Partial-limit solutions and rational solutions with parameter for the Fokas-Lenells equation

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Abstract A partial-limit method is developed to understand solutions related to real eigenvalues for the Fokas–Lenells equation. By applying a partial-limit procedure to soliton solutions of the Fokas–Lenells equation, new multiple-pole solutions related to real repeated eigenvalues are obtained. For the envelop $|u|^2$, the simplest solution corresponds to a real double eigenvalue, showing a solitary wave with algebraic decay. Two such solitons allow elastic scattering but asymptotically with zero phase shift. Single eigenvalue with higher multiplicity gives rise to rational solutions which contain an intrinsic parameter, live on a zero background, and have slowly changing amplitudes. The partial-limit procedure may be extended to other integrable systems and generate new solutions.

Keywords Fokas–Lenells equation · Multiple-pole solution · Real eigenvalue · Rational solution · Asymptotic property

1 Introduction

The Fokas–Lenells (FL) equation,

$$u_{xt} + u - 2i\delta|u|^2 u_x = 0, \quad \delta = \pm 1,$$

has both physical and mathematical significance. Here $i$ is the imaginary unit, $|u|^2 = uu^*$ and $*$ stands for complex conjugate. This equation has been used in 1970s [1] (also see [2,3]) to generate solutions to the massive Thirring model (in light-cone coordinates) [4–6]

$$\chi_{1,x} + i\chi_2 + i|\chi_2|^2 \chi_1 = 0,$$
$$\chi_{2,t} + i\chi_1 + i|\chi_1|^2 \chi_2 = 0,$$

which was initially introduced by Thirring [4] to describe the theory of a massive fermion field coupled to a two-component vector field interacting with itself [5] and it was shown integrable by Mikhailov in 1976 [6]. The connection between the FL equation and the massive Thirring model is given as [1–3]

$$\chi_2 = u^*x e^{-i\beta}, \quad \chi_1 = -iu^* e^{-i\beta},$$
$$\beta = \int_{-\infty}^{x} |u_x|^2 dy.$$

The FL equation resulted as a reduction of $v = \delta u^*$ from the Mikhailov model [2,7]

$$u_{xt} + u - 2iuvu_x = 0, \quad v_{xt} + v + 2iuvu_x = 0,$$

which is a negative-order equation related to the Kaup–Newell (KN) spectral problem [1,8]

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \frac{i}{2} \lambda^2 & \lambda q \\ \frac{i}{2} \lambda r & -\frac{i}{2} \lambda^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where $\lambda$ is a spectral parameter (eigenvalue), $q, r$ are functions of $(x, t) \in \mathbb{R}^2$, and $q = u_x, r = v_x$. Mathematically, FL equation (1) (up to some transformations [9]) was rederived by Fokas in 1995 [10] as a generalization of the nonlinear Schrödinger equation using bi-Hamiltonian structures of the Ablowitz-Kaup-Newell-Segur system. Later, it was also derived by Lenells as a
model of propagation of nonlinear pulses in monomode optical fibers [9], where \(|u|^2\) describes the slowly varying envelope of the pulse. Now equation (1) is known as the FL equation.

Several classical methods have been devoted to seeking solutions for the FL equation, such as the inverse scattering transform or the Riemann–Hilbert method [11–13], dressing method [14], algebraic geometry method [15], Darboux transformation [16,17] and bilinear method [18–20], and soliton solution, algebro-geometric solution and rogue waves have been obtained. It is notable that in the analytic approaches, such as the inverse scattering transform and Riemann–Hilbert method [11,12], soliton solutions to equation (1) were obtained by assuming the discrete eigenvalues \(\lambda_j\) are distinct and do not locate on the coordinate axes of the complex plane. In a recent paper [20], FL equation (1) was solved via bilinear approach and solutions were obtained in terms of double Wronskians. In [20], apart from solitons, solutions related to real eigenvalues were found for FL equation (1) with \(\delta = 1\). The simplest case yields a periodic solution, and it is notable that the solution related to two distinct real double eigenvalues yields two nonsingular solitary waves with algebraic decay. Such a solution of the FL equation was not reported before paper [20].

In the present paper, to understand more about the solutions related to real eigenvalues for FL equation (1) with \(\delta = 1\), a partial-limit procedure will be applied to the solutions of the FL equation and in this way those solutions related to real repeated eigenvalues can be recovered. This will provide a better understanding to the scattering of the solution related to two distinct real double eigenvalues which has been obtained in [20]. In addition, rational solutions (in terms of the envelop \(|u|^2\)) will be derived as a new result for the FL equation. Different from the rogue waves of the FL equation [16], the new rational solutions contain an intrinsic real parameter and live on a zero background.

Conventionally, multiple-pole solutions mean those solutions related to the eigenvalues as multiple poles. Therefore, in principle, multiple-pole solutions can be understood as a result of some limits taking from solitons. Early examples of multiple-pole solutions can be referred to the double-pole solutions of the sine-Gordon equation [21] and the nonlinear Schrödinger equation [22]. It is more convenient to implement a limit procedure on a partial-limit procedure will be implemented to the one-soliton solution of the FL equation and a new rational solution will be derived which is a solitary wave with algebraic decay. Next, general formulae of double Wronskians will be presented for the solutions related to real eigenvalues with higher multiplicity and their scattering property will be investigated. Finally, conclusion and discussion will be given in Sect. 4.

2 Double Wronskian solutions of the FL equation

In [20], Mikhailov model (4) was solved by using bilinear approach and solutions were given in terms of double Wronskians. By means of a reduction technique, solutions in double Wronskian form for FL equation (1) were obtained. In this section, some notations and main formulae obtained in [20] for the solutions of the FL equation will be sketched.

Bilinear form of FL equation (1) is

\[
D_x D_t g \cdot f + gf = 0, \quad (6a)
\]

\[
D_x D_t f \cdot f^* + i\delta D_x g \cdot g^* = 0, \quad (6b)
\]

\[
D_t f \cdot f^* + i\delta gg^* = 0, \quad (6c)
\]

where

\[
u = \frac{g}{f}, \quad (7)
\]

and \(D\) is Hirota’s bilinear operator defined as [28]

\[
D_x^m D_t^n f \cdot g \equiv (\partial_x - \partial_x')^m (\partial_t - \partial_t')^n f(x,t) \cdot g(x',t')|_{x'=x,t'=t}. \quad (8)
\]

Introduce double Wronskians (cf. [29])

\[
|\tilde{N}; \tilde{N} - 1| = |\partial_x \phi, \partial_x^2 \phi, \ldots, \partial_x^N \phi, \psi, \partial_x \psi, \ldots, \partial_x^{N-1} \psi|,
\]

\[
|\tilde{N}; \tilde{N} - 1| = |\phi, \partial_x \phi, \ldots, \partial_x^N \phi; \partial_x \psi, \partial_x^2 \psi, \ldots, \partial_x^{N-1} \psi|,
\]
where
\[ \phi = (\phi_1, \phi_2, \ldots, \phi_{2N})^T, \quad \psi = (\psi_1, \psi_2, \ldots, \psi_{2N})^T, \]
and \( \phi_j \) and \( \psi_j \) are functions of \((x, t)\).

Proposition 1 [20] Bilinear FL equation (6) admits solutions
\[ f = |\tilde{N}; \tilde{N} - 1|, \quad g = |\tilde{N}; \tilde{N} - 1|, \]
where
\[ \phi = \exp\left(\frac{i}{2}A^2x + \frac{i}{2}A^{-2}t\right)C, \quad \psi = S\phi^*, \]
\( A, S \in \mathbb{C}_{2N \times 2N}, \ |A| \neq 0, \) and \( A \) and \( S \) satisfy
\[ A^2 = \delta SS^*, \quad \delta = \pm 1. \]
The envelop \(|u|^2\) can be expressed as
\[ |u|^2 = i\delta \left( \frac{\ln f}{f} \right)_t = 2\delta \left( \frac{\text{Re}[f]}{\text{Im}[f]} \right)_t. \]
Note that the Wronskians \( f \) and \( g \) also provide solutions to massive Thirring model (2) by
\[ \chi_1 = -i\frac{g^*}{f}, \quad \chi_2 = \left( \frac{g^*}{f^*} \right)_x \]

A set of special solutions to matrix equation (12) are given in terms of
\[ S = AT = TA^*, \quad TT^* = \delta I_{2N}, \]
\[ T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \quad A = \begin{pmatrix} K_1 \theta_N \\ 0_{N} K_4 \end{pmatrix}, \]
where \(|I_N|\) is the identity matrix of order \( N \). For \( \delta = 1 \), the allowed solutions are presented in Table 1.

Explicit formula for \( \phi \) corresponding to Table 1 is given by
\[ \phi = \left( \phi^+, \phi^- \right), \]
where \( \phi^\pm = (\phi^1, \phi^2, \ldots, \phi_N)^T \) take the forms
\[ \phi^+ = \exp\left[ \frac{i}{2} (K_1^2 x + K_1^{-2} t) \right] C^+, \]
\[ \phi^- = \exp\left[ \frac{i}{2} (K_2^2 x + K_4^{-2} t) \right] C^-, \]
and \( C^\pm \in \mathbb{C}_{N \times 1} \). For case (1), when
\[ K_N = D[k_j]_{j=1}^N \equiv \text{Diag}(k_1, k_2, \ldots, k_N), \quad k_j \in \mathbb{C}, \]
\( \phi^\pm \) can be taken as
\[ \phi^+ = (c_1^+ e^\eta(k_1), c_2^+ e^\eta(k_2), \ldots, c_N^+ e^\eta(k_N))^T, \]
\[ \phi^- = (c_1^- e^\eta(k_1), c_2^- e^\eta(k_2), \ldots, c_N^- e^\eta(k_N))^T, \]
where
\[ \eta(k) = \frac{i}{2} (k^2 x + k^{-2} t). \]
For case (2), note that \( K_N \) and \( H_N \) are independent real matrices. When
\[ K_N = D[k_j]_{j=1}^N, \quad j \in \mathbb{R}, \]
there is
\[ \phi^+ = (c_1^+ e^\eta(k_1), c_2^+ e^\eta(k_2), \ldots, c_N^+ e^\eta(k_N))^T, \]
where \( c_j^+ \in \mathbb{C} \). When
\[ K_N = J_N[k]_{j=1}^N, \quad k \in \mathbb{R}, \]
\( \phi^+ \) takes the form
\[ \phi^+ = \left( c^+ e^\eta(k), \partial_k (c^+ e^\eta(k)), \frac{1}{2!} \partial_k^2 (c^+ e^\eta(k)), \right. \]
\[ \left. \ldots, \frac{1}{(N-1)!} \partial_k^{N-1} (c^+ e^\eta(k)) \right)^T, \]
where \( c \in \mathbb{C} \). When \( H_N = D[h_j]_{j=1}^N, \quad h_j \in \mathbb{R} \), \( \phi^- \) takes the form of (21) with replacement \((k_j, c_j^+) \) by \((h_j, c_j^-)\). When \( H_N = J_N[h], \quad h \in \mathbb{R} \), \( \phi^- \) takes the form of (23) with replacement \((k, c^+) \) by \((h, c^-)\).

Since this paper only focuses on solitons and those solutions related to real eigenvalues, I skip the case of \( K_N = J_N[k] \) of case (1) and the general mixed case. For more details about these formulae one can refer to [20].

3 Partial-limit solutions

In this section, first, a partial-limit procedure will be implemented on the double Wronskians of one-soliton solution of FL equation (1) with \( \delta = 1 \). The one-soliton solution is associated with a pair of complex eigenvalues, \( k \) and \( k^* \), and to get a nonzero solution, neither
the real part nor imaginary part of $k$ can be zero. In the partial-limit procedure, the imaginary part of $k$ will be naively taken to approach to zero so that $k$ becomes a real double eigenvalue. It is surprised that the resulted solution is nonzero and nonsingular. Such a partial-limit procedure will be the starting point of this section. It will not only provide a better understanding for the solution that was reported in [20] generated from two distinct real double eigenvalues, but also give rise to new rational solutions with an intrinsic real parameter for the FL equation.

\[
\begin{aligned}
|u|^2 &= \frac{8a_1^2 b_1^2}{(a_1^2 + b_1^2)^3} \\
&\times \cosh\left(4a_1 b_1 x - \frac{4a_1 b_1 t}{(a_1^2 + b_1^2)^2} + 2 \ln \left|\frac{k_1^*}{k_1}\right|\right) - \frac{a_1^2 - b_1^2}{a_1^2 + b_1^2}.
\end{aligned}
\]

(27)

where $a_1 = \text{Re}[k_1]$, $b_1 = \text{Im}[k_1]$. Note that, obviously, the product $a_1 b_1$ must NOT be zero; otherwise, $|u|$ is null. This means, to generate usual solitons, the eigenvalues $\{k_j\}$ should not locate on coordinate axes.

To implement a partial-limit procedure, one can take $c_1^\pm = 1$ and expand each element in $f$ and $g$ w.r.t. $b_1$ at $b_1 = 0$. The resulted $f$ and $g$ can be written as

\[
f = b_1 \left[ \frac{i a_1^2 e^{\eta(x)}}{a_1^2} + O(b_1) \right] \frac{a_1 e^{-\eta(x)}}{a_1^2} + O(b_1) \Bigg|_{x = 0} \Bigg|_{a_1^2 \to 0}.
\]

(24)

\[
g = b_1 \left[ e^{\eta(x)} + O(b_1) \right] \frac{a_1 e^{-\eta(x)}}{a_1^2} + O(b_1) \Bigg|_{x = 0} \Bigg|_{a_1^2 \to 0}.
\]

3.1 Partial-limit solution and algebraic one-soliton

The one-soliton solution of FL equation (1) with $\delta = 1$ is given by (cf. [20])

\[
|u|^2 = \frac{8a_1^2 b_1^2}{(a_1^2 + b_1^2)^3} \\
\]

\[
\times \cosh\left(4a_1 b_1 x - \frac{4a_1 b_1 t}{(a_1^2 + b_1^2)^2} + 2 \ln \left|\frac{k_1^*}{k_1}\right|\right) - \frac{a_1^2 - b_1^2}{a_1^2 + b_1^2}.
\]

(27)

Then, substituting the above forms into (24) and taking a limit for $b_1 \to 0$, it turns out that

\[
u = \left[ \frac{i a_1^2 e^{\eta(x)}}{a_1^2} + O(b_1) \right] \frac{a_1 e^{-\eta(x)}}{a_1^2} + O(b_1) \Bigg|_{x = 0} \Bigg|_{a_1^2 \to 0}.
\]

(28)

It can be verified that this is a solution to FL equation (1) with $\delta = 1$. The corresponding envelop is

\[
|u|^2 = \frac{8a_1^2 b_1^2}{(a_1^2 + b_1^2)^3} \\
\]

\[
\times \cosh\left(4a_1 b_1 x - \frac{4a_1 b_1 t}{(a_1^2 + b_1^2)^2} + 2 \ln \left|\frac{k_1^*}{k_1}\right|\right) - \frac{a_1^2 - b_1^2}{a_1^2 + b_1^2}.
\]

(27)

This is a nonsingular solitary wave as depicted in Fig. 1a. Different from the usual soliton, for example, (27), which, for given $t$, decays exponentially as $|x| \to +\infty$, here for given $t$, $|u|^2$ decays algebraically as $|x| \to +\infty$.

By examining column vectors in (28), it is found that (28) is equivalently generated by

\[
\phi = \left( \frac{e^{\eta(x)}}{\partial a_1 e^{\eta(x)}} \right), \quad \psi = S \phi^*, \quad S = \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_1 \end{array} \right).
\]

Table 1 $T$ and $A$ for (15) with $\delta = 1$

| Case | $T$ | $A$ |
|------|-----|-----|
| (1)  | $T_1 = T_3 = 0, T_2 = T_3 = I_N$ | $K_1 = K_4^* = K_N \in \mathbb{C}_{N \times N}$ |
| (2)  | $T_1 = \pm T_4 = I_N, T_2 = T_3 = 0$ | $K_1 = K_N \in \mathbb{R}_{N \times N}, K_4 = H_N \in \mathbb{R}_{N \times N}$ |



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This is a new solution to (11) and (12) where
\[
A = S = \begin{pmatrix} a_1 & 0 \\ 1 & a_1 \end{pmatrix},
\]
(30)
since such \(A\) and \(S\) are not included in Table 1 in [20]. Note that (29) provides an algebraic one-soliton solution because it behaves like a “soliton” with a constant amplitude \(\frac{4}{a^2_1}\) and a constant speed \(x'(t) = \frac{1}{a^2_1}\).

Next, scattering of two such algebraic solitons will be explored.

### 3.2 Elastic scattering of two algebraic solitons

For convenience, introduce
\[
A_j = S_j = \begin{pmatrix} a_j & 0 \\ 1 & a_j \end{pmatrix}, \quad a_j \in \mathbb{R},
\]
(32)
and define
\[
A = \text{Diag}(A_1, A_2, \ldots, A_N),
\]
\[
S = \text{Diag}(\epsilon_1 S_1, \epsilon_2 S_2, \ldots, \epsilon_N S_N),
\]
(33)
where \(\epsilon_j\) takes either +1 or −1. Obviously, \(A = SS^*\), satisfying (12) for \(\delta = 1\). For such \(A, \phi\) and \(\psi\) defined in (11) can be taken as the following explicit forms
\[
\phi = (e^{t(a_1)}\phi, \partial x_1 e^{t(a_2)}\phi, \partial x_2 e^{t(a_3)}\phi, \ldots, e^{t(a_N)}\phi),
\]
\[
\partial_x e^{t(a_1)}\psi, \partial_x e^{t(a_2)}\psi, \ldots, e^{t(a_N)}\psi, \psi = S\phi^*,
\]
(34)
where \(\eta(a)\) is defined by (20).

It has been demonstrated that by the envelop \(|u|^2\) the solution \(u\) generated from \(A_1\) behaves like a single soliton. It is natural to investigate the solution \(u\) generated from \(A_1\) and \(A_2\) and see if the corresponding envelop \(|u|^2\) can exhibit elastic scattering of two solitons. For this purpose, consider
\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & -S_2 \end{pmatrix}.
\]
(35)
Correspondingly,
\[
u = g f
\]
\[
= \begin{pmatrix} |\phi| \eta(a_1) \phi, \partial_x |\phi| \eta(a_1) \phi, \partial_x |\psi| \eta(a_1) \psi \\ |\psi| \eta(a_1) \phi, \partial_x |\psi| \eta(a_1) \phi, \partial_x |\psi| \eta(a_1) \psi \end{pmatrix}.
\]
(36)
This is completely the same solution as discussed in Sec. 4.2.2 in [20]. For the sake of completeness, the formula of \(|u|^2\) is presented below [20],
\[
|u|^2 = \frac{4(a^2_2 - a^2_1)^2 G_2}{F_2},
\]
(37)
where
\[
G_2 = M_1^2 + M_2^2 + M_3^2 + M_4^2 + 2(M_1 M_3 + M_2 M_4) \cos \theta_1 + 2(M_1 M_4 - M_2 M_3) \sin \theta_1,
\]
\[
F_2 = N_1^2 + N_2^2 + N_3^2 + N_4^2 + 2 N_1 N_2 \cos 2 \theta_1 - 2 N_3 (N_1 - N_2) \sin \theta_1 + 2 N_4 (N_1 + N_2) \cos \theta_1,
\]
with
\[
\vartheta_1 = (a_1^2 - a_2^2) \left( x - \frac{t}{a_1^2 a_2^2} \right),
\]
\[
M_1 = -2 a_1 (a_1^2 - a_2^2)(a_4^2 x - t),
\]
\[
M_2 = -a_1 a_2^2 (3a_1^2 + a_2^2),
\]
\[
M_3 = 2a_2 (a_1^2 - a_2^2)(a_4^2 x - t),
\]
\[
M_4 = -a_1^2 a_2^2 (3a_2^2 + a_1^2),
\]
\[
N_1 = -4 a_2^5 a_1^3,
\]
\[
N_2 = -4 a_2^3 a_1^5,
\]
\[
N_3 = -2(a_1^2 - a_2^2)(a_1^4 - a_2^4)(a_4^2 a_1^2 x - t),
\]
\[
N_4 = -a_1^2 a_2 (a_2^2 + 6a_1^2 a_1^2 + a_1^4) + 4(a_1^2 - a_2^2)^2 (t - a_1^2 x)(t - a_2^2 x).
\]

For the scattering property, one can refer to Proposition 3 in [20]. In the coordinate frames \((x_j = x - \frac{t}{a_j^2}, t)\), asymptotically,
\[
|u|^2 \sim \frac{4}{a_j^2 (1 + 4a_j^2 x_j^2)}, \quad (t \to \pm \infty),
\]
(38)
for \(j = 1, 2\).

It is notable that such \(|u|^2\) can be considered as the interaction of two single algebraic solitons derived in the previous subsection. In addition, \(|u|^2\) allows either attractive interaction (see also Fig. 6 in [20]) or repulsive interaction, see Fig. 1b, c, and asymptotically, both cases have no phase shift, although near the interesting point phase shifts can be observed (in other words, the two waves will be eventually back to their original tracks). Such a feather of phase shifts is different from the usual soliton interactions.

It is also notable that, when \(N\) is odd, (33) will give rise to new solutions, since such a case cannot be obtained from assumption (15).
3.3 Rational solutions and new asymptotic property

3.3.1 Rational solutions

Consider

\[ S = J_{2N} [a], \quad A = S, \quad a \in \mathbb{R}, \]

where \( J_{2N} [a] \) is a Jordan matrix defined as in (41). In this case, \( \phi \) and \( \psi \) take the forms

\[
\phi = (e^{\eta(a)}, \partial_a e^{\eta(a)}, \frac{1}{2!} \partial_a^2 e^{\eta(a)}, \ldots, \frac{1}{(2N - 1)!} \partial_a^{2N-1} e^{\eta(a)},)^T, \quad \psi = S \phi^*,
\]

where \( \eta \) is defined as in (20). Consequently, \( f = |\tilde{N}; \tilde{N} - 1| \) composed by the above \( \phi \) and \( \psi \) is a pure complex polynomial of \((x, t)\), while \( g = |\tilde{N}; \tilde{N} - 1| \) is a complex polynomial of \((x, t)\) multiplied by \( e^{2\eta(a)} \). As a result, although \( u = g / f \) is not a form of rational solution, the carrier wave \( |u|^2 \) is. In this sense, the solution is a rational solution of the FL equation in terms of \(|u|^2\). In addition, the structure of \( u = g / f \) also indicates that \( e^{-2\eta(a)} u \) has a form of rational solution, which implies that one can assume a proper form of \( u \) and directly calculate low-order rational solutions (see [30] for the modified Korteweg-de Vries equation).

In such a solution, \( a \) is the single parameter and it cannot be zero. More parameters can be introduced by the trick of employing lower triangular Toeplitz matrices, which is defined as the following.
where $s_j \in \mathbb{C}$. Lower triangular Toeplitz matrices commute with the same type of matrices and play useful roles in expressing multiple-pole solutions (e.g., [24–27]). For the above $\phi$ given in (40), noting that \( A T_{2N} = T_{2N} A \), obviously, the double Wronskians \( f = |\tilde{N}; N - 1| \) and \( g = |\tilde{N}; N - 1| \) composed by \( \phi' = T_{2N} \phi \) and \( \psi' = S \phi^* \) are still solutions to bilinear equations (6). In this case, \( \{s_j\} \) will appear in the solution $u$ as parameters. However, $a$ is special as it is the parameter inheriting from the eigenvalue of the KN spectral problem. Compared with those \( \{s_j\} \) in $T_{2N}$, $a$ is considered as an intrinsic parameter.

### 3.3.2 New asymptotic property

Consider solution

\[
u = \frac{g}{f} = \frac{|\phi, \partial_x \phi, \partial_x^2 \phi; \partial_x \psi\rangle}{|\partial_x \phi, \partial_x^2 \phi; \psi, \partial_x \psi\rangle}, \tag{42}\]

where

\[
\phi = (e^{a(a)}, \partial_a e^{a(a)}, \frac{1}{2!} \partial_a^2 e^{a(a)}, \frac{1}{3!} \partial_a^3 e^{a(a)})^T, \quad \psi = S \phi^*, \quad S = J_4[a], \quad A = S. \tag{43}\]

The envelop $|u|^2$ reads

\[
|u|^2 = \frac{G'}{F'}, \tag{44a}\]

where

\[
G = 16a^2[9a^{12} + 64X^6 + 36a^8(4t - 3X)(4t - 5X) + 48a^4X^2(4t - 3X)(12t - 7X)], \tag{44b}\]

\[
F = 9a^{16} + 256X^8 - 256a^4X^4(24t^2 - 24tX - X^2) + 144a^{12}(8t^2 - 8tX + 5X^2) + 288a^8(128t^4 - 256r^3X + 160r^2X^2 - 32rX^3 + 7X^4), \tag{44c}\]

and

\[
X = t - a^4x. \tag{45}\]

The solution is depicted in Fig. 2a, which exhibits an interaction of two curved waves with different amplitudes.

In the following, by means of asymptotic analysis an unusual asymptotic property of such a rational solution will be explored.

For convenience, it is equivalent to consider (44) in the coordinate frame $(X, t)$, as depicted in Fig. 2c. Note that (45) is an linear transformation which does not change asymptotic property of $|u|^2$. Then, in the coordinate frame

\[
(X, T_1 = t - (\frac{\sqrt{3}}{6a^2} + \frac{X}{2} + \frac{\sqrt{3}}{4}a^2)) \tag{46}\]

$|u|^2$ is rewritten as

\[
|u|^2 = \frac{G'}{F'}, \tag{47a}\]

where

\[
G' = 16[117a^{12} + 444\sqrt{3}a^{10}(2T_1 - X) + 96\sqrt{3}a^6X^2(14T_1 - 3X) + 128\sqrt{3}a^2X^4(6T_1 - X) + 256X^6 + 96a^4X^2(24T_1^2 - 8T_1X + 7X^2) + 36a^8(16T_1^2 - 16T_1X + 19X^2)], \tag{47b}\]

\[
F' = a^2[1521a^{12} + 7488\sqrt{3}a^{10}T_1 + 3072 \sqrt{3}a^2T_1X^2(8T_1^2 - X^2) + 144a^8(296T_1^2 + 17X^2) + 384\sqrt{3}a^6T_1(96T_1^2 + 25X^2) + 192a^4(19T_1^4 + 240T_1^2X^2 + 11X^4) + 1024X^4(12T_1^2 + X^2)]. \tag{47c}\]
The curve \( T_1 = 0 \) is a parabola \( t = t(X) \) opening upward, on which \( |u|^2 \) reads
\[
|u|^2 = \frac{16(117a^{12} - 144\sqrt{3}a^{10}X + 684a^8X^2 - 288\sqrt{3}a^6X^3 + 672a^4X^4 - 128\sqrt{3}a^2X^5 + 256X^6)}{a^2(1521a^{12} + 2448a^8X^2 + 2112a^4X^4 + 1024X^6)},
\]
(48)
which is not a constant. Roughly speaking, when \( |X| >> 0 \), from Fig. 3a it can be seen that the value of \( |u|^2 \) slowly increases w.r.t. \( X \) when \( X < 0 \) and also slowly increases w.r.t. \( X \) when \( X > 0 \). On the other hand, considering \( |u|^2 \) (44) in the coordinate frame
\[
\left( X, T_2 = t - \left( -\frac{\sqrt{3}X^2}{6a^2} + \frac{X}{2} - \frac{\sqrt{3}}{4}a^3 \right) \right),
\]
(49)
it is found that, on the curve \( T_2 = 0 \), \( |u|^2 \) reads
\[
|u|^2 = \frac{16(117a^{12} + 144\sqrt{3}a^{10}X + 684a^8X^2 + 288\sqrt{3}a^6X^3 + 672a^4X^4 + 128\sqrt{3}a^2X^5 + 256X^6)}{a^2(1521a^{12} + 2448a^8X^2 + 2112a^4X^4 + 1024X^6)},
\]
(50)
which, when \( |X| >> 0 \), slowly decreases w.r.t. \( X \) when \( X < 0 \) and slowly decreases w.r.t. \( X \) when \( X > 0 \). Eventually, in both coordinate frames (46) and (49), it turns out that
\[
|u|^2 \sim \frac{4}{a^2}, \quad (|X| \to +\infty).
\]

For the waves depicted in Figs. 2c, 3d compares their amplitudes at different time. The above analysis can be summarized as the following.

**Theorem 1** Consider the envelop \( |u|^2 \) given in (44) in the coordinate frame \((X, t)\). Asymptotically, the waves travel along the curves
\[
t = \frac{\sqrt{3}X^2}{6a^2} + \frac{X}{2} + \frac{\sqrt{3}}{4}a^3
\]
(51)
and
\[
t = -\frac{\sqrt{3}X^2}{6a^2} + \frac{X}{2} - \frac{\sqrt{3}}{4}a^3.
\]
(52)
The amplitudes of the waves are not constants but slowly changing, and finally, they approach to \( \frac{4}{a^2} \) when \( |X| \to +\infty \).

**Theorem 2** In the coordinate frame \((x,t)\), for the envelop \( |u|^2 \) given in (44), when \( |x| \) is large enough, the two waves travel, respectively, along the curves (see Fig. 2b)
\[
t = a^4x + \frac{\sqrt{3}}{2}a^2 \pm a^2\sqrt{8\sqrt{3}a^2 x - 3}, \quad (x >> 0)
\]
(53)
and
\[
t = a^4x - \frac{\sqrt{3}}{2}a^2 \pm a^2\sqrt{-8\sqrt{3}a^2 x - 3}, \quad (x << 0).
\]
(54)

The amplitudes of the waves are not constants but slowly changing, and finally, they approach to \( \frac{4}{a^2} \) when \( |t| \to +\infty \). More precisely, asymptotic properties are listed in Table 2.

Note that in the rational solution the two waves asymptotically travel along parabolas, which is different from the asymptotic tracks of double-pole solutions of solitons which are usually governed by logarithmic functions (cf. [22,27,31,32]).

**4 Conclusions and remarks**

In this paper, by implementing a partial-limit procedure solutions related to real repeated eigenvalues for FL equation (1) with \( \delta = 1 \) were investigated. The simplest solution is generated by a real double eigenvalue of the KN spectral problem, and the envelop \( |u|^2 \) of the solution is a rational solution and provides a solitary wave with algebraic decay. Interaction of two such waves, described by the solution generated by two distinct real double eigenvalues, coincides with the solution given in §4.2.2 in [20]. Thus, a better understanding has been provided for such a solution. Note that when \( N \) is odd, the solution generated from \( N \) distinct real double eigenvalues is new since this case is not based on assumption (15) and not included in Table 1. A comparison between some solutions related to real
Fig. 2  Rational solutions and asymptotic property.  

(a) $|u|^2$ given by (44) in coordinates $(x, t)$ with $a = 1.5$.  
(b) Density plot of (a) with larger range $x \in [-15, 15]$, $t \in [-50, 50]$ and overlapped with red asymptotic curves (53) and (54) where $a = 1.5$.  
(c) $|u|^2$ given by (44) in coordinates $(X, t)$ with $a = 1.5$.  
(d) Density plot of (c) with larger range $X \in [-60, 60]$, $t \in [-150, 150]$ and overlapped with red asymptotic curves (51) and (52) where $a = 1.5$.

Table 2  Asymptotic properties of $|u|^2$ given by (44) as depicted in Fig. 2a

| Branches     | Asymptotic curve                                                                 | Amplitude changing with respect to $t$ |
|--------------|-----------------------------------------------------------------------------------|----------------------------------------|
| Down-left    | $t = a^4x - \frac{\sqrt{3}}{2}a^2 + a^2\sqrt{-8\sqrt{3}a^2x - 3}$                | Increase                               |
| Down-right   | $t = a^4x - \frac{\sqrt{3}}{2}a^2 - a^2\sqrt{-8\sqrt{3}a^2x - 3}$                | Decrease                               |
| Up-left      | $t = a^4x + \frac{\sqrt{3}}{2}a^2 + a^2\sqrt{8\sqrt{3}a^2x - 3}$                | Increase                               |
| Up-right     | $t = a^4x + \frac{\sqrt{3}}{2}a^2 - a^2\sqrt{8\sqrt{3}a^2x - 3}$                | Decrease                               |
Fig. 3 Slowly changing amplitudes of rational solution (47). a Plot of $|u|^2$ on $T_1 = 0$, given by (48) with $a = 1.5$. b Amplitudes changing of $|u|^2$, w.r.t. time $t$, given by (44) in coordinates $(X, t)$: red curve, blue dashed curve and black dotted curve stand for the shapes of $|u|^2$ at $t = 30, t = 230$ and $t = 830$, respectively. (Colour figure online)

Table 3 Comparison of some solutions related to real eigenvalues with Ref. [20]

| Case | Related real eigenvalues | Solutions obtained in Ref. [20] | Solutions obtained in the present paper |
|------|--------------------------|---------------------------------|----------------------------------------|
| (1)  | $2N$ distinct real simple eigenvalues | Available | Not available |
| (2)  | $N$ distinct real double eigenvalues, $N$ is even | Available | Available |
| (3)  | $N$ distinct real double eigenvalues, $N$ is odd | Not available | Available |
| (4)  | Single real eigenvalue with multiplicity $2N - 1$ | Not available | Available |

Eigenvalues obtained in [20] and the present paper is listed in Table 3. In case (3) and (4) solutions obtained in the present paper are new to FL equation (1) with $\delta = 1$, but in case (1) those solutions related to real simple eigenvalues cannot be obtained by taking partial limit.

Such type of solutions (related to real repeated eigenvalues) exhibits new asymptotic properties. For example, for solution (36) with envelop (37), which is related to two distinct real double eigenvalues $a_1$ and $a_2$, as demonstrated in [20] the two waves do have apparent phase shifts near the interaction point, but for large time $t$ (i.e., asymptotically), they travel along two straight lines without any phase shifts. This is different from the interaction of usual solitons. Another new feature is the asymptotic property of rational solutions. As analyzed and illustrated in Sect. 3.3, the envelop $|u|^2$ contains two curved waves with different amplitudes that are slowly changing and finally approach to a same value as $|t| \to +\infty$.

For FL equation (1) with $\delta = -1$, there are solutions related to pure imaginary eigenvalues and the partial limit should be implemented on $k = a + ib$ by taking $a \to 0$ in instead of $b \to 0$. The vector $\phi$ will be composed by $e^{\eta(b)}$ and $\frac{1}{\pi} b \partial_b e^{\eta(b)}$ where $b$ is real and $\eta(b)$ is defined by (20). Corresponding to (31), in the simplest case, $A = S = i \begin{pmatrix} b & 0 \\ 1 & b \end{pmatrix}$. However, these solutions can be obtained as complex conjugates of those solutions given in Sect. 3. In fact, if $u$ is a solution of FL equation (1) with $\delta = 1$, then $u^*$ solves the FL equation with $\delta = -1$.

In addition, mixed solutions can be obtained from a more general form of matrix $A$, e.g., a combined matrix $A = \text{Diag}(A_1, \ldots, A_s, J_{N-2s}(a_{l+1}))$, where $A_j$ is defined as in (32). Using formulae (3) and (14) new solutions to massive Thirring model (2) can be obtained from the solutions to the FL equation obtained in this paper. The rational solution can also be obtained using a special Wronskian technique (cf. [33,34]). The
partial-limit procedure should be available as well to Hirota’s form of soliton solutions (cf. [35,36]) and may be extended to other integrable equations and other methods, e.g., the bilinear neural network method [37–40].

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