Physical quantities and arbitrariness in resolving quantum master equation

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Abstract

By proceeding with the idea that the presence of physical (BRST invariant) extra factors in the path integral is equivalent to taking into account explicitly the arbitrariness in resolving the quantum master equation, we consider the field-antifield quantization procedure both with the Abelian and the non-Abelian gauge fixing.

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1 Introduction

Let $W$ be a quantum master action, and let $\mathcal{O}$ be a physical (BRST invariant) quantity. Then, the product, $\exp\{\frac{i}{\hbar}W\}\mathcal{O}$, resolves the quantum master equation, as well. Thus, the presence of physical extra factors in the path integral is equivalent to taking into account explicitly the arbitrariness in resolving the quantum master equation.

In the present paper, by proceeding with the above simple idea, we consider the field-antifield quantization procedure [1, 2, 3, 4, 5, 6, 7, 8], both with the Abelian and the non-Abelian gauge fixing.

In the case of a non-Abelian gauge fixing, a rather non-trivial measure in the path integral is given by the so-called ”square root formula” [9, 10, 11]. There are two aspects in dependence of the measure on the elements of gauge arbitrariness. Formally, the measure does depend both on the gauge fixing functions themselves, and on the functions that complement the latter gauge fixing functions to constitute together an invertible reparametrization as for the total set of the field-antifield variables.

In [10], it has been shown that the path integrand is actually independent of the complement functions. Furthermore, it has then been shown in [10] that the whole path integral is actually independent on-shell of the non-Abelian gauge fixing functions as well. In the present paper, we use essentially these results.

2 Abelian gauge-fixing

Let us consider the equation for a physical quantity $\exp\{\frac{i}{\hbar}X\}$,

$$\sigma \exp\left\{\frac{i}{\hbar}X\right\} = 0,$$

(2.1)

where

$$\sigma := \frac{\hbar}{i} \exp\left\{- \frac{i}{\hbar} W\right\} \Delta \exp\left\{\frac{i}{\hbar} W\right\} = \frac{\hbar}{i} \Delta + \text{ad}(W),$$

(2.2)

is the total BRST-operator, and $W$ is a solution to the quantum master equation,

$$\Delta \exp\left\{\frac{i}{\hbar} W\right\} = 0,$$

(2.3)

with the $\Delta$ being the odd Laplacian,

$$\Delta := (-1)^{\varepsilon_A} \frac{\partial}{\partial \Phi^A} \frac{\partial}{\partial \Phi^*_A},$$

(2.4)

$$\varepsilon(\Phi^A) := \varepsilon_A := \varepsilon(\Phi^*_A) + 1.$$  

(2.5)
It follows from (2.1), (2.2) that the sum \((W + X)\) does also satisfy the equation (2.3). Therefore, it there holds

\[
\exp \left\{ \frac{i}{\hbar} (W + X) \right\} = \exp \{[\Delta, F]\} \exp \left\{ \frac{i}{\hbar} W \right\},
\]

as the operator \(\exp \{[\Delta, F]\}\), with an arbitrary Fermion operator \(F\), does act transitively on the set of solutions to the equation (2.3) under the suitable boundary conditions. In turn, the solution (2.6) rewrites in the form

\[
\exp \left\{ \frac{i}{\hbar} X \right\} = \exp \left\{ - \frac{i}{\hbar} W \right\} \exp \{[\Delta, F]\} \exp \left\{ \frac{i}{\hbar} W \right\} = \exp \left\{ \frac{i}{\hbar} [\sigma, F'] \right\} \cdot 1,
\]

where \(F'\) is the transformed Fermion operator \(F\), to be considered as an arbitrary one, as well,

\[
F' =: \exp \left\{ - \frac{i}{\hbar} W \right\} F \exp \left\{ \frac{i}{\hbar} W \right\}.
\]

Being the \(F\) a function, then \(F' = F\) is a function as well. It is just the operator (2.8) that might depend on the so-called "composite operators", as there is no other arbitrariness in the solution (2.7). Thereby, the expression for the generating functional has the form

\[
Z[J] =: \int [D\Phi][D\Phi^*][DA] \exp \left\{ \frac{i}{\hbar} \left[ W + X + J_A \Phi^A + \left( \Phi^*_A - \Psi(\Phi) \frac{\partial}{\partial \Phi^A} \right) \Lambda^A \right] \right\},
\]

with \(\Psi(\Phi)\) being the gauge fixing Fermion, and \(\varepsilon(\Lambda^A) =: \varepsilon_A + 1\). Obviously, the effective action corresponding to the generating functional (2.9) is gauge independent on-shell \(J = 0\). Indeed, by making in (2.9) the BRST transformation,

\[
\begin{align*}
\delta \Phi^A &=: \Lambda^A \mu, \\
\delta \Phi^*_A &=: \mu \left( W + X \right) \frac{\partial}{\partial \Phi^A}, \\
\delta \Lambda^A &=: 0, \\
\varepsilon(\mu) &=: 1,
\end{align*}
\]

with \(\mu = \text{const}\), we get the Ward identity

\[
J_A \langle (\Phi^A, (W + X)) \rangle = 0,
\]

where the \(\langle \cdots \rangle\) means the functional average value with the weight functional in (2.9), and the \((\cdot, \cdot)\) stands for the antibracket,

\[
(A, B) =: (-1)^{\varepsilon(A)}[[\Delta, A], B] \cdot 1 = \left. \frac{\partial}{\partial \Phi^C} \frac{\partial}{\partial \Phi^*_C} \right|_\Phi B - (A \leftrightarrow B)(\varepsilon(A) + 1)(\varepsilon(B) + 1),
\]

\[
(A, B) =: (-1)^{\varepsilon(A)}[[\Delta, A], B] \cdot 1 = \left. \frac{\partial}{\partial \Phi^C} \frac{\partial}{\partial \Phi^*_C} \right|_\Phi B - (A \leftrightarrow B)(\varepsilon(A) + 1)(\varepsilon(B) + 1),
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\]

\[
(A, B) =: (-1)^{\varepsilon(A)}[[\Delta, A], B] \cdot 1 = \left. \frac{\partial}{\partial \Phi^C} \frac{\partial}{\partial \Phi^*_C} \right|_\Phi B - (A \leftrightarrow B)(\varepsilon(A) + 1)(\varepsilon(B) + 1),
\]
as for two arbitrary functions $A, B$. The latter antibracket satisfies the differentiation property,

$$\Delta(A, B) = (\Delta A, B) - (A, \Delta B)(-1)^{\varepsilon(A)}, \quad (2.16)$$

and the non-polarized Jacobi identity,

$$((B, B), B) = 0, \quad \varepsilon(B) = 0. \quad (2.17)$$

In terms of the antibracket $\Delta$, the quantum master equation (2.3) rewrites naturally in its quadratic form

$$\frac{1}{2}(W, W) + \hbar i \Delta W = 0, \quad (2.18)$$

so that we have for $X$ the respective quadratic equation of the form,

$$\frac{1}{2}(X, X) + \sigma X = 0, \quad (2.19)$$

with $\sigma$ given in (2.2).

On the other hand, by choosing in (2.10)-(2.13),

$$\mu =: \frac{i}{\hbar} \delta \Psi(\Phi), \quad (2.20)$$

we arrive at the gauge independence on-shell,

$$\delta \Psi Z(J = 0) = 0. \quad (2.21)$$

Finally, we give a more explicit form of the solution (2.7) in the case when $F' = F$ is a function,

$$\exp \left\{ \frac{i}{\hbar} X \right\} = \exp \left\{ (E(\text{ad}(F)) \Delta F) \right\} \times$$

$$\times \exp \left\{ - \frac{i}{\hbar} W \right\} \exp \{ -\text{ad}(F) \} \exp \left\{ \frac{i}{\hbar} W \right\}, \quad (2.22)$$

where the notations

$$E(Z) =: \frac{\exp\{Z\} - 1}{Z} \quad (2.23)$$

and

$$\text{ad}(A)B =: (A, B) \quad (2.24)$$

are used.
3 Non-Abelian gauge-fixing

Let us generalize the generating functional (2.9) to cover the case of non-Abelian gauge fixing functions \( G_A \) that satisfy the antibracket involution relations,

\[
(G_A, G_B) = \mathcal{U}_{AB}^C G_C,
\]

(3.1)

together with the conditions,

\[
\varepsilon(G_A) = \varepsilon_A + 1,
\]

(3.2)

the even matrix,

\[
\left\| \frac{\partial G_A}{\partial \Phi_B} \right\|,
\]

(3.3)

is invertible. Let the functions \( F^A \) that satisfy

\[
\varepsilon(F^A) = \varepsilon_A,
\]

(3.4)

do complement the gauge fixing functions \( G_A \) in the sense that the reparametrization,

\[
\Phi^A, \Phi^*_A \Rightarrow F^A, G_A,
\]

(3.5)

is invertible, and let \( \mathcal{J} \) be the super-Jacobian of the latter reparametrization (3.5). Let us consider the generating functional given by the ”square root formula” [10],

\[
\mathcal{Z}[J] =: \int [D\Phi][D\Phi^*][DA] \sqrt{\mathcal{J} \text{sdet} ((F^A, G_B))} \times \exp \left\{ \frac{i}{\hbar} \left[ W + X + J_A \Phi^A + G_A \Lambda^A \right] \right\}.
\]

(3.6)

It has been shown in [10] that the integrand in (3.6) is independent of \( F^A \). Furthermore, it has then been shown in [10] that the path integral (3.6) is independent of \( G_A \) as well on-shell \( J = 0 \).

The factors inside the square root in the integrand in (3.6) can be presented in their integral form

\[
\mathcal{J} =: \int [D\bar{C}][DC] \exp \left\{ \frac{i}{\hbar} (\bar{C}_A F^A + \bar{C}^A G_A) \left( \frac{\partial}{\partial \Phi_B} C^B + \frac{\partial}{\partial \Phi^*_B} C_B \right) \right\},
\]

(3.7)

\[
\text{sdet} ((F^A, G_B)) =: \int [D\bar{B}][DB] \exp \left\{ \frac{i}{\hbar} \bar{B}_A (F^A, G_C) B^C \right\}.
\]

(3.8)
where the respective Grassmann parities are

\[
\begin{align*}
\epsilon(C_A) &= \epsilon_A + 1, \\
\epsilon(\bar{C} A) &= \epsilon_A, \\
\epsilon(C A) &= \epsilon_A + 1, \\
\epsilon(C A) &= \epsilon_A, \\
\epsilon(B A) &= \epsilon_A, \\
\epsilon(B A) &= \epsilon_A.
\end{align*}
\]

(3.9) (3.10) (3.11) (3.12) (3.13) (3.14)

Now, by multiplying the integral (3.7) by the one (3.8), consider, in their product, the "BRST" transformations,

\[
\begin{align*}
\delta C^A &= (\Phi^A, G_D) B^D \mu, \\
\delta C_A &= (\Phi^*_A, G_D) B^D \mu, \\
\delta \bar{B}_A &= \mu \bar{C}_A, \\
\epsilon(\mu) &= 1.
\end{align*}
\]

(3.15) (3.16) (3.17) (3.18)

It follows that the product of the integrals (3.7) and (3.8) is invariant on-shell, \(G_A = 0\), under the transformations (3.15)-(3.18), as for the case of \(\mu = \text{const}\).

On the other hand, when choosing \(\mu = \frac{i}{\hbar} \bar{B}_A \delta F^A (\frac{\partial}{\partial \Phi^D} C^D + \frac{\partial}{\partial \Phi^D} C_D)\),

(3.19)
in (3.15)-(3.18), one reproduces exactly the infinitesimal change

\[
F^A \rightarrow F^A + \delta F^A,
\]

(3.20)
in the product of the integrals (3.7) and (3.8).

Thus, we have shown that the integrand in the path integral (3.6) is independent of \(F^A\). Now, let us choose \(F^A = \Phi^A\). Then, it follows immediately that the path integral (3.6) reduces to its form (2.9) as for the case of the Abelian gauge fixing.

To compare for [10], here we have used the same "BRST" transformations (3.15)-(3.18), although without explicit use of the intrinsic mini-version of the master equation as for the BC- ghost system.

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