A Century of Research in Sedimentation and Thickening†

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Abstract

This paper provides a concise review of the contributions to research in sedimentation and thickening that were made during the 20th century, starting from the invention of the Dorr thickener in 1905. The different steps of progress that were made in understanding batch sedimentation and continuous thickening processes in mineral processing are reviewed. A major breakthrough was Kynch's kinematic sedimentation theory published in 1952. The authors' own contributions to sedimentation and thickening research are summarized, including the development of the appropriate mathematical framework for Kynch's theory and its extensions to continuous sedimentation and to polydisperse suspensions of spheres. An even more general model framework is provided by the phenomenological theory of sedimentation, which permits the description and simulation of the transient behavior of flocculent suspensions.

1. Introduction

The invention of the Dorr thickener in 1905 and its introduction in the mining concentrators of South Dakota can be regarded as the starting point of the modern era of thickening research. Therefore, the history of thickening can be associated with the beginning of the 20th century. Thus the beginning of the 21st century provides a good opportunity to review the steps of progress of those almost one hundred years of research and development of industrial thickeners. A second prominent reason for publishing this review just now is the 50th anniversary of the publication of Kynch's celebrated paper A theory of sedimentation in 1952, which can be considered as the origin of modern sedimentation and thickening research in numerous disciplines.

Thickening is not a modern undertaking and was certainly not discovered as a process in the Americas. Whenever an ore was dressed to obtain a concentrate, two processes were used in an unseparable form: crushing and washing. There is evidence that in the IV Egyptian dynasty, some 2500 years BC, the ancient Egyptians dug for and washed gold. There is also evidence of gold washing in Sudan in the XII dynasty. Nevertheless, the earliest written reference for crushing and washing in Egypt is that of Agatharchides, a Greek geographer who lived 200 years before Christ. Ardaillon, author of the book Les Mines du Laurion dans l'Antiquité, described in 1897 the process used in the extensive installations for crushing and washing ores in Greece between the V to the III century BC. A.J. Wilson (1994) describes in his book The Living Rock mining of gold and copper in the Mediterranean from the fall of the Egyptian dynasties right to the Middle Ages and the Renaissance.

The development of mineral processing from unskilled labor to craftsmanship and eventually to an industry governed by scientific discipline is largely due to the Saxons in Germany and Cornishmen in England in the beginning of the 16th century. An international exchange of technology began between these two countries and continued for a long time. However, it was in Saxony where Agricola wrote his De Re Metallica, the first major contribution to the development and understanding of the mining indus-
try, published in Latin in 1556, and shortly after translated to German and Italian.

Agricola’s book describes several methods of washing gold, silver, tin and other metallic ores. He describes settling tanks used as classifiers, jigs and thickeners and settling ponds used as thickeners or clarifiers, see Figures 1 and 2. These devices operated in a batch or semi-continuous manner. A typical description is as follows: “To concentrate copper at the Neusohl mine in the Carpathians the ore is crushed and washed and passed through three consecutive washersifters. The fine particles are washed through a sieve in a tub full of water, where the undersize settles to the bottom of the tub. At a certain stage of filling of the tub with sediment, the plug is drawn to let the water run away. Then the mud is removed with a shovel and taken to a second tub and then to a third tub where the whole process is repeated. The copper concentrate which has settled in the last tub is taken out and smelted.”

It is evident from these references that, by using the washing and sifting processes, the ancient Egyptians and Greeks and the medieval Germans and Cornishmen knew the practical effect of the difference in specific gravity of the various components of an ore and used sedimentation in operations that can now be identified as classification, clarification and thickening. There is also evidence that in the early days no clear distinction was made between these three operations.

Agricola’s book had a tremendous impact, not only in the mineral industry but also on society in general, and continued to be the leading textbook for miners and metallurgists outside the English speaking world for at least another three hundred years. Apart from its immense practical value as a manual, the greatest influence of De Re Metallica was in preparing the ground for the introduction of a system of mining education which, with various modifications to suit the local conditions, was later to be adopted internationally.

While in Russia the first mining school was opened in 1715 in Petrozavodsk, the most important mining academic institution, the Freiberg Mining Academy, was founded in Germany in 1765. Twenty years later the Ecole des Mines was established in Paris, but it did not become an important institution until several years later. Regardless of the intense and prolonged technological transfer that was established between Saxony and Cornwall, the British were backward in technical mining education and in the 19th century could virtually offer no facilities for the study of this discipline. The Royal School of Mines was founded in London in 1851 and the Camborn School of Mines, whose prospectus was presented in 1829, was not established until 1859. The irony is that by that time the school was properly established, the mining production of Cornwall was declining and emigration of miners was in full swing to North America, Australia, and later to South and Central Africa (Wilson 1994).

The discovery of gold in California in 1848 and in Nevada a few years later caught Americans ill-equipped to make the best of these resources. They worked with shovels and pans, digging out gravel from the stream-beds and washing it in much the same way as Egyptians had done five thousand years before them. The only professional mining engineers
found in the North American continent until after the Civil war were those which came from Freiberg or other European schools. The first American university to establish mining was the Columbia University in New York in 1864 (Wilson 1994).

It is the purpose of this review to describe the development of the science and technology of thickening during the 20th century, and to present the current accepted theory. This paper is therefore organized as follows: in Section 2, we present an overview of the most important contributions to thickening research that were made during the 20th century. This period can be subdivided into five phases: that of the invention of the Dorr thickener and initial research of thickener design (1900-1940, Sect. 2.1), that of the discovery of the operating variables in a continuous thickener (1940-1950, Sect. 2.2), the Kynch Era, during which the theory of sedimentation was developed (1950-1970, Sect. 2.3), the phase of the phenomenological theory (1970-1980, Sect. 2.4), and the current one of the mathematical theory (1980-present, Sect. 2.5). The year numbers are of course to a certain degree arbitrary and not meant in a precise sense. We shall discuss a few example of research work that either lagged behind or was far ahead of its surrounding current status of knowledge, and would therefore have merited to fall within a prior or subsequent phase than determined by the respective publication year. We will also mention a few instances in which historic achievements have turned out to be still important to contemporary research. In Sections 3 and 4, the authors’ contribution to thickening research are summarized. In Section 3, results related to ideal suspensions, including the extension of Kynch’s theory to continuous sedimentation and polydisperse mixtures are presented, while Section 4 focuses on the recent phenomenological theory of sedimentation of flocculated suspensions. In Section 5, we comment some of the results of Sections 3 and 4, and discuss approaches by other current research groups. This paper closes with a concluding remark (Section 6).

2. Historical perspective

2.1 The invention of the Dorr thickener and thickener design (1900-1940)

Classification, clarification and thickening all involve the settling of one substance in solid particulate form through a second substance in liquid form, but the development of each one of these operations has followed different paths. While clarification deals with very dilute suspensions, classification and thickening are forced to use more concentrated pulps. This is probably the reason for which clarification was the first of these operations amenable to a mathematical description. The work by Hazen in 1904 was the first analysis of factors affecting the settling of solid particles from dilute suspensions in water. It shows that detention time is not a factor in the design of settling tanks, but rather that the portion of solid removed was proportional to the surface area of the tank and to the settling properties of the solid matter, and inversely proportional to the flow through the tank. At the beginning, classification equipment mimicked clarification settling tanks, adding devices to discharge the settled sediment. Therefore, the first theories of mechanical classification were based on Hazen’s theory of settling basins. The introduction of hydraulic classifiers led to theories of hindered settling in gravitational fields, and finally the introduction of the hydrocyclone moved away from gravity settling and used hindered settling in centrifugal fields as a mechanism.

The invention of the Dorr thickener in 1905 (Dorr, 1915) can be mentioned as the starting point of the modern thickening era. John V.N. Dorr, D.Sc., a chemist, cyanide mill owner and operator, consulting engineer and plant designer, tells (Dorr, 1936): “The first mill I built and operated was turned into a profitable undertaking by my invention of the Dorr classifier in 1904 and, in remodeling another mill in the same district, the Dorr thickener was born in 1905. […] Its recognition of the importance of mechanical control and continuous operation in fine solid-liquid mixtures and the size of its units have opened the way for advances in sewage and water purification and made wet chemical processes and industrial mineral processes possible.”

The invention of the Dorr thickener made the continuous dewatering of a dilute pulp possible, whereby a regular discharge of a thick pulp of uniform density took place concurrently with overflow of clarified solution. Scraper blades or rakes, driven by a suitable mechanism, rotating slowly over the bottom of the tank, which usually slopes gently toward the center, move the material as fast as it settles without enough agitation to interfere with the settling.

It is well known that in many mineral processing operations, applications appear and are utilized far ahead of their scientific understanding. This is the case in thickening. The first reference on variables affecting sedimentation is due to Nichols (1908). Authors who studied the effects of solids and elec-
thickener concentration, degree of flocculation, and temperature in the process also include Ashley (1909), Forbes (1912), Mishler (1912, 1918), Clark (1915), Ralston (1916), Free (1916), and the frequently cited work of Coe and Clevenger (1916). Several of these studies introduced confusion in the comprehension of the settling process and Mishler (1912), an engineer and superintendent at the Tigre Mining Company in the Sonora desert in Mexico, was the first to show by experiments that the rate of settling of slimes is different for dilute than for concentrated suspensions. While the settling speed of dilute slimes is usually independent of the depth of the settling column, a different law governed extremely thick slimes, and sedimentation increases with the depth of the settling column. He devised a formula by means of which laboratory results could be used in continuous thickeners. These formulas represent macroscopic balances of water and solids in the thickener and can be written as

\[ F = D, \]
\[ F D F = D D D D + O, \] (1)
\[ \text{where } F \text{ and } D \text{ are the solid mass flow rates in the feed and the discharge respectively, } O \text{ is the water mass flow rate in the overflow, and } D F \text{ and } D D \text{ are the dilutions of the feed and the underflow, respectively. The volume flow rate of water at the overflow } Q_D = O / \rho, \] (2)
\[ \text{where } \rho \text{ is the mass density of water, is given by} \]
\[ Q_D = F (D F - D D) / \rho. \] (3)

Mishler (1912) assumed that the flow rate of water per unit area in a continuous thickener should be equal to the rate of formation of the column of water in a batch column, where a suspension with the same concentration as the thickener feed is allowed to settle. Since this rate of water formation is equal to the rate of descent of the water-suspension interface, Mishler equated this rate, \( Q_D / S \), with the settling rate, which we denote by \( \sigma(D F) \). Then the cross-sectional area \( S \) of the thickener required to treat a feed rate of \( F \) is

\[ S = F (D F - D D) / \rho \sigma(D F). \] (4)

Clark (1915) carefully measured concentrations in a thickener with conical bottom, a configuration that clearly gives rise to at least a two-dimensional flow. Clark’s note unfortunately is little more than half a page long but displays three diagrams of thickeners with the sample locations and the respective mixture densities. The brevity of the note reflects that at the time of its writing the phenomena of settling were far from being understood, let alone in more than one space dimension. This also becomes apparent in Clark’s introductory statement: “Whatever interest they may possess lies in the fact that few determinations [the specific gravity of pulp] of this nature have been recorded, rather than in any unusual or unexpected conditions developed by the determinations themselves.” Clark’s measurements have received little attention as presented in his note, but it seems that they in part stimulated the well-known paper by Coe and Clevenger, which appeared in 1916, since Coe and Clevenger explicitly acknowledge Clark’s experimental support.

Coe and Clevenger’s paper was importance in two respects. On one hand, they were the first to recognize that the settling process of a flocculent suspension gives rise to four different and well-distinguished zones. From top to bottom, they determined a clear water zone, a zone in which the suspension is present at its initial concentration, a transition zone and a compression zone, see Figure 3. Coe and Clevenger reported settling experiments with a variety of materials showing this behaviour.

On the other hand, they were also the first to use the observed batch settling data in a laboratory column for the design of an industrial thickener. They argued that the solids handling capacity, today called solids flux density, has a maximum value in the thickener at a certain dilution \( D_k \) between the feed and discharge concentration. They developed, independently from Mishler, an equation similar to (4) but with the feed dilution \( D F \) replaced by a limiting dilution \( D_k \):

\[ S = F (D_k - D D) / \rho \sigma(D_k), \] (5)

where \( \sigma(D_k) \) is the settling velocity of a pulp with the limiting dilution \( D_k \). They recommend the determination of the limiting dilution by batch experiments for the desired underflow dilution \( D D \). Equation (5), with minor sophistications, continues to be the most reliable method of thickener design to date.

After the important contributions made in the development of thickening technology in the first two decades of the 20th century, the invention of the Dorr thickener and the development of thickener design procedures, the next two decades surely saw the expansion of this technology. Several authors made efforts to model the settling of suspensions by extending Stokes’ equation or postulated empirical equations (Adamson and Glasson 1925, Robinson 1926,
Egolf and McCabe 1937, Ward and Kammermeyer 1940, Work and Kohler 1940), but no further important contributions were made until the forties. It is interesting to note that Robinson's paper was of minor influence for the development of sedimentation theory in the context of mineral processing, but that its approach seems to have been of great interest to investigators in the area of blood sedimentation half a century later (Puccini et al. 1977).

Stewart and Roberts (1933) give, in a review paper, a good idea of the state of the art of thickening in the twenties. They say: “The basic theory is old but limitations and modifications are still but partially developed. Especially in the realm of flocculent suspensions the underlying theory incomplete. Practical testing methods for determining the size of machines to be used are available, but the invention and development of new machines will no doubt be greatly stimulated by further investigation of the many interesting phenomena observed in practice and as fresh problems are uncovered.”

2.2 The discovery of the operating variables in a continuous thickener (1940-1950)

The University of Illinois became very active in thickening research in the forties, shortly after Comings' paper Thickening calcium carbonate slurries had been published in 1940. It should be pointed out that this paper is the first to recognize the importance of local solids concentration and sediment composition for the thickening process, since it shows measurements of solids concentration profiles in a continuous thickener, while all previous treatments had been concerned with observations of the suspension-supernate and sediment-suspension interfaces, with the exception of Clark (1915).

At least nine B.Sc. theses were made under Comings' guidance, mainly on the effect of operating variables in continuous thickening. Examples of thesis subjects were: settling and continuous thickening of slurries; continuous thickening of calcium carbonate slurries; thickening of clay slurries and limiting rates of continuous thickening. These theses were summarized in an important paper by Comings, Pruiss and De Bord (1954).

The mechanism of continuous sedimentation was investigated in the laboratory in order to explain the behaviour of continuous thickeners. Comings et al. (1954) show four zones in a continuous thickener: the clarification zone at the top, the settling zone underneath, the upper compression zone further down and the rake action zone at the bottom. The most interesting conclusion, expressed for the first time, was that the concentration in the settling zone is nearly constant for a thickener at steady state, and that it depends on the rate at which the solids are fed into the thickener, and not on the concentration of the feed suspension. At low feed rate, the solids settled rapidly at a very low concentration, regardless of the feed concentration. When the feed rate was increased, the settling zone concentration increased and approached a definite value when the maximum settling capacity of the equipment was reached. If the feed rate was increased further, the settling zone stayed constant and the solids fed in excess of the settling handling capacity left the thickener by the overflow. It was verified that in most cases the feed suspension was diluted to an unknown concentration.

Fig. 3 Settling of a flocculent suspension as illustrated by Coe and Clevenger (1916), showing the clear water zone (A), the zone in which the suspension is at its initial concentration (B), the transition zone (C) and the compression zone (D).
on entering the thickener. Another finding was that, for the same feed rate, increasing or decreasing the sediment depth could adjust the underflow concentration.

It is worth mentioning that the paper by Comings et al. (1954) did not yet take into account Kynch’s sedimentation theory published two years earlier (Kynch 1952), which is discussed below.

Another contribution of practical importance is the work of Roberts (1949), who advanced the empirical hypothesis that the rate at which water is eliminated from a pulp in compression is at all times proportional to that amount left, which can be eliminated up to infinite time:

\[ \frac{D_v}{D_0} = (D_0 - D_v) \exp(-Kt), \]

where \( D_0, D \) and \( D_v \) are the dilutions at times zero and \( t \) and at infinite time, respectively. The equation has been used until today for the determination of the critical concentration.

### 2.3 Theory of sedimentation (1950-1970)

From the invention of the Dorr thickener to the establishment of the variables controlling the equipment, the only quantitative knowledge that was accomplished was Coe and Clevenger’s (1916) design procedure. This method was solely based on a macroscopic balance of the solid and the fluid and on the observation of the different zones in the thickener. No underlying sedimentation theory existed.

The first attempts to formulate a theory of sedimentation in the sense of relating observed macroscopic sedimentation rates to microscopic properties of solid particles were made by Steinour in a series of papers that appeared in 1944 (Steinour 1944a-c). But it was Kynch, a mathematician at the University of Birmingham in Great Britain, who presented in 1952 his celebrated paper A theory of sedimentation. He proposed a kinematical theory of sedimentation based on the propagation of sedimentation waves in the suspension. The suspension is considered as a continuum and the sedimentation process is represented by the continuity equation of the solid phase:

\[ \frac{\partial \phi}{\partial t} + \frac{\partial f_{bk}(\phi)}{\partial z} = 0, \quad 0 \leq z \leq L, \quad t > 0, \]

where \( \phi \) is the local volume fraction of solids as a function of height \( z \) and time \( t \), and

\[ f_{bk}(\phi) = \phi v_s \]

is the Kynch batch flux density function, where \( v_s \) is the solids phase velocity. The basic assumption of Kynch’s theory is that the local solid-liquid relative velocity is a function of the solids volumetric concentration \( \phi \) only, which for batch sedimentation in a closed column is equivalent to stating that \( v_s = v_s(\phi) \).

Equation (7) is considered together with the initial condition

\[ \phi(z, 0) = \begin{cases} 0 & \text{for } z = L, \\ \phi_0 & \text{for } 0 < z < L, \\ \phi_{max} & \text{for } z = 0, \end{cases} \]

where it is assumed that the function \( f_{bk} \) satisfies

\[ f_{bk}(\phi) = \begin{cases} 0 & \text{for } \phi < 0 \text{ or } \phi > \phi_{max}, \\ \text{for } 0 < \phi < \phi_{max}, \end{cases} \]

where \( \phi_{max} \) is the maximum solids concentration. Kynch (1952) shows that knowledge of the function \( f_{bk} \) is sufficient to determine the sedimentation process, i.e. the solution \( \phi = \phi(z, t) \), for a given initial concentration \( \phi_0 \), and that the solution can be constructed by the method of characteristics.

Kynch’s paper had the greatest influence in the development of thickening thereafter. The period of twenty years after this paper may be referred to as the Kynch Era. When Comings moved from Illinois to Purdue, research on thickening continued there for another ten years. Although Comings soon moved on to Delaware, work continued at Purdue under the direction of P.T. Shannon. A Ph.D. thesis by Tory (1961) and M.Sc. theses by Stroupe (1962) and De Haas (1963) analyzed Kynch’s theory and proved its validity by experiments with glass beads. Their results were published in a series of joint papers by these authors (Shannon et al. 1963, 1964, Tory and Shannon 1965, Shannon and Tory 1965, 1966).

Batch and continuous thickening was regarded as the process of propagating concentration changes upwards from the bottom of the settling vessel as a result of the downward movement of the solids. Equations were derived and experimental results for the batch settling of rigid spheres in water were found to be in excellent agreement with Kynch’s theory. Experiments aiming at verifying the validity of Kynch’s theory have repeatedly been conducted up to the present (Davis et al. 1991, Chang et al. 1997).

Kynch’s paper also motivated industry to explore the possibilities of this new theory in thickener design. Again the Dorr Co. went a step further in their contribution to thickening by devising the Talmage and Fitch method of thickener design (Talmage and Fitch 1955). They affirmed that one settling plot contained all the information needed to design a thickener. Since the water-suspension interface slope (in a
z versus (a diagram) gave the settling rate of the suspension, the slope at different times represented the settling velocity at different concentrations. They used this information in conjunction with the cited Mishler-Coe and Clevenger method to derive a formula for the unit area, that is, the thickener area required to produce a sediment of given concentration at a given solids handling rate. Talmage and Fitch’s method is described in detail in our previous review article (Concha and Barrientos 1993) and in Chapter 11 of Bustos et al. (1999), so the resulting formulas and related diagrams are not explicitly stated here. Although Kynch’s theory cannot be regarded as an appropriate model for flocculent suspensions, thickener manufacturers still use and recommend Talmage and Fitch’s method, which is based on this theory, for design calculations (Outukumpu Mintec 1997).

Yoshioka et al. (1957) and Hassett (1958, 1964, 1968) used the solid flux density function to interpret the operation of a continuous thickener and devise a method for thickener design (see Concha and Barrientos 1993 for details).

Experience by several authors, including Yoshioka et al. (1957), Hassett (1958, 1964, 1968), Shannon et al. (1963), Tory and Shannon (1965), Shannon and Tory (1965, 1966) and Scott (1968a,b), demonstrated that while Kynch’s theory accurately predicts the sedimentation behaviour of suspensions of equally sized small rigid spherical particles, this is not the case for flocculent suspensions forming compressible sediments.

An interesting paper that seems to have been overlooked by the thickening literature is that of Behn (1957). This paper has been ahead of its time and would have been well received in the seventies. Behn was the first writer to relate thickening compression with the consolidation process. Therefore he applies the following consolidation equation for the excess pore pressure \( p_e \), that is, the pressure in the pores of the sediment in excess of the hydrostatic pressure:

\[
\frac{\partial p_e}{\partial t} = k \Delta \rho h_c \frac{\partial^2 p_e}{\partial z^2},
\]

where \( k \) is the average sediment permeability, \( \rho_s \) is the solids density, \( \Delta \rho \) is the solid-liquid density difference, and \( h_c \) is the initial height of the consolidating region. Unfortunately without explicitly mentioning the boundary conditions or presenting any detail of the derivation, Behn (1957) obtained the solution

\[
D - D_w = (D_0 - D_w) \exp(-Kt), \quad K = \frac{k \Delta \rho}{\rho_s h_c}.
\]

Equation (10) is identical to Roberts’ equation (6) and gives some theoretical support to it, although Roberts’ equation is essentially empirical (Fitch 1993).

Although Behn’s paper is valuable in providing a concise review of the mathematical models of sedimentation that had been advanced until 1957 and in recognizing that the settling velocity formulas postulated in some of them in fact furnish the Kynch batch flux density function \( f_{bk} \), it reflects at the same time that the implications of Kynch’s theory had still not yet been well understood. This becomes apparent in Behn’s conclusion that “the Kynch theory does extend into the compression zone, although Kynch did not so indicate.” It has already been pointed out that Kynch’s theory does not apply to the compression zone. This becomes obvious by the fact that the compression zone is characterized by rising curved iso-concentration lines, which finally become horizontal, while in the framework of Kynch’s theory concentration values always propagate along straight lines if cylindrical vessels are considered.

To describe the batch settling velocities of particles in a suspension several equations for \( v_s = v_s(\phi) \) or \( f_{bk}(\phi) = \phi v_s(\phi) \) were proposed, all of them extensions of the Stokes equation. The most frequently used was the two-parameter equation of Richardson and Zaki (1954):

\[
f_{bk}(\phi) = u_s (1 - \phi)^n, \quad n > 1,
\]

where \( u_s \) is the settling velocity of a single particle in quiescent, unbounded fluid. This equation has the inconvenience that the settling velocity becomes zero at a solids concentration of \( \phi = 1 \), while experimentally this occurs at a maximum concentration \( \phi_{max} \) between 0.6 and 0.7.

Michaels and Bolger (1962) proposed the following three-parameter alternative:

\[
f_{bk}(\phi) = u_s (1 - \frac{\phi}{\phi_{max}})^n, \quad n > 1,
\]

where the exponent \( n = 4.65 \) turned out to be suitable for rigid spheres.

For equally sized glass spheres, Shannon et al. (1963) determined the following equation by fitting a forth-order polynomial to experimental measurements, see Figure 4:

\[
f_{bk}(\phi) = \phi (-0.33843 + 1.37762 \phi - 1.62275 \phi^2 - 0.11264 \phi^3 + 0.0902253 \phi^4) \times 10^{-2} \text{ m/s}.
\]
A paper that still belongs to the Kynch era, although it was published later, is the one by Petty (1975). He extended Kynch’s theory from batch to continuous sedimentation. If \( q \) is defined as the volume flow rate of the mixture per unit area of the sedimentation vessel, Kynch’s equation for continuous sedimentation can be written as

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}(q \phi + f_{bk}(\phi)) = 0.
\]

Petty (1975) also discussed, for the first time, the proper boundary conditions at the bottom of the settling vessel, which is a non-trivial problem.

2.4 The phenomenological theory (1970-1980)

Although Behn (1957) was the first writer to apply consolidation theory for the settling of compressible slurries, it was Shirato and his co-workers (1970) who solved the consolidation problem taking into account the effect of sediment growth and compression due to deposition of solid particles from the hindered settling zone. The latter is, however, not explicitly modeled due to the use of material coordinates in the compressible sediment. The result were sediment concentration and excess pore pressure profiles.

If Behn’s work had received greater attention, the development of a phenomenological theory would have started about fifteen years earlier. It took another five years for Adorján (1975, 1977) to present his ad-hoc theory of sediment compression, giving the first satisfactory method of thickener design, and for Smiles (1976a,b) to present his integral approach.

At about the same time, a group of researchers in Brazil made great efforts to give the phenomenological sedimentation theory a proper framework. Strong and important research on thickening, and in general in the field of flow through porous media, was going on in Brazil in the decade of the 1970s. At the Engineering Graduate School of the Federal University of Rio de Janeiro, COPPE, several researchers and graduate students, among them Giulio Massarani, Affonso Silva Telles, Rubens Sampaio, I-Shih Liu, José Teixeira Freire, Liu Kay, João D’Avila and Satoshi Tobinaga, were involved in the application of a newly developed mathematical tool, the Theory of Mixtures of continuum mechanics, to particulate systems. Unfortunately these findings were rarely published in mainstream international journals, but are well documented in local publications and conference volumes, see for example D’Avila (1976, 1978), D’Avila and Sampaio (1977), D’Avila et al. (1978), Liu (1978) and Tobinaga and Freire (1980).

With strong ties with the Brazilian researchers, a group led by the first author of this review at the University of Concepción in Chile worked in the same direction. Findings were presented in the B.Sc. theses by O. Bascur (1976) and A. Barrientos (1978), at the XII International Mineral Processing Congress in São Paulo, Brazil in 1977 (Concha and Bascur 1977) and at the Engineering Foundation Conference on Particle Technology in New Hampshire, USA, in 1980.

Independently, Kos (1975) used the theory of mixtures to set up boundary value problems for batch and continuous sedimentation. Thacker and Lavelle (1977) used the same theory for incompressible suspensions.

2.5 Mathematical theory (1980-present)

At the end of the seventies and during the eighties several papers, for example Concha and Bustos (1985), Buscall and White (1987), Auzerais et al. (1988), Landman et al. (1988), Bascur (1989) and Davis and Russel (1989) show that the phenomenological model, based on the theory of mixtures, was well accepted by the international scientific community.

In spite of the fact that the theory of mixtures did a great job in unifying the sedimentation of dispersed and flocculated suspensions and, once appropriate constitutive equations were formulated, gave rise to a robust framework in which the sedimentation of any suspension could be simulated, the mathematical analysis of these models did not exits. Furthermore no adequate numerical method existed for solving the initial-boundary value problems for batch and continuous thickening.

While it is common that mathematics is needed in solving specific problems, many engineers regard the basic principles of mineral processing as purely physical and mathematical treatment as belonging only to a
later stage in the development of a theory. When originally mathematics get started, it could be precise, but setting up the theory is for them an extra-mathematical operation. Truesdell (1966) stated that the characteristics of a good theory are that the physical concepts themselves are made mathematical at the outset, and mathematics is used to formulate the theory and to obtain solutions.

At the beginning of the eighties, following Truesdell’s ideas and convinced that the only way in establishing a rigorous theory of sedimentation consisted in the cooperation of engineers with mathematicians, the first author, together with María Cristina Bustos, then a staff member of the Department of Mathematics at the University of Concepción, and Wolfgang Wendland, professor of mathematics at the University of Darmstadt, started a fruitful cooperation on the topic of mathematical analysis of sedimentation models. After Wendland moved to the University of Stuttgart in Germany in 1986, first Matthias Kunik and then the second author joined this cooperation, which has lasted for two decades now, are summarized in the following Sections 3 and 4.

3. Sedimentation of ideal suspensions

3.1 Kynch’s sedimentation model and mathematical preliminaries

For batch sedimentation, equation (7) is solved together with the initial condition (8). Note that, due to the assumptions on \( f_{\text{bk}} \), the initial condition (8) could be replaced by the initial condition

\[ \phi(z, 0) = \phi_{\text{max}} \text{ for } 0 \leq z \leq L, \]

combined with the boundary conditions

\[ \phi(0, t) = \phi_{\text{max}}, \phi(L, t) = 0 \text{ for } 0 < t \leq T. \]

To construct the solution of the initial value problem (7), (8), the method of characteristics is employed. This method is based on the propagation of \( \phi_0(z_0) \), the initial value prescribed at \( z = z_0 \), at constant speed \( f_{\text{bk}}(\phi_0(z_0)) \) in a \( z \) versus \( t \) diagram. These lines might intersect, which makes solutions of equation (7) discontinuous in general. This is due to the nonlinearity of the flux density function \( f_{\text{bk}} \). In fact, even for smooth initial data, a scalar conservation law with a nonlinear flux density function may produce discontinuous solutions, as the well-known example of Burgers’ equation illustrates, see Le Veque (1992). On the other hand, one particular theoretically and practically interesting initial-value problem for a scalar conservation law is the Riemann problem where an initial function

\[ \phi_0(z) = \begin{cases} \phi_0^+ & \text{for } z > 0, \\ \phi_0^- & \text{for } z < 0 \end{cases} \]

consisting just of two constants is prescribed. Obviously, the initial-value problem (7), (8) consists of two adjacent Riemann problems producing two ‘fans’ of characteristics and discontinuities, which in this case start to interact after a finite time \( t_c \).

At discontinuities, equation (7) is not satisfied and is replaced by the Rankine-Hugoniot condition (Bustos and Concha 1988, Concha and Bustos 1991), which states that the local propagation velocity \( \sigma(\phi^+, \phi^-) \) of a discontinuity between the solution values \( \phi^+ \) above and \( \phi^- \) below the discontinuity is given by

\[ \sigma(\phi^+, \phi^-) = \frac{f_{\text{bk}}(\phi^+) - f_{\text{bk}}(\phi^-)}{\phi^+ - \phi^-}. \]  

(15)

However, discontinuous solutions satisfying (7) at points of continuity and the Rankine-Hugoniot condition (15) at discontinuities are in general not unique. For this reason, an additional selection criterion, or entropy principle, is necessary to select the physically relevant discontinuous solution, the entropy weak solution.

One of these entropy criteria, which determine the unique entropy weak solution, is Oleinik’s jump entropy condition requiring that

\[ f_{\text{bk}}(\phi) - f_{\text{bk}}(\phi^+) \geq \sigma(\phi^+, \phi^-) \geq f_{\text{bk}}(\phi^-) - f_{\text{bk}}(\phi^+) \]

for all \( \phi \) between \( \phi^- \) and \( \phi^+ \)  

(16)

is valid. This condition has an instructive geometrical interpretation: it is satisfied if and only if, in an \( f_{\text{bk}} \) versus \( \phi \) plot, the chord joining the points \((\phi^+, f_{\text{bk}}(\phi^+))\) and \((\phi^-, f_{\text{bk}}(\phi^-))\) remains above the graph of \( f_{\text{bk}} \) for \( \phi^+ < \phi^- \) and below the graph for \( \phi^+ > \phi^- \), see Figure 5.

Discontinuities satisfying both (15) and (16) are called shocks. If, in addition,

\[ f_{\text{bk}}(\phi^-) = \sigma(\phi^+, \phi^-) \text{ or } f_{\text{bk}}(\phi^+) = \sigma(\phi^+, \phi^-) \]

(17)

is satisfied, the shock is called a contact discontinuity. In that case, the chord is tangent to the graph of \( f_{\text{bk}} \) in at least one of its endpoints.

A piecewise continuous function satisfying the conservation law (7) at points of continuity, the initial condition (8), and the Rankine-Hugoniot condition
interval f on a homogeneous suspension and consider a flux density where we set the problem of equation (7) with the initial condition (8),

$$\phi_0$$ and $$\phi_{\infty}$$ are the entropy weak solution of the Riemann problem (MS) MS-1, in which the supernate-suspension and suspension-sediment interfaces are both rarefactions, see Figure 4. These shocks meet at the critical time $$t_c$$ to form a stationary clear water-sediment interface. In an MS-2, the rising shock is replaced, from top to bottom, by a contact discontinuity followed by a rarefaction wave, see Figure 6 b). In the flux plot, the contact discontinuity is represented by a chord joining the points $$(\phi_0, f_{bk}(\phi_0))$$ and $$(\phi_{\infty}, f_{bk}(\phi_{\infty}))$$ which is tangent to the graph of $$f_{bk}$$ in the second point. If we take the same flux function and still increase $$\phi_0$$, this chord can no longer be drawn

(15) and OleÁnk’s jump entropy condition (16) at discontinuities is unique.

Consider equation (7) together with the Riemann data (14). If we assume (for simplicity) that $$\phi_0 < \phi_{\infty}$$ and that $$f_{bk}(\phi) > 0$$ for $$\phi_0 < \phi < \phi_{\infty}$$, it is easy to see that no shock can be constructed between $$\phi_0$$ and $$\phi_{\infty}$$. In that case, the Riemann problem has a continuous solution

$$\phi(z, t) = \begin{cases} 
\phi_0 & \text{for } z > f_{bk}(\phi_0)t, \\
(f_{bk})^{-1}(z/t) & \text{for } f_{bk}(\phi_0) t \leq z \leq f_{bk}(\phi_{\infty}) t, \\
\phi_{\infty} & \text{for } z < f_{bk}(\phi_{\infty}) t, 
\end{cases}$$

(18)

where $$(f_{bk})^{-1}$$ is the inverse of $$f_{bk}$$ restricted to the interval $$[\phi_0, \phi_{\infty}]$$. This solution is called a rarefaction wave and is the entropy weak solution of the Riemann problem.

For details on entropy weak solutions of scalar conservation laws, we refer to the books by Le Veque (1992), Godlewski and Raviart (1991, 1996), Kröner (1997) or Dafermos (2000).

### 3.2 Solutions of the batch sedimentation problem

We now consider entropy weak solutions for the problem of equation (7) with the initial condition (8), where we set $$\phi_0 \equiv \text{const}$$ corresponding to an initially homogeneous suspension and consider a flux density function $$f_{bk}$$ with at most two inflection points. Using the method of characteristics and applying the theory developed by Ballou (1970), Cheng (1981, 1983) and Liu (1978), Bustos and Concha (1988) and Concha and Bustos (1991) construct entropy weak solutions of this problem in the class of piecewise continuous functions, in which zones of constant concentrations are separated by shocks, rarefaction waves or combinations of these. Therefore, it is necessary to solve the two Riemann problems given at $$t=0$$, $$z=L$$ and at $$t=0$$, $$z=0$$ and to treat the interaction of the two solutions at later times. For flux density functions with at most two inflection points, they obtain five qualitatively different entropy weak solutions or modes of sedimentation.

Their classification turned out to be not yet complete, since they had considered only functions with two inflection points similar to our Figures 4, and two inflection points can also be located in a different way, producing two additional modes of sedimentation. In these two new modes of sedimentation, the supernate-suspension interface is not a sharp shock but a rarefaction wave. The mathematically rigorous, detailed construction of the complete set of seven modes of sedimentation M S-1 to M S-7 is presented in Chapter 7 of Bustos et al. (1999). In this review, the construction of the seven different entropy solutions is outlined in Figures 6 and 7.

The solutions constructed by Bustos and Concha (1988) and Concha and Bustos (1991) were not new and had been published decades earlier by Straumann (1962) and Grassmann and Straumann (1963). However, at that time the mathematical concept of entropy solutions had not yet been developed, and Grassmann and Straumann (1963) had to introduce new physical arguments and insights in every case in order to obtain their solutions of the sedimentation process. Their paper certainly would have deserved greater attention, and should have been quoted by those who were active in thickening research in the 1960s, such as P.T. Shannon and his co-workers, but unfortunately was published in German.

The simplest case is that of a mode of sedimentation (MS) M S-1, in which the supernate-suspension and the suspension-sediment interfaces are both shocks, see Figure 6 a). These shocks meet at the critical time $$t_c$$ to form a stationary clear water-sediment interface. In an MS-2, the rising shock is replaced, from top to bottom, by a contact discontinuity followed by a rarefaction wave, see Figure 6 b).

In the flux plot, the contact discontinuity is represented by a chord joining the points $$(\phi_0, f_{bk}(\phi_0))$$ and $$(\phi_{\infty}, f_{bk}(\phi_{\infty}))$$ which is tangent to the graph of $$f_{bk}$$ in the second point. If we take the same flux function and still increase $$\phi_0$$, this chord can no longer be drawn

![Fig. 5 Geometrical interpretation of Oleáknik’s jump entropy condition, applied to five jumps between $$\phi = \phi_0$$ and $$\phi = \phi_{\infty}$$, for $$i=1, \ldots, 5$$. For $$i=1, 3$$ the condition is satisfied (solid chords), for $$i=2, 4, 5$$ it is violated (dashed chords).](image-url)
and the contact discontinuity becomes a line of continuity. This situation corresponds to an MS-3, see Figure 6 c). Note that the modes of sedimentation MS-2 and MS-3 can occur only with a flux density function $f_{bk}$ with exactly one inflection point (we recall that we always assume that $f_{bk}'(0)/L<0$).

With a flux density function $f_{bk}$ having exactly two inflection points, four additional modes of sedimentation are possible, which are collected in Figure 7. Clearly, an MS-1 is also possible with two inflection points. Bustos and Concha (1988) and Concha and Bustos (1991) obtained two of the four additional modes of sedimentation, namely the modes MS-4 and MS-5 shown in Figure 7 a) and b). These modes are similar to an MS-2 and MS-3 respectively, but the second inflection point produces an additional contact discontinuity, denoted by $C_2$ in Figure 7 a) and by $C_1$ in Figure 7 b), which separates the lower rarefaction wave ($R_1$) from the rising sediment.

Of course, the shape of a given flux density function determines which modes of sedimentation are actually possible. In Chapter 7 of Bustos et al. (1999), a corresponding geometrical criterion to decide this is presented. It is quite obvious that flux density functions that are similar to that shown in Figure 4 make an MS-4 or MS-5 possible. However, it is also possible to place the inflection points a and b in such a way that $f_{bk}'(a)<0$, $f_{bk}'(b)<0$, and that there exists a point $a<\phi<b$ such that the tangent to the graph of $f_{bk}$ at $(\phi, f_{bk}(\phi))$ also goes through the point $(0, f_{bk}(0)=0)$. This situation is shown in the flux plots of Figure 7 c) and d). If the initial concentration $\phi_0$ is then

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**Figure 6** Modes of sedimentation MS-1 to MS-3. From the left to the right, the flux plot, the settling plot showing characteristics and shock lines, and a representative concentration profile taken at time $t=t_c$ are shown for each mode. Chords in the flux plots and shocks in the settling plots having the same slopes are marked by the same symbols. Slopes of tangents to the flux plots occurring as slopes of characteristics in the settling plots are also indicated (Bürger and Tory, 2000).
chosen between $\tilde{\phi}$ and the second inflection point $b$, a new mode of sedimentation, called MS-6, is produced, see Figure 7 c): the supernate-suspension interface is no longer sharp, that is, a shock; rather, an upper rarefaction wave $R_1$ emerges from $z=L$ at $t=0$, which is separated from the supernate by a contact discontinuity and from the bulk suspension by a line of continuity. That line of continuity meets the rising sediment-suspension interface, the shock $S_1$, at $t=t_1$.

After that a curved, convex shock $S_2$ forms, separating the rarefaction wave from the sediment. At the critical time $t=t_c$, the shock $S_2$ meets the upper end of the rarefaction wave, that is, the contact discontinuity $C_1$, and the stationary shock $S_3$ forms, which denotes the sediment-supernate interface located at the sediment height $z_c$.

A similar construction applies if $\phi_0>b$ is chosen such that there exists a point $\tilde{\phi} < \phi(b)$ with $f_{\tilde{\phi}}(\tilde{\phi}) = f_\max$.
"...see Figure 7 d). The resulting entropy weak solution is an MS-7, differing from an MS-6 in that the bulk suspension is separated from the upper rarefaction wave by a contact discontinuity instead of a line of continuity.

All modes of sedimentation terminate in a stationary sediment of the maximum concentration \( \phi_{\text{max}} \) and of height \( z_c = \phi_0 L/\phi_{\text{max}} \). This stationary state is attained at the critical time \( t_c \).

3.3 Continuous sedimentation of ideal suspensions

Similar solutions of standardized initial-boundary value problems can be constructed by the method of characteristics for continuous sedimentation of ideal suspensions (Bustos et al. 1990b, Concha and Bustos 1992). These solutions satisfy the equation (13) wherever \( \phi \) is discontinuous. At discontinuities between two solution values \( \phi^+ \) and \( \phi^- \), the Rankine-Hugoniot condition

\[
\sigma(\phi^+, \phi^-) = q(t) + \frac{f_{\text{bk}}(\phi^+)-f_{\text{bk}}(\phi^-)}{\phi^+ - \phi^-} \tag{19}
\]

and Oleinik's jump entropy condition

\[
q(t) + \frac{f_{\text{bk}}(\phi^+)-f_{\text{bk}}(\phi^-)}{\phi^+ - \phi^-} \geq \sigma(\phi^+, \phi^-) \geq q(t) + \frac{f_{\text{bk}}(\phi^+)-f_{\text{bk}}(\phi^-)}{\phi^+ - \phi^-} \tag{20}
\]

for all \( \phi \) between \( \phi^- \) and \( \phi^+ \) are satisfied. Conditions (19) and (20) are natural extensions of conditions (15) and (16) to the case of a flux density function with a convection term added to the Kynch batch flux density function \( f_{\text{bk}}(\phi) \).

The standard initial function for continuous sedimentation, where \( q(t) \) is a negative constant, is

\[
\phi(z, 0) = \phi_0(z) = \begin{cases} 
\phi_L & \text{for } A < z \leq L, \\
\phi_{\text{max}} & \text{for } 0 \leq z < A,
\end{cases} \tag{21}
\]

where \( 0 < A < L \) is the initial sediment height and \( \phi_L \) is the concentration in the thickener to which the feed flux \( f_r \) is diluted. The value \( \phi_L \) is obtained by solving the equation

\[
q_{\text{bk}} + f_{\text{bk}}(\phi_L) = f_r. \tag{22}
\]

In case Eq. (22) admits several solutions \( \phi_L \), the relevant one is selected by the physical argument that the feed suspension should always be diluted on entering the thickener, as shown by Comings et al. (1954), see Chapter 2 of Bustos et al. (1999).

It is assumed that the feed flux \( f_r \) and the volume average flow velocity \( q \) are kept constant during the continuous sedimentation process. This suggests formulating the boundary conditions as

\[
\phi(L, t) = \phi_L, \quad t > 0, \tag{23}
\]
\[
\phi(0, t) = \phi_{\text{max}}, \quad t > 0. \tag{24}
\]

However, the boundary conditions (23) and (24) can be imposed only in that special case where no characteristics carrying solution values intersect the boundary \( z = L \) from below and \( z = 0 \) from above, respectively. This case is exceptional, since the usual solution picture of the Riemann problem given by Eq. (13) and the initial datum (21) will be a centred wave, a so-called Riemann fan, emerging from \( z = A, t = 0 \), consisting of characteristics and discontinuities, which after some finite time reach \( z = L \) or \( z = 0 \). If the entire Riemann fan cuts the boundary \( z = L \), we say that the thickener overflows; if it cuts \( z = 0 \), the thickener empties.

When such intersections with the boundaries occur, the boundary conditions (23) and (24) are no longer valid. The appropriate mathematical concept to maintain well-posedness of the initial-boundary value problem is that of set-valued entropy boundary conditions. This means that boundary conditions (23) and (24) are replaced by

\[
\phi(L, t) \in e_L(\phi_L; f), \quad t > 0, \tag{25}
\]
\[
\phi(0, t) \in e_0(\phi_{\text{max}}; f), \quad t > 0, \tag{26}
\]

where \( e_L(\phi_L; f) \subset [0, \phi_{\text{max}}] \) and \( e_0(\phi_{\text{max}}; f) \subset [0, \phi_{\text{max}}] \) are particular sets of admissible boundary values that can be constructed using the graph of the composite flux function \( f(\phi) \) defined by

\[
f(\phi) := q(\phi) + f_{\text{bk}}(\phi),
\]

see Bustos and Concha (1992), Bustos et al. (1996), Bürger and Wendland (1998b) and Chapters 6 and 8 of Bustos et al. (1999) for details.

For a flux density function \( f_{\text{bk}} \) with exactly one inflection point (note that \( f_{\text{bk}} \) and \( f \) have the same inflection points), there exist three different modes of continuous sedimentation (Concha and Bustos 1992) depending on the structure of the Riemann fan. The solution is called a mode of continuous sedimentation MCS-1 if it consists of two constant states separated by a shock; an MCS-2 if it consists of two constant states separated by a contact discontinuity; and an MCS-3 if the solution is continuous and consists of two constant states separated by a rarefaction wave.

Note that we have to distinguish less modes of sedimentation in the continuous than in the batch case since we have to solve only one Riemann problem,
and no wave interactions occur at positive times. However, in each of these modes the thickener can either overflow, empty, or attain a steady state, i.e. approximates a stationary solution for $t \to \infty$, so that there are nine qualitatively different solution pictures.

For a detailed construction of the complete set of these solutions we refer to Chapter 8 of Bustos et al. (1999), and for a particularly concise overview to Concha and Bürger (1998). In this review, we present three examples from Chapter 8 of Bustos et al. (1999), see Figure 8, in which the left column shows plots of the composite flux function $f$, and the right one the corresponding settling plots. In Figures 8 and 9, $\varphi_0$, $\varphi_\infty$ and $\varphi_{\max}$.

The construction method for exact entropy solutions of the problem of batch and continuous sedimentation of ideal suspensions with standardized initial and boundary data, and in particular the fact that the exact location and propagation speed of the sediment-suspension interface level are always known, have led Bustos et al. (1990b) to formulate a simple control model for continuous sedimentation. It is shown that steady states corresponding to an MCS-1 can always be recovered after a perturbation of the feed flux density $f_F = q\varphi_0 + f_{\text{bk}}(\phi_L)$ by solving two initial-boundary value problems at known times and with parameters $q$ and $\phi_L$ that can be calculated a priori, see Figure 9.

![Fig. 8 Modes of continuous sedimentation (Bustos et al. 1999). Top: MCS-1 with emptying ICT; Middle: MCS-2 attaining a steady state; Bottom: MCS-3 causing the ICT to overflow.](image-url)
where \( \Phi = (\phi_1, \ldots, \phi_N)^T \) denotes the vector of concentration values, together with prescribed initial concentrations and the zero flux conditions

\[
\phi_i(z, 0) = \phi_i^0(z), \quad 0 \leq z \leq L, \quad \phi_i^0(z) + \cdots + \phi_N^0(z) \leq \phi_{\text{max}};
\]

\[
f|_{z=0} = 0, \quad f|_{z=L} = 0.
\]

It is well known that solutions of equation (28) are discontinuous in general. The propagation speed of a discontinuity in the concentration field \( \phi \), is given by the Rankine-Hugoniot condition

\[
\sigma_i(\Phi^+, \Phi^-) = \frac{f_i(\Phi^+)-f_i(\Phi^-)}{\phi_i^+-\phi_i^-}, \quad i=1, \ldots, N,
\]

where \( \Phi^+, \phi_i^+, \Phi^- \) and \( \phi_i^- \) denote the limits of \( \Phi \) and \( \phi_i \) above and below the discontinuity, respectively. This condition can readily be derived from first principles by considering the flows to and from the interface.

For batch sedimentation in a closed column, the volume average velocity

\[
q := (1-\phi)u_t + \phi_1u_1 + \cdots + \phi_N u_N
\]

vanishes, which can be seen easily by summing equation Eq. (27) over \( i=1, \ldots, N \) and by taking into account the continuity equation of the fluid,

\[
\frac{\partial \phi_i}{\partial t} + \frac{\partial \phi_i}{\partial z}((1-\phi)v_t) = 0,
\]

and that \( q=0 \) at \( z=0 \). In terms of the relative velocities

\[
u_i := v_i - v_t, \quad i=1, \ldots, N,
\]

we can rewrite \( q=0 \) as

\[
u_t = -(\phi_1u_1 + \cdots + \phi_N u_N).
\]

Noting that

\[
f_i = f_i(\phi_i(u_t + v_t)),
\]

we obtain

\[
f_i = f_i(\Phi) = \phi_i(\Phi) = \phi_i(\phi_1u_1 + \cdots + \phi_N u_N)).
\]

Including in his analysis the momentum equations for each particle species and that of the fluid and using equilibrium considerations, Masliyah (1979) derived that the constitutive equation for the solid-fluid relative velocity \( u_i \) should be

\[
u_i := \tilde{u}_{si}V(\phi),
\]

where \( \tilde{u}_{si} \) denotes the Stokes settling velocity of a single particle of species \( i \) with respect to a fluid of density

\[
r(\phi) = \rho_0 + (1-\phi)\rho_f,
\]

i.e.,
and where $V(\phi)$ can be chosen as one of the hindered settling functions known in the monodisperse case, for example as the ubiquitous Richardson and Zaki (1954) formula $V(\phi) = V_{RZ}(\phi) = (1-\phi)^n$, with parameters $n > 1$, for $0 \leq \phi \leq \phi_{\text{max}}$. With the parameters

$$\mu = -\frac{gd_i^2}{18\mu_f}, \quad \delta_i = \frac{d_i^2}{d_1^2}, \quad i = 1, \ldots, N,$$

we finally obtain

$$f_i(\Phi) = \mu (1-\phi)V(\phi) \left( \phi_i \sum_{j=1}^{N} \delta_i \phi_j - \delta_i \phi_i \right). \quad (32)$$

The mathematical theory of systems of conservation laws is significantly more complicated, and much less developed, than that of scalar equations. For initial-boundary value problems of nonlinear coupled systems such as that given by (28), supplemented with the flux density function (32), no general existence and uniqueness result is available. In fact, the analysis of these polydisperse sedimentation equations is just starting. It is, however, possible to obtain numerical solutions of these equations by the application of modern shock-capturing finite difference schemes for systems of conservation laws.

We present here a numerical example from Bürger et al. (2000a). Consider the experiment performed by Schneider et al. (1985) with glass beads of the same density $\rho_s = 2790$ kg/m$^3$ and of the diameters $d_1 = 0.496$ mm and $d_2 = 0.125$ mm in a settling column of height $L = 0.3$ m. The fluid density and viscosity are $\rho_f = 1208$ kg/m$^3$ and $\mu_f = 0.02416$ kgm$^{-1}$s$^{-1}$, respectively. The initial concentrations are $\phi_1^0 = 0.2$ and $\phi_2^0 = 0.05$, and the simulated time here is $T = 1200$ s. Following Schneider et al. (1985), we employ Richardson

![Fig. 10](image-url1) Settling of a bidisperse suspension of heavy particles of two different sizes: iso-concentration lines (a) of the larger particles and (b) of the smaller particles, corresponding to the values of $\phi$: 0.02, 0.04, 0.06, 0.08, 0.1, 0.15, 0.25, 0.3, 0.4, 0.5 and 0.6. The circles and the dashed lines correspond to experimental measurements of interface locations and shock lines, respectively, obtained by Schneider et al. (1985). Concentration profiles taken at the times $t_0$, $t_2$ and $t_3$ are given in Figure 11.

![Fig. 11](image-url2) Settling of a bidisperse suspension of heavy particles of two different sizes: concentration profiles (a) of the larger particles and (b) of the smaller particles (index 2) at $t_1 = 51.9$ s, $t_2 = 299.8$ s and $t_3 = 599.7$ s.
and Zakia's flux density function with n=2.7, which is not at $\phi_{\text{max}}=0.68$, see Concha et al. (1992).

Figure 10 displays the simulated iso-concentration lines for each species, together with the experimental measurements of interface locations and computed shock lines made by Schneider and co-authors, while Figure 11 shows concentration profiles of both species at three selected times.

4. Phenomenological theory of thickening

4.1 The Theory of Mixtures

The theory of Mixtures (Bowen 1976) states that a mixture of continuous media with components $\alpha$ may be described by the following quantities: the apparent component density $\rho_\alpha$, the component velocity $\mathbf{v}_\alpha$, the component stress tensor $\mathbf{T}_\alpha$, the component body force, and the interaction force between components $\mathbf{m}_\alpha$. These quantities constitute a dynamic process if, in regions where the field variables are continuous, they obey the following field equations:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0, \quad (33)$$
$$\rho_\alpha \left( \frac{\partial \mathbf{v}_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha \right) = \nabla \cdot \mathbf{T}_\alpha + \rho_\alpha \mathbf{b} + \mathbf{m}_\alpha. \quad (34)$$

In regions with discontinuities, the field equations must be replaced by the following jump conditions:

$$\begin{bmatrix} \rho_\alpha \mathbf{v}_\alpha \cdot \mathbf{e}_i \end{bmatrix} = \begin{bmatrix} \rho_\alpha \mathbf{v}_\alpha \cdot \mathbf{e}_i \end{bmatrix}, \quad (35)$$
$$\begin{bmatrix} \rho_\alpha \mathbf{v}_\alpha \mathbf{v}_\alpha \cdot \mathbf{e}_i \end{bmatrix} = \begin{bmatrix} \rho_\alpha \mathbf{v}_\alpha \mathbf{v}_\alpha \cdot \mathbf{e}_i \end{bmatrix} + \begin{bmatrix} \mathbf{T}_\alpha \cdot \mathbf{e}_i \end{bmatrix}, \quad (36)$$

where $\mathbf{e}_i$ is the normal vector of a jump discontinuity and $[ \cdot ]$ (bold square brackets) denotes the jump of a quantity across a discontinuity. The jump balances (35) and (36) can be derived from the appropriate integral or macroscopic mass and momentum balances (33) and (34) and do not involve nor provide supplementary information. More details can be found in Chapter 1 of Bustos et al. (1999).

4.2 Solid-fluid particulate systems

A particulate system, consisting of a finely divided solid in a fluid, can be regarded as a mixture of continuous media if the following assumptions are met (Concha et al. 1996):

1. The solid particles are small with respect to the containing vessel, and have the same density, size and shape.
2. Particles and fluid are incompressible.
3. There is no mass transfer between the solid and the fluid.
4. Gravity is the only body force. For such a system, we let $\alpha=s$ denote the solid component and $\alpha=f$ the fluid component. Equations (33) and (34) then produce the following balance equations: the solid mass balance

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{v}_f) = 0, \quad (37)$$
the fluid mass balance

$$\frac{\partial \phi}{\partial t} - \nabla \cdot (\phi \mathbf{v}_s) = 0, \quad (38)$$
the solid linear momentum balance

$$\rho_s \left( \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) = \nabla \cdot \mathbf{T}_s + \rho_s \mathbf{b} + \mathbf{m}_s, \quad (39)$$
and the fluid linear momentum balance

$$\rho_f \left( \frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f \cdot \nabla \mathbf{v}_f \right) = \nabla \cdot \mathbf{T}_f + \rho_f \left( \nabla \cdot \mathbf{b} - \mathbf{m} \right). \quad (40)$$

Here $\phi$ denotes the local volume solid concentration, $\mathbf{v}_s$ and $\mathbf{v}_f$ the respective solid and fluid phase velocity, $\rho_s$ and $\rho_f$ the respective solid and fluid mass densities, $\mathbf{T}_s$ and $\mathbf{T}_f$ the corresponding Cauchy stress tensors, $\mathbf{b}$ the body force and $\mathbf{m}$ the solid-fluid interaction force per unit volume.

In the cases of practical interest, the acceleration terms are small and can be neglected; the solid-fluid interaction force $\mathbf{m}$ can be decomposed into a static term $\mathbf{m}_b$ and a dynamic term $\mathbf{m}_d$; and both stresses can be written as a pressure term and a viscous extra stress. For sake of simplicity, we limit the treatment here to one space dimension, in which the viscous extra stress tensors are unimportant, and introduce the constitutive assumption

5. Particles and fluid are contained in an impervious vessel with frictionless walls, in which all variables are constant across any cross-sectional areas.

The viscosity terms of the extra stress tensors play, however, a decisive role for the stability of the resulting model equations in two or three space dimensions (Bürger and Kunik 2001, Bürger et al. 2001d), see also the discussion in Section 5 of this review. Details on the justification of these assumptions and on the following deduction can be found in Concha et al. (1996) and Bürger and Concha (1998) for one space dimensions and in Bürger et al. (2000e) for several space dimensions.

Inserting the present assumptions into the linear momentum balances (39) and (40), we obtain
\[
\frac{\partial}{\partial t} = \rho(1-\phi)g - m_b - m_d, \quad (41)
\]
\[
\frac{\partial p_e}{\partial z} = \rho_s \phi g + m_b + m_d. \quad (42)
\]

The quantities \(p_e\) and \(p_s\) are theoretical variables which we now express in terms of measurable variables, namely the pore pressure \(p\) and the effective solid stress \(\sigma_e\). Assuming that the local surface porosity of every cross-section of the network formed by the solid flocs equals the volume porosity, we can express \(p_e\) and \(p_s\) in terms of \(p\) and \(\sigma_e\) via the formulas
\[
p_e = (1-\phi)p - (1-\phi)\left[p_e + p_s g(\rho - \rho_s)\right],
\]
\[
p_s = \phi \left[p_e + p_s g(\rho - \rho_s)\right] + \sigma_e(\phi),
\]
where \(p_e\) is again the excess pore pressure, that is, the pore pressure minus the hydrostatic pressure,
\[
p_e = p - p_s g(\rho - \rho_s). \quad (43)
\]

It can be shown (Concha et al. 1996) that the hydrostatic interaction force is proportional to the pore pressure and the concentration gradient:
\[
m_b = p_e \frac{\partial \phi}{\partial z}, \quad (44)
\]
and the hydrodynamic interaction force may be modeled by a Stokes-like (or Darcy-like) equation as a linear function of the solid-fluid relative velocity:
\[
m_d = -\alpha(z, t)(v_s - v_l), \quad (45)
\]
where \(\alpha(z, t)\) is the resistance coefficient of the suspension or the sediment.

Collecting all these results, and introducing them into equations (41) and (42), reduces Eqns. (37)-(40) to
\[
\frac{\partial \phi}{\partial t} + \frac{\partial (\phi v_s)}{\partial z} = 0, \quad (46)
\]
\[
\frac{\partial \phi}{\partial t} - \frac{\partial (1-\phi) v_l}{\partial z} = 0, \quad (47)
\]
\[
\frac{\partial \sigma_e}{\partial z} = \Delta \rho g \phi - \frac{\alpha(\phi)}{1-\phi}(v_s - v_l), \quad (48)
\]
\[
\frac{\partial p_e}{\partial z} + \frac{\partial \sigma_e}{\partial z} = -\Delta \rho g \phi. \quad (49)
\]

At discontinuities, equations (46)-(49) have to be replaced by the appropriate jump conditions.

Three important steps have to be taken to transform Equations (46) to (49) into a usable mathematical model. First, observe that we consider six scalar variables, but have only four scalar equations available to specify them. Thus two quantities should be specified by constitutive equations. We therefore assume that the resistance coefficient \(\alpha\) and the effective solid stress function \(\sigma_e\) are given as constitutive functions of the types \(\alpha = \alpha(\phi)\) and \(\sigma_e = \sigma_e(\phi)\).

Second, we observe that the volume average velocity of the mixture,
\[
q := \phi v_s + (1-\phi) v_l \prod v_s - (1-\phi) (v_s - v_l) \quad (50)
\]
satisfies the simple equation
\[
\frac{\partial q}{\partial z} = 0 \quad (51)
\]
obtained from summing Equations (46) and (47). It is useful to replace Eq. (47) by (51).

Finally, we express the solid-fluid relative velocity \(v_s - v_l\) from (50) in the form
\[
\frac{v_s - v_l}{1-\phi} = q. \quad (52)
\]

Substituting (52) into (48) and (49), we can now define a Dynamic Process for a Particulate System as a set of four unknown field variables: the volumetric solids concentration \(\phi\), the excess pore pressure \(p_e\), the volume average velocity \(q\) and the solids volume flux
\[
f := \phi v_s, \quad (53)
\]
if in any regions of continuity, these four variables satisfy the four scalar field equations
\[
\frac{\partial \phi}{\partial t} + \frac{\partial f}{\partial z} = 0, \quad (54)
\]
\[
\frac{\partial q}{\partial z} = 0, \quad (55)
\]
\[
\frac{\partial \sigma_e(\phi)}{\partial z} = -\Delta \rho g \phi - \frac{\alpha(\phi)}{\phi(1-\phi)} (f - q), \quad (56)
\]
\[
\frac{\partial p_e}{\partial z} + \frac{\partial \sigma_e}{\partial z} = -\Delta \rho g \phi, \quad (57)
\]
where \(\sigma_e\) and \(\alpha\) are given as functions of \(\phi\). At discontinuities Equations (54)-(57) have to be replaced by the appropriate jump conditions.

### 4.3 Kinematical sedimentation process

An ideal suspension can be defined as a suspension of equally sized rigid spherical particles, such as glass beads, for which the effective solid stress is constant, i.e.
4.4 Dynamic sedimentation processes

In applying the phenomenological model of a particulate system to the sedimentation of a flocculent suspension forming compressible sediments, the following two assumptions are made:

6. The suspension is entirely flocculated at the beginning of the sedimentation process.
7. The solid and the liquid can perform a one-dimensional simple compression motion only.

For flocculent suspensions we consider again the equations (54)-(57) defining a Dynamic Process for a Particulate System, together with two constitutive equations (54)-(57) defining a Dynamic Process for a Particulate System, together with two constitutive equations for the resistance coefficient \( \alpha = \alpha(\phi) \) (or, equivalently, for the Kynch batch flux density function \( f_{bk} = f_{bk}(\phi) \)), and for the effective solid stress \( \sigma_s = \sigma_s(\phi) \).

Isolating \( f \) from Eq. (56) and using the definition of \( q \) yields

\[
f = q \phi - \frac{\Delta p \phi^2 (1 - \phi)^2}{\alpha(\phi)} \left( 1 + \frac{\sigma_s(\phi)}{\Delta p \phi} \frac{\partial \phi}{\partial z} \right).
\]

In view of Eq. (59), this can be rewritten as

\[
f = q \phi + f_{bk}(\phi) \left( 1 + \frac{\sigma_s(\phi)}{\Delta p \phi} \frac{\partial \phi}{\partial z} \right), \tag{60}
\]

hence the solids mass balance equation can be written as

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z} (q(t) \phi + f_{bk}(\phi)) = \frac{\partial}{\partial z} \left( - \frac{f_{bk}(\phi) \sigma_s(\phi)}{\Delta p \phi} \frac{\partial \phi}{\partial z} \right). \tag{61}
\]

Observe that \( f \) is now a function of \( \phi(z, t) \), \( \partial \phi/\partial z \) \( (z, t) \) and \( t \). However, for the definition of boundary conditions, it is more convenient to simply refer to \( f = f(z, t) \).

Defining the diffusion coefficient

\[
a(\phi) = - \frac{f_{bk}(\phi) \sigma_s(\phi)}{\Delta p \phi},
\]

and its primitive

\[
A(\phi) = \int_0^\phi a(s) \, ds,
\]

we can rewrite Eq. (61) in the form

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z} (q(t) \phi + f_{bk}(\phi)) = \frac{\partial^2 A(\phi)}{\partial z^2}. \tag{62}
\]

We assume here that the effective solid stress is given as a function of the volumetric solids concentration \( \phi \) satisfying

\[
\sigma_s(\phi) \begin{cases} = \text{const} & \text{for } \phi \leq \phi_c, \\
>0 & \text{for } \phi > \phi_c,
\end{cases}
\]

where \( \phi_c \) is the critical concentration or gel point at which the solid flocs begin to touch each other. A common constitutive equation is the power law (Tiller and Leu 1980, Landman and White 1994)

\[
\sigma_s(\phi) = \begin{cases} 0 & \text{for } \phi \leq \phi_c, \\
\sigma_0 ((\phi/\phi_c)^n - 1) & \text{for } \phi > \phi_c, \quad n > 1, \sigma_0 > 0. \tag{63}
\end{cases}
\]

Due to the constitutive assumptions concerning the functions \( f_{bk} \) and \( \sigma_s \), we see that

\[
a(\phi) \begin{cases} = 0 & \text{for } 0 \leq \phi \leq \phi_c \text{ and } \phi = \phi_{max}, \\
>0 & \text{for } \phi_c < \phi < \phi_{max}. \tag{64}
\end{cases}
\]

Consequently, Eq. (61), or equivalently Eq. (62) is of the first-order hyperbolic type for \( \phi = \phi_c \) and \( \phi = \phi_{max} \) and of the second-order parabolic type for \( \phi_c < \phi < \phi_{max} \).

Summarizing, we say that (62) is a quasilinear strongly degenerate parabolic equation, where the attribute strongly states that the degeneration from parabolic to hyperbolic type not only takes place at isolated values, but on the whole interval \([0, \phi_c]\) of concentration values.

4.5 Initial and boundary conditions

In one space dimension, we see that the volume average velocity of the mixture is given by boundary conditions, and we are left with the quasilinear strongly degenerate parabolic equation (62) for \((z, t) \in (0, L) \times (0, T)\), together with the equation (57).

Obviously, only Eq. (62) has actually to be solved, since by Eq. (57) the excess pore pressure can always be calculated from the concentration distribution. It is therefore sufficient to consider initial-boundary value problem of Eq. (62) only.

For batch sedimentation of a flocculated suspen-


**Section 4.6 Mathematical analysis of the initial-boundary value problems**

It is well known that, due to both the type degeneracy and to the nonlinearity of the function \( f_{bs} \), solutions of (62) are discontinuous and have to be defined as entropy solutions. The basic ingredient of the appropriate solution concept of entropy weak solutions is an entropy inequality, which can be formulated by suitably modifying the well-known results of the pioneering papers by Kružkov (1970) and Volpert (1967) for first-order partial differential equations and by Volpert and Hudjaev (1969) for second-order partial differential equations.

By the vanishing viscosity method using a mollifier technique, Bürger et al. (2000b) showed that an entropy solution of the initial-boundary value problem (62), (65) exists, even if the function \( a = a(\phi) \) has a jump at \( \phi = \phi_c \), which in turn is a consequence of the fact that the vast majority of effective solid stress functions \( \sigma_e = \sigma_e(\phi) \) suggested in the literature have discontinuous derivatives \( \sigma'_e \) at \( \phi = \phi_c \). In that case, the integrated diffusion coefficient \( A(\phi) \) will only be continuous, but not differentiable at \( \phi = \phi_c \).

The existence proof for this case is a fairly straightforward extension of the previous proof presented by Bürger and Wendland (1998a). However, to show uniqueness of entropy solutions for a discontinuous diffusion function, new arguments have to be invoked. In fact, the uniqueness proof by Bürger and Wendland (1998a) is based on a particular jump condition derived by Wu and Yin (1989), which is valid only for Lipschitz continuous diffusion functions \( a(\phi) \) and is limited to one space dimension. Fortunately, a recent result by Carrillo (1999) made it possible to avoid these limitations. He utilized a technique known as “doubling of the variables”, introduced by Kružkov (1970) as a tool for proving the \( L^1 \) contraction principle for entropy solutions of scalar conservation laws, in order to establish the uniqueness result. Most notably, Carrillo’s approach is not based on jump conditions and only presupposes that \( A(\phi) \) is Lipschitz continuous, i.e. \( a(\phi) \) may have discontinuities. Carrillo’s results were employed by Bürger et al. (2000c) to show uniqueness of entropy solutions of the initial-boundary value problem (62), (65).

**4.7 Numerical methods for sedimentation-consolidation processes**

A numerical scheme that approximates entropy solutions of the initial-boundary value problem (62), (65) (or one of its variants) should have the built-in property to reproduce discontinuities of the entropy solutions, most notably the suspension/sediment interface where the equation changes type, appropriately without the necessity to track them explicitly, i.e. the scheme should posses the shock capturing property. Moreover, an obvious requirement is that the scheme should approximate (converge to) the correct (entropy) solution of the problem it is trying to solve. This clearly rules out classical schemes based on naive finite differencing for strictly parabolic equations, which otherwise work well for smooth solutions, see Evje and Karlsen (2000).

In this section we present an example of a working finite difference scheme having all these desired properties, and which is moreover easy to implement. This scheme is presented for the application to batch or continuous sedimentation. There also exist variants of the scheme for the simulation of batch centrifugation of a flocculated suspension (Bürger and Concha 2001) or of pressure filtration (Bürger et al. 2001b). For alternative schemes, in particular schemes that are based on operator splitting and front tracking, that are equally suitable for the sedimentation-consolidation model we refer to Bürger et al. (2000d) and the references cited therein.
The scheme considered here can be referred to as a generalized upwind scheme with extrapolation. We consider a rectangular grid on $Q_1$ with mesh sizes $\Delta z = L/j, \Delta t = T/N$ and let $\phi^n_0 = (j \Delta z, n \Delta t)$. The calculation starts by setting
\[
\phi^n_0 = \phi_j^n (j \Delta z), \quad j = 0, \ldots, j.
\]
The partial differential equation to be solved, Eq. (62), is approximated by the explicit scheme
\[
\frac{\phi^{n+1}_j - \phi^n_j}{\Delta t} + q(n \Delta t) \frac{\phi^l_j + \phi^r_j}{2 \Delta z} + \frac{\epsilon_{bk} \phi_j^{EO}(\phi^1_j, \phi^0_j, \phi^1_j) - \epsilon_{bk} \phi_j^{EO}(\phi^{n-1}_j, \phi^n_j)}{\Delta z} = A(\phi^n_j - A(\phi^n_j), \quad j = 1, \ldots, j - 1.
\]
(67)

The boundary condition at $z=0$ is inserted into Eq. (67) for $j=0$:
\[
\frac{\phi^{n+1}_0 - \phi^n_0}{\Delta t} + q(n \Delta t) \frac{\phi^l_0 + \phi^r_0}{2 \Delta z} + \frac{\epsilon_{bk} \phi_0^{EO}(\phi^1_0, \phi^0_0, \phi^1_0) - \epsilon_{bk} \phi_0^{EO}(\phi^{n-1}_0, \phi^n_0)}{\Delta z} = A(\phi^n_0 - A(\phi^n_0).
\]

For boundary condition (65b), we obtain for $j=\phi^n_0 = \phi_1(n \Delta t)$, while boundary condition (65c) turns the interior scheme (67) into the boundary formula
\[
\frac{\phi^{n+1}_j - \phi^n_j}{\Delta t} + q(n \Delta t) \frac{\phi^l_j + \phi^r_j}{2 \Delta z} + \frac{\epsilon_{bk} \phi_j^{EO}(\phi^1_j, \phi^0_j, \phi^1_j) - \epsilon_{bk} \phi_j^{EO}(\phi^{n-1}_j, \phi^n_j)}{\Delta z} = A(\phi^n_j - A(\phi^n_j), \quad j = 1, \ldots, j - 1.
\]

The numerical flux function of the generalized upwind or Engquist-Osher method (Engquist and Osher, 1981) is given by
\[
f_{bk}\phi^{EO}(u,v) = f_{bk}(0) + \int_{0}^{v} \min\{f_{bk}(s), 0\} ds + \int_{v}^{\max}\max\{f_{bk}(s), 0\} ds.
\]

The quantities $\phi^l$ and $\phi^r$ are given by extrapolation:
\[
\phi^l := \phi^n_0 - \frac{\Delta z}{2} s^n_0, \quad \phi^r := \phi^n_0 + \frac{\Delta z}{2} s^n_0,
\]
which makes the scheme second order accurate in space. The slopes are given by $s^n_0 = s^n_{j-1} = s^n_0 = 0$ and are otherwise calculated using a limiter function, e.g.
\[
s^n_0 = \frac{1}{\Delta z} M M \left(\theta(\theta^n_{j-1} - \phi^n_{j-1}), \frac{\phi^n_{j-1} - \phi^n_{j-1}}{2}, \theta(\phi^n_{j-1} - \phi^n_{j-1})\right), \quad \theta \in [0,2], \quad j = 2, \ldots, j - 2.
\]

using the so-called minmod limiter function
\[
M M (a, b, c) = \begin{cases} 
\min \{a, b, c\} & \text{if } a, b, c > 0, \\
\max \{a, b, c\} & \text{if } a, b, c < 0, \\
0 & \text{otherwise},
\end{cases}
\]
in order to ensure that the total variation of the numerical solution is bounded. The scheme converges to the entropy solution if the following stability condition is satisfied:
\[
\max_{i \in [0,T]} \frac{q(t)}{r} + \max_{i \in [0,T]} \frac{f'(i)}{r} \frac{\Delta t}{\Delta z} + \max_{i \in [0,T]} a(i)^2 \frac{\Delta t}{\Delta z} \leq 1.
\]
The Engquist-Osher method has also been used by other authors for the simulation of sedimentation processes, see e.g. Amberg and Dahlkild (1987).

4.8 Numerical examples

4.8.1 Simulation of batch sedimentation

As a first example of the use of the numerical technique, we present in the sequel four different simulations of batch sedimentation in which published settling experiments were utilized to compare the numerically predicted settling behaviour with the respective experimental findings. In every case, the required model functions $\sigma$ and $f_{bk}$ had to be determined or constructed from the published experimental information. We refer to Bürger et al. (2000b) and Garrido et al. (2000) for details.

The simulation of a settling experiment by Holdich and Butt (1997), as presented by Bürger and Karlsen (2001b), shall be discussed here in detail (Figures 12 and 13). For simulations of experiments performed by Shirato et al. (1970), Shih et al. (1986) and Bergström (1992), we present here only the numerical results (Figures 14, 15 and 16), and refer to the paper by Garrido et al. (2000) for details.

Holdich and Butt (1997) performed sedimentation experiments with a suspension of talc in tap water ($\rho = 1690$ kg/m$^3$) and obtained settling plots from conductivity measurements. From the published experimental data, Garrido et al. (2000) obtained the following constitutive functions:
\[
f_{bk}(\phi) = \begin{cases} 
v_{w}(a_2(\phi)^2 + a_1(\phi)) & \text{for } \phi \leq \phi_{\text{M}} := \frac{3b_2 - b_1}{2 b_2}, \\
v_{w}(\frac{(1 - \phi)^3}{b_1 - b_2 \phi}) & \text{for } \phi_{\text{M}} < \phi < \phi_{\text{max}} := b_1 / b_2, \\
2.14 \times 10^{-6} \phi \text{ Pa} & \text{for } \phi > \phi_{\text{M}},
\end{cases}
\]

where the constants have the values $v_w = -4.4 \times 10^{-6}$ m/s, $a_1 = 3.0693$, $a_2 = -28.5004$, $b_1 = 12$ and $b_2 = 32.5$. The remaining parameters are $L = 0.331$ m and $\Delta z = L/300$. We considered three different initial concentrations, $\phi_0 = 0.052, 0.072$ and 0.112. Figures 12 and 13 show the corresponding numerical simulations. Observe that especially due to the presence of the diffusion term, the settling process can be simulated...
Fig. 12  Simulation of batch sedimentation of a talc suspension in a column (Bürger and Karlsen 2001, Garrido et al. 2000, Holdich and Butt 1997): concentration profiles at different times for three different initial concentrations.

Fig. 13  Simulation of batch sedimentation of a talc suspension in a column, after Bürger and Karlsen (2001): settling plots for three different initial concentrations (0.052, top left, 0.072, top right, and 0.112, bottom). The symbols denote the measured iso-concentration lines.
by the numerical algorithm although the flux density function has a singularity at $\phi_0 = 0.3692$.

The simulations of these four experimental cases illustrate that the phenomenological model is able to predict the most important features normally observed during batch settling of initially unnet-worked suspensions in a column, which are the following:

- Before sedimentation begins, the suspension is homogenized by stirring to obtain a homogeneous concentration. When sedimentation starts, all the solid flocs have the same settling velocity so that they form a sharp water-suspension interface in the upper portion of the column. This stage is called hindered settling.
- Particles at the bottom of the column rapidly occupy the entire surface available. Immediately, new flocs start to accumulate over the deposited sediment pressing over them and, in that way, squeezing out some of its retained water. From that point on the sediment is under compression or consolidation. The sediment surface, that is the suspension-sediment interface, moves upward as new particles incorporate into the sediment.
In the case of suspensions with $\phi \leq \phi_c$, at each point in the column under the supernate-suspension interface the concentration either stays constant or increases.

- Utilizing X-ray, $\gamma$-ray or conductivity measurement instruments, one can generate data from which it is possible to track determined concentration values as they move upward in the column. For dilute concentrations, that is in regions where $\phi \leq \phi_c$, these iso-concentration curves form straight lines which in cylindrical vessels are identical to the characteristics of the scalar conservation law (7). In the compression zone, that is in the sediment, the iso-concentration lines for concentrations greater than the critical emerge from the bottom $z=0$ at positive slope, and become horizontal as the consolidation process proceeds. Experiments in which this behaviour is particularly well visible have been reported by Scott (1968a), Been and Sills (1981) and Tiller et al. (1990).

- At a given instant and a certain height in the column, the water-suspension and the suspension-sediment interfaces meet, leaving an area of approximately triangular shape to their left in a settling plot, in which the concentration equals the constant initial concentration $\phi_0$. The coordinates of this event are the critical time $t_c$ and the critical height $z_c$, forming the critical point at which simultaneous hindered settling and consolidation end and only consolidation perdures. The new suspension-sediment interface moves downward at a diminishing (in absolute value) speed.

- After a sufficiently large finite time, consolidation ends and a constant concentration gradient is established from the critical concentration at the top to a greater concentration at the bottom. Consequently, the entire sedimentation-consolidation process terminates after a finite time. In spite of the fact that this observation has been made by a series of numerical experiments, it is still to be shown that it is an inherent property of the mathematical model.

4.8.2 Continuous thickening of flocculated suspensions

The second example presents a simulation of the dynamic behaviour of a flocculated suspension an Ideal Continuous Thickener from Bürger et al. (2000d). The numerical algorithm employed for this simulation is not based on the one presented in Section 4.7, but rather on an equally suited three-step operator splitting method, see Bürger et al. (2000d) and Chapter 9 of Bustos et al. (1999) for details.

We use a Kynch batch flux density function of the well known Richardson and Zaki type (1954),

$$ f_{bk}(\phi) = -6.05 \times 10^{-4} \phi (1 - \phi)^{12.59} \text{ m/s}, $$

and the effective solid stress function

$$ \sigma_s(\phi) \begin{cases} 0 & \text{for } \phi \leq \phi_c = 0.23, \\ 5.35 \times \exp(17.9 \phi) \text{ Pa} & \text{for } \phi > \phi_c \\ \end{cases} $$

determined by Becker (1982) based on experimental measurements on Chilean copper ore tailings. Note that this choice of the function $\sigma_s(\phi)$ leads to a diffusion function $a(\phi)$ that is discontinuous. Furthermore, we use the parameters $\Delta \rho = \rho_s - \rho = 1500 \text{ kg/m}^3$ and $g = 9.81 \text{ m/s}^2$.

We consider continuous sedimentation with piece-wise constant average flow velocity $q(t)$ and feed flux $f_s(t)$. We start with a steady state, that is, a stationary concentration profile, and then attain two new steady states by manipulating $f_s$ and $q$ appropriately. Steady states are obtained as stationary solutions of equation (62). It is assumed that a desired discharge concentration $\phi_0$ is prescribed. Then the discharge flux is $f_D = q \phi_0$. The requirement that at steady state the discharge flux must equal the feed flux, $f_D = f_s$, leads to an equation from which the concentration value $\phi_l$ at $z=L$ can be computed:

$$ q \phi_l + f_{bk}(\phi_l) = q \phi_0. \quad (68) $$

The sediment concentration profile is then calculated from

$$ \frac{d \phi}{dz} = \frac{q \phi(z) + f_{bk}(\phi(z)) - q \phi_0}{a(\phi(z))}, \quad z > 0, \quad \phi(0) = \phi_0. \quad (69) $$

The boundary value problem (69) is solved until the critical concentration is reached at a certain height $z_c$ denoting the sediment level. Above this level, the concentration assumes the constant value $\phi_c$ calculated from (68). The choice of $\phi_0$ is subject to the requirement that the concentration increases downwards. See Bürger et al. (1999) and Bürger and Concha (1998) for details. Consider the three steady states with parameters given in Table 1. We now prescribe the steady state $\phi_1(z)$ as the initial concentration pro-

| Table 1 Parameters of the steady states considered in Figure 17. |
|---------------------------------------------------------------|
| $i$ | $q$ [10^{-4} m/s] | $\phi_0$ | $\phi_c$ | $f_s$ [10^{-4} m/s] | $z_c$ [m] |
|-----|-----------------|---------|--------|-----------------|--------|
| 1   | $-0.10$         | $0.41$  | $0.0072993352$ | $-0.041$ | $3.10$ |
| 2   | $-0.15$         | $0.38$  | $0.0104589127$ | $-0.057$ | $1.77$ |
| 3   | $-0.05$         | $0.42$  | $0.0036012260$ | $-0.021$ | $2.49$ |
After operating the ICT at this steady state for some time, we then change successively to the steady states \( \phi_2(z) \) and \( \phi_3(z) \). The changes in \( \phi_L(t) \) and \( q(t) \) will be described in detail now.

The ICT is operated at the steady state \( \phi_1(z) \) for \( 0 \leq t \leq t_1 = 50000 \) s. At \( t = t_2 \), we change the volume average velocity \( q \) to the next smaller value \( q_2 \). However, the value of the feed flux density should remain constant during this operation, therefore the boundary concentration value \( \phi_L^1 \) has to be changed to a new value \( \phi_L^{12} \), which is calculated from the feed flux continuity condition

\[
q^2 \phi_L^{12} + f_{bk}(\phi_L^{12}) = f^1,
\]

yielding \( \phi_L^{12} = 0.0076397602 \). The change from \( \phi_L^1 \) to \( \phi_L^{12} \) should be performed at such a time that the new value \( \phi_L^{12} \) is present above the sediment level at \( t = t_2 \). The change propagates as a rarefaction wave into the vessel. This rarefaction wave is marked by \( R_1 \) in Figure 17 b).

We assume that the relevant speed is

\[
\sigma_2 = q^1 + f_{bk}(\phi_L^{12}) = -5.0607 \times 10^{-4} \text{ m/s},
\]

![Fig. 17](image-url) Simulation of transitions between three steady states in an ICT: a) prescribed values of \( \phi_L \), b) settling plot (the iso-concentration lines correspond to the annotated values), c) prescribed values of \( q(t) \), d) the numerically calculated discharge concentration, together with the discharge concentrations of the target steady states, e) the numerically computed solids discharge flux, compared with prescribed values of the feed flux.
therefore the change from $\phi_l^2$ to $\phi_l^{12}$ is performed at
$$t_3=t_5+ \frac{L-z_3^2}{\sigma_5} = 595182 \text{ s} = 165.3 \text{ h}.$$  

It should be noted that the feed flux does not remain precisely constant; it is different from $f_f$ in the small time interval $[t_3, t_4]$, during which we have

$$f_f = f_f^{12} = q^2 \phi_l^{12} + f_{bk}(\phi_l^{12}) = f_f^1 + (q^1 - q^2) \phi_l^{12}$$
$$= -0.04138 \times 10^{-4} \text{ m/s}.$$

From Figure 17 e) it becomes apparent that for $t>t_3$, the actual solids discharge flux density $f_d(t) = q^2 \phi_l(t)$ is larger than the feed flux $f_f$ prescribed. This leads to a slow emptying of the vessel and the sediment level falls at almost constant speed. It may therefore be estimated that it will have fallen to the height $z_2$ of the next target steady state by $t = 316560 \text{ s} = 87.9 \text{ h}$. At that time, the concentration value $\phi_l^2$ corresponding to the new feed flux density $f_f^2$ should have propagated to the sediment level. Again, this change propagates downwards as a rarefaction wave (marked by $R_2$ in Figure 17 b)). The relevant propagation velocity may be taken as

$$\sigma_6 = q^2 + f_{bk}(\phi_l^2) = -4.7746 \times 10^{-4} \text{ m/s},$$

hence we change $\phi_l$ again from $\phi_l^{12}$ to $\phi_l^2$ at

$$t_3 = 316560 \text{ s} + \frac{L-z_2^2}{\sigma_6} = 311854 \text{ s} = 86.6 \text{ h}.$$  

For $t>t_3$, both the value of $q$ and the feed flux $f_f$ correspond to the steady state $\phi_l(z)$ given in Table 1. Although this value is not prescribed explicitly, we observe in Figures 17 d) and e) that both the discharge flux and the discharge concentration converge to the appropriate values pertaining to the target steady state $\phi_l(z)$. At $t = 600000 \text{ s} = 166.7 \text{ h}$ we wish to change to the next steady state by changing $q$ from $q^2$ to $q^3$. The feed flux above the sediment is assumed to remain constant. Therefore $\phi_l^2$ is changed to the value $\phi_l^{23} = 0.0113417003$ which is calculated from

$$q^2 \phi_l^{12} + f_{bk}(\phi_l^{12}) = f_f^{12}.$$  

Again, the change from $\phi_l^2$ to $\phi_l^{23}$ propagates as a rarefaction wave into the vessel (marked by $R_3$ in Figure 17 b)). The value $\phi_l^{23}$ propagates at speed

$$\sigma_6 = q^2 + f_{bk}(\phi_l^{23}) = -4.6338 \times 10^{-4} \text{ m/s},$$

hence the change is performed at

$$t_4 = t_3 + \frac{L-z_3^2}{\sigma_6} = 595182 \text{ s} = 165.3 \text{ h}.$$  

Similar to the change from $\phi_l^1$ to $\phi_l^{12}$, the feed flux assumes a value in the time interval $[t_4, t_5]$ which is slightly different from $f_f^3$, namely, there we have

$$f_f = f_f^{23} = q^2 \phi_l^{23} + f_{bk}(\phi_l^{23}) = f_f^2 + (q^1 - q^2) \phi_l^{23}$$
$$= -0.0559 \times 10^{-4} \text{ m/s}.$$

After $t=t_5$, the feed flux exceeds the discharge flux in absolute value, as can be conceived from Figure 17 e). This causes a rise of the sediment level, again taking place at apparently constant speed, and it will have attained the last desired level $z_5$ when $t = 672240 \text{ s} = 186.7 \text{ h}$. At that time, the last value of the feed flux $f_f^3$ should be valid above the sediment level. In contrast to the previous changes of $\phi_l$, the change from $\phi_l^{23}$ to $\phi_l^3$ propagates as a shock (marked by $S_1$ in Figure 17 b)) with the speed

$$\sigma_7 = q^2 + f_{bk}(\phi_l^{23})_1 - f_{bk}(\phi_l^3) = -5.1391 \times 10^{-4} \text{ m/s}.$$  

This discontinuity reaches the sediment level at the desired time if the change from $\phi_l^{23}$ to $\phi_l^3$ is done at

$$t_5 = 672240 \text{ s} + \frac{L-z_5^2}{\sigma_7} = 669301 \text{ s} = 185.9 \text{ h}.$$  

After $t=t_5$, no more changes are made. Figures 17 b), d) and e) indicate convergence to the third steady state. The simulation is terminated after a simulated time of $T = 300 \text{ h} = 1080000 \text{ s}$.

5. Discussion

We now comment on some of our results that are outlined in Sections 3 and 4, and put them in the appropriate perspective of current research efforts, both our own and those of other groups. We begin with a remark related to the way Petty (1975) extended Kynch’s theory to continuous sedimentation by modeling feed and discharge by boundary conditions. This description was the starting point for the treatment presented in Section 3.3 and gave rise to the investigation of conservation laws with boundary conditions. However, this model still has severe shortcomings, among them the neglected clarification zone, that is the zone above the feed level to which solids may pass in the case of overflow, and the violation of conservation of mass principles due to the choice of boundary conditions, since entropy boundary conditions lead to mathematical well-posedness, but may be physically incorrect. More appropriate models for continuous sedimentation within Kynch’s theory were studied by Diehl in a recent series of papers (Diehl 1995, 1996, 1997, 2000, 2001), in which he analyses a model of continuous sedimentation in a
so-called settler-clarifier, in which the feed, discharge and overflow mechanisms are merely expressed by discontinuities of the flux-density function at the bottom, at the (intermediate) feed level and at the top of the vessel, and an additional singular source term at the feed level. Conservation of mass yields jump conditions valid across these discontinuities. The apparent advantages of this elegant model consist in the completeness of the treatment (since the overflow zone above the feed level, z=L, has so far not been considered in our work) and in the avoidance of boundary conditions, such that only initial conditions need to be considered. Although exact solutions can be constructed for all practically relevant cases, a global existence and uniqueness result is not yet available, so we do not yet wish to elaborate on this sedimentation model here, but recommend studying Diehl's original papers.

We now discuss some issues related to the phenomenological theory of sedimentation. It should be mentioned that the postulate of a constitutive equation of the type \( \sigma_e = \sigma_e(\phi) \) follows widespread usage in the engineering literature, see Bustos et al. (1999) and the references cited therein and recent handbooks on solid-liquid separation such as Rushton et al. (2000), Wakeman and Tarleton (1999) and Concha (2001). One should, however, bear in mind that this relationship is a strong (albeit in many cases useful) simplification, and that different approaches for \( \sigma_e \), although altering the nature of the resulting model equation, could possibly describe better observed consolidation behaviour. In fact, several researchers recently proposed alternate equations for the effective solid stress function. Some of them suggest expressing \( \sigma_e \) as an integral (with respect to height) of a new concentration-dependent phenomenological function (Dreher 1997, Toorman and Huysenstruyt 1997, Toorman 1999). It can be easily seen that in the present model framework this will lead again to a first-order equation, i.e. one essentially falls back to the equation (7) of Kynch's kinematic sedimentation theory, with all its well-known shortcomings. A different, and to our view potentially more promising approach was advanced by Zheng and Bagley (1998, 1999), who proposed an effective stress equation (Eq. 18 of Zheng, 1998) that depends on both the value of the local solids concentration and its rate of change. Thereby the necessity to refer to a critical concentration is removed. However the appropriate mathematical framework, in which the resulting mathematical model should be studied, still remains to be explored.

Another natural question arising from the presentation of Section 4 is whether the phenomenological model extends to several space dimensions and if so, under which extensions. The constitutive assumptions introduced in Section 4 remain valid if the one-dimensionality of the motion is no longer imposed, that is, if a truly multidimensional framework is considered. It is then straightforward to derive the following field equations, which replace (54)-(57):

\[
\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{q} + f_{ik}(\phi) \mathbf{k}) = \nabla \cdot (a(\phi) \nabla \phi), \tag{70}
\]

\[
\nabla \cdot \mathbf{q} = 0, \tag{71}
\]

\[
\nabla \mathbf{p}_e = -\alpha \rho g \mathbf{k} - \nabla \sigma_e(\phi). \tag{72}
\]

In two or three space dimensions, the unknown flow variables are the concentration \( \phi \), the volume average flow velocity field \( \mathbf{q} \) and the excess pore pressure \( \mathbf{p}_e \). The latter can be calculated a posteriori from the concentration distribution. In one space dimension, however, \( q=q(t) \) is determined by a boundary condition, and only solving the scalar equation (62) for \( \phi \) requires computational effort.

Schneider (1982, 1985) was the first to observe a remarkable property of equations (70)-(72): taking the curl of equation (72) reveals that \( \phi \) depends only on the vertical space coordinate and on time wherever it is continuous. The same will then be true for the vertical component of \( \mathbf{q} \). Under specific assumptions on the geometry of the vessel considered, the velocity field \( \mathbf{q} \) can be determined from suitable boundary conditions. However, since equation (72) does not depend on \( \mathbf{q} \), equations (70)-(71) are in most circumstances not sufficient to determine that quantity, and in any case the resulting mathematical model is not well posed and seems to be of little practical use (Bürger and Kunik 2001).

The independence of equation (72) from \( \mathbf{q} \) is, of course, a result of the simplifications performed based on the dimensional analysis. As mentioned before, the more general phenomenological sedimentation-consolidation theory, in the sense that not only several space dimensions but also viscous stresses are considered, and that fewer terms are neglected, is developed by Bürger, Wendland and Concha (2000e). That analysis leads to a set of equations that are similar to (70)-(72), but in which the analogue of equation (72) contains additional viscous and advective acceleration terms. That equation, together with (71), represents a nonlinear version of the well-known Navier-Stokes equations for an incompressible fluid, to which (in the proper sense) they reduce in the case of a pure fluid (\( \phi \equiv 0 \)).
by Bürger et al. (2000e) (see also Bürger 2000) are sufficient for the computation of \( \phi, q \) and \( p_e \). However, some new source terms describing the interaction between the evolution of the concentration distribution, or kinematic waves, and the average flow field appear. In one space dimension, these terms affect only the excess pore pressure distribution, so that the interaction they describe becomes effective only in a truly two- or three-dimensional setup. Schneider (1985) pointed out that this interaction makes the two- or three-dimensional treatment qualitatively different from what is known in one space dimension.

The significance of these terms and their actual magnitude has not yet been analysed. Moreover, if the viscous stress tensors are not neglected, additional difficulties arise from the necessity to relate the solid and fluid phase viscosities, which are theoretical variables, to the effective viscosity of the mixture, which can be measured experimentally. Obviously additional steps of progress in the theoretical understanding of the phenomenological framework have to be made before a definite multidimensional model for sedimentation with compression can be advocated. Having said this, some recent results by Bürger et al. (2001d) are available.

Finally, we mention that the phenomenological framework has led to a variety of similar mathematical models of spatially one-dimensional solid-liquid separation processes of flocculent suspensions such as centrifugation (Bürger and Concha 2001, Bürger and Karlsen 2001a) pressure filtration (Bürger et al. 2001b, Garrido et al. 2001) of flocculated suspensions, and can be regarded as the foundation of a robust and unified theory of solid-liquid separation of flocculated suspensions (Bürger et al. 2001a, Garrido et al. 2002).

6. Concluding remark

This paper has outlined the development of research in sedimentation and thickening during the last century from the invention of the Dorr thickener in 1905 onwards. The development that took place in the first half of that period was perhaps best characterized in the introduction of Roberts’ paper (1949): “Prior to 1916, thickening was an art, and any accurate decision as to what size of machine to install to handle a given tonnage of a specific ore must have been one of those intuitive conclusions, based on both intimate and extensive acquaintance with thickeners and ore pulps. Then in 1916 “knowledge of acquaintance,” became “knowledge about” with the publication of the Coe and Clevenger paper. The unit operation of thickening had graduated to the status of an engineering science.” He concludes that “Thickening has long held the status of an engineering science but it is still long way from being an exact science. On the other hand there is considerable “art” involved in deciding certain points; namely, what safety factor to specify to take care of possible changes in feed characteristics; whether to use the compression depth indicated or to increase the area to get a lower compression depth; how to prophesy island formation; what to do about island formation if it does occur, and so on. Careful, planned observation and reporting on part of operators is needed before some of these items can be reduced to the engineering science status.”

Roberts’ statements show that the title of his paper, Thickening — Art or Science?, must be understood as a serious question thrown up from the status of knowledge of 1949. He could not foresee that only three years later, Kynch’s paper would definitely turn sedimentation and thickening not only into an engineering science, but eventually also into a topic of profound and still very active research in a variety of disciplines, such as chemistry, biology medicine and, most notably from the authors’ viewpoint, mechanics and mathematics.

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Raimund Bürger is a scientific assistant at the Institute of Applied Analysis and Numerical Simulation of the University of Stuttgart, Germany. He has specialized in the formulation, analysis and numerics of mathematical models of solid-liquid separation processes. This includes in particular the mathematical analysis of the nonlinear partial differential equations involved, which frequently exhibit non-standard properties such as type degeneracy and discontinuous coefficients. He studied Mathematics at the Technische Universität Darmstadt in Germany and received his diploma degree in 1993. From 1995 to 2001 he held a position as research fellow within the Collaborative Research Center (Sonderforschungsbereich) 404 “Multifield Problems in Continuum Mechanics” at the University of Stuttgart. He obtained his doctoral degree in Applied Mathematics in 1996 under guidance of Professor Wolfgang L. Wendland and finished his habilitation in 2002. In 1997 he was awarded a scholarship by Fundación Andes, Chile, which he used for a post-doctoral research visit to Fernando Concha's group at the Department of Metallurgical Engineering of the University of Concepción during the period from July 1997 to August 1998. Raimund Bürger is co-author of the monograph Sedimentation and Thickening, more than 30 articles in refereed journals and 15 contributions to proceedings volumes. He is married, has a little son, and lives in Stuttgart, Germany.