DEPENDENT ARTIN-SCHREIER DEFECT EXTENSIONS AND STRONG MONOMIALIZATION

SAMAR ELHITTI AND LAURA GHEZZI

ABSTRACT. In this paper we affirmatively answer a question posed by F.-V. Kuhlmann. We show that the first Artin-Schreier defect extension in Cutkosky and Piltant’s counter-example to strong monomialization is a dependent extension. Our main tool is the use of generating sequences of valuations.

1. Introduction

Let $K$ be a field and let $\nu$ be a valuation on $K$. The value group of $(K, \nu)$ is denoted by $\nu K$, and its residue field by $K \nu$. The value of an element $a$ is denoted by $\nu(a)$. Let $L$ be a finite field extension of $K$. If the extension of $\nu$ from $K$ to $L$ is unique (denoted by $(L|K, \nu)$), the Lemma of Ostrowski says that

$$(1) \quad [L : K] = (\nu L : \nu K) \cdot [L \nu : K \nu] \cdot p^s,$$

with $s \geq 0$, where $p$ is the characteristic of $K \nu$ if it is positive, and $p = 1$ if $K \nu$ has characteristic zero. The factor $p^s$ is called the defect (or ramification deficiency) of the extension $(L|K, \nu)$. If $p^s = 1$ we call $(L|K, \nu)$ a defectless extension; otherwise we call it a defect extension. Note that $(L|K, \nu)$ is always defectless if $\text{char } K \nu = 0$. The defect plays a key role in deep open problems in positive characteristic, such as local uniformization (the local form of resolution of singularities). Indeed the existence of the defect makes the problem of local uniformization much harder, as can be seen in [7] and [8]. We refer the reader to [9] for an excellent overview of the valuation theoretical phenomenon of the defect.

Particular types of extensions that are ubiquitous in this set up are Artin-Schreier extensions. From now on we assume that all fields have characteristic $p > 0$. An Artin-Schreier extension of $K$ is an extension of degree $p$ generated over $K$ by a root $\theta$ of a polynomial of the form $X^p - X - a$ with $a \in K$. Such $\theta$ is called an Artin-Schreier generator of $L = K(\theta)$ over $K$. An extension of degree $p$ of a field of characteristic $p$ is a Galois extension if and only if it is an Artin-Schreier extension [11, Theorem VI.6.4]. We assume that $L|K$ has defect. There is a unique extension of $\nu$ from $K$ to $L$ (see [10].

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Lemma 2.31]). It follows from equation (1) that $L$ is an immediate extension of $K$; that is, $\nu_L = \nu_K$ and $L\nu = K\nu$.

In [10] Kuhlmann classifies Artin-Schreier defect extensions and gives the motivation for such a classification. If an Artin-Schreier defect extension is derived from a purely inseparable defect extension of degree $p$, then we call it a *dependent* Artin-Schreier defect extension. If it cannot be derived in this way, then we call it an *independent* Artin-Schreier defect extension. The precise definitions are given in Section 3.

Kuhlmann (see [9, Section 6.1]) raised the question of which of the Artin-Schreier defect extensions are more “harmful”, the dependent or the independent ones? He points out that there are some indications that the dependent ones are more harmful, for instance based on Temkin’s work [13].

In this paper we are interested in Cutkosky and Piltant’s counterexample to strong monomialization. In [4, Theorem 7.38] they give an example of an extension $R \subset S$ of two-dimensional algebraic regular local rings over a field $k$ of positive characteristic, and a valuation on the rational function field $Q(R)$. The extension $Q(S)\mid Q(R)$ is a tower of two Artin-Schreier defect extensions, such that strong monomialization (in the sense of [4, Theorem 4.8]) does not hold for $R \subset S$. We will recall the example in more details in Section 2.

Strong monomialization ([4, Theorem 4.8] in characteristic zero and any dimension) is a very important result, as it implies simultaneous resolution, which is very useful for applications to local uniformization (see [4, 2] for details). Cutkosky and Piltant further prove that strong monomialization holds over algebraically closed fields, in dimension two and positive characteristic, provided the field extension is defectless ([4, Theorem 7.35]). Recently Cutkosky gave a counterexample to local and weak local monomialization in positive characteristic and any dimension greater than or equal to two ([3, Theorem 1.4]).

Kuhlmann asked whether there is at least one dependent extension in the counterexample to strong monomialization – a further indication that dependent extensions are more harmful.

We prove in Theorem 4.6 that the first extension in the tower constructed in [4, Theorem 7.38] is an Artin-Schreier dependent extension.

It is still an open problem whether strong monomialization holds when independent Artin-Schreier defect extensions are involved.

We now state the layout of this paper. In Section 2 we recall the counterexample to strong monomialization. In Section 3 we recall the definition of dependent (and independent) Artin-Schreier defect extensions and establish that, in our context, it is enough to show that there exists $m \in \mathbb{N}_0$ such that for all $f \in K$, $\nu(\theta - f) < -\frac{1}{p^m}$. Finally, in Section 4 we prove that the extension under consideration is dependent (Theorem 1.6). Since the valuation is determined by a generating sequence in the given ring $R$, the main point is to construct generating sequences in appropriate quadratic transforms $R_k$ of $R$ that allow to compute or bound values of certain key elements of $R_k$. We achieve the
construction in Theorem 4.2 and we achieve the bound on the values of critical elements in $R_k$ in Theorem 4.4.

2. Counterexample to strong monomialization

In this section, we use the notation of [4, Section 7.11]. Let $k$ be an algebraically closed field of characteristic $p > 0$. Cutkosky and Piltant prove that there exists an inclusion $R \subset S$ of algebraic regular local rings of dimension 2 over $k$ such that $K^*|K = Q(S)|Q(R)$ is a tower of two Galois extensions of degree $p$, and a $k$-valuation $\nu^*$ of $K^*$ with valuation ring $V^*$ such that $V^*|V$ has defect $p^2$ (where $V$ is the valuation ring of the restriction of $\nu^*$ to $K$). Let $R_r$ and $S_s$ be iterated quadratic transforms of $R$ and $S$ respectively, such that $R_r \subset S_s$. Then there exists a regular system of parameters $(u_r, v_r)$ of $R_r$ and a regular system of parameters $(x_s, y_s)$ of $S$ such with such an expression:

$$u_r = \gamma_{r,s} x^p_s$$
$$v_r = \delta_{r,s} y^p_s$$

where $\gamma_{r,s}, \delta_{r,s}$ are units in $S_s$. Therefore strong monomialization in the sense of [4, Theorem 4.8] fails for the pair $R \subset S$ with respect to the extension of valuation rings $V^*|V$. See [4, Theorem 7.38] for the full statement of their result.

Remark 2.1. Strong monomialization requires that the inclusion $R_r \subset S_s$ be given by

$$u_r = \gamma_{r,s} x^t_s$$
$$v_r = \delta_{r,s} y_s$$

where $\gamma_{r,s}, \delta_{r,s}$ are units in $S_s$ and $t$ is a positive integer.

To construct the extension $K^*|K$, let $c \geq 1$ be an integer such that $(p - 1)|c$ and let $K^* = k(x, y)$ and $K = k(u, v)$ be two dimensional algebraic function fields, where

$$u := x^p/(1 - x^{p-1})$$
$$v := y^p - x^c y.$$ 

Then $K^*|K$ is a finite separable extension of degree $p^2$. It is a tower of two degree $p$ Artin-Schreier extensions

$$K \to K_1 = K(x) \to K^* = K_1(y).$$

We focus on the first extension $K(x)|K$. We have that $\frac{1}{x}$ is an Artin-Schreier generator with minimal polynomial $F(X) = X^p - X - \frac{1}{u}$.

We recall that the field $K^* = k(x, y)$ has the valuation $\nu^*$ determined by the generating sequence $Q_0 = x$, $Q_1 = y$, $Q_2 = y^p - x$ and $Q_{i+1} = Q_i^p - x^{2^i} Q_i$ for $i \geq 2$ in $S = k[x, y]|(x, y)$ and the value group denoted by $\Gamma^* \simeq \bigcup_{i \geq 0} \mathbb{Z}/p^i$ (see [4, Proposition 7.40]).

The field $K = k(u, v)$ has the valuation $\nu$ obtained by restriction of $\nu^*$ to $K$. $\nu$ is determined by the generating sequence $P_0 = u$, $P_1 = v$, $P_2 = v^p - u$ and $P_{i+1} = P_i^p - u^{2^{i-2}} P_{i-1}$ for $i \geq 2$ in $R = k[u, v]|(u, v)$ (see [4, Corollary 7.41]). We have that the
value group $\Gamma = \Gamma^* \simeq \bigcup_{i \geq 0} \mathbb{Z}/p^i$. We denote the unique extension of $\nu$ to $K_1$ by $\nu_1$. Notice that $\nu_1$ is the restriction of $\nu^*$ to $K_1$.

We prove in Theorem 4.6 that $K(x)$ is a dependent Artin-Schreier defect extension of $K$. Our main tool is generating sequences, therefore we briefly recall the definition (as in [12], [4, Section 7.5] or [5, Section 2]).

Definition 2.2. Let $R$ be an algebraic two dimensional regular local ring, let $K$ be the quotient field of $R$, and let $\nu$ be a valuation of $K$ centered in $R$. Let $(\nu_K)_+ = \nu(R \setminus \{0\})$ be the semigroup of $\nu_K$ consisting of the values of nonzero elements of $R$. For $\gamma \in (\nu_K)_+$, let $I_\gamma = \{ f \in R \mid \nu(f) \geq \gamma \}$. A (possibly infinite) sequence $\{P_i\}$ of elements of $R$ is a generating sequence of $\nu$ if for every $\gamma \in (\nu_K)_+$ the ideal $I_\gamma$ is generated by the set

$$\left\{ \prod_i P_i^{a_i} \mid a_i \in \mathbb{N}_0, \sum_i a_i \nu(P_i) \geq \gamma \right\}.$$

3. Artin-Schreier defect extensions and criteria for dependence

In this section we recall the definition of dependent and independent Artin-Schreier defect extensions as in [10, Section 4.1], or [9, Section 6]. Then we assemble results from [9] and [10] to obtain a criterion to detect if an Artin-Schreier defect extension is dependent in the context of the example under study recalled in Section 2.

Definition 3.1. Let $(K(\theta)|K, \nu)$ be an Artin-Schreier defect extension of valued fields of characteristic $p > 0$ with Artin-Schreier generator $\theta$ such that $\theta^p - \theta \in K$. We call $(K(\theta)|K, \nu)$ a dependent extension if there exists an immediate purely inseparable extension $(K(\eta)|K, \nu)$ of degree $p$ such that for all $c \in K$, $\nu(\theta - c) = \nu(\eta - c)$. An Artin-Schreier defect extension which is not dependent is called independent.

See for instance [9, Example 3.17] for an example of a dependent extension, and [9, Example 3.12] for an independent one. Several other examples of dependent and independent Artin-Schreier defect extensions are given in [10, Section 4.6]. We remark that by [10, Lemma 4.1], Definition 3.1 does not depend on the choice of the Artin-Schreier generator $\theta$.

Now assume that the value group $\nu_K$ is Archimedian; that is, it can be embedded by an order preserving isomorphism as a subgroup of $\mathbb{R}$.

Since $K(\theta)$ is an immediate extension of $K$ and $\nu(\theta - c) < 0$ for all $c \in K$ by [10, Corollary 2.30], we have that $\nu(\theta - K) \subseteq (\nu K)^{<0}$, where $\nu(\theta - K) = \{ \nu(\theta - c) \mid c \in K \}$ and $(\nu K)^{<0} = \{ \nu(c) \mid c \in K, \nu(c) < 0 \}$.

By [9, Theorem 6.1] and the discussion following it, we have that the extension $(K(\theta)|K, \nu)$ is independent if and only if $\nu(\theta - K) = (\nu K)^{<0}$.

Theorem 3.2. Let $(K(\theta)|K, \nu)$ be an Artin-Schreier defect extension with $\theta^p - \theta \in K$. Assume that $\nu K = \bigcup_{i \geq 0} \mathbb{Z}/p^i$. Then:
1) \( K(\theta)|K \) is independent if and only if for all \( m \in \mathbb{N}_0 \) there exists \( f_m \in K \) such that 
\[ \nu(\theta - f_m) \geq -\frac{1}{p^m}. \]
2) \( K(\theta)|K \) is dependent if and only if there exists \( m \in \mathbb{N}_0 \) such that for all \( f \in K, \nu(\theta - f) < -\frac{1}{p^m}. \)

**Proof.** By the above discussion \((K(\theta)|K, \nu)\) is independent if and only if \((\nu_K)^{<0} \subseteq \nu(\theta - K)\). Let \(-1/p^m \in (\nu_K)^{<0},\) where \(m \in \mathbb{N}_0\). Let \(f_m \in K\) be such that 
\[ \nu(\theta - f_m) \geq -\frac{1}{p^m}. \]
Since \(\nu(\theta - K)\) is an initial segment of \(\nu K\) (Lemma 1.1), we have that 
\[ -\frac{1}{p^m} \in \nu(\theta - K). \] This proves 1). \(\square\)

**4. Main result**

In this section, setup and notation are as in [4, Theorem 7.38] and Section 2. Recall that the algebraic function field \( K = k(u, v) \) has the valuation \( \nu \) determined by the generating sequence 
\[ P_0 = u, \ P_1 = v, \ P_2 = v^{p^2} - u \]
and 
\[ P_{i+1} = P_i^{p^2} - u^{p^{2i+2}} P_{i-1} \text{ for } i \geq 2 \]
in \( R = k[u, v]|_{(u, v)} \). We denote the unique extension of \( \nu \) to \( K_1 \) by \( \nu_1 \).

We normalize \( \nu \) so that \( \nu(u) = 1 \), from which we obtain the formulas \( \nu(P_0) = 1 \) and 
\[ \nu(P_i) = \sum_{j=0}^{i-1} p^{4j-2i} \text{ for } i \geq 1. \]

Let 
\[ R = R_0 \to R_1 \to R_2 \to \cdots \to R_k \to \cdots \]
be the sequence of iterated quadratic transforms along \( \nu \), where \( R_k \) are the local rings in the sequence of quadratic transforms of \( R \) along \( \nu \) which are free; the divisor of \( u \) in \( R_k \) has a single irreducible component. Let \( m_k \) be the maximal ideal of \( R_k \).

We have that the valuation ring \( V \) of \( \nu \) is \( V = \bigcup_{i=0}^{\infty} R_k \) ([1], see also [4, Corollary 7.41]).

**Remark 4.1.** Let \( t_k \in R_k \) be irreducible such that \( t_k = 0 \) is a local equation of the divisor of \( u \) in \( R_k \). For \( i \geq k + 1 \), let \( n_{k,i} \) be the largest power of \( t_k \) which divides \( P_i \) in \( R_k \). By [12], the discussion after [4, Definition 7.11] or [5, Theorem 7.1], we have that 
\[ \frac{P_{k+1}}{t_{n_{k,k+1}}} \]
is a regular system of parameters in \( R_k \) and 
\[ \frac{P_{k+1}}{t_{n_{k,k+1}}^2}, \frac{P_{k+1}}{t_{n_{k,k+1}}^2}, \cdots, \frac{P_{k+j}}{t_{n_{k,k+j}}^2}, \cdots \]
is a generating sequence in \( R_k \) for the valuation \( \nu \).

Next we explicitly construct generating sequences in the local rings \( R_k \).
Theorem 4.2. With notation as above, for \( k \geq 0 \) the local rings \( R_k \) have generating sequences

\[
    u_k = P_{k,0}, \quad v_k = P_{k,1}, \quad P_{k,2}, \ldots, P_{k,i}, \ldots
\]

which are defined recursively by the initial conditions

\[
    u_0 = u, \quad v_0 = v, \quad P_{0,i} = P_i \text{ for } i \geq 0,
\]

and, for \( k \geq 1 \), by

\[
    u_k = P_{k,0} = P_{k-1,1}, \quad v_k = P_{k,1} = \frac{P_{k-1,2}}{u_k^{p^2}},
\]

and by

\[
    P_{k,i} = \frac{P_{k-1,i+1}}{u_k^{p^2i}} \text{ for } i \geq 1.
\]

We thus have the formulas

\[
    \nu(P_{k,0}) = \frac{1}{p^{2k}}, \quad \nu(P_{k,1}) = \frac{1}{p^{2k+2}} \quad \nu(P_{k,i}) = \sum_{j=0}^{i-1} p^{4j-2i-2k} \text{ for } i \geq 1.
\]

Proof. Define \( u_0 = u, v_0 = v \) and \( P_{0,i} = P_i \) for \( i \geq 0 \).

We will inductively construct

\[
    P_{k,0}, P_{k,1}, \ldots, P_{k,i}, \ldots
\]

in \( R_k \) for \( k \geq 1 \) such that the following three conditions are satisfied:

C(1,k): Set

\[
    u_k = P_{k,0}, \quad v_k = \frac{P_{k-1,2}}{u_k^{p^2}}.
\]

Then \( u_k, v_k \) are regular parameters in \( R_k \) such that \( u = u_k^{p^{2k}} \tau_k \), where \( \tau_k \) is a unit in \( R_k \).

C(2,k): Set

\[
    P_{k,i} = \frac{P_{k-1,i+1}}{u_k^{p^2i}} \text{ for } i \geq 1.
\]

for \( i \geq 1 \). Then

\[
    u_k = P_{k,0}, \quad v_k = P_{k,1}, \quad P_{k,2}, \ldots, P_{k,i}, \ldots
\]

is a generating sequence for \( \nu \) in \( R_k \).

C(3,k): For \( i \geq 2 \), there exist units \( \gamma_{k,i} \in R_k \) such that

\[
    P_{k,i} = \begin{cases} 
        u_k^{p^2} - \gamma_{k,2}u_k = P_{k,1}^2 - \gamma_{k,2}P_{k,0} & \text{if } i = 2 \\
        P_{k,i-1} - \gamma_{k,i}u_k^{p^2(i-2)}P_{k,i-2} & \text{if } i \geq 3
    \end{cases}
\]

and \( \gamma_{k,i} \equiv 1 \mod m_k^2 \).
First, from the sequence \( \{P_{0,i}\} \), we construct the sequence \( \{P_{1,i}\} \) which satisfies C(1,1), C(2,1) and C(3,1), using a simplification of the following argument for constructing \( \{P_{k+1,i}\} \) for \( k \geq 1 \). We note that the simplification in this first step arises from the fact that the units \( \gamma_{k,0} \in R_0 \) are all equal to 1.

Now assume that \( k \geq 1 \) and that we have constructed the sequence \( \{P_{j,i}\} \) for \( j \leq k \), such that C(1,j), C(2,j) and C(3,j) hold for \( j \leq k \). We construct the sequence \( \{P_{k+1,i}\} \) such that C(1,k+1), C(2,k+1) and C(3,k+1) hold.

Define \( u_{k+1}, s_{k+1} \) by

\[
(5) \quad u_k = u_{k+1}^p s_{k+1} + 1, \quad v_k = u_{k+1}.
\]

Equation (5) defines a sequence of quadratic transforms

\[
R_k \rightarrow T_{k+1} := R_k[s_{k+1}]_{(u_{k+1}, s_{k+1})}.
\]

In the factorization of \( R_k \rightarrow T_{k+1} \) by quadratic transforms, \( T_{k+1} \) is the first local ring in which the divisor of \( u_k \) (and of \( u \)) has a single irreducible component. Since

\[
\nu(u_k^p - u_k) = \nu(u_k^p - \gamma_{k,2} u_k + \gamma_{k,2} u_k - u_k) > \nu(u_k) = \nu(u_k^p)
\]

by C(2,k) and C(3,k), we have that \( \nu(s_{k+1}) > 0 \). Therefore \( \nu \) dominates \( T_{k+1} \) and \( R_{k+1} = T_{k+1} \). Thus \( u_{k+1}, s_{k+1} \) are regular parameters in \( R_{k+1} \). By C(1,k), we have that

\[
u = u_k^{p_{2k}} \tau_k = u_{k+1}^{p_{2(k+1)}} \tau_{k+1},
\]

where

\[
\tau_{k+1} = (s_{k+1}^{p_{2k}} + 1) \tau_k.
\]

We have

\[
P_{k,2} = u_k^{p^2} - \gamma_{k,2} u_k
\]

\[
= u_{k+1}^{p^2} - \gamma_{k,2} u_{k+1}^p (s_{k+1} + 1)
\]

\[
= u_{k+1}^{p^2} ((1 - \gamma_{k,2}) - \gamma_{k,2} s_{k+1}).
\]

Thus

\[
v_{k+1} = \frac{P_{k,2}}{u_{k+1}^{p^2}}, \quad u_{k+1} \in R_{k+1}
\]

and \( u_{k+1}, v_{k+1} \) are regular parameters in \( R_{k+1} \). In particular, C(1,k+1) holds.

For \( i \geq 2 \), using C(2,k), C(3,k) and equation (5), we have

\[
u_{k+1}^{p^2} P_{k+1,i} = P_{k,i+1} = \frac{P_{k,i}}{u_k^{p^2}} - \gamma_{k,i+1} u_k^{p^2} P_{k,i-1}
\]

\[
= \frac{P_{k,i} - \gamma_{k,i+1}[u_{k+1}^{p^2} (s_{k+1} + 1)]^{p^2} P_{k,i-1}}{u_k^{p^2}}
\]

\[
= \frac{P_{k,i} - \gamma_{k,i+1} (s_{k+1}^{p^2} + 1)}{u_k^{p^2}} P_{k,i-1}.
\]

Thus for \( i = 2 \) we have

\[
u_{k+1}^{p^4} P_{k+1,2} = [u_{k+1}^{p^2} P_{k+1,1}]^{p^2} - \gamma_{k,3} (s_{k+1}^{p^2} + 1) u_{k+1}^{p^4 + 1}
\]

\[
= u_{k+1}^{p^2} [P_{k+1,1}^{p^2} - \gamma_{k,3} (s_{k+1}^{p^2} + 1) u_{k+1}^{p^4 + 1}].
\]
For $i \geq 3$ we have
\[
u \geq \sum_{j=1}^{i-1} p^{4j-2i-2k} + p^{4-2i-2k}.
\]

\begin{align*}
P_{k,i} &= \begin{cases} 
P_p^2 - P_{k,0} + \Lambda_{k,2} & \text{if } i = 2 \\
P_p^2 - P_{k,0}^{i-2} + \Lambda_{k,i} & \text{if } i \geq 3 \end{cases} \tag{6} 
\end{align*}

where
\[
u(\Lambda_{k,i}) \geq \sum_{j=1}^{i-1} p^{4j-2i-2k} + p^{4-2i-2k}.
\]

\begin{proof}
Assume that $k \geq 1$. Using equations (2) and (3) and the induction hypothesis we have that
\[
P_{k,i} = \frac{P_{k-1,i+1} - (P_{k-1,0}^{i-2} P_{k-1,i-1} + \Lambda_{k-1,i+1})}{u_{k-1}^{2i}} = \frac{P_{k-1,i+1} - (P_{k-1,0}^{i-2} P_{k-1,i-1} + \Lambda_{k-1,i+1})}{u_{k-1}^{2i}}.
\]

Thus we have that
\[
P_{k,2} = P_{k,2} - u_k + u_k P_{k-1,1} - \frac{\Lambda_{k-1,2} P_{k-1,1}}{u_k^{2i}} + \frac{\Lambda_{k-1,2} P_{k-1,1}}{u_k^{2i}}.
\]

and for $i \geq 3$,
\[
P_{k,i} = P_{k,i-1} - P_{k,i-2} u_k + u_k P_{k-1,i-1} - \frac{\Lambda_{k-1,2} P_{k-1,i-1}}{u_k^{2i}} + \frac{\Lambda_{k-1,2} P_{k-1,i-1}}{u_k^{2i}}.
\]

\end{proof}
Set

(8) \[ \Lambda_{k,2} = u_k v_k^{p^2} - \frac{\Lambda_{k-1,2}^p P_{k-1,1}}{u_k^{p^4}} + \frac{\Lambda_{k-1,3}}{u_k^{p^4}}, \]

and for \( i \geq 3, \)

(9) \[ \Lambda_{k,i} = u_k^{p^{2(i-2)}} v_k^{p^{2i-2}} P_{k,i-2} - \frac{\Lambda_{k-1,2}^{p^{2i-2}} P_{k-1,i-1}}{u_k^{p^{2i}}} + \frac{\Lambda_{k-1,i+1}}{u_k^{p^{2i}}} . \]

In order to verify equation (7), it suffices to show that the values of each of the three terms in equations (8) and (9) satisfy that bound. We use equation (11) and the induction hypothesis. We have

\[ \nu(u_k v_k^{p^2}) = 2p^{-2k}. \]

If \( i \geq 3, \)

\[ \nu(u_k^{p^{2(i-2)}} v_k^{p^{2i-2}} P_{k,i-2}) = 2p^{2i-2k-4} + \sum_{j=0}^{i-2} p^{4j-2i+4-2k} \]
\[ = 2p^{2i-2k-4} + \sum_{j=1}^{i-2} p^{4j-2i-2k} \]
\[ = \sum_{j=1}^{i-1} p^{4j-2i-2k} + p^{2i-2k-4} \]
\[ \geq \sum_{j=1}^{i-1} p^{4j-2i-2k} + p^{4-2i-2k} . \]

For \( i \geq 2, \)

\[ \nu\left(\frac{\Lambda_{k-1,2}^{p^{2i-2}} P_{k-1,i-1}}{u_k^{p^{2i}}} \right) = \nu(\Lambda_{k-1,2}^{p^{2i-2}}) + \nu(P_{k-1,i-1}) - p^{2i} \nu(u_k) \]
\[ \geq p^{2i-2}(2p^{-2k+2}) + \sum_{j=0}^{i-2} p^{4j-2i-2k+4} - p^{2i-2k} \]
\[ = p^{2i-2k} + \sum_{j=1}^{i-1} p^{4j-2i-2k} \]
\[ \geq p^{4-2i-2k} + \sum_{j=1}^{i-1} p^{4j-2i-2k} . \]

and

\[ \nu\left(\frac{\Lambda_{k-1,i+1}}{u_k^{p^{2i}}} \right) \geq \sum_{j=1}^{i} p^{4j-2i-2k} + p^{2i-2k} + p^{4-2i-2k} - p^{2i-2k} \]
\[ \geq \sum_{j=1}^{i-1} p^{4j-2i-2k} + p^{4-2i-2k} . \]

This completes the proof.

Set

\[ \Omega = \sum_{i=0}^{\infty} \left( \frac{1}{p^4} \right)^i = \frac{p^4}{p^4 - 1} . \]

Consider the Artin-Schreier extension \( K_1 = K(x) \) of \( K. \) We have that \( x \) is defined by

\[ u = \frac{x^p}{1 - x^{p-1}} . \]

Recall that \( \nu_1 \) is the extension of \( \nu \) to \( K_1. \) Write

(10) \[ x^p = u + h, \]
where $\nu_1(h) = 1 + \frac{p-1}{p} = 2 - \frac{1}{p}$. We have

$$\left(2 - \frac{1}{p}\right) - \frac{p^4}{p^4 - 1} = \frac{p^4(p - 1) - 2p + 1}{p(p^4 - 1)} > \frac{p^4 - 2p}{p(p^4 - 1)} = \frac{p^3 - 2}{(p^4 - 1)} > 0,$$

since $p \geq 2$. Thus

(11) \quad \nu_1(h) > \Omega.

We further have

(12) \quad -\frac{2}{p} + \frac{1}{\Omega} < -\frac{1}{p^2},

since $\frac{1}{p^2} - \frac{2}{p} + \frac{1}{\Omega} = -\frac{1}{p} (\nu_1(h) - \Omega)$.

**Theorem 4.4.** With the above notation, for $k \geq 0$, we have the following:

1) There exists $h_k \in R_k$ and $\Theta_k \in K_1 = K(x)$ such that

$$h_k^p - x^p = (-1)^k P_{k,k+2} + \Theta_k$$

where $\nu_1(\Theta_k) > \Omega$. Thus

$$\nu_1(h_k^p - x^p) = 1 + \frac{1}{p^4} + \cdots + \frac{1}{p^{4k}} + \frac{1}{p^{4(k+1)}}.$$

2) If $g \in R_k$ then

$$\nu_1(g^p - x^p) \leq 1 + \frac{1}{p^4} + \cdots + \frac{1}{p^{4k}} + \frac{1}{p^{4(k+1)}}.$$

**Proof.** We first prove statement 1), using induction on $k$. In the case $k = 0$, we observe from (10) and (11) that $x^p = u + h$ with $\nu_1(h) > \Omega$. Set $h_0 = P_{0,1}^p$ and $\Theta_0 = -h$. Then

$$h_0^p - x^p = P_{0,1}^p - u + \Theta_0 = P_{0,2} + \Theta_0.$$

Now assume that there exists $h_k \in R_k$ such that

$$h_k^p - x^p = (-1)^k P_{k,k+2} + \Theta_k$$

with $\nu_1(\Theta_k) > \Omega$. Using equations (3) and (6) we have

$$P_{k,k+2} = u_{k,k+2}^{p(k+1)} P_{k+1,k+1} = P_{k+1,0} P_{k+1,k+1} = -P_{k+1,k+3} + P_{k+1,k+2}^2 + \Lambda_{k+1,k+3}.$$ 

Set $h_{k+1} = h_k + (-1)^{k+1} P_{k+1,k+2}$. Then

$$h_{k+1}^p - x^p = (-1)^{k+1} P_{k+1,k+2}^2 + (-1)^k P_{k+1,k+3} + (-1)^k P_{k+1,k+2}^2 + (-1)^k \Lambda_{k+1,k+3} + \Theta_k$$

$$= (-1)^{k+1} P_{k+1,k+3} + (-1)^k \Lambda_{k+1,k+3} + \Theta_k.$$
Set $\Theta_{k+1} = (-1)^k \Lambda_{k+1,k+3} + \Theta_k$. In order to show that $\nu_1(\Theta_{k+1}) > \Omega$, we must show that $\nu_1(\Lambda_{k+1,k+3}) > \Omega$. By equation (7) we have

$$\nu_1(\Lambda_{k+1,k+3}) \geq \sum_{j=1}^{k+2} p^{4j-4k-8} + p^{-4k-4} \left( \sum_{j=0}^{k+1} (p^4)^j + 1 \right) \geq \sum_{j=0}^{k+2} \frac{p^{4j} - 4}{1 - p^4} + 1 \geq \frac{p^{4(k+2)} + p^4 - 2}{p^{4k+4}(p^4 - 1)}.$$ 

Thus

$$\nu_1(\Lambda_{k+1,k+3}) - \Omega \geq \frac{p^4 - 2}{p^{4k+4}(p^4 - 1)} > 0.$$ 

We now prove statement 2). Fix $k \geq 0$. Suppose, by way of contradiction, that there exists $g \in R_k$ such that

$$\nu_1(g^p - x^p) > 1 + \frac{1}{p^4} + \cdots + \frac{1}{p^4(k+1)}.$$ 

Set $g' = g - h_k$. Then

$$g^p - x^p = (g')^p + (h_k^p - x^p) = (g')^p + (-1)^k P_{k,k+2} + \Theta_k.$$ 

Now, since $\{P_{k,i}\}_{i \geq 0}$ is a generating sequence in $R_k$, we may write $g' = \lambda G + H$, where $0 \neq \lambda \in k$, $\nu_1(g') = \nu_1(G)$, $\nu_1(H) > \nu_1(G)$ and

$$G = u_k^m P_{k,1}^a P_{k,2}^a \cdots P_{k,n}^a$$

with $m, n, a_i \in \mathbb{N}_0$ and $a_i < p^2$ for all $i$ (see Definition [2.2] and [6, Lemma 5.1]). Set

$$\alpha = \nu_1(P_{k,k+2}) = \sum_{j=0}^{k+1} p^{4j-4k-4}$$

and $\beta = \nu_1(G^p)$. We have $\alpha = \beta$, otherwise $\nu_1(g^p - x^p) \leq \alpha$. By equation (4) we have

$$\beta = mp^{-2k+1} + \sum_{i=1}^{n} a_i \left( \sum_{j=0}^{i-1} p^{4j-2i-2k+1} \right).$$

Notice that if $a_i = 0$ for all $i$, then $\beta = mp^{-2k+1} \neq \alpha$. Therefore we may suppose that $a_n \neq 0$. Then we can write

$$\beta = a_n p^{-2n-2k+1} + c_1 p^{-2n-2k+3}$$

for some $c_1 \in \mathbb{Z}$ and

$$\alpha = p^{-4(k+1)} + b_1 p^{-4k}$$

for some $b_1 \in \mathbb{Z}$. Notice that, if $2n + 2k - 3 \geq 4k + 4$, the equality $\alpha = \beta$ yields that $a_n$ is a multiple of $p^2$, a contradiction since $0 < a_n < p^2$. 

We thus have $2n + 2k - 3 < 4k + 4$; that is, $n \leq k + 3$. By a similar argument we must have $n = k + 3$ and $a_{k+3} = p$. Thus, using equation (3)

\[
0 \leq \beta - p\left(\nu_1(P_{k,k+3})\right) = \beta - p\left(\sum_{j=0}^{k+2} p^{4j - 4k - 5}\right) = \alpha - p\left(\sum_{j=0}^{k+2} p^{4j - 4k - 5}\right) = \sum_{j=0}^{k+1} p^{4j - 4k - 4} - \sum_{j=0}^{k+2} p^{4j - 4k - 5} = -p^4 < 0
\]

giving a contradiction. □

**Theorem 4.5.** With the above notation, suppose that $f \in K$. Then

\[
\nu_1\left(\frac{1}{x} - f\right) < -\frac{2}{p} + \frac{1}{p} \Omega < -\frac{1}{p^2}.
\]

**Proof.** We may assume that $f \in K$ is such that $\nu_1\left(\frac{1}{x} - f\right) > -\frac{1}{p}$. Then $\nu_1(f) = \nu_1\left(\frac{1}{x}\right) = -\frac{1}{p}$. Set $g = \frac{1}{f}$. We have that $\nu(g) = -\frac{1}{p} > 0$, so $g \in V$ (the valuation ring of $\nu$), and thus $g \in R_k$ for some $k \geq 0$. Now we compute, using Theorem 4.4 and equation (12),

\[
\nu_1\left(\frac{1}{x} - f\right) = \nu_1\left(\frac{1}{x} - \frac{1}{g}\right) = \nu_1\left(\frac{g - x}{xg}\right) = \nu_1(g - x) - \nu_1(xg) = \frac{1}{p} \nu_1(g^p - x^p) - \frac{2}{p} \leq \frac{1}{p} \left(1 + \frac{1}{p^2} + \cdots + \frac{1}{p^{t(k+1)}}\right) - \frac{2}{p} < \frac{1}{p} \Omega - \frac{2}{p} < -\frac{1}{p^2}.
\]

□

**Theorem 4.6.** Let $k$ be an algebraically closed field of characteristic $p > 0$, let $K = k(u,v)$ be an algebraic function field, let $K_1 = K(x)$, where $u = x^p/(1 - x^{p-1})$. Let $\nu$ be the valuation on $K$ determined by the generating sequence $P_0 = u$, $P_1 = v$, $P_2 = v^p - u$ and $P_{i+1} = P_i^p - u^{2^{i-2}} P_{i-1}$ for $i \geq 2$ in $R = k[\nu, v]|_{(u,v)}$. Let $\nu_1$ be the unique extension of $\nu$ to $K_1$. Then $(K_1, \nu_1)$ is a dependent Artin Schreier defect extension of $(K, \nu)$.

**Proof.** Recall that $\frac{1}{x}$ is an Artin-Schreier generator of $K_1$ over $K$. The conclusion now follows from Theorem 3.2 and Theorem 4.5. □
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Department of Mathematics, New York City College of Technology-CUNY, 300 Jay Street, Brooklyn, NY 11201, U.S.A.
E-mail address: selhitti@citytech.cuny.edu

Department of Mathematics, New York City College of Technology-CUNY, 300 Jay Street, Brooklyn, NY 11201, U.S.A.
E-mail address: lghezzi@citytech.cuny.edu