DEGENERACY SCHEMES
AND SCHUBERT VARIETIES

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ABSTRACT. A result of Zelevinsky states that an orbit closure in the space of representations of the equioriented quiver of type \( A_h \) is in bijection with the opposite cell in a Schubert variety of a partial flag variety \( SL(n)/Q \). We prove that Zelevinsky’s bijection is a scheme-theoretic isomorphism, which shows that the universal degeneracy schemes of Fulton are reduced and Cohen-Macaulay in arbitrary characteristic.

Among all algebraic spaces, the best understood are the flag varieties and their Schubert subvarieties. They first appear as interesting examples, but acquire a general importance in the theory of characteristic classes of vector bundles.

Fulton has recently given a theory \( \xi \) of “universal degeneracy loci”, characteristic classes associated to maps among vector bundles, in which the role of Schubert varieties is taken by certain degeneracy schemes. The underlying varieties of these schemes arise in the theory of quivers: they are the closures of orbits in the space of representations of the equioriented quiver \( A_h \). The variety of complexes is another example of this class.

By a remarkable but little-known result of Zelevinsky \( \xi \) (c.f. \( \xi \)), the above quiver varieties can be identified set-theoretically with open subsets of Schubert varieties of a partial flag variety. In this paper, we prove a scheme-theoretic strengthening of Zelevinsky’s identification: the “naive” determinantal conditions defining each quiver variety as a set generate the same ideal as the Plucker equations defining (the opposite cell of) the corresponding Schubert variety. Since the latter ideal is well understood via Standard Monomial Theory, we conclude that the quiver schemes defined by the determinantal equations are reduced and their singularities are identical to those of Schubert varieties. In particular, the quiver varieties are Cohen-Macaulay, answering a question of Fulton.

Our results extend those of Abeasis, Del Fra, and Kraft \( \xi \), who showed that the quiver varieties are Cohen-Macaulay with rational singularities over a field of characteristic zero, and that the determinantal conditions generate the reduced ideals of the quiver varieties of codimension one. Also, our methods are similar to those of Gonciulea and Lakshmibai \( \xi \).

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2.4 Degeneracy schemes

1 Zelevinsky’s bijection

1.1 Quiver varieties

Fix an \( h \)-tuple of non-negative integers \( \mathbf{n} = (n_1, \ldots, n_h) \) and a list of vector spaces \( V_1, \ldots, V_h \) over an arbitrary field \( k \) with respective dimensions \( n_1, \ldots, n_h \).

Define the variety of quiver representations (of dimension \( \mathbf{n} \), of the equioriented quiver of type \( A_h \)) to be the affine space \( Z \) of all \((h-1)\)-tuples of linear maps \((f_1, \ldots, f_{h-1})\):

\[
V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{h-2}} V_{h-1} \xrightarrow{f_{h-1}} V_h .
\]

If we endow each \( V_i \) with a basis, we get \( V_i \cong k^{n_i} \) and

\[
Z \cong M(n_2 \times n_1) \times \cdots \times M(n_h \times n_{h-1}) ,
\]

where \( M(k \times l) \) denotes the affine space of matrices over \( k \) with \( k \) rows and \( l \) columns. The group

\[
G_\mathbf{n} = GL(n_1) \times \cdots \times GL(n_h)
\]

acts on \( Z \) by

\[
(g_1, g_2, \ldots, g_h) \cdot (f_1, f_2, \ldots, f_{h-1}) = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \ldots, g_h f_{h-1} g_{h-1}^{-1}),
\]

corresponding to change of basis in the \( V_i \).

Now, let \( \mathbf{r} = (r_{ij})_{1 \leq i \leq j \leq h} \) be an array of non-negative integers with \( r_{ii} = n_i \), and define \( r_{ij} = 0 \) for any indices other than \( 1 \leq i < j \leq h \). Define

\[
Z^\circ(\mathbf{r}) = \{ (f_1, \ldots, f_{h-1}) \in Z \mid \forall i < j, \text{ rank}(f_{j-1} \cdots f_i : V_i \to V_j) = r_{ij} \}.
\]

(This set might be empty for a bad choice of \( \mathbf{r} \).)

Proposition. The \( G_\mathbf{n} \)-orbits of \( Z \) are exactly the sets \( Z^\circ(\mathbf{r}) \) for \( \mathbf{r} = (r_{ij}) \) with

\[
r_{i,j-1} - r_{i,j} - r_{i-1,j-1} + r_{i-1,j} \geq 0, \quad \forall 1 \leq i < j \leq h.
\]

Proof. This is a standard result of algebraic quiver theory [4], [5], [12]. Since this theory is not well known among geometers, we recall it here.

Consider the abelian category \( \mathcal{R} \) of quiver representations whose objects are sequences of linear maps \((V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{h-1}} V_h)\), where the \( V_i \) are \textit{any} vector spaces of arbitrary dimension. A morphism of \( \mathcal{R} \) from the object \((V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{h-1}} V_h)\) to the object \((V'_1 \xrightarrow{f'_1} \cdots \xrightarrow{f'_{h-1}} V'_h)\) is defined to be an \( h \)-tuple of linear maps \((\phi_i : V_i \to V'_i)\) such that each square

\[
\begin{array}{ccc}
V_i & \xrightarrow{f_i} & V_{i+1} \\
\downarrow \phi_i & & \downarrow \phi_{i+1} \\
V'_i & \xrightarrow{f'_i} & V'_{i+1}
\end{array}
\]
commutes.

Direct sum of objects is defined componentwise, and it is known (Krull-
Remak-Schmidt Theorem) that any object \( R \in \mathcal{R} \) can be written uniquely as a
direct sum of the indecomposable objects

\[
R_{ij} = (0 \to \cdots \to k \overset{\sim}{\to} \cdots \overset{\sim}{\to} k \to 0 \to \cdots \to 0)
\]

for \( 1 \leq i < j \leq h + 1 \) (corresponding to the positive roots of the root system
\( A_h \)). That is,

\[
R \cong \bigoplus_{1 \leq i < j \leq h + 1} m_{ij} R_{ij}
\]

for unique multiplicities \( m_{ij} \in \mathbb{Z}^+ \).

Our variety \( Z \) consists of representations with fixed \( (V_i) \) and all possible
\( (f_i) \). Two points of \( Z \) are in the same \( G_n \)-orbit exactly if they are isomorphic as
objects in \( \mathcal{R} \). So the orbits correspond to arrays \( (m_{ij})_{1 \leq i < j \leq h + 1} \) with \( m_{ij} \in \mathbb{Z}^+ \)
and \( n_i = \sum_{k \leq i < l} m_{kl} \).

We can compute the rank numbers \( (r_{ij}) \) from the multiplicities \( (m_{ij}) \):

\[
r_{ij} = \sum_{k \leq i < j < l} m_{kl},
\]

and conversely

\[
m_{ij} = r_{i,j-1} - r_{i,j} - r_{i-1,j-1} + r_{i-1,j}.
\]

Hence the arrays \( (r_{ij}) \) with the stated conditions classify the \( G_n \)-orbits on \( Z \). ●

We define the quiver variety

\[
Z(\mathbf{r}) = \{(f_1, \ldots, f_{h-1}) \in Z \mid \forall i, j, \text{ rank}(f_{j-1} \cdots f_i : V_i \to V_j) \leq r_{ij}\}.
\]

Finally, we have the dimension formula due to Abeasis and Del Fra [3].

Proposition.

\[
\dim Z(\mathbf{r}) = \dim G_n - \sum_{1 \leq i \leq h} (r_{i,j} - r_{i,j+1})(r_{ij} - r_{i-1,j}).
\]

1.2 Schubert varieties

Given \( \mathbf{n} = (n_1, \cdots, n_h) \), for \( 1 \leq i \leq h \) let

\[
a_i = n_1 + n_2 + \cdots + n_i \quad \text{and} \quad n = n_1 + \cdots + n_h.
\]

For positive integers \( i \leq j \), we shall frequently use the notations

\[
[i, j] = \{i, i+1, \ldots, j\}, \quad [i] = [1, i].
\]
Let $k^n$ be a vector space (over our arbitrary field $k$) with standard basis $e_1, \ldots, e_n$. Consider its general linear group $GL(n)$, the subgroup $B$ of upper-triangular matrices, and the parabolic subgroup $Q$ of block upper-triangular matrices

$$Q = \{(a_{ij} \in GL(n) \mid a_{ij} = 0 \text{ whenever } j \leq a_k < i \text{ for some } k}\}.$$

A partial flag of type $(a_1 < a_2 < \cdots < a_h = n)$ (or simply a flag) is a sequence of subspaces $U = (U_1 \subset U_2 \subset \cdots \subset U_h = k^n)$ with $\dim U_i = a_i$. Let $E_i = \langle e_1, \ldots, e_{a_i} \rangle$ the span of the first $a_i$ coordinate vectors, and $E'_i = \langle e_{a_i+1}, \ldots, e_n \rangle$ the natural complementary subspace to $E_i$, so that $E_i \oplus E'_i = k^n$. Call $E_i = (E_1 \subset E_2 \subset \cdots)$ the standard flag. Let $Fl$ denote the set of all flags $U$ as above.

$Fl$ has a transitive $GL(n)$-action induced from $k^n$, and $Q = \text{Stab}_{GL(n)}(E_i)$, so we may identify $Fl \cong GL(n)/Q$, $gE_i \leftrightarrow gQ$. The Schubert varieties are the closures of $B$-orbits on $Fl$. Such orbits are usually indexed by certain permutations of $[n]$, but we prefer to use flags of subsets of $[n]$, of the form

$$\tau = (\tau_1 \subset \tau_2 \subset \cdots \subset \tau_h = [n]), \quad \#\tau_i = a_i.$$

(A permutation $w: [n] \rightarrow [n]$ corresponds corresponds to the subset-flag with $\tau_i = w[a_i] = \{w(1), w(2), \ldots, w(a_i)\}$. This gives a one-to-one correspondence between cosets of the symmetric group $S_n$ modulo the Young subgroup $S_{n_1} \times \cdots \times S_{n_h}$, and subset-flags.)

Given such $\tau$, let $E_i(\tau) = \langle e_j \mid j \in \tau_i \rangle$ be a coordinate subspace of $k^n$, and $E_i(\tau) = (E_1(\tau) \subset E_2(\tau) \subset \cdots) \in Fl$. Then we may define the Schubert cell

$$X^\circ(\tau) = B : E(\tau) = \left\{ (U_1 \subset U_2 \subset \cdots) \in Fl \mid \dim U_i \cap k^j = \# \tau_i \cap [j] \right\}$$

and the Schubert variety

$$X(\tau) = \overline{X^\circ(\tau)} = \left\{ (U_1 \subset U_2 \subset \cdots) \in Fl \mid \dim U_i \cap k^j \geq \# \tau_i \cap [j] \right\}$$

where $k^j = \langle e_1, \ldots, e_j \rangle \subset k^n$.

We define the opposite cell $O \subset Fl$ to be the set of flags in general position with respect to the spaces $E'_1 \supset \cdots \supset E'_{h-1}$:

$$O = \{(U_1 \subset U_2 \subset \cdots) \in Fl \mid U_1 \cap E'_i = 0\}.$$

We also define $Y(\tau) = X(\tau) \cap O$, an open subset of $\mathcal{X}(\tau)$. By abuse of language, we call $Y(\tau)$ the opposite cell of $X(\tau)$, even though it is not a cell.
1.3 The bijection ζ

We define a special subset-flag \( \tau^\text{max} = (\tau^\text{max}_1 \subset \cdots \subset \tau^\text{max}_h = [n]) \) corresponding to \( n = (n_1, \ldots, n_h) \). We want \( \tau^\text{max}_i \) to contain numbers as large as possible given the constraint \( [a_{i-1}] \subset \tau^\text{max}_i \). Namely, we define \( \tau^\text{max}_i \) recursively by

\[
\tau^\text{max}_h = [n]; \quad \tau^\text{max}_i = [a_{i-1}] \cup \{ \text{largest } n_i \text{ elements of } \tau^\text{max}_{i+1} \}.
\]

Furthermore, given \( r = (r_{ij})_{1 \leq i \leq j \leq h} \) indexing a quiver variety, define a subset-flag \( \tau^r \) to contain numbers as large as possible given the constraints

\[
\# [a_j] \cap \tau^r_i = \begin{cases} a_i - r_{i,j+1} & \text{for } i \leq j \\ a_j & \text{for } i > j \end{cases}
\]

Namely,

\[
\tau^r_i = \begin{cases} 1 \ldots a_{i-1} & \ldots & a_i & \ldots & a_{i+1} & \ldots & a_{i+2} & \ldots & n \\ a_{i-1} & a_{i-1} - r_{i,i+1} & a_{i-1} - r_{i,i+2} & a_{i-1} - r_{i,i+3} & \ldots & r_{i,h} \end{cases}
\]

where we use the visual notation

\[
\underbrace{\ldots a}_b = [a - b + 1, a].
\]

Note that \( r_{ij} - r_{i,j+1} \leq n_j \), so that each \( \tau^r_i \) is a list of increasing integers, and that \( r_{ij} - r_{i,j+1} \leq r_{i+1,j} - r_{i+1,j+1} \), so that \( \tau^r_i \subset \tau^r_{i+1} \). Thus \( \tau^\text{max} \) and \( \tau^r \) are indeed subset-flags.

Now define the Zelevinsky map

\[
\zeta : \mathbb{Z} \rightarrow \text{Fl} \quad (f_1, \ldots, f_{h-1}) \rightarrow (U_1 \subset U_2 \subset \cdots)
\]

where

\[
U_i = \{(u_1, \ldots, u_h) \in k^{n_1} \oplus \cdots \oplus k^{n_h} = k^n : \forall j > i, u_{j+1} = f_j(u_j)\}.
\]

In terms of coordinates, if we identify the linear maps \( (f_1, \ldots, f_{h-1}) \) with the matrices \((A_1, \ldots, A_{h-1})\), and identify \( \text{Fl} \cong GL(n)/Q \), we have

\[
\zeta(A_1, \ldots, A_{h-1}) = \begin{pmatrix}
I_1 & 0 & 0 & 0 & \cdots \\
A_1 & I_2 & 0 & 0 & \cdots \\
A_2 A_1 & A_2 & I_3 & 0 & \cdots \\
A_3 A_2 A_1 & A_3 A_2 & A_3 & I_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \mod Q
\]

where \( I_i \) is an identity matrix of size \( n_i \).

**Theorem.** (Zelevinsky [13])

(i) \( \zeta \) is a bijection of \( \mathbb{Z} \) onto its image \( \text{Y}(\tau^{max}) \): \( \zeta : \mathbb{Z} \cong \text{Y}(\tau^{max}) \).
Also,

\[ Y(\tau^\text{max}) = \{(U_1 \subset U_2 \subset \cdots) \mid \forall i, E_{i-1} \subset U_i, U_i \cap E'_i = 0\}. \]

(ii) \( \zeta \) restricts to a bijection from \( Z(\mathfrak{t}) \) onto \( Y(\tau^*) \): \( \zeta : Z(\mathfrak{t}) \setminus \sim Y(\tau^*). \)

Also,

\[ Y(\tau^*) = \left\{(U_1 \subset U_2 \subset \cdots) \mid \forall i \leq j, \dim E_j \cap U_i \geq a_i - r_{i,j+1}, \begin{array}{c} E_{i-1} \subset U_i, U_i \cap E'_i = 0 \end{array} \right\}. \]

Proof. Obviously \( \zeta \) is injective. To prove (i), we first show that \( \zeta(Z) \) is equal to the right hand side of equation (\(*\)). One inclusion is clear.

To show the other inclusion, consider any \( U \) with \( E_{i-1} \subset U_i \) and \( U_i \cap E'_i = 0 \) for all \( i \). Let \( \pi_i : k^n = E_i \oplus E'_i \to E_i \) be the projection. Then \( \pi_{h-1} \) restricts to an isomorphism \( U_{h-1} \to E_{h-1} \), so there exists an inverse linear map \( \text{id} \oplus f_{h-1} : E_{h-1} \to E_{h-1} \oplus k^{n_h} \)

such that

\[ U_{h-1} = \text{Graph}(f_{h-1}) \subset E_{h-1} \oplus k^{n_h} = k^n. \]

Since \( E_{h-2} \subset U_{h-1} \), we have \( f_{h-1}(E_{h-2}) = 0 \). Next, \( \pi_{h-2} \) restricts to an isomorphism \( U_{h-2} \to E_{h-2} \), and there exists a linear map \( \tilde{f}_{h-2} : E_{h-2} \to E'_{h-2} \) with \( \tilde{f}_{h-2}(E_{h-3}') = 0 \) such that

\[ U_{h-2} = \text{Graph}(\tilde{f}_{h-2}) \subset E_{h-2} \oplus E'_{h-2} = k^n. \]

Since \( U_{h-2} \subset U_{h-1} \), we have

\[ \tilde{f}_{h-2} = (f_{h-2}, f_{h-1}f_{h-2}) \]

for some \( f_{h-2} : E_{h-2} \to k^{n_{h-1}} \). Continuing in this way, we find that \( U \in \zeta(Z) \).

Thus it suffices to show that \((*)\) is valid. Again, the inclusion \( \subset \) is clear. Now consider a flag \( U \) satisfying \( E_{i-1} \subset U_i \) for all \( i \). Then we will show that \( U \) must satisfy \( \dim(k^i \cap U_j) \geq \# \{i \cap \tau^\text{max}_j\} \) for all \( 1 \leq i \leq n, 1 \leq j \leq h \). Acting by \( B \) does not change \( \dim U_i \cap k^j \), so we may assume our \( U \) is a flag of coordinate subspaces \( U = E(\mu) \) for some \( \mu = (\mu_1 \subset \cdots \subset \mu_h = [n]) \) with \( \{a_{i-1} \subset \mu_i \} \) for all \( i \), so that \( \dim U_i \cap k^j = \# \mu_i \cap [j] \). Then by the definition of \( \tau^\text{max} \), we must have

\[ \forall j, \quad \# \mu_{h-1} \cap [j] \geq \# \tau^\text{max}_{h-1} \cap [j], \quad \# \mu_{h-2} \cap [j] \geq \# \tau^\text{max}_{h-2} \cap [j], \ldots. \]

This proves \((*)\), and hence part (i).

The proof of (ii) is similar. Clearly \( Y(\tau^*) \subset Y(\tau^\text{max}) = \zeta(Z) \). For any flag \( U = \zeta(f_1, \ldots, f_{h-1}) \), we have

\[ \dim E_j \cap U_i = \dim E_{i-1} + \dim \ker(f_jf_{j-1} \cdots f_i) = \dim E_{i-1} + \dim V_i - \text{rank}(f_jf_{j-1} \cdots f_i) = \dim V_i - \text{rank}(f_jf_{j-1} \cdots f_i). \]

Hence \( \dim E_j \cap U_i \geq a_i - r_{i,j+1} \) if and only if \( U \in \zeta(Z(\mathfrak{r})) \), so that \( \zeta(Z(\mathfrak{r})) \) is equal to the right hand side of (\(\ast\ast\)). But the conditions on the right side of (\(\ast\ast\)) are enough to force the flag \( U \) to lie in the Schubert variety \( X(\tau^*) \) on the left hand side, as in part (i). \( \bullet \)
1.4 The actions of $B$, $Q$ and $G_n$

Let $W = S_n$ and $W_n = S_{n_1} \times \cdots \times S_{n_h}$ a Young subgroup. Let $W_n$ act on the the coset space $W/W_n$ by left multiplication. Then we may consider $\tau^{\text{max}}$ as a coset in $W/W_n$ which is Bruhat-maximal within its $W_n$ orbit. Since $W_n$ is the Weyl group of $Q$, this means that the $B$-action on the Schubert variety $X(\tau^{\text{max}})$ extends to a $Q$-action.

We may embed $G_n$ into $Q$ as the block diagonal matrices, so that $G_n$ acts on $X(\tau^{\text{max}})$ and in fact on the open subvariety $Y(\tau^{\text{max}})$. Then $\zeta : Z \to Y(\tau^{\text{max}})$ is equivariant with respect to the $G_n$-action.

Now we relate our combinatorial formalism to that in Zelevinsky’s original paper [13]. We have just seen that our $\tau^r$ correspond to certain double cosets in $W_n \backslash W/W_n$. Following Zelevinsky, we may index such double cosets by block permutation matrices, which are defined to be the $h \times h$ arrays $T = (t_{ij})$ of non-negative integers with row and column sums equal to the $n_i$, so that for all $1 \leq i, j \leq h$,

$$
\sum_{i=1}^{h} t_{ij} = n_j, \quad \sum_{j=1}^{h} t_{ij} = n_i.
$$

(If all $n_i = 1$, this defines an ordinary permutation matrix.)

A permutation $w \in W$ corresponds to the block permutation matrix $\text{Block}(w)$ defined by partitioning the ordinary permutation matrix of $w$ into blocks, and summing all entries in each block:

$$
\text{Block}(w) = (t_{ij}), \quad t_{ij} = \# [a_{i-1} + 1, a_i] \cap w[a_{j-1} + 1, a_j].
$$

The block map induces a one-to-one correspondence between double cosets $W_n \backslash W/W_n$ and block permutation matrices.

Zelevinsky’s map takes $Z(r)$ to $Y(\tau^r)$ for each $r = (r_{ij})$. Recall from the proof of Proposition 1.1 that the rank numbers $r_{ij}$, $1 \leq i \leq j \leq h$, can be computed from certain multiplicities $m_{ij}$, $1 \leq i < j \leq h + 1$. Then the block permutation matrix corresponding to $\tau^r$ is given by

$$
\begin{pmatrix}
m_{12} & m_{1}^* & 0 & 0 & \cdots \\
m_{13} & m_{23} & m_{2}^* & 0 & \cdots \\
m_{14} & m_{24} & m_{34} & m_{3}^* & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

where

$$
m_{i}^* = \sum_{k<i,1<l} m_{kl}.
$$

2 Plucker coordinates and determinantal ideals

For a variety $X$ embedded in an affine space $V$ over an infinite field $k$, the vanishing ideal $I$ of $X$ is the set of polynomial functions on $V$ which restrict to
zero on $X$. However, if $k$ is a finite field, we modify this definition in the usual way: the vanishing ideal is the set of polynomials on $V$ which are zero on the points of $X$ over the algebraic closure of $k$:
\[
\mathcal{I} = \{ f \in k[V] \mid f(x) = 0 \ \forall x \in X(\overline{k}) \}.
\]
The ideal $\mathcal{I}$ is necessarily reduced (radical).

## 2.1 Coordinates on the opposite big cell

Consider the opposite cell $O \subset GL(n)/Q$. It is easily seen that $O$ consists of those cosets which have a unique representative $A$ of the form
\[
A = (a_{kl}) = \begin{pmatrix}
I_1 & 0 & 0 & \cdots & 0 \\
A_{21} & I_2 & 0 & \cdots & 0 \\
A_{31} & A_{32} & I_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & A_{k3} & \cdots & I_k
\end{pmatrix} \mod Q,
\]
where $I_i$ is the identity matrix of size $n_i$, and $A_{ij}$ is an arbitrary matrix of size $n_i \times n_j$. That is, $O$ is an affine space with coordinates $a_{kl}$ for those positions $(k,l)$ with $1 \leq l \leq a_i < k \leq n$ for some $i$. Its coordinate ring is the polynomial ring
\[
k[O] = k[a_{kl}],
\]
For a matrix $M \in M(k \times l)$ and subsets $\lambda \subset [k], \mu \subset [l]$, let $\det M_{\lambda \times \mu}$ be the minor with row indices $\lambda$ and column indices $\mu$. Now let $\sigma \subset [n]$ be a subset of size $\# \sigma = a_i$ for some $i$. Define the Plucker coordinate $p_{\sigma} \in k[O]$ to be the $a_i$-minor of our matrix $A$ with row indices $\sigma$ and column indices the interval $[a_i]$:
\[
p_{\sigma} = p_{\sigma}(A) = \det A_{\sigma \times [a_i]}.
\]
Define a partial order on Plucker coordinates by:
\[
\sigma \leq \sigma' \iff \sigma = \{ \sigma(1) < \sigma(2) < \cdots < \sigma(a_i) \}, \quad \sigma' = \{ \sigma'(1) < \sigma'(2) < \cdots < \sigma'(a_i) \},
\]
\[
\sigma(1) \leq \sigma'(1), \quad \sigma(2) \leq \sigma'(2), \cdots, \sigma(a_i) \leq \sigma'(a_i).
\]
This is a version of the Bruhat order.

**Proposition.** Let $\tau = (\tau_1 \subset \cdots \subset \tau_h = [n])$ be a subset-flag and $Y(\tau)$ the intersection of the Schubert variety $X(\tau)$ with the opposite cell $O$. Then the vanishing ideal $\mathcal{I}(\tau) \subset k[O]$ of $Y(\tau) \subset O$ is generated by those Plucker coordinates $p_{\sigma}$ which are incomparable with one of the $p_{\tau_i}$:
\[
\mathcal{I}(\tau) = \langle p_{\sigma} \mid \exists i, \ # \sigma = a_i, \ \sigma \not\leq \tau_i \rangle.
\]

**Proof.** This follows from well-known results of Lakshmibai-Musili-Seshadri in Standard Monomial Theory (see e.g. [10], [8]).

8
2.2 The main theorem

Denote a generic element of the quiver space \( Z = M(n_2 \times n_1) \times \cdots \times M(n_h \times n_{h-1}) \) by \((A_1, \ldots, A_{h-1})\), so that the coordinate ring of \( Z \) is the polynomial ring in the entries of all the matrices \( A_i \). Let \( \mathbf{r} = (r_{ij}) \) index the quiver variety \( Z(\mathbf{r}) = \{(A_1, \ldots, A_{h-1}) \mid \text{rank } A_{j-1} \cdots A_i \leq r_{ij}\} \).

Let \( \mathcal{J}(\mathbf{r}) \subset k[Z] \) be the ideal generated by the determinantal conditions implied by the definition of \( Z(\mathbf{r}) \):

\[
\mathcal{J}(\mathbf{r}) = \left\{ \det(A_{j-1}A_{j-2} \cdots A_i)_{\lambda \times \mu} \mid i \leq j, \lambda \subset [n_j], \mu \subset [n_i], \# \lambda = \# \mu = r_{ij} + 1 \right\}.
\]

Clearly \( \mathcal{J}(\mathbf{r}) \) defines \( Z(\mathbf{r}) \) set-theoretically.

**Theorem.** \( \mathcal{J}(\mathbf{r}) \) is a prime ideal and is the vanishing ideal of \( Z(\mathbf{r}) \subset Z \). There are isomorphisms of reduced schemes

\[
Z(\mathbf{r}) = \text{Spec}(k[Z] / \mathcal{J}(\mathbf{r})) \cong \text{Spec}(k[O] / \mathcal{I}(\tau^r)) = Y(\tau^r).
\]

That is, the quiver scheme \( Z(\mathbf{r}) \) defined by \( \mathcal{J}(\mathbf{r}) \) is equal to the reduced, irreducible variety \( Y(\tau^r) \), the opposite cell of a Schubert variety.

**Proof.** Consider the map of \S 1.3, \( \zeta : Z \xrightarrow{\sim} Y(\tau^{\text{max}}) \subset O \). It is clear that \( \zeta \) is an algebraic isomorphism onto its image, since it is injective on points and on tangent vectors. (In fact, in appropriate coordinates \( O \cong Z \times V \) for some affine space \( V \), and for a certain polynomial function \( \phi : Z \to V \), \( \zeta \) is equivalent to the map \( Z \to Z \times V, \ z \mapsto (z, \phi(z)) \).) Thus by Proposition 2.1 we have the exact sequence

\[
0 \to \mathcal{I}(\tau^{\text{max}}) \to k[O] \xrightarrow{\zeta^*} k[Z] \to 0.
\]

Let \( \bar{\mathcal{J}}(\mathbf{r}) \subset k[Z] \) be the (reduced) vanishing ideal of \( Z(\mathbf{r}) \subset Z \). Clearly \( \mathcal{J} \subset \bar{\mathcal{J}} \). Since \( \zeta \) maps \( Z(\mathbf{r}) \) isomorphically onto \( Y(\tau^r) \) by Theorem 1.3, we have \( (\zeta^*)^{-1}\bar{\mathcal{J}}(\mathbf{r}) = \mathcal{I}(\tau^r) \) by Proposition 2.1. Hence

\[
Z(\mathbf{r}) = \text{Spec}(k[Z] / \mathcal{J}(\mathbf{r})) \cong \text{Spec}(k[O] / (\zeta^*)^{-1}\bar{\mathcal{J}}(\mathbf{r})) = \text{Spec}(k[O] / \mathcal{I}(\tau^r)) = Y(\tau^r).
\]

Furthermore, \( \mathcal{J}(\mathbf{r}) = \bar{\mathcal{J}}(\mathbf{r}) \) if and only if \( (\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) = (\zeta^*)^{-1}\bar{\mathcal{J}}(\mathbf{r}) \); and \( \mathcal{J}(\mathbf{r}) \) is prime if and only if \( (\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) \) is prime. Thus to show the Theorem, it suffices to prove

\[
(\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) = \mathcal{I}(\tau^r).
\]

But clearly \( (\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) \subset (\zeta^*)^{-1}\bar{\mathcal{J}}(\mathbf{r}) = \mathcal{I}(\tau^r) \), so we are left with the opposite inclusion

\[
(\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) \supset \mathcal{I}(\tau^r),
\]

which we will prove in the next section.
2.3 Proof of the main theorem: determinant identities

We define ideals \( I_0, I_1, I_2 \subset k[A] \) generated by certain minors of the generic matrix \( A \in k[A] \) at the end of \( \S 1.2. \)

\[
I_0 = (\zeta^*)^{-1}\mathcal{J}(r)
= I(\tau^{max}) + \left\{ \det(A_{j-1,j-2}\cdots A_{i+1,i})_{\lambda \times \mu} \mid i < j, \lambda \subset [n_j], \mu \subset [n_i], \#\lambda = \#\mu = r_{ij} + 1 \right\}
\]

\[
I_1 = I(\tau^{max}) + \left\{ \det(A_{\lambda \times \mu}) \mid i < j, \lambda \subset [a_1+1, n], \mu \subset [a_1], \#\lambda = \#\mu = r_{ij} + 1 \right\}
\]

\[
I_2 = I(\tau^r) = \left\{ \det(A_{\sigma \times [a_i]} \mid 1 \leq i \leq h-1, \sigma \subset [n], \#\sigma = a_i, \sigma \not\subset \tau_i^r \right\}
\]

To finish the proof of Theorem 2.2, we will show

\[
I_0 \supset I_1 \supset I_2.
\]

**Lemma 1.** Let \( X = (x_{ij}) \) and \( Y = (y_{kl}) \) be matrices of variables \( x_{ij}, y_{kl} \) generating a polynomial ring. Let \( \mathcal{J}_X \) (resp. \( \mathcal{J}_Y \)) be the ideal generated by all \( r+1 \)-minors of \( X \) (resp. \( Y \)). Then \( \mathcal{J}_X \) and \( \mathcal{J}_Y \) both contain all \( r+1 \)-minors of the product \( XY \).

**Proof.**

\[
\det(XY)_{\lambda \times \mu} = \sum_\nu \det X_{\lambda \times \nu} \det Y_{\nu \times \mu}.
\]

**Lemma 2.** Let \( (A_1, \ldots, A_{h-1}) \) be a generic element of \( Z \), and for \( i \leq j \) let \( \mathcal{J}_{ij} \) be the ideal generated by all \( r+1 \)-minors of the \( n_j \times n_i \) product matrix \( A_j \cdots A_i \). Then \( \mathcal{J}_{ij} \) contains all \( r+1 \)-minors of the \( (n-a_{j-1}) \times a_i \) matrix

\[
\tilde{A} = \begin{pmatrix}
A_j \cdots A_1 & A_j \cdots A_2 & \cdots & A_j \cdots A_i \\
A_{j+1} \cdots A_1 & A_{j+1} \cdots A_2 & \cdots & A_{j+1} \cdots A_i \\
\vdots & \vdots & \ddots & \vdots \\
A_h \cdots A_1 & A_h \cdots A_2 & \cdots & A_h \cdots A_i
\end{pmatrix}
\]

**Proof.** Note that we can factor the matrix

\[
\tilde{A} = \begin{pmatrix}
I_j \\
A_{j+1} \\
\vdots \\
A_h \cdots A_{j+1}
\end{pmatrix} \cdot A_j \cdots A_i \cdot (A_{i-1} \cdots A_1, A_{i-1} \cdots A_2, \cdots, A_{i-1}, I_i)
\]

Now apply Lemma 1 twice.

**Lemma 3.** \( I_0 \supset I_1 \).
Proof. Let \( \lambda \subset [a_j+1], \mu \subset [a_i], \# \lambda = \# \mu = r_{ij} + 1 \). Then clearly
\[
\det A_{\lambda \times \mu} \in (\zeta^*)^{-1}(\det \tilde{A}_{\lambda \times \mu}).
\]
Hence by Lemma 2, the generators of \( I_1 \) lie in \( I_0 \).

Lemma 4. (Gonciulea-Lakshmibai) Let \( A \) be a generic element of \( O \). Let \( 1 \leq t \leq a_i, \ 1 \leq s \leq n \), and \( \sigma = \{ \sigma(1) < \sigma(2) < \cdots < \sigma(a_i) \} \subset [n] \) with \( \sigma(a_i-t+1) \geq s \). Then \( p_\sigma(A) \) belongs to the ideal of \( k[O] \) generated by \( t \)-minors of \( A \) with row indices \( \geq s \) and column indices \( \leq a_i \).

Proof. Choose \( \sigma' \subset [s, n] \cap \sigma \) with \( \# \sigma' = t \), and let \( \sigma'' = \sigma \setminus \sigma' \). Then the Laplace expansion of \( p_\sigma(A) \) with respect to the rows \( \sigma' \), \( \sigma'' \), gives
\[
p_\sigma(A) = \det A_{\sigma \times [a_i]} = \sum_{\lambda \cup \lambda'' = [a_i]} \pm \det A_{\sigma' \times \lambda} \det A_{\sigma'' \times \lambda''},
\]
where the sum is over all partitions of the interval \([a_i] \). The first factor of each term in the sum is of the form required.

Lemma 5. \( I_1 \supset I_2 \).

Proof. Let \( \sigma \subset [n] \) with \( \# \sigma = a_i, \ \sigma \not\subset \tau_i^s \) for some \( i, 1 \leq i \leq h-1 \). Now, \( \tau_i^s \) has the largest possible entries such that
\[
\tau_i^s(a_i - r_{i,j+1}) \leq a_j, \quad \forall j \geq i,
\]
so \( \sigma \not\subset \tau_i^s \) must violate this condition for some \( j \):
\[
\sigma(a_i - r_{i,j+1}) \geq a_j + 1, \quad \exists j \geq i.
\]
Hence by Lemma 4, \( p_\sigma(A) \) is in \( I_1 \).

The Main Theorem 2.2 is therefore proved.

2.4 Degeneracy schemes

Fulton defines the universal degeneracy scheme \( \Omega_w \) associated to a permutation \( w \in S_{m+1} \) as follows. Fix \( 2m \) vector spaces \( F_1, F_2, \ldots, F_m, E_m, \ldots, E_2, E_1 \) with \( \dim F_i = \dim E_i = 1 \), and let
\[
Z = M_{2 \times 1} \times M_{3 \times 2} \times \cdots \times M_{m \times m-1} \times M_{m \times m} \times M_{m-1 \times m} \times \cdots \times M_{1 \times 2}
\]
be the quiver space of all maps of the form
\[
F_1 \to F_2 \to \cdots \to F_m \to E_m \to \cdots \to E_2 \to E_1.
\]
(For convenience we will refer to these maps and their compositions by symbols such as \( F_i \to F_j \) and \( F_i \to E_j \).) Define rank numbers
\[
r(F_i, E_j) = \# [i] \cap w[j], \quad 1 \leq i, j \leq m
\]
\[ r(F_i, F_j) = i \quad r(E_j, E_i) = i \quad 1 \leq i < j \leq m \]

and let \( r_w \) be the array of these numbers. Then let

\[ \Omega_w = Z(r_w) \subset Z, \]

the variety of all quiver representations satisfying

\[ \text{rank}(F_i \to E_j) \leq \# [i] \cap w[j], \quad 1 \leq i, j \leq m \]

\[ \text{rank}(F_i \to F_j) \leq i, \quad \text{rank}(E_j \to E_i) \leq i, \quad 1 \leq i < j \leq m. \]

(The latter conditions are clearly superfluous.) More precisely, define \( \Omega_w \) as a scheme by the same determinantal equations defining \( Z(r_w) \) in §2.2.

**Proposition.** The scheme \( \Omega_w \) over an arbitrary field \( k \) is reduced and is isomorphic to the opposite cell of a Schubert variety in \( Fl = GL(n)/Q \), a partial flag variety of \( k^n \), where \( n = 2(1 + \cdots + m) = m(m + 1) \). In particular, \( \Omega_w \) is irreducible, Cohen-Macaulay, and normal, and has rational singularities.

**Proof.** This follows since Schubert varieties are known to have these properties (see e.g. [1]).

**Proposition.**

\[ \text{codim}_Z \Omega_w = \ell(w), \]

where

\[ \ell(w) = \# \left\{ (i, j) \mid 1 \leq i, j \leq m+1, \quad i < j, \quad w(i) > w(j) \right\} \]

is the Bruhat length.

**Proof.** By the dimension formula of Abeasis and Del Fra [2] (c.f. §1.1 above), we have

\[
\dim \Omega_w = \dim G_n - \sum_{1 \leq i, j \leq m} (r(F_i, E_j) - r(F_i, E_{j-1}))(r(F_j, E_j) - r(F_i, E_j)) - \sum_{1 \leq i \leq j \leq m} (r(F_i, E_j) - r(F_i, F_{j+1}))(r(F_i, F_i) - r(F_{i-1}, F_j)) - \sum_{1 \leq i \leq j \leq m} (r(E_j, E_i) - r(E_j, E_{i-1}))(r(E_j, E_i) - r(E_{j+1}, E_i)) - (r(F_m, F_m) - r(F_m, F_m))(r(F_m, F_m) - r(F_{m-1}, F_m))
\]

\[ = 2(1^2 + 2^2 + \cdots + m^2) - \sum_{1 \leq i, j \leq m} \# ([i] \cap w[j]) \cdot \# (i \cap w[j]) \cdot \# (i-1 \cap w[j]) = 2(1^2 + 2^2 + \cdots + m^2) - \sum_{1 \leq i, j \leq m} \left\{ (i, j) \mid 1 \leq i, j \leq m+1, \quad w^{-1}(i) \leq j, \quad i \geq w(j) \right\} = 2(1^2 + 2^2 + \cdots + m^2) - m - \ell(w). \]

On the other hand,

\[ \dim Z = 2(1 \cdot 2 + 2 \cdot 3 + \cdots + (m-1) \cdot m) + m^2. \]
Hence
\[ \text{codim}_Z \Omega_w = 2(1 \cdot 2 + \cdots + (m-1) \cdot m) + m^2 - 2(1^2 + \cdots + m^2) + m + \ell(w) \]
\[ = \ell(w). \]

**Concluding remarks**

Denote \( Z = Z(m) \) and \( \Omega_w = \Omega_w(m) \) to emphasize the dependence on \( m \).
Consider \( S_{m+1} \subset S_{m+2} \) in the usual way. Then there is a natural map \( \pi : Z(m+1) \to Z(m) \) given by forgetting the middle two spaces, and we may easily see the stability property:
\[ \Omega_w(m+1) = \pi^{-1} \Omega_w(m). \]

The map \( \pi \) is a fiber bundle over some open set of \( \Omega_w(m) \), and from the previous proposition, the generic fibers of \( \pi : \Omega_w(m+1) \to \Omega_w(m) \) have the same dimension as the generic fibers of \( \pi : Z(m+1) \to Z(m) \).

Finally, we note that \( \Omega_w(m+1) \) is closely related to a Schubert variety of \( \text{Fl}' = GL(m+1)/B \), the complete flag variety of \( k^{m+1} \), a much smaller flag variety than that of Zelevinsky’s bijection (c.f. Fulton §3). Namely, consider the open set \( Z^o(m+1) \) of elements \( F_1 \to \cdots \to E_1 \) with \( F_i \to F_{i+1} \) injective, \( E_{i+1} \to E_i \) surjective, and \( F_{m+1} \to E_{m+1} \) bijective. Then we have a principal fiber bundle
\[ \psi : Z^o \to \text{Fl}' \times \text{Fl}' \]
\[ (F_1 \to \cdots \to E_1) \mapsto (V, U) \]
where
\[ V_i = \text{Im}(F_i \to E_{m+1}) \quad U_i = \text{Ker}(E_{m+1} \to E_{m+1-i}). \]

Now, letting \( \Omega_w^o(m+1) = \Omega_w(m+1) \cap Z^o(m+1) \), an open subset of \( \Omega_w(m+1) \), we have
\[ \Omega_w^o(m+1) = \psi^{-1} \{ (V, U) \in \text{Fl}' \times \text{Fl}' \mid V_i \cap U_j \leq \# wu_0[i] \cap [j] \}. \]
where \( w_0 \) is the longest element of \( S_{m+1} \). It is well-known that the subset of \( \text{Fl}' \times \text{Fl}' \) on the right is a fiber bundle over \( \text{Fl}' \) with fiber equal to the Schubert variety \( X(wu_0) \subset \text{Fl}' \).

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