The Birkhoff’s Theorem in Einstein-Aether Theory

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Abstract. We show the existence of the Birkhoff’s theorem in Einstein-Aether (EA) theory similar to General Relativity (GR) theory. In GR, Birkhoff’s theorem states that any spherically symmetric solution of the vacuum field equations must be static, unique and asymptotically flat. There are two possibilities from the field equation $G^{\text{ether}}_{tt}$: $\dot{B} \neq 0$ or $\dot{B} = 0$. We have shown that the case $\dot{B} \neq 0$ does not give us a solution for the field equations. On the other hand, in the case $\dot{B} = 0$ we have two possibilities, which are $c_{14} = 0$ and $c_{14} \neq 0$. Thus, in this case we have only to consider $c_{14} \neq 0$, since in the case $c_{14} = 0$ we get exactly the same results of the GR theory. For the found solutions, the metric function also depend on an arbitrary function of the time which can be eliminated redefining a new time coordinate. The first of the cases ($c_{14} = 0$) gives us the Schwarzschild solution and the last one ($c_{14} \neq 0$) are solutions of static black holes.
1. Introduction

In 1923, mathematician George Birkhoff proved a fundamental theorem in his textbook *Relativity and Modern Physics* [1] according to which the Schwarzschild spacetime geometry is the unique spherically symmetric solution of the vacuum Einstein field equations. It implies the absence of time-dependent and spherically symmetric vacuum solutions. That is to say that the vacuum spherically symmetric solutions are static and independent of variations in the matter distribution sourcing the gravitational field as long as the spherical symmetry is preserved. This theorem was independently discovered by Jebsen [2] two years earlier. See [3] for a fascinating history of this discovery. The Birkhoff theorem requires that a spherically symmetric solution must be static, given by the Schwarzschild solution. Furthermore, it also implies the absence of gravitational radiation for pulsating or collapsing, spherically symmetric bodies. That means there exists no zero-mode of gravitational wave, correspondingly in quantum mechanical terms, we can say that zero-mode of graviton does not exist [4]. This reiterates the fact that, in general relativity, the lowest multipolar gravitational radiation that propagates is quadrupole radiation. Because of these experimentally verifiable strict constraints, Birkhoff theorem is of great interest in distinguishing alternative theories of gravity from GR. This theorem has been studied in a class of higher curvature theories namely the Lovelock theories which are natural generalizations of Einstein’s theory in higher dimensions [5, 6, 7]. It was proved that the theorem is valid for the HL solutions which admit the general relativity limit in the low-energy (IR) region, while the theorem can be violated in the high-energy (UV) region, due to nonlinear effects [8]. As EA is closely related to HL, it is pertinent to investigate this theorem. Here we prove that the Birkhoff theorem is valid in EA theory not just in the case of $c_{14} = 0$ which reduces EA to GR anyway, but also in a general case of non-zero $c_{14}$. We also show that the empty exterior of a spherically symmetric system has the unique geometry of the Schwarzschild solution characterized by one parameter, the mass when the constant $c_{14}$ is fixed.

Two particularly important aspects of BT are that: (a) outside a spherically symmetric mass distribution, the gravitational potential (i.e. metric) depends on the distribution of the matter density only at second order in the potential, due to the role of binding energy as a source of gravity; and (b) a shell of mass has no effect on the metric in its interior. Without these properties we presumably could not calculate almost any gravitational fields without knowing details of the distribution of matter all over the Universe. Indeed many calculations in gravity would become impossible either in principle or practice, and many others would become enormously more difficult.

The EA theory offers a way to study the effects Lorentz violation in a gravitational sector. The Lorentz invariance (LI) as an exact symmetry in experimentally verified theories such as special relativity, quantum field theories and the standard model of particle physics, while in General Relativity (GR) it is only a local symmetry in freely falling inertial frames [9]. The violation of LI in gravitational sector is not as well explored as in matter interactions where it is highly constrained by several
The Birkhoff’s Theorem in Einstein-Aether Theory

With the advent of precision experiments [10], besides the cases where breaking LI allows us to construct a mathematically consistent quantum gravity [11], Jacobson and his collaborators introduced and analyzed a general class of vector-tensor theories called the Einstein-Aether (EA) theory [12] [13] [14] [15] [16] to study the effects of violation of LI in gravity. Thus, it is of fundamental importance to check whether Birkhoff’s theorem is still valid or needs to be modified for theories that break LI. A brief review of the vector-tensor theories of gravity can be found in [17].

The first spherical static vacuum solutions in the EA theory were obtained by Eling and Jacobson in 2006 [18]. Since then several more solutions have been found including our recent analytical solutions for static aether [19]. Most of the literature on black holes in EA theory can be found in the papers [21]-[43].

The paper is organized as follows. The Section 2 briefly outlines the EA theory, whose field equations are solved for a general spherically symmetric metric in Section 3. In Section 4 we study the case $\dot{B} \neq 0$. In Section 5 we study the case $\dot{B} = 0$. In Subsections 5.1 to 5.2 we present two solutions of the field equations. We summarize our results in Section 6.

2. Field equations in the EA theory

The general action of the EA theory is given by

$$S = \int \sqrt{-g} (L_{\text{Einstein}} + L_{\text{aether}} + L_{\text{matter}}) d^4x, \quad (1)$$

where, the first term is the usual Einstein-Hilbert Lagrangian, defined by $R$, the Ricci scalar, and $G$, the EA gravitational constant, as

$$L_{\text{Einstein}} = \frac{1}{16\pi G} R. \quad (2)$$

The second term, the aether Lagrangian is given by

$$L_{\text{aether}} = \frac{1}{16\pi G} \left[-K_{mn}^{ab} \nabla_a u^m \nabla_b u^n + \lambda (g_{ab} u^a u^b + 1)\right], \quad (3)$$

where the tensor $K_{mn}^{ab}$ is defined as

$$K_{mn}^{ab} = c_1 g^{ab} g_{mn} + c_2 \delta_m^a \delta_n^b + c_3 \delta_m^a \delta_n^b - c_4 u^a u^b g_{mn}, \quad (4)$$

being the $c_i$ dimensionless coupling constants, and $\lambda$ a Lagrange multiplier enforcing the unit timelike constraint on the aether, and

$$\delta_m^a \delta_n^b = g^{ao} g_{am} g^{bo} g_{bn}. \quad (5)$$

Finally, the last term, $L_{\text{matter}}$ is the matter Lagrangian, which depends on the metric tensor and the matter field.

In the weak-field, slow-motion limit EA theory reduces to Newtonian gravity with a value of Newton’s constant $G_N$ related to the parameter $G$ in the action [11] by [23],

$$G = G_N \left(1 - \frac{c_{14}}{2}\right). \quad (6)$$
The constant $c_{14}$ is defined as
\[ c_{14} = c_1 + c_4. \] (7)

The field equations are obtained by extremizing the action with respect to independent variables of the system. The variation with respect to the Lagrange multiplier $\lambda$ imposes the condition that $u^a$ is a unit timelike vector, thus
\[ g_{ab} u^a u^b = -1, \] (8)
while the variation of the action with respect to $u^a$, leads to
\[ \nabla_a J^a_b + c_4 a_a \nabla_b u^a + \lambda u_b = 0, \] (9)
where,
\[ J^a_m = K^a_m \nabla_b u^b, \] (10)
and
\[ a_a = u^b \nabla_b u_a. \] (11)

The variation of the action with respect to the metric $g_{mn}$ gives the dynamical equations,
\[ G_{ab}^{\text{Einstein}} = T_{ab}^{\text{aether}} + 8\pi G T_{ab}^{\text{matter}}, \] (12)
where
\[ G_{ab}^{\text{Einstein}} = R_{ab} - \frac{1}{2} g_{ab} R, \]
\[ T_{ab}^{\text{aether}} = \nabla_c [J^c_{(a} u_{b)} + u^c J_{(ab)} - J_{(a} c_{b)}] - \frac{1}{2} g_{ab} J^c d \nabla_c u^d + \lambda u_a u_b + c_1 [\nabla_a u_c \nabla_b u^c - \nabla^c u_a \nabla_c u_b] + c_4 a_a a_b, \]
\[ T_{ab}^{\text{matter}} = \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_{\text{matter}})}{\delta g_{ab}}. \] (13)

Later, when we solve the field equations (12), we do take into consideration the equations (8)-(11) in the process of simplification. Thus, in this paper (as in the equations (17)-(22) below) we seem to solve only the dynamical equations, but in fact we are also solving the equations arising from the variations of the action with respect $\lambda$ and $u^a$.

In a more general situation, the Lagrangian of GR is recovered, if and only if, the coupling constants are identically zero, e.g., $c_1 = c_2 = c_3 = c_4 = 0$, considering the equations (1) and (8).

3. Spherical Solutions of EA field equations

We start with the most general spherically symmetric static metric
\[ ds^2 = -A(r, t) dt^2 + B(r, t) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \] (14)
In accordance with equation (8), the aether field is assumed to be unitary and timelike, chosen as

\[ u^a = \left( \frac{1}{\sqrt{A}}, 0, 0, 0 \right). \]  

(15)

In order to identify eventual singularities in the solutions, it is useful to calculate the Kretschmann scalar invariant K. For the metric (14), it is given by

\[ K = \frac{1}{4r^4B^4A^4}(-16\dot{B}^2r^2BA^3 + 8A'^2r^2B^2A^2 + r^4A^2B^2A^2 + 4\dot{r}^4A^2B^2A^2 + 4\dot{r}^4A^2B^2A^2 + 32B^3A^4 + 16B^4A^4 + 16B^2A^4 - 2r^4AB'A'BB\dot{A} + r^4A^2\dot{B}^4 + 8\dot{B}^2r^2A^4 + r^4B^2A^4 + 4\dot{r}BB\dot{A}^2B'A' - 4r^4\dot{B}BA\dot{B}\dot{A} - 4r^4A''BA^2B'A' + 4r^4A''B^2\dot{A}A - 8r^4\dot{B}B^2A^2A'' - 4r^4\dot{B}BA^2\dot{B} + 4\dot{r}BB\dot{A}A^2 + 4r^4\dot{B}BA^2A^2 + 4r^4A''BA^2\dot{B}^2 - 4r^4A''B^2AA^2 - 2r^4A^2\dot{B}^2B'A' - 2r^4A\dot{B}^2BA^2 + 2r^4\dot{A}B^3B\dot{A} + 2r^4AB'A'^3B - 2r^4B^2A'^2\dot{B}\dot{A}). \]  

(16)

The aether field equations are given by

\[ G_{tt}^{\text{aether}} = \frac{c_{13}}{8B^2} \dot{B}^2 + \frac{c_2}{8B^2} \dot{B}^2 + \frac{c_{14}}{8r^2B^2A}(-4A''Br^2A - 8Ar'A'B + 3A'^2Br^2 + 2B'A'r^2A) + \frac{1}{8r^2B^2A}(8rB'A^2 + 8B^2A^2 - 8BA^2) = 0, \]  

(17)

\[ G_{tr}^{\text{aether}} = -\frac{c_{13}}{4AB^2r}(-2Ar\dot{B}B' + 2Ar\dot{B}'B + 4\dot{B}AB - rA'\dot{B}B) - \frac{c_2}{4AB^2r}(-rA'\dot{B}B - 2Ar\dot{B}'B + 2Ar\dot{B}'B) + 2Ar\dot{B}'B + \frac{B}{Br} = 0, \]  

(18)

\[ G_{rt}^{\text{aether}} = -\frac{c_{14}}{4BrA^2}(-rA'\dot{B}A + 2rA'\dot{A}B - 2rA'\dot{A}B) + \frac{\dot{B}}{Br} = 0, \]  

(19)

\[ G_{rr}^{\text{aether}} = -\frac{c_{13}}{8A^2r^2B}(4\ddot{B}Br^2A - 3\ddot{B}^2r^2A - 2\dot{B}\dot{A}Br^2) + \frac{c_{14}}{8A^2}A'^2 - \frac{c_2}{8A^2r^2B}(4\ddot{B}Br^2A - 3\ddot{B}^2r^2A) \]
The Birkhoff’s Theorem in Einstein-Aether Theory

where \( G^{\text{aether}}_{\theta\theta} \) is given by

\[
G^{\text{aether}}_{\theta\theta} = \frac{c_{13}}{8 A B^2} r^2 B^2 - \frac{c_{14}}{8 A B^2} r^2 A'^2 \\
- \frac{c_2}{8 B^2 A^2} r (-3 B^2 r A + 4 \dot{B} B r A - 2 \dot{B} A \dot{B} r) \\
- \frac{1}{8 B^2 A^2} r (4 B'^2 A^2 - 4 A' B A - 2 \dot{B} A \dot{B} r) \\
+ 4 \dot{B} B r A + 2 A^2 B r - 4 A'' B r A + 2 B' A' r A \\
- 2 B^2 r A) = 0, 
\]

and the symbol prime and dot denote the differentiation with respect to \( r \) and \( t \), respectively. We can notice here that when \( c_{13} = 0 \), \( c_{14} = 0 \) and \( c_2 = 0 \) we obtain the same field equations of the GR.

If we substitute the equations

\[ A(r, t) \rightarrow e^{2A(r)}, \]
\[ B(r, t) \rightarrow e^{2B(r)}, \]

we obtain exactly the same field equations of our previous paper \[19\]. Thus,

\[
G^{\text{aether}}_{rr} = - \frac{e^{2A-2B}}{2 r^2} \times [c_{14} (A'^2 r^2 + 2 A'' r^2 + 4 A' r \\
- 2 B' A' r^2) - 4 r B' - 2 e^{2B} + 2] = 0, \]

\[
G^{\text{aether}}_{\theta\theta} = \frac{1}{2 r^2} (c_{14} A'^2 r^2 - 2 e^{2B} + 2 + 4 A' r) = 0, \]

\[
G^{\text{aether}}_{\theta\theta} = - \frac{r}{2 e^{2B}} [(c_{14} - 2) A'^2 r^2 + 2 B' - 2 A' - 2 r A'' \\
+ 2 r B' A'] = 0, \]

\[
G^{\text{aether}}_{rt} = 0, \]

\[
G^{\text{aether}}_{tr} = 0, \]

\[
K = \frac{4}{r^4 e^{4B}} (2 B'^2 r^2 + e^{4B} - e^{2B} + 1 + 2 A'^2 r^2 \\
+ r A'' + 2 r^2 A'' A'^2 - 2 r A'' B' A' + r^2 A'^4 \\
- 2 r^2 A'' B' + r^4 B^2 A'^2), \]

where \( A = A(r) \) and \( B = B(r) \) and the bold variables denote our previous work \[19\].

From equation \[18\] we have

\[
G^{\text{aether}}_{tr} = - \frac{c_2 + c_{13}}{4 A B^2} \left( -2 A r \dot{B} B' + 2 A r \dot{B}' B - r A' \dot{B} B \right) \\
+ (1 - c_{13}) \frac{\dot{B}}{B r} = 0. \]

From equation \[25\] follows

\[
G^{\text{aether}}_{tr} = - \frac{\dot{B}}{B} \left[ (c_2 + c_{13}) \left( -\frac{B'}{2 B} - \frac{A'}{4 A} \right) + \frac{1}{r} (1 - c_{13}) \right] 
\]
The Birkhoff’s Theorem in Einstein-Aether Theory

\[ + (c_2 + c_{13}) \frac{\dot{B}'^2}{2B} = 0. \tag{26} \]

There are two possibilities from equation (26): \( \dot{B} \neq 0 \) or \( \dot{B} = 0 \). We will analyze below the both cases in details.

First, we will assume \( \dot{B} \neq 0 \).

4. Case (I): \( \dot{B} \neq 0 \)

Solving equation (26), assuming \( c_2 + c_{13} \neq 0 \) and \( 1 - c_{13} \neq 0 \), thus

\[ A = \left[ \frac{\dot{B}}{B} F_0(t) r - \frac{2(1 + c_{13})}{c_2 + c_{13}} \right]^2, \tag{27} \]

where \( F_0(t) \) is an arbitrary integration function of the time.

From equation (19) we obtain

\[ G_{\text{aether}}^{rt} = - \frac{c_{14}}{4BrA^3} \left( -rA\dot{B}A + 2r\dot{A}A\dot{B} - 2rA\dot{A}\dot{B} \right) \]

\[ + \frac{\dot{B}}{Br} = 0. \tag{28} \]

From equation (28) follows

\[ G_{\text{aether}}^{rt} = c_{14} \frac{A'}{A} \left( \frac{\dot{B}}{4B} + \frac{\dot{A}}{2A} \right) - \frac{c_{14}}{2A} \frac{\dot{A}'}{A} \]

\[ + \frac{\dot{B}}{Br} = 0. \tag{29} \]

Substituting (27) into (29) and assuming that \( c_2 + c_{13}c_{14} + c_{13} - c_{14} = 0 \) we have

\[ B = \frac{2.34^4}{3} \sqrt{2}G_0(r) \left[ C_2(-C_1 + t)^3 \right]^{\frac{1}{2}}, \tag{30} \]

where \( G_0(r) \) is an arbitrary integration function of the radial coordinate and \( C_1 \) and \( C_2 \) are arbitrary integration constants.

Substituting equations (27) and (30) into the field equations (18) and (19) we can easily show that, if \( c_2 + c_{13}c_{14} + c_{13} - c_{14} = 0 \), they are satisfied identically. On the other hand, substituting equations (27) and (30) into the field equations (17), (20) and (21) we can notice that they are not satisfied identically. Thus, we have shown that the field equations do not admit \( \dot{B}=0 \) as a solution.

Now, we will assume \( \dot{B}=0 \).

5. Case (II): \( \dot{B} = 0 \)

From the field equation (26), let us study the case where

\[ \frac{\dot{B}}{B} = 0. \tag{31} \]
Thus, we have

\[ B(r, t) = f(r). \]  

(32)

Solving simultaneously equations (17)-(22), using (32) and Maple 16 we get six possible different solutions (i) \( c_2 = 0 \) and \( c_{14} = 0 \); (ii) \( c_{13} = 0 \) and \( c_{14} = 0 \); (iii) \( c_{14} = 0 \); (iv) \( c_2 = 0 \) and \( c_{13} = 0 \); (v) \( c_2 = 0 \); (vi) \( c_{13} = 0 \).

However, we can notice from field equations (17)-(22) that whenever it appears the term in \( c_2 \) or \( c_{13} \) they multiply also \( \dot{B} \). Thus, in this particular case we have only to consider \( c_{14} \neq 0 \). When \( c_{14} = 0 \) we get exactly the same results of the GR theory.

Let us now analyze these two possible cases in detail, instead of six cases.

5.1. Subcase (A): \( c_{14} = 0 \)

Assuming equation (32) and the field equations (17)-(21) we get

\[ G_{aether}^{tt} = \frac{1}{8r^2f^2A}(8rf'A^2 + 8f^2A^2 - 8fA^2) = 0, \]  

(33)

\[ G_{aether}^{tr} = 0, \]  

(34)

\[ G_{aether}^{rt} = 0, \]  

(35)

\[ G_{aether}^{rr} = -\frac{1}{8A^2r^2f}(8f^2A^2 - 8fA^2 - 8ArA') = 0, \]  

(36)

\[ G_{aether}^{\theta\theta} = -\frac{1}{8f^2A^2}r(4f'A^2 - 4A'fA + 2A'^2f r) - 4A''f r A + 2f'A'r A) = 0, \]  

(37)

whose solution is given by

\[ A = h(t) \left( 1 + \frac{C_0}{r} \right), \]

\[ f = \frac{r}{r + C_0}, \]  

(38)

where \( h(t) \), is an arbitrary integration function and \( C_0 \) is an arbitrary integration constant. We have chosen \( C_0 = -2M \) is in order to have a resemblance with the Schwarzschild solution as in the GR where \( M \), hereinafter, is the Schwarzschild mass. Thus, \( A(r) \) and \( f(r) \) can be rewritten as

\[ A = F_1(t) \left( 1 - \frac{2M}{r} \right), \]

\[ f = \frac{r}{r - 2M}. \]  

(39)
Substituting this solution into the field equations (17)-(21) we can prove that they are satisfied identically. Since \( h(t) \) is an arbitrary integration function, we can notice that we can eliminate redefining the time coordinate, giving the Schwarzschild static solution. Thus, we have proved that the Birkhoff’s theorem is valid also for EA, in this case.

The Kretschmann scalar is given by
\[
K = \frac{48M^2}{r^6}.
\]

We can note that \( r = 0 \) is the unique singularity of this spacetime.

5.2. Subcase (B): \( c_{14} \neq 0 \)

Assuming equation (32) and the field equations (17)-(21) we get
\[
G_{tt}^{\text{aether}} = \frac{c_{14}}{8r^2f^2A}(-4A''rf^2A - 8ArA'f + 3A'^2f r^2
+ 2f'A''A) + \frac{1}{8r^2f^2A}(8rf' A^2 + 8f^2A^2
- 8fA^2) = 0,
\]
\[
G_{rr}^{\text{aether}} = 0,
\]
\[
G_{tt}^{\text{aether}} = -\frac{c_{14}}{4frA^2}(2rA'jf - 2rA'A') = 0,
\]
\[
G_{rr}^{\text{aether}} = \frac{c_{14}}{8A^2}A'^2 - \frac{1}{8A^2r^2f}(8f^2A^2 - 8fA^2
- 8ArA'f) = 0,
\]
\[
G_{\theta\theta}^{\text{aether}} = -\frac{c_{14}}{8fA^2}r^2A'^2 - \frac{1}{8f^2A^2}r(4f'A^2 - 4A'fA
+ 2A'^2fr - 4A''frA + 2f'A'rA) = 0,
\]
whose solution is given by
\[
f = \frac{1}{8A^2}(A'^2r^2c_{14} + 8A^2 + 8ArA'),
\]
\[
A' = \frac{A'A}{A},
\]
\[
A'' = \frac{1}{8rA^2}(-A'^3c_{14}r^2 - 16A'A^2).
\]

From equation (47) we can see that
\[
A(r, t) = g(r)h(t),
\]
where \( g(r) \) and \( h(t) \) are arbitrary functions. Thus, we use this variable separation in order to solve the equations (46) and (48). Thus, the equation (48) reduces to
\[
\frac{h}{8rg^2}(8g''rg^2 + g'^3r^2c_{14} + 16g'g^2) = 0.
\]
Notice that this equation is the same of our previous paper [19] assuming \( g(r) = e^{2A(r)} \) times an arbitrary function of the time \( h(t) \), since we do not have any field equation for its determination.

Solving equation (50) we get

\[
\frac{1}{2} \alpha g^{-\frac{3}{2} + \frac{1}{4} \alpha} - g^{\frac{3}{2} \alpha} C_2 + C_1 = 0, \quad (51)
\]

where \( C_1, C_2 \) are arbitrary integration constants and \( \alpha = \sqrt{-2c_{14} + 4} \). This last transcendental equation can be only be solved analytically using particular values for \( c_{14} \) (3/2, 16/9, 48/25 and −16). These particular values of \( c_{14} \) are the same used in [19]. For the GR case with \( c_{14} = 0 \) we get the Schwarzschild spacetime \( g = 1 - \frac{2M}{r} \) imposing \( C_2 = C_1 = -\frac{1}{2M} \).

Again, since \( h(t) \) is an arbitrary function, we can notice that it we can be eliminated redefining the time coordinate, giving a black hole static solution.

We have found a solution that depends on \( g \) and this is fixed when we choose a given value of \( c_{14} \). However, \( c_{14} \) is not a constant as the mass is for Schwarzschild solution, where the radial functional dependence does not change with this constant. Here \( c_{14} \) defines the theory of EA, i.e., the radial functional dependence does change with this constant, which will be fixed by observational tests. Once this is fixed, the solution is unique.

In order to prove the existence and uniqueness of this solution mathematically we will use the methods described in the reference [20] for non linear differential equations. First, we obtain the first derivative \( g' \) from equation (51). If both \( g \) and \( g' \) are continuous in a given interval of \( r \) then the equation (51) has a unique solution. Thus,

\[
g' = \frac{4Mg^{\frac{3}{2} + \frac{1}{4} \alpha}}{r \left( 2rg^{\frac{1}{2} \alpha} - 2Mg^{-\frac{3}{2} + \frac{1}{4} \alpha} + \alpha Mg^{-\frac{1}{2} + \frac{1}{4} \alpha} \right)}, \quad (52)
\]

whose denominator is null at

\[
g_0 = \left[ \frac{2r}{M (2 - \alpha)} \right]^{-\frac{4}{7\alpha}}, \quad (53)
\]

where \( c_{14} \neq 0 \), implying \( \alpha \neq 2 \). Thus, the solution (51) is unique if only if \( g \neq g_0 \) for given values of \( c_{14} \) and \( M \). Thus, we have proved that the Birkhoff’s theorem is valid also for EA for the aether static.

6. Proof by Killing Vectors

A spherically symmetrical static vacuum spacetime admits a fourth hypersurface orthogonal Killing vector along with the three spacelike Killing vectors. Although the form of the Schwarzschild metric, being a geometrical entity, is the same in all theories of gravity, the metric coefficients \( A \) and \( B \) have different forms because they are constrained by the field equations of the theory. We found that the three spacelike killing vectors exist for arbitrary metric coefficients \( A(r, t) \) and \( B(r, t) \), that is
\[ K_2 = \sin \phi \partial_\theta + \cos \phi \cot \theta \partial_\phi \]
\[ K_2 = \cos \phi \partial_\theta - \sin \phi \cot \theta \partial_\phi \]
\[ K_2 = \partial_\phi. \]

Whereas the timelike hypersurface orthogonal Killing vector,

\[ K_1 = -\partial_t \]

exists only for \( B(r) \) and either \( A(r) \) or when \( A(r, t) = g(r)h(t) \). That is, for the spherical symmetry the metric should have the \( B(r) \), whereas the staticity is obeyed only when \( A \) is either purely radial or can be separated into radial and temporal parts, so that this separation of variables allows us to have a coordinate transformation such that we can write \( A \) as independent of time. Interestingly, the well known fact that a general spherically symmetric metric, by appropriate choice of the time coordinate, can produce the spherically symmetric and static case was already appreciated in 1921 Jebsen’s derivation of Birkhoff’s theorem. But, this redefinition cannot be done at will and should be permitted by the theory of gravity under consideration.

7. Conclusions

We show the existence of the Birkhoff’s theorem in Einstein-Aether (EA) theory similar to General Relativity (GR) theory. We have shown that there are two possibilities from the field equation \( G_{\text{aether}}^{\text{tr}} : \dot{B} \neq 0 \) or \( \dot{B} = 0 \). We have shown that the case \( \dot{B} \neq 0 \) it is not a solution of the field equations. On the other hand, in the case \( \dot{B} = 0 \) solving simultaneously all the field equations we get six possible different solutions (i) \( c_2 = 0 \) and \( c_{14} = 0 \); (ii) \( c_{13} = 0 \) and \( c_{14} = 0 \); (iii) \( c_{14} = 0 \); (iv) \( c_2 = 0 \) and \( c_{13} = 0 \); (v) \( c_2 = 0 \) and (vi) \( c_{13} = 0 \). However, noticing that in all the field equations whenever it appears the term in \( c_2 \) or \( c_{13} \) they multiply also \( \dot{B} \). Thus, in this case we have only to consider \( c_{14} \neq 0 \), since in the case \( c_{14} = 0 \) we get exactly the same results of the GR theory. In all these solutions with different values of \( c_{14} \), the metric function \( g_{tt} \) also depend on an arbitrary function of the time which can be eliminated redefining the time coordinate. The first of the cases \( c_{14} = 0 \) gives us the Schwarzschild static solution and the last one \( c_{14} \neq 0 \) gives us analytical solutions already studied for the static black holes \[19\]. We have also found a solution that depends on \( g(r) \) and this is fixed when we choose a given value of \( c_{14} \). However, \( c_{14} \) is not a constant as the mass is for Schwarzschild solution, where the radial functional dependence does not change with this constant. However, here \( c_{14} \) defines the theory of EA, i.e., the radial functional dependence does change with this constant, which will be fixed by observational tests. Once this is fixed, the solution is unique, proved mathematically using the theorem of uniqueness \[20\]. Thus, we have shown that the Birkhoff’s theorem is valid also for EA for the aether static.

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The Birkhoff’s Theorem in Einstein-Aether Theory

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