Hints for the existence of hexaquark states in the baryon-antibaryon sector

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The discovery of some baryon-antibaryon resonances has led us to consider 3q 3q̄ systems as possible candidates. We predict their spectrum in the framework of a constituent model, where the chromo-magnetic interaction plays the main role. The relevant parameters are fixed by the present knowledge on tetraquarks. The emerging scenario complies well with experiment, besides the description of the baryon-antibaryon resonances, we find evidence for new tetraquark states, namely the a0(Y) in the hidden strangeness sector and, in the c ¯c sector, the Y′(4140) and the X(4350).

A detailed account of the spectra and the decay channels is provided for future comparisons with data.

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I. INTRODUCTION

The presence in the hadron spectrum of mesons consisting of two q’s and two q̄’s [1][3] as well as of baryons consisting of 4q and a q̄ has been considered since many years [4].

A long time ago Jaffe proposed that the lightest scalar states, f0/σ, κ together with the rest of their nonet, should be interpreted as qq̄q̄ states [1].

The simplifying assumption [5] of considering only 2q pairs transforming as a (3c, 1s, 3F) representation of SU(3)c × SU(2)s × SU(3)f, straights the whole spectrum to the lightest scalar nonet, namely f0(600), κ(800) and f0/a0(980) as built with a pair of such a diquark and anti-diquark [6]. This interpretation was recently enforced by experiments confirming the presence of hidden strangeness in both the states f0(980) and a0(980) [7], promoting the tetraquarks to a more solid status.

Candidates with open or hidden charm come from the study of non-leptonic B decays at BABAR and BELLE, as anticipated in [8], and from BES.

In this paper we study the spectrum of the states consisting of three quarks and three antiquarks in S-wave, interacting via chromo-magnetism. Besides strangeness we include also charm and assume for chromo-magnetism its full content [9], treated along the lines of ref. [10].

It happens that this hypothesis can successfully interpret some observed baryon-antibaryon negative parity states in pp [11], Λ, p [12] and Λ, ¯Λc [13], assuming for the parameters (constituent masses and effective couplings) those values obtained from the tetraquarks phenomenology. To study the case of broken flavor symmetry we had to resort to machine computation.

The paper is organized as follows: in section I we introduce the basics of chromo-magnetism with a formulation more suitable for algebraic computation. Section II deals with the formalism for the construction of the tetraquarks states and the study of the open door decays. In section III, IV and V we discuss the phenomenology of tetraquarks states and the parameter fixing of the model. Hexaquark states are introduced in sections VI along with the details entering the calculation. In section VII we present the results we found for the spectrum and compare them with the relevant experimental data. Section VIII contains our conclusions. Finally, the appendix A contains a table with the full spectrum of the baryon-antibaryon systems that were taken under consideration, while in appendix B the crossing matrices required for the study of the decays of tetraquarks are reported. The matrix elements of the chromo-magnetic operator are given, for all cases, in appendix C.

II. THE CHROMO-MAGNETIC INTERACTION

The hyperfine interaction arising from one gluon exchange between constituents leads to a simple Hamiltonian involving the color and spin degrees of freedom:

$$H_{CM} = \sum_i m_i - \sum_{i<j} C_{ij} O_{CM}^{(i,j)}$$

(1)

the index i (j) refers to the ith (jth) quark, m_i its mass and C_{ij} appropriate coupling constants. The kinetic energy is absorbed in the mass term, so it is not surprise that the quarks masses depend on the system under consideration. The C_{ij}’s depend not only on the m_i’s (as 1/m_i*m_j) but also on the wave function at zero distance of the pair (i, j), so depending on the system as well. Chromo-Magnetism (CM) is encoded in O_{CM}^{(i,j)}, the two particles chromo-magnetic operator, which is given by:

$$O_{CM}^{(i,j)} = \frac{1}{4} \sum_{a=1}^{8} \sum_{k=1}^{3} (\lambda_a \otimes \sigma_k)^{(i)} (\lambda_a \otimes \sigma_k)^{(j)}$$

(2)

where λ_a are the Gell-Mann matrices and σ_k the Pauli matrices. It is reminiscent of the well known exchange
interaction and can be expressed in terms of permutation operators for color and spin \(P_{(i,j)}\), \(P_{(i,j)}^c\) respectively. The action on a \((i,j)\) quark-quark (antiquark-antiquark) pair is given by

\[
O_{CM}^{\text{T}} = (P_c - 1/3) \otimes (P_s - 1/2)
\]  

(3)

where \(P_{(i,j)}\) and \(P_{(i,j)}^c\) exchange the colors and spins (acting independently), of the pair \((i,j)\). Eigenvectors of \(\mathbf{3}\) are the diquark states of definite symmetry in color and spin \((6,3)(SS), (6,1)(SA), (3,3)(AS), (3,1)(AA)\) with eigenvalues \((-1/3,1,2/3,-2)\) respectively.

To express the result for a quark-antiquark pair it is useful to define a generic \(T_N\) for the group \(SU(N)\) as the object: \(T_N: \Psi_i \Xi^B \rightarrow 1/N \Psi_i \Xi^B - \delta_i^B \Psi_C \Xi^C\), with \(\Psi_A\) in the representation \(N\) and \(\Xi^B\) in the c.c. representation \(\overline{N}\). Making the identification \(N = 3\) for \(T_c\) and \(N = 2\) for \(T_s\) we can write quite simply:

\[
O_{CM}^{\text{T}} = -T_c \otimes T_s
\]  

(4)

The eigenvectors of \(T_N\) are the singlet representation \((\delta^B_i \Psi_i \Xi^C)\) with eigenvalue \((1/N - N)\) and the adjoint representation \((\Psi_i \Xi^B - 1/N \delta^B_i \Psi_i \Xi^C)\) with eigenvalue \(1/N\). So eigenvectors and eigenvalues of the chromo-magnetic operator in the present case are: \((8,3), (8,1), (1,3), (1,1)\) with eigenvalues \((-1/6, 1/2, 4/3, -4)\) respectively.

By far the more bonded diquark is the \((3,1)(AA)\) whose \(SU(3)_F\) flavor content, as dictated by the Pauli principle, is \(\mathbf{3}_F\). This is the so called good diquark, it transforms as a scalar antiquark. If one assumes the hypothesis of Jaffe and Wilczek [5], the spectrum of the tetraquarks remains restricted to the scalar nonet suggested by Jaffe a long time ago. The vector, or bad diquark \((\mathbf{3},3)(AS)\), allows for higher spin states but, since it is a \(6_F\), it also introduces exotics, i.e., multiplets higher than \(SU(3)_F\) nonets and are excluded from most models. The other two \(6_F\), states, that Jaffe [3, 14] called sometimes “worse” are not in general taken into account either.

In the present approach in searching for the eigenstates of the chromo-magnetic operator we do not truncate the space in any way, such that, in some sense, all four possible diquarks enter the game.

It is easy to see that we have the following spin-flavor multiplets: spin 0 has four nonets and two \(27_F\)’s , spin 1 has two nonets, four octets, one \(27_F\), two decuplets and two antidecuplets, finally spin 2 has two nonets and one \(27_F\). Exotics, as \(I = 2\) states, are not excluded a priori but we think that these states are much less stable and difficult to be observed.

Often, we have found a number of near threshold decays, usually attributed as molecular states, that are well described by chromo-magnetism. In particular the introduction of the \((6,3)\) diquark encompass the dichotomy between diquark and molecular models as clearly argued in [15].

They showed that the molecular state is not an independent state, but is a linear combination of \((3,1)(3,1)\) and \((6,3)(\bar{6},3)\), the later \((6,3)\), by the way, is the only other diquark with negative chromo-magnetic energy \((-1/3)\). Their observation indicates that a minimal diquark model should include both pairs, and interestingly enough, it would comprise all spin cases as S-wave tetraquarks lying in only \(SU(3)_F\) nonets. From the point of view of \(SU(6)\), this means that a diquark should transform as the symmetric representation, \(21\) (so as \(\mathbf{3}_F\)).

A purely phenomenological motivation to include the \((6,3)\) diquark is that the mass of the \(3, S = 0, (ud)_{I=0}\) pair, say \(\mu\), is related to the mass of the \(\Lambda\) hyperon by the relation \(\mu = m_A - m_s\), which for a state consisting of two of these objects which have no mutual chromo-magnetic interaction imply about twice the mass of the \(f_0(600)\). Instead, by considering the vector space consisting of both the \((3,1)(3,1)\) and \((6,3)(6,3)\), \(S = 0\) color singlet states, the lightest state has a binding energy about 2.7 times larger than the diagonal matrix element for \((3,1)(3,1)\) [10].

In the flavor symmetry limit, i.e. when the couplings \(C_{ij}\) are all equal to each other, it is well known that \(O_{CM}\) can be expressed as a combination of Casimirs. This fact has been extensively exploited in the pioneering works of Jaffe [3] and in many other works [4]. In the present paper we shall attack the more complicated issue of considering different masses and couplings, in most of such cases we have to rely on symbolic manipulations that we performed with FORM [16]. The expressions in Eqs. (3) and (4) result quite suitable for computer implementation.

III. “OPEN DOOR” CHANNELS FOR TETRAQUARKS

It has been observed for the first time by Jaffe [1] that \(qqqq\) mesons may decay into two ordinary (i.e. color singlet) mesons PP, PV, VV (P stands for a pseudoscalar and V for a vector) by simply separating from each other, as long as it is kinematically allowed. He called these channels “open door” or “Ozi super-allowed” decays, since they can occur without gluon exchange or quark annihilation. In open door channels, S-wave states have to decay into S-wave mesons with zero relative angular momentum.

In general calculations are performed in the diquark-antidiquark basis, i.e., the tetraquark is represented as \(q_1q_2\bar{q}_4\bar{q}_3\) denoted [12, 34] in the following. Evidently the diquark and the antidiquark cannot separate from each other as they can never be color singlets. So, in order to access the open door channels it is convenient to pass to the meson-meson basis [13, 24] and [14, 23] which, obviously, coincide if antiquarks 3 and 4 have the same
So we have also to consider the non open door channels, if dynamical amplitudes to be the same). For convenience corresponding component of the normalized eigenvector through a specific channel is given by the square of the only open door channel for a tensor meson is, evidently, VV.

For convenience we maintain those of \([10]\). We call the basis for spin 0: as \(\phi\) in \([12,34]\), \(\alpha\) in \([13,24]\) and \(\epsilon\) in \([14,23]\), in the same order one has \(\psi, \beta\) and \(\chi\) for spin 1, while those of spin 2 are called \(\xi, \gamma\) and \(\delta\). To characterize each basis, we have only to specify the color-spin content of the first and second pairs in the brackets, which combine to form the color singlets i.e. the set of physical states.

**Spin 0**

\[
(\phi)[12,34] : [(6,3)(\bar{5},3)]; [(\bar{3},1)(3,1)]; \\
(\alpha)[13,24] : [(1,1)(1,1)]; [(1,3)(1,3)]; \\
(\epsilon)[14,23] : as [13,24]
\]

For \(\alpha\) and \(\epsilon\) the first components are PP and the second VV. The last two are \(P_bP_b\) and \(V_bV_b\), where \(P_b\) is a colored pseudoscalar and \(V_b\) a colored vector.

**Spin 1**

\[
(\psi)[12,34] : [(6,3)(\bar{5},3)]; [(\bar{3},3)(3,3)]; [(\bar{3},1)(3,3)]; \\
(\beta)[13,24] : [(1,1)(1,3)]; [(1,3)(1,1)]; [(1,3)(1,3)]; \\
(\chi)[14,23] : as [13,24]
\]

**Spin 2**

\[
(\xi)[12,34] : [(6,3)(\bar{5},3)]; [(\bar{3},3)(3,3)] \\
(\gamma)[13,24] : [(1,3)(1,3)]; [(8,3)(8,3)] \\
(\delta)[14,23] : as [13,24]
\]

The only open door channel for a tensor meson is, evidently, VV.

The relative probability for the particle decaying through a specific channel is given by the square of the corresponding component of the normalized eigenvector of the state multiplied by phase space (as is assumed all dynamical amplitudes to be the same). For convenience we call the square of the component along the channel the probability factor (PF) for that channel. In some cases we have also to consider the non open door channels, if for instance, the open door have negligible probabilities or are kinematically forbidden, and so violations of the OZI rule would enter the game. In particular the \(P_bP_b\) or \(V_bV_b\) channel can become relevant at order \(O(\alpha_s)\), as the exchange of one gluon in the t-channel converts this object into an ordinary PP or VV pairs.

The so called crossing matrices operating the change of a basis into another, arise from well known Fierz identities for color and spin \([8]\) and are available in many places, for definiteness we will refer to \([10]\). They are reproduced, together with a necessary completion, in Eqs. \([B1-B5]\).

**IV. TETRAQUARK STATES**

It is immediate to realize that the overall chromo-magnetic contribution in Eq. \([1]\) (let us call it \(O_{CM}\) and assume thoroughly \(C_{qq'} = C_{qq'}^{-}\) for any (anti) quarks pair) greatly simplifies for 0\(^+\) and 2\(^+\) states made of at least three constituents with the same flavor, say, of type \(qqq'\) (\(q\) is not necessarily a light quark and \(q'\) can incidentally coincide), since the corresponding matrices depend exclusively on the combination \((C_{qq} + C_{qq'})\), which factorizes out. For 2\(^+\) we have:

\[
O_{CM} = -1/2(C_{qq} + C_{qq'}) 
\]

The eigenvalues of the above matrix are \(\lambda_1 = 1/3(17 + \sqrt{241})\), \(\lambda_2 = 1/3(\sqrt{241} - 1)\), \(\lambda_3 = 1/3(17 - \sqrt{241})\), \(\lambda_4 = -1/3(\sqrt{241} + 1)\), with corresponding eigenvectors (for briefness we give decimal approximations) \((-0.74, 0.04, 0.17, 0.65), (0.64, 0.18, -0.41, 0.62), (0.18, -0.64, 0.62, 0.41)\) and \((0.04, 0.74, 0.64, 0.17)\).

The spectrum is given by \(M^{(1)}_b = 3m_q + m_{q'} - 1/2 \lambda_a (C_{qq} + C_{qq'}), (a = 1, \ldots, 4)\) for 0\(^+\) and by \(M^{(2)}_b = 3m_q + m_{q'} + 4/3 (C_{qq} + C_{qq'}), (b = 1, 2)\) for 2\(^+\). These considerations apply also to the case of three light constituents within the approximation of exact isospin symmetry. It is worth to stress that this phenomenon does not happen for 1\(^+\).

A simple consequence of the fact that the eigenvectors do not depend on the masses and couplings is that the scalar nonet presents an universal pattern of decays, the lowest state has about 55\% probability to decay into PP (negligible in VV) and for the next states, in order of increasing mass: 41\% in VV, 41\% in PP and 55\% in VV. Identifying the lowest state of the light nonet with the \(a/f_0(600)\) and the third one with the \(f_0(1370)\) we get the mass of light quarks \(m_q\) and \(C_{qq'}\), we find \(m_q = 351.65\) MeV and \(C_{qq'} = 74.4\) MeV. Notice that the...
quark mass and the coupling can be expressed in terms of the masses of $\sigma$ and $f_0$ by:

$$4m_q = m_\sigma + \left(1 + \frac{17}{\sqrt{241}}\right) \frac{m_{f_0} - m_\sigma}{2},$$

$$C_{qq} = \frac{3}{\sqrt{241}} \frac{m_{f_0} - m_\sigma}{2}. \quad (9)$$

So it is immediate to realize that, if we would take for $m_\sigma$ a lower value, around 450 MeV, as suggested by some authors, the change in $m_q$ would be negligible but $C_{qq}$ would rise to 89 MeV.

A similar determination of the parameters concerning the $s$ and $c$ quarks is not feasible because presently we dispose only of one strange scalar as a possible candidate for a tetraquark ($\kappa(800)$) and none for charm. For the $s$ quark we choose the parameters in order to reproduce the masses of the $\kappa(800)$ as a $(qqs\bar{s})$ state, the $a_0(980)$ as a $(qs\bar{q})$ and the $f_1(1420)$ as a $1^+(qs\bar{q})$ state, getting $m_s=455.21$ MeV, $C_{qs}=58.04$ MeV and $C_{ss}=43.2$ MeV.

It is quite unexpected the almost exact agreement with the parameters of our previous calculation for the pentaquarks [17], where we found: $m_q=346.8$ MeV, $C_{qq}=74$ MeV, $m_s=480$ MeV and for $C_{qs}$ and $C_{ss}$ we assumed the hyperfine prescription $C_{qq} = C_{qq}/m_s = m_\sigma$ which, as a matter of fact, is also well satisfied by the tetraquark determinations.

The parameters related to charm have been obtained requiring agreement with the masses of the following states: $X(3872)$ as a $1^+(qcs\bar{c})$ state, the pair $D_s(2317)$ and $D_s(2573)$ as $0^+(qcs\bar{s})$ states and finally $D_s(2460)$ as a $1^+(qcs\bar{q})$ state. The values obtained for the parameters are: $m_c=1631$ MeV, $C_{qc}=26$ MeV, $C_{cc}=18$ MeV, $C_{sc}=17.6$ MeV. A direct determination from the $J/\psi$ and $\eta_c$ masses gives $m_c=1534$ MeV, $C_{cc}=21.6$ MeV. Since we expect a bigger kinetic energy for the tetraquark together with a broader wave function, the discrepancy goes in the right direction $[20]$. On the other side if we determine $C_{qc}$ from the $D^*-D$ mass splitting, we get $C_{qc}=26.2$ MeV, in excellent agreement with the determination via the tetraquarks spectrum.

Here it is interesting to notice that the system $Q\bar{q}$ should obey some general property as a consequence that the recoil of $Q$ can be safely neglected. So it should not depend on the mass of $Q$, but only on the radial and orbital quantum numbers of $\bar{q}$. Since $\bar{q}$ is very light the system would have a spatial extension that falls in the region of dominance of the linear part of the confinement potential, (phenomenological analysis demonstrate that the $c\bar{s}$ system falls in the logarithmic dominated region) for which well known scaling laws [19] prescribe that the wave function at the origin does not depend on the $Q$ mass, so we should expect the product $m_QC_{qq}$ to be constant. A law equivalent to the constancy of the product $m_QC_{qq}$ has been inferred some time ago in ref. [20] and verified for a great number of states involving charm or beauty.

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$J^P$ & $qq\bar{q}$ & $qq\bar{s}$ & $q\bar{q}c$ & $qs\bar{s}$ & $ss\bar{s}$ & Decays \\
\hline
$0^+$ & 600.0 $\pm$ 15 & 792.0 $\pm$ 20 & 2141.7 & -- & -- & 0.55($PP$); 1.710 $^{-3}$($VV$) \\
[Exp] & $f_0(000)$ & $\kappa(800)$ & & & & \\
\hline
$1^+$ & 1046.4 $\pm$ 0.1 & 1189.6 $\pm$ 0.2 & 2442.9 & 1472.2 & 1611.6 & 0.41($PP$); 3.110 $^{-2}$($VV$) \\
[Exp] & $f_0(000)$ & & & & & \\
\hline
$0^+$ & 1370.0 $\pm$ 0.1 & 1477.6 $\pm$ 0.2 & 2661.2 & -- & -- & 3.110 $^{-2}$($PP$); 0.41($VV$) \\
[Exp] & $f_0(1370)$ & & & & & \\
\hline
$0^+$ & 1816.4 $\pm$ 0.1 & 1874.9 $\pm$ 0.2 & 2962.4 & 1996.1 & 2058.9 & 1.710 $^{-3}$($PP$); 0.55($VV$) \\
[Exp] & $X(1600)$ $T=2$ & & & & & \\
\hline
$2^+$ & 1605. twice $I=0$ and $I=1$ & 1686.7 $\pm$ 0.2 & 2819.8 & 1852.3 & 1936.1 & 0.5($VV$); 0.5 (light mesons) \\
[Exp] & & & & & & \\
\hline
\end{tabular}
\caption{0$^+$ and 2$^+$ states with 3 light (strange) quarks calculated exactly according to section IV. Values of masses used in the fit are distinguished with a (*). Experimental results, when available, are displayed in the next row; numbers in square brackets give the reference to the experimental data. Pauli principle fixes the isospins of the various states, so $qq\bar{q}$ have the same isospins as $qq\bar{s}$ while $qs\bar{s}$ have $I=1/2$ and $ss\bar{s}$ is the case of the tetraquark sector, since the wave function, due to a much higher mass, is much peaked around the origin, partially compensating the mass powers in the denominator. Actually, recent data on the $\eta_b$ suggest a mass splitting with the $Y$ of the same order of the $\eta_c - \psi$, and not a factor $(m_c/m_b)^2$ smaller [18].}  
\end{table}
In the case of a “neutral” state \((qq'\bar{q}'\bar{q}'')\), as for hidden strangeness or charm, the \(1^+\) CM matrix in the \(\beta\)-basis is block diagonal, with a \(2\times2\) block corresponding to \(C = +\) and the other \(4\times4\) block to \(C = -\). So, independently of the parameters, we have two exact eigenvectors, one along the direction \(\beta_3\) (pair of color singlet vectors) and the other along \(\beta_6\) (pair of color octet vectors). On the other hand all scalars and tensors have the same charge conjugation, \(C = +\).

It is immediate to calculate the masses of the two \(C\)-even states: the first has mass \(2m_q + 2m_{q'} + 4/3(C_{qq} + C_{q'q'})\) and the second \(2m_q + 2m_{q'} - 1/6(C_{qq} + 18C_{q'q'})\). We can also calculate exactly the \(2^+\) state getting for the mass \(2m_q + 2m_{q'} + 4/3(C_{qq} + C_{q'q'})\), the corresponding eigenvector being along \(\gamma_1\) (pair of color singlet vectors), the value of the other mass is \(2m_q + 2m_{q'} - 1/6(C_{qq} - 18C_{q'q'} + C_{q'q'})\) corresponding to \(\gamma_2\) (pair of color octet vectors). A general trend for this case is that the highest \(1^{++}\) state is degenerate with the highest \(2^+\) state, both decaying exclusively into \(V_{qq'} V_{q'q''}\). The other \(1^{++}\) is below the light tensor state and has dominant decay into \(P_{qq'} V_{qq'} + P_{q'q''} V_{q'q''}\) while the light tensor decays into \(V_{qq'} V_{q'q''}\). The states \(0^+\) and \(1^+\) have to be calculated numerically, with the exception of the case \(q = q'\), when the spectrum of the \(1^+\) becomes highly degenerate. In such a case, the \(C\)-even state \(\beta_3\) is paired with a \(C\)-odd state with eigenvector \(\chi_6 = 2/3(-1, 1, 0, 1/(2\sqrt{2}), -1/(2\sqrt{2}), 0)\), the other \(C\)-even state \(\beta_3\) becomes degenerate with the \(C\)-odd state with eigenvector \(\chi_3 = 2/3(-1/(2\sqrt{2}), 1/(2\sqrt{2}), 0, -1, 1, 0)\). As can be seen from the table below, the mass region \(1100 - 1950\ MeV\) could seem to be populated by some controversial peaks with no definite spin or \(C\)-parity, due to states overlapping.

| \(\bar{q}q\bar{q}q\) | \(\pi\) | \(\eta\) | \(\omega\) | \(\rho\) |
|---|---|---|---|---|
| \(1109. I = 0\) | 1158.6 \(I = 1\) | 1158.6 \(I = 1\) | 1406.6 \(I = 0, 1, 2\) | 1605. \(I = 1\) | 1605. \(I = 1\) |
| \(s\bar{q}s\bar{q}\) | \(\pi\) | \(\eta\) | \(\omega\) | \(\rho\) |
| \(1207.5\) | 1302.7 | 1308.6 | 1513.4 | 1672.7 | 1703. |

**TABLE II.** Axial states made of all light (in the limit of exact isospin) or strange (anti) quarks calculated exactly, according to section IV. They have definite charge conjugation. The states forbidden by the Pauli principle are indicated by \((-\)). Masses are given in MeV.

Even if no candidates have been observed let us, for completeness, give the spectrum of strange and charmed axialis:

| \(qq\bar{q}q\) | \(2329.3\) | 2515.6 | 2611.7 | 2727.8 | 2785.8 | 2877.73 |
|---|---|---|---|---|---|---|
| \(Decays\) | 0.55(\(\pi, \eta\))D | 0.39(\(\pi, \eta\))D* | 0.47(\(\omega, \rho\))D | 0.28(\(\omega, \rho\))D | 0.47(\(\omega, \rho\))D* | 0.11(\(\omega, \rho\))D |
| \(qq\bar{q}\bar{q}\) | 1207.5 | 1302.7 | 1308.6 | 1513.4 | 1672.7 | 1703. |
| \(Decays\) | 0.50(\(\pi, \eta\))K* | 0.11(\(\omega, \rho\))K* | 0.52(\(\omega, \rho\))K* | 0.21(\(\omega, \rho\))K* | 0.50(\(\omega, \rho\))K* | 0.50(\(\omega, \rho\))K* |
| Isospin | 1/2 | 1/2, 3/2 | 1/2 | 1/2, 3/2 | 1/2 | 1/2, 3/2 |

**TABLE III.** Charmed and strange axial mesons, calculated numerically. Masses are in MeV. The non negligible decay channels are indicated in the row below.

When an object contains a pair of (anti) quarks, Pauli principle implies the absence of some states or, otherwise, if the pair is made of light quarks, restrictions on the isospin content, according to the correspondence \(I = 0 \rightarrow 2I_{cs}\) and \(I = 1 \rightarrow 15_{cs}\). This has been taken into account in the elaboration of Tables I, II, III, where Pauli forbidden states are indicated by a hyphen. The very interesting cases of hidden strangeness/charm and tetraquarks with \(C = \pm S = 1\) were calculated numerically and are given in the Table \(\n\). The interest for the somewhat chimerical states with \(C = -S = 1\) and \(C = 2\) i.e. of kind \((csq\bar{q}')\) and \((cqq\bar{q}')\), is justified by the fact that they provide a clear signature for tetraquarks. In the case of \(I = 0\) the first decays into \(D^+K^-\) and \(D^0\bar{K}^0\) and the second into \(D^+D^0\). Since in both cases the objects carrying strangeness or charm are necessarily a pair of quarks and obviously they cannot form by themselves color singlets, so the occurrence of such states is possible
only if the pair of quarks combine with at least a pair of antiquarks.

V. DISCUSSION ON THE RESULTS FOR TETRAQUARKS

First of all, let us recall that the information we used in the fit involves only the mass spectrum, so the pattern of decays may be considered as “predictions”. Let us cite the observed dominance of $\pi\pi$ in the $f_0(600)$ decay and of $\rho\rho$ in that of $f_0(1370) [23][24]$, the dominance of the $\pi K$ channel for $\kappa(800)$ (unfortunately, by now, omitted from PDG).

For the axials we obtained the dominance of $\bar{K}K^* + cc$ ($KK\pi$ probably arising from a off-shell $K^*$) for the $f_1(1420)$ and, analogously, the dominance of $\bar{D}D^* + cc$ for the $X(3872)$.

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$J^P$ & $qs\bar{q}q$ & $cs\bar{q}q$ & $qs\bar{q}q$ & $qc\bar{q}q$ & $cc\bar{q}q$ & $cs\bar{q}s$ \\
\hline
$0^+$ & 1981.1$^\ast$ & 2326.7 & 2315.1$^\ast$ & 3562.7 & 3643.1 & 3904.5 \\
\hline
Exp & $a_0(980)$ & $D^\pm_{s0}(2317)$ & $D^\pm_{s0}(2317)$ & $D^\pm_{s0}(2317)$ & $D^\pm_{s0}(2317)$ & $D^\pm_{s0}(2317)$ \\
\hline
$0^+$ & 1330.3 & 2592.3 & 2574.1$^\ast$ & 3793.9 & 3870.8 & 4060.8 \\
\hline
Exp & $a_0(Y)(29)$ & $D^\pm_{s1}(2573)$ & $D^\pm_{s1}(2573)$ & $D^\pm_{s1}(2573)$ & $D^\pm_{s1}(2573)$ & $D^\pm_{s1}(2573)$ \\
\hline
$0^+$ & 1586.1 & 2757.6 & 2773.7 & 3979.3 & 3898.6 & 4181.1 \\
\hline
Exp & & & & & & \\
\hline
$0^+$ & 1934.9 & 3028.1 & 3028.4 & 4148.4 & 4144.5 & 4295.8 \\
\hline
Exp & & & & & & \\
\hline
$1^+$ & 1327.6 & 2503.7 & 2469.3$^\ast$ & 3682.9 & 3795.3 & 4016.41 \\
\hline
Exp & $f_1(1420)$ & $D^\pm_{s1}(2460)$ & $D^\pm_{s1}(2460)$ & $D^\pm_{s1}(2460)$ & $D^\pm_{s1}(2460)$ & $D^\pm_{s1}(2460)$ \\
\hline
$1^+$ & 1420.1$^\ast$ & 2674.5 & 2634.6 & 3871.9$^\ast$ & 3847.8 & 4109.4 \\
\hline
Exp & & & & & & \\
\hline
$1^+$ & 1461.6 & 2692.1 & 2736.4 & 3924.6 & 3927.8 & 4132.1 \\
\hline
$1^+$ & 1618.9 & 2822.8 & 2823.1 & 3980.5 & 3991.6 & 4172.5 \\
\hline
Exp & & & & & & \\
\hline
$1^+$ & 1770.5 & 2857.8 & 2880.3 & 4057.5 & 3992.2 & 4225.9 \\
\hline
Exp & & & & & & \\
\hline
$1^+$ & 1773.5 & 2959.5 & 2951.1 & 4088.5 & 4084.78 & 4254.1 \\
\hline
$2^+$ & 1768.2 & 2889.1 & 2900.2 & 4027.9 & 4021.2 & 4215.1 \\
\hline
$2^+$ & 1770.5 & 2906.9 & 2912.2 & 4088.5 & 4061.6 & 4254.1 \\
\hline
\end{tabular}
\end{center}
\caption{Spectrum of the tetraquarks calculated numerically. States used in the fit are marked with a (*). When experimental data are available they are displayed in the next row, reference to the source are given in square brackets. Masses are in MeV.}
\end{table}

Since they are pure $\beta_0$ states these channels are exclusive. In particular, for $X(3872)$, the observed decays into $\rho(1700)J/\psi$, can be explained by one gluon exchange in the t-channel, since those rates are comparable with the process being $O(\alpha_s)$. For $D_{s0}^\pm(2317)$ the only kinematically allowed open door channel is $\pi^0 D_s^\pm$, it is just below the $DK$ threshold, at 2359 MeV. The relevant components are $\alpha_1 = 0.78$, $\epsilon_1 = 0.70$, so predicting strong dominance of the $\pi^0 D_s^\pm$ decay. In the case of $D_{s1}^\pm(2573)$ that we interpreted to be $0^+$ (even if it is also consistent with a $2^+$) the only observed decay is $D^0 K^\pm$ while $D^{0*}(2070)K^\pm$ is not, so in agreement with PP prescription arising from scalar nature of the state. Nevertheless, also in this case the components are almost equal $\alpha_1 = 0.60$, $\epsilon_1 = 0.68$ ($D^0 K^\pm$) and so we could expect the $\pi^0 D_s^\pm$ to be relevant, as well. Experimental data neither confirm nor disprove this point. Finally the axial state $D_{s1}^\mp(2460)$, that we put at 2469.3 MeV, has a large component along $\beta_1(0.87)\epsilon$, which corresponds to the dominant $\pi^0 D_s^{\pm*}$ channel. The $\omega D_s^\pm$ decay (notice the state $D_{s1}^\pm(2460)$ has $I = 0$) has a tiny component $\beta_2 = 0.024$ and is also kinematically inaccessible. It remains to explain the large branching fraction in $D_s^{\pm*\gamma}$, suggesting that the state is very narrow, albeit the experimental upper bound is not much restrictive, $\Gamma \lesssim 3.5\text{MeV}$.

Concerning the two degenerate states, the isoscalar $f_0$ and the isovector $a_0$, at 980 MeV, they can only decay into $\eta\pi$ and $KK$, other channels being too high. We predict $\alpha_1 = 0.75$, $\epsilon_1 = 0.74$, and if we take the
corrections for the mixing $\eta_0 - \eta_8$ with a mixing angle $\theta = -16^\circ$ (as obtained recently in $\gamma \gamma \rightarrow X$), we find for the ratio of $g^2_{\alpha_0 K\bar{K}}/g^2_{\eta_8 \eta_8} = 2.48$, to be compared with the value recently obtained by the KLOE experiment [21] of $0.67 \pm 0.06 \pm 0.13$. This abnormally large coupling for $\eta \pi$ cannot be obtained by chromo-magnetism alone, it has been explained recently [25] by non-perturbative effects induced by instantons. Analogously for the dominant decay $f_0 \rightarrow \pi \pi$, which violates OZI rule, we have to rely on the above solution, in association with $f_0(980) - \sigma$ mixing.

We predict a companion (which is a mixture of $8_F$ and $27_F$), for the $a_0(980)$ at 1330.3MeV coupled to $\eta \pi$, $\eta' \pi$, and $K K$. It was recently observed [29] in $\gamma \gamma \rightarrow \eta \pi \gamma$ with an observed mass of 1316 $\pm$ 25 MeV.

In the hidden charm-strange sector ($cs\bar{s}\bar{c}$), we have found two candidates for newly discovered states. The first is the pure $\beta_6$ $1^{++}$ state at 4109.4MeV which we propose to identify to the narrow state $Y(4140)$ found at CDF [23] in $B^{+} \rightarrow X K^{+}$, $X \rightarrow J/\psi \phi$, with a mass 4143 $\pm$ 2.9 $\pm$ 1.2MeV and a width of 11.7 $\pm$ 5.0 $\pm$ 3.7MeV. As the $X(3872)$ the later has dominant decays into $D_{s}^{*}D_{s}^{*} + cc$ (threshold at 4080MeV), but can also decay into $J/\psi \phi$ (threshold at 4116.4MeV). The choice of spin one is strongly suggested by the fact that it was not observed in $\gamma \gamma \rightarrow X$ by BELLE [28]. The second state is a $0^{++}$ at 4295 MeV, with predominant decays into $J/\psi \phi$ ($\beta_2 \simeq 0.81$) and $D_{s}^{*} \overline{D}_{s}^{*}$ ($\beta_2 \simeq 0.69$), to be interpreted as the $X(4350)$, discovered in the same experiment [28], with a mass 4350.6 $\pm$ 4.6 $\pm$ 0.7MeV and width 13.3 $\pm$ 5.1 $\pm$ 4.1MeV. Taking into account phase space, we find the $J/\psi \phi$ channel to be twice more probable than the $D_{s}^{*} \overline{D}_{s}^{*}$ one. Among the non well established states there is a $2^{+}$ state $X(1600)$ (with $I=2$) [21] at 1600 $\pm$ 100MeV that, if interpreted as $(qq\bar{q}\bar{q})$, is compatible with our predictions and, according to the previous section, it has to be degenerate with the highest $1^{++}$, the later being possibly hidden by some $(L=1 q\bar{q})$ state of the $a_1$ family.

It is not excluded that presently we have already seen some $(ss\bar{c}\bar{c})$ states, one of these could be the $f_0(2010)$ found around 2011 $\pm$ 70 MeV [22] that is identifiable to our $2^{+}$ state at 1936 MeV. We predict a $1^{++}$, $(qs\bar{q}\bar{q})$ state, with a mass of 1327.6 MeV decaying predominantly into $\pi \phi$ ($\beta_1 \simeq 0.91$) and another at 1773 MeV with important components also $\beta_1 \simeq 0.34$ ($\eta \pi V$) and $\beta_2 \simeq 0.15$ ($\pi \phi, \eta' \phi$), while $\chi_3$ is also very large, the state is below threshold for $K^{+}\overline{K}^{0}$. The later could be, possibly identified to the $X(1835)$ found at BES [30] at 1834 $\pm$ 6 MeV and width 67.7 $\pm$ 20.3 $\pm$ 7.7 MeV, decaying into $\pi^{+}\pi^{-}\eta'$. The spin-parity of the $X(1835)$ is not known and it was, initially, supposed to be related to a $p\bar{p}$ threshold enhancement, due to the strong dominance of the channel $\pi^{+}\pi^{-}\eta'$.

We also predict a $0^{+}$ $ss\bar{s}\bar{s}$ state at 2058.9 MeV, is strongly coupled to $\phi \phi$, so it would arise as a $\phi \phi$ threshold enhancement.

VI. NEGATIVE PARITY STATES BUILT WITH THREE QUARKS AND THREE ANTIQUARKS

Today it seems to exist experimental evidence for the occurrence of baryon-antibaryon states. People could have the tendency to interpret them as molecular states, but as said before, there is no clear distinction between chromo-magnetism and the molecular point of view as long as we do not neglect some configurations of the diquarks. In obtaining the predictions of chromo-magnetism, since the number of candidates is not enough to completely determine the parameters, we will tentatively assume for the masses and chromo-magnetic couplings of the quarks in the baryon-antibaryon system the same as for tetraquarks. As mentioned before, masses could be larger due to the fact that they are defined including the kinetic energy. On the other hand, couplings could be smaller mainly because the wave function is more spread.

A complete calculation is very complex and probably not of immediate utility in view of the scarcity of these states. We treat two cases, the first is related to $pp\bar{p}$ states and concerns $(qqq\bar{q}\bar{q})$ systems, the second deals with the production of a variety of states of the kind $(qqq\bar{q}Q)$ or $(qQ\bar{q}\bar{q}Q)$, where $Q$ denotes an $s$ or $c$ quark.

It is natural to work with what we call the baryon-antibaryon basis. In the first case, since we are interested in a $p\bar{p}$ pair, it is enough to take the sub block $qqq$ in the 70 of $SU(6)_{cs}$ and $q\bar{q}\bar{q}$ in the $\overline{70}$. The decomposition of the 70, under $SU(3)_{c} \otimes SU(2)$, is given by: $70_{a} = \langle 8_{c}, 4_{s} \rangle + \langle 8_{c}, 2_{s} \rangle + \langle 10_{c}, 2_{s} \rangle + \langle 1_{c}, 2_{s} \rangle$. We can construct 4 color singlets of spin 0 and 6 of spin 1, which are below:

Spin 0

$$|1\rangle = [(c_1, 2_s), (c_1, 2_s)]; \quad |2\rangle = [(8_c, 2_s), (8_c, 2_s)];$$

$$|3\rangle = [(8_c, 4_s), (8_c, 4_s)]; \quad |4\rangle = [(10_c, 2_s), (\overline{10}_{c}, 2_s)].$$

Spin 1

$$|1\rangle = [(c_1, 2_s), (c_1, 2_s)]; \quad |2\rangle = [(8_c, 2_s), (8_c, 2_s)];$$

$$|3\rangle = [(8_c, 4_s), (8_c, 4_s)]; \quad |4\rangle = [(10_c, 2_s), (\overline{10}_{c}, 2_s)];$$

$$|5\rangle = [(8_c, 2_s), (8_c, 4_s)]; \quad |6\rangle = [(8_c, 4_s), (8_c, 2_s)].$$

Evaluating the chromo-magnetic operator of Eq. [1] between these states we get the 2 matrices, describing

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3 The interpretation of the $Y(4140)$ as an axial was already contemplated in ref [27], albeit not excluding the $0^{++}$ alternative.
chromo-magnetism in the 2 sectors, given in Eqs. C1C2, where it was assumed the same ordering as above. This has been done using a computer, but since we are in fact in the symmetry limit, it can also be calculated by purely group theoretical means. It furnishes a valuable check of the machine’s symbolic calculation. It is straightforward to obtain the expression in terms of Casimir operators:

\[ O_{CM} = [C_6(R_{3q}) + C_6(R_{3\bar{q}}) - \frac{1}{2} C_3(R_{3q}) - \frac{1}{2} C_3(R_{3\bar{q}}) \]

\[ - \frac{1}{3} S_{3q}(S_{3q} + 1) - \frac{1}{3} S_{3\bar{q}}(S_{3\bar{q}} + 1) - 12 \]

\[ - [C_6(H) - C_6(R_{3q}) - C_6(R_{3\bar{q}}) + \frac{1}{2} C_3(R_{3q})] \]

\[ + \frac{1}{2} C_3(R_{3\bar{q}}) - \frac{1}{3} S_H(S_H + 1) \]

\[ + \frac{1}{3} S_{3q}(S_{3q} + 1) + \frac{1}{3} S_{3\bar{q}}(S_{3\bar{q}} + 1) \]  

where \( H \) stands for the representation of the hexaquark in \( SU(6)c_{3} \), with \( S_H \) being its spin (0 or 1 in the present case), \( R_{3q} \) and \( R_{3\bar{q}} \) the representations of the 3 quarks and 3 antiquarks subsystems, respectively (of both groups, \( SU(6)c_{3} \) and \( SU(3)c_{c} \)), \( S_{3q} \) and \( S_{3\bar{q}} \) being their spins. As before, \( C_6 \) and \( C_3 \) are the quadratic Casimir operators of \( SU(6)c_{3} \) and \( SU(3)c_{c} \). In the first square brackets we have isolated the contribution of the quark-antiquark and antiquark-antiquark interactions, while in the second the contribution for quark-antiquark interactions. Here a severe complication arises: the Casimir operators in the second bracket are not diagonal. As the operator \( O_{CM} \) transforms as the 35 of \( SU(6)c_{3} \), it does not leave the 70 and, thus the Casimir operators present in the first bracket are diagonal, while for the second one, representation mixing remains possible and in fact it occurs.

The hexaquark state \( (qqqqq) \), we have designated by \( H \), transforms under \( SU(6)c_{3} \) as one of irreducible representations (or mixings thereof) arising in the product below: \( 70^c \otimes 70 = 1 + 35_1 + 35_2 + 189 + 280 + 285 + 405 + 465 + 3675 \). For \( 0^- \) we have to select the blocks that contain components transforming as \( (1c, 1s) \), and for the \( 1^- \) as \( (1c, 3s) \). It is indicated below the relevant representations and the number of components of the suitable color singlets contained in each one:

\[ 0^- : (1c, 1s) \subset 1; \quad 189(1); \quad 405(1); \quad 3675(1) \]

\[ 1^- : (1c, 3s) \subset 35_1(1); \quad 35_2(1); \quad 280(1); \quad 285(1); \quad 3675(2). \]

The matrix elements were found through the determination of the appropriate Clebsch-Gordan coefficients for the above decomposition.

Let us now consider states of the kind \( (qqQQQQ) \) (\( Q \) being an \( s \) or \( c \) quark), for which some experimental evidence is available. The Pauli principle implies that the pair of light (anti-)quarks in the (anti-)baryonic block \( qqQ (qq\bar{Q}) \) must transform under \( SU(6)c_{3} \), as a 21\( c_{3} (21\bar{c}_{3}) \) for \( I = 0 \) and as a 15\( c_{3} (15\bar{c}_{3}) \) in the case of \( I = 1 \). States such \( (qq)_{21c_{3}} Q (qq\bar{Q}) / 21c_{3} / \bar{Q} \) have \( I = 0 \) and are relevant for the \( \Lambda \bar{X} (\Lambda_{c}\bar{X}_{c}) \) channels. For shortness, we shall call them the \( (21, \bar{2}1) \) basis. The other case, namely \( (qq)_{15c_{3}} Q (qq\bar{Q}) / 15c_{3} / \bar{Q} \) is the \( (15, 15) \) basis and comprises hexaquarks with \( I = 0, 1, 2 \). This base will be used in the calculation of the \( \Sigma \bar{X} \) channel.

A criterion to build the physical states, i.e. the color singlets of the six quark system, is to combine successively \( qq \) with \( Q \) (and analogously for the antiquarks) in all possible ways regarding the color group \( SU(3)c_{c} \) and then combining with those of the antiquarks. This can be easily done using the decompositions of \( SU(6)c_{3} \rightarrow SU(3)c_{c} \otimes SU(2)c_{c} \):

\[ 21c_{c} = (3_{s}, 1_{s}) + (6_{c}, 3_{s}) \]

and \( 15c_{c} = (6_{c}, 1_{s}) + (\bar{3}_{c}, 3_{s}). \)

Taking into account the genealogy of the states, we get for each basis, a total of 14 color singlets. They are displayed below. The convention we use is the following: the composition of the baryonic \( (qqQ) \) with anti-baryonic blocks \( (qq\bar{Q}) \) is indicated by a (*), each block is enclosed by a square bracket and within each bracket we placed on the left the color-spin content of \( (qq) \) followed by that of \( Q \) (and analogously for the antiquarks).

As will be seen in the next section, we have also interest to build the basis for the system \( \Lambda_{c}\bar{p} \). We use the ordering convention \( \bar{q}_{1} q_{2} q_{3} q_{4} q_{5} c_{6} \). The \( \bar{p} \), as previously, is put in a \( 70^c \) (antisymmetric in 1, 2) and the \( \Lambda_{c} \) (as the Pauli antisymmetry applies only to the pair 4 and 5) in a \( 10_{c} \) (symmetric with respect to 4 and 6) and a \( 5_{c} \) which decomposes under \( SU(3)c_{c} \otimes SU(2)c_{c} \) as: \( (10, 4)^{+} + (8, 2) \). The mandatory anti-symmetrization with respect to flavor of the pair 4 and 5 implies isospin 0 for the \( \Lambda_{c} \).

**Basis** \((21, \bar{2}1)\) for spin 1

\[
[(3, 1)(3, 2)] \ast [(3, 1)(3, 2)] \Rightarrow [1] = (1, 2) \ast (1, 2) \\
[2] = (8, 2) \ast (8, 2)
\]

\[
[(6, 3)(3, 2)] \ast [(3, 1)(3, 2)] \Rightarrow [3] = (8_{sim}, 4) \ast (8, 2) \\
[4] = (8_{sim}, 2) \ast (8, 2)
\]

\[
[(3, 1)(3, 2)] \ast [(6, 3)(3, 2)] \Rightarrow [5] = (8, 2) \ast (8_{sim}, 4) \\
[6] = (8, 2) \ast (8_{sim}, 2)
\]

\[
[(6, 3)(3, 2)] \ast [(6, 3)(3, 2)] \Rightarrow [7] = (8_{sim}, 4) \ast (8_{sim}, 4) \\
[8] = (8_{sim}, 4) \ast (8_{sim}, 2) \\
[9] = (8_{sim}, 2) \ast (8_{sim}, 4) \\
[10] = (8_{sim}, 2) \ast (8_{sim}, 2) \\
[11] = (10, 4) \ast (\bar{T}_{0}, 4) \\
[12] = (10, 4) \ast (\bar{T}_{0}, 2) \\
[13] = (10, 2) \ast (\bar{T}_{0}, 4) \\
[14] = (10, 2) \ast (\bar{T}_{0}, 2)
\]

\[
(13)
\]

---

4 The representation \( 8_{sim} \) is the color octet symmetric under the exchange of the colors of the light quark pair.
Basis (15, $\overline{15}$) for spin 1

$[(\bar{3}, 3)(\bar{3}, 2)]^{\dagger}[(\bar{3}, 3)(\bar{3}, 2)] \Rightarrow [1] = (1, 4) \gamma (1, 4) [2] = (1, 4) \gamma (1, 2) [3] = (1, 2) \gamma (1, 4) [4] = (1, 2) \gamma (1, 2) [5] = (8, 4) \gamma (8, 4) [6] = (8, 4) \gamma (8, 2) [7] = (8, 2) \gamma (8, 4) [8] = (8, 2) \gamma (8, 2) [9] = (8, 4) \gamma (8, 2) _{\text{sim}} [10] = (8, 2) \gamma (8, 2) _{\text{sim}} [11] = (8, 2) \gamma (8, 4) [12] = (8, 2) \gamma (8, 2) [13] = (8, 2) \gamma (8, 2) _{\text{sim}} [14] = (10, 2) \gamma (10, 2) _{\text{sim}}$

(14)

We have 5 states for spin 0 and 9 states for spin 1, they are given below:

Spin 0

$|1\rangle = (1, 2)_{\beta}(1, 2)_{\alpha} |2\rangle = (8, 2)_{\beta}(8, 2)_{\alpha}$
$|3\rangle = (8, 4)_{\beta}(8, 4)_{\alpha} |4\rangle = (10, 2)_{\beta}(10, 2)_{\alpha}$
$|5\rangle = (8, 2)_{\beta}(8, 2)_{56}$

(15)

Spin 1

$|1\rangle = (1, 2)_{\beta}(1, 2)_{\alpha} |2\rangle = (8, 2)_{\beta}(8, 2)_{\alpha}$
$|3\rangle = (8, 4)_{\beta}(8, 4)_{\alpha} |4\rangle = (10, 2)_{\beta}(10, 2)_{\alpha}$
$|5\rangle = (8, 2)_{\beta}(8, 2)_{\alpha} |6\rangle = (8, 4)_{\beta}(8, 2)_{\alpha}$
$|7\rangle = (8, 2)_{\beta}(8, 2)_{56} |8\rangle = (8, 4)_{\beta}(8, 2)_{56}$
$|9\rangle = (10, 2)_{\beta}(10, 4)_{56}$

(16)

With the introduction of appropriate color and spin projectors, it is easy to build explicitly the above basis. Symbolic expressions for the matrix elements of the chromomagnetic operator $O_{\text{CM}}$ were obtained with the help of FORM [10]. The explicit expressions for the CM matrixes for the three mentioned cases are collected in Appendix C. It was assumed for the CM matrixes the same ordering as for the above states. The mass spectrum of the most interesting baryon-antibaryon states are given in appendix A.

**VII. EXPERIMENTAL EVIDENCE FOR HEXAQUARKS**

1) We predict a 0$^-$ state $(qqqqqq)$, strongly coupled to the $p\bar{p}$ channel (the component along $p\bar{p}$ is 0.894), just below the threshold (1876.54 $MeV$), it has a mass of 1874 $MeV$. This is in agreement with the first observation of a narrow enhancement near $p\bar{p}$ threshold by the BES collaboration [11] in $J/\psi \rightarrow p\bar{p}\gamma$, then named $X(1859)$. Until now both the $J^P$ assignments 0$^+$ or 0$^-$ remain equally possible. It was found at a mass $m_X = 1859 \pm ^{+5}_{-10} \pm ^{+25}_{-0} MeV$ having a width smaller than 30 MeV. The state we found is slightly higher, just 7 $MeV$ above the experimental upper limit. They estimated a branching ratio of $B(B \rightarrow \gamma X) B(X \rightarrow p\bar{p}) \sim 7 \times 10^{-3}$.

2) Also relevant for the light hexaquarks $(qqqqqq)$ may be a quite broad 1$^+$ enhancement above $p\bar{p}$ threshold with mass 1935 $MeV$ and width $\Gamma = 215 \pm 30 MeV$ proposed about 30 years ago [31]. We have a very good candidate for this state at a mass 1911.5 $MeV$ with a large component (0.61 ) along the $p\bar{p}$ channel. However here some caution is needed, because the evidence is based on a partial wave analysis and one would have to check if the analysis is compatible with the inclusion of the additional 0$^-$ state just mentioned above.

3) We have also a pretty good candidate for the $Y(2175)$, a 1$^-$ state recently seen at the BaBar detector [32] at a mass 2170 $\pm 10 \pm 15 MeV$ (with a width $\Gamma = 58 \pm 16 \pm 20 MeV$). We predict a singly hidden strangeness state $(qqqqqq)$ strongly coupled to the $\Lambda\bar{\Lambda}$ channel (with a component of 0.6 along this direction) with a mass 2184 $MeV$. Since this state is below the $\Lambda\bar{\Lambda}$ threshold (around 2231 $MeV$) it has to decay mostly into mesons. In fact BaBar observed this state in the decay $Y \rightarrow f_0(980)\phi$ (through $f_0 \rightarrow \pi\pi$). The $Y(2175)$ has been confirmed by the BES collaboration [33] in $J/\psi \rightarrow \eta f_0(980)\phi$ at a mass $m = 2186 \pm 10 \pm 16 MeV$ and a width $\Gamma = 65 \pm 23 MeV$.

4) The peak in $\Lambda_c \bar{p}$ seen at the mass $m = 3350^{+10}_{-20} \pm 29 MeV$ and width $\Gamma = 70^{+40}_{-30} \pm 40 MeV$ in $B^- \rightarrow \Lambda_c \bar{p}$ may be identified with a 0$^-$ strange charmed hexaquark, we predict to be at 3339 $MeV$. There is also a 1$^-$ at lower mass, 3274 $MeV$, with a component of the same order (0.35). All the states strongly coupled to $\Lambda_c \bar{p}$ are below the threshold (3225 $MeV$), on the other side those above the threshold, with the exception of the two above mentioned states, have negligible couplings. This implies that these two states are the only ones observable in the baryonic channel. It is useful to remark that the experiment privileges the spin 0 assignment.

5) In the singly hidden charm sector $(qqcq\bar{q}\bar{q})$, the heaviest states are loosely coupled to the $\Lambda_c \bar{\Lambda}_c$, and the reasonably coupled states are just above or below the threshold (4573 $MeV$). We display these states and the value of the component along the baryonic channel:

| Mass (MeV) | 4533 | 4556 | 4575 | 4614 | 4642 | 4658 | 4670 |
|------------------|----------|----------|----------|----------|----------|----------|----------|
| comp. in $\Lambda_c \bar{\Lambda}_c$ | 0.41 | 0.21 | 0.52 | 0.42 | 0.48 | 0.16 | 0.24 |

As a matter of fact, recently, a resonance decaying into $\Lambda_c \bar{\Lambda}_c$ has been seen by the Belle detector [33] at $m = 4634^{+8+5}_{-7-8} MeV$ and $\Gamma = 92^{+10+20}_{-24-21} MeV$, compati-
ble $|34,35\rangle$ with $Y(4660) \rightarrow \psi' \pi \pi |34,35\rangle$. Anyway the fact that the component along the baryonic channel is not strongly dominant is welcome, since it is opportune to leave some room for the decay into $\psi' \pi \pi$. Recently it was proposed to interpret the above state as an excited $L=1$ tetraquark [37].

We have also calculated the spectrum of the singly hidden strangeness states $(qq\overline{s}s)$ relevant to the $\Sigma\Sigma$ channel, using, along the same lines, the $(15,\overline{15})$ basis. We find only two states strongly coupled to $\Sigma\Sigma$, both are around the threshold, $2380\text{MeV}$, one being below threshold at a mass of $2356\text{MeV}$ the other above, at $2454\text{MeV}$. Until now, there is no experimental evidence for these states.

VIII. CONCLUSION

The full chromo-magnetic Hamiltonian proved to be very effective in providing for an unified treatment of tetraquarks and hexaquarks. Besides reproducing the pattern of decays of currently accepted tetraquarks, it also predicts a companion for the $a_0(980)$ at a mass around $1330\text{MeV}$, which has been confirmed by experiments, as the scalar named $a_0(Y)$ and two $cs\overline{s}\overline{s}$ states, the $Y(4140)$ and the $X(4350)$. A number of candidates were compared with data for the baryon-antibaryon resonances, namely $pp, \Lambda\overline{\Lambda}, \Lambda\rho$ quite successfully.

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Spectrum of some $B\bar{B}$ states

PF is the probability factor for the $B\bar{B}$ open door channel. Masses are given in $MeV$.

| $B\bar{B}$ state | Threshold | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  |
|------------------|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\rho\bar{\rho}$ 0$^-$ | 1876      | 1263| 1874| 2151| 2407|     |     |     |     |     |     |     |     |     |     |
| $\rho\bar{\rho}$ 1$^-$ | 1876      | 1562| 1732| 1911| 2060| 2174| 2624|     |     |     |     |     |     |     |     |
| $\Lambda_c\bar{\rho}$ 0$^-$ | 3225      | 2653| 3028| 3188| 3339| 3595|     |     |     |     |     |     |     |     |     |
| $\Lambda_c\bar{\rho}$ 1$^-$ | 3225      | 2740| 2949| 3064| 3156| 3223| 3274| 3465| 3553| 3759|     |     |     |     |     |
| $\Lambda\bar{\Lambda}$ 1$^-$ | 2231      | 2105| 2125| 2142| 2184| 2231| 2246.8| 2247.2| 2274| 2297| 2303| 2325| 2343.83| 2343.95| 2421|
| $\Lambda\bar{\Lambda}$ 1$^-$ | 2231      |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| $\Lambda_c\Lambda_c$ 1$^-$ | 4573      | 4468| 4510| 4533| 4556| 4575| 4598| 4614| 4642| 4654| 4658| 4669| 4685| 4689| 4736|
| $\Sigma\Sigma$ 1$^-$ | 2380      | 2211| 2236| 2270| 2273| 2283| 2310| 2334| 2346| 2349| 2356| 2415| 2415.6| 2434| 2454|
| $\Sigma\Sigma$ 1$^-$ | 2380      |     |     |     |     |     |     |     |     |     |     |     |     |     |     |

Appendix A: Spectrum of the $B\bar{B}$ states
Appendix B: Crossing Matrices

Spin 0

\[
R_{\phi \rightarrow \alpha} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2}
\end{pmatrix} \quad (B1)
\]

\[
R_{\phi \rightarrow \epsilon} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{6}} & \frac{1}{2}
\end{pmatrix} \quad (B2)
\]

Spin 1

\[
R_{\phi \rightarrow \beta} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2}
\end{pmatrix} \quad (B3)
\]

\[
R_{\phi \rightarrow \chi} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2}
\end{pmatrix} \quad (B4)
\]

Spin 2

\[
R_{\xi \rightarrow \gamma} = \frac{1}{\sqrt{3}} \begin{pmatrix}
\sqrt{2} & 1 & -\sqrt{2}
\end{pmatrix} \quad R_{\xi \rightarrow \delta} = \frac{1}{\sqrt{3}} \begin{pmatrix}
\sqrt{2} & -1 & -\sqrt{2}
\end{pmatrix} \quad (B5)
\]

Appendix C: Chromo-magnetic operator for \[\text{qqqqqq}\] states

We have computed the matrices of chromo-magnetism by inserting the operator Eq. \[1\] between the states at Eqs. \[10,11\]; they are given below, where \( A_0 \) is for \( 0^- \) and \( A_1 \) for \( 1^- \).

\[
A_0 = \begin{pmatrix}
-2 & -\sqrt{2} & -1 & 0
-\sqrt{2} & -1 & -\frac{3}{\sqrt{2}} & -\sqrt{5}
-1 & -\frac{3}{\sqrt{2}} & -2 & -\frac{\sqrt{2}}{2}
0 & -\sqrt{5} & -\sqrt{5/2} & 0
\end{pmatrix} \quad (C1)
\]

\[
A_1 = \begin{pmatrix}
2 & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & 0 & -\frac{2\sqrt{2}}{3} & -2\frac{\sqrt{2}}{3}
-\frac{\sqrt{2}}{3} & \frac{1}{3} & \sqrt{\frac{2}{3}} & -\frac{\sqrt{3}}{3} & -\frac{1}{3} & \frac{1}{3}
\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{4}{3} & \frac{5}{3\sqrt{2}} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3}
0 & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{2\sqrt{5}}{3} & \frac{2\sqrt{5}}{3} & -\frac{1}{2}
2\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{\sqrt{2}}{3} & -2\frac{\sqrt{5}}{3} & \frac{5}{6} & -\frac{1}{2}
-2\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{\sqrt{2}}{3} & -2\frac{\sqrt{5}}{3} & \frac{5}{6} & -\frac{1}{2}
\end{pmatrix} \quad (C2)
\]
CM MATRICES FOR BASIS (15, 15)

\[
A = \begin{pmatrix}
\frac{-4}{3} (r+2s) & 0 & 0 & \frac{11}{27} \sqrt{2} (r-2s+t) & \frac{2}{27} \sqrt{10} (r+s-2t) & \frac{-2}{27} \sqrt{10} (r+s-2t) \\
0 & \frac{-4}{3} (r-s) & 0 & \frac{2}{27} \sqrt{10} (r+s-2t) & \frac{5}{27} \sqrt{2} (2r-s-t) & \frac{-1}{27} \sqrt{2} (r+4(s+t)) \\
0 & 0 & \frac{-4}{3} (r-s) & 0 & \frac{-2}{27} \sqrt{10} (r+s-2t) & \frac{-1}{27} \sqrt{2} (r+4(s+t)) \\
0 & 0 & 0 & \frac{-4}{3} (r-4s) & \frac{2}{27} \sqrt{5} (r+4(s+t)) & \frac{-4}{27} (2r+5s+2t) \\
\frac{11}{27} \sqrt{2} (r-2s+t) & \frac{2}{27} \sqrt{10} (r+s-2t) & \frac{-2}{27} \sqrt{10} (r+s-2t) & \frac{2}{27} \sqrt{5} (r+4(s+t)) & \frac{1}{54} (5r+62s+77t) & \frac{1}{27} \sqrt{5} (7r-2(s+7t)) \\
\frac{2}{27} \sqrt{10} (r+s-2t) & \frac{5}{27} \sqrt{2} (2r-s-t) & \frac{-1}{27} \sqrt{2} (r+4(s+t)) & \frac{-4}{27} (2r+5s+2t) & \frac{1}{54} (5r+62s+77t) & \frac{1}{54} (-r+2s-35t) \\
\frac{-2}{27} \sqrt{10} (r+s-2t) & \frac{-1}{27} \sqrt{2} (r+4(s+t)) & \frac{5}{27} \sqrt{2} (2r-s-t) & \frac{4}{27} (2r+5s+2t) & \frac{1}{54} (-7r+8s-28t) & \frac{1}{54} (-2r+s-35t)
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\frac{1}{54} (-100r-7(4s+t)) & \frac{1}{9} \sqrt{2} (r+8s) & \frac{1}{18} (2r+31s) & \frac{-1}{9} \sqrt{2} (r+8s) & \frac{1}{18} (2r+31s) & \frac{-r}{2} & \frac{-\sqrt{5} r}{3} \\
\frac{1}{9} \sqrt{2} (r+8s) & \frac{1}{18} (-30r+23s-5t) & \frac{-2}{9} \sqrt{2} (2s+t) & \frac{-r}{2} & \frac{1}{18} (-30r+23s-5t) & \frac{2 \sqrt{2} r}{3} & \frac{2 \sqrt{10} s}{3} \\
\frac{1}{18} (2r+31s) & \frac{-2}{9} \sqrt{2} (2s+t) & \frac{1}{18} (-30r-14s-t) & \frac{-\sqrt{2} r}{2} & \frac{1}{18} (-30r-14s-t) & \frac{11s}{6} & \frac{-\sqrt{5} s}{3} \\
\frac{1}{9} \sqrt{2} (r+8s) & \frac{-r}{2} & \frac{-\sqrt{2} r}{2} & \frac{1}{18} (-30r+23s-5t) & \frac{2}{9} \sqrt{2} (2s+t) & \frac{2 \sqrt{2} s}{3} & \frac{2 \sqrt{10} s}{3} \\
\frac{1}{18} (2r+31s) & \frac{\sqrt{2} r}{2} & \frac{-r}{2} & \frac{2}{9} \sqrt{2} (2s+t) & \frac{1}{18} (-30r-14s-t) & \frac{11s}{6} & \frac{-\sqrt{5} s}{3} \\
\frac{-r}{2} & \frac{-2 \sqrt{2} s}{3} & \frac{11s}{6} & \frac{2 \sqrt{2} s}{3} & \frac{11s}{6} & \frac{1}{2} (-4r-t) & \frac{-\sqrt{5} t}{3} \\
\frac{\sqrt{5} r}{3} & \frac{2 \sqrt{10} s}{3} & \frac{-\sqrt{5} s}{3} & \frac{-2 \sqrt{10} s}{3} & \frac{-\sqrt{5} s}{3} & \frac{-\sqrt{5} t}{3} & \frac{2}{3} (3r+t)
\end{pmatrix}
\]
\[
C = \begin{pmatrix}
\frac{2}{27} \sqrt{5} (r+4(s-t)) & \frac{2}{9} \sqrt{10} (-r+s) & -\frac{2}{9} \sqrt{5} (r+2s) & \frac{2}{9} \sqrt{10} (r-s) & -\frac{2}{9} \sqrt{5} (r+2s) & \frac{2}{3} \sqrt{5} r & 0 \\
-\frac{4}{27} (2r+5s+2t) & \frac{5}{9} \sqrt{2} (r-s) & -\frac{4}{9} (r+2s) & \frac{1}{9} \sqrt{2} (r+2s) & \frac{4}{9} (2r+s) & \frac{4r}{3} & 0 \\
\frac{4}{27} (2r+5s+2t) & \frac{1}{9} \sqrt{2} (r+2s) & -\frac{4}{9} (2r+s) & \frac{5}{9} \sqrt{2} (r-s) & \frac{4}{9} (r+2s) & -\frac{4r}{3} & 0 \\
-\frac{1}{27} \sqrt{2} (4r+4s+t) & -\frac{4}{9} (r+2s) & -\frac{1}{9} \sqrt{2} (2r+s) & \frac{4}{9} (r+2s) & -\frac{1}{9} \sqrt{2} (2r+s) & -\frac{\sqrt{2} r}{3} & 0 \\
\frac{1}{27} \sqrt{\frac{5}{2}} (7r-8s+28t) & \frac{1}{9} \sqrt{5} (r-4s) & \frac{1}{9} \sqrt{\frac{5}{2}} (r+8s) & -\frac{1}{9} \sqrt{5} (r-4s) & \frac{1}{9} \sqrt{\frac{5}{2}} (r+8s) & \sqrt{\frac{5}{2}} r & \frac{5\sqrt{2} r}{3} \\
-\frac{2}{27} \sqrt{2} (7r-5s+7t) & \frac{1}{16} (-5r+47s) & \frac{1}{16} \sqrt{2} (r+8s) & \frac{1}{16} (-r-8s) & \frac{2}{9} \sqrt{2} (r+2s) & \sqrt{2} r & \frac{2\sqrt{10} r}{3} \\
\frac{2}{27} \sqrt{2} (7r-5s+7t) & \frac{1}{16} (-r-8s) & \frac{2}{9} \sqrt{2} (r+2s) & \frac{1}{16} (-5r+47s) & \frac{1}{9} \sqrt{2} (r+8s) & -\sqrt{2} r & \frac{2\sqrt{10} r}{3}
\end{pmatrix}
\]

**CM MATRICES FOR BASIS (15,15)**

\[
\begin{pmatrix}
A & C \\
C^T & B
\end{pmatrix}
\]

Parameters are

\[C_{qq} = r, \quad C_{qc} = s, \quad C_{c\bar{c}} = t\]
CM MATRICES FOR BASIS (21, 21)

\[
A = \begin{pmatrix}
4r & -\sqrt{2} s & 4s & \sqrt{2} s & 4s & \sqrt{2} s & 2\sqrt{5} r & 2\sqrt{5} r \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
3 & 3 & 10 & 10 & 10 & 10 & 10 & 10 \\
3 & 3 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 3 & 9 & 9 & 9 & 9 & 9 & 9 \\
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
2\sqrt{5} r & \sqrt{5} r & \sqrt{5} r & \sqrt{5} r & \sqrt{5} r & \sqrt{5} r & \sqrt{5} r & \sqrt{5} r \\
3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
\frac{4r}{3} & -\frac{4r}{3} & -\frac{\sqrt{2}r}{3} & 0 & 0 & 0 \\
\sqrt{2}r & -\sqrt{2}r & \frac{r}{2} & \frac{5\sqrt{2}r}{3} & \frac{2\sqrt{10}r}{3} & \frac{2\sqrt{10}r}{3} \\
25r + 17s & 5r + 2s & -5\sqrt{2}r & 4\sqrt{2}s & 10r - 10s & \frac{5\sqrt{5}r + 5\sqrt{5}s}{3} \\
10 + 16 & 10 + 9 & 9 & 9 & 9 & 9 \\
5\sqrt{2}r - 4\sqrt{2}s & -10\sqrt{2}r & -2\sqrt{2}s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
5\sqrt{2}r + 5\sqrt{2}s & 5\sqrt{2}r + 10\sqrt{2}s & 2\sqrt{10}r + 4\sqrt{10}s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
10\sqrt{2}r + 2\sqrt{2}s & 5\sqrt{2}r + 4\sqrt{2}s & -5\sqrt{2}r + 25s & 0 & 0 & 0 \\
9 & 9 & 9 & 9 & 9 & 9 \\
5\sqrt{5}r - 2\sqrt{5}s & -5\sqrt{5}r + 2\sqrt{5}s & \frac{5\sqrt{2}r}{9} + \frac{3\sqrt{10}r}{9} & \frac{11\sqrt{5}r}{27} + \frac{22\sqrt{5}s}{27} + \frac{11\sqrt{5}r}{27} & \frac{10r + 10s}{27} - \frac{201}{27} & \frac{5\sqrt{2}r}{27} + \frac{20\sqrt{5}s}{27} + \frac{20\sqrt{5}r}{27}
\end{pmatrix}
\]

CM MATRICES FOR BASIS \((21, 21)\)

\[
\begin{pmatrix}
A & C \\
C^T & B
\end{pmatrix}
\]

Parameters are

\[C_{qq} = r, \ C_{qc} = s, \ C_{cc} = t\]
**CM MATRICES FOR BASIS $\Lambda_\rho$**

**Spin 0**

\[
\begin{bmatrix}
16r & 4\sqrt{2} (-s + r) & -8r & 0 & 4\sqrt{2} (s + r) \\
4\sqrt{2} (-s + r) & 3\sqrt{2} (s + r) & 4\sqrt{2} (-s + r) & 5\sqrt{3} (-s + r) & 0 \\
-8r & 5\sqrt{2} (-s + r) & 16r & 3\sqrt{10} (2s + r) & -\frac{2}{3}\sqrt{2} (2s + 7r) \\
0 & 5\sqrt{3} (-s + r) & 4\sqrt{2} (s + r) & -8s + 8r & -\frac{5}{3}\sqrt{3} (s + 5r) \\
4\sqrt{2} (s + r) & 0 & -\frac{2}{3}\sqrt{2} (2s + 7r) & -\frac{5}{3}\sqrt{3} (s + 5r) & 8r
\end{bmatrix}
\]

**Spin 1**

\[
\begin{bmatrix}
16r & 2\sqrt{2} r & 2\sqrt{2} r & 0 & 4\sqrt{2} r \\
2\sqrt{2} r & 5\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 0 \\
-8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & -8\sqrt{2} r & 0 \\
0 & 16\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 0 \\
8\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & -8\sqrt{2} r \\
-8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & -8\sqrt{2} r & 0 \\
0 & 16\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 0 \\
8\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & -8\sqrt{2} r \\
16\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & 0 \\
-8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & -8\sqrt{2} r & 0 \\
0 & 16\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 0 \\
8\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & -8\sqrt{2} r \\
16\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & 0 \\
-8\sqrt{2} r & 8\sqrt{2} r & 16\sqrt{2} r & -8\sqrt{2} r & 0 \\
0 & 16\sqrt{2} r & -8\sqrt{2} r & 8\sqrt{2} r & 0
\end{bmatrix}
\]

**Parameters are**

\[C_{qq} = r, \quad C_{qc} = s\]