Surprises from the Resolution of Operator Mixing in $\mathcal{N} = 4$ SYM

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Abstract

We reexamine the problem of operator mixing in $\mathcal{N} = 4$ SYM. Particular attention is paid to the correct definition of composite gauge invariant local operators, which is necessary for the computation of their anomalous dimensions beyond lowest order. As an application we reconsider the case of operators with naive dimension $\Delta_0 = 4$, already studied in the literature. Stringent constraints from the resummation of logarithms in power behaviours are exploited and the role of the generalized $\mathcal{N} = 4$ Konishi anomaly in the mixing with operators involving fermions is discussed. A general method for the explicit (numerical) resolution of the operator mixing and the computation of anomalous dimensions is proposed.

We then resolve the order $g^2$ mixing for the 15 (purely scalar) singlet operators of naive dimension $\Delta_0 = 6$. Rather surprisingly we find one isolated operator which has a vanishing anomalous dimension up to order $g^4$, belonging to an apparently long multiplet. We also solve the order $g^2$ mixing for the 26 operators belonging to the representation $20'$ of $SU(4)$. We find an operator with the same one-loop anomalous dimension as the Konishi multiplet.

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1 Introduction and summary

In [1] we resolved the mixing among scalar primary operators of naive scale dimension \( \Delta_0 = 4 \) in the \( 20' \) representation of the \( SU(4) \) R-symmetry of the \( \mathcal{N} = 4 \) SYM theory and computed their anomalous dimensions at order \( g^2 \). A similar analysis was independently performed by the authors of ref. [2] in the \( SU(4) \) singlet sector. More involved mixing problems have been studied [3, 4, 5, 6] in the BMN limit [7] (large R-charge sector) of \( \mathcal{N} = 4 \) SYM, conjectured to be dual to type IIB superstring on a pp-wave [8, 9, 10, 11]. Although the very issue of holography remains somewhat mysterious in this setting, the authors of ref. [16] were able to show that tree-level superstring predictions for the anomalous dimensions of BMN operators with two “impurities” are consistent at the planar level \( (N_c >> 1, g^2 N_c = \text{fixed}, J^2 \approx N_c, \text{so that } g^2 N_c / J^2 << 1) \) with field theoretic results. Non-planar contributions that probe string interactions in the pp-wave background are subtler and require analyzing the mixing of operators with different number of traces [3, 4, 5, 6] along the lines of [1, 2]. Some degeneracies in the anomalous dimensions of “renormalized” BMN operators with two impurities, that at first looked puzzling, have been completely clarified in [17] on supersymmetry grounds and the BMN mixing problem has been rephrased in terms of a dilatation operator [18, 19] and extensively investigated [20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

A remarkable twist of the situation was brought about by Minahan and Zarembo [30] who have shown, quite independently of the BMN limit, that the one-loop dilatation operator in the sector of purely scalar operators can be viewed as the Hamiltonian of an integrable \( SO(6) \) spin chain. Their results have been extended in ref. [31], where it was shown that “pure operators” \( (i.e. \) lowest scalar components of naively 1/4 BPS multiplets, with \( \Delta = 2k + l \), belonging to the representation \([k, l, k]\) of the \( SU(4) \) R-symmetry which can only mix among themselves), are governed by an \( SU(2) \) spin chain which is integrable up to at least \( (g^2)^3 \) (three-loops). Evidence that the planar dilatation operator for two-impurity BMN operators is of the form predicted by string theory was also given. Later on it was proven that the full one-loop dilatation operator is the Hamiltonian of a super spin chain [32, 33] and some interesting “closed sectors” have been identified and analyzed in connection with higher spin symmetry enhancement [34, 35], producing an overwhelming set of data that expose some intriguing regularity \(^1\). Several issues related to the question of integrability beyond one-loop have been recently addressed. In particular in [36] it has been clarified how to accommodate mixing of operators with different “length” \( (i.e. \) number of constituents) in the spin chain picture and incorporate the presence of terms with odd powers of \( g \) in the dilatation operator.

Our aim in this note is to shed some light on certain field-theoretical issues that seem to have become topical and put forward few interesting new results that deserve a deeper understanding. In particular, in Section 2 we reexamine the general problem of operator mixing in \( \mathcal{N} = 4 \) SYM, paying special attention to the correct definition of composite gauge invariant local operators. This theoretical step is crucial for the computation of

\(^1\)In particular the “golden ratio” found in [1] reappears several times in [32, 33].
anomalous dimensions and to make sound statements about integrability at order higher than $g^2$ \cite{36, 37}. As an application, we reconsider the case of operators belonging to the irrep $20'$ of $SU(4)$ with naive dimension $\Delta_0 = 4$, already studied in the literature \cite{1}. In Section 3 we discuss the role of the generalized $\mathcal{N} = 4$ Konishi anomaly \cite{38, 39, 40} in the mixing pattern of operators with fermion impurities \cite{28}. In Section 4 stringent constraints from the resummation of logarithms in (conformal) power-like terms are exploited to give a general method for the explicit (perturbative) resolution of operator mixing and the calculation of anomalous dimensions. In Section 5 we resolve the mixing of the 15 (purely scalar) singlet operators of naive dimension $\Delta_0 = 6$ at order $g^2$. Rather surprisingly we find one operator $\mathcal{T}$ that, despite the fact that it is the lowest component of a long multiplet, has vanishing anomalous dimension up to order $g^4$. A similar analysis carried out up to order $g^2$ aimed at resolving the mixing of the 26 operators with naive dimension $\Delta_0 = 6$ in the $20'$ of $SU(4)$ shows the existence of one operator with the same one-loop anomalous dimension as the Konishi multiplet. In Section 6 we draw our conclusions.

2 The definition of the composite operators

In this section we present a general discussion of the problem of constructing renormalizable gauge invariant composite operators in a (super) conformal theory, i.e. operators that, besides being multiplicative renormalizable (m.r.) have well defined transformation properties under dilation, actually under the whole (super) conformal group.

We shall illustrate this issue in the simple context of operators of naive dimension $\Delta_0 = 4$ in the $20'$ irrep of $SU(4)$. There are 4 purely scalar operators of this kind, which belong, at least at tree-level, to semishort multiplets \cite{11, 12, 13, 14, 44}. Three of them have non-vanishing order $g^2$ anomalous dimension, while the fourth operator, $\mathcal{D}^{ij}_{20'}$ (whose components we will denote for short $\mathcal{D}^{ij}_{20'}$ or $\mathcal{D}^{ij}$ depending on the context), has vanishing anomalous dimension \cite{1, 46, 47, 48, 49, 50, 51, 52, 53}. As an illustration of the problems one encounters we choose to discuss the simple case of $\mathcal{D}^{ij}_{20'}$. One might be tempted to write $\mathcal{D}^{ij}_{20'}$ in the form

$$\mathcal{D}^{ij}(x) = : \sum_{k=1, \ldots, 6} Q^{ik}(x)Q^{jk}(x) - \frac{\delta^{ij}}{6} \sum_{k,l=1, \ldots, 6} Q^{kl}(x)Q^{kl}(x) :, \quad (1)$$

where $Q^{ij}_{20'}$ are the lowest component scalars of the ultrashort $1/2$ BPS $\mathcal{N} = 4$ supercurrent multiplet

$$Q^{ij}(x) = : \text{Tr}(\varphi^i(x)\varphi^j(x)) - \frac{\delta^{ij}}{6} \sum_{k=1, \ldots, 6} \text{Tr}(\varphi^k(x)\varphi^k(x)) :, \quad (2)$$

with $\varphi^i$, $i = 1, \ldots, 6$ the 6 fundamental scalars of $\mathcal{N} = 4$ SYM. In both the above formulae the normal product $:$ denotes as usual the omission of self-contractions (contractions of fields sitting at the same point), which are anyway absent in eq. (2). In Section 2.2 we will see how to give a precise definition of these operators in the regularized theory.
2.1 The problem

Suppose that we want to resolve the mixing problem of the set of operators, $O_p$, of the type described above (i.e. with naive dimension $\Delta_0 = 4$ and belonging to the $20'$ irrep of $SU(4)$) among which we have $D_{20'}$. The final renormalized operators (which we shall denote by $\hat{O}_p$) will have to have well defined (anomalous) dimensions and hence will define an orthogonal basis with respect to the inner product represented by the 2-point functions $\langle \hat{O}_p^\dagger(x)\hat{O}_q(0) \rangle$. Furthermore $\hat{O}_p$ should have a vanishing 2-point function with the protected chiral primary operators (CPO’s) $Q^{ij}$ of eq. (2), which have dimension $\Delta = 2$. Thus we must have for any $p$

$$\langle \hat{O}_p^\dagger(x) Q^{kl}(y) \rangle = 0 \quad (3)$$

to all orders in perturbation theory and non-perturbatively. On the other hand general principles of Quantum Field Theory tell us that the final renormalized operators, $\hat{O}_q$, must be linear combinations of the bare $O_p$ operators. In formulae one must have

$$\hat{O}_p = \sum_q Z_{pq} O_q \quad (4)$$

where the mixing matrix, $Z$, is assumed to be an invertible (not necessarily symmetric) matrix, so that one can invert the relation (4), getting

$$O_q = \sum_p Z_{qp}^{-1} \hat{O}_p \quad (5)$$

A simple perturbative computation shows however that $D_{20'}$ of eq. (1) fails to fulfill condition (3) already at order $g^2$, as an explicit computation gives

$$\langle D^{ij}(x) Q^{kl}(y) \rangle|_{g^2} \neq 0 \quad (6)$$

As a result this operator cannot be expressed as suggested by eq. (5). Similar considerations apply to essentially all the naive definitions of $D_{20'}$ used in the literature, despite the fact that it is the lowest component of an exactly semishort multiplet [41, 42, 43, 44, 45].

A careful inspection of the above argument shows that the problem arises from the incompatibility of the (naive) definition of $D_{20'}$ given in eq. (1) with the assumption that the mixing matrix $Z_{pq}$ (4) is invertible and the lack of an explicit regulator in eq. (1). Notice that, on the contrary, protected operators in short BPS multiplets, such as $Q_{20'}$ of eq. (2), do not seem to suffer of the same problem.

In the next section we will show that, as soon as the theory has been properly regularized, the problem of constructing m.r. operators with well defined conformal dimension can be fully solved, so that the vanishing of correlators such as $\langle D_{20'} Q_{20'} \rangle$ will be guaranteed.
2.2 The solution

Let us regularize the theory, say, by point splitting. Given a bare operator \( O \), we shall call \( \tilde{O} \) the subtracted operator which does not mix with any operator of dimension smaller than the naive dimension of \( O \). In the case of \( D_{20}' \) we find for the regularized operator, \( \tilde{D}_{20}' \), the expression

\[
\tilde{D}^{ij}(x) = \lim_{\epsilon \to 0} \left\{ Q^{ik}(x + \frac{\epsilon}{2})Q^{jk}(x - \frac{\epsilon}{2}) - \frac{\delta^{ij}}{6} Q^{kl}(x + \frac{\epsilon}{2})Q^{kl}(x - \frac{\epsilon}{2}) + 
- \frac{5}{3\pi^2\epsilon^2} \left[ \text{Tr} \left( \varphi^i(x + \frac{\epsilon}{2})\varphi^j(x - \frac{\epsilon}{2}) \right) - \frac{\delta^{ij}}{6} \text{Tr} \left( \varphi^k(x + \frac{\epsilon}{2})\varphi^k(x - \frac{\epsilon}{2}) \right) \right] \right\}.
\]

The second line in this equation can be expanded in a power series of the parameter \( \epsilon \) and can be simplified with the help of the identity (valid after averaging over the angular dependence of \( \epsilon \))

\[
\lim_{\epsilon \to 0} \epsilon \eta^\mu \epsilon^\nu \frac{\epsilon^2}{\epsilon^2} = \frac{\delta_{\mu\nu}}{4}.
\]

After redistributing derivatives, we finally obtain

\[
\tilde{D}^{ij}(x) = \lim_{\epsilon \to 0} \left\{ Q^{ik}(x + \frac{\epsilon}{2})Q^{jk}(x - \frac{\epsilon}{2}) - \frac{\delta^{ij}}{6} Q^{kl}(x + \frac{\epsilon}{2})Q^{kl}(x - \frac{\epsilon}{2}) - \frac{5}{96\pi^2} \Box Q^{ij}(x) \right\} + \frac{5}{48\pi^2} \left[ \text{Tr} \left( (\Box \varphi^i(x))\varphi^j(x) \right) + \text{Tr} \left( \varphi^i(x)(\Box \varphi^j(x)) \right) - \frac{\delta^{ij}}{3} \text{Tr} \left( (\Box \varphi^k(x))\varphi^k(x) \right) \right].
\]

The two lines of this equation have rather different physical content, so we shall comment on them separately. The first line is just an explicit though cumbersome way to eliminate self-contractions in the composite operator, i.e. it implements the normal ordering (\( \text{::} \)) as defined after eq. (2). These terms are always present, even in the free field theory at \( g = 0 \). Their explicit form depends on the way the regulator is introduced. For example, if we use an asymmetric point-splitting, the \( \Box Q^{ij} \) term is replaced by \( (\epsilon\partial)Q^{ij}/\epsilon^2 \). If instead of point-splitting, we use dimensional regularization, the quadratically divergent subtraction can be omitted. The situation is different for the terms in the second line. They all contain the \( \Box \varphi^i(x) \) operator, which by the use of the field equations gives rise to terms that are non-vanishing in the interacting theory, and do not depend on the particular regularization scheme adopted in the calculations. These terms, which have no counterpart in the naive definition of eq. (1), are indeed necessary (and sufficient) to get the vanishing of the three-point function of \( \tilde{D}^{ij} \) inserted with two fundamental fields \( \varphi^k \), i.e.

\[
\langle \tilde{D}^{ij}(x) \varphi^k(y_1) \varphi^l(y_2) \rangle_{g^2} = 0,
\]

which in turn implies the correct vanishing of the two-point function

\[
\langle \tilde{D}^{ij}(x) Q^{kl}(y) \rangle_{g^2} = 0.
\]
The result of this analysis is that the m.r. operator $\tilde{\mathcal{D}}^{ij}(x)$ has a hidden $g$ dependence from the terms in the second line of the r.h.s. of eq. (9). This $g$ dependence can be made explicit with the help of the field equations, which read

$$D^2 \varphi^i = \sqrt{2}g(\tau^i_{AB}[\lambda^A, \lambda^B] + h.c.) + g^2 [\varphi^i, [\varphi^i, \varphi^j]]. \quad (12)$$

where $\lambda^A$, $A = 1, \ldots, 4$ denote the 4 gaugini and $\tau^i_{AB}$ the $4 \times 4$ (antisymmetric) chiral blocks of the $D = 6, 8$-matrices. Consequently we can write an expansion of the type

$$\tilde{\mathcal{D}}^{ij}(x) = D^{ij}_{(0)}(x) + gD^{ij}_{(1)}(x) + g^2 D^{ij}_{(2)}(x) + \ldots, \quad (13)$$

where $D^{ij}_{(0)}$ denotes the terms appearing in first line of eq. (9). Note that $D^{ij}_{(1)}$ contains two fermion “impurities”, i.e. $D^{ij}_{(1)} \approx \tau^{ij}_{AB}\text{Tr}(\varphi^j)[\lambda^A, \lambda^B] + h.c.$. This issue will be discussed in connection with the generalized Konishi anomaly in Section 3. Note also that the formula (12) does not really yield an expansion in powers of $g$, because there is an implicit $g$-dependence in each term, $D^{ij}_{(n)}$. In fact supergauge invariance requires the introduction of $g$-dependent super-Wilson lines between each pair of split points. Hence eq. (13) should be rather considered as a partial operator mixing resolution which ensures that the operator $\tilde{\mathcal{D}}^{ij}$ does not mix with operators of naive scale dimension $\Delta_0$ less than 4.

This situation is not peculiar to the case of the operator $\mathcal{D}^{ij}$. The very same problem occurs for all composite operators where self-contractions are not forbidden by symmetries. Indeed, one has to define composite operators so that they do not mix with gauge invariant operators with lower naive scale dimension. A sufficient condition to achieve this goal for a generic purely scalar operator $\mathcal{O}(x)$ of naive scale dimension $\Delta_0^{(\mathcal{O})}$ is that all $n + 1$-point correlation functions of $\mathcal{O}$ inserted with $n$ the fundamental fields at non-coincident arguments

$$\langle \tilde{\mathcal{O}}(x) \varphi^{k_1}(y_1) \varphi^{k_2}(y_2) \ldots \varphi^{k_n}(y_n) \rangle = 0 \quad (14)$$

vanish for all $n < \Delta_0^{(\mathcal{O})}$. Let us also stress that from the point of view of this discussion $1/2$ and (would-be) $1/4$ BPS operators are exceptional, since self-contractions are absent. The former correspond to chiral primary operators (CPO’s) of the type $Tr(Z^i)$, with $Z = (\varphi^3 + i\varphi^6)/\sqrt{2}$ and the latter to “pure scalar” operators of the type $Tr(X^{t+k}Y^k)$, with $X = (\varphi^1 + i\varphi^4)/\sqrt{2}$ and $Y = (\varphi^2 + i\varphi^5)/\sqrt{2}$.

A natural question arises from the previous considerations. Does this modification change the results for the anomalous dimensions and the operator mixing coefficients present in the literature? The answer is that as far as only the order $g^2$ corrections to the anomalous dimensions of the operators are extracted from a perturbative calculation of two-point functions, terms beyond $D^{ij}_{(0)}$ can be neglected in eq. (13). The reason is that we can write the order $g^2$ correction to the two-point function in the form (for illustrative purposes we shall again refer to $\mathcal{D}^{ij}$, but precisely the same argument holds for any operator)

$$\langle \tilde{\mathcal{D}}^{ij}(x) \tilde{\mathcal{D}}^{ij}(y) \rangle|_{g^2} = \langle D^{ij}_{(0)}(x) D^{ij}_{(0)}(y) \rangle|_{g^2} + g\langle D^{ij}_{(0)}(x) D^{ij}_{(1)}(y) \rangle|_g + g\langle D^{ij}_{(1)}(x) D^{ij}_{(0)}(y) \rangle|_g + g^2\langle D^{ij}_{(0)}(x) D^{ij}_{(2)}(y) \rangle|_0 + g^2\langle D^{ij}_{(2)}(x) D^{ij}_{(0)}(y) \rangle|_0 + g^2\langle D^{ij}_{(1)}(x) D^{ij}_{(1)}(y) \rangle|_0, \quad (15)$$
In this expansion only the first term in the r.h.s. can contain divergent logarithmic terms and thus can contribute to the anomalous dimension. Hence all order $g^2$ calculations of anomalous dimensions performed so far remain unchanged. This however is not the case for the finite parts of the two-point correlation functions nor for higher order computations.

As a second example of this kind of problems let us consider, in fact, the order $g^2$ correction to the three-point function $\langle \tilde{D}^{ij}(x_1) Q^{kl}(x_2) Q^{mn}(x_3) \rangle$. This time the naive contribution $\langle D^{ij}(0) Q^{kl}(x_2) Q^{mn}(x_3) \rangle|_{g^2}$ with the insertion of $D^{ij}_{(0)}$ is non-vanishing and violates conformal invariance, while the complete expression, where also $D^{ij}_{(1)}$ contributes, is zero

$$\langle \tilde{D}^{ij}(x_1) Q^{kl}(x_2) Q^{mn}(x_3) \rangle|_{g^2} = 0,$$

(16)
as expected [46, 47, 48, 49, 50]. What is crucial for the present investigation is that at higher orders in perturbation theory the difference between $\tilde{D}^{ij}$ and $D^{ij}_{(0)}$ affects also the divergent logarithmic parts, thus contributing corrections to the anomalous dimension. As an illustration, let us consider the next order counterpart of eq. (15). Again we write formulae for the particular case of the operator $D^{ij}$, but we remind that they are valid in general. We get to order $g^4$

$$\langle \tilde{D}^{ij}(x) \tilde{D}^{ij}(y) \rangle|_{g^4} = \langle D^{ij}_{(0)}(x) D^{ij}_{(0)}(y) \rangle|_{g^4} + g \langle D^{ij}_{(0)}(x) D^{ij}_{(1)}(y) \rangle|_{g^3} + g \langle D^{ij}_{(1)}(x) D^{ij}_{(1)}(y) \rangle|_{g^3} + g^2 \langle D^{ij}_{(1)}(x) D^{ij}_{(2)}(y) \rangle|_{g^2} + g^2 \langle D^{ij}_{(0)}(x) D^{ij}_{(2)}(y) \rangle|_{g^2} + g^2 \langle D^{ij}_{(2)}(x) D^{ij}_{(0)}(y) \rangle|_{g^2} + \ldots ,$$

(17)

where the dots stand for tree-level or $O(g)$ contributions which do not produce divergent logarithmic behaviours. The $\log^2(\epsilon)$ divergent terms can only come from the first term in the r.h.s. of the expansion, while contributions proportional to $\log(\epsilon)$ generally come from all the terms in (17).

One final remark concerns the conditions for an operator to be a conformal and super-conformal primary. One may wonder whether the presence of derivative terms like $\Box Q^{ij}$ in eq. (9) might spoil this property. The point is that in the regularized theory the naive operator $D^{ij}_{(0)}$ is not anymore primary. It is precisely the presence of derivative terms in the regularized (point-split) operator that makes it primary. The same is true for the appearance of the operator $\tau^{ij}_{AB} \text{Tr}(\varphi^j[\lambda^A, \lambda^B]) + h.c.$ in the $D^{ij}_{(1)}$ contribution. Only the complete operator $\tilde{D}^{ij}$ has the correct conformal properties as demonstrated, for example, by the computation of the three-point function (16).

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2Actually in the present instance, since $D^{ij}$ has vanishing anomalous dimension, the fourth and the fifth term of the expansion vanish separately, while the sum of all the others is zero.
3 Generalized Konishi anomaly and mixing with fermions

The analysis presented in the previous section illustrates that mixing with fermions and operators of different “length” (number of constituents) is possible and in fact required beyond one-loop [36]. Explicit resolution of this kind of mixing leads to daunting computations [28, 36], but at least in the planar limit and for certain classes of operators one can rely on anomaly arguments to determine the correct mixing coefficients to lowest non trivial order in $g$. This is our aim in this section. The outcome of the analysis of the generalized Konishi anomaly is a rather compact form for the mixing of certain operators with fermion and boson “impurities” \(^3\).

In any $\mathcal{N} = 1$ supersymmetric gauge theory with vector multiplets $V$ in the adjoint representation and chiral multiplets $\Phi^I$ in some representation $r$ of the gauge group, the Konishi supermultiplets

$$K_{I}^J = \text{Tr}_{r}(\bar{\Phi}^I e^{2gV} \Phi^J)$$

are real vector multiplets that contain flavour currents in their $\bar{\theta} \theta$ component. Naively the superfield equations of motion yield

$$\frac{1}{4} \partial^2 K_{I}^J = \text{Tr}_{r} \left( \frac{\partial W}{\partial \Phi^I} \Phi^J \right),$$

where $W$ is the superpotential. Together with its hermitian conjugate, (19) implies

$$\frac{1}{16} [D^2, D^2] K_{I}^J = \frac{1}{4} D^2 \text{Tr}_{r} \left( \frac{\partial \bar{W}}{\partial \bar{\Phi}^I} \bar{\Phi}^J \right) - \frac{1}{4} D^2 \text{Tr}_{r} \left( \frac{\partial W}{\partial \Phi^I} \Phi^J \right).$$

While kinetic terms are all chirally invariant, this equation expresses the non-invariance of the interactions governed by $W$ under chiral flavour symmetry transformations even at the classical level.

At the quantum level the singlet current is plagued by the chiral anomaly that is part of the “superglueball” multiplet $S$. In superfield notation the divergence equation of the singlet current reads

$$\frac{1}{4} \partial^2 K_{I}^J = \text{Tr}_{r} \left( \frac{\partial W}{\partial \Phi^I} \Phi^J \right) + \delta_{I}^{J} \frac{g^2}{16\pi^2} \text{Tr}_{r}(W^\alpha W_\alpha),$$

where $W_\alpha = \frac{1}{4} D^2 e^{-2gV} D_\alpha e^{2gV}$ is the chiral superfield strength. Assuming the validity of the Adler-Bardeen theorem, one concludes that the anomaly multiplet

$$S = \frac{g^2}{16\pi^2} \text{Tr}_{r}(W^\alpha W_\alpha)$$

\(^3\)The nomenclature is taken from the BMN limit but it is valid in general [17].
should not be affected by renormalization effects, \textit{i.e.} it should yield finite operator insertions. Since $\text{Tr}(W_\alpha W^\alpha)$ by itself is not finite, one can deduce the running properties of the gauge coupling from the renormalization of $\text{Tr}(W_\alpha W^\alpha)$ \cite{39}. Although this is a consistent scenario in $\mathcal{N} = 1$ theories, it leads to a contradiction in $\mathcal{N} = 4$ SYM. Indeed, it is known that the anomalous divergences of the currents in the $\mathcal{N} = 4$ Konishi multiplet $\mathcal{K}$, being proportional to their common anomalous dimension (as dictated by superconformal invariance), receive contributions not only at one-loop but also at two-loops \cite{54, 55, 56, 57, 58, 59}, and higher orders in perturbation theory (though possibly not from instanton effects \cite{60, 61, 62, 63}). These corrections, however, cannot be reabsorbed in the renormalization of the coupling constant, because the $\beta$-function of this exactly conformal theory vanishes.

The $\mathcal{N} = 4$ generalization of (21) is \cite{64, 65}

$$\frac{1}{4} \bar{D}^A \bar{D}^B \mathcal{K} = g \text{Tr}([W^{AE}, W^{BF}]\bar{W}_{EF}) + \frac{g^2}{16\pi^2} D_E D_F \text{Tr}(W^{AE} W^{BF}) + O(g^3),$$

(23)

where $W^{AB} = \bar{r}^{iAB} W_i$ is the twisted chiral $\mathcal{N} = 4$ SYM multiplet that starts with $\varphi^{AB} = \bar{r}^{iAB} \varphi^i$. Order $g^3$ corrections and higher are expected in (23), since $\mathcal{E}^{AB} = D_E D_F \text{Tr}(W^{AE} W^{BF})$ is a protected operator, being a superdescendant at level two of $Q_{20\nu}$ in the $\mathcal{N} = 4$ supercurrent multiplet. Analogously the tree-level term

$$\mathcal{K}^{AB} = \text{Tr}([W^{AE}, W^{BF}]\bar{W}_{EF}),$$

(24)

is the lowest component of a short 1/8 BPS supermultiplet in free theory, but satisfies

$$\frac{1}{4} \bar{D}^E \bar{D}^F \mathcal{K}^{AB} = g\epsilon_{CDHG} \text{Tr}([W^{AC}, W^{BD}][W^{EG}, W^{FH}]) + O(g^3),$$

(25)

when interactions are turned on. In free theory the first term in the r.h.s. is a 1/4 BPS short multiplet whose lowest component is a “pure scalar” operator $\text{Tr}([X, Y]^2)$ of naive dimension $\Delta_0 = 4$ belonging to the representation $\mathbf{84}$ of $SU(4)$ \cite{52}.

Recently \cite{40} the Konishi anomaly equation has been generalized to encompass the case of an $\mathcal{N} = 1$ gauge theory with a chiral multiplet in the adjoint representation of the gauge group, $\Phi = \Phi^a T_a$, where $T_a$ are the generators of the gauge group in the fundamental representation. In this situation one can envisage the possibility of constructing supergauge invariant generalized Konishi multiplets according to the formula

$$\mathcal{K}_{(n,n)} = \text{Tr}(e^{-2gV} \Phi^n e^{2gV} \Phi^n),$$

(26)

with the “standard” Konishi multiplet being the $\mathcal{K}_{(1,1)}$ instance of the above series. The authors of \cite{40} have mostly, if not exclusively, concentrated their attention on the supermultiplets $\mathcal{K}_{(1,n)}$ that are in a sense $\mathcal{N} = 1$ relatives of the BMN multiplets with two impurities \cite{17}. The generalized super-anomaly equation for $\mathcal{K}_{(1,n)}$ reads

$$\frac{1}{4} \bar{D}^2 \mathcal{K}_{(1,n)} = \text{Tr} \left( \frac{\partial W}{\partial \Phi} \Phi^n \right) + \frac{g^2}{16\pi^2} \text{Tr}([W_\alpha, T^\alpha][T_a, W^{\alpha}]\Phi^{n-1}) + O(g^3).$$

(27)
In its $\mathcal{N} = 1$ decomposition, $\mathcal{N} = 4$ SYM comprises three chiral multiplets $\Phi^I$, so one can consider gauge invariant operators of the form

$$ \mathcal{K}^{I_1 \ldots I_{n-1}}_{(1,n)} = \text{Tr}(e^{-2g\Phi} K e^{2g\Phi} \Phi^I \cdots \Phi^{I_{n-1}}), $$

that are chiral supermultiplets belonging to higher $SU(4)$ “harmonics”. In the AdS/CFT correspondence $[66, 67, 68, 62]$ $\mathcal{K}^{I_1 \ldots I_{n-1}}_{(1,n)}$ should correspond to K-K excitations of the Konishi operator and belong to semi-short multiplets in the free theory that become long when interactions are turned on $[34, 35, 42, 45]$. They satisfy a generalized anomaly equation which, in $\mathcal{N} = 1$ notation, looks almost identical to (27), namely

$$ \frac{1}{4} \partial^2 \mathcal{K}^{I_1 \ldots I_{n-1}}_{(1,n)} = \text{Tr} \left( \frac{\partial \mathcal{W}}{\partial \Phi} \Phi^I \Phi^I \cdots \Phi^{I_{n-1}} \right) 
+ \frac{3g^2}{16\pi^2} \text{Tr}(|W_\alpha| T^a) \left| T_\alpha, W^{\alpha} \right| \Phi^I \cdots \Phi^{I_{n-1}} + O(g^3). 
$$

It is amusing to observe that the factor of 3 in (29) (which is the number of chiral supermultiplets in $\mathcal{N} = 4$ SYM) determines the one-loop anomalous dimension of the singlet current in the $\mathcal{N} = 4$ Konishi multiplet and, as a consequence of superconformal invariance, of the full multiplet. Another consequence of the generalized anomaly is the mixing of operators that are multi-linear in the bosons with operators containing fermion “impurities”. Indeed the four gauginos decompose under $SU(2) \times SU(2) \times U(1)_F$ with operators containing fermion “impurities”. Indeed the four gauginos decompose under $SU(2) \times SU(2) \times U(1)_F$ according to

$$ \lambda^A_\alpha \rightarrow \{ \psi^{\alpha(J=+1/2)}, \chi^{\dot{\alpha}}_{\alpha(J=-1/2)} \} $$

with $r, \dot{r} = 1, 2$ and similarly for the hermitian conjugate fields. Thus the only possibilities to build scalar operators in the singlet and antisymmetric tensor representation of the $SO(4)$ subgroup of $SU(4)$ commuting with $U(1)_J$ can mix with operators containing fermion “impurities”. Indeed the four gauginos decompose under $SU(2) \times SU(2) \times U(1)_F$ according to

$$ \mathcal{F}^{(rs)}_{(Jp)} = \text{Tr}(\psi^{\alpha(r+1/2)} Z^p \psi^{\alpha(s+1/2)} Z^{J-p-1}) , 
\bar{\mathcal{F}}^{(\dot{r}s)}_{(Jp)} = \text{Tr}(\chi^{\dot{\alpha}(+1/2)} Z^p \chi^{\dot{\alpha}(s+1/2)} Z^{J-p-1}). 
$$

$\mathcal{F}^{(rs)}$ and $\bar{\mathcal{F}}^{(\dot{r}s)}$ transform as singlets, while $\mathcal{F}^{(rs)}$ and $\bar{\mathcal{F}}^{(\dot{r}s)}$ transform as $3_L$ and $3_R$ of $SO(4)$, respectively. In particular it is not possible to construct a traceless symmetric tensor (i.e. the $(3_L, 3_R)$ of $SO(4)$).

For the antisymmetric tensors, which are superdescendant at level two of the above singlets $[17, 28]$, we thus expect

$$ O^{(ab)+}_{(Jp)} = B^{(ab)+}_{(Jp)} + c_{(Jp)} \frac{g}{16\pi^2} \sigma^{rs}_{[ab]} \mathcal{F}^{(rs)}_{(Jp)}. $$

\[10\]
\[ \mathcal{O}^{[ab]}_{(J\mid p)} = B^{[ab]}_{(J\mid p)} + c_{(J\mid p)} \frac{g}{16\pi^2} \sigma^{[ab]}_{J\mid p} \mathcal{F}^{(J\mid p)} \]

where \( B^{[ab]}_{(J\mid p)} = \text{Tr}(\phi^a Z_p \phi^b Z^{-p}) \) with \( a, b = 1, \ldots, 4 \) and \( +(-) \) denotes projection onto the (anti)self-dual \((3_L, 3_R)\) component.

The anomaly argument only determines the form of the superdescendants once the superprimary are known and amounts to the addition of an extra anomalous term to the naive second order variation under the “dynamical supercharges”, \textit{i.e.} the ones commuting with \( \Delta - J \) and annihilating \( Z \). The situation for the \( SO(4) \) singlets, which are superprimary, \textit{i.e.} lowest components, of two-impurity BMN multiplets \(^{17}28\), is different. For these operators one has to rely on the methods of Section 2 where the case \( J = 2 \) corresponding to the operator \( \mathcal{D}_{20} \) has been reanalyzed. At any rate, our present analysis suggests that the two-impurity “dilatation operator” of \(^{31}\) should contain terms with odd powers of \( g \) even in the BMN limit, except in those sectors, such as the \((3_L, 3_R)\) of \( SO(4) \), where mixing with fermion impurities is forbidden on symmetry grounds.

Similar arguments should help resolving the mixing for the operators described by the \( SU(2|3) \) super spin chain of \(^{36}\). The simplest instance is the well studied case of the mixing between \( \text{Tr}([X,Y]Z) \) and \( \text{Tr}(\lambda\lambda) \) that is resolved in terms of the operators \( \mathcal{E}_{10}; \mathcal{K}_{10} \), belonging to the Konishi multiplet \(^{55}\). The study can be easily generalized to the mixing between putative \( 1/8 \) BPS operators of the form \( \text{Tr}(Z^l X^k Y) \) that are either primary and thus protected or superdescendants of operators of the form \( \text{Tr}(Z^{l-1} X^{k-1} YY) \) and thus mix with \( \text{Tr}(Z^{l-1} X^{k-1} \lambda\lambda) \). An analysis of the mixing in this sector should help clarifying some of the issues left open in \(^{36}\).

### 4 The operator mixing resolution

Let the operators \( \tilde{O}_p(x, \epsilon) \), with \( p = 1, \ldots, n \), be a basis of bare point-split regularized operators of naive dimension \( \Delta_0 \), which are properly defined as discussed in Section 2. We would like to diagonalize the matrix of their two-point functions and find the corresponding anomalous dimensions. Assume that we have computed the two-point functions to some order in perturbation theory \(^4\). The result of the calculation has the form

\[ \langle \tilde{O}_p(x, \epsilon) \tilde{O}_q(y, \epsilon) \rangle = f_{pq} \left( \frac{\epsilon^2}{(x - y)^2}, g \right) \frac{1}{[(x - y)^2]^{\Delta_0}} , \]  

where \( f_{pq} \) is an hermitian matrix depending on the operator basis we have chosen. In fact, since complex operators come in pairs with the same anomalous dimension we can always choose a basis in which \( f_{pq} \) is real and symmetric.

The renormalized operators \( \hat{O}_p \) which have well defined anomalous dimensions \( \gamma_p(g^2) \)

\(^4\)To this purpose, we can safely neglect any possible dependence on the vacuum angle \( \vartheta \).
are linear combinations of the operators $\tilde{O}_q$

$$
\tilde{O}_p(x, \mu) = \sum_q Z_{pq}(\epsilon^2 \mu^2, g) \tilde{O}_q(x, \epsilon),
$$

where the auxiliary scale $\mu$ is an artifact of the perturbative expansion and plays the role of subtraction point. Its presence contradicts neither scale nor conformal invariance \[54\]. As discussed in Section 2, we shall assume that the matrix $Z$ has an inverse. Scale invariance completely determines the two-point functions of $\tilde{O}_p(x, \mu)$ to be

$$
\langle \tilde{O}_p(x, \mu) \tilde{O}_q^\dagger(y, \mu) \rangle = \frac{\delta_{pq}}{[(x - y)^2 \Delta_0][(x - y)^2 \mu^2]^\gamma_{pq}(g^2)},
$$

where we have separately indicated the dependence on the naive and the anomalous dimension. Let us stress that, while $f_{pq}$ and $Z_{pq}$ can in general depend on both even and odd powers of the coupling constant $g$, the physical anomalous dimensions, $\gamma$, can only be function of $g^2$. Compatibility among the above three equations implies (dropping indices)

$$
Z(\epsilon^2 \mu^2, g) f \left( \frac{\epsilon^2 \mu^2}{(x - y)^2 \mu^2}, g \right) Z^\dagger(\epsilon^2 \mu^2, g) = \left[(x - y)^2 \mu^2\right]^{-\Gamma(g^2)},
$$

where $\Gamma(g^2)$ is the diagonal matrix of anomalous dimensions. For future convenience we introduced a $\mu$ dependence in both the numerator and the denominator of the argument of $f$. Since there exists a basis in which both $f$ and $Z$ are real, the (diagonal) elements of $\Gamma(g^2)$, which represent the sought for anomalous dimensions, are also all real.

It is useful to compute eq. (35) at two special points, namely

1) $\epsilon^2 \mu^2 = 1$ and $(x - y)^2 \mu^2 = 1/u$ which yields

$$
Z(1, g) f(u, g) Z^\dagger(1, g) = u^{\Gamma(g^2)},
$$

2) $\epsilon^2 \mu^2 = u$ and $(x - y)^2 \mu^2 = 1$ which yields

$$
Z(u, g) f(u, g) Z^\dagger(u, g) = 1.
$$

Note that these two equations have to be simultaneously fulfilled. Their consistency implies that the function $f(u, g)$ has to satisfy (for any choice of basis for the set of regularized operators $\tilde{O}_i$ such that the matrix $Z$ has an inverse)

$$
f(u, g) = f(1, g) f^{-1} \left( \frac{1}{u}, g \right) f(1, g),
$$

where $f^{-1}$ is the inverse of the matrix $f$. Assume that we have found a solution, $Z(1, g)$, of eq. (36). Then it is immediate to see that

$$
Z(u, g) = u^{-\frac{1}{2} \Gamma(g^2)} Z(1, g)
$$

solves eq. (37). The last relation has a simple intuitive explanation, one first defines by means of $Z(1, g)$ operators with well defined scale dimension, then the field-theoretical renormalization step amounts to a simple rescaling by the factor $(\epsilon^2 \mu^2)^{-\frac{1}{2} \Gamma(g^2)}$. Since in general (for non-degenerate $\Gamma(g^2)$) the solution for $Z$ is unique, it will be given by eq. (39). Hence the $u$ dependence in $Z(u, g)$ factorizes and we have to solve only eq. (36) for the unknown $Z(1, g)$ and $\Gamma(g^2)$ once the function $f(u, g)$ is known.
4.1 Order by order analysis

In perturbation theory $f(u, g)$, $Z(u, g)$ and $\Gamma(g^2)$ admit an expansion in powers of the coupling constant. Introducing for short the definition $\ell = \log(u)$, we get the obvious expansions

$$f(u, g) = f_{00} + g f_{10} + g^2 (f_{20} + \ell f_{21}) + g^3 (f_{30} + \ell f_{31}) + g^4 (f_{40} + \ell f_{41} + \ell^2 f_{42}) + \ldots , \quad (40)$$

$$Z(1, g) = Z_0 + g Z_1 + g^2 Z_2 + g^3 Z_3 + g^4 Z_4 + \ldots , \quad (41)$$

$$\Gamma(g^2) = g^2 \Gamma_1 + g^4 \Gamma_2 + \ldots . \quad (42)$$

Let us substitute these expressions in eq. (36) and consider for the moment only tree-level terms and the terms proportional to $g^2 \cdot \ell$. We get the equations

$$\begin{align*}
\text{tree :} & \quad Z_0 f_{00} Z_0^\dagger = 1 \\
g^2 \cdot \ell : & \quad Z_0 f_{21} Z_0^\dagger = \Gamma_1
\end{align*} \quad (43)$$

which determine $Z_0$ and $\Gamma_1$. In fact we have two matrix equations involving hermitian $n \times n$ matrices. Hence they lead to $n(n+1)$ scalar equations for $n^2 [Z_0] + n [\Gamma_1]$ unknown. In the next subsection we shall obtain the general solution of the system (43). In order to go to higher order it is convenient to make a change of operator basis which significantly simplifies the formulae. Let us rotate the original basis, by a solution of eqs. (43). In this way tree-level and $g^2 \cdot \ell$ terms will be diagonal and we get in the new basis

$$Z_0 = 1 \quad , \quad f_{00} = 1 \quad , \quad f_{21} = \Gamma_1. \quad (44)$$

The remaining conditions coming from terms up to order $g^4$ now give the following set of relations

$$\begin{align*}
g : & \quad f_{10} + Z_1 + Z_1^\dagger = 0 \quad \rightarrow Z_1^H \quad (45) \\
g^3 \cdot \ell : & \quad f_{31} + Z_1 \Gamma_1 + \Gamma_1 Z_1^\dagger = 0 \quad \rightarrow Z_1^A \quad (46) \\
g^2 : & \quad f_{20} + Z_1 f_{10} + f_{10} Z_1^\dagger + Z_1 Z_1^\dagger + Z_2 + Z_2^\dagger = 0 \quad \rightarrow Z_2^H \quad (47) \\
g^4 \cdot \ell : & \quad f_{41} + Z_1 f_{31} + f_{31} Z_1^\dagger + Z_2 \Gamma_1 + \Gamma_1 Z_2^\dagger + Z_1 \Gamma_1 Z_1^\dagger = \Gamma_2 \quad \rightarrow Z_2^A, \Gamma_2 \quad (48) \\
g^4 \cdot \ell^2 : & \quad f_{42} = \frac{1}{2}(\Gamma_1)^2 \quad (49)
\end{align*}$$

The first pair of equations determine the hermitian, $Z_1^H$, and the anti-hermitian, $Z_1^A$, part of $Z_1$, respectively, as indicated by the symbols after the arrow. The second pair of equations determine the hermitian, $Z_2^H$, and the anti-hermitian, $Z_2^A$, part of $Z_2$, as well as the order $g^4$ correction to the anomalous dimensions, $\Gamma_2$. The counting of equations vs parameters is as before: we have $n(n+1)$ equations for $n^2 [Z_2] + n [\Gamma_2]$ unknown. The last equation is a consistency condition and is automatically satisfied if $f$ satisfies (43).

The pattern repeats itself to every order. The constant term at order $g^n$ and the term with coefficient $g^{n+2} \cdot \ell$ determine $Z_n$ and (for even $n$) also $\Gamma_n$. All the terms containing higher powers of $\ell$ give just consistency conditions which are trivially satisfied if $Z_0$ is true. Let us stress that once $Z_0$ is found, all the remaining equations are linear. Thus in
general the only residual freedom is an arbitrary rotation in the subspaces of operators
with the same (anomalous) dimension, if there are such degenerate operators that cannot
be discriminated by other “good” quantum numbers. Actually we will see in sect. 5.1
that this phenomenon takes place.

4.2 Solving tree-level mixing

In this subsection we shall give the general solution of the tree-level mixing problem
defined by eqs. (43) for the unknown \( Z_0 \) and \( \Gamma_1 \), where \( f_{00} \) and \( f_{21} \) are given hermitian
matrices.

Let \( E_0 \) be the matrix of the (orthonormal) eigenvectors of the hermitian matrix \( f_{00} \).
By definition \( E_0 \) diagonalizes \( f_{00} \),

\[
E_0 \, f_{00} \, E_0^\dagger = h_0 ,
\]

where \( h_0 \) is a diagonal matrix. Since \( f_{00} \) is the matrix of the tree-level two-point functions,
unitarity implies that all the diagonal entries of \( h_0 \) are strictly positive. Hence \( h_0 \) is
invertible and we can unambiguously define the matrix \( (h_0)^{-\frac{1}{2}} \) by taking e.g. its positive
square root. The formulae

\[
(h_0)^{-\frac{1}{2}} E_0 \, f_{00} \, E_0^\dagger (h_0)^{-\frac{1}{2}} = 1 ,
\]
\[
(h_0)^{-\frac{1}{2}} E_0 \, f_{21} \, E_0^\dagger (h_0)^{-\frac{1}{2}} = F_{21} ,
\]

where \( F_{21} \) is again a hermitian matrix, then follow.

Let now \( E_1 \) be the matrix of the (orthonormal) eigenvectors of the hermitian matrix \( F_{21} \).
By definition \( E_1 \) diagonalizes \( F_{21} \), so sandwiching the system \( (51) \) between \( E_1 \) and \( E_1^\dagger \) we obtain

\[
E_1 (h_0)^{-\frac{1}{2}} E_0 \, f_{00} \, E_0^\dagger (h_0)^{-\frac{1}{2}} E_1^\dagger = E_1 E_1^\dagger = 1 ,
\]
\[
E_1 (h_0)^{-\frac{1}{2}} E_0 \, f_{21} \, E_0^\dagger (h_0)^{-\frac{1}{2}} E_1^\dagger = E_1 F_{21} E_1^\dagger = \Gamma_1 .
\]

Putting together the previous results, we conclude that the matrix

\[
Z_0 = E_1 \, (h_0)^{-\frac{1}{2}} \, E_0
\]

diagonalizes simultaneously \( f_{00} \) and \( f_{21} \), hence it is the solution to the tree-level mixing
problem posed by eqs. (43). The order \( g^2 \) correction to the anomalous dimension, \( \Gamma_1 \), can
then be read off from the second of eqs. (52).

Let us note that, if one is interested only in the values of the anomalous dimensions
\( \Gamma_1 \), they can be obtained in a much simpler way by computing the eigenvalues of the matrix

\[
(f_{00})^{-1} \, f_{21} .
\]
We prefer to use the more complicated method described in this section, since it gives us also the mixing matrix \( Z_0 \), i.e. the explicit form of the corresponding operators.

We end the section with two remarks. Note that the role of tree level and order \( g^2 \) terms is not symmetric in the above formulae. Since in general there is no positivity constraint for the order \( g^2 \) contributions, \( f_{21} \) and \( \Gamma_1 \) may have vanishing eigenvalues and may not be invertible. It should also be said that in most cases the tree-level mixing matrix \( Z_0 \) can be computed only numerically, because the eigenvalue problem is highly nontrivial and cannot be solved analytically.

5 Anomalous dimensions of singlet operators with \( \Delta_0 = 6 \)

Let us illustrate some applications of the general method developed in Section 4. We start with the computation of the anomalous dimensions of the purely scalar \( SU(4) \) singlet operators with \( \Delta_0 = 6 \). This example is interesting for several reasons. On the one hand \( \Delta_0 = 6 \) is the first case in which single, double and triple trace operators come into play, so one can study the pattern of large \( N_c \) suppression of the mixing between these three different types of operators. On the other hand the number of operators is sufficiently large (fifteen), so that one can really test the effectiveness of the mixing resolution methods proposed in this paper.

Last, but not least, we find a rather surprising result in this sector of the theory, namely we find one operator which has vanishing order \( g^2 \) correction to its anomalous dimension. Since the corresponding supermultiplets is long even in the free limit \( g = 0 \)

\(^5\) there is no known mechanism which should protect this operator. Although the vanishing of the anomalous dimension of this operator may look as a one-loop accident, we have confirmed it by a two-loop computation, i.e. to order \( g^4 \). At present we don’t know of any simple explanation for this remarkable result. It might point to some deep and as yet uncovered property of \( \mathcal{N} = 4 \) SYM. Certainly this discovery deserves a better understanding.

For \( N_c \geq 6 \) there are 15 distinct \( SU(4) \) singlet operators of naive conformal dimension \( \Delta_0 = 6 \) built in terms of only the elementary scalar fields, \( \varphi^i \), without derivatives. For \( N_c = 2, 3, 4 \) and 5 the number of operators of this kind is 3, 8, 13 and 14 respectively.

As explained in Section 4 for the task of computing the order \( g^2 \) anomalous dimensions of these operators, we shall have only to consider their tree-level mixing and compute to order \( g^2 \) the matrix of their two-point correlation functions. Furthermore, for the reasons discussed after eq. (15), we can ignore at the order we work the mixing of operators built

\(^5\) We thank B. Eden and E. Sokatchev for discussions on this point.
only in terms of elementary scalar fields with operators containing derivatives, fermions, or the gauge field strength, $F_{\mu\nu}$.

We can choose as a basis in the 15 dimensional space of the scalar operators of interest the following set of operators (summation over repeated indices goes from 1 to 6)

\[
\begin{align*}
O_1 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_2 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_3 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_4 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_5 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_6 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_7 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_8 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_9 &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_{10} &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_{11} &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_{12} &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_{13} &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_{14} &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) \\
O_{15} &= \text{Tr}(\phi^i \phi^j \phi^k \phi^l \phi^m) 
\end{align*}
\]

5.1 The order $g^2$ calculation

Perturbative calculations are performed in the $\mathcal{N} = 1$ formulation of $\mathcal{N} = 4$ SYM, as described in [1]. We rewrite all the above operators in the $\mathcal{N} = 1$ language with the help of the identity ($a, b$ are adjoint colour indices)

\[
\sum_{i=1}^{6} \phi^i_a \phi^i_b = \sum_{I=1,2,3} (\phi^I_{a,\uparrow} \phi^I_{b,\uparrow} + \phi^I_{a,\downarrow} \phi^I_{b,\downarrow}) .
\]

As we do not make use of the Wess-Zumino gauge fixing, all the operators have to be made gauge invariant by including the appropriate vector field exponents. This amounts to the substitution

\[
\phi^I(x) \rightarrow e^{-gc(x)} \phi^I(x) \text{e}^{gc(x)} , \quad \phi^I_{\uparrow}(x) \rightarrow e^{gc(x)} \phi^I_{\uparrow}(x) \text{e}^{-gc(x)} ,
\]

where $c(x)$ is the lowest component of the $\mathcal{N} = 1$ vector field. We regularize the operators by point-splitting, which, as already explained, allows us to keep also track of quadratic divergences and consistently subtract them out.
All the order $g^2$ perturbation theory integrals entering the calculation are proportional to the standard massless box integral [54], so the most complicated part of the whole procedure is the evaluation of colour traces. Once this is done, one obtains in the basis [55] the explicit expression of the two $15 \times 15$ matrices, $f_{00}$ and $f_{21}$, which are needed in eq. (13). Using eqs. (53) and (52), we resolve the tree-level mixing by identifying the m.r. operators $\hat{O}_p$ and compute the order $g^2$ corrections to their anomalous dimensions. In Figure 1 we plot the ratio of the order $g^2$ anomalous dimension of the operators $\hat{O}_p$ to the order $g^2$ anomalous dimension of the Konishi multiplet as function of the number of colours $N_c$ from 2 to 30. The points at the extreme right of the figure correspond to $N_c = \infty$. The labels $s$, $d$ and $t$ denote mostly single, double and triple trace operators in the large $N_c$ limit. The anomalous dimensions (in units of $\frac{g^2 N_c}{4\pi^2}$) are the roots of the polynomial

$$
\gamma \left[ 8\gamma^{14} - 488\gamma^{13} - 4(-3369 + 691\nu^2)\gamma^{12} + (-222905 + 136548\nu^2)\gamma^{11} + (2461929 - 3029657\nu^2 + 219960\nu^4)\gamma^{10} - (8841500\nu^4 + 19153227 - 3989871\nu^2)\gamma^9 - (347174976\nu^2 - 107844797 - 157896850\nu^4 + 7954400\nu^6)\gamma^8 + (-1656062605\nu^4 - 444279889 + 2102167847\nu^2 + 240075800\nu^6)\gamma^7 + 2(668238125 + 5674358030\nu^4 - 1578288500\nu^6 - 4542787849\nu^2 + 78212000\nu^8)\gamma^6 - (3264508000\nu^8 + 
$$

Figure 1: The ratio $\frac{\hat{O}_p}{\gamma_1^K}$, $p = 1, \ldots, 15$, of the $SU(4)$ singlet operators as functions of $N_c$. $\gamma_1^K = \frac{3N_c}{4\pi^2}$ is the order $g^2$ anomalous dimension of the Konishi multiplet.
where $\nu = 1/N_c$. The values of the anomalous dimensions for the 5 single trace operators at $N_c = \infty$ agree with the ones obtained in ref. [31] by diagonalizing the dilatation operator.

Two observations which are not obvious from the above picture are the following. First of all, although some of the solutions become rather close to each other, in particular in the range for $N_c$ from 6 to 10, they never intersect. Secondly, as soon as $N_c \geq 6$ (to avoid accidental degeneracies) the ratio of the sum of the order $g^2$ anomalous dimensions to the Konishi one is independent of $N_c$, i.e.

$$\frac{1}{N_c} \sum_{p=1}^{15} \tilde{\gamma}_p^{O_p} \gamma_1^{K_1} = \frac{61}{3}. \quad (59)$$

This $N_c$ independence seems to be a rather general property. In fact the matrix $(f_{00})^{-1} f_{21}$ of eq. (54), whose eigenvalues give the anomalous dimensions has a particularly simple $N_c$-dependence. If we denote by (6) the single trace, by (2|4) and (3|3) the two different types of double trace and by (2|2|2) the triple trace operators defined in eq. (55) (in parenthesis the number of scalar fields in each trace is indicated), then the form of the matrix $(f_{00})^{-1} f_{21}$ is particularly illuminating as it has the block structure shown below

$$
\begin{array}{cccc}
(6) & (2|4) & (3|3) & (2|2|2) \\
(6) & \#N_c & \# & \# & 0 \\
(2|4) & \# & \#N_c & 0 & \# \\
(3|3) & \# & 0 & \#N_c & 0 \\
(2|2|2) & 0 & \# & 0 & \#N_c \\
\end{array}
$$

where $\#$ denotes $N_c$ independent numeric entries. One immediately concludes that the trace of this matrix (which is equal to the sum of the (order $g^2$) anomalous dimensions) will be proportional to $N_c$, while the ratio $\frac{1}{N_c}$ will be independent of $N_c$, as the factor $N_c$ cancels between numerator and denominator.

This suggestive form of the matrix (60) has a simple interpretation in terms of ’t Hooft double-line notation or better in a dual string description [3, 4, 5, 6]. In the latter, where
operators are represented as closed strings, the $N_c$ dependence in the matrix (60) is dictated by the number of string splittings/joinings necessary to get a connected worldsheet diagram representing the corresponding two-point function. In fact, each closed string splitting/joining event costs a factor $g_s \approx 1/N_c$. Notice that some entries which appear as zeroes are actually suppressed by $1/N_c^2$ factors. These corrections are, however, not visible at the order we are working, which is $g^2 \approx g_s \approx 1/N_c$.

The most surprising feature in Figure 1 is, however, the presence of one operator which has no order $g^2$ correction to its anomalous dimension. It is a triple trace operator which we shall denote by $\mathcal{T}(x)$. In the operator basis introduced in eq. (55) it takes the form

$$\mathcal{T}(x) = \mathcal{O}_{13} - \frac{1}{2} \mathcal{O}_{14} + \frac{1}{18} \mathcal{O}_{15}.$$ (61)

Two further equivalent, but much more suggestive (since they yield $\mathcal{T}(x)$ in terms of protected operators only) representations of $\mathcal{T}(x)$ are

$$\mathcal{T}(x) = \sum_{i,j,k=1,\ldots,6} :Q^{ij}(x) Q^{ik}(x) Q^{jk}(x) : = \sum_{i,j=1,\ldots,6} :Q^{ij}(x) D^{ij}(x) : ,$$ (62)

where $Q^{ij}$ are the lowest components of the supercurrent multiplet defined in eq. (2), while $D^{ij}(x)$ is the dimension $\Delta = 4$ double trace protected operator defined in eq. (1). We remind also that as far as one is interested only in the order $g^2$ anomalous dimensions, the naive prescription of the normal ordering can be used (i.e. $: : \equiv$ no self-contractions).

The vanishing of the order $g^2$ anomalous dimension of the operator $\mathcal{T}(x)$ is completely unexpected, because, as we already pointed out, it belongs, even in free theory, to a long supermultiplet and there is no known mechanism at work which might protect it. An exhaustive search throughout the whole set of $\Delta_0 = 6$ purely scalar operators [71] shows that $\mathcal{T}(x)$ is the only such operator with vanishing order $g^2$ anomalous dimension which is not protected by any known shortening condition. This puzzling result is confirmed and made more dramatic by the fact that one also finds a vanishing order $g^4$ correction to the anomalous dimension of $\mathcal{T}(x)$.

We end this section by noticing that the procedure illustrated before can be used to resolve the order $g^2$ mixing also for sets of operators belonging to $SU(4)$ representations other than the singlet. As an example we wish to discuss the case of the operators of naive conformal dimension $\Delta_0 = 6$ in the representation $20'$ of $SU(4)$. A straightforward but tedious analysis shows that for $N_c \geq 6$ there are 26 operators made of scalars. For $N_c = 2, 3, 4$ and 5 the number of operators of this kind is 4, 13, 23 and 25 respectively. In Figure 2 we plot the ratio of the order $g^2$ anomalous dimension of these operators to the order $g^2$ anomalous dimension of the Konishi multiplet as function of the number of colours $N_c$ from 2 to 30. The points at the extreme right of the figure correspond to $N_c = \infty$. As can be seen, for finite $N_c$ all operators have non-vanishing order $g^2$ anomalous dimensions. Another surprise is however in store for us. In fact, we find that one operator, which in the large $N_c$ limit is dominantly double trace, hence cannot belong to the Konishi supermultiplet, has exactly the same order $g^2$ anomalous dimension as the
The ratio \( \gamma_{\hat{O}^p}/\gamma_{K}^{p}, p = 1, \ldots, 26 \), of the operators in 20' as functions of \( N_c \). \( \gamma_{K}^{p} = \frac{3N_c}{4\pi^2} \) is the order \( g^2 \) anomalous dimension of the Konishi multiplet.

Konishi supermultiplet for all values of \( N_c \). What is the origin of this degeneracy and whether it persists at higher orders is still an open problem.

### 5.2 The order \( g^4 \) calculation

In this subsection we sketch the argument which leads to the conclusion that the order \( g^4 \) correction to the anomalous dimension of the operator \( \mathcal{T} \) vanishes. To this end let us first note that the order \( g^2 \) correction to the four-point function of \( \tilde{T}(x) \) (the definition of the operators \( \tilde{O}(x) \) is as in eq. (14) i.e. they have vanishing 2-point functions with all lower dimensional operators) and three \( Q^{ij} \)'s also vanishes, namely

\[
\langle \tilde{T}(x_1) \ Q^{ij}(x_2) \ Q^{ik}(x_3) \ Q^{jk}(x_4) \rangle |_{g^2} = 0.
\]

To realize how unusual this result is we recall that in general even the four-point functions of four protected 1/2 BPS operators are corrected at order \( g^2 \).
Furthermore, one finds

\[ \langle \tilde{O}_\ell(x_1) \ Q^{ij}(x_2) \ Q^{jk}(x_3) \ Q^{ik}(x_4) \rangle |_{g^2} = 0 , \tag{64} \]

where \( \tilde{O}_\ell(x_1) \) is an arbitrary (not necessarily purely scalar, like the operators in eqs. (55)) \( SU(4) \) singlet scalar operator of naive dimension \( \Delta_0 = 6 \). This means that the operator \( \tilde{T} \) is the only \( SU(4) \) singlet scalar conformal primary operator of naive dimension \( \Delta_0 = 6 \) which appears in the OPE of three operators \( Q^{ij} \) up to order \( g^2 \). The importance of this conclusion is that we can exploit the OPE of three \( Q^{ij} \)'s to give a rigorous definition of the renormalized operator \( \hat{T}(x) \) through the formula

\[ \hat{T}(x) = \text{OPE} \left( \sum_{i,j,k} Q^{ij}(x+\epsilon) \ Q^{ik}(x) \ Q^{jk}(x-\epsilon) \right) \bigg|_{\Delta_0=6} , \tag{65} \]

where the projection on the dimension \( \Delta_0 = 6 \) contribution in the OPE means the subtraction of all subleading operators (i.e. those with naive dimension \( \Delta_0 < 6 \)) together with their conformal descendants (derivatives). In particular one has to subtract the Konishi singlet \( K_1 \) (\( \Delta_0 = 2 \)), as well as all singlet scalar operators of naive dimension \( \Delta_0 = 4 \). All these operators have non-vanishing anomalous dimensions, thus the coefficients of the subtractions implicit in the notation of eq. (65) will depend on the coupling constant \( g \).

This technical complication is, however, largely compensated by the following nice property inherent in the OPE definition (65). We do not have to know the explicit mixing of the (naively purely scalar) operator \( T \) with the operators containing fermions, \( F_{\mu\nu} \) and derivatives, since the triple OPE of eq. (65) embodies them implicitly (at least up to order \( g^2 \) which is relevant for the calculation of the order \( g^4 \) correction to the anomalous dimension of \( \hat{T}(x) \)). It should be noted that a similar compact definition can be given also for the protected operator \( D_{20}' \) discussed in Section 2, namely

\[ \hat{D}^{ij}_{20'}(x) = \text{OPE} \left( Q^{ik}(x+\epsilon) \ Q^{jk}(x-\epsilon) \right) |_{20', \Delta_0=4} . \tag{66} \]

It is instructive to verify that this definition is equivalent to eq. (9) up to order \( g^2 \).

Let us stress that eq. (64) implies not only the vanishing of the logarithmically divergent piece (which is related to the anomalous dimension), but also of the associated finite part. This allows us to prove the following Theorem.

Suppose there exists some \( SU(4) \) singlet operator of naive dimension \( \Delta_0 = 6 \), \( \tilde{O}_S(x_1) \), with non-vanishing tree-level 2-point function with \( T \), i.e. such that

\[ \langle \tilde{O}_S(x_1) \ T(x_2) \rangle |_0 \neq 0 . \tag{67} \]

Suppose also that at order \( g^4 \) the divergent part of its 4-point function with three \( Q^{ij} \)'s vanishes

\[ \langle \tilde{O}_S(x_1) \ Q^{ij}(x_2) \ Q^{ik}(x_3) \ Q^{jk}(x_4) \rangle |_{g^4, \log} = 0 , \tag{68} \]

then it follows that the order \( g^4 \) anomalous dimension of \( \hat{T}(x) \) is zero.
In fact, eq. (64) implies that any singlet scalar $\Delta = 6$ operator different from $\hat{T}$ appearing in the OPE of three $Q^{ij}$ must be multiplied by at least a factor $g^3$. Hence it cannot give logarithmic corrections at order $g^4$. Thus the product of three $Q^{ij}$'s in eq. (68) acts as a projector on $\hat{T}(x)$, which we already know has a vanishing order $g^2$ anomalous dimension $\gamma_1 = 0$, hence the function in eq. (68) must be proportional to

$$\gamma_2 \cdot \langle T(x_1) Q^{ij}(x_2) Q^{ik}(x_3) Q^{jk}(x_4) \rangle |_{0}. \quad (69)$$

This tree-level 4-point function is indeed non-vanishing by the very definition of $\hat{T}$, so we conclude that

$$\gamma_2 = 0. \quad (70)$$

To complete the proof we have to find an operator $\tilde{O}_{S}$ with the properties (67) and (68). The details of this very long calculation will be presented elsewhere [71]. Here let us only note that a possible choice for it is the triple trace totally colour symmetric and purely scalar operator

$$O_{S}(x) = O_{13} + \frac{3}{4} O_{14} + \frac{1}{8} O_{15}. \quad (71)$$

The motivation for this choice is that when inserted in the 4-point function $\tilde{O}_{S}$ coincides with $O_{S}$ so one can avoid all the complications stemming from $g$ dependent subtractions. As a last remark we note that with this choice for $O_{S}$ the whole function (not only its divergent part) in eq. (68) is zero, because one can express it in terms of the order $g^2$ and $g^4$ corrections to 2-point and 3-point functions of only protected operators, like $\langle Q^{ij} Q^{ij} \rangle$ and $\langle D^{ij} Q^{ik} Q^{jk} \rangle$, each of which vanishes.

6 Conclusions and summary

Let us summarize our results. We have reexamined the issue of operator mixing in $\mathcal{N} = 4$ SYM and argued that particular care should be exerted in defining regularized composite (gauge invariant) operators, as soon as one is willing to go beyond one-loop in perturbation theory.

Exact superconformal invariance puts stringent constraints that imply the resummation of logarithms in power-like behaviours and allows us to construct a systematic procedure for the explicit (numerical) resolution of the operator mixing problem and the calculation of anomalous dimensions.

Our strategy is in a sense complementary to the one advocated in ref. [31, 32] that emphasizes, instead, the role of the dilatation operator, possibly viewed as the Hamiltonian of a super spin chain [30, 33, 36]. Although the latter approach seems very efficient for one-loop computations in “closed sectors”, we can’t help spending a word of caution on the effectiveness of the method beyond one-loop. At any rate the two approaches should give equivalent results and it is reassuring that they indeed do so in the limited number of
cases where they can be compared. It would be very interesting to consider and possibly
clarify the role of the generalized $\mathcal{N} = 4$ Konishi anomaly \cite{38,39,40} in the mixing of
operators involving fermion “impurities” in the approach of ref. \cite{31,32} in view of the
presence of odd powers of $g$ in the expansion of the dilatation operator, even in the BMN
limit.

More importantly, it would be nice if one could resolve the order $g^4$ mixing for the
singlet operators of naive dimension $\Delta_0 = 6$ using the dilatation operator method \cite{31,32}
and confirm our surprising result of the vanishing of the anomalous dimension of the
(purely scalar) operator $\mathcal{T}$ up to $g^4$ and possibly beyond. Understanding whether this is
an accident of the low orders in the perturbative expansion, that is an exact but isolated
case, or rather the first instance of a class of kinematically unprotected but yet dynamically
“unrenormalized” operators is a challenge for any future investigation in this field, as it
might point towards some hidden dynamical symmetry that could be at the heart of the
conjectured integrability of $\mathcal{N} = 4$ SYM \cite{30,31,33,37}. The story of non-renormalization
theorems in $\mathcal{N} = 4$ SYM theory, e.g. for extremal correlators \cite{72,73,74,75}, is suggestive
in this respect. The holographic correspondence, after the subtle issues of renormalization
of composite operators \cite{76,77,78} has been carefully taken care of, could provide a guide
to the understanding of our puzzling result. Instanton calculus \cite{60,61,79,63} can give
further insights into this issue or even lead to a non-vanishing contribution to $\gamma_T$ at the
non-perturbative level, much as it happens for some non-local observables \cite{80,81}.

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References

[1] M. Bianchi, B. Eden, G. C. Rossi and Ya. S. Stanev, Nucl. Phys. B 646 (2002) 69
\[arXiv:hep-th/0205321].

[2] G. Arutyunov, S. Penati, A. C. Petkou, A. Santambrogio and E. Sokatchev, Nucl.
Phys. B 643 (2002) 49 \[arXiv:hep-th/0206020].

[3] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, Nucl. Phys. B 643
(2002) 3 \[arXiv:hep-th/0205033].
[4] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, JHEP 0207 (2002) 017 [arXiv:hep-th/0205089].

[5] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, Nucl. Phys. B 650 (2003) 125 [arXiv:hep-th/0208178].

[6] N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, JHEP 0210 (2002) 068 [arXiv:hep-th/0209002].

[7] D. Berenstein, J. M. Maldacena and H. Nastase, JHEP 0204 (2002) 013 [arXiv:hep-th/0202021].

[8] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, JHEP 0201 (2002) 047 [arXiv:hep-th/0110242].

[9] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, Class. Quant. Grav. 19 (2002) L87 [arXiv:hep-th/0201081].

[10] M. Blau, J. Figueroa-O’Farrill and G. Papadopoulos, Class. Quant. Grav. 19 (2002) 4753 [arXiv:hep-th/0202111].

[11] J. Figueroa-O’Farrill and G. Papadopoulos, JHEP 0303 (2003) 048 [arXiv:hep-th/0211089].

[12] M. Blau, M. O’Loughlin, G. Papadopoulos and A. A. Tseytlin, Nucl. Phys. B 673 (2003) 57 [arXiv:hep-th/0304198].

[13] R. R. Metsaev, Nucl. Phys. B 625 (2002) 70 [arXiv:hep-th/0112044].

[14] R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D 65 (2002) 126004 [arXiv:hep-th/0202109].

[15] R. R. Metsaev and A. A. Tseytlin, Russ. Phys. J. 45 (2002) 719 [Izv. Vuz. Fiz. 2002N7 (2002) 67].

[16] A. Santambrogio and D. Zanon, Phys. Lett. B 545 (2002) 425 [arXiv:hep-th/0206079].

[17] N. Beisert, Nucl. Phys. B 659 (2003) 79 [arXiv:hep-th/0211032].

[18] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, Phys. Lett. B 558 (2003) 229 [arXiv:hep-th/0212269].

[19] N. W. Kim, T. Klose and J. Plefka, Nucl. Phys. B 671 (2003) 359 [arXiv:hep-th/0306054].

[20] C. S. Chu, V. V. Khoze and G. Travaglini, JHEP 0206 (2002) 011 [arXiv:hep-th/0206005].

[21] C. S. Chu, V. V. Khoze and G. Travaglini, JHEP 0209 (2002) 054 [arXiv:hep-th/0206167].
[22] C. S. Chu, M. Petrini, R. Russo and A. Tanzini, Class. Quant. Grav. 20 (2003) S457 arXiv:hep-th/0211188.

[23] C. S. Chu, V. V. Khoze and G. Travaglini, JHEP 0306 (2003) 050 arXiv:hep-th/0303107.

[24] C. Chu, M. Petrini, R. Russo and A. Tanzini, Fortsch. Phys. 51 (2003) 684.

[25] U. Gursoy, JHEP 0307 (2003) 048 arXiv:hep-th/0208041.

[26] U. Gursoy, JHEP 0310 (2003) 027 arXiv:hep-th/0212118.

[27] D. Z. Freedman and U. Gursoy, JHEP 0308 (2003) 027 arXiv:hep-th/0305016.

[28] B. Eden, arXiv:hep-th/0307081.

[29] T. Klose, JHEP 0303 (2003) 012 arXiv:hep-th/0301150.

[30] J. A. Minahan and K. Zarembo, JHEP 0303 (2003) 013 arXiv:hep-th/0212208.

[31] N. Beisert, C. Kristjansen and M. Staudacher, Nucl. Phys. B 664 (2003) 131 arXiv:hep-th/0303060.

[32] N. Beisert, arXiv:hep-th/0307015

[33] N. Beisert and M. Staudacher, Nucl. Phys. B 670 (2003) 439 arXiv:hep-th/0307042.

[34] M. Bianchi, J. F. Morales and H. Samtleben, JHEP 0307 (2003) 062 arXiv:hep-th/0305052.

[35] N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, arXiv:hep-th/0310292.

[36] N. Beisert, arXiv:hep-th/0310252

[37] T. Klose and J. Plefka, arXiv:hep-th/0310232

[38] K. Konishi, Phys. Lett. B 135, 439 (1984).

[39] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi and G. Veneziano, Phys. Rept. 162 (1988) 169.

[40] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, JHEP 0212 (2002) 071 arXiv:hep-th/0211170.

[41] L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, Adv. Theor. Math. Phys. 3 (1999) 1149 arXiv:hep-th/9912007.

[42] L. Andrianopoli and S. Ferrara, Lett. Math. Phys. 48 (1999) 145 arXiv:hep-th/9812067.

[43] L. Andrianopoli and S. Ferrara, Lett. Math. Phys. 46 (1998) 265 arXiv:hep-th/9807150.
[44] L. Andrianopoli and S. Ferrara, Phys. Lett. B 430 (1998) 248 [arXiv:hep-th/9803171].
[45] F. A. Dolan and H. Osborn, Annals Phys. 307 (2003) 41 [arXiv:hep-th/0209056].
[46] G. Arutyunov, S. Frolov and A. C. Petkou, Nucl. Phys. B 586 (2000) 547 [Erratum-ibid. B 609 (2001) 539] [arXiv:hep-th/0005182].
[47] G. Arutyunov, B. Eden, A. C. Petkou and E. Sokatchev, Nucl. Phys. B 620 (2002) 380 [arXiv:hep-th/0103230].
[48] G. Arutyunov, S. Frolov and A. Petkou, Nucl. Phys. B 602 (2001) 238 [Erratum-ibid. B 609 (2001) 540] [arXiv:hep-th/0010137].
[49] B. Eden, A. C. Petkou, C. Schubert and E. Sokatchev, Nucl. Phys. B 607 (2001) 191 [arXiv:hep-th/0009106].
[50] B. Eden and E. Sokatchev, Nucl. Phys. B 618 (2001) 259 [arXiv:hep-th/0106249].
[51] F. A. Dolan and H. Osborn, Nucl. Phys. B 629 (2002) 3 [arXiv:hep-th/0112251].
[52] F. A. Dolan and H. Osborn, Nucl. Phys. B 599 (2001) 459 [arXiv:hep-th/0011040].
[53] S. Penati and A. Santambrogio, Nucl. Phys. B 614 (2001) 367 [arXiv:hep-th/0107071].
[54] M. Bianchi, S. Kovacs, G. C. Rossi and Ya. S. Stanev, JHEP 9908 (1999) 020 [arXiv:hep-th/9906188].
[55] M. Bianchi, S. Kovacs, G. C. Rossi and Ya. S. Stanev, JHEP 0105 (2001) 042 [arXiv:hep-th/0104016].
[56] M. Bianchi, S. Kovacs, G. C. Rossi and Ya. S. Stanev, Nucl. Phys. B 584 (2000) 216 [arXiv:hep-th/0003203].
[57] B. Eden, P. S. Howe, C. Schubert, E. Sokatchev and P. C. West, Nucl. Phys. B 557 (1999) 355 [arXiv:hep-th/9811172].
[58] B. Eden, P. S. Howe, C. Schubert, E. Sokatchev and P. C. West, Phys. Lett. B 466 (1999) 20 [arXiv:hep-th/9906051].
[59] B. U. Eden, P. S. Howe, A. Pickering, E. Sokatchev and P. C. West, Nucl. Phys. B 581 (2000) 523 [arXiv:hep-th/0001138].
[60] M. Bianchi, M. B. Green, S. Kovacs and G. C. Rossi, JHEP 9808 (1998) 013 [arXiv:hep-th/9807033].
[61] M. Bianchi and S. Kovacs, [arXiv:hep-th/9811060]
[62] M. Bianchi, Nucl. Phys. Proc. Suppl. 102 (2001) 56 [arXiv:hep-th/0103112].
[63] S. Kovacs, [arXiv:hep-th/0310193].
