Quantum loop groups for symmetric Cartan matrices

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ABSTRACT. We introduce a quantum loop group associated to a general symmetric Cartan matrix, by imposing just enough relations between the usual generators \( \{ e_{i,k}, f_{i,k} \}_{i \in I, k \in \mathbb{Z}} \) in order for the natural Hopf pairing between the positive and negative halves of the quantum loop group to be perfect. As an application, we describe the localized \( K \)-theoretic Hall algebra of any quiver without loops, endowed with a particularly important \( \mathbb{C}^* \) action.

To Ivo and Elio, my favorite perfect pairing

1. Introduction

1.1. Let us fix any symmetric Cartan matrix, i.e. \( C = \{ d_{ij} \}_{i,j \in I} \) with

\[
\begin{cases}
  d_{ii} = 2 & \text{if } i = j \\
  d_{ij} = d_{ji} & \text{if } i \neq j
\end{cases}
\]

for some finite set \( I \). Let \( g \) denote the Kac-Moody Lie algebra associated to the Cartan matrix \( C \). A well-known and very important Hopf algebra deformation of \( U(g) \) is the Drinfeld-Jimbo quantum group, which may be defined by the following procedure. We will work over a ground field \( K \) of characteristic 0 (usually taken to be \( \mathbb{C}(q) \)), endowed with an element \( q \in K^\times \) which is not a root of unity.

(1) Start from the algebra

\[
\bar{U}_q^+(g) = K \langle e_i \rangle_{i \in I}
\]

(2) To make \( \bar{U}_q^+(g) \) into a bialgebra, we first enlarge it

\[
\bar{U}_q^\geq(g) = \bar{U}_q^+(g)[h_i^{\geq 1}]_{i \in I} / \left( h_i e_j = q^{d_{ij}} e_j h_i \right)
\]

and define the coproduct

\[
\Delta(e_i) = h_i \otimes e_i + e_i \otimes 1 \\
\Delta(h_i) = h_i \otimes h_i
\]

(3) Define \( \bar{U}_q^\leq(g) = \bar{U}_q^\geq(g)^{\text{coop}} \), with generators denoted by \( f_i, h'_i \) instead of \( e_i, h_i \). Then there is a bialgebra pairing (see (1.48)–(1.49))

\[
\bar{U}_q^\geq(g) \otimes \bar{U}_q^\leq(g) \xrightarrow{\langle , \rangle} K
\]
completely determined by
\[ \langle e_i, f_j \rangle = \frac{1}{q^{-1} - q}, \quad \langle h_i, h'_j \rangle = q^{d_{ij}} \]
and all other pairings between generators being 0.

(4) Consider the radical of the pairing (1.2), namely
\[ I^+_o \subset \tilde{U}^+_q(\mathfrak{g}), \quad x \in I^+_o \Leftrightarrow \langle x, \tilde{U}^-_q(\mathfrak{g}) \rangle = 0 \]
and define \( I^-_o \subset \tilde{U}^-_q(\mathfrak{g}) \) analogously. Since \( I^\pm_o \) are ideals, the quotients
\[ U^\pm_q(\mathfrak{g}) = \tilde{U}^\pm_q(\mathfrak{g}) / I^\pm_o \]
are algebras.

(5) We may define the extended bialgebras \( U^\geq q(\mathfrak{g}) \) and \( U^\leq q(\mathfrak{g}) \) by removing all the tildes in items (2)–(3). Then the pairing (1.2) descends to a bialgebra pairing
\[ \langle \cdot, \cdot \rangle : U^\geq_q(\mathfrak{g}) \otimes U^\leq_q(\mathfrak{g}) \rightarrow K \]
(1.3)

(6) Define the quantum group as the vector space
\[ U_q(\mathfrak{g}) = U^\geq_q(\mathfrak{g}) \otimes U^\leq_q(\mathfrak{g}) \] made into an algebra by imposing the Drinfeld double relations (1.53) between the subalgebras \( U^\geq_q(\mathfrak{g}) \otimes 1 \) and \( 1 \otimes U^\leq_q(\mathfrak{g}) \) of \( U_q(\mathfrak{g}) \). Note that the Drinfeld double relation only takes as input the bialgebra structures and the pairing defined in item (5) and it gives rise to the well-known relation
\[ [e_i, f_j] = \delta_{ij} \cdot \frac{h_i - h'_i}{q - q^{-1}} \]
in \( U_q(\mathfrak{g}) \), for all \( i, j \in I \) (we identify \( h_i \) with \( h_i \otimes 1 \) and \( h'_i \) with \( 1 \otimes h'_i \) in (1.4)).

Although somewhat dry, the procedure above yields beautiful formulas, when one asks to describe the radicals \( I^\pm_o \) explicitly. For example, it was shown in [9] that \( I^+_o \) is generated as a two-sided ideal by the relation
\[ \sum_{k=0}^{1-d_{ij}} (-1)^k \binom{1-d_{ij}}{k}_q e_i^k e_j e_i^{1-d_{ij} - k} = 0 \] (1.5)
for all \( i \neq j \in I \). The analogous result holds in \( I^-_o \) if one replaces all the \( e \)'s by \( f \)'s. Relation (1.5) is not too surprising, in light of the fact that as \( q \to 1 \), it converges to the usual Serre relation that holds in \( \mathfrak{g} \)
\[ [e_i, [e_i, \ldots, [e_i, e_j] \ldots]]] = 0 \] (1.6)
where the number of brackets is \( 1 - d_{ij} \). The fact that relations (1.6) generate the \( (q \to 1 \text{ limit of the) ideal } I^+_o \) was proved by Serre for Lie algebras \( \mathfrak{g} \) of finite type, and by Gabber-Kac for general Kac-Moody Lie algebras \( \mathfrak{g} \).

1All the bialgebras studied in the present paper are also Hopf algebras, but we will not need the antipode map (and thus will not explicitly describe it, although it is straightforward to do so).
2We use the notation \( \binom{n}{k}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q} \) where \([n]!_q = [1]_q \cdots [n]_q \) and \([n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \).

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1.2. The main purpose of the present paper is to carry out the procedure above for quantum loop groups, i.e. to appropriately define a deformation of the universal enveloping algebra of $Lg = g[t^\pm1]$ with Lie bracket given by

\[(1.7) \quad [xt^k, yt^l] = [x, y]t^{k+l}\]

for all $x, y \in g$ and all $k, l \in \mathbb{Z}$. Quantum loop groups are important players in the theory of quantum integrable systems and quantum field theory. Moreover, together with their Yangian degenerations and their elliptic versions, quantum loop groups have played big roles in geometric representation theory in recent decades.

The natural analogue of the six-step procedure in Subsection 1.1 is the following. The construction of quantum loop groups that we are about to recall is due to \cite{2}, who used it to produce an alternate construction of quantum affine groups (this corresponds to the case when $C$ is a Cartan matrix of finite type).

(1) Start from the algebra

\[(1.8) \quad \tilde{U}_q^+(Lg) = K\langle e_{i,k} \rangle_{i \in I, k \in \mathbb{Z}} \text{ relation } (1.9)\]

where we consider the formal series $e_i(z) = \sum_{k \in \mathbb{Z}} e_{i,k} z^k$ and impose the relations

\[(1.9) \quad e_i(z) e_j(w)(z - w q^{d_{ij}}) = e_j(w) e_i(z)(z q^{d_{ij}} - w)\]

for all $i, j \in I$. To motivate relation (1.9), note that as $q \to 1$ it converges to

$[e_i(z), e_j(w)](z - w) = 0$

Unpacking this relation says that the commutator $[e_{i,k}, e_{j,l}]$ only depends on $k + l$, which is to be expected from the Lie bracket equality (1.7).

(2) Enlarge $\tilde{U}_q^+(Lg)$ as follows

\[
\tilde{U}_q^\geq(Lg) = \frac{\tilde{U}_q^+(Lg)[h_{i,0}, h_{i,1}, h_{i,2}, \ldots]_{i \in I}}{\left(h_i(z)e_j(w) = e_j(w)h_i(z)\frac{z q^{d_{ij}} - w}{z - w q^{d_{ij}}}\right)}
\]

where $h_i(z) = \sum_{k=0}^{\infty} \frac{h_{i,k}}{z^k}$, and define the topological coproduct

\[
\Delta(e_i(z)) = h_i(z) \otimes e_i(z) + e_i(z) \otimes 1
\]

\[
\Delta(h_i(z)) = h_i(z) \otimes h_i(z)
\]

(3) Define $\tilde{U}_q^{\leq}(Lg) = \tilde{U}_q^\geq(Lg)^{\text{coop}}$, with generators denoted by $f_{i,-k}, h_{i,-k}'$ instead of $e_{i,k}, h_{i,k}$. Then there is a topological bialgebra pairing

\[(1.10) \quad \tilde{U}_q^\geq(Lg) \otimes \tilde{U}_q^{\leq}(Lg) \overset{(\cdot, \cdot)}{\longrightarrow} K\]

completely determined by

\[\langle e_{i,k}, f_{i,-k} \rangle = \frac{1}{q^i - q}, \quad \langle h_i(z), h'_j(w) \rangle = \frac{z q^{d_{ij}} - w}{z - w q^{d_{ij}}}\]

and all other pairings between generators being 0.

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3Here, “it” more appropriately refers to a central extension of $\tilde{U}_q(Lg)$. 
(4) Consider the radical
\[ I^± \subset \tilde{U}_q^±(\mathfrak{Lg}) \]
of the pairing (1.10), and define the algebras \( U^±_q(\mathfrak{Lg}) = \tilde{U}_q^±(\mathfrak{Lg}) / I^± \).

(5) We may define the extended bialgebras \( U^≥_q(\mathfrak{g}) \) and \( U^≤_q(\mathfrak{g}) \) by removing all the tildes in items (2)-(3). Then the pairing (1.10) descends to a bialgebra pairing
\[ U^≥_q(\mathfrak{Lg}) \otimes U^≤_q(\mathfrak{Lg}) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{K} \]
with the multiplication governed by (1.53). It implies the well-known relation
\[ [e_i(z), f_j(w)] = \delta_{ij} \delta \left( \frac{z}{w} \right) \cdot \frac{h_i(z) - h'_i(w)}{q - q^{-1}} \]
where \( \delta(x) = \sum_{k \in \mathbb{Z}} x^k \) is a formal series and \( \delta_{ij} \) is the Kronecker delta function.

1.3. In the present paper, we will complete the definition of the quantum loop group \( U_q(\mathfrak{Lg}) \) by explicitly describing the radical \( I^+ \) (the negative part \( I^- \) will be analogous, with \( e_{i,k} \)'s replaced by \( f_{i,-k} \)'s); specifically, we develop generators for the ideal \( I^+ \) that provide quantum loop group versions of relations (1.5).

The first combinatorial tool that we need for this is the notion of distinguished zig-zag \( Z \), developed in Subsection 2.14. In brief, for any \( i \neq j \in I \) and any arithmetic progressions \( s, \ldots, t \) and \( s', \ldots, t' \) with ratio 2, such that \( s + t = s' + t' \), we draw the following oriented graph

\[ \text{Figure 1. A distinguished zig-zag } Z \]

The vertices of the graph are placed on two horizontal lines at coordinates prescribed by the chosen arithmetic progressions, and the diagonal edges correspond to all \( (a, b) \in \{s, \ldots, t\} \times \{s', \ldots, t'\} \) such that \( a = b \pm d_{ij} \).

The second combinatorial tool that we need is that of a refined selection, developed in Subsection 3.8, which is a particular type of multiset \( \mathcal{F} \) of edges in the oriented graph of Figure 1, whose removal leaves the graph without any oriented cycles. We consider the formal series
\[ \rho_Z(x_s, \ldots, x_t, y_s', \ldots, y_{t'}) = \sum_{\text{refined selection}} (-1)^{\sigma(\mathcal{S})} \prod_{\text{vertices } c < c'} \left[ (-1)^{\delta_{i(c)c} \delta_{i'(c')} (z_c - z_{c'} q^{-d_i(c)c'})} \right] \prod_{\text{edges } c' \to c} (z_{c'} - z_c)^{1 - \mu_{\mathcal{S}}(c' \to c)} \prod_{\text{vertices } c \text{ in descending order}} e_{i(c)}(z_c) \]

in \( \bar{U}_q^+(Lg)[x_s^{\pm 1}, \ldots, x_t^{\pm 1}, y_s'^{\pm 1}, \ldots, y_{t'}^{\pm 1}] \). Above, \( \delta \) denote Kronecker symbols, and

- \( c \) and \( c' \) go over the vertex set of the graph in Figure 1, i.e. \( \{s, \ldots, t\} \cup \{s', \ldots, t'\} \)
- \( x_s, \ldots, x_t, y_s', \ldots, y_{t'} \) are formal variables, and all other notations in the formula above are defined explicitly in \((3.2) - (3.5)\)
- the sign \( (-1)^{\sigma(\mathcal{S})} \) is defined in \((3.11)\), and
  \[ \mu_{\mathcal{S}}(c' \to c) \in \{0, 1, \ldots\} \]
  denotes the multiplicity of the edge \( c' \to c \) in the multiset \( \mathcal{S} \).
- \( c < c' \) denotes any total order of the vertex set, such that there is no edge in the complement of \( \mathcal{S} \) that points from a smaller vertex to a larger vertex.

Note that the second line of \((1.15)\) is a Laurent polynomial in the variables \( x_s, \ldots, x_t, y_s', \ldots, y_{t'} \) times various series \( e_i \) or \( j(x_a \text{ or } y_b) \), despite the apparent denominators.

**Theorem 1.4.** The ideal \( I^+ \) is generated by the coefficients of the series \( \rho_Z \) of \((1.15)\), as \( Z \) goes over all distinguished zig-zags, for any \( i \neq j \) in \( I \). Thus, the positive half of the quantum loop group may be defined as

\[ U_q^+(Lg) = \mathbb{K} \left\langle e_{i,k} \right\rangle_{i \in I, k \in \mathbb{Z}} \left/ \left( \text{relation (1.9) and } \rho_Z = 0 \right) \right\rangle \forall Z \text{ distinguished zig-zag} \]

The negative half \( U_q^-(Lg) \) can be defined analogously (with \( f_{i,-k} \)'s instead of \( e_{i,k} \)'s).

Since \( \rho_Z \) are formal series in variables \( x_s, \ldots, x_t, y_s', \ldots, y_{t'} \), one might be concerned that \((1.16)\) requires setting all the coefficients of said formal series to 0, thus yielding “too many” relations. In fact, \((1.16)\) still holds if we only ask for the vanishing of a single generic coefficient of \( \rho_Z \) of any given homogeneous degree in the variables \( x_s, \ldots, x_t, y_s', \ldots, y_{t'} \), for any distinguished zig-zag \( Z \) as in Figure 1. The exact meaning of the word “generic” in the previous sentence is given in Remark 3.14.

Distinguished zig-zags with a single northwest-pointing (equivalently, southwest-pointing) diagonal arrow are called **minimal**, and they are in one-to-one correspondence with pairs of non-negative integers \( k, l \) such that

\[ k + l = -d_{ij} \]

where \( k + 1 \) (respectively \( l + 1 \)) is the number of vertices on the top (respectively bottom) row of the zig-zag. For example, when \( k = 2 \) and \( l = 5 \), the minimal zig-zag takes the form in Figure 2.

\[ k + l = -d_{ij} \]

This choice is made possible by the fact that removing \( \mathcal{S} \) leaves no oriented cycles in the graph; the element on the second line of \((1.15)\) does not depend on the choice of total order associated to a refined selection \( \mathcal{S} \), as shown in Proposition 3.3.
Note that $t = s + 2k$ and $t' = s' + 2l$. When $Z$ is the minimal zig-zag associated to $k, l$ as in (1.17), then the selections $\mathcal{S}$ that appear in (1.15) are simply all one-edge subsets of the oriented graph in Figure 2. The signs $(-1)^{\sigma(\mathcal{S})}$ that appear on the first line (1.15) are all 1, and the possible orderings of the vertices $c \in \{s, \ldots, t\} \sqcup \{s', \ldots, t'\}$ that appear on the second line of (1.15) are:

$$\begin{align*}
x_{s+2a} &> x_{s+2(a+1)} > \cdots > x_{t-2} > x_t > \\
&> y_{s'} > y_{s'+2} > \cdots > y_{t'-2} > y_{t'} > \\
&> x_a > x_{s+2} > \cdots > x_{s+2(a-2)} > x_{s+2(a-1)}
\end{align*}$$

for all $a \in \{1, \ldots, k\}$,

$$\begin{align*}
y_{s'+2b} &> y_{s'+2(b+1)} > \cdots > y_{t'-2} > y_{t'} > \\
&> x_a > x_{s+2} > \cdots > x_{t-2} > x_t > \\
&> y_{s'} > y_{s'+2} > \cdots > y_{s'+2(b-2)} > y_{s'+2(b-1)}
\end{align*}$$

for all $b \in \{1, \ldots, l\}$, as well as

$$x_s > x_{s+2} > \cdots > x_{t-2} > x_t > y_{s'} > y_{s'+2} > \cdots > y_{t'-2} > y_{t'}$$

and

$$y_{s'} > y_{s'+2} > \cdots > y_{t'-2} > y_{t'} > x_s > x_{s+2} > \cdots > x_{t-2} > x_t$$

We believe that the relations $\rho_Z = 0$ were not known before for general $Z$, but in the particular case of a minimal zig-zag corresponding to $k = -d_{ij}$, $l = 0$.

**Figure 2. A minimal zig-zag $Z$**

**Figure 3. A particular minimal zig-zag $Z$**
we will show in Proposition 3.19 that the relations $\rho_Z = 0$ are equivalent to the previously known loop version of relations (1.5), namely

\begin{equation}
\text{Sym} \left[ \sum_{k=0}^{1-d_{ij}} (-1)^k \binom{1-d_{ij}}{k} q e_i(z_1) \ldots e_i(z_k) e_j(w) e_i(z_{k+1}) \ldots e_i(z_{1-d_{ij}}) \right] = 0
\end{equation}

where Sym denotes symmetrization in the variables $z_1, \ldots, z_{1-d_{ij}}$. The aforementioned equivalence underscores the fact that our formulas for $\rho_Z$ are not unique; one could add arbitrary multiples of relation (1.9) to $\rho_Z$ and obtain equivalent formulas.

1.5. Our main technical tool to proving Theorem 1.4 is the following trigonometric degeneration of the Feigin-Odesskii [4] shuffle algebra

\begin{equation}
\mathcal{V}^+ = \mathcal{V}^- = \bigoplus_{n \in \mathbb{N}} \mathbb{K}[z \pm 1, \ldots, z \pm 1_{\text{sym}}]_{i \in I} \prod_{i<j \in I} \prod_{n \leq n_i, b \leq n_j} (z_{ia} - z_{jb})
\end{equation}

with the multiplication in $\mathcal{V}^\pm$ defined in Subsection 2.3 (the superscript “sym” in the numerator of (1.19) denotes symmetric Laurent polynomials in $z_{i1}, z_{i2}, \ldots$ for all $i \in I$ separately; meanwhile, the symbol $<$ in the denominator of (1.19) refers to an arbitrary fixed total order on $I$). The following facts were noted in [3].

- There are algebra homomorphisms $\bar{\Upsilon}^\pm: \tilde{U}^\pm (Lg) \to \mathcal{V}^\pm$ given by $e_{i,k}, f_{i,k} \mapsto z_{i1}^k$
- There are bialgebra pairings\(^5\)

\begin{equation}
\tilde{U}^+_q (Lg) \otimes \mathcal{V}^- \rightarrow \mathbb{K}
\end{equation}

\begin{equation}
\mathcal{V}^+ \otimes \tilde{U}^-_q (Lg) \rightarrow \mathbb{K}
\end{equation}

which are non-degenerate in both arguments.

The pairings (1.12) and (1.20)–(1.21) are compatible, in the sense that

\begin{align*}
\left\langle x^+, R^- \right\rangle &= (q^{-1} - q)^{-|\text{deg } x^+|} \left\langle x^+, \bar{\Upsilon}^- (R^-) \right\rangle_{\mathcal{V}^-} \\
\left\langle R^+, x^- \right\rangle &= (q^{-1} - q)^{|\text{deg } x^-|} \left\langle \bar{\Upsilon}^+ (R^+), x^- \right\rangle_{\mathcal{V}^+}
\end{align*}

for any $R^\pm \in \mathcal{V}^\pm$ and $x^\pm \in \tilde{U}^\pm (Lg)$ (see (2.4)–(2.5) for the notation $|\text{deg } x^\pm|$).

1.6. In Definition 2.16 we will construct subalgebras

\begin{equation}
\mathcal{S}^\pm \subset \mathcal{V}^\pm
\end{equation}

We refer the reader to Subsection 2.15 for the full details and notation, but in brief, $\mathcal{S}^\pm$ consists of those rational functions

$$\frac{r(z_{i1}, \ldots, z_{in})_{i \in I} \prod_{i<j \in I} \prod_{n \leq n_i, b \leq n_j} (z_{ia} - z_{jb})}{\mathcal{V}^\pm}$$

\(^5\)Strictly speaking, to claim that the pairings below have the bialgebra property requires one to replace $\tilde{U}^\pm_q (Lg)$, $\mathcal{V}^\pm$ by $\tilde{U}^\pm_q \otimes (Lg), \tilde{U}^\pm_q (Lg), \mathcal{V}^\pm \otimes, \mathcal{V}^\pm \otimes$, where the latter two bialgebras are defined by adding generators $h_{i,k}, h_{i,k}'$ to $\mathcal{V}^\pm$, akin to items (2)–(3) of Subsection 1.2; see Subsection 2.8.
such that for any distinguished zig-zag as in Figure 1, the specialization of the Laurent polynomial \( r \) at the variables

\[
\begin{align*}
  z_{i1} &= xq^s, \\
  z_{i2} &= xq^{s+2}, \\
  z_{i1,t-x} &= xq^{t-2}, \\
  z_{i1,t+x+1} &= xq^t
\end{align*}
\]

\[(1.23)\]

\[
\begin{align*}
  z_{j1} &= yq^{s'}, \\
  z_{j2} &= yq^{s'+2}, \\
  z_{j,t-x'} &= yq^{t'-2}, \\
  z_{j,t+x'+1} &= yq^{t'}
\end{align*}
\]

\[(1.24)\]

(with \( \{s, \ldots, t\} \sqcup \{s', \ldots, t'\} \) as in Subsection 1.3) has the property that

\[
\left| \frac{r}{(x-y)} \right| \quad \text{is divisible by} \quad (x-y) \quad \text{number of southwest pointing arrows in Figure 1}
\]

\[(1.25)\]

When the zig-zag is the specific one displayed in Figure 3, the aforementioned divisibility condition was known to [3, 4], but we believe the divisibility conditions provided by general distinguished zig-zags are new.

**Theorem 1.7.** (Proposition 2.24) We have \( S^\pm = \text{Im} \tilde{\Upsilon}^\pm \).

The connection between the subalgebras \( S^\pm \) and Theorem 1.4 is the following.

**Theorem 1.8.** The subalgebras \( S^\pm \subset V^\pm \) and the quotients \( U^\pm_q(\mathfrak{g}) \twoheadrightarrow \tilde{U}^\pm_q(\mathfrak{g}) \) are mutually dual, i.e. the pairings \( (1.20) \to (1.21) \) descend to pairings given by the diagonal arrows in the diagram below

\[
\begin{array}{ccc}
  \tilde{U}^+_q(\mathfrak{g}) & \otimes & V^- \\
  U^+_q(\mathfrak{g}) & \otimes & S^-
\end{array}
\]

\[
\begin{array}{ccc}
  \langle \cdot, \cdot \rangle_{US} & \quad \text{K} & \quad \langle \cdot, \cdot \rangle_{US} \\
  \langle \cdot, \cdot \rangle_{SU} & \quad \text{K} & \quad \langle \cdot, \cdot \rangle_{SU}
\end{array}
\]

The pairings \( \langle \cdot, \cdot \rangle_{US} \) and \( \langle \cdot, \cdot \rangle_{SU} \) are non-degenerate in both arguments.

Another way to rephrase Theorem 1.8 is that the generators of the ideals \( I^\pm \) can be constructed as linear functionals on \( V^\pm \) which realize the conditions \( (1.25) \) that define the subalgebra \( S^\pm \).

**Theorem 1.9.** The induced homomorphisms

\[
\Upsilon^\pm : U^\pm_q(\mathfrak{g}) \to S^\pm
\]

are isomorphisms. The pairings \( \langle \cdot, \cdot \rangle_{US} \) and \( \langle \cdot, \cdot \rangle_{SU} \) both coincide with a pairing

\[
S^+ \otimes S^- \quad \langle \cdot, \cdot \rangle_{SS} \quad \text{K}
\]

which is non-degenerate in both arguments.

1.10. The present paper follows the philosophy of [12, 13], with the following important observation: various specializations of the parameters that appear in shuffle algebras lead to wildly different generators-and-relations presentations of the associated quantum loop groups. In more detail, consider a quiver with vertex set \( I \) and \( -d_{ij} \) total arrows between any distinct vertices \( i \neq j \). The shuffle algebra considered in relation to this quiver in [12, 13] involved “generic” parameters \( t_e \) associated to the aforementioned arrows; this led to the corresponding shuffle algebra being
determined by 3-variable wheel conditions ([12, equation (2.15)]), and the corresponding quantum loop group being determined by cubic relations ([13, equation (1.6)]). In the present paper, we consider the specialization of the parameters to
\begin{equation}
q^{d_{ij} + 2}, \ldots, q^{−d_{ij}−2}, q^{−d_{ij}}
\end{equation}
for the $−d_{ij}$ arrows between $i$ and $j$. This choice yields the more complicated conditions (1.25) and the more complicated relations $\rho_Z = 0$. We will study the situation of more general specializations in upcoming work.

1.11. An important feature of quantum loop groups in geometric representation theory is their relation to $K$-theoretic Hall algebras of quivers (we follow the presentation of [16], wherein the interested reader may also find an overview of the theory and its history). Specifically, to any quiver $Q$ (without loops), as in the previous Subsection, we may associate the symmetric Cartan matrix $C$ given by
\begin{equation}
−d_{ij} = |\text{arrows between } i \text{ and } j|
\end{equation}
In Section 5, we will recall the construction of the $K$-theoretic Hall algebra
\begin{equation}
K^{\text{nilp}}_{C^*, \text{loc}} = \bigoplus_{n \in \mathbb{N}^I} K_{C^*}((T^*Z_n) \wedge_n \otimes \mathbb{Z}[q^{±1}])
\end{equation}
One defines a convolution product on $K^{\text{nilp}}_{C^*, \text{loc}}$, and a linear map
\begin{equation}
K^{\text{nilp}}_{C^*, \text{loc}} \longrightarrow V^+, \text{geom} := \bigoplus_{n \in \mathbb{N}^I} Q(q)[z_{i1}^{±1}, \ldots, z_{in}^{±1}]_{i \in I}^{\text{sym}}
\end{equation}
which is an algebra morphism, once $V^+, \text{geom}$ is made into an algebra using the shuffle product in Subsection 5.7. We have the following diagram of homomorphisms
\[\begin{array}{ccc}
V^+ & \xrightarrow{Q} & V^+, \text{geom} \\
\downarrow & & \downarrow \\
S^+ & \xrightarrow{\sim} & S^+, \text{geom}
\end{array}\]
with the notation as in (5.14)–(5.16). Denote the image of the map (1.32) by $K^{\text{nilp}}_{C^*, \text{loc}} \subset V^+, \text{geom}$

In simply-laced types (i.e. $d_{ij} \in \{0, −1\}$ for all $i \neq j$), we have $K^{\text{nilp}}_{C^*, \text{loc}} = K^{\text{nilp}}_{C^*, \text{loc}}$; this is not true in general due to the failure of the map (1.32) to be injective.

Theorem 1.12. For any quiver $Q$, we have
\begin{equation}
K^{\text{nilp}}_{C^*, \text{loc}} = S^+, \text{geom} \cong S^+ \cong U^+_q(Lq)
\end{equation}
The isomorphism $K^{\text{nilp}}_{C^*, \text{loc}} \cong U^+_q(Lq)$ was known in finite and affine types ([16]).

---

6 The torus $C^*$ in (1.31) acts on the linear maps corresponding to the $−d_{ij}$ arrows from $i$ to $j$ in the doubled quiver via the characters (1.29); these are known as “normal weights” in [10].

7 The word “one” here conceals decades of foundational work in geometric representation theory; the constructions herein have appeared in many contexts in the work of many mathematicians, but the most relevant to our purposes are [6, 15, 16].
1.13. The plan of the paper is the following.

- In Section 2, we define the algebras $\mathring{S}^{\pm} \subseteq S^{\pm} \subset \mathcal{V}^{\pm}$. The fact that $\mathring{S}^{\pm} = S^{\pm}$ is stated in Proposition 2.24 and will be proved in Section 4.

- In Section 3, we realize the quotient $\mathring{U}^{\pm}_{q}(L\mathfrak{g}) \twoheadrightarrow U^{\pm}_{q}(L\mathfrak{g})$ as dual to $S^{\pm} \subset \mathcal{V}^{\pm}$, and prove Theorems 1.4, 1.8 and 1.9 (modulo a technical result, Proposition 3.18, that will be proved in Section 4).

- In Section 4, we use the combinatorics of words to prove two outstanding technical results, namely Propositions 2.24 and 3.18.

- In Section 5, we discuss $K$-theoretic Hall algebras in relation to shuffle algebras, and prove Theorem 1.12.

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1.14. Let us summarize the main notations used in the Introduction above, as they will be used repeatedly throughout the paper. The main algebras involved are

\begin{align}
\tilde{\Upsilon}^{\pm} : \mathring{U}^{\pm}_{q}(L\mathfrak{g}) &\rightarrow S^{\pm} \subset \mathcal{V}^{\pm} \\
U^{\pm}_{q}(L\mathfrak{g}) &\twoheadrightarrow S^{\pm}
\end{align}

A priori, the image of $\tilde{\Upsilon}^{\pm}$ will be denoted by $\mathring{S}^{\pm} \subset \mathcal{V}^{\pm}$; one of our most important technical results is the fact that $\mathring{S}^{\pm} = S^{\pm}$, to be given in Proposition 2.24. One of our main results is the fact that (1.34) descends to an isomorphism

\begin{align}
\Upsilon^{\pm} : U^{\pm}_{q}(L\mathfrak{g}) &\sim \rightarrow S^{\pm}
\end{align}

where $U^{\pm}_{q}(L\mathfrak{g})$ is the explicit quotient of (1.16). Throughout the present paper, we will encounter pairings

\begin{align}
&U^{\pm}_{q}(L\mathfrak{g}) \otimes \mathring{U}^{\pm}_{q}(L\mathfrak{g}) \langle \cdot, \cdot \rangle_{\mathring{U}^{\pm}_{q}(L\mathfrak{g})} \rightarrow \mathbb{K} \\
&U^{\pm}_{q}(L\mathfrak{g}) \otimes \mathcal{V}^{\pm} \langle \cdot, \cdot \rangle_{\mathcal{V}^{\pm}} \rightarrow \mathbb{K} \\
&\mathcal{V}^{\pm} \otimes U^{\pm}_{q}(L\mathfrak{g}) \langle \cdot, \cdot \rangle_{\mathcal{V}^{\pm}} \rightarrow \mathbb{K} \\
&U^{\pm}_{q}(L\mathfrak{g}) \otimes \mathring{S}^{\pm} \langle \cdot, \cdot \rangle_{\mathring{S}^{\pm}} \rightarrow \mathbb{K} \\
&\mathring{S}^{\pm} \otimes U^{\pm}_{q}(L\mathfrak{g}) \langle \cdot, \cdot \rangle_{\mathring{S}^{\pm}} \rightarrow \mathbb{K} \\
&\mathring{U}^{\pm}_{q}(L\mathfrak{g}) \otimes S^{\pm} \langle \cdot, \cdot \rangle_{S^{\pm}} \rightarrow \mathbb{K} \\
&S^{\pm} \otimes \mathring{U}^{\pm}_{q}(L\mathfrak{g}) \langle \cdot, \cdot \rangle_{S^{\pm}} \rightarrow \mathbb{K} \\
&S^{\pm} \otimes S^{\pm} \langle \cdot, \cdot \rangle_{S^{\pm}} \rightarrow \mathbb{K} \\
&\mathring{S}^{\pm} \otimes \mathring{S}^{\pm} \langle \cdot, \cdot \rangle_{\mathring{S}^{\pm}} \rightarrow \mathbb{K} \\
&S^{\pm} \otimes \mathring{S}^{\pm} \langle \cdot, \cdot \rangle_{\mathring{S}^{\pm}} \rightarrow \mathbb{K}
\end{align}
In all cases, the indices under $\langle \cdot, \cdot \rangle$ are meant to reflect the domain of the pairing in question (which will be important when we discuss issues like non-degeneracy). All the pairings above are compatible with each other under the natural inclusion maps, quotient maps, as well as the homomorphisms (1.34)–(1.35).

One can extend all the algebras above by adding generators $h_{i,k}, h'_{i,k}$ in a compatible way. The resulting objects are topological bialgebras, and they will be denoted by

$$\tilde{U}^\ge_q(Lg), \tilde{U}^\le_q(Lg), U^\ge_q(Lg), U^\le_q(Lg), \tilde{S}^\ge, \tilde{S}^\le, S^\ge, S^\le, \nu^\ge, \nu^\le,$$

All the pairings (1.36)–(1.45) extend to bialgebra pairings between the bialgebras (1.46), in the sense that relations (1.48)–(1.49) hold.

1.15. Let us now introduce some general notation and terminology pertaining to bialgebras, that will be used throughout the paper. We will work over a base field $K$, endowed with an element $q \in K^\times$ which is not a root of unity. All our algebras will be unital, associative algebras over $K$. All our coalgebras will be counital and coassociative over $K$, and all our bialgebras $A$ will be such that the coproduct

$$\Delta : A \to A \hat{\otimes} A$$

is an algebra homomorphism. The hat on top of the $\otimes$ sign in the formula above denotes completion with respect to a certain topology; in all cases studied in the present paper, it will be obvious which completion one needs to take in order to obtain a well-defined coproduct. Thus, we will henceforth drop the term “topological” in front of notions such as “coproduct”, “bialgebra” etc.

1.16. Given bialgebras $A^\ge$ and $A^\le$, we call

$$A^\ge \otimes A^\le \xrightarrow{\langle \cdot, \cdot \rangle} K$$

a bialgebra pairing if it satisfies the properties

$$\langle x, y y' \rangle = \langle \Delta(x), y \otimes y' \rangle$$

$$\langle x x', y \rangle = \langle x \otimes x', \Delta^{\text{op}}(y) \rangle$$

for any $x, x' \in A^\ge$, $y, y' \in A^\le$, where $\Delta^{\text{op}}$ denotes the opposite coproduct. The radical(s) of the pairing (1.47) are the subsets

$$A^\ge \supset I^\ge = \left\{ x \in A^\ge \text{ s.t. } \langle x, A^\le \rangle = 0 \right\}$$

$$A^\le \supset I^\le = \left\{ y \in A^\le \text{ s.t. } \langle A^\ge, y \rangle = 0 \right\}$$

Because of (1.48) and (1.49), it is easy to see that $I^\ge, I^\le$ are two-sided ideals.

**Definition 1.17.** If $I^\ge = 0$ (respectively $I^\le = 0$), then we call the pairing (1.47) non-degenerate in the first (respectively second) argument.
1.18. Given a bialgebra pairing such as (1.47), one defines the Drinfeld double as

\[ A = A^> \otimes A^< \]

where the multiplication on the tensor factors is governed by the relation

\[
\sum_i \sum_j x_{i\,i\,j} \langle x_{i\,i\,j\,2}, y_{j\,j\,2} \rangle = \sum_i \sum_j \langle x_{i\,i\,j\,1}, y_{j\,j\,1} \rangle y_{j\,j\,2}, x_{i\,i\,2}
\]

for any \( x \in A^> \) and \( y \in A^< \) with \( \Delta(x) = \sum_i x_{i\,i\,2} \otimes x_{i\,i\,1} \) and \( \Delta(y) = \sum_j y_{j\,j\,2} \otimes y_{j\,j\,1} \).

Remark 1.19. Relation (1.53) is easy to work with, but we need to first explain how it defines an algebra structure on the vector space \( A \) of (1.52). First of all, one identifies \( x \in A^> \) with \( x \otimes 1 \in A \) and \( y \in A^< \) with \( 1 \otimes y \in A \) for all \( x \) and \( y \). Then one defines for all \( x, x' \in A^> \) and \( y, y' \in A^< \)

\[
(x' \otimes y) \cdot (x \otimes y') = \sum_i \sum_j \langle S(x_{i\,i\,1}), y_{j\,j\,1} \rangle x'_{x,2i} \otimes y'_{x,2j} y_{y,1\,3j} x_{x,3i}
\]

where if \( \Delta^{(2)}(x) = \sum_i x_{i\,i\,2} \otimes x_{i\,i\,1} \) and \( \Delta^{(2)}(y) = \sum_j y_{j\,j\,2} \otimes y_{j\,j\,1} \otimes y_{j\,j\,3} \), the relation

\[
\sum_i \sum_j \langle S(x_{i\,i\,1}), y_{j\,j\,1} \rangle x_{x,2i} y_{y,2j} \langle x_{x,3i}, y_{y,3j} \rangle = yx
\]

is equivalent to (1.53) by general Hopf algebra properties. Above, \( S \) denotes the antipode map, which exists and satisfies all the required properties for all bialgebras considered in the present paper (although we will not write it down explicitly).

Once (1.52) is made into an algebra as above, it is easily seen to be a Hopf algebra by requiring that the natural inclusion maps \( A^>, A^< \to A \) be Hopf algebra homomorphisms.

2. Shuffle algebras

In the present Section, we will define and study the shuffle algebras \( V^\pm \) (and their all-important subalgebras \( S^\pm \)), which will provide models for quantum loop groups.

2.1. Fix a symmetrizable Cartan matrix \( C \) as in (1.1), and set

\[
\zeta_{ij}(x) = \frac{x - q^{-d_{ij}}}{x - 1}
\]

for all \( i, j \in I \). We will now recall the algebra \( \tilde{U}^+_q(Lg) \) of (1.8).

Definition 2.2. Let \( \tilde{U}^+_q(Lg) = \mathbb{K}\langle e_i, k_i \rangle_{i \in I, k \in \mathbb{Z}} \), modulo the relation

\[
e_i(z)e_j(w)\zeta_{ji}(\frac{w}{z}) = e_j(w)e_i(z)\zeta_{ij}(\frac{z}{w})
\]

where

\[
e_i(z) = \sum_{k \in \mathbb{Z}} \frac{e_i(z)}{z^k}
\]
The meaning of (2.2) is that one clears out the denominators of the \( \zeta \) functions, and then equates the coefficients of any \( z^k w^l \) in the two sides of the relation.

Similarly, we define \( \tilde{U}_q^- (Lg) = \mathcal{K} (f_{i,k})_{i \in I, k \in \mathbb{Z}} \) modulo the relation

\[
(2.3) \quad f_i(z) f_j(w) \zeta_{ij} \left( \frac{z}{w} \right) = f_j(w) f_i(z) \zeta_{ji} \left( \frac{w}{z} \right)
\]

where

\[
f_i(z) = \sum_{k \in \mathbb{Z}} f_{i,k} z^k
\]

It is easy to see that \( e_i, k \mapsto f_{i,-k} \) induces an isomorphism

\[
\tilde{U}_q^+ (Lg) \xrightarrow{\sim} \tilde{U}_q^- (Lg)
\]

Note that the algebra \( \tilde{U}_q^\pm (Lg) \) is graded by \( \pm \mathbb{N}^I \times \mathbb{Z} \), via

\[
(2.4) \quad \deg e_i, k = (\varsigma^i, k) \quad \text{and} \quad \deg f_{i,-k} = (-\varsigma^i, -k)
\]

for all \( i \in I, k \in \mathbb{Z} \). Above and henceforth, \( \mathbb{N} \) is considered to contain 0, and \( \varsigma^i \in \mathbb{N}^I \) denotes the \( I \)-tuple of integers with a 1 on position \( i \) and 0 everywhere else. Let

\[
(2.5) \quad |n| = \sum_{i \in I} n_i
\]

for any \( n = (n_i)_{i \in I} \in \mathbb{Z}^I \).

2.3. Fix a total order < on the finite set \( I \). Consider an infinite collection of variables \( z_{i1}, z_{i2}, \ldots \) for all \( i \in I \). For any \( n = (n_i)_{i \in I} \in \mathbb{N}^I \), let \( n! = \prod_{i \in I} n_i! \).

**Definition 2.4.** (34) The big shuffle algebra associated to \( C \) is

\[
(2.6) \quad \mathcal{V}^+ = \bigoplus_{n \in \mathbb{N}^I} \mathcal{K} [z_{i1}^{\pm 1}, \ldots, z_{i_n}^{\pm 1}]_{i \in I} / \prod_{i < j \in I} \prod_{a \leq n_i, b \leq n_j} (z_{ia} - z_{jb})
\]

endowed with the multiplication

\[
(2.7) \quad R(\ldots, z_{i1}, \ldots, z_{in}, \ldots) * R'(\ldots, z_{i1}, \ldots, z_{in}', \ldots) = \frac{1}{n!n'!}.
\]

\[
\text{Sym } \begin{bmatrix} R(\ldots, z_{i1}, \ldots, z_{in}, \ldots) R'(\ldots, z_{i1}, \ldots, z_{in+1}, \ldots, z_{in+n'}, \ldots) \prod_{1 \leq a \leq n_i} \zeta_{ij} \left( \frac{z_{ia}}{z_{jb}} \right) \end{bmatrix}
\]

Above and henceforth, “sym” (resp. “Sym”) denotes symmetric functions (resp. symmetrization) with respect to the variables \( z_{i1}, z_{i2}, \ldots \) for each \( i \in I \) separately.

Define \( \mathcal{V}^- \) to be the same vector space as (2.6), but endowed with the multiplication given by the analogue of formula (2.7), with

\[
\zeta_{ij} \left( \frac{z_{ia}}{z_{jb}} \right) \quad \text{replaced by} \quad \zeta_{ji} \left( \frac{z_{ia}}{z_{jd}} \right)
\]

The algebra \( \mathcal{V}^\pm \) is graded by \( \pm \mathbb{N}^I \times \mathbb{Z} \), via the following assignment for any \( R^\pm \in \mathcal{V}^\pm :
\[
\deg R^\pm(\ldots, z_{i1}, \ldots, z_{in}, \ldots) = (\pm n, \hom \deg R^\pm)
\]
where “hom deg” denotes total homogeneous degree. Denote the graded pieces by
\[ \mathcal{V}^\pm = \bigoplus_{n \in \mathbb{N}} \mathcal{V}_{\pm n} = \bigoplus_{(n,k) \in \mathbb{N} \times \mathbb{Z}} \mathcal{V}_{\pm n,k} \]

2.5. The following are straightforward exercises, which we leave to the reader.

**Proposition 2.6.** There is an algebra isomorphism
\[ \mathcal{V}^+ \xrightarrow{\sim} \mathcal{V}^- \]
given by \( R(\ldots, z_{i_0}, \ldots) \mapsto R(\ldots, z_{i_0}^{-1}, \ldots) \).

**Proposition 2.7.** There exist homomorphisms of \( \pm \mathbb{N} \times \mathbb{Z} \) graded algebras
\begin{align*}
\tilde{U}_q^+ (L \mathfrak{g}) & \xrightarrow{\tilde{\gamma}^+} \mathcal{V}^+, \quad e_{i,k} \mapsto z_{i_1}^k \\
\tilde{U}_q^- (L \mathfrak{g}) & \xrightarrow{\tilde{\gamma}^-} \mathcal{V}^-, \quad f_{i,k} \mapsto z_{i_1}^k
\end{align*}
The maps \( \tilde{\gamma}^\pm \) are neither injective nor surjective, and one of the main purposes of the present paper is to describe
\[ \tilde{S}^\pm = \text{Im} \tilde{\gamma}^\pm \]
(i.e. \( \tilde{S}^\pm \) is the \( \mathbb{K} \)-subalgebra of \( \mathcal{V}^\pm \) generated by \( \{ z_{i_1}^k \}_{i \in I, k \in \mathbb{Z}} \)) and
\[ K^\pm = \text{Ker} \tilde{\gamma}^\pm \]
as a two-sided ideal of \( \tilde{U}_q^\pm (L \mathfrak{g}) \).

2.8. Let us now enlarge the algebras \( \tilde{U}_q^\pm (L \mathfrak{g}) \) and \( \mathcal{V}^\pm \), with the goal of making them into bialgebras. We follow the procedure in Subsection 1.2, items (2)–(3).

**Definition 2.9.** Consider the algebras
\begin{align*}
\tilde{U}_q^\pm (L \mathfrak{g}) &= \tilde{U}_q^\pm (L \mathfrak{g})[h_{i_0}^{\pm 1}, h_{i_1}, h_{i_2}, \ldots]_{i \in I} \\
&= \left( h_i(z) e_j(w) = e_j(w) h_i(z) \frac{\zeta_j(\frac{z}{\zeta_i})}{\zeta_i(\frac{w}{\zeta_i})} \right) \\
\tilde{U}_q^\pm (L \mathfrak{g}) &= \tilde{U}_q^\pm (L \mathfrak{g})[h_{i_0}^{\prime \pm 1}, h_{i_1}^{\prime -1}, h_{i_2}^{\prime -2}, \ldots]_{i \in I} \\
&= \left( h'_i(z) f_j(w) = f_j(w) h'_i(z) \frac{\zeta_j(\frac{z}{\zeta_i})}{\zeta_i(\frac{w}{\zeta_i})} \right)
\end{align*}
where \( h_i(z) = \sum_{k=0}^\infty h_{i,k} z^{-k} \) and \( h'_i(z) = \sum_{k=0}^\infty h'_{i,-k} z^k \). Similarly, define
\begin{align*}
\mathcal{V}^\geq &= \mathcal{V}^+ [h_{i_0}^{\pm 1}, h_{i_1}, h_{i_2}, \ldots]_{i \in I} \\
&= \left( h_i(z) R(\ldots, z_{j_b}, \ldots) = R(\ldots, z_{j_b}, \ldots) h_i(z) \prod_{(j,b)} \frac{\zeta_j(\frac{z}{\zeta_i})}{\zeta_i(\frac{w}{\zeta_i})} \right) \\
\mathcal{V}^\leq &= \mathcal{V}^- [h_{i_0}^{\prime \pm 1}, h_{i_1}^{\prime -1}, h_{i_2}^{\prime -2}, \ldots]_{i \in I} \\
&= \left( h'_i(z) R(\ldots, z_{j_b}, \ldots) = R(\ldots, z_{j_b}, \ldots) h'_i(z) \prod_{(j,b)} \frac{\zeta_j(\frac{z}{\zeta_i})}{\zeta_i(\frac{w}{\zeta_i})} \right)
\end{align*}
One makes sense of the denominators of \( (2.12), (2.13), (2.14), (2.15) \) by expanding in negative, positive, negative, positive powers of the variable \( z \), respectively.
The following Proposition is straightforward, and proved just like [11 Proposition 4.1], so we leave the details as an exercise to the reader.

**Proposition 2.10.** There are coproducts on \( \tilde{U}_q^\geq(Lg) \), \( \tilde{U}_q^\leq(Lg) \), \( \mathcal{V}^\geq \), \( \mathcal{V}^\leq \), given by the following formulas
\[
\Delta(h_i(z)) = h_i(z) \otimes h_i(z)
\]
\[
\Delta(h'_i(z)) = h'_i(z) \otimes h'_i(z)
\]
\[
\Delta(c_i(z)) = h_i(z) \otimes e_i(z) + e_i(z) \otimes 1
\]
\[
\Delta(f_i(z)) = 1 \otimes f_i(z) + f_i(z) \otimes h'_i(z)
\]
for all \( i \in I \), while for all \( R^\pm \in \mathcal{V}^\pm \) we set
\[
\Delta(R^+(\ldots, z_{i_1}, \ldots, z_{i_m}, \ldots)) = \sum_{\{k_i \in \{0, \ldots, n_i\}\}, i \in I} \left[ \prod_{j}^{\mu} h_j(z_{ik_j}) \right] \cdot R^+(\ldots, z_{i_1}, \ldots, z_{ik_1} \otimes z_{ik_1+1}, \ldots, z_{i_m}, \ldots) \prod_{i=1}^{\mu} \prod_{k < l} h_i(z_{ik_l}) \prod_{k < \ell} z_{ik_\ell} \prod_{l} \zeta_{ij} \left( \frac{z_{ik_i}}{z_{ik_l}} \right)
\]
\[
\Delta(R^- (\ldots, z_{i_1}, \ldots, z_{i_m}, \ldots)) = \sum_{\{k_i \in \{0, \ldots, n_i\}\}, i \in I} R^-(\ldots, z_{i_1}, \ldots, z_{ik_1} \otimes z_{ik_1+1}, \ldots, z_{i_m}, \ldots) \prod_{i=1}^{\mu} \prod_{k < \ell} h'_i(z_{ik_\ell}) \prod_{k < \ell} z_{ik_\ell} \prod_{l} \zeta_{ij} \left( \frac{z_{ik_i}}{z_{ik_l}} \right)
\]
In the latter two formulas above, we expand the denominator as a power series as \( |z_{ia}| \ll |z_{jb}| \), and place all the powers of \( z_{ia} \) to the left of the \( \otimes \) sign and all the powers of \( z_{jb} \) to the right of the \( \otimes \) sign (for all \( i, j \in I \), \( 1 \leq a \leq k_i \), \( k_j < b \leq n_j \)).

It is easy to see that (2.8)–(2.9) extend to bialgebra homomorphisms:
\[
\tilde{U}_q^\geq(Lg) \xrightarrow{\tilde{\mathcal{V}}^\geq} \mathcal{V}^\geq \quad \text{and} \quad \tilde{U}_q^\leq(Lg) \xrightarrow{\tilde{\mathcal{V}}^\leq} \mathcal{V}^\leq
\]

2.11. Let \( Dz_a = \frac{dz_a}{2z_1z_n} \). For any rational function \( F(z_1, \ldots, z_n) \), we will write
\[
(2.16) \quad \int_{|z_1| \gg \cdots \gg |z_n|} \frac{F(z_1, \ldots, z_n)}{\prod_{a=1}^{n} Dz_a}
\]
for the constant term in the expansion of \( F \) as a power series in
\[
\frac{z_2}{z_1}, \ldots, \frac{z_n}{z_{n-1}}.
\]
The notation in (2.16) is motivated by the fact that if \( \mathbb{K} = \mathbb{C} \), then one could compute this constant term as a contour integral (with the contours being concentric circles, situated very far from each other compared to the absolute value of \( q \)).

**Definition 2.12.** There exist bialgebra pairings
\[
(2.17) \quad \tilde{U}_q^\geq(Lg) \otimes \mathcal{V}^\leq \xrightarrow{\langle \cdot | \cdot \rangle_\mathcal{V}^\leq} \mathbb{K}
\]
\[
(2.18) \quad \mathcal{V}^\geq \otimes \tilde{U}_q^\leq(Lg) \xrightarrow{\langle \cdot | \cdot \rangle_\tilde{U}_q^\leq} \mathbb{K}
\]
given by
\[ \langle h_i(z), h'_j(w) \rangle_{\tilde{U}V} = \langle h_i(z), h'_j(w) \rangle_{VU} = \frac{\zeta_{ij} \left( \frac{z}{w} \right)}{\zeta_{ji} \left( \frac{w}{z} \right)} \]
while for all \( R^\pm \in V_{\pm n} \) and all \( i_1, \ldots, i_n \in I, k_1, \ldots, k_n \in \mathbb{Z} \), we set
\[ \langle e_{i_1, k_1} \cdots e_{i_n, k_n}, R^- \rangle_{\tilde{U}V} = \int_{|z_1| \gg \ldots \gg |z_n|} \frac{z_1^{k_1} \cdots z_n^{k_n} R^-(z_1, \ldots, z_n)}{\prod_{1 \leq a < b \leq n} \zeta_{i_a i_b} \left( \frac{z_a}{z_b} \right)} \prod_{a=1}^{n} Dz_a \]
and
\[ \langle R^+, f_{i_1, -k_1} \cdots f_{i_n, -k_n} \rangle_{V\tilde{U}} = \int_{|z_1| \ll \ldots \ll |z_n|} \frac{z_1^{-k_1} \cdots z_n^{-k_n} R^+(z_1, \ldots, z_n)}{\prod_{1 \leq a < b \leq n} \zeta_{i_a i_b} \left( \frac{z_a}{z_b} \right)} \prod_{a=1}^{n} Dz_a \]
if \( \zeta^{i_1} + \cdots + \zeta^{i_n} = n \), and 0 otherwise. Implicit in the notation \( (2.19) - (2.20) \) is that we plug the variable \( z_a \) into one of the variables \( z_{i_a \bullet} \) of \( R^\pm \), for every \( a \in \{1, \ldots, n\} \). The choice of such \( \bullet \) is immaterial, due to the symmetry of \( R^\pm \).

The proof is straightforward, and proved just like [11, Proposition 4.2], so we leave the details as an exercise to the reader. The following result will be proved in Section 4.

**Proposition 2.13.** The restriction of the pairings \( (2.17) - (2.18) \) to \( (2.21) \)
\[ \tilde{U}_q^+ (Lg) \otimes V^- \xrightarrow{\langle \cdot, \cdot \rangle_{\tilde{U}V}} \mathbb{K} \quad \text{and} \quad V^+ \otimes \tilde{U}_q^- (Lg) \xrightarrow{\langle \cdot, \cdot \rangle_{V\tilde{U}}} \mathbb{K} \]
are non-degenerate in both arguments.

2.14. We will now define certain subalgebras of \( V^\pm \) which pair trivially with the kernels \( (2.11) \). For any \( i \in I \) and any integers \( s \leq t \) congruent modulo 2, we write
\[ \{s, \ldots, t\}_i = \{s, s+2, \ldots, t-2, t\} \]
(the index \( i \in I \) will play an important role shortly). If \( t = s - 2 \), the progression above is defined to be empty. Given \( i \neq j \) in \( I \), a pair
\[ Z = \{s, \ldots, t\}_i \times \{s', \ldots, t'\}_j \]
will be called a zig-zag (the choice of \( i \) and \( j \) will be part of the datum of the zig-zag). To such a zig-zag, we will associate the number
\[ m_Z = \min \left( \left| (a, b) \in \{s, \ldots, t\}_i \times \{s', \ldots, t'\}_j \text{ s.t. } a = b + d_{ij} \right|, \left| (a, b) \in \{s, \ldots, t\}_i \times \{s', \ldots, t'\}_j \text{ s.t. } a = b - d_{ij} \right| \right) \geq 0 \]
It can be represented graphically as the minimum of the number of northwest-pointing arrows and the number of southwest-pointing arrows in Figure 4.
The figure explains the terminology “zig-zag”. We will call a zig-zag **minimal** if it takes the form

$$t = s + 2k, \quad t' = s' + 2l, \quad t' = s - d_{ij}, \quad t = s' - d_{ij}$$

It is easy to see that for a minimal zig-zag, the right-hand side of (2.22) is \(\min(1, 1) = 1\). We will call a zig-zag **distinguished** if it can be obtained by “repeating” a minimal zig-zag \(m\) times, for some natural number \(m \geq 1\)

$$t = s + 2(k + m - 1), \quad t' = s' + 2(l + m - 1)$$

$$t' = s - d_{ij} + 2(m - 1), \quad t = s' - d_{ij} + 2(m - 1)$$

It is easy to see that for such a distinguished zig-zag \(Z\), the right-hand side of (2.22) is \(\min(m, m) = m\). We will call the number \(m\) the **multiplicity** of \(Z\).
2.15. Using the combinatorics developed in the preceding Subsection, we will now define certain important subalgebras of the shuffle algebra.

**Definition 2.16.** Consider the vector space \( S^{\pm} \subset V^{\pm} \) consisting of symmetric rational functions (recall that \( r \) in formula (2.25) is a Laurent polynomial)

\[
R(\ldots, z_{i_a}, \ldots) = \frac{r(\ldots, z_{i_a}, \ldots)}{\prod_{i<j} \prod_{a\leq n_i, b\leq n_j} (z_{i_a} - z_{j_b})}
\]

where for any distinguished zig-zag of multiplicity \( m \) (as in Figure 6), we have

\[
(x - y)^m \text{ divides } r |_{z_1 = xq^s, z_2 = xq^{s+2}, \ldots, z_{1-s+1} = xq^t, z_{j1} = yq^{s'}, z_{j2} = yq^{s'+2}, \ldots, z_{j-t-s'+2} = yq^{t'}}
\]

If \( d_{ij} = 0 \), then condition (2.26) simply says that \( r \) is divisible by \( z_{i_a} - z_{j_b} \) for all \( a, b \); thus, we could simply cancel these factors against the corresponding factors in the denominator of (2.25). If \( d_{ij} = -1 \), it is easy to show that condition (2.26) for \( m = 1 \) implies conditions (2.26) for all \( m \geq 2 \). However, as soon as \( d_{ij} \leq -2 \), conditions (2.26) for various natural numbers \( m \) are independent.

**Remark 2.17.** When \( m = 1 \) (i.e. the distinguished zig-zag is minimal) and \( k = -d_{ij}, l = 0 \), then property (2.26) is precisely the “wheel condition” of [3, 4].

**Proposition 2.18.** For any \( R \in S^{\pm} \), and any zig-zag \( Z \) as in Figure 4, we have

\[
(x - y)^{m_Z} \text{ divides } r |_{z_1 = xq^s, z_2 = xq^{s+2}, \ldots, z_{1-s+1} = xq^t, z_{j1} = yq^{s'}, z_{j2} = yq^{s'+2}, \ldots, z_{j-t-s'+2} = yq^{t'}}
\]

where \( r \) is the Laurent polynomial in the numerator of (2.25).

**Proof.** We need to show that if property (2.26) holds for all distinguished zig-zags, then it holds for all zig-zags. This is an immediate consequence of the following claim: any zig-zag \( Z \) contains a distinguished zig-zag of multiplicity \( m_Z \). It remains to prove this claim. Let us assume without loss of generality that \( m_Z \) is the number of northwest-pointing arrows in Figure 4. It is easy to see that the left-most northwest-pointing arrow \( A_\downarrow \) must intersect the left-most southwest-pointing arrow \( A_\uparrow \). Thus, the two arrows in question determine a minimal zig-zag as in Figure 5. Any other arrows in the zig-zag \( Z \) are obtained by translating either \( A_\downarrow \) and \( A_\uparrow \) by an even integer to the right. By assumption, the translates by \( 2, 4, \ldots, 2(m_Z - 1) \) to the right of \( A_\downarrow \) and \( A_\uparrow \) are still contained in the zig-zag \( Z \). Therefore, \( Z \) contains a distinguished zig-zag of multiplicity \( m_Z \).

**Proposition 2.19.** \( S^{\pm} \) is a subalgebra of \( V^{\pm} \).
Proof. Let us prove the case \( \pm = + \), as the case \( \pm = - \) is analogous. We need to show that if \( R, R' \in S^\pm \), then \( R \ast R' \) satisfies (2.27) for any zig-zag \( Z \) of the form in Figure 4. In fact, we will prove the stronger fact that every term in the symmetrization (2.7) satisfies the divisibility property (2.27). In other words, consider the expression on the second line of (2.7) and specialize the variables

\[
\text{specialize the variables } \begin{align*}
\zeta_{ii} & \left( \frac{z_{ia}}{z_{ib}} \text{ a variable of } R \right) \\
\zeta_{jj} & \left( \frac{z_{ja}}{z_{jb}} \text{ a variable of } R' \right)
\end{align*}
\]

on the second line of (2.7), the specialization in question is non-vanishing only if \( z_{i,\alpha}, z_{j,\beta} \) are variables of \( R' \) and \( z_{i,\alpha+1}, \ldots, z_{i,\frac{t-s}{2}+1}, z_{j,\beta+1}, \ldots, z_{j,\frac{t'-s'}{2}+1} \) are variables of \( R \) for some \( \alpha \in \{0, \ldots, \frac{t-s}{2}+1\} \) and \( \beta \in \{0, \ldots, \frac{t'-s'}{2}+1\} \). In other words, the zig-zag \( Z \) is partitioned into the two “consecutive” zig-zags

\[
\begin{align*}
Z_1 &= \left( \{s, \ldots, u\}_i, \{s', \ldots, u'\}_j \right) \\
Z_2 &= \left( \{u+2, \ldots, t\}_i, \{u'+2, \ldots, t'\}_j \right)
\end{align*}
\]

where \( u = s + 2(\alpha - 1) \) and \( u' = s' + 2(\beta - 1) \). However, the presence of

\[
\text{specialize the variables } \begin{align*}
\zeta_{ij} & \left( \frac{z_{ia}}{z_{ib}} \text{ a variable of } R \right) \\
\zeta_{ji} & \left( \frac{z_{ja}}{z_{jb}} \text{ a variable of } R' \right)
\end{align*}
\]

implies that the (specialization of) the second line of (2.7) is divisible by a factor of \( x - y \) for every arrow in the zig-zag that points from one of the variables in (2.28) to one of the variables in (2.29). Thus, the required property (2.27) is an immediate consequence of the fact that

\[
m_Z \leq m_{Z_1} + m_{Z_2} + \left| \text{arrows pointing from } Z_2 \text{ to } Z_1 \right|
\]

As shown in the proof of Proposition 2.18 the zig-zag \( Z \) contains a pair of intersecting arrows \( (A_\alpha, A_\beta) \) together with their translates by \( 2, 4, \ldots, 2(m_Z - 1) \) units to the right. Any one of these \( m_Z \) pairs of arrows either lies completely in \( Z_1 \), or it lies completely in \( Z_2 \), or at least one of the arrows in the pair points from \( Z_2 \) to \( Z_1 \). This establishes the inequality (2.32).

\[
\square
\]

2.20. It is easy to note that:

\[
\hat{S}^\pm \subseteq S^\pm \subset \mathcal{V}^\pm
\]

on account of the fact that \( S^\pm \) are subalgebras, and they contain the generators \( \{ z_{ik} \} \in \ell, k \in \mathbb{Z} \) of \( \hat{S}^\pm \). Our choice of the conditions (2.26) that define \( S^\pm \) was motivated by the following result.
Proposition 2.21. The pairings $\langle 2.21 \rangle$ trivially pair anything in the kernels $\langle 2.11 \rangle$ with anything in the subalgebras $S^\pm$, i.e.  

\[(2.33) \quad \left\langle K^+, S^- \right\rangle_{\tilde{U}S} = 0 = \left\langle S^+, K^- \right\rangle_{\tilde{U}S} \]

Therefore, the pairings $\langle 2.21 \rangle$ descend to  

\[(2.34) \quad \tilde{S}^+ \otimes S^- \xrightarrow{(\cdot)_{\tilde{S}S}} K \]
\[(2.35) \quad S^+ \otimes \tilde{S}^- \xrightarrow{(\cdot)_{\tilde{S}S}} K \]

which obviously have the same restriction to  

\[(2.36) \quad \tilde{S}^+ \otimes S^- \xrightarrow{(\cdot)_{\tilde{S}S}} K \]

Akin to Definition 2.9, let us define the extended bialgebras  

\[(2.37) \quad \tilde{S}^\geq, \tilde{S}^\leq \quad \text{and} \quad S^\geq, S^\leq \]

by adding generators $h_{i,k}, h'_{i,k}$ to $\tilde{S}^\pm$ and $S^\pm$, as in Definition 2.9. Then (2.34)–(2.35) extend to bialgebra pairings between the various bialgebras in (2.37). This also explains the word “obviously” before (2.36) to show that two bialgebra pairings are identical, one need only check that they match on the generators, in which case the check is trivial.

Proof. We will prove the first equality in $\langle 2.33 \rangle$, as the second one is analogous. Let us start from the right-hand side of $\langle 2.19 \rangle$ for some $R^2 \in S_{-n}$, and study how the integral changes as we move the contours of integration toward $|z_1| = \cdots = |z_n|$. The following explanation is phrased for $K = \mathbb{C}$ and $q \in \mathbb{C}^*$ satisfying $|q| > 1$; however, this is just a linguistic tool to keep track of various residues involved as we move contours around (since these residues are all rational functions in $q$, they make sense over any field $K$). For any $m \in \{1, \ldots, n\}$, consider the expression  

\[(2.38) \quad X_m = \sum_{\text{fair partition}} \int_{|z_1| \gg \cdots \gg |z_m| \ll |w_1| \cdots \ll |w_t|}^\frac{z_1^{k_1} \cdots z_n^{k_n} R^{-}(z_1, \ldots, z_n)}{\prod_{1 \leq a < b \leq n} \sigma_{ib_a} \left( \frac{z_a}{z_b} \right)} \prod_{s=1}^{t} Dz_a \prod_{a=1}^{m-1} Dw_a \]

In the notation above, a partition is called fair if all of its constituent sets  

\[(2.39) \quad A_s = \{a_s^{(1)} < \cdots < a_s^{(n_s)}\} \]

have the property that $i(A_s) := i_s^{(1)} = \cdots = i_s^{(n_s)}$ for all $s \in \{1, \ldots, t\}$, and the notation on the second line of (2.38) is shorthand for the following iterated residue  

\[
\begin{align*}
\text{Res}_{z_a^{(1)} = z_a^{(2)} = z_a^{(3)} = \cdots = z_a^{(n_s)}} & \left[ \text{Res}_{z_a^{(1)} = z_a^{(2)} = z_a^{(3)} = \cdots = z_a^{(n_s)}} \right] \cdots \\
\end{align*}
\]

followed by relabeling the variable $z_a^{(s)}$ to $w_a q^{l-n_s}$.

Remark 2.22. It is a key part of the argument (and substantially different from the analogous setup studied in [12, Proposition 3.3]) that the new variable $w_s$ whose contour is considered in $\langle 2.38 \rangle$ is the geometric mean of the variables $z_a^{(1)} \cdots z_a^{(n_s)}$. 

Claim 2.23. We have $X_m = X_{m-1}$ for all $m \in \{2, \ldots, n\}$.

Let us first show how Claim 2.23 implies (2.33). By iterating Claim 2.23 a number of $n-1$ times, we conclude that $X_n = X_1$, or more explicitly

$$
\left\langle e_{i_1, k_1} \cdots e_{i_n, k_n}, R^{-} \right\rangle_{\bar{U}S} = \sum_{\{1, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_t} \left[ \left( \frac{z_1^{k_1} \cdots z_n^{k_n}}{\prod_{1\leq a < b \leq n} \zeta_{ib_i} \left( \frac{zb}{z_a} \right)} \right) \prod_{s=1}^t Dw_s \right]
$$

However, for any fixed fair partition $\{1, \ldots, n\} = \bar{A}_1 \sqcup \cdots \sqcup \bar{A}_t$, we claim that

$$
\left[ \left( \frac{z_1^{k_1} \cdots z_n^{k_n}}{\prod_{1\leq a < b \leq n} \zeta_{ib_i} \left( \frac{zb}{z_a} \right)} \right) \prod_{s=1}^t Dw_s \right] = \sum_{\{1, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_t} \left[ \frac{z_1^{k_1} \cdots z_n^{k_n}}{\prod_{1\leq a < b \leq n} \zeta_{ib_i} \left( \frac{zb}{z_a} \right)} \right] \text{Sym} \left( \frac{z_1^{k_1} \cdots z_n^{k_n}}{\prod_{1\leq a < b \leq n} \zeta_{ib_i} \left( \frac{zb}{z_a} \right)} \right)
$$

In (2.41), we write Sym for symmetrization with respect to all pairs of variables $z_a$ and $z_b$ such that $i_a = i_b$, and we denote the elements of $\bar{A}_s$ in any order by

$$
\bar{A}_s = \{ \bar{a}_s^{(1)}, \ldots, \bar{a}_s^{(n_s)} \}
$$

for all $s \in \{1, \ldots, t\}$. Formula (2.41) is an immediate consequence of the fact that the only poles of the integrand involving variables $z_a$ and $z_b$ with $a < b$ and $i_a = i_b$ are $z_a - z_b q^2$. With (2.41) in mind, relation (2.40) yields

$$
\left\langle e_{i_1, k_1} \cdots e_{i_n, k_n}, R^{-} \right\rangle_{\bar{U}S} = \sum_{\{1, \ldots, n\} = \bar{A}_1 \sqcup \cdots \sqcup \bar{A}_t} \left[ \left( \frac{z_1^{k_1} \cdots z_n^{k_n}}{\prod_{1\leq a < b \leq n} \zeta_{ib_i} \left( \frac{zb}{z_a} \right)} \right) \prod_{s=1}^t Dw_s \right]
$$

where in the right-hand side of (2.43) we choose a fixed fair partition $\bar{A}_1 \sqcup \cdots \sqcup \bar{A}_t$ for any unordered sum

$$
n = \sum_{s=1}^t n_s \cdot \zeta^{i(\bar{A}_s)}
$$

(the implication being that $\bar{A}_s$ of (2.42) is chosen to be an arbitrary set of $n_s$ variables among $z_{1s}, \ldots, z_{ns}$, where $i = i(\bar{A}_s)$). However, it is clear that the right-hand side of (2.43) only depends on

$$
\text{Sym} \left( \frac{z_1^{k_1} \cdots z_n^{k_n}}{\prod_{1\leq a < b \leq n} \zeta_{ib_i} \left( \frac{zb}{z_a} \right)} \right) = \bar{T}^+(e_{i_1, k_1} \cdots e_{i_n, k_n}) \prod_{a \neq b} \zeta_{ib_i} \left( \frac{zb}{z_a} \right)
$$
so if a certain linear combination of $e_{i_1,k_1} \cdots e_{i_n,k_n}$ lies in $K^+ = \text{Ker } \overline{T}^+$, then that linear combination will pair trivially with any $R^- \in S^-$. This establishes (2.33).

It remains to prove Claim 2.23 To this end, consider the residue theorem

$$\int_{|z|>|w|} f(z,w) Dz Dw = \int_{|z|=|w|} f(z,w) Dz Dw + \sum_{|c|>1} \int_{z=wc} \left[ \text{Res } f(z,w) \right] Dw$$

for any homogeneous rational function $f$, all of whose poles are of the form $z - wc$. Consider formula (2.38), and let us zoom in on the summand corresponding to a given fair partition $\{m, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_t$. As we move the (larger) contour toward the (smaller) contours of the variables $w_1, \ldots, w_t$, one of two things can happen. The first thing is that the larger contour reaches the smaller ones, which leads to the fair partition

$$\{m - 1, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_t \sqcup \{m - 1\}$$

in formula (2.38) for $m$ replaced by $m - 1$. The second thing is that the variable $z_{m-1}$ gets “caught” in a pole of the form $z_{m-1} = wc$ for some $s \in \{1, \ldots, t\}$ and some $|c| > 1$. However, the apparent poles of the rational function on the second line of (2.38) that involve both $z_{m-1}$ and some $w_s$ for $s \in \{1, \ldots, t\}$ are of the form

$$\begin{cases} 
1 \\
\prod_{\bullet \in \{n_s - 1, n_s - 3, \ldots, 3-n_s, 1-n_s\}} \frac{1}{z_{m-1} - w_s q^{\bullet + t - \lambda_s(A_s)}} 
\end{cases} \begin{array}{ll}
\text{if } i_{m-1} = \lambda(A_s) \\
\text{if } i_{m-1} \neq \lambda(A_s)
\end{array}$$

Let us start with the second option above, and write $d = d_{i_{m-1}(A_s)}$. The apparent simple pole at

$$z_{m-1} = w_s q^{\bullet + d}$$

for some $\bullet \in \{n_s - 1, \ldots, 1 - n_s\}$ such that $\bullet + d \geq 0$ is precisely canceled by the fact that $R$ vanishes at the specialization

$$z_{\lambda(A_s)1} = w_s q^{\bullet + 2d}, z_{\lambda(A_s)2} = w_s q^{\bullet + 2d + 2}, \ldots, z_{\lambda(A_s),1-d} = w_s q^{\bullet}$$

(all the powers of $q$ on the first line lie in the arithmetic progression $\{n_s - 1, \ldots, 1 - n_s\}$) due to the condition (2.26) for the distinguished triangle in Figure 3. As for the first option in (2.44), it leads to the fair partition:

$$\{m - 1, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_{s-1} \sqcup (A_s \sqcup \{m - 1\}) \sqcup A_{s+1} \sqcup \cdots \sqcup A_t$$

in formula (2.38) for $m$ replaced by $m - 1$. However, there is a catch: in this new fair partition, the variables that correspond to the $s$-th part are specialized to

$$w_s q^{n_s+1}, w_s q^{n_s-1}, \ldots, w_s q^{1-n_s}$$

In order to match this with formula (2.38) for $m$ replaced by $m - 1$, we need to move the contour of the variable $w_s$ from

$$|w_s| = |w_r|, \forall r \neq s \quad \text{to} \quad |w_s q| = |w_r|, \forall r \neq s$$

(2.45)
It remains to show that no new poles involving \( w_s \) and \( w_r \) (for an arbitrary \( r \neq s \)) are produced in the rational function (2.46)

\[
\text{Res}_{(z_{a_1}(1), \ldots, z_{a_n}(m)) = (w_r q^{n_r-1}, \ldots, w_r q^{1-n_r})}
\]

\[
\begin{bmatrix}
\frac{z^{k_1}}{1}, \ldots, \frac{z^{k_n}}{n}, R^n (z_1, \ldots, z_n) \\
\prod_{1 \leq a < b \leq n} \frac{\zeta_{b} - \zeta_{a}}{z_{a}}
\end{bmatrix}
\]

as we move the contours according to (2.45). Because all the denominators of (2.46) in the variables \( w_r, w_s \) are of the form \( w_s - w_r \), it suffices to show that the residue (2.46) is regular at both \( w_s q - w_r \) and \( w_s - w_r \).

**Case 1:** regularity at \( w_s q - w_r \). Consider the distinguished zig-zag \( Z \) given in the following picture

![Distinguished Zig-Zag](image)

The denominator of (2.46) will contain a factor of \( w_s q - w_r \) for every diagonal arrow above whose source \( z_b \) and target \( z_a \) satisfy \( b > a \). However, the order (2.39) implies that from every pair of intersecting arrows \( A_{\downarrow} \) and \( A_{\uparrow} \), at most one of them could have the property that its source \( z_b \) has a larger subscript than its target \( z_a \). Since a distinguished zig-zag consists of \( m_Z \) disjoint pairs of intersecting arrows \( (A_{\downarrow}, A_{\uparrow}) \), we conclude that there can be at most \( m_Z \) factors of \( (w_s q - w_r) \) in the denominator of (2.46). However, property (2.26) implies that the numerator of (2.46) is divisible by \( (w_s q - w_r)^{m_Z} \), thus canceling the apparent pole.

**Case 2:** regularity at \( w_s - w_r \). Consider the zig-zag \( Z \) given in the following picture

![Distinguished Zig-Zag](image)

If we removed the bottom right vertex, the resulting zig-zag \( Z' \) would be distinguished. It is easy to see from the picture above that

\[
m_Z = \begin{cases} 
    m_{Z'} + 1 & \text{if } n_s + 1 - d_{(A_s)(A_r)} \in \{n_r - 1, \ldots, 1 - n_r\} \\
    m_{Z'}, & \text{otherwise} 
\end{cases}
\]

The denominator of (2.46) will contain at most \( m_{Z'} \) factors \( w_s - w_r \) which do not involve the bottom right vertex in the Figure above, by the same argument as in **Case 1**. Since only those arrows whose source \( z_b \) and target \( z_a \) satisfy \( b > a \)
produce factors $w_s - w_r$ in the denominator, and since the variable $z_{m-1}$ associated to the bottom right vertex has lower subscript than all of the other ones, we see that the bottom right vertex can give rise to at most one factor $w_s - w_r$ and this only happens if $n_s + 1 - d_{(A_s)(A_r)} \in \{n_r - 1, \ldots, 1 - n_r\}$. Thus, the number of factors $w_s - w_r$ in the denominator is at most equal to the right-hand side of (2.47). However, property (2.27) implies that the numerator of (2.46) is divisible by $(w_s - w_r)^m z$, thus canceling the apparent pole due to (2.47).

In Section 4, we will use the combinatorics of words to prove the following result.

**Proposition 2.24.** We have the identity

\begin{equation}
\tilde{S}^\pm = S^\pm
\end{equation}

Moreover, the pairings (2.34)–(2.35) coincide, thus yielding a pairing

\begin{equation}
S^+ \otimes S^- \xrightarrow{(\cdot, \cdot)_S} \mathbb{K}
\end{equation}

which is non-degenerate in both arguments.

3. Quantum loop groups

The main goal of the present Section is to construct a quotient that fits into the bottom left corner of the diagram below, which will be our desired quantum loop group

\begin{equation}
\tilde{U}_q^+ (L\mathfrak{g}) \otimes V^- \xrightarrow{(\cdot, \cdot)_{\tilde{U}V}} \mathbb{K}
\end{equation}

\begin{equation}
U_q^+ (L\mathfrak{g}) \otimes S^-
\end{equation}

The gist of the construction is that the two vertical arrows above should be dual to each other, i.e. the $\mapsto$ map precisely annihilates those elements $\phi$ such that the linear functionals $(\phi, -)_{\tilde{U}V}$ cut out the inclusion $\hookrightarrow$. The point will be to describe such $\phi$’s (or more precisely a set of generators for the two-sided ideal of such $\phi$’s) and to also prove that the homomorphism (2.8) descends to an isomorphism

\begin{equation}
\tilde{U}_q^+ (L\mathfrak{g}) \xrightarrow{\tilde{r}^+} S^+
\end{equation}

\begin{equation}
U_q^+ (L\mathfrak{g}) \xrightarrow{r^+} S^+
\end{equation}

The analogous results also hold with $+$ replaced by $-$. 
3.1. Let us consider a distinguished zig-zag \( Z \), as in Subsection 2.14.

Figure 7. A distinguished zig-zag \( Z \), decorated with variables

As an oriented graph, its vertex set will be denoted by \( Z_v = \{ s, \ldots, t \}_i \sqcup \{ s', \ldots, t' \}_j \), and its edge set will be denoted by \( Z_e \). We decorate the vertices of the zig-zag with the variables \( x_s, \ldots, x_t \) and \( y_{s'}, \ldots, y_{t'} \), as depicted in Figure 7. Consider the notation

\begin{equation}
(3.2) \quad z_c = \begin{cases} 
  x_c & \text{if } c \in \{ s, \ldots, t \}_i \\
  y_c & \text{if } c \in \{ s', \ldots, t' \}_j
\end{cases}
\end{equation}

and

\begin{equation}
(3.3) \quad \iota(c) = \begin{cases} 
  i & \text{if } c \in \{ s, \ldots, t \}_i \\
  j & \text{if } c \in \{ s', \ldots, t' \}_j
\end{cases}
\end{equation}

for any \( c \in Z_v = \{ s, \ldots, t \}_i \sqcup \{ s', \ldots, t' \}_j \). Then we define

\begin{equation}
(3.4) \quad e_{\iota(c)}(z_c) = \begin{cases} 
  e_i(x_c) & \text{if } c \in \{ s, \ldots, t \}_i \\
  e_j(y_c) & \text{if } c \in \{ s', \ldots, t' \}_j
\end{cases}
\end{equation}

Finally, we will write

\begin{equation}
(3.5) \quad \bar{x}_c = x_c q^c, \quad \bar{y}_c = y_c q^c, \quad \bar{z}_c = z_c q^c
\end{equation}

for all applicable indices \( c \).

**Definition 3.2.** Consider any collection of edges \( H \subset Z_e \) which contains no oriented cycle. Choose a total order of the set of vertices \( Z_v = \{ s, \ldots, t \}_i \sqcup \{ s', \ldots, t' \}_j \) that is compatible with \( H \), i.e. there is an arrow only from a larger vertex to a smaller vertex. Then we consider the formal series

\[ e_H (x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \in \bar{U}^+_q (Lq)[x_s^\pm, \ldots, x_t^\pm, y_{s'}^\pm, \ldots, y_{t'}^\pm] \]

given by the formula

\begin{equation}
(3.6) \quad \prod_{c < c' \in Z_v} \frac{(-1)^{\delta_i(c) \delta_i(c')} (z_c - z_{c'} q^{-d_i(c) - d_i(c')})}{\prod_{c' \rightarrow c \in H} (z_{c'} - z_c)} \prod_{c \in Z_v \text{ in descending order}} e_{\iota(c)}(z_c)
\end{equation}

where \( \delta_{ij} \) is the Kronecker delta symbol, for any \( i, j \in I \). Note that the ratio in (3.6) is a Laurent polynomial in the variables \( x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'} \).

**Proposition 3.3.** The right-hand side of (3.6) does not depend on the chosen total order compatible with \( H \).
Proof. Any two total orders compatible with $H$ can be related by a sequence of transpositions compatible with $H$, i.e. we switch the order of $c$ and $c'$ which are next to each other in the total order and not connected by an edge in $H$. The fact that the right-hand side of (3.6) is unaffected by this transposition is a consequence of

$$
(z_c - z_c^{-d_i(c,c')}) e(z_c) e(z_c) = - \left( z_{c'} - z_c^{-d_i(c,c')} \right) e(z_c) e(z_{c'})
$$

which is precisely (2.2).

3.4. For $H \subset Z_e$ as above, let us compute

$$
E_H(x_s, \ldots, x_t, y_{s'}', \ldots, y_{t'}') = \tilde{\Upsilon}^+(e_H(x_s, \ldots, x_t, y_{s'}', \ldots, y_{t'}'))
$$
as elements of $S^+[x_s^\pm 1, \ldots, x_t^\pm 1, y_{s'}'^\pm 1, \ldots, y_{t'}'^\pm 1]$. We will consider the formal series

$$
\delta(u) = \sum_{k \in \mathbb{Z}} u^k
$$

known as the delta function, and note that it has the property that

$$
\delta \left( \frac{u}{v} \right) P(u) = \delta \left( \frac{u}{v} \right) P(v)
$$

for any Laurent polynomial $P(u)$. Because of this, we might say that the delta function “matches” the variables $u$ and $v$.

**Proposition 3.5.** For any $H$ as above, we have

$$
E_H(x_s, \ldots, x_t, y_{s'}', \ldots, y_{t'}') = \text{Sym} \left[ \prod_{a=1}^{t-s+1} \delta \left( \frac{x_{a+2} - z_{ia}}{z_{ia}} \right) \prod_{b=1}^{t'-s'}+1 \delta \left( \frac{y_{b'+2} - z_{jb}}{z_{jb}} \right) \prod_{(c' \rightarrow c) \in H} (\tilde{z}_{c'} - \tilde{z}_c) \prod_{a,b} \left( z_{ia} - z_{ia} q^{-d_i} \right) \prod_{a \neq a'} \left( z_{ia} - z_{ia'} q^{-d_i} \right) \prod_{b \neq b'} \left( z_{jb} - z_{jb'} q^{-d_i} \right) \right]
$$

On the second line, the indices $a, a'$ (resp. $b, b'$) go from 1 to $\frac{t-s}{2}$ (resp. $\frac{t'-s'}{2}$).

In order for the RHS of formula (3.8) to be an appropriately defined formal series with values in $S^+$, we would need to replace in the denominator of the first line

$$
\tilde{z}_c \text{ by } z_{(c)\Box(c)} q^c
$$

for every $c \in \{s, \ldots, t\} \sqcup \{s', \ldots, t'\}$, where the natural number $\Box(c)$ is chosen such that the delta functions on the first line of (3.8) “match” $z_{(c)\Box(c)}$ with $\tilde{z}_c$. We prefer to keep our formulas in the form (3.8) for better legibility henceforth.
Proof. Since \( \bar{\Upsilon}^+(e_i(x)) = \delta \left( \frac{x_i}{x} \right) \), it is an immediate consequence of (2.7) that

\[
\bar{\Upsilon}^+ \left( \prod_{c \in Z_e \text{ in descending order}} e_{i(c)}(z_c) \right) = \text{Sym} \left[ \prod_{c < c' \in Z_e} \zeta_{i(c')i(c)} \left( \frac{z_{i(c')\square(c')}}{z_{i(c)\square(c)}} \right) \prod_{a=1}^{\frac{t-2}{2}+1} \delta \left( \frac{x_s+2a}{z_{ia}} \right) \prod_{b=1}^{\frac{t'-2}{2}+1} \delta \left( \frac{y_{s'2b+2}}{z_{jb}} \right) \right]
\]

where \( \square(c) \in \mathbb{N} \) is defined in (3.9). By the same token, we have

\[
\bar{\Upsilon}^+(e_H) = \text{Sym} \left[ \prod_{c < c' \in Z_e} \left( (-1)^{d_{i(c),i(c')}} (z_{i(c)\square(c)} - z_{i(c')\square(c')} q^{-d_{i(c),i(c')}}) \right) \prod_{i(c) \rightarrow c} \left( z_{i(c)\square(c')} q^{i(c')} - z_{i(c)\square(c)} q^{i(c)} \right) \right]
\]

where we used (3.7) to match a Laurent polynomial in the variables \( z_c \) to the same Laurent polynomial in the variables \( z_{i(c)\square(c)} \). The formula above matches (3.8), according to the paragraph immediately below the statement of Proposition 3.6.

3.6. We will now define a specific linear combination of the elements \( E_H \) (as \( H \) runs over the subsets of \( Z_e \) without oriented cycles) which is 0, and thus the corresponding linear combination of elements \( e_H \) will lie in the kernel of \( \bar{\Upsilon}^+ \). Recall that the distinguished zig-zag \( Z \) consists of \( m \) southwest pointing arrows (below, we indicate the source and target of the arrows in terms of the associated variables)

\[
\left\{ x_{t-2a} \to y_{t-2a+d_{ij}} \right\}_{0 \leq \alpha < m}
\]

and \( m \) northwest pointing arrows

\[
\left\{ y_{t'-2a} \to x_{t'-2a+d_{ij}} \right\}_{0 \leq \alpha < m}
\]

It is thus natural to cover the zig-zag \( Z \) by the \( m \) overlapping trapezoids

\[
\begin{align*}
x_{t-2a-2k} & \to x_{t-2a} \\
y_{t'-2a-2l} & \to y_{t'-2a}
\end{align*}
\]

as \( \alpha \) goes from 0 to \( m - 1 \) (recall that \( t' - t = d_{ij} + 2l = -d_{ij} - 2k \) as a consequence of (2.23)–(2.24), so the diagonal arrows above shift indices by \( d_{ij} \)). The four edges of a trapezoid will be marked by the symbols \( \nearrow, \searrow, \nwarrow, \swarrow \), where the notation \( \uparrow \) and \( \downarrow \) is reserved for the top and bottom horizontal arrows of (3.10), respectively. A selection is a choice

\[
S = \{ \varepsilon_0, \ldots, \varepsilon_{m-1} \}
\]

of an edge \( \varepsilon_\alpha \) from the \( \alpha \)-th trapezoid, for all \( \alpha \in \{0, \ldots, m - 1\} \), subject to the following conditions for all \( \alpha \in \{0, \ldots, m - 2\} \)
• If $\varepsilon_\alpha$ is $\check{\searrow}$ or $\downarrow$, then $\varepsilon_{\alpha+1}$ can only be $\check{\searrow}$ or $\downarrow$
• If $\varepsilon_\alpha$ is $\searrow$ or $\downarrow$, then $\varepsilon_{\alpha+1}$ can only be $\searrow$ or $\downarrow$

The sign of a selection $S$ is $(-1)^{\sigma(S)}$, where

$$\sigma(S) = \left| 1 \leq \alpha < m \text{ s.t. } \varepsilon_\alpha \text{ is } \searrow \text{ or } \downarrow \right|$$

and for the chosen edge $\varepsilon_\alpha$ in the $\alpha$-th trapezoid, we write

$$\delta(\varepsilon_\alpha) = \begin{cases} 
\bar{x}_{t-2\alpha} - \bar{y}_{t-2\alpha} - 2l & \text{if } \varepsilon_\alpha \text{ is } \check{\searrow} \\
\bar{y}_{t-2\alpha} - \bar{y}_{t-2\alpha} - 2l & \text{if } \varepsilon_\alpha \text{ is } \downarrow \\
\bar{y}_{t-2\alpha} - \bar{x}_{t-2\alpha} - 2k & \text{if } \varepsilon_\alpha \text{ is } \searrow \\
\bar{x}_{t-2\alpha} - \bar{x}_{t-2\alpha} & \text{if } \varepsilon_\alpha \text{ is } \uparrow 
\end{cases}$$

(the variables in question are as in (3.10), see also (3.5) for the bar notation).

**Proposition 3.7.** With the terminology above, we have the following combinatorial identity

$$\sum_{S \text{ selection}} (-1)^{\sigma(S)} \prod_{\alpha=0}^{m-1} \delta(\varepsilon_\alpha) = 0$$

for any distinguished zig-zag $Z$.

**Proof.** We will prove the required statement by induction on $m$ (the base case $m = 1$ is trivial, as all the signs are 1 and all products in (3.13) consist of a single linear factor). For any selection $S = \{\varepsilon_0, \ldots, \varepsilon_{m-2}, \varepsilon_{m-1}\}$ let $S' = \{\varepsilon_0, \ldots, \varepsilon_{m-2}\}$. Once we have chosen $S'$, then the two possible choices for $\varepsilon_{m-1}$ yield the following contribution to $\delta(\varepsilon_{m-1})$ in (3.13):

• $\bar{x}_{t-2(m-1)} - \bar{y}_{t-2(m-1)} + \bar{y}_{t-2(m-1)} - \bar{y}_{t-2(m-1)} - \bar{y}_{t-2(m-1)}$ if $\varepsilon_{m-2}$ is $\check{\searrow}$ or $\downarrow$
• $\bar{x}_{t-2(k+m-1)} - \bar{x}_{t-2(m-1)} + \bar{y}_{t-2(m-1)} - \bar{x}_{t-2(m-1)} - \bar{x}_{t-2(m-1)}$ if $\varepsilon_{m-2}$ is $\searrow$ or $\downarrow$

In both cases, the quantity above is $\pm(\bar{x}_{t-2(m-1)} - \bar{y}_{t-2(m-1)})$. The sign is precisely accounted for by the discrepancy between $\sigma(S)$ and $\sigma(S')$, so we conclude that

$$\left( \text{LHS of (3.13) for } m \right) = \left( \text{LHS of (3.13) for } m-1 \right) \cdot \left( \bar{x}_{t-2(m-1)} - \bar{y}_{t-2(m-1)} \right)$$

As the RHS is zero by the induction hypothesis, so is the LHS. This completes the proof of the induction step. □
3.8. A small issue with trapezoids, as defined above, is that their \( \uparrow \) and \( \downarrow \) edges typically consist of more that one arrow in the oriented graph \( Z \). Thus, let us define the notion of refined selection

\[
\mathcal{S} = \{ \varepsilon_0, \ldots, \varepsilon_{m-1} \}
\]

as a choice of a single edge of \( Z \) for each trapezoid (intuitively, whereas a selection might choose an entire group of \( k \), respectively \( l \), consecutive edges from the upper, respectively lower part of \( Z \), a refined selection only chooses a single edge from such a group). So a refined selection involves the choice of one of the following types of edges from the \( \alpha \)-th trapezoid, for each \( \alpha \in \{0, \ldots, m - 1\} \)

\[
\varepsilon_\alpha = \begin{cases} 
    x_{t-2\alpha} \to y_{t'-2\alpha-2l} & \text{or} \\
    y_{t'-2\alpha-2v} \to y_{t'-2\alpha-2(v-1)} & \text{or} \\
    y_{t'-2\alpha} \to x_{t-2\alpha-2k} & \text{or} \\
    x_{t-2\alpha-2u} \to x_{t-2\alpha-2(u-1)} 
\end{cases}
\]

for some \( u \in \{1, \ldots, k\} \) and \( v \in \{1, \ldots, l\} \). For each type of edge above, we define \( \delta(\varepsilon_\alpha) \) as the difference of the variable on the left minus the variable on the right, with bars on top (see (3.12)). Also define \( \sigma(\mathcal{S}) \) by the same formula (3.11) as \( \sigma(S) \).

It is easy to observe that (3.13) implies

\[
\sum_{\mathcal{S} \text{ refined selection}} (-1)^{\sigma(\mathcal{S})} \prod_{\alpha=0}^{m-1} \delta(\varepsilon_\alpha) = 0
\]

simply because \( \sum_{u=1}^k (\bar{x}_{t-2\alpha-2u} - \bar{x}_{t-2\alpha-2(u-1)}) = \bar{x}_{t-2\alpha-2k} - \bar{x}_{t-2\alpha} \) (as well as the analogous formula for \( \bar{y}'s \) instead of \( \bar{x}'s \)).

**Proposition 3.9.** Let \( H \subset Z_e \) be the complement of a refined selection. Then \( H \) does not contain any oriented cycles.

**Proof.** Assume the contrary, that the complement \( H \) of a refined selection \( \mathcal{S} \) contained an oriented cycle. Because of the specific shape of the oriented graph \( Z \), this can only happen if there are two intersecting diagonal edges in \( H \), together with all the horizontal edges connecting the endpoints of the diagonal ones. In other words, the complement \( H \) of \( \mathcal{S} \) must contain a cycle of the form

\[
x_u \to y_{u'} - 2u' \to y_{u'-2(v-1)} \to \ldots y_{u'-2} \to y_{u'} \to \\
\to x_{u-2k'} \to x_{u-2(k'-1)} \to \cdots \to x_{u-2} \to x_u
\]

for some non-negative integers \( k', l' \) such that \( k' + l' = -d_{ij} \), and some \( u = t - 2\alpha, u' = t' - 2\alpha' \) with \( \alpha, \alpha' \in \{0, \ldots, m - 1\} \) satisfying

\[
u' - 2l' = u + d_{ij} \iff u - 2k' = u' + d_{ij}
\]

The assumptions above, together with \( t' - 2l = t + d_{ij} \iff t - 2k = t' + d_{ij} \), yield

\[
\alpha \geq \alpha' \iff k \geq k' \iff l \leq l'
\]

We have three cases to analyze.
Case 1: If $\alpha = \alpha'$, then $\varepsilon_\alpha$ cannot be $\not\leftarrow$ or $\searrow$, due to the presence of these edges in the cycle (3.15). However, if $\varepsilon_\alpha$ were $\uparrow$ or $\downarrow$, the fact that $k = k'$ and $l = l'$ implies that one of the edges
\[
x_{t-2\alpha-2k} \to x_{t-2\alpha-2(k-1)} \to \cdots \to x_{t-2\alpha-2} \to x_{t-2\alpha}
\]
\[
y_{t'-2\alpha-2l} \to y_{t'-2\alpha-2(l-1)} \to \cdots \to y_{t'-2\alpha-2} \to y_{t'-2\alpha}
\]
lies in $\mathcal{S}$. Then the edge in question does not lie in $H$, which contradicts the presence of the cycle (3.15) in $H$.

Case 2: If $\alpha > \alpha'$, then $l < l'$ and for the same reason as in Case 1

\[
\varepsilon_{\alpha'+1}, \ldots, \varepsilon_\alpha \text{ cannot be } \downarrow
\]

But $\varepsilon_{\alpha'}$ also cannot be $\searrow$, which leaves only two choices: $\varepsilon_{\alpha'}$ must be $\not\leftarrow$ or $\uparrow$. In either of these two cases, the rules of selections imply that

\[
(\varepsilon_{\alpha'+1}, \ldots, \varepsilon_\alpha) = (\not\leftarrow, \ldots, \not\leftarrow)
\]

The fact that $\varepsilon_\alpha$ must be $\not\leftarrow$ contradicts the existence of the cycle (3.15) in the complement of the selection.

Case 3: If $\alpha < \alpha'$, then $k < k'$ and for the same reason as in Case 1

\[
\varepsilon_{\alpha}, \ldots, \varepsilon_{\alpha'} \text{ cannot be } \uparrow
\]

But $\varepsilon_\alpha$ also cannot be $\not\leftarrow$, which leaves only two choices: $\varepsilon_\alpha$ must be $\searrow$ or $\downarrow$. In either of these two cases, the rules of selections imply that

\[
(\varepsilon_{\alpha+1}, \ldots, \varepsilon_{\alpha'}) = (\searrow, \ldots, \searrow)
\]

The fact that $\varepsilon_{\alpha'}$ must be $\searrow$ contradicts the existence of the cycle (3.15) in the complement of the selection.

\[
\square
\]

3.10. If we divide (3.14) by
\[
\prod_{\langle c' \to c \rangle \in Z_c} (\bar{z}_{c'} - \bar{z}_c)
\]
then together with (3.8) we have

\[
\sum_{\mathcal{S}} (-1)^{\sigma(\mathcal{S})} E_{Z_c \setminus \mathcal{S}} = 0
\]

However, we owe the reader a clarification about the notation $E_{Z_c \setminus \mathcal{S}}$. If $\mathcal{S}$ were simply a set of edges of $Z_c$, then the complement $H = Z_c \setminus \mathcal{S}$ has no oriented cycles by Proposition 3.9 and we may use formula (3.8). However, in general $\mathcal{S}$ is a multiset of edges of $Z_c$, because a given horizontal edge in Figure 7 may appear more than once in $\mathcal{S}$. If we denote the multiplicity of an edge in the multiset $\mathcal{S}$ by $\mu(\mathcal{S}(c' \to c) \in \{0, 1, \ldots\}$, then we define the following analogue of (3.8)

\[
E_{Z_c \setminus \mathcal{S}}(x_s, \ldots, x_t, y_s' \ldots, y_t') = \text{Sym} \left[ \prod_{a=1}^{l-1} \delta \left( \frac{\bar{z}_{a+2a}}{\bar{z}_{ia}} \right) \prod_{b=1}^{l-1} \delta \left( \frac{\bar{y}_{b+2b}}{\bar{z}_{jb}} \right) \prod_{\langle c' \to c \rangle \in H} \left( \bar{z}_{c'} - \bar{z}_c \right)^{1 - \mu(\mathcal{S}(c' \to c))} \right]
\]

\[
\prod_{a,b} \frac{z_{ia} - z_{jb}q^{-d_i}}{z_{ia} - z_{jb}} \cdot \prod_{a \neq a'} \frac{z_{ia} - z_{ia}q^{-d_i}}{z_{ia} - z_{ia'}} \cdot \prod_{b \neq b'} \frac{z_{jb} - z_{jb}q^{-d_i}}{z_{jb} - z_{jb'}}
\]
as an element of $S^+[x_s^\pm 1, \ldots, x_t^\pm 1, y_{s'}^\pm 1, \ldots, y_{t'}^\pm 1]$. Similarly, we write
\begin{equation}
(3.19) \quad e_{Z_s\setminus S}(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) = \frac{1}{\prod_{c' \to c} \epsilon(\bar{z}_{c'} - \bar{z}_c)} \prod_{c \in Z_e, \text{ descending order}} e_{c}(z_c)
\end{equation}

for any refined selection $S$ (above, $c < c'$ denotes any total order on $Z_e$ compatible with $Z_s \setminus S$, defined just like the analogous notion in (3.6)).

**Proposition 3.11.** For any distinguished zig-zag $Z$, the formal series
\begin{equation}
(3.20) \quad \rho_Z := \sum_{S \text{ refined selection}} (-1)^{\sigma(S)} e_{Z_s\setminus S}(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) = \prod_{c \in Z_e, \text{ descending order}} e_{c}(z_c)
\end{equation}

lies in $K^+[x_s^\pm 1, \ldots, x_t^\pm 1, y_{s'}^\pm 1, \ldots, y_{t'}^\pm 1]$, where $K^+ = \ker \tilde{\Upsilon}^+$. The Proposition is an immediate consequence of (3.18), as it is easy to note that
\[
\tilde{\Upsilon}^+(e_{Z_s\setminus S}) = E_{Z_e\setminus S}
\]

For any Laurent polynomial $\tau(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'})$, we define
\[
e_{Z_e\setminus S, \tau} = \left[ \tau(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) e_{Z_s\setminus S}(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \right]_{\text{constant term}}
\]
\[
\rho_{Z, \tau} = \left[ \tau(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \rho_Z(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \right]_{\text{constant term}}
\]

and Proposition 3.11 implies that $\rho_{Z, \tau} \in K^+$. 

3.12. Our interest in the series $\rho_Z$ is motivated by the following result.

**Proposition 3.13.** An element $R \in \mathcal{V}^-$ lies in $S^-$ if and only if
\begin{equation}
(3.21) \quad \left[ \tilde{U}_q^+(L\varrho) \cdot \rho_Z(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \cdot \tilde{U}_q^+(L\varrho), R \right]_{\tilde{U}\mathcal{V}} = 0
\end{equation}

for all distinguished zig-zags $Z$.

**Remark 3.14.** In fact, the “if” statement of Proposition 3.13 remains true if one replaced the entire formal series $\rho_Z$ by the particular elements
\[
\{ \rho_{Z, \tau_k} \}_{k \in \mathbb{Z}} \subset \tilde{U}_q^+(L\varrho)
\]

where $\tau_k$ is any degree $k$ homogeneous polynomial in the variables $x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}$ satisfying
\[
\tau_k \left( q \frac{t-s}{2}, \ldots, q \frac{x-s}{2}, q \frac{t'-s'}{2}, \ldots, q \frac{x'-t'}{2} \right) \neq 0
\]

Proof. We will use the following statement as a stepping stone.
Claim 3.15. Consider a distinguished zig-zag $Z$ as in Figure 7, and an element
\[ R \in \mathcal{V} - e^\chi((\frac{-1}{a} + 1) - e^\chi((\frac{-1}{a} + 1)) \]
which satisfies \((2.26)\) for all distinguished zig-zags $Z' \subsetneq Z$. Then $R$ satisfies \((2.26)\) for the zig-zag $Z$ if and only if
\[
\left< \rho_Z(x_1, \ldots, x_{i'}, y_{s'}, \ldots, y_{t'}), R \right>_{\mathcal{V}} = 0
\]
The “if” statement can be weakened according to Remark 3.14.

Let us show how Claim 3.15 allows us to prove Proposition 3.13. The “only if” statement of \((3.21)\) is an immediate consequence of \((2.33)\) and Proposition 3.11. For the “if” statement, consider a distinguished zig-zag $Z$ as in Figure 7, and let us fix a partition of the variables of $R$ into two groups
\[
\bullet \text{ The big variables } u_1, \ldots, u_p \text{ of some colors } i_1, \ldots, i_p
\]
\[
\bullet \text{ The small variables } x_1, \ldots, x_{i'}, y_{s'}, \ldots, y_{t'} \text{ of colors } i, j, \ldots, j
\]
With respect to this choice of variables, we may expand
\[
R = \sum_{k_1, \ldots, k_p \in \mathbb{Z}} \frac{u_1^{-k_1} \cdots u_p^{-k_p} R_{k_1, \ldots, k_p}(x_1, \ldots, x_{i'}, y_{s'}, \ldots, y_{t'})}{\prod_{a < b} \left(1 - \frac{u_a}{u_b}\right) \prod_{b, c} \left(1 - \frac{u_a}{u_b}\right)}
\]
(the products in the denominator run over various pairs of indices: $a$ and $b$ over \{1, \ldots, p\}, and $c$ over \{s, \ldots, t\} $\cup$ \{s', \ldots, t'\}; $z_c$ are defined as in \((3.2))\). It is very important that only finitely many of the $R_{k_1, \ldots, k_p}$ are non-zero, which is a consequence of the fact that any $R \in \mathcal{V}^-$ is a Laurent polynomial divided by a specific collection of linear factors. Then, formula \((2.19)\) takes the following form
\[
\left< e_1, k_1' \cdots e_p, k_p' \cdot \rho_Z, R \right>_{\mathcal{V}} = \int_{|u_1| > \cdots > |u_p| > |x_{i'}| > |y_{s'}|} \sum_{k_1, \ldots, k_p \in \mathbb{Z}} \frac{u_1^{-k_1} \cdots u_p^{-k_p} R_{k_1, \ldots, k_p}(x_1, \ldots, x_{i'}, y_{s'}, \ldots, y_{t'})}{\prod_{a < b} \left(1 - \frac{u_a}{u_b}\right) \prod_{b, c} \left(1 - \frac{u_a}{u_b}\right)}
\]
where the denominator goes over all allowable indices (in particular, $c$ runs over \{s, \ldots, t\} $\cup$ \{s', \ldots, t'\}); and $z_c$ is given by \((3.2)\) and \bullet denotes various integers that will not be important in the subsequent argument. To show that $R$ satisfies \((2.26)\) for the given zig-zag $Z$, we must show that
\[
(x - y)^{m_Z} \mid R_{k_1, \ldots, k_p}(x q^a, \ldots, x q^l, y q^{s'}, \ldots, y q^{t'})
\]
for all $k_1, \ldots, k_p$. We will do so by induction on the finite set
\[
\left\{(k_1, \ldots, k_p) \text{ s.t. } R_{k_1, \ldots, k_p} \neq 0\right\}
\]
in lexicographic order. Formula \((3.21)\) implies that the left-hand side of \((3.24)\) is zero for all $k_1', \ldots, k_p'$. Therefore, the same is true for the right-hand side, and we obtain
\[
0 = \left< \rho_Z, R_{k_1', \ldots, k_p'} \right>_{\mathcal{V}} + \sum_{(k_1, \ldots, k_p) < (k_1', \ldots, k_p')} \left< \rho_Z, b \cdot R_{k_1, \ldots, k_p} \right>_{\mathcal{V}}\text{ lexicographically}
\]
where \( b \) stands for Laurent polynomials in \( x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'} \) that arise from the power series expansion of the second line of (3.24). The induction hypothesis implies that (3.25) holds for all terms in the underlined sum in the right-hand side of (3.26); thus Claim 3.15 implies that the underlined sum is 0. Therefore, so is the first pairing in (3.26), and invoking Claim 3.15 again implies that

\[
(x - y)^{m_x} R_{k_1', \ldots, k_p'}(x q^s, \ldots, x q^t, y q^{s'}, \ldots, y q^{t'})
\]

The induction step is thus complete.

**Remark 3.16.** The argument above shows that (3.21) would still remain valid if one replaced the two-sided ideal generated by the \( \rho_Z \)'s by the corresponding left ideal (and by an analogous proof, one could use instead the corresponding right ideal).

It remains to prove Claim 3.15. The “only if” statement follows from (2.33) and Proposition 3.11. For the “if” statement, note that (2.19) and (3.19) imply the following formula for any refined selection \( \mathcal{S} \) and any Laurent polynomial \( \tau \)

\[
\left( e_{Z_s \setminus \mathcal{S}, \tau}, R \right)_{U \mathcal{V}} = \int_{\{z_{c'} \mid \exists z_c \mid (c' \to c) \in \mathcal{S}}} \omega(\tau) (x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \prod_{(c' \to c) \in \mathcal{S}} (z_c - z_{c'})^{1 - \mu(c' \to c)} \prod_c Dz_c
\]

where

\[
\omega(\tau) = (\tau \cdot r)(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \prod_{a < a' \in \{s, \ldots, t\}} (x_a - x_{a'}) \prod_{b < b' \in \{s', \ldots, t'\}} (y_{b'} - y_b)
\]

is a Laurent polynomial.\(^8\) Running the argument in the proof of Proposition 2.21 (with the terminology therein), we deduce the following analogue of (2.43)

\[
\left( e_{Z_s \setminus \mathcal{S}, \tau}, R \right)_{U \mathcal{V}} \text{ for } \mathcal{S} \text{ a fixed fair partition of } \{1, \ldots, n\} \text{ of } A_1 \sqcup \cdots \sqcup A_t \\text{ with each set } A_s = \{z_{e_s(1)}, \ldots, z_{e_s(n)}\}
\]

where each set \( A_s = \{z_{e_s(1)}, \ldots, z_{e_s(n)}\} \) consists either of all \( x \)'s or all \( y \)'s.

The “\( \sim \)” symbol in (3.29) is due to the following slight imprecision. In proving (2.43), we used Claim 2.23 which used the fact that \( R \) satisfies (2.26) for the distinguished zig-zag \( Z \) and all of its sub zig-zags. In the present situation, our assumption tells us that \( R \) satisfies (2.26) for all proper sub zig-zags of \( Z \), but we do not have this property for \( Z \) itself. This means that the discrepancy in “\( \sim \)” stems from the residue when \( t = 2 \) and

\[
A_1 = \{x_s, \ldots, x_t\} \quad \text{and} \quad A_2 = \{y_{s'}, \ldots, y_{t'}\}
\]

\( \text{Res} \) is the connection between the rational function \( R \) and the Laurent polynomial \( r \)

\[
R(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) = \frac{r(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'})}{\prod_{a \in \{s, \ldots, t\}, b \in \{s', \ldots, t'\}} (x_a - y_b)}
\]

see (2.25). In all formulas involving \( R \) and \( r \), we plug \( x_s, \ldots, x_t \) (respectively \( y_{s'}, \ldots, y_{t'} \)) into the variables of color \( i \) (respectively \( j \)) of \( R \) and \( r \).
However, the existence of the residue above in (3.29) requires the refined selection $\mathcal{S}$ to not contain any horizontal edge in Figure 7. Therefore, the only two refined selections which produce the troublesome residues (3.30) are

$$\mathcal{S}_{\ell} = \{ \ell', \ldots, \ell \} \quad \text{and} \quad \mathcal{S}_{\ell} = \{ \ell, \ldots, \ell \}$$

It is easy to compute the corresponding residues, and we obtain

$$\text{Res}_{(x, \ldots, x) = (xq^{-\frac{1}{2}}, \ldots, xq^{-\frac{1}{2}})} \text{Sym} \left( \frac{\omega_x(x_1, \ldots, x_t, y_1', \ldots, y_t')}{\prod_{(c' \to c) \in Z_s} (z_{c'} - z_c)^{1-N_{\mathcal{S}'(c' \to c)}}} \right)$$

(3.31)

(recall that $s + t = s' + t'$ for a distinguished zig-zag). The case of $\mathcal{S}_{\ell}$ is given by formula (3.31) with $x$ replaced by $y$ and $s, t$ replaced by $s', t'$. Therefore, in order to correct (3.29), we need to add the following term to its right-hand side

$$\left( \delta_{\mathcal{S}_{\ell}, \mathcal{S}_{\ell}} + (-1)^{m_z - 1} \delta_{\mathcal{S}_{\ell} \mathcal{S}_{\ell}} \right) \left[ \text{Res}_{y = x} \frac{\omega_x(xq^{-\frac{1}{2}}, \ldots, xq^{-\frac{1}{2}}, yq^{-\frac{1}{2}}, \ldots, yq^{-\frac{1}{2}})}{(y - x)^{m_z x^{s-1} y^{t'-s-1} q^{\frac{1}{2}m_z}}} \right]$$

constant term in $x$

Taking the sum of (3.31) thus corrected over all refined selections $\mathcal{S}$, weighted by the sign $(-1)^{\sigma(\mathcal{S})}$, yields

$$\left\langle \rho_{Z, \tau}, R \right\rangle_{UV} = \sum_{\text{fixed fair partition}} \sum_{\{1, \ldots, n\} = A_1 \cup \cdots \cup A_t} \int_{|w_1| = \cdots = |w_t| = 1} \prod_{s = 1}^{n} Dw_s$$

$$\left[ \text{Res}_{(z_{c_1}, \ldots, z_{c_n}) = (w_1 q^{n-1}, \ldots, w_n q^{n-1})} \text{Sym} \left( \sum_{\mathcal{S}} (-1)^{\sigma(\mathcal{S})} \frac{\omega_x(x_1, \ldots, x_t, y_1', \ldots, y_t')}{\prod_{(c' \to c) \in Z_s} (z_{c'} - z_c)^{1-N_{\mathcal{S}'(c' \to c)}}} \right) \right]_{\forall s \in \{1, \ldots, t\}}$$

$$+ 2 \left[ \text{Res}_{y = x} \frac{\omega_x(xq^{-\frac{1}{2}}, \ldots, xq^{-\frac{1}{2}}, yq^{-\frac{1}{2}}, \ldots, yq^{-\frac{1}{2}})}{(y - x)^{m_z x^{s-1} y^{t'-s-1} q^{\frac{1}{2}m_z}}} \right]$$

constant term in $x$

Since the second line of the expression above vanishes (due to (3.18)) we conclude that the left-hand side is 0 if and only if the third line of the expression above vanishes. However, because $R$ satisfies (2.26) for all proper sub zig-zags of $Z$, we already have

$$(y - x)^{m_z - 1} \rho_{x, y}$$

Therefore, we conclude that $\left\langle \rho_{Z, \tau}, R \right\rangle_{UV} = 0$ if and only if

$$\frac{\omega_x(xq^{-\frac{1}{2}}, \ldots, xq^{-\frac{1}{2}}, yq^{-\frac{1}{2}}, \ldots, yq^{-\frac{1}{2}})}{(y - x)^{m_z x^{s-1} y^{t'-s-1} q^{\frac{1}{2}m_z}}}$$

In the formula above, the sign $(-1)^{m_z - 1}$ comes from two contributions: first we have $(-1)^{m_z}$ because of the discrepancy between $(x - y)^{m_z}$ and $(y - z)^{m_z}$; secondly, we have $(-1)$ because of the discrepancy between taking the residue at $x = y$ and the residue at $y = x$. 


is a Laurent polynomial in $x$ and $y$, whose specialization at $y = x$ is a non-zero constant. This implies that
\[(y - x)^m z \big| r \left( xq^{-r}, \ldots, xq^{-s+t}, yq^{t'-s'}, \ldots, yq^{s'-t'} \right)\]
as long as $\tau$ is chosen homogeneous of degree $t - s - 1 + t' - s' - 1 + mZ - \deg r$ (if $r$ is not homogeneous, then we perform the argument herein for its homogeneous components) and with
\[\tau \left( q^{-r}, \ldots, q^{-s+t}, q^{t'-s'}, \ldots, q^{s'-t'} \right) \neq 0\]

3.17. Consider the two-sided ideal
\[J^+ = \text{(coefficients of } \rho Z)Z \text{ distinguished zig-zag} \subset \tilde{U}_q^+ (L\mathfrak{g})\]
(by Remark 3.14 we do not need to take all coefficients of $\rho Z$; it suffices to take a single generic coefficient of $\rho Z$ of every homogeneous degree) and define
\[U_q^+ (L\mathfrak{g}) = \tilde{U}_q^+ (L\mathfrak{g}) / J^+\]
By Proposition 3.13 we have $(J^+)^\perp = S^-$ with respect to the pairing $\langle \cdot, \cdot \rangle_{\tilde{U}^+ \mathfrak{g} \mathfrak{V}}$, and thus we obtain the pairing $\langle \cdot, \cdot \rangle_{U^+ \mathfrak{S}}$ in diagram (3.1). If all the vector spaces in said diagram were finite-dimensional (or at least finite-dimensional in every $\pm \mathbb{N}^I \times \mathbb{Z}$ degree), then the non-degeneracy of $\langle \cdot, \cdot \rangle_{\tilde{U}^+ \mathfrak{g} \mathfrak{V}}$ would imply that $(S^-)^\perp = J^+$ and yield the non-degeneracy of $\langle \cdot, \cdot \rangle_{U^+ \mathfrak{S}}$. However, because all these vector spaces are in general infinite-dimensional, care must be taken when proving the following result.

**Proposition 3.18.** The pairing on the bottom of diagram (3.1), namely
\[U_q^+ (L\mathfrak{g}) \otimes S^- \xrightarrow{\langle \cdot, \cdot \rangle_{U^+ \mathfrak{S}}} \mathbb{K}\]
is non-degenerate in both arguments.

The natural analogue of Proposition 3.18 where we switch the roles of $+$ and $-$ also holds. We will prove Proposition 3.18 in the next Section using the combinatorics of words. But for now, let us deduce from it the proofs of our main Theorems.

**Proof. of Theorem 1.8:** As explained in the paragraph preceding the statement of Proposition 3.18, the pairings on top of either diagram (1.26) descend to pairings on the bottom. The non-degeneracy of the latter is established in Proposition 3.18.

**Proof. of Theorems 1.4 and 1.9:** We showed in (3.20) that
\[J^+ \subseteq K^+\]
where $K^+ = \text{Ker } \Upsilon^+$. Thus, we obtain a surjective algebra homomorphism
\[U_q^+ (L\mathfrak{g}) = \tilde{U}_q^+ (L\mathfrak{g}) / J^+ \xrightarrow{\Upsilon^+} \tilde{U}_q^+ (L\mathfrak{g}) / K^+ \xrightarrow{\text{Prop. 2.23}} S^+\]
which obviously intertwines the pairings (3.32) and (2.49). As such, any element in the kernel of $\Upsilon^+$ would also have to be in the radical of the pairing (3.32). As the latter is non-degenerate due to Proposition 3.18, we conclude that $\Upsilon^+$ is an isomorphism, i.e.
\[J^+ = K^+\]
This implies Theorem 1.9. As for the radical $I^+$ of the pairing $\langle \cdot, \cdot \rangle_{U^V}$, we showed in Proposition 2.21 that

$$K^+ \subseteq I^+$$

The compatibility of the pairings in diagram (3.1) and the non-degeneracy of (3.32) imply that

$$I^+ \subseteq J^+$$

We conclude that $I^+ = J^+ = K^+$, thus implying Theorem 1.4.

□

Proposition 3.19. When $Z$ is the particular minimal zig-zag in Figure 3, the relation $\rho_Z = 0$ is equivalent to relation (1.18) in the algebra $\tilde{U}_q^+(L\mathfrak{g})$.

Proof. Let $\mathcal{L} = $ LHS of (1.18). Since $Z$ is minimal, Claim 3.15 shows that

$$\langle \rho_Z(x_s, \ldots, x_t, y_s' = t'), R \rangle_{\tilde{U}^V} \text{ is non-zero multiple of } R(q^s, \ldots, q^t, q^{s' = t'})$$

for any homogeneous rational function $R \in \mathcal{V}_{(1-d_{ij})s'-c^j}$. Above, $s' = t' = \frac{s + t}{2}$ and $s = t + 2d_{ij}$. By the non-degeneracy of the pairing $\langle \cdot, \cdot \rangle_{\tilde{U}^V}$, the fact that (3.33)

$$\langle \mathcal{L}, R \rangle_{\tilde{U}^V} \text{ is non-zero multiple of } R(q^s, \ldots, q^t, q^{s' = t'})$$

would imply that $\mathcal{L}$ is proportional to $\rho_Z$, which is precisely what we need to prove. To establish formula (3.33), we note that $\mathcal{L} \in K^+$ (as shown in [11, Formula (1.1)]). Then the proof of Claim 3.15 shows that only the $k = 0$ and $k = 1 - d_{ij}$ summands in $\mathcal{L}$ contribute non-trivially to the pairing in (3.33). Moreover, just like in the proof of Claim 3.15 one can see that the contribution of the $k = 0$ and $k = 1 - d_{ij}$ summands is proportional to $R(q^s, \ldots, q^t, q^{s' = t'})$, as required.

□

4. Words

In the present Section, we will use techniques stemming from the combinatorics of words in order to prove Proposition 2.24 which will also allow us to construct bases of the shuffle algebras $S^\pm \subset V^\pm$. We will also prove Proposition 3.18 by “covering” our infinite-dimensional quantum loop groups and shuffle algebras by finite-dimensional vector subspaces, also indexed by words.

4.1. The treatment of the present Section is inspired (in chronological order) by [7, 8, 14] and most closely [12]. Let us consider the set of letters

$$\{i^{(k)}\}_{i \in I, k \in \mathbb{Z}}$$

A word is simply a sequence of letters

$$w = \left[ i^{(k_i)}_1 \ldots i^{(k_n)}_n \right]$$

We will call $|w| := n$ the length of a word as above, and

$$\bar{w} = (k_1, \ldots, k_n)$$
the sequence of exponents of \( w \). By analogy with the constructions of Subsection 2.1, the degree of \( w \) is defined as
\[
\deg w = (\varsigma^i_1 + \cdots + \varsigma^i_n, k_1 + \cdots + k_n) \in \mathbb{N}^I \times \mathbb{Z}
\]

Denote the set of all words by \( \mathcal{W} \). To \( w \in \mathcal{W} \) as above, we associate the element
\[
e_w = e_{i_1,k_1} \cdots e_{i_n,k_n} \in \tilde{U}^+_q(L\mathfrak{g})
\]

In the opposite algebra, we will use the notation
\[
f_w = f_{i_1,-k_1} \cdots f_{i_n,-k_n} \in \tilde{U}^-_q(L\mathfrak{g})
\]

4.2. Let us fix a total order \( < \) on \( I \), and associate to it the following total order on the set of letters:
\[
i^{(k)} < j^{(l)} \quad \text{if} \quad \begin{cases} k > l \\
k = l \text{ and } i < j \end{cases}
\]

Then we have the corresponding total lexicographic order on the set of words:
\[
\begin{bmatrix} i^{(k_1)}_1 \cdots i^{(k_n)}_n \\ j^{(l_1)}_1 \cdots j^{(l_m)}_m \end{bmatrix} < \begin{bmatrix} i^{(k_1)}_1 \cdots i^{(k_n)}_n \\ j^{(l_1)}_1 \cdots j^{(l_m)}_m \end{bmatrix}
\]

if \( i_1^{(k_1)} = j_1^{(l_1)}, \ldots, i_x^{(k_x)} = j_x^{(l_x)} \) and either \( i_{x+1}^{(k_{x+1})} < j_{x+1}^{(l_{x+1})} \) or \( x = n < m \).

**Definition 4.3.** A word \( \text{(4.1)} \) is called *non-increasing* if
\[
\begin{cases} k_a < k_b + | s \in \{a, \ldots, b-1\} \text{ s.t. } i_s \neq i_b | \\
or \\
k_a = k_b + | s \in \{a, \ldots, b-1\} \text{ s.t. } i_s \neq i_b | \text{ and } i_a \geq i_b
\end{cases}
\]

for all \( 1 \leq a < b \leq n \). Let \( \mathcal{W}_{\text{non-inc}} \) denote the set of non-increasing words.

**Lemma 4.4.** There are finitely many non-increasing words of given degree, which are bounded above by any given word \( v \).

**Proof.** Let us assume we are counting non-increasing words \( \begin{bmatrix} i^{(k_1)}_1 \cdots i^{(k_n)}_n \end{bmatrix} \) with \( k_1 + \cdots + k_n = k \) for fixed \( n \) and \( k \). The fact that such words are bounded above implies that \( k_1 \) is bounded below. But then the inequality \( \text{(4.6)} \) implies that \( k_2, \ldots, k_n \) are also bounded below. The fact that \( k_1 + \cdots + k_n \) is fixed implies that there can only be finitely many choices for the exponents \( k_1, \ldots, k_n \). Since there are also finitely many choices for \( i_1, \ldots, i_n \in I \), this concludes the proof.

\[\square\]

**Proposition 4.5.** The set \( \{e_w\}_{w \in \mathcal{W}_{\text{non-inc}}} \) is a linear basis of \( \tilde{U}^+_q(L\mathfrak{g}) \).
Proof. By running the proof of [12] Proposition 3.11 for the quiver $Q$ with vertex set $I$ and a single arrow from $i$ to $j$ if and only if $i < j$, we can prove that
\[ \{e_w\}_{w \in W_{\text{non-inc}}} \]
linearly spans $\overline{U}_q^+(Lq)$. Indeed, the only thing this fact requires in loc. cit. is relation [12 (3.20)], which takes the same form as our (1.9) (albeit with different parameters instead of $q^{d/\nu}$, though these do not affect the validity of the argument). We will actually need the stronger statement, which is established in [12 Proposition 3.11], that for any $i_1, \ldots, i_n \in I$ and $k_1, \ldots, k_n \in \mathbb{Z}$ we have
\[ (4.7) \quad e_{i_1, k_1} \cdots e_{i_n, k_n} \in \sum_{\{j(t)^{(k_n)}\} \text{ non-increasing and } \min(k_0) - \beta(n) \leq \min(l) \leq \max(l) \leq \max(k_n) + \beta(n)} K \cdot e_{j_1, l_1} \cdots e_{j_n, l_n} \]
for some constant $\beta(n)$ that only depends on $n$. The stronger statement above says that, as we use (1.9) to reorder the $e_{i,k}$’s so that the right-hand side of the expression above consists only of non-increasing words, we only increase/decrease the $k_n$’s by a bounded amount during the whole procedure.

In order to show that the elements (4.7) are linearly independent in $\overline{U}_q^+(Lq)$, we invoke [13 Proposition 2.14], for the same quiver $Q$, as above. We will explain the details (without proof) in Subsection 4.7 see the paragraph after (4.18).

\[ \square \]

4.6. Let us give an application of property (4.8). As we saw in Remark 3.14, one may replace $J^+$ by the ideal generated by $\rho_{Z, \tau_k}$, as $Z$ runs over all distinguished zig-zags as in Figure 7, and $\tau_k(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'})$ runs over generic Laurent polynomials of any homogeneous degree $k \in \mathbb{Z}$. For given $Z$, let us write
\[ n = \varsigma^i \cdot \left( t - s \right) + \varsigma^j \cdot \left( t' - s' \right) + 1 \]
Then let us choose $\tau_k$ such that
\[ \tau_{k+d}(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) = \tau_k(x_s, \ldots, x_t, y_{s'}, \ldots, y_{t'}) \cdot (x_s \ldots x_t y_{s'} \ldots y_{t'})^d \]
for all $k, d \in \mathbb{Z}$. The effect that this choice has is that
\[ (4.9) \quad \text{if } \rho_{Z, \tau_k} = \sum_{\substack{i_1, \ldots, i_n \in I \\text{coefficient} \cdot e_{i_1, k_1} \cdots e_{i_n, k_n} \\text{coefficient} \cdot e_{i_1, k_1+d} \cdots e_{i_n, k_n+d} \\text{coefficient} \cdot e_{i_1, k_1} \cdots e_{i_n, k_n} \\text{coefficient} \cdot e_{i_1, k_1} \cdots e_{i_n, k_n}}} \]
for all $k, d \in \mathbb{Z}$. Thus, the finitely many coefficients in (4.9) for $k \in \{1, \ldots, n\}$ determine the coefficients for all $k$. We conclude that
\[ (4.11) \quad \rho_{Z, \tau_k} = \sum_{\substack{i_1, \ldots, i_n \in I \\text{coefficient} \cdot e_{i_1, k_1} \cdots e_{i_n, k_n}}} \]
for all $k \in \mathbb{Z}$, for some large enough $\gamma(n)$ that only depends on $n$. 

\[ \square \]
4.7. Given a monomial in \( \{ z_i \}_{i \in I} \in \mathbb{N} \), we will consider all ways to order its variables
\[
\mu = z_i^{-l_1} \cdots z_i^{-l_n}
\]
(4.12)

The **leading word** of \( \mu \) as above is defined as the (lexicographically) largest word
\[
w_{\mu} = [i_1^{(k_1)} \cdots i_n^{(k_n)}]
\]
(4.13)

where:
\[
k_a = l_a - \left| s < a \text{ s.t. } i_s > i_a \right| + \left| t > a \text{ s.t. } i_t < i_a \right|
\]
(4.14)

among all ways to order the variables in (4.12). Following the proof of [12, Lemma 4.8.] for the quiver \( Q \) defined in the previous Subsection (vertex set \( I \) and a single arrow from \( i \) to \( j \) iff \( i < j \)), the leading word of a monomial is non-increasing. It is easy to see that the leading word only depends on \( \text{Sym} \mu \), namely the symmetrization with respect to \( z_{i_1}, z_{i_2}, \ldots \) for each \( i \in I \) separately.

Conversely, the **associated polynomial** of a non-increasing word (4.13) is defined by \( \text{Sym} \mu \) with the \( l_a \)'s determined by (4.14). The functions “leading word” and “associated polynomial” yield mutually inverse bijections between the sets

\[
\text{symmetric polynomials} \leftrightarrow \text{non-increasing words}
\]
(4.15)

More generally, the leading word of any \( R \in \mathcal{V}_n \), denoted by \( \text{lead}(R) \), will be the lexicographically largest of the leading words (4.13) for all the monomials which appear with non-zero coefficient in the Laurent polynomial
\[
R(\ldots, z_{i_a}, \ldots) = \prod_{1 \leq a \leq n, 1 \leq b \leq n_j}^{i < j \in I} \left( 1 - \frac{z_{j_b}}{z_{i_a}} \right)
\]
(4.16)

Conversely, any non-increasing word \( w \) appears as the leading word
\[
w = \text{lead} \left[ \text{Sym} \mu \prod_{1 \leq a \leq n, 1 \leq b \leq n_j}^{i < j \in I} \left( 1 - \frac{z_{j_b}}{z_{i_a}} \right) \right]
\]
(4.17)

with \( \mu \) and \( w \) connected by (4.12)–(4.14).

Analogous to [12, Formula (4.13)], one can show by direct inspection that:
\[
\langle e_w, R \rangle_{\mathcal{V}_n} = \begin{cases} 0 & \text{if } w = \text{lead}(R) \\ \neq 0 & \text{if } w > \text{lead}(R) \end{cases}
\]
(4.18)

The formula above immediately implies the linear independence of the elements \( e_w \), as \( w \) runs over non-increasing words. Indeed, if one were able to write such an element \( e_w \) as a linear combination of elements \( e_v \) for various \( v > w \), then we would contradict (4.18) for \( R \) being the rational function in the right-hand side of (4.17).

**Proof. of Proposition 2.13:** With formulas (2.19)–(2.20) in mind, the non-degeneracy of the pairing in the \( \mathcal{V}_n \) argument is simply restating the well-known fact that if a rational function \( F \) in variables \( z_1, \ldots, z_n \) vanishes when expanded as a power series (in some relative order of its variables) then \( F = 0 \).
Let us now prove non-degeneracy in the $\tilde{U}^+_q(Lg)$ argument (we will focus on $\tilde{U}^+_q(Lg)$ without loss of generality). Consider any non-zero element
\[
\phi = \sum_{v \in W_{\text{non-inc}}} \gamma_v e_v
\]
and let $w$ be the smallest word such that $\gamma_w \neq 0$. Since we showed in (4.17) that there exists an element $R \in V^-$ such that $\text{lead}(R) = w$, then (4.18) implies that
\[
\langle \phi, R \rangle_{\tilde{U}^+_V} = \sum_{v \in W_{\text{non-inc}}} \gamma_v \langle e_v, R \rangle_{\tilde{U}^+_V} = \gamma_w \langle e_w, R \rangle_{\tilde{U}^+_V} \neq 0
\]
This implies the non-degeneracy of the pairing in the $\tilde{U}^+_q(Lg)$ argument.

4.8. We are now ready to prove Proposition 2.24. Recall the surjective homomorphisms
\[
\tilde{\Upsilon}^\pm : \tilde{U}^+_q(Lg) \to \mathcal{S}^\pm
\]
and let $E_w = \tilde{\Upsilon}^+(e_w)$, $F_w = \tilde{\Upsilon}^-(f_w)$ for any word $w$.

**Definition 4.9.** A word $w$ is called **standard** if
\[
E_w \notin \sum_{v > w} K \cdot e_v
\]
Let $W_{\text{stan}}$ denote the set of standard words.

Formula (4.8) implies that any standard word is non-increasing. The converse is definitely not true, since the definitions readily imply the fact that
\[
w \in W_{\text{non-inc}} \setminus W_{\text{stan}} \iff e_w \in \sum_{v > w} K \cdot e_v + \ker \tilde{\Upsilon}^+
\]
Thus, the discrepancy between standard and non-increasing words is a measure of $K^+$.

**Proof. of Proposition 2.24.** We will prove (2.48) for $\pm = -$, as the case $\pm = +$ is analogous. All words in the present proof will have fixed length $n$. For any words
\[
v = [i^{(k_1)}_1, \ldots, i^{(k_n)}_n] \quad \text{and} \quad w = [j^{(l_1)}_1, \ldots, j^{(l_n)}_n]
\]
the following formula is easy to deduce from (2.19) (see [12] Remark 3.16) for a proof in an almost identical setup
\[
\langle E_v, F_w \rangle_{\mathcal{S}^\pm} = \int_{|z_1| \gg \cdots \gg |z_n|} \sum_{\sigma \in S(n)} \prod_{i^{(k_a)}_a = j^{(l_b)}_b, \forall a} \zeta_{i^{(k_a)}_a}^{k_a-l_a(1)} \cdots \zeta_{j^{(l_b)}_b}^{j_l(l_n)} \prod_{a < b} \zeta_{i^{(k_a)}_a} \left( \frac{z_a}{z_b} \right) \prod_{a=1}^n Dz_a
\]
Expanding the integrand as $|z_1| \gg \cdots \gg |z_n|$ reveals that $\langle E_v, F_w \rangle_{\mathcal{S}^\pm} \neq 0$ only if there exists $\sigma \in S(n)$ and integers
\[
\{ c_{a,b} \leq 0 \}_{a < b, \sigma(a) > \sigma(b)}
\]
such that
\[
(4.20) \quad k_a + \sum_{\sigma(t) < \sigma(a)} c_{at} - \sum_{\sigma(s) > \sigma(a)} c_{sa} = t_{\sigma(a)}
\]
Consider the infinite directed graph \( G_n \) whose vertices are collections of integers \( (k_1, \ldots, k_n) \) such that \( k_a \leq k_{a+1} + 1 \) and whose edges are
\[
(4.21) \quad (k_1, \ldots, k_n) \to (l_1, \ldots, l_n)
\]
if and only if there exists \( \sigma \in S(n) \) and integers \( \{1.19\} \) such that \( \{1.20\} \) holds (although the graph is defined to be directed, it is easy to see that an edge \( \{1.21\} \) exists if and only if the opposite edge exists). The preceding discussion implies that
\[
(4.22) \quad \langle E_v, F_w \rangle_{SS} \neq 0 \quad \Rightarrow \quad \text{exists edge } v \to w
\]
in \( G_n \), for any non-increasing words \( v, w \) (above, \( \overline{v} \) denotes the sequence of exponents of the word \( v \), see \( \{4.2\} \)). The following result was proved in \[12, \text{Lemma 3.18}].

**Lemma 4.10.** All connected components of \( G_n \) are finite.

With the Lemma above in mind, let us define the following finite-dimensional vector spaces, for any connected component \( H \subset G_n \):
\[
\hat{S}_H^+ = \bigoplus_{w \in W_{\text{non-inc}}, \overline{w} \in H} \mathbb{K} \cdot E_w \subset \hat{S}^+
\]
\[
\hat{S}_H^- = \bigoplus_{w \in W_{\text{non-inc}}, \overline{w} \in H} \mathbb{K} \cdot F_w \subset \hat{S}^-
\]
Because of \( \{4.22\} \), the descended pairing \( \{2.36\} \) satisfies
\[
\langle \hat{S}_H^+, \hat{S}_{H'}^- \rangle_{SS} = 0
\]
if \( H \neq H' \). However, since the descended pairing is non-degenerate in both arguments (a property which it inherits from the non-degeneracy of \( \langle \cdot, \cdot \rangle_V \) and \( \langle \cdot, \cdot \rangle_{\overline{V}} \) in the \( V \) argument), then so is its restriction
\[
\{4.23\} \quad \hat{S}_H^+ \otimes \hat{S}_H^- \xrightarrow{\langle \cdot, \cdot \rangle_{SS}} \mathbb{K}
\]
for any connected component \( H \subset G_n \). The following statements are straightforward consequences of the finite-dimensionality of the vector spaces \( \hat{S}_H^\pm \).

**Proposition 4.11.** For any \( n \in \mathbb{N} \), we have
\[
\{4.24\} \quad \bigoplus_{|n| = n} \hat{S}_n^\pm = \bigoplus_{H \text{ a connected component of } G_n} \hat{S}_H^\pm
\]
and
\[
\{4.25\} \quad \hat{S}_H^+ = \bigoplus_{w \text{ standard}} \mathbb{K} \cdot E_w \quad \text{and} \quad \hat{S}_H^- = \bigoplus_{w \text{ standard}} \mathbb{K} \cdot F_w
\]
Proof. Let us prove the statements for $\pm = +$. Because the $E_w$’s span $\tilde{S}_H^+$ as $w$ runs over all non-increasing words, all that we need to do to prove \eqref{4.24} is to show that there are no linear relations among the various direct summands of the RHS. To this end, assume that we had a relation

$$\sum_{H \text{ a connected component of } G_n} \alpha_H = 0$$

for various $\alpha_H \in \tilde{S}_H^+$. Pairing the relation above with a given $\tilde{S}_H^-$ implies that

$$\langle \alpha_H, \tilde{S}_H^- \rangle_{\tilde{S}^+} = 0$$

Because the pairing \eqref{4.23} is non-degenerate, this implies that $\alpha_H = 0$. As for \eqref{4.25}, it holds because any vector space spanned by vectors $\alpha_1, \ldots, \alpha_k$ has a basis consisting of those $\alpha_i$’s which cannot be written as linear combinations of $\{\alpha_j\}_{j>i}$.

$\square$

We are now poised to complete the proof of Proposition \ref{2.24} Consider any $R \in S_{-n}$ with $|n| = n$. From \eqref{2.19}, it is easy to see that

$$\langle E_{[i_1^{(k_1)} \ldots i_n^{(k_n)}]}, R \rangle_{\tilde{S}^+} = 0$$

if $k_1$ is small enough. However, by Lemma \ref{4.4} there are only finitely many non-increasing words $w$ of given degree with $k_1$ bounded below. This implies that

$$\langle E_w, R \rangle_{\tilde{S}^+} \neq 0$$

only for finitely many non-increasing words $w$. Letting $H_1, \ldots, H_t \subset G_n$ denote the connected components which contain the sequences of exponents of the aforementioned words, then \eqref{4.18} and the non-degeneracy of the pairings \eqref{4.23} imply that there exists an element

$$R' \in \tilde{S}_{H_1}^- + \cdots + \tilde{S}_{H_t}^- \subset \tilde{S}^-$$

such that $\langle E_w, R \rangle_{\tilde{S}^+} = \langle E_w, R' \rangle_{\tilde{S}^+}$ for all non-increasing words $w$. Then the non-degeneracy of the descended pairing \eqref{2.36} implies that $R = R' \in \tilde{S}^-$, as we needed to prove. Finally, the statement about the non-degeneracy of the pairing \eqref{2.49} follows from the analogous statement for the pairing \eqref{2.36}.

$\square$

4.12. Throughout the remainder of the present Section, we will only consider words

$$w = \left[ i_1^{(k_1)} \ldots i_n^{(k_n)} \right]$$

of fixed degree $\deg w = (n, k) \in \mathbb{N}^I \times \mathbb{Z}$. For any finite set $T \subset W_{\text{non-inc}}$ of words, we define

$$\bar{U}_q^+(L_\mathfrak{g})^T = \bigoplus_{w \in T} \mathbb{K} \cdot e_w$$

(recall that as $w$ runs over $W_{\text{non-inc}}$, the $e_w$’s yield a basis of $\bar{U}_q^+(L_\mathfrak{g})$). Also let

$$\mathcal{V}^{-;T} \subset \mathcal{V}_{n-k}^{-}$$
denote the subspace of rational functions $R$, such that all monomials appearing in the Laurent polynomial (4.16) have leading word in $T$. Thus, the restriction

$$\tilde{U}_q^+(L\phi)^T \otimes V^{-,T} \xrightarrow{\langle \cdot, \cdot \rangle_{UV}} \mathbb{K}$$

is a non-degenerate pairing of finite-dimensional vector spaces (indeed, because the two vector spaces have the same dimension, it suffices to show non-degeneracy in the second factor, which follows immediately from (4.18)).

**Proof. of Proposition 3.18.** Let us consider a finite set of non-increasing words $T$, of some fixed degree $(n,k)$, to be chosen in (4.32). The only thing we postulate for the time being is that the set $T$ can be chosen “arbitrarily large”, i.e. to contain any given finite set of words. Let $S^{-,T} = S^- \cap \mathcal{V}^{-,T}$. Proposition 3.13 implies that

$$S^{-,T} \subseteq \left( J^+ \cap \tilde{U}_q^+(L\phi)^T \right) ^\perp$$

where the orthogonal complement is defined with respect to the pairing (4.28). Our goal will be to prove the opposite inclusion, namely

$$\left( J^+ \cap \tilde{U}_q^+(L\phi)^T \right) ^\perp \subseteq S^{-,T}$$

Let us first use (4.29) to conclude the proof of Proposition 3.18. Because (4.28) is a non-degenerate pairing of finite-dimensional vector spaces, we would have the following implication

$$S^{-,T} \subseteq \left( J^+ \cap \tilde{U}_q^+(L\phi)^T \right) ^\perp \Rightarrow J^+ \cap \tilde{U}_q^+(L\phi)^T = (S^-)^\perp$$

Since every element $\phi \in \tilde{U}_q^+(L\phi)^T$ lies in $\tilde{U}_q^+(L\phi)^T$ for some large enough finite set $T$, if $\phi$ pairs trivially with the whole of $S^-$, then the formula above implies that $\phi \in J^+$. This establishes the non-degeneracy of the pairing (3.32) in the first argument. Since the non-degeneracy in the second argument is inherited from that of $\langle \cdot, \cdot \rangle_{UV}$ (see Proposition 2.13, proved earlier in this Section), we would be done.

To prove (4.29), we will revisit the proof of Proposition 3.13. Specifically, after we stated Claim 3.15, we showed that given $R \in \mathcal{V}^-$,

$$\left\langle \phi, R \right\rangle_{UV} = 0, \forall \phi \in J^+ \Rightarrow R \in S^-$$

For the task at hand, we need to show that given $R \in \mathcal{V}^{-,T}$,

$$\left\langle \phi, R \right\rangle_{UV} = 0, \forall \phi \in J^+ \cap \tilde{U}_q^+(L\phi)^T \Rightarrow R \in S^{-,T}$$

Concretely, we fix $R \in \mathcal{V}^{-,T}$ and we will show that the only $\phi$’s that Proposition 3.13 needs in order to ensure the implication (4.30) actually lie in $\tilde{U}_q^+(L\phi)^T$; this would yield the implication (4.31) and we would be done. To this end, let us choose

$$T = \left\{ w = [i_{1}^{(k_1)} \ldots i_{n}^{(k_n)}] \text{ such that } \deg w = (n, k) \right\}$$

for natural numbers $m, M$ (since $M$ can be arbitrarily large, this would ensure the fact that any finite set of non-increasing words can be contained in $T$). The reason
we subtract by $|s > t|$ s.t. $s \in A, t \notin A, i_s \neq i_t$ in the inequality above is the straightforward fact (which we leave to the interested reader) that the inequality holds for the leading word (4.13) of a monomial (4.12) if and only if it holds for the analogous word associated to any other order of the variables in the monomial.

As shown in the proof of Proposition 3.13 (we will reuse the notation therein), the only $\phi$’s we need to consider in (4.30) are those of the form

$$e_{i_1,k_1} \cdots e_{i_p,k_p} \rho z, \tau_{k-p-1} \cdots \tau_{k-p}$$

for some monomial that appears in the numerator of $R$

$$R = \frac{\ldots + u_1^{-k_1} \cdots u_p^{-k_p} R_{k_1,\ldots,k_p}(x_1,\ldots,x_t, y_1,\ldots,y_{t'}) + \ldots}{\prod_{a < b} \left(1 - \frac{u_a}{u_b}\right) \prod_{b,c} \left(1 - \frac{z_c}{u_b}\right)}$$

(see (3.23)) with non-zero $R_{k_1,\ldots,k_p}$. Moreover, we may assume the word

$$\left[ i_1(k_1) \ldots i_p(k_p) \right]$$

is non-increasing. This is because in the proof of Proposition 3.13 one can reorder the variables $u_1,\ldots,u_p$ arbitrarily; choosing the order which leads to the maximal word (4.35) ensures that the aforementioned word is non-increasing, as explained in Subsection 4.7.

Using (4.11), we may write (4.33) is a linear combination of products of the form

$$e_{i_1,k_1} \cdots e_{i_p,k_p} e_{i_{p+1},k_{p+1}} \cdots e_{i_n,k_n}$$

where $k_{p+1},\ldots,k_n$ are within $\gamma(n-p)$ from their average value. It remains to show that any product (4.36) can be expressed, using (4.8), as a linear combination of products of $e_{i,k}$’s that correspond to non-increasing words in $T$. Let us consider the constant $b = 2 \max(\beta(1),\ldots,\beta(n)) + 2n$, with the $\beta$’s as in (4.8). In the product (4.36), we consider the largest index $x \in \{0,\ldots,p\}$ such that

$$k_x < k_{x+1} - b$$

(we make the convention that $k_0 = -\infty$). We may use relation (4.8) to write

$$e_{i_{x+1},k_{x+1}} \cdots e_{i_n,k_n} = \sum_{[j_{x+1}^{(l_{x+1})} \cdots j_n^{(l_n)}] \in W_{\text{non-inc}}} \text{coefficient} \cdot e_{j_{x+1},l_{x+1}} \cdots e_{j_n,l_n}$$

We need to make two observations about the $l$’s that appear in the formula above.

- All the numbers $l_{x+1},\ldots,l_n$ are within a global constant away from their average
- The large difference between $k_x$ and $k_{x+1}$ ensures that all concatenated words

$$\left[ i_1(k_1) \ldots i_x(k_x) j_{x+1}^{(l_{x+1})} \cdots j_n^{(l_n)} \right]$$

which arise in the procedure above are non-increasing (recall that the word (4.35) was non-increasing to begin with, and thus so are all of its prefixes)
Thus, it remains to show that the concatenated words that appear in (4.38) are in $T$. Because $R \in V^{-T}$, then (4.32) and (4.34) imply that

\[
\sum_{s \in B} k_s \geq -M|A| + m\sum_{s \in A} l_s \geq -M|A| + m\sum_{s \in A} (s > t) \text{ s.t. } s \in A, t \notin A, s \neq i_t
\]

for $A = B$ or $A = B \cup \{x+1, \ldots, n\}$ respectively, where $B \subseteq \{1, \ldots, x\}$ is arbitrary. The latter formula holds because $l_{x+1} + \cdots + l_n = k_{x+1} + \cdots + k_n$. Assume for the purpose of contradiction that the defining property of $T$ is violated for $A = B \cup C$

\[
\sum_{s \in B} k_s + \sum_{s \in C} l_s < -M|A| + m\sum_{s \in A} (s > t) \text{ s.t. } s \in A, t \notin A, s \neq i_t
\]

We claim that (4.39)–(4.40) and (4.41) are incompatible (for $m$ chosen large enough compared to the constant mentioned in the first bullet above). Indeed, letting $\mu$ be the average mentioned in the first bullet above, relations (4.39)–(4.40) imply

\[
\sum_{s \in B} k_s \geq -M|B| + m|B|^2 - \ldots
\]

while (4.41) gives us

\[
\sum_{s \in B} k_s + \sum_{s \in C} l_s \geq -M(|B| + n - x) + m(|B| + n - x)^2 - \ldots
\]

where $y = |C|$ lies in $\{1, \ldots, n - x - 1\}$, and the ellipses in formulas (4.42)–(4.44) denote global constants. Subtracting (4.42) from (4.44) yields

\[
\mu < -M + m(y + 2|B|) + \ldots
\]

and subtracting (4.44) from (4.43) yields

\[
\mu > -M + m(n - x + y + 2|B|) - \ldots
\]

The two inequalities above are incompatible if $m$ is chosen large enough compared to the global constants that appear among the ellipses, thus yielding the desired contradiction.

\[\square\]

5. $K$-theoretic Hall algebras

We will now study an important incarnation of quantum loop groups in geometric representation theory, namely $K$-theoretic Hall algebras associated to quivers. This is an idea that many mathematicians and physicists have been developing in recent
decades, but the particular incarnation we will be working with follows the philosophy of Schiffmann-Vasserot. We follow [16], where the authors develop this line of thought in the setting of equivariant $K$-theory.

5.1. Let us consider a quiver $Q$, i.e. an oriented graph, with vertex set $I$ and arrow set $E$. We allow multiple arrows between any two vertices, but no loops. Given $n = (n_i)_{i \in I} \in \mathbb{N}^I$, we will call

$$Z_n = \bigoplus_{i,j \in E} \text{Hom}(C^{n_i}, C^{n_j})$$

the vector space of $n$-dimensional representations of the quiver $Q$ (if there are multiple arrows from vertex $i$ to vertex $j$, then there are multiple copies of $\text{Hom}(C^{n_i}, C^{n_j})$ in the direct sum above). Quiver representations will always be taken modulo change of basis, i.e. modulo the action of the algebraic group

$$G_n = \prod_{i \in I} GL_{n_i}(\mathbb{C})$$

by conjugation. Thus, the stack of $n$-dimensional representations of $Q$ is

$$\mathcal{Z}_n = Z_n/G_n$$

In the present paper, we will mostly be concerned with the cotangent stack to $\mathcal{Z}_n$. To describe it, consider the quadratic moment map

$$T^*Z_n \xrightarrow{\mu_n} \bigoplus_{i \in I} \text{End}(C^{n_i})$$

given by

$$\mu_n \left( C^{n_i} \xrightarrow{\phi_i} C^{n_i}, C^{n_j} \xrightarrow{\phi_j} C^{n_j} \right)_{v_e = \gamma_{ij} \in E} = \sum_{e = \gamma_{ij} \in E} \left( \phi_e \phi_e^* - \phi_e^* \phi_e \right)_{\in \text{End}(C^{n_j}) \in \text{End}(C^{n_i})}$$

Then we have the following presentation of the cotangent stack

$$T^*\mathcal{Z}_n = \mu_n^{-1}(0)/G_n$$

for any $n \in \mathbb{N}^I$.

5.2. Fix a total order on $I$, and let us assume that all the arrows in the quiver $Q$ point from $i$ to $j$ if and only if $i < j$. This is not a significant restriction, as points of $T^*Z_n$ always come in pairs: for every linear map corresponding to an arrow, there exists a linear map pointing in the opposite direction. In fact, the only place where the orientation of arrows matters is in a certain monomial twist that defines the convolution product on the $K$-theoretic Hall algebra (see [16, Subsection 2.2.2.] or [12, Remark 2.5.]). Let us consider the torus action

$$C^* \curvearrowright T^*\mathcal{Z}_n$$

10See for example the seminal work [5], where many of the $K$-theoretic constructions considered herein were developed, albeit with respect to a significantly different $C^*$ action. This choice of action leads to quite different notions of quantum loop groups (compare our relation (1.9) with [5, relation (7.6.1i)]). The analogue of the present paper in the case of the quantum loop group considered in [5] falls under the treatment of [12, Sections 4 and 5] and [19, Section 5].
(called a “normal weight function” in [16]) defined as follows. For every pair of vertices \( i < j \) in \( I \), fix an indexing 
\[ e_1, \ldots, e_{-d_{ij}} \]
of the set of edges from \( i \) to \( j \) (this defines the numbers \( d_{ij} = d_{ji} \in \mathbb{Z}_{\leq 0} \) for all \( i \neq j \)). Then we define the action (5.2) by requiring \( t \in \mathbb{C}^* \) to rescale
\[
\phi_e \mapsto \phi_e t^{2c - d_{ij}} \\
\phi^*_e \mapsto \phi^*_e t^{2c + d_{ij}}
\]
for all \( c \in \{1, \ldots, -d_{ij}\} \). It is easy to see that \( \mathbb{C}^* \) preserves \( \mu_n^{-1}(0) \), since the weights of \( \phi_e \) and \( \phi^*_e \) always add up to 2. As the \( \mathbb{C}^* \) action above commutes with the \( G_n \) action by conjugation, this ensures that the action (5.2) is well-defined.

**Definition 5.3.** The \( K \)-theoretic Hall algebra (with respect to the torus) is

\[
K_{\mathbb{C}^*} = \bigoplus_{n \in \mathbb{N}^I} K_{\mathbb{C}^*}(T^*Z_n)
\]

It is a module over \( K_{\mathbb{C}^*}(pt) = \mathbb{Z}[q^{\pm 1}] \), so we may also consider its localization

\[
K_{\mathbb{C}^*}\text{-loc} = K_{\mathbb{C}^*} \otimes \mathbb{Q}(q)
\]

The multiplication on (5.3) and (5.4) is given by a suitable convolution product (see [16 Subsection 2.2], as well as [12 Subsection 2.3] for notation closer to ours).

### 5.4. Following [16], we call a point
\[
\left( \mathbb{C}^n, \phi_e \rightarrow \mathbb{C}^m, \phi^*_e \rightarrow \mathbb{C}^n \right)_{\forall e = \overrightarrow{ij} \in E} \subset T^*Z_n
\]

**nilpotent** if there exist flags of subspaces
\[
\left\{ 0 = F^0_0 \subseteq F^1_1 \subseteq \cdots \subseteq F^i_r = \mathbb{C}^n_i \right\}_{i \in I}
\]
such that
\[
\phi_e(F^p_i) \subseteq F^p_{p-1} \quad \text{and} \quad \phi^*_e(F^p_i) \subseteq F^p_{p-1}
\]
for all \( p \in \{1, \ldots, r\} \) and all arrows \( e = \overrightarrow{ij} \). The set

\[
\Lambda_n \subset T^*Z_n
\]

of nilpotent points is a closed substack.

**Definition 5.5.** The \( K \)-theoretic Hall algebra supported on the substack (5.6) is

\[
K_{\mathbb{C}^*}\text{-nilp} = \bigoplus_{n \in \mathbb{N}^I} K_{\mathbb{C}^*}(T^*Z_n)_{\Lambda_n}
\]

Also consider

\[
K_{\mathbb{C}^*}\text{-loc}\text{-nilp} = K_{\mathbb{C}^*}\text{-nilp} \otimes \mathbb{Q}(q)
\]
by analogy with (5.4). The convolution product that makes (5.3) into algebras works equally well to make (5.7) into algebras. The natural maps

\[ K_{C^*}^{\text{nilp}} \to K_{C^*} \quad \text{and} \quad K_{C^*,\text{loc}}^{\text{nilp}} \to K_{C^*,\text{loc}} \]

are algebra homomorphisms.

5.6. Consider the closed embedding of the origin

\[ 0 \hookrightarrow T^*Z_n \]

and let us consider the following composition

\[ \iota_n : K_{C^*}(T^*Z_n)_{\Lambda_n} \xrightarrow{\text{natural map}} K_{C^* \times G_n}(T^*Z_n) \xrightarrow{\text{restriction to } 0} K_{C^* \times G_n}(pt) \]

where the “natural map” takes a coherent sheaf on \( T^*Z_n \) supported on \( \Lambda_n \) and interprets it as a \( G_n \)-equivariant coherent sheaf on the affine space \( T^*Z_n \). One has

\[ K_{C^* \times G_n}(pt) = \mathbb{Z}[q^{\pm 1}][z_{i_1}^{\pm 1}, \ldots, z_{i_{\text{sym}}}^{\pm 1}] \]

where \( z_{i_n} \) denotes the \( a \)-th elementary character of a maximal torus of \( GL_{n_i}(\mathbb{C}) \subset G_n \). Therefore, putting the maps (5.9) together over all \( n \in \mathbb{N} \) yields a map

\[ \iota : K_{C^*}^{\text{nilp}} \to \mathcal{Y}_{\text{int}^+}^{\text{geom}} := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[q^{\pm 1}][z_{i_1}^{\pm 1}, \ldots, z_{i_{\text{sym}}}^{\pm 1}] \]

Tensoring everything with \( \mathbb{Q}(q) \) yields a map

\[ \iota_{\text{loc}} : K_{C^*,\text{loc}}^{\text{nilp}} \to \mathcal{Y}_{\text{int}^+}^{\text{geom}} := \bigoplus_{n \in \mathbb{N}^t} \mathbb{Q}(q)[z_{i_1}^{\pm 1}, \ldots, z_{i_{\text{sym}}}^{\pm 1}] \]

Note that we could have defined the maps above with \( K_{\text{nilp}}^{\text{nilp}} \) replaced by \( K \) everywhere, simply by removing the subscript \( \Lambda_n \) in (5.9). However, Theorem 1.12 would cease to hold in this new setup, essentially because of the failure of Proposition 5.10.

5.7. The maps \( \iota \) and \( \iota_{\text{loc}} \) are algebra homomorphisms, if the codomain of either map is endowed with the shuffle product (2.7), but with \( \zeta_{ij}(x) \) replaced by \(^{11}\)

\[ \zeta_{ij}^{\text{geom}}(x) = \begin{cases} \frac{x - q^{-2}}{x - 1} & \text{if } i = j \\ \frac{q - d_{ij}^{-1}}{q - d_{ij}} \prod_{c=0}^{d_{ij}^{-1}} (1 - xq^{2c+d_{ij}}) & \text{if } i < j \\ \prod_{c=1}^{d_{ij}} \left(1 - \frac{q^{2c+d_{ij}}}{x}\right) & \text{if } i > j \end{cases} \]

\(^{11}\)Note that our rational function \( \zeta_{ij} \) differs by an overall monomial from the function denoted by \( \zeta'^{ij} \) in [12]; this discrepancy is innocuous, and we made it in order to match with the conventions in the present paper. On the \( K \)-theoretic Hall algebra side, the monomial in question can be implemented by appropriately twisting the convolution product, see [12] Remark 2.5.].
Because
\begin{equation}
\zeta_{ij}^{\text{geom}}(x) = \begin{cases}
1, & \text{if } i = j \\
(1 - x) \prod_{c=1}^{d_{ij}} (1 - xq^{2c+d_{ij}}), & \text{if } i < j \\
(1 - \frac{1}{x}) \prod_{c=1}^{d_{ij}} \left(1 - \frac{q^{2c+d_{ij}}}{x}\right), & \text{if } i > j
\end{cases}
\end{equation}
for all \(i, j \in I\), it is easy to see that the linear map
\begin{equation}
\mathcal{V}^+ \xrightarrow{\Omega} \mathcal{V}^+_{\text{geom}}
\end{equation}
we take \(K = \mathbb{Q}(q)\) in the definition of \(\mathcal{V}^+\) in (2.6) given for any \(R \in \mathcal{V}_n^+\) by
\begin{equation}
\Omega(R) = \prod_{i < j \in I} \left(1 - \frac{z_{ia}}{z_{jb}}\right)^{-d_{ij}} \prod_{c=1}^{d_{ij}} \left(1 - \frac{z_{ia}q^{2c+d_{ij}}}{z_{jb}}\right)
\end{equation}
is an algebra homomorphism. With this in mind, we conclude that
\begin{equation}
\mathcal{S}^+_{\text{geom}} := \Omega(\mathcal{S}^+)
\end{equation}
is a subalgebra of \(\mathcal{V}^+_{\text{geom}}\). We define the subalgebra
\begin{equation}
\mathcal{S}^+_{\text{int}}_{\text{geom}} \subset \mathcal{V}^+_{\text{int}}_{\text{geom}}
\end{equation}
analogously, but using the coefficient ring \(\mathbb{Z}[q^{\pm 1}]\) instead of the coefficient field \(\mathbb{Q}(q)\).

**Proposition 5.8.** A Laurent polynomial \(r(\ldots, z_{ia}, \ldots)\) lies in \(\mathcal{S}^+_{\text{geom}}\) if and only if
\begin{equation}
(x - y)^{M_Z} \text{ divides } r\bigg|_{z_{i1}=xq^s, z_{i2}=xq^{s+2}, \ldots, z_{i,t_i}=xq^{s+t_i-1}, z_{j1}=yq^{s'}, z_{j2}=yq^{s'+2}, \ldots, z_{j,t_j}=yq^{s'+t_j-1}}
\end{equation}
for any zig-zag \(Z\) as in Figure 4, where
\begin{equation}
M_Z = m_Z + \\
+ \# \{(a, b, c) \in \{s, \ldots, t\}_i \times \{s', \ldots, t'\}_j \times \{1, \ldots, -d_{ij} - 1\} \text{ s.t. } a = b + 2c + d_{ij}\}
\end{equation}
and \(m_Z\) is given by (2.22). The analogous result holds for \(\mathcal{S}^+_{\text{int}}_{\text{geom}}\).

**Proof.** Letting \(Z\) be the zig-zag consisting of one top vertex and one bottom vertex situated at distance \(\in \{d_{ij} + 2, \ldots, -d_{ij} - 2\}\) apart, property (5.17) implies the fact that \(r\) is divisible by
\begin{equation}
1 - \frac{z_{ia}q^{2c+d_{ij}}}{z_{jb}}
\end{equation}
for all \(i \neq j\), all \(a, b\) and all \(c \in \{1, \ldots, -d_{ij} - 1\}\). This implies that
\begin{equation}
r = \Omega(R)
\end{equation}
for some \(R \in \mathcal{V}^+\). If we further want \(R \in \mathcal{S}^+\) (which is equivalent to \(r \in \mathcal{S}^+_{\text{geom}}\)) then we would need \(R\) to satisfy conditions (2.27) for any zig-zag. Because of the linear factors involved in formula (5.15), this is precisely equivalent to (5.17).
5.9. Our main technical step in establishing Theorem 1.12 is the following.

**Proposition 5.10.** The image of \( \iota \) (resp. \( \iota_{\text{loc}} \)) lies in \( S^{+,\text{geom}}_{\text{int}} \) (resp. \( S^{+,\text{geom}} \)).

**Proof.** We will prove the required statement for \( \iota \), as the one for \( \iota_{\text{loc}} \) is analogous. Throughout the present proof, we will fix a basis of \( \{ \mathbb{C}^{n_i} \}_{i \in I} \) compatible with the maximal torus \( T_n \subset G_n \) whose weights are the symbols \( z_{i_1}, \ldots, z_{i_m} \). Let us fix a zig-zag as in Figure 4, corresponding to \( i < j \) in \( I \), and we will only focus on a subset of the basis vectors

\[
\begin{align*}
\mathbb{C}^{n_i} &= \cdots \oplus \mathbb{C}v_i \oplus \mathbb{C}v_{i+2} \oplus \cdots \oplus \mathbb{C}v_{t-2} \oplus \mathbb{C}v_t \oplus \cdots \\
\mathbb{C}^{n_j} &= \cdots \oplus \mathbb{C}w_{s'} \oplus \mathbb{C}w_{s'+2} \oplus \cdots \oplus \mathbb{C}w_{t'-2} \oplus \mathbb{C}w_{t'} \oplus \cdots
\end{align*}
\]

(5.19) (5.20)

We will consider the affine subspace \( \mathbb{A} \hookrightarrow T^*Z_n \) which parameterizes collections of linear maps \( (\phi_c,e)_{c \in E} \), whose only non-zero matrix coefficients in the basis (5.19)–(5.20) are

\[
\begin{align*}
\phi_c(v_\bullet) &= x_\bullet^c \cdot w_\bullet, & \text{if } \bullet &= \bullet + 2c + d_{ij} - 2 \\
\phi_c^e(v_\bullet) &= y_\bullet^e \cdot v_\bullet, & \text{if } \bullet &= \bullet - 2c - d_{ij}
\end{align*}
\]

for any \( c \in \{1, \ldots, -d_{ij}\}, \bullet \in \{s, \ldots, t\} \) and \( \bullet \in \{s', \ldots, t'\} \). Let us consider the following closed and open subsets of \( \mathbb{A} \), respectively

\[ V = \mathbb{A} \cap \mu_n^{-1}(0) \cap \Lambda_n \quad \text{and} \quad U = \mathbb{A} \setminus V \]

where \( \mu_n \) is the quadratic map (5.1), and \( \Lambda_n \) is the closed subset of nilpotent points \( (\phi_c,e) \) (we note a slight imprecision: here we are interpreting \( \Lambda_n \) as living inside the affine space \( T^*Z_n \), while in (5.6) it was defined as living inside \( T^*3_n \)).

**Claim 5.11.** We may write \( V = \cup_{k=1}^N W_k \), where each \( W_k \) is the closure of the locus of \( \mathbb{A} \) determined by at least \( M_Z \) independent equations of the form

\[ x_\bullet^c = \text{expression in other } x's \text{ and } y's \]

or

\[ y_\bullet^e = \text{expression in other } x's \text{ and } y's \]

Above, \( M_Z \) is the number from (5.18).

Let us first show how Claim 5.11 implies Proposition 5.10. First of all, specializing the variables \( z_{ia} \) and \( z_{jb} \) as in (5.17) precisely corresponds to reducing the equivariance from the torus \( \mathbb{C}^* \times T_n \) to a subtorus \( H \), where every matrix coefficient

\[ x_\bullet^c \text{ is scaled by the weight } \chi : H \to \mathbb{C}^*, \text{ and} \]

\[ y_\bullet^e \text{ is scaled by the inverse weight } \chi^{-1} \]

where \( \chi = \frac{x}{y} \) in the notation of (5.17). Consider the filtration of \( V \) by closed subsets

\[ \emptyset = V_0 \subset V_1 \subset \cdots \subset V_N = V \quad \text{where} \quad V_k = W_1 \cup \cdots \cup W_k \]

and let \( U_k = \mathbb{A} \setminus V_k \). For every \( k \), we have the excision long exact sequence in equivariant algebraic \( K \)-theory

\[ K_H(U_{k-1} \cap W_k) \xrightarrow{\sigma_k} K_H(U_{k-1}) \xrightarrow{\tau_k} K_H(U_k) \to 0 \]
The assumption on $W_k$ implies that any element of $\text{Im} \sigma_k$ is a multiple of $(1 - \chi)^{M_Z}$, and thus so is any element in $\text{Ker} \tau_k$. Iterating this claim $N$ times implies that any element of

\begin{equation}
\text{Ker} \left( K_H(\mathbb{A}) \xrightarrow{\tau_{N-1} \circ \cdots \circ \tau_1} K_H(U) \right)
\end{equation}

is a multiple of $(1 - \chi)^{M_Z}$. With this in mind, take any element $\tau_n(\alpha)$ and restrict it to the $H$-equivariant $K$-theory of the affine subspace $\mathbb{A}$. Denote the resulting element $\beta$; the fact that $\alpha$ is supported on $\mu^{-1}(0) \cap \Lambda_n$ implies that $\beta$ is supported on the closed subset $V \subset \mathbb{A}$, and thus $\beta$ lies in the kernel (5.21). As all elements in the kernel (5.21) are multiples of $(1 - \chi)^{M_Z}$, we conclude the sought-for divisibility condition in (5.17).

It remains to prove Claim 5.11. To this end, let us draw two kinds of arrows between a vertex $a \in \{s, \ldots, t\}$ and a vertex $b \in \{s', \ldots, t'\}$:

- An arrow from $a$ to $b$ (respectively from $b$ to $a$) if $a = b - d_{ij}$ (respectively if $a = b + d_{ij}$); these will be called “long” arrows.
- An arrow from $a$ to $b$ and an arrow from $b$ to $a$ if $a - b \in \{d_{ij} + 2, \ldots, -d_{ij} - 2\}$; these will be called “short” arrows.

Note that $M_Z$ of (5.18) is equal to $m_Z + v$, where $v$ is the number of pairs of short arrows $a \leftrightarrow b$. For every pair of short arrows $a \leftrightarrow b$, note that one of the equations cutting out the closed subset $V \hookrightarrow \mathbb{A}$ is

\begin{equation}
x^b_a y^a_b = 0
\end{equation}

Otherwise, the fact that $x^b_a \neq 0$ and $y^a_b \neq 0$ simultaneously would violate the nilpotency condition (if $v_a \in F_p^i$ for some $p$, then $x^b_a \neq 0$ implies that $w_b \in F_{p-1}^{i+1}$, but $y^a_b \neq 0$ implies that $v_a \in F_{p-2}^{i+2}$; since this works for all $p$, we would obtain a contradiction). We conclude that $V$ is covered by the closure of the subsets

\begin{equation}
W^a_b = \left\{ f^b_a = 0 \text{ for every pair of short arrows } a \leftrightarrow b \right\}
\end{equation}

as $k$ goes over the $2^v$ sets $(f^b_a \in \{x^b_a, y^a_b\})$ of short arrows $a \leftrightarrow b$. This already yields at least $v$ equations in Claim 5.11 but we need $M_Z = m_Z + v$ equations. To obtain this slightly bigger number of equations, let us use the long arrows; recall from the proof of Proposition 2.18 that the zig-zag $Z$ contains two intersecting long arrows

\begin{equation}
b \not\xrightarrow{a} \quad \text{and} \quad a' \not\xrightarrow{b'}
\end{equation}

and their $m_Z - 1$ successive translates to the right by 2. Let us consider the following equations cutting out $V \hookrightarrow \mathbb{A}$ (all these equations come from $\mu^{-1}(0)$)

\begin{align}
x^b_a y^a_b a^{-2} + \cdots &= 0 \\
x^b_a a^{-4} y^a_b a^{-4} + \cdots &= 0 \\
\cdots
\end{align}

\begin{align}
x^b_a a^{a'+2} y^a_b a' + \cdots &= 0 \\
x^b_a a^{a'+2} y^a_b a' + \cdots &= 0 \\
\cdots
\end{align}
(5.29) \[ x_{a'}^{b'-4} y_{b'-2} + \cdots = 0 \]
(5.30) \[ x_{a'}^{b'-2} y_{b'} + \cdots = 0 \]
(the ellipses stand for other sums of products of \(x\)'s and \(y\)'s, which will not matter in the subsequent argument).

Claim 5.12. Besides the defining equations (5.23), the closed subset \(V \cap W_k^c\) is determined by at least one more equation involving the variables in (5.25)–(5.30).

We may apply Claim 5.12 for each of the \(m_Z\) pairs of long arrows given by (5.24) and its translates. For each of these pairs, the “one more equation” prescribed by Claim 5.12 always involves a variable with an index among the left-most endpoints (i.e. \(a'\) and \(b\)) of the arrows (5.24). Since these endpoints are distinct for all the \(m_Z\) pairs of long arrows, we conclude that the closed subset \(V \cap W_k^c\) is determined by at least \(m_Z\) more equations than the number \(v\) of pairs of short arrows. As \(M_Z = m_Z + v\), this concludes the proof of Claim 5.11.

Proof. of Claim 5.12: Among all the short pairs of variables
(5.31) \((x_{a-2}^b y_{b-2}^b), \ldots (x_{a'+2}^b y_{b'+2}^b), (x_{a'}^b y_{b'}^b), \ldots (x_{a'-2}^{b'-2} y_{b'-2}^{b')}\)
the set \(W_k^c\) involves at least one of the variables being equal to 0. If from any such pair, the other variable were also equal to 0, then we would have obtained the required “one more equation”. Therefore, we may assume that among all the pairs of variables (5.31), only one is equal to 0. However, this requires either
\begin{itemize}
  \item \(y_{b-2}^b \neq 0\), in which case (5.29) gives us the extra equation \(x_{a}^{b} = \ldots\), or
  \item \(x_{a'-2}^{b'} \neq 0\), in which case (5.30) gives us the extra equation \(y_{b'}^{a'} = \ldots\), or
  \item at least one of the equations (5.25)–(5.29) being of the form
(5.32) \[ x_{i'}^{i'} y_{i'}^{i'} + \ldots \neq 0 \]
with both \(x_{i'}^{i'} \neq 0\) and \(y_{i'}^{i'} \neq 0\), in which case (5.32) yields an extra equation.
\end{itemize}

\(\square\)

Proof. of Theorem 1.12: By Proposition 5.10, we have an algebra homomorphism
(5.33) \[ \iota_{\text{loc}} : K_{\text{nilp,loc}}^{\text{nilp}} \rightarrow S^{+,\text{geom}} \]
It remains to be prove that the map above is surjective. However, \(S^{+,\text{geom}} \cong S^+\) by (5.16). Since Proposition 2.24 showed that \(S^+\) is generated by \(\{e_{i,k}\}_{i \in I, k \in \mathbb{Z}}\) as a \(\mathbb{Q}(q)\)-algebra, then the same is true of \(S^{+,\text{geom}}\). However, all of the \(e_{i,k}\)'s are contained inside the image of \(\iota_{\text{loc}}\) (when \(n = c_i\), the stack of quiver representations is pt/\(\mathbb{C}^*\), and its \(K\)-theory is precisely \(\mathbb{Z}[q^{\pm 1}]\langle z_i^{\pm 1}\rangle\)), hence all the generators of \(S^{+,\text{geom}}\) are in the image of the map (5.33). This implies that (5.33) is surjective.
\(\square\)
5.13. Theorem 1.12 describes the image of the $K$-theoretic Hall algebra inside the shuffle algebra. To say something the $K$-theoretic Hall algebra itself, we need to modify the definition of our quantum loop groups.

**Definition 5.14.** Define $\tilde{U}_q^+(Lq)^{geom}$ as in Subsection 1.2 but replacing (1.9) by
\begin{equation}
-e_i(z)e_j(w) = e_j(w)e_i(z) = -q^{2c+d_{ij}}(z - wq^{2c+d_{ij}} - w)
\end{equation}
for any $i \neq j$ (when $i = j$, relation (1.9) is unchanged).

Relation (5.34) appeared in the context of $K$-theoretic Hall algebras in [10] (it also arose in different contexts, e.g. [6] in connection with vertex representations).

**Proposition 5.15.** There exists a surjective algebra homomorphism
\begin{equation}
\tilde{U}_q^+(Lq)^{geom} \twoheadrightarrow K_{\mathfrak{c},loc}^{nilp}
\end{equation}
that sends $e_{i,k}$ to the $k$-th power of the tautological line bundle on $T^*3_{\varsigma_i}$.

**Proof.** Let us show that relation (5.34) holds in $K_{\mathfrak{c},loc}^{nilp}$. To this end, note that
\begin{equation}
T^*3_{\varsigma_i + \varsigma_j} = \left\{ (x_1, \ldots, x_{-d_{ij}}, y_1, \ldots, y_{-d_{ij}}) \mid \sum_{i=1}^{d_{ij}} x_i y_i = 0 \right\} / \mathbb{C}^* \times \mathbb{C}^*
\end{equation}
The nilpotent substack $\Lambda_{\varsigma_i + \varsigma_j}$ consists of those tuples $(x_1, \ldots, x_{-d_{ij}}, y_1, \ldots, y_{-d_{ij}})$ where either all the $x$’s are 0, or all the $y$’s are 0. Thus
\begin{equation}
\Lambda_{\varsigma_i + \varsigma_j} = \left( \mathbb{A}^{-d_{ij}} \cup \mathbb{A}^{-d_{ij}} \right) / \mathbb{C}^* \times \mathbb{C}^*
\end{equation}
with the two copies of $\mathbb{A}^{-d_{ij}}$ intersecting at the origin. By the Mayer-Vietoris sequence ([10] Appendix D), the $K$-theory of the nilpotent substack is the direct sum of two copies of the $K$-theory of $\mathbb{A}^{-d_{ij}}$, modulo the identification of the structure sheaf of the origin in the aforementioned two copies. Since the structure sheaf of the origin can be expressed as an equivariant Koszul complex, we conclude that
\begin{equation}
K_{\mathfrak{c}}(\Lambda_{\varsigma_i + \varsigma_j}) = \frac{\mathbb{Z}[\ell_1^\pm 1, \ell_2^\pm 1]}{\prod_{c=0}^{-d_{ij}-1} \left( 1 - \frac{zq^{2c+d_{ij}}}{\ell_2} \right)} \oplus \frac{\mathbb{Z}[\ell_1^\pm 1, \ell_2^\pm 1]}{\prod_{c=0}^{-d_{ij}-1} \left( 1 - \frac{zq^{2c+d_{ij}}}{\ell_1} \right)}
\end{equation}
where $\ell_1$ and $\ell_2$ denote the elementary characters of the two factors of $\mathbb{C}^* \times \mathbb{C}^*$ in (5.37). The images of the LHS and RHS of (5.34) in the $K$-theoretic Hall algebra are
\begin{align*}
\delta \left( \frac{w}{\ell_1} \right) \delta \left( \frac{z}{\ell_2} \right) \prod_{c=0}^{-d_{ij}-1} \left( 1 - \frac{wq^{2c+d_{ij}}}{z} \right)
\end{align*}
and
\begin{align*}
\delta \left( \frac{w}{\ell_1} \right) \delta \left( \frac{z}{\ell_2} \right) \prod_{c=0}^{-d_{ij}-1} \left( 1 - \frac{zq^{2c+d_{ij}}}{w} \right)
\end{align*}
respectively, and they are manifestly equal in the quotient (5.38).

**Remark 5.16.** The stronger relation (1.9) does not hold in $K_{\mathfrak{c}}(\Lambda_{\varsigma_i + \varsigma_j})$, as it is mapped not to 0, but to a zero-divisor.
The surjectivity of the map \([5.35]\) follows along the lines of [5, Section 7.10]. In fact, the argument of loc. cit. proves the surjectivity of the following map

\[
\tilde{U}_q^+ (Lg)_{\text{geom}} \rightarrow K_{\text{nilp}}^C,
\]

where the left-hand side is the \(\mathbb{Z}[q^{\pm 1}]\)-subalgebra of \(\tilde{U}_q^+ (Lg)_{\text{geom}}\) generated by

\[
e_{i,k}^n \left[ \frac{n!}{[n]_q!} \right]
\]

for all \(i \in I, \, k \in \mathbb{Z}\) and \(n \geq 0\) (recall that \([n]_q! = [1]_q \cdots [n]_q\) with \([n]_q = q^n - q^{n-1} - \cdots - q + 1\)). □

References

[1] Ding J., Jing N., On a combinatorial identity, Int. Math. Res. Not. (2000), no. 6, 325–332
[2] Drinfeld V., A new realization of Yangians and of quantum affine algebras, preprint (1986), FTINT 30–86.
[3] Enriquez B., On correlation functions of Drinfeld currents and shuffle algebras, Transform. Groups 5 (2000), no. 2, 111–120.
[4] Feigin B., Odesskii A., Quantized moduli spaces of the bundles on the elliptic curve and their applications, Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000), 123–137, NATO Sci. Ser. II Math. Phys. Chem., 35, Kluwer Acad. Publ., Dordrecht, 2001.
[5] Grojnowski I., Affinizing quantum algebras: from D-modules to K-theory, preprint (1994), https://www.dpmms.cam.ac.uk/~groj/char.ps
[6] Jing N., Quantum Kac-Moody Algebras and Vertex Representations, Lett. Math. Phys. 44: 261–271 (1998)
[7] Lalonde P., Ram A., Standard Lyndon bases of Lie algebras and enveloping algebras, Trans. Amer. Math. Soc. 347 (1995), no. 5, 1821–1830.
[8] Leclerc B., Dual canonical bases, quantum shuffles and \(q\)-characters, Math. Z. 246 (2004), no. 4, 691–732.
[9] Lusztig G., Quivers, perverse sheaves, and quantized enveloping algebras, J. Amer. Math. Soc. 4 (1991), 365–421
[10] Nakajima H., Quiver varieties and finite dimensional representations of quantum affine algebras, J. Am. Math. Soc., Volume 14, Number 1, 145–238
[11] Negut A., The shuffle algebra revisited, Int. Math. Res. Not., Issue 22 (2014), 6242–6275
[12] Negut A., Shuffle algebras for quivers and wheel conditions, J. für die Reine und Angew. Math., no. 795 (2023), pp. 139–182
[13] Negut A., Sala F., Schiffmann O., Shuffle algebras for quivers as quantum groups, arXiv:2111.00249
[14] Negut A., Tsymbaliuk A., Quantum loop groups and shuffle algebras via Lyndon words, arXiv:2102.11269
[15] Schiffmann O., Vasserot É., The elliptic Hall algebra and the K-theory of the Hilbert scheme of \(\mathbb{A}^2\), Duke Math. J. 162 (2013), no. 2, 279–366
[16] Varagnolo M., Vasserot É., K-theoretic Hall algebras, quantum groups and super quantum groups, Selecta Math. (N.S.) 28 (2022), no. 1, Paper No. 7, 56 pp

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