Spectral C*-categories and Fell bundles with path-lifting

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Abstract

Following Crane’s suggestion that categorification should be of fundamental importance in quantising gravity, we show that finite dimensional even $S^*$-real spectral triples over $\mathbb{C}$ are already nothing more than full C*-categories together with a self-adjoint section of their domain maps, while the latter are equivalent to unital saturated Fell bundles over pair groupoids equipped with a path-lifting operator given by a normaliser. Interpretations can be made in the direction of quantum Higgs gravity. These geometries are automatically quantum geometries and we reconstruct the classical limit, that is, general relativity on a Riemannian spin manifold.

1 Introduction

Complementary to traditional studies towards quantum gravity, Connes’ approach to unification is to first geometrise the other three fundamental forces rather than to directly quantise the gravitational (geometrical) force. Connes’ and Chamseddine’s spectral action principle [CC] [C3] is a classical theory of all four forces on a manifold consisting of a product of the conventional 4-space and a finite noncommutative internal space including their theory of classical gravity (or isospectral [CC] general relativity) on both factors of the product space (the equivalence principle on the noncommutative internal space was studied by Schücker, [S2], [SI].)

In a quantum (spectral) gravity [C1] on the discrete component, the Dirac operator would be an observable for gravity with eigenvalues the fermion masses and the fundamental excitations would be holonomies in internal space.

Following Crane’s suggestion that categorification is an important feature in quantisation and that a replacement of the traditional analysis of the continuum might be replaced by topics in category theory [LC1] [LC2] [LC3], we infer for those two reasons that a categorification program for noncommutative geometries might be interesting. To this end, we explore here a connection between finite spectral triples, C*-categories with an additional item of geometrical data and Fell bundles with path-lifting. Our inspiration comes from the path integral approach, while Aastrup, Grimstrup and Nest [AG1], [AG2], [AGN1], [AGN2] have already constructed a quantisation scheme for the product space using loop quantum gravity techniques.

There are already many features of quantum mechanics in the noncommutative standard model: the Dirac operator is an unbounded operator on a Hilbert space, the fermion sector of the action resembles an expectation value, the spectral action is a topological invariant and resembles a path integral and as we see below, finite spectral triples have a natural categorical representation.

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1 reminiscent of the Kaluza-Klein internal space

2 pointed out by Grimstrup [AG1]
In the formalism we construct, both the algebra representing a deformed phase space and the algebra of functions on the configuration “space” are noncommutative and arise from the sectional algebra of a Banach bundle. Some of the concepts are related to deformation quantisation theories especially the tangent groupoid quantisation and we draw from geometrical notions inherited from the bundle structure. In particular, we show that spectral $\mathbb{C}^*$-categories are equivalent to a certain class of Fell bundle with a notion of path-lifting analogous to a path-lifting in a vector bundle.

Other points of view on categorification of spectral triples include \[MZ, BM, BCL6, BCL7\]. In each of these cases, spectral triples are objects and morphisms are constructed between them. The approach here is different but complementary: we focus on the internal categorical structure of spectral triples and we present some first steps towards a quantisation program for Connes’ non-commutative general relativity.

As well as being natural, viewing spectral triples as categories in themselves (a kind of “inner categorification”) might provide a tool for algebraic generalisations of the path integral formulation of quantum gravity wherein the (space-time) manifold is the entire category and the objects are just its boundaries, so that all the dynamics are formulated by morphisms within one category. Nevertheless, other topics in categorification of spectral triples \[BCL1, BCL3\] also have algebraic quantum gravity motivations and modular spectral triples and categorical spectral triples have a strong intersection with our approach here: a spectral $\mathbb{C}^*$-category aims to interpret the spectrum of the Dirac operator in terms of the generating operator of a non-commutative geodesic \[C4\] and modular spectral triples \[BCL6\] and categorical spectral geometries \[BCL7\] provide a very similar point of view. We aim to incorporate into their description the structure of a (Banach bundle) $\mathbb{C}^*$-bundle reflecting the bundle structure of a Riemannian spin manifold.

Definition 1.1. \[BCL7\] In short, a categorical spectral geometry is a triple $(\mathcal{C}, \mathcal{H}, D)$ given by a pre-$\mathbb{C}^*$-algebra $\mathcal{C}$, a Hilbert $\mathbb{C}^*$-module $\mathcal{H}$, a (object bijective) functor $\rho : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ and a generator $D$ of a unitary 1-parameter group on $\mathcal{H}$ respecting $\mathcal{B}(\mathcal{H})$ with smoothness condition: $[D, \rho(x)]$ is extendable to a bounded operator on $\mathcal{H}$.

The definitions we introduce below of spectral $\mathbb{C}^*$-category and Fell bundle triple provide generalisations and clarifications of Fell bundle geometries \[RM2\], which are slightly more elaborate containing the extra axioms (reality and Poincaré duality) needed to make them into real spectral triples. Viewing spectral triples as Fell bundle geometries allowed us to make predictions about the fermion mass matrix - up to the caveat that one does not have available all the technical details for the final non-commutative standard model. Fell bundle geometries incorporate several desirable features absent from real spectral triples including: a mass matrix approaching that of the empirical fermion mass matrix, a bundle structure that reflects that of a Riemannian spin manifold, a clue to a clearer meaning of orientability in non-commutative geometry and the availability of new connections between physics and non-commutative geometry such as an application of the tangent groupoid quantisation.

1.1 Categorical switch-of-focus

Isham, Crane and others explain that future progress in quantum gravity must involve mathematical progress in the understanding of space-time on the smallest scales \[I, LC5\]. Here we explain two points of view on quantum space-times that underpin the ideas made precise in this paper using algebraic techniques.

Although it is traditional to pass to a non-commutative generalisation of a topological space by invoking the Gelfand-Naimark theorem and viewing non-commutative algebras as generalised algebras of functions, it is also true that $\mathcal{C}_0(X)$ is the algebra of sections vanishing at infinity of a line bundle over $X$, so there is an argument to think of non-commutative algebras as generalised algebras of sections, where the fuzzy points hide in the fibres over (a tangible) $X$ instead of in a fuzzy space “$X$”.

Let $(\mathcal{E}, \pi, \mathcal{G})$ be a Fell line bundle over a pair groupoid $\mathcal{G} = M \times M$ such that $\mathcal{G}$ is interpreted as the deformed tangent bundle over a simply connected compact manifold $M$. Each fibre $\mathcal{C}$ is associated to a point in $M$. Call the enveloping algebra $\mathbb{C}^*(\mathcal{E}^0)$ of $(\mathcal{E}^0, \pi, \mathcal{G}_0)$,
the configuration algebra and call the algebra associated to \( G \), the observable or the coordinate algebra \( C^*(M \times M) = C^*(E) \). Switch the focus away from the points in \( M \) to the space of fibres of \( E^0 \), so that \( E \) becomes a generalised deformed tangent bundle.

Now generalise the above example so that instead of \( C \) the fibres of \( E^0 \) are simple matrix algebras over \( C \). Fibres are not necessarily isomorphic to each other but they are Morita equivalent. Now, \( M \) has completely lost its interpretation as the configuration space, which is now a virtual space and is formalised only through the space of fibres of \( E^0 \). A categorical switch-of-focus (also appearing in [RM2]) is a category in which objects are given by the fibres over the objects of \( G \) whose morphisms are elements of fibres over the morphisms \( G \) of \( G \). Each object and each arbitrary disjoint union of objects models a space-time region. If \( E \) is saturated, then \( G \) is a “weakened” form of groupoid (that is, a groupoid where the inverse and composition and unit axioms hold only up to isomorphism). \( \text{Ob}(G) \) is a discrete Grothendieck site with morphisms given by unions of fibres. This categorical structure is closely related to a quantum geometry [LC5, LC3] and there is a need to reformulate the geometrical data appearing in infinitesimal form ([H]), such as a connection on a vector bundle.

The following are three instances in which information of an infinitesimal type can be replaced by finite data through integration: (i) Barrett and others have shown that given just the holonomy of a bundle, they were able to reconstruct the entire bundle and connection from that data ([JB2], [CP], [L]). (ii) A choice of connection on a vector bundle over \( M \) is equivalent to a choice of representation of the fundamental groupoid \( \Pi_1(M) \). (iii) The tangent bundle \( TM \) over \( M \) is integrated to the \( \Pi_1(M) \) (in this case \( M \) is simply connected, so \( \Pi_1(M) \) is the pair groupoid over \( M \)) where infinitesimal data given by the tangent vectors is replaced by a set of flows. So when an integration is performed, the following two things happen: a deformation or non-localisation of the geometry and a categorification of the geometry. This reflects Ehresmann’s point of view that geometry is the study of differentiable categories and it implies that the following are almost synonyms: deforming to a non-commutative algebra, quantisation and inner categorification.

Below we demonstrate that a finite spectral triple is manifestly already a category and we argue that this means that the non-commutative standard model is not completely classical.

## 2 Preliminaries

Because we will be drawing on material from several mathematical disciplines (and arguing that certain of their basic ideas can be unified through concepts from mathematical physics), readers may be familiar with some topics and not others. Therefore we briefly introduce some of the ideas, recall the main definitions and provide references.

For an introduction to category theory the reader may wish to consult [ML] or refer to several introductory sources available online such as [O].

**Definition 2.1.** [GLR, M] A \( C^* \)-category is a category \( C \) in which for all objects \( A, B \in \text{Ob}_C \), the homsets \( C_{AB} := \text{Hom}_C(B, A) \) are complex Banach spaces, the compositions are bilinear maps such that \( \| xy \| \leq \| x \| \cdot \| y \| \) for all \( x \in C_{AB}, y \in C_{BC} \) and there is an involutive antilinear contravariant functor \( * : C \to C \) preserving objects such that \( \| x^* x \| = \| x \|^2 \) and such that \( x^* x \) is a positive element of the \( C^* \)-algebra \( C_{AA} \) for every \( x \in C_{BA} \) (i.e. \( x^* x = y^* y \) for some \( y \in C_{AA} \)).

Note that each \( C_{AA} \) is a \( C^* \)-algebra and also that it possesses a unit element due to the identity axiom in category theory.

**Example 2.2.** [GLR] The category of Hilbert spaces and bounded linear maps.

**Example 2.3.** [GLR] The category \( \text{Rep}(A) \) of representations of a \( C^* \)-algebra \( A \) on a Hilbert space and intertwinning operators.

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3If \( E \) is saturated, then \( G \) is a “weakened” form of groupoid (that is, a groupoid where the inverse and composition and unit axioms hold only up to isomorphism). \( \text{Ob}(G) \) is a discrete Grothendieck site with morphisms given by unions of fibres. This categorical structure is closely related to a quantum geometry [LC5, LC3] but these details may be explored elsewhere.
2.1 Groupoids

First we recall some notation conventions and basic facts. A groupoid \( \mathcal{G} \) is a small category in which all morphisms or arrows \( g \) are invertible and a group is a groupoid with just one object. The set of objects \( \text{Ob} \), which is often referred to as unit space, is denoted \( G_0 \) or \( \mathcal{G}^0 \). The domain (or source) and range (or target) maps \((d, r) : \mathcal{G} \to G_0 \) control the partial composition rule. To each object in \( G_0 \) there belongs an identity or unit and the set of arrows \( h \) satisfying \( h = g^*g \) can be identified with the set of units, which due to the "arrows-only" picture of categories \([\text{ML}]\), is identified with \( G_0 \). In this context, the set of arrows \( \mathcal{G}^1 \) is identified with \( \mathcal{G} \). A groupoid may be equipped with a topology \( \Omega(\mathcal{G}) \). A global (resp. local) bisectiion of a topological groupoid is a section \( x : \mathcal{G}_0 \to \mathcal{G} \) of \( d \) such that \( r \circ x : \mathcal{G}_0 \to \mathcal{G}_0 \) is a (resp. partial) homeomorphism. Let \( I(\mathcal{G}) \) be the inverse semigroup of local (open) bisections of \( \mathcal{G} \) and let \( \text{Bis}(\mathcal{G}) \) denote the group of global bisections. An \( \text{étalement} \) groupoid is a topological groupoid in which \( d \) is a local homeomorphism. An \( \text{étalement} \) groupoid has a covering obtained from the images of open bisections and the topology generated by them is an involutive quantale \( \Omega(\mathcal{G}) \) called an inverse quantal frame. See \([\text{PR}]\) for a study of inverse quantal frames. An abstract inverse semigroup can be be represented by actions of partial or pseudo homeomorphisms of \( M \), see \([\text{MR}]\) for a study of representations of inverse semigroups.

A smooth or Lie groupoid is a groupoid in which both \( \mathcal{G}_0 \) and \( \mathcal{G} \) are manifolds, \( d \) and \( r \) are submersions and all category operations are smooth. For \( G \) a Lie groupoid over a space \( \mathcal{G}_0 = M \), an identification is often made of an arrow as a path (up to a certain equivalence) of a point particle in a topological space and then a bisection can be used to describe the set of paths in \( M \) transcribed by a system of particles. \( \Pi_1(M) \) denotes the fundamental groupoid over \( M \), which is the set of homotopy equivalence classes of paths on \( M \). In the case that \( M \) is simply connected, \( \Pi_1(M) = M \times M \), the groupoid over \( M \). A principal groupoid is one in which \((r, d) : \mathcal{G} \to \mathcal{G}_0 \times \mathcal{G}_0 \) is injective, in other words it is an equivalence relation on \( \mathcal{G}_0 \). The pair groupoid \( \mathcal{G} = M \times M \) is obviously principal as it is the maximal equivalence relation on \( \mathcal{G}_0 \). The bisections of a pair groupoid are in one to one correspondence to diffeomorphisms of the base.

The tangent groupoid quantisation

Let \( M \) be an \( n \)-dimensional manifold. Recall that the tangent bundle \( TM \) and the cotangent bundle \( T^*M \) over \( M \) are both Lie groupoids over \( M \) locally diffeomorphic to \( \mathbb{R}^n \times \mathbb{R}^n \). As Lie algebroids, \( TM \) and \( T^*M \) both integrate to the groupoid \( M \times M \) \([\text{CF}]\).

Here is a very brief overview of the idea of the tangent groupoid. For the details see \([\text{CI}]\). This is a deformation type quantisation procedure through asymptotic morphisms. It involves a particle system on a manifold \( M \). The cotangent space of \( M \) captures the phase space and its \( C^* \)-completion of sections \( C^*(T^*M) \) is conventionally and classically the algebra of observables. Deformation quantisation programs replace it with a non-commutative algebra. The tangent groupoid is given by

\[ \mathcal{G}M = TM \times \{0\} \cup M \times M \times (0, 1], \]

where \( h \) is a continuous parameter taking values in an interval of the real line \([0, 1]\). The classical limit is obtained as \( h \) ‘goes to zero’. The \( C^* \)-algebra of the tangent groupoid is the algebra of continuous sections vanishing at infinity of the union of a norm continuous field of \( C^* \)-algebras \( A_h \) over the space of \( h \). The asymptotic morphism is a morphism from the algebra \( A_0 \) over \( h = 0 \) to any of those over \( h \neq 0 \). At \( h = 1 \) we have the \( C^* \)-completion \( C^*(M \times M) \) of the non-commutative convolution algebra of the pair groupoid over \( M \), whose elements extend to the compact operators on a certain Hilbert space, \( L^2(M) \). Again, for details see \([\text{CI}]\).

2.2 Banach bundles, \( C^* \)-bundles and Fell bundles

Banach bundles are different from fibre bundles in that the absence of transition functions allows them to be non-locally trivial. Following J. Fell-R. Doran \([\text{FD}]\) Section I.13 or N.
Weaver [MW, Chapter 9.1] we have the following definition of Banach bundle.

**Definition 2.4.** A Banach bundle \((\mathcal{E}, \pi, \mathcal{X})\) is a surjective continuous open map \(\pi : \mathcal{E} \to \mathcal{X}\) such that \(\forall x \in \mathcal{X}\) the fibre \(\mathcal{E}_x := \pi^{-1}(x)\) is a complex Banach space and satisfies the following additional conditions:

- the operation of addition \(+: \mathcal{E} \times \mathcal{E} \to \mathcal{E}\) is continuous on the set \(\mathcal{E} \times \mathcal{E} := \{(e_1, e_2) \in \mathcal{E} \times \mathcal{E} \mid \pi(e_1) = \pi(e_2)\}\),
- the operation of multiplication by scalars: \(\mathbb{C} \times \mathcal{E} \to \mathcal{E}\) is continuous,
- the norm \(\|\cdot\| : \mathcal{E} \to \mathbb{R}\) is continuous,
- for all \(x_0 \in \mathcal{X}\), the family \(U^0_{x_0} = \{e \in \mathcal{E} \mid \|e\| < \epsilon, \pi(e) \in O\}\) where \(O \subset \mathcal{X}\) is an open set containing \(x_0 \in \mathcal{X}\) and \(\epsilon > 0\) is a fundamental system of neighbourhoods of \(0 \in \mathcal{E}_{x_0}\).

For a Hilbert bundle we require that for all \(x \in \mathcal{X}\), the fibre \(\mathcal{E}_x\) is a Hilbert space.

**Definition 2.5.** A \(C^*-\)bundle is a Banach bundle \(\mathcal{E}^0\) in which each fibre is a \(C^*-\)algebra and there is a \(C^*-\)completion \(C^*(\mathcal{E}^0)\) of the algebra of compactly supported sections of \(\mathcal{E}^0\).

A Fell bundle over a topological groupoid (for which we follow [K2]) is a generalisation of a Fell bundle over a group ([FD]), and also a generalisation of a \(C^*-\)bundle over a topological space:-

**Definition 2.6.** [K2] A Banach bundle over a groupoid \(p : \mathcal{E} \to \mathcal{G}\) is said to be a Fell bundle if there is a continuous multiplication \(\mathcal{E}^2 \to \mathcal{E}\), where

\[
\mathcal{E}^2 = \{(e_1, e_2) \in \mathcal{E} \times \mathcal{E} \mid (p(e_1), p(e_2)) \in \mathcal{G}^2\},
\]

(where \(\mathcal{G}^2\) denotes the space of composable pairs of elements of \(\mathcal{G}\)) and an involution \(e \mapsto e^*\) that satisfy the following axioms.

1. \(p(e_1 e_2) = p(e_1) p(e_2)\) \(\forall (e_1, e_2) \in \mathcal{E}^2\);
2. The induced map \(\mathcal{E}_{g_1} \times \mathcal{E}_{g_2} \to \mathcal{E}_{g_1 g_2}\), \((e_1, e_2) \mapsto e_1 e_2\) is bilinear \(\forall (g_1, g_2) \in \mathcal{G}^2\);
3. \((e_1 e_2) e_3 = e_1 (e_2 e_3)\) whenever the multiplication is defined;
4. \(\|e_1 e_2\| \leq \|e_1\| \cdot \|e_2\|\), \(\forall (e_1, e_2) \in \mathcal{E}^2\);
5. \(p(e^*) = p(e)^*\), \(\forall e \in \mathcal{E}\);
6. The induced map \(\mathcal{E}_g \to \mathcal{E}_g^*\), \(e \mapsto e^*\) is conjugate linear for all \(g \in \mathcal{G}\);
7. \(e^{**} = e\), \(\forall e \in \mathcal{E}\);
8. \((e_1 e_2)^* = e_2^* e_1^*\), \(\forall (e_1, e_2) \in \mathcal{E}^2\);
9. \(\|e^* e\| = \|e\|^2\), \(\forall e \in \mathcal{E}\);
10. \(\forall e \in \mathcal{E}\), \(e^* e \geq 0\) as element of the \(C^*-\)algebra \(\mathcal{E}_{p(e^*)}\).

The restriction of a Fell bundle to \(\mathcal{G}_0\) is a \(C^*-\)bundle \(\mathcal{E}^0\) and its algebra \(C^*(\mathcal{E}^0)\) is called the diagonal algebra of the Fell bundle.

A Fell bundle is said to be saturated if \(\mathcal{E}_{g_1} \cdot \mathcal{E}_{g_2}\) is total in \(\mathcal{E}_{g_1 g_2}\) for all \((g_1, g_2) \in \mathcal{G}^2\) from which it follows that the fibres are imprimitivity or Morita equivalence bimodules. A unital Fell bundle is one in which every \(C^*-\)algebra has an identity element. A Fell line bundle is a Fell bundle with fibre \(\mathbb{C}\).

A saturated unital Fell bundle over a maximal equivalence relation \(\mathcal{G} = M \times M\) or pair groupoid on a topological space \(M\) is equivalent to a full \(C^*-\)category \(\mathcal{C}\) fibred over the maximal equivalence relation on its object space \(\text{Ob}(\mathcal{C})\). (See [BCL].)
Example 2.7. Consider a saturated Fell bundle $E$ over the pair groupoid on two objects $G \ni g, g^*, gg^*, g^*g$. Since the fibres are imprimitivity or Morita equivalence bimodules the $C^*$-algebra of sections or ‘enveloping’ algebra of the Fell bundle $C^*(E)$ is a Morita equivalence ‘linking algebra’. For linking algebras and related results on Morita equivalence see [BGR, J]. The fibre space of $E$ defines a Morita category $C$, a full small $C^*$-category. The objects $\text{Ob}(C)$ are a Morita equivalence class of $C^*$-algebras. This can be thought of as a weakened form of a groupoid since $E_g \otimes E_g^* \cong E_{gg^*}$ and $E_g \otimes i(e_g) \cong E_g$.

The object $C^*$-algebras are denoted $E_{gg^*}$ and $E_{g^*g}$ and using lower case for their elements, and element $e$ of the linking algebra is given by:

$$e = \begin{pmatrix} e_{gg^*} & e_g \\ e_{g^*} & e_{g^*g} \end{pmatrix}$$

(1)

Example 2.8. The convolution algebra of the principal groupoid $G = M \times M$ over a discrete space $M$ consisting of $n$ points is given by the matrix algebra $M_n(C)$. $M_n(C)$ is the the sectional algebra of the complex line bundle $E$ over the groupoid $G$ and is the linking algebra of the Morita equivalence bimodules whose elements are the morphisms in the category $E$.

The following definition and example is an extract from [K1].

Definition 2.9. Suppose that $\mathcal{A}$ is a $C^*$-subalgebra of a $C^*$-algebra $\mathcal{B}$. An element $b \in \mathcal{B}$ is said to normalise $\mathcal{A}$ if

- $b^*A \subset \mathcal{A}$
- $bA^* \subset \mathcal{A}$

The collection of all such normalisers is denoted $N(\mathcal{A})$. Evidently, $\mathcal{A} \subset N(\mathcal{A})$; further, $N(\mathcal{A})$ is closed under multiplication and taking adjoints. A normaliser, $b \in N(\mathcal{A})$ is said to be free if $a^2 = 0$. The collection of free normalisers is denoted $N_f(\mathcal{A})$.

Example 2.10. Let $B = M_n(C)$, the algebra of complex $n$ by $n$ matrices. Choose a set of matrix units, $\{e_{ij} : 1 \leq i, j \leq n\}$ (one has $e_{ik} = e_{ij}e_{jk}$ and $e_{ij}^* = e_{ji}$), and let $\mathcal{A}$ denote the diagonal subalgebra (viz, $\mathcal{A}$ is spanned by the $e_{ii}$s). Then $a = \sum \lambda_{ij}e_{ij}$ normalises $\mathcal{A}$ if and only if for each $i$, $\lambda_{ij} \neq 0$ for at most one $j$, and for each $j$, $\lambda_{ij} \neq 0$ for at most one $i$ (i.e., at most one entry is non-zero in each row and column). If $i \neq j$, $e_{ij} \in N_f(\mathcal{A})$. Let $P : \mathcal{B} \to \mathcal{A}$ be given by:

$$P(a) = \sum e_{ii}aa_{ii}.$$  

This defines a faithful conditional expectation for which:

$$\ker P = \text{span}N_f(\mathcal{A}).$$

2.3 Real spectral triples

In short, a real spectral triple [C1] is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is a pre-$C^*$-algebra with a faithful representation on a Hilbert space $H$, the Dirac operator $\mathcal{D}$ is a self-adjoint, unbounded operator on $\mathcal{H}$ with compact resolvent such that $[\mathcal{D}, a]$ is a bounded operator for all $a \in \mathcal{A}$. $\mathcal{H}$ is a left $\mathcal{A}$ or $\mathcal{D}^{opp}$-module (finite projective) with a real structure $J$ and $\mathbb{Z}_2$-grading $\chi$. A real structure on a spectral triple [C2] is given by an antiunitary operator $J$ on $H$ such that $J^2 = \pm 1$, $\mathcal{D}J = \pm JD$, $[a, b^{opp}] = 0$, $[[\mathcal{D}, a], b^{opp}] = 0$ where $b^{opp} = Jb^*J^*$ for all $b \in \mathcal{A}$ is an element of $\mathcal{D}^{opp}$. Real spectral triples are Connes’s noncommutative generalisations of Riemannian spin manifolds, to which purpose $\mathcal{A}$, $\mathcal{H}$, $\mathcal{D}$, $J$, $\chi$ and $\chi$ must satisfy a set of 7 axioms detailed in [C3] such as Poincaré duality and orientability. These axioms were designed to be adapted over time.

Connes’s reconstruction theorem establishes that commutative real spectral triples are equivalent to Riemannian spin manifolds, so a commutative Riemannian spin manifold is just a special case of the set of all not-necessarily-commutative Riemannian spin manifolds.
Fluctuations of the metric

In [S1] Schütter explains that while Einstein derived general relativity from Riemannian geometry, Connes extended this to noncommutative geometry and as a result the other three fundamental forces emerged with the gauge and Higgs fields as fluctuations of the metric. The ‘almost commutative’ spectral triple of the noncommutative standard model includes a commutative space-time factor and a finite noncommutative factor. The latter is reminiscent of Kaluza-Klein internal space, but in this case it is 0-dimensional. Connes’ encodes the metric data in the Dirac operator [C3] and his procedure for unification starts by describing the diffeomorphisms giving rise to the equivalence principle as the spinor lift of the automorphisms of the algebra transforming the Dirac operator. The arising space of fluctuated Dirac operators $D^f$ ([S2]) defines the configuration space of the spectral action:

$$D^f = \sum_{\text{finite}} r_j L(\sigma_j)DL(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in \text{Aut}(A)$$

where $L$ is the double valued lift of the automorphism group to the spinors. For the calculation of the spectral standard model action see [C]. The result is the general form of the Dirac operator with arbitrary curvature and torsion:

$$D = \sum_i c_i \left( \frac{\partial}{\partial x_i} + \omega_i \right)$$

The ‘almost commutative’ algebra of the noncommutative standard model comprises two factors, $A = C^\infty(M) \otimes A_{\text{finite}}$ and ‘fluctuating’ the Dirac operator that probes $M$ in the finite space algebra $A_F$, Connes obtains the standard model gauge fields. In this noncommutative case, this means replacing the spinor lift in the above formula with the unitaries of $A_{\text{finite}}$. Finally, fluctuating the Dirac operator $D_{\text{finite}}$ that probes the finite space in $A_{\text{finite}}$ gives rise to the Higgs field. This gives the Higgs field an interpretation as a gravitational connection on an additional ‘dimension’.

3 Finite spectral triples

The following definition is an interpretation for the finite complex case of an even spectral triple [C1] incorporating the fact ([H]) that an inner derivation $\delta_D : a \mapsto [D, a]$ from a Banach algebra $A$ maps into a module $B$ over $A$ ($a \in A$, $[D, a] \in B$).

**Definition 3.1.** A finite even spectral triple $(A, H, D_f, \gamma)$ over $\mathbb{C}$ consists of a finite dimensional complex algebra $A = \bigoplus_i M_{n_i}(\mathbb{C})$, $i = \{1, \ldots, p\}$ and an enveloping algebra $B = M_{n_{\alpha}}(\mathbb{C})$, both represented (faithfully unless stated otherwise) on a Hilbert space $H$ isomorphic to $\mathbb{C}^{n_\alpha}$, $m = \sum_{i=1}^p$. The triple includes a self-adjoint operator $D_f$ giving an inner derivation from $A$ into $B$ such that $D_f \gamma + \gamma D_f = 0$ where $\gamma$ is a $\mathbb{Z}/2$-grading operator satisfying $\gamma^* = \gamma$, $\gamma^2 = 1$, and with $\gamma^* - \gamma = 0$ for all $a \in A$.

Observe that in examples, $p$ tends to be the number 4 since fermions are predicated by chirality and parity (they are always labelled either right or left and whether they are particles or anti-particles).

It is clear that the set of summands of $A$ fall into a Morita equivalence class of non-commutative simple algebras.

The following real structure allowed Connes to define the analogue of a Riemannian spin manifold as a real spectral triple together with a set of 7 axioms (see [C3]).

**Definition 3.2.** [C2] A real spectral triple is a spectral triple $(A, H, D, \gamma, J)$ (defined in [C1] [C3]) in which the Hilbert space carries a real structure, making it into an $A - A^{\text{opp}}$ bimodule, which is an anti-linear operator $J$ satisfying $JD = JD$, $J^2 = 1$, $J = J^*$, $J^{-1}$, $JaJ = b$, $[J, \gamma] = 0 \forall a$ with $b \in A^{\text{opp}}$,

$$J\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ -\psi_1 \end{pmatrix}, \quad (\psi_1, \psi_2) \in H = H_1 \oplus H_2$$
where the bar indicates complex conjugation.

**Definition 3.3.** Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be a finite dimensional Hilbert space with a further chiral decomposition or $\mathbb{Z}/2$-grading $\mathcal{H}_1 = \mathcal{H}_L \oplus \mathcal{H}_R, \mathcal{H}_2 = \mathcal{H}_{2L} \oplus \mathcal{H}_{2R}$. A finite real spectral triple is a finite spectral triple as defined above with a real structure and satisfying the Poincaré duality condition: $\dim \mathcal{H}_R - \dim \mathcal{H}_L \neq 0$.

**Remarks 3.4.**
1. The form the chirality grading operator $\gamma$ takes depends on the KO-dimension (in physical examples this translates into for example whether the signature is Lorentzian or Euclidean). In the definition we used above we have $J\gamma = \gamma J$ which corresponds to the Euclidean signature. In [JB1] and [C5] for example the chirality operator is defined by a matrix solving the constraints $\gamma^* = \gamma, \gamma^2 = I$ and $J\gamma = \gamma J$ corresponding to the Lorentzian signature.

2. The equation $D_f \gamma + \gamma D_f = 0$ was included in Connes’ original axioms as a condition to ensure a reflection of the orientability of a Riemannian spin manifold. However, there may be examples of twisted Fell line bundles [K1, JR] that may give rise to non-orientable Fell bundle triples (see later) that still satisfy the condition. A discussion on modifying the orientability axiom is included in [CS].

The following condition was put in by hand to make examples respect the phenomenology.

**Definition 3.5.** An $S^o$-real finite spectral triple is one in which the Hilbert space of dimension $2l$ carries an additional grading with operator $\epsilon$ with eigenvalues $(I_l, -I_l)$ where $[D, \epsilon] = 0, [J, \epsilon] = 0$ (if the triple has a real structure), $\epsilon^* = \epsilon, \epsilon^2 = 1$.

**Lemma 3.6.** The Dirac operator $D_f$ for a real, $S^o$-real finite spectral triple (up to Poincaré duality) with $p = 4$ is given by the $n_m$ by $n_m$ matrix,

$$D_f = \begin{pmatrix} 0 & M^* & 0 & 0 \\ M & 0 & 0 & 0 \\ 0 & 0 & 0 & M^T \\ 0 & 0 & M & 0 \end{pmatrix} \tag{4}$$

where $T$ denotes transposition.

(This result is the same for either signature. In the Euclidean non-commutative standard model is associated with the fermion double problem. This was solved in [JB1] on the switch to the Lorentzian signature.) (Caveat: Such a triple does not satisfy the Poincaré duality axiom because $M$ (mass matrix) is a square matrix. To rectify this, we only need to declare that $M$ include a column of zeroes and formally delete one of basis vectors but this is a step that can always been done at the end of a calculation.)

**Proof.** This is shown by applying each of the matrix conditions set out included in the definitions above involving the operators $D, \gamma$ (for either signature) and $J, [BM]$. \qed

### 4 Spectral C*-categories

The next definition is similar to categorical spectral geometries by Bertozzini, Conti and Lewkeeratiyutkul that we quoted in the introduction and they are a first building block of a Fell bundle geometry [RM2]. The main difference compared to categorical spectral geometries is that we are presently only dealing with the bounded operators.

Recall that in a category $\mathcal{C}$, a section (or coretraction) $g$ of a morphism $f : A \to B$ is a right inverse for it, $f \circ g = \text{Id}_B$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ be an endofunctor and let $\mathcal{G}$ be a section for $\mathcal{F}$. By this we mean that $\mathcal{F} \circ \mathcal{G} = \text{Id}_\mathcal{C}$. Let $r$ and $d$ denote the range and domain map of $\mathcal{C}$. In analogy with a bisection of a groupoid, not least, the groupoid given by the cotangent bundle over a manifold, we equip a full C*-category with an additional piece of data.

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5Connes’ original intention was for fluidity, not to set axioms in stone.

6this was to avoid colour symmetry breaking which is unacceptable as for example it predicts that photons have mass

7this data will attain a geometrical interpretation
Definition 4.1. A \textit{spectral C*-category} is a small full C*-category \( \mathcal{C} \) equipped with a continuous global self-adjoint section (or coretraction) \( \sigma \) of its domain map such that \( r \circ \sigma : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{C}) \) is bijective.

Given a full C*-category \( \mathcal{C} \), the set of all such continuous self-adjoint sections \( \sigma \) of the domain map will be denoted \( N(\mathcal{C}) \).

Remark 4.2. There is a groupoid isomorphism from the maximal equivalence relation \( \mathcal{G} = M \times M \) defined on the space of objects of \( \mathcal{C} \) to the Picard groupoid of a full C*-category \( \mathcal{C} \). \( \sigma \) can be described as a set of elements of the imprimitivity bimodules arising from a bisection of this Picard or “weak” groupoid.

Example 4.3. Recall from \([BCL4]\) that a full C*-category is a category of Morita equivalence bimodules, so the objects form a Morita equivalence class and the homsets constitute imprimitivity bimodules. Observe that if \( \mathcal{C} \) is a finite dimensional C*-category, that is, its object algebras are matrix algebras and morphisms are given by elements of the linking algebra obtained from the Morita equivalence bimodule structure of the category (see \([R, J]\)). Define the \textit{diagonal} algebra of \( \mathcal{C} \) to be \( A = \bigoplus_i A_i \) where each \( A_i \) is an object of \( \mathcal{C} \). Then \( \sigma \) is defined by a (self-adjoint) matrix in which there appears precisely one non-zero block in each row and column of blocks.

Example 4.4. Non-commutative Fell bundle geometries defined in \([RM2]\) are variants of spectral C*-categories with the additional axioms of reality structure and Poincaré duality built in. All we know from spectral triple theory and the non-commutative standard model is that the Dirac operator takes the form \( \mathbb{H} \) where the mass matrix \( M \) is just a general matrix over \( \mathbb{C} \). However, with the non-commutative standard model’s finite spectral triple viewed as a Fell bundle geometry, we can make use of the additional mathematical structures available from the Morita category and the result is a Dirac operator \( \sigma \) that is beginning to attain a much closer resemblance to the fermion mass matrix constructed from empirical data. For details see \([RM1]\). Note also that although this has not been studied in detail, this formalism does not automatically preclude Majorana masses, it merely ensures that there is one non-zero block in each row and column of blocks in \( \sigma \), which means that a fermion is allowed either a Dirac mass or a Majorana mass but not both. On the other hand, in \([CG]\) Connes constructs a complete invariant of Riemannian geometries and then (quoting from his paper) he shows that his new invariant played the same role with respect to the spectral invariant as the Cabibbo-Kobayashi-Maskawa (CKM) matrix in the Standard Model plays with respect to the list of masses of the quarks.

Remark 4.5. If \( p \) and \( q \) are operators on an infinite dimensional Hilbert space \( \mathcal{H} \), then the Heisenberg commutation relation \( pq - qp = i\hbar \) is not satisfied whenever \( p \) and \( q \) are bounded operators. For this reason, an observable given by a function of \( p \) and \( q \) is always given by an unbounded self-adjoint operator. It is often considered unfortunate that one has to resort to the mathematical convenience of formally integrating observable operators in order to obtain an algebra of bounded operators so that in particular, one may use techniques from C*-algebras. On the other hand, from the point of view of a quantum space-time, where the Hilbert space must be finite dimensional and where the geometry is of a non-local nature (we cannot accommodate the notion of an infinitesimal in a mathematical framework modelling the smallest things that exist \([I]\)), a bounded observable operator is contextual.

The set \( N(\mathcal{C}) \)

Proposition 4.6. (a) Let \( \mathcal{C} \) be a small full C*-category. The set \( N(\mathcal{C}) \) is an involutive monoid. Let \( N_i(\mathcal{C}) \) denote the group of invertible elements of \( N(\mathcal{C}) \). (b) Let \( \mathcal{C} \) be a small full C*-category with \( p \) objects given by simple finite dimensional complex algebras \( A_i = M_{m_i}(\mathbb{C}) \), \( i = \{1..p\} \), \( n = \sum_{i=1}^p m_i \). The set \( N(\mathcal{C}) \) is equal to the normalising set \( N(A) \) where \( A \) is the “diagonal algebra” \( A = \bigoplus_i M_{m_i}(\mathbb{C}) \).

Proof. (a) Recall from \([BCL4]\) that a full C*-category is a category of Morita equivalence bimodules, so the objects form a Morita equivalence class and the homsets constitute imprimitivity bimodules. Note that the object set \( \text{Ob}(\mathcal{C}) \) is closed under direct sums. Therefore
multiplying elements $b, b^*, c$ in $N(C)$ arised from the composition in the Morita category, hence $b^*b \in \text{Ob}(C)$, $bc \in C$ and so $N(C)$ must be an algebraically closed set and is also of course closed under taking adjoints since $C$ is an involutive category. The unit is just the section of $d$ obtained from the units belonging to all the objects of $C$. (b) Kumjian’s definition of the normaliser of a C*-algebra $A$ was quoted earlier in the preliminaries section. This situation closely resembles the finite dimensional example we also quoted under the definition except that here the diagonal algebra $\bigoplus_i C_i$ is replaced by a larger matrix algebra. Now the proof is clear from the observation that a section of the domain $d$ of $C$ such that $r \circ \sigma : \text{Ob}(C) \to \text{Ob}(C)$ is bijective, is a matrix in which at most one matrix block is allowed to be non-zero in each row and in each column. (Earlier examples and remarks help to clarify the situation.)

**Lemma 4.7.** There is a group homomorphism from the group of global bisections $\text{Bis}(G)$ of the pair groupoid $G = \text{Ob}(C) \times \text{Ob}(C)$ to the group of unitary $u$ (invertible partial isometries) normalisers $N_u(A)$ and a $*$-functor $\pi : \mathcal{E} \to \mathcal{G}$ such that $\pi \circ u = \text{Id}_\mathcal{G}$ for all $u \in N_u(A)$.

**Remark 4.8.** Let $(A, H, D, \gamma)$ be an even spectral triple. The commutators $[D, a] \forall a \in A$ generate an $A$-bimodule $\Omega^1_D$, providing the space of 1-forms over the spectral triple $[C_1, EH]$ Note that any $\sigma \in N_i(A)$ defines an inner derivation $\delta_\sigma$ from $A$ into $B$, and this group, for all $a \in A$, generates the (“observable”) enveloping algebra $B$.

### 4.1 Finite spectral triple categorification

With the following proposition we argue that a (finite) spectral triple is already a category.

**Proposition 4.9** (Finite spectral triple categorification). Finite even $S^0$-real spectral triples over $C$ with $p = 4$ (as defined above) are spectral C*-categories (as defined above).

**Proof.** The algebra $A$ of a finite spectral triple is given by a direct sum of simple matrix algebras over $C$. Let each simple algebra $A_i$ define an object in $\text{Ob}(C)$. The full C*-category $C$ comes from the Morita category of isomorphism classes of Morita equivalence bimodules over these objects wherein the algebra $B$ provides the linking algebra or “enveloping algebra” for $C$. The Dirac operator $D$ is a self-adjoint matrix in which there appears one non-zero block in each row and column of blocks, which is exactly what ensures that it is a section (or coretraction) of the domain map of $C$.

**Remark 4.10.** Note that the $S^0$-reality condition was put in by hand for spectral triple physics but here it loses this sense of artificiality due to the categorical structures we are using. For $p > 4$ we would need a modified version of $S^0$-reality.

**Remark 4.11.** In other work we constructed an example of a finite real spectral triple from a finite dimensional non-commutative Fell bundle geometry adding structure such as Poincaré duality and reality and we used this to make physical predictions about the hitherto ad hoc choice of the mass matrix $M$. (Bearing in mind the fact that the full details of the algebra for a canonical non-commutative standard model is not yet available.)

The following could be investigated in order to make connections with other authors’ approaches towards categorification in non-commutative geometry such as $[BCL5, BCL1, MZ, BM]$ wherein objects are spectral triples.

**Conjecture 4.12.** The category of finite even $S^0$-real spectral triples over $C$ and spectral triple morphisms $[BCL1]$ is equivalent to the category of finite dimensional spectral C*-categories and involutive functors.

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$^8S^0$-reality excludes Majorana masses so we would need to modify these constructions to allow their inclusion.
5 Fell bundles and path-liftings

5.1 Notions of generalised connection and geodesics

For maximal clarity, let us briefly recall some basic facts on the geometrical concepts that we will be generalising to an algebraic context.

Consider a vector bundle \((E^0, \pi, M)\) over a simply connected manifold \(M\) and let \(GL(E^0)\) denote its general linear groupoid, that is, the set of all linear isomorphisms between each pair of fibres: \(GL(E^0) = \{ \alpha : E^0_x \to E^0_y \mid x, y \in M, \alpha \text{ an isomorphism} \}\) together with the canonical composition of isomorphisms, inverses and units \(\iota_x : E^0_x \to E^0_x\). We also recalled earlier that the fundamental groupoid \(\Pi_1(M)\) of homotopy equivalence classes of paths on a simply connected \(M\) is given by the pair groupoid \(\mathcal{G} = M \times M\). Earlier we also recall that \(\mathcal{G} = M \times M\) can be thought of as the deformed or integrated tangent bundle over \(M\).

A representation of a groupoid on \(E^0\) is a groupoid homomorphism \(\rho : \mathcal{G} \to GL(E^0)\). A choice of representation \(\rho\) of the fundamental groupoid is equivalent to a choice of flat connection \(\omega\) on \(E^0\). This is because the transition functions of \(E^0\) force \(E^0\) to be trivial and because since paths or flows in \(M\) that differ by a diffeomorphism have been identified in the homotopy classes which give the arrows of \(\Pi_1(M)\), (any curvature of \(M\) has effectively been divided out). The arrows in \(\mathcal{G} = M \times M\) can be put in one to one correspondence with the set of shortest paths (or geodesic flows) between each pair of points in \(M\). Each morphism \(\rho_y \in \rho(\mathcal{G})\) can be constructed by integrating the connection along a member of the equivalence class of paths given by a groupoid arrow in \(\mathcal{G}\). We say that the connection \(\omega\) is the generator of the representation \(\rho\). An isomorphism in \(GL(E^0)\) is called a parallel transport if it belongs to the representation generated by \(\omega\). The connection is a field of infinitesimals, a smooth assignment of geometrical data to each point in \(M\). Observe that a unique representation \(\rho(\mathcal{G})\) can be constructed from a choice of continuous bisection of \(GL(E^0)\) via the connection data and so it follows that a choice of connection on \(E^0\) is equivalent to a choice of bisection.

To consider curved connections one will require either (a) a finer Lie groupoid that will treat as two different arrows two paths that are not diffeomorphic to each other or alternatively (b) a non-commutative generalisation of \(E^0\). For the latter, one begins with a \(C^*\)-bundle \(E^0\) because these are Banach bundles, which are defined without reference to transition functions and so a \(C^*\)-bundle over the groupoid \(M \times M\) is not necessarily trivial. Moreover, as illustrated in the final section, the curvature may be sourced from the non-commutativity in the fibres.

Next we define a notion of path-lifting for the context of a \(C^*\)-bundle \(E^0\) in an ambient Fell bundle \(E\). A unitary path-lifting will give rise to a generalisation of a parallel transport. We borrow some physics terminology to set up new definitions without making immediate physical interpretations (the fibres of \(E^0\) may be non-commutative algebras, so we are being careful not to use language that interchanges between states of the algebra and states of the Hilbert space). Let \((E, \pi, \mathcal{G})\) be a unital saturated Fell bundle over a pair groupoid \(\mathcal{G} = M \times M\) over a simply connected compact manifold \(M = \mathcal{G}_0\). Let \(E^0\) denote the \(C^*\)-bundle given by the restriction of \(E\) to \(\mathcal{G}_0\). Let \(\mathcal{C}\) be the \(C^*\)-category associated to \(E\). Let \(\mathcal{H}\) be a possibly infinite dimensional Hilbert space carrying a representation of \(C^*(\mathcal{E})\) and \(C^*(E^0)\). Let each \(\Psi \in \mathcal{H}\) be called a “state” of an ensemble of “systems” parametrised by \(\mathcal{G}^0\) (or equivalently, by the space of fibres of the “diagonal” bundle \(E^0\)). Each system \(\psi\) has dimension equal to the dimension of the fibre labelling it or parametrising it. Let \(\psi_x\) denote the system at \(x \in M\) for a given state \(\Psi\).

**Definition 5.1.** A path-lifting for \(\psi_x \in \Psi\) is an assignment of an element \(e\) of the fibre \(E_{(x,y)}\) to a choice of path \((x,y) \in \mathcal{G}\) with \(\psi^e_y := e\psi_x\) in a new state \(\Psi^e\). By consistency, such a path-lifting above fixes a path-lifting for \(\psi_y \in \Psi\) (a second system in the same state as \(\psi_x\)) over \((y,x)\) given by \(e^* \in E_{(y,x)}\).

**Definition 5.2.** A path-lifting operator \(PL : \mathcal{H} \to \mathcal{H}\) on a Hilbert space \(\mathcal{H}\) is a section of \((\mathcal{E}, \pi, \mathcal{G})\) supported on a global bisection of \(\mathcal{G}\), which has been extended to a bounded operator on \(\mathcal{H}\), thus providing a path-lifting for every \(\psi \in \Psi\) for each \(\Psi \in \mathcal{H}\).
Lemma 5.3. Let \((E, \pi, G)\) be a Fell bundle as above whose restriction to \(G_0\) is a trivial C*-bundle \(E^0\) with underlying vector bundle structure \(E^0\). A choice of unitary path-lifting section is equivalent to a choice of a flat connection on the vector bundle \(E^0\).

Proof. This follows from the above discussions including because the unitary path-lifting section gives rise to a bisection of \(GL(E^0)\).

Remarks 5.4. 1. In view of Renault’s paper ([JR], proposition 4.7), we can confirm that the normalisers of \(C^*(E^0)\) in \(C^*(E)\) are exactly the sections of \(\pi\) supported on the local bisections of \(G\) and so the group of invertible path-lifting operators can be identified with the group of invertible normalisers \(N_c(C^*(E^0))\) of \(A = C^*(E^0)\). Since the unitary normalisers \(N_u(A) \cong Bis(GL(E^0))\), a choice of group homomorphism from \(Bis(G)\) to \(N_u(A)\) corresponds to a choice of flat connection on the underlying vector bundle structure of \(E^0\) and it extends to a representation \(\rho\) of \(Bis(G)\) on \(H\).

2. The inverse semigroup homomorphism \(I(G) \to I(GL(E^0))\) extends to a representation on \(H\). The non-zero entries of normalisers of \(N(A)\) that are not sums of other normalisers are elements of \(E\). A unitary operator is an invertible partial isometry \(v\). Since we are working with C*-bundles which do not necessarily have isomorphic fibres, one should generalise the general linear groupoid \(GL(E^0)\) to an inverse semigroup implemented by partial isometries (that is, \(v\) is an element of \(E\) with a unique quasi inverse \(v^*\) satisfying \(vv^*v = v\) and \(v^*vv^* = v^*\)).

Let \(M\) be a simply connected manifold. In view of the discussion above, we can label each arrow in the deformed or integrated tangent bundle \(\tilde{G} = M \times M\) by a geodesic flow. In view of the categorical switch-of-focus (see introduction), where we replace geometrical notions in the base space with algebraic notions in the space of fibres, we now give \(E\) the interpretation of a generalised tangent bundle (in the case that \(E\) is a line bundle, \(C^*(E) = C^*(M \times M)\)) and to each unitary path-lifting, we give the interpretation of a geodesic flow. However, to define a non-commutative geodesic (to make precise the transport of fuzzy points in one fibre to fuzzy points in the next fibre) this does not go quite far enough because in the case that the fibres of \(E^0\) are not isomorphic (such as in a C*-bundle with non-commutative fibres over a discrete space or a Banach bundle that is not locally trivial) or even in any case when \(E^0\) is not identified with its underlying vector bundle \(E^0\), then \(GL(E^0)\) no longer provides a good description of parallel transport. Since the unitary operators are the invertible partial isometries, \(GL(E^0)\) should be generalised in further work by dropping the condition of invertibility of its morphisms, so that isomorphisms between fibres are replaced with linear \(*\)-homomorphisms implemented by partial isometries \(v\).

Definition 5.5. Let \((E, \pi, G)\) be a unital saturated Fell bundle with not necessarily isomorphic fibres, a non-commutative geodesic is given by a path-lifting \(e\) such that \(e\) is a partial isometry.

The non-local (not relying on geometrical data of an infinitesimal nature and not relying on an unbounded operator) object we have to model a connection, that is, a generator of a geodesic flow, is a path-lifting section. This is why unitarity is not required in the definition of a path-lifting introduced above and this is why we defined path-liftings first instead of generalised parallel transports. Note also that a path-lifting operator will always be bounded.

5.2 Fell bundle triples

A Fell bundle triple is the following algebraic generalisation of a geometrical space wherein the “manifold” is the space of fibres of a C*-bundle \(E^0\) and where the basic geometrical data is given in the form of a path-lifting operator, instead of a metric or a connection or a curvature 2-form. Below we show that Fell bundle triples are equivalent to spectral C*-categories, which in turn we have already compared with spectral triples, the definitive non-commutative spaces.

Definition 5.6. A Fell bundle triple \((E, \mathcal{H}, PL)\) consists of a unital saturated orientable Fell bundle \((E, \pi, G)\) over a pair groupoid \(G\) over a simply connected compact manifold \(M\), a Hilbert space carrying (faithful unless stated otherwise) a representation of the algebra \(C^*(E)\) with a subrepresentation of \(C^*(E^0)\), together with a path-lifting operator \(PL : \mathcal{H} \to \mathcal{H}\).
Proposition 5.7. Fell bundle triples and spectral $C^\ast$-categories are equivalent.

Proof. We have already recalled that saturated unital Fell bundles over pair groupoids are equivalent to full $C^\ast$-categories. All that remains is to check that their geometrical data coincide. Let $A, B$ be objects in $\text{Ob}(C)$ and let $x$ be the corresponding unit in $G^0$. A path-lifting for $\psi_x$ can equivalently be viewed as a choice of target pair $(y, B)$ and an assignment of a morphism in $\text{Hom}(A, B)$ to the groupoid arrow (or path) $(x, y)$ such that $\psi_y = e\psi_x$ for a system $\psi_y^\prime$. This fixes a path-lifting for $\psi_y \in \Psi$ (a second system in the same state as $\psi_x$) over $(y, x)$ given by $e^\ast \in \text{Hom}(y, x)$. It follows that the fact that a path-lifting operator $PL: \mathcal{H} \to \mathcal{H}$ is defined from a section supported on a bisection of $G$ means that it must be a coretraction of the domain map of $C$; and so a path-lifting operator is a $\ast$-functor from $G$ specifying an element of a $\ast$-functor from $\text{Bis}(G)$ into $C$ (or specifying an element of the group $N_i(A)$) and a continuous section $\sigma$ of $E$. Lastly, such a path-lifting satisfies $\sigma = \sigma^\ast$ because for every path from $x$ to $y$, there is a geometrically equivalent reflected path from $y$ to $x$, to which belong a common piece of geometrical data defined during any given event. 

We make the following new geometrical interpretation of finite Dirac operators:

Proposition 5.8. Finite even $S^\alpha$-real spectral triples over $C$ are Fell bundle triples wherein a Dirac operator $D_f$ is equivalent to a path-lifting operator $PL$.

Example 5.9. The diagram illustrates the support of $D_f$ as in equation (1).

Remark 5.10. We remark that a matrix $M$ satisfying the equation of motion $M(M^\ast M - I) = 0$ (see [BM]) is a partial isometry and attains the interpretation of a non-commutative geodesic. The mass matrix itself on the other hand, attains the interpretation of the generator of the von Neumann algebras.

Example 5.11. A categorical spectral geometry $(\mathcal{E}, \mathcal{H}, \mathcal{D})$ [BCL7] (quoted in the introduction) is essentially data-equivalent to a spectral $C^\ast$-category and a Fell bundle triple although there are small technical differences such as $\mathcal{E}$ is a pre-$C^\ast$-category. In general, (and if $\text{Ob}(\mathcal{E})$ has an underlying structure of a vector bundle) the generator $\mathcal{D}$ is extended to an unbounded operator on $\mathcal{H}$ and can be constructed from the geometrical data in $PL$ where $\mathcal{D}$ is interpreted as the connection on $\text{Ob}(\mathcal{E})$. Modular spectral triples [BCL6] are similar constructions to categorical spectral triples and involve more details arising from the dynamical quality of von Neumann algebras.

Example 5.12. Before we construct the following example of a Fell bundle triple, first consider an example of a semidirect project Fell bundle $\mathcal{G} \ltimes C^\ast(\mathcal{E}^0)$ similar to an example from [K2]. Let $\mathcal{E}$ be a saturated unital Fell bundle over a topological groupoid $G$. The product of elements $e_1 = (g, a)$ and $e_2 = (h, b)$, $a, b \in C^\ast(\mathcal{E}^0)$, for each pair $(g, h)$ such that $gh = h \circ g \in \mathcal{G}$, $d(g) = \pi(a)$, $r(g) = \pi(b)$, in the Fell bundle is given by:

$$e_1 e_2 = (gh, \alpha_g(a)b)$$

where $\alpha_g(a) = uau^\ast$ wherein $\alpha_g$ is a linear $\ast$-isomorphism of fibres (element of $GL(\mathcal{E}^0)$) and $u$ is a unitary free normaliser over $g$ varying continuously over all $g \in \mathcal{G}$. $\pi(u) = g$, $\pi(u^\ast) = g^\ast$, $e_1^\ast = (g^\ast, a^\ast)$.

To construct an example of a Fell bundle triple, we set $\mathcal{E}$ to be a unital Fell bundle over $\mathcal{G}$ the pair groupoid over a compact simply connected space. In the cases that $\mathcal{E}$ is also a vector bundle (as well as a Banach bundle) then $\mathcal{E}$ is a trivial bundle and so orientable. A path-lifting operator $PL$ is obtained from the unitary elements defined by the field $\alpha_g$ for each $g$ of a global bisection $x$ of $\mathcal{G}$. $PL$ will automatically be self-adjoint since $u_g^\ast = u_g^\ast$ for all $g$ in $x$. 

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Comments 5.13. 1. The algebra $\mathcal{A} = C^*(\mathcal{E}^0)$ we call the configuration algebra and the algebra $C^*(\mathcal{E})$ we call the observable or coordinate algebra. Note that if $\mathcal{E}$ is a line bundle then $C^*(\mathcal{E}) = C^*(M \times M)$, which is the observable algebra in the tangent groupoid quantisation. (As already mentioned, traditionally, unbounded observable operators are exponentiated to obtain a bounded (unitary) operator and element of a C*-algebra of observables.)

2. Other work has involved defining an inner automorphism of a C*-bundle $\mathcal{E}^0$ as an inner automorphism of the algebra commuting with the Banach bundle projection map $\pi$ and defining a representation of Bis$(\mathcal{G})$ as a group homomorphism into this inner automorphism group [RM3]. The context of [RM3] treats the relationship between diffeomorphisms of $M$ and the inner automorphism group of the C*-bundle to illustrate and study Connes’ algebraic analogy of Einstein’s equivalence principle.

3. In further work we hope to consider examples where the C*-completion of the algebra of sections of the C*-bundle is a von Neumann algebra with weight $w$ such that $w$ corresponds to the generator of the inner automorphisms of the C*-bundle in order to give $w$ an interpretation as generalised connection on a C*-bundle. We expect this to involve modular spectral triples [BCL6]. (A quantisation program should involve a characterisation of the states of the system and this context might lead to a clearer description of geometrical states for some form of algebraic quantum gravity.)

6 The classical limit

Recall that a spin structure on an orientable Riemannian manifold $M$ is an equivariant lift of the oriented orthonormal frame bundle respect to the double covering $P : \text{Spin}(n) \to \text{SO}(n)$. In a reconstruction theorem, the non-commutative geometor ch checks that any Riemannian spin manifold can be constructed in full (or up to a possible torsion term) from only the data in a real spectral triple. The material in this paper is towards a quantisation program, so constructing a Riemannian spin manifold from an example of a spectral C*-category or a Fell bundle triple is about finding the classical limit, not a reconstruction theorem. For a recent paper on reconstruction theorems see [BC].

We have already argued that spectral triples have a natural categorical aspect coming from the associated Morita category. Moreover, we have seen that viewing every spectral triple as a spectral C*-category automatically incorporates a quantisation aspect in the sense that the non-commutative algebra of observables $C^*(\mathcal{E})$ is pre-built-in to the formalism, without any need to actively deform. This confirms Crane’s suggestion that in quantum gravity one should think of the whole system in terms of categorical structures. Our examples with non-commutative configuration algebras do not even have classical limits as $\hbar$ tends to zero because the obstruction to the observable algebra tending to become commutative exists in the fact that the configuration algebra itself is non-commutative. (The limit of these geometries in the sense of a continuum limit will be in the sense that as one takes a larger scale view, the fuzzy points in each fibre in $\mathcal{E}^0$ merge to a point as the non-commutative fibres $M_n(\mathbb{C})$ are replaced by $\mathbb{C}$. As the points move closer and closer together, we can go back to using the continuum to describe the space together with the traditional calculus.)

This formalism may help towards solving the problem that the non-commutative standard model is traditionally a classical (general relativity) theory. Another open problem is the difficulty of deformation quantisation on curved space-times, which we are considering by allowing the configuration algebra (not only the observable or coordinate algebra) to be non-commutative. To do quantum gravity on a Fell bundle, first we need to do general relativity on a Fell bundle. In the preliminaries, we briefly explained and gave references to Connes’ non-commutative general relativity with regard to fluctuations of the Dirac operator. We

9We take $C^*(M \times M)$ to be a C*-completion of the convolution algebra of $\mathcal{G}$ and identify it with a C*-completion such as $C^*_red(\mathcal{G}, \pi)$ of the algebra of compactly supported sections of the line bundle over $\mathcal{G}$.  

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should reconstruct geometrically a Riemannian spin manifold from the geometrical data provided by a spectral C*-category and its Fell bundle.

Within the context of the categorical switch-of-focus (see introduction), we have discussed in the previous section how a parallel transport can be thought of as a non-local connection and mentioned the fact that given a holonomy group, a unique bundle and connection can be fully constructed. (So the data given by a holonomy group already tells us all we need to know about the curvature of a manifold even without having access to any infinitesimal data.) Analogously, the Dirac operator also carries geometrical data of an infinitesimal nature about a Riemannian manifold. If the operator $\sigma$ is a (bounded) generalised Dirac operator as claimed, then there will exist a non-commutative geometry given by some spectral C*-category and Fell bundle triple over a Riemannian spin manifold $M$ whose classical limit will provide a Riemannian spin manifold complete with its traditional Dirac operator, an unbounded operator on the Hilbert space of square integrable sections of the spinor bundle satisfying Connes’ spectral triple axioms.

As a bundle of C*-algebras, a Clifford bundle has an equivalent description as a C*-bundle and is denoted $Cl^0$ or $E^0$ below. We begin with the trivial case of a Riemannian manifold $M$ together with a Clifford algebra given by $\mathbb{C}$ and where the Riemannian manifold is flat. We have been and we continue to restrict to compact manifolds due the presence of the unit in the axioms.

**Proposition 6.1.** Let $(\mathcal{E}, \pi, \mathcal{G})$ be a Fell line bundle over the pair groupoid $\mathcal{G}$ over a simply connected compact orientable Riemannian manifold $M$, with diagonal C*-bundle $\mathcal{E}^0$ and diagonal (or configuration) algebra $\mathcal{A} = \mathcal{C}(\mathcal{E})$ and with enveloping (or coordinate) algebra $\mathcal{C}(\mathcal{E}) = \mathcal{C}(M \times M)$. Let $\mathcal{C}$ be the associated C*-category. Now introduce geometrical data given by a choice of $\sigma$ and PL as defined above. We can reconstruct from this non-commutative geometry, the classical limit, which is a trivial (flat) Riemannian spin manifold with Clifford fibre $\mathbb{C}$ with a Dirac operator in a coordinate frame in which the (flat) connection is zero by choosing $\alpha = 1$. That is, the Dirac operator is given by the physicist’s flat Dirac operator $\gamma^\mu \partial_\mu + 0$.

**Proof.** Any Fell line bundle $\mathcal{E}$ can be constructed as a semidirect product bundle (due to it being saturated and locally trivial) and so the context is described by the trivial case of example \[5.12\] This is because any Fell line bundle is saturated and for any saturated Fell bundle, $\mathcal{C}^*(\mathcal{E})$ is regular in $\mathcal{C}^*(\mathcal{E})$ and the “regular” Fell bundles are the semidirect product bundles. (See for example the introduction of [JR]). And so any element of $\mathcal{E}$ can be expressed as $(g, a)$ with $\pi(a) = g \in \mathcal{G}$ and the multiplicative in the Fell bundle is given as in \[5.12\] with $\alpha$ being the identity for all $g$ due to the triviality. To choose $\sigma$, we choose a bisection $x$ of $\mathcal{G}$ and a general element $(g, a)$ of the fibre of $\mathcal{E}$ over each groupoid arrow $g \in \mathcal{G}$ in the bisection $x$ supporting $\sigma$, with $\alpha$ defined by $\alpha(a) = a$ for all elements $a \in \mathcal{E}^0$. The different possible choices of $\alpha$ correspond to the different choices of coordinate frames and flat connections. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the spectral triple associated to the Riemannian manifold. The square integrable sections $\mathcal{H}$ of the Clifford bundle over $M$ carries a representation of the algebras $\mathcal{A}$ and $\mathcal{C}(\mathcal{E})$ and we identify $\mathcal{H}$ with the Fell bundle triple Hilbert space $\mathcal{H}$. All other structures pertaining to the Hilbert space can be copied over from $\mathcal{H}$.

Now we use a technique from the tangent groupoid quantisation (see [C1] and refer to the brief description above in the preliminaries). Recall that the Lie algebroid $TM$ integrates to $M \times M$ and so deriving the classical limit involves the reverse process of this integration. Underlying the tangent groupoid quantisation [C1], [H] is the fact that the limit of a quotient is a derivative (consider a sequence of groupoid arrows $g_n$ where the arrows become shorter and shorter until they have no length at all and become tangent vectors). Briefly, all Cauchy sequences in $\frac{TM}{h\mathbb{H}}$ converge in $\frac{TM}{\mathbb{H}}$: consider a sequence,

$$p_n \to q_n \to p \quad \text{with} \quad (p_n, q_n) \in M \times M, \quad p \in M,$$

then as

\[10\]This is a C*-completion $\mathcal{C}^*(\mathcal{G})$ of the algebra of compactly supported sections on $\mathcal{G}$.
\( h \) goes to zero, \( \frac{p_n - q_n}{h} \to v \) where \( v \in TM \)

and we write \( (p_n, q_n, h) \to (p, v, 0) \), or \( (g_n, h) \to (v, 0) \) if \( g_n = (p_n, q_n) \)

So at each (space-time) point \( d(g) \) in \( M \), the classical limit of \( (g, a) \) is a tangent vector \( v \) paired with (or contracted over space-time indices with) an element \( a \) of the Clifford algebra as a fibre over \( d(g) \).

\[ D = \gamma^\mu \partial_\mu. \]

We do this at every point, that is, extending continuously over the space \( M \). (To explain the reference to a direct multiplication involving the space-time index contraction, first consider a finite dimensional example where \( g \) as in \((g, a)\) can be considered as a matrix unit \( e_{ij} \) with \( d(e_{ij}) = i \) and \( r(e_{ij}) = j \). Now let \( \mathcal{G}^0 \) be a continuous space instead. There is a fibre of \( \mathcal{E}^0 \) assigned to each point in the space-time and therefore each element \( a \) comes with space-time coordinates. So when \( g \) goes to \( v \), the vector \( v \) is contracted with \( a \) over the space-time indices.)

Another view on the above construction comes from the fact that the algebroid of derivations \( \text{Der}(Cl^0) \) on the bundle \( Cl^0 \) integrates to \( GL(Cl^0) \). And therefore applying the above procedure to a sequence in \( GL(Cl^0) \), instead of an element of \( TM \), the limit is an element of the Lie algebroid which we identify with a flat Dirac operator \( D_{RSM} \) as above. With the identification of \( \text{Der}(Cl^0) \) with \( \text{End}(Cl^0) \otimes TM \), the Atiyah (short exact) sequence illustrates the point that \( \text{Der}(Cl^0) \) is the extension of \( TM \) by the spin group, which we identify with \( \ker \rho \), where \( \rho \) is the surjective map \( \text{End}(Cl^0) \otimes TM \to TM \):

\[
0 \to \ker \rho \to \text{End}(Cl^0) \otimes TM \to TM \to 0
\]

Next we consider curved Riemannian spin manifolds.

Normally in order to incorporate curvature into a (simply) connected manifold, we would need to replace the (pair) fundamental groupoid with a groupoid in which the diffeomorphisms are not divided out, as that would be the only way to achieve curvature in view of Einstein’s equivalence principle. However, Connes has provided us with an alternative approach, and so we “fluctuate” (see preliminaries) the flat or initial Dirac operator in the non-commutative fibre of the Clifford bundle to reconstruct a general Dirac operator on a Riemannian manifold \( M \). Examples of Clifford algebras include \( M_n(\mathbb{C}) \), \( M_n(\mathbb{R}) \), \( M_n(\mathbb{H}) \), \( M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \), \( M_n(\mathbb{R}) \oplus M_n(\mathbb{R}) \), \( M_n(\mathbb{H}) \oplus M_n(\mathbb{H}) \) and algebras isomorphic to them. Note that at present the scope is limited to those Clifford bundles that can be equivalently described as locally trivial C*-bundles with Clifford algebra over \( \mathbb{C} \). Of course, it might be possible to extend the formalism to include the Clifford algebras over \( \mathbb{R} \) whose coefficients are in \( \mathbb{R} \) or \( \mathbb{H} \) by generalising to “real C*-bundles” (i.e. Banach bundles fibred by real C*-algebras). Nevertheless, the Clifford algebra \( \mathbb{H} \oplus \mathbb{H} \) is already included up to isomorphism since \( M_2(\mathbb{C}) \cong \mathbb{H} \oplus \mathbb{H} \).

Proposition 6.2. Let \( M \) be a Riemannian spin manifold with Clifford bundle and C*-bundle denoted \( Cl^0 \) or \( \mathcal{E}^0 \). There is a Fell bundle triple \( (\mathcal{E}, \pi, \mathcal{G}, PL) \) or equivalently a spectral C*-category \( (\mathcal{C}, \sigma) \), such that the classical limiting geometry as \( M \times M \to TM \) results in the original Riemannian spin manifold.

The context here is the same as that of the previous theorem except that \( \mathcal{E}^0 \) is a more general C*-bundle and \( PL \) is now a path-lifting corresponding to a curved connection.

Proof. Einstein’s equivalence principle for general relativity involves selecting an initial flat metric and then fluctuating it, producing a curved metric. In non-commutative geometry, these diffeomorphisms are realised as coordinate algebra automorphisms, \( \phi \). A ‘fluctuation’ of the Dirac operator is given by Connes as \( L(\phi)D(L(\phi))^{-1} \) where \( L \) is the spinor lift (a mapping of the automorphism group to its double cover). The image group of \( L \) is the spin group,
which is the inner automorphism group of the Clifford algebra. We define a fluctuation of $\sigma$ to be (locally) $\alpha(\sigma) = u\sigma u^*$ with $u$ an inner automorphism of the Clifford algebra (locally) $\alpha(a) = uau^*$.

$(\alpha)$ induces isomorphisms of the fibres of the underlying vector bundle of $Cl^0$ and a fluctuation induces a groupoid homomorphism,

$$\alpha \circ \rho : \mathcal{G} \to GL(Cl^0)$$

$$g \mapsto \alpha((g, u)) = (g, \alpha(u))$$

since the whole groupoid representation can be constructed from a unitary section supported on a bisection as explained earlier. This carries essentially a concept as described in [IS] in spectral triple gravity. Note that if $C^*(Cl^0)$ is commutative then $\alpha$ is the identity, implying a conceptual link between curvature and non-commutativity. In general non-commutative geometrical contexts, we have Connes’s fluctuations formula, quoted in the preliminaries (2), which is a linear combination of fluctuations or diffeomorphisms and results in a curved Dirac operator $\mathcal{D}$. The fluctuated Fell bundle Dirac operator is given (locally) by the formula:

$$\mathcal{D} = \sum_j r_j U_j (g, a) U_j^*; \quad U_j \in Cl(d(g)), \quad r_j \in \mathbb{R} \quad (6)$$

the $U_j$s can be identified with elements of the spinor group because this is the structure group of the Clifford bundle.

Carrying out the equivalence principle using Connes’s method of taking linear combinations of fluctuations beginning with an initial flat local Dirac operator $\mathcal{D}_{\text{initial}} = (g, a)$, or extended over $M$, $\mathcal{D}_{\text{initial}} = \sigma$ and invoking the same argument as in the flat geometry case about producing the classical limit of the Dirac operator we obtain,

$$\sum_j r_j U_j \mathcal{D}_{\text{initial}} U_j^* = \sum_j r_j U_j \sigma U_j^*$$

and as $\hbar \to 0$,

$$\sum_j r_j U_j \left( \sum_i c_i \frac{\partial}{\partial x_i} \right) U_j^* = \sum_i c_i \left( \frac{\partial}{\partial x_i} + \omega_i \right)$$

The result is a general Dirac operator on a Riemannian spin manifold with arbitrary torsion and spin connection $\omega$. The last line comes from Connes’ non-commutative geometry, see for example [C3], and is a consequence of the fact that if $U$ is unitary then $U DU^* = \mathcal{D} + UU^* - \mathcal{D} = \mathcal{D} + U\mathcal{D}U^* - U^*\mathcal{D} = \mathcal{D} + U[\mathcal{D}, U^*]$ where $U[\mathcal{D}, U^*] \in \Omega^1 \mathcal{D}$, the differential algebra.

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