TAUTOLOGICAL HILBERT SCHEME INVARIABLES OF CALABI-YAU 4-FOLDS VIA VIRTUAL PULLBACK

YALONG CAO AND FENG QU

Abstract. Let $X$ be a Calabi-Yau 4-fold and $D$ a smooth connected divisor on it. We consider tautological bundles of $L = \mathcal{O}_X(D)$ on Hilbert schemes of points on $X$ and their counting invariants defined by integrating the Euler classes against the virtual classes. We relate these invariants to Maulik-Nekrasov-Okounkov-Pandharipande’s invariants of Hilbert schemes of points on $D$ by virtual pullback technique and confirm a conjecture of Cao-Kool. The same strategy is also applied to obtain a virtual pullback formula for tautological invariants of one dimensional stable sheaves. This in turn gives a nontrivial identity on primary and descendent invariants as studied in the works of Cao, Maulik and Toda.

Contents

1. Introduction
1.1. Hilbert schemes and tautological invariants
1.2. Euler class heuristic
1.3. Virtual pullback
1.4. Application to other tautological invariants
1.5. Conventional and Notation
2. Proof of virtual pullback formula
2.1. Compatible diagram of obstruction theories
2.2. Preparation on virtual pullback
2.3. Finish of proof
Step 0: pullback to $B$
Step 1: constructing maximal isotropic subbundle
Step 2: applying virtual pullback
3. Application to other tautological invariants
Acknowledgement
References

1. Introduction

1.1. Hilbert schemes and tautological invariants. Let $X$ be a Calabi-Yau 4-fold\footnote{In this paper, a Calabi-Yau 4-fold is a connected smooth complex projective 4-fold $X$ satisfying $K_X \cong \mathcal{O}_X$.}. For a positive integer $n$, we consider the Hilbert scheme $\text{Hilb}^n(X)$ of zero dimensional subschemes of length $n$ on $X$. Such Hilbert schemes are in general singular and may have different dimensions in different irreducible components.

Remarkably, they can be identified with Gieseker moduli spaces of semistable sheaves with fixed determinant and Chern character $(1, 0, 0, 0, -n) \in H^{even}(X)$, which are fine moduli spaces of ideal sheaves of zero dimensional subschemes on $X$. Hence they can be given $(−2)$-shifted symplectic derived scheme structures in the sense of Pantev-Toën-Vaquier-Vezzosi\cite{PTVV} and have well-defined DT$_4$ virtual classes, constructed as homology classes by Borisov-Joyce\cite{BJ} using derived differential geometry (see also\cite{CL14} for constructions in special cases). Recently, Oh-Thomas\cite{OT} lift Borisov-Joyce’s virtual classes to algebraic cycles when working with $\mathbb{Z}[1/2]$-coefficients. These virtual classes depend on the choice of orientation of certain real line bundles on moduli spaces\cite{CGJ} and on each connected component, there are two choices of orientation, which affect the corresponding contribution to the virtual classes by a sign.

In our setting, $\text{Hilb}^n(X)$ are connected by a result of Fogarty\cite[Prop. 2.3]{F}. Therefore their virtual classes

\begin{equation}
[\text{Hilb}^n(X)]^{vir} \in H_{2n}(\text{Hilb}^n(X), \mathbb{Z})
\end{equation}
are unique only up to a sign. To define counting invariants using these classes, we need to involve insertions. Given a line bundle $L$ on $X$, its tautological bundle on $\text{Hilb}^n(X)$ is

$$L^{[n]} := \pi_M^*(\mathcal{O}_Z \otimes \pi_X^* L),$$

where $Z \subseteq \text{Hilb}^n(X) \times X$ denotes the universal subscheme and $\pi_M, \pi_X$ are projections from $\text{Hilb}^n(X) \times X$ onto its factors. The tautological invariants of $\text{Hilb}^n(X)$ are defined by

$$I_n(X, L) := \int_{[\text{Hilb}^n(X)]^{\text{vir}}} e(L^{[n]}) \in \mathbb{Z},$$

which depend on the choice of orientation up to a sign and only receive contribution from torsion-free part of virtual classes. Such invariants and their variances are studied in [CK18] and have close relations with curve counting invariants as in [CMT18, CMT19, CT19, CT20a, CT20b, CT20c]. In particular, we have the following conjectural formula of their generating series.

**Conjecture 1.1.** (Cao-Kool [CK18]) There exists a choice of orientation such that

$$1 + \sum_{n=1}^{\infty} I_n(X, L) q^n = M(-q)^{\sum c_1(L) e(X)},$$

where

$$M(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n}$$

denotes the MacMahon function.

In [CK18], the authors verified the conjecture for $n \leq 3$ when $L$ corresponds to a smooth (connected) divisor and provided further evidence on the corresponding equivariant conjecture on toric Calabi-Yau 4-folds (ref. [CK18, Conj. 1.6]). In physics literature, Nekrasov gave a $K$-theoretic generalization on $\mathbb{C}^4$ (Nek).

The main purpose of this paper is to give a strong evidence of the above conjecture in the following setting.

**Theorem 1.2.** Let $X$ be a Calabi-Yau 4-fold and $L \in \text{Pic}(X)$ such that

$$c_1(L) = c_1(\mathcal{O}_X(D)) \in H^2(X, \mathbb{Q}),$$

for a smooth connected divisor $D \subseteq X$. Then Conjecture 1.1 holds for $L$.

For instance, this allows us to verify Conjecture 1.1 for any effective divisor in the following:

**Example 1.3.** On a smooth sextic 4-fold $X \subset \mathbb{P}^5$, any effective divisor $L$ satisfies that $c_1(L) = c_1(\mathcal{O}_X(D))$ for some smooth connected divisor $D$ by Bertini’s theorem [Hart, §II, Thm. 8.18]. Therefore Conjecture 1.1 holds for any effective divisor on $X$.

**Remark 1.4.** It is worth mentioning that Arkadij Bojko [Bo] is able to reduce the proof of Conjecture 1.1 to Theorem 1.2 by using Joyce’s recent conjectural wall-crossing framework [Joy].

### 1.2. Euler class heuristic.

From Grothendieck-Riemann-Roch formula, our invariants depend only on the first Chern class of $L$ in rational coefficients. To prove Theorem 1.2 we may assume $L = \mathcal{O}_X(D)$ for a smooth connected divisor $D$, in which setting we have the following:

**Proposition 1.5.** ([CK18 Proposition 1.3]) Let $X$ be a Calabi-Yau 4-fold, $D \subseteq X$ any effective divisor and $L = \mathcal{O}_X(D)$. There exists a tautological section $\sigma$ and an isomorphism of schemes:

$$\begin{array}{ccc}
\sigma^{-1}(0) & \cong & \text{Hilb}^n(D) \\
\pi & \longrightarrow & \text{Hilb}^n(X).
\end{array}$$

With this geometric relation, it is tempting to write down (when $D$ is smooth):

$$\int_{[\text{Hilb}^n(X)]^{\text{vir}}} e(L^{[n]})^{[n]} = \int_{[\text{Hilb}^n(D)]^{\text{vir}}} 1,$$

where $[\text{Hilb}^n(D)]^{\text{vir}}$ is the degree zero virtual class of $\text{Hilb}^n(D)$ constructed by Maulik-Nekrasov-Okounkov-Pandharipande [MNOP] (based on [The]), whose generating series is given by [BF08, LP, L4]:

$$1 + \sum_{n=1}^{\infty} \left( \int_{[\text{Hilb}^n(D)]^{\text{vir}}} 1 \right) q^n = M(-q)^{\sum c_1(D e(TD \otimes K_D))}.$$
It is straightforward to check that (e.g. [CK18 pp. 7, (2.5)]):
\[
\int_D c_3(TD \otimes K_D) = \int_X c_1(L) \cdot c_3(X).
\]
Therefore to prove Conjecture 1.1 (when \(L = \mathcal{O}_X(D)\)), it is enough to prove the “Euler class heuristic” (3).

1.3. Virtual pullback. By Proposition 1.5, one can define pullback:

\[(5) \quad i' : A_*(\text{Hilb}^n(X)) \xrightarrow{\text{incl}} A_*(\text{Hilb}^n(D)/\text{Hilb}^n(X)) \xrightarrow{0} A_{*-n}(\text{Hilb}^n(D)),\]

in Chow groups following [Ful] Ex. 17.6.4. Here the first arrow is the specialization map and \(C_{\bullet, \bullet}\) denotes the normal cone, the second is the inclusion map and the last is the Gysin homomorphism. In loc. cit., this pullback is also described by Segre classes of normal cones and this paper are derived functors. For a morphism \(D \rightarrow B\), convention and notation. On primary and descendent invariants studied in [CMT18, CT20a] (ref. Corollary 3.4). Therefore to prove Conjecture 1.1 (when \(\pi \colon X \rightarrow Y\) by employing the virtual pullback formalism of Manolache [Man], in particular, functoriality of virtual pullbacks [BF97, KKP, Man, Qu] (see Lemma 2.5, Proposition 2.10). To make this strategy work, we need compatibility of obstruction theories as prepared in Section 2.1. Finally, we have

\[
\int_X c_1(L) \cdot c_3(X) = \int_D c_3(TD \otimes K_D) = \int_X c_1(L) \cdot c_3(X).
\]

Therefore in order to show (3), it is enough to prove:

**Proposition 1.6.** We have

\[(7) \quad i'[\text{Hilb}^n(X)] \mapsto i'[\text{Hilb}^n(D)] \subseteq \text{A}_0(\text{Hilb}^n(D), \mathbb{Q}).\]

Here we work with rational coefficient and use Oh-Thomas’ lift of virtual classes [OT] to Chow groups. We remark that Proposition 1.6 should be viewed as a virtual pullback formula, though the LHS is not defined directly using Behrend-Fantechi virtual classes [BF97].

Section 2 is devoted to a proof of Proposition 1.6. More specifically, in Section 2.3, we first recall the virtual class construction of [OT]. Via the commutativity of our pullback (5) with the square root Gysin pullback in [OT], we can reduce the LHS of (7) to a familiar construction using standard Gysin pullback (ref. Proposition 2.8). Then we compare it with the RHS of (7) by employing the virtual pullback formalism of Manolache [Man], in particular, functoriality of virtual pullbacks [BF97, KKP, Man, Qu] (see Lemma 2.5, Proposition 2.10). To make this strategy work, we need compatibility of obstruction theories as prepared in Section 2.1. Finally, we conclude a proof of Theorem 1.2 by combining Proposition 1.6 and MNOP formula (1).

1.4. Application to other tautological invariants. In Section 3 the same proof strategy is applied to study moduli spaces of one dimensional stable sheaves on \(X\) and its smooth divisor \(D\). A virtual pullback formula is obtained (ref. Theorem 3.3), which gives a nontrivial identity on primary and descendent invariants studied in [CMT18, CT20a] (ref. Corollary 3.4).

1.5. Convention and Notation. Unless stated otherwise, all pushforwards and pullbacks in this paper are derived functors. For a morphism \(\pi : X \rightarrow Y\) between schemes and \(\mathcal{F}, \mathcal{G} \in D^b(\text{Coh}(X))\), we denote the functor \(R\text{Hom}_\pi(\mathcal{F}, \mathcal{G}) := \pi_! R\text{Hom}(\mathcal{F}, \mathcal{G})\).

2. Proof of Virtual Pullback Formula

The purpose of this section is to prove the virtual pullback formula (i.e. Proposition 1.6).

For simplicity, we denote \(B := \text{Hilb}^n(D), A := \text{Hilb}^n(X)\) and \(E := L^{[n]}\). Recall we have the following diagram (Proposition 1.5):

\[
\begin{array}{c}
E \\
\pi \\
\sigma \downarrow \\
\sigma^{-1}(0) \cong B \xrightarrow{i} A.
\end{array}
\]

We denote \(Z_X \subseteq A \times X\) and \(Z_D \subseteq B \times D\) to be the universal zero dimensional subschemes and \(\pi_A : A \times X \rightarrow A, \pi_B : B \times D \rightarrow B\) to be the projections.
2.1. Compatible diagram of obstruction theories. We recall that (trace-free) deformation obstruction spaces of ideal sheaves of points are canonically isomorphic to deformation obstruction spaces of structure sheaves of points (see e.g. [CKTS Lem. 2.6]). To relate deformation-obstruction theories of $B$ and $A$, we construct the following:

**Lemma 2.1.** There is a distinguished triangle

$$ R\text{Hom}_{\pi_B}(O_{Z_D}, O_{Z_D}) \cong \pi_A R\text{Hom}_{\pi_A}(O_{Z_X}, O_{Z_X}) \cong \pi_B R\text{Hom}_{\pi_B}(O_{Z_D}, O_{Z_D} \boxtimes K_D[-1]), $$

which fits into a commutative diagram

\begin{equation}
\begin{array}{ccc}
R\text{Hom}_{\pi_B}(O_{Z_D}, O_{Z_D}) & \overset{\alpha}{\rightarrow} & i^* R\text{Hom}_{\pi_A}(O_{Z_X}, O_{Z_X}) \\
\downarrow & & \downarrow \\
R\text{Hom}_{\pi_B}(O_{Z_D}, O_{Z_D} \boxtimes K_D[-1]) & \cong & i^* R\text{Hom}_{\pi_A}(O_{Z_X}, O_{Z_X}) \gamma[-4].
\end{array}
\end{equation}

Here vertical maps are given by Grothendieck-Verdier duality.

**Proof.** From the Cartesian diagram

$$ B \times X \overset{i\times id_X}{\rightarrow} A \times X \xrightarrow{\sim} Z_X $$

we have canonical isomorphisms

$$ i^* \pi_A R\text{Hom}(O_{Z_X}, O_{Z_X}) \cong \pi_B i^* R\text{Hom}(O_{Z_X}, O_{Z_X}) $$

where $j = \text{id}_B \times i_D : X$. Then we have

$$ \pi_B R\text{Hom}(i^* O_{Z_X}, i^* O_{Z_X}) \cong \pi_B R\text{Hom}(j_* O_{Z_D}, j_* O_{Z_D}) $$

$$ \cong \pi_B R\text{Hom}(j^* j_* O_{Z_D}, O_{Z_D}). $$

Adjunction and Grothendieck-Verdier duality provide canonical morphisms

$$ j^* j_* O_{Z_D} \rightarrow O_{Z_D}, \quad O_{Z_D} \boxtimes K_D[-1] \rightarrow j^* j_* O_{Z_D}. $$

Moreover, they fit into a distinguished triangle (e.g. [Huy Cor. 11.4]):

$$ O_{Z_D} \boxtimes K_D[-1] \rightarrow j^* j_* O_{Z_D} \rightarrow O_{Z_D}. $$

Applying $\pi_B R\text{Hom}(-, O_{Z_D})$ to it, we obtain a distinguished triangle

$$ R\text{Hom}_{\pi_B}(O_{Z_D}, O_{Z_D}) \rightarrow R\text{Hom}_{\pi_B}(j^* j_* O_{Z_D}, O_{Z_D}) \rightarrow R\text{Hom}_{\pi_B}(O_{Z_D}, O_{Z_D} \boxtimes K_D[-1]). $$

Combining with (10) and (11), we get (9). As the distinguished triangle is constructed from Grothendieck-Verdier duality for $j$, consider the commutative diagram

$$ B \times D \overset{j}{\rightarrow} B \times X $$

and use functoriality of Grothendieck-Verdier duality, one can get the commutative diagram.

In particular, we can restrict to a closed point $[Z] \in B$ and take cohomology.

**Corollary 2.2.** There is a canonical isomorphism

$$ \text{Ext}^0_D(O_Z, O_Z) \cong \text{Ext}^0_X(O_Z, O_Z) $$

and a long exact sequence

$$ 0 \rightarrow \text{Ext}^1_D(O_Z, O_Z) \rightarrow \text{Ext}^1_X(O_Z, O_Z) \rightarrow \text{Ext}^0_D(O_Z, O_Z \boxtimes K_D) \rightarrow \text{Ext}^2_D(O_Z, O_Z) \rightarrow \text{Ext}^2_X(O_Z, O_Z) \rightarrow \text{Ext}^1_D(O_Z, O_Z \boxtimes K_D) \rightarrow \text{Ext}^2_D(O_Z, O_Z) \rightarrow \text{Ext}^3_X(O_Z, O_Z) \rightarrow \text{Ext}^2_D(O_Z, O_Z \boxtimes K_D) \rightarrow 0, $$

where $\beta$ is the natural map $X \rightarrow Y$. 

\[\square\]
which is self-dual under Serre duality. Moreover, Im(θ) is a maximal isotropic subspace of Ext_{X}^{2}(O_{Z}, O_{Z}) with respect to the Serre duality pairing.

Proof. Let W := Ext_{D}^{2}(O_{Z}, O_{Z}) and V := Ext_{X}^{2}(O_{Z}, O_{Z}). By the self-duality of the exact sequence (which follows from Lemma 2.3), we have

\[ W \xrightarrow{\theta} V \cong V^{\vee} \xrightarrow{\psi = \bar{\theta}^{\vee}} W^{\vee}, \text{ s.t. } \text{Im}(\theta) = \text{Ker}(\theta^{\vee}). \]

Notice that

\[ \text{Ker}(\theta^{\vee}) := \{ x \in V^{\vee} : \theta^{\vee}(x)(W) \equiv Q(\theta(W), x) = 0 \} = \text{Im}(\theta)^{\perp}, \]

which is the orthogonal complement of Im(θ) with respect to the pairing Q. The equality Im(θ) = Im(θ)^{\perp} implies that this is a maximal isotropic subspace of V. □

Therefore we should truncate (12) into exact sequence

\[ 0 \rightarrow \text{Ext}_{D}^{2}(O_{Z}, O_{Z}) \rightarrow \text{Ext}_{X}^{2}(O_{Z}, O_{Z}) \rightarrow \text{Ext}_{D}^{0}(O_{Z}, O_{Z} \otimes K_{D}) \rightarrow \text{Ext}_{D}^{2}(O_{Z}, O_{Z}) \rightarrow \text{Im}(\theta) \rightarrow 0, \]

and (naively speaking) use Im(θ) as new obstruction spaces of A (when restricted to B). To make sense this, we construct the following commutative diagrams.

**Lemma 2.3.** There is a commutative diagram

\[
\begin{array}{c}
W \xrightarrow{\tau^{[0,1]}} (R\text{Hom}_{\pi_{B}}(O_{Z,D}, O_{Z,D})[1])^{\vee} \xrightarrow{\phi} E^{\vee}|_{B[1]} \\
\downarrow \quad \downarrow \quad \downarrow \\
i^{*}L_{A} \quad L_{B} \quad L_{B/A},
\end{array}
\]

where W is a perfect complex with tor-amplitude \([a, b]\) in \([-1, 0]\) and horizontal lines are distinguished triangles. Moreover, \(\phi\) fits into a commutative diagram

\[
\begin{array}{c}
\left(\tau^{[0,1]}(i^{*}R\text{Hom}_{A}(O_{Z}, O_{Z})[1])\right)^{\vee} \xrightarrow{\phi} W \\
\downarrow \quad \downarrow \\
i^{*}L_{A}.
\end{array}
\]

Here \(h^{0}(\bullet)\) is an isomorphism and \(h^{-1}(\bullet)\) is surjective for \(\bullet = \phi_{B}, \phi'\) and \(\phi\).

**Proof.** Our aim is to construct the above commutative diagrams in derived categories. A standard way to do this is to use derived enhancement of corresponding moduli spaces (see e.g. [STV], §1.2) and work with the enhanced dg-category, where cone construction is functorial. We follow this strategy in the proof below.

For Hilbert schemes \(A\) (resp. \(B\)) of points on \(\bullet = X\) (resp. \(D\)), we have morphisms

\[ f_{X} : A \rightarrow M_{n}(X, 0), \quad f_{D} : B \rightarrow M_{n}(D, 0), \quad Z \rightarrow O_{Z}, \]

to the moduli stacks \(M_{n}(\bullet, 0)\) of zero dimensional sheaves of length \(n\) on \(\bullet = X\) (resp. \(D\)).

There is a natural morphism

\[ i_{0} : M_{n}(D, 0) \rightarrow M_{n}(X, 0), \]

by pushforward zero dimensional sheaves on \(D\) to \(X\). Consider its derived enhancement

\[ i_{0}^{\text{der}} : M_{n}(D, 0)^{\text{der}} \rightarrow M_{n}(X, 0)^{\text{der}}, \]

as in Toën-Vaquie [TVa]. This induces a map

\[ (i_{0}^{\text{der}})^{*}L_{M_{n}(X, 0)^{\text{der}}} \rightarrow L_{M_{n}(D, 0)^{\text{der}}} \]

of their cotangent complexes in a pre-triangulated dg-category \(L_{\text{qcoh}}(M_{n}(D, 0)^{\text{der}})\) introduced by Toën [To05] §3.1.7, §4.2.4 (we refer to [Lurie] §1 & Rmk. 1.1.0.2] on the closely related notion of stable \(\infty\)-categories, [Lurie] §7.3, [To14] §4.1, [TVa] Cor. 2.2.3.3] for references on

\[ a \leq b, \]

\[ A \]

A perfect complex has tor-amplitude in \([a, b]\) means it is locally quasi-isomorphic to a complex of locally free sheaves in degree \([a, b]\).
cotangent complexes, and \textbf{[BK, Kc, To08]} for references on dg-categories). Pullback to the classical truncation, we get a commutative diagram

\[
\begin{array}{ccc}
(i_0^! \iota^! \tau_0^\bullet) \Rightarrow & \Rightarrow & \Rightarrow \\
\mathcal{M}_{n,(x,0)} & \mathcal{M}_{n,(x,0)} & \mathcal{M}_{n,(x,0)} \\
\mathcal{L}_{\mathcal{M}_{n,(x,0)}} & \mathcal{L}_{\mathcal{M}_{n,(x,0)}} & \mathcal{L}_{\mathcal{M}_{n,(x,0)}} \\
\end{array}
\]

in a pre-triangulated dg-category of quasi-coherent sheaves on \( M_n(D,0) \).

It is well-known (e.g. \textbf{STV} Prop. 1.2 & 5.2) that the above vertical arrows give obstruction theories for moduli stacks \( M_n(\bullet,0) \) (equivalent to the way defined using Atiyah classes \textbf{[STV, Appendix A]})) and moreover we have

\[
\mathbb{L}_{M_n(\bullet,0)} \cong (\mathcal{R}\text{Hom}_{\mathcal{M}_n(\bullet,0)}(\mathbb{F}_n, \mathbb{F}_n)[1])^\vee, \quad \bullet = X, D,
\]

where \( \pi_{M_n(\bullet,0)} : \bullet \times M_n(\bullet,0) \rightarrow M_n(\bullet,0) \) is projection map and \( \mathbb{F}_n \) is the universal sheaf.

By base change via the commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{i} & A \\
\downarrow f_D & & \downarrow f_x \\
M_n(D,0) & \xrightarrow{i_0} & M_n(X,0),
\end{array}
\]

we obtain a commutative diagram

\[
\begin{array}{ccc}
(i^* \mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1])^\vee & \Rightarrow & \Rightarrow \\
& \mathcal{L}_{\mathcal{A}} & \mathcal{L}_{\mathcal{B}},
\end{array}
\]

in a pre-triangulated dg-category of quasi-coherent sheaves on \( B \). Here through the canonical isomorphisms

\[
\tau^{[0,2]}(\mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1]) \cong \tau^{[0,2]}(\mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{I}_{Z_X}, \mathcal{I}_{Z_X})[0][1]),
\]

\[
\tau^{[0,1]}(\mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{O}_{Z_D}, \mathcal{O}_{Z_D})[1]) \cong \tau^{[0,1]}(\mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{I}_{Z_D}, \mathcal{I}_{Z_D})[0][1]),
\]

(see e.g. \textbf{[CK18, Lem. 2.6]}), we recover trace-free obstruction theories for \( A \) and \( B \) \textbf{[MNOP, HT]} by the naturality of Atiyah classes \textbf{[BF] Cor. 3.15].}

Taking cone (cofiber) and use Lemma \textbf{[21]} we obtain a commutative diagram in dg-category:

\[
\begin{array}{ccc}
(i^* \mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1])^\vee & \Rightarrow & \Rightarrow \\
& \mathcal{L}_{\mathcal{A}} & \mathcal{L}_{\mathcal{B}},
\end{array}
\]

As cotangent complexes are concentrated on nonpositive degrees, we can truncate positive degree terms in the left upper arrow and the diagram remains commutative (see \textbf{[Lurie §1.2.1, To05 §4.2.4]} for extension of t-structure and truncation to dg/-stable \( \infty \)-categories). By taking further truncation to degree \( \geq -1 \) terms, we obtain commutative diagrams in dg-category:

\[
\begin{array}{ccc}
(\tau^{[0,1]}(i^* \mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1]))^\vee & \Rightarrow & \Rightarrow \\
& \tau^{>-1} \mathcal{L}_A & \tau^{>-1} \mathcal{L}_B,
\end{array}
\]

\[
\begin{array}{ccc}
(\tau^{[0,1]}(\mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{O}_{Z_D}, \mathcal{O}_{Z_D})[1]))^\vee & \Rightarrow & \Rightarrow \\
& \tau^{>-1} \mathcal{L}_B & \tau^{>-1} \mathcal{L}_{B/A},
\end{array}
\]

\[
\begin{array}{ccc}
(\tau^{[0,1]}(\mathcal{R}\text{Hom}_{\mathcal{M}_A}(\mathcal{O}_{Z_D}, \mathcal{O}_{Z_D})[1]))^\vee & \Rightarrow & \Rightarrow \\
& \tau^{>-1} \mathcal{L}_B & \tau^{>-1} \mathcal{L}_{B/A},
\end{array}
\]
Note that all upper terms are concentrated in degree $[-1, 0]$, the right upper term in the second square is $E^\vee|_B[1]$, where $E$ is the tautological bundle. So we can extend the diagrams to

\[
\tau^{[0,1]} (i^* \mathcal{R} \text{Hom}_{\pi_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1])^\vee \xrightarrow{\lambda_1} (\tau^{[0,1]} (\mathcal{R} \text{Hom}_{\pi_B}(\mathcal{O}_{Z_D}, \mathcal{O}_{Z_D})[1]))^\vee
\]

\[
i^* L_A \rightarrow L_B.
\]

\[
(\tau^{[0,1]} (\mathcal{R} \text{Hom}_{\pi_B}(\mathcal{O}_{Z_D}, \mathcal{O}_{Z_D})[1]))^\vee \xrightarrow{\lambda_2} E^\vee|_B[1]
\]

\[
L_B \rightarrow L_B/A.
\]

By taking cone of the second square, we get a commutative diagram in dg-category:

\[
W \rightarrow (\tau^{[0,1]} (\mathcal{R} \text{Hom}_{\pi_B}(\mathcal{O}_{Z_D}, \mathcal{O}_{Z_D})[1]))^\vee \rightarrow E^\vee|_B[1]
\]

\[
i^* L_A \rightarrow L_B \rightarrow L_B/A.
\]

As $\lambda_2 \circ \lambda_1 = 0$, we have a commutative diagram

\[
(\tau^{[0,1]} (i^* \mathcal{R} \text{Hom}_{\pi_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1]))^\vee \rightarrow W
\]

\[
i^* L_A \rightarrow L_B \rightarrow L_B/A.
\]

by the functoriality of cone construction in dg-categories (e.g. [To08 §5.1]). By our construction, $\bar{h}^0(\bullet)$ is an isomorphism and $h^{-1}(\bullet)$ is surjective for $\bullet = \phi_B$ and $\phi'$. From a diagram chasing, $\phi$ also satisfies this property. $\square$

The complex $W$ constructed in Lemma 2.3 is a “maximal isotropic subcomplex” of the standard sheaf deformation-obstruction complex in the following sense:

**Lemma 2.4.** Let $E := (\tau^{[0,2]} (i^* \mathcal{R} \text{Hom}_{\pi_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1]))^\vee$ and extend map $\mu$ in Lemma 2.3 to $\bar{\mu} : E \rightarrow W$,

by composition with the obvious truncation functor. Under the Grothendieck-Verdier duality

\[E \cong E^\vee[2],\]

we have a distinguished triangle

\[W^\vee[2] \xrightarrow{\bar{\mu}^\vee[2]} E^\vee[2] \cong E \xrightarrow{\bar{h}} W.\]

**Proof.** We denote

\[V_A := i^* \mathcal{R} \text{Hom}_{\pi_A}(\mathcal{O}_{Z_X}, \mathcal{O}_{Z_X})[1], \quad V_B := \mathcal{R} \text{Hom}_{\pi_B}(\mathcal{O}_{Z_D}, \mathcal{O}_{Z_D})[1].\]

In view of Corollary 2.2, we can truncate the distinguished triangle and obtain a distinguished triangle:

\[
(\tau^{[0,2]} V_B)[2] \xrightarrow{\omega^{[2]}} (\tau^{[0,2]} V_A) \rightarrow (\tau^{[0,2]} V_B)^\vee,
\]
For example, \( f \) is defined as the following composition

\[
\begin{array}{ccc}
\tau^{[0,1]} V_B & \xrightarrow{\eta_1} & (\tau^{[0,1]} V_A)[2] \\
\downarrow & & \downarrow \\
(\tau^{[0,2]} V_B)[2] & \xrightarrow{\omega^{[2]}} & (\tau^{[0,2]} V_A)[2] \cong (\tau^{[0,2]} V_A) \cong (\tau^{[0,2]} V_B)^{\vee} \\
\downarrow & & \downarrow \\
(\tau^{[0,1]} V_A)^{\vee} & \xrightarrow{\omega} & (\tau^{[0,1]} V_B)^{\vee} \\
\end{array}
\]

The middle vertical line determines a map

\[
W^{[2]} \xrightarrow{\h^{[2]}} (\tau^{[0,2]} V_A)^{\vee} \xrightarrow{\beta} W.
\]

We are left to check it is a distinguished triangle. Note that \( W \) (resp. \( W^{[2]} \)) is concentrated at degree \([-1,0]\) (resp. \([-2,-1]\)). By Lemma 2.3 \( h^{-1}(\eta_1) \) is surjective and \( h^{-1}(\zeta_1) \) is injective. By a diagram chasing and \((\omega^{[2]}) \circ \omega = 0\), we can conclude \((\mu^{[2]}) \circ \mu = 0\). Hence we have a commutative diagram:

\[
\begin{array}{ccc}
W^{[2]} & \xrightarrow{\exists} & (\tau^{[0,2]} V_A)^{\vee} \\
\downarrow & & \downarrow \\
\exists & \xrightarrow{\cong} & \exists \cong \\
\end{array}
\]

By Lemma 2.3 and the above commutativity, \( h^{-1}(\epsilon) \), \( h^0(\epsilon) \) are isomorphisms. Therefore \( \epsilon \) is an isomorphism.

2.2. Preparation on virtual pullback. If the morphism \( \phi \) in Lemma 2.3 extends to a perfect obstruction theory on \( A \) (which is of course not true), our virtual pullback formula (i.e. Proposition 1.6) will follow from Lemma 2.4 and the functoriality of virtual pullback (see Man, Cor. 4.9). In our case, in order to apply virtual pullback technique to prove Proposition 1.6, we need to prepare some basics.

Consider a Cartesian square of Artin stacks of finite type over \( C \)

\[
\begin{array}{ccc}
W & \xrightarrow{u} & Y \\
\downarrow \quad v & & \downarrow g \\
X & \xrightarrow{f} & Z,
\end{array}
\]

where \( f \) and \( v \) are Deligne-Mumford. Let \( E_f \) (resp. \( E_v \)) be perfect complexes on \( X \) (resp. \( W \)) with tor-amplitude \([-1,0]\). Assume we have closed embeddings of cones stacks

\[
C_f \hookrightarrow h^i/h^0(E_f^+) \quad \text{and} \quad C_v \hookrightarrow u^* C_g \hookrightarrow h^i/h^0(E_v^+),
\]

where \( C_s \) are normal cones of maps as defined in Kresch, which are isomorphic to the intrinsic normal cones of those maps Man Prop. 2.33.

Then we can construct a closed embedding:

\[
C_{f \circ v} \hookrightarrow v^* C_f \times_W u^* C_g \hookrightarrow v^* \left( h^1/h^0(E_f^+) \right) \times_W h^1/h^0(E_v^+).
\]

From (13), (14), one can define virtual pullback:

\[
f^! : A_*(Z) \to A_{*+\text{rk}(E_f)}(X), \quad v^! : A_*(X) \to A_{*+\text{rk}(E_v)}(W),
\]

\[
(f \circ v)^! : A_*(Z) \to A_{*+\text{rk}(E_f)+\text{rk}(E_v)}(W).
\]

For example, \( f^! \) is defined as the following composition

\[
f^! : A_*(Z) \xrightarrow{\text{sp}} A_*(C_f) \xrightarrow{\text{incl}} A_*(h^1/h^0(E_f^+)) \xrightarrow{\eta_1^!} A_{*+\text{rk}(E_f)}(X),
\]
where \( sp \) is the specialization map (see e.g. [Man, Const. 3.6]), \( incl \) is the inclusion map and \( 0^! \) is the Gysin homomorphism [Kre]. Other maps \( v^! \), \( (f \circ v)^! \) can be defined similarly. The following functoriality property holds.

**Lemma 2.5.** In the above setting, we have
\[
(f \circ v)^! = v^! \circ f^!.
\]

**Proof.** From [13], [14], we have a compatible triple:
\[
\begin{array}{ccc}
C_v & \longrightarrow & C_{fsv} \\
\downarrow & & \downarrow \\
u^*C_g & \longrightarrow & u^*C_g \times_W v^*C_f \\
\downarrow & & \downarrow \\
h^1/h^0(E'_v) & \longrightarrow & h^1/h^0(E'_v \oplus v^*E'_f) \longrightarrow h^1/h^0(v^*E'_f),
\end{array}
\]
where vertical lines are closed embedding and the bottom horizontal line is a short exact sequence of cone stacks. This is equivalent to the compatibility of perfect obstruction theories on \( v, f \) and \( f \circ v \) (e.g. [BF97] Thm. 4.5]). Therefore we can apply [Man, Thm. 4.8] to our setting. \( \square \)

### 2.3. Finish of proof.
In this section, we finish the proof of Proposition 1.6. As discussed in Section 1.2, combining with (4), this will lead to a proof of Theorem 1.2.

**Step 0: pullback to \( B \).** We first recall the construction of virtual classes in [OT]. Let
\[
(\tau^{[0,2]}(R\text{Hom}_{Ax}(O_{\mathcal{Z}_X}, O_{\mathcal{Z}_X}[1])))^{\vee} \rightarrow \tau \geq -1 \mathcal{L}_A
\]
be the obstruction theory of \( A \) given by the truncated Atiyah class. By [OT] Prop. 4.1, it can be represented by a 3-term complex of vector bundles:
\[
(\tau^{[0,2]}(R\text{Hom}_{Ax}(O_{\mathcal{Z}_X}, O_{\mathcal{Z}_X}[1])))^{\vee} \cong (T^{\rho^\vee} V \xrightarrow{\Delta^\vee} T^{\vee}),
\]
such that \( V \) is equipped with a nondegenerate quadratic form \( Q \), and compatibility with Grothendieck-Verdier duality holds. The stupid truncation
\[
(V \xrightarrow{\rho^\vee} T^{\vee})
\]
gives a Behrend-Fantechi perfect obstruction theory [BF97], so the intrinsic normal cone satisfies
\[
\mathcal{E}_A \subset [V^{\vee}/T].
\]
Pullback to \( V^{\vee} \cong V \), one obtains an isotropic cone
\[
C_V \subset V, \quad \text{s.t.} \quad Q|_{C_V} = 0.
\]
Then
\[
[A]^{vir} := \sqrt{[0^!][C_V]} \in A_n(A, \mathbb{Q}),
\]
is given by the square root Gysin pullback ([OT] Definition 3.3)) of the zero section \( 0_V : A \rightarrow V \). A choice of orientation as [CGJ, CL17] is required in the definition of square root Gysin pullback.

We remark that the square root Gysin pullback commutes with pullback \([3]\). To be precise, given any map \( g : C \rightarrow A \), consider the Cartesian square
\[
\begin{array}{ccc}
C' & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & A,
\end{array}
\]
and embeddings of normal cones
\[
C_{C'/C} \hookrightarrow C' \times_B (C_{B/A}) \hookrightarrow h^*(E_{B}),
\]
where \( E \) is the tautological bundle as in \([3]\). We can refine \([3]\) to be:
\[
i^! : A_n(C) \xrightarrow{sp} A_n(C_{C'/C}) \xrightarrow{incl} A_n(h^*(E_{B})) \xrightarrow{0^!} A_{n-\lambda}(C'),
\]
which recovers the pullback \([3]\) when \( g \) is the identity map. This is the same as refined Gysin pullback [Ful] [6.2] for the regular embedding \( 0_E : A \rightarrow E \), so we have:
Lemma 2.6. ([OT, Lem. 3.9]) For an isotropic cone $C$ in the quadratic bundle $V$, we have

$$i^*\sqrt{0_V} [C] = \sqrt{0_V} \circ i^*[C].$$

**Step 1: constructing maximal isotropic subbundle.** When pulled back to $B$, we can construct a maximal isotropic subbundle of $V$ from the “maximal isotropic subcomplex” in Lemma 2.4.

**Lemma 2.7.** (A) Given a self-dual resolution

$$i^* \left( \tau_{[0,2]} \left( \mathbb{R}Hom_{\mathcal{A}}(\mathcal{O}_{Z,X}, \mathcal{O}_{Z,X})[1] \right) \right)^{\vee} \cong (i^* T^\vee \xrightarrow{\rho} i^* V \xrightarrow{\beta} i^* T^\vee)$$

as in (17), then the map $\bar{\mu} : \mathcal{E} \to \mathcal{W}$ in Lemma 2.6 induces a maximal isotropic subbundle $V_{\text{iso}} \subset i^* V$.

(B) Moreover, the pullback cone $i^* C_V$ of (19) sits inside $V_{\text{iso}} \subset i^* V$.

**Proof.** (A) First note that we have an isomorphism in the derived category:

$$i^* \left( \tau_{[0,2]} \left( \mathbb{R}Hom_{\mathcal{A}}(\mathcal{O}_{Z,X}, \mathcal{O}_{Z,X})[1] \right) \right) \cong \tau_{[0,2]} \left( i^* \mathbb{R}Hom_{\mathcal{A}}(\mathcal{O}_{Z,X}, \mathcal{O}_{Z,X})[1] \right).$$

From Lemma 2.4 there is a distinguished triangle

$$(23) \quad \mathcal{W}^\vee [2] \xrightarrow{\beta^{[2]} \vee} \mathcal{E}^\vee [2] \cong \mathcal{E} \xrightarrow{\beta} \mathcal{W}.$$  

By taking a locally free resolution $P_\bullet$ of $\mathcal{W}^\vee [2]$, we can represent $\bar{\mu}^\vee [2]$ by

$$\xymatrix{ & \ldots & P_{-3} \ar[r]^{d_{-3}} & P_{-2} \ar[r]^{d_{-2}} & P_{-1} \ar[r]^{d_{-1}} & P_0 \ar[r]^{d_0} & \ldots \ar[r]^\rho & i^* T \ar[r]^\rho & i^* V \ar[r]^\rho & i^* T^\vee \ar[r]^\rho & 0.}$$

As $\mathcal{W}^\vee [2]$ has tor-amplitude in $[-2,-1]$, we can truncate $P_\bullet$ via replacing $P_{-4}$ and $P_{-3}$ by $P_{-3}/(\text{Ker}(d_{-3}))$ and $\text{Ker}(d_{-1})$. Here $P_{-3}/(\text{Ker}(d_{-3}))$ and $\text{Im}(d_{-1})$ are locally free by [Lie, Lem. 2.1.3], $\text{Ker}(d_{-1})$ is locally free as it is the kernel of a surjection between locally free sheaves.

By taking dual and a degree shift, we have the following map between complexes:

$$\xymatrix{ 0 \ar[r]^{i^* T} & i^* V \ar[r]^\rho & i^* T^\vee \ar[r]^\rho & \ldots \ar[r] & 0 \ar[r]^{(\text{Ker}(d_{-1}))^\vee} & (P_{-2})^\vee \ar[r]^{\text{surj}} & (P_{-3}/(\text{Ker}(d_{-3})))^\vee.}$$

By the distinguished triangle (23), the bottom line is quasi-isomorphic to $\mathcal{W}$, which has tor-amplitude in $[-1,0]$. So we can truncate the bottom line into the following form:

$$\xymatrix{ i^* T \ar[r]^\rho & i^* V \ar[r]^\rho & i^* T^\vee \ar[r]^{\rho} & \ldots \ar[r]^{g_{-1}} & W_{-1} \ar[r]^{w} & W_0.}$$

where $W_\bullet$ is a two term complex of locally free sheaves and $W_\bullet \cong \mathcal{W}$ in the derived category.

By Lemma 2.3 $h^0(\bar{\mu})$ is an isomorphism and $h^{-1}(\bar{\mu})$ is surjective. So the cone of $\mu$, represented by the complex

$$i^* T \xrightarrow{-\rho} i^* V \xrightarrow{-\rho g_{-1}} (i^* T^\vee \oplus W_{-1}) \xrightarrow{g_{0} \oplus w} W_0 \to 0,$$

is exact in the third and fourth terms. Define

$$V_{\text{iso}}^\vee := \text{Ker} \left( (i^* T^\vee \oplus W_{-1}) \xrightarrow{(g_{0} \oplus w)} W_0 \right) = \text{Im}((-\rho, g_{-1})).$$
to be the kernel of the surjection, which is a vector bundle fitting into a commutative diagram

\[
\begin{array}{ccc}
    i^*T & \overset{\rho}{\longrightarrow} & i^*V \\
    \downarrow \rho' & & \downarrow \rho \\
    V_{iso} & \overset{\rho_0}{\longrightarrow} & i^*V^{'V} \\
    \downarrow \rho_1 & & \downarrow g_0 \\
    W_{-1} & \overset{w}{\longrightarrow} & W_0.
\end{array}
\]

Since \((-\rho, g_{-1})\) is surjective onto \(V_{iso}^{'V}\), taking dual determines a subbundle

\[V_{iso} \subset i^*V^{'V} \cong i^*V.\]

It is straightforward to check \((p_2, g_0)\) gives a quasi-isomorphism. By Lemma \ref{lem:iso} it is easy to see that \(V_{iso}\) is maximal isotropic.

(B) By the second diagram in Lemma \ref{lem:iso} Lemma \ref{lem:iso} and the isomorphism \ref{lem:iso}, pullback of the intrinsic normal cone \ref{lem:iso} satisfies

\[i^*C_A \subset [V_{iso}/i^*T] \hookrightarrow [i^*V/i^*T].\]

This is consistent with the embedding \ref{lem:iso} determined by \ref{lem:iso} after pullback, as both the map \(\phi'\) in Lemma \ref{lem:iso} and \ref{lem:iso} come from restriction/truncation of the Atiyah class of \(\mathcal{O}_{\mathbb{P}X}.\)

From the Cartesian diagram

\[
\begin{array}{ccc}
    V_{iso} & \subset & i^*V \\
    \downarrow g & & \downarrow \phi \\
    [V_{iso}/i^*T] & \longrightarrow & [i^*V/i^*T],
\end{array}
\]

the pullback cone \ref{lem:iso} satisfies \(i^*C_V \subset V_{iso} \subset i^*V.\)  

From Lemma \ref{lem:iso} the square root Gysin pullback becomes the familiar refined Gysin homomorphism (when restricted to \(B\)).

**Proposition 2.8.** The pullback \ref{lem:iso} of the virtual class \ref{lem:iso} satisfies

\[i^! [A]^{\text{vir}} = 0_{V_{iso}} [i^![C_V]] \in A_0(B, \mathbb{Q}),\]

for certain choice of orientation. Here \(0_{V_{iso}}\) is the refined Gysin homomorphism \[\text{Pf}\] Sect. 6.1 of the zero section \(0_{V_{iso}} : B \rightarrow V_{iso}\).

**Proof.** As \(D\) is connected, \(B = \text{Hilb}^n(D)\) is also connected \[\text{Pf}\ Prop. 2.3.\ We choose the orientation on \(A\) which gives the orientation determined by the maximal isotropic bundle \(V_{iso} \subset i^*V\) when pullback to \(B\). Then we combine Lemma \ref{lem:iso} Lemma \ref{lem:iso} and \[\text{OT}\] Lem. 3.5. \(\square\)

**Remark 2.9.** When \(D\) (equivalently \(B\)) has several connected components, it is not clear to us whether an orientation on \(A\) (which is connected) will pull back to the orientation determined by the maximal isotropic bundle \(V_{iso} \subset i^*V\) for all connected components of \(B\). In view of Remark \ref{lem:iso}, this will follow from compatibility of orientations in wall-crossing argument \[\text{Bo}\].

**Step 2: applying virtual pullback.**

**Proposition 2.10.** We have

\[ [B]^{\text{vir}} = 0_{V_{iso}} i^! [C_V] \in A_0(B, \mathbb{Q}),\]

where \([B]^{\text{vir}}\) is the virtual class defined in \[\text{MNOP} \text{Thm.}\]

**Proof.** Let \(\mathcal{E}_D := \{\tau^{[0,1]}(R\text{Hom}_{\mathbb{P}X}(\mathcal{O}_{\mathbb{P}X}, \mathcal{O}_{\mathbb{P}X})[1])\}^{'V}.\) From Lemma \ref{lem:iso} there is a compatible diagram of perfect obstruction theories \[\text{Bo}\] (p.o.t. for short) when restricted to \(B:\)

\[
\begin{array}{ccc}
    W & \longrightarrow & \mathcal{E}_D \\
    \phi & \phi & \phi \\
    i^*L_A & \longrightarrow & L_B \longrightarrow L_{B/A}.
\end{array}
\]

\(\text{For \(\phi\), more precisely, we mean } h^0(\phi) \text{ is an isomorphism and } h^{-1}(\phi) \text{ is surjective.}\)
The composition $B \xrightarrow{i} A \xrightarrow{s} \text{Spec} \mathbb{C}$ (where $c$ is the constant map) is endowed with the above p.o.t. $(E_D, \phi_B)$, so

\begin{equation}
C_B \hookrightarrow h^1/h^0(E_D^\vee).
\end{equation}

The composition $B \xrightarrow{i} A \xrightarrow{s} \mathcal{E}_A$ can be given a p.o.t. by the following embedding of cone stacks:

\begin{equation}
C_{\text{soi}} \hookrightarrow i^*C_s \times_B C_i \hookrightarrow h^1/h^0(W^\vee) \times_B h^1/h^0(i^*E[-1]).
\end{equation}

Here $s : A \to \mathcal{E}_A$ is the zero section of the intrinsic normal cone and $C_i, C_s, C_{\text{soi}}$ are normal cones of those maps. By [Qu] Lem. 2.8, we have

\begin{equation}
C_s \cong \mathcal{E}_A,
\end{equation}

so $i^*C_s \hookrightarrow h^1/h^0(W^\vee)$ is given by $\phi : W \to i^*\mathbb{L}_A$.

The embeddings (24), (25) of cone stacks allow us to define corresponding virtual pullbacks as in [KKP]. We relate these pullbacks by a deformation argument. Let $M_0^C$ be the deformation space of constant map $c : A \to \text{Spec} \mathbb{C}$, which is a flat family over $\mathbb{P}^1$ whose fiber over $\{\infty\}$ is the normal cone $C_{\infty} = \mathcal{E}_A$, and over $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$, it is isomorphic to the product $\text{Spec} \mathbb{C} \times \mathbb{A}^1$.

For closed immersions between schemes, such deformation spaces are constructed by [Ful] Sect. 5.1. In general, they are constructed by descent in [KKP, Krec].

Consider the map $U : B \times \mathbb{P}^1 \to M_0^C$ and the following Cartesian diagram as in [KKP] (see also [Qu] Prop. 2.11 & Rmk. 2.12):

\begin{equation}
\begin{array}{c}
B \times \{\infty\} \\
\downarrow \text{soi} \\
C_c = \mathcal{E}_A \\
\downarrow \{\infty\} \\
\mathbb{P}^1 \\
\downarrow \text{coi} \\
\text{Spec} \mathbb{C}
\end{array}
\quad
\begin{array}{c}
B \times \mathbb{P}^1 \\
\downarrow \text{at} \\
M_0^C \\
\downarrow \pi \\
\text{Spec} \mathbb{C}
\end{array}
\quad
\begin{array}{c}
B \times \{0\} \\
\downarrow \text{coi} \\
\text{Spec} \mathbb{C}
\end{array}
\quad
\begin{array}{c}
\{0\} \\
\downarrow \pi' \\
\mathbb{P}^1 \\
\downarrow 0.
\end{array}
\end{equation}

By using [KKP] Proof of Thm. 1], there is a virtual pullback $\mathcal{U}'$ such that

\begin{equation}
\mathcal{U}' = (s \circ i)' = (c \circ i)'.
\end{equation}

Next we use functoriality of virtual pullbacks to rewrite $(s \circ i)'$. From the Cartesian diagram

\begin{equation}
\begin{array}{c}
B \xrightarrow{i} A \\
\downarrow i \\
i^*\mathcal{E}_A \xrightarrow{i} \mathcal{E}_A, \\
\downarrow \pi' \\
B \xrightarrow{i} A
\end{array}
\end{equation}

where $\pi$ is the projection map and $t$ is the zero section, using Lemma 2.15 we get

\begin{equation}
(s \circ i)' = (j \circ t)' = t' \circ i'.
\end{equation}

Here

\begin{equation}
i' : A_*(\mathcal{E}_A) \to A_{*+n}(i^*\mathcal{E}_A),
\end{equation}

is given by 21 and

\begin{equation}
t' : A_*(i^*\mathcal{E}_A) \to A_{*+\text{rk}(W^\vee)}(B),
\end{equation}

is defined using the embedding $C_t \hookrightarrow i^*C_s \hookrightarrow h^1/h^0(W^\vee)$.

Since $t$ is the zero section of $i^*\mathcal{E}_A$, as in (26), we use [Qu] Lem. 2.8 & Rmk. 2.9] and obtain a compatible diagram

\begin{equation}
\begin{array}{c}
C_t \xrightarrow{\cong} i^*C_s \\
\cong \quad \cong \\
i^*\mathcal{E}_A \xrightarrow{\cong} i^*\mathcal{E}_A.
\end{array}
\end{equation}

Therefore the specialization map $A_*(i^*\mathcal{E}_A) \to A_*(C_t)$ is the identity and

\begin{equation}
t' = 0_{h^1/h^0(W^\vee)}.
\end{equation}
From the Cartesian diagram

\[
\begin{array}{ccc}
  i^*C_V & \rightarrow & V_{\text{iso}} \\
p' & \downarrow & p \\
i^*\mathcal{C}_A & \rightarrow & h^1/h^0(W^\vee),
\end{array}
\]

where \( p \) is a smooth morphism, we get

\[0^!_{h^1/h^0(W^\vee)} = 0^!_{V_{\text{iso}} \circ p^*}.\]

This together with the Cartesian diagram

\[
\begin{array}{ccc}
i^*C_V & \rightarrow & C_V \\
p' & \downarrow & \pi' \\
i^*\mathcal{C}_A & \rightarrow & \mathcal{C}_A \\
& \downarrow & \pi \\
B & \rightarrow & A
\end{array}
\]

implies that

\[0^!_{h^1/h^0(W^\vee)} (i^![\mathcal{C}_A]) = 0^!_{V_{\text{iso}}} (p^* (i^![\mathcal{C}_A])) = 0^!_{V_{\text{iso}}} i^![C_V].\] (30)

Combining (27), (28), (29), (30), we obtain

\[\int_{[M_1(X,\beta)]_{\text{vir}}} \epsilon(L^{[1]}) \in \mathbb{Z},\]

Finally, by combining Proposition 2.8 and Proposition 2.10, we obtain a proof of Proposition 1.6. This together with MNOP formula (4) leads to a proof of Theorem 1.2.

\section{Application to other tautological invariants}

In this section, we use the same proof strategy above to give an application to the tautological invariants of one dimensional stable sheaves on Calabi-Yau 4-folds.

Let \( M_1(X, \beta) \) be the moduli scheme of one dimensional stable sheaves \( F \) on \( X \) with \( [F] = \beta \in H_2(X, \mathbb{Z}) \) and \( \chi(F) = 1 \). It is a fine moduli space which is independent of the choice of polarization in defining the stability (e.g. [CMT18, Rmk. 2.1]). Such moduli spaces and their primary and descendent invariants are used in [CMT18, CT20a] to give sheaf theoretic interpretation of Gopakumar-Vafa type invariants of Calabi-Yau 4-folds introduced by Klemm-Pandharipande [KP]. Here we consider their tautological invariants, which are closely related to primary and descendent invariants (see Corollary 3.4).

Let \( F \rightarrow M_1(X, \beta) \times X \) be a universal sheaf of \( M_1(X, \beta) \), which is unique up to a twist of line bundle from \( M_1(X, \beta) \). There exists a distinguished choice \( F_{\text{norm}} \), called the “normalized” one [CT20a (1.7)] such that

\[\det(\pi_M^* F_{\text{norm}}) \cong \mathcal{O}_{M_1(X, \beta)},\]

where \( \pi_M : M_1(X, \beta) \times X \rightarrow M_1(X, \beta) \) is the projection.

For \( L \in \text{Pic}(X) \), we define its tautological complex

\[L^{[1]} := \pi_M^*(F_{\text{norm}} \boxtimes L) \in \text{Perf}(M_1(X, \beta)).\]

Using virtual class

\[[M_1(X, \beta)]_{\text{vir}} \in H_2(M_1(X, \beta), \mathbb{Z}),\]

we define the tautological invariant

\[n_{0, \beta}(X, L) := \int_{[M_1(X, \beta)]_{\text{vir}}} \epsilon(L^{[1]}) \in \mathbb{Z},\] (32)
which depends on the choice of orientation. By projection formula, if we use another universal sheaf $F$ to define tautological insertion, the invariant differs by an integration of the first Chern class of a line bundle on $M_1(X, \beta)$. Note also that 

$$\text{rk}(L^1) = 1 + L \cdot \beta,$$

so invariants \(32\) are nontrivial only if $L \cdot \beta = 0$.

Our invariants \(32\) are closely related to tautological stable pair invariants studied in \[CT20b\]. In view of \[CT20b, Conj. 0.1\], we consider the following geometric setting:

**Setting 3.1.** Let $X$ be a Calabi-Yau 4-fold with a projective surjective morphism

$$\pi: X \to B$$

to a variety $B$, and $D \hookrightarrow X$ be a smooth divisor of the form $D = \pi^* H$ for a Cartier divisor $H$ on $B$. Take $L = \mathcal{O}_X(D)$ to be the associated line bundle.

Consider a curve class $\beta \in H^2(D, \mathbb{Z})$ on fibers of $\pi$ (i.e. $\pi_* \beta = 0$). We assume under the natural embedding

$$i: M_1(D, \beta) \hookrightarrow M_1(X, \beta),$$

connected components of two moduli spaces are in 1-1 correspondence.

**Remark 3.2.** We restrict to this geometric setting so that genus zero Gopakumar-Vafa invariants of $D$ can be defined using sheaf theory \(33\). For most examples, curve classes on fibers of $\pi$ are the same as curve classes satisfying $L \cdot \beta = 0$ (which ensures degrees of insertions and virtual classes match).

The technical assumption on identification of connected components is to make the argument in Proposition \[2.8\] work (see the proof of Theorem \[3.3\] below). Hopefully one will get rid of it by compatibility of orientations in wall-crossing argument (see Remark \[2.9\]).

In Setting \[3.1\] we have $\pi: D \to H$ and define genus zero Gopakumar-Vafa invariants of $D$:

$$n_{0, \beta}(D) := \int_{\overline{[M_1(D, \beta)]}^{vir}} 1 \in \mathbb{Z}, \text{ if } \pi_* \beta = 0,$$

where the virtual class is constructed in \[CT20b, Prop. 1.5\] following the standard way \[Katz\]. This definition recovers Katz’s definition \[Katz\] when $D$ is a Calabi-Yau 3-fold, where one takes $H = \text{Spec}(\mathbb{C})$ in $\pi: D \to H$.

**Theorem 3.3.** In Setting \[3.1\] for certain choice of orientation, we have

$$n_{0, \beta}(X, L) = n_{0, \beta}(D).$$

**Proof.** This proof follows exactly the same proof strategy as above, so we only give a sketch and leave readers to check details.

By \[CMT18, Lem. 2.2\], any $[F] \in M_1(X, \beta)$ (or $M_1(D, \beta)$) is scheme theoretically supported on a fiber of $\pi$. Therefore we have a Cartesian square

$$\begin{array}{ccc}
M_1(D, \beta) & \xrightarrow{i} & M_1(X, \beta) \\
\pi \downarrow & & \pi \downarrow \\
H & \xrightarrow{\bar{s}} & B.
\end{array}$$

So there is a section $s$ of line bundle $\bar{s}^* \mathcal{O}_B(H)$ on $M_1(X, \beta)$ and an isomorphism of schemes:

$$s^{-1}(0) \cong M_1(D, \beta) \xrightarrow{i} M_1(X, \beta).$$

Therefore we are in the same situation as \[35\]. By repeating the argument in Lemma \[2.1\] we obtain a similar distinguished triangle, which at a closed point $[F] \in M_1(D, \beta) \hookrightarrow M_1(X, \beta)$ is:

$$\text{RHom}_D(F, F) \to \text{RHom}_X(F, F) \to \text{RHom}_D(F, F \otimes K_D[-1]).$$

\footnote{Here we can regard $\beta$ as an element in $H_2(X, \mathbb{Z})$ by the obvious pushforward map.}
Note that $\text{Hom}_D(F, F \otimes K_D)$’s form a line bundle on $M_1(D, \beta)$, which is $i^*\pi^*\mathcal{O}_B(H)$. To see this, consider the commutative diagram

\[
\begin{array}{ccc}
M_1(X, \beta) \times X & \xrightarrow{j} & M_1(X, \beta) \times_B X \\
\downarrow{\pi_M} & & \downarrow{\pi_X} \\
M_1(X, \beta) & \xrightarrow{\pi} & B,
\end{array}
\]

from (34), we know a universal sheaf is pushforward from some $\mathcal{F}$, so $\text{Hom}_D(i^!F, F \otimes \pi_X^*\mathcal{O}_X(D))$ is connected on each component of $\mathcal{M}$.

Then by considering the derived enhancement of $\iota : M_1(D, \beta) \to M_1(X, \beta)$, taking truncations and cones, we can obtain similar Lemma 2.3 and Lemma 2.4 for this case.

To make arguments in Section 2.4 work, we note that although $M_1(X, \beta)$ may not be connected, we can consider the argument on its connected components. In Setting 3.1 the subscheme $M_1(D, \beta)$ is also connected on each component of $M_1(X, \beta)$, so Proposition 2.8 works and the rest arguments extend. Therefore we can conclude the virtual pullback formula

\[
i^!M_1(X, \beta) = [M_1(D, \beta)]^{\text{vir}},
\]

for certain choice of orientation, where $i^!$ is defined similarly as (5) using (33).

By (9), we obtain

\[
\int_{[M_1(X, \beta)]^{\text{vir}}} e(\pi^*\mathcal{O}_B(H)) = \int_{[M_1(D, \beta)]^{\text{vir}}} 1.
\]

Finally in (36), using the normalized universal sheaf (31), we have

\[
det(\pi_M^*(j_*F_{\text{norm}} \otimes \pi_X^*\pi^*\mathcal{O}_B(H))) \cong det(\pi_M^*(j_*F_{\text{norm}} \otimes j^!\pi_X^*\pi^*\mathcal{O}_B(H)))
\]

\[
\cong det(\pi_M^*(F_{\text{norm}} \otimes \pi^*\mathcal{O}_B(H)))
\]

\[
\cong det(\pi_M^*(F_{\text{norm}}) \otimes \pi^*\mathcal{O}_B(H))
\]

\[
\cong \pi^*\mathcal{O}_B(H).
\]

Therefore we obtain

\[
n_{0,\beta}(X, L) = \int_{[M_1(X, \beta)]^{\text{vir}}} c_1(L_{[1]})
\]

\[
= \int_{[M_1(X, \beta)]^{\text{vir}}} c_1(\det(\pi_M^*(F_{\text{norm}} \otimes \pi_X^*\mathcal{O}_B(H))))
\]

\[
= \int_{[M_1(X, \beta)]^{\text{vir}}} c_1(\pi^*\mathcal{O}_B(H)).
\]

Combining with (37), we are done. \hfill \Box

Tautological invariants discussed above are related to primary and descendendent invariants in \cite{CMT18, CT20a}. Recall that one can define insertions (using normalized universal sheaf $F_{\text{norm}}$):

\[
\tau_i : H^{4-2i}(X, \mathbb{Z}) \to H^2(M_1(X, \beta), \mathbb{Q}),
\]

\[
\tau_i(\bullet) := (\pi_M)_* (\pi_X^*\bullet) \cup \chi_{3+i}(F_{\text{norm}}) \in H^2(M_1(X, \beta), \mathbb{Q}),
\]

and their integration on the virtual class

\[
\langle \tau_i(\bullet) \rangle_{\beta} := \int_{[M_1(X, \beta)]^{\text{vir}}} \tau_i(\bullet) \in \mathbb{Q},
\]

which gives primary (resp. descendendent) invariants when $i = 0$ (resp. $i > 0$).

**Corollary 3.4.** In the setting of Theorem 3.3, we have

\[
\frac{1}{2} \langle \tau_0(c_1^2(L)) \rangle_{\beta} + \frac{1}{12} \langle \tau_0(c_2(X)) \rangle_{\beta} + \langle \tau_1(c_1(L)) \rangle_{\beta} + \langle \tau_2(1) \rangle_{\beta} = n_{0,\beta}(D).
\]
Proof. By the Grothendieck-Riemann-Roch formula, we have
\[
n_{0,\beta}(X, L) = \int_{[M_1(X, \beta)]^{vir}} \chi(L^{[1]}) = \int_{[M_1(X, \beta)]^{vir}} \pi_{M^*} (\chi(\mathbb{F}^{\text{norm}}) \cdot \pi_X (\chi(L) \cdot \text{td}(X)))
\]
\[
= \int_{[M_1(X, \beta)]^{vir}} \pi_{M^*} \left( \chi_3(\mathbb{F}^{\text{norm}}) \cdot \pi_X \left( \frac{1}{2} c_1^2(L) + \frac{1}{12} c_2(X) \right) \right)
\]
\[
+ \int_{[M_1(X, \beta)]^{vir}} \pi_{M^*} (\chi_4(\mathbb{F}^{\text{norm}}) \cdot \pi_X (c_1(L))) + \int_{[M_1(X, \beta)]^{vir}} \pi_{M^*} (\chi_5(\mathbb{F}^{\text{norm}}))
\]
\[
= \frac{1}{2} \langle \tau_0(c_1^2(L)) \rangle_\beta + \frac{1}{12} \langle \tau_0(c_2(X)) \rangle_\beta + \langle \tau_1(c_1(L)) \rangle_\beta + \langle \tau_2(1) \rangle_\beta.
\]
Combining with Theorem 3.3, we are done. \qed

Remark 3.5. Based on [CMT18, CT20a Conj. 0.2 & §1.7], conjecturally we have
\[
\langle \tau_0(\gamma) \rangle_\beta = n_{0,\beta}(\gamma), \quad \forall \gamma \in H^4(X, \mathbb{Z}), \quad \langle \tau_2(1) \rangle_\beta = -\frac{1}{12} \langle \tau_0(c_2(X)) \rangle_\beta,
\]
\[
\langle \tau_1(\alpha) \rangle_\beta = \frac{n_{0,\beta}(\alpha^2)}{2 (\alpha \cdot \beta)} - \sum_{\beta_1 + \beta_2 = \beta} \frac{(\alpha \cdot \beta_1)(\alpha \cdot \beta_2)}{4 (\alpha \cdot \beta)} m_{\beta_1, \beta_2} - \sum_{k \geq 2, \beta} \frac{(\alpha \cdot \beta)}{k} n_{1, \beta/k}, \quad \forall \alpha \in H^2(X, \mathbb{Z}).
\]
Here $n_{0,\beta}(\gamma)$, $n_{1,\beta}$ are all genus Gopakumar-Vafa (GV) type invariants of $X$, $m_{\beta_1, \beta_2}$ are meeting invariants which can be inductively determined from genus zero GV type invariants $[KP]$.

Combining with Corollary 3.4 and noting that $L \cdot \beta = 0$, we obtain a nontrivial relation between genus zero GV invariants of $X$ and $D$.

Acknowledgement. Y. C. is grateful to Martijn Kool, Davesh Maulik and Yukinobu Toda for exciting and fruitful discussions and collaborations over these years, without which the current work is not possible. We thank Arkadij Bojko for helpful communications on his related work. Y. C. is partially supported by the World Premier International Research Center Initiative (WPI), MEXT, Japan; JSPS KAKENHI Grant Number JP19K23397 and Royal Society Newton International Fellows Alumni 2019 and 2020. F. Q. is partially supported by NSFC grant 11801185.

References

[BF97] K. Behrend and B. Fantechi, The intrinsic normal cone, Invent. Math. 128 (1997), 45–88.

[BF08] K. Behrend and B. Fantechi, Symmetric obstruction theories and Hilbert schemes of points on threefolds, Algebra and Number Theory, vol. 2, 313–345, 2008.

[Bo] A. Bojko, Wall-crossing for zero-dimensional sheaves and Hilbert schemes of points on Calabi–Yau 4-folds, in preparation.

[BK] A. Bondal, M. Kapranov, Enhanced triangulated categories, Math. USSR-Sbornik, 70 (1991), 1, 93–107.

[BJ] D. Borisov and D. Joyce, Virtual fundamental classes for moduli spaces of sheaves on Calabi–Yau four-folds, Geom. Topol. (21), (2017) 3231–3311.

[BFI] R. O. Buchweitz and H. Flenner, A semiregularity map for modules and applications to deformations, Compositio Math. 137, 135–210, 2005.

[CGL] Y. Cao, J. Gross, and D. Joyce, Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves on Calabi–Yau 4-folds, Adv. Math. 368, (2020), 107134.

[CK18] Y. Cao and M. Kool, Zero-dimensional Donaldson-Thomas invariants of Calabi-Yau 4-folds, Adv. Math. 338 (2018), 601–648.

[CK19] Y. Cao and M. Kool, Curve counting and DT/PT correspondence for Calabi-Yau 4-folds, Adv. Math. 375 (2020), 107371.

[CKM19] Y. Cao, M. Kool, and S. Monavari, K-theoretic DT/PT correspondence for toric Calabi-Yau 4-folds, arXiv:1906.07856.

[CL14] Y. Cao and N. C. Leung, Donaldson-Thomas theory for Calabi-Yau 4-folds, arXiv:1407.7299.

[CL17] Y. Cao and N. C. Leung, Orientability for gauge theories on Calabi-Yau manifolds, Adv. Math. 314 (2017), 48–70.

[CMT18] Y. Cao, D. Maulik, and Y. Toda, Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds, Adv. Math. 338 (2018), 41–92.

[CMT19] Y. Cao, D. Maulik, and Y. Toda, Stable pairs and Gopakumar-Vafa type invariants for Calabi-Yau 4-folds, arXiv:1902.00033. To appear in J. Eur. Math. Soc. (JEMS).

[CT19] Y. Cao and Y. Toda, Curve counting via stable objects in derived categories of Calabi-Yau 4-folds, arXiv:1909.04807.

[CT20a] Y. Cao and Y. Toda, Gopakumar-Vafa type invariants on Calabi-Yau 4-folds via descenton insertions, Comm. Math. Phys. (2020). https://doi.org/10.1007/s00220-020-03897-9.
