Invariant Curves for Degenerate Hyperbolic Maps of the Plane

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Abstract

We prove existence and uniqueness of an unstable manifold for a degenerate hyperbolic map of the plane arising in statistics.

AMS Classification 37E30

Note: After posting an earlier version of this paper I learned that the results presented here are special cases of theorems due to I. Baldoma, E. Fontich, R. de la Llave, and M. Pau. An appendix to this paper provides details. I am grateful to Baldoma, Fontich and Pau for supplying the appendix, and to D. Cordoba and R. de la Llave for making me aware of the prior results.

1 Introduction

The standard stable manifold theorem [3] applies in particular to maps of the plane having the form

\[
\Phi: (x, y) \mapsto (X, Y)
\]

with

\[
X = \lambda_1 x + O(\|(x, y)\|^2)
\]

\[
Y = \lambda_2 y + O(\|(x, y)\|^2)
\]

\[
|\lambda_1| > 1 > |\lambda_2|.
\]

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Such a map has an invariant unstable manifold tangent to the x-axis. Here we study a degenerate case in which $\lambda_1 = +1$ and $\lambda_2 = -1$. More precisely, we study smooth maps $\Phi : (x, y) \mapsto (X, Y)$, with

$$
\begin{align*}
X &= x + x^2 + \mu xy + O(|(x, y)|^3) \\
Y &= -y + \lambda xy + O(|(x, y)|^3)
\end{align*}
$$

$\mu \in \mathbb{R}, \lambda > 0$.

Like (1), a map (2) expands $x$ and contracts $y$ in the region of interest (where, in particular, $x > 0$ and $|y| << x$), but the stretching and shrinking arise from second order terms in the Taylor expansion of $\Phi$. We will prove existence and uniqueness of a smooth invariant curve, tangent to the positive x-axis, for maps of the form (2). Note that $\Phi^{-1}$ is defined in a neighborhood of the origin, since $\Phi'(0,0)$ is invertible.

Our interest in maps (2) arises from Lee et al. [4], which proposes critical values for the $t$-ratio associated with the method of instrumental variables regression, which has received a great deal of attention in economics research, and has been widely employed across many empirical disciplines. In the instrumental variable model, there is an outcome of interest $Y$ (e.g., rates of illness), a causal factor of interest $X$ (e.g., receipt of a vaccine) and an “instrument” $Z$ (e.g., random assignment to either receiving or not receiving encouragement to take the vaccine). Under certain conditions, the causal effect of interest (e.g., the impact of vaccine receipt on rates of illness) has been shown to be equal to the ratio of two regression coefficients: the coefficient in a regression of $Y$ on $Z$ divided by that from the regression of $X$ on $Z$. Empirical researchers have typically used a $t$-ratio to test statistical significance using a constant critical value threshold (e.g. $\pm 1.96$ for a 5% test), presuming that the $t$-ratio is approximately standard normal. But as discussed in [4], that approximation has been shown in the economics literature to be quite poor in many empirically-relevant cases. [4] corrects for this by providing critical values that depend on the $F$-statistic associated with the regression of $X$ on $Z$. A piece of this critical value function is the solution to a functional equation, which takes the form (2), after a change of variables described in [4]. Our main result on maps (2) provides a rigorous proof of the existence and uniqueness of such a solution to the functional equation, the numerical solution of which was previously reported in an earlier version of [4].

The precise statement of our result is as follows. Here and below, “smooth” means $C^\infty$.

**Theorem 1.1.** Let $\Phi : U \mapsto \mathbb{R}^2$ be a smooth map defined in a neighborhood $U$ of the origin in $\mathbb{R}^2$. Suppose $\Phi$ has the form (2). Then there exists a smooth curve

$$
\Gamma = \{ (x, F(x)) : x \in [0, \delta] \} \subset U
$$

with the following properties.

(I) Invariance: $\Phi^{-1}(\Gamma) \subset \Gamma$. 

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(II) Tangency: \( F(x) = O(x^3) \) as \( x \to 0 \).

(III) Uniqueness: Let \( \bar{\Gamma} = \{(x, \bar{F}(x)) : x \in [0, \delta]\} \), where \( \Phi^{-1}(\bar{\Gamma}) \subset \bar{\Gamma} \) and \( x^{-2/3} \bar{F}(x) \to 0 \) as \( x \to 0^+ \). Then \( \bar{F} = F \) on \([0, \delta^\#]\) for some \( \delta^\# > 0 \).

We now sketch the proof of the above Theorem. We first make a change of variable to bring \( \Phi \) to the form \( \Phi : (x, y) \mapsto (X, Y) \) with

\[
X = x + x^2 + x^3 \theta_A(x, y) + y \theta_B(x, y),
\]

\[
Y = -y(1 - \lambda x + x^2 \theta_C(x, y) + y \theta_D(x, y)) + x^{N+100} \theta_E(x, y).
\]

Here the \( \theta \)'s are smooth functions, and \( N \) is as large as we please. But for the term \( x^{N+100} \theta_E(x, y) \) in (4), the x-axis would be an invariant curve for \( \Phi \). We look for an invariant curve of the form \( \Gamma = \{(x, F(x)) : x \in [0, \delta]\} \)

with

\[
\left| \left( \frac{d}{dx} \right)^m F(x) \right| \leq K_m x^{N-m}
\]

on \([0, \delta]\), \( m = 0, 1, \ldots, N - 10 \), for carefully selected constants \( K_0, K_1, \ldots, K_{N-10} \).

To produce such an \( F \), we proceed as follows. Fix a small number \( \rho, 0 < \rho << \delta \). Later, we will let \( \rho \) tend to zero. We start with a horizontal line segment

\( \Gamma^0 = \{(x, 0) : x \in [0, \rho]\} \)

and then apply an iterate of \( \Phi \) to produce the image

\( \Gamma_\rho = \Phi^\nu(\Gamma^0_\rho) \)

where we take \( \nu \) to be the least integer for which \( \Phi^\nu(\Gamma^0_\rho) \) contains points \((x,y)\) with \( x > \delta \). Note that \( \nu \to \infty \) as \( \rho \to 0 \). We will show that \( \Gamma_\rho \) has the form:

\( \Gamma_\rho = \{(x, F_\rho(x)) : x \in [0, x_{\rho MAX}]\} \)

with \( x_{\rho MAX} \geq \delta \) and

\[
\left| \left( \frac{d}{dx} \right)^m F_\rho(x) \right| \leq K_m x^{N-m} \text{ on } [0, x_{\rho MAX}], \ m = 0, 1, \ldots, N - 10.
\]

Here, the \( K_m \) are as in (5); they are independent of \( \rho \). Moreover, we will show that \( \Gamma_\rho \) is approximately invariant, in the sense that every point of \( \Phi^{-1}(\Gamma_\rho) \) lies within a distance \( \rho \) of a point of \( \Gamma_\rho \). By Ascoli’s theorem, we can find a sequence \( \rho_1, \rho_2, \ldots \) tending to zero, such that the curves
\( \Gamma_{\rho} \) tend to a limiting curve \( \Gamma \) in the \( C^{N-11} \) topology. That curve is invariant, highly tangent to the \( x \)-axis and \( C^{N-11} \) for \( N \) as large as we please.

The uniqueness and \( C^\infty \) smoothness assertions of our theorem then follow easily.

We now delve slightly deeper by providing a few words about the proof of (7), and that of the approximate invariance of the curves \( \Gamma_{\rho} \). We will successively pick constants

\[
1 = K_0 << K_1 << ... << K_{N-10}
\]

and then pick a small enough \( \delta \) depending on the \( K_m \). For these constants, we study smooth curves of the form

\[
\Gamma = \{(x, f(x)) : x \in [0, x_{\text{MAX}}]\}
\]

such that

\[
\left| \left( \frac{d}{dx} \right)^m f(x) \right| \leq K_m x^{N-m} \text{ on } [0, x_{\text{MAX}}], \ m = 0, 1, ..., N - 10.
\]

We show that if \( \Gamma \) is of the form (8) and \( x_{\text{MAX}} \leq \delta \), then \( \Phi(\Gamma) \), the image of \( \Gamma \) under \( \Phi \), is again of the form (8), with different \( x_{\text{MAX}} \) and \( f \), but with the same \( K_0, ..., K_{N-10} \). Starting from the horizontal line segment \( \Gamma_0 \), which clearly has the form (8), we can therefore repeatedly take the image under \( \Phi \), always preserving (8), until at last \( x_{\text{MAX}} \) in (8) exceeds \( \delta \). Thus, we conclude that \( \Gamma_{\rho} = \Phi^\nu(\Gamma_0) \) is of the form (8). That’s the plan of our proof of (7).

To establish the approximate \( \Phi \)-invariance of our curve \( \Gamma_{\rho} \), our main tool is the following Shadowing Lemma.

**Lemma 1.2.**

Let \((x, y) \), \((\hat{x}, \hat{y}) \) \( \in \mathbb{R}^2 \), with \( 0 < x < \delta \), \( |y| \leq K_0 x^N \), and \( \frac{|x-\hat{x}|+x^{-3}|y-\hat{y}|}{x^8} \leq 1 \).

Then the points \((X, Y) = \Phi(x, y)\) and \((\hat{X}, \hat{Y}) = \Phi(\hat{x}, \hat{y})\) satisfy

\[
\frac{|X - \hat{X}| + X^{-3}|Y - \hat{Y}|}{X^8} \leq \frac{|x - \hat{x}| + x^{-3}|y - \hat{y}|}{x^8} \leq 1.
\]

The shadowing lemma allows us to prove the approximate \( \Phi \)-invariance of \( \Gamma_{\rho} \) by the following argument. Let \((x,y) \in \Gamma_{\rho} \). By definition, \((x, y) \in \Phi^\nu(\Gamma_0) \), i.e, \((x,y) = \Phi^\nu(x_0, 0)\) for some \( x_0 \in [0, \rho] \). Let \((x_\nu, y_\nu) = \Phi^\nu(x_0, 0)\) for \( \nu = 0, 1, ..., \bar{\nu} \). Note that all the \( y_\nu \) satisfy \( |y_\nu| \leq K_0 x_\nu^N \), thanks to the invariance of (8) under the map \( \Phi \). We can easily find a point \((\tilde{x}_0, 0) \in \Gamma_0^0 \) such that \((\tilde{x}_0, \tilde{y}_0) \equiv \Phi(\tilde{x}_0, 0)\) satisfies \( \tilde{x}_0 = x_0 \), \( |\tilde{y}_0| \leq K_0 x_0^N \), hence,

\[
\frac{|x_0 - \tilde{x}_0| + x_0^{-3}|y_0 - \tilde{y}_0|}{x_0^8} \leq K_0 x_0^{N-11} \leq K_0 \rho^{N-11}.
\]

Let \((\hat{x}_\nu, \hat{y}_\nu) = \Phi^\nu(\tilde{x}_0, \tilde{y}_0)\) for \( \nu = 0, 1, .., \bar{\nu} \). Starting from (9), and repeatedly applying the shadowing lemma, we learn that
\[ \frac{|x_\rho - \bar{x}_\rho| + x_\rho^3|y_\rho - \bar{y}_\rho|}{x_\rho^8} \leq K_0 \rho^{N-11}. \]

In particular,

\[ |(x_\rho, y_\rho) - (\bar{x}_\rho, \bar{y}_\rho)| \leq \rho^{N-11}. \]  \hspace{1cm} (10)

We now recall that

\[ (x_\rho, y_\rho) = \Phi^\rho(x_0, 0) = (x, y) \]

and that

\[ \Phi^\rho(x_\rho, y_\rho) = \Phi^\rho(x_0, y_0) = \Phi(\Phi^\rho(\tilde{x}_0, 0)). \]

Letting \( (x^#, y^#) = \Phi^\rho(\tilde{x}_0, 0) \in \Phi^\rho(\Gamma^0_\rho) = \Gamma_\rho \), we see that \( (x^#, y^#) = \Phi(x^#, y^#) \). Thus, (10) shows that \( (x, y) \) lies within distance \( \rho^N \) of a point \( \Phi(x^#, y^#) \), with \( (x^#, y^#) \in \Gamma_\rho \). Since \( (x, y) \) here is an arbitrary point of \( \Gamma_\rho \), this concludes the proof of approximate \( \Phi \)-invariance of \( \Gamma_\rho \). This also concludes our summary of the proof of our theorem.

In the sections below, we provide full details of the proof. We warn the reader that our notation in this introduction is not entirely consistent with the notation in subsequent sections. However, we have accurately summarized the main ideas.

Dynamical systems researchers are aware that good things arising from the linearization of a map may also arise from higher terms in its Taylor expansion; Theorem 1.1 is a case in point.

I thank Lai-Sang Young and Rafael de la Llave for useful comments on invariant manifolds. I’m grateful to the authors of Lee et al. [4] for posing an intriguing problem with a practical application, and to Peter Ozsvath for putting me in touch with them.

2 Proof

2.1 Change of Coordinates

We look at maps

\[ \Phi : (x, y) \mapsto (X, Y) \]  \hspace{1cm} (11)

where

\[ Y = -y(1 - \lambda x + y\theta_1 + x^2\theta_2) + x^N\theta_3, \]  \hspace{1cm} (12)

\[ X = x + x^2 + x^3\theta_4 + y\theta_5. \]  \hspace{1cm} (13)

Here and below, \( \theta \)'s denote smooth functions of \( (x, y) \), \( \lambda > 0 \), and \( N \geq 3 \).

Note that a map of the form (2) has the form (11), (12), (13) with \( N = 3 \), since the \( O(|(x, y)|^3) \) terms in (2) may be expressed as \( \theta_A(x, y)x^3 + \theta_B(x, y)x^2y + \theta_C(x, y)xy^2 + \theta_D(x, y)y^3 \).
We make a change of variables:

\[ \tilde{Y} = Y + \gamma X^N, \quad \tilde{y} = y + \gamma x^N. \quad (\gamma \in \mathbb{R}) \]

This changes \( \Phi \) to a map

\[ \tilde{\Phi} : (x, \tilde{y}) \mapsto (X, \tilde{Y}). \]

We will show that, by picking the correct \( \gamma \), we can arrange that \( \tilde{\Phi} \) has the same form as \( \Phi \) but with \( N + 1 \) in place of \( N \). We write \( \theta_i \) for integers \( i \) to denote smooth functions of \( (x, y) \), or equivalently, smooth functions of \( (x, \tilde{y}) \). Our \( \theta_i \) will be independent of \( \gamma \).

Note, \( y = \tilde{y} - \gamma x^N \), so

\[ Y = -[\tilde{y} - \gamma x^N](1 - \lambda x + [\tilde{y} - \gamma x^N]\theta_1 + x^2\theta_2) + x^N\theta_3, \]

where now the \( \theta \)'s are regarded as smooth functions of \( (x, \tilde{y}) \). So,

\[
Y = -\tilde{y}(1 - \lambda x + \tilde{y}\theta_1 + x^2(\theta_2 - \gamma x^{N-2}\theta_1)) + \gamma x^N \\
+ \gamma x^N(-\lambda x + [\tilde{y} - \gamma x^N]\theta_1 + x^2\theta_2) + x^N\theta_3 \\
= -\tilde{y}(1 - \lambda x + \tilde{y}\theta_1 + x^2(\theta_2 - \gamma x^{N-2}\theta_1 - \gamma x^{N-2}\theta_1)) + \gamma x^N + \theta_3x^N \\
+ x^{N+1}(-\lambda \gamma - \gamma^2 x^{N-1}\theta_1 + x\gamma\theta_2) \\
= -\tilde{y}(1 - \lambda x + \tilde{y}\theta_1 + x^2\theta_6) + (\gamma + \theta_3)x^N + \theta_7x^{N+1} \text{ for smooth functions } \theta_6, \theta_7.
\]

Similarly,

\[
\gamma X^N = \gamma(x + x^2 + x^3\theta_4 + y\theta_5)^N \\
= \gamma(x + x^2 + x^3\theta_4 + [\tilde{y} - \gamma x^N]\theta_5)^N \\
= \gamma(x + x^2 + x^3\theta_8 + \tilde{y}\theta_5)^N \\
= \gamma(x + x^2 + x^3\theta_8)^N + (\text{coeff})\tilde{y}\theta_5(x + x^2 + x^3\theta_8)^{N-1} + \tilde{y}^2\theta_9.
\]

Adding these equations and recalling that \( \tilde{Y} = Y + \gamma X^N \), we find that

\[
\tilde{Y} = -\tilde{y}(1 - \lambda x + \tilde{y}\theta_1 + x^2\theta_6) + (2\gamma + \theta_3)x^N + (\theta_7 + \theta_{11})x^{N+1} + \theta_{10}x^{N-1}\tilde{y} + \theta_9\tilde{y}^2, \\
i.e.,
\]

\[
\tilde{Y} = -\tilde{y}(1 - \lambda x + \tilde{y}[\theta_1 - \theta_9] + x^2[\theta_6 - x^{N-3}\theta_{10}]) + (2\gamma + \theta_3)x^N + (\theta_7 + \theta_{11})x^{N+1} \\
= -\tilde{y}(1 - \lambda x + \tilde{y}\theta_{12} + x^2\theta_{13}) + (2\gamma + \theta_3)x^N + \theta_{14}x^{N+1}.
\]

(Recall \( N \geq 3 \).)
Now, $\theta_3 = \beta + x\theta_{15} + \bar{y}\theta_{16}$ for some number $\beta$ and some smooth functions $\theta_{15}, \theta_{16}$. So,

$$\tilde{Y} = -\bar{y}(1 - \lambda x + \bar{y}\theta_{12} - x^2\theta_{17}) + (2\gamma + \beta)x^N + \theta_{15} + \theta_{14}x^{N+1}.$$  

Hence, $\tilde{Y} = -\bar{y}(1 - \lambda x + \bar{y}\theta_{12} + x^2\theta_{17}) + (2\gamma + \beta)x^N + \theta_{18}x^{N+1}$.

Picking $\gamma = -\frac{\beta}{2}$, we kill the $(2\gamma + \beta)x^N$ term, leaving us with:

$$\tilde{Y} = -\bar{y}(1 - \lambda x + \bar{y}\theta_{12} + x^2\theta_{17}) + x^{N+1}\theta_{18}. \quad (14)$$

Also,

$$X = x + x^2 + x^3\theta_4 + [\bar{y} - \gamma x^N]\theta_5,$$

so,

$$X = x + x^2 + x^3[\theta_4 - \gamma x^{N-3}\theta_5] + \bar{y}\theta_5$$

i.e.,

$$X = x + x^2 + x^3\theta_{19} + \bar{y}\theta_5. \quad (15)$$

(Again, recall that $N \geq 3$.)

Equations (14) and (15) show that the map $\tilde{\Phi} : (x, \bar{y}) \mapsto (X, \tilde{Y})$ has the same form as $\Phi : (x, y) \mapsto (X, Y)$, but with $N$ replaced by $(N + 1)$.

### 2.2 Conditions Preserved by Our Map

Thanks to the preceding section, we may suppose our map has the form $\Phi : (x, y) \mapsto (X, Y)$, where

$$X = x + x^2 + x^3\theta_A(x, y) + y\theta_B(x, y),$$

$$Y = -y(1 - \lambda x + y\theta_C(x, y) + x^2\theta_D(x, y)) + x^{N+100}\theta_E(x, y)$$

with $\theta$’s smooth, $\lambda > 0$, $N$ large. We fix $N \geq 100$ until further notice. We are only interested in $\Phi(x, y)$ when $x \geq 0$.

Let $K_1, \ldots, K_{N-10}$ be constants to be picked later. Initially, $C, C', c$, etc. will denote constants independent of the $K$’s. Later, for each $m$, there will come a time when we will have picked $K_1, K_2, \ldots, K_m$, but have not yet picked $K_{m+1}, \ldots, K_{N-10}$. At that point, $C, C', c$, etc. will denote constants that may depend on $K_1, \ldots, K_m$, but not on $K_{m+1}, K_{m+2}$.\ldots

We spell out our conditions for the constants $K_1, \ldots, K_{N-10}, \delta$.

- $K_1$ is greater than a large enough constant determined by $\Phi$.

- Each $K_m$ ($m \geq 2$) exceeds a large enough constant determined by $\Phi, K_1, \ldots, K_{m-1}$.
• \( \delta \) is less than a small enough constant determined by \( \Phi, K_1, \ldots, K_{N-1} \).

**Assertion 1.** Suppose \( |y| \leq x^N \). Let \((X, Y) = \Phi(x, y)\). Then \(|Y| \leq X^N\), provided \( x \in [0, 10\delta] \).

**Proof.** Letting \( C \) denote a constant determined by \( \Phi \), we have
\[
|Y| \leq x^N (1 - \lambda x + Cx^N + Cx^2) + Cx^{N+100}
\leq x^N \quad \text{for} \quad x \in [0, 10\delta]
\]
while
\[
X \geq x + x^2 - Cx^3 - Cx^N \geq x, \quad \text{so} \quad x^N \leq X^N \quad \text{for} \quad x \in [0, 10\delta].
\]
Therefore, \(|Y| \leq x^N \leq X^N\) completing the proof of Assertion 1.

**Assertion 2.** Let \( y = f(x) \) be a smooth function on \([0, x_{MAX}]\) with \( 0 < x_{MAX} \leq 10\delta \), and with
\[
|f(x)| \leq x^N \quad \text{for} \quad x \in [0, x_{MAX}],
\]
\[
|f'(x)| \leq K_1x^{N-1} \quad \text{for} \quad x \in [0, x_{MAX}].
\]
Define \((X, Y) = \Phi(x, f(x))\) for \( x \in [0, x_{MAX}] \). Then \( \frac{dX}{dx} \geq 1 \) for \( x \in [0, x_{MAX}] \).

**Proof.** In fact,
\[
X = x + x^2 + x^3\theta_A(x, f(x)) + f(x)\theta_B(x, f(x)).
\]
So,
\[
\frac{dX}{dx} = 1 + 2x + 3x^2\theta_A(x, f(x)) + x^3\theta_{A,x}(x, f(x)) + x^3\theta_{A,y}(x, f(x))f'(x)
+ f'(x)\theta_{B,x}(x, f(x)) + f(x)\theta_{B,y}(x, f(x))f'(x).
\]
Because \( |f(x)| \leq x^N \) and \( |f'(x)| \leq K_1x^{N-1} \), all terms on the right-hand side other than 1 and 2x are dominated by \( Cx^2 \) for \( x \in [0, x_{MAX}] \) with \( C \) independent of \( K_1 \), since \( x_{MAX} \leq 10\delta \). (Recall, \( \delta \) may depend on \( K_1 \).) So,
\[
\frac{dX}{dx} \geq 1 + 2x - Cx^2 \geq 1,
\]
proving Assertion 2.
Assertion 3. Let $f(x)$ be a smooth function on $[0, x_{MAX}]$ with $x_{MAX} \leq \delta$, and suppose that on that interval we have

$$|f(x)| \leq x^N \text{ and } |f'(x)| \leq K_1 x^{N-1}.$$  

Then, if we pick $K_1$ large enough, it follows that

$$\Phi(\{(x, y) : x \in [0, x_{MAX}], y = f(x)\}) = \{(X, Y) : X \in [0, x_{MAX}], Y = F(X)\}$$

where $F$ is smooth and

$$|F(X)| \leq X^N, |F'(X)| \leq K_1 X^{N-1} \text{ for } X \in [0, x_{MAX}].$$

Proof. Recall that until further notice, constants $C$ are determined by $\Phi$, independently of $K_1, \ldots, K_{N-10}, \delta$. By Assertion 2, we know that $\Phi(\{(x, f(x)) : x \in [0, x_{MAX}]\})$ is the graph of a smooth function $F$ on an interval $[0, x_{MAX}]$. By Assertion 1, we have $|F(X)| \leq X^N$ on $[0, x_{MAX}]$.

It remains only to estimate $|F'(X)|$. To do so, we note that,

$$F(X) = Y, \text{ with }$$

$$X = x + x^2 + x^3 \theta_A(x, f(x)) + f(x)\theta_B(x, f(x))$$

and

$$Y = -f(x)(1 - \lambda x + f(x)\theta_C(x, f(x)) + x^2\theta_D(x, f(x))) + x^N+100\theta_E(x, f(x)).$$

From the proof of Assertion 2 and elementary calculus, we have

a. $\frac{dX}{dx} \geq 1 + 2x - Cx^2 \geq 1; X \geq x$,

b. $F'(X) \frac{dX}{dx} = \frac{dY}{dx}$, and

c.

$$\frac{dY}{dx} = -f'(x)\{1 - \lambda x + f(x)\theta_C(x, f(x)) + x^2\theta_D(x, f(x))\}$$

$- f(x)\{-\lambda + f'(x)\theta_C(x, f(x)) + f(x)\theta_{C,x}(x, f(x)) + f(x)\theta_{C,y}(x, f(x))f'(x)\}$

$+ 2x\theta_D(x, f(x)) + x^2\theta_{D,x}(x, f(x)) + x^2\theta_{D,y}(x, f(x))f'(x)\}$

$+ (N + 100)x^{N+99}\theta_E(x, f(x)) + x^{N+100}\theta_{E,x}(x, f(x)) + x^{N+100}\theta_{E,y}(x, f(x))f'(x).$

Because $|f(x)| \leq x^N, |f'(x)| \leq K_1 x^{N-1}, x \leq x_{MAX} \leq \delta$ (and $\delta$ may depend on $K_1$), all the terms inside the second pair of curly brackets in (c) are dominated by $C$, and therefore

$$|f(x)\{-\lambda + f'(x)\theta_C(x, f(x)) + f(x)\theta_{C,x}(x, f(x)) + f(x)\theta_{C,y}(x, f(x))f'(x)\}$

$+ 2x\theta_D(x, f(x)) + x^2\theta_{D,x}(x, f(x)) + x^2\theta_{D,y}(x, f(x))f'(x)\}| \leq Cx^N.$
Also because,
\[ |f(x)| \leq x^N, |f'(x)| \leq K_1 x^{N-1} \text{ and } 0 \leq x \leq x_{MAX} \leq \delta, \]
we have
\[ |f(x)\theta_C(x, f(x)) + x^2 \theta_D(x, f(x))| \leq C x^2 \]
and
\[ |(N + 100)x^{N+99}\theta_E(x, f(x)) + x^{N+100}\theta_{E,x}(x, f(x)) + x^{N+100}\theta_{E,y}(x, f(x))f'(x)| \leq C x^{N+99}. \]
Therefore, (c) gives
\[
\left| \frac{dY}{dx} \right| \leq |f'(x)| \left[ 1 - \lambda x + C x^2 \right] + C x^N
\leq K_1 x^{N-1} \left[ 1 - \lambda x + C x^2 \right] + C x^N
= K_1 x^{N-1} \left[ 1 - \left( \frac{C}{K_1} \right) x + C x^2 \right].
\]
We pick $K_1$ large enough that $\lambda > \frac{C}{K_1}$. From now on, $K_1$ is fixed and constants $C$ may depend on $K_1$. Then we have,
\[
\left| \frac{dY}{dx} \right| \leq K_1 x^{N-1} \text{ since } x \leq \delta \text{ and } \delta \text{ may depend on } K_1.
\]
Together with (a) and (b), this implies that
\[ |F'(X)| \leq K_1 x^{N-1} \leq K_1 X^{N-1}, \]
completing the proof of Assertion 3.

We note the relationship of $x_{MAX}$ to $X_{MAX}$. We have
\[ X_{MAX} = x_{MAX} + x_{MAX}^2 + x_{MAX}^3 \theta_A(x_{MAX}, f(x_{MAX})) + f(x_{MAX}) \theta_B(x_{MAX}, f(x_{MAX})) \]
with $|f(x_{MAX})| \leq x_{MAX}^N$, so
\[ |X_{MAX} - (x_{MAX} + x_{MAX}^2)| \leq C x_{MAX}^3. \]

**Assertion 4.** Fix $\bar{m} \geq 2$, $\bar{m} \leq N - 10$. Suppose the smooth function $f$ satisfies $|f(x)| \leq x^N$ and $|f^{(m)}(x)| \leq K_m x^{N-m}$ on $[0, x_{MAX}]$ for $m = 1, ..., \bar{m}$ with $x_{MAX} \leq \delta$. Define $F(X)$ on $[0, X_{MAX}]$ as in Assertion 3, and suppose
\[ |F^{(m)}(X)| \leq K_m X^{N-m} \text{ for } m = 1, \ldots, \bar{m} - 1, \text{ and} \]
\[ |F(X)| \leq X^N \]

on \([0, X_{MAX}]\). Here, \(K_1, \ldots, K_{\bar{m}-1}\) have already been picked, but we have not yet picked \(K_{\bar{m}}\). If \(K_{\bar{m}}\) is large enough, then the above hypotheses imply the estimate, \(|F^{(\bar{m})}(X)| \leq K_{\bar{m}} X^{N-\bar{m}}\).

**Proof.** In this proof, \(C, c, C'\) etc. denote constants determined by \(\Phi, K_1, \ldots, K_{\bar{m}-1}\). To prove Assertion 4, we differentiate the equation \(F(X) = Y \; \bar{m} \) times with respect to \(x\), where
\[
X = x + x^2 + x^3 \theta_A(x, f(x)) + f(x) \theta_B(x, f(x)),
\]
\[
Y = -f(x) [1 - \lambda x + f(x) \theta_C(x, f(x)) + x^2 \theta_D(x, f(x))] + x^{N+100} \theta_E(x, f(x)).
\]

Note that for \(\theta(x, y)\) smooth, and for \(1 \leq p \leq \bar{m}\), the quantity \((\frac{d}{dx})^p \theta(x, f(x))\) is a sum of terms \((\frac{d}{dx})^p \theta(x, f(x)) \left[ \prod_{\nu=1}^{\beta} (\frac{d}{dx})^{p_\nu} f(x) \right]\) with each \(p_\nu \geq 1\) and \(\alpha + \sum \nu p_\nu = p\). The above term is bounded by \(C \cdot \prod \nu (K_{p_\nu} x^{N-p_\nu}) \leq C\), where the last estimate holds because \(x \leq x_{MAX} \leq \delta, \) and \(p_\nu \leq \bar{m} \leq N - 10\). (Recall that \(\delta\) is assumed to be less than a small constant determined by the \(K_p\).)

Therefore,
\[
\left| \left( \frac{d}{dx} \right)^p \theta(x, f(x)) \right| \leq C \text{ for } 1 \leq p \leq \bar{m}.
\]

Together with the estimates we assumed for \((\frac{d}{dx})^p f(x)\) where \((0 \leq p \leq \bar{m})\), this yields the following results:
\[
\left| \left( \frac{d}{dx} \right)^p X \right| \leq C \text{ for } 1 \leq p \leq \bar{m},
\]
\[
\left| \left( \frac{d}{dx} \right)^p [(f(x))^2 \theta_C(x, f(x))] \right| \leq (K_{\bar{m}} + C) \cdot C x^{2N-p} \text{ for } 1 \leq p \leq \bar{m},
\]
\[
\left| \left( \frac{d}{dx} \right)^p [x^{N+100} \theta_E(x, f(x))] \right| \leq C x^{N+100-p} \text{ for } 1 \leq p \leq \bar{m},
\]

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\[
\left( \frac{d}{dx} \right)^p [f(x) \cdot x^2 \theta_D(x, f(x))] \leq C \cdot (K_{\bar{m}} + C)x^{N+2-p} \text{ for } 1 \leq p \leq \bar{m}.
\]

So,
\[
\left( \frac{d}{dx} \right)^{\bar{m}} Y = -f^{(\bar{m})}(x) \cdot [1 - \lambda x] + \lambda \bar{m} f^{(\bar{m}-1)}(x) + \text{Error}_1
\]
where \(|\text{Error}_1| \leq x^{N+1-\bar{m}}\). (17)

(Here, we use the fact that \(x \leq x_{MAX} \leq \delta\), where \(\delta\) is less than a small constant depending on the \(K\)'s.)

Next, note that
\[
\left( \frac{d}{dx} \right)^{\bar{m}} F(X) = F^{(\bar{m})}(X) \cdot \left( \frac{dX}{dx} \right)^{\bar{m}} + \sum_{r_1+\cdots+r_p=\bar{m}} \text{(coeffs)} F^{(p)}(X) \cdot \prod_{\nu=1}^{p} \left[ \left( \frac{d}{dx} \right)^{r_\nu} X \right].
\]

By our assumptions on \(F^{(p)}(X)\) for \(p \leq \bar{m} - 1\), together with our estimates for \(\left( \frac{d}{dx} \right)^r X\) when \(r \leq \bar{m}\), we therefore have
\[
\left( \frac{d}{dx} \right)^{\bar{m}} F(X) = F^{(\bar{m})}(X) \cdot \left( \frac{dX}{dx} \right)^{\bar{m}} + \text{Error}_2
\]
where \(|\text{Error}_2| \leq C X^{N-\bar{m}+1}\). (18)

Because \(\left( \frac{d}{dx} \right)^{\bar{m}} F(X) = \left( \frac{d}{dx} \right)^{\bar{m}} Y\), it follows from (17) and (18) that
\[
F^{(\bar{m})}(X) \cdot \left( \frac{dX}{dx} \right)^{\bar{m}} = -f^{(\bar{m})}(x) \cdot (1 - \lambda x) + \lambda \bar{m} f^{(\bar{m}-1)}(x) + \text{Error}_3
\]
where \(|\text{Error}_3| \leq C X^{N-\bar{m}+1}\).

(Recall that \(0 \leq x \leq X\).
Because \(|f^{(\bar{m}-1)}(x)| \leq C \cdot x^{N-\bar{m}+1}\) (recall that \(K_{\bar{m}-1}\) is a constant \(C\)), it therefore follows that
\[
F^{(\bar{m})}(X) \cdot \left( \frac{dX}{dx} \right)^{\bar{m}} = -f^{(\bar{m})}(x) \cdot (1 - \lambda x) + \text{Error}_4
\]
where \(|\text{Error}_4| \leq C x^{N-\bar{m}+1}\).

(Here, we use the estimate \(X \leq 2x\).)
Consequently,

\[
|F^{(m)}(X)| \cdot \left( \frac{dX}{dx} \right)^{\bar{m}} \leq |f^{(m)}(x)|(1 - \lambda x) + Cx^{N-m+1} \\
\leq K_{\bar{m}} x^{N-m}(1 - \lambda x) + Cx^{N-m+1} \\
= K_{\bar{m}} x^{N-m} \left( 1 - \lambda x + \frac{C}{K_{\bar{m}}} x \right) \\
\leq K_{\bar{m}} x^{N-m},
\]

provided we pick \( K_{\bar{m}} \geq \frac{C}{X} \), which we now do. Thus,

\[
|F^{(m)}(X)| \cdot \left( \frac{dX}{dx} \right)^{\bar{m}} \leq K_{\bar{m}} x^{N-m} \leq K_{\bar{m}} X^{N-m}.
\]

We have seen that \( \frac{dX}{dx} \geq 1 \), hence the above estimate implies that \( |F^{(m)}(X)| \leq K_{\bar{m}} X^{N-m} \), completing the proof of Assertion 4.

\( \square \)

**Assertion 5.** For suitable constants \( K_1, \ldots, K_{N-10} \), and small enough \( \delta \), the following holds. Let \( \Gamma = \{(x, f(x)) : x \in [0, x_{MAX}]\} \) with \( f \) smooth and

\[
0 < x_{MAX} \leq \delta, \\
|f(x)| \leq x^N, \\
|f^{(m)}(x)| \leq K_m x^{N-m} \text{ for } m = 1, \ldots, N-10.
\]

Then \( \Phi(\Gamma) = \{(X, F(X)) : X \in [0, X_{MAX}]\} \) for a smooth function \( F \) that satisfies \( |F(X)| \leq X^N \) and \( |F^{(m)}(X)| \leq K_m x^{N-m} \) for \( m = 1, \ldots, N-10 \). Moreover, \( |X_{MAX} - (x_{MAX} + x_{MAX}^2)| \leq C x_{MAX}^3 \).

**Proof.** We have already seen that \( \Phi(\Gamma) = \{(X, F(X)) : X \in [0, X_{MAX}]\} \) with \( X_{MAX} \) satisfying the estimate in Assertion 5. It remains only to check the estimates asserted for \( F \).

We have seen that \( |F(X)| \leq X^N \) and that \( |F'(X)| \leq K_1 X^{N-1} \). The desired estimates for \( F^{(m)}(X) \) \( (2 \leq m \leq N-10) \) follow from Assertion 4 by an obvious induction on \( m \). This completes the proof of Assertion 5.

\( \square \)
2.3 Shadowing

In this section, we prove Lemma 1.2, stated in the introduction. We keep our assumption that \( \Phi \) has the form (16) for a fixed \( N \geq 100 \). Constants denoted \( C, c, C', \) etc. may now depend on \( \Phi, K_1, \ldots, K_{N-10} \), but not on \( \delta \). Recall that we set \( K_0 = 1 \) in the introduction.

Assume
\[
0 < x \leq \delta, \quad |y| \leq x^N.
\]

Suppose
\[
\frac{|x - \hat{x}| + x^{-3}|y - \hat{y}|}{x^8} \leq 1.
\]

Define \((X, Y)\) and \((\hat{X}, \hat{Y})\) by setting
\[
X = x + x^2 + x^3\theta_A(x, y) + y\theta_B(x, y),
\]
\[
\hat{X} = \hat{x} + \hat{x}^2 + \hat{x}^3\theta_A(\hat{x}, \hat{y}) + \hat{y}\theta_B(\hat{x}, \hat{y}),
\]
\[
Y = -y (1 - \lambda x + y\theta_C(x, y) + x^2\theta_D(x, y)) + x^{N+100}\theta_E(x, y),
\]
\[
\hat{Y} = -\hat{y} (1 - \lambda \hat{x} + \hat{y}\theta_C(\hat{x}, \hat{y}) + \hat{x}^2\theta_D(\hat{x}, \hat{y})) + \hat{x}^{N+100}\theta_E(\hat{x}, \hat{y}),
\]
i.e. \((X, Y) = \Phi(x, y)\) and \((\hat{X}, \hat{Y}) = \Phi(\hat{x}, \hat{y})\).

We must show that
\[
|x - \hat{x}| + x^{-3}|y - \hat{y}| \leq 1.
\]

We start the proof of (25). First of all, equations (19) and (20) give
\[
|\hat{y}| \leq C\hat{x}^8 \quad \text{and} \quad |x - \hat{x}| \leq x^8.
\]

We note that
\[
|\{-\lambda x + y\theta_C(x, y) + x^2\theta_D(x, y)\} - \{-\lambda \hat{x} + \hat{y}\theta_C(\hat{x}, \hat{y}) + \hat{x}^2\theta_D(\hat{x}, \hat{y})\}| \leq C[|x - \hat{x}| + |y - \hat{y}|]
\]
and
\[
|x^{N+100}\theta_E(x, y) - \hat{x}^{N+100}\theta_E(\hat{x}, \hat{y})| \\
\leq |\hat{x}^{N+100} - x^{N+100}| |\theta_E(\hat{x}, \hat{y})| + x^{N+100}|\theta_E(x, y) - \theta_E(\hat{x}, \hat{y})| \\
\leq x^N[|x - \hat{x}| + |y - \hat{y}|].
\]

Also,
\[
|(x + x^2) - (\hat{x} + \hat{x}^2)| = |x - \hat{x}|[1 + x + \hat{x}] \leq [1 + 2x + x^8] \cdot |x - \hat{x}|
\]
by (26).
Similarly to (27) and (28), we have

\[ |x^3 \theta_A(x, y) - \hat{x}^3 \theta_A(\hat{x}, \hat{y})| \leq |x^3 - \hat{x}^3| \cdot |\theta_A(\hat{x}, \hat{y})| + x^3 |\theta_A(x, y) - \theta_A(\hat{x}, \hat{y})| \]
\[ \leq C x^2 |x - \hat{x}| + C x^3 |x - \hat{x}| + |y - \hat{y}| \]
\[ \leq C' x^2 |x - \hat{x}| + C' x^3 |y - \hat{y}| \]  
(30)

and (thanks to (19))

\[ |y \theta_B(x, y) - \hat{y} \theta_B(\hat{x}, \hat{y})| \leq |y - \hat{y}| \cdot |\theta_B(\hat{x}, \hat{y})| + |y| \cdot |\theta_B(x, y) - \theta_B(\hat{x}, \hat{y})| \]
\[ \leq C |y - \hat{y}| + C x^N |x - \hat{x}| + |y - \hat{y}| \]
\[ \leq C x^N |x - \hat{x}| + C |y - \hat{y}|. \]  
(31)

We apply the above to estimate |X - \hat{X}| and |Y - \hat{Y}|.

We have from (21) and (22) that

\[ |X - \hat{X}| \leq |(x + x^2) - (\hat{x} + \hat{x}^2)| + |x^3 \theta_A(x, y) - \hat{x}^3 \theta_A(\hat{x}, \hat{y})| + |y \theta_B(x, y) - \hat{y} \theta_B(\hat{x}, \hat{y})| \]
\[ \leq [1 + 2x + x^8]|x - \hat{x}| + \{C' x^2 |x - \hat{x}| + C' x^3 |y - \hat{y}|\} + \{C x^N |x - \hat{x}| + C |y - \hat{y}|\} \]
\[ \leq [1 + 2x + C x^2]|x - \hat{x}| + C |y - \hat{y}|, \]  
(32)

where the second inequality follows by (29), (30), (31).

From (23) and (24) we have

\[ |Y - \hat{Y}| \leq |y - \hat{y}| \cdot |\{1 - \lambda \hat{x} + \hat{y} \theta_C(\hat{x}, \hat{y}) + \hat{x}^2 \theta_D(\hat{x}, \hat{y})\}| \]
\[ + |y| \cdot |\{- \lambda x + \hat{y} \theta_C(x, y) + x^2 \theta_D(x, y)\} - \{- \lambda \hat{x} + \hat{y} \theta_C(\hat{x}, \hat{y}) + \hat{x}^2 \theta_D(\hat{x}, \hat{y})\}| \]
\[ + |x^N + 100 \theta_E(x, y) - \hat{x}^N + 100 \theta_E(\hat{x}, \hat{y})| \]
\[ \leq |y - \hat{y}| + C x^N |x - \hat{x}| + |y - \hat{y}| \]

where we have used the fact that \( \lambda > 0 \) and

\[ |\hat{y} \theta_C(\hat{x}, \hat{y}) + \hat{x}^2 \theta_D(\hat{x}, \hat{y})| \leq C |\hat{y}| + C \hat{x}^2 \leq C' \hat{x}^2 \]

by (26).

Consequently,

\[ |Y - \hat{Y}| \leq (1 + C x^N)|y - \hat{y}| + C x^N |x - \hat{x}|. \]  
(33)
Note also that
\[ x \leq x + x^2 - Cx^3 \leq X \leq x + x^2 + Cx^3, \] (34)
thanks to (19) and (21). In particular \( X^{-3} \leq x^{-3} \), so (33) yields
\[ X^{-3}|Y - \hat{Y}| \leq Cx^{-3}|x - \hat{x}| + (1 + Cx^N)x^{-3}|y - \hat{y}|. \]
Adding this to (32), we find that
\[ X - 3 \leq x - 3, \] so (33) yields
\[ X - 3 |Y - \hat{Y}| \leq Cx^{N-3}|x - \hat{x}| + (1 + Cx^N)x^{-3}|y - \hat{y}|. \] (35)
From (34) we have also
\[ X^{-8} \leq (x + x^2 - Cx^3)^{-8} = x^{-8}(1 + x - Cx^2)^{-8} \leq x^{-8}(1 - 8x + C'x^2). \] (36)
Multiplying (35) by (36) we have
\[ \frac{|X - \hat{X}| + X^{-3}|Y - \hat{Y}|}{X^8} \leq (1 - 8x + C'x^2)(1 + 2x + Cx^2) \left( \frac{|x - \hat{x}| + x^{-3}|y - \hat{y}|}{x^8} \right) \leq \frac{|x - \hat{x}| + x^{-3}|y - \hat{y}|}{x^8}. \]
In particular, recalling our assumption (20), we see that (25) holds. This completes the proof of our shadowing result (25), thus establishing Lemma 1.2. □

2.4 Approximately Invariant Curves

Fix constants \( K_1, \ldots, K_{N-10}, \delta \) as before. As in the introduction, we set \( K_0 = 1 \). Constants \( C \) will depend on those \( K \)’s, but not on \( \delta \). Let \( 0 < \rho < \delta \) be a small number. Later, we will fix \( \delta \) small enough, and let \( \rho \to 0^+ \). We recall that \( \delta \) is less than a small enough constant determined by \( \Phi, K_1, \ldots, K_{N-10} \).

Let \( x_{MAX}^0 = \rho \) and let \( F_0(x) = 0 \) on \( [0, x_{MAX}^0] \). By induction on \( \nu \), define \( x_{MAX}^\nu \) and \( F_\nu(x) \) on \( [0, x_{MAX}^\nu] \) by setting \( \Phi(\{(x, F_{\nu-1}(x)) : x \in [0, x_{MAX}^{\nu-1}]\}) = \{(x, F_{\nu}(x)) : x \in [0, x_{MAX}^\nu]\} \). We terminate the construction of \( F_\nu, x_{MAX}^\nu \) as soon as we can no longer apply our Assertions 1–5 to keep going. That is, we pass from \( x_{MAX}^{\nu-1} \), \( F_{\nu-1} \) to \( x_{MAX}^{\nu}, F_{\nu} \) provided \( x_{MAX}^{\nu-1} \leq \delta \). If \( x_{MAX}^{\nu-1} > \delta \), then we stop.
As long as the $x^\nu_{\text{MAX}}$, $F_\nu$ are well-defined, we have

$$|F_\nu(x)| \leq x^N \text{ for } x \in [0, x^\nu_{\text{MAX}}]$$

and

$$|F_\nu^{(m)}(x)| \leq K_m x^{N-m} \text{ for } x \in [0, x^\nu_{\text{MAX}}], \ 1 \leq m \leq N - 10.$$ 

Indeed, that holds for $\nu = 0$ since $F_0 \equiv 0$; and it then follows by induction thanks to Assertion 5. Note that $x^\nu_{\text{MAX}} \geq x^{\nu-1}_{\text{MAX}} + (x^{\nu-1}_{\text{MAX}})^2 - C(x^{\nu-1}_{\text{MAX}})^3$. As long as $x^{\nu-1}_{\text{MAX}} \leq \delta$ and $\delta$ is less than a small enough constant, we have $x^\nu_{\text{MAX}} \geq x^{\nu-1}_{\text{MAX}} + \frac{1}{2}(x^{\nu-1}_{\text{MAX}})^2$. Consequently, our induction on $\nu$ will eventually terminate, i.e., $x^\nu_{\text{MAX}} > \delta$ for some $\nu$. Let $\bar{\nu}$ denote the first $\nu$ for which $x^\nu_{\text{MAX}} > \delta$. Thus, our induction defines $F_0, F_1, ..., F_{\bar{\nu}}$ but then terminates. We have $\delta < x^\nu_{\text{MAX}} \leq x^{\bar{\nu}-1}_{\text{MAX}} + (x^{\bar{\nu}-1}_{\text{MAX}})^2 + C(x^{\bar{\nu}-1}_{\text{MAX}})^3 \leq \delta + \delta^2 + C\delta^3 < 2\delta$.

Now suppose $\bar{x} \in [0, \frac{1}{4}]$ is given. Then $(\bar{x}, F_\nu(\bar{x})) = \Phi^\nu(\bar{x}_0, 0)$ for some $\bar{x}_0 \in [0, \rho]$. (That’s because, for any $\nu \in \{0, 1, ..., \bar{\nu}\}$, we have $\Phi^\nu\{(x, 0) : x \in [0, \rho]\} = \{(x, F_\nu(x)) : x \in [0, x^\nu_{\text{MAX}}]\}$.)

There exists $\bar{x}_0 \in [0, \bar{x}_0]$ such that

$$\bar{x}_0 + \bar{x}^2_0 + \bar{x}^3_0 \theta_A(\bar{x}_0, 0) = \bar{x}_0.$$ 

Then $\Phi(\bar{x}_0, 0) = (\bar{x}, \bar{y}_0)$ with $|\bar{y}_0| \leq \bar{x}^N_0 \leq \bar{x}^N_0 \leq \rho^N$. (The first inequality here is immediate from (16)). Let $z_\nu = (x_\nu, y_\nu) = \Phi^\nu(\bar{x}_0, 0)$ and $\hat{z}_\nu = (\bar{x}_\nu, \bar{y}_\nu) = \Phi^\nu(\bar{x}_0, \bar{y}_0)$ for $\nu = 0, ..., \bar{\nu}$. Let’s estimate how close $\hat{z}_\nu$ is to $z_\nu$.

Note that

$$\frac{|x_0 - \bar{x}_0| + x_0^3|y_0 - \bar{y}_0|}{x^8_0} = \frac{|\bar{y}_0|}{\bar{x}^{11}_0} \leq \bar{x}^{N-11}_0 \leq \rho^{N-11}.$$ 

Repeatedly applying Lemma 1.2, we see that

$$\frac{|x_\nu - \bar{x}_\nu| + x_\nu^3|y_\nu - \bar{y}_\nu|}{x^8_\nu} \leq \rho^{N-11}.$$ 

In particular,

$$|x_\nu - \bar{x}_\nu|, |y_\nu - \bar{y}_\nu| \leq \rho^{N-11},$$

so

$$|z_\nu - \hat{z}_\nu| \leq C \rho^{N-11}.$$ 

That is,

$$|\Phi^\nu(\bar{x}_0, 0) - \Phi^\nu(\bar{x}_0, \bar{y}_0)| \leq C \rho^{N-11}.$$ 

Recall that

$$\Phi^\nu(\bar{x}_0, 0) = (\bar{x}, F_\nu(\bar{x}))$$
and
\[(\bar{x}_0, \bar{y}_0) = \Phi(\bar{x}_0, 0), \text{ so} \]
\[\Phi^\theta(\bar{x}_0, \bar{y}_0) = \Phi^{\theta+1}(\bar{x}_0, 0) = \Phi(\Phi^\theta(\bar{x}_0, 0)).\]

Consequently,
\[|(\bar{x}, F^\theta(\bar{x})) - \Phi(\Phi^\theta(\bar{x}_0, 0))| \leq C\rho^{N-11}.\]

Now \(\tilde{x}_0 \in [0, \bar{x}_0] \subset [0, \rho]\), and
\[\Phi^\theta(\{(x, 0) : x \in [0, \rho]\}) = \{(x, F^\theta(x)) : x \in [0, x^\nu_{MAX}]\},\]

hence
\[\Phi^\theta(\bar{x}_0, 0) = (\hat{x}, F^\theta(\hat{x}))\]
for some \(\hat{x} \in [0, x^\nu_{MAX}] \subset [0, 2\delta]\). Thus,
\[|(\bar{x}, F^\theta(\bar{x})) - \Phi(\hat{x}, F^\theta(\hat{x}))| \leq C\rho^{N-11}.\] (37)

We have \(\Phi(\hat{x}, F^\theta(\hat{x})) = (\hat{X}, \hat{Y})\), with \(|\hat{X} - (\hat{x} + \hat{x}^2)| \leq C|\hat{x}|^3 + C|F^\theta(\hat{x})| \leq C'|\hat{x}|^3\) and \(|\hat{Y}| \leq 2\delta\). Hence \(\hat{X} \geq \hat{x}\).

On the other hand, (37) gives \(|\hat{X} - \bar{x}| \leq C\rho^{N-11}\). Therefore,
\[\hat{x} \leq \bar{x} + C\rho^{N-11} \leq \frac{\delta}{2} + C\rho^{N-11} \leq \delta,\]
provided \(C\rho^{N-11} < \frac{\delta}{2}\). We have established the following result.

**Assertion 6.** Suppose \(\rho\) is less than a small enough positive constant determined by \(\Phi, K_1, \ldots, K_{N-10}, \delta\). Then, given \(\bar{x} \in [0, \frac{\delta}{2}]\), there exists \(\hat{x} \in [0, \delta]\) such that
\[\hat{x} \leq \bar{x} + C\rho^{N-11}\]
and
\[|(\bar{x}, F^\theta(\bar{x})) - \Phi(\hat{x}, F^\theta(\hat{x}))| \leq C\rho^{N-11}.\]

Moreover, \(F^\theta\) satisfies
\[\left|\left(\frac{d}{dx}\right)^m F^\theta(x)\right| \leq K_m x^{N-m} \text{ for } 0 \leq m \leq N - 10, \ x \in [0, \delta].\]
2.5 Passing to the Limit

As before, we fix \( N \geq 100 \) and suppose our map \( \Phi \) has the form (16). We fix the constants \( K_1, ..., K_{N-10}, \delta \) and consider a sequence \( \rho_1, \rho_2, ... \) of positive numbers tending to zero. For each \( \rho_j \), we apply Assertion 6.

Thus, we obtain a sequence of functions \( F_j \in C^{N-10}([0, \delta]) \), with the following properties.

\[
|F_j(x)| \leq x^N \text{ for } x \in [0, \delta]. \tag{38}
\]

\[
\left| \frac{d}{dx}^m F_j(x) \right| \leq K_m x^{N-m} \text{ for } x \in [0, \delta], \quad 1 \leq m \leq N - 10. \tag{39}
\]

Given \( \bar{x} \in [0, \frac{\delta}{2}] \) there exists \( \hat{x}_j \in [0, \delta] \cap [0, \bar{x} + C \rho_j^{N-11}] \) such that

\[
|((\bar{x}, F_j(\bar{x})) - \Phi(\hat{x}_j, F_j(\hat{x}_j)))| \leq C \rho_j^{N-11}. \tag{40}
\]

By Ascoli’s Theorem, we may pass to a subsequence to achieve for some \( F \in C^{N-11} ([0, \delta]) \) that

\[
F_j \to F \text{ in } C^{N-11} \text{ norm.} \tag{41}
\]

From (38) and (39), we have

\[
|F(x)| \leq x^N \text{ for } x \in [0, \delta] \tag{42}
\]

and

\[
\left| \frac{d}{dx}^m F(x) \right| \leq K_m x^{N-m} \text{ for } x \in [0, \delta], \quad 1 \leq m \leq N - 11. \tag{43}
\]

Now let \( \bar{x} \in [0, \frac{\delta}{2}] \), and let \( \hat{x}_j \) be as in (40). Passing to a subsequence \( \hat{x}_{j_i} (i = 1, 2, 3, ...) \) depending on \( \bar{x} \), we may achieve,

\[
\hat{x}_{j_i} \to \hat{x} \text{ as } i \to \infty \tag{44}
\]

with,

\[
\hat{x} \in [0, \bar{x}] \subset [0, \frac{\delta}{2}]. \tag{45}
\]

Thanks to (41) and (44), we have

\[
(\hat{x}_{j_i}, F_{j_i}(\hat{x}_{j_i})) \to (\hat{x}, F(\hat{x})) \text{ as } i \to \infty,
\]

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hence,
\[
\Phi(\hat{x}_{j_i}, F_{j_i}(\hat{x}_{j_i})) \to \Phi(\hat{x}, F(\hat{x})) \text{ as } i \to \infty. 
\] (46)

We now have
\[
|\Phi(\hat{x}, F(\hat{x}))) - \Phi(\hat{x}, F(\hat{x}))| 
\leq |\Phi(\hat{x}, F(\hat{x}))) - \Phi(\hat{x}, F(\hat{x}))| + |\Phi(\hat{x}, F(\hat{x}))) - \Phi(\hat{x}, F(\hat{x}))|.
\] (47)

The three terms on the right in (47) all tend to zero as \( i \to \infty \), thanks to (40), (41), and (46). Hence, \((\hat{x}, F(\hat{x})) = \Phi(\hat{x}, F(\hat{x}))\). We have therefore proven the following.

**Assertion 7.** Let \( N \geq 100 \), and suppose \( \Phi \) has the form (16). Then there exist \( \delta > 0 \) and \( F \in C^{N-11}(0, \delta) \) with the following properties,

- \( |F(x)| \leq x^N \) for \( x \in [0, \delta] \).
- \( \left| \left( \frac{d}{dx} \right)^m F(x) \right| \leq K_m x^{N-m} \) for \( x \in [0, \delta] \), \( 1 \leq m \leq N - 11 \).
- Given \( \bar{x} \in [0, \frac{\delta}{2}] \) there exists \( \hat{x} \in [0, \bar{x}] \) such that \((\bar{x}, F_N(\bar{x})) = \Phi(\hat{x}, F_N(\hat{x}))\).

We now pass from the setting of maps (16) back to our original coordinates, in which our map \( \Phi \) has the form (2).

Recall that we pass from (2) to (16) by repeatedly making coordinate changes of the form \((x, y) \to (x, y + \gamma x^n), (X, Y) \to (X, Y + \gamma X^n)\) with \( n \geq 3 \). From Assertion 7, we therefore read off the following conclusion.

**Assertion 8.** Let \( \Phi \) be a mapping of the form (2), and let \( N \geq 100 \) be given. Then there exist positive constants \( \delta_N, C_N \) and a function \( F_N \in C^N([0, \delta_N]) \) with the following properties.

(I) **Tangency:** \( |F_N(x)| \leq C_N x^3 \) for \( x \in [0, \delta_N] \).

(II) **Invariance:** Given \( \bar{x} \in [0, \delta_N] \) there exists \( \hat{x} \in [0, \bar{x}] \) such that \((\bar{x}, F_N(\bar{x})) = \Phi(\hat{x}, F_N(\hat{x}))\).

So far, \( F_N \) and \( \delta_N \) may depend on \( N \). In the next section, we remedy this defect.

**2.6 Uniqueness**

We prove the following local uniqueness result.

**Assertion 9.** Let \( \Phi \) be as in (2), let \( N \geq 100 \), and let \( F_N \in C^N([0, \delta_N]) \) be as in Assertion 8. Suppose \( \hat{F} : [0, \delta] \to \mathbb{R} \) satisfies
• $x^{-2/3} \tilde{F}(x) \to 0$ as $x \to 0^+$

and

• For every $\bar{x} \in [0, \tilde{\delta}]$ there exists $\hat{x} \in [0, \bar{x}]$ such that $(\bar{x}, \tilde{F}(\bar{x})) = \Phi(\hat{x}, \tilde{F}(\hat{x}))$.

Then for some small positive $\delta \leq \min(\delta_N, \tilde{\delta})$ we have $\tilde{F} = F_N$ on $[0, \delta]$.

**Proof.** By making a change of coordinates,

\[
\begin{align*}
y^\# &= y - F_N(x), \quad x^\# = x, \\
Y^\# &= Y - F_N(X), \quad X^\# = X,
\end{align*}
\]

we may assume without loss of generality that $F_N = 0$ on $[0, \delta_N]$. (However, $\Phi$ is now merely $C^N$, not $C^\infty$.) We must show that $\tilde{F}(x) = 0$ for small positive $x$. Thanks to the invariance condition in Assertion 8, with $F_N = 0$, our map $\Phi$ has the form $(x, y) \mapsto (X, Y)$ with

\[
\begin{align*}
X &= x + x^2 + \mu xy + O((|x, y|)^3) \\
Y &= -y(1 - \lambda x + O(|x, y|^2)).
\end{align*}
\]

Hence, $\Phi^{-2}$ has the form $(x, y) \mapsto (X, Y)$ with

\[
\begin{align*}
X &= x - 2x^2 + O(|(x, y)|^3) \\
Y &= y(1 + 2\lambda x + O(|(x, y)|^2)) \quad (48)
\end{align*}
\]

Note that the term $\mu xy$ above contributes only $O(|(x, y)|^3)$ to $\Phi^{-2}(x, y)$. We study $\Phi^{-2}(x, \tilde{F}(x))$ for small positive $x$.

Since $|\tilde{F}(x)| = o(x^{2/3})$, we have

\[
|(x, \tilde{F}(x))|^3 = o(x^2)
\]

and

\[
|(x, \tilde{F}(x))|^2 = o(x).
\]

Consequently, (48) and the invariance property of $\tilde{F}$ together imply for $x > 0$ small enough:

\[
\Phi^{-2}(x, \tilde{F}(x)) = (\hat{x}, \tilde{F}(\hat{x})) \quad (49)
\]

with

\[
0 < \hat{x} < x - \frac{1}{2}x^2 \quad (50)
\]
and
\[ |\tilde{F}(\hat{x})| \geq |\tilde{F}(x)|. \] (51)

Now suppose that for some small enough positive \( x_0 \), we have \( \tilde{F}(x_0) \neq 0 \). Repeatedly applying (49), (50), (51), we learn that \( \Phi^{-2\nu}(x_0, \tilde{F}(x_0)) = (x_{\nu}, \tilde{F}(x_{\nu})) \), with \( x_{\nu} \to 0 \) as \( \nu \to \infty \), but \( |\tilde{F}(x_{\nu})| \geq |\tilde{F}(x_0)| > 0 \) for all \( \nu \). This contradicts our hypothesis \( |\tilde{F}(x)| = o(x^{2/3}) \). Thus, \( \tilde{F}(x_0) = 0 \) for all small enough \( x_0 > 0 \), completing the proof of Assertion 9.

\[ \Box \]

2.7 Endgame

Let \( \Phi \) be as in the statement of Theorem 1.1. For each \( N \geq 100 \), let \( F_N \in C^N([0, \delta_N]) \) be as in Assertion 8.

Assertion 9 tells us that \( F_N = F_{N'} \) in an interval \([0, \delta_{N,N'})\) for all \( N, N' \geq 100 \). In particular, \( F_N = F_{100} \) on an interval \([0, \delta_{N,100}]\) for each \( N \geq 100 \).

Consequently, \( F_{100} \in C^N([0, \delta_{N,100}]) \) for each such \( N \). Repeatedly applying the invariance condition in Assertion 8 to \( F_{100} \), we learn that, for any \( \nu \geq 1 \), the graph \( \Gamma = \{(x, F_{100}(x)) : x \in [0, \delta_{100}]\} \) is equal to the image of the graph \( \{(x, F_{100}(x)) : x \in [0, \delta_{100}]\} \) under the map \( \Phi^\nu \), for some \( \delta(\nu) > 0 \).

We have \( \delta(\nu+1) \leq \frac{1}{2} \delta(\nu)^2 \), hence \( \delta(\nu) \to 0 \) as \( \nu \to \infty \). Taking \( \nu \) so large that \( \delta(\nu) < \delta_{N,100} \), we see that \( \Gamma \) is the image of a \( C^N \) curve under the smooth map \( \Phi^\nu \). Therefore, \( F_{100} \in C^\infty([0, \delta_{100}]) \) for all \( N \geq 100 \).

Thus, \( F_{100} \in C^\infty([0, \delta_{100}]) \). Together with Assertion 8 for \( F_{100} \), this proves the existence claimed in Theorem 1.1. Finally, the uniqueness claimed in Theorem 1.1 is precisely Assertion 9.

The proof of Theorem 1.1 is complete. ■

Appendix

Theorem 1.1 follows from Theorem 2.1 in [1] applied to the inverse map of \( \Phi^2 \). Indeed, Theorem 2.1 provides, under appropriate conditions, a stable manifold of the origin, tangent to the \( x \) axis, for a map \( \Psi \) such that \( \Psi(0,0) = (0,0) \) and \( D\Psi(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Given \( \Phi \) as in (1) in your preprint, we have
\[ \Phi^2(x,y) = \begin{pmatrix} x + 2x^2 + O((|x,y|^3) \\ y - 2\lambda xy + yO(|(x,y)|^2) + O(|(x,y)|^4) \end{pmatrix} \]

(An important point here is that the second component of \( \Phi^2(x,y) \) has no term of the form \( cx^3 \).)

Then
\[ \Psi(x,y) = \begin{pmatrix} \Psi_1(x,y) \\ \Psi_2(x,y) \end{pmatrix} = \Phi^{-2}(x,y) = \begin{pmatrix} x - 2x^2 + O((|x,y|^3) \\ y + 2\lambda xy + yO(|(x,y)|^2) + O(|(x,y)|^4) \end{pmatrix}. \]

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Taking $F = \Psi$, $N = M = 2$ in the statement of Theorem 2.1 we check that
\[
\frac{\partial^2 \Psi_1}{\partial x^2}(0, 0) = -2 < 0, \quad \frac{\partial^2 \Psi_2}{\partial x^2}(0, 0) = 0, \quad \frac{\partial^2 \Psi_1}{\partial x \partial y}(0, 0) = 2\lambda > 0.
\]
Then there exists a $C^\infty$ map $K : [0, t_0] \to \mathbb{R}^2$ and a polynomial (of degree 3) $R : \mathbb{R} \to \mathbb{R}$ such that
\[
\Psi \circ K = K \circ R.
\]
The image of $K$ is the stable manifold of $\Psi$. The polynomial approximation of $K$ given in Section 3 of [1] provides, using that there is no term of the form $cx^3$ in $\Psi_2$,
\[
K(t) = \left( t + \frac{O(t^3)}{O(t^3)} \right), \quad R(t) = t - 2t^2 + dt^3, \quad d \in \mathbb{R}.
\]
$K$ is not unique but its image is the graph of a unique function $\varphi$ (see Remark 2.3). Also $\varphi(x) = O(x^3)$. The uniqueness is among all Lipschitz functions from $[0, x_0]$ to $\mathbb{R}$ satisfying $|\varphi(x)| \leq C|x|$ for arbitrary constant, changing if necessary the value of $x_0$. Our uniqueness statement is not exactly yours.

Now we have that graph $\varphi$ is invariant by $\Psi$. We take $\tilde{\varphi}$ such that
\[
\text{graph } \tilde{\varphi} = \Phi^{-1}(\text{graph } \varphi)
\]
(in a slightly smaller domain). Since
\[
\Phi^{-1}\left(\begin{array}{c} x \\ \varphi(x) \end{array}\right) = \left(\begin{array}{c} x - x^2 + O(x^3) \\ -\varphi(x) - \lambda x \varphi(x) + O(x^3) \end{array}\right) = \left(\begin{array}{c} x - x^2 + O(x^3) \\ O(x^3) \end{array}\right)
\]
we have that $\tilde{\varphi}(x) = O(x^3)$.

Moreover, since
\[
\Psi(\text{graph } \tilde{\varphi}) = \Phi^{-3}(\text{graph } \varphi) = \Phi^{-1}(\Psi(\text{graph } \varphi)) \subset \Phi^{-1}(\text{graph } \varphi) = \text{graph } \tilde{\varphi},
\]
graph $\tilde{\varphi}$ is also invariant by $\Psi$. Then, by the uniqueness property, $\varphi = \tilde{\varphi}$ in the common domain.

**Final remark 1** In [2] we deal with invariant manifolds of arbitrary (finite) dimension. One could deduce the main part of your result Theorem 1.1 from Corollary 2.5 of [2] except the smoothness at 0 because in higher dimension, in general, the manifold is not smooth at the origin. However it does provide smoothness in $(0, t_0)$.

**Final remark 2** We are aware of the applications of our results in Celestial Mechanics and Chemistry. We are really happy to hear that there are also applications in Economics.
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