A NOTE ON THE $K$–STABILITY ON TORIC MANIFOLDS

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Abstract. In this note, we prove that on polarized toric manifolds the relative $K$-stability with respect to Donaldson’s toric degenerations is a necessary condition for the existence of Calabi’s extremal metrics, and also we show that the modified $K$-energy is proper in the space of $G_0$-invariant Kähler metrics for the case of toric surfaces which admit the extremal metrics.

0. Introduction

Around the existence of Calabi’s extremal metrics, there is a well-known conjecture (cf. [Ya], [Ti]):

Conjecture 0.1. A polarization Kähler manifold $M$ admits a Kähler metric with constant scalar curvature (or more generally an extremal metric) if and only if $M$ is stable in a certain sense of GIT.

For “only if” part of this conjecture, the first breakthrough was made by Tian [Ti]. By introducing the concept of $K$-stability, he gave an answer to “only if” part for Kähler-Einstein manifolds. A remarkable progress was made by Donaldson [D1] who showed the Chow-Mumford stability for a polarized Kähler manifold with constant scalar curvature when the holomorphic automorphisms group $\text{Aut}(M)$ of $M$ is finite. The Donaldson’s result was lately generalized by Mabuchi [M1], [M2] to any polarized Kähler manifold $M$ which admits an extremal metric without any assumption for $\text{Aut}(M)$. However it is still unknown whether there is a generalization of Tian’s result for the $K$-stability on the Kähler-Einstein manifolds analogous to Donaldson-Mabuchi’s result for the Chow-Mumford stability.

As we know the definition of $K$-stability on a polarization Kähler manifold is associated to degenerations (or test configurations in the sense of Donaldson [D2]) on the underlying manifold. In order to study the relation between the $K$-stability and $K$-energy on a polarized toric manifold, Donaldson in [D2] introduced a class of special degenerations induced by rational, piecewise linear functions, called toric degenerations, and proved

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that for the surfaces’ case the $K$-energy is bounded from below in the space of $G_0$-invariant Kähler metrics under the assumption of $K$-stability for any toric degeneration, where $G_0$ is a maximal compact torus group. In this note, we focus on polarized toric manifolds as in [D2] and shall give an answer to “only if” part in Conjecture 0.1 in the sense of relative $K$-stability with respect to toric degenerations for the extremal metrics. Furthermore, we show that the modified $K$-energy is proper in the space of $G_0$-invariant Kähler metrics on a toric surface which admits an extremal metric. The relative $K$-stability is a generalization of $K$-stability which was first introduced by Székelyhidi [Sz], as well as the modified $K$-energy is a generalization of the $K$-energy.

1. Statement of main theorems

Let $(M, L)$ be an $n$-dimensional polarized Kähler manifold. Recall the definition of relative $K$-stability.

**Definition 1.1** ([D2], [Sz]). A test configuration for the polarized Kähler manifold $(M, L)$ of exponent $r$ consists of a $\mathbb{C}^*$-equivariant flat family of schemes $\pi: W \rightarrow \mathbb{C}$ (where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication) and a $\mathbb{C}^*$-equivariant ample line bundle $L$ on $W$. We require that the fibres $(W_t, L|_{W_t})$ are isomorphic to $(M, L^r)$ for any $t \neq 0$. A test configuration is called trivial if $W = M \times \mathbb{C}$ is a product.

Let $(M, L)$ be equipped with $\mathbb{C}^*$-actions $\beta$. We say that a test configuration $(W, L)$ is compatible with $\beta$, if there are $\mathbb{C}^*$-actions $\tilde{\beta}$ on $(W, L)$ such that $\pi: W \rightarrow \mathbb{C}$ is $\tilde{\beta}$-equivariant with trivial $\mathbb{C}^*$-actions on $\mathbb{C}$ and the restriction of $\tilde{\beta}$ to $(W_t, L|_{W_t})$ for nonzero $t$ coincides with that of $\beta$ on $(M, L^r)$ under the isomorphism.

Note that $\mathbb{C}^*$-actions on $W$ induce $\mathbb{C}^*$-actions on the central fibre $M_0 = \pi^{-1}(0)$ and the restricted line bundle $L|_{M_0}$. We denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the induced $\mathbb{C}^*$-actions of $\alpha$ and $\beta$ on $(M_0, L|_{M_0})$, respectively. The relative $K$-stability is based on the modified Futaki invariant on the central fibre,

\begin{equation}
F_{\tilde{\beta}}(\tilde{\alpha}) = F(\tilde{\alpha}) - \frac{(\tilde{\alpha}, \tilde{\beta})}{(\beta, \tilde{\beta})} F(\tilde{\beta}),
\end{equation}

where $F(\tilde{\alpha})$ and $F(\tilde{\beta})$ are generalized Futaki invariants of $\tilde{\alpha}$ and $\tilde{\beta}$ defined in [D2], respectively, and $(\tilde{\alpha}, \tilde{\beta})$ and $(\tilde{\beta}, \tilde{\beta})$ are inner products defined in [Sz].

**Definition 1.2** ([Sz]). The polarized Kähler manifold $(M, L)$ with $\mathbb{C}^*$-actions $\beta$ is $K$-semistable relative to $\beta$ if $F_{\tilde{\beta}}(\cdot) \leq 0$ for any test-configuration compatible with $\beta$. It is called relative $K$-stable in addition that the equality holds if and only if the test-configuration is trivial.
An n-dimensional polarized toric manifold $M$ corresponds to an integral polytope $P$ in $\mathbb{R}^n$ which is described by a common set of some half-spaces,
\begin{equation}
\langle l_i, x \rangle < \lambda_i, \ i = 1, ..., d.
\end{equation}
Here $l_i$ are $d$--vectors in $\mathbb{R}^n$ with all components in $\mathbb{Z}$, which satisfy the Delzant condition ([Ab]). Without loss of generality, we may assume that the original point 0 lies in $P$, so all $\lambda_i > 0$. Recall that a piecewise linear (PL) function $f$ on $P$ is of the form
\[ u = \max\{f^1, ..., f^r\}, \]
where $f^\lambda = \sum a_i^\lambda x_i + c^\lambda$, $\lambda = 1, ..., r$, for some vectors $(a_i^\lambda) \in \mathbb{R}^n$ and some numbers $c^\lambda \in \mathbb{R}$. $f$ is called a rational PL-function if components $a_i^\lambda$ and numbers $c^\lambda$ are all rational. According to [D2], a rational PL-function induces a test configuration, called a toric degeneration. Moreover, one can show that a toric degeneration is always compatible to $\mathbb{C}^*$--actions induced by an extremal vector field on $M$ (cf. [ZZ1]).

Now we can state our first main theorem in this note.

**Theorem 1.3.** Let $M$ be a polarized toric manifold which admits an extremal metric in this polarized Kähler class. Then $M$ is $K$-stable relative to $\mathbb{C}^*$-actions induced by an extremal vector field on $M$ for any toric degeneration.

We will give a proof of Theorem 1.3 in the next section. The proof can be also used to discuss the properness of the modified $K$-energy $\mu(\phi)$. As same as the $K$-energy, $\mu(\phi)$ is defined on a Kähler class whose critical point is an extremal metric while the critical point of $K$-energy is a Kähler metric with constant scalar curvature.

Let $\omega_g$ be a Kähler form of Kähler metric $g$ on a compact Kähler manifold $M$ and $\omega_\phi = \omega_g + \sqrt{-1} \partial \bar{\partial} \phi$ be a Kähler form associated to a potential function $\phi$ in Kähler class $[\omega_g]$. Let $\theta_X$ be a normalized potential function associated to an extremal vector field $X$ and the metric $\omega_g$ on $M$ [ZZ1]. Then the $\mu(\phi)$ is given by
\[ \mu(\phi) = -\frac{1}{V} \int_0^1 \int_M \dot{\phi}_t [R(\omega_{\phi_t}) - \overline{R} - \theta_X(\omega_{\phi_t})] \omega_{\phi_t}^n \wedge dt, \]
where $\phi_t$ ($0 \leq t \leq 1$) is a family of potential functions connecting 0 to $\phi$, $R(\omega_{\phi_t})$ denote the scalar curvatures of $\omega_{\phi_t}$, $\theta_X(\omega_{\phi_t}) = \theta_X + X(\phi)$ are normalized potential functions associated to $X$ and $\omega_{\phi_t}$, and $\overline{R}$ is the average of scalar curvature of $\omega_g$. It can be showed that the functional $\mu(\phi)$ is well-defined, i.e., it is independent of the choice of path $\phi_t$ ([Gu]). Thus one sees that $\phi$ is a critical point of $\mu(\cdot)$ iff the corresponding metric $\omega_{\phi}$ is extremal.
Definition 1.4. Let

\[ I(\phi) = \frac{1}{V} \int_M \phi(\omega^n_g - \omega^n_\phi), \]

where \( V = \int_M \omega^n_g \). \( \mu(\phi) \) is called proper associated to a subgroup \( G \) of the automorphisms group \( \text{Aut}(M) \) in Kähler class \( [\omega_g] \) if there is a continuous function \( p(t) \) in \( \mathbb{R} \) with the property

\[ \lim_{t \to +\infty} p(t) = +\infty, \]

such that

\[ \mu(\phi) \geq \inf_{\sigma \in G} p(I(\phi_\sigma)), \]

where \( \phi_\sigma \) is defined by

\[ \omega_g + \sqrt{-1} \partial \bar{\partial} \phi_\sigma = \sigma^*(\omega_g + \sqrt{-1} \partial \bar{\partial} \phi). \]

The above definition was first introduced by Tian for the \( K \)-energy ([Ti]). He proved that the properness of \( K \)-energy is a sufficient and necessary condition for the existence of Kähler-Einstein metrics. In a very recent paper [ZZ2], the authors proved the existence of minimizing weak solution of extremal metrics on toric manifolds under the assumption of properness of \( \mu(\phi) \). This weak solution will be an extremal metric if one can further prove some regularities of the solution.

The following is our second main theorem in this note.

Theorem 1.5. Let \( M \) be a toric surface which admits an extremal metric \( \omega_E \). Suppose that

\[ \bar{R} + \theta_X(\omega_E) > 0. \]

Then \( \mu(\phi) \) is proper associated to \( T \) in the space of \( G_0 \)-invariant Kähler metrics. Here \( T \) is a torus actions group of \( M \) and \( G_0 \) is a maximal compact subgroup of \( T \).

We note that in Theorem 1.5 we need not to assume that the surface is polarized.

2. Proof of Theorem 1.3

Let \( P \) be an integral polytope in \( \mathbb{R}^n \) associated to a polarized toric manifold \( M \) as in Section 1. We choose a \( G_0 \)-invariant Kähler metric \( \omega_g \) on \( M \). Then there exists a convex function \( \psi_0 \) in \( \mathbb{R}^n \) such that

\[ \omega_g = \sqrt{-1} \partial \bar{\partial} \psi_0. \]

Let \( u_0 \) be a Legendre function of \( \phi_0 \) which is a convex function on \( P \). Set

\[ C = \{ u \mid u \text{ is a convex function on } P \text{ with } u - u_0 \in C^\infty(P) \}. \]
Then one can show that functions in $\mathcal{C}$ are corresponding to $G_0$-invariant Kähler potential functions on $M$ by one-to-one (cf. [Ab],[D2]).

Let $d\sigma_0$ be the Lebesgue measure on the boundary $\partial P$ and $\nu$ be the outer normal vector field on $\partial P$. Let $d\sigma$ be an induced measure on $\partial P$ such that $d\sigma = \lambda_i^{-1}(\nu, x)d\sigma_0$ on each face $\langle l_i, x \rangle = \lambda_i$ of $P$. We define a linear functional on the space of continuous function on $P$ by

$$L(u) = \int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X) ud\nu,$$

where $\theta_X$ is the normalized potential function associated to the extremal vector field $X$ as in Section 1, which is equal to $<A, x> + a$ for some vector $A \in \mathbb{R}^n$ and constant $a$ in the coordinates $x$ in $P$ ([ZZ1]). The following result was proved in [ZZ1].

**Lemma 2.1 ([ZZ1]).** Let $\tilde{\alpha}$ and $\tilde{\beta}$ are two $C^*$-actions induced by a rational PL-function $f$ and the extremal vector field $X$. Then

$$F_{\tilde{\beta}}(\tilde{\alpha}) = -\frac{1}{2Vol(P)} L(f).$$

By the above lemma, to prove Theorem 1.3 we suffice to prove

**Proposition 2.2.** Let $M$ be a toric manifold which admits an extremal metric. Then for any PL-function $f$ on $P$, we have

$$L(f) \geq 0.$$  

Moreover the equality holds if and only if $f$ is an affine linear function on $P$.

**Proof.** By definition, we may assume that

$$f = \max\{f^1, ..., f^r\},$$

and each $f^\alpha = \sum c_i^\alpha x_i + c^\alpha$ is an affine linear function on $P$. Then $P$ can be divided into $m(\geq r)$ small polytopes $P^1, ..., P^m$ such that for each $P^\lambda$ there exists a $f^{\alpha\lambda}$ such that $f = f^{\alpha\lambda}$ on $P^\lambda$. By the assumption of the existence of extremal metric, we see that there exists a $u \in \mathcal{C}$ which satisfies the Abreu’s equation ([Ab]),

$$-\sum_{i,j} u^{ij}_{ij} = \bar{R} + \theta_X,$$

where $(u^{ij})^{-1}$ and $u^{ij}_{kl} = \frac{\partial^2 u^{ij}}{\partial x^k \partial x^l}$. Thus

$$L(f) = \int_{\partial P} f d\sigma + \int_P \sum_{i,j} u^{ij}_{ij} f dx.$$
For any small $\delta > 0$, we let $P_\delta$ be the interior polygon with faces parallel to those of $P$ separated by distance $\delta$. Integrating by parts, one sees

$$\int_{P_\delta} u_{ij}^i f dx = \sum_{\lambda} \int_{P^\lambda \cap P_\delta} \sum_{i,j} u_{ij}^i f^{\alpha \lambda} dx$$

$$= \sum_{\lambda} \int_{\partial (P^\lambda \cap P_\delta)} \sum_{i,j} u_{ij}^i f^{\alpha \lambda} n_j^{\alpha \lambda} d\sigma_0 - \sum_{\lambda} \int_{P^\lambda \cap P_\delta} \sum_{i,j} u_{ij}^i c_j^{\alpha \lambda} n_i^{\alpha \lambda} d\sigma_0$$

$$= \sum_{\lambda} \int_{\partial (P^\lambda \cap P_\delta)} \sum_{i,j} u_{ij}^i f^{\alpha \lambda} n_j^{\alpha \lambda} d\sigma_0 - \sum_{\lambda} \int_{\partial (P^\lambda \cap P_\delta)} \sum_{i,j} u_{ij}^i c_j^{\alpha \lambda} n_i^{\alpha \lambda} d\sigma_0$$

$$= \sum_{\lambda} \int_{\partial P_\delta} \sum_{i,j} u_{ij}^i f^{\alpha \lambda} n_j^{\alpha \lambda} d\sigma_0 - \sum_{\lambda} \int_{\partial P_\delta} \sum_{i,j} u_{ij}^i c_j^{\alpha \lambda} n_i^{\alpha \lambda} d\sigma_0$$

where $(n_1^{\alpha \lambda}, \ldots, n_n^{\alpha \lambda})$ is the unit outer normal vector on $\partial P^\lambda$. By [D2] we know that the first term goes to $-\int_{\partial P} f d\sigma$ and the second term goes to $0$ at the last equality as $\delta$ goes to zero. On the other hand, if $\partial P^\lambda \cap \partial P^\mu$ is a common $(n-1)$-dimensional face of $P^\lambda$ and $P^\mu$, then $(c_1^{\alpha \lambda} - c_1^{\alpha \mu}, \ldots, c_n^{\alpha \lambda} - c_n^{\alpha \mu})$ is a nonzero vector, and

$$n_i^{\alpha \lambda} = -\frac{1}{\sqrt{\sum_i(c_i^{\alpha \lambda} - c_i^{\alpha \mu})^2}}(c_i^{\alpha \lambda} - c_i^{\alpha \mu}), n_i^{\alpha \mu} = -\frac{1}{\sqrt{\sum_i(c_i^{\alpha \mu} - c_i^{\alpha \lambda})^2}}(c_i^{\alpha \mu} - c_i^{\alpha \lambda}).$$

Substituting them into the third term at the last equality in (2.5) and letting $\delta$ go to $0$, we derive

$$\int_P \sum_{i,j} u_{ij}^i f dx$$

(2.6) $= -\int_{\partial P} f d\sigma + \sum_{\lambda<\mu} \int_{\partial P^\lambda \cap \partial P^\mu} \frac{\sum_{i,j} u_{ij}^i (c_i^{\alpha \lambda} - c_i^{\alpha \mu})(c_j^{\alpha \lambda} - c_j^{\alpha \mu})}{\sqrt{\sum_i(c_i^{\alpha \lambda} - c_i^{\alpha \mu})^2}} d\sigma_0.$

Hence, combining (2.4) and (2.6),

$$L(f) = \sum_{\lambda<\mu} \int_{\partial P^\lambda \cap \partial P^\mu} \frac{\sum_{i,j} u_{ij}^i (c_i^{\alpha \lambda} - c_i^{\alpha \mu})(c_j^{\alpha \lambda} - c_j^{\alpha \mu})}{\sqrt{\sum_i(c_i^{\alpha \lambda} - c_i^{\alpha \mu})^2}} d\sigma_0 \geq 0.$$

Note that the equality holds if only if there is no common $(n-1)$-dimensional face for any $P^\lambda$ and $P^\mu$, which implies that $f$ is just an affine linear function. \qed
3. Proof of Theorem 1.5

To prove Theorem 1.5, we need to use a Donaldson’s version of the modified $K$-energy ([D2], [ZZ1]).

**Lemma 3.1.** Let $\omega_{\phi}$ be a $G_0$-invariant Kähler metric on $M$ and $u$ be a Legendre function of $\phi$ in $C$. Then

$$\mu(\phi) = \frac{2^n n! (2\pi)^n}{Vol(M)} F(u),$$

where

$$F(u) = -\int_P \log(\det(D^2 u)) dx + L(u)$$

Let $p \in P$. We set

$C_{\infty} = \{ u \in C^0(\overline{P}) \cup C^\infty(P) | u \text{ is a convex function on } P \text{ with } \inf_P u = u(p) = 0 \}.$

The following proposition was proved in [ZZ1].

**Proposition 3.2.** Suppose that there exists a $\lambda > 0$ such that

$$L(u) \geq \lambda \int_{\partial P} u d\sigma,$$

for any $u \in C_{\infty}$. Then there exist two uniform constants $\delta, C > 0$ such that for any $G_0$-invariant Kähler metric $\omega_{\phi}$ it holds

$$\mu(\phi) \geq \delta \inf_{\sigma \in T} I(\phi_\sigma) - C.$$

In particular, $\mu(\phi)$ is proper associated to subgroup $T$ in the space of $G_0$-invariant Kähler metrics.

**Proof of Theorem 1.5.** According to Proposition 3.2, we suffice to verify the condition (3.2). We use an argument by the contradiction. First note that by the convexity of functional $F(u)$ on $C$, one sees that $F(u)$ is bounded from below by the existence of extremal metrics. Then one can show (cf. [D2], [ZZ1]),

$$L(u) \geq 0, \forall u \in C_{\infty}.$$

Furthermore one concludes that

$$L(u) \geq 0, \forall u \in C_1,$$

where $C_1$ is a set of positive convex functions $u$ on $P \cup \partial P$ such that

$$\int_{\partial P} u d\sigma_0 dx < \infty.$$
On the other hand, suppose that (3.2) is not true, then by (3.4) it is easy to see that there exists a $u_\infty \in \mathcal{C}_1$, which is not affine linear, such that

$$L(u_\infty) = 0.$$ (3.5)

Recall that a simple PL-function is a form of

$$u = \max\{0, \sum a_ix_i + c\}$$

for some vector $(a_i) \in \mathbb{R}^n$ and number $c \in \mathbb{R}$. We call the hyperplane $\sum a_ix_i + c = 0$ a crease of $u$. Then applying Proposition 5.3.1 in [D2], we see that under the relations (3.4), (3.5) and the assumption (1.3) in Theorem 1.5 for the case of toric surfaces there exists a simple PL-function $v_0$ with crease intersecting the interior of $P$ such that $L(v_0) = 0$. But the late is contradict to Proposition 2.3. Thus Proposition 3.2 is true and so is theorem 1.5.

\[\square\]

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