Well-posedness, energy and charge conservation
for nonlinear wave equations in discrete space-time

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Abstract

We consider the problem of discretization for $U(1)$-invariant nonlinear wave equations in any dimension. We show that the classical finite-difference scheme used by Strauss and Vazquez [SV78] conserves the positive-definite discrete analog of the energy if the grid ratio is $dt/dx \leq 1/\sqrt{n}$, where $dt$ and $dx$ are the mesh sizes of the time and space variables and $n$ is the spatial dimension. We also show that if the grid ratio is $dt/dx = 1/\sqrt{n}$, then there is the discrete analog of the charge which is conserved.

We prove the existence and uniqueness of solutions to the discrete Cauchy problem. We use the energy conservation to obtain the a priori bounds for finite energy solutions, thus showing that the Strauss – Vazquez finite-difference scheme for the nonlinear Klein-Gordon equation with positive nonlinear term in the Hamiltonian is conditionally stable.

Keywords: Nonlinear wave equation, nonlinear Klein-Gordon equation, discrete space-time, finite-difference schemes, grid ratio, $U(1)$-invariance, energy conservation, charge conservation, a priori estimates.

1 Introduction

We study the $U(1)$-invariant nonlinear wave equation discretized in space and time. Our objective has been to find a stable finite-difference scheme for numerical simulation of the nonlinear wave processes, which corresponds to a well-posed Cauchy problem and provides us with the a priori energy bounds.

The discretized models are widely studied in applied mathematics and in theoretical physics. Such models originally appeared in the condensed matter theory, due to atoms in a crystal forming a lattice. Now these models occupy a prominent place in theoretical physics, in part due to some of these models (such as the Ising model) being exactly solvable. Lattice models are also used for the description of polymers.

The paper [SV78] set the ground for considering the energy-conserving difference schemes for the nonlinear Klein-Gordon equations and nonlinear wave equations. The importance of having conserved quantities in the numerical scheme was illustrated by noticing that instability occurs for a finite-difference scheme which does not conserve the energy. The authors gave the implicit difference scheme and wrote down the expression for the energy conserved by that scheme. This finite-difference scheme was favorably compared to three other schemes in [JV90]. The higher dimensional analog of the Strauss-Vazquez scheme and the corresponding energy-momentum tensor was written in [YHHH95]. The general theory of finite-difference schemes for the nonlinear Klein-Gordon equation aimed at the energy conservation was developed in [LVQ95] and [Fur01]. In the paper [CJ10] the energy preserving schemes are constructed for a wide class of second order nonlinear Hamiltonian systems of wave equations.

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Equation (2.1) can be written in the Hamiltonian form, with the Hamiltonian
\[ \psi \]
where \( \psi \) is the charge functional for the discrete nonlinear Klein-Gordon equation. Moreover, under the assumption that the conserved discrete energy is positive-definite, providing one with the a priori energy estimates in the case of the continuous limits of our discrete versions for the energy and charge coincide with the energy and charge in the continuous case.

Let us emphasize that the positive definiteness of the energy allows one to have the a priori bounds on the norm of the solution. Such a priori bounds are of utmost importance for applications. Numerically, such bounds indicate the stability of the finite-difference scheme.

The discrete charge conservation does not seem to be particularly important on its own, but could be considered as an indication that the positive definiteness of the energy allows one to have the a priori bounds on the norm of the solution. Such a priori bounds are of utmost importance for applications. Numerically, such bounds indicated the stability of the finite-difference scheme.

Importance of the Strauss – Vazquez finite-difference scheme over schemes from [LVQ95, Fur01] is in allowing for an easier solution algorithm. Namely, the discrete scheme involves the value of the unknown function at the “next” moment of time only at a single lattice point, and can be solved (numerically) with respect to the value at that point. See Remark 2.2 below. At the same time, the corresponding discrete energy contains quadratic terms which are not positive-definite, showing that the scheme is not unconditionally stable.

We use the same finite-difference scheme by Strauss and Vazquez [SV78]. We show that under the assumption that the energy is positive-definite, providing one with the a priori energy estimates in the case of the discrete nonlinear Klein-Gordon equation. Moreover, under the assumption that the energy is positive-definite, showing that the scheme is not unconditionally stable.

In the case of the discrete energy and charge is positive on its own, but could be considered as an indication that the positive definiteness of the energy allows one to have the a priori bounds on the norm of the solution. Such a priori bounds are of utmost importance for applications. Numerically, such bounds indicate the stability of the finite-difference scheme.

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In Section 2 we formulate our main results. We establish the existence of the solutions to the corresponding discrete Cauchy problem and analyze the uniqueness of solutions. We show that for confining polynomial nonlinearities the uniqueness will follow when the mesh size is sufficiently small. Besides, we describe a class of potentials the uniqueness will follow when the mesh size is sufficiently small. Besides, we describe a class of potentials the uniqueness will follow when the mesh size is sufficiently small. Besides, we describe a class of potentials the uniqueness will follow when the mesh size is sufficiently small.

## 2 Main results

### 2.1 Continuous case

Let us first consider the U(1)-invariant nonlinear wave equation
\[
\dot{\psi}(x, t) = \Delta \psi(x, t) - 2\partial_x v(x, |\psi(x, t)|^2)\psi(x, t), \quad x \in \mathbb{R}^n,
\]
where \( \psi(x, t) \in C^N, N \geq 1, \) and \( v(x, \lambda) \) is such that \( v \in C(\mathbb{R}^n \times \mathbb{R}) \) and \( v(x, \cdot) \in C^2(\mathbb{R}) \) for each \( x \in \mathbb{R}^n \). Equation (2.1) can be written in the Hamiltonian form, with the Hamiltonian
\[
\mathcal{H}(\psi, \dot{\psi}) = \int_{\mathbb{R}^n} \left[ \frac{\dot{\psi}^2}{2} + \frac{\nabla \psi^2}{2} + v(x, |\psi(x, t)|^2) \right] dx,
\]
where for \( \psi \in C^N \) we define \( |\psi|^2 = \bar{\psi} \cdot \psi \). The value of the Hamiltonian functional \( \mathcal{H} \) and the value of the charge functional
\[
\mathcal{Q}(\psi, \dot{\psi}) = \frac{i}{2} \int_{\mathbb{R}^n} (\bar{\psi} \cdot \dot{\psi} - \dot{\bar{\psi}} \cdot \psi) dx
\]
are formally conserved for solutions to (2.1). A particular case of (2.1) is the nonlinear Klein-Gordon equation, with \( v(x, \lambda) = \frac{m^2}{2} \lambda + z(x, \lambda) \), with \( m > 0 \):
\[
\dot{\psi} = \Delta \psi - m^2 \psi - 2\partial_x z(x, |\psi|^2)\psi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}.
\]
In the case of (2.4) the conservation of the energy \( \int_{\mathbb{R}^n} \left[ \frac{\dot{\psi}^2}{2} + \frac{\nabla \psi^2}{2} + \frac{m^2 |\psi|^2}{2} + z(x, |\psi|^2) \right] dx \) yields an a priori estimate on the norm of the solution:
\[
\int_{\mathbb{R}^n} |\psi(x, t)|^2 dx \leq \frac{2}{m^2} \mathcal{H}(\psi_{t=0}, \dot{\psi}_{t=0}).
\]
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2.2 Discretized equation

Let us now describe the discretized equation. Let \((X, T) \in \mathbb{Z}^n \times \mathbb{Z}\) denote a point of the space-time lattice. We will always indicate the temporal dependence by superscripts and the spatial dependence by subscripts. Fix \(\varepsilon > 0\), and let \(V_X(\lambda) = v(\varepsilon X, \lambda)\) be a function on \(\mathbb{Z}^n \times \mathbb{R}\), so that \(V_X \in C^2(\mathbb{R})\) for each \(X \in \mathbb{Z}^n\). We introduce

\[
B_X(\lambda, \mu) := \begin{cases} 
    \frac{V_X(\lambda) - V_X(\mu)}{\lambda - \mu}, & \lambda \neq \mu, \\
    \lambda \in \mathbb{R}, \quad \lambda \mu \in \mathbb{R}, \quad X \in \mathbb{Z}^n,
\end{cases}
\]

(2.6)

and consider the standard implicit finite-difference scheme for (2.1) [SV78]:

\[
\frac{\psi^{T+1}_X - 2\psi^T_X + \psi^{T-1}_X}{\tau^2} = \sum_{j=1}^n \psi^T_{X+e_j} - 2\psi^T_X + \psi^T_{X-e_j} - B_X(|\psi^{T+1}_X|^2, |\psi^{T-1}_X|^2)(\psi^{T+1}_X + \psi^{T-1}_X),
\]

(2.7)

where \(\psi^T_X \in \mathbb{C}^N, N \geq 1\), is defined on the lattice \((X, T) \in \mathbb{Z}^n \times \mathbb{Z}\). Above,

\[
e_1 = (1, 0, 0, 0, \ldots) \in \mathbb{Z}^n, \quad e_2 = (0, 1, 0, 0, \ldots) \in \mathbb{Z}^n, \quad \text{etc.}
\]

Remark 2.1. The continuous limit of (2.7) is given by (2.1), with \(\varepsilon X\) corresponding to \(x \in \mathbb{R}^n\) and \(\tau T\) corresponding to \(t \in \mathbb{R}\). Since \(\partial_x V_X(\lambda) = B_X(\lambda, \lambda)\), the continuous limit of the last term in the right-hand side of (2.7) coincides with the right-hand side in (2.1).

We assume that \(\psi^T_X\) takes values in \(\mathbb{C}^N\) with \(N \geq 1\).

Remark 2.2. An advantage of the Strauss-Vazquez finite-difference scheme (2.7) over other energy-preserving schemes discussed in [LVQ95, Fur01] is the fact that at the moment \(T + 1\) the relation (2.7) only involves the function \(\psi\) at the point \(X\), allowing for a simple realization of the solution algorithm even in higher dimensional case.

2.3 Well-posedness

We will denote by \(\psi_T^T\) the function \(\psi\) defined on the lattice \((X, T) \in \mathbb{Z}^n \times \mathbb{Z}\) at the moment \(T \in \mathbb{Z}\).

Theorem 2.3 (Existence of solutions). Assume that

\[
k_1 := \inf_{X \in \mathbb{Z}^n, \lambda \geq 0} \partial_\lambda V_X(\lambda) > -\infty.
\]

Define

\[
\tau_1 = \begin{cases} 
    \sqrt{-1/k_1}, & k_1 < 0; \\
    +\infty, & k_1 \geq 0.
\end{cases}
\]

Then for any \(\tau \in (0, \tau_1)\) and any \(\varepsilon > 0\) there exists a global solution \(\psi^T_T, T \in \mathbb{Z}\), to the Cauchy problem for equation (2.7) with arbitrary initial data \(\psi^0, \psi^1\) (which stand for \(\psi_T^T\) at \(T = 0\) and \(T = 1\)).

Moreover, if \((\psi^0, \psi^1) \in L^2(\mathbb{Z}^n) \times L^2(\mathbb{Z}^n)\), one has \(\psi^T_T \in L^2(\mathbb{Z}^n)\) for all \(T \in \mathbb{Z}\).

Remark 2.4. We do not claim in this theorem that \(\|\psi^T_T\|_{L^2(\mathbb{Z}^n)}\) is uniformly bounded for all \(T \in \mathbb{Z}\). For the a priori estimates on \(\|\psi^T_T\|_{L^2(\mathbb{Z}^n)}\), see Theorem 2.13 below.

One can readily check that any \(X\)-independent polynomial potential of the form

\[
V_X(\lambda) = V(\lambda) = \sum_{q=0}^p C_q \lambda^{q+1}, \quad C_q \in \mathbb{R}, \quad C_p > 0
\]

(2.10)

satisfies (2.9). Note that since \(\lim_{\lambda \to +\infty} V(\lambda) = +\infty\), this potential is confining.
Remark 2.5. Note that in the case of $V_X(\lambda)$ given by (2.10), by the little Bézout theorem, $B_X(\lambda, \mu)$ defined in (2.6) is a polynomial of $\lambda$ and $\mu$ with real coefficients.

Theorem 2.6 (Uniqueness of solutions). Assume that the functions $K^\pm_X(\lambda, \mu) = B_X(\lambda, \mu) + 2\partial_\lambda B_X(\lambda, \mu)(\lambda \pm \sqrt{\lambda \mu})$ are bounded from below:

$$k_2 := \inf_{\pm, X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} K^\pm_X(\lambda, \mu) > -\infty. \quad (2.11)$$

Define

$$\tau_2 = \begin{cases} \sqrt{-1/k_2}, & k_2 < 0; \\ +\infty, & k_2 \geq 0. \end{cases}$$

Then for any $\tau \in (0, \tau_2)$ and any $\varepsilon > 0$ there exists a solution to the Cauchy problem for equation (2.7) with arbitrary initial data $(\psi^0, \psi^1)$, and this solution is unique.

Remark 2.7. Note that since $K_X(\lambda, \lambda) = B_X(\lambda, \lambda) = \partial_\lambda W_X(\lambda)$, the values of $k_1$ and $k_2$ from Theorem 2.3 and Theorem 2.6, whether $k_2 > -\infty$, are related by $k_2 \leq k_1$, and then the values of $\tau_1$ and $\tau_2$ from these theorems are related by $\tau_2 \leq \tau_1$.

Theorem 2.8 (Existence and uniqueness for polynomial nonlinearities).

(i) The condition (2.11) holds for any confining polynomial potential (2.10).

(ii) Assume that

$$V_X(\lambda) = \sum_{q=0}^4 C_{X,q} \lambda^{q+1}, \quad X \in \mathbb{Z}^n, \quad \lambda \geq 0, \quad (2.12)$$

where $C_{X,q} \geq 0$ for $X \in \mathbb{Z}^n$ and $1 \leq q \leq 4$, and $C_{X,0}$ are uniformly bounded from below:

$$k_3 := \inf_{X \in \mathbb{Z}^n} C_{X,0} > -\infty. \quad (2.13)$$

Define

$$\tau_3 = \begin{cases} \sqrt{-1/k_3}, & k_3 < 0; \\ +\infty, & k_3 \geq 0. \end{cases}$$

Then for any $\tau \in (0, \tau_3)$ and any $\varepsilon > 0$ there exists a solution to the Cauchy problem for equation (2.7) with arbitrary initial data $(\psi^0, \psi^1)$, and this solution is unique.

Thus, even though the potential (2.10) satisfies conditions (2.9) and (2.11) in Theorem 2.3 and Theorem 2.6, the corresponding values $\tau_1$ and $\tau_2$ could be hard to specify explicitly. Yet, the second part of Theorem 2.8 gives a simple description of a class of $X$-dependent polynomials $V_X(\lambda)$ for which the range of admissible $\tau > 0$ can be readily specified.

We will prove existence and uniqueness results stated in Theorems 2.3, 2.6 and 2.8 in Section 3.

2.4 Energy conservation

Theorem 2.9 (Energy conservation). Let $\psi$ be a solution to equation (2.7) such that $\psi^T \in l^2(\mathbb{Z}^n)$ for all $T \in \mathbb{Z}$. Then the discrete energy

$$E^T = \sum_{X \in \mathbb{Z}^n} \left[ \left( \frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) \frac{|\psi_X^{T+1} - \psi_X^T|^2}{2} + \sum_{j=1}^n \sum_{\pm} \frac{|\psi_X^{T+1} - \psi_X^{T+1}|^2}{4\varepsilon^2} + \frac{V_X(|\psi_X^{T+1}|^2) + V_X(|\psi_X^T|^2)}{2} \right] \varepsilon^n \quad (2.14)$$

is conserved.
Remark 2.10. The discrete energy is positive-definite if the grid ratio satisfies

\[ \frac{\tau}{\varepsilon} \leq \frac{1}{\sqrt{n}}. \]

(2.15)

Remark 2.11. The continuous limit of the discrete energy \( E^T \) defined in (2.14) coincides with the energy functional (2.22) of the continuous nonlinear wave equation (2.1).

Remark 2.12. If \( \psi^0 \) and \( \psi^1 \in l^2(\mathbb{Z}^n) \), then, by Theorem 2.3, one also has \( \psi^T \in l^2(\mathbb{Z}^n) \) for all \( T \in \mathbb{Z} \) as long as

\[ \inf_{x \in \mathbb{Z}^n, \lambda \geq 0} \partial_x V_X(\lambda) > -\infty. \]

Proof. For any \( u, v \in \mathbb{C}^N \), there is the identity

\[ |u|^2 - |v|^2 = \text{Re} \ [(\bar{u} - \bar{v}) \cdot (u + v)]. \]

(2.16)

Applying (2.16), one has:

\[ \sum_{X \in \mathbb{Z}^n} (|\psi^{T+1}_X - \psi^{T}_X|^2 - |\psi^{T}_X - \psi^{T-1}_X|^2) = \text{Re} \ \sum_{X \in \mathbb{Z}^n} (\bar{\psi}^{T+1}_X - \bar{\psi}^{T-1}_X) \cdot (\psi^{T+1}_X - 2\psi^T_X + \psi^{T-1}_X). \]

(2.17)

Using (2.16), we also derive the following identity for any function \( \psi^T_X \in \mathbb{C}^N \):

\[ \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \left[ |\psi^{T+1}_X - \psi^{T}_X - \psi^{T-1}_X|^2 + |\psi^{T+1}_X - \psi^{T}_X + \psi^{T-1}_X|^2 \right] \]

\[ = \text{Re} \ \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \left[ (\bar{\psi}^{T+1}_X - \bar{\psi}^{T-1}_X) \cdot (\psi^{T+1}_X - 2\psi^T_X + \psi^{T-1}_X) + (\psi^{T+1}_X - \bar{\psi}^{T-1}_X) \cdot (\psi^{T+1}_X - 2\psi^T_X + \psi^{T-1}_X) \right] \]

\[ = \text{Re} \ \sum_{X \in \mathbb{Z}^n} \left[ 2n(\psi^{T+1}_X - 2\psi^T_X + \psi^{T-1}_X) - 2 \sum_{j=1}^n (\psi^{T+1}_X \pm \psi^T_X \pm \psi^{T-1}_X) \right]. \]

(2.18)

Further, (2.16) together with (2.16) imply that

\[ V_X(|\psi^{T+1}_X|^2) - V_X(|\psi^{T-1}_X|^2) = \text{Re} \ [(\bar{\psi}^{T+1}_X - \bar{\psi}^{T-1}_X) \cdot (\psi^{T+1}_X + \psi^{T-1}_X)] B_X(|\psi^{T+1}_X|^2, |\psi^{T-1}_X|^2). \]

(2.19)

Taking into account (2.17), (2.18), and (2.19), we compute:

\[ \frac{E^T - E^{T-1}}{\varepsilon^n} = \sum_{X \in \mathbb{Z}^n} \left[ \left( \frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) \frac{|\psi^{T+1}_X - \psi^T_X|^2 - |\psi^T_X - \psi^{T-1}_X|^2}{2} \right] \]

\[ + \sum_{j=1}^n \sum_{\pm} \left[ (\bar{\psi}^{T+1}_X - \bar{\psi}^{T-1}_X) \cdot \left( \left( \frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) \psi^{T+1}_X + 2\psi^T_X + \psi^{T-1}_X \right) \right] \]

\[ + \sum_{j=1}^n \left[ (\psi^{T+1}_X - 2\psi^T_X + \psi^{T-1}_X) - 2 \sum_{j=1}^n (\psi^{T+1}_X \pm \psi^T_X \pm \psi^{T-1}_X) \right]. \]

(2.20)

The expression in the square brackets adds up to zero due to (2.7). It follows that \( E^T = E^{T-1} \) for all \( T \in \mathbb{Z} \). □
2.5 A priori estimates

**Theorem 2.13** (A priori estimates). Assume that $\varepsilon > 0$ and $\tau > 0$ satisfy

$$\frac{\tau}{\varepsilon} \leq \frac{1}{\sqrt{n}}.$$  

Assume that

$$V_X(\lambda) = \frac{m^2}{2} \lambda + W_X(\lambda),$$  

where $m > 0$, and for each $X \in \mathbb{Z}^n$ the function $W_X \in C^2(\mathbb{R})$ satisfies $W_X(\lambda) \geq 0$ for $\lambda \geq 0$. Then any solution $\psi_X^T$ to the Cauchy problem (2.7) with arbitrary initial data $(\psi^0, \psi^1) \in l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n)$ satisfies the a priori estimate

$$\|\psi^T\|_{L^2}^2 \varepsilon^n \leq \frac{4\varepsilon^0}{m^2},$$

where $\varepsilon^0$ is the energy (2.14) of the solution $\psi_X^T$ at the moment $T = 0$.

**Proof.** This immediately follows from the conservation of the energy (2.14) with $V_X(\lambda)$ given by (2.20),

$$E^T = \sum_{X \in \mathbb{Z}^n} \left[ \left( \frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) |\psi_X^{T+1} - \psi_X^T|^2 \right] + \sum_{j=1}^n \left[ \frac{|\psi_X^{T+1} - \psi_X^T - \psi_{X-e_j}^T|^2 + |\psi_X^{T+1} - \psi_{X+e_j}^T|^2}{4\varepsilon^2} \right] + \sum_{X \in \mathbb{Z}^n} W_X(|\psi_X^{T+1}|^2) + W_X(|\psi_X^T|^2) \varepsilon^n.$$  



\[ \frac{\tau}{\varepsilon} = \frac{1}{\sqrt{n}}. \]  

**Remark 2.14.** In the continuous limit $\varepsilon \to 0$, the relation (2.21) is similar to the a priori estimate (2.5) for the solutions to the continuous nonlinear Klein-Gordon equation (2.4).

**Remark 2.15.** In [SV78], in the case $\psi_X^T \in \mathbb{R}$, $(X, T) \in \mathbb{Z} \times \mathbb{Z}$ (in the dimension $n = 1$), the following expression for the discretized energy was introduced:

$$E^T_{SV} = \frac{1}{2} \sum_{X \in \mathbb{Z}^n} \left[ \left( \frac{\psi_X^{T+1} - \psi_X^T}{\tau} \right)^2 + \frac{(\psi_X^{T+1} - \psi_X^{T-1})(\psi_X^{T+1} - \psi_X^T)}{\varepsilon^2} + V(|\psi_X^{T+1}|^2) + V(|\psi_X^T|^2) \right].$$  

The presence of the second term which is not positive-definite deprives one of the a priori $l^2$ bound on $\psi$, such as the one stated in Theorem 2.13. In view of this, the Strauss-Vazquez finite-difference scheme for the nonlinear Klein-Gordon equation is not unconditionally stable. Other schemes (conditionally and unconditionally stable) were proposed in [LVQ95, Fur01]. Now, due to the a priori bound (2.21), we deduce that, as the matter of fact, the Strauss-Vazquez scheme is stable in $n$ dimensions under the condition that the grid ratio is $\tau/\varepsilon \leq 1/\sqrt{n}$. Note that in the case $\psi \in \mathbb{R}$, the Strauss-Vazquez energy (2.22) agrees with the energy defined in (2.14).

2.6 The charge conservation

Let us consider the charge conservation. We will define the discrete charge under the following assumption:

**Assumption 2.16.**

$$\frac{\tau}{\varepsilon} = \frac{1}{\sqrt{n}}.$$  

Under Assumption 2.16 $\psi_X^T$ drops out of equation (2.7); the latter can be written as

$$\left( \psi_X^{T+1} + \psi_X^{T-1} \right) \left( 1 + \frac{\tau^2}{\varepsilon^2} B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2) \right) = \frac{1}{n} \sum_{j=1}^n \left( \psi_X^{T+1} + \psi_X^{T-1} \right).$$  

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**Theorem 2.17** (Charge conservation). Let Assumption [2.16] be satisfied. Let $\psi$ be a solution to equation (2.24) such that $\psi_T \in \ell^2(\mathbb{Z}^n)$ for all $T \in \mathbb{Z}$ (see Theorem 2.3). Then the discrete charge

\[
Q^T = \frac{i}{4\tau} \sum_{X \in \mathbb{Z}^n} \left[ \bar{\psi}_{X+e_j} \cdot \psi_{X+e_j}^T + \bar{\psi}_{X-e_j} \cdot \psi_{X-e_j}^T - \psi_{X+e_j} \cdot \psi_{X+e_j}^T - \psi_{X-e_j} \cdot \psi_{X-e_j}^T \right] e^n
\]

is conserved.

**Remark 2.18.** The continuous limit of the discrete charge $Q$ defined in (2.25) coincides with the charge functional (2.3) of the continuous nonlinear wave equation (2.1).

**Proof.** Let us prove the charge conservation. One has:

\[
\frac{4\tau}{i\varepsilon^n} Q^T = \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \left[ \bar{\psi}^T_{X\pm e_j} \cdot \psi^T_{X\pm e_j} - C. \right],
\]

\[
\frac{4\tau}{i\varepsilon^n} Q^{T-1} = \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \left[ \bar{\psi}^{T-1}_{X\pm e_j} \cdot \psi^{T-1}_{X\pm e_j} - C. \right] = - \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \left[ \bar{\psi}^T_{X\pm e_j} \cdot \psi^T_{X\pm e_j} - C. \right].
\]

Therefore,

\[
\frac{4\tau}{i\varepsilon^n} (Q^T - Q^{T-1}) = \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \bar{\psi}^T_{X\pm e_j} \cdot (\psi^T_{X\pm e_j} + \psi^{T-1}_{X\pm e_j}) - C. - \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \bar{\psi}^T_{X\pm e_j} \cdot \psi^T_{X\pm e_j} - C. = 0.
\]

To get to the second line, we used the complex conjugate of (2.24). This finishes the proof of Theorem 2.17

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**3 Proof of well-posedness and uniqueness results**

**Proof of Theorem 2.3** We rewrite equation (2.7) in the following form:

\[
(\psi_{X+1}^T + \psi_{X-1}^T) (1 + \tau^2 B_X(|\psi_{X+1}^T|^2, |\psi_{X-1}^T|^2)) = \frac{\tau^2}{\varepsilon^2} \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_{X+e_j}^T + \psi_{X-e_j}^T + 2\psi_{X}^T), \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}.
\]

(3.1)

By (2.9) and the choice of $\tau_1$ in Theorem 2.3 for $\tau \in (0, \tau_1)$ one has

\[
\inf_{X \in \mathbb{Z}^n, \lambda \geq 0} \left( 1 + \tau^2 \partial_\lambda V_X(\lambda) \right) > 0.
\]

(3.2)

Since

\[
\inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} B_X(\lambda, \mu) = \inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \neq \lambda} \frac{V_X(\lambda) - V_X(\mu)}{\lambda - \mu} = \inf_{X \in \mathbb{Z}^n, \lambda \geq 0} \partial_\lambda V_X(\lambda),
\]

(3.3)

inequality (3.2) yields

\[
c := \inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} \left( 1 + \tau^2 B_X(\lambda, \mu) \right) > 0.
\]

(3.4)

Let us show that equation (3.1) allows us to find $\psi_{X+1}^T$, for any given $X \in \mathbb{Z}^n$ and $T \in \mathbb{Z}$, once one knows $\psi_T^T$ and $\psi_T^{T-1}$. Equation (3.1) implies that

\[
(1 + \tau^2 B_X(|\psi_{X+1}^T|^2, |\psi_{X-1}^T|^2)) (\psi_{X+1}^T + \psi_{X-1}^T) = \xi_X^T,
\]

(3.5)

\[
\xi_X^T := \frac{\tau^2}{\varepsilon^2} \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_{X+e_j}^T + \psi_{X-e_j}^T + 2\psi_{X}^T) \in \mathbb{C}^N.
\]

(3.6)
If $\xi^T_X = 0$, then there is a solution to (3.5) given by $\psi^{T+1}_X = -\psi^{T-1}_X$. Due to (3.4), this solution is unique. Now let us assume that $\xi^T_X \neq 0$. We see from (3.5) that we are to have
\[ \psi^{T+1}_X + \psi^{T-1}_X = s\xi^T_X, \quad \text{with some } s \in \mathbb{R}. \] (3.7)

Let us introduce the function
\[ f(s) := (1 + \tau^2 B_X(|s\xi^T_X - \psi^{T-1}_X|^2, |\psi^{T-1}_X|^2))s. \] (3.8)

We do not indicate dependence of $f$ on $\psi^{T-1}_X, \xi^T_X,$ and $X$, treating them as parameters. For $\xi^T_X \neq 0$, we can solve (3.5) if we can find $s \in \mathbb{R}$ such that
\[ f(s) = 1. \] (3.9)

Since $f(0) = 0$, while $\lim_{s \to \infty} f(s) = +\infty$ by (3.4), one concludes that there is at least one solution $s > 0$ to (3.9).

Let us prove that once $(\psi^0, \psi^1) \in L^2(\mathbb{Z}^n) \times L^2(\mathbb{Z}^n)$, then one also knows that $||\psi^T||_{L^2(\mathbb{Z}^n)}$ remains finite (but not necessarily uniformly bounded) for all $T \in \mathbb{Z}$. As it follows from (3.4) and (3.5),
\[ |\psi^{T+1}_X| \leq \frac{1}{e} |\xi^T_X| + |\psi^{T-1}_X|. \] (3.10)

Since $||\xi^T||_{L^2(\mathbb{Z}^n)} \leq \left( \frac{4\tau^2}{\varepsilon^2} + 2 \right) ||\psi^T||_{L^2(\mathbb{Z}^n)}$ by (3.6), the relation (3.10) implies the estimate
\[ ||\psi^{T+1}||_{L^2(\mathbb{Z}^n)} \leq \frac{1}{e} \left( \frac{4\tau^2}{\varepsilon^2} + 2 \right) ||\psi^T||_{L^2(\mathbb{Z}^n)} + ||\psi^{T-1}||_{L^2(\mathbb{Z}^n)}, \] (3.11)

and, by recursion, the finiteness of $||\psi^T||_{L^2(\mathbb{Z}^n)}$ for all $T \geq 0$. The case $T \leq 0$ is finished in the same way.

Now we turn to the uniqueness of solutions to the Cauchy problem for equation (2.7).

**Proof of Theorem 2.6** First, note that, by Remark 2.7, $\tau_1$ from Theorem 2.3 and $\tau_2$ from Theorem 2.6 are related by $\tau_2 \leq \tau_1$. Therefore, the existence of a solution $\psi^T_X$ to the Cauchy problem for equation (2.7) follows from Theorem 2.3.

Let us prove that this solution $\psi^T_X$ is unique. When in (3.6) one has
\[ \xi^T_X := \frac{\tau^2}{\varepsilon^2} \sum_{j=1}^n (\psi^T_{X+e_j} - 2\psi^T_X + \psi^T_{X-e_j}) + 2\psi^T_X = 0, \]
then, by (3.4), the only solution $\psi^{T+1}_X$ to (3.5) is given by $\psi^{T+1}_X = -\psi^{T-1}_X$. We now consider the case $\xi^T_X \neq 0$. By (3.5), (3.7), and (3.8), it suffices to prove the uniqueness of the solution to (3.9). This will follow if we show that $f(s)$ satisfies
\[ f'(s) > 0, \quad s \in \mathbb{R}. \] (3.12)

The explicit expression for $f'(s)$ is
\[ 1 + \tau^2 B_X(|s\xi^T_X - \psi^{T-1}_X|^2, |\psi^{T-1}_X|^2) + \tau^2 \partial_X B_X(|s\xi^T_X - \psi^{T-1}_X|^2, |\psi^{T-1}_X|^2) (-2 \text{Re}(\psi^{T-1}_X \cdot \xi^T_X) + 2|\xi^T_X|^2) s. \] (3.13)

Using the relation (3.7), we derive the identity
\[ (-2 \text{Re}(\psi^{T-1}_X \cdot \xi^T_X) + 2|\xi^T_X|^2) s = 2|s\xi^T_X - \psi^{T-1}_X|^2 - |\psi^{T-1}_X|^2 = 2|\psi^{T+1}_X + \psi^{T-1}_X|^2 - |\psi^{T-1}_X|^2 \]
and rewrite the expression (3.13) for $f'(s)$ as
\[ f'(s) = 1 + \tau^2 \left[ B_X(|\psi^{T+1}_X|^2, |\psi^{T-1}_X|^2) + 2\partial_X B_X(|\psi^{T+1}_X|^2, |\psi^{T-1}_X|^2) \left( |\psi^{T+1}_X + \psi^{T-1}_X|^2 - |\psi^{T-1}_X|^2 \right) \right]. \] (3.14)

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We denote \( \lambda = |\psi_{X}^{T+1}|^2, \mu = |\psi_{X}^{T-1}|^2 \). Since \( \lambda - \sqrt{\lambda \mu} + \frac{\tau}{4} \leq |\psi_{X}^{T+1} + \psi_{X}^{T-1}|^2 \leq \lambda + \sqrt{\lambda \mu} + \frac{\tau}{4} \), we see that

\[
f'(s) \geq 1 + \tau^2 \min_{x \in \mathbb{Z}_n} \inf_{\lambda, \mu \geq 0} K_{X}^\tau(\lambda, \mu), \quad \text{with} \quad K_{X}^\tau(\lambda, \mu) = B_{X}(\lambda, \mu) + 2\partial_{\lambda}B_{X}(\lambda, \mu)(\lambda \pm \sqrt{\lambda \mu}).
\]  

(3.15)

By (2.11) and by our choice of \( \tau_2 \) in Theorem 2.6, for any \( \tau \in (0, \tau_2) \) we have

\[
\kappa := \inf_{x \in \mathbb{Z}_n, \lambda, \mu \geq 0} \{1 + \tau^2 K_{X}^\tau(\lambda, \mu)\} > 0;
\]

then, by (3.15), \( f'(s) \geq \kappa \), where \( \kappa > 0 \). It follows that for \( \xi_{X}^T \neq 0 \) there is a unique \( s \) which solves (3.9). Hence, there is a unique solution \( \psi_{X}^{T+1} \) to equation (3.5) for given values \( \psi_{X}^{T-1} \) and \( \xi_{X}^T \). This finishes the proof of the theorem.

**Proof of Theorem 2.8** Let us prove that the condition (2.11) in Theorem 2.6 is satisfied by any polynomial potential of the form (2.10). The inequality (2.11) will be satisfied if the highest order term from \( V(\lambda) \) contributes a strictly positive expression. More precisely, we need to prove the following result.

**Lemma 3.1.** Let \( V(\lambda) = \lambda^{p+1} \), so that \( B(\lambda, \mu) = \frac{\lambda^{p+1} - \mu^{p+1}}{\lambda - \mu}, p \geq 0 \). Then the following inequality takes place:

\[
\inf_{\lambda, \mu \geq 0, \lambda^2 + \mu^2 \geq 1} [B(\lambda, \mu) + 2\partial_{\lambda}B(\lambda, \mu)(\lambda \pm \sqrt{\lambda \mu})] > 0.
\]  

(3.16)

**Proof.** Since \( B \) and \( \partial_{\lambda}B \) are strictly positive for \( \lambda^2 + \mu^2 > 0 \), the inequality (3.16) is nontrivial only for the negative sign in (3.16) and only when \( \mu > \lambda \). First we note that

\[
B(\lambda, \mu) = \frac{\mu^{p+1} - \lambda^{p+1}}{\mu - \lambda}, \quad \partial_{\lambda}B(\lambda, \mu) = \frac{-(p + 1)\lambda^p(\mu - \lambda) - \lambda^{p+1} + \mu^{p+1}}{(\mu - \lambda)^2} = \frac{\mu^{p+1} - (p + 1)\lambda^p\mu + p\lambda^{p+1}}{(\mu - \lambda)^2}.
\]

Let \( z \geq 0 \) be such that \( z^2 = \lambda/\mu \). To prove the lemma, we need to check that

\[
1 - \frac{z^{2p+2}}{1 - z^2} + 2 \frac{1 - (p + 1)z^{2p} + pz^{2p+2}}{(1 - z^2)^2}(z^2 - z) > 0, \quad 0 \leq z < 1,
\]  

(3.17)

or equivalently,

\[
(1 + z)(1 - z^{2p+2}) - 2z(1 - (p + 1)z^{2p} + pz^{2p+2}) > 0.
\]

The left-hand side takes the form

\[
(1 + z)(1 - z^{2p+2}) - 2z(1 - z^{2p+2} - (p + 1)(z^{2p} - z^{2p+2})) = (1 - z) - (z^{2p+2}) + 2z(p + 1)(z^{2p} - z^{2p+2}),
\]

which is clearly strictly positive for all \( 0 \leq z < 1 \) and \( p \geq 0 \), proving (3.17).

This finishes the proof of the first part of Theorem 2.8; now we turn to the second part.

**Lemma 3.2** (Uniqueness criterion). Assume that for a particular \( \tau > 0 \) and for all \( \lambda \geq 0, \mu \geq 0, X \in \mathbb{Z}_n \), the following inequalities hold:

\[
1 + \tau^2 \inf_{x \in \mathbb{Z}_n, \lambda, \mu \geq 0} \left( B_{X}(\lambda, \mu) - \partial_{\lambda}B_{X}(\lambda, \mu)\frac{\mu}{2} \right) > 0; \tag{3.18}
\]

\[
\inf_{x \in \mathbb{Z}_n, \lambda, \mu \geq 0} \partial_{\lambda}B_{X}(\lambda, \mu) \geq 0. \tag{3.19}
\]

Then the is a solution \( \psi_{X}^{T} \) to the Cauchy problem for equation (2.7) with arbitrary initial data \( (\psi^{0}, \psi^{1}) \), and this solution is unique.
Proof of Lemma 3.2. The inequalities (3.18) and (3.19) lead to

$$1 + \tau^2 \inf_{X \in \mathbb{Z}^n, \lambda \geq 0} B_X(\lambda, \lambda) > 0,$$

hence, by the same argument as in Theorem 2.3, there is a solution $\psi^T_X$. The relation (3.14) shows that $f'(s) \geq c$ for some $c > 0$. The rest of the proof is the same as for Theorem 2.6.

In the second part of Theorem 2.8 we assume that

$$V_X(\lambda) = \sum_{q=0}^{4} C_{X,q} \lambda^{q+1}, \quad X \in \mathbb{Z}^n, \quad \lambda \geq 0,$$

where $C_{X,q} \geq 0$ for $X \in \mathbb{Z}^n$ and $1 \leq q \leq 4$, and

$$k_3 = \inf_{X \in \mathbb{Z}^n} C_{X,0} > -\infty.$$

One can see that the term $C_{X,0}\lambda$ in $V_X(\lambda)$ contributes to $B_X(\lambda, \mu)$ the expression $b_{X,0}(\lambda, \mu) = C_{X,0}$, while each term in $V_X(\lambda)$ of the form $C_{X,q} \lambda^{q+1}$, with $1 \leq q \leq 4$ and $C_{X,q} \geq 0$, contributes to $B_X(\lambda, \mu)$ the expression $C_{X,q} b_q(\lambda, \mu)$, with $b_q(\lambda, \mu) = \sum_{k=0}^{q} \lambda^{q-k} \mu^k$. For $\tau \in (0, \tau_3)$, with $\tau_3 = \sqrt{-1/k_3}$ for $k_3 < 0$ and $\tau_3 = +\infty$ for $k_3 \geq 0$, one has

$$1 + \tau^2 \inf_{X \in \mathbb{Z}^n} C_{X,0} > 0.$$

Lemma 3.3. For $1 \leq q \leq 4$, $b_q(\lambda, \mu) = \sum_{k=0}^{q} \lambda^{q-k} \mu^k$ satisfies the inequality

$$b_q(\lambda, \mu) \geq \partial_{\lambda} b_q(\lambda, \mu) \frac{\mu}{2} \quad \text{for all } \lambda, \mu \geq 0.$$

By (3.22) and Lemma 3.3 condition (3.18) is satisfied. Since $C_{X,q} \geq 0$ for $1 \leq q \leq 4$, each term $C_{X,q} b_q(\lambda, \mu)$ satisfies condition (3.19). Therefore, by Lemma 3.2, there is a unique solution $\psi^T_X$ to the Cauchy problem for equation (2.7). This finishes the proof of Theorem 2.8.

4 Conclusion

We found out that the Strauss – Vazquez finite-difference scheme [SV78] for the $U(1)$-invariant nonlinear wave equation in $n$ spatial dimensions, with the grid ratio $\tau/\varepsilon \leq 1/\sqrt{n}$, admits the positive-definite discrete analog of the energy which is conserved. The result holds in any spatial dimension $n \geq 1$, for the field valued in $\mathbb{C}^N$, $N \geq 1$. In the case of the nonlinear Klein-Gordon equation with positive potential, this provides a priori bounds for the solution, showing that the finite-difference scheme is stable.

We found out that if the grid ratio is $\tau/\varepsilon = 1/\sqrt{n}$, then this finite-difference scheme also preserves the discrete charge.

We proved that the solution of the corresponding Cauchy problem exists and is unique for a broad class of nonlinearities. In particular, this is the case for any confining polynomial potential if the discretization is sufficiently small. Finally, we indicated a class of polynomials for which the size of the discretization could be readily specified.

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