Intrinsic Analysis of the Sample Fréchet Mean and Sample Mean of Complex Wishart Matrices

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Abstract—We consider two types of averaging of complex covariance matrices, a sample mean (average) and the sample Fréchet mean. We analyse the performance of these quantities as estimators for the true covariance matrix via ‘intrinsic’ versions of bias and mean square error, a methodology which takes account of geometric structure. We derive simple expressions for the intrinsic bias in both cases, and the simple average is seen to be preferable. The same is true for the asymptotic Riemannian risk, and for the Riemannian risk itself in the scalar case. Combined with a similar preference for the simple average using non-intrinsic analysis, we conclude that the simple average is preferred overall to the sample Fréchet mean in this context.

Index Terms—complex Wishart matrix, Fréchet mean, intrinsic analysis, Riemannian manifold

I. INTRODUCTION

In this letter we give a performance analysis of the average of complex-valued covariance matrices found by the simple sample mean or via the sample Fréchet mean. Classical measures such as bias and mean square error (MSE) do not take into account the geometrical structure. We consider two types of averaging of complex covariance matrices, the manifold should be taken into account. This leads to outliers. Our motivation to study this topic came from the desire to build improved graphical models from neuroscience data, used in schizophrenia studies [12]; see also [11]. This requires good spectral matrix estimates for different groups of individuals and we want to know whether it is best to average estimated spectral matrices over individuals in each group, or find instead their sample Fréchet mean.

Positive definite and Hermitian complex covariance matrices form a manifold in the space of complex-valued matrices. If such a manifold is equipped with a Riemannian metric, it becomes a Riemannian manifold. In evaluating performance criteria for the different forms of averaging of covariance matrices, the manifold should be taken into account. This leads to considering ‘intrinsic’ versions of bias and MSE [9], [13], [15]. Indeed this was the approach adopted for performance analysis by [7].

In a recent work [17] the averaging of a set \( S_1, \ldots, S_N \) of positive definite complex covariance matrix estimators was studied. The matrices are considered homogeneous in the sense that they are estimating the same true covariance matrix \( \Sigma \), and statistically, they are independent and identically complex-Wishart distributed. Ordinary bias and MSE were used to compare the performance of the ordinary average of the estimated covariance matrices, and of their sample Fréchet mean, (also called the Riemannian mean in this setting). In this letter we derive the intrinsic versions of bias and MSE to enable a ‘geometry-aware’ performance appraisal. In contrast to the classical measures, intrinsic evaluation of an estimator is invariant under reparametrization, and it is dependent on the model only [9].

The main contribution of this letter is to prove the simple average of \( N \) Wishart matrices outperforms their Fréchet mean in terms of intrinsic bias for \( N \geq 2 \). We also show that the same is true for the asymptotic Riemannian risk, and for the Riemannian risk itself in the scalar case for \( N \geq 2 \). These results are enabled by the derivation of simple expressions for the intrinsic bias for both types of average.

II. PRELIMINARIES

A. Complex Wishart matrices

Let \( \mathbf{X}_0, \ldots, \mathbf{X}_{K-1} \) be \( K \) independent \( p \)-dimensional complex-Gaussian random vectors with zero means and covariance matrix \( \Sigma \). Then the maximum likelihood estimator for \( \Sigma \) is the covariance matrix estimator \( \mathbf{S} = \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{X}_k \mathbf{X}_k^H \), where \( \mathbf{H} \) denotes Hermitian transpose and \( K \mathbf{S} \) has the complex Wishart distribution [10] with \( K \) complex degrees of freedom and mean \( K \Sigma \), denoted by

\[
K \mathbf{S} \overset{d}{\sim} \mathcal{W}_p^C(K, \Sigma). \quad (1)
\]

Such matrices arise frequently, e.g., [3], [12]. We assume \( K \geq p \), since then \( \mathbf{S} \) has full rank \( p \), and \( \mathbf{S} \) is positive definite, with probability one.

B. Positive Definite Hermitian matrices

Let \( \mathcal{H}^+(p) \) denote the set of \( p \times p \) positive definite Hermitian matrices, a differentiable manifold. Denote the set of all \( p \times p \) invertible matrices by \( GL(p) \). The group action of \( GL(p) \) on \( \mathcal{H}^+(p) \) is the transformation \( \phi : GL(p) \times \mathcal{H}^+(p) \to \mathcal{H}^+(p) \) given by \( \phi_L(P) = LPL^H \).

C. Riemannian manifold

A manifold equipped with a Riemannian metric \( g \) is a Riemannian manifold. \( \mathcal{H}^+(p) \) can be turned into a Riemannian manifold by defining at every point in \( \mathcal{H}^+(p) \) an inner product
for elements in the tangent space that varies differentiably along the manifold. The minimum length curve connecting two points on the manifold is called the geodesic and the Riemannian distance \( d_g \) between the points is given by the length of this curve. We use a scaled version of the Frobenius inner product that is invariant under the group action of \( GL(p) \). For any \( A, B \in T_p \mathcal{M} \), where \( T_p \mathcal{M} \) denotes the tangent space at the point \( p \), \( \langle A, B \rangle_p = \text{tr}(P^{-1} A P^{-1} B) \), and the corresponding norm is

\[
\| A \|_p = \langle A, A \rangle_p^{1/2} = \| P^{-1/2} A P^{-1/2} \|_{F, \text{Fr}}.
\]

\[ (\langle A, B \rangle_p = \text{tr}(P^{-1} A P^{-1} B), \quad (\text{2}) \]

and the corresponding norm is

\[ \| A \|_p = \langle A, A \rangle_p^{1/2} = \| P^{-1/2} A P^{-1/2} \|_{F, \text{Fr}}. \quad \] (3)

D. Riemannian distance

For \( P_0 \in GL(p) \) the matrix logarithm is any \( p \times p \) matrix \( Q \) such that \( \exp(Q) = P_0 \), where \( \exp(\cdot) \) is the usual power series expansion. For \( P_1 \in \mathcal{H}^+(p) \), \( \mathcal{H}^+(p) \), the Frobenius inner product takes the form \([8\text{p.}\; 1729])

\[
\langle A, B \rangle_p = \text{tr}(P^{-1} A P^{-1} B),
\]

and the corresponding norm is

\[ \| A \|_p = \langle A, A \rangle_p^{1/2} = \| P^{-1/2} A P^{-1/2} \|_{F, \text{Fr}}. \]

(3)

\[ \text{D. Riemannian distance} \]

For \( P_0 \in GL(p) \) the matrix logarithm is any \( p \times p \) matrix \( Q \) such that \( \exp(Q) = P_0 \), where \( \exp(\cdot) \) is the usual power series expansion. For \( P_1 \in \mathcal{H}^+(p) \), the matrix logarithm function is \([1\text{p.}\; 429] \]

\[ \log \rho_1 = U \text{diag}(\log \rho_1(1), \ldots, \log \rho_p(1)) U^H, \quad (4) \]

where \( \rho_j(1) > 0 \) is the \( j \)-th eigenvalue of \( P_1 \), \( (\rho_j(1))^H = \rho_j(1) \), so \( P_1 \) is Hermitian. For any invertible matrix \( A \) and matrix \( B \) having real positive eigenvalues,

\[ \log(A^{-1} B A) = A^{-1} \log(B) A. \]

The Riemannian distance \( d_g(P_1, P_2) \) is defined by \([8\text{p.}\; 1729] \]

\[ d_g^2(P_1, P_2) = \text{tr}(\log^2(P^{-1/2} P_2 P_1^{-1/2})), \quad (6) \]

\[ = \| \log(P^{-1/2} P_2 P_1^{-1/2}) \|_{F, \text{Fr}}. \quad (7) \]

We will denote the Riemannian manifold by \( \mathcal{M} \).

E. Fréchet Mean

Let \( S_1, \ldots, S_N \) be independent random matrices with common distribution \( F \) on \( \mathcal{M} \). Let \( \bar{F} \) be the empirical distribution. The sample Fréchet mean of \( \bar{F} \) is the minimizer of \([2\text{p.}\; 629] \]

\[ (1/N) \sum_{j=1}^N d_g^2(P, S_j), \quad \bar{P} \in \mathcal{H}^+(p). \quad (8) \]

The Riemannian metric space has negative sectional curvature so the sample Fréchet mean is unique, \([6\text{p.}\; 6] \).

F. Mappings for Riemannian Manifolds

The exponential map \( \text{Emap}: T_p \mathcal{M} \rightarrow \mathcal{M} \) is a function mapping a vector \( U \) (starting from \( P \in \mathcal{M} \)) in the tangent space, to a point \( S \) on the Riemannian manifold:

\[ S = \text{Emap}_P(U) = P^{1/2} \exp(P^{-1/2} U P^{-1/2}) P^{1/2}. \]

The (inverse) logarithmic map is:

\[ U = \text{Lmap}_P(S) = P^{1/2} \log(P^{-1/2} S P^{-1/2}) P^{1/2}, \]

and takes \( S \) on the Riemannian manifold to \( U \) in the tangent space, \( \mathcal{M} \rightarrow T_p \mathcal{M} \); the nonpositive curvature of the manifold guarantees the unique inverse mapping.

III. Intrinsic Versions of Bias and MSE

Before proceeding we need to state some definitions.

Definition 1: \([9\text{p.}\; 128] \) For a fixed sample size \( N \), and an estimator \( \hat{S} \) of \( \Sigma \), the estimator vector field is \( \text{Lmap}_\Sigma(\hat{S}) \).

Definition 2: \([15\text{p.}\; 1615] \) The expectation of the estimator \( S \) on the manifold, \( \mathcal{M} \), is defined as \( E \{ S \} = \text{Emap}_\Sigma(\hat{S}) \).

Remark 1: An estimator \( S \) is an intrinsically unbiased estimator of \( \Sigma \) if and only if \( E \{ \text{Lmap}_\Sigma(S) \} = 0 \), since then \( E \{ S \} = \Sigma^{1/2} \exp(0) \Sigma^{1/2} = \Sigma \). In fact \( E \{ \text{Lmap}_\Sigma(S) \} \) is called the bias vector field, \([15\text{p.}\; 1615] \).

Definition 3: \([9\text{p.}\; 129], [13\text{p.}\; 1568] \) The intrinsic (or invariant) bias of an estimator \( \hat{S} \) of \( \Sigma \) is a scalar defined as

\[ \text{ibias}(\hat{S}) \overset{\text{def}}{=} \| E \{ \text{Lmap}_\Sigma(\hat{S}) \} \|_\Sigma^2. \]

(9)

Hence, calculating the intrinsic bias of an estimator involves (i) mapping the estimator to a vector on the tangent space specified by the true parameter; (ii) finding the squared norm of the expectation of the vector.

Definition 4: \([9\text{p.}\; 129], [13\text{p.}\; 1568] \) The Riemannian risk of \( \hat{S} \) is a scalar defined as

\[ E \{ \| \text{Lmap}_\Sigma(\hat{S}) \|_\Sigma^2 \}. \]

(10)

Remark 2: The Frobenius inner product is a Riemannian metric in the Euclidean domain and makes \( \mathcal{M} \) a Riemannian manifold, then the Riemannian risk is exactly the MSE.

IV. Intrinsic Bias of the Estimators

A. Intrinsic Bias of the Sample Fréchet Mean

Theorem 1: Let \( R_1, R_2, \ldots, R_N \) be independent and identically distributed (IID) random matrices with \( K R \overset{\text{d}}{=} W_p^C(K, I) \), and let \( \hat{P}_I \) be their sample Fréchet mean. Then

\[ \text{ibias}(\hat{P}_I) = p \log^2(a), \]

(11)

where

\[ a = 1/K \left( \exp \left( \sum_{j=1}^p \psi(K - p + j) \right) \right)^{1/p}, \]

(12)

and \( \psi(\cdot) \) denotes the digamma function.

Proof: From \([9\text{p.}\; 129] \), when \( \Sigma = I \), the intrinsic bias is given by

\[ \text{ibias}(\hat{P}_I) \overset{\text{def}}{=} \| E \{ \text{Lmap}_I(\hat{P}_I) \} \|_I^2, \]

(13)

where

\[ \text{Lmap}_I(\hat{P}_I) = I^{1/2} \log(I^{-1/2} \hat{P}_I I^{-1/2}) I^{1/2} = \log(\hat{P}_I). \]

(14)

From \([8\text{p.}\; 1729] \), \( \hat{P}_I \) is positive definite and Hermitian matrix. From \([4\text{p.}\; 128] \), \( \log(\hat{P}_I) \) is Hermitian. Hence, \( E \{ \log(\hat{P}_I) \} \) is Hermitian. Let

\[ E \{ \log(\hat{P}_I) \} \overset{\text{def}}{=} \begin{bmatrix} b_1 & b_2 & \cdots & b_{1p} \\ b_2 & b_2 & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1p} & b_{2p} & \cdots & b_{pp} \end{bmatrix}, \]

(15)
where \( b_{ij}^* \) is the complex conjugate of \( b_{ij} \). For fixed
1 \( \leq i < j \leq p \), we define the elementary matrix \( C_{ij} \) as the matrix formed from \( I \) by exchanging row \( i \) with row \( j \). Then, \( C_{ij} \) is Hermitian and fulfills the property that the sample Fréchet mean of \( C_{ij} \) is \( \hat{C}_{ij} \). So,

\[
C_{ij} \hat{P}_1 C_{ij}^H = \hat{P}_1. \tag{16}
\]

The expression on the left of (16) is clearly Hermitian, and it is
equivalent to \( \hat{P}_1 \) and \( C_{ij} \). So, \( C_{ij} \) is Hermitian and fulfills
the property that the sample Fréchet mean of \( C_{ij} \) is \( \hat{C}_{ij} \). So,

\[
C_{ij} \hat{P}_1 C_{ij}^H = \hat{P}_1. \tag{16}
\]

We now demonstrate that \( \hat{P}_1 \) is the equivariant property that the sample Fréchet mean of \( C_{ij} \) is \( \hat{C}_{ij} \). Note that \( \hat{C}_{ij} \) is Hermitian and fulfills the property that the sample Fréchet mean of \( C_{ij} \) is \( \hat{C}_{ij} \). So,

\[
C_{ij} \hat{P}_1 C_{ij}^H = \hat{P}_1. \tag{16}
\]

Using (16) we have

\[
E\{\log(\hat{P}_1)\} = E\{\log(\hat{C}_{ij}^{\text{H}})\} = (d_i, \ldots, d_i, \ldots, d_i, \ldots, d_i, \ldots, d_i, \ldots, d_i, \ldots, d_i),
\]

and diagonalizing \( C_{ij} \) we get

\[
\hat{C}_{ij} C_{ij}^H = \hat{P}_1. \tag{16}
\]

To make the step from (17) to (18) we used (5) with \( A \) equaled to \( C_{ij} \). Now

\[
diag(E\{\log(\hat{P}_1)\}) = \{d_i, \ldots, d_i, \ldots, d_i, \ldots, d_i\},
\]

and diagonalizing \( C_{ij} \) we get

\[
\hat{C}_{ij} C_{ij}^H = \hat{P}_1. \tag{16}
\]

But the first \( i - 1 \) elements of the vector column of \( E\{\log(\hat{P}_1)\} \) are

\[
\begin{bmatrix}
\log(\lambda_1) \\
\vdots \\
\log(\lambda_i)
\end{bmatrix}
\]

and the last \( i - 1 \) elements of the vector column of \( E\{\log(\hat{P}_1)\} \) are

\[
\begin{bmatrix}
\log(\lambda_{i+1}) \\
\vdots \\
\log(\lambda_p)
\end{bmatrix}
\]

with row \( i \).

We now turn to the more general case. Let \( S_1, \ldots, S_N \) be IID samples with \( KS \overset{d}{=} W_C^P(K, \Sigma) \) and consider their sample Fréchet mean \( \hat{S}_N \). Then \( \text{bias}(\hat{S}_N) \) is given by

\[
\text{bias}(\hat{S}_N) = \left\| E\{\text{map}(\hat{S}_N)\} - \Sigma \right\|^2 = a^2_0 \Sigma.
\tag{22}
\]

Using the result \( \psi(z + 1) = \psi(z) + (1/z) \) it can be shown
after some manipulation that the form of (22) matches that in
(13) eqns. (100), (102).

2) Scalar Case: Consider the scalar case, \( (p = 1) \), where
\( KS \overset{d}{=} W_C^P(K, 1) \), i.e., the single entry of the covariance matrix is unity. From (21) we have that \( \text{bias}(S) = a^2_0 \) so that, using (12),

\[
\text{bias}(S) = \log^2(a) = [\psi(K) - \log(K)]^2. \tag{24}
\]

It is straightforward to show that \( S \overset{d}{=} \frac{1}{2\pi} \chi^2_{2K} \), (scaled chi-square with \( 2K \) degrees of freedom), so that (24) gives the intrinsic bias in this case. Now suppose \( Y_1, \ldots, Y_K \overset{d}{=} \chi^2_{\frac{1}{2}} \). Then, \( Y \), the mean, has distribution \( Y \overset{d}{=} \frac{1}{2\pi} \chi^2_{2K} \), and so \( \text{bias}(\sqrt{\Sigma}) \) is given by (24); this result agrees with (13) p. 1569.

B. Intrinsic Bias of the Sample Mean

Let \( S_1, \ldots, S_N \) be IID samples with \( KS \overset{d}{=} W_C^P(K, \Sigma) \) and consider their ordinary sample mean \( S \). We know that

\[
\text{bias}(\bar{S}) = \left\| E\{\text{map}(\hat{S})\} - \Sigma \right\|^2 = a^2_0 \Sigma. \tag{25}
\]

The intrinsic bias of the sample mean can be inherited from
that of the Fréchet mean by taking the sample size of the
Fréchet mean to be \( N = 1 \), and then replacing the degrees of freedom \( K \) by \( KN \) in (21),

\[
\text{bias}(\bar{S}) = \left\| E\{\text{map}(\hat{S})\} - \Sigma \right\|^2 = a^2_0 (KN). \tag{26}
\]
V. INTRINSIC RISK OF THE ESTIMATORS

A. Intrinsic Risk of the Sample Fréchet Mean

The Riemannian or intrinsic risk was defined in (10). We now examine this for the sample Fréchet mean.

\[
E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\}
\]

where for the last step we have used (2) and the cyclic nature of trace. From (9) and (7) we alternatively can write

\[
E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\}
\]

Hence the Riemannian risk is independent of the underlying true covariance matrix. The Riemannian risk can be decomposed into two parts, (i) the intrinsic bias which does not depend on the sample size, and (ii) the sum of the variances of every entry of \( \log(\Sigma) \). For the last step we have used (2) and the cyclic nature of trace. From (9) and (7) we alternatively can write

\[
E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\}
\]

For the sample mean we know \( \hat{S}_I \) converges almost surely to the population mean \( I \). So we have

\[
\lim_{N \to \infty} E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = \lim_{N \to \infty} \text{bias}(\hat{S}_I) = \lim_{N \to \infty} \text{bias}(\hat{S}_I)
\]

B. Intrinsic Risk of the Sample Mean

If we set \( N = 1 \), and replace the degrees of freedom \( K \) by \( KN \), we can replace \( P \) by \( S \) and \( P \) by \( S \) to give

\[
E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\}
\]

VI. COMPARISONS

A. Intrinsic Bias

We firstly examine \( a_0(K) \) for \( K \geq p \). From (12) we have

\[
a_0(K) = (1/p) \sum_{j=1}^{p} \psi(K + 1 - j) - \log(K).
\]

Clearly \( (1/p) \sum_{j=1}^{p} \frac{1}{K + 1 - j} > (1/K) \) and \( (1/K) - \log(1 + \frac{1}{K}) > 0 \) for \( K \geq 1 \), so \( a_0(K + 1) = a_0(K) \), and since \( a_0(K) \) is negative, it is monotonically increasing and bounded above by zero, and \( a_0(K) \) decreases in magnitude with increasing \( K \). So, for \( N \geq 2 \),

\[
\text{bias}(P) = \text{bias}(\hat{P}) = pa_0(K) > pa_0(KN) = \text{bias}(\tilde{S}),
\]

i.e., the simple average is preferable in terms of intrinsic bias for \( N \geq 2 \).

1) Scalar case: Set \( p = 1 \) to obtain for \( N \geq 2 \),

\[
\text{bias}(S) = a_0(K) > a_0(KN) = \text{bias}(\tilde{S}),
\]

Since \( \text{bias}(\tilde{S}) = [\psi(KN) - \log(KN)]^2 \), (see (24)), and \( \psi(z) \to \log z \) as \( z \to \infty \), we see that \( \lim_{N \to \infty} \text{bias}(S) = 0 \).

B. Riemannian Risk

We know that the sample Fréchet mean \( \hat{P}_I \) converges almost surely to the population Fréchet mean \( P \) [7] p. 4555. So we can conclude that

\[
\lim_{N \to \infty} E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = \text{bias}(\hat{P}_I) = pa_0(K).
\]

For the sample mean we know \( \hat{S}_I \) converges almost surely to the population mean \( I \). So we have

\[
\lim_{N \to \infty} E\{\|\Sigma^{1/2} \log(\Sigma^{-1/2} P \Sigma^{-1/2}) \Sigma^{1/2} \|_2^2\} = \lim_{N \to \infty} \text{bias}(\hat{S}_I) = \lim_{N \to \infty} pa_0(KN).
\]

From (27) we thus see that, asymptotically, the Riemannian risk will be smaller for the simple average.

1) Scalar case: For \( p = 1 \) and finite \( N \), the Riemannian risk is \( \psi(K) - \log(K) + \text{var}(\log(K)) \), where \( \psi(K) \) is the trigamma function. So the Riemannian risk of \( P_1, Rr(\hat{P}_1) \), say, is given by

\[
Rr(\hat{P}_1) = \frac{1}{N} [\psi(K) + \log(K)].
\]

When \( N = 1 \), (29) gives \( Rr(S) = \psi(K) + \log(K) \)

so, replacing \( K \) by \( KN \),

\[
Rr(\tilde{S}) = \psi(KN) + \log(KN).
\]

Plotting the difference \( Rr(\hat{P}_1) - Rr(\tilde{S}) \) as a function of \( N \) and \( K \), we find that the risk is always higher for \( Rr(\hat{P}_1) \).

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VII. CONCLUSIONS

We have proved that the simple average of \( N \) Wishart matrices is preferred over the Fréchet mean (i) in terms of intrinsic bias for \( N \geq 2 \), and (ii) likewise for the asymptotic Riemannian risk, and for the Riemannian risk itself in the scalar case. Simple expressions were given for the intrinsic bias for both types of average. Non-intrinsic performance measures in [17] also favoured the simple average. There is thus strong evidence to prefer the sample mean in this context.
REFERENCES

[1] D. S. Bernstein, *Matrix Mathematics*. Princeton, NJ: Princeton University Press, 2005.
[2] R. Bhattacharya and V. Patrangenaru, “Large sample theory of intrinsic and extrinsic sample means on manifolds. I,” *The Annals of Statistics*, vol. 31, pp. 1–29, 2003.
[3] D. R. Brillinger, *Time Series: Data Analysis and Theory (Expanded Edition)*. New York: McGraw-Hill Inc., 1981.
[4] P. S. Bullen, *A Dictionary of Inequalities*, Essex, U.K.: Longman, 1998.
[5] K. Conradsen, A. A. Nielsen, J. Schou & H. Skriver, “A test statistic in the complex Wishart distribution and its application to change detection in polarimetric SAR data,” *IEEE Trans. Geosci. Remote Sensing*, vol. 41, pp. 4–19, 2003.
[6] I. L. Dryden, A. Koloydenko and D. Zhou, “Non-Euclidean statistics for covariance matrices, with applications to diffusion tensor imaging,” *The Annals of Applied Statistics*, vol. 3, pp. 1102–1123, 2009.
[7] I. Ilea, L. Bombrun, R. Terebes, M. Borda and C. Germain, “An M-estimator for robust centroid estimation on the manifold of covariance matrices,” *IEEE Signal Proc. Lett.*, vol. 23, pp. 1255–1259.
[8] X. Jiang, L. Ning and T. T. Georgiou, “Distances and Riemannian metrics for multivariate spectral densities,” *IEEE Trans. Autom. Control*, vol. 57, pp. 1723–1735, 2012.
[9] G. Garcia and J. M. Oller, “What does intrinsic mean in statistical estimation,” *Statist. Oper. Res. Trans. (SORT)*, vol. 30, pp. 125-70, 2006.
[10] N. R. Goodman, “Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction),” *Ann. Math. Statist.*, vol. 34, pp. 152–77, 1963.
[11] T. Medkour, A. T. Walden and A. Burgess, “Graphical modelling for brain connectivity via partial coherence,” *J. Neurosci. Meth.*, vol. 180, pp. 374–383, 2009.
[12] T. Medkour, A. T. Walden, A. P. Burgess & V. B. Strelets, “Brain connectivity in positive and negative syndrome schizophrenia,” *Neuroscience*, vol. 169, pp. 1779–88, 2010.
[13] J. M. Oller and J. M.Corcuera, “Intrinsic analysis of statistical estimation,” *The Annals of Statistics*, vol. 23, pp. 1562–81, 1995.
[14] D. B. Percival and A. T. Walden, *Wavelet Methods for Time Series Analysis*. New York: Cambridge University Press, 2000.
[15] S. T. Smith, “Covariance, subspace, and intrinsic Cramér-Rao bounds,” *IEEE Trans. Signal Proc.*, vol. 53, pp. 1610–1630, 2005.
[16] F. Yger, F. Lotte & M. Sugiyama, “Averaging covariance matrices for EEG signal classification based on the CSP: an empirical study.” 23rd European Signal Processing Conference (EUSIPCO), pp. 2721–2725, 2015.
[17] L. Zhuang and A. T. Walden, “Sample mean versus sample Fréchet mean for complex Wishart matrices: a statistical study,” *IEEE Trans. Signal Proc.*, vol. 65, pp. 4551–4561, 2017.