Ordered mechanical systems typically have one or only a few stable rest configurations, and hence are not usually considered useful for encoding memory [1,2]. On the other hand, frustrated magnetic systems can espouse an extensive manifold of quasi-degenerate configurations [3,4]. Multistable and history-dependent responses can occur in mechanical systems with quenched disorder, such as amorphous materials [5], crumpled sheets [6-8], or random spring networks. We introduce an ordered, periodic mechanical metamaterial that exhibits an extensive set of spatially disordered prestressed distortions as a consequence of geometric frustration. This mechanical system encompasses continuous degrees of freedom, and is hence richer than the magnetic artificial spin ice whose topological structure it reproduces [9,13]. We show how this system exhibits history-dependent response, as the state in which the system reaches following a set of external manipulations depends on the order in which they were applied [14,15]. Thus multistability and potential to store complex memory emerge from geometric frustration in an ordered mechanical lattice that creates its own disorder.

When interactions between different components of a complex system cannot simultaneously be minimized [16,17], the lowest-energy state is a compromise in which some elements of the system are left “unhappy” [9]. This can lead to constrained disorder, degeneracy, and multistability [18]. Recent research translated these concepts to engineered soft-matter systems, such as acoustic channels [19], buckled elastic beams [1], and monolayers of colloidal spheres [20]. Fracture in magnetic spin systems is related to incompatibility of soft modes in mechanical systems [21,24]. Mapping of metamaterial architectures to spin systems has provided insight into the mechanical consequences of frustration, such as stress control [25] and domain-wall topology [2,26]. Exploring irreversibility and history dependence through these analogies has opened new routes for programmable elastic responses and mechanical memory storage [7,8,14,15,27,28].

In artificial magnetism, the frustration-based designs of artificial spin ice (ASI) [29,30] lead to constrained disorder, and thus to a degenerate manifold of configurations often captured by interesting emergent descriptions [12,51]. There are strong similarities between ASI and mechanical metamaterials realized by repeated arrangements of simple units endowed with a soft mode: In ASI, nano-islands are arranged along the edges of a lattice, and their magnetization is described by binary arrows pointing toward or from the vertices of the lattice, with certain vertex configurations locally minimizing the energy. Identically, in a mechanical metamaterial, displacements point into or out of the repeating units, and the softest deformation mode of each unit is related to a certain mutual arrangement of these displacements. The similarity extends to frustration and incompatibility: In vertex-frustrated ASI lattices not all vertices can simultaneously be in their lowest-energy state [9,32], as demonstrated for the Shakti-lattice ASI in Fig. 1(a). Equivalently, in non-periodic mechanical metamaterials, one can arrange the units so that they cannot all simultaneously deform according to their soft mode [21,23,24]. Vertex frustration in magnetism thus corresponds to incompatibility in mechanical metamaterials. However, the way these two classes of systems resolve that frustration or incompatibility is only analogous. The continuous nature of elastic degrees of freedom can lift the degeneracy present in the corresponding magnetic system, leading to a well-defined ground state [1], but possibly endowed with a manifold of quasi-degenerate, disordered metastable states [14].

Here we introduce an ordered periodic mechanical metamaterial, inspired by ASI, that exhibits a large multiplicity of disordered metastable states. This leads to multistability of internal degrees of freedom and to mechanical memory, as configurations can depend on their preparation history. The topology of irreversible transitions between states renders this metamaterial non-Abelian, as its state depends not only on the external manipulations it underwent, but also on their precise sequence.

In the Shakti ASI [2,13], vertices have coordination numbers (or number of impinging arrows) $z = 2,3,4$. Vertices where $z = 3$ or $z = 4$ spins meet have two distinct energy-minimizing configurations [Fig. 1(b)], and $z = 2$ vertices prefer to have their spins aligned. Because of the frustrated lattice geometry, it is impossible to set all vertices in these lowest-energy configurations [see rectangular loop (yellow) in Fig. 1(a)], without resulting in a conflict (red arrows). Excitations of the $z = 3$ vertices cost less energy than excitations of the $z = 4$ vertices. Consequently, the extensively-degenerate, disordered ground state of the Shakti ASI is achieved by any distribution of $z = 3$ excitations such that each excitation resolves the frustration for two minimal loops of spins. The remaining $z = 3$ vertices and all the $z = 4$ vertices adopt minimum-energy configurations.

In Fig. 1(c) we show the Chaco mechanical metamateria-
FIG. 1: Design principle - (a) The Shakti lattice artificial spin ice (ASI) exhibits vertex frustration: A conflict (red arrows) results when trying to place all vertices in a rectangular loop in their lowest-energy states (as individually shown in b). (c) The Chaco lattice mechanical metamaterial inherits the vertex frustration of the Shakti ASI. The five units surrounding any node in the metamaterial may not simultaneously deform according to their softest mode (as individually shown in d). Thick lines indicate $k_1$ springs forming the edges of the square and triangular units. Thin lines indicate $k_2$ springs responsible for the interactions between neighboring edges within each unit. Spontaneous edge displacements are generated by pinning the corners of all units (white dots) to distances smaller than the relaxed edge length. (e) Experimental realization of the Chaco metamaterial, and (f) the softest deformations of its units. In the triangular units, the different stiffnesses at the corners connecting adjacent edges give rise to the softest modes obtained by the $k_2$ springs in the theoretical model.

rial, whose incompatibility corresponds to the frustration of

the Shakti ASI. The $z = 3$ and $z = 4$ vertices of the

magnetic Shakti [Fig. 1(b)] correspond to the triangular and square units, respectively, comprising the mechanical Chaco [Fig. 1(d)]. The arrows describing the magnetic moments in the lowest-energy states of the former correspond to the softest, or lowest-energy deformations of the latter. We model the mechanical Chaco as a network of linear springs. Each edge of the squares and triangles consists of two springs with stiffness $k_1$ and rest length $\ell$. These are connected by internal coupling springs of stiffness $k_2$. In ASI, nanoislands are naturally magnetized. In metamaterials, one can induce an equivalent spontaneous displacement of all edges by prestressing the system [2]. Namely, pinning an edge to a distance $2\alpha\ell$, with $\alpha < 1$, imposes a compression, which causes it to buckle by an amount $\delta = \ell\sqrt{1 - \alpha^2}$. When applying this uniform compression to the mechanical Chaco, the units may not all simultaneously adopt their zero-energy deformations, due to the inherent incompatibility. Nevertheless, in the weak coupling limit, $k_3 \ll k_1$, we expect the free nodes to behave in an approximately binary way [2], adopting displacements $\delta_i \approx \pm \delta z_i$ from their rest position, where $z_i$ is the local direction perpendicular to the edge. In this limit, the $k_1$ bonds are approximately relaxed and energy focuses to the $k_2$ bonds. The overall compression applied to the lattice breaks the symmetry between stretching and compressing, such that compressing a $k_2$ bond costs more energy than stretching it.

Experimentally, we create the elastic network shown in Fig. 1(e), top. Similarly to the spring model, a thin enough elastic beam of length $2\ell$ compressed by a factor $\alpha$ tends to buckle via its first deformation mode. We prestress the rubber network by constraining the corners of all square and triangular units to an array of metal pins connected to a rigid substrate, and spaced such that the network is uniformly compressed by factor $\alpha$ [Fig. 1(e), bottom]. The geometry of the units leads to mechanical soft modes [Fig. 1(f)], which reproduce those of the theoretical springs model [Fig. 1(d)].

Similarly to the magnetic Shakti, the most energy-efficient way to resolve the frustration in the mechanical Chaco is to localize stress on half of the triangles. This leaves the remaining triangles and all squares close to their zero-energy states. However, the continuous nature of the mechanical degrees of freedom allows minimizing the energy by long-range pairing of the excited triangles. In a perfect realization, this leads to an ordered ground state [Fig. 2(a)], which has a four-fold degeneracy, since stresses may be placed on either vertical or horizontal pairs of triangles and since spin reversal leaves the energy unchanged. For sufficiently large $k_2/k_1$, the ground state is the only mechanically-stable configuration. However, for small $k_2/k_1$, we also find an extensive manifold of metastable states. Namely, any configuration with all squares in one of their individual zero-energy states becomes metastable. This holds for configurations with stresses localized to exactly half of the triangles, equivalent to any ground state of the Shakti [Fig. 2(b)]. It also holds for configurations where more than half of the triangles are excited [Fig. 2(c)]. Although these states have higher energy than the ground state, the excess
FIG. 2: Multiple stable configurations - Square and triangle unit energies for sample configurations with increasing energy in a $4 \times 4$ Chaco lattice with periodic boundary conditions, all for $k_2/k_1 = 0.033$ and $\alpha = 0.9$: (a) Ordered ground state with energy localized on vertical pairs of triangles. (b) Complex stress distribution of a metastable state derived from a magnetic Shakti ground state, in which each stressed triangle is adjacent to a relaxed triangle. (c) Metastable configuration with all squares in relaxed random orientations and stress localized to more than half of the triangles.

stress in the triangles does not suffice to overcome the energy barrier for flipping the squares. Thus, since thermal fluctuations are negligible, if the system is in such a disordered metastable state, it will remain there and will not relax to the ordered ground state.

We distinguish between two types of degrees of freedom within the Chaco metamaterial - the square units, and the central edges between pairs of back-to-back triangles. For small $k_2/k_1$, we describe both of them as approximately binary variables, stuck in one their stable states. Namely, squares adopt one of their soft configurations, and each central edge buckles to one of two directions. These degrees of freedom constitute hysterons [7, 33, 34], bistable elements whose state is history dependent. Hence, they are a natural representation for memory in the metamaterial. As described above, both states of the squares are always stable. In contrast, the stability of the central edge between two triangles depends on the configuration of the four squares surrounding it. We label the $2^4 = 16$ states of these squares by $2 \times 2$ matrices with binary entries, where 0 indicates a displacement into the double triangle and 1 out of it. By symmetry, these 16 states reduce to seven distinct configurations [Fig. 3(a)].

We now fix all squares to their zero-energy states, and compute minimal-energy states of the central edge as function of $k_2/k_1$, for each configuration. The horizontal displacement of the central point is negligible, thus we focus on the vertical displacement $y$. We find it may exhibit monostable, metastable or bistable behavior for the different configurations [Fig. 3(b)]: Due to their vertical asymmetry, configurations $i - iii$ are typically monostable, with $y > 0$: configuration $i$ allows both triangles to adopt zero-energy states, such that $y = \delta$ for any $k_2/k_1$; configurations $ii, iii$ show a reduced amplitude $0 < y < \delta$, which decreases as $k_2/k_1$ increases. Configurations $iv - vii$ are vertically symmetric. As a result, the central edge is bistable for small $k_2/k_1$; for larger $k_2/k_1$, the central point adopts the value $y = 0$. For very small values of $k_2/k_1$, metastable solutions appear for configurations $i - iii$. From here on, we focus on $0.022 \lesssim k_2/k_1 \lesssim 0.057$, where configurations $i - iii$ are strictly monostable and configurations $iv - vii$ are bistable. Our experimental design is similarly aimed to exhibit such behavior.

Since the configuration of the squares governs the stability of the central edge, we can precisely control its state by manipulating the squares around it. Interestingly, the final state of the central edge depends not only on the configuration of squares, but on the exact order in which they were flipped, and consequently transition pathways are sequence dependent. The transitions between states are best understood using the directed graph [5, 21] shown in Fig. 3(c), where we separately mark the two possible states of the central edge for the six bistable configurations, resulting in a total of 22 states. All transitions between states with the same direction of the central edge are reversible (gray). An irreversible transition (green) occurs whenever a bistable configuration is connected to a monostable configuration with the opposite direction of the central edge.

We propose a protocol, shown in simulations and experiments in Fig. 3(d,e), respectively, that demonstrates the aforementioned control: We start from the bistable configuration $(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix})$, with the central edge pointing upwards (top panels). Next, we flip the lower-left square, thus switching to the monostable state $(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$. This indirectly forces the central edge to flip downwards (middle panels). Finally, we flip the lower-left square back to its initial state, which brings us back to the original bistable configuration $(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix})$. Interestingly, the central edge does not revert back, but remains down (bottom panels). After this irreversible path, if we continue flipping the lower-left square, the system reversibly switches back and forth between the latter two states.

An extended lattice exhibits an emergent sequence-dependent, non-Abelian behavior. Even though the dynamics of each double triangle is governed by the transition graph shown in Fig. 3(c), flipping of a single square in the lattice affects up to the four double triangles surrounding it. Given a set of squares that can be externally manipulated, different configurations of the central edges between triangles may be
FIG. 3: Local memory and irreversibility - (a) The seven distinct configurations of a pair of back-to-back triangles and their surrounding four squares within the Chaco lattice, with 1 (0) indicating displacement out (in) of the triangle. (b) Stable vertical positions of the central point between them as a function of $k_2/k_1$ for $\alpha = 0.9$ and assuming that the four external edges are fixed with ideal displacements $\mathbf{s}_i = \pm \delta \mathbf{z}_i$. (c) Transition graph between the possible states. Bistable configurations are shown twice to include the possible deflections of the central edge. Reversible transitions are plotted in grey. Irreversible transitions, which flip the central edge, are plotted in green. (d,e) Irreversible control sequence for the internal bistable edge in simulations (d) and experiment (e). The starting configuration (top) is bistable with the internal edge up. Flipping the lower-left square creates a monostable configuration, so the internal edge flips down (middle). The lower-left square is returned to its original bistable state, but the internal edge remains pointing down (bottom). The pathway corresponding to panels (d,e) is highlighted in red on the transition graph (c).

reached by manipulating the squares in different sequences. We illustrate this by simulations of a $4 \times 4$ system with open boundaries; starting from the initial configuration [Fig. 4(a)], each of the four central squares is individually flipped, according to two different sequences of operations [Fig. 4(b,c)]. The final configuration of the central edges is sequence dependant, as shown on the right. See supplementary video [35].

The same non-Abelian sequence dependence is exhibited by the experimental lattice [Fig. 4(d-f)], and agrees with the theoretical model, except for double triangles that are in configuration vi of Fig. 3(a). As opposed to the springs model, in the experimental system the squares apply torque to the edges of the internal beam. In configuration vi the torques are not symmetric under up-down reflection, and consequently they excite the second bending mode of the internal beam. Hence, this configuration is bistable only in the theoretical springs model, while it is monostable in the experimental network of elastic beams.

The mechanism underlying this non-Abelian response sheds light on the generic emergence of sequence-dependent responses in frustrated mechanical systems [14, 15, 36]. The order in which bistable degrees of freedom are manipulated follows a pathway of states, which favors one of the possible final states. In essence, the sequence directs the otherwise spontaneous symmetry breaking of the system at its final state. This concept may be used to cleverly manipulate mechanical systems between multiple functionalities [37, 38].

Furthermore, a structurally-ordered system that can create its own disorder, and that responds to probes in non-Abelian ways, might exhibit non-monotonic and even discontinuous, hysteretic mechanical response; the allocation of the strained units could change under external load, thus these effects would be history-dependent for interesting future vistas on hysteretic viscoelasticity and viscoplasticity. Following the vertex-frustration analogy back to the magnetic realm, non-Abelian protocols may increase the memory capacity in storage devices based on frustrated geometries. Due to the sequence sensitivity, manipulating $n$ binary degrees of freedom may encode more than $2^n$ unique states. Thus the manifold of reachable states may be enriched by frustrated interactions.
FIG. 4: Non-Abelian control sequence in an extended lattice - Sequence dependence for a lattice of $4 \times 4$ squares in simulations (a-c) and experiments (d-f). a,d) Initial configurations with the central pair of triangles monostable, and the eight surrounding triangle pairs bistable. Resulting states after each one of the four central squares is each flipped in clockwise (b,e) or counterclockwise (c,f) sequences, as indicated by the numbers inside the squares. The monostable central edge is determined by the state of the four squares, while the surrounding bistable pairs end in different configurations for different sequences. Red squares unit indicate external manipulation, cyan arrows indicate the initial displacements, yellow arrows the opposite displacements, and blue arrows indicate an elastic beam in its second bending mode.
These authors contributed equally to this work

[shokef@tau.ac.il]

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METHODS

Experimental Methods

To realize the frustrated Chaco lattice, we design a network of elastic beams, tracing the geometry of the Hookean springs in our theoretical model. Using a 3D printer (Prusa MK3 i3) we manufacture molds for the network, which are then used to cast samples made from silicone rubber (Mold Max™) with a Shore hardness of 30 A. The geometric dimensions of the rubber samples are shown in Fig. S1. Specifically, the basic beam length, constituting half the edge of each square or triangular unit in the Chaco lattice is equal to \( \ell = 15 \text{mm} \). To induce prestress in the samples, we constrain the elastic network to dimensions smaller than its rest configuration, using a set of pins, \( 2 \pm 0.01 \text{mm} \) in diameter. Each pin acts as a rotation axis fixed in space. The spacing between neighboring pins is set to induce a uniform compression factor of \( \alpha = 0.92 \pm 0.01 \). To limit the deformation of the central beam between two triangles to its first buckling mode, we remove from the triangles the elastic beams which trace the \( k_2 \) springs. Instead, the \( k_2 \) springs are not removed from the squares, however, to ensure the squares do not exhibit additional metastable states. The control sequence protocols are performed manually. The beams corresponding to the \( k_2 \) springs allow easy manipulation of the squares between their two stable states. We force the squares to flip by pushing one of their outer beams. See supplementary video [S5]. Experiments are documented using a digital camera (Sony alpha3). In Figs. 3[4] and S5 the background is digitally removed for clarity.

Computational Methods

We simulate the harmonic springs theoretical model for the Chaco metamaterial with overdamped dynamics, or the method of steepest energy descent. The equation of motion for a given point with position \( \vec{r}_i \) is given by

\[
\frac{d\vec{r}_i}{dt} = \frac{1}{\gamma} \sum_{(ij)} -k_{ij}(e_{ij} - l_{ij})\hat{e}_{ij},
\]

where the spring connecting point \( i \) to point \( j \) has spring constant \( k_{ij} \), relaxed length \( l_{ij} \), and current extension \( e_{ij} = |\vec{r}_i - \vec{r}_j| \), and \( \gamma \) is a linear drag coefficient, which sets the relaxation time scale in the system, which we set to \( \gamma/k_1 = 1 \). However all the results we present are for the final state after the system has fully relaxed, thus this time scale is not relevant for the result we present. Energy is minimized by integrating the equations of motion with variable time step \( \Delta t = 0.01 - 0.1 \).

For Figs. 2[3] and S3 we define the energy contained within each of the mechanical units, accounting for contributions from both \( k_1 \) and \( k_2 \) bonds, so that summing up all unit energies gives the total lattice energy. Each \( k_1 \) bond in the Chaco lattice is shared between two mechanical units, either between a square and a triangle or between two triangles. All the \( k_2 \) bonds are internal to a single unit. Thus we define the energy of each unit as the sum of harmonic spring energies of all its internal \( k_2 \) bonds plus one half the harmonic spring energy for each \( k_1 \) bond along the boundary of the unit.

The overall compression breaks the \( Z_2 \) spin-reversal symmetry between nearest-neighbor degrees of freedom. The mechanical degrees of freedom behave in a spin-like binary way for \( k_2 \ll k_1 \), whereas elastic, continuous deformations are expected otherwise. Consider the interaction energies when ideal displacements \( \pm \delta \) are imposed on pairs of neighboring edges: One in and one out displacement allow the internal \( k_2 \) bonds to rotate and remain in their relaxed length. Two out displacements result in a stretched internal \( k_2 \) bond, with lengths \( l^+ \) and \( l^0 \), and energies \( E^+ \) and \( E^0 \). Two in displacements result in a compressed internal \( k_2 \) bond, with lengths \( l^- \) and \( l^0 \), and energies \( E^- \) and \( E^0 \). Due to the overall compression, stretching the internal bonds costs less energy than compressing them further, as shown in Fig. S2. An important consequence of this breaking of the \( Z_2 \) symmetry is that excited triangular and square mechanical units have distinct energy hierarchies than the \( z = 3 \) and \( z = 4 \) Shakti lattice ASI vertices [9].

Multistability and collapse to the ground state

Figure S3 compares the relative energy contributions from the square and triangular units as \( k_2/k_1 \) is gradually increased starting from four classes of initial configuration: the mechanical ground state [Fig. 2(a)], a configuration derived from a Shakti-lattice ASI ground state without paired defects [Fig. 2(b)], a state with randomly chosen relaxed orientations from both the squares [Fig. 2(c)], and finally a state with random displacements \( s_i(0) \) of all edges. Once the starting metastable state becomes unstable, some edges within the lattice flip, creating sharp drops in the energy.

In the mechanical ground state [Fig. 2(a)], energy is localized to the triangles, with full localization \( E^-/E_0 \rightarrow 1/2 \) as \( k_2/k_1 \rightarrow 0 \). This behavior is also seen for the Shakti-lattice ASI ground state [Fig. 2(b)], which implies that the energy is also localized to the triangles, and further implies that the gap between the Shakti ASI ground state and the true mechanical ground state closes as \( k_2/k_1 \rightarrow 0 \). For larger \( k_2/k_1 \) the triangles and the squares in the Shakti ASI ground state have higher energy compared to the mechanical ground state. The energy of the squares in the Shakti ground state increases smoothly until \( k_2/k_1 \approx 0.15 \), where sharp jumps start to ap-
pear, indicating that some squares flip so that excited triangles can become paired.

For relaxed squares in random orientations [Fig. 2(c)], the energy is also localized to the triangles, but the presence of higher-order excitations on the triangles causes the energy stored to be nearly twice that of the triangle energies in the mechanical ground state, giving $E^\Delta/E_0 \rightarrow 1$ as $k_2/k_1 \rightarrow 0$. Finally, the state with random edge displacements has large energy stored in both the squares and the triangles. As $k_2/k_1$ increases, edges become unstable and flip, taking the configuration towards the ordered ground state. The three metastable states shown all reach the ground state for $k_2/k_1 > 0.5$. In larger systems, the true ground state is not always fully reached because several domains with different ground-state orientations may form.

Figure S3 plots the relative energy curves for two system sizes and also shows three separate initial conditions for the different classes considered. Other than stochastic variations in the positions of the energy jumps, due to different initial conditions, the intensive relative energy curves show the same behaviors for each class. The important major difference that appears in the larger system is some of the energy curves do not collapse to merge with the mechanical ground state curves as $k_2/k_1$ increases. Inspection of the real space evolution of the configurations for the larger system in Fig. S4 shows that the failure to collapse completely to the ground state curves is a result of the presence of multiple competing ground-state domains.

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FIG. S1: Geometric dimensions of the experimental Chaco lattice. All lengths are measured in millimeters.

FIG. S2: Energetics of the ideal nearest-neighbor interactions for the square and triangular mechanical units. The edge bonds are springs with relaxed length $\ell = 1$ and stiffness $k_1$. The internal bonds have relaxed lengths $\ell_0 = \sqrt{2}$, $\ell_\Delta = 1$ and stiffness $k_2$. Ideal displacements of amplitude $\delta = \sqrt{1 - \alpha^2}$ place all energy on the internal $k_2$ bonds, and spin inversion symmetry is broken in the mechanical system: bond extension costs less energy than bond compression.
FIG. S3: Relative energy of the squares, $E_{\Box}/E_0$ (solid) and of the triangles, $E_{\Delta}/E_0$ (dotted), both normalized by the average unit energy in the ground state $E_0 = E_{\Box} + 2E_{\Delta}$ (shown in the insets) as $k_2/k_1$ is slowly increased for systems of $4 \times 4$ (a) and $8 \times 8$ (b) squares with periodic boundary conditions. The line colors correspond to four classes of initial state: mechanical ground-state (blue), Shakti ASI ground-state with unpaired defects (red), randomly-oriented relaxed squares (yellow), and random displacements (purple). Three distinct initial conditions are shown for each class to illustrate the stochastic variations between runs. In (b), not all runs collapse to the ground state curves as $k_2/k_1$ increases because of competing ground-state domains in the larger system.

FIG. S4: Real space evolution of the metamaterial configurations for a system of $8 \times 8$ squares, for the three classes of initial condition, unpaired defects corresponding to a magnetic Shakti ground state, random initial choice of squares in their minimum energy orientations, and random choice of the edge displacements. As $k_2/k_1$ increases, edges inside the system start to flip to bring the configuration closer to the mechanical ground state. However, for the $k_2/k_1$ values shown here, there are still several competing ground-state domains, separated by domain walls.
FIG. S5: Experimental states of a pair of back to back triangles corresponding to the seven distinct theoretical configurations shown in Fig. 3a. Configurations (i-iii) are monostable both theoretically and experimentally. Configurations (iv), (v), (vii) are bistable both theoretically and experimentally, while configuration (vi) is bistable in the theoretical springs model, while experimentally its central beam exhibits a single stable state with a second mode of buckling.