On the packing measure of self-similar sets

Tuomas Orponen

Department of Mathematics and Statistics, University of Helsinki, POB 68, FI-00014 Helsinki, Finland

E-mail: tuomas.orponen@helsinki.fi

Received 1 February 2013, in final form 1 August 2013
Published 3 October 2013
Online at stacks.iop.org/Non/26/2929

Abstract

Building on a recent result of Hochman (2012 arXiv:1212.1873), we give an example of a self-similar set \( K \subset \mathbb{R} \) such that \( \dim_H K = s \in (0, 1) \) and \( P^s(K) = 0 \). This answers a question posed by Peres and Solomyak.

Mathematics Subject Classification: 28A80, 28A78, 37C45

1. Introduction

In 1998, Solomyak [8] showed that there exist self-similar sets \( K \subset \mathbb{R} \) of Hausdorff dimension \( \dim_H K = s \in (0, 1) \), which have a zero \( s \)-dimensional Hausdorff measure. In 2000, Peres et al [7] proved that such examples are in fact not all that rare: for certain natural families of self-similar sets, a large proportion of all sets in the family have this property.

For packing measure, things are different. In fact, the same theorem in [7] contained the further statement that almost all sets in these families have positive and finite \( s \)-dimensional packing measure. Until the recent breakthrough article by Hochman [2], it appeared to be difficult to determine whether this statement could be further strengthened as follows: all self-similar sets \( K \) with \( \dim_H K = s \in (0, 1) \) have positive \( s \)-dimensional packing measure, denoted by \( P^s \). The question is explicitly stated in [6, question 2.3]. In this note, building on Hochman’s paper, we answer the question in the negative by exhibiting an explicit counterexample. In fact, we find a self-similar set \( K \subset \mathbb{R} \) with similarity dimension \( s := \log 3/\log 4 \) such that \( K \) has ‘no total overlaps’ and \( P^s(K) = 0 \). Then, we employ Hochman’s result to conclude that \( K \) has Hausdorff dimension \( s \). More precisely, we use the following theorem, the proof of which is the same as [2, theorem 1.6], apart from changing some numerical values:
Theorem 1.1. Let \( K_u \subset \mathbb{R} \) be the self-similar set generated by the three similitudes
\[
\psi_0(x) := \frac{x}{4}, \quad \psi_1(x) := \frac{x + 1}{4}, \quad \psi_u(x) := \frac{x + u}{4},
\]
where \( u \in [0, 1] \). Then \( \dim_H K_u = \log 3/\log 4 \) for every \( u \in \mathbb{R} \setminus \mathbb{Q} \).

2. Self-similar sets and packing measures

A set \( K \subset \mathbb{R} \) is self-similar, if \( K \) is compact and satisfies the equation
\[
K = \bigcup_{j=1}^{q} \psi_j(K),
\]
where the mappings \( \psi_j: \mathbb{R} \to \mathbb{R}, 1 \leq j \leq q \), are contracting similitudes. This means that \( \psi_j \) has the form \( \psi_j(x) = r_j x + a_j \) for some contraction ratio \( r_j \in (-1, 1) \) and translation vector \( a_j \in \mathbb{R} \). For a given set of contracting similitudes, there is one and only one non-empty compact set \( K \) satisfying (2.1); this foundational result is due to Hutchinson [4]. The similarity dimension of a self-similar set \( K \), as in (2.1), is the unique number \( s \geq 0 \) satisfying
\[
\sum_{j=1}^{q} r_j^s = 1.
\]
The Hausdorff dimension of \( K \), denoted by \( \dim_H K \), is always bounded from above by the similarity dimension, see [1, theorem 9.3]. In general, however, determining when the two dimensions coincide poses a difficult problem. A recent breakthrough in this respect is Hochman’s paper [2], containing many satisfactory answers, including, but not limited to, theorem 1.1.

On a self-similar set \( K \), as in (2.1), there a natural self-similar measure \( \mu \) is supported, satisfying the equation
\[
\mu = \sum_{j=1}^{q} r_j \cdot \psi_j^\ast \mu.
\]
Here \( \psi_j^\ast \mu \) is the push-forward of \( \mu \) under \( \psi_j \), defined by \( \psi_j^\ast \mu(B) = \mu(\psi_j^{-1}(B)) \) for \( B \subset \mathbb{R} \).

We are concerned with the question of when \( \mu \) is (or is not) absolutely continuous with respect to the \( s \)-dimensional packing measure on \( \mathbb{R} \), denoted by \( P^s \). The definition of \( P^s \) is not directly used in the paper, but we include it here for completeness. First, one defines the \( s \)-dimensional packing pre-measure \( P^s \) by
\[
P^s(K) = \lim_{\delta \to 0} \sup \left\{ \sum_{i=1}^{\infty} d(B_i)^s : B_i \in D_\delta(K) \text{ are disjoint} \right\},
\]
where \( D_\delta(K) \) is the collection of balls centred at \( K \), with \( d(B_i) \leq \delta \). The pre-measure \( P^s \) is not countably additive, unfortunately, and this is the reason for defining
\[
\mathcal{P}^s(K) := \inf \left\{ \sum_{i=1}^{\infty} P^s(B_i) : K \subset \bigcup_{i=1}^{\infty} B_i \right\}.
\]
For more information on packing measures, see [5, section 5.10] and [1, section 3.4].

We will occasionally use the notation \( A \lesssim B \) to mean that \( A \leq C B \) for some absolute constant \( C \geq 1 \). The two-sided inequality \( A \lesssim B \lesssim A \) is abbreviated to \( A \asymp B \).
3. The construction of \( K \) and some reductions

The self-similar set \( K \), which answers \cite{6, question 2.3} negatively, is generated by the three similitudes \( \psi_0, \psi_1 \) and \( \psi_u \) as introduced in theorem 1.1. The parameter \( u \in [0, 1] \) is chosen as follows: pick natural numbers \( \lambda_j \in \{3^j, 3^j + 1\} \) in such a manner that
\[
\begin{align*}
\lambda_j &= \sum_{j=1}^{\infty} 4^{-\lambda_j} \in \mathbb{R} \setminus \mathbb{Q}.
\end{align*}
\]
(3.1)

This is certainly possible, since there are uncountably many admissible sequences \((\lambda_1, \lambda_2, \ldots)\), and no two sequences produce the same number \( u \). In fact, as pointed out by one of the referees, the choice \( \lambda_j = 3^j \) is admissible, since the base-4 expansion of \( u \) so obtained is not eventually periodic. Theorem 1.1 now implies that \( \dim_H K = \log 3 / \log 4 : s \), so it remains to prove that \( \mathcal{P}^s(K) = 0 \).

According to \cite[corollary 2.2]{7}, if \( \mathcal{P}^s(K) > 0 \), then the natural self-similar measure \( \mu \) supported on \( K \) coincides with a normalised version of the restriction of \( \mathcal{P}^s \) to \( K \). In particular, this means that \( \mu \ll \mathcal{P}^s \). Using \cite[chapter 6, exercise 5]{5}, we can then infer that
\[
\Theta_1^s(\mu, x) = \liminf_{r \to 0} \frac{\mu(B(x, r))}{(2r)^s} < \infty
\]
for \( \mu \) almost every \( x \in \mathbb{R} \). Thus, in order to show that \( \mathcal{P}^s(K) = 0 \), we need to verify the following theorem.

**Theorem 3.2.** Let \( \mu \) be the natural self-similar measure on \( \mathbb{R} \) associated with the system \( \{\psi_0, \psi_1, \psi_u\} \). Then \( \Theta_1^s(\mu, x) = \infty \) at \( \mu \) almost every point \( x \in \mathbb{R} \).

The condition \( \Theta_1^s(\mu, x) = \infty \) has a natural geometric interpretation, which will be formulated in the next lemma. First we need to introduce some notation. Let \( I_0 = {[0, 1]} \), and, for \( n \geq 1 \), define the collection of intervals
\[
I_n := \{\psi_{i_1} \circ \ldots \circ \psi_{i_n}([0, 1]) : (i_1, \ldots, i_n) \in \{0, 1, u\}^n\}
\]
for \( n \geq 1 \). Then, it is easy to verify that the natural self-similar measure \( \mu \) on \( K \) has the property that
\[
\mu(J) \geq \frac{\# \{I \in I_n : I \subset J\}}{3^n}
\]
for any interval \( J \subset \mathbb{R} \) and any \( n \in \mathbb{N} \).

**Lemma 3.3.** Let \( x \in \mathbb{R} \). Assume that there exists \( C \geq 1 \) such that
\[
\liminf_{n \to \infty} \# \{(i_1, \ldots, i_n) \in {0, 1, u}^n : \psi_{i_1} \circ \ldots \circ \psi_{i_n}(0) \in B(x, C4^{-n})\} = \infty.
\]
(3.4)

Then \( \Theta_1^s(\mu, x) = \infty \).

**Proof.** Given \( M > 0 \), we find \( n_M \in \mathbb{N} \) such that at least \( M \) distinct points of the form \( \psi_{i_1} \circ \ldots \circ \psi_{i_n}(0) \) are contained in \( B(x, C4^{-n}) \) for \( n \geq n_M \). The corresponding intervals in \( I_n \) have length \( 4^{-n} \), and are thus contained in \( B(x, (C + 1)4^{-n}) \). This shows that
\[
\frac{\mu(B(x, (C + 1)4^{-n}))}{(2(C + 1)4^{-n})^s} \geq [2(C + 1)]^{-s} \cdot \frac{M}{3^n} \cdot 4^{ns} = \frac{M}{[2(C + 1)]^s}, \quad n \geq n_M,
\]
which completes the proof. \( \square \)
4. Proof of $P^*(K) = 0$

The goal of this section is to demonstrate that the condition in (3.4) is met at $\mu$ almost every point $x \in \mathbb{R}$. Let us begin by introducing some further notation and terminology. Write $\Omega := [0, 1, u]^3$, and let $\pi : \Omega \to \text{spt} \mu = K$ be the projection

$$\pi(\omega_1, \omega_2, \ldots) := \lim_{n \to \infty} \psi_{\omega_1} \circ \cdots \circ \psi_{\omega_n}(0) = \sum_{n=1}^{\infty} \omega_n \cdot 4^{-n}.$$ 

Then $\mu = \pi^* P$, where $P$ is the equal-weights product measure on $\Omega$. Let $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, and let $i, j \in \mathbb{N}$ be indices with $i \leq j$. We say that $(\omega, j)$ is influenced by $(\omega, i)$, if there exists $k \in \mathbb{N}$ such that $i + \lambda_k < j \leq i + \lambda_{k+1}$, and

$$(\omega_i, \omega_{i+\lambda_1}, \ldots, \omega_{i+\lambda_k}) = (u, 0, \ldots, 0).$$ 

Then, define

$$S(\omega, j) := \#\{1 \leq i \leq j : (\omega, j) \text{ is influenced by } (\omega, i)\}.$$ 

The point of this definition is, as we shall see later, that (3.4) holds for all points $x = \pi(\omega) \in K$ such that

$$\lim\inf_{j \to \infty} S(\omega, j) = \infty.$$ 

(4.1)

This in mind, we need to establish the following.

**Lemma 4.2.** The equation (4.1) is valid $P$ almost surely.

**Proof.** We will be done as soon as we show that

$$P\left[ \omega : \lim\inf_{j \to \infty} S(\omega, j) \leq M \right] = 0$$

for any given $M \in \mathbb{N}$. We note that

$$\left\{ \omega : \lim\inf_{j \to \infty} S(\omega, j) \leq M \right\} \subset \lim\sup_{j \to \infty} B_{j,M},$$

where $B_{j,M}$ is the set

$$B_{j,M} = \{ \omega : S(\omega, j) \leq M \}.$$ 

Using the Borel–Cantelli lemma, we get $P[\lim\sup B_{j,M}] = 0$, if we manage to prove that

$$\sum_{j=1}^{\infty} P[B_{j,M}] < \infty.$$ 

(4.3)

Thus, (4.3) will imply lemma 4.2.

We are aiming at an upper bound for $P[B_{j,M}]$. Fix $j \geq \lambda_1$ and choose $k = k(j) \in \mathbb{N} \cup \{0\}$ so that $\lambda_{k+1} \leq j < \lambda_{k+2}$. Then divide the natural numbers between $j - \lambda_{k+1}$ and $j - 1$ into consecutive blocks $I_1, \ldots, I_N$ of length $|I_j| = [\lambda_k + 1, 2\lambda_k]$. Let $i_1, \ldots, i_N$ be the smallest numbers in these blocks. Then

$$N \geq \frac{\lambda_{k+1}}{\lambda_k} \geq 3^{\lambda_1 - 3^j} \geq 3^j.$$ 

Now, the blocks $I_n$ are disjoint, so the random variable $X_j : \Omega \to \mathbb{N}$ defined by

$$X_j(\omega) := \#\{1 \leq n \leq N : (\omega_i, \omega_{i+\lambda_1}, \ldots, \omega_{i+\lambda_k}) = (u, 0, \ldots, 0)\}$$

is independent for each $j \geq 1$. Moreover, $X_j$ is almost surely finite, and we have

$$P[X_j = \infty] = 0, \quad \text{for } j \geq 1.$$ 

Using the Borel–Cantelli lemma, we get $P[X_j = \infty] = 0$, if we manage to prove that

$$\sum_{j=1}^{\infty} P[X_j = \infty] < \infty.$$ 

Thus, (4.3) will imply lemma 4.2.
has distribution $X_j \sim \text{Bin}(N, p_j)$, where the ‘success probability’ $p_j$ equals

$$p_j := P[(\omega : (\omega_0, \omega_{\lambda_1}, \ldots, \omega_{\lambda_k})) = (u, 0, \ldots, 0)] = 3^{-k-1}.$$

We claim that $B_{j,M} \subset \{X_j \leq M\}$. Indeed, suppose that $\omega \in B_{j,M}$. Then, in particular, there are at most $M$ among the numbers $i_n$, $1 \leq n \leq N$, such that $(\omega, j)$ is influenced by $(\omega, i_n)$.

For the rest of the numbers $i_n$ either

$$(\omega_0, \omega_{\lambda_1}, \ldots, \omega_{\lambda_k}) \neq (u, 0, \ldots, 0) \quad (4.4)$$

or $i_n + \lambda_{k+1} < j$, or $j \leq i_n + \lambda_k$, by definition of the notion of ‘influence’. The latter two possibilities are absurd, since $i_n \geq j - \lambda_{k+1}$ and $i_n + \lambda_k \in \{1, \ldots, j - 1\}$. Thus, (4.4) must hold for all but at most $M$ numbers $i_n$, which specifically means that $X_j(\omega) \leq M$.

The probability $P[X_j \leq M]$ can be estimated by a standard tail bound for the binomial distribution (or see Hoeffding’s inequality [3]):

$$P[X_j \leq M] \leq \exp\left(-\frac{2(N3^{-k-1} - M)^2}{N}\right) \leq C_M \exp(-3^{j-2k}),$$

so that finally

$$\sum_{j=1}^{\infty} P[B_{j,M}] \leq \sum_{j=1}^{\lambda_1-1} P[B_{j,M}] + \sum_{k=0}^{\infty} \sum_{\lambda_{k+1} \leq j < \lambda_{k+2}} P[X_j \leq M]
\leq \lambda_1 + C_M \sum_{k=0}^{\infty} 3^{3k^2} \exp(-3^{j-2k}) < \infty.$$

This proves lemma 4.2.

Finally, it is time to check that our definition of ‘influence’ is useful:

**Lemma 4.5.** Assume that $\omega \in \Omega$ satisfies (4.1). Then $\pi(\omega) \in K$ satisfies (3.4).

**Proof.** Fix $\omega \in \Omega$, $j > 1$, and write $x = \pi(\omega)$. Our task is to find many sequences $(i_1, \ldots, i_j) \in \{0, 1, u\}^j$ such that

$$\psi_{i_1} \circ \ldots \circ \psi_{i_j}(0) = \sum_{n=1}^{j} i_n \cdot 4^{-n} \in B(x, C4^{-j}),$$

where $C \geq 1$ is some absolute constant. One such sequence is always obtained by taking $(i_1, \ldots, i_j) = (\omega_1, \ldots, \omega_j)$, since

$$\sum_{n=1}^{\infty} \omega_n \cdot 4^{-n} = x = \sum_{n=1}^{\infty} \omega_n \cdot 4^{-n} \asymp 4^{-j}. \quad (4.6)$$

The following describes how we find other sequences. Suppose that $(\omega, j)$ is influenced by $(\omega, i)$ for some $1 \leq i \leq j$. Once again, this means that $i + \lambda_k < j \leq i + \lambda_{k+1}$ and $(\omega_{\lambda_1}, \omega_{\lambda_1+\lambda_k}, \ldots, \omega_{\lambda_k+\lambda_k}) = (u, 0, \ldots, 0)$. Consider the modified sequence $\tilde{\omega}$, which is otherwise identical with $\omega$, except that the symbol $u$ at index $i$ is replaced by 0, and the zeroes at the indices $i + \lambda_1, \ldots, i + \lambda_k$ are replaced by 1. Then, using the definition of $u$, we have

$$\left| \sum_{n=1}^{j} \omega_n \cdot 4^{-n} - \sum_{n=1}^{j} \tilde{\omega}_n \cdot 4^{-n} \right| = |(u - 0) \cdot 4^{-i} + (0 - 1) \cdot 4^{-(i+\lambda_1)} + \ldots + (0 - 1) \cdot 4^{-(i+\lambda_k)}|$$

$$= \left| \sum_{n=\lambda_k+1}^{\infty} 4^{-i-\lambda_{\lambda_k}} - \sum_{n=1}^{\lambda_k} 4^{-i-\lambda_{\lambda_k}} \right| = \sum_{n=\lambda_k+1}^{\infty} 4^{-i-\lambda_{\lambda_k}} \lesssim 4^{-j},$$

and

$$\sum_{n=1}^{\infty} \omega_n \cdot 4^{-n} = x = \sum_{n=1}^{\infty} \omega_n \cdot 4^{-n} \asymp 4^{-j}. \quad (4.6)$$
since \(i + \lambda_{k+1} \geq j\). It follows from this and (4.6) that \(\tilde{\psi}_{\omega_0} \circ \cdots \circ \tilde{\psi}_{\omega_j}(0) \in B(x, C4^{-j})\) for some absolute constant \(C \geq 1\). Thus, for each pair \((\omega, i)\) influencing the pair \((\omega, j)\), the construction just described produces a sequence \((i_1, \ldots, i_j)\) with \(\tilde{\psi}_{i_1} \circ \cdots \circ \tilde{\psi}_{i_j}(0) \in B(x, C4^{-j})\).

Moreover, no sequence \((i_1, \ldots, i_j)\) is obtained twice in this manner. For if \((\omega, j)\) is influenced by both \((\omega, i)\) and \((\omega, i')\) with \(i < i'\), say, then

- \(\omega_i = u\), by definition of \((\omega, i)\) influencing \((\omega, j)\), and
- both \(i\) and \(i'\) give rise to modified sequences \(\tilde{\omega}\) and \(\tilde{\omega}'\), as above.

Recalling how these sequences were constructed, we see that \(\tilde{\omega}_i = 0\). On the other hand, \(\tilde{\omega}'\) coincides with \(\omega\) for all indices smaller than \(i'\), so in particular \(\tilde{\omega}'_i = \omega_i = u\). This means that \(\tilde{\omega} \neq \tilde{\omega}'\) and completes the proof of the claim. \(\square\)

To conclude the proof of \(P^s(K) = 0\), we note that, by lemma 4.5, the set \(G = \{x : (3.4)\) holds at \(x\}\) contains the \(\pi\)-images of all those sequences \(\omega \in \Omega\) where (4.1) holds. The set consisting of such sequences has full \(P\)-measure according to lemma 4.2. Hence, the equation \(\mu = \pi^\sharp P\) implies full \(\mu\)-measure for \(G\).

Acknowledgments

The author is grateful to the referees for their constructive suggestions.

The research was supported by the Finnish National Doctoral Programme in Mathematics and its Applications.

References

[1] Falconer K 1990 Fractal Geometry: Mathematical Foundations and Applications (New York: Wiley)
[2] Hochman M 2012 On self-similar sets with overlaps and inverse theorems for entropy arXiv:1212.1873
[3] Hoeffding W 1963 Probability inequalities for sums of bounded random variables J. Am. Stat. Assoc. 58 13–30
[4] Hutchinson J 1981 Fractals and self-similarity Indiana Univ. Math. J. 30 713–47
[5] Mattila P 1995 Geometry of Sets and Measures in Euclidean Spaces (Cambridge: Cambridge University Press)
[6] Peres Y and Solomyak B 2000 Problems on self-similar sets and self-affine sets: an update Prog. Probab. 46 95–106
[7] Peres Y, Simon K and Solomyak B 2000 Self-similar sets of zero Hausdorff measure and positive packing measure Israel J. Math. 117 353–79
[8] Solomyak B 1998 Measure and dimension for some fractal families Math. Proc. Camb. Phil. Soc. 124 531–46