ON THE CARDINALITY AND WEIGHT SPECTRA OF COMPACT SPACES, II

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Abstract. Let $B(\kappa, \lambda)$ be the subalgebra of $\mathcal{P}(\kappa)$ generated by $[\kappa]^{\leq \lambda}$. It is shown that if $B$ is any homomorphic image of $B(\kappa, \lambda)$ then either $|B| < 2^\lambda$ or $|B| = |B|^\lambda$, moreover if $X$ is the Stone space of $B$ then either $|X| \leq 2^{2^\lambda}$ or $|X| = |B| = |B|^\lambda$.

This implies the existence of 0-dimensional compact $T_2$ spaces whose cardinality and weight spectra omit lots of singular cardinals of "small" cofinality.

1. Introduction

It was shown in [J] that for every uncountable regular cardinal $\kappa$, if $X$ is any compact $T_2$ space with $w(X) > \kappa$ ($|X| > \kappa$) then $X$ has a closed subspace $F$ such that $\kappa \leq w(F) \leq 2^{<\kappa}$ (resp. $\kappa \leq |F| \leq \sum\{2^{2^\lambda} : \lambda < \kappa\}$). In particular, the weight or cardinality spectrum of a compact space may never omit an inaccessible cardinal, moreover under GCH the weight spectrum cannot omit any uncountable regular cardinal at all.

In the present note we prove a theorem which implies that for singular $\kappa$ on the other hand there is always a 0-dimensional compact $T_2$ space whose cardinality and weight spectra both omit $\kappa$.

We formulate our main result in a boolean algebraic framework. The topological consequences easily follow by passing to the Stone spaces of the boolean algebras that we construct.

2. The Main Result

We start with a general combinatorial lemma on binary relations. In order to formulate it, however, we need the following definitions.

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Definition 1. Let $\prec$ be an arbitrary binary relation on a set $X$ and $\tau, \mu$ be cardinal numbers. We say that $\prec$ is $\tau$-full if for every subset $a \subset X$ with $|a| = \tau$ there is some $x \in X$ such that $|\{y \in a : y \prec x\}| = \tau$. Moreover, $\prec$ is said to be $\mu$-local if for every $x \in X$ we have $|\text{pred}(x, \prec)| \leq \mu$, where $\text{pred}(x, \prec) = \{y \in X : y \prec x\}$.

Now, our lemma is as follows.

Lemma 2. Let $\prec$ be a binary relation on the cardinal $\kappa$ that is both $\tau$-full and $\mu$-local. Then for every almost disjoint family $A \subset [\kappa]^{\tau}$ we have

$$|A| \leq \kappa \cdot \mu^{\tau}.$$ 

Proof. For every set $a \in A$ there is a $\xi_a \in \kappa$ such that $g(a) = a \cap \text{pred}(\xi_a, \prec)$ has cardinality $\tau$ because $\prec$ is $\tau$-full. This map $g$ is clearly one-to-one for $A$ is almost disjoint. But the range of $g$ is a subset of $\cup\{|\text{pred}(\xi, \prec)|^{\tau} : \xi \in \kappa\}$ whose cardinality does not exceed $\kappa \cdot |\mu|^\tau$, and this completes the proof.

Before we formulate our main result we need some notation. Given the cardinals $\kappa$ and $\lambda$ (we may assume $\lambda \leq \kappa$) we denote by $B(\kappa, \lambda)$ the boolean subalgebra of the power set algebra $\mathcal{P}(\kappa)$ generated by all subsets of $\kappa$ os size $\leq \lambda$. In other words

$$B(\kappa, \lambda) = [\kappa]^{\leq \lambda} \cup \{x \subset \kappa : x \in [\kappa]^{\leq \lambda} \}.$$ 

What we can show is that the size of a homomorphic image of $B(\kappa, \lambda)$ (as well as the size of its Stone space) has to satisfy certain restrictions, namely it is either “small” or cannot have “very small” cofinality.

Theorem 3. Let $h : B(\kappa, \lambda) \to B$ be a homomorphism of $B(\kappa, \lambda)$ onto the boolean algebra $B$. Then (i) either $|B| < 2^\lambda$ or $|B|^\lambda = |B|$; (ii) if $X = \text{St}(B)$ is the Stone space of $B$ then either $|X| \leq 2^{2^\lambda}$ or $|X| = |B| = |B|^\lambda$.

Proof. Set $|B| = \varrho$ and assume that $\varrho \geq 2^\lambda$. Since $[\kappa]^{\leq \lambda}$ generates $B(\kappa, \lambda)$ therefore $A = h''[\kappa]^{\leq \lambda}$ generates $B$ and thus we have $|A| = \varrho$ as well. We claim that the relation $\leq_B$ is

(a) $\tau$-full on $A$ for each $\tau \leq \lambda$;
(b) $2^\lambda$-local on $A$.

Indeed, if $a \in [A]^{\tau}$ where $\tau \leq \lambda$ then there is a set $x \in [[\kappa]^{\leq \lambda}]^{\tau}$ such that $a = h''x$. But then $b = \cup x \in [\kappa]^{\leq \lambda}$ as well, hence $h(b) \in A$ and clearly $a \subset \text{pred}(h(b), \leq_B)$ because $h$ is a homomorphism. This, of course, is much more than what we need for (a).
To see (b), let us first note that if \( b, c \in [\kappa]^{\leq \lambda} \) and \( h(b) \leq h(c) \) then \( b \cap c \in [\kappa]^{\leq \lambda} \) as well and \( h(b \cap c) = h(b) \wedge h(c) = h(b) \) using that \( h \) is a homomorphism again. But this implies \( \text{pred}(h(c), \leq_B) = h''(\mathcal{P}(c)) \) for any \( c \in [\kappa]^{\leq \lambda} \), consequently \( |\text{pred}(h(c), \leq_B)| \leq |\mathcal{P}(c)| \leq 2^\lambda \) and this completes the proof of (b).

Applying Lemma 2 we may now conclude that for every cardinal \( \tau \leq \lambda \) and for every almost disjoint family \( \mathcal{A} \subset [\varnothing]^{\tau} \) we have

\[
|\mathcal{A}| \leq \varrho \cdot (2^\lambda)^\tau = \varrho.
\]

This, in turn, implies \( \varrho^\tau = \varrho \). Indeed, assume that \( \varrho^\lambda > \varrho \) and \( \tau \) be the smallest cardinal with \( \varrho^\tau > \varrho \). Then \( \tau \leq \lambda \) and \( \varrho^{< \tau} = \varrho \), and as is well-known, there is an almost disjoint family \( \mathcal{A} \subset [\varnothing]^{\tau} \) of size \( \varrho^\tau > \varrho \), namely \( \mathcal{A} = \{A_f : f \in \tau \varrho\} \) where \( A_f = \{f \upharpoonright \xi : \xi < \tau\} \) for any \( f \in \tau \varrho \).

Now, to prove (ii) first note that if \( |B| \leq 2^\lambda \) then trivially \( |X| \leq 2^{2^\lambda} \). So assume \( |B| > 2^\lambda \) and in this case we prove that actually

\[
|X| = 2^{2^\lambda} \cdot |B|.
\]

We first show that \( |X| \geq 2^{2^\lambda} \cdot |B| \), which, as \( |X| \geq |B| \) is always valid, boils down to showing that \( |X| \geq 2^{2^\lambda} \).

Using that \( |B| = |h''[\kappa]^{\leq \lambda}| = \varrho > 2^\lambda \) we may select a collection \( \{a_\alpha : \alpha \in (2^\lambda)^+\} \subset [\kappa]^{\leq \lambda} \) such that \( \alpha \neq \beta \) implies \( h(a_\alpha) \neq h(a_\beta) \) and by a straight forward \( \Delta \)-system argument we may also assume that \( \{a_\alpha : \alpha \in (2^\lambda)^+\} \) is a \( \Delta \)-system with root \( a \). Then, as \( h \) is a homomorphism, we also have \( h(a_\alpha) \wedge h(a_\beta) = h(a) \) for distinct \( \alpha \) and \( \beta \) and so \( \{h(a_\alpha) - h(a) : \alpha \in (2^\lambda)^+\} \) are pairwise disjoint and distinct elements \( B \), all but at most one of which is non-zero. However the existence of \( 2^\lambda \) many pairwise disjoint non-zero elements in a boolean algebra clearly implies the existence of \( 2^{2^\lambda} \) ultrafilters in it, hence we are done with showing \( |X| \geq 2^{2^\lambda} \).

Next, to see \( |X| \leq 2^{2^\lambda} \cdot |B| \) note that, again as \( h \) is a homomorphism, \( h''[\kappa]^{\leq \lambda} \) is a (not necessarily proper) ideal in \( B \), hence there is no more than one ultrafilter \( u \) on \( B \) such that \( u \cap h''[\kappa]^{\leq \lambda} = \emptyset \). If, on the other hand, \( u \in X \) is such that \( b \in u \cap h''[\kappa]^{\leq \lambda} \) then \( u \) is generated by its subset \( u \cap \text{pred}(b, \leq_B) \). However \( \leq_B \) is clearly \( 2^\lambda \)-local on \( h''[\kappa]^{\leq \lambda} \), and so we conclude that

\[
|X| \leq 1 + |\{\mathcal{P}(\text{pred}(b, \leq_B)) : b \in h''[\kappa]^{\leq \lambda}\}| \leq 1 + 2^{2^\lambda} \cdot |B| = 2^{2^\lambda} \cdot |B|.
\]

This completes the proof of our theorem.
Now let $X(\kappa, \lambda)$ be the Stone space of the boolean algebra $B(\kappa, \lambda)$. Using Stone duality and the notation of [J] the above result has the following reformulation about the weight and cardinality spectra of the 0-dimensional compact $T_2$ space $X(\kappa, \lambda)$.

**Corollary 4.**

(i) For every $\mu \in Sp(w, X(\kappa, \lambda))$ we have either $\mu < 2^\lambda$ or $\mu^\lambda = \mu$, hence $\text{cf}(\mu) > \lambda$;

(ii) if $\mu \in Sp(||, X(\kappa, \lambda))$ then either $\mu < 2^{2^\lambda}$ or $\mu^\lambda = \mu$.

In fact, for every closed subspace $Y$ of $X(\kappa, \lambda)$ we have either $w(Y) \leq 2^\lambda$ or $w(Y)^\lambda = w(Y)$ and $|Y| = 2^{2^\lambda} \cdot w(Y)$.

It follows from this immediately that if $2^{2^\lambda} < \kappa$ then the cardinality and weight spectra of the space $X(\kappa, \lambda)$ omit every cardinal $\mu \in (2^{2^\lambda}, \kappa]$ with $\text{cf}(\mu) \leq \lambda$. In particular, if GCH holds then $\lambda < \kappa$ implies that both $Sp(||, X(\kappa, \lambda))$ and $Sp(w, X(\kappa, \lambda))$ omit all cardinals $\mu \in (\lambda, \kappa]$ with $\text{cf}(\mu) \leq \lambda$.

Note that similar omission results were obtained by van Douwen in [vD] for the case $\lambda = \omega$ and $\kappa$ strong limit.

An interesting open problem arises here that we could not settle: Can one find for every cardinal $\kappa$ a compact $T_2$ space $X$ such that the cardinality and/or weight spectra of $X$ omit every singular cardinal below $\kappa$?

**References**

[vD] E. van Douwen, *Cardinal functions on compact F-spaces and on weakly countably compact boolean algebras*, Fund. Math. 114 (1981), 236-256.

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