Gravitational Lamb Shift of Bose-Einstein Condensates due to Spacetime Fluctuations

Charles H.-T. Wang, Robert Bingham and J. Tito Mendonça

1SUPA Department of Physics, University of Aberdeen, King’s College, Aberdeen AB24 3UE, UK
2STFC Rutherford Appleton Laboratory, Chilton, Didcot, Oxfordshire OX11 0QX, UK
3SUPA Department of Physics, University of Strathclyde, Glasgow G4 0NG, UK
4IPFN, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

We investigate the oscillation of the center of mass of trapped Bose-Einstein condensates coupled to the zero-point fluctuations of the gravitational field. A semiclassical analysis is performed that allows to calculate the mean square amplitude of the oscillation. In analogy with the Lamb shift in quantum electrodynamics, this gives rise to an upshift of the energy of the trapped condensates. We show that for an elongated trap, the energy shift scales quadratically with the length as well as cubically with the total number of atoms, leading to an energy increase of 1 % using a 5 cm long trap with $10^8$ rubidium atoms. This could potentially lead to the first observable effect of low energy quantum gravity and provide a stringent test for any viable theory of quantum gravity.

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Introduction and summary.—Physics at the turn of the 21st century has seen tremendous advances in confirming and controlling the quantum behavior of complex matter. The unprecedented precision of atom interferometry holds out new prospects for laboratory tests of general relativity. The development of large molecule interferometry is providing an arena to challenge our understanding of the foundations of quantum mechanics. Furthermore, there has been a surge of interest in detecting signatures of low energy quantum gravity and unified theories using quantum matter waves.

The lowest energy quantum gravity effect one would hope to probe is the zero-point quantum fluctuations of spacetime. A possible scenario is the decoherence of matter waves through coupling to a fluctuating metric. Unfortunately, such efforts have been hampered by the smallness of the effect and hence high sensitivity requirements, the need to suppress other environmental decoherence effects within the interferometer, and the ambiguity of predictions in the absence of a consistent quantum theory of gravity.

Skeptics of quantum gravity phenomenology often argue that its experiments require accessing energies up to the Planck scale of $10^{19}$ GeV, which is far unattainable from our low energy environment. However, this view must be assessed with caution. After all, quantum electrodynamics (QED) as a highly successful quantum field theory, has demonstrated how renormalization involving high energy states can influence low energy physical observables. The Casimir effect testifies to the observable nature of the modified renormalization of quantum vacuum through boundary conditions set by conducting plates at low energy.

It is interesting to note that the decoherence of electrons due to vacuum fluctuations of the electromagnetic field has been predicted for some time, but is still far below measurable levels. One important consequence of the vacuum fluctuations of a quantum field is the energy shift of a bound state coupled to the field. The measurement of the energy or corresponding frequency shift through spectroscopy or resonance techniques can generally be done more accurately than the visibility measurement for quantum decoherence. Indeed, the resulting Lamb shift of electron energy levels in an atom provides the first experimental evidence for EM vacuum fluctuations and agrees with predictions from the renormalization procedures in QED.

In this paper we investigate the energy shift of quantum bound states due to spacetime fluctuations. We shall refer to this phenomenon as the “gravitational Lamb shift”, even though the quantum states may be confined by non-gravitational interactions. The possibility of energy shift due to spacetime fluctuations has been previously noticed. Here we proposed a realistic scenario where such an effect could be measured. We have in mind a large number of trapped cold atom and but trapped molecules may also be relevant. For such a system, the gravitational Lamb shift is most important for its center-of-mass (CM) wavefunction. However the size of the effect is generally quite small.

A natural analogy with gravitational wave detectors fortuitously emerges in this work, which makes possible significant amplifications of the effect by elongated coherent atom traps. The trap acts as a quantum detector for zero-point energy gravitational. The resulting gravitational Lamb shift not only scales cubically with the number of atoms, but also quadratically with the longitudinal extension of the trap, leading to an energy increase of 1 % using a 5 cm long trap with $10^8$ rubidium atoms. The required number of cold atoms and the physical size of the atom trap are not far-fetched from currently being achieved in the laboratory.

Position oscillations under vacuum EM fluctuations.—Let us begin by recapitulating Welton’s theory of the Lamb shift of a non-relativistic electron in a hydrogen atom. The phenomenon has a neat semiclassical explana-
ticular wave component, say in the $z$ direction, modulates the distance between any point on the $x$-$y$ plane and a chosen origin along a macroscopic straight line, defined as those on the averaged flat space. For a particular polarization the instantaneous displacements on macroscopic concentric circles are illustrated in the diagram on the left. As in the standard description of weak GWs, the radial displacements are proportional to the radius. The position oscillations along e.g. the $x$-axis is illustrated in the diagram on the right. Eq. (6) is obtained by integrating the resulting mean square deviations in all wave directions. In both diagrams the dashed lines represent displaced positions relative to the origin.

The EM fluctuations at a finite $T$ are homogeneous and isotropic with a mean square electric field given by the statistical average

$$\langle E^2 \rangle_s = \frac{4}{\pi c^3} \int_0^\infty E(\omega, T) \omega^2 \, d\omega$$

in terms of the thermal energy distribution of a harmonic oscillator with angular frequency $\omega$:

$$E(\omega, T) = \frac{\hbar \omega}{2} + \frac{\hbar \omega}{\exp(h\omega/kT) - 1}.$$ (2)

Relations (1) and (2) follow from the fundamental fluctuation-dissipation theorem applied to EM interactions.

For isotropic fluctuations, the lowest order averaged change of the potential energy $V(r)$ at location $r$ is given by [21]:

$$\langle \Delta V(r) \rangle_s = \frac{1}{6} \langle \Delta r^2 \rangle_s \nabla^2 V(r)$$

where $\langle \Delta r^2 \rangle_s$ is the mean square deviation of the electron at $r$ executing a Brownian motion as driven by the random electric field satisfying. The resulting motion is divergent if the integration in Eq. (1) is unregularized.

However, the semiclassical coupling between the non-relativistic electron and the Fourier components of stochastic EM field becomes invalid as the corresponding frequency exceeds the Compton frequency of the particle. Therefore the upper bound of integration is cut off near the Compton frequency $\omega_c = mc^2/\hbar$ at $\sigma \omega_c$ when coupled to a particle of mass $m$. Here $\sigma$ is an order one cutoff parameter that can be fixed either by experiment or renormalizable quantum calculation, yielding $\sigma \approx 3/4$ for the EM Lamb shift. This then leads to a finite overall energy shift through the expectation value:

$$\Delta E = \langle \langle \Delta V(r) \rangle_s \rangle \Phi$$ (4)

where

$$\langle f(r) \rangle = \int f(r) |\Psi(r)|^2 \, d^3 r$$

using the time-independent wavefunction $\Psi(r)$ of the quantum state.

**Position oscillations under spacetime fluctuations.**— Despite the heuristicity of the semiclassical approach to the position oscillation of a non-relativistic electron in vacuum summarized above, it is supported by experiment and is justified in the subsequent full-fledged QED. To proceed, we wish to construct a gravitational analog that may one day be fully justified by a consistent, or sufficiently effective, theory of quantum gravity. To go a step further, we would like to outline a new experiment that could verify the resulting prediction and to hopefully guide theoretical development of quantum gravity.

As a relatively simple, yet generic, model for spacetime fluctuations let us consider the perturbed metric $g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}$ in the transverse traceless gauge of linearized gravity, where $\mu, \nu = 0, 1, 2, 3$ and $\eta_{\mu \nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric and the metric perturbation $h_{\mu \nu}$ is purely spatial, i.e. $h_{0\mu} = 0$. For $i, j = 1, 2, 3$, the metric perturbation $h_{ij}$ may be interpreted as the gravitational radiation field. Up on applying the fundamental fluctuation-dissipation theorem in the fashion of [22], the thermal fluctuations of this field satisfies

$$\langle h^{ij} h_{ij} \rangle_s = \frac{32G}{\pi c^2} \int_0^\infty E(\omega, T) \, d\omega$$

where $E(\omega, T)$ is the same as given by Eq. (2). (See e.g. [23].)

The effect of the fluctuating field $h_{ij}$ is to induce a homogeneous and isotropic strain fluctuation of geometry that in turn induces a proper distance fluctuation between two free fall trajectories. If the two trajectories have a mean relative spatial displacement vector $r$ with a constant norm $r$, then their relative position oscillation due to the fluctuating field $h_{ij}$ can be described as an isotropic inhomogeneous fluctuation with a mean square deviation given by

$$\langle \Delta r^2 \rangle_s = \frac{1}{12} \langle h^{ij} h_{ij} \rangle_s r^2.$$ (6)
See Fig. 1 for illustration. In stark contrast to the homogenous position oscillation induced by fluctuating EM field \cite{21}, the position oscillation induced by fluctuating gravitational field increases with the mean separation.

This is analogous to the increased effect of normal gravitational waves on a detector using a pair of free masses with a large separation. For zero temperature \( T = 0 \), substituting Eq. (5) into Eq. (6), and imposing \( \sigma \omega_C \) as the high frequency cutoff following a similar argument as the EM case to ensure the consistency of the semiclassical treatment, we get

\[
\langle \Delta r^2 \rangle_s = \frac{4\sigma^2 m^2}{3\pi m_P} r^2
\]

where \( m_P = \sqrt{\hbar c/G} \) is the Planck mass. Here the free cutoff parameter \( \sigma \) is also expected to be order one.

**Is there any gravitational Lamb shift in a free hydrogen atom?**—Having established the mean square derivation for the position oscillation due to zero point fluctuations of spacetime \( \omega_C \), we are now in a position to apply it to physically interesting cases. A natural first case to study is to compare the gravitational Lame shift with the EM Lamb shift of the electron in a hydrogen atom using the same formula \( \omega_c \), but with different position oscillation characteristics.

The potential energy in question takes the Coulomb form \( V(r) = -e^2/r \) where \( r \) is the position of the electron relative to the CM of the atom with norm \( r \), and \( e \) is the electron charge. Since \( \nabla^2 V(r) = 4\pi e^2 \delta^3(r) \) it follows immediately from Eq. (4) and Eq. (7) that \( \Delta E = 0 \).

Therefore, to the lowest order there is no gravitational Lamb shift for electron energy levels in a simple atom. The intuitive reason is that the position oscillation simply vanishes at \( r = 0 \) in this case, but \( \nabla^2 V(r) \) vanishes everywhere except \( r = 0 \). It is instructive to compare \( \nabla^2 V(r) \) to the material distribution of a gravitational wave detector. Analogously, one cannot detect gravitational waves with a free point mass alone!

**Gravitational Lamb shift of BEC in a compact trap.**—To obtain a nonvanishing gravitational Lamb shift, it is necessary to consider a non-Coulomb confining potential \( V(r) \) with \( \nabla^2 V(r) \neq 0 \) for \( r > 0 \), assuming \( r = 0 \) is the classical center of mass of the system.

A physically interesting case fulfilling this condition is the BEC in a harmonic trap. In this case, the CM motion of the condensate decouples from its internal degrees of freedom \cite{22}. This CM wavefunction satisfies a single particle Schrödinger equation with mass \( m \) as the total mass of the condensate. The crucial property being invoked here is that the total Hamiltonian \( H_{\text{tot}} \) of the trapped BEC is completely separable into two non-interacting part: \( H_{\text{tot}} = H_{\text{CM}} + H_{\text{int}} \), where \( H_{\text{CM}} \) is the Hamiltonian for the CM motion and \( H_{\text{int}} \) is that for internal motions. To first approximation the total energy shift is the sum of the shift of the two parts. However, the mass term associated with internal modes is of the order of the individual atomic mass implying negligible energy shifts according to Eq. (7).

Let us consider a 3-dimensional spherically symmetric harmonic potential \( V(r) = \frac{1}{2} m \omega^2 r^2 \) with an isotropic trap frequency \( \omega/2\pi \). Clearly we have a positive constant \( \nabla^2 V(r) = 3m\omega^2 \) which promises a positive gravitational Lamb shift. It then follows from Eq. (7) that the local shift of potential energy \( \Delta V \) yields

\[
\Delta V(r) = \frac{2\sigma^2 m^3}{3\pi m_P^2} \omega^2 r^2 \tag{8}
\]

where \( r = \sqrt{x^2 + y^2 + z^2} \). Evidently, the higher shifts occur further away from the center of the potential.

It is straightforward to see from Eq. (11) that, for the CM quantum state \( \Psi(r) = \Psi_{n_x, n_y, n_z}(r) \) with the conventional 3D harmonic oscillator quantum numbers \( n_x, n_y, n_z = 0, 1, 2, \cdots \) in the Cartesian basis, the gravitational Lamb shift is given by

\[
\Delta E = \frac{2\sigma^2 m^2}{3\pi m_P^2} \left( n_x + n_y + n_z + \frac{3}{2} \right) \hbar \omega. \tag{9}
\]

From this expression we see that the resulting gravitational Lamb shift scales quadratically with the total mass \( m \) of the trapped BEC and increases with larger quantum numbers \( n_x, n_y, n_z \). The reason for the mass scaling is twofold: the potential energy increases with mass according to Eq. (12) for the same trap frequency, and cutoff for integrating the fluctuating gravitational field using Eq. (5) has also increased with a larger Compton frequency \( \omega_C \). The energy shift increases with the larger quantum number due to a more off-center expectation value for the CM position of the BEC, allowing higher levels of the fluctuating tidal force relative to the center of the trap.

For a realistic ground state BEC in a harmonic trap with \( n_x = n_y = n_z = 0 \), however, the gravitational Lamb shift is fantastically small, as the total trapped mass \( m \lesssim 10^8 \text{ GeV}/c^2 \) falls far short of the Planck mass \( m_P \sim 10^{19} \text{ GeV}/c^2 \). This situation can be compared to detecting gravitational waves with a small continuum, where the effect could be calculated, but might not be measured.

**Gravitational Lamb shift of BEC in an extended trap.**—In order to obtain a measurable effect, it is necessary to envisage an experiment with a longer baseline which allows the interactions with larger tidal forces associated with the zero point gravitational waves. Consider now a cylindrically symmetric harmonic potential \( V(r) = \frac{1}{2} m \omega^2 (x^2 + y^2) \) with a transverse trap frequency \( \omega/2\pi \) in the \( (x, y) \)-plane and a zero longitudinal trap frequency along the \( z \)-axis. However, the atoms are “boxed” to move only freely between \(-L/2 < z < L/2\). In other words, the wavefunction is subject to zero boundary conditions at \( z = \pm L/2 \). Thus we have \( \nabla^2 V(r) = 2m\omega^2 \) within the trap.
It then follows from Eq. (7) that the local shift of potential energy \( \Delta V(r) \) yields
\[
\Delta V(r) = \frac{4\sigma^2 m^3}{9\pi m_r^3} \omega^2 r^2. \tag{10}
\]

The idea is to allow the wavefunction to extend to macroscopically large value of \( r^2 \) along the \( z \)-axis, where the gravitational fluctuations become much larger relative to the vicinity of center of the potential. The CM wavefunction in this case takes the form \( \Psi(r) = \Psi_{n_x,n_y}(x,y)\Psi_{n_z}(z) \) where \( n_x, n_y = 0, 1, 2, \cdots \) are the quantum numbers for the effective 2D harmonic oscillator wavefunction \( \Psi_{n_x,n_y}(x,y) \) in the \( (x, y) \)-plane, and \( n_z = 0, 1, 2, \cdots \) is the quantum number for the effective boxed particle state \( \Psi_{n_z}(z) \) in the \( z \)-direction.

The amount of shift is also calculated using \( \Delta \) to be
\[
\Delta E = \frac{4\sigma^2 m^3}{9\pi m_r^3} \omega^2 \times \left\{ (n_x + n_y + 1) \frac{\hbar}{m \omega} + \frac{L^2}{12} \left[ 1 - \frac{6}{(n_z + 1)^2 \pi^2} \right] \right\}. \tag{11}
\]

For a trap with a long length-to-width aspect ratio, \( L \gg \sqrt{\hbar/ma} \), the energy shift is dominated by the second term in (11). Furthermore, for \( T \gg nK \), a BEC containing \( N \) atoms satisfying
\[
E_n = \frac{\pi^2 (n_x + 1)^2 \hbar^2}{2mL^2} \approx \frac{1}{2} k_B T
\]
implies a large value \( n_z \gtrsim 10^3 \times \sqrt{N} \times (L/m) \gg 1 \) for a sufficiently large value of \( L \). It is however important to note that this restriction is not severe, as even at absolute zero temperature with \( n_z = 0 \), it only affects the energy shift by an order one factor. In these limits, the energy shift reduces to
\[
\Delta E = \frac{\sigma^2 m^3}{27\pi m_r^3} \omega^2 L^2. \tag{12}
\]

For example, assuming \( \sigma \approx 1 \), if \( \omega = 2\pi \) kHz and \( N \) is the number of trapped rubidium atoms, then the ratio of the energy shift to the difference of transverse energy levels is
\[
\frac{\Delta E}{\hbar \omega} \approx 4 \times 10^{-24} N^3 (L/m)^2. \tag{13}
\]

For \( N = 10^6 \) and \( L = 1 \) cm, the gravitational Lamb shift is vanishingly small, with \( \Delta E/\hbar \omega \approx 4 \times 10^{-10} \). With a relatively moderate improvement of atom number to \( N = 10^8 \) while keeping \( L = 1 \) cm, the gravitational Lamb shift yields \( \Delta E/\hbar \omega \approx 0.04\% \). For \( N = 10^8 \) and a further increase of the trap length to \( L = 5 \) cm, the gravitational Lamb shift has a potentially measurable value of \( \Delta E/\hbar \omega \approx 1\% \).

The discussion presented here is readily extended to other trap geometries. We expect that such traps will be available in the near future and the proposed experiment could potentially lead to the first observable effect of low energy quantum gravity. Furthermore the determination of the free cutoff parameter \( \sigma \) through the energy shift experiment provides a stringent criterion for any viable theory of quantum gravity to reproduce its measured value.

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