APPLICATIONS OF $p$-ADIC ANALYSIS FOR BOUNDING PERIODS OF SUBVARIETIES UNDER ´ETALE MAPS

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Abstract. Using methods of $p$-adic analysis we give a different proof of Burnside’s problem for automorphisms of quasiprojective varieties $X$ defined over a field of characteristic 0. More precisely, we show that any finitely generated torsion subgroup of $\text{Aut}(X)$ is finite. In particular this yields effective bounds for the size of torsion of any semiabelian variety over a finitely generated field of characteristic 0. More generally, we obtain effective bounds for the length of the orbit of a preperiodic subvariety $Y \subset X$ under the action of an ´etale endomorphism of $X$.

1. Introduction

In [MS94], Morton and Silverman conjecture that there is a constant $C(N, d, D)$ such that for any morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d$ defined over a number field $K$ with $[K : \mathbb{Q}] \leq D$, the number of preperiodic points of $f$ over $K$ is less than or equal to $C(N, d, D)$. This conjecture remains very much open, but in the case where $f$ has good reduction at a prime $p$, a great deal has been proved about bounds depending on $p$, $N$, $d$, $D$ (see [Zieg, Pez05, Hut09]).

In this paper, we study the more general problem of bounding periods of subvarieties of any dimension. We prove the following.

**Theorem 1.1.** Let $K$ be a finite extension of $\mathbb{Q}_p$, let $\mathfrak{o}_v$ be the ring of integers of $K$, let $k_v$ be its residue field and let $e$ be the ramification index of $K/\mathbb{Q}_p$. Let $\mathcal{X}$ be a smooth $\mathfrak{o}_v$-scheme whose generic fiber $\mathcal{X}$ has dimension $g$, let $\Phi : \mathcal{X} \to \mathcal{X}$ be ´etale, let $\mathcal{Y}$ be a subvariety of $\mathcal{X}$, and assume there is a point on $\mathcal{Y}(\mathfrak{o}_v)$ which is smooth on the generic fiber of $\mathcal{Y}$. If $\mathcal{Y}$ is preperiodic under the action of $\Phi$, then the length of its orbit is bounded by $p^{1+r} \cdot \# \text{GL}_g(k_v) \cdot \# \mathcal{X}(k_v)$, where $\mathcal{X}$ is the special fiber of $\mathcal{X}$, and $r$ is the smallest nonnegative integer larger than $(\log(e) - \log(p-1))/\log(2)$.

Theorem 1.1 is proved using $p$-adic analytic parametrization of forward orbits under the action of an ´etale endomorphism of a quasiprojective variety. The same method can be used to study finitely generated torsion subgroups of $\text{Aut}(X)$, when $X$ is a quasi-projective variety defined over a field $K$ of characteristic zero. Theorem 1.1 gives an upper bound on the size of the largest finitely generated torsion subgroup in $\text{Aut}(X)$ when $X$ has a smooth model over a finite extension of the $p$-adic integers; the bound depends only on the dimension of $X$ and on the number of points in the special fiber of this model. This gives rise to a new proof of the following theorem of Bass and Lubotzky [BL83].

**Theorem 1.2.** Let $X$ be a geometrically irreducible quasiprojective variety defined over a field of characteristic 0. Then each finitely generated torsion subgroup $H$ of $\text{Aut}(X)$ is finite.
Theorem 1.2 shows in particular that the Burnside problem has an affirmative
solution for automorphism groups of quasiprojective varieties. We recall that the
Burnside problem is said to have a positive answer for a group $G$ if every finitely
generated torsion subgroup of $G$ is finite. The first substantial result in this area
was due to Burnside (c.f. [Lam01, §9]), who showed that if $H$ is a (not necessarily
finitely generated) torsion subgroup of $\text{GL}_n(\mathbb{C})$ of exponent $d$ then the order of
$H$ could be bounded in terms of $d$ and $n$. Using a specialization argument and
applying Burnside’s result, Schur [Sch11] later showed that every finitely generated
torsion subgroup of $\text{GL}_n(\mathbb{C})$ is finite. Proofs of geometric Burnside-type results
generally proceed along similar lines as that of the Burnside-Schur theorem: one
first uses specialization to reduce to the case that the base field is a finitely generated
extension of the prime field; one then shows that in this case a torsion subgroup
necessarily has bounded exponent and is finite. An interesting problem that arises
naturally is to then bound the exponent and the order of a torsion subgroup
of $\text{Aut}(X)$ in terms of geometric data and the field $k$ of definition for $X$ when $k$ is
a finitely generated extension of $\mathbb{Q}$. Some work in this direction has been done
by Serre [Ser09], who gave sharp upper-bounds on the sizes of torsion subgroups
of the group of birational transformations of $\mathbb{P}^2(k)$ when $k$ is a finitely generated
extension of $\mathbb{Q}$.

The $p$-adic analytic parametrization of forward orbits under the action of an
automorphism of a quasiprojective variety can be used in different directions as
well. We can prove the following result.

Theorem 1.3. Let $X$ be an irreducible quasiprojective variety defined over $\mathbb{Q}$ of
dimension larger than 1, and let $\Phi$ be an automorphism of $X$ for which there exists
no nonconstant $f \in \mathbb{Q}(X)$ such that $f \circ \Phi = f$. Then there exists a codimension-2
subvariety $Y$ whose orbit under $\Phi$ is Zariski dense in $X$.

For any subvariety $Y$, its orbit under $\Phi$ is the union of all $\Phi^n(Y)$ for $n \in \mathbb{N}$.

Theorem 1.3 yields positive evidence to two conjectures in arithmetic geometry.
On one hand, we have the potential density question, i.e. describe the class of vari-
eties $X$ defined over $\mathbb{Q}$ for which there exists a number field $K$ such that $X(K)$ is
Zariski dense in $X$ (see [ABR11, BT00, HT00] for various results regarding poten-
tially dense varieties). Our Theorem 1.3 yields that if $Y$ is potentially dense, then
$X$ is potentially dense. On the other hand, if $X$ is a surface, then Theorem 1.3
yields the existence of a point $x \in X(\mathbb{Q})$ whose orbit is Zariski dense in $X$ (note that
in this case, $Y$ is a finite collection of points and since $X$ is irreducible, we obtain
that a single orbit under $\Phi$ must be Zariski dense in $X$). Hence, Theorem 1.3 yields
a positive answer for automorphisms of surfaces for the following conjecture (pro-
posed independently by Amerik, Bogomolov and Rovinsky [ABR11], and Medvedev
and Scanlon [MS]).

Conjecture 1.4. Let $X$ be a quasiprojective variety defined over an algebraically
closed field $K$ of characteristic 0. Let $\Phi : X \rightarrow X$ be an endomorphism defined
over $K$ such that there exists no positive dimensional variety $Y$ and no dominant
rational map $\Psi : X \rightarrow Y$ such that $\Psi \circ \Phi = \Psi$ generically. Then there exists
$x \in X(K)$ such that $O_\Phi(x)$ is Zariski dense in $X$.

Alternatively one can formulate the hypothesis in Conjecture 1.4 by asking that
$\Phi$ does not preserve a rational fibration, i.e. there exists no nonconstant $f \in K(X)$
such that $f \circ \Phi = f$. It is immediate to see that if $\Phi$ preserves a rational fibration,
then there is no point with Zariski dense orbit under \( \Phi \). Conjecture 1.4 strengthens a conjecture of Zhang, which was also the motivation for both Amerik, Bogomolov and Rovinsky, respectively for Medvedev and Scanlon for formulating the above Conjecture 1.4. Motivated by the dynamics of endomorphisms on abelian varieties, Zhang [Zha06] proposed the following question: given a projective variety \( X \) defined over a number field \( K \) endowed with a polarizable endomorphism \( \Phi \), is there a point \( x \in X(\overline{K}) \) whose orbit under \( \Phi \) is Zariski dense in \( X \)? We say that \( \Phi \) is polarizable if there exists an ample line bundle \( L \) on \( X \) such that \( \Phi^*(L) = L^{\otimes d} \) (in \( \text{Pic}(X) \)) for some integer \( d > 1 \).

In [AC08], Amerik and Campana prove Conjecture 1.4 for projective varieties of trivial canonical bundle defined over an uncountable field \( K \). However, if \( K = \overline{\mathbb{Q}} \), then the problem seems much more difficult. Only recently, Medvedev and Scanlon [MS] proved Conjecture 1.4 when \( \Phi = (f_1, \ldots, f_N) \) is an endomorphism of \( \mathbb{A}^N \) given by \( N \) one-variable polynomials \( f_i \) defined over \( \overline{\mathbb{Q}} \). Also, Junyi Jun [Jun, Theorem 1.4] proved Conjecture 1.4 for birational maps on projective surfaces. Finally, connected to Conjecture 1.4 we mention Amerik’s result [Ame11] who proved (using the \( p \)-adic approach introduced in [BGT10]) that most orbits of algebraic points are infinite under the action of an arbitrary rational self-map (of infinite order).

Our proof of Theorem 1.3 uses a result (see Theorem 4.1) which gives an upper bound for the period of codimension-1 subvarieties of \( X \) which are periodic under \( \Phi \); in particular this yields that the union of all periodic hypersurfaces is Zariski closed. We note that Cantat [Can10, Theorem A] proved a similar bound for the number of periodic hypersurfaces under the stronger hypothesis that there exist no nonconstant rational function \( f \) and no constant \( \alpha \) such that \( f \circ \Phi = \alpha \cdot f \). Our Theorem 1.3 yields that each periodic subvariety with a point over some complete \( v \)-adic field has bounded period. So, if \( Y \) is a codimension-2 subvariety of \( X \) which is neither periodic, nor contained in one of the finitely many codimension-1 periodic subvarieties, then its orbit under \( \Phi \) is Zariski dense. Using the same approach, it is immediate to get the existence of codimension-1 subvarieties with a Zariski dense orbit in \( X \).

Using the hypothesis that \( X \) contains a Zariski dense orbit, and also using Vojta’s proof of the Mordell-Lang Theorem for semialbelian varieties (see [Voj96]) we obtain the following stronger bound for the number and period of codimension-1 periodic subvarieties.

**Theorem 1.5.** Let \( X \) be a quasi-projective variety and let \( \sigma : X \to X \) be an automorphism defined over a number field \( K \) and suppose that there is a point \( x \in X(K) \) such that the orbit of \( x \) under \( \sigma \) is Zariski dense in \( X \). Then any \( \sigma \)-invariant closed subset \( W \) of \( X \) has at most \( \dim X - h^1(Y, \mathcal{O}_Y) + \rho(Y) \) geometric components of codimension one, where \( Y \) is a projective closure of \( X \) and \( \rho(Y) \) is the Picard number of \( Y \).

Of course, Conjecture 1.4 for an automorphism \( \sigma : X \to X \) follows immediately whenever one knows that the union of all \( \sigma \)-invariant subvarieties is Zariski closed. Hence, the following equivalence is of interest here.

**Definition 1.6.** Let \( X \) be a quasi-projective variety over a field \( K \) and let \( \sigma : X \to X \) be an automorphism of \( X \). We say that \((X, \sigma)\) satisfies the geometric Dixmier-Moeglin equivalence if the following are equivalent for each \( \sigma \)-stable subvariety \( Y \) of \( X \):
(1) there exists a point $y \in Y$ such that $\{\sigma^n(y) : n \in \mathbb{Z}\}$ is Zariski dense in $Y$;
(2) the union of all proper $\sigma$-invariant subvarieties of $Y$ is Zariski closed;
(3) there does not exist a non-trivial $f \in k(Y)$ such that $f \circ \sigma = f$.

We note that the geometric Dixmier-Moeglin equivalence does not hold in general—for example, there are Hénon maps of $\mathbb{A}^2$ with the property that (3) holds but (2) does not (c.f. Devaney and Nitecki [DN79] and Bedford and Smillie [BS04, Theorem 1])—but it is conjectured to hold when $X$ is smooth and projective and $\sigma$ has zero entropy. As before, for $X$ a complex variety, we have the implications (2) $\implies$ (1) $\implies$ (3) [BRS10]. Theorem 1.2 proves that the equivalences from Definition 1.6 hold for any surface.

Here is the plan of our paper: in Section 2 we prove some preliminary results (see Proposition 2.1) for rigid analytic functions, which we use then in Section 3 for proving Theorems 1.1 and 1.2 and their corollaries. In Section 4 we find an upper bound for the period of codimension-1 periodic subvarieties under the action of an automorphism $\Phi$ of a quasiprojective variety which does not preserve a rational fibration (see Theorem 4.1). In Section 5, using Theorem 4.1, we prove Theorem 1.3 and Theorem 1.5. Finally, we conclude with Section 6 in which we discuss related questions (in the spirit of Poonen’s conjectures [Poo11]) about uniform boundedness for periods of points in algebraic families of endomorphisms.

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2. Nonarchimedean analysis

2.1. Power series. The setup for this section is as follows: $p$ is a prime number, $K_v/\mathbb{Q}_p$ is a finite extension, while the $v$-adic norm $| \cdot |_v$ satisfies $|p|_v = 1/p = |p|^{1/e}$ (i.e., $e$ is the ramification index for this extension). We let $\mathfrak{o}_v$ be the ring of $v$-adic integers of $K_v$, let $\pi$ be a uniformizer of $\mathfrak{o}_v$, and we let $k_v$ be its residue field.

We let $g$ be a positive integer, and let $c$ be a positive real number. For two power series $F, G \in \mathfrak{o}_v[[z_1, \ldots, z_g]]$, we write $F \equiv G \pmod{p^c}$ if each coefficient $a_{\alpha}$ of $F - G$ satisfies $|a_{\alpha}|_v \leq p^{-c}$. Alternatively, for some $m \in \mathbb{N}$ we use the notation $F \equiv G \pmod{p^m}$ if $F - G \in \mathfrak{o}_v[[z_1, \ldots, z_g]]$. More generally, for power series $\mathcal{F} := (F_1, \ldots, F_g)$ and $\mathcal{G} := (G_1, \ldots, G_g)$ we write $\mathcal{F} \equiv \mathcal{G} \pmod{p^c}$ if $F_i \equiv G_i \pmod{p^c}$ for each $i$; similarly, $\mathcal{F} \equiv \mathcal{G} \pmod{p^m}$ if $F_i \equiv G_i \pmod{p^m}$ for each $i$.

Finally, for each $n \in \mathbb{N}$ we denote by $\mathbb{F}^n$ the composition of $\mathcal{F}$ with itself $n$ times.

We use the following result in Section 3.

Proposition 2.1. Let $C \in \mathfrak{o}_v^g$, let $L \in \text{GL}_g(\mathfrak{o}_v)$, and let $F_1, \ldots, F_g \in \mathfrak{o}_v[[z_1, \ldots, z_g]]$, such that for $z := (z_1, \ldots, z_g)$ we have

$$F(z) := (F_1, \ldots, F_g)(z) \equiv C + Lz \pmod{p^r}.$$ Let $m = p^{1+r} \cdot \# \text{GL}_g(k_v)$ where $r$ is any nonnegative integer larger than $(\log(c) - \log(p - 1))/\log(2)$. Then $\mathcal{F}^m(z) \equiv z \pmod{p^r}$ for some $c > 1/(p - 1)$.

Proof. Let $s := \# \text{GL}_g(k_v)$; then it is immediate that $\mathcal{F}^s(z) \equiv D + z \pmod{p^r}$ for some $D \in \mathfrak{o}_v^g$. So, $\mathcal{F}^r \equiv z \pmod{p^r}$. Hence we are left to show that if $\mathcal{F}(z) \equiv z \pmod{p^r}$ and if $r$ is the least nonnegative integer greater than $(\log(c) - \log(p - 1))/\log(2)$. Since $s > 1/(p - 1)$, we have $\mathcal{F}^s(z) \equiv D + z \pmod{p^r}$. Then $\mathcal{F}^m(z) \equiv z \pmod{p^r}$.
there exists a finite set \( X \) of number fields. For each quasiprojective variety \( X \): 

\[
\gamma \in \gamma
\]

now on, we assume \( e \geq p - 1 \).

We let \( F(z) = z + H(z) \), where each coefficient of \( H \) is in \( \pi \cdot o_v \). Then \( F'(z) = z + pH(z) + H_1(z) \), where \( H_1 = 0 \mod \pi^2 \). Thus \( F'(z) = z \mod \pi^2 \). By induction we obtain that

\[
F'(z) \equiv z \mod \pi^{\min\{e, 1\}}
\]

So, if \( r > (\log(e) - \log(p - 1)) / (\log(2)) \) then \( |\pi|^{2r} = p^{-\frac{2r}{e-p-1}} < p^{-1} \), while \( |\pi|^{e+1} < |\pi| \), and so indeed

\[
F'(z) \equiv z \mod p^c
\]

which yields the desired conclusion. \( \square \)

2.2. Algebraic geometry. We need the following application of the implicit function theorem on Banach spaces.

**Proposition 2.2.** Let \((K_v, |\cdot|_v)\) be a finite extension of \( \mathbb{Q}_p \) with residue field \( k_v \), and let \( o_v \) be the ring of \( v \)-adic integers of \( K_v \). Let \( X \) be a quasiprojective variety defined over \( K_v \), let \( \mathcal{X} \) be a \( o_v \)-scheme whose generic fiber is isomorphic to \( X \), let \( r : \mathcal{X}(K_v) \to \overline{\mathcal{X}}(k_v) \) be the usual reduction map to the special fiber \( \overline{\mathcal{X}} \) of \( \mathcal{X} \), and let \( \iota : \mathcal{X}(o_v) \to X(K_v) \) be the usual map coming from base extension. Let \( \alpha \in \mathcal{X}(o_v) \) such that \( \iota(\alpha) \) is a smooth point on \( X \) and let \( U_{\alpha} = \{ \beta \in \mathcal{X}(o_v) : r(\alpha) = r(\beta) \} \) and let \( U = \iota(U_{\alpha}) \). Then \( U \) is Zariski dense in \( X \).

**Proof.** Let \( x = \iota(\alpha) \). We consider an affine chart containing the point \( x \in X \) after viewing \( X \) as a subset of the \( n \)-dimensional projective space defined over \( K_v \). So, letting \( d = \dim(X) \), then there exist \((n-d)\) polynomials \( f_i \), which we may suppose are defined over \( o_v \) in \( n \) variables \( z_1, \ldots, z_n \) such that locally at \( x \) the variety \( X \) is the zero set of the polynomials \( f_i \). Furthermore, since \( x \) is a nonsingular point for \( X \), the Jacobian matrix \( (df_i/\text{d}z_j)_{1 \leq i,j \leq n-d} \) has rank \( n-d \). Without loss of generality we may assume the minor \( (df_i/\text{d}z_j)_{1 \leq i,j \leq n-d} \) is invertible.

We let \( x = (x_1, \ldots, x_n) \) be the coordinates of the point \( x \) in the above affine chart; each \( x_i \in o_v \). Then \( U_{\alpha} \) is identified with points \((z_1, \ldots, z_n) \in o_v^n \) such that \( z_i \equiv x_i \mod \pi_v \) for \( \pi_v \) a generator for the maximal ideal in \( o_v \). Using the Implicit Function Theorem (see [Lan99, Theorem 5.9, page 19]), we see that there exists a sufficiently small \( p \)-adic neighborhood \( U_0 \) of \((x_{n-d+1}, \ldots, x_n)\), there exists a \( p \)-adic neighborhood \( V_0 \) of \((x_1, \ldots, x_{n-d})\), and there exists a \( p \)-adic analytic function \( g : U_0 \to V_0 \) such that \( g(x_{n-d+1}, \ldots, x_n) = (x_1, \ldots, x_{n-d}) \) and moreover for each \( \gamma \in U_0 \) we have \( (g(\gamma), \gamma) \in X(K_v) \). Furthermore, at the expense of shrinking both \( U_0 \) and \( V_0 \) we may assume that for each \( \gamma \in U_0 \), the point \((g(\gamma), \gamma) \) is in \( U \). Since \( U_0 \subset k^d \) is a \( d \)-dimensional \( K_v \)-manifold we conclude that \( U \) is Zariski dense in \( X \). \( \square \)

The following result is a consequence of Proposition 2.2 for varieties defined over number fields. For each quasiprojective variety \( X \) defined over a number field \( K \), there exists a finite set \( S \) of places (containing all archimedean places) and there exists a \( o_{K,S} \)-scheme \( \mathcal{X} \) whose generic fiber is isomorphic to \( X \) (where \( o_{K,S} \) is the
subring containing all $u \in K$ such that $|u|_v \leq 1$ for all $v \notin S$). In particular, we can prove the following result for $(\mathfrak{o}_K)_v$-schemes, where $(\mathfrak{o}_K)_v$ is the localization at the nonarchimedean place $v$ of the ring of algebraic integers $\mathfrak{o}_K$ of $K$.

**Proposition 2.3.** Let $K$ be a number field, let $v$ be a nonarchimedean place of $K$, and let $(\mathfrak{o}_K)_v$ be the localization of $\mathfrak{o}_K$ at the place $v$. Let $X$ be a quasiprojective variety defined over $K$, and let $\mathcal{X}$ be an $(\mathfrak{o}_K)_v$-scheme whose generic fiber is isomorphic to $X$, let $r : \mathcal{X}(\mathfrak{o}_K)_v \rightarrow \mathcal{X}(K_v)$ be the usual reduction map, and let $\iota : \mathcal{X}(\mathfrak{o}_K)_v \rightarrow X(K)$ be the usual map coming from base extension. Let $\alpha \in \mathcal{X}(\mathfrak{o}_K)_v$ such that $\iota(\alpha)$ is a smooth point on $X$, let and let $U$ be the set of all $y \in X(K)$ such that the Zariski closure of $y$ intersects $\mathcal{X}$ at $\alpha_v$. Then $U$ is Zariski dense in $X$.

**Proof.** Let $K_v$ be the completion of $K$ with respect to $| \cdot |_v$, and let $\mathfrak{o}_v$ be the ring of $v$-adic integers of $K_v$. Then using Proposition 2.2 there exists a set $U_1 \subset X_\mathfrak{o}_v(\mathfrak{o}_v)$ whose intersection with the generic fiber $X_{K_v}$ is a Zariski dense subset of $X_{K_v}$ (where $X_{\mathfrak{o}_v}$ is the base extension of $X$ to Spec$(\mathfrak{o}_v)$, while $X_{K_v}$ is its generic fiber).

We identify $U_1$ with its intersection with the generic fiber $X_{K_v}$. Arguing as in the proof of Proposition 2.2 we consider a system of coordinates for an affine subset $X_1 \subset X$ containing $x$ (also defined over $K$), and find an open set $U_0 \subset \mathcal{A}^d(K_v)$ and a $v$-adic analytic function $g : U_0 \rightarrow K_v^{n-d}$ such that for each $z \in U_0$, we have $(g(z), z) \in X_1(K_v) \subset X(K_v)$. Furthermore, for each such point $(g(\gamma), \gamma) \in X(K_v)$ there exists a section $\beta$ of $X_{\mathfrak{o}_v}$ whose intersection with the special fiber is $\mathfrak{o}_v$, while its intersection with the generic fiber is $(g(\gamma), \gamma)$.

We let $\pi : X_1 \rightarrow \mathcal{A}^d$ be the projection on the last $d$ coordinates. Then using the Fiber Dimension Theorem [Sha94, Section 6.3] we conclude that there exists an open Zariski subset $U_2 \subseteq \mathcal{A}^d$ such that for each $\gamma \in U_2(K) \cap U_0$, the fiber $\pi^{-1}(\gamma)$ is a $K$-variety of dimension 0 (here we use that $X$ and also $X_1$ are defined over $K$).

Since $U_0 \subset \mathcal{A}^d$ is a $d$-dimensional $K_v$-manifold and $U_2$ is the complement in $\mathcal{A}^d$ of a proper algebraic subvariety defined over $K$, we conclude that $U_2(K) \cap U_0$ is Zariski dense in $\mathcal{A}^d$. For each $\gamma \in U_2(K) \cap U_0$, we have

$$(g(\gamma), \gamma) \in U_1 \cap \pi^{-1}(\gamma) \subset U_1 \cap X_1(K).$$

Let $h$ denote the map from $U_2(K) \cap U_0$ to $X_1$ sending $\gamma$ to $(g(\gamma), \gamma)$. Then the dimension of the closure of $U_2(K) \cap U_0$ is equal to the dimension of the closure of $h(U_2(K) \cap U_0)$ since $\pi \circ h$ is the identity on $U_2(K) \cap U_0$ and $\pi$ is finite-to-one on $h(U_2(K) \cap U_0)$. Since this dimension is $d$, which is also the dimension of $X_1$, we see that $h(U_2(K) \cap U_0) \subseteq U$ is Zariski dense in $X_1$ and thus $U$ is Zariski dense in $X$.

\[\square\]

3. Burnside’s problem

In this section we continue with the notation from Section 2 for $g$, $p$, $(K_v, | \cdot |_v)$, $\mathfrak{o}_v$, $\pi$, $k_v$, $e$ and $r$. In addition, assume $p > 2$.

Our first result gives an upper bound for the size of torsion of the automorphism group of a quasiprojective variety $X$ defined over a local field. So, our setup is as follows: for a $\mathfrak{o}_v$-scheme $\mathcal{X}$, we let $\overline{\mathcal{X}}$ be its special fiber (over $k_v$). For a point $\alpha \in \mathcal{X}(\mathfrak{o}_v)$, we let its residue class $\mathcal{U}_\overline{\alpha} = \{ \beta \in \mathcal{X}(\mathfrak{o}_v) : \overline{\beta} = \overline{\alpha} \}$, where $\overline{\gamma} \in \overline{\mathcal{X}}(k_v)$ is the reduction modulo $v$ of $\gamma \in \mathcal{X}(\mathfrak{o}_v)$. Finally, we note that if $\overline{\alpha}$ is a smooth point, then each $\beta \in \mathcal{U}_\overline{\alpha}$ is also a smooth point.
Theorem 3.1. Let $\mathcal{X}$ be a $\mathfrak{o}_v$-scheme whose generic fiber is a $K$-variety of dimension $g$, and let $G \subseteq \text{Aut}(\mathcal{X})$ be a torsion group. If $\mathcal{X}(\mathfrak{o}_v)$ contains a smooth point, then $G$ is finite and $\#G \leq (\#k_v)^{g(1+\epsilon)-(g+\epsilon+1)} \cdot \#GL_g(k_v) \cdot \#\mathcal{X}(k_v)$.

Proof. We let $\alpha \in \mathcal{X}(k_v)$ be a smooth point and let $G_\alpha$ be the subgroup of $G$ consisting of all $\sigma$ such that $\sigma\bar{\alpha} = \bar{\alpha}$. Since $[G : G_\alpha] \leq \#\mathcal{X}(k_v)$, it will suffice to bound the size of $G_\alpha$.

Let $\mathcal{O}_\alpha$ denote the local ring of $\mathcal{X}$ at $\alpha$, let $\mathfrak{m}_\alpha$ denote its maximal ideal, and let $\tilde{\mathcal{O}}_\alpha$ denote the completion of $\mathcal{O}_\alpha$ at $\mathfrak{m}_\alpha$. Since $\alpha \in \mathcal{X}$ is smooth, the quotient $\tilde{\mathcal{O}}_\alpha/(\pi)$ is also regular. By the Cohen structure theorem for regular local rings (see [Coh46, Theorem 9] or [Mat86, Theorem 29.7]), the quotient ring $\tilde{\mathcal{O}}_\alpha/(\pi)$ can be written as formal power series $k_v[[y_1, \ldots, y_g]]$. Choosing $z_i \in \mathfrak{m}_v$ for $i = 1, \ldots, g$ such that the residue class of each $z_i$ is equal to $y_i$, we obtain a minimal basis $(\pi, z_1, \ldots, z_g)$ for $\mathfrak{m}_v$ (see [Coh46]). Thus, we see that $\tilde{\mathcal{O}}_\alpha$ is naturally isomorphic to a formal power series ring $\mathfrak{o}_v[[z_1, \ldots, z_g]]$.

Arguing exactly as in the proof of [BGT10, Proposition 2.2] we then obtain that there is a $v$-adic analytic isomorphism $\iota : U_\alpha \to \sigma^g_v$, such that for any $\sigma \in G_\alpha$, there are power series $F_1, \ldots, F_g \in \sigma_v[[z_1, \ldots, z_g]]$ with the properties that

(i) each $F_i$ converges on $\sigma^g_v$;
(ii) for all $(\beta_1, \ldots, \beta_g) \in \sigma^g_v$, we have

$$\iota(\sigma^{-1}(\beta_1, \ldots, \beta_g)) = (F_1(\beta_1, \ldots, \beta_g), \ldots, F_g(\beta_1, \ldots, \beta_g));$$

(iii) each $F_i$ is congruent to a linear polynomial mod $v$ (in other words, all the coefficients of terms of degree greater than one are in the maximal ideal $\mathfrak{m}_v$ of $\sigma_v$). Moreover, for each $i$, we have

$$F_i(z_1, \ldots, z_g) = \frac{1}{\pi} \cdot H_i(\pi z_1, \ldots, \pi z_g),$$

for some $H_i \in \sigma_v[[z_1, \ldots, z_g]]$.

Denoting $\tilde{\beta} = (\beta_1, \ldots, \beta_g)$ and $\sigma\alpha^{-1}$ as $F_\sigma$, we thus have

$$F_\sigma(\tilde{\beta}) \equiv C_\sigma + L_\sigma(\tilde{\beta}) \pmod{v}$$

for a $C_\sigma \in \sigma^g_v$ and a $g \times g$ matrix $L_\sigma$ with coefficients in $\sigma_v$. Let $\mathcal{T}_\sigma$ be the reduction of $L_\sigma$ modulo $\pi$. Since $\sigma$ is an étale map of $\mathfrak{o}_v$-schemes, $\mathcal{T}_\sigma$ must be invertible. We define $D_\alpha : G_\alpha \to \mathbb{G}_a^g(k_v) \rtimes GL_g(k_v)$ by $D_\alpha(\sigma) = (\mathcal{T}_\sigma, \mathcal{L}_\sigma)$, where $\mathbb{G}_a^g(k_v) \rtimes GL_g(k_v)$ is the group of affine transformations of $k_v^g$.

We clearly have $F_{\sigma_1, \sigma_2} = F_{\sigma_1} F_{\sigma_2}$ for $\sigma_1, \sigma_2 \in G_\alpha$. Reducing modulo $\pi$, it follows from (3.1.2) that $D_\alpha(\sigma_1 \sigma_2) = D_\alpha(\sigma_1) D_\alpha(\sigma_2)$. Thus, $D_\alpha$ is a group homomorphism; let $G_{\alpha, 1}$ be the kernel of $D_\alpha$.

Next we bound $\#G_{\alpha, 1}$. We consider the map

$$E_\alpha : G_{\alpha, 1} \to Y_g := ((\mathfrak{o}_v/\pi^{e+1}\mathfrak{o}_v)[[z_1, \ldots, z_g]]/(z_1, \ldots, z_g)^{e+2})^g,$$

given by reducing each component of $F_\sigma \in G_{\alpha, 1}$ modulo $\pi^{e+1}$. Using property (iii) above we observe that $E_\alpha$ is indeed well-defined and that it satisfies $E_\alpha(\sigma_1 \sigma_2) = E_\alpha(\sigma_1) E_\alpha(\sigma_2)$. Furthermore, because each $E_\alpha(\sigma)$ for $\sigma \in G_{\alpha, 1}$ is an invertible power series, we conclude that $E_\alpha$ restricts to a group homomorphism from $G_{\alpha, 1}$ into the subgroup of units $F$ of $Y_g$ (with respect to the composition of functions) which satisfy the congruence $F(z) \equiv z \pmod{\pi}$. Since this group of units has at
most \((\#k_e)c^{g\left(\frac{s+t+1}{s}\right)}\) elements, we are left to show that if \(\sigma \in \ker E_\alpha\), then \(\sigma\) is the identity.

Indeed, if \(\mathcal{F}_\sigma(z) \equiv z \pmod{\pi^{t+1}}\) for each \(z \in \mathfrak{o}_e\), then \(\mathcal{F}_\sigma(\vec{\beta}) \equiv \vec{\beta} \pmod{p^c}\) for some \(c > 1/(p-1)\). Now fix \(\vec{\beta}\); by \cite{Poo13} Theorem 1, there are \(v\)-adic analytic power series \(\theta_1, \ldots, \theta_g \in \mathfrak{o}_v[z]\), convergent on \(\mathfrak{o}_v\), such that

\[
\mathcal{F}_\sigma^u(\vec{\beta}) = (\theta_1(u), \ldots, \theta_g(u))
\]

for all \(n \in \mathbb{N}\). Since \(\sigma\) has finite order, there is an \(N_\sigma\) such that \(\mathcal{F}_{N_\sigma}^N\) is the identity, we so have \(\theta_i(kN_\sigma) = \beta_i\) for all \(k \in \mathbb{N}\). Hence \(\theta_i(u) - \beta_i\) has infinitely many zeros \(u \in \mathfrak{o}_v\). Therefore, \(\theta_i(u) - \beta_i\) is identically zero since any nonzero convergent power series on \(\mathfrak{o}_v\) has finitely many zeros in \(\mathfrak{o}_v\). Thus, \(\mathcal{F}_\sigma(\vec{\beta}) = \vec{\beta}\) for all \(\vec{\beta} \in \mathfrak{o}_v\).

Hence, we have \(\sigma(z) = z\) for all \(z \in \mathcal{U}_v\). Since \(\mathcal{U}_v\) is Zariski dense in \(\mathcal{X}\) (they are both \(g\)-dimensional \(K_v\)-manifolds), we have that \(\sigma\) acts on identically on all of \(\mathcal{X}\). This concludes our proof. 

The following result is an immediate corollary of Theorem 3.1 since each torsion point of a semiabelian variety \(X\) induces a torsion element of \(\text{Aut}(X)\).

**Corollary 3.2.** Let \(X\) be a semiabelian \(\mathfrak{o}_v\)-scheme whose generic fiber has dimension \(g\). Then \(\#\mathcal{X}_{tor}(\mathfrak{o}_v) \leq (\#k_e)c^{g(1+c)\cdot \left(\frac{s+t+1}{s}\right)} \cdot \#\text{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)\).

If \(G\) is cyclic, then we can give a much better bound for \(\#G\). In fact, Theorem 3.1 yields an upper bound for the order of any \(\mathfrak{o}_v\)-subscheme \(\mathcal{Y}\) of \(\mathcal{X}\) which is preperiodic under the action of an étale endomorphism \(\Phi\) of \(\mathcal{X}\); this gives a higher dimensional generalization of the main result of Hutz \cite{Hut09}. We recall that \(r\) is the smallest nonnegative integer larger than \((\log(e) - \log(p-1))/\log(2)\), where \(e\) is the ramification index of \(K_v/Q_p\).

**Proof of Theorem 3.1.** We use the same setup as in the proof of Theorem 3.1. Let \(\beta \in \mathcal{Y}(\mathfrak{o}_v)\) be a smooth point on \(\mathcal{Y}\). Since \(\bar{\mathcal{X}}(k_v)\) is finite, there is an \(\ell \geq 0\) such that the residue class of \(\Phi^\ell(\beta)\) is periodic under \(\Phi\); we note this residue class as \(U_0\) and we denote \(\Phi^\ell(\mathcal{Y})\) as \(\mathcal{Y}'\). There is then an integer \(k\) such that \(\Phi^k(U_0) = U_0\) and \(k + \ell \leq \#\bar{\mathcal{X}}(k_v)\).

Since \(\Phi^\ell(\beta) \in \mathcal{Y}'(\mathfrak{O}_v) \cap U_0\) is a smooth point on the generic fiber of \(\mathcal{Y}'\), Proposition 2.1 yields that \(\mathcal{Y}'(\mathfrak{O}_v) \cap U_0\) is Zariski dense in \(\mathcal{Y}'\). Let \(x \in U_0 \cap \mathcal{Y}'(\mathfrak{O}_v)\), let \(m := p^{1+r} \cdot \#\text{GL}_g(k_v)\), and let \(\Psi := \Phi^mk\). Arguing as in the proof of Theorem 3.1 (note that in order to apply the strategy from \cite[BGT10 Proposition 2.2]{BGT10} we require that \(\Phi\) is étale and that \(x\) is smooth on \(\mathcal{X}\) only), and also applying Proposition 2.1, we obtain that \(\mathcal{Y}'(z) \equiv z \pmod{p^c}\) for some \(c > 1/(p-1)\). Hence, by \cite[Poo13 Theorem 1]{Poo13}, there exists a \(v\)-adic analytic function \(\mathcal{G}_{\Psi,x} : \mathfrak{o}_v \rightarrow U_0\) such that \(\mathcal{G}_{\Psi,x}(n) = \Psi^n(x)\).

Now, let \(F\) be a polynomial in the vanishing ideal of \(\mathcal{Y}'\). Because \(\mathcal{Y}'\) is periodic, there exists a positive integer \(N\) such that \(\Phi^N(\mathcal{Y}') = \mathcal{Y}'\), and thus \(F(\Phi^{nk}(x)) = 0\) for each \(n \in \mathbb{N}\). On the other hand, \(\mathcal{G}_{\Psi,x}(n) = \Phi^{nk}(x)\) and so, \(F(\mathcal{G}_{\Psi,x}(n)) = 0\) for all \(n \in \mathbb{N}\). Since a nonzero \(v\)-adic analytic function cannot have infinitely many zeros in \(\mathbb{N} \subset \mathfrak{o}_v\), we conclude that \(F(\mathcal{G}_{\Psi,x}(n)) = 0\) for all \(n \in \mathbb{N}\); in particular, \(F(\Phi^{mk}(x)) = 0\). Thus, \(\Phi^{mk}(x) \in \mathcal{Y}'\), and so \(\Phi^{km}(\mathcal{Y}') = \mathcal{Y}'\). Since \(k + \ell \leq \#\bar{\mathcal{X}}(k_v)\) and \(\Psi^{pm} = p^{1+r} \cdot \#\text{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)\), we have that the length of the orbit of \(\mathcal{Y}'\) under \(\Phi\) is bounded by \(km + \ell \leq m \cdot \#\bar{\mathcal{X}}(k_v) = p^{1+r} \cdot \#\text{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)\). 

The following two results are simple consequences of Theorem 3.1.
Corollary 3.3. Let $X$ be a $k_v$-scheme whose generic fiber $X_\kbar$ has dimension $g$, let $\Phi : X \to X$ be étale, and let $\alpha \in X(k_v)$ be a smooth preperiodic point. Then the length of its orbit is bounded by $p^{1+r} \cdot \# \text{GL}_g(k_v) \cdot \# \overline{X}(k_v)$.

Corollary 3.4. Let $X$ be a semiabelian $k_v$-scheme whose generic fiber has dimension $g$. Then each torsion point of $X(k_v)$ has order bounded above by $p^{1+r} \cdot \# \text{GL}_g(k_v) \cdot \# X(k_v)$.

Our arguments above allow us to show that for any field $K$ of characteristic 0, and for any finitely generated extension $L/K$, then each finitely generated torsion subgroup of $\text{Aut}(L)$ fixing $K$ is finite, i.e., Burnside’s problem has a positive answer. At the expense of replacing $L$ by a finite extension and then viewing $L$ as the function field of a geometrically irreducible quasiprojective variety defined over $K$, we obtain the geometric formulation of the Burnside problem from Theorem 1.2.

Proof of Theorem 1.2. Let $\sigma_1, \ldots, \sigma_m$ be a finite set of generators for $H$, and let $K$ be a finitely generated field such that $X, \sigma_1, \ldots, \sigma_m$ are all defined over $K$. After passing to a finite extension of the base, we may assume that $X(K)$ contains a smooth point $\alpha$. Let $R$ be a finitely generated $\mathbb{Z}$-algebra containing all the coefficients of all the polynomials defining $X$ in some projective space, along with all the coefficients of all the polynomials defining all the $\sigma_i$ locally, as in the proof of [BGT10, Theorem 4.1]. By [BGT10, Proposition 4.3], since a finite intersection of dense open subsets is dense, we see that there is a dense open subset $U$ of $\text{Spec} R$ such that:

(i) there is a scheme $X_U$ that is quasiprojective over $U$, and whose generic fiber equals $X$;
(ii) each fiber of $X_U$ is geometrically irreducible;
(iii) each $\sigma_i$ extends to an automorphism $\sigma_i|_U$ of $X_U$; and
(iv) $\alpha$ extends to a smooth section $U \to X_U$.

Now, arguing as in [BGT10, Proposition 4.4], and using [Bel06, Lemma 3.1], we see that there is an embedding of $R$ into $\mathbb{Z}_p$ (for some prime $p \geq 5$), and a $\mathbb{Z}_p$-scheme $X_{\mathbb{Z}_p}$ such that

(i) $X_{\mathbb{Z}_p}$ is quasiprojective over $\mathbb{Z}_p$, and its generic fiber equals $X$;
(ii) both the generic and the special fiber of $X_{\mathbb{Z}_p}$ are geometrically irreducible;
(iii) each $\sigma_i$ extends to an automorphism $(\sigma_i)_{\mathbb{Z}_p}$ of $X_{\mathbb{Z}_p}$; and
(iv) $\alpha$ extends to a smooth section $\text{Spec} \mathbb{Z}_p \to X_{\mathbb{Z}_p}$.

Then Theorem 3.1 finishes our proof. □

4. Bounds on the Number of Periodic Hypersurfaces

In this section we give explicit bounds on the number of $\sigma$-periodic hypersurfaces when $\sigma$ is an automorphism of an irreducible quasi-projective variety $X$ which preserves no rational fibration. In particular, we show the number of $\sigma$-periodic hypersurfaces is finite unless there exists a nonconstant rational function $f$ such that $f \circ \sigma = f$. Moreover, we are able to give a bound for both the lengths of periods and the number of $\sigma$-periodic hypersurfaces in terms of geometric data, although this bound depends upon the field of definition for $\sigma$. We note that Cantat [Can10, Theorem B] proved there exists a bound $N(\sigma)$ (depending on $\sigma$) such that if there exist more than $N(\sigma)$ irreducible periodic hypersurfaces, then $\sigma$ must preserve a nonconstant rational fibration. In the case that $\sigma$ is defined over a number field
K and there is a point \( x \in X(K) \) with a dense orbit under \( \sigma \), we are able to give a bound that depends only upon the dimension of \( X \) and the Picard number of a projective closure (see Theorem 1.5). We begin with a lemma about ranks of multiplicative subgroups of a field that are stable under an automorphism of the field. As a matter of notation, for an automorphism \( \sigma \) of a field \( K \), we denote by \( K^\sigma \) the set of all fixed points of \( \sigma \).

**Proposition 4.1.** Let \( k \) be an algebraically closed of characteristic zero and let \( K \) be a finitely generated field extension of \( k \). Suppose that \( \sigma : K \to K \) is a \( k \)-algebra automorphism with \( K^\sigma = k \). If \( G \) is a finitely generated \( \sigma \)-invariant subgroup of \( K^* \) then the rank of \( G/(G \cap k^*) \) is at most \( \text{trdeg}_k(K) \).

**Proof.** Suppose, towards a contradiction, that the rank of \( G/(G \cap k^*) \) is \( m > \text{trdeg}_k(K) \) and suppose that \( x_1, \ldots, x_m \) are elements of \( G \) whose images in \( G/(G \cap k^*) \) generate a free abelian group of rank \( m \). Then there is some nonzero polynomial \( P(t_1, \ldots, t_m) \in k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \) such that \( P(x_1, \ldots, x_m) = 0 \). We write \( P \) as

\[
\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} t_1^{j_1} \cdots t_m^{j_m}
\]

and we let

\[
N := \# \{(j_1, \ldots, j_m) : c_{j_1, \ldots, j_m} \neq 0\}.
\]

We may take \( P \) so that \( N > 1 \) is minimal. By multiplying \( P \) by an appropriate monomial and nonzero constant, we may also assume that the constant coefficient of \( P \) is equal to one. Then we have

\[
P(\sigma^i(x_1), \ldots, \sigma^i(x_m)) = 0
\]

for all \( i \in \mathbb{Z} \). In other words, for each integer \( i \),

\[
(z_{j_1, \ldots, j_m})_{(j_1, \ldots, j_m)} = (\sigma^i(x_1^{j_1} \cdots x_m^{j_m}))_{(j_1, \ldots, j_m)} \in G^N
\]

is a solution to the \( S \)-unit equation

\[
\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} z_{j_1, \ldots, j_m} = 0.
\]

By minimality of \( N \), each of these solutions is primitive; that is, no proper subsum vanishes. (If some proper non-trivial subsum of

\[
\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} \sigma^i(x_1)^{j_1} \cdots \sigma^i(x_m)^{j_m}
\]

vanished for some \( i \), then we could apply \( \sigma^{-i} \) to this subsum and get a smaller \( N \), contradicting minimality.) By the theory of \( S \)-unit equations for fields of characteristic zero (see Evertse et al. [ESS02]), we know there are only finitely many primitive solutions in \( G^N \) to the equation

\[
\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} z_{j_1, \ldots, j_m} = 0
\]

up to multiplication by elements of \( G \). It follows that there is some \( M > 0 \) and some \( y \in G \) such that

\[
\sigma^M(x_1^{j_1} \cdots x_m^{j_m}) = y x_1^{j_1} \cdots x_m^{j_m}
\]

whenever \( c_{j_1, \ldots, j_m} \neq 0 \). Since \( c_{0, \ldots, 0} \neq 0 \), we see that \( y = 1 \). Thus if we pick \( (j_1, \ldots, j_m) \neq (0, \ldots, 0) \) with \( c_{j_1, \ldots, j_m} \neq 0 \) then \( \sigma^M \) fixes \( x_1^{j_1} \cdots x_m^{j_m} \), which by assumption is not in \( k^* \), and so \( \sigma^M \) has a fixed field of transcendence degree at
least one over $k$. Since the fixed field of $\sigma^K$ is a finite extension of the fixed field of $\sigma$, we see that the fixed field of $\sigma$ has transcendence degree at least one over $k$, a contradiction. The result follows. □

As a corollary, we obtain the following result.

**Theorem 4.1.** Let $K$ be a finitely generated extension of $\mathbb{Q}$ and let $X$ be an irreducible quasi-projective variety defined over $K$. Then there exists a positive constant $N = N(X, K)$ such that whenever $\sigma \in \text{Aut}_K(X)$ has the property that there are no nonconstant $f, g \in \mathcal{K}(X)$ with $f \circ \sigma = g$ there are at most $N$ $\sigma$-periodic hypersurfaces and they all have period at most $N$. Moreover, $N$ can be taken to be $\text{rank}(\text{Cl}(\bar{X})) + \dim(X)$, where $\bar{X}$ is the normalization of $X$.

We note that when $Y$ is a normal quasi-projective variety over a finitely generated extension of $\mathbb{Q}$, we have $\text{Cl}(Y)$ has finite rank [BRS10, Lemma 5.6 (1)]. We will find it convenient to regard $K$ as a subfield of $\mathbb{C}$ throughout.

**Proof of Theorem 4.1.** It is no loss of generality to assume that $X$ is normal. Suppose that there is no nonconstant $f \in \mathcal{K}(X)$ with $f \circ \sigma = f$. Let $N := \text{rank}(\text{Cl}(X)) + \dim(X)$, and suppose that we have $N + 1$ distinct $\sigma$-periodic hypersurfaces $Y_0, \ldots, Y_N$. By replacing $\sigma$ by an iterate, we may assume that $\sigma(Y_i) = Y_i$ for all $i$. By relabeling if necessary, we may assume that there is some $m \leq N - \dim(X) - 1$ such that $[Y_0], \ldots, [Y_m]$ generate a free $\mathbb{Z}$-module of $\text{Cl}(Y)$ and that for $i > m$, $[Y_0], \ldots, [Y_m], [Y_i]$ are dependent in $\text{Cl}(X)$. This means that for $i \in \{N - \dim(X), \ldots, N\}$, there is a principal divisor $(f_i) = c_{i,i}[Y_i] + \sum_{j=0}^{m} c_{i,j}[Y_j]$, where the $c_{i,j}$ are integers and $c_{i,i}$ is nonzero. By construction, we have $f_i \circ \sigma$ has the same divisor as $f_i$ for $i = N - \dim(X), \ldots, N$. Also, the $f_i$ generate a free abelian subgroup of $\mathbb{C}(X)^*$, which can be seen by noting that the valuation on $\mathbb{C}(X)$ induced by $Y_i$, $\nu_{Y_i}$, has the property that $\forall i \in \{N - \dim(X), \ldots, N\}$ \ \nu_{Y_i}(f_i) = 0$ for $j \in \{N - \dim(X), \ldots, N\} \setminus \{i\}$.

Since $f_i \circ \sigma$ has the same divisor as $f_i$, we see that $f_i \circ \sigma / f_i$ is in $\Gamma(X, \mathcal{O}_X)^*$. Let $G$ denote the subgroup of $\mathbb{C}(X)^*/\mathbb{C}^*$ generated by $\Gamma(X, \mathcal{O}_X)^*/\mathbb{C}^*$ and by the images of the $f_i$. Then we have shown that the rank of $G$ is at least $\dim(X) + 1$. Moreover, $G$ is finitely generated since $\Gamma(X, \mathcal{O}_X)^*/\mathbb{C}^*$ is finitely generated [BRS10, Lemma 5.6 (2)]. Furthermore, $\sigma$ induces an automorphism of $G$ since $\Gamma(X, \mathcal{O}_X)^*/\mathbb{C}^*$ is closed under application of $\sigma$ and since $f_i \circ \sigma \in \Gamma(X, \mathcal{O}_X)^* f_i$. We now let $g_1, \ldots, g_s$ be elements of $\mathbb{C}(X)^*$ whose images in $\mathbb{C}(X)^*/\mathbb{C}^*$ generate $G$. Let $G_0$ denote the subgroup of $\mathbb{C}(X)^*$ generated by $g_1, \ldots, g_s$. Then there exist complex numbers $\lambda_1, \ldots, \lambda_s$ such that $g_i \circ \sigma = \lambda_i G_0$. Let $H$ denote the subgroup of $\mathbb{C}(X)^*$ generated by $G_0$ and by $\lambda_1, \ldots, \lambda_s$. Then $H$ is finitely generated and by construction we have $h \circ \sigma \in H$ for all $h \in H$. Furthermore, the rank of $H/(H \cap \mathbb{C}^*)$ is at least $\dim(X) + 1$, since its rank is at least as large as the rank of $G$. Lemma 4.1 gives a contradiction. The result follows. □

We note that for any complex variety $X$ with automorphism $\sigma$, there is some finitely generated extension $K$ of $\mathbb{Q}$ such that $X$ is defined over $K$ and such that $\sigma \in \text{Aut}_K(X)$ and so Theorem 4.1 can be applied using the value of $N(X, K)$ given in the statement of the theorem.

Also as a corollary of Theorem 4.1 we can prove that for any quasiprojective variety $X$ defined over $\mathbb{Q}$ under the action of an automorphism $\Phi$ which does not preserve a rational fibration, there exist non-periodic codimension-1 subvarieties
(defined over \( \overline{\mathbb{Q}} \)). Indeed, using Theorem 1.1 there exist finitely many codimension-
1 periodic subvarieties \( Y_i \); in addition, let \( N_1 \in \mathbb{N} \) such that each \( Y_i \) is fixed by \( \Phi^{N_1} \). So we can find an algebraic point \( x \in X(\overline{\mathbb{Q}}) \) which is not contained in the
above finitely many codimension-1 subvarieties \( Y_i \). Then we simply take \( Y \) to be the
intersection of \( X \) (inside some projective space) with a hyperplane (defined over \( \overline{\mathbb{Q}} \))
passing through \( x \), but not containing \( \Phi^{N_1}(x) \); then \( Y \) is not periodic (since if it
were, then it would be fixed by \( \Phi^{N_1} \) but on the other hand, \( \Phi^{N_1}(x) \notin Y(\overline{\mathbb{Q}}) \)), and
therefore its orbit under \( \Phi \) is Zariski dense in \( X \).

5. Subvarieties with Zariski dense orbits

The setup for this Section is as follows: \( X \) is a quasiprojective variety defined
over \( \mathbb{C} \), and \( \Phi \) is an automorphism of \( X \) which preserves no nonconstant rational
fibration. Our goal is to prove Theorem 1.3; we use Theorems 1.1 and 4.1.

Proof of Theorem 1.3. Arguing as before, for a suitable prime \( p \geq 5 \), we find a
\( \mathbb{Z}_p \)-scheme \( \mathcal{X} \) such that

(i) \( X \) is the generic fiber of \( \mathcal{X} \), while the special fiber \( \overline{\mathcal{X}} \) of \( \mathcal{X} \) is a geometrically
irreducible quasiprojective variety.

(ii) \( \Phi \) extends to an automorphism of \( \mathcal{X} \).

(iii) there exists \( x_0 \in \mathcal{X}(\mathbb{Z}_p) \) such that its reduction \( \overline{x_0} \) modulo \( p \) is a smooth
point of \( \overline{\mathcal{X}} \).

Let \( U_0 := \{ x \in \mathcal{X}(\mathbb{Z}_p) : \overline{x} = \overline{x_0} \} \) be the residue class of \( x_0 \) (since \( \overline{x_0} \) is a smooth
point on \( \overline{\mathcal{X}} \), then each \( x \in U_0 \) is also smooth on \( \mathcal{X} \)). Furthermore, we identify each section in \( U_0 \) with its intersection with the generic fiber \( X \). Using Theorem 1.1
there exists a positive integer \( N_1 \) such that each periodic subvariety \( Y \) which con-
ains a point from \( U_0 \) which is smooth also on \( Y \) has period bounded above by \( N_1 \).

By Theorem 4.1 there exist at most finitely many codimension-1 subvarieties
which are fixed by \( \Phi^{N_1} \). Let \( Y_1 \) be the union of all these codimension-1 subvarieties.
On the other hand, by the definition of \( N_1 \), if \( x \in U_0 \) is (pre)periodic then \( \Phi^{N_1}(x) = x \). Because \( \Phi \) has infinite order (since it preserves no nonconstant rational fibration),
the vanishing locus for the equation \( \Phi^{N_1}(x) = x \) is a proper subvariety \( Y_0 \) of \( \mathcal{X} \).
In conclusion, \( Y_0 \cup Y_1 \) is a proper subvariety of \( \mathcal{X} \) and therefore, there exists a
Zariski dense set of points \( x \in U_0 \setminus (Y \cup Y_1)(\mathbb{Z}_p) \) (because \( U_0 \) is a \( p \)-adic manifold of
dimension larger than \( \dim(Y_0 \cup Y_1) \)). Furthermore, we can choose \( x \in X(\overline{\mathbb{Q}}) \) by
Proposition 2.3 finally note that \( x \) is smooth since it is in \( U_0 \).

For each such point \( x \in X(\overline{\mathbb{Q}}) \cap U_0 \) which is not contained in \( Y_0 \cup Y_1 \), we can
find a codimension-2 subvariety \( Y \) (defined over \( \overline{\mathbb{Q}} \)) whose orbit under \( \Phi \) is Zariski
dense in \( X \). Indeed, we consider \( X \) embedded into a large projective space \( \mathbb{P}^m \) and
then intersect \( X \) with two (generic) hyperplane sections \( H_1 \) and \( H_2 \) (defined over
\( \overline{\mathbb{Q}} \)) which pass through \( x \), but not through \( \Phi^{N_1}(x) \) (note that \( \Phi^{N_1}(x) \neq x \) because \( x \notin Y_0 \)). Furthermore, since \( H_1 \) and \( H_2 \) are generic sections passing through \( x \),
then \( Y := X \cap H_1 \cap H_2 \) is a codimension-2 irreducible subvariety defined over \( \overline{\mathbb{Q}} \)
and moreover \( x \in Y \) is a smooth point. We claim that \( Y \) is not periodic under \( \Phi \). Otherwise since \( Y \) intersects \( U_0 \) then it must be fixed by \( \Phi^{N_1} \) (by Theorem 1.1
and our choice for \( N_1 \)). However, \( x \in Y \), but \( \Phi^{N_1}(x) \notin Y \), which shows that \( Y \) is
not fixed by \( \Phi^{N_1} \), and thus \( Y \) is not periodic under the action of \( \Phi \). Let \( Z \) be the
Zariski closure of the orbit of \( Y \) under the action of \( \Phi \). Since \( Y \) is not periodic under
positive dimensional periodic subvarieties. For example, if $f$ is a two-variable polynomial of degree $n$ because of Northcott's theorem, morphism $\Phi$ varies over families of degree-$n$ varieties rather than just the universal family of degree-$n$ varieties conjecture, where the morphisms vary across a general families of self-maps of $P$ that cannot happen in the case of preperiodic points of morphisms. Poonen's set-up, some fibers may have infinitely many preperiodic points. Although $\Phi$ is a dehomogenized one-variable polynomial in general.

Possibly that one may be able to bound the periods of the $x$-coordinates of the form $[x: y: z] \mapsto [f(x, z): f(y, z): z^n]$ has infinitely many $f$-invariant curves of the form $[xz^{n-1}: f^k(x, z): z^n]$, where $f^k$ is the homogenized $k$-th iterate of the dehomogenized one-variable polynomial $x \mapsto f(x, 1)$. On the other hand, it is possible that one may be able to bound the periods of the $f$-periodic subvarieties in general.

To state our question, we will need a little terminology. To be clear, we will say that $V$ is a $K$-subvariety of $X$ if $V$ is a geometrically irreducible subvariety of $X$ defined over $K$. Since so little is known about this question, we will ask it

\[ \Phi, \text{ then } \dim(Z) > \dim(Y). \] Now, if $\dim(Z) < \dim(X)$, then $Z$ is a codimension-1 subvariety, and in addition it is fixed by $\Phi^N$. Then it has to be contained in $Y$. However this is impossible since $x \in Z$ but $x \not\in Y$. In conclusion, $Z = X$, as desired.

In particular, if the codimension-2 subvariety $Y$ from the conclusion of Theorem 1.3 has the property that $Y(L)$ is Zariski dense in $Y$ (for some number field $L$ containing the field of definition for $\Phi$), then $X(L)$ is Zariski dense in $X$. So our Theorem 1.3 may be used to prove that certain varieties $X$ have a Zariski dense set of rational points, by reducing the problem to finding a potentially dense set of rational points on a codimension-2 subvariety $Y$ of $X$.

In the case that $\sigma : X \to X$ is defined over a number field $K$ and there is a point $x \in X(K)$ with dense orbit under $\sigma$, we obtain a much stronger upper bound (that has no dependence on the number field) for the period of codimension-1 subvarieties of $X$ periodic under the automorphism.

**Proof of Theorem 1.3** We extend $\sigma$ to a map $X' \to X'$ where $X'$ is defined over the ring of integers $\mathcal{O}_K$. Let $R$ be the localization of $\mathcal{O}_K$ away from all the primes of bad reduction. Then we obtain an automorphism of $R$-schemes $\sigma_0 : X \to X'$. Now, let $Y$ be some projective closure for $X'$; then $x$ meets $Y \setminus X'$ over at most finitely many finite primes, call this set $T$, and let $R'$ denote the localization of $R$ away from $T$. Let $Y'$ be the generic fiber of $Y$.

Let $W$ be an invariant subvariety of $X$. Suppose that $W$ has at least $\dim X - h^1(Y, \mathcal{O}_Y) + \rho + 1$ geometric components, where $\rho$ is the Picard number of $Y$ (the rank of its Néron-Severi group). Then clearly $x$ is not in $W$ so there is an at most finite set $T'$ of primes at which $x$ meets $W$. Let $S = T \cup T' \cup (\text{Spec } \mathcal{O}_K \setminus \text{Spec } R)$. Then $x$ is $S$-integral relative to $W$ and, since $\sigma^{-1}(W) = W$, we see that $\sigma^n(x)$ is $S$-integral relative to $W$ for all $n$ (if $\sigma^n(x)$ met $W$ modulo a prime, then $x$ would meet $\sigma^{-n}(W)$ modulo that same prime). But by a result of Vojta [Voj96, Cor. 0.3], this would mean that the orbit of $x$ was not dense, since $W$ has at least $\dim X - h^1(Y, \mathcal{O}_Y) + \rho + 1$ geometric components, which gives a contradiction. □

6. Other Questions

Poonen [Poo11] has proposed a variant of Morton-Silverman’s uniform boundedness conjecture, where the morphisms vary across a general families of self-maps of varieties rather than just the universal family of degree-$d$ self-maps $\mathbb{P}^n \to \mathbb{P}^n$. In Poonen’s set-up, some fibers may have infinitely many preperiodic points. Although that cannot happen in the case of preperiodic points of morphisms $\mathbb{P}^n \to \mathbb{P}^n$ (because of Northcott’s theorem), morphism $\mathbb{P}^n \to \mathbb{P}^n$ can have infinitely many positive dimensional periodic subvarieties. For example, if $f$ is a homogeneous two-variable polynomial of degree $n$, then the morphism $\mathbb{P}^2 \to \mathbb{P}^2$ given by $[x: y: z] \mapsto [f(x, z): f(y, z): z^n]$ has infinitely many $f$-invariant curves of the form $[xz^{n-1}: f^k(x, z): z^n]$, where $f^k$ is the homogenized $k$-th iterate of the dehomogenized one-variable polynomial $x \mapsto f(x, 1)$. On the other hand, it is possible that one may be able to bound the periods of the $f$-periodic subvarieties in general.

To state our question, we will need a little terminology. To be clear, we will say that $V$ is a $K$-subvariety of $X$ if $V$ is a geometrically irreducible subvariety of $X$ defined over $K$. Since so little is known about this question, we will ask it
in slightly less generality than Poonen uses. Given a morphism $\Phi : X \to X$ and a $K$-subvariety $V$ of $X$ such that $V$ is periodic under the action of $\Phi$, we define $\text{Per}_\Phi(V)$ to be the smallest $n$ such that $\Phi^n(V) \subseteq V$.

**Question 6.1.** Let $\pi : \mathcal{F} \to S$ be a morphism of varieties defined over a number field $K$ and let $\Phi : \mathcal{F} \to \mathcal{F}$ be an $S$-morphism. For $s \in S(K)$, we let $\mathcal{F}_s$ be the fiber $\pi^{-1}(s)$ and let $\Phi_s$ be the restriction of $\Phi$ to $\mathcal{F}_s$. Is there a constant $N_{\mathcal{F}}$ such that for any $s \in S(K)$ and any periodic $K$-subvariety $V$ of $\mathcal{F}_s$, we have $\text{Per}_{\Phi_s}(V) \leq N_{\mathcal{F}}$?

Even in the case where one can assign canonical heights to subvarieties of $X$, there may be subvarieties of $X$ of positive dimension having canonical height 0 that are not preperiodic (see [GTZ11]). Thus, we do not even know the answer to Question 6.1 even in the case of a constant family of maps.

**Question 6.2.** Let $\Phi : X \to X$ be a morphism of varieties defined over a number field $K$. Is there a constant $N_X$ such that for any periodic $K$-subvariety $V$ of $X$, we have $\text{Per}_\Phi(V) \leq N_X$?

We may also ask an analog of Question 6.1 for finite subgroups of automorphism groups.

**Question 6.3.** Let $\pi : \mathcal{F} \to S$ be a morphism of varieties defined over a number field $K$. For $s \in S(K)$, we let $\mathcal{F}_s$ denote the fiber $\pi^{-1}(s)$. Must the set

$$\{n \mid \text{there is an } s \in S(K) \text{ such that } \text{Aut}(\mathcal{F}_s) \text{ has a subgroup of order } n\}$$

be finite?

The theorems of Mazur [Maz77] and Merel [Mer96] show that Questions 6.1 and 6.3 has a positive answer when $\mathcal{F}$ is a family of elliptic curves. Similarly, work of Kondo [Kon99] shows that Question 6.3 has a positive answer when $\mathcal{F}$ is a family of K3 surfaces.

As with Question 6.1, we do not know the answer to Question 6.3 even in the constant family case. On the other hand, the bound in Theorem 3.1 depends only on the dimension of $X$ and the number of points in the special fiber of $X$ at the place $v$; by the Weil bounds of Deligne [Del74, Del80], the number of points on this special fiber can be bounded in terms of $\#k_v$, the dimension of $X$, and the Betti numbers of $X$. Thus, one might expect that there is a bound on the the largest finite subgroup of $\text{Aut}(\mathcal{F}_s)$ having good reduction at $v$ as $\mathcal{F}_s$ varies in a family.

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