Schrödinger group and non-linear generalizations of quantum mechanics

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Abstract.
A free particle may be characterized abstractly in terms of its invariance group. Both the Hamilton-Jacobi and Schrödinger equations for classical and quantum free particles are invariant under the same group, the Schrödinger group.

The aim of this paper is to discuss possible restrictions on non-linear generalizations of quantum mechanics. The basic idea is to search for those non-linear generalization of quantum mechanics for which a description of the non-relativistic free particle includes a representation of the Schrödinger group, as in classical and quantum mechanics. This condition imposes strong constraints on possible models of free particles.

To carry out such an investigation it is necessary to select an appropriate mathematical formalism, one which can describe physical systems of non-relativistic particles and allows for the incorporation of non-linearities in a natural way. Here the formalism of ensembles on configuration space is used, which is an approach that is capable of describing both classical and quantum systems and, in addition, allows for more general theories that describe mixed classical-quantum systems. Modifications of the equations involving derivatives of up to fourth order are considered.

1. Introduction
Non-linear extensions of quantum quantum mechanics have an important role to play when it comes to the question of whether linearity is a fundamental aspect of quantum theory or, conversely, whether it may be relaxed. One approach to finding non-linear extensions that has proven extremely fruitful is based on looking at unitary representations of the diffeomorphism group of physical space and at non-linear gauge transformations (see references [1, 2] for a summary of results and further literature). The application of these methods has allowed a unification of various non-linear theories as well as providing some of the clearest indications of the viability of such theories.

The focus of this paper is not on deriving new theories but rather on examining possible restrictions on non-linear generalizations of quantum mechanics. The basic idea is to look at the consequences of requiring that in a particular case, that of a free particle, certain invariance properties are satisfied. This in some sense amounts to assuming that a physical system, such as a free particle, may be characterized abstractly in terms of its invariance group. Since both the Hamilton-Jacobi and Schrödinger equations for classical and quantum free particles are invariant under the same group, the Schrödinger group [3, 4], it seems natural to assume that non-linear generalizations of quantum mechanics, when applied to a free particle, will also have...
this property. It turns out that this condition imposes strong constraints on possible models of free particles.

To carry out the analysis, it is convenient to select an appropriate mathematical formalism which can describe physical systems of non-relativistic particles and allows for the incorporation of non-linearities in a natural way. This is what motivates the use of the formalism of ensembles on configuration space [5] here, rather than say carrying out the analysis in Hilbert space. This formalism, as explained in some detail in the next section, is capable of describing both classical and quantum systems and, in addition, allows for more general theories that describe mixed classical-quantum systems. These features make it attractive for the purpose of the analysis.

The paper is organized as follows. After a review of the essential aspects of the formalism of ensembles on configuration space, the next two sections discuss the Schrödinger Lie algebra and transformations of the space-time coordinates and fields. The section that follows considers possible non-linear modifications of the equations, allowing for derivatives of up to fourth order, given the restrictions introduced in the previous sections. The paper ends with a brief summary and discussion.

2. Ensembles on configuration space

The formalism of ensembles on configuration space [5] is an approach that is capable of describing both classical and quantum systems and, in addition, allows for a theory of mixed classical-quantum systems. The approach introduces very few physical and mathematical assumptions. The basic building blocks are the configuration space of the physical system, an ensemble of configurations, and dynamics generated from an action principle.

To introduce the main ideas, consider a non-relativistic particle with a configuration space with coordinates $x$. The description of such a physical system in terms of ensembles on configuration space requires

(i) a probability density $P(x)$ on configuration space, with $P(x) \geq 0$ and $\int d^3x P(x) = 1$,
(ii) a canonically conjugate quantity $S(x)$,
(iii) an ensemble Hamiltonian $H[P,S]$,

where $H[P,S]$ is a functional of $P$ and $S$ that must satisfy certain requirements (these conditions, which are common to all observables, are discussed below in the section on observables).

The state of the system is described by the functions $P$ and $S$. The time evolution of the state follows Hamiltonian equations of motion determined by $H$ which, when expressed in terms of Poisson brackets, take the form

$$\frac{\partial P}{\partial t} = \{P, H\} = \frac{\delta H}{\delta S}, \quad \frac{\partial S}{\partial t} = \{S, H\} = -\frac{\delta H}{\delta P}, \quad (1)$$

where the Poisson bracket for two arbitrary functionals $A[P,S]$ and $B[P,S]$ is given by

$$\{A, B\} = \int d^3 x \left( \frac{\delta A}{\delta P} \frac{\delta B}{\delta S} - \frac{\delta A}{\delta S} \frac{\delta B}{\delta P} \right). \quad (2)$$

This is a very general formalism which, at this stage of the presentation, essentially describes dynamics of probabilities using Hamilton equations of motion. A physical interpretation is only possible if an appropriate ensemble Hamiltonian is introduced together with a set of appropriate observables.
2.1. Classical and quantum particles
I now introduce a couple of physically relevant examples of ensemble Hamiltonians. The ensemble Hamiltonian that describes a classical particle subject to a force derived from a potential term $V(x)$ is given by

$$H_C[P, S] = \int d^3x \, P \left[ \frac{|\nabla S|^2}{2m} + V \right]. \quad (3)$$

To see this, evaluate the equations of motion,

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V = 0, \quad \frac{\partial P}{\partial t} + \nabla \cdot \left( P \frac{\nabla S}{m} \right) = 0. \quad (4)$$

The first equation is the Hamilton-Jacobi equation, the second equation is the continuity equation which ensures that probability is preserved. $P(x,t)$ is the probability that the particle is at a given location in configuration space at a given time.

The ensemble Hamiltonian that describes a quantum particle subject to a force derived from a potential $V(x)$ is given by

$$H_Q[P, S] = H_C[P, S] + \frac{\hbar^2}{2m} \int d^3x \, \frac{1}{P} \left( \frac{|\nabla P|^2}{2m} \right). \quad (5)$$

The corresponding equations of motion are

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V + \frac{\hbar^2}{2m} \left( \frac{\nabla^2 \sqrt{P}}{\sqrt{P}} \right) = 0, \quad \frac{\partial P}{\partial t} + \nabla \cdot \left( P \frac{\nabla S}{m} \right) = 0. \quad (6)$$

These equations are the Madelung equations [6] which describe a quantum particle using Madelung hydrodynamical variables $P$ and $S$. To see the connection to the Schrödinger equation, introduce the complex canonical transformation given by $\psi = \sqrt{P} \, e^{iS/\hbar}, \bar{\psi} = \sqrt{P} \, e^{-iS/\hbar}$ which leads to

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi, \quad (7)$$

the Schrödinger equation. Thus, in the formulation in terms of ensembles on configurations space, the wavefunction representation given in terms of $\psi, \bar{\psi}$ is related to the representation in terms of hydrodynamical variables $P, S$ via a non-linear canonical transformation.

2.2. Observables
Observables are functionals $A[P, S]$ which satisfy some basic requirements:

(i) An infinitesimal canonical transformation generated by $A$,

$$P \to P + \epsilon \frac{\delta A}{\delta S}, \quad S \to S - \epsilon \frac{\delta A}{\delta P}, \quad (8)$$

must preserve the normalization and the positivity of the probability. This leads to [5]

$$A[P, S + c] = A[P, S], \quad \frac{\delta A}{\delta S} = 0 \text{ if } P(x) = 0. \quad (9)$$

where $c$ is a constant. Note that the first equation implies gauge invariance of the theory under $S \to S + c$. 

(ii) Observables are homogeneous of degree one in \( P \), i.e., \( A[\lambda P, S] = \lambda A[P, S] \), where \( \lambda \) is an arbitrary positive constant [5]. It can be seen from Eqs. (10-11) below that this property is satisfied by all classical and quantum observables.

No further conditions are necessary. It is important to emphasize that the set of observables forms a closed algebra under the Poisson bracket. Observables include for example (i) the ensemble Hamiltonians introduced previously (i.e. \( H_C \) and \( H_Q \)), (ii) the position observable \( X[P, S] := \int d^3 x P x \), and (iii) the momentum observable \( \pi[P, S] := \int d^3 x P \nabla S \).

Classical and quantum observables may be defined as averages. Thus, observables have a dual role, as generators of canonical transformations and as average values. To define a classical observable, consider a function \( f(x, p) \) defined on the standard phase space of classical mechanics and define the corresponding classical observable by:

\[
C_f := \int d^3 x P f(x, \nabla S).
\] (10)

One can show that the Poisson bracket for classical observables is isomorphic to the phase space Poisson bracket [5]. To define a quantum observable, consider a Hermitian operator \( \hat{M} \) defined on Hilbert space. Define the corresponding quantum observable by:

\[
Q_{\hat{M}} := \langle \psi | \hat{M} | \psi \rangle = \int d^3 x d^3 x' (PP')^{1/2} e^{i(S-S')/\hbar} \langle x' | \hat{M} | x \rangle.
\] (11)

One can show that the Poisson bracket for quantum observables is isomorphic to the commutator on Hilbert space [5].

Thus, with classical and quantum observables defined according to Eqs. (10-11) one arrives at the non-trivial result that the algebras of each of these two classes of observables are isomorphic to the algebras that arise naturally in the phase space and Hilbert space representations of classical and quantum mechanics.

2.3. Composite systems and independent subsystems
Consider the case where a system consists of two independent, non-interacting particles, one particle described by a set of coordinates \( x_1 \) and the other by \( x_2 \) (the conditions on \( P \) and \( S \) listed below extend trivially to the case of more independent, non-interacting particles). Thus \( P \) and \( S \) must be of the forms [5]

\[
P(x_1, x_2) = P_1(x_1)P_2(x_2), \quad S(x_1, x_2) = S_1(x_1) + S_2(x_2),
\] (12)

with the second relation being valid up to an additive constant which has no physical significance.

In this case, there is a natural independence condition which may be imposed on the ensemble Hamiltonian of the composite system, which is that it decomposes into additive subsystem contributions. This additivity condition is satisfied by both the classical and quantum ensemble Hamiltonians, implying that the total energy of two non-interacting particles is equal the sum of the energies of each individual particle [5].

3. Schrödinger Lie algebra
After the brief review of the formalism of ensembles on configuration presented in the previous section, I now consider the problem of the representation of the generators of the Schrödinger Lie algebra in terms of observables.

The Schrödinger Lie algebra consists of the ten generators of the Galilei Lie algebra,

(i) \( A_i \) : space displacements,
(ii) $H$ : time displacement,
(iii) $L_i$ : space rotations,
(iv) $G_i$ : Galilean transformations ("boosts"),

with $i = 1, 2, 3$, plus two additional generators,
(v) $C$ : expansions,
(vi) $D$ : dilations,

for a total of 12 generators [7].

These generators satisfy a Lie algebra which consists of the Galiei Lie algebra,

\[
\{H, A_i\} = 0, \quad \{H, L_i\} = 0, \quad \{H, G_i\} = -A_i,
\]

\[
\{L_i, A_j\} = \epsilon_{ijk} A_k, \quad \{L_i, L_j\} = \epsilon_{ijk} L_k, \quad \{L_i, G_j\} = \epsilon_{ijk} G_k,
\]

\[
\{A_i, A_j\} = 0, \quad \{A_i, G_j\} = -m \delta_{ij}, \quad \{G_i, G_j\} = 0,
\]

where $m$ is the mass of the particle, plus the additional relations

\[
\{C, A_i\} = G_i, \quad \{C, L_i\} = 0, \quad \{C, G_i\} = 0, \\
\{D, A_i\} = -A_i, \quad \{D, L_i\} = 0, \quad \{D, G_i\} = G_i, \\
\{C, H\} = D, \quad \{D, H\} = -2H, \quad \{C, D\} = -2C.
\]

3.1. Representation of the generators in terms of observables

There are natural representations for most of the generators of the Schrödinger Lie algebra in terms of observables. The configuration space is that of a non-relativistic particle, Euclidean space $\mathbb{R}^3$.

The generators for space displacements and rotations are given by the observables

\[
A_i = \int d^3x \ P (\partial_i S), \quad L_i = \int d^3x \ P (\epsilon_{ijk} x_j \partial_k S),
\]

which correspond to the momentum and angular momentum observables. A natural choice for the Galilean boost transformations is given by

\[
G_i = \int d^3x \ P (m x_i - t \partial_i S).
\]

This follows from the definition $G_i = (m X_i - t A_i)$ where $X_i = \int d^3x \ P x_i$ is the position observable. The two remaining generators are defined by $C = (t^2 H - t X \cdot A - \frac{m}{2} X \cdot X)$ and $D = (t^2 H - X \cdot A)$ [7], thus they can be written as

\[
C = t^2 H - \int d^3x \ P \left( t x \cdot \nabla S - \frac{m}{2} x \cdot x \right), \quad D = 2tH - \int d^3x \ P (x \cdot \nabla S).
\]

Regarding the representation of $H$, there are clearly at least two choices; i.e., the ensemble Hamiltonians of Eqs. (3) and (5) that correspond to quantum and classical particles. I now want to consider the question of what additional options are available given the restrictions imposed by the Schrödinger Lie algebra. The idea then is to fix the choices of the other generators and look at the freedom that remains in the choice of $H$. 


3.2. Conditions that must be satisfied by the generator of time displacements

Consider first the conditions on the remaining generator \( H \) imposed by the Galilei Lie Algebra. It must transform as a scalar under translations and rotations (i.e., \( \{ H, A_i \} = \{ H, L_i \} = 0 \)) and in addition it must satisfy

\[
\{ H, G_i \} = -m \int d^3 x \frac{\delta H}{\delta S} x_i = -A_i = - \int d^3 x \ P \ \partial_i S.
\]

These condition are not a very stringent and it leads to a family of solutions \( H^{(K)} \) labeled by a functional \( K \),

\[
H^{(K)}[P, S] = \int d^3 x \ P \left[ \frac{\nabla S}{2m} + K[P, S] \right],
\]

where \( K \) is any observable that is invariant under translations, rotations and boosts, i.e., \( K \) is a Galilean scalar.

I now consider the conditions imposed by the additional two generators, \( C \) and \( D \). A calculation of the Poisson brackets \( \{ C, H \}, \{ D, H \} \) and \( \{ C, D \} \) leads to the following further conditions on \( K \),

\[
\left\{ K, \int d^3 x \ P \ \cdot \nabla S \right\} =: \{ K, X\pi \} = -2K, \quad (18)
\]

\[
\left\{ K, \int d^3 x \ P \ \cdot x \right\} =: \{ K, X^2 \} = 0. \quad (19)
\]

It is clear that these conditions are trivially satisfied for \( K_C[P, S] = 0 \), corresponding to the ensemble Hamiltonian of the classical free particle, Eq. (3).

One can show that \( K_Q[P, S] = \frac{\hbar^2}{4} \int d^3 x \frac{1}{P} \left( \frac{\nabla P^2}{2m} \right) \) also solves the conditions imposed on \( K \), corresponding to the ensemble Hamiltonian of the quantum free particle, Eq. (5). Since both \( K_Q \) and \( X^2 \) are functionals of \( P \) only, it immediately follows that \( \{ K_Q, X^2 \} = 0 \), as required. A longer calculation with Poisson brackets confirms that \( \{ K_Q, X\pi \} = -2K_Q \), as expected.

4. Transformations of space-time coordinates and fields

While it is possible to formulate the invariance requirement for the observable \( K \) introduced in Eq. (17) via Poisson brackets equations, it is more convenient to consider the Schrödinger group as a kinematical symmetry group and work instead with the corresponding transformations of the space-time coordinates and fields.

The Schrödinger group, denoted by \( \mathcal{G} \), is of the form [8]

\[
\mathcal{G} = SL(2, \mathbb{R}) \wedge G,
\]

where \( G \) is the connected, static Galilei group which induces the transformations

\[
\mathbf{x} \to R\mathbf{x} + \mathbf{a} + \mathbf{v} t, \quad t \to t
\]

and \( SL(2, \mathbb{R}) \) is the group

\[
\mathbf{x} \to \frac{\mathbf{x}}{\gamma t + \delta}, \quad t \to \frac{\alpha t + \beta}{\gamma t + \delta}; \quad \alpha \gamma - \beta \delta = 1
\]

which includes time translations, dilations and inversions.

The theory of ensembles on configuration space has been formulated in Section 2 via a Hamiltonian formalism, with the advantage that it is straightforward to define an algebra of
observables and, among the observables, to identify a set of generators for the Schrödinger Lie algebra. But to examine invariance under transformations of space-time coordinates and fields it is more convenient to use the Lagrangian formalism. The Lagrangian that corresponds to the ensemble Hamiltonian of Eq. (17) is given by

$$L(K)[P,S] = \int d^3x dt P \left( \dot{S} + \frac{\left| \nabla S \right|^2}{2m} \right) + K[P,S].$$

Setting $K = 0$ in Eq. (23) leads to the Lagrangian of a classical free particle,

$$L_C[P,S] = \int d^3x dt P \left( \dot{S} + \frac{\left| \nabla S \right|^2}{2m} \right),$$

while setting $K[P,S] = \frac{\hbar^2}{4} \int d^3x dt P \left( \frac{\left| \nabla P \right|^2}{2m} \right)$ leads to the Lagrangian $L_Q$ of a quantum free particle.

In their work on the maximal kinematical group of fluid dynamics, O’Raifeartaigh and Sreedhar [8] have shown that the Lagrangian $L_C$ is a particular case of a more general Lagrangian invariant under the following transformations of the coordinates and fields,

$$\xi = \frac{1}{\gamma t + \delta} (R x + a + v t), \quad \tau = \frac{\alpha t + \beta}{\gamma t + \delta}; \quad \alpha v - \beta \gamma = 1$$

$$\tilde{P} = (\gamma t + \delta)^n P, \quad \tilde{S} = S - \frac{\delta v^2}{2} + \frac{|\gamma (R x + a) - \delta v|^2}{2\gamma (\gamma t + \delta)},$$

corresponding to the Schrödinger group. One can also show that the Lagrangian $L_Q$ is also invariant under the same transformations of the coordinates and fields.

Two special cases are of interest. Setting $\beta = \gamma = 0, \alpha = \delta = 1$ leads to the static Galilei transformations. The $SL(2,R)$ transformations (which extend the Galilean group to the Schrödinger group) are obtained by setting $a = v = 0, R = 1$. These will play a role in the following section. They take the form

$$\xi = \frac{x}{\gamma t + \delta}, \quad \tau = \frac{\alpha t + \beta}{\gamma t + \delta}; \quad \alpha v - \beta \gamma = 1$$

$$\tilde{P} = (\gamma t + \delta)^n P, \quad \tilde{S} = S + \frac{\gamma x^2}{2(\gamma t + \delta)}.$$
Under $SL(2, R)$ transformations,
\[
\frac{\partial \xi_j}{\partial x_k} = \eta^{-1} \delta_{jk}, \quad \frac{\partial \xi_j}{\partial t} = -\eta^{-2} \gamma x_j, \quad \frac{\partial \tau}{\partial x_k} = 0, \quad \frac{\partial \tau}{\partial t} = \eta^{-2},
\]
(29)
where $\eta := (\gamma t + \delta)$. The measure transforms as
\[
d\tilde{\mu} := d\tau d^n\xi = \eta^{-(n+2)} dt d^n x =: \eta^{-(n+2)} d\mu
\]
(30)
and the first few spatial derivatives of $P$ and $S$ transform as
\[
\tilde{P} = \eta^n P, \quad \tilde{\nabla}^2 P = \eta^{(n+2)} \nabla^2 P, \quad \tilde{S} = \eta S + \gamma x, \quad |\tilde{\nabla} S|^2 = \eta^2 |\nabla S + \gamma^{-1} \gamma x|^2
\]

### 5.1. Possible choices of $K$

The task of finding the general form of an appropriate functional $K$ simplifies if one first considers how it may depend on $S$. The two Poisson bracket relations $\{C, H\} = D$ and $\{D, H\} = -2H$ imply that $K$, when written as an integral of some function $\kappa$ of $P$ and $S$ and its derivatives (as in Eq. (31)), cannot depend on $S$, $\nabla S$ or $\nabla^2 S$. Furthermore, $K$ must be invariant under rotations and spatial displacements, which are contained in the Galilean transformations. Thus, the most general form for $K$ allowing for spatial derivatives of up to third order, which lead to equations for $P$ and $S$ that are of up to fourth order in the spatial derivatives, is given by
\[
K[P, S] = \int dt d^n x \kappa(P, |\nabla P|^2, \nabla^2 P, \nabla(\nabla^2 P) \cdot \nabla P, \nabla(\nabla^2 S) \cdot \nabla P) =: \int dt d^n x \kappa(a, b, c, d, e)
\]
(31)
where on the second line a more compact notation (i.e., $a = P$, $b = |\nabla P|^2$, etc.) has been introduced. As shown below, it will only be necessary to study the invariance of such an expression for $K$ under $SL(2, R)$ transformations.

It will be instructive to consider possible choices of $K$ with increasing order of spatial derivatives.

### 5.2. Case of no spatial derivatives

I first consider $K^{(0)}$, which contains terms without derivatives. Then I set $\kappa = \kappa^{(0)}(a)$ in Eq. (31). Invariance under $SL(2, R)$ transformations requires
\[
\int d\tilde{\mu} k^{(0)}(\tilde{a}) = \int d\mu \eta^{-(n+2)} k^{(0)}(\eta^n a),
\]
(32)
which forces $\kappa^{(0)}(a) = C_0 a^{(1 + \frac{\pi}{2})}$ implying $K^{(0)}[P, S] = C_0 \int d^n x P^{(1 + \frac{\pi}{2})}$, where $C_0$ is a constant. But in this case $K^0$ depends on the dimensionality $n$ of the configuration space and additivity is not satisfied in arbitrary dimensions. Thus one must set $C_0 = 0$ implying that there is no choice of $K$ without derivatives which satisfies all the required conditions.
5.3. Case of spatial derivatives to first order
I now consider $K^{(1)}$, which contains terms with derivatives of up to first order. Then I set $\kappa = \kappa^{(1)}(a, b)$ in Eq. (31). Under $SL(2, R)$ transformations,

$$\int d\tilde{\mu} k^1(\tilde{a}, \tilde{b}) = \int d\mu \eta^{-(n+2)} k^1(\eta^n a, \eta^{2(n+2)} b)$$  \hspace{1cm} (33)

Invariance under $SL(2, R)$ requires

$$k^{(1)} = C_1 a^{(1+2/n)} \beta^{(1)} \left( a^{-(2+2/n)} b \right)$$

where $C_1$ is a constant and $\beta^{(1)}$ an arbitrary function. The only solution that satisfies the independence condition and is homogeneous of degree 1 in $P$ is given by $\beta^{(1)}(u) = u$, which leads to

$$K^{(1)}[P, S] = C_1 \int d\mu \frac{1}{P} |\nabla P|^2.$$  \hspace{1cm} (34)

The choice $C_1 = h^2/4$ corresponds to the quantum theory of the free particle.

5.4. Case of spatial derivatives to second order
I now consider $K^{(2)}$, which can contain terms with derivatives of up to second order. Then I set $\kappa = \kappa^{(2)}(a, b, c)$ in Eq. (31). Invariance under $SL(2, R)$ requires

$$\int d\tilde{\mu} k^{(2)}(\tilde{a}, \tilde{b}, \tilde{c}) = \int d\mu \eta^{-(n+2)} k^{(2)}(\eta^n a, \eta^{2(n+2)} b, \eta^{(n+2)} c)$$  \hspace{1cm} (35)

Invariance under $SL(2, R)$ requires

$$k^{(2)} = C_2 c \beta^{(2)} \left( a^{-(2+2/n)} b, a^{-(1+2/n)} c \right)$$

where $C_2$ is a constant and $\beta^{(2)}$ an arbitrary function. The choice $\beta^{(2)}(u, v) = uv^{-1}$ leads to the solution found previously, which corresponds to the quantum theory of the free particle. The only new solution that satisfies the independence condition and is homogeneous of degree 1 in $P$ is given by $\beta^{(2)}(u, v) = 1$, which leads to

$$K^{(2)}[P, S] = C_2 \int d\mu \nabla^2 P.$$  \hspace{1cm} (36)

But a term such as $K^{(2)}$ does not contribute to the equation of motion, so nothing new comes out of allowing for spatial derivatives of up to second order.

5.5. Case of spatial derivatives to third order
Finally, I consider $K^{(3)}$, which can contain terms with derivatives of up to third order. Then I set $\kappa = \kappa^{(3)}(a, b, c, d, e)$ in Eq. (31). Invariance under $SL(2, R)$ requires

$$k^{(3)} = C_3 c \beta^{(3)} \left( a^{-(2+2/n)} b, a^{-(1+2/n)} c, a^{-(1+4/n)} d, a^{-(1+4/n)} e \right)$$  \hspace{1cm} (37)

where $C_3$ is a constant and $\beta^{(3)}$ an arbitrary function. The choices $\beta^{(3)}(u, v, w, y, z) = uv^{-1}$ and $\beta^{(3)}(u, v, w, y, z) = 1$ lead to the solutions found previously. There are no new solutions that satisfy the independence condition and are homogeneous of degree 1 in $P$, so nothing new comes out of allowing for spatial derivatives of up to third order.
6. Summary and discussion
The aim of this work was to look at possible restrictions on non-linear generalizations of quantum mechanics that are a consequence of invariance requirements. The basic idea was to examine a particular case, that of a free particle: Since both the Hamilton-Jacobi and Schrödinger equations for classical and quantum free particles are invariant under the same group, the Schrödinger group, it seems natural to assume that non-linear generalizations of quantum mechanics, when applied to a free particle, should also have this property.

While the analysis could have been done using a Hilbert space formulation, it seemed convenient to use instead the formalism of ensembles on configuration space, an approach that is capable of describing both classical and quantum systems and, in addition, allows for more general theories that describe mixed classical-quantum systems. One important advantage is that the formalism allows for the inclusion of non-linearities in a natural way. Another advantage is that two additional conditions of physical importance, i.e. Independence and Homogeneity (see section 5 for the precise definition), can be formulated in a relatively simple way.

It turns out that invariance under the Schrödinger group together with the two requirements introduced in section 5 leads to surprisingly strong constraints on possible non-linear extensions of quantum mechanics, so strong that no non-linear modifications of the equations, allowing for derivatives of up to fourth order, were found. It appears therefore that the assumptions introduced in this paper must be relaxed if one wants to formulate a non-linear theory that goes beyond quantum mechanics and is not equivalent to it. It is important to point out that these results do not necessarily exclude non-linear theories that are physically equivalent to quantum mechanics [1, 2, 9] as it can be shown for particular cases that they may be generated in the formalism of ensembles on configuration space via non-linear canonical transformations [5].

It is of interest to reexamine non-linear theories formulated in Hilbert space [1, 2] using the tools of the formalism of ensembles on configuration space. The goal would be to gain further insight into the physics of these theories. This will be the subject of a future publication.

Acknowledgments
I am grateful to H.-D. Doebner, G. A. Goldin and M. J. W. Hall for fruitful discussions.

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