Circularly Symmetric Tests of Goodness-of-Fit

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Abstract

It is realized that existing powerful tests of goodness-of-fit are all based on sorted uniforms and, consequently, can suffer from the confounded effect of different locations and various signal frequencies in the deviations of the distributions under the alternative hypothesis from those under the null. This paper proposes circularly symmetric tests that are obtained by circularizing reweighted Anderson-Darling tests, with the focus on the circularized versions of Anderson-Darling and Zhang test statistics. Two specific types of circularization are considered, one is obtained by taking the average of the corresponding so-called scan test statistics and the other by using the maximum. To a certain extent, this circularization technique effectively eliminates the location effect and allows the weights to focus on the various signal frequencies. A limited but arguably convincing simulation study on finite-sample performance demonstrates that the circularized Zhang method outperforms the circularized Anderson-Darling and that the circularized tests outperform their parent methods. Large-sample theoretical results are also obtained for the average type of circularization. The results show that both the circularized Anderson-Darling and circularized Zhang have asymptotic distributions that are a weighted sum of an infinite number of independent squared standard normal random variables. In addition, the kernel matrices and functions are circulant. As a result, asymptotic approximations are computationally efficient via the fast Fourier transform.

Key Words: Anderson-Darling; Circulant matrices; Discrete Fourier transform; Reweighted Anderson-Darling.

1 Introduction

The problem of determining whether a sample of $n$ observations, $X_1, ..., X_n$, can be considered as a sample from a given distribution is theoretically fundamental. It is also practically important for model checking in particular and non-parametric inference in general (see [Liu, 2022b] for a most recent investigation and references therein). In this paper, we continue to make efforts to seek satisfactory answers to the question considered in [Liu, 2022b]: ‘what would be a default or all-purpose test that could be considered relatively neutral regarding the locations of deviations from the hypothesized distribution?’ This is known as so-called omnibus testing and is not only practically important in the context of hypothesis testing but also theoretically relevant in non-parametric estimation (see, e.g., [Liu, 2022a]).

With a brief review of reweighted Anderson-Darling tests in Section 2, we introduce necessary notations and focus on two important tests in the class, namely, Anderson-Darling and Zhang. It is argued that as far as a general-purpose test concerns, existing powerful methods suffer from the confounded effect of locations and frequencies in the deviations of the distributions under
the alternative hypothesis from those under the null hypothesis. This motivates the method of
circularization to create circularly symmetric tests by circularizing reweighted Anderson-Darling
tests. Two types of circularization method are considered: one is obtained by taking the average
of the corresponding scan statistics and the other by using the maximum. A simple but arguably
convincing simulation study in Section 3 on finite-sample performance demonstrates that the cir-
cularized Zhang method outperforms the circularized Anderson-Darling and that the circularized
tests outperform their parent methods.

The Gaussian process approximation approach is taken in Section 4 to investigate the large-
sample properties of the average type of circularized Anderson-Darling and Zhang. The asymptotic
distributions are a weighted sum of an infinite number of independent squared standard normal
random variables. The corresponding kernel matrices are circulant and, thereby, have simple an-
alytical eigenvalue decomposition with the eigenvalues that can be efficiently computed via the
discrete Fourier transform or, more exactly, the discrete cosine transform.

Section 5 concludes the paper with a few remarks. These remarks reinforce the importance to
the question “what would be a default test?”, for assessing goodness-of-fit.

2 Circularly Symmetric Tests

The problem to be considered is to test whether an observed sample \(X_1, \ldots, X_n\) has a specified
continuous distribution \(F_0(x), x \in \mathbb{R}\). Formally, let \(F(.)\) stand for the true distribution of the
observed sample and write the null hypothesis

\[ H_0 : F(x) = F_0(x) \quad \text{for all } x \in \mathbb{R} \]

and the alternative

\[ H_1 : F(x) \neq F_0(x) \quad \text{for some } x \in \mathbb{R}. \]

Denote by \(X_{(1)} \leq \ldots \leq X_{(n)}\) the order statistics of the observed sample. Then under the null
hypothesis \(U_{(1)} = F_0(X_{(1)}) \leq \ldots \leq U_{(n)} = F_0(X_{(n)})\) form an ordered sample of size of \(n\) from the
standard uniform distribution. \(\text{Liu (2022)}\) considered the following class of test statistics, called
the reweighted Anderson-Darling tests,

\[ B(U, w) \equiv -2 \sum_{i=1}^{n} w_i \left( \frac{\mu_i \ln U_{(i)}}{\mu_i} + (1 - \mu_i) \ln \frac{1 - U_{(i)}}{1 - \mu_i} \right) \quad (2.1) \]

where \(\mu_i = E(U_{(i)}) = i/(n + 1)\) for \(i = 1, \ldots, n\) and \(w_i\)’s are appropriately scaled nonnegative
numbers. This class contains the slightly modified Anderson-Darling test statistic, obtained by
taking \(w_i = 1\),

\[ W_n^2 \equiv -2 \sum_{i=1}^{n} \left( \frac{\mu_i \ln U_{(i)}}{\mu_i} + (1 - \mu_i) \ln \frac{1 - U_{(i)}}{1 - \mu_i} \right) \quad (2.2) \]

and a slightly modified Zhang test statistic, obtained by taking \(w_i = 1/[2\mu_i(1 - \mu_i) \sum_{k=1}^{n} \frac{1}{k}]\),

\[ R_n^2 \equiv \frac{1}{\sum_{k=1}^{n} \frac{1}{k}} \sum_{i=1}^{n} \frac{1}{\mu_i(1 - \mu_i)} \left( \frac{\mu_i \ln U_{(i)}}{\mu_i} + (1 - \mu_i) \ln \frac{1 - U_{(i)}}{1 - \mu_i} \right) \quad (2.3) \]

as two special cases. The original versions of Anderson-Darling (Anderson and Darling 1952) and
Zhang (Zhang 2002) are obtained from these modified by replacing \(\mu_i\) with \((i - 1/2)/n\).
In comparing Anderson-Darling and Cramér-von Mises (von Mises, 1928), Anderson and Darling (1952) pointed out that a statistician may prefer to use $W_n^2$ when he/she feels that Cramér-von Mises does not give enough weight to the tails of the distribution $F(x)$. While relative, the implicit comparison raises the question ‘what would be a default test?’ considered in Liu (2022b), where $R_n^2$ was proposed and found to be practically similar to the Zhang test (Zhang, 2002). A closer look at these test statistics from a statistical perspective seems to suggest that further improvements are necessary and possible.

Since the distributional deviations in the central areas of $F(x)$ captured by the $U(i)$’s are averaged together with those in the tail areas, efficiency of these methods is questionable in such cases. This implies that the above methods are all based on sorted uniforms and, thereby, suffer from the confounded effect of different locations and various signal frequencies in the deviations of the distributions under the alternative hypothesis from those under the null. This motivates the idea of circularization to eliminate the location effect so that we allow the weights to focus on the various signal frequencies.

The technique of circularization is straightforward and is described as follows. Let $U(0) = 0$ and let $U(n+1) = 1$. Extend the uniform spacings $D_i = U(i) - U(i-1)$ as a circular process at $n+1$ locations equally spaced on a circle or on the index set of $\{0, ..., 2(n+1)\}$. That is,

$$D_{n+1+i} = U((n+1)+i) - U(n+1)+(i-1) = U(i) - U(i-1) = D_i$$

for $i = 1, ..., n+1$. Define the circular counterparts of $U(i) = U(i) - U(0)$:

$$U^{(c)}(i) = U(i+1) - U(c) \quad (c = 0, ..., n; i = 1, ..., n+1).$$

Now we define the circularly symmetric version of (2.1) as the average of the $n+1$ location-shifted $B(U^{(c)}, w)$ test statistics:

$$C(U; w) = -\frac{2}{n+1} \sum_{c=0}^{n} \sum_{i=1}^{n} w_i \left[ \mu_i \ln \frac{U^{(c)}(i)}{\mu_i} + (1 - \mu_i) \ln \frac{1 - U^{(c)}(i)}{1 - \mu_i} \right].$$

(2.5)

Here in this paper we focus on the circularized versions of $W_n^2$ and $R_n^2$:

$$\hat{W}_n^2 \equiv -\frac{2}{n+1} \sum_{c=0}^{n} \sum_{i=1}^{n} w_i \left[ \mu_i \ln \frac{U^{(c)}(i)}{\mu_i} + (1 - \mu_i) \ln \frac{1 - U^{(c)}(i)}{1 - \mu_i} \right]$$

(2.6)

and

$$\tilde{R}_n^2 \equiv -\frac{1}{(n+1)} \sum_{k=1}^{n} \sum_{c=0}^{n} \sum_{i=1}^{n} \frac{1}{\mu_i (1 - \mu_i)} \left[ \mu_i \ln \frac{U^{(c)}(i)}{\mu_i} + (1 - \mu_i) \ln \frac{1 - U^{(c)}(i)}{1 - \mu_i} \right].$$

(2.7)

The normalizing constants in (2.6) and (2.7) are taken in such a way that the expectations of these test statistics are about one for $n$ large; See Remarks 4.1 and 4.2 in Section 4.

Alternative ways of circularization can be also worth consideration. For example, one may consider the alternative way of circularization via the use of the maximum of the scan statistics $B(U^{(c)}, w)$’s instead of the average, that is, $\hat{C}(U, w) = \max_c B(U^{(c)}, w)$, using the notations in Section 2. Accordingly, we introduce the corresponding circularized versions of $W_n^2$ and $R_n^2$:

$$W_n^2 \equiv -\frac{2}{n+1} \max_{c \in \{0, ..., n\}} \sum_{i=1}^{n} \left[ \mu_i \ln \frac{U^{(c)}(i)}{\mu_i} + (1 - \mu_i) \ln \frac{1 - U^{(c)}(i)}{1 - \mu_i} \right]$$

(2.8)
and
\[ \bar{R}_n^2 \equiv -\frac{1}{\sum_{k=1}^{n} \max_{i \in \{0, \ldots, n\}} \sum_{i=1}^{n} \frac{1}{\mu_i(1-\mu_i)} \left[ \mu_i \ln \frac{U^{(c)}_{(i)}}{\mu_i} + (1-\mu_i) \ln \frac{1 - U^{(c)}_{(i)}}{1-\mu_i} \right]. \] (2.9)

A simple simulation study is done in next section to compare the performance of \( \bar{W}_n^2 \) and \( \bar{R}_n^2 \) and their circularized versions. Large sample results for understanding and numerical approximation are given in Section 4 for \( \bar{W}_n^2 \) and \( \bar{R}_n^2 \). Large sample results for \( \bar{W}_n^2 \) and \( \bar{R}_n^2 \) appear challenging and are expected to be considered elsewhere.

## 3 Power Comparison: a Simulation Study

We focus on our investigation on the performance of \( W^2_n, R^2_n \), and their circularized versions, \( \bar{W}_n^2 \) and \( \bar{R}_n^2 \), for a class of situations where the deviations of \( F(.) \) from \( F_0(.) \) are locally smooth but in different locations. For this, we use, without loss of generality, the standard uniform Unif(0, 1) as \( F_0(.) \) and consider the class of \( F(.) \)'s obtained by simple local perturbations. More precisely, \( F(.) \) is given by the following probability density function (pdf)
\[
f_{\eta,\eta',\sigma}(x) = 1 + \tau \mathbf{1}_(\eta-\eta',\eta'])(x) - \tau \mathbf{1}_(\eta,\eta+\sigma](x) \quad (0 < x < 1) \] (3.1)

where \( \sigma > 0, 0 \leq \eta - \sigma, \eta + \sigma \leq 1, 0 \leq \tau \leq 1, \) and \( \mathbf{1}_A(x) \) is the indicator function of the subset \( A \), that is, \( \mathbf{1}_A(x) = 1 \) if \( x \in A \) and \( \mathbf{1}_A(x) = 0 \) otherwise. The corresponding \( F(.) \) has the pdf
\[
F_{\eta,\eta',\sigma}(x) = \begin{cases} 
x, & \text{if } p \in (0, \eta - \sigma]; 
x + \tau(x - \eta + \sigma), & \text{if } x \in (\eta - \sigma, \eta]; 
x - \tau(x - \eta - \sigma), & \text{if } x \in (\eta, \eta + \sigma); 
x, & \text{if } x \in [\eta + \sigma, 1), 
\end{cases} \quad (0 < x < 1) \] (3.2)

with the inverse cdf
\[
F^{-1}_{\eta,\eta',\sigma}(p) = \begin{cases} 
p, & \text{if } p \in (0, \eta - \sigma]; 
p - \frac{\tau(p-\eta+\sigma)}{1+\tau}, & \text{if } p \in (\eta - \sigma, \eta + \tau\sigma]; 
p + \frac{\tau(p-\eta-\sigma)}{1-\tau}, & \text{if } p \in (\eta + \tau\sigma, \eta + \sigma); 
p, & \text{if } p \in [\eta + \sigma, 1), 
\end{cases} \quad (0 < p < 1). \] (3.3)

For a simple but representative numerical comparison of \( W^2_n, R^2_n \) and their circularized versions \( W^2_n \) and \( R^2_n \), we consider two scenarios. In the first scenario, we have a small interval of length 0.1 with an appropriate magnitude \( \tau = 0.99 \) large enough for visible differences of the performance. The tail and central locations are specified by \( \eta = 0.05 \) and 0.5. The performance of \( W^2_n \) (dotted curve) and \( R^2_n \) (solid curve) in terms of the cumulative distribution function (CDF) of their Fisher’s P-value is shown by the the first two plots on the upper panel of Figure \( \square \) The performance of their circularized versions is shown by the the first two plots on the lower panel of Figure \( \square \). We see a dramatcial improved balance of performance with respect to the unknown locations of deviations. Clearly, the performance of the circularized versions are location invariant, as expected.

In the second scenario, we have a large interval of length 0.5 with an appropriate magnitude \( \tau = 0.25 \). One may reach to the same conclusion as that in the first scenario. In summary, the results in Figure \( \square \) shows clearly that the circularized Zhang method outperforms the circularized Anderson-Darling and that the circularized tests outperform their parent methods.
The title \( \{\tau, (\eta - \sigma, \eta + \sigma)\} \) of each plots represents the magnitude \( \tau \) and location \( (\eta - \sigma, \eta + \sigma) \) where \( F(\cdot) \) deviates from \( F_0(\cdot) \). The performance in terms of CDF of Fisher’s P-value are shown by solid curves for \( R_n^2 \) and dotted for \( W_n^2 \) in the four plots in the upper panel. The four plots in the lower penal show the corresponding results for \( \tilde{R}_n^2 \) and \( \tilde{W}_n^2 \). The diagonal line is the uniform reference, corresponding to tests that have no power.

The corresponding comparison is made and shown in Figure 2 on the performance of \( \tilde{W}_n^2 \) and \( \tilde{R}_n^2 \), defined in (2.8) and (2.9), respectively. The results seem to suggest that \( \tilde{W}_n^2 \) and \( \tilde{R}_n^2 \) improve the performance when signals are of high frequency, while they are comparable to \( W_n^2 \) and \( R_n^2 \) for signals of low frequency.

4 Large-Sample Results for \( \tilde{W}_n^2 \) and \( \tilde{R}_n^2 \)

4.1 Gaussian process approximation and kernel matrices

Familiar asymptotic results for the \( U(i) \) process can be conveniently applied by writing:

\[
\sqrt{n+2}(U_{(i)} - \mu_i) \approx B(t) = W(t) - tW(1) \text{ with } t = i/(n + 1),
\]

where “\( \approx \)” means that in the limit with \( i/(n + 1) \to t \), the random variable on its left-hand side converges in distribution to the random variable on its right-hand side; See, e.g., Anderson and Darling (1952). We can work with this Gaussian process approximation effectively as follows. Define \( Z_i = \sqrt{n+1} [W(i/(n+1)) - W((i-1)/(n+1))] \), \( i = 1, ..., n+1 \), and let \( \tilde{Z} = \frac{1}{n+1} \sum_{i=1}^{n+1} Z_i \).
Then $Z_i$ are iid $N(0,1)$. It follows that

$$B(i/(n+1)) - B((i-1)/(n+1))$$

$$= [W(i/(n+1)) - W((i-1)/(n+1))] - [i/(n+1) - (i-1)/(n+1)]W(1)$$

$$= [W(i/(n+1)) - W((i-1)/(n+1))] - \frac{1}{n+1}\sum_{j=1}^{n+1}[W(j/(n+1)) - W((j-1)/(n+1))]$$

$$= (Z_i - \bar{Z})/\sqrt{n+1}.$$

Let

$$C_n = \left[ I - \frac{1}{n+1} \mathbf{1}\mathbf{1}' \right], \quad (4.1)$$

a symmetric $(n+1) \times (n+1)$ matrix and the centering operator for $Z = (Z_1, ..., Z_n, Z_{n+1})'$. Thus for all $c = 0, ..., n$,

$$\sqrt{n+2}[U(c)_{(i)} - \mu_i] \approx \frac{1}{\sqrt{n+1}}\tau'_{c+1:i}C_nZ \quad (4.2)$$

where $Z = (Z_1, ..., Z_n, Z_{n+1})' \sim N_{n+1}(\mathbf{0}, \mathbf{I})$ and $\tau'_{c+1:i}$ is the index vector for the elements of $Z$ at $c+1, ..., c+i$, defined circularly. Let $A_i$ be the $(n+1) \times (n+1)$ matrix obtained by stacking these index vectors. That is, $A_k$ is circulant with its first row consisting of $k$ ones and $n+1-k$ zeros (in that order). So

$$\sum_{c=0}^{n}(U(c)_{(i)} - \mu_i)^2 \approx \frac{1}{(n+1)(n+2)}Z'C_nA_i'A_iC_nZ$$

Figure 2: The same legend as that of Figure 1 but for $\tilde{R}_{n}^{2}$ and $\tilde{W}_{n}^{2}$ instead of $\tilde{R}_{n}^{2}$ and $\tilde{W}_{n}^{2}$.
and, thereby,

\[
\sum_{c=0}^{n} \sum_{i=1}^{n} \psi_i (U_{(c)}^{(i)} - \mu_i)^2 \approx \frac{n+1}{n+2} Z' K_n(\psi) Z, \quad (4.3)
\]

where

\[
K_n(\psi) = \frac{1}{(n+1)^2} \sum_{i=1}^{n} \psi_i C_n A'_k A_k C_n \quad (4.4)
\]

is the \((n+1) \times (n+1)\) kernel matrix with \(\psi_i = w_i/\left[\mu_i(1-\mu_i)\right]\) for \(i = 1, ..., n\). To obtain the corresponding approximation to \(C(U, w)\) using the above Gaussian process, we consider the following Taylor expansion of the \(i\)-th summand of Eq. (2.2) at \(\mu_i\):

\[
-2 \left[ \mu_i \ln \frac{U_{(i)}}{\mu_i} + (1-\mu_i) \ln \frac{1-U_{(i)}}{1-\mu_i} \right] \approx \frac{1}{\mu_i(1-\mu_i)} (U_{(i)} - \mu_i)^2.
\]

This suggests the following approximation to the circularly symmetric test statistic \(C(U, w)\):

\[
C(U, w) \approx \frac{1}{n+2} Z' K_n(\psi) Z
\]

or, simply, \(C(U, w) \approx \frac{1}{n} Z' K_n(\psi) Z\).

Following Anderson and Darling (1952), we denote by \(1/\lambda_i\) the eigenvalues of \(\frac{1}{n+2} K_n(\psi)\) and the corresponding eigenvectors \(V_i\). That is, we can write

\[
C(U, w) \approx \sum_{i=1}^{n+1} \frac{1}{\lambda_i} Z V_i V'_i Z \quad (4.5)
\]

a weighted sum of independent \(\chi_1^2\)'s with weights \(1/\lambda_i\)'s; See Anderson and Darling (1952), Stephens (1974), Sinclair and Spurr (1988), Zolotarev (1961), Davies (1980), and Duchesne and De Micheaux (2010) for the case of its continuum limit, and numerical methods if desirable.

Computationally, the large-sample approximation relies on the eigenvalue decomposition of the kernel matrix \(K_n(\psi)\). The easy-to-prove results summarized into the following proposition show that the kernel matrix \(K_n(\psi)\) is circulant (see, e.g., Gray 2006).

**Proposition 1.** Consider the \((n+1) \times (n+1)\) matrix \(C_n\) defined in \((4.2)\) and the \((n+1) \times (n+1)\) matrices \(A_k, k = 1, ..., n\), defined above. Then

(a) \(C_n\) and \(A_k\)'s are all circulant matrices, so are \(A'_k A_k\) and \(C_n A'_k A_k C_n\);

(b) The matrix \(A'_k A_k\) is a symmetric Toeplitz matrix, with the \((i, j)\)'s elements:

\[
\max(0, k+i-j) + \max(0, k-i+j-(n+1))
\]

for \(1 \leq i \leq j \leq n+1\);

(c) for all \(k = 1, ..., n\),

\[
C'_n A'_k A_k C_n = A'_k A_k - \frac{k^2}{n+1} 11';
\]

and
(d) the kernel matrix $K_n(\psi)$ is symmetric and circulant with elements:

$$
\frac{1}{(n+1)^2} \sum_{k=1}^{n} \psi_k \left[ \max(0, k+i-j) + \max(0, k-i+j-(n+1)) - \frac{k^2}{n+1} \right]
$$

(4.6)

for $1 \leq i \leq j \leq n+1$.

The following theorem summarizes the properties regarding eigenvalues and eigenvectors of circulant matrices (Theorem 7 of Gray, 2006).

**Theorem 1.** Let $C$ be a $(n+1) \times (n+1)$ circulant matrix with its first row denoted by $c = (c_0, ..., c_n)'$, i.e., the $(k,j)$ entry of $C$ is given by $C_{k,j} = c_{(j-k) \mod (n+1)}$. Then $C$ has eigenvectors

$$
\psi(m) = \frac{1}{\sqrt{n+1}} (1, e^{-2\pi i m/(n+1)}, e^{-2\pi i 2m/(n+1)}, \ldots, e^{-2\pi i n m/(n+1)})', \quad m = 0, 1, \ldots, n,
$$

and corresponding eigenvalues

$$
\phi_m = \sum_{k=0}^{n} c_k e^{-2\pi i km/(n+1)}
$$

and can be expressed in the form $C = V\Phi V^*$, where $i$ is the unit imaginary number, $V$ has the eigenvectors as columns in order, the asterisk * denotes conjugate transpose, and $\Phi$ is diag($\phi_k$). In particular all circulant matrices share the same eigenvectors, the same matrix $U$ works for all circulant matrices, and any matrix of the form $C = V\Phi V^*$ is circulant. Furthermore, for any two $(n+1) \times (n+1)$ circulant matrices $C$ and $B$, $C$ and $B$ commute, i.e., $CB = BC$, and $CB$, $\alpha C$, and $C + B$ are circulant matrices, where $\alpha$ is a scalar.

Since the eigenvalues of any real symmetric matrix are real, the symmetric circulant matrix $K_n(\psi)$ has $(n+1)$ real eigenvalues

$$
\phi_m = \sum_{k=0}^{n} c_k \cos \left( \frac{2\pi km}{n+1} \right).
$$

(4.7)

Moreover, a $(n+1) \times (n+1)$ symmetric circulant matrix $C$ satisfies the extra condition that $c_{n-i} = c_i$ and is thus determined by $[(n+1)/2] + 1$ elements. The corresponding eigenvalues can be written as

$$
\phi_m = c_0 + 2 \sum_{k=1}^{(n+1)/2-1} c_k \cos \left( \frac{2\pi km}{n+1} \right) + c_{(n+1)/2} \cos \left( \frac{2\pi (n+1)m}{n+1} \right)
$$

for $(n+1)$ even, and

$$
\phi_m = c_0 + 2 \sum_{k=1}^{n/2} c_k \cos \left( \frac{2\pi km}{n+1} \right)
$$

for $(n+1)$ odd. These properties allow for efficient computation via fast discrete Fourier transform (Cooley and Tukey, 1965). The large-sample approximation to (4.6) for $\tilde{W}_n^2$ and $\tilde{R}_n^2$ is given in the next two subsections.
4.2 Circularized $W_n^2$

The continuum limit of the kernel structure (4.6) for the test statistic $W_n^2$ can be obtained and is summarized into the kernel function in the following theorem; See Appendix A.1 for the proof.

**Theorem 2.** If $w_k = 1$, i.e., $\psi_k \propto 1/\mu_k(1 - \mu_k)$, then in its continuum limit the kernel matrix $K_n(\psi)$ with elements (4.6) is given by

$$
\kappa(t, s) = \begin{cases} 
2 \left[ (s - t) \ln(s - t) + [1 - (s - t)] \ln(1 - (s - t)) \right] + 1, & \text{if } t \leq s; \\
2 \left[ (t - s) \ln(t - s) + [1 - (t - s)] \ln(1 - (t - s)) \right] + 1, & \text{if } s < t.
\end{cases}
$$

(4.8)

Figure 3: Quantile-Quantile plot of the large-sample approximation to the distribution of $\tilde{W}_n^2$ versus the true distribution in the cubic-root scale, obtained on 1,000,000 Monte Carlo samples. The quantile points are obtained for 1,000 equally spaced probabilities from $1/1001$ to $1 - 1/1001$.

**Remark 4.1.** In case with $w_k = 1$, i.e., $\psi_k = 1/\mu_k(1 - \mu_k)$, the diagonal elements of the kernel matrix are given by

$$
\sum_{k=1}^{n} \frac{k - \frac{k^2}{n+1}}{k(n + 1 - k)} = \frac{n}{n + 1}.
$$

(4.9)

This implies that the sum of all the eigenvalues of the kernel matrix is $n$. Because the rank of the circulant matrix $C_n$ is $n$ and the circulant matrix $\sum_{k=1}^{n} \psi_k A_k^t A_k$ is full rank, the kernel matrix is a rank-$n$ matrix and thus has $n$ non-zero eigenvalues with the zero eigenvalue given by

$$
\phi_0 = \sum_{k=0}^{n} c_k,
$$

where $c = (c_0, ..., c_n)$ denotes the first row of the kernel matrix.

Remark 4.1 implies that $E(C(U, w)) \approx 1$ and, naturally, suggests $\tilde{W}_n^2$ defined in (2.6) for the preference of $E(\tilde{W}_n^2) \approx 1$. A numerical evaluation of the large sample-based approximation to the distribution of $\tilde{W}_n^2$ is shown by the quantile-quantile plots in Figure 3 for a selected cases of $n = 10$, 20, and 50. The quantile points are obtained for 1,000 equally spaced CDF values from $1/1001$ to
1−1/1001 based on a Monte Carlo approximation of 1,000,000 replicates. The asymptotic kernel function (4.8) is used to compute \(n + 1\) values as the first row of a corresponding kernel matrix:

\[
\begin{align*}
    c_0 &= \frac{1}{n + 1}, \\
    c_k &= 2 \left[ \frac{k}{n+1} \ln \frac{k}{n+1} + \frac{n+1-k}{n+1} \ln \frac{n+1-k}{n+1} \right] + 1, \\
    &\quad \text{for } k = 1, \ldots, n.
\end{align*}
\]

One may see from Figure 3 that the approximation is satisfactory.

### 4.3 Circularized \(R_n^2\)

The continuum limit of the kernel structure (4.6) for the test statistic \(R_n^2\) can also be obtained and is summarized into the kernel function in the following theorem; See Appendix A.2 for the proof.

**Theorem 3.** If \(w_k = 1/\mu_k(1 - \mu_k)\), i.e., \(\psi_k \propto 1/\mu_k(1 - \mu_k)^2\), then in its continuum limit the kernel matrix \(K_n(\psi)\) with elements (4.6) is given by

\[
\kappa(t, s) = \begin{cases} 
2 \left[ 2(s - t) - 1 \right] \ln \frac{s-t}{1-s-t}, & \text{if } t \leq s; \\
2 \left[ 2(t - s) - 1 \right] \ln \frac{t-s}{1-t-s}, & \text{if } s < t.
\end{cases}
\]

(4.10)

**Remark 4.2.** In case with \(w_k = 1/\mu_k(1 - \mu_k)\), i.e., \(\psi_k = 1/\mu_k(1 - \mu_k)^2\), the diagonal elements of the kernel matrix are given by

\[
\begin{align*}
    \sum_{k=1}^{n} \psi_k C_n' A_k A_k C_n &= \left( n + 1 \right) \sum_{k=1}^{n} \frac{k - k^2}{k(n+1-k)}^2 \\
    &= 2 \sum_{k=1}^{n} \frac{1}{k}.
\end{align*}
\]

This implies that the sum of all the eigenvalues of the kernel matrix is is \([2(n+1)]\sum_{k=1}^{n} \frac{1}{k}\).

Remark 4.2 implies that \(E(B(U, w)) = 2 \sum_{k=1}^{n} \frac{1}{k}\) and, naturally, suggests the test statistic \(\tilde{R}_n^2\) defined in (2.7) for the preference of \(E(\tilde{R}_n^2) \approx 1\). A numerical evaluation of the large sample-based approximation to the distribution of \(\tilde{R}_n^2\) was conducted in the same way as that for \(\tilde{W}_n^2\) in the
previous subsection. The asymptotic kernel function (4.10) is used to compute \( n + 1 \) values as the first row of a corresponding kernel matrix:

\[
c_0 = \frac{1}{2(n+1) \sum_{k=1}^{n} \frac{1}{k}}, \quad c_k = \frac{\left(2k \frac{n+1}{n+1-k} - 1\right) \ln \frac{k}{n+1-k} - 1}{(n+1) \sum_{k=1}^{n} \frac{1}{k}}, \quad k = 1, \ldots, n.
\]

The results are displayed in Figure 4, which shows that the approximation is satisfactory, although this can be further improved by small sample corrections.

5 Concluding Remarks

It was argued and supported by numerical results that existing methods for assessing goodness-of-fit can suffer from the confounded effect of different locations and various signal frequencies in the deviations of distributions under the alternative hypothesis from those under the null hypothesis. This paper proposed the method of circularization to create circularly symmetric test statistics. The limited finite-sample power comparison showed that circularized methods perform better than their parent methods, as general-purpose tests or, more precisely producing balanced performance with respect to the unknown location of distributional deviations.

One may notice that technically, the use of the circular statistics \( U^{(c)}(i) \) shares with the familiar scan statistics and the use of wavelets to capture signal of different frequencies, especially in the context of dealing with circular processes. While the fundamental statistical problem appears to be the same, one of them needs to be fully understood. Nevertheless, such connections would help motivate ideas worth further investigations. Moreover, while this paper chose to focus on assessing goodness-of-fit, it is expected to see more research on such general ‘goodness-of-test’ problems in broader contexts.

It is also expected that circularized test statistics can be very useful for non-parametric inference when sorted uniforms are used as pivotal quantities (see, e.g., Liu, 2022a). Nevertheless, the original test statistic \( R_n^2 \) can also be useful when the deviations of the true distribution are known \( a \ priori \) to be in tails of the distribution. Of course, in that case, one may want put even more weights than \( R_n^2 \) in tail areas of the distribution. On the other hand, when deviations are known \( a \ priori \) to be in the central areas, our numerical example seems to suggest that existing methods are less powerful than their circular versions. In this case, more powerful tests could be constructed by re-weighting summands of \( C(U, w) \) indexed by \( c \). To summarize, this discussion reinforces the importance of considering the question ‘what would be a good default test?’, for the sake of quality theory and practice of scientific inference.

A Proofs of Theorems

A.1 Proof of Theorem 2

Due to the symmetry property, it is sufficient to consider the case \( s \geq t \). In this case, the element in the continuum limit with \( i/(n+1) \rightarrow t \) and \( j/(n+1) \rightarrow s \), where \( 0 < t < s < 1 \), is obtained as
follows: Note that $n + 1 - (j - i) > 0$,

$$
\lim_{n \to \infty} \frac{1}{(n + 1)^2} \left[ \sum_{k=j-i}^{n} \psi_k[k - (j - i)] + \sum_{k=n+1-(j-i)}^{n} \psi_k[k - i + j - (n + 1)] - \sum_{k=1}^{n} \psi_k^2 \right]
$$

$$
= \lim_{n \to \infty} \left[ \sum_{k=j-i}^{n} \frac{k - (j - i)}{k(n + 1 - k)} + \sum_{k=n+1-(j-i)}^{n} \frac{k - i + j - (n + 1)}{k(n + 1 - k)} - \frac{1}{n + 1} \sum_{k=1}^{n} \frac{k}{n + 1 - k} \right]
$$

$$
= \lim_{n \to \infty} \left[ -\frac{j - i}{n + 1} \sum_{k=1}^{n+1-(j-i)} \frac{1}{k} + \frac{1}{n + 1} \sum_{k=j-i}^{n} \frac{1}{k} + j - i \sum_{k=1}^{n} \frac{1}{k} + \frac{j - i}{n + 1} \sum_{k=n+1-(j-i)}^{n} \frac{1}{k}
+ \sum_{k=1}^{n+1-(j-i)} \frac{1}{k} \sum_{k=n+1-(j-i)}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} + \frac{n}{n + 1} \right]. \tag{A.1}
$$

If $j - i \geq (n + 1)/2$, that is, $j - i \geq n + 1 - (j - i)$ and $s - t \geq \frac{1}{2}$, then (A.1) becomes

$$
\lim_{n \to \infty} \left[ \frac{2(j - i)}{n + 1} \sum_{k=n+1-(j-i)+1}^{j-i} \frac{1}{k} + \frac{(j - i)}{n + 1} \frac{1}{n + 1 - (j - i)} - 2 \sum_{k=n+1-(j-i)+1}^{n} \frac{1}{k} + \frac{n}{n + 1} \right]
$$

$$
= 2(s - t) \lim_{n \to \infty} \left[ \log |j - i| - \log (n + 1 - (j - i)) \right] - 2 \lim_{n \to \infty} \left[ \log (n) - \log (n + 1 - (j - i)) \right] + 1
$$

$$
= 2(s - t) \log \frac{s - t}{1 - (s - t)} - 2 \log \frac{1}{1 - (s - t)} + 1
$$

$$
= 2 \left[ (s - t) \log (s - t) + [1 - (s - t)] \log (1 - (s - t)) \right] + 1,
$$

where the method of series estimation with integrals is used. If $j - i < (n + 1)/2$, that is, $j - i < n + 1 - (j - i)$ and $s - t < \frac{1}{2}$, then (A.1) becomes

$$
\lim_{n \to \infty} \left[ -\frac{j - i}{n + 1} \sum_{k=j-i+1}^{n+1-(j-i)} \frac{1}{k} + \frac{j - i}{n + 1} \sum_{k=j-i}^{n+1-(j-i)-1} \frac{1}{k} - \sum_{k=n+1-(j-i)+1}^{n} \frac{1}{k} - \sum_{k=n+1-(j-i)}^{n} \frac{1}{k} + \frac{n}{n + 1} \right]
$$

$$
= -2(s - t) \lim_{n \to \infty} \frac{n + 1 - (j - i)}{j - i} - 2 \lim_{n \to \infty} \log \frac{n}{n + 1 - (j - i)} + 1
$$

$$
= -2(s - t) \log \frac{1 - (s - t)}{s - t} - 2 \log \frac{1}{1 - (s - t)} + 1
$$

$$
= 2 \left[ (1 - (s - t)) \log (1 - (s - t)) + (s - t) \log (s - t) \right] + 1,
$$

where the method of series estimation with integrals is used again. For the $s = t = i/(n + 1)$ case, it is easy to see that a simplified version of the above derivation gives the claimed result. To summarize, we have Equation (4.8) and complete the proof.
A.2 Proof of Theorem 3

Due to the symmetry property, it is sufficient to consider the case $s \geq t$. In this, the element in the continuum limit with $t = i/(n + 1)$ and $s = j/(n + 1)$, where $0 < t < s < 1$, is obtained as follows:

$$
limit_{n \to \infty} \frac{1}{(n+1)^2} \left[ \sum_{k=j-i}^{n} \psi_k [k - (j - i)] + \sum_{k=n+1-(j-i)}^{n} \psi_k [k - i + j - (n + 1)] - \sum_{k=1}^{n} \psi_k \frac{k^2}{n + 1} \right]
$$

$$
= \ limit_{n \to \infty} \frac{1}{(n+1)^2} \left[ \sum_{k=j-i}^{n} \frac{k - (j - i)}{[k(n + 1 - k)]^2} + \sum_{k=n+1-(j-i)}^{n} \frac{k - i + j - (n + 1)}{[k(n + 1 - k)]^2} - \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{(n + 1 - k)^2} \right]
$$

$$
= \ limit_{n \to \infty} \left[ \sum_{k=n+1-(j-i)}^{n} \frac{1}{n + 1 - k} \sum_{i=1}^{n} \frac{1}{n + 1 - k} - \frac{j - i}{n + 1} \sum_{k=j-i}^{n} \frac{1}{k + n + 1 - k} \right] \left[ \frac{1}{k + n + 1 - k} \right] - \sum_{k=1}^{n} \frac{1}{k^2} \right]. \quad (A.2)
$$

If $j - i < (n + 1)/2$, that is, $j - i < n + 1 - (j - i)$ and $s - t < \frac{1}{2}$, then (A.2) becomes

$$
limit_{n \to \infty} (n+1) \left[ \sum_{k=j-i}^{n+1-(j-i)-1} \frac{1}{n + 1 - k} - \frac{j - i}{n + 1} \sum_{k=j-i}^{n+1-(j-i)-1} \frac{1}{n + 1 - k} \right] \left[ \frac{1}{k + n + 1 - k} \right] - 2 \sum_{k=1}^{n} \frac{1}{k^2} \right)
$$

$$
= 2 \ln \frac{1 - (s - t)}{s - t} - (s - t) \int_{s-t}^{1-(s-t)} \left( \frac{1}{x} + \frac{1}{1 - x} \right)^2 \, dx - 2 \int_{1-(s-t)}^{1} \frac{1}{x^2} \, dx
$$

$$
= 4 \left( s - t - \frac{1}{2} \right) \ln \frac{s - t}{1 - (s - t)} - 2,
$$

where the method of series estimation with integrals is used. If $j - i \geq (n + 1)/2$, that is, $j - i \geq n + 1 - (j - i)$ and $s - t \geq \frac{1}{2}$, then (A.2) becomes

$$
limit_{n \to \infty} (n+1) \left[ - \frac{2}{n + 1} \sum_{k=n+1-(j-i)+1}^{j-i} \frac{1}{n + 1 - k} - 2 \sum_{k=1}^{j-i} \frac{1}{(n + 1 - k)^2}

+ \frac{j - i}{n + 1} \sum_{k=n+1-(j-i)}^{j-i-1} \left( \frac{1}{k + n + 1 - k} \right)^2 \right]
$$

$$
= - 2 \ln \frac{s - t}{1 - (s - t)} - 2 \int_{0}^{s-t} \frac{1}{(1-x)^2} \, dx + (s - t) \int_{1-(s-t)}^{s-t} \left( \frac{1}{x} + \frac{1}{1 - x} \right)^2 \, dx
$$

$$
= 4 \left( s - t - \frac{1}{2} \right) \ln \frac{s - t}{1 - (s - t)} - 2,
$$

the same as that in the $s - t < \frac{1}{2}$ case, where the method of series estimation with integrals is used again. For the $s = t = i/(n + 1)$ case, it is easy to see that a simplified version of the above derivation gives the claimed result. To summarize, we have Equation (4.10) and complete the proof.
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