Topological Gauged WZW Models and 2D Gravity

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Abstract

We study the “topological gauged WZW model associated with $SU(2)/U(1)$”, which is defined as the twisted version of the corresponding supersymmetric gauged WZW model. It is shown that this model is equivalent to a topological conformal field theory characterized by two independent topological conformal algebras, one of which is the “twisted Kazama-Suzuki type” and the other is “twisted Coulomb gas type”. We further show that our formalism of this gauged WZW model naturally reduces to the well-known formulations of 2D gravity coupled with conformal matter; one of the gauge choices leads to the K.Li’s theory, and the alternative choices lead to the KPZ theory or the DDK (Liouville) theory. In appendix we argue on a possibility of deriving such topological conformal models from the $G/G$-gauged WZW models.
1 Introduction

Conformal models in two-dimensional space-time possessing $N = 2$ world-sheet supersymmetry form a special class of conformal field theories, namely they are believed to comprise the only known solutions to string theory at the perturbative level.

Recently a large class of new $N = 2$ superconformal models were constructed by Y.Kazama and H.Suzuki [1]. They studied coset models which have $N = 1$ superconformal symmetry, and then determined the precise conditions under which these $N = 1$ models actually possess $N = 2$ supersymmetry. Some algebraic structures of these models were studied in [2], [3] emphasizing their geometrical back-grounds.

Subsequently, in the light of the work [4], T.Eguchi and S.Yang pointed out [6] that by “twisting” the energy-momentum tensor of these $N = 2$ superconformal models with respect to their $U(1)$-current, they can be interpreted as a kind of topological conformal field theories with central charge $c = 0$. In the topological conformal models of this type, one of the supercharges of the original $N = 2$ models can be reinterpreted as the BRST charge, and the chiral primary fields become the physical observables which are the cohomology classes determined by this BRST charge.

In general, topological field theories are introduced by E.Witten [4] as the physical theories describing some mathematical objects, that is, the geometries of various moduli spaces. It is then interesting to ask if it is possible to describe the geometrical meanings of the topological conformal models of this type, in other words the moduli spaces of these models. Particularly the geometrical interpretation of the BRST cohomology should be clarified.

The topological conformal models of this type play an important role in the recent understanding of two-dimensional quantum gravity. Namely the system of topological matter (the topological conformal model obtained by twisting the $N = 2$ minimal model) coupled with topological gravity [7], [8], [10] is equivalent [11] to the $N$ matrix model [13] which is a powerful method for doing non-perturbative calculations in two-dimensional gravity [12].

On the other hand there is still a big gap in the understanding of the relation between
the matrix model formulation and the conventional approach to two-dimensional gravity based on conformal field theory coupled to the Liouville theory [16] (the David-Distler-Kawai (DDK) theory [15]) or the Knizhnik-Polyakov-Zamolodchikov (KPZ) theory (gravity in the light-cone gauge) [14]. In this sense it is important to clarify the relation between the topological formulation and other continuum approaches.

With these motivations we intend to investigate the topological conformal models of this type, that is, obtainable by twisting $N=2$ models, from the standpoint of two-dimensional Lagrangian field theory. The reason why we will do is that our interests are intensively turned to the global structures of the models, not to the local fluctuations, since all local degrees of freedom are canceled out in any topological field theory. The traditional algebraic techniques of conformal field theories, in which we don’t write the Lagrangian explicitly, are thought to be not so powerful for our purpose.

We begin in section 2 by formulating the “topological gauged WZW models”, by twisting the corresponding $N=2$ supersymmetric gauged WZW models [21] [11]. The quantization of the systems is given by the path-integrations of chiral fields $g$, gauge fields $A$, and two-dimensional metric $g$. The technique of these path-integrations is an application of the formalism developed by K.Gawedzki, A.Kupiainen [17], and D.Karabali, H.J.Schnitzer [18]. Then we specialize to the case of $SU(2)/U(1)$. In this section we shall study the path-integration only for the “matter part”, that is, the quantization with the moduli of two-dimensional metric and the gauge fields fixed. The local operator formalism, which is consistent with the path-integration, is presented. The matter part is shown to be characterized by two kind of topological conformal algebras (TCAs). One of them is a topological conformal algebra obtained by twisting a $N=2$ superconformal algebra realized by the Kazama-Suzuki (KS) formalism [1], and the other is found to be equivalent to that obtained by twisting the one realized in terms of the “$N=2$ Coulomb gas formalism” [23]. These topological algebras are not equivalent, and define respectively the “Kazama-Suzuki (KS) sector”, the “Coulomb gas (CG) sector” of the matter part. Investigations of the geometrical meanings of these two sectors are given.

In section 3 we shall discuss the relation with the well-known formulation of two-dimensional gravity coupled to conformal matter. For this aim, our interests will be poured
into the Coulomb Gas sector rather than the Kazama-Suzuki sector. There we argue on the path-integrations of the residual moduli, i.e. the moduli of the gauge field and the metric. They lead to the gauge fixed action of the famous formulation of pure topological gravity by E.Verlinde and H.Verlinde [8]. This implies that our model is equivalent to the model considered by K.Li [10]. By taking alternative gauge fixing procedures, our model is shown to be equivalent to other continuum approaches to two-dimensional gravity. Namely we shall intend to perform the path-integral over the mode of the Virasoro anomaly, in place of the chiral anomaly, of the matter sector. This leads to the KPZ theory [14] or the DDK theory [15].

Finally in section 4 we give several discussions.

In appendix A we discuss a possibility of constructing the topological gauged WZW models from the $G/G$-gauged WZW models, with a careful restriction on the functional space of the gauge field over which the path-integral is performed. Moreover we give some comments on the geometrical back-grounds of the model.

2 Topological Gauged WZW Model associated with $SU(2)/U(1)$

2.1 Topological Gauged WZW Models - the definition of the model

First of all, we shall present a brief review on gauged WZW models. Let $(\Sigma, \mathcal{J})$ be a compact connected Riemann surface ($\mathcal{J}$ is a complex structure), and $G$ be a compact semi-simple Lie group with its Lie algebra $g$. Suppose $H$ be a closed subgroup of $G$ and the corresponding Lie subalgebra $h$. The action of the “$G/H$-gauged WZW model with level $k$” is defined as follows;

$$kS_G(g,A) = \frac{ik}{4\pi} \int_{\Sigma} (g^{-1} \tilde{\partial} g, g^{-1} \partial g) - \frac{ik}{24\pi} \int_{B} ([g^{-1} d\tilde{g}, g^{-1} d\tilde{g}])$$

$$+ \frac{ik}{2\pi} \int_{\Sigma} \{- (g^{-1} \tilde{\partial} g, A^{10}) + (A^{01}, \tilde{\partial} g g^{-1}) - (A^{01}, \text{Ad}(g) A^{10}) + (A^{01}, A^{10})\}. \quad (2.1)$$
where the chiral field $g$ is $G$-valued and the gauge field $A = A^{10} + A^{01}$ ($A^{10}$, $A^{01}$ are respectively the holomorphic and anti-holomorphic components of $A$) is $h$-valued, i.e. $A^{10}$ ($A^{01}$) is $h^{\mathbb{C}}$-valued (locally defined) $(1,0)$-form (resp. $(0,1)$-form) satisfying $A^{10\dagger} = -A^{01}$ ("$\dagger$" is the canonical "hermitian conjugation" so that $g = \{u \in g^{\mathbb{C}}; u^{\dagger} = -u\}$). The inner product $(\ , \ )$ is the Cartan-Killing form normalized by $(\theta, \theta) = 2$ ($\theta$ is the highest root of $g$), and $d = \partial + \bar{\partial}$ is the canonical splitting defined by $J$. It is well-known that if $\pi_3(G) = \mathbb{Z}$, that is indeed our case since $G$ is compact semi-simple, we must restrict the value of the parameter $k$ in $\mathbb{Z}_{\geq 0}$ \[19\]. The non-negativity of $k$ is of course required in order to define the theory positive-definitely. This action is manifestly conformally invariant because the above definition needs only the complex structure $J$, does not need any metric structure on $\Sigma$.

The most important property of this action is the following identity (what is called the "Polyakov-Wiegmann identity" \[20\]):

$$S_G(\Omega g, \Omega A) = S_G(g, A) - S_G(\Omega^\dagger \Omega, A),$$ \hspace{1cm} (2. 2)

where we introduce the concept of “chiral gauge transformation” (“complex gauge transformation”) defined for any $H^{\mathbb{C}}$-valued $\Omega$,

$$\Omega g = \Omega^{\dagger -1} g \Omega^{-1},$$

$$(\Omega A)^{10} = Ad(\Omega) A^{10} - \partial \Omega \Omega^{-1}. \hspace{1cm} (2. 3)$$

This identity is nothing but the cocycle condition of chiral anomaly, and can be proved by straightforward calculations. If the chiral gauge transformation $\Omega$ is “unitary” (i.e. $H$-valued), this identity (2. 2) immediately implies the relation $S_G(\Omega g, \Omega A) = S_G(g, A)$. Because for any unitary $\Omega$, $\Omega^\dagger \Omega = 1$ holds and this means the 2nd term of the RHS of (2. 2) vanishes. In other words our action $S_G$ has the gauge invariance for the “vectorial direction”. But when $\Omega$ is $H^{\mathbb{C}}/H$-valued (i.e. an “axial gauge transformation”), we suffer the chiral anomaly at the classical level.

The quantization of this model from the standpoint of path-integration is fully investigated in \[17\] \[18\]. It is shown there that this gauged WZW model describes the so-called “$G/H$-coset conformal field theory (CFT)”. Namely the $G/H$-gauged WZW model is one of the solution of the problem what Lagrangian field theory describes this coset model.
Our quantization scheme of the “topological gauged WZW models” (defined below) will be an application of those works. It is worthwhile to comment on the mechanism of appearing the coset CFTs from these models: The conformal anomaly of the WZW model is originated in its chiral anomaly. The path-integration of the $H$-gauge field, especially the integral along the orbits of axial gauge transformations “absorbs” the $H$-part of chiral anomaly, which gives the result; $c_{\text{gauged WZW}} = c_{G,k} - c_{H,k}$. This mechanism is very similar as that given in [13]. Namely the Weyl anomaly of the conformal matter is canceled out by means of the path-integration of the Liouville field. This is a suggestive point for our later discussions on 2D gravity.

In the papers [21], [11] N=2 supersymmetric extensions of the above gauged WZW models were presented. Those can be defined associated with a general compact Kähler homogeneous space $G/H$ with $H$ being a closed subgroup of $G$ including a maximal torus. It is known that the coset space of this type corresponds to a “parabolic decomposition” of $g^C$;

$$g^C = h^C \oplus m_+ \oplus m_-,$$

$$= Z(h^C) \oplus h^C_0 \oplus m_+ \oplus m_-.$$ (2. 4)

Here $Z(h^C)$ is the center of $h^C$, $h^C_0$ is the semi-simple part of $h^C$. We have further set $m_\pm = \sum_{\alpha \in \Delta_\pm} g_\alpha$, where $\Delta_\pm = \Delta(g^C)_\pm \setminus \Delta(h^C)_\pm$, $\Delta(g^C)_\pm$, $\Delta(h^C)_\pm$ are respectively the system of positive (negative) roots of $g^C$, $h^C_0$. Associated with this palabolic decomposition, we need to prepare $m_+ \oplus m_-$-valued Weyl fermions $\psi, \bar{\psi}$ (defined with respect to some spin structure compatible with the complex structure $J$) and a $h$-valued gauge field $A$, in order to define the desired supersymmetric model. So the “N=2 supersymmetric gauged WZW model associated with $G/H$” is given by;

$$Z = \int Dg DAD\psi D\bar{\psi} \exp \left[ -kS_G(g, A) - \frac{1}{\pi} \int_{\Sigma} dv(g) \left\{ (\psi, \partial_A \bar{\psi}) + (\bar{\psi}, \partial_A \psi) \right\} \right],$$ (2. 5)

where $dv(g)$ is the canonical volume element defined by a fixed Kähler metric $g$, and $D_A = \partial_A + \bar{\partial}_A$ is the canonical splitting of the covariant exterior derivative $D_A$. We can show that this model describes the N=2 superconformal model (SCF) obtained by the Kazama-Suzuki supercoset construction for $G/H$ [21], [11], [34]. In particular, in the case that $G/H$ is hermitian symmetric we can reproduce those given in [1], [2].
However, the models we are wanting now are topological conformal models, namely the “twisted” version [3] of these N=2 SCFs. Therefore we shall regard the fermionic fields \( \psi, \bar{\psi} \) as a BRST ghost system. That is to say, we shall replace the \( m_+ \), \( m_- \)-components of \( \psi \) by a \( m_+ \)-valued \((0,0)\)-form \( \psi \) (ghost), a \( m_- \)-valued \((1,0)\)-form \( \chi \) (anti-ghost) respectively and similarly the \( m_- \), \( m_+ \)-components of \( \bar{\psi} \) by a \( m_- \)-valued \((0,0)\)-form \( \bar{\psi} \), \( m_+ \)-valued \((0,1)\)-form \( \bar{\chi} \). We shall call the supersymmetric gauged WZW model (2.5) with this “twist” of the fermionic fields as the “topological gauged WZW model associated with \( G/H \)”, or briefly the “topological \( G/H \)-model”. The following part of this paper will be devoted to the most simple case; \( G = SU(2), H = U(1) \). Full investigations of the above models for general homogeneous spaces \( G/H \), along the scheme of path-integral quantization performed in the following sections, will be presented in [34], [35] on both the untwisted (N=2 SCF) and the twisted (topological) version. Especially it can be shown [34] that the topological \( G/H \)-model indeed gives the twisted Kazama-Suzuki model for \( G/H \).

Another derivation of the topological gauged WZW model from the \( G/G \)-gauged WZW model (manifestly a theory of \( c = 0 \)) is speculated in appendix A, which needs a careful restriction of the domain over which the path-integration is performed in order to get non-trivial (finite) physical degrees of freedom.

2.2 Quantization of the Topological \( SU(2)/U(1) \)-Model

Now let us begin investigations of the topological \( SU(2)/U(1) \)-model, i.e. \( G = SU(2), H = U(1), g = su(2), h = u(1) \). Let

\[
g^C = h^C \oplus g_+ \oplus g_-
\]

be the usual Cartan decomposition, which is the parabolic decomposition (2.4) in this case. Therefore, the ghost fields \( \psi, \bar{\psi} \) should be \( g_+ \), \( g_- \)-valued \((0,0)\)-forms, and the anti-ghost fields \( \chi, \bar{\chi} \) are \( g_- \)-valued \((1,0)\)-form, \( g_+ \)-valued \((0,1)\)-form respectively. The gauge field \( A \) should be \( U(1) \) (Cartan) valued. Let us introduce the standard canonical Cartan-Weyl
base $e, f, t$ of $g^C$:

$$
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7}$$

They satisfy the relations;

$$(t,t) = 2, \quad (e,f) = 1,$$

other combinations vanish,

with the usual definition of the Cartan-Killing metric for $g^C$;

$$(A,B) \equiv \text{Tr}(AB). \tag{2.9}$$

In terms of this base, let us write the ghost system and the gauge field introduced above by components; $\psi e, \bar{\psi} f, \chi f, \bar{\chi} e$, and $A^i t$ (so that $A$ becomes a real 1-form on $\Sigma$). Then the model we begin with is defined by\footnote{In this paper we shall take the convention that fermionic fields and fermionic (BRST) transformations should \textit{anti-commute} with 1-forms on the space-time $\Sigma$ (i.e. $dz, d\bar{z}$). The motivation of it is due to the following fact; In field theory, BRST ghost fields should be identified with differential forms on some functional space, and particularly in topological field theory it is often convenient to consider the product space of the space-time and some moduli space.}:

$$Z = \int \mathcal{D}g \mathcal{D}g \mathcal{D}A \mathcal{D}(\chi, \bar{\chi}, \psi, \bar{\psi}) \exp \left[-kS_G(g,A) - \frac{1}{2\pi i} \int_{\Sigma} (\partial_A \psi \chi - \bar{\chi} \partial_A \bar{\psi}) \right] \tag{2.10}$$

$$\equiv \int \mathcal{D}g \ Z[g], \tag{2.11}$$

where $g$ denotes the metric on $\Sigma$. Although the gauged WZW action $S_G(g,A)$ and the ghost’s action does not depend on the metric $g$ (since they depend only on $\mathcal{J}$), the path-integral measures $\mathcal{D}A, \mathcal{D}(\chi, \bar{\chi}, \psi, \bar{\psi})$ implicitly depend on $g$, which appears as the Virasoro anomalies (the Weyl or the gravitational anomalies). In the following, we suppose that \textit{the metric $g$ defines the complex structure $\mathcal{J}$ so that $g$ is a Kähler metric with respect to $\mathcal{J}$}. The covariant exterior derivatives are explicitly written as;

$$D_A \psi = d\psi + iA\psi, \quad D_A \bar{\psi} = d\bar{\psi} - iA\bar{\psi}, \tag{2.12}$$
Our fundamental standpoint of quantization is that we shall perform the path-integration of all fields appearing in the theory, i.e. the chiral field $g$, the gauge field $A$, the ghost fields $\chi, \bar{\chi}, \psi, \bar{\psi}$ and the metric $g$. However, in this section we shall perform the path-integration only for the “matter part”, that is, quantize this model with the moduli of the metric and the gauge field fixed. On these integrals of the moduli we will discuss in the next section.

The action (2.10) has a following “on-shell” BRST symmetry, which is originated in the N=2 SUSY of the untwisted model;

$$\delta_{G/H} \chi = k(e, \partial_agg^{-1}), \quad \bar{\delta}_{G/H} \bar{\chi} = -k(f, g^{-1}\bar{\partial}_ag),$$

$$\delta_{G/H} g = (\psi e)g, \quad \bar{\delta}_{G/H} g = -g(f\bar{\psi}),$$

other combinations are defined to vanish. (2.13)

This action further has the $U(1)$-gauge symmetry (along the vectorial direction), but suffer the chiral anomaly for axial gauge transformations, in the similar manner as the usual $SU(2)/U(1)$-gauged WZW model. We must perform the gauge fixing for these $H^C$-chiral gauge transformations, not only for vectorial gauge transformations. It may be somewhat confusing to call it “gauge fixing”, since the chiral anomaly exists. More properly we should call it a transformation of the path-integration variables, namely from the measure of $U(1)$-gauge field to the product of the measure along the chiral gauge orbits and the measure of the modulus. According to the Fadeev-Popov (FP) like prescription, we shall insert the following identity into (2.10);

$$1 = \int \mathcal{D}a \mathcal{D}X \mathcal{D}Y \delta(A - h^a)\Delta_{FP}(a)$$

where $h = e^{X+iY}$ is a chiral gauge transformation for $H^C$ ($X, Y$ are real scalar fields, and now we choose $\frac{1}{2}t$ as the $U(1)$-generator), $a$ is the “back-ground gauge field” which describes the modulus of the $U(1)$-gauge field $A$, and $\int \mathcal{D}a$ is in effect nothing but a finite-dimensional integral. “$\Delta_{FP}(a)$” is a well-known FP determinant (the Jacobian between the path-integral measures $\mathcal{D}(h^a)$ and $\mathcal{D}X\mathcal{D}Y$), which can be rewritten by additional FP ghosts; $\xi \frac{1}{\sqrt{2}}t$ ($h^C$-valued (0,0)-form, ghost), $\bar{\xi} \frac{1}{\sqrt{2}}t$ ($h^C$-valued (0,0)-form, ghost), $\zeta \frac{1}{\sqrt{2}}t$ ($h^C$-valued (1,0)-form, anti-ghost), $\bar{\zeta} \frac{1}{\sqrt{2}}t$ ($h^C$-valued (0,1)-form, anti-ghost),

$$\Delta_{FP}(a) = \int \mathcal{D}(\zeta, \bar{\zeta}, \xi, \bar{\xi}) \exp \left[-\frac{1}{2\pi i} \int_{\Sigma} (\bar{\partial} \xi \zeta - \bar{\zeta} \partial \xi) \right].$$

(2.15)
Notice that $\Delta_{FP}$ is indeed independent of the back-ground gauge field $a$, since $H^C$ is abelian.

We should remark that the integral $\int \mathcal{D}Y$ (the vectorial direction) is a gauge volume, but $\int \mathcal{D}X$ (the axial direction) must be performed because of the existence of chiral anomaly. We need to estimate the chiral anomalies for the chiral field $g$ and the ghost system $\chi, \bar{\chi}, \psi, \bar{\psi}$ independently.

For $g$, by the Polyakov-Wiegmann identity (2.2), we obtain

$$S_G(g, h^a) = S_G(h^{-1}g, a) - S_G(h^1h, a) = S_G(h^{-1}g, a) - \frac{i}{2\pi} \int_\Sigma \{\partial X \partial X + iXF(a)\}$$

(2.16)

where $F(a) = da$ is the curvature of the back-ground gauge field $a$. Moreover since the measure $\mathcal{D}g$ has no anomaly, it holds that

$$\int \mathcal{D}g \exp\{-kS_G(h^{-1}g, a)\} = \int \mathcal{D}g \exp\{-kS_G(g, a)\}.$$  

(2.17)

On the other hand, for the ghost system $\chi, \bar{\chi}, \psi, \bar{\psi}$,

$$Z_{\chi\psi} = \int \mathcal{D}(\chi, \bar{\chi}, \psi, \bar{\psi}) \exp \left[ -\frac{1}{2\pi i} \int_\Sigma (\bar{\partial}_a \psi \chi - \bar{\chi} \partial_a \bar{\psi}) \right]$$

(2.18)

$$= \int \mathcal{D}(\chi, \bar{\chi}, \psi, \bar{\psi}) \exp \left[ -\frac{1}{2\pi i} \int_\Sigma (\bar{\partial}_a \psi \chi - \bar{\chi} \partial_a \bar{\psi}) \right] \times \exp \left[ \frac{i}{\pi} \int_\Sigma \{\partial X \partial X + iXF(a) + \frac{1}{2i}XR(g)\} \right],$$

(2.19)

where $\begin{pmatrix} 0 & R(g) \\ -R(g) & 0 \end{pmatrix}$ is the curvature of the Levi-Civita connection $\begin{pmatrix} 0 & \omega(g) \\ -\omega(g) & 0 \end{pmatrix}$ with respect to $g$. This abelian anomaly appears as the “back-ground charge”. It can be easily estimated by direct computations or more elegantly by making use of the index theorem associated with the twisted Dolbeault complex (refer for example [22]); $O \longrightarrow \Omega^{(0,0)} \longrightarrow \Omega^{(0,1)} \longrightarrow O$ with respect to the ghost system $\chi, \psi$. This leads that the back-ground charge of the ghost’s gauge current (which is nothing but the “− ghost number current”) is equal to $\frac{1}{2} \chi(\Sigma) + C_1$. Here $\chi(\Sigma)$ stands for the Euler number of $\Sigma$ and $C_1$ means the 1st Chern number of the holomorphic line bundle such that $\psi$ is one of the section of it. The terms $\frac{1}{2} \chi(\Sigma), C_1$ correspond respectively to the terms $\frac{1}{2\pi} \int_\Sigma XR(g)$, $-\frac{1}{\pi} \int_\Sigma XF(a)$ in (2.19).
Taking all things into account, we can arrive at the gauge fixed model;

\[ Z_{\text{gauge}}[g] = \int D a Z_{\text{gauge}}[g, a] \]
\[ Z_{\text{gauge}}[g, a] = \int D(g, X, \chi, \psi, \bar{\psi}, \zeta, \bar{\zeta}, \xi, \bar{\xi}) \]
\[ \times \exp \{-kS_G(g, a) - S_{\chi\psi}(\chi, \bar{\chi}, \psi, \bar{\psi}, a)\} \times \exp\{-S_X(X, a, \omega(g)) - S_{\zeta\xi}(\zeta, \bar{\zeta}, \xi, \bar{\xi})\}. \] (2.20)

In (2.20) we introduce the following notations;

\[ S_{\chi\psi}(\chi, \bar{\chi}, \psi, \bar{\psi}, a) = \frac{1}{2\pi i} \int_{\Sigma} (\bar{\partial}_{\alpha} \psi \chi - \bar{\chi} \partial_{\alpha} \bar{\psi}), \] (2.21)
\[ S_{\zeta\xi}(\zeta, \bar{\zeta}, \xi, \bar{\xi}) = \frac{1}{2\pi i} \int_{\Sigma} (\bar{\partial}_{\alpha} \zeta - \bar{\xi} \partial_{\alpha} \bar{\xi}), \] (2.22)
\[ S_X(X, a, \omega(g)) = \frac{1}{2\pi i} \int_{\Sigma} \{ \bar{\partial} X \partial X + i \alpha_{+} X F(a) + i \alpha_{-} X R(g) \}; \] (2.23)

\[ (\alpha_{+} = \sqrt{k + 2}, \alpha_{-} = -\frac{1}{\sqrt{k + 2}}). \]

where we have properly rescaled the scalar field \( X \) in (2.23).

Let us turn to the local operator formalism. We fix a coordinate neighborhood \( U \subset \Sigma \) and a holomorphic coordinate system \( z \) on \( U \), in which we work. For the time being we shall set \( a = 0, \omega(g) = 0 \) on \( U \) in order to make things easy, which is of course always possible if \( U \) is sufficiently small.

From the gauge fixed action (2.20) we can immediately compute the total energy-momentum (EM) tensor \( T_{\text{tot}} \) of the matter part (we treat only the holomorphic sector);

\[ T_{\text{tot}} = T_{g} + T_{X} + T_{\chi\psi} + T_{\zeta\xi}. \] (2.24)

In (2.24) \( T_{g} \) is the EM tensor of the \( SU(2) \)-WZW model (the Sugawara EM tensor) and \( T_{X} \) is that of the real scalar field \( X \). \( T_{\chi\psi}, T_{\zeta\xi} \) are those of the ghost fields. Their explicit forms are given by;

\[ T_{g} = \frac{1}{2(k + 2)} : (J_g, J_g) :, \] (2.25)
\[ T_{X} = - : (\partial_{\bar{z}} X)^2 + \alpha_{-} \partial_{\bar{z}}^2 X, \] (2.26)
\[ T_{\chi\psi} = - : \chi_{\bar{z}} \partial_{\bar{z}} \psi :, \] \[ T_{\zeta\xi} = - : \zeta_{\bar{z}} \partial_{\bar{z}} \xi :, \] (2.27)
where \( J_g = -k \partial_z g^{-1} \) is the chiral current of the \( SU(2) \)-WZW model, and "\[\]" stands for the usual normal ordering. The free scalar field \( X \) and the ghost fields \( \chi, \psi, \zeta, \xi \) satisfy the following operator product expansions (OPEs):

\[
\partial_z X(z) \partial_w X(w) \sim -\frac{1}{2(z-w)^2}, \quad \chi(z) \psi(w) \sim \frac{1}{z-w}, \quad \zeta(z) \zeta(w) \sim \frac{1}{z-w}.
\]

(2.28)

Moreover we express the chiral current \( J_g \) by components for later convenience;

\[
J_g = J^0_g t + J^+_g f + J^-_g e,
\]

(2.29)

with the following OPEs satisfied;

\[
J^0_g(z) J^\pm_g(w) \sim \frac{\pm 1}{z-w} J^\pm_g(w), \quad J^0_g(z) J^0_g(w) \sim \frac{k}{2(z-w)^2}, \quad J^+_g(z) J^-_g(w) \sim \frac{k}{(z-w)^2} + \frac{2}{z-w} J^0_g(w),
\]

(2.30)

other combinations have no singular OPEs.

From these explicit forms it is easy to show that \( T_{tot} \) has vanishing central charge;

\[
c_{tot} = c_g + c_X + c_{\chi \psi} + c_{\zeta \xi} = \frac{3k}{k+2} + (1 + 6\alpha_-^2) + (-2) + (-2) = 0.
\]

(2.31)

where \( c_g, c_X, c_{\chi \psi}, c_{\zeta \xi} \) are the central charges of \( T_g, T_X, T_{\chi \psi}, T_{\zeta \xi} \) respectively. Thus the matter part of our model is a topological conformal model as was expected.

In order to extract the physical degrees of freedom from our total matter system, it is important to construct the BRST complex of which cohomology classes describe them. From our treatment of the gauge symmetry our BRST complex should be characterized by two kind of BRST charges.

The on-shell BRST symmetry (supersymmetry) (2.13) should be characterized by the following BRST charge;

\[
Q_{G/H} = \frac{1}{2\pi i} \oint dz G^+_g, \quad (2.32)
\]

where the BRST current \( G^+_g \) is defined as

\[
G^+_g = -\alpha_- \psi J^+_g. \quad (2.33)
\]
Similarly the gauge fixing for the \( H^C \)-gauge transformations should be performed by

\[
Q_{H^C} = -\frac{\alpha_-}{2\pi i} \oint dz \xi J^0_{\text{tot}} = \frac{1}{2\pi i} \oint dz G^+_C, \tag{2.34}
\]

where we define

\[
G^+_C = -\alpha_- (\xi J^0_{\text{tot}} - \partial_z \xi). \tag{2.36}
\]

(Of course the term “\( \partial_z \xi \)” does not contribute, since it is a total derivative.) In (2.34), (2.36) \( J^0_{\text{tot}} \) is the “total \( U(1) \)-current” of the model;

\[
J^0_{\text{tot}} = J^0_\chi + J^0_\chi \psi + J^0_X \equiv \hat{J}^0 + J^0_X, \tag{2.37}
\]

where \( \hat{J}^0 = J^0_\chi + J^0_\chi \psi \) is the \( U(1) \) current associated with the \( U(1) \)-gauge symmetry, and \( J^0_X \) is that of the gauge field. Their explicit forms are easily computed from (2.21), (2.23);

\[
J^0_\chi \psi = :\chi_z \psi :, \tag{2.38}
\]

\[
J^0_X = \alpha_+ \partial_z X. \tag{2.39}
\]

Now it may be useful to give some comments on the \( U(1) \)-currents appeared above. The \( U(1) \)-currents \( \hat{J}^0, J^0_{\text{tot}} \) are invariant under the \( Q_{G/H} \)-transformation, and are not exact with respect to this BRST charge. Especially the total \( U(1) \)-current \( J^0_{\text{tot}} \) has no Schwinger term in its current-current OPE. This guarantees the nilpotency of the BRST charge \( Q_{H^C} \). (c.f. \[18\]) It is of course \( Q_{H^C} \)-trivial;

\[
J^0_{\text{tot}} = \{ Q_{H^C}, \alpha_+ \zeta \}, \tag{2.40}
\]

as is a common feature of the standard BRST theory.

We can also check that \( Q_{G/H} \) and \( Q_{H^C} \) anti-commute with each other. Hence our BRST complex, whose differential is given by the total BRST charge \( Q_{\text{tot}} = Q_{G/H} + Q_{H^C} \), has a double complex structure. The physical observables are constructed as the cohomology classes of this double complex.
In order to have some insight into the above cohomological problem we give attention to the fact that $T_{\text{tot}}$ can be written as the following BRST exact forms:

$$T_{\text{tot}} = \{Q_{\text{tot}}, G_{G/H}^{-} + G_{H^C}^{-}\}$$

$$= \{Q_{G/H}, G_{G/H}^{-}\} + \{Q_{H^C}, G_{H^C}^{-}\}$$

$$= T_{G/H} + T_{H^C},$$

where $G_{G/H}^{-}, G_{H^C}^{-}$ are defined as

$$G_{G/H}^{-} = -\alpha_\chi z J_g,$$

$$G_{H^C}^{-} = -\alpha_\zeta (\hat{j}_0^0 - J_X^0) - \partial_z \zeta_z,$$

and we set

$$T_{G/H} = \{Q_{G/H}, G_{G/H}^{-}\},$$

$$= \frac{1}{2(k+2)} \{ (J_g, J_g) : - 2(\hat{j}_0^0)^2 : \}$$

$$+ \frac{1}{k+2} \partial_z \hat{j}_0^0 : \chi_z \partial_z \psi :,$$

$$T_{H^C} = \{Q_{H^C}, G_{H^C}^{-}\}$$

$$= \frac{1}{(k+2)} (\hat{j}_0^0)^2 : - \frac{1}{k+2} \partial_z \hat{j}_0^0$$

$$- (\partial_z X)^2 : + \alpha_\zeta \partial_z \zeta :.$$

From the look of (2.33), (2.42), and (2.44) it is clear that \{$T_{G/H}, \ J_{KS}, \ G_{G/H}^{\pm}, \ G_{G/H}^{-}\}$ generate the topological conformal algebra (TCA) constructed by twisting the Kazama-Suzuki model for $SU(2)/U(1) = \mathbb{CP}^1$ ($c = c_k \equiv \frac{3k}{k+2}$) [9], with the familiar definition of $U(1)$-current;

$$J_{KS} = : \psi \chi_z : + \frac{2}{k+2} \hat{j}_0^0. \quad (2.46)$$

Here we notice that the generators $T_{G/H}, \ J_{KS}, \ G_{G/H}^{\pm}$ are invariant under the $Q_{H^C}$-transformations. We shall call this TCA as the “Kazama-Suzuki (KS) sector” of the topological $SU(2)/U(1)$-model.

On the other hand, suppose we bosonize the $U(1)$-current $\hat{j}_0^0$ by introducing a real scalar boson $\varphi$ compactified in the circle with the radius $\alpha_+$ (normalized by $\partial_z \varphi(z) \partial_w \varphi(w) \sim -\frac{1}{2(z-w)^2}$);

$$\hat{j}_0^0 \equiv i \alpha_+ \partial_z \varphi , \quad (2.47)$$
and further define a complex boson $\hat{\phi}$ as $\hat{\phi} = \varphi - iX$. We can easily obtain the following expressions for $T_{HC}$ (2.45), $G_{HC}^+$ (2.36) and $G_{HC}^-$ (2.43) in terms of $\hat{\phi}$;

\[
T_{HC} = - \partial_z \hat{\phi}^\dagger \partial_z \hat{\phi} : + i \alpha_- \partial^2_z \hat{\phi} - : \zeta_z \partial_z \xi ;, 
\]

\[
G_{HC}^+ = i \xi \partial_z \hat{\phi} + \alpha_- \partial_z \xi, 
\]

\[
G_{HC}^- = i \zeta_z \partial_z \hat{\phi}^\dagger + \alpha_- \partial_z \zeta_z. 
\]

(Notice the relation $J_{tot}^0 = i \alpha_+ \partial_z \hat{\phi}$.)

Surprisingly this EM tensor (2.48), the fermionic currents $G_{HC}^+$ (2.49), $G_{HC}^-$ (2.50), and the $U(1)$-current;

\[
J_{CG} = : \xi \zeta : + i \alpha_- \partial_z \hat{\phi} - i \alpha_- \partial_z \hat{\phi}^\dagger 
\]

generate another TCA, which precisely coincides with that constructed by twisting the so-called “Coulomb gas representation” of the $N = 2$ minimal model [23]. That is to say, they can be identified with that of the topological matter system in the K.Li’s theory of 2D gravity!! [10] We shall call this TCA the “Coulomb gas (CG) sector” in contrast to the Kazama-Suzuki sector.

These two TCAs are completely isomorphic to each other because their untwisted $N = 2$ superconformal algebras are both of $c = \frac{3k}{k + 2}$ (i.e. the minimal model). But it should be noticed that they are “essentially” independent. To be specific, $G_{G/H}^+$, $G_{G/H}^-$, $T_{G/H}$, $q_{KS} \equiv \frac{1}{2\pi i} \oint dz J_{KS}$ (not $J_{KS}$ itself!) commute (or anti-commute for fermionic currents) with the corresponding operators of the CG sector and they are not BRST-equivalent. Therefore these two independent TCAs should correspond to different geometrical degrees of freedom, that is, describe different moduli spaces.

However, as will be seen below, in the set of physical observables of the total system we can find the objects naturally supposed to be common objects of the two sectors, which are no other than the “chiral primary fields”. This aspect is one of the interesting problems in studies of the topological gauged WZW models. The similar phenomena in the higher rank cases will be seen in [34].

Let us study the physical observables of the model, anticipating the appearance of two kinds of moduli spaces. Of course they are defined as the BRST cohomology classes with respect to the total BRST charge $Q_{tot}$. If degeneration occurs at the “$E_2$ term” of the
spectral sequence defined by this total double complex, one can precisely compute the total cohomology, by taking cohomology first by $Q_{G/H}$ and then by $Q_{H^C}$ or vice versa. We can see that, when taking cohomology first by $Q_{G/H}$, the system is characterized by the CG sector, while, when taking cohomology first by $Q_{H^C}$, it is characterized by the KS sector. Assuming the above degeneration, we may conclude that there exists a kind of “duality” between those two sectors. Related with this observation it is appropriate to note that the following simple relation between $J_{KS}$ and $J_{CG}$ holds;

$$J_{KS^-} : \psi_{xz} := J_{CG^-} : \xi_z : \pmod{\text{BRST}}. \quad (2.52)$$

This suggests that the bosonic parts of the physical observables for the two sectors are completely common objects. The difference between them consists in the ghost sector.

Although we have not yet succeeded in the complete resolution of this total BRST complex, we can present the following set of physical observables;

$$O_{k,l,n}(z) = : \psi(z)^k e^{-ik\alpha_-\hat{\varphi}(z)} \xi(z)^l e^{in\alpha_-\hat{\varphi}(z)} :, \quad (2.53)$$

where $k, l = 0, 1, n \in \mathbb{Z}$. Here we omitted the “super partners”, which can be constructed from $(2.53)$ by making use of $G_{G/H}^-, G_{H^C}^-$. It should be remarked that the scalar field $\hat{\varphi}$ and the ghost $\psi$ are not independent;

$$i\alpha_+ \partial_z \hat{\varphi}(z) \psi(w) \sim -\frac{1}{z - w} \psi(w), \quad (2.54)$$

thus we need the term $e^{-ik\alpha_-\hat{\varphi}(z)}$ in order to restore the BRST invariance.

At first glance, it seems that there exist infinite dimensional physical degrees of freedom. However, more strictly we must further restrict the physical Hilbert space, since the Hilbert space characterizing the theory should be constructed from that of the original $SU(2)$ WZW model, which is smaller than the Fock space of $\varphi$. This leaves at most finite physical degrees of freedom, because the WZW model with level $k \in \mathbb{Z}_{\geq 0}$ possesses only finite primary fields. Considering the no-ghost sector, we can take the following subset of $(2.53)$;

$$O_j(z) = : e^{-ij\alpha_-\hat{\varphi}(z)} :, \quad (j = 0, 1, \cdots, k). \quad (2.55)$$

These are precisely the same as the “chiral primary fields” (of the “$A_{k+1}$-type”) in the untwisted $N = 2$ minimal model. It is worth remarking that these chiral primary fields
correspond to the primary states having the forms; \(|j,j\rangle_g \otimes \mid -j\rangle_X \ (j = 0, 1, \cdots, k)\), where “\(|j,j\rangle_g\)” means the “highest weight component of the primary states with spin \(j/2\)” of \(g\) and “\(|-j\rangle_X\)” stands for the primary state of \(X\) having a value \(-j\) as the \(U(1)\)-charge for \(J^h_X\).

Let us argue on the geometrical interpretation of the model. The moduli space (of the matter system) is thought to be a Kähler space;

\[
G/H \times H^C \cong \mathbb{C}P^1 \times (\mathbb{C}P^1 - \{0, \infty\}),
\]

(roughly speaking, the “target space” of our gauged WZW model.) The coset space \(G/H\) corresponds to the KS sector, and \(H^C\) corresponds to the CG sector. It is clear from the construction that the moduli space for the CG sector is indeed so, since it should correspond to the constant modes of chiral gauge transformations. However, as regards the KS sector it is not so manifest what geometrical meanings are included from our starting point, i.e. the twisted N=2 supersymmetric gauged WZW model. In fact the BRST transformations of the KS sector are the twisted version of the supersymmetric transformations, so naively it is unclear what meanings its BRST cohomology should have. But, as was already commented, our topological gauged WZW action (2.10) can be also derived from the \(G/G\)-gauged WZW model (see appendix), in which the BRST transformations for the KS sector (2.13) is obtained by performing the gauge fixing corresponding to the gauge degrees of freedom of the \(G/H\)-part of gauge transformations. Therefore we can regard its moduli space as above, and further give the following simple geometrical observation: The zero-modes of \(\psi, \bar{\psi}\) (the ghosts for the KS sector) are identified with \((1,0)-\) and \((0,1)-\)forms on the moduli space \(G/H\), the BRST charges \(Q_{G/H}, \bar{Q}_{G/H}\) are respectively identified with the holomorphic and anti-holomorphic parts of the exterior derivative on this Kähler space. The similar observation is possible with respect to the \(H^C\)-part of the moduli space, the ghosts for the CG sector \(\xi, \bar{\xi}\), and the BRST charges \(Q_{H^C}, \bar{Q}_{H^C}\).

It is a challenging problem how we should interpret geometrically the bosonic part (i.e. the \(\hat{\varphi}\)-part) of the chiral primary fields. Remarkably they have fractional \(U(1)\)-charges, and so it is difficult to interpret them as differential forms on some moduli spaces like as ghost fields. In this respect it may be worthwhile to keep the following fact in mind: Our Lagrangian defining the model (2.1) is neither zero (nor BRST exact) nor some topological.
invariants, namely our theory is not topologically invariant at the classical level. This is the main distinction of our model from the topological field theories which start with zero Lagrangians (for example see [3], [4], [5]). In these theories all the quantum effects are canceled out, and all the concepts can be translated into mathematical languages. In particular the objects with fractional $U(1)$-charges as above do not come into sight. But our model is a theory with an essentially non-zero Lagrangian. It is thought that the quantum effects are not completely canceled out, although we have still only topological invariants as the physical observables. We might be able to say that the appearance of these objects is a purely quantum effect.

On this problem, the formulation presented in [11] is interesting and suggestive from geometrical viewpoints. The author introduces the concept such as the “$(k+2)$th root of the canonical line bundle over the Riemann surface” in order to deal with the above objects with fractional $U(1)$-charges in the topological matter. The difference between the formulation given in [11] and ours mainly consists in the choice of the gauge fixing prescriptions: In [11] the gauge fixing is performed by a kind of “fixed point manipulation”, in which the physical degrees of freedom are directly extracted by some simple geometrical argument, and then the model reduces to “the abelian model”, that is, the topological $U(1)$ gauge theory coupled to 2D gravity. Meanwhile our prescription given above is the standard BRST-procedure, in which we design to confine the unphysical (gauge) degrees of freedom by introducing the new unphysical ones, that is, the (non-zero modes of) ghost fields. It may be plausible that the “fixed point manipulation” in [11] corresponds to the Lefschetz formula applied to the BRST complex with $Q_{G/H}$, and that “the abelian model” in [11] is related with the CG sector obtained above [33].

Taking account of this consideration and the fact that the twisted Coulomb-gas model is no other than the K.Li’s minimal topological matter [10], we may well say that the CG sector will play an important role in order to specify the relation with 2D gravity. For this purpose we think it convenient to drop the KS sector out in our model.

3 Relation with 2D Gravity

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3.1 Relation with the K.Li’s Theory of 2D Gravity

In the previous section we have found out that the topological \( SU(2)/U(1) \)-model is characterized by two independent topological conformal algebras; the KS sector ("the coset part") and the CG sector ("the Cartan part"). In this section we would like to discuss the coupling to 2D gravity of our model. As was already mentioned, the CG sector is thought more important than the KS sector for this aim. Therefore, in the following arguments of this paper we will kill the KS sector out. This only means to drop the zero-modes of the ghosts \( \chi, \psi \) by taking the equivariant cohomology in defining the physical states, since the non zero-modes of the ghost fields are already suffered quartet confinements \[33\].

To complete the quantization of our model, we must integrate the residual degrees of freedom, i.e. the back-ground gauge field \( a \) and the metric \( g \). Introducing a zweibein \((e^+, e^-) \) \((e^-) = e^- \) instead of \( g \) and then defining an \( ISO(2) \)-gauge field \( A = (a, e^+, e^-) \), the partition function \( (2.10) \) can be rewritten as follows;

\[
Z = \int \mathcal{D}a \mathcal{D}g \ Z_{\text{gauge}}[a, g] \equiv \int \mathcal{D}A \ Z_{\text{gauge}}[A]. \tag{3.1}
\]

Notice that, as we saw in the previous section, the partition function of the matter \( Z_{\text{gauge}}[A] \) is topologically invariant. Namely it is invariant (modulo BRST) under any infinitesimal variation of \( A \);

\[
- \frac{1}{Z_{\text{gauge}}[A]} \frac{\delta Z_{\text{gauge}}[A]}{\delta A} = - \frac{1}{Z_{\text{gauge}}[A]} \left( \frac{\delta Z_{\text{gauge}}[A]}{\delta a_z}, \frac{\delta Z_{\text{gauge}}[A]}{\delta e^+_z}, \frac{\delta Z_{\text{gauge}}[A]}{\delta e^-_z} \right) \approx \left\{ Q_{\text{tot}}, \bar{Q}_{\text{tot}}, (\alpha_+, \alpha_-), (\alpha_+ \bar{\zeta}_z, \alpha_- \bar{\zeta}_\bar{z}), (G^-_{G/H} + G^-_{H/C}, \bar{G}^-_{G/H} + \bar{G}^-_{H/C}) \right\}.
\]

So we can perform an additional gauge fixing for the residual degrees of freedom according to the standard techniques of topological gauge theory \[4\] \[5\]. The BRST transformations for this topological symmetry are defined as follows;

\[
\delta_s A = \eta, \quad \delta_s \lambda = \pi, \quad \delta_s \eta = \delta_s \pi = 0, \tag{3.3}
\]

\[2\]The identification rule of \( h \cong u(1) \) and \( so(2) \) we are choosing is

\[
\begin{pmatrix}
\frac{i}{2} & \sqrt{2} \tan \frac{\theta}{2} \\
0 & 1
\end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
where we introduce ghosts \( \eta = (\eta^0, \eta^+, \eta^-) \), anti-ghosts \( \lambda = (\lambda_0, \lambda_+, \lambda_-) \), and \( B \)-fields \( \pi = (\pi_0, \pi_+, \pi_-) \).

Then we will determine the gauge condition for this topological symmetry. It will select the \( U(1) \)-bundle in which the \( U(1) \)-gauge field \( a \) lives. As we will see in the following the most convenient gauge condition is

\[
a = \omega(g) \quad \iff \quad D_a e^+ = D_a e^- = 0 \quad \text{(the torsion free condition)}, \\
F(a)(\equiv da) = F_0 \quad \text{(the gauge condition for the Weyl symmetry)},
\]

where \( \omega(g) \) is the Levi-Civita connection (as was already introduced) and \( F_0 \) is a fixed real 2-form. It should be remarked that this gauge condition implies the torsion free condition and the gauge condition for the Weyl symmetry. With this gauge condition we obtain the total gauge fixed form of the model;

\[
Z_{\text{tot}} = \int \mathcal{DA} Z_{\text{gauge}}[A] \exp \left[ -\frac{1}{2\pi i} \int_\Sigma \delta_s \left\{ \lambda_0 (da - F_0) + \lambda_+ D_a e^+ + \lambda_- D_a e^- \right\} \right]
= \int \mathcal{DA} Z_{\text{gauge}}[A] \exp \left\{ -S^{\text{VV}}(A) \right\}.
\]

(3.5)

Here \( S^{\text{VV}}(A) \) is no other than the Verlinde-Verlinde’s (1st step) gauge fixed action of the pure topological gravity \[^3\];

\[
S^{\text{VV}}(A) = \frac{1}{2\pi i} \int_\Sigma \left\{ \pi_0 (da - F_0) + \pi_+ D_a e^+ + \pi_- D_a e^- \right\} \\
+ \frac{1}{2\pi i} \int_\Sigma (\lambda_0 d\eta^0 + \lambda_+ D_A \eta^+ + \lambda_- D_A \eta^-),
\]

(3.6)

\[
(D_A \eta^+) = d\eta^+ + i a \wedge \eta^+ - i \eta^0 \wedge e^+, \\
(D_A \eta^-) = d\eta^- - i a \wedge \eta^- + i \eta^0 \wedge e^-.
\]

From the look of (3.5) we can conclude that our model can be naturally identified with the K.Li’s model of topological matter coupled with topological gravity!! \[^7\] (Of course, we must further fix the residual gauge symmetries in (3.3), i.e. the symmetries of the diffeomorphisms and the local Lorentz, in order to accomplish the gauge fixing. (See \[^8\] for the detail of this procedure.))

One may be afraid that the above gauge condition (3.4) might affect the topological invariance of the matter part, because under this gauge condition the back-ground gauge

\[^3\]In the following, we assume that the \( U(1) \)-bundle we are working on is the unitary frame bundle defined by \( g \) (or \((e^+, e^-))\).
field \( a \) is no longer independent of the metric. However, our case is not so, because this effect only adds BRST-exact terms to the matter part. In fact, the total EM tensor \( T_{\text{tot}} \) (2.24) is deformed into \( T'_{\text{tot}} \) by this effect;

\[
T'_{\text{tot}} = T_{\text{tot}} + \partial_z J^0 \tag{3.7}
\]

but it is equal to the original one modulo BRST-exact terms because of (2.40);

\[
T'_{\text{tot}} = T_{\text{tot}} + \{ Q_{HC}, \alpha_x \partial_z \zeta \}. \tag{3.8}
\]

So this effect will not affect the topological invariance of the matter system.

The coupling of our topological matter and topological gravity is realized as the well-known form [10]. That is, the physical observables are nothing but the tensor products of the chiral primary fields (2.53) (with some suitable shift of the scalar field \( \hat{\varphi} \), see [10]) and the observables of the pure topological gravity (3.6).

### 3.2 Relations with the KPZ and the DDK Theories

We shall try to take alternative gauge fixing procedures. Substantially, imposing the gauge condition (3.4) is equivalent to inserting the following identity into (2.10):

\[
1 = \int D X D Y \delta(A - h(\omega(g))) \Delta_{\text{FP}}, \quad (h = e^{X+iY}). \tag{3.9}
\]

This leads to

\[
Z = \int \mathcal{D} g Z_{\text{matter}}[g] \tag{3.10}
\]

where \( Z_{\text{matter}}[g] \) is given by;

\[
Z_{\text{matter}}[g] = \int \mathcal{D}(g, \chi, \bar{\chi}, \psi, \bar{\psi}) D X D Y \Delta_{\text{FP}} \\
\times \exp \left[ -kS_G(g, e^X \omega(g)) - S_{\chi\bar{\psi}}(\chi, \bar{\chi}, \psi, \bar{\psi}; e^X \omega(g)) \right]. \tag{3.11}
\]

Observing the simple fact;

\[
e^X \omega(g) = \omega(e^X g), \tag{3.12}
\]

4Here the path-integration (2.10) are performed with the domain of the gauge field restricted to the chiral gauge orbit of \( \omega(g) \).
we find that we may integrate the mode of the anomaly of the matter as the Virasoro anomaly (the Weyl or the gravitational anomaly) instead of the chiral anomaly. Namely we may perform the path-integration of the metric $g$ before completing the path-integration of the gauge field. For this aim, we must define $Z_{\text{matter}}[g]$ as the functional of the metric $g$. It is a non-trivial problem of renormalization, because now $Z_{\text{matter}}[g]$ is still a non-topological matter with $c \neq 0$ before completing the path-integration of the gauge field, contrary to the previous gauge choice in which the matter $Z_{\text{gauge}}[g,a]$ was already topological. There are two natural choices of the renormalization condition, i.e. “the KPZ type” (renormalize preserving the Weyl invariance) and “the DDK type” (renormalize preserving the diffeomorphism invariance).

If we choose the renormalization condition of the KPZ type, we can absorb the $X$-dependence of the action of (3.11) into the integration measure of the metric $g$ making use of the relation (3.12). Namely we may transform the integration variable from $g$ to $e^X g$ because of the Weyl invariance. Hence we can drop the gauge volume $\int \mathcal{D}X \mathcal{D}Y \Delta_{FP}$ off. It should be remarked the following fact: In the previous discussion we dropped the integral $\int \mathcal{D}Y$ independently, since this measure has no chiral anomaly. However, in this case we must drop them with this combination preserved, because now we would like to integrate the mode of the Virasoro anomaly and so we must always drop the gauge volume with the combination of $c = 0$ preserved.

From the above discussions we obtain the following gauge fixed form of the model;

$$Z = \int \mathcal{D}g \ Z_{\text{matter}}[g]^W, \quad (3.13)$$

where

$$Z_{\text{matter}}[g]^W = \int \mathcal{D}(g, \chi, \bar{\chi}, \psi, \bar{\psi})^W \times \exp \left[ -kS_G(g, \omega(g)) - S_{\chi \psi}(\chi, \bar{\chi}, \psi, \bar{\psi}; \omega(g)) \right]. \quad (3.14)$$

The superscript “W” indicates the Weyl invariant definition. Remark that, in this situation, the gauge fixed action (3.14) has an extra dependence on the metric $g$ through the Levi-Civita connection $\omega(g)$. Taking account of this fact, the EM tensor for the matter sector should get the following forms:

$$T_{\text{matter}} = T_g + \partial_z J^0_g + T_{\chi \psi} + \partial_z J^0_{\chi \psi} \quad (3.15)$$
\[ T_{(1,p)}^{FF} = \mathcal{T} + T_{G/H}, \quad (3.16) \]

where we set

\[ T_{(1,p)}^{FF} = - (\partial_z \varphi)^2 + i \alpha_0 \partial^2_z \varphi, \quad (3.17) \]

\[ (\alpha_0 = \alpha_+ - \alpha_- \equiv \frac{p-1}{\sqrt{p}}, \ p = k + 2). \]

\( T_{(1,p)}^{FF} \) is no other than the well-known Feigin-Fuchs representation of the \((1,p)\)-conformal matter with central charge \( c_{1,p} = 1 - \frac{6(p-1)^2}{p} \) \cite{24}!! In fact the “twist” in the expression of the RHS of (3.15) completely coincides with that of the quantum Hamiltonian reduction \( \text{à la} \) Drinfeld-Sokolov \cite{25}. Remembering the BRST-exactness of \( T_{G/H} \), which is indeed the EM tensor of the KS sector we are intending to kill out, we can conclude that our model is essentially equivalent to the \((1,p)\)-conformal matter coupled to gravity, with the correspondence; \( p = k + 2 \).

To complete the quantization we must perform the residual integral, i.e. of the metric \( g \). Since we have renormalized the matter part with the Weyl invariance preserved, we necessarily suffer the gravitational anomaly;

\[ Z_{\text{matter}}^W = Z_{\text{matter}}^W \exp \left\{ \frac{c_{1,p}}{24} S_{\text{KPZ}}(f; g) \right\}, \quad (3.18) \]

where \( f \) is an arbitrary diffeomorphism on \( \Sigma \) and \( f g \equiv f^{-1} \). The “KPZ action” \cite{14} in (3.18) is defined as follows:

\[ S_{\text{KPZ}}(f; g) = \frac{1}{2\pi} \int_{\Sigma} dv f(\bar{g}) \langle (\bar{g})^*, S(f^{-1}) \rangle \quad (3.19) \]

where

\[ S(F) = S_{zz}(F)dz \otimes dz + S_{z\bar{z}}(F)d\bar{z} \otimes d\bar{z}, \]

\[ S_{zz}(F) \equiv \frac{F_{zzz} - \frac{3}{2} \left( \frac{F_{z\bar{z}}}{F_z} \right)^2}{F_z}, \quad S_{z\bar{z}}(F) \equiv \frac{S_{z\bar{z}}(F)}{S_{zz}(F)}. \quad (3.20) \]

In the above definition “\( dv(g) \)” denotes the canonical volume element defined by the metric \( g \), “\( g^* \)” denotes the dual metric of \( g \), and “\( \langle , \rangle \)” means the usual contraction of tensor fields.

\[ ^5 \text{This definition is the same as the well-known one. If we take the “light-cone gauge” (on the 2D space-time with the metric of the Lorentzian signature), our definition (3.19) reduces to the form given in [14].} \]
Needless to say, this KPZ action is related with the gravitational anomaly as the gauged WZW action \((2.1)\) and the Liouville action are related with the chiral and the Weyl anomalies respectively. Here the Liouville action is defined as \( [15], [16] \) (with the cosmological term omitted);

\[
S_L(\sigma; g) = \frac{1}{2\pi i} \int_{\Sigma} \{ \bar{\partial}\sigma \partial\sigma + 2iR(g)\sigma \}.
\]  

\((3.21)\)

The RHS \((3.21)\) is defined with respect to the complex structure so that the metric \(g\) becomes Kähler. The KPZ action is manifestly Weyl invariant, but behaves as the 1-cocycle with respect to diffeomorphisms. On the contrary, the Liouville action is clearly invariant under any diffeomorphism, but is the 1-cocycle with respect to the Weyl rescaling.

In order to integrate the metric \(g\), we need to clarify the definition of the measure \(Dg\) (the measure of the “dynamical metric”). It should be remarked that the measure \(Dg\) should possess no Virasoro anomaly, because, if this is not the case, it is contrary to the existence of topological gravity. Therefore, if we parametrize the dynamical metric \(g\) as \(g = e^{\sigma}(\hat{g})\), where \(e^{\sigma}, f\) of course mean respectively the “Liouville mode”, the mode of the diffeomorphisms, and \(\hat{g}\) is the “back-ground metric” (the modulus of \(g\)), we must define this measure as follows;

\[
Dg = D\hat{g} D(\sigma; \hat{g}) \exp \left\{ \frac{1}{24} S_L(\sigma; \hat{g}) \right\} D(f; \hat{g})
\times \exp \left\{ -\frac{26}{24} S_{KPZ}(f; \hat{g}) \right\} \Delta_{FP}(\hat{g})^W.
\]  

\((3.22)\)

Here \(D\hat{g}, D(\sigma; \hat{g}), D(f; \hat{g})\) denote respectively the measure of the modulus, that of the Liouville mode, and that of the mode of diffeomorphisms, which are defined associated with the back-ground metric \(\hat{g}\). \(\Delta_{FP}(\hat{g})^W\) is the usual FP determinant (i.e. the Jacobian between the functional measure \(D(f; \hat{g})\) and \(D^f(\hat{g})\) renormalized to be Weyl invariant. In fact this definition has no anomaly: The measure of the modulus \(D\hat{g}\), which is nothing but a finite dimensional measure, does not contribute, and \(D(f; \hat{g})\) has clearly no anomaly. Moreover \(D(\sigma; \hat{g})\) and \(\Delta_{FP}(\hat{g})^W\) possess respectively \(c = 1, c = -26\), which are precisely canceled with the (classical) anomalies in the functionals \(\exp \left\{ \frac{1}{24} S_L(\sigma; \hat{g}) \right\} (c = -1)\) and \(\exp \left\{ -\frac{26}{24} S_{KPZ}(f; \hat{g}) \right\} (c = 26)\). (c.f. \([13], [14], [25]\) Alternatively we may say that the following identity should hold;

\[
1 = \int D\hat{g} D(\sigma; \hat{g}) \exp \left\{ \frac{1}{24} S_L(\sigma; \hat{g}) \right\} D(f; \hat{g})
\]  

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Inserting this identity into (3.13), we obtain the final gauge fixed form;

\[ Z = \int \mathcal{D}\hat{g} \mathcal{D}(f, b, \bar{b}, c, \bar{c}; \hat{g})^W \exp \left\{ \frac{1}{24} S_L(\sigma; \hat{g}) - \frac{c_1}{24} S_{\text{KPZ}}(f; \hat{g}) - S_{bc}(b, \bar{b}, c, \bar{c}; \hat{g}) \right\} . \]  

(3.24)

Here we have introduced the usual \( b, c \)-ghosts with \( c = -26 \) to rewrite the functional determinant \( \Delta_{\text{FP}}(\hat{g})^W \). This implies that by choosing the renormalization condition of the “KPZ type”, our model reproduces the famous formulation of the 2D induced gravity by Knizhnik-Polyakov-Zamolodchikov [14].

The total central charge is of course equal to zero [14].

\[ c^{\text{tot}} = c_{1,p} + c_f + c_{bc} = c_{1,p} + (26 - c_{1,p}) + (-26) = 0. \]  

(3.25)

Next let us turn to the case that the renormalization condition of the DDK type, i.e. renormalizing without the gravitational anomaly, is chosen. We shall express the partition function of the matter part under this renormalization condition by “\( Z_{\text{matter}}[g]^D \)”. (The superscript “D” indicates the diffeomorphism invariance corresponding to the superscript “W” indicating the Weyl invariance.) The transformation from \( Z_{\text{matter}}[g]^W \) to \( Z_{\text{matter}}[g]^D \) is essentially the well-known procedure of the “finite renormalization” with the bare Lagrangian fixed, and so we can likewise regard the matter \( Z_{\text{matter}}[g]^D \) as the \((1,p)\)-conformal matter. However, in this case the effect of non-zero central charge of the matter part appears as the Weyl anomaly in place of the gravitational anomaly;

\[ Z_{\text{matter}}[e^\sigma g]^D = Z_{\text{matter}}[g]^D \exp \left\{ \frac{c_1}{24} S_L(\sigma; g) \right\} . \]  

(3.26)

In the similar way as the KPZ type, inserting into (3.14) the identity;

\[
1 = \int \mathcal{D}\hat{g} \mathcal{D}(\sigma; \hat{g}) \exp \left\{ \frac{1}{24} S_L(\sigma; \hat{g}) \right\} \mathcal{D}(f; \hat{g}) \\
\times \exp \left\{ -\frac{26}{24} S_L(\sigma; \hat{g}) \right\} \Delta_{\text{FP}}(\hat{g})^D \delta(g - e^\sigma(\hat{g})),
\]  

(3.27)

we arrive at the final gauge fixed form;

\[ Z = \int \mathcal{D}\hat{g} \mathcal{D}(\sigma, b, \bar{b}, c, \bar{c}; \hat{g})^D Z_{\text{matter}}[g]^D \exp \left\{ -\frac{25}{24} S_L(\sigma; \hat{g}) - S_{bc}(b, \bar{b}, c, \bar{c}; \hat{g}) \right\} . \]  

(3.28)
This means that under this “DDK type” renormalization condition, our model reduces to the famous formulation by David-Distler-Kawai [15].

The total central charge also vanishes [15]:

\[ c^{\text{tot}} = c_{1,p} + c_{\sigma} + c_{bc} = c_{1,p} + (1 + 24 \times \frac{25 - c_{1,p}}{24}) + (-26) = 0. \] (3. 29)

4 Conclusions and Discussions

We have quantized the topological gauged WZW model associated with \( SU(2)/U(1) \) by path-integration. Our quantization scheme can be immediately generalized to the higher rank cases, i.e. to the cases for general compact Kähler homogeneous spaces \( G/H \) (not necessarily hermitian symmetric) in the almost similar manner. This will be one of the main subject of the subsequent paper [34].

We have also shown that the topological \( SU(2)/U(1) \)-model naturally reduces to the known theory of 2D gravity with matter. There were three types of the gauge fixing procedures which include the designation of the renormalization condition; the K.Li type, the KPZ type, and the DDK type. The crucial difference among these gauge conditions consists in the choice of the methods to cancel out the anomaly of the matter originated in the gauged WZW action, that is, the choice of whether one integrates the mode of the chiral anomaly, the mode of the gravitational anomaly, or the mode of the Weyl anomaly.

In the K.Li gauge we can get the topological matter before performing the path-integration of the metric, and hence automatically make topological gravity couple with it. On the contrary, in the KPZ or the DDK gauge we must integrate the metric in order to make the central charge of the system vanish. It has not yet been known how we should perform the integration on the moduli space in these gauges.

Let us recall that in the the K.Li gauge the physical operators (of the matter part) are given as the chiral ring generated by the elements having “vertex operator” forms; \( e^{i\beta \hat{\phi}(z)} \), where \( \beta \) takes a value in \( \{ (1 - s)\alpha_+ ; s = 1, \ldots, k + 1 \equiv p - 1 \} \). This can be rewritten in the form \( e^{i\beta \hat{\phi}(z)} e^{\beta X(z)} \), and the “\( \hat{\phi} \)-part” \( e^{i\beta \hat{\phi}(z)} \) is no other than one of the primary fields of the (1, \( p \))-conformal matter. So we can naturally consider the chiral primary fields in the
K.Li gauge to be the “primary fields dressed with gauge field (or gravity)” of the conformal matter.

There are similar situations in the KPZ or the DDK gauge, too. In fact, it is familiar that in the DDK theory any primary field is dressed with the Liouville field $\sigma$ so that the total conformal weight is equal to $(1, 1)$ in the influence of gravity [15]. In the KPZ theory the appearance of the gravity dressing is not so manifest, since this effect is described by the “hidden $SL(2, \mathbb{R})$-current algebra”, which can be understood by making related to the coadjoint orbits of the Virasoro group (the group of diffeomorphisms on $S^1$ with central extension), or from the viewpoint of the Hamiltonian reduction of the $SL(2, \mathbb{R})$-WZW model. [26] [25] But this is thought to have intrinsically the same origin as above.

It is well-known that topological gravity is equivalent to the intersection theory on the moduli space of $\Sigma$ [7]. The physical observables (usually denoted by “$\sigma_n$”) are identified with the de Rham cohomology classes on the moduli space. The degrees of freedom for infinitesimal deformations of the modulus are carried by zero-modes of the $\eta$-ghost (it is in our notation, of course it is written as “$\psi$” in [8]), which are identified with 1-forms on the moduli space.

On the other hand, in the KPZ or the DDK gauge the degrees of freedom for the deformations of the modulus are carried by the zero-modes of the $b$-(anti-)ghost, which correspond to vector fields on the moduli space. The correspondence of the physical observables between the K.Li gauge and the KPZ or the DDK gauge is only a partially solved problem even now [29].

It is also interesting to search the correspondence between these theories for the general $(q, p)$-conformal matter. It might be suggestive that in our formalism the $(1, p)$-conformal matter comes into sight by suffering the same twist which occurs in the quantum Hamiltonian reduction à la Drinfeld-Sokolov [25]. So naively it might be plausible that the correspondence can be extended to the case of the general $(q, p)$-matter such as $k + 2 = \frac{p}{q}$, since this correspondence is the well-known one in the theory of the quantum Hamiltonian reduction. In topological gravity we must suitably place some physical operators as the back-ground sources in order to treat the general $(q, p)$-matter [9], while, in the free field realization of the WZW model [28], the parameter $k$ appears as the back-ground charge of the model. Hence there might exist some kind of connection between changing the
parameter $q$ and making the level $k$ fractional.

The extensions of our investigation on 2D gravity to the cases of general flag manifolds $G/H$ are also interesting subjects. The quantum theory of these cases will include “topological Yang-Mills” besides topological gravity. In other words we may say that in these systems gravity and gauge fields coexist and both couple with some conformal matters (a continuum limit of the “lattice gauge theory on the random lattice”!). Especially we can make the following observation in the topological $G/T$-model ($T$ is the maximal torus of $G$). \cite{34}: If we quantize this model in the same way as the $SU(2)/U(1)$-case with the KPZ or the DDK like gauge, we find out that the model suffers the same twist as the Hamiltonian reduction \cite{27} as above. From this observation it is plausible that we can get the $W_G$-conformal matter from the topological $G/T$-model. Therefore it will be meaningful to study the “$W_G$-gravity” \cite{30}, \cite{31} based on the topological $G/T$-model. The detail will be given in the subsequent paper \cite{34}.

As the last remark of this paper, we comment on the phenomenon that happens when we set the back-ground gauge field $a$ as $a = \omega_{\text{spin}}(g)$ (“the spin connection”, symbolically we can write it as \(\frac{1}{2}\omega(g)\)), instead of fixing as $a = \omega(g)$. In this situation we find that the KS sector of the model suffer the “inverse twist”, and we can regain the (untwistd) $N = 2$ KS model for $CP^1$, which is a theory of $c \neq 0$. But the CG sector is not affected, since the back-ground gauge field $a$ is decoupled from the ghosts of the CG sector $\xi, \zeta$.

The similar phenomenon exists in the higher rank cases, too. In the case $G/H$ is hermitian symmetric we can get the corresponding KS model. But in a generic case, although we can easily show that the KS sector becomes some non-topological CFT as well, we cannot reproduce the original supersymmetric gauged WZW model, especially it is not likely that there exists a $N = 2$ SUSY.

These studies will be given in the paper \cite{35}.
Appendix

A Another Derivation of the Topological Gauged WZW Model

In this appendix we intend to derive the topological $G/H$-model, which was defined as the twisted version of the supersymmetric gauged WZW model \((2.3)\), from the $G/G$-gauged WZW model. The $G/G$-gauged WZW model is nothing but the one regarding the gauge field $A$ as $g$-valued in \((2.1)\):

$$Z_{G/G} = \int \mathcal{D}g \mathcal{D}g \mathcal{D}A \exp\{-kS_G(g, A)\} \equiv \int \mathcal{D}g \ Z_{G/G}[g]. \quad (A. 1)$$

Of course this model is equivalent to the $G/G$-coset CFT, which is obviously a theory of $c = 0$, so naively a trivial theory with no physical degrees of freedom.\(^6\) As was already commented, the mechanism of this type cancellation of central charge can be understood by means of the similar logic given in \([15]\). The path-integrations of “geometrical fields”, such as metric (or complex structure), gauge fields, are capable to cancel the anomalies of the matter coupled with them. Particularly, in the $G/H$-gauged WZW model such cancellation of the chiral anomaly is “partial”, while, in the $G/G$-case the cancellation occurs completely. Since the model we want is topologically invariant but has non-trivial finite physical degrees of freedom, we must try to avoid this triviality.

Now we shall give the following simple observation; Let $H$ be a closed subgroup of $G$ such that the coset space $G/H$ becomes Kähler, which implies in particular that $H$ includes the maximal torus $T$ of $G$. We denote its Lie subalgebra by $h$. In this situation any element $g$ of $G$ can be expressed as $g = fh, \ h \in H, \ f \in G$, since $H$ includes the maximal torus. Making use of this fact pointwisely, we can connect necessarily the chiral field $g$ to some $H$-valued field along a vectorial (so, non-anomalous) gauge orbit. Hence we may suppose that the chiral anomaly of the gauged WZW action in effect exists only along the (axial) $H^C$-direction. Taking this observation into account, it is plausible that

\(^6\)More strictly, we can find that this model becomes equivalent to topological gravity plus topological Yang-Mills $Z = \int \mathcal{D}g \mathcal{D}A$ \([32]\).
the \( G/G \)-gauged WZW model leaves the topological invariance (i.e. the property \( c = 0 \)), even if we restrict the domain of the path-integration of \( A \) to the subspace composed of the elements expressible in the form \( A = \Omega A' \), where \( \Omega \) is a \( G \)-valued gauge transformation and \( A' \) has only \( h \)-components. In fact, from the above observation the integral of the gauge field \( A \) so restricted is sufficient to cancel out the chiral anomaly.

In the following arguments we shall impose this restriction on the functional space of \( A \) integrated, that is, define the path-integration space of \( A \) as follows;

\[
\mathcal{A}_{\text{rest}} = \{ A : A = \Omega(h), \forall \Omega \in \mathcal{G}_G, \forall h \in \mathcal{G}_H^C, \\
a : \text{the modulus of the gauge field} \},
\]

where \( \mathcal{G}_G \) denotes the space of \( G \)-valued (vectorial) gauge transformations and \( \mathcal{G}_H^C \) denotes the space of \( H^C \)-valued chiral gauge transformations.

Let us try to perform the gauge fixing of the model (A.1) for the \( G/H \)-part, keeping the above assumption in mind. We shall follow the standard BRST prescription, with the gauge condition;

\[
(m_+, A^{10}) = 0, \quad (m_-, A^{01}) = 0.
\]

\( m_\pm \) was introduced in section 2, see (2.4). Actually it is sufficient to impose only the 1st condition, since our gauge field \( A \) has the property \( A^{10} = -A^{01} \). We introduce the following BRST ghost system and the corresponding BRST transformations;

\[
\begin{align*}
ghosts \quad \psi & : m_+\text{-valued (0,0)-form}, \quad \bar{\psi} \equiv \psi^\dagger \ : m_-\text{-valued (0,0)-form}, \\
\text{anti-ghosts} \quad \chi & : m_-\text{-valued (1,0)-form}, \quad \bar{\chi} \equiv \chi^\dagger \ : m_+\text{-valued (0,1)-form}, \\
B\text{-fields} \quad B & : m_-\text{-valued (1,0)-form}, \quad \bar{B} \equiv B^\dagger \ : m_+\text{-valued (0,1)-form}, \\
\text{BRST transformations} \quad \delta_{G/H} A &= D_A \psi \equiv d\psi + [A, \psi], \\
\tilde{\delta}_{G/H} A &= D_A \bar{\psi} \equiv d\bar{\psi} + [A, \bar{\psi}], \\
\delta_{G/H} g &= [\psi, g], \quad \tilde{\delta}_{G/H} g = [\bar{\psi}, g], \\
\delta_{G/H} \chi &= B, \quad \tilde{\delta}_{G/H} \bar{\chi} = \bar{B}, \\
\delta_{G/H} \psi &= \frac{1}{2} [\psi, \psi], \quad \tilde{\delta}_{G/H} \bar{\psi} = \frac{1}{2} [\bar{\psi}, \bar{\psi}], \\
\delta_{G/H} \bar{\psi} &= \tilde{\delta}_{G/H} \psi = \frac{1}{2} [\psi, \bar{\psi}],
\end{align*}
\]

(other combinations are defined to vanish).
We obtain the partition function with the gauge fixing as follows;

\[
Z_{G/G, \text{gauge}}[g] = \int \mathcal{D}(g, A, B, \bar{B}, \chi, \bar{\chi}, \psi, \bar{\psi}) \times \exp \left[-kS_G(g, A) - \frac{1}{2\pi i} \int (\delta_{G/H} + \delta_{\bar{G}/H})(\chi - \bar{\chi}, A)\right],
\]

(A. 5)

\[
= \int \mathcal{D}(g, A', \chi, \bar{\chi}, \psi, \bar{\psi}) \times \exp \left[-kS_G(g, A') - \frac{1}{2\pi i} \int \{(\partial_{A'}\psi, \chi) - (\bar{\chi}, \partial_{A'}\bar{\psi})\}\right],
\]

(A. 6)

where \(A'\) has only h-components. In the 2nd line of the above equation we have integrated the \(B\)-fields out, which impose on the gauge field \(A\) the constraints (A. 3). Solving this constraints on the restricted space (A. 2), we obtain the gauge field \(A'\) possessing only h-components. In fact, since we have already used the gauge degrees of freedom of the \(G/H\)-part, \(A'\) cannot have the \(G/H\)-components by the assumption; \(A \in \mathcal{A}_{\text{rest}}\). However, strictly speaking, this consideration is somewhat too naive. Actually there still exist the residual gauge degrees of freedom of the \(G/H\)-part which correspond to the zero-modes of the ghosts \(\psi, \bar{\psi}\). But to make things easy, we shall neglect these zero-modes for the time being.

From the look on (A. 6), we find that it can be identified with the topological gauged WZW model associated with \(G/H\) defined by twisting the corresponding supersymmetric model in section 2!! We can further show that the BRST symmetry (2. 13) introduced in section 2, which was originated in the N=2 SUSY of the untwisted model, is thought to be the on-shell counterparts of the “off-shell” BRST symmetry for the \(G/H\)-gauge transformations (A. 4). In this respect we should notice the following fact: In the standard local operator formalism of the WZW model, the Hilbert space is factorized into the holomorphic and the anti-holomorphic sectors; \(\mathcal{H} = \mathcal{H}^+ \otimes \mathcal{H}^-\), and the right (left) action of the loop group on \(\mathcal{H}^+\) (resp. \(\mathcal{H}^-\)) is defined to be trivial.

In the \(SU(2)/U(1)\)-case, as was already shown in section 2, after performing the residual path-integration of the model (i.e. \(H^C\)-part or the CG part), we indeed get a topological conformal model with non-trivial (finite) physical degrees of freedom, that is, the chiral primary ring. In general cases we can get the similar results, too. This fact will be shown in the paper [34].

The above derivation of the topological gauged WZW model may be somewhat tech-
nical. But it is expected to give a transparent interpretation as regards the geometrical back-grounds of the model, especially of the KS sector. Now it is clear that the moduli space of the KS sector is indeed the Kähler space $G/H$, which corresponds to the space of constant modes of gauge transformations. At this stage, we cannot neglect the zero-modes of the ghost fields $\psi$, $\bar{\psi}$. The bosonic partners of these zero-modes must exist in both the space of the gauge field $A$ and the space of the chiral field $g$. For $A$, this degrees of freedom of modulus is no other than the “residual components of $G/H$-part” omitted in the above discussion. While, the modulus of $g$ is considered as generated by the action of zero-modes of the $m_\pm$-components of the chiral current of $g$. Of course these degrees of freedom are at most finite dimensional, so do not affect in essence the above arguments of the derivation. They may correspond to the holomorphic and anti-holomorphic coordinates on the moduli space, i.e. on the Kähler manifold $G/H$. Correspondingly the zero-modes of ghosts $\psi$, $\bar{\psi}$ are identified with the (1,0)- and (0,1)-forms on this space, which was already mentioned in section 2, and will generate the Dolbeault cohomology algebra on $G/H$ (c.f. [2]).

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