaks solution for any positive integer \( m \geq 2 \), which blow up and concentrate at some points in \( M \). Esposito and Pistoia [6] obtained the existence of bubbling solutions for the problem when \( h = \frac{N-2}{4(N-1)} \text{Scal}_g \), the solutions blow-up at a maximum point of the Weyl curvature tensor of \( g \).

We refer also to [2, 15, 16] where it has been proven that given any non degenerate critical point \( p_0 \) of scalar curvature function \( s \) then problem (1) possesses solutions which concentrate near \( p_0 \). In this paper, we are interested in the cases that are not covered by these results, namely the cases where the scalar curvature has degenerate critical points. This is for example the case when \((M; g)\) is an Einstein manifold or more generally when the scalar curvature of \( g \) is constant. Similar result has been proven in a different setting: families of constant mean curvature hypersurfaces foliating geodesic spheres has been obtained by Pacard and Xu [21] extending previous works of R. Ye [27, 28]. In fact, R. Ye proved the existence of branches of constant mean curvature hypersurfaces each of which is associated to non-degenerate critical points of the scalar curvature. Moreover, he proved that the elements of these branches form a local foliation of a neighborhood of \( p \) by constant mean curvature hypersurfaces. Pacard and Xu [21] relaxed the condition on the point concentration, showing that the same construction hold provided \( p \) is a critical point of some function \( \phi : M \times (0, \rho_0) \to \mathbb{R} \) close to the scalar curvature function in some Holder norms.

Our aim is then, to prove similar result for the problem (1) relaxing the results in [2, 15, 16, 17]. To this purpose, assuming that the manifold \((M, g)\) has constant scalar curvature, we define

\[
\Xi(\xi) = \frac{1}{\text{120}(N+2)} \left( 8\|\text{Ric}_\xi\|^2 + 5\text{Scal}_g^2 - 3\|R_\xi\|^2 \right) \\
- \frac{1}{9} \left( \frac{1}{N+2} c_3 \|\text{Ric}_\xi\|^2 + \frac{1}{2N} c_4 \text{Scal}_g^2 \right)
\]  

where \( \text{Scal}_g \) denotes the scalar curvature of \( g \) (which is assumed to be constant), \( \text{Ric}_\xi \) denotes the Ricci tensor and \( R_\xi \) is the Riemannian tensor at the point \( \xi \) and where \( c_3 \) and \( c_4 \) are two constants given in (71) which can be computed explicitly. Our main result is the following.

**Theorem 1.1.** Assume that the manifold \( M \) has constant scalar curvature and let \( \xi_0 \) be a \( C^1 \)-stable critical point of \( \Xi \). Then there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), problem (1) has a solution \( u_\varepsilon \) which concentrates near \( \xi_0 \) as \( \varepsilon \) tends to zero.

**Remark 1.** It is worthmentionning that the class of compact Riemannian manifolds with constant scalar curvature is an important class of Riemannian manifolds which contains Einstein manifolds, Einstein-like manifolds, as well as the fact that the metric on a compact Riemannian manifold (with dimension \( \geq 3 \)) can be conformally deformed to a metric of constant scalar curvature, see for instance [22, 26].
In manifolds with constant scalar curvature, our result provides existence of solutions concentrating near any $C^1$ stable critical point $\xi_0$ of $\Xi$. In the particular case where the metric $g$ is Einstein, one gets existence of solutions concentrating near any $C^1$ stable critical point of the function $\xi \mapsto \|R\xi\|^2$.

We point out that without assuming $\text{Scal}_g$ to be constant we can define
\[
\Phi(\xi) = \text{Scal}_g(\xi) - \varepsilon^2 \tilde{\Xi}(\xi) + o(\varepsilon^2),
\]
where $\tilde{\Xi}$ is given by
\[
\tilde{\Xi}(\xi) = \frac{1}{120(N + 2)} \left( -18\Delta_g \text{Scal}_g(\xi) + 8\|\text{Ric}_g\|^2 + 5\text{Scal}_g(\xi)^2 - 3\|R\xi\|^2 \right) - \frac{1}{9} \left( \frac{1}{N + 2} c_3 \|\text{Ric}_\xi\|^2 + \frac{1}{2N} c_4 \text{Scal}_g(\xi)^2 \right).
\]

Any critical point of $\Phi$ gives rise to solution to our problem. In particular for $\varepsilon$ small enough, one can find critical points near any local strict maximum (or minimum) of $\text{Scal}_g$, see [2, 15, 16, 17], or any critical point of $\text{Scal}_g$ for which the Browder degree of $\nabla \text{Scal}_g$ at $\xi$ is not zero. The latter case provides $\Lambda_M + 1$ solutions, where $\Lambda_M$ is the Ljusternik-Shnirelman category of $M$. We refer the reader to the paper [1] for more details about this fact. When the scalar curvature is constant (and in this case $\tilde{\Xi} \equiv \Xi$ ) we have existence of solutions concentrating near any $C^1$ stable critical point of $\Xi$ by the above Theorem 1.1.

The first step in proving our main result is a high order expansions of the metric near geodesic normal coordinates. This is done in Section 2 together with the expansion of the volume element and some preliminary results. In Section 3, we consider the approximate solution and we give the existence result. Section 4 will be devoted to the finite dimensional reduction procedure while in Section 5 we prove Proposition 1 as well as some preliminary lemmas. In Section 6 (Appendix), we give the expansion of the energy functional.

2. Some preliminary results. In this subsection we first introduce Fermi coordinates (geodesic normal coordinates) in a neighborhood of a point $\xi \in M$. Let $E_i$, $i = 1, \ldots, N$, be an orthonormal basis of $T_\xi M$. We next, choose normal geodesic coordinates in a neighborhood of $\xi$ in $M$ through the map
\[
F(z) := \exp_\xi(z_i E_i), \quad z := (z_1, \ldots, z_N),
\]
where $\exp_\xi$ is the exponential map on $M$ at $\xi$ and where we have used Einstein’s convention of summation over repeated indices. This yields the coordinate vector fields $X_i := F_*(\partial_{z_i})$. Recall that the Fermi coordinates above are defined so that the metric coefficients $g_{ij} = g(X_i, X_j) = \delta_{ij}$ at $\xi$ and $X_k g_{ij} = X_k g(X_i, X_j) = 0$ at $\xi$. We now compute higher terms in the Taylor expansions of the functions $g_{ij}$. The metric coefficients at $q := F(z)$ are given in terms of geometric data at $\xi := F(0)$ and $|z| := \left( z_1^2 + \ldots + z_N^2 \right)^{1/2}$.

We now give the well known expansion for the metric in normal coordinates, we refer the reader to [12, 21] and some references therein for the expansion of the metric coefficients. The expansions of the inverse of the metric and the volume element follows then from classical taylor expansions.
Lemma 2.1. In a normal coordinates neighborhood of $\xi \in M$, the Taylor series of $g$ around $\xi$ is given by

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{klji} z_k z_l - \frac{1}{6} \nabla_m R_{klji} z_k z_l z_m + \left( \frac{1}{20} \nabla_{pq} R_{klji} + \frac{2}{45} R_{klir} R_{pjqr} \right) z_k z_l z_p z_q + O(|z|^5),$$

as $|z| \to 0$. Moreover,

$$g^{ij} = \delta_{ij} - \frac{1}{3} R_{ijkl} z_k z_l - \frac{1}{6} \nabla_m R_{ijkl} z_k z_l z_m - \left( \frac{1}{20} \nabla_{pq} R_{ijkl} + \frac{2}{45} R_{ijir} R_{pjqr} \right) z_k z_l z_p z_q + O(|z|^5).$$

Furthermore, the volume element for the normal coordinate system has the following expansion

$$\sqrt{\det(g)} = 1 - \frac{1}{6} R_{klz_k z_l} - \frac{1}{12} \nabla_m R_{klz_k z_l z_m}$$

$$- \left( \frac{1}{40} \nabla_{pq} R_{kl} + \frac{1}{180} R_{klir} R_{pqir} - \frac{1}{72} R_{kl} R_{pq} \right) z_k z_l z_p z_q + O(|z|^5).$$

Here all curvature terms are evaluated at $\xi$. Convention over repeated indices is understood and where the symbol $O(|z|^r)$ indicates an analytic function such that it and its partial derivatives of any order, with respect to the vector fields $z_j X_i$, are bounded by a constant times $|z|^r$ in some fixed neighborhood of 0.

Let $H_\varepsilon$ be the Hilbert space $H^1_\varepsilon(M)$ equipped with the inner product

$$\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^N} \left( \varepsilon^2 \int_M \nabla g \nabla u \nabla v d\mu_g + \int_M u v d\mu_g \right),$$

which induces the norm

$$\| u \|_2^2 := \frac{1}{\varepsilon^N} \left( \varepsilon^2 \int_M |\nabla g u|^2 d\mu_g + \int_M u^2 d\mu_g \right).$$

Let $L^p_\varepsilon$ be the Banach space $L^p_\varepsilon(M)$ with the norm

$$|u|_{p, \varepsilon} = \left( \frac{1}{\varepsilon^N} \int_M |u|^p d\mu_g \right)^{1/p}.$$

It is clear that the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L^p_\varepsilon$ is a compact continuous map. We let

$$i^*_\varepsilon : L^{p'}_\varepsilon \hookrightarrow H_\varepsilon$$

be the adjoint operator of the embedding $i_\varepsilon$, where $p' = \frac{p}{p-1}$. The embedding $i^*_\varepsilon$ is a continuous map such that for any $w$ in $L^{p'}_\varepsilon$, the function $u = i^*_\varepsilon(w)$ in $H_\varepsilon$ is the unique solution of the equation

$$-\varepsilon^2 \Delta_g u + u = w \quad \text{in } M,$$

that is

$$\langle i^*_\varepsilon(w), \phi \rangle_\varepsilon = \frac{1}{\varepsilon^N} \int_M w \phi d\mu_g, \quad \phi \in H_\varepsilon.$$

By the continuity of the embedding $H_\varepsilon$ into $L^p_\varepsilon$, for any $w \in L^{p'}_\varepsilon$, one has

$$\| i^*_\varepsilon(w) \|_\varepsilon \leq C |w|_{p', \varepsilon}$$

for some positive constant $C$ independent of $w$. 

(7)
We can rewrite problem (1) in the equivalent way
\[ u^* = \epsilon f(u), \quad u \in H_\epsilon, \]  
where \( f(u) = (u^+)^{p-1} \).

Next, we introduce the following equation which correspond to limiting equation to problem (1). It is well known that there exists a unique positive spherically symmetric function \( U \in H_1(\mathbb{R}^N) \) such that
\[ -\Delta U + U = U^{p-1} \quad \text{in } \mathbb{R}^N. \]  
Moreover, the function \( U \) and its derivatives are exponentially decaying at infinity, namely
\[ \lim_{|z| \to \infty} U(z) |z|^{-N+1} / c > 0, \quad \lim_{|z| \to \infty} U'(z) |z|^{-N+1} / c = -c. \]  

Let us define a smooth cut-off function \( \chi_r \) by
\[ \chi_r(z) := \begin{cases} 1 & \text{if } z \in B(0, \frac{r}{2}); \\ \epsilon & \text{if } z \in (0, 1) \setminus B(0, \frac{r}{2}); \\ 0 & \text{if } z \in \mathbb{R}^N \setminus B(0, r), \end{cases} \]  
and which satisfies
\[ |
abla \chi_r(z)| \leq \frac{2}{r}, \quad |\nabla^2 \chi_r(z)| \leq \frac{2}{r^2}. \]  

For any point \( \xi \) in \( M \) and for any positive real number \( \lambda \), we define the function \( W_{\lambda, \xi} \) on \( M \) by
\[ W_{\lambda, \xi}(x) := \begin{cases} \chi_r \left( \exp^{-1}\xi(x) \right) U_\epsilon \left( \exp^{-1}_\xi(x) \right) & \text{if } x \in B_\theta(\xi, r); \\ 0 & \text{otherwise}, \end{cases} \]  
where \( U_\epsilon(z) \) is defined by
\[ U_\epsilon(z) := U \left( \frac{z}{\epsilon} \right). \]  

It is well known that every solution to the linear equation
\[ -\Delta \psi + \psi = (p-1)U^{p-2} \psi, \quad \text{in } \mathbb{R}^N \]  
is a linear combination of the functions
\[ \psi^i(z) = \frac{\partial U(z)}{\partial z_i}, \quad i = 1, 2, \ldots, N. \]  

Let us define on \( M \) the functions
\[ Z^i_{\epsilon, \xi}(x) := \begin{cases} \chi_r \left( \exp^{-1}_\xi(x) \right) \psi^i_\epsilon \left( \exp^{-1}_\xi(x) \right) & \text{if } x \in B_\theta(\xi, r); \\ 0 & \text{otherwise}, \end{cases} \]  
where \( \psi^i_\epsilon(z) = \psi^i \left( \frac{z}{\epsilon} \right) \).

We next define the eigenspace
\[ K_{\epsilon, \xi} := \text{Span} \left\{ Z^i_{\epsilon, \xi} : i = 1, 2, \ldots, N \right\}, \]  
and
\[ K^\perp_{\epsilon, \xi} = \left\{ \phi \in H_\epsilon : \langle \phi, Z^i_{\epsilon, \xi} \rangle_\epsilon = 0, \forall i = 1, 2, \ldots, N \right\}. \]  

3. **Improving the approximate solution.** The aim of this section is to improve the approximate solution \( W_{\epsilon, \xi} \) in order to get smaller error terms.
3.1. Expansion of the Laplace-Beltrami operator in Fermi coordinates.

Using the above expansions of the coefficients of the metric $g$, the coefficients of its inverse and its determinant in Fermi coordinates given in Lemma 2.1, we give the expansion of the Laplace-Beltrami operator. We first recall the expression of the Laplace-Beltrami operator

$$\Delta_g u = \frac{1}{\sqrt{\det(g)}} \partial_i \left( \sqrt{\det(g)} g^{ij} \partial_j u \right)$$

where we have used Einstein convention for repeated indices. These expansions and the properties of the curvature tensor yield.

**Lemma 3.1.** At the point $q = F(z)$, the following expansion holds

$$\Delta_g u = \Delta_{\mathbb{R}^n} u - \frac{\varepsilon^2}{3} R_{kij\ell} z_k \partial_{ij\ell} u + \frac{2\varepsilon^2}{3} R_{kssj} z_k \partial_j u$$

$$+ O_{ij}(\varepsilon^3 |z|^3) \partial_{ij} u + O_{j}(\varepsilon^3 |z|^2) \partial_j u$$

(15)

where we have identified $u(F(z)) = u(z)$.

Now, we look for an approximate solution of the form

$$u_{1,\varepsilon,\xi} = W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi}$$

(16)

where the correction $V_{\varepsilon,\xi}$ is a function defined on $M$ and will be chosen so that the terms of order $\varepsilon^2$ in the equation $-\Delta_g u_{1,\varepsilon,\xi} + u_{1,\varepsilon,\xi} - |u_{1,\varepsilon,\xi}|^{p-1}$ vanishes. We choose $V_{\varepsilon,\xi}$ of the form

$$V_{\varepsilon,\xi}(x) := \begin{cases} \chi_r \left( \exp \frac{-1}{\varepsilon} \right) V_{\varepsilon} \left( \exp \frac{-1}{\varepsilon} \right) & \text{if } x \in B_g(\xi, r); \\ 0 & \text{otherwise,} \end{cases}$$

(17)

where $V_{\varepsilon}(z)$ is defined by $V_{\varepsilon}(z) := V(z)$ with $V : \mathbb{R}^n \to \mathbb{R}$ is the unique bounded solution of the problem

$$L_0 v := -\Delta_{\mathbb{R}^n} v + v - (p - 1)|U|^{p-2} v = -\frac{1}{3} R_{kij\ell} z_k \partial_{ij\ell} U + \frac{2}{3} R_{kssj} z_k \partial_j U.$$  

(18)

We note that the above equation is always solvable since the right hand side is orthogonal to the kernel of the linearised operator by oddness. Moreover the function $V$ and its derivatives are exponentially decaying at infinity.

Observe that since $U$ is radial, we have that

$$\partial_i U = z_i \frac{U'}{|z|}$$

$$\partial_{ij} U = \delta_{ij} \frac{U'}{|z|} + z_i z_j \frac{U''}{|z|^2} - z_i z_j \frac{U'}{|z|^2}$$

then

$$R_{kij\ell} z_k \partial_{ij\ell} U = R_{kij\ell} z_k \left( \frac{U'}{|z|} + R_{kij\ell} z_k z_i z_j \frac{U''}{|z|^2} - R_{kij\ell} z_k z_i z_j \frac{U'}{|z|^2} \right).$$

The second and third terms at the right hand side are zero by the antisymmetry of the curvature tensor.

Then equation (18) can be rewritten as

$$L_0 v = \frac{1}{3} R_{kssj} z_k \partial_j U = \frac{1}{3} R_{kij\ell} z_k \partial_{ij\ell} U.$$  

(19)
Now, we decompose 
\[ R_{kl} z_k \partial_l U = R_{kl} \frac{z_k z_l}{|z|^2} \partial_l |z| U' = \left( \sum_{k \neq l} R_{kl} \frac{z_k z_l}{|z|^2} + \frac{1}{N} \text{Scal}_g \right) |z| U'. \]

Then, we can look for the solution of (19) of the form
\[ V = \frac{1}{3} \left( \sum_{k \neq l} R_{kl} z_k z_l \psi_1(|z|) + \frac{1}{N} \text{Scal}_g \psi_2(|z|) \right) \]

(20)

where \( \psi_1 \) and \( \psi_2 \) are two radial functions.

We will look for a solution to (8), or equivalently to (1), of the form
\[ u_\varepsilon = W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi_{\varepsilon, \xi} \]

(21)

where the rest term \( \phi_{\varepsilon, \xi} \) belongs to the space \( K_{\varepsilon, \xi} \), the functions \( W_{\varepsilon, \xi} \) and \( V_{\varepsilon, \xi} \) are defined in (12) and (17) respectively.

Let \( \Pi_{\varepsilon, \xi} : H_{\varepsilon} \to K_{\varepsilon, \xi} \) and \( \Pi_{\varepsilon, \xi}^+ : H_{\varepsilon} \to K_{\varepsilon, \xi}^+ \) be the orthogonal projections. In order to solve problem (8) we will solve the system
\[ \Pi_{\varepsilon, \xi} \{ W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi - i_\varepsilon^* \left[ f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) \right] \} = 0, \]
\[ \Pi_{\varepsilon, \xi} \{ W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi - i_\varepsilon^* \left[ f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) \right] \} = 0. \]

(22)

(23)

4. The existence result. Let us introduce the linear operator \( L_{\varepsilon, \xi} : H_{\varepsilon} \cap K_{\varepsilon, \xi}^+ \to K_{\varepsilon, \xi}^+ \) defined by
\[ L_{\varepsilon, \xi}(\phi) := \Pi_{\varepsilon, \xi}^+ \{ \phi - i_\varepsilon^* \left[ f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) \right] \}. \]

(24)

This operator is well defined by using (7). Therefore equation (22) is equivalent to
\[ L_{\varepsilon, \xi}(\phi) = N_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi} \]

(25)

where
\[ N_{\varepsilon, \xi}(\phi) = \Pi_{\varepsilon, \xi} \left\{ i_\varepsilon^* \left[ f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) + \phi \right] - f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) - f'(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) \phi \right\} \]

(26)

and
\[ R_{\varepsilon, \xi} = \Pi_{\varepsilon, \xi} \left\{ i_\varepsilon^* \left[ f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) \right] - W_{\varepsilon, \xi} - \varepsilon^2 V_{\varepsilon, \xi} \right\}. \]

(27)

We first give the following result whose proof is postponed to Section 5 to solve equation (22).

**Proposition 1.** There exists \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for any \( \xi \in M \) and for any \( \varepsilon \in (0, \varepsilon_0) \), there exists a unique \( \phi_{\varepsilon, \xi} = \phi(\varepsilon, \xi) \) which solves equation (22), which is continuously differential with respect to \( \xi \), moreover,
\[ \| \phi_{\varepsilon, \xi} \|_\varepsilon \leq C \varepsilon^3. \]

(28)

We now introduce the functional \( J_\varepsilon : H_{\varepsilon} \to R \) defined by
\[ J_\varepsilon(u) = \frac{1}{\varepsilon^N} \left( \frac{1}{2} \int_M \varepsilon^2 |\nabla_g u|^2 \, d\mu_g + \frac{1}{2} \int_M u^2 \, d\mu_g - \frac{1}{p} \int_M u_+^p \, d\mu_g \right). \]

(29)

It is well known that any critical point of \( J_\varepsilon \) is solution to problem (1). We also define the functional \( \tilde{J}_\varepsilon : M \to R \) by
\[ \tilde{J}_\varepsilon(\xi) := J_\varepsilon \left( W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi_{\varepsilon, \xi} \right), \]
where $W_{\varepsilon, \xi}$ and $V_{\varepsilon, \xi}$ are defined in (12) and (17) respectively and $\phi_{\varepsilon, \xi}$ is given by Proposition 1.

Next, we prove that the critical points of $\tilde{J}_\varepsilon$ are the solutions to problem (23).

**Proposition 2.** For $\varepsilon$ small, if $\xi$ is a critical point of the functional $\tilde{J}_\varepsilon$, then $W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}$ is a solution of (8), or equivalently of problem (1).

**Proof.** We argue as in Lemma 4.1 in [15].

The problem is thus reduced to finding critical points of $\tilde{J}_\varepsilon$ and so it is necessary to compute the asymptotic expansion of $\tilde{J}_\varepsilon$.

**Proposition 3.** (i) For $\varepsilon > 0$ small enough, one has

$$
\tilde{J}_\varepsilon(\xi) = J_\varepsilon(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) = J_\varepsilon(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) + O(\varepsilon^6),
$$

uniformly with respect to $\xi$.

Moreover, setting $\xi(y) = \exp_\xi(y), y \in B(0, r)$, we have

$$
\left. \left( \frac{\partial}{\partial y_h} \tilde{J}_\varepsilon(\xi(y)) \right) \right|_{y=0} = \left. \left( \frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon, \xi(y)} + \varepsilon^2 V_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) \right) \right|_{y=0}
$$

$$
= \left. \left( \frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon, \xi(y)} + \varepsilon^2 V_{\varepsilon, \xi(y)}) \right) \right|_{y=0} + O(\varepsilon^5),
$$

uniformly with respect to $\xi$ as $\varepsilon$ goes to zero, $h = 1, 2, \cdots, N$.

(ii) It holds that

$$
J_\varepsilon(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) = c_0 - c_1 \Phi(\xi) \varepsilon^2 + o(\varepsilon^4),
$$

where $\Phi(\xi)$ is defined in (6), $c_0$ and $c_1$ are two constants given by

$$
c_0 = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2)dz - \frac{1}{p} \int_{\mathbb{R}^N} U^p dz,
$$

and

$$
c_1 = \frac{1}{6} \int_{\mathbb{R}^N} \left( \frac{U(|z|)}{|z|} \right)^2 z_1^4 dz.
$$

**Proof.** The proof is postponed to Appendix.

**Proof of Theorem 1.1.** Let us recall the definitions of $W_{\varepsilon, \xi}$ and $V_{\varepsilon, \xi}$ in (12) and (17) and let $\phi_{\varepsilon, \xi}$ be the unique solution of problem (22) given by Proposition 1. From Proposition 2, the function $W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}$ is a solution of (8) or equivalently of problem (1) if $\xi$ is a critical point of the function $\tilde{J}_\varepsilon$, defined by (29). Using (6), (30) and (32) we can write

$$
\tilde{J}_\varepsilon(\xi) = c_0 - c_1 \text{Scal}_g(\xi) \varepsilon^2 + c_1 \varepsilon^4 \tilde{\Xi}(\xi) + o(\varepsilon^4).
$$

Then clearly critical points of $\tilde{J}_\varepsilon$ can be found near any $C^1$ stable critical points of $\text{Scal}_g$ or if the latter is constant near any $C^1$ stable critical points of $\Xi$. 

5. The finite dimensional reduction. The section is devoted to the proof of Proposition 1. Let us recall that the linear operator $L_{\varepsilon, \xi}$ which is given in (24). As a first step, we want to study the invertibility of $L_{\varepsilon, \xi}$.

Lemma 5.1. For any $\phi \in H_{\varepsilon} \cap K_{\varepsilon, \xi}^\perp$ and $\xi \in M$, if $\varepsilon$ is small enough, there holds
\[
\|L_{\varepsilon, \xi}(\phi)\|_{\varepsilon} \geq C\|\phi\|_{\varepsilon},
\] (33)
where $C$ is a positive constant.

Proof. The proof follows that of Proposition 3.1 in [15], but we give here a sketch of proof for completeness. We argue by contradiction, assuming that there exist sequences $\varepsilon_n \to 0$, $\xi_n \in M$ such that (up to a subsequence) $\xi_n \to \xi$, $\phi_n \in K_{\varepsilon_n, \xi_n}^\perp$, with $\|\phi_n\|_{\varepsilon_n} = 1$ such that
\[
L_{\varepsilon_n, \xi_n}(\phi_n) = \psi_n \quad \text{and} \quad \|\psi_n\|_{\varepsilon_n} \to 0.
\]
Thus, there exists $\zeta_n \in K_{\varepsilon_n, \xi_n}$ such that
\[
\phi_n - i_{\varepsilon_n}^*[f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})\phi_n] = \psi_n + \zeta_n.
\] (34)
We will divide the proof into the following three steps to get a contradiction.

\begin{itemize}
\item [Step 1. We claim that]
\[
\|\zeta_n\|_{\varepsilon_n} \to 0 \quad \text{as} \quad n \to \infty.
\] (35)
\end{itemize}

In fact, let $\zeta_n := \sum_{k=1}^{N} a_k^k Z_{\varepsilon_n, \xi_n}^k$. For any $h = 1, \cdots, N$, we multiply (34) by $Z_{\varepsilon_n, \xi_n}^h$, then since $\phi_n, \psi_n \in K_{\varepsilon_n, \xi_n}^\perp$, we find
\[
\sum_{k=1}^{N} a_k^k \langle Z_{\varepsilon_n, \xi_n}^k, Z_{\varepsilon_n, \xi_n}^h \rangle_{\varepsilon_n} = \langle i_{\varepsilon_n}^*[f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})\phi_n], Z_{\varepsilon_n, \xi_n}^h \rangle_{\varepsilon_n}
\]
\[
= \frac{1}{\varepsilon_n^2} \int_M f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})\phi_n Z_{\varepsilon_n, \xi_n}^h \varepsilon_n.
\] (36)
By a direct computations, we have
\[
\langle Z_{\varepsilon_n, \xi_n}^k, Z_{\varepsilon_n, \xi_n}^h \rangle_{\varepsilon_n} = \begin{cases} c + o(1) & \text{if } h = k; \\ o(1) & \text{if } h \neq k, \end{cases}
\] (37)
where $c$ is a positive constant. Moreover, set
\[
\tilde{\phi}_n(x) = \begin{cases} \phi_n(\exp_{\varepsilon_n}(z)\chi_r(z)) & \text{if } z \in B(0, r/\varepsilon_n); \\ 0 & \text{otherwise}, \end{cases}
\]
then we have that $\|\tilde{\phi}_n\|_{H^1(\mathbb{R}^N)} \leq C\|\phi_n\|_{\varepsilon_n} \leq C$. Therefore, we can assume that $\tilde{\phi}_n$ converges to some $\tilde{\phi}$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^q_{\text{loc}}(H^1(\mathbb{R}^N))$ for any $q \in [2, 2^*)$ if $N \geq 3$ or $q \geq 2$ if $N = 2$. Since $\phi_n \in K_{\varepsilon_n, \xi_n}^\perp$, we have
\[
- \frac{1}{\varepsilon_n} \int_M f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})\phi_n Z_{\varepsilon_n, \xi_n}^h \varepsilon_n
\]
\[
= \frac{1}{\varepsilon_n} \int_M \left[\varepsilon_n^2 \nabla \cdot \nabla Z_{\varepsilon_n, \xi_n}^h - \nabla Z_{\varepsilon_n, \xi_n}^h \phi_n - f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})\phi_n Z_{\varepsilon_n, \xi_n}^h \phi_n \right] \varepsilon_n
\]
\[
= \int_{\mathbb{R}^N} (\nabla \psi_h \nabla \tilde{\phi} + \psi_h \tilde{\phi} - f'(U)\psi_h \tilde{\phi}) dz + o(1) = o(1).
\] (38)
From (36)-(38), we get that $a_n^k \to 0$ for any $k = 1, \cdots, N$, and then (35) follows.
Step 2. Let us write \( u_n := \phi_n - \psi_n - \zeta_n \), there holds
\[
\frac{1}{\varepsilon_n} \int_M f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n}) u_n^2 \to 1 \quad \text{as} \quad n \to \infty. \tag{39}
\]

In fact, since \( \|\phi_n\|_{\varepsilon_n} = 1, \|\psi_n\|_{\varepsilon_n} \to 0 \), and (35), we have that
\[
\|u_n\|_{\varepsilon_n} \to 1. \tag{40}
\]

Moreover, \( u_n \) satisfies the following equation
\[
-\varepsilon_n^2 \Delta u_n + u_n = f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})(\psi_n + \zeta_n) \quad \text{in} \ M. \tag{41}
\]

Multiplying above equation by \( u_n \), we deduce that
\[
\|u_n\|_{\varepsilon_n}^2 = \frac{1}{\varepsilon_n} \int_M f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n}) u_n^2
+ \frac{1}{\varepsilon_n} \int_M f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})(\psi_n + \zeta_n)u_n \tag{42}
\]

By Hölder’s inequality and Sobolev embedding, we can find
\[
\left| \frac{1}{\varepsilon_n} \int_M f'(W_{\varepsilon_n, \xi_n} + \varepsilon_n^2 V_{\varepsilon_n, \xi_n})(\psi_n + \zeta_n)u_n \right| = o(1). \tag{43}
\]

Then we get (39).

Step 3. Get a contradiction.

Indeed, we observe that \( u_n \) is compactly supported in \( B_g(\xi_n, r) \). Set
\[
\tilde{u}_n = u_n(\exp_\varepsilon(\varepsilon_n z) \chi_r(\varepsilon_n z)) \quad z \in B(0, r/\varepsilon_n).
\]

We note that \( \|\tilde{u}_n\|_{H^1(\mathbb{R}^N)} \leq C\|u_n\|_{\varepsilon_n}^2 \leq C \). Then, up to a subsequence, \( \tilde{u}_n \to \tilde{u} \) weakly in \( H^1(\mathbb{R}^N) \) and strongly in \( L^q_{loc}(\mathbb{R}^N) \) for any \( q \geq 2 \) if \( N = 2 \) or \( q \geq 2 \) if \( N = 2 \). By (41) we can get that \( \tilde{u} \) solves the problem
\[
-\Delta \tilde{u} + \tilde{u} = f'(U)\tilde{u} \quad \text{in} \ \mathbb{R}^N. \tag{44}
\]

Since \( \phi_n, \psi_n \in K^\perp_{\varepsilon_n, \xi_n} \), we find
\[
(Z^h_{\varepsilon_n, \xi_n} u_n)_{\varepsilon_n} = -(Z^h_{\varepsilon_n, \xi_n} \xi_n)_{\varepsilon_n} \leq \|Z^h_{\varepsilon_n, \xi_n}\|_{\varepsilon_n} \|\xi_n\|_{\varepsilon_n} = o(1). \tag{45}
\]

On the other hand, we have
\[
(Z^h_{\varepsilon_n, \xi_n} u_n)_{\varepsilon_n} = \frac{1}{\varepsilon_n^N} \int_M \varepsilon_n^N \nabla g Z^h_{\varepsilon_n, \xi_n} \nabla g u_n + Z^h_{\varepsilon_n, \xi_n} u_n \right| d\mu_g
\]
\[
= \int_{B(0, r/\varepsilon_n)} \left[ \sum_{i,j=1}^N g_{\xi_n}^{ij}(h)(\varepsilon_n z) \frac{\partial}{\partial z_i} (\psi^h(z) \chi_r(\varepsilon_n z)) \frac{\partial}{\partial z_i} (\tilde{u}_n(z))
+ \psi^h(z) \chi_r(\varepsilon_n z) \tilde{u}_n(z) u_n \right] g_{\xi_n}(\varepsilon_n z)^{1/2} dz \tag{46}
\]

From (45) and (46) we obtain that
\[
\int_{\mathbb{R}^N} |\nabla \psi^h \nabla \tilde{u} + \psi^h \tilde{u}|^2 dz + o(1). \tag{47}
\]
Therefore, by (44) and (47) we get that \( \tilde{u}_n \to 0 \) weakly in \( H^1(\mathbb{R}^N) \) and strongly in \( L^q_{\text{loc}}(\mathbb{R}^N) \) for any \( q \in [2, 2^*) \) if \( N \geq 3 \) or \( q \geq 2 \) if \( N = 2 \). Then

\[
\frac{1}{\varepsilon^N} \int_M f'(W_{\varepsilon,n,\xi} + \varepsilon^2 V_{\varepsilon,n,\xi_n})u_n^2 \leq \frac{1}{\varepsilon^N} \int_{B_g(\xi_{n,\varepsilon})} f'(U_{\varepsilon,n} + \varepsilon^2 V_{\varepsilon,n})(\exp_{\xi_n}^{-1}(x))u_n^2(x)dx \\
\leq C \int_{B(0,r/\varepsilon_n)} f'(U(z))\tilde{u}_n^2(z)dz = o(1)
\]

which gives a contradiction because of (39).

Next, we have the following estimate of \( R_{\varepsilon,\xi} \).

**Lemma 5.2.** For any \( \xi \in M \), if \( \varepsilon \) is small enough, there holds

\[
\|R_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq C\varepsilon^3,
\]

where \( C \) is a positive constant.

**Proof.** The proof follows that of Lemma 3.3 in [15], we sketch it here for completeness. Let us introduce the function \( \Gamma_{\varepsilon,\xi} \) defined by \( W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi} := i^*(\Gamma_{\varepsilon,\xi}) \), that is,

\[
-\varepsilon^2 \Delta_g(W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi}) + W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi} = \Gamma_{\varepsilon,\xi} \quad \text{on} \quad M.
\]

We remark that \( W_{\varepsilon,\xi}(x) = 0 \) and \( V_{\varepsilon,\xi} = 0 \) if \( x \notin B_g(\xi, r) \). Therefore, we have \( V_{\varepsilon,\xi}(x) = 0 \), if \( x \notin B_g(\xi, r) \) and if \( x \in B_g(\xi, r) \),

\[
\Gamma_{\varepsilon,\xi} = -\varepsilon^2 \Delta_g[(U_{\varepsilon} + \varepsilon^2 V_{\varepsilon})\chi_r] + U_{\varepsilon}\chi_r + \varepsilon^2 V_{\varepsilon}\chi_r
\]

\[
\quad = U_{\varepsilon}^{p-1}(z)\chi_r(z) - \varepsilon^2 U_{\varepsilon}(z)\Delta\chi_r(z) - 2\varepsilon^2 \nabla U_{\varepsilon}(z)\nabla\chi_r(z)
\]

\[
\quad - \varepsilon^4 V_{\varepsilon}(z)\Delta\chi_r(z) - 2\varepsilon^4 \nabla V_{\varepsilon}(z)\nabla\chi_r(z)
\]

\[
\quad + \varepsilon^2 \left[(p-1)|U_{\varepsilon}|^{p-2}U_{\varepsilon} - \frac{1}{3} R_{kij}\varepsilon z_k\partial_jz_l U_{\varepsilon} + \frac{2}{3} R_{kassj}\varepsilon z_k\partial_j U_{\varepsilon}\right]\chi_r(z)
\]

\[
\quad - \varepsilon^2 g_{\varepsilon}^{ij}
\]

\[
\quad - \delta_{ij}(U_{\varepsilon}\chi_r + \varepsilon^2 V_{\varepsilon}\chi_r) + \varepsilon^2 g_{\varepsilon}^{jk}
\]

\[
\quad \Gamma_{\varepsilon,\xi}(z)(\exp_{\xi}^{-1}(z)) + \varepsilon^2 g_{\varepsilon}^{jk}\partial_k(U_{\varepsilon}\chi_r + \varepsilon^2 V_{\varepsilon}\chi_r)(\exp_{\xi}^{-1}(z))
\]

(49)

From (7) and (49), we then have

\[
\|i^*(f(W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi})) - W_{\varepsilon,\xi} - \varepsilon^2 V_{\varepsilon,\xi}\|_\varepsilon = \|i^*(f(W_{\varepsilon,\xi})) - i^*(\Gamma_{\varepsilon,\xi})\|_\varepsilon
\]

\[
\leq C \left| f(W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi}) - \Gamma_{\varepsilon,\xi} \right|_{\nu',\varepsilon}
\]

\[
= C \left( \frac{1}{\varepsilon^N} \int_M f(W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi}) - \Gamma_{\varepsilon,\xi} \right)^{1/\nu'} \]

\[
= C \left( \frac{1}{\varepsilon^N} \int_{B(0,r)} [(U_{\varepsilon}(z) + \varepsilon^2 V_{\varepsilon}(z))^{p-1}\chi_r^{p-1}(z) - \Gamma_{\varepsilon,\xi}(\exp_{\xi}(z))]^{1/\nu'} |g_{\varepsilon}(z)|^{1/2} dz \right)^{1/\nu'}
\]

\[
\leq C \left( \frac{1}{\varepsilon^N} \int_{B(0,r)} U_{\varepsilon}^{p-1}(z) (\chi_r^{p-1}(z) - \chi_r(z)) |\nabla\chi_r(z)|^{1/\nu'} dz \right)^{1/\nu'}
\]

\[
+ C\varepsilon^2 \left( \frac{1}{\varepsilon^N} \int_{B(0,r)} U_{\varepsilon}^{p}(z)|\nabla\chi_r(z)|^{1/\nu'} dz \right)^{1/\nu'}
\]

\[
\]
\[ + C\varepsilon^4 \left( \frac{1}{\varepsilon^N} \int_{B(0,r)} V^p_\varepsilon(z) |\Delta \chi_{r}(z)|^p \, dz \right)^{\frac{1}{p}} \]
\[ + C\varepsilon^2 \left( \frac{1}{\varepsilon^N} \int_{B(0,r)} (\nabla U_\varepsilon(z) \nabla \chi_{r}(z))^p \, dz \right)^{\frac{1}{p}} \]
\[ + C\varepsilon^4 \left( \frac{1}{\varepsilon^N} \int_{B(0,r)} (\nabla V_\varepsilon(z) \nabla \chi_{r}(z))^p \, dz \right)^{\frac{1}{p}} \]
\[ + C \left( \frac{1}{\varepsilon^N} \int_{B(0,r)} |(U_\varepsilon + t\varepsilon^2 V_\varepsilon)^p - 2V_\varepsilon^p| \, dz \right)^{\frac{1}{p}} \]
\[ + C\varepsilon^3 \left( \int_{B(0,r/\varepsilon)} |z|^{3} \partial_j(U \chi_r)(z)|^p \, dz \right)^{\frac{1}{p}} \]
\[ + C\varepsilon^3 \left( \int_{B(0,r/\varepsilon)} |z|^{3} \partial_k(U \chi_r)(z)|^p \, dz \right)^{\frac{1}{p}} \]
\[ = O(\varepsilon^3) \] (51)

by using the fact that the function \( U \) and \( V \) and their derivatives decay exponentially. This concludes the proof of Lemma.

**Proof of Proposition 1:** In order to solve (22) or equivalently equation (25), we need to find a fixed point for the operator \( T_{\varepsilon, \xi} : H_{\varepsilon} \cap K_{\varepsilon, \xi}^{+} \to H_{\varepsilon} \cap K_{\varepsilon, \xi}^{+} \) defined
\[ T_{\varepsilon, \xi}(\phi) = L_{\varepsilon, \xi}^{-1}(N_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi}) , \]
for \( \varepsilon \) small and for any \( \xi \in M \). From Lemma 5.1, we have
\[ \| T_{\varepsilon, \xi}(\phi) \| \leq C(\| N_{\varepsilon, \xi}(\phi) \| + \| R_{\varepsilon, \xi} \| ) \]
and
\[ \| T_{\varepsilon, \xi}(\phi_1) - T_{\varepsilon, \xi}(\phi_2) \| \leq C \| N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2) \| . \]

By (7), we deduce
\[ \| N_{\varepsilon, \xi}(\phi) \| \leq C \| f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi) - f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) - f'(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi})\phi \| , \]
and
\[ \| N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2) \| \leq C \| f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi_1) - f(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi} + \phi_2) - f'(W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi})(\phi_1 - \phi_2) \| . \]

Thus we obtain that \( T_{\varepsilon, \xi} \) is a contraction map on suitable ball of \( H_{\varepsilon} \), which has a fixed point in the ball centered at 0 with radius \( c\varepsilon^3 \) in \( K_{\varepsilon, \xi}^{+} \) for a suitable constant
c. Furthermore, the map $\xi \to \phi_{\varepsilon, \xi}$ is the $C^1$--function by the Implicit Function Theorem.

6. **Appendix: Proof of Proposition 3.** In this section, we give the proof of Proposition 3.

**Proof of (i) in Proposition 3.** We recall the definition of the approximate solution given in (16)

$$u_{1, \varepsilon, \xi} = W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}$$

We have

$$J_{\varepsilon} (u_{1, \varepsilon, \xi} + \phi_{\varepsilon, \xi}) - J_{\varepsilon} (u_{1, \varepsilon, \xi}) = \frac{1}{2} \|\phi_{\varepsilon, \xi}\|^2 + \frac{1}{\varepsilon^N} \int_M \left[ \varepsilon^2 \nabla_y u_{1, \varepsilon, \xi} \nabla_y \phi_{\varepsilon, \xi} + u_{1, \varepsilon, \xi} \phi_{\varepsilon, \xi} - f(u_{1, \varepsilon, \xi}) \phi_{\varepsilon, \xi} \right] \mu_g$$

$$- \frac{1}{\varepsilon^N} \int_M \left[ F(u_{1, \varepsilon, \xi} + \phi_{\varepsilon, \xi}) - F(u_{1, \varepsilon, \xi}) - f(u_{1, \varepsilon, \xi}) \phi_{\varepsilon, \xi} \right] \mu_g$$

$$= - \frac{1}{2} \|\phi_{\varepsilon, \xi}\|^2 + \frac{1}{\varepsilon^N} \int_M \left[ f(u_{1, \varepsilon, \xi} + \phi_{\varepsilon, \xi}) - f(u_{1, \varepsilon, \xi}) \right] \phi_{\varepsilon, \xi} \mu_g$$

$$- \frac{1}{\varepsilon^N} \int_M \left[ F(u_{1, \varepsilon, \xi} + \phi_{\varepsilon, \xi}) - F(u_{1, \varepsilon, \xi}) - f(u_{1, \varepsilon, \xi}) \phi_{\varepsilon, \xi} \right] \mu_g$$

$$= - \frac{1}{2} \|\phi_{\varepsilon, \xi}\|^2 + \frac{1}{\varepsilon^N} \int_M f'(u_{1, \varepsilon, \xi} + t \phi_{\varepsilon, \xi}) \phi^2_{\varepsilon, \xi} \mu_g - \frac{1}{2\varepsilon^N} \int_M f'(u_{1, \varepsilon, \xi} + 2 \phi_{\varepsilon, \xi}) \phi^2_{\varepsilon, \xi} \mu_g$$

for some $t_1, t_2 \in [0, 1]$. Since for any $t \in [0, 1]$, we have

$$\frac{1}{\varepsilon^N} \int_M f'(u_{1, \varepsilon, \xi} + t \phi_{\varepsilon, \xi}) \phi^2_{\varepsilon, \xi} \mu_g \leq \frac{1}{\varepsilon^N} \int_M u^{p-2}_{1, \varepsilon, \xi} \phi^2_{\varepsilon, \xi} \mu_g + c \frac{1}{\varepsilon^N} \int_M \phi^p_{\varepsilon, \xi} \mu_g$$

$$\leq c \frac{1}{\varepsilon^N} \int_M \phi^2_{\varepsilon, \xi} \mu_g + c \frac{1}{\varepsilon^N} \int_M \phi^p_{\varepsilon, \xi} \mu_g \leq c (\|\phi_{\varepsilon, \xi}\|^2 + \|\phi_{\varepsilon, \xi}\|^p) \right). \tag{53}$$

Thus, (30) follows from (28), (52) and (53).

On the other hand

$$\frac{\partial}{\partial y_h} J_{\varepsilon} (u_{1, \varepsilon, \xi}(y) + \phi_{\varepsilon, \xi}(y)) - \frac{\partial}{\partial y_h} J_{\varepsilon} (u_{1, \varepsilon, \xi})$$

$$= J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y) + \phi_{\varepsilon, \xi}(y)) \left[ \frac{\partial}{\partial y_h} u_{1, \varepsilon, \xi}(y) + \frac{\partial}{\partial y_h} \phi_{\varepsilon, \xi}(y) \right] - J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y)) \left[ \frac{\partial}{\partial y_h} u_{1, \varepsilon, \xi}(y) \right]$$

$$= \left[ J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y) + \phi_{\varepsilon, \xi}(y)) - J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y)) \right] \left[ \frac{\partial}{\partial y_h} u_{1, \varepsilon, \xi}(y) \right]$$

$$+ J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y) + \phi_{\varepsilon, \xi}(y)) \left[ \frac{\partial}{\partial y_h} \phi_{\varepsilon, \xi}(y) \right]. \tag{54}$$

Using the fact that $-\Delta_y u_{1, \varepsilon, \xi} + u_{1, \varepsilon, \xi} - |u_{1, \varepsilon, \xi}|^{p-1} = O(\varepsilon^3)$ and arguing as in Lemma 5.1 in [15] (with obvious modifications)

$$\left[ J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y) + \phi_{\varepsilon, \xi}(y)) - J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y)) \right] \left[ \frac{\partial}{\partial y_h} u_{1, \varepsilon, \xi}(y) \right] = O(\varepsilon^5),$$

and

$$J'_{\varepsilon} (u_{1, \varepsilon, \xi}(y) + \phi_{\varepsilon, \xi}(y)) \left[ \frac{\partial}{\partial y_h} \phi_{\varepsilon, \xi}(y) \right] = O(\varepsilon^5).$$

Thus (31) holds.
Proof of (ii) in Proposition 3. Let us compute the energy \( J_\varepsilon(u_{1,\varepsilon}) = J_\varepsilon(W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi}) \).

For more generality, we will not assume here that the scalar curvature of \((M, g)\) is constant. All the computations will be made for general manifolds and to conclude the proof of (ii) in Proposition 3 we will take \( \text{Scal}_g = \text{constant} \).

\[
J_\varepsilon(W_{\varepsilon,\xi} + \varepsilon^2 V_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi}) + \varepsilon^2 J_\varepsilon'(W_{\varepsilon,\xi})[V_{\varepsilon,\xi}] + \frac{1}{2}\varepsilon^4 J_\varepsilon''(W_{\varepsilon,\xi})[V_{\varepsilon,\xi}, V_{\varepsilon,\xi}] + O(\varepsilon^6)
\]

\[
= J_\varepsilon(W_{\varepsilon,\xi}) + \varepsilon^2 \int_M \left( - \Delta_g W_{\varepsilon,\xi} + W_{\varepsilon,\xi} - |W_{\varepsilon,\xi}|^{p-1} \right) V_{\varepsilon,\xi} \, d\mu_g
\]

\[
+ \frac{1}{2}\varepsilon^4 \int_M \left( - \Delta_g V_{\varepsilon,\xi} + V_{\varepsilon,\xi} - (p-1)|W_{\varepsilon,\xi}|^{p-2}V_{\varepsilon,\xi} \right) V_{\varepsilon,\xi} \, d\mu_g + O(\varepsilon^6).
\]

We first compute \( J_\varepsilon(W_{\varepsilon,\xi}) \). To this win we recall the expansions given in Lemma 2.1, we have

\[
\frac{\varepsilon^2}{\varepsilon^n} \frac{1}{2} \int_M |\nabla g_{\varepsilon,\xi}(x)|^2 \, d\mu_g
\]
\[-\frac{\varepsilon^4}{2} \left( \frac{1}{40} \nabla_{pq} R_{kl} + \frac{1}{180} R_{kiqr} R_{pqr} - \frac{1}{72} R_{kl} R_{pq} \right) \int_{\mathbb{R}^N} U^2 z_k z_l z_p z_q dz + o(\varepsilon^4), \]

and

\[
\frac{1}{\varepsilon^N} \frac{1}{p} \int_M W_{c,\xi}(x)^p d\mu_g = \frac{1}{p} \int_{B(0,r/\varepsilon)} (U(z) \chi_{r/\varepsilon}(z))^p \sqrt{\det(g_{\xi}(\varepsilon z))} dz
\]

\[
= \frac{1}{p} \int_{\mathbb{R}^N} U^p dz - \frac{\varepsilon^2}{6p} R_{kl} \int_{\mathbb{R}^N} U^p z_k z_l dz
\]

\[
- \frac{\varepsilon^4}{p} \left( \frac{1}{40} \nabla_{pq} R_{kl} + \frac{1}{180} R_{kiqr} R_{pqr} - \frac{1}{72} R_{kl} R_{pq} \right) \int_{\mathbb{R}^N} U^p z_k z_l z_p z_q dz + o(\varepsilon^4). \]

Then we get

\[
J_\varepsilon(W_{c,\xi}(x)) = c_0 + \varepsilon^2 \Theta_1(\xi) + \varepsilon^4 \Theta_2(\xi) + o(\varepsilon^4), \tag{56}
\]

where

\[
c_0 = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2) dz - \frac{1}{p} \int_{\mathbb{R}^N} U^p dz,
\]

\[
\Theta_1(\xi) = -\frac{1}{6} R_{kijl} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_l dz - \frac{1}{12} R_{kl} \int_{\mathbb{R}^N} |\nabla U|^2 z_k z_l dz
\]

\[
- \frac{1}{12} R_{kl} \int_{\mathbb{R}^N} U^2 z_k z_l dz + \frac{1}{6p} R_{kl} \int_{\mathbb{R}^N} U^p z_k z_l dz
\]

\[
= -\frac{1}{6} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_l dz
\]

\[
- \frac{1}{6} R_{kk} \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla U|^2 z_k^2 dz + \frac{1}{2} \int_{\mathbb{R}^N} U^2 z_k^2 dz - \frac{1}{p} \int_{\mathbb{R}^N} U^p z_k^2 dz \right]
\]

\[
= -\frac{1}{6} R_{kijl} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_l dz - \frac{1}{6} R_{kk} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_k^2 dz
\]

\[
= -\frac{1}{6} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_l dz - \frac{1}{6} R_{kk} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_k^2 dz
\]

\[
= -\frac{1}{6} \text{Scal}_g(\xi) \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^4 dz
\]

\[
= -c_1 \text{Scal}_g(\xi), \tag{57}
\]

where \(c_1 = \frac{1}{6} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^4 dz\). Moreover

\[
\Theta_2(\xi) = -\frac{1}{2} \frac{1}{20} \nabla_{pq} R_{kl} + \frac{2}{45} R_{kiqr} R_{pqr} - \frac{1}{9} R_{ki} R_{pq} \times
\]

\[
\times \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_l z_p z_q dz
\]

\[
+ \frac{1}{36} R_{kijl} R_{pq} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_l z_p z_q dz
\]

\[
- \frac{1}{2} \left( \frac{1}{40} \nabla_{pq} R_{kl} + \frac{1}{180} R_{kiqr} R_{pqr} - \frac{1}{72} R_{kl} R_{pq} \right) \int_{\mathbb{R}^N} |\nabla U|^2 z_k z_l z_p z_q dz
\]

\[
- \frac{1}{2} \left( \frac{1}{40} \nabla_{pq} R_{kl} + \frac{1}{180} R_{kiqr} R_{pqr} - \frac{1}{72} R_{kl} R_{pq} \right) \int_{\mathbb{R}^N} U^2 z_k z_l z_p z_q dz
\]
Here we have used Lemma 6.2. Moreover, we then deduce that

\[
\Theta_2(x) = -\frac{1}{2} \left( \frac{1}{20} \nabla_R R_{ijkl} + \frac{2}{45} R_{klij} R_{pqr} - \frac{1}{9} R_{kisrl} R_{pqj} \right) \times
\int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_p z_q d\eta(z)
\]

\[
+ \frac{1}{36} R_{kijkl} R_{pq} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_p z_q d\eta(z)
\]

\[
- \left( \frac{1}{40} \nabla_R R_{kilr} + \frac{1}{180} R_{kilr} R_{pqr} - \frac{1}{72} R_{kijl} R_{pq} \right) \times
\int_{\mathbb{R}^N} \left| \nabla U \right|^2 z_i z_j z_k z_p z_q d\eta(z) + \frac{1}{2} \int_{\mathbb{R}^N} U^2 z_i z_j z_k z_p z_q d\eta(z) - \frac{1}{p} \int_{\mathbb{R}^N} U^p z_i z_j z_k z_p z_q d\eta(z).
\]

Since

\[
\nabla_R R_{kijl} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_p z_q d\eta(z) = 0.
\]

here we have used Lemma 6.2. Moreover,

\[
R_{kilr} R_{pqr} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_p z_q d\eta(z) = 0,
\]

\[
R_{kisrl} R_{pqj} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_p z_q d\eta(z) = 0,
\]

\[
R_{kijl} R_{pq} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_p z_q d\eta(z) = 0.
\]

Now, for a radial function \( f(x) \), we have

\[
\nabla_R R_{kijl} \int_{\mathbb{R}^N} f(x) z_i z_j z_k z_p z_q d\eta(z)
\]

\[
= \nabla_R R_{kijl} \int_{\mathbb{R}^N} f(x) z_i^4 d\eta(z) + \sum_{p \neq q} \nabla_R R_{kijl} \int_{\mathbb{R}^N} f(x) z_i^2 z_j^2 d\eta(z)
\]

\[
+ 2 \sum_{p \neq q} \nabla_R R_{kijl} \int_{\mathbb{R}^N} f(x) z_i^2 z_j^2 d\eta(z)
\]

\[
= \left( 3 \nabla_R R_{kijl} + \sum_{p \neq q} \nabla_R R_{kijl} + 2 \sum_{p \neq q} \nabla_R R_{kijl} \right) \int_{\mathbb{R}^N} f(x) z_i^2 z_j^2 d\eta(z)
\]

\[
= \left( \nabla_R R_{kijl} + \sum_{p \neq q} \nabla_R R_{kijl} + 2 \nabla_R R_{kijl} \right) \int_{\mathbb{R}^N} f(x) z_i^2 z_j^2 d\eta(z)
\]

\[
= \left( \Delta \text{Scal}_g(x) + 2 \sum_{k,l} \nabla_R R_{kijl} \right) \int_{\mathbb{R}^N} f(x) z_i^2 z_j^2 d\eta(z)
\]

\[
= \frac{1}{N + 2} \left( \Delta \text{Scal}_g(x) + 2 \sum_{k,l} \nabla_R R_{kijl} \right) \int_{\mathbb{R}^N} f(x) z_i^2 d\eta(z),
\]

\[
U^p z_i z_j z_k z_p z_q d\eta(z).
\]

\[
= \frac{1}{N + 2} \left( \Delta \text{Scal}_g(x) + 2 \sum_{k,l} \nabla_R R_{kijl} \right) \int_{\mathbb{R}^N} f(x) z_i^2 d\eta(z),
\]
Here we used the fact
\[ R_{ijkl}R_{piqr} \int_{\mathbb{R}^N} f(r)z_kz_lz_pz_qdz \]
where in the last equality we have used the fact that
\[ \sum_{k,l} 2\nabla_{kl}R_{kl}(\xi) = \sum_{k,l} \nabla_{kk}R_{ll}(\xi) = \Delta \mathcal{S} \]
which follows from Bianchi identity.

We next compute the term
\[ R_{kilr}R_{piqr} \int_{\mathbb{R}^N} f(r)z_kz_lz_pz_qdz \]
\[ = \sum_k R_{kikr}R_{kikr} \int_{\mathbb{R}^N} f(r)z_1^2dz + \sum_{k\neq i} R_{kikr}R_{piqr} \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz \]
\[ + \sum_{k\neq l} R_{kilr}R_{kilr} \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz + \sum_{k\neq l} R_{kilr}R_{likr} \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz \]
\[ = \left( 3 \sum_k R_{kikr}R_{kikr} + \sum_{k\neq i} \left[ R_{kikr}R_{ili} + R_{kilr}R_{kilr} + R_{kilr}R_{likr} \right] \right) \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz \]
\[ = \frac{1}{N+2} \left( \|\nabla f\|^2 + 3\|R\|^2 \right) \int_{\mathbb{R}^N} f(r)z_1^2dz. \] (64)
Here we used the fact \( R_{kilr}R_{kilr} = 2R_{kilr}R_{krl} \) by the first Bianchi identity. This yields
\[ R_{kilr}R_{kilr} + R_{kilr}R_{likr} = 3R_{kilr}R_{krl} = \frac{3}{2} R_{kilr}R_{kilr} = \frac{3}{2} \|R\|^2. \]

Finally
\[ R_{kl}R_{pp} \int_{\mathbb{R}^N} f(r)z_kz_pz_pdz \]
\[ = \sum_k R_{kk}R_{kk} \int_{\mathbb{R}^N} f(r)z_1^2dz + \sum_{k\neq p} R_{kk}R_{pp} \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz \]
\[ + \sum_{k\neq q} R_{kk}R_{qq} \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz + \sum_{k\neq l} R_{kl}R_{kl} \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz \]
\[ = \left( 3 \sum_k R_{kk}R_{kk} + \sum_{k\neq p} R_{kk}R_{pp} + 2 \sum_{k\neq l} R_{kl}R_{kl} \right) \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz \]
\[ = \left( \mathcal{S} + 2\sum_{k,l} R_{kl}R_{kl} \right) \int_{\mathbb{R}^N} f(r)z_1^2z_2^2dz \]
\[ = \frac{1}{N+2} \left( \mathcal{S} + 2\sum_{k,l} R_{kl}R_{kl} \right) \int_{\mathbb{R}^N} f(r)z_1^2dz. \] (65)
From (58) to (65), we obtain
\[
\Theta_2(\xi) = \frac{1}{2(N+2)} \left[ -\frac{1}{20} \Delta \text{Scal}_g(\xi) - \frac{1}{180} \left( \|\text{Ric}_\xi\|^2 + \frac{3}{2} \|R_\xi\|^2 \right) 
+ \frac{1}{72} \left( \text{Scal}_g(\xi)^2 + 2\|\text{Ric}_\xi\|^2 \right) \right] 
\times \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla U|^2 z^2 \, dz + \frac{1}{2} \int_{\mathbb{R}^N} U^2 z^2 \, dz - \frac{1}{p} \int_{\mathbb{R}^N} U^p z^2 \, dz \right).
\]
Using Lemma 6.1 and Lemma 6.2 below, we find
\[
\Theta_2(\xi) = \frac{3}{2(N+2)^2} \left[ -\frac{1}{20} \Delta \text{Scal}_g(\xi) - \frac{1}{180} \left( \|\text{Ric}_\xi\|^2 + \frac{3}{2} \|R_\xi\|^2 \right) 
+ \frac{1}{72} \left( \text{Scal}_g(\xi)^2 + 2\|\text{Ric}_\xi\|^2 \right) \right] \times \left( \int_{\mathbb{R}^N} U'(|z|) \frac{\text{Scal}_g(\xi)}{|z|} \, dz \right)
= \frac{1}{240(N+2)^2} \Xi(\xi) \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 \, dz
= \frac{1}{120(N+2)} c_1 \Xi(\xi),
\]
where
\[
\Xi(\xi) = -18 \Delta_g \text{Scal}_g(\xi) + 8 \|\text{Ric}_\xi\|^2 + 5 \text{Scal}_g(\xi)^2 - 3 \|R_\xi\|^2.
\]
Thus, (32) follows from (56), (57), (66) and (67).

Now we compute the remaining term in (55), namely
\[
\mathcal{A} := \varepsilon^2 \int_{M_w} \left( -\Delta_g W_{\varepsilon,\xi} + W_{\varepsilon,\xi} - |W_{\varepsilon,\xi}|^{p-1} \right) V_{\varepsilon,\xi} \, d\mu_g 
+ \frac{1}{\varepsilon^4} \int_{M_w} \left( -\Delta_g V_{\varepsilon,\xi} + V_{\varepsilon,\xi} - (p-1)|W_{\varepsilon,\xi}|^{p-2} V_{\varepsilon,\xi} \right) V_{\varepsilon,\xi} \, d\mu_g + O(\varepsilon^6).
\]
Recalling the expression of $V_{\varepsilon,\xi}$ given in (17)
\[
V_{\varepsilon,\xi}(x) := \begin{cases} 
\chi_r \left( \exp_{\xi}^{-1} (x) \right) V \left( \exp_{\xi}^{-1} (x) \right) & \text{if } x \in B_g(\xi, r), \\
0 & \text{otherwise},
\end{cases}
\]
where $V_\varepsilon(z) := V(z)$ with $V$ satisfying (19), using the fact that
\[
\Delta_g V - \Delta_{R^N} V = \mathcal{O}_r(\varepsilon^2 |z|^2) \partial_{ij}^2 V + \mathcal{O}_r(\varepsilon^2 |z|^2) \partial_j V
\]
and reasoning as before, the above quantity $\mathcal{A}$ is then given by
\[
\mathcal{A} = -\frac{\varepsilon^4}{2} \int_{\mathbb{R}^N} L_0 V \, V + O(\varepsilon^5).
\]
Recall the explicit formula of $V$ given in (20) which satisfies (19), we can compute
\[
\int_{\mathbb{R}^N} L_0 V \, V = \frac{1}{9} \int \left( \sum_{k \neq \ell} R_{ik} z_k \psi_1(|z|) + \frac{1}{N} \text{Scal}_g \psi_1(|z|) \right) \sum_{i,j} R_{ij} z_i z_j \frac{U'}{|z|}
= \frac{1}{9} \sum_{k \neq \ell} \sum_{i,j} R_{ki} R_{ij} \int_{\mathbb{R}^N} z_i z_j z_k \psi_1(|z|) \frac{U'(|z|)}{|z|}
\]
(70)
\[
+ \frac{1}{9N} \text{Scal}_g \sum_{i,j} R_{ij} \int_{\mathbb{R}^N} z_i z_j \psi_2(|z|) \frac{U'(|z|)}{|z|}. 
\]

No using the fact that
\[
\int_{\mathbb{R}^N} z_i z_j \psi_2(|z|) \frac{U'(|z|)}{|z|} \, dz = \delta_{ij} \int_{\mathbb{R}^N} z_1^2 \psi_2(|z|) \frac{U'(|z|)}{|z|} \, dz
\]

and for \( k \neq \ell \)
\[
\int_{\mathbb{R}^N} z_i z_j z_k z_\ell \psi_1(|z|) \frac{U'(|z|)}{|z|} \, dz = \left( \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk} \right) \int_{\mathbb{R}^N} z_1^2 z_2^2 \psi_1(|z|) \frac{U'(|z|)}{|z|} \, dz
\]

\[
= \frac{\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}}{N + 2} \int_{\mathbb{R}^N} z_1^2 \psi_1(|z|) \frac{U'(|z|)}{|z|} \, dz.
\]

Here we have used Lemma 6.2 below. We deduce that
\[
\int L_0 V = \frac{1}{9} \left( \frac{2}{N + 2} \| \text{Ric}_\xi \|^2 \int_{\mathbb{R}^N} z_1^2 \psi_1(|z|) \frac{U'(|z|)}{|z|} \, dz
\]

\[
+ \frac{1}{N} \text{Scal}_g(\xi)^2 \int_{\mathbb{R}^N} z_1^2 \psi_2(|z|) \frac{U'(|z|)}{|z|} \, dz \right) + \frac{1}{9} c_1 \left( \frac{2}{N + 2} c_3 \| \text{Ric}_\xi \|^2 + \frac{1}{N} c_4 \text{Scal}_g(\xi)^2 \right)
\]

where
\[
c_3 := \frac{1}{c_1} \int_{\mathbb{R}^N} z_1^2 \psi_1(|z|) \frac{U'(|z|)}{|z|} \, dz \quad c_4 := \frac{1}{c_1} \int_{\mathbb{R}^N} z_1^2 \psi_2(|z|) \frac{U'(|z|)}{|z|} \, dz \quad (71)
\]

are two constants depending only on the dimension \( N \). Putting together this and (56), we deduce that
\[
J_\varepsilon (W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) = c_\Theta \varepsilon^2 \Theta_1(\xi) + \varepsilon^4 \tilde{\Theta}_2(\xi) + O(\varepsilon^4), \quad (72)
\]

where \( \Theta_1 = -c_1 \text{Scal}_g(\xi) \) with \( c_1 = \frac{1}{6} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^2 \, dz \) and where
\[
\tilde{\Theta}_2(\xi) = \Theta_2(\xi) - \frac{1}{9} c_1 \left( \frac{1}{N + 2} c_3 \| \text{Ric}_\xi \|^2 + \frac{1}{2N} c_4 \text{Scal}_g(\xi)^2 \right)
\]

\[
= \frac{1}{120(N + 2)} c_1 \left( -18 \Delta \text{Scal}_g(\xi) + 8 \| \text{Ric}_\xi \|^2 + 5 \text{Scal}_g(\xi)^2 - 3 \| R_\xi \|^2 \right)
\]

\[
- \frac{1}{9} c_1 \left( \frac{1}{N + 2} c_3 \| \text{Ric}_\xi \|^2 + \frac{1}{2N} c_4 \text{Scal}_g(\xi)^2 \right). \quad (73)
\]

Now reasoning as above, we can compute
\[
\frac{\partial}{\partial y_\ell} J_\varepsilon (W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) - \frac{\partial}{\partial y_\ell} J_\varepsilon (W_{\varepsilon, \xi})
\]

\[
= J_\varepsilon' (W_{\varepsilon, \xi} + \varepsilon^2 V_{\varepsilon, \xi}) \left[ \frac{\partial W_{\varepsilon, \xi}}{\partial y_\ell} + \varepsilon^2 \frac{\partial V_{\varepsilon, \xi}}{\partial y_\ell} \right] - J_\varepsilon' (W_{\varepsilon, \xi}) \left[ \frac{\partial W_{\varepsilon, \xi}}{\partial y_\ell} \right]
\]

\[
= J_\varepsilon' (W_{\varepsilon, \xi}) \left[ \frac{\partial W_{\varepsilon, \xi}}{\partial y_\ell} + \varepsilon^2 \frac{\partial V_{\varepsilon, \xi}}{\partial y_\ell} \right] + J_\varepsilon'' (W_{\varepsilon, \xi}) \left[ \varepsilon^2 V_{\varepsilon, \xi} \cdot \frac{\partial W_{\varepsilon, \xi}}{\partial y_\ell} + \varepsilon^2 \frac{\partial V_{\varepsilon, \xi}}{\partial y_\ell} \right]
\]

\[
+ O(\varepsilon^4 |V_{\varepsilon, \xi}|^2) - J_\varepsilon' (W_{\varepsilon, \xi}) \left[ \frac{\partial W_{\varepsilon, \xi}}{\partial y_\ell} \right] \quad (74)
\]

\[
= \int_{\mathcal{M}_g} \left( - \Delta_g W_{\varepsilon, \xi} + W_{\varepsilon, \xi} - |W_{\varepsilon, \xi}|^{p-1} \right) \frac{\partial V_{\varepsilon, \xi}}{\partial y_\ell} \, d\mu_g
\]
\[+ \int_{M_{\varepsilon}} \left( - \Delta g V_{\varepsilon, \xi} + V_{\varepsilon, \xi} - (p-1)|W_{\varepsilon, \xi}|^{p-2} V_{\varepsilon, \xi} \right) \left[ \frac{\partial W_{\varepsilon, \xi}}{\partial y_h} + \frac{\partial V_{\varepsilon, \xi}}{\partial y_h} \right] d\mu_g + O(\varepsilon^4|V_{\varepsilon, \xi}|^2) = O(\varepsilon^4)\]

since \(V_{\varepsilon, \xi}\) satisfies (19) while the main term in

\[\int_{M_{\varepsilon}} \left( - \Delta g V_{\varepsilon, \xi} + V_{\varepsilon, \xi} - (p-1)|W_{\varepsilon, \xi}|^{p-2} V_{\varepsilon, \xi} \right) \left[ \frac{\partial W_{\varepsilon, \xi}}{\partial y_h} + \frac{\partial V_{\varepsilon, \xi}}{\partial y_h} \right] d\mu_g\]

is equals zero since \(\frac{\partial W_{\varepsilon, \xi}}{\partial y_h}\) is in the kernel of the linearised operator.

Using the above computations and the fact that the scalar curvature of \((M, g)\) is constant, we get the desired result.

Next, we prove the following two key lemmas.

**Lemma 6.1.** For any \(k = 1, 2, \ldots, N\), we have

\[\frac{1}{2} \int_{\mathbb{R}^N} |\nabla U|^2 z_k^2 dz + \frac{1}{2} \int_{\mathbb{R}^N} U^2 z_k^2 dz - \frac{1}{p} \int_{\mathbb{R}^N} U^p z_k^2 dz = \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_k^2 dz.\]

**Proof.** Multiply both sides of the equation \(-\Delta U + U = U^{p-1}\) in \(\mathbb{R}^N\) by \(\frac{1}{3} \frac{\partial u}{\partial x_k}\) and then integrate by parts over \(\mathbb{R}^N\).

**Lemma 6.2.** It holds that

\[\alpha := \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^2 dz = 3 \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^2 z_2^2 dz = \frac{3}{N+2} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^2 dz,\]

\[\int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^2 z_2^2 z_3^2 dz = \frac{1}{N(N+2)} \int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^2 dz,\]

and

\[\int_{\mathbb{R}^N} \left( \frac{U'(|z|)}{|z|} \right)^2 z_i z_j z_k z_l dz = \frac{\alpha}{3} \left( \delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl} \right).\]

**Proof.** The result follows from the fact that

\[\int_{S^{N-1}} z_1^4 = 3 \int_{S^{N-1}} z_1^2 z_2^2 = \frac{3}{N+2} \int_{S^{N-1}} z_1^2,\]

\[\int_{S^{N-1}} z_1^2 z_2^2 z_3^2 = \frac{1}{N(N+2)} \int_{S^{N-1}} z_1^2,\]

and

\[\int_{S^{N-1}} z_i z_j z_k z_l = \frac{1}{3} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \int_{S^{N-1}} z_1^2.\]

This proves the lemma.

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E-mail address: shbdeng@swu.edu.cn
E-mail address: ziedkhemiri314@yahoo.fr
E-mail address: fmahmoudi@dim.uchile.cl