Z2-INDEX OF THE GRASSMANIAN $G_{2n}^n$

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Abstract. We study the real Grassmann manifold $G_{2n}^n$ (of $n$-subspaces in $\mathbb{R}^{2n}$), and the action of $Z_2$ on it by taking the orthogonal complement. The homological index of this action is estimated from above and from below. In case $n$ is a power of two it is shown that $\text{ind} \ G_{2n}^n = 2^n - 1$.

1. Introduction

The topology of real Grassmannians has many applications in the discrete and convex geometry. For example, the Schubert calculus and other topological facts (e.g. from [3, 5]) can be applied to obtain some existence theorems for flat transversals (affine flats intersecting all members of a given family of sets), see [4, 15, 8, 12] for example.

In this paper we consider the Grassmannian $G_{2n}^n$ of $n$-dimensional subspaces of $\mathbb{R}^{2n}$. This space has a natural $Z_2$-action (involution) by taking the orthogonal complement of the subspace. The well-known invariant of $Z_2$-spaces is homological index, introduced and studied in [9, 13, 3], see also the book [10] for a simplified introduction to the index and its many applications to combinatorics and geometry.

The following theorem gives an estimate for the index of the Grassmannian.

**Theorem 1.** If $n = 2^l(2m+1)$, then

$$2^{l+1} - 1 \leq \text{ind} \ G_{2n}^n \leq 2n - 1,$$

for $n = 2m + 1$ the index equals 1, for $n = 2(2m + 1)$ the index equals 3.

The lower and the upper bounds coincide for $n = 2^l$, odd $n$, $n = 2(2m + 1)$. In other cases there is still some gap between them. This result easily produces some geometric consequences. Here is one example (it also uses Lemma 1 below).

**Corollary 2.** Let $n = 2^l(2m+1)$, $k = 2^{l+1} - 1$. Consider some $k$ continuous (in the Hausdorff metric) $O(n)$-invariant functions $\alpha_1, \ldots, \alpha_k$

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on (convex) compacts in $\mathbb{R}^n$. Then for any (convex) compact $K \subseteq \mathbb{R}^{2n}$ there exist a pair of orthogonal $n$-dimensional subspaces $L$ and $M$, such that for their respective orthogonal projections $\pi_L$ and $\pi_M$ we have

$$\forall i = 1, \ldots, k \alpha_i(\pi_L(K)) = \alpha_i(\pi_M(K)).$$

In this corollary $\alpha_i$ can be the Steiner measures (volume, the boundary measure, the mean width, etc.) for example. The same statement holds if we consider a point $x \in K$ and sections of $K$ by mutually orthogonal affine $n$-subspaces $L$ and $M$ through $x$, instead of projections to $L$ and $M$.

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2. Preliminary observations

Let us state some topological definitions on spaces with group action, see [6] for more detailed discussion.

**Definition 1.** Let $G$ be a compact Lie group or a finite group. A space $X$ with continuous action of $G$ is called a $G$-space. A continuous map of $G$-spaces, commuting with the action of $G$ is called a $G$-map or an equivariant map. A $G$-space is called free if the action of $G$ is free.

There exists the universal free $G$-space $EG$ such that any other $G$-space maps uniquely (up to $G$-homotopy) to $EG$. The space $EG$ is homotopy trivial, the quotient space is denoted $BG = EG/G$. For any $G$-space $X$ and an Abelian group $A$ the equivariant cohomology $H^*_G(X, A) = H^*(X \times_G EG, A)$ is defined, and for free $G$-spaces the equality $H^*_G(X, A) = H^*(X/G, A)$ holds.

Consider the case $G = Z_2$. Note that

$$H^*_Z(pt, Z_2) = H^*(\mathbb{R}P^\infty, Z_2) = Z_2[c] = \Lambda,$$

where the dimension of the generator is $\dim c = 1$. Since any $G$-space $X$ can be mapped to the point $\pi_X : X \to pt$, we have a natural map $\pi_X^* : \Lambda \to H^*_G(X, Z_2)$, the image $c$ under this map will be denoted by $c$, if it does not make a confusion. The generator element of $Z_2$ will be denoted by $\sigma$.

**Definition 2.** The cohomology index of a $Z_2$-space $X$ is the maximal $n$ such that the power $c^n \neq 0$ in $H^*_G(X, Z_2)$. If there is no maximum, we consider the index equal to $\infty$. Denote the index of $X$ by $\text{ind} X$.

Let us state the following well-known lemma.

**Lemma 1** (The generalized Borsuk-Ulam theorem for odd maps). If there exists an equivariant map $f : X \to Y$, then $\text{ind} X \leq \text{ind} Y$.

Now we are ready to prove the upper bound in Theorem [1].
Lemma 2.
\[ \text{ind } G_{2n}^n \leq 2n - 1. \]

Proof. Let us parameterize \( G_{2n}^n \) by the orthogonal projection matrices \( P \). These matrices are characterized by the equations
\[ P^t = P, \quad P^2 = P, \quad \text{tr} \ P = n. \]
The action of \( Z_2 \) is given by (\( E \) is the unit matrix)
\[ \sigma(P) = E - P. \]
Now consider the map \( f : G_{2n}^n \to \mathbb{R}^{2n} \), defined by the coordinates
\[ f_1(P) = P_{11} - 1/2, \quad f_i(P) = P_{1i}, \quad (i = 2, \ldots, 2n). \]
This map is \( Z_2 \)-equivariant, if the action on \( \mathbb{R}^{2n} \) is antipodal, i.e. \( \sigma : x \mapsto -x \). Note also that \( f(P) \) is never zero, otherwise \( P \) would have an eigenvalue \( 1/2 \), which is not true. Hence \( f \) composed with the projection \( \mathbb{R}^{2n} \setminus \{0\} \to S^{2n-1} \) gives the equivariant map
\[ \tilde{f} : G_{2n}^n \to S^{2n-1}, \]
and the result follows by Lemma 1.

Lemma 3. Suppose \( n = ds \) for some positive integers \( d, s \). Then
\[ \text{ind } G_{2n}^n \geq \text{ind } G_{2d}^d. \]

Proof. Let us decompose
\[ \mathbb{R}^{2n} = \mathbb{R}^{2d} \oplus \cdots \oplus \mathbb{R}^{2d} \]
into \( s \) summands. Consider a \( d \)-subspace \( L \in G_{2d}^d \), and define with the above decomposition
\[ f(L) = L \oplus \cdots \oplus L \subset \mathbb{R}^{2n}. \]
The map \( f : G_{2d}^d \to G_{2n}^n \) is evidently equivariant and by Lemma 1 we obtain the inequality.

In order to prove Theorem 1 it remains to prove the following lemmas.

Lemma 4. If \( n \) is odd, then \( \text{ind } G_{2n}^n = 1 \), if \( n = 2 \mod 4 \), then \( \text{ind } G_{2n}^n = 3 \).

Lemma 5. If \( n = 2^l \), then \( \text{ind } G_{2n}^n = 2n - 1 \).
3. External Steenrod squares

In order to prove Lemma 5, we have to describe the cohomology of the subgroup \( G \subset O(2n) \), generated by the subgroup \( O(n) \times O(n) \) (from some decomposition \( \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n \)), and \( Z_2 \) that interchanges the summands \( \mathbb{R}^n \). This group is the wreath product \( O(n) \wr Z_2 = (O(n) \times O(n)) \rtimes Z_2 \).

In order to describe the cohomology of a wreath product, we have to use the construction of external Steenrod squares. We mostly follow [2, Ch. V], where the Steenrod squares were defined in the unoriented cobordism. The cobordism was defined using mock bundles, if we allow the mock bundles to have codimension 2 singularities, we obtain ordinary cohomology modulo 2. In the sequel we consider the cohomology modulo 2 and omit the coefficients in notation. This construction is known and was used in [7] to describe the modulo 2 cohomology of the symmetric group and configuration spaces. Still, for reader’s convenience we give a short and self-contained explanation here.

The construction of the external Steenrod squares on a polyhedron \( K \) starts with the fiber bundle (for some integer \( n > 0 \))

\[
\sigma_K : (K \times K \times S^n)/Z_2 \to S^n/Z_2 = \mathbb{R}P^n.
\]

The group \( Z_2 \) acts by permuting \( K \times K \), and antipodally on \( S^n \). Consider a cohomology class \( \xi \in H^*(K) \), represented by a mock bundle \( \xi : E(\xi) \to K \). Then the mock bundle

\[
(\xi \times \xi \times S^n)/Z_2 \to (K \times K \times S^n)/Z_2
\]

is the external Steenrod square \( \text{Sq}_e \xi \). The operation \( \text{Sq}_e \) is evidently multiplicative, in [2, Ch. V, Proposition 3.3] it is claimed that \( \text{Sq}_e \) is also additive. We are going to show that it is not true, first we need a definition.

**Definition 3.** The difference \( \text{Sq}_e(\xi + \eta) - \text{Sq}_e \xi - \text{Sq}_e \eta \) is represented by the mock bundle

\[
\xi \odot \eta = (\xi \times \eta \times S^n + \eta \times \xi \times S^n)/Z_2,
\]

where \( Z_2 \) exchanges the components \( \xi \times \eta \) and \( \eta \times \xi \).

Since the fiber of \( \sigma_K \) is \( K \times K \), the restriction of \( \xi \odot \eta \) to the fiber is \( \xi \times \eta + \eta \times \xi \), which is nonzero if \( \eta \neq \xi \) as cohomology classes. Thus the operation \( \odot \) is not trivial.

We need a lemma about the \( \odot \)-multiplication.

**Lemma 6.** Denote \( c \) the hyperplane class in \( H^1(\mathbb{R}P^n) \). Then for any \( \xi, \eta \in H^*(K) \) the product

\[
(\xi \odot \eta) \odot \sigma_K^*(c) = 0
\]

in \( H^*((K \times K \times S^n)/Z_2) \).
**Proof.** Consider the mock bundle
\[
\alpha = \xi \times \eta \times S^{n-1} + \eta \times \xi \times S^{n-1},
\]
which has the natural $\mathbb{Z}_2$-action, it represents $(\xi \odot \eta) \sim \sigma_K^*(c)$ after taking the quotient by the $\mathbb{Z}_2$-action.

Now divide $S^n$ into the upper and the lower half-spheres $H^+$ and $H^-$. Consider the mock bundle (with boundary)
\[
\beta = \xi \times \eta \times H^+ + \eta \times \xi \times H^-,
\]
over $K \times K \times S^n$. The action of $\mathbb{Z}_2$ on $\beta$ is defined by permuting the summands and the antipodal identification of $H^+$ and $H^-$. Now it is clear that $\alpha$ is the boundary of $\beta$, and $\alpha/\mathbb{Z}_2$ is the boundary of $\beta/\mathbb{Z}_2$. Hence it is zero in the cohomology, and the similar statement is true for the unoriented bordism. □

We have to introduce another operation.

**Definition 4.** Let $\xi : E(\xi) \to K$, $\eta : E(\eta) \to K$ be two mock bundles. Let $p_+, p_-$ be the north and the south poles of $S^n$. Denote the mock bundle over $(K \times K \times S^n)/\mathbb{Z}_2$
\[
\iota(\xi \times \eta) = (\xi \times \eta \times \{p_+\} + \eta \times \xi \times \{p_+\})/\mathbb{Z}_2.
\]
It is obvious from the definition that we have relation
\[
\iota(\xi \times \eta) \sim \sigma_K^*(c) = 0,
\]
it is also obvious that
\[
\iota(\xi \times \xi) = \mathrm{Sq}_e \xi \sim \sigma_K^*(c)^n.
\]
Let us describe the $\sim$-multiplication of the Steenrod squares, $\odot$, and $\iota(\ldots)$ classes. The following formulas are obvious from the definition:
\[
(\xi \odot \eta) \sim (\xi \odot \chi) = (\xi \sim \zeta) \odot (\eta \sim \chi) + (\xi \sim \chi) \odot (\eta \sim \zeta),
\]
\[
(\xi \odot \eta) \sim (\mathrm{Sq}_e \zeta) = (\xi \sim \zeta) \odot (\eta \sim \zeta),
\]
\[
(\xi \odot \eta) \sim \iota(\xi \odot \chi) = \iota((\xi \sim \zeta) \times (\eta \sim \chi)) + \iota((\xi \sim \chi) \times (\eta \sim \zeta)),
\]
\[
\mathrm{Sq}_e \xi \sim \mathrm{Sq}_e \xi \eta = \mathrm{Sq}_e(\xi \sim \eta),
\]
\[
\mathrm{Sq}_e \xi \sim \iota(\eta \times \xi) = \iota((\xi \sim \eta) \times (\xi \sim \zeta)),
\]
\[
\iota(\xi \times \eta) \sim \iota(\xi \times \chi) = 0.
\]
Now we can describe the structure of the cohomology $H^*((K \times K \times S^n)/\mathbb{Z}_2)$.

**Definition 5.** Consider a $\mathbb{Z}_2$-algebra $A$ with linear basis $v_1, \ldots, v_n$. Denote $A \odot A$ the subalgebra of $A \otimes A$, invariant w.r.t. $\mathbb{Z}_2$-action by permutation. The linear base of $A$ is
\[
\{v_i \odot v_j\}_{i=1}^n, \quad \{v_i \odot v_j + v_j \odot v_i\}_{i<j}.
\]

**Definition 6.** Consider a $\mathbb{Z}_2$-algebra $A$ with linear basis $v_1, \ldots, v_n$. Denote $\iota(A \otimes A)$ the quotient vector space $A \otimes A/(v_i \odot v_j + v_j \odot v_i)$. As $\mathbb{Z}_2$-algebra it has zero multiplication.
Lemma 7. The maps $Sq_e$, $\otimes$, map the algebra $H^*(K) \otimes H^*(K)$ to $H^*((K \times K \times S^n)/Z_2)$. The map $\iota$ maps $\iota(H^*(K) \otimes H^*(K))$ to $H^*((K \times K \times S^n)/Z_2)$. The images of these maps generate the cohomology $H^*((K \times K \times S^n)/Z_2)$.

The latter cohomology can be described as the quotient of $H^*(K) \otimes H^*(K) \otimes \mathbb{Z}[c] \oplus \iota(H^*(K) \otimes H^*(K))$ by the relations
\[ c^{n+1} = 0, \ (\xi \otimes \eta) \otimes c = 0, \ Sq_e \xi \otimes c^n = \iota(\xi \otimes \xi). \]

The $c$ is the preimage of the hyperplane class in $H^1(\mathbb{R}P^n)$.

Note the important particular case: if $n \to \infty$, we image of $\iota(\ldots)$ disappears, and we also can take the quotient of $H^*(K) \otimes H^*(K)$ by the linear span of all $\xi \otimes \eta$ for $\xi, \eta \in H^*(K)$. Hence, the cohomology $H^*((K \times K \times S^\infty)/Z_2)$ has a quotient isomorphic to $Sq_e(H^*(K)) \otimes \mathbb{Z}[c]$. Here $Sq_e(H^*(K))$ is the same algebra as $H^*(K)$, but with twice larger degrees.

Proof. The Leray-Serre spectral sequence for $\sigma_K$ starts with
\[ E_2^{p,q} = H^p(\mathbb{R}P^n, \mathcal{H}^q(K \times K)). \]

Let us describe the sheaf $\mathcal{H}^*(K \times K)$. If $v_1, \ldots, v_n$ is the linear basis of $H^*(K)$, then an element $v_i \otimes v_i$ gives a subsheaf, isomorphic to the constant sheaf $\mathbb{Z}_2$. The two elements $v_i \otimes v_j$ and $v_j \otimes v_i$ generate a non-constant sheaf $\mathcal{A} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with permutation action of $\pi_1(\mathbb{R}P^n)$. The cohomology $H^*(\mathbb{R}P^n, \mathcal{A}) = H^*(S^n, \mathbb{Z}_2)$, since $\mathcal{A}$ is the direct image of $\mathbb{Z}_2$ under the natural projection $\pi: S^n \to \mathbb{R}P^n$. Thus we know the structure of $E_2^{p,q}$.

The first column of $E_2$ is the $\mathbb{Z}_2$-invariant elements of $H^*(K \times K)$, and all these elements are the restrictions of either $Sq_e \xi$ or $\xi \otimes \eta$ to the fiber. Hence all the differentials of the spectral sequence are zero on the first column. The columns between the first and the last ($n$-th) are generated by multiplication with $c$, and the differentials are zero on them too. The last column is isomorphic to $\iota(H^*(K) \otimes H^*(K))$, the differentials are zero on it from the dimension considerations.

Hence in this spectral sequence $E_2 = E_\infty$. Denote $v_1, \ldots, v_n$ the linear base of $H^*(K)$. The first column of $E_2$ has the linear base
\[ \{v_i \times v_i\}_{i=1}^n, \ \{v_i \times v_j + v_j \times v_i\}_{i<j}, \]
the columns $j = 1, 2, \ldots, n-1$ have the linear base
\[ \{(v_i \times v_i)c^j\}_{i=1}^n, \]
and the last column has the linear base
\[ \{\iota(v_i \times v_j)\}_{i,j=1}^n. \]

From the definition of $Sq_e$, $\otimes$, and $\iota(\ldots)$ the final cohomology $H^*((K \times K \times S^n)/Z_2)$ is described the same way with $v_i \times v_i$ replaced by $Sq_e v_i$, and $v_i \times v_j + v_j \times v_i$ replaced by $v_i \otimes v_j$. \qed
Now consider a vector bundle \( \nu : E(\nu) \to K \) and define
\[
\text{Sq}_e \nu : (E(\nu) \times E(\nu) \times S^n)/\mathbb{Z}_2 \to (K \times K \times S^n)/\mathbb{Z}_2.
\]
The Stiefel-Whitney classes of \( \text{Sq}_e \nu \) are described by the following lemma.

**Lemma 8.** Let \( \dim \nu = k \), and let the Stiefel-Whitney class of \( \nu \) be
\[
w(\nu) = w_0 + w_1 + \ldots + w_k.
\]
Then
\[
w(\text{Sq}_e \nu) = \sum_{0 \leq i < j \leq k} w_i \odot w_j + \sum_{i=0}^{k} (1 + c)^{k-i} \text{Sq}_w w_i,
\]
where \( c \) is the image of the hyperplane class in \( H^1(\mathbb{R}P^n) \).

**Proof.** Consider the case of one-dimensional \( \nu \) first. Taking \( n \) large enough we do not have to consider the image of \( \iota(\ldots) \), then we can return to lesser \( n \) by the natural inclusion
\[
(K \times K \times S^n)/\mathbb{Z}_2 \to (K \times K \times S^{n+m})/\mathbb{Z}_2.
\]
The restriction of \( \text{Sq}_e \nu \) to the fiber \( K \times K \) has the Stiefel-Whitney class
\[
w(\nu \times \nu) = 1 + w_1(\nu) \times 1 + 1 \times w_1(\nu) + w_1(\nu) \times w_1(\nu).
\]
Hence \( w(\text{Sq}_e \nu) \) is either \( 1 + w_1(\nu) \odot 1 + \text{Sq}_w w_1(\nu), \) or \( 1 + w_1(\nu) \odot 1 + c + \text{Sq}_w w_1(\nu) \).
Any point \( x \in K \) gives a natural section
\[
s : S^n/\mathbb{Z}_2 \to (\{x\} \times \{x\} \times S^n)/\mathbb{Z}_2
\]
of the bundle \( \sigma_K \), and the bundle \( s^* (\text{Sq}_e \nu) \) over \( \mathbb{R}P^n \) is isomorphic to \( \gamma \oplus \varepsilon \), where \( \gamma \) is the canonical bundle of the projective space, \( \varepsilon \) is the trivial bundle. Hence we should have
\[
w(\text{Sq}_e \nu) = 1 + w_1(\nu) \odot 1 + c + \text{Sq}_w w_1(\nu).
\]
The general formula for \( k > 1 \) follows from the splitting principle, suppose \( \nu = \tau_1 \oplus \cdots \oplus \tau_k \), then
\[
w(\text{Sq}_e \nu) = \prod_{i=1}^{k} (1 + w_1(\tau_i) \odot 1 + c + \text{Sq}_w w_1(\tau_i)),
\]
and the result follows by removing parentheses. \( \square \)

**4. THE PROOF OF LEMMAS 4 AND 5**

In order to calculate the index of \( G_{2n}^n \), we describe the cohomology of \( G_{2n}^n/\mathbb{Z}_2 \). Consider the subgroup \( G = O(n) \wr \mathbb{Z}_2 \) of \( O(2n) \), that is generated by two copies of \( O(n) \) for some decomposition \( \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n \), and by the operator \( \sigma \) that interchanges the summands of the decomposition. It is clear that \( G_{2n}^n/\mathbb{Z}_2 = O(2n)/G \).
The cohomology of $BO(n)$ is the polynomial algebra in Stiefel-Whitney classes
$$H^*(BO(n)) = \mathbb{Z}_2[w_1, \ldots, w_n].$$

The group cohomology $H^*(BG)$ (by Lemma 7) is generated by the external Steenrod squares $Sq e w_1, \ldots, Sq e w_n$, the generator $c \in H^1(B\mathbb{Z}_2)$, and some combinations $x \odot y$ for $x, y \in H^*(BO(n))$, the relations are $(x \odot y)c = 0$.

Let us find the kernel of the natural map $\pi^* : H^*(BG) \to H^*(O(2n)/G)$. The cohomology $H^*(O(2n)/G)$ can be calculated by considering the Leray-Serre spectral sequence with the term $E_2^{p,q} = H^p(BG, H^q(O(2n)))$, see [11, Section 11.4]. The kernel of $\pi^*$ is given by the images of the differentials $d_r$ of this spectral sequence in its bottom row.

Note that the action of $G$ on $O(2n)$ is induced by the inclusion $G \subset O(2n)$, and the cohomology of $O(2n)$ is acted on by $G$ through its factor group $G/G^+$ of order 2. Here $G^+$ denotes the elements of $G$ with positive determinant. Hence we can replace $G$ by $G^+$ and simultaneously pass from the sheaf $\mathcal{H}^q(O(2n))$ to the cohomology $H^q(SO(2n))$ (see [1], Ch. III, Proposition 6.2), thus obtaining
$$E_2^{p,q} = H^p(BG, H^q(O(2n))) = H^p(BG^+, H^q(SO(2n))).$$

In order to find the images of $d_r$’s, note that the fiber bundle
$$SO(2n) \longrightarrow SO(2n) \times_G EG^+ \longrightarrow BG^+$$
is induced from the fiber bundle
$$SO(2n) \longrightarrow ESO(2n) \longrightarrow BSO(2n)$$
by the inclusion $G^+ \to SO(2n)$. In the spectral sequence of the latter fiber bundle all the primitive generators of $H^*(SO(2n))$ are transgressive. They are mapped to the bottom row by the corresponding differentials $d_r$, their images being the Stiefel-Whitney classes of $O(2n)$. Thus, in the considered spectral sequence, the differentials $d_r$ are generated by the transgressions that send the primitive generators of $H^*(SO(2n))$ to the Stiefel-Whitney classes of the representation of $G^+$ on $\mathbb{R}^{2n}$. Denote this representation $W_{2n}$.

Let us summarize as follows.

**Lemma 9.** The kernel of the natural map $\pi^* : H^*(BG) \to H^*(O(2n)/G)$ is generated by the homogeneous components of positive degree of the
expression
\[ \sum_{0 \leq i < j \leq n} w_i \odot w_j + \sum_{i=0}^{n} (1 + c)^{n-i} \text{Sq}_e w_i. \]

**Proof.** In the bottom row of the spectral sequence passing from $H^*(BG)$ to $H^*(BG^+)$ “kills” the element $w_1(W_{2n})$ and the ideal generated by it. The other differentials “kill” the other classes $w_r(W_{2n})$ by the above considerations.

It remains to calculate the Stiefel-Whitney classes of $W_{2n}$. Remind that by the Stiefel-Whitney classes of a representation we mean the Stiefel-Whitney classes of the vector bundle $\eta : (W_{2n} \times EG)/G \to BG$. Denote $V_n$ the natural representation of $O(n)$, and consider its corresponding bundle $\xi : (V_n \times EO(n))/O(n) \to BO(n)$. It can be checked by definition that $\eta = \text{Sq}_e \xi$ and the claim follows by applying Lemma 8. $\square$

Now the proof of Lemma 4 is finished as follows: we have to find the nilpotency degree of $c$ in $H^*(BG)/\ker \pi^*$. If $n$ is odd, then the one-dimensional generator of $\ker \pi^*$ is $c + w_1 \odot 1$,

hence $c \neq 0$, $c^2 = 0$ by Lemma 6 and $\text{ind} G_{2n}^n = 1$ in this case.

If $n \equiv 2 \mod 4$, then we have the relations in dimensions 2 and 3
\[ c^2 + \text{Sq}_e w_1 + 1 \odot w_2 = 0 \]
\[ c \text{Sq}_e w_1 + 1 \odot w_3 + w_1 \odot w_2 = 0. \]

Substituting $\text{Sq}_e w_1 = c^2 + 1 \odot w_2$ from the first relation to the second we obtain
\[ c^3 = 1 \odot w_3 + w_1 \odot w_2, \]

hence $c^4 = 0$ by Lemma 6 and $\text{ind} G_{2n}^n = 3$ in this case.

Now let us turn to Lemma 5. Let $n = 2^l$, and let us add the additional relations of the form $w_i = 0$ for all $i$ except $i = 2^l - 2^k$ ($k = 0, \ldots, l$) and $i = 2^l$. In this case the remaining relations in $\ker \pi^*$ are
\[ c^{2^l} = \text{Sq}_e w_{2^l-2^{l-1}} + 1 \odot w_{2^l} \]
\[ c^{2^{l-1}} \text{Sq}_e w_{2^l-2^{l-1}} = \text{Sq}_e w_{2^l-2^{l-2}} + w_{2^l-2^{l-1}} \odot w_{2^l} \]
\[ \ldots \]
\[ c^2 \text{Sq}_e w_{2^l-2} = \text{Sq}_e w_{2^{l-1}} + w_{2^l-2} \odot w_{2^l} \]
\[ c \text{Sq}_e w_{2^{l-1}} = w_{2^{l-1}} \odot w_{2^l} \]
\[ \text{Sq}_e w_{2^l} = 0, \]

along with the relations of the form
\[ w_{2^l-2^k} \odot w_{2^l-2^m} = 0, \ 0 \leq k < m \leq l. \]
Thus we obtain $c^{2l+1-1} = c^{2n-1} = w_2 \otimes w_{2l-1} \neq 0$. Also, we must have $c^{2n} = 0$ by the upper bound $\text{ind} G_{2n}^m \leq 2n - 1$, without any additional relations. Therefore, $\text{ind} G_{2n}^m = 2n - 1$ in this case.

References

[1] K. Brown. Cohomology of groups. Graduate Texts in Mathematics, 87, New York: Springer-Verlag, 1982.
[2] S. Buoncristiano, C.P. Rourke, B.J. Sanderson. A geometric approach to homology theory. Cambridge University Press, 1976.
[3] P.E. Conner, E.E. Floyd. Fixed point free involutions and equivariant maps. // Bull. Amer. Math. Soc., 66(6), 1960, 416–441.
[4] V.L. Dol’nikov. Transversals of families of sets in $\mathbb{R}^n$ and a connection between the Helly and Borsuk theorems (In Russian). // Sb., Math. 79(1), 1994, 93–107; translation from Mat. Sb., 184(5), 1993, 111–132.
[5] H.L. Hiller. On the cohomology of real Grassmannians. // Trans. Amer. Math. Soc., 257(2), 1980, 521–533.
[6] Wu Yi Hsiang. Cohomology theory of topological transformation groups. Berlin-Heidelberg-New-York, Springer Verlag, 1975.
[7] Nguyên H.V. Hung. The mod 2 equivariant cohomology algebras of configuration spaces. // Pacific Jour. Math., 143(2), 1990, 251–286.
[8] R.N. Karasev. Theorems of Borsuk-Ulam type for flats and common transversals (In Russian). // Math. Sbornik, 200(10), 2009, 39–58; translated in arXiv:0905.2747.
[9] M.A. Krasnosel’skii. On the estimation of the number of critical points of functionals (In Russian). // Uspehi Mat. Nauk, 7(2), 1952, 157–164.
[10] J. Matoušek. Using the Borsuk-Ulam theorem. // Berlin-Heidelberg, Springer Verlag, 2003.
[11] J. McCleary. A user’s guide to spectral sequences. Cambridge University Press, 2001.
[12] L. Montejano, R.N. Karasev. Topological transversals to a family of convex sets. // arXiv:1006.0103 2010.
[13] A.S. Schwarz. Some estimates of the genus of a topological space in the sense of Krasnosel’skii. (In Russian) // Uspehi Mat. Nauk, 12:4(76), 1957, 209–214.
[14] N.E. Steenrod, D.B. Epstein. Cohomology operations. Princeton University Press, 1962.
[15] R. Živaljević. Topological methods. // Handbook of Discrete and Computational Geometry, ed. by J.E. Goodman, J. O’Rourke, CRC, Boca Raton, 2004.

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