Wilsonian Approximated Renormalization Group for Matrix and Vector Models in $2 < d < 4$

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Abstract

Wilson’s approximation scheme of RG recursion formula dropping momentum dependence of the propagators is applied to large-$N$ vector and matrix models in dimensions $2 < d < 4$ by making use of their exact solutions in zero dimension. In spite of apparent dependence of critical exponents upon the dilatational parameter $\rho$ involved by the approximation, the exact exponents are reproduced for vector models in the limit $\rho \to 0$. Application to matrix models is then reexamined after the same fashion. It predicts critical exponents $\nu = 2/d$ and $\eta = 2 - d/2$ for the $\text{tr} \Phi^4$ matrix model.
1. Introduction

The study of non-gaussian random matrix models initiated by the pioneering papers [1] have been combined with Weingarten’s idea of discretized quantum gravity [2] and yielded a thorough understanding of $c \leq 1$ noncritical strings [3]. There the ‘$c = 1$ barrier’ [4], traditionally attributed to the tachyonic nature of ground state of bosonic strings, obscures itself among the technical difficulty with the matrix model in dimensions $d > 1$: that it does not allow one to perform angular $U(N)$ integration so that the system is no more reduced into free fermionic eigenvalues confined in a potential well. Despite the importance of uncovering the nature of phase transition of random surfaces around $c = 1$, several previous attempts proposed to circumvent this difficulty, including the Brézin–Zinn-Justin program [5, 6] and the light-cone quantization [7], still fail to provide us with a reliable prediction even in the ‘planar’ large-$N$ limit.

This letter is aimed to present an insight into this long-standing problem, by viewing the $\text{tr} \Phi^4$ matrix model as a Landau-Ginzburg hamiltonian and exploiting Wilson’s treatment of the renormalization group [8]. It consists of the following steps to derive an RG recursion formula:

i) to separate $\Phi(x)$ into its high/low-frequency parts with respect to an arbitrarily chosen mass scale $\rho$

ii) to perform functional integration over the high-frequency field by using an approximation of propagators $1/(p^2 + r)$ and $-d/dp^2|_{p=0} \to 1/(\text{const.} + r)$, as well as truncation of induced interactions

iii) to rescale coordinates and the low-frequency field to pull back the renormalized action into the same form as the original action.

The approximation ii) is equivalent to substituting all the loop integrations by zero-dimensional combinatorics, which was readily put in our disposal by ref.[1]. This program was previously applied by Ferretti [9] to a $d = 3$ matrix model, although he relied upon an assumption of universality for the choice of $\rho$ which proves incorrect in the sequel.

In this letter I first apply the program to $O(N)$-symmetric vector models in the large-$N$ limit [10, 11], which have served as a probe to matrix models due to their resemblance to and their simplicity relative to the latter [12, 6]. I shall show an apparent $\rho$-dependence of the exponents involved by the approximation, and then extract the known exact result for $2 < d < 4^*$ by taking the maximal dilatation limit $\rho \to 0$. Next I reexamine the application to matrix models and obtain the mass exponent $\nu = 2/d$ and the anomalous dimension $\eta = 2 - d/2$ in the same limit.

2. Wilsonian approximated RG for vector models

In this section we consider a Euclidean field theory of an $N$-component scalar $\Phi(x)$ in dimensions $2 < d < 4$ with a cutoff, which we choose as the unit of mass. The action is

$$S[\Phi] = \int d^d x \left[ \frac{1}{2} (\nabla \Phi)^2 + \frac{r}{2} \Phi^2 + \frac{u}{N} (\Phi^2)^2 \right]$$

Due to the IR divergence this approximation is known to be inapplicable for $d \leq 2$. 

$$\frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + r} \Phi(p) \cdot \Phi(-p)$$

$$\quad$$

* Due to the IR divergence this approximation is known to be inapplicable for $d \leq 2$. 

1
\[
\phi(x) = \int_{0 \leq |p| \leq \rho} \frac{d^d p}{(2\pi)^d} e^{ipx} \Phi(p) + \int_{\rho < |p| \leq 1} \frac{d^d p}{(2\pi)^d} e^{ipx}(p). \tag{2}
\]

Now we introduce an arbitrary mass scale \(0 < \rho < 1\) and separate \(\Phi\) into its low and high frequency parts accordingly,

\[
\Phi(x) = \int_{0 \leq |p| \leq \rho} \frac{d^d p}{(2\pi)^d} e^{ipx} \Phi(p) + \int_{\rho < |p| \leq 1} \frac{d^d p}{(2\pi)^d} e^{ipx} \phi(p).
\]

We aim to integrate over the high frequency \(\phi\) in the large-\(N\) limit and incorporate its effect as a renormalized action of the low frequency \(\Phi\). \(\rho\) is then to play the role of a new cutoff. Substituting \(\Phi = \bar{\Phi} + \phi\) into the action (1) it reads

\[
S[\Phi] = S[\bar{\Phi}] + \sigma[\bar{\Phi}, \phi] + S[\phi],
\]

\[
\sigma[\bar{\Phi}, \phi] = \frac{u}{N} \int d^d x \left[ 2 \Phi^2 \phi^2 + 4 \Phi^4 (\Phi \cdot \phi) + 4 \left( \bar{\Phi} \cdot \phi \right) \phi^2 + 4 \left( \bar{\Phi} \cdot \phi \right)^2 \right]. \tag{3}
\]

The quadratic term separates into \(S[\bar{\Phi}]\) and \(S[\phi]\) due to momentum conservation. Integration over \(\phi\) yields the induced action \(\tilde{S}[\bar{\Phi}]\), determined by

\[
e^{-S[\bar{\Phi}]} \equiv \left( e^{-\sigma[\bar{\Phi}, \phi]} \right) = 1 - \frac{u}{N} \int d^d x \left[ 2 \Phi^2(x) \langle \phi^2(x) \rangle + \cdots \right] \\
+ \frac{1}{2} \left( \frac{u}{N} \right)^2 \int d^d x d^d y \left[ 4 \Phi^2(x) \langle \phi^2(x) \phi^2(y) \rangle \Phi^2(y) + \cdots \right] - \cdots. \tag{4}
\]

Here \(\langle \cdots \rangle\) denotes an average with respect to the measure \(D\phi e^{-S[\phi]}\).

It is easy to confirm that the last three terms in eq.(3) containing \(\Phi \cdot \phi\) do not contribute to \(\tilde{S}\) in the large-\(N\) limit. Furthermore we truncate induced interactions to those already present in the original action (1). This truncation to first three relevant terms is a natural approximation for handling the RG transformation, although it is justified only a posteriori. Then the terms exhibited explicitly in eq.(1) suffice. By reexponentiating the rhs of eq.(1) we obtain

\[
\tilde{S}[\bar{\Phi}] = 2 \frac{u}{N} \int \bar{\Phi}^2(x) \langle \phi^2(x) \rangle - 2 \left( \frac{u}{N} \right)^2 \int \bar{\Phi}^2(x) \bar{\Phi}^2(y) \left( \langle \phi^2(x) \phi^2(y) \rangle - \langle \phi^2(x) \rangle \langle \phi^2(y) \rangle \right). \tag{5}
\]

Now we are in a position to apply the Wilsonian approximation to \(\phi\)-correlators: to replace all propagators \(1/(p^2 + r)\) appearing in the loop integrals with \(1/(\text{const.} + r)\). Since the final result (eqs.(16,18)) is insensitive to the numerical value of the constant, we set it equal to unity. This approximation virtually reduces correlators to zero-dimensional ones, which are exactly calculable using the saddle point method (11). The (contracted) two-point function in eq.(7),

\[
\langle \phi^2 \rangle \approx \frac{c_d}{1 + r} \left( \frac{u c_d^2}{(1 + r)^3} + \cdots \right) = N \frac{c_d}{1 + r} C_2 \left( \frac{u c_d}{(1 + r)^2} \right), \tag{6}
\]

2
where $c_d = \int_{|p|\leq 1} \frac{d^d p}{(2\pi)^d}$ and $C_2$ denotes the zero-dimensional two-point function in eq. (A6). Similarly the connected four-point function at zero-external momenta is approximated using its zero-dimensional counterpart $C_4$ in eq. (A7),

\[
\langle \phi^2 \phi^2 \rangle - \langle \phi^2 \rangle \langle \phi^2 \rangle = \langle \phi \phi \rangle + \langle \phi \phi \phi \rangle + \langle \phi \phi \phi \phi \rangle + \cdots
\]

\[
= 2 \int \frac{d^d p}{(2\pi)^d} \frac{N}{(p^2 + r)^2} - 8 \frac{u}{N} \int \frac{d^d p}{(2\pi)^d} \frac{N}{(p^2 + r)^2} \int \frac{d^d q}{(2\pi)^d} \frac{N}{(q^2 + r)^2} \approx \frac{2 c_d}{(1 + r)^2} - 24 \frac{u c_d^2}{(1 + r)^4} + \cdots = N \frac{c_d}{(1 + r)^2} C_4 \left( \frac{u c_d}{(1 + r)^2} \right). 
\]

A characteristic feature of vector models is that the momentum-dependent self-energy diagrams which could have contributed to wave function renormalization vanish in the large-$N$ limit. Thus the renormalized action for low frequency $\Phi$ reads

\[
S' [\Phi] = S [\Phi] + \bar{S} [\Phi]
\]

\[
= \int d^d x \left[ \frac{1}{2} (\nabla \Phi)^2 + \frac{1}{2} [r + 4(1 + r)g C_2(g)] \Phi^2 + \frac{u}{N} [1 - 2g C_4(g)] (\Phi^2)^2 \right]
\]

with $g = u c_d/(1+r)^2$. Note that the coefficient functions of $\Phi^2$ and $(\Phi^2)^2$ by construction have an interpretation as normalized 1PI vertices \cite{9},

\[
r + 4(1 + r)g C_2(g) = (1 + r) \Gamma_2(g) - 1, \quad (9a)
\]

\[
1 - 2g C_4(g) = \frac{\Gamma_4(g)}{8g}. \quad (9b)
\]

They can be confirmed by Schwinger-Dyson equations\cite{10}; eq. (9b) \times 8g \) is by

\[
\langle \phi \phi \rangle + \langle \phi \phi \phi \rangle = \langle \phi \phi \phi \rangle
\]

(10)

(the solid/crossed blobs represent Green/1PI-vertex functions $C_4/\Gamma_4$ respectively) and eq. (9b) is by eq. (A4).

Finally we must rescale $x \rightarrow \rho^{-1} x$ ($p \rightarrow \rho p$) and $\Phi \rightarrow \rho^{d/2-1} \Phi$ so that the renormalized action \cite{11} has the same momentum range $0 \leq p \leq 1$ as the original one \cite{12},

\[
S' [\Phi] = \int d^d x \left[ \frac{1}{2} (\nabla \Phi)^2 + \frac{\rho^2}{2} [r + 4(1 + r)g C_2(g)] \Phi^2 + \frac{u}{N} \rho^{d-4} [1 - 2g C_4(g)] (\Phi^2)^2 \right].
\]

(11)

Therefore the RG recursion equation takes the form ($\epsilon \equiv 4 - d$)

\[
r' = \rho^{-2} [r + 4(1 + r)g C_2(g)] = \rho^{-2} \left[ r + (1 + r) (1 + 16g)^{1/2} \right] \quad (12a)
\]

\[
u' = u \rho^{-\epsilon} [1 - 2g C_4(g)] = u \rho^{-\epsilon} \frac{1 + (1 + 16g)^{1/2}}{2}. \quad (12b)
\]

\[
\text{I thank G. Ferretti for pointing out this observation.}
\]
Now we are ready to solve the RG equation following the general scheme. The non-gaussian fixed point is determined by $u' = u \neq 0$ in eq.\((12)\), that is

$$
\rho^* = \frac{1 + (1 + 16g_s)^{-1/2}}{2}.
$$

(13)

For $\epsilon \ll 1$ eq.\((13)\) always has a solution $g_s \sim \epsilon/4 \log(1/\rho) > 0$ as it should. For $\epsilon \approx 1$, however, it ceases to have a solution on the perturbative sheet $(1 + 16g)^{1/2} = +\sqrt{1 + 16g}$ for a sufficiently small $\rho$ because its rhs always exceeds $1/2$. Since whether $\rho^* > 2^{-1/\epsilon}$ has no physical significance we are obliged to continue the fixed point to the second sheet $(1 + 16g)^{1/2} = -\sqrt{1 + 16g}$. Consequently the fixed point moves from $g = 0$ to $g = +\infty$ on the first sheet and then turns back to $g = 0$ on the second sheet as $\rho$ decreases from 1 to 0. Under this agreement the fixed point $r = r^* = r_s$, $u = u^* = u_s$, $g_s = u_s c_d/(1 + r_s)^2$ in eq.\((12)\) is given by

\begin{align}
g_s &= \frac{\rho^* (1 - \rho^*)}{4 (1 - 2 \rho^*)^2}, \quad \text{(14a)} \\
r* &= \frac{-\rho^*}{1 - \rho^2 - \rho^* + 2 \rho^{2+\epsilon} < 0}, \quad \text{(14b)} \\
u* &= \frac{\rho^* (1 - \rho^*) (1 - \rho^2)^2}{4 c_d (1 - \rho^2 - \rho^* + 2 \rho^{2+\epsilon})^2} > 0. \quad \text{(14c)}
\end{align}

The signs of $r_s$ and $u_s$ are in accord with the general feature of the Wilson-Fisher fixed point for $2 < d < 4$. In order to calculate critical exponents we need to linearize the RG equation \((12)\) around $(r_s, u_s)$,

\[
\begin{pmatrix}
r' - r_s \\
u' - u_s
\end{pmatrix} = \begin{pmatrix}
\rho^{-2+\epsilon} & \frac{\rho^* (-1 + \rho^2) (-1 + \rho^*)^2}{2 c_d (1 - \rho^2 - \rho^* + 2 \rho^{2+\epsilon})} \\
\frac{4 c_d (1 - \rho^2 - \rho^* + 2 \rho^{2+\epsilon})}{\rho^2 (1 + \rho^2)} & 2 - 3 \rho^* + 2 \rho^2 \epsilon
\end{pmatrix} \begin{pmatrix}
r - r_s \\
u - u_s
\end{pmatrix}.
\]

(15)

The eigenvalues of the matrix above,

$$
\lambda_{1,2} = 1 + \rho^{-2}/2 - 3 \rho^*/2 + \rho^2 \epsilon \pm \sqrt{(1 + \rho^{-2}/2 - 3 \rho^*/2 + \rho^2 \epsilon)^2 - \rho^{2-2}},
$$

(16)

determines the first two of the series of scaling indices $y_m (m = 1, 2, \cdots)$ by \[8\]

$$
y_m = -\frac{\log \lambda_m}{\log \rho},
$$

(17)

the greatest of which is related to the mass exponent $\nu$

$$
\nu = \frac{1}{y_1} = \frac{1}{2} + \frac{\epsilon}{4} + \left(1 + \frac{\rho^2 \log \rho}{2 (1 - \rho^2)}\right) \epsilon^2 + \left(1 + \frac{\rho^2 \log \rho}{2 (1 - \rho^2)}\right) \epsilon^3 + O(\epsilon^4).
$$

(18)

Due to the approximation $1/(\rho^2 + r) \rightarrow 1/(1 + r)$, $O(\epsilon^2)$ or higher order terms depend on $\rho$, the portion of integrated momentum region by one step and scaling is apparently broken. However we can confirm that $y_{1,2}$ are smooth under the switchover of the sheets
at $\rho = 2^{-1/\epsilon}$. Moreover, in the $\rho \to 0$ limit the eigenvalues approach $\lambda_1 \to \rho^{-2}$, $\lambda_2 \to \rho^\epsilon$ and provide us with the exact values for large-$N$ vector/spherical models in $2 < d < 4$, $y_m = d - 2m$, $\nu = 1/(d - 2)$ [10, 13]. We expect that $y_m$ for $m \geq 3$ be reproduced by relaxing the truncation of induced interactions. This exactness might be attributed to that in the limit $\rho \to 0$ the cutoff theory is so strongly course-grained by a single step of RG transformation that it flows into the limiting IR theory quickly enough to exceed the accumulation of errors in the approximation. We will exploit this observation to calculate critical exponents of matrix models in the subsequent section.

3. Wilsonian approximated RG for matrix models

Application of the Wilsonian approximation to matrix models was already considered in ref.[9]; here the outline of derivation of the RG equation is briefly recalled.

We start from a Euclidean action of an $N \times N$ hermitian matrix field $\Phi(x)$,

$$S[\Phi] = \int d^d x \ tr \left[ \frac{1}{2} (\nabla \Phi)^2 + \frac{r}{2} \Phi^2 + \frac{u}{N} \Phi^4 \right]$$

(19)
equipped with a cutoff=1 as before. We separate $\Phi$ with respect to a momentum $\rho$ into low/high-frequency parts $\Phi = \bar{\Phi} + \phi$, whose coupling reads

$$\sigma[\bar{\Phi}, \phi] = \frac{u}{N} \int d^d x \ tr \left[ 4 \bar{\Phi}^3 \phi + 4 \bar{\Phi}^2 \phi^2 + 2 \bar{\Phi} \phi \Phi \phi + 4 \bar{\Phi} \phi^3 \right].$$

(20)

In the case of matrix models, $\phi$-integration also induces products of traces such as $(\tr \bar{\Phi}^2)^2$ which is as relevant as the single trace, $\tr \bar{\Phi}^4$. However we can still truncate induced interactions to those present in the action (19) consistently because the $(\tr \bar{\Phi}^2)^2$ term is induced always with a suppression factor $1/N^2$ relative to $\tr \bar{\Phi}^4$ and thus negligible in the $N \to \infty$ limit. Taking into account planarity of large-$N$ matrix models, $Z_2$ symmetry $\phi \leftrightarrow -\phi$ and momentum conservation, the induced action reads

$$\tilde{S}[\bar{\Phi}] = 4 \frac{u}{N} \int \langle \tr \bar{\Phi}^2 \phi^2(x) \rangle$$

$$- 8 \left( \frac{u}{N} \right)^2 \iiint \left( \langle \tr \Phi \phi^3(x) \tr \Phi \phi^3(y) \rangle_{\text{conn}} + \langle \tr \Phi^2 \phi^2(x) \tr \Phi^2 \phi^2(y) \rangle_{\text{conn}} \right)$$

$$+ 32 \left( \frac{u}{N} \right)^3 \iiint \langle \tr \Phi^2 \phi^2(x) \tr \Phi \phi^3(y) \tr \Phi \phi^3(z) \rangle_{\text{conn}}$$

$$- \frac{32}{3} \left( \frac{u}{N} \right)^4 \iiint \langle \tr \Phi \phi^3(x) \tr \Phi \phi^3(y) \tr \Phi \phi^3(z) \tr \Phi \phi^3(w) \rangle_{\text{conn}}.$$
their contribution are summarized into \((g = uc_d/(1 + r)^2)\)

\[
\tilde{S}_r[\Phi] = (1 + r) \frac{\Gamma_2(g) - 1}{2} \int d^dx \text{tr } \Phi^2.
\]  

(23)

Similarly, the induced interaction terms (the last three in eq. (22)) are neatly compiled into

\[
\tilde{S}_u[\Phi] = \frac{u}{N} \left( \frac{\Gamma_4(g)}{4g} - 1 \right) \int d^dx \text{tr } \Phi^4
\]  

(24)

under our approximation.

In the case of large-\(N\) matrix models, the second term in the lhs of eq. (22) contributes also to wave function renormalization. To incorporate its contribution we need to differentiate it by the external momentum \(p^2\) at \(p = 0\), which can not be treated in the original ultra-local approximation. Following Golner’s modification \([14]\) justified on the dimensional ground, we approximate this procedure simply by replacing with multiplication of a propagator

\[
\left. \frac{d}{dp^2} \right|_{p=0} \approx \frac{-1}{1 + r}.
\]  

(25)

Then the induced kinetic term is evaluated as

\[
\tilde{\mathcal{S}}_{p^2}[\Phi] = 2g \Gamma_4(g)C_2(g)^3 \int d^dx \text{tr } \left( \nabla \Phi \right)^2.
\]  

(26)

To recapitulate, the renormalized action for low-frequency \(\bar{\Phi}\) reads

\[
S'[\Phi] = S[\Phi] + \tilde{S}[\Phi] = \int d^dx \text{tr } \left[ \frac{1}{2} \left( \frac{1}{2} + 4g \Gamma_4(g)C_2(g)^3 \right) (\nabla \Phi)^2 + \frac{1}{2} [(1 + r)\Gamma_2(g) - 1] \Phi^2 + \frac{u}{N} \frac{\Gamma_4(g)}{4g} \Phi^4 \right].
\]  

(27)

After rescaling the kinetic term to a standard form and then \(x \to r^{-1}x, \Phi \to \rho^{d/2-1}\Phi\), the RG recursion equation takes the form

\[
r' = \rho^{-2} \frac{(1 + r)\Gamma_2(g) - 1}{1 + 4g \Gamma_4(g)C_2(g)^3};
\]  

\[
u' = u \rho^{-\epsilon} \frac{\Gamma(g)/4g}{[1 + 4g \Gamma_4(g)C_2(g)^3]^2}.
\]  

(28a)

(28b)

We are now ready to solve the RG eq. utilizing zero-dimensional Green functions \([1]\),

\[
\Gamma_2 = \frac{1}{C_2} = \frac{3}{a^2(4 - a^2)}, \quad \Gamma_4 = \frac{C_4}{(C_2)^4} = \frac{9(1 - a^2)(5 - 2a^2)}{a^4(4 - a^2)^4}
\]  

(29)
with $12g^4 + a^2 - 1 = 0$. For $0 < \epsilon < 2$ there exists a unique non-gaussian fixed point determined by eq.(28b),

$$\rho^\epsilon = \frac{\Gamma_4(g_*)/4g_*}{[1 + 4g_* \Gamma_4(g_*) C_2(g_*)^3]^2},$$

which turns out to move from $g_* = 0$ to $\infty$ as $\rho = 1 \to 0$, always on the perturbative sheet $a^2 = (-1 + \sqrt{1 + 48g^2})/(24g)$. We can again confirm $r_* < 0$ and $u_* > 0$. The $y_{1,2}$ indices are obtained by following the same procedure as in the previous section. For any $\rho$ they turn out to lie in the range $y_1 > 0 > y_2$ (and are equal to the mean field values $y_1 = 2$, $y_2 = 0$ for $\epsilon \to 0$ as should be), in accordance with the fact that the Wilson-Fisher fixed point for $2 < d < 4$ is associated with a single relevant perturbation. The $O(\epsilon^2)$ and higher order terms of the $\nu$ exponent depend on $\rho$ as before, and converges to $\nu \to 1/(2 - \epsilon/2) = 2/d$ in the $\rho \to 0$ limit (Fig.1).

![Fig.1: Plot of $1/\nu$ for $2 < d < 4$.](image)

From top to bottom (at $d = 2$): $\rho = 1$, $1/2$, $1/4$, $10^{-1}$, $10^{-2}$, $10^{-3}$, $10^{-4}$ and 0.

On the other hand, the anomalous dimension $\eta$, determined by the wave function renormalization factor via

$$\eta = -\frac{\log [1 + 4g_* \Gamma_4(g_*) C_2(g_*)^3]}{\log \rho} = \frac{\epsilon}{2} - \frac{\log(\Gamma_4(g_*)/4g_*)}{2 \log \rho}$$

can be shown to converge to $\epsilon/2 = 2 - d/2$ in the $\rho \to 0$ limit (Fig.2).

\footnote{The wave function renormalization factor is responsible for this fact: without it $g_*$ would proceed to the second sheet as in the case of vector models and fail to possess a meaningful $\rho \to 0$ limit for critical exponents this time.}
Although the exactness of the $\rho \to 0$ limit in the case of vector models does not necessarily imply that in matrix models, it nevertheless provides us with a strong supporting ground for these limiting values of the critical exponents.

4. Concluding remarks

In this letter I have identified the Wilson-Fisher fixed point (in the stable $u > 0$ region) both for large-$N$ vector and matrix models using the Wilson’s scheme and computed critical exponents. The essential approximations employed are three-term truncation of induced interactions and zero-dimensionalization of the propagators, combined with the $\rho \to 0$ limit. This program is, in a sense, complementary to the standard approach using the gap equation where the approximation $1/(p^2 + r) \to 1/(1 + r)$ is known to yield the full series of exact $y_m$ indices for large-$N$ vector models without ambiguity in $\rho$, after taking all the induced interactions into account. What is remarkable for matrix models is that the critical exponents for magnetization and specific heat, derived from (non-mean-field) $\nu$ and $\eta$ via (hyper-)scaling relations, are predicted to stay at the classical mean field values $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$, despite $d$ is below the upper critical dimension 4. This consequence is nontrivial and may not be attributed to the roughness of the approximation when we recall its exactness for vector models.

Generalization to higher-order truncation and criticality as well as to non-hermitian matrix models is straightforward. Direct calculation of various magnetic exponents will be made possible by relaxing the $\mathbb{Z}_2$ symmetry $\phi \leftrightarrow -\phi$, and serve for the check of consistency. I hope to discuss these points in a subsequent publication. Application of our program to large-$N$ QCD utilizing its low-dimensional exact solution is another interesting subject, although a special care is required for a cutoff procedure in order to maintain gauge invariance.

Finally I list a few points yet to be clarified. The precise mechanism for the three-term truncation in the $\rho \to 0$ limit to work out successfully for vector models must be fully explained in order to justify the matrix model results. Turning back to the original
motivation, the relationship of these field-theoretic (spacetime) exponents to geometrical (world-sheet) exponents \( d_H, \gamma_{\text{str}} \) etc., measured numerically for \( c > 1 \) candidates [13], is also unclear to the author at present.

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**Appendix: Green functions of 0D vector model**

Here I summarize the derivation of Green functions of zero-dimensional vector model following ref.[11]. The partition function is

\[
Z = \int d^N \phi \ e^{-\left[\frac{1}{2} \phi^2 + \frac{g}{N} \phi^2 \right]}
\]

\[
= \int_{-\infty}^{\infty} dt \ e^{-N \left[\frac{1}{2} e^t + g e^{2t} - \frac{1}{2} t \right]} \equiv \int dt \ e^{-N F(t)}
\]

\[
e^{-N F(t_s) - \frac{1}{2} \log F''(t_s) + O(1/N)} \tag{A1}
\]

where \( e^t \equiv \phi^2/N \). The saddle point \( t_s \) in the above is determined by

\[
F'(t_s) = \frac{1}{2} e^{t_s} + 2g \ e^{2t_s} - \frac{1}{2} = 0. \tag{A2}
\]

The four-point function is

\[
\langle (\phi^2)^2 \rangle = -N \frac{1}{Z} \frac{dZ}{dg} = N^2 \ e^{2t_s} + N \ \frac{d}{dg} \left( \frac{1}{2} \log F''(t_s) \right) + O(1). \tag{A3}
\]

On the other hand, making use of the SD equation

\[
0 = \frac{1}{Z} \int d^N \phi \sum_{i=1}^{N} \frac{d}{d\phi_i} \left[ \phi_i e^{-\left[\frac{1}{2} \phi^2 + \frac{g}{N} \phi^2 \right]} \right] = N - \langle \phi^2 \rangle - 4g \frac{1}{N} \langle (\phi^2)^2 \rangle \tag{A4}
\]

the two-point function is given by

\[
\langle \phi^2 \rangle = N - 4g \frac{1}{N} \langle (\phi^2)^2 \rangle
\]

\[
= N \ e^{t_s} - 4g \ \frac{d}{dg} \left( \frac{1}{2} \log F''(t_s) \right) + O \left( \frac{1}{N} \right). \tag{A5}
\]

To recapitulate,

\[
C_2(g) \equiv \lim_{N \to \infty} \frac{1}{N} \langle \phi^2 \rangle = e^{t_s}
\]

\[
= \frac{-1 + (1 + 16g)^{1/2}}{8g} = 1 - 4g + 32g^2 - \cdots \tag{A6}
\]

\[
C_4(g) \equiv \lim_{N \to \infty} \frac{1}{N} \left[ \langle (\phi^2)^2 \rangle - \langle \phi^2 \rangle^2 \right] = \left(1 + 8g \ e^{t_s} \right) \frac{d}{dg} \left( \frac{1}{2} \log F''(t_s) \right)
\]

\[
= \frac{1 - (1 + 16g)^{-1/2}}{4g} = 2 - 24g + 320g^2 - \cdots. \tag{A7}
\]
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