Kalman Filters and Homography: Utilizing the Matrix $A$

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Abstract
Many problems in Computer Vision can be reduced to either working around a known transform, or given a model for the transform computing the inverse problem of the transform itself. We will look at two ways of working with the matrix $A$ and see how transforms are at the root of image processing and vision problems.

Keywords Homography · Kalman Filters · Computer Vision

1 Introduction
As stated by a great teacher of linear algebra identifying the $A$ is crucially important in many Applied Mathematics problems because linear transformation is a key abstraction, we know everything when we know what happens to a basis. Many applications, from stress analysis to network analysis involves the construction of an $A$, i.e. a transformation that takes certain inputs and maps them to certain outputs.

2 Direct Linear Transform
In computer vision a transform between two 2D pictures that can involve a translation, scaling and a rotation is named a homography, and can be stated simply as;

$$x' = Hx$$

where $x, x'$ 2D pixel coordinates. In expanded form we have

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Homogeneous coordinates are used. The use of letter $H$ is the custom, but the transformation above in this problem is our $A$.

A special case of transformation, so-called affine transformation is shown below,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & t_x \\ a_3 & a_4 & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad x' = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} x$$

This transformation preserves $w = 1$ and for example the translation vector inside it is at $[t_x, t_y]$.

A similarity transformation is,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s \cos(\theta) & -s \sin(\theta) & t_x \\ s \sin(\theta) & s \cos(\theta) & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad x' = \begin{bmatrix} sR & t \\ 0 & 1 \end{bmatrix} x$$
Similarity transformations can include scale changes as well, rotation is captured with the cell elements containing sin, cos.

Now we arrive at the reverse problem; how, based on a few matching point between two images, one original the other transformed, do we find the homography, the transformation that led to the resulting matrix?

We assume the matching points are in the form of \(x_i, y_i\) and \(x'_i, y'_i\). If we expand \(x' - Hx = 0\) for every such matches we obtain,

\[
\begin{bmatrix}
-x_1 & -y_1 & -1 & 0 & 0 & 0 & x'_1 & x'_1 & x'_1 \\
0 & 0 & 0 & -x_1 & -y_1 & -1 & x_1 y_1 & y_1 & y_1 \\
-x_2 & -y_2 & -1 & 0 & 0 & 0 & x'_2 & x'_2 & x'_2 \\
0 & 0 & 0 & -x_2 & -y_2 & -1 & x_2 y_2 & y_2 & y_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
h_1 \\
h_2 \\
h_3 \\
h_4 \\
h_5 \\
h_6 \\
h_7 \\
h_8 \\
h_9 \\
\end{bmatrix} = 0
\]

More matches as data points would allow the expansion of the matrix vertically. The reason for representing \(x' - Hx = 0\) instead of \(x' = Hx\) is we can see the former as an optimization problem, even if we can’t solve for equality to zero, we can attempt to approach zero, by using a singular value decomposition method.

A fantastic and simple application of DLT can be demonstrated even without even known matches per se between two images, if we want certain section of an image to be extracted and scaled, rotated into a full-blown straight image, we could simply define four corners of the selection area to be the outermost corners of an imaginary target output. Let’s consider the Sudoku image below,

If we want to extract the sudoku section of that image,

```python
from scipy import ndimage
from PIL import Image
im = np.array(Image.open('sudoku81.JPG').convert('L'))
corners = [[257.4166, 14.9375],
           [510.8489, 197.6145],
           [59.30208, 269.65625],
           [325.598958, 469.05729]]
corners = np.array(corners)
plt.plot(corners[:,0], corners[:,1], 'rd')
plt.imshow(im, cmap=plt.cm.Greys_r)
```

Here picked corners of the image area are shown in red. Using the SVD calculation shown in the Appendix, we can obtain an \(H\). Now we can warp the selection using the newly found \(H\) and retrieve the full-blown extracted image.

```python
def warpfcn(x):
    x = np.array([x[0], x[1], 1])
    xt = np.dot(H, x)
    xt = xt/xt[2]
    return xt[0], xt[1]
im_g = ndimage.geometric_transform(im, warpfcn, (300,300))
```
3 Kalman Filters

Another domain where we ask “where is the $A$?” is filtering. But to be exact, in this case we are asking where is the $A$ and $H$? Kalman filters use two transformations in succession to compute a final model. The Kalman Filter model is,

$$x_{t+1} = Ax_t + Q$$
$$y_t = Hx_t + R$$

where $Q$ and $R$ are multivariate noise. Kalman filters are widely used for object tracking through noisy measurements. The $x_t$ could be the location of an object plus noise, $y_t$ could be measurement through visual, radar, sonar methods and is on a different domain than $x_t$. One obvious example is visual tracking of an object that moves in 3D space; we see its projection onto a 2D plane through a certain transformation (image projection) while successive $x_t$'s are transformed through a displacement transition due to the known movement mechanics of the object.

In filtering case both $A$, $H$ can be assumed as known, the aim is usually trying to “reverse the arrow”, meaning for given $y_t$ deducing a hidden $x_t$. In this sense Kalman filters are examples of recursive filters and $x_t$ and $y_t$ can be assumed to be distributed as Gaussians due to the added noise at both steps.

We can implement a visual tracking using the approaches outlined above. We move a chessboard plane on a flat surface (table) on constant speed toward our camera while tracking a single reference point on this image where $x_t$ is the 3D location of our chessboard plane. The transition equation of the Kalman filters only needed to account for constant velocity, along one axis. Matrix $A$ would look like,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that matrix $A$ is 4x4, not 3x3. We used homogenous coordinates to represent 3D location, which made $x_t$ a 4x1 vector, therefore the transition equation captured in $A$ had to be 4x4 so we could multiply it with $x_t$.

$A$ is the identity matrix $I$ with one difference; its $(3,4)$'th item is a constant, $d$, that is our displacement. To verify if this $A$ can be used for calculating displacement, one could perform a sample multiplication using
\( x_t = [a_1 \ a_2 \ a_3 \ a_4] \) and see \( Ax_t \), or \( x_{t+1} \), becomes \([a_1 \ a_2 \ a_3 + d \ a_4]\). We can see displacement \( d \) is added to the z-coordinate, depth. For our testing we picked \( d = -0.5 \), meaning for each frame chessboard plane moved 0.5 cm toward the camera. Each transition for \( x_t \) adds to uncertainty, and that is reflected in \( Q \).

Observation equation \( y_t \) handles the calculation where a 3D coordinate is projected onto 2D (pixel measurements) on screen with added noise. For standard pinhole camera model, camera matrix \( P \) (the \( H \) of the KF) is responsible for 2D projection. \( P \) is unique for each camera, and a process called camera calibration can determine camera matrix \( P \), and various methods for calibration exist in literature. OpenCV library contained a function for calibration which we tested, but we were not happy with the results. At the end, we calibrated the camera manually; by simply deciding on a 3D real world location (middle point at the end of our flat surface / table) and tested various \( P \) values while at the same time drawing an imaginary chessboard image on screen using this projection matrix \( P \). This was repeated until the imaginary board was located at the desired location.

```python
import cv2
dim = 3
if __name__ == '__main__':
    fin = sys.argv[1]
cap = cv2.VideoCapture(fin)
N = int(cap.get(cv2.CAP_PROP_FRAME_COUNT))
fps = int(cap.get(cv2.CAP_PROP_FPS))
kalman = Kalman(util.K, mu_init=array([1., 1., 165., 0.5]))
for i in range(N):
    ret, frame = cap.read()
    h, w = frame.shape[:2]
    gray = cv2.cvtColor(frame, cv2.COLOR_BGR2GRAY)
    status, corners = cv2.findChessboardCorners(gray, (dim,dim))
    is_x = []; is_y = []
    if status:
        cv2.drawChessboardCorners(gray, (dim,dim), corners, status)
        for p in corners:
            is_x.append(p[0][0])
            is_y.append(p[0][1])
        if len(is_x) > 0 :
            kalman.update(array([is_x[5], h-is_y[5], 1.]))
        util.proj_board(gray,
                        kalman.mu_hat[0],
                        kalman.mu_hat[1],
                        kalman.mu_hat[2])
        cv2.imshow('frame',gray)
        cv2.waitKey(20)
```

The tracking is computed with the code above, standard KF tracking. We used a chessboard image so that `cvFindChessboardCorners` and `cvDrawChessboardCorners` OpenCV functions could be used. Using these two methods, a chessboard image can be detected and marked (on screen, real-time) with great accuracy and speed. The chessboard image had 9 squares on it, giving us 9 points of which, we only used the 5th, middle point. At each frame `cvFindChessboardCorners` detected the 2D pixel locations, and we passed these values over to the Kalman filter that recursively updated its hidden state. In each case initial condition for the filter is the middle end point, somewhat away from both starting locations, therefore the uncertainty \( Q \) of the filter at time = 0 had to be large. Hence we used \( Q = I \cdot 150 \text{ cm} \) for the Kalman filter.

## 4 Appendix: Code

### 4.1 Homography, SVD

```python
import scipy, numpy.linalg as lin
```
from scipy import ndimage

def H_from_points(fp, tp):
    if fp.shape != tp.shape:
        raise RuntimeError('number of points do not match')

    m = np.mean(fp[:2], axis=1)
    maxstd = np.max(np.std(fp[:2], axis=1)) + 1e-9
    C1 = np.diag([1/maxstd, 1/maxstd, 1])
    C1[0][2] = -m[0]/maxstd
    C1[1][2] = -m[1]/maxstd
    fp = np.dot(C1, fp)

    m = np.mean(tp[:2], axis=1)
    maxstd = np.max(np.std(tp[:2], axis=1)) + 1e-9
    C2 = np.diag([1/maxstd, 1/maxstd, 1])
    C2[0][2] = -m[0]/maxstd
    C2[1][2] = -m[1]/maxstd
    tp = np.dot(C2, tp)

    nbr_correspondences = fp.shape[1]
    A = np.zeros((2*nbr_correspondences, 9))
    for i in range(nbr_correspondences):
        A[2*i] = [-fp[0][i], -fp[1][i], -1, 0, 0, 0,
                   tp[0][i]*fp[0][i], tp[0][i]*fp[1][i], tp[0][i]]
        A[2*i+1] = [0, 0, 0, -fp[0][i], -fp[1][i], -1,
                    tp[1][i]*fp[0][i], tp[1][i]*fp[1][i], tp[1][i]]

    U, S, V = lin.svd(A)
    H = V[8].reshape((3, 3))
    H = np.dot(lin.inv(C2), np.dot(H, C1))

    # normalize and return
    return H / H[2,2]

fp = [ [p[1],p[0],1] for p in corners]
fp = np.array(fp).T
tp = np.array([[0,0,1],[0,300,1],[300,0,1],[300,300,1]]).T
H = H_from_points(tp, fp)

4.2 Kalman Filter

from numpy import *

class Kalman:
    # T is the translation matrix

from scipy import ndimage

def H_from_points(fp, tp):
    if fp.shape != tp.shape:
        raise RuntimeError('number of points do not match')

    m = np.mean(fp[:2], axis=1)
    maxstd = np.max(np.std(fp[:2], axis=1)) + 1e-9
    C1 = np.diag([1/maxstd, 1/maxstd, 1])
    C1[0][2] = -m[0]/maxstd
    C1[1][2] = -m[1]/maxstd
    fp = np.dot(C1, fp)

    m = np.mean(tp[:2], axis=1)
    maxstd = np.max(np.std(tp[:2], axis=1)) + 1e-9
    C2 = np.diag([1/maxstd, 1/maxstd, 1])
    C2[0][2] = -m[0]/maxstd
    C2[1][2] = -m[1]/maxstd
    tp = np.dot(C2, tp)

    nbr_correspondences = fp.shape[1]
    A = np.zeros((2*nbr_correspondences, 9))
    for i in range(nbr_correspondences):
        A[2*i] = [-fp[0][i], -fp[1][i], -1, 0, 0, 0,
                   tp[0][i]*fp[0][i], tp[0][i]*fp[1][i], tp[0][i]]
        A[2*i+1] = [0, 0, 0, -fp[0][i], -fp[1][i], -1,
                    tp[1][i]*fp[0][i], tp[1][i]*fp[1][i], tp[1][i]]

    U, S, V = lin.svd(A)
    H = V[8].reshape((3, 3))
    H = np.dot(lin.inv(C2), np.dot(H, C1))

    # normalize and return
    return H / H[2,2]

fp = [ [p[1],p[0],1] for p in corners]
fp = np.array(fp).T
tp = np.array([[0,0,1],[0,300,1],[300,0,1],[300,300,1]]).T
H = H_from_points(tp, fp)

4.2 Kalman Filter

from numpy import *

class Kalman:
    # T is the translation matrix
# K is the camera matrix calculated by calibration

def __init__(self, K, mu_init):
    self.ndim = 3
    self.Sigma_x = eye(self.ndim+1)*150
    self.Phi = eye(4)
    self.Phi[2,3] = -0.5
    self.H = append(K, [[0], [0], [0]], axis=1)
    self.mu_hat = mu_init
    self.cov = eye(self.ndim+1)
    self.R = eye(self.ndim)*1.5

def normalize_2d(self, x):
    return array([x[0]/x[2], x[1]/x[2], 1.0])

def update(self, obs):
    # Make prediction
    self.mu_hat_est = dot(self.Phi, self.mu_hat)
    prod = dot(self.Phi, dot(self.cov, transpose(self.Phi)))
    self.cov_est = prod + self.Sigma_x

    # Update estimate
    prod = self.normalize_2d(dot(self.H, self.mu_hat_est))
    self.error_mu = obs - prod
    prod = dot(self.cov, transpose(self.H))
    prod = dot(self.H, prod)
    self.error_cov = prod + self.R
    prod = dot(self.cov_est, transpose(self.H))
    self.K = dot(prod, linalg.inv(self.error_cov))
    self.mu_hat = self.mu_hat_est + dot(self.K, self.error_mu)

    prod = dot(self.K, self.H)
    left = eye(self.ndim+1)
    diff = left - prod
    self.cov = dot(diff, self.cov_est)

References

[1] Solem, E. *Programming Computer Vision with Python*, 2012

[2] Strang, *Too Much Calculus*, [pdf](http://web.mit.edu/18.06/www/Essays/too-much-calculus.pdf)