The weak Frenet frame of non-smooth curves with finite total curvature and absolute torsion

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Abstract. We deal with a notion of weak binormal and weak principal normal for non-smooth curves of the Euclidean space with finite total curvature and total absolute torsion. By means of piecewise linear methods, we first introduce the analogous notation for polygonal curves, where the polarity property is exploited, and then make use of a density argument. Both our weak binormal and normal are rectifiable curves which naturally live in the projective plane. In particular, the length of the weak binormal agrees with the total absolute torsion of the given curve. Moreover, the weak normal is the vector product of suitable parameterizations of the tangent indicatrix and of the weak binormal. In the case of smooth curves with positive curvature, the weak binormal and normal yield (up to a lifting) the classical notions of binormal and normal.

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In classical differential geometry, it sometimes happens that the geometry of a proof can become obscured by analysis. This statement by M. A. Penna [10], which may be referred e.g. to the classical proof of the Gauss-Bonnet theorem, suggests to apply piecewise linear methods in order to make the geometry of a proof completely transparent.

For this purpose, by using the geometric description of the torsion of a smooth curve, Penna [10] gave in 1980 a suitable definition of torsion for a polygonal curve of the Euclidean space \( \mathbb{R}^3 \), and used piecewise linear methods and homotopy arguments to produce an illustrative proof of the well-known property that the total torsion of any closed unit speed regular curve of the unit sphere \( S^2 \) is equal to zero.

Differently to the smooth case, the polygonal torsion is a function of the segments. His definition, in fact, relies on the notion of binormal vector at the interior vertexes. Since the angle between consecutive discrete binormals describes the movements of the “discrete osculating planes” of the polygonal, binormal vectors naturally live in the projective plane \( \mathbb{RP}^2 \), see Sec. 1.

We recall here that J. W. Milnor [6, 7] defined the tangent indicatrix of a polygonal \( P \) as the geodesic polygonal \( t_P \) of the Gauss sphere \( S^2 \) obtained by connecting with oriented geodesic arcs the consecutive points given by the direction of the oriented segments. Therefore, the total curvature \( TC(P) \), i.e., the sum of the turning angles of the polygonal, agrees with the length \( L_{S^2}(t_P) \) of the tantrix, and the total absolute torsion \( TAT(P) \) agrees with the sum of the shortest angles in \( S^2 \) between the geodesic arcs meeting at the edges of \( t_P \), i.e., with the total curvature of the tantrix in \( S^2 \). Of course, the two above definitions of total absolute torsion are equivalent, compare Remark 1.3.

From another viewpoint, W. Fenchel [4] in the 1950’s exploited the spherical polarity of the tangent and binormal indicatrix in order to analyze the differential geometric properties of smooth curves in \( \mathbb{R}^3 \). In his survey, Fenchel proposed a general method that gathers several results on curves in a unified scheme. We point out that Fenchel deals with \( C^4 \) rectifiable curves (parameterized by arc-length) such that at each point it is well-defined the osculating plane, that is, a plane containing the vectors \( \mathbf{t} := \dot{c} \) and \( \mathbf{\tau} := \ddot{c} \), such that its suitably oriented normal unit vector \( \mathbf{b} \), the binormal vector, is of class \( C^2 \), and the two vectors \( \mathbf{t} \) and \( \mathbf{b} \) never vanish simultaneously. He then defines the principal normal by the vector product

\[
\mathbf{n} := \mathbf{b} \times \mathbf{t} .
\]

\[\text{(0.1)}\]

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Since the derivatives of $t$ and $b$ are perpendicular to both $t$ and $b$, the curvature $k$ and torsion $\tau$ are well-defined through the formulas:

$$i = kn, \quad b = -\tau n.$$  

As a consequence, one has

$$\hat{n} = -k t + \tau b$$

and hence the Frenet-Serret formulas hold true, but Fenchel allows both the curvature and torsion to be zero or negative. Related arguments have been treated in [11, 12, 3, 5, 14].

**Content of the paper.** We deal with curves in the Euclidean space $\mathbb{R}^3$ with finite total curvature and total absolute torsion. We address to J. M. Sullivan [12] for the analysis of curves with finite total curvature, and also to our paper [9] for the BV-properties of the unit normal, in the case of planar curves.

By melting together the approaches by Penna and Fenchel previously described, in this paper we firstly define the binormal indicatrix $b_P$ of a polygonal $P$ in $\mathbb{R}^3$ as the arc-length parameterization $b_P$ of the polar in $\mathbb{RP}^2$ of the tangent indicatrix $t_P$, see Definition 1.7 and Figure 1. We remark that a similar definition has been introduced by T. F. Banchoff in his paper [1] on space polygons.

As a consequence, by means of a density argument, a good notion of weak binormal indicatrix for a non-smooth curve with finite total curvature and absolute torsion is obtained in our first main result, Theorem 3.1.

For this purpose, we recall that similarly to the length $L(c)$, the total curvature $TC(c)$ and total absolute torsion $TAT(c)$ of a curve $c$ in $\mathbb{R}^3$ are defined in terms of any sequence of inscribed polygonals with infinitesimal meshes, compare e.g. Sullivan [12] or Sec. 2. Furthermore, for smooth curves, the total absolute torsion, which agrees with the length in the Gauss sphere of the smooth binormal curve $b$, actually agrees with the spherical curvature of the smooth tangent $t$ in $S^2$. This property may be seen in Example 2.2 referring to a helicoidal curve, where we exploit piecewise linear methods in the computation.

In Theorem 3.1, in fact, we show the existence of a curve $b_c$ of $\mathbb{RP}^2$, parameterized by arc-length, whose length is equal to the total absolute torsion, i.e.,

$$L_{\mathbb{RP}^2}(b_c) = TAT(c).$$

Furthermore, for smooth curves whose torsion $\tau$ (almost) never vanishes, our weak binormal $b_c$ in $\mathbb{RP}^2$, when suitably lifted to $S^2$, agrees with the arc-length parameterization of the smooth binormal $b$, Theorem 3.2.

For future use, the analogous properties concerning the weak tangent indicatrix $t_c$ are collected in Propositions 3.3 and 3.5. In particular, we recover the well-known equality $L_{S^2}(t_c) = TC(c)$.

Now, when looking for a possible weak notion of principal normal, a drawback appears. In fact, in Penna’s approach [10], the curvature of an open polygonal $P$ is a non-negative measure $\mu_P$ concentrated at the interior vertexes, whereas the torsion is a signed measure $\nu_P$ concentrated at the interior segments, see Remark 1.4. Since these two measures are mutually singular, in principle there is no way to extend Fenchel’s formula (1.1) in order to define the principal normal.

To overcome this problem, in Sec. 4 we proceed as follows. Firstly, we choose two suitable curves $\tilde{t}_P, \tilde{b}_P : [0, C + T] \rightarrow \mathbb{RP}^2$, where $C = TC(P)$ and $T = TAT(P)$, which on one side inherit the properties of the tangent and binormal indicatrix $t_P$ and $b_P$, respectively, and on the other side take account of the order in which curvature and torsion are defined along $P$. More precisely, one of the two curves is constant when the other one parameterizes a geodesic arc, whose length is equal to the curvature or to the (absolute value of the) torsion at one vertex or segment of $P$, respectively. As in Fenchel’s approach, by exploiting the polarity of the curves $\tilde{t}_P$ and $\tilde{b}_P$, the weak normal of the polygonal is well-defined by the inner product

$$n_P(s) := \tilde{b}_P(s) \times \tilde{t}_P(s) \in \mathbb{RP}^2, \quad s \in [0, T + C]$$

compare Remark 1.1 and Figure 2. Notice that by our Definition 4.2 we infer that

$$L_{\mathbb{RP}^2}(n_P) = TC(P) + TAT(P).$$

As a consequence, in our second main result, Theorem 4.5, using again an approximation procedure, the weak principal normal of a curve $c$ with finite total curvature and absolute torsion is well-defined as a
rectifiable curve $n_i$ in $\mathbb{R}^2$. It turns out that the product formula (1.1) continues to hold in a suitable sense, and we also have:

$$L_{\mathbb{RP}^2}(n_e) = TC(c) + TAT(c).$$

In particular, for smooth curves whose curvature (almost) never vanishes, it turns out that the principal normal $n$ agrees with a lifting of a suitable parameterization of the weak normal $n_e$. More precisely, in Proposition 4.7 we obtain that

$$[n(s(t))] = n_e(t) \in \mathbb{RP}^2 \quad \forall t \in [0, TC(c) + TAT(c)]$$

where $s(t)$ is the inverse of the increasing and bijective function

$$t(s) := \int_0^s (k(\lambda) + |\tau(\lambda)|)\,d\lambda, \quad \lambda \in [0, L(c)].$$

Finally, in Sec. 5 we make use of an analytical approach in order to define the binormal and principal normal of smooth regular curves with inflection points. Namely, if $|\dot{c}(s_0)| = 1$ but $\ddot{c}(s_0) = 0_{\mathbb{R}^3}$, in terms of the first non-zero higher order derivative $c^{(n)}(s_0)$ of $c$ at $s_0$, in Proposition 5.1 we get:

$$t(s_0) = \dot{c}(s_0), \quad b(s_0) = \frac{\ddot{c}(s_0) \times c^{(n)}(s_0)}{|\dot{c}(s_0) \times c^{(n)}(s_0)|}, \quad n(s_0) = b(s_0) \times t(s_0)$$

where by smoothness we have $\dot{c}(s_0) \perp c^{(n)}(s_0)$, and hence

$$|c(s_0) \times c^{(n)}(s_0)| = |c^{(n)}(s_0)|, \quad n(s_0) = \frac{c^{(n)}(s_0)}{|c^{(n)}(s_0)|}.$$

In general, the binormal and normal fail to be continuous at inflection points, see Example 5.2. However, according to our previous results, it turns out that they are both continuous when seen as functions in $\mathbb{RP}^2$. Thus the natural ambient of definition of both the binormal and principal normal is indeed the projective plane $\mathbb{RP}^2$.

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## 1 Weak binormal and total torsion of polygonals

In this section, we introduce a weak notion of binormal indicatrix $b_P$ for a polygonal $P$ in $\mathbb{R}^3$, Definition 1.7. It is a rectifiable curve in the projective plane $\mathbb{RP}^2$ whose length is equal to the total absolute torsion of $P$.

Let $P$ be a polygonal curve in $\mathbb{R}^3$ with consecutive vertices $v_i, i = 0, \ldots, n$, where $n \geq 3$ and $P$ is not closed, i.e., $v_0 \neq v_n$. Without loss of generality, we assume that every oriented segment $\sigma_i := [v_{i-1}, v_i]$ has positive length $L(\sigma_i) := |v_i - v_{i-1}|$, for $i = 1, \ldots, n$, and that two consecutive segments are never aligned.

Finally, we recall that the **mesh** of the polygonal is defined by $\text{mesh} P := \sup \{L(\sigma_i) \mid i = 1, \ldots, n\}$.

**Binormal vectors and torsion.** In the definition by Penna [10], the **discrete unit binormal** is the unit vector given at each interior vertex $v_i$ of $P$ by the formula:

$$b_i := \frac{\sigma_i \times \sigma_{i+1}}{|\sigma_i \times \sigma_{i+1}|}, \quad i = 1, \ldots, n - 1.$$  (1.1)

The **torsion** of $P$ is a function $\tau(\sigma_i)$ of the interior oriented segments $\sigma_i$ defined as follows. Let $i = 2, \ldots, n - 1$. If the three segments $\sigma_{i-1}, \sigma_i, \sigma_{i+1}$ are co-planar, i.e., if the vector product $b_{i-1} \times b_i = 0_{\mathbb{R}^3}$, one sets $\tau(\sigma_i) = 0$. Otherwise, one sets

$$\tau(\sigma_i) := \frac{\theta_i}{L(\sigma_i)}.$$

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where \( \theta_i \) denotes the angle between \(-\pi/2\) and \(\pi/2\) whose magnitude is the undirected angle between the binormals \(b_{i-1}\) and \(b_i\), and whose sign is equal to the sign of the scalar product between the linearly independent vectors \(b_{i-1} \times b_i\) and \(\sigma_i\). The total torsion and total absolute torsion are respectively defined by Penna through the formulas:

\[
\text{TT}(P) := \sum_{i=2}^{n-1} \tau(\sigma_i) \cdot \mathcal{L}(\sigma_i) = \sum_{i=2}^{n-1} \theta_i, \quad \text{TAT}(P) := \sum_{i=2}^{n-1} |\tau(\sigma_i)| \cdot \mathcal{L}(\sigma_i) = \sum_{i=2}^{n-1} |\theta_i|.
\]

**Remark 1.1** In the above definition, one actually considers angles between unoriented (osculating) planes. In fact, with the above notation we have:

\[
|\theta_i| = \min\{\arccos(b_{i-1} \cdot b_i), \arccos(-b_{i-1} \cdot b_i)\} \in [0, \pi/2].
\]

**AN EQUIVALENT DEFINITION.** In the classical approach by Milnor \cite{1, 2}, one considers the tangent indicatrix of \(P\), i.e., the polygonal \(t_P\) in the Gauss sphere \(S^2\) obtained by letting \(t_i := \sigma_i / \mathcal{L}(\sigma_i) \in S^2\), for \(i = 1, \ldots, n\), and connecting with oriented geodesic arcs \(\gamma_i\) the consecutive points \(t_i\) and \(t_{i+1}\), for \(i = 1, \ldots, n-1\). Therefore, one has \(\mathcal{L}(\gamma_i) = d_{S^2}(t_i, t_{i+1})\), where \(d_{S^2}\) denotes the geodesic distance on \(S^2\).

**Remark 1.2** The total curvature \(\text{TC}(P)\) of \(P\) is the sum of the turning angles \(\alpha_i\) at the interior vertexes of \(P\), compare e.g. \cite{12}, and it is therefore equal to the length of \(t_P\), i.e.,

\[
\text{TC}(P) = \sum_{i=1}^{n-1} \mathcal{L}(\gamma_i) = \mathcal{L}_{S^2}(t_P).
\]

In particular, the arc-length parameterization \(t_P : [0, |C|] \to S^2\), where \(C := \mathcal{L}(t_P) = \text{TC}(P)\), is Lipschitz-continuous and piecewise smooth, with \(|t_P| = 1\) everywhere except to a finite number of points, the edges of the tangent indicatrix \(t_P\), which correspond to the interior segments of the polygonal \(P\).

Milnor then defined the total absolute torsion of \(P\) through the formula:

\[
\text{TAT}(P) := \sum_{i=2}^{n-1} \tilde{\theta}_i
\]

where \(\tilde{\theta}_i \in [0, \pi/2]\) is the shortest angle in \(S^2\) between the un-oriented geodesic arcs \(\gamma_{i-1}\) and \(\gamma_i\) meeting at the edge \(t_i\) of \(t_P\).

**Remark 1.3** It turns out that \(\tilde{\theta}_i = 0\) exactly when \(b_{i-1} \times b_i = 0_{R^3}\), i.e., when \(b_{i-1} = b_i\) or \(b_{i-1} = -b_i\), so that \(\tau(\sigma_i) = 0\), and actually that \(\tilde{\theta}_i = |\theta_i|\) for each \(i = 1, \ldots, n-1\), whence the above definitions are equivalent. In fact, by similarity, and up to a rotation, we can assume that \(\sigma_i = (1, 0, 0)\). Setting \(\sigma_{i-1} = (\alpha_1, \beta_1, \gamma_1)\) and \(\sigma_{i+1} = (\alpha_2, \beta_2, \gamma_2)\), one has \(\sigma_{i-1} \times \sigma_i = (0, \gamma_1, -\beta_1)\) and \(\sigma_i \times \sigma_{i+1} = (0, -\gamma_2, \beta_2)\), so that

\[
b_{i-1} = \frac{(0, \gamma_1, -\beta_1)}{\sqrt{\beta_1^2 + \gamma_1^2}}, \quad b_i = \frac{(0, -\gamma_2, \beta_2)}{\sqrt{\beta_2^2 + \gamma_2^2}}
\]

where \(\sigma_{i-1}, \sigma_i, \sigma_{i+1}\) are not co-planar provided that \(\sigma_{i-1} \times \sigma_i \neq 0_{R^3}\), \(\sigma_i \times \sigma_{i+1} \neq 0_{R^3}\), and \(b_{i-1} \times b_i \neq 0_{R^3}\). Now, the shortest angle \(\tilde{\theta}_i\) between the geodesic arcs \(\gamma_{i-1}\) and \(\gamma_i\) meeting at \(t_i\) is equal to the angle between the planes \(\pi_i^{i-1}\) and \(\pi_i^{i+1}\) spanned by the vectors \((\sigma_{i-1}, \sigma_i)\) and \((\sigma_i, \sigma_{i+1})\), respectively. But the corresponding unit normals are \(b_{i-1}\) and \(b_i\), whence \(\tilde{\theta}_i = |\theta_i|\), where \(|\theta_i|\) is given by \cite{12}, as required.

**Remark 1.4** In an analytical approach, it turns out that the total curvature and absolute torsion of a polygonal \(P\) can be seen as the total variation of mutually singular Radon measures \(\mu_P\) and \(\nu_P\) in \(R^3\). In fact, with the above notation we have:

\[
\text{TC}(P) = |\mu_P|(R^3), \quad \mu_P := \sum_{i=1}^{n-1} \alpha_i \delta_{v_i}, \quad \text{TAT}(P) = |\nu_P|(R^3), \quad \nu_P := \sum_{i=2}^{n-1} \theta_i \mathcal{H}^1 \cup \sigma_i
\]
where $\delta_i$ is the unit Dirac mass at the vertex $v_i$ and $\mathcal{H}^1 \searrow \sigma_i$ is the restriction to the segment $\sigma_i$ of the 1-dimensional Hausdorff measure $\mathcal{H}^1$.

**Remark 1.5** If the polygonal $P$ is closed, i.e., $v_0 = v_n$, the above notation is modified in a straightforward way: the torsion is defined at all the $n$ segments $\sigma_i$, whereas the tangent indicatrix $t_P$ is a closed polygonal curve in $S^2$, so that $n$ angles are to be considered in both the definitions of $\text{TAT}(P)$.

**The Projective Plane.** We have seen that the torsion is computed in terms of angles between undirected unit normal vectors $b_i$ of $\mathbb{R}^3$, see Remarks [1,1] and [1,3]. This implies that any reasonable notion of binormal (for non-smooth curves) naturally lives in the real projective plane $\mathbb{R}P^2$.

For this purpose, we recall that $\mathbb{R}P^2$ is defined by the quotient space $\mathbb{R}P^2 := S^2/\sim$, the equivalence relation being $y \sim \tilde{y} \iff y = \tilde{y}$ or $y = -\tilde{y}$, and hence the elements of $\mathbb{R}P^2$ are denoted by $[y]$. The projective plane $\mathbb{R}P^2$ is naturally equipped with the induced metric

$$d_{\mathbb{R}P^2}([y], [\tilde{y}]) := \min \{d_{S^2}(y, \tilde{y}), d_{S^2}(y, -\tilde{y})\}.$$ 

Similarly to $(S^2, d_{S^2})$, the metric space $(\mathbb{R}P^2, d_{\mathbb{R}P^2})$ is complete, and the projection map $\Pi : S^2 \to \mathbb{R}P^2$ such that $\Pi(y) := [y]$ is continuous. Let $u : A \to \mathbb{R}P^2$ be continuous map defined on an open set $A \subset \mathbb{R}^n$. If $A \subset \mathbb{R}^n$ is simply connected, by the lifting theorem, see e.g. [11] p. 34, there are exactly two continuous functions $v_i : A \to S^2$ such that $[v_i] := \Pi \circ v_i = u$, for $i = 1, 2$, with $v_2(x) = -v_1(x)$ for every $x \in A$.

The manifold $\mathbb{R}P^2$ is non-orientable. Moreover, the mapping $g : S^2 \to \mathbb{R}^6$

$$g(y_1, y_2, y_3) = \left(\frac{\sqrt{2}}{2} y_1^2, \frac{\sqrt{2}}{2} y_2^2, \frac{\sqrt{2}}{2} y_3^2, y_1 y_2, y_2 y_3, y_3 y_1\right)$$

induces an embedding $\tilde{g} : \mathbb{R}P^2 \to \mathbb{R}^6$, $\mathbb{R}P^2 := g(S^2) \subset \mathbb{R}^6$, $\tilde{g}([y]) := g(y)$.

Notice that $\mathbb{R}P^2$ is a non-orientable, smooth, compact, connected submanifold of $\mathbb{R}^6$ without boundary, such that $|z| = \sqrt{2}/2$ for every $z \in \mathbb{R}P^2$. Also, $g$ maps the equator $S^2 \cap \{y^3 = 0\}$ into a circle $C$ of radius $1/2$, covered twice, with constant velocity equal to one. The circle $C$ is a minimum length generator of the first homotopy group $\pi_1(\mathbb{R}P^2) \simeq \mathbb{Z}_2$. We also have $\mathcal{H}^2(\mathbb{R}P^2) = 2\pi$, where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure, compare e.g. [8] Prop. 2.3. Moreover, $g$ is an isometric embedding. If e.g. a map $u : A \to \mathbb{R}P^2$ is given by $u = g \circ v$ for some smooth map $v : A \to S^2$, we in fact have

$$|D_i v|^2 = |v|^2 \cdot |D_i v|^2 + (v \cdot D_i v)^2$$

for each partial derivative $D_i$. Therefore, since $|v| = 1$ and $2 (v \cdot D_i v) = D_i |v|^2 = 0$ a.e. for every $i$, we infer that $|D_i u| = |D_i v|$. 

**Polar Curve.** Using the above notation, and following Fenchel’s approach [4], we now introduce the polar of the tangent indicatrix $t_P$, a curve supported in the projective plane $\mathbb{R}P^2$, in such a way that the length in $\mathbb{R}P^2$ of the polar is equal to the total absolute torsion $\text{TAT}(P)$.

For this purpose, we recall that the support of $t_P$ is the union of $n - 1$ geodesic arcs $\gamma_i$, where $\gamma_i$ has initial point $t_i$ and end point $t_{i+1}$, for $i = 1, \ldots, n - 1$. Since we assumed that consecutive segments of $P$ are never aligned, each arc $\gamma_i$ is non-trivial and well-defined. According to the definition [1,1], it turns out that the discrete unit normal $b_i \in S^2$ is the “north pole” corresponding to the great circle passing through $\gamma_i$ and with the same orientation as $\gamma_i$.

For any $i = 2, \ldots, n - 1$, we denote by $\Gamma_i$ the geodesic arc in $\mathbb{R}P^2$ with initial point $[b_{i-1}]$ and end point $[b_i]$. Then $\Gamma_i$ is degenerate when $b_{i-1} = \pm b_i$, i.e., when the three segments $\sigma_{i-1}$, $\sigma_i$, $\sigma_{i+1}$ are co-planar. We thus have $L_{\mathbb{R}P^2}(\Gamma_i) = \theta_i = |\theta_i|$ for each $i$, and hence that

$$\sum_{i=2}^{n-1} L_{\mathbb{R}P^2}(\Gamma_i) = \text{TAT}(P).$$

Furthermore, for $i < n - 2$, the end point of $\Gamma_i$ is equal to the initial point of $\Gamma_{i+1}$. Finally, if $\text{TAT}(P) = 0$, i.e., if the polygonal $P$ is coplanar, all the arcs $\Gamma_i$ degenerate to a point $[b] \in \mathbb{R}P^2$, which actually identifies the binormal to $P$. 

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Figure 1: An example of a polygonal curve with tangent indicatrix moving as in the left figure. The weak binormal indicatrix moves as in the right figure. Since the weak binormal indicatrix lives in the projective space $\mathbb{RP}^2$, in the figure we have drawn one of its two possible liftings into the sphere $S^2$.

**Definition 1.6** Polar of the tangent indicatrix $t_P$ is the oriented curve in $\mathbb{RP}^2$ obtained by connecting the consecutive geodesic arcs $\Gamma_i$, for $i = 2, \ldots, n - 1$.

**Weak binormal.** Therefore, the polar of $t_P$ connects by geodesic arcs in $\mathbb{RP}^2$ the consecutive discrete binormals $[b_i]$ of the polygonal $P$, and its total length is equal to the total absolute torsion $\text{TAT}(P)$ of $P$. In particular, it is a rectifiable curve. This property allows us to introduce a suitable weak notion of binormal.

**Definition 1.7** We denote binormal indicatrix of the polygonal $P$ the arc-length parameterization $b_P$ of the polar in $\mathbb{RP}^2$ of the tangent indicatrix $t_P$ (see Figure 1).

We thus have $b_P : [0, T] \to \mathbb{RP}^2$, where $T := L_{\mathbb{RP}^2}(b_P) = \text{TAT}(P)$. Moreover, $b_P$ is Lipschitz-continuous and piecewise smooth, with $|\dot{b}_P| = 1$ everywhere except to a finite number of points.

**Remark 1.8** An important monotonicity property holds true. If $P$ and $P'$ are two polygonal curves in $\mathbb{R}^3$, and $P'$ is obtained by replacing a segment $\sigma$ of $P$ with the two segments joining the end points of $\sigma$ with a new vertex, it turns out that

$$L_{\mathbb{S}^2}(t_P) \leq L_{\mathbb{S}^2}(t_{P'}) \quad \text{and} \quad L_{\mathbb{RP}^2}(b_P) \leq L_{\mathbb{RP}^2}(b_{P'}).$$

The first inequality is trivial. Moreover, looking at the tangent indicatrix and weak binormal corresponding to the polygonals, see Definition 1.7, the triangle inequality in $\mathbb{S}^2$ and in $\mathbb{RP}^2$, respectively, yields that their lengths satisfy the inequalities:

$$L_{\mathbb{S}^2}(t_P) \leq L_{\mathbb{S}^2}(t_{P'}) \quad \text{and} \quad L_{\mathbb{RP}^2}(b_P) \leq L_{\mathbb{RP}^2}(b_{P'}).$$

**Remark 1.9** For future use, we point out that the polar of the binormal indicatrix curve $b_P$ agrees (up to the extremal geodesic segments of $t_P$) with the tangent indicatrix $t_P$. In fact, for closed polygonals in the Gauss sphere, polarity is an involutive transformation. This property implies in particular that the total curvature of $b_P$ in $\mathbb{RP}^2$ is bounded by the length of $t_P$, i.e.,

$$\text{TC}_{\mathbb{RP}^2}(b_P) \leq L_{\mathbb{S}^2}(t_P) = \text{TC}(P).$$

### 2 Curves with finite total absolute torsion

In this section, we collect some notation concerning the total absolute torsion of curves in $\mathbb{R}^3$. We thus let $c$ be a simple curve in $\mathbb{R}^3$ parameterized by $c : I \to \mathbb{R}^3$, where $I := [a, b]$ and $c$ is continuous and one-to-one.
Any polygonal curve $P$ inscribed in $c$, say $P \ll c$, is obtained by choosing a finite partition $D := \{a = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = b\}$ of $I$, say $P = P(D)$, and letting $P : I \to \mathbb{R}^3$ such that $P(\lambda_i) = v_i := c(\lambda_i)$ for $i = 0, \ldots, n$, and $P(\lambda)$ affine on each interval $I_i := [\lambda_{i-1}, \lambda_i]$ of the partition, so that $P(I_i) = \sigma_i = [v_{i-1}, v_i]$. The length $L(c)$, the total curvature $TC(c)$, and the total absolute torsion $TAT(c)$ of $c$ are respectively defined through the formulas:

\[
\begin{align*}
L(c) &:= \sup\{L(P) \mid P \ll c\} \\
TC(c) &:= \sup\{TC(P) \mid P \ll c\} \\
TAT(c) &:= \sup\{TAT(P) \mid P \ll c\}.
\end{align*}
\]

Let $c$ be a curve in $\mathbb{R}^3$ with finite total curvature, i.e., $TC(c) < \infty$. Then it is rectifiable, too, see e.g. [12]. Assume that $c : [0, L] \to \mathbb{R}^3$ is its arc-length parameterization, whence $L = L(c) < \infty$. Then $c$ is Lipschitz-continuous, hence by Rademacher’s theorem it is differentiable a.e. in $[0, L]$. Assume that $c$ is a BV function and moreover that its essential variation in a.e. discontinuity point of $c$ appears at any edge point of $c$. Conversely, by taking $v$, where we denote solutions to the Frenet-Serret system (2.1) are rectifiable curves (a planar curve). Conversely, by taking $c$ in $\mathbb{R}^2$, see (3.1) below. For this purpose, we first discuss here the regular case, i.e., when $c$ is the (positive) curvature and $\tau$ the torsion of the curve.

**Remark 2.1** Notice that a rectifiable curve may have unbounded total curvature but zero torsion (just consider a planar curve). Conversely, by taking $s \in [0, 1]$ and letting $c(s) \equiv 1$ and $\tau(s) = (1 - s)^{-1}$, solutions to the Frenet-Serret system (2.1) are rectifiable curves $c$ such that $\int_c k \, ds = 1$ but $\int_c |\tau| \, ds = +\infty$.

As the following example shows, the (absolute value of the) torsion may be seen as the curvature of the tangent (or tangent indicatrix), when computed in the sense of the spherical geometry.

**Example 2.2** Given $R > 0$ and $K \geq 0$, we let $c : [-L/2, L/2] \to \mathbb{R}^3$ denote the helicoidal curve

\[
c(s) := (R \cos(s/v), R \sin(s/v), Ks/(2\pi v)), \quad s \in [-L/2, L/2]
\]

where we denote $v := (R^2 + (K/2\pi)^2)^{1/2}$ and choose $L := 2\pi v$, so that $|c| \equiv 1$ and the length $L(c) = L$. Moreover, $c(\pm L/2) = (\pm R, 0, \pm K/2)$, and $c(0) = (R, 0, 0)$. We thus have

\[
\begin{align*}
t(s) &= v^{-1}(-R \sin(s/v), R \cos(s/v), K/2\pi) \\
n(s) &= (-\cos(s/v), -\sin(s/v), 0) \\
b(s) &= v^{-1}((K/2\pi) \sin(s/v), -(K/2\pi) \cos(s/v), R)
\end{align*}
\]

so that both curvature and torsion are constant, $k \equiv Rv^{-2}, \tau \equiv v^{-2}(K/2\pi)$. Therefore, the integral of the curvature and of the torsion of $c$ are readily obtained:

\[
\int_c k \, ds = L \cdot k = \frac{2\pi R}{v}, \quad \int_c |\tau| \, ds = L \cdot \tau = \frac{K}{v}, \quad v := (R^2 + (K/2\pi)^2)^{1/2}.
\]
We now compute the spherical curvature $k_{S^2}(t)$ of the tantrix $t$, a closed curve embedded in the Gauss sphere $S^2$ and parameterizing (when $K > 0$) a small circle whose radius depends on $R$ and $K$. We consider a sequence of (strongly converging) polygonal curves $\{t_n\}$ in $S^2$ inscribed in the tantrix $t$. The total curvature of $t_n$ is equal to the sum of the width in $S^2$ of the angles between consecutive segments. When $n \to \infty$, by uniform convergence we obtain the total curvature of $t$ in $S^2$. Actually, it agrees with the integral of the absolute torsion of $c$, i.e.,

$$\int_{S^2} k_{S^2}(t) \, ds = \frac{K}{v} = \int_c |\tau| \, ds.$$ 

To this purpose, for each $n \in \mathbb{N}^+$, we let $t_n(i) := t(s_i)$, where $s_i = (L/n)i$ and $i \in \mathbb{Z} \cap [-n,n]$, and we consider the closed spherical polygonal generated by the consecutive points $t_n(i) \in S^2$.

The turning angle in $S^2$ of two consecutive geodesic segments $t_n(i-1)t_n(i)$ and $t_n(i)t_n(i+1)$, agrees with the angle between the two planes in $\mathbb{R}^3$ spanned by $0_{\mathbb{R}^3}$ and the end points of the above segments, i.e., between the normals $t_n(i-1) \times t_n(i)$ and $t_n(i) \times t_n(i+1)$. By symmetry, such an angle $\theta_n$ does not depend on the choice of $i$, and will be computed at $i = 0$. The total spherical curvature of the polygonal being equal to $n \cdot \theta_n$, we check:

$$\lim_{n \to \infty} n \cdot \theta_n = \frac{K}{v}.$$ 

In fact, in correspondence to the middle point we have

$$t_n(0) = v^{-1}(0, R, K/2\pi), \quad t_n(\pm 1) = v^{-1}(\mp R \sin(2\pi/n), R \cos(2\pi/n), K/2\pi)$$

so that we get

$$t_n(0) \times t_n(\pm 1) = \frac{1}{v^2} \cdot \left(R(K/2\pi)(1 - \cos(2\pi/n)), \mp R(K/2\pi) \sin(2\pi/n), \pm R^2 \sin(2\pi/n)\right).$$

Denoting for simplicity

$$M_n := |t_n(0) \times t_n(\pm 1)| = \frac{R}{v^2} \cdot \left(\left(K/2\pi\right)^2 2(1 - \cos(2\pi/n)) + R^2 \sin^2(2\pi/n)\right)^{1/2}$$

and setting $N_n^+ := \pm (t_n(0) \times t_n(\pm 1))/M_n$, we compute

$$N_n^+ \times N_n^- = \frac{R^2}{M_n^2} \left(K/2\pi\right) \sin(2\pi/n) 2(1 - \cos(2\pi/n)) \cdot (0, -R, (K/2\pi))$$

$$|N_n^+ \times N_n^-| = \frac{R^2}{M_n^2} \left(K/2\pi\right) \sin(2\pi/n) 2(1 - \cos(2\pi/n)) v.$$ 

By symmetry, the turning angle of the geodesic arcs connecting two consecutive points $t_n(i)$ does not depend on the choice of $i$ and is equal to

$$\theta_n := \arcsin |N_n^+ \times N_n^-|.$$ 

Since for $n \to \infty$ we have $2(1 - \cos(2\pi/n)) \sim (2\pi/n)^2$ and $\sin(2\pi/n) \sim 2\pi/n$, we get $M_n \sim R(2\pi/n)v$ and finally $n \cdot \theta_n \sim n \cdot |N_n^+ \times N_n^-| \to K/v$ where, we recall, $\int_c |\tau| = K/v$.

**Remark 2.3** In the previous example, we have considered a sequence $\{t_n\}$ of polygonal curves in $S^2$ inscribed in the tantrix $t$ of $c$ and converging to $t$ in the sense of the Hausdorff distance. In general, each $t_n$ is not the tangent indicatrix of a polygonal inscribed in $c$. However, the total spherical curvature $n \cdot \theta_n$ of $t_n$ clearly agrees with the length in $\mathbb{R}P^2$ of the polar of $t_n$, which is constructed as in Sec. 1, see Definition 1.6.

Now, one may similarly consider a sequence $\{P_h\}$ of polygons inscribed in $c$, each one made of $h$ segments with the same length, so that mesh $P_h \to 0$. The total absolute torsion $\text{TAT}(P_h)$ of $P_h$, i.e., the total spherical curvature of the tangent indicatrix $t_{P_h}$, agrees with the length in $\mathbb{R}P^2$ of the binormal indicatrix $b_{P_h}$, see Definition 1.7. By means of a similar computation (that we shall omit), one can show that $L_{\mathbb{R}P^2}(b_{P_h}) \to K/v$ as $h \to \infty$. This implies the expected formula:

$$\text{TAT}(c) = \int_c |\tau| \, ds.$$
3 Weak binormal of a non-smooth curve

In this section, we consider rectifiable curves $c$ in $\mathbb{R}^3$ with finite (and non zero) total curvature $TC(c)$ and finite total absolute torsion $TAT(c)$. Using a density approach by polygons, we shall see, Theorem 3.1, that a weak notion of binormal indicatrix of $c$ is well-defined. For smooth curves, we shall recover the classical binormal, see Theorem 3.2, and Remark 3.3. For future use, a similar property concerning the tangent indicatrix is briefly discussed, see Proposition 3.4 and 3.5.

More precisely, we shall define a Lipschitz-continuous function $b_c : [0, T] \rightarrow \mathbb{R}^2$, where $T = TAT(c)$, satisfying $|b_c| = 1$ a.e. in $[0, T]$. Therefore, $b_c$ is a curve in $\mathbb{R}P^2$ with length equal to the total absolute torsion of $c$, i.e.,

$$L_{\mathbb{R}P^2}(b_c) = TAT(c).$$

This is the content of our first main result:

**Theorem 3.1** Let $c : [0, L] \rightarrow \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ with finite total curvature $TC(c)$ and finite (and non-zero) total absolute torsion $T := TAT(c)$. There exists a rectifiable curve $b_c : [0, T] \rightarrow \mathbb{R}P^2$ parameterized by arc-length, so that $L_{\mathbb{R}P^2}(b_c) = TAT(c)$, satisfying the following property. For any sequence $\{P_h\}$ of inscribed polyhedral curves, let $b_h : [0, T] \rightarrow \mathbb{R}P^2$ denote for each $h$ the parameterization with constant velocity of the binormal indicatrix $b_{P_h}$ of $P_h$, see Definition 3.1. If mesh $P_h \rightarrow 0$, then $b_h \rightarrow b_c$ uniformly on $[0, T]$ and $L_{\mathbb{R}P^2}(b_h) \rightarrow L_{\mathbb{R}P^2}(b_c)$.

Furthermore, we shall see that if $c$ is smooth in the sense of the previous section (so that the Frenet-Serret formulas (2.1) hold), the binormal $b(s)$ of $c$ agrees with the value of a suitable lifting of the weak binormal $b_c$ in $\mathbb{S}^2$, when computed at the expected point.

**Theorem 3.2** Let $c : [0, L] \rightarrow \mathbb{R}^3$ be a rectifiable curve of class $C^3$ parameterized in arc-length, so that $L = L(c)$. Assume that $\ddot{c}(s) \neq 0$ for each $s \in [0, L]$, so that the spherical frame $(t(n, b)$ of $c$ is well-defined. Let $b_c : [0, T] \rightarrow \mathbb{R}P^2$ be the rectifiable curve in $\mathbb{R}P^2$ defined in Theorem 3.1 so that $T = TAT(c)$. Then, for each $s \in [0, L]$ there exists $t(s) \in [0, T]$ such that

$$b(s) = \tilde{b}_c(t(s))$$

for a unique lifting $\tilde{b}_c$ of $b_c$ in $\mathbb{S}^2$. Moreover, $t(s)$ is equal to the total absolute torsion $TAT(c_{|[0,s]})$ of the curve $c_{|[0,s]} : [0, s] \rightarrow \mathbb{R}^3$. In particular, we have:

$$t(s) = \int_0^s |\tau(\lambda)| \, d\lambda \quad \forall \, s \in [0, L]$$

where $\tau(\lambda)$ is the torsion of the curve $c$ at the point $c(\lambda)$.

**Remark 3.3** Notice that if the torsion $\tau$ of $c$ (almost) never vanishes, the function $t(s) : [0, L] \rightarrow [0, T]$ in equation (3.2) is strictly increasing, and its inverse $s(t) : [0, T] \rightarrow [0, L]$ gives

$$\tilde{b}_c(t) = b(s(t)) \quad \forall \, t \in [0, T], \quad T = TAT(c).$$

Therefore, in this case, the weak binormal $b_c$ in $\mathbb{R}P^2$, when suitably lifted to $\mathbb{S}^2$, agrees with the arc-length parameterization of the binormal $b$ of the given curve.

**Weak tangent indicatrix.** Arguing as in the proof of Theorems 3.1 and 3.2 we correspondingly obtain the following properties concerning the tantrix.

**Proposition 3.4** Let $c$ be a curve in $\mathbb{R}^3$ with finite total curvature $TC(c)$. There exists a rectifiable curve $t_c : [0, C] \rightarrow \mathbb{S}^2$, where $C := TC(c)$, parameterized by arc-length, so that $L_{\mathbb{S}^2}(t_c) = TC(c)$, satisfying the following property. For any sequence $\{P_h\}$ of inscribed polyhedral curves such that mesh $P_h \rightarrow 0$, denoting by $t_h : [0, T] \rightarrow \mathbb{R}P^2$ the parameterization with constant velocity of the tangent indicatrix $t_{P_h}$ of $P_h$, then $t_h \rightarrow t_c$ uniformly on $[0, C]$ and $L_{\mathbb{S}^2}(t_h) \rightarrow L_{\mathbb{S}^2}(t_c)$. 


Proposition 3.5 Let $c : [0, L] \to \mathbb{R}^3$ be a rectifiable curve of class $C^2$ parameterized in arc-length, so that $L = L(c)$, and let $c_0 : [0, C] \to \mathbb{S}^2$ be the rectifiable curve in $\mathbb{S}^2$ defined in Proposition 3.4, so that $C = TC(c)$. Then, for each $s \in [0, L]$ there exists $k(s) \in [0, C]$ such that the tangent indicatrix $t := c$ satisfies $$t(s) = t_c(k(s)).$$ Moreover, $k(s)$ is equal to the total curvature $TC(c|_{[0, s]})$ of the curve $c|_{[0, s]} : [0, s] \to \mathbb{R}^3$, whence:

$$k(s) = \int_0^s k(\lambda) \, d\lambda \quad \forall s \in [0, L]$$

(3.3)

where $k(\lambda) := |\dot{c}(\lambda)|$ is the (non-negative) curvature of the curve $c$ at the point $c(\lambda)$.

Remark 3.6 As before, if the curvature $k$ of $c$ (almost) never vanishes, the function $k(s) : [0, L] \to [0, C]$ in equation (3.3) is strictly increasing, and its inverse $s(k) : [0, C] \to [0, L]$ gives

$$t_c(k) = t(s(k)) \quad \forall k \in [0, C], \quad C = TC(c).$$

Whence, the weak tangent $t_c$ agrees with the arc-length parameterization of the tantrix $t$.

**Proof.** We now give the proofs of the previous results.

Proof of Theorem 3.1. Choose the approximating sequence $\{P_h\}$ of polygons inscribed in $c$. For $h$ large, so that $T_h := TAT(P_h)_h > 0$, the binormal indicatrix of the polygon $P_h$ has been defined by the arc-length parameterization $b_{P_h} : [0, T_h] \to \mathbb{R}P^2$ of the curve in $\mathbb{R}P^2$ given by the polar of the tangent indicatrix $t_{P_h}$, see Definition 1.7. Hence it is a rectifiable curve such that $T_h = L_{\mathbb{R}P^2}(b_{P_h}) = TAT(P_h)$ and $|b_{P_h}| = 1$ a.e. on $[0, T_h]$. Since mesh $P_h \to 0$, we also know that $T_h \not\to T := TAT(c)$.

Define $b_h : [0, T] \to \mathbb{R}P^2$ by $b_h(s) := b_{P_h}((T_h/s)/T_h)$, so that $|\dot{b}_h(s)| = T_h/T$ a.e., where $T_h/T \not\to 1$. By Ascoli-Arzelà’s theorem, we can find a subsequence $\{b_{h_k}\}$ that uniformly converges in $[0, T]$ to some Lipschitz continuous function $b : [0, T] \to \mathbb{R}P^2$. Moreover, by a standard argument in analysis, it turns out that the limit function $b$ does not depend on the choice of the approximating sequence $\{P_h\}$ of polygons. As a consequence, by a contradiction argument one infers that all the sequence $\{b_h\}$ uniformly converges to $b$. In particular, the curve $b$ is identified by $c$, and we thus denote $b_c = b$.

We claim that $b_h \to b$ strongly in $L^1$. As a consequence, we deduce that $|\dot{b}_c| = 1$ a.e. on $[0, T]$, and hence that

$$L_{\mathbb{R}P^2}(b_c) = \int_0^T |\dot{b}_c(s)| \, ds = T = TAT(c).$$

In order to prove the claim, recalling from Sec. 1 that $\tilde{g} : \mathbb{R}P^2 \to \mathbb{R}^2 \subset \mathbb{R}^6$ is the isometric embedding of the projective plane, we shall denote here $f := \tilde{g} \circ f$, for any function $f$ with values in $\mathbb{R}P^2$, and we consider the tantrix $\tau_h$ of the curve $b_h : [0, T] \to \mathbb{R}P^2$, i.e., $\tau_h(s) = \tilde{b}_h(s)/|\dot{b}_h(s)|$. We have $L_{\mathbb{R}P^2}(b_h) = TAT(P_h)$ and $|\dot{b}_h(s)| = T_h/T$, whereas by Remark 1.9

$$TC_{\mathbb{R}P^2}(b_h) \leq L_{\mathbb{R}^2}(t_{P_h}) = TC(P_h).$$

Therefore, it turns out that the essential total variation of $\tau_h$ in $\mathbb{R}P^2$ is lower than the sum $TC(P_h) + TAT(P_h)$. We thus get:

$$\sup_h \text{Var}_{\mathbb{R}P^2}(\tau_h) \leq TC(c) + TAT(c) < \infty.$$

As a consequence, by compactness, possibly passing to a subsequence we infer that $\dot{b}_h$ converges weakly in the BV-sense to some BV-function $v : [0, T] \to \mathbb{R}P^2$.

We claim that $v(s) = \tilde{b}(s)$ for a.e. $s \in [0, T]$, which clearly yields that the whole sequence $\{b_h\}$ converges strongly in $L^1$ (and hence a.e. on $[0, T]$) to the function $b$.

In fact, using that by Lipschitz-continuity

$$b_h(s) = b_h(0) + \int_0^s \dot{b}_h(\lambda) \, d\lambda \quad \forall s \in [0, T]$$
and setting

\[ V(s) := \dot{b}(0) + \int_0^s v(\lambda) d\lambda, \quad s \in [0, T] \]

by the weak BV convergence \( b_h' \to v \), which implies the strong \( L^1 \)-convergence, we have \( b_h \to V \) in \( L^\infty \), hence \( b_h' \to V = v \) a.e. in \( [0, L] \). But we already know that \( b_h \to b \) in \( L^\infty \), thus \( v = \dot{b} \). \( \square \)

**Proof of Theorem 3.2** For any given \( s \in [0, L] \), since \( |\dot{c}(s)| = 1 \) and \( \ddot{c}(s) \neq 0 \), the binormal is defined by \( b(s) := t(s) \times n(s) \), with \( t(s) := \dot{c}(s) \) and \( n(s) : = \ddot{c}(s)/|\ddot{c}(s)| \), so that \( \dot{c}(s) \times \ddot{c}(s) \neq 0 \) and

\[ b(s) = \frac{\dot{c}(s) \times \ddot{c}(s)}{|\ddot{c}(s)|}. \]

We thus may and do choose a sequence of polygonals \( \{P_h\} \) inscribed in \( c \) such that \( \text{mesh } P_h \to 0 \) and (with the notation from Sec. 1 for \( P = P_h \) the following properties hold for any \( h \in \mathbb{N}^+ \) large enough:

i) the four points \( v_{i-2} = c(s - 2h) \), \( v_{i-1} = c(s - h) \), \( v_i = c(s + h) \), \( v_{i+1} = c(s + 2h) \) are consecutive (and interior) vertexes of \( P_h \);

ii) the three segments \( \sigma_{i-1} = v_{i-1} - v_{i-2} \), \( \sigma_i = v_i - v_{i-1} \), \( \sigma_{i+1} = v_{i+1} - v_i \) satisfy \( \sigma_{i-1} \times \sigma_i \neq 0_{\mathbb{R}^3} \) and \( \sigma_i \times \sigma_{i+1} \neq 0_{\mathbb{R}^3} \).

By taking the second order expansions of \( c \) at \( s \), we get

\[
\begin{align*}
\sigma_{i-1} &= -\ddot{c}(s) h + \frac{3}{2} \dddot{c}(s) h^2 + o(h^2), \\
\sigma_i &= 2 \ddot{c}(s) h^2 + o(h^2), \\
\sigma_{i+1} &= \dddot{c}(s) h + \frac{3}{2} \ddot{c}(s) h^2 + o(h^2)
\end{align*}
\]

and hence

\[
\sigma_{i-1} \times \sigma_i = 2h^2 \dddot{c}(s) \times \dddot{c}(s) + o(h^3), \quad \sigma_i \times \sigma_{i+1} = 2h^2 \dddot{c}(s) \times \dddot{c}(s) + o(h^3).
\]

On account of (1.1), we thus get for any \( h \) large:

\[ b_{i-1}(h) := \frac{\sigma_{i-1} \times \sigma_i}{|\sigma_{i-1} \times \sigma_i|} = -b(s) + o(h^3), \quad b_i(h) := \frac{\sigma_i \times \sigma_{i+1}}{|\sigma_i \times \sigma_{i+1}|} = -b(s) + o(h^3) \]

so that in particular \( b_i(h) \to -b(s) \) as \( h \to \infty \).

Now, consider the polygonal \( P_h(s) \) given by the union of the segments \( \sigma_1, \ldots, \sigma_{i-1}, \sigma_i \) of \( P_h \). It turns out that the total absolute torsion of \( P_h(s) \) satisfies \( \text{TAT}(P_h(s)) = t_h(s) \) for some number \( t_h(s) \in [0, \text{TAT}(P_h)] \). Since \( \text{TAT}(P_h) \to \text{TAT}(c) \in \mathbb{R}^+ \), possibly passing to a subsequence, the sequence \( \{t_h(s)\} \) converges to some number \( t(s) \in [0, T] \). By Theorem 3.1 we thus infer that \( b_i(h) \to b_c(t(s)) \) as \( h \to \infty \), whence we obtain \( b(s) = -b_c(t(s)) \).

Moreover, since both the end points of the segment \( \sigma_i \) of \( P_h \) converge to \( c(s) \) as \( h \to \infty \), whereas mesh \( P_h(s) \to 0 \), we deduce that \( \text{TAT}(P_h(s)) \to \text{TAT}(c_{[0,s]}) \), which yields the equality \( t(s) = \text{TAT}(c_{[0,s]}) \). Since by smoothness of the curve \( c \)

\[ \text{TAT}(c_{[0,s]}) = \int_0^s |\dot{b}(\lambda)| d\lambda \]

recalling that \( \dot{b}(\lambda) = -\tau(\lambda) n(\lambda) \), we finally obtain the equality (3.2). \( \square \)

**Proof of Proposition 3.4** As in the proof of Theorem 3.1 but with \( t_{P_h}, c_h, S^2, C, h, t, \) and \( c_h \) instead of \( b_{P_h}, T_h, \mathbb{R}^2, T, b, \) and \( b_c \), respectively, this time using that \( L_{S^2}(t_h) = \text{TC}(P_h) \) and \( \text{TC}_{S^2}(t_h) = \text{TAT}(P_h) \) to obtain that the essential total variation in \( S^2 \) of the tantrix \( \tau_h \) of \( t_h \) is lower than the sum \( \text{TC}(P_h) + \text{TAT}(P_h) \). We omit any further detail. \( \square \)

**Proof of Proposition 3.5** Similarly to the proof of Theorem 3.2 for any \( s \in [0, L[ \) we choose \( \{P_h\} \) inscribed in \( c \) such that mesh \( P_h \to 0 \) and for any \( h \in \mathbb{N}^+ \) the two points \( v_{i-1} = c(s - h) \) and \( v_i = c(s + h) \)
are consecutive (and interior) vertexes of $P_h$. We thus get $\sigma_i := v_i - v_{i-1} = 2\dot{c}(s) h + o(h)$, whence $t_i(h) := \sigma_i/|\sigma_i| \to \dot{c}(s) = t(s)$ as $h \to \infty$. Also, denoting again by $P_h(s)$ the polygonal corresponding to the segments $\sigma_1, \ldots, \sigma_{i-1}, \sigma_i$ of $P_h$, we have $\text{TC}(P_h(s)) = k_h(s) \in [0, \text{TC}(P_h)]$, where $\text{TC}(P_h) \to \text{TC}(c) \in \mathbb{R}_0^+$, whence a subsequence of $\{k_h(s)\}$ converges to some $k(s) \in [0, C]$. Proposition 3.4 yields that $t_i(h) \to t_c(k(s))$ as $h \to \infty$, whence we get $t(s) = t_c(k(s))$. We clearly have $\text{TC}(P_h(s)) \to \text{TC}(c_{[0,s]})$, which implies that
\[
  k(s) = \text{TC}(c_{[0,s]}) = \int_0^s |\dot{t}(\lambda)| \, d\lambda.
\]

Recalling that $\dot{t} = k \mathbf{n}$, we finally obtain the equality (3.3). \hfill \Box

4 Weak normal of a non-smooth curve

We have seen that the curvature of an open polygonal $P$ is a non-negative measure $\mu_P$ concentrated at the interior vertexes of $P$, whereas the torsion is a signed measure $\nu_P$ concentrated at the interior segments, see Remark 1.4. Since these two measures are mutually singular, in principle there is no analogous to the classical formula by Fenchel for the (principal) normal of smooth curves in $\mathbb{R}^3$, namely
\[
  \mathbf{n} = b \times t. \tag{4.1}
\]

In this section, following Banchoff [1], a weak notion of normal indicatrix of a polygonal is introduced, Definition 4.2, in such a way that formula (4.1) continues to hold. As a consequence, according to the cited Fenchel’s approach, the principal normal of a curve with finite total curvature and absolute torsion is well-defined in a weak sense, Theorem 4.5.

WEAK NORMAL OF POLYGONALS. Let $P$ be an open polygonal in $\mathbb{R}^3$ such that two consecutive (and non-degenerate) segments are never aligned. Denoting $C = \text{TC}(P)$ and $T = \text{TAT}(P)$, we first choose two suitable curves
\[
  \bar{t}_P : [0, C + T] \to \mathbb{R}P^2, \quad \bar{b}_P : [0, C + T] \to \mathbb{R}P^2
\]
which on one side inherit the properties of the tangent indicatrix and of the binormal indicatrix of $P$, respectively, and on the other side take account of the order in which curvature and torsion are defined along $P$. More precisely, we shall recover the properties
\[
  L_{\mathbb{R}P^2}(\bar{b}_P) = \text{TC}_{\mathbb{R}P^2}(\bar{t}_P) = \text{TAT}(P), \quad \text{TC}_{\mathbb{R}P^2}(\bar{b}_P) \leq L_{\mathbb{R}P^2}(\bar{t}_P) = \text{TC}(P), \tag{4.2}
\]
(while all equalities hold in the case of closed polygonals), which are satisfied (up to a lifting) by the curves $t_P$ and $b_P$ defined in Sec. 4. Moreover, in accordance to the mutual singularities of the measures $\mu_P$ and $\nu_P$, see Remark 1.4, one curve is constant when the other one parameterizes a geodesic arc, whose length is equal to the curvature or to the (absolute value of the) torsion at one vertex or segment of $P$, respectively.

Recalling the notation from Sec. 1, we let $v_i$, $i = 0, \ldots, n$, denote the vertexes, and $\sigma_i := [v_{i-1}, v_i]$, $i = 1, \ldots, n$, the oriented segments of $P$. Also, we let $t_i := \sigma_i/|\sigma_i| \in \mathbb{S}^2$, for $i = 1, \ldots, n$, and $\gamma_i$ is the oriented geodesic arc in $\mathbb{S}^2$ connecting the consecutive points $t_i$ and $t_{i+1}$, for $i = 1, \ldots, n - 1$. Finally, $\Gamma_i$ is the geodesic arc in $\mathbb{R}P^2$ with initial point $[b_{i-1}]$ and end point $[b_i]$, for any $i = 2, \ldots, n - 1$, where $b_i$ is the discrete binormal (1.11). We thus have
\[
  \text{TC}(P) = \sum_{i=1}^{n-1} L_{\mathbb{S}^2}(\gamma_i), \quad \text{TAT}(P) = \sum_{i=2}^{n-1} L_{\mathbb{R}P^2}(\Gamma_i).
\]

Remark 4.1 In order to explain our construction below, let us choose a lifting $\widehat{b}_P : [0, T] \to \mathbb{S}^2$ of the (continuous) curve $b_P$ from Definition 1.7 and let $\widehat{b}_i$ and $\Gamma_i$ denote the points and geodesic arcs corresponding to $[b_i]$ and $\Gamma_i$. For $i = 1, \ldots, n - 1$, we let $\widehat{\gamma}_i = \widehat{b}_i \times \gamma_i$, i.e., $\widehat{\gamma}_i$ is the oriented geodesic arc in $\mathbb{S}^2$ obtained by means of the vector product of the lifted discrete binormal $\widehat{b}_i$ with each point in the support of the arc $\gamma_i$. 12
For $i = 2, \ldots, n - 1$, we also let $\Gamma_i = \tilde{\Gamma}_i \times t_{i+1}$, i.e., $\Gamma_i$ is the oriented geodesic arc in $S^2$ obtained by means of the vector product of each point in the support of the lifted arc $\tilde{\Gamma}_i$, with the direction $t_{i+1}$.

It turns out that for $i = 1, \ldots, n - 2$, the final point of $\tilde{\gamma}_i$ agrees with the initial point of $\Gamma_{i+1}$, and that the final point of $\Gamma_{i+1}$ agrees with the initial point of $\tilde{\gamma}_{i+1}$. Using this order to join the geodesic arcs, one obtains a rectifiable curve in $S^2$ whose total length is equal to the sum of the lengths of $t_P$ and of $b_P$, i.e., to $TC(P) + TAT(P)$. However, since the curve depends on the chosen lifting of the binormal, it is more natural to work in the projective plane. Therefore, we shall consider the geodesic arcs $[\gamma_i] := \Pi(\tilde{\gamma}_i)$ with end points $[t_i] := \Pi(t_i)$, where $\Pi : S^2 \to \mathbb{R}P^2$ is the canonical projection.

Recalling that $C := TC(P)$ and $T = TAT(P)$, we shall denote for brevity $C_0 := 0$, $T_1 := 0$, and

$$
C_i := \sum_{j=1}^{i} L_{\mathbb{R}P^2}(\gamma_j), \quad i = 1, \ldots, n - 1, \quad T_i := \sum_{j=2}^{i} L_{\mathbb{R}P^2}(\Gamma_j), \quad i = 2, \ldots, n - 1.
$$

We define $\bar{t}_P : [0, C + T] \to \mathbb{R}P^2$ and $\bar{b}_P : [0, C + T] \to \mathbb{R}P^2$ as follows:

i) $\bar{t}_P$ parameterizes with velocity one the oriented geodesic arc $[\gamma_i]$ on the interval $[C_{i-1} + T_i, C_i + T_i]$, for $i = 1, \ldots, n - 1$;

ii) $\bar{t}_P$ is constantly equal to $[t_i]$ on the interval $[C_{i-1} + T_{i-1}, C_{i-1} + T_i]$, for $i = 2, \ldots, n - 2$;

iii) $\bar{b}_P$ is constantly equal to $[b_i]$ on the interval $[C_{i-1} + T_i, C_i + T_i]$, for $i = 1, \ldots, n - 1$;

iv) $\bar{b}_P$ parameterizes with velocity one the oriented geodesic arc $\Gamma_i$ on the interval $[C_i + T_i, C_i + T_{i+1}]$, for $i = 2, \ldots, n - 2$.

The functions $\bar{t}_P$ and $\bar{b}_P$ are both continuous, and property (4.2) is readily checked. Furthermore, it turns out that the unit vectors $t_P(s)$ and $b_P(s)$ are orthogonal, for each $s \in [0, C + T]$. As a consequence, we are able to define the weak normal according to the formula (4.1).

**Definition 4.2** Normal indicatrix of the polygonal $P$ is the curve $n_P : [0, C + T] \to \mathbb{R}P^2$ (see Figure 2) given by the pointwise vector product

$$
n_P(s) := \bar{b}_P(s) \times \bar{t}_P(s) \in \mathbb{R}P^2, \quad s \in [0, T + C].
$$

For closed polygonals, the above notation is modified in a straightforward way, arguing as in Remark 1.3.

**Remark 4.3** By the definition, it turns out that

$$
L_{\mathbb{R}P^2}(n_P) = L_{\mathbb{R}P^2}(\bar{t}_P) + L_{\mathbb{R}P^2}(\bar{b}_P) = TC(P) + TAT(P).
$$

Notice that, the curvature and torsion of $P$ being mutually singular measures, see Remark 1.4, the above equality is the analogous in the category of polygonals to the integral formulas

$$
\int_c |\hat{n}(s)| \, ds = \int_c \sqrt{k^2(s) + \tau^2(s)} \, ds, \quad \int_c |k(s)| \, ds = TC(c), \quad \int_c |\tau(s)| \, ds = TAT(c)
$$

for smooth curves $c$, which clearly follow from the Frenet-Serret formulas (2.1).

Moreover, we have $|\hat{n}_P(s)| = 1$ for a.e. $s \in [0, C + T]$. In fact, by the definition of $\bar{t}_P$ and $\bar{b}_P$, we get:

i) for $i = 1, \ldots, n - 1$ and $s \in [C_{i-1} + T_i, C_i + T_i]$, we have $\bar{b}_P(s) \equiv [b_i] \in \mathbb{R}P^2$ and hence $\hat{n}_P(s) = [b_i] \times \bar{t}_P(s)$, where $|\hat{n}_P(s)| = 1$ and $[b_i]$ is orthogonal to $\bar{t}_P(s)$;

ii) for $i = 2, \ldots, n - 2$ and $s \in [C_{i-1} + T_{i-1}, C_{i-1} + T_i]$, we have $\bar{t}_P(s) \equiv [t_i]$ and hence $\hat{n}_P(s) = \hat{b}_P(s) \times [t_i]$, where $|\hat{b}_P(s)| = 1$ and $[t_i]$ is orthogonal to $\bar{t}_P(s)$.  

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Remark 4.4 Notice that the turning angle in \( \mathbb{RP}^2 \) of the curve \( \mathbf{n}_P \) is equal to \( \pi/2 \) at each “non-trivial” vertex of \( \mathbf{n}_P \). Indeed, from a vertex of \( \mathbf{n}_P \) we move by rotating either around \( t_a \) or \( b_\beta \) (\( \beta = \alpha \) or \( \beta = \alpha - 1 \)), where \( t_a \perp b_\beta \), hence the two curves are orthogonal. More precisely, for \( i = 1, \ldots, n - 1 \), if the geodesic arcs \( \gamma_i \) and \( \Gamma_{i+1} \) are non-degenerate, they meet orthogonally at the vertex \( \mathbf{n}_P(C_i + T_i) \) of \( \mathbf{n}_P \). Similarly, for any \( i = 2, \ldots, n - 2 \) such that both the geodesic arcs \( \Gamma_{i+1} \) and \( \gamma_{i+1} \) are non-degenerate, they meet orthogonally at the vertex \( \mathbf{n}_P(C_i + T_{i-1}) \).

Weak normal of curves. In the same spirit as in Theorem 3.1 for non-smooth curves we now obtain our second main result.

Theorem 4.5 Let \( c \) be a curve in \( \mathbb{R}^3 \) with finite total curvature \( C := \text{TC}(c) \) and total absolute torsion \( T := \text{TAT}(c) \). There exists a rectifiable curve \( \mathbf{n}_c : [0, C + T] \to \mathbb{RP}^2 \) parameterized by arc-length, so that \( \mathcal{L}_{\mathbb{RP}^2}(\mathbf{n}_c) = \text{TC}(c) + \text{TAT}(c) \), satisfying the following property. For any sequence \( \{P_h\} \) of inscribed polyhedral curves, let \( n_h : [0, C + T] \to \mathbb{RP}^2 \) denote the parameterization with constant velocity of the normal indicatrix \( \mathbf{n}_{P_h} \) of \( P_h \), see Definition 4.2. If \( \text{mesh } P_h \to 0 \), then \( n_h \to n_c \) uniformly on \( [0, C + T] \) and \( \mathcal{L}_{\mathbb{RP}^2}(b_h) \to \mathcal{L}_{\mathbb{RP}^2}(n_c) \).

Proof: By Definition 4.2 the normal indicatrix of \( P_h \) is the curve \( \mathbf{n}_{P_h} : [0, C_h + T_h] \to \mathbb{RP}^2 \) given by
\[
\mathbf{n}_{P_h}(s) := b_{P_h}(s) \times \mathbf{i}_{P_h}(s),
\]
so that \( \mathcal{L}_{\mathbb{RP}^2}(\mathbf{n}_{P_h}) = C_h + T_h \), where \( C_h = \text{TC}(P_h) \) and \( T_h = \text{TAT}(P_h) \), and \( |\mathbf{n}_{P_h}(s)| = 1 \) a.e. on \( [0, C_h + T_h] \). Also, condition \( \text{mesh } P_h \to 0 \) yields that \( C_h \nearrow C \) and \( T_h \nearrow T \).

Setting \( n_h : [0, C + T] \to \mathbb{RP}^2 \) by \( n_h(s) := \mathbf{n}_{P_h}((C_h + T_h)s/(C + T)) \), as before we deduce that the sequence \( \{n_h\} \) uniformly converges in \( [0, C + T] \) to some Lipschitz continuous function \( n : [0, C + T] \to \mathbb{RP}^2 \), and that the limit function \( n = n_c \) does not depend on the choice of the approximating sequence \( \{P_h\} \).

We claim that \( |\mathbf{n}_c| = 1 \) a.e. in \( [0, C + T] \). This yields that
\[
\mathcal{L}_{\mathbb{RP}^2}(\mathbf{n}_c) = \int_0^{C + T} |\mathbf{n}_c(s)| \, ds = C + T = \text{TC}(c) + \text{TAT}(c).
\]
For this purpose, we note that by Definition 4.2 we have \( n_h(s) = \mathbf{b}_h(s) + \mathbf{i}_h(s) \) for each \( s \in [0, T + C] \), where
\[
\mathbf{b}_h(s) := b_{P_h}(((C_h + T_h)s/(C + T))), \quad \mathbf{i}_h(s) := \mathbf{i}_{P_h}((C_h + T_h)s/(C + T)).
\]
Arguing as in Theorem 3.1 and Proposition 3.4 using that (by Remark 1.9) we again have:
\[
\mathcal{L}_{\mathbb{RP}^2}(\mathbf{b}_h) = \text{TC}_{\mathbb{RP}^2}(\mathbf{i}_h) = \text{TAT}(P_h), \quad \text{TC}_{\mathbb{RP}^2}(\mathbf{b}_h) \leq \mathcal{L}_{\mathbb{RP}^2}(\mathbf{i}_h) = \text{TC}(P_h),
\]
we deduce that (possibly passing to a subsequence) \( \tilde{b}_h \to \tilde{b} \) and \( \tilde{t}_h \to \tilde{t} \) strongly in \( W^{1,1} \) (and uniformly) to some continuous functions \( \tilde{b}, \tilde{t} : [0, C + T] \to \mathbb{R}P^2 \). This yields that \( n_c(s) = \tilde{b}(s) \times \tilde{t}(s) \) and hence that

\[
\lim_{h \to \infty} \hat{n}_h(s) = \lim_{h \to \infty} (\hat{b}_h(s) \times \hat{t}_h(s) + \hat{b}_h(s) \times \hat{t}_h(s)) = (\tilde{b}(s) \times \tilde{t}(s) + \tilde{b}(s) \times \tilde{t}(s)) = \hat{n}_c(s)
\]

for a.e. \( s \in [0, C + T] \). But we already know that \( |\hat{n}_h(s)| = (C_h + T_h)/(C + T) \) for a.e. \( s \), where \( C_h \not\rightarrow C \) and \( T_h \not\rightarrow T \), whence the claim is proved, as required. \( \square \)

Remark 4.6 On account of Remark 4.4 denoting by \( \tau_h \) the tantrix of the curve \( n_h := \tilde{g} \circ n_h \) in \( \mathbb{R}P^2 \), in general we have \( \text{sup}_{\tau_h} \text{Var}_{\mathbb{R}P^2}(\tau_h) = +\infty \). Therefore, we cannot argue as in Theorem 3.1 to conclude that the sequence \( \hat{n}_h \) converges weakly in the BV-sense (and hence strongly in \( L^1 \)) to the function \( \hat{n}_c \). Actually, the derivative \( \hat{n}_c \) of the weak normal \( n_c \) is not a function with bounded variation, in general.

The case of smooth curves. We finally have:

Proposition 4.7 Let \( c : [0, L] \to \mathbb{R}^3 \) be a smooth curve satisfying the hypotheses of Theorem 3.2 so that \( L = \mathcal{L}(c), C = \text{TC}(c), \) and \( T = \text{TAT}(c) \) are finite. Let \( s : [0, C + T] \to [0, L] \) be the inverse of the increasing and bijective function \( t : [0, L] \to [0, C + T] \) given by

\[
t(s) := \int_0^s (k(\lambda) + |\tau(\lambda)|) d\lambda, \quad s \in [0, L]
\]

where \( k(\lambda) \) and \( \tau(\lambda) \) are the curvature and torsion of the curve \( c \) at the point \( c(\lambda) \). Then the principal normal \( n \) in \( S^2 \) of the curve \( c \), and the curve \( n_c \) in \( \mathbb{R}P^2 \) given by Theorem 3.1 are linked by the formula:

\[
[n(s(t))] = n_c(t) \in \mathbb{R}P^2 \quad \forall t \in [0, C + T].
\]

Proof: For any given \( s \in [0, L] \), we choose a sequence \( \{P_h\} \) as in the proof of Theorem 3.2 and we correspondingly denote:

\[
b_i(h) := \frac{\sigma_i \times \sigma_{i+1}}{|\sigma_i \times \sigma_{i+1}|}, \quad t_i(h) := \frac{\sigma_i}{|\sigma_i|}.
\]

Letting \( t_h(s) := \text{TC}(P_h(s)) + \text{TAT}(P_h(s)) \), this time we infer that (possibly passing to a subsequence) \( t_h(s) \to t(s) := \text{TC}(c|[0,s]) + \text{TAT}(c|[0,s]) \), so that \( t(s) \) satisfies the formula (4.3). As a consequence, arguing as in the proofs of Theorem 3.2 and Proposition 3.3, on account of Theorem 4.5 this time we get:

\[
\lim_{h \to \infty} [b_i(h) \times t_i(h)] = n_c(t(s)) .
\]

Since \( b_i(h) \to b(s) \) and \( t_i(h) \to t(s) \), we also have \( b_i(h) \times t_i(h) \to n(s) \), so that formula (4.4) holds. We omit any further detail. \( \square \)

5 On the spherical indicatrices of smooth curves

The trihedral \((t, n, b)\) is well-defined everywhere in the case of regular curves \( \gamma \) in \( \mathbb{R}^3 \) of class \( C^2 \) such that \( \dot{\gamma}(t) \) is non-zero everywhere, and the Frenet-Serret formulas (2.1) hold true if in addition \( \gamma \) is of class \( C^3 \).

Fenchel in [4] used a geometric approach in order to define (under weaker hypotheses on the curve) the osculating plane. He chooses the binormal \( b \) as a smooth function. Therefore, the principal normal is the smooth function given by \( n = b \times t \). The Frenet-Serret formulas continue to hold, but this time the curvature may vanish and even be negative. He also calls \( k \)-inflection or \( \tau \)-inflection a point of the curve where the curvature or the torsion changes sign, respectively.

By using an analytical approach, we recover some of the ideas by Fenchel in order to define the binormal (and principal normal). In general, it turns out that the binormal and normal fail to be continuous at the inflection points (see Example 5.2). However, both the binormal and normal are continuous when seen as functions in the projective plane \( \mathbb{RP}^2 \).

For this purpose, in the sequel we shall assume that \( \gamma : [a, b] \to \mathbb{R}^3 \) satisfies the following properties:
i) $\gamma$ is differentiable at each $t \in [a, b]$ and $\gamma'(t) \neq 0_{\mathbb{R}^3}$, i.e., $\gamma$ is a regular curve;

ii) for each $t_0 \in [a, b]$, the function $\gamma$ is of class $C^n$ in a neighborhood of $t_0$, for some $n \geq 2$, and $\gamma^{(n)}(t_0) \neq 0_{\mathbb{R}^3}$, but $\gamma^{(k)}(t_0) = 0_{\mathbb{R}^3}$ for $2 \leq k \leq n - 1$, if $n \geq 3$.

We thus denote by $c(s) := \gamma(t(s))$ the arc-length parameterization of the curve $\gamma$, i.e., $t(s) = s(t)^{-1}$, with $s(t) := \int_{t_0}^t |\gamma'(\lambda)|\,d\lambda \in [0, L]$, where $L := L(\gamma)$.

**Proposition 5.1** Under the above assumptions, the Frenet-Serret frame $(t, b, n)$ is well-defined by:

\[
t(s_0) := \dot{c}(s_0), \quad b(s_0) := \frac{\dot{c}(s_0) \times c^{(n)}(s_0)}{|c^{(n)}(s_0)|}, \quad n(s_0) := b(s_0) \times t(s_0) = \frac{c^{(n)}(s_0)}{|c^{(n)}(s_0)|}
\]

(5.1)

for each $s_0 \in [0, L]$, where $s_0 = s(t_0)$ and $n \geq 2$ is given as above. Furthermore, $\dot{c}(s_0) = 0_{\mathbb{R}^3}$ at a finite or countable set of points, and if $\ddot{c}(s_0) \neq 0_{\mathbb{R}^3}$, then $n(s_0) = \dot{c}(s_0)/|\dot{c}(s_0)|$. Finally, $[b]$ and $[n]$ are continuous functions with values in $\mathbb{R}^2$.

**Proof:** We set $t(s) := \dot{c}(s)$ for each $s$. If $\ddot{c}(s_0) = 0_{\mathbb{R}^3}$, then for some $n \geq 3$ and for $h$ small (and non-zero) we have

\[
\ddot{c}(s_0 + h) = \frac{h^{n-2}}{(n-2)!} + o(h^{n-2}).
\]

This implies that $\ddot{c}(s) = 0_{\mathbb{R}^3}$ only at isolated points $s \in [0, L]$, hence at an at most countable set.

If $\ddot{c}(s_0) \neq 0_{\mathbb{R}^3}$, one defines as usual $n(s_0) := \dot{c}(s_0)/|\dot{c}(s_0)|$ and $b(s_0) := \dot{c}(s_0) \times \dot{c}(s_0)/|\dot{c}(s_0)|$. In fact, the orthogonality property $\dot{c}(s_0) \bullet \ddot{c}(s_0) = 0$ yields that $(\dot{c}(s_0) \times \ddot{c}(s_0)) \times \dot{c}(s_0) = \ddot{c}(s_0)$.

If $\ddot{c}(s_0) = 0_{\mathbb{R}^3}$, with the above notation, for $h$ small we have $\ddot{c}(s_0 + h) \bullet \ddot{c}(s_0 + h) = 0_{\mathbb{R}^3}$. Letting $h \to 0$, we obtain that $\dot{c}(s_0) \bullet c^{(n)}(s_0) = 0_{\mathbb{R}^3}$, whence $\dot{c}(s_0) \times c^{(n)}(s_0) \neq 0_{\mathbb{R}^3}$ and $|\dot{c}(s_0) \times c^{(n)}(s_0)| = |c^{(n)}(s_0)| > 0$. As a consequence, the binormal is well-defined at $s_0$ such that $\ddot{c}(s_0) = 0_{\mathbb{R}^3}$ by the limit

\[
\dot{b}(s_0) = \lim_{h \to 0} \dot{b}(s_0 + h) = \lim_{h \to 0} \frac{\dot{c}(s_0 + h) \times \ddot{c}(s_0 + h)}{|\dot{c}(s_0 + h) \times \ddot{c}(s_0 + h)|} = \frac{\dot{c}(s_0) \times c^{(n)}(s_0)}{|c^{(n)}(s_0)|}
\]

and the principal normal is consequently defined by letting $n(s_0) := \dot{b}(s_0) \times t(s_0)$, where this time the orthogonality property $\dot{n}(s_0) \bullet c^{(n)}(s_0) = 0_{\mathbb{R}^3}$ yields that

\[
n(s_0) = \frac{(\dot{c}(s_0) \times c^{(n)}(s_0)) \times \dot{c}(s_0)}{|c^{(n)}(s_0)|} = \frac{c^{(n)}(s_0)}{|c^{(n)}(s_0)|}.
\]

Finally, we observe where $\ddot{c} \neq 0_{\mathbb{R}^3}$ both $n$ and $b$ are continuous (as functions valued in $\mathbb{S}^2$, hence also as functions valued in $\mathbb{R}^2$), hence the problematic points are where $\ddot{c} = 0_{\mathbb{R}^3}$, which is a set of isolated points.

At one of these point, $n(s_0)$ is ideally given by the limit of $\ddot{c}(s_0 + h)/|\ddot{c}(s_0 + h)|$, as $h \to 0$. Using equation (5.2), it is easy to see that, depending on the parity of the derivative order $n$, either the right and left limits coincide (thus the limit exists, and $n$ is continuous at $s_0$) or they are opposite to one another. Hence $n$ (and $b$) may not be continuous as sphere-valued functions, but are continuous as projective-valued function, since their directions are well defined and continuous.

**Example 5.2** If $\gamma(t) = (t, \sqrt{2}t^3/3, t^5/5)$, where $t \in [-1, 1]$, we have $|\gamma'(t)| = 1 + t^4$ and $s(t) = t + t^5/5 + s_0$, with $s_0 = 6/5$. Therefore, letting $t = t(s)$ we compute

\[
t(s) = \dot{c}(s) = \frac{1}{1 + t^4} (1, \sqrt{2}t^2, t^4), \quad \ddot{c}(s) = \frac{1}{(1 + t^4)^3} (-4t^3, 2\sqrt{2}t(1 - t^4), 4t^3)
\]

and hence $\ddot{c}(s) = 0 \iff s = s_0 = s(0)$. For $s \neq s_0$, i.e., for $t \neq 0$, we get $k(s) = |\ddot{c}(s)| = 2\sqrt{2}|t|(1 + t^4)^{-2}$, whereas $k(s_0) = 0$, so that $c(s_0)$ is a $k$-inflection point in the sense of Fenchel \cite{5}. As a consequence, for $s \neq s_0$ we obtain:

\[
n(s) = \frac{1}{1 + t^4} \frac{t}{|t|} (-\sqrt{2}t^2, 1 - t^2, \sqrt{2}t^2), \quad b(s) = \frac{1}{1 + t^4} \frac{t}{|t|} (t^4, -\sqrt{2}t^2, 1).
\]
Arguing as in the proof of Proposition 5.1, we compute

\[ e^{(3)}(s) = \frac{1}{(1 + t^4)^2} \left( -12t^2(1 - 3t^4), 2\sqrt{2}(1 - 16t^4 + 7t^8), 12t^2(1 - 3t^4) \right) \]

and hence \( e^{(3)}(s_0) = (0, 2\sqrt{2}, 0) \). By using formulas (5.1), we thus get \( b(s_0) = (0, 0, 1) \) and \( n(s_0) = (0, 1, 0) \). As to the torsion, using that \( \dot{c}(s) \times \ddot{c}(s) = k(s) b(s) \), for \( s \neq s_0 \) we get

\[ \mathbf{\tau}(s) = \frac{(\dot{c}(s) \times \ddot{c}(s)) \cdot e^{(3)}(s)}{|\dot{c}(s) \times \ddot{c}(s)|^2} = \frac{b(s) \cdot e^{(3)}(s)}{k(s)} = 2\sqrt{2} \frac{t}{1 + t^4} \]

and hence we infer that \( \mathbf{\tau}(s) \to 0 \) as \( s \to s_0 \). Finally, recalling that \( |\dot{\gamma}(t)| = 1 + t^4 \), we compute:

\[ \text{TC}(c) = \int_c k(s) \, ds = \int_{-1}^1 2\sqrt{2} \frac{|t|}{(1 + t^4)^2} |\dot{\gamma}(t)| \, dt = \frac{\sqrt{2}}{2} \pi, \]

\[ \text{TAT}(c) = \int_c |\mathbf{\tau}(s)| \, ds = \int_{-1}^1 2\sqrt{2} \frac{|t|}{1 + t^4} |\dot{\gamma}(t)| \, dt = 2\sqrt{2}. \]

**Remark 5.3** Finally, we point out that with the above assumptions, the statements of Theorem 3.2, Proposition 3.3, and Proposition 4.7 continue to hold. More precisely, using that the non-negative curvature \( k(\lambda) \) and torsion \( \mathbf{\tau}(\lambda) \) may vanish only at a negligible set of inflection points, with our previous notation one readily obtains the following relations concerning the trihedral \((t, b, n)\) from Proposition 5.1:

i) \( t(s_1(k)) = t_c(k) \in S^2 \) for \( k \in [0, C] \), where \( s_1 : [0, C] \to [0, L] \) is the inverse of the function

\[ k(s) := \int_0^s k(\lambda) \, d\lambda, \quad s \in [0, L]; \]

ii) \( [b(s_2(t))] = b_c(t) \in \mathbb{RP}^2 \) for \( t \in [0, T] \), where \( s_2 : [0, T] \to [0, L] \) is the inverse of the function

\[ t(s) := \int_0^s |\mathbf{\tau}(\lambda)| \, d\lambda, \quad s \in [0, L]; \]

iii) \( [n(s_3(\rho))] = n_c(\rho) \in \mathbb{RP}^2 \) for \( \rho \in [0, C + T] \), where \( s_3 : [0, C + T] \to [0, L] \) is the inverse of the function

\[ \rho(s) := \int_0^s (k(\lambda) + |\mathbf{\tau}(\lambda)|) \, d\lambda, \quad s \in [0, L]. \]

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