A new multi-component two dimensional Toda lattice hierarchy and two dimensional Toda lattice with self-consistent sources

Xiaojun Liu*  
Department of Applied Mathematics,  
China Agricultural University, Beijing, 100083, PRC

Yunbo Zeng† and Runliang Lin‡  
Department of Mathematical Sciences,  
Tsinghua University, Beijing, 100084, PRC

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Abstract

We propose a new multi-component two-dimensional Toda lattice hierarchy (mc2dTLH) which includes two-dimensional Toda lattice equation with self-consistent sources (2dTLSCS) as the first non-trivial equation. The Lax representations for this mc2dTLH are presented. We construct a non-auto-Bäcklund Darboux transformation (DT) for 2dTLSCS by applying the method of variation of constant (MVC) to ordinary DT of 2dTLSCS. This non-auto-Bäcklund DT enables us to obtain various solutions such as solitons, rational solutions etc., to 2dTLSCS.

1 Introduction

Multi-component generalizations of soliton equations attract a lot of attention from both physical and mathematical points of view [1–8]. The multi-component KP (mcKP) hierarchy given by [1] contains physically relevant nonlinear integrable systems such as Davey-Stewartson equation, two-dimensional Toda lattice and three-wave resonant interaction equation. The multi-component Toda lattice hierarchy [8] contains non-abelian Toda lattice equation. There exist several equivalent formulations of this multi-component soliton equations. For example, there are matrix pseudo-differential operator (Sato) formulation, τ-function approach via matrix Hirota bilinear identities, multi-component free fermion formulation for mcKP hierarchy. For two dimensional Toda lattice hierarchy (2dTLH), a similar matrix-difference operator approach to multi-component hierarchy was also presented by [8].

Another kind of multi-component generalizations to soliton equations are the so-called soliton equations with self-consistent sources (SECS), which were initiated by V.K. Mel’nikov [9–11]. For two dimensional Toda lattice equation (2dTL), the corresponding 2dTL with self-consistent sources (2dTLSCS) was first presented in [12] by source generating method as follows

\[ q_{xy} = e^{q - q^{(-1)}} - e^{q^{(1)} - q} + \sum (w_i w_i^*) y, \]
\[ w_{i,y} = e^{q - q^{(-1)}} w_i^{(-1)} \]  
\[ (i = 1, \ldots, N), \]
\[ w_{i,y}^* = -e^{q^{(1)} - q} w_i^{(1)}. \]
In [13], we proposed a method to construct a new multi-component KP hierarchy which includes two kinds of KP equation with self-consistent sources presented by Mel’nikov [9]. In this paper, we will present a new multi-component 2dTLH which includes 2dTLSCS as the first non-trivial equation.

We briefly recall the framework of [8] as follows. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two series of variables. Let $n \in \mathbb{Z}$ be a discrete variable. Then Lax equation of 2dTLH is given by

\begin{align}
L_x &= [B_m, L], \\
L_y &= [C_m, L], \\
M_x &= [B_m, M], \\
M_y &= [C_m, M],
\end{align}

where

\begin{align}
L &= \Lambda + u_0 + u_1 \Lambda^{-1} + u_2 \Lambda^{-2} + \cdots, \\
M &= v_{-1} \Lambda^{-1} + v_0 + v_1 \Lambda + v_2 \Lambda^2 + \cdots,
\end{align}

$\Lambda$ is a shift operator such that $\Lambda f(n) = f(n+1)\Lambda$, $u_i$ and $v_i$ are functions of $x$, $y$ and $n$, $B_m = L^m$ stands for the positive part ($\geq 0$) of $L^m$ with respect to the powers of $\Lambda$ and $C_m = M^m$ stands for the negative part ($< 0$) of $M^m$. The commutativity of (2) gives rise to zero-curvature equations of 2dTLH

\begin{align}
B_{k,x} - B_{m,x} + [B_k, B_m] &= 0, \\
C_{k,y} - C_{m,y} + [C_k, C_m] &= 0, \\
B_{k,y} - C_{m,x} + [B_k, C_m] &= 0
\end{align}

When $m = k = 1$, (3) leads to following 2-dimensional Toda equation

\begin{align}
u_y &= v - v^{(1)}, \\
v_x &= v \left( u - u^{(-1)} \right),
\end{align}

and Lax pair for (4) is

\begin{align}
\psi_x &= B(\psi) = (\Lambda + u)(\psi), \\
\psi_y &= C(\psi) = (v\Lambda^{-1})(\psi),
\end{align}

where $x := x_1$, $y := y_1$, $B := B_1$, $C := C_1$, $u := u_0$, $v := v_{-1}$. Eliminating $u$ from (4) and introducing $q := q(n, x, y)$ which satisfies

\begin{align}v := \exp \left( q - q^{(-1)} \right),
\end{align}

then (4) gives the so-called two dimensional Toda lattice equation:

\begin{align}q_{xy} = \exp \left( q - q^{(-1)} \right) - \exp \left( q^{(1)} - q \right).
\end{align}

Our multi-component generalization to 2dTLH can be presented as follows. We first introduce a new vector field $\partial_{\bar{y}_k}$, which is a linear combination of all vector fields $\partial_{y_m}$. Then we get a new Lax type equation which consists of $\partial_{\bar{y}_k}$-flow and evolutions of wave functions. Under the evolutions of wave functions, the commutativity of $\partial_{\bar{y}_k}$, $\partial_{y_m}$ and $\partial_{x_k}$ flow give rise to the new multi-component 2dTL hierarchy (mc2dTLH). This hierarchy enables us to derive the 2dTLSCS in different way from [12, 14] and to obtain their Lax representations. This hierarchy is also different from mc2dTLH given by [8].

In the second part of our paper, we solve 2dTLSCS by means of Darboux transformations (DT). Since the Lax representation for 2dTLSCS is obtained, we can construct an auto-Bäcklund DT for 2dTLSCS,
which transforms between the solution of 2dTLSCS with same number of source terms. However, such auto-Bäcklund transformation can not be used to construct non-trivial solution from the trivial solution. The idea for us to construct non-auto-Bäcklund DTs is to consider 2dTLSCS as 2dTL with non-homogeneous terms (i.e. self-consistent source terms). Inspired by ODE method, we can apply the method of variation of constant (MVC) to DT to find a new non-auto-Bäcklund DT which transforms original solution of 2dTLSCS with \( N \) self-consistent sources to a new solution of 2dTLSCS with \( (N+1) \) self-consistent sources. This non-auto-Bäcklund DT enables us to find various solutions for 2dTLCs. Furthermore, we obtained the \( m \)-time repeated non-auto-Bäcklund DTs formula, and exhibit some solutions of 2dTLSCS which include solitons, rational solutions and etc.

Our paper will organized as follows. In section 2 we propose resolvent identities and present new mc2dTLH which includes 2dTLSCS. In section 3 we first construct auto-Bäcklund DT for 2dTLSCS. Then by applying method of variation of constant to this DT, we find a non-auto-Bäcklund DT which can increase the number of source term by 1. We obtain \( m \)-time repeated non-auto-Bäcklund DT formula which can be expressed in compact form with Casoratian determinants. In section 4 we present some solutions to 2dTLSCS by using this \( m \)-time repeated DTs.

### 2 New Multi-component 2-dimensional Toda Lattice Hierarchy

#### 2.1 Sato approach and resolvent identities

First we introduce some useful notations and definitions which can be found in [8].

**Definition 1** (The residue of shift operator). Let \( P = \sum_{i \in \mathbb{Z}} P_i \Lambda^i \), then residue of \( P \) is

\[
\text{Res}_\Lambda P = P_0.
\]

**Definition 2** (The adjoint operator \( \ast \)). \( P^* = \sum_{i \in \mathbb{Z}} \Lambda^{-i} P_i \).

**Definition 3** (Shift operator’s action). The shift operator action \( P(\lambda^n) \) can be defined as

\[
P(\lambda^n) = \sum_{i \in \mathbb{Z}} P_i \Lambda^i(\lambda^n) = (\sum_{i \in \mathbb{Z}} P_i \Lambda^i) \cdot \lambda^n.
\]

**Definition 4** (Formal inversion of difference operator \( \Delta \)). For difference operator \( \Delta = \Lambda - 1 \), the formal inversion are given by

\[
\Delta_+^{-1} = -\sum_{i \geq 0} \Lambda^i \quad \text{or} \quad \Delta_-^{-1} = \sum_{i \leq -1} \Lambda^i.
\]

Introduce wave operators

\[
\hat{W}^{(\infty)} = b_0 + b_1 \Lambda^{-1} + b_2 \Lambda^{-2} + \cdots,
\]

\[
\hat{W}^{(0)} = c_0 + c_1 \Lambda + c_2 \Lambda^2 + \cdots,
\]

where \( b_0 = 1 \). Ueno and Takasaki proved [8] that if \( L \) and \( M \) are solutions to (2) then there exist wave operators, such that \( L \) and \( M \) can be written as

\[
L = \hat{W}^{(\infty)} \Lambda \hat{W}^{(\infty)}^{-1}, \quad M = \hat{W}^{(0)} \Lambda^{-1} \hat{W}^{(0)}^{-1},
\]

and

\[
\partial_{x_m} \hat{W}^{(\infty)} = -L_{\leq 0}^{m} \hat{W}^{(\infty)}, \quad \partial_{x_m} \hat{W}^{(0)} = L_{\geq 0}^{m} \hat{W}^{(0)},
\]

\[
\partial_{y_m} \hat{W}^{(\infty)} = M_{\geq 0}^{m} \hat{W}^{(\infty)}, \quad \partial_{y_m} \hat{W}^{(0)} = -M_{\leq 0}^{m} \hat{W}^{(0)}.
\]
Define wave function
\[ w^{(\infty)} = \hat{W}^{(\infty)}(\lambda^0) e^{\xi(x,\lambda)} = \left( \sum_{i \geq 0} b_i \lambda^{-i} \right) \lambda^n e^{\xi(x,\lambda)} := \hat{w}^{(\infty)}(\lambda^n) e^{\xi(x,\lambda)}, \]
\[ w^{(0)} = \hat{W}^{(0)}(\lambda^0) e^{\xi(y,\lambda^{-1})} = \left( \sum_{i \geq 0} c_i \lambda^i \right) \lambda^n e^{\xi(y,\lambda^{-1})} := \hat{w}^{(0)}(\lambda^n) e^{\xi(y,\lambda^{-1})}, \]
and adjoint wave function
\[ w^{(\infty)*} = \hat{W}^{(\infty)*} - 1(\lambda^{-n}) e^{-\xi(x,\lambda)} := \hat{w}^{(\infty)*}(\lambda^{-n}) e^{-\xi(x,\lambda)}, \]
\[ w^{(0)*} = \hat{W}^{(0)*} - 1(\lambda^{-n}) e^{-\xi(y,\lambda^{-1})} := \hat{w}^{(0)*}(\lambda^{-n}) e^{-\xi(y,\lambda^{-1})}, \]
where \( \xi(x,\lambda) = \sum_{i \geq 1} x_i \lambda^i, \xi(y,\lambda^{-1}) = \sum_{i \geq 1} y_i \lambda^{-i} \). Then (9) can also be given as the compatibility condition of the following linear evolution equations
\[ L w^{(\infty)} = \lambda w^{(\infty)}, \quad M w^{(0)} = \lambda^{-1} w^{(0)}, \]
\[ \partial x_m w^{(\infty)} = B_m \hat{w}^{(\infty)}, \quad \partial x_m w^{(0)} = B_m \hat{w}^{(0)}, \]
\[ \partial y_m w^{(\infty)} = C_m w^{(\infty)}, \quad \partial y_m w^{(0)} = C_m w^{(0)}. \]

**Lemma 1** (Ueno and Takasaki [8]). Suppose \( P = \sum P_i \Lambda^i, Q = \sum Q_j \Lambda^j \), then
\[ \text{Res}_A P \cdot Q^* = \text{Res}_A \lambda^{-1} P(\lambda^n) \cdot Q(\lambda^{-n}). \]

**Proof.** Only need to show for \( P = P_i \Lambda^i, Q = Q_j \Lambda^j \). \( \text{Res}_A P Q^* = \text{Res}_A P_i \Lambda^{i-j} Q_j = \delta_{i,j} P_i Q_j \), while \( \text{Res}_A \lambda^{-1} P(\lambda^n)Q(\lambda^{-n}) = \text{Res}_A \lambda^{-1} P_i \Lambda^{n+i} Q_j \lambda^{-n-j} = \delta_{i,j} P_i Q_j. \)

Similar to the KP theory, in which the principle part of resolvent can be expressed in terms of a quadratic form of wave function and adjoint wave function [15], we have the following resolvent identities for 2dTLH.

**Proposition 1** (Resolvent identities).
\[ \sum_{k \geq 0} L_{<0}^k \lambda^{-k} = -w^{(\infty)} \Delta_+^{-1} w^{(\infty)*}, \quad \sum_{k \geq 0} M_{>0}^k \lambda^k = -w^{(0)} \Delta_+^{-1} w^{(0)*}, \]
\[ \sum_{k \in \mathbb{Z}} L_{<0}^k \lambda^{-k} = w^{(\infty)} \Delta_-^{-1} w^{(\infty)*}, \quad \sum_{k > 0} M_{\leq 0}^k \lambda^k = w^{(0)} \Delta_-^{-1} w^{(0)*}. \]

**Proof.** We prove one of them, others are similar. Since \( L = \hat{W}^{(\infty)} \hat{W}^{(\infty)*} - 1, \)
\[ L_{<0}^k = \left( \hat{W}^{(\infty)} \Lambda^k \hat{W}^{(\infty)*} - 1 \right)_{<0} \]
\[ = \sum_{m \geq 1} \text{Res}_A \left( \hat{W}^{(\infty)} \Lambda^k \hat{W}^{(\infty)*} - 1 \lambda^m \right) \Lambda^{-m} \]
\[ = \sum_{m \geq 1} \text{Res}_A \lambda^{-1} \left( \hat{W}^{(\infty)} \Lambda^k (\lambda^n) e^{\xi(x,\lambda)} \right) \left( \Lambda^{-m} \hat{W}^{(\infty)*} - 1 (\lambda^{-n}) e^{-\xi(x,\lambda)} \right) \Lambda^{-m} \]
\[ = \sum_{m \geq 1} \text{Res}_A \lambda^{-1} w^{(\infty)} \Lambda^{-m} w^{(\infty)*} \]
\[ = \text{Res}_A \lambda^{-1} w^{(\infty)} \Delta_+^{-1} w^{(\infty)*}. \]
So \( \sum_{k \in \mathbb{Z}} L_{<0}^k \lambda^{-k} = w^{(\infty)} \Delta_-^{-1} w^{(\infty)*}. \)
2.2 New mc2dTL hierarchy

For fixed $k \geq 1$, $N > 0$, we define a new time variable $\bar{y}_k$ such that the corresponding vector field is

$$\partial_{\bar{y}_k} = \partial_{y_k} + \sum_{i=1}^{N} \sum_{j>1} \lambda_i^j \partial_{y_j}$$

(10)

where $\lambda_i$ are distinct arbitrary non-zero parameters. Then the $\bar{y}_k$ flow is given by

$$\partial \bar{C}_k = \partial \bar{C}_k$$

(11)

where

$$\bar{C}_k = C_k + \sum_{i=1}^{N} \sum_{j>1} \lambda_i^j C_j,$$

which, according to Proposition 1, can be rewritten as

$$\bar{C}_k = C_k + \sum_{i=1}^{N} w_i^{(0)} \Delta^{-1} w_i^{(0)*}.$$

By setting $w_i = w_i^{(0)}$, $w^*_i = w_i^{(0)*}$, the compatibilities of (2) and (11) give rise to the following new multi-component two dimensional Toda lattice hierarchy.

Proposition 2. We have the following new mc2dTLH, for $m \neq k$:

$$B_{m,x_k} - B_{k,x_m} + [B_m, B_k] = 0,$$

(12a)

$$C_{m,y_k} - \bar{C}_{k,y_m} + [C_m, \bar{C}_k] = 0,$$

(12b)

$$B_{m,y_k} - \bar{C}_{k,x_m} + [B_m, \bar{C}_k] = 0,$$

(12c)

$$B_{k,y_m} - C_{m,x_k} + [B_k, C_m] = 0,$$

(12d)

$$w_{i,x_m} = B_m(w_i), \quad w_{i,y_m} = C_m(w_i) \quad (i = 1, \ldots, N),$$

(12e)

$$w^*_{i,x_m} = -B^*_m(w^*_i), \quad w^*_{i,y_m} = -C^*_m(w^*_i),$$

(12f)

for $m = k$:

$$B_{k,y_k} - \bar{C}_{k,x_k} + [B_k, \bar{C}_k] = 0,$$

(13a)

$$\partial_{x_k} w_i = B_k(w_i), \quad \partial_{x_k} w^*_i = -B^*_k(w^*_i) \quad (i = 1, \ldots, N)$$

(13b)

where $\bar{C}_k = C_k + \sum_{i=1}^{N} w_i \Delta^{-1} w_i^*$.

It is worth noting that $w_i$ and $w_i^*$ need not necessarily to be the wave function and adjoint wave function. In fact, the equations (12c) and (12d) (or (13b)) ensure the closeness of (12a)-(12d) (or (13a)). Furthermore, under the conditions (12e) and (12f) (or (13b)), one can easily obtains the Lax representations of (12a)-(12d) as

$$\psi_{x_m} = B_m(\psi), \quad \psi_{x_k} = B_k(\psi),$$

(14a)

$$\psi_{y_m} = C_m(\psi), \quad \psi_{y_k} = \bar{C}_k(\psi),$$

(14b)

or get the Lax representation of (13a) as

$$\psi_{x_k} = B_k(\psi), \quad \psi_{y_k} = \bar{C}_k(\psi).$$

(15)
**Example 1** (Two dimensional Toda lattice equation with Self-consistent sources). When \( m = k = 1 \), let \( u = u_0, \ v = v_{-1}, \ x = x_1, \ y = y_1 \)

\[
B_1 = \Lambda + u, \quad C_1 = v\Lambda^{-1}.
\]

then (13) becomes

\[
\begin{align*}
uy &= -\Delta(v + \sum_{i=1}^{N} w_i w_i^{(-1)}), \quad vx = v(u - u^{(-1)}), \\
w_{i,x} &= B_1(w_i), \quad w_i^{*} = -B_1^*(w_i^*), \quad i = 1, \ldots, N
\end{align*}
\]

Under \( u = q_x, \ v = \exp(q - q^{(-1)}) \), (16) yields

\[
\begin{align*}
q_{xy} &= e^q - q^{(-1)} - e^q - q + \sum_{i=1}^{N} (w_i w_i^*) x, \\
w_{i,x} &= w_i^{(1)} + q_x w_i, \quad (i = 1, \ldots, N) \\
w_i^{*} &= -w_i^{(-1)} - q_x w_i^*.
\end{align*}
\]

This is two dimensional Toda lattice equation with \( N \) self-consistent sources (2dTLSCS).

Analogously, let us introduce \( \bar{x}_k \), such that

\[
\partial_{\bar{x}_k} = \partial_{x_k} + \sum_{i=1}^{N} \sum_{j \geq 1} \lambda_i^{-j} \partial_{x_k},
\]

then we will get another new multi-component two dimensional Toda lattice hierarchy.

**Proposition 3.** We have another new mc2dTLH as follows, for \( m \neq k \)

\[
\begin{align*}
B_{m,\bar{x}_k} - \bar{B}_{k,m} + [B_{m}, \bar{B}_k] &= 0, \quad (18a) \\
C_{m,\bar{x}_k} - \bar{B}_{k,y_m} + [C_{m}, \bar{B}_k] &= 0, \quad (18b) \\
C_{m,y_k} - C_{k,\bar{y}_m} + [C_{m}, \bar{C}_k] &= 0, \quad (18c) \\
B_{m,y_k} - C_{k,x_m} + [B_{m}, \bar{C}_k] &= 0, \quad (18d) \\
w_{i,y_m} &= C_{m}(w_i), \quad w_{i,x_m} = B_{m}(w_i), \quad i = 1, \ldots, N, \quad (18e) \\
w_i^{*},y_m &= -C^*_m(w_i^*), \quad w_i^{*},x_m = -B^*_m(w_i^*). \quad (18f)
\end{align*}
\]

for \( m = k \)

\[
\begin{align*}
C_{k,\bar{x}_k} - \bar{B}_{k,y_k} + [C_{k}, \bar{B}_k] &= 0, \quad (19a) \\
\partial_{y_k} w_i &= C_{k}(w_i), \quad \partial_{y_k} w_i^{*} = -C_{k}^*(w_i^*). \quad i = 1, \ldots, N, \quad (19b)
\end{align*}
\]

where

\[
\bar{B}_k = B_k - \sum_{i=1}^{N} w_i \Delta_i^{-1} w_i^*.
\]

**Example 2** (2dTLSCS [12, 14]). When \( m = k = 1 \), (19) leads to (11). It is interesting to see that (11) is equivalent to (17) under

\[
\begin{align*}
x &\rightarrow -y, \quad y \rightarrow -x, \quad q \rightarrow q, \\
w_i &\rightarrow -e^q w_i, \quad w_i^{*} \rightarrow e^{-q} w_i.
\end{align*}
\]

So hereafter we may concentrate on 2dTLSCS (16). This transformation was discovered by Prof. Hu Xingbiao.

6
3 Darboux transformation for 2dTLSCS

In the second part of our paper, we concentrate on 2dTLSCS (16). First recall the Lax pair of 2dTL equation (5). For convenience, hereafter we denote $B = B_1$, $C = C_1$, $u = u_0$, $v = v_1$, $x = x_1$, $y = y_1$.

3.1 Applying the method of variation of constant to DT of 2dTLSCS

Let us first introduce the notion of Casoratian determinant: for $m$ discrete variable functions $h_1, \ldots, h_m$, the Casoratian determinant

$$\text{cas}(h_1, \ldots, h_m) = \left| \begin{array}{ccc} h_1 & \cdots & h_m \\ h_1^{(1)} & \cdots & h_m^{(1)} \\ \vdots & \ddots & \vdots \\ h_1^{(m-1)} & \cdots & h_m^{(m-1)} \end{array} \right|.$$

Darboux transformation for 2dTLSCS (4) was given in [16]. Let us first recall this DT and its proof as following Lemma.

**Lemma 2.** Let $h$ be special solution to (4). Let $D = \Lambda + \sigma, \sigma := -h^{(1)}/h$, then DT

$$\tilde{u} := u^{(1)} + \sigma^{(1)},$$

$$\tilde{v} := v\sigma/\sigma^{(-1)},$$

$$\tilde{\psi} := D(\psi) = \frac{\text{cas}(h,\psi)}{h},$$

gives a new solution to (4). Thus $\tilde{u}, \tilde{v}$ are new solution for (4).

**Proof.** Since $\tilde{B} := \Lambda + \tilde{u}$, $\tilde{C} := \tilde{u}\Lambda^{-1}$, $\tilde{\psi} = D(\psi)$, a sufficient condition such that (21) holds is

$$D_x + \tilde{B}D = 0,$$
$$D_y + \tilde{C}D = 0.$$

Notice that

$$D(h) = 0,$$

take partial derivative $\partial_x, \partial_y$ to (22), one gets

$$D_x(h) + D(h_x) = (D_x + \tilde{B})D(h) = \tilde{B}D(h) = 0,$$
$$D_y(h) + D(h_y) = (D_y + \tilde{C})D(h) = \tilde{C}D(h) = 0,$$

which mean

$$D_x(h) + \tilde{B}D(h) = 0,$$
$$D_y(h) + \tilde{C}D(h) = 0.$$

From (20a) and (20b) one knows the operators acting on $h$ in (23) are scalar functions multiplications. So (21) holds.

The Lax representation for 2dTLSCS (16) is

$$\psi_x = B(\psi),$$
$$\psi_y = (C + \sum_{i=1}^{N} w_i \Delta_1^{-1} w^*_i)(\psi).$$
Note that Lax representation (24) holds under following equations
\[
\begin{align*}
    w_{i,x} &= B(w_i), \quad i = 1, \ldots, N, \\
    w^*_{i,x} &= -B^*(w_i).
\end{align*}
\]

(24c) (24d)

**Proposition 4** (Darboux transformation for 2dTLSCS (24)). Let \( h \) be a special solution to (24), \( D = \Lambda + \sigma \), \( \sigma := -h^{-1}/h \). Based on the Darboux transformation (20), define
\[
\begin{align*}
    \tilde{w}_i &:= D(w_i) = \frac{\text{cas}(h, w_i)}{h}, \\
    \tilde{w}_i^* &:= D^*^{-1}(w_i^*) = -\frac{S(hw_i^*)}{h^{(1)}},
\end{align*}
\]
(25a) (25b)

where \( S := \Lambda \Delta^{-1} \). Then (20) and (25) together give a new solution to (24). Thus one gets a new solution to (16).

**Proof.** From Lemma 2 it is easy to see that \( \tilde{w}_i \) defined by (25a) satisfies (24c). It is necessary to prove \( \tilde{w}_i^* \) satisfies (24d). From the proof of Lemma 2 we know
\[
(\partial_x - \tilde{B})D = D(\partial_x - B).
\]
Taking formal adjoint * to this equality and rewrite it as
\[
(-\partial_x - \tilde{B}^*)D^*^{-1} = D^*^{-1}(-\partial_x - B^*).
\]
This is a sufficient condition for \( D^*^{-1} \) to be the Darboux transformation for (24). Thus we have proved (25b). At last we need to prove that Darboux transformation given by (20) and (25) fulfills (24b). That is
\[
D_y + D_C + \sum_{i=1}^N Dw_i \Delta^{-1} w_i^* - C_D - \sum_{i=1}^N \tilde{w}_i \Delta^{-1} \tilde{w}_i^* D = 0.
\]
(26)

Based on (21a), we have to prove the extra terms w.r.t. \( w_i, w_i^* \) in (26) are equal. For every \( i \), we have
\[
\begin{align*}
    - \frac{w_i^{(1)} S(hw_i^*)}{h} + \frac{h^{(1)}}{h^2} w_i \Delta^{-1}(hw_i^*) &+ w_i^{(1)} \Lambda \Delta^{-1} w_i^* \\
    - \frac{h^{(1)}}{h} w_i \Delta^{-1} w_i^* - \tilde{w}_i \Delta^{-1} \tilde{w}_i^* &+ \tilde{w}_i \Delta^{-1} \tilde{w}_i^* \frac{h^{(1)}}{h} \\
    = -\tilde{w}_i \Delta^{-1} \tilde{w}_i^* &+ \frac{h^{(1)}}{h} w_i \Delta^{-1} w_i^* + w_i^{(1)} \Lambda \Delta^{-1} w_i^* \\
    - \frac{h^{(1)}}{h} w_i \Delta^{-1} w_i^* &+ \tilde{w}_i \Delta^{-1} \tilde{w}_i^* \frac{h^{(1)}}{h} \\
    = -\tilde{w}_i \Delta^{-1} \tilde{w}_i^* &+ w_i^{(1)} \Lambda \Delta^{-1} w_i^* - \frac{h^{(1)}}{h} w_i \Lambda \Delta^{-1} w_i^* \\
    + \tilde{w}_i \Delta^{-1} \tilde{w}_i^* &- \tilde{w}_i \Delta^{-1} \tilde{w}_i^* = 0.
\end{align*}
\]
\]

(27a) (27b)

**Theorem 1** (Darboux transformation and method of variation of constant for 2dTLSCS (16)). Let \( f \) and \( g \) be two linear independent solutions to (24). Suppose \( a(y) \) is arbitrary functions of time \( y \). Let \( h := f + a(y)g \),
\[
\begin{align*}
    \tilde{w}_{N+1} &= c(y) D(f), \\
    \tilde{w}_{N+1} &= \frac{d(y)}{h^{(1)}},
\end{align*}
\]
(27a) (27b)
then (20), (23) and (24) give a new solution for (16) and (24) with \( N + 1 \) self-consistent sources, where \( c(y), d(y) \) satisfy \( c(y)d(y) = \partial_y \log a(y) \).

**Proof.** It is easy to see \( \tilde{w}_{N+1} \) satisfies (24). To prove that \( \tilde{w}_{N+1}^* \) satisfies (24d), we have

\[
\tilde{w}_{N+1}^* = \frac{2}{h} \frac{d(y)}{h} \frac{\partial_y \log a(y)}{h} - \frac{d(y)}{h} \frac{\partial_y \log a(y)}{h} = \frac{d(y)}{h} \frac{\partial_y \log a(y)}{h}.
\]

Based on proposition 4, we want to show that extra terms come out from

\[
\mathcal{D}_y + \mathcal{D} \left( \tilde{w}_{N+1} - \sum_{i=1}^{N} w_i \Delta_* \tilde{w}_i^* \right) - \left( \tilde{C} + \sum_{i=1}^{N+1} \tilde{w}_i \Delta_* \tilde{w}_i^* \right) \mathcal{D}
\]

can be canceled out. It because

\[
- \frac{a_y y}{h} + \frac{h}{h^2} - \frac{\tilde{w}_{N+1} \Delta_1 \tilde{w}_{N+1} \Lambda - h}{h} = \frac{a_y h^2}{h^2} \frac{y}{h} + \frac{h}{h^2} \frac{\partial_y \log a(y)}{h} = \frac{a_y h^2}{h^2} \frac{y}{h} = 0.
\]

\[\square\]

### 3.2 \( m \)-time repeated non-auto-Bäcklund DTs

**Theorem 2.** Let \( f_j \) and \( g_j \) (\( j = 1, 2, \ldots, m \)) be \( m \) pairs of independent solutions to (24). Suppose \( a_j(y) \) are arbitrary functions of time. Let

\[ h_j := f_j + a_j(y)g_j. \]

Then after \( m \)-time repetition of Theorem 4 we find a solution for (24) with \( N + m \) self-consistent sources, which is

\[
\begin{align*}
\tilde{u}[m] &= \tilde{u} + \frac{\tilde{c}(h_1, \ldots, h_m)}{\tilde{c}(h_1, \ldots, h_m)} \tilde{c}(h_1, \ldots, h_m), \\
\tilde{v}[m] &= \tilde{v} \frac{\tilde{c}(h_1, \ldots, h_m)}{\tilde{c}(h_1, \ldots, h_m)}, \\
\tilde{w}_i[m] &= \tilde{w}_i \frac{\tilde{c}(h_1, \ldots, h_m, w_i)}{\tilde{c}(h_1, \ldots, h_m)}, \quad i = 1, \ldots, N, \\
\tilde{w}_i^*[m] &= (-1)^m \frac{\tilde{c}(h_1, \ldots, h_m, w_i^*)}{\tilde{c}(h_1, \ldots, h_m)}, \quad i = 1, \ldots, N, \\
\tilde{w}[m + j] &= c_j(y)f_j[m] = c_j(y) \frac{\tilde{c}(h_1, \ldots, h_m, f_j)}{\tilde{c}(h_1, \ldots, h_m)}, \quad j = 1, \ldots, m, \\
\tilde{w}[m + j] &= (-1)^{m-j} d_j(y) \frac{\tilde{c}(h_1, \ldots, h_m, f_j)}{\tilde{c}(h_1, \ldots, h_m)}, \quad j = 1, \ldots, m.
\end{align*}
\]

where

\[
\tilde{c}(h_1, \ldots, h_m) = \left| \begin{array}{ccc}
h_1 & \cdots & h_m \\
\vdots & \ddots & \vdots \\
h_1^{(m-2)} & \cdots & h_m^{(m-2)} \\
h_1^{(m-2)} & \cdots & h_m^{(m-2)} \\
h_1^{(m)} & \cdots & h_m^{(m)} \\
h_1^{(m)} & \cdots & h_m^{(m)}
\end{array} \right|, \quad \tilde{c}(h_1, \ldots, h_m, f) = \left| \begin{array}{ccc}
S(h_1) & \cdots & S(h_m) \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
S(h_1) & \cdots & S(h_m)
\end{array} \right|
\]
and \( c_j(y) d_j(y) = \partial_y \log a_j(y) \).

**Proof.** Since each time Darboux transformation has the form \( D = \Lambda + \sigma \), after \( m \)-time repetition, corresponding operator has the form

\[
D(m) = \Lambda^m + \sigma_{m-1} \Lambda^{m-1} + \cdots + \sigma_0.
\]

There are \( m \) indetermined coefficients \( \sigma_i, \ i = 0, \ldots, m-1 \). From (22) we know

\[
D(m) h_i = 0, \quad i = 1, 2, \ldots, m.
\]

So the indeterminded coefficients satisfies

\[
\begin{bmatrix}
  h_1 & h_1^{(1)} & \cdots & h_1^{(m-1)} \\
  h_2 & h_2^{(1)} & \cdots & h_2^{(m-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_m & h_m^{(1)} & \cdots & h_m^{(m-1)} \\
\end{bmatrix}
\begin{bmatrix}
  \sigma_0 \\
  \sigma_1 \\
  \vdots \\
  \sigma_{m-1} \\
\end{bmatrix} =
\begin{bmatrix}
  h_1^{(m)} \\
  h_2^{(m)} \\
  \vdots \\
  h_m^{(m)} \\
\end{bmatrix}.
\]

By Cramer rule, it is easy to see

\[
\sigma_0 = (-1)^m \frac{\text{cas}^{(1)}(h_1, \ldots, h_m)}{\text{cas}(h_1, \ldots, h_m)}, \quad \sigma_{m-1} = -\frac{\text{cas}(h_1, \ldots, h_m)}{\text{cas}(h_1, \ldots, h_m)}.
\]

Note that DT of \( u, v \) are the same for 2dTL and 2dTLSCS. So we may omit source term temporary. That is, assuming that \( D(m) \) transforms \( \psi \) to \( \tilde{\psi} = \psi\lbrack m\rbrack \), and satisfying \( \tilde{\psi}_x = \tilde{B}\psi \) and \( \tilde{\psi}_y = \tilde{C}\psi \), then we have

\[
\begin{align*}
D(m)_x + D(m)B - \tilde{B}D(m) &= 0, \quad (29a) \\
D(m)_y + D(m)C - \tilde{C}D(m) &= 0. \quad (29b)
\end{align*}
\]

Comparing the coefficient of \( \Lambda^m \) in (29a), we have (28a). Comparing the coefficient of \( \Lambda^{-1} \) in (29b), we have (28c). For arbitrary eigenfunction \( w \), its DT \( \tilde{w} = D(m)(w) \) can be expressed in a compact form (28a) according to the Laplace expansion formula. For (28c), we need induction. Suppose for any adjoint eigenfunction \( w^* \), the \( m \)-time DT formula is correct, then by (25b), the \( m + 1 \)-th DT is

\[
w^*[m + 1] = - \frac{S(h_{m+1}[m + 1]w^*[m + 1])}{h_{m+1}[m]^{(1)}}
\]

\[
= \frac{S \left[ \Delta \left( \frac{h_{m+1}[m + 1]}{h_m[m+1]^{(1)}} S(h_m[m - 1]w^*[m - 1]) \right) - \frac{h_m[m - 1]^{(1)}w^*[m - 1]}{h_m[m-1]^{(1)}} \right]}{h_{m+1}[m]^{(1)}}
\]

\[
= \frac{h_m[m - 1]^{(1)} S(h_m[m - 1]w^*[m - 1])}{h_{m+1}[m]^{(1)} h_m[m - 1]^{(1)}} - \frac{S(h_{m+1}[m - 1]w^*[m - 1])}{h_{m+1}[m]^{(1)}}.
\]

By assumption

\[
\frac{S(h_m[m - 1]w^*[m - 1])}{h_m[m - 1]^{(1)}} = (-1)^{m+1} \frac{\text{cas}(h_1, \ldots, h_m, w^*)}{\text{cas}^{(1)}(h_1, \ldots, h_m)},
\]

\[
\frac{S(h_{m+1}[m - 1]w^*[m - 1])}{h_{m+1}[m - 1]^{(1)}} = (-1)^{m+1} \frac{\text{cas}(h_1, \ldots, h_{m+1}, w^*)}{\text{cas}^{(1)}(h_1, \ldots, h_{m+1}, h_{m+1})},
\]

we know

\[
w^*[m + 1] = \frac{(-1)^{m+1}}{\text{cas}^{(1)}(h_1, \ldots, h_m) \text{cas}^{(1)}(h_1, \ldots, h_{m+1})}
\]
Let us start from trivial solutions. We have:

\[ w_{\cdots, h_m, w^*} \text{cas}^{(1)} (h_1, \cdots, h_m, h_{m+1}) \]

Next, we want to prove the formula for source terms.

Next, we want to prove the formula for source terms. Firstly, it is easy to see

\[ w_{N+j}[m] = c_j(y)f_j[m] \]

is given by (28f), then

\[ w_{N+j}[m+1] = \frac{S(h_{m+1}[m] w_{N+j}[m])}{h_{m+1}[m]} \]

Because \( h_{m+1}[m] \) is obtained by \( m \)-time repetition of DT by using \( h_1, \cdots, h_m \) sequentially, which is equivalent to \( m \)-time repetition of DT by successively using \( h_1, \cdots, h_{j-1}, h_j+1, \cdots, h_m \) and at last \( h_j \),

\[ h_{m+1}[m] = h_{m+1}[m-1]^{(1)} - \frac{h_j[m-1]^{(1)}}{h_j[m-1]} h_{m+1}[m-1] \]

Note that (28f), we have \( w_{N+j}[m] = \frac{d_j(y)}{h_{m+1}[m]} \), so

\[ w_{N+j}[m+1] = \frac{d_j(y)}{h_{m+1}[m]} S \Delta \left( \frac{h_{m+1}[m-1]}{h_j[m-1]} \right) \]

When \( j = m+1 \),

\[ w_{N+m+1}[m+1] = \frac{d_{m+1}(y)}{h_{m+1}[m]+1} = \frac{d_{m+1}(y) \text{cas}^{(1)} (h_1, \cdots, h_m)}{\text{cas}^{(1)} (h_1, \cdots, h_{m+1})} \]

\[ \square \]

4 Solutions for 2dTLSCS

Let us start from trivial solution \( q = 1, \nu = 1, u = 0, N = 0 \) for 2dTLSCS (24). The Lax pair reads

\[ \psi_x = \psi^{(1)} \]

\[ \psi_y = \psi^{(-1)} \]
4.1 Solitons

Equations (30) have two linearly independent solutions

\[ f(n,x,y) = \exp(n\omega + zx + z^{-1}y), \quad g(n,x,y) = \exp(-n\omega + z^{-1}x + zy), \]

where \( z = e^\omega \). Let \( a(y) = e^{\alpha(y)} \), then

\[ h = f + a(y)g = 2\exp \Omega \cdot \cosh Z, \]

where \( \Omega = \cosh \omega \cdot x + \cosh \omega \cdot y + \alpha/2, \quad Z = n\omega + \sinh \omega \cdot x - \sinh \omega \cdot y - \alpha/2. \)

By (28), taking \( m = 1 \) we have the following 1-soliton solution for (16)

\[ u_1 = \cosh(Z + 2\omega) \cosh(Z + \omega) - \cosh(Z + \omega) \cosh Z, \]
\[ v_1 = \frac{\cosh(Z + \omega) \cosh(Z - \omega)}{\cosh^2 Z}, \]
\[ w_1 = c(y) \frac{\sinh \omega \cdot e^\Omega}{\cosh Z}, \]
\[ w^*_1 = \frac{d(y) e^{-\Omega}}{2 \cosh(Z + \omega)}, \]

where \( c(y)d(y) = \dot{\alpha}. \)

If take two pairs of independent solutions, with respect to \( z_j = e^{\omega_j} (j = 1,2) \), i.e.

\[ f_j = \exp(n\omega_j + z_j x + z_j^{-1}y), \quad g_j = \exp(-n\omega_j + z_j^{-1}x + z_jy) \quad j = 1,2. \]

Let \( a_j(y) = e^{\alpha_j(y)} \), then

\[ h_j = f_j + a_jg_j = 2\exp \Omega_j \cdot \cosh Z_j, \]

where \( \Omega_j = \cosh \omega_j \cdot x + \cosh \omega_j \cdot y + \alpha_j/2, \quad Z_j = n\omega_j + \sinh \omega_j \cdot x - \sinh \omega_j \cdot y - \alpha_j/2. \)

To simplify the notion, for \( k \in \mathbb{Z} \), define

\[
H_k = \begin{vmatrix}
\cosh Z_1 & \cosh Z_2 \\
\cosh(Z_1 + k\omega_1) & \cosh(Z_2 + k\omega_2)
\end{vmatrix}
= \sinh \frac{k(\omega_1 - \omega_2)}{2} \sinh \left( Z_1 + Z_2 + \frac{k}{2}(\omega_1 + \omega_2) \right)
+ \sinh \frac{k(\omega_1 + \omega_2)}{2} \sinh \left( Z_1 - Z_2 + \frac{k}{2}(\omega_1 - \omega_2) \right).\]
Then 2-soliton solution for (16) is
\[ u[2] = \frac{H_1^{(1)}}{H_1^{(-1)}} - \frac{H_2}{H_1}, \]
\[ v[2] = \frac{H_1^{(1)} H_1^{(-1)}}{H_1^2}, \]
\[ w_1[2] = c_1(y) a_1(y) \frac{2 \sinh \omega_1 (\cosh \omega_1 - \cosh \omega_2) \exp \Omega_1 \cosh(Z_2 + \omega_2)}{H_1}, \]
\[ w_2[2] = c_2(y) a_2(y) \frac{2 \sinh \omega_2 (\cosh \omega_1 - \cosh \omega_2) \exp \Omega_2 \cosh(Z_1 + \omega_1)}{H_1}, \]
\[ w_3^*[2] = \frac{d_1(y) \exp(-\Omega_1) \cosh(Z_2 + \omega_2)}{H_1^{(1)}}, \]
\[ w_4^*[2] = \frac{d_2(y) \exp(-\Omega_2) \cosh(Z_1 + \omega_1)}{H_1^{(1)}}, \]
where \( c_j(y) d_j(y) = \alpha_j \).

### 4.2 Rational solution

In equation (30), noticing that \( \partial^k \psi \) is another solution. Since \( g(n, x, y) = z^n \exp(zx + z^{-1}y) \) and \( f_k(n, x, y) := \partial^k \psi \) \((k \geq 1)\) are all independent solutions for (30). Let \( \xi := zx + z^{-1}y \), then
\[ f_1(n, x, y) = \partial_x g = z^{n-1} e^\xi (n + z \xi_x), \]
\[ f_2(n, x, y) = \partial^2_x g = z^{n-2} e^\xi (n^2 + 2n z \xi_x - n + z^2 \xi_x^2 + z^2 \xi_{xx}), \]
\[ f_3(n, x, y) = \cdots \]
Let \( h_k = f_k + a(y) g \). Take \( k = 1, m = 1 \) in (28), one yields
\[ u[1] = -\frac{z}{(\eta + za + 1/2)^2 - 1/4}, \]
\[ v[1] = 1 - \frac{1}{(\eta + za)^2}, \]
\[ w[1] = c(y) \frac{z^{n+1} e^\xi a}{\eta + za}, \]
\[ w^*[1] = d(y) \frac{z^{-n} e^{-\xi}}{\eta + za + 1}, \]
where \( \eta = n + z \xi_x, c(y) d(y) = \frac{d}{dy} \log a \). This is a rational solution for (16).

If take \( k = 2, m = 1 \), we find another rational solution for (16)
\[ u[1] := z \left( \frac{\eta^{(2)} + z^2 a}{\eta^{(1)} + z^2 a} - \frac{\eta^{(1)} + z^2 a}{\eta + z^2 a} \right), \]
\[ v[1] := \frac{(\eta^{(1)} + z^2 a)(\eta^{(-1)} + z^2 a)}{(\eta + z^2 a)^2}, \]
\[ w[1] := 2 c(y) a z^{n+1} e^\xi \frac{n + z \xi_x}{\eta + z^2 a}, \]
\[ w^*[1] := d(y) \frac{z^{-n+1} e^{-\xi}}{\eta^{(1)} + z^2 a}, \]
where \( \eta = n^2 + 2nz \xi_x - n + z^2 \xi_x^2 + z^2 \xi_{xx}, c(y) d(y) = \frac{d}{dy} \log a. \)
4.3 Other solutions

Let
\[ f = z^n e^{x+z^{-1}y} := z^n e^{F(x,y,z)}, \quad g = z^{-n} e^{x+z^{-1}y} := z^{-n} e^{G(x,y,z)}, \]
be pair of solutions to (30), then \( f_z \) and \( g_z \) are another pair of solutions to (30). Let
\[ h = f + a(y)g = 2 \exp \Omega \cdot \cosh Z, \]
where \( \Omega \) and \( Z \) are defined in subsection 4.1. Then
\[ h_z = f_z + a(y)g_z = 2 \Omega z \cosh Z + 2 e^{\Omega} Z \sinh Z. \]

From (28), taking \( m = 2 \), we construct solutions with singularities. For simplicity, we define
\[ \left| \begin{array}{cc} h & h_z \\ h_z & h \end{array} \right| = 4 e^{2\Omega} C_k, \quad \text{where} \quad C_k = (z + \frac{k}{2z}) \sinh(k\omega) + \frac{k}{2z} \sinh(2Z + k\omega). \]

\[ \cosh(h, h_z, f) = -8a(x)e^{3\Omega} \frac{\sinh^2 \omega}{z} \cosh(Z + \omega), \]
\[ \cosh(h, h_z, f) = 4a(x)e^{3\Omega} \left( D_1^{(1)} \cosh Z + D_1 \cosh(Z + 2\omega) - D_2 \cosh(Z + \omega) \right), \]
where
\[ D_k = -\left( \frac{n}{z} + \frac{k}{2z} + F_z \right) \frac{n}{z} + \frac{k}{2z} + G_z \sinh(k\omega) + \frac{k^2}{4z^2} \sinh(k\omega) + \frac{k\Omega}{z} \cosh(k\omega), \quad k = 1, 2. \]
then find the following solution for (16)
\[ u[2] = \frac{C_2^{(1)}}{C_1^{(1)}} - \frac{C_2}{C_1}, \quad v[2] = \frac{C_1^{(1)}}{C_1^{(-1)}}, \]
\[ w_1[2] = -2c_1(y) a(y) \frac{\sinh^2 \omega e^{\Omega}}{z C_1} \cosh(Z + \omega), \]
\[ w_2[2] = c_2(y) a(y) \frac{e^{\Omega}}{C_1} \left( D_1^{(1)} \cosh Z + D_1 \cosh(Z + 2\omega) - D_2 \cosh(Z + \omega) \right), \]
\[ w_1^*[2] = -d_1(y) \frac{\Omega z \cosh(Z + \omega) + (Z_z + 1/z) \sinh(Z + \omega)}{2 e^{\Omega} C_1}, \]
\[ w_2^*[2] = d_2(y) \frac{\cosh(Z + \Omega)}{2 e^{\Omega} C_1^{(1)}}. \]

where \( c_j(y) d_j(y) = \frac{d}{dy} \log a(y). \)

Conclusion

We present a new multi-component two dimensional Toda lattice hierarchy, which enables us to find the two dimensional Toda lattice equation with self-consistent sources in different way from [9–12, 14] as well as their Lax representations. Since two dimensional Toda lattice equation with self-consistent sources can be considered as a two dimensional Toda lattice equation with non-homogeneous terms, method of variation of constant can be applied to the ordinary Darboux transformations for 2dTLSCS to construct a non-auto-Bäcklund Darboux transformations. Then it offers a different way to solve 2dTLSCS in contrast with [12, 14].

The 2dTLH offers various types of reductions, for example periodic reductions and reductions to Toda lattice equation. It is an interesting question does this new mc2dTLH offers similar reductions. We may discuss such problems elsewhere.

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