GENERIC UNIQUENESS OF THE MINIMAL MOULTON CENTRAL CONFIGURATION

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Abstract. We prove that, for generic values of the masses, the Newtonian potential function of the collinear N-body problem has \( N! / 2 \) critical values when restricted to a fixed inertia level. In particular, we prove that for generic masses, there is only one minimal Moulton configuration.

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1. Introduction

In the N-body problem there is a family of solutions that conserve the shape in the evolution in time. Between this motions, those with zero angular momentum are called homothetic motions. They have the form

\[ x(t) = (r_1(t), \ldots, r_N(t)) = \phi(t) x_0 \]

where \( \phi(t) > 0 \) is a solution of a one center problem in the line \( \mathbb{R}_+ \), and \( x_0 \) a central configuration. This kind of configurations can be defined in many equivalent ways, say for instance as the critical points of the restrictions of the potential function

\[ U(x) = \sum_{i < j} m_i m_j r_{ij}, \]

to the level sets of the moment of inertia

\[ I(x) = \sum_{i=1}^N m_i r_i^2. \]

In this paper we will be interested in the collinear N-body problem, therefore a configuration \( x = (r_1, r_2, \ldots, r_N) \in \mathbb{R}^N \) will represent the vector of positions of the bodies, which are supposed to be punctual, each with mass \( m_i > 0 \), and contained in a straight line. As usual, \( r_{ij} = |r_i - r_j| \) will denote the distance between the bodies \( r_i \) and \( r_j \).

When the bodies evolve in a space of dimension \( k > 1 \) not much is know about the geometry of central configurations. Not even know in general if there exist only a finite number – modulo similitude – of central configurations. One of the most recent works on this topic, due to Albouy and Kaloshin [2], shows the generic finiteness in the case of five bodies in the plane, that is, excluding the situation in which the vector of masses \( m = (m_1, \ldots, m_5) \) belongs to a given subvariety of \( \mathbb{R}^5_+ \).

In contrast, for dimension \( k = 1 \), Moulton proved in [13] that if we identify configurations which are homothetic by a positive factor, then there are exactly \( N! \) equivalence classes of critical points, each one corresponding to an order \( \sigma \in S_N \) of the bodies in the line. They are all non degenerate local minima. The case of three bodies was first considered by Euler, see [4].

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Among all critical points minima play an important and special role. Indeed the second author and Venturelli have proved in [10] that if \( \alpha \) is a given minimal configuration normalized in the sense that \( I(\alpha) = 1 \), then for any configuration \( x_0 \) there is at least one motion \( x(t) \) starting from \( x_0 \) which is completely parabolic for \( t \to +\infty \), and whose normalized configuration \( x(t) I(x(t))^{-1/2} \) converges to \( \alpha \).

More recently, Percino and Sánchez-Morgado [15] built the Busemann functions associated to each minimal configuration. These facts can also be viewed (see [9]) in terms of weak KAM theory: from this viewpoint we can say that the set of minimal central configurations plays the role of the Mather set.

Following the analogy with the Aubry-Mather theory on compact manifolds, we can ask for the generic structure of the minimal objects. In his last work Mañé showed that generically a Tonelli Lagrangian on a compact manifold has a unique minimizing measure, see [11]. He conjectured that generically, the unique minimizing measure must be supported on a single periodic orbit or a fixed point; in particular, as a consequence of a previous work of the first author with Contreras [3], it must be generically hyperbolic. In all these results, the notion of generic property is the one introduced by Mañé and meaning that a property is generic if, given any Lagrangian \( L \), the property holds for every Lagrangian \( L + \varphi \) with \( \varphi \) in a residual subset of \( C^2(M) \). In the context of the \( N \)-body problem, a natural notion of generic property is to be satisfied for generic values of the masses.

The mass vector \( m = (m_1, \ldots, m_N) \in \mathbb{R}^N_+ \) is a parameter which determines the potential function \( U \) and the moment of inertia \( I \). Thus the mass vector also determines the central configurations, which can be defined as the critical points of the homogeneous function (of zero degree) \( \tilde{U} = U I^{1/2} \). The main result of the present note is the following theorem and his corollary.

**Theorem 1.** There is an open and dense set of mass vectors \( A \subset \mathbb{R}^N_+ \) such that, if \( m \in A \) then the function \( \tilde{U} \) has \( N!/2 \) critical values.

**Corollary 2.** There is an open and dense set of mass vectors for which the collinear \( N \)-body problem has only one minimal configuration.

Of course, the uniqueness in the statement of the corollary refers to the similarity classes of central configurations, that is to say, once we identify configurations which are homothetic by a non zero factor. Thus there are \( N!/2 \) different central configurations in this sense, and generically only one of them is minimal.

Let us explain briefly what can happen in some situations which are not generic. It is clear that if two masses are equal, then commutation of the corresponding bodies gives an extra symmetry of the problem which is not induced by an spacial isometry. In that case it is also clear that we must have at least two non similar minimal configurations, and at most \( N!/4 \) critical values of the potential function restricted to any inertia level. If all the masses are equal, the action of the full symmetric group preserves the set of central configurations, which implies that the restriction of the potential function to any inertia level has only one critical value, that is, the potential takes the same value at every normalized central configuration.

**2. Proof**

The proof of theorem 1 is divided in several lemmas. The first one that we prove characterizes the fact that two permutations are not equal nor symmetric.

Recall that \( S_N \) denotes the group of bijections of the set \( \{1, \ldots, N\} \) into itself. Each element of \( S_N \) is therefore identified with an ordering of the \( N \) bodies in the oriented straight line. If \( \sigma \in S_N \) is a given permutation, then \( \hat{\sigma} \in S_N \) will denote the permutation corresponding to the inverse order. More precisely, \( \hat{\sigma} \) is defined by \( \hat{\sigma}(k) = \sigma(N + 1 - k) \) for all \( k = 1, \ldots, N \). Moreover, given a permutation \( \sigma \) and
three numbers \(i, j, k \in \{1, \ldots, N\}\), we will say that \(\sigma(i)\) is between \(\sigma(j)\) and \(\sigma(k)\) if either \(\sigma(j) < \sigma(i) < \sigma(k)\) or \(\sigma(k) < \sigma(i) < \sigma(j)\).

**Lemma 1.** If \(\sigma\) and \(\tau\) are two given permutations then we have the following alternative: either \(\sigma = \tau\), \(\sigma = \bar{\tau}\), or there are three numbers \(i, j, k \in \{1, \ldots, N\}\) such that \(\sigma(i)\) is between \(\sigma(j)\) and \(\sigma(k)\) but \(\tau(i)\) is not between \(\tau(j)\) and \(\tau(k)\).

**Proof.** Clearly, each one of the first two possibilities in the triple alternative excludes the others. Thus it suffices to show that if the third possibility is not satisfied then one of the two first must be true.

Let \(\sigma\) and \(\tau\) be two permutations such that for any choice of \(i, j, k \in \{1, \ldots, N\}\) we have that \(\sigma(i)\) is between \(\sigma(j)\) and \(\sigma(k)\) if and only if \(\tau(i)\) is between \(\tau(j)\) and \(\tau(k)\). We will prove that either \(\sigma = \tau\) or \(\sigma = \bar{\tau}\). Let \(l = \sigma^{-1}(1)\). Since \(\sigma(l) = 1\) is not between any other two values of \(\sigma\), the same must happen for \(\tau(l)\). Therefore \(\tau(l) \in \{1, N\}\).

Suppose that \(\tau(l) = 1\). Then define

\[\alpha = \max \{ r \in \{1, \ldots, N\} \mid \tau \circ \sigma^{-1}(s) = s \text{ for all } s \leq r \}.\]

If \(\alpha = N\) then \(\sigma = \tau\). Otherwise we must have \(\alpha \leq N - 2\), and we can define \(j = \sigma^{-1}(\alpha), i = \sigma^{-1}(\alpha + 1)\) and \(k = \tau^{-1}(\alpha + 1)\). Thus we have \(\sigma(j) = \tau(j) = \alpha, \tau(i) > \alpha + 1 = \sigma(i)\) and \(\sigma(k) > \alpha + 1 = \tau(k)\). Therefore \(\sigma(j) < \sigma(i) < \sigma(k)\) but \(\tau(j) < \tau(k) < \tau(i)\). Since these inequalities contradict our hypothesis on the permutations \(\sigma\) and \(\tau\), we conclude that \(\alpha = N\) and that \(\sigma = \tau\).

If \(\tau(l) = N\) we have \(\bar{\tau}(l) = 1\), and the same argument implies \(\sigma = \bar{\tau}\). \(\square\)

Our second lemma proves the analytical dependence, on the masses, of the unique normal central configuration of a given ordering with respect to the masses. Let us introduce before a convenient notation.

For \(\sigma \in S_N\) we define the open set \(\Omega_\sigma\) as the set of configurations of \(N\) bodies in the oriented line with the ordering prescribed by \(\sigma\), that is to say,

\[\Omega_\sigma = \{ x = (r_1, \ldots, r_n) \mid r_{\sigma(1)} < \cdots < r_{\sigma(N)} \} .\]

It is clear that the set \(\Omega \subset \mathbb{R}^N\) of configurations without collisions is the disjoint union of the above sets. Thus \(\Omega\) has \(N!\) connected components.

Since we will consider varying masses, it is convenient to use the notation \(U_m(x) = U(x, m)\) and \(I_m(x) = I(x, m)\) for the values at \(x\) of the potential function and the moment of inertia respect to the origin respectively. Note that both functions \(U\) and \(I\) are real analytic functions in \(\Omega \times \mathbb{R}_+^N\).

We also need to recall a very useful and well known way to normalize central configurations which was suggested by Yoccoz. It is clear that \(z \in \Omega\) is a central configuration if and only if there exists \(\lambda \in \mathbb{R}\) such that

\[\nabla U_m(z) + \lambda \nabla I_m(z) = 0.\]

Since the functions \(U_m\) and \(I_m\) are homogeneous of degree \(-1\) and \(2\) respectively, we deduce that

\[0 = \langle \nabla U_m(z), z \rangle + \lambda \langle \nabla I_m(z), z \rangle = -U_m(z) + 2\lambda I_m(z)\]

hence that \(\lambda = \lambda_m(z) = U_m(z)/2I_m(z)\). We also see that \(z\) is a central configuration if and only if \(\mu z\) is also for all \(\mu > 0\), and that \(\lambda_m(\mu z) = \mu^{-3}\lambda_m(z)\). Therefore we conclude that there are two natural ways to normalize the size of a central configuration: fixing the value of the moment of inertia, or fixing the value of \(\lambda\).

The advantage of the second one is that the normalized configuration is a critical point of the function \(U_m + \lambda I_m\) in the open set \(\Omega\) rather than a critical point of the restriction of \(U_m\) to some level set of the moment of inertia.
Lemma 2. For each $\sigma \in S_N$ there is a real analytic function

$$x_\sigma : \mathbb{R}^N_+ \to \Omega_\sigma$$

such that $x_\sigma(m)$ is the unique central configuration in $\Omega_\sigma$ for the collinear N-body problem with mass vector $m$ such that $I_m(x_\sigma(m)) = 1$.

Proof. We will prove that for any value of $m \in \mathbb{R}^N_+$ and any $\sigma \in S_N$ the function

$$V_m = U_m + I_m$$

has a unique critical point in $\Omega_\sigma$. Clearly $V_m$ is a proper function over each convex set $\Omega_\sigma$ since $U_m(x) \leq K$ implies that $x$ is in the closed set

$$C_m(K) = \{ (r_1, \ldots, r_N) \in \mathbb{R}^N \mid K r_{ij} \geq m_i m_j > 0 \text{ for all } 1 \leq i < j \leq N \} \subset \Omega$$

and $\{ x \mid I_m(x) \leq K \}$ is a compact subset of $\mathbb{R}^N$. On the other hand $V_m$ is strictly convex in $\Omega$. A simple computation shows that

$$\frac{\partial^2 V_m}{\partial r_i^2}(x) = 2m_i + \sum_{k \neq i} 2m_i m_k r_{ik}^{-3}$$

and that

$$\frac{\partial^2 V_m}{\partial r_i \partial r_j}(x) = -2m_i m_j r_{ij}^{-3}$$

when $i \neq j$. By applying the Gershgorin circle theorem (see for instance [2]) we can conclude that the spectrum of the Hessian matrix is uniformly bounded from below by $2m_0$ where $m_0 = \min \{ m_1, \ldots, m_N \} > 0$. Therefore, the function $V_m$ has one and only one critical point at each component $\Omega_\sigma$ of $\Omega$. We will call $c_\sigma(m)$ this critical point. We have that $c_\sigma(m)$ is the unique central configuration in $\Omega_\sigma$ such that $\lambda_m(c_\sigma(m)) = 1$.

The map $c_\sigma : \mathbb{R}^N_+ \to \Omega_\sigma$ is real analytic because it is also defined by the real analytic implicit function theorem (see for instance chapter 6 in [8]), applied to the real analytic function

$$F_\sigma : \Omega_\sigma \times \mathbb{R}^N_+ \to \mathbb{R}^N$$

given by

$$F_\sigma(x, m) = \frac{\partial U}{\partial x}(x, m) + \frac{\partial I}{\partial x}(x, m) = \nabla V_m(x).$$

We know that the necessary condition to apply the implicit function theorem is satisfied since

$$\frac{\partial F_\sigma}{\partial x}(x, m) = D^2 V_m(x)$$

is the Hessian matrix of the function $V_m$ and we already know that is positive definite at every point.

In order to finish the proof, we write as a function of $m$ the corresponding central configuration with unitary moment of inertia. Indeed, since $\lambda_m(c_\sigma(m)) = 1$ we have that

$$2 I(c_\sigma(m), m) = U(c_\sigma(m), m),$$

and therefore

$$x_\sigma(m) = \sqrt{2} c_\sigma(m) U(c_\sigma(m), m)^{-1/2}$$

defines a real analytic function which gives, for each value of the mass vector $m$ the unique central configuration in $\Omega_\sigma$ with moment of inertia equal to 1. □

Now we will prove that the collinear central configurations, also called Moulton configurations, are local minima of $U_m = U_m I_m^{1/2}$. Note that $\tilde{U}_m(x)$ is the value of the potential $U_m$ at the normalized configuration $I_m(x)^{-1/2}x$. Moreover, if we call

$$S_m = \{ x \in \mathbb{R}^N \mid I_m(x) = 1 \}$$

then every central configurations in \( S_m \) is a nondegenerate local minima of the restriction \( U_m \mid_{S_m} \), and a global minima on each component \( \Omega_{\sigma} \cap S_m \). We give the proof of this well known fact for the sake of completeness. We will use the arguments in the proof of the previous lemma.

**Lemma 3.** Given \( m \in \mathbb{R}^N \) and \( \sigma \in S_N \) let us write \( \Sigma = \Omega_{\sigma} \cap S_m \) for the set of normal configurations with order \( \sigma \). The function \( U_m \mid_{\Sigma} \) has a unique global minimum which is nondegenerate.

**Proof.** We already know that \( U_m \mid_{\Sigma} \) has a unique critical point, thus we only have to prove that it is a nondegenerate minimum. The critical point is the point \( x_{\sigma} \) in the previous lemma, so we have

\[
x_{\sigma} = \sqrt{2c_{\sigma}} U_m(c_{\sigma})^{-1/2}
\]

where \( c_{\sigma} \) is the unique critical point of \( U_m + I_m \) in \( \Omega_{\sigma} \). Now we consider the map \( \varphi : \Sigma \to \Omega_{\sigma} \) given by

\[
\varphi(x) = \left( \frac{U_m(x)}{2} \right)^{1/3} x.
\]

This smooth map is a smooth embedding which satisfies \( \varphi(x_{\sigma}) = c_{\sigma} \). Moreover, for any \( x \in \Sigma \) we have

\[
U_m(\varphi(x)) = 2^{1/3} U(x)^{2/3}, \quad I_m(\varphi(x)) = (1/4) U(x)^{2/3}
\]

hence

\[
U(x) = k V_m(\varphi(x))^{3/2}
\]

for some constant \( k > 0 \). This proves that \( x_{\sigma} \) is a nondegenerate minimum of \( U_m \mid_{\Sigma} \) because \( V_m \) has a nondegenerate minimum at \( c_{\sigma} = \varphi(x_{\sigma}) \) and \( V_m(c_{\sigma}) > 0 \). \( \square \)

From now on, we will denote \( M_N(\sigma, m) \) the minimal value of the potential function \( U_m \) restricted to \( \Sigma = \Omega_{\sigma} \cap S_m \), the set of normal configurations of \( N \) bodies in the oriented line with a given order prescribed by a permutation \( \sigma \in S_N \). Thus we have \( M_N(\sigma, m) = U_m(x_{\sigma}) \).

We will say that \( \sigma \in S_{N+k} \) is compatible with \( \sigma_0 \in S_N \) whenever for every

\[
x = (r_1, \ldots, r_N, r_{N+1}, \ldots, r_{N+k}) \in \Omega_{\sigma} \subset \mathbb{R}^{N+k}
\]

we have

\[
y = (r_1, \ldots, r_N) \in \Omega_{\sigma_0} \subset \mathbb{R}^N.
\]

Of course the condition can be written in terms of \( \sigma \) and \( \sigma_0 \) exclusively. More precisely, taking into account that the value \( \sigma(n) \) is the number of the body in the \( n \)-th place from, it is easy to see that \( \sigma \) is compatible with \( \sigma_0 \) if and only if the function

\[
\sigma \mid_{\{1, \ldots, N\}} \circ \sigma_0^{-1} : \{1, \ldots, N\} \to \{1, \ldots, N, N+1, \ldots, N+k\}
\]

is increasing.

**Lemma 4.** Assume that \( m_0 = (m_1, \ldots, m_N) \in \mathbb{R}_+^N \) and \( \sigma \in S_N \) are given and that \( \tau \in S_{N+K} \) is compatible with \( \sigma \). If for \( \epsilon > 0 \) we define the mass vector

\[
m(\epsilon) = (m_1, \ldots, m_N, \epsilon m_{N+1}, \ldots, \epsilon m_{N+K}) \in \mathbb{R}_+^{N+K}
\]

then we have

\[
\lim_{\epsilon \to 0} M_{N+K}(\tau, m(\epsilon)) = M_N(\sigma, m_0).
\]
Proof. Let \((\epsilon_n)_{n>0}\) be a minimizing sequence for \(\mathcal{M}_{N+K}(\tau, m(\epsilon))\). This means that \(\epsilon_n \to 0\) and that
\[
\liminf_{\epsilon \to 0} \mathcal{M}_{N+K}(\tau, m(\epsilon)) = \lim_{n \to \infty} \mathcal{M}_{N+K}(\tau, m(\epsilon_n)).
\]
Now, for each \(n > 0\), we define \(x_n \in \mathbb{R}^{N+K}\) as the unique normalized central configuration of the \(N + K\) bodies given by lemma \([2]\) for the mass vector \(m(\epsilon_n)\) and the ordering given by \(\tau\). Thus, for each \(n > 0\) we have
\[
I(x_n, m(\epsilon_n)) = 1 \quad \text{and} \quad \mathcal{M}_{N+K}(\tau, m(\epsilon_n)) = U(x_n, m(\epsilon_n)),
\]
where the last equality is due to lemma \([3]\). Moreover, if we write
\[
x_n = (r_1^n, \ldots, r_N^n, r_{N+1}^n, \ldots, r_{N+K}^n), \quad \text{and} \quad y_n = (r_1^n, \ldots, r_N^n),
\]
then the compatibility of \(\tau\) with \(\sigma\) says that the configuration \(y_n\) has the ordering given by the permutation \(\sigma\). The configuration \(y_n\) is not normalized for the vector mass \(m_0\) as it is clear that \(I(y_n, m_0) < 1\). However, if we define
\[
\alpha_n = m_{N+1} (r_{N+1}^n)^2 + \cdots + m_{N+K} (r_{N+K}^n)^2,
\]
we can write
\[
I(y_n, m_0) = I(x_n, m(\epsilon_n)) - \epsilon_n \alpha_n = 1 - \epsilon_n \alpha_n
\]
so the normalization of \(y_n\) gives the configuration \(z_n = (1 - \epsilon_n \alpha_n)^{-1/2} y_n\), and we have
\[
U(z_n, m_0) = (1 - \epsilon_n \alpha_n)^{1/2} U(y_n, m_0).
\]
On the other hand, we have that
\[
U(x_n, m(\epsilon_n)) = U(y_n, m_0) + \sum_{i=1}^{N+K} \sum_{j=N+1}^{N+K} \frac{\epsilon_n m_i m_j}{r_i^n - r_j^n} + \sum_{N+1 \leq j < N+K} \frac{\epsilon_n^2 m_i m_j}{r_i^n r_j^n}.
\]
Since \((1 - \epsilon_n \alpha_n)^{1/2} < 1\), we deduce that
\[
U(z_n, m_0) < U(y_n, m_0) < U(x_n, m(\epsilon_n)).
\]
Thus, given that \(\mathcal{M}_N(\sigma, m_0) \leq U(z_n, m_0)\), we conclude that
\[
\mathcal{M}_N(\sigma, m_0) < U(x_n, m(\epsilon_n)) = \mathcal{M}_{N+K}(\tau, m(\epsilon_n)).
\]
Taking the limit for \(n \to \infty\) we obtain the inequality
\[
\mathcal{M}_N(\sigma, m_0) \leq \liminf_{\epsilon \to 0} \mathcal{M}_{N+K}(\tau, m(\epsilon)).
\]
We fix now \(\delta > 0\) and we define \(z = (r_1, \ldots, r_N)\) as the unique normal central configuration for the mass vector \(m_0\) and ordering prescribed by \(\sigma\). In particular we have \(\mathcal{M}_N(\sigma, m_0) = U(z, m_0)\) by lemma \([3]\). We will prove that the inequality
\[
\mathcal{M}_{N+K}(\tau, m(\epsilon)) \leq U(z, m_0) + \delta
\]
is satisfied whenever \(\epsilon > 0\) is small enough. This will finish the proof, since it implies that
\[
\limsup_{\epsilon \to 0} \mathcal{M}_{N+K}(\tau, m(\epsilon)) \leq \mathcal{M}_N(\sigma, m_0).
\]
Since \(\tau\) is compatible with \(\sigma\), we can add to the configuration \(z = (r_1, \ldots, r_N)\) the positions of \(K\) bodies, in such a way that the ordering of the resulting extended configuration \(y = (r_1, \ldots, r_N, r_{N+1}, \ldots, r_{N+K})\) is given by \(\tau\). We shall call \(r_0\) the minimal distance between the positions in the configuration \(y\), that is to say,
\[
r_0 = \min \{ |r_i - r_j| \mid 1 \leq i < j \leq N + K \} > 0.
\]
We will also consider the moment of inertia of the configuration \(y\) with respect to the mass vector \(m(\epsilon)\), and we will denote it by \(I^\epsilon\). Thus we can write
\[
I^\epsilon = I(y, m(\epsilon)) = I(z, m_0) + \epsilon \alpha,
\]
implies that the central configurations in $\Omega \times \mathbb{R}^2$ that the configurations have the form

$$\text{potential function and the moment of inertia we get}$$

Proof. Let us first compute $I_\epsilon$ whenever $\epsilon < \epsilon_0$ where $\epsilon_0$ is the unique positive root of the polynomial

$$M_{N+K}(\tau, m(\epsilon)) \leq U(x_\epsilon, m(\epsilon)),$$

where $\mu = \max \{ m_1, \ldots, m_{N+K} \}$. Therefore, since the right hand of the previous inequality is a continuous function of $\epsilon$, and $M_{N+K}(\tau, m(\epsilon)) \leq U(x_\epsilon, m(\epsilon))$, we conclude that there is $\epsilon_0 > 0$ such that

$$M_{N+K}(\tau, m(\epsilon)) < U(z, m_0) + \delta$$

whenever $\epsilon < \epsilon_0$, as we wanted to prove. \hfill \box

Lemma 5. There is $\mu > 0$ for which

$$M_3(id, (1, \mu, 1)) \neq M_3((2, 3), (1, \mu, 1)) = M_3(id, (1, 1, \mu)).$$

Proof. Let us first compute $M_3(id, (1, \mu, 1))$. The symmetry of the mass vector implies that the central configurations in $\Omega id$ are also symmetric. This means that the configurations have the form $x_\epsilon = (-r, 0, r)$ with $r > 0$. Computing the potential function and the moment of inertia we get

$$I(x_\epsilon) = 2r^2 \text{ and } U(x_\epsilon) = \frac{1}{2r} + \frac{2\mu}{r}.$$

So the normal central configuration for this order of the masses is $(-1/\sqrt{2}, 0, 1/\sqrt{2})$. We deduce that

$$M_3(id, (1, \mu, 1)) = \frac{\sqrt{2}}{2} + 2\sqrt{2\mu}.$$

The second distribution of masses is not symmetric. However, Euler has showed (see \cite{4}, or \cite{1} for a modern reference) that up to a translation and rescale, a central configuration for the mass vector $(m_1, m_2, m_3)$ and order $\sigma = id$ is $(0, 1, 1+s)$, where $s$ is the unique positive root of the polynomial

$$p(s) = -(m_1 + m_2)s^5 - (3m_1 + 2m_2)s^4 - (3m_1 + m_2)s^3 +$$

$$+(m_2 + 3m_3)s^2 + (2m_2 + 3m_3)s + (m_2 + m_3).$$

Since in our case we have $m_1 = m_2 = 1$ and $m_3 = \mu$ the polynomial becomes

$$p(s) = -2s^5 - 5s^4 - 4s^3 + (1 + 3\mu)s^2 + (2 + 3\mu)s + (1 + \mu).$$

We claim that there is $\mu > 0$ for which $(0, 1, 3)$ is a translated central configuration. Therefore $s = 2$ must be a root of this polynomial, which gives rise to the linear equation

$$p(2) = 19\mu - 171 = 0$$

whose solution is $\mu = 9$. We conclude that, the central configurations, for the mass vector $(1, 1, 9)$ and the ordering given by $\sigma = id$, have the form $y_\epsilon = (0, r, 3r)$ with
\( r > 0 \). Using the Leibnitz formula for the moment of inertia with respect to the center of mass we avoid to translate the configuration. More precisely, we have
\[
I_G(y_r) = \frac{1}{m_1 + m_2 + m_3} \left( m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2 \right)
\]
\[= \frac{1}{11} \left( r^2 + 9(3r)^2 + 9(2r)^2 \right) = \frac{118}{11} r^2.\]
In particular, the central configuration with moment of inertia \( I_G = 1 \) is, up to a translation, the configuration \( y_r \) for \( r = (11/118)^{1/2} \). Now we can compute the value of the potential function in this configuration, and we get
\[
U(y_r) = \frac{1}{r^2} + \frac{9}{2r} + \frac{9}{3r} = \frac{51}{6} \left( \frac{118}{11} \right)^{1/2}.
\]
Therefore the lemma is proved, since for \( \mu = 9 \) we have computed
\[
\mathcal{M}_3(id, (1, 9, 1)) = \frac{37}{\sqrt{2}}
\]
and
\[
\mathcal{M}_3((2, 3), (1, 9, 1)) = \mathcal{M}_3(id, (1, 1, 9)) = \frac{51}{6} \left( \frac{118}{11} \right)^{1/2}.
\]

\[\square\]

**Proof of theorem** [1] The set of critical values of the function \( \tilde{U} \) is exactly
\[
V_c(m) = \{ U(x_\sigma(m), m) \mid \sigma \in S_N \}.
\]
Therefore the number of values is a lower semicontinuous function of \( m \in \mathbb{R}_+^N \), so in particular it is a continuous function over the set of maxima
\[
A = \{ m \in \mathbb{R}_+^N \text{ such that } |V_c(m)| = N!/2 \}
\]
from which we conclude that this set is open.

In order to prove that \( A \) is dense in \( \mathbb{R}_+^N \) suppose by contradiction that it is not. As a consequence of the analyticity proved in lemma [2] we know that for each pair of permutations \( \sigma, \tau \in S_N \) the set
\[
\mathcal{M}_{\sigma, \tau} = \{ m \in \mathbb{R}_+^N \mid U(x_\sigma(m), m) \neq U(x_\tau(m), m) \}
\]
is either open and dense, or empty. The last situation happens for instance when \( \sigma = \tau \) or \( \sigma = \bar{\tau} \). Therefore we conclude that there must be a pair of non symmetric permutations \( \sigma \) and \( \tau \) (that is, such that \( \sigma \neq \tau \) and \( \sigma \neq \bar{\tau} \)), for which \( U(x_\sigma(m), m) = U(x_\bar{\tau}(m), m) \) for all \( m \in \mathbb{R}_+^N \).

On the other hand, since \( \sigma \neq \tau \) and \( \sigma \neq \bar{\tau} \) by lemma [1] (applied to the inverse permutations \( \sigma^{-1} \) and \( \tau^{-1} \)) we can assume without loss of generality that there are numbers \( 1 \leq i < j < k \leq N \) such that
\[
\sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k),
\]
and
\[
\tau^{-1}(i) < \tau^{-1}(k) < \tau^{-1}(j).
\]
We can also assume, renumbering the bodies if necessary, \( i = 1, j = 2 \) and \( k = 3 \). Now consider for small \( \epsilon > 0 \) the mass vector \( m_\epsilon = (1, \mu, 1, \epsilon, \ldots, \epsilon) \) given by where \( \mu \) is the value of the mass given by lemma [3]. By lemma [2] we have
\[
\mathcal{M}_N(\sigma, m_\epsilon) = U(x_\sigma(m_\epsilon), m_\epsilon) = U(x_\tau(m_\epsilon), m_\epsilon) = \mathcal{M}_N(\tau, m_\epsilon)
\]
for all \( \epsilon > 0 \). Moreover applying lemma [4] with \( N = 3, \sigma_0 = id \) and \( \tau_0 = (2, 3) \) we have
\[
\lim_{\epsilon \to 0} \mathcal{M}_N(\sigma, m_\epsilon) = \mathcal{M}_3(id, (1, \mu, 1))
\]
and
\[ \lim_{\epsilon \to 0} M_{N}(\tau, m_\epsilon) = M_{3}((2, 3), (1, \mu, 1)). \]
This is impossible since it contradicts lemma 5. \hfill \Box

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