Activation of entanglement in teleportation

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We study the activation of entanglement in teleportation protocols. To this end, we present a derivation of the average fidelity of teleportation process with noisy classical channel for qudits. In our work we do not make any assumptions about the entangled states shared by communicating parties. Our result allows us to specify the minimum amount of classical information required to beat the classical limit when the protocol is based on the Bell measurements. We also compare average fidelity of teleportation obtained using noisy and perfect classical channel with restricted capacity. The most important insight into the intricacies of quantum information theory that we gain is that though entanglement, obviously, is a necessary resource for efficient teleportation it requires a certain threshold amount of classical communication to be more useful than classical communication. Another interesting finding is that the amount of classical communication required to activate entanglement for teleportation purposes depends on the dimension \( d \) of the system being teleported but is not monotonic reaching maximum for \( d = 4 \).

In the first part of this article we give a description of the teleportation protocol and the notation we use. We also derive the value of the average fidelity of the teleportation process in the case where perfect channel is replaced with a noisy one. Later we give formula, based on the Shannon entropy, for a minimum capacity of the noisy classical channel and using it we move to our main result. The last part is devoted to the comparison of two channels: perfect with limited capacity and noisy.

I. INTRODUCTION

The teleportation protocol is a widely used and tested tool \[1,2\]. It became main part of many quantum communication protocols and still is an interesting research field \[3,4\]. It allows to transmit an unknown quantum state from a sender traditionally named "Alice" to a receiver "Bob" who are spatially separated. This protocol consists of a classical and a quantum channel. The presence of noise in these channels introduces imperfections in the process. A popular way to describe efficiency of teleportation is through the average fidelity \[13\]. When the protocol works perfectly, fidelity is equal to 1, which is the maximum value. Otherwise it is less. A lot of works has been devoted to this subject \[8,12\], but very little is said in them about the efficiency of teleportation, depending only on the classical channel’s noise. While we have perfect classical communication we do not have to care about it and this is what people usually do. Things change when we are interested in the amount of information needed to have a non classical process. In one of the recent papers \[1,2\] authors considered a teleportation protocol of qubits with imperfect classical channel where no restrictions on the quantum channel were made. They also derived a minimal capacity of a classical channel needed to have fidelity grater than the classical maximum of \( \frac{2}{3} \).

The motivation for this work is to determine the minimum capacity of a classical channel in a more general case. We ask what is the minimum amount of information sent via classical channel if the average fidelity is to exceed the classical limit in case of qudits. This provides us with a threshold value of classical communication. Without any amount of entanglement is, for teleportation purposes, less useful than classical communication. Another interesting finding is that this threshold value for entanglement activation depends on the dimension \( d \) of the system being teleported but is not monotonic reaching maximum for \( d = 4 \).

In the first part of this article we give a description of the teleportation protocol and the notation we use. We also derive the value of the average fidelity of the teleportation process in the case where perfect channel is replaced with a noisy one. Later we give formula, based on the Shannon entropy, for a minimum capacity of the noisy classical channel and using it we move to our main result. The last part is devoted to the comparison of two channels: perfect with limited capacity and noisy.

II. TELEPORTATION PROTOCOL

Here we briefly present the teleportation protocol for qudits which we are operating with to establish the notation.

Alice has to teleport a qudit \( |\Psi\rangle = \sum_{i=0}^{d-1} a_i |i\rangle \). Both Alice and Bob share the maximally entangled state \( |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle \). To start the teleportation Alice measures the states (two particles) possessed by her in Bell basis

\[
|\Psi_{mn}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{mk} |k+n\rangle \otimes |k\rangle
\]

where \( \omega = e^{\frac{2\pi i}{d}} \).

There are \( d^2 \) possible results where each one is encoded into a bit string of length \( \log d^2 \). We label them by \( (m,n) \). Measurement made by Alice demolishes her state \( |\Psi\rangle \) and the only thing she can do is to send a message with information gained by it to Bob. After having received from Alice the outcome of her measurement, Bob performs particular unitary transformation

\[
U_{mn} = \sum_{p=0}^{d-1} \omega^{mp} |n+p\rangle \langle p|
\]
on his particle and recreates the state $|\Psi\rangle$. This is a
standard procedure which for $d = 2$ gives teleportation of qubits [1]. More general one is presented in [2].

III. AVERAGE FIDELITY WITH A NOISY CLASSICAL CHANNEL

Consider a situation in which Alice has to teleport an
arbitrary qudit state $|\Psi\rangle$ to Bob. The density matrix od
this state is simply $\rho_A = |\Psi\rangle\langle\Psi|$ while Bob’s state (not
pure in general) after the teleportation we label by $\rho_B$. The
efficiency of this process due to the fact that the
state of Alice is pure, can be described as the average
fidelity calculated over all states $|\Psi\rangle$.

$$f_{\text{avg}} = \int \! d\Psi \text{tr}(\rho_A \rho_B) = \int \! d\Psi \langle \Psi | \rho_B |\Psi\rangle$$

where $d\Psi$ is a Haar measure.

When there is no classical communication between the
parties the average fidelity is simply $\frac{1}{2}$ whether they share
some entangled state or not. It is straightforward to show
it by using (3) and $\rho_B = \frac{1}{2}I$.

On the other hand, one can assume that Alice and Bob
have perfect classical communication and no shared en-
tangled state. In this case Alice measures her state in
some basis spanned by vectors $|i\rangle$ and sends the result
(classical information about the basis and the measure-
ment outcome) to Bob who simply prepares the state of Alice collapsed to. So with probability $|\langle i | \Psi\rangle|^2$
Bob will produce the state $|i\rangle$ ($\rho_B = \sum_i |\langle i | \Psi\rangle|^2 |i\rangle \langle i|$).

Then again using (3) and properties of trace we have

$$f_{\text{avg}} = \text{tr} \int \! d\Psi \rho_A \sum_i |\langle i | \Psi\rangle|^2 |i\rangle \langle i|$$

$$= \text{tr} \left( \int \! d\Psi |\Psi\rangle \langle \Psi| \sum_i |i\rangle \langle i| |i\rangle \langle i| \right).$$

Here, integration under the trace goes over all symmetric
elements of $d^2$ dimensional Hilbert space and it commutes
with all the operators representing such states. Due to
Schur’s lemma it is proportional to the symmetrical projector $P_{\text{SYM}}$ where the proportionality factor is determined by the trace condition $\text{tr} \int \! d\Psi |\Psi\rangle \langle \Psi| |\Psi\rangle \langle \Psi| = 1$. In this case we have

$$f_{\text{avg}} = \text{tr} \left( \frac{P_{\text{SYM}}}{d_{\text{SYM}}} \sum_i |i\rangle \langle i| \right) = \frac{2}{d+1}$$

where $d_{\text{SYM}} = \frac{d(d+1)}{2}$ is a dimension of symmetric space.

This is a well known limit for the teleportation proto-
coloc which means that greater values can only be
achieved by quantum processes based on entanglement
between Alice and Bob.

Finally, we may ask what is the maximal fidelity of the
process with the use of a noisy classical channel without
any restrictions on shared entanglement? Let us assume,
that we are allowed one use of a channel and Bob by us-
ing it can read one out of $d^2$ messages built with $\log d^2$
bits. If we assume that Alice sends to Bob a message
$(m, n)$, what takes place in the teleportation protocol,
and that Bob receives $(a, b) = (n + i, m + j)$ with proba-
bility $p_{ij}(mn)$ then the channel can be characterized by
error probabilities as

$$p_{ij}(mn) = p(m + i, n + j | m, n).$$

where the addition is modulo $d$ and no error event oc-
curs with probability $p_{00}(mn)$. This is the most general
description of a channel with $d^2$ inputs and outputs.

After receiving the information Bob performs unitary transformation $U_{ab}$ on his particle. This transformation,
by using (2), can be split into two parts and written as

$$U_{ab} = \omega^{-mj} U_{ij} U_{mn}.$$  

Operation $U_{mn}$ reproduces the state $\rho_A$ and $U_{ij}$ introduces an error with probability $p_{ij}(mn)$. Finally, after
performing his operation and by using

$$\tilde{p}_{ij} = \frac{1}{d^2} \sum_{m, n = 0}^{d-1} p_{ij}(mn)$$

Bob’s state can be described as

$$\rho_B = \sum_{i, j = 0}^{d-1} \tilde{p}_{ij} U_{ij} U_{ij}^\dagger.$$  

To compute the average fidelity, the formula [3] has to be used again. Thus we obtain

$$f_{\text{avg}} = \text{tr} \sum_{i, j = 0}^{d-1} \tilde{p}_{ij} \int \! d\Psi \rho_A U_{ij} \rho_A U_{ij}^\dagger.$$  

Because $U_{00}$ is identity we can simplify it as

$$f_{\text{avg}} = \tilde{p}_{00} + \text{tr} \sum_{i, j > 0}^{d-1} \tilde{p}_{ij} \int \! d\Psi \rho_A U_{ij} \rho_A U_{ij}^\dagger.$$  

When $\tilde{p}_{00} = 1$ then, of course, we have a perfect tele-
portation fidelity. To calculate the more general case we use
the following lemma.

**Lemma 1** For $\rho_A = |\Psi\rangle\langle\Psi|$ and $U_{ij}$ described by (3)
value of $\text{tr} \int \! d\Psi \rho_A U_{ij} \rho_A U_{ij}^\dagger$ for all $i$ and $j$ (excluding $U_{00}$) is equal to $\frac{1}{d+1}$.

**Proof** Main tools of this proof are again Schur’s lemma
and properties of symmetric and antisymmetric projectors.
Using them we can write

$$\text{tr} \int \! d\Psi \rho_A U_{ij} \rho_A U_{ij}^\dagger = \text{tr} \left( U_{ij} \otimes U_{ij}^\dagger \text{tr} \left( \int \! d\Psi |\Psi\rangle \langle \Psi| \right) \right)$$

$$= \text{tr} \left( U_{ij} \otimes U_{ij}^\dagger \frac{P_{\text{SYM}}}{d_{\text{SYM}}} \right) = \text{tr} \left( U_{ij} \otimes U_{ij}^\dagger \frac{\bar{I} - \bar{I}}{2d_{\text{SYM}}} \right) =$$

$$= \frac{1}{d(d+1)} \left( \text{tr}(U_{ij} U_{ij}^\dagger) + \text{tr}(U_{ij})\text{tr}(U_{ij}^\dagger) \right) = \frac{1}{d+1}$$

(11)
where $V = F_{SYM} - F_{ASYM}$ and for all $i$ and $j$ operators $U_{ij}$ (excluding unity) are traceless. □

Now it is easy to derive the average fidelity depending on the probability $\bar{p}_{00}$

$$f_{avg} = \frac{d \bar{p}_{00} + 1}{d + 1}. \quad (12)$$

So to have the fidelity of the teleportation process better or equal to $\frac{2}{d+1}$ the average probability $\bar{p}_{00}$ of sending correct message through noise channel should be greater or equal $\frac{1}{2}$.

Below we present another derivation of the same result which we include because it shows how imperfections of classical channel can be moved to the shared state.

One can write (6) differently as

$$U_{ab} = \omega^{-i m} U_{mn} U_{ij}. \quad (13)$$

This situation is equivalent to teleportation protocol with a perfect classical channel in which Bob interferes with his part of quantum source by preforming on it unitary operation $U_{ij}$ with probability $p_{ij}$. This action will change pure maximally entangled state $\rho_\Phi = |\Phi>|\Phi|$ into a mixed state $\rho_\Phi^B = \sum_{ij} p_{ij} |i>|j> \otimes U_{ij} \rho_\Phi \otimes U_{ij}^\dagger$. Using the previously accepted definitions is easy to show that

$$\rho_\Phi^B = \bar{p}_{00} \rho_\Phi + \frac{1}{d} \sum_{i+j>0} p_{ij} \sum_{kl} |k>|l> \otimes U_{ij} \rho_\Phi \otimes U_{ij}^\dagger. \quad (14)$$

To calculate the average fidelity in this case one of the results from (12) can be used. In that paper authors gave formula $f_{max} = (d F_{max} + 1)/(d + 1)$ where $F_{max}$ is the maximal overlap between the state (built by LOCC) used in the protocol and the state giving a perfect teleportation. Here

$$F_{max} = tr \rho_\Phi^B \rho_\Phi. \quad (15)$$

Because operators $U_{ij}$ are traceless we have $F_{max} = p_{00}$ which gives (12) as well. Of course the result will be the same if instead of Bob Alice will disturb her part of the singlet state (in this case $\rho_\Phi^A = \sum_{ij} p_{ij} U_{ij} \otimes \rho_\Phi U_{ij}^\dagger \otimes I$).

IV. MINIMAL CAPACITY OF THE CLASSICAL CHANNEL

The amount of classical communication is given by the mutual information between Alice’s input and Bob’s output [13]. Capacity then can be a function of the dimension of the state being teleported and probabilities [13] as

$$C = \log d^2 - \sum_{mn} p_{mn} H(p_{ij}(mn))$$

$$= \log d^2 + \sum_{mn} p_{mn} \sum_{ij} p_{ij}(mn) \log p_{ij}(mn) \quad (16)$$

where $H(p_{ij}(mn))$ is the Shannon entropy and $p_{mn}$ is the probability that Alice will get the result $(m,n)$. We know of course that in a typical scheme all $p_{mn} = \frac{1}{d}$ but we can use more a general distribution if we are interested in minimum of $C$ taken over all errors probabilities. Here we can’t use (14) but we can split it into

$$C = \log d^2 + \sum_{mn} p_{mn} p_{00}(mn) \log p_{00}(mn)$$

$$+ \sum_{mn} p_{mn} \sum_{i+j>0} p_{ij}(mn) \log p_{ij}(mn). \quad (17)$$

Because $H(p_{ij}(mn))$ is a convex function $C$ has minimum if all probabilities $p_{ij}$ are independent of information $(m,n)$ sent through classical channel and any error occurs with the same probability. This can be written as

$$\forall_{mn} \quad p_{00}(mn) = p_0 \quad (18)$$

$$\forall_{mij;i+j>0} \quad p_{ij}(mn) = p \quad (19)$$

where $p_{00}$ and $p$ are constant values related by

$$p = \frac{1 - p_{00}}{d^2 - 1}. \quad (20)$$

After that it is straightforward to show that the minimal capacity does not depend on $p_{mn}$ and is given by

$$C_{min}(d, p_0) = \log d^2 + p_0 \log p_{00}$$

$$+ (1 - p_{00}) \log \frac{1 - p_{00}}{d^2 - 1}. \quad (21)$$

Now we can answer the question what is the minimal capacity of classical channel needed to reach the limit [14]. Because, we are using a channel in which $\bar{p}_{00} = p_{00}$ and [12] it is enough to substitute $p_{00} = \frac{1}{d}$ to (21) which gives

$$C_{min}(d, \frac{1}{d}) = \log d - (1 - \frac{1}{d}) \log(d + 1).$$

For $d = 2$ we reproduce the result ($C_{min} = 0.208$) for 2-bit noisy classical channel derived in [10]. What is interesting here is the fact that the function is not monotonically increasing and has a maximum ($C_{min} = 0.259$) for $d = 4$ (see FIG. 1).

![FIG. 1. Threshold channel capacity for entanglement activation as a function $C_{min}(d, \frac{1}{d})$ of the dimension of the state being teleported.](image-url)
V. COMPARISON OF CHANNELS

So far we studied the case of a general noisy channel with \( d^2 \) inputs and outputs. One might ask if, for a given capacity, it is more beneficial to use this channel or the one with less inputs and outputs but no noise. Now we answer this question. We will call these two channels perfect and noisy respectively.

For the perfect channel we have to find the optimal strategy for Alice and Bob. It is not a difficult task. Without loss of generality we can assume that Alice will send a particular part of her result. For example, if they are allowed to use 2-bit channel in teleportation of 4-dimensional states Bob after being received information from Alice has \( \frac{1}{4} \) probability to perform correct unitary transformation.

To be more precise assume that Alice’s result was 0110 and she agreed with Bob to always send two first bits of her measurement outcome. After that Bob will reject 12 out of 16 possibilities and pick with equal probability one out of four (0100, 0101, 0110, 0111) to perform unitary transformation \( \mathbf{2} \).

For the generalization it is sufficient to put \( p_{00} = \frac{2^C}{d^2} \) use again \( \mathbf{10} \) and lemma1. Fidelity for perfect channel can be than written as

\[
f_{\text{avg}}^P(C, d) = \frac{2^C}{d^2} + 1 \frac{1}{d + 1}.
\]

It’s easy to see that for \( C = \log d^2 \) this fidelity is 1 and that beyond \( C = \log d \) it is greater than \( \mathbf{1} \).

VI. CONCLUSIONS

We studied the threshold amount of the classical communication required for the teleportation protocol to exceed maximal classical fidelity. We have shown its amount depends on the dimension of the teleported state but is, interestingly, not monotonic and reaches maximum for \( d = 4 \). We have also compared different channels of the same capacity and found that, for teleportation purposes, the one with white noise is optimal. We have restricted ourselves to the standard teleportation protocols involving maximally entangled states and measurements in Bell basis. We conjecture that our results hold also in the general case but proving this is an open avenue of research. It would also be interesting to see how the threshold value of communication changes if some restrictions on shared states are put.
ACKNOWLEDGMENTS

We thank Michal Horodecki for helpful discussions. This work is a part of the Foundation for Polish Science TEAM project cofinanced by the EU European Regional Development Fund. M.P. is supported by ERC grant QOLAPS and U.K. EPSRC. W.L. is supported by the National Centre for Research and Development (Chist-Era Project QUASAR).

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