Fully De-Amortized Cuckoo Hashing for Cache-Oblivious Dictionaries and Multimaps

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Abstract

A dictionary (or map) is a key-value store that requires all keys be unique, and a multimap is a key-value store that allows for multiple values to be associated with the same key. We design hashing-based indexing schemes for dictionaries and multimaps that achieve worst-case optimal performance for lookups and updates, with a small or negligible probability the data structure will require a rehash operation, depending on whether we are working in the the external-memory (I/O) model or one of the well-known versions of the Random Access Machine (RAM) model. One of the main features of our constructions is that they are fully de-amortized, meaning that their performance bounds hold without one having to tune their constructions with certain performance parameters, such as the constant factors in the exponents of failure probabilities or, in the case of the external-memory model, the size of blocks or cache lines and the size of internal memory (i.e., our external-memory algorithms are cache oblivious). Our solutions are based on a fully de-amortized implementation of cuckoo hashing, which may be of independent interest. This hashing scheme uses two cuckoo hash tables, one “nested” inside the other, with one serving as a primary structure and the other serving as an auxiliary supporting queue/stash structure that is super-sized with respect to traditional auxiliary structures but nevertheless adds negligible storage to our scheme. This auxiliary structure allows the success probability for cuckoo hashing to be very high, which is useful in cryptographic or data-intensive applications.
1 Introduction

A dictionary (or map) is a key-value store that requires all keys be unique and a multimap [3] is a key-value store that allows for multiple values to be associated with the same key. Such structures are ubiquitous in the “inner-loop” computations involved in various algorithmic applications. Thus, we are interested in implementations of these abstract data types that are based on hashing and use $O(n)$ words of storage, where $n$ is the number of items in the dictionary or multimap.

In addition, because such solutions are used in real-time applications, we are interested in implementations that are de-amortized, meaning that they have asymptotically optimal worst-case lookup and update complexities, but may have small probabilities of overflowing their memory spaces. Moreover, we would like these lookup and update bounds to hold without requiring that we build such a data structure specifically “tuned” to certain performance parameters, since it is not always possible to anticipate such parameters at the time such a data structure is deployed (especially if the length $p(n)$ of the sequence of operations on the structure is not known in advance). For instance, if we wish for a failure probability that is bounded by $1/n^c$ or $1/n^c\log n$, for some constant $c > 0$, we should not be required to build the data structure using an amount of space or other components that are parameterized by $c$. (For example, our first construction gives a single algorithm parameterized only by $n$, and for which, for any $s > 0$, lookups take time at most $s$ with probability that depends on $s$. Previous constructions are parameterized by $c$ as well, and lookups take time $c$ with probability $1 − 1/n^c$ for fixed constants $c, c’$.) Likewise, in the internal-memory model, we would like solutions that achieve their performance bounds without being tuned for the parameters of the memory hierarchy, like the size, $B$, of disk blocks, or the size, $M$, of internal memory. We refer to solutions that avoid such parameterized constructions as fully de-amortized.

By extending and combining various ideas in the algorithms literature, we show how to develop fully de-amortized data structures, based on hashing, for dictionaries and multimaps. Without specifically tuning our structures to constant factors in the exponents, we provide solutions with performance bounds that hold with high probability, meaning that they hold with probability $1 − 1/poly(n)$, or with overwhelming probability, meaning that they hold with probability $1 − 1/n^{\omega(1)}$. We also use the term with small probability to mean $1/poly(n)$, and with negligible probability to mean $1/n^{\omega(1)}$. Briefly, we are able to achieve the following bounds:

- For dictionaries, we present two fully de-amortized constructions. The first is for the Practical RAM model [33], and it performs all standard operations (lookup, insert, delete) in $O(1)$ steps in the worst case, where all guarantees hold for any polynomial number of operations with high probability. The second works in the external-memory (I/O) model, the standard RAM model, and the $AC^0$ RAM model [40], and also achieves $O(1)$ worst-case operations, where these guarantees hold with overwhelming probability.

- For multimaps, we provide a fully de-amortized scheme that in addition to the standard operations can quickly return or delete all values associated with a key. Our construction is suitable for external memory and is cache oblivious in this setting. For instance, our external-memory solution returns all $n_k$ values associated with a key in $O(1 + n_k/B)$ I/Os, where $B$ is the block size, but it is not specifically tuned with respect to the parameter $B$.

Our algorithms use linear space and work in the online setting, where each operation must be completed before performing the next.

Our solutions for dictionaries and multimaps include the design of a variation on cuckoo hash tables, which were presented by Pagh and Rodler [36] and studied by a variety of other researchers (e.g., see [16, 28, 34]). These structures use a freedom to place each key-value pair in one of two hash tables to achieve worst-case constant-time lookups and removals and amortized constant-time insertions, where operations fail with polynomially small probability. We obtain the first fully de-amortized variation on cuckoo hash
tables, with negligible failure probability to boot.

1.1 Motivations and Models

Key-value associations are used in many applications, and hash-based dictionary schemes are well-studied in the literature (e.g., see [14]). Multimaps [3] are less studied, although a multimap can be viewed as a dynamic inverted file or inverted index (e.g., see Knuth [26]). Given a collection, $\Gamma$, of documents, an inverted file is an indexing strategy that allows one to list, for any word $w$, all the documents in $\Gamma$ where $w$ appears. Multimaps also provide a natural representation framework for adjacency lists of graphs, with nodes being keys and adjacent edges being values associated with a key. For other applications, please see Angelino et al. [3].

Hashing-based implementations of dictionaries and multimaps must necessarily be designed in a computational model that supports indexed addressing, such as the Random Access Machine (RAM) model (e.g., see [14]) or the external-memory (I/O) model (e.g., see [1, 42]). Thus, our focus in this paper is on solutions in such models. There are, in fact, several versions of the RAM model, and, rather than insist that our solutions be implemented in a specific version, we consider solutions for several of the most well-known versions:

- **The standard RAM**: all arithmetic and comparison operations, including integer addition, subtraction, multiplication, and division, are assumed to run in $O(1)$ time.
- **The Practical RAM [33]**: integer addition, as well as bit-wise Boolean and SHIFT operations on words, are assumed to run in $O(1)$ time, but not integer multiplication and division.
- **The $AC^0$ RAM [40]**: any $AC^0$ function can be performed in constant time on memory words, including addition and subtraction, as well as several bit-level operations included in the instruction sets of modern CPUs, such as Boolean and SHIFT operations. This model does not allow for constant-time multiplication, however, since multiplication is not in $AC^0$ [20].

Thus, the $AC^0$ RAM is arguably the most realistic, the standard RAM is the most traditional, and the Practical RAM is a restriction of both. As we are considering dictionary and multimap solutions in all of these models, we assume that the hash functions being used are implementable in the model in question and that they run in $O(1)$ time and are sufficiently random to support cuckoo hashing. This assumption is supported in practice, for instance, by the fact that one of the most widely-used hash functions, SHA-1, can be implemented in $O(1)$ time in the Practical RAM model. See also Section 3.5 for discussion of the theoretical foundations of this assumption.

A framework that is growing in interest and impact for designing algorithms and data structures in the external-memory model is the cache-oblivious design paradigm, introduced by Frigo et al. [21]. In this external-memory paradigm, one designs an algorithm or data structure to minimize the number of I/Os between internal memory and external memory, but the algorithm must not be explicitly parameterized by the block size, $B$, or the size of internal memory, $M$. The advantage of this approach is that one such algorithm can comfortably scale across all levels of the memory hierarchy and can also be a better match for modern compilers that perform predictive memory fetches.

Our notion of a “fully de-amortized” data structure extends the cache-oblivious design paradigm in two ways. First, it requires that all operations be characterized in terms of their worst-case performance, not its amortized performance. Second, it extends to internal-memory models the notion of avoiding specific tuning of the data structure in terms of non-essential parameters. Formally, we say that a data structure is fully de-amortized if its performance bounds hold in the worst case and the only parameter its construction details depend is $n$, the number of items it stores. Thus, a fully de-amortized data structure implemented in external-memory is automatically cache-oblivious and its I/O bounds hold in the worst case.

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1 Most, but not all, researchers also assume that the standard RAM supports bit-wise Boolean operations, such as AND, OR, and XOR; hence, we also allow for these as constant-time operations in the standard RAM model.
In addition to being fully de-amortized, our strongest constructions have performance bounds that hold with overwhelming probability. While our aim of achieving structures that provide worst-case constant time operations with overwhelming probability instead of with high probability may seem like a subtle improvement, there are many applications where it is essential. In particular, it is common in cryptographic applications to aim for negligible failure probabilities. For example, cuckoo hashing structures with negligible failure probabilities have recently found applications in oblivious RAM simulations [22]. Moreover, a significant motivation for de-amortized cuckoo hashing is to prevent timing attacks and clocked adversaries from compromising a system [5]. Finally, guarantees that hold with overwhelming probability allow us to handle super-polynomially long sequences of updates, as long as the total number of items resident in the dictionary is bounded by \( n \) at all times. This may be useful in long-running or data-intensive applications.

1.2 Previous Related Work

Since the introduction of the cache-oblivious framework by Frigo et al. [21], several cache-oblivious algorithms have subsequently been presented, including cache-oblivious B-trees [7], cache-oblivious binary search trees [9], and cache-oblivious sorting [10]. Pagh et al. [37] describe a scheme for cache-oblivious hashing, which is based on linear probing and achieves \( O(1) \) expected-time performance for lookups and updates, but it does not achieve constant time bounds for any of these operations in the worst case.

As mentioned above, the multimap ADT is related to the inverted file and inverted index structures, which are well-known in text indexing applications (e.g., see Knuth [26]) and are also used in search engines (e.g., see Zobel and Moffat [44]). Cutting and Pedersen [15] describe an inverted file implementation that uses B-trees for the indexing structure and supports insertions, but doesn’t support deletions efficiently. More recently, Luk and Lam [31] describe an internal-memory inverted file implementation based on hash tables with chaining, but their method also does not support fast item removals. Lester et al. [29, 30] and Büttrich et al. [13] describe external-memory inverted file implementations that support item insertions only. Büttrich and Clarke [12] consider trade-offs for allowing for both item insertions and removals, and Guo et al. [23] give a solution for performing such operations by using a B-tree variant. Finally, Angelino et al. [3] describe an efficient external-memory data structure for the multimap ADT, but like the above-mentioned work on inverted files, their method is not cache-oblivious; hence, it is not fully de-amortized.

Also as mentioned above, our solutions include the design of a variation on cuckoo hash tables, which were presented by Pagh and Rodler [36] and studied by a variety of other researchers (e.g., see [16, 28, 34]). These structures use a freedom to place each key-value pair in one of two hash tables to achieve worst-case constant-time lookups and removals and amortized constant-time insertions with high probability. Kirsch and Mitzenmacher [24] and Arbitman et al. [5] study a method for de-amortizing cuckoo hashing, which achieves constant-time lookups, insertions, and deletions with high probability, and uses space \( (2 + \epsilon)n \) for any constant \( \epsilon > 0 \) (as is standard in cuckoo hashing). These methods are not fully de-amortized, however, since, in order to achieve a failure probability of \( 1/n^\alpha \), they construct an auxiliary structure consisting of \( O(c) \) small lookup tables. In contrast, neither of our dictionary constructions are parameterized by \( c \); furthermore our second construction provides guarantees that hold with overwhelming probability rather than with high probability. In a subsequent paper, Arbitman et al. [6] study a hashing method that achieves worst-case constant-time lookups, insertions, and removals with high probability while maintaining loads very close to 1, but their method also is not fully de-amortized.

Kirsch, Mitzenmacher, and Wieder [25] introduced the notion of a stash for cuckoo hashing, which allows the failure probability to be reduced to \( O(1/n^\alpha) \), for any \( \alpha > 0 \), by using a constant-sized adjunct memory to store items that wouldn’t otherwise be able to be placed. One of our novel additions in this paper is to demonstrate that by using super-constant sized stashes, along with a variation of the q-heap data structure, we can ensure failures happen only with negligible probability while maintaining constant time lookup and delete operations.
1.3 Our Results

In this paper we describe efficient hashing-based implementations of the dictionary and multimap ADTs. Our constructions are fully de-amortized and are alternately designed for the external-memory (I/O) model and the well-known versions of the RAM model mentioned above. Because they are fully de-amortized, our external-memory algorithms are cache-oblivious.

We begin by presenting two new fully de-amortized cuckoo hashing schemes in Section 2. Both of our constructions provide $O(1)$ worst-case lookups, insertions, and deletions, where the guarantees of the first construction (in the Practical RAM model) hold with high probability, and the guarantees of the second construction (in the external-memory model, the standard RAM model, and the AC$^0$ RAM model) hold with overwhelming probability. Moreover, these results hold even if we use polylog($n$)-wise independent hash functions. Like the construction of Arbitman et al. [5], both of our dictionaries use space $(2 + \epsilon)n$ for any constant $\epsilon > 0$, though when combined with another result of Arbitman et al. [6], we can achieve $(1 + \epsilon)n$ words of space for any constant $\epsilon > 0$ (see Section 3.5). Our second dictionary can be seen as a quantitative improvement over the previous solution for de-amortized cuckoo tables [5], as the guarantees of [5] only hold with high probability (and their solution is not fully de-amortized).

Both of our dictionary constructions utilize a cuckoo hash table that has another cuckoo hash table as an auxiliary structure. This secondary cuckoo table functions simultaneously as an operation queue (for the sake of de-amortization [5][6][24]) and a stash (for the sake of improved probability of success [25]).

Our second construction also makes use of a data structure for the AC$^0$ RAM model we call the atomic stash. This structure maintains a small dictionary, of size at most $O(\log^{1/2} n)$, so as to support constant-time worst-case insertion, deletions, and lookups, using $O(\log^{1/2} n)$ space. This data structure is related to the q-heap or atomic heap data structure of Fredman and Willard [20] (see also [43]), which requires the use of lookup tables of size $O(n^\epsilon)$ and is limited to sets of size $O(\log^{1/6} n)$. Andersson et al. [2] and Thorup [40] give alternative implementations of these data structures in the AC$^0$ RAM model, but these methods still need precomputed functions encoded in table lookups or have time bounds that are amortized instead of worst-case. Our methods instead make no use of precomputed lookup tables, hold in the worst case, and use techniques that are simpler than those of Anderson et al. and Thorup. We emphasize that our results do not depend on our specific atomic stash implementation; one could equally well use q-heaps, atomic heaps, or other data structures that allow constant-sized lookups into data sets of size $\omega(1)$ (under suitable assumptions, such as keys fitting into a memory word) in order to obtain bounds that hold with overwhelming probability in the standard or AC$^0$ RAM models. We view this combination of cuckoo hash tables with atomic stashes (or other similar data structures) as an important contribution, as they allow super-constant sized stashes for cuckoo hashing while still maintaining constant time lookups.

In Section 4 we also show how to build on our fully de-amortized cuckoo hashing scheme to give an efficient cache-oblivious multimap implementation in the external memory model. Our multimap implementation utilizes two instances of the nested cuckoo structure of Section 2 together with arrays for storing collections of key-value pairs that share a common key. This implementation assumes that there is a cache-oblivious mechanism to allocate and deallocate power-of-two sized memory blocks with constant-factor space and I/O overhead; this assumption is theoretically justified by the results of Brodal et al. [11]. A lower bound of Verbin and Zhang [41] implies that our bounds are optimal up to constant factors in the external memory model, even if we did not support fast findAll and removeAll operations.

In addition to a theoretical analysis of our data structures, we have performed preliminary experiments with an implementation, and a later writeup will include full details of these.

The time bounds we achieve for the dictionary and multimap ADT methods are shown in Table 1. Our space bounds are all $O(n)$, for storing a dictionary or multimap of size at most $n$. 

\[ \]
Dictionary I/O Performance        Multimap I/O Performance

\[ \text{add}(k, v) \quad O(1) \quad O(1) \]
\[ \text{containsKey}(k) \quad O(1) \quad O(1) \]
\[ \text{containsItem}(k, v) \quad O(1) \quad O(1) \]
\[ \text{remove}(k, v) \quad O(1) \quad O(1) \]
\[ \text{get}(k)/\text{getAll}(k) \quad O(1) \quad O(1 + n_k/B) \]
\[ \text{removeAll}(k) \quad - \quad O(1) \]

Table 1: Performance bounds for our dictionary and multimap implementations, which all hold in the worst-case with overwhelming probability, except for implementations in the Practical RAM model, in which case the above bounds hold with high probability. These bounds are asymptotically optimal. We use \( B \) to denote the block size, and \( n_k \) to denote the number of items with key equal to \( k \).

2 Nested Cuckoo Hashing

In this section, we describe both of our nested cuckoo hash table data structures, which provide fully de-amortized dictionary data structures with worst-case \( O(1) \)-time lookups and removals and \( O(1) \)-time insertions with high and overwhelming probability, respectively. At a high level, this structure is similar to that of Arbitman et al. \[5\] and Kirsch and Mitzenmacher \[24\], in that our scheme and these schemes use a cuckoo hash table as a primary structure, and an auxiliary queue/stash structure to de-amortize insertions and reduce the failure probability. But our approach substantially differs from prior methods in the details of the auxiliary structure. In particular, our auxiliary structure is itself a full-fledged cuckoo hash table, with its own (much smaller) queue and stash, whereas prior methods use more traditional queues for the auxiliary structure.

Our two dictionary constructions are identical, except for the implementation of the “inner” cuckoo hash table’s queue and stash. Below, we describe both constructions simultaneously, highlighting their differences when necessary.

2.1 The Components of Our Structure

Our dictionaries maintain a dynamic set, \( S \), of at most \( n \) items. Our primary storage structure is a cuckoo hash table, \( T \), subdivided into two subtables, \( T_0 \) and \( T_1 \), together with two random hash functions, \( h_0 \) and \( h_1 \). Each table \( T_i \) stores at most one item in each of its \( m \) cells, and we assume \( m \geq (1 + \epsilon)n \) where \( n \) is the total number of items, for some fixed constant \( \epsilon > 0 \). For any item \( x = (k, v) \) that is stored in \( T \), \( x \) will either be stored in cell \( T_0[h_0(k)] \) or in cell \( T_1[h_1(k)] \). Some of the items in \( S \) may not be stored in \( T \), however. They will instead be stored in an auxiliary structure, \( Q \).

The structure \( Q \) is simultaneously two double-ended queues (deques), both of which support fast enqueue and dequeue operations at either the front or the rear, and a cuckoo hash table with its own queue and stash. Because \( Q \) is a cuckoo hash table, it supports worst-case constant-time lookups for items based on their keys. The two deques correspond to the queue and stash of the primary structure; we call them OuterQueue and OuterStash respectively. Each item \( x \) that is stored in \( Q \) is therefore also augmented with a prev pointer, which points to the predecessor of \( x \) in its deque, and a next pointer, which points to the successor of \( x \) in its deque. We also augment both deques with front and rear pointers, which respectively point to the first element in the deque and the last element in the deque.

The “inner” cuckoo hash structure consists of two tables, \( R_0 \) and \( R_1 \), each having \( m^{2/3} \) cells, together with two random hash functions, \( f_0 \) and \( f_1 \), as well as a small list, \( L \), which is used to implement two deques that we call InnerQueue and InnerStash respectively. Each item \( x = (k, v) \) that is stored in \( Q \) will
be located in cell \( R_0[f_0(k)] \) or in cell \( R_1[f_1(k)] \), or it will be in the list \( L \). The only manner in which our two constructions differ is in the implementation of \( L \).

In summary, our dictionary data structure consists simply of the “outer” cuckoo table \( T \) with a queue and stash, and the “inner” cuckoo table \( Q \), which contains its own queue and stash.

### 2.2 Operations on the Primary Structure

To perform a lookup, that is, a get \((k, v)\), we try each of the cells \( T_0[h_0(k)] \), \( T_1[h_1(k)] \), \( R_0[f_0(k)] \), and \( R_1[f_1(k)] \), and perform a lookup in \( L \), until we either locate an item, \( x = (k, v) \), or we determine that there is no item in these locations with key equal to \( k \), in which case we conclude that there is no such item in \( S \).

Likewise, to perform a remove \((k, v)\) operation, we first perform a get \((k, v)\) operation. If an item \( x = (k, v) \) is found in one of the locations \( T_0[h_0(k)] \) or \( T_1[h_1(k)] \), then we simply remove this item from this cell. If such an \( x \) is found in \( R_0[f_0(k)] \), or \( R_1[f_1(k)] \), or in the structure \( L \), then we remove \( x \) from this cell and we remove \( x \) from its deque(s) by updating the prev and next pointers for \( x \)’s neighbors so that they now point to each other.

Performing an add \((k, v)\) operation is somewhat more involved. We begin by performing an enqueueLast \((x, 0, \text{OuterQueue})\) operation, which adds \( x = (k, v) \) at the end of the deque OuterQueue, together with the bit 0 to indicate that \( x \) should next be inserted in \( T_0 \). Then, for constants \( \alpha, \alpha’ \geq 1 \), set in the analysis, we perform \( \alpha \) (outer) insertion substeps, followed by \( \alpha’ \) (inner) insertion substeps. Each outer insertion substep begins by performing a dequeueFront(OuterQueue) operation to remove the pair \(((k, v), b)\) at the front of the deque. If the cell \( T_0[h_0(k)] \) is empty, then we add \((k, v)\) to this cell and this ends this substep. Otherwise, we evict the current item, \( y \), in the cell \( T_0[h_0(k)] \), replacing it with \((k, v)\). If this insertion substep just created a second cycle in the insertion process (which we can detect by a simple marking scheme, using \( O(\log^2 n) \) space with overwhelming probability\footnote{\cite{5} refers to this as a cycle-detection mechanism, and notes there are many possible instantiations. See also \cite{25}.} then we complete the insertion substep by performing an enqueueLast\((y, b’, \text{OuterStash})\) operation, where \( b’ = (b + 1) \mod 2 \). If this insertion substep has not created a second cycle, however, then we complete the insertion substep by performing an enqueueFirst\((y, b’, \text{OuterQueue})\) operation, where \( b’ = (b + 1) \mod 2 \), which adds the pair \((y, b’)\) to the front of the primary structure’s queue. Thus, by design, an add operation takes \( O(\alpha) = O(1) \) time in the worst case, assuming the operations on \( Q \) run in \( O(1) \) time and succeed.

Finally, similar to \cite{4}, every \( m^{1/4} \) operations we try to insert an element from the stash by performing a dequeueFront(OuterStash) operation to remove the pair \(((k, v), b)\) at the front of the deque and then spending 2 moves trying to insert \((k, v)\). If no free slot is found, we return the current element from the connected component of \((k, v)\) to the front of the stash. This ensures that items belonging to connected components that have had cycles removed via deletions do not remain in the stash long after the deletions occur. All constants mentioned throughout are chosen for theoretical convenience; no effort has been made to optimize them.

### 2.3 Operations on the Auxiliary Structure

As mentioned above, the auxiliary structure, \( Q \), is a standard cuckoo table of size \( m’ = m^{2/3} \) augmented with its own (inner) queue and stash maintained via the structure \( L \), and pointers to give \( Q \) the functionality of two double-ended queues, called OuterQueue and OuterStash. The enqueue and dequeue operations to OuterQueue and OuterStash, therefore, involve standard \( O(1) \)-time pointer updates to maintain the deque property, plus insertion and deletion algorithms for the inner cuckoo table. Our inner insertion algorithm is different from our outer one in that we do not immediately place items on the back of the inner queue. Instead, on an insertion of item \((k, v)\), we spend 16 steps trying to insert \((k, v)\) into the inner cuckoo table.
immediately. If the insertion does not complete in 16 steps, we place the current element from the connected component of \((k,v)\) in the back of InnerQueue. The reason for this policy is that, as we will show, with overwhelming probability almost every item in the inner table can be inserted in 16 steps, due to the extreme sparsity of the table. Finally, every \(m^{1/6}\) “inner” operations we additionally spend 16 moves trying to insert an element from the front of InnerQueue and a single move trying to insert an element from the front of InnerStash, returning elements to the back of InnerQueue or InnerStash in the event that a vacant slot has not been found. The purpose of this is to ensure that items whose connected components have shrunk due to deletions do not remain in the inner queue or stash unnecessarily.

The only manner in which our two constructions differ is in the implementation of \(L\), which supports the inner stash and inner queue. In our first construction, for the Practical RAM model, \(L\) is simply two double-ended queues (one for InnerQueue, and one for InnerStash), and when performing a lookup in \(L\), we simply look at all the cells in both deques. In our second construction, we implement \(L\) using a data structure we call the atomic stash. This structure maintains a small dictionary, of size at most \(O(\log^{1/2} n)\), so as to support constant-time worst-case insertion, deletions, and lookups, using \(O(\log^{1/2} n)\) space. Thus, in our second construction, the structure \(L\) is simultaneously two deques and an atomic stash to enable fast lookups into \(L\). Details are in Appendix A.

3 Analysis

The critical insights into why such a standard approach is sufficient to support fully de-amortized constant-time updates and lookups for the auxiliary structure, \(Q\), are based on enhancing the analysis of previous work for cuckoo hash tables.

**Definition 1:** A sequence \(\pi\) of insert, delete, and lookup operations is \(n\)-bounded if at any point in time during the execution of \(\pi\) the data structure contains at most \(n\) elements.

We prove the following two theorems.

**Theorem 1:** Let \(C\) be the inner cuckoo hash table consisting of two tables of size \(m^{2/3}\), together with a queue/stash \(L\) as described above. For any polynomial \(p(n)\), any \(m^{1/3}\)-bounded sequence of operations \(\pi'\) on \(C\) of length \(p(n)\), and any \(s \leq m^{1/6}\), every insert into \(C\) completes in \(O(1)\) steps, and \(L\) has size at most \(s\), with probability at least \(1 - p(n)/m^{O(\sqrt{s})}\).

Theorem 1 states that with overwhelming probability, all insertions performed on the auxiliary structure \(Q\) will run in \(O(1)\) time, and all of the inner table’s internal data structures will stay small (ensuring fast lookups), provided we never try to put more than \(m^{1/3}\) items in \(Q\). The following theorem addresses this size condition.

**Theorem 2:** Let \(T\) be a cuckoo hashing scheme consisting of two tables of size \(m\). Let \(m \geq (1+\epsilon)n\), for some constant \(\epsilon > 0\), and let \(Q\) be the queue/stash as described above. For any polynomial \(p(n)\) and any \(n\)-bounded sequence of operations \(\pi\) of length \(p(n)\), the number of items stored in \(Q\) will never be more than \(2 \log^6 n\), with probability at least \(1 - 1/n^{\Omega(\log n)}\).

3.1 Notation and Preliminary Lemmata

A standard tool in the analysis of cuckoo hashing is the cuckoo graph \(G\). Specifically, given a cuckoo hash table \(T\), subdivided into two tables \(T_0\) and \(T_1\), each with \(m\) cells, and hash functions \(h_0\) and \(h_1\), the cuckoo graph for \(T\) is a bipartite multigraph (without self-loops), in which left vertices represent the table cells in \(T_0\) and right vertices represent the cells in \(T_1\). Each key \(x\) inserted into the hash table corresponds to an edge connecting \(T_0[h_0(x)]\) to \(T_1[h_1(x)]\). Thus, if \(S\) denotes the set of \(n\) items in \(T\), and each of \(T_0\) and \(T_1\) have \(m\) cells, the cuckoo graph \(G(S,h_0,h_1)\) contains \(2m\) nodes (\(m\) on each side) and \(n\) edges. Given a
cuckoo graph $G(S, h_0, h_1)$, we use $C_{S,h_0,h_1}(v)$ to denote the number of edges in the connected component that contains $v$.

In our analysis, there will actually be two cuckoo graphs $G$ and $G'$. $G$ corresponds to the outer table, and has vertex set $[m] \times [m]$ and at most $n$ edges at any point in time. $G'$ corresponds to the inner table and has vertex set $[m'] \times [m']$ for $m' = m^{2/3}$, and (we will show) $n' \leq m^{1/3}$ edges with overwhelming probability at any point in time.

We will make frequent use of the following lemmata.

**Lemma 3:** Let $G(S, h_0, h_1)$ be a cuckoo graph with $n$ edges and vertex set $[m] \times [m]$. Then

$$\Pr(C_{S,h_0,h_1}(v) \geq k) \leq \left( \frac{nk}{m} \right)^k \frac{1}{k!},$$

where the probability is over the choice of $h_0$ and $h_1$.

**Proof:** By standard arguments, (e.g. [17] Lemma 1), $\Pr(C_{S,h_0,h_1}(v) \geq k) \leq \Pr(\text{Bin}(nk, 1/m) \geq k)$. The lemma follows by an easy calculation.

We now present some basic facts about the stash. The first relates the time it takes to determine whether an element needs to be put in the stash on an insertion.

**Lemma 4:** [4] Claim 5.4 For any element $x$ and any cuckoo graph $G(S, h_0, h_1)$, the number of moves required before $x$ is either placed in the stash or inserted into the cuckoo table is at most $2C_{S,h_0,h_1}(x)$.

[25] Lemma 2.2] shows that the number of keys that must be stored in the stash corresponds to the quantity $\bar{e}(G(S, h_0, h_1))$, defined as follows. For a connected component $H$ of $G(S, h_0, h_1)$, define the *excess* $e(H) := \#\text{edges}(H) - \#\text{nodes}(H)$, and define

$$\bar{e}(G(S, h_0, h_1)) := \sum_H \max(e(H), 0),$$

where the sum is over all connected components $H$ of $G(S, h_0, h_1)$.

Given a vertex in this random graph, recall that $C_{S,h_0,h_1}(v)$ denotes the number of edges in the connected component that contains $v$, and let $B_v$ be the number of edges in the component of $v$ that need to be removed to make it acyclic ($B_v$ is also called the *cycloptic number* of the component containing $v$). Notice that $B_v = e(H) + 1$, that is, $B_v$ is 1 more than the number of keys from $v$’s component that need to be placed in the stash.

We use the following lemma from [25].

**Lemma 5:** Let $|S| = n$ with $(1 + \epsilon)n \leq m$ for some constant $\epsilon$, and let $G(S, h_0, h_1)$ be a cuckoo graph with vertex set $[m] \times [m]$. Then

$$\Pr(B_v \geq t \mid C_{S,h_0,h_1}(v) = k) \leq \left( \frac{3e^5k^3}{m} \right)^t.$$

### 3.2 Proving Theorem 1

Throughout this section, we use the following notation, following that in [5]. Let $\pi'$ be an $m^{1/3}$-bounded sequence of $p(n)$ operations. Denote by $(x_1, \ldots, x_{p(n)})$ the elements inserted by $\pi'$. For any integer $0 < i \leq p(n) - m^{1/3}$, let $S_i$ denote the set of elements that are stored in the data structure just before the insertion of $x_i$, and let $\tilde{S}_i$ denote $S_i$ together with the elements $\{x_i, x_{i+1}, \ldots, x_{i+m^{1/3}}\}$, ignoring any deletions between time $i$ and time $i + m^{1/3}$. Since $\pi'$ is an $m^{1/3}$ bounded sequence, we have $|S_i| \leq m^{1/3}$ and $|\tilde{S}_i| \leq 2m^{1/3}$ for all $i$.

We need the following lemma, which states that all components of $G'$ are tiny.
Lemma 6: Let \(|S| = n'|S| = n'\), and let \(G'(S, f_0, f_1)\) be any cuckoo graph with vertex set \([m'] \times [m']\). Assume \(n'/m' \leq 2m^{1/3}\). Then for any \(k \leq m^{1/4}\), and for any node \(v\), \(\Pr[C_{S, f_0, f_1}(v) \geq k] \leq m^{-\Omega(k)}\).

Proof: Lemma 3 implies that \(\Pr(C_{S, f_0, f_1}(v) \geq k) \leq \left(\frac{2k}{m^{1/3}}\right)^k \frac{1}{k!}\).

For \(k \leq m^{1/4}\) the probability that \(C_{S, f_0, f_1}(v) \geq k\) is at most \(m^{-\Omega(k)}\); applying a union-bound over all \(v\), the probability some connected component has size greater than \(k\) is still at most \(m^{-\Omega(k)}\).

3.2.1 Showing InnerStash Stays Small

We begin by showing that InnerStash has size at most \(s\) at all points in time with probability at least \(1 - 1/m^{\Omega(s)}\). This requires extending the analysis of prior works \([25, 27]\) to superconstant sized stashes, while leveraging the sparsity of the inner cuckoo table \(C\).

Lemma 7: Let \(|S| = n'|S| = n'\), and let \(G'(S, f_0, f_1)\) be any cuckoo graph vertex set \([m'] \times [m']\). Assume \(n' \leq 2m^{1/3}\) and \(m' = m'^{2/3}\). For any \(s < m'^{1/6}\), with probability \(1 - 1/m^{\Omega(s)}\) the total number of vertices that reside in the stash of \(G'(S, f_0, f_1)\) is at most \(s\).

The proof is entirely the same as that for the outerstash, which we prove in Lemma 12. We defer the proof until then.

We are in a position to show formally that the inner stash is small at all points in time.

Lemma 8: Let \(\pi'\) be any \(m^{1/3}\)-bounded sequence of operations of length \(p(n)\) on the inner cuckoo table \(C\). For any \(s < m^{1/12}\), with probability \(1 - p(n)/m^{\Omega(s)}\) over the choice of hash functions \(f_0, f_1\), InnerStash has size at most \(s\) at all points in time.

Proof: Following the approach of [5], we define a good event \(\xi_1\) that ensures that InnerStash has size at most \(s\). Namely, let \(\xi_1\) be the event that at all points \(i\) in time, the number of vertices \(v\) in the stash of \(G'(S_i, f_0, f_1)\) is at most \(s\). By Lemma 7 and a union bound over all \(p(n)\) operations in \(\pi'\), \(\xi_1\) occurs with probability at least \(1 - p(n)/m^{\Omega(s)}\).

We distinguish between the real stash and the effective stash. The real stash at any point \(i\) in time is the set of nodes that reside in the stash of \(G'(S_i, f_0, f_1)\); the effective stash refers to the real stash, together with elements that used to be in the real stash but have since had cycles removed from their components due to deletions, and have not yet been inserted in the inner cuckoo table. Let \(E_i\) denote the set of items in the effective stash but not the real stash of \(G'(S_i, f_0, f_1)\).

Clearly the event \(\xi_1\) guarantees that at any point in time, the size of the real stash is at most \(s\). To see that the size of the effective stash never exceeds \(s\), assume by way of induction that for \(j \geq m^{1/3}\), the effective stash has size at most \(s\) at all times less than \(j\) (clearly event \(\xi_1\) guarantees that this is true for all \(j \leq m^{1/3}\) as a base case). In particular, the inductive hypothesis ensures that the effective stash has size at most \(s\) at time \(j - m^{1/3}\). Observe that during the execution of operations \(\{x_{j-m^{1/3}}, x_{j-m^{1/3}+1}, \ldots, x_j\}\), we spend \(m^{1/3} \geq s^2\) moves in total on the elements of the effective stash. By Lemma 6 and a union bound of all \(i \leq p(n)\), we may assume all connected components of \(G'(S_j, f_0, f_1)\) have size at most \(s/2\); this adds a failure probability of at most \(p(n)/m^{\Omega(s)}\) to the result. Thus, by Lemma 4, the insertion of any item \(x\) in the effective stash never requires more than \(s\) moves before it succeeds or causes \(x\) to be returned to the back of the stash. It follows that by time \(j\), we have spent at least \(s\) moves on each of the items in \(E_{j-m^{1/3}}\), and hence all elements in \(E_{j-m^{1/3}}\) are inserted into the table by time \(j\). Thus, at time \(j\) the size of the effective stash is at most \(s\), and this completes the induction.
3.2.2 Showing InnerQueue Stays Small

The bulk of our analysis relies on the following technical lemma.

**Lemma 9:** Let \(|S| = n'\), and let \(G'(S, f_0, f_1)\) be any cuckoo graph with vertex set \([m'] \times [m']\). Assume \(n'/m' \leq 2m^{1/3}\). For any \(0 < s \leq m^{1/6}\), with probability \(1/m^{\Omega(\sqrt{s})}\) the total number of vertices \(v \in G'(S, f_0, f_1)\) with \(C_{S,f_0,f_1}(v) > 8\) is at most \(s\).

**Proof:** Lemma 6 implies that the probability that all connected components have size at most \(\sqrt{s}\) is at least \(1 - m^{-\Omega(\sqrt{s})}\). We assume for the remainder that there are no components of size greater than \(\sqrt{s}\); this adds at most a probability \(1/m^{\Omega(\sqrt{s})}\) of failure in the statement of the lemma.

Pick a special vertex \(v_i\) in each component \(C_i\) of size greater than \(8\) at random, and assign to \(v_i\) a “weight” equal to the number of edges in \(C_i\). We show with probability at least \(1 - m^{-\Omega(\sqrt{s})}\) we cannot find a set of weight \(s\); the lemma follows.

Since no component has size more than \(\sqrt{s}\), we would need to find at least \(j = s/\sqrt{s} = \sqrt{s}\) vertices that are in components of size greater than \(8\) and are the special vertex for that component. We use the fact that for any fixed set of \(j\) distinct vertices \(v_1 \ldots v_j\), the probability (over both the choice of \(G'\) and the choice of the special vertex for each component) that all \(j\) vertices are the special vertices for their component is upper bounded by \(\Pr[C_{S,f_0,f_1}(v) \geq 8]^j\). Indeed, write \(E_i\) for the event \(v_1, \ldots, v_{i-1}\) are all special. We may write

\[
\Pr[v_1 \ldots v_j \text{ are all special}] \leq \prod_{i=1}^{j} \Pr[v_i \text{ is special } | E_i].
\]

To bound the right hand side, notice that

\[
\Pr[v_i \text{ is special } | E_i] \leq \Pr[v_i \text{ is in a different component from } v_1, \ldots, v_{i-1} \text{ and } C_{S,f_0,f_1}(v) \geq 8 | E_i]
\]

\[
\leq \Pr[C_{S,f_0,f_1}(v) \geq 8],
\]

where the last inequality holds because because the density of edges after taking out the earlier (larger than expected) components is less than the density of edges a priori.

Thus, taking a union bound over every set \(\{v_1, \ldots, v_j\}\) of \(j = \sqrt{s}\) vertices, the probability that we can find such a set is at most

\[
\left(\frac{n'}{\sqrt{s}}\right) \left(\Pr[v_1, \ldots v_j \text{ are all special}]\right) \leq \left(\frac{n'}{\sqrt{s}}\right) \left(\Pr[C_{S,f_0,f_1}(v) \geq 8]\right)^\sqrt{s}
\]

\[
\leq \left(\frac{n'}{\sqrt{s}}\right) \left(\frac{8^8}{8^{1/\sqrt{s}/3}}\right)^\sqrt{s} \leq m^{-\Omega(\sqrt{s})}.
\]

where the second inequality follows by Lemma 3.

We are in a position to show formally that the inner queue is small at all points in time.

**Lemma 10:** Let \(\pi'\) be any \(m^{1/3}\)-bounded sequence of operations of length \(p(n)\) on the inner cuckoo table \(C\). For any \(s \leq m^{1/6}\), with probability \(1 - p(n)/m^{\Omega(\sqrt{s})}\) over the choice of hash functions \(f_0, f_1\), InnerQueue has size at most \(s\) at all points in time.

**Proof:** Following the approach of [5], we define a good event \(\xi_2\) that ensures that InnerQueue has size at most \(s\). Namely, let \(\xi_2\) be the event that at all points \(i\) in time, the number of vertices \(v \in G'\) with \(C_{S_i,f_0,f_1}(v) > 8\) is at most \(s\). By Lemma 9 and a union bound over all \(p(n)\) operations in \(\pi'\), \(\xi_2\) occurs with probability at least \(1 - p(n)/m^{\Omega(\sqrt{s})}\).
We distinguish between the real queue and the effective queue. The real queue at any point \(i\) in time is the set of nodes \(v\) such that \(C_{S_i, f_0, f_1}(v) > 8\); the effective queue refers to the real queue, together with elements that used to be in the real queue but have since had their components shrunk to size at most 8 via deletions, and have not yet been inserted in the inner cuckoo table. Let \(E_i\) denote the set of items in the effective queue but not the real queue at time \(i\).

Clearly the event \(\xi_2\) guarantees that at any point in time, the size of the real queue is at most \(s\). To see that in fact the size of the effective queue never exceeds \(s\), assume by way of induction that for \(j \geq m^{1/3}\), the effective queue has size at most \(s\) at all times less than \(j\) (clearly event \(\xi_2\) guarantees that this is true for all \(j \leq m^{1/3}\) as a base case). In particular, the inductive hypothesis ensures that the effective queue has size at most \(s\) at time \(j - m^{1/3}\). Observe that during the execution of operations \(\{x_{j-m^{1/3}}, x_{j-m^{1/3}+1}, \ldots, x_{j}\}\), we spend \(16m^{1/3}/m = 16s\) moves on the elements of the effective queue, with at least 16 moves devoted to each element. By definition of the real queue, combined with Lemma \(4\) at most 16 moves are required to insert each element of \(E_{j-m^{1/3}}\), and thus all elements in \(E_{j-m^{1/3}}\) are inserted into the table by time \(j\). Thus, at time \(j\) the size of the effective queue is at most \(s\), and this completes the induction.

### 3.2.3 Putting it together

By design, every insert into \(C\) terminates in \(O(1)\) steps. Combining Lemmata \(8\) and \(10\) with probability \(1 - p(n)/m^{-\Omega(\sqrt{s})}\), both InnerQueue and InnerStash have size at most \(s/2\) at all times. Since \(L\) only contains items from the inner queue and inner stash, it follows that \(L\) never contains more than \(s\) items. This proves Theorem \(1\).

### 3.3 Proving Theorem \(2\)

Throughout this section, we use the following notation, following that in Section \(3.2\). Let \(\pi\) be an \(n\)-bounded sequence of \(p(n)\) operations. Denote by \(\pi_{i, \ldots, j}\) the elements inserted by \(\pi\) for any integer \(0 < i \leq p(n)\), let \(S_i\) denote the set of elements that are stored in the data structure just before the insertion of \(x_i\), let \(\hat{S}_i\) denote \(S_i\) together with the elements \(\{x_i, x_{i+1}, \ldots, x_{i+\log^6 n}\}\), ignoring any deletions between time \(i\) and time \(i + \log^6 n\), and let \(\bar{S}_i\) denote \(S_i\) together with the elements \(\{x_i, x_{i+1}, \ldots, x_{i+m/2}\}\), ignoring any deletions between time \(i\) and time \(i + m/2\) (treating any operations past time \(p(n)\) as empty). Since \(\pi\) is an \(n\) bounded sequence, we have \(|S_i| \leq n\), \(|\hat{S}_i| \leq n + \log^6 n\), and \(|\bar{S}_i| \leq n + m/2\) for all \(i\).

In proving Theorem \(2\) it suffices to show that neither the stash nor the queue of the primary structure will grow too large. We will use the following lemma.

**Lemma 11**: Let \(|S| = n\), and let \(G(S, h_0, h_1)\) be a cuckoo graph with vertex set \(V = \{m\} \times \{m\}\), with \(m \geq (1 + \epsilon)n\) for some constant \(\epsilon > 0\). There exists a constant \(c_2 > 0\) such that with probability \(1 - 1/n^{\Omega(\log n)}\) over the choice of \(h_0\) and \(h_1\), all components of \(G(S, h_0, h_1)\) are of size at most \(c_2^2 n^2\).

**Proof**: It is a standard calculation that for any node \(v\) in the cuckoo graph, \(\Pr[C_{S,h_0,h_1}(v) \geq k] \leq \beta^k\) for some constant \(\beta \in (0, 1)\) (see e.g. \(25\) Lemma 2.4). The conclusion follows by setting \(k = O(\log^2 n)\), and applying the union bound over all vertices.

### 3.3.1 Showing OuterStash Stays Small

**Lemma 12**: Let \(|S| = n\), and let \(G(S, h_0, h_1)\) be any cuckoo graph with vertex set \(V = \{m\} \times \{m\}\), where \((1 + \epsilon)n \leq m\) for some constant \(\epsilon > 0\). For any \(s\) such that \(s \leq m^{1/6}\), with probability \(1/m^{\Omega(s)}\) the total number of vertices that reside in the stash of \(G(S, h_0, h_1)\) is at most \(s\).
The choice of $s \leq m^{1/6}$ is fairly arbitrary but convenient and sufficient for our purposes. Notice Lemma \[\ref{lem:optimization} \] readily implies Lemma \[\ref{lem:beta} \].

**Proof:** Here we follow the work of [22], which considers the analysis of super-constant sized stashes, extending the work of [25].

The starting point is Lemmata [3] and [5] above. Specifically, in [22] it is shown that these lemmata imply that

$$\text{Pr}(B_v \geq t) \leq \sum_{k=1}^{\infty} \min \left( \left( \frac{3e^5 k^3}{m} \right)^t, 1 \right) \beta^k$$

for some constant $\beta$. In particular, we can concern ourselves with values of $k$ that are $O(m^{1/5})$, since the summation over $\beta^k$ terms for larger values of $k$ is dominated by $2^{-\Omega(m^{1/5})} = m^{-\Omega(m^{1/6})}$.

It follows that $\text{Pr}(B_v \geq t)$ is at most $\max \left( m^{-\Omega(t)}, m^{-\Omega(m^{1/6})} \right)$. We therefore claim that $\text{Pr}(B_v \geq j + 1) \leq m^{-\alpha j}$ for some constant $\alpha$ for $j \leq m^{1/6}$.

Now, following the derivation of Theorem 2.2 of [25], we have the probability that the stash exceeds size $s$ is given by the probability that $2m$ independently chosen components have an excess of more than $s$ edges, which can be bounded as:

$$\text{Pr}(\bar{e}(g) \geq s) \leq \sum_{k=1}^{2m} \binom{2m}{k} s^k m^{-\alpha s - k}$$

$$\leq \sum_{k=1}^{2m} m^{-\alpha s} \left( \frac{2e s}{k} \right)^k$$

$$\leq (2m)m^{-\alpha s} e^{2s}$$

$$= m^{-\Omega(s)}.$$

Here the second to last line follows from a straightforward optimization to find the the maximum of $(x/k)^k$ (which occurs at $k = x/e$).

We now show formally that the outer stash is small at all points in time.

**Lemma 13:** Let $\pi$ be any $n$-bounded sequence of operations of length $p(n)$. For $s \leq m^{1/5}$, with probability $1 - p(n)/m^{\Omega(s)}$ over the choice of hash functions $h_0, h_1$, OuterStash has size at most $s$ at all points in time.

**Proof:** We define a good event $\xi_3$ that ensures that OuterStash has size at most $s$. Namely, let $\xi_3$ be the event that at all points $i$ in time, the number of vertices in the stash of $G(\bar{S}_i, h_0, h_1)$ is at most $s$, and additionally all connected components of $G(\bar{S}_i, h_0, h_1)$ have size at most $\log^2 n$. By Lemmata [11] and [12], as well as a union bound over all $p(n)$ operations in $\pi$, $\xi_3$ occurs with probability at least $1 - p(n)/m^{\Omega(s)}$. As in [5], a minor technical point in applying Lemma [12] is that $\bar{S}_i$ is $n + m^{1/2}$-bounded, not $n$-bounded. But we can handle this by applying Lemma [12] with $\epsilon' = \epsilon/2$, since for large enough $m$, $(1 + \epsilon/2)(n + m^{1/2}) \leq (1 + \epsilon)n \leq m$.

We distinguish between the real stash and the effective stash. The real stash at any point $i$ in time is the set of nodes $v \in V$ such that reside in the stash of $G(S_i, h_0, h_1)$; the effective stash refers to the real stash, together with elements that used to be in the real stash but have since had cycles removed from their components due to deletions, and have not yet been inserted in the outer cuckoo table. Let $E_i$ denote the set of items in the effective stash but not the real stash of $G(S_i, h_0, h_1)$.

Clearly the event $\xi_3$ guarantees that at any point in time, the size of the real stash is at most $s$. To see that the size of the effective stash never exceeds $s$, assume by way of induction that for $j \geq m^{1/2}$, the effective stash has size at most $s$ at all times less than $j$ (clearly event $\xi_3$ guarantees that this is true for all $j \leq m^{1/2}$ as a base case). In particular, the inductive hypothesis ensures that the effective stash has size at most $s$ at time
Lemma 14: Let \( j - m^{1/2} \). Observe that during the execution of operations \( \{ x_{j - m^{1/2}}, x_{j - m^{1/2} + 1}, \ldots, x_j \} \), we spend at least \( m^{1/2}/m^{1/4} = m^{1/4} \) moves in total on the elements of the effective stash. Since all connected components have size at most \( \log^2 n \), Lemma 4 implies the insertion of any item \( x \) in the effective stash never requires more than \( 2 \log^2 n \) moves before it succeeds or causes \( x \) to be returned to the back of the stash. Thus, all elements in the effective stash at time \( j - m^{1/2} \) require at most \( 2m^{1/5} \log^2 n \leq m^{1/4} \) operations in total to process, and it follows that all elements in \( E_{j - m^{1/2}} \) are inserted into the table by time \( j \). Thus, at time \( j \) the size of the effective stash is at most \( s \), and this completes the induction.

3.3.2 Showing OuterQueue Stays Small

Let \( |S| = n \), and let \( G(S, h_0, h_1) \) be any cuckoo graph with vertex set \([m] \times [m]\). It is well-known that, for any node \( v \), there is significant probability that an insertion of \( v \) into \( G \) takes \( \Omega(\log n) \) time. But one might hope that for a sufficiently large set of distinct vertices \( \{v_1, \ldots, v_N\} \), the average size of the connected components of the \( v_i \)’s is constant with overwhelming probability over choice of \( G \), and thus any sequence of \( N \) insertions will take \( O(N) \) time in total. Indeed, Lemma 4.4 of Arbitman et al. [5] establishes that, for any distinct vertices \( \{v_1, \ldots, v_N\} \) with \( N \leq \log n \), \( \sum_{i=1}^N C_{S,h_0,h_1}(v_i) = O(N) \) with probability \( 1 - 2^{-\Omega(N)} \) over the choice of \( h_0 \) and \( h_1 \). Roughly speaking, Arbitman et al. use this result to conclude that a logarithmic sized queue suffices for deanonymizing cuckoo hashing (where all guarantees hold with high probability), since any sequence of \( \log n \) insertions can be processed in \( O(\log n) \) time steps.

Our goal is to achieve guarantees which hold with overwhelming probability. To achieve this, we use the fact that we can afford to keep a queue of super-logarithmic size without affecting the asymptotic space usage of our algorithm. We show that any sequence of, say, \( N = \log^6 n \) operations can be cleared from the queue in \( O(N) \) time with probability \( 1 - 1/n^{\Omega(1)} \). It follows that with overwhelming probability the queue does not overflow. Unfortunately, the techniques of [5] do not generalize to values of \( N \) larger than \( O(\log n) \), so we generalize their result using different methods.

Intuitively, one should picture the random process we wish to analyze as a standard queueing process, where the time to handle each job is a random variable. In our case, the random variable for a job – which is a key \( k \) to be placed – is the time to find a spot for \( k \) in the cuckoo table, which is proportional to the size of the connected component in which \( k \) is placed in the cuckoo graph by Lemma 4. This random variable is known to be constant on average and have exponentially decreasing tails, so if the job times were independent, this would be a normal queue (more specifically, a Galton-Watson process), and the bounds would follow from standard analyses.

Unfortunately, the job times in our setting are not independent. Roughly speaking, our analysis proceeds by showing that with overwhelming probability on a given instance of the cuckoo graph, the expected value of the size of the connected component of a randomly chosen vertex is close to its expectation if the graph was chosen randomly.

Our main technical tool will be the following lemma.

Lemma 14: Let \( |S| = n \), and let \( G(S, h_0, h_1) \) be a cuckoo graph with vertex set \([m] \times [m]\). Let \( \{v_1, \ldots, v_N\} \) be a set of \( N > \log^6(n) \) vertices chosen uniformly at random (with replacement) from the cuckoo graph \( G(S, h_0, h_1) \). There is a constant \( c \) such that with probability \( 1 - n^{-\Omega(\log n)} \) (over both the choice of the \( v_i \)’s and the generation of the cuckoo graph \( G \)), \( \sum_{i=1}^N C_{S,h_0,h_1}(v_i) \leq cN \).

Proof: Let \( X_i \) be the number of vertices in \( G(S, h_0, h_1) \) in components of size \( i \), and let \( \mu_i = \mathbb{E}[X_i] \) be the expected number of vertices in components of size \( i \) in a random cuckoo graph, and let \( \mu = \frac{1}{2m} \sum_{i=1}^{2m} i \mu_i \) be the expected size of the connected component of a random node in a random cuckoo graph. By standard calculations (see e.g. [25] Lemma 2.4), there is a constant \( \beta \in (0, 1) \) such that for any \( v \), \( \mathbb{E}[C_{S,h_0,h_1}(v)] \leq \sum_{k=1}^\infty \beta^k = O(1) \), where the expectation is taken over choice of \( h_0 \) and \( h_1 \), and hence \( \mu = O(1) \).
For fixed hash functions \( h_0 \) and \( h_1 \), we can write

\[
\mathbb{E}_{v \in V} C_{S,h_0,h_1}(v) = \frac{1}{2m} \sum_{v \in V} C_{S,h_0,h_1}(v) = \frac{1}{2m} \sum_{i=1}^{2m} iX_i.
\]

(1)

Our goal is to show that with overwhelming probability over the choice of \( h_0 \) and \( h_1 \), the right hand side of Equation \( 1 \) is close to \( \mu \). We will do this by showing that for small enough \( i \), \( X_i \) is tightly concentrated around \( \mu_i \), and that larger \( i \) do not contribute significantly to the sum.

**Lemma 15:** The following properties both hold.

1. Suppose \( \mu_i \geq n^{2/3} \). Then \( \Pr(|X_i - \mu_i| \geq n^{2/3}) = 2e^{-\Omega(n^{1/3})} \). (The \( \Omega \) notation hides factors polylogarithmic in \( n \).)

2. Let \( i^* \) be the smallest value of \( i \) such that \( \mu_i < n^{2/3} \), and let \( X_* \) be the number of vertices in components of size at least \( i^* \). Then \( \Pr[X_* \geq \gamma n^{2/3}] \leq 1/n^{O(\log n)} \) for some constant \( \gamma \).

**Proof:**

1. This follows from a standard application of Azuma’s inequality, applied to the edge exposure martingale that reveal the edges of the cuckoo graph one at a time. More specifically, we reveal the edges of \( G \) one at a time in an arbitrary order, say \( e_1, \ldots, e_n \) and let \( Z_j = E[X_j|e_1, \ldots, e_{j-1}] \). Then \( Z_j \) is a martingale, and changing a single edge can only change the number of components of size \( j \) by a constant (specifically, two), and hence \( |Z_j - Z_{j-1}| \leq 2j \) for all \( j \). Thus, by Azuma’s inequality

\[
\Pr(|X_i - E[X_i]| \geq 2i\lambda\sqrt{n}) = \Pr(|Z_n - Z_0| \geq 2i\lambda\sqrt{n}) \leq 2e^{-\lambda^2/2}.
\]

Setting \( \lambda = \frac{n^{2/3}}{2\lambda\sqrt{n}} = \Omega(n^{1/6}/i) \), we see

\[
\Pr(|X_i - E[X_i]| \geq n^{2/3}) \leq e^{-\Omega(n^{1/3}/i^2)}.
\]

By a standard calculation, \( E[X_i] \geq n^{2/3} \) implies \( i \leq c_1 \log n \) for some constant \( c_1 \), and the theorem follows.

2. Since \( \Pr[C_{S,h_0,h_1}(v) \geq k] \leq \beta^k \) for some constant \( \beta \in (0,1) \), it follows easily that \( E[X_*] \leq \sum_{i=1}^{n^{2/3}} \frac{n^{2/3}}{i} \leq n^{2/3} \). It follows that \( E[X_*] = O(n^{2/3}) \).

In order to get concentration of \( X_* \) about its mean, we use a slight modification of the edge exposure martingale, which will essentially allow us to assume that all connected components have size \( O(\log^2 n) \) when attempting to bound the differences between martingale steps, which happens with very high probability by Lemma \( 1 \). This technique is formalized for example in Theorem 3.7 of [32].

Let \( Q \) be the event that all connected components are of size at most \( \log^2 n \). We reveal the edges of \( G \) one at a time in an arbitrary order, say \( e_1, \ldots, e_n \) and let \( Z_j = E[X_*|e_1, \ldots, e_{j-1}] \) if the edges \( e_1, \ldots, e_{j-1} \) do not include a component of size greater than \( \log^2 n \) and \( Z_j = Z_{j-1} \) if the edges \( e_1, \ldots, e_{j-1} \) do include a component of size greater than \( \log^2 n \). Then \( Z_j \) is a martingale, and since changing a single edge can only change the number of components of size \( i \) by at most two, we see that \( |Z_i - Z_{i-1}| \leq 2e^2\log^2 n \). Now as \( Z_n \) will equal \( X_* \) except in the case where event \( Q \) does not hold, we can apply Azuma’s inequality to the above martingale to conclude that

\[
\Pr(|X_* - E[X_*]| \geq 2c\lambda\sqrt{n}\log n) \leq 2e^{-\lambda^2/2} + \Pr(-Q).
\]

Setting \( \lambda = \frac{n^{1/3}}{c\log^2(n)\sqrt{n}} \), we obtain

\[
\Pr(|X_* - E[X_*]| \geq n^{1/3}) \leq e^{-\Omega(n^{2/3})} + \frac{1}{n^{O(\log n)}}.
\]
Noting that \( E[X_*] = O(n^{2/3}) \), we conclude

\[
\Pr(X_* \geq \gamma n^{1/3}) \leq \frac{1}{n^{\Omega(\log n)}}
\]

for some constant \( \gamma \).

Properties 1 and 2 of Lemma 15 together imply that with very high probability over choice of \( h_0 \) and \( h_1 \),

\[
\sum_{i=1}^{2m} i X_i = \sum_{i=1}^{i^* - 1} (\mu_i \pm n^{2/3}) + O(n^{2/3}) \leq \sum_{i=1}^{2m} 2\mu_i + O(n^{2/3} \log n).
\]

Combining the above with Equation 1, with overwhelming probability over choice of \( h_0 \) and \( h_1 \) it holds that

\[
E_{v \in V} C_{S,h_0,h_1}(v) = \frac{1}{2m} \sum_{i=1}^{2m} i X_i \leq 2\mu + o(1) = O(1).
\]

Thus, we have shown that with very high probability over the choice of \( G \), there is a constant \( c_3 \) such that

\[
E_{v \in V}[C_{S,h_0,h_1}(v)] \leq c_3.
\]

Our last step in proving Lemma 14 is to show that if we choose a set of vertices at random, the sum of the component sizes is concentrated around its mean. Indeed, by applying a similar argument as in the proof of Lemma 15 Property Two, we can assume for all \( v \in V \), \( C_{S,h_0,h_1}(v) \leq c_2 \log^2 n \), as long as we add an additional term of \( \frac{1}{n^{\Omega(\log n)}} \) in the bound on the probability obtained using Azuma’s inequality. This yields

\[
\Pr(|\sum_{i=1}^{S} C_{S,h_0,h_1}(v_i) - c_3| \geq \lambda \sqrt{S} c_2 \log^2 n) \leq 2e^{-\lambda^2/2} + \frac{1}{n^{\Omega(\log n)}}.
\]

Setting \( \lambda = \sqrt{S}/\log^2 n \) yields

\[
\Pr(|\sum_{i=1}^{S} C_{S,h_0,h_1}(v_i) - c_3| \geq c_2) \leq e^{-\Omega(S/\log^4 n)} + \frac{1}{n^{\Omega(\log n)}}.
\]

The conclusion follows, with \( c = c_3 + c_2 \).

Notice that for any set of distinct items \( \{x_1, \ldots, x_N\} \) to be inserted, the sets \( \{h_0(x_1), \ldots, h_0(x_N)\} \) and \( \{h_1(x_1), \ldots, h_1(x_N)\} \) are uniformly distributed set of vertices in \( G(S,h_0,h_1) \). Thus, Lemma 14 ensures that for distinct items \( \{x_1, \ldots, x_N\} \) with \( N \geq \log^6 n \), and for any set \( S \) of size \( n \), with probability at least \( 1 - 1/n^{\log n} \) over the choice of \( h_0 \) and \( h_1 \), \( \sum_{i=1}^{N} C_{S,h_0,h_1}(v_i) \leq cN \).

With this in hand, we are ready to show that with overwhelming probability OuterQueue does not exceed size \( \log^6 n \) over any sequence of \( \text{poly}(n) \) operations.

**Lemma 16:** With probability \( 1 - n^{-\Omega(\log n)} \), the queue of the primary structure has size at most \( \log^6 n \) at all times.

**Proof:** We define a good event \( \xi_4 \) that ensures that OuterQueue has size at most \( \log^6 n \). Namely, let \( \xi_4 \) be the event that for all times \( \log^6 n \leq j \leq p(n) \), it holds that \( \sum_{i=j-\log^6 n}^{j} C_{S_{j-1},f_0,f_1}(x_i) \leq c \log^6 n \). By Lemma 14 and a union bound over all \( p(n) \) operations in \( \pi \), \( \xi_4 \) occurs with probability at least \( 1 - 1/n^{\log n} \).

For \( j > \log^6 n \), suppose by induction there are at most \( \log^6 n \) items in the effective queue at all times less than \( j \) (as a base case, this is clearly true for all \( j \leq \log^6 n \)). In particular, this holds at time \( j - \log^6 n \).
We can assume all items in the effective queue at time \( j - \log^6 n \) are distinct because we process deletions immediately. Since all the (at most) \( \log^6 n \) items in the effective queue are distinct, event \( \xi_4 \) guarantees that all of these items can be cleared from the queue in \( c \log^6 n \) steps. Setting the number of steps expended on elements of the queue every operation to \( \alpha = c \), all (at most) \( \log^6 n \) items in the effective queue at time \( j - \log^6 n \) will be cleared from the queue before time \( j \). Thus, at time \( j \), the queue contains at most \( \log^6 n \) items, and this completes the induction.

\[ Q \] only contains items from OuterStash and OuterQueue, and we have shown both deques contain \( \log^6 m \) items with overwhelming probability. Theorem 2 follows.

### 3.4 Putting it All Together

For any constant \( \epsilon > 0 \), our nested cuckoo construction uses \( (2 + \epsilon)n \) words for the outer cuckoo table, \( O(m^{2/3}) \) for the inner structure \( Q \), and (with overwhelming probability) \( O(\log^2 m) \) words for the cycle-detection mechanisms. The latter two space costs are dominated by the first, and so our total space usage is \( (2 + \epsilon)n \) words in total for constant \( \epsilon > 0 \).

We derive our final theoretical guarantees on the running time of each operation for both of our constructions.

**Construction One:** Inserts take \( O(1) \) time by design. Lookups and deletions require examining \( T_0[h_0(k)] \), \( T_1[h_1(k)] \), \( R_0[f_0(k)] \), and \( R_1[f_1(k)] \), and performing a lookup in \( L \), which in Construction 1 potentially requires examining all elements in \( L \). Theorems 1 and 2 together imply that for \( 0 < s < \log^2 n \), with probability at least \( 1 - 1/m^{\Omega(\sqrt{s})} \), \( L \) contains only \( s \) items. Thus, lookups and removals take time \( s \) with probability at least \( 1 - 1/m^{\Omega(\sqrt{s})} \), even though \( s \) is not a tuning parameter of our construction.

To clarify, the hidden constant in the exponent is fixed, i.e. independent of all parameters. Thus, for any constant \( c \), there is some larger constant \( s \) such that the probability lookups and removals take time more than \( s \) is bounded by \( 1 - 1/m^c \). We also remark that it is straightforward to extend our analysis to all \( s \leq m^{1/12} \), but for clarity we have not presented our results in this generality.

**Construction Two:** Again, inserts take \( O(1) \) time by design. As in Construction One, lookups and deletions require examining \( T_0[h_0(k)] \), \( T_1[h_1(k)] \), \( R_0[f_0(k)] \), and \( R_1[f_1(k)] \), and performing a lookup in \( L \); assuming \( L \) has size \( s = O(\log^{1/2} n) \), such a lookup can be performed in constant time using our atomic stash. Theorems 1 and 2 therefore imply that lookups and removes take \( O(1) \) time with probability \( 1 - 1/m^{\Omega(\log^{1/4} n)} \).

### 3.5 Extensions

#### 3.5.1 \( \text{poly} \log(n) \)-wise Independent Hash Functions

We remark that an argument of Arbitman et al. [5] implies almost without modification that in Construction Two, insertions, deletions, and lookups take \( O(1) \) time with overwhelming probability even if the hash functions \( h_0, h_1, f_0 \), and \( f_1 \) are chosen from \( \text{poly} \log(n) \)-wise independent hash families. For completeness, we reproduce this argument in our context.

In the analysis above, the only places we used the independence of our hash functions were in Lemmata 7, 9, 12, and 14 above. These lemmata allowed us to define four events that occur with high or overwhelming probability, whose occurrence guarantee that the our time bounds hold. Specifically, for any fixed \( s \), our time bounds for Construction Two hold if none of the following “bad” events occur:

1. Event 1: There exists a set \( \{v_1, \ldots, v_N\} \) of \( N = O(\log^6 n) \) vertices in the outer cuckoo graph, such that \( \sum_{i=1}^N c_{S,h_0,n_1(v_i)} > cN \) (this is the complement of event \( \xi_4 \) from Section 3.3.2).
2. Event 2: There exists a set of at most $2m$ vertices in the outer cuckoo graph, such that the number of stashed elements from the set exceeds $O(\log^6 n)$ (this is the complement of event $\xi_3$ from Section 3.3.1).

3. Event 3: There exists a set of at least $O(\log^{1/2} n)$ vertices in the inner cuckoo graph, all of whose connected components at some point in time have size greater than 8 (this is the complement of the event $\xi_2$ defined in Section 3.2.2).

4. Event 4: There exists a set of vertices in the cuckoo graph, such that the number of stashed elements from the set at some point in time exceeds $O(\log^{1/2} n)$ (this is the complement of the event $\xi_1$ defined in Section 3.2.1).

Lemmas 7, 9, 12, and 14 ensure that if $h_0, h_1, f_0, \text{ and } f_1$ are fully random, none of the four events occur with probability at least $1 - 1/n^{\Omega(\log^{1/4}(n))}$. In order to show that the conclusion holds even if the hash functions are polylog($n$)-wise independent, we apply a recent result of Braverman [8] stating that polylogarithmic independence fools constant-depth boolean circuits.

**Theorem 17:** (8) Let $s \geq \log m$ be any parameter. Let $F$ be a boolean function computed by a circuit of depth $d$ and size $m$. Let $\mu$ be an $r$-independent distribution where

$$r \geq 3 \cdot 60^{d+3} \cdot (\log m)^{(d+1)(d+3)} \cdot s^{d(d+3)},$$

then $|E_\mu[F] - E[F]| < 0.82s \cdot 15m$.

Theorem 17 implies that if we can develop constant-depth boolean circuits of quasi-polynomial size that recognize Events 1-4 above, then the probability any of the events occur under polylogarithmic independent hash functions will be very close to the probability the events occur under fully random hash functions. The circuits that recognize our events are similar to those used in Arbitman et al. [5]; the input wires to the first two circuits contain the values $h_0(x_1), h_1(x_1), ..., h_0(x_n), h_1(x_n)$ (where the $x_i$’s represent the elements inserted into the outer cuckoo table), while the input wires to the second two circuits contain the values $f_0(x_1), f_1(x_1), ..., f_0(x_j), f_1(x_j)$, where $j$ is the number of items in the inner cuckoo table.

1. **Identifying Event 1:** Just as in [5], this event occurs if and only if the graph contains at least one forest from a specific set of forests of the bipartite graph on $[m] \times [m]$, where $m = (1 + \epsilon)n$. We denote this set of forests by $F_n$, and observe that $F_n$ is a subset of all forests with at most $cN = O(\log^6 (n))$ vertices, which implies that $|F_n| = n^{\text{poly log}(n)}$. Therefore, the event can be identified by a constant-depth circuit of size $n^{\text{poly log}(n)}$ that simply enumerates all forests in $F_n$, and for every such forest checks whether it exists in the graph.

2. **Identifying Event 2:** A constant-depth circuit identifying this event enumerates over all $S \subseteq [m] \times [m]$ of size $\log^{1/2}(n)$ and checks whether all of elements of $S$ are stashed. As in [5], a minor complication is that we must define a canonical set of stashed elements for each set $S$; this is only for simplifying the analysis, and does not require modifying our actual construction. Our circuit checks whether all elements of $S$ are stashed by, for each $x \in S$, enumerating over all connected components in which edge $(h_0(x), h_1(x))$ is stashed according to the canonical set of stashed items for $S$ and checking if the component exists in the graph. We may assume Event 1 does not occur, and thus we need only iterate over components of $O(\log^6 (n))$ vertices. The circuit thus has $O(n^{\text{poly log}(n)})$ size.

3. **Identifying Events 3 and 4:** We may assume Events 1 and 2 do not occur. Then there are at most $O(\log^6 n)$ edges in the inner cuckoo table, so a constant depth circuit of quasipolynomial size simply enumerates over all possible edge sets $E'$ of size $O(\log^6 n)$ satisfying Event 3 or Event 4 and checks if $E'$ equals the input to the circuit.

Thus, we can set $s = \text{poly log}(n)$ and $m = \text{poly log}(n)$ in the statement of Theorem 17 to conclude that, even if we use hash functions from a poly log($n$)-wise independent family of functions, Events 1-
4 still only occur with negligible probability. We remark that similar arguments demonstrate that when poly \( \log(n) \)-wise independent hash functions are used, all operations under Construction 1 still take \( O(s) \) time with probability \( 1 - 1/n^{\Omega(\sqrt{s})} \) when \( s = \text{poly} \log(n) \) (the amount of independence required depends on \( s \)).

### 3.5.2 Sufficiently Independent Hash Functions Evaluated in \( O(1) \) time

Unfortunately, all known constructions of \( \text{poly} \log(n) \)-wise independent hash families that can be evaluated in the RAM model while maintaining our \( O(n) \) space bound come with important caveats. The classic construction of \( k \)-wise independent hash functions due to Carter and Wegman based on degree-\( k \) polynomials over finite fields requires time \( O(k) \) to evaluate; ideally we would like \( O(1) \) evaluation time to maintain our time bounds. The work of Siegel [39] is particularly relevant; for polynomial-sized universes, he proves the existence of a family of \( n^\epsilon \)-wise independent hash functions for \( \epsilon > 0 \) (which is super-logarithmic), which can be evaluated in \( O(1) \) time using look up tables of size \( n^\delta \) for some \( \delta < 1 \). However, his construction is non-uniform in that he relies on the existence of certain expanders for which we do not possess explicit constructions. Subsequent works that improve and/or simplify [39] (e.g., [18,19,35]) all possess polynomial probabilities of failure, which render them unsuitable when seeking guarantees that hold with overwhelming probability. The development of uniformly computable hash families which can be evaluated in \( O(1) \) time using \( o(n) \) words of memory remains an important open question.

### 3.5.3 Achieving Loads Close to One

We remark that we can achieve \( O(1) \) worst-case operations with overwhelming probability using \( (1 + \epsilon)n \) words of memory for any constant \( \epsilon > 0 \) by substituting our fully de-amortized nested cuckoo hash tables in for the “backyard” cuckoo table of Arbitman et al. [6]. The construction of [6] uses a main table consisting of \( (1 + \epsilon/2)d \) buckets, each of size \( d \) for some constant \( d \), and uses a de-amortized cuckoo table as a “backyard” to handle elements from overflowing buckets. They show that for constant \( \epsilon > 0 \), with overwhelming probability the backyard cuckoo table must only store a small constant fraction of the elements. Note that \( n^\alpha \)-wise independent hash functions for some \( \alpha > 0 \) are required to map items to buckets in the main table, and therefore the technique cuts our space usage by a factor of about 2, but increases the amount of independence we need to assume in our hash functions for theoretical guarantees to hold.

### 4 Cache-Oblivious Multimaps

In this section, we describe our cache-oblivious implementation of the multimap ADT. To illustrate the issues that arise in the construction, we first give a simple implementation for a RAM, and then give an improved (cache-oblivious) construction for the external memory model. Specifically, we describe an amortized cache-oblivious solution and then we describe how to de-amortize this solution.

In the implementation for the RAM model, we maintain two nested cuckoo hash tables, as described in Section 2. The first table enables fast \texttt{containsItem}(\( k, v \)) operations; this table stores all the \( (k, v) \) pairs using each entire key-value pair as the key, and the value associated with \( (k, v) \) is a pointer to \( v \)'s entry in a linked list \( L(k) \) containing all values associated with \( k \) in the multimap. The second table ensures fast \texttt{containsKey}(\( k \)), \texttt{getAll}(\( k \)), and \texttt{removeAll}(\( k \)) operations: this table stores all the unique keys \( k \), as well as a pointer to the head of \( L(k) \).

**Operations in the RAM implementation.**

1. \texttt{containsKey}(\( k \)): We perform a lookup for \( k \) in Table 2.
2. \texttt{containsItem}(\( k, v \)): We perform a lookup for \( (k, v) \) in Table 1.
3. add\((k, v)\): We add \((k, v)\) to Table 1 using the insertion procedure of Section 2. We perform a lookup for \(k\) in Table 2, and if \(k\) is not found we add \(k\) to Table 2. We then insert \(v\) as the head of the linked list corresponding to Table 2.

4. remove\((k, v)\): We remove \((k, v)\) from Table 1, and remove \(v\) from the linked list \(L(k)\); if \(v\) was the head of \(L(k)\), we also perform a lookup for \(k\) in Table 2 and update the pointer for \(k\) to point to the new head of \(L(k)\) (if \(L(k)\) is now empty, we remove \(k\) from Table 2.)

5. getAll\((k)\): We perform a lookup for \(k\) in Table 2 and return the pointer to the head of \(L(k)\).

6. removeAll\((k)\): We remove \(k\) from Table 2. In order to achieve unamortized \(O(1)\) I/O complexity, we do not update the corresponding pointers of \((k, v)\) pairs in Table 1; this creates the presence of “spurious” pointers in Table 1, but Angelino et al. [3] explain how to handle the presence of such spurious pointers while increasing the cost of all other operations by \(O(1)\) factors.

All operations above are performed in \(O(1)\) time in the worst case with overwhelming probability by the results of Section 2. Two major issues arise in the above construction. First, the space-usage remains \(O(n)\) only if we assume the existence of a garbage-collector for leaked memory, as well as a memory allocation mechanism, both of which must run in \(O(1)\) time in the worst case. Without the memory allocation mechanism, inserting \(v\) into \(L(k)\) cannot be done in \(O(1)\) time, and without the garbage collector for leaked memory, space cannot be reused after remove and removeAll operations. Second, in order to extract the actual values from a getAll\((k)\) operation, one must actually traverse the list \(L(k)\). Since \(L(k)\) may be spread all over memory, this suffers from poor locality.

We now present our cache-oblivious multimap implementation. Our implementation avoids the need for garbage collection, and circumvents the poor locality of the above getAll operation. We do require a cache-oblivious mechanism to allocate and deallocate power-of-two sized memory blocks with constant-factor space and I/O overhead; this assumption is theoretically justified by the results of Brodal et al. [11].

**Amortized Cache-Oblivious Multimaps.** As in the RAM implementation, we keep two nested cuckoo tables. In Table 1, we store all the \((k, v)\) pairs using each entire key-value pair as the key. With each such pair, we store a count, which identifies an ordinal number for this value \(v\) associated with this key, \(k\), starting from 0. For example, if the keys were \((4, Alice), (4, Bob), (4, Eve)\), then \((4, Alice)\) might be pair 0, \((4, Bob)\) pair 1, and \((4, Eve)\) pair 2, all for the key, 4.

In Table 2, we store all the unique keys. For each key, \(k\), we store a pointer to an array, \(A_k\), that stores all the key-value pairs having key \(k\), stored in order by their ordinal values from Table 1. With the record for a key \(k\), we also store \(n_k\), the number of pairs having the key \(k\), i.e., the number of key-value pairs in \(A_k\). We assume that each \(A_k\) is maintained as an array that supports amortized \(O(1)\)-time element access and addition, while maintaining its size to be \(O(n_k)\).

**Operations.**

1. containsKey\((k)\): We perform a lookup for \(k\) in Table 2.

2. containsItem\((k, v)\): We perform a lookup for \((k, v)\) in Table 1.

3. add\((k, v)\): After ensuring that \((k, v)\) is not already in the multimap by looking it up in Table 1, we look up \(k\) in Table 2, and add \((k, v)\) at index \(n_k\) of the array \(A_k\), if \(k\) is present in this table. If there is no key \(k\) in Table 2, then we allocate an array, \(A_k\), of initial constant size. Then we add \((k, v)\) to \(A_k[0]\) and add key \(k\) to Table 2. In either case, we then add \((k, v)\) to Table 1, giving it ordinal \(n_k\), and increment the value of \(n_k\) associated with \(k\) in Table 2. This operation may additionally require the growth of \(A_k\) by a factor of two, which would then necessitate copying all elements to the new array location and updating the pointer for \(k\) in Table 2.

4. remove\((k, v)\): We look up \((k, v)\) in Table 1 and get its ordinal count, \(i\). Then we remove \((k, v)\) from Table 1, and we look up \(k\) in Table 2, to learn the value of \(n_k\) and get a pointer to \(A_k\). If \(n_k > 1\), we
swap \((k', v') = A_k[n_k - 1]\) and \((k, v) = A_k[i]\), and then remove the last element of \(A_k\). We update the ordinal value of \((k', v')\) in Table 1 to now be \(i\). We then decrement the value of \(n_k\) associated with \(k\) in Table 2. If this results in \(n_k = 0\), we remove \(k\) from Table 2. This operation may additionally require the shrinkage of the array \(A_k\) by a factor of 2, so as to maintain the \(O(n)\) space bound.

5. get\(All(k)\): We look up \(k\) in Table 2, and then list the contents of the \(n_k\) elements stored at the array \(A_k\) indexed from this record.

6. remove\(All(k)\): For all entries \((k, v)\) of \(A_k\), we remove \((k, v)\) from Table 1. We also remove \(k\) from Table 2 and deallocate the space used for \(A_k\). As in the RAM implementation, in order to achieve unamortized \(O(1)\) I/O cost, we do not update the pointers of \((k, v)\) pairs in Table 1; this creates the presence of “spurious” pointers in Table 1 which are handled the same as in the RAM case.

In terms of I/O performance, contains\(Key(k)\) and contains\(Item(k, v)\) clearly require \(O(1)\) I/Os in the worst case. get\(All(k)\) operations use \(O(1 + n_k/B)\) I/Os in the worst case, because scanning an array of size \(n_k\) uses \(O(\lceil n_k/B \rceil)\) I/Os, even though we don’t know the value of \(B\). remove\(All(k)\) utilizes \(O(n_k)\) I/Os in the worst-case with overwhelming probability, but these can be charged to the insertions of the \(n_k\) values associated with \(k\), for \(O(1)\) amortized I/O cost. add\((k, v)\) and remove\((k, v)\) operations also require \(O(1)\) amortized I/Os with overwhelming probability; the bound is amortized because there is a chance this operation will require a growth or shrinkage of the array \(A_k\), which may require moving all \((k, v)\) values associated with \(k\) and updating the corresponding pointers in Table 1.

In the next section, we explain how to de-amortize add\((k, v)\) and remove\((k, v)\) operations.

**De-Amortizing the Key-Value Arrays.** To de-amortize the array operations, we use a rebuilding technique, which is standard in de-amortization methods (e.g., see [38]).

We consider the operations needed for insertions to an array; the methods for deletions are similar. The main idea is that we allocate arrays whose sizes are powers of 2. Whenever an array, \(A\), becomes half full, we allocate an array, \(A'\), of double the size and start copying elements \(A\) into \(A'\). In particular, we maintain a crossover index, \(i_A\), which indicates the place in \(A\) up to which we have copied its contents into \(A'\). Each time we wish to access \(A\) during this build phase, we copy two elements of \(A\) into \(A'\), picking up at position \(i_A\), and updating the two corresponding pointers in Table 1. Then we perform the access of \(A\), as would otherwise, except that if we wish access an index \(i < i_A\), then we actually perform this access in \(A'\). Since we copy two elements of \(A\) for every access, we are certain to complete the building of \(A'\) prior to our needing to allocate a new, even larger array, even if all these accesses are insertions. Thus, each access of our array will now complete in worst-case \(O(1)\) time with overwhelming probability. It immediately follows that add\((k, v)\) and remove\((k, v)\) operations run in \(O(1)\) worst-case time. All time bounds in Table 1 follow.

5 Conclusion

In this paper, we have studied fully de-amortized dictionary and multimap algorithms that support worst-case constant-time operations with high or overwhelming probability. At the core of our result is a “nested” cuckoo hash construction, in which an inner cuckoo table is used to support fast lookups into a queue/stash structure for an outer cuckoo table, as well as a simplified and improved implementation of an atomic stash, which is related to the atomic heap or q-heap data structure of Fredman and Willard [20]. We gave fully de-amortized constructions with guarantees that hold with high probability in the Practical RAM model, and with overwhelming probability in the external-memory (I/O) model, the standard RAM model, or the AC^0 RAM model.

Several interesting questions remain for future work. First, lookups in our structure may require four or more I/Os in external-memory; it would be interesting to develop fully de-amortized structures supporting lookups in as few as two I/Os. A prime possibility suited for external memory is random-walk cuckoo hashing with two hash functions and super-constant bucket sizes. Second, it would be interesting to develop
a fully-deamortized dictionary for the Practical RAM model where all operations take $O(1)$ time with overwhelming probability.

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A Our Atomic Stash Structure

In this section, we describe our deterministic atomic stash implementation of the dictionary ADT. This structure dynamically maintains a sparse set, $S$, of at most $O(w^{1/2})$ key-value pairs, using $O(w)$ words of memory, so as to support insertions, deletions, and lookups in $O(1)$ worst-case time, where $w$ is our computer’s word size. Our construction is valid in the I/O model and the AC0 RAM model [40], that is, the RAM model minus constant-time multiplication and division, but augmented with constant-time AC0 instructions that are included in modern CPU instruction sets. (See previous work on the atomic heap [20] data structure for a solution in the standard RAM model, albeit in a way that limits the size of $S$ to be $O(w^{1/6})$ and is less efficient in terms of constant factors.) We assume that keys and values can each fit in a word of memory.

Our construction builds on the fusion tree and atomic heap implementations of previous researchers [2, 20, 40], but improves the simplicity and capacity of these previous implementations by taking advantage of modern CPU instruction sets and the fact that we are interested here in maintaining only a simple dictionary, rather than an ordered set. For instance, our solution avoids any lookups in pre-computed tables.

A.1 Components of an Atomic Stash

Our solution is based on associating with each key $k$ in $S$, a compressed key, $B_k$, of size $w' = \lfloor w^{1/2} \rfloor - 1$, which we store as a representative of $k$. In addition, for each compressed key, $B_k$, we store a binary mask, $M_k$, of equal length, such that

$$B_k \land M_k \neq B_j \land M_k,$$

for $j \neq k$, with $j$ in $S$, where “$\land$” denotes bit-wise AND. That is, all of the masked keys in $S$ are unique.

A critical element in our data structure is an associative cache, $X$, which is stored in a single word, which we view as being divided into $w'$ fields of size $w'$ each, plus a (high-order) indicator bit for each field,
so \( w \geq w'(w' + 1) \). We denote the field with index \( i \) in \( X \) as \( X(i) \) and the indicator bit with index \( i \) (for field \( i \)) in \( X \) as \( X[i] \), and we assume that indices are increasing right-to-left and begin with index 1. Thus, the bit position of \( X[j] \) in \( X \) is \( j(w' + 1) \). Note that with standard SHIFT, OR, and AND operations, we can read or write any field or indicator bit in \( X \) in \( O(1) \) time given its index and either a single bit or a bit mask, \( \bar{1} \), of \( w' \) 1’s. Each field \( X(i) \) is either empty or it stores a compressed key, \( B_k \), for some key \( k \) in \( S \). In addition, we also maintain a word, \( Y \), with indices corresponding to those in \( X \), such that \( Y(i) \) stores the mask, \( M_k \), if \( X(i) \) stores the binary key \( B_k \). We also maintain key and value arrays, \( K \) and \( V \), such that, if \( B_k \) is stored in \( X(i) \), then we store the key-value pair \((k, v)\) in \( K \) and \( V \), so that \( K[i] = k \) and \( V[i] = v \). To keep track of the size of \( S \), we maintain a count, \( n_S \), which is the number of items in \( S \). Finally, we maintain a “used” mask, \( U \), which has all 1’s in each field of \( X \) that is used. That is, \( U(i) = \bar{1} \) iff \( X(i) \) holds some key, \( B_k \), which is equivalent to saying \( Y(i) \neq 0 \).

The reason we use both a compressed key and a mask for each key stored in \( S \) is that, as we add keys to \( S \), we may need to sometimes expand the function that compresses keys so that the masked values of keys remain unique, while still fitting in a field of \( X \). Even in this environment, we would like for previously-compressed keys to still be valid. In particular, our method generates a sequence of compression functions, \( p_1, p_2, \) etc., so that \( p_i \) returns a \( w' \)-bit string whose first \( i \) bits are significant (and can be either 0 or 1) and whose remaining bits are all 0’s. In addition, if we let \( M_d \) denote a bit mask with \( d \) significant 1 bits and \( w' - d \) following 0 bits, then, for \( d \geq 2 \),

\[
p_d(k) \land M^{d-1} = p_{d-1}(k),
\]

for any key \( k \).

### A.2 Operations in an Atomic Stash

Let us now describe how we perform the various update and query operations on an atomic stash. Assume we may use the following primitive operations in our methods:

- A binary operator, “⊕,” which denotes the bit-size XOR operation.
- **DUPLICATE**(*B*): return a word \( Z \) having the binary key \( B \) (of size \( w' \)) stored in each of its fields. (Note: we can implement DUPLICATE either using a single multiplication or using \( O(1) \) instructions of a modern CPU.)
- **VecEQ**(*W*, *B*): given a word \( W \) and a binary key \( B \) (of size \( w' \)), set the indicator bit \( W[i] = 1 \) iff \( B = W(i) \). This operation can be implemented using standard AC0 operations [2][40].
- **MSB**(*W*): return the index of the most significant 1 bit in the word \( W \), or 0 if \( W = 0 \). As Thorup observed [40], this operation can be implemented, for example, by converting \( W \) to floating point and returning the exponent plus 1.

We perform a **getIndex**(*k*) operation, which returns the index of key \( k \) in \( X \), or 0 if \( k \) is not a key in \( S \), as follows. We assume in this method that we have access to the current compression function, \( p_d \).

**getIdx**(*k*):

\[
B \leftarrow p_d(k) \\
Z \leftarrow \text{DUPLICATE}(B) \\
T \leftarrow Y \land (X \oplus Z) \\
R \leftarrow U \oplus T \\
\text{VecEQ}(R, \bar{1}) \\
\text{return } \text{MSB}(R)/(w' + 1)
\]

The correctness of this method follows from the fact that \( B \) is the key in \( S \) at index \( i \) associated with \( k \) iff \( T(i) \) is all 0’s and index \( i \) is being used to store a key from \( S \), since all masked keys in \( S \) are unique (and
if $B$ is not in $X$, then $\text{MSB}(R) = 0$). Also, if one desires that we should avoid integer division, then we can define $w'$ so that $(w' + 1)$ is a power of 2, while keeping $w \geq w'(w' + 1)$, so that above division can be done as a SHIFT. In addition, note that we can implement a get($k$) operation, by performing a call to getIndex($k$) and, if the index, $i$, returned is greater than 0, then returning $(K[i], V[i])$.

To remove the key-value pair from $S$ associated with a key $k$ in $S$, we perform the following operation:

- remove($k$):
  
  $i \leftarrow$ get($k$)
  $X(i) \leftarrow X(n_S)$
  $Y(i) \leftarrow Y(n_S)$
  $U(n_S) \leftarrow 0$
  $K[i] \leftarrow K[n_S]$
  $V[i] \leftarrow V[n_S]$
  $n_S \leftarrow n_S - 1$

Our insertion method, which follows, assumes that we have access to the current compression function, $p_d$, as well as the mask, $M^d$, of $d$ 1’s followed by $(w' - d)$ 0’s. We also assume we know the current value of (the global variable) $d$ and that we have a function, P_EXPAND($k_1$, $k_2$), which takes two distinct keys, $k_1$ and $k_2$, and expands the compression function from $p_d$ to $p_{d+1}$ so that $p_{d+1}(k_1) \neq p_{d+1}(k_2)$. This method also defines $M^{d+1}$ to consist of $(d + 1)$ significant 1 bits and $w' - (d + 1)$ trailing 0’s, and it increments $d$.

- add($k$, $v$):
  
  $i \leftarrow$ get($k$)
  if $i > 0$ then {We had a collision.}
  P_EXPAND($k$, $K[i]$) {Also increments $d$}
  $X(i) \leftarrow p_d(K[i])$
  $Y(i) \leftarrow M^d$
  $n_S \leftarrow n_S + 1$
  $X(n_S) \leftarrow p_d(k)$
  $Y(n_S) \leftarrow M^d$
  $U(n_S) \leftarrow 1$
  $K[n_S] \leftarrow k$
  $V[n_S] \leftarrow v$

Since the compressed keys in $S$ form a set of unique masked keys, the new key, $k$, will collide with at most one of them, even when compressed. So, it is sufficient in this case that we expand the compression function so that it distinguishes $k$ and this one colliding field. Thus, a simple inductive argument, we maintain the property that all the masked keys in $S$ are unique.

### A.3 Compressing Keys and De-Amortization

Let us now describe how we compress keys and expand the compression function, as well as how we perform the necessary de-amortization so that the size of compressed keys is never more than $w'$.

Our method makes use of the following primitive AC0 operation [2][40], which can also be computed from included primitives in modern CPU instruction sets.

- SELECT($W$, $k$): Given a word $W$ and key $k$, the fields of $W$ are viewed as bit pointers. A length $w'$ bit string, $B$, is returned so that the $i$-th bit of $B$ equals the $W(i)$-th bit of $k$.

Our compression function is encoded in terms of the counter, $d$, and a word, $W$, that encodes the bits to be selected from keys, so that $W(i)$ is the index of the $i$th bit of $k$ in the output of $p_d(k)$, for $i \leq d$. For $i > d$, $W(i) = 0$. Thus, we define $p_d$ simply as follows.
• \texttt{pd}(k):
  \textbf{return} \texttt{SELECT}(W, k)

Thus, we also have a simple definition for \texttt{P\_EXPAND}:

• \texttt{P\_EXPAND}(k_1, k_2):
  \begin{align*}
  d & \leftarrow d + 1 \\
  W(d) & \leftarrow \text{MSB}(k_1 \oplus k_2) \\
  M^d & \leftarrow M^{d-1} \lor (1 \text{ SHIFT } d)
  \end{align*}

Note that in the way we are using the expansion function, \texttt{pd}, we only ever add bits to the ends of our compressed keys. We never remove bits. This allows keys compressed under previous instantiations of the compression function to continue to be used. But this also implies that, as we continue to add keys to \textit{S}, we might run out of fields to use, even if we keep the size, \textit{nS}, of \textit{S} to be at most, say, \textit{w'}/2. Of course, we could use a standard amortized rebuilding scheme to maintain \textit{d} to be at most \textit{w'}, but this would require using amortization instead of achieving true worst-case constant-time bounds for our updates and lookups.

As a de-amortization technique, therefore, let us revise our construction, so that, whenever \textit{d} \succ \textit{w'}/2, we create a new, initially empty, atomic stash. For each additional atomic-stash operation after this initialization, we remove two items from the old atomic stash and add them to this new atomic stash. In performing accesses and updates during this time, we also keep a crossover index, \textit{x}, so that references to fields and indicator indices less than or equal to \textit{x} are done in the new stash and references to fields and indicator indices greater than \textit{x} are done in the old stash. Thus, after \textit{w'}/2 additional operations, we can discard the old stash (for which \textit{d} \leq \textit{w'}) and fully replace it with the new one (for which \textit{d} \leq \textit{w'}/2). Therefore, so long as \textit{nS} \leq \textit{w'}/2, we can perform all insertion, deletion, and lookup operations in worst-case \textit{O}(1) time, which in the I/O model corresponds to \textit{O}(1) I/Os.