THE GENERALIZED LINEAR PERIODS

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Abstract. Let $F$ be a local field of characteristic zero. Let $\mu$ be a good character of $\text{GL}_n(F) \times \text{GL}_{p+1}(F)$. We study the generalized linear period problem for the pair $(G, H_{p,p+1}) = (\text{GL}_{2p+1}(F), \text{GL}_p(F) \times \text{GL}_{p+1}(F))$ and we prove that any bi-($H_{p,p+1}, \mu$)-equivariant tempered generalized function on $G$ is invariant under the matrix transpose. We also show that any $P \cap H_{p,p+1}$-invariant linear functional on an $H_{p,p+1}$-distinguished irreducible smooth representation of $G$ is also $H_{p,p+1}$-invariant if $F$ is nonarchimedean, where $P$ is the standard mirabolic subgroup of $G$ consisting of matrices with last row vector $(0, \ldots, 0, 1)$.

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1. Introduction

Let $F$ be a local field of characteristic zero. Let $p, q, n$ be positive integers and $n = p + q$. Let $\theta_{p,q}$ be the involution defined on $\text{GL}_n(F)$ given by

$$\theta_{p,q}(g) = \omega_{p,q} \cdot g \cdot \omega_{p,q}$$

for $g \in \text{GL}_n(F)$ where $\omega_{p,q} = \begin{pmatrix} 1_p & \ 0 \\ 0 & 1_q \end{pmatrix}$ and $1_p$ (resp. $1_q$) is the identity matrix in the $p \times p$ (resp. $q \times q$) matrix space $\text{Mat}_{p,p}(F)$ (resp. $\text{Mat}_{q,q}(F)$). Let $H_{p,q}$ be the fixed points of $\theta_{p,q}$ in $\text{GL}_n(F)$. Then $H_{p,q} \cong \text{GL}_p(F) \times \text{GL}_q(F)$. It is well known that the pair $(\text{GL}_n(F), \text{GL}_p(F) \times \text{GL}_q(F))$ satisfies the Gelfand-Kazhdan criterion [AG09a, §7] with respect to the inverse map; see [JR96] for the non-archimedean case and [AG09a, Theorem 7.1.3] for the archimedean case. It implies that $\dim \text{Hom}_{H_{p,q}}(\pi, \mathbb{C}) \leq 1$ for all irreducible admissible smooth representation $\pi$ of $\text{GL}_n(F)$. Jacquet-Rallis [JR96] proved that if $\dim \text{Hom}_{H_{p,q}}(\pi, \mathbb{C}) = 1$ and $F$ is non-archimedean, then $\pi \cong \pi^\vee$ where $\pi^\vee$ denotes the representation of $\text{GL}_n(F)$ contragredient to $\pi$. When $p = q$, it is closely related to the Shalika period problems (see [CS20]). Furthermore, Friedberg-Jacquet [FJ93] have studied the relation between the linear period of $\pi$ and the exterior square $L$-function $L(s, \pi, \Lambda^2)$.

This paper studies the twisted version of the linear period. We say that a character of $F^\times$ is pseudo-algebraic if it has the form

$$t \mapsto \begin{cases} 1, & \text{if } F \text{ is nonarchimedean,} \\
 t^n, & \text{if } F = \mathbb{R}, \\
 i(t)^m i'(t)^{m'}, & \text{if } F \cong \mathbb{C}, 
\end{cases}$$

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where $m$ and $m'$ are non-negative integers and $\iota$ and $\iota'$ are two distinct topological isomorphisms from $F$ to $\mathbb{C}$. Let $\mu_F$ be a character of $F^\times$ and $\mu_F \circ \det$ be a character of $GL_p(F)$. Let $\mu_F \circ \det \otimes \mathbb{C}$ be a character of $H_{p,q}$, denoted by $\mu$. We say that $\mu$ is a good character of $H_{p,q}$ if

- $\mu^{2r} - |^{-s}$ is not pseudo-algebraic for all $r \in \{±1,±2,\cdots,±p\}$ and all $s \in \{1,2,\cdots,2p^2\}$. (See [CS20] for more details.) From now on, we assume that $q = p + 1$ throughout this paper, unless otherwise specified. One of the main results in this paper is the following:

**Theorem 1.1.** Suppose that $n = 2p + 1$. Let $f$ be a tempered generalized function on $GL_n(F)$. If for every $h \in H_{p,p+1}$,

$$f(hx) = f(xh) = \mu(h)f(x)$$

for $x \in GL_n(F)$ and any good character $\mu$, as generalized functions on $GL_n(F)$, then

$$f(x) = f(x').$$

Here and as usual, a superscript “$t$” indicates the transpose of a matrix. Then the pair $(GL_{2p+1}(F), GL_p(F) \times GL_{p+1}(F))$ satisfies the generalized Gelfand-Kazhdan criterion (see [SZ11] Theorem 2.3) with respect to the matrix transpose, which implies that

$$\dim \text{Hom}_{H_{p,p+1}}(\pi, \mu) \cdot \dim \text{Hom}_{H_{p,p+1}}(\pi', \mu^{-1}) \leq 1$$

for any irreducible admissible smooth representation $\pi$ of $GL_{2p+1}(F)$ and any good character $\mu$ of $H_{p,p+1}$.

The analogue for the pair $(GL_{2p}(F), GL_p(F) \times GL_{p+1}(F))$ has been proved by Chen-Sun in [CS20]. We will use a similar idea appearing in [CS20] to prove Theorem 1.1.

Define $I_{p,p+1} := Mat_{p+1}(F) \oplus Mat_{p+1}(F)$ and $N_{p,p+1} := \{(x,y) \in I_{p,p+1}|(xy)^p = 0\}$. Denote by $\mathcal{C}_{N_{p,p+1}}(I_{p,p+1})$ the space consisting of tempered generalized functions on $I_{p,p+1}$ supported on $N_{p,p+1}$. By linearization, Theorem 1.1 is reduced to the following theorem.

**Theorem 1.2.** Let $f$ be a tempered generalized function on $I_{p,p+1}$ supported on the nilpotent cone $N_{p,p+1}$ such that for $h = \left( \begin{array}{ll} a & 0 \\ b & 1 \end{array} \right) \in H_{p,p+1},$

$$f(axb^{-1}, bya^{-1}) = f(x,y)$$

holds for any $(x,y) \in I_{p,p+1}$. Then $f(x,y) = f(y^\iota, x^\iota)$.

There is a brief introduction to the proof of Theorem 1.2. We will regard the $n$-dimensional vector space as a graded $sl_2(F)$-module. Chen-Sun [CS20] used the graded modules and Fourier transform to prove that there does not exist any $H_{p,p'}$-invariant generalized function $f$ on $Mat_{p,p}(F) \times Mat_{p,p}(F)$ such that both $f$ and its Fourier transform $\mathcal{F}(f)$ are supported on the nilpotent cone of $I_{p,p+1}$, where $\mathcal{F}$ is the Fourier transform of $f$ on $I_{p,p+1}$ such that both $f_0$ and its Fourier transform $\mathcal{F}(f_0)$ are supported on the orbit $H_{p,p+1} e$ (the regular nilpotent orbit), where $e^{Sp} \neq 0$. There is a key observation due to Dmitry Gourevitch that $e^\iota \in H_{p,p+1} e$ and so if $f_0 \in \mathcal{C}_{N_{p,p+1}}(I_{p,p+1})^{H_{p,p+1}, e}$ (see Theorem 3.1) then $f_0 = 0$. Therefore,

$$\mathcal{C}_{N_{p,p+1}}(I_{p,p+1})^{H_{p,p+1}, e} = 0$$

i.e. Theorem 1.2 holds. (All the techniques in this paper work for the pair $(GL_{2p+1}(F), GL_{p+1}(F) \times GL_p(F))$ as well. But they do not work for the pair $(GL_{2p+2}(F), GL_p(F) \times GL_{p+2}(F))$ because Proposition 6.10 fails; see Remark 6.11.) In fact, we will prove a stronger result that any $H_{p,p}$-invariant generalized function on $I_{p,p+1}$ is also invariant under transposition, where $H_{p,p}$ is a proper subgroup of $H_{p,p+1}$. (See the proof of Theorem 6.4.)

In a similar way, we can prove the following.

**Theorem 1.3.** Let $f$ be a tempered generalized function on $GL_n(F)$. Let $\mu_F$ be any character (not necessarily good) of $F^\times$. If for every $h \in H_{1,n-1},$

$$f(hx) = f(xh) = \mu_F(a)f(x)$$
Theorem 2.1. [AG09a, Theorem 3.1.1] Then let $G$ be the stabilizer subgroup of $G$ consisting of those tempered generalized functions $X$ on $a$ Nash manifold (see [AG09a, §2.3]) if $F$ is archimedean. Let $\mathcal{C}(X)$ denote the space of tempered generalized functions on $X$. Let a reductive group $G(F)$ act on an affine variety $X$. Let $x \in X$ such that its orbit $G(F) x$ is closed in $X$. We denote the normal bundle by $N^X_{G(F)x}$ and denote its fiber (the normal space) at the point $x$ by $N^X_{G(F)x,x}$. Let

$$G_x := \{ g \in G(F) | gx = x \}$$

be the stabilizer subgroup of $x$. Let $\chi$ be a character of $G(F)$. Denote by $\mathcal{C}(X)^{G(F),x}$ the subspace in $\mathcal{C}(X)$ consisting of those tempered generalized functions $f$ satisfying

$$g \cdot f = \chi(g)f$$

for all $g \in G(F)$. If $\chi$ is trivial, then it will be denoted by $\mathcal{C}(X)^{G(F)}$.

**Theorem 2.1.** [AG09a, Theorem 3.1.1] Let $G(F)$ act on a smooth affine variety $X$. Let $\chi$ be a character of $G(F)$. Suppose that for any closed orbit $G(F)x$ in $X$, we have

$$\mathcal{C}(N^X_{G(F)x,x})^{G_x,x} = 0.$$

Then

$$\mathcal{C}(X)^{G(F),x} = 0.$$

If $V$ is a finite dimensional representation of $G(F)$, then we denote the nilpotent cone in $V$ by

$$\Gamma(V) := \{ x \in V | [G(F)x \ni 0] \}.$$

Let $Q_G(V) := V/V^G$. There is a canonical embedding $Q_G(V) \hookrightarrow V$ (see [AG09a, Notation 2.3.10]). Set $R_G(V) := Q(V) \setminus \Gamma(V)$. There is a stronger version of Theorem 2.1.
Theorem 2.2. [AG09a Corollary 3.2.2]. Let $X$ be a smooth affine variety. Let $G(F)$ act on $X$. Let $K \subset G(F)$ be an open subgroup and let $\chi$ be a character of $K$. Suppose that for any closed orbit $G(F)x$ such that
\[ \mathcal{C}(R_{G,x}(N_{G(F)}x,x))^{Kx,\chi} = 0 \]
we have
\[ \mathcal{C}(Q_{G,x}(N_{G(F)}x,x))^{Kx,\chi} = 0. \]
Then $\mathcal{C}(X)^{K,\chi} = 0$.

3. A vanishing result of generalized functions

In this section, we shall use $q$ to denote $p + 1$. Let
\[ I_{p,q} = \text{Mat}_{p,q}(F) \oplus \text{Mat}_{q,p}(F) = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x \in \text{Mat}_{p,q}(F), y \in \text{Mat}_{q,p}(F) \right\} \subset \mathfrak{g}l_n(F). \]
Denote by
\[ N_{p,q} := \{(x,y) \in I_{p,q} | xy \text{ is a nilpotent matrix in } \text{Mat}_{p,p}(F) \} \]
the nilpotent cone in $I_{p,q}$. Denote $\tilde{H}_{p,q} := H_{p,q} \rtimes \langle \sigma \rangle$ where $\sigma$ acts on $H_{p,q}$ by the involution
\[ \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} (a^{-1})^t \\ (b^{-1})^t \end{pmatrix}. \]
The group $\tilde{H}_{p,q}$ acts on $I_{p,q}$ by
\[ \begin{pmatrix} a \\ b \end{pmatrix} \cdot (x,y) = (axb^{-1}, bya^{-1}) \]
and
\[ \sigma \cdot (x,y) = (y^t, x^t) \]
for $(x,y) \in I_{p,q}$. Let $\chi$ be the sign character of $\tilde{H}_{p,q}$, i.e. $\chi|_{H_{p,q}}$ is trivial and $\chi(\sigma) = -1$.

Denoted by $\mathcal{C}_{N_{p,q}}(I_{p,q})$ the space of tempered generalized functions on $I_{p,q}$ supported on $N_{p,q}$. Set
\[ \mathcal{C}_{N_{p,q}}(I_{p,q})^{\tilde{H}_{p,q},\chi} := \{ f \in \mathcal{C}_{N_{p,q}}(I_{p,q}) | g \cdot f = \chi(g)f \text{ for all } g \in \tilde{H}_{p,q} \}. \]

Theorem 3.1. We have $\mathcal{C}_{N_{p,q}}(I_{p,q})^{\tilde{H}_{p,q},\chi} = 0$.

The rest part of this section is devoted to proving Theorem 3.1. Then Theorem 1.2 follows from Theorem 3.1 directly by definition.

Define a non-degenerate symmetric $F$-bilinear form on $\mathfrak{g}l_n(F)$ by
\[ \langle z, w \rangle_{\mathfrak{g}l_n(F)} := \text{the trace of } zw \text{ as a } F\text{-linear operator.} \]
Note that the restriction of this bilinear form on $I_{p,q}$ is still non-degenerate. Fix a non-trivial unitary character $\psi$ of $F$. Denote by
\[ \mathcal{F} : \mathcal{C}(I_{p,q}) \rightarrow \mathcal{C}(I_{p,q}) \]
the Fourier transform which is normalized such that for every Schwartz function $\varphi$ on $I_{p,q}$,
\[ \mathcal{F}(\varphi)(z) = \int_{I_{p,q}} \varphi(w)\psi(\langle z, w \rangle_{\mathfrak{g}l_n(F)})dw \]
for $z \in I_{p,q}$, where $dw$ is the self-dual Haar measure on $I_{p,q}$. If $I_{p,q}$ can be decomposed into a direct sum of two quadratic subspaces $U_1 \oplus U_2$ such that each $U_i$ is non-degenerate with respect to $\langle -, - \rangle|_{U_i}$, then we may define the partial Fourier transform
\[ \mathcal{F}_{U_1}(\varphi)(x,y) = \int_{U_1} \varphi(z,y)\psi(\langle x, z \rangle|_{U_1})dz \]
for $x \in U_1, y \in U_2$ and $\varphi \in \mathcal{C}(U_1 \oplus U_2)$. Similarly for $\mathcal{F}_{U_2}(\varphi)$. It is clear that the Fourier transform $\mathcal{F}$ intertwines the action of $\tilde{H}_{p,q}$. Thus we have the following lemma.
Lemma 3.2. The Fourier transform \( \mathfrak{F} \) preserves the space \( \mathcal{C}_{\mathcal{N}_{p,q}}(I_{p,q})^{H_{p,q}} \).

3.1. Reduction within the null cone. Recall
\[
\mathcal{N}_{p,q} = \{(x, y) \in I_{p,q} | xy \text{ is a nilpotent matrix in } \text{Mat}_{p,p}(F)\}.
\]
Let \( \mathcal{O} \) be an \( H_{p,q} \)-orbit in \( \mathcal{N}_{p,q} \). Recall that every \( e \in \mathcal{O} \) can be extended to a graded \( \mathfrak{sl}_2 \)-triple \( \{h, e, f\} \) (see [KR71, Proposition 4]) in the sense that
\[
[h, e] = 2e, \ [h, f] = -2f \text{ and } [e, f] = h
\]
where \( f \in \mathcal{N}_{p,q} \) and \( h \in \mathfrak{h}_{p,q} \), where \( \mathfrak{h}_{p,q} = \mathfrak{gl}_n(F) \oplus \mathfrak{gl}_n(F) \) is the Lie algebra of \( H_{p,q} \). Let \( I_{p,q}^{f} \) denote the elements in \( I_{p,q} \) annihilated by \( f \) under the adjoint action of the \( \mathfrak{sl}_2 \)-triple \( \{h, e, f\} \) on \( I_{p,q} \subset \mathfrak{gl}_n(F) \). Then
\[
I_{p,q} = [\mathfrak{h}_{p,q}, e] + I_{p,q}^{f}.
\]

Following [CS20, Proposition 3.9], we shall prove the following proposition in this subsection.

**Proposition 3.3.** Let \( f \) be a \( H_{p,q} \)-invariant tempered generalized function on \( I_{p,q} \) such that \( f \) and its Fourier transforms \( \mathfrak{F}(f) \) are all supported on an orbit \( \mathcal{O} = H_{p,q} e \subset \mathcal{N}_{p,q} \). If \( \text{tr}(2 - h)|_{I_{p,q}^{f}} \neq 2pq \) and \( F \) is non-archimedean, then \( f = 0 \). If \( F \) is archimedean and \( \text{tr}(2 - h)|_{I_{p,q}^{f}} \) is not of the form \( 2pq - e \) with \( e \geq 0 \), then \( f = 0 \).

Denote by \( \mathcal{C}_O(I_{p,q}) \) the space of tempered generalized functions on \( I_{p,q} \setminus (\partial \mathcal{O}) \) with support in \( \mathcal{O} \), where \( \partial \mathcal{O} \) is the complement of \( \mathcal{O} \) in its closure in \( I_{p,q} \). (See [AG09a, Notation 2.5.3] for more details.) We will use similar notation without further explanation.

Let \( F^\times \) act on \( \mathcal{C}(I_{p,q}) \) by
\[
(t \cdot f)(x, y) = f(t^{-1}x, t^{-1}y)
\]
for \( t \in F^\times \), \( (x, y) \in I_{p,q} \) and \( f \in \mathcal{C}(I_{p,q}) \). The orbit \( \mathcal{O} \) is invariant under dilation and so \( F^\times \) acts on \( \mathcal{C}_O(I_{p,q})^H_{p,q} \) as well.

**Lemma 3.4.** [CS20, Lemma 3.13] Let \( \eta : F^\times \to \mathbb{C}^\times \) be an eigenvector for the action of \( F^\times \) on \( \mathcal{C}_O(I_{p,q})^H_{p,q} \). Then \( \eta^2 = | - |^{\text{tr}(2 - h)|_{I_{p,q}^{f}} - \kappa } \) for some pseudo-algebraic character \( \kappa \) of \( F^\times \).

Let \( Q \) be a quadratic form on \( I_{p,q} \) defined by
\[
Q(x, y) = \text{tr}(x \circ y) + \text{tr}(y \circ x)
\]
for \( (x, y) \in I_{p,q} \). Denote by \( Z(Q) \) the zero locus of \( Q \) in \( I_{p,q}(F) \). Then \( \mathcal{N}_{p,q} \subset Z(Q) \subset I_{p,q} \). Recall the following homogeneity result on tempered generalized functions.

**Theorem 3.5.** [AG09a, Theorem 5.1.7] Let \( L \) be a non-zero subspace of \( \mathcal{C}_{Z(Q)}(I_{p,q}) \) such that for every \( f \in L \), one has that \( \mathfrak{F}(f) \in L \) and \( (\psi \circ Q) \cdot f \in L \) for all unitary character \( \psi \) of \( F \). Then \( L \) is a completely reducible \( F^\times \)-subrepresentation of \( \mathcal{C}(I_{p,q}) \), and it has an eigenvalue of the form
\[
| - |^{\frac{1}{2} \dim I_{p,q} \kappa^{-1}}
\]
where \( \kappa \) is a pseudo-algebraic character of \( F^\times \).

Now we are prepared to prove Proposition 3.3. The basic idea is due to Chen-Sun in [CS20].

**Proof of Proposition 3.3** Denote by \( L \) the space of all tempered generalized functions \( f \) on \( I_{p,q} \) with the properties in Proposition 3.3. Assume by contradiction that \( L \) is nonzero. Then by Lemma 3.4 and Theorem 3.5, one has
\[
| - |^{\text{tr}(2 - h)|_{I_{p,q}^{f}} - \kappa_1} = \eta^2 = | - |^{\dim I_{p,q} \kappa_2^{-2}}
\]
where \( \kappa_1 \) and \( \kappa_2 \) are two pseudo-algebraic characters of \( F^\times \). This finishes the proof. \( \square \)
3.2. Proof of Theorem 3.1 In this subsection, we will give the proof of Theorem 3.1. We need the following definition and lemmas.

**Definition 3.6.** We fix a grading on \(\mathfrak{sl}_2(F)\) given by \(h \in \mathfrak{sl}_2(F)_0\) and \(e, f \in \mathfrak{sl}_2(F)_1\) where \(\{h, e, f\}\) is the \(\mathfrak{sl}_2\)-triple defined in \((3.1)\). A graded representation of \(\mathfrak{sl}_2(F)\) is a representation of \(\mathfrak{sl}_2(F)\) on a graded vector space \(V = V_0 \oplus V_1\) such that

\[ \mathfrak{sl}_2(F)_i(V_j) \subset V_{i+j} \]

for \(i, j \in \mathbb{Z}/2\mathbb{Z}\). Then \(V_0\) (resp. \(V_1\)) is called the even (resp. odd) part of \(V\).

**Lemma 3.7.** Every irreducible graded representation of \(\mathfrak{sl}_2(F)\) is irreducible (as a usual representation of \(\mathfrak{sl}_2(F)\)).

Denote by \(V^\lambda_\omega\) the irreducible graded representation of \(\mathfrak{sl}_2(F)\) with highest weight \(\lambda\) and highest weight vector of parity \(\omega \in \mathbb{Z}/2\mathbb{Z}\). Let \(V = V_0 \oplus V_1\) such that \(\dim V_0 = p\) and \(\dim V_1 = q = p + 1\). Consider \(V\) as a graded representation of \(\mathfrak{sl}_2(F)\).

**Lemma 3.8.** If \(V = V_0 \oplus V_1\) is irreducible as a graded representation of \(\mathfrak{sl}_2(F)\), then \(e\) is regular nilpotent, i.e., \(\dim O\) is the biggest dimension among the nilpotent orbits of \(I_{p,p+1}\).

In general, there is a decomposition of \(\mathfrak{sl}_2(F)\)-graded modules

\[ V = V^\lambda_1 \oplus V^\lambda_2 \oplus \cdots \oplus V^\lambda_d \]

for \(d \geq 1\). (See [CS20].) There is an isomorphism

\[ I_{p,q} = \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0) = \text{Hom}(V_0 \oplus V_1, V_0 \oplus V_1)_1 \]

of \(F\)-vector spaces, where \(\text{Hom}(V, V)_1\) is the odd part of \(\text{Hom}(V, V)\) as a graded \(\mathfrak{sl}_2(F)\)-module.

**Lemma 3.9.** Let

\[ m_{i,j} := tr(2 - h)|_{I_{p,q}} - \dim I_{p,q} = \frac{1}{2} \sum_{1 \leq i \leq d} m_{i,j} + \frac{1}{2}(p-q)^2 \]

and

\[ m_{i,j} = \begin{cases} \min\{\lambda_i, \lambda_j\} + 1, & \text{if } \lambda_i \neq \lambda_j \pmod{2}; \\ 2 \min\{\lambda_i, \lambda_j\} + 2, & \text{if } \lambda_i \equiv \lambda_j \equiv 1 \pmod{2} \text{ and } \omega_i = \omega_j; \\ 0, & \text{if } \lambda_i \equiv \lambda_j \equiv 1 \pmod{2} \text{ and } \omega_i \neq \omega_j; \\ -|\lambda_i - \lambda_j| - 1, & \text{if } \lambda_i \equiv \lambda_j \equiv 0 \pmod{2} \text{ and } \omega_i = \omega_j; \\ \lambda_i + \lambda_j + 3, & \text{if } \lambda_i \equiv \lambda_j \equiv 0 \pmod{2} \text{ and } \omega_i \neq \omega_j. \end{cases} \]

**Proposition 3.10.** If \(tr(2 - h)|_{I_{p,p+1}} = 2p(p+1)\), then there exists an \(h \in H_{p,p+1}\) such that \(\sigma \cdot e = heh^{-1}\).

**Proof.** Suppose \(V = \oplus_{i=1}^d V^\lambda_i\) with \(d \geq 1\). If \(\lambda_i\) is odd, then \(\dim V^\lambda_i \cap V_0 = \dim V^\lambda_i \cap V_1\). If \(\lambda_i\) is even, then \(\dim V^\lambda_i \cap V_0 + \dim V^\lambda_i \cap V_1 = (1)\omega_i\). Since \(V_1 = \dim V_0 + 1\), we obtain that the number of indices \(i\) such that \(\lambda_i\) is even and \(\omega_i = 0\) equals 1. Denote by \(t\) the number of indices \(i\) such that \(\lambda_i\) is even and \(\omega_i = 0\). Assume that \(tr(2 - h)|_{I_{p,p+1}} = 2p(p+1)\).

It is easy to see that \(e = \begin{pmatrix} 0 & 1_p & 0 \\ 0 & 0 & 0 \\ 1_p & 0 & 0 \end{pmatrix}\) and \(h = \begin{pmatrix} \omega_{1} & \omega_{p+1} \\ \omega_{p+1} & \omega_{1} \end{pmatrix}\) where \(\omega_1 = (1)\) is the \(1 \times 1\) matrix and \(\omega_{i+1} := \begin{pmatrix} 0 \\ \omega_{i} \\ 1 \end{pmatrix}\) for \(i = 1, 2, \cdots, p\). In this case, \(V\) is irreducible as a \(\mathfrak{sl}_2(F)\)-graded representation.
In general, if $V$ is reducible, then $(q - p)^2 + \sum_{1 \leq i \leq d} m_{i,j} > 0$. The following proof is similar to Lemma 7.7.5. Reorder the space $V_{\lambda_i}^\omega$ so that $\omega_i = 0$ for $1 \leq i \leq t$ and $\omega_i = 1$ for $i > t$. Furthermore, we require that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$ and $\lambda_{t+1} \geq \lambda_{t+2} \geq \cdots \geq \lambda_{2t+1} = \lambda_d$. Then

\[
(q - p)^2 + \sum_{1 \leq i \leq t} \sum_{1 \leq j \leq d} m_{i,j} = 1 + \sum_{1 \leq i \leq t} (1 - |\lambda_i - \lambda_j| - 1) + \sum_{t+1 \leq i \leq t+1} (\lambda_i + \lambda_j + 3)
\]

\[
+ \sum_{t+1 \leq i \leq t+1} (\lambda_i + 3) + \sum_{t+1 \leq i \leq t+1} (1 - |\lambda_i - \lambda_j| - 1)
\]

\[
= 4t(t + 1) - \sum_{1 \leq i \leq t} |\lambda_i - \lambda_j| + 2 \sum_{t+1 \leq i \leq t} |\lambda_i - \lambda_j| - \sum_{t+1 \leq i \leq t} |\lambda_i - \lambda_j|
\]

\[
= 4t(t + 1) + 4 \sum_{i=1}^t (\lambda_i + \lambda_{t+1+i})i
\]

which is positive unless $t = 0$.

If there is another $\mathfrak{sl}_2(F)$-triple $\{h', e', f'\}$ such that $V$ is irreducible, then Lemma 3.8 implies that both $e$ and $e'$ are regular nilpotent and so they are $H_{p,p+1}$-conjugate due to Kostant-Rallis’ result that the regular nilpotent elements are in the same $H_{p,p+1}$-orbit (see [KR71] Theorem 6). This finishes the proof. □

**Remark 3.11.** The above proposition does not hold for general $p$ and $q$. For instance, let $(p, q) = (6, 8)$. There does exist a unipotent element $e = (x, y)$ such that

\[
\sum_{1 \leq i \leq t} m_{i,j} + 4 = 0
\]

where $d = 3, V = V_1^1 \oplus V_1^2 \oplus V_8^1$ is the decomposition of $\mathfrak{sl}_2(F)$-graded modules, $\text{rank}(x) = 6$ and $\text{rank}(y) = 5$. Therefore $\sigma \cdot e \notin \mathcal{H}_{6,8} e$.

Finally, we can give a proof of Theorem 3.11.

**Proof of Theorem 3.11.** It suffices to show that $\mathcal{C}_{N_{p,q}}(I_{p,q})_{\mathcal{H}_{p,q}} = 0$. Due to Proposition 3.3, assume that $tr(2 - h)|_{t_{p,q}} = 2pq$. Suppose that $f \in \mathcal{C}_{O}(I_{p,q})_{\mathcal{H}_{p,q}}$ is a tempered generalized function on $I_{p,q}$ supported on the orbit $\mathcal{O} = H_{p,q} e \subset N_{p,q}$. Then its Fourier transform $\hat{\mathcal{F}}(f)$ is supported on $\mathcal{O}$ due to Lemma 3.2. Thanks to Proposition 3.10, it implies that $f = 0$ which means that every element in $\mathcal{C}_{N_{p,q}}(I_{p,q})_{\mathcal{H}_{p,q}}$ is zero, as required.

If $F$ is archimedean and $\kappa_1 \kappa_2 = \frac{|2pq - tr(2 - h)|}{t_{p,q}}$, then $\kappa_1 = \kappa_2 = 1$ and $tr(2 - h)|_{t_{p,q}} = 2pq$. Otherwise, it contradicts Lemma 3.9. However, $\sigma \cdot e \in H_{p,q} e$ in this case. Thus $f = 0$. This finishes the proof. □

4. **Proof of Theorem 1.1**

Recall that $n = p + q$ and $q = p + 1$. Let $\mathcal{H}_{p,q} := H_{p,q} \times H_{p,q}$ be a reductive group. Define

\[
\hat{\mathcal{H}}_{p,q} := \mathcal{H}_{p,q} \rtimes \langle \sigma \rangle
\]

where $\sigma$ acts on $\mathcal{H}_{p,q}$ by the involution $(h_1, h_2) \mapsto ((h_2^{-1})^t, (h_1^{-1})^t)$. Let $\hat{\mathcal{H}}_{p,q}$ act on $\text{GL}_n(F)$ by

\[
(h_1, h_2) \cdot g = h_1 g h_2^{-1}
\]

and $\sigma \cdot g = g^t$ for $h_i \in H_{p,q}$ and $g \in \text{GL}_n(F)$. Let $\chi$ be the sign character of $\hat{\mathcal{H}}_{p,q}$. Let $\mu \otimes \mu^{-1}$ be a character of $\mathcal{H}_{p,q}$. Let $\tilde{\mu}$ be a character of $\hat{\mathcal{H}}_{p,q}$ twisted by the sign character, i.e. $\tilde{\mu}|_{\mathcal{H}_{p,q}} = \mu \otimes \mu^{-1}$ and $\tilde{\mu}(\sigma) = -1$.

This section is devoted to a proof of the following theorem.
Theorem 4.1. Assume that $\mu$ is a good character of $H_{p,q}$. We have

$$\mathcal{C}(\text{GL}_n(F))\tilde{\mathcal{H}}_{p,q,\tilde{\nu}} = 0.$$ 

Then Theorem 4.1 will follow from Theorem 4.1 immediately.

Suppose that

\[
x_{p,k} = \begin{pmatrix}
1_k & 1_{p-k} \\
1_k & 1_{p+1-k}
\end{pmatrix}
\]

$k = 0, 1, \cdots, p$. Then the orbits $\tilde{\mathcal{H}}_{p,q} x_{p,k}$ are closed orbits in $\text{GL}_n(F)$ (see [JR96 Proposition 4.1]).

Lemma 4.2. [JR96 proposition 4.1] The following double cosets

$$H_{p,p+1} \left( \begin{array}{ccc} g_{11} & 0 & g_{12} \\ 0 & x_{p-\nu,k} & 0 \\ g_{21} & 0 & g_{22} \end{array} \right) H_{p,p+1}$$

exhaust all closed orbits in $H_{p,p+1}\backslash \text{GL}_{2p+1}(F)/H_{p,p+1}$, where $x_{p-\nu,k}$ (for $\nu = 0, 1, \cdots, p - k$) is defined in (4.1). $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ satisfies

$$g \begin{pmatrix} 1_{\nu} \\ -1_{\nu} \end{pmatrix} g^{-1} \begin{pmatrix} 1_{\nu} \\ -1_{\nu} \end{pmatrix} = \begin{pmatrix} A & \nu \\ A^2 - \nu & A \end{pmatrix}$$

and $A \in \text{Mat}_{p,p}(F)$ is a semisimple matrix without eigenvalues $\pm 1$.

Proof. See [Car15 Theorem 4.13].

Thanks to Theorem 2.2 if

$$\mathcal{C}(R(N_G^{\text{GL}_n(F)}))\tilde{\mathcal{H}}_{p,q,x,\tilde{\nu}} = 0$$

implies

$$\mathcal{C}(Q(N_G^{\text{GL}_n(F)}))\tilde{\mathcal{H}}_{p,q,x,\tilde{\nu}} = 0$$

for any $\tilde{\mathcal{H}}_{p,q}$-closed orbit $O = \tilde{\mathcal{H}}_{p,q} x$, where $\tilde{\mathcal{H}}_{p,q,x}$ is the stabilizer of $x$, then Theorem 4.1 holds.

At first, let us consider the simple case: $\nu = 0$.

Lemma 4.3. We have

$$\mathcal{C}(R(N_{\tilde{\mathcal{H}}_{p,q,x_{p,k}}}^{\text{GL}_n(F)}))\tilde{\mathcal{H}}_{p,q,x_{p,k},\tilde{\nu}} = 0 \implies \mathcal{C}(Q(N_{\tilde{\mathcal{H}}_{p,q,x_{p,k}}}^{\text{GL}_n(F)}))\tilde{\mathcal{H}}_{p,q,x_{p,k},\tilde{\nu}} = 0$$

where $\tilde{\mathcal{H}}_{p,q,x_{p,k}} = \{ h \in \tilde{\mathcal{H}}_{p,q} | h \cdot x_{p,k} = x_{p,k} \}$ is the stabilizer of $x_{p,k}$ in $\tilde{\mathcal{H}}_{p,q}$.

Proof. By easy computation, we have $\tilde{\mathcal{H}}_{p,q,x_{p,k}} \cong (\text{GL}_k(F) \times \text{GL}_k(F) \times H_{p-k,q-k}) \rtimes \langle \sigma \rangle$, where $\sigma$ acts on $\text{GL}_k(F) \times \text{GL}_k(F) \times H_{p-k,q-k}$ by the involution

$$(g_1, g_2, h) \mapsto ((g_{1,1}^{-1})^t, (g_{2,1}^{-1})^t, (h^{-1})^t)$$

for $g_i \in \text{GL}_k(F)$ and $h \in H_{p,q}$. The normal bundle (see [CS20 Lemma 4.3]) is given by

$$N_{\tilde{\mathcal{H}}_{p,q,x_{p,k}}}^{\text{GL}_n(F)} = \frac{\mathcal{g}_{l_4}(F)}{b_{p,q} + Ad_{x_{p,k}} b_{p,q}} \cong I_{k,k} \oplus I_{p-k,q-k}.$$ 

The action of $\tilde{\mathcal{H}}_{p,q,x_{p,k}}$ on $N_{\tilde{\mathcal{H}}_{p,q,x_{p,k}}}^{\text{GL}_n(F)}$ is given by

$$(g_1, g_2, h) \cdot (x, y, z) = (g_2 x g_1^{-1}, g_1 y g_2^{-1}, h z h^{-1})$$

and $\sigma \cdot (x, y, z) = (x^t, y^t, z^t)$ for $g_i \in \text{GL}_k(F), h \in H_{p-k,q-k}, (x, y) \in I_{k,k}$ and $z \in I_{p-k,q-k}$. By [AG09a Proposition 2.5.8], $\mathcal{C}(N_{\tilde{\mathcal{H}}_{p,q,x_{p,k}}}^{\text{GL}_n(F)}\tilde{\mathcal{H}}_{p,q,x_{p,k},\tilde{\nu}})$ is a product of

$$\mathcal{C}(I_{k,k})^{(\text{GL}_k(F) \times \text{GL}_k(F)) \rtimes \langle \sigma \rangle, \mu_{k,\tilde{\nu}} \otimes \mu_{k,\tilde{\nu}}^{-1}}$$

and

$$\mathcal{C}(I_{p-k,q-k})^{(\text{GL}(F) \times \text{GL}(F)) \rtimes \langle \sigma \rangle, \mu_{k,\tilde{\nu}} \otimes \mu_{k,\tilde{\nu}}^{-1}}.$$
where $\mu_k = \mu_F \circ \text{det}$ is the character of $GL_k(F)$, $\mu_k \otimes \mu_k^{-1}$ is the character of $GL_k(F) \times GL_k(F)$, $\mu_k \otimes \mu_k^{-1}$ is the character of $(GL_k(F) \times GL_k(F)) \rtimes \langle \sigma \rangle$ twisted by the sign character. Thanks to Theorem 3.1

\[
\mathcal{C}_{\nu,k}(I_{p-k,q-k}) \tilde{H}_{p-k,q-k} = 0.
\]

Thus it suffices to show that

\[
\mathcal{C}_{\nu,k}(I_{k,k}) (GL_k(F) \times GL_k(F)) \rtimes \langle \sigma \rangle, \mu_k \otimes \mu_k^{-1} = 0.
\]

It follows from [CS20 Proposition 3.9] since $\mu_F$ is a good character. We have finished the proof. \qed

Remark 4.4. If $x \in H_{p,q}$, then $\mathcal{O} = \tilde{H}_{p,q} x = H_{p,q}$ is a closed orbit in $GL_n(F)$. The group embedding from the stabilizer subgroup $\tilde{H}_{p,q} \cong H_{p,q}$ of $x$ to $H_{p,q}$ is given by

\[
(h, \delta) \mapsto \begin{cases} (h, x^{-1}h) & \text{if } \delta = 1; \\ (h, x^{-1}h') & \text{if } \delta = \sigma. \end{cases}
\]

(See [CS20 Page 12].) Similarly, we have

\[
\mathcal{C}(R(N^{GL_n(F)})) \tilde{H}_{p,q,x} = 0 \implies \mathcal{C}(Q(N^{GL_n(F)})) \tilde{H}_{p,q,x} = 0
\]

for $x \in H_{p,q}$.

Now we can give a proof of Theorem 4.1

Proof of Theorem 4.1. Applying Theorem 4.2, we only need to prove that there does not exist any $(\tilde{H}_{p,q}, \tilde{\mu})$-equivariant tempered generalized function on the normal bundle of $\tilde{H}_{p,q}$-closed orbits. Thanks to Lemma 4.3 we have proved Theorem 4.1 if $\nu = 0$. Thus applying Lemma 4.2 it is reduced to prove that

\[
\mathcal{C}(R(N^{GL_n(F)})) \tilde{H}_{p,q,x} = 0 \implies \mathcal{C}(Q(N^{GL_n(F)})) \tilde{H}_{p,q,x} = 0
\]

for the closed orbit $\tilde{H}_{p,q,x}$, where

\[
x = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in GL_n(F)
\]

and $x\theta_{p,q}(x^{-1}) = \begin{pmatrix} A & -1_k \\ 1_{p-k} & 1_{q-k} \end{pmatrix} A^{-1}$, $A$ is a semisimple element in $Mat_{\nu,\nu}(F)$

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

for $\nu = 1, 2, \cdots, p-k$, without eigenvalues $\pm 1$. Furthermore, we may assume that $x\theta_{p,q}(x) = \theta_{p,q}(x)x$ (where $x$ is called normal in the sense of [AG09a §7.4]), $A$ is a scalar matrix and $A^2 \neq 1$. Then

\[
\tilde{H}_{p,q,x} \cong (GL_{\nu}(F) \times GL_{k}(F) \times GL_{k}(F)) \rtimes H_{p-k,q-k} \rtimes \langle \sigma \rangle \cong (GL_{\nu}(F) \times \langle \sigma \rangle) \rtimes \tilde{H}_{p-k,q-k}.
\]

\[
\tilde{\mu} | (GL_{\nu}(F) \rtimes \langle \sigma \rangle) = \chi
\]

is the sign character and

\[
N^{GL_n(F)}_{\tilde{H}_{p,q,x}} = Mat_{\nu,\nu}(F) \oplus I_{k,k} \oplus I_{p-k,q-k} \cong Mat_{\nu,\nu}(F) \oplus N_{\tilde{H}_{p-k,q-k}}^{GL_n(F)}.
\]

where $GL_{\nu}(F)$ acts on $Mat_{\nu,\nu}(F)$ by inner conjugation and $\sigma$ acts on $Mat_{\nu,\nu}(F)$ by the matrix transpose. Therefore (4.2) follows from Lemma 4.3 and

\[
\mathcal{C}(Mat_{\nu,\nu}(F))^{GL_{\nu}(F) \rtimes \langle \sigma \rangle, x} = 0.
\]

(See [CS20 Theorem D].) This finishes the proof. \qed
5. Proof of Theorem 1.3

The method in this paper does not work for arbitrary $p$ and $q$ (see Remark 3.11). However, we can still prove several cases if $p$ is small, such as $p = 1$. The main purpose in this section is to study the case for the pair $(\text{GL}_n(F), \text{GL}_1(F) \times \text{GL}_{n-1}(F))$. Recall that $H_{1,n-1} = \text{GL}_1(F) \times \text{GL}_{n-1}(F)$. We can define $I_{1,n-1}, N_{1,n-1}, \tilde{H}_{1,n-1}, \tilde{H}_{1,n-1}$ and $\tilde{H}_{1,n-1}$ similarly. Given a closed orbit $\tilde{H}_{1,n-1} x$ in $\text{GL}_n(F)$, we denote $\tilde{H}_{1,n-1} x$ the stabilizer of $x$ in $\tilde{H}_{1,n-1}$.

We follow the method in the previous section to give a proof of Theorem 1.3.

**Proof of Theorem 1.3** The case for $n = 2$ is trivial. Assume that $n \geq 3$. Applying Theorem 2.2 and Lemma 4.2 we only need to prove that

$$\mathscr{E}(R(N_{\tilde{H}_{1,n-1} x,x}^{\text{GL}_n(F)}))\tilde{H}_{1,n-1} x = 0 \implies \mathscr{E}(Q(N_{\tilde{H}_{1,n-1} x,x}^{\text{GL}_n(F)}))\tilde{H}_{1,n-1} x = 0$$

for $x = \begin{pmatrix} 1 \\ \mathbf{1}_{n-2} \end{pmatrix}$ ($k = 0, 1$) or $x$ satisfying

$$x \omega_{1,n-1} x^{-1} \omega_{1,n-1} = \begin{pmatrix} A & 1 \\ A^2 - 1 & A \end{pmatrix}$$

where $A$ is a scalar in $\text{Mat}_{n,n}(F) = F$ and $A^2 \neq 1$. Now we separate them into three cases.

- Assume $\nu = 0$ and $k = 0$. Then $N_{\tilde{H}_{1,n-1} x,x}^{\text{GL}_n(F)} \cong I_{1,n-1}$ and the stabilizer of $x = \mathbf{1}_n$ is isomorphic to $\tilde{H}_{1,n-1}$. Then it is enough to show that

$$\mathscr{E}_{N_{\tilde{H}_{1,n-1} x,x}}(I_{1,n-1})\tilde{H}_{1,n-1} x = 0.$$

In fact, we will prove that a stronger result

$$\mathscr{E}_{N_{\tilde{H}_{1,n-1} x,x}}(I_{1,n-1})\tilde{H}_{1,n-1} x = 0.$$

(See the equality (6.1) which will be proved later.) Then we are done.

- Assume $\nu = 0$ and $k = 1$. Then $N_{\tilde{H}_{1,n-1} x,x}^{\text{GL}_n(F)} \cong F \oplus F$ and

$$\tilde{H}_{1,n-1} x \cong (\text{GL}_1(F) \times \text{GL}_1(F) \times \text{GL}_{n-2}(F)) \rtimes \langle \sigma \rangle.$$

The action of $\tilde{H}_{1,n-1} x$ on $F \oplus F$ is given by

$$(g_1, g_2, h) \cdot (x, y) = (g_2 x g_1^{-1}, g_1 y g_2^{-1})$$

and $\sigma \cdot (x, y) = (x, y)$ for $g_i \in \text{GL}_1(F), h \in \text{GL}_{n-2}(F)$ and $x, y \in F$. Moreover,

$$\mathscr{E}_{N_{\tilde{H}_{1,n-1} x,x}}(I_{1,1})\tilde{H}_{1,n-1} x \cdot \mu F \otimes \overline{\mu F} = 0$$

since the element $\sigma$ fixes $N_{1,1}$ pointwisely.

- Assume $k = 0$ and $\nu = 1$. Then $N_{\tilde{H}_{1,n-1} x,x}^{\text{GL}_n(F)} \cong F$ and $\tilde{H}_{1,n-1} x \cong (\text{GL}_1(F) \times \text{GL}_{n-2}(F)) \rtimes \langle \sigma \rangle$. The action on $F$ is trivial. This implies $\mathscr{E}(F)\tilde{H}_{1,n-1} x \cdot \chi = 0$.

We have finished the proof of Theorem 1.3.

6. Applications

In this section, we use a similar idea to give another application in the representation theory.

In [Gur17], assuming that $F$ is non-archimedean, Gurevich investigated the role of the mirabolic subgroup on the symmetric variety $\text{GL}_n(F)/H_{p,n-1}$ where $H_{p,n-1} = \text{GL}_p(F) \times \text{GL}_{n-1}(F)$. More precisely, let $P$ be a mirabolic subgroup of $\text{GL}_n(F)$ consisting of matrices with last row vector $(0, \cdot \cdot \cdot , 0, 1)$. Let $\text{GL}_n(F)$ act on $\text{Mat}_{n,n}(F)$ by inner conjugation. Bernstein [Ber84] proved that any $P$-invariant generalized function on $\text{Mat}_{n,n}(F)$ must be $\text{GL}_n(F)$-invariant. We expect that there is a more general phenomenon related to the mirabolic subgroup $P$. 

Theorem 3.6], it suffices to show that

\[ n \]

We shall show that

Any

Lemma 6.1.\]

Assume

(6.1)

\[ f(x) = f(x^1) \]

for any \((x, y) \in N_{n-1, 1}\). Thus \(f\) is \(H_{n-1, 1}\)-invariant.

Gurevich proved that any \(P \cap H_{1, n-1}\)-invariant tempered generalized function on \(N_{1, n-1}\) is also \(H_{1, n-1}\)-invariant (see [Gur17, Theorem 4.2]). Then by [Gur17, Theorem 3.9] and [Gur17, Corollary 5.1], he proved that any \(P \cap H_{1, n-1}\)-invariant linear functional on an \(H_{1, n-1}\)-distinguished irreducible smooth representation of \(GL_n(F)\) is also \(H_{1, n-1}\)-invariant (see [Gur17, Theorem 1.1]). We will give a new and shorter proof to [Gur17 Theorem 4.2] here, including the archimedean case.

Proposition 6.2. Let \(P\) be the standard mirabolic subgroup of \(GL_n(F)\) consisting of matrices with last row vector \((0, \cdots, 0, 1)\). Let \(I_{n-p, n-p}\) and \(N_{n-p, n-p}(I_{n-p, n-p})\) be as before. Then

\[ \mathcal{K}(N_{n-1, 1})P(H_{1, n-1}) = \mathcal{K}(N_{n-1, 1})(I_{1, n-1})H_{1, n-1}. \]

Proof. Assume \(n-1 \geq 2\). We will prove that any generalized function \(f \in \mathcal{K}(N_{n-1, 1})(I_{1, n-1})P(H_{1, n-1})\) satisfies \(f(x) = f(x^1)\) for all \(x \in I_{1, n-1}\). Then \(f\) is invariant with respect to \(P \cap H_{1, n-1}\) and so \(f\) is invariant under

\[ \langle P \cap H_{1, n-1}, P \cap H_{1, n-1}^{H_{1, n-1}} \rangle = H_{1, n-1}. \]

It is known that \(H_{1, n-2}\) is a proper subgroup in \(P \cap H_{1, n-1}\). Let \(H_{1, n-1}\) be as usual and \(\chi\) be its sign character. We will show that

(6.1)

Note that \(I_{1, n-1} = I_{1, n-2} \oplus V \oplus V^*\) with \(\text{dim} V = 1\). Let \((e, v, v^*) \in I_{1, n-2} \oplus V \oplus V^*\) be a unipotent element in \(N_{1, n-1}\). Then \(v^*(v) = 0\) (see [Aiz13, §6.1]). Thus either \(v = 0\) or \(v^* = 0\). Without loss of generality, assume \(v = 0\). Take any \(f \in \mathcal{K}(N_{1, n-1})(I_{1, n-2} \oplus V \oplus V^*)^{H_{1, n-2} \chi}\) such that \((e, 0, v^*) \in \text{supp}(f)\). Then the partial Fourier transform \(\mathfrak{F}_{V \times V^*}(f)\) is also supported on \(H_{1, n-2}(e, 0, v^*)\); see [Aiz13 §4.2]. Thanks to [Aiz13 Lemma 6.3.4] that \((I_{1, n-2} \oplus V \oplus \{0\}) \cup (I_{1, n-2} \oplus \{0\} \oplus V^*)\) does not support any nonzero \(H_{1, n-2}\)-invariant generalized functions. \(f = 0\). Then any \(H_{1, n-2}\)-invariant generalized function on \(I_{1, n-1}\) is invariant under transposition. This finishes the proof.

Proposition 6.2 implies that any \(P \cap H_{1, n-1}\)-invariant linear functional on an \(H_{1, n-1}\)-distinguished irreducible smooth admissible representation of \(GL_n(F)\) is also \(H_{1, n-1}\)-invariant.

Corollary 6.3. Any \(P \cap H_{1, n-1}\)-invariant linear functional on an \(H_{1, n-1}\)-distinguished irreducible smooth admissible representation of \(GL_n(F)\) is also \(H_{1, n-1}\)-invariant.

Proof. Denote by \(\mathcal{D}(GL_n(F))\) the distributions on \(GL_n(F)\). Following [Gur17 Corollary 5.1], and [Kem15 Theorem 3.6], it suffices to show that

\[ \mathcal{D}(GL_n(F))(P \cap H_{1, n-1}) \times H_{1, n-1} = \mathcal{D}(GL_n(F))^{H_{1, n-1} \times H_{1, n-1}}. \]

Note that \(\mathcal{D}(GL_n(F))(P \cap H_{1, n-1}) \times H_{1, n-1} \cong \mathcal{D}(GL_n(F)/H_{1, n-1}^{P \cap H_{1, n-1}});\) see [Kem15 Lemma 3.7]. Thus it is equivalent to proving

\[ \mathcal{D}(GL_n(F)/H_{1, n-1})^{P \cap H_{1, n-1}} = \mathcal{D}(GL_n(F)/H_{1, n-1})^{H_{1, n-1}}. \]

We shall show that

\[ \mathcal{K}(GL_n(F)/H_{1, n-1})^{H_{1, n-2} \chi} = 0. \]
which will imply $\mathcal{D}(GL_n(F)/H_{1,n-1}) = 0$ due to a general principle of "distribution versus Schwartz distribution" (see [AG09a Theorem 4.0.2]). Then we are done since $P \cap H_{1,n-1}$ and its transpose generate the whole group $H_{1,n-1}$.

Suppose that $n \geq 3$. Applying Theorem 2.2 and Lemma 4.2 it is enough to show that

\[ \mathcal{C}_{\lambda_1}(I_{1,1})^{H_{1,1},\chi} = 0 = \mathcal{C}_{\lambda_1,n-1}(I_{1,n-1})^{H_{1,n-2},\chi}. \]

which follows from [CS20 Theorem D] and (6.1). Here the action of $H_{1,1}$ on $I_{1,1}$ is given by

\[ (a,b) \cdot (x,y) = (bxa^{-1},ayb^{-1}) \text{ and } \sigma(x,y) = (x,y). \]

This finishes the proof. \(\square\)

In fact, we can prove a bit more. Let $P$ be the standard mirabolic subgroup of $GL_{2p+1}(F)$ consisting of matrices with last row vector $(0, \cdots, 0, 1)$. Then $H_{p,p}$ is a proper subgroup of $P \cap H_{p,p+1}$.

**Theorem 6.4.** Let $F$ be nonarchimedean. Any $P \cap H_{p,p+1}$-invariant linear functional on an $H_{p,p+1}$-distinguished irreducible smooth representation of $GL_{2p+1}(F)$ is also $H_{p,p+1}$-invariant.

Before we give the proof of Theorem 6.4, we shall apply Theorem 6.4 to study the relation between the exterior square $L$-function and the $H_{p,p+1}$-invariant spherical representation $\pi$.

Let $F$ be a finite field extension of $\mathbb{Q}_p$. Let $O_F$ be the ring of integers of $F$. Let $\varpi$ be the uniformizer in $O_F$ and $|\varpi| = q^{-1}$ where $q$ is the cardinality of the residue field $O_F/\varpi O_F$ of $F$. Let $B(F)$ be the standard Borel subgroup of $GL_{2p+1}(F)$ with unipotent radical $N(F)$. Let $\pi$ be a unitary spherical principal series representation of $GL_{2p+1}(F)$ distinguished by $H_{p,p+1}$ such that

\[ \pi = Ind^F_{B}(\chi_1 \otimes \cdots \otimes \chi_{2p+1})(\text{normal induction}) \]

where $\chi_i$ are unitary characters of $F^\times$. Let $K$ be a maximal open compact subgroup of $GL_{2p+1}(F)$.

**Lemma 6.5.** [Shi76] Up to a scalar there exists a unique right $K$-invariant Whittaker function $W$, in $\pi$ given by $W(\varpi^\lambda) = 0$ unless $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{2p+1})$ in $\mathbb{Z}^{2p+1}$ satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2p+1}$, where

\[ W(\varpi^\lambda) = \delta_B^{1/2}(\pi^\lambda) \det((\chi_i(\varpi^{\lambda_j+2p+1-j}))_{i,j}) \]

and $\delta_B$ is the modular function of $B(F)$.

Let $H'_{p,p+1}$ be the image of $H_{p,p+1}$ in $GL_{2p+1}(F)$ under the following embedding

\[ \begin{pmatrix} a_{i,j} \\ b_{i,j} \end{pmatrix} \mapsto (c_{i,j}) \in GL_{2p+1}(F) \]

where $(a_{i,j}) \in GL_p(F)$, $(b_{i,j}) \in GL_{p+1}(F)$ and $c_{i,j} = \begin{cases} b_{s,t} & \text{if } i = 2s - 1, j = 2t - 1; \\ a_{s,t} & \text{if } i = 2s, j = 2t; \\ 0 & \text{otherwise}. \end{cases}$

Consider the integral

\[ W \mapsto \int_{N(F) \cap H'_{p,p+1} \setminus P \cap H'_{p,p+1}} W(p) \det(p)^s dp. \]

It will give us a $P \cap H'_{p,p+1}$-invariant linear functional on $\pi$ when $s = 0$ which is $H'_{p,p+1}$-invariant as well due to Theorem 6.4. Moreover, the integral is related to the exterior square $L$-function $L(s, \pi, \Lambda^2)$ which sets up a relation between $L(s, \pi, \Lambda^2)$ and the distinction problem of $\pi$. Denote by $L(s, \pi)$ the standard $L$-function of $\pi$. Take a measure $dp$ on $P$ such that the volume of the compact subset $P \cap H'_{p,p+1} \cap K$ is 1.

**Theorem 6.6.** Let $\pi = Ind^F_{B}(\chi_1 \otimes \cdots \otimes \chi_{2p+1})$ be a unitary spherical representation of $GL_{2p+1}(F)$. If $\pi$ is distinguished by $H'_{p,p+1}$, then $L(s, \pi, \Lambda^2) L(s, \pi)$ has a pole at $s = 0$. 

Proof. Suppose that \( B(F) = AN(F) \) where \( A \) is the split torus and so \( A \subset H_{p,p+1}^r \). Then \( P \cap H_{p,p+1}^r = (N(F) \cap H_{p,p+1}^r)(P \cap A)(P \cap H_{p,p+1}^r \cap K) \). Denote by \( \delta_{P \cap H_{p,p+1}^r} \) the modular character of \((N(F) \cap H_{p,p+1}^r)(P \cap A)\). Note that \( \delta_{B(\mathcal{O}^F)}^{1/2} = \delta_{P \cap H_{p,p+1}^r} \). Then

\[
\int_{N(F) \cap H_{p,p+1}^r \setminus P \cap H_{p,p+1}^r} W(p) |\det(p)|^s \, dp = \sum_{\lambda_1 \geq \cdots \geq \lambda_2 \geq 0} \frac{\det((\chi_i(\mathcal{O}^F)^{\lambda_1+2p+1-j}))_{i,j}}{\det((\chi_i(\mathcal{O}^F)^{2p+1-j}))_{i,j}} (q^{-s})^{tr \lambda}
\]

which equals

\[
\left( 1 - q^{-(2p+1)s} \prod_i \chi_i(\mathcal{O}^F) \right) \prod_{i<j} (1 - \chi_i(\mathcal{O}^F) \chi_j(\mathcal{O}^F) q^{-2s})^{-1} \cdot \prod_i (1 - \chi_i(\mathcal{O}^F) q^{-s})^{-1}.
\]

(See [Mac95 §1.5 Example 4.]) Since \( \pi \) is distinguished by \( H_{p,p+1}^r \), \( \prod_i \chi_i \) is trivial. Thus

\[
\lim_{s \to 0} (1 - q^{-(2p+1)s}) L(s, \pi, \Lambda^2) L(s, \pi) \neq 0.
\]

Therefore \( L(0, \pi, \Lambda^2) L(0, \pi) = \infty. \)

\[\square\]

6.1. Proof of Theorem 6.4. This subsection is devoted to a proof of Theorem 6.4. The basic ideas come from [Sun12, CS20]. The generalized function \( f \) will be restricted to a smaller open subset which can be handled easily. It will give us a very strict condition for the support of \( f \). Then we will show that any \( H_{p,p+1}^r \)-invariant tempered generalized function on \( I_{p,p+1} \) is invariant under transposition, which will imply Theorem 6.4.

From Proposition 6.2 we have seen that

\[
I_{p,p+1} = I_{p,p} \oplus V \oplus V^*
\]

where \( V \oplus V^* \) is equipped with a natural non-degenerate quadratic form

\[
(v, v^*) \mapsto v^*(v)
\]

for \( v \in V \) and \( v^* \in V^* \), which induces a symmetric bilinear form \( \langle - , - \rangle \). Let \( F^x \) act on \( \mathcal{O}_{N_{p,p+1}}(I_{p,p+1}) \) by

\[
t \cdot f(x, y, v, v^*) = f(t^{-1}x, t^{-1}y, t^{-1}v, t^{-1}v^*)
\]

for \( (x, y) \in I_{p,p}, v \in V \) and \( v^* \in V^* \). Recall that

\[
\left( \begin{array}{c} a \\ 0 \\ b \\ 0 \end{array} \right) \cdot f(x, y, v, v^*) = f(a^{-1}xb, b^{-1}ya, a^{-1}v, v^*)
\]

for \( a, b \in GL_p(F) \) and

\[
\sigma \cdot f(x, y, v, v^*) = f(y^t, x^t, (v^*)^t, v^t).
\]

Let \( (e, v_0, v_0^*) \in N_{p,p+1} \) and \( D = (H_{p,p} \times F^x) (e, v_0, v_0^*) \) be a \( H_{p,p} \times F^x \)-orbit in \( N_{p,p+1} \). Then \( e = \left( \begin{array}{c} 0 \\ x_0 \\ 0 \\ 0 \end{array} \right) \in N_{p,p} \) and \( v_0^*(x_0 \circ y_0)^k v_0 = 0 \) for any non-negative integer \( k \). (See [Aiz13 §6.1].)

Let \( \{ h, e, f \} \) be a graded \( \mathfrak{sl}_2(F) \)-triple (see 3.1) related to \( I_{p,p} \), which integrates to an algebraic homomorphism

\[
\mathrm{SL}_2(F) \to GL_{2p}(F).
\]

Denote by \( D_t \) the image of \( \left( \begin{array}{c} t \\ t^{-1} \end{array} \right) \) in \( H_{p,p} \). Let

\[
T = \{ (D_t, t^{-2}) \in H_{p,p} \times F^x \mid t \in F^x \}
\]

be a closed subgroup in \( H_{p,p} \times F^x \) which fixes the element \( e \). Define

\[
E(e) := \{ (v, v^*) \in V \times V^* \mid v^*(x_0 \circ y_0)^k v = 0 \text{ for all non-negative integers } k \},
\]

and

\[
V(e) := \{ (v, v^*) \in E(e) \mid h \cdot (v, v^*) = (v, v^*) \text{ for any } h \in \langle D_t \rangle \}.
\]

The following lemma is similar to [CS20 Lemma 3.13].
Lemma 6.7. Let $\eta$ be an eigenvalue for the action of $F^\times$ on $\mathcal{C}_D(I_{p,p+1})^{H_{p,p}}$. Then
\[
\eta^2 = | - t^{(\frac{1}{2} - h)}|_{t^{r-p}+2p} \kappa_1 \kappa_2^{-2}
\]
for some pseudo-algebraic characters $\kappa_1$ and $\kappa_2$ of $F^\times$.

Proof. Consider the map
\[
H_{p,p} \times F^\times \times (I_{p,p}^T \oplus V \oplus V^*) \longrightarrow I_{p,p} \oplus V \oplus V^*
\]
via $(h, \xi, v, v^*) \mapsto h.(e + \xi + v + v^*)$ for $h \in H_{p,p}, \xi \in H_{p,p} \times F^\times, v \in V$ and $v^* \in V^*$, which is submersive at every point of $H_{p,p} \times F^\times \times \{(0, v_0, v_0^*)\}$. Moreover, $H_{p,p} \times F^\times \times \{(0, v_0, v_0^*)\}$ is open in the inverse image of $\mathcal{O} = (H_{p,p} \times F^\times).(e, v_0, v_0^*)$ under the map (6.2). (See [CS20, Page 18].) Thanks to [JSZ11, Lemma 2.7], the restriction map yields an injective linear map
\[
\mathcal{C}_D(I_{p,p} \oplus V \oplus V^*)^{H_{p,p} \times F^\times, 1 \times \eta} \rightarrow \mathcal{C}_D(I_{p,p}^T \oplus V \oplus V^*)^{T, 1 \times \eta, t^r}
\]
where $1 \times t^r((D_t, t^{-2})) = \eta(t)^{-2}$. It is easy to see that the representation $\mathcal{C}_D(I_{p,p}^T)$ of $T$ is completely reducible and every eigenvalue has the form
\[
(D_t, t^{-2}) \mapsto |t|^{tr(h-2)}|_{t^{r-p} \kappa_1}(t)^{-1}
\]
where $\kappa_1$ is a pseudo-algebraic character of $F^\times$. Thus
\[
\eta(t)^2 = |t|^{tr(h-2)}|_{t^{r-p} \kappa_1}(t)^{-1}
\]
for any $t \in F^\times$, where $\eta_0$ is an eigenvalue for the action of $T$ on $\mathcal{C}_D(V \oplus V^*)$. In order to compute $\eta_0$, we will restrict $\eta_0$ to a smaller subspace $\mathcal{C}_D(V \oplus V^*)$ of $\mathcal{C}_D(V \oplus V^*)$.

Define a symplectic form on $(V \oplus V^*) \times (V \oplus V^*)$ as follow
\[
\langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle = \langle x_1, y_2 \rangle - \langle y_1, x_2 \rangle
\]
where $x_i, y_i \in V \oplus V^*$. Then $V \oplus V^*$ is a maximal isotropic subspace. Consider the Weil representation on $Mp_{4p}(F) = Mp((V \oplus V^*) \times (V \oplus V^*), < -, >)$. Under the Weil representation $\omega_{\psi}$,
\[
\left\{
\begin{array}{l}
\omega_{\psi} \left( A (A^t)^{-1} \right) \varphi(x) = |\det A|^{1/2} \varphi(A^{-1}x), \quad \text{for } A \in GL_{2p}(F), \\
\omega_{\psi} \left( 1_{2p} N 1_{2p} \right) \varphi(x) = \psi((Nx,x)) \varphi(x), \quad \text{for } N = N^t \in Mat_{2p,2p}(F),
\end{array}
\right.
\]
for $\varphi \in S(V \oplus V^*)$ and $x \in V \oplus V^*$. We may extend $\omega_{\psi}$ from the Schwartz space $S(V \oplus V^*)$ to the generalized function space $\mathcal{G}(V \oplus V^*)$. Note that
\[
\begin{pmatrix} X & \end{pmatrix} \begin{pmatrix} X \end{pmatrix}^{-1} = \begin{pmatrix} 1_n & -X \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix} \begin{pmatrix} 1_n & \end{pmatrix}
\]
holds for any $X \in GL_n(F)$. Here we only need the case that $X$ is a diagonal matrix. Denote $D_t = \begin{pmatrix} A_t & B_t \\ A_t^{-1} & \end{pmatrix}$
and $X_t = \begin{pmatrix} A_t & \end{pmatrix}$. Then the action of $D_t$ on $V \oplus V^*$ is given by
\[
(v, v^*) \mapsto (A_t v, v^* A_t^{-1}).
\]
It is obvious that
\[
\omega_{\psi} \left( 1_{2p} X_t 1_{2p} \right) f(v, v^*) = \psi((A_t v, v^* A_t^{-1}), (v, v^*)) f(v, v^*)
\]
\[
= f(v, v^*)
\]
for any $f \in \mathcal{C}_{V(e)}(V \oplus V^*)$. Then $\begin{pmatrix} 1_{2p} & X_i \\ 1_{2p} \end{pmatrix}$ acts on $\mathcal{C}_{V(e)}(V \oplus V^*)$ trivially and so is $\begin{pmatrix} 1_{2p} & X_i^{-1} \\ 1_{2p} \end{pmatrix}$. Thus $D_i$ does not contribute to $\eta_0$. Therefore $\eta_0$ has the form

$$(D_t, t^{-2}) \mapsto |t|^{-1} \left| \dim(V \oplus V^*) \right| \kappa_2(t)^{-2} = |t|^{-2p} \kappa_2(t)^2$$

and so $\eta(t)^2 = |t|^{tr(2-h)}|t|^{-2p} \kappa_2(t)^{-2}$ for some pseudo-algebraic characters $\kappa_i$ of $F^\times$. \hfill \Box

Let $e$ be a nilpotent element in $I_{p,p}$. Let $\{h, e, f\}$ be the $\mathfrak{sl}_2(F)$-triple (see (3.1)). Then Chen-Sun [CS20, Lemma 3.12] proved

$$(6.3) \quad 2p^2 < tr(2 - h)|t|_{p,p} \leq 4p^2.$$ 

Now we can give a proof of Theorem 6.4

**Proof of Theorem 6.4** Following [Gur17, Corollary 5.1], it suffices to show that

$$(6.4) \quad \mathcal{C}(GL_{2p+1}(F)/(P \cap H_{p,p+1}) \times H_{p,p+1}) = \mathcal{C}(GL_{2p+1}(F)/(H_{p,p+1})).$$

Note that $\mathcal{C}(GL_{2p+1}(F)/(P \cap H_{p,p+1}) \times H_{p,p+1}) \cong \mathcal{C}(GL_{2p+1}(F)/(H_{p,p+1})).$ Thus (6.4) is equivalent to

$$(6.5) \quad \mathcal{C}(GL_{2p+1}(F)/(H_{p,p+1})^{P \cap H_{p,p+1}} = \mathcal{C}(GL_{2p+1}(F)/(H_{p,p+1})\times H_{p,p+1}).$$

We shall show that

$$\mathcal{C}(GL_{2p+1}(F)/(H_{p,p+1})\times H_{p,p+1}) = 0.$$ 

Then the identity (6.3) follows from the fact that $P \cap H_{p,p+1}$ and its transposition generate the whole group $H_{p,p+1}$. Applying Theorem 2.2 and Lemma 4.2, it is enough to show that

$$(6.6) \quad \mathcal{C}_{N_{p+1}}(I_{p,p} \oplus V \oplus V^*) = 0$$

with $\dim V = \dim V^* = p$.

Now the rest of this part is devoted to proving the equality (6.6). Take $(e, v_0, v_0^*) \in N_{p+1}$ and the $\mathfrak{sl}_2(F)$-triple $\{h, e, f\}$ related to $I_{p,p}$. Denote $\mathfrak{O} := H_{p,p}(e, v_0, v_0^*) \subset N_{p+1}$. Recall that $F^\times$ acts on $\mathcal{C}_{\mathfrak{O}}(I_{p,p} \oplus V \oplus V^*)$ by

$$t.f(x, y, v, v^*) = f(t^{-1}x, t^{-1}y, t^{-1}v, t^{-1}v^*)$$

for $t \in F^\times, (x, y) \in I_{p,p}$ and $f \in \mathcal{C}_{\mathfrak{O}}(I_{p,p} \oplus V \oplus V^*)$. Let $\eta$ be an eigenvalue for the action of $F^\times$ on $\mathcal{C}_{\mathfrak{O}}(I_{p,p} \oplus V \oplus V^*)$. By Lemma 6.7, $\eta^2 = | - |^{tr(2-h)}|t|_{p,p}^{+2p}$. By Theorem 6.3 one has $\eta^2 = | - |^{\dim I_{p,p+1}}$ and so

$$tr(2 - h)|t|_{p,p} + 2p = 2p(p + 1),$$

which contradicts the inequality (6.3). This finishes the proof. \hfill \Box

**Remark 6.8.** It seems that our method fails for Theorem 6.3 when $F$ is archimedean. The reason is that there are many possible solutions for $\eta^2 = | - |^{\dim I_{p,p+1}} \kappa^{-2}$ due to Lemma 6.7

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References

[AG09a] Avraham Aizenbud and Dmitry Gourevitch, Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis’s theorem, Duke Math. J. 149 (2009), no. 3, 509–567, With an appendix by the authors and Eitan Sayag. MR 2553879

[AG09b] Avraham Aizenbud, Dmitry Gourevitch, Stephen Rallis, and Gérard Schiffmann, Multiplicity one theorems, Ann. of Math. (2) 172 (2010), no. 2, 1407–1434. MR 2680495

[AGS08] Avraham Aizenbud, Dmitry Gourevitch, and Eitan Sayag, (GL_{n+1}(F), GL_n(F)) is a Gelfand pair for any local field F, Compos. Math. 144 (2008), no. 6, 1504–1524. MR 2474319

[Aiz13] Avraham Aizenbud, A partial analog of the integrability theorem for distributions on p-adic spaces and applications, Israel J. Math. 193 (2013), no. 1, 233–262. MR 3038552

[Ber84] J. N. Bernstein, Le “centre” de Bernstein, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32. MR 771671

[Car15] Shachar Carmeli, On the stability and Gelfand property of symmetric pairs, arXiv preprint arXiv:1511.01381 (2015).

[CS20] Fulin Chen and Binyong Sun, Uniqueness of twisted linear periods and twisted Shalika periods, Sci. China Math. 63 (2020), no. 1, 1–22. MR 4047168

[FJR93] Solomon Friedberg and Hervé Jacquet, Linear periods, J. Reine Angew. Math. 443 (1993), 91–139. MR 1241129

[GRS10] Avraham Aizenbud, Dmitry Gourevitch, and Gérard Schiffmann, Multiplicity one theorems, Ann. of Math. (2) 172 (2010), no. 2, 1407–1434. MR 2680495

[JSZ11] Dihua Jiang, Binyong Sun, and Chen-Bo Zhu, Uniqueness of Ginzburg-Rallis models: the Archimedean case, Trans. Amer. Math. Soc. 363 (2011), no. 5, 2763–2802. MR 2763736

[Kem15] Alexander Kemarsky, Distinguished representations of GL_n(C), Israel J. Math. 207 (2015), no. 1, 435–448. MR 3358053

[KR71] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces, Amer. J. Math. 93 (1971), 753–809. MR 311837

[Mac95] I. G. Macdonald, Symmetric functions and Hall polynomials, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 1354144

[Mez21] Dor Mezer, Multiplicity one theorem for (GL_{n+1}(F), GL_n(F)) over a local field of positive characteristic, Math. Zeit. 297 (2021), no. 3–4, 1383–1396.

[Shi76] Takuro Shintani, On an explicit formula for class-1 “Whittaker functions” on GL_n over p-adic fields, Proc. Japan Acad. 52 (1976), no. 4, 180–182. MR 407208

[Sun12] Binyong Sun, Multiplicity one theorems for Fourier-Jacobi models, Amer. J. Math. 134 (2012), no. 6, 1655–1678. MR 2999291

[SZ11] Binyong Sun and Chen-Bo Zhu, A general form of Gelfand-Kazhdan criterion, Manuscripta Math. 136 (2011), no. 1–2, 185–197. MR 2820401

[SZ12] Binyong Sun and Chen-Bo Zhu, Multiplicity one theorems: the Archimedean case, Ann. of Math. (2) 175 (2012), no. 1, 23–44. MR 2874638

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