Non–orientable string one–loop corrections in the presence of a B field

L. Bonora, M. Salizzoni

International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2–4, 34014 Trieste, Italy, and INFN, Sezione di Trieste
E-mail: bonora@sissa.it, sali@sissa.it

ABSTRACT: We discuss the problem of noncommutative $SO(N)$ gauge field theories from the string one–loop point of view. To this end we propose an expression for the string propagator on the boundary of the Möbius strip in the presence of a constant $B$ field. We discuss in detail the problems related to its derivation. Then we use it to compute the one–loop corrections to two–, three– and four–gluon amplitudes in an open string theory with orthogonal Chan–Paton factors. We show that these corrections in the field theory limit in 4D are compatible with the one–loop corrections of a renormalizable noncommutative $SO(N)$ gauge field theory.
1. Introduction

An interesting problem that has been raised in connection with the recent attention on non-commutative field theories as effective field theories of open strings attached to D–branes in the presence of a constant B field, is the existence of noncommutative gauge theories with gauge transformations valued in a Lie subalgebra of $u(N)$. There are several reasons why the existence of at least some of them is expected and desirable. From an abstract point of view, [1], there should not be any obstruction to constructing a noncommutative gauge field theory with any Lie algebra (even though this may not imply that these theories are effective field theories of the strings). On the other hand we know that noncommutative field theories retain certain features of string theory better than ordinary theories do, [2-31]. We have in mind here the ultraviolet convergence properties of noncommutative theories but, even more, the possibility of having soliton solutions in situations where ordinary theories are unfit to support them, [32]. This is particularly important in connection with tachyon condensation. In this regard, another important property is the possibility of embedding the Moyal product into the star product of open string field theory in a factorized way,
It would be rather disappointing if such remarkable properties could not be extended, for example, to string theories or string field theories with orthogonal Chan-Paton factors.

Recently there have been a few attempts at defining and studying noncommutative versions of gauge field theories with orthogonal and symplectic, \cite{34,35}, or even more general Lie algebras, \cite{36}. These noncommutative theories have been defined at the semiclassical (tree) level. As soon as one tries to go beyond the tree level one has to face an unexpected result: in four dimensions they look (at least naively) nonrenormalizable. One is tempted to dismiss this fact as a non–problem. After all, these are effective field theories, which are nonlocal as ordinary theories. However the right question we should ask is whether this corresponds to some feature (perhaps ill–definiteness) of the string theory the gauge field theory is supposed to represent in the low energy limit. To know the answer we have to study the one–loop corrections of the relevant string theory. This is what we want to do in this paper for an unoriented open string theory with orthogonal Chan–Paton factors in the presence of a background $B$ field. We would like to specify from the very beginning that our approach is not unproblematic, and we will list below the aspects of our treatment that may appear controversial.

In order to compute one–loop corrections one needs the string Green functions on the relevant world–sheets, which are the annulus and the Möbius strip. While the Green function for the former case in the presence of a $B$ field is well–known, the latter case has not been studied yet. For our purposes we need the propagator on the boundary of the strip, but the presence of the $B$ field requires the knowledge of the propagator on the whole Möbius strip in order for us to be able to take an unambiguous limit for the propagator on the boundary. Extending the propagator outside the boundary of the Möbius strip is a non–trivial operation, even in the ordinary case ($B = 0$). For this reason we devote section 2 to the construction of the string propagator on the Möbius strip in the ordinary case, a subject which does not seem to have been carefully analyzed in the literature. In section 3 we use this propagator in order to compute one–loop 2–, 3– and 4–point gluon amplitudes and their field theory limit in the absence of a $B$ field.

Subsequently we turn on a constant $B$ field. In section 4 we compute the string propagator on the Möbius strip, we discuss the problems raised by this calculation and finally we find the expression of the propagator on the boundary, which is what we actually need since we intend to compute amplitudes of vertex operators inserted at the boundary of the world–sheet. With this tool it is then elementary, in section 5, to extend the one–loop results of section 3 to the case of a constant $B$ field. On the basis of the results obtained in section 5 we would be led to conclude that the string one–loop corrections entails that the noncommutative limiting field theory corrections (in 4D) are those of a one–loop renormalizable field theory with the same renormalization constants as the corresponding ordinary $SO(N)$ gauge field theory. However, apparently, this does not correspond to what one gets from one–loop corrections in the corresponding noncommutative $SO(N)$ gauge field theory. In section 6 this issue is discussed at length.

Finally, as promised above, we would like to list the most problematic aspects of our paper. These are: (i) the possibility of a non–vanishing $B$ field in the string context considered here, (ii) the continuation to the bulk of the Möbius strip of our string propagator,
(iii) the one–loop non–renormalizability of the noncommutative $SO(N)$ gauge theories. In the course of the paper we argue that all these aspects may not be unsurmountable obstacles. In any case we do not see decisive arguments in favor of the contrary. Altogether we believe that the approach presented here, although not accompanied by uncontroversial arguments, represents nevertheless a concrete possibility.

2. The string propagator on a non-orientable world-sheet. Case $B = 0$

One–loop contributions in open unoriented string theory come from the annulus and the Möbius strip world–sheet. Henceforth for conciseness we denote a Möbius band by $\mathcal{M}$ and an annulus by $\mathcal{A}$. As for the parametrization of the latter we will use the notation of [11]. The annulus will be represented either in the $z$–plane or in the $\rho$–plane. In the first case the annulus is represented in the most obvious way as the region $q \leq |z| \leq 1$, where $q$ is the modulus. In the $\rho$–plane the annulus will be identified with the region $w \leq |ho| \leq 1$ of the lower half plane with the lower and upper semicircle identified in such a way as to preserve the orientation of the surface (the two semicircles are ‘parallel’). The map between the two representations is given by:

$$z = e^{2\pi i \ln w}, \quad \ln q = \frac{2\pi^2}{\ln w}$$

Alternatively the modulus is represented by the imaginary number $\tilde{\tau}$ defined by:

$$q = e^{i\pi \tilde{\tau}}, \quad \tilde{\tau} = -\frac{2\pi i}{\ln w}$$

It is convenient to perform the modular transformation $\tilde{\tau} \rightarrow -1/\tilde{\tau}$. After this operation, following [12], we will parametrize the above variables as follows

$$w = e^{-2\tau}, \quad \rho = e^{-2\nu}$$

where $\tau = -i\pi \tilde{\tau}$.

The representation of the Möbius band is the same except that the upper semicircle in the $\rho$–plane is identified with the lower one in an antiparallel way (see figure 1). The field theory limit corresponds to an infinitely thin annulus or band, i.e. $q \rightarrow 1$, which corresponds to $w \rightarrow 0$ or $\tau \rightarrow \infty$.

Our purpose in this paper is to compute amplitudes involving several gluon vertices inserted at the boundary of the annulus $\mathcal{A}$ or of the Möbius strip $\mathcal{M}$. To this end we need to know the string propagator on both surfaces. The string propagator in the annulus, in the presence of a $B$ field, was calculated long ago in [40] and elaborated on in [20].

As for the string propagator on the boundary of $\mathcal{M}$, in the absence of $B$, it can be found, for example, in [41]. However, as explained in the introduction, when in presence of a $B$ field one needs to know the propagator in the bulk of $\mathcal{M}$ in order to be able to take the correct limit to the boundary. In view of this it is a good propedeutical exercise to find the string propagator on the bulk of $\mathcal{M}$ without $B$ field. This exercise does not seem to have been done previously in the literature.
2.1 The Green function method

We start from the sigma model action of open strings attached to a D–brane

\[
\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2x \left( \sqrt{h} \alpha^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j g_{ij} \right)
\]

(2.2)

where \(g_{ij}\) is the (closed) string metric. Our problem is to find, on the surface \(\Sigma\) of interest (either \(A\) or \(M\)), a solution (the Neumann function) of the equation

\[
\nabla^2 G^{ij}(x, x') = 2\pi\alpha' g^{ij} \delta(x - x')
\]

(2.3)

satisfying the boundary conditions

\[
\partial_\perp G^{ij}(x, x') \bigg|_{\partial\Sigma} = 0
\]

(2.4)

In these equations \(x\) stands for either \(z\) or \(\rho\) (see Appendix A for some auxiliary formulas).

However with the (2.4) boundary conditions it would be impossible to satisfy Gauss’s theorem. Therefore we modify them so that in \(\rho, \bar{\rho}\) coordinates the above equations become

\[
4\partial_\rho \partial_{\bar{\rho}} G^{ij}(\rho, \rho') = 2\pi\alpha' g^{ij} \delta(\rho - \rho')
\]

(2.5)

and

\[
\partial_\theta G^{ij}(\rho, \rho') \bigg|_{\partial\Sigma} = Kg^{ij}
\]

(2.6)

The constant \(K\) will be determined below. The boundary \(\partial\Sigma\) corresponds to real \(\rho\) with \(w \leq |\rho| \leq 1\).

Following the notation of \([10, 20]\) we write the solution for the Möbius strip as follows:

\[
\frac{1}{\alpha'} G^{ij}_{\mathcal{M}}(\rho, \rho') = g^{ij} \left( I^\mathcal{M}(\rho, \rho') + J^\mathcal{M}(\rho, \rho') + f^\mathcal{M}(\rho, \rho') \right)
\]

(2.7)
where

\[ I_M(\rho, \rho') = \ln \left( \frac{|\tilde{\tau}|}{2} \right) + \frac{(\ln \rho'')^2 + (\ln \rho'')^2}{4 \ln w} + \ln \left| \frac{\rho}{\rho'} - \frac{\rho'}{\rho} \right| \]

\[ + \ln \prod_{n=1}^{\infty} \frac{1 - (-w)^n \frac{\rho}{\rho'}}{(1 - (-w)^n)^2} \]

\[ J_M(\rho, \rho') = \ln \left( \frac{|\tilde{\tau}|}{2} \right) + \frac{(\ln \rho''')^2 + (\ln \rho''')^2}{4 \ln w} + \ln \left| \frac{\rho}{\rho'} - \frac{\rho'}{\rho} \right| \]

\[ + \ln \prod_{n=1}^{\infty} \frac{1 - (-w)^n \frac{\rho}{\rho'}}{(1 - (-w)^n)^2} \]

\[ f_M(\rho, \rho') = -\frac{i\pi}{2\ln w} \ln \frac{\rho \rho'}{\rho \rho'}. \]

There is a subtlety in the above definition: the log square terms must be understood as

\[ (\ln \rho \rho')^2 = \frac{1}{4} (\ln (\frac{\rho}{\rho'})^2)^2 \]

and so on.

Notice that \( G_{ijM}(\rho, \rho') = G_{ijM}(\rho', \rho) \). It is now quite a standard matter to verify that eqs. (2.5) and (2.6) are satisfied, with \( K \) being

\[ K = \frac{2\pi}{\ln w} \]

(2.11)

It is also easy to verify that the continuity conditions on the boundary of the Möbius band are satisfied:

\[ G_{ijM}(1, \rho') = G_{ijM}(-w, \rho'), \quad G_{ijM}(-1, \rho') = G_{ijM}(w, \rho'), \quad \forall \rho' \]

However one should notice that the propagator is not everywhere continuous. In fact, while it is easy to verify that

\[(I_M + J_M)(-w\rho, \rho') = (I_M + J_M)(\rho, \rho') \]

(2.12)

so that the combination \((I_M + J_M)\) satisfies everywhere the periodicity conditions for the Möbius strip, a similar identity is not satisfied by \( f_M \), for which we have

\[ f_M(e^{-i\theta}, \rho') - f_M(-we^{i\theta}, \rho') = \ln(e^{-4i\theta}) \]

(2.13)

so that there is a line of discontinuity that does not permit an exact matching along the arcs \( A'B' \) and \( B'A \). One simple solution would be to drop \( f_M \) in the definition (2.7). But in this case we would not satisfy Gauss’s theorem, see below. On the other hand \( f_M \) does not contribute to the propagator on the boundary, thus we prefer to keep \( f_M \) in the definition (2.7) (for further comments on this point see below and section 4).
It is now time to discuss Gauss’s theorem, i.e. the integrated version of (2.5), which says that the integral of \( \partial \perp G_M \) along the boundary \( \partial M \) equals the integral of the RHS of (2.5). The latter equals \( 2\pi \alpha' \). As for the former one has to integrate over the boundary \( AA' \) and \( BB' \) of \( M \), but, due to the above mentioned discontinuity, also along the arcs \( A'B \) and \( B'A \). Now the normal derivative of \( G_M \) along the arcs \( A'B \) and \( B'A \) vanishes as well as the normal derivative of \( f_M + f_M \) along \( AA' \) and \( BB' \). The only non–vanishing contribution comes from the normal derivative of \( f_M \) along the boundary of \( M \). This is given by (see Appendix A for notation)

\[
\int_{A'} \partial_{\perp} f + \int_{B'} \partial_{\perp} f = 2\pi
\]

and we have used the fact that \((\rho \partial - \bar{\rho} \partial_{\bar{\rho}}) \ln \frac{\rho}{\bar{\rho}} = 2\pi \alpha' \). Therefore Gauss’s theorem is verified.

However so far we have been somewhat cavalier in integrating over \( M \). The point is that the Möbius strip is a nonorientable surface and integration theory on nonorientable surfaces takes on a peculiar twist. The reason is that on nonorientable manifolds only densities can be integrated, see [39]. A density is an expression that, under a coordinate change, gets multiplied by the inverse modulus of the Jacobian of the partial derivatives (not just by the inverse Jacobian). When we integrate both sides of (2.5) we are precisely integrating two densities. However, for this reason, the measure on \( M \) cannot induce an oriented measure on the boundary. The consequence is that the relative signs on the two sides of Gauss’s theorem remains undetermined. This question is purely technical and can be settled by means of a physical analog: consider an electrostatic analog in which \( G \) is proportional to the electrostatic potential generated by a charge placed in \( \rho' \). The integral of the normal derivative (electric field) along the boundary equals the total charge. Therefore we know how to fix the relative sign in Gauss’s theorem. It has also to be remarked that the normal derivative across the line \( A'B \) and \( B'A \) is continuous, notwithstanding the discontinuity mentioned above.

In the following we will need the propagator on the boundary of \( M \). This is obtained by taking the limit for \( \bar{\rho} \) and \( \bar{\rho}' \) approaching the real axis: \( G_{ij}^{ij} \to G_M^{ij} \). We get

\[
G_M^{ij}(\rho, \rho') = 2\alpha' \gamma^{ij} G_M(\rho, \rho')
\]

where \( G_M(\rho, \rho') \) is given by

\[
G_M^{+}(\rho, \rho') = \ln \left( -\frac{\pi}{2 \ln w} \right) + \frac{(\ln \frac{\rho}{\rho'})^2}{2 \ln w} + \ln \left| \sqrt{\frac{\rho}{\rho'}} - \sqrt{\frac{\rho'}{\rho}} \right| + \ln \prod_{n=1}^{\infty} \frac{1 - \left( -w \right)^n \rho' \rho}{1 - \left( -w \right)^n}, \quad \text{if } \rho \rho' > 0
\]

\[
G_M^{-}(\rho, \rho') = \ln \left( -\frac{\pi}{2 \ln w} \right) + \frac{(\ln \frac{\rho}{\rho'})^2}{2 \ln w} + \ln \left| \sqrt{\frac{\rho'}{\rho}} + \sqrt{\frac{\rho}{\rho'}} \right| + \ln \prod_{n=1}^{\infty} \frac{1 + \left( -w \right)^n \rho' \rho}{1 - \left( -w \right)^n}, \quad \text{if } \rho \rho' < 0
\]
This is the propagator we will use for our calculations in the following section.

Finally we notice that by replacing \((-w)^n\) with \(w^n\) in (2.7) we get the Green function for the annulus, from which one can extract the planar and nonplanar propagators.

3. Field theory limit of gluon amplitudes without \(B\) field

We wish to calculate string theory amplitudes and to extract from them information concerning the low energy effective field theory. In particular we are interested in the renormalization properties (in 4D) of the latter. For this reason in this section we intend to compute two-, three- and four-gluon one-loop amplitudes from string theory with \(SO(N)\) CP factors and evaluate their field theory limit, more specifically the UV divergent contributions of the various amplitudes, in order to compare them with the field theory ones. While this has been done in detail in theories with unitary CP factors, \([42]\), to our best knowledge nothing similar has been done for theories with orthogonal or symplectic CP factors. Therefore working out the field theory limit in the latter case without \(B\) field is a necessary preparation to the next section and a calculation interesting in itself. The novelty in this case is that, beside the annulus amplitudes, one has to consider also the Möbius strip ones.

The method we adopt here was developed over the years by several people, see \([42]\) and references therein. It is based on calculations carried out in the framework of the bosonic string theory. Indeed it is enough to embed the gauge field theory we want to regularize in the bosonic string theory. It is not even necessary that the string theory be critical. As a regulator of a field theory a bosonic string theory in generic dimensions will do. For these and other considerations on the method used here, we refer to \([42]\).

We start by writing down the tree level gluon amplitudes with CP factors belonging to the Lie algebra \(SO(N)\) at the lowest order in \(\alpha'\).

\[
A^{(0)}(p_1, p_2) = 2i \text{tr}(t^{a_1}t^{a_2}) \epsilon_1 \cdot \epsilon_2 p_1 \cdot p_2 \tag{3.1}
\]

\[
A^{(0)}(p_1, p_2, p_3) = 4g_D \text{tr}(t^{a_1}t^{a_2}t^{a_3}) \left( \epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3 + \epsilon_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1 + \epsilon_3 \cdot \epsilon_1 p_1 \cdot \epsilon_2 \right) \tag{3.2}
\]

\[
A^{(0)}(p_1, p_2, p_3, p_4) = 4ig_D^2 \text{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}) \left( \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 \frac{p_1 \cdot p_3}{p_1 \cdot p_2} \right. \\
+ \left. \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 \frac{p_1 \cdot p_3}{p_2 \cdot p_3} \right) \tag{3.3}
\]

To give a meaning to eq.(3.1) it is useful to introduce a small mass for the gluon: \(p_1^2 = m^2\) (which is anyhow necessary as an IR cutoff, although we will not need it explicitly in the following). The above amplitudes have been normalized in such a way as to coincide with the corresponding tree level amplitudes in field theory. In particular, \(g_D\) is the D-dimensional gauge coupling, the \(t^{a}\)'s are the generators of \(SO(N)\) in the fundamental representation, the \(\epsilon_i\)'s are gluon polarizations and \(p \cdot q = p_i \hat{g}^{ij} q_j\). Later on we will use the above formulas for \(D = 4\). In that case \(g_D = g_4 \equiv g\). We recall that (3.3) contains, in field theory terms, also one–particle reducible contributions.
We write down now general form of the one–loop amplitudes (which, for later reference, is valid in general, also when a $B$ field is switched on):

\[
A^{(1)}(p_1, \ldots, p_M) = \frac{1}{2} \chi_M f^{a_1, a_2, \ldots, a_M}_N \frac{g_D^M}{(4\pi)^2} (2\alpha')^{-\frac{D}{2}} \int \prod_{r=2}^M dv_\nu d\tau e^{2\tau^* - \frac{D}{2}}
\]

\[
\times \prod_{n=1}^\infty \left( 1 - \eta_n e^{-2n\tau} \right)^{2-D} \exp \left[ \sum_{r < s} p_r G(\nu_{rs}) p_s \right]
\]

\[
\times \exp \left[ \sum_{r \neq s} \left( p_s \partial_r G(\nu_{sr}) \epsilon_r + \frac{1}{2} \epsilon_r \partial_r p_s \partial_s G(\nu_{sr}) \epsilon_s \right) \right]_{\text{m.l.}}
\]

where $\chi_M = i$ (1) for $M$ even (odd). $f^{a_1, a_2, \ldots, a_M}_N$ is the group theory factor. It equals $N \text{tr}(t^{a_1} \ldots t^{a_N})$ in the annulus case for planar amplitudes and $\text{tr}(t^{a_1} \ldots t^{a_N})$ in the M"obius strip case. Moreover $pGq$ stands for $p_i G_{ij} q_j$, $\nu_{rs} = \nu_r - \nu_s$ and $\partial_r = \frac{\partial}{\partial \nu_r}$. The factor $\eta_n = 1$ in the orientable case, $= (-1)^n$ in the non-orientable case. The suffix m.l. stands for multilinear, meaning that in the series expansion of the exponential we keep only the terms that are linear in each polarization. The propagator $G$ is either the annulus or the M"obius strip propagator, and the integrals over the $\nu$ variables are evaluated in the appropriate regions of integration (moduli space).

The constants in front of the tree and one–loop amplitudes have been defined in such a way as to agree in the zero slope limit with the corresponding field theory results.

The strategy now consists in replacing in eq.(3.4) the appropriate propagators and singling out the regions of the moduli space which give rise to divergent contributions in the $\alpha' \to 0$ limit. This will be done explicitly below for the M"obius amplitudes. As for the annulus amplitudes, since their evaluation does not depend on the CP factors, we can borrow for them the analysis already carried out in [42] and [20, 22] in the case of unitary CP factors. These amplitudes split in general into planar and non–planar contributions. As for the latter we can rely on the results of [20], which, as expected, tells us that they do not give rise to UV divergences in the field theory limit. The planar amplitudes do give rise to divergent contributions in the field theory limit. They have been analyzed in detail in [42]: assuming dimensional regularization in the $\alpha' \to 0$ limit, they reproduce exactly the results obtained in field theory with the background field method [43]. More precisely, one can single out the divergent part that corresponds in field theory to one–particle irreducible diagrams. The result can be written

\[
A^{(1)}(p_1, \ldots) \big|_{\text{div}} = -\frac{N}{2} \frac{g^2}{(4\pi)^2} \frac{11}{3} \epsilon A^{(0)}(p_1, \ldots) \tag{3.4}
\]

for two–, three– and four–point functions, with $\epsilon = 2 - D/2$. Throughout the paper the label $\text{div}$ stands for irreducible divergent part, in the sense that in field theory these divergences correspond to one-particle irreducible diagrams. It is also possible to extract from string theory the one–loop one–particle reducible contributions but here we will not

\[\text{A more careful statement is needed when a constant } B \text{ field is present because of the UV/IR mixing. However in this paper we will not deal with this problem.}\]
be concerned with them. In the remaining part of this section we will show how to extract relations similar to (3.4) for the Möbius amplitudes. Following [42], we will use two different methods. Since these two methods have already been carefully spelt out in [42] for the annulus amplitudes, we skip many details and focus on the peculiarities introduced by a non-orientable world-sheet.

3.1 Möbius amplitudes: first method

This method is based on the ‘doubling trick’, [11]. One can show that a large amount of information contained in a Möbius amplitude is captured by doubling the integration region. Let us start from the propagator along the boundary of $M$, written as follows:

$$G_M(\rho, \rho') = \ln \left( \frac{1 - c}{\sqrt{c}} \exp \left( \frac{\ln^2 c}{2 \ln w} \right) \prod_{n=1}^{\infty} \frac{(1 - (-w)^n c)(1 - (-w)^n c)}{(1 - (-w)^n)^2} \right),$$

where $c = \rho/\rho'$. This coincides with (2.16) provided $c \leq 1$. Following [41], it can be recast in the form:

$$G_M(\nu - \nu') = \ln \left( \frac{4\pi}{\ln q} \sin \left( \frac{\pi(\nu - \nu')}{2} \right) \prod_{n=1}^{\infty} \frac{1 - 2(-\sqrt{q})^n \cos(\pi(\nu - \nu')) + q^n}{(1 - 2(-\sqrt{q})^n)^2} \right), \quad (3.5)$$

where $q = \exp[-\pi^2]$ and $\nu - \nu' = -\frac{1}{2} \ln c$. The form (3.5) of the Green function is periodic in the insertion coordinates $\nu$'s with a period double (4 instead of 2) with respect to the annulus case: this is because the boundary of the Möbius strip can be viewed as having double length with respect to one of the two boundaries in the annulus. For our purposes we will need another form of $G_M(\nu)$, first proposed by Fradkin and Tseytlin [44]. Using

$$\ln[1 + b^2 - 2b \cos x] = -2 \sum_{n=1}^{\infty} \frac{b^n}{n} \cos nx$$

and

$$\sum_{n=1}^{\infty} b^n = \frac{b}{1 - b},$$

we obtain

$$G_M(\nu - \nu') = -\sum_{n=1}^{\infty} \frac{1}{n} \cos \left( \frac{\pi n(\nu - \nu')}{\tau} \right) \left[ 1 + \frac{(-\sqrt{q})^n}{1 - (-\sqrt{q})^n} \right], \quad (3.6)$$

where we have used the regularization $\sum_{n=1}^{\infty} 1 = -\frac{1}{2}$ and we have neglected the terms that do not depend on $\nu$. The effect of this regularization is that no negative powers of $\alpha'$ are generated in the integration over the variables $\nu$'s and $\tau$. [42]. In this way we can replace the exponentials of the Green function simply by an infrared cutoff and extract from the amplitude only the terms proportional to $(\alpha')^{2-D/2}$. Keeping this fact in mind we rewrite the amplitude (3.4) as

$$A^{(1)}(p_1, \ldots, p_M) = \frac{1}{2} \chi_M f_N^{a_1a_2\ldots a_M} \frac{g_M^D}{(4\pi)^{D/2}} (2\alpha')^{2-D/2} \int_0^{\infty} D^M \tau I^{(1)}_M(\tau) \quad (3.7)$$
where

\[
I^{(1)}_{\mathcal{M}}(\tau) = (2\alpha')^{-2} \int_0^\tau d\nu_M \int_0^{\nu_M} d\nu_{M-1} \cdots \int_0^{\nu_2} d\nu_2 \\
\times \exp \left[ \sum_{r<s} p_r G_{\mathcal{M}}(\nu_{rs}) p_s \right] \\
\times \exp \left[ \sum_{r\neq s} \left( p_r \partial_s G_{\mathcal{M}}(\nu_{sr}) \epsilon_s + \frac{1}{2} \epsilon_r \partial_r \partial_s G_{\mathcal{M}}(\nu_{sr}) \epsilon_s \right) \right]
\] (3.8)

and

\[
\mathcal{D}^{\mathcal{M}} = d\tau w^{-1} \tau^{-D/2} \prod_{n=1}^\infty (1 - (-w)^n)^{2-D}
\] (3.9)

Going to the variables \( \hat{\nu} = \nu/\tau \) it is easier to implement the non-orientability of the Möbius band. We noticed above that the Green function \( G_{\mathcal{M}} \) has double period in \( \hat{\nu} \). The integration region must be chosen accordingly: the integration range is now \([0, 2]\) instead of \([0, 1]\), because we need to make two complete revolutions to go around the boundary back to the starting point.

For the two point function, after a partial integration with null boundary terms, we obtain

\[
I^{(1)}_{\mathcal{M}}(\tau) = (2\alpha')^{-2} \tau^{M-1} \int_0^2 d\hat{\nu}_M \int_0^{\hat{\nu}_M} d\hat{\nu}_{M-1} \cdots \int_0^{\hat{\nu}_2} d\hat{\nu}_2 \\
\times \exp \left[ \sum_{r<s} p_r G_{\mathcal{M}}(\hat{\nu}_{rs}) p_s \right] \\
\times \exp \left[ \sum_{r\neq s} \left( p_r \frac{1}{\tau} \partial_s G_{\mathcal{M}}(\hat{\nu}_{sr}) \epsilon_s + \frac{1}{2} \frac{1}{\tau^2} \epsilon_r \partial_r \partial_s G_{\mathcal{M}}(\hat{\nu}_{sr}) \epsilon_s \right) \right]
\] (3.10)

The partial integration has yielded the appropriate powers of \( \alpha' \), so we can disregard the exponentials of the Green functions, and perform the \( \hat{\nu} \) integration with the help of the formula

\[
\int_0^2 dx \sin(\pi n x) \sin(\pi m x) = \delta_{nm}
\] (3.13)

and we are left with

\[
I^{(1)}_2 = \epsilon_1 \cdot \epsilon_2 p_1 \cdot p_2 \tau \int_0^2 d\hat{\nu} \left( \frac{1}{\tau} \partial G_{\mathcal{M}}(\hat{\nu}) \right)^2 e^{2\alpha' p_1 \cdot p_2 G_{\mathcal{M}}(\hat{\nu})} \\
= \epsilon_1 \cdot \epsilon_2 p_1 \cdot p_2 \int_0^2 \left[ \sum_{m=1}^\infty \frac{\pi}{\tau} \sin(\pi m \hat{\nu}) \frac{1 + (-\sqrt{q})^m}{1 - (-\sqrt{q})^m} \right] \\
\times \left[ \sum_{n=1}^\infty \frac{\pi}{\tau} \sin(\pi n \hat{\nu}) \frac{1 + (-\sqrt{q})^n}{1 - (-\sqrt{q})^n} \right]
\] (3.12)
Since the integration over $\tau$ will be shared by the 3– and 4–point functions, let us define

$$Z_M = \pi^2 \int_0^{\infty} \frac{D^M \tau}{\tau} \sum_{m=1}^\infty \left( \frac{1 + (\sqrt{q})^m}{1 - (\sqrt{q})^m} \right)^2$$

(3.15)

The sum present in $Z_M$ can be rewritten as

$$\sum_{n=1}^\infty \left( \frac{1 + (\sqrt{q})^m}{1 - (\sqrt{q})^m} \right)^2 = -4(-\sqrt{q}) \frac{d}{d(-\sqrt{q})} \ln \left[ (-\sqrt{q})^{1/8} \prod_{n=1}^{\infty} (1 - (\sqrt{q})^n) \right]$$

(3.16)

then, using the relation (8.A.27) of [1], we can go to the $k$ representation which is more suitable for the field theory limit $\tau = 0$

$$f(-\sqrt{q}) = \prod_{n=1}^{\infty} \left( 1 - (\sqrt{q})^n \right)$$

$$= w^{1/24} q^{-1/48} \left( \frac{-\ln w}{\pi} \right)^{1/2} f(-w)$$

$$= w^{1/24} q^{-1/48} \left( \frac{-\ln w}{\pi} \right)^{1/2} \prod_{n=1}^{\infty} \left( 1 - (-w)^n \right).$$

(3.17)

and find the following expression for $Z_M$

$$Z_M = \pi^2 \int_0^{\infty} \frac{D^M \tau}{\tau} \frac{d}{d(-\sqrt{q})} \ln \left[ (-\sqrt{q})^{1/8} \left( \frac{-\ln w}{\pi} \right) w^{1/24} \cdot q^{-1/48} f(-w) \right]$$

$$= 4 \int_0^{\infty} \frac{D^M \tau}{\tau} w(\ln w)^2 \left[ \frac{\pi^2}{12 w(\ln k)^2} + \frac{1}{2w \ln w} + \frac{1}{24} \sum_{n=1}^{\infty} \frac{n(-w)^{n-1}}{(1 - (-w)^n)} \right]$$

Now we expand the partition function present in $D^M \tau$ in powers of $w = e^{-\tau}$, and keep only the power $\tau^{1-D/2}$, that is the only one that gives rise to divergences in the dimensional regularization

$$Z_M = \frac{2}{3} (26 - D) \int_0^{\infty} d\tau \tau^{1-D/2} e^{-2\alpha_m \tau^2} = \frac{2}{3} (26 - D) \Gamma \left( 2 - \frac{D}{2} \right) (2\alpha_m^2)^{D/2-2}$$

After setting $\epsilon = 2 - D/2$, we obtain for the two point function

$$A^{(1)}_{M}(p_1, p_2) \bigg|_{\text{div}} = \frac{i}{2} \tr(t^{a_1} t^{a_2}) \frac{g_D^2}{(4\pi)^{D/2}} (2\alpha')^{2-D/2} (2\alpha_m^2)^{D/2-2}$$

$$\times \epsilon_1 \cdot \epsilon_2 p_1 \cdot p_2 \frac{2}{3} (26 - D) \Gamma \left( 2 - \frac{D}{2} \right)$$

$$= \frac{g^2}{(4\pi)^2} \frac{11}{3} \epsilon \cdot A^{(0)}_{2}(p_1, p_2)$$

(3.18)

where $g \equiv g_4$. For three gluons we have

$$I_3^M = \frac{1}{\tau} \int_0^{\infty} d\tilde{\nu}_3 \int_0^{\infty} d\tilde{\nu}_2 \left\{ -\epsilon_1 \cdot \epsilon_2 \tilde{\nu}_3 \tilde{\nu}_2 \left[ \epsilon_1 \epsilon_2 \epsilon_3 \tilde{\nu}_3 \left[ \epsilon_3 \tilde{\nu}_3 \tilde{\nu}_3 \right] + \ldots \right\}$$
where the dots stand for the terms obtained by cyclic symmetry and for terms of higher order in \( \alpha' \). The power of \( \alpha' \) in the expression above is the correct one, without partial integration: also in this case we can neglect the exponentials, because they are irrelevant for ultraviolet divergencies. The integral over \( \tilde{\nu} \)'s coordinates is done again using the formula (3.13):

\[
I^M_3 = 2 \left( \epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3 + \epsilon_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1 + \epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 \right) \frac{\pi^2}{\tau} \sum_{n=1}^{\infty} \left( \frac{1 - (\sqrt{q})^n}{1 - (\sqrt{q})^n} \right)^2
\]

\[
A^{(1)}(p_1, p_2, p_3) \bigg|_{\text{div}} = \frac{1}{2} \text{tr} (t^{a_1} t^{a_2} t^{a_3}) \frac{g_D^2}{(4\pi)^{D/2}} (2\alpha')^{2-D/2}(2\alpha' m^2)^{D/2-2} \\
\times 2 \left( \epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3 + \epsilon_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1 + \epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 \right) \frac{2}{3} (26 - D) \Gamma \left( 2 - \frac{D}{2} \right)
\]

\[
= \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A^{(0)}(p_1, p_2, p_3)
\]

Finally for four gluons we have

\[
I^M_4 = \frac{1}{\tau} \int_0^1 d\tilde{\nu}_4 \int_0^{\tilde{\nu}_4} d\tilde{\nu}_3 \int_0^{\tilde{\nu}_3} d\tilde{\nu}_2 \left( \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 \tilde{\partial}_2^2 G(\tilde{\nu}_2) \tilde{\partial}_3^2 G(\tilde{\nu}_3) \right) \]

\[
+ \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 \tilde{\partial}_3^2 G(\tilde{\nu}_3) \tilde{\partial}_4^2 G(\tilde{\nu}_4) \\
+ \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_2 \tilde{\partial}_4^2 G(\tilde{\nu}_4) \tilde{\partial}_3^2 G(\tilde{\nu}_3) + \ldots
\]

Again we have the correct power of \( \alpha' \) without partial integration and we can discard the exponential; the dots denotes terms proportional to the external momenta that will play no role because they are not present in the IPI tree level diagrams.

\[
I^M_4 = 2 \left( -\frac{1}{2} \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \frac{1}{2} \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_2 \right) \frac{\pi^2}{\tau} \sum_{n=1}^{\infty} \left( \frac{1 + (\sqrt{q})^n}{1 - (\sqrt{q})^n} \right)^2
\]

Therefore

\[
A^{(1)}(p_1, p_2, p_3, p_4) \bigg|_{\text{div}} = \frac{i}{2} \text{tr} (t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \frac{g_D^2}{(4\pi)^{D/2}} (2\alpha')^{2-D/2}(2\alpha' m^2)^{D/2-2} \\
\times 2 \left( -\frac{1}{2} \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \frac{1}{2} \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_2 \right) \frac{2}{3} (26 - D) \Gamma \left( 2 - \frac{D}{2} \right)
\]

\[
= \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A^{(0)}(p_1, p_2, p_3, p_4)
\]

We can now summarize our results by collecting together the planar amplitudes (3.4) and the Möbius ones. The final result is

\[
A^{(1)}(p_1, \ldots) \bigg|_{\text{div}} = -\frac{N-2}{2} \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A^{(0)}(p_1, \ldots)
\]
3.2 Möbius amplitudes: second method

The second method is more laborious, but it has the advantage that one can single out more explicitly the regions of the moduli space corresponding to the different divergent contributions and thus provides a better understanding of the field theory limit. The \( \alpha' \to 0 \) limit corresponds to the parameters \( \tau \) and \( \nu \), going to infinity, or, more precisely, to \( \tau \to \infty \) and \( \nu = \frac{\nu}{\tau} \) finite. Therefore, on the basis of (3.4), we need the corresponding asymptotic expansion of \( G_M(\nu) \) and its derivatives. The latter is given, up to \( \mathcal{O}(e^{-4\tau}) \) terms, by (from now on we drop the subscript \( \nu \) from the propagator):

\[
G^+(\nu) = -\hat{\nu}^2\tau + \hat{\nu} \tau - e^{-2\hat{\nu}^2\tau + e^{-2\hat{\nu}^2\tau} - e^{2\hat{\nu}^2\tau} - 1} \\
\partial_\nu G^+(\nu) = -2\hat{\nu} + 1 + 2e^{-2\hat{\nu}^2\tau + e^{2\hat{\nu}^2\tau} - e^{-2\hat{\nu}^2\tau} + 1} \\
\partial^2_\nu G^+(\nu) = -\frac{2}{\tau} - 4e^{-2\hat{\nu}^2\tau + e^{2\hat{\nu}^2\tau} - e^{-2\hat{\nu}^2\tau} + 1} \\
\partial^2_\nu G^+(\nu) = -\frac{2}{\tau} - 4e^{-2\hat{\nu}^2\tau + e^{2\hat{\nu}^2\tau} - e^{-2\hat{\nu}^2\tau} + 1}
\]

and

\[
G^- (\nu) = -\hat{\nu}^2\tau + \hat{\nu} \tau + e^{-2\hat{\nu}^2\tau + e^{2\hat{\nu}^2\tau} - e^{-2\hat{\nu}^2\tau} + 1} \\
\partial_\nu G^- (\nu) = -2\hat{\nu} + 1 - 2e^{-2\hat{\nu}^2\tau - e^{2\hat{\nu}^2\tau} - e^{-2\hat{\nu}^2\tau} - 1} \\
\partial^2_\nu G^- (\nu) = -\frac{2}{\tau} - 4e^{-2\hat{\nu}^2\tau - e^{2\hat{\nu}^2\tau} - e^{-2\hat{\nu}^2\tau} - 1} \\
\partial^2_\nu G^- (\nu) = -\frac{2}{\tau} - 4e^{-2\hat{\nu}^2\tau - e^{2\hat{\nu}^2\tau} - e^{-2\hat{\nu}^2\tau} - 1}
\]

To compute the one–loop amplitude we have to specify which partial propagator \( G^+ \) or \( G^- \) we have to insert in eq.(3.4). To this end we split the boundary of \( \mathcal{M} \) into two parts \( AA' \) lying in the positive real \( \rho \) axis, and \( BB' \) along the negative \( \rho \) axis (see figure). One has to consider all the configurations which are compatible with any given ordering of the gluon insertions along the boundary of \( \mathcal{M} \).

3.2.1 Two–gluon amplitude

In the two–gluon amplitude only one propagator is involved. Therefore the two–gluon amplitude on \( \mathcal{M} \) contains two contributions, one with \( G^+ \) corresponding to the gluon insertions in the same interval \( AA' \) or \( BB' \) and the other with \( G^- \) corresponding to one insertion in \( AA' \) and the other in \( BB' \). We will use translational invariance in order to fix the insertion 1 at the point \( A' \), i.e. \( \rho_1 = 1 \) or \( \nu_1 = 0 \). After these preliminaries we insert all the data in eq.(3.4) and find

\[
A^{(1)}_{\mathcal{M}}(p_1,p_2) = \frac{i}{2} \text{tr}(t^{a_1} t^{a_2}) \frac{g_3^2}{4\pi} (2\alpha')^2 \frac{D}{2} \int_0^\infty d\tau e^{2\tau} \tau^{-\frac{D}{2}} \prod_{n=1}^\infty (1 - (-1)^n e^{-2\pi \nu})^{2-D} \\
\times (-\epsilon_1 \cdot \epsilon_2) \int_0^\tau dv \left( e^{2\alpha' p_1 \cdot p_2 G^+(\nu)} \partial_\nu^2 G^+(\nu) + e^{2\alpha' p_1 \cdot p_2 G^-(\nu)} \partial_\nu^2 G^-(\nu) \right)
\]

where \( \nu = \nu_2 \).

Now we integrate by parts in \( \nu \) and disregard the contributions at \( \nu = 0, \tau \), since, as was noticed in \([12]\), they correspond in field theory to massless tadpole contributions,
which are defined to vanish in dimensional regularization. Therefore the RHS of (3.25) can be replaced by:

\[
\frac{i}{2} \text{tr}(t^{a_1} t^{a_2}) \frac{g^2_D}{(4\pi)^2} (2\alpha')^{2-D} \int_0^\infty d\tau e^{2\tau} \tau^{-D/2} \prod_{n=1}^\infty (1 - e^{-2n\tau})^{2-D} \\
(\epsilon_1 \cdot \epsilon_2 p_1 \cdot p_2) \int_0^\tau d\nu \left( e^{2\alpha' p_1 \cdot p_2 G^+(\nu)} (\partial_\nu G^+(\nu))^2 + e^{2\alpha' p_1 \cdot p_2 G^-(\nu)} (\partial_\nu G^-(\nu))^2 \right)
\] (3.26)

At this point we insert the expansions (3.23) and (3.24) and evaluate the \(\nu\) integral first. One notices that the two exponentials \(e^{2\alpha' p_1 \cdot p_2 G^\pm(\nu)}\) for large \(\tau\) can be written as \(e^{2\alpha' p_1 \cdot p_2 (\hat{\nu} - \hat{\nu}'^2)\nu}\) and play the role of a cutoff factor. Therefore, for large \(\tau\), the \(\nu\) integral in (3.26) is determined by

\[
\int_0^\tau d\nu \left( (1 - 2\hat{\nu}^2 + 8e^{-2\tau}) e^{2\alpha' p_1 \cdot p_2 (\hat{\nu} - \hat{\nu}'^2)\tau} \right)
\]

Now inserting this equation back into (3.26), we see that there are contributions to the \(\tau\) integral proportional to \(e^{2\tau}\). These are recognized to be contributions from the tachyon and must be discarded by hand (this ad hoc operation is the price we have to pay for having embedded our gauge theory in the bosonic string rather than in a superstring theory). The terms of zeroth order in \(e^{2\tau}\) are the relevant ones for our purposes. As shown in [42], these integrals can be exactly evaluated and the pole in \(\epsilon = \frac{4-D}{2}\) easily extracted. The result is

\[
A_{\mathcal{M}}^{(1)}(p_1, p_2) \bigg|_{\text{div}} = \text{itr}(t^{a_1} t^{a_2}) \frac{g^2}{(4\pi)^2} \epsilon_1 \cdot \epsilon_2 p_1 \cdot p_2 \frac{11}{3} \frac{1}{\epsilon}
\] (3.27)

which, if we forget the factor of \(N\), is twice the planar contribution with opposite sign. If we put together the results for the planar annulus amplitude and the Möbius strip we finally obtain for the 1PI divergent part of the two–gluon amplitude

\[
A^{(1)}(p_1, p_2) \bigg|_{\text{div}} = -\frac{N - 2}{2} \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A^{(0)}(p_1, p_2)
\] (3.28)

This is exactly what is expected from renormalization theory in the background field method formalism.

### 3.2.2 Three–gluon amplitude

The three–gluon amplitude involves two propagators and four possible configurations for any given ordering of the external legs, [15]. The four configurations can be classified as follows. The orientation of the boundary of \(\mathcal{M}\) is chosen from \(A'\) to \(A\) and from \(B\) to \(B'\). We call it the standard orientation. We consider the three insertions at \(\rho_1, \rho_2, \rho_3\) ordered according to the standard orientation and set \(\rho_1 = A'\), see figure. Now we append by convention \(a + \) or \(a - \) to \(\rho\) according to whether \(\rho\) falls in the interval \(AA'\) or in \(BB'\). The four configurations are then specified as follows

- s1: \((\rho_1^+, \rho_2^+, \rho_3^+)\)
- s2: \((\rho_1^+, \rho_2^+, \rho_3^-)\)


• s3: \((\rho_1^+, \rho_2^+, \rho_3^-)\)

• s4: \((\rho_1^+, \rho_3^-, \rho_2^+)\)

Each triple is in order of decreasing modulus. For instance, s4 means \(|\rho_1| \geq |\rho_3| \geq |\rho_2|\) and that \(\rho_1\) and \(\rho_2\) are in \(AA'\) while \(\rho_3\) is in \(BB'\). s1–s4 specify distinct sectors of the integration region (moduli space).

The amplitude given by (3.24) contains three pieces, which are proportional to the three terms contained in the RHS of (3.2). We will consider here the one proportional to \(\epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3\). The corresponding coefficient in \(A^{(1)}(p_1, p_2, p_3)\) is given by

\[
\frac{1}{2} \text{tr}(\ell^{a_1} \ell^{a_2} \ell^{a_3}) \frac{g_D^3}{(4\pi)^2} (2\alpha')^{1-D} \int_0^\infty \int_0^\infty e^{-2\tau \tau - \frac{D}{2}} \prod_{n=1}^\infty (1 - (-1)^n e^{-2n\tau})^{2-D} (3.29)
\]

The last four lines in this equation correspond to the contributions from the four configurations listed above, in the same order. As analyzed in [42], the divergent contributions corresponding to 1PI diagrams in the \(\alpha' \to 0\) limit, come from two different regions of the moduli space, which we call type I and type II.

The type I region corresponds to the three insertion points being kept widely separated while \(\tau \to \infty\), i.e. while the Möbius strip shrinks to zero size (\(w \to 0, q \to 1\)). Intuitively, this corresponds in the field theory language to Feynman diagrams with three propagators and three three–point vertices. This means that \(\nu_3\) and \(\nu_{32}\) are of order \(\tau\) while \(\tau \to \infty\). It is possible to show that these contributions come only from the first terms (those not containing exponentials) in the asymptotic expansions (3.23, 3.24). We seem to have four contributions of this type, corresponding to the four configurations s1–s4. However this is not the case. Only two of them contribute to type I, precisely s1 and s3. In s2 and s4, point 2 and point 3 are bound to lie on opposite sides of the band; in the field theory limit these contributions do not flow toward the expected Feynman diagrams. In a sense they are analogous to the nonplanar ones.

To evaluate the type I contribution we remark that the exponentials in eq. (3.29) play simply the role of dumping factors. Therefore we simplify things by replacing them with a universal dumping factor \(e^{-2\alpha' m^2 \tau}\). After discarding the tachyon contribution one can see that the relevant UV divergent part from region of type I in eq. (3.29) is contained in

\[
\frac{1}{2} \text{tr}(\ell^{a_1} \ell^{a_2} \ell^{a_3}) \frac{g_D^3}{(4\pi)^2} (2\alpha')^{1-D} \int_0^\infty d\tau \tau \tau - \frac{D}{2} (2 - D) \int_0^\tau d\nu_3 \int_0^{\nu_3} d\nu_2 e^{-2\alpha' m^2 \tau} 8 \left( \frac{\nu_2}{\tau^2} \right)
\]
After a standard integration, this becomes

\[-\text{tr}(t^{a_1} t^{a_2} t^{a_3}) \frac{g^3_D}{(4\pi)^2} (4 \epsilon^2) \frac{4}{3} m^{D-4}\Gamma(\epsilon)\]

Collecting the above results and setting $D = 4$ one finds the type I contribution to the divergent part of the three–gluon amplitude is:

\[A_{\mathcal{M}}^{(1)}(p_1, p_2, p_3)\big|_I = -\text{tr}(t^{a_1} t^{a_2} t^{a_3}) \frac{g^3_D}{(4\pi)^2} \left(\epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3\right) \frac{4}{3} \frac{1}{\epsilon} \quad (3.30)\]

Let us pass now to the type II region. It is the region in the moduli space where two insertion points come close together like $1/\tau$ as $\tau \to \infty$. In field theory such terms correspond to one–loop three–gluon diagrams with one four–point vertex. There are three possibilities: either $\rho_1 \to 1$, or $\rho_1 \to \rho_2$, or $\rho_3 \to -w$. These correspond to either $\nu_2 \sim O(\tau^{-1})$ or $\nu_3 \sim O(\tau^{-1})$. In field theory terms this corresponds to Feynman diagrams with one internal propagators and one four–point vertex.

Using the asymptotic expansions (3.23, 3.24) into (3.29) one can see that the type II contributions can only come from the exponential terms in (3.23, 3.24). Once again, however, we should not apply the formulas mechanically. The type II contributions of the sectors $s_1$–$s_4$ must be carefully evaluated. For instance it is evident that in $s_4$ the punctures $2$ and $3$ cannot approach each other because they are confined to lie on opposite sides of the band. On the other hand $\rho_3$ cannot go to $-w$ because $|\rho_3| \geq |\rho_2|$, and, for the same reason $\rho_2$ cannot go to $1$. Therefore neither $3$ nor $2$ can get close to $1$. Thus sector $s_4$ is not going to contribute to type II. On the other hand, in $s_1$ we have the possible collapses $2 \to 1$ and $2 \to 3$, in $s_2$ we have the only possible collapse $3 \to 1$, while in $s_4$ we can have both $3 \to 1$ and $2 \to 3$. As it turns out, $2 \to 1$ does not contributes to the divergent part. Carrying out the explicit calculations, the divergent part of (3.29), as far as type II is concerned, is contained in

\[
\frac{1}{2} \text{tr}(t^{a_1} t^{a_2} t^{a_3}) \frac{g^3_D}{(4\pi)^2} (2\alpha^2) \frac{e^\frac{4-D}{2}}{2} \int_0^\infty d\tau \frac{e^{2\tau}}{\tau} \int_0^{\nu_3} d\nu_2 e^{-2\nu_2 m^2 \tau} \frac{1}{\tau} \left[ 8e^{-2\tau+2(\nu_3-\nu_2)} - 8^{-2\tau+2(\nu_2-\nu_3)} + 8e^{-2\nu_3} - 8e^{-4\tau+2\nu_3} + 8e^{-2\tau+2(\nu_3-\nu_2)} - 8^{-2\tau+2(\nu_3-\nu_2)} + 8e^{-2\nu_3} - 8e^{-4\tau+2\nu_3} \right] \quad (3.31)
\]

where the last three lines correspond to the $s_1$, $s_2$ and $s_3$ contributions, respectively. The calculation now is straightforward. Setting $D = 4-2\epsilon$ one finds that the type II contribution to the divergent part of the three–gluon amplitude is:

\[A_{\mathcal{M}}^{(1)}(p_1, p_2, p_3)\big|_{II} = \text{tr}(t^{a_1} t^{a_2} t^{a_3}) \frac{g^3_D}{(4\pi)^2} \left(\epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3\right) \frac{1}{\epsilon} \quad (3.32)\]

Finally the total divergent part for the three–gluon amplitude is

\[A_{\mathcal{M}}^{(1)}(p_1, p_2, p_3)\big|_{I+II} = \text{tr}(t^{a_1} t^{a_2} t^{a_3}) \frac{g^3_D}{(4\pi)^2} \left(\epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3\right) \frac{4}{3} \frac{1}{\epsilon} \quad (3.33)\]
Therefore

\[
\left. A^{(1)}(p_1, p_2, p_3) \right|_{1+II} = -\frac{N - 2}{2} \frac{g^2}{(4\pi)^2} \frac{11}{3} \epsilon \ A^{(0)}(p_1, p_2, p_3) \quad (3.34)
\]

### 3.2.3 Four–gluon amplitude

The four–gluon amplitude involves three propagators and eight possible configurations for any given ordering of the external legs, see [41]. The eight configurations can be classified as above. We consider the four insertions at \(\rho_1, \rho_2, \rho_3, \rho_4\) ordered according to the standard orientation of the boundary of \(\mathcal{M}\) and set \(\rho_1 = A'\). The corresponding eight sectors of integration are then specified as follows

- s1: \((\rho_1^+, \rho_2^+, \rho_3^+, \rho_4^+\))
- s2: \((\rho_1^+, \rho_2^+, \rho_3^+, \rho_4^-\))
- s3: \((\rho_1^+, \rho_2^+, \rho_3^-, \rho_4^-\))
- s4: \((\rho_1^+, \rho_2^-, \rho_3-, \rho_4^-\))
- s5: \((\rho_1^+, \rho_4^-, \rho_2^+, \rho_3^-\))
- s6: \((\rho_1^+, \rho_4^-, \rho_2^+, \rho_3^+\))
- s7: \((\rho_1^+, \rho_2^+, \rho_4^-, \rho_3^-\))
- s8: \((\rho_1^+, \rho_3^-, \rho_4^-, \rho_2^+\))

Each quadruple is written in order of decreasing modulus.

Now we single out in (3.4) the piece proportional to \(\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4\) (the other two pieces can be dealt with similarly, see [42]) and simplify the resulting expression as in the three–gluon case. In particular we replace the exponential factors with a unique dumping factor \(e^{-2a'^2m^2\tau}\).

Next we discuss the contributions from region I and II. To this end we avoid explicitly writing down encumbering equations. Let us recall that type I contributions come from well separated configurations of the punctures in the limit \(\tau \to \infty\), they correspond in field theory to Feynman diagram with four internal propagators. The only two sectors that can contribute are s1 and s4. All the other sectors are non–planar–like in that they contain at least two points on opposite sides of the band. Their field theory limit is different from that expected for type I contributions.

As for type II contributions they correspond to two separate couples of points coming simultaneously together like \(\mathcal{O}(1/\tau)\) as \(\tau \to \infty\). In field theory this correspond to Feynman diagram with two internal propagators. Sector by sector we find: in s1 we can have \(2 \to 1\) and \(3 \to 4\); in s2 we can have \(2 \to 3\) and \(\rho_4 \to -w\), i.e. \(4 \to 1\); in s3 we can have \(2 \to 1\) and \(3 \to 4\); in s4 we can have \(2 \to 3\) and \(\rho_4 \to -w\), i.e. \(4 \to 1\); no two separate couples of points can come simultaneously together in the remaining sectors. So sectors s5–s8 do not contribute neither to type I nor to type II divergences.
Now, going to explicit formulas, we find that the relevant multiplicative factor of 
\( \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 \) in \( A^{(1)}(p_1, p_2, p_3, p_4) \) is

\[
\frac{i}{2} \text{tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \frac{g_4^{D}}{(4\pi)^2} (2\alpha')^{\frac{D-2}{2}} \int_{0}^{\infty} d\tau e^{2\tau} e^{\frac{-D}{2}} \prod_{n=1}^{\infty} (1 - (-1)^n e^{-2n\tau})^{2-D} \int_{0}^{\tau} dv_4 \int_{0}^{v_4} dv_3 \int_{0}^{v_3} dv_2 \int_{0}^{v_2} d\nu e^{-2\alpha' \nu^2} \left[ \partial^2_{\nu^3} G_+(\nu_3) \partial^2_{\nu^4} G_+(\nu_4) + \partial^2_{\nu^3} G_-(\nu_3) \partial^2_{\nu^4} G_-(\nu_4) + \partial^2_{\nu^3} G_-(\nu_3) \partial^2_{\nu^4} G_+ \right] (3.35)
\]

where the terms in square brackets refer to sector s1 down to s4, respectively.

It remains for us to evaluate the above integral for type I and II. As pointed out above the type I contributions come only from the first terms (those not containing exponentials) in the asymptotic expansions (3.23, 3.24).

\[
i \text{tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \frac{g_4^{D}}{(4\pi)^2} (2\alpha')^{\frac{D-2}{2}} \int_{0}^{\infty} d\tau e^{2\tau} e^{\frac{-D}{2}} \int_{0}^{\tau} dv_4 \int_{0}^{v_4} dv_3 \int_{0}^{v_3} dv_2 \int_{0}^{v_2} d\nu e^{-2\alpha' \nu^2} \left( \frac{8}{\tau^2} \right)
\]

Proceeding as above this gives rise to the following divergent part of the four–point amplitude

\[
A^{(1)}_{\mathcal{M}}(p_1, p_2, p_3, p_4) = -i \text{tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \frac{g_4^{D}}{(4\pi)^2} \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 \frac{4}{3} \frac{1}{\epsilon} (3.36)
\]

Type II contributions come from the terms containing exponentials in (3.23, 3.24). From (3.35) one gets

\[
-\frac{i}{2} \text{tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \frac{g_4^{D}}{(4\pi)^2} (2\alpha')^{\frac{D-2}{2}} \int_{0}^{\infty} d\tau e^{2\tau} e^{\frac{-D}{2}} \int_{0}^{\tau} dv_4 \int_{0}^{v_4} dv_3 \int_{0}^{v_3} dv_2 \int_{0}^{v_2} d\nu e^{-2\alpha' \nu^2} \left[ 32 e^{-2\tau} - 2(\nu_2 + \nu_3 - \nu_4) + 32 e^{-2\tau} + 2(\nu_2 + \nu_3 - \nu_4) + 32 e^{-2\tau} - 2(\nu_2 - \nu_3 - \nu_4) \right]
\]

whose evaluation leads to the divergent part

\[
A^{(1)}_{\mathcal{M}}(p_1, p_2, p_3, p_4) = i \text{tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \frac{g_4^{D}}{(4\pi)^2} \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 \frac{16}{3} \frac{1}{\epsilon} (3.37)
\]

Summing type I and type II we get

\[
A^{(1)}_{\mathcal{M}}(p_1, p_2, p_3, p_4) = \text{tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \frac{g_4^{D}}{(4\pi)^2} \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 \frac{44}{3} \frac{1}{\epsilon} (3.38)
\]

Once again we obtain

\[
A^{(1)}(p_1, p_2, p_3, p_4) = -\frac{N - 2}{2} \frac{g_4^{D}}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A^{(0)}(p_1, p_2, p_3, p_4) (3.39)
\]

Eqs. (3.28, 3.34, 3.39) coincide with the results of the previous subsection. We remark that they are the one–loop quantum corrections expected in an SO(N) gauge field theory in the background field formalism, \[43, 42\]. They correspond to a renormalization constant

\[
Z_A = 1 + \frac{N - 2}{2} \frac{g_4^{D}}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} (3.40)
\]

This amounts to one–loop renormalizability (in 4D) of the low energy effective action of the string theory with so(N) Chan–Paton factors, that is the well–known fact that SO(N) gauge field theory in 4D is renormalizable.
4. The string propagator on a non-orientable world-sheet. Case $B \neq 0$

We turn now to the same problems considered in section 2 and 3, but in the presence of a constant $B$ field. A few words of caution are in order.

A D-brane with an $SO(N)$ (or $Sp(N)$) gauge theory on it can be found in correspondence with an orientifold: it corresponds to a set of branes and mirror branes which collapse on the orientifold. This fact entails a problem when we want to consider such a system in the presence of a $B$ field. In fact the orientifold projection contains a space inversion which seems to exclude the presence of a $B$ field in the final configuration. It was however argued in [34] that this is not a cogent difficulty, a way out can be found. Here we add an alternative simple argument to the one presented in [34], which seems to be more appropriate to the type of problems we consider in this paper. In the original (before projection) theory one can always add to the $B$ field a constant part without changing the equations of motion of (super)gravity. This constant part is not directly affected by the string oscillators (which determine the equations of motion of the low energy effective action via the string amplitudes). On the other hand the orientifold projection operator is defined through the action on the string oscillators, so that a constant ‘relic’ $B$ field may conceivably not be affected by the projection. For similar considerations, see [37].

In this paper we give all this for granted and consider a set of D-branes collapsed over an orientifold with orthogonal (or symplectic) Chan–Paton factors in the presence of a constant $B$ field. This is expected to give rise to a noncommutative $SO(N)$ ($Sp(N)$) gauge field theory. The tree level analysis of such theories has been carried out in [34]. As explained above, in this paper we wish to do the one–loop analysis. But this entails a new problem. In fact the sigma–model action for open strings attached to a D–brane is (we adopt the conventions of [38])

$$\frac{1}{4\pi\alpha'} \int_{\Sigma} d^{2}x \left( \sqrt{\hbar} h^{\alpha\beta} \partial_{\alpha}X^{i}\partial_{\beta}X^{j}g_{ij} - 2\pi\alpha' \int_{\Sigma} d^{2}x \epsilon^{\alpha\beta} \partial_{\alpha}X^{i}\partial_{\beta}X^{j}B_{ij} \right)$$  \hspace{1cm} (4.1)

where $\Sigma$ is the string world–sheet, $g_{ij}$ is the closed string metric and $B_{ij}$ are the components of the constant $B$ field. At tree level the relevant world–sheet is the disk, while at one–loop the relevant world–sheets are the annulus and the Möbius band. Disk and annulus are orientable and the integrals in (4.1) are well–defined on such surfaces. But the Möbius strip in nonorientable and, while the first term in (4.1) is well defined on it, the second is not. The reason is that, as we have recalled in section 2, on nonorientable manifolds only densities can be integrated, see [38]. Now, the first integrand in (4.1) is a density, while the second is not (it is the component of the pull–back of a two–form). Therefore the second part of (4.1) is meaningless when $\Sigma$ is the Möbius band. However, since $B$ is constant, in

\footnote{In $SO(N)$ there is no global $U(1)$ factor as in $U(N)$. Therefore one may wonder whether the $B$ field, which is not protected by the gauge invariant combination $B - dA$, might be gauged away. The answer is no, because the $B$ field after the orientifold projection is not dynamical anymore, it does not appear in the effective action, so also its gauge properties disappear. Said differently, away from the orientifold every brane has a $U(1)$ field on it which guarantees the existence of a nonvanishing gauge invariant combination $B - dA$; it is natural to assume that in the collapsing limit, by continuity, the value of the gauge invariant combination $B - dA$ will be unchanged even though a (global) $U(1)$ $A$ has disappeared.}
general we can replace (4.1) with
\[
\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 x \sqrt{h} \partial_\alpha X^i \partial_\beta X^j g_{ij} - \frac{i}{2} \int_{\partial\Sigma} dt X^i \partial_t X^j B_{ij}
\] (4.2)
where \(\partial_t\) is the derivative tangent to the boundary \(\partial\Sigma\). This expression is now well-defined also for the M"obius strip since its boundary (a circle) is orientable. From now on we will use (4.2) instead of (4.1).

Let us turn now to the string propagator. The problem is to find, on the surface \(\Sigma\) of interest (in our case \(\mathcal{M}\)), the solution of the equation
\[
4\partial_\rho \partial_{\bar{\rho}} G^{ij}(\rho, \rho') = 2\pi\alpha' g^{ij} \delta(\rho - \rho')
\] (4.3)
and
\[
[(g + F)^i_k \partial_\rho - (g - F)^i_k \partial_{\bar{\rho}}] G^{kij}(\rho, \rho') \bigg|_{\partial\Sigma} = K g^{ij}
\] (4.4)
where \(K\) is the same as in section 2. Moreover we require that \(G^{ij}(\rho, \rho') = G^{ji}(\rho', \rho)\).

The solution we propose for the M"obius strip is as follows.
\[
\frac{1}{\alpha'} G^{ij}_{\mathcal{M}}(\rho, \rho') = g^{ij} (I_{\mathcal{M}}(\rho, \rho') + f_{\mathcal{M}}(\rho, \rho')) + (2g^{ij} - g^{ij})J_{\mathcal{M}}(\rho, \rho') + \frac{\theta^{ij}}{\alpha'} K_{\mathcal{M}}(\rho, \rho')
\] (4.5)
where
\[
\hat{g}^{ij} = \left(\frac{1}{g + F} \frac{1}{g - F}\right)^{ij}, \quad \theta^{ij} = -2\pi\alpha' \left(\frac{1}{g + F} \frac{1}{g - F}\right)^{ij}
\] (4.6)
are the open string metric and the deformation parameter, respectively, \(I_{\mathcal{M}}, f_{\mathcal{M}}\) and \(J_{\mathcal{M}}\) are the same as in section 2, and
\[
K_{\mathcal{M}}(\rho, \rho') = \frac{(\ln \frac{\rho}{\rho'})^2 - (\ln \frac{\rho}{\rho'})^2}{2\ln w} + \ln \frac{\rho - \rho'}{\rho - \rho'} + \frac{1}{2} \ln \frac{\rho'}{\rho'}
\]
\[
+ \ln \prod_{n=1}^{\infty} \frac{(1 - (-w)^n \frac{\rho}{\rho'})(1 - (-w)^n \frac{\rho'}{\rho})}{(1 - (-w)^n \frac{\rho}{\rho})(1 - (-w)^n \frac{\rho'}{\rho})}
\] (4.7)
As in section 2 the log square terms must be understood as
\[
\left(\ln \frac{\rho}{\rho'}\right)^2 = \frac{1}{4}\left(\ln \left(\frac{\rho}{\rho'}\right)^2\right)^2
\]
and so on.

Notice that \(G^{ij}_{\mathcal{M}}(\rho, \rho') = G^{ij}_{\mathcal{M}}(\rho', \rho)\). It is now quite a standard matter to verify that eqs. (4.3) and (4.4) are satisfied. It is also easy to verify that the continuity condition on the boundary of the Möbius band is satisfied:
\[
G^{ij}_{\mathcal{M}}(1, \rho') = G^{ij}_{\mathcal{M}}(-w, \rho'), \quad G^{ij}_{\mathcal{M}}(-1, \rho') = G^{ij}_{\mathcal{M}}(w, \rho'), \quad \forall \rho'
\]
It remains for us to discuss Gauss’s theorem. The normal derivatives along the boundary \( AA'B'B' \) of fig.1 are very complicated. We limit ourselves here to reporting the result (see Appendix A for notation):

\[
\begin{align*}
\int_{A'}^B dl \partial_\perp G(\rho, \rho') + \int_{B'}^A dl \partial_\perp G(\rho, \rho') &= 0 \\
\int_{A}^{A'} dl \partial_\perp G(\rho, \rho') + \int_{B'}^{B} dl \partial_\perp G(\rho, \rho') &= \int_{A}^{A'} dl \partial_\perp f(\rho, \rho') + \int_{B'}^{B} dl \partial_\perp f(\rho, \rho') = 2\pi
\end{align*}
\] (4.8)

Therefore Gauss’s theorem is verified (see the analogous proof in section 2). We notice however that, as in the case of \( B = 0 \), \( G \) is not continuous across the junction line \( A'B' \), which is identified with \( B'A \) of fig.1. In fact, instead of

\[
G_{ij}^M(-w\rho, \rho') = G_{ij}^M(\rho, \rho'),
\] (4.9)

which would be needed in order to satisfy the Möbius periodicity conditions, we only have

\[
G_{ij}^M(-w\rho, \rho') = G_{ij}^M(\rho, \rho'),
\] (4.10)

This means that \( G \) is single–valued on the double covering \( \hat{M} \) of \( M \). \( \hat{M} \) is obtained by adding to the half annulus \( AA'B'B' \) of fig.1 its complex conjugate region in the upper half plane and identifying \( e^{i\theta} \) with \( we^{i(\pi+\theta)} \) for \( 0 \leq \theta \leq 2\pi \). The covering projection is obtained by identifying \( \rho \) and \( \bar{\rho} \). The resulting figure is a torus with a Möbius strip inscribed in it.

When we restrict our consideration to the half annulus \( AA'B'B' \), \( G \) satisfies all the requirements, including Gauss’s theorem, but has a finite discontinuity along the junction line. We should therefore ask ourselves if this discontinuity may have any physical consequences. In string theory open string amplitudes depend on the propagator on the boundary of \( M \), not on the values taken by the propagator in the bulk. Now the limit to the boundary of \( M \) is well defined and the discontinuity disappears. Therefore the discontinuity across the junction line does not seem to entail any physical consequence. On the other hand, if we consider the electrostatic analog of section 2, we see that the electric field turns out to be discontinuous along the junction, and, in this case, a physical interpretation is possible only on the double covering \( \hat{M} \).

Since in this paper we are interested in open string amplitudes we will assume that the right object to be considered is the restriction of \( G \) to the boundary of \( M \). By taking the limit for \( \rho \) and \( \rho' \) approaching the real axis we get: \( G_{ij}^M \rightarrow G_{ij}^{\hat{M}} \), where

\[
G_{ij}^{\hat{M}}(\rho, \rho') = 2\alpha'\hat{g}^{ij}G_M(\rho, \rho') - \frac{i}{2}g^{ij}\epsilon(\rho - \rho')
\] (4.11)

and \( G_M(\rho, \rho') \) is the same as in section 2, eqs.(2.16, 2.17). This is the propagator we will use for our calculations in the following section.

Finally we notice that by replacing \((-w)^n\) with \(w^n\) in (2.7) we get the Green function for the annulus, from which one can extract the planar and nonplanar propagators. This was done in \[40\] and in \[20\] and we will rely on those results.

\[\text{This, of course, does not mean that the propagator } G \text{ is the string propagator on the torus, because of the boundary conditions [14].}\]
To complete this section we write down the expression of the above Möbius propagator in the $z$ plane. The latter is obtained from (4.7, 2.8, 2.9, 2.10, 1.5), passing from $\rho$ to $z$, changing $\tau \to -1/\tau$ and using well-known identities for the Jacobi theta–functions, [11]:

$$
\frac{1}{\alpha'} \mathcal{G}^{ij}_{\mathcal{M}}(z, z') = \delta^{ij} (I_{\mathcal{M}}(z, z') + f_{\mathcal{M}}(z, z')) + (2\delta^{ij} - \delta^{ij}) J_{\mathcal{M}}(z, z') + \frac{\theta^{ij}}{\alpha'} \mathcal{K}_{\mathcal{M}}(z, z')
$$

(4.12)

where

$$
I_{\mathcal{M}}(z, z') = \ln \left| \left( \frac{z}{\bar{z}'} \right)^{1/4} - \left( \frac{\bar{z}}{z'} \right)^{1/4} \right| + \ln \prod_{n=1}^{\infty} \frac{1 - (-\sqrt{q})^n \sqrt{\frac{z}{z'}}}{1 - (-\sqrt{q})^n \sqrt{\frac{\bar{z}}{\bar{z}'}}} \frac{1 - (-\sqrt{q})^n \sqrt{\frac{\bar{z}}{z'}}}{1 - (-\sqrt{q})^n \sqrt{\frac{z}{\bar{z}')}}
$$

$$
f_{\mathcal{M}}(z, z') = -\ln |zz'|
$$

$$
J_{\mathcal{M}}(z, z') = \ln \left| (z\bar{z}')^{1/4} - (\bar{z}z')^{-1/4} \right| + \ln \prod_{n=1}^{\infty} \frac{1 - (-\sqrt{q})^n \sqrt{z\bar{z}'}}{1 - (-\sqrt{q})^n \sqrt{\bar{z}z'}} \frac{1 - (-\sqrt{q})^n \sqrt{\bar{z}z'}}{1 - (-\sqrt{q})^n \sqrt{z\bar{z}'}}
$$

$$
K_{\mathcal{M}}(z, z') = \ln \left| \left( \frac{z\bar{z}'}{(z\bar{z}')^{1/4} - (\bar{z}z')^{-1/4}} \right)^{1/4} - \left( \frac{(z\bar{z}')^{1/4} - (\bar{z}z')^{-1/4}}{z\bar{z}'} \right)^{1/4} \right| + \ln \prod_{n=1}^{\infty} \frac{1 - (-\sqrt{q})^n \sqrt{z\bar{z}'}}{1 - (-\sqrt{q})^n \sqrt{\bar{z}z'}} \frac{1 - (-\sqrt{q})^n \sqrt{\bar{z}z'}}{1 - (-\sqrt{q})^n \sqrt{z\bar{z}'}}
$$

where $q = \exp[-\pi^2/\tau]$. Actually the expression for $K_{\mathcal{M}}(z, z')$ differs from (4.7) by a constant term, which is within the ambiguity allowed by the Green function’s defining equations. If $F = 0$ and we restrict the above expressions to the boundary, i.e. $|z| = |z'| = 1$, $I_{\mathcal{M}}$ becomes identical to $J_{\mathcal{M}}$ and the propagator reduces (up to an additive constant) to the expression one can find in [11]. The expression (4.12) of the Green function shows that it is indeed defined on the Möbius band since it can be thought as the “bulk counterpart” of (3.7). If we express the $z$ coordinate in terms of $\hat{v}$, obtaining $z = \exp 2\pi i \hat{v}$, we see that (4.12) has double period with respect to the analogous expression for the Green function on the annulus in presence of a $B$ field, presented for instance in eq. (2.21) of [20].

5. Field theory limit of gluon amplitudes with $B$ field

Switching on a constant $B$ field, on the basis of the discussion in previous section, amounts to replacing the propagator used in section 3 with the full propagator (1.11). Inserting it into the general formula (1.4) has a simple effect. The addition of the second term $-\frac{1}{\alpha'} \theta^{ij} (\rho - \rho')$ does not affect derivatives of propagators, while it modifies the term $\prod_{r<s} e^{p_r G(r - s) p_s}$. This modification turns out to be very simple since the insertion points along the boundary of $\mathcal{M}$ are ordered, so that the relevant $\epsilon$ function is always either $+1$ or $-1$. As a consequence the corresponding exponential factors can be extracted from the moduli integral. In other words, the gluon amplitudes are multiplied by a global (noncommutative) factor

$$
A^{(1)}(p_1, \ldots, p_m) \to \prod_{r<s} e^{p_r \times p_s} A^{(1)}(p_1, \ldots, p_m)
$$

(5.1)
where $A^{(1)}(p_1, \ldots, p_m)$ are the $B = 0$ amplitudes and $p \times q = \frac{i}{2} p_i \theta^{ij} q_j$. The same is true also at tree level, [28, 24], and, on the basis of [22], it is likely to hold at any loop order, although we do not try to prove it here.

We can now infer that the analysis of the singularities in the field theory limit does not change with respect to the previous section, except for the global noncommutative factor in (5.1). We can therefore conclude that the structure of the divergent terms, as well as the renormalization constants, are the same as in the ordinary $SO(N)$ gauge theories. Therefore, if there exists a noncommutative gauge field theory that represents the low energy effective action of open strings with orthogonal CP factors in the presence of a constant $B$ field, this noncommutative gauge field theory is one–loop renormalizable.

6. Discussion

The above conclusion seems to imply that a renormalizable noncommutative gauge field theory with $so(N)$ Chan–Paton factors should exist. We recall that, even without resorting to an action, we can extract the gluon Feynman rules for this low energy field theory from the string tree amplitudes. They are as follows

**gluon propagator.**

$$A, i \quad \quad \quad \quad B, j \quad \quad = \quad \frac{i}{p^2} \delta_{ab} \hat{g}_{ij} \quad (6.1)$$

**3–gluon vertex.** The external gluons carry labels $(a, i, p), (b, j, q)$ and $(c, k, r)$ for the Lie algebra, momentum and Lorentz indices and are ordered in anticlockwise sense:

$$-g f^{abc} \cos(p \times q) (\hat{g}_{ij} (p - q)_k + \hat{g}_{jk} (q - r)_i + \hat{g}_{ki} (r - p)_j) \quad (6.2)$$

**4–gluon vertex.** The gluons carry labels $(a, i, p), (b, j, q), (c, k, r)$ and $(d, l, s)$ for Lie algebra, Lorentz index and momentum. They are clockwise ordered:
We recall that this last vertex can be obtained from the string four–gluon amplitude only after subtracting two suitable tree one–particle reducible diagrams.

One can verify that the above Feynman diagrams can be obtained from the action suggested in [34]. From that action, which was called $NCSO(N)$, one can in addition extract the Feynman rules for the ghost fields. A natural question that arises is whether by applying these Feynman rules to compute one–loop amplitudes one gets the same results as the ones we obtained in the previous section. The surprising answer is that, if we apply Feynman rules in the ordinary way, we get a different result.

To illustrate the problem the simple $NCSO(2)$ case will do. From the string theory point of view it is rather easy to argue that the theory should not have UV divergences. Let us summarize our previous analysis. The one–loop contributions to open string amplitudes with $SO(N)$ Chan–Paton factors are of three types: planar (P) and nonplanar (NP) with the world–sheet of the annulus, and nonorientable (NO) with the world–sheet of the Möbius strip. Due to the structure of the string propagators on the annulus and on the Möbius strip, the contributions in the presence and in the absence of the B field for P and NO differ only by overall noncommutative factors. It follows that those contributions which become divergent in the field theory limit are the same whether B is there or not. Now in the ordinary $SO(N)$ case the divergent part comes from the planar contribution with a factor of $N$ in front, and from the NO contribution with a factor of $-2$. So altogether the divergent field theory part is proportional to $N - 2$, and therefore vanishes in the case $N = 2$. This is obvious from the ordinary field theory side, because the theory is free. However, as we noticed above, this conclusion holds also in the noncommutative case. Therefore the $NCSO(2)$ theory should not give rise to UV divergences.
Now let us look at the one–loop order on the noncommutative field theory side. The Feynman rules are very simple in this case since only the four–point vertex is nonvanishing. Let us rewrite the four–gluon vertex adapted to this case

\[-2ig^2 \left[ \cos(p \times r - q \times s) (\hat{g}_{ik}\hat{g}_{jl} + \hat{g}_{ij}\hat{g}_{kl} - 2\hat{g}_{il}\hat{g}_{jk}) \\
+ \cos(p \times s + q \times r) (\hat{g}_{il}\hat{g}_{jk} + \hat{g}_{ik}\hat{g}_{jl} - 2\hat{g}_{ij}\hat{g}_{kl}) \right] \]  

(6.4)

The one–loop correction is infinite. So the theory needs a renormalization. What is worse is that the divergent part is not of the form (6.4), but

\[\sim \frac{g^4}{\epsilon} \left[ \cos(p \times r - q \times s) (7\hat{g}_{ik}\hat{g}_{jl} + 7\hat{g}_{ij}\hat{g}_{kl} - 8\hat{g}_{il}\hat{g}_{jk}) \\
+ \cos(p \times s + q \times r) (7\hat{g}_{il}\hat{g}_{jk} + 7\hat{g}_{ik}\hat{g}_{jl} - 8\hat{g}_{ij}\hat{g}_{kl}) \right] \]  

(6.5)

In order to eliminate this divergence we need a counterterm of the form

\[\sim (7A_i \ast A^i \ast A_j \ast A^j - 4A_i \ast A_j \ast A^i \ast A^j) \]  

(6.6)

Therefore not only the $NCSO(2)$ gauge field theory is not finite, but the divergent part breaks the gauge symmetry. One might argue that $NCSO(N)$ gauge theories are nonlocal theories and it is perhaps too much hoping for another miracle like the renormalizability of noncommutative $U(N)$ theories to happen also in this case. However the fact the string theory with $so(N)$ CP factors in the presence of a $B$ field is well–behaved and its field theory limit is well–defined, suggests another possible solution to the puzzle. After a moment’s thought one realizes that the element where field theory and string theory diverge is not the Feynman rules themselves (or the action they come from) but their application in the one–loop calculation. We have applied them in the usual way, but that may be too naive. We would need a suitably modified set of rules. However so far we have not been able to modify the Feynman rules in such a way as to reconcile noncommutative field theory with the results from string theory. It should be recalled at this point that this reconciliation is certainly desirable but it might not be possible (without violating any fundamental principle, like locality, since the theory we are dealing with is nonlocal). If this turns out to actually be the case, it means that we have found an example of a discrepancy between string theory and the corresponding effective (noncommutative ) field theory at one–loop.

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25
A. Möbius strip notation

In section 2 and 4 we use the following notation for differentiation and integration on $\mathcal{M}$. We write $\rho = x + iy = re^{i\theta}$. Then

$$\partial_r = \frac{1}{r}(\rho \partial_\rho + \bar{\rho} \partial_{\bar{\rho}}), \quad \partial_\theta = i(\rho \partial_\rho - \bar{\rho} \partial_{\bar{\rho}})$$

and

$$\partial^2_r + \partial^2_\theta = 4\partial_\rho \partial_{\bar{\rho}} = \partial^2_r + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial^2_\theta$$

The normal derivatives and line elements along the boundary of fig.1 are defined as follows

$$\partial_\perp = \frac{1}{r} \partial_\theta, \quad dl = dr \quad \text{along} \quad AA'$$
$$\partial_\perp = -\frac{1}{r} \partial_\theta, \quad dl = -dr \quad \text{along} \quad B'B$$
$$\partial_\perp = \partial_\theta, \quad dl = -d\theta \quad \text{along} \quad A'B$$
$$\partial_\perp = -\partial_\theta, \quad dl = w d\theta \quad \text{along} \quad B'A$$

B. $SO(N)$ tensors

In this Appendix we collect the conventions relevant for the $so(N)$ Lie algebra tensors and traces. We denote the Lie algebra generators by $t^a$, where $a = 1, \ldots, \frac{N(N-1)}{2}$. They are real antisymmetric matrices with Lie bracket and normalization defined by

$$[t^a, t^b] = f_{abc} t^c, \quad \text{tr}(t^a t^b) = -\frac{1}{2} \delta^{ab}$$

(B.1)

$\text{tr}$ is the trace in the fundamental representation and summation over repeated indices is understood. With these conventions we find

$$\text{tr}(t^a t^b t^c) = -\frac{1}{4} f^{abc}$$

Unlike the $u(N)$ Lie algebra, $so(N)$ does not possess a third order invariant symmetric tensor. The fourth order invariant symmetric tensor is defined by means of

$$\text{Sym}(t^a t^b t^c) \equiv \frac{1}{6} \left( t^a t^b t^c + 5 \text{ permutations} \right) \equiv d^{abcd} t^d$$

(B.2)

We find

$$\text{tr}(t^a t^b t^c t^d) = -\frac{1}{2} d^{abcd} - \frac{1}{6} f^{abc} f^{xcd} + \frac{1}{12} f^{xac} f^{xbd}$$

(B.3)

Evaluating one–loop Feynman diagrams in field theory requires the corresponding traces in the adjoint representation. Let us denote by $F^a$ the matrices

$$(F^a)_{bc} = f^{abc}$$
and by Tr the traces in the vector space of the adjoint representation. Then one finds

$$\text{Tr}(F^a F^b) = -\frac{1}{2}(N-2)\delta^{ab}, \quad \text{Tr}(F^a F^b F^c) = \frac{1}{4}(N-2)f^{abc}$$  \hspace{1cm} (B.4)

and

$$\text{Tr}(F^a F^b F^c F^d) = \frac{N-2}{2}d^{abcd} + \frac{1}{4}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) + \frac{N-2}{12}\left(f^{abcd}f^{abcd} - f^{abc}f^{bcd}\right)$$  \hspace{1cm} (B.6)

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