Repulsive Casimir force in Bose–Einstein Condensate

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Abstract. We study the Casimir effect for a three dimensional system of ideal free massive Bose gas in a slab geometry with Zaremba and anti-periodic boundary conditions. It is found that for these type of boundary conditions the resulting Casimir force is repulsive in nature, in contrast with usual periodic, Dirichlet or Neumann boundary condition where the Casimir force is attractive (Martin and Zagrebnov 2006 Europhys. Lett. \textbf{73} 15). Casimir forces in these boundary conditions also maintain a power law decay function below condensation temperature and exponential decay function above the condensation temperature albeit with a positive sign, identifying the repulsive nature of the force.

Keywords: Bose–Einstein condensation, Casimir effect, quantum gases
1. Introduction

In his original paper, Casimir described a nonclassical attractive force related to quantum vacuum fluctuations in the electromagnetic field between two uncharged parallel conducting plates [1]. Since then, the Casimir effect for quantum vacuum fluctuation has been extensively studied for various types of geometries and boundary conditions (see [2, 3] and the references therein) using Quantum Field Theory (QFT) techniques. But the Casimir type force due to thermal fluctuation in ideal free Bose gas in vacuum was first reported in the seminal work of Martin and Zagrebnov [4] and since then it has been thoroughly studied [5–10] in statistical mechanics (SM). It is quite well known that the Casimir force depends upon boundary conditions and is attractive for scalar fields (as well as free Bose gas) in either case of vacuum or thermal fluctuation and is reported to be an attractive force for the usual case of Dirichlet (D), Neumann (N) as well as periodic boundary (P) conditions on both sides. However, it is of significant interest to get physical configurations where the Casimir force is repulsive instead of attractive, not only for its relevance for technical applications to nano devices [3, 11–15], but also because the existence of repulsive or null Casimir forces allows a more accurate analysis of micro-gravity effects [16] as well as the study of cosmic strings [17]. It has been recently reported that in QFT approach one can achieve repulsive Casimir force due to quantum fluctuation using Zaremba\(^3\) and anti-periodic boundary condition [18, 19] (see section 3.4 and 3.5 of [19]). But Casimir type force due to thermal fluctuation in free Bose gas with these type of boundary condition is yet to be reported. Casimir-type interactions are nowadays identified in several types of systems spanning from biology [20] to cosmology [21] but the QED and condensed-matter contexts are those where the theoretical predictions concerning the existence and properties of Casimir forces found firm experimental confirmation [22, 23]. Therefore, it is of significant importance to figure out if Zaremba and anti-periodic boundary conditions can be responsible for repulsive Casimir force in a free Bose gas due to thermal fluctuations in SM. In this manuscript we have considered an ideal free Bose gas

\(^{3}\) Zaremba boundary condition indicates Dirichlet condition in one side and Neumann condition in another side, also known as hybrid or mixed boundary condition.
confined in three dimensional slab like geometry $L \times L \times d$ (where $L \gg d$), subjected to Zaremba/anti-periodic boundary condition in $z$ direction and calculated the repulsive Casimir force due to thermal fluctuation in vacuum. Point to note, imperfect Bose gas with the repulsive microscopic interparticle interactions subjected to periodic boundary conditions generate an effective Casimir force of repulsive nature [10]. But in that case [10], the repulsive behaviour of Casimir force is due to the interparticle interaction whereas in the present endeavor the repulsive Casimir force is solely due to boundary condition. Relation between the decay length characterizing the Casimir force (due to these boundary condition) and the bulk correlation lengths are also discussed.

2. Model

Let us consider ideal free massive Bose gas confined between two infinitely large square shaped plates of area $A$. The plates are along the $xy$ plane and are separated by distance $d$ along the $z$-axis. For the slab geometry we consider $\sqrt{A} \gg d$ as well the system is in thermodynamic equilibrium with its surroundings at temperature $T$. At this temperature the thermal de Broglie wavelength of a single particle of mass $m$ is $\lambda = \hbar \sqrt{\beta/m}$ where $\beta = \frac{1}{k_B T}$ and $k_B$ is the Boltzmann’s constant. In the thermodynamic limit, we consider $\lambda \ll d$. Thus the energy of the single particle is $E = \frac{q_x^2}{2m} + \frac{q_y^2}{2m} + \frac{p_z^2}{2m}$, where for Zaremba (Z) and antiperiodic (A) boundary condition [18, 19] we have respectively,

$$p_z = \left(n + \frac{1}{2}\right) \frac{\hbar \pi}{d}, \quad n = 0, 1, 2, 3, ...$$  \hspace{1cm} (1a) \\

$$p_z = \left(n + \frac{1}{2}\right) \frac{2\pi\hbar}{d}, \quad n = 0, \pm 1, \pm 2, \pm 3, ....$$  \hspace{1cm} (1b) \\

Based on the assumptions described above the grand-canonical potential per unit area can be written as

$$\Phi_d(T, \mu) = \beta^{-1} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dq_x dq_y}{(2\pi\hbar)^2} \Omega(q_x, q_y, p_z)$$  \hspace{1cm} (2) \\

where,

$$\Omega = \left\{ \begin{array}{ll} 
\ln \left[ 1 - e^{-\beta \left( \frac{q_x^2}{2m} + \frac{q_y^2}{2m} + \frac{2\hbar^2(n+\frac{1}{2})^2}{m d^2} - \mu \right)} \right] & (Z) \\
\ln \left[ 1 - e^{-\beta \left( \frac{q_x^2}{2m} + \frac{q_y^2}{2m} + \frac{2\hbar^2(n+\frac{1}{2})^2}{m d^2} - \mu \right)} \right] & (A) 
\end{array} \right.$$

Here $\mu$ is the chemical potential. Representing (2) by its low-activity series for $\mu < 0$ and performing the integration we obtain,
We write equation \( \Phi_d(T, \mu) \). Now let us consider both condensed and noncondensed phases subjected to the Zaremba boundary condition separately. Bose gas and where we have defined \( \psi_n(x) = \frac{e^{-2x^2/\lambda^2}}{\sqrt{\pi \lambda^2}} \). We now use the fact. Note that, the surface term is absent for both cases just like periodic boundary condition [4]. Nevertheless, in any general system the bulk as well the surface term, do not contribute to the Casimir force, because the force due to the bulk term is counterbalanced by the same contribution acting from outside the slabs when they are immersed in the critical medium [4, 6] and the surface term does not change with the change of thickness of the slab. Now the Casimir force can be obtained from Casimir potential through \( -\partial_\mu \Phi_{\text{Cas}}^{(Z)} \). Next let us consider both condensed and noncondensed phases subjected to the Zaremba boundary condition separately.

In the condensed phase \( T \leq T_c \) (\( \mu = 0 \)), Bose–Einstein condensate (BEC) occurs in Bose gas and \( \mu = 0 \). We write equation (9) as

\[
\Phi_{\text{Cas}}^{(Z)}(T, 0) = -\frac{2}{\beta(\sqrt{2\pi} \lambda)^3} \sum_{n=1}^{\infty} (-1)^n (\lambda/d)^2 \sum_{r=1}^{\infty} \psi_n((\lambda/d)^2 r) \]

where we have defined \( \psi_n(x) = \frac{e^{-2x^2/\lambda^2}}{\sqrt{\pi \lambda^2}} \). We now use the fact. \( \lambda/d \ll 1 \) and the sum \( \sum_{r=1}^{\infty} \) can be converted into an integral. Therefore we obtain,

\[
\Phi_{\text{Cas}}^{(Z)}(T, 0) = -\frac{2}{\beta(\sqrt{2\pi} \lambda)^3} \int_0^{\infty} \frac{(-1)^n (\lambda/d)^2}{\pi d^2} \psi_n((\lambda/d)^2 r) dr.
\]
\[
\Phi_{\text{Cas}}(T, 0) = -\frac{1}{8\pi\beta d^2} \sum_{n=1}^{\infty} (-1)^n \frac{(\eta(3))}{n^3} = \frac{\eta(3)}{8\pi\beta d^2}.
\]

(11)

Where \( \eta(s) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n^s} \) is the Dirichlet eta function. Now, using the relation, \( \eta(3) = \frac{3}{4} \zeta(3) \), where \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) is the Riemann zeta function. We finally obtain

\[
\Phi_{\text{Cas}}(T, 0) = \frac{3\zeta(3)}{32\pi} \frac{k_B T}{d^2}.
\]

(12)

Note that the sign of \( \Phi_{\text{Cas}}(T, 0) \) is positive in contrast to the situation described in [4]. Finally we have the Casimir force per unit area:

\[
F_c = -\partial_d \Phi_{\text{Cas}} = \frac{3\zeta(3)}{16\pi} \frac{k_B T}{d^3}.
\]

(13)

which is repulsive.

Finally we consider the non-condensed phase with \( T > T_c \) (\( \mu < 0 \)). The double sums in equation (6) for \( d \gg \lambda \) can be estimated following [4]

\[
\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} e^{\beta r \mu} r^{5/2} (-1)^n e^{-2(n d / \lambda)^2 / r} \leq \frac{\zeta(5/2)}{\sqrt{\pi} \sqrt{2^{-5/2} \beta^{-5/2} \mu d / \lambda}} + O \left( e^{-\sqrt{8 \beta \mu d / \lambda}} \right).
\]

(14)

As a result, the leading contributing to Casimir potential (equation (6)) is decaying exponentially just like Periodic, Dirichlet or Neumann boundary condition [4], but the force is positive unlike those boundary condition. The Casimir force in noncondensed phase is therefore \( F_C \propto \exp(-d / \kappa(Z)) \), where

\[
\kappa(Z) = \lambda \frac{2}{\sqrt{(-\mu)\beta}}.
\]

(15)

The Casimir decay length, \( \kappa \) for mixed boundary condition is exactly equal to the Dirichlet/Neumann [8] case. Let's turn our attention towards the other case with

\[
\text{Figure 1. The Casimir force of ideal Bose gas in condensed phase (} T < T_c \).
\]
anti-periodic boundary condition. Following the same procedure we find out the Casimir force for \( T < T_c \) is,

\[
F_c^{(A)} = -\partial_d \Phi_{\text{Cas}}^{(A)} = \frac{3\zeta(3) k_B T}{2\pi d^3}.
\] (16)

And in the non condensed phase the Casimir force is \( F_c^{(A)} \propto \exp(d/\kappa^{(A)}) \) with

\[
\kappa^{(A)} = \lambda \sqrt{\frac{1}{2(-\mu)\beta}}.
\] (17)

In [8], it has been established that decay length of Casimir force is related directly to the Bulk correlation length for Dirichlet (D), periodic (P) and Neumann (N) boundary conditions, \( \frac{1}{2}\kappa^{(P)} = \kappa^{(D)} = \kappa^{(N)} = \xi = \frac{\lambda}{4} \sqrt{\frac{2}{(-\mu)\beta}} \). We have thus extended their relation in case of Zaremba and anti-periodic boundary condition \( \frac{1}{2}\kappa^{(P)} = \kappa^{(D)} = \kappa^{(N)} = \kappa^{(Z)} = \frac{1}{4}\kappa^{(AP)} \).

A plot of Casimir force for different boundary conditions is shown in figure 1. But, most significant point of this calculation is the repulsive nature of critical Casimir force, both below and above critical temperature. Nevertheless from here we can see that, like the other boundary conditions upon approaching the phase containing the condensate \( (\mu \to 0) \) the range of force and the correlation length diverges with the critical exponent \( \nu = 1/2 \) in Zaremba and anti-periodic boundary condition, identifying the characteristic nature of ideal gas.

3. Concluding remarks

In conclusion, we have identified the repulsive nature of Casimir force in both phases of an ideal free Bose gas for thermal fluctuations in vacuum subjected to Zaremba and anti-periodic boundary conditions using the techniques of SM. Extracting the final result in the current study is possible due to the identities of equations (5) and (6) (referred to as Jacobi identity in equation (16) of [4]). If one compares these two identities with the corresponding identities for Neumann, Dirichlet and periodic case, one can immediately notice the \( (-1)^n \) in the second term with a sum over \( n \) in equations (5) and (6). This results in a positive contribution to the Casimir term in grand canonical potential (equation (12)) unlike the cases with Neumann, Dirichlet or periodic boundary condition [4, 5]. The reason behind the \( (-1)^n \) term in our identities of equations (5) and (6) is the quantised momenta in equation (1), which is proportional to the half integers in this case whereas for Dirichlet, Neumann or periodic scenario quantised momenta which are proportional to the integers. As a consequence, these boundary conditions result in a repulsive Casimir force. In three dimensions, the magnitude of Casimir force with Zaremba boundary condition is \( \frac{3}{32} \) times the Casimir force in periodic boundary condition, while for antiperiodic scenario the Casimir force is attenuated by a factor of \( \frac{3}{4} \) in comparison to the periodic case for any temperature\(^5\). But these values will of course change for trapped bosonic systems [6] which needs to be investigated as they

\(^4\) The definition of thermal wavelength with us and [8] is different by a factor of \( \frac{1}{\sqrt{2\pi}} \).

\(^5\) The Casimir force for Dirichlet/Neumann boundary condition are \( \frac{1}{4} \) times than the periodic one [6].
are substantially related with experimental detection of BEC [27, 28]. Point to note, this analysis is done solely for ideal gas without any sort of interaction, which proves that the repulsive behaviour is solely due to boundary condition. But at the same time the effect of interaction needs to be checked under these boundary conditions to find out if any interesting scenario arises like [10]. This is the next program that we wish to take up. The current study unraveling repulsive Casimir force in BEC could be a good prospect to different aspects of physics including nanotechnology [15, 16], biology [20], and complex networks [29]. However, we should investigate the case of dynamical Casimir [2, 3, 24] effect in BEC, which has not yet been reported. Such a study can not only disentangle new features of BEC but also shed new light on the relationship between Casimir force and the cosmological constant, dark energy and dark matter [25, 26], especially in those models where scalar field BEC [30] and Axion BEC [31] are possible Dark matter candidates.

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Appendix

In this section we derive the mathematical identities described in equations (5) and (6). For appropriate functions \( f \), the Poisson summation formula can be stated as

\[
\sum_{n=-\infty}^{\infty} f(n) = \sum_{\nu=\infty}^{\infty} \hat{f}(\nu)\quad (A.1)
\]

where, \( \hat{f}(\nu) \) is the Fourier transform of \( f(n) \).

\[
\hat{f}(\nu) = \int_{-\infty}^{\infty} dn \, f(n)e^{i2\pi \nu n}. \quad (A.2)
\]

Then using equation (A.1),

\[
\sum_{n=-\infty}^{\infty} e^{-\pi \alpha(n+1/2)^2} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} dn \, e^{-\pi \alpha(n+1/2)^2+i2\pi \nu n} = \sum_{\nu=-\infty}^{\infty} \frac{1}{\sqrt{a}} e^{-2\pi \nu^2/a} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{a}} (-1)^n e^{-\pi n^2/a} = \frac{1}{\sqrt{a}} + 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{a}} (-1)^n e^{-\pi n^2/a} \quad (A.3)
\]
which is equation (6). Now the left hand side of equation (A.3) can be rewritten as,

\[ \sum_{n=-\infty}^{\infty} e^{-\pi a(n+1/2)^2} = e^{-\pi a/4} + e^{-\pi a(3/2)^2} + e^{-\pi a(5/2)^2} + e^{-\pi a(7/2)^2} + e^{-\pi a(9/2)^2} + \ldots \]

\[ + e^{-\pi a(1/2)^2} + e^{-\pi a(3)^2} + e^{-\pi a(5)^2} + e^{-\pi a(7)^2} + e^{-\pi a(9)^2} + \ldots \]  \hspace{1cm} (A.4)

Therefore one can see the n = 0 term matches with \( n = -1 \), \( n = 1 \) term matches with \( n = -2 \), \( n = 2 \) term matches with \( n = -3 \), \( n = 3 \) term matches with \( n = 4 \) and so on.

As a result the equation (A.4) can be written as,

\[ \sum_{n=-\infty}^{\infty} e^{-\pi a(n+1/2)^2} = 2(e^{-\pi a/4} + e^{-\pi a(3/2)^2} + e^{-\pi a(5/2)^2} + e^{-\pi a(7/2)^2} + e^{-\pi a(9/2)^2} + \ldots) \]

\[ = 2 \sum_{n=0}^{\infty} e^{-\pi(n+\frac{1}{2})^2} a. \] \hspace{1cm} (A.5)

Combining equations (A.5) and (A.3) we obtain,

\[ \sum_{n=0}^{\infty} e^{-\pi(n+\frac{1}{2})^2} a = \frac{1}{2\sqrt{a}} + \frac{1}{\sqrt{a}} \sum_{n=1}^{\infty} (-1)^n e^{-\pi n^2/a} \]

which is equation (5).

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