SOME PROPERTIES OF MINIMAL $S(\alpha)$ AND $S(\alpha)FC$ SPACES

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Abstract. A $S(n)$-space is $S(n)$-functionally compact ($S(n)FC$) if every continuous function onto a $S(n)$-space is closed. $S(n)$-closed, $S(n)$-$\theta$-closed, minimal $S(n)$ and $S(n)FC$ spaces are characterized in terms of $\theta(n)$-complete accumulation points. In paper we also give new characteristics of $R$-closed and regular functionally compact spaces. Results obtained to answer some questions raised by D.Dikranjan, E.Giuli, L.Friedler, M.Girou, D.Pettey and J.Porter.

1. Introduction

Dikranjan and Giuli [3] introduced a notion of the $\theta^n$-closure operator and developed a theory of $S(n)$-closed and $S(n)$-$\theta$-closed spaces. Jiang, Reilly, and Wang [7] used the $\theta^n$-closure in studying properties of minimal $S(n)$-spaces.

In work [10] continues the study of properties inherent in $S(n)$-closed and $S(n)$-$\theta$-closed spaces, using the $\theta^n$-closure operator; in addition, wider classes of spaces (weakly $S(n)$-closed and weakly $S(n)$-$\theta$-closed spaces) are introduced.

In this paper we continue the investigation of $S(n)$-closed, $S(n)$-$\theta$-closed, minimal $S(n)$ spaces with the use of $\theta(n)$-complete accumulation points. As we introduce new classes of $S(n)$-spaces — $S(n)$-functionally compact spaces and to answer some questions raised in [3, 7].

Section 2 acquaints the reader with main definitions and known properties in the theory of $S(n)$-spaces. Section 3 is completely devoted to the study of weakly $S(n)$-closed and weakly $S(n)$-$\theta$-closed
spaces. It is proved that any \( S(n) \)-closed (\( S(n) \)-\( \theta \)-closed) space is weakly \( S(n) \)-closed (weakly \( S(n) \)-\( \theta \)-closed). In the remaining sections, we characterize \( S(n) \)-closed, \( S(n) \)-\( \theta \)-closed, minimal \( S(n) \) spaces, \( S(n) \)-functionally compact, \( R \)-closed, minimal regular and regular functionally compact spaces with the use of \( \theta(n) \)-complete accumulation and \( \theta(\omega) \)-complete accumulation points.

2. Main definitions and notation

Let \( X \) be a topological space, \( M \subseteq X \), and \( x \in X \). For any \( n \in \mathbb{N} \), we consider the \( \theta(n) \)-closure operator: \( x \notin \overline{\text{cl}}_{\theta(n)}M \) if there exists a set of open neighborhoods \( U_1, U_2, ..., U_n \) of the point \( x \) such that \( \overline{\text{cl}} U_i \subseteq U_{i+1} \) for \( i = 1, 2, ..., n-1 \) and \( \overline{\text{cl}} U_n \cap M = \emptyset \) if \( n > 1 \); \( \overline{\text{cl}}_{\theta(0)}M = \overline{\text{cl}}M \) if \( n = 0 \); and, for \( n = 1 \), we get the \( \theta \)-closure operator, i.e., \( \overline{\text{cl}}_{\theta(1)}M = \overline{\text{cl}}_\theta M \). A set \( M \) is \( \theta(n) \)-closed if \( M = \overline{\text{cl}}_{\theta(n)}M \). Denote by \( \text{Int}_{\theta(n)}M = X \setminus \overline{\text{cl}}_{\theta(n)}(X \setminus M) \) the \( \theta(n) \)-interior of the set \( M \). Evidently, \( \overline{\text{cl}}_{\theta(n)}(\overline{\text{cl}}_{\theta(s)}M) = \overline{\text{cl}}_{\theta(n+s)}M \) for \( M \subseteq X \) and \( n, s \in \mathbb{N} \). For \( n \in \mathbb{N} \) and a filter \( \mathcal{F} \) on \( X \), denote by \( \overline{\text{ad}}_{\theta(n)}\mathcal{F} \) the set of \( \theta(n) \)-adherent points, i.e., \( \overline{\text{ad}}_{\theta(n)}\mathcal{F} = \bigcap \{ \overline{\text{cl}}_{\theta(n)}\mathcal{F}_\alpha : F_\alpha \in \mathcal{F} \} \). In particular, \( \overline{\text{ad}}_{\theta(0)}\mathcal{F} = \overline{\text{ad}}\mathcal{F} \) is the set of adherent points of the filter \( \mathcal{F} \). For any \( n \in \mathbb{N} \), a point \( x \in X \) is \( S(n) \)-separated from a subset \( M \) if \( x \notin \overline{\text{cl}}_{\theta(n)}M \). For example, \( x \) is \( S(0) \)-separated from \( M \) if \( x \notin \overline{\text{cl}}M \). For \( n > 0 \), the relation of \( S(n) \)-separability of points is symmetric. On the other hand, \( S(0) \)-separability may be not symmetric in some not \( T_1 \)-spaces. Therefore, we say that points \( x \) and \( y \) are \( S(0) \)-separated if \( x \notin \overline{\{ y \}} \) and \( y \notin \overline{\{ x \}} \).

Let \( n \in \mathbb{N} \) and \( X \) be a topological space.

1. \( X \) is called an \( S(n) \)-space if any two distinct points of \( X \) are \( S(n) \)-separated.

2. A filter \( \mathcal{F} \) on \( X \) is called an \( S(n) \)-filter if every point, not being an adherent point of the filter \( \mathcal{F} \), is \( S(n) \)-separated from some element of the filter \( \mathcal{F} \).

3. An open cover \( \{ U_\alpha \} \) of the space \( X \) is called an \( S(n) \)-cover if every point of \( X \) lies in the \( \theta(n) \)-interior of some \( U_\alpha \).

It is obvious that \( S(0) \)-spaces are \( T_0 \)-spaces, \( S(1) \)-spaces are Hausdorff spaces, and \( S(2) \)-spaces are Urysohn spaces. It is clear that every filter is an \( S(0) \)-filter, every open cover is an \( S(0) \)-cover, and every open filter is an \( S(1) \)-filter. Open \( S(2) \)-filters are called Urysohn filters. For \( n > 1 \), open \( S(n) \)-filters were defined in [12].
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S(1)-covers are called Urysohn covers. In a regular space, every filter (every cover) is an S(n)-filter (S(n)-cover) for any \( n \in \mathbb{N} \).

S(n)-closed and S(n)-θ-closed spaces are S(n)-spaces, closed and, respectively, θ-closed in any S(n)-space containing them.

Porter and Votaw [12] characterized S(n)-closed spaces by means of open S(n)-filters and S(n−1)-covers.

Let \( n \in \mathbb{N}^+ \) and \( X \) be an S(n)-space. Then the following conditions are equivalent:

1. \( ad_{S(n)} F \neq \emptyset \) for any open filter \( F \) on \( X \);
2. \( ad F \neq \emptyset \) for any open S(n)-filter \( F \) on \( X \);
3. for any S(n−1)-cover \( \{U_{\alpha}\} \) of the space \( X \) there exist \( \alpha_1, \alpha_2, ..., \alpha_k \) such that \( X = \bigcup_{i=1}^{k} U_{\alpha_i} \);
4. \( X \) is an S(n)-closed space.

Dikranjan and Giuli [3] characterized S(n)-θ-closed spaces in terms of S(n−1)-filters and S(n−1)-covers.

Let \( n \in \mathbb{N}^+ \) and \( X \) be an S(n)-space. Then the following conditions are equivalent:

1. \( ad F \neq \emptyset \) for any closed S(n−1)-filter \( F \) on \( X \);
2. any S(n−1)-cover of \( X \) has a finite subcover;
3. \( ad_{g(n−1)} F \neq \emptyset \) for any closed filter \( F \) on \( X \);
4. \( X \) is an S(n)-θ-closed space.

Note that, for \( n = 1 \), S(1)-closedness and S(1)-θ-closedness are H-closedness and compactness, respectively. For \( n = 2 \), S(2)-closedness and S(2)-θ-closedness are U-closedness and U-θ-closedness, respectively. From characteristics themselves, it follows that any S(n)-θ-closed subspace of an S(n)-space is an S(n)-closed space.

Recall that a open cover \( \mathcal{V} \) is a shrinkable refinement of open cover \( \mathcal{U} \) if and only if for each \( V \in \mathcal{V} \), there is a \( U \in \mathcal{U} \) such that \( V \subseteq U \). A open cover \( \mathcal{V} \) is a regular refinement of \( \mathcal{U} \) if and only if \( \mathcal{V} \) refines \( \mathcal{U} \) is a shrinkable refinement of itself. An open cover is regular if and only if it has an open refinement.

An open filter base \( \mathcal{F} \) in \( X \) is a regular filter base if and only if for each \( U \in \mathcal{F} \), there exists \( V \in \mathcal{F} \) such that \( V \subseteq U \).

A R-closed space is a regular space closed in any regular space containing them.

Berri, Sorgenfrey [2] characterized R-closed spaces by means of regular filters and regular covers.

Let \( X \) be a regular space. The following are equivalent:

1. \( X \) is R-closed.
(2) Every regular filter base in $X$ is fixed.
(3) Every regular cover has a finite subcover.
For undefined notions and related theorems, we refer readers to [3].

3. Weakly $S(n)$-closed and weakly $S(n)$-$\theta$-closed spaces

In the Aleksandrov and Urysohn memoir on compact spaces [1],
the notion of a $\theta$-complete accumulation point was introduced. A
point $x$ is called a $\theta$-complete accumulation point of a set $F$ if
$|F \cap U| = |F|$ for any neighborhood $U$ of the point $x$. It was noted
that any $H$-closed space has the following property:
(*) any infinite set of regular power has a $\theta$-complete accumula-
tion point. However, the converse is not true. The first example of
a space possessing property (*) and not being $H$-closed was con-
structed by Kirtadze [8]. Simple examples in [10, 11] also sh ows
the converse is not true.

Example 1. (Example 1 in [10].) Let $T_1$ and $T_2$ be two copies
of the Tychonoff plane $T = (\omega_1 + 1) \times (\omega_0 + 1) \setminus \{\omega_1, \omega_0\}$, whose
elements will be denoted by $(\alpha, n, 1)$ and $(\alpha, n, 2)$, respectively.
On the topological sum $T_1 \oplus T_2$, we consider the identifications
$(\omega_1, k, 1) \sim (\omega_1, 2k, 2)$ for every $k \in \mathbb{N}$; and we identify all points
$(\omega_1, 2k - 1, 2)$ for any $k \in \mathbb{N}$ with the same point $b$. Adding,
to the obtained space, a point $a$ with the base of neighborhoods
$U_{\beta,k}(a) = \{a\} \cup \{(\alpha, n, 1) : \beta < \alpha < \omega_1, k < n \leq \omega_0\}$ for arbitrary
$\beta < \omega_1$ and $k < \omega_0$, we get a Urysohn space $X$.

Note that space $X$ is an example of a non-$H$-closed, Urysohn
space with the property that for every chain of non-empty sets, the
intersection of the $\theta$-closures of the sets is nonempty, every infinite
set has a $\theta$-complete accumulation point. J.Porter investigated the
space with the same properties in [11].
Definition 3.1. A neighborhood $U$ of a set $A$ is called an $n$-hull of the set $A$ if there exists a set of neighborhoods $U_1, U_2, ..., U_n = U$ of the set $A$ such that $clU_i \subseteq U_{i+1}$ for $i = 1, ..., n - 1$.

Definition 3.2. A point $x$ from $X$ is called a $\theta^0(n)$-complete accumulation ($\theta(n)$-complete accumulation) point of an infinite set $F$ if $|F \cap U| = |F|$ ($|F \cap U| = |F|$) for any $U$, where $U$ is an $n$-hull of the point $x$.

Note that, for $n = 1$, a $\theta^0(1)$-complete accumulation point is a point of complete accumulation, and a $\theta(1)$-complete accumulation point is a $\theta$-complete accumulation point.

Definition 3.3. A topological space $X$ is called weakly $S(n)$-$\theta$-closed (weakly $S(n)$-closed) if any infinite set of regular power of the space $X$ has a $\theta^0(n)$-complete accumulation ($\theta(n)$-complete accumulation) point.

Note that any $\theta^0(n)$-complete accumulation point is a $\theta(n)$-complete accumulation point; hence, any weakly $S(n)$-$\theta$-closed space is weakly $S(n)$-closed. Moreover, since a $\theta(n)$-complete accumulation point is a $\theta^0(n+1)$-complete accumulation point, it follows that a weakly $S(n)$-closed space will be weakly $S(n+1)$-$\theta$-closed. For $n = 1$, weakly $S(1)$-$\theta$-closed and weakly $S(1)$-closed spaces are compact Hausdorff spaces and spaces with property (*), respectively.

Theorem 3.4. Let $X$ be an $S(n)$-closed $S(n)$-space. Then $X$ is weakly $S(n)$-closed.
Proof. Suppose the contrary. Let $X$ be $S(n)$-closed but not weakly $S(n)$-closed. Then in the spaces $X$ there exists an infinite set $F$ of regular power that has no $\theta(n)$-complete accumulation point. For any point $x \in X$, there exists an $n$-hull $U$ of the point $x$ with the property $|F \cap U| < |F|$. If we take such $n$-hull for every point $x \in X$, we derive an $S(n-1)$-cover of the space $X$. By $S(n)$-closedness, there exists a finite family $U$ of $n$-hulls such that $|F \cap U| < |F|$ for every $U \in U$ and $\bigcup U = X$. This contradicts the fact that $F$ is infinite set of regular power.

\[ \square \]

**Theorem 3.5.** Let $X$ be an $S(n)$-$\theta$-closed $S(n)$-space. Then $X$ is weakly $S(n)$-$\theta$-closed space.

A proof of Theorem 3.5 is analogous to that of Theorem 3.4.

It was proved in [3] that $S(n)$-closedness implies $S(n+1)$-$\theta$-closedness. Thus, for $S(n)$-spaces, classes of the considered spaces are presented in the following diagram:

- compact Hausdorff space $\iff$ weakly $S(1)$-$\theta$-closed $\downarrow \downarrow$
- $H$-closed $\implies$ weakly $H$-closed $\downarrow$
- $U$-$\theta$-closed $\implies$ weakly $U$-$\theta$-closed $\downarrow$
- $U$-closed $\implies$ weakly $U$-closed $\downarrow$
- $\ldots$ $\ldots$ $\ldots$ $\downarrow$
- $S(n-1)$-$\theta$-closed $\implies$ weakly $S(n-1)$-$\theta$-closed $\downarrow$
- $S(n-1)$-closed $\implies$ weakly $S(n-1)$-closed $\downarrow$
- $S(n)$-$\theta$-closed $\implies$ weakly $S(n)$-$\theta$-closed $\downarrow$
- $S(n)$-closed $\implies$ weakly $S(n)$-closed $\downarrow$

Note that all implications in the diagram are irreversible. Examples of $S(n)$-closed but not $S(n)$-$\theta$-closed spaces and $S(n)$-$\theta$-closed but not $S(n-1)$-closed spaces are considered in [3]. Examples
are considered in [10], showing that the remaining implications are irreversible.

**Theorem 3.6.** Let $X$ be a Lindelöf (finally compact) weakly $S(n)$-closed $S(n)$-space. Then $X$ is $S(n)$-closed space.

**Proof.** Suppose the contrary. Let $X$ be a Lindelöf weakly $S(n)$-closed but not $S(n)$-closed. Then in the spaces $X$ there exists open filter $\mathcal{F}$ such that $ad_{\mathcal{F}} \mathcal{F} = \emptyset$. For each point $x \in X$ there are $F_x \in \mathcal{F}$ and an $n$-hull $U_x$ of $F_x$ such that $x \notin U_x$. Note that $\bigcap_{x \in X} U_x = \emptyset$. Since $X$ is a Lindelöf, there exists a countable family $\{U_x\}$ such that $\bigcap_{i=1} U_{x_i} = \emptyset$. Consider a sequence $\{y_j\}$ such that $y_j \in \bigcap_{i=1} F_{x_i}$. Clearly, the infinite set $\{y_j\}$ has not a $\theta(n)$-complete accumulation point. This contradicts the fact that $X$ is a weakly $S(n)$-closed space. \qed

**Corollary 3.7.** Let $X$ be a countable weakly $S(n)$-$\theta$-closed $S(n)$-space. Then $X$ is $S(n)$-closed.

**Corollary 3.8.** Let $X$ be second-countable weakly $S(n)$-$\theta$-closed $S(n)$-space. Then $X$ is $S(n)$-closed.

Recall, that a space is linearly Lindelöf (finally compact in the sense of accumulation points) if every increasing open cover $\{U_\alpha : \alpha \in \kappa\}$ has a countable subcover (by increasing, we mean that $\alpha < \beta < \kappa$ implies $U_\alpha \subseteq U_\beta$).

**Theorem 3.9.** Let $n > 1$ and $X$ be a linearly Lindelöf weakly $S(n)$-$\theta$-closed $S(n)$-space. Then $X$ is weakly $S(n-1)$-closed.

**Proof.** Suppose the contrary. Then there is a countable set $S$ such that the set $S$ has not a $\theta(n-1)$-complete accumulation point. Let $x \in X$ and $U$ be a $(n-1)$-hull of $x$ such that $\overline{U} \cap S = \emptyset$. For every $y \in \overline{U}$ there exists a neighborhood $W_y$ of $y$ such that $W_y \cap S = \emptyset$. Consider a open set $W = \bigcup_{y \in \overline{U}} W_y$. Then $\overline{W} \subseteq W$ and $W$ is an $n$-hull of the point $x$. Note that $W \cap S = \emptyset$. It is follows that $x$ is not an $\theta^0(n)$-complete accumulation point of $S$. This contradicts fact that $X$ is a weakly $S(n)$-$\theta$-closed. \qed

**Corollary 3.10.** Let $n > 1$ and $X$ be a Lindelöf weakly $S(n)$-$\theta$-closed $S(n)$-space. Then $X$ is $S(n-1)$-closed.
In [3] raised the question (Problem 5) about the product of $U$-$\theta$-closed spaces. Namely, it is required to prove or to disprove that the product of $U$-$\theta$-closed spaces is feebly compact. In particular, it was not known if every Lindelöf $U$-$\theta$-closed space is $H$-closed.

In [9], two Urysohn $U$-$\theta$-closed spaces whose product is not feebly compact are constructed. Thus, the question is negatively solved.

**Corollary 3.11.** Let $X$ be a Lindelöf $U$-$\theta$-closed Urysohn space. Then $X$ is $H$-closed.

**Remark 3.12.** Observe that every $H$-closed space is feebly compact. By corollary 3.11, product of Lindelöf $U$-$\theta$-closed spaces is feebly compact.

**Corollary 3.13.** Let $n > 1$ and $X$ be a Lindelöf weakly $S(n)$-$\theta$-closed $S(n)$-space. Then $X$ is $S(n)$-$\theta$-closed.

Thus, for Lindelöf $S(n)$-spaces, classes of the considered spaces are presented in the following diagram:

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compact Hausdorff space ⇐⇒ weakly $S(1)$-$\theta$-closed
\downarrow
H-closed ⇐⇒ weakly $H$-closed
\downarrow
U-$\theta$-closed ⇐⇒ weakly U-$\theta$-closed
\downarrow
U-closed ⇐⇒ weakly U-closed
\downarrow
\ldots
\downarrow
\ldots
\downarrow
S(n-1)$-$\theta$-closed ⇐⇒ weakly $S(n-1)$-$\theta$-closed
\downarrow
S(n-1)-closed ⇐⇒ weakly $S(n-1)$-closed
\downarrow
S(n)$-$\theta$-closed ⇐⇒ weakly S(n)$-$\theta$-closed
\downarrow
S(n)-closed ⇐⇒ weakly S(n)-closed
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**Question 1.** Does there exists a non $S(n)$-$\theta$-closed Lindelöf S(n)-closed space ($n > 1$)?
4. Characterizations $S(n)$-closed and $S(n)$-$\theta$-closed spaces

Now for every $n \in \mathbb{N}$ we introduce an operator of $\theta^n_0$-closure; for $M \subseteq X$ and $x \in X$ $x \notin cl_{\theta^n_0} M$ if there is a $n$-hull $U$ of $x$ such that $U \cap M = \emptyset$. A set $M \subseteq X$ is $\theta^n_0$-closed if $M = cl_{\theta^n_0} M$.

**Definition 4.1.** A subset $M$ of a topological space $X$ is an $S(n)$-$\theta^n_0$-set if every $S(n)$-cover $\gamma$ with respect to $M \{ M \subseteq \bigcup \{ Int_{\theta^n} U_\alpha : U_\alpha \in \gamma \} \}$ by open sets of $X$ has a finite subfamily which covers $M$ with the $\theta^n_0$-closures of its members.

**Definition 4.2.** The set $A$ is weakly $\theta(n)$-converge to the set $B$ if for any $S(n-1)$-cover $\gamma = \{ U_\alpha \}$ of $B$ there exists a finite family $\{ U_{\alpha i} \}_{i=1}^k \subseteq \gamma$ such that $|A \setminus \bigcup_{i=1}^k \bigcap_{j=1}^k U_{\alpha i} | < |A|$.

**Theorem 4.3.** For $n \in \mathbb{N}$, a $S(n)$-space $X$ is $S(n)$-closed if and only if any infinity set $A \subseteq X$ weakly $\theta(n)$-converge to the set $B$ of its $\theta(n)$-complete accumulation points.

**Proof.** Necessary. Let $X$ be $S(n)$-closed space and $A \subseteq X$. Take any $S(n-1)$-cover $\gamma$ of $B$ where $B$ is the set of $\theta(n)$-complete accumulation points of $A$. For each point $x \notin B$ we take an $n$-hull $O(x)$ such that $|O(x) \cap A| < |A|$. Then we have an $S(n-1)$-cover $\gamma' = \gamma \cup \{ O(x) : x \notin B \}$ of $X$. As the space $X$ is $S(n)$-closed there are finite families $\{ U_i \}_{i=1}^k \subseteq \gamma$ and $\{ O(x_j) \}_{j=1}^k$ such that $\bigcup_{i=1}^k U_i \cup \bigcup_{j=1}^k O(x_j) = X$. Note that $A \setminus \bigcup_{i=1}^k \overline{O(x_i)} \subseteq \bigcup_{j=1}^k O(x_j)$. As $|A \setminus \bigcup_{j=1}^k O(x_j) | < |A|$ we have $|A \setminus \bigcup_{i=1}^k \overline{O(x_i)} | < |A|$. Thus $A$ weakly $\theta(n)$-converge to the set $B$.

Note that $B$ is an $S(n)$-$\theta^n_0$-set. Really, $(A \cap \bigcup_{i=1}^k \overline{U_i}) \cap \overline{S(x)} \neq \emptyset$ for every $x \in B$ and for any $n$-hull $S(x)$ of the point $x$. It is follows that $S(x) \cap (\bigcup_{i=1}^k U_i) \neq \emptyset$ and $x$ is contained in $\theta^n_0$-closure of $\bigcup_{i=1}^k U_i$. Thus $B \subseteq cl_{\theta^n_0} \bigcup_{i=1}^k U_i$.

Sufficiency. Let $\varphi = \{ V_\alpha \}$ be open $S(n)$-filter on $X$. Assume that $ad \varphi = \emptyset$. Choose $V_0 \in \varphi$ such that $|V_0| = inf \{ |V_\alpha| : V_\alpha \in \varphi \}$. Since $\bigcap_{\alpha} \overline{V_\alpha} = \emptyset$ we have $\xi = \{ U_\alpha : U_\alpha = X \setminus \overline{V_\alpha} \}$ is an $S(n-1)$-cover of $B$ where $B$ is the set of $\theta(n)$-complete accumulation points of $V_0$. By condition, there exists a finite family $\{ U_\alpha \}_{i=1}^k \subseteq \xi$ such that $|V_0 \setminus \bigcup_{i=1}^k \overline{U_\alpha} | < |V_0|$. Consider $V_\alpha_i \in \varphi$ such that $U_\alpha_i = X \setminus V_\alpha_i$. Let $V = \bigcap_{i=1}^k V_\alpha_i$ then $V \cap V_0 \subseteq V_0 \setminus \bigcup_{i=1}^k \overline{U_\alpha_i}$ and

\[ |V \cap V_0| < |V_0| \]
$|V \cap V_0| < |V_0|$. This contradicts our choice of $V_0$. Thus $X$ is
$S(n)$-closed space.

\[\square\]

**Corollary 4.4.** Let $X$ be a $S(n)$-closed space and $A$ be an infinity
set of $X$. Then a set $B$ of $\theta(n)$-complete accumulation points of $A$
is an $S(n)$-$\theta^0_n$-set.

**Definition 4.5.** The set $A$ is $\theta^0(n)$-converge to the set $B$ if for
any $S(n - 1)$-cover $\gamma = \{U_\alpha\}$ of $B$ there exists a finite family
$\{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$ such that $|A \setminus \bigcup_{i=1}^k U_{\alpha_i}| < |A|$.  

**Theorem 4.6.** For $n \in \mathbb{N}$, a $S(n)$-space $X$ is $S(n)$-$\theta$-closed if and
only if any infinity set $A \subseteq X$ $\theta^0(n)$-converge to the set $B$ of its
$\theta^0(n)$-complete accumulation points.

5. **Characterization minimal $S(n)$-spaces**

A $\mathcal{P}$ space is minimal $\mathcal{P}$ if it has no strictly coarser $\mathcal{P}$ topology.
The terms minimal Urysohn and minimal regular are abbreviated as $MU$ and $MR$, respectively.

**Definition 5.1.** The set $A$ is $\theta(n)$-converge to the set $B$ if for
any $S(n - 1)$-cover $\gamma = \{U_\alpha\}$ of $B$ there exists a finite family
$\{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$ such that $|A \setminus \bigcup_{i=1}^k U_{\alpha_i}| < |A|$.  

**Theorem 5.2.** For $n \in \mathbb{N}$, a $S(n)$-space $X$ is minimal $S(n)$-space
if and only if any infinity set $A \subseteq X$ $\theta(n)$-converge to the set $B$ of its
$\theta(n)$-complete accumulation points, and if there exists a point $x$
such that $A$ does not $\theta(n)$-converge to $X \setminus \{x\}$, then $x$ is a complete
accumulation point of $A$.

**Proof.** Necessary. Let $X$ be minimal $S(n)$-space and $A \subseteq X$. Then
$X$ is an $S(n)$-closed space (Corollary 2.3. in [7]) and $A$ (weakly)
$\theta(n)$-converge to the set $B$ of its $\theta(n)$-complete accumulation points.
Let $x \in X$ such that $A$ does not $\theta(n)$-converge to $X \setminus \{x\}$. Consider
$S(n - 1)$-cover $\gamma = \{U_\alpha\}$ of $X \setminus \{x\}$ such that $|A \setminus \bigcup_{i=1}^k U_{\alpha_i}| = |A|$
holds for any $\gamma' = \{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$.

Let $\omega$ open $S(n)$-filter generated by $\{X \setminus \overline{U_\alpha} : U_\alpha \in \gamma\}$. Then $\omega$
has unique adherent point $x$. Since $X$ is minimal $S(n)$-space, we have
that open $S(n)$-filter $\omega$ converge to $x$. Thus for every open
neighborhood $O(x)$ of $x$ there is $V \in \omega$ such that $V \subseteq O(x)$. So
$|V \cap A| = |A|$ we have $x$ is a complete accumulation point of $A$.  


Sufficiency. We only need show that any open \(S(n)\)-filter \(\varphi\) with unique adherent point \(x\) is convergent.

Suppose that \(ad \varphi = \{x\}\), but \(\varphi\) does not converge. Then there is an open neighborhood \(O(x)\) of \(x\) such that \(W_\alpha = V_\alpha \setminus O(x) \neq \emptyset\), for any \(V_\alpha \in \varphi\). Choose \(W_{\alpha_0}\) such that \(|W_{\alpha_0}| = \inf\{|W_\alpha| : V_\alpha \in \varphi\}\). Let \(B\) be the set of \(\theta(n)\)-complete accumulation points of \(W_{\alpha_0}\). Note that \(x \in B\). On a contrary, assume that \(x \notin B\) then for every \(y \in B\) there are \(n\)-hull neighborhood \(O(y)\) of \(y\) and \(W_\alpha\) such that \(O(y) \cap W_\alpha = \emptyset\). Consider \(S(n-1)\)-cover \(\gamma = \{O(y) : y \in B\}\) of \(B\). For every finite family \(\{O(y_i)\}_{i=1}^k \subseteq \gamma\) there is \(W_\alpha\) such that \((U_{i=1}^k \overline{O(y_i)}) \cap (W_\alpha \cap W_{\alpha_0}) = \emptyset\). By the choice of \(W_{\alpha_0}\), we have \(|W_\alpha \cap W_{\alpha_0}| = |W_{\alpha_0}|\). Thus \(W_{\alpha_0}\) does not \(\theta(n)\)-converge to \(B\). It follows that \(x \in B\) and \(W_{\alpha_0}\) does not \(\theta(n)\)-converge to \(B \setminus \{x\}\). For each point \(y \in X \setminus B\) we take an \(n\)-hull \(O_1(y)\) such that \(|O_1(y) \cap A| < |A|\). Consider \(S(n-1)\)-cover \(\gamma_1 = \gamma \cup \{O_1(y) : y \in X \setminus B\}\) of \(X \setminus \{x\}\). For every finite family \(\{V_i\}_{i=1}^k \subseteq \gamma_1\) there is \(W_\alpha\) such that \((U_{i=1}^k \overline{V_i}) \cap (W_\alpha \cap W_{\alpha_0}) = \emptyset\). Thus \(W_{\alpha_0}\) does not \(\theta(n)\)-converge to \(X \setminus \{x\}\). By the condition, \(x\) is a complete accumulation point of \(W_{\alpha_0}\). This contradicts the fact that \(W_{\alpha_0} = V_{\alpha_0} \setminus O(x)\).

Clearly, that the weakly \(\theta(n)\)-converge implies \(\theta(n)\)-converge. By Theorems 4.3 and 5.2, we have

**Theorem 5.3.** For \(n \in \mathbb{N}\), a \(S(n)\)-space \(X\) is minimal \(S(n)\)-space if and only if \(X\) is a \(S(n)\)-closed, and if there exists a point \(x\) such that infinity set \(A\) does not \(\theta(n)\)-converge to \(X \setminus \{x\}\), then \(x\) is a complete accumulation point of \(A\).

In [5] raised the question (Q40) about the characterization of \(MU\) spaces. Namely, find a property \(Q\) which does not imply \(U\)-closed for which a space is \(U\)-closed and has property \(Q\) if and only if it is \(MU\). The following theorem answers this question.

**Theorem 5.4.** An Urysohn space \(X\) is \(MU\) if and only if \(X\) is a \(U\)-closed, and if there exists a point \(x\) such that infinity \(A\) does not \(\theta(2)\)-converge to \(X \setminus \{x\}\), then \(x\) is a complete accumulation point of \(A\).

In [5] raised the question (Q35): Does every \(MU\) space have a base of open sets with \(U\)-closed complements?
Note, that the negative answer to this question is the following example [6]. This is an example of a $MU$ space that has no open base with $U$-closed complements.

Example 2. (Herrlich) For any ordinal number $\alpha$, let $W(\alpha)$ be the set of all ordinals strictly less than $\alpha$. Let $\omega_0$ be the first infinite ordinal and $\omega_1$ the first uncountable ordinal. Let $R = (W(\omega_1 + 1) \times W(\omega_0 + 1)) \setminus \{(\omega_1, \omega_0)\}$ and $R_n = R \times \{n\}$ where $n = 0, \pm 1, \pm 2, \ldots$. Denote the elements of $R_n$ by $(x, y, n)$. Identify $(\omega_1, y, n + 1)$ if $n$ is odd and $(x, \omega_0, n)$ with $(x, \omega_0, n + 1)$ if $n$ is even. Call the resulting space $T$. To the subspace $E = R_1 \cup R_2 \cup R_3$ of $T$ add two points $a$ and $b$, and let $X = E \cup \{a, b\}$. A set $V \subset X$ is open if and only if

1. $V \cap E$ is open in $E$,
2. $a \in V$ implies there exist $\alpha_0 < \omega_0$ such that $\{(\alpha, \beta, 1) : \beta_0 < \beta \leq \omega_0, \alpha_0 < \alpha < \omega_1\} \subset V$, and
3. $b \in V$ implies there exist $\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$ such that $\{(\alpha, \beta, 3) : \beta_0 < \beta < \omega_0, \alpha_0 < \alpha \leq \omega_1\} \subset V$.

Really, for any open $V \ni a$, if $a \in O(a) = \{(\alpha, \beta, 1) : \beta_0 < \beta \leq \omega_0, \alpha_0 < \alpha < \omega_1\} \subset V$ then $X \setminus O(a)$ is not a $U$-closed. Infinity set $\{(\alpha, \omega_0, 2) : \alpha_0 < \alpha < \omega_1\}$ do not weakly $\theta(2)$-converge to the set of its $\theta(2)$-complete accumulation points.

6. Characterization $S(n)FC$ spaces

Definition 6.1. A $S(n)$-space is $S(n)$-functionally compact ($S(n)FC$) if every continuous function onto a $S(n)$-space is closed.

A set $C$ will be called complete accumulation set of a set $A$ if $|U \cap A| = |A|$ for any open set $U \supseteq C$.

Theorem 6.2. For $n \in \mathbb{N}$, a $S(n)$-space $X$ is $S(n)FC$ if and only if any infinity set $A \subseteq X$ $\theta(n)$-converge to the set $B$ of its $\theta(n)$-complete accumulation points, and if there exists a $\theta^n$-closed set $C$ such that $A$ does not $\theta(n)$-converge to $X \setminus C$, then $C$ is a complete accumulation set of $A$.

Proof. Necessary. Let $X$ be $S(n)FC$ and $A \subseteq X$. Then $X$ is an $S(n)$-closed space and $A$ (weakly) $\theta(n)$-converge to the set $B$ of its $\theta(n)$-complete accumulation points. Let $\theta^n$-closed set $C$ such that $A$ does not $\theta(n)$-converge to $X \setminus C$. 


Consider $S(n-1)$-cover $\gamma = \{U_\alpha\}$ of $X \setminus C$ such that $|A \setminus \bigcup_{i=1}^k \overline{U_{\alpha_i}}| = |A|$ holds for any $\gamma' = \{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$.

Let $\omega$ open $S(n)$-filter generated by $\{X \setminus U_\alpha : U_\alpha \in \gamma\}$.

Suppose that there exists an open set $W \supseteq C$ such that $|A \cap W| < |A|$. Consider the quotient space $(X/C, \tau)$ of $X$ with $C$ identified to a point $e$. Now $\tau_1 = \{V \in \tau : c \in V \implies V \in \omega\}$ is a topology on $X/C$. In $(X/C, \tau_1)$ we have $ad_\theta^c N_x$ for any $x$ where $N_x$ is the neighbourhood filter at the point $x$, and thus $(X/C, \tau_1)$ is an $S(n)$-space. It is clear that $\tau_1$ is strictly coarser than $\tau$. The quotient function from $X$ to $X/C$ is denoted as $p_C$, and $q_C$ denotes $s \circ p_C$ where $s : (X/C, \tau) \to (X/C, \tau_1)$ is the identity function. Note that $q_C(X \setminus W)$ is not closed in $(X/C, \tau_1)$. Thus, $q_C$ is continuous but is not closed. This is a contradiction that $X$ is a $S(n)FC$ space.

Sufficiency. Suppose that $X$ is not $S(n)FC$ space. Then there is a continuous function $f$ from $X$ onto an $S(n)$-space $Y$ such that $f$ is not closed. Consider the closed set $A \subseteq X$ such that $f(A)$ is not closed. Let $y \in \overline{f(A)} \setminus f(A)$ and $N_y = \{V_\alpha\}$ is the neighbourhood $S(n)$-filter at the point $y$. Then $B = f^{-1}(y) = \bigcap \{f^{-1}(V_\alpha)\}$ and $B$ is a $\theta^n$-closed subset of $X$. Note that $X \setminus A$ is an open set containing $B$ such that $W_\alpha = U_\alpha \setminus (X \setminus A) \neq \emptyset$ for any $U_\alpha \in \{f^{-1}(V_\alpha)\}$.

Choose $W_{\alpha_0}$ such that $|W_{\alpha_0}| = \inf\{|| W_\alpha ||\}$.

Let $D$ be the set of $\theta(n)$-complete accumulation points of $W_{\alpha_0}$. By the condition, set $W_{\alpha_0}$ $\theta(n)$-converge to the set $D$. We claim that $\theta^n$-closed set $B$ such that $W_{\alpha_0}$ does not $\theta(n)$-converge to $X \setminus B$. Indeed, for any $x \in X \setminus B$ there is $U_{\alpha_x}$ such that $x$ is $S(n)$-separated from $U_{\alpha_x}$. Let $O(x)$ be a $n$-hull neighbourhood of $x$ such that $\overline{O(x)} \setminus U_{\alpha_x} = \emptyset$. Consider a $S(n-1)$-cover $\gamma = \{O(x) : x \in X \setminus B\}$ of $X \setminus B$. For any finite family $\{O(x_i)\}_{i=1}^k \subseteq \gamma$ there is $U = \bigcup_{i=1}^k U_{\alpha_{x_i}}$ such that $\bigcup_{i=1}^k \overline{O(x_i)} \setminus (U \cap W_{\alpha_0}) = \emptyset$. By the choice of $W_{\alpha_0}$, we have $|U \cap W_{\alpha_0}| = |W_{\alpha_0}|$. It follows that $W_{\alpha_0}$ does not $\theta(n)$-converge to $X \setminus B$. By the condition, $B$ is a complete accumulation set of $W_{\alpha_0}$. This contradicts the fact that $X \setminus A$ is an open set containing $B$.

\textit{Corollary 6.3.} An Urysohn space $X$ is $UFC$ if and only if $X$ is an $U$-closed, and if there exists a $\theta^2$-closed set $C$ such that infinity set $A$
does not \( \theta(2) \)-converge to \( X \setminus C \), then \( C \) is a complete accumulation set of \( A \).

**Definition 6.4.** A \( S(n) \)-space \( X \) is \( S(n)FFC \) (\( S(n)CFC \)) if every continuous function \( f \) onto a \( S(n) \)-space \( Y \) with \( f^{-1}(y) \) finite (compact) is a closed function.

**Theorem 6.5.** For \( n \in \mathbb{N} \), a \( S(n) \)-space \( X \) is \( S(n)FFC \) (\( S(n)CFC \)) if and only if \( X \) is a \( S(n) \)-closed, and if there exists a finite (compact) set \( C \) such that infinity set \( A \) does not \( \theta(n) \)-converge to \( B \setminus C \), then \( C \) is a complete accumulation set of \( A \).

**Proof.** A proof of Theorem 6.5 is analogous to that of Theorem 6.2. □

**Question 2.** Is every \( S(n)FC \) (\( S(n)FFC \), \( S(n)CFC \)) space necessarily compact \( (n > 1) \)?

7. \( S(\omega) \)-CLOSED AND MINIMAL \( S(\omega) \) SPACES

Two filters \( F \) and \( Q \) on a space \( X \) are \( S(\omega) \)-separated if there are open families \( \{ U_\beta : \beta < \omega \} \subseteq F \) and \( \{ V_\beta : \beta < \omega \} \subseteq Q \) such that \( U_0 \cap V_0 = \emptyset \) and for \( \gamma + 1 < \omega \), \( cU_{\gamma+1} \subseteq U_{\gamma} \) and \( cV_{\gamma+1} \subseteq V_{\gamma} \).

A space \( X \) is \( S(\omega) \) if for distinct points \( x, y \in X \), the neighborhood filters \( N_x \) and \( N_y \) are \( S(\omega) \)-separated.

A \( S(\omega) \)-closed space is a \( S(\omega) \) space closed in any \( S(\omega) \) space containing them.

In 1973, Porter and Votaw \[12\] established next results.

1. A minimal \( S(\omega) \) space is \( S(\omega) \)-closed and semiregular.
2. A minimal \( S(\omega) \) space is regular.
3. A space is \( R \)-closed if and only if it is \( S(\omega) \)-closed and regular.
4. A space is \( MR \) if and only if it is minimal \( S(\omega) \).

**Definition 7.1.** A neighborhood \( U \) of a point \( x \) is called an \( \omega \)-hull of the point \( x \) if there exists a set of neighborhoods \( \{ U_i \}_{i=1}^\infty \) of the point \( x \) such that \( cU_i \subseteq U_{i+1} \) and \( U_i \subseteq U \) for every \( i \in \mathbb{N} \).

**Definition 7.2.** A point \( x \) from \( X \) is called a \( \theta(\omega) \)-accumulation point of an infinite set \( F \) if \( |F \cap U| = |F| \) for any \( U \), where \( U \) is an \( \omega \)-hull of the point \( x \).

**Definition 7.3.** The set \( A \) is \( \theta(\omega) \)-converge to the set \( B \) if for any regular cover \( \gamma = \{ U_\alpha \} \) of \( B \) there exists a finite family \( \{ U_\alpha \}_{i=1}^s \subseteq \gamma \) such that \( |A \setminus \bigcup_{i=1}^s U_\alpha| < |A| \).
Theorem 7.4. A regular space $X$ is $R$-closed if and only if any infinity set $A \subseteq X$ $\theta(\omega)$-converge to the set $B$ of its $\theta(\omega)$-accumulation points.

Proof. A proof of Theorem 7.4 is analogous to that of Theorem 4.3. □

In [5] raised the question (Q39) about the characterization of $MR$ spaces. Namely, find a property $P$ which does not imply $R$-closed for which a space is $R$-closed and has property $P$ if and only if it is $MR$. The following theorem answers this question.

Theorem 7.5. A regular space $X$ is $MR$ space if and only if $X$ is a $R$-closed, and if there exists a point $x \in B$ such that infinite set $A$ does not $\theta(\omega)$-converge to $X \setminus \{x\}$, then $x$ is a complete accumulation point of $A$.

Proof. A proof of Theorem 7.5 is analogous to that of Theorem 5.2. □

We introduce an operator of $\theta_\omega$-closure; for $M \subseteq X$ and $x \in X$ $x \notin cl_{\theta_\omega} M$ if there is a $\omega$-hull $U$ of $x$ such that $U \cap M = \emptyset$. A set $M \subseteq X$ is $\theta_\omega$-closed if $M = cl_{\theta_\omega} M$.

Definition 7.6. A regular space is regular functionally compact (RFC) if every continuous function onto a regular space is closed.

Theorem 7.7. A regular space $X$ is RFC if and only if $X$ is $R$-closed, and if there exists a $\theta_\omega$-closed set $C$ such that infinite set $A$ does not $\theta(\omega)$-converge to $X \setminus C$, then $C$ is a complete accumulation set of $A$.

Proof. A proof of Theorem 7.7 is analogous to that of Theorem 6.2. □

Question 3. Is every RFC space necessarily compact?

Remark 7.8. Note that a negative answer to the question (Q27 in [5]) of compactness of $U$-closed space in which the closure of any open set is $U$-closed be any non-compact $H$-closed Urysohn space.

Question 4. (Q26 in [5]) Is an $R$-closed space in which the closure of every open set is $R$-closed necessarily compact?
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