Divisible designs from twisted dual numbers

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Dedicated to Helmut M"aurer on the occasion of his 70th birthday

Abstract

The generalized chain geometry over the local ring $K(\varepsilon; \sigma)$ of twisted dual numbers, where $K$ is a finite field, is interpreted as a divisible design obtained from an imprimitive group action. Its combinatorial properties as well as a geometric model in 4-space are investigated.

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1 Preliminaries

This paper deals with a special class of divisible designs, namely, those that are chain geometries over certain finite local rings, and their representation in projective space.

A finite geometry $\Sigma = (\mathcal{P}, \mathcal{B}, \|)$, consisting of a set $\mathcal{P}$ of points, a set $\mathcal{B}$ of blocks, and an equivalence relation $\|$ (parallel) on $\mathcal{P}$, is called a $t$-$(s, k; \lambda_t)$-divisible design ($t$-DD for short), if there exist positive integers $t, s, k, \lambda_t$ such that the following axioms hold:

- Each block $B$ is a subset of $\mathcal{P}$ containing $k$ pairwise non-parallel points.
- Each parallel class consists of $s$ points.
- For each set $Y$ of $t$ pairwise non-parallel points there exist exactly $\lambda_t$ blocks containing $Y$.
- $t \leq k \leq v/s$, where $v := |\mathcal{P}|$.

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Note that sometimes DDs are called “group divisible designs”.

A DD with trivial parallel relation, i.e. with \( s = 1 \), is an ordinary design. A DD with \( k = v/s \) is called transversal. In the subsequent sections we shall deal with transversal 3-DDs.

A method to construct DDs with a large group of automorphisms is due to A.G. Spera [11], using imprimitive group actions: Let \( G \) be a group acting on a (finite) set \( \mathcal{P} \) of “points” and leaving invariant an equivalence relation \( \parallel \) (“parallel”). Let \( t \) be a positive integer such that there are at least \( t \) parallel classes, and let \( B_0 \) be a set of \( k \geq t \) pairwise non-parallel points (the “base block”). Assume that \( G \) acts transitively on the set of \( t \)-tuples of pairwise non-parallel points. Let \( B \) be the orbit of \( B_0 \) under \( G \), i.e. \( B = \{ B_0^g \mid g \in G \} \).

Then \( \Sigma = (\mathcal{P}, B, \parallel) \) is a \( t \)-DD with

\[
\lambda_t = \frac{|G|}{|G_{B_0}|} \cdot \frac{{t \choose t}}{v/s} s^t
\]

where \( G_{B_0} \) is the (setwise) stabilizer of \( B_0 \) in \( G \) (see [11, Prop. 2.3]).

The projective line \( \mathbb{P}(R) \) over a finite local ring \( R \) is endowed with an equivalence relation (usually denoted by \( \parallel \)). It is invariant under the action of the general linear group \( GL_2(R) \) on \( \mathbb{P}(R) \). Since \( GL_2(R) \) acts transitively on the set of triples of non-parallel points, any \( k \)-set \( (k \geq 3) \) of mutually non-parallel points of \( \mathbb{P}(R) \) can be chosen as base block \( B_0 \) in order to apply Spera’s construction of a DD. This is, of course, a very general approach. Therefore, it is not surprising that not too much can be said about the corresponding 3-DDs. It is straightforward to express their parameters \( v \) and \( s \), as well as the order of the group \( GL_2(R) \), in terms of \( |R| \) and \( |I| \), i.e. the cardinality of the unique maximal ideal \( I \) of the given ring \( R \). However, in order to calculate the parameter \( \lambda_3 \) by virtue of (1), one needs to know the order of the stabilizer of \( B_0 \) in \( GL_2(R) \). But it seems hopeless to calculate this order without further information about the base block \( B_0 \).

If \( R \) is even a finite local algebra over a field \( F \), say, then the projective line \( \mathbb{P}(F) \) over \( F \) can be considered as a subset of \( \mathbb{P}(R) \), and it can be chosen as a base block. All 3-DDs obtained in this way satisfy \( \lambda_3 = 1 \); they are—up to notational differences—precisely the (classical) chain geometries \( \Sigma(F, R) \); see [1], [6] or [9]. This was pointed out by Spera [11, Example 2.5]. In the cited paper also a series of interesting DDs are constructed from base blocks which are certain subsets of \( \mathbb{P}(F) \). See also [7] for similar results. We mention in passing that higher-dimensional projective spaces over local algebras give rise to 2-DDs [12].

The divisible designs which are constructed in the present paper arise also from chain geometries. However, we use this term in a more general form...
which was introduced in [3] just a few years ago. The essential difference is
as follows: We consider a finite local ring $R$ containing a subfield $K$ which
is not necessarily in the centre of $R$. Thus $R$ need not be a $K$-algebra, but
of course it is an algebra over some subfield of $K$. As before, we can define
$\mathbb{P}(K) \subseteq \mathbb{P}(R)$ to be the base block. This gives a 3-DD which coincides
with the (generalized) chain geometry $\Sigma(K, R)$. It is possible to express the
parameter $\lambda_3$ of this DD in algebraic terms (see [3, Theorem 2.4]), but this
is not very explicit in the general case. Therefore, we focus our attention on
a particular class of local rings, namely twisted dual numbers. If the “twist”
is non-trivial, then 3-DDs with parameter $\lambda_3 = |K|$ are obtained.
In Section 4 we present an alternative description of our 3-DDs in a finite
projective space over $K$.

2 Twisted dual numbers

Let $R$ be a (not necessarily commutative) local ring containing a (not neces-
sarily central) subfield $K$. In view of our objective to construct DDs, we will
later restrict ourselves to finite rings and fields, and hence we assume from
the beginning that $K$ is commutative. As usual, we denote by $R^*$ the group
of units (invertible elements) of $R$. We set $I := R \setminus R^*$; since $R$ is local we
have that $I$ is an ideal.

The ring $R$ is in a natural way a left vector space over $K$, sometimes written
as $KR$. We assume that $\dim(KR) = 2$. Moreover, we assume that $R$ is not a
field. We want to determine the structure of $R$. The ideal $I$ is a non-trivial
subspace of the vector space $KR$. So $\dim(KI) = 1$, and $I = K\varepsilon$ for some
$\varepsilon \in R \setminus K$. Then $1, \varepsilon$ is a basis of $KR$, and we may write $R = K + K\varepsilon$.

In order to describe the multiplication in $R$ we first observe that $\varepsilon^2 \in I$, so
$\varepsilon^2 = b\varepsilon$ for some $b \in K$. This implies $(\varepsilon - b)\varepsilon = 0$, whence also $\varepsilon - b \in I$
and so $b = 0$. For each $x \in K$ we have $\varepsilon x \in I$, so there is a unique $x' \in K$
such that $\varepsilon x = x'\varepsilon$. One can easily check that $\sigma : x \mapsto x'$ is an injective field
endomorphism.

Conversely, given a field $K$ and an injective endomorphism $\sigma$ of $K$ we obtain
a ring of twisted dual numbers $R = K(\varepsilon; \sigma) = K + K\varepsilon$ with multiplication

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc\sigma)\varepsilon.$$ 

In the special case that $\sigma = \text{id}$ this is the well known commutative ring $K(\varepsilon)$
of dual numbers over $K$.

The subfield $\text{Fix}(\sigma)$ of $K$ fixed elementwise by $\sigma$ will be called $F$. So $F = K$
if, and only if, $\sigma = \text{id}$. 
The units of \( R \) are exactly the elements of \( R \setminus I = K^* + K\varepsilon \). One can easily check that the inverse of a unit \( u = a + b\varepsilon \) (with \( a, b \in K, a \neq 0 \)) is

\[
u^{-1} = a^{-1} - a^{-1}b(a^\sigma)^{-1}\varepsilon. \tag{2}\]

Later we shall need the following algebraic statements on \( R = K(\varepsilon; \sigma) \).

**2.1 Lemma.** The multiplicative group \( R^* \) is the semi-direct product of \( K^* \) and the normal subgroup

\[U = 1 + K\varepsilon = \{1 + b\varepsilon \mid b \in K\}. \tag{3}\]

*Proof:* Direct computation, using (2) for showing that \( U \) is normal in \( R^* \). \( \square \)

**2.2 Lemma.** Let \( N \) be the normalizer of \( K^* \) in \( R^* \), i.e.,

\[N = \{n \in R^* \mid n^{-1}K^*n = K^*\}. \tag{4}\]

Then \( N = R^* \) if \( \sigma = \text{id} \) and \( N = K^* \) otherwise.

*Proof:* For \( \sigma = \text{id} \) the assertion is clear. So let \( \sigma \neq \text{id} \) and \( n = a + b\varepsilon \in N \). Take an element \( x \in K \setminus F \). Using (2) we get \( n^{-1}xn = x + a^{-1}b(x - x^\sigma)\varepsilon \), which must belong to \( K \) since \( n \in N \). Because of our choice of \( x \) we have \( x - x^\sigma \neq 0 \), whence \( b = 0 \), as desired. \( \square \)

## 3 The associated DD

In this section we construct a 3-DD using the ring \( R = K(\varepsilon; \sigma) \). The construction is a special case of Spera’s construction method described in Section 1 (see also [8, Section 2.3]). On the other hand, the resulting DD is nothing else than the (generalized) chain geometry over \( (K, R) \) (compare [3], for details on ordinary chain geometries see [6], [9]).

From now on we assume that \( R \), and hence also \( K \) and \( F \), are finite. Then \( F = \text{GF}(m) \) for some prime power \( m \), and \( K = \text{GF}(q) \) with \( q \) a power of \( m \). Moreover, \( \sigma \) now is an automorphism of \( K \), namely, \( \sigma : x \mapsto x^m \).

The construction is based on the action of the group \( G = \text{GL}_2(R) \) of invertible \( 2 \times 2 \)-matrices with entries in \( R \) on the projective line over \( R \), i.e., on the set

\[
P(R) = \{R(a, b) \leq R^2 \mid \exists c, d \in R : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\}. \tag{5}\]

Since \( R \) is local, each pair \((a, b)\) as in (5) has the property that at least one of the two elements \( a, b \) is invertible, because otherwise the existence of an
inverse matrix \((\begin{pmatrix} x & y \\ y & x \end{pmatrix})\) would lead to the contradiction \(1 = ax + by \in I\). So \(\mathbb{P}(R)\) is the disjoint union

\[
\mathbb{P}(R) = \{R(x, 1) \mid x \in R\} \cup \{R(1, z) \mid z \in I\}.
\] (6)

On \(\mathcal{P} = \mathbb{P}(R)\) we have an equivalence relation \(\|\) given by

\[
R(a, b) \| R(c, d) : \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin G.
\] (7)

More explicitly, this means for arbitrary \(x, y \in R, z, w \in I\):

\[
R(1, z) \| R(1, w); \ R(x, 1) \not\| R(1, z); \ (R(x, 1) \| R(y, 1) \iff x - y \in I).
\] (8)

Using the description in (8) one can see that \(\|\) in fact is an equivalence relation.

Let us recall two facts (see [9, 1.2.2] and [9, Prop. 1.3.3], where non-parallel points are called “distant”: The group \(G\) acts on \(\mathcal{P}\) leaving \(\|\) invariant. Moreover, \(G\) acts transitively on the set of triples of pairwise non-parallel points of \(\mathcal{P}\). By virtue of this action of \(G\) and (8), any two parallel classes have the same cardinality \(s = |I|\).

In order to apply Spera’s method we now need a base block consisting of pairwise non-parallel points. As usual for chain geometries, we use the projective line over \(K\).

Since \(K\) is a subfield of \(R\), the projective line \(\mathbb{P}(K)\) can be seen as a subset \(B_0\) of \(\mathcal{P} = \mathbb{P}(R)\) as follows:

\[
B_0 = \mathbb{P}(K) = \{R(x, 1) \mid x \in K\} \cup \{R(1, 0)\}.
\] (9)

Let \(B = B_0^G\). Then we get the following.

**3.1 Theorem.** The structure \(\Sigma = (\mathcal{P}, B, \|)\) is a transversal 3-DD with parameters \(v = q^2 + q, s = q, k = q + 1 (= v/s)\) and

\[
\lambda_3 = \begin{cases} 
1 & \text{if } \sigma = \text{id,} \\
q & \text{if } \sigma \neq \text{id.}
\end{cases}
\]

**Proof:** From Spera’s theorem we know that \(\Sigma\) is a DD. The values of \(v, s,\) and \(k\) are obtained from (6), (8), and (9), respectively. By [3, Theorem 2.4], we have \(\lambda_3 = |R^*|/|N|\), where \(N\) is the normalizer defined in (4). By Lemma 2.2 we have two cases: If \(\sigma = \text{id}\), the normalizer \(N\) coincides with \(R^*\) and so \(\lambda_3 = 1\). If \(\sigma \neq \text{id}\), the normalizer \(N\) equals \(K^*\), whence \(\lambda_3 = |R^*|/|N| = (q - 1)q/(q - 1) = q\). □
The equation $\sigma = \text{id}$ holds precisely when $K$ lies in the centre of $R$; in this case our DD is an ordinary chain geometry, namely, the \textit{Miquelian Laguerre plane} over the algebra of dual numbers (see [1, I.2, II.4]).

We mention here that the parameter $\lambda_3$ could also be computed directly using the formula

$$\lambda_3 = \frac{|G|}{|G_{B_0}|} \cdot \frac{1}{s^3}$$

(see (1), note that $k = v/s$).

We add without proof that the 3-DD $\Sigma$ can also be described as a \textit{lifted DD} in the sense of [5, Theorem 2.5], using the point set $P$, the equivalence relation $\parallel$, the group $H = \left\{ \left( \begin{array}{cc} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{array} \right) \mid a, b, c, d \in K \right\}$ which acts on $P$, and the base block $B_0$ as (trivial) base DD. However, this alternative approach does not immediately show the large group of automorphisms given by the action of $G$ on $P$.

We now have a closer look at the case $\sigma \neq \text{id}$. We want to determine the $q$ blocks through three given pairwise non-parallel points more explicitly. Because of the transitivity properties of $G$ it suffices to consider the points $\infty = R(1, 0)$, $0 = R(0, 1)$, $1 = R(1, 1)$. From [3, Theorem 2.4] we know the following: The blocks through $\infty$, $0$, $1$ are exactly the images of $B_0$ under the group

$$\hat{R}^* = \left\{ \left( \begin{array}{cc} u & 0 \\ 0 & u \end{array} \right) \mid u \in R^* \right\},$$

(11)

and two elements $\omega = \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right)$ and $\omega' = \left( \begin{array}{cc} a' & 0 \\ 0 & a' \end{array} \right)$ of $\hat{R}^*$ determine the same block if, and only if, $Nu = Nu'$, with $N$ as in (4). So from Lemmas 2.2 and 2.1 we obtain:

\textbf{3.2 Lemma.} \textit{Let $\sigma \neq \text{id}$. Then the blocks containing $\infty = R(1, 0)$, $0 = R(0, 1)$, $1 = R(1, 1)$ are exactly the $q$ sets}

$$B_0', \text{ with } \omega = \left( \begin{array}{cc} 1 + b\epsilon & 0 \\ 0 & 1 + b\epsilon \end{array} \right), \quad b \in K.$$  

(12)

We now give an explicit description of the action of the group

$$\hat{U} = \left\{ \left( \begin{array}{cc} u & 0 \\ 0 & u \end{array} \right) \mid u \in U \right\} = \left\{ \left( \begin{array}{cc} 1 + b\epsilon & 0 \\ 0 & 1 + b\epsilon \end{array} \right) \mid b \in K \right\},$$

(13)

associated to $U$ (see (3)), on $P = \mathbb{P}(R)$.  

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A direct calculation shows that each $\omega = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$, with $u \in R^*$, acts on $P$ via “conjugation” as follows:

$$\omega : R(x, 1) \mapsto R(u^{-1}xu, 1), \quad R(1, z) \mapsto R(1, u^{-1}zu),$$

(14)

where, as before, $x \in R, z \in I$. For $u = 1 + b\varepsilon \in U$ this yields, using (2),

$$\omega : R(x, 1) \mapsto R(x + b(x_1 - x_1^\sigma)\varepsilon, 1), \quad R(1, z) \mapsto R(1, z),$$

(15)

where $x = x_1 + x_2\varepsilon$. So the mapping $\omega \in \hat{U}$ of (15) maps each point to a parallel one. Moreover, it fixes exactly those elements of the base block $B_0 = \mathbb{P}(K)$ that belong to the subset $\mathbb{P}(F)$. This subset in turn is the intersection of all blocks through $\infty, 0, 1$ (compare (12)); such intersections are also called traces (in German: “Fährten”, see [1], [3]).

We consider a parallel class on which $\hat{U}$ does not act trivially. By (15) this is the parallel class of some point $p = R(x_1, 1)$, where $x_1 \in K \setminus F$ and consequently $p \in B_0 \setminus \mathbb{P}(F)$. Then $\hat{U}$ acts regularly on the parallel class under consideration. As a matter of fact, for each $p'$ parallel to $p$, which has the form $p' = R(x_1 + x_2\varepsilon, 1)$, there is a unique $b \in K$ with $x_2 = b(x_1 - x_1^\sigma)$, so $p^\omega = p'$, with $\omega$ as in (15). This means that for each $p' \parallel p$ there is exactly one block through $\infty, 0, 1$ that contains $p'$ (and each block through $\infty, 0, 1$ is obtained in this way, as each block meets all parallel classes).

All these results can be carried over to an arbitrary triple of pairwise non-parallel points, using the action of $G$. So we have the following.

3.3 Proposition. Let $\sigma \neq \text{id}$. Let $p_1, p_2, p_3 \in P$ be pairwise non-parallel. Let $T$ be the intersection of all blocks through $p_1, p_2, p_3$, and let $C$ be a parallel class not meeting $T$. Then the following hold.

(a) There is a $g \in G$ such that $T = \mathbb{P}(F)^g$.

(b) Each block through $p_1, p_2, p_3$ meets $C$, and for each $x \in C$ there is a (unique) block through $p_1, p_2, p_3, x$.

3.4 Corollary. Let $p_1, p_2, p_3$ be pairwise non-parallel, let $T$ be the intersection of all blocks through $p_1, p_2, p_3$, and let $x \parallel p_1, p_2, p_3$. Then the number of blocks through $p_1, p_2, p_3, x$ is

- $q$, if $x \in T$,
- $0$, if $x \notin T$, but $x \parallel x'$ for some $x' \in T$,
- $1$, otherwise.
Finally, let us point out a particular case:

**3.5 Corollary.** Let \( q \) be even and let \( m = 2 \), i.e., \( x^q = x^2 \) for all \( x \in K \). Then \( \Sigma = (\mathcal{P}, \mathcal{B}, \|) \) is a 4-divisible design with parameter \( \lambda_4 = 1 \).

This result is immediate from Corollary 3.4, since \( F = GF(2) \) implies now \( |T| = |\mathbb{P}(F)| = 3 \).

## 4 A geometric model

Now we are looking for a geometric point model of the DD \( \Sigma \) defined above, i.e., a DD isomorphic to \( \Sigma \) whose points are points of a suitable projective space. We find such a model on the Klein quadric \( \mathcal{K} \) in \( PG(5, K) \) by using H. Hotje’s representation [10].

**4.1 Remark.** One could also first find a line model of \( \Sigma \) in \( PG(3, K) \) (where the points of \( \Sigma \) are certain lines in 3-space) and then apply the Klein correspondence. For details on such line models see [4], in particular Examples 5.2 and 5.4, and [2].

We embed the ring \( R = K(\varepsilon; \sigma) \) in the ring \( M = M(2, K) \) of \( 2 \times 2 \)-matrices with entries in \( K \) via the ring monomorphism

\[
    a + b\varepsilon \mapsto \begin{pmatrix} a & b \\ 0 & a^\sigma \end{pmatrix}.
\]

(16)

From now on we identify the ring \( R \) with its image under this embedding.

The projective line \( \mathbb{P}(M) \) is defined, mutatis mutandis, according to (5). The points of \( \mathbb{P}(M) \) are of the form \( M(A, B) \), where \((A, B)\) are the first two rows of an invertible \( 4 \times 4 \)-matrix over \( K \), because (up to notation) \( GL_2(M) \) equals \( GL_4(K) \). Then (16) allows to identify the point set \( \mathbb{P}(R) \) of \( \Sigma \) with a subset of \( \mathbb{P}(M) \).

Now we establish the existence of a bijection \( \Phi \) from \( \mathbb{P}(M) \) onto the Klein quadric \( \mathcal{K} \). For this we notice that \( M \) is a \( K \)-algebra, with \( K \) embedded in \( M \) via \( x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \), and that this algebra is kinematic, i.e., each element of \( M \) satisfies a quadratic equation over \( K \). Note that this embedding of \( K \) in \( M \) is different from the one obtained from (16), unless \( \sigma = id \). In [10] Hotje embeds the projective line over an arbitrary kinematic algebra in an appropriate quadric. For the matrix algebra \( M \) this quadric is \( \mathcal{K} \), and the embedding, which here is a bijection, is the following:

\[
    \Phi : \mathbb{P}(M) \to \mathcal{K} : M(A, B) \mapsto K(\bar{B}A, \det A, \det B),
\]

(17)
where \( A, B \) are matrices in \( M \), and for \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we set \( \tilde{B} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \).

The image of \( \Phi \) is indeed the Klein quadric, because \( M \times K \times K \) is a 6-dimensional vector space over \( K \) endowed with the hyperbolic quadratic form \((C, x, y) \mapsto \det C - xy\).

We need the following additional statements:

4.2 Proposition. Consider the bijection \( \Phi : \mathbb{P}(M) \to \mathcal{K} \) given in (17), and its restriction to \( \mathbb{P}(R) \). Then

(a) The bijection \( \Phi \) induces a homomorphism of group actions, mapping \( \text{GL}_2(M) \), acting on \( \mathbb{P}(M) \), to a subgroup of the group of collineations of \( \text{PG}(5, K) \) leaving \( \mathcal{K} \) invariant.

(b) This homomorphism maps the subgroup \( \text{GL}_2(R) \), acting on \( \mathcal{P} = \mathbb{P}(R) \), to a subgroup of the group of collineations of \( \text{PG}(5, K) \) leaving \( \mathcal{P}^\Phi \) invariant.

(c) Two points of \( \mathbb{P}(R) \) are parallel if, and only if, their \( \Phi \)-images are joined by a line contained in \( \mathcal{K} \).

Proof: For (a) see [10, (7.1/2/3)]; (b) follows from (a).

(c): This follows from [10, (7.5)] and [4, Prop. 3.2]. □

Writing \( K(x_1, x_2, x_3, x_4, x_5, x_6) \) instead of \( K\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}\right) \), we obtain by a direct computation that the mapping \( \Phi \) given in (17) acts on the points of \( \mathcal{P} = \mathbb{P}(R) \subseteq \mathbb{P}(M) \) as follows:

\[
R(a + b \epsilon, 1) \mapsto K(a, b, 0, a^\sigma, aa^\sigma, 1); \quad R(1, c \epsilon) \mapsto K(0, -c, 0, 0, 1, 0) \quad (18)
\]

We shall identify the elements of \( \mathbb{P}(M) \) with their \( \Phi \)-images. Then, in particular, we have

\[
B_0 = \{K(a, 0, 0, a^\sigma, aa^\sigma, 1) \mid a \in K\} \cup \{K(0, 0, 0, 0, 1, 0)\}. \quad (19)
\]

In the next lemma we collect some observations, which can be seen directly using (18) and (19).

4.3 Lemma. Let \( \mathcal{P} \) and \( B_0 \) be the point sets in \( \text{PG}(5, K) \) from above. Then the following hold:

(a) \( \mathcal{P} = \mathcal{C} \setminus \{S\} \), where \( \mathcal{C} \) is the cone with vertex \( S = K(0, 1, 0, 0, 0, 0) \) over \( B_0 \), i.e. the union of all lines joining \( S \) with \( B_0 \).
(b) $\mathcal{P}$ is entirely contained in the hyperplane $H$ with equation $x_3 = 0$, which is the tangent hyperplane to $\mathcal{K}$ at $S$.

(c) Two points of $\mathcal{P}$ are parallel if, and only if, they lie on a generator of $\mathcal{C}$, i.e. a line through $S$ contained in $\mathcal{C}$.

Now we describe the (image of) the base block $B_0$ more closely:

4.4 Lemma. Let $B_0$ be as in (19). Then the following hold:

(a) $B_0$ is a cap, i.e. a set of points no three of which are collinear.

(b) If $\sigma = \text{id}$, then $B_0$ is a regular conic; in particular, $B_0$ is contained in a plane.

(c) If $\sigma \neq \text{id}$, then $B_0$ spans the 3-space $U_0$, given by $x_2 = 0 = x_3$, complementary to $S$ in $H$.

Proof: (a): Assume that the line $L$ carries three points of $B_0$. Then $L \subseteq \mathcal{K}$. From Proposition 4.2(c) we see that the three points are pairwise parallel, a contradiction.

(b): Here $B_0 = \{K(a, 0, 0, a, a^2, 1) \mid a \in K\} \cup \{K(0, 0, 0, 1, 0)\}$, which obviously is a regular conic in the plane spanned by the points $K(1, 0, 0, 1, 0)$, $K(0, 0, 0, 1, 0)$, $K(0, 0, 0, 0, 1)$ (namely, the intersection of this plane with the Klein quadric).

(c): In this case, the four vectors

$$(0, 0, 0, 1, 0), \ (0, 0, 0, 0, 1), \ (1, 0, 0, 1, 1), \ \text{and} \ (a, 0, 0, a^{\sigma}, a a^{\sigma}, 1),$$

with $a \in K \setminus F$, are linearly independent, so the point set $B_0$ spans $U_0$. \qed

In case $\sigma = \text{id}$, our geometric model is nothing else than the “cylinder model” of the Miquelian Laguerre plane $\Sigma$: The points are the points of a cylinder in 3-space (a quadratic cone minus its vertex), and the blocks are the regular conics on the cylinder (the intersections with planes complementary to the vertex). See, e.g., [1, I.2] for the real case.

We have a closer look at the special case that $\sigma^2 = \text{id}$, $\sigma \neq \text{id}$. Then $q = m^2$ and $K$ is a quadratic extension of $F$. In this case there are Baer subspaces, i.e. spaces coordinatized by $F$, in each projective space over $K$.

4.5 Proposition. Let $\sigma^2 = \text{id}$, $\sigma \neq \text{id}$. Then $B_0$ is an elliptic quadric in the Baer subspace $\mathbb{B} \cong \text{PG}(3, m)$ of $U_0 \cong \text{PG}(3, q)$ defined by the $F$-subspace

$$\{(x, 0, 0, x^{\sigma}, f_1, f_2) \mid x \in K, f_i \in F\}.$$  \(20\)
Proof: Obviously, the set in (20) is a 4-dimensional subspace of $K^6$, seen as a vector space over $F$, satisfying the equations $x_2 = 0 = x_3$ and hence giving rise to a Baer subspace $B$ of $U_0$. The elements of $B_0$ all lie in $B$. Moreover, by (19), $B_0$ equals the quadric in $B$ determined by $N(x) = f_1f_2$, where $N(x) = xx^\sigma$ is the norm of $x \in K$ with respect to the field extension $K : F$ and, in particular, $N$ is a quadratic form on the vector space $FK$. Since $B_0$ is a cap by 4.4 (a), the quadric must be elliptic. □

The quadratic form used in the above is just the restriction to $B$ of the quadratic form describing the Klein quadric. The intersection of the Klein quadric and $U_0$ is a hyperbolic quadric.

For the rest of this section we consider the case that $\sigma \neq \text{id}$. We try to describe the geometric model of the DD $\Sigma$ more explicitly. From the above we know that our base block $B_0$ is a certain cap that spans a 3-space $U_0$ complementary to $S$ in the tangent hyperplane $H \cong PG(4, K)$ of $K$ at $S$. In the next proposition we describe all blocks. Together with Lemma 4.3 this gives a description of $\Sigma$ in terms of $PG(4, K)$.

4.6 Proposition. Let $\sigma \neq \text{id}$. Then the blocks of $\Sigma$ are exactly the intersections of the cone $C$ with the 3-spaces complementary to $S$ in $H$.

Proof: We know that $B_0 = C \cap U_0$, with $U_0$ complementary to $S$ in $H$. Let $B$ be any block. Then $B = B_0^g$ for some $g \in G = GL_2(R) \leq GL_2(M)$. By Proposition 4.2(b), $g$ induces a collineation, say $\tilde{g}$, of $PG(5, K)$ leaving $K$ and $P$ invariant. This collineation fixes $S$ (which is the intersection of the lines corresponding to parallel classes) and its tangent hyperplane $H$. So $B$, seen as a set of points in $H$, is $B = B_0^g = C \cap U_0^g$, where $U_0^g$ is a 3-space complementary to $S$, as desired. The 3-space $U_0^g$ is independent of the choice of $g$, as it is nothing else than the span of $B$.

So we have a mapping from the set of blocks to the set of complements of $S$ in $H$, which is injective since each complement contains exactly one point of each generator of $C$, i.e. of each parallel class of $P$, and hence cannot belong to more than one block. A simple counting argument shows that the mapping is also surjective: The number of blocks is $b = |G|/|G_{B_0}| = q^4$ (this can be computed directly, or from (10) using $\lambda_3 = q$), and the number of complements of $S$ in $H$ also is $q^4$, because they form an affine 4-space of order $|K| = q$. □

4.7 Remark. The projective model of $\Sigma$ studied in this section is a special case of the lifted $t$-DDs described in [5, Cor. 3.3]. There, the following geometries are described as $t$-DDs obtained via the lifting process: Consider an
arbitrary finite projective space $\text{PG}(n, q)$ and a set $B_0$ of $k$ points spanning a subspace $U_0$ and having the property that any $t$ points of $B_0$ are independent. Let $S$ be a complement of $B_0$. The point set of the $t$-DD is the cone with basis $B_0$ and vertex $S$, minus $S$. The blocks are the intersections of the cone with subspaces complementary to $S$, and two points are parallel if, and only if, together with $S$ they span the same subspace.

The following is an obvious geometric analogue of Proposition 3.3 and Corollary 3.4.

4.8 Corollary. Let $p_1, p_2, p_3$ be pairwise non-parallel, let $T$ be the intersection of all blocks through $p_1, p_2, p_3$, and let $x \parallel p_1, p_2, p_3$. Then

(a) $T$ is the intersection of the cone $C$ with the plane $E$ spanned by $p_1, p_2, p_3$.

(b) The blocks through $p_1, p_2, p_3, x$ are exactly the intersections of $C$ with 3-spaces through $E$ complementary to $S$. The number of such 3-spaces is

- $q$, if $x \in T$,
- $0$, if $x \notin T$, but $x \parallel x'$ for some $x' \in T$,
- $1$, otherwise.

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