Parabolic regularization of the gradient catastrophes for the Burgers–Hopf equation and Jordan chain

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Abstract
Non-standard parabolic regularization of the gradient catastrophes for the Burgers–Hopf equation is proposed. It is based on the analysis of all (generic and higher order) gradient catastrophes and their step-by-step regularization by embedding the Burgers–Hopf equation into the multi-component parabolic systems of the quasilinear PDEs with the most degenerate Jordan block. The probabilistic realization of such a procedure is presented. The complete regularization of the Burgers–Hopf equation is achieved by embedding it into an infinite parabolic Jordan chain. It is shown that the Burgers equation is a particular reduction of the Jordan chain. Gradient catastrophes for the parabolic Jordan systems are also studied.

Keywords: non-generic gradient catastrophes, parabolic regularization, Jordan chain

1. Introduction

The problem of gradient catastrophe (the unbounded growth of derivatives) and the mechanisms of its regularization have been addressed in a number of papers in various contexts. Although the gradient catastrophe (GC) shows up in quite different ways in different branches of physics and mathematics, its essence is encoded in the Burgers–Hopf (BH) (or Riemann) equation

\[ u_t = uu_x \]  

(1)

where \( u = u(x, t) \) denotes the function of two independent variables \( (x, t) \) and the subscripts stand for partial derivatives. The BH equation is the most simplified one-dimensional version...
of the Navier–Stokes equation in hydrodynamics and the best studied prototype of nonlinear PDEs exhibiting the GC (see e.g. [1–3]). The hodograph equation \( x + ut + f(u) = 0 \) for the BH equation provides us a simple and almost explicit formulae describing the behavior of \( u \) near to the point of the first GC, where \( u \) is finite but its derivatives \( u_t \) and \( u_{tt} \) are unbounded [1–3].

A standard method of regularizing this GC is to add higher order derivatives of \( u \) on the rhs of equation (1) [1, 2]. Adding \( u_{xx} \), one gets the Burgers equation, which mimics the effects of dissipation. With the \( u_{xx} \) term on the rhs of (1) one obtains the Korteweg–de Vries (KdV) equation, which describes the dispersive regularization of the BH equation. Various extensions of such types of regularizations have been discussed, for instance, in [4–6].

In the present paper, we propose a method of regularization consisting of the step-by-step regularization of all (generic and higher order) GCs of the BH equation by introducing new degrees of freedom. At each step, these new variables obey the parabolic system of quasi-linear partial differential equations of the Jordan block type. Complete regularization is achieved by embedding the BH equation into the infinite Jordan chain

\[
  u_{kl} = u_l u_{k-1} + u_{k+1}, \quad k = 1, 2, 3, \ldots
\]

The basic point of our approach is that the complete study of the GC phenomenon for the BH equation has to include not only analysis of the so-called generic GC but also an investigation of the higher GCs with more singular behavior of \( u_t \) and \( u_{tt} \). The GC for the BH equation corresponds to the critical point of the function \( \phi(u) = x + ut + f(u) \), where \( f \) is the function inverse to the initial datum \( u(x, 0) \). According to the general approach formulated by Poincaré in [7] (see also [8]), one does not just study a single situation (even a generic one) but a whole family of close situations in order to get a complete and deep understanding of certain phenomena. In our case, this means that analysis of the critical points of the infinite family of functions \( \phi(u) \) corresponding to all possible initial data is required. The presence of unremovable degenerate critical points of different orders is a characteristic feature of families of functions. This is a typical situation in the theory of singularities (or the theory of catastrophes) (see e.g. [8–12]): ‘So, the non-generic degenerations become unremovable if one considers not an individual object but a family’ ([11], section 8) and ‘Thus, in the investigation of critical points of functions depending on parameters, we have to consider degenerate critical points as well as non-degenerate. The larger the number of parameters, the more complicated critical points that can occur’ [12].

For the BH equation (1), the GC of order \( k \) corresponds to a critical point of order \( k + 1 \) of the function \( W = xu + ut^2/2 + W(u) \), \( dW/du = f(u) \), which satisfies \( \partial W/\partial u^l = 0 \) for \( l = 0, 1, \ldots, k + 1 \) and is associated with the \( A_{k+1} \) singularity. The derivative \( u_t \) behaves like \( u_t \sim (x - x_0)^{-(k+1)/(k+1)} \) near the critical point \( x_0 \). Such types of singularities, called secondary ones in the singularity theory [11], have important implications in physics. Zeldovich’s theory of the large-scale structure of the Universe [13] provides us with notable evidence of their relevance.

The second point is that the regularization of the GC at each step is performed by adding one new dependent variable in a way that in such a large space the degree of degeneration of the critical point is reduced by one. Moreover in this procedure, one requires that: (1) at each step the system is parabolic in order to respect the priority role of dissipation and diffusion phenomena in the regularization of GCs; (2) the extended systems should be minimal, in a sense that they should have only one characteristic speed as a BH equation; (3) the extended systems should include the transport equation for the BH equation, which is the equation, for example, for the entropy \( s \) in fluid-dynamics; (4) all regularizing systems should be equations governing the dynamics of critical points of certain functions.
Within this approach, the first step in the regularization of the BH equation is
\[
\begin{cases}
u_t = uu_x \\
s_t = u_s
\end{cases}
\Rightarrow
\begin{cases}
u_t = uu_{1x} + u_{2x} \\
u_{2t} = uu_{2x}.
\end{cases}
\tag{3}
\]
This extension regularizes the first GC of the BH equation. In order to regularize the Nth order GC, one adds in total N – 1 variables and passes to the parabolic N-component system of quasilinear PDEs with the most degenerate Jordan block. The singular sectors and associated GCs for these N-component Jordan systems are studied. It is shown that the dominant behavior of \(u_{1x}\) at the kth order GC for the N-component case coincides with that of the \(k = 1\)th order for the \(N + 1\)-component Jordan system.

In this regularization process the conditions defining the \(N = 1\)th gradient catastrophe of the BH equation become the equations for the critical point of a certain function \(W^{(N)}(u_1, \ldots, u_N)\) of \(N\) variables. It is shown that this formal procedure has an explicit realization as the averaging of all quantities associated with the BH equation with the one-dimensional distribution
\[
G^{(N)}(u_1-u, u_2, \ldots, u_N) = \int_{-\infty}^{+\infty} d\lambda \exp \left(i\lambda(u_1-u) + \sum_{k=2}^{N}(i\lambda)^k u_k\right),
\tag{4}
\]
where \(u_1, u_2, \ldots, u_N\) are parameters. At \(N = 2\) it is the standard Gaussian distribution, which can also be interpreted as the Maxwell distribution. At \(N = 3\) and \(N = 4\), \(G^{(3)}\) and \(G^{(4)}\) are expressed in terms of the Airy and Peacys functions, respectively.

The regularization of all GCs for the BH equation within this approach requires the Jordan chain (2), viewed as the infinite-component parabolic system with a single eigenvalue \(u_1(x,t)\).

The importance of the Jordan chain as the regularization of the BH equation is also confirmed by its relation with the well-known regularizing equations. First, the chain (2) admits the reduction \(u_2 = \frac{\nu}{2} + \nu u_{1x}, \ u_3 = \frac{\nu}{3}u_1^3 + 2\nu u_1 u_{1x} + \nu^2 u_{1xx}\) and so on, under which it becomes the Burgers equation
\[
u_{1t} = 2u_1 u_{1x} + \nu u_{1xx},
\tag{5}
\]
which is the most canonical parabolic regularization of the BH equation [1]. Under this reduction, the probability density (4) at \(N \to \infty\) is essentially the Fourier transform of the resolvent of the adjoint Lax operator \(L^* = \nu \partial_x + u_1\) for the Burgers equation. Moreover, the chain (2) considered formally—forgetting its parabolic origin—admits many other differential reductions as, for example, to the kdv equation. This phenomenon, which indicates the universality of the Jordan chain as a regularizer, is of interest and requires further study.

The paper is organized as follows. In section 2 the basic facts concerning GCs for the BH equation are presented. In section 3, the GCs for the N-component Jordan system and associated singularities are studied. Regularization of the GCs for the Jordan systems is considered in section 4. Parabolic regularizations of all GCs for the BH equation by the Jordan chain and its relation to the Martínez Alonso–Shabat universal hierarchy of hydrodynamic type [14] are discussed in section 5. The probabilistic realization of this regularization process is presented in section 6. The reduction of the Jordan chain to the Burgers equation is considered in section 7. The universality of the Jordan chain is briefly discussed in section 8. The appendix contains some explicit formulae relating the BH equation and the Jordan system with two components as well as a discussion of the physical model of regularization with \(G^{(2)}\) being the Maxwell distribution for velocities for the ideal Boltzmann gas with the temperature \(T = 2u_2\).
2. Higher gradient catastrophes for the Burgers–Hopf equation

The behavior of solutions of the BH equation near the generic GC (when \(u(x,t)\) is finite but the derivatives \(u_t\) and \(u_x\) are unbounded) is given by the simple and almost explicit well-known formulae [1–3]. On the other hand, the study of nongeneric or higher GCs has attracted much less attention.

Here, we present an analysis of the generic and higher GCs for the BH equation in a form that is appropriate for our purposes. The BH equation (1) can be viewed (trivially) as the equation governing the dynamics of the critical point of the function

\[
W(u, x, t) = xu + \frac{t u^2}{2} + \tilde{W}(u).
\]

Indeed

\[
x + u_t + \tilde{W}_u(u) = 0
\]

is the hodograph equation for the BH equation where \(\tilde{W}_u(u)\) is connected with the initial value of \(u\) at \(t = 0\).

Infinite families of functional \(W\) corresponding to the family of all possible initial data \(u(x, 0)\) can be viewed as the family of functions depending on the infinite set of parameters with \(W\) of the form

\[
W(x, t, t_2, \ldots) = xu + \frac{t u^2}{2} + \sum_{n=3}^{\infty} \frac{t_{n-1} u^n}{n!} + \tilde{W}(u).
\]

So the appearance of degenerate critical points is very natural [8–12].

Generic and higher GCs are associated with the strata of a singular sector of the BH equation studied in [15]. The singular sector \(S\) is the union of strata \(S_k\) defined (in a simplified way) by the conditions

\[
S_k = \{(x, t, u) : W_u = W_{uu} = \ldots \partial_u^{k+1} W = 0, \partial_u^{k+2} W \neq 0\}, \quad k = 1, 2, \ldots.
\]

Calculating the differential of (7) or differentiating (7) w.r.t. \(x\) and \(t\), one gets in the regular sector \(S_0 (W_{uu} \neq 0)\)

\[
u_x = -\frac{1}{W_{uu}}, \quad u_t = -\frac{u}{W_{uu}}.
\]

In the strata \(S_k\) it holds

\[
u_x = -(k + 1)! \partial_u^{k+2} W, \quad u_t = -(k + 1)! u \partial_u^{k+2} W.
\]

The first GC (\(W_{uu} = 0\)) happens on the submanifold of variables \((x, t, t_1, t_2, \ldots)\) of codimension one, \(u_x \to \infty\) and \(u_t \to \infty\). Let us denote the points of this submanifold with \(x_0, t_0, u_0\). Higher GCs happen on the submanifolds of codimension \(k\).

Let us consider the multiscaling expansion around this point. Subtracting from the general variation, its infinitesimal Galilean transformation (see e.g. [5, 6])

\[
x \rightarrow x' = x_0 - u_0(t - t_0),
\]

one has

\[
u = u_0 + \epsilon^\alpha v, \quad t = t_0 + \epsilon^\beta \tau, \quad x = x_0 - u_0 \epsilon^\beta \tau + \epsilon^\gamma y,
\]

\[
W(u, x, t) = xu + \frac{t u^2}{2} + \tilde{W}(u).
\]
where $\epsilon$ is a small parameter and the constants $\alpha$, $\beta$ and $\gamma$ are fixed. Substituting (13) in $W(u)$, one gets at $S_k$

$$W = W(x_0, t_0, u_0) - \frac{1}{2} \epsilon^2 u_0^2 \tau + \epsilon^\gamma u_0 y + \epsilon^{\alpha+\gamma} y^2 + \frac{1}{2} \epsilon^{2+2\alpha} \tau y^2 + \epsilon^{\alpha(k+2)} A_{k+2} u^{k+2} + \ldots,$$

(14)

where $A_{k+2} = \frac{1}{k+2} \frac{\partial^k w}{\partial u^{k+2}}|_{u=0}$. The balance of all three nontrivial terms in (14) implies that

$$\alpha + \gamma = \beta + 2\alpha = \alpha(k + 2),$$

and, hence,

$$\beta = \alpha k, \quad \gamma = \alpha(k + 1).$$

Choosing $\alpha = 1$, one has

$$W = W(x_0, t_0, u_0) - \frac{1}{2} \epsilon^2 u_0^2 \tau + \epsilon^{k+1} u_0 y + \epsilon^{k+2} W_k^* + o(\epsilon^{k+2}),$$

(17)

where

$$W_k^* = y u + \frac{1}{2} \tau y^2 + A_{k+2} u^{k+2}.$$ (18)

For other balances of two (instead of three) nontrivial terms in (14), namely, $\alpha + \gamma = \alpha(k + 2)$, $\beta + 2\alpha > \alpha + \gamma$ or $\beta + 2\alpha = \alpha(k + 2)$, $\beta + 2\alpha < \alpha + \gamma$ (19),

one gets the formula (17) with, respectively, $\tau = 0$ or $y = 0$.

If instead of (13) one considers the naive expansion

$$u = u_0 + \epsilon^\alpha v, \quad t = t_0 + \epsilon^\beta \tau, \quad x = x_0 + \epsilon^\gamma y,$$

(20)

one gets

$$W = W(x_0, t_0, u_0) - \frac{1}{2} \epsilon^2 u_0^2 \tau + \epsilon^\gamma u_0 y + \epsilon^{\alpha+\gamma} y^2 + \epsilon^{\alpha+\beta} u_0 \tau y + \frac{1}{2} \epsilon^{2+2\alpha} \tau y^2 + \epsilon^{\alpha(k+2)} A_{k+2} u^{k+2} + \ldots.$$ (21)

For $W$ of the form (21), one has two different cases. The first is the generic one with $u_0 \neq 0$. In this case, since $\alpha + 2\gamma > \beta + \gamma$, the balance of the other terms implies

$$\gamma = \beta = \alpha(k + 1),$$

and, at $\alpha = 1$

$$\beta = \gamma = k + 1.$$ (23)

The potential $W$ becomes

$$W = W(x_0, t_0, u_0) + \epsilon^{k+1} \left( u_0 y + \frac{1}{2} (u_0)^2 \tau \right) + \epsilon^{k+2} W_k^*$$

(24)

with

$$W_k^*(\xi, \nu) = \xi u + A_{k+2} u^{k+2}$$ (25)

and $\xi = y + u_0 \tau$.

In the particular case $u_0 = 0$, the balance is given by

$$\alpha + \gamma = 2\alpha + \beta + \alpha(k + 2),$$

(26)

and, hence, one has for $\alpha = 1,$

...
\[ W = W(x_0, t_0, u_0) + \epsilon^{k+2} W^*_k \]  

(27)

with

\[ W^*_k(\xi, \nu) = \xi \nu + \frac{1}{2} \nu^2 \tau + A_{k+2} \nu^{k+2}. \]  

(28)

Comparing (17), (13) and (27), (20), one observes that within the naive expansion (27), the choice \( u_0 = 0 \) is equivalent to the elimination of the contribution of the Galilean transformation from the variation of \( x \).

The condition for the critical point for the function \( W^*_k \) is

\[ \frac{\partial W^*_k}{\partial \nu} = y + \nu \tau + (k + 2) A_{k+2} \nu^{k+1} = 0, \quad k = 1, 2, 3, \ldots . \]  

(29)

These equations are well-known in the cases \( k = 1, 2 \). At \( k = 2 \), it corresponds to the first time for the so-called generic GC for the BH equation (see e.g. [1, 2, 5]). Within our approach, this describes first a higher \( (k = 2) \) GC with an additional condition \( W_{uuu}(u_0) < 0 \).

Equation (29) defines the behavior of \( \nu \) at the GCs. Taking \( \tau = 0 \), one gets at \( S_k \)

\[ \nu \sim (y)^{1/(k+1)} , \quad k = 1, 2, \ldots. \]  

(30)

So at higher GCs, one has more singular behavior of the derivatives. The behavior (30) also follows from the simple observation that at \( S_k \)

\[ u_x \sim (\delta x)^{-k/(k+1)} . \]  

(31)

Since for a pure variation of \( x \) we have \( \delta x \sim \epsilon^\gamma y \sim \epsilon^{\alpha(k+1)} \), one gets

\[ u_x \sim (\delta x)^{-k/(k+1)} . \]  

(32)

The function \( W^*_k \) (17) and equation (29) are the same as (6) and (7) with the particular choice \( \tilde{W}_k^* = A_{k+2} \nu^{k+2} \). For each \( k \), solutions of (29) obey the BH equation \( \nu_\tau = \nu \nu_\nu \), and provide its solutions exhibiting a GC of order \( k \).

In a similar manner, one can analyze higher GCs for each member

\[ u_{n-1} = \frac{1}{n!}(u^n)_x, \quad n = 2, 3, 4, \ldots . \]  

(33)

of the BH hierarchy. The \( n \)th equation of the hierarchy (33) describes the dynamics of the critical points \( \partial W_n/\partial \nu = 0 \) of the function

\[ W_n(x, t_{n-1}, u) = xu + \frac{t_{n-1}}{n!} u^n + \tilde{W}_n(u) . \]  

(34)

One has

\[ \frac{\partial W_n}{\partial u} = x + \frac{t_{n-1}}{(n - 1)!} u^{n-1} + \frac{\partial \tilde{W}_n}{\partial u} = 0 \]

\[ \frac{\partial^2 W_n}{\partial u^2} = \frac{t_{n-1}}{(n - 2)!} u^{n-2} + \frac{\partial^2 \tilde{W}_n}{\partial u^2} \]  

(35)

and the analogue of the formula (11) is
\[ u_t = -\frac{1}{\partial_t^2} W_n \quad \text{and} \quad u_{n-1} = -\frac{u^{n-1}}{(n-1)!\partial_t^2 W_n}. \]  

(Singular sectors \( S_k \) and \( k \)th order GCs are again defined by the formula (9). Performing the expansion \[ u = u_0 + \epsilon^\alpha \upsilon, \quad t = t_{n-10} + \epsilon^\beta \tau_{n-1}, \quad x = x_0 - \epsilon^\gamma \tau_0, \quad \text{and} \quad y = \epsilon^\gamma, \] 

with the balancing condition (16), one gets (\( \alpha = 1 \)) \[ W_n = W_n(x_0, t_{n-10}, u_0) = \frac{1}{2} \epsilon^k(u_0)^{n-1}t_{n-1}^* + \epsilon^{k+1}u_0y + \epsilon^{k+2}W_{n-1}^*, \] 

where \[ W_{n-1}^* = yu + \frac{1}{2(n-1)!} \epsilon^{n-1}u^{n-1}t^{k+2} + A_{k+2}u^{k+2}. \] 

So, generically, one has the same behavior of \( u \) and its derivatives \( u_t, u_r \) at the GC points for all the equations of the BH hierarchy. For the case \( k = 2 \), see also [5]. In the particular case \( u_0 = 0 \) one instead has \[ W = W(x_0, t_0, u_0) + \epsilon^{k+2}y + \frac{1}{n!} \epsilon^{n+2}t^{n+2} + \epsilon^{k+2}A_{k+2}u^{k+2} + \ldots. \] 

So, the balance is \( \alpha + \gamma = n\alpha + \beta = \alpha(k + 2) \) and for \( \alpha = 1 \) we obtain \[ W = W(x_0, t_0, u_0) + \epsilon^{k+2} \left( y + \frac{1}{n!} \epsilon^{n+1}t^{n+1} + A_{k+2}u^{k+2} \right) + \ldots. \] 

The function \( W_{n}^* \) (18) represents a particular family of deformations for the singularities of \( A_{k+1} \)-type (see e.g. [8--12]).

The relation between the GCs for the BH equation and the singularities (catastrophes) of the \( A_{k+1} \)-type has been known since the paper [16], and has been discussed many times. A very interesting and important implication of the higher GCs of the BH equation was discovered by Zel’ dovich [13]. In his model of matter distribution, in the Universe the GCs are responsible for the formation of compact objects, since the density of matter \( n \) is proportional to the gradient of the particle velocity near point \( x_0 \) of the GC. In the one-dimensional approximation \[ n(x)|_{x_0} \sim u_t(x)|_{x_0} \sim (x - x_0)^{-k/(k+1)}. \] 

The cases \( k = 1, 2, 3, \infty \) are those considered in [13] (see also [8]). For the recent developments of Zel’ dovich’s model, see e.g. [17, 18] and the references therein.

### 3. The gradient catastrophe for the \( N \)-component Jordan system

The BH equation (1) can be extended in many different ways following the choice of physical effects to be incorporated or the general mathematical scheme in which it should be embedded.

Our choice is to consider the BH equation as the simplest instance of systems of quasi-linear PDEs of the first-order (hydrodynamic-type equation) \[ u_i = \sum_k A_{ik}(u)u_k, \quad i = 1, 2, \ldots, N. \]
In addition, it is required that the characteristic speeds of the system (43), i.e. the eigenvalues of the matrix $A$, are all coincident. With such a choice, the system (43) with a single characteristic speed is a pure parabolic system of PDEs closest to the BH equation (1).

In the 2-component case, the simplest example of such a system is given by

$$u_1 t = u_1 u_1 x + u_2 x, \quad u_2 t = u_1 u_2 x.$$  \hfill (44)\

It is noted that the second equation (44) has a simple physical meaning. Namely, it is the transport equation for the BH equation (1) \[1, 2\].

The systems (43), with the matrix $A(u)$ being the most degenerate Jordan block of order $N$, i.e. the systems

$$\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}_t = \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}_x$$  \hfill (45)\

were introduced in \[19\]. The $N$-component system (45) has a common property with the BH equation: namely, it only has one family of characteristics

$$\frac{dx}{dt} = -u_1(x, t).$$  \hfill (46)\

Consequently, it can be represented in the form

$$\mathcal{D}_1 u_1 = \mathcal{D}_2 u_{i+1}, \quad i = 1, 2, \ldots, N - 1,$$

$$\mathcal{D}_1 u_N = 0,$$  \hfill (47)\

with the vector fields $\mathcal{D}_1 = \partial_t - u_1 \partial_x$ (derivative along the characteristic) and $\mathcal{D}_2 = \partial_x$ which obey the relation $[\mathcal{D}_1, \mathcal{D}_2] = u_1 \mathcal{D}_2$.

The system (45) describes the dynamics of the critical points of the function \[19\]

$$W^{(N)} = xu_1 + t \left( \frac{u_1^2}{2} + u_2 \right) + \tilde{W}^{(N)}(u_1, u_2, \ldots, u_N)$$  \hfill (48)\

which obey the PDEs

$$\frac{\partial W^{(N)}}{\partial u_k} = \frac{\partial^k W^{(N)}}{\partial u_1^k}, \quad k = 1, 2, \ldots, N.$$  \hfill (49)\

The hodograph equations are

$$\frac{\partial W^{(N)}}{\partial u_k} = 0, \quad k = 1, 2, \ldots, N,$$  \hfill (50)\

i.e.

$$\frac{\partial W^{(N)}}{\partial u_1} = x + t u_1 + \frac{\partial \tilde{W}^{(N)}}{\partial u_1} = 0,$$

$$\frac{\partial W^{(N)}}{\partial u_2} = t + \frac{\partial^2 \tilde{W}^{(N)}}{\partial u_1^2} = 0,$$

$$\frac{\partial W^{(N)}}{\partial u_k} = \frac{\partial^k \tilde{W}^{(N)}}{\partial u_1^k}, \quad k = 3, 4, \ldots, N.$$  \hfill (51)
Let us denote $\partial W^{(N)} / \partial u_k = W^{(N)}_k$. One has the relation (see [19] section 6, lemma 2)

$$\begin{pmatrix}
W^{(N)}_{k+1} \\
W^{(N)}_{k+2} \\
\vdots \\
W^{(N)}_{2N}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{pmatrix}
= 
\begin{pmatrix}
p_{k-N+1} \\
p_{k-N+2} \\
\vdots \\
p_k
\end{pmatrix}, \quad k = 0, 1, 2, \ldots ,
$$

(52)

where $t_0 = x$, $t_1 = t$ and $p_l$ are the standard elementary Schur polynomials defined, for non-negative indices, by the relation

$$\exp \left( \sum_{j \geq 1} N u_j z^j \right) = \sum_{i \geq 0} p_i(u) z^j$$

(53)

and $p_i = 0$ if $i < 0$. So

$$\frac{\partial u_N}{\partial t_k} = \frac{p_k}{W^{(N)}_{N+1}},$$

$$\frac{\partial u_{N-1}}{\partial t_k} = \frac{p_k W^{(N)}_{N+2}}{(W^{(N)}_{N+1})^2} - \frac{p_{k-1}}{W^{(N)}_{N+1}},$$

$$\vdots$$

$$\frac{\partial u_l}{\partial t_k} = C_{N-l+1} \frac{1}{(W^{(N)}_{N+1})^{N-l+1}} + C_{N-l} \frac{1}{(W^{(N)}_{N+1})^{N-l}} + \cdots + C_{1} \frac{1}{(W^{(N)}_{N+1})^{N}},$$

(54)

with some suitable coefficient $C_m$ depending on $u$.

The GC of order $k$ is defined by

$$\partial_w^{s} W^{(N)} = 0, \quad s = 1, 2, \ldots , N + k, \quad \partial_{w_0}^{N+k+1} W^{(N)} \neq 0.$$  

(55)

The expansion around the points $x_0$, $t_0$ and $u_0$ for $i = 1, 2, \ldots , N$ (with subtraction of the Galilean transformation) is given by

$$u_i = u_{0,i} + \alpha_i t_1, \quad i = 1, 2, \ldots , N, \quad t = t_0 + \epsilon^\beta \tau, \quad x = x_0 - u_1 t^\beta \tau + \epsilon^\gamma y.$$  

(56)

One gets

$$W^{(N)} = W^{(N)}_0 + \epsilon^\gamma u_{10} + \epsilon^{\alpha_1 + \alpha_2} v_1 y + \frac{1}{2} \epsilon^\beta v_1^2 \tau + \frac{1}{2} \epsilon^{\beta + 2\alpha_2} v_1^2 \tau - \frac{1}{2} \epsilon^\beta u_{10}^2 \tau + \epsilon^\beta u_2 \tau + \sum_{L} A_{N+k+1}^{(N)} l_1^{\alpha_1} l_2^{\alpha_2} \cdots l_n^{(N)\tau} \epsilon y$$

(57)

where $L = \{ L : \sum_{k=1}^N L_k = N + k + 1 \}$ and $A_L = \partial_{w_0}^{L} W^{(N)} |_{w_0 = u_{00}}$. Complete balance is achieved if

$$\alpha_1 + \gamma = \beta + 2\alpha_1 = \beta + \alpha_2 = \alpha_1 (N + k + 1).$$

(58)

At $\alpha_1 = 1$ we have

$$\alpha_k = k \quad s = 1, 2, \ldots , N, \beta = N + k - 1, \quad \gamma = N + k$$

and
\[ W^{(N)} = W_0^{(N)} + e^{N+1}u_{10}^3 + e^{N+1} \left( -\frac{1}{2}(u_{10})^2 + u_{20} \right) \tau + e^{N+k+1}W_k^{(N)+} + o(e^{N+k+1}), \]  

(60)

where

\[ W_k^{(N)+} = yP_1(v) + \tau P_2(v) + A_{N+k+1}P_{N+k+1}(v). \]  

(61)

At the critical point of \( W^* \) one has

\[ \frac{\partial W^{(N)*}}{\partial v_1} \bigg|_{v_1=0} = y + \tau v_1 + A_{N+k}P_{N+k}(v) = 0, \]  

(62)

\[ \frac{\partial W^{(N)*}}{\partial v_1} \bigg|_{v_1=0} = \tau + A_{N+k}P_{N+k-1}(v) = 0, \]  

\[ \frac{\partial W^{(N)*}}{\partial v_1} \bigg|_{v_1=0} = A_{N+k+1}P_{N+k-l+1}(v) = 0, \quad l = 3, \ldots, N. \]

In particular, for \( N = 2 \)

\[ W^{(2)*} = v_1y + \epsilon^2 \left( \frac{1}{2}(v_1)^2 + v_2 \right) \tau + \epsilon^3 P_{k+3}(v_1, v_2) \]  

(63)

and the hodograph equations are

\[ y + v_1\tau + A_{k+3}P_{k+2}(v_1, v_2) = 0, \]  

\[ \tau + A_{k+3}P_{k+1}(v_1, v_2) = 0, \quad k = 1, 2, 3, \ldots. \]  

(64)

The function \( (W^{(N)})^* \) and equation (62) coincide with (48) and (49) with the choice \( W^{(N)} = A_{N+k+1}P_{N+k+1}(v) \). For each \( k \), the solutions of equation (62) obey the Jordan system (45) with the substitution \( x \rightarrow y, t \rightarrow \tau \) and \( u_l \rightarrow v_l \), and provide us with the solutions exhibiting the \( k \)th order GC.

The equation (64) defines the mapping \( (v_1, v_2) \rightarrow (y, \tau) \) given by

\[ \begin{cases} 
  y = A_{k+3}(v_1P_{k+1}(v_1, v_2) - P_{k+2}(v_1, v_2)), \\
  \tau = -A_{k+3}P_{k+1}(v_1, v_2).
\end{cases} \]  

(65)

At \( k = 1 \) one has the mapping

\[ \begin{cases} 
  \frac{y}{\tau} = \frac{1}{\tau}(v_1)^3, \\
  \frac{\tau}{\tau} = -\frac{1}{2}(v_1)^2 - v_2.
\end{cases} \]  

(66)

while at \( k = 2 \) it is

\[ \begin{cases} 
  \frac{y}{\tau} = \frac{1}{\tau}(v_1)^4 + \frac{1}{2}(v_1)^2v_2 - \frac{1}{2}(v_2)^2, \\
  \frac{\tau}{\tau} = -\frac{1}{2}(v_1)^3 - v_1v_2.
\end{cases} \]  

(67)

One can rewrite equation (64) as a single equation for \( v_1 \). At \( k = 1 \) one gets

\[ y = \frac{A_3}{3}(v_1)^3 = 0, \]  

(68)

and at \( k = 2 \) one has
Then, \( u = \frac{A_{5}}{2v_{4}A_{5}} + \frac{(\tau)^{2}}{2v_{4}} + \frac{2}{3}v_{1}\tau = 0. \) (69)

The formula (62) implies that near to the GC point \( x_{0}, t_{0}, u_{0} \) one has

\[
\frac{\partial u_{l}}{\partial y} \sim (y)^{-\frac{(N+k-l)}{(N+k)}} \quad l = 1, 2, \ldots, N. \quad (70)
\]

This formula gives us the dominant (most singular) behavior of \( u_{l} \) at the point of the \( k \)th order GC. Dominant and non-dominant terms can be obtained from the formulae (52) and (54). Since at the \( k \)th order GC \( W_{N+m} \sim \epsilon^{k-(m-1)} \) for \( m = 1, 2, \ldots, N \) and as \( \delta \tau \to 0 \), with \( \alpha_{1} = 1 \), we have \( \delta x \sim \epsilon^{N+k} \), one gets from (54)

\[
u_{N} \sim \epsilon^{-k} \sim (\delta x)^{-k/(N+k)},
\]

\[
u_{N-1} \sim d_{1} \epsilon^{-k} \sim d_{2} \epsilon^{-(k+1)} \sim \tilde{d}_{1}(\delta x)^{-k/(N+k)} + \tilde{d}_{2}(\delta x)^{-(k+1)/(N+k)},
\]

\[
u_{l} \sim \sum_{i=1}^{N-l} c_{i} \epsilon^{-(k+l-1)} \sim \sum_{i=1}^{N-l} \tilde{c}_{i}(\delta x)^{-(k+l-1)/(N+k)}, \quad l = 1, 2, \ldots, N,
\]

where \( d, \tilde{d}, c, \tilde{c} \) are certain constants.

The presence of nondominant (less singular terms) in the formulae describing the behavior of derivatives \( u_{l} \) at the point of the GCs allows us to equilibrate the singular terms of different orders in equation (45). This is a novel and important feature of the multi-component system (45) in comparison with the BH equation (1).

In order to recover these nondominant terms within the multiscale expansion one has to modify the formulae (56) and (59). Namely, in the generic case \( u_{10} \neq 0 \), one should consider the expansion near the GC point of order \( k \) in the form

\[
x = x_{0} + \epsilon^{N+k} y, \quad t = t_{0} + \epsilon^{N+k} \tau, \quad u_{l} = u_{0} + \sum_{m=l}^{N} \epsilon^{m} u_{lm},
\]

for \( l = 1, 2, \ldots, N \). Using (72), one immediately gets the formulae (71) for \( u_{l} \sim \frac{d_{l}}{\epsilon^{N+k}} \). Then, straightforward calculations show that the nonleading terms \( u_{lm}, m = l + 1, \ldots, N \) in the formulae (72) for \( u_{l} \) only give contributions for terms of order \( \epsilon^{N+k+2} \) and higher in the expansion of \( W(x, t, u) \) near to the point \( x_{0}, t_{0}, u_{0} \), leaving unchanged the dominant term of order \( \epsilon^{N+k+1} \), namely \( \epsilon^{N+k+1} W_{k}^{(N)}(u_{11}, \ldots, u_{N0}) \).

For example, in the simplest case \( N = 2 \) and \( k = 1 \) one has

\[
x = x_{0} - u_{10} \epsilon^{2} \tau + \epsilon^{3} y, \quad t = t_{0} + \epsilon^{2} \tau, \quad u_{1} = u_{10} + \epsilon u_{11} + \epsilon^{2} u_{12}, \quad u_{2} = u_{20} + \epsilon^{2} u_{22},
\]

and

\[
W = W(x_{0}, t_{0}, u_{10}, u_{20}) + \epsilon^{3} \left( u_{10} y + \left( \frac{1}{2} u_{10}^{2} + u_{20} \right) \tau \right) + \epsilon^{4} \left( \xi u_{11} + A_{3} P_{3}(u_{11}, u_{22}) \right)
\]

\[
+ \epsilon^{5} \left( (\xi + A_{4} P_{4}(u_{11}, u_{22})) u_{12} + A_{5} P_{5}(u_{11}, u_{22}) \right) + \ldots \quad (74)
\]

One has a similar situation in the particular case \( u_{10} = 0 \).

Finally, we note that due to the formulae (70) and (71), the dominant (most singular) behavior of the derivative \( u_{l} \) near the point of the GC is the same for a given \( N + k \). In particular, the dominant singularity of \( u_{l} \) at the \( k \)th order GC for the \( N \)-component case coincides with that of the \( k - 1 \)th order GC for the \( N + 1 \)-component case. At the \( k \)th order GC for the
two-component case, the derivative $u_{1x}$ dominantly behaves like that of $u_{1x}$ for the first-order GC in the $k + 1$-component system.

4. Regularization of gradient catastrophes for the Jordan systems

The observation made at the end of the previous section on the disappearance of the first and subsequent GCs after increasing the number of dependent variables $u_i$ suggests a way to regularize GCs for Jordan systems.

Let us start with the two-component system

$$u_{1t} = u_1 u_{1x} + u_{2x}, \quad u_{2t} = u_1 u_{2x}.$$  \hspace{1cm} (75)

This describes the dynamics of the critical points of the function

$$W^{(2)}(x, t, u_1, u_2) = xu + t \left( \frac{u_1^2}{2} + u_2 \right) + \tilde{W}^{(2)}(u_1, u_2)$$  \hspace{1cm} (76)

where $W^{(2)}(x, t, u_1, u_2)$ (and consequently $\tilde{W}^{(2)}(u_1, u_2)$) satisfies the heat equation

$$W^{(2)}_{u_2} = W^{(2)}_{u_1 u_1}.$$  \hspace{1cm} (77)

The first GC happens when

$$W^{(2)}_{u_1 u_1} = W^{(2)}_{u_1 u_2} = 0,$$  \hspace{1cm} (78)

and hence the second differential

$$d^2 W^{(2)} = \sum_{i,k=1}^{2} \frac{\partial^2 W^{(2)}}{\partial u_i \partial u_k} du_i du_k$$  \hspace{1cm} (79)

is degenerate, i.e.

$$\begin{bmatrix} W^{(2)}_{u_1 u_1} & W^{(2)}_{u_1 u_2} \\ W^{(2)}_{u_1 u_1} & W^{(2)}_{u_1 u_2} \end{bmatrix} = 0.$$  \hspace{1cm} (80)

A way to avoid this degeneracy is to expand our two-dimensional space $(u_1, u_2)$ to a three-dimensional one with the local coordinates $(u_1, u_2, u_3)$ and to consider there a function $W^{(3)}(u_1, u_2, u_3)$, which in addition to equation (77), i.e.

$$W^{(3)}_{u_1} = W^{(3)}_{u_1 u_1},$$  \hspace{1cm} (81)

also obeys the equation

$$W^{(3)}_{u_2} = W^{(3)}_{u_1 u_1}.$$  \hspace{1cm} (82)

In this 3-dimensional space, the condition (78) of the first GC in the case $N = 2$ becomes part of the equations

$$W^{(3)}_{u_1} = W^{(3)}_{u_2} = W^{(3)}_{u_3} = 0,$$  \hspace{1cm} (83)

defining the critical point of function $W^{(3)}$. The second differential

$$d^2 W^{(3)} = \sum_{i,k=1}^{3} \frac{\partial^2 W^{(3)}}{\partial u_i \partial u_k} du_i du_k.$$

\hspace{1cm} (84)
is nondegenerate since
\[
\begin{vmatrix}
W_{u_1 u_1}^{(3)} & W_{u_1 u_2}^{(3)} & W_{u_1 u_3}^{(3)} & W_{u_1 u_4}^{(3)} \\
W_{u_2 u_1}^{(3)} & W_{u_2 u_2}^{(3)} & W_{u_2 u_3}^{(3)} & W_{u_2 u_4}^{(3)} \\
W_{u_3 u_1}^{(3)} & W_{u_3 u_2}^{(3)} & W_{u_3 u_3}^{(3)} & W_{u_3 u_4}^{(3)} \\
W_{u_4 u_1}^{(3)} & W_{u_4 u_2}^{(3)} & W_{u_4 u_3}^{(3)} & W_{u_4 u_4}^{(3)}
\end{vmatrix} = \begin{vmatrix} 0 & 0 & \partial^4_{u_1} W^{(3)} \\
0 & \partial^4_{u_2} W^{(3)} & \partial^4_{u_2} W^{(3)} \\
0 & \partial^4_{u_3} W^{(3)} & \partial^4_{u_3} W^{(3)} \\
0 & \partial^4_{u_4} W^{(3)} & \partial^4_{u_4} W^{(3)} \end{vmatrix}_{ij} \neq 0,
\]

(85)
in the regular sector for the 3-component Jordan system defined by \( \partial^4_{u_1} W^{(3)} \neq 0 \).

So the addition of the new variable \( u_3 \) and the transition from the function \( W^{(2)} \) to the function \( W^{(3)} \) obeying (81) and (82), and consequently from the system (75) to the system
\[
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = \begin{pmatrix} u_1 & 1 & 0 \\
0 & u_1 & 1 \\
0 & 0 & u_1
\end{pmatrix} \begin{pmatrix} u_1 \\
u_2 \\
u_3
\end{pmatrix},
\]

(86)
regularizes the first GC for the system (75).

However, the system (86) has its own GCs. The first one appears when \( \partial^4_{u_1} W^{(3)} = 0 \). In order to regularize it we add a new variable \( u_4 \) and consider the functions \( W^{(4)}(u_1, u_2, u_3, u_4) \), obeying the conditions
\[
W_{u_4 u_4}^{(4)} = \partial^4_{u_4} W^{(4)}, \quad k = 2, 3, 4.
\]

(87)
The condition \( \partial^4_{u_1} W^{(3)} = 0 \) becomes the criticality condition \( W_{u_4 u_4}^{(4)} = 0 \) and the second differential is nondegenerate in the regular sector \( \partial^4_{u_4} W^{(4)} \neq 0 \). Repeating this process iteratively, one arrives at the \( N \)-component Jordan system (45) for which the condition for critical points is (49), i.e.
\[
\partial^k_{u_1} W^{(N)} = 0, \quad k = 1, 2, \ldots, N.
\]

(88)
The system (45) represents the regularization of the first-order GCs for all the Jordan systems with a smaller number of fields (from two to \( N - 1 \)).

Moreover, this provides us with the regularization of the higher order GCs for these systems. Indeed, let us consider again the system (75). Its second-order GC is verified if
\[
\partial^3_{u_1} W^{(2)} = \partial^4_{u_1} W^{(2)} = 0.
\]

(89)
Its regularization is achieved by expanding the two-dimensional space \((u_1, u_2)\) to the four-dimensional space with coordinates \((u_1, u_2, u_3, u_4)\), and considering the critical powers of the function \( W^{(4)}(u_1, u_2, u_3, u_4) \), obeying the equations
\[
W_{u_4 u_4}^{(4)} = \partial^4_{u_4} W^{(4)}, \quad k = 2, 3, 4.
\]

(90)
The critical points of such functions are governed by the system (45) with \( N = 4 \) and the second differential of the potential \( W^{(4)} \) is regular when \( \partial^4_{u_4} W^{(4)} \neq 0 \). In order to regularize the \( k \)th order GCs for the system (75), one introduces \( k \) new variables \( u_5, \ldots, u_{k+2} \) and a function \( W^{(k+2)}(u_1, \ldots, u_{k+2}) \) obeying the equations
\[
W_{u_4 u_4}^{(k+2)} = \partial^4_{u_4} W^{(k+2)}, \quad l = 2, \ldots, k + 2.
\]

(91)
Its critical points are governed by the system (45) with \( N = k + 2 \), which provides us with the regularization of all GCs for a system (75) of orders 1, \ldots, \( k \). In a similar way, one regularizes higher GCs for the system (86) and so on. Thus, the system (45) gives us the regularization of GCs of different orders for the Jordan systems with a smaller number of components.
In order to regularize the GCs of all orders one should proceed to the formal limit \( N \to \infty \). In this limit one has the infinite Jordan system [19]

\[ u_{lt} = u_1 u_{lx} + u_{l+1}, \quad l = 1, 2, 3, \ldots \]  

(92)
i.e. the Jordan chain (2).

We would like to emphasize that the method of regularizing singularities of different types by expanding the space of variables is well-known and has been widely used in various branches of mathematics and physics (e.g. blow-ups). Here it is realized for a specific system and under specific requirements.

5. Parabolic regularization of gradient catastrophes for the Burgers–Hopf equation by the Jordan chain

Here we turn back to the BH equation (1) and will apply to it the method of regularization of GCs by adding new degrees of freedom (expansion of the space of dependent variables). We intend to apply this method within a sort of minimalistic approach; minimalistic in the following sense:

1. In physics, dissipation and viscosity are the principal phenomena which regularize the GC [1, 2]. Both of them are of a parabolic nature. So, our first requirement is that the extensions (regularizations) of the BH equation to larger systems should belong to the class of parabolic systems of PDEs of the first order.

2. The extended systems should incorporate the transport equation for the BH equation, i.e. the equation \( s_t = u s_x \), which is one of the important equations in hydrodynamics. For example, it is the equation for entropy in adiabatic processes [1, 2].

3. The extended systems should be minimal in the sense that they should have the same number of characteristic speeds as the BH equation, i.e. only one.

4. Such extended systems can be viewed as those governing the dynamics of the critical points of certain functions.

Within this approach, the first and simplest step to extend the BH equation is the following one:

\[ \begin{cases} u_t = u u_x \\ s_t = u s_x \end{cases} \rightarrow \begin{cases} u_t = u_1 u_{1x} + u_2 \\ s_t = u_1 u_{2x} \end{cases} \]  

(93)

Note that due to the invariance of the transport equation under the transformation \( u_2 \to f(u_2) \) with the arbitrary function \( f(u_2) \), the equation \( u_t = u_1 u_{1x} + a(u_2) u_{2x} \) is equivalent to that in (93). The system (93) describes the dynamics of the critical points of the functions

\[ W^{(2)}(u_1, u_2) = xu_1 + \left( \frac{1}{2} u_1^2 + u_2 \right) + \tilde{W}^{(2)}(u_1, u_2) \]  

(94)

where \( W_2 \) obeys the parabolic equation

\[ W^{(2)}_{u_2} = W^{(2)}_{u_1 u_1}. \]  

(95)

The condition \( W_{uu} = 0 \) of the first GC for the BH equation becomes the condition \( W^{(2)}_{uu} = W^{(2)}_{u_1 u_1} = 0 \) of the critical point for \( W^{(2)} \) and the degenerate second-order differential \( dW = W_{uu} (du)^2 \) becomes nondegenerate
$$dW^{(2)} = 2 \sum_{i,j=1}^2 W_{u_i u_j}^2 du_i du_j. \quad (96)$$

Thus, the first gradient catastrophe of the BH equation is regularized by the passage given in (93).

However, this extension does not regularize the second GC of the BH equation. Comparing (32) at $k = 2$ and (70), (71) with $N = 2$, $l = 1$, we observe that $u_{1x}$ at the second GC for the BH equation has the same singular behavior $(\delta x)^{-2/3}$ as $u_{1x}$ at the first GC for the two-component Jordan system. So, in order to regularize the second GC for the BH equation one should add two new dependent variables to the BH equation or one new variable to the two component system (93). In both ways, one ends up with the 3-component Jordan system.

Continuing this process, one regularizes the first $N - 1$ GCs of the BH equation passing to the $N$-component Jordan system (45). In order to regularize all GCs for the BH equation, one should pass formally to the infinite Jordan system (92)

$$u_{ll} = u_1 u_{lx} + u_{l+1} x, \quad l = 1, 2, 3, \ldots \quad (97)$$

This Jordan chain represents the minimal parabolic regularization of the BH equation (1).

In fact, the infinite Jordan system (97) was derived in a different context fifteen years ago. As an infinite chain for moments of the finite component $\epsilon$-systems, it was obtained in [20]. It also appeared as a very particular example within the classification of integrable hydrodynamic chains [21]. However, as an infinite system (92), it appeared for the first time (in an equivalent form) in [14] as the first flow

$$\frac{\partial y_n}{\partial t_1} = y_1 \frac{\partial y_n}{\partial x} - y_n \frac{\partial y_1}{\partial x} + \frac{\partial y_{n+1}}{\partial x}, \quad n = 1, 2, 3, \ldots \quad (98)$$

of the universal hierarchy of hydrodynamical type. Indeed, one can show that the chains (92) and (98) are connected by the simple invertible change of variables

$$y_n = p_n(u_1, u_2, \ldots), \quad n = 1, 2, 3, \ldots \quad (99)$$

where $p_n$ are the elementary Schur polynomials with an infinite set of variables. The relation (99) relates both infinite hierarchies too.

The Martínez Alonso–Shabat (MAS) hierarchy has been widely studied [14, 22, 23]. For example, it was shown that the finite-component reductions of the MAS chain obtained by imposing $y_l = 0$ for $l > N$ represent diagonalizable integrable hydrodynamic systems themselves. It should be noted that due to the relation $\exp \left( \sum_{n=1}^\infty u_n z^n \right) = \sum_{n=0}^\infty y_n z^n$, the above finite-component reduction of the MAS chain is equivalent to the algebraic reduction of the infinite Jordan chain. Vice versa, the $N$-component Jordan system is an algebraic reduction of the infinite MAS chain. It seems that such a reduction of the MAS chain has not been studied before.

### 6. Parabolic regularizations averaging with the generalized airy distribution

The formal regularization procedure described in the previous sections has an explicit probabilistic realization. First, one observes that the general solution of the system of PDEs (49) can be written as

$$W^{(N)}(u_1, \ldots, u_N) = \int_{-\infty}^{+\infty} d\lambda f(\lambda) \exp \left( \sum_{k=1}^N (i\lambda)^k u_k \right). \quad (100)$$
where \( f(\lambda) \) is an arbitrary function. Equation (49) represents a subsystem of PDEs which define the generalized Airy function considered in [24]. We will refer to the function (100) as generalized Airy-type functions. Since

\[
W^{(1)}(u) \equiv W^{(N)}(u, 0, \ldots, 0) = \int_{-\infty}^{+\infty} d\lambda f(\lambda) \exp(i\lambda u),
\]

one can rewrite (100) as

\[
W^{(N)}(u_1, \ldots, u_N) = \int_{-\infty}^{+\infty} du \ W^{(1)}(u) G^{(N)}(u_1 - u, u_2, \ldots, u_N),
\]

where

\[
G^{(N)}(u_1 - u, \ldots, u_N) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \exp \left( i\lambda(u_1 - u) + \sum_{k=2}^{N}(i\lambda)^k u_k \right).
\]

For even \( N \), the function \( G^{(N)}(u_1 - u, u_2, \ldots, u_N) \) is well-defined for \( u_N > 0 \) if \( N = 4n + 2 \) and \( u_N < 0 \) if \( N = 4n \) with \( n = 0, 1, 2, \ldots \).

One immediately has

\[
\int_{-\infty}^{+\infty} du \ G^{(N)}(u_1 - u, u_2, \ldots, u_N) = 1, \quad N = 1, 2, \ldots.
\]

So, all \( G^{(N)} \) represent densities of one-dimensional probability distributions depending on the \( N \) parameters \( u_1, u_2, \ldots, u_N \). The function \( W^{(N)}(u_1, \ldots, u_N) \) is the mean value of \( W^{(1)}(u) \)

\[
W^{(N)}(u_1, \ldots, u_N) = \langle W^{(1)}(u) \rangle_N.
\]

In particular

\[
\left\langle \frac{u^n}{n!} \right\rangle_N = p_n(u_1, \ldots, u_N).
\]

Indeed

\[
\left\langle \frac{u^n}{n!} \right\rangle_N = \frac{1}{2\pi n!} \int_{-\infty}^{+\infty} du (u^n) \int_{-\infty}^{+\infty} d\lambda \exp \left( -i\lambda u + \sum_{k=1}^{N}(i\lambda)^k u_k \right)
\]

\[
= \frac{1}{2\pi n!} \int_{-\infty}^{+\infty} d\lambda \exp \left( \sum_{k=1}^{N}(i\lambda)^k u_k \right) i^n \frac{\partial^n}{\partial \lambda^n} \int_{-\infty}^{+\infty} du \exp(-i\lambda u)
\]

\[
= \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d\lambda \delta(\lambda) i^n \frac{\partial^n}{\partial \lambda^n} \exp \left( \sum_{k=1}^{N}(i\lambda)^k u_k \right) = p_n(u_1, \ldots, u_N)
\]

where \( \delta(\lambda) \) is the Dirac-delta function. Consequently, for

\[
W^{(1)}_m(u) = \sum_{k=1}^{m} \frac{u^k}{k!} t_k + \tilde{W}_1(u)
\]

one has

\[
\left\langle W^{(1)}_m \right\rangle_N = \sum_{k=1}^{m} t_k p_k(u_1, \ldots, u_N) + \tilde{W}_m(u_1, \ldots, u_N) = W^{(N)}_m(u_1, \ldots, u_N).
\]
where

\[ \tilde{W}(u_1, \ldots, u_N; m) \equiv \langle W_m^{(1)} \rangle_N. \quad (110) \]

Moreover, using as before the integration by parts, one obtains

\[ \frac{\partial W_m^{(N)}(u_1, \ldots, u_N)}{\partial u_k} = \langle \frac{\partial W^{(1)}(m)}{\partial u_k} \rangle_N, \quad k = 1, \ldots, N, \quad (111) \]

for the integrable functions \( W^k(u) \) such that \( W^k(u)G^N(u_1 - u, u_2, \ldots, u_N) \to 0 \) as \( |u| \to \infty \). The equations (105) and (111) show that the GC conditions of order \( N - 1 \) (9) for the BH equation after averaging with the probability \( G^{(N)} \) become the equations defining the critical points of the function \( W^{(N)} \). The dynamics of these critical points given by the \( N \)-component Jordan system represent the averaging of the BH equation and of the whole BH hierarchy (33).

This regularization of the BH equation by averaging considered here is similar to the classical Whitham averaging method [1, 25]. Again, analogously to Whitham’s averaging of conservation laws [1], we note that if one performs the averaging of equation \( u_t = (u^2/2)_x \), i.e. the equation (1) by the substitution \( u \to \langle u \rangle_N, u^2/2 \to \langle u^2/2 \rangle_N \), one gets only the first equation of the Jordan system (45), while the averaging of the function \( W^{(1)}(u) \) given by (105) provides us with a complete system (45).

The density \( G^{(N)} \) of the probability distribution plays a central role in such parabolic regularizations. It can be written in different ways. Using the formal relation (53), one gets

\[ G^{(N)}(u_1 - u, \ldots, u_N) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \exp(-i\lambda u) \sum_{k=0}^{\infty} (-1)^k p_k = \sum_{k=0}^{\infty} (-1)^k p_k(u_1, \ldots, u_N) \delta^{(k)}(u) \quad (112) \]

where \( \delta^{(k)}(u) \) is the \( k \)-th derivative of the Dirac-delta function. So

\[ W^{(N)}(u_1, \ldots, u_N) = \sum_{k=0}^{\infty} \frac{\partial^k W^{(1)}(u)}{\partial (u^k)} \bigg|_{u=0} p_k(u_1, \ldots, u_N). \quad (113) \]

For \( N = 1, 2, 3 \) one has simple explicit expressions for \( G^{(N)} \). At \( N = 1 \) one obviously has

\[ G^{(1)}(u_1 - u) = \delta(u_1 - u). \quad (114) \]

For \( N = 2 \) and \( u_2 > 0 \) one easily gets

\[ G^{(2)}(u_1 - u, u_2) = \frac{1}{\sqrt{4\pi u_2}} \exp \left( -\frac{(u_1 - u)^2}{4u_2} \right). \quad (115) \]

So, in this case one obtains the Gauß distribution, with \( u_1 \) being the maximum and \( 2u_2 \) being the squared width. Thus, the regularization of the first GC for the BH equation is achieved by averaging with the Gauß distribution with finite width \( \sqrt{2u_2} \).

In the case \( N = 3 \) one proceeds in the standard manner using the identity

\[ i(u_1 - u)\lambda - \lambda^2 u_2 - i\lambda u_1 = \frac{2u_1^3}{27 u_2^3} - \frac{(u_1 - u) u_2^2}{3u_3} + i \left( \frac{u_1 - u}{(3u_3)^{2/3}} - \frac{u_2}{(3u_3)^{2/3}} \right) \lambda' - i \frac{\lambda'^3}{3} \quad (116) \]

where

\[ \lambda' = (3u_3)^{1/3} \lambda - \frac{i u_2}{(3u_3)^{2/3}}. \quad (117) \]

One gets
\[ G^{(3)} = \frac{1}{(3u_3)^{1/3}} \exp \left( \frac{2u_2^3}{27u_3^2} - \frac{(u_1 - u)u_2^3}{3u_3} \right) \times \text{Ai} \left( \frac{u_2}{(3u_3)^{4/3}} - \frac{u_1 - u}{(3u_3)^{1/3}} \right) \]  

(118)

where Ai is the Airy function

\[ \text{Ai}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \exp \left( -i\lambda z - i\lambda^3/3 \right) \]  

(119)

For \( N = 4 \) one gets

\[ G^{(4)}(u_1 - u, u_2, u_3, u_4) = \frac{1}{a} \exp \left( \frac{3u_4^4}{256a^4} + \frac{u_2u_3^2}{16a^3} + \frac{(u_1 - u)u_2}{4a^2} \right) \times \Lambda \left( \frac{3u_2^2}{8a^2} + \frac{u_3^3}{8a} + \frac{u_2u_3}{2a^2} + \frac{u_4}{a} \right) \]  

(120)

where \( a = (-u_4)^{1/4} \) and

\[ \Lambda(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \exp \left( i\lambda y - \lambda^2 x - \lambda^4 \right) \]  

(121)

is a Pearcey function. So, the regularization of the second- and third-order GCs for the BH equation is achieved by averaging with the Airy and Pearcey distributions (118) and (120).

The functions \( G^{(N)}(u_1 - u, u_2, \ldots, u_N) \) decay exponentially like \( \exp(-u^N/(N-1)) \mu^a \) (with a suitable \( a \)) at one of the infinities (depending on \( N \)). So, the regularization of the \( k \)th order GC for the BH equation with \( u_t \sim (\delta x)^{-k/(k+1)} \) (32) requires the averaging with the probability densities, which decrease at one of the infinities as \( \exp(-u^{(k+1)/k}) \).

### 7. Jordan chain and Burgers equation

The Burgers equation

\[ \phi_t = 2\phi\phi_x + \nu\phi_{xx}, \]  

(122)

is the most known and straightforward regularization for the BH equation due to the effects of viscosity and dissipation [1, 2]. Here, we will show that the regularization of the BH equation provided by the Jordan chain (92) and Burgers equation are strictly connected.

Consider the hierarchy of infinite Jordan systems [15], the first member of which is given by (92).

Note that for higher Jordan flows one has

\[ \frac{\partial u_k}{\partial t_k} = p_k u_{1x} + p_{k-1} u_{2x} + \cdots + u_{k+1} = \frac{\partial p_{k+1}}{\partial x}, k = 2, 3, 4, \ldots, \]  

(123)

where \( p_k \) are elementary Schur polynomials. The first equation (45)

\[ u_{1t} = u_1 u_{1x} + u_2, \]  

(124)

is obviously converted into the Burgers equation (122) under the differential constraint

\[ u_2 = \frac{u_1^2}{2} + \nu u_{1x}. \]  

(125)

The rest of equation (45) defines all other \( u_t \) fields in terms of \( u_1 \), namely,
\[ u_3 = \frac{1}{3} u_1^3 + 2 \nu u_1 u_{1x} + \nu^2 u_{1xx} \]
\[ u_4 = \frac{1}{4} u_1^4 + 3 \nu u_1^2 u_{1x} + 3 \nu^2 u_1 u_{1xx} + \frac{5}{2} \nu^2 u_{1x}^2 + \nu^3 u_{1xxx} \]  
(126)

\[ \ldots \]

So the constraints (124) and (125) define an admissible reduction of the Jordan chain (45) into the single Burgers equation.

Further, the elementary Schur polynomials under this reduction become the differential polynomials defined by recursion

\[ p_{k+1} = (\nu \partial_x + u_1)^k u_1, \quad k = 1, 2, 3, \ldots \]  
(127)

Hence, under the constraints (124) and (125) the equation (123) becomes

\[ \frac{\partial u_1}{\partial t_k} = \left( (\nu \partial_x + u_1)^k u_1 \right)_x, \quad k = 1, 2, 3, \ldots \]  
(128)

which is nothing but the well-known infinite Burgers hierarchy. Thus, the whole hierarchy of Jordan chains is converted into the unconstrained hierarchy of Burgers equations. The fact that the universal hydrodynamic type hierarchy admits the reduction to the Burgers hierarchy was observed for the first time in [14].

Note that the generating function (53) of the elementary Schur polynomial for the Jordan chain under the reduction (125) becomes

\[ G = \sum_{k=0}^{\infty} (z(\nu \partial_x + u_1))^k \cdot 1 = (1 - z(\nu \partial_x + u_1))^{-1} \cdot 1. \]  
(129)

In this Burgers reduction, the distribution \( G^{(\infty)}(u_1 - u) \) is given by

\[ G^{(\infty)}(u_1 - u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \, e^{-i\lambda u} \left( 1 - \nu(\nu \partial_x + u_1) \right)^{-1} \cdot 1, \]  
(130)

and

\[ W^{(\infty)}(u_1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda f(\lambda) \left( 1 - \nu(\nu \partial_x + u_1) \right)^{-1} \cdot 1. \]  
(131)

The operator \( (\nu \partial_x + u_1)^{-1} \) is the resolvent of the operator \( L^* = \nu \partial_x + u_1 \), which is adjoint to the ‘Lax’ operator \( L = -\nu \partial_x + u_1 \) for the well-known ‘Lax pair’ for the Burgers equation, i.e.

\[ \begin{cases} L \psi = (-\nu \partial_x + u_1) \psi, \\ \psi_t = \nu \psi_{xx}. \end{cases} \]  
(132)

The formulae (130) and (131) establish the relation between the parabolic regularization by the Jordan chain and the method of the resolvent developed a long time ago for integrable systems [26, 27].

Now let us proceed in a converse direction. Consider a solution of the Burgers equation (122) and define the infinite set of functions

\[ u_1 = \phi, \quad u_2 = \frac{\phi^2}{2} + \nu \phi_x, \quad u_3 = \frac{\phi^3}{3} + 2 \nu \phi \phi_x + \nu^2 \phi_{xx}, \ldots \]  
(133)
constructed recursively via
\[(u_{n+1})_x = (u_n)_t - \phi(u_n)_x.\]  
(134)

In terms of these ‘momenta’ \(u_k\), the Burgers equation becomes the Jordan chain (92). So the Burgers equation (122) provides us with a class of solutions for the Jordan chain with functional dependencies \(u_1, u_2, u_3, \ldots\). The system (92) is the form of equation (122) in terms of a countable set of ‘momenta’.

The momenta \(u_k\) defined by (133) and (134) have a very simple meaning in terms of the variable \(\psi\) converting the Burgers equation by the Cole–Hopf transformation \(\phi = \nu \psi_{xx}/\psi\) into the heat equation \(\psi_t = \nu \psi_{xx}\). Indeed, it is simple to check that the definitions (133) are equivalent to the following one (see equation (132))
\[\nu \partial^3 \psi / \partial x^3 \equiv p_n(u_1, u_2, \ldots),\]  
(135)

where \(p_n\) are elementary Schur polynomials evaluated at (133). For \(n = 1\), it is the Cole–Hopf transformation, while for \(n \geq 2\) it can be viewed as an extension to the jet space. Substituting \(\psi = \frac{1}{\nu} \int \phi(x', t) \, dx'\) into (135), one recovers the formulae (133).

The \(N\)-component Jordan system (45) is related to the Burgers equation too, however with constraints. Let us take the two-component Jordan system and impose the constraint (125). The first equation of the Jordan system is reduced to the Burgers equation, while the second equation becomes the constraint
\[\frac{1}{3} u_1^3 + 2\nu u_1 u_{1x} + \nu^2 u_{1xx} = \text{const}.\]  
(136)

Starting with the Burgers equation, introducing \(u_1\) and \(u_2\) as in (133) and requiring that \(u_1\) and \(u_2\) satisfy the two-component Jordan system, one ends up with the Burgers equation together with the constraint
\[\frac{1}{3} \phi^3 + 2\nu \phi_x + \nu^2 \phi_{xx} = \text{const}.\]  
(137)

So the two-component Jordan system under the constraint (125) is equivalent to the Burgers equation under the constraint (137). In a similar way, one shows that the 3-component Jordan system under the constraint (133) is equivalent to the Burgers equation constrained by
\[\frac{1}{3} \phi^3 + 2\nu \phi_x + \nu^2 \phi_{xx} = \text{const}.\]  
(138)

In general, the \(N\)-component Jordan system under the constraint (133) is equivalent to the Burgers equation with the constraint
\[u_{N+1}(\phi) = \text{const},\]  
(139)

where \(u_{N+1}(u_1)\) are given in (134). The constraint (139) has a simple form in terms of the function \(\psi\). For instance, at \(N = 2\) it is
\[\frac{\psi_{xxx}}{\psi} + \frac{\psi_x \psi_{xx}}{\psi^2} + \frac{\psi^3_x}{3\psi^3} = \text{const}.\]  
(140)

It is noted that these reductions of the Burgers equation are different from the \(N\)-phase reductions considered in [14].
8. Jordan chain as the universal regularization of the BH equation

In our construction of the Jordan chain, it arose as the inductive limit when \( N \to \infty \) of the \( N \)-component parabolic system with the most degenerate Jordan blocks. Viewed in this way, the Jordan chain represents the pure parabolic regularization of all GCs for the BH equation including, in particular, the Burgers equation.

However, if one treats the infinite system (92) as a formal chain without insisting on parabolicity (like for the MAS chain (98)), then the formal Jordan chain offers many other possibilities. Having in mind (122) it is natural to consider a constraint with a second-order derivative. Namely, let us impose the constraint

\[
u^2 = u_{1xx},
\]

(141)

The first equation (92) becomes the KdV equation

\[
u_1 = 3u_1u_{1x} + \frac{1}{4}u_{1xxx},
\]

(142)

while the other equations define the other fields

\[
u_3 = \frac{4}{3}u_1^3 + \frac{5}{8}(u_1)^2 + u_1u_{1xx} + \frac{1}{16}u_{1xxxx}, \quad u_4 = u_1^4 + \ldots,
\]

Thus, under the constraint (141), the Jordan chain becomes the KdV equation. One can show that under this constraint, higher Jordan chains are converted into higher KdV equations. This fact was first observed in [14] for the MAS chain.

On the other hand, starting with the KdV equation (142), and introducing the ‘momenta’ \((v = u_1)\)

\[
u_2 = v^2 + \frac{1}{4}v_{xx}, \quad u_3 = \frac{4}{3}v^3 + \frac{5}{8}(v)^2 + vv_{xx} + \frac{1}{16}v_{xxxx}, \quad \ldots,
\]

(144)

via the recursion formula

\[
u_{n+1x} = \nu_{nt} - \nu_{nxx}, \quad n = 1, 2, 3, \ldots
\]

(145)

one gets the Jordan chain. So, the KdV equation, which provides us a dispersive regularization of the BH equation, is also a particular reduction of the Jordan chain. Many other integrable reductions of the hierarchy of the MAS chain have been found in [14, 22, 23]; however, most of them are not gradient catastrophe free.

The Jordan chain contains other known regularizations of the BH equation, for instance, those of the Hamiltonian [5] and dissipative [6] variety. For a general dissipative regularization the constraint is [6]

\[
u_2 = \epsilon (b_1(u_1)u_{1xx} + c_1(u_1)(u_1)^2) + \epsilon^2 (b_2(u_1)u_{1xxx} + c_2(u_1)u_{1x}u_{1xx} + d_2(u_1)(u_1)^3) + \ldots
\]

(146)

while for a Hamiltonian regularization [5]

\[
u_2 = \epsilon (b_1(u_1)u_{1xx} + c_1(u_1)(u_1)^2) + \epsilon^2 (b_2(u_1)u_{1xxx} + c_2(u_1)u_{1x}u_{1xx} + d_2(u_1)(u_1)^3) + \ldots
\]

(147)

where \( \epsilon \) is a small parameter and \( b_i, c_i, d_i, \ldots \) are certain functions [5, 6].

Other methods of regularizing GCs for the BH equation are also related to the Jordan chain. One of the approaches is to incorporate the derivatives and pass from the equation for the critical points of functions to the Euler–Lagrange equations for functionals (see e.g. [28, 29]).
In order to regularize the $k$th order GC for the BH equation, one passes from the function $W_k^*$ (18) to the ‘Lagrangian’

$$\mathcal{L}_k = \frac{a}{2} (v_1)^2 + W_k^*, \quad (148)$$

where $a$ is a constant. The Euler–Lagrange equation for $\mathcal{L}_k$ is

$$au_{yy} - y - \tau v - (k + 2)A_{k+2}t^{k+1} = 0. \quad (149)$$

Solutions of this equation obviously do not exhibit GC. For $k = 1$, the equation (149) is a sort of deformed Painlevé I equation. At $k = 2$ this equation, i.e.

$$au_{yy} - y - \tau v - 4A_4 u^3 = 0, \quad (150)$$

is the equation obtained in [30] within the singularity analysis of the diffusion-type equation and then studied in [31]. For the $k > 2$ equation (149), it was obtained in [28] as a particular PDE system case.

Equation (149) represents the regularization of the GC for the BH equation of orders $k = 1, 2, 3, \ldots$. On the other hand, all of them provide us with particular classes of solutions (with different $k$) of a single PDE. Similarly to the hodograph equation (7), one obtains this equation differentiating (149) w.r.t. $y$ and $\tau$ and eliminating $\tau$ from the obtained system. In the case of equation (149), one gets the equation

$$v_\tau - vv_y + a(v_{yy}\tau_y - v_{y\tau}v_y) = 0 \quad (151)$$

or, equivalently

$$(1 - a\partial_y(v_{yy} - v_1\partial_y))v_\tau = vv_y. \quad (152)$$

For a small $a$ equation (152), at the first order in $a$, looks like

$$v_\tau = vv_y - 3a(v_y^2)v_{yy}. \quad (153)$$

Equation (153) is a very special case of general Hamiltonian deformations of the BH equation studied in [5]. The complete equation (152) is a Jordan chain under the constraint

$$u_1 = v_1, \quad u_{2x} = (au_{1xx} - au_1\partial_y^2)u_1, \quad (154)$$

One has the different regularization of the GCs for the BH equation passing from $W_k^*$ (18) to the Lagrangian

$$\mathcal{L}_k = \frac{a}{2} v_1 v_\tau + W_k^* \quad (155)$$

The Euler–Lagrange equations for $\mathcal{L}_k$ give

$$au_{\tau\tau} - y - \tau v - (k + 2)A_{k+2}t^{k+1} = 0. \quad (156)$$

All equations (156) with $(k = 1, 2, 3, \ldots)$ provide a subclass of solutions for the equation

$$v_\tau - vv_y + a(v_{yy}\tau_y - v_{y\tau}v_y) = 0 \quad (157)$$

which is again a suitable reduction of the Jordan chain.

A similar type of regularization can also be performed for $N$-component Jordan systems. One deforms the functions $W_1^{(N)*}$ (61) to the Lagrangian

$$\mathcal{L}^{(N)} = \frac{a}{2} (v_{1y})^2 + W^{(N)*}, \quad (158)$$
taking into account that the derivative $v_1$ is the most singular one. One gets the following equations:

$$
\begin{align}
&av_{1yy} - y - \tau v_1 + AN_{k+1}P_{n+k}(v) = 0, \\
&\tau - AN_{k+1}P_{n+k-1}(v) = 0, \\
&P_{n+k-l}(v) = 0, \quad l = 3,4,\ldots,N.
\end{align}
$$

These equations represent a completely degenerate parabolic regularization of the $k$th order GC for the $N$-component Jordan system. Of course, there are a number of other regularizations according to the different choices of the differential part for the Lagrangian $\mathcal{L}^{(N)}$. 

So, the Jordan chain can be viewed as a universal regularization of the BH equation. Specific regularizations correspond to its particular reductions. Some of these reductions lead to integrable PDEs, like the Burgers and the KdV equations, for which the Jordan chain hierarchy is reduced to the infinite family of commuting flows. The characterization of such reductions and the associated physical phenomena is an interesting open problem.

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Appendix. First step of the regularization procedure

Here we present some explicit formulae relating to the BH equation and the two-component Jordan system. Let us consider the BH equation with the initial data

$$
\begin{equation}
\begin{split}
&u(x,0) = u_0(x).
\end{split}
\end{equation}
$$

The solution of this Cauchy problem is given implicitly by the hodograph equation

$$
\begin{equation}
W_u \equiv x + ut + \tilde{W}_u(u) = 0
\end{equation}
$$

where $\tilde{W}_u(u)$ is the inverse function of $u_0(x)$. Differentiating (A.2) w.r.t. $x$ and $t$ one gets

$$
\begin{align}
1 + W_{uu}u_t &= 0, \\
\tilde{W}_u &= 0.
\end{align}
$$

At the regular sector ($W_{uu} \neq 0$) one has the formulae (10). The first-order GC appears when

$$
\begin{equation}
W_{uu} = t + \tilde{W}_{uu} = 0.
\end{equation}
$$

Equations (A.2) and (A.4) define the submanifold of codimension one in the space $x, t$ and the parameters $t_2, t_3, \ldots$ (see [8]).

In order to regularize the first GC of the BH equation, we enlarge the space of variables $u \rightarrow (u_1, u_2)$ and pass to the system (44). This system describes the dynamics of the critical points for the functions (76), i.e.

$$
W^{(2)}(x, t, u_1, u_2) = xu_1 + t \left( \frac{u_1^2}{2} + u_2 \right) + \tilde{W}^{(2)}(u_1, u_2)
$$

which obey the equation (77)

$$
W^{(2)}_{u_2} = W^{(2)}_{u_1 u_1}.
$$
The equations for the critical points are
\[ x + \dot{u}_1 + \ddot{W}^{(2)}_{u_1} = 0, \quad t + \ddot{W}^{(2)}_{u_2} = 0. \]  
(A.7)

Differentiating these equations w.r.t. \( x \) and \( t \) one gets
\[
1 + \dot{W}^{(2)}_{u_1} u_{1x} + \dot{W}^{(2)}_{u_2} u_{2x} = 0, \\
\dot{u}_1 + \dot{W}^{(2)}_{u_1} u_{1t} + \dot{W}^{(2)}_{u_2} u_{2t} = 0, \\
1 + \ddot{W}^{(2)}_{u_1} u_{1x} + \ddot{W}^{(2)}_{u_2} u_{2x} = 0, \\
\ddot{u}_1 + \ddot{W}^{(2)}_{u_1} u_{1t} + \ddot{W}^{(2)}_{u_2} u_{2t} = 0. \]  
(A.8)

Taking into account the second equation (A.7), one finally obtains
\[
1 + \dot{W}^{(2)}_{u_1} u_{1x} + \dot{W}^{(2)}_{u_2} u_{2x} = 0, \\
\dot{u}_1 + \dot{W}^{(2)}_{u_1} u_{1t} + \dot{W}^{(2)}_{u_2} u_{2t} = 0, \\
1 + \ddot{W}^{(2)}_{u_1} u_{1x} + \ddot{W}^{(2)}_{u_2} u_{2x} = 0, \\
\ddot{u}_1 + \ddot{W}^{(2)}_{u_1} u_{1t} + \ddot{W}^{(2)}_{u_2} u_{2t} = 0. \]  
(A.9)

At the regular sector, when \( \dot{W}^{(2)}_{u_1} \neq 0 \), one has
\[
\begin{align*}
\dot{u}_{2x} &= -\frac{1}{\dot{W}^{(2)}_{u_1}}, \\
\dot{u}_{1x} &= \frac{\dot{W}^{(2)}_{u_2}}{(\dot{W}^{(2)}_{u_1})^2}, \\
\dot{u}_{2t} &= -\frac{\dot{u}_1}{\dot{W}^{(2)}_{u_1}}, \\
\dot{u}_{1t} &= \frac{\dot{u}_1 \dot{W}^{(2)}_{u_2}}{(\dot{W}^{(2)}_{u_1})^2} - \frac{1}{\dot{W}^{(2)}_{u_1}}.
\end{align*} \]  
(A.10)

Comparing equations (A.3) and (A.9), (A.10), one notes that in the transition from the one-component to the two-component case, even if the condition \( W_{u_2} = 0 \) (defining the first-order \( \mathcal{G} \mathcal{C} \) for the \( \mathcal{B} \mathcal{H} \) equation) is substituted by a similar condition \( W_{u_1} = 0 \), the later one does not lead to a blow up of the derivatives of \( u_1 \) and \( u_2 \). In contrast to equation (A.3), the second equation (A.7) does not define the singular sector, but together with the first equation serves to calculate the two functions \( u_1(x,t) \) and \( u_2(x,t) \). The first \( \mathcal{G} \mathcal{C} \) for the system (44) happens when \( W^{(2)}_{u_1} = 0 \) and the derivative blow up according to the formulae (A.10).

The transition from the \( \mathcal{B} \mathcal{H} \) equation to the two-component Jordan system can be performed, averaging with the probability distribution (115), namely
\[
\begin{align*}
\dot{u}_1 &= (u)_2 = \frac{1}{\sqrt{4\pi u_2}} \int_{-\infty}^{+\infty} du \exp \left( \frac{(u_1 - u)^2}{4u_2} \right), \\
\dot{u}_2 &= \frac{1}{2} ((u_2^2) - (u_2^2)) = \frac{1}{2\sqrt{4\pi u_2}} \int_{-\infty}^{+\infty} du u^2 \exp \left( \frac{(u_1 - u)^2}{4u_2} \right) - \frac{1}{2} u_1^2, \\
W^{(2)}_{u_1} u_2 &= (W(u)_2 = \frac{1}{\sqrt{4\pi u_2}} \int_{-\infty}^{+\infty} du u \exp \left( \frac{(u_1 - u)^2}{4u_2} \right), \\
\frac{\partial^k W^{(2)}(u_1, u_2)}{\partial u_1^k} &= \left( \frac{\partial^k W(u)}{\partial u^k} \right)_2 = \frac{1}{\sqrt{4\pi u_2}} \int_{-\infty}^{+\infty} du \frac{\partial^k W(u)}{\partial u^k} \exp \left( -\frac{(u_1 - u)^2}{4u_2} \right), \quad k = 1, 2, 3, \ldots.
\end{align*} \]  
(A.11)

The averaging of equations (A.2) and (A.4) for the \( \mathcal{B} \mathcal{H} \) equation gives equation (A.7), defining the critical points of \( W^{(2)}(u_1, u_2) \). Thus, this averaging regularizes the first-order \( \mathcal{G} \mathcal{C} \) for the \( \mathcal{B} \mathcal{H} \) equation.

Then, the last equation of (A.11) with \( k = 3 \), i.e.
\[
\frac{\partial^3 W^{(2)}(u_1, u_2)}{\partial u_1^3} = \left( \frac{\partial^3 W(u)}{\partial u^3} \right)_2 = \frac{1}{\sqrt{4\pi u_2}} \int_{-\infty}^{+\infty} du \frac{\partial^3 W(u)}{\partial u^3} \exp \left( -\frac{(u_1 - u)^2}{4u_2} \right),
\]  
(A.12)
implies that $W_{uuu}(u)$ vanishes at some point $(u_0, v_0)$ only if $W_{uuu}(u)$ takes positive and negative values. Assuming its smoothness, one concludes that there exists a point $u_0$ at which $W_{uuu}(u_0) = 0$ and $W_{uuu}(u) \neq 0$. For the BH equation, such a condition is realizable at the second stratum $S_2$, which exhibits the second-order GC. So the first-order GC for the system (44) is related to the second-order GC of the BH equation. Analogously, the last formula (A.11) with $n > 3$ establishes the relation between the $n$th order GC for the BH equation and the GC of order $n − 1$ for the two-component Jordan system.

The process of averaging described above admits a simple physical interpretation. Let us consider the BH equation as the equation which describes the collisionless motion of a cloud of particles of dust with mass $m = 1$ (see e.g. [13, 17]). In addition to the usual picture, we assume that the cloud is an ideal Boltzmann gas at temperature $T$. At $T = 0$ one has the flow of particles with the velocities $u(x, t)$ and a GC of first order. At $T > 0$, the particles have different values of velocities, with the probability given by the Maxwell distribution

$$
\rho(u, u_1, T) = \frac{1}{\sqrt{2\pi T}} \exp \left(-\frac{(u − u_1)^2}{2T}\right).
$$

(A.13)

This is the Gaussian distribution (115) with $u_2 = T/2$. The first two formulae (A.11) are the classical statistical physics formulae; namely, $u_1 = \langle u \rangle$ is the velocity of the macroscopic motion of the cloud and $E = \langle u^2 \rangle = \frac{1}{2}u_1^2 + \frac{T}{2}$ is the mean value of the single particle energy. We also assume that both macroscopic quantities $u_1$ and $T$ depend on $x$ and $t$ in a way that the function $W^{(2)} = xu_1 + tE + \tilde{W}(u_1, T)$ has the extremum. So, one has the system (44) with $u_2 = T/2$. Thus, the first-order GC for dust particle motion is regularized by the thermal effect of the nonzero and variable temperature $T$.

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