Well-posedness by noise for linear advection of $k$-forms

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Abstract

In this work, we extend existing well-posedness by noise results for the stochastic transport and continuity equations by treating them as special cases of the linear advection equation of $k$-forms, which arises naturally in geometric fluid dynamics. In particular, we prove the existence and uniqueness of weak $L^p$-solutions to the stochastic linear advection equation of $k$-forms that is driven by a H"older continuous, $W^{1,1}_{loc}$ drift and smooth diffusion vector fields, such that the equation without noise admits infinitely many solutions.

1 Introduction

The aim of this paper is to extend the well-posedness by noise result shown in [FGP10] to a broader class of SPDEs of the form

$$\begin{align*}
    dK(t, x) + \mathcal{L}_b K(t, x) dt + \sum_{i=1}^{N} \mathcal{L}_{\xi_i} K(t, x) \circ dW_t^i &= 0, \\
    K(0, x) &= K_0(x), \quad x \in \mathbb{R}^n,
\end{align*}$$

(1)

where

- $K$ is a stochastic process that takes values in the space of differential $k$-forms in $\mathbb{R}^n$ (i.e. the natural objects that one can integrate over $k$-dimensional submanifolds in $\mathbb{R}^n$).
- $b(t, x), \{\xi_i(t, x)\}_{i=1,...,N}$ are time-dependent vector fields satisfying Assumption A below.
- $\mathcal{L}_v$ denotes the Lie derivative with respect to the vector field $v$, generalising the notion of directional derivative to tensor fields (see Appendices A and B for supplementary notes on tensor calculus).

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We also note that \( \{W^j_i\}_{i=1,\ldots,N} \) is a family of i.i.d. Brownian motions, and \( \circ \) indicates that the stochastic integral is understood in the Stratonovich sense. The multiplicative noise term considered here can be referred to as “transport noise”, since the noise appears in the transport velocity of the Lie derivative (see [Hol15, CFH17]).

Without noise, that is, \( \xi_i \equiv 0 \) for all \( i = 1, \ldots, N \), one can construct a drift vector field \( b \) satisfying condition A (1), such that the Cauchy problem (1) admits infinitely many solutions, as illustrated in Section 5 of the present work. Thus, the noise term in (1) has the effect of restoring the uniqueness of solutions. This phenomenon, typically referred to as “well-posedness by noise”, has been observed in various PDE systems such as the stochastic linear transport equation [FGP10, AF11, CO13, MNP+15] and the continuity equation [NO15, Oli17, LFMM18, GS19], and has attracted attention in recent years (see [Ges16] for a recent overview and various references). Our work aims to extend the aforementioned results by interpreting these systems (transport, continuity, etc.) as particular cases of (1).

**Background.** From a physical viewpoint, our motivation for considering equation (1) resides in its importance in stochastic fluid modelling, where the equation represents the advection of a locally conserved quantity \( K \) along stochastic paths traversed by fluid particles. That is, (1) is equivalent to the stochastic local conservation law

\[
\int_{\Omega_0} K(0, x) = \int_{\Omega(t)} K(t, x),
\]

where \( \Omega_0 \) is any \( k \)-dimensional (oriented) submanifold of \( \mathbb{R}^n \), and \( \Omega(t) = \phi_t(\Omega_0) \) is the image of \( \Omega_0 \) under a sufficiently smooth stochastic flow map

\[
\phi_t(x) = x + \int_0^t b(t, \phi_s(x)) \, ds + \sum_{i=1}^N \int_0^t \xi_i(t, \phi_s(x)) \circ dW^i_s,
\]

representing the Lagrangian trajectory of a stochastic fluid particle with label \( x \) (see [Hol15, CGH17, BdLHLT19]). For example, if \( K(t, x) = \rho(t, x) d^n x \) is the mass density, which is a volume (top) form, conservation of mass (2) implies that \( \rho \) satisfies the stochastic continuity equation

\[
d\rho(t, x) + \text{div}(b(t, x) \rho(t, x)) \, dt + \sum_{i=1}^N \text{div}(\xi_i(t, x) \rho(t, x)) \circ dW^i_t = 0,
\]

since the Lie derivative of volume forms is given by \( \mathcal{L}_v(\rho(t, x) d^n x) = \text{div}(\rho v) d^n x \). Moreover, if \( K \) is a zero-form (i.e., a scalar field), (1) becomes

\[
dK(t, x) + b(t, x) \cdot DK(t, x) \, dt + \sum_{i=1}^N \xi_i(t, x) \cdot DK(t, x) \circ dW^i_t = 0,
\]

which is equivalent to (4) when \( b \) and \( \{\xi_i\}_{i=1,\ldots,N} \) are divergence-free. Another well-known example in physics is the advection of a magnetic field \( B \), commonly interpreted as a two-form in 3D ideal magnetohydrodynamics, where we have \( \mathcal{L}_v B = \text{curl}(v \times B) \). We refer the readers to [HMR98] for more instances of deterministic fluid models with advected quantities, and to [Hol15] for their stochastic counterparts.
In [FGP10], it is shown that the stochastic linear transport equation (5) admits a unique $L^\infty$-weak solution, strong in the probabilistic sense, under the assumptions that the vector field $b$ is bounded, $\alpha$-Hölder continuous for $0 < \alpha < 1$, with $\text{div} \, b \in L^p([0, T] \times \mathbb{R}^n)$ for some $p > 2$, and $\xi_i(t, x) \equiv e_i, i = 1, \ldots, n = N$, providing the first example of an ill-posed PDE that becomes well-posed under the addition of a multiplicative noise. More precisely, one can construct examples where $b$ has unbounded divergence, such that the stochastic equation (5) is well-posed but the corresponding deterministic counterpart is not. ¹ The well-posedness proof in [FGP10] is based on a characteristic argument instead of a standard PDE argument, making extensive use of the regularity of the flow $\phi_t$ of the SDE (3) and its Jacobian $J\phi_t$. As we will show, this technique can be extended to prove well-posedness of the general equation (1), thanks to the underlying geometric structure of the equation. In subsequent years following [FGP10], effort has been made to weaken the conditions on the drift $b$ such as in [MNP⁺15], where only boundedness and measurability are assumed, and PDE arguments for the proof of well-posedness have also been developed in [AF11], providing an alternative explanation as to why well-posedness by noise occurs. Furthermore, in [FF13], the transport noise has been shown not only to restore uniqueness but also to have a regularising effect on the transport equation.

Well-posedness by noise for the stochastic continuity equation (4) has also been shown in several works in the literature. For instance, [LFMNM18] treats the case where $b$ and $\text{div} \, b$ satisfy the so-called Ladyzhenskaya-Prodi-Serrin condition by employing a PDE-based argument, and [Oli17] uses a characteristic argument to show well-posedness in the case where $b$ is random and satisfies some integrability conditions, without any assumptions on the divergence (see [NO15, GS19, PS17] for other related works on the stochastic continuity equation). Interestingly, the case corresponding to $k = 2, n = 3$ (i.e. two-form in three dimensional space) in our general equation (1), known as the vector advection equation, is also considered in [FMN14], where well-posedness is proved when the drift vector field is bounded, Hölder continuous, and divergence-free, again by employing a characteristic argument.

**Strategy of proof.** The class of equations (1) that we consider in this work includes all of the equations above (stochastic linear transport, continuity, and vector advection) as special cases. In particular, they have the same geometric interpretation as: a differential $k$-form being advected by a stochastic vector field. By exploiting this geometric structure, we are able to prove the well-posedness of (1), and therefore of all of the equations that belong to this class, for initial data $K_0$ of class $L^p \cap L^\infty_{\text{loc}}$, and drift and diffusion vector fields satisfying the following conditions.

**Assumption A.**

1. $b(t, x)$ is of class $L^\infty([0, T]; W^{1,1}_{\text{loc}} \cap C^\alpha_b(\mathbb{R}^n, \mathbb{R}^n))$ for some $0 < \alpha < 1$.

2. $\{\xi_k(t, x)\}_{k=1,\ldots,N}$ is of class $L^\infty([0, T]; C^{4+\beta}_b(\mathbb{R}^n, \mathbb{R}^n))$ for some $0 < \beta < 1$ such that for all $v \in \mathbb{R}^n \setminus \{0\}$, there exist $0 < \delta < K < \infty$ such that

$$\delta |v|^2 \leq \sum_{i,j=1}^n \sum_{k=1}^N \xi^i_k(t, x) \xi^j_k(t, x) v_i v_j \leq K |v|^2. \quad (6)$$

¹Note that one of the essential conditions for the well-posedness of the transport equation in [DL89] is that the divergence must be bounded.
The basic strategy is the following. Given a flow \( \phi_t \) satisfying (3), the solution to (1) can be expressed as the push-forward (in the sense of differential geometry) of the initial data
\[
K(t, x) = (\phi_t)_* K_0(x),
\]
provided the drift and diffusion vector fields are sufficiently smooth. This representation of the solution allows us to carry out a characteristic argument similar to the one appearing in [FGP10], by first proving the existence and uniqueness of a \( C^1 \)-flow that solves (3), and then applying an extension of DiPerna-Lions commutator argument (see [DL89]) to prove uniqueness. Since we consider a more general noise, we need to extend the flow regularity results in [FGP10] to Hölder continuous drift \( b \) and general diffusion profiles \( \{\xi_i(t, x)\}_{i=1, \ldots, N} \) satisfying Assumption A (2). This is effected by invoking classical estimates for general parabolic PDEs and then using the so-called Itô-Tanaka trick to transform (3) into an SDE with more regular coefficients in order to deduce the existence, uniqueness, stability, and \( C^1 \)-regularity of the flow.

Now let us highlight some differences between this and previous works. As mentioned before, one novel aspect of this paper is that we employ concepts from differential geometry to extend the techniques based on characteristics and prove well-posedness of the stochastic transport, continuity and advection equations. However, to accommodate this generality, some adjustments had to be made in the assumptions, as we list below:

- When proving existence for non-smooth \( b \), it is more convenient to work with \( L^p \)-solutions \((1 \leq p < \infty)\) rather than \( L^\infty \)-solutions as treated in [FGP10], since in general, the push-forward (7) contains derivatives of the backward flow \( \phi_t^{-1} \), which a-priori is not clear how to bound in \( L^\infty \), but can be bounded in \( L^p \).

- The additional assumption \( b \in W^{1,1}_{\text{loc}} \) was made since the general form of \( L_b K \) requires a derivative on \( b \), and it is also convenient when computing the commutator estimates in the uniqueness proof, since it avoids having to control the derivative of the Jacobian, as carried out in [FGP10]. For this reason, we do not require assumptions on \( \text{div} \ b \).

- Similarly, the additional assumption that \( K \) is of class \( L^\infty_{\text{loc}} \) was made for convenience when computing estimates to prove uniqueness.

Due to the generality of our noise profile \( \{\xi_i(t, x)\}_{i=1, \ldots, N} \), we also have to treat the Stratonovich-to-Itô correction term \( [L_{\xi_i}, [L_{\xi_j}, \rho^i]] K(t, x) \) of the commutator \( [L_{\xi}, \rho^i] K(t, x) \) (which vanishes in the special case \( \xi_i(t, x) \equiv e_i, i = 1, \ldots, N \) considered in [FGP10]) appearing in the noise term. A novelty of this paper consists in showing that this extra term also satisfies a DiPerna-Lions type estimate, and converges to zero as \( \epsilon \to 0 \). This is a-priori far from being obvious since it gives rise to second-order terms, which at first glance seem too singular; however, convergence estimates are made possible by arranging terms in a fashion such that DiPerna and Lions’ argument [DL89] can be applied. We remark that in the special case \( L_{\xi} = \xi \cdot \nabla \), a similar estimate for the double commutator is obtained in [PSS18], Proposition 3.4. We note as well that in order to treat the weak formulation of equation (1) in the sense of Definition 2.15 below, weak versions of Lie derivatives need to be developed, and notions such as mollifiers need to be extended to differential forms (more generally, to tensor fields), which play an important role in our analysis.
What next? We note that our work is exploratory, with a focus on generalising the characteristic argument to a broader class of systems, hence we did not treat the case when $b$ is of extremely low regularity that is considered state-of-the-art in the literature. Naturally, this leads us to ask whether the well-posedness result shown here also holds true for drifts of lower regularity, such as the Ladyzhenskaya-Prodi-Serrin condition. On a related note, it would also be interesting to investigate whether the transport noise can regularise the equation as observed for the linear transport equation in [FF13]. Another possible line of research would be to investigate whether the same phenomenon can be observed when the underlying space is a smooth manifold instead of $\mathbb{R}^n$, which is a natural setting for our equation due to its coordinate-free nature. This would require extending several intermediate results such as the commutator estimates to manifolds, which is not immediately clear at this point. As noted by several other authors, a significant step forward in the analysis of fluid PDEs may also be achieved if one could obtain similar well-posedness/regularisation by noise results for nonlinear fluid PDEs. However, this is known to be much more challenging than the linear case. For instance [Fla11, AOdT18] show how transport noise does not improve well-posedness even in the simplest nonlinear example of the inviscid Burgers’ equation.

Plan of this paper.

◊ In Section 2, we present important background material that will be needed in the proofs of our main results.

◊ Section 3 establishes the regularity of the solution to the flow equation (3) under Assumption (A), by extending the flow regularity results demonstrated in [FGP10] to our more general case.

◊ Section 4 proves the main results of this work. More concretely, we show existence and uniqueness of solutions to equation (1).

◊ In Section 5, we construct an example of drift profile $b$ satisfying Assumption A (1) such that the deterministic version of (1) admits infinitely many solutions for the same initial profile. For this, we provide a generalisation to $n$-dimensions of the scalar counterexample considered in [FGP10].

2 Preliminaries and Notation

In this section, we state some preliminary results and definitions that will be necessary for proving our main theorems. In Subsection 2.1, we provide some background in PDE and stochastic analysis, such as classical estimates that will be employed during our proofs. In Subsection 2.2, we define standard Banach spaces of tensor fields, which constitute the natural spaces in which our equations are defined. In Subsection 2.3, we state the Kunita-Itô-Wentzell formula for $k$-forms, which will be an essential tool in our uniqueness proofs. Finally, in Subsection 2.4, we define the concept of classical and weak solutions of our SPDE (1). The readers familiar with these topics may move directly to Section 3.
2.1 Background in analysis

Since we will often use mollifiers, we provide a definition. Let \( \rho : \mathbb{R}^n \to \mathbb{R} \) be a smooth function such that \( 0 \leq \rho(x) \leq 1 \), \( \rho(x) = \rho(-x) \), \( x \in \mathbb{R}^n \), \( \int_{\mathbb{R}^n} \rho(x) dx = 1 \), \( \text{Supp}(\rho) \subset B(0,1) \), and \( \rho(x) = 1 \), \( x \in B(0,1/2) \). For any \( \epsilon > 0 \), let \( \rho^\epsilon \) be defined as \( \rho^\epsilon(x) = \epsilon^{-n} \rho(x/\epsilon) \).

We state certain parabolic estimates that will be useful in our analysis.

**Theorem 2.1** ([KP09]). Let \( b, f \in L^\infty([0,\infty); C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n)) \), and \( \xi_k \in L^\infty([0,\infty); C_b^{1+\alpha}(\mathbb{R}^n, \mathbb{R}^n)) \), \( k = 1, \ldots, N \) be measurable. Assume that there exist \( 0 < \delta < K < \infty \) such that the uniform ellipticity condition

\[
\delta \|v\|^2 \leq \sum_{i,j} \xi_k^i(x) \xi_k^j(x) v_i v_j \leq K \|v\|^2
\]

is satisfied for all \( v \in \mathbb{R}^n \setminus \{0\} \), \( x \in \mathbb{R}^n \), \( k = 1, \ldots, N \). Then there exists a unique solution \( u_\lambda \in L^\infty([0,\infty); C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n)) \) of the backward system

\[
\partial_t u_\lambda + Lu_\lambda - \lambda u_\lambda = f, \quad \lambda > 0,
\]

where the differential operator \( L \) is defined by

\[
Lu = \sum_k \left( \frac{1}{2} \xi_k^i(t,x) \xi_k^j(t,x) \partial_i \partial_j u + (b^i(t,x) + \xi_k^i(t,x) \partial_i \xi_k^j(t,x)) \partial_j u \right),
\]

where Einstein convention of summing over repeated indices is assumed. Moreover, the following estimate holds:

\[
\sup_{t \geq 0} \|u(t,\cdot)\|_{C_b^{2+\alpha}} \lesssim \sup_{t \geq 0} \|f(t,\cdot)\|_{C_b^\alpha}.
\]

We state a result we will need, which can be found in detail in [ST18].

**Lemma 2.2** (Generalised heat kernel estimates). Consider the Cauchy problem

\[
\begin{align*}
\partial_t u(t,x) + u(t,x) - \sum_{i,j=1}^n a^{ij}(t) \partial_{ij} u(t,x) &= f(x), \quad t > 0, \quad x \in \mathbb{R}^n, \\
u(0,x) &= g(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where the coefficients \( a(t) = (a^{ij}(t)) \) are bounded, measurable, symmetric, and uniformly elliptic. Then there exists \( c > 0 \) such that the function \( p(t,\tau,x) \geq 0 \), representing the kernel of the solution of (11), satisfies the following inequality:

\[
|D_x p(t,\tau,x)| \lesssim \chi_{\tau > 0} \exp(-\tau \frac{|x| \exp(-|x|^2/(c\tau))}{\tau^{n/2+1}}).
\]

Throughout the paper, we assume that we are working with a stochastic basis of the form \((\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P}, (W_t))_{t \in [0,T]}\) with the usual conditions, where \( W_t \) is an \( n \)-dimensional Brownian motion. \( (\mathcal{F})_{s,t} \) represents the completed \( \sigma \)-algebra generated by \( W_r - W_u, s \leq u \leq r \leq t \), for \( 0 \leq s < t \). We present a version of the Fubini Theorem in the stochastic case appearing in [Kry11].

**Theorem 2.3** (Stochastic Fubini). Let \( T \in \mathbb{R}^+, \mathcal{P} \) be the predictable sigma-algebra on \([0,\infty) \times \Omega, \) and \( G_t(x), H_t(x) \) be real functions defined on \([0,T] \times \mathbb{R}^n \times \Omega\) satisfying the following properties:
• $G_t(x)$ and $H_t(x)$ are $\mathcal{P}_T -$measurable, where $\mathcal{P}_T$ is the restriction of $\mathcal{P}$ to $[0, T] \times \Omega$.

• For every $(x, \omega) \in \mathbb{R}^n \times \Omega \setminus A$, where $A$ is a set of measure zero, we have

$$\int_0^T (|G_t(x)| + |H_t(x)|^2)dt < \infty.$$ 

• $G_t(x)$ and $H_t(x)$ also satisfy

$$\int_0^T \int_{\mathbb{R}^n} |G_t(x)|dxdt + \int_0^T \left( \int_{\mathbb{R}^n} |H_t(x)|^2dx \right)^{1/2} dt < \infty, \ a.s.$$ 

Then we have that the stochastic process

$$\int_0^t G_s(x)ds + \int_0^t H_s(x)dB_s, \ t \in [0, T], \quad (13)$$

is well-defined, $\mathcal{P}_T \otimes \mathcal{B}([0, T] \times \mathbb{R}^n)$-measurable, and can be modified into a continuous stochastic process by only changing its values in a set of measure zero. Moreover, the stochastic integral

$$\int_0^t \int_{\mathbb{R}^n} H_s(x)dx dB_s$$

is well-defined, and the following equality holds:

$$\int_{\mathbb{R}^n} \int_0^t G_s(x)ds dx + \int_{\mathbb{R}^n} \int_0^t H_s(x)dB_s dx = \int_0^t \int_{\mathbb{R}^n} G_s(x)dx ds + \int_0^t \int_{\mathbb{R}^n} H_s(x)dx dB_s, \ a.s. \ t \in [0, T].$$

We introduce an important definition.

**Definition 2.4 (Stochastic flow of diffeomorphisms).** Let $k \in \mathbb{N}$. A continuous random field $\varphi_{s,t} : \mathbb{R}^n \to \mathbb{R}^n$, $s, t \in [0, T]$, is called a stochastic flow of $C^k$-diffeomorphisms if it satisfies the following properties:

• $\varphi_{s,r} = \varphi_{t,r} \circ \varphi_{s,t}$, for all $s, t, r \in [0, T]$, such that $s < t < r$ a.s.

• $\varphi_{s,s} = \text{id}_{\mathbb{R}^n}$, for any $s \in [0, T]$, a.s.

• $\varphi_{s,t}(\cdot)$ is $k$-times differentiable, and its derivatives are continuous in $(s, t, x)$. Furthermore, the map $\varphi_{s,t} : \mathbb{R}^n \to \mathbb{R}^n$ is an onto $C^k$-diffeomorphism, for $s, t \in [0, T]$.

We will also need extensions of Itô’s formula to tensor fields, which can be found in [Kun97].

**Theorem 2.5.** Let $K$ be a $C^3$-smooth tensor field and let $0 \leq s \leq t \leq T$. Then Itô’s second formula (i.e. Itô’s formula with respect to the final time variable) for tensor fields reads

$$\begin{align*}
(\varphi_{s,t})^*K - K &= \int_s^t (\varphi_{s,r})^*(\mathcal{L}_bK)dr + \int_s^t (\varphi_{s,r})^*(\mathcal{L}_\xi K) \circ dB_r, \\
(\varphi_{t,s})^*K - K &= -\int_s^t (\varphi_{s,r})^*(\mathcal{L}_bK)dr - \int_s^t (\varphi_{s,r})^*(\mathcal{L}_\xi K) \circ d\hat{B}_r.
\end{align*} \quad (14)$$
Moreover, Itô’s first formula (i.e. Itô’s formula with respect to the initial time variable) reads

\[(\varphi_{s,t})^*K - K = \int_s^t \mathcal{L}_b((\varphi_{r,s})^*K)\,dr + \int_s^t \mathcal{L}_\xi((\varphi_{r,s})^*K) \circ d\hat{B}_r,\]

(16)

\[(\varphi_{t,s})^*K - K = -\int_s^t \mathcal{L}_b((\varphi_{r,s})^*K)\,dr - \int_s^t \mathcal{L}_\xi((\varphi_{r,s})^*K) \circ dB_r,\]

(17)

where \(\circ d\hat{B}_r\) denotes Stratonovich integration with respect to the backward Brownian motion.

2.2 Some Banach spaces of tensor fields

Here, we introduce some Banach spaces of \((r, s)\)-tensor fields \(\Gamma(T^{(r,s)}(\mathbb{R}^n))\) that will be employed throughout this paper. For a general background on tensors and tensor analysis, see Appendices A and B. We only consider tensor fields on the Euclidean space \(\mathbb{R}^n\) equipped with the standard Euclidean metric. However, these notions can be extended to general Riemannian manifolds (see for instance [Sco95, Dod81, BHM10, BHM+13, AD03]).

**L\(p\)-tensor fields.** For every \(x \in \mathbb{R}^n\), one can induce a natural metric on \(\Gamma(T^{(r,s)}(\mathbb{R}^n))\) by making use of the standard inner product on \(\mathbb{R}^n\), which we denote by angle brackets \(\langle \cdot, \cdot \rangle_x : T^{(r,s)}_x(\mathbb{R}^n) \times T^{(r,s)}_x(\mathbb{R}^n) \to \mathbb{R}\), and the corresponding norm is denoted by \(|\cdot|_x : T^{(r,s)}_x(\mathbb{R}^n) \to \mathbb{R}\) (see Appendix A for more details about the construction of a bundle metric). For \(1 \leq p < \infty\), we define the \(L^p\)-norm of \(K \in \Gamma(T^{(r,s)}(\mathbb{R}^n))\) by

\[\|K\|_{L^p} = \left(\int_{\mathbb{R}^n} |K(x)|_x^p \,d^n x\right)^{\frac{1}{p}},\]

and for \(p = \infty\), we define the supremum norm

\[\|K\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} |K(x)|_x.\]

For \(1 \leq p \leq \infty\), the \(L^p\)-section of \(T^{(r,s)}(\mathbb{R}^n)\), denoted by \(L^p(T^{(r,s)}(\mathbb{R}^n))\), is an equivalence class of tensor fields \(K\) with \(\|K\|_{L^p} < \infty\), where two tensor fields \(K_1, K_2\) are identified if \(\|K_1 - K_2\|_{L^p} = 0\).

Given a compact subset \(U \subset \mathbb{R}^n\), we define the local \(L^p\)-norm of \(K\) on \(U\) by

\[\|K\|_{L^p(U)} = \left(\int_U |K(x)|_x^p \,d^n x\right)^{\frac{1}{p}},\]

for \(1 \leq p < \infty\), and

\[\|K\|_{L^\infty(U)} := \sup_{x \in U} |K(x)|_x,\]

(18)

for \(p = \infty\). If \(\|K\|_{L^p(U)} < \infty\) for any compact subset \(U \subset \mathbb{R}^n\), then we say that \(K\) is of class \(L^p_{\text{loc}}(T^{(r,s)}(\mathbb{R}^n))\). For convenience, we will denote by \(\|\cdot\|_{L^p_{U \eta}}\) the local \(L^p\)-norm on the closed ball \(B(0, R)\).
In the special case $p = 2$, the space $L^2(T^{(r,s)}(\mathbb{R}^n))$ is furthermore equipped with an inner product $\langle\langle \cdot, \cdot \rangle\rangle$, defined by

$$\langle\langle F, K \rangle\rangle_{L^2} = \int_{\mathbb{R}^n} \langle F(x), K(x) \rangle_x d^n x,$$

for any $F, K \in L^2(T^{(r,s)}(\mathbb{R}^n))$, making it a Hilbert space. More generally, for $F \in L^p(T^{(r,s)}(\mathbb{R}^n))$ and $G \in L^q(T^{(r,s)}(\mathbb{R}^n))$, where $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$, we have

$$\langle\langle F, K \rangle\rangle_{L^p} = \int_{\mathbb{R}^n} |F(x)|_x |K(x)|_x d^n x \leq \|F\|_{L^p} \|G\|_{L^q} < \infty,$$

where we have made use of the Cauchy-Schwartz and Hölder inequalities.

**$C^k$-tensor fields.** We can also define the differentiability class $C^k$ of tensor fields. If every component $K^{i_1,\ldots,i_r}_{i_{r+1},\ldots,i_{r+s}}(x)$ of an $(r, s)$-tensor field $K$ is $k$-times differentiable, where $k \in \mathbb{N}$, then we say that $K$ is a $C^k$-section of the tensor bundle, denoted by $C^k(T^{(r,s)}(\mathbb{R}^n))$ (see [AM78]). Given an $(r, s)$-tensor field $K$ of class $C^1$, we define its derivative to be the $(r, s + 1)$-tensor field

$$\nabla K(x) = \frac{\partial K^{i_1,\ldots,i_r}_{i_{r+1},\ldots,i_{r+s}}(x)}{\partial x_l} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^l \otimes dx^{i_{r+1}} \otimes \cdots \otimes dx^{i_{r+s}}.$$

Analogously, we can define the $k$-th derivative $\nabla^k K$ for $C^k$-tensor fields $K$, which will be an $(r, s + k)$-tensor. For a general Riemannian manifold, $\nabla$ is the covariant derivative with respect to the Levi-Civita connection.

For $0 < \alpha \leq 1$, we say that $K$ is $\alpha$-Hölder continuous if

$$\sup_{x, y \in \mathbb{R}^n} \frac{|K(x) - \tau^y_x K(y)|}{|x - y|^\alpha} < \infty,$$

where $\tau^y_x$ is the parallel transport from point $y$ to $x$ along the geodesic with respect to the metric (which we will take to be Euclidean in this paper, in which case $\tau^y_x K(y) = K(y)$). We denote the space of $C^\alpha$-Hölder tensor fields by $C^\alpha(T^{(r,s)}(\mathbb{R}^n))$. In the special case $\alpha = 1$, we say that $K$ is (globally) Lipschitz continuous.

**Other related tensor classes.**

- We denote by $C^{m+\alpha}(T^{(r,s)}(\mathbb{R}^n))$, for $m \in \mathbb{N}$ and $0 < \alpha < 1$, the class of tensor fields that are $m$-times differentiable, such that its $m$-th covariant derivative $\nabla^m K$ is $\alpha$-Hölder continuous.

- We denote by $C^b_m(T^{(r,s)}(\mathbb{R}^n))$, for $m \in \mathbb{N}$, the class of $C^k$ tensor fields such that $\nabla^k K \in L^\infty$, $k = 0, \ldots, m$, and the $m$-th covariant derivative $\nabla^m K$ is $\alpha$-Hölder continuous.

- We denote by $C^k_b(T^{(r,s)}(\mathbb{R}^n))$, for $k \in [0, \infty]$, the class of $C^k$ tensor fields with compact support.

We note that the space $C^b_m(T^{(r,s)}(\mathbb{R}^n))$ is Banach under the norm

$$\|K\|_{C^b_m} = \sum_{k=0}^m \|\nabla^k K\|_{L^\infty} + \sup_{x, y \in \mathbb{R}^n} \frac{\|\nabla^m K(x) - \tau^y_x\nabla^m K(y)\|}{|x - y|^\alpha}.$$
**$W^{k,p}$-tensor fields.** One can also define Sobolev spaces of tensor fields in the same way one defines Sobolev spaces of scalar fields. First we provide the notion of weak derivatives of tensor fields.

**Definition 2.6.** Given $K \in L^1_{loc}(T^{(r,s)}(\mathbb{R}^n))$, we say that $S \in L^1_{loc}(T^{(r,s+1)}(\mathbb{R}^n))$ is a weak derivative of $K$ if it satisfies $\langle \langle S_i, \theta \rangle \rangle_{L^2} = -\langle \langle K, \nabla_i \theta \rangle \rangle_{L^2}$, for all $i = 1, \ldots, n$, where

$$S_i(x) = S^{i_1, \ldots, i_r}_{i_{r+1}, \ldots, i_{r+s}}(x) \frac{\partial}{\partial x_i} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{i_{r+1}} \otimes \cdots \otimes dx^{i_{r+s}} \in L^1_{loc}(T^{(r,s)}(\mathbb{R}^n)),$$

$$\nabla_i \theta(x) = \frac{\partial \theta^{i_1, \ldots, i_r}_{i_{r+1}, \ldots, i_{r+s}}}{\partial x_i}(x) \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{i_{r+1}} \otimes \cdots \otimes dx^{i_{r+s}} \in L^1_{loc}(T^{(r,s)}(\mathbb{R}^n)),$$

for any test tensor field $\theta \in C_0^\infty(T^{(r,s)}(\mathbb{R}^n))$. We denote the weak derivative of a tensor field by $S = \nabla_w K$, where “$w$” is usually omitted if it is understood from the context.

In coordinates, this is the standard notion of weak derivatives component-wise. We can also generalise this notion to Riemannian manifolds by taking $\nabla_i \theta$ to be the $i$-th component of the covariant derivative with respect to the Levi-Civita connection, and one can also define $k$-th weak derivatives in a similar manner. Finally, we define the Sobolev norm $\| \cdot \|_{W^{k,p}}$ of a tensor field, for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, by

$$\|K\|_{W^{k,p}} = \sum_{m=0}^{k} \|\nabla_m^K\|_{L^p},$$

and we say that $K$ belongs to the Sobolev space $W^{k,p}(T^{(r,s)}(\mathbb{R}^n))$ if $\nabla_1^K, \ldots, \nabla_k^K$ exist and $\|K\|_{W^{k,p}} < \infty$. We also identify $K_1, K_2$ in $W^{k,p}(T^{(r,s)}(\mathbb{R}^n))$ if $\|K_1 - K_2\|_{W^{k,p}} = 0$. If the tensor field $K_U := 1_U K$ is of class $W^{k,p}$ for every compact subset $U \subset \mathbb{R}^n$, then we say that $K$ is of class $W^{k,p}_loc(T^{(r,s)}(\mathbb{R}^n))$.

**Distribution-valued tensor fields.** Given the space $D(T^{(r,s)}(\mathbb{R}^n)) := C_0^\infty(T^{(r,s)}(\mathbb{R}^n))$ of smooth $(r,s)$-tensor fields with compact support, a distribution-valued tensor field $K$ is a continuous linear functional $K : D(T^{(r,s)}(\mathbb{R}^n)) \to \mathbb{R}$. We denote the space of all distribution-valued tensor fields by $D'(T^{(r,s)}(\mathbb{R}^n))$, which is equipped with the weak-* topology.

**Definition 2.7 (Mollification of tensor fields).** Consider a tensor field $K \in L^1_{loc}(T^{(r,s)}(\mathbb{R}^n))$ and a mollifier $\rho^\varepsilon \in C_0^\infty(\mathbb{R}^n)$. We define the mollified tensor field $K^\varepsilon$ as the smooth $(r,s)$-tensor field satisfying

$$K^\varepsilon(x)(\alpha^1(x), \ldots, \alpha^r(x), v_1(x), \ldots, v_s(x))$$

$$= \int_{\mathbb{R}^n} \rho^\varepsilon(x - y)K(y)(\tau_y^\varepsilon \alpha^1(x), \ldots, \tau_y^\varepsilon \alpha^r(x), \tau_y^\varepsilon v_1(x), \ldots, \tau_y^\varepsilon v_s(x)) \, dy,$$

for any $\alpha^i \in \Omega^1(\mathbb{R}^n)$, $i = 1, \ldots, r$, $v_j \in \mathfrak{X}(\mathbb{R}^n)$, $j = 1, \ldots, s$, where $\tau_y^\varepsilon$ denotes the parallel transport from $x$ to $y$ along the geodesic with respect to the metric. We remind that in the Euclidean case $\tau_y^\varepsilon \alpha(x) = \alpha(x)$. We employ the standard convolution notation $K^\varepsilon = \rho^\varepsilon * K$ to denote the mollification of a tensor.
Definition 2.8 (Mollification of distribution-valued tensor fields). Given a distribution-valued tensor field \( K \in \mathcal{D}'(T^{(r,s)}(\mathbb{R}^n)) \) and a mollifier \( \rho^c \in C_0^\infty(\mathbb{R}^n) \), we define the mollified tensor field \( K^c \) as a smooth \((r,s)\)-tensor field satisfying
\[
\langle \langle K^c, \theta \rangle \rangle_{L^2} = K(\rho^c * \theta),
\]
for any test tensor field \( \theta \in \mathcal{D}(T^{(r,s)}(\mathbb{R}^n)) \).

Distributional Lie derivatives. We now introduce weaker notions of Lie derivatives to be able to properly define some of the operators that will appear in this work.

Definition 2.9 (Weak Lie derivatives). For \( K \in C^1(T^{(r,s)}(\mathbb{R}^n)) \) and \( b \in W_{loc}^{1,1}(T\mathbb{R}^n) \), we say that \( \mathcal{L}_b^w K \) is the Lie derivative in the weak sense with respect to \( b \) if the following limit exists:
\[
\mathcal{L}_b^w K = \lim_{c \to 0} \mathcal{L}_{b^c} K,
\]
in \( L_{loc}^1 \), where \( \mathcal{L}_b \) is the Lie derivative with respect to the smooth vector field \( b^c = \rho^c * b \). We will omit the superscript “\( w \)” if it is clear from the context.

Example 2.10 (Lie derivative of \( k \)-forms). The explicit formula for the Lie derivative of a \( C^1 \)-smooth \( k \)-form \( K \) with respect to a smooth vector field \( b \in \mathfrak{X}(\mathbb{R}^n) \) can be computed as (see [MR13])
\[
\mathcal{L}_b K(x)(v_1, \ldots, v_k) = b^l(x) \frac{\partial K_{i_1 \ldots i_k}}{\partial x^l}(x) v_1^{i_1} \cdots v_k^{i_k} + \sum_{j=1}^k K_{i_1 \ldots i_k}(x) \frac{\partial b^j}{\partial x^l}(x) v_1^{i_1} \cdots v_j^{i_j} \cdots v_k^{i_k}, \quad \forall v_1, \ldots, v_k \in \mathfrak{X}(\mathbb{R}^n),
\]
and its corresponding \( L^2 \)-adjoint operator on a smooth \( k \)-form \( \theta \) is given by
\[
\mathcal{L}_b^T \theta(x)(v_1, \ldots, v_k) = - \frac{\partial}{\partial x^l} (b^l(x) \theta_{i_1 \ldots i_k}(x)) v_1^{i_1} \cdots v_k^{i_k} - \sum_{j=1}^k \delta^{l_j} \delta_{mp} \theta_{i_1 \ldots i_j \ldots i_k}(x) \frac{\partial b^m}{\partial x^l}(x) v_1^{i_1} \cdots v_j^{i_j} \cdots v_k^{i_k},
\]
for all \( v_1, \ldots, v_k \in \mathfrak{X}(\mathbb{R}^n) \). Taking into account Definition 2.9, we see that these formulas also hold for \( b \in W_{loc}^{1,1}(T\mathbb{R}^n) \).

Definition 2.11 (Distributional Lie derivatives). For \( K \in L_{loc}^\infty(T^{(r,s)}(\mathbb{R}^n)) \) and \( b \in W_{loc}^{1,1}(T\mathbb{R}^n) \), we say that \( \mathcal{L}_b^{dist} K \in \mathcal{D}'(T^{(r,s)}(\mathbb{R}^n)) \) is a Lie derivative with respect to \( b \) in the sense of distributions if
\[
\mathcal{L}_b^{dist} K(\theta) = \langle \langle K, (\mathcal{L}_b^w)^T \theta \rangle \rangle,
\]
for any \( \theta \in \mathcal{D}(T^{(r,s)}(\mathbb{R}^n)) \), where \( (\mathcal{L}_b^w)^T \) denotes the \( L^2 \)-adjoint of the weak Lie derivative \( \mathcal{L}_b^w \). We will usually omit the superscript “\( dist \)” if it is understood from the context.
2.3 Kunita-Itô-Wentzell formula for tensor fields

We will need a generalisation of the Itô-Wentzell formula to k-form-valued processes, which is provided in [BdLHL19].

**Theorem 2.12** (Kunita-Itô-Wentzell (KIW) formula for k-forms: Itô version). Let $K(t, x) \in L^\infty \left( [0, T]; C^2 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \right)$ be a continuous adapted semimartingale taking values in the k-forms

$$K(t, x) = K(0, x) + \int_0^t G(s, x) \, ds + \sum_{i=1}^M \int_0^t H_i(s, x) \, dW^i_s, \quad t \in [0, T],$$

(24)

where $W^1_t, \ldots, W^M_t$ are i.i.d. Brownian motions, $G \in L^1 \left( [0, T]; C^2 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \right)$, and $H_i \in L^2 \left( [0, T]; C^2 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \right), i = 1, \ldots, M$ are k-form-valued continuous adapted semimartingales. Let $\{\phi_t\}_{t \in [0, T]}$ be a continuous adapted solution of the diffusion process

$$d\phi_t(x) = b(t, \phi_t(x)) \, dt + \sum_{i=1}^N \xi_i(t, \phi_t(x)) \circ dB^i_t, \quad \phi_0(x) = x,$$

(25)

which is assumed to be a $C^1$-diffeomorphism, where $B^1_t, \ldots, B^N_t$ are i.i.d. Brownian motions, $b(t, \cdot) \in W^{1,1}_t(\mathbb{R}^n), \xi(t, \cdot) \in C^2(\mathbb{R}^n), i = 1, \ldots, N$ for all $t \in [0, T]$, and $\int_0^T |b(s, \phi_s(x)) + \frac{1}{2} \sum_i \xi_i \cdot \nabla \xi_i(s, \phi_s(x)) + \sum_j |\xi_j(s, \phi_s(x))|^2 |ds < \infty$ for all $x \in \mathbb{R}^n$. Then, the following formula holds:

$$\phi_t^* K(t, x) = K(0, x) + \int_0^t \phi_s^* G(s, x) \, ds + \sum_{i=1}^M \int_0^t \phi_s^* H_i(s, x) \, dW^i_s$$

$$+ \int_0^t \phi_s^* \mathcal{L}_b K(s, x) \, ds + \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} K(s, x) \, dB^j_s$$

$$+ \sum_{j=1}^N \int_0^t \phi_s^* H_j(s, x) \, d |W^i, B^j|_s + \sum_{j=1}^N \frac{1}{2} \int_0^t \phi_s^* \mathcal{L}_{\xi_j} \mathcal{L}_{\xi_j} K(s, x) \, ds.$$

(26)

**Theorem 2.13** (Kunita-Itô-Wentzell (KIW) formula for k-forms: Stratonovich version). Let $K(t, x) \in L^\infty \left( [0, T]; C^3 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \right)$ be a continuous adapted semimartingale taking values in the k-forms

$$K(t, x) = K(0, x) + \int_0^t G(s, x) \, ds + \sum_{i=1}^M \int_0^t H_i(s, x) \circ dW^i_s, \quad t \in [0, T],$$

(27)

where $W^1_t, \ldots, W^M_t$ are i.i.d. Brownian motions, $G \in L^1 \left( [0, T]; C^3 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \right)$, and $H_i \in L^\infty \left( [0, T]; C^3 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \right), i = 1, \ldots, M$ are k-form-valued continuous adapted semimartingales such that

$$H_i(t, x) = H_i(0, x) + \int_0^t g(s, x) \, ds + \sum_{j=1}^S \int_0^t h_{ij}(s, x) \, dN^{ij}_s, \quad t \in [0, T], \quad i = 1, \ldots, M$$

(28)
satisfies the assumptions in Theorem 2.12 and $N^ij$ are i.i.d. Brownian motions. Let $\{\phi_t\}_{t \in [0,T]}$ be a continuous adapted solution of the diffusion process

$$d\phi_t(x) = b(t, \phi_t(x)) \, dt + \sum_{i=1}^{N} \xi_i(t, \phi_t(x)) \circ dB^i_t, \quad \phi_0(x) = x, \tag{29}$$

which is assumed to be a $C^1$-diffeomorphism, where $B^1_t, \ldots, B^N_t$ are i.i.d. Brownian motions, $b(t, \cdot) \in W^{1,1}_c(\mathbb{R}^n, \mathbb{R}^n)$ for all $t \in [0,T]$, $\xi_i \in L^\infty([0,T]; C^3(T\mathbb{R}^n))$, and $\int_0^T |b(s, \phi_s(x)) + \frac{1}{2} \sum_i \xi_i \cdot \nabla \xi_i(s, \phi_s(x))| + \sum_i |\xi_i(s, \phi_s(x))|^2 \, ds < \infty$ for all $x \in \mathbb{R}^n$. Then, the following holds:

$$\phi^*_t K(t, x) = K(0, x) + \int_0^t \phi^*_s G(s, x) \, ds + \sum_{i=1}^{M} \int_0^t \phi^*_s H^i(s, x) \circ dW^i_s$$

$$+ \int_0^t \phi^*_s \mathcal{L}_u K(s, x) \, ds + \sum_{j=1}^{N} \int_0^t \phi^*_s \mathcal{L}_{\xi_j} K(s, x) \circ dB^j_s. \tag{30}$$

2.4 Notions of solutions

Definition 2.14 (Classical solution). A classical solution $K \in C \left([0,T]; C^3 \left(\wedge^k (T^*\mathbb{R}^n)\right)\right)$ to equation (1) is a $k$-form-valued stochastic process adapted to $(\mathcal{F}_t)_{t \geq 0}$, such that for any $k$ vector fields $u = (u_1, \ldots, u_k) \in \mathfrak{X}(\mathbb{R}^n)^k$, it holds almost surely

$$K_i(x)(u) - K_0(x)(u) + \int_0^t \mathcal{L}_u K(s, x)(u) \, ds + \sum_{k=1}^{N} \int_0^t \mathcal{L}_{\xi_k} K(s, x)(u) \circ dW^k_s = 0,$$

In coordinates, choosing $u_i = \partial/\partial x_i$, for $i = 1, \ldots, k$, this is equivalent to saying that the SPDE (1) is satisfied component-wise.

Definition 2.15 (Weak $L^p$-solution). We say that a $k$-form-valued process $K$ satisfies equation (1) in the $L^p$ sense if

- $K \in L^\infty \left([0,T]; L^p \left(\Omega \times \wedge^k (T^*\mathbb{R}^n)\right)\right)$,
- $K(t, \cdot) \in L^\infty_{\text{loc}} \left(\wedge^k (T^*\mathbb{R}^n)\right)$ for all $t \in [0,T]$, $\mathbb{P}$-almost surely,
- For any test $k$-form $\theta \in \mathcal{D} \left(\wedge^k (T^*\mathbb{R}^n)\right)$, the process $\langle \langle K_t, \theta \rangle \rangle$ has a continuous, $\mathcal{F}_t$-adapted modification and for any $t \in [0,T]$, we have almost surely

$$\langle \langle K_t, \theta \rangle \rangle - \langle \langle K_0, \theta \rangle \rangle + \int_0^t \langle \langle K_s, \mathcal{L}^T_{\xi_k} \theta \rangle \rangle \, ds + \sum_{k=1}^{N} \int_0^t \langle \langle K_s, \mathcal{L}_{\xi_k}^T \theta \rangle \rangle \circ dW^k_s = 0. \tag{31}$$
In Itô form, this reads
\[
\langle\langle K_t, \theta \rangle\rangle - \langle\langle K_0, \theta \rangle\rangle + \int_0^t \langle\langle K_s, \mathcal{L}^T_{\xi_k} \theta \rangle\rangle \, ds + \sum_{k=1}^N \int_0^t \langle\langle K_s, \mathcal{L}^T_{\xi_k} \theta \rangle\rangle \, dW_s^k \\
- \sum_{k=1}^N \frac{1}{2} \int_0^t \langle\langle K_s, \mathcal{L}^T_{\xi_k} \mathcal{L}^T_{\xi_k} \theta \rangle\rangle \, dt = 0, \quad t \in [0, T], \quad a.s.
\]

We remind that \(\langle\langle \cdot, \cdot \rangle\rangle\) stands for the \(L^2\)-inner product of tensor fields defined in Subsection 2.2.

Remark 2.16 (Remarks on notation). We mention some aspects regarding the notation we employ along the article. \(a \lesssim b\) means there exists \(C > 0\) such that \(a \leq Cb\), where \(C\) is a universal constant that may depend on fixed parameters, constant quantities, and the domain itself. Note also that this constant might differ from line to line. It is also important to remind that the condition “almost surely” is not always indicated, since in some cases it is clear from the context. For a given function \(f\) depending on \(t \in [0, T]\) and \(x \in \mathbb{R}^n\), we sometimes adopt the notation \(f_t(x) = f(t, x)\). For \(r > 0\) and \(x \in \mathbb{R}^n\), we define \(B(x, r)\) as the open ball centred at \(x\) with radius \(r\). If \(A\) is a matrix, we understand by \(|A|\) its Frobenius norm.

3 Regularity of the flow of the characteristic equation

We state our main theorem establishing the regularity properties of the flow of equation (3), extending the result presented in [FGP10], which covers the special case \(N = n, \xi_k = e_k, k = 1, \ldots, n\). We sketch the main ingredients that need to be added to the proof of the original result in [FGP10] for its extension to the more general type of noise that we consider.

Theorem 3.1 (Flow regularity). Consider equation (3) with \(b \in L^\infty([0, T]; C^1_b(\mathbb{R}^n, \mathbb{R}^n))\). We assume the noise coefficients \(\xi_k \in L^\infty([0, T]; C^{1+\alpha}(\mathbb{R}^n, \mathbb{R}^n))\) to be measurable, and that there exist \(0 < \delta < K < \infty\) such that the uniform ellipticity condition \(\delta |v|^2 \leq \sum_{i,j} \xi_k^i(x)\xi_k^j(x)v_i v_j \leq K |v|^2\) is satisfied for all \(v \in \mathbb{R}^n \setminus \{0\}, x \in \mathbb{R}^n, k = 1, \ldots, N\). Then we have the following:

1. For every \(s \in [0, T], x \in \mathbb{R}^n\), the stochastic equation (3) has a unique continuous adapted solution \(\{X_t^{n,x}(\omega)\}_{t \in [s,T]}\), where \(X_s^{n,x}(\omega) = x\) for all \(\omega \in \Omega\).

2. There exists a stochastic flow of diffeomorphisms \(\phi_{s,t}\) (see Definition 2.4) for equation (3). This flow is of class \(C^{1+\alpha'}\), for any \(\alpha' < \alpha\).

3. Let \(\{b_n\}_{n \in \mathbb{N}}\) be a sequence of vector fields such that \(b_n \to b\) in \(L^\infty([0, T]; C^0_b(T\mathbb{R}^n))\), and \(\phi^n\) be the corresponding stochastic flow. Then for any \(p \geq 1\), we have

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[ \sup_{s \leq t \leq T} |\phi_{s,t}^n(x) - \phi_{s,t}(x)|^p \right] = 0,
\]

\[
\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[ \sup_{s \leq t \leq T} |D\phi_{s,t}^n(x)|^p \right] < \infty,
\]

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[ \sup_{s \leq t \leq T} |D\phi_{s,t}^n(x) - D\phi_{s,t}(x)|^p \right] = 0.
\]
We proceed to sketch the proof of Theorem 3.1, indicating which auxiliary tools are required to extend the results in [FGP10] to our more general case, and at which parts of the proof they are needed in order to achieve this extension. Note that since this proof is based on the results in [FGP10], this reference can be checked for completeness and details.

**Sketch proof of Theorem 3.1.** The essence of the argument is the following: we show that the flow $X_t$ of equation (3) can be expressed as a composition of functions which are regular enough. More concretely, one can prove that the process $Y_t = \Psi_\lambda(t, X_t) = X_t + \psi_\lambda(t, X_t)$, where $\psi_\lambda$ is a solution of the parabolic equation (36), solves an equation of the type (3), where the new coefficients $\tilde{b}, \tilde{\xi}_k$ enjoy more regularity than the original ones (this is called the Itô-Tanaka trick). By applying results treating the regularity of solutions of parabolic equations and estimation lemmas for $\psi_\lambda$, which can be found in Subsection 2.1, one can show that $\Psi_\lambda$ enjoys enough regularity and is invertible. Finally, it can be proven that the regularity of $\psi_\lambda, \Psi_\lambda$ is inherited by the flow $X_t$, since $\phi_{s,t} = \Psi_\lambda^{-1} \circ \psi_{s,t}^\lambda \circ \Psi_\lambda$, and $X_{t}^x = \phi_{s,t}(x)$. This yields points 1 and 2. For point 3, we take into account the previous steps and closely follow the techniques in [FGP10].

In order to facilitate the understanding of the main ideas, we break up this sketch into three steps. In Step 1 and Step 2, we treat the regularity of the functions $\psi_\lambda$ and $\Psi_\lambda$, whereas Step 3 is devoted to the Itô-Tanaka trick and the regularity of the flow, which gives points 1 and 2. Step 4 treats the third and last point.

Note that, for simplicity and without loss of generality, we only discuss the case $N = 1$.

**Step 1.** For $\lambda > 0$, consider the backward parabolic equation

$$\partial_t \psi_\lambda + L\psi_\lambda - \lambda \psi_\lambda = -b, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

with (assume Einstein’s convention of summing over repeated indices)

$$L\psi_\lambda = \frac{1}{2} \xi^i(t, x) \xi^j(t, x) \partial_i \partial_j \psi_\lambda + \left( b^i(t, x) + \xi^i(t, x) \partial_i \xi^j(t, x) \right) \partial_j \psi_\lambda.$$

Then, due to Theorem 2.1 (making the choice $f = -b$), there exists a unique solution $\psi_\lambda$ of (36) in the space $L^\infty([0, \infty); C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$. Moreover, there exists $C > 0$ such that

$$\sup_{t \geq 0} \| \psi_\lambda(t, \cdot) \|_{C_b^{2+\alpha}} \lesssim \sup_{t \geq 0} \| b(t, \cdot) \|_{C_b^\alpha}.$$

**Step 2.** Construct the function

$$\Psi_\lambda(t, x) = x + \psi_\lambda(t, x).$$

By means of the following lemma, to be found in [FGP10], we can establish that the function $\Psi_\lambda$ gains extra regularity with respect to $\psi_\lambda$.

**Lemma 3.2.** For $\lambda$ large enough such that $\sup_{t \geq 0} \| D\psi_\lambda(t, \cdot) \|_0 < 1$, we have the following properties:

1. Uniformly in $t \in [0, \infty)$, $\Psi_\lambda$ has bounded first and second spatial derivatives, and its derivative $D^2_x \Psi_\lambda$ is globally $\alpha$-Hölder continuous.
2. For any \( t \geq 0 \), \( \Psi_\lambda(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is a non-singular diffeomorphism of class \( C^3 \).

3. Uniformly in \( t \in [0, \infty) \), \( \Psi_\lambda^{-1}(\cdot, x) \) has bounded first and second derivatives for every \( x \in \mathbb{R}^n \).

In order to be able to apply Lemma 3.2 to our function \( \psi_\lambda \), we need to show that for \( \lambda \) large enough, \( \psi_\lambda \) satisfies the assumption. In the original argument in [FGP10], the fact that \( \sup_{t \geq 0} \| D \Psi_\lambda(t, \cdot) \|_0 < 1 \) for \( \lambda \) large enough is established by means of an auxiliary result, which states that the solution \( \psi_\lambda \) to equation (36) satisfies \( \sup_{t \geq 0} \| D \psi_\lambda(t, \cdot) \|_0 \to 0 \), as \( \lambda \to +\infty \). Moreover, in the original argument, the key to proving this auxiliary result lies in some well-known estimates for the heat semigroup, which read

\[
\| Du_\lambda \|_0 \lesssim t^{-1/2} \sup_{x \in \mathbb{R}^n} |g(x)|, \quad g \in C_b(\mathbb{R}^n), \quad t > 0.
\]  

(40)

We argue that estimates of the type (40) are also available in the general \( \xi \) case (see Lemma 2.2).

**Step 3.** The crucial point for deriving Theorem 3.1 by making use of Step 1 and Step 2 consists in carrying out the so-called Itô-Tanaka trick. This is the most fundamental idea of this proof. Define

\[
\tilde{b}(t, y) = -\lambda \psi_\lambda(t, \Psi_\lambda^{-1}(t, y)), \quad \tilde{\xi}(t, y) = D\Psi_\lambda(t, \Psi_\lambda^{-1}(t, y)),
\]  

\[
(41)
\]

and consider, for \( s \in [0, T] \), \( y \in \mathbb{R}^n \), the following SDE in Itô form:

\[
Y_t = y + \int_s^t \tilde{\xi}(u, Y_u) dB_u + \int_s^t \tilde{b}(u, Y_u) du, \quad t \in [s, T].
\]  

(42)

We note that equation (42) is equivalent to the flow equation (3) in the following sense: if \( X_t \) solves (3), then \( Y_t := \Psi_\lambda(t, X_t) \) solves (42) with \( y = \Psi_\lambda(s, x) \) (for this, apply Itô’s formula to the function \( \psi_\lambda(t, X_t) \)). Letting \( \varphi_{s,t} \) be the corresponding \( C^{1+\alpha'} \) flow of diffeomorphism (for some \( \alpha' < \alpha \)) owing to the regularity shown for \( \psi_\lambda \) in Step 1 and for \( \Psi_\lambda \) in Step 2, and using the fact that \( \phi_{s,t} = \Psi_\lambda^{-1} \circ \varphi_{s,t} \circ \Psi_\lambda \), \( X^\alpha = \phi_{s,t}(x) \), we conclude that \( X_t \) enjoys the extra regularity specified in Theorem 3.1, which proves points 1 and 2.

**Step 4.** To prove point 3 in this lemma, let \( \psi_{\lambda, n} \) be the solution of (36) with \( b \) replaced by \( b^n \), \( \Psi_{\lambda, n} \) the function (39) constructed from \( \psi_{\lambda, n} \), and \( \tilde{b}^n, \tilde{\xi}^n \) the vector fields (41) derived from \( \psi_{\lambda, n} \) and \( \Psi_{\lambda, n} \). By arguing as in [FGP10], one can show that \( \psi_n \to \psi \) in \( L^\infty([0, T]; C^{1+\alpha}_b(\mathbb{R}^n)) \), and therefore \( \tilde{b}^n \to \tilde{b}, \tilde{\xi}^n \to \tilde{\xi} \) in \( L^\infty([0, T]; C^{1+\alpha}_b(\mathbb{R}^n)) \), which allows us to deduce (33). Letting \( \varphi_{s,t}^n \) be the flow of (42) with vector fields \( \tilde{b}^n \) and \( \tilde{\xi}^n \), we observe that \( D\varphi_{s,t}^n \) satisfies

\[
D\varphi_{s,t}^n = I + \int_s^t D\varphi_{s,t}^n (u, \varphi_{s,u}^n) D\varphi_{s,u}^n du + \int_s^t D\tilde{b}_{s,t}^n (u, \varphi_{s,u}^n) D\varphi_{s,u}^n du.
\]

By taking into account Grönwall’s inequality, uniform boundedness of \( \{D\tilde{b}^n\}_{n \in \mathbb{N}} \), \( \{D\tilde{\xi}^n\}_{n \in \mathbb{N}} \), and Burkholder-Davis-Gundy inequality, we obtain the estimate

\[
\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[ \sup_{s \leq u \leq T} |D\varphi_{s,t}^n(x)|^p \right] < \infty,
\]

for any \( p \geq 1 \), and using the uniform boundedness of \( \psi_{\lambda, n} \), \( \Psi_{\lambda, n} \), and \( (\Psi_{\lambda, n})^{-1} \), we deduce (34). Finally, combining (34) and (35), we can obtain (35) by following the same procedure as in [FGP10].

**Remark 3.3.** We note that Theorem 3.1 also holds for the backward equation, by simply replacing \( b \) and \( \xi \) with \( -b \) and \( -\xi \) respectively.
4 Well-posedness of the stochastic equation

In this section, we prove the existence and uniqueness of solutions to the SPDE (1), which constitutes our main result. In Subsection 4.1, we establish the existence and uniqueness of classical and weak solutions in the case when the drift vector field $b$ is smooth. In Subsection 4.2, we show existence in the case when $b$ is only $\alpha$-Hölder continuous and $W^{1,1}_{loc}$, by taking the limit of a regularised version of the equation considered in Subsection 4.1. Finally, in Subsection 4.3, we prove uniqueness in the non-smooth $b$ case by means of an extension of Di-Perna-Lions type commutator estimates [DL89, Lio96] to $k$-forms.

4.1 The smooth $b$ case: Existence and Uniqueness

**Proposition 4.1.** Let $b$, $\xi_k \in C_b^{4+\beta}(T\mathbb{R}^n)$, $k = 1, \ldots, N$ for some $0 < \beta < 1$ be continuous semimartingales, and let $K_0 \in C^3(\wedge^k(T^*\mathbb{R}^n))$ be measurable. Then there exists a unique classical solution to equation (1) of the form $\bar{K}(t, x) = (\phi_t)_*K_0(x)$, where $\phi_t$ is the flow of (3).

**Proof.** For the sake of simplicity, we assume without loss of generality that $N = 1$. Let $\phi_{s,t}$ be the flow map from time $s$ to time $t$, for $s \leq t$, which is a $C^4$ stochastic flow of diffeomorphisms due to classical results [Kun97]. Then, the corresponding backward equation (see pages 113 and 117 in [Kun97])

$$\phi_{t,s}(x) = x - \int_s^t b(r, \phi_{t,r}(x))dr - \int_s^t \xi(r, \phi_{t,r}(x)) \circ d\hat{W}_r,$$

where $d\hat{W}_r$ denotes backward integration against $W_r$. By Itô’s first formula on tensor fields (17), we obtain (taking $s = 0$)

$$\phi_{t,0}^*K_0(x) = K_0(x) - \int_0^t L_b(\phi_{r,0}^*K_0(x)) dr - \int_0^t L_\xi(\phi_{r,0}^*K_0(x)) \circ dW_r, \quad (44)$$

Setting $K(t, x) = \phi_{t,0}^*K_0(x) = (\phi_t)_*K_0(x)$, which is $C^3(\wedge^k(T^*\mathbb{R}^n))$, we see that (44) is identical to (1), and therefore $K(t, x)$ solves (1) in the strong sense.

Due to the linearity of equation (1), to prove uniqueness one only has to show that $K(t, x) \equiv 0$, for all $t > 0$, when $K_0(x) \equiv 0$. Applying the Itô-Wentzell-Kunita formula for $k$-forms (see Theorem 2.13), we have that for any $x \in \mathbb{R}^n$ and $t > 0$,

$$\phi_t^*K(t, x) = -\int_0^t \phi_s^*L_b K(s, x) \circ dW_s - \int_0^t \phi_s^*L_\xi K(s, x) \circ dW_s + \int_0^t \phi_s^*L_b K(s, x) ds + \int_0^t \phi_s^*L_\xi K(s, x) ds \circ dW_s = 0.$$

Next, we show the existence of weak solutions of the linear transport of $k$-forms when the driving vector fields $b$ and $\xi$ are smooth. First, we state the following useful lemma.
Lemma 4.2. Given a $k$-form $K \in L^\infty _{\text{loc}} \left( \bigwedge ^k (T^* \mathbb{R}^n) \right)$, a test $k$-form $\theta \in \mathcal{D} \left( \bigwedge ^k (T^* \mathbb{R}^n) \right)$, and a $C^1$-diffeomorphism $\phi$, we have the following identity

$$
\int _{\mathbb{R}^n} \langle \phi _* K(x), \theta (x) \rangle \, d^n x = \int _{\mathbb{R}^n} \left\langle K(y), J\phi (y) \left( \phi ^* \theta ^*(y) \right) \right\rangle \, d^n y. \tag{45}
$$

Here, for a $k$-form $\theta$, the corresponding $k$-vector field $\theta ^z$ is defined as

$$
\theta ^z = \delta ^{i_1 j_1} \cdots \delta ^{i_k j_k} \theta _{j_1 \cdots j_k} (x) \frac{\partial }{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial }{\partial x^{i_k}},
$$

and similarly for a $k$-vector field $v$, the $k$-form $v^\flat$ is defined as

$$
v^\flat = \delta _{i_1 j_1} \cdots \delta _{i_k j_k} v^{j_1 \cdots j_k} (x) d x^{i_1} \wedge \cdots \wedge d x^{i_k}.
$$

Proof. By standard tensor identities, we have

$$
\langle \phi _* K(x), \theta (x) \rangle = \phi _* K(x) (\theta ^z (x)) = K(\phi ^{-1} (x)) \left( \phi ^* \theta ^z (\phi ^{-1} (x)) \right).
$$

Hence, by the change of coordinates $x = \phi (y)$, we obtain

$$
\int _{\mathbb{R}^n} \langle \phi _* K(x), \theta (x) \rangle \, d^n x = \int _{\mathbb{R}^n} \left( K(\phi ^{-1} (x)) \left( \phi ^* \theta ^z (\phi ^{-1} (x)) \right) \right) \, d^n x
$$

$$
= \int _{\mathbb{R}^n} K(y) \left( \phi ^* \theta ^z (y) \right) J\phi (y) \, d^n y
$$

$$
= \int _{\mathbb{R}^n} \left\langle K(y), J\phi (y) \left( \phi ^* \theta ^z (y) \right) \right\rangle \, d^n y. 
$$

Remark 4.3. When changing coordinates, we observe that the term $J\phi (y) \left( \phi ^* \theta ^z (y) \right)$ appears. For simplicity, we will denote it by $\tilde{\theta}$.

Proposition 4.4. Let $b$, $\xi _k \in L^\infty \left( [0, T] ; C^4 _b (T \mathbb{R}^n) \right)$, $k = 1, \ldots, N$ for some $0 < \beta < 1$ be continuous semimartingales, and $K_0 \in L^p \cap L^\infty _{\text{loc}} \left( \bigwedge ^k (T^* \mathbb{R}^n) \right)$ for $p \geq 1$ be measurable. Then there exists a weak solution to equation (1) of the form $K(t, x) = (\phi _t)_* K_0 (x)$.

Proof. Again, for the sake of simplicity, we assume without loss of generality that $N = 1$. Consider the mollified initial data $K_\epsilon ^\ast := \rho ^\ast K_0$. Then by Proposition 4.1, for almost all $\omega \in \Omega$, $K_\epsilon ^\ast (t, x, \omega) := (\phi _t(\omega))_* K_0 (x)$ is a unique strong solution to (1) and therefore also a weak solution. Hence, for any test $k$-form $\theta \in \mathcal{D} \left( \bigwedge ^k (T^* \mathbb{R}^n) \right)$, we have

$$
\langle (\phi _t)_* K_\epsilon ^\ast (\theta) \rangle - \langle K_0 ^\ast (\theta) \rangle = - \int _0 ^t \langle (\phi _s)_* K_0 ^\ast (\mathcal{L}_\xi ^T \theta) \rangle \, d W_s - \int _0 ^t \langle (\phi _s)_* K_0 ^\ast (\mathcal{L}_\xi ^T \theta) \rangle \, d W_s
$$

$$
+ \frac{1}{2} \int _0 ^t \langle (\phi _s)_* K_0 ^\ast (\mathcal{L}_\xi ^T \mathcal{L}_\xi ^T \theta) \rangle \, d s.
$$

We analyse term-by-term convergence in the last equality:
By the weak convergence $K^n_t \to K_0$, the LHS converges as
\[
\langle \langle K^n_t, \theta \rangle \rangle \to \langle \langle K_0, \theta \rangle \rangle, \quad \langle \langle \phi_t, K^n_t, \theta \rangle \rangle = \langle \langle \phi_t, K_0, \theta \rangle \rangle \to \langle \langle \phi_t, K_0, \theta \rangle \rangle = \langle \langle \phi_t K_0, \theta \rangle \rangle,
\]
where $\tilde{\theta}(x) := J\phi_t(x) ((\phi_t)^* \theta^b)(x)$ as in Lemma 4.2.

By the dominated convergence theorem, the first term in the RHS converges as
\[
\int_0^t \langle \langle \phi_s, K^n_t, L^T_\theta \rangle \rangle ds = \int_0^t \langle \langle K_0, L^T_\theta \rangle \rangle ds
\]
\[
\to \int_0^t \langle \langle K_0, L^T_\theta \rangle \rangle ds = \int_0^t \langle \langle \phi_s, K_0, L^T_\theta \rangle \rangle ds.
\]

Similarly, the Itô-correction term and the martingale term converge as
\[
\frac{1}{2} \int_0^t \langle \langle \phi_s, K^n_t, L^T_\xi \rangle \rangle ds \to \frac{1}{2} \int_0^t \langle \langle \phi_s, K_0, L^T_\xi \rangle \rangle ds,
\]
\[
\int_0^t \langle \langle \phi_s, K^n_t, L^T_\xi \rangle \rangle dW_s \to \int_0^t \langle \langle \phi_s, K_0, L^T_\xi \rangle \rangle dW_s.
\]

Now, noting that
\[
\langle \langle \phi_t, K_0 \rangle \rangle_{j_1, \ldots, j_k}(x) = (K_0)_{i_1, \ldots, i_k}(\psi_t(x)) \frac{\partial \psi_t}{\partial x^{j_1}}(x) \cdots \frac{\partial \psi_t}{\partial x^{j_k}}(x),
\]
where we have employed the notation $\psi := \phi^{-1}$ (which is a $C^4$-diffeomorphism), we obtain
\[
\sup_{0 \leq t \leq T} \|K\|_{L^n} < \infty \text{ for any } R > 0.
\]
Furthermore, by (34), and changing coordinates $x = \phi_t(y)$, we conclude
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^n} |K(t, x)|^p d^n x dt \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^n} |(\phi_t)_* K_0(x)|^p d^n x dt \right]
\]
\[
\leq \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^n} |K_0(y)|^p |D\psi_t(x)|^p k J\phi_t(y) d^n y dt \right]
\]
\[
\leq \left( \sup_{x \in \mathbb{R}^n} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |D\psi_t(x)|^{2pk} \right] \right)^{\frac{1}{2}} \left( \sup_{y \in \mathbb{R}^n} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |J\phi_t(y)|^2 \right] \right)^{\frac{1}{2}} \int_{\mathbb{R}^n} |K_0(y)|^p d^n y < \infty,
\]
so $K \in L^p(\Omega \times [0, T] \times \bigwedge^k (T^* \mathbb{R}^n))$. Hence, $K(t, x) = (\phi_t)_* K_0(x)$ is a weak solution to (1).

4.2 The non-smooth $b$ case: Existence of weak solutions

Proposition 4.5. Let $b \in L^\infty([0, T]; W^{1,1}_{\text{loc}} \cap C^\alpha_0 (T \mathbb{R}^n))$, $\xi_k \in L^\infty([0, T]; C^{4+\beta}_b (T \mathbb{R}^n))$, $k = 1, \ldots, N$, for some $0 < \alpha, \beta < 1$, and $K_0 \in L^p \cap L^\infty_{\text{loc}} \left( \bigwedge^k (T^* \mathbb{R}^n) \right)$ for $p \geq 1$ be measurable. Then there exists a weak solution to (1) of the form $K(t, x) = (\phi_t)_* K_0(x)$.
Proof. As usual, for simplicity, we treat the case \( N = 1 \). First, we take the mollified vector field \( b^\epsilon = \rho^\epsilon * b \), and consider equation (1) with drift vector field \( b^\epsilon \) instead of \( b \). Let \( \phi^\epsilon_t \) be the \( C^1 \)-diffeomorphic flow of the characteristic equation (3) with drift vector field \( b^\epsilon \) and define \( K^\epsilon(t, x) = (\phi^\epsilon_t)_*K_0(x) \), which is a weak solution of (1) by Proposition 4.4. By (34) applied to the inverse flow, we have

\[
\sup_{0 < \epsilon < 1} \sup_{x \in \mathbb{R}^n} \mathbb{E} \left[ \sup_{s \leq t \leq T} |D\psi^\epsilon_t(x)|^p \right] < \infty, \tag{46}
\]

for any \( p \geq 1 \). As before, by changing coordinates \( x = \phi^\epsilon_t(y) \), we obtain

\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^n} |K^\epsilon(t, x)|^p d^n x \, dt \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^n} |(\phi^\epsilon_t)_*K_0(x)|^p d^n x \, dt \right] \\
\leq \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^n} |K_0(y)|^p |D\psi^\epsilon_t(x)|^p k J\phi^\epsilon_t(y) d^n y \, dt \right] \\
\leq \left( \sup_{0 < \epsilon < 1} \sup_{x \in \mathbb{R}^n} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |D\psi^\epsilon_t(x)|^{2pk} \right] \right)^{\frac{1}{2}} \left( \sup_{0 < \epsilon < 1} \sup_{y \in \mathbb{R}^n} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |J\phi^\epsilon_t(y)|^2 \right] \right)^{\frac{1}{2}} \int_{\mathbb{R}^n} |K_0(y)|^p d^n y < \infty.
\]

Therefore, there exists a subsequence \( \epsilon_n \to 0 \) and some \( K \in L^p \left( \Omega \times [0, T] \times \wedge^k(T^*\mathbb{R}^n) \right) \) such that \( K^{\epsilon_n}(t, x) \) converges weakly to \( K(t, x) \). We shall employ the notation \( \epsilon \) instead of \( \epsilon_n \) for the sake of simplicity. Now, we examine the term-by-term convergence in the weak formulation of the Lie transport equation (32):

- **The LHS converges to**

\[
\int_{\mathbb{R}^n} \langle K^\epsilon(t, x), \theta(x) \rangle \, dx \to \int_{\mathbb{R}^n} \langle K(t, x), \theta(x) \rangle \, dx,
\]

due to weak convergence of \( K^\epsilon \).

- **The Itô diffusion term on the RHS.** Similarly to the previous case, we have

\[
\left| \int_{\mathbb{R}^n} \langle K^\epsilon(t, x) - K(t, x), \mathcal{L}_x^T \theta \rangle \, d^n x \right| \to 0,
\]

due to weak convergence of \( K^\epsilon \). Note that if \( K^\epsilon \) is adapted, then also \( K \) is adapted (since the set of adapted processes is closed), so the Itô integral is well-defined.

- **The Stratonovich-Itô correction term on the RHS.** Again,

\[
\left| \int_{\mathbb{R}^n} \langle K^\epsilon(t, x) - K(t, x), \mathcal{L}_x^T \mathcal{L}_x^T \theta \rangle \, d^n x \right| \to 0,
\]

due to weak convergence of \( K^\epsilon \).

- **The drift term on the RHS.** We can check that

\[
\left| \int_{\mathbb{R}^n} \langle K^\epsilon(t, x), \mathcal{L}_x^T \theta \rangle \, d^n x - \int_{\mathbb{R}^n} \langle K(t, x), \mathcal{L}_x^T \theta \rangle \, d^n x \right| \leq \left| \int_{\mathbb{R}^n} \langle K^\epsilon(t, x) - K(t, x), \mathcal{L}_x^T \theta \rangle \, d^n x \right| + \left| \int_{\mathbb{R}^n} \langle K(t, x), \mathcal{L}_x^T \theta - \mathcal{L}_x^T \theta \rangle \, d^n x \right| \to 0,
\]

\]
where the first term converges due to weak convergence of $K^\epsilon$ and the uniform boundedness of $\mathcal{L}_b^\epsilon \theta$, and for the second term we have used the fact that $b \in W^{1,1}_{\text{loc}}$ and Definition 2.9 for weak Lie derivatives.

- Finally, by the dominated convergence theorem for Itô integrals, check that the time integral of the above terms vanishes in the limit.

Now, we claim that $K(t, x) = (\phi_t)_* K_0(x)$ and so $K \in L_{\text{loc}}^\infty([0, T] \times \bigwedge^k (T^* \mathbb{R}^n))$. We have

$$\langle \langle K^\epsilon(t, x), \theta \rangle \rangle = \langle \langle (\phi_t^\epsilon)_* K_0, \theta \rangle \rangle = \int_{\mathbb{R}^n} \left\langle K_0(y), J\phi_t^\epsilon(y) \left( (\phi_t^\epsilon)^\ast \theta^\sharp(y) \right) \right\rangle \, dy. \quad (47)$$

One can check that $J\phi_t^\epsilon(y) \left( (\phi_t^\epsilon)^\ast \theta^\sharp(y) \right) \to J\phi_t(y) \left( \phi_t^\ast \theta^\sharp(y) \right)$ in $L^p$ by using (33) and (35), so $\langle \langle (\phi_t^\epsilon)_* K_0, \theta \rangle \rangle \to \langle \langle (\phi_t)_* K_0, \theta \rangle \rangle$, and therefore $K(t, x) = (\phi_t)_* K_0(x)$ as expected. \qed

### 4.3 The non-smooth $b$ case: Uniqueness of weak solution

We now wish to show uniqueness of weak solutions of equation (1). For simplicity, we treat the case $N = 1$. From the linearity of the equation, it is sufficient to show that if $K(0, x) \equiv 0$, then $K(t, x) \equiv 0$, a.s., for all times. First, consider the mollified $k$-form $K^\epsilon(t, x) = \rho^\ast K(t, x)$ with initial condition $K(0, x) = 0$, which (by (1)), satisfies

$$K^\epsilon(t, x) = - \int_0^t (\rho^\ast \mathcal{L}_b K)(s, x) \, ds - \int_0^t (\rho^\ast \mathcal{L}_\xi K)(s, x) \circ dW_s, \quad (48)$$

where $\mathcal{L}_b K$, $\mathcal{L}_\xi K$ are understood as Lie derivatives in the sense of distributions (see Definition 2.11). Since all the terms in (48) are now smooth, one can apply the Itô-Wentzell formula for $k$-forms (see Theorem 2.13) to obtain

$$\phi_t^\ast K^\epsilon(t, x) = - \int_0^t \phi_s^\ast (\mathcal{L}_b K)^\epsilon(s, x) \, ds - \int_0^t \phi_s^\ast (\mathcal{L}_\xi K)^\epsilon(s, x) \circ dW_s$$

$$+ \int_0^t \phi_s^\ast \mathcal{L}_b K^\epsilon(s, x) \, ds + \int_0^t \phi_s^\ast \mathcal{L}_\xi K^\epsilon(s, x) \circ dW_s$$

$$= \int_0^t \phi_s^\ast [\mathcal{L}_b, \rho^\ast] K(s, x) \, ds + \int_0^t \phi_s^\ast [\mathcal{L}_\xi, \rho^\ast] K(s, x) \circ dW_s, \quad (49)$$

where

$$[\mathcal{L}_b, \rho^\ast] K := \mathcal{L}_b K^\epsilon - (\mathcal{L}_b K)^\epsilon \quad (50)$$

denotes the commutator between the linear operators $\mathcal{L}_b$ and $\rho^\ast (\cdot)$. We will show that the right-hand side of (49) vanishes weakly as $\epsilon \to 0$, thus proving uniqueness of weak solutions.

**Theorem 4.6.** Assuming that the vector fields $b$ and $\xi$ satisfy the assumption $A$, there exists a unique weak solution of (1) for any initial condition $K_0 \in L^p \cap L_{\text{loc}}^\infty \left( \bigwedge^k (T^* \mathbb{R}^n) \right)$.

The proof of this theorem relies on a commutator estimate for (50), which we will detail below.
Lemma 4.7 (Stratonovich-to-Itô correction). Given a stochastic $k$-form-valued process of the form

$$\alpha(t, x) = \alpha(s, x) + \int_s^t G(r, x)dr + \sum_{i=1}^M \int_s^t H_i(r, x)dW^i_r, \quad (51)$$

satisfying the conditions in Theorem 2.12, and a stochastic flow

$$\phi_t(x) = x + \int_0^t b(s, \phi_s(x))ds + \sum_{k=1}^N \int_0^t \xi_k(s, \phi_s(x)) \circ dB^k_s,$$

which also satisfies the conditions in Theorem 2.12, then we have for $i = 1, \ldots, M$,

$$\int_0^t \phi^*_s \alpha(s, x) \circ dW^i_s = \frac{1}{2} \int_0^t \phi^*_s H_i(s, x) ds + \frac{1}{2} \sum_{k=1}^N \int_0^t \phi^*_s \mathcal{L}_\xi \alpha(s, x) d[W^i, B^k]_s + \int_0^t \phi^*_s \alpha(s, x) dW^i_s.$$

Proof. We consider the case $M = N = 1$. Applying the Kunita-Itô-Wentzell formula (Theorem 2.12) for $k$–forms, we obtain

$$\phi^*_s \alpha(t, x) = \alpha(0, x) + \int_0^t \phi^*_s G(s, x) ds + \int_0^t \phi^*_s H(s, x) dW_s + \int_0^t \phi^*_s \mathcal{L}_\xi \alpha(s, x) ds + \int_0^t \phi^*_s \mathcal{L}_\xi \mathcal{L}_\xi \alpha(s, x) ds,$$

so we have

$$[\phi^*_s \alpha(\cdot, x), W]_t = \int_0^t \phi^*_s H(s, x) ds + \int_0^t \phi^*_s \mathcal{L}_\xi \alpha(s, x) d[W, B]_s.$$

The final result follows easily by noting that

$$\int_0^t \phi^*_s \alpha(s, x) \circ dW_s = \frac{1}{2} [\phi^*_s \alpha(\cdot, x), W]_t + \int_0^t \phi^*_s \alpha(s, x) dW_s.$$

The following is a straightforward corollary.

Corollary 4.8. Given a $k$-form valued process $K$ satisfying (1), a smooth vector field $\xi$ and a stochastic flow $\phi_t$ that solves (3), we have the following identity:

$$\int_0^t \phi^*_s [\mathcal{L}_\xi, \rho^*] K(s, x) \circ dW_s = \frac{1}{2} \int_0^t \phi^*_s [\mathcal{L}_\xi [\mathcal{L}_\xi, \rho^*]] K(s, x) ds + \int_0^t \phi^*_s [\mathcal{L}_\xi, \rho^*] K(s, x) dW_s.$$
4.3.1 Commutator estimates

Consider the integral
\[
\int_{\mathbb{R}^n} \langle [L_b, \rho^\epsilon]K(x), \theta(x) \rangle \, d^n x = \int_{\mathbb{R}^n} \langle L_b K^\epsilon(x) - (L_b K)^\epsilon(x), \theta(x) \rangle \, d^n x, \tag{52}
\]
where \( L_b K \) is understood as a Lie derivative in the sense of distributions, and \( \theta \) is a test \( k \)-form in \( C^\infty_0 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \). We prove a generalisation to \( k \)-forms of the classical Di Perna-Lions type commutator estimates (see [DL89, Lio96, FGP10, Fla11]) for terms of type (52).

**Lemma 4.9.** Given a \( k \)-form \( K \in L^\infty_{loc} \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \), a vector field \( b \in W^{1,1}_{loc} (T \mathbb{R}^n) \), and a test function \( \theta \in C^\infty_0 \left( \bigwedge^k (T^* \mathbb{R}^n) \right) \) such that \( \text{Supp}(\theta) \subset B(0, R) \) for some \( R > 0 \), then for \( \epsilon < 1 \), we have the following estimate on the commutator (52):
\[
\left| \int_{\mathbb{R}^n} \langle [L_b, \rho^\epsilon]K(x), \theta(x) \rangle \, d^n x \right| \lesssim \| \theta \|_{L^\infty_R} \| K \|_{L^\infty_{R+1}} \| b \|_{W^{1,1}_{R+1}}. \tag{53}
\]
Moreover, \( \langle [L_b, \rho^\epsilon]K(x), \theta(x) \rangle \) converges to zero in \( L^1_{loc} \) as \( \epsilon \to 0 \).

**Proof.** By the definition of the commutator (50), we have
\[
\int_{\mathbb{R}^n} \langle [L_b, \rho^\epsilon]K(x), \theta(x) \rangle \, d^n x = \int_{\mathbb{R}^n} \langle L_b K^\epsilon(x), \theta(x) \rangle \, d^n x - \int_{\mathbb{R}^n} \langle (L_b K)^\epsilon(x), \theta(x) \rangle \, d^n x. \tag{54}
\]
The first integral becomes
\[
\int_{\mathbb{R}^n} \langle L_b K^\epsilon(x), \theta(x) \rangle \, d^n x = \int_{\mathbb{R}^n} \langle K^\epsilon(x), L^T_b \theta(x) \rangle \, d^n x
\]
\[
= \int_{\mathbb{R}^n} \delta^{i_1 j_1} \cdots \delta^{i_k j_k} K^\epsilon_{i_1 \ldots, i_k}(x) (L^T_b \theta)_{j_1 \ldots, j_k}(x) \, d^n x \int_{\mathbb{R}^n} K^\epsilon(x)(e_{i_1}, \ldots, e_{i_k})(L^T_b \theta)_{i_1 \ldots, i_k}(x) \, d^n x,
\]
where \( (L^T_b \theta)_{i_1 \ldots, i_k} := \delta^{i_1 j_1} \cdots \delta^{i_k j_k} (L^T_b \theta)_{j_1 \ldots, j_k} \), which, by (22) can be expressed as
\[
(L^T_b \theta)_{i_1 \ldots, i_k}(x) = - \frac{\partial}{\partial x^l} (b^i(x) \theta^{i_1 \ldots, i_k}(x)) - \sum_{j=1}^k \theta^{i_1 \ldots, i_{j-1}, j} \frac{\partial b^{i_j}}{\partial x^l}(x). \tag{55}
\]
Hence by Definition 2.7, we obtain
\[
\int_{\mathbb{R}^n} \langle L_b K^\epsilon(x), \theta(x) \rangle \, d^n x = - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho^\epsilon(x - y) K_{i_1 \ldots, i_k}(y) \frac{\partial}{\partial x^l} (b^i(x) \theta^{i_1 \ldots, i_k}(x)) \, d^n y \, d^n x
\]
\[
- \sum_{j=1}^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho^\epsilon(x - y) K_{i_1 \ldots, i_k}(y) \theta^{i_1 \ldots, i_{j-1}, j} \frac{\partial b^{i_j}}{\partial x^l}(x) \, d^n x \, d^n y
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial \rho^\epsilon}{\partial x^l}(x - y) K_{i_1 \ldots, i_k}(y) b^i(x) \theta^{i_1 \ldots, i_k}(x) - \sum_{j=1}^k \rho^\epsilon(x - y) K_{i_1 \ldots, i_k}(y) \frac{\partial b^{i_j}}{\partial x^l}(x) \theta^{i_1 \ldots, i_{j-1}, j}(x) \right) \, d^n y \, d^n x.
\]
For the second integral in (54), we take into account the property $D_y \rho^\varepsilon(x - y) = -D_x \rho^\varepsilon(x - y)$ and apply the definition of mollifiers on distribution-valued $k$-forms (see Definition 2.8), obtaining
\[
\int_{\mathbb{R}^n} \langle (L_b K)^{\varepsilon} (x), \theta(x) \rangle \, d^n x = \int_{\mathbb{R}^n} \langle K(y), L_b^T (\rho^\varepsilon \ast \theta)(y) \rangle \, d^n y
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial \rho^\varepsilon}{\partial x^l} (x - y) K_{\varepsilon}^{l} (y)_{\varepsilon}^{l} \theta^{i_1, \ldots, i_k} (x) - \rho^\varepsilon (x - y) K_{\varepsilon}^{l} (y) \frac{\partial b_j^l}{\partial y^l} (y) \theta^{i_1, \ldots, i_k} (x) \right) \, d^n y \, d^n x
\]
\[
- \sum_{j=1}^k \int_{\mathbb{R}^n} \left( \frac{\partial \rho^\varepsilon}{\partial x^l} (x - y) K_{\varepsilon}^{l} (y) \frac{\partial b_j^l}{\partial y^l} (y) \theta^{i_1, \ldots, i_k} (x) \right) \, d^n y \, d^n x.
\]
Now, combining the expressions we have obtained, we derive
\[
\int_{\mathbb{R}^n} \langle [L_b, \rho^\varepsilon \ast] K(x), \theta(x) \rangle \, d^n x
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varepsilon^n \frac{\partial}{\partial y^l} \rho \left( \frac{y - x}{\varepsilon} \right) K_{\varepsilon}^{l} (y) \left( \frac{b_j^l (y) - b_j^l (x)}{\varepsilon} \right) \theta^{i_1, \ldots, i_k} (x) \, d^n y \, d^n x
\]
\[
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho^\varepsilon (x - y) \left( \frac{\partial b_j^l}{\partial y^l} (y) \theta^{i_1, \ldots, i_k} (x) K_{\varepsilon}^{l} (y) \right)
\]
\[
\sum_{j=1}^k \left( \frac{\partial b_j^l}{\partial y^l} (y) - \frac{\partial b_j^l}{\partial x^l} (x) \right) \theta^{i_1, \ldots, i_k} (x) K_{\varepsilon}^{l} (y) \right) \, d^n y \, d^n x
\]
\[
=: \int I_1^\varepsilon (x) \, d^n x + \int I_2^\varepsilon (x) \, d^n x. \tag{56}
\]
Applying Young’s inequality for convolutions, we have
\[
|I_2^\varepsilon| \lesssim \|\theta\|_{L^1_R} \|K\|_{L^1_{R+1}} \|b\|_{W^{1,1}_{R+1}},
\]
since $\varepsilon < 1$, and where we have also taken into account that $\|\rho^\varepsilon\|_{L^1} = 1$, independently of $\varepsilon$, and that $\theta$ has support in $B(0, R)$. To obtain an analogous estimate for the first integral, we follow the proof of Lemma 2.3 in [Lio96]. First, we note that by Taylor’s theorem,
\[
b_j^l (y) - b_j^l (x) = \int_0^1 (y - x) \cdot D b_j^l (x + \lambda (y - x)) \, d\lambda,
\]
hence,
\[
I_1^\varepsilon (x) = \varepsilon^n \int_{B(x, \varepsilon) \cap \mathbb{R}^n} \frac{\partial}{\partial y^l} \rho \left( \frac{y - x}{\varepsilon} \right) K_{\varepsilon}^{l} (y) \left( \frac{b_j^l (y) - b_j^l (x)}{\varepsilon} \right) \theta^{i_1, \ldots, i_k} (x) \, d^n y
\]
\[
= \int_{B(0, 1)} \frac{\partial}{\partial w} \rho (w) K_{\varepsilon}^{l} (x + \varepsilon w) \left( \int_0^1 w \cdot D b_j^l (x + \lambda \varepsilon w) \, d\lambda \right) \theta^{i_1, \ldots, i_k} (x) \, d^n w,
\]
and therefore
\[
|I_1^\varepsilon (x)| \leq \|D \rho\|_{L^1_R} \|K\|_{L^1_{R+1}} \|\theta\|_{L^1_R} \int_{B(0, 1)} \int_0^1 |D b_j^l (x + \lambda \varepsilon w)| \, d\lambda \, d^n w.
\]
where \( \chi'(z) = \int_0^1 \frac{1}{(\lambda^2)(z)} \mathrm{1}_{B(0,\lambda)}(z) \mathrm{d}\lambda \), and that \( \|\chi'\|_{L^2} = \text{meas}(B(0,1)) \), we obtain
\[
\int_{B(0,R)} |I_1'(x)| \mathrm{d}^n x \leq \|D\rho\|_{L^1} \|K\|_{L^{\infty}_{R+1}} \|\theta\|_{L^\infty_R} \|Db\|_{L^1_{R+1}} \|\chi'\|_{L^1} \lesssim \|K\|_{L^{\infty}_{R+1}} \|\theta\|_{L^\infty_R} \|b\|_{W^{1,1}_{R+1}}, \tag{57}
\]
where we have made use of Young’s convolution inequality and the constant in the last inequality is independent of \( K, b, \theta \) and \( \epsilon \).

We now show the \( L^1_{\text{loc}} \) convergence of \( \langle [L_b, \rho^\epsilon \ast] K(x), \theta(x) \rangle \). First, one can check that
\[
I_2'(x) \to \langle K(x), \theta(x) \rangle \text{ div } b(x)
\]
in \( L^1_{\text{loc}} \) as \( \epsilon \to 0 \). Next, taking into account the density of \( C_0^\infty \) functions in \( W^{1,1}_{\text{loc}} \), we consider a sequence of \( C_0^\infty \) vector fields \( \{b_\alpha\} \) such that \( b_\alpha \to b \) in \( W^{1,1}_{\text{loc}} \). For every \( b_\alpha \in C_0^\infty(T\mathbb{R}^n) \), we have
\[
I_1'[b_\alpha](x) := \epsilon^{-n} \int_{B(x,\epsilon)} \frac{\partial}{\partial y_i} \rho \left( \frac{y - x}{\epsilon} \right) K_{i_{\beta_1}, \ldots, i_{\beta_k}}(y) \left( \frac{b_{\alpha_i}'(y) - b_{\alpha_i}'(x)}{\epsilon} \right) \theta^{i_{\beta_1}, \ldots, i_{\beta_k}}(x) \mathrm{d}^n y
= \int_{B(x,\epsilon)} \frac{\partial}{\partial y_i} (w) K_{i_{\beta_1}, \ldots, i_{\beta_k}}(x + \epsilon w) \left( \frac{b_{\alpha_i}'(x + \epsilon w) - b_{\alpha_i}'(x)}{\epsilon} \right) \theta^{i_{\beta_1}, \ldots, i_{\beta_k}}(x) \mathrm{d}^n w
\to \langle K(x), \theta(x) \rangle \int_{B(x,\epsilon)} w_i \frac{\partial b_{\alpha_i}'}{\partial x_i}(x) \frac{\partial}{\partial w_j}(w) \mathrm{d}^n w = - \langle K(x), \theta(x) \rangle \text{ div } b_\alpha(x),
\]
in \( L^1_{\text{loc}} \) as \( \epsilon \to 0 \). Then, by the linearity of \( I_1' \) in the \( b \)-argument and using estimate (57), we have
\[
\|I_1'[b] + \langle K, \theta \rangle \text{ div } b\|_{L^1_R}
\leq \|I_1'[b] - I_1'[b_\alpha]\|_{L^1_R} + \|I_1'[b_\alpha] + \langle K, \theta \rangle \text{ div } b_\alpha\|_{L^1_R} + \|\langle K, \theta \rangle \text{ div } b - \langle K, \theta \rangle \text{ div } b_\alpha\|_{L^1_R}
\leq C \|K\|_{L^{\infty}_{R+1}} \|\theta\|_{L^\infty_R} \left( \|b - b_\alpha\|_{W^{1,1}_{R+1}} + \|\text{div } b - \text{div } b_\alpha\|_{L^1_R} \right) + \|I_1'[b_\alpha] + \langle K, \theta \rangle \text{ div } b_\alpha\|_{L^1_R}
\to 0,
\]
as \( \alpha, \epsilon \to 0 \). Hence \( I_1' + I_2' \to - \langle K, \theta \rangle \text{ div } b + \langle K, \theta \rangle \text{ div } b = 0 \) in \( L^1_{\text{loc}} \) as \( \epsilon \to 0 \).

\[\Box\]

**Lemma 4.10.** Given a \( k \)-form \( K \in L^\infty_{\text{loc}} \left( \bigwedge^k (T^*\mathbb{R}^n) \right) \), a vector field \( \xi \in C^2(\mathbb{R}^n) \), and a test function \( \theta \in C_0^\infty \left( \bigwedge^k (T^*\mathbb{R}^n) \right) \) such that \( \text{Supp}(\theta) \subset B(0, R) \) for some \( R > 0 \), then for \( \epsilon < 1 \), we have the following estimate on the Stratonovich to Itô correction of the commutator:
\[
\left| \int_{\mathbb{R}^n} \langle [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^\epsilon \ast]] K(x), \theta(x) \rangle \mathrm{d}^n x \right|
\lesssim \|\theta\|_{L^\infty_R} \|K\|_{L^\infty_{R+1}} \left( \|\xi\|_{L^{\infty}_{R+1}} \|\nabla^2 \xi\|_{L^1_{R+1}} + \|\nabla \xi\|_{L^\infty_{R+1}} \|\nabla \xi\|_{L^1_{R+1}} \right). \tag{58}
\]
Moreover, \( \langle [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^\epsilon \ast]] K(x), \theta(x) \rangle \) converges to zero in \( L^1_{\text{loc}} \) as \( \epsilon \to 0 \).

By a straightforward but lengthy calculation using the explicit formula (55) and the property that $D_x \rho(x - y) = -D_y \rho(x - y)$, we find

$$\langle [L_\xi, [L_\xi, \rho^*]] K, \theta \rangle = \int_{\mathbb{R}^n} \frac{\partial^2 \rho^T}{\partial y^m \partial y^n}(y - x) K_{i_1, \ldots, i_k}(y) \theta^{i_1, \ldots, i_k}(x) (\xi^l(y) - \xi^l(x)) (\xi^m(y) - \xi^m(x)) \, d^n y$$

$$+ \int_{\mathbb{R}^n} \frac{\partial \rho}{\partial y^i}(y - x) \left[ K_{i_1, \ldots, i_k}(y) \theta^{i_1, \ldots, i_k}(x) \left( 2 \frac{\partial \xi^m}{\partial y^m}(y) (\xi^l(y) - \xi^l(x)) + \xi^m(y) \left( \frac{\partial \xi^l}{\partial y^m}(y) - \frac{\partial \xi^l}{\partial x^m}(x) \right) \right) \right. $$

$$+ \left. \frac{\partial \xi^l}{\partial x^m}(x) (\xi^m(y) - \xi^m(x)) \right) \right] d^n y$$

$$+ \theta^{i_1, \ldots, i_k}(x) \sum_{j=1}^k K_{i_1, \ldots, i_k, m}(y) \left( \frac{\partial \xi^m}{\partial y^m}(y) \frac{\partial \xi^l}{\partial y^l}(y) \right)$$

$$+ \sum_{j=1}^k K_{i_1, \ldots, i_k, l}(y) \left( \frac{\partial \xi^m}{\partial y^m}(y) \frac{\partial \xi^l}{\partial x^l}(x) - \xi^m(y) \frac{\partial \xi^l}{\partial x^l}(x) + \frac{\partial \xi^m}{\partial y^m}(y) \frac{\partial \xi^l}{\partial x^l}(x) \right)$$

$$- \frac{\partial \xi^m}{\partial x^j}(x) \left( \frac{\partial \xi^l}{\partial y^l}(y) - \frac{\partial \xi^l}{\partial x^l}(x) \right) + \frac{\partial \xi^m}{\partial x^l}(x) \left( \frac{\partial \xi^l}{\partial y^l}(y) - \frac{\partial \xi^l}{\partial x^l}(x) \right) \right] d^n y$$

$$=: I_1^l(x) + I_2^l(x) + I_3^l(x).$$

By Young’s convolution inequality, one can verify that for the third integral

$$|I_3^l(x)| \leq C \| \theta \|_{L^p_{\mathbb{R}}} \| K \|_{L^p_{\mathbb{R}^+}} \left( \| \xi \|_{L^p_{\mathbb{R}^+}} \| \nabla^2 \xi \|_{L^p_{\mathbb{R}^+}} + \| \nabla \xi \|_{L^p_{\mathbb{R}^+}} \| \nabla \xi \|_{L^p_{\mathbb{R}^+}} \right),$$

where $C$ is independent of $\epsilon$. For the second integral, we apply a similar strategy as carried out in the proof of the previous lemma to show that

$$|I_2^l(x)| \leq C \| \theta \|_{L^p_{\mathbb{R}}} \| K \|_{L^p_{\mathbb{R}^+}} \left( \| \xi \|_{L^p_{\mathbb{R}^+}} \| \nabla^2 \xi \|_{L^p_{\mathbb{R}^+}} + \| \nabla \xi \|_{L^p_{\mathbb{R}^+}} \| \nabla \xi \|_{L^p_{\mathbb{R}^+}} \right).$$

For the first integral, using

$$\xi^l(y) - \xi^l(x) = \int_0^1 (y - x) \cdot \nabla \xi^l(x + \lambda(y - x)) \, d\lambda,$$

we observe the following:

$$\int_{\mathbb{R}^n} I_1^l(x) \, d^n x$$

$$= \epsilon^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{i_1, \ldots, i_k}(x) \theta^{i_1, \ldots, i_k}(x) \frac{\partial^2 \rho}{\partial y^i \partial y^m}(y) \left( \frac{y - x}{\epsilon} \right) \left( \frac{\xi^l(y) - \xi^l(x)}{\epsilon} \right) \left( \frac{\xi^m(y) - \xi^m(x)}{\epsilon} \right) \, d^n y \, d^n x$$

$$+ \int_{B(0,R)} \int_{B(0,1)} K_{i_1, \ldots, i_k}(x + \epsilon \lambda w) D_{lm}^2 \rho(w) \left( \int_0^1 w \cdot \nabla \xi^l(x + \epsilon \lambda w) \, d\lambda \right)$$

$$\times \left( \int_0^1 w \cdot \nabla \xi^m(x + \epsilon \lambda w) \, d\lambda \right) \theta^{i_1, \ldots, i_k}(x) \, d^n w \, d^n x.$$
Hence, we have
\[
\left| \int_{\mathbb{R}^n} I^1_\epsilon(x) d^n x \right| \leq C \| \theta \|_{L^\infty_R} \| K \|_{L^\infty_{R+1}} \| \nabla \xi \|^2 \|_{L^1_{R+1}} \leq C \| \theta \|_{L^\infty_R} \| K \|_{L^\infty_{R+1}} \| \nabla \xi \|_{L^\infty_{R+1}} \| \nabla \xi \|_{L^1_{R+1}},
\]
where \( \chi'(z) = \int_0^1 \frac{1}{(z \lambda)^m} 1_{B(0, \epsilon \lambda)}(z) d\lambda \) as before and the constant \( C \) does not depend of \( \epsilon \). Combining all of the above, we obtain the estimate (58).

To show the \( L^1_{loc} \) convergence, first it is not hard to show that
\[
I^*_{3, \epsilon} \to \langle K(x), \theta(x) \rangle \frac{\partial}{\partial x^m} \left( \xi^m(x) \frac{\partial \xi^l}{\partial x^l}(x) \right)
\]
\[
+ \sum_{j=1}^k K_{i_1, \ldots, i_k}(x) \theta^{i_1 \ldots i_k}(x) \left( \frac{\partial \xi^m}{\partial x^m}(x) \frac{\partial \xi^l}{\partial x^l}(x) - \xi^m(x) \frac{\partial^2 \xi^l}{\partial x^m \partial x^l}(x) \right)
\]
in \( L^1_{loc} \) as \( \epsilon \to 0 \). Now, since \( \xi \in C^2 \), by using a similar technique as in the previous lemma, one can show that
\[
I^1_\epsilon \to \langle K(x), \theta(x) \rangle \frac{\partial}{\partial x^m} \left( \xi^m(x) \frac{\partial \xi^l}{\partial x^l}(x) \right)
\]
\[
I^2_\epsilon \to - \langle K(x), \theta(x) \rangle \left( 2 \frac{\partial^2 \xi^l}{\partial x^m \partial x^l}(x) \xi^m(x) + \delta^m_l \frac{\partial^2 \xi^l}{\partial x^m \partial x^l}(x) + \delta^m_l \frac{\partial \xi^m}{\partial x^m}(x) - \xi^m(x) \frac{\partial^2 \xi^l}{\partial x^m \partial x^l}(x) \right)
\]
in \( L^1_{loc} \) as \( \epsilon \to 0 \). Finally, it is easy to check that \( I^1_\epsilon + I^2_\epsilon + I^3_\epsilon \to 0 \) in \( L^1_{loc} \) as \( \epsilon \to 0 \).

**Corollary 4.11.** Given a \( C^1 \) diffeomorphism \( \phi \), a \( k \)-form \( K \in L^\infty_{loc} \left( \wedge^k (T^* \mathbb{R}^n) \right) \), vector fields \( b \in \mathcal{W}^1_{loc} (T \mathbb{R}^n) \), \( \xi \in C^2 (T \mathbb{R}^n) \), and a test \( k \)-form \( \theta \in C^\infty_0 \left( \wedge^k (T^* \mathbb{R}^n) \right) \) such that \( \text{Supp}(\bar{\theta}) \subset B(0, R) \), where \( \bar{\theta}(y) := J \phi^{-1}(y) (\phi_* \theta)(y) b \) for some \( R > 0 \), then for \( \epsilon < 1 \), we have the following estimates on the commutator (50), pulled-back by \( \phi \).

- \[
\left| \int_{\mathbb{R}^n} \langle \phi^* [\mathcal{L}_b, \rho^*] K(x), \theta(x) \rangle d^n x \right| \lesssim \| \bar{\theta} \|_{L^\infty_R} \| K \|_{L^\infty_{R+1}} \| b \|_{\mathcal{W}^1_{R+1}},
\]
- \[
\left| \int_{\mathbb{R}^n} \langle \phi^* [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^*]] K(x), \theta(x) \rangle d^n x \right| \lesssim \| \bar{\theta} \|_{L^\infty_R} \| K \|_{L^\infty_{R+1}} \left( \| \xi \|_{L^\infty_{R+1}} \| \nabla^2 \xi \|_{L^1_{R+1}} + \| \nabla \xi \|_{L^\infty_{R+1}} \| \nabla \xi \|_{L^1_{R+1}} \right).
\]

Moreover, it holds
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \langle \phi^* [\mathcal{L}_b, \rho^*] K(x), \theta(x) \rangle d^n x = 0,
\]
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \langle \phi^* [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^*]] K(x), \theta(x) \rangle d^n x = 0.
\]
Proof. Using Lemma 4.9, we get
\[
\left| \int_{\mathbb{R}^n} \langle \phi^* [\mathcal{L}_b, \rho^*] K(x), \theta(x) \rangle \ d^n x \right| = \left| \int_{\mathbb{R}^n} \left\langle [\mathcal{L}_b, \rho^*] K(y), \tilde{\theta}(y) \right\rangle \ d^n y \right|
\]
where \( \tilde{\theta}(y) = J\phi^{-1}(y) (\phi_* \theta^*)(y) \). Now, applying Lemma 4.9, we obtain our first inequality (59). The second inequality (60) is obtained in the same way, using Lemma 4.10.

4.3.2 Proof of uniqueness

We are now ready to prove our uniqueness result.

Proof of Theorem 4.6. We showed earlier by applying the Itô-Wentzell-Kunita formula that the pull-back of the mollified \( k \)-form \( K^\epsilon \) along the stochastic flow \( \phi_t \) satisfies
\[
\phi_t^* K^\epsilon(t, x) = \int_0^t \phi_s^* [\mathcal{L}_b, \rho^*] K(s, x) \ ds + \frac{1}{2} \int_0^t \phi_s^* [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^*]] K(s, x) \ ds + \int_0^t \phi_s^* [\mathcal{L}_\xi, \rho^*] K(s, x) \ dW_s.
\]
As discussed at the beginning of this subsection, in order to prove the uniqueness of equation (1), it suffices to show that \( \phi_t^* K_t \) is zero in the weak sense almost surely for all \( t \in [0, T] \), since the equation for \( K_t \) is linear and \( \phi_t \) is a \( C^1 \)-diffeomorphism for all \( t \in [0, T] \).

Fixing a test function \( \theta \in C_0^\infty \left( \bigwedge^k (T^*\mathbb{R}^n) \right) \) with \( \text{Supp}(\theta) \subset B(0, r) \) and using the stochastic Fubini theorem, we have
\[
\int_{\mathbb{R}^n} \langle \phi_t^* K(t, x), \theta(x) \rangle \ d^n x
\]
\[
= \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_b, \rho^*] K(s, x), \theta(x) \rangle \ d^n x \ ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^*]] K(s, x), \theta(x) \rangle \ d^n x \ ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_\xi, \rho^*] K(s, x), \theta(x) \rangle \ d^n x \ dW_s.
\]
We check term-by-term convergence:

- **The LHS.** Using Lemma 4.2, we obtain
\[
\int_{\mathbb{R}^n} \langle \phi_t^* K^\epsilon(t, x), \theta(x) \rangle \ d^n x = \int_{\mathbb{R}^n} \langle K^\epsilon(t, y), J\phi_t^{-1}(y)((\phi_t)_* \theta^*)(y) \rangle \ d^n y
\]
\[
\to \int_{\mathbb{R}^n} \langle K(t, y), J\phi_t^{-1}(y)((\phi_t)_* \theta^*)(y) \rangle \ d^n y = \int_{\mathbb{R}^n} \langle \phi_t^* K(t, x), \theta(x) \rangle \ d^n x,
\]

- **The first term on the RHS.** By (59), we have
\[
\left| \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_b, \rho^*] K(s, x), \theta(x) \rangle \ d^n x \right| \lesssim \| \tilde{\theta} \|_{L_R^n} \| K_s \|_{L_{R+1}^\infty} \| b_s \|_{W_{R+1}^{1,1}}.
\]
uniformly in \( \epsilon \), where \( \tilde{\theta} := J\phi^{-1}_s(y)((\phi_s)_\ast \theta^p) \) and \( R = \sup_{s \in [0,T]} x \in B(\gamma) |\phi_s(x)| \) is its support. Arguing as in [FGP10], since \( R < \infty \) almost surely, we have \( \|\tilde{\theta}\|_{L_R^\infty} \in L^\infty([0,T]) \) almost surely. By assumption, we also have \( \|K_s\|_{L^\infty_{R+1}}, \|b_s\|_{W^{1,1}_{R+1}} \in L^\infty([0,T]) \). Hence, by dominated convergence theorem, we obtain

\[
\lim_{\epsilon \to 0} \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^\ast [\mathcal{L}_\xi, \mathcal{L}_\xi], \theta(x) \rangle \, d^n x \, ds = 0, \quad a.s.
\]

- **The second term on the RHS.** By (60), we have

\[
\frac{1}{2} \left| \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^\ast [\mathcal{L}_\xi, \mathcal{L}_\xi], \theta(x) \rangle \, d^n x \, ds \right| \lesssim \|\tilde{\theta}\|_{L_R^\infty} \|K_s\|_{L^\infty_{R+1}} \left( \|\xi_s\|_{L^\infty_{R+1}} \|\nabla^2 \xi_s\|_{L^1_{R+1}} + \|\nabla \xi_s\|_{L^\infty_{R+1}} \|\nabla \xi_s\|_{L^1_{R+1}} \right).
\]

Again, it is easy to show that the bound is in \( L^1([0,T]) \), and hence by dominated convergence theorem we have

\[
\lim_{\epsilon \to 0} \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^\ast [\mathcal{L}_\xi, \mathcal{L}_\xi], \theta(x) \rangle \, d^n x \, ds = 0, \quad a.s.
\]

- **The martingale term on the RHS.** By the dominated convergence theorem for Itô integrals, we have

\[
\lim_{\epsilon \to 0} \left| \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^\ast \rho^\ast \mathcal{L}_\xi, \theta(x) \rangle \, d^n x \, dW_s \right| = 0,
\]

in probability.

\[
\square
\]

5 **Ill-posedness of the deterministic equation**

In this section, we explain why the weak deterministic transport equation

\[
\partial_t K(t, x) + \mathcal{L}_b K(t, x) = 0
\]

is not well-posed in \( L^p \cap L^\infty_{loc}(\Lambda^k(T^*\mathbb{R}^n)) \). To this purpose, we provide an example of a vector field \( b \in C^\alpha(T^*\mathbb{R}^n) \cap W^{1,1}_{loc}(T^*\mathbb{R}^n) \) such that for the same initial profile \( K_0 \in L^p \cap L^\infty_{loc}(\Lambda^k(T^*\mathbb{R}^n)) \), infinitely many solutions in \( L^p \cap L^\infty_{loc}(\Lambda^k(T^*\mathbb{R}^n)) \) of equation (63) can be constructed. To this end, we consider the deterministic flow equation

\[
\frac{d}{dt} \phi_t(X) = b(t, \phi_t(X)).
\]

As in the stochastic case, we know that for initial data \( K_0 \in L^p \cap L^\infty_{loc}(\Lambda^k(T^*\mathbb{R}^n)) \) and \( b \) smooth enough, the pushed-forward \( k \)-form-valued function \( K_t(x) = (\phi_t)_\ast K_0 \) solves the deterministic transport equation (63), where \( \phi_t \) represents the flow of (64). We select a particular vector field
b ∈ C_0(T R^n) ∩ W_{loc}^{1,1}(T R^n) such that equation (64) does not possess a flow, since it does not satisfy uniqueness of solutions. Then we show how to construct a solution of equation (63) out of the solutions of (64).

Aiming to generalise the example provided in [FGP10], we consider the function

\[ b(x) = \frac{1}{1 - \alpha \left| x \right|} (\left| x \right| R)^\alpha, \]

where \( R > 0 \) is introduced only to obtain boundedness. The function \( b \) can be checked to be \( \alpha \)-Hölder and to pertain to the space \( W_{loc}^{1,1} \). For verifying that \( b \) is \( \alpha \)-Hölder, it is sufficient to check the condition for the function \( \frac{1}{\left| x \right|} x^\alpha \). Assume without loss of generality that \( \left| x \right| > \left| y \right| \). Then:

\[
\left| \left| x \right|^\alpha x - \left| y \right|^\alpha y \right| \leq \left| x \right|^\alpha \left| \frac{x}{\left| x \right|} - \frac{y}{\left| y \right|} \right| + \left| \frac{y}{\left| y \right|} \left( \left| x \right|^\alpha - \left| y \right|^\alpha \right) \right|.
\]

Now simply note that the second term on the right-hand side

\[
\left| \frac{y}{\left| y \right|} (\left| x \right|^\alpha - \left| y \right|^\alpha) \right| \leq \left| \left| x \right|^\alpha - \left| y \right|^\alpha \right| \leq \left| x - y \right|^\alpha.
\]

The first term can be bounded by

\[
\left| x \right|^\alpha \left| \frac{x}{\left| x \right|} - \frac{y}{\left| y \right|} \right| = \left| x \right|^{\alpha - 1} \left| x - y + \left( \frac{y}{\left| y \right|} - \frac{x}{\left| x \right|} \right) \right| \lesssim \left| x \right|^{\alpha - 1} \left| x - y \right| \leq \left| x - y \right|^\alpha,
\]

where in the last step we have taken into account that \( \left| x \right|^{\alpha - 1} \leq \left| x - y \right|^{\alpha - 1} \), due to the assumption \( \left| x \right| > \left| y \right| \).

Under substitution of (65), the flow equation (64) becomes

\[ \dot{x} = \frac{1}{1 - \alpha \left| x \right|} (\left| x \right| R)^\alpha. \]  

(66)

Now, it can be checked that:

- For \( x_0 \in \mathbb{R}^n \setminus \{0\} \), equation (66) admits a well-defined unique solution \( \phi_t(x_0) \). The function \( \phi_t(\cdot) \) maps \( \mathbb{R}^n \setminus \{0\} \) one-to-one onto \( \mathbb{R}^n \setminus B \left( 0, t^{\frac{1}{1 - \alpha}} \right) \).

- For \( x_0 = 0 \), and given a unitary vector \( v \in \mathbb{R}^n \), the functions \( x_v(t) = vt^{\frac{1}{1 - \alpha}} \) satisfy equation (66). The functions \( x_v(t - t_0) \) solve equation (66) with initial value cero at \( t_0 \).

- Also, extending the argument in [FGP10], observe that given \( x \in B \left( 0, t^{\frac{1}{1 - \alpha}} \right) \), there is a unique number \( t_0(t, x) \geq 0 \), and a unique unitary direction \( v \) such that \( x_v(t - t_0(t, x)) = x \).

Consider, for \( |v| = 1 \), a bounded measurable function \( \gamma_v : [0, \infty) \to \wedge^k(T^* \mathbb{R}^n) \), and define:

\[
u_{\gamma_v}(t, x) = \begin{cases} (\phi_t)_* u_0, & x \not\in B \left( 0, t^{\frac{1}{1 - \alpha}} \right), \\ \gamma_v(t_0(t, x)), & x \in B \left( 0, t^{\frac{1}{1 - \alpha}} \right). \end{cases}
\]

These are weak solutions in \( L^p \cap L_{loc}^\infty(\wedge^k(T^* \mathbb{R}^n)) \) of the Lie transport equation (63).
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Appendix A  Background in tensors

Let $V$ be a finite dimensional vector space with basis $\{e_1, \ldots, e_n\}$, and $V^*$ be its corresponding dual space with basis $\{e^1, \ldots, e^n\}$, defined via the relation $\langle e_i, e_j \rangle = \delta_{ij}$, which is one when $i = j$ and zero otherwise, where $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$ is the natural pairing $\langle v, \alpha \rangle := \alpha(v)$. For a contravariant vector $v \in V$, we express its components as $v = v^i e_i$ with the upper indices, and for a covariant vector (covector) $\alpha \in V^*$, we express its components as $\alpha = \alpha^i e_i^*$ with the lower indices, where we have assumed Einstein’s convention of summing over repeated indices.

Definition A.1. Given a vector space $V$ and its dual $V^*$, we define an $(r,s)$-tensor $T$ as a multilinear map

$$T : V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}. $$

Fixing a basis $\{e_1, \ldots, e_n\}$ for $V$ and $\{e^1, \ldots, e^n\}$ for $V^*$, we may interpret tensors as vectors with basis $\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{i_{r+1}} \otimes \cdots \otimes e^{i_{r+s}}\}_{i_k \in \{1, \ldots, n\}}$, where

$$e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{i_{r+1}} \otimes \cdots \otimes e^{i_{r+s}} (e^{j_1}, \ldots, e^{j_r}, e_{j_{r+1}}, \ldots, e_{j_{r+s}}) = \delta_{i_1}^{j_1} \cdots \delta_{i_r}^{j_r} \delta_{j_{r+1}}^{i_{r+1}} \cdots \delta_{j_{r+s}}^{i_{r+s}}. $$

We can express $T$ as

$$T = T_{i_1 \ldots i_r}^{j_1 \ldots j_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_{r+1}} \otimes \cdots \otimes e^{j_{r+s}},$$

where $T_{i_1 \ldots i_r}^{j_1 \ldots j_r} = T(e_{i_1}, \ldots, e_{i_r}, e^{i_{r+1}}, \ldots, e^{i_{r+s}})$ are its components. We denote the space of $(r,s)$-tensors over $V$ by $T_V^{(r,s)}$.

A.1  Raising/lowering indices and the bundle metric

Consider a symmetric, non-degenerate, positive definite bilinear form $g : V \times V \to \mathbb{R}$, which is a $(0,2)$-tensor whose components are given by $g_{ij} = g(e_i, e_j)$. We call $g$ a metric tensor or just a metric. Fixing a vector $v \in V$, one can construct an associated covector $v^\flat \in V^*$ through the relation $v^\flat(w) = g(v, w)$, for any $w \in V$, and we let $g^\flat : V \to V^*$ be the map $g^\flat : v \mapsto v^\flat$, for $v \in V$. This procedure is called lowering the index, since the components of $v^\flat$ now have lower indices $((v^\flat)_i = g_{ij} v^j)$. We also introduce the inverse operation $g^\flat = (g^\flat)^{-1} : V^* \to V$, which maps a covector $\alpha \in V^*$ to a vector $\alpha^\flat \in V$ by $\alpha^\flat = g^\flat(\alpha)$. This is called raising the index. One can then define a $(2,0)$-tensor $g^*_{\flat}$ on $V^*$ by $g^*_{\flat}(\alpha, \beta) = g(\alpha^\flat, \beta^\flat)$, whose components we simply denote by $g^*_{ij} = g^*_{\flat}(e^i, e^j)$. We call $g^*$ the co-metric tensor or just the co-metric. Combining the metric and co-metric tensors, we can construct a metric on tensors, defined as follows (see for instance [BHM10]).
Definition A.2. Given a metric tensor $g$ on $V$, a bundle metric $\bar{g}$ on $T_V^{(r,s)}$ is defined by
\[
\bar{g}(T, S) = g_{i_1j_1} \cdots g_{i_rj_r} g^{i_{r+1}j_{r+1}} \cdots g^{i_{r+s}j_{r+s}} T^i_{i_1 \ldots i_r} \cdots T^i_{i_r \ldots i_r} S_{j_{r+1} \ldots j_r} \cdots S_{j_{r+s} \ldots j_{r+s}},
\]
for any $S, T \in T_V^{(r,s)}$. This also provides us with a norm $|T|_g$ on $T_V^{(r,s)}$, defined by
\[
|T|_g = \sqrt{\bar{g}(T, T)}.
\]

We note that this generalises the Frobenius inner product for matrices to general tensors.

A.2 Section of a tensor bundle on $\mathbb{R}^n$

An $(r, s)$-tensor bundle on $\mathbb{R}^n$ is a tuple $(T^{(r,s)}(\mathbb{R}^n), \pi)$, where $T^{(r,s)}(\mathbb{R}^n) := \mathbb{R}^n \times T_{\mathbb{R}^n}^{(r,s)}$, and $\pi$ is the natural projection map $\pi : \mathbb{R}^n \times T_{\mathbb{R}^n}^{(r,s)} \to \mathbb{R}^n$. For every point $x \in \mathbb{R}^n$, we have $\pi^{-1}(x) \cong T_x^{(r,s)}$, which we call the fibres of the bundle. We will often refer to $T^{(r,s)}(\mathbb{R}^n)$ as the tensor bundle $(T^{(r,s)}(\mathbb{R}^n), \pi)$, omitting the projection map for simplicity. For example, taking $(r, s) = (1, 0)$ we get the tangent bundle $T\mathbb{R}^n$, and considering $(r, s) = (0, 1)$ we obtain the cotangent bundle $T^*\mathbb{R}^n$. Although we will not describe it in details here, the notion of tensor bundles can be generalised to manifolds, which locally “look like” a tensor bundle on $\mathbb{R}^n$, but requires additional structure on the fibres.

A section $s$ of the tensor bundle $(T^{(r,s)}(\mathbb{R}^n), \pi)$ is a map $s : \mathbb{R}^n \to T^{(r,s)}(\mathbb{R}^n)$ such that $\pi \circ s : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. We denote the space of all sections by $\Gamma(T^{(r,s)}(\mathbb{R}^n))$ and the set of all smooth sections by $C^\infty(T^{(r,s)}(\mathbb{R}^n))$, or simply $\mathfrak{T}^{(r,s)}(\mathbb{R}^n)$. Geometrically, a section is a map that assigns a unique tensor to each point on $\mathbb{R}^n$, giving us a tensor field. In the next section, we describe in more details two common examples of tensor fields, namely, vector fields and differential forms.

Appendix B  Vector fields and differential forms

Two fundamental examples of tensor fields (i.e., sections of a tensor bundle) are vector fields and differential one-forms, which we will discuss in more details here.

Vector fields. A vector field $X$ is a $(1, 0)$-tensor field $X : \mathbb{R}^n \to V$, where $V \cong \mathbb{R}^n$ consists of contravariant vectors (i.e., it is a section of the tangent bundle). We denote by $\mathfrak{X}(\mathbb{R}^n) := C^\infty(T\mathbb{R}^n)$ the set of all smooth vector fields on $\mathbb{R}^n$, where $T\mathbb{R}^n := T^{(1,0)}(\mathbb{R}^n)$.

Differential one-forms. A differential one-form $\alpha$ is a $(0, 1)$-tensor field $\alpha : \mathbb{R}^n \to V^*$, where $V \cong \mathbb{R}^n$ consists of covariant vectors (i.e., it is a section of the cotangent bundle). We denote by $\Omega^1(\mathbb{R}^n) := C^\infty(T^*\mathbb{R}^n)$ the set of all differential one-forms on $\mathbb{R}^n$, where $T^*\mathbb{R}^n := T^{(0,1)}(\mathbb{R}^n)$.

As a standard convention, the basis of the vector fields evaluated at a point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ is expressed as $\{\partial/\partial x^1(x), \ldots, \partial/\partial x^n(x)\}$ and its corresponding dual basis is denoted by
\{dx^1(x), \ldots, dx^n(x)\}. Note that these are related by \(\langle \partial/\partial x^i(x), dx^j(x) \rangle = \delta^j_i\). In coordinates, a vector field \(X\) and a one-form \(\alpha\) can be expressed explicitly as

\[
X(x) = X^i(x) \frac{\partial}{\partial x^i}(x), \quad \alpha(x) = \alpha_i(x) \, dx^i(x),
\]

and more generally, we can express an \((r, s)\)-tensor field \(T\) in coordinates as

\[
T(x) = T^{i_1, \ldots, i_r}_{\sigma_1, \ldots, \sigma_s}(x) \frac{\partial^r}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_r}} \otimes_d \frac{\partial^s}{\partial x^{\sigma_1}} \cdots \frac{\partial}{\partial x^{\sigma_s}} (x) \otimes dx^{i_{r+1}} \otimes \cdots \otimes dx^{i_{r+s}}(x).
\]

**Differential \(k\)-forms.** Other fundamental examples of tensor fields in differential geometry are **differential \(k\)-forms**, which generalise the notion of a differential one-form. Let \(0 \leq k \leq n\) be an integer. A differential \(k\)-form \(\alpha\) on \(\mathbb{R}^n\) is a \((0, k)\)-tensor field such that for any permutation \(\sigma \in S_k\), we have

\[
\alpha(x)(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \text{sgn}(\sigma) \alpha(x)(v_1, \ldots, v_k), \quad \text{for all } v_1, \ldots, v_k \in V,
\]

where \(\text{sgn}(\sigma)\) is the signature of the permutation \(\sigma \in S_k\). We denote the space of \(k\)-forms by \(\Omega^k(\mathbb{R}^n)\), which is a smooth section of the alternating \((0, k)\)-tensor bundle \(\bigwedge^k T^* \mathbb{R}^n \rightarrow \mathbb{R}^n\), whose fibres are given by \(\mathbb{R}^n \wedge \ldots \wedge \mathbb{R}^n\) \((k\text{ times}) := \sum_{\sigma \in S_k} \text{sgn}(\sigma) \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n\) \((k\text{ times})\). In coordinates, one can express a \(k\)-form as

\[
\alpha(x) = \sum_{i_1 < \ldots < i_k} \text{sgn}(\sigma) \alpha_{i_1, \ldots, i_k}(x) \, dx^{i_{\sigma(1)}}(x) \otimes \cdots \otimes dx^{i_{\sigma(k)}}(x)
\]

\[
= \sum_{i_1 < \ldots < i_k} \alpha_{i_1, \ldots, i_k}(x) \, dx^{i_1}(x) \wedge \cdots \wedge dx^{i_k}(x),
\]

where \(dx^{i_1} \wedge \cdots \wedge dx^{i_k} := \sum_{\sigma \in S_k} \text{sgn}(\sigma) \, dx^{i_{\sigma(1)}} \otimes \cdots \otimes dx^{i_{\sigma(k)}}\). Given a \(k\)-form \(\alpha\) and an \(l\)-form \(\beta\), one can construct a \((k + l)\)-form \(\alpha \wedge \beta\), defined as

\[
(\alpha \wedge \beta)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).
\]

Differential \(k\)-forms are especially important in differential geometry since they allow us to extend the notion of integration to general oriented manifolds. For a \(k\)-form \(\alpha \in \Omega^k(\mathbb{R}^n)\), we define its integral over \(\mathbb{R}^k\) by

\[
\int_{\mathbb{R}^k} \alpha = \sum_{i_1 < \cdots < i_k} \int_{\mathbb{R}^k} \alpha_{i_1, \ldots, i_k}(x) \, dx^{i_1} \cdots dx^{i_k},
\]

where \(dx^{i_1} \cdots dx^{i_k}\) is the Lebesgue measure. We note that all of the above notions can be defined in coordinate free form on arbitrary \(k\)-dimensional smooth oriented manifolds, however, we will not discuss this further here. Instead, we refer the reader to [MR13] for more details.

**B.1 Exterior calculus**

We also review some elementary operations defined on tensor fields, and in particular, on vector fields and \(k\)-forms.
Exterior derivative. Given a $k$-form $\alpha \in \Omega^k(\mathbb{R}^n)$, which in coordinates is written as $\alpha(x) = \alpha_{i_1,\ldots,i_k}(x)dx^{i_1}(x) \wedge \ldots \wedge dx^{i_k}(x)$, its exterior derivative $d\alpha$ is the $(k+1)$-form

$$d\alpha(x) = \frac{\partial \alpha_{i_1,\ldots,i_k}}{\partial x^l}(x)dx^l(x) \wedge dx^{i_1}(x) \wedge \ldots \wedge dx^{i_k}(x),$$

where $i_1 < \cdots < i_k$, and we are implicitly summing over $l = 1, \ldots, n$. Hence, the exterior derivative is a map $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$.

Contraction. Given a vector field $X = X^i(x)\partial/\partial x^i(x)$ and a $k$-form $\alpha = \alpha_{i_1,\ldots,i_k}(x)dx^{i_1}(x) \wedge \ldots \wedge dx^{i_k}(x)$, we define the contraction $\iota_X \alpha$ of $\alpha$ with $X$ as the $(k-1)$-form

$$\iota_X \alpha(x) = X^i(x)\alpha_{i,i_2,\ldots,i_k}(x)dx^{i_2}(x) \wedge \ldots \wedge dx^{i_k}(x),$$

where $i_2 < \cdots < i_k$, and sum over $l = 1, \ldots, n$ is assumed. For a vector field $X$, the contraction is as a map $\iota_X : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$.

Pull-back and push-forward. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. The push-forward $\varphi_*X$ of a vector field $X = X^i(x)\partial/\partial x^i(x)$ with respect to $\varphi$ is defined as

$$(\varphi_*X)(\varphi(x)) = X^i(x)\frac{\partial \varphi^i}{\partial x^j}(\varphi(x)), \quad (67)$$

and the pull-back $\varphi^*\alpha$ of a one-form $\alpha = \alpha_i(y)dy^i(y)$ with respect to $\varphi$ is defined as

$$(\varphi^*\alpha)(x) = \alpha_i(\varphi(x))\frac{\partial \varphi^i}{\partial x^j}(x)dx^j(x). \quad (68)$$

One can also define the pull-back of a vector field as $\varphi^*X := (\varphi^{-1})_*X$, and the push-forward of a one-form as $\varphi_*\alpha := (\varphi^{-1})^*\alpha$.

For a general $(r, s)$-tensor field $T$, we define its pull-back $\varphi^*T$ with respect to $\varphi$ by

$$(\varphi^*T)(x)(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_s) = T(\varphi(x))(\varphi_*\alpha^1, \ldots, \varphi_*\alpha^r, \varphi_*v_1, \ldots, \varphi_*v_s), \quad (69)$$

for any $\alpha^1, \ldots, \alpha^r \in \Omega^1(\mathbb{R}^n)$, and $v_1, \ldots, v_s \in \mathcal{X}(\mathbb{R}^n)$. Similarly, we define the push-forward of an $(r, s)$-tensor field as $\varphi_*T := (\varphi^{-1})^*T$.

Lie derivative. Let $T \in \mathcal{T}^{(r,s)}(\mathbb{R}^n)$, $X \in \mathcal{X}(\mathbb{R}^n)$, and $\varphi_t$ be the flow of the vector field $X$. The Lie derivative of $T$ with respect to $X$ is defined by

$$\mathcal{L}_X T = \lim_{t \to 0} \frac{1}{t}(\varphi_t^*T - T),$$

which is a map $\mathcal{L}_X : \mathcal{T}^{(r,s)}(\mathbb{R}^n) \to \mathcal{T}^{(r,s)}(\mathbb{R}^n)$. Below, we list some common examples of Lie derivatives.

- For a function $f \in C^\infty(\mathbb{R}^n)$ and a vector field $X \in \mathcal{X}(\mathbb{R}^n)$, we have

$$\mathcal{L}_X f(x) = Xf(x) = X^i(x)\frac{\partial f}{\partial x^i}(x).$$
• Given two vector fields \( X, V \in \mathfrak{X}(\mathbb{R}^n) \), we have
\[
\mathcal{L}_X V(x) = [X, V](x) = \left( X^i(x) \frac{\partial V^j}{\partial x^i}(x) - V^i(x) \frac{\partial X^j}{\partial x^i}(x) \right) \frac{\partial}{\partial x^j}(x),
\]
where \([\cdot, \cdot]\) is the standard commutator of vector fields (Jacobi-Lie bracket).

• For a \( k \)-form \( \alpha \in \Omega^k(\mathbb{R}^n) \) and a vector field \( X \in \mathfrak{X}(\mathbb{R}^n) \), we have the identity
\[
\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X d\alpha.
\]
This is called Cartan’s magic formula.

• In coordinates, the Lie derivative of a \( k \)-form with respect to a vector field is given by
\[
\mathcal{L}_X \alpha(x)(v_1, \ldots, v_k) = X^l(x) \frac{\partial \alpha_{i_1, \ldots, i_k}}{\partial x^l}(x)v_1^{i_1} \cdots v_k^{i_k} + \sum_{j=0}^{k} \alpha_{i_1, \ldots, i_{j-1}, j, i_{j+1}, \ldots, i_k}(x) \frac{\partial X^j}{\partial x^l}(x)v_1^{i_1} \cdots v_j^{i_j} \cdots v_k^{i_k}, \quad \forall v_1, \ldots, v_k \in \mathfrak{X}(\mathbb{R}^n). \tag{70}
\]

**Hodge-star operator.** To complete this section, we introduce the Hodge-star operator \( \ast \) on \( \mathbb{R}^n \), which maps a \( k \)-form to an \((n-k)\)-form. Given a \( k \)-form \( \alpha \), we define its Hodge-star \( \ast \alpha \) as the \((n-k)\)-form satisfying
\[
\int_{\mathbb{R}^n} \beta \wedge \ast \alpha = \int_{\mathbb{R}^n} (\beta(x), \alpha(x)) \, d^n x,
\]
for any \( \beta \in \Omega^k(\mathbb{R}^n) \). In coordinates, the Hodge-star can be expressed as
\[
\ast \alpha(x) = \frac{1}{k! (n-k)!} \varepsilon_{i_1, \ldots, i_n} \delta^{i_1 j_1} \cdots \delta^{i_k j_k} \alpha_{j_1, \ldots, j_k}(x) dx^{i_1+1} \wedge \cdots \wedge dx^n, \quad \alpha \in \bigwedge^k(\mathbb{R}^n), \tag{71}
\]
where \( \varepsilon_{i_1, \ldots, i_n} \) is the Levi-Civita symbol \( \varepsilon_{\sigma(1), \ldots, \sigma(n)} = \text{sgn}(\sigma) \), for every \( \sigma \in S_n \). Its inverse \( \ast^{-1} : \Omega^{n-k}(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n) \) can be computed to be \( \ast^{-1} \beta = (-1)^{k(n-k)}(\ast \beta) \in \Omega^k(\mathbb{R}^n) \), for any \( \beta \in \Omega^{n-k}(\mathbb{R}^n) \). We can also extend the definition of the Hodge-star operator to a general Riemannian manifold.

**Appendix C Alternative geometric proof of existence**

In the present appendix, we provide an alternative geometric proof of Proposition 4.4, which is one of the main results employed to prove existence of weak solutions to equation (1) in Section 4. We first state the following auxiliary lemma.

**Lemma C.1.** Let \( \alpha \in L^\infty(\bigwedge^k(\mathbb{R}^n)) \), \( \beta \in C^\infty_0(\bigwedge^k(\mathbb{R}^n)) \) be \( k \)-forms, and \( b \in \mathfrak{X}(\mathbb{R}^n) \) be a differentiable vector field. Then we have the following identity:
\[
\int_{\mathbb{R}^n} \alpha \wedge \mathcal{L}_b(\ast \beta) = -\int_{\mathbb{R}^n} \langle \alpha, \mathcal{L}_b^T \beta \rangle \, d^n x. \tag{72}
\]
Proof. First, consider the mollification $\alpha^\epsilon = \rho^\epsilon \ast \alpha$. Applying the standard product rule for Lie derivatives, we have

$$\int_{\mathbb{R}^n} \alpha^\epsilon \wedge \mathcal{L}_b(\ast \beta) = \int_{\mathbb{R}^n} \mathcal{L}_b(\alpha^\epsilon \wedge \ast \beta) - \int_{\mathbb{R}^n} (\mathcal{L}_b \alpha^\epsilon) \wedge \ast \beta$$

$$= \int_{\mathbb{R}^n} d(\mu_b(\alpha^\epsilon \wedge \ast \beta)) - \int_{\mathbb{R}^n} \langle \mathcal{L}_b \alpha^\epsilon, \beta \rangle \, d^n x = - \int_{\mathbb{R}^n} \langle \alpha^\epsilon, \mathcal{L}_b^T \beta \rangle \, d^n x.$$ 

Now, the left-hand side can be rewritten as

$$\int_{\mathbb{R}^n} \alpha^\epsilon \wedge \mathcal{L}_b(\ast \beta) = \int_{\mathbb{R}^n} \langle \alpha^\epsilon, \ast^{-1} \mathcal{L}_b(\ast \beta) \rangle \, d^n x. \tag{73}$$

Hence, by the $L^1_{\text{loc}}$ convergence $\alpha^\epsilon \to \alpha$, we conclude that indeed

$$\int_{\mathbb{R}^n} \alpha \wedge \mathcal{L}_b(\ast \beta) = - \int_{\mathbb{R}^n} \langle \alpha, \mathcal{L}_b^T \beta \rangle \, d^n x. \tag{74}$$

\[ \square \]

**Proof of Proposition 4.4.** Consider the integral

$$\int_{\mathbb{R}^n} \langle (\phi_t)_* K_0, \theta \rangle \, d^n x = \int_{\mathbb{R}^n} (\phi_t)_* K_0 \wedge \ast \theta = \int_{\mathbb{R}^n} K_0 \wedge \phi_t^*(\ast \theta). \tag{75}$$

Using Itô’s second formula for tensor fields (14), we have

$$\phi_t^*(\ast \theta) = \ast \theta + \int_0^t \phi_s^* \mathcal{L}_b(\ast \theta) \, ds + \int_0^t \phi_s^* \mathcal{L}_\xi(\ast \theta) \circ dW_s, \tag{76}$$

so we obtain

$$\int_{\mathbb{R}^n} \langle (\phi_t)_* K_0, \theta \rangle \, d^n x = \int_{\mathbb{R}^n} K_0 \wedge \ast \theta + \int_0^t \int_{\mathbb{R}^n} K_0 \wedge \phi_s^* \mathcal{L}_b(\ast \theta) \, ds + \int_0^t \int_{\mathbb{R}^n} K_0 \wedge \phi_s^* \mathcal{L}_\xi(\ast \theta) \circ dW_s$$

$$= \int_{\mathbb{R}^n} \langle K_0, \theta \rangle \, dx + \int_0^t \int_{\mathbb{R}^n} \langle (\phi)_* K_0 \wedge \mathcal{L}_b(\ast \theta) \rangle \, ds + \int_0^t \int_{\mathbb{R}^n} \langle (\phi)_* K_0 \wedge \mathcal{L}_\xi(\ast \theta) \rangle \circ dW_s$$

$$= \int_{\mathbb{R}^n} \langle K_0, \theta \rangle \, dx - \int_0^t \int_{\mathbb{R}^n} \langle (\phi)_* K_0, L^T_b \theta \rangle \, dx \, ds - \int_0^t \int_{\mathbb{R}^n} \langle (\phi)_* K_0, L^T_\xi \theta \rangle \, dx \circ dW_s, \tag{77}$$

where we have applied Lemma C.1 and stochastic Fubini (see Theorem 2.3). To check that the conditions in Theorem 2.3 are satisfied, we note that

$$\int_{\mathbb{R}^n} K_0 \wedge \left( \int_0^t \phi_s^* \mathcal{L}_b(\ast \theta) \, ds \right) = \int_{\mathbb{R}^n} \langle \ast^{-1} K_0(x), \left( \int_0^t \phi_s^* \mathcal{L}_b(\ast \theta) \, ds \right) \rangle \, d^n x$$

$$= \int_{\mathbb{R}^n} \int_0^t \langle \ast^{-1} K_0(x), \phi_s^* \mathcal{L}_b(\ast \theta) \rangle \, ds \, d^n x.$$
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for the deterministic integral, and

\[
\int_{\mathbb{R}^n} K_0 \wedge \left( \int_0^t \phi_s^* \mathcal{L}_\xi (\ast \theta) \circ dW_s \right) = \int_{\mathbb{R}^n} \left( \ast^{-1} K_0 (x), \phi_s^* \mathcal{L}_\xi (\ast \theta) \mathcal{L}_\xi (\ast \theta) + \int_0^t \phi_s^* \mathcal{L}_\xi (\ast \theta) dW_s \right) \right) d^n x
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^n} \left( \ast^{-1} K_0 (x), \phi_s^* \mathcal{L}_\xi (\ast \theta) \right) ds d^n x + \int_{\mathbb{R}^n} \int_0^t \left( \ast^{-1} K_0 (x), \phi_s^* \mathcal{L}_\xi (\ast \theta) \right) dW_s d^n x,
\]

for the stochastic integral. Note that owing to the boundedness of $K_0$, the smoothness of $\mathcal{L}_b (\ast \theta)$, $\mathcal{L}_\xi (\ast \theta)$, and $\mathcal{L}_\xi \mathcal{L}_\xi (\ast \theta)$ together with the fact that $\theta$ has compact support, it can be checked that the integrand satisfies the integrability assumptions for Fubini’s theorem (Theorem 2.3). □

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