NON-SEMISIMPLE QUANTUM INVARIANTS AND TQFTS FROM SMALL AND UNROLLED QUANTUM GROUPS

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Abstract. We show that unrolled quantum groups at odd roots of unity give rise to relative modular categories. These are the main building blocks for the construction of 1+1+1-TQFTs extending CGP invariants, which are non-semisimple quantum invariants of closed 3-manifolds decorated with ribbon graphs and cohomology classes. When we consider the zero cohomology class, these quantum invariants are shown to coincide with the renormalized Hennings invariants coming from the corresponding small quantum groups.

The goal of this paper is two-fold: first of all, we provide a new family of concrete examples of relative modular categories. These are ribbon categories which can be used as fundamental bricks for the construction of non-semisimple quantum invariants of closed 3-manifolds [5] and Extended Topological Quantum Field Theories (ETQFTs) in dimension 1+1+1 [6]. They should be thought of as a non-semisimple analogue to standard modular categories, although differences with respect to their semisimple counterparts are many: first of all, a relative modular category \( \mathcal{C} \) comes equipped with a structure group \( G \) that provides a grading on its objects; secondly, it enjoys finiteness properties only up to the action of a periodicity group \( Z \) of transparent objects; more importantly, it is only generically semisimple, with non-semisimple part confined to a critical set \( X \subset G \) whose complement is dense in \( G \). In Theorem 1.3 we prove that categories \( \mathcal{C}^h \) of finite-dimensional weight representations over unrolled quantum groups \( U_q^g \) associated with simple complex Lie algebras \( g \) at an odd root of unity \( q \) are relative modular. These categories were already known to induce topological invariants \( N_{\mathcal{C}^h} \) of certain decorated closed 3-manifolds \( (M, T, \omega) \), where \( T \subset M \) is a \( \mathcal{C}^h \)-colored ribbon graph, where \( \omega \in H^1(M \setminus T; G) \) is a cohomology class, and where the triple \( (M, T, \omega) \) is subject to a crucial admissibility condition. The definition of these so-called Costantino-Geer-Patureau (CGP) quantum invariants relies on the notion of computable surgery presentation introduced in [5]. The results of this paper imply these invariants can be extended to graded 1+1+1-TQFTs for all simple complex Lie algebras \( g \). In the case of \( \mathfrak{sl}_2 \), these invariants contain the Akutsu-Deguchi-Ohtsuki invariants of colored links and the abelian Reidemeister torsion of closed 3-manifolds, and they were already known to extend to graded 1+1+1-TQFTs.

The second main result of this paper is a Hennings type formula for these CGP quantum invariants. More precisely, every simple complex Lie algebra \( g \) also determines a corresponding small quantum group \( \hat{U}_q^g \) for every odd root of unity \( q \). These finite-dimensional factorizable quotients have been studied a lot in literature [19, 17, 18], and they induce renormalized Hennings TQFTs in dimension 2+1, see [7]. In particular, their categories of finite-dimensional representations \( \mathcal{C} \) yield quantum invariants \( H^r_{\mathcal{C}} \) of certain admissible closed 3-manifolds \( (M, T) \), where \( T \subset M \) is a \( \mathcal{C} \)-colored bichrome graph, which is a very mild generalization of standard ribbon graphs obtained by specifying special components which correspond to surgery presentations. Then, we prove in Theorem 1.4 that, for a fixed simple complex Lie algebra \( g \) at an odd root of unity \( q \), the CGP invariant \( N_{\mathcal{C}^h} \) of decorated closed 3-manifolds with zero cohomology classes coincides with
the corresponding renormalized Hennings invariant $H'_g$. This result builds a bridge between the two theories, and in this setting it gives us a way of computing the CGP quantum invariant which bypasses computable surgery presentations.

1. Overview of non-semisimple constructions

In this section, we quickly review the two main constructions this paper deals with. References are provided by [5, 2, 6] for the CGP theory, and by [7] for the renormalized Hennings one. Both of these constructions use the notion of $m$-trace, see [8, 9, 11]. All the manifolds we consider are always assumed to be oriented.

1.1. 3-Manifold invariants from non-degenerate relative pre-modular categories. We begin by fixing some terminology which will be extensively used throughout the paper: every time we have a ribbon linear category $C$ over a field $k$, we have an associated notion of skein equivalence between formal linear combinations of $C$-colored ribbon graphs. Indeed, if $R_C$ denotes the category of $C$-colored ribbon graphs, and if $(\xi, V)$ and $(\xi', V')$ are objects of $R_C$, we say two formal linear combinations $\sum_{i=1}^m \alpha_i \cdot T_i$ and $\sum_{i'=1}^{m'} \alpha'_{i'} \cdot T'_{i'}$ of morphisms of $R_C$ from $(\xi, V)$ and $(\xi', V')$ are skein equivalent, and we write

$$\sum_{i=1}^m \alpha_i \cdot T_i \doteq \sum_{i'=1}^{m'} \alpha'_{i'} \cdot T'_{i'},$$

if we have the equality $\sum_{i=1}^m \alpha_i \cdot F_{\xi}(T_i) = \sum_{i'=1}^{m'} \alpha'_{i'} \cdot F_{\xi'}(T'_{i'})$ under the Reshetikhin-Turaev functor $F_{\xi} : R_C \to G$. Then, let us recall the definition of relative pre-modular categories, which, as we mentioned earlier, are ribbon linear categories carrying additional structures. First of all, if $G$ is an abelian group, a compatible $G$-structure on a ribbon category $C$ is an equivalence $C \cong \bigoplus_{g \in G} C_g$ for a family $\{C_g \mid g \in G\}$ of full subcategories of $C$ satisfying the following conditions: If $V \in C_g$, then $V^\ast \in C_{-g}$; if $V \in C_g$ and $V' \in C_{g'}$, then $V \otimes V' \in C_{g+g'}$; if $V \in C_g$ and $V' \in C_{g'}$ with $g \neq g'$, then $\text{Hom}_C(V, V') = 0$. Next, if $Z$ is an abelian group, a free realization of $Z$ in a ribbon category $C$ is a monoidal functor $\sigma : Z \to C$, where $Z$ also denotes the discrete category over $Z$ with tensor product given by the group operation $+$, which induces a free action on isomorphism classes of simple objects of $C$ by tensor product on the right, and which satisfies $\vartheta(gk) = \text{id}_{\sigma(k)}$ for every $k \in Z$, where $\vartheta$ denotes the twist of $C$. Next, we say a subset $X$ of $G$ is small symmetric if $X = -X$, and if $G \not\subseteq \bigcup_{i=1}^m (g_i + X)$ for all $m \in \mathbb{N}$ and all $g_1, \ldots, g_m \in G$.

Definition 1.1 ([6]). If $G$ and $Z$ are abelian groups, and if $X \subseteq G$ is a small symmetric subset, then a pre-modular $G$-category relative to $(Z, X)$ is a ribbon linear category $C$ over a field $k$ together with a compatible $G$-structure on $C$, a free realization $\sigma : Z \to C_0$, and a non-zero $m$-trace $t$ on $\text{Proj}(C)$. These data are subject to the following conditions:

1. Generic semisimplicity. For every $g \in G \times X$ the homogeneous subcategory $C_g$ is semisimple and dominated by $\Theta(\sigma(g))$ for some finite set $\Theta(\sigma(g)) = \{V_i \in C_g \mid i \in I_g\}$ of simple projective objects with epic evaluation;

2. Compatibility. There exists a bilinear map $\psi : G \times Z \to k^\ast$ such that $c_{g,k}, V \circ c_{\sigma(k), V} = \psi(g, k) \cdot \text{id}_V \otimes \sigma(k)$ for every $g \in G$, for every $V \in C_g$, and for every $k \in Z$.

Relative pre-modular $G$-categories are a slight generalization of the notion of relative $G$-modular category introduced for the first time in [5], see Section 1.5 of [6] for a full discussion of the relation between the two definitions. The change in terminology is motivated by the semisimple theory, where quantum invariants...
are defined for any non-degenerate pre-modular category, and modularity is an additional condition ensuring the invariant extends to a TQFT. For the rest of this paper, we approach the original definition of [5] by further requiring $\dim_C(\sigma(k)) = 1$ for every $k \in \mathbb{Z}$. Indeed, this happens in all the examples we will consider, and assuming this condition simplifies some of the notation that follows. The general case is discussed in detail in [6]. If $\mathcal{C}$ is a pre-modular $G$-category relative to $(\mathbb{Z}, X)$ then the associated Kirby color of index $g \in G \setminus X$ is the formal linear combination of objects

$$\Omega_g := \sum_{i \in I_g} d(V_i) \cdot V_i.$$  

It follows from Lemma 5.10 of [5] that there exist constants $\Delta_{-\Omega}, \Delta_{+\Omega} \in \mathbb{k}$, called stabilization coefficients, which realize the skein equivalences of Figure 1, and which are independent of both $V \in \mathcal{C}_g$ and $g \in G \setminus X$. Then we say the relative pre-modular category $\mathcal{C}$ is non-degenerate if $\Delta_{-\Omega} \Delta_{+\Omega} \neq 0$.

**Figure 1.** Skein equivalences defining $\Delta_{-\Omega}$ and $\Delta_{+\Omega}$. 

In [5, 6] it is shown that every non-degenerate relative pre-modular category $\mathcal{C}$ gives rise to a topological invariant $N_{\mathcal{C}}$ of admissible triples $(M, T, \omega)$, where $M$ is a closed 3-manifold, $T \subset M$ is a $\mathcal{C}$-colored ribbon graph, and $\omega \in H^1(M \setminus T; G)$ is a compatible cohomology class, meaning that every edge $e \subset T$ is colored with an object of $\mathcal{C}(\omega, m_e)$ for the homology class $m_e$ of a positive meridian of $e$. The CGP invariant $N_{\mathcal{C}}$ is defined only for admissible triples $(M, T, \omega)$, which are triples such that every component of $M$ contains either a projective edge of $T$, that is an edge of $T$ whose color is a projective object of $\mathcal{C}$, or a generic curve for $\omega$, that is an embedded closed oriented curve whose homology class is sent to $G \setminus X$ by $\omega$. Its definition uses computable surgery presentations in $S^3$, which are surgery presentations $L = L_1 \cup \ldots \cup L_\ell$ of $M$ satisfying $(\omega, m_i) \in G \setminus X$ for all integers $1 \leq i \leq \ell$, where $m_i$ denotes the homology class of a meridian of the component $L_i$. We interpret a surgery presentation of $M$ which is computable with respect to some decoration $(T, \omega)$ as a $\mathcal{C}$-colored ribbon graph by arbitrarily choosing orientations, and by labeling every component with the corresponding Kirby color, with index prescribed by the evaluation of $\omega$ against the homology class of a positive meridian. This is a technical complication, because arbitrary surgery presentations are not computable in general. Computable surgery presentations do exist for admissible closed 3-manifolds, but only up to replacing admissible decorations via certain operations called projective and generic stabilizations, see Section 3.1 of [6]. The idea is to build $N_{\mathcal{C}}$ out of a renormalized invariant $F'_\mathcal{C}$ of admissible closed $\mathcal{C}$-colored ribbon graphs which combines the Reshetikhin-Turaev functor $F_\mathcal{C}$ on $\mathcal{R}_{\mathcal{C}}$ with the m-trace $t$ on $\text{Proj}(\mathcal{C})$. We say a closed $\mathcal{C}$-colored ribbon graph is admissible if one of his edges is projective. The formula

$$F'_\mathcal{C}(T) := t_V(F_\mathcal{C}(T_V))$$

defines a topological invariant of the admissible closed $\mathcal{C}$-colored ribbon graph $T$ thanks to Theorem 3 of [12], where $V$ is a projective object of $\mathcal{C}$, and where $T_V$ is a
cutting presentation of $T$, meaning an endomorphism of $(+, V)$ in $\mathcal{R}_\sigma$ whose trace is $T$. Then, for a fixed choice of a square root $\mathcal{D}_\Omega \in \mathcal{K}$ of $\Delta_\Omega \Delta_{+\Omega}$, the formula

$$N_{\chi}(M, T, \omega) := \mathcal{D}_\Omega^{-1-t\delta_\Omega^{\sigma(L)}} F_{\chi}(L \cup T)$$

defines a topological invariant of the admissible triple $(M, T, \omega)$ thanks to Proposition 3.1 of [6], where $L = L_1 \cup \ldots \cup L_T \subset S^3$ is a surgery presentation of $M$ of signature $\sigma(L)$ which is computable with respect to an admissible decoration $(T, \omega)$ obtained from $(T, \omega)$ by performing projective or generic stabilization, and where $\delta_\Omega = \mathcal{D}_\Omega / \Delta_{-\Omega}$.

1.2. **2+1-TQFTs from relative modular categories.** A stronger non-degeneracy condition is required in order to extend CGP invariants to graded TQFTs.

**Definition 1.2 ([6]).** A pre-modular $G$-category $\mathcal{C}$ relative to $(Z, X)$ is relative modular if there exists a relative modularity parameter $\zeta_\Omega \in k^*$ realizing the skein equivalence of Figure 2 for all $g, h \in G \setminus X$ and for all $i, j \in I_g$.

![Figure 2. Relative modularity condition.](image)

This condition automatically implies non-degeneracy, because the relative modularity parameter satisfies $\zeta_\Omega = \Delta_{-\Omega} \Delta_{+\Omega}$, see [6]. When $\mathcal{C}$ is relative modular then, as explained in Section 6.2 of [6], $N_{\chi}$ extends to a $Z$-graded 2+1-TQFT $\mathcal{V}_\chi : \mathcal{C}_\chi \rightarrow \text{Vect}_{\chi}^Z$ via a $Z$-graded refinement of the universal construction of [3], where $\mathcal{C}_\chi$ is the category of admissible cobordisms of dimension 2+1, and where $\text{Vect}_{\chi}^Z$ is the category of $Z$-graded vector spaces. More precisely, an object of $\mathcal{C}_\chi$ is a 5-tuple $\Sigma = (\Sigma, P, \vartheta, B, \mathcal{Z})$, where $\Sigma$ is a closed surface, where $P \subset \Sigma$ is a $\mathcal{C}$-colored ribbon set, where $\vartheta \in H^1(\Sigma \setminus P; G)$ is a compatible cohomology class, where $B \subset \Sigma \setminus P$ is a finite set composed of exactly one base point in every connected component of $\Sigma$, and where $\mathcal{Z} \subset H_1(\Sigma; \mathbb{R})$ is a Lagrangian subspace. A morphism of $\mathcal{C}_\chi$ from $(\Sigma, P, \vartheta, B, \mathcal{Z})$ to $(\Sigma', P', \vartheta', B', \mathcal{Z}')$ is an equivalence class of admissible 4-tuples $M = (M, T, \omega, n)$, where $M$ is a 3-dimensional cobordism from $\Sigma$ to $\Sigma'$, where $T \subset M$ is a $\mathcal{C}$-colored ribbon graph from $P$ to $P'$, where $\omega \in H^1(M \setminus T, \mathcal{Z} \cup B'; G)$ is a compatible relative cohomology class restricting to $\vartheta$ and $\vartheta'$ on the incoming and outgoing boundary of $M$ respectively, and where $n \in \mathbb{Z}$ is a signature defect. A 4-tuple $(M, T, \omega, n)$ is admissible if every component of $M$ which is disjoint from the incoming boundary $\partial_- M$ contains either a projective edge of $T$, or a generic curve for $\omega$, and two 4-tuples $(M, T, \omega, n)$ and $(M', T', \omega', n')$ are equivalent if $n = n'$, and if there exists a positive diffeomorphism $f : M \rightarrow M'$ which preserves boundary identifications and satisfies $f(T) = T'$ and $f^{*}(\omega) = \omega$.

Then, $N_{\chi}$ can be extended to an invariant of closed morphisms of $\mathcal{C}_\chi$ by setting

$$N_{\chi}(M, T, \omega, n) := \delta_\Omega^{\sigma_\Omega} N_{\chi}(M, T, \omega).$$

Remark that the category $\mathcal{C}_\chi$ thus obtained is not rigid, as objects $(\Sigma, P, \vartheta, \mathcal{Z})$ such that $P$ does not contain any projective point and such that $\vartheta$ does not admit any generic curve are not dualizable.

State spaces associated with objects of $\mathcal{C}_\chi$ by the $Z$-gradeds TQFT $\mathcal{V}_\chi$ can be described in skein theoretical terms in all degrees. We have two relevant notions of skein equivalence between morphisms of $\mathcal{C}_\chi$ from $\Sigma$ to $\Sigma'$, one which is local,
the other which is not. Indeed, we say a formal linear combination of morphisms of \( \tilde{\text{Cob}} \) from \( \Sigma \) to \( \Sigma' \) is a local skein relation if it can be written in the form

\[
\sum_{i=1}^{m} \alpha_i \cdot M_P \circ ((D^3, T_i, \omega_i, 0)) \sqcup \text{id}_\Sigma
\]

for some coefficients \( \alpha_1, \ldots, \alpha_m \in k \), for some \( \mathcal{C} \)-colored ribbon set \( P \subset S^2 \) with at least one point labeled by a projective object, for some morphism \( M_P \) of \( \tilde{\text{Cob}} \) from \( (S^2, P, \vartheta, \{0\}) \sqcup \Sigma \) to \( \Sigma' \), where the cohomology class \( \vartheta \) is uniquely determined by \( P \), and for some \( \mathcal{C} \)-colored ribbon graphs \( T_1, \ldots, T_m \subset D^3 \) from \( \varnothing \) to \( P \) satisfying

\[
\sum_{i=1}^{m} \alpha_i \cdot f_{D^3}(T_i) = 0
\]

for some embedding \( f_{D^3} : D^3 \to \mathbb{R}^2 \times I \) mapping \( P \) into \( \mathbb{R}^2 \times \{1\} \). On the other hand, we say a formal linear combination of morphisms of \( \tilde{\text{Cob}} \) from \( \Sigma \) to \( \Sigma' \) is a non-local skein relation if it can be written in the form

\[
(M, T \cup K, j^*(\omega), n) - \psi(\langle \omega, \ell_K \rangle, k) \cdot (M, T, \omega, n)
\]

for some \( k \in Z \) and for some framed knot \( K \subset M \setminus T \) of color \( \sigma(k) \), where \( j^* \) is induced by inclusion, and where \( \ell_K \) denotes the homology class of \( K \) in \( H_1(M \setminus T; \mathbb{Z}) \). Then, if \( \Sigma = (\Sigma, P, \vartheta, B, \mathcal{Z}) \) is an object of \( \tilde{\text{Cob}} \), and if \( M \) is a 3-dimensional cobordism from \( \varnothing \) to \( \Sigma \), the admissible skein module \( \hat{S}(M; \Sigma) \) is the finite-dimensional quotient, induced by both local and non-local skein relations, of the free vector space \( \mathcal{T}(M; \Sigma) \) generated by all pairs \((T, \omega)\) such that \((M, T, \omega, 0)\) is a morphism of \( \tilde{\text{Cob}} \) from \( \varnothing \) to \( \Sigma \). Analogously, if \( M' \) is a 3-dimensional cobordism from \( \Sigma \) to \( \varnothing \), then the admissible skein module \( \hat{S}(M'; \Sigma) \) is the finite-dimensional quotient, induced by both local and non-local skein relations, of the free vector space \( \mathcal{T}(M'; \Sigma) \) generated by all pairs \((T', \omega')\) such that \((M', T', \omega', 0)\) is a morphism of \( \tilde{\text{Cob}} \) from \( \Sigma \) to \( \varnothing \). Then, for every \( k \in Z \), the degree \( k \) state space of a connected object \( \Sigma \) of \( \tilde{\text{Cob}} \) satisfies

\[
\mathcal{V}^k(\Sigma) \cong \hat{S}(M; \Sigma \sqcup S^2_k)/\hat{S}(M'; \Sigma \sqcup S^2_k)^\perp
\]

with respect to the pairing \( \langle \cdot, \cdot \rangle_{\Sigma \sqcup S^2_k} : \hat{S}(M'; \Sigma \sqcup S^2_k) \otimes \hat{S}(M; \Sigma \sqcup S^2_k) \to k \) defined by

\[
\langle [T', \omega'], [T, \omega] \rangle_{\Sigma \sqcup S^2_k} := \text{N}_\# ((M', T', \omega', 0) \circ (M, T, \omega, n)),
\]

where \( M \) is a connected 3-dimensional cobordism from \( \varnothing \) to \( \Sigma \sqcup S^2 \), where \( M' \) is a connected 3-dimensional cobordism from \( \Sigma \sqcup S^2 \) to \( \varnothing \), and where the object

\[
S^2_k = (S^2, P(+, V_0, (+, \sigma(-k)), (-, V_0), \theta((+, V_0), (+, \sigma(-k)), (-, V_0)), B, \{0\})
\]

of \( \tilde{\text{Cob}} \) is determined by the \( \mathcal{C} \)-colored ribbon set \( P((+, V_0), (+, \sigma(-k)), (-, V_0)) \subset S^2 \) composed of three points in standard positions with orientations and colors specified by their subscript for some \( V_0 \in G \times X \) and for some \( V_0 \in \Theta(\mathcal{C}) \). Remark that an explicit characterization of these quotients can sometimes be achieved. This is the case for certain objects of \( \tilde{\text{Cob}} \), called generic surfaces, whose state spaces are described in Proposition 7.16 [6] in terms of homogeneous colorings of trivalent graphs.

1.3. 3-Manifold invariants from finite-dimensional non-degenerate unimodular ribbon Hopf algebras. Next, let us move on to the renormalized Hennings theory. We start by fixing our notation for Hopf algebras, and by recalling some crucial definitions and results. If \( k \) is a field, a finite-dimensional ribbon Hopf algebra \( H \) is a finite-dimensional vector space over \( k \) endowed with a multiplication \( m : H \otimes H \to H \), a unit \( \eta : k \to H \), a coproduct \( \Delta : H \to H \otimes H \), a counit
$\varepsilon : H \to k$, an antipode $S : H \to H$, an R-matrix $R = \sum_{i=1}^{r} a_i \otimes b_i \in H \otimes H$, and a ribbon element $v$ in the center of $H$. We use the notation $m(x \otimes y) = xy$ for every $x \otimes y \in H \otimes H$ and $\eta(1) = 1$, and we denote with $u = \sum_{i=1}^{r} S(b_i)a_i \in H$ the Drinfeld element, and with $g = uv^{-1} \in H$ the pivotal element associated with the ribbon structure of $H$. As a consequence of finite-dimensionality, $H$ admits a right integral $\lambda \in H^*$ and a left cointegral $\Lambda \in H$ which are unique up to scalar, and we can fix a pair satisfying $\lambda(\Lambda) = 1$. The Hopf algebra $H$ is non-degenerate if the stabilization coefficients $\Delta_{-\lambda} := \lambda(v)$ and $\Delta_{+\lambda} := \lambda(v^{-1})$ satisfy $\Delta_{-\lambda}\Delta_{+\lambda} \neq 0$, and it is unimodular if the left cointegral $\Lambda$ is two-sided, meaning that it is also a right cointegral. The category $H$-mod of finite-dimensional left $H$-modules is a ribbon category which, thanks to Theorem 1 of [1], admits an $m$-trace on the ideal of projective $H$-modules, which is unique up to scalar and uniquely determined by

$$t_H(f) = \lambda(gf(1))$$

for all $f \in \End_{\mathcal{C}}(H)$, where $H \in \Proj(\mathcal{C})$ denotes the regular representation of $H$. Furthermore, this $m$-trace is non-degenerate, meaning that the pairing

$$t_V(\cdot \circ \cdot) : \Hom_{\mathcal{C}}(V', V) \otimes \Hom_{\mathcal{C}}(V, V') \to k$$

is non-degenerate for all $V \in \Proj(\mathcal{C})$ and $V' \in \mathcal{C}$.

In [7] it is shown that every finite-dimensional non-degenerate ribbon Hopf algebra $H$ gives rise to a topological invariant $H'_{\mathcal{C}}$ of admissible pairs $(M, T)$, where $M$ is a closed 3-manifold, and $T \subset M$ is a $\mathcal{C}$-colored bichrome graph. The latter are $\mathcal{C}$-colored ribbon graphs carrying a set of specified edges, and their name comes from the fact that we think about special edges as being red, while the rest of the graph is blue. Red edges can only be colored with the regular representation $H$, and they can only intersect coupons in a prescribed way: for every coupon of a bichrome graph there exists an integer $k \geq 0$ such that the first $k$ input edges are incoming and red, the first $k$ output edges are outgoing and red, while all the other ones are blue. Such a coupon is colored with a morphism in the $k$-th stabilized subcategory $[k]\mathcal{C}$ of $\mathcal{C}$, which is the category whose objects have the form $[k]V := H^{\otimes k} \otimes V \in \mathcal{C}$ with $V \in \mathcal{C}$, and whose morphisms have the form $\sum_{i=1}^{m} x_i \otimes f_i \in \Hom_{\mathcal{C}}([k]V, [k]V')$ with $x_i \in H^{\otimes k}$ and $f_i \in \Hom_{\mathcal{C}}(V, V')$. The category $\mathcal{R}_{\lambda}$ of $\mathcal{C}$-colored bichrome graphs provides a graphical calculus which is formalized by the Hennings-Reshetikhin-Turaev functor $F_{\lambda} : \mathcal{R}_{\lambda} \to \mathcal{C}$ introduced in Proposition 2.5 of [7]. By definition, $F_{\lambda}$ coincides with the Reshetikhin-Turaev functor in the absence of red edges, it coincides with the Hennings invariant in the absence of blue edges, and it coherently combines the two behaviors for general $\mathcal{C}$-colored bichrome graphs. Remark that $F_{\lambda}$ yields a notion of skein equivalence between between formal linear combinations of $\mathcal{C}$-colored bichrome graphs in the same way $F_{\mathcal{C}}$ does for $\mathcal{C}$-colored ribbon graphs. This way, $\mathcal{R}_{\lambda}$ is naturally identified with the subcategory of $\mathcal{R}_{\lambda}$ whose morphisms are entirely blue. The renormalized Hennings invariant $H'_{\mathcal{C}}$ is then defined only for admissible pairs $(M, T)$, which are pairs such that every component of $M$ contains a projective blue edge of $T$. Its definition uses surgery presentations in $S^3$, which we interpret as $\mathcal{C}$-colored bichrome graphs by arbitrarily choosing orientations, by labeling every component with the regular representation $H$, and by taking them to be red. The idea is to build $H'_{\mathcal{C}}$ out of a renormalized invariant $F'_{\lambda}$ of admissible closed $\mathcal{C}$-colored bichrome graphs which combines the Hennings-Reshetikhin-Turaev functor $F_{\lambda}$ on $\mathcal{R}_{\lambda}$ with the $m$-trace $t$ on $\Proj(\mathcal{C})$. We say a closed $\mathcal{C}$-colored bichrome graph is admissible if one of his blue edges is projective. The formula

$$F'_{\lambda}(T) := t_V(F_{\lambda}(TV))$$
defines a topological invariant of the admissible closed \( \mathcal{C} \)-colored bichrome graph \( T \) thanks to Theorem 2.7 of [7], where \( V \) is a projective object of \( \mathcal{C} \), and where \( TV \) is a cutting presentation of \( T \), meaning an endomorphism of \((+,V)\) in \( \mathcal{A}_A \) whose trace is \( T \). Then, for a fixed choice of a square root \( \varnothing_{h} \in \kappa \) of \( \Delta_{-}\Delta_{+} \), the formula

\[
\mathcal{H}_{V}(M,T) := \varnothing_{h}^{-1-\mathcal{L}_{\lambda}}(\mathcal{L}_{\lambda}) F_{\mathcal{L}}(L \cup T)
\]

defines a topological invariant of the admissible pair \((M,T)\) thanks to Theorem 2.9 of [7], where \( L = L_1 \cup \ldots \cup L_{k} \subset S^{3} \) is a surgery presentation of \( M \) of signature \( \sigma(L) \), and where \( \delta_{h} = \mathcal{H}_{\lambda}/\Delta_{-}\lambda \).

1.4. 2+1-TQFTs from finite-dimensional factorizable ribbon Hopf algebras. A finite-dimensional ribbon Hopf algebra \( H \) with R-matrix \( R = \sum_{i,j=1}^{r} a_i \otimes b_i \) is factorizable if the Drinfeld map \( \psi_{H} : H^{*} \rightarrow H \), which is defined by

\[
\psi_{H}(f) := \sum_{i,j=1}^{r} f(b_{j}a_{i}) \cdot a_{j}b_{i}
\]

for every \( f \in H^{*} \), is an isomorphism. This condition implies both non-degeneracy and unimodularity, see [14] and [21]. When \( H \) is factorizable then, as explained in Section 3 of [7], \( \mathcal{H}_{\mathcal{C}} \) extends to a 2+1-TQFT \( \mathcal{V}_{H} : \mathcal{C}_{\mathcal{H}} \rightarrow \text{Vect}_{k} \) via the universal construction of [3], where \( \mathcal{C}_{\mathcal{H}} \) is the category of admissible cobordisms of dimension 2+1. More precisely, an object of \( \mathcal{C}_{\mathcal{H}} \) is a triple \( \Sigma = (\Sigma,\mathcal{P},\mathcal{L}) \), where \( \Sigma \) is a closed surface, where \( \mathcal{P} \subset \Sigma \) is a blue \( \mathcal{C} \)-colored ribbon set, and where \( \mathcal{L} \subset H_{1}(\Sigma;\mathbb{R}) \) is a Lagrangian subspace. A morphism of \( \mathcal{C}_{\mathcal{H}} \) from \((\Sigma,\mathcal{P},\mathcal{L})\) to \((\Sigma',\mathcal{P}',\mathcal{L}')\) is an equivalence class of admissible triples \( M = (M,T,n) \), where \( M \) is a 3-dimensional cobordism from \( \Sigma \) to \( \Sigma' \), where \( T \subset M \) is a \( \mathcal{C} \)-colored bichrome graph from \( P \) to \( P' \), and where \( n \in \mathbb{Z} \) is a signature defect. A triple \((M,T,n)\) is admissible if every component of \( M \) which is disjoint from the incoming boundary \( \partial M \) contains a projective blue edge of \( T \), and two triples \((M,T,n)\) and \((M',T',n')\) are equivalent if \( n = n' \), and if there exists a positive diffeomorphism \( f : M \rightarrow M' \) which preserves boundary identifications and satisfies \( f(T) = T' \). Then, \( \mathcal{H}_{\mathcal{C}} \) can be extended to an invariant of closed morphisms of \( \mathcal{C}_{\mathcal{H}} \) by setting

\[
\mathcal{H}_{\mathcal{C}}(M,T,n) := \delta_{n}^{H}(M,T).
\]

Remark that the category \( \mathcal{C}_{\mathcal{H}} \) thus obtained is not rigid, as objects \((\Sigma,\mathcal{P},\mathcal{L})\) such that \( P \) does not contain any projective blue point are not dualizable.

State spaces associated with objects of \( \mathcal{C}_{\mathcal{H}} \) by the TQFT \( \mathcal{V}_{\mathcal{C}} \) can be presented as quotients of admissible skein modules, just like we did in the CGP case, and of course this time only local skein relations are needed. However, they can also be efficiently described in terms of the dual coadjoint \( H \)-module \( X \), which is the vector space \( H \) equipped with the action \( \rho_{X}(h)(x) := h_{(2)} x S^{-1}(h_{(1)}) \) for all \( h \in H \) and \( x \in X \). Indeed, if \( V \) is a left \( H \)-module with action \( \rho_{V} : H \rightarrow \text{End}_{k}(V) \), we can consider its subspace of \( H \)-invariant vectors, which is defined as

\[
V^{H} := \{ v \in V \mid \rho_{V}(h)(v) = \varepsilon(h) \cdot v \ \forall h \in H \}.
\]

Remark that we have an obvious isomorphism between \( \text{Hom}_{\mathcal{C}}(\mathbb{1},V) \) and \( V^{H} \) sending \( f \) to \( f(1) \). Then, if \( \Sigma_{g} \) is a closed surface of genus \( g \in \mathbb{N} \), and if \( PV \subset \Sigma_{g} \) is a single positive framed blue point of color \( V \in \mathcal{C} \), it follows directly from Corollary 3.21 of [7] that the state space of the object \( \Sigma_{g,V} = (\Sigma_{g},PV,\mathcal{L}) \) of \( \mathcal{C}_{\mathcal{H}} \) determined by any arbitrary Lagrangian \( \mathcal{L} \subset H_{1}(\Sigma_{g};\mathbb{R}) \) satisfies

\[
\mathcal{V}_{\mathcal{C}}(\Sigma_{g,V}) \cong \left( (V^{*} \otimes X^{*g})^{H} \right)^{*}.
\]
1.5. Main results. As we mentioned earlier, this paper contains two main results related to the non-semisimple constructions we just recalled. The first one concerns the existence of a family of graded TQFTs, as well as graded ETQFTs, for the CGP theory. The setting is provided by unrolled quantum groups at odd roots of unity. More precisely, in Subsection 2.2 we recall the definition, for every simple complex Lie algebra \( \mathfrak{g} \) of rank \( n \) and dimension \( 2N + n \), of a particular quantization, denoted \( U_q(\mathfrak{g}) \), of the enveloping algebra \( U(\mathfrak{g}) \) for \( q = e^{2\pi i} \), where \( r \geq 3 \) is an odd integer which is required not to be a multiple of 3 when \( \mathfrak{g} = \mathfrak{g}_2 \). These \textit{unrolled} quantum groups are quite different from the ones which usually provide the basic building blocks for quantum constructions in low dimensional topology. For instance, they are infinite dimensional: indeed, they are generated by \{\( E_i, F_i, K_i, K_i^{-1} \mid 1 \leq i \leq n \)\}, but while generators \( E_i \) and \( F_i \), as well as the induced root vectors, are set to be nilpotent, generators \( K_i \) and \( K_i^{-1} \) are not required to be idempotent. This produces a representation theory where weights are allowed to take complex values, instead of integral ones. However, complex weights generate problems when it comes to defining a braiding on representations. In order to overcome this difficulty, we can add a generator \( H_i \) corresponding to a logarithm of \( K_i \) for every integer \( 1 \leq i \leq n \), as the superscript in \( U_q^{(i)}(\mathfrak{g}) \) suggests. This exponential relation is not set at the level of the quantum group, but we restrict to representations where it is satisfied. More precisely, we focus on the full subcategory \( \mathcal{C}^{H} \) of finite-dimensional representations of \( U_q^{(i)}(\mathfrak{g}) \) where the action of generators \( H_i \) is diagonalizable, and where the action of generators \( K_i \) is obtained by exponentiating. This category is non-semisimple, and it was studied in detail in [10]: a full subcategory \( \mathcal{D}^{H} \) of \( \mathcal{C}^{H} \) was proven to be ribbon, and the equality \( \mathcal{D}^{H} = \mathcal{C}^{H} \) was conjectured. In [5], it was shown that \( \mathcal{D}^{H} \) is relative pre-modular, and thus yields a quantum invariant \( \mathcal{N}_{g_0^H} \) of admissible decorated 3-manifolds. In [11], the conjecture was proven: \( \mathcal{C}^{H} \) is a relative pre-modular category. Its structure group \( G \) is given by the quotient \( \mathfrak{n}^*/\Lambda_R \), where \( \mathfrak{n} \) is a Cartan subalgebra of \( \mathfrak{g} \) with root lattice \( \Lambda_R \), its periodicity group \( Z \) is given by the intersection \( \Lambda_R \cap (r \cdot \Lambda_W) \), where \( \Lambda_W \) denotes the weight lattice, and its critical set \( X \) is given by \{\( \{\xi \in \mathfrak{n}^*/\Lambda_R \mid \exists \alpha \in \Phi_+ : 2(\alpha, \xi) \in \mathbb{Z} \} \}. The following is our first main result, which implies, as an immediate consequence, the existence of a \( Z \)-graded ETQFT in dimension \( 1+1+1 \) extending the invariant \( \mathcal{N}_{g_0^H} \).

**Theorem 1.3.** The category \( \mathcal{C}^{H} \) is relative modular.

The second main result of this paper builds a bridge between this family of quantum invariants and the renormalized Hennings ones coming from the corresponding small quantum groups. Indeed, in Subsection 2.1 we recall the definition of a more classical quantization of \( U(\mathfrak{g}) \), denoted \( \bar{U}_q(\mathfrak{g}) \), again for \( q = e^{2\pi i} \). These \textit{small} quantum groups are far better known: they are finite-dimensional, as generators \( K_i \) and \( K_i^{-1} \) are set to be idempotent, they are ribbon and factorizable, and thus they yield TQFTs in dimension \( 2+1 \). The category \( \mathcal{C} \) of finite-dimensional representations of \( \bar{U}_q(\mathfrak{g}) \) is still non-semisimple, but all weights take integral values. Indeed, we have a very natural forgetful functor \( \Phi_{\mathcal{C}} \) from the full subcategory \( \mathcal{C}^{H}_{[0]} \) of \( \mathcal{C}^{H} \) whose objects have all weights in \( \Lambda_R \) to \( \mathcal{C} \): the image \( V \) of an object \( V \) of \( \mathcal{C}^{H}_{[0]} \) is simply defined by forgetting the action of generators \( H_i \). This functor immediately induces a functor \( \Phi_{\mathcal{C}} \) from \( \mathcal{R}_{g_0^H}^H \) to \( \mathcal{R}_{g_0} \): the image \( \tilde{T} \) of a morphism \( T \) of \( \mathcal{R}_{g_0^H}^H \) is simply defined by applying the forgetful functor \( \Phi_{\mathcal{C}} \) to all its colors.

**Theorem 1.4.** If \( M \) is a closed 3-manifold and \( T \subset M \) is an admissible \( \mathcal{C}^{H}_{[0]} \)-colored ribbon graph, then

\[
\mathcal{N}_{g_0^H}(M, T, 0) = H^0_{g}(M, \tilde{T}).
\]

Remark that each invariant depends on the choice of a square root of the product of the stabilization coefficients for the corresponding version of the quantum group.
2. Quantum groups at odd roots of unity

In this section we recall definitions of small and unrolled quantum groups associated with arbitrary simple complex Lie algebras $\mathfrak{g}$, and we prove our first result: categories of finite-dimensional weight representations of unrolled quantum groups at odd roots of unity are relative modular, and can therefore be used to construct ETQFTs in dimension $1+1+1$.

2.1. Small quantum groups. Let $\mathfrak{g}$ be a simple complex Lie algebra of rank $n$ and dimension $2N+n$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let $\Phi_+$ be a set of positive roots of $\mathfrak{g}$, and let $\mathcal{U}_q\mathfrak{g}$ be the associated quantum group, over a formal parameter $q$, introduced in Appendix A. Let us fix an odd integer $r \geq 3$ with the further condition that $r \not\equiv 0$ modulo 3 if $\mathfrak{g} = \mathfrak{g}_2$, and let us specialize $q$ to $e^{\frac{2\pi i}{r}}$. Let $\bar{U}_q\mathfrak{g}$ denote the small quantum group of $\mathfrak{g}$, which is the C-algebra obtained from $\mathcal{U}_q\mathfrak{g}$ by adding relations

$$K_\mu = 1, \quad E^r_\alpha = F^r_\alpha = 0$$

for every $\mu \in \Lambda_R \cap r \cdot \Lambda_W$ and every $\alpha \in \Phi_+$. Then $\bar{U}_q\mathfrak{g}$ inherits from $\mathcal{U}_q\mathfrak{g}$ the structure of a Hopf algebra, and we denote with $\bar{U}_q\mathfrak{h}$, with $\bar{U}_q\mathfrak{n}_+$, and with $\bar{U}_q\mathfrak{n}_-$ the subalgebras of $\bar{U}_q\mathfrak{g}$ generated by $\{K_i \mid 1 \leq i \leq n\}$, by $\{E_i \mid 1 \leq i \leq n\}$, and by $\{F_i \mid 1 \leq i \leq n\}$, respectively. As it follows from Theorem 30 of [10], a Poincaré-Birkhoff-Witt basis is given by

$$\left\{ \prod_{k=1}^N F_{\beta_k}^{c_k} K_\mu \left( \prod_{k=1}^N E_{\beta_k}^{b_k} \right) \mid \mu \in \Lambda_R/(\Lambda_R \cap r \cdot \Lambda_W), \right.$$

$$0 \leq b_1, \ldots, b_N < r, \quad 0 \leq c_1, \ldots, c_N < r \left. \right\},$$

so $\bar{U}_q\mathfrak{g}$ is finite-dimensional. A pivotal element is given by $K_2, \rho \in \bar{U}_q\mathfrak{g}$ where $\rho := \frac{1}{2} \sum_{k=1}^N \beta_k$.

Furthermore, if we consider $R_0 \in \bar{U}_q\mathfrak{h} \otimes \bar{U}_q\mathfrak{h}$ given by

$$R_0 := \frac{1}{[\Lambda_R/(\Lambda_R \cap r \cdot \Lambda_W)]} \sum_{\mu, \mu' \in \Lambda_R/(\Lambda_R \cap r \cdot \Lambda_W)} q^{-\langle \mu, \mu' \rangle} : K_\mu \otimes K_{\mu'}$$

and $\Theta \in \bar{U}_q\mathfrak{n}_+ \otimes \bar{U}_q\mathfrak{n}_-$ given by

$$\Theta := \sum_{b_1, \ldots, b_N = 0}^{r-1} \left( \prod_{k=1}^N \frac{1}{b_k! b_{k-1}!} \right) \left( \prod_{k=1}^N \frac{F_{\beta_k}^{b_k}}{E_{\beta_k}^{b_{k-1}}} \otimes \prod_{k=1}^N \frac{E_{\beta_k}^{b_k}}{F_{\beta_k}^{b_{k-1}}} \right),$$

then $R := R_0 \Theta \in \bar{U}_q\mathfrak{g} \otimes \bar{U}_q\mathfrak{g}$ is an R-matrix for $\bar{U}_q\mathfrak{g}$, as explained in [18]. Next, thanks to Proposition A.5.1 of [19], a right integral $\lambda$ of $\bar{U}_q\mathfrak{g}$ is given by

$$\lambda \left( \prod_{k=1}^N F_{\beta_k}^{b_k} \right) K_\mu \left( \prod_{k=1}^N E_{\beta_k}^{b_k} \right) = q^{-4\langle \rho, \rho \rangle} \delta_{\mu,2}^N \prod_{k=1}^N \delta_{b_k,r-1} \prod_{k=1}^N \delta_{c_k,r-1}. $$
This formula can be deduced from the one in [19] by remarking that Lyubashenko
uses Luszitg’s coproduct \( \Delta := (\omega \otimes \omega) \circ \Delta^{op} \circ \omega \), where \( \omega \) denotes the involutive
algebra automorphism of \( \bar{U}_q \mathfrak{g} \) defined by \( \omega(E_i) = F_i \) and by \( \omega(K_i) = K_i^{-1} \) for all
integers \( 1 \leq i \leq n \), and by remarking that \( \lambda \) is a right integral for \( \Delta \) if and only if
\( \lambda \circ \omega \) is a left integral for \( \Delta \). Finally, thanks to Proposition A.5.2 of [19], a two-sided
cointegral \( \Lambda \) of \( \bar{U}_q \mathfrak{g} \) is given by

\[
\Lambda := \sum_{\mu \in \Lambda_n/(\Lambda \cap \nu \Lambda W)} q^{2(\mu, \rho)} \cdot \left( \prod_{k=1}^N F_{\beta_k}^{r-1} \right) K_{\mu} \left( \prod_{k=1}^N E_{\beta_k}^{r-1} \right).
\]

**Proposition 2.1.** The Hopf algebra \( \bar{U}_q \mathfrak{g} \) is factorizable and ribbon.

This result is proved in [19], see also [18]. We denote with \( \mathcal{C} \) the ribbon category
of finite-dimensional \( \bar{U}_q \mathfrak{g} \)-modules, and with \( \tilde{\mathcal{C}} \) the m-trace on \( \text{Proj}(\mathcal{C}) \) given by
Theorem 1 of [1], which satisfies \( \tilde{\mathcal{C}}(\Lambda \circ \epsilon) = 1 \) for the regular representation \( \tilde{\mathcal{U}} \) of \( \bar{U}_q \mathfrak{g} \).

**Corollary 2.2.** The renormalized Hennings invariant \( H'_{\mathfrak{g}} \) extends to a TQFT

\[ V_{\mathfrak{g}} : \mathcal{C}ob_{\mathfrak{g}} \to \text{Vect}_{\mathbb{C}}. \]

**2.2. Unrolled quantum groups.** Let \( U^H_q \mathfrak{g} \) denote the unrolled quantum group of \( \mathfrak{g} \), which is the \( \mathbb{C} \)-algebra obtained from \( \bar{U}_q \mathfrak{g} \) by adding generators

\[ \{ H_i \mid 1 \leq i \leq n \} \]

and relations

\[ [H_i, H_j] = [H_j, K_j] = 0, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j, \quad E_\alpha = F_\alpha = 0 \]

for every integer \( 1 \leq i, j \leq n \) and every positive root \( \alpha \in \Phi_+ \). Then \( U^H_q \mathfrak{g} \) can be
made into a pivotal Hopf algebra by setting

\[ \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \epsilon(H_i) = 0, \quad S(H_i) = -H_i \]

for every integer \( 1 \leq i \leq n \), and we denote with \( U^H_q \mathfrak{h} \), with \( U^H_q \mathfrak{n}_+ \), and with \( U^H_q \mathfrak{n}_- \)
the subalgebras of \( U^H_q \mathfrak{g} \) generated by \( \{ H_i \mid 1 \leq i \leq n \} \), by \( \{ E_i \mid 1 \leq i \leq n \} \), and
by \( \{ F_i \mid 1 \leq i \leq n \} \) respectively. For every \( z \in \mathbb{C} \) let us introduce the notation

\[ q^z := e^{\frac{2\pi i z}{r}}, \quad \{ z \} := q^z - q^{-z}. \]

A \( U^H_q \mathfrak{g} \)-module \( V \) with action \( \rho_V : U^H_q \mathfrak{g} \to \text{End}_q(V) \) is a weight module if it is a
semisimple \( U^H_q \mathfrak{h} \)-module and if for every \( \mu \in \mathfrak{h}^* \) and every \( v \in V \) we have

\[ \rho_V(H_i)(v) = \mu(H_i) \cdot v \quad \forall 1 \leq i \leq n \quad \Rightarrow \quad \rho_V(K_i)(v) = q_\mu(H_i) \cdot v \quad \forall 1 \leq i \leq n, \]

where we are identifying \( \mathfrak{h} \) with the corresponding linear subspace of \( U^H_q \mathfrak{h} \) in the
obvious way. We denote with \( \mathcal{C}^H \) the full subcategory of the category of finite-dimensional
\( U^H_q \mathfrak{g} \)-modules whose objects are weight modules. Then \( \mathcal{C}^H \) can be made into a ribbon
category as follows: first of all, a pivotal element is given by \( K^{-1} \in U^H_q \mathfrak{g} \), where the choice of the exponent is explained in Remark 4 of [11].
Furthermore, if \( V \) and \( V' \) are objects of \( \mathcal{C}^H \), their braiding morphism is given by

\[ c_{V,V'} : V \otimes V' \to V' \otimes V \]

\[ v \otimes v' \mapsto \tau_{V,V'}(R_{0,V,V'}((\rho_V \otimes \rho_{V'})(\Theta)(v \otimes v'))), \]

for the linear maps \( R_{0,V,V'} : V \otimes V' \to V \otimes V' \) and \( \tau_{V,V'} : V \otimes V' \to V' \otimes V \)
determined by

\[ R_{0,V,V'}(v \otimes v') := q^{(v,v')}, \quad \tau_{V,V'}(v \otimes v') := v' \otimes v. \]
for all $v \in V$, $v' \in V'$ satisfying
\[ \rho_V(H_i)(v) = \nu(H_i) \cdot v, \quad \rho_V(H_i)(v') = \nu'(H_i) \cdot v' \]
for every integer $1 \leq i \leq n$, and for the element $\Theta \in U_q^H \mathbb{N}_+ \otimes U_q^H \mathbb{N}_-$ given by
\[
\Theta := \sum_{b_1, \ldots, b_N = 0}^{r-1} \left( \prod_{k=1}^N \left\{ b_k \right\}_{|b_k|_{\beta_k}^+}^{-1} \right) \left( \prod_{k=1}^N \rho_{b_k} \right) \otimes \left( \prod_{k=1}^N \rho_{\beta_k} \right).
\]
Thanks to Theorem 4 of [11], $G^H$ is a ribbon category.

If we set $G := h^*/\Lambda_R$, then $G^H$ supports the structure of a $G$-category: indeed, for every $\gamma \in h^*$ we can define the homogeneous subcategory $G^H_\gamma$ to be the full subcategory of $G^H$ with objects given by modules whose weights are all of the form $\gamma + \mu$ for some $\mu \in \Lambda_R$. Furthermore, if we set $Z := \Lambda_R \cap (r \cdot \Lambda_W)$, then we have a free realization $\sigma : Z \to G^H_\gamma$ mapping every $\kappa \in Z$ to the object $\sigma(\kappa) \in G^H_\gamma$ given by the vector space $\mathbb{C}$ with $U_q^H \mathfrak{g}$-action specified by
\[
\rho_{\sigma(\kappa)}(H_i)(1) := \kappa(H_i), \quad \rho_{\sigma(\kappa)}(E_i)(1) := 0, \quad \rho_{\sigma(\kappa)}(F_i)(1) := 0
\]
for every integer $1 \leq i \leq n$. Remark that, since $\langle \Lambda_R, r \cdot \Lambda_W \rangle \subset r\mathbb{Z}$, then
\[
\dim G(\sigma(\kappa)) = \rho_{\sigma(\kappa)}(K_2, \rho)(1) = q^{2(r, \kappa)} = 1
\]
for every $\kappa \in Z$. Now the bilinear map
\[
\psi : G \times Z \to \mathbb{C}^*, \quad ([\gamma], \kappa) \mapsto q^{2(\gamma, \kappa)}
\]
satisfies $c_{\sigma(\kappa), V} \circ c_{V, \sigma(\kappa)} = \psi([\gamma], \kappa) \cdot \text{id}_V \otimes \sigma(\kappa)$ for every $\gamma \in h^*$, every $V \in G^H_\gamma$, and

then, as explained in Section 7 of [5], the category $G^H_{\gamma}$ is semisimple for every $[\gamma] \in G \setminus X$. Therefore, the last ingredient we are missing is an $m$-trace. In order to define it, let us introduce typical $U_q^H \mathfrak{g}$-modules. First of all, we say a vector $v_+$ of a $U_q^H \mathfrak{g}$-module $V$ is a highest weight vector if $\rho_V(E_i)(v_+) = 0$ for every integer $1 \leq i \leq n$. Analogously, we say a vector $v_-$ of $V$ is a lowest weight vector if $\rho_V(F_i)(v_-) = 0$ for every integer $1 \leq i \leq n$. Then for every weight $\mu \in h^*$ there exists a simple finite-dimensional weight $U_q^H \mathfrak{g}$-module $V_\mu$ featuring a highest weight vector of weight $\mu$. This module is unique up to isomorphism, and every simple $U_q^H \mathfrak{g}$-module is of this form, see Proposition 33 of [10]. Every such module also has a lowest weight vector, and it is called typical if its lowest weight is given by $\mu - 2(r - 1) \cdot \rho$. If we consider the set
\[
\hat{h}^* := \{ \gamma \in h^* \mid 2(\alpha, \gamma + \rho) + m(\alpha, \alpha) \not\in r\mathbb{Z} \ \forall \alpha \in \Phi_+, \ \forall 1 \leq m \leq r - 1 \}
\]
then, thanks to Proposition 34 of [10], $V_\gamma$ is typical if and only if $\gamma \in \hat{h}^*$. Remark that if $\gamma \in \hat{h}^*$ satisfies $2(\alpha, \gamma) \not\in Z$ for every $\alpha \in \Phi$, then $\gamma \in \hat{h}^*$. This means that if $\gamma \in \hat{h}^*$ satisfies $[\gamma] \not\in \Xi$, then $V_\gamma$ is typical. We also point out that, although $[(r - 1) \cdot \rho] \in \Xi$, the module $V_{(r - 1) \cdot \rho}$ is always typical, because
\[
2(\alpha, (r - 1) \cdot \rho + \rho) + m(\alpha, \alpha) = 2r(\alpha, \rho) + md_\alpha
\]
is not in $r\mathbb{Z}$ for any integer $1 \leq m \leq r - 1$. Now, thanks to Lemma 7.1 of [5] and Theorem 38 of [10], every typical $U_q^H \mathfrak{g}$-module is projective and ambidextrous. Then, by combining Theorem 3.3.2 of [8] with Lemma 17 of [13], there exists a non-zero $m$-trace on the ideal $\text{Proj}(G^H)$ of projective objects of $G^H$ which is unique.
up to scalar. Here is the normalization we choose: for every \( \mu \in \mathfrak{h}^* \) we define the \( m \)-dimension \( d^H(V_\mu) \) of \( V_\mu \) to be
\[
t^H_\mu (\text{id}_{V_\mu}) = \frac{1}{N} \prod_{k=1}^N r\{ (\mu - (r-1) \cdot \rho, \beta_k) \}.
\]
Remark that this normalization is \( r^N \) times the one given in Formula (51) of [10].
For all \( \mu, \nu \in \mathfrak{h}^* \), if \( f^+_{\mu, \nu} := F_{\mathfrak{g}H} (T^H_{\mu, \nu}) \) and \( f^-_{\mu, \nu} := F_{\mathfrak{g}H} (T^H_{\mu, \nu}) \) for the \( \mathfrak{g}H \)-colored ribbon graphs \( T^H_{\mu, \nu} \) and \( T^H_{\mu, \nu} \) represented in Figure 3, Proposition 45 of [10] gives
\[
t^H_\mu (f^+_{\mu, \nu}) = r^N e^{2(\mu - (r-1) \cdot \rho, \nu - (r-1) \cdot \rho)}.
\]
and, since \( V^+_{\mu} \cong V_{2(r-1) \rho - \mu} \), it also gives
\[
t^H_\mu (f^-_{\mu, \nu}) = r^N e^{-2(\mu - (r-1) \cdot \rho, \nu + (1-r) \cdot \rho)} = r^N t^H_{\mu, \nu} (f^+_{\mu, \nu})^{-1}.
\]

![Figure 3. \( \mathfrak{g}H \)-colored ribbon graphs \( T^H_{\mu, \nu} \) and \( T^H_{\mu, \nu} \).](image)

2.3. Relative modularity. In this subsection we will prove the category \( \mathfrak{g}H \) is relative modular, and thus yields a Z-graded TQFT. In order to do this, we will first need a preliminary definition. We say an endomorphism \( f \in \text{End}_{\mathfrak{g}H}(V) \) of an object \( V \) of \( \mathfrak{g}H \) is transparent in \( \mathfrak{g}H \) if for all objects \( U, W \in \mathfrak{g}H \) we have
\[
\text{id}_U \otimes f = c_{V, U} \circ (f \otimes \text{id}_U) \circ c_{U, V}, \quad f \otimes \text{id}_W = c_{V, W} \circ (\text{id}_W \otimes f) \circ c_{V, W}.
\]

**Lemma 2.3.** If \( f \in \text{End}_{\mathfrak{g}H}(V) \) is transparent in \( \mathfrak{g}H \), then there exist some integer \( m \) and some morphisms \( g_i \in \text{Hom}_{\mathfrak{g}H}(V, \sigma(\kappa_i)) \) and \( h_i \in \text{Hom}_{\mathfrak{g}H}(\sigma(\kappa_i), V) \) for every integer \( 1 \leq i \leq m \) such that
\[
f = \sum_{i=1}^m h_i \circ g_i.
\]

**Proof.** If \( v_+ \) is a highest weight vector of \( V_{(r-1) \rho} \) and \( v \) is a weight vector of \( V \) then \( c_{V_{(r-1) \rho}, V}(v_+ \otimes v) \) is proportional to \( v \otimes v_+ \) because
\[
\rho_{V_{(r-1) \rho}} \left( \prod_{k=1}^N \mathcal{E}_{x_k, y_k} \right) (v_+) = 0
\]
for all integers \( 0 \leq b_1, \ldots, b_N < r \) whose sum is strictly positive. Furthermore, \( c_{V_{(r-1) \rho}, V}(f(v) \otimes v_+) \) is proportional to \( v_+ \otimes f(v) \) because \( f \) is transparent in \( \mathfrak{g}H \). But now
\[
\left\{ \rho_{V_{(r-1) \rho}} \left( \prod_{k=1}^N \mathcal{E}_{x_k, y_k} \right) (v_+) \bigg| 0 \leq b_1, \ldots, b_N < r \right\}
\]
is a basis of \( V_{(r-1) \rho} \) thanks to Proposition 34 of [10]. This means that
\[
\rho_V \left( \prod_{k=1}^N \mathcal{E}_{x_k, y_k} \right) (f(v)) = 0
\]
for every weight vector \( v \in V \) and for all integers \( 0 \leq b_1, \ldots, b_N < r \) whose sum is strictly positive.
Analogously, if $v_-$ is a lowest weight vector of $V_{(r-1)\rho}$ and $v$ is a weight vector of $V$ then $c_{V_{(r-1)\rho}}(v \otimes v_-)$ is proportional to $v_- \otimes v$ because
\[ \rho_{V_{(r-1)\rho}} \left( \prod_{k=1}^{N} E^{b_k}_{\beta_k} \right) (v_-) = 0 \]
for all integers $0 \leq b_1, \ldots, b_N < r$ whose sum is strictly positive. Furthermore, $c_{V_{(r-1)\rho}}(v_- \otimes f(v))$ is proportional to $f(v) \otimes v_-$ because $f$ is transparent in $\mathcal{E}^H$.

But now
\[ \left\{ \rho_{V_{(r-1)\rho}} \left( \prod_{k=1}^{N} E^{b_k}_{\beta_k} \right) (v_+) \right\} \quad 0 \leq b_1, \ldots, b_N < r \]
is a basis of $V_{(r-1)\rho}$ thanks to Proposition 34 of [10]. This means that
\[ \rho_V \left( \prod_{k=1}^{N} F^{b_k}_{\beta_k} \right) (f(v)) = 0 \]
for every weight vector $v \in V$ and for all integers $0 \leq b_1, \ldots, b_N < r$ whose sum is strictly positive.

Now, since $K_i - K_i^{-1} = (q_i - q_i^{-1}) \cdot [E_i, F_i]$ for every integer $1 \leq i \leq n$, we get the equality $\rho_V(K_i)(f(v)) - \rho_V(K_i^{-1})(f(v)) = 0$ for every $v \in V$, which implies
\[ \rho_V(K_i)^2(f(v)) = f(v). \]

Then, if $f(v)$ is a weight vector of weight $\kappa$, this tells us that $2(\kappa, \alpha_i) \in r\mathbb{Z}$ for every integer $1 \leq i \leq n$. Since $r$ is odd and coprime with $d_i$ for every integer $1 \leq i \leq n$, this means precisely that $\kappa \in \mathbb{Z}$. Therefore, each weight vector of $\text{im} f$ determines a 1-dimensional submodule which is isomorphic to $\sigma(\kappa)$ for some $\kappa \in \mathbb{Z}$. Since $\text{im} f$ is a direct sum of its weight spaces, this means that
\[ \text{im} f \cong \bigoplus_{i=1}^{m} \sigma(\kappa_i) \]
for some integer $m \geq 1$ and some $\kappa_1, \ldots, \kappa_m \in \mathbb{Z}$. Let $\pi_i \in \text{Hom}_{\mathcal{E}^H}(\text{im} f, \sigma(\kappa_i))$ and $\iota_i \in \text{Hom}_{\mathcal{E}^H}(\sigma(\kappa_i), \text{im} f)$ denote the corresponding projection and injection morphisms for every integer $1 \leq i \leq m$. We can factorize $f = \iota_f \circ \pi_f$ where $\pi_f \in \text{Hom}_{\mathcal{E}^H}(V, \text{im} f)$ is naturally induced by $f$ and $\iota_f \in \text{Hom}_{\mathcal{E}^H}(\text{im} f, V)$ denotes inclusion. Then the result follows by setting $g_i := \pi_i \circ \pi_f$ and $h_i := \iota_f \circ \iota_i$ for every integer $1 \leq i \leq m$.

Let us complete $\{0\} \subset \Lambda_R$ to a set $\mathcal{F}_r \subset \Lambda_R$ of representatives of equivalence classes in $\Lambda_R/\mathcal{E}$. Then for every $\gamma \in \mathfrak{h}^*$ satisfying $[\gamma] \in G \setminus X$ the set
\[ \Theta(\mathcal{E}_r) := \{ V_{\gamma + \mu} \in \mathcal{E}_r \mid \mu \in \mathcal{F}_r \} \]
is a set of representatives of $\mathbb{Z}$-orbits of isomorphism classes of simple objects of $\mathcal{E}_r$. We have now everything in place to prove Theorem 1.3.

\[ \begin{array}{c}
\begin{array}{c}
\text{d}^H(V_\mu) \uparrow \\
\Omega_{[\gamma]} \downarrow \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
V_\mu \\
\delta_{\mu, \nu} r^{2N} \text{[}\mathcal{F}_r\text{]} \uparrow \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
V_\mu \\
\bigcup \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
V_\nu \\
\bigcup \\
\end{array}
\end{array} \]

**Figure 4.** Relative modularity of $\mathcal{E}^H$. 

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**NON-SEMI-SIMPLE QUANTUM INVARIANTS AND TQFTS**

13
Proof of Theorem 1.3. We know $\mathcal{C}^H$ is a non-degenerate relative pre-modular category thanks to Theorem 7.2 of [5] and to Theorem 4 of [11]. Therefore, we only need to prove that $\mathcal{C}^H$ satisfies the relative modularity condition of Definition 1.6 of [6]. We will do this by showing the skein equivalence of Figure 4 for every $\gamma \in G \setminus X$, for every $\mu \in \hat{H}^*$, and for every $\nu \in \{\mu\} + \mathcal{K}_r$, so let $f_{[\gamma],\mu,\nu}$ denote the morphism of $\mathcal{C}^H$ obtained by applying the Reshetikhin-Turaev functor $F_{\mathcal{C}}^H$ to the $\mathcal{C}^H$-colored ribbon graph represented in the left hand part of Figure 4, ignoring the coefficient.

Thanks to the handle slide property, we have the skein equivalence of Figure 5, which means $f_{[\gamma],\mu,\nu}$ is transparent in $\mathcal{C}^H$. Now, thanks to Lemma 2.3, we have

$$f_{[\gamma],\mu,\nu} = \sum_{i=1}^{m} h_{[\gamma],\mu,\nu,i} \circ g_{[\gamma],\mu,\nu,i}$$

with $g_{[\gamma],\mu,\nu,i} \in \text{Hom}_{\mathcal{C}^H}(V_\mu \otimes V_\nu^*, \sigma(\kappa_i))$, with $h_{[\gamma],\mu,\nu,i} \in \text{Hom}_{\mathcal{C}^H}(\sigma(\kappa_i), V_\mu \otimes V_\nu^*)$, and with $\kappa_i \in Z$ for every integer $1 \leq i \leq m$. But now $h_{[\gamma],\mu,\nu,i} = 0$ unless $\mu = \nu$ and $\kappa_i = 0$, because

$$\text{Hom}_{\mathcal{C}^H}(V_\mu \otimes V_\nu^*, \sigma(\kappa_i)) \cong \text{Hom}_{\mathcal{C}^H}(V_\mu, V_\nu \otimes \sigma(\kappa_i))$$

and because $\mathcal{K}_r$ is a set of representatives of equivalence classes in $\Lambda_R/Z$. This means that $f_{[\gamma],\mu,\mu}$ factors through the tensor unit 1. But now, since $V_\mu$ is simple, both $\text{Hom}_{\mathcal{C}^H}(V_\mu \otimes V_\nu^*, 1)$ and $\text{Hom}_{\mathcal{C}^H}(1, V_\mu \otimes V_\nu^*)$ are 1-dimensional. This means $f_{[\gamma],\mu,\mu}$ is a scalar multiple of $\text{coev}_{V_\mu} \circ \text{ev}_{V_\mu}$. In order to compute the proportionality coefficient let us compare the m-traces of $\text{coev}_{V_\mu} \circ \text{ev}_{V_\mu}$ and of $f_{[\gamma],\mu,\mu}$. The first one is easily seen to be

$$t^H_{\nu_\mu} \circ \text{ev}_{V_\mu} = t^H_{\nu_\mu} \circ \text{ev}_{V_\mu} = t^H_{\nu_\mu} (\text{id}_{V_\mu}) = d^H(V_\mu).$$

On the other hand, if

$$f_{[\gamma],\mu,\mu} = \sum_{\nu \in \{\gamma\} + \mathcal{K}_r} d^H(V_\nu) \cdot f_{\nu,\mu,\mu},$$

where $f_{\nu,\mu,\mu}$ is obtained from $f_{[\gamma],\mu,\mu}$ by replacing the label $\Omega_{[\gamma]}$ of the meridian with $V_\nu$, then the second one is given by

$$t^H_{\nu \otimes V_\nu} (f_{[\gamma],\mu,\mu}) = \sum_{\nu \in \{\gamma\} + \mathcal{K}_r} d^H(V_\nu) t^H_{\nu \otimes V_\nu} (f_{\nu,\mu,\mu})$$

$$= \sum_{\nu \in \{\gamma\} + \mathcal{K}_r} d^H(V_\nu) t^H_{\nu} (f_{\nu,\mu} \circ f_{\mu,\nu}^-)$$

$$= \sum_{\nu \in \{\gamma\} + \mathcal{K}_r} d^H(V_\nu) d^H(V_\nu)^{-1} t^H_{\nu} (f_{\mu,\nu}^-) t^H_{\nu} (f_{\mu,\nu}^-)$$

$$= r^{2N} |\mathcal{K}_r|,$$

where the morphisms $f_{\mu,\nu}^-$ and $f_{\mu,\nu}^+$ are represented in Figure 3. \qed
Corollary 2.4. The CGP invariant $N\subset n$ extends to a $Z$-graded TQFT

$$V_{\subset n} : \text{Cob}_{\subset n} \to \text{Vect}_Z.$$ 

2.4. Projective generators. In this subsection we prove some key technical results which will be later used for the proof Theorem 1.4. We say an object $P$ of $\mathcal{C}$ is a projective generator of $\mathcal{C}$ if for every object $V$ of $\text{Proj}(\mathcal{C})$ there exist some integer $m$ and some morphisms $f_i \in \text{Hom}_\mathcal{C}(V, P)$ and $g_i \in \text{Hom}_\mathcal{C}(P, V)$ for every integer $1 \leq i \leq m$ such that

$$\text{id}_V = \sum_{i=1}^m g_i \circ f_i.$$ 

Remark that a natural choice for a projective generator of $\mathcal{C}$ is the regular representation $\bar{U}$ of $\mathcal{U}_q\mathfrak{g}$. Analogously, we say an object $P$ of $\mathcal{C}_Z^H$ is a projective generator of $\mathcal{C}_Z^H$ if for every object $V$ of $\text{Proj}(\mathcal{C}_Z^H)$ there exist some integer $m$, some $\kappa_i \in Z$, and some morphisms $f_i \in \text{Hom}_{\subset n}(V, P \otimes \sigma(\kappa_i))$ and $g_i \in \text{Hom}_{\subset n}(P \otimes \sigma(\kappa_i), V)$ for every integer $1 \leq i \leq m$ such that

$$\text{id}_V = \sum_{i=1}^m g_i \circ f_i.$$ 

In order to construct a projective generator of $\mathcal{C}_Z^H$ we consider, for every $\mu \in \Lambda_R$, the projective cover $P_\mu$ of the simple weight $U_q^H\mathfrak{g}$-module $V_\mu$ of highest weight $\mu$, which is an indecomposable projective weight $U_q^H\mathfrak{g}$-module. Every indecomposable projective object of $\mathcal{C}_Z^H$ is of this form, and thus every projective object of $\mathcal{C}_Z^H$ is a direct sum of projective covers of simple objects of $\mathcal{C}_Z^H$. Therefore, if $\mathcal{K}_r$ denotes the set of representatives of equivalence classes in $\Lambda_R/Z$ of Subsection 2.3, then

$$P := \bigoplus_{\mu \in \mathcal{K}_r} P_\mu$$ 

is by construction a projective generator of $\mathcal{C}_Z^H$. 

Lemma 2.5. $\dim_C(\text{Hom}_{\subset n}(P, \sigma(\kappa))) = \dim_C(\text{Hom}_{\subset n}(\sigma(\kappa), P)) = \delta_{0,\kappa}$ for every $\kappa \in Z$. 

Proof. Since the vector space of $U_q^H\mathfrak{g}$-module morphisms from a projective indecomposable weight $U_q^H\mathfrak{g}$-module to its unique simple quotient is 1-dimensional, we have

$$\dim_C(\text{Hom}_{\subset n}(P, \sigma(\kappa))) = \delta_{0,\kappa}.$$ 

Furthermore, since projective objects of $\mathcal{C}_Z^H$ are also injective, and since duals of indecomposable objects of $\mathcal{C}_Z^H$ are indecomposable, $P^*$ is also a direct sum of representatives of $Z$-orbits of isomorphism classes of indecomposable projective objects of $\mathcal{C}_Z^H$, and we have

$$\dim_C(\text{Hom}_{\subset n}(\sigma(\kappa), P)) = \dim_C(\text{Hom}_{\subset n}(P^*, \sigma(-\kappa))) = \delta_{0,\kappa}.$$ 

The following result establishes a cutting property for Kirby meridians which is analogous to Lemma 3.6 of [7]. 

Lemma 2.6. There exist generators $\varepsilon \in \text{Hom}_{\subset n}(P, 1)$ and $\Lambda \in \text{Hom}_{\subset n}(1, P)$ realizing the skein equivalence of Figure 6. Furthermore, these morphisms satisfy $1_P^H(\Lambda \circ \varepsilon) = 1$. 


Proof. Thanks to Lemma 2.5, and thanks to the non-degeneracy of $t^H$, the composition of a non-trivial morphism of $\text{Hom}_{CH}(P, 1)$ with a non-trivial morphism of $\text{Hom}_{\mathcal{C}}(1, P)$ has non-zero m-trace. Therefore, let us fix a pair of generators $\varepsilon \in \text{Hom}_{CH}(P, 1)$ and $\Lambda \in \text{Hom}_{\mathcal{C}}(1, P)$ satisfying $t^H(P \circ \varepsilon) = 1$, and let us prove they realize the skein equivalence of Figure 6. If $h_\gamma$ is the morphism of $\mathcal{C}^H$ obtained by applying the Reshetikhin-Turaev functor $F_{\mathcal{C}}$ to the $\mathcal{C}^H$-colored ribbon graph represented in the left hand part of Figure 6, the handle slide property yields the skein equivalence represented in Figure 7. This means that, thanks to Lemmas 2.3 and 2.5, the morphism $h_\gamma$ factors through the tensor unit. But then, since both $\text{Hom}_{CH}(P, 1)$ and $\text{Hom}_{\mathcal{C}}(1, P)$ are 1-dimensional, we must have $h_\gamma = \alpha \cdot \Lambda \circ \varepsilon$ for some $\alpha \in \mathbb{C}$. In order to show $\alpha = r^{2N} |\mathcal{F}|$, we use the $\mathbb{Z}$-graded TQFT $\mathcal{V}_{\mathcal{C}}$. Indeed, let us consider the object $S^2(\mathbb{P}, P)$ of $\hat{\text{Cob}}_{\mathcal{C}}$ defined by $$(S^2, P(\mathbb{P}, 0, B, \{0\}),$$ where the $\mathcal{C}^H$-colored ribbon set $P(\mathbb{P}, 0, B, \{0\})$ is given by a single framed point with positive orientation and color $\mathbb{P}$, and let us consider the closed morphism $S^2(\mathbb{P}, P) \times S^1$ of $\hat{\text{Cob}}_{\mathcal{C}}$ defined by $$(S^2 \times S^1, P(\mathbb{P}, 0, B, \{0\}).$$ The strategy will be to compute the CGP invariant of $S^2(\mathbb{P}, P) \times S^1$ in two different ways. On one hand, the isomorphism $\mathcal{V}_{\mathcal{C}}(S^2(\mathbb{P}, P)) \cong \text{Hom}_{\mathcal{C}}(1, P)$ gives $$N_{\mathcal{C}}(S^2(\mathbb{P}, P) \times S^1) = \sum_{\kappa \in \mathbb{Z}} \dim_{\mathbb{C}}(\mathcal{V}_{\mathcal{C}}(S^2(\mathbb{P}, P))) = \sum_{\kappa \in \mathbb{Z}} \dim_{\mathbb{C}}(\text{Hom}_{\mathcal{C}}(1, P \otimes \sigma(-\kappa))) = \sum_{\kappa \in \mathbb{Z}} \dim_{\mathbb{C}}(\text{Hom}_{\mathcal{C}}(\sigma(\kappa), P)) = 1,$$ where the last equality follows from Lemma 2.5. On the other hand, up to performing projective stabilization of index $[\gamma]$ on $P(\mathbb{P}, 0, B, \{0\})$, as explained in Section

$$V_{(r-1)\rho} \quad P \quad V_{(r-1)\rho}$$

$\Omega_\gamma \quad \Omega_\rho$

Figure 7. Transparency of $h_\gamma$. 

\begin{figure}
\centering
\includegraphics{figure6}
\caption{Cutting property for Kirby-colored meridians.}
\end{figure}
for all \(id \in C\). Forgetful functor. We single unknot of framing 0. This choice determines the \(C^H\)-colored ribbon graph \(T_{P, \gamma}\) represented in Figure 8, where \(sp_{P, \gamma} \in \text{Hom}_{g,n}(P, P \otimes V, V')\) is a section of \(id_P \otimes e_{V, \gamma}\), that is a morphism satisfying \((id_P \otimes e_{V, \gamma}) \circ sp_{P, \gamma} = id_P\). Thus we get

\[
N_{g,n} \left( S^2_{(+, P)} \times S^1 \right) = \frac{\alpha |H^H(\Lambda \circ \varepsilon)|}{r^{2N}N_r} = \frac{\alpha}{r^{2N}N_r}. \quad \square
\]

2.5. Forgetful functor. Let us consider the forgetful functor \(\Phi_\varepsilon : C^H_{[0]} \to \mathcal{C}\) which forgets the action of \(H_i\) for all integers \(1 \leq i \leq n\). If \(V\) is an object of \(C^H_{[0]}\) we denote with \(\tilde{V}\) its image under \(\Phi_\varepsilon\), and if \(f\) is a morphism of \(C^H_{[0]}\) we denote with \(\tilde{f}\) its image under \(\Phi_\varepsilon\). This induces a ribbon functor \(\Phi_\varepsilon : \mathcal{R}_{\mathfrak{g}, [0]} \to \mathcal{R}_\varepsilon\) from the category of \(C^H_{[0]}\)-colored ribbon graphs to the category of \(\mathcal{C}\)-colored ribbon graphs. If \(T\) is a morphism of \(\mathcal{R}_{\mathfrak{g}, [0]}\) we denote with \(\tilde{T}\) its image under \(\Phi_\varepsilon\).

**Lemma 2.7.** The forgetful functor \(\Phi_\varepsilon : C^H_{[0]} \to \mathcal{C}\) is ribbon, and it satisfies

\[
\Phi_\varepsilon \circ F_{\mathfrak{g}, [0]} = F_\varepsilon \circ \Phi_\varepsilon.
\]

**Proof.** The result follows immediately from the equality

\[
R^H_{0, V, V'} = (\rho_V \otimes \rho_{V'})(R_0)
\]

for all \(V, V' \in C^H_{[0]}\), where \(R^H_{0, V, V'} : V \otimes V' \to V \otimes V'\) is defined in Subsection 2.2, and where \(R_0 \in \mathcal{U}_0 \otimes \mathcal{U}_0\) is defined in Subsection 2.1. To show the claim, remark

\[
\sum_{\mu' \in \Lambda_R/Z} q^{\mu, \mu'} = |\Lambda_R/Z| \delta_{\mu, 0}
\]

for every \(\mu \in \Lambda_R\). This means that

\[
(\rho_V \otimes \rho_{V'})(R_0) (v \otimes v') = \frac{1}{|\Lambda_R/Z|} \sum_{\mu, \mu' \in \Lambda_R/Z} q^{-\langle \mu, \mu' \rangle} \cdot \rho_V(K_\mu)(v) \otimes \rho_{V'}(K_{\mu'})(v')
\]

\[
= \frac{1}{|\Lambda_R/Z|} \sum_{\mu, \mu' \in \Lambda_R/Z} q^{\langle \mu, \mu' \rangle + \langle \mu', \mu \rangle - \langle \mu, \mu' \rangle} \cdot v \otimes v'
\]

\[
= \sum_{\mu \in \Lambda_R/Z} \delta_{\mu, 0} q^{\langle \nu, \nu' \rangle} \cdot v \otimes v'
\]

for all \(v, v' \in V\) satisfying

\[
\rho_V(H_i)(v) = \nu(H_i) \cdot v, \quad \rho_{V'}(H_i)(v') = \nu'(H_i) \cdot v'
\]

for every integer \(1 \leq i \leq n\). \(\square\)
The forgetful functor $\Phi_\mathfrak{g}$ preserves the property of being projective.

**Lemma 2.8.** If $P$ is a projective object of $\mathcal{C}^H_{[0]}$, then $\bar{P}$ is a projective object of $\bar{\mathcal{C}}$.

*Proof.* The typical $U_q^H \mathfrak{g}$-module $V_{(r-1), \rho}$ introduced in Subsection 2.2 generates $\text{Proj}(\mathcal{C}^H_{[0]})$ thanks to Lemma 17 of [13]. Then $P$ must be a direct summand of a tensor product $V_{(r-1), \rho} \otimes W$ for some $W \in \mathcal{C}^H_{[0]}$. Now the proof of Lemma 7.1 of [5] can be repeated to show the image $\bar{V}_{(r-1), \rho}$ of $V_{(r-1), \rho}$ under the forgetful functor $\Phi_\mathfrak{g}$ is projective. This means $\bar{V}_{(r-1), \rho}$ generates $\text{Proj}(\bar{\mathcal{C}})$, and thus $\bar{P}$, which is a direct summand of $\bar{V}_{(r-1), \rho} \otimes \bar{W}$, is projective. \(\square\)

In particular, the image $\bar{P}$ of the projective generator $P$ of $\mathcal{C}^H_{[0]}$ is a projective object of $\bar{\mathcal{C}}$.

**Lemma 2.9.** $\dim_C (\text{Hom}_{\mathfrak{g}}(1, P)) = \dim_C (\text{Hom}_{\mathfrak{g}}(\bar{P}, 1)) = 1$.

*Proof.* Let $P$ be a projective $U_q^H \mathfrak{g}$-module in $\mathcal{C}^H_{[0]}$, and remark that its image $\bar{P}$ under $\Phi_\mathfrak{g}$ coincides with $P$ as a vector space. Let us consider the space $P_{U_q^H \mathfrak{g}}$ of $U_q^H \mathfrak{g}$-invariants vectors of $P$. Remark that $\text{Hom}_{\mathfrak{g}}(1, \bar{P})$ is naturally isomorphic to $P_{U_q^H \mathfrak{g}}$, simply by identifying every morphism $f \in \text{Hom}_{\mathfrak{g}}(1, \bar{P})$ with the image $f(1) \in P_{U_q^H \mathfrak{g}}$. We claim the subspace $P_{U_q^H \mathfrak{g}}$ of $P$ formed by vectors of $P_{U_q^H \mathfrak{g}}$ is a $U_q^H \mathfrak{g}$-submodule of $P$. Indeed, this follows from the commutation relations satisfied by the additional generators $H_1, \ldots, H_n$. But now remark that every weight vector of $P_{U_q^H \mathfrak{g}}$ with respect to the action of $U_q^H \mathfrak{g}$ determines a split 1-dimensional submodule of $P_{U_q^H \mathfrak{g}}$ which is isomorphic to $\sigma(\kappa)$ for some $\kappa \in Z$. This means that $P_{U_q^H \mathfrak{g}} \cong \bigoplus_{i=1}^{\text{dim}_C(P_{U_q^H \mathfrak{g}})} \sigma(\kappa_i)$ with $\kappa_i \in Z$ for every integer $1 \leq i \leq \text{dim}_C(P_{U_q^H \mathfrak{g}})$. Thus we get

$$\dim_C(P_{U_q^H \mathfrak{g}}) \leq \dim_C \left( \bigoplus_{\kappa \in Z} \text{Hom}_{U_q^H \mathfrak{g}}(\sigma(\kappa), P) \right),$$

and the converse inequality follows from the equality $\Phi_\mathfrak{g}(\sigma(\kappa)) = \mathbb{1}$ for every $\kappa \in Z$.

First, let us consider $P = P$. Thanks to Lemma 2.5, we have

$$\dim_C \left( \bigoplus_{\kappa \in Z} \text{Hom}_{U_q^H \mathfrak{g}}(\sigma(\kappa), P) \right) = 1.$$

Thus, the space $P_{U_q^H \mathfrak{g}}$ is 1-dimensional. This means $\dim_C(\text{Hom}_{\mathfrak{g}}(1, P)) = 1$.

Next, let us consider $P = P^*$. Thanks to Lemma 2.5, we have

$$\dim_C \left( \bigoplus_{\kappa \in Z} \text{Hom}_{U_q^H \mathfrak{g}}(\sigma(\kappa), P^*) \right) = \dim_C \left( \bigoplus_{\kappa \in Z} \text{Hom}_{U_q^H \mathfrak{g}}(P, \sigma(-\kappa)) \right) = 1.$$

Thus, the space $(P^*)_{U_q^H \mathfrak{g}}$ is 1-dimensional. This means $\dim_C(\text{Hom}_{\mathfrak{g}}(P^*, \mathbb{1})) = \dim_C(\text{Hom}_{\mathfrak{g}}(\mathbb{1}, P^*)) = 1$. \(\square\)
3. Equality of 3-manifold invariants

The goal for this section will be to prove Theorem 1.4. We will use as a key ingredient the fact that meridians labeled with Kirby colors have the cutting property with respect to the projective generator $P$ of $\mathcal{C}_H^{[0]}$, while red meridians labeled with the regular representation have the cutting property with respect to its image $\overline{P}$ in $\mathcal{C}$. The proof will require a comparison of all the ingredients that correspond to each other in the two theories.

3.1. Stabilized surgery presentations. In this subsection we introduce special surgery presentations of admissible decorated closed 3-manifolds which are tailored for the comparison between the CGP and the renormalized Hennings invariants. In order to do so, we need to start by comparing the m-trace $t^H$ on $\text{Proj}(\mathcal{C}_H)$ with the m-trace $t$ on $\text{Proj}(\mathcal{C})$.

Remark 3.1. Both $t^H$ and $t$ are unique up to scalar, but the chosen normalizations do not agree, as they are determined by the conditions

$$t^H_0(\Lambda \circ \varepsilon) = 1, \quad t_0(\Lambda \circ \varepsilon) = 1$$

respectively, where $P$ is the projective generator of $\mathcal{C}_H^{[0]}$ introduced in Subsection 2.4, where $\varepsilon \in \text{Hom}_{\mathcal{C}^{[0]}}(P, 1)$ and $\Lambda \in \text{Hom}_{\mathcal{C}^{[0]}}(1, P)$ are the morphisms introduced in Lemma 2.6, where $\overline{U}$ is the regular representation of $\tilde{U}_q$, and where $\varepsilon$ and $\Lambda$ are the counit and the cointegral respectively. Therefore, thanks to Lemma 2.8, there exists a non-zero coefficient $\alpha \in \mathbb{C}^*$ such that

$$t^H|_{\mathcal{C}_H^{[0]}} = \alpha \cdot t \circ \Phi_\varepsilon,$$

meaning that $t^H(f) = \alpha t_V(f)$ for every $V \in \text{Proj}(\mathcal{C}_H^{[0]})$ and every $f \in \text{End}_{\mathcal{C}_H}(V)$.

Lemma 3.2. The renormalized invariants $F_{\mathcal{C}_H^{[0]}}'$ and $F_\hat{\varepsilon}'$ satisfy

$$F_{\mathcal{C}_H^{[0]}}' = \alpha \cdot F_\hat{\varepsilon}' \circ \Phi_\varepsilon,$$

meaning that $F_{\mathcal{C}_H^{[0]}}'(T) = \alpha F_{\hat{\varepsilon}'}(\tilde{T})$ for every closed admissible $\mathcal{C}_H^{[0]}$-colored ribbon graph $T$, where $\alpha$ is the coefficient introduced in Remark 3.1.

Proof. If a closed admissible $\mathcal{C}_H^{[0]}$-colored ribbon graph $T$ admits a projective edge of color $V$, we can consider a cutting presentation $T_V$ of $T$, which is an endomorphism of $(+, V)$ in $\mathcal{R}_H^{[0]}$ satisfying

$$\tilde{\omega}_{(+, V)} \circ (T_V \otimes \text{id}_{(-, V)}) \circ \text{coev}_{(+, V)} = T.$$

Then we have

$$t^H(F_{\mathcal{C}_H^{[0]}}'(T)) = \alpha \tilde{\omega}_V(\Phi_\varepsilon(F_{\mathcal{C}_H^{[0]}}'(T)))) = \alpha \tilde{\omega}_V(\Phi_\varepsilon(T_V(\tilde{T}))),$$

where the second equality follows from Lemma 2.7. \qed

Now, let us recall the formulas defining the CGP and the renormalized Hennings invariants in the setting of Theorem 1.4. If $(M, T, 0, 0)$ is a closed connected morphism of $\tilde{\text{Cob}}_{\mathcal{C}}$, if $L = L_1 \cup \ldots \cup L_\ell \subset S^3$ is a surgery presentation of $M$, and if we replace $(T, 0)$ with some $(\tilde{T}, \tilde{\omega})$ obtained by projective stabilization of sufficiently generic index ensuring $L$ becomes computable, as explained in Subsection 1.1 and, in greater detail, in Section 3.2 of [6], then we have

$$N_{\mathcal{C}_H^{[0]}}(M, T, 0, 0) = \mathcal{D}_{\tilde{T}}^{-1-\delta_\omega} \mathcal{D}_{\tilde{\omega}}^{-\sigma(L)} F_{\mathcal{C}_H^{[0]}}'(L \cup \tilde{T}).$$

On the other hand, if $(M, \tilde{T}, 0)$ is the closed connected morphism of $\tilde{\text{Cob}}_{\mathcal{C}}$ obtained by applying the functor $\Phi_\varepsilon$ to $T \subset M$, then we have

$$H_\varepsilon'(M, \tilde{T}, 0) = \mathcal{D}_{\lambda}^{-1-\delta_\lambda} \mathcal{D}_{\lambda}^{-\sigma(L)} F_\lambda'(L \cup \tilde{T}).$$
The surgery presentation $L$ is utilized in different ways by the two constructions. In the first case, $L$ is labeled with Kirby colors, and thus it is not a morphism in the domain of $\Phi_{g'}$. In the second case, $L$ is taken to be red, and thus it is not a morphism in the image of $\Phi_{g'}$. In order to compare the two formulas, we introduce special morphisms of $\mathcal{C}$ which encode these two different procedures.

First, for all weights $\mu, \nu \in \mathcal{h}^*$ satisfying $[\mu] = [\nu] \in G \setminus X$ the tensor product $W_{\mu, \nu} := V_{\mu} \otimes V_{\nu}^*$ is an object of $\mathcal{C}$. Therefore, since $U$ is a projective generator of $\mathcal{C}$, we can fix a decomposition

$$\text{id}_{W_{\mu, \nu}} = \sum_{i=1}^{m_{\mu, \nu}} g_{\mu, \nu, i} \circ f_{\mu, \nu, i}$$

for some morphisms $f_{\mu, \nu, i} \in \text{Hom}_{\mathcal{C}}(W_{\mu, \nu}, U)$ and $g_{\mu, \nu, i} \in \text{Hom}_{\mathcal{C}}(U, W_{\mu, \nu})$. Let us also set

$$d_{\mu, \nu} := (m_{\mu, \nu} \otimes \text{id}_{W_{\mu, \nu}}) \circ (\text{id}_{V_{\mu}} \otimes \text{coev}_{V_{\mu}} \otimes \text{id}_{V_{\nu}}) \circ s_{\nu} \in \text{Hom}_{\mathcal{C}}(P, W_{\mu, \nu}^* \otimes W_{\mu, \nu}),$$

where $s_{\nu} \in \text{Hom}_{\mathcal{C}}(P, W_{\mu, \nu})$ is a morphism satisfying $\text{ev}_{V_{\nu}} \circ s_{\nu} = \varepsilon$, and where $m_{\mu, \nu} \in \text{Hom}_{\mathcal{C}}(V_{\mu} \otimes V_{\mu}^*, (V_{\mu} \otimes V_{\mu}^*)^*)$ is the isomorphism coming from the pivotal structure of $\mathcal{C}$. Now, let us fix once and for all a weight $\gamma \in \mathcal{h}^*$ satisfying $[\gamma] \in G \setminus X$. Then we denote with $h_{\Omega} \in \text{Hom}_{\mathcal{C}}(P, \bar{U}^* \otimes U)$ the morphism

$$h_{\Omega} := \sum_{\mu \in \{\gamma\} \cup \mathcal{X}} m_{\mu, \gamma} \sum_{i=1}^{m_{\mu, \gamma}} d_{\mu, \gamma} \cdot ((g_{\mu, \gamma, i})^* \otimes f_{\mu, \gamma, i}) \circ d_{\mu, \gamma}.$$

Next, we denote with $f_{\lambda \otimes 1} \in \text{Hom}_{\mathcal{C}}(\bar{U}, \bar{U}^* \otimes U)$ the unique morphism which sends the generator $1 \in U$ to $\lambda \otimes 1 \in \bar{U}^* \otimes U$, where $\lambda$ is the right integral of $U_{g'}$, and we consider a morphism $s_P \in \text{Hom}_{\mathcal{C}}(P, \bar{U}^* \otimes \bar{P})$ satisfying $(\varepsilon \otimes \text{id}_{\bar{P}}) \circ s_P = \text{id}_{\bar{P}}$. Then we denote with $h_{\lambda} \in \text{Hom}_{\mathcal{C}}(P, \bar{U}^* \otimes U)$ the morphism

$$h_{\lambda} := (f_{\lambda \otimes 1} \otimes \varepsilon) \circ s_P.$$

This allows us to review the recipe for the computation of the two invariants. If $(M, T, 0, 0)$ is a closed morphism of $\text{Cob}_{g, n}$, if $e \subset T$ is a projective edge of color $V$, and if $L = L_1 \cup \ldots \cup L_\ell \subset S^3$ is a surgery presentation of $M$, then let us fix disjoint paths $\gamma_i \subset S^3 \setminus (L \cup T)$ connecting $e$ to $L_i$ for every integer $1 \leq i \leq \ell$. Before starting, we perform special projective stabilizations both on $e \subset T$ and on $\bar{e} \subset \bar{T}$ at the intersection point with $\gamma_i$ for every integer $1 \leq i \leq \ell$, as shown in Figure 9, where $s_V \in \text{Hom}_{\mathcal{C}}(V, P \otimes V)$ is a section of $\varepsilon \otimes \text{id}_{V}$, where $s_V \in \text{Hom}_{\mathcal{C}}(V, P \otimes V)$ is its image under $\Phi_{g'}$, and where $s_V \in \text{Hom}_{\mathcal{C}}(P, W_{\gamma, \gamma})$ is a morphism satisfying $\text{ev}_{V_{\gamma}} \circ s_{\gamma} = \varepsilon$. Next, we isotope the $s_{\gamma}$-colored and the $\varepsilon$-colored coupons along the path $\gamma_i$ until the intersection point with $L_i$. This is our initial configuration.

**Figure 9.** Projective stabilizations on $e \subset T$ and on $\bar{e} \subset \bar{T}$. 
Let us start from the CGP invariant. First, we need to slide every $V_\gamma$-colored edge along the corresponding component $L_i$, so to turn the surgery presentation $L$ into a computable one. This produces the $\mathcal{C}^H$-colored ribbon graph represented in the top-left corner of Figure 10. Up to skein equivalence of $\mathcal{C}^H$-colored ribbon graphs, we can replace a tubular neighborhood of $L_i$, as shown in the top-right corner of Figure 10, thus obtaining a $\mathcal{C}^H[0]$-colored ribbon graph. This means we can apply the functor $\Phi_R$ which, up to skein equivalence of $\mathcal{C}$-colored ribbon graphs, produces the $\mathcal{C}$-colored ribbon graph represented in the bottom-left corner of Figure 10. Again up to skein equivalence of $\mathcal{C}$-colored ribbon graphs, we can replace a tubular neighborhood of $L_i$ as shown in the bottom-right corner of Figure 10. The resulting $\mathcal{C}$-colored ribbon graph is denoted $(L \cup \bar{T})_{\Omega}$, and is said to be obtained from the surgery presentation $L$ and from the admissible $\mathcal{C}^H[0]$-colored ribbon graph $T$ by $\Omega$-stabilization along the paths $\gamma_1, \ldots, \gamma_{\ell}$. By construction, using Lemma 3.2, we have

$$N_{\mathcal{C}^H}(M, T, 0, 0) = \alpha \mathcal{D}_{\Omega}^{-1 - \ell \sigma(L)} F_{\Phi}^R ((L \cup \bar{T})_{\Omega}).$$

Let us move on to discuss the renormalized Hennings invariant. First, we need to interpret every component $L_i$ as a red edge, and to label it with the regular representation $\bar{U}$. This produces the $\mathcal{C}$-colored bichrome graph represented in the left-hand part of Figure 11. Up to skein equivalence of $\mathcal{C}$-colored bichrome graphs, we can turn every red component blue by replacing a tubular neighborhood of $L_i$ as shown in the right-hand side of Figure 11. The resulting $\mathcal{C}$-colored ribbon graph

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Skein equivalences of $\mathcal{C}^H$-colored and $\mathcal{C}$-colored ribbon graphs defining $(L \cup \bar{T})_{\Omega}$.}
\end{figure}
is denoted \((L \cup \bar{T})_{h_\lambda}\), and is said to be obtained from the surgery presentation \(L\) and from the admissible \(C^0_{\text{bdy}}\)-colored ribbon graph \(T\) by \(\lambda\)-stabilization along the paths \(\gamma_1, \ldots, \gamma_6\). By construction, using Lemma 3.8 of [7], we have

\[
H'_\varphi(M, \bar{T}, 0) = \mathcal{D}_\lambda^{-1} \cdot \delta_{\lambda}^{-\sigma(L)} F_{\varphi}^L \left((L \cup \bar{T})_{h_\lambda}\right).
\]

3.2. Stabilization coefficients. Next, we need to compare \(\mathcal{D}\) with \(\mathcal{D}_\lambda\), \(\delta\) with \(\delta\), and \(h_\Omega\) with \(h_\lambda\). In order to do this, we will prove a key technical result. Let \((S^1 \times S^1)_{(-, P)}\) be the object of \(\text{Cob}_\varphi\) defined by

\[
(S^1 \times S^1)_{(-, P)} := (S^1 \times S^1, P_{(-, P)}, \mathcal{X}),
\]

where the blue \(\mathcal{G}\)-colored ribbon set \(P_{(-, P)}\) is given by a single framed point with negative orientation and color \(P\), and where the Lagrangian subspace \(\mathcal{X}\) is generated by the homology class of the curve \(\{1, 0\} \times S^1\). Analogously, let \(S^2_{2((-,-),(+))}\) be the object of \(\text{Cob}_\varphi\) defined by

\[
S^2_{2((-,-),(+))} := (S^2, P_{((-,-),(+))}, \{0\}),
\]

where the blue \(\mathcal{G}\)-colored ribbon set \(P_{((-,-),(+))}\) is given by three framed points, two with negative, one with positive orientation, and all with color \(P\). Let us also consider the morphism \((\mathbb{D}^3 \setminus N^3)_P : (S^1 \times S^1)_{(-, P)} \to S^2_{2((-,-),(+))}\) of \(\text{Cob}_\varphi\) defined by

\[
(\mathbb{D}^3 \setminus N^3)_P := (\mathbb{D}^3 \setminus N^3, T_P, 0),
\]

where \(N^3 \subset D^3\) is an open tubular neighborhood of the curve \(\{0\} \times \frac{1}{2} \cdot S^1 \subset D^3\), and where the \(\mathcal{G}\)-colored framed tangle \(T_P\) is represented in Figure 12.

**Lemma 3.3.** The linear map

\[
V_\varphi ((\mathbb{D}^3 \setminus N^3)_P) : V_\varphi ((S^1 \times S^1)_{(-, P)}) \to V_\varphi \left(S^2_{2((-,-),(+))}\right)
\]

is injective.

**Proof.** As we will show, the proof follows rather directly from the surjectivity of

\[
V_\varphi ((\mathbb{D}^3 \setminus N^3)_P) : V_\varphi \left(S^2_{2((-,-),(+))}\right) \to V_\varphi ((S^1 \times S^1)_{(-, P)}).
\]

Indeed, a vector of the form

\[
\sum_{i=1}^m \alpha_i \cdot \left[(\mathbb{D}^3 \setminus N^3)_P \circ M_i\right] \in V_\varphi \left(S^2_{2((-,-),(+))}\right)
\]
for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and some $[M_1], \ldots, [M_m] \in V_{\bar{g}}((S^1 \times S^1)(-\mathcal{P}))$ is trivial if and only if

$$\sum_{i=1}^{m} \alpha_i \langle M'_i, ([D^3 \setminus N^3]_\mathcal{P} \circ M)_g(S^2_{((-\mathcal{P}),(-\mathcal{P}),(+\mathcal{P}))} \rangle = 0$$

for every $[M'] \in V'_{\bar{g}}(S^2_{((-\mathcal{P}),(-\mathcal{P}),(+\mathcal{P}))}$, where for every object $\Sigma$ of $\text{Cob}_{\bar{g}}$ the linear map

$$\langle \cdot, \cdot \rangle_\Sigma : V'_{\bar{g}}(\Sigma) \otimes V_{\bar{g}}(\Sigma) \rightarrow \mathbb{C}$$

denotes the non-degenerate pairing induced by the universal construction in Section 3.3 of [7]. Then, since $\langle M', ([D^3 \setminus N^3]_\mathcal{P} \circ M)_g(S^2_{((-\mathcal{P}),(-\mathcal{P}),(+\mathcal{P}))} \rangle = 0$ for every $[M] \in V_{\bar{g}}((S^1 \times S^1)(-\mathcal{P}))$ and every $[M'] \in V'_{\bar{g}}(S^2_{((-\mathcal{P}),(-\mathcal{P}),(+\mathcal{P}))}$, the injectivity of

$$V_{\bar{g}}((D^3 \setminus N^3)_\mathcal{P}) : V_{\bar{g}}((S^1 \times S^1)(-\mathcal{P})) \rightarrow V_{\bar{g}}(S^2_{((-\mathcal{P}),(-\mathcal{P}),(+\mathcal{P}))})$$

is equivalent to the surjectivity of

$$V'_{\bar{g}}((D^3 \setminus N^3)_\mathcal{P}) : V'_{\bar{g}}(S^2_{((-\mathcal{P}),(-\mathcal{P}),(+\mathcal{P}))} \rightarrow V'_{\bar{g}}((S^1 \times S^1)(-\mathcal{P})).$$

In order to prove that $V'_{\bar{g}}((D^3 \setminus N^3)_\mathcal{P})$ is surjective we remark that, as soon as an object $\Sigma = (\Sigma, \mathcal{P}, \mathcal{Z})$ of $\text{Cob}_{\bar{g}}$ features a projective blue point of $\mathcal{P}$ in every connected component of $\Sigma$, the proof of Proposition 3.13 of [7] can be repeated to show that the linear map

$$\pi'_\Sigma : \mathcal{T}'(M'; \Sigma) \rightarrow V'_{\bar{g}}(\Sigma) \quad T' \mapsto [M', T', 0]$$

is surjective for every connected 3-dimensional cobordism $M'$ from $\Sigma$ to $\emptyset$. This means that every vector in $V'_{\bar{g}}((D^3 \setminus N^3)_\mathcal{P})$ can be described by a linear combination of $\bar{g}$-colored bichrome graphs inside $(D^3 \setminus N^3) \cup_{\Sigma} \tilde{D}^3$ from $P_\mathcal{P}$ to $\emptyset$. But now every such $\bar{g}$-colored bichrome graph is skein equivalent to a $\bar{g}$-colored ribbon graph like the one represented in Figure 13 for some $f' \in \text{Hom}_{\bar{g}}(\mathcal{P} \otimes \mathcal{P}^*, \mathcal{P})$. Clearly every vector of this form lies in the image of $V'_{\bar{g}}((D^3 \setminus N^3)_\mathcal{P})$. \hfill $\Box$
Let us consider now the morphisms
\((S^1 \times D^2)_{\Omega} : \emptyset \rightarrow (S^1 \times S^1)(-,p)\) and
\((S^1 \times D^2)_{\lambda} : \emptyset \rightarrow (S^1 \times S^1)(-,p)\) of \(\check{\text{Cob}}_E\) defined by
\[
(S^1 \times D^2)_{\Omega} := (S^1 \times D^2, T_{\Omega}, 0),
\]
\[
(S^1 \times D^2)_{\lambda} := (S^1 \times D^2, T_{\lambda}, 0)
\]
where the \(\check{\text{C}}\)-colored ribbon graphs \(T_{\Omega}\) and \(T_{\lambda}\) are represented in the left hand part and in the right hand part of Figure 14 respectively.

**Lemma 3.4.** The morphisms \((S^1 \times D^2)_{\Omega}\) and \((S^1 \times D^2)_{\lambda}\) of \(\check{\text{Cob}}_E\) satisfy
\[
[(S^1 \times D^2)_{\Omega}] = \alpha \cdot [(S^1 \times D^2)_{\lambda}] \in V_{\check{\text{C}}}(S^2((-,p),(-,p),(+,p)))
\]
where \(\alpha\) is the coefficient introduced in Remark 3.1. Furthermore, there exist compatible choices for the coefficients \(D_{\Omega}\) and \(D_{\lambda}\) yielding
\[
D_{\Omega} = \alpha D_{\lambda}, \quad \delta_{\Omega} = \delta_{\lambda}.
\]

**Proof.** We start by proving \([(S^1 \times D^2)_{\Omega}]\) and \([(S^1 \times D^2)_{\lambda}]\) are linearly dependent in \(V_{\check{\text{C}}}(S^2((-,p),(-,p),(+,p)))\). This is done by using Lemma 3.3. Indeed, on one hand, the proof of Lemma 2.6 gives the equality
\[
[(D^3 \setminus N^3)_{p}] \circ (S^1 \times D^2)_{h_{\Omega}} = r^2 N |\mathcal{H}_r| \cdot [(D^3, T_{D^3}, 0)] \in V_{\check{\text{C}}}(S^2((-,p),(-,p),(+,p)))
\]
Figure 15. The $\bar{C}$-colored ribbon graph $T_{D^3} \subset D^3$.

where the $\bar{C}$-colored ribbon graph $T_{D^3}$ is represented in Figure 15. On the other hand, $\bar{U}$ is a projective generator of $\bar{C}$, which means $
abla \id_{\bar{P}} = \sum_{i=1}^{m} g_{\bar{P},i} \circ f_{\bar{P},i}$

for some morphisms $f_{\bar{P},i} \in \text{Hom}_{\bar{C}}(\bar{P}, \bar{U})$ and $g_{\bar{P},i} \in \text{Hom}_{\bar{C}}(\bar{U}, \bar{P})$. Then, thanks to Lemma 3.6 of [7] combined with Lemma 2.9, we know there exists a non-zero coefficient $\beta \in \mathbb{C}^*$ giving the skein equivalence of Figure 16. This gives

$\left( (D^3 \setminus N^3) \circ (S^1 \times D^2)_{h_{\lambda}} \right) = \beta \cdot \left( (D^3, T_{D^3}, 0) \right) \in V_{\bar{C}} \left( S^2 \left( (-, \bar{P}), (-, \bar{P}), (+, \bar{P}) \right) \right)$.

Therefore, we get

$\beta \cdot \left( [S^1 \times D^2]_{h_{\lambda}} \right) = r^{2N} |\mathcal{F}_r| \cdot \left( [S^1 \times D^2]_{h_{\lambda}} \right)$.

Next, this relation allows us to compare the stabilization coefficients. Indeed, if $T_{\pm \Omega}$ and $T_{\pm \lambda}$ denote the $\bar{C}$-colored ribbon graphs represented in Figure 17, then we have

$F'_{\bar{C}}(T_{\pm \Omega}) = \Delta_{\pm \Omega} f_{\bar{P}}(\Lambda \circ \bar{e}) = \alpha \Delta_{\pm \Omega} H_{\bar{P}}(\Lambda \circ \bar{e}) = \alpha \Delta_{\pm \Omega}$,

and analogously

$F'_{\bar{C}}(T_{\pm \lambda}) = \Delta_{\pm \lambda} f_{\bar{P}}(\Lambda \circ \bar{e}) = \alpha \Delta_{\pm \lambda} H_{\bar{P}}(\Lambda \circ \bar{e}) = \alpha \Delta_{\pm \lambda}$.

This means

$H'_{\bar{C}}(S^3, T_{\pm \Omega}, 0) = \alpha \mathcal{F}_r^{-1} \Delta_{\pm \Omega}, \quad H'_{\bar{C}}(S^3, T_{\pm \lambda}, 0) = \alpha \mathcal{F}_r^{-1} \Delta_{\pm \lambda}$.

But now, thanks to the previous equality, we have

$\beta H'_{\bar{C}}(S^3, T_{\pm \Omega}, 0) = r^{2N} |\mathcal{F}_r| H'_{\bar{C}}(S^3, T_{\pm \lambda}, 0)$.

Figure 16. Cutting property for red meridians.
This gives
$$\beta \Delta_{\pm \Omega} = r^{2N} |\mathcal{H}_r| \Delta_{\pm \lambda}.$$  
In particular, combining this equality with the explicit value of $\Delta_{-\Omega} \Delta_{+\Omega}$ given by Figure 4, we can choose
$$\mathcal{D}_\Omega = r^N \sqrt{|\mathcal{H}_r|}, \quad \mathcal{D}_\lambda = \frac{\beta}{r^N \sqrt{|\mathcal{H}_r|}}.$$  
This immediately implies
$$\delta_\Omega = \frac{\mathcal{D}_\Omega}{\Delta_{-\Omega}} = \frac{\mathcal{D}_\lambda}{\Delta_{-\lambda}} = \delta_\lambda.$$  
Furthermore, we can now compute the equality
$$\beta = \frac{r^{2N} |\mathcal{H}_r|}{\alpha}.$$  
Indeed, let us consider the object $S^2_+(\mathcal{P})$ and the closed morphism $S^2_+(\mathcal{P}) \times S^1$ of $\text{Cob}_\mathcal{F}$ introduced in the proof of Lemma 2.6. By applying the functor $\Phi_{\mathcal{F}}$ to their decorations, we obtain an object $S^2_+(\mathcal{P})$ and a closed morphism $S^2_+(\mathcal{P}) \times S^1$ of $\text{Cob}_\mathcal{F}$. Then, this time the strategy will be to compute the renormalized Hennings invariant of $S^2_+(\mathcal{P}) \times S^1$ in two different ways. On one hand, the isomorphism $V_{\mathcal{F}}(S^2_+(\mathcal{P})) \cong \text{Hom}_\mathcal{F}(1, \mathcal{P})$ gives
$$H_{\mathcal{F}}(S^2_+(\mathcal{P}) \times S^1) = \text{dim}_\mathcal{C}(V_{\mathcal{F}}(S^2_+(\mathcal{P}))) = \text{dim}_\mathcal{C}(\text{Hom}_\mathcal{F}(1, \mathcal{P})) = 1,$$  
where the last equality follows from Lemma 2.9. On the other hand, we can choose a surgery presentation of $S^2 \times S^1$ composed of a single unknot of framing 0. This choice determines the $\mathcal{F}$-colored bichrome graph $H_{\mathcal{P}}$ given by a positive Hopf link of framing 0, with one red component colored with $\bar{U}$ and one blue component colored with $\mathcal{P}$. Therefore, we get
$$H'_{\mathcal{F}}(S^2_+(\mathcal{P}) \times S^1) = \mathcal{D}_\lambda^{-2} F_{\mathcal{F}}(H_{\mathcal{P}}) = \frac{r^{2N} |\mathcal{H}_r| \text{tr}_\mathcal{P}(\mathcal{L} \circ \mathcal{F})}{\beta} = \frac{r^{2N} |\mathcal{H}_r| \text{tr}_\mathcal{P}(\mathcal{L} \circ \mathcal{F})}{\alpha \beta} = \frac{r^{2N} |\mathcal{H}_r|}{\alpha \beta}. \quad \square$$  

3.3. **Proof of Theorem 1.4.** We are now ready to prove Theorem 1.4. As explained in the proof of Lemma 3.4, using the explicit value of $\Delta_{-\Omega} \Delta_{+\Omega}$ given by Figure 4, we can choose the square roots $\mathcal{D}_\Omega$ and $\mathcal{D}_\lambda$ to be of the form
$$\mathcal{D}_\Omega = r^N \sqrt{|\mathcal{H}_r|}, \quad \mathcal{D}_\lambda = \frac{r^N \sqrt{|\mathcal{H}_r|}}{\alpha},$$
Also, let us set
\[ \delta := \delta_{\Omega} = \delta_{\lambda}. \]

**Proof of Theorem 1.4.** If \( M \) is a closed 3-manifold, and \( T \subset M \) is an admissible \( \mathcal{C}^H_0 \)-colored ribbon graph, then let \( e \subset T \) be a projective edge of color \( V \), let \( L = L_1 \cup \ldots \cup L_\ell \subset S^3 \) be a surgery presentation of \( M \), let \( \gamma_i \subset S^3 \setminus (L \cup T) \) be disjoint paths connecting \( e \) to \( L_i \) for every \( 1 \leq i \leq \ell \), and let \( (L \cup T)_{h_0} \) and \( (L \cup T)_{h_1} \) be \( \mathcal{C} \)-colored ribbon graphs obtained by \( \Omega \)-stabilization and by \( \lambda \)-stabilization along \( \gamma_1, \ldots, \gamma_\ell \), as explained in Subsection 3.1. Then we have

\[
N_{\mathcal{C}}(M, T, 0, 0) = \alpha \mathcal{Z}_0^{-1-\ell} \delta^{-\sigma(L)} F^i_\mathcal{C}((L \cup T)_{h_0})
= \alpha^{-\ell} \mathcal{Z}_\lambda^{-1-\ell} \delta^{-\sigma(L)} F^i_\mathcal{C}((L \cup T)_{h_0})
= \alpha^{-\ell} \mathcal{Z}_\lambda^{-\ell} \delta^{-\sigma(L)} F^i_\mathcal{C}(S^3, (L \cup T)_{h_0}, 0)
= \mathcal{Z}_\lambda^{-\ell} \delta^{-\sigma(L)} F^i_\mathcal{C}(S^3, (L \cup T)_{h_3}, 0)
= \mathcal{Z}_\lambda^{-\ell} \delta^{-\sigma(L)} F^i_\mathcal{C}((L \cup T)_{h_3})
= F^i_\mathcal{C}(M, T, 0),
\]
where the second and the fourth equalities follow from Lemma 3.4.

**Appendix A. Quantum groups**

In this appendix we collect some standard definitions related to quantum groups, see \([4, 15, 16, 20]\) for more details. Let \( \mathfrak{g} \) be a simple complex Lie algebra of rank \( n \) and dimension \( 2N + n \), let \( B \) be its Killing form, let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \), let \( \Phi \) be the corresponding root system, let \( \Phi_+ \) be a choice of a set of positive roots of \( \mathfrak{g} \), and let \( \{\alpha_1, \ldots, \alpha_n\} \) be an ordering of its set of simple roots. Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be the corresponding Cartan matrix, which is the integral matrix given by

\[
a_{ij} := \frac{2B^*(\alpha_i, \alpha_j)}{B^*(\alpha_i, \alpha_i)},
\]
where \( B^* \) is the symmetric bilinear form on \( \mathfrak{h}^* \) determined by the restriction of \( B \) to \( \mathfrak{h} \) under the isomorphism which identifies a vector \( H \in \mathfrak{h} \) with the linear form \( B(H, \cdot) \in \mathfrak{h}^* \), and let \( \{H_1, \ldots, H_n\} \) be the basis of \( \mathfrak{h} \) determined by \( \alpha_j(H_i) = a_{ij} \) for all integers \( 1 \leq i, j \leq n \). For every \( \alpha \in \Phi_+ \) we set

\[
d_\alpha := \frac{B^*(\alpha, \alpha)}{\min\{B^*(\alpha_i, \alpha_i) \mid 1 \leq i \leq n\}},
\]
and for every integer \( 1 \leq i \leq n \) we use the short notation \( d_i := d_{\alpha_i} \). We denote with \( \langle \cdot, \cdot \rangle \) the symmetric bilinear form on \( \mathfrak{h}^* \) determined by \( \langle \alpha_i, \alpha_j \rangle = d_i d_j \) for all integers \( 1 \leq i, j \leq n \), and we denote with \( \lambda_1, \ldots, \lambda_n \) the corresponding fundamental dominant weights, which are the vectors of \( \mathfrak{h}^* \) determined by the condition \( \langle \lambda_i, \alpha_j \rangle = d_i \delta_{ij} \) for every \( i, j = 1, \ldots, n \). We denote with \( \Lambda_R \) the root lattice, which is the subgroup of \( \mathfrak{h}^* \) generated by simple roots, and we denote with \( \Lambda_W \) the weight lattice, which is the subgroup of \( \mathfrak{h}^* \) generated by fundamental dominant weights. If \( q \) is a formal parameter, then for every \( \alpha \in \Phi_+ \) we set \( q_\alpha := q^{d_\alpha} \), for all \( k \geq \ell \in \mathbb{N} \) we define

\[
\{k\}_\alpha := q_\alpha^k - q_\alpha^{-k}, \quad [k]_\alpha := \frac{\{k\}_\alpha}{\{1\}_\alpha}, \quad [k]!_\alpha := [k]_\alpha [k-1]_\alpha \cdots [1]_\alpha,
\]

\[
\left[ \begin{array}{c} k \\ \ell \end{array} \right]_\alpha := \frac{[k]!_\alpha [k-\ell]!_\alpha}{[k-\ell]!_\alpha}.
\]
and for every integer $1 \leq i \leq n$ we use the short notation
\[ q_i := q_{\alpha_i}, \quad \{ k \}_{i} := \{ k \}_{\alpha_i}, \quad [ k ]_{i} := [ k ]_{\alpha_i}, \quad [ k ]_{i}^{!} := [ k ]_{\alpha_i}!. \]

Let $\mathcal{U}_q\mathfrak{g}$ denote the quantum group of $\mathfrak{g}$, which is the $\mathbb{C}(q)$-algebra with generators
\[ \{ K_i, K_i^{-1}, E_i, F_i \mid 1 \leq i \leq n \} \]
and relations
\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad [ K_i, K_j ] = 0, \\
K_i E_j K_i^{-1} = q_i^{a_{ij}} \cdot E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} \cdot F_j, \\
[E_i, F_j] = \delta_{ij} \cdot K_i - K_i^{-1} \quad \frac{q_i}{q_i - q_i^{-1}}
\]
for all integers $1 \leq i, j \leq n$ and
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right]_i \cdot E_i^k E_j F_i^{1-a_{ij}} E_i^{-k} = 0, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right]_j \cdot F_i^k F_j F_i^{1-a_{ij}} E_i^{-k} = 0
\]
for all integers $1 \leq i, j \leq n$ with $i \neq j$. Then $\mathcal{U}_q\mathfrak{g}$ can be made into a Hopf algebra by setting
\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \\
\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad S(K_i) = K_i^{-1}, \\
S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i
\]
for all integers $1 \leq i \leq n$. For every
\[
\mu = \sum_{i=1}^{n} m_i \cdot \alpha_i \in \Lambda_R
\]
we use the notation
\[
K_\mu := \prod_{i=1}^{n} K_i^{m_i},
\]
and for every $\alpha \in \Phi_+$, we define root vectors $E_\alpha$ and $F_\alpha$ as follows: first, we consider the Weyl group $W$ of $\mathfrak{g}$ associated with $\mathfrak{h}$, which is the subgroup of $\text{GL}(\mathfrak{h}^\ast)$ generated by reflections
\[
s_i : \mathfrak{h}^\ast \to \mathfrak{h}^\ast, \quad \alpha_j \mapsto \alpha_j - a_{ij} \cdot \alpha_i
\]
for every integer $1 \leq i \leq n$. Next, we consider the unique element $w_0 \in W$ corresponding to a word of maximal length in the generators. The choice of a decomposition $w_0 = s_{i_1} \circ \cdots \circ s_{i_k}$ determines a total order on the set of positive roots
\[
\Phi_+ = \{ \alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, (s_{i_2} \circ \cdots \circ s_{i_{k-1}})(\alpha_{i_k}) \}. \]
Then, for every integer \(1 \leq i \leq n\), we consider the automorphism \(T_i\) of \(\mathcal{U}_q\mathfrak{g}\) determined by

\[
T_i(K_j) := K_j K_i^{-a_{ij}},
\]

\[
T_i(E_j) := \begin{cases} 
-F_i K_i^{-a_{ij}} & i = j, \\
\displaystyle \sum_{k=0} (-1)^k \frac{q^k}{[k]_i![-a_{ij} - k]_j!} \cdot E_i^k E_j E_i^{a_{ij}-k} & i \neq j,
\end{cases}
\]

\[
T_i(F_j) := \begin{cases} 
-K_i^{-1} E_i & i = j, \\
\displaystyle \sum_{k=0} (-1)^k \frac{q^k}{[k]_i![-a_{ij} - k]_j!} \cdot F_i^k F_j F_i^{a_{ij}-k} & i \neq j.
\end{cases}
\]

Now, for every integer \(1 \leq k \leq N\), we set

\[
\beta_k := (s_1 \circ \cdots \circ s_{k-1})(\alpha_{ik}) \in \Phi_+
\]

and

\[
E_{\beta_k} := (T_1 \circ \cdots \circ T_{k-1})(E_{ik}) \in \mathcal{U}_q\mathfrak{g}, \quad F_{\beta_k} := (T_1 \circ \cdots \circ T_{k-1})(F_{ik}) \in \mathcal{U}_q\mathfrak{g}.
\]

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