One Loop Gluon Gluon Scattering in Light Cone Gauge

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Abstract
We calculate the one loop amplitudes for the two gluon scattering process in light cone gauge with fermions and scalars circulating in the loop. This extends the work of [1, 2], in which only the gluon circulates the loop. By putting all fields in the adjoint representation with $N_f = 2, N_s = 6$, the scattering amplitude of gluon by gluon in the special case of $\mathcal{N} = 4$ Super Yang-Mills can be obtained. The massive fermion and scalar with arbitrary representations are also considered.

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1 Introduction

In the previous work of [1, 2], we calculated the one loop gluon scattering amplitude in the light cone world sheet setup. The main ingredients of light cone world sheet are explained in [1] and the references therein. This setup shows us how to organize the calculation in order to cancel the infamous $1/q^+$ (gauge artificial) divergences in the light cone gauge propagator. In the cases we have calculated, we were able to show that the artificial divergences actually cancel in physical processes, hence any prescription (e.g. principle value, Mandelstam-Leibbrandt prescription [6]) should work. For the helicity conserving amplitudes, infrared divergences were regulated by cutting off $1/q^+$ at $q^+ = 0$. Notwithstanding this non-covariant regulation, we still got a covariant probability by including soft Bremsstrahlung and collinear emission (absorption).

This paper extends the work of [1, 2]: in addition to the gauge field itself, scalar and fermion fields are put into the loop. This paper is organized as follows: in section 2, we review the derivation of light cone gauge Feynman rules suitable for massless fields and the light cone world sheet. In section 3, the calculational techniques of [2] are briefly reviewed. In section 4, the results are listed without the calculational details, since most of calculation is done by computers. When we specialise to the case $N_g = 1$, $N_f = 2$ (Dirac fermions) and $N_s = 6$, we find that the gluon scattering amplitude in $N = 4$ SYM takes on a very simple form.

In the last part of the paper (section 5 and 6), we deal with massive matter fields. Some of the techniques that are designed for massless fields in the first part of the calculation are not useful anymore. So we simply follow the textbook procedure, but spinors in light cone are extensively used to organize the calculation (section 5). Section 6 lists the results.

2 The Light Cone Setup

The light cone gauge Feynman rules are usually obtained by the lagrangian method, namely, we first set $A_- = 0$ and integrate out $A^-$. Then the Feynman rules can be read off from the lagrangian, which is a function of $A^1$ and $A^2$.

Here, a more flexible method is used. The gluon propagator in light cone gauge is:

$$-i(g^{\mu\nu} - \frac{g^{\mu\nu} + k^{\nu} - g^{\mu+}k^\nu}{k^2 + i\epsilon})$$

(1)

Note our metric is $diag(1, -1, -1, -1)$. The numerator can be factored into:

$$(g^{\mu\nu} - \frac{g^{\mu+}k^{\nu}}{k^+} - \frac{g^{\nu+}k^\mu}{k^+}) = -(\epsilon_\mu\epsilon_{\nu}^* + \epsilon_\mu^*\epsilon_{\nu}^*) - \frac{g^{\mu+}g^{\nu+}}{k^{+2}}$$

(2)

Where $\epsilon_{\lambda, \nu}$ are light cone gauge polarization vectors given by

$$\epsilon_\mu^\nu = \frac{1}{\sqrt{2}}(p^1 - ip^2, p^0 + p_T, 1, -i, -p^1 - ip^2) = \frac{1}{\sqrt{2}}(p^\nu, 1, -i, -p^\nu) : \epsilon_{\lambda} = \epsilon_\nu$$

(3)

They satisfy $\epsilon \cdot \epsilon^* = -1$ and $k \cdot \epsilon = 0$. These polarization vectors are defined both on-shell and off-shell.

The Feynman rules are obtained by dotting the polarisation vectors into the covariant three or four point vertices. The tri-gluon vertex Fig.1, for example, becomes

$$V_{ggg} = g f^{abc}(-\epsilon^*_{1\mu})(-\epsilon^*_{2\nu})(-\epsilon^*_{3\rho})[g^{\mu\nu}(p_1 - p_2)\rho + g^{\rho\nu}(p_2 - p_3)\mu + g^{\rho\mu}(p_3 - p_1)\nu]$$

Setting $\epsilon_1 = \epsilon_2$, $\epsilon_3 = \epsilon_\nu$, the above becomes:

$$g f^{abc}[(p_2 - p_3)\nu + (p_1 - p_3)^\lambda + (p_3 - p_1)^\mu + p_2^\rho - (p_3 - p_1)] = 2 g f^{abc}(p_1 + p_2)\nu + p_1^\mu p_2^\rho K_{21}$$

(4)
where $K_{i,j}^\mu := (p_1^\mu p_j^\mu - p_j^\mu p_i^\mu)$. They are related to spinor products according to:

$$K_{i,j}^{\vee} = p_1^\mu p_j^\mu [p_i | p_j] = p_1^\mu p_j^\mu \delta^\alpha_i \delta^\alpha_j$$

The spinor notation here is also different from the conventional one [4]. The reader can refer to the appendix for an explanation of the spinor notation.

The gluon propagator Eq.1 almost factorises into the product of two polarisation vectors. While the third term on the rhs of Eq.2 will make an extra contribution to the four point vertex. For example, consider the t-channel diagram Fig.2, the first two terms of Eq.2 can be associated with the two tri-gluon vertices, the third term, which describes the mediation of $A^-$, gives:

$$
\epsilon_1^{\star \mu} \epsilon_4^{\star \nu} V^{\nu \mu a} \frac{i}{(p_1 + p_4)^2} \epsilon_2^{\star \nu} \epsilon_3^{\star \mu} V_{\nu \rho \delta} \frac{\delta_1^{\star \rho} \delta_3^{\star \delta} (p_1 + p_4)^2}{(p_1^\mu + p_4^\mu)^2} = \frac{i \epsilon_1^{\star \mu} \epsilon_4^{\star \nu} (p_1^\mu - p_4^\nu) \epsilon_2^{\star \nu} \epsilon_3^{\star \mu} (p_1^\mu - p_4^\nu)}{(p_1^\mu + p_4^\mu)^2} \tag{5}
$$

What’s happening here is that the explicit factor of $k^2$ in the third term of Eq.2 cancels the propagator, effectively making a four point contact vertex.

2 Calling polarisations by $\vee$ or $\wedge$ is potentially confusing, especially if you are looking at the diagram up-side-down. So, sometimes it is clearer to associate $\wedge$ with 'in' and $\vee$ with 'out'.

The fermion-gluon vertex is the usual $ig \gamma^\mu t^a$. We can set $\mu = \wedge$ or $\vee$ by dotting $-\epsilon_1^{\star \nu} \epsilon_4^{\star \nu}$ into $\mu$.2
Multiplying the spinors to the gamma matrix, we get (assuming now the fermion is left-handed):
\[
ig(t^a)_{bc} \sqrt{2p_1^+ p_2^+} \left( -\epsilon^a_{\mu} \right) \left[ \begin{array}{cc} 0 & p_2^\alpha \\ (\sigma_\mu)_{\alpha\dot{\alpha}} & 0 \end{array} \right] \left[ \begin{array}{c} p_1^\alpha \\ 0 \end{array} \right] = -ig(t^a)_{bc} \sqrt{2p_1^+ p_2^+} (-\sqrt{2}|p_2| |q| |p_1|)
\]
\[
= -2ig(t^a)_{bc} \sqrt{2p_1^+ p_2^+} \left( \frac{q^\alpha}{q^\mu} - \frac{p_2^\alpha}{p_1^\mu} \right) = -2igt^a \frac{\sqrt{p_1^+ p_2^+}}{q^\mu} K_{p_1,q} \rightarrow -2igt^a \frac{p_2^+}{q^\mu} K_{p_1,q} \quad (6)
\]

In Eq.6, in order to avoid defining what is $\sqrt{p^+}$, we choose to associate $p_2^+$ instead of $\sqrt{p_1^+ p_2^+}$ to a vertex. This won’t cause any problem, since fermion loop always closes, so the square root always appears in pairs.

The fermion propagator is given by: $ip_\mu \gamma^\mu / (p^2 + i\epsilon)$. We can decompose $p \cdot \gamma$ according to
\[
p \cdot \sigma = \sqrt{2} \left| \begin{array}{cc} -p^- & -p^\land \\ -p^\lor & p^+ \end{array} \right| = \sqrt{2} \left| \begin{array}{c} \frac{p^-}{p^+} \left( \frac{p^\lor}{p^\land} \right) \end{array} \right| = \sqrt{2} \left| \begin{array}{c} \frac{p}{p^+} \left( \frac{p^\lor}{p^\land} \right) \end{array} \right| = \frac{2p^+ p^-}{p^2} \left| \begin{array}{c} p \rangle \langle p \end{array} \right| = \frac{2p^+ p^-}{p^2} \left| \begin{array}{c} p \rangle \langle p \end{array} \right| (7)
\]

So the fermion propagator almost factorises too. It is also possible to contract a pair of vertices here: the second term of Eq.7 will again cancel a propagator. For example:

\[
\begin{align*}
-92(t^d)_{ce} (t^a)_{eb} \sqrt{2p_1^+ p_2^+} |p_2| (-\epsilon^3 \cdot \sigma) \frac{i(p_1 + p_4) \cdot \sigma (-\epsilon^4 \cdot \sigma) |p_1|}{(p_1 + p_4)^2} \sqrt{2p_1^+} \\
\rightarrow -92(t^d)_{ce} (t^a)_{eb} \sqrt{2p_1^+ p_2^+} |p_2| (\sqrt{2}|q| |p_3|) \frac{i}{(p_1 + p_4)^2} \frac{(p_1 + p_4)^2}{\sqrt{2}(|p_1^- + p_4^-|)} |q| |p| \frac{2p_1^+}{p_1^+ + p_4^+} \rightarrow -92(t^d)_{ce} (t^a)_{eb} \frac{p_2^+}{p_1^+ + p_4^+} \quad (8)
\end{align*}
\]

The scalar Feynman rules have no suspense in them at all, and can be read off from any field theory book. The Feynman rules that pertain to our calculation will be summarised in the appendix. The main property of the Feynman rules above is the absence of $p^-$.  

3 Brief Description of Calculational Procedure(massless case)

In the above figure 3, the $k$’s and $q$ are the dual momenta. They are related to the real momenta coming into the three legs according to: $p_1 = k_1 - k_0$, $p_2 = k_2 - k_1$ and $p_3 = k_0 - k_2$. The unregulated integrands have a symmetry under $k_1 \rightarrow k_1 + a$, which would ensure that each diagram depends only on the real momenta. But here, as in [1, 2], we use a regulator $\exp \{-\delta(q^1)^2 + (q^2)^2\}$ that breaks this symmetry. Hence, a regulated amplitude can depend on the individual dual momenta. This seemingly unwieldy regulator is designed for the world sheet purpose, which doesn’t concern us here. The result doesn’t differ too much when a cut off regulator is used.

The calculation roughly goes as follows,
1. Exponentiate all propagators according to
\[
\frac{i}{p^2 + i\epsilon} = \int_0^\infty dT e^{iT p^2}
\]
For the schematic diagram Fig.3. We have
\[
\Gamma = \int \frac{d^4q}{(2\pi)^4} \frac{i}{(q-k_0)^2 (q-k_1)^2 (q-k_2)^2}
\]
\[
= \int \frac{d^4q}{(2\pi)^4} dT_0 dT_1 dT_2 e^{iT_0(q-k_0)^2 + iT_1(q-k_1)^2 + iT_2(q-k_2)^2}
\]
2. Integrate out $q^-$, leaving a delta function relating $q^+$ to the Feynman parameters (this step requires the absence of $q^-$ in all Feynman rules).
\[
\Gamma = \int \frac{dq^+ dq}{(2\pi)^3} \frac{1}{2} dT_0 dT_1 dT_2 \delta(\sum T_i k_i^+ - \sum T_i q^+) e^{iT_0(q-k_0)^2 + iT_1(q-k_1)^2 + iT_2(q-k_2)^2}
\]
3. Integrate $q^1$, $q^2$ using $\exp[-\delta q^2]$ as a damping factor.
\[
\Gamma = \int dT_0 dT_1 dT_2 \frac{dq^+}{(2\pi)^3} \frac{\pi}{2i(\sum T_i - i\delta)} \delta(\sum T_i k_i^+ - \sum T_i q^+) e^{iT_0(q-k_0)^2 + iT_1(q-k_1)^2 + iT_2(q-k_2)^2}
\]
\[
= \int \frac{dq^+}{16\pi^2} dT_0 dT_1 dT_2 \delta(\sum T_i - 1) \delta(\sum T_i k_i^+ - q^+) \frac{1}{T_0 T_2(q-k_0)^2}
\]
Here is the rub: we cannot simply integrate over $T_0, T_1$ and $T_2$, because the prefactor of this diagram will have up to second order poles at $q^+ = k_i^+$, $i = 0,1,2$. In order to show the dependence of gauge artificial divergences, we proceed as follows: first eliminate one Feynman parameter in favour of $q^+$:
\[
k_0^+ < q^+ < k_1^+ : T_1 = \frac{x(q^+ - k_1^+)}{k_1^+ - k_0^+}, \quad T_2 = \frac{(1-x)(q^+ - k_1^+)}{k_2^+ - k_0^+}, \quad T_0 = 1 - T_1 - T_2
\]
\[
k_1^+ < q^+ < k_2^+ : T_1 = \frac{(1-x)(q^+ - k_2^+)}{k_1^+ - k_2^+}, \quad T_2 = \frac{x(q^+ - k_2^+)}{k_0^+ - k_2^+}, \quad T_0 = 1 - T_1 - T_2
\]
Now $dT_0 dT_1 dT_2 \delta(\sum T_i - 1) \delta(\sum T_i k_i^+ - q^+) = dq^+ dx J$. After integrating out $x$, we are left with a function of $q^+$ which is defined differently in different regions: $k_0^+ < q^+ < k_1^+$ and $k_1^+ < q^+ < k_2^+$. All these can be visualized very clearly when we represent a Feynman diagram on the light cone world sheet. The details can be found in [1, 2].

Our observation is that: in each region, all poles cancel.\(^3\) Hence we can perform the final $q^+$ integral and obtain the results.

\(^3\)In the case helicity conserving amplitude, all poles cancel up to infrared terms, but since infrared divergence is always proportional to a tree, they are easy to recognize and deal with.
4 Results

Since the Feynman rules of fermions and scalars are similar to the pure gluon case, the evaluation of scalar and fermion loop is similar to [2]. In fact, for the helicity conserving amplitudes, the reader needs only work out the prefactor of each diagram (as a rational function of $q^+$) and read off the results from [2]. All of the calculation is done by MATLAB program, so we only list the results here.

We will be following [3] and decompose an $n$-particle amplitude into:

$$M_n = \sum_{\text{perm}'} \text{Tr}(t^{a_1}t^{a_2}...t^{a_n})M(p_1, \epsilon_1; p_2, \epsilon_2;...; p_n, \epsilon_n)$$ (9)

where $\text{perm}'$ is over non cyclic permutations for complex representation, non cyclic and non reflexive permutations for real representations.

4.1 Self-energy Diagrams

$$p \quad k_1$$

\begin{align*}
\wedge, a \\
q \\
k_0 \\
\wedge, b
\end{align*}

Figure 4: A helicity violating self-mass diagram

The helicity violating self-energy diagram is given by:

$$\Pi_S^\wedge = \frac{1}{3} \frac{ig^2}{16\pi^2} \text{Tr}[t^{a_1}t^{b}] \left[ k_0^\wedge k_1^\wedge + k_0^2 + k_1^2 \right]$$

$$\Pi_F^\wedge = -\frac{4}{3} \frac{ig^2}{16\pi^2} \text{Tr}[t^{a_1}t^{b}] \left[ k_0^\wedge k_1^\wedge + k_0^2 + k_1^2 \right]$$

$$\Pi_G^\wedge = \frac{2}{3} \frac{ig^2}{16\pi^2} \text{Tr}[t^{a_1}t^{b}] \left[ k_0^\wedge k_1^\wedge + k_0^2 + k_1^2 \right]$$ (10)

This diagram describes the amplitude of a left handed gluon flip into a right handed gluon. It is only nonzero because the regulator we had used doesn’t respect Lorentz invariance. This is purely an artifact, and it has to be cancelled by a counter term. Also, we observe that $\Pi_S^\wedge \times N_a + \Pi_F^\wedge \times N_f + \Pi_G^\wedge \times N_g = 0$ in the $\mathcal{N} = 4$ SYM case.

$$p \quad k_1$$

\begin{align*}
\wedge, a \\
q \\
k_0 \\
\vee, b
\end{align*}

Figure 5: A helicity conserving self-mass diagram

$$\Pi_S^\vee = -\frac{ig^2}{16\pi^2} \text{Tr}[t^{a_1}t^{b}] \left[ -\frac{1}{\delta} + \int_0^1 dx x(1-x)p^2 \log x(1-x)p^2 \delta \epsilon \gamma \right]$$

$$\Pi_F^\vee = -2 \frac{ig^2}{16\pi^2} \text{Tr}[t^{a_1}t^{b}] \left[ \frac{2}{\delta} + \int_0^1 dx \left[ x^2 + (1-x)^2 \right] p^2 \log x(1-x)p^2 \delta \epsilon \gamma \right]$$

$$\Pi_G^\vee = -2 \frac{ig^2}{16\pi^2} \text{Tr}[t^{a_1}t^{b}] \left[ -\frac{1}{\delta} + \int_0^1 dx \left[ \frac{x}{1-x} + \frac{1-x}{x} + x(1-x) \right] p^2 \log x(1-x)p^2 \delta \epsilon \gamma \right]$$ (11)
In the above list, we have omitted part of the gluon energy terms that is non-covariant (proportional to \( p^+ \)), which was interpreted as a world sheet boundary cosmological constant in [1].

When we try to fit the self-energy diagrams into the big picture as in Fig.6, the trace factor becomes

\[
\Gamma^{\wedge\wedge\vee} = \frac{i g^3}{8\pi^2} \text{Tr}[t^a t^b t^c] = \frac{2p_+^3}{p_1 p_2} K_{2,1}^{\wedge} \left[ \frac{1}{6} \log p_0^2 \delta e^\gamma - \frac{1}{9} \alpha \frac{1}{6} \frac{p_1^+ p_2^+}{p_3^2} \right]
\]

\[
\Gamma^{\wedge\wedge\vee}_F = \frac{-i g^3}{4\pi^2} \text{Tr}[t^a t^b t^c] = \frac{-2p_3^+}{p_1^2 p_2^2} K_{2,1}^{\wedge} \left[ \frac{2}{3} \log p_0^2 \delta e^\gamma + \frac{16}{9} \alpha \frac{1}{3} \frac{p_1^+ p_2^+}{p_3^2} \right]
\]

\[
\Gamma^{\wedge\wedge\vee}_G = \frac{i g^3}{8\pi^2} \text{Tr}[t^a t^b t^c] = \frac{-2p_3^+}{p_1^2 p_2^2} K_{2,1}^{\wedge} \left[ \frac{11}{3} \log p_0^2 \delta e^\gamma + \frac{70}{9} \alpha \frac{1}{3} \frac{p_1^+ p_2^+}{p_3^2} \right] + \text{infrared terms}
\]

where \( \alpha = 1 \) if leg 3 is off shell and 0 otherwise, and \( p_o \) is the off shell momentum. The so called infrared terms present in the gluon triangle diagram will eventually be combined with infrared terms from other diagrams to become proportional to a tree. They are given in [2] and are not repeated here.

All of the above also contain an anomalous term:

\[
\Gamma^{\wedge\wedge\vee}_{S^{\text{An}}} = \frac{i g^3}{8\pi^2} \text{Tr}[t^a t^b t^c] \frac{1}{3} (k_0^\wedge + k_1^\wedge + k_2^\wedge)
\]

\(^4\)In the light cone setup, certain tadpole diagrams are ill defined, and are discarded from the beginning. This is the reason of the non-covariance in the result.
\[
\Gamma_F^{\mu\nu} = -\frac{ig^3}{4\pi^2} \text{Tr}[e^a t^b t^c] \frac{2}{3}(k_0^\mu + k_1^\mu + k_2^\mu)
\]
\[
\Gamma_G^{\mu\nu} = \frac{ig^3}{8\pi^2} \text{Tr}[e^a t^b t^c] \frac{2}{3}(k_0^\mu + k_1^\mu + k_2^\mu)
\]

here the \(k\)'s are dual momenta, they arise because \(\delta\) is an exponential damping factor of the transverse dual momenta. Had we used a cut off regulator ((\(q - k\))^2 < \(\Lambda\) for example) instead of \(\delta\), the anomalous terms would change according to: \((k_0 + k_1 + k_2) \rightarrow (k_0 - k_1) + (k_2 - k_1)\). These polynomial terms must be cancelled by counter terms.

The MHV triangles (with all three legs having the same helicity) give:

\[
\begin{align*}
\Gamma_S^{----} &= \frac{ig^4}{6\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{p_1^\mu_1 p_2^\mu_2 p_3^\mu_3 p_4^\mu_4} \left( \frac{2}{3} p_0^\mu \right) \\
\Gamma_F^{----} &= -\frac{ig^4}{12\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{p_1^\mu_1 p_2^\mu_2 p_3^\mu_3 p_4^\mu_4} \left( \frac{4}{3} p_0^\mu \right) \\
\Gamma_G^{----} &= \frac{ig^4}{24\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{p_1^\mu_1 p_2^\mu_2 p_3^\mu_3 p_4^\mu_4} \left( \frac{4}{3} p_0^\mu \right)
\end{align*}
\]

\[\tag{14}\]

### 4.3 Scattering Amplitude

Finally, we simply list the results for the gluon scattering amplitude with massless fermions and scalars.

The tree level amplitude for four gluons with the same helicity is zero. At one loop, the amplitude is

\[
\begin{align*}
A_S^{----} &= \frac{ig^4}{48\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] (s + t) \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}} \\
A_F^{----} &= -\frac{ig^4}{24\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] (s + t) \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}} \\
A_G^{----} &= \frac{ig^4}{48\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] (s + t) \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}
\end{align*}
\]

\[\tag{15}\]

The tree level amplitude with only one unlike helicity is zero too. At one loop level, the amplitude is

\[
\begin{align*}
A_S^{----} &= \frac{ig^4}{48\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] (s + t) \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}} \\
A_F^{----} &= -\frac{ig^4}{24\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] (s + t) \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}} \\
A_G^{----} &= \frac{ig^4}{48\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] (s + t) \frac{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1}}
\end{align*}
\]

\[\tag{16}\]

The helicity conserving amplitude is non-zero at tree level. They are given by [4]:

\[
\begin{align*}
A^{----} &= ig^2 f_{abc} f_{cde} \frac{-2K^{\mu_4} p_1^\mu p_3^\mu}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1} p_1^\mu p_2^\mu} \\
A^{----} &= ig^2 f_{abc} f_{cde} \frac{-2K^{\mu_4} p_1^\mu p_3^\mu}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1} p_1^\mu p_2^\mu}
\end{align*}
\]

\[\tag{17}\]

The factor \(f_{abc} f_{cde}\) can be converted to \(-1/C(G)\text{Tr}[t^a t^b][t^c, t^d] \rightarrow -2/C(G)\text{Tr}[t^a t^b t^c t^d]\).

At one loop level, the amputated Green’s function is (with infrared terms omitted)

\[
A_S^{----} = \frac{ig^4}{8\pi^2} \text{Tr}[\epsilon^a \epsilon^b \epsilon^c \epsilon^d] \left\{ -\frac{2K^{\mu_4} p_1^\mu p_3^\mu}{K^{\mu_4} K^{\mu_3} K^{\mu_2} K^{\mu_1} p_1^\mu p_2^\mu} \left[ \frac{1}{18} + \frac{1}{6} \log \delta e^c t \right] - \frac{1}{6} \times \frac{1}{3} \right\}
\]

7
A^{\wedge,\wedge,\wedge}_{F} = \frac{-ig^{4}}{8\pi^{2}} \text{Tr}[\tau^{a}t^{b}t^{c}t^{d}] \left\{ \frac{-2K_{13}^{\perp}p_{3}^{\perp}p_{4}^{\perp}}{K_{13}^{\perp}K_{32}^{\perp}K_{21}^{\perp}K_{14}^{\perp}p_{2}^{\perp}} \left[ \frac{19}{9} - \frac{2}{3} \log \delta\epsilon \gamma t \right] - \frac{1}{3} \times \frac{2}{3} \right\} \\
A^{\wedge,\wedge,\wedge}_{G} = \frac{ig^{4}}{8\pi^{2}} \text{Tr}[\tau^{a}t^{b}t^{c}t^{d}] \left\{ \frac{-2K_{13}^{\perp}p_{3}^{\perp}p_{4}^{\perp}}{K_{13}^{\perp}K_{32}^{\perp}K_{21}^{\perp}K_{14}^{\perp}p_{2}^{\perp}} \left[ -(\log^{2} \frac{s}{t} + \pi^{2}) - \frac{11}{3} \log \delta\epsilon \gamma t + \frac{73}{9} \right] - \frac{1}{3} \times \frac{2}{3} \right\} \\
(18)

A^{\wedge,\wedge,\wedge}_{S} = \frac{ig^{4}}{8\pi^{2}} \text{Tr}[\tau^{a}t^{b}t^{c}t^{d}] \left\{ \frac{-2K_{13}^{\perp}p_{3}^{\perp}p_{4}^{\perp}}{K_{13}^{\perp}K_{32}^{\perp}K_{21}^{\perp}K_{14}^{\perp}p_{2}^{\perp}} \left[ -\frac{s^{2}t^{2}}{2(s+t)^{2}} \log \frac{s}{t} + \frac{1}{6} \log \delta\epsilon \gamma s + \frac{1}{(s+t)^{2}} \log \frac{s}{t} - \frac{1}{3} \times \frac{1}{3} \right] \\
+ \frac{s(2t^{2} - 5st - s^{2})}{6(s+t)^{3}} \log \frac{s}{t} + \frac{1}{6} \log \delta\epsilon \gamma s + \frac{ts}{(s+t)^{2}} + \frac{19}{18} \right\} - \frac{1}{3} \times \frac{1}{3} \\
A^{\wedge,\wedge,\wedge}_{F} = \frac{-ig^{4}}{4\pi^{2}} \text{Tr}[\tau^{a}t^{b}t^{c}t^{d}] \left\{ \frac{-2K_{13}^{\perp}p_{3}^{\perp}p_{4}^{\perp}}{K_{13}^{\perp}K_{32}^{\perp}K_{21}^{\perp}K_{14}^{\perp}p_{2}^{\perp}} \left[ \frac{st(t^{2} + s^{2})}{2(s+t)^{4}} \log \frac{s}{t} + \frac{19}{9} \right] - \frac{1}{3} \times \frac{2}{3} \right\} \\
+ \frac{s(5t^{2} + st + 2s^{2})}{3(s+t)^{3}} \log \frac{s}{t} - \frac{2}{3} \log \delta\epsilon \gamma s + \frac{ts}{(s+t)^{2}} + \frac{19}{9} \right\} - \frac{1}{3} \times \frac{2}{3} \\
A^{\wedge,\wedge,\wedge}_{G} = \frac{ig^{4}}{8\pi^{2}} \text{Tr}[\tau^{a}t^{b}t^{c}t^{d}] \left\{ \frac{-2K_{13}^{\perp}p_{3}^{\perp}p_{4}^{\perp}}{K_{13}^{\perp}K_{32}^{\perp}K_{21}^{\perp}K_{14}^{\perp}p_{2}^{\perp}} \left[ \frac{(s^{2} + st + t^{2})^{2}}{(s+t)^{4}} \log \frac{s}{t} + \frac{73}{9} \right] - \frac{1}{3} \times \frac{2}{3} \right\} \\
\left( 19 \right)

The symbol \times above is the relevant four point vertex: \(-2(p_{1}^{\perp}p_{3}^{\perp} + p_{2}^{\perp}p_{4}^{\perp})/(p_{1}^{\perp} + p_{3}^{\perp})(p_{2}^{\perp} + p_{4}^{\perp})\) or \(2(p_{2}^{\perp}p_{3}^{\perp} + p_{1}^{\perp}p_{4}^{\perp})(p_{1}^{\perp} + p_{3}^{\perp})(p_{2}^{\perp} + p_{4}^{\perp})\). The four point vertex is not a valid counter term, because it is not a polynomial of momenta. But we can add a term proportional to \(p^{2}\) to the self-energy term Eq.11, this only changes the field strength renormalisation by a constant, hence is perfectly allowed. With this term, the coefficient of \(s\) and \(t\) channel exchange diagram is shifted. So if we pick the numerical factor in front of \(p^{2}\) to be \(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\) for scalar, fermion and gluon respectively, then they will match the coefficient of the lone four point vertex, completing it to a full tree. This brings about a change in the numerical factor:\(\frac{1}{18} \rightarrow -\frac{5}{18}, \frac{49}{9} \rightarrow -\frac{13}{9}, \frac{73}{9} \rightarrow -\frac{73}{9} \) [2]. In fact, the numerical factor is quite unimportant, as it can be altered by a redefinition of coupling constant.

It is tempting to go to \(\mathcal{N} = 4\) SYM by putting \(N_{s} = 6, N_{f} = 2\)(Dirac fermions) and \(N_{g} = 1\). The result is amusing(up to infrared terms):

\[ A^{\wedge,\wedge,\wedge}_{SYM} = A^{\wedge,\wedge,\wedge}_{SYM} = 0 \]
\[ A^{\wedge,\wedge,\wedge}_{SYM} = \frac{ig^{4}}{8\pi^{2}} \text{Tr}[\tau^{a}t^{b}t^{c}t^{d}] \left\{ \frac{-2K_{13}^{\perp}p_{3}^{\perp}p_{4}^{\perp}}{K_{13}^{\perp}K_{32}^{\perp}K_{21}^{\perp}K_{14}^{\perp}p_{2}^{\perp}} \left[ -(\log^{2} \frac{s}{t} + \pi^{2}) \right] \right\} \\
A^{\wedge,\wedge,\wedge}_{SYM} = \frac{ig^{4}}{8\pi^{2}} \text{Tr}[\tau^{a}t^{b}t^{c}t^{d}] \left\{ \frac{-2K_{13}^{\perp}p_{3}^{\perp}p_{4}^{\perp}}{K_{13}^{\perp}K_{32}^{\perp}K_{21}^{\perp}K_{14}^{\perp}p_{2}^{\perp}} \left[ -(\log^{2} \frac{s}{t} + \pi^{2}) \right] \right\} \\
\left( 20 \right)

Quite remarkably, in this case, all counter terms cancel.

4.4 A Word on Infrared Terms

In the above list of results, we have omitted the infrared sensitive terms. First, as we can see that the self-energy diagram always contains a term \(\log \delta\epsilon \gamma p^{2}x(1 - x)\). This gives a multi-particle branch cut on the positive real axis, and stops us from doing wave function renormalisation. This can be cured by summing over collinear emissions or absorptions. The analysis of [2] showed that doing so is equivalent to replacing
log $\delta e^\gamma p^2 x(1-x)$ with $\log \delta e^\gamma \Delta^2 x(1-x)$, $\Delta$ being the jet resolution. For fermion loop or scalar loop, this substitution alone is enough to regulate the infrared divergence (the triangle or box diagrams involving fermions or scalars are devoid of further infrared divergences). While for gluons, we refer to [2] for a complete treatment of infrared terms. Here we only list the infrared terms from triangle and box diagrams from the gluon loop. The $k$ below is dual momentum, with $p_1^+ = k_1^+ - k_3^+$, $p_2^+ = k_2^+ - k_1^+$, $p_3^+ = k_3^+ - k_2^+$, $p_4^+ = k_0^- - k_3^-$ and $k_1^+ < k_3^+ < k_1^+ < k_1^+$. $p_1$, $p_2$ are incoming legs while $p_3$, $p_4$ are outgoing.

$$\frac{ig^4}{8\pi^2} \text{Tr} \left[ t^a t^b t^c t^d \right] \times$$

$k_0^+ < q^+ < k_3^+$

$$\log \left( \frac{k_2^+ - q^+}{k_0^+ - k_2^+} \right) \left( -k_0^+ + q^+ \right) s \delta e^\gamma \left[ \frac{1}{q^+ - k_3^+} + \frac{1}{q^+ - k_1^+} \right]$$

$k_3^+ < q^+ < k_1^+$

$$\log \left( \frac{k_2^+ - q^+}{k_0^+ - k_2^+} \right) \left( -k_0^+ + q^+ \right) s \delta e^\gamma \left[ -\frac{1}{q^+ - k_3^+} + \frac{1}{q^+ - k_1^+} \right]$$

$k_1^+ < q^+ < k_2^+$

$$\log \left( \frac{k_2^+ - q^+}{k_0^+ - k_2^+} \right) \left( -k_0^+ + q^+ \right) s \delta e^\gamma \left[ -\frac{1}{q^+ - k_3^+} + \frac{1}{q^+ - k_1^+} \right]$$

The above is multiplied by the corresponding tree amplitude (either $-2K_{12}^+ p_3^+ p_4^+/(K_{14}^+ K_{16}^+ K_{21}^+ K_{14}^+ p_4^+)$ or $-2K_{12}^+ p_2^+ p_4^+/(K_{43}^+ K_{32}^+ K_{21}^+ K_{43}^+ p_4^+)$ for $\wedge \wedge \vee \vee$ and $\wedge \vee \wedge \vee$).

The infrared terms above will be combined with soft Bremsstrahlung and collinear emission (absorptions)
along with the self-energy insertions on the external legs to give a finite result. Note also that the \( \log \delta e^\gamma \) part of the above infrared terms actually cancel the divergent \( x \) integral in the \( \log \delta e^\gamma \) part of the the gluon self-energy diagrams on the external legs. Thus the coefficients of \( \log \delta e^\gamma \) become \( 1/6, -2/3 \) and \(-11/3\) (times tree) for scalar fermion and scalar respectively. Here we quote the scattering probability with general massless gauge and matter fields, with initial and final states treated as in [2].

\[
P^{\wedge \wedge \wedge} = |A_{\text{tree}}^{\wedge \wedge \wedge}|^2 \left[ 1 + \frac{g^2 C(G)}{8\pi^2} \left[ -2 \log^2 \frac{\Delta^2}{s} - 2 \log^2 \frac{\Delta^2}{|t|} + \frac{2\pi^2}{3} + \left( \frac{67}{9} - \frac{5}{18} N_s - \frac{26}{9} N_f \right) \right] \right]
\]

\[
P^{\wedge \wedge \wedge} = |A_{\text{tree}}^{\wedge \wedge \wedge}|^2 \left[ 1 + \frac{2C(G)}{8\pi^2} \left[ -2 \log^2 \frac{\Delta^2}{s} - 2 \log^2 \frac{\Delta^2}{|t|} + \frac{2\pi^2}{3} + \left( \frac{67}{9} - \frac{5}{18} N_s - \frac{26}{9} N_f \right) \right] \right]
\]

where \( \Delta \) is the jet resolution.

### 5 Massive Matter Fields

The main difficulty for massive field calculation is the Feynman parameter integrals. Indeed, we can only reduce all Feynman parameter integrals into a set of three definitive integrals. They are:

\[
I(s) := \int_0^1 dx \frac{s}{sx(1-x) + M}
\]

\[
J(s) := \int_0^1 dx \frac{1}{x} \log \frac{sx(1-x) + M}{M}
\]

\[
K(s, t) := \int_0^1 dx \frac{1}{stx(1-x) + M(s+t)} \log \left( \frac{(sx(1-x) + M)(tx(1-x) + M)}{M^2} \right)
\]

Where \( M \) is in fact \(-m^2 + i\epsilon\). As \( M \to 0 \):

\[
I(s) \sim 2 \log \frac{s}{M}
\]

\[
J(s) \sim \frac{1}{2} \log^2 \frac{s}{M}
\]

\[
K(s, t) \sim -\pi^2 + 2 \log \frac{s}{M} \log \frac{t}{M}
\]

\[
K(s, t) - 2J(s) - 2J(t) \sim -\pi^2 - \log^2 \frac{s}{t}
\]

For example, a simple integral:

\[
\int_0^1 \delta(T_1 + T_0 + T_3 + T_2 - 1)dt_1 dt_0 dt_3 dt_2 \frac{T_1 T_0 (1 - T_1 - T_0)}{(T_1 T_3 t + T_0 T_2 s + M)^2}
\]

can be reduced to:

\[
-\frac{M}{s^2(s+t)} J(s) - \frac{M}{t^2(s+t)} J(t) - \frac{M}{2st(s+t)} K(s, t) + \frac{1}{2st}
\]
The Feynman rules for scalar fields remain good, but the decomposition Eq.7 does not, since it introduces $1/q^+$ factors into the Feynman rules. This will complicate the already complicated Feynman parameter integral. Also $q^+$ and $q^-$ will be treated the same in contrast to the massless case, where $q^-$ is integrated out and $q^+$ is given by $\sum T_i k^+_i / \sum T_i$.

In order to organize the gamma matrix algebra in the fermion part of the calculation, we make use of the factorisability of gluon polarisation vectors Eq.38 to reduce products of gamma matrices to products of $K_{ij} = p_i^- p_j^- - p_i^- p_j^+$, which had been proved to be quite handy. For example, if we are to calculate the diagram Fig.4. We would write down:

$$(ig)^2 \text{Tr}(\tau^a \lambda^b) \text{Tr} \left( \left( \gamma^\mu - m \gamma^\nu \right) \left( -\epsilon^\nu_\alpha \right) \right) \left( -\epsilon^\alpha_\beta \right) \frac{i}{(q-k_0)^2 - m^2} \frac{i}{(q-k_1)^2 - m^2} \tag{25}$$

The numerator can be written:

$$\text{Tr} \left[ \begin{array}{ccc}
m & m & 0 \ (q-k_0) \cdot \sigma \ (q-k_0) \cdot \sigma \\
0 & -\sqrt{2}|k_0 - k_1| \eta \ -\sqrt{2}|k_0 - k_1| \eta \\
-\sqrt{2}|k_0 - k_1| \eta & 0 & -\sqrt{2}|k_0 - k_1| \eta \\
\end{array} \right] \times$$

$$= |\eta|^2 |q-k_0| |k_0 - k_1| + |\eta|^2 |q-k_0| |k_0 - k_1| + \langle k_0 - k_1 \rangle |q-k_0| |k_0 - k_1|$$

$$= (1-x)\sqrt{2}(k_0 - k_1) + \sqrt{2}\frac{x(k_0 - k_1)^2}{2(k_0 - k_1)^2} + |\eta|^2 |k_0 - k_1| |q-k_0| |k_0 - k_1|$$

$$= -x(1-x)(k_0 - k_1)^2 + \frac{q^2}{2} \tag{26}$$

For the notation of spinor, see the appendix. The standard procedure of momentum integration tell us to shift $q \rightarrow q + xk_0 + (1-x)k_1$, and to replace $q^\mu q^\nu$ by $q^2 g^{\mu\nu}/4$, etc. $^5$:

$$\overline{\sigma}_{\alpha \beta}^a \sigma_{\mu \nu}^{ab} g_{\mu \nu} = 2 \epsilon_{ah} \epsilon_{ab}; \quad \sigma_{\mu \nu}^{ab} g_{\mu \nu} = 2 \epsilon^{ab} \epsilon^{cd} \tag{28}$$

For a more complicated example, a string of spinor products becomes,

$$\left[ p_1 |q-k_0| p_4 \right] |q-k_3| p_3$$

$$= \left[ p_1 |q + T_1(k_1 - k_0) + T_2(k_2 - k_0) + T_3(k_3 - k_0) | p_4 \right] |q + T_0(k_0 - k_3) + T_1(k_1 - k_3) + T_2(k_2 - k_3) | p_3$$

$$= \left[ p_1 | q | p_3 | p_4 \frac{q^2}{2} + p_1 | T_2 p_2 | p_4 \right] | q | T_0 p_4 - T_1 p_2 | p_3$$

$$= (-1) \frac{1}{p_3 p_4} K_{H} \frac{q^2}{2} + \frac{\sqrt{2}T_2}{p_3 p_4} K_{H} \frac{K}{2} \left( \frac{\sqrt{2}T_0}{p_3} K_{34} - \frac{\sqrt{2}T_1}{p_3} K_{32} \right)$$

Basically, all of the spinor products can be reduced to one of the $K_{ij}$’s, and the reader can find in the appendix some practical details as to how to organize the products of $K_{ij}$’s. After the momentum integral is done, we can perform the Feynman parameter integrals using Eq.22.

$^5$We have omitted the complication due to the $\delta$ regulator. For one thing, the momentum integration is no longer homogeneous, hence the replacement of $q^\mu q^\nu$ by $q^2 g^{\mu\nu}/4$ is problematic. Some extra work is required to sort this out, which we will not bore the reader with.
6 Results

6.1 Self-energy Diagrams

For Fig. 4, the results are:

\[
\Pi_S^{\gamma} = -\frac{ig^2}{16\pi^2} \text{Tr}[t^at^b] \left[ \frac{1}{6} p^2 \log \delta p^m m^2 + \frac{1}{12} \left( \frac{p^2 - 4m^2}{p^2} \right)^2 I(p^2) - \frac{5}{18} p^2 + \frac{4}{3} m^2 \right]
\]

\[
\Pi_F^{\gamma} = \frac{ig^2}{8\pi^2} \text{Tr}[t^at^b] \left[ -\frac{2}{3} p^2 \log \delta p^m m^2 - \frac{1}{3} \left( \frac{p^2 - 4m^2}{p^2} \right)(p^2 + 2m^2) I(p^2) + \frac{13}{9} p^2 + \frac{8}{3} m^2 \right]
\]

The gluon mass counter term in the expressions above has been removed already.

6.2 Triangle Diagrams

The results for triangle diagram Fig. 7 are:

\[
\Gamma_S^{\gamma\gamma} = \frac{ig^3}{8\pi^2} \text{Tr}[t^at^b] \times \frac{-2p^+_{1p}}{p_1^+p_2^-p_3^-} K_{21} \left[ \frac{1}{6} \log m^2 \delta p^m p^2 + \frac{4m^2}{p_0^2} - \frac{1}{3} \left( \frac{p^2 - 4m^2}{p^2} \right)^2 I(p^2) \right] - \frac{2}{3} \left( \frac{p^2 - 4m^2}{p^2} \right) \frac{2}{3} p^2 + \frac{4m^2}{p_0^2}
\]

\[
\Gamma_F^{\gamma\gamma} = -\frac{ig^3}{4\pi^2} \text{Tr}[t^at^b] \times \frac{-2p^+_{1p}}{p_1^+p_2^-p_3^-} K_{21} \left[ \frac{2}{3} \log m^2 \delta p^m p^2 + \frac{8m^2}{p_0^2} - \frac{1}{3} \left( \frac{p^2 - 4m^2}{p^2} \right)^2 I(p^2) \right] - \frac{2}{3} \left( \frac{p^2 - 4m^2}{p^2} \right) \frac{2}{3} p^2 + \frac{4m^2}{p_0^2}
\]

where \( \alpha = 1 \) if leg 3 is off shell and 0 otherwise, and \( p_o \) is the off shell momentum. Their anomalous terms are identical with the massless result, which agrees with the fact that anomalous terms are UV effects.

The result for the MHV triangle is:

\[
\Gamma_S^{\gamma\gamma} = \frac{ig^3}{8\pi^2} \text{Tr}[t^at^b] \frac{(K_{21}^\gamma)^3}{p_1^+p_2^-p_3^-} K_{21} \left[ -\frac{4m^2}{p_0^2} I(p^2) + \frac{4m^2}{p_0^2} J(p^2) + \frac{24m^2}{3p_0^2} \right]
\]

\[
\Gamma_F^{\gamma\gamma} = -\frac{ig^3}{4\pi^2} \text{Tr}[t^at^b] \frac{(K_{21}^\gamma)^3}{p_1^+p_2^-p_3^-} K_{21} \left[ -\frac{8m^2}{p_0^2} I(p^2) + \frac{8m^2}{p_0^2} J(p^2) + \frac{24m^2}{3p_0^2} \right]
\]

6.3 Scattering Amplitudes

\[
A_S^{\gamma\gamma\gamma\gamma} = \frac{ig^4}{\pi^2} \text{Tr}[t^at^b] \frac{(K_{21}^\gamma)^3}{p_1^+p_2^-p_3^-} K_{21} \left[ \frac{1}{6} \frac{m^4}{st} K(s,t) \right]
\]

\[
A_F^{\gamma\gamma\gamma\gamma} = -\frac{2ig^4}{\pi^2} \text{Tr}[t^at^b] \frac{(K_{21}^\gamma)^3}{p_1^+p_2^-p_3^-} K_{21} \left[ \frac{1}{3} \frac{m^4}{st} K(s,t) \right]
\]

\[
A_S^{\gamma\gamma\gamma\gamma} = \frac{ig^4}{8\pi^2} \text{Tr}[t^at^b] \frac{(K_{21}^\gamma)^3}{p_1^+p_2^-p_3^-} K_{21} \left[ -\frac{(2t + s)stm^2}{2(s + t)st} (1 - \frac{4m^2}{st}) I(s) - \frac{(2s + t)stm^2}{2(s + t)st} (1 - \frac{4m^2}{st}) I(t) \right]
\]

\[
+ \frac{(2t + s)^2 m^2}{(s + t)^2st} J(s) + \frac{(2s + t)^2 m^2}{(s + t)^2st} J(t) + \frac{stm^2}{2(s + t)^2} (1 - \frac{2m^2(s + t)}{st}) K(s,t)
\]
\[
A_{\gamma \gamma \gamma \gamma}^F = \frac{-i g^4}{4\pi^2} \text{Tr} [t^a t^b t^c t^d] \frac{K_{13} K_{24} p_4^+ p_1^+}{K_{13} K_{24} K_{21} K_{14}} \times \\
\left[ -\frac{(s+t)tm^2}{(s+t)s} (1 - \frac{4m^2}{s}) I(s) - \frac{(2s+t)sm^2}{(s+t)t} (1 - \frac{4m^2}{t}) I(t) \right. \\
+ \frac{2(2s+t)t^2m^2}{(s+t)^2(s+t)t} J(s) + \frac{2(2s+t)^2s^2m^2}{(s+t)^2(s+t)t} J(t) + \frac{stm^2}{(s+t)^2} (1 - \frac{2m^2(s+t)}{st}) K(s,t) \\
+ \frac{4(2s^2 - st + 2t^2)m^2}{st} + \frac{(s + t)}{3} \right] (33)
\]

In contrast to the massless case, the external leg factors are included in the results below for the helicity conserving amplitude.

\[
A_{\gamma \gamma \gamma \gamma}^S = \frac{i g^4}{8\pi^2} \text{Tr} [t^a t^b t^c t^d] \frac{-2K_{13}^2 p_3^+ p_4^+}{K_{13}^2 K_{24} K_{14} K_{21}} \times \\
\left\{ \frac{m^2}{2s} - \frac{2m^2}{3t} - \frac{4m^4}{3s^2} + \frac{1}{12} I(t) + \frac{m^4}{s^2} K(s,t) - \frac{2m^2}{s} + \frac{4m^4}{3t} + \frac{1}{18} \\
\left[ -\frac{1}{6} \log \delta \gamma^m \right] - \frac{1}{3} \times + \frac{1}{3} \right\} (34)
\]

\[
A_{\gamma \gamma \gamma \gamma}^S = \frac{i g^4}{8\pi^2} \text{Tr} [t^a t^b t^c t^d] \frac{-2K_{13}^2 p_3^+ p_4^+}{K_{13}^2 K_{24} K_{14} K_{21}} \times \\
\left\{ \frac{m^2}{2s} - \frac{2m^2}{3t} - \frac{4m^4}{3s^2} + \frac{1}{12} I(t) + \frac{m^4}{s^2} K(s,t) - \frac{2m^2}{s} + \frac{4m^4}{3t} + \frac{19}{9} \\
\left[ -\frac{1}{6} \log \delta \gamma^m \right] - \frac{2}{3} \times + \frac{2}{3} \right\}
\]

\[
A_{\gamma \gamma \gamma \gamma}^F = \frac{-i g^4}{4\pi^2} \text{Tr} [t^a t^b t^c t^d] \frac{-2K_{13}^2 p_3^+ p_4^+}{K_{13}^2 K_{24} K_{14} K_{21}} \times \\
\left[ \frac{tm^2(5s^2 + 2t^2 - 5st)}{3(s + t)^3} + \frac{4tm^2(2t^2 + 5s)}{3(s + t)^2s^2} - \frac{t(5s^2 + 2t^2 + st)}{6(s + t)^3} I(s) \\
+ \frac{tm^2(5^2 + 2s^2 - 5st)}{3(s + t)^3} I(s) + \frac{4sm^2(5t + 2s)}{3(s + t)^2t^2} - \frac{s(5t^2 + 2s^2 + st)}{6(s + t)^3} I(t) \right] \\
\left\{ -\frac{1}{6} \log \delta \gamma^m - \frac{1}{3} \right\} - \frac{1}{6} \times + \frac{1}{3} \right\}
\]
\begin{align}
& \frac{4stm^2}{(s+t)^3} + \frac{st(s^2 + t^2)}{(s+t)^4} (J(s) + J(t)) + \frac{m^2(s-t)^2}{(s+t)^3} + \frac{2m^4}{(s+t)^2} - \frac{st(s^2 + t^2)}{2(s+t)^2} K(s,t) \\
& + \frac{4n^2(2r^2 + 2s^2 + 3st)}{3(s+t)st} + \frac{19s^2 + 19t^2 + 47st}{9(s+t)^2} \\
& + \frac{2}{3} \log \delta e^\gamma m^2 - \frac{2}{3} \left\{ - \frac{1}{3} \times + \frac{2}{3} \right\}
\end{align}

The last numerical factors $-1/3$ and $-2/3$ inside each square bracket is due to the amputation of external legs: $\lim_{p^2 \rightarrow 0} \Pi(p^2/\mu^2)^{1/2}$.

## 7 Concluding Remarks

In summary, we have calculated the next to leading order gluon scattering amplitudes with fermions and scalars in the loop, as an extension of [1, 2]. They are all part of the project of showing that summing planar diagrams on the light cone world sheet does give correct results (at least at one loop order). The light cone world sheet tells us to organize diagrams such that an explicit cancellation of gauge artificial divergence can be seen. There is no gauge artificial divergences in the fermion and scalar part of the calculation, and the method developed in [2] works out as expected. And the same counter terms required in [2] appear here too.

In the special case of $\mathcal{N} = 4$ SYM, all of these counter terms cancel among different sectors. This is a mere additional bonus from the field theory point of view, but quite crucial for the project proposed in [1, 2]. More specifically, in [2], the dimension of the target space was increased to accommodate extra world sheet fields (spurions), these fields couple to gluons in such a way that will produce the counter terms needed (the four point contact vertex and a $2/3$ in Eq.19). In order to formulate $\mathcal{N} = 4$ SYM on the world sheet we need to increase the space time dimension to ten, so as to have six more world sheet fields in addition to the original $q^i(\sigma, \tau), \quad i = 1, 2, 3, 4$ [5]. These eight fields correspond to the eight bosonic degrees of freedom in $\mathcal{N} = 4$ SYM. The AdS/CFT duality says this theory is dual to the string theory in 10 dimensions, meaning we no longer have room for extra dimensions to produce counter terms.

The massive matter contribution to the gluon scattering was also calculated for completeness. An exponential damping factor was used instead of the dimension reduction as UV regulator, so the box integral is slightly different from [7, 8], so no machinery of hypergeometric functions need be used.

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## A Notation

\begin{align}
\gamma^\mu &= \left| \begin{array}{cc}
0 & \sigma^\mu \\
\bar{\sigma}^\mu & 0
\end{array} \right| \quad \sigma^\mu := (I, \bar{\sigma}) \quad \bar{\sigma}^\mu := (I, -\sigma) \\
\epsilon_{ab} &= \epsilon^{ab} = \epsilon^{i\bar{a}} = \epsilon_{i\bar{b}} = i\sigma_2 \\
p_a &= \epsilon_{ab} p^b \quad p^a = p_b \epsilon^{ba} \\
|p| &= p_a |p| := p^a |p| := p_\hat{a} |p| := p_a \\
\eta^\alpha &= \eta_{\alpha} = \left| \begin{array}{c}
1 \\
0
\end{array} \right| \\
\eta_a &= \eta_{\bar{a}} = \left| \begin{array}{c}
0 \\
-1
\end{array} \right.
\end{align}

(36)
We can define the light cone version spinor as:

\[ p^\alpha = \begin{vmatrix} -e^-_\alpha \\ 1 \end{vmatrix}, \quad p_\alpha = \begin{vmatrix} p^-_\alpha \\ 1 \end{vmatrix} \]  

(37)

The spinors satisfy the Dirac equation if \( p \) is light like. Note that they don’t have the correct normalisation, namely \( p \cdot \sigma^\alpha \bar{\sigma}_\alpha \neq p^\alpha \bar{p}^\alpha \), but they have the merit that \( p^\alpha = (-p)_\alpha \). The polarisation vectors of gluon can be written as:

\[ \epsilon_{\wedge \hat{a} \hat{a}} = \sqrt{2} \frac{|p| q}{q |p|}, \quad \epsilon_{\vee \hat{a} \hat{a}} = -\sqrt{2} |q |p| \]  

(38)

\[ K_\wedge^{ij} = p_j^+ p_i^+ \langle p_i | p_j \rangle, \quad K_\vee^{ij} = p_j^+ p_i^+ [p_i | p_j \rangle \]  

The \( K_{ij} \)'s satisfy:

\[ \sum_j K_{ij}^{\mu} = 0 \]

\[ p_j^+ K_{jk}^{\mu} + p_k^+ K_{ij}^{\mu} = 0 \]

\[ K_\wedge^{i k} K_\wedge_{jk} + K_\wedge^{j k} K_\wedge_{ki} = 0 \]

\[ \sum_j K_\wedge^{i k} K_\wedge_{jk} p_j^2 = p_i^+ p_k^+ \sum_j \frac{p_j^2}{2p_j^2} \]  

(39)

In the current case we are dealing with, \( i, j \) run from 1 to 4, but only two of the six \( K_{ij} \)'s are independent, say \( K_{43} \) and \( K_{32} \). And any product of \( K_{ij} \)'s with total helicity 4 can be reduced to either \( (K_\wedge^{43})^4 \) or \( (K_\wedge^{32})^4 \). Product of helicity 2 can be reduced to \( (K_\wedge^{43})^2 \) and \( K_\wedge^{43} K_\wedge^{32} \). Product of helicity 0 can be reduced to 1 and \( K_\wedge^{43} K_\wedge^{32} \). The reduction is in general a formidable task for human, but quite a piece of cake for computers, as all our calculations are done with computers.

B  Feynman Rules

![Figure 8: gluon-fermion-fermion 3 point vertex](image)

The fermion gluon vertex Fig.8 is given by:

\[ V_\wedge^l = -2ig(t^a)_{bc} \frac{p_j^2}{q^+ p_l^+} K_\wedge^{p_l,q} \]

\[ V_\vee^l = -2ig(t^a)_{bc} \frac{1}{q^+ p_l^+} K_\wedge^{p_l,q} \]

\[ V_\wedge^r = -2ig(t^a)_{bc} \frac{1}{q^+ p_l^+} K_\vee^{p_l,q} \]

\[ V_\vee^r = -2ig(t^a)_{bc} \frac{p_j^2}{q^+ p_l^+} K_\vee^{p_l,q} \]  

(40)

where the subscript \( l, r \) denotes whether the fermion is left or right handed.

The gluon scalar vertex Fig.9 is given by:

\[ V_{\wedge(\vee)} = -2ig(t^a)_{bc} \frac{1}{p_b^- + p_c^-} K_{p_c,p_b}^{\wedge(\vee)} \]  

(41)
Here we have used real scalar fields and hence it transforms in a real representation.

The gluon scalar four point vertex Fig.10 is given by:

\[ V^{\wedge\wedge}_l = -2ig^2 (t^d)_{ce} (t^e)_{cb} \frac{p_2^+}{p_1^+ + p_4^+} + 2g^2 f^{daec} (t^e)_{cb} \frac{(p_3^+ - p_4^+) p_2^+}{(p_3^+ + p_4^+)^2} \]

\[ V^{\wedge\wedge}_r = 2g^2 f^{daec} (t^e)_{cb} \frac{(p_3^+ - p_4^+) p_2^+}{(p_3^+ + p_4^+)^2} \]

\[ V^{\wedge\wedge}_l = 2g^2 f^{daec} (t^e)_{cb} \frac{(p_3^+ - p_4^+) p_2^+}{(p_3^+ + p_4^+)^2} \]

\[ V^{\wedge\wedge}_r = -2ig^2 (t^d)_{ce} (t^e)_{cb} \frac{p_2^+}{p_1^+ + p_4^+} + 2g^2 f^{daec} (t^e)_{cb} \frac{(p_3^+ - p_4^+) p_2^+}{(p_3^+ + p_4^+)^2} \]

(42)

where the first super index on \( V \) refer to the polarisation of \( p_3 \), the second \( p_4 \).

The gluon scalar four point vertex Fig.11 is given by:

\[ V_{ggs} = -ig^2 \left[(t^a)_{ce}(t^b)_{ed} + (t^b)_{ce}(t^a)_{ed}\right] + g^2 f^{abc} (t^e)_{cd} \frac{(p_3^+ - p_4^+) (p_2^+ - p_1^+)}{(p_1^+ + p_2^+)^2} \]

(43)

For tri-gluon vertex Fig.1

\[ V_{ggs} = -gf^{abc} [\epsilon_1^* \cdot \epsilon_2^* (p_1 - p_2) \cdot \epsilon_3^* + \epsilon_2^* \cdot \epsilon_3^* (p_2 - p_3) \cdot \epsilon_1^* + \epsilon_3^* \cdot \epsilon_1^* (p_3 - p_1) \cdot \epsilon_2^*] \]

Setting \( \mu, \nu = \wedge, \rho = \vee \), the above becomes:

\[ g f^{abc} [(p_2 - p_3)^+ \frac{p_1^+}{p_1^+} - (p_2 - p_3)^\wedge + (p_3 - p_1)^+ \frac{p_2^+}{p_2^+} - (p_3 - p_1)^\wedge] \]

\[ = 2gf^{abc} (p_1 + p_2)^+ K_{21}^\wedge \]

(44)
The gluon four point vertex receives contribution from two sources: the left diagram in Fig. 13 is simply the covariant four point vertex:

\[
V_1 = ig^2 f^{ace} f^{ebd} \left[ g^\mu_\sigma g^{\nu_\rho} - g^\mu_\rho g^{\nu_\sigma} \right] + ig^2 f^{dae} f^{ebc} \left[ g^{\mu_\nu} g^{\rho_\sigma} - g^{\mu_\rho} g^{\nu_\sigma} \right] + ig^2 f^{dce} f^{ebd} \left[ g^{\mu_\nu} g^{\rho_\sigma} - g^{\mu_\sigma} g^{\nu_\rho} \right] \tag{45}
\]

The second is obtained by shrinking a propagator.

\[
V_2 = g^2 f^{dae} f^{ebc} [\epsilon^*_1 \cdot \epsilon^*_3(p_4 - p_1) + \frac{ig^{\alpha_+} g^{\beta_+} g^\beta(p_1 + p_4)^2}{\left(p_1^* + p_4^*\right)^2(p_1 + p_4)^2} \left[ \epsilon^*_2 \cdot \epsilon^*_3(p_2 - p_3)_\beta \right)]
\]

\[
= ig^2 f^{dae} f^{ebc} (\epsilon^*_1 \cdot \epsilon^*_3)(\epsilon^*_2 \cdot \epsilon^*_3)(p_4 - p_1)^+(p_2 - p_3)^+
\]

There are two cases in which the gluon four point vertex is nonzero:

\[
V_{\land \lor \lor} = -2ig^2 f^{dae} f^{ebc} \frac{p_1^+ p_3^+ + p_2^+ p_4^+}{(p_1^* + p_4^*) (p_2^* + p_3^*)} + 2ig^2 f^{ace} f^{ebd} \frac{p_3^+ p_2^+ + p_1^+ p_4^+}{(p_1^* + p_3^*) (p_2^* + p_4^*)}
\]
\[ V_{\Lambda \Lambda \Lambda \Lambda} = 2i g^2 f^{abc} f^{ced} \frac{p_2^+ p_3^+ + p_4^+ p_1^+}{(p_1^+ + p_2^+)(p_3^+ + p_4^+)} + 2i g^2 f^{dae} f^{xbc} \frac{p_1^+ p_2^+ + p_3^+ p_4^+}{(p_1^+ + p_4^+)(p_2^+ + p_3^+)} \] (46)

When using these vertices, we need to watch the indices of structure constants closely: not all terms are going to make contributions to \( \text{Tr}[\epsilon^{[a} t^b t^c t^d]\).

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