Ordinal and Horizontal Sums
Constructing PBZ∗–Lattices

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Abstract

PBZ∗–lattices are algebraic structures related to quantum logics, which consist of bounded lattices endowed with two kinds of complements, named Kleene and Brouwer, such that the Kleene complement satisfies a weakening of the orthomodularity condition and the De Morgan laws, while the Brouwer complement only needs to satisfy the De Morgan laws for the pairs of elements with their Kleene complements. PBZ∗–lattices form a variety PBZL∗, which includes the variety OML of orthomodular lattices (considered with an extended signature, by letting their two complements coincide) and the variety V(AOL) generated by the class AOL of antiortholattices.

We investigate the congruences of antiortholattices, in particular of those obtained through certain ordinal sums and of those whose Brouwer complements satisfy the De Morgan laws, infer characterizations for their subdirect irreducibility and prove that even the lattice reducts of antiortholattices are directly irreducible. Since the two complements act the same on the lattice bounds in all PBZ∗–lattices, we can define the horizontal sum of any nontrivial PBZ∗–lattices, obtained by gluing them at their smallest and at their largest elements; a horizontal sum of two nontrivial PBZ∗–lattices is a PBZ∗–lattice exactly when at least one of its summands is an orthomodular lattice. We investigate the algebraic structures and the congruence lattices of these horizontal sums, then the varieties they generate.

We obtain a relative axiomatization of the variety V(OML ⊞ AOL) generated by the horizontal sums of nontrivial orthomodular lattices with nontrivial antiortholattices w.r.t. PBZL∗, as well as a relative axiomatization of the join of varieties OML ∨ V(AOL) w.r.t. V(OML ⊞ AOL).

Keywords: PBZ∗–lattice, orthomodular lattice, antiortholattice, ordinal sum, horizontal sum, subdirect irreducibility, lattice of subvarieties, relative axiomatization.

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1 Introduction

Let H be complex separable Hilbert space, and let \( \mathcal{E}(H) \) be the set of all effects of H, i.e., the set of all positive linear operators of H that are bounded by the identity operator I. Within the unsharp approach to quantum logic, it has been argued at length (see e.g. [10, Ch. 4]) that effects are a more adequate mathematical counterpart than projection operators of the notion of quantum event, in that the latter do not form the largest set of operators that can be assigned a probability value according to the Born rule. However, if we order
the effects in $\mathcal{E}(H)$ under the natural order determined by the set of all density operators of $H$ via the trace functional — namely, if we let, for all $E, F \in \mathcal{E}(H)$,

$$E \leq F \text{ iff for all density operators } \rho \text{ of } H, \quad Tr(\rho E) \leq Tr(\rho F),$$

the poset $(\mathcal{E}(H), \leq)$ has the drawback of failing, in general, to be a lattice.

On the other hand, consider the structure

$$\mathbf{E}(H) = (\mathcal{E}(H), \land_s, \lor_s, \sim, \mathcal{O}, I),$$

where:

- $\land_s$ and $\lor_s$ are the meet and the join, respectively, of the spectral ordering $\leq_s$ so defined for all $E, F \in \mathcal{E}(H)$:

$$E \leq_s F \text{ iff } \forall \lambda \in \mathbb{R} : M^E(\lambda) \leq M^F(\lambda),$$

where for any effect $E$, $M^E$ is the unique spectral family [21, Ch. 7] such that $E = \int_{-\infty}^{\infty} \lambda \ dM^E(\lambda)$ (the integral is here meant in the sense of norm-converging Riemann-Stieltjes sums [25, Ch. 1]);

- $\mathcal{O}$ and $I$ are the null and identity operators, respectively;

- $E' = I - E$ and $E^\sim = P_{\ker(E)}$ (the projection onto the kernel of $E$).

The operations in $\mathbf{E}(H)$ are well-defined. The spectral ordering is indeed a lattice ordering [24, 15] that coincides with the natural order when both orderings are restricted to the set of projection operators of the same Hilbert space.

The papers [12], [13] and [14] contain the beginnings of an algebraic investigation of a variety of lattices with additional structure, the variety $\text{PBZL}^*$ of $\text{PBZ}^*$-lattices. A $\text{PBZ}^*$-lattice can be viewed as an abstraction from this concrete physical model, much in the same way as an orthomodular lattice can be viewed as an abstraction from its substructure consisting of projection operators only. The faithfulness of $\text{PBZ}^*$-lattices to the physical model whence they stem is further underscored by the fact that they reproduce at an abstract level the "collapse" of several notions of sharp physical property that can be observed in $\mathbf{E}(H)$.

Further motivation for the study of $\text{PBZL}^*$ comes from its universal algebraic properties. For a start, $\text{PBZ}^*$-lattices can be seen as a common generalisation of orthomodular lattices [1] and of Kleene algebras [20] with an additional unary operation. In the lattice of subvarieties of $\text{PBZL}^*$, moreover, we happen to encounter many situations of intrinsic interest in universal algebra: to name a few, subtractive varieties with equationally definable principal ideals that fail to be point-regular [2]; binary discriminator varieties [8, 2]; ternary discriminator varieties generated by a single finite non-primal algebra.

Regarding their similarity type, $\text{PBZ}^*$-lattices have, in addition to their bounded lattice structure, two unary operations, out of which one is a lattice involution, called Kleene complement, and the second is called Brouwer complement; the bounded involution lattice reduct of a $\text{PBZ}^*$-lattice has to satisfy a weakening of the orthomodularity condition, which is called paranormorthomodularity; and, while the Brouwer complement does not satisfy the De Morgan laws, as the involution does, it is required to satisfy them for all pairs of elements with their Kleene complements; this latter property is called condition ($*$).
This paper is concerned with the study of ordinal and horizontal sums producing PBZ*-lattices. Informally, the ordinal sum of a lattice \( A \) with a largest element \( 1_A \) and a lattice \( B \) with a smallest element \( 0_B \) is a lattice \( A \oplus B \) obtained by gluing \( A \) and \( B \) at the \( 1_A \) and \( 0_B \), while the horizontal sum of two non–trivial bounded lattices \( L \) and \( M \) is the non–trivial bounded lattice \( L \boxplus M \) obtained by gluing \( L \) and \( M \) at their smallest elements, as well as at their largest elements. If \( H \) is a non–trivial bounded lattice, \( H^d \) is the dual of \( H \) and \( K \) is a pseudo–Kleene algebra (that is a bounded involution lattice in which any meet of an element and its involution is smaller than any join of an element and its involution), then the ordinal sum \( H \oplus K \oplus H^d \) can be organized as an antiortholattice, that is a PBZ*-lattice with no other sharp elements beside 0 and 1, with the clear definition for the involution and the trivial Brouwer complement, which takes 0 to 1 and all other elements to 0. Since both complements take 0 to 1 and 1 to 0, we can define the horizontal sum of two non–trivial PBZ*-lattices \( L \) and \( M \), obtained by defining the Kleene and Brouwer complements on the horizontal sum of the bounded lattice reducts of \( L \) and \( M \) by restriction, that is such that \( L \) and \( M \) become subalgebras of \( L \boxplus M \); however, while, in this way, \( L \boxplus M \) becomes an algebra of the same similarity type as PBZ*-lattices, it does not become a PBZ*-lattice unless at least one of \( L \) and \( M \) is an orthomodular lattice (organized as a PBZ*-lattice by letting its Brouwer complement equal its involution). We study the algebraic structures and congruences of these glued sums producing PBZ*-lattices. There is a well-developed theory of horizontal sums in the context of orthomodular lattices (see e.g. [1, 7, 17]), but the case of PBZ*-lattices differs substantially from this particular one, as we learn by examining the congruences, the singleton-generated subalgebras and the sets of sharp and of dense elements of horizontal sums of PBZ*-lattices.

In the final section of this paper, we study the subvarieties generated by horizontal sums of the variety \( \text{PBZL}^* \) of the PBZ*-lattices. We axiomatize the variety \( \text{V}(\text{OML} \boxplus \text{AOL}) \) generated by the horizontal sums of orthomodular lattices with antiortholattices with respect to \( \text{PBZL}^* \), as well as the varietal join \( \text{OML} \lor \text{V}(\text{AOL}) \) of the variety of orthomodular lattices with the variety generated by the class of antiortholattices with respect to \( \text{V}(\text{OML} \boxplus \text{AOL}) \). These results yield an alternate proof for the axiomatization of \( \text{OML} \lor \text{V}(\text{AOL}) \) relative to \( \text{PBZL}^* \) that we have obtained in [14].

2 Preliminaries

We will often use the results in this section without referencing them.

2.1 Notations for Lattices and Universal Algebras

We refer the reader to [4, 16] for the following universal algebra notions and to [15] for the lattice–theoretical ones.

We will denote by \( \mathbb{N} \) the set of the natural numbers and by \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). For any class \( \mathcal{C} \) of algebras of the same type, \( V(\mathcal{C}) \) will denote the variety generated by \( \mathcal{C} \); so \( V(\mathcal{C}) = HSP(\mathcal{C}) \), where \( H \), \( S \) and \( P \) denote the usual class operators. For any algebra \( A \) and any class operator \( O \), \( O(A) \) will be shorthand for \( O(\{A\}) \) and, whenever we need to specify the type, if \( \mathcal{C} \) is a subclass of a variety \( \mathcal{V} \) of algebras of the same type and \( A \in \mathcal{V} \), we will denote \( O_\mathcal{V}(\mathcal{C}) \) and \( O_\mathcal{V}(A) \) instead of \( O(\mathcal{C}) \) and \( O(A) \), respectively. The join of varieties will be denoted by \( \lor \). We will consider only algebras with a nonempty universe; by trivial algebra
we mean a one–element algebra. For brevity, we denote by $A \cong B$ the fact that two algebras $A$ and $B$ of the same type are isomorphic. We make the following convention: if $A$ is an algebra, then $A$ will denote the universe of $A$, with the exception of partition, equivalence and congruence lattices, that will be designated by their universes; other such exceptions will be specified later; sometimes, for brevity, algebras will be designated by their set reducts without being specified as such. If $A$ is a member of a variety $V$ and $S \subseteq A$, then $\langle S \rangle_V$ or $\langle S \rangle_{V,A}$ will denote the subalgebra of $A$ generated by $S$, as well as its universe; if $a \in A$, then $(\langle a \rangle)_V$ will simply be denoted by $\langle a \rangle_V$ or $\langle a \rangle_V$. For any subalgebras $B$ and $C$ of an algebra $A$, $B \cap C$ will denote the subalgebra of $A$ with universe $B \cap C$.

The dual of any (bounded) lattice $L$ will be denoted by $L^d$. For any bounded lattice $L$, $\text{At}(L)$ and $\text{CoAt}(L)$ will denote the set of the atoms and that of the coatoms of $L$, respectively. For any lattice $L$ and any $a, b \in L$, $[a]$ and $[a]$ will be the principal filter, respectively principal ideal of $L$ generated by $a$, and $[a, b] = [a] \cap [b]$ will be the interval of $L$ bounded by $a$ and $b$. For any $n \in \mathbb{N}^+$, $D_n$ denotes the $n$–element chain, regardless of the algebraic structure with a bounded lattice reduct we consider on it, and $D_n$ denotes its universe. We will use Grätzer’s notation for lattices [15].

It will denote the disjoint union of sets and, for any set $S$, $|S|$ will be the cardinality of $S$, $\mathcal{P}(S)$ will be the set of the subsets of $S$, $\langle (\{x\} : x \in S), \{S\} \rangle$ and $\langle \{x\} : x \in S, \{S\} \rangle$ will be the bounded lattices of the partitions and the equivalences of $S$, respectively, and $eq : \text{Part}(S) \rightarrow \mathcal{E}(S)$ will be the canonical lattice isomorphism; for any $n \in \mathbb{N}^+$ and any $\{S_1, \ldots, S_n\} \in \text{Part}(S)$, $\mathcal{E}(\langle S_1, \ldots, S_n \rangle)$ will simply be denoted by $\mathcal{E}(S_1, \ldots, S_n)$. Also, for any $U \subseteq S^2$ and any $\sigma \in \mathcal{E}(S)$, we denote by $U/\sigma = \{\{x, y\}/\sigma : (x, y) \in U\}$.

Let $\mathcal{V}$ be a variety of algebras of a similarity type $\tau$ and $A \in \mathcal{V}$. Then, for any $n \in \mathbb{N}^+$, any terms $t(x_1, \ldots, x_n)$ and $u(x_1, \ldots, x_n)$ over $\tau$ with at most the variables $x_1, \ldots, x_n$ and any $M_1, \ldots, M_n \in \mathcal{P}(A)$, we will use the following notation: $A \models_{M_1, \ldots, M_n} t(x_1, \ldots, x_n) \approx u(x_1, \ldots, x_n)$ iff, for all $a_1 \in M_1, \ldots, a_n \in M_n$, $t^A(a_1, \ldots, a_n) = u^A(a_1, \ldots, a_n)$. If the variables in an equation are not numbered, then, by convention, we consider the set of these variables ordered by their order of appearance in that equation in its current writing, from left to right. As usual, if $k \in \mathbb{N}^+$ and, for all $i \in \{1, k\}$, $\gamma_i = (t_i(x_1, \ldots, x_n) \approx u_i(x_1, \ldots, x_n))$ for some terms $t_i(x_1, \ldots, x_n)$ and $u_i(x_1, \ldots, x_n)$ over $\tau$ with at most the variables $x_1, \ldots, x_n$, then $A \models_{M_1, \ldots, M_n} \{\gamma_1, \ldots, \gamma_k\}$ will be short for $A \models_{M_1, \ldots, M_n} \gamma_i$ for all $i \in \{1, k\}$.

We will denote by $\text{Con}(A)$ the congruence lattice of $A$ (with respect to $\tau$); if, for some $n \in \mathbb{N}^+$, $\tau$ contains constants $\kappa_1, \ldots, \kappa_n$, then we will denote $\text{Con}_{\kappa_1, \ldots, \kappa_n}(A) = \{\theta \in \text{Con}(A) : (\forall i \in \{1, n\})(\kappa_i^A/\theta = \{\kappa_i^A\})\}$. Recall that $A$ is subdirectly irreducible in $V$ iff $A$ is trivial or $\Delta_A$ is strictly meet–irreducible in the lattice $\text{Con}(A)$. For any $U \subseteq A^2$, we will denote by $CG_U(A)$ the congruence of $A$ (with respect to $\tau$) generated by $U$; for any $a, b \in A$, the principal congruence $CG_U(\{\langle a, b \rangle\})$ will simply be denoted by $CG_U(a, b)$. For any $S \subseteq A$, the $\tau$–subalgebra of $A$ generated by $S$ will be denoted by $\langle S \rangle_V$ and so will its universe. If $\mathcal{V}$ is the variety of bounded lattices, then the index $\mathcal{V}$ in the previous notations will be omitted.

Let $L = (L, \leq)$ be a lattice with greatest element $1^L$, and $M = (M, \leq)$ be a lattice with least element $0^M$. Also, let $\varepsilon$ be the equivalence on $L \Pi M$ defined by:

$$\varepsilon = eq(\{1^L, 0^M\} \cup \{x : x \in (L \Pi M) \setminus \{1^L, 0^M\}\}),$$

and let $L \oplus M = (L \Pi M)/\varepsilon$. Note that $\varepsilon \cap L^2 = \Delta_L \in \text{Con}(L)$, thus $L \cong$
\[ L/\Delta_L = L/(\varepsilon \cap L^2), \text{ and } \varepsilon \cap M^2 = \Delta_M \in \text{Con}(M), \text{ thus } M \cong M/\Delta_M = M/(\varepsilon \cap M^2), \text{ so we can identify } L \text{ with } L/(\varepsilon \cap L^2) = (L/\varepsilon, \leq L/\varepsilon) \text{ and } M \text{ with } M/(\varepsilon \cap M^2) = (M/\varepsilon, \leq M/\varepsilon), \text{ by identifying } x \text{ with } x/\varepsilon \text{ for each } x \in L \text{ and each } x \in M; \text{ with this identification, we get } 1_L = 0_M \text{ and } L \cap M = \{1_L\} = \{0_M\}. \]

Then the \textit{ordinal sum} of \( L \) and \( M \) is the lattice \( L \oplus M = (L \oplus M, \leq L \oplus M) \), where:

\[ \leq L \oplus M \leq L \cup \leq M \cup \{(x, y) : x \in L, y \in M\}. \]

For any \( \alpha \in \text{Con}(L) \) and \( \beta \in \text{Con}(M) \), we let:

\[ \alpha \oplus \beta = eq((L/\alpha \setminus \{1^L/\alpha\}) \cup (M/\beta \setminus \{0^M/\beta\}) \cup \{1^L/\alpha \cap 0^M/\beta\}) \in \text{Con}(L \oplus M). \]

Clearly, the ordinal sum of bounded lattices and the attendant operation on congruences are both associative operations, and the map \( (\alpha, \beta) \mapsto \alpha \oplus \beta \) is a lattice isomorphism from \( \text{Con}(L) \times \text{Con}(M) \) to \( \text{Con}(L \oplus M) \).

Let \((L_i)_{i \in I}\) be a non–empty family of nontrivial bounded lattices, with \( L_i = (L_i, \leq L_i, 0^{L_i}, 1^{L_i}) \) for all \( i \in I \). Also, let \( \varepsilon \) be the equivalence on \( \prod_{i \in I} L_i \) defined by:

\[ \varepsilon = eq(\{\{0^{L_i} : i \in I\}, \{1^{L_i} : i \in I\}\} \cup \{\{x\} : x \in \prod_{i \in I} (L_i \setminus \{0^{L_i}, 1^{L_i}\})\}), \]

and let \( \boxplus_{i \in I} L_i = (\prod_{i \in I} L_i)/\varepsilon \). Note that, for all \( i \in I, \varepsilon \cap L_i^2 = \Delta_L \in \text{Con}(L_i) \), thus \( L_i \cong L_i/\Delta_L = L_i/(\varepsilon \cap L_i^2) \). For each \( i \in I \), we identify \( L_i \) with \( L_i/(\varepsilon \cap L_i^2) = (L_i/\varepsilon, \leq L_i/\varepsilon, 0^{L_i}/\varepsilon, 1^{L_i}/\varepsilon) \), by identifying \( x \) with \( x/\varepsilon \) for each \( x \in L_i \).

The \textit{horizontal sum} of the family \((L_i)_{i \in I}\) is the bounded lattice:

\[ \boxplus_{i \in I} L_i = (\prod_{i \in I} L_i, \leq \boxplus_{i \in I} L_i, 0^{\boxplus_{i \in I} L_i}, 1^{\boxplus_{i \in I} L_i}), \]

where \( 0^{\boxplus_{i \in I} L_i} = 0^{L_i} \) and \( 1^{\boxplus_{i \in I} L_i} = 1^{L_i} \) for each \( j \in I \), and \( \leq \boxplus_{i \in I} L_i = \bigcup_{i \in I} \leq L_i \).

If \( \alpha_i \in \text{Eq}(L_i) \setminus \{\Delta_{L_i}\} \) for all \( i \in I \), then we denote by \( \boxplus_{i \in I} \alpha_i \), the equivalence on \( \prod_{i \in I} L_i \) defined by:

\[ \boxplus_{i \in I} \alpha_i = eq(\bigcup_{i \in I} (L_i/\alpha_i \setminus \{0^{L_i}/\alpha_i, 1^{L_i}/\alpha_i\}) \cup \bigcup_{i \in I} (0^{L_i}/\alpha_i, 1^{L_i}/\alpha_i)). \]

Note that, for any nontrivial bounded lattice \( L \), \( D_2 \boxplus L = L \) and \( \Delta_{D_2} \boxplus \alpha = \alpha \) for any \( \alpha \in \text{Eq}(L) \setminus \{\Delta_L\} \). Clearly, the binary operation \( \boxplus \) on nontrivial bounded lattices is associative and commutative, and so is the attendant operation on proper equivalences of the universes of those lattices.

### 2.2 Congruences with Singleton Classes and Generated Subalgebras

**Theorem 1** [16, Corollary 2, p. 51] \( \text{The congruence lattice of any algebra is a complete sublattice of the equivalence lattice of its set reduct.} \)

**Corollary 2** \( \text{The congruence lattice of any algebra is a complete sublattice of the congruence lattice of any of its reducts.} \)

**Lemma 3** (i) \( \text{If } M \text{ is a set, } \emptyset \neq S \subseteq M \text{ and } \sigma \in \text{Part}(S), \text{ then } P = \{\pi \in \text{Part}(M) : \sigma \subseteq \pi\} \text{ and } E = \{\varepsilon \in \text{Eq}(M) : \sigma \subseteq M/\varepsilon\} \text{ are complete sublattices of } \text{Part}(M) \text{ and } \text{Eq}(M), \text{ respectively, in particular they are bounded lattices.} \)

(ii) \( \text{If } A \text{ is an algebra from a variety } \mathcal{V}, \emptyset \neq S \subseteq A \text{ and } \sigma \in \text{Part}(S) \text{ is such that the set } C = \{\emptyset \in \text{Conv}(A) : \sigma \subseteq A/\emptyset\} \text{ is non–empty, then } C \text{ is a complete sublattice of } \text{Conv}(A), \text{ in particular it is a bounded lattice.} \)
(iii) Let $\mathbb{V}$ be a variety of algebras of a similarity type $\tau$, $n \in \mathbb{N}^*$ and $\kappa_1, \ldots, \kappa_n$ be constants in $\tau$. If $A$ is a member of $\mathbb{V}$ such that $\text{Conv}_{\kappa_1, \ldots, \kappa_n}(A)$ is non–empty, then $\text{Conv}_{\kappa_1, \ldots, \kappa_n}(A)$ is a complete sublattice of $\text{Conv}(A)$, in particular it is a bounded lattice.

**Proof.**

Note that, in the statement of the lemma, by $\subseteq$ we mean set inclusion, not the partitions ordering, so that, for any $\pi \in \text{Part}(M)$, $\sigma \subseteq \pi$ means that, for each $x \in S$, $x/eq(\sigma) = x/eq(\pi)$. $S/\sigma \cup \{(M \setminus S) \setminus \{0\}\} \subseteq P$, thus $P \neq \emptyset$, hence $E = eq(P) \neq \emptyset$.

If $S = M$, then, for any $\pi \in \text{Part}(M)$, $\sigma \subseteq \pi$ is equivalent to $\sigma = \pi$, thus, in this case, $P = \{\sigma\}$ and $E = \{eq(\sigma)\}$, therefore $P$ and $E$ are trivial, thus complete sublattices of $\text{Part}(M)$ and $\text{Eq}(M)$, respectively.

If $S \subseteq M$, then, for any $\emptyset \neq \{\pi_i : i \in I\} \subseteq P$ and any $j \in I$, the fact that $\sigma \subseteq \pi_j$ shows that $\pi_j \setminus \sigma \in \text{Part}(M \setminus S)$, and hence $\bigwedge_{i \in I} \pi_i = \sigma \cup \bigvee (\pi_j \setminus \sigma) \supseteq \sigma$ and $\bigvee_{i \in I} \pi_i = \sigma \cup \bigwedge (\pi_j \setminus \sigma) \supseteq \sigma$, therefore $\bigwedge_{i \in I} \pi_i, \bigvee_{i \in I} \pi_i \in P$, hence $P$ is a complete sublattice of $\text{Part}(M)$, thus $E = eq(P)$ is a complete sublattice of $\text{Eq}(M)$.

$\{\pi \in \text{Part}(M) : \sigma \subseteq \pi\} \subseteq E$, so that, if $C \neq \emptyset$, then, by (i) and Theorem 1, for any $\emptyset \neq \{\gamma_i : i \in I\} \subseteq C \subseteq E$, we have $\bigwedge_{i \in I} \gamma_i, \bigvee_{i \in I} \gamma_i \in E \cap \text{Conv}(A) = C$, hence $C$ is a complete sublattice of $\text{Conv}(A)$.

This is a particular case of (ii). $\blacksquare$

**Lemma 4**

(i) If $\mathbb{V}$ is a variety and $(A_i)_{i \in I} \subseteq \mathbb{V}$ is a non–empty family such that $\prod_{i \in I} A_i$ has no skew congruences, then $\text{Conv}_{\kappa_1, \ldots, \kappa_n}(\prod_{i \in I} A_i) = \prod_{i \in I} \alpha_i : (\forall i \in I) (\alpha_i \in \text{Conv}_{\kappa_1, \ldots, \kappa_n}(A_i)) \cong \prod_{i \in I} \text{Conv}_{\kappa_1, \ldots, \kappa_n}(A_i)$.

(ii) If $\mathbb{V}$ is a variety of bounded lattice–ordered structures, then, for all $A, B \in \mathbb{V}$, $\text{Conv}_{01}(A \times B) = \{\alpha \times \beta : \alpha \in \text{Conv}_{01}(A), \beta \in \text{Conv}_{01}(B)\} \cong \text{Conv}_{01}(A) \times \text{Conv}_{01}(B)$.

**Proof.**

(i) Routine. $\blacksquare$

Let $\mathbb{V}$ be a variety of similar algebras, $A$ a member of $\mathbb{V}$, $\theta \in \text{Conv}(A)$, $S \subseteq A$, $B \in Sy(A)$, $(A_i)_{i \in I}$ a non–empty family of members of $\mathbb{V}$ and, for all $i \in I$, $S_i \subseteq A_i$. Then:

(i) $\langle \prod_{i \in I} S_i \rangle_{\mathbb{V}, \prod_{i \in I} A_i} = \prod_{i \in I} \langle S_i \rangle_{\mathbb{V}, A_i}$;

(ii) $\langle S \cap B \rangle_{\mathbb{V}, B} = \langle S \cap B \rangle_{\mathbb{V}, A} \cap B \subseteq \langle S \rangle_{\mathbb{V}, A} \cap B$, where the converse of the inclusion doesn’t always hold;

(iii) $\langle S \rangle_{\mathbb{V}, A/\theta} = \langle S/\theta \rangle_{\mathbb{V}, A/\theta}$.

Indeed, (i) is clear and so is the inclusion in (iii), while, if we replace $\mathbb{V}$ by the variety of lattices, $A$ by the five–element modular non–distributive lattice $M_3$, $B$ by the sublattice of $M_3$ with universe $B = \{0, a, b, 1\}$ and $S$ by the set $\{b, c\}$, where $a, b, c$ are the three atoms of $M_3$, then we get a counter–example for the converse of the inclusion in (iii). The fact that $S \cap B \subseteq \langle S \cap B \rangle_{\mathbb{V}, A} \cap B$ and $(S \cap B)_{\mathbb{V}, A} \cap B \in Sy(B)$ proves that $(S \cap B)_{\mathbb{V}, B} \subseteq \langle S \cap B \rangle_{\mathbb{V}, A} \cap B$ and, if we consider an element $b \in \langle S \cap B \rangle_{\mathbb{V}, A} \cap B$, then, for some $n \in \mathbb{N}$, $b_1, \ldots, b_n \in S \cap B$ and some term $t$ in the language of $\mathbb{V}$, we have $b = t^A(b_1, \ldots, b_n) = t^B(b_1, \ldots, b_n) \in \langle S \cap B \rangle_{\mathbb{V}, B}$, thus $(S \cap B)_{\mathbb{V}, A} \cap B \subseteq \langle S \cap B \rangle_{\mathbb{V}, B}$, so (iii) holds.

Finally, $S/\theta \subseteq \langle S \rangle_{\mathbb{V}, A/\theta} \in Sy(A/\theta)$ and thus $\langle S/\theta \rangle_{\mathbb{V}, A/\theta} \subseteq \langle S \rangle_{\mathbb{V}, A/\theta}$, while, if we consider an $a \in \langle S \rangle_{\mathbb{V}, A}$, then, for some $n \in \mathbb{N}$, $a_1, \ldots, a_n \in S$
and some term $t$ in the language of $\forall$, we have $a = t^A(a_1, \ldots, a_n)$, thus $a/\theta = t^{A/\theta}(a_1/\theta, \ldots, a_n/\theta) \in \langle S/\theta \rangle_{\forall, A/\theta}$, hence $\langle S \rangle_{\forall, A/\theta} \subseteq \langle S/\theta \rangle_{\forall, A/\theta}$, which concludes the proof of (iii).

2.3 PBZ*-Lattices: Definitions, Notations and Previously Established Properties

We recall some preliminary notions on PBZ*-lattices and related structures only to such an extent as is necessary for the purposes of the present paper. For additional information on bounded involution lattices and pseudo-Kleene algebras, see [10, 6]; for Kleene lattices, a locus classicus is [20]; for BZ-lattices, see [10, 5]; finally, for PBZ*-lattices, see [12, 13, 14].

Definition 5 A bounded involution lattice (in brief, BI-lattice) is an algebra $L = (L, \land, \lor, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \land, \lor, 0, 1)$ is a bounded lattice with induced partial order $\leq$, $a'' = a$ for all $a \in L$, and $a \leq b$ implies $b' \leq a'$ for all $a, b \in L$.

A pseudo–Kleene algebra is a BI-lattice $L$ satisfying, for all $a, b \in L$: $a \land a' \leq b \lor b'$. Distributive pseudo–Kleene algebras are called Kleene algebras or Kleene lattices.

Note that, for any BI-lattice $L$, $'$ : $L \rightarrow L$ is a dual lattice automorphism of $L$, called involution. The involution of a pseudo–Kleene algebra is called Kleene complement. If $L$ is a BI-lattice, then for any $U \subseteq L$ and any $X \subseteq L^2$ we set $U' = \{ u' : u \in U \}$ and $X' = \{ (x', y') : (x, y) \in X \}$.

For every algebra $A$, if $A$ has a (bounded) lattice reduct, then this reduct will be denoted by $A_i$, and, if $A$ has a BI-lattice reduct, then such a reduct will be denoted by $A_{bi}$. Let $L$ be an algebra having a BI-lattice reduct. We say that an element $a \in L$ is Kleene–sharp or, simply, sharp\footnote{See however the introduction to the present paper or [12] for the several distinct notions of sharp element that collapse in the context of PBZ*-lattices.} iff $a \land a' = 0$, or, equivalently, iff $a \lor a' = 1$. We will denote the set of the sharp elements of $L$ by $S(L)$.

Definition 6 Let $L$ be a BI-lattice. Then:

- $L$ is an ortholattice iff $S(L) = L$;
- $L$ is a paraorthomodular BI-lattice iff, for all $a, b \in L$, if $a \leq b$ and $a' \land b = 0$, then $a = b$;
- $L$ is an orthomodular lattice iff $L$ is an ortholattice and, for all $a, b \in L$, if $a \leq b$, then $b = (b \land a') \lor a$.

If an algebra $A$ has a BI-lattice reduct and $A_{bi}$ is paraorthomodular, then $A$ is said to be paraorthomodular, as well.

Clearly, any ortholattice is a pseudo–Kleene algebra. Note that, if a BI-lattice $L$ is orthomodal, then it is paraorthomodular; however, if $L$ is an ortholattice, then $L$ is orthomodal iff it is paraorthomodular [3, Prop. 2.1].

We denote $MO_0 = D_2$ and, for any non–empty set $I$, $MO_{|I|} = \bigoplus_{i \in I} D_2$. Clearly, for any cardinal number $\kappa$, $MO_\kappa$ is an orthomodular lattice (and a Boolean algebra iff $\kappa \in \{0, 1\}$).
Definition 7 A Brouwer–Zadeh lattice (in brief, BZ–lattice) is an algebra $L = (L, \land, \lor, ', ~, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \land, \lor, ', 0, 1, 0, 1)$ is a pseudo–Kleene algebra and, for all $a, b \in L$:

1. $a \land a' = 0$;
2. $a \leq a''$;
3. $a \leq b$ implies $b'' \leq a''$;
4. $a^* = a''$.

A BZ$^*$–lattice is a BZ–lattice $L$ satisfying the condition:

(*) for all $a \in L$, $(a \land a')'' \leq a' \lor a''$.

A PBZ$^*$–lattice is a paraorthomodular BZ$^*$–lattice.

An antiortholattice is a PBZ$^*$–lattice $L$ with $S(L) = \{0, 1\}$.

The operation $\sim$ of a BZ–lattice is called Brouwer complement. If $L$ is a BZ–lattice, then for any $U \subseteq L$ we set $U^\sim = \{u^\sim : u \in U\}$.

Lemma 8 [10] [12] If $L$ is a BZ–lattice, then, for all $a, b \in L$:

(i) $a' \leq a'$;
(ii) $a'''' = a''$;
(iii) $(a \lor b)'' = a'' \lor b''$;
(iv) $(a \land b)''' \geq a''' \lor b'''$.

Lemma 8(i) shows that, in any BZ–lattice $L$, for any $a \in L$, $a'' = 1$ iff $a = 0$. By Lemma 8(iv), in any BZ–lattice $L$, condition (*) is equivalent to 

$(a \land a')'' = a'' \lor a''$ for all $a \in L$.

In any BZ–lattice $L$, we set $\diamondsuit a = a'''$ and $\Box a = a''$ for all $a \in L$. Note that, if $L$ is a BZ$^*$–lattice, then $L$ is paraorthomodular iff it satisfies the following equational condition: for all $a, b \in L$, $(a'' \lor (\diamondsuit a \land \Box b)) \land \Box a \leq \Box b$, therefore

PBZ$^*$–lattices form a variety.

If $L$ is a PBZ$^*$–lattice, then $S(L) = \{a : a \in L\} = \{a \in L : a = a''\} = \{a \in L : a' = a'\}$ [12]. For every PBZ$^*$–lattice $L$, $S(L)$ is the universe of the largest subalgebra of $L$ which is an orthomodular lattice, denoted by $S(L)$.

Lemma 9 [12] Let $L$ be a PBZ$^*$–lattice. Then:

- $L_{bi}$ is an ortholattice iff $S(L) = L$ iff $L_{bi}$ is an orthomodular lattice;
- $L$ is an antiortholattice iff $x'' = 0$ for all $x \in L \setminus \{0\}$.

The Brouwer complement of an antiortholattice $L$, given by Lemma 9, is called the trivial Brouwer complement: $0'' = 1$ and $x'' = 0$ for all $x \in L \setminus \{0\}$.

We denote by BA, BI, PKA, OL, OML, BZL and PBZL$^*$ the varieties of Boolean algebras, BI–lattices, pseudo–Kleene algebras, ortholattices, orthomodular lattices, BZ–lattices and PBZ$^*$–lattices, respectively. As shown by Lemma 9, OL and OML can be viewed as classes of algebras of type $(2, 2, 1, 0, 0)$, with a repeat occurrence of the unary operation symbol. The proper universal class of antiortholattices will be denoted by $\mathcal{A}OL$. By [14], $V(\mathcal{A}OL)$ is axiomatized relative to $\mathcal{PBZL}^*$ by the equation:

\[ J_0 \ x \approx (x \land y') \lor (x \land y). \]
Clearly, $\text{AO\!L}$ is closed w.r.t. subalgebras and quotients, but not w.r.t. direct products, since Definition 6 ensures us that every antiortholattice is directly indecomposable.

Note that antiortholattices are exactly the paraorthomodular pseudo–Kleene algebras endowed with the trivial Brouwer complement which satisfy condition ($\ast$), or, equivalently, exactly the pseudo–Kleene algebras $\text{L}_2$ with $S(\text{L}) = \{0, 1\}$, endowed with the trivial Brouwer complement. Thus any pseudo–Kleene algebra where 0 is meet–irreducible becomes an antiortholattice once endowed with the trivial Brouwer complement. Hence any $\text{BZ}$–lattice with a meet–irreducible bottom element is an antiortholattice. In particular, any $\text{BZ}$–chain is an antiortholattice and, of course, any self–dual bounded chain becomes an antiortholattice if endowed with its dual lattice automorphism as Kleene complement, and with the trivial Brouwer complement.

Let $\text{M}$ be a bounded lattice, $f : \text{M} \rightarrow \text{M}^d$ be a dual lattice isomorphism and $\text{K}$ be a BI-lattice, with involution $^\text{K}$. In the ordinal sum $\text{M} \oplus \text{K}_1 \oplus \text{M}^d$, we will denote by $\text{M}^d$ the universe of the sublattice $\text{M}^d$. Then the bounded lattice $\text{M} \oplus \text{K}_1 \oplus \text{M}^d$ can be made into a BI-lattice $\text{M} \oplus \text{K} \oplus \text{M}^d$ by defining its involution as follows:

$$x' = \begin{cases} f(x), & \text{for all } x \in \text{M}; \\ x'^\text{K}, & \text{for all } x \in \text{K}; \\ f^{-1}(x), & \text{for all } x \in \text{M}^d. \end{cases}$$

In this BI-lattice, $M' = M^d$. Clearly, $\text{M} \oplus \text{K} \oplus \text{M}^d$ is a pseudo–Kleene algebra iff $\text{K}$ is a pseudo–Kleene algebra. The pseudo–Kleene algebra $\text{M} \oplus \text{D}_1 \oplus \text{M}^d$ will be denoted by $\text{M} \oplus \text{M}^d$, as its underlying bounded lattice.

Let $\text{A}$ and $\text{B}$ be nontrivial BI-lattices, with involutions $^\text{A}$ and $^\text{B}$, respectively. Then $\text{A}_1 \oplus \text{B}_1$ can be organized as a BI-lattice $\text{A} \oplus \text{B}$ by letting the involution of $\text{A} \oplus \text{B}$ be $t : \text{A} \oplus \text{B} \rightarrow \text{A} \oplus \text{B}$, $a' = a^\text{A}$ for all $a \in \text{A}$ and $b' = b^\text{B}$ for all $b \in \text{B}$. This makes $\text{A}$ and $\text{B}$ subalgebras of the BI-lattice $\text{A} \oplus \text{B}$.

Now let $\text{A}$ and $\text{B}$ be BZ–lattices, with Brouwer complements $^\sim\text{A}$ and $^\sim\text{B}$, respectively. Then $\text{A}_{bi} \oplus \text{B}_{bi}$ can be organized as an algebra $\text{A} \oplus \text{B}$ of type $(2, 2, 1, 1, 0, 0)$ by defining $^\sim : \text{A} \oplus \text{B} \rightarrow \text{A} \oplus \text{B}$, $a^\sim = a^{^\sim\text{A}}$ for all $a \in \text{A}$ and $b^\sim = b^{^\sim\text{B}}$ for all $b \in \text{B}$. This makes $\text{A}$ and $\text{B}$ subalgebras of $\text{A} \oplus \text{B}$, which is an algebra of type $(2, 2, 1, 1, 0, 0)$, but not necessarily a BZ–lattice.

If $\text{C}$ and $\text{D}$ are subclasses of the variety of bounded lattices or of one of the varieties $\text{BI}$ and $\text{BZL}$, then we let:

$$\text{C} \oplus \text{D} = \{\text{D}_1\} \cup \{\text{A} \oplus \text{B} : \text{A} \in \text{C} \setminus \{\text{D}_1\}, \text{B} \in \text{D} \setminus \{\text{D}_1\}\}.$$  

Clearly, the operation $\oplus$ on classes is associative and, if $\text{A}$ is a nontrivial bounded lattice or BI-lattice or BZ–lattice, then $\text{D}_2 \oplus \text{A} = \text{A}$, hence, in the notation above:

- if $\text{D}_2 \in \text{C}$, then $\text{D} \subseteq \text{C} \oplus \text{D}$;
- thus, if $\text{D}_2 \in \text{C} \cap \text{D}$, then $\text{C} \cup \text{D} \subseteq \text{C} \oplus \text{D}$.

For future reference, we consider the following identities in the language of BZ–lattices:

**SDM** (the Strong de Morgan law) $(x \land y)^\sim \approx x^\sim \lor y^\sim$;

**WSDM** (weak SDM) $(x \land y^\sim)^\sim \approx x^\sim \lor \Diamond y$;

**S1** $(x \land (x \land y)^\sim)^\sim \approx x^\sim \lor \Diamond (x \land y)$;

...
Lemma 10 of Stone algebras and other related structures (see e.g. [9]). The case of dense elements in PBZ\(^*\)–lattices. Then:

(i) \( L \) is an orthomodular lattice iff \( S(L) = L \) iff \( T(L) = \{0, 1\} \).

(ii) \( L \) is an antiortholattice iff \( S(L) = \{0, 1\} \) iff \( T(L) = L \).

Remark that \( J_0 \) above implies \( J_1 \) and \( J_2 \) and that SDM implies WSDM, which in turn implies \( S_1, S_2, S_3 \). Also note that, for any BZ–lattice \( L \), \( L \vdash J_1 \) iff, for all \( x, y \in L \), \( y \leq x \) implies \( x = (x \vee y) \wedge (x \wedge y) \). Hence an equivalent form of \( J_1 \) is as follows and, similarly, \( S_1 \) can be written equivalently in the following form:

\( J_1' \) \( x \vee y \cong ((x \lor y) \wedge y) \lor ((x \lor y) \wedge \lor y) \).

\( S_1' \) \((x \lor y) \wedge y \cong (x \lor y) \lor \lor y\).

Clearly, \( \text{OML} \vdash \text{SDM} \), thus \( \text{OML} \vdash \{ \text{WSDM}, S_1, S_2, S_3 \} \), and, in \( \{ L \in \text{EBZL} : L \models x \cong x' \} \), \( J_1 \) is equivalent to the orthomodularity condition, thus \( \text{OML} \vdash J_1 \). Also, clearly, \( \text{OML} \vdash J_2 \).

Trivially, \( \text{AOL} \vdash \{ \text{WSDM}, S_1, S_2, S_3 \} \) and \( \text{AOL} = J_0 \), hence \( \text{AOL} \models \{ J_1, J_2 \} \). The fact that \( J_0 \) axiomatizes \( V(\text{AOL}) \) over \( \text{PBZL}^* \) and \( V(\text{AOL}) \models \text{WSDM} \) shows that \( J_0 \) implies WSDM. Clearly, an antiortholattice \( L \) satisfies SDM iff \( 0 \) is meet–irreducible in \( L \); for instance, \( D_2^1 \oplus D_2^4 \) can be organized as an antiortholattice (see Section 4) that fails SDM. Therefore \( \text{AOL} \not\vdash \text{SDM} \).

Note from the above that \( \text{OML} \lor V(\text{AOL}) \models \{ \text{WSDM}, J_1, J_2, S_1, S_2, S_3 \} \).

3 Dense elements in PBZ\(^*\)–lattices

Whenever a bounded lattice \( L \) is endowed with a closure operator \( C \), important information on the structure of \( L \) is encoded not only in its set \( \{ x \in L : C(x) = x \} \) of closed elements, but also in its set \( \{ x \in L : C(x) = 1 \} \) of dense elements.

Under optimal circumstances, like for Stone algebras, knowledge of both sets — plus some information concerning their distribution in the lattice ordering — is sufficient to fully reconstruct \( L \). This is the idea behind the representation by triples of Stone algebras and other related structures (see e.g. [9]). The case of PBZ\(^*\)–lattices falls somewhat short of such an ideal situation — still, the study of dense elements provides useful insights into their structure.

For any PBZ\(^*\)–lattice \( L \), we call dense any \( a \in L \) such that \( a \cong 0 \). The set of all dense elements of \( L \) will be denoted by \( D(L) \); we also set \( T(L) = \{0\} \cup D(L) \). Clearly, \( S(L) \cap T(L) = \{0, 1\} \) and a subalgebra of \( L \) is included in \( T(L) \) iff it is an antiortholattice; in particular, if \( T(L) \) is the universe of a subalgebra of \( L \), then this subalgebra, that we will denote by \( T(L) \), is the largest subalgebra of \( L \) which is an antiortholattice.

Since \( S(L) \cap T(L) = \{0, 1\} \) and \( S(L)' = S(L) \), we also have \( S(L) \cap T(L)' = S(L) \cap (T(L) \cup T(L)') = \{0, 1\} \).

Lemma 10 Let \( L \) be a PBZ\(^*\)–lattice. Then:

(i) \( L \) is an orthomodular lattice iff \( S(L) = L \) iff \( T(L) = \{0, 1\} \).

(ii) \( L \) is an antiortholattice iff \( S(L) = \{0, 1\} \) iff \( T(L) = L \).
Proof. The only nontrivial item is the right-to-left direction of the second equivalence in (i). Let \( L \in \text{PBZ}^* \setminus \text{OML} \). If \( L \) is isomorphic to \( D_3 \), then \( T(L) \neq \{0,1\} \). Otherwise, there is an \( a \in L \) s.t.

\[
0 < a \land a' = (a \lor a')' \leq a \lor a' < 1,
\]

so, by Lemma 8(v), \( a \lor a' \in T(L) \setminus \{0,1\} \). \( \blacksquare \)

For any \( \text{PBZ}^* \)-lattice \( L \), \( \{0,1\} \) is the universe of the smallest subalgebra of \( L \), which belongs to \( \text{OML} \cap \text{AOl} \), so note from the previous proposition that, if \( L \in \text{OML} \cup \text{AOl} \), then \( S(L) \) and \( T(L) \) are subuniverses of \( L \).

On the other hand, let \( L \) be a generic \( \text{PBZ}^* \)-lattice. Observe that:

- \( L \setminus S(L) \) is closed w.r.t. to the Kleene complement;
- \( T(L) \) is closed w.r.t. to the Brouwer complement, as well as joins, hence \( T(L)' \) is closed w.r.t. meets and: if \( T(L) \) is closed under the Kleene complements, then it is also closed under meets and \( T(L) = T(L)' \);
- if \( D(L) \) is closed w.r.t. meets, then \( D(L) \) is a lattice filter of \( L \) and \( T(L) \) is closed w.r.t. meets;
- \([u] \subseteq D(L) \) for all \( u \in D(L) \), hence \([u] \subseteq D(L)' \) for all \( u \in D(L)' \).

Clearly, if \( L \) satisfies the SDM, then \( D(L) \) is closed w.r.t. meets, thus so is \( T(L) \), hence \( T(L) \) is the universe of a bounded sublattice of \( L \) in which \( 0 \) is meet–irreducible, thus \( T(L)' \) is the universe of a bounded sublattice of \( L \) in which \( 1 \) is join–irreducible. Recall from Subsection 2.3 that, in any antiortholattice which satisfies the SDM, \( 0 \) is meet–irreducible, and that any BZ–lattice with \( 0 \) meet–irreducible is an antiortholattice which satisfies the SDM. Hence, for any BZ–lattice \( L \), the following are equivalent:

- \( L \) satisfies the SDM and \( 0 \) is meet–irreducible in \( L \);
- \( L \) is an antiortholattice and it satisfies the SDM;
- \( L \) is an antiortholattice and \( 0 \) is meet–irreducible in \( L \).

Let us also retain, from the above:

Lemma 11 For any \( \text{PBZ}^* \)-lattice \( L \):

- \( T(L) \) is closed w.r.t. meets iff \( T(L) \) is the universe of a bounded sublattice of \( L \) iff \( T(L)' \) is closed w.r.t. joins iff \( T(L)' \) is the universe of a bounded sublattice of \( L \) iff \( (T(L))_0 = T(L) \cup T(L)' \);
- If \( L \) satisfies the SDM, then \( T(L) \) is the universe of a bounded sublattice of \( L \) in which \( 0 \) is meet–irreducible, so \( D(L) \) is a lattice filter of \( L \);
- \( T(L) \) is closed w.r.t. the Kleene complement iff \( T(L) \) is a subuniverse of \( L \) iff \( T(L) = T(L)' \).

Observe that, if \( L \) is a \( \text{PBZ}^* \)-lattice, then all subalgebras of \( L \) which are closed w.r.t. the Brouwer complement are bounded subalgebras of \( L \). Moreover, for any subsemilattice \( S \) of the underlying join–semilattice of \( L \) or the underlying meet–semilattice of \( L, S \) if \( S \) is closed w.r.t. the Brouwer complement of \( L \), then \( \{0,1\} \subseteq S \). Thus, the only interval of \( L \) which is closed w.r.t. the Brouwer complement is \( [0,1] = L \), so \( L \) has no proper convex subalgebras.

The next lemma will be useful in what follows.
Lemma 12 Let $L$ be a nontrivial PBZ$^*$-lattice that satisfies $J1$ and $u, v \in L \setminus \{0\}$ such that $u \leq v$ and $v$ is join–irreducible in $L_i$. Then:

(i) either $v \lor v^\sim \leq u^\sim$ or $u^\sim = v^\sim$;

(ii) if $v \in S(L)$, then $v = \hat{\Diamond} u$;

(iii) if $u, v \in S(L)$, then $u = v$;

(iv) if $v \in T(L)$, then $u \in T(L)$;

(v) if $v \in T(L)$ and $u \in S(L)$, then $u = v = 1$ and $L$ is an antiortholattice that satisfies the SDM.

Proof. (i) $u \leq v$ implies $v^\sim \leq u^\sim$. Since $L$ satisfies $J1$, we have $v = (v \land u^\sim) \lor (v \land \hat{\Diamond} u)$, so that, by the join–irreducibility of $v$, either $v = v \land u^\sim$ or $v = v \land \hat{\Diamond} u$, so either $v \leq u^\sim$ or $v \leq \hat{\Diamond} u$, hence either $v \lor v^\sim \leq u^\sim$ or $u^\sim = u^\sim \sim v^\sim$, the latter of which implies $u^\sim = v^\sim$.

(ii) If $v \in S(L)$, then $v \lor v^\sim = 1 \not\leq u^\sim$, so, by (i), we have $u^\sim = v^\sim$, thus $v = \hat{\Diamond} v = \hat{\Diamond} u$.

(iii) By (i) and the fact that $\hat{\Diamond} u = u$.

(iv) By (i), we have either $v \leq u^\sim$ or $u^\sim = v^\sim = 0$, so that either $u^\sim \in T(L)$ or $u \in D(L)$, that is either $u^\sim = 1$, which would contradict $u \neq 0$, or $u \in S(L)$, so we have $u^\sim = 1$, thus 1 is join–irreducible in $L_i$, hence $L$ is an antiortholattice and it satisfies the SDM.

Proposition 13 Let $L$ be a nontrivial PBZ$^*$–lattice that satisfies $J1$ and $v \in D(L)$ such that $v$ is join–irreducible in $L_i$. Then:

- all elements of $L$ which are comparable with $v$ belong to $T(L)$;
- for any $x \in L \setminus T(L)$, we have $x \land v = 0$, $x \lor x' \not\geq v$ and $x$ and $x \land x'$ are incomparable to $v$.

Proof. By Lemma 12(iii) and the fact that $D(L)$ is closed w.r.t. upper bounds.

Proposition 14 Let $L$ be a nontrivial orthomodular lattice. Then the only join–irreducible elements of $L_i$ are its atoms and, dually, its only meet–irreducible elements are its coatoms.

Proof. Since $L \in OM_{\Pi}$, we have $L \vdash J1$, so, by Lemma 12(iii), for every $v \in L \setminus \{0\}$ such that $v$ is join–irreducible in $L_i$, there exists no $u \in L$ with $0 < u < v$, hence $v$ is an atom of $L_i$.

Corollary 15 Let $L$ be a nontrivial PBZ$^*$–lattice, $u \in S(L) \setminus \{1\}$ and $v \in S(L) \setminus \{0\}$. Then:

- if $v$ is join–irreducible in $L_i$, then $v$ is an atom of $S(L)$;
- if $u$ is meet–irreducible in $L_i$, then $u$ is a co–atom of $S(L)$.
4 Ordinal Sums and Congruences of Antiortholattices

Note, from Corollary 2 and the characterization of subdirect irreducibility in Subsection 2.1 that, if a reduct of an algebra $A$ is subdirectly irreducible, then so is $A$. We will often use the following lemmas and propositions without referencing them.

Lemma 16 If $M$ is a nontrivial bounded lattice and $K$ is a pseudo-Kleene algebra, then the canonical pseudo-Kleene algebra $L = M \oplus K \oplus M^d$, endowed with the trivial Brouwer complement, becomes an antiortholattice.

Proof. Clearly, for any $x, y \in L$, $x \land y = 0$ implies $x = 0$ or $y = 0$ or $x, y \in M$. Thus, for any $a, b \in L$ such that $a \leq b$ and $a' \land b = 0$, we have one of the following situations:

- $a' = 0$, so that $a = 1$ and thus $b = 1 = a$;
- $b = 0$, so that $a = 0 = b$;
- $a', b \in M$, so that $a \in M^d$, thus $K \cong D_1$, with $K = \{a\} = \{b\}$, so $a = b$.

Therefore $L$ is paraorthomodular.

Now let $\sim : L \to L$ be the trivial Brouwer complement and let $a \in L$. If $a \in K$, then $a' \in K$, so $a \land a' \in K$, thus $0 \notin \{a', a \land a'\}$, hence $(a \land a')^\sim = 0 = a^\sim \lor a'^\sim$. If $a \in M$, then $a' \in M^d$, thus $a \leq a'$, so $a^\sim \leq a'^\sim$, hence $(a \land a')^\sim = a^\sim \lor a'^\sim$. If $a \in M^d$, then $a' \in M$, so $(a \land a')^\sim = a^\sim \lor a'^\sim$, by duality from the previous case. Thus the pseudo-Kleene algebra $L$, endowed with the trivial $\sim$, fulfills condition $(*)$, hence it becomes an antiortholattice. $\Box$

We call the antiortholattice $M \oplus K \oplus M^d$ in the previous lemma the canonical antiortholattice with lattice reduct $M \oplus K \oplus M^d$.

Let $L$ be a BI-lattice, $S = (S, \land, \lor)$ a sublattice of $L$ and $S' = (S', \land, \lor)$. Then $S'$ is also a sublattice of $L$ and, in the particular case when $S = L$, we have $S = S' = L$. The map $\rho : S' \to S$ is a dual lattice isomorphism between $S'$ and $S$, hence $S' \cong S$ and the map $\theta \to \theta'$ from $\Con(S')$ to $\Con(S') = \Con(S)$ is a lattice isomorphism, thus $(\theta \land \zeta)' = \theta' \land \zeta'$ and $(\theta \lor \zeta)' = \theta' \lor \zeta'$ for all $\theta, \zeta \in \Con(S')$. Note, also, that $\theta'' = \theta$ for all $\theta \in \Con(S')$.

If $L = M \oplus K \oplus M^d$ for some bounded lattice $M$ and some BI-lattice $K$, so that, with the notation above, $M^d = M^{\theta}$, then, for any $\theta \in \Con(M)$, $\alpha \in \Con(K)$ and $\zeta \in \Con(M^d)$, we have: $\theta' \in \Con(M^d)$, $\zeta' \in \Con(M)$ and $\theta \oplus \alpha \oplus \zeta \in \Con(L)$, in particular $\theta \oplus \alpha \oplus \theta' \in \Con(L)$. In particular, if $K$ is trivial, thus $\Con(K) = \{\Delta_K\} = \{\nabla_K\} \cong D_1$ and $M \oplus K \oplus M^d = M \oplus M^d$, we have $\theta \oplus \Delta_K \oplus \theta' = \theta \oplus \theta' \in \Con(M \oplus M^d)$ for all $\theta \in \Con(M)$.

Lemma 17 If $L$ is a nontrivial BI-lattice and $\theta \in \Con(L) \setminus \{\nabla_L\}$, then: $\theta$ preserves the trivial Brouwer complement on $L$ iff $0/\theta = \{0\}$ iff $1/\theta = \{1\}$.

Proof. Since $\theta$ preserves the involution, we have $0/\theta = \{0\}$ iff $1/\theta = \{1\}$. Now let $\sim : L \to L$ be the trivial Brouwer complement. If $0/\theta = \{0\}$, then clearly $\theta$ preserves $\sim$. Finally, assume that $\theta$ preserves $\sim$, let $a \in 0/\theta$ and assume by absurdum that $a \neq 0$. Then $(0, 1) = (a^\sim, 1) = (a^\sim, 0^\sim) \in \theta$, which contradicts the fact that $\theta \neq \nabla_L$. Therefore $0/\theta = \{0\}$. $\Box$

Proposition 18 (i) For any BI-lattice $L$, $\Con_{BI}(L) = \{\theta \in \Con(L) : \theta = \theta'\}$ and $\Con_{BI}(L) = \Con_{BI}(L)$.
(ii) For any antiortholattice \( L \), \( \text{Con}_{\text{BZL}}(L) = \text{Con}_{\text{BI}}(L) \) and \( \text{Con}_{\text{BZL}}(L) = \text{Con}_{\text{BI}}(L) \cup \{ \vee L \} \), which is a complete bounded sublattice of \( \text{Con}_{\text{BI}}(L) \).

If \( L \) is nontrivial, then \( \text{Con}_{\text{BZL}}(L) \cong \text{Con}_{\text{BI}}(L) \oplus D_2 \) and has the top element of \( \text{Con}_{\text{BI}}(L) \) as a unique co-atom.

**Proof.** (ii) \( \text{Con}_{\text{BI}}(L) \subseteq \text{Con}(L) \) and, for any \( \theta \in \text{Con}(L) \), we have \( \theta \in \text{Con}_{\text{BI}}(L) \) exactly when, for all \( a, b \in L \): \((a, b) \in \theta \) iff \((a', b') \in \theta \) iff \((a, b) \in \theta' \), that is exactly when \( \theta = \theta' \). Thus \( \text{Con}_{\text{BI}}(L) = \text{Con}_{\text{BZL}}(L) \).

(iii) By Lemma 17, \( \text{Con}_{\text{BZL}}(L) = \text{Con}_{\text{BI}}(L) \) and \( \text{Con}_{\text{BZL}}(L) = \text{Con}_{\text{BI}}(L) \cup \{ \vee L \} \). \( \text{Con}_{\text{BI}}(L) \) is a bounded lattice, with smallest element \( D_L \), and a complete sublattice of \( \text{Con}_{\text{BI}}(L) \), therefore \( \text{Con}_{\text{BI}}(L) \cup \{ \vee L \} \) is a complete bounded sublattice of \( \text{Con}_{\text{BI}}(L) \). If \( L \) is nontrivial, then \( \vee L \notin \text{Con}_{\text{BI}}(L) \), hence \( \text{Con}_{\text{BI}}(L) \cup \{ \vee L \} \cong \text{Con}_{\text{BI}}(L) \oplus D_2 \) and has max(\( \text{Con}_{\text{BI}}(L) \)) as a unique co-atom. ■

**Theorem 19** Let \( M \) be a bounded lattice, \( K \) be a bounded involution lattice and \( L = M \oplus K \oplus M^d \). Then:

(i) \( \text{Con}_{\text{BI}}(L) = \{ \alpha \oplus \beta \oplus \alpha' : \alpha \in \text{Con}(M), \beta \in \text{Con}_{\text{BI}}(K) \} \cong \text{Con}(M) \times \text{Con}_{\text{BI}}(K) \);

(ii) if \( M \) is nontrivial and \( K \) is a pseudo-Kleene algebra, then \( \text{Con}_{\text{BZL}}(L) = \{ \alpha \oplus \beta \oplus \alpha' : \alpha \in \text{Con}_0(M), \beta \in \text{Con}_{\text{BI}}(K) \} \cup \{ \vee L \} \cong (\text{Con}_0(M) \times \text{Con}_{\text{BI}}(K)) \oplus D_2 \);

(iii) \( L_{bi} \) is subdirectly irreducible iff one of the following holds:

- \( M \) is trivial and \( K \) is subdirectly irreducible;
- \( M \) is subdirectly irreducible and \( K \) is trivial.

(iv) if \( M \) is nontrivial and \( K \) is a pseudo-Kleene algebra, then the antiortholattice \( L \) is subdirectly irreducible iff one of the following holds:

- \( \text{Con}_0(M) = \{ D_M \} \) and \( K \) is subdirectly irreducible;
- \( K \) is trivial and the set \( \text{Con}_0(M) \setminus \{ D_M \} \) has a minimum.

**Proof.** (i) For any \( \alpha, \gamma \in \text{Con}(M) \) and any \( \beta \in \text{Con}(K) \), we have, according to the definition of the involution of \( L \): \((\alpha \oplus \beta \oplus \gamma)' = \gamma' \oplus \beta' \oplus \alpha' \), hence, by Proposition 13: \( \alpha \oplus \beta \oplus \gamma = (\alpha \oplus \beta \oplus \gamma)' \) iff \( \alpha = \gamma', \beta = \beta' \) and \( \gamma = \alpha' \) iff \( \alpha = \gamma' \) and \( \beta \in \text{Con}_{\text{BI}}(K) \), hence \( \text{Con}_{\text{BZL}}(L) = \{ \alpha \oplus \beta \oplus \alpha' : \alpha \in \text{Con}(M), \beta \in \text{Con}_{\text{BI}}(K) \} \), which is isomorphic to \( \text{Con}(M) \times \text{Con}_{\text{BI}}(K) \), because the map \( (\alpha, \beta) \mapsto \alpha \oplus \beta \oplus \alpha' \) for all \( \alpha \in \text{Con}(M) \) and \( \beta \in \text{Con}_{\text{BI}}(K) \) sets a lattice isomorphism between these lattices, since it is clearly bijective and preserves the join and the intersection.

(iii) By (i), Proposition 13 and the clear fact that, if \( M \) is nontrivial, then, for any \( \alpha \in \text{Con}(M) \) and any \( \beta \in \text{Con}(K) \), we have: \( 0/(\alpha \oplus \beta \oplus \alpha') = \{ 0 \} \) iff \( 0/\alpha = \{ 0 \} \) iff \( 1/\alpha' = \{ 1 \} \) iff \( 1/(\alpha \oplus \beta \oplus \alpha') = \{ 1 \} \).

Corollary 20 Let \( K \) be a pseudo-Kleene algebra. Then:

(i) for any \( 0 \)-regular, in particular any simple nontrivial bounded lattice \( M \),

if \( L = M \oplus K \oplus M^d \), then \( \text{Con}_{\text{BZL}}(L) = (\Delta_M \oplus \vee K \oplus \Delta_M) \cup \{ \vee L \} \cong \text{Con}_{\text{BI}}(K) \oplus D_2 \), and \( L \) is subdirectly irreducible as an antiortholattice if \( K \) is subdirectly irreducible;
(ii) if \( L = D_2 \oplus K \oplus D_2 \), then \( \text{Con}_{L}(L) = (eq(\{0\}, K, \{1\}) \cup \{\nabla L\}) \cong \text{Con}_{3}(K) \oplus D_2 \), and \( L \) is subdirectly irreducible as an antiortholattice iff \( K \) is subdirectly irreducible.

**Proof.** By Theorem [19] [11] and the fact that, in this case, \( \text{Con}_{0}(M) = \{\Delta_M\} \cong D_1 \), so \( \{\alpha \oplus \beta \oplus \alpha' : \alpha \in \text{Con}_{0}(M), \beta \in \text{Con}_{0}(K)\} = (\Delta_M \oplus \nabla K \oplus \Delta_M) \), as a principal filter of \( \text{Con}_{BL}(L) \).

By [10], the equality \( \Delta_D \oplus \nabla K \oplus \Delta_D = eq(\{0\}, K, \{1\}) \) and the fact that \( D_2 \).

We take advantage of this opportunity to correct a mistake in [13] Lm.3.3.(2). There, it had been claimed that, if \( L \) is a subdirectly irreducible algebra in \( V(\Delta \alpha L) \), then every \( a \in L \) is comparable with \( a' \).

However, the canonical antiortholattice on \( D_2 \oplus MO_2 \oplus D_2 \), where \( MO_2 = D_2 \oplus D_2 \) is the smallest orthomodular lattice which is not a Boolean algebra \[3\], contains two pairs of incomparable elements \( a, a' \) and \( b, b' \), corresponding to the four atoms of \( MO_2 \).

**Corollary 21** Let \( K \) be a BI-lattice. Then:

(i) the BI-lattice \( D_2 \oplus K \oplus D_2 \) is subdirectly irreducible iff \( K \) is trivial;

(ii) if \( K \) is a pseudo-Kleene algebra, then the antiortholattice \( D_3 \oplus K \oplus D_3 \) is subdirectly irreducible iff \( K \) is trivial.

**Proof.** By Theorem [19] [11] and the fact that \( D_2 \) is nontrivial and simple, thus subdirectly irreducible.

By [10], Corollary [20] [11] and the fact that \( D_3 \oplus K \oplus D_3 \cong D_2 \oplus D_2 \oplus K \oplus D_2 \oplus D_2 \).

**Corollary 22** The only simple antiortholattices that satisfy SDM are \( D_1 \), \( D_2 \) and \( D_3 \).

**Proof.** Recall that an antiortholattice satisfies SDM iff it has the 0 meet-irreducible, so any antiortholattice chain satisfies SDM. The antiortholattices \( D_1 \) and \( D_2 \) are simple and, by Corollary [20] [11], so is \( D_3 = D_2 \oplus D_1 \oplus D_2 \).

Now let \( L \) be a simple antiortholattice which satisfies SDM and assume ex absurdum that \( |L| > 3 \). By Proposition [15] [11], \( \text{Con}_{BL}(L) \cong \text{Con}_{BIO}(L) \oplus D_2 \), thus \( \text{Con}_{BIO}(L) \cong D_1 \) since \( L \) is simple. But 0 is meet-irreducible in \( L \), so 1 is join-irreducible in \( L \), from which it easily follows that \( \alpha = eq(\{0\}, L \setminus \{0, 1\}, \{1\}) \in \text{Con}_{0}(L) \) (see also [22]); clearly, \( \alpha \) preserves the Kleene complement of \( L \), hence \( \alpha \in \text{Con}_{BIO}(L) \). Therefore \( \Delta_L, \alpha \in \text{Con}_{BIO}(L) \), and, since \( |L| > 3 \), it follows that \( \Delta_L \neq \alpha \), which contradicts the fact that \( |\text{Con}_{BIO}(L)| = 1 \). ■

Another proof of the previous corollary can be obtained from the results in [13] Subsection 4.2.

**Lemma 23** Any infinite chain \( C \) is subdirectly reducible. Moreover, \( \Delta_C \) is meet-reducible in \( \text{Con}(C) \), as well as in \( \text{Con}_{0}(C) \) in the case when \( C \) has a bottom element.

**Proof.** Let \( C \) be an infinite chain. Then there exist \( a, b, c \in C \) such that \( a < b < c \). If we denote by \( \theta = eq(\{a, b\} \cup \{x : x \in C \setminus [a, b]\}) \) and by \( \zeta = eq(\{b, c\} \cup \{x : x \in C \setminus [b, c]\}) \), then, clearly, \( \theta, \zeta \in \text{Con}(C) \setminus \{\Delta_C\} \) and \( \theta \cap \zeta = \Delta_C \). If \( C \) has a 0, then we may take \( \alpha \neq 0 \), and then \( \theta, \zeta \in \text{Con}(C) \). ■

Note that, for any \( n \in \mathbb{N}^* \), \( \text{Con}(D_n) \cong D_n^{-1} \). Indeed, \( \text{Con}(D_0) \cong D_1 \cong D_0 \), while, if \( n \geq 2 \), then \( D_n = \bigoplus_{i=1}^{n-1} D_2 \), so that \( \text{Con}(D_n) \cong \text{Con}(D_2)^{n-1} \cong D_2^{n-1} \).
Corollary 24  
(i) For any \( k \in \mathbb{N}^* \) and any \( n \in \{2k, 2k+1\} \), \( \text{Con}_{\mathcal{BI}}(D_n) \cong D_2^k \) and \( \text{Con}_{\mathcal{BI}}(D_n) \cong D_2^{k-1} \oplus D_2 \).

(ii) The only subdirectly irreducible (bounded) involution chains are \( D_1, D_2 \) and \( D_3 \). The only subdirectly irreducible antiotholattice chains are \( D_1, D_2, D_3, D_4 \) and \( D_5 \).

Proof.  
(i) Note that \( D_{2k} \cong D_k \oplus D_2 \oplus D_k \) and \( D_{2k+1} \cong D_{k+1} \oplus D_{k+1} \cong D_{k+1} \oplus D_1 \oplus D_{k+1} \). By Theorem [19], it follows that:
\[
\text{Con}_{\mathcal{BI}}(D_{2k}) \cong \text{Con}(D_k) \times \text{Con}_{\mathcal{BI}}(D_2) \cong D_2^{k-1} \times D_2 \cong D_2^k;
\]
\[
\text{Con}_{\mathcal{BI}}(D_{2k+1}) \cong \text{Con}(D_{k+1}) \times \text{Con}_{\mathcal{BI}}(D_1) \cong D_2^k \times D_1 \cong D_2^k.
\]

Let us denote by \([r]\) the integer part of any real number \( r \), so that \( k = [n/2] \).

Of course, \( \text{Con}_{\mathcal{BI}}(D_1) \cong D_1 \cong D_2^{[1/2]} \) and \( \text{Con}_{\mathcal{BI}}(D_2) \cong D_2 \cong D_2^{[2/2]} \) if \( n \geq 3 \), then \( D_n \cong D_2 \oplus D_{n-2} \oplus D_2 \) and \( n - 2 \in \{2(k-1), 2(k-1)+1\} \), hence, by Corollary [20], \( \text{Con}_{\mathcal{BI}}(D_{n-2}) \cong D_2 \cong D_2^{k-1} \oplus D_2 \).

(iii) \( D_1 \) is trivial, thus subdirectly irreducible both as a BI–lattice and as an antiotholattice, while (ii) ensures us that, for any \( n \in \mathbb{N} \) with \( n \geq 2 \), if \( k = [n/2] \in \mathbb{N}^* \), then:

- \( \text{Con}_{\mathcal{BI}}(D_n) \cong D_2^k \), which has exactly \( k \) atoms, so that: \( (D_n)_bi \) is subdirectly irreducible iff \( k = 1 \) iff \( n \in \{2, 3\} \);

- \( \text{Con}_{\mathcal{BI}}(D_n) \cong D_2^{k-1} \oplus D_2 \), which has one atom if \( k \leq 2 \) and \( k-1 \) atoms if \( k > 2 \), so that the antiotholattice \( D_n \) is subdirectly irreducible iff \( k-1 \leq 1 \) iff \( k \leq 2 \) iff \( n \in \{2, 5\} \).

Hence the only subdirectly irreducible finite bounded involution chains are \( D_1, D_2 \) and \( D_3 \), while the only subdirectly irreducible finite antiotholattice chains are \( D_1, D_2, D_3, D_4 \) and \( D_5 \).

If \( C \) is an infinite BI–chain, then its 0 is meet–irreducible, hence \( C \) is an antiotholattice, and there exists a \( u \in C \) such that \( u < u' \) and the filter \( \langle u \rangle \) is infinite, thus \( \langle u \rangle \) is an infinite bounded chain and \( C = \langle u \rangle \oplus \langle u' \rangle \oplus \langle u' \rangle = \langle u \rangle \oplus \langle u, u' \rangle \oplus \langle u' \rangle \), where \( \langle u, u' \rangle \) is clearly a bounded lattice. Then, by Lemma [23] and Theorem [11], \( \Delta_{\langle u \rangle} \) is meet–irreducible in \( \text{Con}([u]) \), as well as in \( \text{Con}([u]) \), \( \text{Con}_{\mathcal{BI}}([u]) \cong \text{Con}_{\mathcal{BI}}([u, u']) \), as well as in \( \text{Con}_{\mathcal{BI}}([u, u']) \), \( \text{Con}_{\mathcal{BI}}([u, u']) \cong \text{Con}_{\mathcal{BI}}([u, u']) \oplus D_2 \), and hence \( C \) is subdirectly reducible both in \( \mathcal{BI} \) and in \( \mathcal{PEZL}^* \).

Note that the argument above can be adapted for the subdirect irreducibility of any infinite involution chain \( C \), not necessarily bounded, in the class \( L \) of involution lattices, that is self-dual lattices \( L \) endowed with a unary operation \( \prime \) given by a dual lattice automorphism of \( L \), because, for any lattice \( M \) with a 1 and any BI–lattice \( K, L = M \oplus K \oplus M^d \) is an involution lattice with \( \text{Con}(L) \cong \text{Con}(M) \times \text{Con}_{\mathcal{BI}}(K) \); see also [23].

Corollary 25  
Let \( C \) be a bounded chain, \( K \) a BI–lattice, and \( L = C \oplus K \oplus C^d \).

- If \( |C| \geq 3 \), then the BI–lattice \( L \) is subdirectly reducible.
- If \( |C| \geq 4 \) and \( K \) is a pseudo-Kleene algebra, then the antiotholattice \( L \) is subdirectly reducible.

Proof. If \( C \) is finite, then the statements follow by Corollary [23], and the fact that, if \( n \geq 3 \), then \( n - 1 \geq 2 \), so that the BI–lattice \( D_n \oplus K \oplus D_n \) is
nontrivial, and we have $D_n \oplus K \oplus D_n \cong D_2 \oplus D_{n-1} \oplus K \oplus D_{n-1} \oplus D_2$ and $D_{n+1} \oplus K \oplus D_{n+1} \cong D_3 \oplus D_{n-1} \oplus K \oplus D_{n-1} \oplus D_3$.

If $C$ is infinite, then the subdirect reducibility of $L$ follows by the argument at the end of the proof of Corollary 24 (i) in which we replace $(u)$ by $C$ and $[u, u']$ by $K$.

5 Horizontal Sums of PBZ*–Lattices

There is a well-developed theory of horizontal sums in the context of orthomodular lattices: see e.g. [11, 7, 17]. In the present section, we follow in the footsteps of [15] and try to broaden our scope to the context of PBZ*-lattices.

Of the next two results, the former is straightforward and the latter is implicit in [12, Ex. 5.3]:

**Lemma 26** (i) For any nontrivial BI-lattices $A$ and $B$: $A \oplus B$ is paraorthomodular iff $A$ and $B$ are paraorthomodular.

(ii) For any nontrivial BI-lattices $A$ and $B$: $A \oplus B$ is an ortholattice, respectively an orthomodular lattice, iff $A$ and $B$ are ortholattices, respectively orthomodular lattices.

(iii) For any nontrivial BZ–lattices $A$ and $B$: $A \oplus B$ satisfies condition (*) iff $A$ and $B$ satisfy condition (*).

(iv) For any nontrivial BZ–lattices $A$ and $B$ such that $A \oplus B$ is a BZ–lattice: $A \oplus B$ is a $\text{PBZ}^*$–lattice iff $A$ and $B$ are $\text{PBZ}^*$–lattices.

**Proposition 27** (i) If $A$ and $B$ are nontrivial pseudo–Kleene algebras, then:

$A \oplus B$ is a pseudo–Kleene algebra iff at least one of $A$ and $B$ is an ortholattice.

(ii) If $A$ and $B$ are nontrivial BZ–lattices, then: $A \oplus B$ is a BZ–lattice iff at least one of $A_{bi}$ and $B_{bi}$ is an ortholattice.

(iii) If $A$ and $B$ are nontrivial $\text{PBZ}^*$–lattices, then: $A \oplus B$ is a $\text{PBZ}^*$–lattice iff at least one of $A$ and $B$ is an orthomodular lattice.

The following corollaries ensue. The lesson we learn from the latter is that classes of the form $\forall \oplus \mathcal{W}$, for $\forall, \mathcal{W}$ subvarieties of $\text{PBZL}^*$, are sometimes varieties in their own right, and in particular, well-known subvarieties of $\text{PBZL}^*$.

**Corollary 28**

- If $n \in \mathbb{N} \setminus \{0, 1\}$ and $A_1, \ldots, A_n$ are nontrivial pseudo–Kleene algebras, then: $\bigoplus_{i=1}^{n} A_i$ is a pseudo–Kleene algebra iff, for some $k \in [1, n]$ and every $i \in [1, n] \setminus \{k\}$, $A_i$ is an ortholattice.

- If $n \in \mathbb{N} \setminus \{0, 1\}$ and $A_1, \ldots, A_n$ are nontrivial BZ–lattices, then: $\bigoplus_{i=1}^{n} A_i$ is a BZ–lattice iff, for some $k \in [1, n]$ and every $i \in [1, n] \setminus \{k\}$, $(A_i)_{bi}$ is an ortholattice.

- If $n \in \mathbb{N} \setminus \{0, 1\}$ and $A_1, \ldots, A_n$ are nontrivial $\text{PBZ}^*$–lattices, then: $\bigoplus_{i=1}^{n} A_i$ is a $\text{PBZ}^*$–lattice iff, for some $k \in [1, n]$ and every $i \in [1, n] \setminus \{k\}$, $A_i$ is an orthomodular lattice.

**Corollary 29** (i) $\text{OL} \oplus \text{PKA} = \text{PKA}$ and $\text{OML} \oplus \text{PBZL}^* = \text{PBZL}^*$.

(ii) $\text{OL} \oplus \text{OL} = \text{OML} \oplus \text{OL} = \text{OL}$ and $\text{OML} \oplus \text{OML} = \text{OML}$.
(iii) For any classes C and D of BZ–lattices such that \( C \uplus D \subseteq \text{BZL} \), \( (C \uplus D) \cap \text{PBZL}^* = (C \cap \text{PBZL}^*) \uplus (D \cap \text{PBZL}^*) \).

Proof. The right-to-left inclusions follow from the fact that \( D_2 \in \text{OL} \subseteq \text{OML} \). The left-to-right inclusions are consequences of Lemma 26 and Proposition 27.

Lemma 30 If \( \mathcal{V} \) is the variety of bounded lattices or one of the varieties \( \mathbb{H} \) and \( \mathbb{BZL} \) and \( A \) and \( B \) are nontrivial members of \( \mathcal{V} \) such that \( A \uplus B \in \mathcal{V} \), then, for any subalgebra \( M \) of \( A \uplus B \) and any \( \theta \in \text{ConV}(A \uplus B) \), we have:

- \( M = (M \cap A) \uplus (M \cap B) \);
- \( (A \uplus B)/\theta = A/\theta \uplus B/\theta \).

Proof. Since \( A \) and \( B \) are subalgebras of \( A \uplus B \), it follows that \( M \cap A \) and \( M \cap B \) are subalgebras of \( M \). We have \( M = M \cap (A \uplus B) = M \cap (A \cup B) = (M \cap A) \cup (M \cap B) \). Since \( A \cap B = \{0, 1\} \), it follows that \( (M \cap A) \cap (M \cap B) \neq \emptyset \). Therefore \( M = (M \cap A) \uplus (M \cap B) \) by the definition of a horizontal sum.

Since \( \theta \in \text{ConV}(A \uplus B) \) and \( A \) and \( B \) are subalgebras of \( A \uplus B \), it follows that \( \theta \cap A^2 \in \text{ConV}(A) \), \( \theta \cap B^2 \in \text{ConV}(B) \) and \( A/\theta = A/(\theta \cap A^2) \) and \( B/\theta = B/(\theta \cap B^2) \) are subalgebras of \( (A \uplus B)/\theta \). \( A \cap B = \{0, 1\} \), hence \( A/\theta \cap B/\theta = (A \cap B)/\theta = \{0/\theta, 1/\theta\} \). By the definition of the horizontal sum, it follows that \( (A \uplus B)/\theta = A/\theta \uplus B/\theta \).

Lemma 31 For any PBZ\(^*\)–lattice \( L \), any subalgebra \( M \) of \( L \), any \( \theta \in \text{ConBZL}(L) \), any nontrivial orthomodular lattice \( A \), any nontrivial PBZ\(^*\)–lattice \( B \) and any non–empty family \( \{L_i\}_{i \in I} \) of PBZ\(^*\)–lattices, we have:

- \( S(M) = S(L) \cap M \);
- \( S(L)/\theta = S(L)/\theta \);
- \( S(A \uplus B) = A \uplus S(B) \);
- \( S\left( \prod_{i \in I} L_i \right) = \prod_{i \in I} S(L_i) \).

Proof. \( S(M) = \{x \in M : x' = x^\sim\} = S(L) \cap M \), so \( S(M) = S(L) \cap M \).

Clearly, \( S(L)/\theta \subseteq S(L)/\theta \). Now let \( x \in L \) be such that \( x/\theta \in S(L)/\theta \), that is \( (x/\theta)' = (x/\theta)^\sim \). Then \( x/\theta = x''/\theta = (x'/\theta)' = (x^\sim/\theta)' = x^\sim/\theta = x^\sim/\theta \in S(L)/\theta \). Hence \( S(L)/\theta \subseteq S(L)/\theta \). Therefore \( S(L)/\theta = S(L)/\theta \), so \( S(L)/\theta = S(L)/\theta \).

On the other hand, by Proposition 27, Lemma 30 and the definition of the subalgebra of sharp elements of a PBZ\(^*\)–lattice:

\( S(A \uplus B) = (S(A \uplus B) \cap A) \uplus (S(A \uplus B) \cap B) = S(A) \uplus S(B) = A \uplus S(B) \).

Next, let us give a direct proof of a result from [12], to the effect that the orthomodular lattice of sharp elements in a member of \( V(A \Theta \text{OL}) \) is always Boolean.

Proposition 32 If \( L \in V(A \Theta \text{OL}) \), then \( S(L) \) is a Boolean algebra.
Proof. We will apply Lemma 31. If \( L \in V(\text{AOL}) \) = HSP(\( \text{AOL} \)), then there exists a non–empty family \( (A_i)_{i \in I} \subseteq \text{AOL} \setminus \{D_1\} \), a subalgebra \( A \) of \( \prod_{i \in I} A_i \) and a \( \theta \in \text{CNBZL}(A) \) such that \( L = A/\theta \). Then, for all \( i \in I \), \( S(A_i) = \{0, 1\} \), so the orthomodular lattice \( S(A_i) \cong D_2 \), thus \( S(\prod_{i \in I} A_i) = \prod_{i \in I} S(A_i) \cong D_2 \), which is a Boolean algebra, hence \( S(A) = S(\prod_{i \in I} A_i) \cap A \) is embedded in the Boolean algebra \( S(\prod_{i \in I} A_i) \), therefore \( S(A) \) is a Boolean algebra, thus \( S(L) = S(A/\theta) = S(A)/\theta = S(A)/(\theta \cap (S(A))^2) \) is a Boolean algebra. ■

Note that, since \( S(L) \) is the largest orthomodular subalgebra in any \( L \in \text{PBZL}^* \), Proposition 32 shows that, for any \( L \in V(\text{AOL}) \), any orthomodular subalgebra of \( L \) is Boolean.

Corollary 33

(i) For any \( L \in \Omega \text{ML} \oplus V(\text{AOL}) \), \( S(L) \) is a horizontal sum of an orthomodular lattice with a Boolean algebra.

(ii) \( \{L \in \Omega \text{ML} \oplus \text{AOL} : S(L) \in \text{BA}\} = \text{BA} \oplus \text{AOL} \) and \( \{L \in \Omega \text{ML} \oplus V(\text{AOL}) : S(L) \in \text{BA}\} = (\text{BA} \oplus \text{AOL}) \cup V(\text{AOL}) \), but \( \{L \in \text{PBZL}^* : S(L) \in \text{BA}\} \not\subseteq V(\Omega \text{ML} \oplus V(\text{AOL})) \).

Proof. (i) Let \( L = A \oplus B \), with \( A \in \Omega \text{ML} \) and \( B \in V(\text{AOL}) \), thus, by Lemma 31 and Proposition 32, \( S(L) = A \oplus S(B) \), with \( A \in \Omega \text{ML} \) and \( S(B) \in \text{BA} \).

Clearly, for any bounded lattices \( A \) and \( B \) with \( |A| > 2 \) and \( |B| > 3 \), \( A \oplus B \) has the diamond or the pentagon (the latter if \( A \) or \( B \) has length at least 4) as a bounded subalattice, thus \( A \oplus B \) is non–distributive. The horizontal sum of BI-lattices \( D_3 \oplus D_3 \not\cong D_2^* \), in fact \( D_3 \oplus D_3 \not\in \text{PKA} \). Hence, for any BI-lattices \( A \) and \( B \) with \( |A| > 2 \) and \( |B| > 2 \), \( A \oplus B \) is not a Boolean algebra, more precisely, for any nontrivial BI-lattices \( A \) and \( B \), \( A \oplus B \in \text{BA} \) iff \( A \cong D_2 \) and \( B \in \text{BA} \) or vice–versa.

Now let \( A \in \Omega \text{ML} \setminus \{D_1\} \), \( B \in \text{PBZL}^* \) and \( L = A \oplus B \), so that \( S(L) = A \oplus S(B) \) by Lemma 31 hence, by the above, \( S(L) \in \text{BA} \) iff \( A \cong D_2 \) and \( S(B) \in \text{BA} \) or \( A \in \text{BA} \) and \( S(B) = \{0, 1\} \). Now apply the fact that \( \{B \in \text{PBZL}^* : S(B) = \{0, 1\}\} = \text{AOL} \) and Proposition 32.

See below the PBZ\(^*\)–lattice \( M \) in Example 70 which has \( S(M) = \{0, a, a', 1\} \), so \( D_2^* \cong S(M) \in \text{BA} \), but \( M \not\in V(\Omega \text{ML} \oplus V(\text{AOL})) \). ■

Lemma 34 If \( A \) is an ortholattice with \( |A| > 2 \) and \( B \) is a non–trivial BI-lattice, then \( A \oplus B \), endowed with the trivial Brouwer complement, fails condition (\( * \)).

Proof. Since \( |A| > 2 \), there exists an \( a \in A \setminus \{0, 1\} \), so that \( a' \in A \setminus \{0, 1\} \), as well. Since \( A \) is an ortholattice, we have \( a \land a' = 0 \). Thus, if \( \sim : A \oplus B \to A \oplus B \) is the trivial Brouwer complement, then \( a \land a' \sim = 0 \sim = 1 \neq 0 = 0 \lor a = a'' \lor a'\sim \).

■

Lemma 35 Let \( A \) and \( B \) be PBZ\(^*\)–lattices with \( |A| > 2 \) and \( |B| > 2 \). Then:

(i) \( A \oplus B \) is not an antiortholattice;

(ii) \( A \oplus B \) is an orthomodular lattice iff \( A \) and \( B \) are orthomodular lattices.

Proof. (i) By Proposition 27 and Lemma 34.

(ii) By Lemma 34 and the fact that \( \{0, 1\} \subseteq S(A) \cap S(B) \), we have: \( A \) and \( B \) are orthomodular lattices iff \( S(A) = A \) and \( S(B) = B \) iff \( S(A \oplus B) = A \oplus B \) iff \( A \oplus B \) is an orthomodular lattice. ■

The following lemma clarifies the relationships between dense elements on the one hand, and subalgebras, products and congruences on the other, in PBZ\(^*\)–lattices.
Lemma 36 For any PBZ∗-lattice $L$, any subalgebra $M$ of $L$, any $\theta \in \text{Con}_{\text{PBZL}}(L)$, any nontrivial orthomodular lattice $A$, any nontrivial PBZ∗-lattice $B$ and any non-empty family $(A_i)_{i \in I}$ of PBZ∗-lattices, we have:

- $T(M) = T(L) \cap M$ and $\langle T(M) \rangle_{\text{PBZL}} = \langle T(L) \rangle_{\text{PBZL}} \cap M$;
- $T(L)/\theta \subseteq T(L)/\theta$;
- if $\theta \in \text{Con}_{\text{PBZL}}(L)$, then $T(L)/\theta = T(L)/\theta$;
- $T(A \oplus B) = T(B)$, so $T(A \oplus B) = T(B)$ if $T(B)$ is a subalgebra of $B$;
- $D(\prod_{i \in I} L_i) = \prod_{i \in I}(D(L_i))$, so $T(\prod_{i \in I} L_i) = \{0 \prod_{i \in I} L_i\} \cup \prod_{i \in I}(D(L_i))$;
- $\langle T(\prod_{i \in I} L_i) \rangle_{\text{PBZL}} = \prod_{i \in I}(T(L_i))_{\text{PBZL}}$.

Proof. $T(M) = \{0\} \cup D(M) = T(L) \cap M$, hence $\langle T(M) \rangle_{\text{PBZL}} = \langle T(L) \rangle_{\text{PBZL}} \cap M$.

Clearly, $T(L)/\theta \subseteq T(L)/\theta$ and, if $\theta \in \text{Con}_{\text{PBZL}}(L)$, then, for any $x \in L$ such that $x/\theta \in T(L)/\theta$, we have $x \in 0/\theta = \{0\}$ or $x \in 0/\theta = \{0\}$, so $x \in T(L)$, thus $T(L)/\theta \subseteq T(L)/\theta$.

By Proposition 27, $A \oplus B$ is a PBZ∗-lattice. Since $A, B \in S(A \oplus B)$, we have, by Lemma 10, $T(A \oplus B) = T(A) \cup T(B) = \{0, 1\} \cup T(B) = T(B)$, which is a subalgebra of $A \oplus B$ if it is a subalgebra of $B$.

Clearly, $D(\prod_{i \in I} L_i) = \prod_{i \in I}(D(L_i))$, whence the rest of the statement follows. Therefore

$$\langle T(\prod_{i \in I} L_i) \rangle_{\text{PBZL}} = \langle D(\prod_{i \in I} L_i) \rangle_{\text{PBZL}} = \prod_{i \in I}(D(L_i))_{\text{PBZL}}.$$

\[\blacksquare\]

Let us strengthen the property mentioned at the end of Subsection 23 which characterizes antiortholattices with SDM:

Proposition 37 Let $(L_i)_{i \in I}$ be a non-empty family of nontrivial PBZ∗-lattices, $L = \prod_{i \in I} L_i$ and $A$ be a subalgebra of $L$ such that $A$ is an antiortholattice. Then:

- if $a_i \in L_i$ for all $i \in I$ such that $a = (a_i)_{i \in I} \in A$, then: $a = 0$ or $a_i \neq 0$ for all $i \in I$, and, dually: $a = 1$ or $a_i \neq 1$ for all $i \in I$;
- if, for every $i \in I$ and all $x_i, y_i \in T(L_i) \setminus \{0\}$, we have $x_i \wedge y_i \neq 0$, in particular if $T(L_i) \setminus \{0\}$ is closed w.r.t. the meet or 0 is meet-irreducible in $L_i$ for every $i \in I$, then 0 is not a finite meet of elements of $T(L) \setminus \{0\}$, in particular 0 is meet-irreducible in $A$.

Proof. By Lemma 28, which ensures us that $A = T(A) \subseteq T(L) = \{0\} \cup \prod_{i \in I}(T(L_i) \setminus \{0\})$ and thus $A \setminus \{0\} \subseteq T(L) \setminus \{0\} = \prod_{i \in I}(T(L_i) \setminus \{0\})$. \[\blacksquare\]

We now focus on the properties of the sets of sharp elements and of dense elements in some particular horizontal sums. We show that in any horizontal sum of an orthomodular lattice and of an antiortholattice, the former includes all the sharp elements and the latter all the dense elements; moreover, horizontal sums of orthomodular lattices and of antiortholattices are exactly the PBZ∗-lattices $L$ such that $S(L) \cup T(L) = L$.

Lemma 38 If $A$ is an orthomodular lattice and $B$ is an antiortholattice, then $S(A \oplus B) = A$ and $T(A \oplus B) = B$.
Proof. A ⊕ B ∈ PBZL∗ by Proposition 27\textsuperscript{[31]}. By Lemmas 31 and 10 S(A ⊕ B) = A ⊕ S(B) = A. By Lemmas 10 and 30 T(A ⊕ B) = B. ■

Proposition 39 Let A be a nontrivial orthomodular lattice, B a nontrivial PBZ∗-lattice and L = A ⊕ B. Then: A = S(L) iff B = T(L) iff B ∈ AOL, and, if so, then L ∈ OML ⊕ AOL.

Proof. By Lemmas 38 and 10 T(L) = T(B), hence: B = T(L) iff B = T(B) iff B ∈ AOL, which in turn implies L ∈ OML ⊕ AOL.

By Lemmas 11 and 10 S(L) = A ⊕ S(B), so S(L) = A ∪ S(B), hence: A = S(L) iff A = A ∪ S(B) iff S(B) ⊆ A iff S(B) ⊆ A ∩ B = {0, 1} iff S(B) = {0, 1} iff B ∈ AOL. ■

Theorem 40 For any nontrivial PBZ∗-lattice L, the following are equivalent:

(i) L ∈ OML ⊕ AOL;
(ii) T(L) is a subuniverse of L and L = S(L) ⊕ T(L);
(iii) L = S(L) ∪ T(L);
(iv) T(L) is a subuniverse of L;
(v) T(L) is closed w.r.t. the Kleene complement;
(vi) T(L)′ is closed w.r.t. the Kleene complement;
(vii) T(L) = T(L)′;
(viii) T(L)′ is closed w.r.t. the Brouwer complement;
(ix) T(L) ∪ T(L)′ is closed w.r.t. the Brouwer complement.

Proof. (i) ⇒ (ii). If L ∈ OML ⊕ AOL, then, by Lemma 38 T(L) is a subuniverse of L and L = S(L) ⊕ T(L).

Clearly, (ii) implies (iii) and (iv).

(iii) ⇒ (i). If L = S(L) ∪ T(L), then, if a ∈ {0, 1}, then a′ ∈ {0, 1} ⊆ T(L); if a ∈ T(L) \ {0, 1} = T(L) \ S(L) = (S(L) ∪ T(L)) \ S(L) = L \ S(L), then it follows that a′ ∈ L \ S(L) = T(L) \ {0, 1} ⊆ T(L); therefore T(L) is closed w.r.t. the Kleene complement, hence T(L) is the universe of a subalgebra of L by Lemma 11. So L = S(L) ⊕ T(L), whence (i) follows.

(iv) ⇔ (v) follows from Lemma 11

(v) ⇔ (vi) ⇔ (vii) are clear.

(viii) ⇔ (viii). T(L)′ is closed w.r.t. the Brouwer complement iff T(L)′ ⊆ T(L)′, which is equivalent to T(L)′ ⊆ S(L) ∩ T(L)′ = {0, 1} since T(L)′ ⊆ S(L). But, since \{x ∈ L : x′ ∈ \{0, 1\}\ = T(L), the inclusion T(L)′ ⊆ \{0, 1\} is equivalent to T(L)′ ⊆ T(L) and then to T(L)′ = T(L).

(viii) ⇔ (viii) is also obvious. ■

Proposition 41 Let L be a nontrivial PBZ∗-lattice. Then the following are equivalent:

(i) all elements of L \ {0, 1} are join–irreducible in L.

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(ii) all elements of $L \setminus \{0, 1\}$ are meet–irreducible in $L_4$;

(iii) $L = \text{MO}_\kappa \uplus A$ for a cardinal number $\kappa$ and an antiortholattice chain $A$.

Proof. Trivially, (iii) implies (i), which is equivalent to (ii). To prove that (ii) implies (iii), assume that all elements of $L \setminus \{0, 1\}$ are join–irreducible in $L_4$.

Then, by Corollary 13, $S(L) \setminus \{0, 1\} \subseteq \text{At}(L) \cap \text{CoAt}(L) \subseteq \text{At}(L) \cup \text{CoAt}(L) \subseteq S(L) \setminus \{0, 1\}$, so that $\text{At}(L) = \text{CoAt}(L) = S(L) \setminus \{0, 1\}$, thus $S(L)_0$ has length 3 if $S(L) \setminus \{0, 1\} \neq \emptyset$ (and, of course, 2 otherwise), hence $S(L) = \biguplus_{u \in S(L) \setminus \{0, 1\}} \{0, u, u', 1\} \cong \text{MO}_\kappa$ for a cardinal number $\kappa$ (such that $|S(L) \setminus \{0, 1\}| = 2\kappa$).

Now let $x, y \in L \setminus S(L) \subseteq L \setminus \{0, 1\}$, so that $x', y' \in L \setminus S(L)$, as well. Assume by absurdum that $x$ and $y$ are incomparable, so that $x'$ and $y'$ are also incomparable, thus $x \lor y \neq 0 = x' \lor y'$ are join–irreducible, hence $x \lor y = 1 = x' \lor y'$. If $x$ and $x'$ would be incomparable, then $0 \neq x \lor x'$ would be join–irreducible, so that $x \lor x' = 1$, which would contradict the fact that $x \not\in S(L)$. So $x$ and $x'$ are comparable, thus $x \land x' \in \{x, x'\}$. Analogously, $y$ and $y'$ are comparable, so $y \lor y' \in \{y, y'\}$. Since $L$ is a pseudo–Kleene algebra, it follows that $x \land x' \leq y \lor y'$. But $x \not\leq y$, hence either $x \land x' \neq x$ or $y \lor y' \neq y$, so that either $x' \geq x \leq y' \geq y$ or $x \geq x' \leq y \geq y'$. In the first of these two situations, we obtain $1 = x \lor y \leq x \lor y' = y'$, which contradicts the fact that $y' \neq 1$, while, in the second situation, we obtain $1 = x' \lor y' \leq y' \lor y = y$, which contradicts the fact that $y \neq 1$. Therefore $x$ and $y$ are comparable, hence $L \setminus S(L)$ is linearly ordered, so, by the structure of $S(L)$, $x$ is comparable with at most one element of $S(L) \setminus \{0, 1\}$ and, if $x$ and $x' \in S(L)$ would be incomparable, then $x \lor x' = 1$, which would contradict the fact that $x \not\in S(L)$. Hence $x$ and $x'$ are comparable, thus $0 = x \lor x' \in \{x, x'\}$, thus $x' = 0$ since $x \neq 0$. Hence $L \setminus S(L) \subseteq T(L)$, thus $L = S(L) \cup T(L)$, so, by Theorem 13, $T(L)$ is a subuniverse of $L$, thus $T(L)$ is an antiortholattice chain, and $L = S(L) \uplus T(L) = \text{MO}_\kappa \uplus T(L)$.

Proposition 42 Let $L$ be a nontrivial antiortholattice. Then the following are equivalent:

(i) all elements of $L \setminus \{0, 1\}$ are join–irreducible in $L_4$;

(ii) all elements of $L \setminus \{0, 1\}$ are meet–irreducible in $L_4$;

(iii) $L_4$ is a chain.

Proof. By Proposition 14 and the fact that $S(L) \cong D_2$.

Corollary 43 Let $L$ be a nontrivial PBZ$^*$–lattice that satisfies J1. Then: all elements of $T(L) \setminus \{0, 1\}$ are join–irreducible in $L_4$ if $L$ is a horizontal sum of an orthomodular lattice with an antiortholattice chain.

Proof. The converse is trivial. For the direct implication, let $a \in L \setminus S(L)$, so that $a \leq a \lor a' \in T(L) \setminus \{0, 1\}$, hence $a \in T(L)$ by Proposition 13, so $L = S(L) \cup T(L)$, therefore $L \in \text{OML} \uplus \text{AL}$ and $T(L)$ is a subalgebra of $L$ and thus an antiortholattice by Theorem 10. Furthermore, Proposition 42 ensures us that $T(L)$ is an antiortholattice chain.

Corollary 44 Let $L$ be a nontrivial PBZ$^*$–lattice.
Example 79 below disproves the converse inclusion.

(i) If all elements of $T(L) \setminus \{0, 1\}$ are join-irreducible in $L$, then $L = S(L) \cup T(L) \cup T(L)'$ and $T(L) \setminus T(L)' \subseteq \{x \in L : x \geq x'\}$. The converse is not true, even if $L$ fulfills $J_1$.

(ii) All elements of $D(L)$ are join-irreducible in $L$ iff $L$ is an antiortholattice chain.

**Proof.** [1] For the direct implication, consider $a \in L \setminus S(L)$, so that $a \lor a' \in T(L) \setminus \{0, 1\}$, hence $a = a \lor a' \in T(L) \setminus \{0, 1\}$ or $a' = a \lor a' \in T(L) \setminus \{0, 1\}$, so that $a \in T(L) \setminus \{0, 1\}$ and, if $a \notin T(L)'$, then $a = a \lor a'$, that is $a \geq a'$. To disprove the converse, see below the PBZ*-lattice $K$ in Example [79] which fulfills $K = S(K) \cup T(K) \cup T(K)'$, but in which the element $t' \in T(K) \setminus \{0, 1\}$ is join-reducible. Moreover, $T(K) \setminus T(K)' = \{t'\}$ and $t' \geq t$. Also, $K \in \mathcal{OML} \uplus V(\mathcal{AL})$, so $K \vdash J_1$ by Corollary [77].

[1] If all elements of $D(L)$ are join–irreducible in $L$, then 1 is join–irreducible in $L$, hence $L$ is an antiortholattice, thus $L$ is an antiortholattice chain by Proposition [42]. The converse is trivial. ■

**Proposition 45** $V(AOL) \subseteq \{L \in PBZL^*: L = \langle T(L)\rangle_{BZL}\}$.

**Proof.** By Lemma [10] $L = T(L) = \langle T(L)\rangle_{BZL}$ for any $L \in AOL$.

By Lemma [35] it follows that, for any non-empty family $(L_i)_{i \in I} \subseteq AOL$, $\prod_{i \in I} L_i = \prod_{i \in I} (T(L_i))_{BZL} = \langle \prod_{i \in I} L_i \rangle_{BZL}$, hence $L = \langle T(L)\rangle_{BZL}$ for any $L \in P(AOL)$.

Again by Lemma [36] we obtain that, for any $A \in P(AOL)$ and any subalgebra $B$ of $A$, $\langle T(B)\rangle_{BZL} = \langle T(A)\rangle_{BZL} \cap B = A \cap B = B$, hence $L = \langle T(L)\rangle_{BZL}$ for any $L \in SP(AOL)$.

We apply Lemma [36] once again and obtain that, for any $A \in SP(AOL)$ and any $\theta \in Con_{BZL}(A)$, $A/\theta = \langle T(A)\rangle_{BZL}/\theta = \langle T(A/\theta)\rangle_{BZL} \subseteq \langle A/\theta\rangle_{BZL}$, hence $A/\theta = \langle T(A/\theta)\rangle_{BZL}$, therefore $L = \langle T(L)\rangle_{BZL}$ for any $L \in HSP(AOL) = V(AOL)$.

Hence $V(AOL) \subseteq \{L \in PBZL^*: L = \langle T(L)\rangle_{BZL}\}$. The PBZ*-lattice $M$ in Example [79] below disproves the converse inclusion. ■

**Corollary 46** Let $L \in V(AOL)$. Then the following are equivalent:

(i) $T(L)$ is a subuniverse of $L$;

(ii) $T(L)$ is closed w.r.t. the Kleene complement;

(iii) $L$ is an antiortholattice.

**Proof.** (i) $\iff$ (ii). By Lemma [11]

(i) $\iff$ (iii). For any BZ-lattice $L$, $T(L)$ is the universe of a subalgebra of $L$ iff $\langle T(L)\rangle_{BZL} = T(L)$, hence, by Proposition [45] and Lemma [10] if $L \in V(AOL)$, then: $T(L)$ is a subuniverse of $L$ if $L = T(L)$ iff $L \in AOL$. ■

**Corollary 47** If $A \in OML \setminus \{D_1\}$, $B \in V(AOL) \setminus \{D_1\}$ and $L = A \uplus B$, then:

- $A = (L \setminus \langle T(L)\rangle_{BZL}) \cup \{0, 1\} \subseteq S(L)$ and $B = \langle T(L)\rangle_{BZL}$;
- $L = S(L) \cup B = S(L) \cup T(L)_{BZL}$;
- $L \in OML \uplus AOL$ iff $A = S(L)$ iff $B = T(L)$ iff $B \in AOL$.

$OML \uplus V(AOL) \subseteq \{L \in PBZL^* : L = S(L) \cup \langle T(L)\rangle_{BZL}\} \not\subseteq V(OML \uplus V(AOL))$. Moreover, $\{L \in PBZL^* : L = \langle T(L)\rangle_{BZL}\} \not\subseteq V(OML \uplus V(AOL))$. 23
6 Direct Irreducibility in Certain Varieties of PBZ*-lattices

Recall from [13] that antiortholattices are directly irreducible, and from [14] that, moreover, the class of the directly irreducible members of \( V(\mathbb{AOL}) \) is \( \mathbb{AOL} \). Now let us see that even the lattice reducts of antiortholattices are directly irreducible. In relation to this property, let us investigate pseudo–Kleene algebras with directly reducible lattice reducts, as well as bounded lattice complements in lattice reducts of antiortholattices.

**Proposition 48** Let \( A \) and \( B \) be bounded lattices. Then:

(i) if \( A, B \in \mathbb{BI} \) and they are non–trivial, then the direct product of BI–lattices \( A \times B \), endowed with the trivial Brouwer complement, fails condition (*);

(ii) if \( L \in \mathbb{PKA} \) is such that \( L_0 = A \times B \), then \( (0^A, 1^B)^L = (1^A, 0^B) \).

**Proof.** In the following, for brevity, we will drop the superscripts.

(i) If \( ^\sim : A \times B \to A \times B \) is the trivial Brouwer complement, then, in \( A \times B \), we have: \( (0, 1)' = (0', 1') = (1, 0) \) and \( (0, 1) \neq (1, 0) \neq (1, 0) \), hence: \((1, 1) = (0, 0)^\sim = ((0, 1) \wedge (1, 0))^\sim = (0, 1)^\sim \wedge (0, 1)^\sim = (0, 1) \vee (0, 1) \neq (0, 0) \neq (1, 1) \).

(ii) Let \( (0, 1)' = (a, b) \in L = A \times B \) and \( (1, 0)' = (c, d) \in L = A \times B \). Since \( L \in \mathbb{PKA} \), we have \( (0, b) = (b, 0) \leq (0, 1) \wedge (a, c) = (1, 0) \) and \( (1, 1) = (0, 0) \vee (a, b) \geq (1, 0) \wedge (c, d) = (c, 0) \), so that \( b \leq d \) in \( B \) and \( a \geq c \) in \( A \). Hence \( (a, d) = (a, b) \vee (c, d) = (0, 1)^\prime \vee (1, 0)^\prime = ((0, 1) \wedge (1, 0))^\prime = (0, 0)^\prime = (1, 1) \) and \( (c, b) = (a, b) \wedge (c, d) = (0, 1)^\prime \wedge (1, 0)^\prime = ((0, 1) \vee (1, 0))^\prime = (1, 1)^\prime = (0, 0) \), thus \( c = 0 \) and \( a = 1 \) in \( A \), while \( b = 0 \) and \( d = 1 \) in \( B \). Therefore \( (0, 1)^\prime = (a, b) = (1, 0) \).

Note that a BI–lattice \( L \) can be directly irreducible while \( L_0 \) is directly reducible; indeed, the BI–lattice \( \mathbb{D}_3 \equiv \mathbb{D}_3 \), in which the incomparable elements equal their involutions, is directly irreducible, but its lattice reduct is isomorphic to \( \mathbb{D}_2^2 \).

**Proposition 49** The lattice reduct of any antiortholattice is directly irreducible.

**Proof.** Let \( L \in \mathbb{AO\mathbb{L}} \) and assume ex absurdo that \( L_0 = A \times B \) for some non–trivial bounded lattices \( A \) and \( B \). Then \( (0, 1)' = (1, 0) \) by Proposition 48(ii), hence \( (0, 1) \in S(L) \), which contradicts the fact that \( L \) is an antiortholattice, since \( (0, 1) \notin \{(0, 0), (1, 1)\} \).
Proposition 50  The only complemented elements of the lattice reduct of a distributive antiortholattice are 0 and 1.

Proof. Let \( L \) be a distributive antiortholattice and assume by absurdum that, for some \( a, b \in L \setminus \{0, 1\} \), \( a \lor b = 1 \) and \( a \land b = 0 \), so that \( a' \land b' = (a \lor b)' = 1' = 0 \).

Since \( L_{ab} \in P(\mathcal{K}) \), we have \( b \land b' \leq a \lor a' \), hence \( b \land b' = (a \lor a') \land b \land b' = (a \land b \land b') \lor (a' \land b \land b') = 0 \lor 0 = 0 \), thus \( b \in S(L) \), which contradicts the fact that \( L \) is an antiortholattice. \( \blacksquare \)

Example 51  Here is a non–modular antiortholattice with other complemented elements beside 0 and 1, namely, in the following Hasse diagram, \( a \) and \( a' \) are bounded lattice complements of both \( b \) and \( b' \):

![Hasse diagram](image)

Lemma 52  If \( L, A \) and \( B \) are bounded lattices such that \( L = A \oplus B \), \( |A| > 2 \), \( |B| > 2 \) and \( |L| \geq 5 \), then \( L \) is directly irreducible.

Proof. Let \( L = (L, \land, \lor, 0, 1) \), and assume ex absurdo that \( L = K \times M \) for some nontrivial bounded lattices \( K \) and \( M \). Since \( |L| > 4 \), we may assume, w.l.g., that there exists \( a, u \in K \setminus \{0, 1\} \), so that \( (u, 1) \not\in \{(0, 1), (0, 1), (1, 1), (1, 1)\} \), hence \( (u, 0) \not\in \{(0, 1), (1, 0), (1, 0), (1, 0)\} \), hence \( (u, 0) \in A \setminus \{0, 1\} \). Now let \( v, w \in B \setminus \{0, 1\} \). Then \( u \land v, w = (u, 1) \land (v, 0) = 0 = (0, 1), (0, 0), (1, 0), (1, 0) \), thus \( 0 = w = 1 \), which contradicts the fact that \( M \) is nontrivial.

Hence \( L \) is directly irreducible. \( \blacksquare \)

Let \( L_1 \in \mathcal{O}ML \setminus \{D_1\} \) and \( L_2 \in \mathcal{A}OL \setminus \{D_2\} \), be such that \( L = L_1 \oplus L_2 \in \mathcal{O}ML \), whence \( L_2 \neq D_2 \). If \( L_1 = D_2 \), then, by Proposition 49, \( L \) is directly irreducible. If \( L_1 \neq D_2 \), then by Lemma 52, \( L_1 \) is likewise directly irreducible.

So, we obtain that:

Proposition 53  (i) If \( L \in (\mathcal{O}ML \oplus \mathcal{A}OL) \setminus \mathcal{O}ML \), then \( L \) is directly irreducible, thus \( L \) is directly irreducible.

(ii) If \( L \in (\mathcal{O}ML \oplus \mathcal{A}OL) \setminus (\mathcal{O}ML \setminus \mathcal{A}OL) \), then \( L \) is directly irreducible, thus \( L \) is directly irreducible.

Also, recall that any ortholattice \( L \) with more than 2 elements is such that 0 is meet–reducible and 1 is join–reducible in \( L \). Thus:

Corollary 54  If \( A \) is a finite ortholattice and \( B \) is a finite pseudo–Kleene algebra with \( |A| > 2 \) and \( |B| > 2 \), then \( A \oplus B \) has at least three distinct atoms (thus at least three distinct co–atoms).

Proof. Since \( A \) is not an antiortholattice, \( A_1 \) has at least two distinct atoms. Since \( B \) is finite and has \( |B| > 2 \), it follows that \( B_1 \) has at least one atom, which is not equal to 1. Our conclusion follows. \( \blacksquare \)
7 Singleton–Generated Subalgebras of PBZ*–lattices

Lemma 55 Let \( \mathcal{V} \) be a variety and \( \mathbb{C}, \mathbb{D} \) subclasses of \( \mathcal{V} \) such that, for all \( \mathbb{M} \in \mathbb{C} \) and all \( x \in M \), we have \( \langle x \rangle_{V,M} \in \mathbb{D} \). Let \( \mathbb{A} \in \mathcal{V} \) and \( a \in A \). Then:

(i) if \( \mathbb{A} \in \mathcal{P}_V(\mathbb{C}) \), then \( \langle a \rangle_{V,A} \in \mathcal{P}_V(\mathbb{D}) \);
(ii) if \( \mathbb{A} \in \mathcal{S}_V(\mathbb{C}) \), then \( \langle a \rangle_{V,A} \in \mathcal{S}_V(\mathbb{D}) \);
(iii) if \( \mathbb{A} \in \mathcal{H}_V(\mathbb{C}) \), then \( \langle a \rangle_{V,A} \in \mathcal{H}_V(\mathbb{D}) \);
(iv) if \( \mathbb{A} \in \mathcal{V}_V(\mathbb{C}) \), then \( \langle a \rangle_{V,A} \in \mathcal{V}_V(\mathbb{D}) \).

Proof. (i) For some non–empty family \( \langle A_i \rangle_{i \in I} \subseteq \mathbb{C} \), we have \( \mathbb{A} = \prod_{i \in I} A_i \), so that \( a = (a_i)_{i \in I} \), with \( a_i \in A_i \) for all \( i \in I \). Then \( \langle a \rangle_{V,A} = \prod_{i \in I} (a_i)_{V,A_i} = \prod_{i \in I} \langle a_i \rangle_{V,A_i} \in \mathcal{P}_V(\mathbb{D}) \).

(ii) For some \( \mathbb{B} \subseteq \mathbb{C} \) with \( \mathbb{A} \subseteq \mathbb{B} \), we have \( \mathbb{A} \subseteq \mathcal{S}_V(\mathbb{B}) \), therefore \( \langle a \rangle_{V,A} = \langle a \rangle_{V,B} \cap \mathbb{A} \subseteq \mathcal{S}_V(\mathbb{B}) \).

(iii) For some \( \mathbb{B} \subseteq \mathbb{C} \) and some \( \theta \in \mathcal{C}_V(\mathbb{B}) \), we have \( \mathbb{A} = \mathbb{B} \theta / \mathbb{C} \), so \( a = b / \theta \) for some \( b \in B \). Since \( b / \theta \in \mathcal{B}_V(\mathbb{B} \theta / \mathbb{C}) \), it follows that \( \langle b / \theta \rangle_{V,B / \theta} \subseteq \langle b \rangle_{V,B} / \theta \). If \( u \in \langle b \rangle_{V,B} \), then \( u = t_B(b) \) for some \( t \) over the type of \( V \), thus \( u / \theta = t_B \theta (b / \theta) \in \langle b / \theta \rangle_{V,B / \theta} \), therefore \( \langle b / \theta \rangle_{V,B / \theta} \subseteq \langle b / \theta \rangle_{V,B / \theta} \). Hence \( \langle a \rangle_{V,A} = \langle b / \theta \rangle_{V,B / \theta} = \langle b / \theta \rangle_{V,B / \theta} \in \mathcal{H}_V(\mathbb{D}) \).

(iv) By (i), (ii) and (iii), if \( \mathbb{A} \in \mathcal{V}_V(\mathbb{C}) = \mathcal{H}_V(\mathbb{C}) \), then \( \langle a \rangle_{V,A} \in \mathcal{H}_V(\mathbb{D}) \).

Note that, in any orthomodular lattice \( L \), \( \langle x \rangle_{\text{BZL},L} = \{0, x, x', 1\} \in \{D_1, D_2, D_3\} \subseteq \mathbb{B}A = V(D_2) \subseteq V(D_3) = V(D_4) \), where the latter equality follows from the easy to notice facts that \( D_3 \in \mathcal{H}_{\text{BZL}}(D_4) \) and \( D_4 \in S_{\text{BZL}}(D_2 \times D_3) \). If \( M \) is an antithollattice and \( x \in M \), then, clearly, \( \langle x \rangle_{\text{BZL},M} = \{0, x \wedge x', x, x' \vee x', 1\} \in \{D_1, D_2, D_1 \wedge D_2 \vee D_2 \} \subseteq \mathcal{A}_L \cap V(D_3) \) since \( D_2 \vee D_2 \vee D_2 \in S_{\text{BZL}}(D_1 \times D_1) \), more precisely: if \( M \cong D_1 \), then \( \langle x \rangle_{\text{BZL},M} = M \cong D_1 \), while, if \( M \) is non–trivial:

- if \( x \in \{0, 1\} \), then \( \langle x \rangle_{\text{BZL},M} = \{0, 1\} \cong D_2 \);
- if \( x \notin \{0, 1\} \), but \( x \) and \( x' \) are comparable, then \( \langle x \rangle_{\text{BZL},M} = \{0, x, x', 1\} \cong D_4 \);
- if \( x \notin \{0, 1\} \) and \( x \not\approx x' \), then \( \langle x \rangle_{\text{BZL},M} = \{0, x \wedge x', x, x' \vee x', 1\} \cong D_2 \vee D_2 \vee D_2 \).

Clearly, for any non–trivial orthomodular lattice \( L \), any non–trivial \( \text{PBZ}^* \)–lattice \( M \) and any \( x \in L \oplus M \), we have:

- if \( x \in \{0, 1\} \), then \( \langle x \rangle_{\text{BZL},L,M} = \{0, 1\} \cong D_2 \);
- if \( x \in L \setminus \{0, 1\} \), then \( \langle x \rangle_{\text{BZL},L,M} = \langle x \rangle_{\text{BZL},L} = \{0, x, x', 1\} \cong D_2 \);
- if \( x \in M \setminus \{0, 1\} \), then \( \langle x \rangle_{\text{BZL},L,M} = \langle x \rangle_{\text{BZL},M} \).

From the above, we obtain:

Proposition 56 Let \( \mathbb{A} \in \text{PBZL}^* \) and \( a \in A \). Then:

- if \( \mathbb{A} \in \text{OML} \), then \( \langle a \rangle_{\text{BZL},A} \in \{D_1, D_2, D_3\} \subseteq \mathbb{B}A \);
Lemma 55. (iv), it follows that, if $A \in AOL$, then $(a)_{BZL,A} \in \{D_1, D_2, D_4, D_2 \oplus D_2 \oplus D_2\} \subseteq AOL \cap V(D_3)$.

Lemma 60. Let $C, D$ be subclasses of $PBZ^*$ such that $C$ contains non-trivial algebras and, for all $M \in C$ and all $a \in M$, we have $(a)_{BZL,M} \in D$. Then, for all $A \in V(OML \oplus V(C))$ and all $a \in A$, we have $(a)_{BZL,A} \in V(D)$.

Proof. In any non-trivial PBZ*-lattice $M$, $(0)_{BZL,M} = \{0, 1\} \cong D_2$, hence $D_2 \in D$, therefore $BA \subseteq V(D)$. Now let $A \in PBZ^*$ and $a \in A$.

By Lemma 55 (iv), if $A \in V(C)$, then $(a)_{BZL,A} \in V(D) \supseteq BA$, therefore, by the above, if $A \in OML \oplus V(C)$, then $(a)_{BZL,A} \in BA \cup V(D) = V(D)$. Again by Lemma 55 (iv), it follows that, if $A \in V(OML \oplus V(C))$, then $(a)_{BZL,A} \in V(D)$.

Theorem 58. For any $A \in V(OML \oplus V(AOL))$ and all $a \in A$, we have $(a)_{BZL,A} \subseteq V(D_3)$.

Proof. By Proposition 56 and Lemma 57.

8 Congruences of Horizontal Sums

In order to better understand the properties of horizontal sums of PBZ*-lattices, it is crucial to investigate the structure of their congruence lattices — in particular, to find convenient descriptions of simple and subdirectly irreducible algebras. Thus, clearly, it is crucial to investigate the structure of their congruence lattices — in particular, to find convenient descriptions of simple and subdirectly irreducible.

Let $V$ be the variety of bounded lattices or one of the varieties $BI, BZL$, and let $L$ and $M$ be nontrivial members of $V$. Since $L$ and $M$ are subalgebras of $L \oplus M$, for any $\theta \in \text{Conv}(L \oplus M)$, we have $\theta \cap L^2 \in \text{Conv}(L)$ and $\theta \cap M^2 \in \text{Conv}(M)$; additionally, if $\theta \neq \nabla_{L \oplus M}$, then $\theta \cap L^2 \neq \nabla_L$, $\theta \cap M^2 \neq \nabla_M$ and $\theta = (\theta \cap L^2) \oplus (\theta \cap M^2)$.

Lemma 59 (23). For any bounded lattices $L$ and $M$ with $|L| > 2$ and $|M| > 2$,

$$\text{Con}_{01}(L \oplus M) = \{\alpha \oplus \beta : \alpha \in \text{Con}_{01}(L), \beta \in \text{Con}_{01}(M)\}$$

$$\cong \text{Con}_{01}(L) \times \text{Con}_{01}(M);$$

and

$$\text{eq}(L \setminus \{0\}, M \setminus \{1\}), eq(L \setminus \{1\}, M \setminus \{0\}), \nabla_{L \oplus M}.\]

Lemma 60. Let $V$ be one of the varieties $BI, BZL$, and let $A$ and $B$ be nontrivial members of $V$. Then, for any $\alpha \in \text{Conv}(A) \setminus \{\nabla_A\}$ and any $\beta \in \text{Conv}(B) \setminus \{\nabla_B\}$, we have: $\alpha \oplus \beta \in \text{Conv}(A \oplus B)$ iff $\alpha \oplus \beta \in \text{Con}(A \oplus B)$.

Proof. If $\alpha \in \text{Conv}(A) \setminus \{\nabla_A\}$ and $\beta \in \text{Conv}(B) \setminus \{\nabla_B\}$, then $\alpha$ preserves the involution of $A$, $\beta$ preserves the involution of $B$, $0/\alpha \neq 0/\beta$ and $1/\alpha \neq 1/\beta$, thus, clearly, $\alpha \oplus \beta$ preserves the involution of $A \oplus B$, and the same holds for the Brouwer complement in the case when $V = BZL$; hence, whenever $\alpha \oplus \beta$ is a lattice congruence of $A \oplus B$, it is a full congruence.

For $V = BI$, this result was proven in [23], but, for the sake of completeness, we provide a new proof for it here.
Proposition 61 Let $\mathcal{V}$ be one of the varieties $\mathbb{B}$ and $\mathbb{BZL}$ and $A$ and $B$ be members of $\mathcal{V}$ with $|A| > 2$ and $|B| > 2$ such that $A \oplus B \in \mathcal{V}$. Then:

- $\text{Con}_{\text{V01}}(A \oplus B) = \{ \alpha \oplus \beta : \alpha \in \text{Con}_{\text{V01}}(A), \beta \in \text{Con}_{\text{V01}}(B) \} \cong \text{Con}_{\text{V01}}(A) \times \text{Con}_{\text{V01}}(B)$;
- $\text{Con}_{\text{V}}(A \oplus B) = \text{Con}_{\text{V01}}(A \oplus B) \cup \{ \Delta_{AB} \} \cong (\text{Con}_{\text{V01}}(A) \times \text{Con}_{\text{V01}}(B)) \oplus D_2$.

Proof. Every proper congruence $\theta$ of $A \oplus B$ satisfies $\theta = (\theta \cap A^2) \oplus (\theta \cap B^2)$, with $\theta \cap A^2 \in \text{Con}_{\text{V}}(A) \setminus \{ \Delta_A \}$ and $\theta \cap B^2 \in \text{Con}_{\text{V}}(B) \setminus \{ \Delta_B \}$. Let $D = \{ \alpha \oplus \beta : \alpha \in \text{Con}_{\text{V01}}(A), \beta \in \text{Con}_{\text{V01}}(B) \}$. By Lemmas 50 and 59 it follows that:

$$\{ \Delta_{AB} \} \cup D \subseteq \text{Con}_{\text{V}}(A \oplus B)$$

$$\subseteq \{ \Delta_{AB}, \text{eq}(A \setminus \{0\}, B \setminus \{1\}), \text{eq}(A \setminus \{1\}, B \setminus \{0\}) \} \cup D.$$

Since $|A| > 2$, there exists an $a \in A \setminus \{0, 1\}$, so that $a' \in A \setminus \{0, 1\}$. Note that $\text{eq}(A \setminus \{0\}, B \setminus \{1\}) \cap A^2$ contains $(a, 0)$ but not $(a', 1)$, while $\text{eq}(A \setminus \{1\}, B \setminus \{0\}) \cap A^2$ contains $(a, 1)$ but not $(a', 0)$. Hence $\text{eq}(A \setminus \{0\}, B \setminus \{1\}) \notin \text{Con}_{\text{V}}(A \oplus B)$ and $\text{eq}(A \setminus \{1\}, B \setminus \{0\}) \notin \text{Con}_{\text{V}}(A \oplus B)$.

Therefore $\text{Con}_{\text{V}}(A \oplus B) = \{ \Delta_{AB} \} \cup \{ \alpha \oplus \beta : \alpha \in \text{Con}_{\text{V01}}(A), \beta \in \text{Con}_{\text{V01}}(B) \}$, so, clearly,

$$\text{Con}_{\text{V01}}(A \oplus B) = \{ \alpha \oplus \beta : \alpha \in \text{Con}_{\text{V01}}(A), \beta \in \text{Con}_{\text{V01}}(B) \} \cong \text{Con}_{\text{V01}}(A) \times \text{Con}_{\text{V01}}(B).$$

Hence

$$\text{Con}_{\text{V}}(A \oplus B) = \{ \Delta_{AB} \} \cup \text{Con}_{\text{V01}}(A \oplus B)$$

$$\cong \text{Con}_{\text{V01}}(A \oplus B) \oplus D_2$$

$$\cong (\text{Con}_{\text{V01}}(A) \times \text{Con}_{\text{V01}}(B)) \oplus D_2.$$

$
$

Corollary 62 Let $\mathcal{V}$ be one of the varieties $\mathbb{B}$ and $\mathbb{BZL}$, $n \in \mathbb{N} \setminus \{0, 1\}$ and $A_1, \ldots, A_n$ be members of $\mathcal{V}$ with $|A_i| > 2$ for all $i \in [1, n]$. Then:

- $\text{Con}_{\text{V01}}(\bigoplus_{i=1}^n A_i) = \{ \bigoplus_{i=1}^n \alpha_i : (\forall i \in [1, n]) (\alpha_i \in \text{Con}_{\text{V01}}(A_i)) \} \cong \prod_{i=1}^n \text{Con}_{\text{V01}}(A_i)$;
- $\text{Con}(\bigoplus_{i=1}^n A_i) = \text{Con}_{\text{V01}}(\bigoplus_{i=1}^n A_i) \cup \{ \Delta_{AB} \} \cong (\prod_{i=1}^n \text{Con}_{\text{V01}}(A_i)) \oplus D_2$;
- $\bigoplus_{i=1}^n A_i$ is subdirectly irreducible as a member of $\mathcal{V}$ if, for some $k \in [1, n]$, either $\text{Con}_{\text{V01}}(A_k) = \{ \Delta_A \}$ or $\text{Con}_{\text{V01}}(A_k)$ has a single atom, and, for every $i \in [1, n] \setminus \{k\}$, $\text{Con}_{\text{V01}}(A_i) = \{ \Delta_A \}$.

Recall from [3, Prop. 4.3] that if $L$ is an orthomodular lattice, then $L$ is congruence–regular and $\text{Con}_{\text{BZL}}(L) = \text{Con}_{\text{BZ1}}(L) = \text{Con}(L)$.

Proposition 63 Let $A$ be an orthomodular lattice and $B$ a $\text{BZ}$-lattice with $|A| > 2$ and $|B| > 2$. Then:

(i) $\text{Con}_{\text{BZL101}}(A \oplus B) = \{ \Delta_A \oplus \beta : \beta \in \text{Con}_{\text{BZL01}}(B) \} \cong \text{Con}_{\text{BZL01}}(B)$ and $\text{Con}_{\text{BZL}}(A \oplus B) = \text{Con}_{\text{BZL01}}(A \oplus B) \cup \{ \Delta_{AB} \} \cong \text{Con}_{\text{BZL01}}(B) \oplus D_2$.
Corollary 64 Let $A$ be an orthomodular lattice and $B$ a BZ–lattice with $|A| > 2$ and $|B| > 2$. Then:

(i) $A \boxplus B$ is simple iff $\text{Con}_{\text{BZL}_01}(B) = \{\Delta_B\}$;

(ii) $A \boxplus B$ is subdirectly irreducible iff either $\text{Con}_{\text{BZL}_01}(B) = \{\Delta_B\}$ or $\text{Con}_{\text{BZL}_01}(B)$ has a single atom;

(iii) if $B$ is an antiortholattice, then: $A \boxplus B$ is subdirectly irreducible iff $B$ is subdirectly irreducible;

(iv) if $B$ is an antiortholattice chain, then: $A \boxplus B$ is subdirectly irreducible iff $|B| \leq 5$.

Proof. (i) By Proposition 61.

(ii) By (i) and Proposition 63 (ii). By (i) and Corollary 24 (ii).

(iii) By Proposition 63 (ii).

(iv) By (iii) and Corollary 24 (ii). By (iii) and Corollary 24 (ii).

By Proposition 61 the PBZ∗–lattices $L$ with all elements in $L \setminus \{0,1\}$ join–irreducible belong to the subvariety $\text{HPBZL}^*$ of $\text{PBZL}^*$ generated by the horizontal sums of antiortholattice chains with arbitrary horizontal sums of Boolean algebras, which is generated by its finite members according to [13, Corollary 4.1], so the subvariety generated by these PBZ∗–lattices is generated by its finite subdirectly irreducible members, hence Corollary 64 gives us:

Corollary 65 $V(\{L \in \text{PBZL}^* : \text{all elements of } L \setminus \{0,1\} \text{ are join–irreducible in } L_i\}) = V(\{\text{MO}_k \boxplus D_n : k \in \mathbb{N}^*, n \in \{2,5\}\})$.

Corollary 66 Let $A \in \text{OML} \setminus \{D_1, D_2\}$, and let $(B_i)_{i \in I}$ be a nonempty family of nontrivial antiortholattices such that $B = \bigsqcup_{i \in I} B_i \neq D_2$. Then:

(i) if $A \boxplus B$ is simple, then $B_i$ is simple for each $i \in I$;

(ii) if $\prod_{i \in I} B_i$ has no skew congruences, in particular if $I$ is finite, then: $A \boxplus B$ is simple iff $B_i$ is simple for each $i \in I$;

(iii) if $\prod_{i \in I} B_i$ has no skew congruences, in particular if $I$ is finite, then: $A \boxplus B$ is subdirectly irreducible iff $B_i$ is simple for all $i \in I$ or, for some $j \in I$, $B_j$ is subdirectly irreducible, but not simple, and $\text{Con}_{\text{BZL}_01}(B_j)$ has no atoms for any $i \in I \setminus \{j\}$;

(iv) if $\prod_{i \in I} B_i$ has no skew congruences, in particular if $I$ is finite, and $\text{Con}_{\text{BZL}}(B_i)$ is finite for all $i \in I$, then: $A \boxplus B$ is subdirectly irreducible iff, for some $j \in I$, $B_j$ is subdirectly irreducible and $B_i$ is simple for all $i \in I \setminus \{j\}$.
we let $B \in A \cup \{D_1\}$, hence $\text{Con}_{\text{BZL}}(B_i) = \text{Con}_{\text{BZLO1}}(B_i) \cup \{\nabla_i\} \cong \text{Con}_{\text{BZLO1}}(B_i) \oplus D_2$, so that $B_i$ is simple iff $\text{Con}_{\text{BZLO1}}(B_i) = \{\Delta_{B_i}\}$, which has no atoms, and $B_i$ is subdirectly irreducible iff either $\text{Con}_{\text{BZLO1}}(B_i) = \{\Delta_{B_i}\}$ or $|\text{At}(\text{Con}_{\text{BZLO1}}(B_i))| = 1$.

3. Clearly, $\text{Con}_{\text{BZLO1}}(B) \supseteq \{\prod_{i \in I} \beta_i \mid (\forall i \in I) (\beta_i \in \text{Con}_{\text{BZLO1}}(B_i))\}$, so, if $B_k$ is not simple for some $k \in I$, then $\text{Con}_{\text{BZLO1}}(B) \supseteq \{\Delta_B\}$, so by Corollary 64.(i) $A \sqcup B$ is not simple.

4. If $\prod_{i \in I} B_i$ has no skew congruences, then $\text{Con}_{\text{BZLO1}}(B) = \{\prod_{i \in I} \beta_i \mid (\forall i \in I) (\beta_i \in \text{Con}_{\text{BZLO1}}(B_i))\}$, so by Corollary 64.(i) $A \sqcup B$ is simple iff $\text{Con}_{\text{BZLO1}}(B) = \{\Delta_B\}$ iff $\text{Con}_{\text{BZLO1}}(B_i) = \{\Delta_{B_i}\}$ for each $i \in I$ if $B_i$ is simple for each $i \in I$.

5. If $\prod_{i \in I} B_i$ has no skew congruences, then $\text{Con}_{\text{BZLO1}}(B) = \{\prod_{i \in I} \beta_i \mid (\forall i \in I) (\beta_i \in \text{Con}_{\text{BZLO1}}(B_i))\}$, from which it is easy to derive that

$$\text{At}(\text{Con}_{\text{BZLO1}}(B)) = \bigcup_{j \in I} \{\alpha_j \times \prod_{i \notin I \cup \{j\}} \Delta_{B_i} : \alpha_j \in \text{At}(\text{Con}_{\text{BZLO1}}(B_j))\}.$$ 

Set $\kappa = |\text{At}(\text{Con}_{\text{BZLO1}}(B))|$, and $\kappa_i = |\text{At}(\text{Con}_{\text{BZLO1}}(B_i))|$, for all $i \in I$. Thus $\kappa = \sum_{i \in I} \kappa_i$, and hence, by (ii): 

- $A \sqcup B$ is s.i. if $A \sqcup B$ is simple or $\kappa = 1$
  - if $B_i$ is simple for all $i \in I$ or, for some $j \in I$, $\kappa_j = 1$
    and $\kappa_i = 0$ for any $i \in I \setminus \{j\}$
  - if $B_i$ is simple for all $i \in I$ or, for some $j \in I$, $B_j$ is s.i.,
    but not simple, and $\kappa_i = 0$ for any $i \in I \setminus \{j\}$.

By (ii) and the fact that, if, for some $j \in I$, $\text{Con}_{\text{BZLO1}}(B_j)$ is finite, then: 

- $B_j$ is simple iff $\text{Con}_{\text{BZLO1}}(B_j) = \{\Delta_{B_j}\}$ iff $\kappa_j = 0$. 

**Corollary 67** Let $A \in \text{OML}\setminus\{D_1, D_2\}$, $I$ be a non-empty set and $(K_i)_{i \in I} \subseteq \text{PKA}$. For all $i \in I$, we consider the antiortholattice $B_i = D_2 \oplus K_i \oplus D_2$, and we let $B = \prod_{i \in I} B_i$. Then:

(i) if $A \sqcup B$ is simple, then $K_i \cong D_1$ for each $i \in I$;

(ii) if $\prod_{i \in I} B_i$ has no skew congruences, in particular if $I$ is finite, then: $A \sqcup B$ is simple iff $K_i \cong D_1$ for each $i \in I$;

(iii) if $\prod_{i \in I} B_i$ has no skew congruences, in particular if $I$ is finite, then: $A \sqcup B$ is subdirectly irreducible iff $K_i \cong D_1$ (that is $B_i \cong D_3$) for all $i \in I$ or, for some $j \in I$, the BI-lattice $K_j$ is nontrivial and subdirectly irreducible and $\text{Con}_{\text{BZ}}(K_i)$ has no atoms for any $i \in I \setminus \{j\}$;

(iv) if $\prod_{i \in I} B_i$ has no skew congruences, in particular if $I$ is finite, and $\text{Con}_{\text{BZ}}(K_i)$ is finite for all $i \in I$, then: $A \sqcup B$ is subdirectly irreducible iff, for some $j \in I$, the BI-lattice $K_j$ is subdirectly irreducible and $K_i \cong D_1$ for all $i \in I \setminus \{j\}$.

**Proof.** By Corollary 20 and Corollary 66.

**9 Varieties of PBZ*-Lattices Generated by Horizontal Sums**

The aim of this final section is to investigate the structures of varieties of PBZ*-lattices generated by horizontal sums and to provide axiomatic bases for some of
them. In particular, we will give a basis for \( V(\text{OML} \oplus \text{AOL}) \) relative to \( \text{PBZL}^* \), while the problem of finding a basis for \( V(\text{OML} \oplus V(\text{AOL})) \) is left open. In the process, we give a different proof to the axiomatization of the varietal join \( \text{OML} \lor V(\text{AOL}) \) relative to \( \text{PBZL}^* \), established in [13].

**Proposition 68** Let \( V \) be the variety of bounded lattices or one of the varieties \( \text{BI} \) and \( \text{BZL} \) and \( C \) and \( D \) be subclasses of \( V \). Then:

(i) if \( C \) and \( D \) are closed under subalgebras, then \( C \oplus D \) is closed under subalgebras;

(ii) if \( C \) and \( D \) are closed under quotients, then \( C \oplus D \) is closed under quotients.

**Proof.** By Lemma 30.

**Corollary 69** \( \text{OML} \oplus \text{AOL} \) and \( \text{OML} \oplus V(\text{AOL}) \) are closed w.r.t. subalgebras and quotients.

Observe that for any \( L \in \text{OML} \oplus \text{AOL} \), \( \text{Con}_{\text{BZL}}(L) = \text{Con}_{\text{BZL}}(L) \cup \{ \nabla \} \), so, by Lemma 36, \( T(L/\theta) = T(L)/\theta \) for any \( \theta \in \text{Con}_{\text{BZL}}(L) \).

The next batch of results is about the De Morgan laws and quotients. In particular, we show that \( \text{WSDM} \) is satisfied in \( \text{OML} \oplus \text{AOL} \) only in limit cases.

**Proposition 70** If \( A \in \text{PBZL}^* \setminus \text{AOL} \) and \( B \in \text{PBZL}^* \setminus \text{OML} \), then the algebra \( A \oplus B \) fails \( \text{WSDM} \).

**Proof.** It follows from our assumptions that neither algebra is \( D_1 \) or \( D_2 \), hence \( |A| > 2 \) and \( |B| > 2 \). Thus there exist an \( x \in B \setminus \{ 0, 1 \} \) and a \( y \in A \setminus \{ 0, 1 \} \) and, moreover, we can choose \( y \) such \( y \neq 0 \), because \( A \) is not an antiortholattice. But then \( y \neq 1 \neq \nabla y \), so we have \( \{ y, y, \nabla y \} \cap \{ 0, 1 \} = \emptyset \). We obtain: \( (x \land y)^\lor = 0^\lor = 1 \neq \nabla y = 0 \lor \nabla y = x \lor \nabla y \), so \( A \oplus B \) fails \( \text{WSDM} \).

Some corollaries follow.

**Corollary 71**

- If \( A \) is an orthomodular lattice with \( |A| > 2 \) and \( B \in \text{PBZL}^* \setminus \text{OML} \), then the PBZ* –lattice \( A \oplus B \) fails \( \text{WSDM} \).

- If \( A \) is an orthomodular lattice and \( B \) is an antiortholattice such that \( |A| > 2 \) and \( |B| > 2 \), then the PBZ* –lattice \( A \oplus B \) fails \( \text{WSDM} \).

**Corollary 72**

- Let \( L \in \text{OML} \oplus \text{AOL} \). Then: \( L \vdash \text{WSDM} \) iff \( L \in \text{OML} \cup \text{AOL} \).

- For all \( A \in \text{OML} \setminus \{ D_1, D_2 \} \) and all \( B \in \text{PBZL} \setminus \text{OML} \), we have \( A \oplus B \notin \text{OML} \lor V(\text{AOL}) \).

- For all \( A \in \text{OML} \setminus \{ D_1, D_2 \} \) and all \( B \in \text{AOL} \setminus \{ D_1, D_2 \} \), we have \( A \oplus B \notin \text{OML} \lor V(\text{AOL}) \).

**Proof.** By Corollary 71 and the fact that \( \text{OML} \lor V(\text{AOL}) \vdash \text{WSDM} \).

**Corollary 73**

- \((\text{OML} \oplus \text{AOL}) \cap V(\text{AOL}) = \text{AOL} \).

- \((\text{OML} \oplus \text{AOL}) \cap (\text{OML} \lor V(\text{AOL})) = \text{OML} \cup \text{AOL} \).

**Corollary 74** If a PBZ* –lattice \( L \) satisfies the SDM, then:
(i) \( \langle T(L) \rangle_{\text{BL}} = T(L) \cup T(L)' \);

(ii) \( \langle T(L) \rangle_{\text{BZL}} = T(L) \cup T(L)' \) iff either \( L \in \text{OML} \) or \( L \in \text{AOL} \) and 0 is meet-irreducible in \( L \) iff \( \langle T(L) \rangle_{\text{BZL}} = T(L) \) iff \( T(L) \) is a subuniverse of \( L \).

Proof. \( \Box \) By Lemma 11
\( \Box \) By (i), Lemma 11 Theorem 40 Corollary 72 and Lemma 10 ■

We now examine condition \( J2 \). It turns out that this weakened form of orthomodularity characterizes horizontal sums of an orthomodular lattice and of an antorhlolattice among all horizontal sums of \( \text{PBZ}^* \)-lattices.

Proposition 75 \( \text{OML} \oplus \text{AOL} \models J2 \).

Proof. We know that \( \text{OML} \models J2 \) and \( \text{AOL} \models J2 \). Now let \( A \in \text{OML} \) and \( B \in \text{AOL} \) with \( |A| > 2 \) and \( |B| > 2 \), and let \( L = A \oplus B \). Then by Lemma 38 \( S(L) = A \), from which it easily follows that \( L \models_{L,A} J2 \). Since \( B \) is an antortholattice, we have \( S(B) = \{0,1\} \), so, for every \( y \in L \setminus A = B \setminus \{0,1\} = B \setminus S(B) \), we have \( y \land y' \neq 0 \), thus \( (y \land y')^\sim = 0 \), so \( (y \land y') = 1 \), from which it easily follows that \( L \models_{L,L\cup A} J2 \). Therefore \( L \models J2 \). ■

Theorem 76 Let \( A \in \text{OML} \setminus \{D_1\} \) and \( B \in \text{PBZL}^* \setminus \{D_1\} \). Then:

- \( A \oplus B \models S1 \) iff \( B \models S1 \);
- \( A \oplus B \models S2 \) iff \( B \models S2 \);
- \( A \oplus B \models S3 \) iff \( B \models S3 \);
- \( A \oplus B \models J1 \) iff \( B \models J1 \);
- \( A \oplus B \models J2 \) iff \( B \in \text{OML} \oplus \text{AOL} \).

Proof. For the first four equivalences, the left-to-right implications are trivial, recalling that \( \text{OML} \models \{J1,S1,S2,S3\} \).

Denote by \( L = A \oplus B \), which is a \( \text{PBZ}^* \)-lattice by Proposition 27(11), and note that \( L \setminus A = B \setminus \{0,1\} \) and \( L \setminus B = A \setminus \{0,1\} \). By the above, to prove the right-to-left implications in the first four equivalences, it suffices to show that \( L \models_{A\setminus\{0,1\},B\setminus\{0,1\}} \{J1,S1,S2,S3\} \) and \( L \models_{B\setminus\{0,1\},A\setminus\{0,1\}} \{J1,S1,S2,S3\} \).

For all \( a \in A \setminus \{0,1\} \) and all \( b \in B \setminus \{0,1\} \), we have \( a \land b = 0 \), thus \( (a \land b)^\sim = 1 \), so \( (a \land b) = 0 \). Since \( A = S(A) \), \( a \land a' = 0 \), hence \( (a \land a')^\sim = 1 \), thus \( (a \land a') = 0 \), and, if \( b \in S(B) \), then \( b \land b' = 0 \), hence \( (b \land b')^\sim = 1 \), thus \( (b \land b') = 0 \). It immediately follows that \( L \models_{A\setminus\{0,1\},B\setminus\{0,1\}} \{J1,S1\} \), \( L \models_{B\setminus\{0,1\},A\setminus\{0,1\}} \{J1,S1\} \), \( L \models_{B\setminus\{0,1\},A\setminus\{0,1\}} \{J2,S2,S3\} \) and \( L \models_{A\setminus\{0,1\},S(B)\setminus\{0,1\}} \{J2,S2,S3\} \).

Now let \( a \in A \setminus \{0,1\} \) and \( b \in B \setminus S(B) \subseteq B \setminus \{0,1\} \), so that \( b \land b' \notin \{0,1\} \), thus \( (b \land b')^\sim \neq 1 \), hence \( a \land (b \land b')^\sim = 0 \), so \( (a \land (b \land b'))^\sim = 1 \). Since \( A = S(A) \) and \( a \notin \{0,1\} \), it follows that \( a^\sim = a' \notin \{0,1\} \). Since \( 0 \neq b \land b' \leq (b \land b')^\sim \), we have \( (b \land b') = 0 \), so \( a^\sim \lor b \land b' \neq 1 \), thus \( L \models_{A\setminus\{0,1\},B\setminus S(B)} S2 \), therefore \( L \models_{A\setminus\{0,1\},B\setminus S(B)} S2 \). Also, if \( (b \land b') \neq 1 \), then \( (b \land b')^\sim = 0 \) and \( a \land (b \land b') = 0 \), so that \( (a \land (b \land b')^\sim = 1 \) implies \( (b \land b') = 0 \), hence \( L \models_{A\setminus\{0,1\},B\setminus S(B)} S3 \), therefore \( L \models_{A\setminus\{0,1\},B\setminus S(B)} S3 \).

By Corollary 25(11), Proposition 75 and the commutativity and associativity of horizontal sums, \( \text{OML} \oplus \text{OML} \oplus \text{AOL} = \text{OML} \oplus \text{AOL} \models J2 \), which proves the right-to-left implication in the last equivalence. Now assume that \( L \models J2 \), so that \( L \models_{A\setminus\{0,1\},B\setminus S(B)} J2 \), thus, for all \( a \in A \setminus \{0,1\} \) and all \( b \in B \setminus S(B) \),

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\[ a = a \land \Diamond (b \land b'), \text{ hence } a \leq \Diamond (b \land b'), \text{ thus } \Diamond (b \land b') = 1, \text{ so } (b \land b')^\sim = 0, \text{ whence } b^\sim \leq b^\sim \lor \Box b = 0. \] Therefore \( B \setminus S(B) \subseteq T(B) \), hence \( B = S(B) \cup T(B) \), thus \( B \in OML \sqcup AOL \) by Theorem 40.

**Corollary 77**

- \( OML \sqcup AOL \not\equiv WSDM \);
- \( OML \sqcup V(AOL) \equiv \{S1, S2, S3, J1\} \);
- \( OML \sqcup AOL \not\equiv J2 \);
- \( OML \sqcup V(AOL) \not\equiv J2 \).

**Proof.** By Corollary 71 \( OML \sqcup AOL \not\equiv WSDM \).

By Theorem 76 and the fact that \( AOL \), and thus \( V(AOL) \), satisfies \( J1, J2, S1, S2 \) and \( S3 \), we have: \( OML \sqcup AOL \not\equiv J2 \) and \( OML \sqcup V(AOL) \not\equiv \{S1, S2, S3, J1\} \).

The PBZ*-lattice \( K \in OML \sqcup V(AOL) \) in Example 79 below fails \( J2 \), thus \( OML \sqcup V(AOL) \not\equiv J2 \).

**Corollary 78**

- \( OML \cup V(AOL) \subseteq V(OML \sqcup AOL) \subseteq V(OML \sqcup V(AOL)) \subseteq PBZ^* \);
- \( \{L \in OML \cup V(AOL) : L \not\equiv J2\} = OML \sqcup AOL \).

**Proof.** We will use Corollary 77 \( OML \sqcup AOL \not\equiv WSDM \) by Proposition 27 (ii), thus \( OML \cup V(AOL) \subseteq V(OML \sqcup AOL) \subseteq V(OML \sqcup V(AOL)) \subseteq PBZ^* \).

All these inclusions are proper. Indeed, \( OML \sqcup AOL \), and thus \( V(OML \sqcup AOL) \), fails WSDM, while \( OML \sqcup V(AOL) \) satisfies WSDM, therefore \( OML \cup V(AOL) \subseteq V(OML \sqcup AOL) \). \( OML \sqcup AOL \), and thus \( V(OML \sqcup AOL) \), satisfies \( J2 \), while \( OML \sqcup V(AOL) \), and thus \( V(OML \sqcup V(AOL)) \), fails \( J2 \), hence \( V(OML \sqcup AOL) \subseteq V(OML \sqcup V(AOL)) \).

The PBZ*-lattice \( L \) in Example 79 below fails \( S2 \), while \( OML \sqcup V(AOL) \) and thus \( V(OML \sqcup V(AOL)) \), satisfies \( S2 \). So \( L \in PBZ^* \setminus V(OML \sqcup V(AOL)) \), hence \( V(OML \sqcup V(AOL)) \subseteq PBZ^* \).

For the last bullet, the right-to-left inclusion follows from Corollary 77 and the other inclusion from Theorem 76 and Corollary 29 (i).

**Example 79** Let us consider the PBZ*-lattices \( M_3, K, L \) and \( M \), with the lattice orderings, elements and Kleene complements given by the diagrams below and: \( S(K) = \{0, u, u', s, s', 1\} \), \( T(K) = \{0, t', 1\} \) and \( t^\sim = s \) in \( K \). \( S(L) = \{0, s, s', 1\} \), \( T(L) = \{0, u', t\} \) and \( t^\sim = s \) in \( L \). \( S(M) = \{0, a, a', 1\} \), \( T(M) = \{0, z, t, u', v', z', 1\} \), \( u^\sim = a \) and \( v^\sim = a' \) in \( M \). By Corollaries 72 and 77 \( M_3 = D_2 \sqcup D_3 \) fails WSDM, thus \( M_3 \in (OML \sqcup AOL) \setminus (OML \sqcup V(AOL)) \).

\( K = D_2 \sqcup (D_2 \times D_3) \in (OML \sqcup V(AOL)) \setminus V(OML \sqcup AOL) \), because \( OML \sqcup AOL \), and thus \( V(OML \sqcup AOL) \), satisfies \( J2 \) by Proposition 73 while \( K \not\equiv J2 \), because, in \( K \), \( (u \land (t \land t')^\sim) \lor (u \land (t \lor t')) = (u \land t^\sim) \lor (u \lor t) = (u \land s) \lor (u \land s') = 0 \lor 0 = 0 \neq u \). Note, also, that \( K \equiv \{S2, S3\} \), by Corollary 77 and the fact that \( K \in OML \sqcup V(AOL) \).

In \( L \), \( u \lor t = u \neq t = 0 \lor t = (u \land s) \lor (u \land s') = ((u \lor t) \land t^\sim) \lor ((u \lor t) \lor t^\sim) \), therefore \( L \) fails \( J1 \), and \( (u \land (t \land t')^\sim) = (u \land t^\sim) \lor (u \land s) = 0 \lor 1 = 1 \neq s' = 0 \lor s^\sim = u \lor t^\sim = u \lor (t \lor t') \), therefore \( L \) fails \( S3 \). Furthermore, easy verifications establish that \( L \) satisfies \( S3 \) and fails \( J2 \).
We notice that \( M \) satisfies J1. Notice, also, that \( M \) fails S1, because, in \( M \), \((z' \land (z' \land a))^- = (z' \land a^-) = u^- = a \neq a' = a^- = \neg u = 0 \lor \neg (z' \land a) = z' \lor \neg (z' \land a)\). Note, also, that \((T(M))_{\text{BL}} = M\), so \( M = (T(M))_{\text{BL}} \cup (T(M))_{\text{E}} \). Since \( M \neq S1 \), while \( \text{OML} \supset V(\text{AOL}) \), and thus \( V(\text{OML} \supset V(\text{AOL})) \), satisfies S1 by Corollary, it follows that \( M \notin V(\text{OML} \supset V(\text{AOL})) \), in particular \( M \notin \text{OML} \supset V(\text{AOL}) \), so \( M \notin V(\text{AOL}) \). Easy verifications establish, furthermore, that \( M \) fails each of J2, S2 and S3.

\[ M_3 = D_2 \oplus D_3 : \quad K = D_2 \oplus (D_2 \times D_3) \]

![Diagram](image)

Recall that, if \( A \) is an algebra from a double–pointed variety \( V \) with constants 0, 1 then, if an element \( e \in A \) is central iff \( C_{g'}, A(e, 0) \) and \( C_{g'}, A(e, 1) \) are complementary factor congruences of \( A \). Let us denote by \( C(L) \) the set of the central elements of any PBZ–lattice \( L \).

**Lemma 80** [13] For any PBZ–lattice \( L \), \( C(L) = \{ a \in S(L) : (\forall b \in L) ((a \land b) = a \lor b, (a' \land b) = a' \lor b, b = (a \land b) \lor (a' \land b)) \} = \{ a \in S(L) : L \models L, \{ a \} \} SDM, L \models L, \{ a \} \{ SDM, J0 \} \}.

**Lemma 81** Any PBZ–lattice that satisfies J2, S2 and S3 and does not belong to \( \text{OML} \supset \text{AOL} \) is directly reducible.

**Proof.** Part of this argument has been applied in a result in [13] in a slightly different context; for the sake of completeness, we provide a complete proof of the present lemma.

Let \( L \) be a directly irreducible PBZ–lattice that satisfies J2, S2 and S3. Then the only central elements of \( L \) are 0 and 1. We want to show that \( S(L) \cup T(L) = L \).

Let \( x \in L \). Then the element \((x \land x')^- \) is sharp and, by J2, we have that \( L \models L, \{ x \land x' \} \lor \approx \approx (v \land u) \lor (v' \land u) \). Furthermore, by S2, we have that \( L \models L, \{ x \land x' \} \) SDM, while S3 gives us \( L \models L, \{ x \land x' \} \) SDM. By Lemma 81 \((x \land x')^- \) is a central element of \( L \), so \( \lor (x \land x') \) is central and thus \( \lor (x \land x') \in \{ 0, 1 \} \). If \( \lor (x \land x') = 0 \), then \( x \land x' \leq \lor (x \land x') = 0 \), whence \( x \in S(L) \). If \( \lor (x \land x') = 1 \), then \( x \land x' \lor \bot x = (x \land x') \approx = 0 \), so that \( x \in T(L) \).

Our claim is therefore settled, and by Theorem 40 \( L \) belongs to \( \text{OML} \supset \text{AOL} \).

**Proposition 82** All members of \( V(\text{OML} \supset \text{AOL}) \setminus (\text{OML} \supset \text{AOL}) \) are directly reducible. In particular, all subdirectly irreducible members of \( V(\text{OML} \supset \text{AOL}) \) belong to \( \text{OML} \supset \text{AOL} \).

**Proof.** By Lemma 81 and Corollary 77 which ensures us that \( V(\text{OML} \supset \text{AOL}) = \{ J2, S2, S3 \} \).
Theorem 83 \{L \in V(\text{OML} \oplus \text{AOL}) : L \vDash \text{WSDM}\} = \text{OML} \lor V(\text{AOL}).

Proof. We have \text{OML} \lor V(\text{AOL}) \subseteq \{L \in V(\text{OML} \oplus \text{AOL}) : L \vDash \text{WSDM}\}. Now let \(L \in V(\text{OML} \oplus \text{AOL})\) be such that \(L\) satisfies WSDM and is subdirectly irreducible. Then, by Proposition 82 we have \(L \in \text{OML} \oplus \text{AOL}\), and by Corollary 72 \(L \in \text{OML} \cup \text{AOL} \subseteq \text{OML} \lor V(\text{AOL})\), which completes the proof.

Theorem 84 \(\{L \in P_{\text{BZL}}^* : L \vDash \{J_2, S_2, S_3\}\} = V(\text{OML} \oplus \text{AOL})\).

Proof. By Corollary 77 all members of \(\text{OML} \oplus \text{AOL}\) satisfy the identities \(S_2, S_3\) and \(J_2\), hence the right-to-left inclusion is established. Lemma 81 gives us the converse inclusion.

Note that, since \(V(\text{OML} \oplus \text{AOL}) \models \{J_1, S_1\}\) according to Corollary 77, Theorem 83 shows that \(\{J_2, S_2, S_3\} \models \{J_1, S_1\}\). By Corollary 77 \(\text{OML} \oplus V(\text{AOL})\) satisfies \(J_1, S_1, S_2\) and \(S_3\) and fails \(J_2\), so \(\{J_1, S_1, S_2, S_3\} \not\models J_2\).

Theorem 84 and the fact that WSDM implies \(S_2\) and \(S_3\) give us a new proof for the following result from [14]:

Corollary 85 \(\{L \in P_{\text{BZL}}^* : L \vDash \{J_2, \text{WSDM}\}\} = \text{OML} \lor V(\text{AOL}).\)

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