A generalized precision matrix for t-Student distributions in portfolio optimization

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Abstract

The Markowitz model is still the cornerstone of modern portfolio theory. In particular, when focusing on the minimum-variance portfolio, the covariance matrix or better its inverse, the so-called precision matrix, is the only input required. So far, most scholars worked on improving the estimation of the input, however little attention has been given to the limitations of the inverse covariance matrix when capturing the dependence structure in a non-Gaussian setting. While the precision matrix allows to correctly understand the conditional dependence structure of random vectors in a Gaussian setting, the inverse of the covariance matrix might not necessarily result in a reliable source of information when Gaussianity fails. In this paper, exploiting the local dependence function, different definitions of the generalized precision matrix (GPM), which holds for a general class of distributions, are provided. In particular, we focus on the multivariate t-Student distribution and point out that the interaction in random vectors does not depend only on the inverse of the covariance matrix, but also on additional elements. We test the performance of the proposed GPM using a minimum-variance portfolio set-up by considering S&P 100 and Fama and French industry data. We show that portfolios relying on the GPM often generate statistically significant lower out-of-sample variances than state-of-art methods.

Keywords: Generalized Precision Matrix, t-Student distribution, heavy tails, portfolio optimization, minimum-variance portfolio

JEL classification: C46, C58, G11

1. Introduction

Even after 70 years, the cornerstone of many sophisticated portfolio approaches is still centered around the risk-return optimization framework developed by [Markowitz 1952]. Especially, the minimum-variance framework is very popular due to its simplicity as the only input necessary for the analytical solution is the precision matrix, which is defined as the inverse of the covariance matrix in the Gaussian setting [Stevens 1998]. More specifically, when considering a $d$-dimensional multivariate Gaussian vector $X = [X_1, X_2, \cdots, X_d]^T$ with mean $\mu$ and covariance $\Sigma$, the precision matrix is defined as $\Omega = \Sigma^{-1}$ and $\omega_{pq}$ is its $(p, q)$-th element.
While most scholars focus on improving the estimation of the precision matrix as it received much criticism (see Bloomfield et al. (1977); DeMiguel et al. (2009); Kritzman et al. (2010) for poor out-of-sample performance; Michaud (1989); Ledoit & Wolf (2004a); Black & Litterman (1990, 1992) for unreliable and extreme weights and Ledoit & Wolf (2004b); Meucci (2009); Won et al. (2013) for the ill-conditioning of the covariance matrix), not much attention has been put on discussing the flaws and limitations of the inverse covariance matrix in correctly capturing the dependence structure when the Gaussian assumption fails.

While in a Gaussian setting, the precision matrix can give information on the conditional independence, in fact $\omega_{pq} = 0$ if and only if $X_p \perp \perp X_q$ conditional on $X_{\setminus (p,q)}$, where $X_{\setminus (p,q)}$ indicates the vector $X$ with the $(p,q)$-th elements removed (Lauritzen, 1996; Koller & Friedman, 2009), this is not true when dealing with non-Gaussian data. More specifically, in the case of non-Gaussian data, zero elements in the precision matrix do not necessarily imply conditional independence but just zero partial correlation (Baba et al., 2004).

By questioning the appropriateness of using the inverse of the covariance matrix to capture the complete dependence structure in a non-Gaussian setting, we contribute to dependence literature and provide further discussion. More precisely, in this research, we provide different definitions of a generalized precision matrix (GPM), which hold for a large class of distributions by building on the local dependence function (LDF) defined by Holland & Wang (1987).

Since its introduction, the LDF has received attention as it better allows to capture hidden structures of dependence compared to a single scalar (Jones & Koch, 2003). While Holland & Wang (1987) define the LDF as the mixed partial derivative of the logarithmic density, in 1996, Jones proposed an alternative motivation for the LDF starting from the correlation curve, which was previously developed as a generalization of the Pearson correlation coefficient (see Bjerve & Doksum (1993); Doksum et al. (1994); Blyth (1994a,b)). In his work, Jones (1996) localized the linear correlation coefficient using the kernel method, which then allowed to estimate the LDF. Then, Jones (1998) showed that the LDF is constant for the bivariate normal distribution. Additionally, Bairamov et al. (2000) developed a local dependence measure as a localized version of the Galton correlation coefficient, which has been further motivated and extended to an application to the elliptically symmetric distributions by Kotz & Nadarajah (2003). This has then been adapted to the bivariate extreme value distributions by Nadarajah et al. (2003). Motivated by Jones (1996, 1998), Bairamov et al. (2003) develop a LDF based on the regression concept and discuss its properties focusing on many bivariate distributions, including normal, Farlie Gumbel Morgenstern, bivariate exponential conditionals, and Gumbel’s bivariate exponential distribution. The LDF has also been considered in graphical models; for example, Whittaker (2009) used the LDF as a measure of interaction while Capitanio et al. (2003) focused on the conditional independence graphs for skew-normal variables. Recently, Morrison et al. (2017)
and Spantini et al. (2018) used the absolute values of the LDF for modeling non-Gaussian graphical models and defining a GPM.

Here, we contribute to the literature by adding alternative definitions to the GPM that we discuss in the context of the most appropriate application depending on the aim of the researcher. Additionally, we apply them to the multivariate t-Student distribution. This is especially valuable as generally asset returns do not follow a Gaussian distribution, which is an assumption of the Markowitz model, and are often characterized by fat tails, which can increase the potential errors in estimation. In particular, we show that conditional independence between elements of $X$ depends not just on the inverse of the covariance matrix but also on additional elements. Furthermore, we provide a graphical representation of the LDF to show that degree and direction of dependence varies in different plane regions. Additionally, an empirical application of the minimum-variance portfolio using daily log-arithmetic returns of constituents of the S&P 100 and also the Fama and French industry portfolios are considered. As this approach depends only on the precision matrix, it allows us to show the effect of using different GPMs as the input parameter. Following a rolling window approach, we show that often our portfolio computed using the newly defined GPMs has statistically significantly lower out-of-sample variance versus portfolios using the inverse sample covariance. Furthermore, we are able to show that the new GPM is more stable than the inverse covariance matrix by using the Frobenious norm between consecutive estimates.

The paper takes the following form: Section 2 introduces a measure of dependence for continuous densities and defines different GPMs, while Section 3 applies this measure to the t-Student distribution and provides a Taylor approximation. Section 4 discusses the dependence in a bivariate case using contour plots of the LDF, while Section 5 focuses on the minimum-variance portfolio application using the S&P 100 and Fama and French industry data. Lastly, Section 6 concludes.

2. A measure of dependence for continuous densities

The notion of local dependence function (LDF) was introduced by Holland & Wang (1987) and Jones (1996). While Holland & Wang (1987) tried to understand the dependence between two random variables using a limiting argument for the cross product ratio of probabilities, Jones (1996) obtains the same measure starting from a local (using kernel smoothing) definition of the correlation coefficient.

The LDF can be defined, as discussed by Holland & Wang (1987) and Jones (1996), given two random variables $(X_1, X_2)$ with continuous density $f(x_1, x_2)$ with support $K = \{(x_1, x_2) : f(x_1, x_2) > 0\}$ as the mixed partial derivative
of the logarithm of the density; such that

$$\gamma(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \log f(x_1, x_2) \quad \forall (x_1, x_2) \in K.$$ (1)

Equation (1) can be defined for any positive and mixed-differentiable function. \(\gamma(x_1, x_2)\) allows to identify the dependence structure including different degrees and direction of dependence by varying \(x_1, x_2\). Independence between \(x_1\) and \(x_2\) holds if and only if \(\gamma(x_1, x_2) = 0, \forall x_1, x_2\). We refer to Section 4 for the discussion of dependence, as we visualize Equation (1) in Figure (1).

Consider now what could be a possible extension to the multivariate case. Given a \(d\)-dimensional multivariate vector \(X = [X_1, X_2, \ldots, X_d]^\top\) with continuous and strictly positive density \(f(x_1, \ldots, x_d)\) with support \(K = \{(x_1, \ldots, x_d) : f(x_1, \ldots, x_d) > 0\}\), one can define \(\gamma(x_p, x_q)\) conditionally on the remaining variables \(X_{\setminus \{p,q\}}\), where \((p, q) = 1, \ldots, d\); such that

$$\gamma(x_p, x_q | X_{\setminus \{p,q\}}) = \frac{\partial^2}{\partial x_p \partial x_q} \log f(x) \quad \forall (x) \in K.$$ (2)

Here, we can observe conditional independence of \(x_p\) and \(x_q\) given all remaining variables \(X_{\setminus \{p,q\}}\) if and only if \(\gamma(x_p, x_q | X_{\setminus \{p,q\}}) = 0, \forall (x_p, x_q)\). This has recently been shown by [Whittaker (2009)] in the context of graphical models.

By taking expectation with respect to the conditioning variables, one can define an average measure of dependence as

$$\overline{\gamma}(x_p, x_q) = E_{X_{\setminus \{p,q\}}} [\gamma(x_p, x_q | X_{\setminus \{p,q\}})].$$ (3)

While Equations (2) and (3) allow us to understand the direction and degree of dependence by varying \((x_p, x_q)\), defining a GPM, say \(\Omega\), requires a further expectation step that could be defined in different ways.

Let the first one be defined by simple expectation, i.e.

$$\Omega = -E_{X_p, X_q} [\overline{\gamma}(X_p, X_q)] = -E_X \left[ \frac{\partial^2}{\partial x_p \partial x_q} \log f(X) \right].$$ (4)

Note that the minus sign allows recovering exactly the precision matrix \(\Sigma^{-1}\) in the Gaussian case as shown in Appendix [A]. The above definition of \(\Omega = [\omega_{pq}]\) provides an estimate of the average structure of dependence over the whole space of the distribution. Therefore, note that if \(\omega_{pq} = 0\) it does not necessarily imply conditional independence between \(x_p, x_q\). However, the converse is true, so if a pair of \(\{x_p, x_q\}\) are conditional independent, then \(\omega_{pq} = 0\). Even if we cannot clearly read the conditional independence from \(\Omega\), it allows us to focus on analyzing the interactions.
Additionally, one could consider decomposing $\Omega$ over different regions defined by the couple $\{x_p, x_q\}$. For example, to analyze the strength of dependence on the tails one could define a region $A_t = \{(x_p, x_q) : x_p^2 + x_q^2 \geq t\}$ and then compute

$$\Omega_{A_t} = -\left[ \mathbb{E}_{X_p, x_q} \left[ \gamma(X_p, X_q) \right] 1_{A_t}(X_p, X_q) \right].$$

(5)

If one is interested in obtaining a precision matrix which provides null elements if and only if conditional independence is present, one needs to consider taking the expectation of the absolute value, i.e.

$$\Omega_{Abs} = \mathbb{E}_{X_p, x_q} \left[ |\gamma(X_p, X_q)| \right] = \mathbb{E}_X \left[ \left| \frac{\partial^2}{\partial x_p \partial x_q} \log f(X) \right| \right].$$

(6)

$\Omega_{Abs}$ was also used by Spantini et al. (2018) and Morrison et al. (2017) for modeling non-Gaussian graphical models. In their works, Spantini et al. (2018) focused on investigating the low-dimensional structure of transformations between random variables, which allows characterizing complex probability distributions while Morrison et al. (2017) introduced a new algorithm, namely SING (Sparsity Identification in Non-Gaussian distributions), which uses transport maps to spot sparse dependence structures in continuous and non-Gaussian probability distributions.

While both definitions of the GPM, $\Omega$ and $\Omega_{Abs}$, are useful, they differ in their appropriate application. If the aim is to estimate a conditional independence graph, one should use $\Omega_{Abs}$; $\Omega$, on the other hand, allows to better understand the direction and strengths of the relationships and interaction between two variables, given the others.

In the following, an application using the t-Student distribution is discussed.

3. t-Student distribution and its GPMs

It is generally accepted that financial data are characterized by excessive kurtosis which is often well captured by the multivariate t-Student distribution (Cont, 2001).

As an application of the GPM, the starting point is the density of a $d$-variate t-Student distribution with the zero mean $\mu$, scatter matrix $\Sigma^{-1}$ and $\nu$ degrees of freedom:

$$f(x) = \frac{\Gamma[(\nu + d)/2]}{\Gamma(\nu/2)\nu^{d/2}\pi^{d/2}|\Sigma|^{1/2}} \left[ 1 + \frac{1}{\nu} (x - \mu)^\top \Sigma^{-1} (x - \mu) \right]^{-\frac{\nu+d}{2}}. $$

(7)

Looking at the probability density function in Equation (7), one can notice that the tail probability decays at a polynomial rate, resulting in heavy tails (Ding, 2016). Setting $k = \frac{\Gamma[(\nu + d)/2]}{\Gamma(\nu/2)\nu^{d/2}\pi^{d/2}|\Sigma|^{1/2}}$ and $\delta(x) = (x - \mu)^\top \Sigma^{-1} (x - \mu);$
then (7) reduces to

\[ f(x) = k [1 + v^{-1} \delta(x)]^{-\frac{v+d}{2}}. \]  

Following Holland & Wang (1987) and Jones (1996) we can derive the LDF of the t-Student distribution as the second partial derivative of the logarithmic density in Equation (2) such that

\[ \gamma(x_p, x_q | \Sigma_{p,q}) = -\frac{v + d}{v} \left( \frac{\Sigma^{-1}}{1 + v^{-1} \delta(x)} - \frac{\Sigma^{-1} xx^\top \Sigma^{-1}}{v(1 + v^{-1} \delta(x))^2} \right). \]  

See Appendix 9 for full derivation.

Taking a closer look at the Equation (2), we notice that there are two main terms that define the dependence structure. A first term that considers only the inverse covariance matrix \( \Sigma^{-1} \) and a second term which is a combination of \( x \) and \( \Sigma^{-1} \) in the numerator. More accurately, we can see that the first term is a scaled version of \( \Sigma^{-1} \), while the second term could be defined as an index accounting for the thickness of the tails of the distribution. If \( x \to 0 \), positioned at the center of the distribution, considering \( \Sigma^{-1} \) could still give a reliable estimate as it is very similar to the Gaussian case (see Appendix 8 for complete derivation); however, moving away from the center, additional elements should be considered.

### 3.1. A Taylor Expansion

Approximating a function as a polynomial is often simpler than considering the exact function. In the following, we expand \( \log f(x) \) around \( \delta(x) = 0 \) under the condition \( |\delta(x)| < 1 \). We aim to define an approximate GPM that could be simply estimated and which could give more insights into the structure of the GPM. By expanding the logarithm of Equation (8) up to the third order, we get

\[ M[\log f(x)] = -\frac{v + d}{2} \left[ v^{-1} \delta(x) - \frac{1}{2} v^{-2} \delta(x)^2 + \frac{1}{3} v^{-3} \delta(x)^3 - O(x)^4 \right]. \]  

By taking derivatives, we have an approximate expression for the LDF as

\[ \gamma(x_p, x_q | \Sigma_{p,q}) \approx -\frac{v + d}{2} \left[ v^{-2} \delta'(x) \delta''(x) \left[ 1 - v^{-1} \delta(x) + v^{-2} \delta^2(x) \right] \right. \]

\[ \left. - v^{-2} \delta'(x) \delta'(x)^\top \left[ 1 + 2 v^{-1} \delta(x) \right] \right]. \]  

Letting \( Y \) be the standardized version of the random variable \( X \), i.e. \( Y = \Sigma^{-1/2}(X - \mu) \). Then by taking the expectation with respect to \( \Sigma \), we can define another GPM, \( \Omega_{Taylor} \), as

\[ \tilde{\Omega}_{Taylor} = -(v + d) \left[ v^{-1} \Sigma^{-1} c + 4 v^{-3} \Sigma^{-1} \left( K(Y) + (d + 2) I_d \right) \Sigma^{-1/2} \right]. \]
where $K(Y)$ is the Móri et al. (1994) matrix of kurtosis (see e.g. example 9 in Jammalamadaka et al. (2021)) and the constant $c$ is equal to $[-1 + 2v^{-1} + v^{-1}(d) - v^{-2} \text{tr}(K(Y) + (d+2)I_d)]$. Overall, we find that this expansion boils down to two components: the inverse covariance matrix times a constant $(v^{-1}c\Sigma^{-1})$ and a second part which relies on the matrix of kurtosis $K(Y)$. See Appendix 10 for complete derivation.

### 3.2. Estimation of the GPM for the t-Student distribution

While in Section 2 we discussed the general formulas for the GPM, we now provide data estimation formulas for the case of the multivariate t-Student distribution.

Therefore, given a $d$-dimensional multivariate random sample $X_1, \cdots, X_n$ and given a consistent estimate of $\hat{\Sigma}^{-1}$ and $\hat{\nu}$, consistent estimation of $\Omega$ for the symmetric t-Student distribution can simply be obtained by computing the empirical estimate as

$$ \hat{\Omega} = \hat{\nu} + \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\hat{\Sigma}^{-1}}{1 + \hat{\nu}^{-1}\delta(X_i)} - \frac{\hat{\Sigma}^{-1}X_i X_i^\top \hat{\Sigma}^{-1}}{\hat{\nu}(1 + \hat{\nu}^{-1}\delta(X_i))} \right]. $$

(13)

Additionally, one could consider decomposing the single elements of $\hat{\Omega}$ over different regions defined by the (single) couple $x_p, x_q$. For example, to analyze the strength of dependence on the tails one could define a region $A_t = \{(x_p, x_q) : x_p^2 + x_q^2 \geq t \}$ and then compute

$$ \hat{\Omega}_{A_t} = -\left[ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_p \partial x_q} \log f(X_i) \right]|_{A_t}(X_p, X_q). $$

(14)

Clearly $\hat{\Omega} = \hat{\Omega}_{A_t} + \hat{\Omega}\setminus A_t$

While the above allows to create a better understanding of the interaction between variables, considering the definition of $\hat{\Omega}_{Abs}$ below allows to study the independence structure in the vector $X_i$. Therefore, if one is interested in clearly reading the independence structure, one could consider

$$ \hat{\Omega}_{Abs} = \frac{\hat{\nu} + d}{n} \left[ \frac{\hat{\Sigma}^{-1}}{1 + \hat{\nu}^{-1}\delta(X_i)} - \frac{\hat{\Sigma}^{-1}X_i X_i^\top \hat{\Sigma}^{-1}}{\hat{\nu}(1 + \hat{\nu}^{-1}\delta(X_i))} \right]. $$

(15)

Here, by using the absolute values, we can find the independence structure by looking for the nulls in the matrix. Additionally, we can use these estimators as our input for the minimum-variance portfolio. Compared to the sample inverse covariance $\Sigma^{-1}$, these newly defined estimators take into account the fat tails of the distribution.
4. Visualization of the LDF

In Figure (1) below, we provide a plot of the LDF of the bivariate t-Student distribution with $\nu = 6$ as defined previously in Equation (9). We include a positive and negative linear correlation of each $\rho = |0.7|$ and $\rho = |0.5|$. By providing a visualization, we aim to investigate the dependence when the degree and direction of the dependence are divergent in different plane regions.

Regardless of the choice of $\rho$ one can identify circularly symmetric contours. Unlike the constant dependence obtained when using the sample inverse covariance matrix, the LDF provides varying dependence in degree and direction. In order to correctly interpret the dependence, we follow Jones & Koch (2003) who suggested focusing on a “regional” analysis when working with contour plots of LDFs or even adding dependence maps if necessary.

Focusing on the diagonal elements, we find a positive association between $X_p$ and $X_q$ when $\rho = 0.5, 0.7$ in the first and third quadrant and a negative association when $\rho = -0.5, -0.7$ in the second and fourth quadrant. This finding is in line with Jones (1996) who displays the LDF of the Cauchy distribution, and Jones (2002) who focuses on a bivariate t-Student distribution with marginal distributions in different degrees of freedom. Note, regardless of the $\rho$ value, we find a certain symmetry on the diagonal, which is expected as we consider the LDF of the symmetric t-Student distribution. The strongest dependence, negative and positive, is found in the center of the plots where the values of $x_p$ and $x_q$ are small, almost equal to null. In Figure (1), we also identify dependence on the off-diagonal, which is strongest at the center and diminishing when the values of $x_p$ and $x_q$ are either decreasing or increasing symmetrically for a positive value of $\rho$ or behave oppositely for $-\rho$. Dependence on the off-diagonal could be explained by the quadratic form $\delta(x)$ in Equation (9). Our findings, therefore, agree with Holland & Wang (1987) who found analyzing the Cauchy distribution that as $\gamma(\cdot)$ “changes sign in $K$ most measures of association can be inadequate or even misleading” (p. 869).
5. Financial Portfolio Application

Financial portfolio optimization has attracted large attention over the years. Still today, the Markowitz framework is the backbone of modern portfolio theory, due to its simplicity of considering only risk and return. Here we focus on the minimum variance (MV) problem and analyze the out-of-sample performance of the following estimators, namely $\hat{\Sigma}^{-1}$, $\hat{\Omega}$, $\hat{\Omega}_{Taylor}$, $\hat{\Omega}_{Abs}$.

The minimization problem of the MV can be stated as such

$$\min_w w^\top \Sigma w \text{ s.t. } 1^\top w = 1$$

which then admits the following analytical solution

$$w_{MV} = \frac{\Sigma^{-1} 1}{1^\top \Sigma^{-1} 1}$$
where \( w \) is the \( n \times 1 \) vector of asset weights, \( 1 \) a \( n \times 1 \) unit vector, and \( \Sigma \) is the \( n \times n \) covariance matrix. The analytical solution of the optimization \( w_{MV} \) is then the vector of weights of the optimal minimum-variance portfolio. As \( \Sigma \) is unknown and moving beyond the Gaussian assumptions, we can use the different estimators, namely \( \hat{\Sigma}^{-1}, \hat{\Omega}, \hat{\Omega}_{Taylor}, \hat{\Omega}_{Abs} \), as our inputs.

To test the performance of the proposed estimators, we create a portfolio with \( N \) assets and follow a rolling window approach. We use different window sizes (\( ws \)), decide to re-balance the portfolios monthly (\( \tau = 21 \)), and consider various degrees of freedom \( v \). For example, if our window size \( ws = 250 \), we use the first 250 returns to compute the different estimates for the inverse covariance matrix using our alternative definitions of the GPM. The different estimates of the inverse covariance matrix are then used as inputs to compute the optimal weights \( w \) for the MV portfolios. The resulting portfolios are assumed to be held for the following \( \tau = 21 \) days. Then, we roll the lock-back window forward by 21 days. By doing so, we discard the oldest 21 observations and include 21 new observations. This process is repeated until the end of the time series is reached. Using \( M \) out-of-sample returns, where \( L_{t+1} \) is defined as the vector of returns of \( \tau = 21 \) days, we evaluate the resulting portfolios in terms of risk/return profile and portfolio composition. We compute the following measures including the out-of-sample mean (\( \hat{\mu}_p \)) and out-of-sample variance (\( \hat{\sigma}_p^2 \)) (see Equations (18-19)). We also report the average total turnover (TO), the 95% Value at Risk (VaR 95%) computed as the empirical quantile at a 95% confidence level, and the Sharpe Ratio assuming a risk-free rate of zero, (\( \hat{SR} \)), which serves as a risk-adjusted performance indicator (see Equations (20-21)). We then check whether the annualized out-of-sample variances of \( \hat{\Omega}, \hat{\Omega}_{Taylor}, \hat{\Omega}_{Abs} \) are statistically significantly different to the inverse covariance matrix \( \hat{\Sigma}^{-1} \) using the robust performance hypothesis test by [Ledoit & Wolf (2011)](\ref{eq:20}).

\[
\hat{\mu}_p = \frac{1}{M} \sum_{t=1}^{M} \hat{w}_t L_{t+1}, \tag{18}
\]

\[
\hat{\sigma}_p^2 = \frac{1}{M-1} \sum_{t=1}^{M} \left( \hat{w}_t L_{t+1} - \hat{\mu}_p \right)^2 \tag{19}
\]

\[
\hat{SR} = \frac{\hat{\mu}_p}{\hat{\sigma}_p} \tag{20}
\]

\[
TO = \frac{1}{M} \sum_{t=1}^{M} \sum_{i=1}^{N} \left| \hat{w}_{i,t-1} - \hat{w}_{i,t} \right| \tag{21}
\]

We consider two datasets to test the performance. More specifically, we use daily logarithmic return data of
In Table 1, we present a summary of the descriptive statistics, where we notice that the asset return series clearly exhibit leptokurtic behavior.

| DATASET | \(\bar{T}\) | \(N\) | \(\bar{\mu}\) | \(\bar{\sigma}\) | skew | kurt | period | freq. |
|---------|-------|-----|-----|-----|-----|------|--------|-------|
| FF30    | 25100 | 30  | 0.00046 | 0.014 | 0.489 | 38.434 | 01/07/1926 - 29/10/2021 | DAILY |
| SP100   | 5400  | 80  | 0.00031 | 0.022 | -0.509 | 22.576 | 03/01/2000 - 09/12/2020 | DAILY |

Table 1: The table reports a summary of the descriptive statistics for the 30 Fama and French industry portfolios (FF30) and the S&P 100 (SP100), respectively. Columns 1-9 report the number of observations (\(\bar{T}\)), the number of constituents (\(k\)), the average mean (\(\bar{\mu}\)), the average standard deviation (\(\bar{\sigma}\)), the average skewness (skew) the average kurtosis (kurt) of the asset returns, the time period (period) and the sampling frequency (freq.).

We find that out of all four estimators, \(\hat{\Omega}\) has consistently the lowest annualized out-of-sample return variance \(\hat{\sigma}_p^2\) for all degrees of freedom and window sizes. All of these are statistically significantly different to the estimator \(\hat{\Sigma}^{-1}\). All other estimators are sometimes statistically significant; however, their mean annualized variance is never always lower. For window size \(ws = 170\) the portfolio with the largest annualized mean return, which is not the statistics to be optimized, is always computed with the estimator \(\hat{\Sigma}^{-1}\), while portfolios built with \(\hat{\Omega}_{Abs}\) and \(\hat{\Omega}_{Taylor}\) even encounter losses. For window size \(ws = 250\), it is the portfolio with estimator \(\hat{\Omega}_{Abs}\), which shows the highest annualized mean return while again portfolios built with \(\hat{\Omega}_{Taylor}\) encounter losses for \(\nu = 3, 6\). The turnover is found to be lowest for \(\hat{\Omega}_{Abs}\) throughout all window sizes and degrees of freedom. The Sharpe Ratio is highest for \(\hat{\Omega}_{Abs}\) in window size \(ws = 250\) throughout all degrees of freedom while in window size \(ws = 170\) it is highest for \(\hat{\Sigma}^{-1}\). Looking at the 95% VaR, we find that \(\hat{\Omega}_{Abs}\) and \(\hat{\Omega}_{Taylor}\) deliver the worst values and values for \(\hat{\Sigma}\) and \(\hat{\Omega}\) are very similar.

Overall, using the S&P 100 data, we show that using the GPM, more specifically, \(\hat{\Omega}\), instead of the simple inverse covariance matrix \(\hat{\Sigma}\) in a portfolio setting allows to significantly reduce the out-of-sample-annualized variance of such portfolio, making it valuable for investors.

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1 https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
Table 2: The table reports the out-of-sample risk and return measures of $N = 80$ assets of the S&P 100 considering different window sizes different degrees of freedom and re-balancing the portfolio every month over the period from 03/01/2000-09/12/2020. Reported are from Column 1-8: the window size, the estimators, the degrees of freedom, the annualized out-of-sample variance, the annualized out-of-sample mean, out-of-sample Sharpe Ratio, 95% VaR and the average total turnover. Furthermore, we report the significance for the test of the difference in the annualized variance with regard to $\hat{\Sigma}^{-1}$, at the 10%, 5% and 1% level with *,**,***, respectively. The values in bold indicate the lowest annualized variance for that window size and degrees of freedom.

We now repeat the analysis using the Fama and French industry portfolios. Table 3 reports the results for the Fama and French industry portfolios over the period of 01/07/1926 - 29/10/2021. As we re-balance the portfolio monthly ($\tau = 21$) and use different window sizes we get the following number of out-of-sample observations $M$: with $ws = 250$ we get $M = 1183$ and with $ws = 170$ we get $M = 1187$ observations. Looking at the annualized out-of-sample return variance, we find that portfolios with $\hat{\Omega}$ present the lowest variance for window size $ws = 170$ and throughout all degrees of freedom. In the case of $ws = 250$, we show that the out-of-sample return variance of $\hat{\Omega}$ and $\hat{\Sigma}^{-1}$ is almost equally small resulting in no statistically significant difference. These findings are different compared to the larger portfolio presented in Table 2.

Nevertheless, we find that using $\hat{\Omega}$ reduces the portfolio turnover, shows a higher Sharpe Ratio and annualized mean return, compared to $\hat{\Sigma}^{-1}$. However, we notice that $\hat{\Omega}_{Abs}$ and $\hat{\Omega}_{Taylor}$ often have even lower turnover. However, portfolios build with estimators $\hat{\Omega}_{Abs}$ and $\hat{\Omega}_{Taylor}$ never allow for the lowest variance. Portfolios built with $\hat{\Omega}$ show the highest annualized mean return, which not the statistics to be optimized, over all $ws = 250$ and all degrees of freedom while for $ws = 170$ this is true for $\hat{\Omega}_{Abs}$. The largest Sharpe Ratio is computed for portfolios
Table 3: The table reports the out-of-sample risk and return measures of $N = 30$ Fama and French industry portfolios considering different window sizes, different degrees of freedom and re-balancing the portfolio every month over the period from 01/07/1926 - 29/10/2021. Reported are from Column 1-8: the window size, the estimators, the degrees of freedom, the annualized out-of-sample variance, the annualized out-of-sample mean, out-of-sample Sharpe Ratio, 95% VaR and the average total turnover. Furthermore, we report the significance for the test of the difference in the annualized variance with regard to $\hat{\Sigma}^{-1}$, at the 10%, 5% and 1% level with *,**,***, respectively. The values in bold indicate the lowest annualized variance for that window size and degrees of freedom.

| $ws$ | Estimator | $v$ | $\hat{\delta}_{\hat{\mu}}^2$ | $\hat{\mu}_P$ | $SR$ | 95% VAR | $TO$ |
|------|------------|-----|-------------------------------|-----------------|------|---------|------|
| 170  | $\hat{\Sigma}^{-1}$ | 9    | 0.01007                      | 0.06132         | 0.03849 | -0.00847 | 1.00125     |
|      | $\hat{\Omega}$ | 9    | 0.00977                       | 0.08145         | 0.05192 | -0.00837 | 0.87157     |
|      | $\hat{\Omega}_{Taylor}$ | 9    | 0.01315**                    | 0.13214         | 0.07258 | -0.01124 | 0.41116     |
|      | $\hat{\Omega}_{Abs}$ | 9    | 0.02425**                    | 0.20723         | 0.08383 | -0.01398 | 0.06816     |
|      | $\hat{\Omega}$ | 6    | 0.00977                       | 0.08162         | 0.05204 | -0.00838 | 0.87011     |
|      | $\hat{\Omega}_{Taylor}$ | 6    | 0.01982**                    | 0.18141         | 0.08117 | -0.01303 | 0.13638     |
|      | $\hat{\Omega}_{Abs}$ | 6    | 0.02425**                    | 0.20725         | 0.08383 | -0.01398 | 0.06792     |
|      | $\hat{\Omega}$ | 3    | 0.00977                       | 0.08162         | 0.05204 | -0.00838 | 0.87011     |
|      | $\hat{\Omega}_{Taylor}$ | 3    | 0.02689**                    | 0.21714         | 0.08342 | -0.01453 | 0.12614     |
|      | $\hat{\Omega}_{Abs}$ | 3    | 0.02425**                    | 0.20725         | 0.08383 | -0.01398 | 0.06792     |
| 250  | $\hat{\Sigma}^{-1}$ | 9    | 0.01073                      | -0.00318        | -0.00193 | -0.00881 | 0.68229     |
|      | $\hat{\Omega}$ | 9    | 0.01078                       | 0.00539         | 0.00327 | -0.00910 | 0.59838     |
|      | $\hat{\Omega}_{Taylor}$ | 9    | 0.01544**                    | 0.00289         | 0.00147 | -0.01159 | 0.26554     |
|      | $\hat{\Omega}_{Abs}$ | 9    | 0.02677**                    | 0.00607         | 0.00234 | -0.01572 | 0.04691     |
|      | $\hat{\Omega}$ | 6    | 0.01077                       | 0.00593         | 0.00360 | -0.00907 | 0.59734     |
|      | $\hat{\Omega}_{Taylor}$ | 6    | 0.02291**                    | 0.00644         | 0.00268 | -0.01395 | 0.08484     |
|      | $\hat{\Omega}_{Abs}$ | 6    | 0.02677**                    | 0.00610         | 0.00235 | -0.01572 | 0.04672     |
|      | $\hat{\Omega}$ | 3    | 0.01077                       | 0.00593         | 0.00360 | -0.00907 | 0.59734     |
|      | $\hat{\Omega}_{Taylor}$ | 3    | 0.02914**                    | 0.00909         | 0.00336 | -0.01648 | 0.08601     |
|      | $\hat{\Omega}_{Abs}$ | 3    | 0.02677**                    | 0.00610         | 0.00235 | -0.01572 | 0.04672     |

using $\hat{\Omega}$ in case of window size of $ws = 250$ and in case of window size of $ws = 170$ it is $\hat{\Omega}_{Abs}$. Looking at the 95% VaR we find that $\hat{\Omega}_{Abs}$ and $\hat{\Omega}_{Taylor}$ represent the worst values and again the portfolios build with estimator $\hat{\Omega}$ and $\hat{\Sigma}^{-1}$ are very similar. To summarize, using both datasets, we find the portfolios using estimator $\hat{\Omega}$ often show the smallest or at least a very similar annualized variance compared to portfolios using $\hat{\Sigma}^{-1}$ while also often showing additional performance benefits including lower turnover, larger Sharpe Ratio and annualized mean return. The effect is stronger when we look at larger portfolios and shorter time periods, as we have shown in Table 2.

Additionally, by plotting the wealth evolution of the FF30 over the rolling window $ws = 250$ with $v = 6$ in Figure 2, we can see all portfolios develop quite similarly apart from the $\hat{\Omega}_{Taylor}$ and $\hat{\Omega}_{Abs}$ which are more volatile and these portfolios tend to carry higher losses and gains. This finding is as expected as the $\hat{\Omega}_{Taylor}$ is only an approximation and $\hat{\Omega}_{Abs}$ is considering only absolute values making this matrix valuable in understanding the conditional independence between random variables but not ideal in a portfolio setting. Portfolios built considering $\hat{\Omega}$ often shows a slightly larger wealth than portfolios built considering $\hat{\Sigma}^{-1}$. Additionally as seen from Table
2 Table 3, these often carry statistically significantly lower out-of-sample return variance. Other windowsizes and degrees of freedom show similar behavior.

Figure 2: FF30 Wealth Evolution over Rolling Window with $ws = 250$ and $\nu = 6$

In order to understand the stability of our estimators, we consider the Frobenius norm between each individual estimator $\hat{\Omega}$, $\hat{\Omega}_{Taylor}$, $\hat{\Omega}_{Abs}$ and $\hat{\Sigma}^{-1}$ for each rolling window $M$. We compute the norm as distance between two matrices for each estimator individually. An example for $\hat{\Sigma}^{-1}$ is given in Equation (22) below. We plot the values as boxplots for each estimator and also over the time periods to see its evolution.

$$D_{F_{t-1}} = (\hat{\Sigma}_{t-1}, \hat{\Sigma}_{t}) = \hat{\Sigma}_{t-1}^{-1} \hat{\Sigma}_{t}^{-1}, \|F\| = \sqrt{\text{tr}\left((\hat{\Sigma}_{t-1}^{-1}, \hat{\Sigma}_{t}) (\hat{\Sigma}_{t-1}^{-1}, \hat{\Sigma}_{t})^\top\right)}$$  \hspace{1cm} (22)

An example using the Fama and French data with windowsize $ws = 170$ and degrees of freedom $\nu = 6$ is given in Figures 3-4. Using the boxplots, we can see that $\hat{\Omega}_{Taylor}$ and $\hat{\Sigma}^{-1}$ seem to estimate the most volatile estimates in consecutive periods. Additionally, throughout the whole rolling window period, $\hat{\Omega}$ seem to be more stable. Again, $\hat{\Omega}_{Taylor}$ is very volatile and seems to be an exaggeration of $\hat{\Sigma}^{-1}$. The fact that $\hat{\Omega}_{Taylor}$ is most volatile is expected as it is only an approximation.
Figure 3: Frobenius norm for each estimator with $w_{s} = 170$ and $v = 6$ of the Fama and French dataset on FF30

Figure 4: Frobenius norm for each estimator with $w_{s} = 170$ and $v = 6$ for each rolling window on FF30

6. Conclusion

In this work, we question whether a single precision matrix, defined as the inverse covariance matrix $\Sigma^{-1}$, can comprehend the complete dependence structure of random assets when the Gaussian assumption fails.

We contribute by discussing the unconditional LDF developed by [Holland & Wang (1987)] and [Jones (1996)] as a dependence measure for continuous densities. Then we use this information and build on [Morrison et al. (2017)]
and Spantini et al. (2018) by providing different definitions of alternative GPMs, which hold for a large class of distributions. We discuss their most appropriate application arguing that $\Omega$ allows to better understand the strengths of the relationships and interaction between two variables, given the others, while $\Omega_{Abs}$ defined by Morrison et al. (2017) and Spantini et al. (2018) is useful to estimate the conditional independence graph. Additionally, we discuss possible decomposition over different regions and introduce $\Omega_{Taylor}$ as an approximation. Considering that when working with financial data, the assumption of Gaussianity often fails, adapting the GPMs to different distributions is of utmost importance. As an application, we take a closer look at the t-Student distribution and provide different numerical estimations for our estimators.

By including a graphical representation of the LDF we are able to investigate the dependence when the degree and direction of the dependence are divergent in different plane regions. We agree with Holland & Wang (1987) and show that the dependence is strongest at the center and diminishes when the values of $x_p$ and $x_q$ are either decreasing or increasing symmetrically. This change in direction and strength might make a monotonic measure of dependence inadequate or even misleading.

Lastly, considering a minimum-variance portfolio application, we are able to analyze the out-of-sample performance of our estimators. We find that for the S &P 100, the portfolios with $\hat{\Omega}$ have a statistically significantly lower annualized variance compared to portfolios built with $\hat{\Sigma}^{-1}$. When working with the Fama and French industry portfolios, for which we consider a much longer time frame and a smaller number of assets, we find that the outcome in terms of annualized out-of-sample variance using either $\hat{\Omega}$ and $\hat{\Sigma}^{-1}$ is very similar, as their values are almost equally small which resulted in no statistically significant difference. Nevertheless, portfolios built with $\hat{\Omega}$ show often additional performance benefits including lower turnover and larger Sharpe ratio and mean annualized return compared to portfolios built with $\hat{\Sigma}^{-1}$. Additionally, by plotting the wealth evolution and the Frobenius norm we find that our new $\hat{\Omega}$ is less volatile and more stable throughout time.

To summarize, these findings allow us to argue that $\hat{\Omega}$ seems to be able to better capture the dependencies when dealing with t-Student data and often create portfolios with lower annualized variance and additional benefits. This is especially pronounced when dealing with a shorter time periods (here 20 years) and larger number of assets. The effect diminishes when we work with a time period of almost 100 years. Additionally, $\hat{\Omega}$ also to create more stable estimates when considering consecutive periods.

Possible adaptation to the asymmetric t-Student distribution could improve these findings and is high on the agenda. Additionally, not setting a priori degrees of freedom in the portfolio analysis can also be considered.
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7. Appendices

8. Derivatives and Expectation of Gaussian distribution

The density of a $d$-variate Gaussian distribution with the zero mean $\mu$ and scatter matrix $\Sigma^{-1}$ is defined as such

$$f(x) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\left[-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right]$$

$$= \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\left[-\frac{1}{2} \delta(x) \right]$$

$$= k \exp\left[-\frac{1}{2} \delta(x) \right]$$  \hspace{1cm} (23)

By taking the derivatives of $\log f(x)$ we get

$$\frac{\partial}{\partial x} \log f(x) = -\delta'(x) = -2\Sigma^{-1} x$$

$$\frac{\partial^2}{\partial x \partial x^T} \log f(x) = -\Sigma^{-1}$$  \hspace{1cm} (24)

Then, by defining the $GPM = -\mathbb{E}_X \left( \frac{\delta^2}{\partial x \partial x^T} \log f(X) \right)$ we get

$$\Omega_{Gaussian} = \Sigma^{-1}$$  \hspace{1cm} (25)
9. Derivatives of \( t \)-Student distribution

\[
f(x) = k [1 + v^{-1}(x - \mu)^\top \Sigma^{-1}(x - \mu)]^{-(v+d)/2}
\]

\[
\log f(x) = \log k - \frac{(v + d)}{2} \log [1 + v^{-1} \delta(x)]
\]

\[
\frac{\partial}{\partial x} \log f(x) = -\frac{(v + d)}{2} v^{-1} \Sigma^{-1} x \frac{1}{1 + v^{-1} \delta(x)}
\]

Now taking the second partial derivative we get:

\[
\frac{\partial^2}{\partial x \partial x^\top} \log f(x) = -\frac{v + d}{2} \left( 2v^{-1} \Sigma^{-1} (1 + v^{-1} \delta(x)) - 2v^{-2} \Sigma^{-1} xx^\top \Sigma^{-1} \right)
\]

\[
\frac{\partial^2}{\partial x \partial x^\top} \log f(x) = -\frac{v + d}{2} \left( 2v^{-1} \Sigma^{-1} (1 + v^{-1} \delta(x)) - 2v^{-2} \Sigma^{-1} xx^\top \Sigma^{-1} \right)
\]

As we define \( GP M = -E_X \left( \frac{\partial^2}{\partial x \partial x^\top} \log f(X) \right) \), then given the estimates \( \hat{\nu} \) and \( \hat{\Sigma} \) and given a multivariate random sample \( X_1, \ldots, X_n \), we have

\[
\hat{\Omega} = \frac{\hat{\nu} + d}{\hat{\nu}} \frac{1}{n} \sum_{i=1}^{n} \left| \frac{\hat{\Sigma}^{-1}}{1 + \hat{\nu}^{-1} \delta(X)} - \frac{\hat{\Sigma}^{-1} XX^\top \hat{\Sigma}^{-1}}{\hat{\nu}(1 + \hat{\nu}^{-1} \delta(X))^2} \right|
\]

10. Expectations and Taylor Approximation

In the following the expectation of Equation (10) is taken. Let \( Y \) be the standardized version of the random variable \( X \), i.e. \( Y = \Sigma^{-1/2}(X - \mu) \).
For this note that

\[
E\delta(X) = E\left(\text{tr} X^\top \Sigma^{-1} X\right) \\
= E\left(\text{tr} \Sigma^{-1} XX^\top\right) \\
= \text{tr} \Sigma^{-1} E\left(XX^\top\right) \\
= \text{tr}(\Sigma^{-1} \Sigma) = \text{tr} I_d = d 
\]  

(31)

where we have used the facts that, the trace (defined to be the sum of elements on the main diagonal) of a scalar is the scalar itself, \(\text{tr}(AB) = \text{tr}(BA)\) (if products exist), the trace is a linear operator, and \(E(XX^\top) = \Sigma\). We know that \(E\delta(X) = E\left(X^\top \Sigma^{-1} X\right)\) can be re-written as \(E(Y^\top Y)\).

Therefore,

\[
E\delta^2(X) = E\left(\text{tr} Y^\top Y Y^\top Y\right) \\
= E\left(\text{tr} YY^\top YY^\top\right) \\
= \text{tr}\left(E(Y Y^\top YY^\top)\right) \\
= \text{tr}(K(Y) + (d + 2)I_d) 
\]  

(32)

where \(K(Y)\) is Móri et al. [1994] matrix of Kurtosis as shown in Example 9 in Jammalamadaka et al. [2021].

Then continuing we get

\[
E\delta'(X)\delta'(X)^\top = E\left(2\Sigma^{-1} XX^\top 2\Sigma^{-1}\right) = 2\Sigma^{-1} E(XX^\top)2\Sigma^{-1} = 4\Sigma^{-1} \Sigma \Sigma^{-1} = 4\Sigma^{-1} 
\]  

(33)

\[
E\delta'(X)\delta'(X)^\top \delta(X) = E\left(\Sigma^{-1} XX^\top 2\Sigma^{-1} XX^\top 2\Sigma^{-1} X\right) \\
= 4E\left(\Sigma^{-1} XX^\top \Sigma^{-1} X\Sigma^{-1}\right) \\
= 4E\left(\Sigma^{-1/2} YY^\top YY^\top \Sigma^{-1/2}\right) \\
= 4\left(\Sigma^{-1/2} (K(Y) + (d + 2)I_d) \Sigma^{-1/2}\right) 
\]  

(34)

Then, we can use the results of this section and compute the expectation of the approximated LDF defined
previously in Equation (11); such that

\[
\hat{\Omega}_{\text{Taylor}} = -\frac{1}{2} \left[ -2v^{-1} \Sigma^{-1} + 2v^{-2} \Sigma^{-1} (d) - 2v^{-3} \Sigma^{-1} tr \{ K(Y) + (d + 2) I_d \} \\
+ 4v^{-2} \Sigma^{-1} + 2v^{-3} 4\Sigma^{-1/2} \{ K(Y) + (d + 2) I_d \} \Sigma^{-1/2} \right] \\
= -(v + d) \left[ -v^{-1} \Sigma^{-1} + v^{-2} \Sigma^{-1} (d) - v^{-3} \Sigma^{-1} tr \{ K(Y) + (d + 2) I_d \} \\
+ 2v^{-2} \Sigma^{-1} + 4v^{-3} \Sigma^{-1/2} \{ K(Y) + (d + 2) I_d \} \Sigma^{-1/2} \right] \quad (35)
\]