Quantum 2+1 Evolution Model

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Abstract

A quantum evolution model in 2+1 discrete space–time, connected with 3D fundamental map \( R \), is investigated. Map \( R \) is derived as a map providing a zero curvature of a two dimensional lattice system called “the current system”. In a special case of the local Weyl algebra for dynamical variables the map appears to be canonical one and it corresponds to known operator-valued \( R \) – matrix. The current system is a kind of the linear problem for 2+1 evolution model. A generating function for the integrals of motion for the evolution is derived with a help of the current system. The subject of the paper is rather new, and so the perspectives of further investigations are widely discussed.

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1 Introduction

1.1 3D integrable models

In 3D integrable models the Tetrahedron equation (TE) takes place of the Yang – Baxter equation (YBE) in 2D. Having got a solution of TE, one may hope to construct a 3D integrable model. In the case of finite number of states one may construct usual layer – to – layer transfer matrices \( T \), so that TE provides the commutability of them [1, 2, 3]. Such finite states models are interpreted usually as statistical mechanics models. Really only one such model still exists, the Zamolodchikov – Bazhanov – Baxter model [1, 2, 3, 4]. The uniqueness does not mean that 3D world has no interest.

When 3D \( R \)-matrices have infinitely many states, which is more usual in 3D, very natural is to investigate a kind of transfer matrices that has no hidden space. We denote such transfer matrices as \( U \) versus the notation for usual transfer matrix \( T \). Matrices \( U \) commute with the set of \( T \)-s, but have no degrees of freedom when the set of \( T \) is fixed. Thus \( U \) resemble a hamiltonian. Conventionally models with infinitely many states are regarded as field theory ones.

The structure of \( U \) is clarified in Fig. 1 for 2D case. Here \( p \) and \( q \) stand for the spectral parameters of the vertices of \( T \), \( p/q \) consequently is the argument of the vertices of \( U \), and \( \sigma_j \) and \( \sigma'_j \) are the indices taking values in a finite set. This 2D picture we give just an
example for the sake of clearness. The 1 + 1 evolution models, connected with classical or quantum bilinear Hirota or Hirota-Miwa equitations on the lattice, are always formulated in terms of $U$-type evolution operators, see for example [8, 8] and references therein.

In 3D, $U$-matrix appears as an element of a cubic lattice included between two nearest inclined planes. We do not draw the graphical representation of 3D $U$ here, we will consider sections of the cubic lattice by two, in- and out-, inclined planes mentioned. A two dimensional lattice appearing in such sections is called the lagome lattice and we will consider it in details below. The first who considered $U$ – matrices in 3D, constructed with a help of finite – state $R$ – matrix, and constructed some eigenvectors for it, was I. Korepanov [8, 8, 8].

### 1.2 3D integrability: usual approach

So, the origin of 3D integrability is a solution of TE. Those who dealt with it know that it is practically impossible to find it directly. For example, even to prove TE analytically for an ansatz given and tested numerically is bloody complicated [12, 13]. This means, we guess, there must be an alternative way of a 3D Boltzmann weights' derivation.

Primitive way is to find a solution of TE is to consider the intertwining relation for a triple sets of 2D $L$-matrices,

$$
\sum_{j_1,j_2,j_3} R_{j_1,j_2,j_3}^{k_1,k_2,k_3} (L_{j_1}^{k_1})_{1,2} (L_{j_2}^{k_2})_{1,3} (L_{j_3}^{k_3})_{2,3} = \sum_{j_1,j_2,j_3} (L_{j_1}^{k_1})_{1,3} (L_{j_2}^{k_2})_{1,3} (L_{j_3}^{k_3})_{2,3} R_{j_1,j_2,j_3}^{k_1,k_2,k_3},
$$

where the structure of $L_{1,2}L_{1,3}L_{2,3}$ versus $L_{1,2}L_{1,3}L_{2,3}$ is the Yang – Baxter structure, and the extra indices correspond to the possibility to consider coefficients $R_{j_1,j_2,j_3}^{k_1,k_2,k_3}$ as 3D $R$-matrix. TE appears as the admissibility condition for

$$
L_{1,2} L_{1,3} L_{2,3} L_{1,4} L_{2,4} L_{3,4} \mapsto L_{3,4} L_{2,4} L_{1,4} L_{2,3} L_{1,3} L_{1,2}.
$$

Another scenario is the Zamolodchikov tetrahedral algebra

$$
\Psi_{1,2}^{k_1} \Psi_{1,3}^{k_2} \Psi_{2,3}^{k_3} = \sum_{j_1,j_2,j_3} \Psi_{2,3}^{j_2} \Psi_{1,3}^{j_3} \Psi_{1,2}^{j_1} R_{j_1,j_2,j_3}^{k_1,k_2,k_3}.
$$

In the compact form, introducing the formal basis for the indices of $R$, $e(i,j) \equiv |i j|$, 

$$
R_{a,b,c} = \sum_{j,k} R_{j_1,j_2,j_3}^{k_1,k_2,k_3} e_a(j_1,k_1) e_b(j_2,k_2) e_c(j_3,k_3).
$$

TE looks like

$$
R_{1,2,3} \cdot R_{1,4,5} \cdot R_{2,4,6} \cdot R_{3,5,6} = R_{3,5,6} \cdot R_{2,4,6} \cdot R_{1,4,5} \cdot R_{1,2,3},
$$

where the alphabetical indices, labelling the numbers of the spaces, are conventionally changed to the numerical indices, such change we will make frequently.

Most amusing thing is that all these really give a 3D $R$-matrix: Korepanov’s $R$-matrix [8, 14], Korepanov’s $R$-matrix as well as Hietarinta’s one are some special cases of more general, complete $R$-matrix derived by Sergeev, Mangazeev and Stroganov [8], and complete $R$-matrix is equivalent to Zamolodchikov – Bazhanov – Baxter’s weights in the thermodynamic limit.
1.3 3D integrability: functional approach

A way to get something else in 3D is to refuse the finite number of states in the previous approach. Namely, 3D models appear in the local Yang – Baxter equation (LYBE) approach. LYBE (i.e. a Yang – Baxter equation with different “spectral” parameters in the left and right hand sides) can be adapted to a discrete space – time evolution of the triangulated two dimensional oriented surface as a kind of zero curvature condition [15, 16, 17].

In few words, if a matrix $L_{i,j}(x)$, acting as usual in a tensor product of two finite dimensional spaces labelled by numbers $i$ and $j$, with some fixed functional structure and depending on a set of parameters $x$, obeys the equation

$$L_{1,2}(x_a) L_{1,3}(x_b) L_{2,3}(x_c) = L_{2,3}(x'_c) L_{1,3}(x'_b) L_{1,2}(x'_a), \tag{1.6}$$

called the local Yang–Baxter equation, so that parameters $x_a$, $x_b$ and $x_c$ are independent and $x'_a$, $x'_b$ and $x'_c$ can be restored from (1.6) without any ambiguity,

$$x'_a = f_a(x_a, x_b, x_c), \quad x'_b = f_b(x_a, x_b, x_c), \quad x'_c = f_c(x_a, x_b, x_c), \tag{1.7}$$

then the functional map $R$ is introduced:

$$R_{a,b,c} \cdot \varphi(x_a, x_b, x_c) \cdot R^{-1}_{a,b,c} = \varphi(x'_a, x'_b, x'_c) \quad \forall \varphi(...). \tag{1.8}$$

Due to the difference of the “spectral” parameters in the left and right hand sides of LYBE, any shift of a line of a two dimensional lattice, constructed with a help of $L_{i,j}(x_{i,j})$, changes the set of parameters $x_{i,j}$. Partially, if any shift of the lines can be decomposed into primitive shifts like (1.6) in different ways, then corresponding different products of $R$-s coincide. The basic example of this is the functional Tetrahedron equation.
Suppose we move all the lines of the lattice in some regular way, conserving a structure of the lattice. Then the change of parameters $x_{i,j}$ can be considered as an one-step evolution of the dynamical variables $x_{i,j}$ governed by an appropriately defined evolution operator $U = \prod_{\text{triangles}} R$. This evolution is integrable due to uniqueness of LYBE and (1.7). The partition function for the lattice becomes the natural integral of motion. In terms of the transfer matrices, the partition function is the $T$-type transfer matrix. Being functional, these R-operators correspond to something infinitely dimensional. Contrary to the previous finite dimensional $R$-matrices, there are known a lot of such R-operators. The reader can find an interesting set of such simplest functional R-s in [19].

A quantization of known functional R-s is still open problem. Simplest functional R-s are to be regarded as some functional limits of multivariable R-s with a symplectic structure conserving. The problem is to rise known R-s to the complete phase space case, this is done just for a couple of R-s.

1.4 3D integrability: general concept of evolution

Here we discuss, what else can be invented to get a 3D integrability.

The main observation is that the relations like tetrahedral Zamolodchikov algebra and LYBE have usual graphical interpretation as the equality of the objects assigned to two similar graphs. These graphs are the triangles, and we will denote them briefly as $\triangle$ and $\triangledown$. Left hand side type graph $\triangle$ corresponds to a product like $L_{1,2} L_{1,3} L_{2,3}$, and right hand side type graph $\triangledown$ corresponds to $L_{2,3} L_{1,3} L_{1,2}$. Algebraic objects are assigned to the elements of these graphs. In the case of LYBE these algebraical objects are matrix $L$ with the indices assigned to the edges (in the form of subscript, for $L_{1,2}$ 1 and 2 stand for the edges), and parameter $x$ assigned to the vertex. From 3D point of view $x$-s are the dynamical variables, whereas $L$ and its indices are auxiliary objects. The equality of l.h.s. of LYBE and r.h.s. of LYBE gives the notion of the algebraic equivalence of $\triangle$ and $\triangledown$. Note, this form of the algebraic equivalence is not obligatory!

3D integrability we can get from any other decent definition of an algebraic equivalence.

In this paper we consider a system, associated with a set of equivalent planar graphs. We propose another notion of an algebraic equivalence of equivalent graphs.

We will deal with all elements of the cw-complex, so we start from recalling the relationship between the elements of a planar graph and repeating some definitions.

Consider a graph $G_n$ formed by $n$ straight intersecting lines. The elements of its cw-complex are the vertices, the edges and the sites. $G_n$ consists on $N_V = \frac{n(n-1)}{2}$ vertices, $N_S = \frac{(n-1)(n-2)}{2}$ closed inner sites and $N_S^* = 2n$ outer open sites, $N_E = n(n-2)$ closed inner edges and $N_E^* = 2n$ outer edges. If two graphs $G_n$ and $G'_n$ have the same outer structure, i.e. $G'_n$ can be obtained from $G_n$ by appropriate shift of the lines, then call $G'_n$ and $G_n$ equivalent.

Suppose we assign to the elements of a graph some elementary algebraic (maybe, the term “arithmetical” is more exact) objects. These objects are divided into two classes: dynamical variables and auxiliary objects (see the interpretation of $L_{1,2}(x)$ above in this
subsection). Dynamical variables are parameters of graph $G_n$, and auxiliary objects give some two dimensional rules of a game (like the summation over all intermediate indices in the product of $L$-s). Dynamical variables plus a rule of game give an algebraic object corresponding to the whole graph (like the partition function for $L$-s). This algebraic (arithmetical) object for whole $G_n$ we call the observable object. Denote it $O(G_n)$. It depends on the set of the dynamical variables.

Consider now two equivalent graphs, $G_n$ and $G'_n$. The algebraic problem of the equivalence arises,

$$O(G_n) = O(G'_n).$$

(1.9)

If, according to the two dimensional rules of the game, we can get (1.9) choosing the dynamical variables for $G'_n$ appropriately for the variables of $G_n$ given, then the algebraic equivalence makes a sense. If, moreover, parameters of $G'_n$ can be restored from the algebraic equivalence condition (1.9) without any ambiguity, then this equivalence is decent and the integrability is undoubted.

The algebraic equivalence usually called zero curvature, and LYBE as the zero curvature condition as well as functional evolution models was considered in [15, 16, 17]. Another formulation of the algebraic equivalence, different to the LYBE approach, is Korepanov’s matrix model (see [8, 31] and references therein). The formulation of the matrix model differs from the usual assigning the vertex – type Boltzmann weights to the vertices of a lattice, but functional evolution models probably are the same.

We chose another way.

The method we use was formulated originally in [20], the classical (i.e. functional) evolution model was described in [22], and the quasiclassical case was investigated in [21]. This paper contains the overview of the method, and the description of the quantum evolution model. The main new result is the generating function for integrals of motion for this evolution.

2 Auxiliary Linear Problem

In this section we give some rules allowing one to assign an algebraic system to a graph. The elements to which we assign something are vertices and sites. First, we give most general rules, which do not give an algebraic equivalence of equivalent graphs in general, due to a sort of a “gauge ambiguity”. As a special case we find the rules which contain not a gauge ambiguity, and so a notion of an algebraic equivalence will be introduced. Then we describe the map of the dynamical variables given by the equivalence of 2-simplices, and discuss other similar approaches giving this map.

2.1 General approach

Choose as a game the following rules:

- Assign to each oriented vertex $V$ an auxiliary “internal current” $\phi$. Suppose this current produces four “site currents” flowing from the vertex into four adjacent faces, and proportional to the internal current with some coefficients $a, b, c, d$, called the dynamical variables, as it is shown in Fig. 2. All this variables, $\phi$ and $a, ..., d$ for different vertices are independent for a while.

- Define the complete site current as an algebraic sum of the contributions of vertices surrounding this site.
• For any closed site of a lattice let its complete current is zero. Such zero relations we regard as the linear equations for the internal currents.

• For any graph $G_n$ the site currents assigned to outer (open) sites we call the “observable currents”. In part, two equivalent graphs $G_n$ and $G'_n$ must have the same observable currents – this is the algebraic meaning of the equivalence.

Clarify these rules on the example of equivalence of $G_3$. As it was mentioned, this is usual Yang – Baxter equivalence graphically, $\triangle = \triangledown$, shown in Fig. 3. Assign to the vertices $W_j$ of the left hand side graph $\triangle$ the currents $\phi_j$ and the dynamical variables $a_j, b_j, c_j, d_j$, and to the vertices $W'_j$ of the right hand side graph $\triangledown$ – the currents $\phi'_j$ and the dynamical variables $a'_j, b'_j, c'_j, d'_j$. Six currents of outer sites denote as $\phi_b, \ldots, \phi_g$, and two zero valued currents of closed sites – as $\phi_h$ and $\phi_a$ as it is shown in Fig. 3. Then, using the rules described above, we obtain the following system of eight linear (with respect to the currents) relations:
\[ \phi_h \equiv c_1 \cdot \phi_1 + a_2 \cdot \phi_2 + b_4 \cdot \phi_3 = 0, \quad (2.1) \]
\[
\begin{cases}
\phi_b \equiv \phi_1' = c_2 \cdot \phi_2 + d_3 \cdot \phi_3,
\phi_c \equiv a_2' \cdot \phi_2' = a_1 \cdot \phi_1 + a_3 \cdot \phi_3,
\phi_d \equiv b_3' \cdot \phi_3' = d_1 \cdot \phi_1 + b_2 \cdot \phi_2,
\phi_e \equiv b_2' \cdot \phi_2' + a_3' \cdot \phi_3' = b_1 \cdot \phi_1,
\phi_f \equiv d_1' \cdot \phi_1' + d_4' \cdot \phi_4' = d_2 \cdot \phi_2,
\phi_g \equiv a_1' \cdot \phi_1' + c_2' \cdot \phi_2' = c_3 \cdot \phi_3,
\phi_a \equiv b_1' \cdot \phi_1' + d_2' \cdot \phi_2' + c_4' \cdot \phi_3' = 0. \quad (2.2)
\end{cases}
\]

Given are the currents and the dynamical variables for the left hand side graph. Due to \( \phi_h = 0 \), eq. (2.1), only two currents are independent, let them be \( \phi_1 \) and \( \phi_3 \). All the variables for the right hand side graph we try to restore via the linear system. First, use \( \phi_b, \phi_c \) and \( \phi_d \) to express all \( \phi_j' \). Substitute \( \phi_j' \) into relations for \( \phi_e, \phi_f \) and \( \phi_g \), then it will appear three homogeneous linear relations for two arbitrary \( \phi_1 \) and \( \phi_3 \), so six coefficients of \( \phi_1 \) and \( \phi_3 \) must vanish. Solving this six equations with respect to the primed variables, we obtain
\[
\begin{align*}
b_2' a_2'^{-1} &= \Lambda_1^{-1} \cdot b_3 a_3^{-1}, \quad a_3' b_3'^{-1} = \Lambda_1^{-1} \cdot a_2 b_2^{-1}, \\
d_1' c_1'^{-1} &= \Lambda_2^{-1} \cdot b_3 d_3^{-1}, \quad d_3' b_3'^{-1} = \Lambda_2^{-1} \cdot c_1 d_1^{-1}, \\
a_1' c_1'^{-1} &= \Lambda_3^{-1} \cdot a_2 c_2^{-1}, \quad c_2' a_2'^{-1} = \Lambda_3^{-1} \cdot c_1 a_1^{-1},
\end{align*}
\] (2.5)

where three polynomials arisen:
\[
\begin{align*}
\Lambda_1 &= b_3 a_3^{-1} a_1 b_1^{-1} - c_1 b_1^{-1} + a_2 b_2^{-1} d_1 b_1^{-1}, \\
\Lambda_2 &= b_3 d_3^{-1} c_2 d_2^{-1} - a_2 b_2^{-1} d_1 b_2^{-1} + c_1 d_1^{-1} b_2 d_2^{-1}, \\
\Lambda_3 &= a_2 c_2^{-1} d_3 c_3^{-1} - b_3 c_3^{-1} + c_1 a_1^{-1} a_3 c_3^{-1}.
\end{align*}
\] (2.6)

Substituting \( \phi_j' = 0 \) (2.4), we obtain the homogeneous linear equation for \( \phi_1, \phi_3 \) again, and the coefficients of them vanish if
\[
\begin{align*}
b_1' c_1'^{-1} &= \Lambda_a \Lambda_1 (c_2 b_2^{-1} d_1 b_1^{-1} + h_3 a_3^{-1} a_1 b_1^{-1})^{-1}, \\
d_2' a_2'^{-1} &= \Lambda_a \Lambda_2 (a_1 b_1^{-1} b_2 d_2^{-1} + a_3 d_3^{-1} c_2 d_2^{-1})^{-1}, \\
c_3' b_3'^{-1} &= \Lambda_a \Lambda_3 (d_1 a_1^{-1} a_3 c_3^{-1} + h_2 c_2^{-1} d_3 c_3^{-1})^{-1},
\end{align*}
\] (2.7)

where \( \Lambda_a \) is arbitrary. The origin of \( \Lambda_a \) technically is \( \phi_a = \Lambda_a \cdot \phi_h \).

This \( \Lambda_a \) is a sort of a gauge. The origin of it is that due to \( \phi_a = 0 \) we may change it \( \phi_a \mapsto \lambda_a \phi_a \), this gives \( \Lambda_a \mapsto \lambda_a \Lambda_a \), or equivalent
\[
\begin{align*}
b_1' &\mapsto \lambda_a b_1', \\
d_2' &\mapsto \lambda_a d_2', \\
c_3' &\mapsto \lambda_a c_3'.
\end{align*}
\] (2.8)
As the consequence of simple re-scaling of the currents almost nothing changes if
\[ a \mapsto a \lambda, \quad b \mapsto b \lambda, \quad c \mapsto c \lambda, \quad d \mapsto d \lambda \quad (2.10) \]
partially in all vertices \( W_j \) and \( W'_j \) with six different \( \lambda_j \) and \( \lambda'_j \).

Thus in the most general interpretation: the map \( W_1, W_2, W_3 \mapsto W'_1, W'_2, W'_3 \) is defined up to projective ambiguity \( \lambda_1, \lambda_2, \lambda_3, \lambda_h \mapsto \lambda'_1, \lambda'_2, \lambda'_3, \lambda_h \).

Very important feature of all these calculations is that

\[
\text{we never tried to commute anything!}
\]

Return to a general case of graph \( G_n \). \( 4 N_V = 2 n(n - 1) \) free invertible variables \( a_V, b_V, c_V, d_V \), assigned to the vertices \( V \) of \( G_n \), we regard as the generators of a body \( \mathcal{B}(G_n) \). Let \( \mathcal{B}_P(G_n) \) be the set of functions on \( \mathcal{B}(G_n) \) invariant with respect to the vertex ambiguity \( (2.10) \). Note in general, for an open graph \( G_n \) one may consider \( \mathcal{B}'_P(G_n) \) - set of functions invariant with respect to both vertex and closed site ambiguities. But such general considerations of \( \mathcal{B}'_P \) for the closed graphs, i.e. the graphs defined on the torus, needs a notion of a trace (or of a characteristic polynomials), or equivalent, of an algebra. The algebra will be introduced in the subsequent section.

Consider a little change of \( G_n \), so that only one \( \triangle \) in \( G_n \) transforms into \( \bigtriangledown \). Call the resulting graph \( G'_n \). Let the vertices involved into this change are marked as \( W_1, W_2, W_3 \) for \( \triangle \) and \( W'_1, W'_2, W'_3 \) for \( \bigtriangledown \) arranged as in Fig. \( 3 \). Introduce a functional operator \( R = R_{1,2,3} \) making the corresponding map on \( \mathcal{B}_P \):

\[
R_{1,2,3} \cdot \varphi(W_1, W_2, W_3, ...): R^{-1}_{1,2,3} = \varphi(W'_1, W'_2, W'_3, ...), \quad \varphi \in \mathcal{B}_P, \quad (2.11)
\]

where \( W_j \) stands for \( \{ a_j, b_j, c_j, d_j \} \) forever, and all other vertices except \( W_1, W_2, W_3 \) and their variables remain untouched. This \( R \) we call the \textbf{fundamental map}.

Let now \( G'_n \) be an arbitrary graph equivalent to \( G_n \). \( G'_n \) can be obtained from \( G_n \) by different sequences of elementary \( \triangle \mapsto \bigtriangledown \) in general. Thus the corresponding different sequences of \( R \)-s must coincide, this is natural admissibility (or associability) condition for \( G_n \mapsto G'_n \).

Note that in terms of functional operators the sequence of naïve geometrical transformations is antihomomorphic to the sequence of corresponding functional maps.

The simplest case is the equivalence of two quadrilaterals, \( G_4 \), and the admissibility condition is nothing but the Tetrahedron equation \( (1.1) \). And due to the ambiguity of \( R \), \( (2.8, 2.9) \), any admissibility condition is still equation for \( \Lambda_n \)-th involved. Note, \( \mathcal{B}'_P(G_n) \) introduced previously, is gauge invariant subspace of \( \mathcal{B}_P \). \( R \) acts on \( \mathcal{B}'_P \) uniquely. Unfortunately the basis of \( \mathcal{B}'_P \) is not local, and it is simpler to introduce an algebra constraints removing the projective ambiguities then to consider \( \mathcal{B}'_P \) formally.
A way to remove $\Lambda_q$-ambiguity from the definition of $R$, (2.3,2.7), is to impose some additional conditions for the elements of $W$, $a,b,c,d$, such that (2.7) would become a consequence of (2.5) and the additional conditions.

Complete classification of these additional conditions is still the open problem, and this is the main mathematical problem of this approach.

2.2 Local case: the Weyl algebra

Here we consider a special local case: suppose first that the elements of two different $W_i$ and $W_j$ for the given $G_n$ commute. Destroy also the vertex projective invariance choosing $a_j \equiv 1$ for any $j$ forever. Then (2.3) give the expressions for $b'_2, b'_3, d'_j c'_j^{-1}, d'_j b'_j^{-1}, c'_1, c'_2$. Suppose also any pair of the variables from $W$ are linearly independent, then

- the commutability of the elements for different $W'_j$ from $\nabla$ gives (after some calculations) $bc = qcb$ with the same $C$-number $q$ for any vertex,
- these relations conserve by the map $R$, i.e. $b'c' = qc'b'$.
- Also $b^{-1}c^{-1}d$ appear to be centres, depending on the vertex.

The gauge ambiguity becomes the ambiguity for these centres. We are going to get a sort of quantum theory, $b$ and $c$ are already quantized, so we have to keep all centres to be invariant, $b_j^{-1}c_j^{-1}d_j = b'_j^{-1}c'_j^{-1}d'_j$. This is possible, and further we will threat these centres as a kind of spectral parameters.

Change now notations for the dynamical variables to more conventional, and write down the resulting expressions for the map $R$. New notations for the site currents are shown in Fig.4.

Proposition. Let the vertex dynamical variables are given by

$$a = 1, \quad b = q^{1/2} u, \quad c = w, \quad d = \kappa u w, \quad (2.12)$$

here $u, w$ obey the local Weyl algebra relation,

$$u \cdot w = q \cdot w \cdot u, \quad (2.13)$$
and \( u \) and \( w \) for different vertices commute, and number \( \kappa \) is the invariant of the vertex, i.e. \( \kappa_{i,j} \), assigned to the intersection of lines \( i \) and \( j \), is the same for all equivalent graphs.

Then the problem of the algebraic equivalence (i.e. equality of the outer currents) of two graphs: \( G \) with the data \( \phi, u, w \), and \( G' \) with the data \( \phi', u', w' \), can be solved without any ambiguity with respect to all \( \phi', u', w' \), and the local Weyl algebra structure for the set of \( u', w' \) is the consequence of the local Weyl algebra relations for the set of \( u, w \).

Write the fundamental simplex map for \( \Delta = \nabla \) explicitly. The map \( R = R_{1,2,3} : W_1, W_2, W_3 \mapsto W'_1, W'_2, W'_3 \)

\[
R \cdot u_j = u'_j \cdot R, \quad R \cdot w_j = w'_j \cdot R, \quad j = 1, 2, 3,
\]

(2.14)

is given by

\[
\begin{align*}
    w'_1 &= w_2 \cdot \Lambda_3, \quad u'_1 = \Lambda_2^{-1} \cdot w_3^{-1}, \\
    w'_2 &= \Lambda_3^{-1} \cdot w_1, \quad u'_2 = \Lambda_1^{-1} \cdot u_3, \\
    w'_3 &= \Lambda_2^{-1} \cdot u_1^{-1}, \quad u'_3 = u_2 \cdot \Lambda_1,
\end{align*}
\]

(2.15)

where

\[
\begin{align*}
    \Lambda_1 &= u_1^{-1} \cdot u_3 - q^{1/2} u_1^{-1} \cdot w_1 + \kappa_1 u_1 \cdot u_2^{-1}, \\
    \Lambda_2 &= \frac{\kappa_1}{\kappa_2} u_2^{-1} \cdot w_3^{-1} + \frac{\kappa_3}{\kappa_2} u_1^{-1} \cdot w_2^{-1} - q^{-1/2} \frac{\kappa_1 \kappa_3}{\kappa_2} u_2^{-1} \cdot w_2^{-1}, \\
    \Lambda_3 &= w_1 \cdot w_3^{-1} - q^{1/2} u_3 \cdot w_3^{-1} + \kappa_3 w_2^{-1} \cdot u_3.
\end{align*}
\]

(2.16)

Reverse formulae, giving \( R^{-1} \), look similar:

\[
\begin{align*}
    \Lambda_1^{-1} &= \frac{\kappa_1}{\kappa_2} u'_1 \cdot u'_3^{-1} - q^{1/2} \frac{\kappa_3}{\kappa_2} u'_1 \cdot w'_1^{-1} + \kappa_3 w'_1^{-1} \cdot u'_2, \\
    \Lambda_2^{-1} &= u'_2 \cdot w'_3 + u'_1 \cdot w'_2 - q^{-1/2} \kappa_2 u'_2 \cdot w'_2, \\
    \Lambda_3^{-1} &= \frac{\kappa_3}{\kappa_2} w'_1^{-1} \cdot w'_3 - q^{1/2} \frac{\kappa_1}{\kappa_2} u'_3^{-1} \cdot w'_3 + \kappa_1 w'_2 \cdot u'_3^{-1}.
\end{align*}
\]

(2.17)

The conservation of the Weyl algebra structure

\[
    u_j \cdot w_j = q w_j \cdot u_j \quad \mapsto \quad u'_j \cdot w'_j = q w'_j \cdot u'_j
\]

(2.18)

means that \( R \) is the canonical map, hence \( R_{1,2,3} \) can be regarded as an usual operator depending on \( u_1, w_1, u_2, w_2, u_3, w_3 \). The structure of \( R \) will be described in the next subsection.

Now the projective ambiguity is removed, and the current system game gives the unique correspondence between the elements of equivalent graphs. This is well defined meaning of the algebraic equivalence. Hence all the admissibility conditions (and surely the Tetrahedron relation) become trivial consequences of this unambiguity, and we get them gratis!
Mention now a couple of useful limits of our fundamental map $R_{1,2,3}$. The first one is the limit when $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$, and then $\kappa \to 0$. Denote such limiting procedure via

$$\kappa_1 = \kappa_2 = \kappa_3 \ll 1.$$  \hfill (2.19)

Corresponding map we denote $R_{1,2,3}^{pl}$. The conditions for $\kappa$-s are uniform for whole Tetrahedron relation,

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = \kappa_6 \ll 1,$$  \hfill (2.20)

so $R_{1,2,3}^{pl}$ obeys TE. The other case is the limit of $R_{1,2,3}$ when

$$\kappa_1 \ll \kappa_2 = \kappa_3 \ll 1.$$  \hfill (2.21)

These conditions are uniform for TE again,

$$\kappa_1 \ll \kappa_2 = \kappa_3 \ll \kappa_4 = \kappa_5 = \kappa_6 \ll 1.$$  \hfill (2.22)

Corresponding map we call $r_{1,2,3}$, and due to the uniformness it also obeys TE. Recall, all these maps, $R$ with $\kappa_1 = \kappa_2 = \kappa_3 = 1$, $R_{1,2,3}^{pl}$ and $r$ were derived previously as a hierarchy of $R$–operators solving TE, see $[29, 23, 32, 24]$.

2.3 Structure of $R$

Remarkable feature of $R$ is its spatial invariance. Change a little the operators on which $R$ depends:

$$\Gamma_1 = \kappa_1^{-1} u_2 \cdot u_5^{-1} \cdot \Lambda_1, \quad \Gamma_2 = \kappa_2 u_1 \cdot w_3 \cdot \Lambda_2, \quad \Gamma_3 = \kappa_3^{-1} w_1^{-1} \cdot w_2 \cdot \Lambda_3.$$ \hfill (2.23)

Then for $\alpha, \beta, \gamma$ being the cyclic permutations of $1, 2, 3$,

$$(\Gamma_{\beta} \cdot \Gamma_\alpha - q \Gamma_\alpha \cdot \Gamma_{\beta}) \cdot \Gamma_\gamma - \Gamma_\gamma \cdot (\Gamma_{\beta} \cdot \Gamma_\alpha - q \Gamma_\alpha \cdot \Gamma_{\beta})$$

$$- q^{-1} (1 - q) (1 - q^2) (\Gamma_\alpha - \Gamma_{\beta}) = 0,$$  \hfill (2.24)

and

$$q \Gamma_\alpha \cdot \Gamma_{\beta} - q^{-1} \Gamma_{\beta} \cdot \Gamma_\alpha - \Gamma_\gamma (q^{1/2} \Gamma_\alpha \cdot \Gamma_{\beta} - q^{-1/2} \Gamma_{\beta} \cdot \Gamma_\alpha)$$

$$+ q^{-1/2} (1 - q) (q^{-1} \Gamma_\alpha + q \Gamma_{\beta} - \Gamma_\gamma) = 0.$$  \hfill (2.25)

It resembles $SO(3)$ invariance.

Give now a realisation of $R$ in terms of more simple functions. First, recall the definition and properties of the quantum dilogarithm. Let conventionally

$$(x; q)_n = (1 - x) (1 - qx) (1 - q^2x) \ldots (1 - q^{n-1}x).$$ \hfill (2.26)

Then the quantum dilogarithm (by definition) $[25, 26]$

$$\psi(x) \overset{df}{=} (q^{1/2}x; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q; q)_n} x^n,$$ \hfill (2.27)

and

$$\psi(x)^{-1} = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q; q)_n} x^n.$$ \hfill (2.28)
This function is useful for the rational transformations of the Weyl algebra:

\[ \psi(qx) = (1 - q^{1/2}x)^{-1} \psi(x), \quad \psi(q^{-1}x) = (1 - q^{-1/2}x) \psi(x), \] (2.29)
hence

\[ \psi(u) \cdot w = w \cdot (1 - q^{1/2}u)^{-1} \cdot \psi(u), \quad \psi(w) \cdot u = u \cdot (1 - q^{-1/2}w) \cdot \psi(w). \] (2.30)

\( \psi \) is called the quantum dilogarithm due to the pentagon identity \[25\]

\[ \psi(w) \cdot \psi(u) = \psi(u) \cdot \psi(-q^{-1/2}u w) \cdot \psi(w), \] (2.31)
this corresponds to Roger’s five term relation for the usual dilogarithm. From the other side \( \psi \) is the quantum exponent due to

\[ \psi(u) \cdot \psi(w) = \psi(u + w). \] (2.32)

Recall, everywhere the Weyl algebra relation \( u w = q w u \) is implied.

Introduce now a generalised permutation function. Let \( P(x,y), x \cdot y = q^2 y \cdot x, \) is defined by the following relations:

\[
\begin{align*}
P(q x, y) &= y^{-1} P(x, y) = P(x, y) y, \\
P(x, q y) &= P(x, y) x^{-1} = x P(x, y),
\end{align*}
\] (2.33)
and

\[ P(x, y)^2 = 1. \] (2.34)

For \( z \) obeying

\[ x \cdot z = q^{f_x} z \cdot x, \quad y \cdot z = q^{f_y} z \cdot y \] (2.35)
it follows

\[ P(x, y) \cdot z = q^{f_x f_y} z \cdot x^{f_x} \cdot y^{-f_x} \cdot P(x, y). \] (2.36)

This function we call the generalised permutation because of usual permutation operator of the tensor product is

\[ P \equiv P(u \otimes u^{-1}, w \otimes w^{-1}) \] (2.37)

Considering three independent \( \Gamma_\alpha \) (2.23), \( \alpha = 1, 2, 3, \) one may see that all them depends on three operators \( U, W \) and \( s: \)

\[ U = w_2^{-1} \cdot w_3, \quad W = w_1 \cdot u_3^{-1}, \quad -q^{1/2} s \cdot U \cdot W^{-1} = u_1 \cdot u_2^{-1}. \] (2.38)

\[ U W = q W U \text{ and } s \text{ is the center. One can directly verify that} \]

\[ R = \psi(\kappa_3 U) \cdot \psi(W^{-1}) \cdot P\left(\sqrt[3]{\kappa_3} U, s^{-1} \cdot W^2\right) \cdot \psi(\kappa_1 \kappa_3 W^{-1}) \cdot \psi(\kappa_2 U^{-1})^{-1}, \] (2.39)

being substituted into \[2.14\], gives \[2.13.216\]. On \( U \) and \( W, \) \( R \) acts as follows.

\[
\begin{align*}
R \cdot U \cdot R^{-1} &= \frac{\kappa_2}{\kappa_3} U^{-1} \cdot \left( W - q^{-1/2} + \kappa_3 U \right), \\
R \cdot W \cdot R^{-1} &= s \cdot W^{-1} \cdot \left( W - q^{1/2} + \kappa_3 U \right) \cdot \left( W - q^{1/2} + \kappa_1 s \cdot U \right)^{-1}.
\end{align*}
\] (2.40)
When \( \kappa_1 = \kappa_2 = \kappa_3 = 1 \), expression (2.39) for \( R \) coincides with the operator solution of the Tetrahedron equation from [23, 24]. This is the generalisation of the finite dimensional 3D \( R \)-matrix from \( q^N = 1 \) to general \( q \), and the finite dimensional \( R \)-matrix corresponds to the Zamolodchikov–Bazhanov–Baxter model.

We don’t discuss this correspondence here, the reader may find the details concerning Zamolodchikov–Bazhanov–Baxter model in [1, 3, 4, 13], the details concerning the finite \( R \)-matrix – in [5], the details concerning the quantum dilogarithm – in original papers [25, 26], and operator valued \( R \) as the generalisation of finite \( R \)– in [29, 23, 24, 30].

Few words concerning the meaning of (2.39). All \( \psi \)-s can be decomposed into the seria with respect to their arguments. Substitute these \( R \)-s into the Tetrahedron relation (1.5) and move all the generalized permutations \( P \) out. \( P \)-s thesirself obey the Tetrahedron equation and so can be cancelled from TE for \( R \)-s. Then twelve \( \psi \)-s rest in the left hand side of TE, and twelve \( \psi \)-s rest in the right hand side. The Tetrahedron equation in this case becomes a relation resembling the braid group relation in 2D. This twenty-four terms relation can be proved directly via the seria decomposition of all 24 quantum dilogarithms. The proof is based on several finite \( q \)-re-summations (like \( q \)-binomial theorems). This is the first value of the formula (2.39). The second one is that relation (2.39) gives a nice way to derive the finite dimensional complete \( R \)-matrix (just replacing \( \psi \)-s and \( P \) by their finite dimensional counterparts, [26, 23, 24, 30]).

Generalised permutation \( P(x, y) \) has no good series realization. Note, if we abolish condition \( P^2 = 1 \) for a moment, then formally
\[
P(x, y) \sim \sum_{\alpha, \beta \in \mathbb{Z}} q^{-\alpha \beta} x^\alpha \cdot y^\beta.
\]

This \( P \) obeys \( P^2 = 1 \) if one takes the Euler definition \( \sum_{n \in \mathbb{Z}} q^n \delta_{m, 0} = [33, 34] \). Note that in the manipulations with \( q \)-seria the Euler principle “A sum of any infinite series is the value of an expression, which expansion gives this series” was never failed. Actually \( P(x, y) \) is to be defined specially for every realisation of the Weyl algebra. As an example mention Kashaev and Faddeev’s non invariant realisation of the Weyl generators as shifts on the space of appropriately defined functions \( \varphi([t]) \):
\[
w_{j} \cdot \varphi([t]) = [t]_{j} \varphi([t]), \quad u_{j} \cdot \varphi([t]) = -q^{-1/2} [t]_{j} \varphi([t]) : [t]_{j} \mapsto q^{-1} [t]_{j} \right). \quad (2.42)
\]

Where \([t] \) is a list of the arguments of \( \varphi \) and \([t]_{j} \) is its \( j \)-th component. Thus \( u_{j} \) and \( w_{j} \) refer to the \( j \)-th “pointer” of the list of arguments and hence are not functional operators in usual sense. Actually the action of \( u_{j} \), \( w_{j} \) on \( \varphi([t]) \) would be given symbolically by the following correspondence:
\[
\varphi([t]) \leftrightarrow |t_1 \rangle \otimes |t_2 \rangle \otimes |t_3 \rangle \otimes ..., \quad (2.43)
\]

if the eigenvectors \(|t_j \rangle \) of the operators \( w_j \) might be defined.

Generalised permutation introduced
\[
P_{1,2,3} = P(\sqrt{\frac{\kappa_3}{\kappa_2}} U, s^{-1} W^2) = P(\sqrt{\frac{\kappa_3}{\kappa_2}} w_{2}^{-1} w_{3}, -q^{-1/2} u_{1}^{-1} w_{1}^{-1} u_{2} w_{3} u_{3}^{-1}) \quad (2.44)
\]
act of $u_j, w_j, j = 1, 2, 3$, as follows:

$$
\begin{align*}
P_{1,2,3} \cdot w_1 &= \sqrt{\frac{\kappa_2}{\kappa_3}} w_1 w_2 w_3^{-1} \cdot P_{1,2,3}, \\
P_{1,2,3} \cdot w_2 &= \sqrt{\frac{\kappa_3}{\kappa_2}} w_3 \cdot P_{1,2,3}, \\
P_{1,2,3} \cdot w_3 &= \sqrt{\frac{\kappa_2}{\kappa_3}} w_2 \cdot P_{1,2,3},
\end{align*}
$$

and

$$
\begin{align*}
P_{1,2,3} \cdot u_1 &= \sqrt{\frac{\kappa_2}{\kappa_3}} u_1 w_2 w_3^{-1} \cdot P_{1,2,3}, \\
P_{1,2,3} \cdot u_2 &= -q^{1/2} \sqrt{\frac{\kappa_3}{\kappa_2}} u_1 w_1^{-1} u_3 \cdot P_{1,2,3}, \\
P_{1,2,3} \cdot u_3 &= -q^{1/2} \sqrt{\frac{\kappa_2}{\kappa_3}} u_1^{-1} w_1 u_2 \cdot P_{1,2,3}.
\end{align*}
$$

This gives the following action of $P$ on $\varphi(t_1, t_2, t_3)$:

$$
P_{1,2,3} \cdot \varphi(t_1, t_2, t_3) = \varphi(\sqrt{\frac{\kappa_2}{\kappa_3}} t_1 t_2 \sqrt{\frac{\kappa_3}{\kappa_2}} t_3, \sqrt{\frac{\kappa_3}{\kappa_2}} t_3 \sqrt{\frac{\kappa_2}{\kappa_3}} t_2),
$$

where $t_1, t_2, t_3$ stand on the positions corresponding 1, 2, 3 of $R_{1,2,3}$.

Another thing to be mentioned is the case of $|q| = 1$. In this case the quantum dilogarithmic functions should be replaced by Faddeev’s integral [28]. In few words, it appears when one considers the Jacobi imaginary transformation of an argument of $\psi$ and $q$:

$$
u = e^{i z}, \quad -q^{1/2} = e^{i \pi \theta} \mapsto \tilde{\nu} = e^{i z/\theta}, \quad -\tilde{q}^{1/2} = e^{-i \pi/\theta}.
$$

Then

$$
\psi_F(u) = \left( \frac{q^{1/2} u : q}{\tilde{q}^{1/2} \tilde{u} : \tilde{q}} \right)_\infty,
$$

and the following expression for $\psi_F(u)$ is valid in the limit of real $\theta$ [28]:

$$
\psi_F(u) (= s(z)) = \exp \frac{1}{4} \int_\infty^\infty \frac{e^{z \xi}}{\sinh \pi \xi \sinh \pi \theta \xi} \frac{d \xi}{\xi},
$$

where the singularity at $\xi = 0$ is circled from above.

Return now to map [2.39]. The map $R$ conserves four independent operators:

$$
w_1 \cdot w_2, \quad u_2 \cdot u_3, \quad s
$$
Let \( \sigma \). It is easy to check
\[
R = \kappa_1 q^{-1/2} u_1 \cdot u_2^{-1} \cdot u_3 - \kappa_3 u_1 \cdot u_2^{-1}
\]
\[
- \kappa_2 q^{-1/2} w_1 \cdot w_2 \cdot u_3^{-1} \cdot w_3 - \kappa_2 w_1 \cdot w_2^{-1}
\]
\[
= \left( W^{-1} + \kappa_1 U - q^{1/2} \kappa_3 U W^{-1} \right) + s^{-1} \left( W + \kappa_2 U^{-1} - q^{1/2} \kappa_2 U^{-1} W \right).
\]
(2.52)

Actually \( R \) depends only on two of them, \( s \) and \( H \).

Consider the following product
\[
\sigma = \psi(a w^{-1}) \cdot \psi(b u) \cdot \psi(-q^{-1/2} c u w) \cdot \psi(a' w) \cdot \psi(b' u^{-1}).
\]
(2.53)

Let
\[
\chi = aw^{-1} + a' w + bu + b' u^{-1} - q^{-1/2} c u w - q^{-1/2} ab' u^{-1} w^{-1}.
\]
(2.54)

It is easy to check \( \sigma \cdot \chi = \chi \cdot \sigma \). Hence \( \sigma \) as an operator is a function on \( \chi \):
\[
\sigma = \sigma(a', bb', c_{a'b}) \left| \chi \right).
\]
(2.55)

I did not find explicit form of function \( \sigma \), only a special case of \( \sigma \) when \( c = b' = 0 \), then
\[
\psi(a w^{-1}) \psi(b u) \psi(a' w) = \psi(a \theta^{-1}) \psi(a' \theta)
\]
(2.56)

where
\[
a \theta^{-1} + a' \theta = a w^{-1} + b u + a' w.
\]
(2.57)

Nevertheless direct calculations give \( R^2 \) in terms of \( \sigma \) introduced. First, it is convenient to rewrite \( R \):
\[
R = \psi(W^{-1}) \psi(-q^{1/2} \kappa_3 U W^{-1}) \psi\left(\sqrt{\kappa_3} U \cdot s^{-1} W^2 \right) \psi\left(-q^{1/2} \kappa_1 \kappa_2 U^{-1} W^{-1} \right) \psi(\kappa_1 W)^{-1}.
\]
(2.58)

Then
\[
R^2 = N \cdot D^{-1},
\]
(2.59)

where
\[
N = \psi(W^{-1}) \psi(-q^{1/2} \kappa_3 U W^{-1}) \psi(\kappa_1 U) \psi(s^{-1} W) \psi(-q^{1/2} \kappa_2 s^{-1} U^{-1} W),
\]
(2.60)

and
\[
D = \psi\left(\kappa_1 W \right) \psi(-q^{1/2} \kappa_1 \kappa_2 U^{-1} W) \psi\left(\kappa_1 \kappa_2 U^{-1} \right) \psi\left(\kappa_1 W \right) \psi\left(-q^{1/2} \kappa_1 s U W^{-1} \right).
\]
(2.61)

Comparing these with the definition of \( \sigma \), we obtain
\[
N = \sigma(s^{-1}, \kappa_2 \kappa_3 s^{-1}, {\kappa_1 s}_H), \quad D = \sigma(\kappa_1^2 s, \kappa_1^2 \kappa_2 s, \kappa_3 s^{-1} \kappa_1 s_H),
\]
(2.62)

where \( H \) is given by \( 2.52 \).
2.4 Fusion

One more remarkable feature of the current model is a sort of a fusion. As an example consider a planar graph formed by two pairs of the parallel lines. Four vertices arise as the intersection points of these two pairs of the lines. This is shown in Fig. 5.

The vertices are labelled by the pairs of the indices, \( W_{1,1}, W_{1,2}, W_{2,1}, \) and \( W_{2,2} \). Single closed site means that there are three independent currents. Let them be the internal current \( \phi_{1,1} \), assigned to north-west corner, and two currents \( x \) and \( y \) assigned to southern and western semi-strips, \( x \) and \( y \) are observable currents for this cross considered as an alone graph.

Applying the linear system rules, we obtain step by step

\[
\begin{align*}
\phi_{1,1} &= \phi, \\
\phi_{1,2} &= y - w_{1,1} \cdot \phi, \\
\phi_{2,2} &= w_{2,2}^{-1} \cdot x - \kappa_{1,2} u_{1,2} w_{1,2}^{-1} \cdot y + \kappa_{1,2} w_{1,1} u_{1,2} w_{1,2}^{-1} \cdot \phi,
\end{align*}
\]

and from zero value of the closed site current

\[
\begin{align*}
\phi_{2,1} &= -w_{2,1}^{-1} w_{2,2}^{-1} \cdot x + (\kappa_{1,2} u_{1,2} w_{1,2}^{-1} w_{2,2}^{-1} - q^{1/2} u_{1,2} w_{2,1}^{-1}) \cdot y \\
&+ (q^{1/2} w_{1,1} u_{1,2} w_{2,1}^{-1} - \kappa_{1,1} u_{1,1} w_{1,1}^{-1} - \kappa_{1,2} w_{1,1} u_{1,2} w_{2,1}^{-1} w_{2,2}^{-1}) \cdot \phi.
\end{align*}
\]

Let further \(-x'\) and \(-y'\) are the edge variables assigned to the northern and eastern semistips. In general they are

\[
\begin{align*}
x' &= \alpha \cdot x + \beta \cdot y + f_x \cdot \phi, \\
y' &= \gamma \cdot x + \delta \cdot y + f_y \cdot \phi,
\end{align*}
\]
where

\[ \begin{align*}
\alpha &= w_{2,1}^{-1} w_{2,2}^{-1}, \\
\beta &= q^{1/2} u_{1,2} w_{2,1}^{-1} - \kappa_{1,2} u_{1,2} w_{1,2}^{-1} w_{2,1}^{-1}, \\
\gamma &= -q^{1/2} u_{2,2} w_{2,2}^{-1} + \kappa_{2,1} u_{2,1} w_{2,2}^{-1}, \\
\delta &= q^{1/2} \kappa_{2,1} u_{1,2} u_{2,1} + q^{1/2} \kappa_{1,2} u_{1,2} u_{2,2} w_{2,2}^{-1} \\
&\quad - \kappa_{1,2} \kappa_{2,1} u_{1,2} u_{2,1} w_{2,2}^{-1},
\end{align*} \]

(2.66)

and

\[ \begin{align*}
f_x &= -q^{1/2} u_{1,1} - q^{1/2} w_{1,1} u_{1,2} w_{2,1}^{-1} \\
&\quad + \kappa_{1,1} u_{1,1} w_{1,1} w_{2,1}^{-1} + \kappa_{1,2} w_{1,1} u_{1,2} w_{2,1}^{-1} w_{2,2}^{-1}, \\
f_y &= -q^{1/2} \kappa_{2,1} w_{1,1} u_{1,2} u_{2,1} - q^{1/2} \kappa_{1,2} w_{1,1} u_{1,2} u_{2,2} w_{2,2}^{-1} \\
&\quad + \kappa_{1,1} \kappa_{2,1} u_{1,1} w_{1,1} u_{2,1} + \kappa_{1,2} \kappa_{2,1} w_{1,1} u_{1,2} u_{2,1} w_{2,2}^{-1}.
\end{align*} \]

(2.67)

Currents \(x, y, x', y'\) become the edge currents when we rewrite the cross in Fig. 3 as a single vertex with modified (thick) lines; denote it as \(W \mapsto \{W_{1,1}, W_{1,2}, W_{2,1}, W_{2,2}\}\). Suppose we combine such crosses \(\tilde{W}\) (vertices with thick lines) in any way, then zero value conditions for the restricted strips (closed thick edges) look very simply: due to the signs \((-\)\) in the definition of outgoing \(x'\) and \(y'\) these zero value conditions becomes “outgoing edge current of one thick vertex = incoming edge current of another thick vertex”. Thus the strip variables just transfer from one combined (thick) vertex to another, and therefore they look like edge variables of the thick vertices.

In the case when for any thick vertex the map \(x, y \mapsto x', y'\) (2.65) does not contain extra \(\phi\), i.e. \(f_x = f_y = 0\) in any sense, then the part of the linear system corresponding to the edge variables factorises from the whole current system. If such factorisation exists for a graph \(G\) then it exists for any equivalent graph \(G'\), so sub-manifold of \(B_P\) given by \(f_x = f_y = 0\) is invariant of a map \(G \mapsto G'\).

On this sub-manifold we can delete all the edge currents \(x = y = \ldots = 0\). In this case all corner currents of cross \(\tilde{W}\) are proportional to \(\phi = \phi_{1,1}\), and the structure of the “thick” vertex becomes the structure of usual vertex. Thus one may define “thick” analogies of \(u, w\) and \(\kappa\). This phenomenon resembles the usual two-dimensional fusion.

Write now explicit formulae. Introduce

\[ K^{-1} = \frac{1}{\kappa_{1,2} \kappa_{2,1} \kappa_{2,2}} \left( u_{1,1} u_{1,2}^{-1} + w_{1,1} w_{2,1}^{-1} - q^{-1/2} \kappa_{1,1} u_{1,1} u_{1,2}^{-1} w_{1,1} w_{2,1}^{-1} \right), \]

\[ k = q^{1/2} \kappa_{2,1} \kappa_{2,2} w_{1,1}^{-1} w_{1,2}^{-1} w_{2,1} w_{2,2}, \]

\[ \tilde{k} = q^{-1/2} \kappa_{1,2} \kappa_{2,2} u_{1,1}^{-1} u_{1,2} u_{2,1}^{-1} u_{2,2}. \]

(2.68)

Without mentioning of a representation of the Weyl algebra, its right module etc., suppose \(\phi\) in (2.65) obeys \(f_x \cdot \phi = f_y \cdot \phi = 0\). From this, it follows that

\[ K^{-1} \cdot \phi = k^{-1} \cdot \phi = \tilde{k}^{-1} \cdot \phi = K^{-1} \phi, \]

(2.69)
where $K$ is introduced as an “eigenvalue”. On this “subspace” the fusion is defined as

$$\Delta(w) = -w_{1,1} w_{1,2}, \quad \Delta(u) = -q^{1/2} u_{1,1} u_{2,1}, \quad \Delta(\kappa) = K.$$ (2.70)

The meaning of all these is the following. Consider three “thick” crosses $\tilde{W}_j \mapsto \tilde{W}'_j$, $j = 1, 2, 3$, arranged into “thick” Yang – Baxter – type graphs, $\triangle$ and $\triangledown$. Solving the complete problem of the equivalence (with twelve vertices in each hand side) one obtains the set of relations like

$$\Delta(w'_j) \cdot \phi'_j - \Delta(w_j)' \cdot \phi'_j = \sum_{k=1}^3 X_k \cdot f_{x,k} \phi_k + Y_k \cdot f_{y,k} \phi_k,$$ (2.71)

eq \begin{align*}
\text{etc., with some } X_k \text{ and } Y_k, f_{x,k} \text{ and } f_{y,k} \text{ given by (2.67). } \Delta(w'_j) \text{ we obtain from } \Delta(w_j) \\
\text{applying all eight } R \text{-s repeatedly, and } \Delta(w_j)' \text{ is the result of the application of single } R \text{ in terms of } \Delta(u_j), \Delta(w_j) \text{ and } \Delta(K).}
\end{align*}

2.5 Matrix part

Few words concerning the matrix variables $\alpha, \beta, \gamma, \delta$ in (2.65). This remark is not important for our current approach, but the structure of matrix variables is very interesting. First, the map of edge auxiliary variables

$$\begin{align*}
x' &= \alpha \cdot x + \beta \cdot y, \\
y' &= \gamma \cdot x + \delta \cdot y,
\end{align*}$$ (2.72)

appeared in Korepanov’s matrix models [8, 31]. The Yang – Baxter equivalence in Korepanov’s interpretation is the Korepanov equation: admissibility of the map of three edge variables $(x, y, z)$ assigned to three lines of the Yang – Baxter graph. Let

$$X_1 = \begin{pmatrix}
\alpha_1 & \beta_1 & 0 \\
\gamma_1 & \delta_1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad X_2 = \begin{pmatrix}
\alpha_2 & 0 & \beta_2 \\
0 & 1 & 0 \\
\gamma_2 & 0 & \delta_2
\end{pmatrix}, \quad X_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \alpha_3 & \beta_3 \\
0 & 0 & \gamma_3
\end{pmatrix},$$ (2.73)

Then the admissibility is

$$X_1 \cdot X_2 \cdot X_3 = X'_3 \cdot X'_2 \cdot X'_1,$$ (2.74)

where the primed $X$-s consist on primed $\alpha, \beta, \gamma, \delta$. Korepanov’s equation is equivalent to the usual local Yang – Baxter equation for the so-called ferroelectric weights:

$$X = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \mapsto L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \alpha & \beta & 0 \\
0 & \gamma & \delta & 0 \\
0 & 0 & 0 & \zeta
\end{pmatrix},$$ (2.75)

where in the numeric case $\zeta = \alpha \cdot \delta - \beta \cdot \gamma$, and the conventional $2^2 \times 2^2 = 4 \times 4$ matrix form for Yang – Baxter matrix $L$ is used (for the equivalence see [13] for example). In our case the elements of different $X$-s commute, and the elements of one $X$ obey the algebra

$$\begin{align*}
\alpha \cdot \beta &= \beta \cdot \alpha, \quad \gamma \cdot \delta = \delta \cdot \gamma, \\
\alpha \cdot \gamma &= q \gamma \cdot \alpha, \quad \alpha \cdot \delta = q \delta \cdot \alpha, \quad \beta \cdot \delta = q \delta \cdot \beta,
\end{align*}$$ (2.76)

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and
\[ \zeta \overset{\text{def}}{=} \alpha \cdot \delta - \beta \cdot \gamma = \delta \cdot \alpha - \gamma \cdot \beta \, . \tag{2.77} \]

From (2.76) and (2.77) it follows that \( \zeta \beta = \beta \zeta, \zeta \gamma = \gamma \zeta \), and consequently
\[ \beta^2 \cdot \gamma + q \gamma \cdot \beta^2 - (1 + q) \beta \cdot \gamma \cdot \beta = 0 \, , \tag{2.78} \]
\[ \beta \cdot \gamma^2 + q \gamma^2 \cdot \beta - (1 + q) \gamma \cdot \beta \cdot \gamma = 0 \, . \]

Hence
\[ \alpha \cdot \delta = - \frac{q}{1 - q} (\beta \cdot \gamma - \gamma \cdot \beta) \, , \tag{2.79} \]
and
\[ \zeta = - \frac{1}{1 - q} (\beta \cdot \gamma - q \gamma \cdot \beta) \, . \tag{2.80} \]

Call the algebra of \( \alpha, \beta, \gamma, \delta \), given by (2.76) and (2.77), as \( \mathcal{X} \). Interesting is the following **Proposition.** Korepanov's equations are nine equation for twelve variables, so \( X_j^i \) are defined ambiguously: in general one can't fix one element from \( \alpha'_1, \alpha'_2 \), one from \( \delta'_1, \delta'_2 \), and one from \( \alpha'_3, \delta'_3 \). Impose on these three arbitrary elements the simple part of \( \mathcal{X} \), given by (2.76) and (2.77), as \( \mathcal{X} \). Then all other relations of \( \mathcal{X} \), namely relations (2.76) for the other primed elements and all three relations (2.77) (or, equivalent, (2.79)) for the elements of \( \{X'_1, X'_2, X'_3\} \), hold automatically as the consequence of Korepanov's equation.

This observation, we guess, is a way of a quantization of Korepanov's matrix model. Remarkably is that \( \mathcal{X} \) is the nontrivial algebra.

As an example consider the case when \( \delta = 0 \). Corresponding algebra, \( \mathcal{X}_{\delta=0} \), contains only one nontrivial relation, \( \alpha \cdot \gamma = q \gamma \cdot \alpha \), and \( \beta \) is a center. Being a \( \mathcal{C} \)–number, \( \beta_j \) are to be conserved by the map \( X_j \mapsto X'_j \). Equations (2.74) contain \( \beta_2 = \beta_1 \beta_3 \). Hence \( \beta \) is the pure gauge and one may put \( \beta \equiv 1 \). The solution of (2.74) is:
\[
\begin{cases}
\alpha'_1 &= (\alpha_3 + \alpha_1 \cdot \gamma_3)^{-1} \cdot \alpha_1 \cdot \alpha_2 \\
\gamma'_1 &= f \cdot \gamma_1 \cdot \alpha_3 \cdot (\alpha_3 + \alpha_1 \cdot \gamma_3)^{-1} \\
\alpha'_2 &= \alpha_3 + \alpha_1 \cdot \gamma_3 \\
\gamma'_2 &= \gamma_1 \cdot \gamma_3 \\
\alpha'_3 &= \alpha_2 \cdot f^{-1} \\
\gamma'_3 &= (\alpha_3 + \alpha_1 \cdot \gamma_3) \cdot \gamma_1^{-1} \cdot \gamma_2 \cdot \alpha_3^{-1} \cdot f^{-1}
\end{cases} \tag{2.81}
\]

where \( f \) is not fixed by (2.74), this is the ambiguity mentioned. Permutation relations for the combinations of the primed elements, which do not contain \( f \), namely for \( \alpha'_1, \alpha'_2, \gamma'_2, \alpha'_3 \gamma'_1 \) and \( \gamma'_3 \gamma'_1 \), do not contradict the set of the local Weyl algebrae \( \alpha'_j \cdot \gamma'_j = q \gamma'_j \cdot \alpha'_j \). This corresponds to the statement of the proposition above. Consider now the Weyl algebrae for all primed elements. From this, it follows immediately
\[ f \cdot \alpha_j = \alpha_j \cdot f \, , \quad f \cdot \gamma_j = \gamma_j \cdot f \, . \tag{2.82} \]

Hence \( f \) is a \( \mathcal{C} \)–number, and therefore we may put \( f = 1 \). Thus the conservation of \( \mathcal{X}_{\delta=0} \) fixes the ambiguity.
The map \( \alpha_j, \gamma_j \to \alpha_j', \gamma_j' \) we’ve obtained is nothing but \( r_{1,2,3} \), given by the limiting procedure (2.21). The identification is \( \alpha = w^{-1} \) and \( \gamma = -q^{1/2} u \cdot w^{-1} \). This case,

\[
X = \begin{pmatrix} w^{-1} & 1 \\ -q^{1/2} u \cdot w^{-1} & 0 \end{pmatrix}
\]

(2.83)
is the quantization of the case (\( \eta \)) from the list of simple functional maps in [4].

The case of general \( \mathcal{X} \) is rather complicated technically, it is a subject of a separate investigation.

### 2.6 Co-current system and \( L \)-operator

In this subsection we give another form of the current approach.

Consider the whole linear system for a graph \( G \) defined on a torus (boundary conditions assumed). This system is the set of zero equations

\[
\phi_{\text{site}} \overset{def}{=} \sum_{\text{vertices}} W_{\text{vertex}} \cdot \phi_{\text{vertex}} = 0,
\]

(2.84)

where such equation we write for each site of \( G \), the sum is taken over all vertices surrounded this site, and contribution from vertex \( V \) denoted as \( W_V \cdot \phi_V \), is one of \( \phi_V, q^{1/2} u_V \cdot \phi_V, w_V \cdot \phi_V \) or \( \kappa_V u_V w_V \cdot \phi_V \) according to Fig. 2 and the orientation of \( V \). The toroidal structure means that all the sites are closed and the number of the sites equals to the number of the vertices. Gathering all zero equations (2.84) together, we obtain the matrix form of them,

\[
L \cdot \Phi = 0,
\]

(2.85)

where we combine the internal currents \( \phi_V \) into the column \( \Phi \) and the matrix of the coefficients \( L \) consists of

\[
1, \quad q^{1/2} u_V, \quad w_V \quad \text{and} \quad \kappa_V u_V w_V
\]

(2.86)

for all vertices \( V \) of the lattice. \( L \) is the square matrix, and explicit form of it depends on the geometry \( G \).

(2.85) can be interpreted as an equation of motion for the action \( \mathcal{A} = \Phi^* \cdot L \cdot \Phi \), where the row co-current vector \( \Phi^* \) does not depend on \( \Phi \) and its components \( \phi_S^* \) are assigned to the sites \( S \) of \( G \). The equation of motion for \( \Phi^* \) is \( \Phi^* \cdot L = 0 \). Corresponding zero equations now are assigned to the vertices, and each such equation connects the site co-currents from the sites surrounding this vertex. The problem of the equivalence of \( \triangle \) and \( \nabla \) in terms of co-currents can be formulated as follows: the co-current system for the left hand side graph \( \triangle \) of Fig. 3 is

\[
\begin{align*}
\phi_1^* & \equiv \phi_c^* + \phi_e^* \cdot q^{1/2} u_1 + \phi_h^* \cdot w_1 + \phi_d^* \cdot \kappa_1 u_1 w_1 = 0, \\
\phi_2^* & \equiv \phi_h^* + \phi_d^* \cdot q^{1/2} u_2 + \phi_h^* \cdot w_2 + \phi_f^* \cdot \kappa_2 u_2 w_2 = 0, \\
\phi_3^* & \equiv \phi_c^* + \phi_h^* \cdot q^{1/2} u_3 + \phi_g^* \cdot w_3 + \phi_b^* \cdot \kappa_3 u_3 w_3 = 0,
\end{align*}
\]

(2.87)

and co-current system for the right hand side graph \( \nabla \) is

\[
\begin{align*}
\phi_1'' & \equiv \phi_c'' + \phi_a'' \cdot q^{1/2} u_1' + \phi_b'' \cdot w_1' + \phi_f'' \cdot \kappa_1 u_1' w_1' = 0, \\
\phi_2'' & \equiv \phi_e'' + \phi_e'' \cdot q^{1/2} u_2' + \phi_g'' \cdot w_2' + \phi_a'' \cdot \kappa_2 u_2' w_2' = 0, \\
\phi_3'' & \equiv \phi_c'' + \phi_d'' \cdot q^{1/2} u_3' + \phi_a'' \cdot w_3' + \phi_f'' \cdot \kappa_3 u_3' w_3' = 0.
\end{align*}
\]

(2.88)
The equivalence means that when we remove $\phi^*_h$ from (2.87) and $\phi^*_a$ from (2.88), then the resulting systems as the systems for $\phi^*_b, ..., \phi^*_f$ are equivalent.

Consider now co-current equation for single vertex, as in Figs. 4 or 2. Let the co-currents be $\phi^*_a, \phi^*_b, \phi^*_c$ and $\phi^*_d$, where the indices $a, b, c, d$ are arranged as in Fig. 2. The co-current equation for this vertex is

$$\phi^* \equiv \phi^*_a + \phi^*_b \cdot q^{1/2} u + \phi^*_c \cdot w + \phi^*_d \cdot \kappa u w = 0.$$ (2.89)

Suppose we have solved a part of such equations for whole graph $G$, and obtain $\phi^*_a, \phi^*_c, \phi^*_d$ in the form usual for homogeneous linear equations:

$$\phi^*_a = - \phi^*_c \cdot q^{1/2} y, \quad \phi^*_c = - \phi^*_d \cdot q^{1/2} x$$ (2.90)

with some multipliers $x$ and $y$. Then from (2.89) we get

$$\phi^*_b = - \phi^*_a \cdot q^{1/2} y', \quad \text{or} \quad \phi^*_a = - \phi^*_b \cdot q^{1/2} x',$$ (2.91)

where

$$x' = \omega^{-1} \cdot y, \quad \text{and} \quad y' = x \cdot \omega$$ (2.92)

with

$$\omega = \omega(x, y | u, w) = y \cdot u^{-1} - q^{1/2} u^{-1} \cdot w + \kappa x^{-1} \cdot w.$$ (2.93)

Now we may change the interpretation completely. Assign $x, y, x', y'$ to the edges which separates corresponding sites. These edge variables are shown in Fig. 6.

Now we can introduce the auxiliary functional operator $L$, giving the map $x, y \mapsto x', y'$, as we used to be:

$$L_{x,y}(\kappa, u, w) \cdot x = \omega(x, y | u, w)^{-1} \cdot y \cdot L_{x,y}(\kappa, u, w),$$

$$L_{x,y}(\kappa, u, w) \cdot y = x \cdot \omega(x, y | u, w) \cdot L_{x,y}(\kappa, u, w).$$ (2.94)
With the definition (2.14), \( L \) operators obey

\[
L_y z(\kappa_3, u_3, w_3) \cdot L_x z(\kappa_2, u_2, w_2) \cdot L_x y(\kappa_1, u_1, w_1) \cdot R_{1,2,3} = \\
= R_{1,2,3} \cdot L_x y(\kappa_1, u_1, w_1) \cdot L_x z(\kappa_2 u_2, w_2) \cdot L_y z(\kappa_3, u_3, w_3) .
\]

Moreover, Local Yang-Baxter relation

\[
L_y z(\kappa_3, u_3, w_3) \cdot L_x z(\kappa_2, u_2, w_2) \cdot L_x y(\kappa_1, u_1, w_1) = \\
= L_x y(\kappa_1, u'_1, w'_1) \cdot L_x z(\kappa_2, u'_2, w'_2) \cdot L_y z(\kappa_3, u'_3, w'_3)
\]

as a set of relations for \( u'_k, w'_k \), with \( u_k, w_k \) given and with \( x, y, z \) arbitrary, gives again the map (2.13) uniquely! Thus the kind of the local Yang – Baxter relation appears and for our current approach.

Conclude this section by few remarks concerning the functional maps. All the maps introduced are connected to several graphical manipulations. Usually we combine such manipulations (\( \triangle \rightarrow \triangledown \) of \( x, y \rightarrow x', y' \) etc.), and write the sequence of the dynamical variables’ sets obtained \( \Sigma \mapsto \Sigma' \), in the direct form

\[
\Sigma = \Sigma_0 \xrightarrow{A_1} \Sigma_1 \xrightarrow{A_2} \Sigma_2 \ldots \Sigma_{n-1} \xrightarrow{A_n} \Sigma_n ,
\]

where \( A_j \) stands for \( j \)-th manipulation, which allows us to calculate \( \Sigma_j \) in terms of previous variables \( \Sigma'_{j-1} \). The same result, \( \Sigma_0 \mapsto \Sigma_n \), can be obtained as

\[
A_1 A_2 \ldots A_n \cdot \Sigma_0 = \Sigma_n \cdot A_1 A_2 \ldots A_n ,
\]

where \( A_j \) is a functional operator corresponding the manipulation \( A_j \). Remarkable is the reverse order of the operators with respect to the naïve manipulations. Note that the direct order we obtain considering the “pointer” action of the operators, as it was mentioned in the previous subsection, but the “pointer” action is not suitable for the quantization.

### 3 Evolution system

In this section we apply operator \( R \) defined in the previous section to construct an evolution model explicitly. Due to the current system’s background we formulate this model in terms of the regular lattice defined on the torus, its motion, its current system and so on.

The main result of our paper is the generating function for the integrals of motion for the evolution. The derivation of the integrals is based on the auxiliary linear problem.

#### 3.1 Kagome lattice on the torus

An example of a regular lattice which contains both \( \triangle \) and \( \triangledown \) – type triangles is so-called kagome lattice. As it was mentioned in the introduction, the kagome lattices appear in the sections of the regular 3D cubic lattices by inclined planes. Thus the kagome lattice and its evolution corresponds actually to the rectangular 3D lattice and thus is quite natural. The kagome lattice consists on three sets of parallel lines, usual situation shown in Fig. 7. The sites of the lattice are both \( \triangle \) and \( \triangledown \) triangles, and hexagons.

For given lattice introduce the labelling for the vertices. Mark the \( \triangle \) triangles by the point notation \( P \), and let \( a \) and \( b \) are the multiplicative shifts in the northern and eastern
directions, so that the elementary shift in the south-east direction is \( c = a^{-1}b \). Nearest to
triangle \( P \) are triangles \( aP, bP, cP, a^{-1}P, b^{-1}P \) and \( c^{-1}P \). Some of them are shown in Fig 7.

For three vertices surrounding the \( \triangle \)-type triangle \( P \) introduce the notations \((1, P)\),
\((2, P)\) and \((3, P)\). These notations we will use as the subscripts for everything assigned to
the vertices.

This kagome lattice we define on the torus of size \( M \), formally this means the following
equivalence:
\[
a^M P \sim b^M P \sim c^M P \sim P.
\] (3.1)

Since the notion of the equivalence, we may consider the shifts of all inclined lines through
the rectangular vertices into north-eastern direction as it is shown in Fig 7. It is easy to
see that Fig. 8 is equivalent to Fig. 3. The structure of the kagome lattice conserves by
such shifts being made simultaneously for all \( \triangle \)-s, but the marking of the vertices changes
a little. This is visible in Fig. 8.

Give now pure algebraic definition of the evolution. The phase space of the system is
the set of \( 3M^2 \) Weyl pairs \( u_{j,P} \) and \( w_{j,P} \), \( j = 1, 2, 3, P = a^\alpha b^\beta P_0 \), where \( P_0 \) is some frame
of the reference’s distinguished point, and the toroidal boundary conditions mean
\[
\begin{align*}
 u_{j,a^M P} &= u_{j,b^M P} = u_{j,P}, \\
 w_{j,a^M P} &= w_{j,b^M P} = w_{j,P}.
\end{align*}
\] (3.2)

The phase space is quantized by the definition. Let \( u_{j,P}', w_{j,P}' \) for \( P \) fixed are given by (2.13),
so that the map \( \{u_{j,P}, w_{j,P}\} \mapsto \{u_{j,P}', w_{j,P}'\} \) is given by the operator
\[
\mathcal{R} = \prod_p R_p,
\] (3.3)
where $R_{P'}$ acts trivially on the variables of any triangle $P \neq P'$. Note, we suppose $\kappa_{j,P}$ do not depend on $P$, 
\begin{equation}
\kappa_{j,P} = \kappa_j, \tag{3.4}
\end{equation}
so that with respect to $\kappa$-s the translation invariance of the lattice is assumed. Define the superscript '⋆' as follows:
\begin{equation}
\begin{aligned}
u_{1,P}^\star &= u'_{1,P}, \\
u_{aP}^\star &= u'_{aP}, \\
 u_{bP}^\star &= u'_{bP}, \\
v_{1,P}^\star &= v'_{1,P}, \\
v_{aP}^\star &= v'_{aP}, \\
v_{bP}^\star &= v'_{bP}.
\end{aligned}
\tag{3.5}
\end{equation}

This identification means following: $u_{j,P}^\star, v_{j,P}^\star$ are the variables which appear on the places of previous $u_{j,P}, v_{j,P}$ according to Fig. 8. The evolution operator $U : \{u_{j,P}, v_{j,P}\} \mapsto \{u_{j,P}^\star, v_{j,P}^\star\}$ we define as usual:
\begin{equation}
U \cdot u_{j,P} \cdot U^{-1} = u_{j,P}^\star, \quad U \cdot v_{j,P} \cdot U^{-1} = v_{j,P}^\star. \tag{3.6}
\end{equation}

Regard the primary variables $\{u_{j,P}, v_{j,P}\}$ of the given lattice as the initial data for the discrete time evolution,
\begin{equation}
u_{j,P} = u_{j,P}(0), \quad v_{j,P} = v_{j,P}(0). \tag{3.7}
\end{equation}

The evolution from $t = n$ to $t = n + 1$ is just
\begin{equation}
\begin{aligned}
u_{j,P}(n+1) &= U \cdot u_{j,P}(n) \cdot U^{-1}, \\
v_{j,P}(n+1) &= U \cdot v_{j,P}(n) \cdot U^{-1}.
\end{aligned} \tag{3.8}
\end{equation}

Surely, the map $U$ is the canonical map for the Weyl algebrae, so that $U$ is the quantum evolution operator. Further we’ll consider mainly the situation for $t = 0$ and the map from $t = 0$ to $t = 1$. We will omit the time variable and write $f$ instead of $f(0)$ and $f^\star = U \cdot f \cdot U^{-1}$ instead of $f(1)$ for any object $f$. Due to the homogeneity of evolution (3.5, 3.6, 3.7) our considerations appear to be valid for a situation with $t = n$ and the map from $t = n$ to $t = n + 1$. 

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3.2 Linear system

Investigate now the linear system for the quantum system obtained.

Assign to the vertex \((j, P)\) of the primary \((t = 0)\) kagome lattice the internal current \(\phi_{j,P}\). The linear system is the set of \(3M^2\) linear homogeneous equation for \(3M^2\) internal currents

\[
\begin{cases}
  f_1.P & = w_1.P \cdot \phi_{1,P} + \phi_{2,P} + q^{1/2}u_3.P \cdot \phi_{3,P} = 0, \\
  f_2.P & = q^{1/2}u_1.P \cdot \phi_{1,P} + \kappa_1u_{2,a}Pw_{2,a}P \cdot \phi_{2,a}P + w_{3,b}P \cdot \phi_{3,b}P = 0, \\
  f_3.P & = \phi_{1,a^{-1}P} + \kappa_1u_{1,b^{-1}P}w_{1,b^{-1}P} \cdot \phi_{1,b^{-1}P} + w_{2,P} \cdot \phi_{2,P} \\
                + q^{1/2}u_{b,b^{-1}P} \cdot \phi_{2,b^{-1}P} + \phi_{3,a^{-1}P} + \kappa_3u_{3,P}w_{3,P} \cdot \phi_{3,P} = 0.
\end{cases}
\]  

(3.9)

Here we have introduced absolutely unessential notations \(f_{j,P}\) just in order to distinguish these equations. \(f_{j,P}\) are assigned to the sites. Due to the homogeneity we may impose the \textbf{quasiperiodical boundary conditions} for \(\phi_{j,P}\):

\[
\phi_{j,a,M}P = A \phi_{j,P}, \quad \phi_{j,b,M}P = B \phi_{j,P}. \tag{3.10}
\]

It is useful to rewrite this system in the matrix form as \((\ref{2.85})\): \(F \equiv L \cdot \Phi = 0\).

First combine \(\phi_{j,P}\) with the same \(j\) into the column vector \(\Phi_j\) with \(M^2\) components, so as \((\Phi_j)_P = \phi_{j,P}\). Introduce matrices \(T_a\) and \(T_b\) as

\[(T_a \cdot \Phi_j)_P = \phi_{j,a,P}, \quad (T_b \cdot \Phi_j)_P = \phi_{j,b,P}. \tag{3.11}\]

Due to \((\ref{3.10})\)

\[T_a^P = A, \quad T_b^P = B. \tag{3.12}\]

Combine further \(u_{j,P}\) and \(w_{j,P}\) with the same \(j\) into diagonal matrices \(u_j\) and \(w_j\) with the same ordering of \(P\) as in the definition of \(\Phi_j\),

\[
u_j = \text{diag}_P u_{j,P}, \quad w_j = \text{diag}_P w_{j,P}. \tag{3.13}\]

Obviously,

\[(T_a \cdot u_j \cdot T_a^{-1})_P = u_{j,a,P}, \quad (T_b \cdot u_j \cdot T_b^{-1})_P = u_{j,b,P}. \tag{3.14}\]

and the same for \(w_j\).

Combine further \(\Phi_1, \Phi_1, \Phi_3\) into \(3M^2\) column \(\Phi\). Then from \((\ref{3.3})\) the matrix \(L\) can be extracted in the \(3 \times 3M^2 \times 3M^2\) block form:

\[
L = \begin{pmatrix}
  w_1 & 1 & q^{1/2}u_3 \\
  q^{1/2}u_1 & T_a \kappa_2 u_2 w_2 & T_b w_3 \\
  T_a^{-1} + T_b^{-1} \kappa_1 u_1 w_1 & w_2 + T_b^{-1} q^{1/2}u_2 & T_a^{-1} + \kappa_3 u_3 w_3
\end{pmatrix}
\]  

(3.15)

Recall, system \(L \cdot \Phi = 0\) is \(3M^2\) equations for \(3M^2\) components of \(\Phi\).

Introduce now co-currents. As it was mentioned, \(L \cdot \Phi = 0\) we regard as the equations of motion for 2D system with the action

\[
A \equiv \Phi^* \cdot L \cdot \Phi. \tag{3.16}
\]
The block form of the co-currents $\Phi^*$ is thus fixed from the form of $L$, or from $(\ref{eq:3.18})$. Equations of motion for $\Phi^*$ are $F^* \equiv \Phi^* \cdot L = 0$, and in the component form

$$
\begin{align*}
&f_{1,P}^* = \phi_{1,P}^* \cdot q^{1/2} u_{1,P} + \phi_{2,1-P}^* + \phi_{2,a-1}^* \cdot \kappa_1 u_{1,P} w_{1,P} + \phi_{3,1-P}^* \cdot w_{1,P}, \\
&f_{2,P}^* = \phi_{1,P}^* \cdot \kappa_2 u_{2,P} w_{2,P} + \phi_{2,2-P}^* \cdot q^{1/2} u_{2,P} + \phi_{3,2-P}^* \cdot w_{2,P} + \phi_{3,3-P}^* \cdot \kappa_3 u_{3,P} w_{3,P} + \phi_{3,b-P}^* \cdot q^{1/2} u_{3,P}.
\end{align*}
$$

Here $f_{j,P}^*$ corresponds to $(j,P)$-th vertex. The assignment of the co-currents is shown in Fig. [3].

Elements of $F^* = \Phi^* \cdot L$ have the following remarkable feature: coefficients in $f_{j,P}^*$ belong to the algebra of $u_{j,P}, w_{j,P}$ only. We will use this in the next subsection.

### 3.3 Properties of $L$ and the quantum determinant

Consider first the general properties of equation $\Phi^* \cdot V = 0$ for a matrix $V$ similar to $L$ introduced:

$$
V = \|v_{j,k}\|,
$$

with the commutative columns,

$$
\forall j, j' : v_{j,k} \cdot v_{j',k'} - v_{j',k'} \cdot v_{j,k} = 0 \text{ if } k' \neq k.
$$

Such matrices have the following properties.

**Property 1:** Consider a system

$$
\sum_j z_j \cdot v_{j,k} = \alpha_k
$$

with $\alpha_k$ being $C$-numbers, as the system for $z_j$. Then for $k \neq k'$

$$
\alpha_k \alpha_{k'} - \alpha_{k'} \alpha_k = \sum_{j'} z_{j'} \alpha_k v_{j',k'} - \sum_{j} z_{j} \alpha_{k'} v_{j,k} = \sum_{j,j'} (z_{j'} z_j - z_j z_{j'}) v_{j,k} v_{j',k'} = 0.
$$

$$
(3.21)
$$
Matrix $||\mathbf{v}_{j,k} \cdot \mathbf{v}_{j',k'}||$ is non-degenerate in general, so the last equality gives immediately
\[ z_j \cdot z_{j'} = z_{j'} \cdot z_j . \] (3.22)

**Consequence:** Let $||\mathbf{v}_{i,j}||$ is the inverse to $||\mathbf{v}_{j,k}||$ matrix:
\[ \sum_j \mathbf{v}_{i,j} \cdot \mathbf{v}_{j,k} = \sum_j \mathbf{v}_{i,j} \cdot \mathbf{v}_{j,k} = \delta_{i,k} , \] (3.23)
\[ \forall i \quad \mathbf{v}_{i,j} \cdot \mathbf{v}_{i,j'} - \mathbf{v}_{i,j'} \cdot \mathbf{v}_{i,j} = 0 . \] (3.24)

**Property 2:** Because of in $||\mathbf{v}_{j,k}||$ non-commutative elements belong to the same column, the algebraic supplements $V_{k,j}$ as well as the quantum determinant $\det (\mathbf{v})$ are well defined. Here we’ve used the notation “$\det$” as the formal operator-valued determinant
\[ \det ||\mathbf{v}_{i,j}|| = \sum_\sigma (-1)^\sigma \prod_j \mathbf{v}_{j,\sigma(j)} . \] (3.25)

$V_{i,j}$ and $\det (\mathbf{v})$ are polynomials of $\mathbf{v}_{j,k}$ such that in each summand all multipliers belong to different columns and thus commute. Moreover, if in $||\mathbf{v}||$ two rows coincide, then $\det (\mathbf{v}) \equiv 0$. Hence
\[ \sum_k \mathbf{v}_{j,k} \cdot V_{k,l} = \delta_{j,l} \det (\mathbf{v}) . \] (3.26)

Note, $\mathbf{v}_{j,k} \cdot V_{k,l} = V_{k,l} \cdot \mathbf{v}_{j,k}$.

As it was mentioned previously, sometimes it is useful to introduce formally a module for the body of $||\mathbf{v}_{j,k}||$. Here we do this, introducing $\phi_j^*$ and $\phi_0^*$ which belong to such formal module. This allows us to formulate the following
**Consequence:** Consider now the system of co-vector equations
\[ (\Phi^* \cdot \mathbf{V})_k = \sum_j \phi_j^* \cdot \mathbf{v}_{j,k} = 0 . \] (3.27)
Due to property 2 all $\phi_j^*$ belong to the null space of $\det (\mathbf{v})$:
\[ \phi_j^* \cdot \det (\mathbf{v}) = 0 . \] (3.28)
From the other hand side, $\phi_j^*$-s are connected by $z_{j,j'}$ – some rational functions of $\mathbf{v}_{j,k}$:
\[ \phi_j^* = \phi_j^* \cdot z_{j,j'} . \] (3.29)

Property 1 provides the commutability of $z_{j,j'}$, hence a solution of (3.27) can be written as
\[ \phi_j^* = \phi_0^* \cdot z_j , \quad z_j \cdot z_{j'} = z_{j'} \cdot z_j , \quad \phi_0^* \cdot \det (\mathbf{v}) = 0 , \quad z_j \cdot \det (\mathbf{v}) = \det (\mathbf{v}) \cdot z_j , \] (3.30)
where in general $z_j \neq \bar{z}_j$.

Apply now both properties and their consequences to $\mathbf{L}$ given by (3.15). First, for any representation of the Weyl algebra the null subspace $\phi_0^*$ of whole Gilbert space is defined,
\[ \phi_0^* \cdot \det (\mathbf{L}) = 0 . \] (3.31)

The existence of $\phi_0^*$ means the solvability of $\Phi^* \cdot \mathbf{L} = 0$. Corresponding $z_j$ have the lattice structure, $z_{j,p}$. These commutative elements are assigned to the sites of the kagome lattice, and observable are $z_{j,p} \cdot \bar{z}_{j',p}$. These operators connect the co-currents in different sites, and thus $z_{j,p}$ actually give the realisation of the path group on the kagome lattice.

Another important thing is that due to $T^A_a = A$ and $T^B_b = B$, $\det(\mathbf{L})$ is a Laurent polynomial with respect to the quasimomenta $A$ and $B$. 

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3.4 Evolution of the co-currents and integrals of motion

Consider now the shift of the inclined lines giving the evolution. The internal currents as well as the co-currents change, and we can trace these changes.

Introduce two extra matrices, $K$ and $M$:

$$
K = \begin{pmatrix}
0 & \Lambda_0 & 0 \\
0 & 0 & T_a T_b \\
1 & K_{3,2} & 0
\end{pmatrix}
$$

(3.32)

where

$$
\Lambda_0 = \frac{\kappa_1}{\kappa_2} q^{-1/2} w_1 u_2^{-1} w_3^{-1} + \frac{\kappa_3}{\kappa_2} u_1^{-1} w_2^{-1} u_3,
$$

(3.33)

$$
K_{3,2} = T_a^{-1} q^{-1/2} \Lambda_2 + \frac{\kappa_5}{\kappa_2} \Lambda_1 + T_b^{-1} \frac{\kappa_1}{\kappa_2} \Lambda_3.
$$

(3.34)

with $\Lambda_j$ standing for the diagonal matrices with the entries given by (2.14) correspondingly, and

$$
M = \begin{pmatrix}
0 & u_1^{-1} u_2' T_a & q^{-1/2} u_1^{-1} T_b \\
\frac{\kappa_1}{\kappa_2} w_2^{-1} u_2^{-1} u_1' w_1' & 0 & \frac{\kappa_3}{\kappa_2} w_2^{-1} u_2^{-1} u_3' w_3' T_b \\
w_3^{-1} & w_3^{-1} w_2' T_a & 0
\end{pmatrix}
$$

(3.35)

Apply the evolution operator $U$ to $L$: $L^* \equiv U \cdot L \cdot U^{-1}$,

$$
L^* = \begin{pmatrix}
w_1' & 1 & q^{1/2} T_b^{-1} u_3' T_b \\
q^{1/2} u_1' & \kappa_2 u_2' w_2' T_a & w_3' T_b \\
T_a^{-1} + T_b^{-1} \kappa_1 u_1' w_1' & T_a^{-1} (w_2' + T_b^{-1} q^{1/2} u_2') T_a & T_a^{-1} + T_b^{-1} \kappa_3 u_3' w_3' T_b
\end{pmatrix}
$$

(3.36)

The following relation can be verified directly:

$$
K \cdot L^* = L \cdot M.
$$

(3.37)

$M$ in general is the matrix making $\phi_{k,p}^* \mapsto \phi_{k,p}$, and $K$ makes $\phi_{k,P}^* \mapsto \phi_{k,P}^*$. Also $K$ and $M$ admit

$$
K \mapsto K + L \cdot N, \quad M \mapsto M + N \cdot L^*
$$

(3.38)

with arbitrary $N$. One can prove the following

**Proposition:** $\bullet \quad K \cdot \det(L) = \det(L) \cdot K \quad \bullet$

One can understand this in other terms. Since $\Phi^* \cdot L = 0$ can be solved for $t = 0$, then for $t = 1$ equation $\Phi^* \cdot L^* = 0$ must also be solved because they are bounded by simple linear relations. Hence subspace $\phi_0^n$ must coincide with $(\phi_0^n)^*$, i.e.

$$
\det(L^*) = \det(L) \cdot D,
$$

(3.39)
with some operator $D$. One may hope, $D$ is not too complicated, and (3.39) is not trivial.

Careful analysis of $K$ and $M$ shows that this $D$ does not depend on the quasimomenta $A$ and $B$. In the functional limit $q^{1/2} \rightarrow \pm 1$ one may easily calculate the determinants of $K$ and $M$, both them are proportional to $A^M B^M$, and this term cancels from the determinants of the left and right hand sides of (3.33). This is so and in the quantum case.

Hence $D$ in (3.39) is a ratio of any $A, B$ - monomials from $\text{det}(L)$ and $\text{det}(L^\star)$. Element $D$ can be extracted, say, from $A^M B^M$ component of $\text{det}(L)$:

$$D = \prod_P u_{1,P}^{-1} \cdot \prod_P u_{1,P}^*.$$

This means that we can introduce a simple operator $d$:

$$D = d \cdot d^*-1.$$

Thus

$$J = \text{det}(L) \cdot d$$

is the invariant of the evolution, $J^\star = J$, i.e.

$$U \cdot J = J \cdot U.$$

Decompose $J$ as a series of $A$ and $B$,

$$J = \sum_{\alpha, \beta \in \Pi} A^\alpha B^\beta J_{\alpha,\beta},$$

where $\alpha$ and $\beta$ are integers and their domain (Newton’s polygon) $\Pi$ is defined by $|\alpha| \leq M$, $|\beta| \leq M$ and $|\alpha + \beta| \leq M$. Quasimomenta $A$ and $B$ are arbitrary $C$ - numbers, and the invariance of $J$ means the invariance of each $J_{\alpha,\beta}$. From the other side, $J$ is a functional of the dynamical variables of the lattice, i.e.

$$J_{\alpha,\beta} = J_{\alpha,\beta}(\{u_{j,P}, w_{j,P}\}).$$

Surely, due to the homogeneity of the lattice these functionals does not depend on time layer, and hence the conservation of $J$, $J^\star = J$, means

$$J_{\alpha,\beta}(\{u_{j,P}, w_{j,P}\}) = J_{\alpha,\beta}(\{u^\star_{j,P}, w^\star_{j,P}\}),$$

i.e. functionals $J_{\alpha,\beta}$ give the integrals of motion in usual sense.

Note further,

$$\Phi^\star \cdot K \sim \Phi^\star \cdot U^{-1},$$

where it is supposed $\phi^\star_0 \cdot U^{-1} \sim \phi^\star_0$, and (3.47) gives the linear action of $U$ on $z_{j,P}$. To get the equality from (3.47), one has to normalize only one component of $\Phi^\star$.

Some elements of $\text{det}(L)$, corresponding to the border of the Newton polygon $\Pi$ of $J(A,B)$, can be easily calculated. Operator $d$ introduced is defined up to any integral of motion. The simplest integrals are

$$j_1 = \prod_P u_{2,P} \cdot u_{3,P}, \quad j_2 = \prod_P u_{1,P} w_{3,P}^{-1}, \quad j_3 = \prod_P w_{1,P} w_{2,P},$$

(3.48)
and the convenient choice of \( d \) is

\[
d = \prod_p \left( q^{1/2} u_{2,p} \cdot u_{3,p} \cdot w_{3,p} \right)^{-1}.
\]

(3.49)

d can be absorbed into \( \det \),

\[
J = \det (L^{(0)}),
\]

(3.50)

where

\[
L^{(0)} = \begin{pmatrix}
w_1 & q^{-1/2} u_2^{-1} & q^{-1/2} w_3^{-1} \\
q^{1/2} u_1 & T_a q^{1/2} \kappa_2 w_2 & T_b u_3^{-1} \\
T_a^{-1} + T_b^{-1} \kappa_1 u_1 w_1 & q^{1/2} u_2^{-1} w_2 + T_b^{-1} & T_a^{-1} w_3^{-1} u_3^{-1} + \kappa_3
\end{pmatrix}.
\]

(3.51)

Whole number of \( J_{\alpha,\beta} \) is \( 3M^2 + 3M + 1 \), and there are \( 3M^2 + 1 \) independent between them, and between these one can choose only \( 3M^2 \) commutative, so \( J \) gives the complete set of integrals. (As to whole number of summands in \( J \), e. g. for \( M = 2 \) it is \( 1536 = 2^3 \times 3 \).) The existence of \( 3M^2 \) abelian integrals is the hypothesis tested for small \( M \).

All integrals corresponding to the boundary of domain \( \Pi \), \(|\alpha| = M, |\beta| = M, |\alpha + \beta| = M \), are equivalent to the following \( 3M \) elements:

\[
\begin{aligned}
\bar{U}_j &= \prod_\sigma w_{1,\alpha^\sigma b/P_0} w_{2,\alpha^\sigma b/P_0}^{-1}, \\
\bar{V}_j &= \prod_\sigma u_{2,\alpha^\sigma b - \sigma P_0} u_{3,\alpha^\sigma b - \sigma P_0}^{-1}, \\
\bar{W}_j &= \prod_\sigma u_{1,\alpha^\sigma b \sigma P_0} w_{3,\alpha^\sigma b \sigma P_0}^{-1},
\end{aligned}
\]

(3.52)

where \( P_0 \) is some frame of reference’s point as previously. Note, \( \bar{V}_j \) are not \( T_a, T_b \) – invariant, but restoring this invariance in any way, one obtains the invariants of \( U \). Between \( \bar{W}_j, \bar{U}_j, \bar{V}_j \) one may choose \( 3M - 1 \) commutative elements. Inner part of \( \Pi \) gives \( 3M^2 - 3M + 1 \) highly complicated independent integrals, which gives \( g = 3M^2 - 3M + 1 \) commutative (up to (3.52)) independent elements. Note, \( g \) is the formal genus of the curve \( J(A, B) = const. \)

### 3.5 Walks on the lattice and the integrals of motion

Give now a geometrical interpretation of the integrals of motion. This interpretation follows directly from the analysis of the determinant. Every integral of motion is a sum of monomials associated with walks on the lattice such that all the walks have the same homotopy class with respect to the torus on which the kagome lattice is defined.

It is useful to formulate the walks in terms of general vertex variables \( a, b, c \) and \( d \) as in Fig. 3. Recall the shorter notation \( W = \{a, b, c, d\} \) for the dynamical variables’ set. Consider the matrix \( L \) in this general case. Each row in \( L \) corresponds to a vertex of the lattice, and each column of \( L \) corresponds to a polygon (i.e. to a site) of the lattice. Thus \( \det(L) \) consists on the monomials, each of them corresponds (up to a sign) to a product of different \( W_{j,p} \) such that:
Figure 10: Fixed outlets for the lattice walks

- for any vertex \((j, P)\) only one of \(a_{j,P}, b_{j,P}, c_{j,P}, d_{j,P}\) is taken in this monomial, and
- for any site only one of surrounding \(a, ..., d\) is taken in this monomial.

Take the lattice and mark the places of the vertex variables \(a, ..., d\), corresponding to the monomial, by the arrows, ingoing to the corresponding vertices. Thus for any site and for any vertex we have painted only one arrow.

In order to get a purely invariant functional, we have to multiply \(\det(L)\) by an integrating monomial, in general case this monomial is, for example, \(\prod P a_{1,P}^{-1}, d_{2,P}^{-1}, b_{3,P}^{-1}\). This choice of the integrating multiplier corresponds to element \(d\) given by (3.49). It is easy to see that this monomial has the same structure as described above. But due to the power \(-1\) we may interpret geometrically this monomial as the set of outgoing arrows.

The system of the outgoing arrows is thus fixed and shown in Fig. 10 for each \(\triangle\) – type triangle of the lattice. For the system of the outgoing arrows and any system of ingoing arrows the following is valid:

- for any site there exist exactly one outgoing arrow and exactly one ingoing arrow, and they may touch the same vertex, and
- for any vertex there exist exactly one outgoing arrow and exactly one ingoing arrow, and they may belong to the same site.

Hence there is the unique way to connect all the arrows inside each site so that a walk appears.

So, the walks we consider, obey the following demands:
- the system of outlets of the walk is fixed and given by Fig. 10,
- the walk visits any site only once,
- the walk must visit all the sites and
- the walk must visit all the vertices.

For any walk \(W\) let \(\sigma(W)\) be the number of the components of the connectedness (i.e. the number simply connected subwalks).
Let now walk \( W \) belongs to a given homotopy class \( \alpha \mathcal{A} + \beta \mathcal{B} \) of the torus, where cycle \( \mathcal{A} \) corresponds to \( T^M_a \) and cycle \( \mathcal{B} \) corresponds to \( T^M_b \), and denote such walk as \( W_{\alpha,\beta} \).

To a walk given assign a monomial according to the following rules: let the walk pass through vertex \((j, P)\) so that the walk goes the vertex from the side \(x \in W_{j,P}\), and outgoes the vertex from the side \(y \in W_{j,P}\). Then the multiplier corresponding to \((j, P)\) is \(x \cdot y^{-1}\).

The monomial \( J_W \) is the product of such multipliers corresponding to all the vertices. Thus the reader may see that each monomial we construct gains the structure of an element of \( \mathcal{B}'_P \), described in section “Auxiliary linear problem”, subsection “General approach”: monomial \( J_W \),

\[
J_W = \cdots x \cdot y^{-1} \cdot x' \cdot y'^{-1} \cdots ,
\]

(3.53)

where \(x\) and \(y\) are assigned to a same vertex, so \(x \cdot y^{-1}\) does not contain the vertex projective ambiguity, and \(y\) and \(x'\) belong to a same site, so \(y^{-1} \cdot x'\) does not contain the site ambiguity. Finally we have to provide the projective invariance of \( J_W \) with respect to the start and end points of each simply connected subwalk. In our case of the local Weyl algebrae this invariance is obvious, because of elements \(x \cdot y^{-1}\) for different vertices commute.

With the structure of the walks introduced, the simple analysis of the determinant gives immediately

\[
J_{\alpha,\beta} = \sum_{\text{all } W_{\alpha,\beta}} (-)^\sigma(W_{\alpha,\beta}) \cdot J_{W_{\alpha,\beta}} ,
\]

(3.54)

where the sum is taken over all the walks of the homotopy class \( \alpha \mathcal{A} + \beta \mathcal{B} \) given and the system of the outlets of the walks fixed.

### 3.6 Monodromy operator

Consider now another interpretation of the two dimensional kagome lattice.

Let now to each vertex of the lattice the local \( \mathcal{L} \)-operator is assigned. Instead of \( \omega \) in the definition of \( L \), use

\[
x' = C^{-1} \cdot x , \quad y' = C \cdot y ,
\]

(3.55)

where

\[
C = C(x^{-1}u, y^{-1}w)
\]

(3.56)

and

\[
C(u,w) = u^{-1} - q^{1/2} u^{-1}w + \kappa w ,
\]

(3.57)

For the \( \triangle \)-type triangle \( P \) let the in - edge variables be \( x_P, y_P, z_P \) so that out edge variables are \( x_aP, y_bP \) and \( z_cP \). These notations are shown in Fig. 11. Surely LYBE for \( \mathcal{L} \)-operators means that for \( x_P, y_P, z_P \) given, \( x_aP, y_bP \) and \( z_cP \) are the same for the right hand side YBE graph \( \nabla \).

Consider now the whole toroidal kagome lattice. We are going to assign \( x, y \) and \( z \) to some minimal set of the edges so that the variables of all other edges can be restored via functions \( C_{i,P} \).

To do this, cut the torus along some line, shown as the dashed line in Fig. 12. Call this line ‘the string’. The edge variables along the string we’ll denote as \( x_j, y_j \) and \( z_j \). It is useful to draw the string so that it intersects all \( x \) and \( z \) lines once, and \( y \)-lines twice (i.e. the homotopy class of the string is \( \pm(2 \mathcal{A} - \mathcal{B}) \), an orientation of the string and so a sign are unessential) Note, \( x_j, y_j \) and \( z_j \) introduced we assign to the edges which are right-touched to the string.
Figure 11: Edge variables of the triangle

Figure 12: The string (dashed) on the toroidal kagome lattice.
Now switch on the $L$-operator game with the edge variables. We interpret it as the shift of the string. We enumerate the lines so that the triangle $P = a^j b^i P_0$ is surrounded by the lines $x_i$, $y_j$ and $z_{i+j}$, so as the $L$-operators are

$$L_{x_i}y_j (\{u,w\}_{1,a^j b^i P_0}) , \quad L_{x_i}z_k (\{u,w\}_{2,a^k-ib^i P_0}) , \quad L_{y_j}z_k (\{u,w\}_{3,a^j b^{k-1} P_0}) . \quad (3.58)$$

The $L$-operator game allows us to restore all the edge variables for the lattice, including the left-touched to the original string variables $x'_i$, $y'_j$, $z'_k$.

Thus for given variables from the right side of the string we obtain the analogous values from the left side of the string as functionals of the given variables. Thus the map corresponding to the kagome lattice and the string chosen appears:

$$T (L) : \{x_i, y_j, z_k\} \mapsto \{x'_i, y'_j, z'_k\} . \quad (3.59)$$

As the operator, $T(L)$ is ordered product of all $L$ (3.58). Define $A < B$ if the ordered product of $A$ and $B$ is $A \cdot B$. Then in $T(L)$

$$Ly_jz_{i+j} < Lx_i, z_{i+j} < Lx_i, y_j , \quad (3.60)$$

$$Lx_i, y_j < Lx_i, z_{i+j+1} < Ly_j, z_{i+j+1} . \quad (3.61)$$

These relations are enough to restore $T(L)$. Operator $T(L)$ resembles the monodromy matrix in $2D$. The difference is that instead of the distinguished point in $2D$ monodromy matrix (i.e. the point where the transfer matrix is torn), in $3D$ we have the distinguished string.

Now, what should stand for a "trace" of $T(L)$. Consider the system

$$x'_j = x_j , \quad y'_j = y_j , \quad z'_j = z_j \quad (3.62)$$

on some left module element $\phi^*_0$. Here are $3M$ equations, $3M - 1$ from them are independent due to

$$\prod_j x_j y_j z_j = \prod_j x'_j y'_j z'_j , \quad (3.63)$$

where it is implied that all $x_P, y_P, z_P \forall P$ are commutative, this is the consequence of the commutability of primary $x_j, y_j, z_j$. Then solve $3M - 2$ equations of (3.62) leaving two variables, up to unessential signs and powers of $q :$

$$A = \prod_j q y_j z_j , \quad B = \prod_j x_j^{-1} z_j . \quad (3.64)$$

A single equation rests for $A$ and $B$, and amusingly this equation coincides with the quantum determinant relation $\phi^*_0 \cdot J(A, B) = 0$. So in this sense $J(A, B) = 0$ is the trace of the monodromy operator.

Note however, $J(A, B)$ is the invariant curve, this was established in the previous section, so it is not necessary to consider $\phi^*_0 \cdot J(A, B) = 0$. The actual problem for the further investigations is to diagonalize $J(A, B)$ for $A$ and $B$ arbitrary.

4 Discussion

Conclude this paper by an overview of the problems to be solved and the aims to be reached. The approach proposed gives a way for their solution.
First, mention the problems of the classification of the map

\[ R : \{ a_j, b_j, c_j, d_j \} \mapsto \{ a'_j, b'_j, c'_j, d'_j \}, \quad j = 1, 2, 3, \quad (4.1) \]

in general. The aim is to classify all conserving symplectic structures of the body \( B \). We have discussed only the local case, when the variables, assigned to different vertices commute, and the scalars (spectral parameters) are conserved. We suppose, such case is not unique, and there are another ways to remove the projective ambiguity. The simplest case to be investigated is to consider all the variables \( a, b, c, d \) for each vertex as matrices with, for example, non-commutative entries, but with this entries commutative for any two vertices. The matrix structure may be common for all vertices, and thus we would have no commutation between different vertices in general. Another simple possibility is another kind of locality, the case when the dynamical variables commute while do not belong to a same site.

Note, once our locality is imposed, the Weyl structure appears immediately. Thus the Weyl algebra is the consequence of the locality technically, but a principal origin of the Weyl algebra is mysterious.

Next fundamental problem is the quantization of Korepanov’s matrix model mentioned above. The conservation of complete algebra \( X, (2.74, 2.77) \), means that we can not use \( (2.79) \) to fix all the ambiguity of Korepanov’s equation. Analysis of \( (2.74) \) plus some extra (but natural) symmetry conditions allows to fix the functional map \( r : X_j \mapsto X'_j \) up to one unknown function of three variables. The problem of the Tetrahedron equation for these \( r \) is open. All these are a subject of a separate paper.

Pure technical problem to be mentioned is the investigation of \( q - \) hypergeometrical function \( \sigma \), eqs. \( (2.53, 2.55) \).

Another interesting thing is functional \( L \) – operators and LYBE related to them. The map given by \( L \), eqs. \( (2.93, 2.94) \), is a bi-rational one. Note, the case of linear \( L \) coincides with Korepanov’s \( X \). Thus the rational case of it as well as the general case of the bi-rational transformation have good perspectives for the investigation.

The main set of problems for further investigations is connected with the integrals of \( U, J(A, B) \) seems to be not constructive. The aim is to calculate the spectrum of it, and to calculate \( U \) as a function of its integrals. Possible approach is functional equations for the integrals of motion, that should follow from the determinant or topological representation of \( J \).

Another possibility is a way resembling the Bethe ansatz in 2D should exist in 3D, i.e. a way of a triangulation of \( U \) with a help of some artificial operators. If such way exists, it must based on the linear problem derived.

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