Symplectic Novikov Lie Algebra

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Abstract

It is well known that a symplectic Lie algebra admit a left symmetric product. In this work, we study the case where this product is Novikov, we show that the left-symmetric product associated to the symplectic Lie algebra is Novikov if and only if it is associative. In this case the symplectic Lie algebra is called symplectic Novikov Lie algebra (SNLA). We show that any SNLA is completely reducible two-step solvable. The classification of 4-dimensional and nilpotent 6-dimensional, also some methods for building large classes of examples are presented. Finally, we give a geometric study of the affine connection associated with an SNLA.

1 Introduction

Novikov algebras arise in many areas of mathematics and physics. The Novikov structures appearing in connection with the Poisson brackets of hydrodynamic type [10]. The study of Novikov algebras was initiated by Zelmanov [22] then developed by several authors [6] and for the their classification problem see [8], [9] and [7]. The study of symplectic Lie groups (algebras) was developed by Bon-Yao Chu [11], Lichnerowitz, Medina and Ph. Revoy in [18] and [20].

A finite-dimensional algebra \((\mathfrak{g}, \cdot)\) over \(\mathbb{R}\) is called left-symmetric if it satisfies the identity

\[
\text{ass}(x, y, z) = \text{ass}(y, x, z) \quad \forall x, y, z \in \mathfrak{g},
\]

where \(\text{ass}(x, y, z)\) denotes the associator \(\text{ass}(x, y, z) = (x.y).z - x.(y.z)\). In this case, the commutator \([x, y] = x.y - y.x\) defines a bracket that makes \(\mathfrak{g}\) a Lie algebra. Clearly, each associative algebra product (i.e. \(\text{ass}(x, y, z) = 0, \forall x, y, z \in \mathfrak{g}\)) is a left symmetric product.

The algebra is called Novikov, if in addition

\[
(x.y).z = (x.z).y \quad \forall x, y, z \in \mathfrak{g},
\]

is satisfied. Let \(L_x\) and \(R_x\) denote the left and right multiplications by the element \(x \in \mathfrak{g}\), respectively. The identity (1) is now equivalent to the formula

\[
[L_x, L_y] = L_{[x,y]} \quad \forall x, y \in G
\]
or in other words, the linear map $L : \mathfrak{g} \to \text{End}(\mathfrak{g})$ is a representation of Lie algebras.

The identity (2) is equivalent to each of the following identities

$$[R_x, R_y] = 0 \quad \forall x, y \in G \quad (3)$$

$$L_{x,y} = R_y \circ L_x \quad \forall x, y \in G. \quad (4)$$

For more details on left-symmetric algebras, we refer the reader to the survey article [2] and the references therein (See as well [16]).

A symplectic Lie algebra $(\mathfrak{g}, \omega)$ is a real Lie algebra with a skew-symmetric non-degenerate bilinear form $\omega$ such that for any $x, y, z \in \mathfrak{g}$,

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0$$

this is to say, $\omega$ is a non-degenerate 2-cocycle for the scalar cohomology of $\mathfrak{g}$. We call $\omega$ a symplectic form of $\mathfrak{g}$. Note that in such case, $\mathfrak{g}$ must be even-dimensional. A fundamental example of symplectic Lie algebras are the Frobenius Lie algebras, i.e. Lie algebras admitting a non-degenerate exact 2-form. Symplectic Lie algebras are in one-to-one correspondence with simply connected Lie groups with left invariant symplectic forms. The geometry of symplectic Lie groups is an active field of research. There are several results on the construction of symplectic Lie group (see [20], [12], [4], [14]) and some classifications in low dimension ([21], [17], [14]).

It is known that (see [11] and [20]) the product given by

$$\omega(x, y, z) = -\omega(y, [x, z]) \quad \forall x, y \in \mathfrak{g} \quad (5)$$

induces a left symmetric algebra structure that satisfies $x.y - y.x = [x, y]$ on $\mathfrak{g}$, we say that the left symmetric product is associated with symplectic Lie algebra $(\mathfrak{g}, \omega)$. Geometrically, this is equivalent to existence in a symplectic Lie group a left-invariant affine structure (a left-invariant linear connection with zero torsion and zero curvature). It is easy to see (see [20]) that the left symmetric product associated with symplectic Lie algebra $(\mathfrak{g}, \omega)$ satisfies

$$\omega(x, y, z) = \omega(x, z, y) \quad \forall x, y, z \in \mathfrak{g} \quad (6)$$

and

$$\omega(x, y, z) + \omega(y, z, x) + \omega(z, x, y) = 0 \quad \forall x, y, z \in \mathfrak{g}. \quad (7)$$

The aim of this work is to study the case where the left symmetric product associated with symplectic Lie algebra $(\mathfrak{g}, \omega)$ is of Novikov.

Notations: For $\{e_i\}_{1 \leq i \leq n}$ a basis of $\mathfrak{g}$, we denote by $\{e^i\}_{1 \leq i \leq n}$ the dual basis on $\mathfrak{g}^*$ and $e^{ij}$ the two-form $e^i \wedge e^j \in \wedge^2 \mathfrak{g}^*$. For an endomorphism $f : \mathfrak{g} \to \mathfrak{g}$ we denote by $f^t : \mathfrak{g}^* \to \mathfrak{g}^*$ its dual and by $f^* : \mathfrak{g} \to \mathfrak{g}$ the symplectic adjoint endomorphism defined by

$$\omega(f(x), y) = \omega(x, f^*(y)) \quad \forall x, y \in \mathfrak{g}.$$
The paper is organized as follows: In section 2, we show that the left-symmetric product associated to \((g, \omega)\) is Novikov if and only if it is associative and we give some algebraic consequence. In section 3, we give a classification of 4-dimensional and nilpotent 6-dimensional SNLA, then we characterize the left-symmetric algebras \(h\) which defines a NSLA structures on symplectic cotangent Lie algebra \((h^* \times \ell h, \omega)\), this construction provides many examples of SNLA. In section 4, we study the effect of symplectic reduction as well as the effect of symplectic oxidation on NSLAs and we show that every SNLA is completely reducible. In section 4, the last section studies the affine structure \(\nabla\) associated with an SNLA we show that she is bi-invariant and completed if and only if the Lie algebra \(g\) is nilpotent. We also give a geometric study of the symplectic connection associated with \(\nabla\).

2 Definition and first properties

**Definition 1.** A symplectic Novikov Lie algebra (SNLA), is a symplectic Lie algebra \((g, \omega)\) with the associated left symmetric product is Novikov.

Our first result is the following

**Theorem 1.** Let \((g, \omega)\) be a symplectic Lie algebra. Then the left symmetric product associated with \((g, \omega)\) is Novikov if and only if it is associative.

**Proof.** Suppose that the left symmetric product \(\cdot\) associated with \((g, \omega)\) is Novikov. By using \((5)\) and \((6)\) we have for every \(x, y, z, t \in g\)

\[
\omega((x.y).z, t) = \omega(x.(z.y), t) \iff -\omega(z, [x.y, t]) = \omega(x, z.y) \\
\iff \omega(z, [x.y, t]) = \omega(z, [x, t.y]) \\
\iff \omega(z, (x.y).t - t.(x.y) - x.(t.y) + (t.y).x) = 0 \\
\iff \omega(z, (x.y).t - x.(t.y) + (t.x).y - t.(x.y)) = 0 \\
\iff \omega(z, 2\text{ass}(t, x, y)) = 0.
\]

Then the product \(\cdot\) is associative. Conversely suppose that the left symmetric product \(\cdot\) associated with \((g, \omega)\) is associative. By using \((5)\) and \((6)\) we have for every \(x, y, z, t \in g\)

\[
\omega((x.y).z, t) = \omega(x.(y.z), t) \iff \omega(y, [x, t.z]) = \omega(y, [x, t].z) \\
\iff \omega(y, (x.(t.z) - (t.z).x - (x.t).z + (t.x).z) = 0 \\
\iff \omega(y, (t.x).z - (t.z).x) = 0.
\]

Then the product \(\cdot\) is Novikov. \(\square\)

**Remark 1.** The product \(\cdot\) is Novikov and associative this results in the formulas

\[
R_{[x,y]} = 0, \quad \forall x, y \in g. \quad (8) \\
ad_{[x,y]} = I_{[x,y]} = [I_x, I_y], \quad \forall x, y \in g. \quad (9)
\]
Note that a Lie algebra admitting a Novikov structure must be solvable, see [6]. In the case of SNLA we have the result.

**Corollary 1.** Let \((\mathfrak{g}, \omega)\) be an SNLA. Then, the associated Lie algebra is two-step solvable.

**Proof.** For any \(x, y, z, t \in \mathfrak{g}\), by use that the product associated with \((\mathfrak{g}, \omega)\) is Novikov and associative we have

\[
[[x, y], [z, t]] = [x.y - y.x, z.t - t.z] = (x.y - y.x).(z.t - t.z) - (z.t - t.z).(x.y - y.x)
\]

\[
= x.y.z.t - x.y.t.z - y.x.z.t - y.t.x.z - z.t.x.y + z.t.y.x - t.z.y.x - t.z.y.x
\]

\[
= 0.
\]

This shows that the Lie algebra \(\mathfrak{g}\) is two-step solvable. \(\square\)

Recall that an algebra \((\mathfrak{g}, .)\) is called an LR-algebra, if the product satisfies the identities

\[
x.(y.z) = y.(x.z) \quad \text{and} \quad (x.y).z = (x.z).y
\]

for all \(x, y, z \in \mathfrak{g}\). Indicating that in an LR-algebra the left and right multiplication operators commute i.e. \([L_y, L_x] = 0\) and \([R_y, R_x] = 0\). From (8) and (9) we obtain the following corollary

**Corollary 2.** The SNLA \((\mathfrak{g}, \omega)\), is LR-algebra if and only if the Lie algebra \(\mathfrak{g}\) is two-step nilpotent.

### 3 Examples and Low dimensions SNLA

We give a complete classification of the four-dimensional SNLA and six-dimensional nilpotent SNLA.

#### 3.1 Low dimensions SNLA

Start with the non-abelian two-dimensional Lie algebra \(\text{aff}(\mathbb{R}) = \text{span}\{e_1, e_2\}\), the symplectic Lie algebra \((\text{aff}(\mathbb{R}), e^{12})\) with \([e_1, e_2] = e_2\) is SNLA and the Novikov product is \(e_1.e_1 = -e_1, \ e_2.e_1 = -e_2\).

We use the classification of four-dimensional symplectic Lie algebras given by Ovando in [11] and the classification of six-dimensional symplectic nilpotent Lie algebras given in [17] (see also [14] for a more recent list). One obtains after a direct computation.

**Proposition 1.** The four-dimensional SNLAs are listed below:
1. The symplectic Lie algebras \( \mathfrak{d}_{4,1} \) and \( \mathfrak{t}'_2 \) are Frobenius Lie algebras (their symplectic forms are exacts).

2. The Lie algebras \( L_{6,18}, L_{6,23} \) and \( L_{6,25} \) are the only two-step nilpotent SNLA. It is well known that [8] all two-step nilpotent Lie algebra admits a Novikov structure, we notice that this result is no longer correct in the case of SNLA.

3. Being NSLA is a property that depends on both the Lie structure and the symplectic structure. For example the left-symmetric product associated with the symplectic Lie algebra \( (\mathfrak{d}_{4,1}, e^{12} - e^{34} + e^{24}) \) is not Novikov.
We obtain the following corollary

**Corollary 3.** Any nilpotent SNLA of dimension less than or equal to 6 is at most a 3-step nilpotent Lie algebra.

### 3.2 Symplectic Novikov cotangent Lie algebra

Let \((h, \cdot)\) be a left-symmetric algebra, for any \(x \in h\), \(\ell_x, r_x : h \to h\) denote the left and the right multiplication by \(x\) given by \(\ell_x y = x y\) and \(r_x y = y x\), respectively and \(\text{ad}_x\) is the endomorphism of \(h\) given by \(\text{ad}_x y = [x, y]\).

A **symplectic cotangent Lie algebra** \((h^* \times \ell h, \omega)\) of \(h\), is the vector space \(h^* \times h\) endowed with the Lie bracket

\[
[(\alpha, x), (\beta, y)] = (\ell^t_x \alpha - \ell^t_x \beta, [x, y]), \quad x, y \in h, \text{ and } \alpha, \beta \in h^*
\]

and non-degenerate 2-cocycle

\[
\omega((\alpha, x), (\beta, y)) = \beta(x) - \alpha(y), \quad x, y \in h, \text{ and } \alpha, \beta \in h^*.
\]

**Proposition 2.** The symplectic cotangent Lie algebra \((h^* \times \ell h, \omega)\) is SNLA if and only if \((h, \cdot)\) is Novikov associative algebra.

**Proof.** We denote by \(\odot\) the left symmetric product associated with \((h^* \times \ell h, \omega)\). For any \(x, y, z \in h\) and for any \(\alpha, \beta, \gamma \in h\), we have

\[
\omega((\alpha, x) \odot (\beta, y), (\gamma, z)) = -\omega((\beta, y), ((\alpha, x), (\gamma, z)))
\]

\[
= -\omega((\beta, y), (\ell^t_x \alpha - \ell^t_x \gamma, [x, z]))
\]

\[
= \text{ad}^t_x \beta(z) + \ell^t_x \gamma(y) - r^t_y \alpha(z).
\]

By asking \((\alpha, x) \odot (\beta, y) = (\lambda, s)\) and for \(\gamma = 0\), we get that \(\lambda = -\text{ad}^t_x \beta + r^t_y \circ \alpha\), similarly, if \(z = 0\), we obtain that \(s = \ell^t_x y = x y\). Therefore

\[
(\alpha, x) \odot (\beta, y) = (-\text{ad}^t_x \beta + r^t_y \circ \alpha, x y).
\]

On the other hand, the product \(\odot\) is Novikov then

\[
((\gamma, z) \odot (\beta, y)) \odot (\alpha, x) = ((\gamma, z) \odot (\alpha, x)) \odot (\beta, y)
\]

a direct computation yields

\[
(-\text{ad}^t_{x, y} \alpha - r^t_x \circ \text{ad}^t_x \beta + r^t_y \circ r^t_x (\gamma), (z, y), x) = (-\text{ad}^t_{x, y} \beta - r^t_y \circ \text{ad}^t_x \alpha + r^t_y \circ r^t_x (\gamma), (z, x), y)
\]

then

\[
[r_x, r_y] = 0 \quad \text{and} \quad \text{ad}_{x, y} z = \text{ad}_y \circ r_x (z),
\]

for all \(x, y, z \in h\). This is equivalent to,

\[
[r_x, r_y] = 0 \quad \text{and} \quad \text{ass}(x, z, y) = 0,
\]

for all \(x, y, z \in h\). Hence the result. \(\square\)
Example 1. 1. All n-dimensional Novikov algebra with an abelian Lie algebra define an 2n-dimensional symplectic Novikov cotangent Lie algebra.

2. The following table gives all three-dimensional Novikov associative algebras $A_{3i}$:

- $A_{3,2}$: $e_1^2 = e_2$
- $A_{3,3}$: $e_1^2 = e_2, e_1.e_2 = e_2.e_1 = e_3$
- $A_{3,4}$: $e_1^2 = e_3, e_2^2 = e_3$
- $A_{3,5}$: $e_1^2 = -e_3, e_2^2 = e_3$
- $g_{3,1}$: $e_1^2 = e_2, e_1.e_2 = (a + 1)e_3, e_2.e_1 = ae_3$
- $g_{3,2}$: $e_1^2 = ae_3, e_1.e_2 = e_3, e_2^2 = e_3$
- $g_{3,3}$: $e_1.e_2 = \frac{1}{2}e_3, e_2.e_1 = -\frac{1}{2}e_3$.

For the list of three-dimensional real Novikov algebras see [7].

Recall the following well-known construction of Novikov algebra (see [7]). This construction can generate many examples of NSLAs.

**Proposition 3.** Let $(g,.)$ be an associative, commutative algebra and $D$ a derivation of $(g,.)$, i.e., satisfying $D(x.y) = D(x).y + x.D(y)$. Then the product $x*y = x.D(y)$ is Novikov.

In particular, it defines a Novikov structure on the Lie algebra given by

$$[x,y] := x*y - y*x = x.D(y) - y.D(x), \quad \forall x, y \in g$$

Note that $D \in \text{Der}(g,.)$ implies $D \in \text{Der}(g,[,])$. Thus, we obtain the following result.

**Corollary 4.** Let $(h,.)$ be an associative, commutative algebra and $D \in \text{Der}(h,.)$. Then the symplectic cotangent Lie algebra $(h^* \times h, \omega)$ is SNLA if and only if

$$x.y.D^2(z) = 0, \quad \forall x, y, z \in h.$$ (10)

In particular if $D^2 = 0$, then $(h^* \times h, \omega)$ is an SNLA.

## 4 Reduction and SNLA Oxidation

Let $(g, \omega)$ be a symplectic Lie algebra and $h$ an isotropic (i.e. $\omega_{D(g) \times D(g)} = 0$ or in other words $h \subseteq h^+$) ideal. The orthogonal $h^\perp$ is a subalgebra of $g$ which contains $h$, and therefore $\omega$ descends to a symplectic form $\overline{\omega}$ on the quotient Lie algebra $\overline{g} = g^\perp / h$. The symplectic Lie algebra $(\overline{g}, \overline{\omega})$ is called the *symplectic reduction of $(g, \omega)$ with respect to the isotropic ideal $h$. Recall that, a symplectic Lie algebra is called *completely reducible* if it can be symplectically reduced (in several steps) to the trivial symplectic algebra. Note that, every four-dimensional symplectic Lie algebras, nilpotent symplectic Lie algebras and completely solvable symplectic Lie algebras are completely reducible. Moreover,
an irreducible symplectic Lie algebra is a symplectic Lie algebra which does not admit a reduction, that is, if it does not have a non-trivial isotropic ideal (for more details see [4]).

**Lemma 1.** Let \((g, \omega)\) be an SNLA then all symplectic reductions \((\mathfrak{g}, \mathfrak{w})\) are SNLA.

**Proof.** It suffices to remark that the left-symmetric product associated with \(\mathfrak{w}\) is given by \(x.y = \mathfrak{w}y\) for \(x, y \in \mathfrak{h}^\perp\). \(\square\)

**Theorem 2.** An irreducible no trivial symplectic Lie algebra is never SNLA.

**Proof.** Let \((g, \omega)\) be a irreducible symplectic Lie algebra, Theorem 3.16 [3] gives a characterization of irreducible symplectic Lie algebra, more precisely it shows that the commutator ideal \(D(g)\) is a maximal abelian ideal of \(g\), which is non-degenerate with respect to \(\omega\), the symplectic Lie algebra \((g, \omega)\) is an orthogonal semi-direct sum of an abelian symplectic subalgebra \((\mathfrak{h}, \omega|_{\mathfrak{h}})\) and the ideal \((D(g), \omega|_{D(g)})\) and we can choose a basis \(\{f_1, f_1, \ldots, f_h, f_h, e_{1,1}, e_{1,2}, \ldots, e_{1,m}, e_{2,m}\}\) of \(g\), such that \(\{f_1, f_1, \ldots, f_h, f_h\}\) is a basis of \(\mathfrak{h}\), \(\{e_{1,1}, e_{1,2}, \ldots, e_{1,m}, e_{2,m}\}\) is a basis of \(D(g)\). In this basis the Lie bracket is given for \(\lambda_i^k, \lambda_i^k \in \mathbb{R}\) by

\[
[f_i, e_{1,k}] = -\lambda^k_i e_{2,k} \quad [f_i, e_{2,k}] = \lambda^k_i e_{1,k}
\]

\[
[f_i, e_{1,k}] = -\lambda^k_i e_{2,k} \quad [f_i, e_{2,k}] = \lambda^k_i e_{1,k}
\]

and the symplectic form is given by

\[
\omega = \sum_{i=1}^{h} f^i \wedge \overline{f^i} + \sum_{i=1}^{m} e^{1,k} \wedge e^{2,k}.
\]

From [3] a direct computation yields the left-symmetric product associated with symplectic Lie algebra \((g, \omega)\).

\[
\left\{
\begin{align*}
\text{for } 1 \leq j \leq h \text{ and } 1 \leq k \leq m \text{ we have}
\end{align*}
\right.
\]

\[
(f_j, e_{1,k}) - (f_j, e_{1,k}) = \lambda^j_i e_{1,k} - \lambda^j_i e_{2,k} = \sum_{i=1}^{m} \lambda^j_i \overline{f^i} - \lambda^j_i f_i
\]

\[
(f_i, e_{1,k}) = -\lambda^k_i e_{2,k}, \quad f_i, e_{1,k} = -\lambda^k_i e_{2,k}, \quad f_i, e_{2,k} = \lambda^k_i e_{1,k}.
\]

For \(1 \leq j \leq h\) and \(1 \leq k \leq m\) we have

\[
(f_j, e_{2,k}) - (f_j, e_{1,k}) = \lambda^j_i e_{1,k} - \lambda^j_i e_{2,k} = 2\lambda^j_i \left( \sum_{i=1}^{m} \lambda^j_i \overline{f^i} - \lambda^j_i f_i \right)
\]

\[
= 2 \left( \sum_{i=1}^{m} \lambda^j_i \overline{f^i} - \lambda^j_i f_i \right) + 2\lambda^j_i \left( \sum_{i=1}^{m} \lambda^j_i \overline{f^i} - \lambda^j_i f_i \right).
\]
Thus if $(\mathfrak{g}, \omega)$ is SNLA we obtain $\lambda_j^k = 0$ for $1 \leq j \leq h$ and $1 \leq k \leq m$, a seminar calculation of $(\overline{f}_j,e_{2,k}).e_{1,k} - (\overline{f}_j,e_{1,k}).e_{2,k}$ implies that $\lambda_j^k = 0$ for $1 \leq j \leq h$ and $1 \leq k \leq m$ then an SNLA is irreducible if and only if it is trivial. \(\square\)

According to lemma we obtain the following corollary

**Corollary 5.** Every SNLA is completely reducible in particular an SNLA admits an isotropic ideal.

The following Lemma is an immediate consequence of (8).

**Lemma 2.** Let $(\mathfrak{g}, \omega)$ be an SNLA. Then the commutator ideal $D(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is isotrope.

**Proposition 4.** Let $(\mathfrak{g}, \omega)$ be a $p$-step nilpotent SNLA. Then we have the estimate

$$p \leq \dim D(\mathfrak{g}) + 1 \leq \frac{1}{2} \dim \mathfrak{g} + 1.$$  

**Proof.** Let $(\mathfrak{g}, \omega)$ be a $p$-step nilpotent SNLA and let

$$\mathfrak{g} = C^0(\mathfrak{g}) \supset C^1(\mathfrak{g}) = D(\mathfrak{g}) \supset \ldots \supset C^p(\mathfrak{g}) = 0$$

with $C^k(\mathfrak{g}) = [\mathfrak{g}, C^{k-1}(\mathfrak{g})]$ be a the descending central sequence of $\mathfrak{g}$, which shows that

$$\dim D(\mathfrak{g}) > \ldots > 0.$$  

Then $p \leq \dim D(\mathfrak{g}) + 1$ and we deduce from Lemma 2 that $\dim D(\mathfrak{g}) \leq \frac{1}{2} \dim \mathfrak{g}$. \(\square\)

This Proposition with Proposition 1 shows in particular the following corollary

**Corollary 6.** Any filiform and quasi-filiform Lie algebra is never SNLA.

There exists a process to construct the symplectic Lie algebras. It is often called the construction by double extension and it was described by Medina and Revoy [20]. This construction also called central symplectic oxidation was studied and developed by Baues and Cortés in [4].

Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and let $\varphi$ be a derivation of this Lie algebra. Then the bilinear map $\omega_\varphi$ on $\mathfrak{g}$ given by

$$\omega_\varphi(x, y) = \omega(\varphi x, y) + \omega(x, \varphi y)$$

is a closed 2-form on $\mathfrak{g}$. Then we can define a one-dimensional central extension $\mathfrak{g}_1 = \mathfrak{g} \oplus \langle h \rangle$ (vector space direct sum of $\mathfrak{g}$ with one-dimensional vector spaces $\langle h \rangle$) of $\mathfrak{g}$ by

$$[x + t_1 h, y + t_2 h]_{\mathfrak{g}_1} = [x, y] + \omega_\varphi(x, y) h. \quad x, y \in \mathfrak{g}, \quad t_1, t_2 \in \mathbb{R}.$$
It is easy to verify that the bilinear form \( \omega_{\varphi, \varphi} \) defined by
\[
\omega_{\varphi, \varphi}(x, y) = \omega_{\varphi}(\varphi x, y) + \omega_{\varphi}(x, \varphi y)
\]
is also a closed 2-form on \( g \). Assume that a closed form is an exact form, that is there exists \( \lambda \in g^* \) with \( \omega_{\varphi, \varphi}(x, y) = -\lambda[x, y] \). We can now define a new derivation \( \varphi_1 \) of the Lie algebra \( g_1 \) by
\[
\varphi_1(h) = 0 \quad \text{and} \quad \varphi_1(x) = -\varphi(x) - \lambda(x)h, \quad \forall x \in g.
\]
We consider the one-dimensional extension by derivation \( g_2 = \langle \xi \rangle \oplus g_1 \) of \( g_1 \). We define a non-degenerate two-form \( \Omega \) on \( g_2 \) by
\[
\begin{cases}
\Omega(x, y) = \omega(x, y) & \forall x, y \in g \\
\Omega(\xi, x) = \Omega(h, x) = 0 & \forall x \in g \\
\Omega(\xi, h) = 1.
\end{cases}
\]

**Definition 2.** Let \((g, \omega)\) be a symplectic Lie algebra, \( \varphi \in \text{Der}(g) \) a derivation and \( \lambda \in g^* \) an one form such that \( \omega_{\varphi, \varphi}(x, y) = -\lambda([x, y]) \) for \( x, y \in g \). The symplectic Lie algebra \((g_2, \Omega)\) is called central symplectic oxidation with respect \((\varphi, \lambda)\).

**Remark 3.**
1. It is clear that the Lie bracket in \( g_2 = \langle \xi \rangle \oplus g \oplus \langle h \rangle \) is given by
\[
\begin{align*}
[x, y]_{g_2} &= [x, y] + \omega_{\varphi}(x, y)h, & \forall x, y \in G \\
[\xi, x]_{g_2} &= \varphi(x) + \lambda(x)h, & \forall x \in G.
\end{align*}
\]
In particular, \( \langle h \rangle \) is a central ideal of \( g_2 \).

2. Let \((g, \omega)\) be a symplectic Lie algebra and \( \varphi \in \text{Der}(g) \) a derivation. Then there exists a symplectic oxidation if and only if the cohomology class \([\omega_{\varphi, \varphi}] \in H^2(g)\) vanishes.

3. Any nilpotent symplectic Lie algebra is a symplectic oxidation of a symplectic nilpotent Lie algebra.

**Lemma 3.** Let \((g, \omega)\) be a symplectic Lie algebra, \( \varphi \in \text{Der}(g) \) and \( \zeta \in g \) such that \( \text{ad}_\zeta = \varphi^2 \) and \( \varphi \circ \varphi^* = 0 \). Then the obstruction \([\omega_{\varphi, \varphi}] \in H^2(g)\) to the existence of a symplectic oxidation with respect to \((\varphi, \omega(\zeta, \cdot))\) vanishes.

**Proof.** Let \( x \) and \( y \in g \). Then
\[
\omega_{\varphi, \varphi}(x, y) = \omega(\varphi^2 x, y) + \omega(\varphi x, \varphi y) + \omega(\varphi x, \varphi y) + \omega(x, \varphi^2 y) = \omega(\text{ad}_\zeta x, y) + \omega(x, \text{ad}_\zeta y) = -\omega(\zeta, [x, y]) = \lambda([x, y]).
\]
with \( \lambda \in g^* \) given by \( \lambda(x) = \omega(-\zeta, x) \).

\[\square\]
Let’s keep the notations above, the following theorem gives conditions on \((\varphi, \lambda)\) so that the symplectic oxidation is SNLA.

**Theorem 3.** Let \((g, \omega)\) be a symplectic Lie algebra. The central symplectic oxidation \((\xi) \oplus g \oplus (\hbar)\) with respect to \((\varphi, \lambda)\) is SNLA if \((g, \omega)\) does and the following properties are verified

1. \(\varphi \circ L_x = ad_x \circ \varphi^* \text{ and } R_x \circ \varphi^* = L_{\varphi^*(x)} \quad \forall x \in g\)
2. \(R_x \circ (\varphi + \varphi^*) = (\varphi + \varphi^*) \circ R_x \quad \forall x \in g\)
3. \(\varphi \circ \varphi^* = 0, \text{ad}_{\zeta} = \varphi^2 \text{ and } \zeta \in \ker(\varphi)\)

with \(\zeta \in g\) such that \(\lambda = \omega(\zeta, \cdot)\).

**Proof.** Let “\(*\)” be the left-symmetric product associated with \((g, \omega)\) and “\(\cdot\)” the one associated with the symplectic oxidation. We have

\[
\begin{align*}
x.y &= x \ast y + \omega(\varphi x, y)h \\
\xi x &= -\varphi^* x \\
x \xi &= -(\varphi + \varphi^*) x - \lambda(x)h \\
\xi \xi &= -\zeta.
\end{align*}
\]

Using this we obtain

\[
\begin{align*}
(x.y).z &= (x \ast y) \ast z + \omega(\varphi(x \ast y), z)h \quad (12) \\
(\xi x).y &= -\varphi^*(x) \ast y - \omega(\varphi^*(x), y)h \quad (13) \\
(x \xi).y &= -(\varphi + \varphi^*)(x) \ast y - \omega(\varphi + \varphi^*)(x), y)h \quad (14) \\
(x.y).\xi &= -(\varphi + \varphi^*)(x \ast y) - \lambda(x \ast y)h \quad (15) \\
(\xi x).\xi &= -\zeta \ast x - \omega(\varphi(\zeta), x)h \quad (16) \\
(\xi x).\xi &= (\varphi^*)^2(x) + \lambda((\varphi^*(x))h. \quad (17)
\end{align*}
\]

By using (11), we can see that \((x.y).z = (x.z).y\) is equivalent to

\[(x \ast y) \ast z = (x \ast z) \ast y \quad \text{and} \quad \omega(\varphi(x \ast y), z) = \omega(\varphi(x \ast z), y).\]

for any \(x, y, z \in g\), which is equivalent to \((g, \omega)\) is SNLA and \(L_x = ad_x \circ \varphi^*\).

By using (12), we can see that \((\xi x).y = (\xi y).x\) is equivalent to

\[\varphi^*(x) \ast y = \varphi^*(y) \ast x \quad \text{and} \quad \omega(\varphi^*(x), y) = \omega(\varphi^*(y), x)\]

or any \(x, y \in g\), which is equivalent to \(R_y \circ \varphi^* = L_{\varphi^*(y)}\) and \(\varphi \circ \varphi^* = 0\).

Now, by using (13) and (14), we can see easily that \((x \xi).y = (x.y)\xi\) is equivalent to

\[(\varphi + \varphi^*)(x \ast y) = (\varphi + \varphi^*)(x) \ast y \quad \text{and} \quad \omega(\varphi^2(x), y) = \lambda(x \ast y) = \omega(\text{ad}_{\zeta} x, y)\]
for any $x$ and $y \in \mathfrak{g}$, which is equivalent to $(\varphi + \varphi^*) \circ R_y = R_y \circ (\varphi + \varphi^*)$ and $\varphi^2 = \text{ad}_\zeta$.

Finally, by using (15) and (16), we can see that $(\xi, \xi).x = (\xi \cdot x).\xi$ is equivalent to

$$(\varphi^*)^2 = -L_{\zeta} = \text{ad}_\zeta^* \quad \text{and} \quad \varphi(\zeta) = 0.$$ 

Thus the proposition follows. $\square$

**Proposition 5.** Let $(\mathfrak{g}, \omega)$ be a 2n-dimensional SNLA with not trivial center. Then, $(\mathfrak{g}, \omega)$ is a central SNLA oxidation of a $(2n - 2)$-dimensional SNLA.

**Proof.** Let $(\mathfrak{g}, \omega)$ be a 2n-dimensional SNLA with not trivial center, $\langle h \rangle$ a one dimensional central ideal and $(\mathfrak{g}, \mathfrak{z})$ the symplectic reduction with respect to $\langle h \rangle$. We may choose $\xi \in \mathfrak{g}$ such that $\omega(\xi, h) = 1$ and $(\mathfrak{g} = \langle \xi \rangle \oplus \mathfrak{z} \oplus \langle h, \mathfrak{z} \rangle)$ is the central symplectic oxidation with respect $(\varphi_\xi, \lambda_\omega)$, with $\varphi_\xi = \text{ad}_\xi \mid \mathfrak{g}$ is the restriction of adjoint operator and $\lambda_\omega(x) = \omega(\xi, [\xi, x])$ for $x \in \mathfrak{g}$ (see [4] Proposition 2.16.). Remark that $\varphi_\xi = -L_{\xi} \mid \mathfrak{g}$, then (12) becomes for all $x, y \in \mathfrak{g}$

$$\begin{cases}
  x.y = x \ast y + \omega(\text{ad}_\xi \mid \mathfrak{g}x, y)h \\
  \xi.x = L_{\xi} \mid \mathfrak{g}x \\
  x.\xi = R_{\xi} \mid \mathfrak{g}x - \lambda_\omega(x)h \\
  \xi.\xi = -\zeta
\end{cases}$$

The proof is completed by verifying that $(\varphi_\xi, \lambda_\omega)$ verified the conditions of Theorem 3. $\square$

## 5 Geometry of symplectic Novikov Lie group

Let $(G, \omega^+)$ be a symplectic Lie group, (i.e., a Lie group $G$ endowed with a left invariant symplectic form $\omega^+$). We denote by $\mathfrak{g}$ the Lie algebra of $G$, $\omega = \omega^+(e)$, with $e$ the unit of $G$. Let $\nabla$ be a left invariant linear connection given by the formula

$$\nabla_{x^+} y^+ = (x, y)^+$$

with $x^+$ denotes the left invariant vector field on $G$ whose value at $e$ is $x \in \mathfrak{g}$ and "\ast" is a left symmetric product associated with $(\mathfrak{g}, \omega)$. It is well known that $\nabla$ defines a left invariant affine structure on $G$, (i.e a left invariant flat and torsion free connection). This affine structure is called associated to the symplectic structure.

We will say subsequently that a symplectic Lie group $(G, \omega^+)$ is symplectic Novikov Lie group if the symplectic Lie algebra $(\mathfrak{g}, \omega)$ is SNLA.

**Proposition 6.** Let $(G, \omega^+)$ be a symplectic Novikov Lie group. The affine structure associated to $\omega^+$, noted by $\nabla$, satisfies the following properties

$$\nabla_{x^+} y^+ = (x, y)^+$$

with $x^+$ denotes the left invariant vector field on $G$ whose value at $e$ is $x \in \mathfrak{g}$ and "\ast" is a left symmetric product associated with $(\mathfrak{g}, \omega)$. It is well known that $\nabla$ defines a left invariant affine structure on $G$, (i.e a left invariant flat and torsion free connection). This affine structure is called associated to the symplectic structure.

We will say subsequently that a symplectic Lie group $(G, \omega^+)$ is symplectic Novikov Lie group if the symplectic Lie algebra $(\mathfrak{g}, \omega)$ is SNLA.
1. $\nabla$ is bi-invariant.

2. $\nabla$ is completed if and only if the Lie algebra $\mathfrak{g}$ is nilpotent.

Proof. 1. Follows from a general result (see, for example, [19] Proposition 1-1) which states that in a Lie group an affine connection is bi-invariant if and only if its associated left-symmetric product is associative.

2. It is known, that a left-invariant affine structure on a Lie group $G$ is complete if and only if the right multiplications $R_x$ in the corresponding left symmetric algebra $(\mathfrak{g}, .)$ are nilpotent for all $x \in \mathfrak{g}$. Iterating (5) we have

$$\omega(L_x^k y, z) = (-1)^k \omega(y, \text{ad}_x^k z)$$

for all positive integers $k$, where $x, y, z \in \mathfrak{g}$. In particular the Lie algebra $\mathfrak{g}$ is nilpotent if and only if the operators $L_x$ are nilpotent for all $x \in \mathfrak{g}$.

On the other hand, a direct calculation gives

$$L_x \circ \text{ad}_x = \text{ad}_x \circ L_x = 0$$

for all $x \in \mathfrak{g}$. Furthermore we have

$$R_x^k = \text{ad}_x^k + (-1)^k L_x^k$$

then (2) holds.

Let $(M, \Omega)$ be a smooth $2n$-dimensional symplectic manifold (i.e. $\Omega$ is a closed nondegenerate 2-form on $M$). A symplectic connection on $(M, \Omega)$ is a torsion free linear connection $\nabla$ on $M$ for which the symplectic 2-form $\omega$ is parallel i.e. $\nabla \Omega = 0$. Recall that [5] from any linear connection $\nabla$, we can build a symplectic connection. More precisely, let the tensor field $N$ given by the relation

$$\nabla_X \Omega(Y, Z) = \Omega(N(X, Y), Z)$$

for all vector field $X$, $Y$ and $Z$ on $M$. Then the linear connection given by

$$\nabla^N_X Y = \nabla_X Y + \frac{1}{3} N(X, Y) + \frac{1}{3} N(Y, X)$$

is torsion free and preserves $\Omega$.

Now, let $(G, \omega^+)$ be a symplectic Lie group. By applying the above construction to symplectic Lie group, we get the following proposition

**Proposition 7.** Let $(G, \omega^+)$ be a symplectic Lie group and $\nabla$ be the affine structure associated to $\omega^+$. The symplectic connection $\nabla^\omega$ associated with $\nabla$ is given (in the neutral element) by

$$\nabla^\omega_x y = \frac{1}{3} (\text{ad}_x y - \text{ad}_x^* y)$$

(19)

for all $x, y \in \mathfrak{g}$.
The connection $\nabla^\omega$ depends only on the Lie group and $\omega^+$ was first introduced by Benayadi and Boucetta in \cite{3} then appears in another article (see for example \cite{15} and \cite{1}).

**Remark 4.** All Lie algebras two-step nilpotent admits $x.y = \frac{1}{2}[x, y]$ as a Novikov structure. This structure is not SNLA. In fact, Note that the affine structure $\nabla$ associated to $\omega^+$ is symplectic unless $G$ is abelian. On the other hand a simple calculation gives $\nabla^\omega = \nabla = \frac{1}{2}[x^+, y^+]$.

Recall that, the curvature tensor is then described in terms of the map

$$K : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

$$(X, Y) \mapsto K(x, y) = [\nabla_X, \nabla_Y] - \nabla_{[x, y]}$$

and $(\mathfrak{g}, \nabla)$ is called flat if $K = 0$. The Ricci tensor is the symmetric tensor $ric$ given by

$$ric(X, Y) = tr(Z \mapsto K(X, Z)Y).$$

**Theorem 4.** Let $(G, \omega^+)$ be a symplectic Novikov Lie group. For any $x, y \in \mathfrak{g}$, we have

1. The curvature tensor of the connection $\nabla^\omega$ is given (in the neutral element) by

$$K(x, y) = \frac{2}{9}ad_{[x, y]}.$$

In particular, $(G, \omega^+)$ is flat if and only if $\mathfrak{g}$ is two-step nilpotent.

2. The Ricci tensor is given (in the neutral element) by

$$ric(x, y) = \frac{2}{9}tr(ad_x \circ ad_y).$$

**Proof.**

1. For any $x, y$ and $z \in \mathfrak{g}$, we have

$$K(x, y)z = ([\nabla_X^\omega, \nabla_Y^\omega] - \nabla_{[x, y]}^\omega)z$$

$$= \nabla_X^\omega(\frac{2}{3}y.z - \frac{1}{3}z.y) - \nabla_Y^\omega(\frac{2}{3}x.z - \frac{1}{3}z.x) - \nabla_{x.y}^\omega z + \nabla_{y.x}^\omega z$$

$$= \frac{4}{9}(x.y.z - y.x.z) + \frac{2}{9}(x.z.y - x.y.z) - \frac{2}{9}(y.z.x - y.z.x)$$

$$\frac{1}{9}(z.y.x - z.x.y) + \frac{2}{3}(y.x.z - x.y.z) + \frac{1}{3}(z.x.y - y.z.x)$$

$$= -\frac{2}{9}L_{[x, y]} z + \frac{1}{9}R_{[x, y]} z.$$
2. We fix a symplectic basis \((e_i, f_i)\) of \(g\) (i.e. \(\omega = \sum_i e_i \wedge f^i\)) with dual basis \((e^i, f^i)\), use the Einstein summation convention we have

\[
\text{ric}(x, y) = \text{tr}(z \mapsto K(x, z)y) = \omega(K(x, e_i)y, e^i) - \omega(K(x, f_i)y, f^i) = -\frac{2}{9}(\omega([x, e_i]y, e^i) - \omega([x, f_i]y, f^i)) = -\frac{2}{9}(\omega(x, y,e_i - e_i, x.y, e^i) - \omega(x, y,f_i - f_i, x.y, f^i)) = -\frac{2}{9}(\text{tr}(L_{x,y}) - \text{tr}(R_{x,y})).
\]

By contrast, a direct computation yields \(\text{tr}(R_{x,y}) = 2\text{tr}(L_{x,y})\) and the associativity gives \(L_{x,y} = L_x \circ L_y\), for all \(x, y \in g\) then

\[
\text{ric}(x, y) = \frac{2}{9}\text{tr}(L_x \circ L_y) = \frac{2}{9}\text{tr}(\text{ad}_x \circ \text{ad}_y).
\]

\(\blacksquare\)

**Remark 5.** For a general study of the Ricci curvature of the symplectic connection associated with the affine connection of a symplectic Lie group see [7].

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