Higher-spin Gauge and Trace Anomalies in Two-dimensional Backgrounds

Satoshi Iso$^1$, Takeshi Morita$^2$ and Hiroshi Umetsu$^3$

$^1$Institute of Particle and Nuclear Studies
High Energy Accelerator Research Organization (KEK)
Oho 1-1, Tsukuba, Ibaraki 305-0801, Japan

$^2$Yukawa Institute for Theoretical Physics, Kyoto University
Kyoto 606-8502, Japan

$^3$Okayama Institute for Quantum Physics
Kyoyama 1-9-1, Okayama 700-0015, Japan

Abstract

Two-dimensional quantum fields in electric and gravitational backgrounds can be described by conformal field theories, and hence all the physical (covariant) quantities can be written in terms of the corresponding holomorphic quantities. In this paper, we first derive relations between covariant and holomorphic forms of higher-spin currents in these backgrounds, and then, by using these relations, obtain higher-spin generalizations of the trace and gauge (or gravitational) anomalies up to spin 4. These results are applied to derive higher-moments of Hawking fluxes in black holes in a separate paper $^{15}$. 

$^*$satoshi.iso@kek.jp
$^\dagger$mtakeshi@yukawa.kyoto-u.ac.jp, takeshi@theory.tifr.res.in
Present address: Department of Theoretical Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India.
$^\ddagger$hiroshi.umetsu@pref.okayama.jp
1 Introduction

Hawking radiation is a universal quantum effect which arises in the background spacetime with event horizons [1, 2]. Such universal behavior arises because fields in black hole backgrounds can be reduced to an infinite set of two-dimensional conformal fields near the horizon. The emergence of conformal symmetries near the horizon was first emphasized in [3] and used to derive the Hawking radiation based on gauge or gravitational anomalies [4, 5]. The anomaly method has been applied to rotating black holes [6, 7] and various others. Such conformal structure near the horizon is also used to derive the higher-spin (HS) currents of Hawking radiation [8, 9] by examining conformal transformation properties of these HS currents.

The above derivation of the HS fluxes is based on the fact that two-dimensional quantum fields can be described by conformal field theories even in the presence of the electric and gravitational backgrounds, and hence all the physical quantities are written in terms of the conformal, i.e. holomorphic and anti-holomorphic, quantities. The HS currents used in [8, 9] are the holomorphic currents. They are holomorphic functions and different from the \((u \cdots u)\)-component of the ordinary covariant currents by some functions of the electric and gravitational backgrounds. These differences are responsible for the conformal transformation properties of the conformal currents. In the simplest case of the energy-momentum (EM) tensor in the gravitational background, it is well-known that we can define the holomorphic EM tensor \(t(u)\) from the original covariant EM tensor \(T_{\mu \nu}\) by

\[
t(u) = T_{uu} - \frac{c}{24\pi} \left( \partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right),
\]

(1.1)

where \(c\) is the central charge and \(\varphi\) is the conformal factor of the gravitational background. This relation gives the transformation property of \(t(u)\) under conformal transformations.

In this paper, we generalize this relation to all the HS currents in electric and gravitational backgrounds. This gives a further justification of our analysis in [8] and [9].

In section 2, we first review the relation between covariant and holomorphic forms of the \(U(1)\) current and the EM tensor in electric and gravitational backgrounds. For the case of EM tensor, this relation can be obtained from the conservation equations for EM tensor \(\nabla_\mu T^\mu_\nu = F_{\mu \nu} J^\mu\) and the trace anomaly \(T^\mu_\mu = cR/24\pi\). Note that \(T_{\mu \nu}\) denotes the matter EM tensor and therefore it is not conserved by itself in the electric background.

In section 3, we generalize them to HS currents. Here we construct higher-spin \(W_{1+\infty}\) currents from two-dimensional fermion fields in the electric and gravitational backgrounds. Since we do not know either conservation equations or trace anomalies for HS currents at the be-
ginning, we cannot start from these equations. Instead we will take the following procedure to obtain the relations between covariant and holomorphic HS currents. The original fermion field $ψ$ transforms covariantly under gauge and local Lorentz transformations. We will construct covariant HS currents by regularizing the fermion bilinears $\partial^m ψ^\dagger(x)\partial^m ψ(x)$ in the covariant way under gauge and general coordinate transformations. On the other hand, we can define a new fermion field $Ψ$ which is holomorphic in the electric and gravitational backgrounds, and by using it, we construct a holomorphic form of the HS currents. After defining these two types of HS currents, we give relations between the covariant and holomorphic forms of HS currents.

In section 4, by using the relations between covariant and conformal HS currents in section 3, we obtain conservation equations and trace anomalies for the HS currents. This is the inverse step compared to the derivations of the holomorphic $U(1)$ current and EM tensor in section 2. We show that the relations in section 3 and some assumptions for the currents are sufficient to determine the explicit forms of trace anomalies for HS currents. For the classically traceless spin 3 current $J^{(3)}_{µρλ}$, it acquires the following quantum correction:

$$J^{(3)}_{µν} = \frac{h}{12π} ∇_µ F^µ_ν.$$  \hspace{1cm} (1.2)

This is considered as a spin-3 generalization of the trace anomaly for EM tensor. For the spin 4 current $J^{(4)}_{µρσν}$, it is classically traceless but it acquires the quantum anomaly given by

$$J^{(4)}_{µ µρσ} = -\frac{h}{160π} ∇_ρ R + h g_{µρ} \left[ \frac{1}{160π} ∇^2 R + \frac{1}{24π} \left( 5 - \frac{13}{120} R^2 \right) \right].$$ \hspace{1cm} (1.3)

A generalization to higher spins than 4 is also possible but the calculation becomes more complicated.

In section 5, we consider a chiral theory where the central charges in the left and right handed sectors are different. In this case, we can obtain a generalization of the gauge(or gravitational) anomalies for higher-spin currents. We first review how we get the gravitational anomaly from the relations obtained in section 3, and then generalize it to spin 3 and 4 currents. For the spin 3 current, the generalization of the gauge anomaly becomes

$$∇_µ J^{(3)}_{νρ} = \cdots ± \frac{h}{96π} \left( ϵ_{νσ} ∇^σ ∇_µ F^µ_ρ + ϵ_{ρσ} ∇^σ ∇_µ F^µ_ν - g_{µρ} ϵ ασ ∇^σ ∇_µ F^µ_α \right).$$ \hspace{1cm} (1.4)

Here $\cdots$ represents classical violation of the conservation equation for matter currents in the electric and gravitational background. $+(−)$ corresponds to the right (left) handed fermion.

These results can be applied to derive the HS fluxes of Hawking radiation. The relations between covariant and conformal HS currents obtained in section 3 provide another derivation
of fluxes of HS currents in Hawking radiation. These relations are equivalent to solving the conservation equations and trace anomaly equations for HS currents. Hence the derivation gives a generalization of the Christensen and Fulling’s method [11], in which the conservation equation of the EM tensor and the trace anomaly equation are solved with the regularity condition at the horizon. On the other hand, as we see in section 5, these relations can be rewritten as a generalization of the gauge anomaly. By applying these anomaly equations to black holes, it gives a generalization of the anomaly method [4, 5] (see also appendix of [13] and [14]). These two derivations also clarify some points which were obscure in the previous papers [8, 9]. We will discuss these applications in a separate paper [15].

In appendix A, we summarize the relations between holomorphic and covariant HS currents up to spin 4.

2 U(1) current and EM tensor

In this section, we review a derivation of the holomorphic \( U(1) \) and EM tensor in the background of \( U(1) \) gauge and gravitational fields. These holomorphic quantities are obtained by solving conservation equations together with the anomaly equations.

Throughout this paper we employ the conformal gauge \( ds^2 = e^{\varphi} du dv \) for the gravitational background and the Lorenz gauge \( \nabla^\mu A_\mu = 0 \) for the gauge field background.

First we derive the holomorphic \( U(1) \) current. The \( U(1) \) current \( J^\mu \) satisfies the conservation equation

\[
\nabla_\mu J^\mu = 0,
\]

and the chiral anomaly for the chiral current \( J^{5\mu} \) is given by

\[
\nabla_\mu J^{5\mu} = \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}.
\]

Here the charge of the field is set \( e = 1 \). \( F_{\mu\nu} \) is the field strength of the background gauge field and \( \epsilon^{\mu\nu} \) is the covariant antisymmetric tensor, \( \epsilon^{\mu\nu} = 2e^{-\varphi} = g^{\mu\nu} \). The chiral current is related to the gauge current by \( J^{5\mu} = \epsilon^{\mu\nu} J_\nu \). From eqs. (2.1) and (2.2), we find

\[
\partial_v \left( J_u - \frac{1}{\pi} A_u \right) = 0, \quad \partial_u \left( J_v - \frac{1}{\pi} A_v \right) = 0,
\]

where the gauge conditions are used. Hence we define the (anti-)holomorphic \( U(1) \) currents as follows:

\[
j(u) \equiv J_u - \frac{1}{\pi} A_u, \quad \tilde{j}(v) \equiv J_v - \frac{1}{\pi} A_v.
\]
The holomorphic $U(1)$ currents generate a combination of the holomorphic gauge transformation, which is a combination of gauge and chiral transformations. Note that these currents are not covariant under the $U(1)$ gauge transformations.

Next we derive the holomorphic EM tensor. The conservation equation of the matter EM tensor is given by

$$\nabla_\mu T^\mu_\nu = F_{\mu\nu}J^\mu.$$  \hspace{1cm} (2.5)

The r.h.s. represents dissipation of the energy in the matter sector to the background gauge field. The trace anomaly of the EM tensor is given by

$$T^\mu_\mu = \frac{c}{24\pi}R,$$  \hspace{1cm} (2.6)

where $c$ is the central charge of the matter field and $R$ is the Ricci scalar $R = -4e^{-\varphi}\partial_u\partial_v\varphi$. From these equations, we obtain

$$\partial_v \left(T_{uu} - \frac{c}{24\pi} \left( \frac{\partial^2_u \varphi}{(\partial_u \varphi)^2} - \frac{1}{2} \left( \partial_u \varphi \right)^2 \right) - \frac{1}{\pi} A^2_u - 2A_u j(u) \right) = 0.$$  \hspace{1cm} (2.7)

Thus we define the holomorphic energy-momentum tensor as

$$t(u) \equiv T_{uu} - \frac{c}{24\pi} \left( \frac{\partial^2_u \varphi}{(\partial_u \varphi)^2} - \frac{1}{2} \left( \partial_u \varphi \right)^2 \right) - \frac{1}{\pi} A^2_u - 2A_u j(u).$$  \hspace{1cm} (2.8)

The anti-holomorphic one is defined similarly. These currents play a central role in conformal field theories since they generate conformal transformations, which is a combination of the general coordinate, Weyl and chiral transformations.

3 Holomorphic and covariant HS currents

In the previous section the relations between holomorphic and covariant quantities are obtained in the cases of the $U(1)$ current and energy-momentum tensor. In this section, we give a generalization of such relations to higher-spin (HS) currents. We consider fermionic fields in the gravitational and electric backgrounds, and construct holomorphic and covariant currents from them. Then we investigate the relations between these currents.

3.1 Holomorphic HS currents

In order to construct the holomorphic higher-spin currents from fermionic fields, let us recall some properties of the fermion in the two dimensions. The equation of motion for the right-handed fermion with unit charge is given by

$$\left( \partial_v - iA_v + \frac{1}{4}i\partial_v\varphi \right) \psi(u,v) = 0,$$  \hspace{1cm} (3.1)
in the gravitational and electric backgrounds \((\varphi, A_\mu)\). In the Lorentz gauge, the gauge field can be written locally as

\[
A_u = \partial_u \eta(u, v), \quad A_v = -\partial_v \eta(u, v),
\]

where \(\eta(u, v)\) is a scalar field. Since gravitational fields and gauge fields are not generally holomorphic, \(\psi(u, v)\) is not holomorphic either. In order to construct holomorphic quantities from the fermion field, we define a new field \(\Psi\) as

\[
\Psi \equiv \exp \left( \frac{1}{4} \varphi(u, v) + i\eta(u, v) \right) \psi(u, v).
\]

Then the equation (3.1) becomes \(\partial_v \Psi = 0\) and hence \(\Psi\) is holomorphic. Similarly we can define \(\Psi^\dagger\) as

\[
\Psi^\dagger \equiv \exp \left( \frac{1}{4} \varphi(u, v) - i\eta(u, v) \right) \psi^\dagger(u, v),
\]

so that \(\Psi^\dagger\) also becomes holomorphic.

Regularized holomorphic currents are constructed from these holomorphic fields. For example, the holomorphic \(U(1)\) current can be defined as

\[
j(u) =: \Psi^\dagger(u) \Psi(u) := \lim_{\epsilon \to 0} \left[ \Psi^\dagger(u + \frac{\epsilon}{2}) \Psi \left( u - \frac{\epsilon}{2} \right) + \frac{i}{2\pi \epsilon} \right],
\]

where the point splitting regularization is used and \(\Psi\) has the following operator product expansion,

\[
\Psi^\dagger(u) \Psi(w) \sim -\frac{i}{2\pi} \frac{1}{u - w}.
\]

Note that we have not attached a Wilson line phase in the regularization, since the gauge field is not holomorphic and the Wilson line phase breaks the holomorphy. As a result, the holomorphic \(U(1)\) current is not gauge invariant. We can also construct holomorphic currents: \(\partial_u^a \Psi^\dagger \partial_u^a \Psi(u)\) in the same way.

In order to clarify the difference between the holomorphic current and the ordinary covariant current, let us consider the covariant \(U(1)\) current \(J_u\) in the electric background. We here omit the gravitational background for simplicity. \(J_u\) can be defined as

\[
J_u \equiv \lim_{\epsilon \to 0} \left[ \psi^\dagger(u + \frac{\epsilon}{2}, v) e^{i \int_{u - \frac{\epsilon}{2}}^{u + \frac{\epsilon}{2}} A_u(u', v) du'} \psi(u - \frac{\epsilon}{2}, v) + \frac{i}{2\pi \epsilon} \right].
\]

In contrast with the holomorphic \(U(1)\) current, we have attached the Wilson line phase in the regularization, so this current is gauge invariant but not holomorphic. By using (3.3), (3.4)
and the operator product expansion (3.6), the covariant \( U(1) \) current can be related to the holomorphic \( U(1) \) current (3.5) as follows,

\[
J_u = \lim_{\epsilon \to 0} \left[ \epsilon \int_{u-\epsilon/2}^{u+\epsilon/2} A_u(u',v)du' + i\eta(u+\epsilon/2,v) - i\eta(u-\epsilon/2,v) \right] \left( : \Psi^\dagger(u+\epsilon/2)\Psi(u-\epsilon/2) : - \frac{i}{2\pi\epsilon} \right) + \frac{i}{2\pi\epsilon}
\]

(3.8)

This equation reproduces the equation (2.4) which was originally derived from the conservation equation (2.1) and the chiral anomaly (2.2). Hence this evaluation of the covariant current is equivalent to solving the conservation and anomaly equations. Similarly the relations between the covariant and the holomorphic HIS currents contain the full information of conservation equations and anomalies for these HIS currents. We will discuss it in the next section.

In the following subsections, we will consider a generalization of the relation (3.8) to the HIS currents. Instead of considering each HIS currents separately, it turns out that it is useful to introduce the following generating function of the holomorphic currents

\[
G_{\text{hol}}(u + a, u + b) \equiv \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m! n!} \partial_u^m \Psi^\dagger(u) \partial_u^n \Psi(u) : = \Psi^\dagger(u + a)\Psi(u + b) + \frac{i}{2\pi(a-b)}.
\]

(3.9)

This should be understood as a formal power series in terms of the parameters \( a \) and \( b \) around the position \( u \). This function is holomorphic but not gauge covariant. In the subsection 3.3, we will construct a generating function for the covariant currents, and then give a relation between these two functions.

We here comment on the transformation property of the fermion field \( \Psi \) under (holomorphic) gauge transformations. In the Lorentz gauge, there remains residual holomorphic gauge symmetry,

\[
\psi'(u,v) = e^{i\Lambda(u)}\psi(u,v), \quad \eta'(u,v) = \eta(u,v) + \Lambda(u).
\]

(3.10)

Under this transformation, \( \Psi(u) \) transforms as a field with twice the charge of \( \psi \),

\[
\Psi'(u) = e^{2i\Lambda(u)}\Psi(u).
\]

(3.11)

This clarifies a point which we did not explain explicitly in [9], i.e., we there used this transformation property for the holomorphic field under the holomorphic transformation connecting a suitable gauge at infinity and a suitable one near the horizon.
3.2 Covariant HS currents

Now we will define the \((u \cdots u)\)-component of the covariant HS currents constructed from the fermion \(\psi\) in the electric and gravitational backgrounds. Since these currents are covariant under holomorphic general coordinate transformations, \(u \rightarrow \tilde{u} = f(u)\), it is convenient to define the coordinate which is invariant under these transformations.

In the rest of this section, we consider \(v\) to be a fixed coordinate and treat the system as a one-dimensional one with the coordinate \(u\). Then, under the holomorphic general coordinate transformations, we can define an “invariant coordinate” \(x\) which satisfies \(\partial_x = e^{-\varphi} \partial_u\) and regard \(u\) as a function of \(x\), i.e. \(u = u(x)\). Since \(dx\) is invariant under the above holomorphic transformations, the point splitting regularization is also invariant if it is defined in the \(x\) coordinate, not in the \(u\) coordinate. \(u(x + \epsilon)\) is now expanded as a formal power series of \(\epsilon\) as

\[
  u(x + \epsilon) = u(x) + \epsilon \partial_x u(x) + \frac{\epsilon^2}{2} \partial_x^2 u(x) + \frac{\epsilon^3}{3!} \partial_x^3 u(x) + \cdots 
\]

where we used only the relation \(\partial_x = e^{-\varphi} \partial_u\). It is important that the last expression does not explicitly depend on \(x\). A field \(\phi\) located at \(u(x + \epsilon)\) is defined as the following expansion,

\[
  \phi(u(x + \epsilon)) = \phi(u(x)) + \epsilon \partial_x \phi(u(x)) + \frac{\epsilon^2}{2} \partial_x^2 \phi(u(x)) + \cdots 
  = \phi(u) + \epsilon e^{-\varphi} \partial_u \phi(u) + \frac{\epsilon^2}{2} (e^{-\varphi} \partial_u)^2 \phi(u) + \cdots. \tag{3.13}
\]

Let’s first consider the relation between the holomorphic and covariant EM tensor in the electric and gravitational backgrounds. As explained above, covariant regularization can be defined in the \(x\) coordinate as follows,

\[
  T_{uu} = e^{2\varphi(x,v)} \lim_{\epsilon \to 0} \left\{ -i \frac{e^{i j_{u(x-\epsilon/2)} du A_u(u',v)} }{2} \right. 
  \left. \left[ (e^{-\frac{i}{2} \varphi(u(x+\epsilon/2),v)} \nabla_u \psi^\dagger(u(x+\epsilon/2),v) \right) \left( e^{-\frac{i}{2} \varphi(u(x-\epsilon/2),v)} \psi(u(x-\epsilon/2),v) \right) 
  - (e^{-\frac{i}{2} \varphi(u(x+\epsilon/2),v)} \psi^\dagger(u(x+\epsilon/2),v) \right) \left( e^{-\frac{i}{2} \varphi(u(x-\epsilon/2),v)} \nabla_u \psi(u(x-\epsilon/2),v) \right) 
  - \frac{1}{2\pi \epsilon^2} \right. \right\}. \tag{3.14}
\]

* Since the coordinate \(x\) is formally introduced as a function of \(u\), \(v\) should be kept fixed if a formula contains \(x\) explicitly. A derivative with respect to \(v\) must be taken only after the \(x\) coordinate is removed.
Hence the Wilson line phase is introduced to guarantee the $U(1)$ gauge invariance. The covariant derivative is defined by

$$
\nabla_u \psi(u(x + \epsilon/2)) = \left( \partial_u - \frac{1}{4} \partial_u \varphi(u(x + \epsilon/2)) - iA_u(u(x + \epsilon/2)) \right) \psi(u(x + \epsilon/2)).
$$

(3.15)

Hence $e^{-\frac{1}{2}\varphi \nabla_u \psi^\dagger}$ and $e^{-\frac{1}{2}\varphi \psi^\dagger}$ transform as scalars under holomorphic coordinate transformations. Therefore $T_{uu}$ transforms as a weight 2 tensor.

This EM tensor can be rewritten in terms of the holomorphic fields by using eqs. (3.3) and (3.4) as

$$
T_{uu} = e^{2\varphi(u,v)} \lim_{\epsilon \to 0} \left\{ -\frac{i}{2} e^{2i(\eta(u(x+\epsilon/2)) - \eta(u(x - \epsilon/2)))} \left[ e^{-\frac{1}{2}\varphi(u(x+\epsilon/2),v) - \frac{1}{2}\varphi(u(x - \epsilon/2),v)} \nabla_u \Psi^\dagger(u(x + \epsilon/2),v) \Psi(u(x - \epsilon/2),v) \right. \\
- \left. e^{-\frac{1}{2}\varphi(u(x+\epsilon/2),v) - \frac{1}{2}\varphi(u(x - \epsilon/2),v)} \nabla_u \Psi^\dagger(u(x + \epsilon/2),v) \Psi(u(x - \epsilon/2),v) \right] \\
- \frac{1}{2\pi^2} \right\},
$$

(3.16)

where the covariant derivative of $\Psi$ is $\nabla_u \Psi = (\partial_u - \frac{1}{2} \partial_u \varphi - 2iA_u) \Psi$ and the gauge field is written by the scalar field $\eta$, eq. (3.2). By using the following operator product expansion

$$
\Psi^\dagger(u(x + \epsilon/2))\Psi(u(x - \epsilon/2)) + \frac{i}{2\pi} \frac{1}{u(x + \epsilon/2) - u(x - \epsilon/2)} =: \Psi^\dagger(u)\Psi(u) ;
$$

(3.17)

we find

$$
T_{uu} = -\frac{i}{2} : \left( \partial_u \Psi^\dagger(u)\Psi(u) - \Psi^\dagger(u)\partial_u \Psi(u) \right) : + 2A_u(u,v) : \Psi^\dagger(u)\Psi(u) : \\
+ \frac{1}{\pi} A_u^2(u,v) + \frac{1}{24\pi} \left( \partial_u^2 \varphi(u,v) - \frac{1}{2}(\partial_u \varphi(u,v))^2 \right).
$$

(3.18)

This relation is equivalent to eq. (2.8) with the following identification of the holomorphic EM tensor,

$$
t(u) = -\frac{i}{2} : \left( \partial_u \Psi^\dagger(u)\Psi(u) - \Psi^\dagger(u)\partial_u \Psi(u) \right) :
$$

(3.19)

Generalizing these definitions of the covariant currents, we define the covariant HS current $J^{(n)}_{u\cdots u}$ as follows,

$$
J^{(n+1)}_{u\cdots u} = e^{(n+1)\varphi(u,v)} \lim_{\epsilon \to 0} \left[ \sum_{m=0}^{n} \frac{n!}{2^n m!(n-m)!} e^{i \int_{u(x - \epsilon/2)}^{u(x + \epsilon/2)} du' A_u(u',v)} \right. \\
\times e^{-\frac{m}{3} \varphi(u(x + \epsilon/2),v)} \left( i \nabla_u \right)^{n-m} \psi^\dagger(u(x + \epsilon/2),v) \\
\times e^{-\frac{m}{3} \varphi(u(x - \epsilon/2),v)} \left( i \nabla_u \right)^{m} \psi(u(x - \epsilon/2),v) \\
+ \frac{i^{n+1} n!}{2\pi e^{n+1}} \right] .
$$

(3.20)
This current is symmetric with respect to $\psi$ and $\psi^\dagger$ and invariant under the $U(1)$ gauge transformations thanks to the Wilson line phase. The point splitting regularization is performed in the $x$ coordinate. Furthermore, we have multiplied the conformal factors at $u(x \pm \epsilon/2)$ to make the combinations to be scalars under holomorphic general coordinate transformations. Therefore, because of the factor $e^{(n+1)\varphi}$, this current transforms as a tensor with a weight $(n + 1)$ under holomorphic general coordinate transformations. By rewriting this quantity in terms of the holomorphic fields and using the operator product expansion (3.17), we can take the limit $\epsilon \to 0$ and get a formula which no more contains the formally introduced $x$ coordinate. Then a relation between the covariant and the holomorphic HS currents is obtained. The holomorphic HS currents are defined as

$$j^{(n+1)}(u) = \sum_{m=0}^{n} \frac{n!}{2^m m!(n - m)!} : ( -i \partial_u )^{n-m} \psi^\dagger (i \partial_u )^m \psi : .$$

(3.21)

In these notations, we denote the $U(1)$ currents and the EM tensors as $J_u = J^{(1)}_u$, $T_{uu} = J^{(2)}_{uu}$, $j(u) = j^{(1)}(u)$ and $t(u) = j^{(2)}(u)$.

The explicit forms of the relations between the covariant and holomorphic currents with spin 3 and 4 will be given in section 4.

### 3.3 Generating functions of HS currents

Instead of studying each HS current separately, it is simpler and more systematic to consider a generating function of the HS currents. We define the following generating function of the covariant HS currents as a formal power series with respect to a parameter $a$,

$$G_{\text{cov}}(a) \equiv \sum_{n=0}^{\infty} \frac{(2ia)^n}{n!} e^{-(n+1)\varphi(u,v)} J_{u^{(n+1)}}^{(n+1)} u .$$

(3.22)

In substituting the definition of $J_{u^{(n+1)}}^{(n+1)}$, we use the following relation,

$$e^{i \int_{u_0}^{u} du'A_u(u',v) e^{-(k+\frac{1}{4})\varphi(u(x+\epsilon/2)) \nabla^k u \psi^\dagger(u(x+\epsilon/2))}} e^{i \int_{u_0}^{u} du'A_u(u',v) e^{-(k+\frac{1}{4})\varphi(u(x+\epsilon/2)) \psi^\dagger(u(x+\epsilon/2))}} .$$

(3.23)

\footnote{Our definitions of the HS currents are different from those of the $W_{1+\infty}$ algebra in [12]. Their HS currents are given by combining our HS currents and the derivative of them.
where \( u_0 \) is a fixed value on the \( u \) coordinate. A similar calculation can be done for the term including \( \psi \). Then \( G_{\text{cov}}(a) \) can be represented as

\[
G_{\text{cov}}(a) = \lim_{\epsilon \to 0} \left[ \sum_{m=0}^{\infty} \frac{a^m}{m!} \left( e^{-\varphi(u+\epsilon/2)} \frac{\partial}{\partial u(x+\epsilon/2)} \right)^m e^{i\eta(u(x+\epsilon/2))-i(k+\frac{1}{2})\varphi(u+\epsilon/2)} \psi(u(x+\epsilon/2)) + \frac{i}{4\pi(a + \epsilon/2)} \right]
\]

\[
e^{-\frac{1}{2}(\varphi(u(x+a),v)+\varphi(u(x-a),v))} e^{i\int u(x-a) \partial u(x) \psi(u(x),v) \psi(u(x-a),v)} + \frac{i}{4\pi a}.
\]

(3.24)

In the last expression we have naively taken the limit \( \epsilon \to 0 \) for notational simplicity, but the precise meaning of \( G_{\text{cov}}(a) \) is given by the first expression. By similar procedures to those used in the previous subsection, i.e. employing eqs. (3.3), (3.4) and (3.17), the generating function can be described in terms of the holomorphic fields,

\[
G_{\text{cov}}(a) = e^{-\frac{1}{2}(\varphi(u(x+a),v)+\varphi(u(x-a),v))} e^{i\int u(x-a) \partial u(x) \psi(u(x),v) \psi(u(x-a),v)}
\]

\[
\times \left( G_{\text{hol}}(u(x+a), u(x-a)) - \frac{i\hbar}{2\pi} \frac{1}{u(x+a) - u(x-a)} \right) + \frac{i\hbar}{4\pi a},
\]

(3.25)

where the gauge field is represented by using \( \eta \), eq. (3.2). In this expression, in order to distinguish the quantum contributions from the classical ones, we recovered the Planck's constant \( \hbar \). By expanding the relation with respect to the parameter \( a \), we can obtain equations which relate the covariant HS currents with a sum of the holomorphic currents in the electric and gravitational background.

We can also define a generating function for the holomorphic HS currents \( j^{(n)}(u) \),

\[
G_{\text{hol}}(u + \alpha, u - \alpha) = \sum_{n=0}^{\infty} \frac{(2i\alpha)^n}{n!} j^{(n+1)}(u)
\]

(3.26)

By using eqs. (3.3) and (3.4), \( G_{\text{hol}}(\alpha) \) can be written in terms of \( \psi \) and \( \psi^\dagger \) as

\[
G_{\text{hol}}(u + \alpha, u - \alpha) = e^{\frac{i}{2}(\varphi(u + \alpha) + \varphi(u - \alpha)) - 2i(\eta(u + \alpha) - \eta(u - \alpha))}
\]

\[
\times \left[ e^{-\frac{i}{2}(\varphi(u + \alpha) + \varphi(u - \alpha)) + i(\eta(u + \alpha) - \eta(u - \alpha))} \psi^\dagger(u + \alpha, v) \psi(u - \alpha, v) + \frac{i\hbar}{2\pi} \frac{1}{x(u + \alpha) - x(u - \alpha)} \right] - \frac{i\hbar}{2\pi} \frac{1}{x(u + \alpha) - x(u - \alpha)}
\]

\[
+ \frac{i\hbar}{4\pi \alpha}.
\]

(3.27)
The first term in (3.27) can be described in terms of the covariant HS currents and their derivatives. By expanding (3.27) with respect to the parameter \( \alpha \), the equations relating the holomorphic HS currents with a sum of the covariant ones can be derived. The obtained relations are summarized in the appendix A.

In the next section, we investigate the relations for spin 3 and 4 currents, and discuss the conservation equations, trace anomalies and generalizations of gauge (or gravitational) anomalies.

4 Trace anomalies for HS currents

In section 2, we derived the relation between \( T_{uu} \) and \( t(u) \) from the conservation equation (2.5) and the anomaly equation (2.6). In this section, we take an inverse step for the HS currents. We first provide relations between the holomorphic and covariant HS currents from equation (3.25), and then, by using these relations, evaluate their conservation equations and trace anomalies. In order to fix the definitions of the currents, we impose the following assumptions about the currents:

1. Anomalies appear in the trace parts of the currents only.
2. The covariant currents are classically traceless.
3. The covariant currents are totally symmetric.

Under these assumptions, we will obtain the conservation equations and trace anomalies for the spin 3 and 4 currents. The derivation can be straightforwardly applied to general HS currents, though calculations become more complicated.

We use the following notations of the currents in this section. \( J_{\mu}^{(1)} \) denotes the covariant \( U(1) \) current \( J_\mu \), \( J_{\mu\nu}^{(2)} \) denotes the covariant energy-momentum tensor \( T_{\mu\nu} \) and \( J_{\mu_1 \ldots \mu_n}^{(n)} \) does the spin \( n \) covariant current.

4.1 Trace anomaly for spin 3 current

We here consider the spin 3 current. From eq.(3.20), the covariant spin 3 current is given by

\[
\begin{align*}
J_{uuu}^{(3)} &= -\frac{1}{4} e^{\frac{\alpha}{2}} \lim_{\epsilon \to 0} e^{i J_{u(x+\epsilon/2)} A_{u}(u',v)} \\
& \times \left[ e^{-\frac{\alpha}{4} \phi(u(x+\epsilon/2),v)} - \frac{\alpha}{4} \phi(u(x-\epsilon/2),v) \nabla_u^2 \psi^\dagger (u(x + \epsilon/2), v) \psi (u(x - \epsilon/2), v) \\
& - 2 e^{-\frac{\alpha}{4} \phi(u(x+\epsilon/2),v)} - \frac{\alpha}{4} \phi(u(x-\epsilon/2),v) \nabla_u \psi^\dagger (u(x + \epsilon/2), v) \nabla_u \psi (u(x - \epsilon/2), v) \\
& + e^{-\frac{\alpha}{4} \phi(u(x+\epsilon/2),v)} - \frac{\alpha}{4} \phi(u(x-\epsilon/2),v) \psi^\dagger (u(x + \epsilon/2), v) \nabla_u^2 \psi (u(x - \epsilon/2), v) \right].
\end{align*}
\] (4.1)
The corresponding holomorphic spin 3 current is

\[ j^{(3)}(u) \equiv -\frac{1}{4} \left( : \Psi^\dagger \partial_u^2 \Psi - 2 \partial_u \Psi^\dagger \partial_u \Psi + \partial_u^2 \Psi^\dagger \Psi : \right). \quad (4.2) \]

The following relation between these currents can be derived by expanding \( G_{\text{cov}}(a) \) and taking the \( a^2 \) terms as

\[
J_{uuu}^{(3)} = j^{(3)}(u) + 4A_u j^{(2)}(u) + \left( \frac{1}{4} (\partial_u^2 \varphi - (\partial_u \varphi)^2) + 4A_u^2 \right) j^{(1)}(u) + \frac{1}{4} \partial_u \varphi \partial_u j^{(1)}(u) \\
+ \frac{\hbar}{4\pi} \left( A_u (\partial_u^2 \varphi - (\partial_u \varphi)^2) + \partial_u \varphi \partial_u A_u - \frac{1}{3} \partial_u^2 A_u + \frac{16}{3} A_u^3 \right). \tag{4.3}
\]

From this equation, we derive the conservation equation and anomaly equation for the covariant spin 3 current. First let us consider the \( uu \) component of the conservation equations \( \nabla_\mu J^{(3)}(u)_{\mu uu} \).

By taking the derivative of eq. (4.3) with respect to \( v \), we find

\[
\nabla_v J^{(3)}_{uuu} = -2F_{uv} J^{(2)}(u)_{uu} - \frac{1}{8} \nabla_u \left( g_{uv} R J^{(1)}(u) \right) + \frac{\hbar}{24\pi} \nabla_u^2 F_{uv}. \tag{4.4}
\]

Here we have used the equations in appendix A to describe the holomorphic currents in terms of the covariant currents. Since, as mentioned above, we assume that anomalies arise only in the trace part of the currents, we regard the last term in eq. (4.4), which is a quantum contribution, as the covariant derivative of the trace anomaly,

\[
\nabla_u J^{(3)}_{uu} = -\frac{\hbar}{24\pi} \nabla_u^2 F_{uv}. \tag{4.5}
\]

Thus the \( uu \) component of the conservation equation becomes

\[
\nabla_\mu J^{(3)}(u)_{\mu uu} = -2g^{uu} F_{uv} J^{(2)}(u)_{uu} - \frac{1}{8} \nabla_u \left( R J^{(1)}(u) \right). \tag{4.6}
\]

Here we have multiplied (4.4) and (4.5) by \( g^{uu} \). From this equation we may naively guess the general components of the conservation equation as follows,

\[
\nabla_\mu J^{(3)}(u)_{\mu \nu \rho} = -F_{\nu \mu} J^{(2)}(u)_{\rho \mu} - F_{\rho \mu} J^{(2)}(u)_{\nu \mu} - \frac{1}{16} \nabla_\mu \left( R J^{(1)}(u) \right) - \frac{1}{16} \nabla_\nu \left( R J^{(1)}(u) \right) - \frac{1}{16} \nabla_\rho \left( R J^{(1)}(u) \right), \tag{4.7}
\]

where the indices \( \nu, \rho \) are symmetrized. But this is not traceless at the classical level, which contradicts with the second assumption. Note that terms proportional to \( g_{\nu \rho} \) can be added to the conservation law without affecting eq. (4.6). Thus by using this freedom, we can make the r.h.s. of the conservation equation traceless with respect to \( \nu \) and \( \rho \),

\[
\nabla_\mu J^{(3)}(u)_{\mu \nu \rho} = -F_{\nu \mu} J^{(2)}(u)_{\rho \mu} - F_{\rho \mu} J^{(2)}(u)_{\nu \mu} - \frac{1}{16} \nabla_\mu \left( R J^{(1)}(u) \right) - \frac{1}{16} \nabla_\nu \left( R J^{(1)}(u) \right) + \frac{1}{16} g_{\nu \rho} \nabla_\mu \left( R J^{(1)}(u) \right). \tag{4.8}
\]
This satisfies the three conditions we require.

Next the trace anomaly \( J_{vu}^{(3)} \) can be read from equation (4.5),

\[
J_{vu}^{(3)} = -\frac{\hbar}{24\pi} \nabla_v F_{uv}.
\]

(4.9)

This can be covariantized as

\[
J_{\mu \nu}^{(3)} = \frac{\hbar}{12\pi} \nabla_\mu F_{\nu \nu}.
\]

(4.10)

In order to check the consistency with the conservation law (4.8), we calculate \( J_{uv}^{(3)} \). Since \( J_{uv}^{(3)} \) is given by

\[
J_{uv}^{(3)} = -\frac{\hbar}{24\pi} \nabla_v F_{vu},
\]

(4.11)

we can show \( \nabla_\mu J_{\mu \nu}^{(3)} = 0 \) by using the identity in two dimensions, \([\nabla_\mu, \nabla_\nu] F_\rho \sigma = 0\). Hence (4.10) is consistent with (4.8).

The conservation equation (4.8) implies that the theory can possess symmetry associated with the spin 3 current if the corresponding spin 3 gauge field is included in it. Let us consider an action containing linear couplings of the HS currents to general higher-spin gauge fields \( B_{\mu_1 \cdots \mu_n}^{(n)} \),

\[
S[A, g, B^{(n)}] = \int d^2 x \sqrt{-g} \left( \mathcal{L}_0 + \sum_{n=3}^{\infty} \frac{1}{n!} B_{\mu_1 \cdots \mu_n}^{(n)} J^{(n)}_{\mu_1 \cdots \mu_n} \right),
\]

(4.12)

where \( \mathcal{L}_0 \) is the Lagrangian for the free fermion in the electric and gravitational backgrounds, \( A_\mu \) and \( g_{\mu \nu} \), respectively. We also introduce the following effective action for these gauge fields,

\[
ed \Gamma[A, g, B^{(n)}] = \int D\bar{\psi} D\psi e^{iS[A, g, B^{(n)}]},
\]

(4.13)

The expectation values of the HS currents in these backgrounds are given by

\[
\langle J^{(1)}_\mu \rangle = \langle J_\mu \rangle = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta A_\mu} \Gamma[A, g, B^{(n)}],
\]

\[
\langle J^{(2)}_{\mu \nu} \rangle = \langle T_{\mu \nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu \nu}^{\phi}} \Gamma[A, g, B^{(n)}],
\]

\[
\langle J^{(n)}_{\mu_1 \cdots \mu_n} \rangle = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta B_{\mu_1 \cdots \mu_n}^{(n)}} \Gamma[A, g, B^{(n)}], \quad (n \geq 3).
\]

(4.14)

Then the conservation equation (4.8) indicates that the effective action is invariant under the following infinitesimal transformations of the background fields,

\[
\delta_\xi B^{(3)}_{\mu \nu \rho} = \frac{1}{3} (\nabla_\mu \xi^{\nu \rho} + \nabla_\nu \xi^{\rho \mu} + \nabla_\rho \xi^{\mu \nu})
\]

(4.15)

\[
\delta_\xi g^{\mu \nu} = -2\xi^{\mu \sigma} F_{\sigma \nu} - 2\xi^{\nu \sigma} F_{\sigma \mu},
\]

(4.16)

\[
\delta_\xi A_\mu = \frac{1}{8} R_{\nu \lambda} \xi^{\nu \mu},
\]

(4.17)
where $\xi^{\mu\nu}$ is a symmetric traceless parameter.

This transformation law is valid only for the weak $B^{(3)}$ field limit, i.e. we have assumed that the rank 3 gauge field $B^{(3)}$ was originally absent. Since the OPE between spin 3 currents generate higher-spin currents, they no longer form a closed algebra, contrary to the spin 1 or spin 2 currents. Hence higher-spin gauge symmetries larger than 2 and their backgrounds must be considered as $W_\infty$ gauge symmetry and gauge fields as a whole. This is beyond the scope of the present paper.

4.2 Trace anomaly for spin 4 current

The covariant spin 4 current is given from eq. (3.22) by

$$J^{(4)}_{uuu} = \frac{i}{8} e^{A\phi} \lim_{\epsilon \to 0} e^{i F_{u(x-\epsilon/2)}^{(x-\epsilon/2)}} du' A_u (u',v) \times \left[ e^{-\frac{13}{4} \phi(u(x+\epsilon/2),v)}} - \frac{1}{4} \phi(u(x-\epsilon/2),v)} \right. \nabla^3 u \psi^\dagger (u(x+\epsilon/2),v) \psi(u(x-\epsilon/2),v) \\
-3e^{-\frac{2}{4} \phi(u(x+\epsilon/2),v)}} - \frac{1}{4} \phi(u(x-\epsilon/2),v)} \nabla^2 u \psi^\dagger (u(x+\epsilon/2),v) \psi(u(x-\epsilon/2),v) \\
+3e^{-\frac{5}{4} \phi(u(x+\epsilon/2),v)}} - \frac{8}{4} \phi(u(x-\epsilon/2),v)} \nabla u \psi^\dagger (u(x+\epsilon/2),v) \nabla^2 u \psi(u(x-\epsilon/2),v) \\
-3e^{-\frac{13}{4} \phi(u(x+\epsilon/2),v)}} - \frac{1}{4} \phi(u(x-\epsilon/2),v)} \nabla^3 u \psi^\dagger (u(x+\epsilon/2),v) \nabla^3 u \psi(u(x-\epsilon/2),v),$$

(4.18)

and the corresponding holomorphic current is

$$j^{(4)}(u) = \frac{i}{8} : \partial^3 u \Psi^\dagger \Psi - 3 \partial^2 u \Psi^\dagger \partial \Psi + 3 \partial_\alpha \Psi^\dagger \partial_\alpha \Psi - \Psi^\dagger \partial_\alpha \Psi : .$$

(4.19)

The relation between these two currents are obtained from $a^3$ terms of the equation (3.25) as

$$J^{(4)}_{uuu} = j^{(4)}(u) + 6A_u j^{(3)}(u) + \frac{3}{4} \partial_\alpha \phi \partial_\alpha j^{(2)}(u) + \left[ \frac{1}{4} (4 \partial_\alpha^2 \phi - 5 (\partial_\alpha \phi)^2) + 12 A_u \right] j^{(2)}(u)$$

$$+ \frac{3}{2} A_u \partial_\alpha \phi \partial_\alpha j^{(1)}(u) + \left[ 2 A_u \left( \partial_\alpha^2 \phi - \frac{5}{4} (\partial_\alpha \phi)^2 \right) + \frac{3}{2} \partial_\alpha A_u \partial_\alpha \phi - \frac{1}{2} \partial_\alpha^2 A_u + 8 A^3_u \right] j^{(1)}(u)$$

$$- \frac{h}{2\pi} A_u (\partial_\alpha - 2 \partial_\alpha \phi) (\partial_\alpha - \partial_\alpha \phi) A_u + \frac{h}{2\pi} A_u^2 \left( \partial_\alpha^2 \phi - \frac{1}{2} (\partial_\alpha \phi)^2 \right) + \frac{2h}{\pi} A^4_u$$

$$- \frac{h}{160\pi} (\partial_\alpha - 3 \partial_\alpha \phi) (\partial_\alpha - 2 \partial_\alpha \phi) \left( \partial_\alpha^2 \phi - \frac{1}{2} (\partial_\alpha \phi)^2 \right) + \frac{7h}{480\pi} \left( \partial_\alpha^2 \phi - \frac{1}{2} (\partial_\alpha \phi)^2 \right)^2 .$$

(4.20)

First we derive the $uuu$ component of the conservation equation from this equation by taking the derivative of (4.20) with respect to $v$ and multiplying $g^{uv}$,

$$\nabla^u J^{(4)}_{uuu} = 3g^{uv} F_{vu} J^{(3)}_{uuu} - \frac{3}{8} R \nabla^u J^{(2)}_{uu} - \frac{1}{2} J^{(2)}_{uu} \nabla^u R - \frac{1}{4} g^{uv} J^{(1)}_{uu} \nabla^2 F_{vu} + \frac{h}{320\pi} \nabla^3 u R .$$

(4.21)
As in the case of the spin 3 current, we regard the last term as the contribution of the trace anomaly because it is proportional to \( \hbar \) and quantum. From the assumptions 1 and 2, \( J^{(4)}_{uvuu} \) is given by

\[
J^{(4)}_{uvuu} = -\frac{\hbar}{320\pi} g_{uv} \nabla^2 R.
\] (4.22)

From (4.21), we guess the covariant conservation equation as

\[
\nabla^\mu J^{(4)}_{\mu \nu \rho \sigma} = F_{\mu \nu} J^{(3)}_{\rho \sigma} + F_{\mu \rho} J^{(3)}_{\nu \sigma} + F_{\mu \sigma} J^{(3)}_{\nu \rho} - \frac{1}{8} R \left( \nabla_\nu J^{(2)}_{\rho \sigma} + \nabla_\rho J^{(2)}_{\sigma \nu} + \nabla_\sigma J^{(2)}_{\nu \rho} \right)
\]

\[
- \frac{1}{6} \left( J^{(2)}_{\nu \rho} \nabla_\sigma R + J^{(2)}_{\rho \sigma} \nabla_\nu R + J^{(2)}_{\sigma \nu} \nabla_\rho R \right)
\]

\[
- \frac{1}{24} \left( J^{(1)}_{\nu \rho} \nabla_\sigma F^{\mu \sigma} + J^{(1)}_{\rho \sigma} \nabla_\mu F^{\mu \nu} + J^{(1)}_{\sigma \nu} \nabla_\mu F^{\mu \rho} + J^{(1)}_{\rho \nu} \nabla_\sigma F^{\mu \rho} \right).
\] (4.23)

Next we add appropriate terms proportional to \( g_{\mu \nu} \) so that this conservation equation becomes classically traceless. In general, one can construct a rank 3 traceless symmetric tensor from any rank 3 symmetric tensor \( B_{\nu \rho \sigma} \) by subtracting the trace part \( (g_{\nu \rho} B_{\mu \mu \sigma} + g_{\sigma \rho} B_{\mu \mu \nu} + g_{\sigma \nu} B_{\mu \mu \rho})/4 \).

We define \( C_\nu \) as the trace of (4.23),

\[
C_\nu \equiv g^{\rho \sigma} \nabla^\mu J^{(4)}_{\mu \rho \sigma} = F_{\mu \nu} J^{(3)}_{\rho \sigma} - \frac{1}{8} R \nabla_\nu J^{(2)}_{\rho \sigma} - \frac{1}{4} R \nabla_\rho J^{(2)}_{\nu \sigma} - \frac{1}{3} J^{(2)}_{\nu \rho} \nabla_\sigma R - \frac{1}{6} J^{(2)}_{\rho \sigma} \nabla_\nu R
\]

\[
- \frac{1}{12} \left( J^{(1)}_{\nu \rho} \nabla_\sigma F^{\mu \sigma} + J^{(1)}_{\rho \sigma} \nabla_\mu F^{\mu \nu} + J^{(1)}_{\sigma \nu} \nabla_\mu F^{\mu \rho} \right).
\] (4.24)

Note that \( C_\nu \) includes the traces of the covariant spin 2 and 3 currents which vanish classically but not at the quantum level due to the trace anomalies. According to our assumption 1, we treat such anomalous quantities as contributions of the trace anomaly. Therefore we define \( \tilde{C}_\nu \) as \( C_\nu \) without the anomalous terms,

\[
\tilde{C}_\nu \equiv -\frac{1}{4} R \nabla_\rho J^{(2)}_{\nu \rho} - \frac{1}{3} J^{(2)}_{\nu \rho} \nabla_\sigma R - \frac{1}{12} \left( J^{(1)}_{\nu \rho} \nabla_\sigma F^{\mu \sigma} + J^{(1)}_{\rho \sigma} \nabla_\mu F^{\mu \nu} \right).
\] (4.25)
and construct a new conservation equation, which are classically traceless,

\[ \nabla^\mu J^{(4)}_{\mu \rho \sigma} = F_{\mu \nu} J^{(3)}_{\nu \rho} + F_{\mu \rho} J^{(3)}_{\rho \sigma} + F_{\mu \sigma} J^{(3)}_{\sigma \nu} - \frac{1}{8} R \left( \nabla_\nu J^{(2)}_{\rho \sigma} + \nabla_\rho J^{(2)}_{\nu \sigma} + \nabla_\sigma J^{(2)}_{\nu \rho} \right) \]

Next the \( uu \) component of the trace anomaly can be read from (4.22),

\[ J^{(4)}_{\mu \mu uu} = -\frac{\hbar}{160\pi} \nabla^2 R. \]  

Then general components of the trace anomaly have the following form,

\[ J^{(4)}_{\mu \nu \rho} = -\frac{\hbar}{160\pi} \nabla_\nu \nabla_\rho R + g_{\nu \rho} A, \]

where \( A \) is not fixed from (4.27) only. We can determine \( A \) by imposing consistency of (4.28) with (4.26). The trace of (4.26) becomes

\[ \nabla_\mu J^{(4)}_{\mu \rho \sigma} = -\frac{\hbar}{160\pi} \nabla_\rho \nabla_\sigma R + \frac{1}{8} J^{(2)}_{\rho \sigma} \nabla R - \frac{1}{6} R \left( \nabla_\nu J^{(2)}_{\rho \sigma} + \nabla_\sigma J^{(2)}_{\rho \nu} - \nabla_\rho J^{(2)}_{\nu \sigma} \right) \]

where \( \tilde{F} \equiv \epsilon^{\mu \nu} F_{\mu \nu} / 2 = g^{uv} F_{uv} \). On the other hand, the divergence of the (4.28) is

\[ \nabla_\mu J^{(4)}_{\mu \nu \rho} = -\frac{\hbar}{160\pi} \left( \nabla_\rho \nabla_\nu R + \frac{1}{4} \nabla_\nu R^2 \right) + \nabla_\rho A. \]

By comparing these two equations, \( A \) is determined as

\[ A = \frac{\hbar}{160\pi} \nabla^2 R + \frac{\hbar}{24\pi} \left( \tilde{F}^2 - \frac{7}{48} R^2 \right). \]

As a result, we obtain the trace anomaly of the spin 4 current,

\[ J^{(4)}_{\mu \nu \rho} = -\frac{\hbar}{160\pi} \nabla_\nu \nabla_\rho R + g_{\nu \rho} \left[ \frac{\hbar}{160\pi} \nabla_\sigma R + \frac{\hbar}{24\pi} \left( \tilde{F}^2 - \frac{13}{120} R^2 \right) \right]. \]
fields from the conservation equation (4.26),
\[
\delta \xi B^{(4)\mu\nu\rho\sigma} = \frac{1}{4} (\nabla_\mu \xi_{\nu\rho\sigma} + \nabla_\nu \xi_{\rho\sigma\mu} + \nabla_\rho \xi_{\sigma\mu\nu} + \nabla_\sigma \xi_{\mu\nu\rho}) \tag{4.33}
\]
\[
\delta \xi B^{(3)\mu\nu\rho} = -3 \xi_{\mu\nu\sigma} F^{\sigma \rho} - 3 \xi_{\nu\rho\sigma} F^{\rho \mu} - 3 \xi_{\rho\mu\sigma} F^{\sigma \nu}, \tag{4.34}
\]
\[
\delta \xi g^{\mu\nu} = \frac{3}{4} \nabla_\rho (\xi^{\mu\nu\rho} R) - \xi^{\mu\nu\rho} \nabla_\rho R - \frac{3}{16} \nabla_\mu (R \xi_\rho^{\ \rho\nu}) - \frac{3}{16} \nabla_\nu (R \xi_\rho^{\ \rho\mu}) + \frac{1}{4} \xi_\rho^{\mu\nu} \nabla_\mu R + \frac{1}{4} \xi_\rho^{\rho\mu} \nabla_\mu R \tag{4.35}
\]
\[
\delta \xi A_\mu = -\frac{1}{4} \xi^{\mu\nu\rho} \nabla_\rho \nabla_\nu F^{\nu \sigma} + \frac{1}{16} \xi_\rho^{\ \rho\sigma} [\nabla_\mu \nabla_\nu F^{\nu \sigma} + \nabla_\sigma \nabla_\nu F^{\mu \nu}], \tag{4.36}
\]
where $\xi^{\mu\nu\rho}$ denotes a symmetric traceless parameter.

5 Higher-spin gauge anomalies

In the previous section, we have obtained the conservation equations and trace anomalies in the HS currents by considering non-chiral theories; i.e. the anomaly coefficients are the same between the holomorphic and anti-holomorphic sectors.

In this section, we consider a chiral fermionic theory where we have $c_L$ left-handed fermions and $c_R (\neq c_L)$ right-handed fermions. In this case, the conservation equation becomes anomalous. This is a generalization of the gauge or gravitational anomalies to the HS currents. If these HS currents are coupled to HS gauge fields, these violation of conservation equations lead to quantum violation of HS local symmetries.

We here remark that, in the presence of $c_R$ right-handed and $c_L$ left-handed fermions, the coefficients of the anomalous terms in the $(u \cdots u)$ sector are multiplied by $c_R$, and those in $(v \cdots v)$ sector by $c_L$.

In the following of this section, we will derive the anomalous conservation equations for the currents up to rank 4.

5.1 $U(1)$ gauge and gravitational anomalies

In this subsection, we reproduce the gauge and gravitational anomalies from the relations between the (anti-) holomorphic and covariant $U(1)$ and spin 2 currents.

First we consider the $U(1)$ current. The relations in the present case with $c_L \neq c_R$ become
\[
J_u = j(u) + \frac{c_R h}{\pi} A_u, \quad J_v = j(v) + \frac{c_L h}{\pi} A_v. \tag{5.1}
\]
By taking derivatives of these equations, we obtain

$$\nabla_v J_u + \nabla_u J_v = \frac{(c_R - c_L)}{2} \frac{\hbar}{\pi} F_{vu}, \tag{5.2}$$

$$\nabla_v J_u - \nabla_u J_v = \frac{(c_R + c_L)}{2} \frac{\hbar}{\pi} F_{vu}, \tag{5.3}$$

where we have used the Lorenz gauge condition \( \partial_u A_v = -\partial_v A_u \). They can be written in the covariant forms as

$$\nabla_\mu J^\mu = -\frac{(c_R - c_L)}{2} \frac{\hbar}{\pi} \epsilon^{\mu\nu} F_{\mu\nu}, \tag{5.4}$$

$$\nabla_\mu J^{5\mu} = \frac{(c_R + c_L)}{2} \frac{\hbar}{\pi} \epsilon^{\mu\nu} F_{\mu\nu}. \tag{5.5}$$

Thus, if \( c_L \neq c_R \), the gauge symmetry is broken by the anomaly.

Next we consider the energy-momentum tensor. Now the relation (2.8) is modified as

$$t(u) = T_{uu} - 2A_u j(u) - \frac{c_R \hbar}{\pi} A_u^2 - \frac{c_R \hbar}{24 \pi} \left( \partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right). \tag{5.6}$$

We can also obtain a similar equation for the right-handed fermion. By taking derivatives of them, we obtain

$$\nabla^u T_{uu} = F_{vu} g^{vu} J_u - \frac{c_R \hbar}{48 \pi} \partial_u R$$

$$= F_{vu} g^{vu} J_u - \frac{\hbar}{48 \pi} \left( \frac{c_R - c_L}{2} + \frac{c_L + c_R}{2} \right) \partial_u R \tag{5.7}$$

$$\nabla^v T_{vv} = F_{uv} g^{uv} J_v - \frac{\hbar}{48 \pi} \left( -\frac{c_R - c_L}{2} + \frac{c_L + c_R}{2} \right) \partial_v R \tag{5.8}$$

In the case of the non-chiral theory \( (c_L = c_R) \), we can regard the anomalous terms as the contribution of the trace anomaly. However, in the case \( c_L \neq c_R \), the terms proportional to \( (c_L - c_R) \) cannot be regarded as the contribution of the trace anomaly. As a result, we obtain the following anomalous conservation equation and trace anomaly equation:

$$\nabla^\mu T_{\mu\nu} = F_{\mu\nu} J^\mu - \frac{\hbar}{48 \pi} \frac{c_R - c_L}{2} \epsilon_{\mu\nu} \nabla^\mu R, \tag{5.9}$$

$$T^\mu_\mu = \frac{\hbar}{24 \pi} \frac{c_L + c_R}{2} R. \tag{5.10}$$

The first equation reproduces the gravitational anomaly for the covariant EM tensor.

### 5.2 Spin 3 and 4 gauge anomalies

We have shown that our method reproduces the correct anomaly equations for the rank 1 and 2 currents in the chiral theory. We further consider a generalization to higher-spin currents.
First we study the rank 3 current. The equation [4.4] now becomes
\[
\nabla_v J^{(3)}_{uu} = -2F_{uv}J^{(2)}_{uu} - \frac{1}{8} \nabla_u \left( g_{uv} R J^{(1)}_u \right) + \frac{\hbar}{24\pi} \left( \frac{c_R - c_L}{2} + \frac{c_L + c_R}{2} \right) \nabla^2 F_{uv}. \tag{5.11}
\]

We also obtain a similar equation for \( J^{(3)\nu}_{vv} \). As in the case of the energy-momentum tensor, we cannot regard the anomalous term proportional to \( (c_R - c_L) \) as the contribution of the trace anomaly.

Equations consistent with (5.11) can be given as follows:
\[
\nabla_\mu J^{(3)\mu\nu\rho} = -F_{\nu\rho} J^{(2)\mu\rho} - F_{\rho\mu} J^{(2)\mu\nu} - \frac{1}{16} \nabla_\nu \left( R J^{(1)}_\rho \right) - \frac{1}{16} \nabla_\rho \left( R J^{(1)}_\nu \right) + \frac{1}{16} g_{\nu\rho} \nabla_\mu \left( R J^{(1)\mu} \right)
\]
\[+ \frac{\hbar}{48\pi} \frac{c_R - c_L}{2} \left( \epsilon_{\nu\sigma} \nabla^\sigma F^{\mu}_\rho + \epsilon_{\rho\sigma} \nabla^\sigma F^{\mu}_\nu - g_{\nu\rho} \epsilon_{\sigma\alpha} \nabla^\sigma \nabla_\mu F^{\mu\alpha} \right), \tag{5.12}
\]
\[J^{(3)\mu\nu} = \frac{\hbar}{12\pi} \frac{c_L + c_R}{2} \nabla_\mu F^{\mu\nu}. \tag{5.13}
\]

This is the spin 3 generalization of the gauge or gravitational anomaly. Note that, the conservation equation has been modified, but the transformation properties of the background gauge fields [4.13] - [4.17] are not changed, since they are classical properties.

We can similarly obtain a generalization to the rank 4 current;
\[
\nabla_\mu J^{(4)\mu\nu\rho\sigma} = F_{\nu\rho} J^{(3)\mu\nu\rho} + F_{\rho\mu} J^{(3)\mu\nu\sigma} + F_{\nu\sigma} J^{(3)\mu\nu\rho} - \frac{1}{8} R \left( \nabla_\nu J^{(2)\rho\sigma} + \nabla_\rho J^{(2)\sigma\nu} + \nabla_\sigma J^{(2)\nu\rho} \right)
\]
\[ - \frac{1}{6} \left( J^{(2)\nu}_\nu \nabla_\sigma R + J^{(2)\rho}_\rho \nabla_\nu R + J^{(2)\sigma}_\sigma \nabla_\nu R \right)
\]
\[ - \frac{1}{24} \left( J^{(1)\nu \rho}_\nu \nabla_\sigma F^{\mu}_\sigma + J^{(1)\nu \sigma}_\nu \nabla_\rho F^{\mu}_\sigma + J^{(1)\rho \nu}_\rho \nabla_\sigma F^{\mu}_\sigma \nabla_\nu F^{\mu}_\rho + J^{(1)\rho \sigma}_\rho \nabla_\nu F^{\mu}_\sigma \nabla_\nu F^{\mu}_\rho \right)
\]
\[+ \frac{\hbar}{960\pi} \frac{c_R - c_L}{2} \left( \epsilon_{\nu\rho\sigma} \nabla^\sigma F^{\mu}_\rho \nabla_\sigma R + \epsilon_{\rho\sigma} \nabla^\sigma F^{\mu}_\sigma \nabla_\nu R + \epsilon_{\sigma\rho} \nabla^\sigma F^{\mu}_\rho \nabla_\nu R \right)
\]
\[+ \frac{1}{4} \left( g_{\nu\rho} \hat{C}_\sigma + g_{\rho\sigma} \hat{C}_\nu + g_{\sigma\nu} \hat{C}_\rho \right), \tag{5.14}
\]

\[J^{(4)\mu\nu}_\rho = \frac{\hbar}{160\pi} \frac{c_L + c_R}{2} \nabla_\rho \nabla_\nu R + \frac{c_L + c_R}{2} \left[ \frac{\hbar}{160\pi} \nabla^2 R + \frac{\hbar}{24\pi} \left( \vec{F}^2 - \frac{13}{120} R^2 \right) \right]. \tag{5.15}
\]

Here we have modified \( \hat{C}_\nu \) to \( \hat{C}_\nu \) including the anomalous terms as follows,
\[
\hat{C}_\nu \equiv - \frac{1}{4} R \nabla_\rho J^{(2)\rho \nu}_\nu - \frac{1}{3} J^{(2)\rho \nu}_\nu \nabla_\rho R - \frac{1}{12} \left( J^{(1)\rho \nu \rho}_\rho \nabla_\mu F^{\mu\nu} + J^{(1)\rho \nu}_\rho \nabla_\mu F^{\mu\nu} \right)
\]
\[+ \frac{1}{2} \frac{\hbar}{960\pi} \frac{c_R - c_L}{2} \left( \epsilon_{\nu\rho\sigma} \nabla^\sigma F^{\rho\nu} + 2 \epsilon_{\rho\sigma} \nabla^\sigma \nabla_\rho F^{\rho\nu} \right). \tag{5.16}
\]

This is the spin 4 generalization of the gauge and gravitational anomalies. The r.h.s. of (5.14) contains both of classical and quantum parts. The classical parts arises due to the same reason as in the non-chiral case in section 4. The quantum parts are the anomalies.
6 Summary

In this paper, we considered a two-dimensional theory of fermions in the electric and gravitational backgrounds and obtained a generalization of the gauge, gravitational and trace anomalies for higher-spin (HS) currents up to spin 4. In order to derive these anomalies, we started from the relation between holomorphic and covariant forms of HS currents in the electric and gravitational backgrounds.

These anomaly equations can be applied to derive the higher-spin fluxes of Hawking radiation. This will be discussed in a separate paper [15].

In the cases of spins 1 and 2, the forms of anomalies can be determined by the descent equations and they have nice geometrical meanings. It will be interesting to investigate higher-spin anomalies than 4, and examine whether there are any systematic structures in the form of anomalies.

Finally we notice that the anomalies we obtained are specific to HS currents constructed from fermions. If they are constructed from bosons, their anomalies have different combinations with different coefficients.

A Holomorphic and covariant currents up to spin 4

In section 3 we represented the covariant currents in terms of the holomorphic currents as in (4.3). However, when we calculate the conservation equation, it is more convenient to describe the holomorphic currents in terms of the covariant currents and thus we give their explicit expressions by expanding the generating function [3.27].

\[ j^{(1)}(u) =: \Psi \dagger \Psi := J^{(1)}_u - \frac{1}{\pi} A_u \] (A.1)

**spin 2 current**

\[ j^{(2)}(u) = \frac{i}{2} : \Psi \dagger \partial_u \Psi - \partial_u \Psi \dagger \Psi := J^{(2)}_{uu} - 2A_u J^{(1)}_u + \frac{1}{\pi} A^2 - \frac{1}{24\pi} \left( \partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right) \] (A.2)

**spin 3 current**

\[ j^{(3)}(u) = -\frac{1}{4} : \Psi \dagger \partial^2_u \Psi - 2\partial_u \Psi \dagger \partial_u \Psi + \partial^2_u \Psi \dagger \Psi : \]

\[ = J^{(3)}_{uuu} - 4A_u J^{(2)}_{uu} - \frac{1}{4} \partial_u \varphi \partial_u J^{(1)}_u - \left( -4A^2 + \frac{1}{4} \left( \partial_u^2 \varphi - (\partial_u \varphi)^2 \right) \right) J^{(1)}_u \]

\[ + \frac{1}{6\pi} \left( \partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right) A_u + \frac{1}{12\pi} \partial_u^2 A_u - \frac{4}{3\pi} A^3 \] (A.3)
spin 4 current

\[ j^{(4)}(u) = \frac{i}{8} : \partial^2_u \Psi \Psi - 3 \partial^2_u \Psi \partial_u \Psi + 3 \partial_u \Psi \partial^2_u \Psi - \Psi \partial^2_u \Psi : \]

\[ = J^{(4)}_{uuuu} - 6 A_u J^{(3)}_{uuu} + \frac{3}{4} \partial_u \varphi \partial_u J^{(2)}_{uu} - \left[ \partial^2_u \varphi - \frac{5}{2} (\partial_u \varphi)^2 - 12 A_u \right] J^{(2)}_{uu} \]

\[ + \frac{3}{2} A_u \partial_u \varphi \partial_u J^{(1)}_u \left[ -\frac{3}{2} A_u \left( \partial^2_u \varphi - (\partial_u \varphi)^2 \right) - \frac{1}{2} \partial^2_u A_u + 8 A_u^3 \right] J^{(1)}_u \]

\[ - \frac{1}{2 \pi} A_u \left[ \partial^2_u A_u + \left( \partial^2_u \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right) A_u - 4 A_u^3 \right] \]

\[ + \frac{1}{160 \pi} \left( \partial^4_u \varphi - \partial_u \varphi \partial^3_u \varphi + \frac{4}{3} (\partial^2_u \varphi)^2 - \frac{7}{3} (\partial_u \varphi)^2 \partial^2_u \varphi + \frac{7}{12} (\partial_u \varphi)^4 \right) . \]  

(A.4)

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