THE SYMPLECTIC MAPPING CLASS GROUP OF $\mathbb{CP}^2\# n\mathbb{CP}^2$
WITH $n \leq 4$

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Abstract. In this paper we prove that the Torelli part of the symplectomorphism groups of the $n$-point ($n \leq 4$) blow-ups of the projective plane is trivial. Consequently, we determine the symplectic mapping class group. It is generated by reflections on $K_\omega$—spherical class with zero $\omega$ area.

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1. Introduction

A symplectic manifold $(X, \omega)$ is an even dimensional manifold $X$ with a closed, nondegenerate two form $\omega$. The symplectomorphism group of $(X, \omega)$, denoted by $\text{Symp}(X, \omega)$, is the group of diffeomorphisms $\phi$ of $M$ which preserve $\omega$, and is given the $C^\infty$-topology. $\text{Symp}(X, \omega)$ is an infinite dimensional Fréchet Lie group.

For a closed 4-dimensional symplectic manifold $(X, \omega)$, since Gromov’s work [Gro85], the homotopy type of $\text{Symp}(X, \omega)$ has attracted much interest over the past 30 years. For the special case of some monotone 4-manifolds, the (rational) homotopy of $\text{Symp}(X, \omega)$ was fully computed in [Gro85, AM99, Eva11]. However, for an arbitrary symplectic 4 manifold, the complication grows drastically: for $S^2 \times S^2$, see [Abr98, AM99, Anj02]; and [AP12] for other instances.

The goal of this note is modest: for some rational 4-manifolds, we compute $\pi_0(\text{Symp}(X, \omega))$, which is the symplectic mapping class group (denoted as SMC for short). In the cases we consider, the homological action of $\text{Symp}(X, \omega)$ is already known in [LW11]. Therefore it suffices to describe $\pi_0(\text{Symp}_h(X, \omega))$, which is the subgroup of $\text{Symp}(X, \omega)$ acting trivially on homology, namely, its Torelli part.
Theorem 1.1. \( \text{Symp}(X, \omega) \) is connected for \( X = \mathbb{C}P^2 \# 4 \mathbb{C}P^2 \) with arbitrary symplectic form \( \omega \).

The cases \( S^2 \times S^2 \) and \( (\mathbb{C}P^2 \# k \mathbb{C}P^2) \) with \( k \leq 3 \) are known before. Our approach actually works in a uniform way for all \( k \leq 4 \) (See discussions in remark 3.5). One also note that Theorem 1.1 is not true in general for \( k \geq 5 \), see Seidel’s famous example in [Sei08].

Our strategy is based on Evans’ beautiful approach in [Eva11] by systematically exploring the geometry of certain stable configuration of symplectic spheres (a related approach first appeared in Abreu’s paper [Abr98]). It is summarized by the following diagram:

\[
\begin{align*}
\text{Symp}(U) \rightarrow & \rightarrow \text{Stab}^1(C) \rightarrow \text{Stab}^0(C) \rightarrow \text{Stab}(C) \rightarrow \text{Symp}(X) \\
& \downarrow \downarrow \downarrow \downarrow \\
& \mathcal{G}(C) \rightarrow \text{Symp}(C) \rightarrow C_0
\end{align*}
\]

Here \( C_0 \) is the space of a full stable standard configuration of fixed homological type. Every other term in diagram (1) is a group associated to \( C \in C_0 \), and \( U = X \setminus C \). Now we give the definition of stable standard spherical configurations and the groups will be discussed later in section 2.1.

Definition 1.2. Given a symplectic 4-manifold \((X, \omega)\), we call an ordered finite collection of symplectic spheres \( \{C_i, i = 1, \ldots, n\} \) a spherical symplectic configuration, or simply a configuration if

1. for any pair \( i, j \) with \( i \neq j \), \( [C_i] \neq [C_j] \) and \( [C_i] \cdot [C_j] = 0 \) or \( 1 \).
2. they are simultaneously \( J \)-holomorphic for some \( J \in J_\omega \).
3. \( C = \bigcup C_i \) is connected.

We will often use \( C \) to denote the configuration. The homological type of \( C \) refers to the set of homology classes \( \{[C_i]\} \).

Further, a configuration is called
- standard if the components intersect \( \omega \)-orthogonally at every intersection point of the configuration. Denote by \( C_\text{st} \) the space of standard configurations having the same homology type as \( C \).
- stable if \( [C_i] \cdot [C_i] \geq -1 \) for each \( i \).
- full if \( H^2(X, C; \mathbb{R}) = 0 \).

It is shown in [LW11] that for a rational manifold, the homological action of \( \text{Symp}(X, \omega) \) is generated by Lagrangian Dehn twists. Therefore, Theorem 1.1 implies:

Corollary 1.3. For a rational manifold with Euler number up to 7, the SMC is a finite group generated by Lagrangian Dehn twists. Moreover, a generating set corresponds to a finite set of \( K_\omega \)-null spherical classes with zero \( \omega \)-area. In particular, SMC is trivial for generic choice of \( \omega \).

It is shown in [BLW12] that the following proposition holds:

Proposition 1.4. Suppose \((X^4, \omega)\) is a symplectic rational manifold. Then \( \text{Symp}(X, \omega) \) acts transitively on the space of
- homologous Lagrangian spheres
• homologous symplectic $-2$-spheres
• $\mathbb{Z}_2$-homologous Lagrangian $\mathbb{R}P^2$’s and homologous symplectic $-4$-spheres if $b_2^-(X) \leq 8$

Hence we also have the following corollary:

**Corollary 1.5.** For a rational manifold with Euler number up to $7$, the space of
• homologous Lagrangian spheres,
• $\mathbb{Z}_2$-homologous Lagrangian $\mathbb{R}P^2$,
• homologous $-2$ symplectic spheres,
• homologous $-4$ symplectic spheres,
is connected.

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2. Analyzing the diagram

We analyze the diagram (1) and derive a criterion for the connectedness of $\text{Symp}_h(X, \omega)$ in Corollary 2.10.

2.1. Groups associated to a configuration. Let $C$ be a configuration in $X$. We first introduce the groups appearing in (1):

**Subgroups of $\text{Symp}_h(X, \omega)$**

Recall that $\text{Symp}_h(X, \omega)$ is the group of symplectomorphisms of $(X, \omega)$ which acts trivially on $H_*(X, \mathbb{Z})$.

• $\text{Stab}(C) \subset \text{Symp}_h(X, \omega)$ is the subgroup of symplectomorphisms fixing $C$ setwise, but not necessarily pointwise.
• $\text{Stab}^0(C) \subset \text{Stab}(C)$ is the subgroup the group fixing $C$ pointwise.
• $\text{Stab}^1(C) \subset \text{Stab}^0(C)$ is subgroup fixing $C$ pointwise and acting trivially on the normal bundles of its components.

**$\text{Symp}_c(U)$ for the complement $U$**

$\text{Symp}_c(U)$ is the group of compactly supported symplectomorphisms of $(U, \omega|_U)$, where $U = X \setminus C$ and the form $\omega|_U$ is the inherited form on $U$ from $X$. It is topologised in this way: let $(U, \omega)$ be a non-compact symplectic manifold and let $K$ be the set of compact subsets of $U$. For each $K \in \mathcal{K}$ let $\text{Symp}_K(W)$ denote the group of symplectomorphisms of $U$ supported in $K$, with the topology of $C^\infty$-convergence. The group $\text{Symp}_c(U, \omega)$ of compactly-supported symplectomorphisms of $(U, \omega)$ is topologised as the direct limit of $\text{Symp}_K(W)$ under inclusions.

**$\text{Symp}(C)$ and $G(C)$ for the configuration $C$**

Given a configuration of embedded symplectic spheres $C = C_1 \cup \cdots \cup C_n \subset X$ in a 4-manifold, let $I$ denote the set of intersection points amongst the components.
Suppose that there is no triple intersection amongst components and that all intersections are transverse. Let $k_i$ denote the cardinality of $I \cap C_i$, which is the number of intersection of points on $C_i$.

The group $\text{Symp}(C)$ of symplectomorphisms of $C$ fixing the components of $C$ is the product $\prod_{i=1}^n \text{Symp}(C_i, I \cap C_i)$. Here $\text{Symp}(C_i, I \cap C_i)$ denotes the group of symplectomorphisms of $C_i$ fixing the intersection points $I \cap C_i$. Since each $C_i$ is a 2–sphere and $\text{Symp}(S^2)$ acts transitively on $N$–tuples of distinct points in $S^2$, we can write $\text{Symp}(C_i, I \cap C_i)$ as $\text{Symp}(S^2, k_i)$. Thus

\begin{equation}
\text{Symp}(C) \cong \prod_{i=1}^n \text{Symp}(S^2, k_i)
\end{equation}

As shown in [Eva11] we have:

\begin{equation}
\text{Symp}(S^2, 1) \cong S^1; \quad \text{Symp}(S^2, 2) \cong S^1; \quad \text{Symp}(S^2, 3) \cong \ast;
\end{equation}

where $\cong$ means homotopy equivalence. And when $k = 1, 2$, the $S^1$ on the right can be taken to be the loop of a Hamiltonian circle action fixing the $k$ points.

The symplectic gauge group $\mathcal{G}(C)$ is the product $\prod_{i=1}^n \mathcal{G}_k(C_i)$. Here $\mathcal{G}_k(C_i)$ denotes the group of symplectic gauge transformations of the symplectic normal bundle to $C_i \subset X$ which are equal to the identity at the $k_i$ intersection points. Also shown in [Eva11]:

\begin{equation}
\mathcal{G}_0(S^2) \cong S^1; \quad \mathcal{G}_1(S^2) \cong \ast; \quad \mathcal{G}_k(S^2) \cong \mathbb{Z}^{k-1}, \quad k > 1.
\end{equation}

Since we assume the configuration is connected, each $k_i \geq 1$. Thus by (4), we have

\begin{equation}
\pi_0(\mathcal{G}(C)) = \bigoplus_{i=1}^n \pi_0(\mathcal{G}_k(C_i)) = \bigoplus_{i=1}^n \mathbb{Z}^{k_i - 1}
\end{equation}

It is useful to describe a canonical set of $k_i$ generators for $\mathcal{G}_k(C_i)$. For each intersection point $y \in I \cap C_i$, the evaluation map

\[ ev_y : \mathcal{G}_k(C_i) \to SL(2, \mathbb{R}) \]

is a homotopy fibration, and hence it induces a map $\mathbb{Z} = \pi_1(SL(2, \mathbb{R})) \to \pi_0(\mathcal{G}_k(C_i))$. Let $g_{C_i}(y) \in \pi_0(\mathcal{G}_k(C_i))$ denote the image of $1 \in \mathbb{Z}$.

2.2. Reduction to the connectedness of $\text{Stab}(C)$. The aim of this subsection is to show

**Proposition 2.1.** $\text{Symp}_h(X, \omega)$ is connected if there is a full, stable, standard configuration $C$ with connected $\text{Stab}(C)$.

This is derived from the right end of diagram (1) for a full, stable, standard configuration $C$:

\begin{equation}
\text{Stab}(C) \to \text{Symp}_h(X, \omega) \to \mathcal{C}_0
\end{equation}

Recall that $\mathcal{C}_0$ is the space of standard configurations having the homology type of $C$. We will show (1) is a homotopy fibration and $\mathcal{C}_0$ is connected.

We first review certain general facts regarding these configurations which are well-known to experts. By [LW11], we have the following fact.

**Lemma 2.2.** Let $(M, \omega)$ be a symplectic 4-manifold and $C$ a stable configuration $\cup_i C_i$. Then there is a path connected Baire subset $\mathcal{T}_D$ of $\mathcal{J}_\omega \times M_{d(C_i)}$ such that a pair $(J, \Omega)$ lies in $\mathcal{T}_D$ if and only if there is a unique embedded $J$–holomorphic
configuration having the same homological type as $C$ with the $i$–th component containing $\Omega_i$.

**Lemma 2.3.** Assume $C$ is a stable, standard configuration. The space $C_0$ of standard configurations having the homology type of $C$ is path connected.

**Proof.** Consider $C$, the space of configurations as in Definition 1.2. By Lemma 2.2 we see that the space $\mathcal{C}$ is connected. Using a Gompf isotopy argument, it is shown in [Eva11] that the inclusion $i : C_0 \to \mathcal{C}$ is a weak homotopy equivalence. Therefore, $C_0$ is also connected. 

With $C$ being full, the following lemma holds:

**Lemma 2.4.** If the stable, standard configuration $C$ is also full, then $\text{Symp}_h(X,\omega)$ acts transitively on $C_0$. In particular, (6) is a homotopy fibration.

**Proof.** From Lemma 2.3 any $C_1, C_2 \in C_0$ are isotopic through standard configurations. The property that the configurations are symplectically orthogonal where they intersect, together with the vanishing of $H^2(X; \mathbb{R})$, allows us to extend such an isotopy to a global homologically trivial symplectomorphism of $X$ (by Banyaga’s symplectic isotopy extension theorem, see [MS05], Theorem 3.19). So we have shown that the action of $\text{Symp}_h(X,\omega)$ on the connected space $C_0$ is transitive by establishing the 1–dimensional homotopy lifting property of the map $\text{Symp}_h(X,\omega) \to C_0$. By a finite dimensional version of this argument (or Theorem A in [Pai60]), we conclude that (6) is a homotopy fibration. 

**Proof of Proposition 2.1**

Since (6) is a homotopy fibration by Lemma 2.4, we have the associated homotopy long exact sequence. Because of the connectedness of $C_0$ as shown in Lemma 2.3, the connectedness of $\text{Stab}(C)$ implies the connectedness of $\text{Symp}_h(X,\omega)$. Therefore, we have 2.1 as the reduction of our problem.

2.3. **Reduction to the surjectivity of $\psi$:** $\pi_1(\text{Symp}(C)) \to \pi_0(\text{Stab}^0(C))$. To investigate the connectedness of $\text{Stab}(C)$, considering the action of $\text{Stab}(C)$ on $C$ and the following portion of diagram 1 which appeared in [Eva11] and [AP12]:

\[
\text{Stab}^0(C) \to \text{Stab}(C) \to \text{Symp}(C)
\]

The following lemma already appeared in [Eva11] and was explained to the authors by J. D. Evans\(^1\). We here include more details for readers’ convenience.

**Lemma 2.5.** This diagram (7) is a homotopy fibration when $C$ is a simply-connected standard configuration.

**Proof.** First we show $\text{Stab}(C) \to \text{Symp}(C)$ is surjective.

Recall that at each intersection point between two different components $\{x_{ij}\} = C_i \cap C_j$, the two components are symplectically orthogonal to each other in a Darboux chart containing $x_{ij}$. For convenience of exposition define the level of components as follows: let $C_1$ be the unique component of level 1, and the level-$k$ components are defined as those intersects components in level $k - 1$ but does

\(^1\)Private communications.
not belong to any lower levels. This is well-defined again because of the simply-connectedness assumption.

An element in $\text{Symp}(C)$ is the composition of Hamiltonian diffeomorphism $\phi_i$ on each component $C_i$, because of the simply connectedness of sphere. We start with endowing $C_1$ with a Hamiltonian function $f_1$ generating $\phi_1$. Let $C^2_1$ be curves on level 2. Because $C^2_1$ intersects $C_1$ $\omega$-orthogonally, we can find a symplectic neighborhood $U_1$ of $C_1$, identified as a neighborhood of zero section of the normal bundle, so that $U_1 \cap C_1$ consists of finitely many fibers. Pull-back $f_1$ by the projection $\pi$ of the normal bundle and multiply a cut-off function $\rho(r), \rho(r) = 1, r \leq \epsilon \ll 1; \rho(r) = 0, r \geq 2\epsilon$. Here $r$ is the radius in the fiber direction. Denote by $\bar{\phi}_1$ the symplectomorphism generated by this cut-off. Notice that $\bar{\phi}_1$ creates an extra Hamiltonian diffeomorphism $\epsilon_j$ on each component $C_j$ of level 2, and we denote $\epsilon_j' = \phi_j \circ \epsilon_j^{-1}$ for $C_j$ belonging to level 2.

One proceeds by induction on the level $k$. Notice one could always choose a Hamiltonian function $f_i$ on a component $C_i$ on level $k$ which generates $\phi_i'$ with the property that $f_i(x_i) < 0$. Here $C_i$ is the component of level $k - 1$ intersecting $C_i$. We emphasize this can be done because the component $C_i$ on level $k - 1$ which intersects $C_i$ is unique (and that the intersection is a single point) due to the simply connectedness assumption, and we do not restrict the value on any other intersections of $C_i$ and components of level $k + 1$. Therefore we only fix the value of $f_i$ at a single point.

One then again use the pull-back on the symplectic neighborhood and cut-off along the fiber direction to get a Hamiltonian function $H_i$ which generates a diffeomorphism $\phi_i$ supported on the neighborhood of $C_i$. We note that $d(\pi^*f_1, \rho(r))|_{F_x} = 0$ whenever $f_1(x) = 0$, where $F_x$ is the normal fiber over the point $x \in C_1$. Hence $dH_i|_{C_1} = 0$ since $f_i(x_i) = 0$ as prescribed earlier, which means action of $\phi_i$ on $C_1$ is trivial. Taking the composition $\phi$ of all these $\bar{\phi}_i'$’s, $\phi$ is supported on a neighborhood of $C_i$ and equals $\phi_i$ when restricted to $C_i$.

The transitivity of the action of $\text{Stab}(C)$ on $\text{Symp}(C)$ follows easily. For any two maps $\phi_1, \phi_2 \in \text{Symp}(C), \phi_2 \phi_1^{-1} \in \text{Symp}(C)$. We can extend $\phi_2 \phi_1^{-1}$ to $\text{Stab}(C)$. Then this extended $\phi_2 \phi_1^{-1}$ maps $\phi_1$ to $\phi_2$.

Now symplectic isotopy theorem (or Theorem A in [Pai60]) for the surjective map $\text{Stab}(C) \to \text{Symp}(C)$ proves the diagram (7) is a fibration.

Now we can establish the connectedness of $\text{Stab}(C)$ under the following assumptions:

**Proposition 2.6.** Let $(X, \omega)$ be a symplectic 4-manifold, and $C$ a simply-connected, full, stable, standard configuration. If each component of $C$ has no more than 3 intersection points, then the surjectivity of the connecting map $\psi: \pi_1(\text{Symp}(C)) \to \pi_0(\text{Stab}^0(C))$ implies the connectedness of $\text{Stab}(C)$.

**Proof.** Since we assume that each component of $C$ has no more than 3 intersection points, it follows from (3) and (2) that $\pi_0(\text{Symp}(C)) = 1$.

By Lemma 2.5 we have the homotopy long exact sequence associated to (7),

$$\cdots \to \pi_1(\text{Symp}(C)) \xrightarrow{\psi} \pi_0(\text{Stab}^0(C)) \to \pi_0(\text{Stab}(C)) \to \pi_0(\text{Symp}(C))$$

Then the surjectivity of $\psi$ implies that $\text{Stab}(C)$ is connected. 

$\square$
2.4. Three types of configurations. Next we investigate when the map \( \psi : \pi_1(\text{Symp}(C)) \to \pi_0(\text{Stab}^0(C)) \) is surjective. For this purpose we observe that an element of \( \text{Stab}^0(C) \) induces an automorphism on the normal bundle of \( C \). Thus we further have the following homotopy fibration appeared in [Eva11] and [AP12]:

\[
\text{Stab}^1(C) \to \text{Stab}^0(C) \to \mathcal{G}(C)
\]

In particular, there is the associated map \( \iota : \pi_0(\text{Stab}^0(C)) \to \pi_0(\mathcal{G}(C)) \). Consider the composition map

\[
\bar{\psi} = \iota \circ \psi : \pi_1(\text{Symp}(C)) \to \pi_0(\text{Stab}^0(C)) \to \pi_0(\mathcal{G}(C)).
\]

Notice that \( \pi_0(\mathcal{G}(C)) \) inherits a group structure from \( \mathcal{G}(C) \) and \( \bar{\psi} \) is a group homomorphism. As shown in [Eva11], \( \bar{\psi} \) can be computed explicitly.

When \( k_i \geq 3, \pi_1(\text{Symp}(S^2, k_i)) \) is trivial by (3). When \( k_i = 1, 2, \text{Symp}(C_i, I \cap C_i) \) is homotopic to the loop of a Hamiltonian circle action on \( C_i \) fixing the \( k_i \) points. Denote such a loop by \((\phi_i)_{k_i}\). Observe that \((\phi_i)_{k_i}\) is a generator of \( \pi_1(\text{Symp}(C_i, I \cap C_i)) = \mathbb{Z} \). Recall that for each component \( C_j \) there is a canonical set of generators \( \{g_{C_j}(y), y \in I \cap C_j\} \) for \( \mathcal{G}_{k_j}(C_j) \), introduced at the end of 2.1. The following is Lemma 4.1 in [Eva11]

**Lemma 2.7.** Suppose \( C_i \) is a component with \( k_i = 1, 2 \). The image of \((\phi_i)_{2\pi} \in \text{Symp}(C_i, I \cap C_i)\) under \( \bar{\psi} \) is described as follows.

- if \( k_i = 1 \) and \( C_j \) is the only component intersecting \( C_i \) with \( \{x\} = C_i \cap C_j \), then \((\phi_i)_{2\pi}\) is sent to
  \[g_{C_j}(x)\]
  in the factor subgroup \( \pi_0(\mathcal{G}_{k_j}(C_j)) \) of \( \pi_0(\mathcal{G}(C)) \).
- if \( k_i = 2 \) and \( x \in C_i \cap C_j, y \in C_i \cap C_j \), then \((\phi_i)_{2\pi}\) is sent to
  \[(g_{C_j}(x), g_{C_j}(y))\]
  in the factor subgroup \( \pi_0(\mathcal{G}_{k_j}(C_j)) \times \pi_0(\mathcal{G}_{k_j}(C_j)) \) of \( \pi_0(\mathcal{G}(C)) \).

Use Lemma 2.7 we will show that \( \bar{\psi} \) is surjective for the following configurations.

**Definition 2.8.** Introduce three types of configurations (see Figure 1 for examples).

- **(type I)** \( C = \bigcup_{i=1}^{n} C_i \) is called a chain, or a type I configuration, if \( k_1 = k_n = 1 \) and \( k_j = 2, 2 \leq j \leq n - 1 \).
- **(type II)** Suppose \( C = \bigcup_{i=1}^{n} C_i \) is a chain. \( C' = C \cup \overline{C_p} \) is called a type II configuration if the sphere \( \overline{C_p} \) is attached to \( C_p \) at exactly one point for some \( p \) with \( 2 \leq p \leq n - 1 \).
- **(type III)** Suppose \( C'' = C \cup \overline{C_q} \) is a type II configuration. \( C''' = C'' \cup \overline{C_q} \) is called a type III configuration if the sphere \( \overline{C_q} \) is attached to \( C_q \) at exactly one point for some \( q \) with \( 2 \leq q \leq n - 1 \) and \( q \neq p \).

**Lemma 2.9.** \( \bar{\psi} \) is surjective for a type I or II configuration and an isomorphism for a type III configuration.

**Proof.** We first prove the surjectivity for a type I configuration \( C = \bigcup_{i=1}^{n} C_i \). In this case, there are \( n - 1 \) intersection points \( x_1, \ldots, x_{n-1} \) in total with

\[I \cap C_1 = \{x_1\}, \quad I \cap C_n = \{x_{n-1}\}, \quad I \cap C_i = \{x_{i-1}, x_i\}, \quad i = 2, \ldots, n.\]
Notice that $\pi_1(Symp(C_i, k_i)) = \mathbb{Z}$ for each $i = 1, \ldots, n$. Notice also that $\pi_0(G_{k_i}(C_i)) = \mathbb{Z}$ for each $i$ for $i = 2, \ldots, n-1$, and $\pi_0(G_{k_1}(C_1))$ and $\pi_0(G_{k_n}(C_n))$ are trivial. Thus the homomorphism $\bar{\psi}_C$ associated to $C$ is of the form $\mathbb{Z}^n \to \mathbb{Z}^{n-2}$.

For each $i = 1, \ldots, n$, denote the generator $(\phi_i)_t$ of $\pi_1(Symp(C_i, k_i)) = \mathbb{Z}$ by $\text{rot}(i)$. For each $i = 2, \ldots, n-1$, denote by $g_i(i-1)$ and $g_i(i)$ the generators $g_{C_i}(x_{i-1})$ and $g_{C_i}(x_i)$ of $\pi_0(G_2(C_i)) = \mathbb{Z}$.

Then by Lemma 2.7 the homomorphism $\bar{\psi}_C$ is described by

$$\bar{\psi}_C: \begin{align*}
\text{rot}(1) &\to g_2(1), \\
\text{rot}(2) &\to (0, g_3(2)), \\
g_3(j) &\to (g_{j-1}(j-1), g_{j+1}(j)), \quad 3 \leq j \leq n-2 \\
g_{n-1}(j) &\to (g_{n-2}(n-2), 0) \\
g_{n-1}(n) &\to g_{n-1}(n-1)
\end{align*}$$

Choose the bases of $\pi_1(Symp(C_i))$ and $\pi_0(G(C))$ to be

$$\{\text{rot}(1), \ldots, \text{rot}(n)\}$$

and

$$\{g_2(2), g_3(3), g_4(4), \ldots, g_{n-1}(n-1)\},$$

respectively. Notice that $g_i(i-1) = \pm g_i(i)$, then by (9), $\bar{\psi}_C$ is represented by the following $(n-2) \times n$ matrix if we drop the possible negative sign for each entry,

$$\begin{bmatrix}
1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 1 \\
& \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}$$

Observe that the first $n-2$ minor as a $(n-2) \times (n-2)$ is upper triangular matrix whose determinant is $\pm 1$. This shows that $\bar{\psi}_C$ is surjective.

For a type II configuration $C' = C \cup \overline{C_p}$, let $\overline{x}_p$ be the intersection of $C_p$ and $\overline{C_p}$. Notice that $\pi_1(Symp(C')) = \mathbb{Z}^n$ as in the case of $C$, with the $\mathbb{Z}$ summand from $C_p$ replaced by a $\mathbb{Z}$ summand from $\overline{C_p}$. Notice also that $\pi_0(G(C')) = \mathbb{Z}^{n-1}$ with the extra $\mathbb{Z}$ summand coming from the new intersection point $\overline{x}_p$ in $C_p$. Denote by
rot(\bar{p}) the generator of \( \pi_1(\text{Symp}(\mathbb{C}P_\mathbb{n}^2, \bar{x}_p)) \). Denote by \( g'_{p}(p) \) the generator \( g_{C'}(\bar{x}_p) \) of \( \pi_0(\mathbb{G}(C_p)) \). By Lemma 2.7, the homomorphism \( \psi_{C'} \) is of the form \( \mathbb{Z}^n \to \mathbb{Z}^{n-1} \), and it differs from \( \psi_{C} \) as in (9):

\[
\begin{align*}
\text{rot}(p) &= 0 \\
\text{rot}(\bar{p}) &\to g'_{p}(p)
\end{align*}
\]

It is not hard to see that \( \psi_{C'} \) is again surjective. We illustrate by the type II configuration in Figure 1. With respect to the bases

\[
\{\text{rot}(1), \text{rot}(2), \text{rot}(3), \text{rot}(4), \text{rot}(5)\} \quad \text{and} \quad \{g_2(2), g'_2(2), g_3(3), g_4(4)\},
\]

\( \psi_{C'} \) is represented by the following 4 \times 5 matrix (if we drop the possible negative sign),

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

For a type III configuration \( C'' = C' \cup \mathbb{C}q = C \cup \mathbb{C}_p \cup \mathbb{C}_q \), observe first that \( \pi_1(\text{Symp}(C'')) = \mathbb{Z}^n \) and \( \pi_0(\mathbb{G}(C')) = \mathbb{Z}^n \). By Lemma 2.7, we can describe \( \psi_{C''} : \mathbb{Z}^n \to \mathbb{Z}^{n} \) similar to the case of the type II configuration \( C' \). Precisely, \( \psi_{C''} \) differs from \( \psi_{C} \) in (9) as follows:

\[
\begin{align*}
\text{rot}(p) &= \text{rot}(q) = 0 \\
\text{rot}(\bar{p}) &\to g'_{p}(p) \\
\text{rot}(\bar{q}) &\to g'_{q}(q)
\end{align*}
\]

It is easy to see that \( \psi_{C''} \) is an isomorphism in this case. We illustrate by the type III configuration in Figure 1. With respect to the bases

\[
\{\text{rot}(1), \text{rot}(2), \text{rot}(3), \text{rot}(4), \text{rot}(5)\} \quad \text{and} \quad \{g_2(2), g'_2(2), g_3(3), g'_{4}(4), g_4(4)\},
\]

\( \psi_{C''} \) is represented by the following square matrix (if we drop the possible negative sign),

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\( \blacksquare \)

2.5. Criterion. Finally, we arrive at the following criterion for the connectedness of \( \text{Symp}_{h}(X, \omega) \).

**Corollary 2.10.** Suppose a stable, standard configuration \( C \) is type I, II or III, and it is full. If \( \text{Symp}_{c}(U) \) is connected, then \( \text{Symp}_{h}(X, \omega) \) is connected.

**Proof.** By Lemma 5.2 in [Eva11], \( \text{Symp}_{c}(U) \) is weakly homotopy equivalent to \( \text{Stab}^1(C) \). So by our assumption that \( \text{Symp}_{c}(U) \) being connected, \( \text{Stab}^1(C) \) is also connected. Therefore the map \( \epsilon : \pi_0(\text{Stab}^0(C)) \to \pi_0(\mathbb{G})(C) \) associated to the homotopy fibration (8) is a group isomorphism. Now we have \( \psi_{C} = \tilde{\psi}_{C} \).
Since $C$ is type I, II or III, by Lemma 2.9, $\psi_C$ is surjective. Notice that any type I, II, or III configuration is simply-connected. By the assumption of $C$ being full, we can apply Proposition 2.6 and Proposition 2.1 to conclude that $\text{Symp}_h(X, \omega)$ is connected.

3. Proof in the case of $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$

3.1. The configuration for $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$. Let $X = \mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$ and $\omega$ an arbitrary symplectic form on $X$. We consider a configuration $C$ in [Eva11], consisting of symplectic spheres in homology classes $S_{12} = H - E_1 - E_2$, $S_{34} = H - E_3 - E_4$, $E_1$, $E_2$, $E_3$ and $E_4$. Here $\{H, E_i\}$ is the standard basis of $H_2(X; \mathbb{Z})$ with positive pairing with $\omega$. In Figure 2 we label the spheres by their homology classes.

![Figure 2](image)

To apply the criterion in Corollary 2.10, we need to check that we can always find a configuration $C$ of such a homology type, so that

- $C$ is stable.
- $C$ is a type I, II or III configuration.
- $C$ is full.
- $\text{Symp}_c(U)$ is connected.

Existence of such a configuration is a direct consequence of Gromov-Witten theory and the first three statements follows from definition. Note also that the actual choice of configuration will not affect the last statement because $\text{Symp}_h(X)$ acts transitively on $C_0$, which means $U$ is well-defined up to symplectomorphism for any choice of $C \in C_0$.

It thus remains to prove the connectedness of $\text{Symp}_c(U)$. We will actually show that $\text{Symp}_c(U)$ is weakly contractible in the next subsection.

3.2. Contractibility of $\text{Symp}_c(U)$. Let us first recall the following result of Evans (Theorem 1.6 in [Eva11]):

**Theorem 3.1.** If $\mathbb{C}^* \times \mathbb{C}$ is equipped with the standard (product) symplectic form $\omega_{\text{std}}$ then $\text{Symp}_c(\mathbb{C}^* \times \mathbb{C})$ is weakly contractible.

This is relevant since Evans observed in [Eva11] that, if $(\omega, J_0)$ is Kähler with $\omega$ monotone and $C$ holomorphic, then $(U, J_0)$ has a finite type Stein structure $f$ with $\omega|_U = -dd^c f$, and there is a biholomorphism $\Psi$ from $(U, J_0)$ to $\mathbb{C}^* \times \mathbb{C}$ (In addition, $\Psi$ satisfies $\Psi^* \omega_{\text{std}} = \omega|_U$).

Let us also recall the next result of Evans (Proposition 2.2 in [Eva11]):
Proposition 3.2. If \((W, J_0)\) is a complex manifold with two finite type Stein structures \(\phi_1\) and \(\phi_2\), then \(\text{Symp}_c(W, -dd^c \phi_1)\) and \(\text{Symp}_c(W, -dd^c \phi_2)\) are weakly homotopy equivalent.

Now we complete our proof of the connectedness of \(\text{Symp}_c(CP^2 \# nCP^2, \omega)\) for an arbitrary \(\omega\) by proving the following

Proposition 3.3. \(\text{Symp}_c(U, \omega|_U)\) is weakly contractible.

Proof. We first choose a specific configuration \(C\) convenient for our purpose (as we explained in Section 3.1 this does not affect our result). According to [Li08] Proposition 4.8, we can always pick an integrable complex structure \(J_0\) compatible with \(\omega\), so that \((X, J_0)\) is biholomorphic to a generic blow up of 4 points on \(CP^2\) (the genericity here means that no 3 points lies on the same line, and indeed this can always be done for less than 9 point blow ups). For such a generic holomorphic blow up, there is a unique smooth rational curve in each class in the homology type of \(C\). Thus we canonically obtain a configuration \(C\) associated to \(J_0\). Observe that the complement \(U = X \setminus C\) is biholomorphic to \(C^* \times \mathbb{C}\). That is because the configuration \(C\) is the total transformation of two lines blowing up at four points. Removing \(C\) gives us a biholomorphism from \((U, J_0)\) to \(CP^2\) with two lines removed, which is \(C^* \times \mathbb{C}\).

Now we construct a Stein structure \(\phi\) on \((U, J_0)\) with \(-dd^c \phi = \omega|_U\), whenever \(\omega\) is a rational symplectic form on \(CP^2 \# nCP^2\). Since \((U, J_0)\) is biholomorphic to \(C^* \times \mathbb{C}\) equipped with the standard finite type Stein structure \((J_{std}, \omega_{std} = -dd^c |z|^2)\), we can then apply Proposition 3.2 and Theorem 3.1 in this case to conclude the weak contractibility of \(\text{Symp}_c(U, \omega|_U)\).

So we assume that \(|\omega| \in H^2(X; \mathbb{Q})\). Up to rescaling, we can write \(PD([l\omega]) = aH - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4\) with \(a, b_i \in \mathbb{Z}^{\geq 0}\). Further, we assume \(b_1 \geq b_2, b_3 \geq b_4\). Since \(H - E_1 - E_3\) is an exceptional class we also have \(\omega(H - E_1 - E_3) > 0\). This means that \(a > b_1 + b_3\), namely, \(2a \geq 2b_1 + 2b_3 + 2\). Rewrite

\[
PD(2l|\omega|) = (2b_1 + 1)(H - E_1 - E_2 + E_1 + (2b_1 - 2b_2 + 1)E_2 + (2a - 2b_1 - 1)(H - E_3 - E_4) \\
+ (2a - 1 - 2b_1 - 2b_3)E_3 + (2a - 1 - 2b_1 - b_4)E_4.
\]

Notice that the coefficients are all in \(\mathbb{Z}^{\geq 0}\). In this way we represent \(PD(2l|\omega|)\) as a positive integral combination of all elements in the set \(\{H - E_1 - E_2, H - E_3 - E_4, E_1, E_2, E_3, E_4\}\), which is the homology type of \(C\).

Denote the symplectic sphere with homology class \(E_i\) in \(C\) by \(C_{E_i}\), and similarly for the two remaining spheres. Notice that each sphere is a smooth divisor. Consider the effective divisor

\[
F = (2b_1 + 1)C_{H - E_1 - E_2} + C_{E_1} + (2b_1 - 2b_2 + 1)C_{E_2} + (2a - 2b_1 - 1)C_{H - E_3 - E_4} \\
+ (2a - 1 - 2b_1 - 2b_3)C_{E_3} + (2a - 1 - 2b_1 - b_4)C_{E_4}.
\]

There is a holomorphic line bundle \(\mathcal{L}\) with a holomorphic section \(s\) whose zero divisor is exactly \(F\). Notice that \(c_1(\mathcal{L}) = [F] = [2l|\omega|]\). By [GH94] section 1.2, we can take an hermitian metric \(|\cdot|\) and a compatible connection on \(\mathcal{L}\) such that the curvature form is just \(2l|\omega|\). Moreover, for the holomorphic section \(s\), the function \(\phi = -\log |s|^2\) is plurisubharmonic on the complement \(U\) with \(-d(d\phi \circ J_0) = 2l|\omega|\). Notice that \(F\) and \(C\) have the same support so the complement
of $F$ is the same as $U$. Thus we have endowed $(U, J_0)$ with a finite type Stein structure $\phi$.

As argued above, this implies that $\text{Symp}_c(U, \omega|_U) = \text{Symp}_c(U, 2\omega|_U)$ is weakly contractible when $[\omega] \in H^2(X, \mathbb{Q})$ by the biholomorphism from $(U, J_0)$ to $(\mathbb{C}^* \times \mathbb{C}, J_{\text{std}})$.

Finally, suppose $\omega$ is not rational, but we assume $\omega(H) \in \mathbb{Q}$ without loss of generality by rescaling. We take a base point $\varphi_0 \in \text{Symp}_c(U, \omega|_U)$, and a $S^n(n \geq 0)$ family of symplectomorphisms determined by a based map $\iota : S^n \to \text{Symp}_c(U, \omega|_U)$.

Denote the union of support of this $S^n$ family by $V_\iota$, which is a compact subset of $U$.

Note the following fact:

**Claim 3.4.** There exists an $\omega'$ symplectic on $X$ such that:

1. $[\omega'] \in H^2(X, \mathbb{Q})$,
2. $\omega'(E_i) \geq [\omega](E_i), [\omega'](H) = [\omega](H)$
3. The configuration $C$ is $\omega'$-symplectic
4. $(X \setminus C, \omega') \cong (X \setminus C, \omega)$ in such a way that the image contains $V_\iota$.

**Proof.** Recall that to blow up an embedded ball $B$ in a symplectic manifold $(M, \omega)$, one removes the ball and collapses the boundary by Hopf fibration which incurs an exceptional divisor. The reverse of this procedure is a blowdown.

Now take $E_i$ in the configuration $C$ and blow them down to get a disjoint union of balls $B_i$ in the blown-down manifold, which is a symplectic $\mathbb{C}P^2$ with line area equal $\omega(H)$. One then enlarge $B_i$ by a very small amount to $B'_i$ so that the sizes of $B'_i$ become rational numbers. After the enlargement, blow up $B'_i$. This produces a symplectic form on $X$ which clearly satisfies (1) and (2). (3) can be achieved as long as the enlarged ball has boundary intersecting proper transformation of $S_{12}$ and $S_{34}$ on a big circle. This is always possible: perturb $S_{12}$ and $S_{34}$ slightly so that they are symplectically orthogonal to $E_i$ before blow-down. Then in a neighborhood of the resulting balls $B_i$, one has a Darboux chart where $B_i$ is the standard ball, while the portion of $S_{12}$ and $S_{34}$ inside this chart is the $x_1 - x_2$ plane. This is guaranteed by symplectic neighborhood theorem near $E_i$. Hence the (3) is obtained when the enlargement stays inside the Darboux chart. For more details one is referred to [MW96].

To see (4), we note that from the above description, $(X \setminus C, \omega')$ is symplectomorphic to the complement of $\bigcup B'_i$ union two lines (the proper transforms of $S_{12}$ and $S_{34}$) in the symplectic $\mathbb{C}P^2$ from blowing down. The same thus applies to $(X \setminus C, \omega)$, while $B'_i$ are replaced by $B_i \subset B'_i$. Therefore, the statement regarding embedding holds in (4). Since $V_\iota$ is compact and embeds in $(X \setminus C, \omega)$, as long as the amount of enlargement from $B_i$ to $B'_i$ is small enough, the embedded image contains $V_\iota$ as claimed.

Therefore we can find an isotopy in $\text{Symp}_c(U, \omega'|_U) \hookrightarrow \text{Symp}_c(U, J_0), \omega|_U)$, from the $S^n$ family of maps to the base point $\varphi_0$ by the proved case when $\omega$ is rational. We emphasize in the above proof, the choice of $\omega'$ depends on $i$, but this is irrelevant for our purpose. This concludes that for arbitrary symplectic form $\omega$ on $X$, $\text{Symp}_c(U, \omega|_U)$ is weakly contractible and hence $\text{Symp}_c(\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2})$ is connected for any symplectic form. 

\[\square\]
Remark 3.5. The approach we adopt in this note in fact provides a uniform way to establish the connectedness of the Torelli part of SMC for all symplectic rational 4-manifolds with $\chi \leq 7$. This can be viewed as a continuation of the techniques first introduced by Gromov in [Gro85] and further developed by many others in [Abr98, AM99, LP04, Eva11, AP12] etc.

Here we just list the configurations for the 1,2,3-point blow up of $\mathbb{C}P^2$ equipped with an arbitrary symplectic form:

- $\mathbb{C}P^2 \# \mathbb{C}P^2, \{E_1, H - E_1\}$ (with a marked point).
- $\mathbb{C}P^2 \# 2\mathbb{C}P^2, \{E_1, E_2, H - E_1 - E_2\}$.
- $\mathbb{C}P^2 \# 3\mathbb{C}P^2, \{E_1, E_2, H - E_1 - E_2, H - E_1 - E_3, H - E_2 - E_3\}$.

The configurations are all of type I. Combined with our argument verbatim, one can recover the connectedness of $\text{Symp}_h(\mathbb{C}P^2 \# n\mathbb{C}P^2, \omega), n \leq 3$. However, such a result for these manifolds is not new, see [Abr98, AM99, LP04, Eva11].

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