Quantitative Form of Ball’s Cube Slicing in $\mathbb{R}^n$ and Equality Cases in the Min-Entropy Power Inequality

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Abstract. We prove a quantitative form of the celebrated Ball’s theorem on cube slicing in $\mathbb{R}^n$ and obtain, as a consequence, equality cases in the min-entropy power inequality. Independently, we also give a quantitative form of Khintchine’s inequality in the special case $p = 1$.

1. Introduction

In his seminal paper [1], Keith Ball proved that the maximal $(n - 1)$-dimensional volume of the section of the cube $C_n := [-\frac{1}{2}, \frac{1}{2}]^n$ by an hyperplane is $\sqrt{2}$. Therefore proving a conjecture by Hensley [10].

More precisely, for $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ with $|a| := \sqrt{a_1^2 + \cdots + a_n^2} = 1$, put $\sigma(a, t) = |C_n \cap H_{a,t}[n-1]$ for the volume of the intersection of the cube with the hyperplane $H_{a,t} = \{x \in \mathbb{R}^n : \langle x, a \rangle = t\}$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$ and $| \cdot |_{n-1}$ stands for the $(n-1)$-dimensional volume.

**Theorem 1** (Ball [1]). For all unit vector $a$ and all $t \in \mathbb{R}$, it holds $\sigma(a, t) \leq \sqrt{2}$. Moreover, equality holds only if $t = 0$ and $a$ has only two non-zero coordinates having value $\frac{1}{\sqrt{2}}$.

Ball’s result means that the maximal volume of the sections of the cube by hyperplanes are achieved when the section is a product of a $(n - 2)$-dimensional cube $C_{n-2}$ with the diagonal of a 2-dimensional cube $C_2$. The original proof is based on Fourier transform and series expansion. Alternative proofs can be found in [23] (based on distribution functions) and very recently in [22] (by mean of a transport argument).

Ball used Theorem 1 to give a negative answer to the famous Busemann-Petty problem in dimension 10 and higher [2]. His paper has inspired many research in convex geometry and is still very current. We refer to [11, 14, 16, 15, 6] to quote just a few of the most recent papers in the field and refer to the reference therein for a more detailed description of the literature.

Our first main result is the following quantitative version of Ball’s theorem.

**Theorem 2.** Fix $\varepsilon \in (0, \frac{1}{10})$. Let $a \in \mathbb{R}^n$ with $|a| = 1$ and $t \in \mathbb{R}$ be such that $\sigma(a, t) \geq (1 - \varepsilon)\sqrt{2}$. Then, there exists two indices $j_0, j_1$ such that

$$\frac{1}{\sqrt{2}}(1 - 37.5\varepsilon) \leq |a_{j_0}|, |a_{j_1}| \leq \frac{1}{\sqrt{2}}(1 + 2\varepsilon).$$

Moreover, $\sum_{j \neq j_0, j_1} a_j^2 \leq 50\varepsilon$ and in particular, for all $j \neq j_0, j_1$, $|a_j| \leq \sqrt{50}\varepsilon$. 

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Ball’s slicing theorem, combined with a result of Rogozin [27], was used by Bobkov and Chistyakov [3] to derive an optimal inequality for min-entropy power. Namely, they proved that
\[ N_N(X_1 + \cdots + X_n) \geq \frac{1}{2} \sum_{i=1}^\infty N_N(X_i) \]
for any independent random variables \( X_1, \ldots, X_n \), with \( N_N \) the min-entropy power we now define. We may call the latter Bobkov-Chistyakov’s min-entropy power inequality.

For a \((\mathbb{R} \text{ valued})\) random variable \( X \), the min-Entropy power is defined as
\[ N_N(X) = M^{-2}(X). \]
when
\[ M(X) := \inf \{ c : \mathbb{P}(X \in A) \leq c \ | A| \text{ for all Borel } A \} < \infty \]
and \( N_N(X) = 0 \) otherwise. When \( X \) is absolutely continuous with respect to the Lebesgue measure, with density \( f \), then \( M(X) = \| f \|_\infty \) is the essential supremum of \( f \) with respect to the Lebesgue measure.

The nomenclature “min-entropy power” is information theoretic. In that field the entropy power inequality refers to the fundamental inequality due to Shannon [28] which demonstrates that \( X_i \) independent random variables with densities \( f_i \) satisfy
\[ N(X_1 + \cdots + X_n) \geq \sum_i N(X_i), \]
where \( N(X) = e^{2h(X)} \) denotes the “entropy power”, with the Shannon entropy \( h(X) = -\int f(x) \log f(x) dx \). The Rényi entropy [25], for \( \alpha \in [0, \infty] \) defined as \( h_\alpha(X) = \frac{\int f^{\alpha}(x) dx}{1-\alpha} \) for \( \alpha \in (0, 1) \cup (1, \infty) \) and through continuous limits otherwise, gives a parameterized family of entropies that includes the usual Shannon entropy as a special case (by taking \( \alpha = 1 \)). It can be easily seen (through Jensen’s inequality, and the expression \( h_\alpha(X) = (\mathbb{E} f^{\alpha-1}(X))^{\frac{1}{1-\alpha}} \)) that for a fixed variable \( X \), the Rényi entropy is decreasing in \( \alpha \). Thus for a fixed variable \( X \), the parameter \( \alpha = \infty, h_\infty(X) = -\log \| f \|_\infty \), furnishes the minimizer of the family \( \{ h_\alpha(X) \}_\alpha \), and is often referred to as the “min-entropy”. Hence the terminology and notation min-entropy power used \( N_N(X) = e^{2h_\infty(X)} \) is in analogy with the Shannon entropy power \( N(X) = e^{2h(X)} \). Entropy power inequalities for the full class of Rényi entropies have been a topic of recent interest in information theory, see e.g. [4, 5, 17, 18, 21, 24, 26], and for more background we refer to [19] and references therein.

In [3] it was observed in a closing remark that the constant \( \frac{1}{2} \) in (1) is sharp. Indeed by taking \( n = 2 \) and \( X_1 \) and \( X_2 \) to be i.i.d. uniform on an interval (1) is seen to hold with equality. In the following theorem, we demonstrate that this is (essentially) the only equality case. In fact, thanks to the quantitative form of Ball’s slicing theorem above, we can derive a quantitative form of Bobkov-Chistyakov’s min entropy power inequality, see Corollary 6 below, that, in turn, allows us to characterize equality cases in (1) which constitutes our second main theorem.

**Theorem 3.** For \( X_1, \ldots, X_n \) independent random variables,
\[ N_N(X_1 + \cdots + X_n) \geq \frac{1}{2} \sum_{i=1}^n N_N(X_i) \]
with equality if and only if there exists \( i_1 \) and \( i_2 \) and \( x \in \mathbb{R} \) such that \( X_{i_1} \) is uniform on a set \( A \), and \( X_{i_2} \) is a uniform distribution on \( x-A \) and for \( i \neq i_1, i_2 \), \( X_i \) is a point mass.
Note that this is distinct from the $d$-dimensional case, see [20], where sharp constants can be approached asymptotically for $X_i$ i.i.d. and uniform on a $d$-dimensional ball. More explicitly, for $d \geq 2$, if $\Lambda$ denotes all finite collections of independent $\mathbb{R}^d$-valued random variables

$$
\sup_{X \in \Lambda} \frac{N_{\infty}(X_1 + \cdots + X_n)}{\sum_{i=1}^n N_{\infty}(X_i)} = \lim_{n \to \infty} \frac{N_{\infty}(Z_1 + \cdots + Z_n)}{\sum_{i=1}^n N_{\infty}(Z_i)},
$$

where $Z_i$ are i.i.d. and uniform on a $d$-dimensional Euclidean unit ball.

We end with a quantitative Khintchine's inequality. Though our result is independent, we stress that, as it is well known in the field and as it was pointed out by Ball himself in [1, Additional remarks], the inequality $\sigma(a, t) \leq \sqrt{2}$ of Theorem 1 is however related to Khintchine's inequalities.

Denote by $B_1, B_2, \ldots$ symmetric $-1, 1$-Bernoulli variables. Khintchine's inequalities assert that, for any $p \in (0, \infty)$ there exist some constant $A_p, A'_p$ such that for all $n$ and all $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ it holds

$$
A_p \left( \sum_{i=1}^n a_i^2 \right)^{\frac{p}{2}} \leq R_p(a) := \mathbb{E} \left[ \left| \sum_{i=1}^n a_i B_i \right|^p \right] \leq A'_p \left( \sum_{i=1}^n a_i^2 \right)^{\frac{p}{2}}.
$$

Such inequalities were proved by Khintchine in a special case [13], and studied in a more systematic way by Littlewood, Paley and Zygmund.

The best constants in (3) are known. This is due to Haagerup [8], after partial results by Steckin [29], Young [31] and Szarek [30]. In particular, Szarek proved that $A_1 = 1/\sqrt{2}$, that was a long outstanding conjecture of Littlewood, see [9].

The connection between Theorem 1 and Khintchine's inequalities goes as follows: as fully derived in [6], Ball's theorem can be rephrased as

$$
\mathbb{E} \left[ \left| \sum_{i=1}^n a_i \xi_i \right|^{-1} \right] \leq \sqrt{2} \left( \sum_{i=1}^n a_i^2 \right)^{-\frac{1}{2}},
$$

where $\xi_i$ are i.i.d. random vectors in $\mathbb{R}^3$ uniform on the centered Euclidean unit sphere $S^2$. As a result Ball's slicing of the cube can be seen as a sharp $L_1 - L_2$ Khintchine-type inequality.

Our last main result is a quantitative version of (the lower bound in) Khintchine's inequality for $p = 1$, that has the same flavour of Theorem 2 (thought being independent).

**Theorem 4.** Fix $\varepsilon \in (0, 1/100)$, an integer $n$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $|a| = 1$, satisfying

$$
R_1(a) \leq \frac{1 + \varepsilon}{\sqrt{2}}.
$$

Then, there exists two indices $i_1, i_2$ such that

$$
\frac{1 - 30\varepsilon}{\sqrt{2}} \leq |a_{i_1}|, |a_{i_2}| \leq \frac{1 + \varepsilon}{\sqrt{2}},
$$

Also, it holds $\sum a_i^2 \leq 57\varepsilon$ and in particular, for any $i \neq i_1, i_2$, $|a_i| \leq \sqrt{57\varepsilon}$.

The proofs of Theorem 2 and Theorem 4 are based on a careful analysis of Ball's integral inequality

$$
\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du \leq \sqrt{\frac{2}{s}}, \quad s \geq 2
$$
and, respectively, Haagerup’s integral inequality

\[ \int_0^{\infty} \left( 1 - \left| \cos \left( \frac{u}{\sqrt{s}} \right) \right| \right)^s \frac{du}{u^{p+1}} \geq \int_{-\infty}^{\infty} \left( 1 - e^{-u^2/2} \right) \frac{du}{u^{p+1}}, \quad s \geq 2 \]

in the special case \( p = 1 \). It is worth mentioning that Theorem 4 is restricted to \( p = 1 \) because the latter integrals can be made explicit only in that case. In order to deal with general \( p \) (at least \( p \in [p_0, 2] \), say, with \( p_0 \approx 1.85 \) implicitly defined through the Gamma function, see [8]), one would need to study very carefully the map \( F_p : s \mapsto \int_0^{\infty} \left( 1 - \left| \cos \left( \frac{u}{\sqrt{s}} \right) \right| \right)^s \frac{du}{u^{p+1}} \) and prove that it is increasing and then decreasing on \( [2, \infty) \) with careful control of its variations. The difficulty is also coming from the fact that, at \( p = p_0 \), \( F_p(2) = F_p(\infty) \). This in particular makes the quantitative version difficult to state properly. Indeed, for \( 0 < p < p_0 \), the extremizers in the lower bound of (3) are those \( a \) with two indices equal to \( 1/\sqrt{2} \) and the others vanishing. While for \( p > p_0 \), there are no extremizers for finite \( n \) (the "extremizer" is \( a = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \) in the limit (by the central limit theorem)). At \( p = p_0 \) the two "extremizers" coexist. Theorem 4 is therefore only a first attempt in the understanding of quantitative forms of Khintchine’s inequalities.

The next sections are devoted to the proof of Theorem 2, Theorem 3 and Theorem 4.

2. Quantitative slicing: Proof of Theorem 2

In this section, we give a proof of Theorem 2. We need first to recall part of the original proof by Ball, based on Fourier and anti-Fourier transform. We may omit some details that can be found in [1].

By symmetry we can assume without loss of generality that \( a_j \geq 0 \) for all \( j \). Reducing the dimension of the problem if necessary, we will further reduces to \( a_j \neq 0 \) for all \( j \).

In [1] it is proved that \( \sigma(a, t) \leq \frac{1}{a_j} \) for all \( j \) (see also [23, step 1]). The argument is geometric. Put \( e_j := (0, \ldots, 0, 1, 0, \ldots, 0) \) for the \( j \)-th unit vector of the canonical basis. Then it is enough to observe that the volume of \( C_n \cap H_{a,t} \) equals the volume of its projection to the hyperplane \( H_{e_j,0} \) (orthogonal to the \( j \)-th direction) divided by the cosine of the angle of \( a \) and \( e_j \), that is precisely \( a_j \), while the projection of \( C_n \) on \( H_{e_j,0} \) has volume 1. Therefore \( a_j \leq \frac{1}{\sqrt{2(1+\varepsilon)}} \leq \frac{1}{\sqrt{2}}(1+2\varepsilon) \) for all \( j \), which proves one inequality of Theorem 2.

We follow the presentation of [23, step 2]. Let \( \hat{S} \) be the Fourier transform of \( S : t \mapsto \sigma(a, t) \). By definition, we have

\[ \hat{S}(u) = \int_{\mathbb{R}} \sigma(a, t) e^{-2\pi i u t} dt = \int_{C_n} e^{-2\pi i u(x,a)} dx = \prod_{j=1}^{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i a_j u} dx_j = \prod_{j=1}^{n} \frac{\sin(\pi a_j u)}{\pi a_j u}. \]
Therefore, taking the anti-Fourier transform, Ball obtained the following explicit formula\(^1\) for \(\sigma(a, t)\):

\[
\sigma(a, t) = \int_{-\infty}^{\infty} \hat{S}(u)e^{2i\pi ut}du = \int_{-\infty}^{\infty} e^{2i\pi ut} \prod_{j=1}^{n} \frac{\sin(\pi a_j u)}{\pi a_j u} du.
\]

Applying Holder’s inequality, since \(a_1^2 + \cdots + a_n^2 = 1\), one gets

\[
\sigma(a, t) \leq \int_{-\infty}^{\infty} \prod_{j=1}^{n} \left( \int_{-\infty}^{\infty} \left| \frac{\sin(\pi a_j u)}{\pi a_j u} \right|^{1/a_j^2} du \right)^{a_j^2} du.
\]

(4)

Ball’s theorem follows from the fact that \(I(a_j) := \int_{-\infty}^{\infty} \left| \frac{\sin(\pi a_j u)}{\pi a_j u} \right|^{1/a_j^2} du \leq \sqrt{2}\) with equality only if \(a_j = 1/\sqrt{2}\). Changing variable, this is equivalent to proving that

\[
\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du \leq \sqrt{\frac{2}{s}}
\]

(5)

for every \(s > 2\) (for \(s = 2\) this is an identity). The latter is known as Ball’s integral inequality and was proved in [1]\(^2\) (see [23, 22] for alternative approaches).

One key ingredient in the proof of Theorem 2 is a reverse form of Ball’s integral inequality given in Lemma 5 below.

Turning to our quantitative question, observe that if for all \(j = 1, \ldots, n\), \(I(a_j) < (1 - \epsilon)\sqrt{2}\), then (4) would imply that \(\sigma(a, t) < (1 - \epsilon)\sqrt{2}\), a contradiction. Therefore, there must exist \(j_0\) such that \(I(a_{j_0}) \geq (1 - \epsilon)\sqrt{2}\). The aim is now to prove that \(a_{j_0}\) is close to \(1/\sqrt{2}\). In fact, changing variables \((s = 1/a_{j_0}^2 \geq 2(1 - \epsilon))\), we observe that

\[
I(a_{j_0}) = \int_{-\infty}^{\infty} \left| \frac{\sin(\pi a_{j_0} u)}{\pi a_{j_0} u} \right|^{1/a_{j_0}^2} du = \sqrt{s} \int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du.
\]

Hence, \(I(a_{j_0}) \geq (1 - \epsilon)\sqrt{2}\) is equivalent to saying that

\[
\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du \geq (1 - \epsilon) \sqrt{\frac{2}{s}}.
\]

Lemma 5 guarantees that, if \(s \geq 2\), then \(s = \frac{1}{a_{j_0}^2} \leq 2 + 50\epsilon\). If \(s \leq 2\) then \(\frac{1}{a_{j_0}^2} \leq 2\) which amounts to \(a_{j_0} \geq \frac{1}{\sqrt{2}}\). In any case

\[
a_{j_0} \geq \frac{1}{\sqrt{2} + 50\epsilon} \geq \frac{1}{\sqrt{2}(1 - \frac{25}{2}\epsilon)}
\]

\(^1\)An alternative explicit formula is given by Franck and Riede [7] (with different normalization). The authors ask if there could be an alternative proof of Ball’s theorem based on their formula.

\(^2\)An asymptotic study of such integrals can be found in [12].
since $\frac{1}{\sqrt{1+t^2}} \geq 1 - \frac{1}{2}t$ for any $t \in (0,1)$.

Iterating the argument, assume that for all $j \neq j_0$, $I(a_j) < (1-3\varepsilon)\sqrt{2}$. Since $I(a_{j_0}) \leq \sqrt{2}$, (4) would imply that

$$\sigma(a, t) < (1-3\varepsilon)^{1-a_{j_0}^2}\sqrt{2}$$
$$\leq (1-3\varepsilon)^{1-\frac{1}{2(1-\varepsilon)}}\sqrt{2}$$
$$\leq (1-\varepsilon)\sqrt{2}$$

where we used that $a_{j_0} \leq 1/(\sqrt{2}(1-\varepsilon))$ and some algebra. This is a contradiction. Therefore, there exists a second index $j_1 \neq j_0$ such that $I(a_{j_1}) \geq (1-3\varepsilon)\sqrt{2}$. Proceeding as for $j_0$, we can conclude that necessarily

$$a_{j_1} \geq \frac{1}{\sqrt{2}}(1 - \frac{75}{2}\varepsilon).$$

The expected result concerning $a_{j_0}$, $a_{j_1}$ follows.

Since $a_1^2 + \cdots + a_n^2 = 1$ we can conclude that

$$\sum_{j \neq j_0, j_1} a_j^2 \leq 1 - \frac{1}{2}(1 - \frac{25}{2}\varepsilon)^2 - \frac{1}{2}(1 - \frac{75}{2}\varepsilon)^2 \leq 50\varepsilon.$$ 

Thus, $a_j^2 \leq 50\varepsilon$ for all $j \neq j_0, j_1$. This ends the proof of the theorem.

**Lemma 5.** Let $s \geq 2$ be such that

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du \geq (1-\delta)\sqrt{\frac{2}{s}}$$

for some small $\delta > 0$. Then, $s \leq 2 + 50\delta$.

**Proof.** Set $\sigma = \frac{s}{2} - 1$. We use the technology developed in [1] where it is proved that

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du = \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sin^2(t)}{t^2} \right|^{1+\sigma} dt = 1 - \sum_{n=1}^{\infty} \frac{|\sigma(\sigma-1)\ldots(\sigma-n+1)|}{n!} \alpha_n$$

and

$$\sqrt{s} = 1 + \frac{1}{1+\sigma} = \frac{1}{\pi} \int_{-\infty}^{\infty} (e^{-t^2/\pi})^{1+\sigma} dt = 1 - \sum_{n=1}^{\infty} \frac{|\sigma(\sigma-1)\ldots(\sigma-n+1)|}{n!} \beta_n$$

with

$$\alpha_n := \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-t^2/\pi} \left(1 - e^{-t^2/\pi}\right)^n dt, \quad \beta_n := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(t)}{t^2} \left(1 - \frac{\sin^2(t)}{t^2}\right)^n dt.$$ 

Therefore, the assumption

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du \geq (1-\delta)\sqrt{\frac{2}{s}}$$

can be recast

$$\sum_{n=1}^{\infty} \frac{|\sigma(\sigma-1)\ldots(\sigma-n+1)|}{n!} (\beta_n - \alpha_n) \leq \delta \sqrt{\frac{1}{1+\sigma}}.$$ 

Note that, in [1], it is proved that $\alpha_n < \beta_n$ so that the left hand side of the latter is positive and in fact an infinite sum of positive terms. Hence, the first term of the sum must not exceed the right hand side. Since $\beta_1 = \frac{1}{3}$ and $\alpha_1 = \frac{\sqrt{2}-1}{\sqrt{2}}$, it holds

$$\sigma \frac{3 - 2\sqrt{2}}{3\sqrt{2}} = \sigma (\beta_1 - \alpha_1) \leq \delta \sqrt{\frac{1}{1+\sigma}} \leq \delta.$$
Returning to the variable $s$ it follows that $s \leq 2 + \delta \frac{6\sqrt{2}}{3-2\sqrt{2}}$ from which the expected result follows since $\frac{6\sqrt{2}}{3-2\sqrt{2}} \simeq 49.46 \leq 50$. \hfill \Box

3. MIN-ENTROPY POWER INEQUALITY

In this section we extend the quantitative slicing results for the unit cube, to a quantitative version (Corollary 6 below) of Bobkov and Chistyakov’s min-entropy power inequality (Inequality (1)) for random variables in $\mathbb{R}$. Then we prove the full characterization of extremizers of this min-entropy power inequality, i.e. we prove Theorem 3.

The quantitative version of Bobkov and Chistyakov’s min-entropy power inequality reads as follows.

**Corollary 6.** For $X_i$ independent random variables and $\varepsilon \in (0, 1/75)$ if

$$N_\infty \left( (1 - \varepsilon) \sum_{i=1}^{n} X_i \right) \leq \frac{1}{2} \sum_{i=1}^{n} N_\infty (X_i),$$

then there exists indices $i_o$ and $i_1$ such that

$$\left(1 - 37.5\varepsilon\right)^2 \left( \frac{1}{2} \sum_{i=1}^{n} N_\infty (X_i) \right) \leq N_\infty (X_{i_o}) \leq (1 + 2\varepsilon)^2 \left( \frac{1}{2} \sum_{i=1}^{n} N_\infty (X_i) \right)$$

while

$$\sum_{i \neq i_o, i_1} N_\infty (X_i) \leq 50\varepsilon \sum_{i=1}^{n} N_\infty (X_i).$$

Its proof relies on the following result by Rogozin.

**Theorem 7** (Rogozin [27]). For $X_i$ independent random variables, let $Z_i$ be independent random variables uniform on an origin symmetric interval chosen such that $N_\infty (X_i) = N_\infty (Z_i)$, with the interpretation that $Z_i$ is deterministic, and equal to zero, in the case that $N_\infty (X_i) = 0$. Then,

$$N_\infty (X_1 + \cdots + X_n) \geq N_\infty (Z_1 + \cdots + Z_n).$$

Note that our framework here is formally more general than [27] and [3].

**Proof of Corollary 6.** Suppose that, for $\delta > 1$

$$N_\infty (X_1 + \cdots + X_n) \leq \frac{\delta}{2} \sum_{i=1}^{n} N_\infty (X_i),$$

then by Theorem 7,

$$N_\infty (Z_1 + \cdots + Z_n) \leq \frac{\delta}{2} \sum_{i=1}^{n} N_\infty (X_i).$$

Writing $U_i = \frac{Z_i}{\sqrt{N_\infty (Z_i)}}$ and $\theta_i = \frac{N_\infty (X_i)}{\sum_j N_\infty (Z_j)}$ we can re-write this inequality as

$$N_\infty (\theta_1 U_1 + \cdots + \theta_n U_n) \leq \frac{\delta}{2}$$
where we observe that \( \theta = (\theta_1, \ldots, \theta_n) \) is a unit vector and \( U = (U_1, \ldots, U_n) \) is the uniform distribution on the unit cube. Moreover since \( U_i \) are log-concave and symmetric, \( \sum_i \theta_i U_i = \langle \theta, U \rangle \) is as well, and hence \( N_\infty(\theta_1 U_1 + \cdots + \theta_n U_n) = f_{\langle \theta, U \rangle}^{-2}(0) = \sigma^{-2}(\theta, 0) \). Thus, we have

\[
\sigma(\theta, 0) \geq \sqrt{\frac{2}{\delta}}.
\]

Now observe that the min-entropy is 2-homegeneous, i.e. \( N_\infty(\lambda X) = \lambda^2 N_\infty(X) \). Therefore, (6) reads as (7) with \( \delta = (1 - \varepsilon)^{-2} \). Hence

\[
\sigma(\theta, 0) \geq (1 - \varepsilon) \sqrt{2}.
\]

Thus by Theorem 2, there exist \( i_o \) and \( i_1 \) such that

\[
\frac{1}{\sqrt{2}}(1 - 37.5\varepsilon) \leq \theta_{i_o}, \theta_{i_1} \leq \frac{1}{\sqrt{2}}(1 + 2\varepsilon)
\]

while

\[
\sum_{i \neq i_o, i_1} \theta_i^2 \leq 50\varepsilon.
\]

Interpreting this in terms of the definition \( \theta_j = \sqrt{N_\infty(X_j)}/\sum_i N_\infty(X_i) \). This gives,

\[
(1 - 37.5\varepsilon)^2 \left( \frac{1}{2} \sum_{i = 1}^n N_\infty(X_i) \right) \leq N_\infty(X_{i_o}), N_\infty(X_{i_1}) \leq (1 + 2\varepsilon)^2 \left( \frac{1}{2} \sum_{i = 1}^n N_\infty(X_i) \right),
\]

while,

\[
\sum_{i \neq i_o, i_1} N_\infty(X_i) \leq 50\varepsilon \sum_{i = 1}^n N_\infty(X_i).
\]

This ends the proof of the Corollary. \( \square \)

**Proof of Theorem 3.** We distinguish between sufficiency and necessity. The former being simpler.

- **Necessity:**

Writing for convenience \( N_\infty(X_1) \geq N_\infty(X_2) \geq \cdots \geq N_\infty(X_n) \) by Corollary 6 when \( N_\infty(X_1) > 0 \), equality in (2) implies that

\[
N_\infty(X_1) = N_\infty(X_2), \quad N_\infty(X_k) = 0 \text{ for } k \geq 3.
\]

That is

\[
N_\infty(X_1 + X_2 + X_3 \cdots + X_n) = N_\infty(X_1 + X_2) = N_\infty(X_1).
\]

and since symmetric rearrangement preserves min-entropy and reduces the entropy of independent sums, \( N_\infty(X_1 + X_2) \geq N_\infty(X_1^* + X_2^*) \geq \frac{1}{2} (N_\infty(X_1^*) + N(X_2^*)) = N_\infty(X_1) = N_\infty(X_1 + X_2) \). Letting \( f, g \) represent the densities of \( X_1^* \) and \( X_2^* \) respectively, this implies

\[
\|f * g\|_\infty = \|f \|_\infty = f * g(0) = \int f(y)g(y)dy = \int_{\{f = \|f\|_\infty\}} \|f\|_\infty g(y)dy + \int f(y)g(y)dy = \|f\|_\infty
\]

which can only hold if \( \{g > 0\} \subseteq \{f = \|f\|_\infty\} \). Reversing the roles of \( f \) and \( g \), we must also have \( \{f > 0\} \subseteq \{g = \|g\|_\infty\} \). Since \( \{f = \|f\|_\infty\} \subseteq \{f > 0\} \) obviously holds, we have the following chain of inclusions,

\[
\{g > 0\} \subseteq \{f = \|f\|_\infty\} \subseteq \{f > 0\} \subseteq \{g = \|g\|_\infty\} \subseteq \{g > 0\}.
\]

For this it follows that \( X_1^* \) and \( X_2^* \) are i.i.d. uniform distributions.

Thus, \( X_1 \) and \( X_2 \) are uniform distributions as well. Without loss of generality we may assume that \( X_1 \) and \( X_2 \) are uniform on sets of measure 1, \( K_1 \) and \( K_2 \). Denote \( f_i = \mathbb{1}_{K_i} \),
Then $f_1 \ast f_2$ is uniformly continuous and $f_1 \ast f_2(x) \to 0$ with $|x| \to \infty$. Indeed, because continuous compactly supported functions are dense in $L^2$, it follows\footnote{Given an $\varepsilon > 0$, there exists $\varphi$ continuous and compactly supported such that $\|\varphi - g\|_2 < \varepsilon/3$. Since $\varphi$ is continuous and compactly supported, it is uniformly continuous, and hence for small enough $y$, $\|\varphi_{\tau_y} - y\|_2 < \varepsilon/3$. Thus $\|g_{\tau_y} - g\| \leq \|g_{\tau_y} - \varphi_{\tau_y}\| + \|\varphi_{\tau_y} - \varphi\| + \|\varphi - g\| < \varepsilon.$} that for $g_{\tau_y}(x) := g(x + y)$, $\|g_{\tau_y} - g\|_2 \to 0$ for $y \to 0$. Further $\|g_{\tau_{y_1}} - g_{\tau_{y_2}}\|_2 = \|g_{\tau_{y_1-y_2}} - g\|_2$, so that for $|y_1 - y_2|$ sufficiently small, $\|g_{\tau_{y_1}} - g_{\tau_{y_2}}\|_2$ can be made arbitrarily small as well. Thus,

$$|f_1 \ast f_2(x) - f_1 \ast f_2(x')| \leq \int |f_1(-y)||f_2(x+y) - f_2(x'+y)|dy$$

$$\leq \|f_1\|_2 \|(f_2)_{\tau_x} - (f_2)_{\tau_{x'}}\|_2$$

$$= \|(f_2)_{\tau_{x-x'}} - f_2\|_2$$

hence $f_1 \ast f_2$ is indeed uniformly continuous.

Taking $\varphi_i$ to be continuous, compactly supported functions approximating $f_i$ in $L^2$, we have

$$\|\varphi_1 \ast \varphi_2 - f_1 \ast f_2\|_\infty \leq \|f_1 \ast (\varphi_2 - f_2)\|_\infty + \|\varphi_2 \ast (\varphi_1 - f_1)\|_\infty$$

$$\leq \|f_1\|_2 \|\varphi_2 - f_2\|_2 + \|\varphi_2\|_2 \|\varphi_1 - f_1\|_2.$$

Since the right hand side goes to zero, and $\varphi_1 \ast \varphi_2$ is compactly supported, it must be true that $f_1 \ast f_2(x)$ tends to zero for large $|x|$. Thus $f_1 \ast f_2$ attains its maximum value at some point $x$, and thus we can rewrite the equality of the min-entropies of $X_1 + X_2$, $X_1$ and $X_2$, as $f_1 \ast f_2(x) = |K_1 \cap (x - K_2)| = |K_1| = |K_2| = 1$. Thus almost surely $x - K_1 = K_2$.

Put $Y = X_2 + \cdots + X_n$. By the same argument, since $N_\infty(X_1 + Y) = \frac{1}{2}(N_\infty(X_1) + N_\infty(Y))$, $Y$ is uniform on a set $x' - K_1$. Thus, $\text{Var}(Y) = \sum_{i=2}^n \text{Var}(X_i) = \text{Var}(X_2)$. Hence, for $i > 2$, $\text{Var}(X_i) = 0$ and the $X_i$ are deterministic. Letting $A = K_1$, the proof of necessity is complete.

\textbf{- Sufficiency:}

To prove sufficiency, assume that $X_1$ is uniform on a set $A$, $X_2$ uniform on $x - A$ and $X_i$ a point mass for $i \geq 3$ then,

$$N_\infty(X_1 + X_2 + X_3 + \cdots + X_n) = N_\infty(X_1 + X_2)$$

$$= \left\| \frac{1_A}{|A|} + \frac{1_{x-A}}{|A|} \right\|_\infty^{-2}.$$

Observe that

$$\frac{1_A}{|A|} \ast \frac{1_{x-A}}{|A|} = \frac{1}{|A|^2} \int 1_A(y)1_{x-A}(x-y)dy$$

$$= \frac{1}{|A|^2},$$

Thus $|A|^2 \geq N_\infty(X_1 + X_2)$ and it follows that $|A|^2 = N_\infty(X_1 + X_2) = N_\infty(X_1) = N_\infty(X_2)$. \hfill \Box

\section{Quantitative Khintchine’s Inequality}

In this section we prove Theorem 4 that resembles the proof of Theorem 2. We need to recall some results from [8].

Assume without loss of generality that $a_k \neq 0$ for all $k$. Put

$$F(s) = \frac{2}{\pi} \int_0^\infty \left( 1 - \left| \cos \left( \frac{t}{\sqrt{s}} \right) \right| \right)^s \frac{dt}{t^2}, \quad s > 0.$$
From [8, Lemma 1.4 (and its proof)], we can extract that

\[ F(s) = \frac{2}{\sqrt{\pi s}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \sqrt{\frac{2}{\pi}} \prod_{k=0}^{\infty} \left(1 - \frac{1}{(s+2k+1)^2}\right)^{\frac{1}{2}} \]

is an increasing function of \( s \), with \( F(2) = 1/\sqrt{2} \) and \( \lim_{s \to \infty} F(s) = \sqrt{2/\pi} \). Haagerup also proved [8, Lemma 1.3] that

(8) \[ R_1(a) \geq \sum_{k=1}^{n} a_k^2 F\left(\frac{1}{a_k^2}\right) \]

with the convention that \( a_k^2 F\left(\frac{1}{a_k^2}\right) = 0 \) if \( a_k = 0 \) (recall the definition of \( R_1 \) from (3)). For completeness, let us reproduce the argument using Nazarov and Podkorytov’s presentation [23]. From the identity

\[ |s| = \frac{2}{\pi} \int_{0}^{\infty} (1 - \cos(st)) \frac{dt}{t^2} \]

applied to \( s = \sum_{k=1}^{n} a_k B_k \), we have

\[
R_1(a) = \mathbb{E}\left(\left| \sum_{k=1}^{n} a_k B_k \right| \right) \\
= \frac{2}{\pi} \int_{0}^{\infty} \left(1 - \mathbb{E}\left(\cos\left(t \sum_{k=1}^{n} a_k B_k\right)\right)\right) \frac{dt}{t^2} \\
= \frac{2}{\pi} \int_{0}^{\infty} \left(1 - \prod_{k=1}^{n} \cos(a_k t)\right) \frac{dt}{t^2}
\]

where at the last line we used that

\[ \mathbb{E}\left(\cos\left(t \sum_{k=1}^{n} a_k B_k\right)\right) = \text{Re}\left(\mathbb{E}\left(e^{it \sum_{k=1}^{n} a_k B_k}\right)\right) = \prod_{k=1}^{n} \cos(a_k t). \]

Since \( \sum a_k^2 = 1 \), the following Young’s inequality \( \prod_{k=1}^{n} a_k^2 s_k \leq \sum a_k^2 s_k \) holds for any non-negative \( s_1, \ldots, s_n \). Therefore, (take \( s_k = |\cos(a_k t)|a_k^{-2} \)), it holds

\[
R_1(a) \geq \frac{2}{\pi} \int_{0}^{\infty} \left(1 - \prod_{k=1}^{n} |\cos(a_k t)|\right) \frac{dt}{t^2} \\
\geq \frac{2}{\pi} \int_{0}^{\infty} \left(1 - \sum_{k=1}^{n} a_k^2 |\cos(a_k t)|a_k^{-2}\right) \frac{dt}{t^2} \\
= \sum_{k=1}^{n} a_k^2 \frac{2}{\pi} \int_{0}^{\infty} \left(1 - |\cos(a_k t)|a_k^{-2}\right) \frac{dt}{t^2}
\]

which amounts to (8).
Now observe that \( R_1(a) \geq \max_k |a_k| \). Indeed, given \( k_o \), multiplying by \( B_{k_o} \), that satisfies \(|B_{k_o}| = 1\), it holds

\[
R_1(a) = E \left( |B_{k_o}| \left| \sum_{k=1}^{n} a_k B_k \right| \right) \\
= E \left( a_{k_o} + \sum_{k=1}^{n} a_k B_k B_{k_o} \right) \\
\geq E \left( a_{k_o} + \sum_{k=1}^{n} a_k B_k B_{k_o} \right) \\
= |a_{k_o}|.
\]

It follows by assumption that \(|a_k| \leq \frac{1 + \varepsilon}{\sqrt{2}}\) for any \( k \).

Assume that \( F(1/a_k^2) > \frac{1 + 3\varepsilon}{\sqrt{2}} \) for all \( k \). Then, by (8) and monotonicity of \( F \), it would hold

\[
R_1(a) \geq \sum_{k=1}^{n} a_k^2 F \left( \frac{1}{a_k^2} \right) > \frac{1 + \varepsilon}{\sqrt{2}}.
\]

This contradicts the starting hypothesis \( R_1(a) \leq \frac{1 + \varepsilon}{\sqrt{2}} \). Therefore, there exists at least one index, say \( k_o \), such that \( F(1/a_{k_o}^2) \leq \frac{1 + \varepsilon}{\sqrt{2}} \). Using Lemma 8 we can conclude that

\[
|a_{k_o}| \geq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 + 20\varepsilon}} \geq \frac{1 - 10\varepsilon}{\sqrt{2}}
\]

since \( \frac{1}{\sqrt{1 + t}} \geq 1 - \frac{t}{2} \) for any \( t \in (0, 1) \).

We iterate the argument. Assume that \( F(1/a_k^2) > \frac{1 + 3\varepsilon}{\sqrt{2}} \) for all \( k \neq k_o \). From (8) and monotonicity of \( F \), it would hold (recall that \(|a_k| \leq \frac{1 + \varepsilon}{\sqrt{2}} \) for any \( k \) and in particular for \( k_o \))

\[
R_1(a) \geq \sum_{k=1}^{n} a_k^2 F \left( \frac{1}{a_k^2} \right) > \frac{1 + 3\varepsilon}{\sqrt{2}} \sum_{k \neq k_o} a_k^2 + a_{k_o}^2 F \left( \frac{1}{a_{k_o}^2} \right) \geq \frac{1 + 3\varepsilon}{\sqrt{2}} \sum_{k \neq k_o} a_k^2 + a_{k_o}^2 F \left( \frac{2}{(1 + \varepsilon)^2} \right).
\]

Now Lemma 9 guarantees that \( F \left( \frac{2}{(1 + \varepsilon)^2} \right) \geq \frac{1 - \alpha \varepsilon}{\sqrt{2}} \), with \( \alpha = \frac{\varepsilon^2}{12} \), so that, since \( \sum_{k \neq k_o} a_k^2 = 1 - a_{k_o}^2 \) and \(|a_{k_o}| \leq \frac{1 + \varepsilon}{\sqrt{2}} \), it holds

\[
R_1(a) > \frac{1 + 3\varepsilon}{\sqrt{2}} \sum_{k \neq k_o} a_k^2 + a_{k_o}^2 \frac{1 - \alpha \varepsilon}{\sqrt{2}}
\]

\[
= \frac{1 + 3\varepsilon}{\sqrt{2}} + a_{k_o}^2 \left( \frac{1 - \alpha \varepsilon}{\sqrt{2}} - \frac{1 + 3\varepsilon}{\sqrt{2}} \right)
\]

\[
\geq \frac{1 + 3\varepsilon}{\sqrt{2}} - \left( \frac{1 + \varepsilon}{\sqrt{2}} \right)^2 \frac{(3 + \alpha)\varepsilon}{\sqrt{2}}
\]

\[
= \frac{1 + \varepsilon}{\sqrt{2}} + \frac{\varepsilon}{4\sqrt{2}} (4 - (3 + \alpha)(1 + \varepsilon)^2)
\]

\[
> \frac{1 + \varepsilon}{\sqrt{2}}
\]

since for \( \varepsilon \in (0, 1/100) \), \( 4 > (3 + \alpha)(1 + \varepsilon)^2 \). This again contradicts the hypothesis \( R_1(a) \leq \frac{1 + \varepsilon}{\sqrt{2}} \). Therefore, there exists a second index \( k_1 \neq k_o \), such that \( F(1/a_{k_1}^2) \leq \frac{1 + 3\varepsilon}{\sqrt{2}} \). Lemma 8 then
implies that
\[ |a_k| \geq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 + 60\varepsilon}} \geq \frac{1 - 30\varepsilon}{\sqrt{2}} \]
(since, again, \(1/\sqrt{1 + t} \geq 1 - \frac{t}{2}\)). This proves the first part of the Theorem.

For the second part we use the previous results together with \(\sum_{k=1}^{n} a_k^2 = 1\) to ensure that
\[ \sum_{k \neq k_0, k_1} a_k^2 = 1 - a_{k_0}^2 - a_{k_1}^2 \leq 1 - \left( \frac{1 - 10\varepsilon}{\sqrt{2}} \right)^2 - \left( \frac{1 - 30\varepsilon}{\sqrt{2}} \right)^2 \leq \frac{80}{\sqrt{2}} \varepsilon \leq 57\varepsilon. \]
This ends the proof of the theorem.

**Lemma 8.** Fix \(\varepsilon \in (0, 3/100)\) and \(s > 0\) such that \(F(s) \leq \frac{1 + \varepsilon}{\sqrt{2}}\). Then
\[ s \leq 2(1 + 20\varepsilon). \]

**Proof.** Assume that \(s \geq 2\) (otherwise there is nothing to prove). By expansion, \(F(s) = F(2) + \int_2^s F'(t)dt \leq \frac{1 + \varepsilon}{\sqrt{2}}\). Therefore, since \(F(2) = 1/\sqrt{2}\),
\[ \int_2^s F'(t)dt \leq \frac{\varepsilon}{\sqrt{2}}. \]
Observe that \(F(3) = \frac{4}{\sqrt{3}} \simeq 0.74 \geq 0.71 \simeq \frac{101}{\sqrt{2}} \geq \frac{1 + \varepsilon}{\sqrt{2}}\). Hence, since \(F\) is increasing, necessarily \(s \leq 3\). It follows that
\[ (s - 2) \inf_{2 \leq t \leq 3} F'(t) \leq \frac{\varepsilon}{\sqrt{2}} \]
and we are left with estimating \(\inf_{2 \leq t \leq 3} F'(t)\). Using the expression of \(F\) above as a product, we deduce that, for \(t \in (2, 3)\)
\[ F'(t) = F(t) \sum_{k=0}^{\infty} \frac{1}{(t + 2k)(t + 2k + 1)(t + 2k + 2)} \geq F(2) \sum_{k=0}^{\infty} \frac{1}{(2k + 3)(2k + 4)(2k + 5)} \geq \frac{1}{40\sqrt{2}} \]
where in the last inequality we used that \(F(2) = 1/\sqrt{2}\) and estimated from below the infinite sum by the first 5 terms\(^4\). The expected result follows. \(\square\)

**Lemma 9.** Fix \(\varepsilon \in (0, 1/100)\). Then
\[ F\left( \frac{2}{(1 + \varepsilon)^2} \right) \geq \frac{1 - \alpha \varepsilon}{\sqrt{2}} \]
with \(\alpha = \frac{\pi^2}{12}\).

**Proof.** By expansion,
\[ F\left( \frac{2}{(1 + \varepsilon)^2} \right) = F(2) + \int_2^{(1+\varepsilon)^2} F'(t)dt \geq \frac{1}{\sqrt{2}} - \left( \frac{2}{(1 + \varepsilon)^2} \right) \sup_{(1+\varepsilon)^2 \leq t \leq 2} F'(t). \]

\(^4\)Alternatively one can argue that \(\sum_{k=0}^{\infty} \frac{1}{(2k + 3)(2k + 4)(2k + 5)} \geq \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^3} = \frac{1}{8}(\zeta(3) - 1)\) where \(\zeta(3) \simeq 1.202 \geq 1.2\) is the Riemann zeta function, from which we deduce that the infinite series is bounded below by \(1/40\).
Now, as in the proof of Lemma 8, for any \( t \in \left( \frac{2}{1+\varepsilon^2}, 2 \right) \), it holds
\[
F'(t) = F(t) \sum_{k=0}^{\infty} \frac{1}{(t+2k)(t+2k+1)(t+2k+2)} \leq F(2) \sum_{k=0}^{\infty} \frac{1}{8(k+1)^2} = \frac{\pi^2}{48\sqrt{2}}
\]
where the inequality follows from the rough estimate \( (t+2k)(t+2k+1)(t+2k+2) \geq 8(k+1)^2 \), valid for any \( k \) and any \( t \in \left( \frac{2}{1+\varepsilon^2}, 2 \right) \) (this is trivial for \( t \geq 1 \) and \( k \geq 1 \), the case \( k = 0 \) has to be treated separately, details are left to the reader).

Combining with the previous estimate, we get
\[
F \left( \frac{2}{1+\varepsilon^2} \right) \geq \frac{1}{\sqrt{2}} \left( 1 - \frac{\pi^2}{48} \left( 2 - \frac{2}{(1+\varepsilon^2)^2} \right) \right) = \frac{1}{\sqrt{2}} \left( 1 - \frac{\pi^2}{24} \frac{2\varepsilon + \varepsilon^2}{(1+\varepsilon^2)^2} \right) \geq \frac{1}{\sqrt{2}} \left( 1 - \frac{\pi^2}{12} \varepsilon \right)
\]
which is the expected result. \( \square \)

**Remark 10.** The range \( \varepsilon \in (0, 1/100) \) in Theorem 4 is technical and here to guarantee that \( (1 + \varepsilon)/\sqrt{2} \leq \sqrt{2}/\sqrt{\pi} = \lim_{s \to \infty} F(s) \) and also that \( F(3) \geq (1 + \varepsilon)/\sqrt{2} \) (see the proof of Lemma 8 above).

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