INTEGRABLE STRUCTURES FOR A GENERALIZED MONGE–AMPÈRE EQUATION

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We consider a third-order generalized Monge–Ampère equation

\[ u_{yyy} - u_{xxy}^2 + u_{xxx}u_{xyy} = 0, \]

which is closely related to the associativity equation in two-dimensional topological field theory. We describe all integrable structures related to it: Hamiltonian, symplectic, and also recursion operators. We construct infinite hierarchies of symmetries and conservation laws.

Keywords: Monge–Ampère equation, integrability, Hamiltonian operator, symplectic structure, symmetry, conservation law, jet space, WDVV equation, two-dimensional topological field theory

1. Introduction

The Monge–Ampère equations [1] are the most interesting objects of the applications of the geometric theory of differential equations. Generalizations of the classical Monge–Ampère equations are discussed, for example, in [2], and one such generalization is the equation

\[ u_{yyy} - u_{xxy}^2 + u_{xxx}u_{xyy} = 0. \]  \hspace{1cm} (1)

This equation is closely related to the associativity equation arising in two-dimensional topological field theory [3] and has been studied in many papers (see, e.g., [4]–[8]), where its integrability was established (in the sense of the existence of a bi-Hamiltonian structure). We note that in all these papers, the equation was considered not in its initial form (1) but in the form of a three-component system of hydrodynamic type:

\[ a_y = b_x, \quad b_y = c_x, \quad c_y = (b^2 - ac)_x. \]  \hspace{1cm} (2)

Of course, Eqs. (1) and (2) are closely related, but they are not identical and even not equivalent, being associated by the differential substitution \( a = u_{xxx}, \ b = u_{xxy}, \) and \( c = u_{xyy} \) (similarly to how the KdV and mKdV equations are related by the Miura map or the Burgers and heat equations are related by the Cole–Hopf transformation).

Our aim here is to investigate Eq. (1) directly without reducing it to the evolutionary form and to study the structures that arise on this equation. For this, we use geometric and cohomological methods described initially in [9] and discussed in detail in [10]. These methods have been successfully used to analyze several differential equations (see, e.g., [11], [12]).

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Prepared from an English manuscript submitted by the authors; for the Russian version, see Teoreticheskaya i Matematicheskaya Fizika, Vol. 171, No. 2, pp. 208–224, May, 2012.
In Sec. 2, we briefly recall basic notions from the geometry of jet spaces and infinitely prolonged equations. Section 3 contains the main results obtained for Monge–Ampère equation (1) (including a description of the Hamiltonian, symplectic, and recursion operators and also hierarchies of symmetries and conservation laws). In particular, we show that Eq. (1) admits a symplectic structure of the form $D_x$ (this is the only local operator among those responsible for the integrability of the equation, and it corresponds to the symplectic structure described in [13], [14]). A nonlocal Hamiltonian structure $D^{-1}$ corresponds to this operator. The other operators are quite complicated and are described in Secs. 3.4.1, 3.4.2, 3.5.1 and 3.5.2.

All computations were done using CDIFF, a specialized package of programs for computations in the geometry of differential equations, written in the program language REDUCE (see http://gdeq.org).

2. Theoretical background

2.1. Jets and equations. We recall that the geometric approach to nonlinear partial differential equations [15] consists in regarding an equation $E$ together with all its prolongations (i.e., differential consequences) as a submanifold in the manifold $J^\infty(\pi)$ of infinite jets of a locally trivial bundle $\pi: E \to M$, where $M$ and $E$ are smooth manifolds of the respective dimensions $n$ and $n + m$.

The manifold $M$ “contains” independent variables, and sections of $\pi$ play the role of unknown functions (fields) in $E$. If $U \subset M$ is a coordinate neighborhood such that $\pi|_U$ is trivial, then we choose local coordinates $x^1, \ldots, x^n$ in $U$ and $u^1, \ldots, u^m$ in the fiber of $\pi|_U$. The corresponding adapted coordinates $u^j_\sigma$, where $\sigma$ is a multi-index, then arise naturally in $J^\infty(\pi)$ and are defined as follows. If $f = (f^1, \ldots, f^m)$ is a local section, then we set

$$f^j(u^j_\sigma) = \frac{\partial |\sigma| f^j}{\partial x^\sigma}.$$ 

Functions on $J^\infty(\pi)$ can depend on $x^i$ and only a finite number of $u^j_\sigma$.

The vector fields

$$D_i = \frac{\partial}{\partial x^i} + \sum_{j, \sigma} u^j_\sigma \frac{\partial}{\partial u^j_\sigma}, \quad i = 1, \ldots, n,$$

are called total derivatives, and differential operators in total derivatives are called $\mathcal{C}$-differential operators.

If an equation is given by the system $F = 0$, where $F = (F^1, \ldots, F^r)$ is a vector function on $J^\infty(\pi)$, then its infinite prolongation $E$ is given by $D_\sigma(F) = 0$, $\sigma \geq 0$, where $D_\sigma = D_{\sigma_1} \circ \cdots \circ D_{\sigma_s}$ for $\sigma = \sigma_1 \ldots \sigma_s$. Total derivatives can be restricted to $E$ (we preserve the same notation for these restrictions) and generate the Cartan distribution $\mathcal{C}$. This distribution is integrable in the formal sense, i.e., $[X, Y] \in \mathcal{C}$ for any $X, Y \in \mathcal{C}$, and its $n$-dimensional integral manifolds are solutions of $E$.

2.2. Symmetries. Let $\pi_\infty: E \to M$ denote the natural projection. A $\pi_\infty$-vertical vector field $X$ on $E$ is called a symmetry of $E$ if it preserves the Cartan distribution, i.e., if $[X, \mathcal{C}] \subset \mathcal{C}$. Every symmetry of $E$ has the form

$$E_\varphi = \sum D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma},$$

where the summation ranges the internal coordinates on $E$ and the vector function $\varphi = (\varphi^1, \ldots, \varphi^m)$ satisfies the equation $\ell_E(\varphi) = 0$. Here, $\ell_E$ is the linearization operator of the vector function $F$ restricted to $E$. More precisely, we take the functions $F^\alpha$ that define the equation $E$ and construct the matrix
The group $E_{p,q-1}(\mathcal{E})$ plays a special role in the theory. Its elements are called the conservation laws of $\mathcal{E}$, and the group itself is denoted by $\text{Cl}(\mathcal{E})$. We also need the group $E_{1,n-1}^1$, whose elements are called cosymmetries and which is denoted by $\text{cosym}(\mathcal{E})$.  

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We need an additional construction in what follows. Let $P$ and $Q$ be spaces of sections of vector bundles over $E$. Let $\hat{P} = \text{Hom}(P, \Lambda^n(E))$ and similarly for $\hat{Q}$. Then for any $\mathcal{C}$-differential operator $\Delta: P \to Q$, its formal adjoint $\Delta^*: \hat{Q} \to \hat{P}$ is defined by the Green formula

$$\langle \Delta^*(\hat{q}), p \rangle - \langle \hat{q}, \Delta(p) \rangle = d_h \omega(\hat{q}, p),$$

where $\omega: \hat{Q} \times P \to \Lambda^{n-1}(E)$ is a map that is a $\mathcal{C}$-differential operator in both arguments and $\langle \cdot, \cdot \rangle$ denotes the natural pairing. If $\Delta$ is given by the matrix $(\Delta_{ij})$, where $\Delta_{ij} = \sum_\alpha a_{ij}^\alpha D_\alpha$, then $\Delta^* = (\Delta_{ji}^*)$, where

$$\Delta^*_{ji} = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_{ji}^\sigma.$$

Everywhere below, we assume that the considered equation satisfies the following conditions:

1. The differentials $dF^i$ of the functions that define $E$ are linearly independent at all points of $E$.
2. If $\Delta$ is a $\mathcal{C}$-differential operator such that $\Delta \circ \ell_E = 0$, then $\Delta = 0$.
3. If $\Delta$ is a $\mathcal{C}$-differential operator such that $\Delta \circ \ell^*_E = 0$, then $\Delta = 0$.

If an equation satisfies these conditions, then the following statements hold:

1. The differential $\delta: \text{Cl}(E) \to \text{cosym}(E)$ is monomorphic, i.e., $\delta(\omega) = 0$ if and only if $\omega = 0$.
2. The group of cosymmetries coincides with the kernel of $\ell^*_E$, i.e., $\psi \in \text{cosym}(E)$ if and only if $\ell^*_E(\psi) = 0$.

If $\omega \in \text{Cl}(E)$ is a conservation law, then the cosymmetry $\delta(\omega)$ is called its generating function (or generating section).

### 2.4. Differential coverings

Let $E$ and $\hat{E}$ be two equations. A smooth map $\tau: \hat{E} \to E$ is called a morphism if it takes the Cartan distribution on $\hat{E}$ to the corresponding distribution on $E$. A surjective morphism $\tau$ is said to be a covering if for any point $\theta \in \hat{E}$, the differential $d\tau|_\theta$ maps the Cartan plane $\mathcal{C}_\theta(\hat{E})$ to $\mathcal{C}_{\tau(\theta)}(E)$ isomorphically. Coordinates along the fibers of $\tau$ are called nonlocal variables in the covering under consideration. Let $\tau': \hat{E}' \to E$ be another covering. We say that it is equivalent to $\tau$ if there exists a morphism $f: \hat{E} \to \hat{E}'$ that is a diffeomorphism and such that $\tau = \tau' \circ f$.

If $D_1, \ldots, D_n$ are total derivatives on $E$ and $w^1, \ldots, w^r, \ldots$ are nonlocal variables, then the covering structure is given by the vector fields

$$\tilde{D}_i = D_i + X_i, \quad i = 1, \ldots, n,$$

where $X_i = \sum_\alpha X_i^\alpha \partial/\partial w^\alpha$ are $\tau$-vertical fields satisfying the condition

$$D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \quad 1 \leq i < j \leq n. \quad (5)$$

A covering is Abelian if the coefficients $X_i^\alpha$ are independent of nonlocal variables. In this case, (5) is equivalent to

$$D_i(X_j) - D_j(X_i) = 0, \quad 1 \leq i < j \leq n. \quad (6)$$

In the particular case of one-dimensional coverings, conditions (6) define a $d_h$-closed horizontal 1-form $\omega_\tau = \sum_i X_i \, dx^i$ on $E$, and two coverings of this type are equivalent if and only if the corresponding forms are in the same cohomology class, i.e., $\omega_\tau - \omega_{\tau'} = d_h(g)$ for some function $g$. If $n$ (the number of

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1They follow from Vinogradov’s 2-line theorem [18].
independent variables) is two, then this establishes a one-to-one correspondence between the group \( \text{Cl}(\mathcal{E}) \) and the equivalence classes of one-dimensional Abelian coverings over \( \mathcal{E} \).

Let \( \tau: \tilde{\mathcal{E}} \to \mathcal{E} \) be a covering. Then symmetries of \( \tilde{\mathcal{E}} \) are called nonlocal \( \tau \)-symmetries of \( \mathcal{E} \). We also note that any \( \mathcal{C} \)-differential operator \( \Delta \) on \( \mathcal{E} \) can be lifted to a \( \mathcal{C} \)-differential operator \( \tilde{\Delta} \) on \( \tilde{\mathcal{E}} \). This procedure consists of replacing the total derivatives \( D_i \) in the local representation of \( \Delta \) with \( \tilde{D}_i \) using (4).

In particular, the linearization operator \( \ell_\mathcal{E} \) can be thus lifted, and solutions of the equation \( \tilde{\ell}_\mathcal{E}(\varphi) = 0 \) are called (nonlocal) shadows of symmetries in the covering \( \tau \). Similarly, solutions of \( \tilde{\ell}_\mathcal{E}^*(\psi) = 0 \) are (nonlocal) shadows of cosymmetries in the covering \( \tau \).

### 2.5. The \( \ell \)-covering

Let \( \mathcal{E} \) be an equation. We consider a new set of dependent variables \( q = (q^1, \ldots, q^m) \) (in many respects, it is convenient to consider \( q \) an odd variable), where \( m \) is the number of unknown functions in \( \mathcal{E} \), and augment the initial equation with

\[
\ell_\mathcal{E}(q) = 0.
\]

(7)

The resulting system consisting of \( \mathcal{E} \) and Eq. (7) is called the \( \ell \)-covering of \( \mathcal{E} \). It is an analogue of the tangent bundle for the equation \( \mathcal{E} \). The \( \ell \)-covering is important for the subsequent computations because of the following properties.

#### 2.5.1. Recursion operators for symmetries

We consider a vector function \( \Phi = (\Phi^1, \ldots, \Phi^m) \), where \( \Phi^j = \sum_{\alpha,\sigma} \Phi^j_{\alpha,\sigma} q^\alpha_\sigma \), which is a symmetry shadow in the \( \ell \)-covering. This means that it satisfies the equation

\[
\tilde{\ell}_\mathcal{E}(\Phi) = 0,
\]

(8)

where \( \tilde{\ell}_\mathcal{E} \) is the linearization operator lifted to the \( \ell \)-covering. It can then be shown that the matrix \( \mathcal{C} \)-differential operator \( R_\Phi = (\sum_{\sigma} \Phi^j_{\alpha,\sigma} D_\sigma) \) takes symmetries of \( \mathcal{E} \) to symmetries. In other words, \( R_\Phi \) is a recursion operator for symmetries.

A recursion operator \( R \) is said to be hereditary if

\[
\{R_\varphi_1, R_\varphi_2\} - R(\{\mathcal{R}_\varphi_1, \varphi_2\} + \{\varphi_1, \mathcal{R}_\varphi_2\} - \mathcal{R}(\varphi_2, \varphi_2)) = 0
\]

holds for any \( \varphi_1 \) and \( \varphi_2 \). Hereditary operators have the following property important for integrability: if \( \varphi \) is a symmetry such that \( E_\varphi(R) - [\mathcal{E}_\varphi, R] = 0 \), then all symmetries \( R^k \varphi \) pairwise commute, i.e., form a commutative hierarchy.

#### 2.5.2. Symplectic operators

Similarly, we now consider a vector function \( \Psi = (\Psi^1, \ldots, \Psi^r) \) with the components \( \Psi^j = \sum_{\alpha,\sigma} \Psi^j_{\alpha,\sigma} q^\alpha_\sigma \) (we recall that \( r \) is the number of functions \( F^j \) defining \( \mathcal{E} \)) satisfying the equation

\[
\tilde{\ell}_\mathcal{E}^*(\Psi) = 0,
\]

(9)

where \( \tilde{\ell}_\mathcal{E}^* \) is the lift of the operator \( \ell_\mathcal{E}^* \) to the \( \ell \)-covering. Then the operator \( S_\Psi = (\sum_{\sigma} \Psi^j_{\alpha,\sigma} D_\sigma) \) takes symmetries of the equation \( \mathcal{E} \) to its cosymmetries.

Let an operator \( S \) satisfy the condition

\[
S^* \circ \ell_\mathcal{E} = \ell_\mathcal{E}^* \circ S.
\]

(10)

Then \( S \) can be understood as a variational 2-form \( \Omega_S \) on \( \mathcal{E} \) whose values on symmetries are given by \( \Omega_S(\varphi_1, \varphi_2) = (S \varphi_1, \varphi_2) \). This form can be considered an element of the group \( E_1^{2,n-1}(\mathcal{E}) \) in the term \( E_1 \) of the \( \mathcal{C} \)-spectral sequence.
We now consider two conservation laws \( \omega_1 \) and \( \omega_2 \) such that \( \delta \omega_i = \mathcal{S} \varphi_i, \ i = 1, 2 \), for some \( \varphi_1, \varphi_2 \in \text{sym} \mathcal{E} \). Then the bracket \( \{ \omega_1, \omega_2 \}_\mathcal{S} = \Omega_\mathcal{S}(\varphi_1, \varphi_2) \) is defined. This bracket is skew-symmetric by (10) and satisfies the Jacobi identity if
\[
\delta \Omega_\mathcal{S} = 0. \tag{11}
\]
Operators that have properties (10) and (11) are said to be symplectic.

**2.5.3. Nonlocal covectors.** Solving equations (8) or (9) often leads to only trivial results because symplectic and especially recursion operators in many cases are nonlocal, i.e., contain terms such as \( D_x^{-1} \). Such terms are incorporated into the solution by introducing nonlocal variables, which amounts to constructing appropriate coverings. One way to construct them is based on the fact (see [10]) that a conservation law on the \( \ell \)-covering corresponds to any cosymmetry of the equation \( \mathcal{E} \). We call these conservation laws nonlocal covectors. Consequently, if \( n = 2 \), then an Abelian covering corresponds to a cosymmetry. Numerous computations (see, e.g., [9], [11], [12]) show that nonlocal variables arising in such a way suffice for finding the necessary structures.

**2.6. The \( \ell^* \)-covering.** We again consider an equation \( \mathcal{E} \) and introduce a new set of dependent variables \( p = (p^1, \ldots, p^r) \), where \( r \) is the number of functions \( F^j \) defining the equation \( \mathcal{E} \). We augment the initial equation with
\[
\ell^*_\mathcal{E}(p) = 0. \tag{12}
\]
The resulting system consisting of \( \mathcal{E} \) and Eq. (12) is called the \( \ell^* \)-covering of \( \mathcal{E} \). It is an analogue of the cotangent bundle for the equation \( \mathcal{E} \). The \( \ell^* \)-covering is also important for the subsequent computations because of the following properties. Like the variable \( q \) in the \( \ell \)-covering, it is convenient to consider \( p \) an odd variable.

**2.6.1. Hamiltonian operators.** To find the Hamiltonian operators, we consider a vector function \( \Phi = (\Phi^1, \ldots, \Phi^n) \), where \( \Phi^j = \sum_{\alpha, \sigma} \Phi^j_{\alpha, \sigma} p^\sigma \), and assume that it is a symmetry shadow in the \( \ell^* \)-covering. This means that it satisfies the equation
\[
\tilde{\ell}_\mathcal{E}(\Phi) = 0, \tag{13}
\]
where \( \tilde{\ell}_\mathcal{E} \) is the linearization operator lifted to the \( \ell^* \)-covering. It can then be shown that the matrix \( \mathcal{E} \)-differential operator \( \mathcal{H}_\Phi = (\sum_{\sigma} \Phi^j_{\alpha, \sigma} D_\alpha) \) takes cosymmetries of \( \mathcal{E} \) to symmetries.

Solutions of (13) of a special type are identified with variational bivectors \( \Lambda_{\mathcal{H}} \) on \( \mathcal{E} \). These solutions must satisfy the condition \( \ell_\mathcal{E} \circ \mathcal{H} = \mathcal{H}^* \circ \ell^*_\mathcal{E} \). If the condition is satisfied, then the operation
\[
\{ \omega_1, \omega_2 \}_{\mathcal{H}} = \langle \mathcal{H}(\delta \omega_1), \delta \omega_2 \rangle, \quad \omega_1, \omega_2 \in \text{Cl}(\mathcal{E}),
\]
defines a skew-symmetric bracket on the space of conservation laws. This bracket satisfies the Jacobi identity if and only if \( [\Lambda_{\mathcal{H}}, \Lambda_{\mathcal{H}}] = 0 \), where \( [\cdot, \cdot] \) is the variational Schouten bracket [9], [10] on the space of variational multivectors. In this case, we have \( \mathcal{H} \delta \{ \omega_1, \omega_2 \}_{\mathcal{H}} = \{ \mathcal{H} \delta \omega_1, \mathcal{H} \delta \omega_2 \} \).

**2.6.2. Recursion operators for cosymmetries.** We consider a vector function \( \Psi = (\Psi^1, \ldots, \Psi^r) \), where \( \Psi^j = \sum_{\alpha, \sigma} \Psi^j_{\alpha, \sigma} p^\sigma \), and assume that it is a cosymmetry shadow in the \( \ell^* \)-covering. This means that it satisfies the equation
\[
\tilde{\ell}^*_\mathcal{E}(\Psi) = 0. \tag{14}
\]
In this case, we can show that the matrix \( \mathcal{E} \)-differential operator \( \tilde{\mathcal{R}}_\Psi = (\sum_{\sigma} \Psi^j_{\alpha, \sigma} D_\alpha) \) takes cosymmetries of \( \mathcal{E} \) to cosymmetries. In other words, \( \tilde{\mathcal{R}} \) is a recursion operator for cosymmetries of \( \mathcal{E} \).
2.6.3. Nonlocal vectors. As in Sec. 2.5, Eqs. (13) and (14) often lead to only trivial results for the same reason: the Hamiltonian and recursion operators are nonlocal in many cases. Nonlocal variables, introduced by constructing the appropriate coverings, correspond to them. One way to construct the coverings is based on the fact that a conservation law on the \( \ell^* \)-covering corresponds to any symmetry of the equation \( \mathcal{E} \). We call these conservation laws nonlocal vectors. Consequently, if \( n = 2 \), then an Abelian covering corresponds to a symmetry. As computations in [9], [11], [12] show, nonlocal variables arising in such a way suffice for finding the necessary structures.

2.7. General computational scheme. In all the computations we did to analyze concrete equations (in particular, Eq. (1)), we adhered to the following scheme:

1. Extend the initial equation with a minimal set of nonlocal variables (usually associated with conservation laws) to ensure the existence of nontrivial solutions of the main equations defining integrable structures.

2. Compute a minimal set of (local and nonlocal) symmetries and cosymmetries necessary both for generating a hierarchy and for constructing nonlocal vectors and covectors.

3. Extend the \( \ell \)-covering and construct symplectic structures and recursion operators for symmetries.

4. Extend the \( \ell^* \)-covering and construct Hamiltonian structures and recursion operators for cosymmetries.

3. Main results

We choose the functions

\[
u_{k,i} = \frac{\partial^{k+i}u}{\partial x^k \partial y^i}, \quad i = 0, 1, 2, \quad k = 0, 1, \ldots,
\]

as the internal coordinates on \( \mathcal{E} \). With this choice of coordinates, the total derivatives on \( \mathcal{E} \) become

\[
\begin{align*}
D_x &= \frac{\partial}{\partial x} + \sum_{k \geq 0} \left( u_{k+1,0} \frac{\partial}{\partial u_{k,0}} + u_{k+1,1} \frac{\partial}{\partial u_{k,1}} + u_{k+1,2} \frac{\partial}{\partial u_{k,2}} \right), \\
D_y &= \frac{\partial}{\partial y} + \sum_{k \geq 0} \left( u_{k,1} \frac{\partial}{\partial u_{k,0}} + u_{k,2} \frac{\partial}{\partial u_{k,1}} + D_x^k \left( u_{2,1}^2 - u_{3,0} u_{1,2} \right) \frac{\partial}{\partial u_{k,2}} \right).
\end{align*}
\]

We note that Eq. (1) becomes homogeneous if the variables in it are assigned the weights (gradings) \(|x| = 0\), \(|x| = -1\), and \(|y| = -4\).

3.1. Conservation laws and Abelian coverings. In our further calculations, we need a certain set of nonlocal variables, which are denoted by \( Q_{i,j} \). The second subscript here indicates the weight of the variable, and the first corresponds to the nonlocality level, by which we mean the following. The zeroth-level variables are determined by local functions on \( \mathcal{E} \):

\[
\begin{align*}
\frac{\partial Q_{0,7}}{\partial x} &= -u_{0,1} u_{4,0} + u_{0,2}, \quad & \frac{\partial Q_{0,7}}{\partial y} &= -u_{0,1} u_{3,1} - u_{0,2} u_{3,0} + u_{1,1} u_{2,1}, \\
\frac{\partial Q_{0,9}}{\partial x} &= 2u_{0,2} u_{2,0} + u_{1,1} - u_{2,0} u_{2,1}, \quad & \frac{\partial Q_{0,9}}{\partial y} &= 2u_{0,2} u_{1,1} - u_{1,2} u_{2,0}, \\
\frac{\partial Q_{0,12}}{\partial x} &= u_{1,1} (u_{0,2} - u_{2,0} u_{2,1}), \quad & \frac{\partial Q_{0,12}}{\partial y} &= \frac{1}{2} (u_{0,2}^2 - 2, u_{1,1}, u_{1,2} u_{2,0}).
\end{align*}
\]
The first-level variables are determined by local functions and zeroth-level variables:

\[
\frac{\partial Q_{1,3}}{\partial x} = 2u_{0,1} - u_{2,0}^2, \quad \frac{\partial Q_{1,3}}{\partial y} = 2(Q_{0,7} + u_{0,1}u_{3,0} - u_{1,1}u_{2,0}),
\]

\[
\frac{\partial Q_{1,6}}{\partial x} = Q_{0,7} + u_{0,1}u_{3,0}, \quad \frac{\partial Q_{1,6}}{\partial y} = \frac{1}{2}u_{1,1}^2,
\]

and

\[
\frac{\partial Q_{1,8}}{\partial x} = Q_{0,9} - 2u_{0,1}u_{1,0}u_{4,0} - 4u_{0,1}u_{2,0}u_{3,0} + 2u_{0,2}u_{1,0},
\]

\[
\frac{\partial Q_{1,8}}{\partial y} = 4Q_{0,12} - 2u_{0,1}u_{1,0}u_{3,1} - 2u_{0,1}u_{1,1}u_{3,0} - 2u_{0,1}u_{2,0}u_{2,1} -
\]

\[
- 2u_{0,2}u_{1,0}u_{3,0} - u_{0,2}u_{2,0}^2 + 2u_{1,0}u_{1,1}u_{2,1} + 2u_{1,1}^2u_{2,0}.
\]

There exist deeper-level nonlocalities, such as

\[
\frac{\partial Q_{2,5}}{\partial x} = -18Q_{1,6} - 2u_{0,1}u_{2,0} + 4u_{1,0}u_{1,1} + u_{2,0}^3,
\]

\[
\frac{\partial Q_{2,5}}{\partial y} = -3Q_{0,9} - 2u_{0,1}u_{1,1} + 4u_{0,2}u_{1,0},
\]

and

\[
\frac{\partial Q_{2,7}}{\partial x} = -40Q_{0,7}u_{1,0} - 10Q_{1,8} - 10u_{0,1}^2 - 60u_{0,1}u_{1,0}u_{3,0} +
\]

\[
+ u_{1,0}u_{2,0}^2u_{3,0} - \frac{1}{2}u_{2,0}^4,
\]

\[
\frac{\partial Q_{2,7}}{\partial y} = -40Q_{0,7}u_{0,1} - 10Q_{1,11} - 30u_{0,1}^2u_{3,0} + 20u_{0,1}u_{1,1}u_{2,0} +
\]

\[
+ 10u_{0,1}u_{2,0}^2u_{3,0} - 30u_{1,0}u_{1,1} + u_{1,0}u_{2,0}^2u_{2,1} - 3u_{1,1}u_{2,0}^3.
\]

and also

\[
\frac{\partial Q_{3,4}}{\partial x} = \frac{1}{3}Q_{2,5} + Q_{1,3}u_{2,0} - \frac{4}{3}u_{0,1}u_{1,0},
\]

\[
\frac{\partial Q_{3,4}}{\partial y} = 2Q_{0,7}u_{1,0} + Q_{1,3}u_{1,1} - Q_{1,8} - 2u_{0,1}^2 - u_{0,1}u_{2,0}^2.
\]

**Remark 1.** The zeroth-level nonlocal variables are associated with conservation laws of Eq. (1). For example, the conservation law

\[
\omega_{0,7} = (-u_{0,1}u_{4,0} + u_{0,2}) dx + (-u_{0,1}u_{3,1} - u_{0,2}u_{3,0} + u_{1,1}u_{2,1}) dy
\]

corresponds to \(Q_{0,7}\). The first-level nonlocal variables are associated with conservation laws of the Abelian coverings determined by the zeroth-level variables, and so on.

**Remark 2.** Of course, the list of nonlocal variables above is not at all exhaustive. We listed only those variables that are used to construct the necessary nonlocal symmetries (Sec. 3.2) and cosymmetries (Sec. 3.3; also see Sec. 2.7). New nonlocalities do arise under the actions of recursion operators, but there is no need to describe them explicitly here.
3.2. Symmetries. The further results of direct computation are needed, first, for constructing non-local vectors (see Sec. 2.6.3) and, second, for “seeding” the symmetry hierarchies.

The linearization of Eq. (1) has the form

\[ D^3_y(\varphi) - 2u_{2,1}D^2_y(\varphi) + u_{1,2}D^3_y(\varphi) + u_{3,0}D_xD^2_y(\varphi) = 0, \]  

(17)

where the total derivatives \( D_x \) and \( D_y \) are given by (16). Solving (17) in “lower dimensions,” we obtain the following symmetries.\(^2\)

Symmetries of degree 0:

\[ \varphi^0_0 = 1, \quad \varphi^0_1 = u_{1,0}, \quad \varphi^0_4 = u_{0,1}, \quad \varphi^0_5 = Q_{2,5} + 8u_{0,1}u_{1,0}, \]

\[ \varphi^0_8 = -2Q_{0,7}u_{1,0} + Q_{1,8} - 2u^2_{0,1} + u_{0,1}u^2_{2,0}. \]

Symmetries of degree 1:

\[ \varphi^1_{-4} = y, \quad \varphi^1_{-1} = x, \]

\[ \varphi^1_{0,1} = xu_{1,0} - 4u, \quad \varphi^1_{0,2} = yu_{0,1} + u, \]

\[ \varphi^1_{3} = 4xu_{0,1} - Q_{1,3}, \]

\[ \varphi^1_{4} = x(Q_{2,5} + 8u_{0,1}u_{1,0}) - 3(8Q_{3,4} - 3Q_{1,5}u_{1,0} + 16uu_{0,1}). \]

Symmetries of degree 2:

\[ \varphi^2_{-4} = y, \quad \varphi^2_{-8} = y^2, \quad \varphi^2_{-2} = x^2, \quad \varphi^2_{-5} = xy, \quad \varphi^2_{-2} = x^2, \]

\[ \varphi^2_{-1} = x^2u_{1,0} + 4xyu_{0,1} - 4xu - yQ_{1,3}, \]

\[ \varphi^2_{2} = 2x^2u_{0,1} - xQ_{1,3} - u^2_{1,0}. \]

Symmetries of degree 3:

\[ \varphi^3_{-3} = x^3 - 2yu_{1,0}, \]

\[ \varphi^3_{1} = 12x^3u_{0,1} - 9x^2Q_{1,3} - 18xu^2_{1,0} - 2y(Q_{2,5} + 8u_{0,1}u_{1,0}) + 24uu_{1,0}. \]

We also need one symmetry of degree 4, which has the form

\[ \varphi^4_{-4} = x^4 - 8xyu_{1,0} - 8y^2u_{0,1} + 16yu. \]

3.3. Cosymmetries. The reasons for computing explicit cosymmetries are similar to those indicated in Sec. 3.2.

To find cosymmetries, we must solve the equation adjoint to (17):

\[ D^3_y(\psi) - 2D^2_y(u_{2,1}\psi) + D^3_y(u_{1,2}\psi) + D_xD^2_y(u_{3,0}\psi) = 0. \]  

(18)

\(^2\)In the notation for symmetries, the superscript indicates the polynomial degree of a symmetry with respect to \( x \) and \( y \), the first subscript equals the weight, and the second subscript, if any, is the number of the symmetry in the set of symmetries of the given weight and order.
We write the computational results needed in what follows using notation similar to that in Sec. 3.2.

Cosymmetries of degree 0:

\[
\psi^0_0 = 1, \quad \psi^0_2 = u_{2,0}, \quad \psi^0_5 = u_{1,1},
\]

\[
\psi^0_6 = -18Q_{1,6} + 6u_{0,1}u_{2,0} + 12u_{1,0}u_{1,1} + u_{2,0}^2,
\]

\[
\psi^0_9 = 2Q_{0,7}u_{2,0} - Q_{0,9} + 4u_{0,1}u_{1,1} + 2u_{0,1}u_{1,0}u_{3,0} - u_{1,1}u_{2,0}^2.
\]

Cosymmetries of degree 1:

\[
\psi^1_{-4} = y, \quad \psi^1_{-1} = x, \quad \psi^1_{1,1} = xu_{2,0} - 3u_{1,0},
\]

\[
\psi^1_{1,2} = yu_{1,1} + u_{1,0}, \quad \psi^1_k = 4xu_{1,1} + 2u_{0,1} + u_{2,0}^2.
\]

Cosymmetries of degree 2:

\[
\psi^2_{-2} = 3x^2 - 2yu_{2,0},
\]

\[
\psi^2_0 = x^2u_{2,0} + 4xyu_{1,1} - 2xu_{1,0} + y(2u_{0,1} + u_{2,0}^2) - 4u,
\]

\[
\psi^2_3 = 2x^2u_{1,1} + x(2u_{0,1} + u_{2,0}^2) - Q_{1,3} - 2u_{1,0}u_{2,0}.
\]

Cosymmetries of degree 3:

\[
\psi^3_{-3} = 2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3,
\]

\[
\psi^3_2 = x^3u_{1,1} + x^2\left(\frac{3}{2}u_{0,1} + \frac{3}{4}u_{2,0}^2\right) - x\left(\frac{3}{2}Q_{1,3} + 3u_{1,0}u_{2,0}\right) +
\]

\[
+ y\left(3Q_{1,6} - u_{0,1}u_{2,0} - 2u_{1,0}u_{1,1} - \frac{1}{6}u_{2,0}^3\right) + \left(2u_{2,0}^2 + \frac{1}{2}u_{2,0}^4\right).
\]

### 3.4. The ℓ-covering.

The ℓ-covering is determined by the system of equations

\[
u_{yy} - u_{xxy} + u_{xxy}u_{xyy} = 0,
\]

\[
q_{yyy} - 2u_{xxy}q_{xyy} + u_{xyy}q_{xxx} + u_{xxx}q_{xyy} = 0,
\]

where \( q \) is an odd variable along the fiber of the covering. The internal coordinates on the space of the ℓ-covering are functions (15) together with

\[
q_{k,i} = \frac{\partial^{k+i}q}{\partial x^k \partial y^i}, \quad i = 0, 1, 2, \quad k = 0, 1, \ldots,
\]

and the total derivatives in these coordinates become

\[
D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} \left( u_{k+1,0} \frac{\partial}{\partial q_{k,0}} + u_{k+1,1} \frac{\partial}{\partial q_{k,1}} + u_{k+1,2} \frac{\partial}{\partial q_{k,2}} +
\right.
\]

\[
+ q_{k+1,0} \frac{\partial}{\partial q_{k,0}} + q_{k+1,1} \frac{\partial}{\partial q_{k,1}} + q_{k+1,2} \frac{\partial}{\partial q_{k,2}} \right),
\]

\[
D_y = \frac{\partial}{\partial y} + \sum_{k \geq 0} \left( u_{k,1} \frac{\partial}{\partial q_{k,0}} + u_{k,2} \frac{\partial}{\partial q_{k,1}} + D^k_x(u_{2,1}^2 - u_{3,0}u_{1,2}) \frac{\partial}{\partial q_{k,2}} +
\right.
\]

\[
+ q_{k,1} \frac{\partial}{\partial q_{k,0}} + q_{k,2} \frac{\partial}{\partial q_{k,1}} + D^k_x(2u_{2,1}q_{2,1} - u_{1,2}q_{3,0} - u_{3,0}q_{1,2}) \frac{\partial}{\partial q_{k,2}} \right).
\]

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By the general theory [10], a conservation law on the $\ell$-covering (a nonlocal form) corresponds to any cosymmetry $\hat{\psi}$ of the initial equation. We let $\Omega_{\psi}$ denote the corresponding nonlocal variable. In the case of Eq. (1), this variable is defined by the relations

\begin{align}
\frac{\partial \Omega_{\psi}}{\partial x} &= \psi q_{0,2} + a_{0,1}q_{0,1} + a_{0,0}q, \\
\frac{\partial \Omega_{\psi}}{\partial y} &= b_{0,2}q_{0,2} + b_{1,1}q_{1,1} + b_{2,0}q_{2,0} + b_{0,1}q_{0,1} + b_{1,0}q_{1,0} + b_{0,0}q,
\end{align}

where

\begin{align}
b_{0,2} &= -u_{3,0}\psi, & b_{1,1} &= 2u_{2,1}\psi, & b_{2,0} &= -u_{1,2}\psi, \\
b_{0,1} &= -D_x(b_{1,1}), & b_{1,0} &= -D_x(b_{2,0}), \\
b_{0,0} &= -D_x(b_{1,0}),
\end{align}

and

\begin{align}
a_{0,1} &= D_x(b_{0,2}) - D_y(\psi), & a_{0,0} &= D_x(b_{0,1}) - D_y(a_{0,1}).
\end{align}

Below, we set $\Omega^{k}_{i,j} = \Omega_{\psi_{i,j}}$.

### 3.4.1. Recursion operators for symmetries.

To find recursion operators for symmetries of Eq. (1), we solve the equation $\hat{\ell}_{\psi}(\Phi) = 0$, where $\hat{\ell}_{\psi}$ is linearization operator (17) in the $\ell$-covering extended by nonlocal forms and $\Phi$ is a function on this extension. The simplest nontrivial solution has the form

\begin{align}
\Phi_1 &= \Omega^{3}_{-3} - x\Omega^{2}_{-2} + 4y\Omega^{1}_{1,2} + 2y\Omega^{1}_{1,1} + 3x^2\Omega^{1}_{-1} - 2u_{1,0}\Omega^{1}_{-4} - \\
&\quad - 2y^2\Omega^{1}_{0} = 2xy\Omega^{0}_{1} + (2u_{1,0}y - x^3)\Omega^{0}_{0}.
\end{align}

Using the first equation in (20), we set the operator

\begin{equation}
D_{\psi} = D^{-1}_x \circ (\psi D^2_y + a_{0,1}D_y + a_{0,0})
\end{equation}

into correspondence with each nonlocal form $\Omega_{\psi}$, where the coefficients are determined using relations (21) and (22), i.e.,

\begin{align}
a_{0,1} &= -D_x(u_{3,0}\psi) - D_y(\psi), \\
a_{0,0} &= -2D_x^2(u_{2,1}\psi) + 2D_xD_y(u_{3,0}\psi) + D^2_y(\psi).
\end{align}

Then the recursion operator

\begin{align}
R_1 &= D_{\psi}^2 - xD_{\psi}^2 + 4yD_{\psi_{1,2}} + 2yD_{\psi_{1,1}} + 3x^2D_{\psi_{-1}} - 2u_{1,0}D_{\psi_{-4}} - \\
&\quad - 2y^2D_{\psi_0} = 2xyD_{\psi_{1}} + (2u_{1,0}y - x^3)D_{\psi_{0}}
\end{align}

corresponds to solution (23).

**Remark 3.** Because the variables $x$ and $y$ in Eq. (1) “enjoy equal rights,” we can set the operator

\begin{equation}
D'_{\psi} = D^{-1}_x \circ (b_{0,2}D^2_y + b_{1,1}D_xD_y + b_{2,0}D^2_x + b_{0,1}D_y + b_{1,0}D_x + b_{0,0})
\end{equation}

into correspondence with the nonlocal form $\Omega_{\psi}$ using the second equality in (20) and construct a recursion operator $R'_1$ similar to operator (25). The action of the obtained operators on symmetries of Eq. (1) is the same.
3.4.2. Symplectic structures. To find symplectic structures, we solve the equation \( \tilde{\ell}^*_E(\Psi) = 0 \), where \( \tilde{\ell}^*_E \) as in Sec. 3.4.1 is operator (18) on the \( \ell \)-covering extended by the nonlocal forms and \( \Psi \) is a function on this extension.

Below, we present the first two solutions of this equation. The simplest is \( \Psi_1 = \Omega_{1,0} \), and the corresponding symplectic structure is

\[
S_1 = D_x.
\]

The next solution is nonlocal:

\[
\Psi_2 = \Omega^2_{-1} - 6x\Omega^1_{-1} + 2u_{2,0}\Omega^1_{-4} + 2y\Omega^0_2 - (2u_{2,0}y - 3x^2)\Omega^0_0.
\]

The corresponding symplectic operator \( S_2 \) is

\[
S_2 = D_{\Psi^2_{-1}} - 6x\mathcal{D}_{\Psi^1_{-1}} + 2u_{2,0}\mathcal{D}_{\Psi^1_{-4}} + 2y\mathcal{D}_{\Psi^0_2} - (2u_{2,0}y - 3x^2)\mathcal{D}_{\Psi^0_0},
\]

where the operators \( \mathcal{D}_{\Psi} \) are defined by Eq. (24).

3.5. The \( \ell^* \)-covering. As already indicated above, the \( \ell^* \)-covering is obtained by augmenting the initial equation with an equation adjoint to the linearization. In other words, this covering is described by the equations

\[
u_{yy} - u^2_{xy} + uu_{xx}u_{xy} = 0,
\]

\[
u_{xy}u_{xy} - 2u_{xxx}u_{xy} + u_{xxx}p_{xy} + u_{xxy}p_{xx} - 2u_{xxy}p_{xy} - u_{xxx}p_{xy} + p_{yyy} = 0,
\]

where \( p \) is a new odd variable. The internal coordinates in the space of the \( \ell^* \)-covering are functions (15) together with the functions

\[
p_k, i = \frac{\partial^{k+i} p}{\partial x^k \partial y^i}, \quad i = 0, 1, 2, \quad k = 0, 1, \ldots,
\]

and the total derivatives in these coordinates have the forms

\[
D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} \left( u_{k+1,0} \frac{\partial}{\partial u_{k,0}} + u_{k+1,1} \frac{\partial}{\partial u_{k,1}} + u_{k+1,2} \frac{\partial}{\partial u_{k,2}} + p_k \left( u_{k+1,0} \frac{\partial}{\partial p_{k,0}} + u_{k+1,1} \frac{\partial}{\partial p_{k,1}} + u_{k+1,2} \frac{\partial}{\partial p_{k,2}} \right) \right),
\]

\[
D_y = \frac{\partial}{\partial y} + \sum_{k \geq 0} \left( u_{k,1} \frac{\partial}{\partial u_{k,0}} + u_{k,2} \frac{\partial}{\partial u_{k,1}} + D^k_x(u^2_{2,1} - u_{3,0}u_{1,2}) \frac{\partial}{\partial u_{k,2}} + p_k \left( u_{k,1} \frac{\partial}{\partial p_{k,0}} + u_{k,2} \frac{\partial}{\partial p_{k,1}} + u_{k,3} \frac{\partial}{\partial p_{k,2}} \right) \right).
\]

We consider a symmetry \( \varphi \) of Eq. (1). Then [10] a conservation law on the \( \ell^* \)-covering and consequently a nonlocal variable, denoted by \( \Pi_\varphi \) and called a nonlocal vector, corresponds to this symmetry. For Eq. (1), the correspondence \( \varphi \mapsto \Pi_\varphi \) is given by the relations

\[
\frac{\partial \Pi_\varphi}{\partial x} = \varphi_{p_0,2} + a_{0,1}p_{0,1} + a_{0,0}p,
\]

\[
\frac{\partial \Pi_\varphi}{\partial y} = b_{0,2}p_{0,2} + b_{1,1}p_{1,1} + b_{2,0}p_{2,0} + b_{0,1}p_{0,1} + b_{1,0}p_{1,0} + b_{0,0}p,
\]

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where
\[ b_{0,2} = -u_{3,0}\varphi, \quad b_{1,1} = 2u_{2,1}\varphi, \quad b_{2,0} = -u_{1,2}\varphi, \]
\[ b_{0,1} = -D_x(b_{1,1}) + 2u_{3,1}\varphi, \quad b_{1,0} = -D_x(b_{2,0}) - u_{2,2}\varphi, \]
\[ b_{0,0} = -D_x(b_{1,0}), \]
and
\[ a_{0,1} = D_x(b_{0,2}) - D_y(\varphi) + u_{4,0}\varphi, \quad a_{0,0} = D_x(b_{0,1}) - D_y(a_{0,1}). \]

In what follows, we let \( \mathbf{\Pi}_{\varphi_{i,j}} \) denote \( \mathbf{\Pi}_{\varphi_{i,j}} \).

To describe our obtained results further, we set the operator
\[ D\varphi = D_x^{-1} \circ (\varphi D_y^2 + a_{0,1}D_y + a_{0,0}) \tag{29} \]
(whose form follows from the first equation in (27)) into correspondence with each nonlocal vector \( \mathbf{\Pi}_{\varphi} \). Here, because of relations (28) and (29), the coefficients \( a_{0,0} \) and \( a_{0,1} \) have the forms
\[ a_{0,0} = -2u_{2,1}D_x^2(\varphi) + u_{3,0}D_xD_y(\varphi) + D_y^2(\varphi) - u_{3,1}D_x(\varphi), \]
\[ a_{0,1} = -u_{3,0}D_x(\varphi) - D_y(\varphi). \]

### 3.5.1. Hamiltonian structures

As in Secs. 3.4.1 and 3.4.2, Hamiltonian structures are solutions of the equation
\[ \tilde{\mathcal{E}}(\Phi) = 0, \tag{30} \]
where the operator \( \tilde{\mathcal{E}} \) is the linearization lifted to the \( \tilde{\mathcal{E}}^* \)-covering extended by nonlocal vectors. The simplest solution of Eq. (30) is
\[ \Phi_0 = \mathbf{\Pi}_{-8}^2 - 2y\mathbf{\Pi}_{-4}^1 + y^2\mathbf{\Pi}_{0}^0, \]
to which the operator \( \mathcal{H}_0 : \text{cosym} \mathcal{E} \rightarrow \text{sym} \mathcal{E} \)
\[ \mathcal{H}_0 = \mathcal{D}_{\varphi_{-3}}^2 - 2y\mathcal{D}_{\varphi_{-1}}^4 + y^2\mathcal{D}_{\varphi_0}^0 \]
corresponds. The next solution is much more complicated and has the form
\[ \Phi_1 = \mathbf{\Pi}_{-4}^1 - 4x\mathbf{\Pi}_{-3}^1 + 6x^2\mathbf{\Pi}_{-2}^2 - 8u_{1,0}\mathbf{\Pi}_{-1}^2 - 8u_{0,1}\mathbf{\Pi}_{-1}^3 + 16y\mathbf{\Pi}_{0,2}^1 + \\
+ (8xu_{1,0} + 16yu_{0,1} - 16u)\mathbf{\Pi}_{1,1}^1 + 8y\mathbf{\Pi}_{1,0}^1 - (4x^3 - 8yu_{1,0})\mathbf{\Pi}_{-1}^2 - \\
- 8y^2\mathbf{\Pi}_{-4}^0 - 8xy\mathbf{\Pi}_{0}^0 + (x^4 - 8xyu_{1,0} - 8y^2u_{0,1} + 16yu)\mathbf{\Pi}_{0}^0, \]
and the operator
\[ \mathcal{H}_1 = \mathcal{D}_{\varphi_{-4}}^4 - 4x\mathcal{D}_{\varphi_{-3}}^3 + 6x^2\mathcal{D}_{\varphi_{-2}}^2 - 8u_{1,0}\mathcal{D}_{\varphi_{-1}}^1 - 8u_{0,1}\mathcal{D}_{\varphi_{-1}}^2 + 16y\mathcal{D}_{\varphi_{0,2}}^1 + \\
+ (8xu_{1,0} + 16yu_{0,1} - 16u)\mathcal{D}_{\varphi_{1,1}}^1 + 8y\mathcal{D}_{\varphi_{1,0}}^1 - (4x^3 - 8yu_{1,0})\mathcal{D}_{\varphi_{-1}}^2 - \\
- 8y^2\mathcal{D}_{\varphi_0}^0 - 8xy\mathcal{D}_{\varphi_0}^0 + (x^4 - 8xyu_{1,0} - 8y^2u_{0,1} + 16yu)\mathcal{D}_{\varphi_0}^0 \]
corresponds to this solution. Here and below, the operators \( \mathcal{D}_{\varphi} \) are defined by Eq. (29).
Remark 4. The form of the solutions presented above is determined by the choice of nonlocal vectors in the $\ell^*$-covering. They are determined in turn by the choice of the basis in the space of symmetries. In fact, Eq. (30) has a simpler solution $\Psi'_0 = \Pi'_1$, where the nonlocal variable $\Pi'_1$ is defined by the system
\[
\frac{\partial \Pi'_1}{\partial x} = p, \quad \frac{\partial \Pi'_1}{\partial y} = \Pi'_2,
\]
\[
\frac{\partial \Pi'_2}{\partial x} = p_{0,1}, \quad \frac{\partial \Pi'_2}{\partial y} = \Pi'_3,
\]
\[
\frac{\partial \Pi'_3}{\partial x} = p_{0,2}, \quad \frac{\partial \Pi'_3}{\partial y} = -u_{3,0}p_{0,2} + 2u_{2,1}p_{1,1} - u_{1,2}p_{2,0}.
\]
The Hamiltonian operator $\mathcal{H}'_0 = D_x^{-1}$, which is the inverse of symplectic operator (26), corresponds to this solution. This operator corresponds to the Hamiltonian operator $J_0$ in [7]. The operator $\mathcal{H}_1$ explicitly depending on $x$ and $y$ seems new.

Remark 5. There is a relation $\mathcal{H}_1 = \mathcal{R}_1 \circ \mathcal{H}_0$ between the two Hamiltonian structures, where $\mathcal{R}_1$ is the recursion operator given by (25).

3.5.2. Recursion operators for cosymmetries. In conclusion, we consider the equation $\tilde{\ell}_E'(\Psi) = 0$ on the $\ell^*$-covering extended by nonlocal vectors. Its simplest nontrivial solution is
\[
\Psi_0 = \Pi_{-3}^3 - 3x\Pi_{-2}^2 + 2u_{2,0}\Pi_{-5}^2 + 2u_{1,1}\Pi_{-8}^2 +
+ 2(u_{1,0} + 2u_{1,1}y - u_{2,0}x)\Pi_{-4}^1 - (2u_{2,0}y - 3x^2)\Pi_{-1}^1 + 2y\Pi_{0}^0 -
- (2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3)\Pi_{0}^0.
\]
The recursion operator $\hat{R}_0$: cosym $\mathcal{E} \to$ cosym $\mathcal{E}$ of the form
\[
\hat{R}_0 = D_{\varphi_{-3}} - 3x D_{\varphi_{-2}} + 2u_{2,0} D_{\varphi_{-5}} + 2u_{1,1} D_{\varphi_{-8}} +
+ 2(u_{1,0} - 2u_{1,1}y - u_{2,0}x) D_{\varphi_{-4}} - (2u_{2,0}y - 3x^2) D_{\varphi_{-1}} + 2y D_{\varphi_{0}} -
- (2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3) D_{\varphi_{0}}
\]
corresponds to the solution $\Psi_0$.

3.6. Hierarchies. Unfortunately, because of the extreme complexity of the computations, we failed to completely describe the Lie symmetry algebra of the considered equation. We can currently make the following assertions:

1. The action of the operator $\mathcal{R}_1$ preserves the polynomial degree of the symmetries with respect to $x$ and $y$.

2. There exist at least nine hierarchies of symmetries generated by the operator $\mathcal{R}_1$ from the symmetries
\[
\varphi_{0}^0, \varphi_{1}^0, \varphi_{-4}^1, \varphi_{0,1}^1, \varphi_{-1}^1, \varphi_{-8}^2, \varphi_{-5}^2, \varphi_{-3}^3, \varphi_{-4}^4.
\]

3. Symmetries of the forms $\mathcal{R}_1^m \varphi_{0}^0$ and $\mathcal{R}_1^m \varphi_{1}^1$ pairwise commute.
4. It seems (although we do not yet have a rigorous proof) that there is an infinite hierarchy of symmetries with an arbitrarily high polynomial degree in $x$ and $y$.

5. Because the local symplectic structure presented in equality (26) exists, there exists an infinite hierarchy of cosymmetries corresponding to the described hierarchies of symmetries.

6. The conservation laws corresponding to the hierarchies $R_{1,1}^n \phi_0^0$ and $R_{1,1}^m \phi_0^0$ are in involution with respect to the Poisson bracket defined by the symplectic structure $S_1 = D_x$.

**Remark 6.** In the case of evolution equations, the existence of a commutative hierarchy means that the initial equation has “higher analogues” that are obtained by the action of a recursion operator. In the nonevolutionary case, there is no well-defined action of recursion operators on the equation, and such a construction is therefore impossible. Nevertheless, we can consider an alternative scheme: (1) pass (if possible) to the evolutionary representation, (2) construct “higher analogues” in this representation, and (3) return to the initial variables. We did not pose the question whether such a scheme is invariant, i.e., whether the result is completely determined by the initial equation or depends on the intermediate steps. Possibly, an answer to this question will lead to a deeper understanding of the integrability of equations of general form.

**Acknowledgments.** This work was supported in part by the Russian Foundation for Basic Research (Joint Grants NWO-RFBR No. 047.017.015, P. K., I. S. K., and A. M. V.; RFBR-Consortium E.I.N.S.T.E.IN No. 09-01-92438, I. S. K., A. M. V., and R. V.; and RFBR-CNRS No. 08-07-92496, I. S. K. and A. M. V.).

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