SIGN-CHANGING SOLUTIONS FOR FOURTH ORDER ELLIPTIC EQUATIONS WITH KIRCHHOFF-TYPE

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(Communicated by Shoji Yotsutani)

ABSTRACT. In this paper, we study the following fourth-order elliptic equation with Kirchhoff-type
\[
\begin{aligned}
\Delta^2 u - (a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx) \Delta u + V(x) u &= f(u), \\
&\quad x \in \mathbb{R}^N, \\
&\quad u \in H^2(\mathbb{R}^N),
\end{aligned}
\]
where the constants \(a > 0, b \geq 0\). By constraint variational method and quantitative deformation lemma, we obtain that the problem possesses one least energy sign-changing solution \(u_b\). Moreover, we also prove that the energy of \(u_b\) is strictly larger than two times the ground state energy. Finally, we give a convergence property of \(u_b\) when \(b\) as a parameter and \(b \to 0\).

1. Introduction and main results. This paper is concerned with the existence of sign-changing solutions to the following fourth order elliptic equations of Kirchhoff type
\[
\begin{aligned}
\Delta^2 u - (a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx) \Delta u + V(x) u &= f(u), \\
&\quad x \in \mathbb{R}^N, \\
&\quad u \in H^2(\mathbb{R}^N),
\end{aligned}
\] (1.1)
where \(a, b\) are positive constants, \(\Delta^2\) is the biharmonic operator.

Problem (1.1) arises in the study of travelling waves in suspension bridge and the study of the static deflection of an elastic plate in a fluid, see [23]. For the case \(b = 0\), there are many results focused on the existence, multiplicity and concentration of solutions, see for instance [17, 18, 34, 35, 36, 39, 40, 41, 42]. For
the case $b \neq 0$, problem (1.1) is a nonlocal one as the appearance of the term $\int_{\mathbb{R}^N} |\nabla u|^2 \, dx$ implies that (1.1) is not a pointwise identity. This causes some mathematical difficulties which make the study of the problem particularly interesting and has received considerable attention in mathematical analysis in the last years, see [8, 9, 10, 13, 15, 24, 25, 28, 31, 32, 33, 38] and the references therein.

From the discussion in these literatures, we know that problem (1.1) has deep mathematical and physical background. In fact, if we set $V(x) = 0$, replace $\mathbb{R}^N$ by a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and set $u = \Delta u = 0$ on $\partial \Omega$, then problem (1.1) is reduced to the following equation

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\Delta^2 u - (a + b \int_{\Omega} |\nabla u|^2 \, dx) \Delta u = f(u), & x \in \Omega, \\
u = \Delta u = 0 & \text{on } \partial \Omega,
\end{array} \right.
\end{aligned}
\] (1.2)

which is related to the following stationary analogue of the equation of Kirchhoff type

\[
\begin{aligned}
u_{tt} - \Delta^2 u - (a + b \int_{\Omega} |\nabla u|^2 \, dx) \Delta u = f(u), & x \in \Omega,
\end{aligned}
\] (1.3)

this problem is used to describe some phenomena appeared in different physical, engineering and other sciences because it is regarded as a good approximation for describing nonlinear vibrations of beams or plates, see [2, 3]. Later, in [19], by the fixed point theorems in cones of ordered Banach spaces, Ma et.al. considered existence and multiplicity of positive solutions for the following fourth order equation

\[
\begin{aligned}
\left\{ \begin{array}{ll}
u''' - M(\int_{\Omega} |u|^2 \, dx) u'' = q(x)f(x, u', u), & \\
u(0) = u(1) = u''(0) = u''(1) = 0.
\end{array} \right.
\end{aligned}
\] (1.4)

Recently, in [20, 21], Ma and Wang considered problem (1.4) and the following fourth order equation of Kirchhoff type

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\Delta^2 u - M(\int_{\Omega} |\nabla u|^2 \, dx) \nabla u = f(x, u), & x \in \Omega, \\
u = \Delta u = 0 & \text{on } \partial \Omega
\end{array} \right.
\end{aligned}
\]

and obtained the existence and multiplicity of solutions. In [29], by inserting a parameter $\lambda$, the authors considered the existence of nontrivial solutions for (1.2) with $\lambda$. Recently, Xu and Chen [31, 32, 33] obtained infinitely many large energy solutions and negative energy solutions to (1.1), respectively. However, up to our knowledge, there seem no any results on the existence of sign-changing solutions to (1.1).

In this paper, the first aim is to establish the existence result of sign-changing solutions to problem (1.1). In what follows, we make the following assumptions:

(V1) $V(r) \in C([0, +\infty), (0, +\infty))$ is bounded from below by a positive constant $V_0$;
(V2) there exists a constant $d_0 > 0$ such that

\[
\lim_{|y| \to \infty} \text{meas} \left( \{ x \in \mathbb{R}^N : |x - y| \leq d_0, V(x) \leq L \} \right) = 0, \ \forall L > 0,
\]

where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;
(F1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(s) = o(|s|)$ as $s \to 0$;
(F2) for some constant $p \in (2, 2_*)$, $\lim_{s \to -\infty} \frac{f(s)}{s^{2_*/p}} = 0$, where $2_* = \frac{2N}{N-4}$ if $N > 4$,

$2_* = \infty$ if $N \leq 4$;
(F3) $\lim_{s \to \infty} \frac{F(s)}{s} = +\infty$, where $F(s) = \int_0^s f(t) \, dt$;
(F_4) $f(s)$ is a non-decreasing function of $s \in \mathbb{R} \setminus \{0\}$.

To state our main results, for $a > 0$ fixed, define the Sobolev space

when $(V_1)$ holds, $E = H^2_r(\mathbb{R}^N) = \{ u \in H^2(\mathbb{R}^N) | u(x) = u(|x|) \}$,

or

when $(V_2)$ holds, $E = \left\{ u \in D^{2,2}(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)u^2dx < +\infty \right\}$,

with the inner product as

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a\nabla u \cdot \nabla v + V(x)uv) \, dx, \forall u, v \in E,$$

and the norm

$$\|u\| = \left( \int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2 + V(x)|u|^2) \, dx \right)^{\frac{1}{2}}, \forall u \in E.$$

When $(V_1)$ or $(V_2)$ holds, we know that the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 2_*$ is compact and is continuous for $2 \leq p \leq 2_*$ (see [4, 7]), then

$$\|u\|_p \leq \gamma_p \|u\|, \text{ for } 2 \leq p \leq 2_*.$$  \hspace{1cm} (1.5)

Weak solutions to problem (1.1) are critical points of the following functional $\Phi_b : E \to \mathbb{R}$ defined by

$$\Phi_b(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^N} F(u) \, dx,$$

where $F(u) = \int_0^u f(t) \, dt$. Obviously, the functional $\Phi_b$ is well-defined for every $u \in E$ and belongs to $C^2(E, \mathbb{R})$. Moreover, for any $u, v \in E$, we have

$$\langle \Phi'_b(u), v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a\nabla u \cdot \nabla v + V(x)uv) \, dx$$

$$+ b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} f(u)v \, dx.$$  \hspace{1cm} (1.7)

We call $u$ a least energy sign-changing solution to (1.1) if $u$ is a solution of (1.1) with $u^\pm \neq 0$ and

$$\Phi_b(u) = \inf \{ \Phi_b(u) | u^\pm \neq 0, \Phi'_b(u) = 0 \},$$

where

$$u^+(x) = \max\{u(x), 0\} \text{ and } u^-(x) = \min\{u(x), 0\}.$$  \hspace{1cm} (1.8)

When $b = 0$, problem (1.1) turns to be the following one

$$\begin{cases}
\Delta^2 u - a\Delta u + V(x)u = f(u), & x \in \mathbb{R}^N, \\
u \in H^2(\mathbb{R}^N),
\end{cases}$$

with the energy functional

$$\Phi_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2) \, dx - \int_{\mathbb{R}^N} F(u) \, dx.$$  \hspace{1cm}

As we know, in the past decades, there are some powerful methods developed to study the existence and multiplicity of sign-changing solutions for nonlinear elliptic problems, see [22, 16, 8, 37, 12, 5, 1, 26] and the references therein. In [16, 11], the authors used the descended flow methods, in [12], the authors used super and
sub solution combining with truncation techniques. From discussions on the existence of sign-changing solutions to (1.8), we know, methods of finding sign-changing solutions to (1.8) heavily rely on the following decompositions:
\[
\langle \Phi_0'(u), u^+ \rangle = \langle \Phi_0'(u^+), u^+ \rangle, \quad \langle \Phi_0'(u), u^- \rangle = \langle \Phi_0'(u^-), u^- \rangle, \quad (1.9)
\]
\[
\Phi_0(u) = \Phi_0(u^+) + \Phi_0(u^-). \quad (1.10)
\]
However, for the case \( b > 0 \), since the effect of the nonlocal term, the functional \( \Phi_b \) no longer satisfies (1.9) and (1.10). Indeed, we have
\[
\Phi_b(u) = \Phi_b(u^+) + \Phi_b(u^-) + b \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \int_{\mathbb{R}^N} |\nabla u^-|^2 dx, \quad (1.11)
\]
if \( u^+ \neq 0 \),
\[
\langle \Phi_b'(u), u^+ \rangle = \langle \Phi_b'(u^+), u^+ \rangle + b \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \int_{\mathbb{R}^N} |\nabla u^+|^2 dx > \langle \Phi_b'(u^+), u^+ \rangle,
\]
if \( u^- \neq 0 \),
\[
\langle \Phi_b'(u), u^- \rangle = \langle \Phi_b'(u^-), u^- \rangle + b \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \int_{\mathbb{R}^N} |\nabla u^-|^2 dx > \langle \Phi_b'(u^-), u^- \rangle.
\]

In this paper, the nonlocal term \( \int_{\mathbb{R}^N} |\nabla u|^2 dx \Delta u \) is involved in the equation, we borrow some ideas from [5, 1, 26] and first try to seek a minimizer of the energy functional \( \Phi_b \) over the following constraint:
\[
\mathcal{M}_b := \{ u \in E, \ u^\pm \neq 0 \text{ and } \langle \Phi_b'(u), u^\pm \rangle = \langle \Phi_b'(u), u^- \rangle = 0 \}. \quad (1.14)
\]

Our main results are as follows.

**Theorem 1.1.** Assume that assumptions \((V_1)\) (or \((V_2)\)) and \((F_1)-(F_4)\) hold, then the problem (1.1) has one least-energy sign-changing solution \( u_b \), which has precisely two nodal domains.

Another aim of this paper is to show that the energy of any sign-changing solution of (1.1) is strictly larger than two times the ground state energy. This is trivial for the typical equation (1.8).

**Theorem 1.2.** If the assumptions of Theorem 1.1 hold, then \( c_b := \inf_{u \in \mathcal{N}_b} \Phi_b(u) > 0 \) is achieved and
\[
\Phi_b(u_b) > 2c_b,
\]
where \( \mathcal{N}_b = \{ u \in E \setminus \{0\} : \langle \Phi_b'(u), u \rangle = 0 \} \), \( u_b \) is the least energy sign-changing solution obtained in Theorem 1.1. In particular, \( c_b \) will be achieved either by a positive or a negative function.

In the following, we will show a convergence property of \( u_b \) as \( b \to 0 \), this result can be stated in the following theorem.

**Theorem 1.3.** If the assumptions of Theorem 1.1 hold, for any sequence \( \{b_n\} \) with \( b_n \to 0 \) as \( n \to \infty \), there exists a subsequence, still denoted by \( \{b_n\} \), such that \( u_{b_n} \) convergent to \( u_0 \) strongly in \( E \) as \( n \to \infty \), where \( u_0 \) is a least energy sign-changing solution of the problem (1.8), which changes sign only once.

This paper is organized as follows. In section 2, we prove some crucial lemmas. In section 3, we obtain that the minimizer of the constrained problem is a sign-changing solution by quantitative deformation lemma and degree theory. Furthermore, we use some energy estimations and comparisons to prove Theorem 1.2 and 1.3.
2. Preliminaries. We first show that the set $\mathcal{M}_b$ is nonempty in $E$ and to seek a critical point of $\Phi_b$ by constraint minimization on $\mathcal{M}_b$.

For fixed $u \in E$ with $u^\pm \neq 0$, we denote $B := \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \int_{\mathbb{R}^N} |\nabla u^-|^2 dx$ for convenience. Then, $su^+ + tu^- \in \mathcal{M}_b$ if and only if

$$\begin{cases}
    s^2\|u^+\|^2 + bs^4 \left( \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \right)^2 + bBs^2t^2 - \int_{\mathbb{R}^N} f(su^+)su^+ dx = 0, \\
    t^2\|u^-\|^2 + bt^4 \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 + bBs^2t^2 - \int_{\mathbb{R}^N} f(tu^-)tu^- dx = 0.
\end{cases} \tag{2.1}$$

Hence, the problem $su^+ + tu^- \in \mathcal{M}_b$ is reduced to verify that there is only one solution $(s, t) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ of system (2.1). In order to solve this problem, we consider the solvability of the following system with a parameter $\mu \in [0,1]$

$$\begin{cases}
    s^2\|u^+\|^2 + bs^4 \left( \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \right)^2 + \mu bBs^2t^2 - \int_{\mathbb{R}^N} f(su^+)su^+ dx = 0, \\
    t^2\|u^-\|^2 + bt^4 \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 + \mu bBs^2t^2 - \int_{\mathbb{R}^N} f(tu^-)tu^- dx = 0.
\end{cases} \tag{2.2}$$

Define

$$\mathcal{G} := \{ \mu | 0 \leq \mu \leq 1 \text{ such that (1.1) is uniquely solvable in } \mathbb{R}^+ \times \mathbb{R}^+ \}. \tag{2.3}$$

Set

$$g_\mu(s, t) = s^2\|u^+\|^2 + bs^4 \left( \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \right)^2 + \mu bBs^2t^2 - \int_{\mathbb{R}^N} f(su^+)su^+ dx = 0,$$

$$h_\mu(s, t) = t^2\|u^-\|^2 + bt^4 \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 + \mu bBs^2t^2 - \int_{\mathbb{R}^N} f(tu^-)tu^- dx = 0. \tag{2.4}$$

Lemma 2.1. The set $\mathcal{G}$ contains $0$, i.e. $0 \in \mathcal{G}$.

Proof. Since $g_0(s, t)$ is independent of $t$ and $h_0(s, t)$ is independent of $s$, without loss of generality, we need only to prove that there is a unique $t > 0$ such that $h_0(s, t) = 0$. Since $u^- \neq 0$, from $(F_1)$-$(F_4)$, we get that $h_0(s, 0) = 0$, $h_0(s, t) > 0$ for $t > 0$ small and $h_0(s, t) < 0$ for $t$ large. Thus, there exists $t_1 > 0$ such that $h_0(s, t_1) = 0$. Next, we prove the uniqueness of $t_1$. Arguing by contradiction, suppose that there exist $t_1, t_2$ such that $0 < t_1 < t_2$ and $h_0(s, t_1) = h_0(s, t_2) = 0$, then

$$\frac{\|u^-\|^2}{t_1^2} + b \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 = \int_{\mathbb{R}^N} \frac{f(t_1u^-)t_1u^-}{t_1^4} dx.$$

Similarly, we have

$$\frac{\|u^-\|^2}{t_2^2} + b \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 = \int_{\mathbb{R}^N} \frac{f(t_2u^-)t_2u^-}{t_2^4} dx.$$

Therefore

$$\left( \frac{1}{t_1^4} - \frac{1}{t_2^4} \right) \|u^-\|^2 = \int_{\mathbb{R}^N} \left( \frac{f(t_1u^-)}{(t_1u^-)^3} - \frac{f(t_2u^-)}{(t_2u^-)^3} \right) (u^-)^4 dx,$$

which is absurd in view of $(F_4)$ and $0 < t_1 < t_2$. The proof of Lemma 2.1 is completed. \hfill \Box

Lemma 2.2. The system (2.2) is solvable, for particularly, $\mu = 1$, (2.2) is solvable.
Proof. Set $\tilde{F}(\mu, s, t) = (g_\mu(s, t), h_\mu(s, t))$, $V = (0, \infty) \times (0, \infty)$ and $U = \mathbb{R}$. It is obvious that $\tilde{F}, \frac{\partial \tilde{F}}{\partial s}, \frac{\partial \tilde{F}}{\partial t}$ are continuous in $U \times V$. Suppose $\mu_0 \in \mathcal{G}$ and $(\hat{s}, \hat{t}) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ is the unique solution of (2.2) with $\mu = \mu_0$, then, we have

$$
\frac{\partial g_\mu(s, t)}{\partial s}|_{(\hat{s}, \hat{t})} = \hat{s}||u^+||^2 + 3b\hat{s}^3 \left( \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \right)^2 
+ \mu_0 bB\hat{s}\hat{t}^2 - \int_{\mathbb{R}^N} f'(\hat{s}u^+)\hat{s}(u^+)^2 dx,
$$

(2.5)

$$
\frac{\partial h_\mu(s, t)}{\partial t}|_{(\hat{s}, \hat{t})} = 2\mu_0 bB\hat{s}\hat{t}, \quad \frac{\partial g_\mu(s, t)}{\partial s}|_{(\hat{s}, \hat{t})} = 2\mu_0 bB\hat{s}\hat{t}^2,
$$

(2.6)

$$
\frac{\partial h_\mu(s, t)}{\partial t}|_{(\hat{s}, \hat{t})} = \hat{t}||u^-||^2 + 3b\hat{t}^3 \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 
+ \mu_0 bB\hat{s}\hat{t}^2 - \int_{\mathbb{R}^N} f'(\hat{t}u^-)\hat{t}(u^-)^2 dx.
$$

(2.7)

From $(F_4)$, for $s \neq 0$, we have

$$
f'(s)s^2 - 3f(s)s \geq 0.
$$

(2.8)

Then

$$
\frac{\partial g_\mu(s, t)}{\partial s}|_{(\hat{s}, \hat{t})} \leq -2\hat{s}||u^+||^2 - 2\mu_0 bB\hat{s}\hat{t}^2
$$

and

$$
\frac{\partial h_\mu(s, t)}{\partial t}|_{(\hat{s}, \hat{t})} \leq -2\hat{t}||u^-||^2 - 2\mu_0 bB\hat{s}\hat{t}^2.
$$

Set the matrix

$$
M = \begin{pmatrix}
\frac{\partial g_\mu(\hat{s}, \hat{t})}{\partial s} & \frac{\partial g_\mu(\hat{s}, \hat{t})}{\partial t} \\
\frac{\partial h_\mu(\hat{s}, \hat{t})}{\partial s} & \frac{\partial h_\mu(\hat{s}, \hat{t})}{\partial t}
\end{pmatrix}.
$$

Therefore, we have

$$
\det M \geq (2\hat{s}||u^+||^2 + 2\mu_0 bB\hat{s}\hat{t}^2) (2\hat{t}||u^-||^2 + 2\mu_0 bB\hat{s}\hat{t}^2) - (2\mu_0 bB\hat{s}\hat{t}^2) (2\mu_0 bB\hat{s}\hat{t}^2) > 0.
$$

Then, from the implicit function theorem, we can find open neighborhoods $U_0 \subset U$ of $\mu_0$, $V_0 \subset V$ of $(\hat{s}, \hat{t})$ such that system (2.2) is uniquely solvable in $U_0 \times V_0$, that is, we can find a unique function $\varphi : U_0 \to V_0$ satisfies $(s, t) = \varphi(\mu)$, $\mu \in U_0$ and

$$
\begin{cases}
g_{\mu_0}(s, t) = 0, \\
h_{\mu_0}(s, t) = 0.
\end{cases}
$$

Extend $U_0$ as much as possible, we regard this extended domain as $\tilde{U}_0$, set $\tilde{U}_0 = (\delta_1, \delta_2)$. If $\delta_2 < +\infty$, then there exists $\{\mu_n\} \subset \tilde{U}_0$ such that

$$
\mu_n \to \delta_2 \text{ as } n \to \infty,
$$

From the above proof, for $(s_n, t_n) = \varphi(\mu_n)$, we have

$$
s_n \to \hat{s}, \quad t_n \to \hat{t} \text{ as } n \to \infty,
$$

and

$$
\begin{cases}
g_{\mu_n}(s_n, t_n) = 0, \\
h_{\mu_n}(s_n, t_n) = 0.
\end{cases}
$$

It follows from the continuity of $g_\mu$ and $h_\mu$ that

$$
\begin{cases}
g_{\delta_2}(\hat{s}, \hat{t}) = 0, \\
h_{\delta_2}(\hat{s}, \hat{t}) = 0.
\end{cases}
$$
This contradicts the fact that \( \delta_2 \notin \tilde{U}_0 \). Thus \( \delta_2 = +\infty \). Similarly, we have \( \delta_1 = +\infty \). Hence, we know the system (2.2) is solvable, particularly, when \( \mu = 1 \), that is, the system (2.1) is solvable.

Next, we prove the uniqueness of the solution.

**Lemma 2.3.** If \( u \in \mathcal{M}_b \), then \((1,1)\) is the unique solution of system (2.1).

**Proof.** Set \((s_0,t_0)\) be the solution of (2.1), without loss of generality, we assume that \(0 < s_0 \leq t_0\), then
\[
s_0^2 \|u^+\|^2 + b s_0^4 \left( \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \right)^2 + b B s_0^2 t_0^2 = \int_{\mathbb{R}^N} f(s_0 u^+)s_0 u^+ dx
\]
and
\[
t_0^2 \|u^-\|^2 + b t_0^4 \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 + b B t_0^2 s_0^2 = \int_{\mathbb{R}^N} f(t_0 u^-)t_0 u^- dx.
\]
From (2.10), we have
\[
\frac{\|u^-\|^2}{t_0^2} + b \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 + b B \frac{s_0^2}{t_0^2} = \int_{\mathbb{R}^N} f(t_0 u^-) \frac{1}{(t_0 u^-)^3} (u^-)^4 dx,
\]
which, since \(0 < s_0 \leq t_0\), thus \(0 < \frac{s_0}{t_0} \leq 1\), hence
\[
\frac{\|u^-\|^2}{t_0^2} + b \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 + b B \geq \int_{\mathbb{R}^N} f(t_0 u^-) \frac{1}{(t_0 u^-)^3} (u^-)^4 dx.
\]
From \(u \in \mathcal{M}_b\), we have
\[
\|u^-\|^2 + b \left( \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \right)^2 + b B = \int_{\mathbb{R}^N} f(u^-) \frac{1}{(u^-)^3} (u^-)^4 dx,
\]
Combining (2.11) with (2.12), we have
\[
\left( \frac{1}{t_0^2} - 1 \right) \|u^-\|^2 \geq \int_{\mathbb{R}^N} \left( \frac{f(t_0 u^-)}{(t_0 u^-)^3} - \frac{f(u^-)}{(u^-)^3} \right) (u^-)^4 dx,
\]
From this inequality, we can conclude that \( t_0 \leq 1 \). In fact, arguing by contradiction, assume that \( t_0 > 1 \), the left side of the above inequality is negative, while the right side is non-negative. This is absurd, thus, \( t_0 \leq 1 \). Similarly, we can prove \( s_0 \geq 1 \).

Since \(0 < s_0 \leq t_0\), we have \(s_0 = t_0 = 1\).

Assume that \((s_1,t_1)\) and \((s_2,t_2)\) are solutions of system (2.1), then
\[
\omega_1 := s_1 u^+ + t_1 u^- \in \mathcal{M}_b \quad \text{and} \quad \omega_2 := s_2 u^+ + t_2 u^- \in \mathcal{M}_b,
\]
thus
\[
\omega_2 = \left( \frac{s_2}{s_1} \right) s_1 u^+ + \left( \frac{t_2}{t_1} \right) t_1 u^- = \left( \frac{s_2}{s_1} \right) \omega_1^+ + \left( \frac{t_2}{t_1} \right) \omega_1^- \in \mathcal{M}_b.
\]
From the above proof and the fact that \(\omega_1 \in \mathcal{M}_b\), we have
\[
\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1,
\]
thus,
\[
\left\{
\begin{array}{l}
    s_1 = s_2, \\
    t_1 = t_2.
\end{array}
\right.
\]
The uniqueness of solution is obtained, hence, the proof of this lemma is completed.

From Lemmas 2.1, 2.2 and 2.3, we can easily get the following lemma.
Lemma 2.4. Assume that assumptions (V₁) (or (V₂)) and (F₁)-(F₄) hold, if \( u \in E \) with \( u^{±} \neq 0 \), then there is a unique pair \((s_u, t_u)\) of positive numbers such that \( s_u u^{+} + t_u u^{-} \in \mathcal{M}_b \).

Lemma 2.5. Assume that assumptions (V₁) (or (V₂)) and (F₁)-(F₄) hold, suppose that \( u \in E \) such that \( g_1(1,1) \leq 0 \) and \( h_1(1,1) \leq 0 \), where \( g_1(1,1), h_1(1,1) \) are given as (2.4) with \( \mu = 1 \). Then the unique pair \((s_u, t_u)\) of positive numbers obtained in Lemma 2.3 satisfies \( 0 < s_u, t_u \leq 1 \).

Proof. Without loss of generality, we assume \( 0 < t_u \leq s_u \), since \( s_u u^{+} + t_u u^{-} \in \mathcal{M}_b \), then

\[
\begin{aligned}
&\quad s_u^2 \|u^{+}\|^2 + bs_u^4 \left( \int_{\mathbb{R}^N} |\nabla u^{+}|^2 dx \right)^2 + bBs_u^4 \\
&\geq s_u^2 \|u\|^2 + bs_u^4 \left( \int_{\mathbb{R}^N} |\nabla u^{+}|^2 dx \right)^2 + bBs_u^4 u^{+} u^{-} \\
&= \int_{\mathbb{R}^N} f(s_u u^{+}) s_u u^{+} dx.
\end{aligned}
\]

The assumption \( g_1(1,1) \leq 0 \) gives that

\[
\|u^{+}\|^2 + b \left( \int_{\mathbb{R}^N} |\nabla u^{+}|^2 dx \right)^2 + bB \leq \int_{\mathbb{R}^N} f(u^{+}) u^{+} dx.
\]

Combining (2.13) with (2.14), we get

\[
\left( \frac{1}{s_u^2} - 1 \right) \|u^{+}\|^2 \geq \int_{\mathbb{R}^N} \left( \frac{f(s_u u^{+})}{(s_u u^{+})^3} - \frac{f(u^{+})}{(u^{+})^3} \right) (u^{+})^4 dx.
\]

If \( s_u > 1 \), then the left side of this inequality is negative. But from (F₄), the right is positive, hence, we must have \( s_u \leq 1 \). Then, the proof is completed.

Lemma 2.6. For fixed \( u \in E \) with \( u^{±} \neq 0 \), then the vector \((s_u, t_u)\) which obtained in Lemma 2.3 is the unique maximum point of the function \( \hat{\phi} : (\mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{R} \) defined as \( \hat{\phi}(s, t) = \Phi_b(s u^{+} + t u^{-}) \).

Proof. From the proof of Lemma 2.1, 2.2 and 2.3, we know that \((s_u, t_u)\) is the unique critical point of \( \hat{\phi} \) in \((\mathbb{R}^+ \times \mathbb{R}^+)\). From the assumption (F₃), we have

\[
\hat{\phi}(s, t) \to -\infty \text{ uniformly as } |(s, t)| \to \infty,
\]

thus, it is sufficient to check that a maximum point cannot be achieved on the boundary of \((\mathbb{R}^+ \times \mathbb{R}^+)\). Without loss of generality, we may assume that \((0, \bar{t})\) is a maximum point of \( \hat{\phi} \). By calculation, we have

\[
\hat{\phi}(s, \bar{t}) = \Phi_b(s u^{+} + \bar{t} u^{-}) \\
= \frac{s^2}{2} \int_{\mathbb{R}^N} |\Delta u^{+}|^2 dx + \frac{as^2}{2} \int_{\mathbb{R}^N} |\nabla u^{+}|^2 dx + \frac{s^2}{2} \int_{\mathbb{R}^N} V(x)(u^{+})^2 dx \\
+ \frac{bs^4}{4} \left( \int_{\mathbb{R}^N} |\nabla u^{+}|^2 dx \right)^2 - \int_{\mathbb{R}^N} F(s u^{+}) dx \\
+ \frac{\bar{t}^2}{2} \int_{\mathbb{R}^N} |\Delta u^{-}|^2 dx + \frac{a\bar{t}^2}{2} \int_{\mathbb{R}^N} |\nabla u^{-}|^2 dx + \frac{\bar{t}^2}{2} \int_{\mathbb{R}^N} V(x)(u^{-})^2 dx \\
+ \frac{b\bar{t}^4}{4} \left( \int_{\mathbb{R}^N} |\nabla u^{-}|^2 dx \right)^2 - \int_{\mathbb{R}^N} F(\bar{t} u^{-}) dx.
\]
From the above equation, we know \( \hat{\phi}(s,t) \) is an increasing function with respect to \( s \) if \( s \) is small enough, the pair \((0, \hat{t})\) is not a maximum point of \( \hat{\phi} \) in \((\mathbb{R}^+ \times \mathbb{R}^+)\). □

By Lemma 2.4, we can define the following minimization problem

\[
m_b := \inf \{ \Phi_b(u) : u \in \mathcal{M}_b \}.
\]  

(2.15)

**Lemma 2.7.** Assume that assumptions \((V_1)\) (or \((V_2)\)) and \((F_1)-(F_4)\) hold, then \( m_b > 0 \) can be achieved.

**Proof.** For every \( u \in \mathcal{M}_b \), we have \( \langle \Phi'_b(u), u \rangle = 0 \). Then from \((F_1), (F_2), (1.5)\) and Sobolev embedding theorem, we get

\[
\|u\|^2 \leq \int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^2 dx + b \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2
\]

\[
= \int_{\mathbb{R}^N} f(u)udx
\]

\[
\leq \varepsilon \int_{\mathbb{R}^N} |u|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |u|^p dx
\]

\[
\leq \varepsilon \gamma_2^2 \|u\|^2 + C_\varepsilon \gamma_p^p \|u\|^p,
\]

(2.16)

thus, there exists a constant \( \alpha > 0 \) such that \( \|u\|^2 \geq \alpha \). From \((F_4)\), we have

\[
f(s)s - 4F(s) \geq 0.
\]

(2.17)

Then

\[
\Phi_b(u) = \Phi_b(u) - \frac{1}{4} \langle \Phi'_b(u), u \rangle \geq \frac{1}{4} \|u\|^2 \geq \frac{\alpha}{4} > 0.
\]

This implies that \( m_b \geq \frac{\alpha}{4} > 0 \).

Let \( \{u_n\} \subset \mathcal{M}_b \) be such that \( \Phi_b(u_n) \rightarrow m_b \). Then \( \{u_n\} \) is bounded in \( E \), and there exists \( u_b \in E \) such that \( u_n^\pm \rightharpoonup u_b^\pm \) in \( E \). Since \( \{u_n\} \subset \mathcal{M}_b \), we have

\[
\int_{\mathbb{R}^N} |\Delta u_n^\pm|^2 dx + a \int_{\mathbb{R}^N} |\nabla u_n^\pm|^2 dx + \int_{\mathbb{R}^N} V(x)(u_n^\pm)^2 dx + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} |\nabla u_n^\pm|^2 dx
\]

\[
= \int_{\mathbb{R}^N} f(u_n^\pm)u_n^\pm dx.
\]

(2.18)

Similar as the proof of (2.16), there exists a constant \( \beta > 0 \) such that \( \|u_n^\pm\|^2 \geq \beta \) for all \( n \in \mathbb{N} \). Since \( \{u_n\} \subset \mathcal{M}_b \), thus

\[
\beta \leq \|u_n^\pm\|^2 < \int_{\mathbb{R}^N} f(u_n^\pm)u_n^\pm dx \leq \varepsilon \int_{\mathbb{R}^N} |u_n^\pm|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |u_n^\pm|^p dx.
\]

It follows from the boundedness of \( u_n \) that there is \( C_1 > 0 \) such that

\[
\beta \leq \varepsilon C_1 + C_\varepsilon \int_{\mathbb{R}^N} |u_n^\pm|^p dx.
\]

Choosing \( \varepsilon = \frac{\beta}{2C_1} \), we get

\[
\int_{\mathbb{R}^N} |u_n^\pm|^p dx \geq \frac{\beta}{2C_\varepsilon}.
\]

From the above inequality and the compactness of the embedding \( E \hookrightarrow L^p(\mathbb{R}^N) \) for \( 2 < p < 2^* \), we have

\[
\int_{\mathbb{R}^N} |u_b^\pm|^p dx \geq \frac{\beta}{2C_\varepsilon},
\]

(2.19)
thus, \( u_b^+ \neq 0 \). Combining (\( F_1 \))-(\( F_2 \)) with the compactness lemma of Strauss [27], we get
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} f(u_n^+) u_n^+ dx = \int_{\mathbb{R}^N} f(u_b^+) u_b^+ dx, \\
\lim_{n \to \infty} \int_{\mathbb{R}^N} F(u_n^+) dx = \int_{\mathbb{R}^N} F(u_b^+) dx.
\] (2.20)

By the weak semicontinuity of norm, we have
\[
\|u_n^+\|^2 + b \int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx \int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx \\
\leq \liminf_{n \to \infty} \left\{ \|u_n^+\|^2 + b \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right\}.
\] (2.21)

Then from (2.20), we get
\[
\|u_b^+\|^2 + b \int_{\mathbb{R}^N} |\nabla u_b^+|^2 dx \int_{\mathbb{R}^N} |\nabla u_b^+|^2 dx \leq \int_{\mathbb{R}^N} f(u_b^+) u_b^+ dx,
\] (2.22)

that is
\[
\left\{ \begin{array}{l}
g_1(1,1) \leq 0, \\
h_1(1,1) \leq 0.
\end{array} \right.
\]

From the above inequalities and Lemma 2.5, there exists \((s_{u_b}, t_{u_b}) \in (0,1] \times (0,1]\) such that
\[
\bar{u}_b := s_{u_b} u_b^+ + t_{u_b} u_b^- \in \mathcal{M}_b.
\]

Since condition (\( F_4 \)) implies that \( G(s) := sf(s) - 4F(s) \) is a non-negative function, increasing in \(|s|\), we have
\[
m_b \leq \Phi_b(\bar{u}_b) = \Phi_b(\bar{u}_b) - \frac{1}{4} \langle \Phi_b'(\bar{u}_b), \bar{u}_b \rangle \\
= \frac{1}{4} \|\bar{u}_b\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (f(\bar{u}_b)\bar{u}_b - 4F(\bar{u}_b)) dx \\
= \frac{1}{4} \|s_{u_b} u_b^+\|^2 + \frac{1}{4} \|t_{u_b} u_b^-\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (f(s_{u_b} u_b^+) s_{u_b} u_b^+ - 4F(s_{u_b} u_b^+)) dx \\
+ \frac{1}{4} \int_{\mathbb{R}^N} (f(t_{u_b} u_b^-) t_{u_b} u_b^- - 4F(t_{u_b} u_b^-)) dx \\
\leq \frac{1}{4} \|u_b\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (f(u_b)u_b - 4F(u_b)) dx \\
\leq \liminf_{n \to \infty} \left[ \Phi_b(u_n) - \frac{1}{4} \langle \Phi_b'(u_n), u_n \rangle \right] = m_b.
\] (2.23)

Then, we have \( s_{u_b} = t_{u_b} = 1 \). Thus, \( \bar{u}_b = u_b \) and \( \Phi_b(u_b) = m_b \).

3. **Proof of main results.** The main aim of this section is to prove our main results. We first prove that the minimizer \( u_b \) for the minimization problem (2.15) is indeed a sign-changing solution of (1.1), that is \( \Phi'(u_b) = 0 \).

**Proof of Theorem 1.1.** By the quantitative deformation lemma, we prove that \( \Phi'(u_b) = 0 \).

It is clear that \( \langle \Phi_b'(u_b), u_b^+ \rangle = 0 = \langle \Phi_b'(u_b), u_b^- \rangle \). If follows from Lemma 2.6 that for \((s,t) \in (\mathbb{R}^+ \times \mathbb{R}^+) \) and \((s,t) \neq (1,1)\),
\[
\Phi_b(s u_b^+ + t u_b^-) < \Phi_b(u_b^+ + u_b^-) = m_b.
\] (3.1)
If $\Phi'(u_b) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that
$$\|\Phi'(b)(u_b)\| \geq \theta, \quad \forall \|v-u_b\| \leq 3\delta.$$ Let $D := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g(s, t) := su_b^+ + tu_b^-$. It follows from Lemma 2.6 again that
$$\bar{m}_b := \max_{\partial D} \Phi_b \circ g < m_b. \quad (3.2)$$

For $\varepsilon := \min \left(\frac{(m_b - \bar{m}_b)}{2}, \frac{\delta_0}{8}\right)$ and $S := B(u_b, \delta)$. It follows from Lemma 2.3 in [30] that there exists a deformation $\eta$ such that
(a) $\eta(1, u) = u$ if $u \not\in \Phi_b^{-1}([m_b - 2\varepsilon, m_b + 2\varepsilon]) \cap S_{2\varepsilon}$;
(b) $\eta(1, \Phi_b^{m_b+\varepsilon} \cap S) \subset \Phi_b^{m_b-\varepsilon}$;
(c) $\Phi_b(\eta(1, u)) \leq \Phi_b(u)$ for all $u \in E$.

It is clear that
$$\max_{(s, t) \in D} \Phi_b(\eta(1, g(s, t))) < m_b. \quad (3.3)$$

We prove that $\eta(1, g(D)) \cap \mathcal{M}_b \neq \emptyset$, contradicting to the definition of $m_b$. Define $h(s, t) := \eta(1, g(s, t))$ and
$$\Psi_0(s, t) := \langle \Phi_b'(g(s, t)), u_b^+ \rangle, \langle \Phi_b'(g(s, t)), u_b^- \rangle \rangle = \langle \Phi_b'(su_b^+ + tu_b^-), u_b^+ \rangle, \langle \Phi_b'(su_b^- + tu_b^-), u_b^- \rangle \rangle,
$$
$$\Psi_1(s, t) := \langle \Phi_b'(h(s, t)), h^+(s, t) \rangle, \frac{1}{t}(\Phi_b'(h(s, t)), h^+(s, t)) \rangle.$$

Lemma 2.4 and the degree theory now yields $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$. Therefore, $\Psi_1(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, so that $\eta(1, g(s_0, t_0)) = h(s_0, t_0) \in \mathcal{M}_b$, which is a contradiction. Thus, $u_b$ is a critical point of $\Phi_b$, and so, a sign-changing solution for problem (2.1).

Now, we show that $u_b$ has exactly two nodal domains, arguing by contradiction, we assume that $u$ has at least three nodal domains $\Omega_1, \Omega_2, \Omega_3$. Without loss of generality, we may assume that $u > 0$ a.e. in $\Omega_1$ and $u < 0$ a.e. in $\Omega_2$. Set
$$u_i = \chi_{\Omega_i} u_i, \quad i = 1, 2, 3,$$
where
$$\chi_{\Omega_i} = \begin{cases} 1, & x \in \Omega_i, \\ 0, & x \in \mathbb{R}^N \setminus \Omega_i, \end{cases}$$
then $u_i \in E$ and $u_i \neq 0$ and $\langle \Phi_b'(u_i), u_i \rangle = 0$ for $i = 1, 2, 3$. Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^+ \neq 0$. Then from Lemma 2.4, there is a unique pair $(s_v, t_v) \in D$ of positive numbers such that $s_v v^+ + t_v v^- \in \mathcal{M}_b$, or equivalently, $s_v u_1 + t_v u_2 \in \mathcal{M}_b$. Then
$$\Phi_b(s_v u_1 + t_v u_2) \geq m_b. \quad (3.4)$$
Moreover, combining with the fact $\langle \Phi_b'(u_b), u_i \rangle = 0$ that $\langle \Phi_b'(v), v^\pm \rangle < 0$. From Lemma 2.5, we have that
$$(s_v, t_v) \in (0, 1) \times (0, 1).$$
On the other hand,
\[
0 = \frac{1}{4}\Phi_b'(u_b) = u_3 \frac{1}{4}\int_{\mathbb{R}^N} |\Delta u_3|^2dx + \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_3|^2dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)|u_3|^2dx \\
+ \frac{b}{4}\int_{\mathbb{R}^N} |\nabla u_1|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx + \frac{b}{4}\int_{\mathbb{R}^N} |\nabla u_2|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx \\
+ \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u_3|^2dx \right)^2 - \frac{1}{4}\int_{\mathbb{R}^N} F(u_3)dx \\
< \Phi_b(u_3) + \frac{b}{4}\int_{\mathbb{R}^N} |\nabla u_1|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx \\
+ \frac{b}{4}\int_{\mathbb{R}^N} |\nabla u_2|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx.
\]

Thus, similar as (2.23), we can get
\[
\Phi_b(s_vu_1 + t_vu_2) = \Phi_b(s_vu_1) + \Phi_b(t_vu_2) + \frac{bs_v^2t_v^2}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2dx \int_{\mathbb{R}^N} |\nabla u_2|^2dx \\
= \frac{s_v^2}{4}||u_1||^2 + \frac{1}{4}\int_{\mathbb{R}^N} (f(s_vu_1)s_vu_1 - 4F(s_vu_1))dx \\
+ \frac{t_v^2}{4}||u_2||^2 + \frac{1}{4}\int_{\mathbb{R}^N} (f(t_vu_2)t_vu_2 - 4F(t_vu_2))dx \\
\leq \frac{1}{4}||u_1||^2 + \frac{1}{4}\int_{\mathbb{R}^N} (f(u_1)u_1 - 4F(u_1))dx \\
+ \frac{1}{4}||u_2||^2 + \frac{1}{4}\int_{\mathbb{R}^N} (f(u_2)u_2 - 4F(u_2))dx \\
= \Phi_b(u_1) + \Phi_b(u_2) + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2dx \int_{\mathbb{R}^N} |\nabla u_2|^2dx \\
+ \frac{b}{4}\int_{\mathbb{R}^N} |\nabla u_1|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx \\
+ \frac{b}{4}\int_{\mathbb{R}^N} |\nabla u_2|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx.
\]

Then, combining (3.4), (3.5) with (3.6), we have
\[
m_b \leq \Phi_b(s_vu_1 + t_vu_2) < \Phi_b(u_1) + \Phi_b(u_2) + \Phi_b(u_3) + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2dx \int_{\mathbb{R}^N} |\nabla u_2|^2dx \\
+ \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u_2|^2dx \int_{\mathbb{R}^N} |\nabla u_3|^2dx \\
= \Phi_b(u_b) = m_b,
\]
which is a contradiction. Hence, \( u_3 = 0 \) and \( u_b \) has exactly two nodal domains. \( \square \)

From Theorem 1.1, we know that the problem (1.1) has a least energy sign-changing solution \( u_b \) which changes sign only once. Now, we show that the energy of \( u_b \) is strictly larger than two times the ground state energy.

**Proof of Theorem 1.2.** Let \( \mathcal{N}_b \) and \( c_b \) be given in Theorem 1.2. Similar as the proof of Lemma 2.7, for each \( b > 0 \), there exists \( v_b \in \mathcal{N}_b \) such that \( \Phi_b(v_b) = c_b > 0 \). From Corollary 2.9 in [14], we know that the critical points of the functional \( \Phi_b \) on
$N_b$ are critical points of $\Phi_b$ in $E$, we can conclude that $\Phi_b'(v_b) = 0$. Thus, $v_b$ is a ground state solution of (1.1). From Theorem 1.1, we know that the problem (1.1) has a least energy sign-changing solution $u_b$ which changes sign only once. Suppose that $u_b = u^+_b + u^-_b$. As the proof of Lemma 2.1, there is unique $t^+_b > 0$ such that $t^+_bu^+_b \in N_b$.

From (1.12), we can get $\langle \Phi'_b(u_b^+_b), u_b^+_b \rangle < 0$. Thus, $(F_1)$-$(F_4)$ implies that $t^+_b \in (0, 1)$. By the similar arguments, we can prove that there is unique $t^-_b \in (0, 1)$ such that $t^-_bu^-_b \in N_b$.

Then from Lemma 2.6, we have

$$2c_b \leq \Phi_\lambda(t^+_bu^+_b) + \Phi_\lambda(t^-_bu^-_b) \leq \Phi_\lambda(t^+_bu^+_b + t^-_bu^-_b) < \Phi_\lambda(u^+_b + u^-_b) = m_b,$$

that is, $\Phi_b(u_b) > 2c_b$, which implies that $c_b > 0$ can not be achieved by a sign-changing function. This completes the proof.

In the following, we will analyze the convergence property of $u_b$ as $b > 0$ by regarding $b > 0$ as a parameter in problem (1.1).

**Proof of Theorem 1.3.** For any $b > 0$, let $u_b \in E$ be the least energy sign-changing solution of (1.1) obtained in Theorem 1.1, which changes sign only once. Next, by proving the following Lemma 3.1, 3.2 and 3.3 to complete the proof of Theorem 1.3.

**Lemma 3.1.** For any sequence $\{b_n\}$ with $b_n \to 0$ as $n \to \infty$, $\{u_{b_n}\}$ is bounded in $E$.

**Proof.** Choose a nonzero function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi^\pm \neq 0$. From (2.17) and the assumption $(F_3)$, for any $b \in [0, 1]$, there exists a pair $(\theta_1, \theta_2)$ of positive numbers, which does not depend on $b$, such that

$$\begin{cases}
\theta_1^2\|\varphi^+\|^2 + b\theta_1^2 \left( \int_{\mathbb{R}^N} |\nabla \varphi^+|^2 dx \right)^2 + bB_\varphi \theta_1^2 \theta_2^2 - \int_{\mathbb{R}^N} f(\theta_1 \varphi^+)\theta_1 \varphi^+ dx < 0, \\
\theta_2^2\|\varphi^-\|^2 + b\theta_2^2 \left( \int_{\mathbb{R}^N} |\nabla \varphi^-|^2 dx \right)^2 + bB_\varphi \theta_1^2 \theta_2^2 - \int_{\mathbb{R}^N} f(\theta_2 \varphi^-)\theta_2 \varphi^- dx < 0,
\end{cases}$$

where $B_\varphi = \int_{\mathbb{R}^N} |\nabla \varphi^+|^2 dx \int_{\mathbb{R}^N} |\nabla \varphi^-|^2 dx$. From Lemma 2.4 and Lemma 2.5, for any $b \in [0, 1]$, there exists a unique pair $((s_\varphi(b), t_\varphi(b))) \in (0, 1) \times (0, 1]$ such that

$$\tilde{\varphi} := s_\varphi(b)\theta_1 \varphi^+ + t_\varphi(b)\theta_2 \varphi^- \in \mathcal{M}_b, \quad (3.7)$$

Thus, for any $b > 0$, we have

$$\Phi_b(u_b) \leq \Phi_b(\tilde{\varphi}) = \Phi_b(\tilde{\varphi}) - \frac{1}{4}\langle \Phi_b'(\tilde{\varphi}), \tilde{\varphi} \rangle$$

$$= \frac{1}{4}\|\tilde{\varphi}\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (f(\tilde{\varphi})\tilde{\varphi} - 4F(\tilde{\varphi})) \, dx$$

$$\leq \frac{1}{4}\|\tilde{\varphi}\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (C_2\tilde{\varphi}^2 + C_3\tilde{\varphi}^p) \, dx$$

$$\leq \left\{ \frac{1}{4}\|\theta_1 \varphi^+\|^2 + \frac{1}{4}\|\theta_2 \varphi^-\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (C_2\theta_1^2|\varphi^+|^2 + C_3\theta_2^2|\varphi^-|^2) \, dx \right\} := C_0, \quad (3.8)$$

where $C_0$ does not depend on $b$. For $n$ large enough, we have

$$C_0 + 1 \geq \Phi_{b_n}(u_{b_n}) = \Phi_{b_n}(u_{b_n}) - \frac{1}{4} \Phi'_{b_n}(u_{b_n}), u_{b_n}) \geq \frac{1}{4} \|u_{b_n}\|^2. \quad (3.9)$$

Hence, $\{u_{b_n}\}$ is bounded in $E$. □

From Lemma 3.1, there exists a subsequence of $b_n$, still denoted by $b_n$, such that $u_{b_n} \rightharpoonup u_0$ weakly in $E$. Then, $u_0$ is a weak solution of (1.8).

**Lemma 3.2.** $u_0$ is a least energy sign-changing solution of (1.8) which changes sign only once.

**Proof.** Suppose that $v_0$ is a least energy sign-changing solution of (1.8). From [6], the existence of $v_0$ can be proved. By Lemma 2.4, for each $b_n > 0$, there is a unique pair $(s_{b_n}, t_{b_n})$ of positive numbers such that

$$s_{b_n} v_0^+ + t_{b_n} v_0^- \in \mathcal{M}_{b_n}.$$

Then, we have

$$(s_{b_n})^2 \|v_0^+\|^2 + b_n(s_{b_n})^4 \left( \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \right) + b_n(s_{b_n} t_{b_n})^2 \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx = \int_{\mathbb{R}^N} f(s_{b_n} v_0^+) s_{b_n} v_0^+ dx \quad (3.10)$$

and

$$(t_{b_n})^2 \|v_0^-\|^2 + b_n(t_{b_n})^4 \left( \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx \right) + b_n(s_{b_n} t_{b_n})^2 \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx = \int_{\mathbb{R}^N} f(t_{b_n} v_0^-) t_{b_n} v_0^- dx. \quad (3.11)$$

Recall that $v_0^\pm$ satisfies

$$\|v_0^\pm\|^2 = \int_{\mathbb{R}^N} f(v_0^\pm) v_0^\pm dx. \quad (3.12)$$

We can get $\{s_{b_n}\}, \{t_{b_n}\}$ are bounded. In fact, arguing by contradiction, assume that $\{s_{b_n}\}$ is unbounded, that is, there exists a subsequence of $\{s_{b_n}\}$, still denoted by $\{s_{b_n}\}$, such that $s_{b_n} \to +\infty$ as $n \to \infty$. From (3.10), we have

$$\frac{\|v_0^+\|^2}{(s_{b_n})^2} + b_n \left( \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \right) + b_n \left( \frac{t_{b_n}}{s_{b_n}} \right)^2 \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx = \int_{\mathbb{R}^N} f(s_{b_n} v_0^+) (s_{b_n} v_0^+)^2 dx. \quad (3.13)$$

Observe that from (3.13), the assumption $s_{b_n} \to +\infty$ as $n \to \infty$ and the assumption $(F_4)$, we have

$$b_n \left( \frac{t_{b_n}}{s_{b_n}} \right)^2 \to +\infty, \text{ as } n \to \infty,$$

thus,

$$t_{b_n} \to +\infty \text{ and } \left( \frac{s_{b_n}}{t_{b_n}} \right)^2 \to 0^+, \text{ as } n \to \infty. \quad (3.14)$$
Combining (3.14) with (3.11), we have
\[
\frac{\|v_0^+\|^2}{(t_{b_n})^2} + b_n \left( \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \right)^2 + b_n \left( \frac{s_{b_n}}{t_{b_n}} \right)^2 \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx
= \int_{\mathbb{R}^N} \frac{f(t_{b_n}v_0^+)}{(t_{b_n}v_0^+)^3} (v_0^-)^4 dx.
\] (3.15)

The left side of (3.15) goes to zero as \( n \to \infty \), while the right side goes to negative infinity as \( n \to \infty \). This is absurd. Thus, \( \{s_{b_n}\} \) is bounded. Similarly, we can get \( \{t_{b_n}\} \) is bounded.

By selecting subsequences of \( \{s_{b_n}\} \) and \( \{t_{b_n}\} \), still denoted by \( \{s_{b_n}\} \) and \( \{t_{b_n}\} \), such that \( (s_{b_n}, t_{b_n}) \to (s_0, t_0) \) as \( n \to \infty \). Next, we will show that \( s_0 = t_0 = 1 \). In fact, from (3.10), we have
\[
s_0^2\|v_0^+\|^2 = \int_{\mathbb{R}^N} f(s_0v_0^+)s_0v_0^+ dx,
\]
thus
\[
\frac{\|v_0^+\|^2}{s_0^2} = \int_{\mathbb{R}^N} \frac{f(s_0v_0^+)}{(s_0v_0^+)^2} (v_0^+)^4 dx, \quad (3.16)
\]
Similarly, from (3.11), we have
\[
\frac{\|v_0^-\|^2}{t_0^2} = \int_{\mathbb{R}^N} \frac{f(t_0v_0^-)}{(t_0v_0^-)^2} (v_0^-)^4 dx, \quad (3.17)
\]
(3.16) together with (3.12) yield that
\[
\left( \frac{1}{s_0^2} - 1 \right)\|v_0^+\|^2 = \int_{\mathbb{R}^N} \left( \frac{f(s_0v_0^+)}{(s_0v_0^+)^2} - \frac{f(v_0^+)}{(v_0^+)^2} \right) (v_0^+)^4 dx. \quad (3.18)
\]
If \( s_0 > 1 \), the left side of (3.18) is negative, but the right side is positive, this is a contradiction. If \( s_0 < 1 \), the left side of (3.18) is positive, but the right side is negative, this is also a contradiction. Therefore, \( s_0 = 1 \). Similarly, we can get \( t_0 = 1 \), that is,
\[
(s_{b_n}, t_{b_n}) \to (1, 1), \quad \text{as} \quad n \to \infty. \quad (3.19)
\]
Now, we can prove \( u_0 \) is a least energy sign-changing solution of (1.8) which changes sign only once. From (3.19) and Lemma 2.6, we have
\[
\Phi_0(v_0) \leq \Phi_0(u_0) = \lim_{n \to \infty} \Phi_{b_n}(u_{b_n}) = \lim_{n \to \infty} \Phi_{b_n}(u_{b_n}^+ + u_{b_n}^-)
\leq \lim_{n \to \infty} \Phi_{b_n}(s_{b_n}v_0^+ + t_{b_n}v_0^-) = \Phi_0(v_0^+ + v_0^-) = \Phi_0(v_0). \quad (3.20)
\]

Combining Lemma 3.1, 3.2 with 3.3, we complete the proof of Theorem 1.3.

Acknowledgments. The authors would like to thank the referee for giving valuable comments and suggestions, which make us possible to improve the paper.
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Received December 2015; revised March 2016.