MARKOV CHAINS RELATED TO FUNDAMENTAL WHITTAKER FUNCTIONS

NEIL O’CONNELL

Abstract. We consider a Markov chain on non-negative integer arrays of a given shape (and satisfying certain constraints) which is closely related to fundamental $\text{SL}(r+1, \mathbb{R})$ Whittaker functions and the Toda lattice. In the index zero case the arrays are reverse plane partitions. We show that this Markov chain has non-trivial Markovian projections and a unique entrance law starting from the array with all entries equal to $+\infty$.

1. Introduction

Fundamental $\text{SL}(r+1, \mathbb{R})$ Whittaker functions are series solutions to the eigenvalue equation

\begin{equation}
\mathcal{H}^r \phi = -\lambda^2 \phi,
\end{equation}

where

\begin{equation}
\mathcal{H}^r = -\frac{1}{2} \sum_{i=1}^{r+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{r} e^{x_i-x_{i+1}}
\end{equation}

is the Hamiltonian of the $(r+1)$-particle quantum Toda chain. They were first introduced by Hashizume \cite{13} in a more general context.

Let $\alpha_1, \alpha_2, \ldots$ be a fixed sequence of complex numbers (independent of $r$) and denote $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $i \leq j$. Let $r \geq 1$ and define $\nu \in \mathbb{C}^{r+1}$ by $\sum_{i=1}^{r+1} \nu_i = 0$ and $\alpha_i = \nu_i - \nu_{i+1}$ for $i = 1, \ldots, r$. Then

\begin{equation}
\phi_r(x) = \sum_{n \in \mathbb{Z}_+^r} a_r(n) \prod_{i=1}^{r} e^{n_i(x_i-x_{i+1})+\nu_i x_i}
\end{equation}

satisfies (1) with $\lambda^2 = \sum_{i=1}^{r+1} \nu_i^2 / 2$ provided the coefficients $a_r(n)$ satisfy

\begin{equation}
\left[ \sum_{i=1}^{r} \nu_i^2 - \sum_{i=1}^{r-1} n_i n_{i+1} + \sum_{i=1}^{r} \alpha_i n_i \right] a_r(n) = \sum_{i=1}^{r} a_r(n - e_i),
\end{equation}

where $e_1, \ldots, e_r$ denote the standard basis vectors in $\mathbb{Z}^r$ and with the convention $a_r(n) = 0$ for $n \notin \mathbb{Z}_+^r$. Ishii and Stade \cite{15} obtained the following recursive formula for the coefficients $a_r(n)$. For $n \in \mathbb{Z}_+^r$ and $k \in \mathbb{Z}_+^{r-1}$, define

\begin{equation}
q_r(n, k) = \prod_{i=1}^{r} \frac{1}{(n_i - k_i)!((n_i - k_{i-1} + \alpha_{ir}))!},
\end{equation}

Research supported by the European Research Council (Grant number 669306).
with the convention $k_0 = k_r = 0$. Define $a_r(n)$, $r \geq 1$, $n \in (\mathbb{Z}_+)^r$, recursively by

$$a_1(n) = \frac{1}{n!(n + a_1)!}$$

and, for $r \geq 2$,

$$a_r(n) = \sum_k q_r(n, k)a_{r-1}(k).$$

where the sum is over $0 \leq k_i \leq n_i$, $i = 1, \ldots, r - 1$. Then, for each $r \geq 1$, $a_r(n)$ satisfies the difference equation (4) with

$$a_r(0) = \prod_{1 \leq i \leq j \leq r} \frac{1}{a_{ij}!}.$$  

For example,

$$\phi_1(x_1, x_2) = \frac{1}{\pi a_1} \left(2e^{(x_1 - x_2)/2}\right),$$

where $I_a(z)$ is the modified Bessel function of the first kind $I_a(z) = \sum_{n=0}^{\infty} \frac{1}{n!(n+a)!}(\frac{z}{2})^{2n+a}.$

For $r = 2$, the coefficients are given by

$$a_2(n, m) = \frac{(n + m + a + b)!}{n!(n + a)!(n + a + b)!m!(m + b)!(m + a + b)!},$$

where $a = \alpha_1$ and $b = \alpha_2$. This formula is due to Bump [7].

We remark that, in the case $\alpha \equiv 0$, the recursive formula (7) agrees with a special case (for complete flag manifolds) of a formula given in Batyrev, Ciocan-Fontanine, Kim and van Straten [3, Theorem 5.1.6] for the coefficients of hypergeometric series of partial flag manifolds (see also Remark 5.1 below).

The formula (7) is quite similar to Givental’s integral formula [11, 16, 9] for another family of eigenfunctions of the quantum Toda lattice known as class one Whittaker functions. It is this similarity which motivated the present work. Gerasimov, Kharchev, Lebedev and Oblezin [9] showed that Givental’s formula may be understood in terms of some intertwining relations between the Hamiltonians of quantum Toda chains with different numbers of particles. These intertwining relations were extended and given a probabilistic interpretation, in terms of Brownian motion, in [19]. In this paper we will show that Ishii and Stade’s formula (7) admits a similar development, in a discrete setting.

When $\alpha \equiv 0$, the natural setting for this is in the context of reverse plane partitions. Given an integer partition $\lambda$, a reverse plane partition $\pi$ with shape $\lambda$ is a filling of $\lambda$ with non-negative integers $(\pi_{ij}, (i, j) \in \lambda)$ which is weakly increasing across rows and down columns. Let $\text{RPP}(\lambda)$ denote the set of reverse plane partitions of shape $\lambda$. Fix $\lambda$, and consider the Markov chain on $\text{RPP}(\lambda)$, defined as follows: for each $(i, j) \in \lambda$, subtract one from $\pi_{ij}$ at rate

$$b_{ij}(\pi) = (\pi_{ij} - \pi_{i,j-1})(\pi_{ij} - \pi_{i-1,j}),$$

with the convention $\pi_{i,0} = \pi_{0,j} = 0$. The infinitesimal generator of this Markov chain is given by the difference operator

$$G^\lambda = \sum_{(i,j) \in \lambda} b_{ij}(\pi)D_{\pi_{ij}},$$
where $D_n$ denotes the backward difference operator $D_n f(n) = f(n-1) - f(n)$.

Note that if $π$ is a Markov chain on $\text{RPP}(\lambda)$ with generator $G^λ$ and $µ ⊂ λ$, then the restriction of $π$ to $µ$ is a Markov chain on $\text{RPP}(µ)$ with generator $G^µ$. In particular, the first row of $π$ is a Markov chain in its own right, and it is natural to think of it as an interacting particle system on the non-negative integers: the values $n_j := π_{1,j}$, $j = 1, \ldots, λ_1$ are the positions of $λ_1$ particles; the left-most particle at position $n_1$ jumps to the left at rate $n_1^2$, while for each $j > 1$, the particle at position $n_j$ jumps to the left at rate $n_j(n_j - n_{j-1})$.

We may also consider restrictions to certain skew diagrams $λ/µ$. For this we require that $µ ⊂ λ^0$, where $λ^0$ denotes the set of $(i, j) ∈ λ$, such that $(i + 1, j) ∈ λ$ and $(i, j + 1) ∈ λ$. Remarkably, if the initial law on $\text{RPP}(λ)$ is chosen correctly, then the restriction $π|_{λ/µ}$ will evolve as a Markov chain in its own right.

The simplest non-trivial example is related to Vandermonde’s identity

\[
\binom{n+m}{n} = \sum_k \binom{n}{k} \binom{m}{k}.
\]

Let $λ = (2, 1)$, $µ = (1)$, and write $π_{11} = k$, $π_{12} = n$, $π_{21} = m$. In this notation,

\[G^{(2,1)} = k^2D_k + n(n-k)D_n + m(m-k)D_m.\]

Suppose that, at time zero, $π_{12} = n$, $π_{21} = m$ and $π_{11}$ is chosen at random according to the probability distribution

\[p_{n,m}(k) = \binom{n+m}{n}^{-1} \binom{n}{k} \binom{m}{k}, \quad 0 ≤ k ≤ n ∧ m.\]

Then, if $π$ evolves according to $G^{(2,1)}$, the restriction $π|_{λ/µ} = (π_{12}, π_{21})$ is also a Markov chain, in its own filtration, with generator

\[L = \frac{n^3}{n + m}D_n + \frac{m^3}{n + m}D_m.\]

More generally, if $λ$ is the staircase shape $δ_{r+1} = (r, r-1, \ldots, 1)$ and $µ = δ_r$, then the restriction $π|_{λ/µ}$ represents the ‘boundary values’ $n_i = π_{i, r-i+1}$, $i = 1, \ldots, r$. Suppose that, initially, the conditional law of $π|_{δ_r}$, given these boundary values, is proportional to

\[W_r(π) = \prod_{(i,j) ∈ δ_r} \binom{π_{i,j+1}}{π_{ij}} \binom{π_{i+1,j}}{π_{ij}}.\]

Then, if $π$ evolves according to $G^{δ_{r+1}}$, the boundary values $(π_{1,r}, \ldots, π_{r,1})$ will evolve as a Markov chain on $\mathbb{Z}_r^+$ with generator

\[L' = \sum_{i=1}^r \frac{a_r(n - ε_i)}{a_r(n)} D_{n_i},\]

where $a_r(n)$ are the series coefficients defined by with $α = 0$.

We will also show that the Markov chain on $\text{RPP}(λ)$ with generator $G^λ$ has a unique entrance law starting from $π_{ij} = +∞$ for all $(i, j) ∈ λ$. An important ingredient for proving this is a law of large numbers (via a large deviation principle) for the distribution $W_r$, as the boundary values go to infinity.

The outline of the paper is as follows. In the next section we present the main results in the staircase setting. In Section 3 we extend the range of parameters which may be considered and describe some invariance properties of the associated
Markov chains. In Section 4 we discuss the Markov chain with generator $L^r$, with some explicit calculations for $r = 1, 2$. In Section 5 we state and prove the main results in the context of general shapes and in the final section we outline some extensions to other root systems.

2. Main results for staircase shapes

Denote by $L_k$ and $R_k$ the shift operators defined, for functions $f$ on $\mathbb{Z}_+$, by

$$(L_k f)(k) = \begin{cases} f(k - 1) & k > 0 \\ 0 & k = 0 \end{cases}$$

and

$$(R_k f)(k) = f(k + 1).$$

The difference equation (4) may be written as $h^r a_r = 0$, where

$$h^r = \sum_{i=1}^{r} L_{n_i} - \sum_{i=1}^{r} n_i^2 + \sum_{i=1}^{r-1} n_i n_{i+1} - \sum_{i=1}^{r} \alpha_i n_i.$$  \hspace{1cm} (11)

For $q_r$ defined by (5) and functions $f$ on $\mathbb{Z}_+^{r-1}$ define $q_r f$ on $\mathbb{Z}_+^r$ by

$$(q_r f)(n) = \sum_k q_r(n, k) f(k),$$

where the sum is over $0 \leq k_i \leq n_i$, $i = 1, \ldots, r - 1$.

**Proposition 2.1.** The following intertwining relation holds:

$$h^r \circ q_r = q_r \circ h^{r-1}. \hspace{1cm} (12)$$

**Proof.** Let

$$P_r(n) = \sum_{i=1}^{r} n_i^2 - \sum_{i=1}^{r-1} n_i n_{i+1} + \sum_{i=1}^{r} \alpha_i n_i.$$  \hspace{1cm} (13)

It suffices to show that

$$h^r q_r(n, k) = h^{r-1} q_r(n, k),$$

where

$$h^r = \sum_{i=1}^{r} R_{n_i} - P_r(n).$$

With the convention $k_0 = k_r = 0$, we compute

$$L_{n_i} q_r(n, k) = (n_i - k_i)(n_i - k_{i-1} + \alpha_{ir}) q_r(n, k),$$

hence

$$h^r q_r(n, k) = S_{n, k} q_r(n, k),$$

where

$$S_{n, k} = \sum_{i=1}^{r} (n_i - k_i)(n_i - k_{i-1} + \alpha_{ir}) - P_r(n)$$

$$= \sum_{i=1}^{r} (k_i k_{i-1} - k_i n_i - n_i k_{i-1} + \alpha_{i+1, r} n_i - \alpha_{ir} k_i) + \sum_{i=1}^{r-1} n_i n_{i+1}. $$

Similarly,

$$R_{k_j} q_r(n, k) = (n_j - k_j)(n_{j+1} - k_j + \alpha_{j+1, r}) q_r(n, k),$$

$$= \sum_{i=1}^{r} (k_j k_{j-1} - k_j n_i - n_i k_{j-1} + \alpha_{i+1, r} n_i - \alpha_{ir} k_i) + \sum_{i=1}^{r-1} n_i n_{i+1}. $$
hence
\[ h^{r-1}_k q_r(n, k) = T_{n,k} q_r(n, k), \]
where
\[
T_{n,k} = \sum_{j=1}^{r-1} (n_j - k_j)(n_{j+1} - k_j + \alpha_{j+1,r}) - P_{r-1}(k)
\]
\[
= \sum_{j=1}^{r-1} (n_j n_{j+1} - k_j n_{j+1} - n_j k_j + \alpha_{j+1,r} n_j - \alpha_{j,r} k_j) + \sum_{j=1}^{r-2} k_j k_{j+1}.
\]
Thus \( S_{n,k} = T_{n,k} \), as required. \( \square \)

In the following we will assume that \((\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_+^r\) and denote
\[ \beta_{ij} = \alpha_{i,i+j-1} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+j-1}. \]
Let \( D = L - I \) denote the backward difference operator
\[ D_k f(k) = \begin{cases} f(k-1) - f(k) & k > 0 \\ -f(0) & k = 0. \end{cases} \]
Since \( h^r a_r = 0 \) and \( a_r(n) > 0 \) for \( n \in \mathbb{Z}_+^r \), the corresponding Doob transform
\[ L^r = a_r(n)^{-1} \circ h^r \circ a_r(n) = \sum_{i=1}^r \frac{a_r(n - e_i)}{a_r(n)} D_{n_i} \]
generates a Markov chain on \( \mathbb{Z}_+^r \).

If \( r = 1 \) then, writing \( n = n_1 \) and \( a = a_1 \), \( L^1 = n(n+a)D_n \). If \( r = 2 \) then, writing \( n = \pi_{12}, m = \pi_{21}, k = \pi_{11}, a = \alpha_1, b = \alpha_2 \), and using the formula (9),
\[ L^2 = \frac{n(n+a)(n+a+b)}{n+m+a+b} D_n + \frac{m(m+b)(m+a+b)}{n+m+a+b} D_m. \]

Let \( \Pi^r \) denote the set of non-negative integer arrays \((\pi_{ij}, 1 \leq i + j \leq r + 1)\) satisfying
\[ \pi_{ij} \geq \pi_{i,i-1} \lor (\pi_{i-1,j} - \beta_{ij}), \quad 1 \leq i + j \leq r + 1 \]
with the convention \( \pi_{i,0} = \pi_{0,j} = 0 \). Note that if \( \alpha_i = 0 \) for \( 1 \leq i \leq r \) then \( \Pi^r \) is the set of reverse plane partitions with staircase shape \( \delta_{r+1} = (r, r-1, \ldots, 1) \).

For \( \pi \in \Pi^r \), set
\[ w_r(\pi) = \prod_{1 \leq i+j \leq r} \frac{1}{(\pi_{i,j+1} - \pi_{ij})!(\pi_{i+1,j} - \pi_{ij} + \beta_{i+1,j})!}. \]
By (7), we can write
\[ a_r(n) = \sum_{\pi \in \Pi_n^r} w_r(\pi), \]
where \( \Pi_n^r \) is the set of \( \pi \in \Pi^r \) with \( \pi_{i,r-i+1} = n_i, 1 \leq i \leq r \).

For \( n \in \mathbb{Z}_+^r \), let \( K^r_n \) be the probability distribution on \( \Pi_n^r \) defined by
\[ K^r_n(\pi) = w_r(\pi)/a_r(n). \]
For \( \pi \in \Pi^r \) and \( 1 \leq i + j \leq r + 1 \), set
\[ b_{ij}(\pi) = (\pi_{ij} - \pi_{i,j-1})(\pi_{ij} - \pi_{i-1,j} + \beta_{ij}), \]
with the convention $\pi_{i,0} = \pi_{0,j} = 0$. Let
\[
G^r = \sum_{1 \leq i+j \leq r+1} b_{ij}(\pi)D_{\pi_{ij}}.
\]
For example, if $r = 2$ then, writing $n = \pi_{12}$, $m = \pi_{21}$, $k = \pi_{11}$, $a = \alpha_1$, $b = \alpha_2$,
\[
G^2 = k(k+a)D_k + n(n-k+a+b)D_n + m(m-k+b)D_m.
\]

**Theorem 2.2.** Let $\pi(t), \ t \geq 0$ be a Markov chain on $\Pi'$ with generator $G^r$ and initial law $K^r_n$, for some $n \in \mathbb{Z}_+^r$. Then $N(t) = (\pi_{1r}(t), \ldots, \pi_{rr}(t))$ is a Markov chain on $\mathbb{Z}_+^r$ with generator $L^r$ and, for all $t > 0$, the conditional law of $\pi(t)$ given $\{N(s), s \leq t\}$ is $K^r_{N(t)}$.

**Proof.** For functions $f$ on $\mathbb{Z}_+^r \times \mathbb{Z}_+^{r-1}$, define $\tilde{q}_r f$ on $\mathbb{Z}_+^r$ by
\[
(\tilde{q}_r f)(n) = \sum_k q_r(n,k)f(n,k),
\]
where the sum is over $0 \leq k_i \leq n_i$, $i = 1, \ldots, r - 1$. Let
\[
g^r = h_k^{r-1} + \sum_{i=1}^r L_n q_r(n,k)D_n = h_k^{r-1} + \sum_{i=1}^r (n_i-k_i)(n_i-k_{i-1}+\alpha_{ir})D_n.
\]
By Proposition 2.1,
\[
[h^r(\tilde{q}_r f)](n) = \sum_k h_k^r [q_r(n,k)f(n,k)]
= \sum_k \left( h_k^r q_r(n,k) f(n,k) + \sum_{i=1}^r L_n q_r(n,k) D_n f(n,k) \right)
= \sum_k \left( q_r(n,k) h_k^{r-1} f(n,k) + \sum_{i=1}^r L_n q_r(n,k) D_n f(n,k) \right)
= \sum_k q_r(n,k) g^r f(n,k)
= [\tilde{q}_r(g^r f)](n).
\]
The statement of theorem follows, by induction and the theory of Markov functions \[18\] \[24\]. The application of the latter in this context (and indeed for all the examples considered in this paper) is free of technical considerations since, given the initial law, the relevant part of the state space is finite. \qed

In Section 5 we will present a more general version of Theorem 2.2, valid for arbitrary shapes. We will also prove, again in a more general setting, the existence of a unique entrance law for the Markov chain with generator $G^r$, starting from the array with all entries equal to $+\infty$. The following is a special case of Theorem 5.3.

**Theorem 2.3.** The Markov chain on $\Pi'$ with generator $G^r$ has a unique entrance law starting from $\pi_{ij} = +\infty$ for $1 \leq i + j \leq r + 1$. Under this entrance law, $N(t) = (\pi_{1r}(t), \ldots, \pi_{rr}(t))$ is a Markov chain on $\mathbb{Z}_+^r$ with generator $L^r$ and, for all $t > 0$, the conditional law of $\pi(t)$ given $\{N(s), s \leq t\}$ is $K^r_{N(t)}$. 

Remark 2.1. When \( \alpha \equiv 0 \), the normalised coefficients
\[
A_r(n) = \left( \prod_{i=1}^{r} n_i^{-2} \right) a_r(n)
\]
are given, for \( n \in \mathbb{Z}_+^r \), by the binomial sum formula
\[
A_r(n) = \sum_{\pi \in \Pi_n^\tau} W_r(\pi), \quad W_r(\pi) = \prod_{(i,j) \in \delta_r} \left( \frac{\pi_{i,j+1}}{\pi_{ij}} \right) \left( \frac{\pi_{i+1,j}}{\pi_{ij}} \right).
\]
These are the unique solution to \( H^r A_r = 0 \) on \( \mathbb{Z}_+^r \) with \( A_r(0) = 1 \), where
\[
H^r = \sum_{i=1}^{r} n_i^2 D_{ni} + \sum_{i=1}^{r-1} n_i n_{i+1}.
\]

Remark 2.2. The diagonal values \( a_n = A_3(n,n,n) \) are the Apéry numbers
\[
a_n = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2.
\]
associated with \( \zeta(3) \). This sequence satisfies the three-term recurrence
\[
n^3 a_n = (34n^3 - 51n^2 + 27n - 5)a_{n-1} - (n - 1)^3 a_{n-2},
\]
with \( a_0 = 1 \) and \( a_1 = 5 \). We remark that this recurrence may be derived, in an elementary way, using the connection to the Toda lattice. The latter implies that \( A_3(n,m,l) \) is annihilated by the three commuting difference operators
\[
n^2 D_n + m^2 D_m + l^2 D_l + nm + ml, \quad mn^2 D_n + (n-l)m^2 D_m + ml^2 D_l,
\]
\[
n^2 l^2 D_n \cdot D_{nl} - l(l-m)n^2 D_n - nm^2 D_m + n(m-n)^2 D_l,
\]
where \( D_{nl} f(n,l) = f(n-1,l-1) - f(n,l) \). The corresponding difference equations may be combined to obtain (14).

3. More general parameters and symmetries

The assumption \((\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_+^r \) is not necessary for some version of Theorem 2.2 to hold. For example, if \((\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r \) then the statement may be modified as follows. Denote by \( \Pi^{\alpha,\beta} \) the set of non-negative integer arrays \( (\pi_{ij}, 1 \leq i+j \leq r+1) \) satisfying
\[
\pi_{ij} \geq \omega_{ij} \lor \pi_{i,j-1} \lor (\pi_{i-1,j} - \beta_{ij}), \quad 1 \leq i+j \leq r+1,
\]
where \( \omega = (\omega_{ij}, 1 \leq i+j \leq r+1) \) is the unique solution to
\[
\omega_{ij} = \omega_{i,j-1} \lor (\omega_{i-1,j} - \beta_{ij}), \quad 1 \leq i+j \leq r+1,
\]
with the conventions \( \pi_{i,0} = \pi_{0,j} = \omega_{i,0} = \omega_{0,j} = 0 \). This agrees with the previous definition when \((\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_+^r \), since \( \omega \equiv 0 \) in that case. Similarly, let \( \mathbb{Z}_+^{\alpha,\beta} \) denote the set of \( n \in \mathbb{Z}_+^r \) satisfying \( n_i \geq \omega_{i,r-i+1}, i = 1, \ldots, r \). Then the statement and proof of Theorem 2.2 remain valid, as written, with \( \Pi^\tau \) and \( \mathbb{Z}_+^\tau \) replaced by \( \Pi^{\alpha,\beta} \) and \( \mathbb{Z}_+^{\alpha,\beta} \), respectively.

The array \( \omega \in \Pi^{\alpha,\beta} \) is an absorbing state for the Markov chain with generator \( G^\tau \). It is given explicitly as follows. Consider the triangular array \( \left( \nu_k^i, 1 \leq i \leq k \leq r+1 \right) \) defined by \( \alpha_i = \nu_i^k - \nu_{i+1}^k \) for \( 1 \leq i \leq k \leq r+1 \) and \( \nu_1^1 + \nu_2^2 + \cdots + \nu_k^k = 0 \) for
where

\[ \omega_{i,k-1} = \hat{\nu}_i^k + \cdots + \hat{\nu}_i^k - \nu_{1}^k - \cdots - \nu_i^k. \]

It is straightforward to verify that this satisfies (15), by induction over \( k \).

Note that if \((\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_r^\tau\), then

\[ \omega_{ij} = -\beta_{1j} - \beta_{2j} - \cdots - \beta_{ij}, \quad 1 \leq i + j \leq r + 1. \]

This leads us to observe the following basic symmetry. For \((\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_r^\tau\), define

\[ \gamma_{ij} = -\beta_{1j} - \beta_{2j} - \cdots - \beta_{ij}, \quad 1 \leq i + j \leq r + 1. \]

From the definitions, if \( \pi \in \Pi^r = \Pi^{r^\alpha} \) then \( \hat{\pi} \in \Pi^{r^\alpha} \), where

\[ \hat{\pi}_{ij} = \pi_{ji} - \gamma_{ji}, \quad 1 \leq i + j \leq r + 1. \]

Moreover, if \( \pi(t) \) is a Markov chain on \( \Pi^{r^\alpha} \) with generator \( G^r = G^{r^\alpha} \) then \( \hat{\pi}(t) \) is a Markov chain on \( \Pi^{r^\alpha} \) with generator \( G^{r^\alpha} \). If \((\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_r^\tau\), then \( \omega = \gamma \), and the transformation \( \pi \mapsto \hat{\pi} \) takes us back to the simplest case with \( \omega \equiv 0 \).

Now consider the Markov chain with state space \( \mathbb{Z}_r^\alpha \) and generator \( L' = L^{r,\alpha} \), where \( \alpha \in \mathbb{Z}_r^\tau \). Let us write \( a_t = \nu_i - \nu_{i+1}, \quad i = 1, \ldots, r \), where \( \nu \in (\mathbb{Z}/(r+1))^{r+1} \) with \( \sum_i \nu_i = 0 \), and denote \( a_r(n) = a_{r,\nu}(n) \). From the above,

\[ n'_i = n_i + \nu_1 + \cdots + \nu_i, \quad j = 1, \ldots, r. \]

Let

\[ S^{r,\nu} = (\mathbb{Z}_+^{r,\alpha})' = (\hat{\nu}_1, \hat{\nu}_1 + \hat{\nu}_2, \ldots, \hat{\nu}_1 + \cdots + \hat{\nu}_r) + \mathbb{Z}_+^r, \]

and note that this set is invariant under permutations of the parameters \( \nu_1, \ldots, \nu_{r+1} \).

For \( n \in \mathbb{Z}_+^{r,\alpha} \), define \( a_{r,\nu}(n') = a_{r,\nu}(n) \). For example,

\[ \tilde{a}_{1,\nu}(n') = \frac{1}{(n' - \nu_1)!/(n' - \nu_2)!}, \]

and, using (4),

\[ \tilde{a}_{2,\nu}(n', m') = (n' + m')! \prod_{i=1}^{3} \frac{1}{(n' - \nu_i)!/(n' + \nu_i)!}. \]

**Proposition 3.1.** For any fixed \( n' \in S^{r,\nu} \), \( a_{r,\nu}(n') \) is invariant under permutations of the parameters \( \nu_1, \ldots, \nu_{r+1} \).

**Proof.** This follows from a more general form of the recursion (7) given in [15, Theorem 15]. Let \( t \in \{1, 2, \ldots, r+1\} \) and define, for \( n \in \mathbb{Z}_+^r \) and \( k \in \mathbb{Z}_+^{r-1} \),

\[ q_{r,\nu}(n, k) = \prod_{i=1}^{t-1} \frac{1}{(n_i - k_i)!/(n_i - k_{i-1} + \nu_i - \nu_{i+1})!} \prod_{i=t}^{r} \frac{1}{(n_i - k_i - 1)!/(n_i - k_i - \nu_{i+1} + \nu_{i+1})!}. \]

Then

\[ a_{r,\nu}(n) = \sum_{k \in \mathbb{Z}_+^{r-1}} q_{r,\nu}(n, k) a_{r-1,\mu}(k), \]

where \( \mu = (\nu_1 + \nu_1/r, \ldots, \nu_{r-1} + \nu_{r-1}/r, \nu_{r+1} + \nu_{r+1}/r, \ldots, \nu_{r+1} + \nu_{r+1}/r). \)
Changing variables to
\[ n'_i = n_i + \nu_1 + \cdots + \nu_i, \quad i = 1, \ldots, r + 1, \]
\[ k'_i = k_i + \mu_1 + \cdots + \mu_i, \quad i = 1, \ldots, r, \]
we can write \( q_{r, \nu}(n, k) = \tilde{q}_{r, \nu}(n', k'), \) where
\[
\tilde{q}_{r, \nu}(n', k') = \prod_{i=1}^{r} \frac{1}{(n'_i - k'_i + j \theta/r)!((n'_i - k'_i - 1) - (r - i + 1)\theta/r)!}.
\]
In this notation,
\[
\tilde{a}_{r, \nu}(n') = \sum_{k} \tilde{q}_{r, \nu}(n', k') \tilde{a}_{r-1, \mu}(k'),
\]
where the sum is over \( k' \) such that \( k \in \mathbb{Z}^{r-1}_+. \) Since \( t \) is arbitrary, the claim follows by induction over \( r. \)

**Corollary 3.2.** If \( N \) is a Markov chain in \( \mathbb{Z}^r_+ \) with generator \( L^{r, \alpha} \) then the law of the Markov chain \( N' \), which has state space \( S^{r, P} \), is invariant under permutations of the parameters \( \nu_1, \ldots, \nu_{r+1}. \)

4. Transition probabilities and hitting times

Let \( (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r_+ \) and consider the Markov chain on \( \mathbb{Z}^r_+ \) with generator \( L^{r, \alpha} \). Denote by \( P_n \) the law of this process started from \( n \) and \( E_n \) the corresponding expectation. By Theorem 2.3 this law is also well defined for \( n = +\infty \), in this case we will write \( P = P_+ \) and \( E = E_+ \). Let \( T_n \) denote the first hitting time of \( n \).

The transition probabilities are given by
\[
p_i^r(n, m) = \frac{a_r(m)}{a_r(n)} \tilde{p}_i^r(n, m),
\]
where \( \tilde{p}_i^r(n, m) \) is the heat kernel associated with \( h^r \). Let
\[
g_r(n, m) = \int_0^\infty \tilde{p}_i^r(n, m) dt,
\]
and note that, since the rate at which the chain leaves a state \( m \in \mathbb{Z}^r_+ \) is \( P_r(m) \),
\[
P_n(T_m < \infty) = P_r(m)g_r(n, m).
\]

By Theorem 2.3 we may define
\[
p_i^r(m) = \lim_{n \to \infty} p_i^r(n, m), \quad h^r_m = P(T_m < \infty)
\]
and for \( m \in \mathbb{Z}^r_+ \setminus \{0\} \),
\[
g_r(m) = \lim_{n \to \infty} g_r(n, m), \quad a^*_r(m) = g_r(m)/a_r(m).
\]
Note that \( h^r_0 = 1 \) and, for \( n \in \mathbb{Z}^r_+ \setminus \{0\} \),
\[
h^r_n = P_r(n)a_r(n)a^*_r(n).
\]

**Proposition 4.1.** For \( r \geq 2 \) and \( k \in \mathbb{Z}^{r-1}_+ \setminus \{0\} \),
\[
a^*_r(k) = \sum_n q_r(n, k)a^*_r(n),
\]
where the sum is over \( n \in \mathbb{Z}^r_+ \) such that \( n_i \geq k_i \) for \( i = 1, \ldots, r. \)
4.1. The case $r = 1$. Writing $n = n_1$ and $a = a_1$, $L^1 = n(n + a)D_n$ and

$$a_1(n) = \frac{1}{n!(n + a)!}, \quad g_1(n) = \frac{1}{n(n + a)}, \quad a_1^*(n) = \Gamma(n)\Gamma(n + a).$$

When $a = 1$ this is essentially Kingman’s coalescent [17].

Proposition 4.2. For $k \geq n$, $\Re(s) > -1 - a$ and $t \geq 0$,

$$\mathbb{E}_k e^{-sT_n} = \sum_{j=n+1}^k \frac{j(j + a)}{j(j + a) + s}$$

and

$$\tilde{p}_t(k, n) = \sum_{j=n}^k \prod_{j=n, l \neq j}^k \frac{1}{(l - j)(l + j + a)} e^{-j(j + a)t}.$$

Proof. The first claim is immediate from the fact that, under $P_k$, $T_n$ is distributed as a sum of independent exponential random variables with respective parameters $j(j + a), j = n + 1, \ldots, k$.

For the second claim, define

$$\varphi_j(n) = \frac{1}{(n - j)(n + a + j)!}, \quad \varphi_j^*(n) = (-1)^j [(1 + j)(j + a)](-j)_n(a + j)_n,$$

and note that

$$h^1\varphi_j(n) = -j(j + a)\varphi_j(n), \quad h^1\varphi_j^*(n) = -j(j + a)\varphi_j^*(n),$$

where $h^1 = L_n - n(n + a)$ and $h^1 = R_n - n(n + a)$ as defined previously. The expression given for $\tilde{p}_t(k, n)$ in the statement of the proposition is equivalent to

$$\tilde{p}_t(k, n) = \sum_{j=n}^k \varphi_j(k)\varphi_j^*(n)e^{-j(j + a)t}.$$

This satisfies the required forward and backward equations

$$\partial_t\tilde{p}_t(k, n) = h^1_k\tilde{p}_t(k, n) + h^1_n\tilde{p}_t(k, n),$$

so it only remains to show that $f(k, n) = \delta_{kn}$, where

$$f(k, n) = \sum_{j=n}^k \prod_{j=n, l \neq j}^k \frac{1}{(l - j)(l + j + a)}.$$
Clearly \( f(n, n) = 1 \). Writing \((l - j)(l + j + a) = l(l + a) - j(j + a)\), the fact that \( f(k, n) = 0 \) for \( k > n \) may be seen as a consequence of the more general identity (see for example [20, Eq. 8.17])

\[
\sum_{a=1}^{N} \prod_{b=1, b \neq a}^{N} \frac{1}{\lambda_a - \lambda_b} = 0.
\]

(17)

\[\square\]

**Remark 4.1.** The functions \( \tilde{p}_l(k, n) \) are also related to the classical Toda chain, see for example [20, Theorem 8.5]. In particular, they satisfy the Toda equations

\[
\tilde{p}_l(k, n)\partial_k^2 \tilde{p}_l(k, n) - (\partial_k \tilde{p}_l(k, n))^2 = \tilde{p}_l(k - 1, n + 1)\tilde{p}_l(k, n) - \tilde{p}_l(k - 1, n)\tilde{p}_l(k, n + 1).
\]

4.2. The case \( r = 2 \) and \( \alpha = 0 \). In this case, writing \((n, m) = (n_1, n_2)\),

\[
a_2(n, m) = \frac{(n + m)!}{n!^2 m!^2}, \quad L^2 = \frac{n^3}{n + m}D_n + \frac{m^3}{n + m}D_m.
\]

We first note the following decomposition for the absorption time. For the Markov chain with generator \( L^2 \), let \( \tau_{n,m} \) be the first hitting time of \((0, 0)\) for the chain started at \((n, m)\). For the Markov chain with generator \( n^2D_n \), let \( \tau_n \) be the first hitting time of 0 for the chain started at \( n \).

**Proposition 4.3.** The absorption time \( \tau_{n,m} \) has the same law as \( \tau_n + \tau'_m \), where \( \tau'_m \) is an independent copy of \( \tau_m \). This identity in law remains valid for \( n = m = +\infty \).

**Proof.** It suffices to show that \( H_s(n, m) = G_s(n)G_s(m) \), where

\[
H_s(n, m) = \mathbb{E}e^{-s\tau_{n,m}}, \quad G_s(n) = \mathbb{E}e^{-s\tau_n}.
\]

The function \( H_s(n, m) \) is the unique solution to \( L^2H = sH \) on \( \mathbb{Z}_+^2 \) with \( H(0,0) = 1 \), and \( G_s(n) \) satisfies \( n^2D_nG_s(n) = sG_s(n) \) for \( n > 0 \). Let \( \tilde{H}_s(n, m) = G_s(n)G_s(m) \). Clearly \( \tilde{H}_s(0,0) = 1 \). For \( n > 0 \) and \( m = 0 \) we have \( L^2 = m^2D_m \), hence \( L^2\tilde{H}_s = s\tilde{H}_s \) on this set. Similarly, \( L^2\tilde{H}_s = s\tilde{H}_s \) for \( n = 0 \) and \( m > 0 \). For \( n, m > 0 \), we have

\[
L^2\tilde{H}_s(n, m) = \frac{n^3}{n + m}D_n[G_s(n)G_s(m)] + \frac{m^3}{n + m}D_m[G_s(n)G_s(m)]
\]

\[
= s \frac{n}{n + m} [G_s(n)G_s(m)] + s \frac{m}{n + m} [G_s(n)G_s(m)] = s\tilde{H}_s(n, m),
\]

hence \( \tilde{H}_s = H_s \), as required. \( \square \)

**Remark 4.2.** The random variables \( S_1 = 2\tau_\infty/\pi^2 \) and \( S_2 = 2(\tau_\infty + \tau'_\infty)/\pi^2 \) have many interesting interpretations and applications [6, 21].

**Corollary 4.4.**

\[
\sum_{(n,m)\neq(0,0)} a_2(n, m)a_2^*(n, m) = 2\zeta(2), \quad \sum_{(n,m)\neq(0,0)} \frac{1}{n!^2 m!^2} a_2^*(n, m) = \zeta(2).
\]

**Proof.** The expected time spent at \((n, m) \neq (0, 0)\) is given by \( a_2(n, m)a_2^*(n, m) \), so the first identity follows from Proposition 4.3. By Proposition 4.1

(18)

\[
\sum_{n,m \geq k} q_2((n, m), k)a_1(k)a_2^*(n, m) = \frac{1}{k^2}.
\]
Summing over $k \geq 1$ gives
\[
\sum_{n,m \geq 1} \left[ a_2(n,m) - q_2((n,m),0) \right] a_2^*(n,m) = \zeta(2)
\]
or, equivalently,
\[
\sum_{(n,m) \neq (0,0)} \left[ a_2(n,m) - q_2((n,m),0) \right] a_2^*(n,m) = \zeta(2).
\]
Combined with the first identity this gives the second. \qed

**Remark 4.3.** Proposition 4.3 may be extended to the case $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$ with $\alpha_1 + \alpha_2 = 0$ as follows. Suppose $a = \alpha_1 \geq 0$ and let $E = \{ (n,m) \in \mathbb{Z}_+^2 : m \geq a \}$. Then, for $(n,m) \in E$,
\[
a_2(n,m) = \frac{(n+m)!}{n!^2(n+a)!m^2(m-a)!}, \quad L^2 = \frac{n^2(n+a)}{n+m} D_n + \frac{m^2(m-a)}{n+m} D_m.
\]
Let $\tau_{n,m}$ be the first hitting time of $(0,a)$ for the Markov chain with generator $L^2$ started at $(n,m) \in E$. Let $\tau_n$ be the first hitting time of 0 for the Markov chain with generator $n(n+a)D_n$ started at $n \geq 0$, and note that $\tau_{n-a}$ has the same law as the first hitting time of $a$ for the Markov chain with generator $m(m-a)D_m$ started at $m \geq a$. Then it holds that $\tau_{n,m}$ has the same law as $\tau_n + \tau_{n-a}'$, where $\tau_{n-a}'$ is an independent copy of $\tau_{n-a}$. This identity remains valid for $n = m = +\infty$, and the first identity of Corollary 4.3 extends to
\[
\sum_{(n,m) \in E \setminus \{0,a\}} a_2(n,m)a_2^*(n,m) = \sum_{k=1}^{\infty} \frac{2}{k(k+a)}.
\]

**Proposition 4.5.** For $n, m \geq 1$,
\[
\mathbb{P}(k,l)(T_{n,m} < \infty) = \frac{n^3 + m^3}{n^3 m^3} \sum_{j=n}^{k} j^3 \prod_{a=n,a \neq j}^{k} \frac{a^3}{a^3 - j^3} \prod_{b=m}^{l} \frac{b^3}{b^3 + j^3}.
\]
For $n \geq 1$,
\[
\mathbb{P}(k,l)(T_{n,0} < \infty) = \sum_{j=n}^{k} \prod_{a=n,a \neq j}^{k} \frac{a^3}{a^3 - j^3} \prod_{b=1}^{l} \frac{b^3}{b^3 + j^3}.
\]
**Proof.** Let
\[
*L^2 = B_n \circ \frac{n^3}{n+m} + B_m \circ \frac{m^3}{n+m},
\]
where $B_k$ denotes the forward difference operator $B_k f(k) = f(k+1) - f(k)$.

For $\nu \in \mathbb{C}$ with $\sum_i \nu_i = 0$, the function
\[
\psi_{\nu}(n,m) = \frac{n!^3 m!^3}{\prod_{i=1}^{n} (n-\nu_i)! (m+\nu_i)!}
\]
is an eigenfunction of $L^2$ with eigenvalue $-\sum_i \nu_i^2 / 2$, and
\[
\psi_{\nu}^*(n,m) = \frac{n+m}{n!^3 m!^3} \prod_{i=1}^{\nu_i} (-\nu_i)_n \nu_i m
\]
is an eigenfunction of $*L^2$ with the same eigenvalue. These claims, which may be anticipated from the formula (11), are easily verified.
Let $\nu = (1, \omega, \omega^2)$, where $\omega$ is the primitive cube root of unity $\omega = -1/2 + i\sqrt{3}/2$. Note that $\sum \nu_i = \sum \nu_i^2 = 0$, and
\[ x^3 + j^3 = (x + j)(x + j\omega)(x + j\omega^2). \]
Define $f^*_0(n, m) = \delta_{n, 0}$ and, for $j \geq 1$,
\[ f^*_j(n, m) = \psi^*_j\nu(n, m) = \frac{n + m}{n!^3m!^3} \prod_{a = 0}^{n-1} (a^3 - j^3) \prod_{b = 0}^{m-1} (b^3 + j^3). \]
Define $f_0(k, l) = 1$ and, for $j \geq 1$,
\[ f_j(k, l) = k!^3l!^3 \prod_{a = 0, a \neq j}^{k} \frac{1}{a^3 - j^3} \prod_{b = 0}^{l} \frac{1}{b^3 + j^3}. \]
Then $L^2f_j = *L^2 f^*_j = 0$, for all $j \geq 0$. We will show that
\[ \mathbb{P}_{(k, l)}(T_{n,m} < \infty) = \sum_{j=n}^{k} f_j(k, l)f^*_j(n, m), \]
which implies the statement of the proposition. The function
\[ F(n, m) = \sum_{j=n}^{k} f_j(k, l)f^*_j(n, m) \]
satisfies $*L^2F = 0$ and, for $(k, l) \neq (0, 0)$,
\[ F(k, l) = \frac{1}{k^2 + l^2 - kl}. \]
It therefore suffices to show that
\[ \sum_{j=n}^{k} f_j(k, l)f^*_j(n, l + 1) = 0, \quad 0 \leq n < k, \]
or equivalently
\[ \sum_{j=n}^{k} \prod_{a = n, a \neq j}^{k} \frac{1}{a^3 - j^3} = 0, \quad 0 \leq n < k. \]
This follows from (17). \qed

Let $R_n(x) = \prod_{a=1}^{n}(a^3 + x^3)$ with $R_0(x) = 1$, and set $h_{nm} = \mathbb{P}(T_{n,m} < \infty)$.

**Corollary 4.6.** For $n, m \geq 1$,
\[ (19) \quad h_{nm} = \frac{n^3 + m^3}{n!^3m!^3} \sum_{j=n}^{\infty} R_{n-1}(-j)R_{m-1}(j) \frac{3\pi^2j^5}{\sin^2(\pi\omega j)} \]
and, for $n \geq 1$,
\[ h_{n0} = h_{0n} = \frac{1}{\Gamma(n)^3} \sum_{j=n}^{\infty} R_{n-1}(-j) \frac{3\pi^2j^2}{\sin^2(\pi\omega j)}. \]
The above corollary also provides formulas for $a^*_2(n,m)$, using
\begin{equation}
 h_{nm} = (n^2 + m^2 - nm)a_2(n,m)a^*_2(n,m).
\end{equation}
We note that the hitting probabilities satisfy
\begin{equation}
 h_{nm} = h_{mn}, \quad h_{n,0} + h_{n-1,1} + \cdots + h_{0,n} = 1.
\end{equation}
In the above formulas, noting that $R_{n-1}(-j) = 0$ for $j = 1, \ldots, n-1$, the summations may be extended to $j \geq 1$. Thus, if we define, for $k \geq 0$,
\begin{equation}
 S_k = \sum_{j=0}^{\infty} \frac{6\pi^2 j^{2+3k}}{\sin^2(\pi\omega j)}.
\end{equation}
then each hitting probability can be expressed as a finite rational linear combination of these series. By Proposition 2.24 of [5] we have $S_0 = 1$ and, by (a corrected version of) Corollary 2.23 in that paper, $S_k = 0$ for all positive, even values of $k$.

The identity $S_{2k} = \delta_{k0}$ may also be inferred from Corollary 4.6 together with the relations (21). For example, by Corollary 4.6 we have $h_{10} = S_0/2; \text{ together with } h_{10} = h_{01} \text{ and } h_{01} + h_{10} = 1, \text{ this implies } S_0 = 1$.

The first few hitting probabilities are given by
\begin{align*}
 h_{00} &= 1, \quad h_{10} = h_{01} = 1/2, \quad h_{11} = S_1 = 0.87987 \ldots, \quad h_{20} = (1 - S_1)/2, \\
 h_{21} &= 9S_1/16, \quad h_{30} = (8 - 9S_1)/16, \quad h_{22} = (S_1 - S_3)/8.
\end{align*}

**Remark 4.4.** We can write $S_k = 24\pi^2 T_{2+3k}$, where
\begin{equation}
 T_r = -\sum_{n=1}^{\infty} \frac{n^r q^n}{(1 - q^n)^2}, \quad q = e^{2\pi i \omega} = -e^{-\pi \sqrt{3}}.
\end{equation}
These are well known $q$-series and may be evaluated for even values of $r$ using formulas due to Ramanujan [12]. For example,
\begin{align*}
 T_2 &= \frac{1}{24\pi^2}, \quad T_4 = \frac{3\Gamma(\frac{1}{3})^{18}}{40960\pi^{12}}, \quad T_6 = \frac{3\sqrt{3} \Gamma(\frac{1}{3})^{18}}{28672\pi^{18}}, \quad T_8 = 0, \quad \ldots
\end{align*}

**Remark 4.5.** The summation formula (18) may be verified directly using (19), (20) and [22, Equation (34)].

4.3. **Imaginary exponential functionals.** Let $\alpha \equiv 0$ and denote by $p_t(n,m)$ the transition kernel of the Markov chain with generator $L_r = a_r(n)^{-1} \circ h^r \circ a_r(n)$.

Let $B = (B_1, \ldots, B_{r+1})$ be a standard Brownian motion in $\mathbb{R}^{r+1}$, started at the origin, and set
\begin{align*}
 Y_k(t) &= e^{i(B_k(t) - B_{k+1}(t))}, \quad Z_k(t) = \int_0^t Y_k(s)ds, \quad k = 1, \ldots, r.
\end{align*}
We will use the following notation. For $0 \neq y, z \in \mathbb{C}^r$ and $n \in \mathbb{Z}^r$,
\begin{align*}
 y^{-n} &= \prod_{k=1}^r y_k^{-n_k}, \quad z^n = \prod_{k=1}^r z_k^m, \quad n! = \prod_{k=1}^r n_k!,
\end{align*}
Proposition 4.7. For $n, m \in \mathbb{Z}_+^r$ with $n \geq m$,
\[ p_t(n, m) = \frac{a_r(m)}{a_r(n)(n-m)!} \mathbb{E} \left[ Y(t)^{-n} Z(t)^{n-m} \right]. \]
In particular,
\[ P_n(T_0 \leq t) = p_t(n, 0) = \frac{\mathbb{E} Z(t)^n}{a_r(n)n!}. \]

Proof. It is straightforward to show that $h_r^n F(n, x) = I_r x F(n, x)$, where
\[ I_r = \frac{1}{2} \sum_{j=1}^{r+1} \frac{\partial^2}{\partial x_j^2} + \sum_{k=1}^{r} y_k, \]
$F(n, x) = y^{-n}$, $y_k = e^{i(x_k - x_{k+1})}$, $k = 1, \ldots, r$.
Setting $H(n, x) = a_r(n)^{-1} F(n, x)$, this implies the Markov duality relation
\[ L_r H(n, x) = I_r H(n, x). \]
It follows, for example using [8, Theorem 4.4.11], that, if $N_t$ is a Markov chain with
generator $L^r$, started at $n$ then
\[ \mathbb{E} H(N_t, x) = \mathbb{E} \left[ H(n, x + B_t) \exp \left( \sum_{k=1}^{r} y_k Z_k(t) \right) \right]. \]
Equivalently,
\[ \sum_{0 \leq m \leq n} p_u(n, m)a_r(m)^{-1} y^{-m} = a_r(n)^{-1} y^{-n} \mathbb{E} \left[ Y(t)^{-n} \exp \left( \sum_{k=1}^{r} y_k Z_k(t) \right) \right]. \]
Since $|Y_k(t)| = 1$ and $|Z_k(t)| \leq t$ almost surely, for each $k$, the statement of the
proposition follows, using bounded convergence and Cauchy’s theorem. \qed

Remark 4.6. The above proof also shows that, if $n, l \in \mathbb{Z}_+^r$ and $l \not\preceq n$, then
\[ \mathbb{E} \left[ Y(t)^{-n} Z(t)^l \right] = 0. \]

Corollary 4.8.
\[ a_r(n) = \lim_{t \to \infty} \frac{1}{n!} \mathbb{E} Z(t)^n. \]

Corollary 4.9. For any $y \in \mathbb{C}^r$,
\[ \Phi_r(y) = \sum_{n \in \mathbb{Z}_+^r} a_r(n) y^n = \lim_{t \to \infty} \mathbb{E} \exp \left( \sum_{k=1}^{r} y_k Z_k(t) \right). \]
Recalling Propositions 4.2 and 4.3, we also deduce the following.
Corollary 4.10. If $r = 2$ and $T$ is an independent exponentially distributed random variable with parameter $\lambda^2$, independent of $B$, then
\[
\mathbb{E}Z_1(T)^n = \mathbb{E}Z_2(T)^n = \frac{1}{(1+i\lambda)_n(1-i\lambda)_n},
\]
\[
\mathbb{E}[Z_1(T)^n Z_2(T)^m] = \binom{n+m}{n} \mathbb{E}Z_1(T)^n \mathbb{E}Z_2(T)^m.
\]

5. More general shapes

Let $\alpha_1, \alpha_2, \ldots$ be a sequence of non-negative integers and denote
\[
\beta_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+j-1}.
\]

Let $\Pi^{\lambda/\mu}$ denote the set of non-negative integer arrays $(\pi_{ij}, (i, j) \in \lambda/\mu)$ which satisfy $\pi_{ij} \geq \pi_{i,j-1} \lor (\pi_{i-1,j} - \beta_{ij})$, $(i, j) \in \lambda/\mu$, with the convention $\pi_{ij} = 0$ for $(i, j) \notin \lambda/\mu$. We will write $\Pi^{\lambda} = \Pi^{\lambda/\phi}$. Note that if $\alpha \equiv 0$ then $\Pi^{\lambda/\mu}$ is the set of reverse plane partitions of shape $\lambda/\mu$.

Fix $\lambda$ and $\mu \subset \lambda^\circ$. Denote by $\bar{\mu}$ the extension of $\mu$ to include $(i, j) \in \lambda/\mu$ such that either $(i - 1, j) \notin \mu$ or $(i, j - 1) \notin \mu$. For $\sigma \in \Pi^{\lambda/\mu}$, let $\Pi^{\lambda}_{\sigma}$ denote the set of $\pi \in \Pi^{\lambda}$ with $\pi|_{\lambda/\mu} = \sigma$. For $\pi \in \Pi^{\lambda}$, let
\[
W_{\lambda,\mu}(\pi) = \prod_{(i, j) \in \bar{\mu}} \left( \frac{\pi_{ij}}{\pi_{i,j-1}} \right) \left( \frac{\pi_{ij} + \beta_{ij}}{\pi_{i-1,j}} \right) \prod_{(i, j) \in \mu} \frac{\pi_{ij}!}{(\pi_{ij} + \beta_{ij})!}.
\]

For $\sigma \in \Pi^{\lambda/\mu}$, let $K^{\lambda,\mu}_{\sigma}$ denote the probability distribution on $\Pi^{\lambda}_{\sigma}$ defined by
\[
K^{\lambda,\mu}_{\sigma}(\pi) = W_{\lambda,\mu}(\pi)/A_{\lambda,\mu}(\sigma),
\]
where
\[
A_{\lambda,\mu}(\sigma) = \sum_{\pi \in \Pi^{\lambda}_{\sigma}} W_{\lambda,\mu}(\pi).
\]

For $\sigma \in \Pi^{\lambda/\mu}$ and $(i, j) \in \lambda/\mu$, set
\[
b_{ij}(\sigma) = (\sigma_{ij} - \sigma_{i,j-1})(\sigma_{ij} - \sigma_{i-1,j} + \beta_{ij}),
\]
with the convention $\sigma_{ij} = 0$ for $(i, j) \notin \lambda/\mu$. Let $C(\mu)$ denote the set of external corners of $\mu$, that is, the set of $(i, j) \in \mu$ such that $(i, j + 1) \notin \mu$ and $(i + 1, j) \notin \mu$. Define
\[
G^{\lambda,\mu} = \sum_{(i, j) \in \lambda/\mu} b_{ij}(\sigma) D_{\sigma_{ij}}, \quad G^\lambda = G^{\lambda/\phi}.
\]
\[
H^{\lambda,\mu} = G^{\lambda,\mu} + V_{\lambda,\mu}(\sigma), \quad V_{\lambda,\mu}(\sigma) = \sum_{(i, j) \in C(\mu)} \sigma_{i+1,j} \sigma_{i,j+1} + \sum_{i=1}^{l(\mu)} \beta_{i+1,\mu,i+1,\mu+1}.
\]

In proving the following theorem, we will also show that $H^{\lambda,\mu} A_{\lambda,\mu} = 0$, so that the corresponding Doob transform
\[
L^{\lambda,\mu} = A_{\lambda,\mu}^{-1} \circ H^{\lambda,\mu} \circ A_{\lambda,\mu}(\sigma)
\]
generates a Markov chain on $\Pi^{\lambda/\mu}$.

Theorem 5.1. Let $\pi(t), t \geq 0$ be a Markov chain on $\Pi^{\lambda}$ with generator $G^\lambda$ and initial law $K_{\lambda}^{\lambda,\mu}$, for some $\mu \subset \lambda^\circ$ and $\sigma \in \Pi^{\lambda/\mu}$. Then $\sigma(t) = \pi(t)|_{\lambda/\mu}$ is a Markov chain on $\Pi^{\lambda/\mu}$ with generator $L^{\lambda,\mu}$ and, for $t > 0$, the conditional law of $\pi(t)$ given
\[
\{\sigma(s), s \leq t\} \text{ is } K_{\pi(t)}^{\lambda,\mu}.
\]
Proof. For notational convenience, let us fix \( \lambda \) and \( \mu \subset \lambda^\circ \) and write \( W = W_{\lambda,\mu} \), \( H = H^{\lambda,\mu} \) and \( V = V_{\lambda,\mu} \). For functions \( f \) on \( \Pi^\lambda \), define

\[
(\Lambda f)(\sigma) = \sum_{\pi \in \Pi^\lambda} W(\pi)f(\pi).
\]

We will show that

\[
(22) \quad H \circ \Lambda = \Lambda \circ G^\lambda,
\]

which implies both \( HA_{\lambda,\mu} = 0 \) and the statement of the theorem.

For \( \pi \in \Pi^\lambda \) and \( v \in \lambda/\mu \),

\[
[L_{\sigma_v}W(\pi)]b_v(\sigma) = W(\pi)b_v(\pi).
\]

Thus,

\[
[H(\Lambda f)](\sigma) = \sum_{\pi \in \text{RPP}_\sigma(\lambda)} \left( [HW(\pi)]f(\pi) + \sum_{v \in \lambda/\mu} [L_{\sigma_v}W(\pi)]b_v(\sigma)D_{\pi_v}f(\pi) \right)
\]

\[
= \sum_{\pi \in \text{RPP}_\sigma(\lambda)} \left( [HW(\pi)]f(\pi) + \sum_{v \in \lambda/\mu} W(\pi)b_v(\sigma)D_{\pi_v}f(\pi) \right).
\]

To complete the proof, we will show that, for \( \pi \in \Pi^\lambda \),

\[
(23) \quad HW(\pi) = G^\mu W(\pi),
\]

where

\[
G^\mu = \sum_{u \in \mu} B_{\pi_u} \circ b_u(\pi),
\]

and \( B_n \) denotes the forward difference operator \( B_nf(n) = f(n+1) - f(n) \).

For \( u = (i,j) \in \mu \), define

\[
b'_u(\pi) = (\pi_{i,j+1} - \pi_{i,j})(\pi_{i+1,j} - \pi_{ij} + \beta_{i+1,j}),
\]

and note that

\[
B_{\pi_u}(b_u(\pi)W(\pi)) = [b'_u(\pi) - b_u(\pi)]W(\pi).
\]

On the other hand, for \( \pi \in \Pi^\lambda \) and \( v \in \lambda/\mu \),

\[
b_v(\sigma)D_{\pi_v}W(\pi) = [b_v(\pi) - b_v(\sigma)]W(\pi).
\]

Thus \( (23) \) reduces to the identity, for \( \pi \in \Pi^\lambda \),

\[
(24) \quad \sum_{v \in \lambda/\mu} [b_v(\pi) - b_v(\sigma)] + V(\sigma) = \sum_{u \in \mu} [b'_u(\pi) - b_u(\pi)].
\]

For \( u = (i,j) \), let us write \( u \to v \) is \( v \) is either \( (i + 1, j) \) or \( (i, j + 1) \). For \( (i,j) \in \mu \),

\[
b'_{ij}(\pi) - b_{ij}(\pi) = \pi_{i+1,j}\pi_{i,j+1} - \pi_{ij}\pi_{i+1,j} - \pi_{ij}\pi_{i,j+1} + \pi_{ij}\pi_{i-1,j} + \pi_{ij}\pi_{i,j-1} - \pi_{i-1,j}\pi_{i,j-1} - \beta_{ij}(\pi_{ij} - \pi_{i,j-1}) + \beta_{i+1,j}(\pi_{i,j+1} - \pi_{ij}).
\]

Summing over \( (i,j) \in \mu \) gives

\[
\sum_{u \in \mu} [b'_u(\pi) - b_u(\pi)] = \sum_{(i,j) \in \mu} \pi_{i+1,j}\pi_{i,j+1} - \sum_{u \in \mu, v \in \lambda/\mu} \pi_u \pi_v - T(\pi) + U(\pi),
\]
where
\[ T(\pi) = \sum_{i=1}^{l(\mu)} \beta_i \pi_i \pi_{i+1} \quad \text{and} \quad U(\pi) = \sum_{i=1}^{l(\mu)} \beta_i \pi_i \pi_{i+1}. \]

On the other hand,
\[
\sum_{v \in \lambda/\mu} [b_v(\pi) - b_v(\sigma)] = \sum_{(i,j) \in \lambda/\mu} \pi_{i-1,j} \pi_{i,j-1} - \sum_{u \in \mu, v \in \lambda/\mu} \pi_u \pi_v - T(\pi)
\]
\[
= \sum_{(i,j) \in \mu} \pi_{i+1,j} \pi_{i,j+1} - \sum_{(i,j) \in C(\mu)} \pi_{i+1,j} \pi_{i,j+1} - \sum_{u \in \mu, v \in \lambda/\mu} \pi_u \pi_v - T(\pi)
\]
\[
= \sum_{(i,j) \in \mu} \pi_{i+1,j} \pi_{i,j+1} - \sum_{u \in \mu, v \in \lambda/\mu} \pi_u \pi_v - T(\pi) + U(\pi) - V(\sigma),
\]
as required.

**Remark 5.1.** The main content of Theorem 5.1 is when \( \tilde{\mu} = \lambda \). In this case, when \( \alpha \equiv 0 \), the numbers \( A_{\lambda,\mu}(\sigma) \) are, up to a trivial factor, the coefficients of the hypergeometric series of the partial flag manifold corresponding to \( \mu \) as defined in [3, Definition 5.1.5], see also Theorem 5.1.6]

**Remark 5.2.** Theorem 5.1 may be generalised to allow general integer-valued \( \alpha_i \) by modifying the state spaces \( \Pi^\lambda \) and \( \Pi^{\lambda/\mu} \) as in the type \( A_r \) case, see [3]. The basic symmetry observed there also extend naturally to the general setting.

**Proposition 5.2.** Let \( \pi \) be distributed according to \( K^\lambda_{\mu} \), where \( \sigma \in \Pi^{\lambda/\mu} \). Suppose that in the limit as \( N \to \infty \), \( \sigma/N \to a \), where \( a = (a_{ij}) \in (\mathbb{R}_{>0})^{\lambda/\mu} \). Then, in the same limit, \( \pi/N \to x^a \) in probability, where \( x^a = (x_{ij}^a) \in (\mathbb{R}_{>0})^{\lambda} \) is the unique solution to the equations
\[
(25) \quad (x_{i+1,j} - x_{ij})(x_{ij} + x_{ij+1} - x_{ij}) = (x_{ij} - x_{i-1,j})(x_{ij} - x_{ij+1}), \quad (i,j) \in \mu
\]
satisfying \( x_{ij} = a_{ij} \) for \( (i,j) \in \lambda/\mu \), \( 0 \leq x_{ij} \leq x_{i+1,j} + x_{i,j+1} \) for \( (i,j) \in \mu \), and with the convention \( x_{i,0} = x_{0,j} = 0 \).

**Proof.** The proof is similar to that of [23, Theorem 10.2], see also [21, Lemma 5.4]. Let \( X_a \) be the set of \( x = (x_{ij}) \in \mathbb{R}_{>0}^{\lambda} \) satisfying \( x_{ij} = a_{ij} \) for \( (i,j) \in \lambda/\mu \) and \( x_{ij} \leq x_{i+1,j} \land x_{i,j+1} \) for \( (i,j) \in \mu \). By Stirling’s formula, for \( b \in \mathbb{Z}_+ \),
\[
\lim_{N \to \infty} \frac{1}{N} \log \left( \frac{yN + b}{xN} \right) = -y h\left( \frac{x}{y} \right),
\]
uniformly on any compact set \( 0 \leq x \leq y \leq K \), where
\[
h(p) = p \log p + (1 - p) \log (1 - p).
\]

Thus,
\[
(26) \quad \lim_{\pi/N \to x} \frac{1}{N} \log W_{\lambda,\mu}(\pi) = -F(x),
\]
uniformly for \( x \in X_a \), where
\[
F(x) = \sum_{(i,j) \in \mu} \left[ x_{i+1,j} h\left( \frac{x_{ij}}{x_{i+1,j}} \right) + x_{i,j+1} h\left( \frac{x_{ij}}{x_{i,j+1}} \right) \right].
\]
We will show that $F$ is strictly convex on $(X_n)^\circ$. Let us write $u \to v$ if $u = (i, j)$ and $v$ is either $(i + 1, j)$ or $(i, j + 1)$. In this notation,

$$F(x) = \sum_{u \in \mu} \sum_{v \in \lambda; u \to v} x_{uv} \left( \frac{x_u}{x_v} \right).$$

Suppose that $x \in (X_n)^\circ$ so that $x_u > 0$ for all $u \in \lambda$ and $x_v > x_u$ for all $u, v \in \lambda$ with $u \to v$. For $u \in \mu$, we compute

$$\partial_{x_u} F(x) = \sum_{t \to u} \log \left( 1 - \frac{x_t}{x_u} \right) - \sum_{v \in \lambda; u \to v} \log \left( \frac{x_v}{x_u} - 1 \right),$$

$$\partial_{x_u}^2 F(x) = \sum_{t \to u} \frac{x_t}{x_u} - x_t + \sum_{v \in \lambda; u \to v} \frac{x_v}{x_u} - x_u.$$

For $u, v \in \mu$ with $u \to v$,

$$\partial_{x_u} \partial_{x_v} F(x) = -\frac{1}{x_v - x_u}.$$

If $u, v \in \mu$ and neither $u \to v$ nor $v \to u$, then $\partial_{x_u} \partial_{x_v} F(x) = 0$. The associated quadratic form is thus given by

$$\sum_{u, v \in \mu} \xi_u \xi_v \partial_{x_u} \partial_{x_v} F(x) = \sum_{u, v \in \lambda; u \to v} \frac{1}{x_u - x_v} \left( \frac{x_v}{x_u} \xi_u^2 + \frac{x_u}{x_v} \xi_v^2 - 2 \xi_u \xi_v \right),$$

with the convention $\xi_u = 0$ for $u \in \lambda/\mu$. This is clearly non-negative and, moreover, vanishes if, and only if,

$$\left( \frac{x_v}{x_u} \xi_u^2 + \frac{x_u}{x_v} \xi_v^2 - 2 \xi_u \xi_v \right) = \left( \sqrt{\frac{x_v}{x_u}} \xi_u - \sqrt{\frac{x_u}{x_v}} \xi_v \right)^2 = 0,$$

or, equivalently, $\xi_u/x_u = \xi_v/x_v$, for all $u, v \in \lambda$ with $u \to v$. Recalling that $\xi_v = 0$ for $v \in \lambda/\mu$, this implies that $\xi_u = 0$ for all $u \in \lambda$. The Hessian is therefore positive definite on $(X_n)^\circ$.

Now, $F$ is continuous on the compact set $X_n$ and therefore has at least one global minimiser $x^a \in X_n$. If $x^a \in \partial X_n$ then, since $a_v > 0$ for $v \in \lambda/\mu$, there exists $u = (i, j) \in \mu$ such that either

$$x^a_{i-1,j} \vee x^a_{i,j+1} = x^a_{ij} < x^a_{i+1,j} \wedge x^a_{i,j+1}$$

or

$$x^a_{i-1,j} \vee x^a_{i,j+1} < x^a_{ij} = x^a_{i+1,j} \wedge x^a_{i,j+1},$$

with the convention $x^a_{ij} = 0$ for $(i, j) \notin \mathbb{N}^2$. In the first case, $\partial_{x^a} F(x^a) = -\infty$ and in the second case $\partial_{x^a} F(x^a) = +\infty$, both contradicting the minimising property of $x^a$. Thus $x^a \in (X_n)^\circ$ and, by the strict convexity of $F$ on $(X_n)^\circ$, it is the unique minimiser and hence also the unique solution in $(X_n)^\circ$ to the critical point equations (26). Finally, it is easy to see that the critical point equations cannot hold at a point on the boundary $\partial X_n$, since $a_v > 0$ for $v \in \lambda/\mu$.

Since $F$ is continuous and the limit (26) holds uniformly for $x \in X_n$, it follows that the sequence $\pi/N$ satisfies a large deviation principle in $X_n$ with rate function $I_\alpha(x) = F(x) - F(x^a)$, hence the statement of the proposition. □
For $\rho, \pi \in \Pi^\lambda$, write $\rho \leq \pi$ if $\rho_{ij} \leq \pi_{ij}$, for all $(i,j) \in \lambda/\mu$. Let $\Pi^\lambda \star$ denote the set of $\pi \in \Pi^\lambda$ whose entries may take the value $+\infty$ while still respecting the required inequalities $\pi_{ij} \geq \pi_{i-1,j-1} \vee (\pi_{i-1,j} - \beta_{ij})$. We will write $\pi \to +\infty$ (resp. $\pi = +\infty$) to mean $\pi_{ij} \to +\infty$ (resp. $\pi_{ij} = +\infty$) for all $(i,j) \in \lambda/\mu$.

**Theorem 5.3.** The Markov chain on $\Pi^\lambda$ with generator $G^\lambda$ has a unique entrance law starting from $\pi = +\infty$. Moreover, under this entrance law, for each $\mu \subset \lambda^\circ$, $\sigma(t) = \pi(t)|_{\lambda/\mu}$ is a Markov chain on $\Pi^{\lambda/\mu}$ with generator $L^{\lambda/\mu}$ and, for all $t > 0$, the conditional law of $\pi(t)$, given $\{\sigma(s) , s \leq t\}$, is $K^{\lambda/\mu}_\sigma(t)$.

**Proof.** The proof of the first claim is in two steps. First we note the following monotonicity property. Given two different starting positions $\rho(0), \pi(0) \in \Pi^\lambda$ with $\rho(0) \leq \pi(0)$, it is clear from the definition of $G^\lambda$ that we may construct a coupling between two realisations $\rho(t)$ and $\pi(t)$ of the Markov chain with generator $G^\lambda$ and these starting positions, such that, almost surely, $\rho(t) \leq \pi(t)$ for all $t > 0$. Indeed, we simply allow the jumps at $(i,j)$ to occur independently unless $\rho_{ij} = \pi_{ij}$, in which case we note that $b_{ij}(\rho) \geq b_{ij}(\pi)$ and couple the next jump so that either $\rho_{ij}$ decreases by one or both $\rho_{ij}$ and $\pi_{ij}$ decrease by one, thus preserving the order $\rho \leq \pi$. It follows that the law of the process $\pi(t)$, $t \geq 0$ is stochastically increasing in the initial position $\pi(0)$. We can therefore let $\pi(0) \to +\infty$ in $\Pi^\lambda$ to obtain a unique (in law) limiting process $\pi(t)$, $t \geq 0$ in $\Pi^{\lambda/\mu}$. It only remains to show that $\pi(t) \to +\infty$ almost surely. This is straightforward, by induction on $\lambda$.

For the second claim, let $\tau, \sigma \in \Pi^{\lambda/\mu}$ with $\tau \leq \sigma$. By Theorem 5.1, we can construct random starting positions $\rho(0), \pi(0)$ taking values in $\Pi^\lambda$ such that: the distribution of $\pi(0)$ is $K^\lambda_\sigma$; the distribution of $\rho(0)$ is $K^\lambda_\tau$; $\rho(0) \leq \pi(0)$, almost surely. Indeed, we start with a realisation $\pi(0)$ distributed according to $K^\lambda_\sigma$; then let this evolve according to $G^\lambda$ for a fixed time $s > 0$ and condition on its restriction to $\lambda/\mu$ being $\tau$ at time $s$. By Theorem 7.1, this construction has the required properties. The monotonicity property therefore extends to such random initial conditions, as follows: the law of the process $\pi(t)$, $t \geq 0$, with initial distribution $K^\lambda_\sigma$, is stochastically increasing in $\sigma$. By uniqueness of the entrance law, it remains to show that if $\pi$ is distributed according to $K^\lambda_\sigma$, then $\pi \to +\infty$ in probability as $\sigma \to +\infty$. Since $K^\lambda_\sigma$ is stochastically increasing in $\sigma$, it suffices to show this if, say, $N \to +\infty$ and $\sigma_{ij}/N \to 1$ for all $(i,j) \in \lambda/\mu$. This follows from Proposition 5.2. $\Box$

6. Extensions to other root systems

6.1. **Type $B_r$**. Define $H^B_r = n^2 D_n/2$ and, for $r \geq 2$,

$$H^{B_r} = \sum_{i=1}^{r-1} n_i^2 D_{ni} + \frac{1}{2} n_i^2 D_{nr} + \sum_{i=1}^{r-1} n_i n_{i+1}.$$ 

Denote by $B_r(n)$ the unique solution to $H^{B_r} B_r = 0$ on $\mathbb{Z}_+^r$ with $B_r(0) = 1$. The numbers $B_r(n)$ are, up to a trivial factor, coefficients of a fundamental Whittaker function associated with the group $SO_{2r+1}(\mathbb{R})$. A recursive (over $r$) formula for these coefficients is given in [14], and may be interpreted as providing a formula for $B_r(n)$ as a sum over reverse plane partitions, as follows.

Let $\delta^r_i$ denote the shifted staircase shape

$$\delta^r_i = \{(i,j) : 1 \leq i \leq j \leq 2r - i\}.$$
For \( \pi \in \text{RPP}(\delta'_r) \), define
\[
W_{B_r}(\pi) = \prod_{i=1}^{r-1} \left( \frac{\pi_{i+1,i+1}}{\pi_{i,i}} \right) \prod_{(i,j) \in \delta'_r} \left( \frac{\pi_{i,j}}{\pi_{i,j-1}} \right) \left( \frac{\pi_{i,j}}{\pi_{i-1,j}} \right),
\]
with the convention \( \pi_{i,0} = \pi_{0,j} = 0 \). For \( n \in \mathbb{Z}_+^r \), denote by \( \text{RPP}_n(\delta'_r) \) the set of \( \pi \in \text{RPP}(\delta'_r) \) with \( \pi_{i,2r-i} = n_i, i = 1, \ldots, r \). In this notation, Theorem 3.1 of [14] yields the formula
\[
(27) \quad B_r(n) = \sum_{\pi \in \text{RPP}_n(\delta'_r)} W_{B_r}(\pi).
\]
Since \( H' B_r = 0 \) and \( B_r > 0 \) on \( \mathbb{Z}_+^r \), we may consider the Doob transform
\[
L^{B_r} = B_r(n)^{-1} \circ H' B_r \circ B_r(n) = \sum_{i=1}^{r-1} \frac{B_r(n - e_i)}{B_r(n)} n_i D_n + \frac{1}{2} \frac{B_r(n - e_r)}{B_r(n)} n_r^2 D_n.
\]
For \( \pi \in \text{RPP}(\delta'_r) \), we define
\[
b_{ij}(\pi) = \begin{cases} 
(\pi_{ij} - \pi_{i-1,j})(\pi_{ij} - \pi_{i,j-1}) & i \neq j \\
(\pi_{ii} - \pi_{i-1,i})(\pi_{ii} - \pi_{i-1,i-1})/2 & i = j
\end{cases}
\]
with the convention \( \pi_{i,0} = \pi_{0,j} = 0 \), and set
\[
G^{B_r} = \sum_{(i,j) \in \delta'_r} b_{ij}(\pi) D_n.
\]
For \( n \in \mathbb{Z}_+^r \), let \( K^{B_r}_n \) be the probability distribution on \( \text{RPP}_n(\delta'_r) \) defined by \( K^{B_r}_n(\pi) = W_{B_r}(\pi)/B_r(n) \).

**Theorem 6.1.** Let \( \pi(t), t \geq 0 \) be a Markov chain on \( \text{RPP}(\delta'_r) \) with generator \( G^{B_r} \) and initial distribution \( K^{B_r}_n \), for some \( n \in \mathbb{Z}_+^r \). Then
\[
N(t) = (\pi_{1,2r-1}(t), \ldots, \pi_{r,r}(t))
\]
is a Markov chain on \( \mathbb{Z}_+^r \) with generator \( L^{B_r} \) and, for all \( t > 0 \), the conditional law of \( \pi(t) \) given \( \{N(s), s \leq t\} \) is \( K^{B_r}_{N(t)} \).

**Proof.** For \( n \in \mathbb{Z}_+^r \) and functions \( f \) on \( \text{RPP}(\delta'_r) \), define
\[
(\Lambda_{B_r} f)(n) = \sum_{\pi \in \text{RPP}_n(\delta'_r)} W_{B_r}(\pi) f(\pi).
\]
We will show that \( H^{B_r} \circ \Lambda_{B_r} = \Lambda_{B_r} \circ G^{B_r} \), from which the statement of the theorem follows. Note that this intertwining relation also implies \( H^{B_r} B_r = 0 \), for \( B_r \) defined by (27). The proof is similar to the proof of Theorem 5.1.

For \( n \in \mathbb{Z}_+^r \), let \( b_{i,2r-i}(n) = n_i^2, i = 1, \ldots, r - 1 \) and \( b_{r,r}(n) = n_r^2/2 \). Note that
\[
H^{B_r} = \sum_{i=1}^{r} b_{i,2r-i}(n) D_n + \sum_{i=1}^{r-1} n_i n_{i+1}.
\]
For \( \pi \in \text{RPP}_n(\delta'_r) \) and \( i = 1, \ldots, r \), we have
\[
[L_n, W_{B_r}(\pi)] b_{i,2r-i}(n) = W_{B_r}(\pi) b_{i,2r-i}(\pi).
\]
Thus, as in the type $A$ case, it suffices to show that, for $\pi \in \text{RPP}_n(\delta'_r)$,

\begin{equation}
H^{B_r}W_{B_r}(\pi) = \sum_{1 \leq i \leq j < 2r - i} B_{\pi_{ij}}(b_{ij}(\pi))W_{B_r}(\pi).
\end{equation}

For $u = (i, j)$, with $1 \leq i \leq j < 2r - i$, define

\[ b'_u(\pi) = \begin{cases} 
(\pi_{i+1,j} - \pi_{ij})(\pi_{i,j+1} - \pi_{ij}) & i \neq j \\
(\pi_{i+1,i} - \pi_{ii})(\pi_{ii+1} - \pi_{ii})/2 & i = j
\end{cases} \]

and note that $B_{\pi_u}(b_u(\pi))W_{B_r}(\pi) = [b'_u(\pi) - b_u(\pi)]W_{B_r}(\pi)$.

On the other hand, for $\pi \in \text{RPP}_n(\delta'_r)$ and $i = 1, \ldots, r$,

\[ b_{i,2r-i}(n)D_{n}W_{B_r}(\pi) = [b_{i,2r-i}(\pi) - b_{i,2r-i}(n)]W_{B_r}(\pi). \]

Thus (28) reduces to the identity, for $\pi \in \text{RPP}_n(\delta'_r)$,

\[
\sum_{i=1}^{r} [b_{i,2r-i}(\pi) - b_{i,2r-i}(n)] + \sum_{i=1}^{r-1} n_i n_{i+1} = \sum_{1 \leq i \leq j < 2r - i} [b'_{ij}(\pi) - b_{ij}(\pi)].
\]

This is readily verified, as in the type $A$ case. \hfill \Box

**Remark 6.1.** The diagonal values $b_n = B_2(n, n)$ are the Apéry numbers

\[ b_n = \sum_k \binom{n}{k}^2 \binom{n+k}{k} \]

associated with $\zeta(2)$. This sequence satisfies the recurrence

\[ n^2 b_n = (11n^2 - 11n + 3)b_{n-1} + (n - 1)^2 b_{n-2}, \]

with $b_0 = 1$ and $b_1 = 3$. We note that these Apéry numbers are also given, in the notation of the previous section, by the diagonal values $A_{(2,2,1),(1,1)}(n,n,n)$.

### 6.2. Type $BC_r$

There is a more refined structure incorporating types $BC_r$ which naturally interpolate, via intertwining relations, between the root systems of type $B_r$. The intertwining relations we discuss here are analogous to those presented in [17] in the context of class one Whittaker functions.

For example, if we let

\[ H_k^{BC_1} = \frac{1}{2}kD_k + \frac{1}{2}k(k-1)D_k^{(2)} \]

where $D_k^{(2)}f(k) = f(k-2) - f(k)$, and

\[ Q_{BC_1}^{B_1}(n,k) = 2^{-n} \binom{n}{k}, \]

then one can easily check that

\[ H^{B_1} \circ Q_{BC_1}^{B_1} = Q_{BC_1}^{B_1} \circ H^{BC_1}. \]

This intertwining relation yields the following. Let $E = \{n \geq k \geq 0\}$. 


Proposition 6.2. Let \((X_t, Y_t)\) be a Markov chain on \(E\) with generator

\[
G = H^{BC_1}_k + \frac{1}{2}n(n-k)D_n.
\]

Suppose \(X_0 = n\) and \(Y_0 \sim \text{Binomial}(n,1/2)\). Then \(X_t\) is a Markov chain on \(\mathbb{Z}_+\) with generator \(H^{B_1} = n^2D_n/2\) and, for all \(t > 0\), the conditional law of \(Y_t\) given \(X_s\), \(0 \leq s \leq t\), is Binomial\((X_t,1/2)\).

For \(r = 2\), we define

\[
H^{BC_2} = n^2D_n + \frac{1}{2}mD_m + \frac{1}{2}m(m-1)D_m^{(2)} + nm,
\]

and denote by \(BC_2(n,m)\) the unique solution to \(H^{BC_2} = 0\) on \(\mathbb{Z}_+^2\) with \(BC_2(0,0) = 1\). The numbers \(BC_2(n,m)\) are positive integers and given by

\[
BC_2(n,m) = \sum_k \binom{n}{k} \binom{m}{k} 2^k.
\]

These are the Delannoy numbers. They also satisfy \(\tilde{H}^{BC_2} = 0\), where

\[
\tilde{H}^{BC_2} = \frac{1}{2}n^2D_n + \frac{1}{2}m^2D_m + \frac{1}{2}nmD_{n,m} + nm,
\]

and \(D_{n,m}f(n,m) = f(n-1,m-1) - f(n,m)\). These claims follow from the intertwining relations

\[
H^{BC_2} \circ Q = Q \circ H^{B_1}, \quad \tilde{H}^{BC_2} \circ Q = Q \circ H^{B_1}, \quad Q((n,m),k) = \binom{n}{k} \binom{m}{k} 2^k.
\]

Denote the corresponding Doob transforms by

\[
L^{BC_2} = BC_2(n,m)^{-1} \circ H^{BC_2} \circ BC_2(n,m),
\]

\[
\tilde{L}^{BC_2} = BC_2(n,m)^{-1} \circ \tilde{H}^{BC_2} \circ BC_2(n,m).
\]

Let \(P\) be the set of \((n,m,k)\) in \(\mathbb{Z}_+^3\) satisfying \(0 \leq k \leq n \land m\) and let \(P_{n,m}\) be the set of \((a,b,c)\) in \(P\) with \(a = n\) and \(b = m\). Denote by \(K^{BC_2}_{n,m}\) the probability distribution supported on \(P_{n,m}\) and defined by \(K^{BC_2}_{n,m}(n,m,k) = Q((n,m),k)/BC_2(n,m)\). Let

\[
G = n(n-k)D_n + \frac{1}{2}(m-k)D_m + \frac{1}{2}(m-k)(m-k-1)D_m^{(2)} + \frac{1}{2}k^2D_k,
\]

\[
\tilde{G} = \frac{1}{2}n(n-k)D_n + \frac{1}{2}m(m-k)D_m + (n-k)(m-k)D_{n,m} + \frac{1}{2}k^2D_k.
\]

The above intertwining relations yield the following.

Proposition 6.3. Suppose that \(X = (X_1, X_2, X_3)\) is a Markov chain in \(P\) with initial law \(K^{BC_2}_{n,m}\) and generator \(G\) (resp. \(\tilde{G}\)). Then \((X_1, X_2)\) is a Markov chain with generator \(L^{BC_2}\) (resp. \(\tilde{L}^{BC_2}\)) and, for all \(t > 0\), in both cases, the conditional law of \(X(t)\) given \((X_1(s), X_2(s))\), \(0 \leq s \leq t\), is \(K^{BC_2}_{X_1(t),X_2(t)}\).
6.3. Type $G_2$. Let

\[ H^{G_2} = n^2D_n + 3m^2D_m + 3nm \]

and denote by $G_2(n,m)$ the solution to $H^{G_2}G_2 = 0$ on $\mathbb{Z}_2^6$ with $G_2(0,0) = 1$. The numbers $G_2(n,m)/(n!m!)^2$ are the series coefficients of the fundamental Whittaker function, with index zero, associated with the group $G_2(\mathbb{R})$ [13, 14].

Let $\Pi$ denote the set of $(n,m,i,j,k,l) \in \mathbb{Z}_2^6$ satisfying

\[ k \leq i \land j, \quad i \lor j \leq l, \quad l \leq n \land m, \quad i + j \leq n. \]

For $(n,m,i,j,k,l) \in \Pi$, set

\[ W(n,m,i,j,k,l) = \binom{n}{i,j} \binom{m}{l} \binom{m}{i} \binom{m}{j} \binom{l}{i} \binom{l}{j} \binom{i}{k} \binom{j}{k}. \]

For functions $f$ on $\Pi$, define

\[ (\Lambda f)(n,m) = \sum_{i,j,k,l} W(n,m,i,j,k,l) f(n,m,i,j,k,l). \]

Let

\[ G = (n-l)(n-i-j)D_n + 3m(m-l)D_m + 3(l-i)(l-j)D_l + i(i-k)D_i + j(j-k)D_j + k^2D_k. \]

Then one can check that

\[ H^{G_2} \circ \Lambda = \Lambda \circ G. \]

This immediately yields the binomial sum formula

\[ G_2(n,m) = \sum_{i,j,k,l} W(n,m,i,j,k,l), \]

which may be simplified to obtain, for example,

\[ G_2(n,m) = \sum_{i,j} \binom{n}{i} \binom{m}{j} \binom{m}{i} \binom{m}{j} \binom{n+m-i-j}{m} \binom{i+j}{m}. \]

One can check that this agrees with [14, Theorem 5.1].

Since $H^{G_2}G_2 = 0$ and $G_2 > 0$ on $\mathbb{Z}_2^6$, the corresponding Doob transform

\[ L^{G_2} = G_2(n,m)^{-1} \circ H^{G_2} \circ G_2(n,m) \]

generates a Markov chain on $\mathbb{Z}_2^6$. For $(n,m) \in \mathbb{Z}_2^6$, let $K_{n,m}$ denote the probability distribution on $\Pi$ which is supported on the set of $(p_1,p_2,\ldots,p_6) \in \Pi$ with $p_1 = n$ and $p_2 = m$, and defined on this set by

\[ K_{n,m}(n,m,i,j,k,l) = W(n,m,i,j,k,l)/G_2(n,m). \]

The intertwining relation (29) then yields the following.

**Proposition 6.4.** Let $X = (X_1, X_2, \ldots, X_6)$ be a Markov chain on $\Pi$ with initial law $K_{n,m}$ and generator $G$. Then $(X_1, X_2)$ is a Markov chain on $\mathbb{Z}_2^2$ with generator $L^{G_2}$ and moreover, for all $t > 0$, the conditional law of $X(t)$, given $\{X_1(s), X_2(s) : 0 \leq s \leq t\}$, is $K_{X_1(t),X_2(t)}^{G_2}$. 

Remark 6.2. The coefficient $G_2(n, m)$ is the constant term of $P^n Q^m$, where

$$P = \frac{(1 + x + y + xz)(xz + yz + yw)}{xyz}, \quad Q = \frac{(1 + y + z + w)}{w}. $$

This is easily verified using (30). One can also check that the Newton polyhedron of the Laurent polynomial $f = PQ$ is reflexive. Setting $x = z_1/z_0, y = z_2/z_0, z = z_3/z_0$ and $w = z_4/z_0$, the equation $f = \psi$ may be written as

$$(z_0^2 + z_0 z_1 + z_0 z_2 + z_1 z_3)(z_1 z_4 + z_2 z_3 + z_2 z_4)(z_0 + z_2 + z_3 + z_4) = \psi z_0 z_1 z_2 z_3 z_4,$$

and defines a family of quintic Calabi-Yau threefolds in $\mathbb{P}^4$. We note that the constant term series coefficients of $f$, given by the diagonal values

$$G_2(n, n) = \sum_{i,j} \binom{n}{i}^2 \binom{n}{j}^2 \binom{i+j}{j} (2n-i-j),$$

agree with the holomorphic function coefficients associated with the Calabi-Yau equation listed as #212 in the database [1].

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**School of Mathematics and Statistics**  
**University College Dublin**  
**Dublin 4, Ireland**