ON A P-CURL SYSTEM ARISING IN ELECTROMAGNETISM

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Abstract. We prove existence of solution of a \( p \)-curl type evolutionary system arising in electromagnetism with a power nonlinearity of order \( p \), \( 1 < p < \infty \), assuming natural tangential boundary conditions. We consider also the asymptotic behaviour in the power obtaining, when \( p \) tends to infinity, a variational inequality with a curl constraint. We also discuss the existence, uniqueness and continuous dependence on the data of the solutions to general variational inequalities with curl constraints dependent on time, as well as the asymptotic stabilization in time towards the stationary solution with and without constraint.

1. Introduction

We consider a nonlinear electromagnetic field in a bounded domain \( \Omega \) of \( \mathbb{R}^3 \). The electric and the magnetic fields, respectively \( e = e(x, t) \) and \( h = h(x, t) \), and the electric and magnetic inductions, respectively \( d(x, t) \) and \( b = b(x, t) \), satisfy the Maxwell’s equations (\( \partial_t = \frac{\partial}{\partial t}, \nabla \times = \text{curl}, \nabla \cdot = \text{div} \))

\[
\begin{align*}
\partial_t d + j &= \nabla \times h, \\
\partial_t b + \nabla \times e &= f, \\
\nabla \cdot d &= q, \\
\nabla \cdot b &= 0
\end{align*}
\]  \tag{1}

where \( j \) denotes the total current density, \( q \) is the electric charge and \( f \), which is zero in the classical setting, is here a given internal magnetic current (see [3].
6). Denoting by μ the magnetic permeability constant, we assume the following constitutive law

\[ b = \mu h \]

and the following nonlinear extension of Ohm’s law,

\[ |j|^{p-2}j = \sigma e, \]

where \( \sigma \) is the electric conductivity.

If in the first equation of (1) we neglect the term \( \partial_t d \), the magnetic field \( h \) is then divergence free and

\[ \mu \partial_t h + \nabla \times (\frac{1}{\sigma} \nabla \times h |^{p-2} \nabla \times h) = f. \]

Denoting \( \Gamma = \partial \Omega \) and \( \Sigma_T = \Gamma \times (0, T) \), we impose the following natural tangential boundary conditions

\[ h \cdot n = 0 \quad \text{and} \quad e \times n = g \quad \text{on} \ \Sigma_T, \]

where \( n \) denotes the external unitary normal vector to the boundary \( \Gamma \). The boundary condition \( h \cdot n = 0 \) is naturally associated with \( \nabla \cdot h = 0 \) in \( Q_T = \Omega \times (0, T) \) and \( e \times n = g \) corresponds to consider a superconductive wall, i.e., a tangent current field.

Recalling the relation between \( e \) and \( h \), if we set \( \nu = \frac{1}{\sigma} > 0 \), we are lead to the problem

\[ \begin{align*}
\nabla \cdot h &= 0 \quad \text{and} \quad \mu \partial_t h + \nabla \times (\nu |\nabla \times h|^{p-2} \nabla \times h) = f \quad \text{in} \ Q_T, \\
h \cdot n &= 0 \quad \text{and} \quad \nu |\nabla \times h|^{p-2} (\nabla \times h) \times n = g \quad \text{on} \ \Sigma_T, \\
h(0) &= h_0 \quad \text{in} \ \Omega. \quad (2c)
\end{align*} \]

As a necessary condition for the existence of solution of this problem, the external field \( f \) must satisfy \( \nabla \cdot f = 0 \). Besides, the given field \( g \) on \( \Sigma_T \) must be tangential and compatible with \( f \), more precisely, \( \nabla_T \cdot g = f \cdot n \) on \( \Gamma \), where \( \nabla_T \cdot \) denotes the surface divergent (see [9, 10, 8]).

We may also consider another constitutive law that arises in type-II superconductors and is known as an extension of the Bean critical-state model presented in [11]. In this case the current density cannot exceed the critical value \( \Psi > 0 \) and we have

\[ e = \begin{cases} 
\nu |\nabla \times h|^{p-2} \nabla \times h & \text{if} \ |\nabla \times h| < \Psi(x, t), \\
(\nu \Psi^{p-2} + \lambda) \nabla \times h & \text{if} \ |\nabla \times h| = \Psi(x, t),
\end{cases} \]

where the parameters \( \nu = \nu(x) \geq 0 \) is a given function and \( \lambda = \lambda(x, t) \geq 0 \) can be regarded as a (unknown) Lagrange multiplier.

Some easy calculations (see [11, 8] for details) leads to the variational inequality, for a.e. \( t \in (0, T) \),

\[ \int_{\Omega} \partial_t h(t) \cdot (v - h(t)) + \mu \int_{\Omega} \nabla \times h(t) |^{p-2} \nabla \times h(t) \cdot \nabla \times (v - h(t)) \]

\[ \geq \int_{\Omega} f(t) \cdot (v - h(t)) + \int_{\Gamma} g(t) \cdot (v - h(t)), \quad (3) \]

for any test function \( v = v(x) \) such that \( |\nabla \times v(x)| \leq \Psi(x, t) \). This leads to search the solution in the time dependent convex set

\[ K(t) = \{ v = v(x) : |\nabla \times v(x)| \leq \Psi(x, t), \ x \in \Omega \} \quad \text{for a.e.} \ t \in (0, T). \quad (4) \]
In Section 2 we study the evolutionary problem (2), showing the existence of a unique solution in the variational framework of quasilinear monotone operators in the appropriate functional subspace of $W^{1,p}(\Omega)^3$. We notice that in the case of normal boundary condition ($h \times n = 0$ on $\Sigma_T$) existence results for similar nonlinear Maxwell’s system have been obtained in [18, 19]. But these results with tangential boundary condition ($h \cdot n = 0$ on $\Sigma_T$) are presented here for the first time. We also prove the asymptotic convergence, as $t \to \infty$ to the stationary solution of the problem already considered in [8].

In Section 3 we derive the Bean-type superconductivity variational inequality model with critical value $\Psi = 1$ as the limit case $p \to \infty$, extending a previous scalar case by [2] and a vectorial case with normal boundary condition due to [19].

Finally, in Section 4, we solve the evolutionary variational inequality (3) with the time dependent convex set (4), showing the existence, uniqueness and continuous dependence on the data $f$, $g$, $h_0$ and $\Psi$ of the solution, in the appropriate setting. We also discuss the asymptotic convergence of the solution in $L^2(\Omega)$, as $t \to \infty$, towards the corresponding stationary solution obtained in [8], for $p \geq \frac{6}{5}$.

2. The variational equation

In what follows $\Omega$ is a bounded, simply connected domain of $\mathbb{R}^3$ with a $C^1$ boundary $\Gamma$. If $E$ denotes a vectorial space, we denote by $E$ the space $E^3$.

2.1. The functional framework. We introduce the functional space $W^p(\Omega) = \{ v \in W^{1,p}(\Omega) : \nabla \cdot v = 0, \ v \cdot n|_{\Gamma} = 0 \}$, $1 \leq p \leq \infty$, which is a closed subspace of the Sobolev space $W^{1,p}(\Omega)$.

Proposition 1. For $1 < p < \infty$, $W^p(\Omega)$ is a reflexive Banach space where the semi-norm $\| \nabla \times \cdot \|_{L^p(\Omega)}$ is a norm, equivalent to the $W^{1,p}(\Omega)$-norm.

Proof. For $p > \frac{6}{5}$ the proof can be found in Theorem 2.1 and Lemma 2.1 of [8] and for general $1 < p < \infty$ in Theorem 2.2 and Corollary 3.3 of [1].

Remark 1. Two immediate consequences follow from this proposition: there exist positive constants $C_q$ and $C_r$ such that, given $v \in W^p(\Omega)$, the Sobolev inequality

$$\| v \|_{L^q(\Omega)} \leq C_q \| \nabla \times v \|_{L^p(\Omega)},$$

holds with $q \leq \frac{3p}{4-p}$ if $1 < p < 3$, any $q < \infty$ if $p = 3$ and $q = \infty$ for $p > 3$ and the trace theorem

$$\| v|_{\Gamma} \|_{L^r(\Gamma)} \leq C_r \| \nabla \times v \|_{L^p(\Omega)},$$

holds with $r \leq \frac{2p}{4-p}$ if $1 < p < 3$, any $r < \infty$ if $p = 3$ and $r = \infty$ for $p > 3$.

In particular, $v \in L^2(\Omega)$ if $p \geq \frac{6}{5}$.

In what follows the exponents $p$, $q$ and $r$ are related by these Sobolev and trace inequalities. 

We denote

$$\tilde{W}^p(\Omega) = W^p(\Omega) \cap L^2(\Omega)$$

and

$$L^2_\sigma(\Omega) = \tilde{W}^p(\Omega)^{\prime} = \left\{ v \in L^2(\Omega) : \forall \eta \in C^1(\Omega), \int_\Omega v \cdot \nabla \eta = 0 \right\},$$

and we observe that, if $p \geq \frac{6}{5}$, $\tilde{W}^p(\Omega)^{\prime} = W^p(\Omega)^{\prime}$. 

2.2. Existence of solution in the evolution problem. Let $a : Q_T \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a Carathéodory function satisfying the structural conditions
\begin{align}
    a(x, t, u) \cdot u &\geq a_* |u|^p, \quad (7a) \\
    |a(x, t, u)| &\leq a^* |u|^{p-1}, \quad (7b) \\
    (a(x, t, u) - a(x, t, v)) \cdot (u - v) &> 0, \text{ if } u \neq v, \quad (7c)
\end{align}
for given constants $a_*, a^* > 0$, for all $u, v \in \mathbb{R}^3$ and a.e. $(x, t) \in Q_T$.

We consider the following problem:
\begin{align}
    \nabla \cdot h &= 0 \quad \text{and} \quad \partial_t h + \nabla \times (a(x, t, \nabla \times h)) = f \quad \text{in} \quad Q_T, \quad (8a) \\
    h \cdot n &= 0 \quad \text{and} \quad a(x, t, \nabla \times h) \times n = g \quad \text{on} \quad \Sigma_T, \quad (8b) \\
    h(0) &= h_0 \quad \text{in} \quad \Omega. \quad (8c)
\end{align}

Taking (5) and (6) into account we assume that
\begin{align}
    f &\in L^{q'}(Q_T) \quad \text{and} \quad g \in L^{r'}(\Sigma_T), \quad (9)
\end{align}
where $q'$ and $r'$ denote the conjugate exponents of $q$ and $r$ respectively, and
\begin{align}
    h_0 \in L^2(\Omega). \quad (10)
\end{align}

Hence the following formula of integration by parts
\begin{align}
    \int_\Omega \nabla \cdot a \cdot \varphi - \int_\Omega a \cdot \nabla \varphi = \int_\Gamma a \times n \cdot \varphi \quad \forall \varphi \in W^{1,p}(\Omega) \quad (11)
\end{align}
holds with $a \in L^{q'}(\Omega)$, $\nabla \times a \in L^{q'}(\Omega)$ and, in the sense of traces, $a \times n |_{\Gamma} \in L^{r'}(\Gamma)$ (see [4] and [9]).

Whenever $\partial_t h(t) \in \widetilde{W}^p(\Omega)'$, interpreting the integral $\int_\Omega \partial_t h \cdot \varphi$ in the duality sense, the above formula yields the following weak formulation of the problem (8): to find $h \in L^p(0, T; \widetilde{W}^p(\Omega))$ such that, for a.e. $t \in (0, T),
\begin{align}
    \int_\Omega \partial_t h(t) \cdot \varphi + \int_\Omega a(x, t, \nabla \times h(t)) \cdot \nabla \varphi \\
    = \int_\Omega f(t) \cdot \varphi + \int_\Gamma g(t) \cdot \varphi \quad \forall \varphi \in \widetilde{W}^p(\Omega) \\
    \quad h(0) = h_0. \quad (12)
\end{align}

**Proposition 2.** Suppose that the operator $a$ satisfies the assumptions (7a-c) and the data and the initial condition satisfy (9) and (10). Then the problem (8) has a unique solution $h \in L^p(0, T; \widetilde{W}^p(\Omega)) \cap C(0, T; L^2(\Omega))$ and $\partial_t h \in L^{p'}(0, T; \widetilde{W}^{p'}(\Omega))$.

In addition, there exists a positive constant $C$ such that
\begin{align}
    \|h\|^2_{L^\infty(0, T; L^2(\Omega))} + \|\nabla \times h\|^p_{L^p(\Omega_T)} \\
    \leq C \left( \|f\|^{p'}_{L^{q'}(Q_T)} + \|g\|^{r'}_{L^{r'}(\Sigma_T)} + \|h_0\|^2_{L^2(\Omega)} \right). \quad (13)
\end{align}

**Proof.** The operator $A(t) : \mathbb{W}^p(\Omega) \rightarrow \mathbb{W}^p(\Omega)'$ defined for a.e. $t \in (0, T)$ by
\begin{align}
    \langle A(t)h, \varphi \rangle = \int_\Omega a(x, t, \nabla \times h) \cdot \nabla \times \varphi \quad \forall h, \varphi \in \mathbb{W}^p(\Omega), \quad (14)
\end{align}
is a uniformly bounded (independently of \( t \)), hemi-continuous, monotone and coercive operator, due to the structural properties (7a-c). Defining, for a.e. \( t \in (0, T) \), \( L(t) \in \widetilde{W}^p(\Omega)' \) by

\[
\langle L(t), \varphi \rangle = \int_\Omega f(t) \cdot \varphi + \int_\Gamma g(t) \cdot \varphi \quad \forall \varphi \in \widetilde{W}^p(\Omega),
\]

and adapting a well-known existence theorem to monotone operators independent of \( t \) (see [7]), we easily prove that problem (12) has a solution in \( L^p(0, T; \widetilde{W}^p(\Omega)) \cap C(0, T; L^2(\Omega)) \).

The uniqueness of solution results directly from the strict monotonicity (7c) of the operator \( A \).

To obtain the estimate (13) choose \( h(t) \) as test function in (12). Denoting \( Q_t = \Omega \times (0, t) \) and \( \Sigma_t = \Gamma \times (0, t) \), we have

\[
\frac{1}{2} \int_\Omega |h(t)|^2 + a_s \int_{Q_t} |\nabla \times h|^p \leq \int_{Q_t} f \cdot h + \int_{\Sigma_t} g \cdot h + \frac{1}{2} \int_\Omega |h_0|^2.
\]

Applying Hölder and Young inequalities and the Remark 1, we obtain

\[
\frac{1}{2} \|h(t)\|_{L^2(\Omega)}^2 + a_s \|\nabla \times h\|_{L^p(Q_t)}^2 \leq \|f\|_{L^p(\Omega)} \|h\|_{L^p(Q_t)} + \|g\|_{L^p(\Omega)} \|h\|_{L^p(\Omega)} + \frac{1}{2} \|h_0\|_{L^2(\Omega)}^2
\]

\[
\leq C_1 \left( \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} \right) \|\nabla \times h\|_{L^p(Q_t)} + \frac{1}{2} \|h_0\|_{L^2(\Omega)}^2
\]

and the conclusion follows.

**Remark 2.** The functional framework we introduced provides a general variational setting for the stationary solutions of (8). Indeed, for instance for arbitrary \( f \in L^q(\Omega) \), \( g \in L^\nu(\Gamma) \) and \( \nu \in L^\infty(\Omega) \), \( \nu(x) \geq a_* > 0 \) for a.e. \( x \in \Omega \), the unique minimum of the functional in \( W^p(\Omega) \),

\[
J(h) = \int_\Omega \frac{\nu}{p} |\nabla \times h|^p - \int_\Omega f \cdot h - \int_\Gamma g \cdot h,
\]

provides the weak stationary solution to (8). However, as remarked in [8] in the stationary problem, for the existence of solution of the strong boundary value problem (8) with given data \((f, g)\), it is necessary that \( f \) is divergence free and \( g \) is tangential and compatible with \( f (\nabla \cdot g = f \cdot n) \) on \( \Gamma \). But the weak formulation (12) of the problem (8) has a unique solution with no restrictions on the data.

Usually, a weak equation is also a strong one, as long as it has enough regularity. The situation here requires also additional compatibility conditions, since we are working with strongly coupled systems and the test functions have strong restrictions (they are divergence free and tangential on the boundary). Indeed, given \( f \in L^q(\Omega) \), the Helmholtz decomposition (see [14]) gives us that \( f = f_0 + \nabla \xi \), where \( f_0 \) is divergence free. On the other hand, if \( g \in L^\nu(\Gamma) \), \( g = g_T + g_N \), where \( g_T \) and \( g_N \) are, respectively, the tangential and the normal components of \( g \). So, the set of test functions \( \widetilde{W}^p(\Omega) \) only takes into account \( f_0 \) (the divergence free component of \( f \)) and \( g_T \) (the tangential component of \( g \)) and consequently the problems (12) with data \((f, g)\) and \((f_0, g_T)\) yields the same solution and both correspond to the weak formulation of the problem (8) with data \((f_0, g_T)\).

In the particular case where

\[
a(x, t, u) = \nu(x)|u|^{p-2}u, \quad \text{with } 0 < a_* \leq \nu \leq a^*, \quad x \in \Omega,
\]

(15)
we can improve the Proposition 2 assuming more regularity on the data.
In what follows we denote \( \alpha \lor \beta = \max\{\alpha, \beta\} \) and \( \alpha \land \beta = \min\{\alpha, \beta\} \).

**Proposition 3.** Let \( f \in L^{q/2}(Q_T) \), \( g \in L^{\infty}(0,T,L^r(\Gamma)) \cap W^{1,r}(0,T,L^r(\Gamma)) \) and \( h_0 \in \overline{W}^r(\Omega) \). Then the solution \( h \) of the problem (8) for \( a \) defined in (15) verifies

\[
\partial_t h \in L^2(Q_T) \quad \text{and} \quad \nabla \times h \in L^\infty(0,T,L^p(\Omega)).
\]

**Proof.** Using Galerkin approximations (see for instance [7] or Chapter 3 of [20]), we may set formally \( \partial_t h(t) \) as test function in (12). Integrating between 0 and \( t \) leads to

\[
\int_0^t \int_\Omega |\partial_t h|^2 + \int_0^t \int_\Omega \nu |\nabla \times h|^p |\nabla \times h| \cdot \partial_t \nabla \times h = \int_0^t \int_\Omega f \cdot \partial_t h + \int_0^t \int_\Gamma g \cdot \partial_t h.
\]

But

\[
\int_0^t \int_\Gamma g(t) \cdot \partial_t h(t) = \int_\Gamma g(0) \cdot h(0) - \int_0^t \int_\Gamma \partial_t g \cdot h(t)
\]

so we conclude that

\[
\left| \int_0^t \int_\Gamma g(t) \cdot \partial_t h(t) \right| \leq C_1 \|g\|_{L^{\infty}(0,T,L^r(\Gamma))} \|\nabla \times h(t)\|_{L^p(\Omega)} + \|\partial_t g\|_{L^r(\Sigma_T)} \|\nabla \times h\|_{L^p(Q_T)} + C_2.
\]

Noting that

\[
\int_0^t \int_\Omega \nu |\nabla \times h|^p |\nabla \times h(t)| \cdot \partial_t \nabla \times h(t) = \frac{1}{p} \int_\Omega \nu |\nabla \times h(t)|^p - \frac{1}{p} \int_\Omega \nu |\nabla \times h_0|^p,
\]

we have

\[
\int_0^t \int_\Omega \nu |\nabla \times h|^p |\nabla \times h(t)|^p \leq \|f\|_{L^2(Q_T)} \|\partial_t h\|_{L^2(Q_T)} + C_1 \|g\|_{L^{\infty}(0,T,L^r(\Gamma))} \|\nabla \times h(t)\|_{L^p(\Omega)}
\]

\[
\quad + \|\partial_t g\|_{L^r(\Sigma_T)} \|\nabla \times h\|_{L^p(Q_T)} + C_2 + \frac{\alpha}{p} \|\nabla \times h_0\|_{L^p(\Omega)}^p
\]

and so

\[
\|\partial_t h\|_{L^2(Q_T)}^2 + \|\nabla \times h\|_{L^{\infty}(0,T,L^p(\Omega))}^p \leq C(\|f\|_{L^2(Q_T)}^2 + \|g\|_{L^{\infty}(0,T,L^r(\Gamma))}^p + \|\partial_t g\|_{L^r(\Sigma_T)}^p) + C_3.
\]

\( \square \)

2.3. The asymptotic behaviour when \( t \to \infty \). In this section we give sufficient conditions in order to establish that

\[
\|h(t) - h_\infty\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty,
\]

where \( h \) denotes the solution of the problem (8) and \( h_\infty \) solves the stationary problem
\[ \nabla \cdot h_\infty = 0 \quad \text{and} \quad \nabla \times (a_\infty(x, \nabla \times h_\infty)) = f_\infty \quad \text{in} \quad \Omega, \]
\[ h_\infty \cdot n = 0 \quad \text{and} \quad a_\infty(x, \nabla \times h_\infty) \times n = g_\infty \quad \text{on} \quad \Gamma, \]

where
\[ f_\infty \in L^{q'}(\Omega) \quad \text{and} \quad g_\infty \in L^{r'}(\Gamma), \]

obtaining the variational formulation
\[ \int_\Omega a_\infty(x, \nabla \times h_\infty) \cdot \nabla \varphi = \int_\Omega f_\infty \cdot \varphi + \int_\Gamma g_\infty \cdot \varphi \quad \forall \varphi \in W^{1,p}_0(\Omega) \tag{18} \]

by applying the integration by parts (11).

Let
\[ f \in L^\infty(0, \infty; L^{q'}(\Omega)) \quad \text{and} \quad g \in L^\infty(0, \infty; L^{r'}(\Gamma)), \]

and denote
\[ \xi(t) = \|f(t) - f_\infty\|^2_{L^{q/(p)}(\Omega)} + \|g(t) - g_\infty\|^2_{L^{r/(p)}(\Gamma)} \quad \text{with} \quad s = \begin{cases} \frac{2}{q'} & \text{if} \quad \frac{2}{q'} \leq p < 2 \\ \frac{p}{2} & \text{if} \quad p \geq 2 \end{cases} \tag{19} \]

and
\[ \zeta(t) = \|a(x, t, u) - \tilde{a}_\infty(x, \tilde{u})\|^p_{L^p(\Omega \times \mathbb{R}^3)}. \]

2.3.1. The degenerate case \( p > 2. \)

**Theorem 2.1.** Let \( p > 2, \) suppose that the operators \( a \) and \( a_\infty \) satisfy (7a, b, c') and
\[ \int_0^t (\xi(\tau) + \xi(\tau)) \, d\tau \xrightarrow[t \to \infty]{} 0. \]

Then we have
\[ \|h(t) - h_\infty\|_{L^2(\Omega)} \xrightarrow[t \to \infty]{} 0. \]

**Proof.** Choosing for test function in (18), for a.e. \( t \in \mathbb{R}^+, \) \( w(t) = h(t) - h_\infty, \) we have
\[ \int_\Omega a_\infty(x, \nabla \times h_\infty) \cdot \nabla \times w(t) = \int_\Omega f_\infty \cdot w(t) + \int_\Gamma g_\infty \cdot w(t). \tag{20} \]

Taking \( w(t) \) as test function in (12), for a.e. \( t \in \mathbb{R}^+, \)
\[ \int_\Omega \partial_t h(t) \cdot w(t) + \int_\Omega a(x, t, \nabla \times h(t)) \cdot \nabla \times w(t) = \int_\Omega f(t) \cdot w(t) + \int_\Gamma g(t) \cdot w(t), \]

we conclude that
\[ \int_\Omega \partial_t h(t) \cdot w(t) + \int_\Omega \left( a(x, t, \nabla \times h(t)) - a(x, t, \nabla \times h_\infty) \right) \cdot \nabla \times w(t) \]
\[ = \int_\Omega (f(t) - f_\infty) \cdot w(t) + \int_\Gamma (g(t) - g_\infty) \cdot w(t) \]
\[ + \int_\Omega \left( a_\infty(x, \nabla \times h_\infty) - a(x, t, \nabla \times h_\infty) \right) \cdot \nabla \times w(t). \tag{21} \]

Since, by (7c'),
\[ \int_\Omega \left( a(x, t, \nabla \times h(t)) - a(x, t, \nabla \times h_\infty) \right) \cdot \nabla \times w(t) \geq a_* \int_\Omega |\nabla \times w(t)|^p, \]

subtracting (20) from (21), using Hölder and Young inequalities and the Remark 1, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |w(t)|^2 + C \left( \int_{\Omega} |w(t)|^2 \right)^{\frac{p}{2}} \\
\leq D_1 \left( \|f(t) - f_\infty\|_{L^{p'}(\Omega)} + \|g(t) - g_\infty\|_{L^{p'}(\Gamma)} \right) + D_2 \zeta(t).
\]

Denoting \( \phi(t) = \int_{\Omega} |w(t)|^2 \) and \( l(t) = 2D_1 \xi(t) + 2D_2 \zeta(t) \), the inequality (22) is written as follows

\[
\phi'(t) + 2C \phi(t)^{\frac{p}{2}} \leq l(t).
\]

So, applying Lemma 2.2 bellow with \( t_0 = \frac{t_0}{2} \), the theorem follows from

\[
\int_{\Omega} |h(t) - h_\infty|^2 \leq \left( \frac{2}{p-2} \int_{t_0}^t c(\sigma) d\sigma \right)^{\frac{p-2}{2}} + \int_{t_0}^t l(\sigma) d\sigma.
\]

\[\square\]

**Lemma 2.2** ([15], p 600). Let \( \phi \) be a real, continuous, positive function, a.e. differentiable in an interval \( I \subseteq \mathbb{R} \), such that

\[
\phi'(t) + c(t) \phi(t)^{\frac{p}{2}} \leq l(t) \quad \text{for a.e. } t \in I,
\]

being \( p > 2 \), \( c \geq 0 \) and \( l \) integrable in \( I \). Then

\[
\forall t_0, t \in I : t_0 \leq t \quad \phi(t) \leq \left( \frac{2}{p-2} \int_{t_0}^t c(\sigma) d\sigma \right)^{\frac{p-2}{2}} + \int_{t_0}^t l(\sigma) d\sigma.
\]

2.3.2. The case \( p = 2 \).

**Theorem 2.3.** Let \( p = 2 \) and suppose that the operators \( a \) and \( a_\infty \) verify (7 a, b, c') and

\[
\int_{t}^{t+1} (\zeta(\tau) + \xi(\tau)) d\tau \xrightarrow{t \to \infty} 0.
\]

Then we have

\[
\|h(t) - h_\infty\|_{L^2(\Omega)} \xrightarrow{t \to \infty} 0.
\]

**Proof.** Arguing as in the previous theorem, calling \( w(t) = h(t) - h_\infty \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |w(t)|^2 + a_\sigma \int_{\Omega} |\nabla \times w(t)|^2 \leq \int_{\Omega} (f(t) - f_\infty) \cdot w(t) + \int_{\Gamma} (g(t) - g_\infty) \cdot w(t) \\
+ \int_{\Omega} (a_\infty(x, \nabla \times h_\infty) - a(x, \nabla \times h_\infty)) \cdot \nabla \times w(t),
\]

from which we obtain, using Hölder and Young inequalities and the Remark 1,

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |w(t)|^2 + C_1 \int_{\Omega} |w(t)|^2 \\
\leq D_1 \left( \|f(t) - f_\infty\|_{L^{p'}(\Omega)}^2 + \|g(t) - g_\infty\|_{L^{p'}(\Gamma)}^2 \right) + D_2 \zeta(t).
\]

So

\[
\frac{d}{dt} \int_{\Omega} |w(t)|^2 + C \int_{\Omega} |w(t)|^2 \leq l(t) \leq l_0,
\]

(23)
where \( C = 2 C_1 \),

\[
l(t) = 2 D_1 \xi(t) + 2 D_2 \zeta(t),
\]

and \( l_0 \) is a constant which exists by the assumptions on \( a, a_\infty, f, f_\infty, g \) and \( g_\infty \).

In order to prove that \( w \in L^\infty(0, \infty; L^2(\Omega)) \), we multiply (23) by \( e^{Ct} \) and integrate in time, between \( \sigma \) and \( \tau \), \( l \leq \tau \leq \sigma \). Then

\[
\int_\sigma^\tau \int_\Omega e^{Ct} |w(t)|^2 \, dx \, dt + C \int_\sigma^\tau \int_\Omega e^{Ct} |w(t)|^2 \, dx \, dt \leq \int_\sigma^\tau l_0 e^{Ct}.
\]

But

\[
\int_\sigma^\tau \int_\Omega e^{Ct} \partial_t |w(t)|^2 = e^{C\tau} \int_\Omega |w(\tau)|^2 - e^{C\sigma} \int_\Omega |w(\sigma)|^2 - C \int_\sigma^\tau \int_\Omega e^{Ct} |w(t)|^2.
\]

Combining (24) and (25) we get

\[
e^{C\tau} \int_\Omega |w(\tau)|^2 \leq \frac{l_0}{C^2} \left( e^{C\tau} - e^{C\sigma} \right) + e^{C\sigma} \int_\Omega |w(\sigma)|^2
\]

and taking \( \tau = t \) and \( \sigma = 0 \), there exists a positive constant \( l_1 \) such that, for all \( t \),

\[
\int_\Omega |w(t)|^2 \leq \frac{l_0}{C^2} + \int_\Omega \left| h_0 - h_\infty \right|^2 \leq l_1.
\]

Applying Lemma 2.4 below, fixing \( t_0 > 0 \), for all \( t > t_0 \) we have

\[
\int_\Omega \left| h(t) - h_\infty \right|^2 \leq e^{C(t_0 - t)} l_1 + \frac{1}{1 - e^{Ct_0}} \sup_{\sigma \geq t_0} \int_{\tau}^{\tau+1} l(\sigma) \, d\sigma.
\]

\[\square\]

**Lemma 2.4** ([5], p 286). Let \( \phi(t) \) be a nonnegative function, absolutely continuous in any compact interval of \( \mathbb{R}^+ \), \( l(t) \) a nonnegative function belonging to \( L^1_{\text{loc}}(\mathbb{R}^+) \) and \( c \) a positive function such that

\[
\phi'(t) + c \phi(t) \leq l(t), \quad \forall t \geq 0.
\]

Then

\[
\forall t_0, t \in \mathbb{R}^+ : t_0 \leq t \quad \phi(t) \leq e^{c(t_0 - t)} \phi(t_0) + \frac{1}{1 - e^{Ct_0}} \sup_{\tau \geq t_0} \int_{\tau}^{\tau+1} l(\sigma) \, d\sigma.
\]

2.3.3. The singular case for \( \frac{6}{5} \leq p < 2 \) and \( a(x, t, u) = a_\infty(x, u) = \nu(x)|u|^{p-2}u \).

**Theorem 2.5.** Let (15) hold, \( \frac{6}{5} \leq p < 2 \) and \( \partial_t g \in L^\infty(0, \infty; L^r(\Gamma)) \). Suppose that

\[
\int_{t_0}^{t+1} \xi(\tau) \, d\tau \xrightarrow{t \to \infty} 0.
\]

Then we have

\[
\|h(t) - h_\infty\|_{L^2(\Omega)} \xrightarrow{t \to \infty} 0.
\]

**Proof.** By the property (7 c'),

\[
\nu \left( |\nabla \times h(t)|^{p-2} \nabla \times h(t) - |\nabla \times h_\infty|^{p-2} \nabla \times h_\infty \right) \cdot \left( \nabla \times h(t) - \nabla \times h_\infty \right)
\geq a_* \left( |\nabla \times h(t)| + |\nabla \times h_\infty| \right)^{p-2} |\nabla \times (h(t) - h_\infty)|^2.
\]
Setting \( w(t) = h(t) - h_\infty \), recalling (21) and using the above inequality, we obtain

\[
\frac{d}{dt} \int_\Omega |w(t)|^2 + a_s \int_\Omega (|\nabla \times h(t)| + |\nabla \times h_\infty|)^{p-2} |\nabla \times w(t)|^2 \\
\leq \int_\Omega (f(t) - f_\infty) \cdot w(t) + \int_\Gamma (g(t) - g_\infty) \cdot w(t).
\]

We recall now the inverse H"older inequality (see [17], p 8): let \( 0 < s < 1 \) and \( s' = \frac{2}{s-1} \). If \( F \in L^s(\Omega) \), \( FG \in L^1(\Omega) \) and \( \int_\Omega |G(x)|^{s'} \, dx < \infty \) then

\[
\left( \int_\Omega |F(x)|^s \, dx \right)^{\frac{1}{s}} \leq \int_\Omega |F(x)|^s \, dx \left( \int_\Omega |G(x)|^{s'} \, dx \right)^{-\frac{1}{s'}}
\]

and we apply it, with \( s = \frac{2}{p} \) (so \( s' = \frac{2}{p-2} \)), \( F = |\nabla \times (h(t) - h_\infty)|^2 \) and \( G = (|\nabla \times h(t)| + |\nabla \times h_\infty|^2)^{-1} \), in \( \hat{\Omega} = \{ x \in \Omega : |\nabla \times h(x, t)| + |\nabla \times h_\infty(x)| \neq 0 \} \).

So,

\[
\left( \int_{\hat{\Omega}} (|\nabla \times (h(t) - h_\infty)|^2)^{\frac{1}{p}} \right)^{\frac{p}{2}} \leq \int_{\hat{\Omega}} |\nabla \times (h(t) - h_\infty)^2 (|\nabla \times h(t)| + |\nabla \times h_\infty|)^{p-2} \left( \int_{\hat{\Omega}} (|\nabla \times h(t)| + |\nabla \times h_\infty|)^{p} \right)^{\frac{p-2}{p}}
\]

and

\[
\int_{\hat{\Omega}} |\nabla \times (h(t) - h_\infty)|^2 (|\nabla \times h(t)| + |\nabla \times h_\infty|)^{p-2} \left( \int_{\hat{\Omega}} (|\nabla \times h(t)| + |\nabla \times h_\infty|)^{p} \right)^{\frac{p-2}{p}}
\]

\[
\geq \left( \int_{\hat{\Omega}} |\nabla \times (h(t) - h_\infty)|^p \right)^{\frac{1}{2}} \left( \int_{\hat{\Omega}} (|\nabla \times h(t)| + |\nabla \times h_\infty|)^p \right)^{\frac{p-2}{p}}. \tag{26}
\]

From (18) and the assumptions we have

\[
\int_{\Omega} |\nabla \times h_\infty|^p \leq C_1.
\]

Simple calculations allows us to rewrite the inequality (17) in the form

\[
||\partial_t h||_{L^2(\Omega \times (0, \infty))} + ||\nabla \times h||_{L^p(\Omega \times (0, \infty))}^{\frac{2}{p}} \\
\leq C_2 \left( ||f||_{L^\infty(0, \infty; L^2(\Omega))} + ||g||_{L^\infty(0, \infty; L^{s'}(\Gamma))} + ||\partial_t g||_{L^\infty(0, \infty; L^{s'}(\Gamma))} \right) + C_3,
\]

where \( C_2 \) and \( C_3 \) are positive constants.

We get, using the Proposition 3,

\[
\left( \int_{\hat{\Omega}} (|\nabla \times h(t)| + |\nabla \times h_\infty|)^{p} \right)^{\frac{2}{p}} \leq C_4,
\]

and, from (26),

\[
\int_{\hat{\Omega}} |\nabla \times (h(t) - h_\infty)|^2 (|\nabla \times h(t)| + |\nabla \times h_\infty|)^{p-2} \geq \left( \int_{\hat{\Omega}} |\nabla \times (h(t) - h_\infty)|^p \right)^{\frac{1}{p}} \frac{1}{C_4}.
\]

By the Remark 1 we know, since \( p \geq \frac{4}{3} \), that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |w(t)|^2 + C_5 \int_\Omega |w(t)|^2 \leq D_1 \left( ||f(t) - f_\infty||_{L^2(\Omega)}^2 + ||g(t) - g_\infty||_{L^{s'}(\Gamma)}^2 \right),
\]

where \( D_1 \) is a constant.
and so, for $C = 2C_5$ and $l(t) = 2D_1 \xi(t)$ we deduce that
\[
\frac{d}{dt} \int_\Omega |w(t)|^2 + C \int_\Omega |w(t)|^2 \leq l(t),
\]
and the proof is concluded exactly as the previous one. □

3. A LIMIT PROBLEM WHEN $n \to \infty$

Given $p > 1$ let $\delta : Q_T \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathéodory function satisfying (7b), the monotonicity condition
\[
(\delta(x, t, u) - \delta(x, t, v)) \cdot (u - v) \geq 0,
\]
and also
\[
\delta \in L^1(Q_T \times \mathbb{R}^3), \quad \partial \delta \in L^1(Q_T \times \mathbb{R}^3) \quad \text{and} \quad \delta(x, t, 0) = 0. \tag{28}
\]
Let
\[
\mathbb{K} = \{ v \in \mathbb{W}^p(\Omega) : |\nabla \times v| \leq 1 \text{ a.e. in } \Omega \}
\]
and assume that
\[
f \in L^2(Q_T), \quad g \in L^\infty(0, T; L^1(\Gamma)) \cap W^{1,1}(0, T; L^1(\Gamma)) \quad \text{and} \quad h_0 \in \mathbb{K}. \tag{29}
\]
For $n \in \mathbb{N}$, $n > 3 \lor p$, define
\[
a_n(x, t, u) = |\nabla \times u|^{n-2} \nabla \times u + \delta(x, t, u) \tag{30}
\]
and consider the following problem: to find $h_n \in L^n(0, T; \mathbb{W}^n(\Omega)) \cap C([0, T]; L^2_\#(\Omega))$ and $h_t \in L^{n}(0, T; \mathbb{W}^n(\Omega'))$ such that, for a.e. $t \in (0, T)$,
\[
\int_\Omega \partial_t h_n(t) \cdot \varphi + \int_\Omega |\nabla \times h_n(t)|^{n-2} \nabla \times h_n(t) \cdot \nabla \varphi + \int_\Omega \delta(x, t, \nabla \times h_n(t)) \cdot \nabla \varphi = \int_\Omega f(t) \cdot \varphi + \int_\Gamma g(t) \cdot \varphi \quad \forall \varphi \in \mathbb{W}^n(\Omega), \tag{31}
\]
h_n(0) = h_0.

We define the variational inequality: to find $h_* \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; \mathbb{W}^\infty(\Omega))$ such that for a.e. $t \in (0, T)$, $h_*(t) \in \mathbb{K}$,
\[
\int_\Omega \partial_t h_*(t) \cdot (v - h_*(t)) + \int_\Omega \delta(x, t, \nabla \times h_*(t)) \cdot \nabla \times (v - h_*(t)) \geq \int_\Omega f(t) \cdot (v - h_*(t)) + \int_\Gamma g(t) \cdot (v - h_*(t)) \quad \forall v \in \mathbb{K}, \tag{32}
\]
h_*(0) = h_0.

Remark 3. Note that (32) has at most one solution and observe that the operator $\delta$ may be the null operator. □

Proposition 4. With the assumptions (27), (28) and (29), let $h_n$ be the solution of the problem (31).

Then there exists a positive constant $C$, independent of $n$, such that
\[
\|h_n\|_{L^\infty(0, T; L^2(\Omega))} \leq C, \quad \|\nabla \times h_n\|_{L^1(Q_T)} \leq C, \quad \|\partial_t h_n\|_{L^2(Q_T)} \leq C. \tag{33}
\]
Proof. Choosing $h_n$ as test function in (31) and using the Remark 1, we obtain

\[
\frac{1}{2} \int_{\Omega} |h_n(t)|^2 + \int_0^t \int_{\Omega} \nabla \times h_n|^n + \int_0^t \delta(x,t,\nabla \times h_n) \cdot \nabla \times h_n \\
= \int_0^t \int_{\Omega} f \cdot h_n + \int_0^t \int_{\Gamma} g \cdot h_n + \frac{1}{2} \int_{\Omega} |h_0|^2 \\
\leq \frac{2C}{n} \left( \int_0^t \int_{\Omega} |f| + \int_0^t \int_{\Gamma} |g| \right) + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla \times h_n|^n + \frac{1}{2} \int_{\Omega} |h_0|^2,
\]

where $C_1$ is a positive constant independent of $n$.

Since, for a.e. $t \in (0,T)$,

\[
\delta(x,t,\nabla \times h_n(t)) \cdot \nabla \times h_n(t) \geq 0,
\]

the first two inequalities of (33) follow immediately.

On the other hand, formally we have from (31), with $\varphi = \partial_t h_n(t)$,

\[
\int_{\Omega} |\partial_t h_n(t)|^2 + \int_{\Omega} |\nabla \times h_n(t)|^{n-2} \nabla \times h_n(t) \cdot \partial_t \nabla \times h_n(t) \\
+ \int_{\Omega} \delta(x,t, h_n(t)) \cdot \partial_t \nabla \times h_n(t) = \int_{\Omega} f(t) \cdot \partial_t h_n(t) + \int_{\Gamma} g(t) \cdot \partial_t h_n(t).
\]

Since

\[
\int_0^t \int_{\Omega} \delta(x,t, \nabla \times h_n) \cdot \partial_t \nabla \times h_n = \int_{\Omega} \delta(x,t, \nabla \times h_n(t)) \cdot \nabla \times h_n(t) \\
- \int_{\Omega} \delta(x,0, \nabla \times h_n(0)) \cdot \nabla \times h_n(0) - \int_{\Omega} \partial_t \delta(x,t, \nabla \times h_n) \cdot h_n \\
\geq - \int_{\Omega} \delta(x,0, \nabla \times h_0) \cdot \nabla \times h_0 - \int_0^t \int_{\Omega} \partial_t \delta(x,t, \nabla \times h_n) \cdot h_n,
\]

and, on the other hand,

\[
\int_0^t \int_{\Gamma} g \cdot \partial_t h_n = \int_{\Gamma} g(t) \cdot h_n(t) - \int_{\Gamma} g(0) \cdot h_0 - \int_0^t \int_{\Gamma} \partial_t g \cdot h_n,
\]

we have

\[
\frac{1}{2} \int_0^t \int_{\Omega} |\partial_t h_n|^2 \leq \frac{1}{n} \frac{\|\nabla \times h_0\|_{L^n(\Omega)}}{} + \frac{C_2}{n} \left( \int_{\Gamma} |g(t)| \right)^{n'} + \frac{1}{2} \int_{\Omega} |\nabla \times h_n(t)|^n.
\]

Theorem 3.1. Let $h_n$ be the solution of the problem (31). Then, at least for subsequences, we have

\[
h_n \rightharpoonup h_* \quad \text{in } C([0,T]; L^2(\Omega))-\text{strong},
\]

\[
\partial_t h_n \rightharpoonup \partial_t h_* \quad \text{in } L^2(Q_T)-\text{weak},
\]

\[
\nabla \times h_n \rightharpoonup \nabla \times h_* \quad \text{in } L^q(Q_T)-\text{weak},
\]

for any fixed $3 < q < \infty$, where $h_*$ is the solution of the problem (32).
Proof. By the uniform estimates in (33) we only need to check that \( h_* \) solves (32).

Let \( \varphi \in W^p(\Omega) \) be such that \( |\nabla \times \varphi| < 1 \) a.e.. Taking \( \varphi - h_n \) as a test function in (31), we have

\[
\int_{Q_T} \partial_t h_n \cdot (\varphi - h_n) + \int_{Q_T} |\nabla \times h_n|^n - 2 \nabla \times h_n \cdot \nabla \times (\varphi - h_n) + \int_{Q_T} \delta(x, t, \nabla \times h_n) \cdot \nabla \times (\varphi - h_n) = \int_{Q_T} f \cdot (\varphi - h_n) + \int_{\Sigma_T} g \cdot (\varphi - h_n).
\]

By the monotonicity of the operator \( a_n \) defined in (30) we have

\[
\int_{Q_T} \partial_t h_n \cdot (\varphi - h_n) + \int_{Q_T} |\nabla \times \varphi|^n - 2 \nabla \times \varphi \cdot \nabla \times (\varphi - h_n) + \int_{Q_T} \delta(x, t, \nabla \times \varphi) \cdot \nabla \times (\varphi - h_n) \geq \int_{Q_T} f \cdot (\varphi - h_n) + \int_{\Sigma_T} g \cdot (\varphi - h_n). \tag{36}
\]

Applying limit in \( n \) to both members of (36) we get

\[
\int_{Q_T} \partial_t h_* \cdot (\varphi - h_*) + \int_{Q_T} \delta(x, t, \nabla \times \varphi) \cdot \nabla \times (\varphi - h_*) \geq \int_{Q_T} f \cdot (\varphi - h_*) + \int_{\Sigma_T} g \cdot (\varphi - h_*). \tag{37}
\]

Since \( \varphi \) is an arbitrary function of \( W^p(\Omega) \) satisfying \( |\nabla \varphi| < 1 \), the inequality (37) still holds, by density, for all \( \varphi \in \mathbb{K}_* \). We also have \( h_*(0) = h_0 \).

Given \( p < q < n \), by (33),

\[
||\nabla \times h_n||_{L^q(\Omega)} \leq |Q_T|^\frac{q}{q-p} ||\nabla \times h_n||_{L^n(\Sigma_T)} \leq |Q_T|^\frac{q}{q-p} C_0^\frac{q}{q-p}
\]

and

\[
||\nabla \times h_*||_{L^q(\Omega)} \leq \liminf_n ||\nabla \times h_n||_{L^q(\Omega)} \leq |Q_T|^\frac{q}{q-p} \forall q > p,
\]

so \( \nabla \times h_* \in L^\infty(\Omega) \) and \( ||\nabla \times h_*||_{L^\infty(\Omega)} \leq 1 \) which proves that \( h_*(t) \) belongs to the convex set \( \mathbb{K}_* \) for a.e. \( t \in (0, T) \).

Choosing \( \varphi = h_* + \lambda(w - h_*) \), with \( \lambda \in (0, 1] \) and \( w \) any element of \( \mathbb{K}_* \), we have

\[
\int_{Q_T} \partial_t h_* \cdot (w - h_*) + \int_{Q_T} \delta(x, t, \nabla \times h_* + \lambda(w - h_*)) \cdot \nabla \times (w - h_*) \geq \int_{Q_T} f \cdot (w - h_*) + \int_{\Sigma_T} g \cdot (w - h_*).
\]

Letting \( \lambda \to 0 \), we get

\[
\int_{Q_T} \partial_t h_* \cdot (w - h_*) + \int_{Q_T} \delta(x, t, \nabla \times h_*) \cdot \nabla \times (w - h_*) \geq \int_{Q_T} f \cdot (w - h_*) + \int_{\Sigma_T} g \cdot (w - h_*).
\]

Standard arguments imply that, for a.e. \( t \in (0, T) \),

\[
\int_{\Omega} \partial_t h_*(t) \cdot (w - h_*(t)) + \int_{\Omega} \delta(x, t, \nabla \times h_*(t)) \cdot \nabla \times (w - h_*(t)) \geq \int_{\Omega} f \cdot (w - h_*(t)) + \int_{\Sigma_T} g \cdot (w - h_*(t)).
\]
Remark 4. If $\delta = \delta(x,u)$ is independent of $t$ (in particular $\delta = 0$) in the corresponding stationary problem (31) with stationary data $f(x,t) = f_\infty(x)$ and $g(x,t) = g_\infty(x)$, i.e.

\[
\int_\Omega |\nabla \times h_n\omega - \nabla \times h_\omega\omega \cdot \nabla \varphi \, dx + \int_\Omega \delta(x,\nabla \times h_n\omega) \cdot \nabla \varphi = \int_\Omega f_\infty \cdot \varphi + \int_\Gamma g_\infty \cdot \varphi \quad \forall \varphi \in W^n(\Omega),
\]

it was shown in [8] that there exists subsequences $n' \to \infty$ and $h_\omega\omega \in K_\omega$ such that

\[
h_n\omega\omega \rightharpoonup h_\omega\omega \quad \text{in} \quad W^q(\Omega)-\text{weak}, \quad \text{for} \quad n' \to \infty,
\]

for any fixed $3 < q < \infty$, where $h_\omega\omega$ is a solution in $K_\omega$ of

\[
\int_\Omega \delta(x,\nabla \times h_\omega\omega) \cdot \nabla \times (\varphi - h_\omega\omega) \geq \int_\Omega f_\infty \cdot (\varphi - h_\omega\omega) + \int_\Gamma g_\infty \cdot (\varphi - h_\omega\omega) \quad \forall \varphi \in K_\omega.
\]

In general, (39) may have more than one solution if $\delta$ is not strictly monotone, in particular when $\delta = 0$. □

Remark 5. If we apply Theorem 2.1 for each fixed $n > 3$ with

\[
\theta(t) = \int_t^T \left( \|f(\tau) - f_\infty\|_{L^1(\Omega)} + \|g(\tau) - g_\infty\|_{L^1(\Gamma)} \right) \, d\tau \quad \text{to} \quad 0
\]

we have the estimate

\[
\int_\Omega |h_n(t) - h_\omega\omega|^2 \leq (C(n-2)t)^{\frac{2}{n-2}} + \theta(t),
\]

where the constant $C > 0$ is independent of $n$ and $t$.

So, for a subsequence $n'$ satisfying (38), there exists a sequence, $t_{n'} \to \infty$, such that

\[
h_n(t_{n'}) \rightharpoonup h_\omega\omega \quad \text{in} \quad L^2(\Omega)-\text{weak}, \quad \text{for} \quad n' \to \infty.
\]

An interesting open question in the degenerate case is whether there exists a sequence $t_n \to \infty$ such that $h_\omega(t_n)$ converges, in some sense, to $h_\omega\omega$. □

4. The variational inequality with evolutionary curl constraint

Define, for a.e. $t \in (0,T)$, the following closed convex subset of $W^p(\Omega)$,

\[
K(t) = \{ v \in W^p(\Omega) : |\nabla \times v| \leq \Psi(t), \text{ a.e. in } \Omega \},
\]

where $\Psi : Q_T \to \mathbb{R}^+$ is a function such that $\Psi \geq \alpha > 0$.

In this section we assume the following regularity of the data:

\[
f \in L^q(0,T;L^q(\Omega)) \cap L^{1,q'}(0,T;L^{1,q'}(\Gamma)), \quad g \in L^\infty(0,T;L^q(\Gamma)) \cap W^{1,q'}(0,T;L^{1,q'}(\Gamma)), \quad \nu \in L^\infty(\Omega), \quad 0 \leq a_s \leq \nu \leq a^*, \quad \Psi \in L^1(0,T;L^\infty(\Omega)) \quad \text{and} \quad h_0 \in K(0).
\]
We define the variational inequality: to find \( h \), in a suitable class of functions, such that
\[
\int_{\Omega} \partial_t h(t) \cdot (\varphi - h(t)) + \int_{\Omega} \nu |\nabla \times h(t)|^{p-2} \nabla \times h(t) \cdot \nabla \times (\varphi - h(t)) \\
\geq \int_{\Omega} f(t) \cdot (\varphi - h(t)) + \int_{\Gamma} g(t) \cdot (\varphi - h(t)),
\]
\( \forall \varphi \in \mathcal{K}(t), \) for a.e. \( t \in (0, T) \).

4.1. **The approximated problem.** Following a natural constraint penalization also used in a similar scalar parabolic problem [12, 13], we introduce a small positive parameter \( \varepsilon < 1 \).

Let us consider a continuous bounded increasing function \( k_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^+ \), satisfying
\[
k_\varepsilon(s) = \begin{cases} 1, & \text{if } s \leq 0, \\ e^{\varepsilon s}, & \text{if } \varepsilon < s \leq \frac{1}{\varepsilon} - \varepsilon, \\ e^{\frac{s}{\varepsilon}}, & \text{if } s \geq \frac{1}{\varepsilon}. \end{cases}
\]
and define, for \((x, t) \in Q_T\) and \( u \in \mathbb{R}^3 \),
\[
a(x, t, u) = \nu(x) k_\varepsilon(\|u\|^p - \Psi_p(x, t)) |u|^{p-2} u.
\]

The operator \( A \), as defined in (14) with this \( a \), is bounded, monotone, coercive and hemicontinuous and so, by Proposition 2, for each \( \varepsilon > 0 \), the approximated problem
\[
\int_{\Omega} \partial_t h_\varepsilon(t) \cdot \varphi + \int_{\Omega} \nu k_\varepsilon(\|\nabla \times h_\varepsilon(t)\|^p - \Psi(t)^p) |\nabla \times h_\varepsilon(t)|^{p-2} \nabla \times h_\varepsilon(t) \cdot \nabla \times \varphi \\
= \int_{\Omega} f(t) \cdot \varphi + \int_{\Gamma} g(t) \cdot \varphi \quad \forall \varphi \in \widetilde{W}^p(\Omega), \text{ for a.e. } t \in (0, T),
\]
\[
h_\varepsilon(0) = h_0,
\]
has a unique solution, \( h_\varepsilon \in L^p(0, T; \widetilde{W}^p(\Omega)) \cap C([0, T]; L^2_p(\Omega)) \), satisfying the estimate (13), independently of \( \varepsilon \). Since \( k_\varepsilon(s) \geq 1 \), we have the following lemma.

**Lemma 4.1.** There is a positive constant \( C \) such that, for all \( 0 < \varepsilon < 1 \),
\[
\|h_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))} + \|\nabla \times h_\varepsilon\|_{L^p(Q_T)} \leq C.
\]

4.2. **Existence of solution of the variational inequality.** In order to prove that a subsequence of the solutions of the approximate problems converges, with \( \varepsilon \rightarrow 0 \), for the solution of the variational inequality, we need additional a priori estimates.

**Lemma 4.2.** There is a positive constant \( C \) such that, for \( 0 < \varepsilon < 1 \),
\[
\|k_\varepsilon(\|\nabla \times h_\varepsilon\|^p - \Psi_p)\|_{L^1(Q_T)} \leq C.
\]

**Proof.** Choosing in (41) \( \varphi = h_\varepsilon \), we obtain, for a positive constant \( C_1 \),
\[
a_\ast \int_{Q_T} k_\varepsilon(\|\nabla \times h_\varepsilon\|^p - \Psi_p)|\nabla \times h_\varepsilon|^p \\
\leq C_1 \left( \|f\|_{L^{p'}(Q_T)}^{p'} + \|g\|_{L^{p'}(\Sigma_T)}^{p'} + \|h_0\|^2_{L^2(\Omega)} \right).
\]
Observing that \( k_\varepsilon(s) = 1 \) for \( s \leq 0 \) and \( k_\varepsilon(s) \geq 0 \) for \( s \geq 0 \), we have
\[
\int_{Q_T} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)(|\nabla \times h_\varepsilon|^p - \Psi^p) = \int_{\{|\nabla \times h_\varepsilon|^p - \Psi^p \leq 0\}} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)(|\nabla \times h_\varepsilon|^p - \Psi^p) + \int_{\{|\nabla \times h_\varepsilon|^p - \Psi^p > 0\}} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)(|\nabla \times h_\varepsilon|^p - \Psi^p)
\]
\[
\geq \int_{\{|\nabla \times h_\varepsilon|^p - \Psi^p \leq 0\}} \int_{\{|\nabla \times h_\varepsilon|^p - \Psi^p \leq 0\}} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)(|\nabla \times h_\varepsilon|^p - \Psi^p) + \int_{\{|\nabla \times h_\varepsilon|^p - \Psi^p \leq 0\}} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)(|\nabla \times h_\varepsilon|^p - \Psi^p)\Psi^p
\]
\[
\geq -\int_{Q_T} \Psi^p.
\]

Recalling that \( \Psi \geq \alpha > 0 \) we obtain
\[
\int_{Q_T} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p) \leq \int_{Q_T} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)\frac{\Psi^p}{\alpha^p}
\]
\[
= \frac{1}{\alpha^p} \int_{Q_T} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)(\Psi^p - |\nabla \times h_\varepsilon|^p)
\]
\[
+ \frac{1}{\alpha^p} \int_{Q_T} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)|\nabla \times h_\varepsilon|^p.
\]

Applying the relations (42) and (43) to the last inequality we have
\[
\int_{Q_T} k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)
\leq \frac{C}{\alpha^p} \left( \|f\|_{L^{p'}(Q_T)} + \|g\|_{L^{p'}(\Sigma_T)} + \|h_0\|_{L^2(\Omega)} \right) + \frac{1}{\alpha^p} \|\Psi^p\|_{L^p(\Omega)}.
\]

Lemma 4.3. There is a positive constant \( C \) such that, for \( 0 < \varepsilon < 1 \),
\[
\|\partial_t h_\varepsilon\|_{L^2(Q_T)} \leq C.
\]

Proof. Using Galerkin approximations, we can use \( \partial_t h_\varepsilon(t) \) formally as test function in equation (41). Integrating that equation over \((0, t)\) and setting
\[
\phi_\varepsilon(s) = \int_0^s k_\varepsilon(\tau) d\tau,
\]
we have
\[
\int_{Q_t} |\partial_t h_\varepsilon|^2 + \frac{1}{p} \int_{Q_t} \nu \partial_t \left( k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p) \right)
\]
\[
+ \int_{Q_t} \nu k_\varepsilon(|\nabla \times h_\varepsilon|^p - \Psi^p)\Psi^{p-1} \partial_t \Psi = \int_{Q_t} f : \partial_t h_\varepsilon + \int_{\Sigma_t} g \cdot \partial_t h_\varepsilon. \quad (44)
\]

Observing that \( \phi_\varepsilon(s) = s \), if \( s \leq 0 \), and \( \phi_\varepsilon(s) \geq s \), if \( s \geq 0 \),
\[
\frac{1}{p} \int_{Q_T} \nu \partial_t \left( \phi_\varepsilon \left( |\nabla \times h_\varepsilon|^p - \Psi^p \right) \right) \\
= \frac{1}{p} \int_{\Omega} \nu \phi_\varepsilon \left( |\nabla \times h_\varepsilon(t)|^p - \Psi(t)^p \right) - \frac{1}{p} \int_{\Omega} \nu \phi_\varepsilon \left( |\nabla \times h_0|^p - \Psi(0)^p \right) \\
\geq \frac{a_0}{p} \|\nabla \times h_\varepsilon(t)\|_{L^p(\Omega)}^p - \frac{a_0}{p} \|\Psi(t)\|_{L^p(\Omega)}^p. \quad (45)
\]

The Hölder inequality allows us to obtain
\[
\int_{Q_T} \nu k_\varepsilon \left( |\nabla \times h_\varepsilon|^p - \Psi^p \right) \Psi^{p-1} \partial_t \Psi \\
\leq a^* \|k_\varepsilon \left( |\nabla \times h_\varepsilon|^p - \Psi^p \right)\|_{L^1(Q_T)} \|\Psi^{p-1}\|_{L^\infty(Q_T)} \|\partial_t \Psi\|_{L^\infty(Q_T)}. \quad (46)
\]
and
\[
\int_{Q_T} f \cdot \partial_t h_\varepsilon \leq \|f\|_{L^2(Q_T)} \|\partial_t h_\varepsilon\|_{L^2(Q_T)}. \quad (47)
\]

Arguing as in (34) and (35) we have
\[
\int_{\Sigma_T} g \cdot \partial_t h_\varepsilon = \int_{\Gamma} g(t) \cdot h_\varepsilon(t) - \int_{\Gamma} g(0) \cdot h_0 - \int_{\Sigma_T} \partial_t g \cdot h_\varepsilon \\
\leq \frac{C}{p} \|g\|_{L^\infty(0,T;L^p(\Gamma))} + \frac{1}{p} \|\nabla \times h_\varepsilon(t)\|_{L^p(\Omega)} + \|g(0)\|_{L^p(\Gamma)} \|h_0\|_{L^p(\Gamma)} \\
+ C_1 \|\partial_t g\|_{L^1(\Sigma_T)} \|\nabla \times h_\varepsilon\|_{L^p(Q_T)}. \quad (48)
\]

Using the Proposition 3, the relations (45-48) in the equality (44) we obtain the lemma. \qed

**Theorem 4.4.** The variational inequality (40) has a unique solution \( h \) belonging to \( L^p(0,T;W^{\infty}(\Omega)) \cap H^1(0,T;L^2(\Omega)). \)

**Proof.** By the Lemmas 4.1, 4.2 and 4.3 and well-known compactness results (see [16]), there exists a subsequence \( \varepsilon \to 0 \) such that
\[
\begin{align*}
\varepsilon & \quad \longrightarrow \quad h \\
\nabla \times h_\varepsilon & \quad \longrightarrow \quad \nabla \times h \\
\partial_t h_\varepsilon & \quad \longrightarrow \quad \partial_t h
\end{align*}
\]

\( \text{in } L^\infty(0,T;L^2(\Omega))-\text{weak* and } C([0,T];L^1(\Omega))-\text{strong,} \)
\[
\begin{align*}
\nabla \times h_\varepsilon & \quad \text{in } L^p(Q_T)-\text{weak,} \\
\partial_t h_\varepsilon & \quad \text{in } L^2(Q_T)-\text{weak.}
\end{align*}
\]

By the monotonicity of \( k_\varepsilon \), choosing \( \varphi \in \mathbb{K}(t) \) we obtain
\[
\int_{\Omega} \nu k_\varepsilon \left( |\nabla \times h_\varepsilon(t)|^p - \Psi(t)^p \right) |\nabla \times h_\varepsilon(t)|^{p-2} \nabla \times h_\varepsilon(t) \cdot \nabla \times (\varphi - h_\varepsilon(t)) \\
\leq \int_{\Omega} \nu |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \times (\varphi - h_\varepsilon(t)).
\]

Choosing in (41) for test function \( \varphi - h_\varepsilon(t) \), being \( \varphi \in \mathbb{K}(t) \) and integrating in time, we obtain
\[
\int_{Q_T} \partial_t h_\varepsilon \cdot (\varphi - h_\varepsilon) + \int_{Q_T} \nu |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \times (\varphi - h_\varepsilon) \\
\geq \int_{Q_T} f \cdot (\varphi - h_\varepsilon) + \int_{\Sigma_T} g \cdot (\varphi - h_\varepsilon).
\]
Lemma 4.5. Let $\phi$ where conclude next lemma, applying a variant of Minty’s Lemma and standard arguments, we
get

$$\liminf_{\varepsilon \to 0} \int_{Q_T} \partial_t h \cdot (\varphi - h) = \liminf_{\varepsilon \to 0} \left( \int_{Q_T} \partial_t h \cdot \varphi - \frac{1}{2} \int_{\Omega} |h_\varepsilon(t)|^2 + \frac{1}{2} \int_{\Omega} |h_0|^2 \right) \leq \int_{Q_T} \partial_t h \cdot (\varphi - h),$$

we get

$$\int_{Q_T} \partial_t h \cdot (\varphi - h) + \int_{Q_T} \nu |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \times (\varphi - h_\varepsilon) \geq \int_{Q_T} f \cdot (\varphi - h) + \int_{\Sigma_T} g \cdot (\varphi - h).$$

Assuming that $h(t) \in K(t)$ for a.e. $t \in (0, T)$ (this fact will be proved in the next lemma), applying a variant of Minty’s Lemma and standard arguments, we conclude

$$\int_{Q_T} \partial_t h(t) \cdot (\varphi - h) + \int_{Q_T} \nu |\nabla h|^{p-2} \nabla h(t) \cdot \nabla \times (\varphi - h(t)) \geq \int_{Q_T} f(t) \cdot (\varphi - h(t)) + \int_{\Sigma_T} g(t) \cdot (\varphi - h(t)),$$

where $\varphi$ is any function belonging to $K(t)$, for a.e. $t \in (0, T)$.

The uniqueness is immediate. □

Lemma 4.5. Let $h_\varepsilon$ be the solution of the problem (41) and $h$ the weak limit of a subsequence of $\{h_\varepsilon\}_\varepsilon$ in $L^p(0, T; \bar{W}^p(\Omega)) \cap H^1(0, T; L^2(\Omega))$. Then

$$h(t) \in K(t), \text{ for a.e. } t \in (0, T).$$

Proof. Define

$$A_\varepsilon = \{ (x, t) \in Q_T : |\nabla \cdot h_\varepsilon(x, t)|^{p-\Psi} < \varepsilon \},$$

$$B_\varepsilon = \{ (x, t) \in Q_T : \sqrt{\varepsilon} \leq |\nabla \cdot h_\varepsilon(x, t)|^{p-\Psi} \leq \frac{1}{\varepsilon} \},$$

$$C_\varepsilon = \{ (x, t) \in Q_T : |\nabla \cdot h_\varepsilon(x, t)|^{p-\Psi} > \frac{1}{\varepsilon} \}.$$

We have

$$\int_{A_\varepsilon} \sqrt{\varepsilon} \leq \sqrt{\varepsilon}|Q_T| \xrightarrow{\varepsilon \to 0} 0$$

and

$$\int_{C_\varepsilon} \frac{1}{\varepsilon} \int_{C_\varepsilon} \frac{k_\varepsilon(|\nabla \cdot h_\varepsilon(x, t)|^{p-\Psi} - \Psi(x, t))}{e^{1/\varepsilon^2}} \leq C \frac{1}{\varepsilon} e^{-1/\varepsilon^2} \xrightarrow{\varepsilon \to 0} 0.$$

Recalling that, in $B_\varepsilon$, $k_\varepsilon(|\nabla \cdot h_\varepsilon(x, t)|^{p-\Psi} - \Psi(x, t)) \geq e^{1/\varepsilon^2}$, we have

$$|B_\varepsilon| = \int_{A_\varepsilon} 1 \leq \int_{A_\varepsilon} \frac{k_\varepsilon(|\nabla \cdot h_\varepsilon(x, t)|^{p-\Psi} - \Psi(x, t))}{e^{1/\varepsilon^2}} \leq C e^{-1/\varepsilon^2},$$

and so, $|B_\varepsilon| \xrightarrow{\varepsilon \to 0} 0$. 

Since
\[
\int_{Q_T} (|\nabla \times h| - \Psi)^+ = \int_{Q_T} \liminf_{\varepsilon \to 0} (|\nabla \times h_\varepsilon| - \Psi) \wedge \frac{1}{\varepsilon} \vee \sqrt{\varepsilon} \leq \liminf_{\varepsilon \to 0} \int_{Q_T} (|\nabla \times h_\varepsilon| - \Psi) \wedge \frac{1}{\varepsilon} \vee \sqrt{\varepsilon} = \liminf_{\varepsilon \to 0} \left( \int_{A_\varepsilon} \sqrt{\varepsilon} + \int_{B_\varepsilon} (|\nabla \times h_\varepsilon| - \Psi) + \int_{C_\varepsilon} \frac{1}{\varepsilon} \right)
\]
\[
= \liminf_{\varepsilon \to 0} \int_{Q_T} (|\nabla \times h_\varepsilon| - \Psi) \chi_{B_\varepsilon} \leq \liminf_{\varepsilon \to 0} \int_{Q_T} |\nabla \times h_\varepsilon| - \Psi \|_{L^p(Q_T)} |B_\varepsilon|^{\frac{1}{p}} = 0.
\]
we conclude that $|\nabla \times h| \leq \Psi$ a.e. in $Q_T$, completing the proof. \hfill \Box

4.3. Continuous dependence on the data. Consider given data $(f_i, g_i, h_{i0}, \Psi_i)$ and define $\mathbb{K}_i(t) = \{v \in \mathbb{W}^p(\Omega) : |\nabla \times v| \leq \Psi_i(t) \text{ a.e. in } \Omega\}$, for $i = 1, 2$.

**Lemma 4.6.** Given a function $h_1 \in L^p(0, T; \mathbb{W}^p(\Omega))$ such that $h_1(t) \in \mathbb{K}_1(t)$ for a.e. $t \in (0, T)$, there exists a function $\hat{h}_2 \in L^p(0, T; \mathbb{W}^p(\Omega))$, verifying $h_2(t) \in \mathbb{K}_2(t)$ for a.e. $t \in (0, T)$ and a positive constant $C$ such that
\[
|\nabla \times (h_1 - \hat{h}_2)|_{L^p(Q_T)} \leq C \|\Psi_1 - \Psi_2\|_{L^\infty(\Omega)}.
\]

**Proof.** Define
\[
\beta(t) = \|\Psi_1(t) - \Psi_2(t)\|_{L^\infty(\Omega)}, \quad \eta(t) = \frac{\alpha}{\alpha + \beta(t)} \quad \text{and} \quad \hat{h}_2 = \eta h_1.
\]

Then
\[
|\nabla \times \hat{h}_2(t)| = \eta(t)|\nabla \times h_1(t)| \leq \eta(t)\Psi_1(t) \leq \Psi_2(t),
\]

since
\[
\frac{\Psi_1(t)}{\Psi_2(t)} - \frac{\Psi_1(t) - \Psi_2(t)}{\Psi_2(t)} \leq \frac{\beta(t)}{\alpha} + 1 = \frac{1}{\eta(t)},
\]
and so $\hat{h}_2(t) \in \mathbb{K}_2(t)$ for a.e. $t \in (0, T)$.

Now,
\[
|\nabla \times (h_1(t) - \hat{h}_2(t))|_{L^p(Q_T)}^p = \int_{Q_T} |\nabla \times (h_1(t) - \hat{h}_2(t))|^p \]
\[
= \int_{Q_T} (1 - \eta(t))|\nabla \times h_1(t)|^p \]
\[
= \int_{Q_T} \left( \frac{\beta(t)}{\alpha + \beta(t)} \right)^p |\nabla \times h_1(t)|^p \]
\[
\leq \frac{\beta(t)^p}{\alpha^p} \int_{\Omega} |\nabla \times h_1(t)|^p.
\]

Integrating in time we obtain
\[
|\nabla \times (h_1 - \hat{h}_2)|_{L^p(Q_T)}^p \leq \frac{1}{\alpha^p} \|\nabla \times h_1\|_{L^p(Q_T)}^p \|\Psi_1 - \Psi_2\|_{L^\infty(\Omega)}^p.
\]

\hfill \Box

**Remark 6.** If we replace, in the last lemma, the subscript 1 by the subscript 2, the corresponding function we construct will be denoted by $\tilde{h}_1$. \hfill \Box
Theorem 4.7. Let $h_i$ denote the solution of the variational inequality (40) with data $(f_i, g_i, \Psi_i, h_{i0})$, $i = 1, 2$.

Then there exists a positive constant $C$ such that

$$\|h_1 - h_2\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \times (h_1 - h_2)\|^2_{L^p(Q_T)} \leq C(\|f_1 - f_2\|^p_{L^p(Q_T)} + \|g_1 - g_2\|^p_{L^p(\Sigma_T)}$$

$$+ \|h_{i0} - h_{20}\|^2_{L^2(\Omega)} + \|\Psi_1 - \Psi_2\|_{L^\infty(Q_T)}). \quad (50)$$

Proof. We know that, given $\varphi \in \mathcal{K}_2(t)$, $i = 1, 2$, we have

$$\int_{\Omega} \partial_t h_i(t) \cdot (\varphi - h_i(t)) + \int_{\Omega} \nu|\nabla \times h_i(t)|^{p-2} \nabla \times h_i(t) \cdot \nabla \times (\varphi - h_i(t))$$

$$\geq \int_{\Omega} f_i(t) \cdot (\varphi - h_i(t)) + \int_{\Gamma} g_i(t) \cdot (\varphi - h_i(t)),$$

$$h_i(0) = h_{i0}.$$ 

Choose, for $i = 1$, $\hat{h}_1$ as test function. Then,

$$\int_{\Omega} \partial_t h_1(t) \cdot (h_1(t) - \hat{h}_1(t)) + \int_{\Omega} \nu|\nabla \times h_1(t)|^{p-2} \nabla \times h_1(t) \cdot \nabla \times (h_1(t) - \hat{h}_1(t))$$

$$\leq \int_{\Omega} f_1(t) \cdot (h_1(t) - \hat{h}_1(t)) + \int_{\Gamma} g_1(t) \cdot (h_1(t) - \hat{h}_1(t),$$

from which we obtain

$$\int_{\Omega} \partial_t h_1(t) \cdot (h_1(t) - h_2(t)) + \int_{\Omega} \nu|\nabla \times h_1(t)|^{p-2} \nabla \times h_1(t) \cdot \nabla \times (h_1(t) - h_2(t))$$

$$\leq \int_{\Omega} f_1(t) \cdot (h_1(t) - h_2(t)) + \int_{\Gamma} g_1(t) \cdot (h_1(t) - h_2(t))$$

$$+ \int_{\Omega} \partial_t h_1(t) \cdot (\hat{h}_1(t) - h_2(t)) + \int_{\Omega} \nu|\nabla \times h_1(t)|^{p-2} \nabla \times h_1(t) \cdot \nabla \times (\hat{h}_1(t) - h_2(t))$$

$$+ \int_{\Omega} f_1(t) \cdot (h_2(t) - \hat{h}_1(t)) + \int_{\Gamma} g_1(t) \cdot (h_2(t) - \hat{h}_1(t)).$$

We have an analogous expression with $h_1$ substituted by $h_2$ and $\hat{h}_1$ by $\hat{h}_2$. From both expressions we get

$$\int_{\Omega} \partial_t (h_1(t) - h_2(t)) \cdot (h_1(t) - h_2(t))$$

$$+ \int_{\Omega} \nu(|\nabla \times h_1(t)|^{p-2} \nabla \times h_1(t) - |\nabla \times h_2(t)|^{p-2} \nabla \times h_2(t)) \cdot \nabla \times (h_1(t) - h_2(t))$$

$$\leq \int_{\Omega} (f_1(t) - f_2(t)) \cdot (h_1(t) - h_2(t)) + \int_{\Gamma} (g_1(t) - g_2(t)) \cdot (h_1(t) - h_2(t)) + 0(t),$$

(51)
where
\[
\Theta(t) = \int_{\Omega} \partial_t h_1(t) \cdot (\hat{h}_1(t) - h_2(t)) + \int_{\Omega} \nu |\nabla \times h_1(t)|^{p-2} \nabla \times (\hat{h}_1(t) - h_2(t)) \\
+ \int_{\Omega} f_1(t) \cdot (h_2(t) - \hat{h}_1(t)) + \int_{\Gamma_1} g_1(t) \cdot (h_2(t) - \hat{h}_1(t)) \\
+ \int_{\Omega} \partial_t h_2(t) \cdot (\hat{h}_2(t) - h_1(t)) + \int_{\Omega} \nu |\nabla \times h_2(t)|^{p-2} \nabla \times (\hat{h}_2(t) - h_1(t)) \\
+ \int_{\Omega} f_2(t) \cdot (h_1(t) - \hat{h}_2(t)) + \int_{\Gamma_1} g_2(t) \cdot (h_1(t) - \hat{h}_2(t)).
\]

\[\bullet\] \( p \geq 2 \)

From (51) we deduce, using the Remark 1, that there exists a positive constant \( C_1 \) such that
\[
\frac{1}{2} \int_{\Omega} |h_1(t) - h_2(t)|^2 + a_s \int_{Q_T} |\nabla \times (h_1 - h_2)|^p \\
\leq C_1 \left( \|f_1 - f_2\|_{L^p(Q_T)} + \|g_1 - g_2\|_{L^p(\Sigma_T)} \right) \|\nabla \times (h_1 - h_2)\|_{L^p(\Omega)} \\
+ \frac{1}{2} \int_{\Omega} |h_{10} - h_{20}|^2 + \int_{0}^{T} \Theta(t) d\tau.
\]

It is easy to understand, by the expression of \( \Theta \), that
\[
\int_{0}^{T} \Theta(t) d\tau \leq C_2 \left( \|\nabla \times (h_1 - \hat{h}_2)\|_{L^p(\Omega)} + \|\nabla \times (h_2 - \hat{h}_1)\|_{L^p(\Omega)} \right)
\leq C_3 \|\Psi_1 - \Psi_2\|_{L^\infty(\Omega)}.
\]

\( C_2, C_3 \) positive constants and, from this last inequality we conclude that
\[
\|h_1 - h_2\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \times (h_1 - h_2)\|^p_{L^p(\Omega)} \leq C \left( \|f_1 - f_2\|^p_{L^p(\Omega)} \\
+ \|g_1 - g_2\|^p_{L^p(\Omega)} + \|h_{10} - h_{20}\|^2_{L^2(\Omega)} + \|\Psi_1 - \Psi_2\|_{L^\infty(\Omega)} \right).
\]

\[\bullet\] \( 1 < p < 2 \)

Again, using (51), the Remark 1 and arguments similar to (26), defining
\[
\hat{Q}_T = \{(x,t) \in Q_T : \nabla \times h_1(x,t) \neq 0, \nabla \times h_2(x,t) \neq 0\}
\]
we obtain
\[
\frac{1}{2} \int_{\Omega} |h_1(t) - h_2(t)|^2 + a_s \left( \int_{\hat{Q}_T} (|\nabla \times h_1| + |\nabla \times h_2|)^p \right) \\
\leq \|f_1 - f_2\|^p_{L^p(\Omega)} \|h_1 - h_2\|_{L^p(\Omega)} + \|g_1 - g_2\|^p_{L^p(\Omega)} \|h_1 - h_2\|_{L^p(\Omega)} \\
+ \frac{1}{2} \|h_{10} - h_{20}\|^2_{L^2(\Omega)} + \int_{0}^{T} \Theta(t) d\tau.
\]

So, there exists constants \( C_2 \) and \( C_3 \) such that
\[
\|h_1 - h_2\|^2_{L^\infty(0,T;L^2(\Omega))} + C_2 \|\nabla \times (h_1 - h_2)\|_{L^p(\Omega)} \\
\leq C_3 \left( \|f_1 - f_2\|^p_{L^p(\Omega)} + \|g_1 - g_2\|^p_{L^p(\Omega)} \right) + \frac{1}{2} \|h_{10} - h_{20}\|^2_{L^2(\Omega)} + \int_{0}^{T} \Theta(t) d\tau
\]
and the conclusion follows as in the previous case. \( \square \)
4.4. The asymptotic behaviour in time of the solutions of the variational inequality. Consider the stationary variational inequality: to find $h_{\infty} \in K_{\infty}$ such that
\begin{equation}
\int_{\Omega} \nu \cdot \nabla \times (\nabla \times h_{\infty}) \cdot \nabla \times (\varphi - h_{\infty}) \geq \int_{\Omega} f_{\infty} \cdot (\varphi - h_{\infty}) + \int_{\Gamma} g_{\infty} \cdot (\varphi - h_{\infty}) \quad \forall \varphi \in K_{\infty},
\end{equation}
where $K_{\infty} = \{ v \in W^{p}(\Omega) : |\nabla \times v| \leq \Psi_{\infty} \text{ a.e. in } \Omega \}$, and assume
\begin{align*}
f_{\infty} &\in L^{q}(\Omega), \quad g_{\infty} \in L^{r'}(\Gamma) \quad \text{and} \quad \Psi_{\infty} \in L^{\infty}(\Omega), \quad \Psi_{\infty} \geq \alpha > 0.
\end{align*}

**Theorem 4.8.** Let $p \geq \frac{6}{5}$, $h$ be the solution of the variational inequality (40) and $h_{\infty}$ the solution of the (52).

Suppose that $f \in L^{\infty}(0, \infty; L^{q/2}(\Omega))$,
\begin{align*}
g &\in L^{\infty}(0, \infty; L^{r'}(\Gamma)) \cap W^{1,r'}(0, \infty; L^{r'}(\Gamma)), \\
\Psi &\in W^{1,\infty}(0, \infty; L^{\infty}(\Omega)).
\end{align*}

Suppose in addition that, for $\xi$ defined in (19),
\begin{align*}
\int_{t}^{t+1} \xi(\tau) d\tau \xrightarrow{t \to \infty} 0 &\quad \text{if } p > 2 \quad \text{and} \quad \int_{t}^{t+1} \xi(\tau) d\tau \xrightarrow{t \to \infty} 0 &\quad \text{if } \frac{6}{5} \leq p \leq 2
\end{align*}
and
\begin{align*}
\exists D > 0 \quad \exists \gamma \quad \|\Psi(t) - \Psi_{\infty}\|_{L^{\infty}(\Omega)} \leq \frac{D}{t} \quad \text{with} \quad \gamma > \begin{cases} 
\frac{3}{2} &\quad \text{if } p > 2, \\
\frac{1}{2} &\quad \text{if } \frac{6}{5} \leq p \leq 2.
\end{cases}
\end{align*}

Then we have
\begin{equation}
\|h(t) - h_{\infty}\|_{L^{2}(\Omega)} \xrightarrow{t \to \infty} 0.
\end{equation}

**Proof.** Let
\begin{align*}
\beta(t) &= \|\Psi(t) - \Psi_{\infty}\|_{L^{\infty}(\Omega)}, \quad \text{and} \quad \eta(t) = \frac{\alpha}{\alpha + \beta(t)}.
\end{align*}

Define
\begin{equation}
\overline{h}(t) = \eta(t)h_{\infty}, \quad \text{and} \quad \overline{h}_{\infty}(t) = \eta(t)h(t).
\end{equation}

As in Lemma 4.6 we have $\overline{h}(t) \in K_{\infty}$ and $\overline{h}_{\infty}(t) \in K(t)$, for a.e. $t \in (0, \infty)$. Substituting, in (51), $h_{1}$ by $h$ and $h_{2}$ by $h_{\infty}$, we obtain
\begin{align*}
\int_{\Omega} \partial_{t}(h(t) - h_{\infty}) \cdot (h(t) - h_{\infty}) \\
+ a_{s} \int_{\Omega} \left( (\nabla \times h(t))^{p-2} \nabla \times h(t) - (\nabla \times h_{\infty})^{p-2} \nabla \times h_{\infty} \right) \cdot \nabla \times (h(t) - h_{\infty}) \\
\leq \int_{\Omega} (f(t) - f_{\infty}) \cdot (h(t) - h_{\infty}) + \int_{\Gamma} (g(t) - g_{\infty}) \cdot (h(t) - h_{\infty}) + \Theta(t),
\end{align*}
where

\[
\Theta(t) = \int_\Omega \partial_t h(t) \cdot (\overline{h}_\infty(t) - h_\infty) + \int_\Omega \nu |\nabla \times h(t)|^{p-2} \nabla \times h(t) \cdot \nabla \times (\overline{h}_\infty(t) - h_\infty)
\]

\[
+ \int_\Omega f(t) \cdot (h_\infty - \overline{h}_\infty(t)) + \int_\Gamma g(t) \cdot (h_\infty - \overline{h}_\infty(t))
\]

\[
+ \int_\Omega \nu |\nabla \times h_\infty|^{p-2} \nabla \times (\overline{h}(t) - h(t))
\]

\[
+ \int_\Omega f_\infty \cdot (h(t) - \overline{h}(t)) + \int_\Gamma g_\infty \cdot (h(t) - \overline{h}(t))
\]

and \(\overline{h}(t)\) and \(\overline{h}_\infty(t)\) are defined in (53).

From Lemma 4.3, we observe that there exists positive constants, \(C_1\) and \(C_2\), independent of \(t\), such that

\[
\| \partial_t h \|_{L^2(\Omega \times (0,t))} \leq C_1 t^\frac{1}{2} + C_2.
\]

Define \(\Phi(t) = \int_\Omega \| h(t) - h_\infty \|^2 \).

Arguing as in Section 2.3, for \(p > 2\), we obtain, for a positive constant \(C\),

\[
\Phi'(t) + C \Phi^\frac{p}{2}(t) \leq l(t),
\]

where, for a positive constant \(C_3\),

\[
l(t) = C_3 \left( \| f(t) - f_\infty \|_{L^p(\Omega)}^{p'} + \| g(t) - g_\infty \|_{L^p(\Gamma)}^{p'} + \| \Psi(t) - \Psi_\infty \|_{L^\infty(\Omega)} \right)
\]

\[
+ C_1 t^\frac{1}{2} \| \Psi(t) - \Psi_\infty \|_{L^\infty(\Omega)}.
\]

But

\[
\int_\frac{t}{2}^t l(\tau) \, d\tau
\]

\[
\leq C_3 \int_\frac{t}{2}^t \left( \| f(\tau) - f_\infty \|_{L^p(\Omega)}^{p'} + \| g(\tau) - g_\infty \|_{L^p(\Gamma)}^{p'} + \| \Psi(\tau) - \Psi_\infty \|_{L^\infty(\Omega)} \right) \, d\tau
\]

\[
+ C_1 \int_\frac{t}{2}^t \tau^\frac{1}{2} \| \Psi(\tau) - \Psi_\infty \|_{L^\infty(\Omega)} \, d\tau \xrightarrow{t \to \infty} 0,
\]

because \(\gamma > \frac{3}{2}\).

For \(\frac{4}{3} \leq p \leq 2\) we have

\[
\Phi'(t) + C \Phi(t) \leq l(t),
\]

where, for a positive constant \(C_3\),

\[
l(t) = C_3 \left( \| f(t) - f_\infty \|_{L^p(\Omega)}^2 + \| g(t) - g_\infty \|_{L^p(\Gamma)}^2 + \| \Psi(t) - \Psi_\infty \|_{L^\infty(\Omega)} \right)
\]

\[
+ C_1 t^\frac{1}{2} \| \Psi(t) - \Psi_\infty \|_{L^\infty(\Omega)}.
\]
But
\[
\int_t^{t+1} l(\tau) \, d\tau 
\leq C_3 \int_t^{t+1} \left( \| f(\tau) - f_\infty \|_{L^{q'}(\Omega)}^2 + \| g(\tau) - g_\infty \|_{L^{r'}(\Gamma)}^2 + \| \Psi(\tau) - \Psi_\infty \|_{L^{\infty}(\Omega)} \right) \, d\tau 
+ C_1 \int_t^{t+1} \tau^{\frac{\gamma}{2}} \| \Psi(\tau) - \Psi_\infty \|_{L^{\infty}(\Omega)} \, d\tau \quad \xrightarrow{t \to \infty} 0,
\]
because $\gamma > \frac{1}{2}$.

Arguing, in both cases, exactly as in the Section 2.3, the conclusion follows.  \(\square\)

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