Analytic Classification of Plane Branches up to Multiplicity 4

Abramo Hefez *
Universidade Federal Fluminense
Instituto de Matemática
R. Mario Santos Braga, s/n
24020-140 Niterói, RJ - Brazil
E-mail: hefez@mat.ufrj.br

Marcelo E. Hernandes †
Universidade Estadual de Maringá
Departamento de Matemática
Av. Colombo, 5790
87020-020 Maringá, PR - Brazil
E-mail: mehernandes@uem.br

Abstract

We perform the analytic classification of plane branches of multiplicity less or equal than four. This is achieved by computing a Standard basis for the modules of Kähler differentials of such branches by means of the algorithm we developed in [9] and then applying the classification method for plane branches presented in [10].

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1 Introduction

Until very recently, the analytic classification of plane branches within an equisingularity class was an open problem (cf. [10]). The first serious attempt to solve this problem was made by S. Ebeey in [5], where he classified branches of multiplicity two and three and very few classes of branches of multiplicity four. Few years later, O. Zariski dedicated the book [17] to the study of this problem without much success.

In this paper we will exploit the general approach of [10], where a general method to perform the effective analytic classification of plane branches within a given equisingularity class is described, to classify all branches with multiplicity less or equal than four. An important invariant in this context, used to stratify the parameter space Σ of any given equisingularity class, is the value set Λ of the associated module of Kähler differentials, which we determine computing a Standard Basis for the module by means of the algorithm we developed in [9].

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Some authors have used other invariants as Tjurina’s number $\tau$ of the branch (cf. [13]) or Zariski’s $\lambda$ invariant (cf. [15]), introduced in [16], to stratify the parameter space $\Sigma$. These invariants express very partial information one can get from the set $\Lambda$ and do not separate properly branches for classification purpose. Another invariant adopted in [7] and [6] is the Hilbert function on the Tjurina algebra of the branch. This also doesn’t work properly and in Section 6 we will give an example in which two branches with two different sets $\Lambda$, have the same Hilbert function.

It should be noted that the differentials obtained by means of the above mentioned algorithm may also serve to determine explicitly the analytic coordinate changes that will reduce a given curve to its normal form (cf. [10] and [11]). It also should be noted that our classification corresponds to the classification by contact equivalence, introduced by J. Mather, and is not the same as Arnold’s (see [1] and [12]), where germs of functions are classified up to right equivalence; i.e., up to changes of coordinates in the source. It is worth noting that the contact equivalence classification is much more difficult than the right equivalence classification.

Finally, we should mention that in [14] the Tjurina’s numbers were computed for all plane branches of multiplicity less or equal than four. The method used there was to determine, by ad-hoc calculations, the cardinality of the set $\Lambda \setminus \Gamma$, where $\Gamma$ is the semigroup of values of the branch, which measures the difference between the conductor of $\Gamma$ and Tjurina’s number (cf. Remark 4.8 of [9]). This can easily be derived from our results since we give explicitly, for such branches, the sets $\Lambda$ and $\Gamma$.

2 Preliminaries

Let $A$ be either the ring of formal or convergent power series in two indeterminates $X$ and $Y$ with coefficients in $\mathbb{C}$ and $B$ the ring of formal or convergent power series in one indeterminate $t$ with coefficients in $\mathbb{C}$. A plane branch $(f)$ is a class in $A$, modulo associates, of an irreducible non-unit $f$ in $A$. Two branches $(f_1)$ and $(f_2)$ are analytically equivalent, or shortly equivalent, if there exist an automorphism $\Phi$ and a unit $u$ of $A$ such that $\Phi(f_1) = uf_2$.

Consider a Puiseux parametrization $\varphi = (x(t), y(t)) \in B \times B$ of the branch $(f)$, and its associated map germ $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$. It is known (see Lemma 2.2 in [2]) that given two plane branches $(f_1)$ and $(f_2)$ parametrized, respectively, by $\varphi_1$ and $\varphi_2$, then $(f_1)$ and $(f_2)$ are analytically equivalent if, and only if, $\varphi_1$ and $\varphi_2$ are $A$-equivalent, where $A$-equivalence means that there exist germs of analytic isomorphisms $\sigma$ and $\rho$ of $(\mathbb{C}^2, 0)$ and $(\mathbb{C}, 0)$, respectively, such that $\varphi_2 = \sigma \circ \varphi_1 \circ \rho^{-1}$.

So, the analytic classification of plane branches reduces to the $A$-classification of parametrizations, which we are going to undertake in this paper.

To the map germ $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ there is associated a ring homomorphism $\varphi^* : A \to B$, determining a natural valuation $v_\varphi$ on $A$. The value set $\Gamma = v_\varphi(A) \subset \mathbb{N} \cup \{\infty\}$ will be called the semigroup of values of the branch. This
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is a well known complete invariant for the topological classification of plane branches.

The semigroup $\Gamma$ of a plane branch has a conductor $c$ and any element in the finite set $\mathbb{N} \setminus \Gamma$ is called a gap of $\Gamma$. If $v_0 < v_1 < \cdots < v_g$ is a minimal set of generators for $\Gamma$, then the multiplicity of the branch is $v$.

The semigroup $\Gamma$ of a plane branch has a conductor $c$, and any element in the finite set $\Lambda$. So, this allows to determine all possible sets $\Lambda$ for the plane branches with given semigroup $\Gamma$, making effective the above theorem.

Moreover, if $\varphi$ and $\varphi'$ (with coefficients $a'_i$ instead of $a_i$) are parametrizations as in (2.2), representing two plane branches $(f)$ and $(g)$ with same semigroup of values and same set of values of differentials, then $(f)$ is equivalent to $(g)$ if, and only if, there exists $r \in \mathbb{C}^*$ such that $r^{\lambda - v_1} = 1$ and $a_i = r^{i-v_1} a'_i$, for all $i$.

In [9] we developed an algorithm to compute a Standard basis for the $\varphi^*(A)$-module $\varphi^*(AdX + AdY)$, whose values added to the elements of $\Gamma$ give the set $\Lambda$. So, this allows to determine all possible sets $\Lambda$ for the plane branches with given semigroup $\Gamma$, making effective the above theorem.
3 Branches of Multiplicity Less than Four

The case of multiplicity one will be disregarded since all such branches are equivalent to each other (cf. [8] Proposition 3.1).

Now suppose that a plane branch with multiplicity \( v_0 = 2 \) is given. Then, its semigroup of values is given by \( \Gamma = \langle 2, v_1 \rangle \), with \( v_1 \) odd. According to formula (2.1), the conductor of \( \Gamma \) is \( c = v_1 - 1 \). So, \( \Lambda \setminus \Gamma = \emptyset \) and by Zariski’s result we have that the given branch is equivalent to one with Puiseux parametrization \((t^2, t^{v_1})\).

This gives the classification of all branches of multiplicity two.

Let a branch of multiplicity \( v_0 = 3 \) be given, then in this case, \( \Gamma = \langle 3, v_1 \rangle \), with \( \text{GCD}(3, v_1) = 1 \), whose conductor is \( c = 2(v_1 - 1) \). The gaps of \( \Gamma \) above \( v_1 \) are the numbers:

\[
2v_1 - 3 \left[ \frac{v_1}{3} \right], 2v_1 - 3 \left( \left[ \frac{v_1}{3} \right] - 1 \right), \ldots, 2v_1 - 3 \cdot 2, 2v_1 - 3.
\]

If \( \Lambda \setminus \Gamma = \emptyset \), then the branch is equivalent to one with a parametrization \((t^3, t^{v_1})\).

If \( \Lambda \setminus \Gamma \neq \emptyset \), then the invariant \( \lambda \) may be any of the following integers:

\[
2v_1 - 3 \left[ \frac{v_1}{3} \right], 2v_1 - 3 \left( \left[ \frac{v_1}{3} \right] - 1 \right), \ldots, 2v_1 - 3 \cdot 2.
\]

Once \( \lambda \) is chosen, in the above set, it follows that any gap \( j > \lambda \), is such that \( j \in \Lambda \). Hence, by the NFT (Normal Forms Theorem) we have that the given branch is equivalent to one with Puiseux parametrization

\[(t^3, t^{v_1} + t^\lambda)\].

Clearly, two parametrizations as above are equivalent if, and only if, they are identical.

To describe the set \( \Lambda \setminus \Gamma \), suppose, for example, that \( \lambda = 2v_1 - 3k \), for \( 2 \leq k \leq \left[ \frac{v_1}{3} \right] \), then

\[
\Lambda \setminus \Gamma = \{2v_1 - 3j; 1 \leq j \leq k - 1\}.
\]

4 Branches of Multiplicity 4

We will describe in this section all possible sets \( \Lambda \) for a plane branch of multiplicity \( v_0 = 4 \). It is easy to verify that the only possible semigroups of values of multiplicity 4 are either of the form \( \langle 4, v_1 \rangle \), with \( \text{GCD}(4, v_1) = 1 \), or of the form \( \langle 4, v_1, v_2 \rangle \), with \( \text{GCD}(4, v_1) = 2 \) and \( \text{GCD}(4, v_1, v_2) = 1 \).

We will consider first the case of semigroups of the form \( \Gamma = \langle 4, v_1 \rangle \). According to (2.1), we have \( c = 3(v_1 - 1) \).

Assume that \( \Lambda \setminus \Gamma \neq \emptyset \); otherwise, the branch would be equivalent to one with Puiseux parametrization \((t^4, t^{v_1})\).
The gaps of $\Gamma$, above $v_1$, are of the form
\[ 2v_1 - 4j, \ 1 \leq j \leq \left\lfloor \frac{v_1}{4} \right\rfloor; \ 3v_1 - 4j, \ 1 \leq j \leq \left\lfloor \frac{v_1}{2} \right\rfloor. \]

So, the $\lambda$ invariant may take any of the following values:
\[ 2v_1 - 4j, \ 2 \leq j \leq \left\lfloor \frac{v_1}{4} \right\rfloor; \ 3v_1 - 4j, \ 2 \leq j \leq \left\lfloor \frac{v_1}{2} \right\rfloor. \]

We will analyze two distinct cases according to the value of $\lambda$:

**Case a)** $\lambda = 3v_1 - 4j$, for some $j = 2, \ldots, \left\lfloor \frac{v_1}{4} \right\rfloor$.

Using that $\lambda + v_0 = \min(\Lambda \setminus \Gamma)$ and the NFT, we may assume that the branch is equivalent to one with the following Puiseux parametrization:
\[
\varphi(t) = \left( t^4, t^{v_1} + t^{3v_1 - 4j} + \sum_{i=1}^{j\lceil \frac{v_1}{4} \rceil - 2} a_i t^{2v_1 - 4j - \lceil \frac{v_1}{4} \rceil - i} \right),
\]
because $\{3v_1 - 4k; \ 1 \leq k \leq j - 1\} \subseteq \Lambda$.

Applying now to these branches Algorithm 4.10 of [9], which will be referred in the sequel as the algorithm, simply, we get only one minimal non-exact differential (MNED) $\omega_1 = xdy - \frac{v_1}{4} ydx$ with $v_\varphi(\omega_1) = 3v_1 - 4(j - 1)$.

Hence,
\[
\Lambda \setminus \Gamma = \{v_\varphi(\omega_1) + \gamma \notin \Gamma; \ \gamma \in \Gamma\} = \{3v_1 - 4s; \ 1 \leq s \leq j - 1\}.
\]

**Case b)** $\lambda = 2v_1 - 4j$, for some $j = 2, \ldots, \left\lfloor \frac{v_1}{4} \right\rfloor$.

Applying the NFT, remembering that $\min(\Lambda \setminus \Gamma) = \lambda + v_0$, we may assume that the branch is equivalent to one with the following Puiseux parametrization:
\[
\varphi(t) = \left( t^4, t^{v_1} + t^{2v_1 - 4j} + \sum_{i=1}^{\lceil \frac{v_1}{4} \rceil + 1 - i} a_i t^{3v_1 - 4j + 1 - i} \right),
\]
because $\{2v_1 - 4k, 3v_1 - 4k; \ 1 \leq k \leq j - 1\} \subseteq \Lambda$.

Applying the algorithm to the above parametrization, we get in the first step the MNED $\omega_1 = xdy - \frac{v_1}{4} ydx$, for which
\[
\frac{\varphi^\ast(\omega_1)}{dt} = (v_1 - 4j)t^{2v_1 - 4j - 1 - 1} + \sum_{i=1}^{\lceil \frac{v_1}{4} \rceil} (2v_1 - 4 \left( \left\lceil \frac{v_1}{4} \right\rceil + j + 1 - i \right)) a_i t^{3v_1 - 4j + 1 - i - 1}.
\]
In the second step of the algorithm, we have
\[
\omega_2 = v_1 x^{j - 1} \omega_1 - (v_1 - 4j)ydy.
\]

Hence,
\[
\frac{\varphi^\ast(\omega_2)}{dt} = \sum_{i=1}^{\lceil \frac{v_1}{4} \rceil - j} v_1 a_i \left( 2v_1 - 4 \left( \left\lceil \frac{v_1}{4} \right\rceil + j + 1 - i \right) \right) t^{3v_1 - 4j + 1 - i - 1}.
\]
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\[ + \left( v_1 a_{[v_2]} + j (2v_1 - 8j) - (v_1 - 4j) (3v_1 - 4j) \right) t^{3v_1 - 4j - 1} + \ldots \]

If, for some \( i = 1, \ldots, \left[ \frac{v_2}{4} \right] - j \), we have \( a_i \neq 0 \), then \( \omega_2 \) is a MNED, with \( v_\varphi(\omega_2) = 3v_1 - 4 \left( \left[ \frac{v_2}{4} \right] + 1 - k \right) \), where \( k = \min(i; a_i \neq 0) \). Hence, the algorithm ends since we already got \( v_0 = 2 = 2 \) MNED’s (cf. Remark 4.11 of [9]).

In this case, we have that
\[
\Lambda \setminus \Gamma = \{ v_\varphi(\omega_1) + \gamma \notin \Gamma; \gamma \in \Gamma \} \cup \{ v_\varphi(\omega_2) + \gamma \notin \Gamma; \gamma \in \Gamma \} = \{ 2v_1 - 4s; 1 \leq s \leq j - 1 \} \cup \{ 3v_1 - 4s; 1 \leq s \leq \left[ \frac{v_2}{4} \right] + 1 - k \}.
\]

If \( a_i = 0 \) for all \( i \) with \( 1 \leq i \leq \left[ \frac{v_2}{4} \right] - j \) and \( a_{\left[ \frac{v_2}{4} \right] + 1 - j} \neq \frac{v_2 + \lambda}{2v_1} \), then \( \omega_2 \) is a MNED with \( v_\varphi(\omega_2) = 3v_1 - 4j. \) Again, the algorithm ends since we already got the maximum number of MNED’s.

In this situation,
\[
\Lambda \setminus \Gamma = \{ 2v_1 - 4s; 1 \leq s \leq j - 1 \} \cup \{ 3v_1 - 4s; 1 \leq s \leq j \}.
\]

On the other hand, if \( a_i = 0 \) for all \( i = 1, \ldots, \left[ \frac{v_2}{4} \right] - j \) and \( a_{\left[ \frac{v_2}{4} \right] + 1 - j} = \frac{v_2 + \lambda}{2v_1} \), then the algorithm ends. In this case, \( \omega_1 \) is the only MNED, and
\[
\Lambda \setminus \Gamma = \{ 2v_1 - 4s; 1 \leq s \leq j - 1 \} \cup \{ 3v_1 - 4s; 1 \leq s \leq j - 1 \}.
\]

We will analyze now the case \( \Gamma = \langle 4, v_1, v_2 \rangle \). It is easy to see that \( \lambda = v_2 + v_1 - 4m_1 \), where \( m_1 = \frac{v_2}{4} \) that is, \( \lambda = v_2 - v_1 \), and that the gaps of \( \Gamma \) above \( v_1 \) are the following:
\[
v_2 - 4j, 1 \leq j \leq \left[ \frac{v_2 - v_1}{4} \right]; \ v_2 + v_1 - 4j, 1 \leq j \leq \left[ \frac{v_2}{4} \right].
\]

By the NFT, a branch with the above \( \Gamma \) is equivalent to one with Puiseux parametrization
\[
\varphi(t) = (t^4, t^{\nu_1} + t^\lambda + \sum_{i=1}^{\left[ \frac{v_2}{4} \right] - 1} a_t t^{v_2 - 4([\frac{v_2}{4}]+1-i)},
\]

because \( \{ v_2 + v_1 - 4k; 1 \leq k \leq m_1 - 1 \} \subset \Lambda \).

Putting \( z = y^2 - x^{m_1} \), we have that \( v_\varphi(z) = v_2 \) and therefore \( \{ x, y, z \} \) is a minimal Standard basis for \( \varphi^*(A) \).

In the first step of the algorithm we get \( \omega_1 = x dy - \frac{\nu_1}{4} y dx \) and \( \omega_2 = 2 y dy - m_1 x^{m_1-1} dx \). But, \( \omega_2 = dz \), hence it is an exact differential. On the other hand, \( \omega_1 \) is a MNED, namely
\[
\frac{\varphi^*(\omega_1)}{dt} = (v_2 - 4m_1) t^{v_2 + v_1 - 4(m_1 - 1) - 1}
\]
\[
+ \sum_{i=1}^{\left[ \frac{v_2}{4} \right] - 1} \left( v_2 - v_1 - 4 \left( \left[ \frac{v_2}{4} \right] + 1 - i \right) \right) a_t t^{v_2 - 4([\frac{v_2}{4}]+1-i)-1}
\]
with \( v_\varphi(\omega_1) = v_2 + v_1 - 4(m_1 - 1) \). Moreover, \( \omega_1 \) is the unique MNED, and we have

\[
\Lambda \setminus \Gamma = \{ v_2 + v_1 - 4s; 1 \leq s \leq m_1 - 1 \}.
\]

Now, a further application of the NFT allows to deduce the following theorem

**Theorem 4.1** A plane branch of multiplicity less or equal than 4 is equivalent to a member \( C_a \) of one of the families described in Table (1). Two branches \( C_a \) and \( C_{a'} \) belonging to distinct families are never equivalent, and if they belong to the same family, they are equivalent if and only if they differ by an homothety; that is, there is \( r \in \mathbb{C}^* \) with \( r^{\lambda - v_1} = 1 \), where \( \lambda \) is Zariski’s invariant of the branch, such that \( a_i' = r^{v_1-a_i} \), for all \( i > \lambda \).

## 5 Examples and Remarks

The method of classification of branches we used in this paper is effective; that is, it is possible to perform the computations in order to put any given branch into normal form. For a computer implementation in MAPLE see [11].

In [17], Section 3, Chapter IV, after a long computation, Zariski concluded that all branches in the equisingularity class determined by the semigroup \( \langle 4, 6, v_2 \rangle \) are equivalent to each other. This follows immediately from the last row in Table (1), from where we see that all such branches are equivalent to the branch with Puiseux parametrization

\[
\varphi(t) = \left( t^4, t^6 + t^{v_1-6} \right).
\]

In [5], Ebey, and subsequently P. Carbonne in [4] studied the equisingularity classes determined by \( \langle 4, v_1 \rangle \), where \( v_1 \leq 11 \). All their results are contained in Table (1).

In [3], the family of branches

\[
\varphi_a(t) = \left( t^4, t^{v_1} + t^{2v_1-8} + \frac{3v_1 - 8}{2v_1} t^{3v_1-16} + at^{3v_1-12} \right),
\]

was considered. At that time, we hadn’t the tools to classify the members of this family, modulo analytic equivalence. Now, since we are in the last case of the semigroup \( \langle 4, v_1 \rangle \) in Table (1), we have that \( \varphi_a \) determines a plane branch equivalent to that determined by \( \varphi_{a'} \) if and only if \( a' = r^{2v_1-12}a \), where \( r^{v_1-8} = 1 \).

As mentioned in the introduction, we will now show that \( \Lambda \), the set of values of Kähler differentials of the local ring of a plane branch, is a finer invariant than the Hilbert function on the Tjurina algebra of that local ring, used in [7] and [6].

To illustrate this, consider the branches given by

\[
f(X, Y) = Y^4 - X^9 + X^7Y, \quad g(X, Y) = Y^4 - X^9 + X^5Y^2.
\]
One can easily verify that both branches determine the same Hilbert function.

The branches \((f)\) and \((g)\) have, respectively, the following Puiseux expansions:

\[
(t^4, t^9 - \frac{1}{4} t^{10} - \frac{1}{32} t^{11} + \frac{7}{2048} t^{13} + \frac{1}{512} t^{14} + \frac{39}{65536} t^{15} + \ldots),
\]

and

\[
(t^4, t^9 - \frac{1}{4} t^{11} + \frac{1}{32} t^{13} + \frac{1}{128} t^{15} + \ldots),
\]

whose normal forms, which may be obtained through \([11]\), are, respectively,

\[
(t^4, t^9 + t^{10} - \frac{1}{2} t^{11}), \quad (t^4, t^9 + t^{11}).
\]

According to Table (1), these branches have distinct sets of values of Kähler differentials, which are, respectively,

\[
\{4, 8, 9, 12, 13, 14, 16, \ldots\} \quad \text{and} \quad \{4, 8, 9, 12, 13, 15, \ldots\}.
\]

Finally, remark that we have the moduli problem for branches with multiplicity at most 4 is solved as follows.

For multiplicity 1, 2 and 3, or when \(\Lambda \setminus \Gamma = \emptyset\), this is trivial and already contained in \([17]\).

Suppose now that an equisingularity class is given by a semigroup \(\Gamma\) of multiplicity 4. Then the normal forms in Table (1) determine a finite family of disjoint constructible sets in affine spaces, each one corresponding to a set \(\Lambda \setminus \Gamma\), modulo a weighted action of the finite group \(G\) of the complex \((\lambda - \nu_1)\)-th roots of unity, where \(\lambda = \min(\Lambda \setminus \Gamma) - 4\).

For example, let \(\Gamma = \langle 4, v_1 \rangle\), and \(\Lambda \setminus \Gamma = \{2v_1 - 4s, 3v_1 - 4s; \quad 1 \leq s \leq j - 1\}\), for some \(j = 2, \ldots, \lfloor \frac{v_1}{4} \rfloor\).

So, \(\lambda = 2v_1 - 4j\). In this case, the corresponding component of the moduli is \(\mathbb{C}^{j-1}/G\), where the action of \(G\) on \(\mathbb{C}^{j-1}\) is as follows:

\[
(a_{\lfloor \frac{v_1}{4} \rfloor - j + 2}, a_{\lfloor \frac{v_1}{4} \rfloor - j + 3}, \ldots, a_{\lfloor \frac{v_1}{4} \rfloor}) \sim (a'_{\lfloor \frac{v_1}{4} \rfloor - j + 2}, a'_{\lfloor \frac{v_1}{4} \rfloor - j + 3}, \ldots, a'_{\lfloor \frac{v_1}{4} \rfloor}) \iff a'_i = r^{i-v_1} a_i, \quad \left[\frac{v_1}{4}\right] - j + 2 \leq i \leq \left[\frac{v_1}{4}\right], \quad r \in G.
\]

Now, if \(\Gamma = \langle 4, v_1, v_2 \rangle\), then the moduli consists of one component, which is a point if \(v_1 = 6\), and \(\mathbb{C}^{\lfloor \frac{v_1}{4} \rfloor - 1}/G\), where \(G = \{r \in \mathbb{C}; \quad r^{v_2 - 2v_1} = 1\}\), if \(v_1 > 6\).

The action of \(G\) is as follows:

\[
(a_1, a_2, \ldots, a_{\lfloor \frac{v_1}{4} \rfloor - 1}) \sim (a'_1, a'_2, \ldots, a'_{\lfloor \frac{v_1}{4} \rfloor - 1}) \iff a'_i = r^{i-v_1} a_i, \quad 1 \leq i \leq \left[\frac{v_1}{4}\right] - 1, \quad r \in G.
\]
The last column of the above table contains the result of \[14\], and is obtained by using the equality \(\tau = c - \frac{1}{2}A \setminus \Gamma\), where \(c\) is the conductor of \(\Gamma\), which was pointed out in Remark 4.8 of \[9\].

**References**

[1] V. I. Arnold, Local Normal Forms of Functions. Invent. Math. 35 (1976) 87-109.

[2] Bruce, J. W. and Gaffney, T. J. - Simple Singularities of Mappings \(\mathbb{C}, 0 \rightarrow \mathbb{C}^2\), 0. J. London Math. Soc. (2), 26 (1982) 465-474.
[3] V. Bayer and A. Hefez, Algebroid Plane Curves whose Milnor and Tjurina Numbers Differ by one or two. Bol. Soc. Brasil. Mat., Vol. 32, No. 1 (2001) 63-81.

[4] P. Carbonne, Sur les Différentielles de Torsion, J. Alg. 202 (1998), 367-403.

[5] S. Ebey, The Classification of Singular Points of Algebraic Curves, Trans. Amer. Math. Soc. 118 (1965) 454-471.

[6] G-M. Greuel, C. Hertling and G. Pfister, Moduli Spaces of semiquasihomogeneous singularities with fixed principal part. J. Alg. Geom. (6) (1997) 169-199.

[7] G-M. Greuel and G. Pfister, On moduli spaces of semiquasihomogeneous singularities. Progress in Math. vol 134 (1996) 171-185.

[8] A. Hefez, Irreducible Plane Curve Singularities. In Real and Complex Singularities, D. Mond and M.J. Saia, Editors, Lecture Notes in Pure and Appl. Math. V. 232, Marcel Dekker (2003) 1-120.

[9] A. Hefez and M.E. Hernandes, Standard bases for local rings of branches and their module of differentials, J. Symb. Comp. 42 (2007) 178-191.

[10] A. Hefez and M. E. Hernandes, The analytic classification of plane branches. Preprint (2007).

[11] M. E. Hernandes, www.dma.uem.br/~hernandes/publications.html

[12] P. Javorski, Normal Forms and Bases of Local Rings of Irreducible Germs of Functions of Two Variables. J. Sov. Math. Vol. 50 No. 1 (1990) 1350-1364.

[13] O. A. Laudal and G. Pfister, Local Moduli and Singularities. Lect. Notes Math. 1310, Springer-Verlag (1998).

[14] K-I. Nishiyama and M. Watari, Tjurina numbers of plane curve singularities whose multiplicities are three and four. Arch. Math. 86 (2006) 529-539.

[15] R. Peraire, Moduli of Plane Curve Singularities with a Single Characteristic Exponent. Proc. Amer. Math. Soc. 126 (1) (1998) 25-34.

[16] O. Zariski, Characterization of plane algebroid curves whose module of differentials has maximum torsion. Proc. Nat. Acad. Sci. USA Vol.56, N. 3 (1966) 781-786.

[17] O. Zariski, Le Problème des Modules pour les Branches Planes. Cours donné au Centre de Mathématiques de L’École Polytechnique. Nouvelle éd. revue par l’auteur. Rédigé par François Kimetry et Michele Merle. Avec un appendice de Bernard Teissier. Paris, Hermann (1986).

English translation by Ben Lichtin: The Moduli Problem for Plane Branches. University Lecture Series, AMS (2006).