Enhancement of Gilbert damping in spin pumping into a two-dimensional electron gas

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We theoretically consider spin pumping from a ferromagnetic insulator (FI) into a two-dimensional electron gas (2DEG) in which the Rashba and Dresselhaus spin-orbit interactions coexist. We show that the Gilbert damping of the FI is strongly enhanced when the ratio of the two spin-orbit interactions is near a special value at which the total spin of the 2DEG is conserved. This strong enhancement, which appears only when we take the vertex correction into account, reflects the divergence of the spin relaxation time in the 2DEG. We also show that the shift in the resonant frequency is enhanced in a similar way.

I. INTRODUCTION

In the field of spintronics112, spin pumping has long been used as a methods of injections spins injection into various materials.8,9 Spin pumping was first used to inject spins from a ferromagnetic metal into an adjacent normal metal (NM).8,9 Subsequently, it was used on ferromagnetic insulator (FI)/NM junctions.10 Because spin injection is generally related to the loss of the magnetization in ferromagnets, it affects the Gilbert damping measured in ferromagnetic resonance (FMR) experiments.11 When we employ spin injection from the FI, the modulation of the Gilbert damping reflects the properties of the spin excitation in the adjacent materials, such as magnetic thin films,12 magnetic impurities on metal surfaces,13 and superconductors.1417 An attractive strategy is to combine spin pumping with spin-related transport phenomena in semiconductor microstructures.1618 A two-dimensional electron gas (2DEG) in a semiconductor heterostructure is an easily controlled physical system that has been used in spintronics devices.1822 A 2DEG system has two types of spin-orbit interaction, i.e., Rashba23,24 and Dresselhaus spin-orbit interactions.2526 In our previous work,27 we theoretically studied spin pumping into a 2DEG in semiconductor heterostructures with both Rashba and Dresselhaus spin-orbit interactions.23,24 In that study, we formulated the modulation of the Gilbert damping in the FI by using the second-order perturbation with respect to the interfacial coupling15,28–32 and related it to the dynamic spin susceptibility of the 2DEG. We further calculated the spin susceptibility. By examining the dependence of the in-plane azimuth angle $\theta$ of the ordered spin in the FI (see Fig. 1(b)), we obtained characteristic features due to elastic spin flipping and magnon absorption. However, we did not take the vertex correction into account, even though it frequently plays an important role in theoretical descriptions of spin transport.3334

In this study, we consider the same setting, i.e., a junction composed of an FI and a 2DEG as shown in Fig. 1(a), and discuss the effect of the vertex correction. We theoretically calculate the modulation of the Gilbert damping and the shift in the FMR frequency by solving the vertex function within the ladder approximation. We show that the vertex correction substantially changes the results, in particular, when the strengths of the Rashba- and Dresselhaus-type spin orbit interactions are chosen...
to be almost equal but slightly different; we show that both of the Gilbert damping and the FMR frequency shift are strongly enhanced at low resonant frequencies. This remarkable feature should be able to be observed experimentally.

The rest of this work is organized as follows. In Sec. II we briefly summarize our model of the FI/2DEG junction and describe a general formulation for the magnon self-energy following Ref. [27]. In Sec. III we formulate the vertex correction that corresponds to the self-energy in the Born approximation. We show the modulation of the Gilbert damping and the shift in the FMR frequency in Sec. IV and Sec. V, respectively, and discuss the effect of the vertex correction in a detail. Finally, we summarize our results in Sec. VI. The four appendices detail the calculation in Sec. III.

II. FORMULATION

Here, we describe a model for the FI/2DEG junction shown in Fig. 1 (a) and formulate the spin relaxation rate in an FMR experiment. Because we have already given a detailed formulation on this model in our previous paper [27], we will briefly summarize it here.

A. Two-dimensional electron gas

We consider a 2DEG whose Hamiltonian is

$$H_{\text{kin}} = \sum_k (c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger) \hat{h}_k(c_{k\uparrow}^\dagger c_{k\downarrow}),$$  \hspace{1cm} (1)

$$\hat{h}_k = \xi_k \hat{I} - h_{\text{eff}} \cdot \sigma,$$  \hspace{1cm} (2)

where $c_{k\sigma}$ is the annihilation operator of conduction electrons with wavenumber $k = (k_x, k_y)$ and $z$ component of the spin, $\sigma$ ($=\uparrow, \downarrow$), $\hat{I}$ is a $2 \times 2$ identity matrix, $\sigma_a$ ($a = x, y, z$) are the Pauli matrices, and $\xi_k = h^2 k^2 / 2m - \mu$ is the kinetic energy measured from the chemical potential.

The spin-orbit interaction is described by the effective Zeeman field,

$$h_{\text{eff}} = k_F (-\alpha \sin \varphi - \beta \cos \varphi, \alpha \cos \varphi + \beta \sin \varphi, 0),$$  \hspace{1cm} (3)

where $\alpha$ and $\beta$ respectively denote the amplitudes of the Rashba- and Dresselhaus-type spin-orbit interactions and $\varphi$ is the azimuth angle of the wavenumber of the conduction electrons. We assume that $k_F \alpha$ and $k_F \beta$ are much smaller than the chemical potential.

We also consider impurities, with the Hamiltonian,

$$H_{\text{imp}} = u \sum_i \sum_{\sigma \in \text{imp}} \Psi_{\sigma}^\dagger(r_i) \Psi_{\sigma}(r_i),$$  \hspace{1cm} (4)

where $\Psi_{\sigma}(r) = A^{-1/2} \sum_k c_{k\sigma} e^{i k \cdot r}$, $A$ is the area of the junction, $u$ is the strength of the impurity potential, and $r_i$ is the position of the impurity site.

The finite-temperature Green’s function for the conduction electrons is defined by a $2 \times 2$ matrix $\hat{g}(k, i\omega_n)$ whose elements are

$$g_{\sigma\sigma'}(k, i\omega_n) = \int_0^{\beta} d\tau e^{i\omega_n \tau} g_{\sigma\sigma'}(k, \tau),$$  \hspace{1cm} (5)

$$g_{\sigma\sigma'}(k, \tau) = -\hbar^{-1} (c_{k\sigma}(\tau)c_{k\sigma'}^\dagger),$$  \hspace{1cm} (6)

where $c_{k\sigma}(\tau) = e^{iH_{\text{kin}}/\hbar} c_{k\sigma} e^{-iH_{\text{imp}}/\hbar}$, $H_{\text{NM}} = H_{\text{kin}} + H_{\text{imp}}$, $\omega_n = \pi(2n+1)/\beta$ is the fermionic Matsubara frequency, and $\beta$ is the inverse temperature. By employing the Born approximation, the finite-temperature Green’s function can be expressed as

$$\hat{g}(k, i\omega_n) = (i\hbar\omega_n - \xi_k + i\Gamma \text{sgn}(\omega_n)/2) \hat{I} - \hbar_{\text{eff}} \cdot \sigma,$$  \hspace{1cm} (7)

where $\Gamma = \xi_k \pm |h_{\text{eff}}(\varphi)|$ is the spin-dependent electron dispersion, $\Gamma = 2\pi n_i \Gamma / \hbar$ (where $n_i$ is the number of impurity sites, and $\Gamma$ is the density of states per spin per unit volume).

Note that the case of $\alpha/\beta = 1$ is a special one because the effective Zeeman field $h_{\text{eff}}$ is always parallel to the direction of the azimuth angle $3\pi/4$ in the $xy$ plane (see also Fig. 3 (d)). Therefore, the spin component in this direction is conserved at $\alpha/\beta = 1$. By defining the spin operator as

$$s_{3\pi/4} = \frac{1}{2} \sum_k (c_{k+}^\dagger c_{k-} - c_{k-}^\dagger c_{k+}),$$  \hspace{1cm} (8)

$$\left( \begin{array}{c} c_{k+}^\dagger \\ c_{k-}^\dagger \end{array} \right) = \left( \begin{array}{cc} 1/\sqrt{2} & e^{-i\pi/4}/\sqrt{2} \\ e^{i\pi/4}/\sqrt{2} & -1/\sqrt{2} \end{array} \right) \left( \begin{array}{c} c_{k+} \\ c_{k-} \end{array} \right),$$  \hspace{1cm} (9)

we can prove $[H_{\text{kin}} + H_{\text{imp}}, s_{3\pi/4}] = 0$. When the value of $\alpha/\beta$ is slightly shifted from 1, the spin conservation law is broken slightly and this leads to a slow spin relaxation. As will be discussed in Sec. IV and Sec. V, this slow spin relaxation, which is a remnant of the spin conservation at $\alpha/\beta = 1$, strongly affects the spin injection from the FI into the 2DEG. To describe this feature, we need to consider the vertex correction to take the conservation law into account in our calculation as explained in Sec. III.

B. Ferromagnetic insulator

We consider the quantum Heisenberg model for the FI and employ the spin-wave approximation assuming that the temperature is much lower than the magnetic transition temperature and the magnitude of the localized spins, $S_0$, is sufficiently large. The resultant Hamiltonian is

$$H_{\text{FI}} = \sum_k \hbar \omega_k b_k^\dagger b_k,$$  \hspace{1cm} (10)

where $b_k$ is the magnon annihilation operator with wavenumber $k$, $\hbar \omega_k = D k^2 + 2\gamma h_{\text{dc}}$ is the energy dispersion of a magnon, $D$ is the spin stiffness, $\gamma$ is the gyromagnetic ratio, and $h_{\text{dc}}$ is the externally applied DC magnetic field. We note that the external DC magnetic field
controls the direction of the ordered spins. By introducing new coordinates \((x', y', z')\) fixed on the ordered spins by rotating the original coordinates \((x, y, z)\) as shown in Fig. \ref{fig:figure1}(b), we find that the magnon annihilation operator is related to the spin ladder as \(S_{k}^{z+} = (2S_0)^{1/2}b_k\). The spin correlation function is defined as

\[
G(k, i\omega_n) = \frac{2S_0}{i\omega_n - \omega_k + i\alpha_G|\omega_n|},
\]
where \(\alpha_G\) is a phenomenological dimensionless parameter that describes the strength of the Gilbert damping in the bulk FI.

\[
\chi(0, \omega_n) = \langle \hat{g}(k, i\omega_n + i\delta) \hat{g}^\dagger(k, i\omega_m + i\omega_n) \rangle.
\]

\[\chi(0, i\omega_n) = \frac{1}{G_0(k, i\omega_n)^{-1} - \Sigma(k, i\omega_n)},\]

where \(\Sigma(k, i\omega_n)\) is the self-energy due to the interfacial exchange coupling and \(\chi(k, i\omega_n)\) is the spin susceptibility for conduction electrons per unit area, defined as

\[
\chi(k, i\omega_n) = \int_0^{\beta} d\tau e^{i\omega_n \tau} \chi(k, \tau),
\]

\[
\chi(k, \tau) = -\frac{1}{i\hbar A} \langle S_{k}^{z+}(\tau)S_{k}^{z-}(0) \rangle.
\]

\[
\delta\tau = \sin \theta \sigma_x + \cos \theta \sigma_y \pm i\sigma_z.
\]

We perform a second-order perturbation theory with respect to the interfacial exchange interaction \(H_{\text{int}}\). Accordingly, the spin correlation function of the FI is calculated as

\[
G(k, i\omega_n) = \frac{1}{G_0(k, i\omega_n)^{-1} - \Sigma(k, i\omega_n)},
\]

where \(\Sigma(k, i\omega_n)\) is the self-energy due to the interfacial exchange coupling and \(\chi(k, i\omega_n)\) is the spin susceptibility for conduction electrons per unit area, defined as

\[
\chi(k, i\omega_n) = \int_0^{\beta} d\tau e^{i\omega_n \tau} \chi(k, \tau),
\]

\[
\chi(k, \tau) = -\frac{1}{i\hbar A} \langle S_{k}^{z+}(\tau)S_{k}^{z-}(0) \rangle.
\]

\[
\delta\tau = \sin \theta \sigma_x + \cos \theta \sigma_y \pm i\sigma_z.
\]

\[\delta\tau = \sin \theta \sigma_x + \cos \theta \sigma_y \pm i\sigma_z.
\]

\[
\delta\omega_0 \simeq \frac{2S_0|T_0|^2A}{\hbar \omega_0} \text{Im} \chi^R(0, \omega_0),
\]

\[
\delta\omega_0 \simeq \frac{2S_0|T_0|^2A}{\hbar \omega_0} \text{Re} \chi^R(0, \omega_0),
\]

where \(s_k^{\pm}(\tau) = e^{H_{NM}\tau/\hbar} s_k^{\pm} e^{-H_{NM}\tau/\hbar}\). The uniform component of the retarded spin correlation function is obtained by analytic continuation \(i\omega_n \rightarrow \omega + i\delta\), as

\[
G^R(0, \omega) = \frac{2S_0}{\omega - (\omega_0 + \delta\omega_0)} + i(\alpha_G + \delta\alpha_G)\omega,
\]

\[
\delta\alpha_G \simeq \frac{2S_0|T_0|^2A}{\hbar \omega_0} \text{Re} \chi^R(0, \omega_0),
\]

\[
\delta\omega_0 \simeq \frac{2S_0|T_0|^2A}{\hbar \omega_0} \text{Im} \chi^R(0, \omega_0),
\]

where the superscript \(R\) indicates the retarded component, \(\omega_0 = \omega_{q=0} (= \hbar \gamma_{\text{dc}})\) is the FMR frequency, and \(\delta\omega_0\) and \(\delta\alpha_G\) are respectively the changes in the FMR frequency and Gilbert damping due to the FI/2DEG interface. In Eqs. \ref{eq:vertex1} and \ref{eq:vertex2}, we made an approximation by replacing \(\omega\) with the FMR frequency \(\omega_0\) by assuming that the FMR peak is sufficiently sharp \((\alpha + \delta\alpha \ll 1)\). Thus, both the FMR frequency shift and the modulation of the Gilbert damping are determined by the uniform spin susceptibility of the conduction electrons, \(\chi(0, \omega)\). In what follows, we include the vertex correction for calculation of \(\chi(0, \omega)\), which was not taken into account in our previous work.

\[
\chi(0, i\omega_n) = \frac{1}{4\beta A} \sum_{k, i\omega_m} \text{Tr} \left[ \hat{g}(k, i\omega_m)\hat{G}(k, i\omega_m, i\omega_n) \right].
\]

III. VERTEX CORRECTION

We calculate the spin susceptibility in the ladder approximation that obeys the Ward-Takahashi relation with the self-energy in the Born approximation. The Feynman diagrams for the corresponding spin susceptibility and the Bethe-Salpeter equation for the vertex function are shown in Fig. \ref{fig:figure2} (a) and (b), respectively. The spin susceptibility is written as

\[
\chi(0, i\omega_n) = \frac{1}{4\beta A} \sum_{k, i\omega_m} \text{Tr} \left[ \hat{g}(k, i\omega_m)\hat{G}(k, i\omega_m, i\omega_n) \right] \hat{g}(k, i\omega_m + i\omega_n)\delta^{\sigma'\sigma}.
\]
where the vertex function \( \hat{\Gamma}(\mathbf{k}, i\omega_m, i\omega_n) \) is a \( 2 \times 2 \) matrix whose components are determined by the Bethe-Salpeter equation (see Fig. 2(b)),

\[
\Gamma_{\sigma'\sigma}(\mathbf{k}, i\omega_m, i\omega_n) = (\hat{\sigma}^x + i\hat{\sigma}^y)_{\sigma'\sigma} + \frac{u^2N_i}{A} \sum \sum g_{\sigma'\sigma_2}(\mathbf{q}, i\omega_m) \\
\times \Gamma_{\sigma_2\sigma_1}(\mathbf{q}, i\omega_m, i\omega_n) g_{\sigma_1\sigma}(\mathbf{q}, i\omega_m + i\omega_n).
\]

(25)

Since the right-hand side of this equation is independent of \( \mathbf{k} \), the vertex function can simply be described as \( \hat{\Gamma}(i\omega_m, i\omega_n) \). We express the vertex function with the Pauli matrices as

\[
\hat{\Gamma}(i\omega_m, i\omega_n) \equiv E\hat{I} + X\hat{\sigma}_x + Y\hat{\sigma}_y + Z\hat{\sigma}_z,
\]

(26)

where \( E, X, Y, \) and \( Z \) will be determined self-consistently later. The Green’s function for the conduction electrons can be rewritten as

\[
\hat{g}(\mathbf{q}, i\omega_m) = \frac{A\hat{I} + B\hat{\sigma}_x + C\hat{\sigma}_y}{D} = \frac{A(i\omega_m) + i\Gamma}{2}\text{sgn}(\omega_m) + \frac{C(i\omega_m)}{2}\text{sgn}(\omega_m),
\]

(27)

\[
A(i\omega_m) = \hbar \omega_m - \xi_0 + \frac{i\Gamma}{2}\text{sgn}(\omega_m),
\]

(28)

\[
B = -\hbar \text{eff} \cos(\phi - \theta),
\]

(29)

\[
C = -\hbar \text{eff} \sin(\phi - \theta),
\]

(30)

\[
D(i\omega_m) = \prod_{\nu = \pm} (i\omega_m - E_\nu + \frac{i\Gamma}{2}\text{sgn}(\omega_m)),
\]

(31)

where \( \phi \) is the azimuth angle by which the effective Zeeman field is written as \( \hbar_{\text{eff}} = (\hbar_{\text{eff}} \cos \phi, \hbar_{\text{eff}} \sin \phi, 0) \). This \( \hbar_{\text{eff}} \) is written as \( \hbar_{\text{eff}} \simeq k_F \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \sin 2\varphi} \) using the Fermi wavenumber \( k_F \). By substituting Eqs. (26) and (27) into the second term of Eq. (25) and by the algebra of Pauli matrices, we obtain

\[
\frac{u^2N_i}{A} \sum \sum g_{\sigma'\sigma_2}(\mathbf{q}, i\omega_m) \\
\times \Gamma_{\sigma_2\sigma_1}(\mathbf{q}, i\omega_m, i\omega_n) g_{\sigma_1\sigma}(\mathbf{q}, i\omega_m + i\omega_n) \\
= E'\hat{I} + X'\hat{\sigma}_x + Y'\hat{\sigma}_y + Z'\hat{\sigma}_z.
\]

(32)

\[
\begin{pmatrix}
E' \\
X' \\
Y' \\
Z'
\end{pmatrix} = \begin{pmatrix}
\Lambda_0 + \Lambda_1 & 0 & 0 & 0 \\
0 & \Lambda_0 + \Lambda_2 & \Lambda_3 & 0 \\
0 & \Lambda_3 & \Lambda_0 - \Lambda_2 & 0 \\
0 & 0 & 0 & \Lambda_0 - \Lambda_1
\end{pmatrix} \begin{pmatrix}
E \\
X \\
Y \\
Z
\end{pmatrix},
\]

(33)

and \( \Lambda_j(i\omega_m, i\omega_n) \) \((j = 0, 1, 2, 3)\) are expressed as

\[
\Lambda_0(i\omega_m, i\omega_n) = \frac{u^2N_i}{A} \sum \frac{AA'}{DD'},
\]

(34)

\[
\Lambda_1(i\omega_m, i\omega_n) = \frac{u^2N_i}{A} \sum \frac{h_{\text{eff}}^2 \cos 2(\phi - \theta)}{DD'},
\]

(35)

\[
\Lambda_2(i\omega_m, i\omega_n) = \frac{u^2N_i}{A} \sum \frac{h_{\text{eff}}^2 \sin 2(\phi - \theta)}{DD'},
\]

(36)

\[
\Lambda_3(i\omega_m, i\omega_n) = \frac{u^2N_i}{A} \sum \frac{h_{\text{eff}}^2 \sin 2(\phi - \theta)}{DD'}. \]

(37)

using the abbreviated symbols, \( A = A(i\omega_m), \ A' = A(i\omega_m + i\omega_n), \ D = D(i\omega_m), \) and \( D' = D(i\omega_m + i\omega_n) \). Here, we have used the fact that the contributions of the first-order terms of \( B \) and \( C \) become zero after replacing the sum with the integral with respect to \( \mathbf{k} \) and performing the azimuth integration. We can solve for \( E, X, Y, \) and \( Z \) by combining Eq. (33) and the Bethe-Salpeter equation (25), which we rewrite as

\[
E\hat{I} + X\hat{\sigma}_x + Y\hat{\sigma}_y + Z\hat{\sigma}_z = \hat{\sigma}^x + i\hat{\sigma}^y + i\hat{\sigma}^z.
\]

(38)

with \( \hat{\sigma}^x = \hat{\sigma}^y + i\hat{\sigma}^z \). The solution is

\[
E = 0,
\]

(39)

\[
X = \frac{\Lambda_3}{(1 - \Lambda_0)^2 - \Lambda_2^2 - \Lambda_3^2},
\]

(40)

\[
Y = \frac{1 - \Lambda_0 - \Lambda_2}{(1 - \Lambda_0)^2 - \Lambda_2^2 - \Lambda_3^2},
\]

(41)

\[
Z = \frac{i}{1 - \Lambda_0 + \Lambda_1}.
\]

(42)

By replacing the sum with an integral,

\[
\frac{1}{A} \sum_{\nu} \cdots \simeq D(\epsilon_F) \int_{-\infty}^{\infty} d\xi \int_0^{2\pi} \frac{d\varphi}{2\pi} \cdots,
\]

(43)

Eqs. (34)-(37) can be rewritten as

\[
\Lambda_j(i\omega_m, i\omega_n) = \theta(-\omega_m)\theta(\omega_m + \omega_n)\tilde{\Lambda}_j(i\omega_n),
\]

(44)

\[
\tilde{\Lambda}_j(i\omega_n) = \frac{i\Gamma}{4} \int_0^{2\pi} \frac{d\varphi}{2\pi} \sum_{\nu'0} \frac{f_j(\nu, \nu', \varphi)}{i\omega_n + (\nu - \nu')\hbar_{\text{eff}}(\varphi) + i\Gamma},
\]

(45)

where \( \theta(x) \) is a step function and

\[
f_0(\nu, \nu', \varphi) = 1,
\]

(46)

\[
f_1(\nu, \nu', \varphi) = \nu\nu',
\]

(47)

\[
f_2(\nu, \nu', \varphi) = \nu\nu' \cos 2(\phi(\varphi) - \theta),
\]

(48)

\[
f_3(\nu, \nu', \varphi) = \nu\nu' \sin 2(\phi(\varphi) - \theta).
\]

(49)
Substituting the Green’s function and the vertex function into Eq. (24), we obtain

\[ \chi(0, i\omega_n) = \frac{1}{4\beta A} \sum_{k, i\omega_m} \frac{2}{DD'} [2BCX + (AA' - B^2 + C^2)Y - i(\omega - B^2 - C^2)Z] \] (50)

By summing over \( k \) and \( \omega_n \) and by analytical continuation, \( i\omega_n \rightarrow \omega + i\delta \), the retarded spin susceptibility is obtained as

\[ \chi^R(0, \omega) = \frac{D(\epsilon_F)\hbar\omega}{2\pi} \left[ \frac{\tilde{\Lambda}_0^R(1 - \tilde{\Lambda}_0^R) - \tilde{\Lambda}_2^R(1 - \tilde{\Lambda}_2^R) + (\tilde{\Lambda}_3^R)^2}{1 - \tilde{\Lambda}_0^R(1 - \tilde{\Lambda}_2^R)} + \frac{\tilde{\Lambda}_1^R}{\tilde{\Lambda}_0^R + \tilde{\Lambda}_1^R} \right] - D(\epsilon_F), \] (51)

where

\[ \tilde{\Lambda}_j^R = \Lambda_j^R(\omega) = \frac{\hbar\omega}{\Delta_0} \int_0^{2\pi} d\varphi \sum_{\nu, \nu'} f_j(\nu, \nu', \varphi) \times \frac{\hbar\omega/\Delta_0 + (\nu - \nu')\hbar\epsilon_{\text{eff}}/\Delta_0 + i\Gamma/\Delta_0}{(\nu - \nu')^2 + (\hbar\epsilon_{\text{eff}}/\Delta_0)^2}. \] (52)

A detailed derivation is given in Appendix A. Here, we have introduced a unit of energy, \( \Delta_0 = k_F\beta \), for the convenience of making the physical quantities dimensionless. Using Eqs. (22) and (23), we finally obtain the shift in the FMR frequency and the modulation of the Gilbert damping as

\[ \frac{\delta\omega_0}{\omega_0} = \alpha_{G,0} \text{Re} F(\omega_0), \] (53)

\[ \delta\alpha_G = -\alpha_{G,0} \text{Im} F(\omega_0), \] (54)

\[ F(\omega) = \frac{\Delta_0}{2\pi\Gamma} \left[ \frac{\tilde{\Lambda}_0^R(1 - \tilde{\Lambda}_0^R) - \tilde{\Lambda}_2^R(1 - \tilde{\Lambda}_2^R) + (\tilde{\Lambda}_3^R)^2}{1 - \tilde{\Lambda}_0^R(1 - \tilde{\Lambda}_2^R)} + \frac{\tilde{\Lambda}_1^R}{\tilde{\Lambda}_0^R + \tilde{\Lambda}_1^R} \right] - \frac{\Delta_0}{\pi\hbar\omega}, \] (55)

where \( \alpha_{G,0} = 2\pi S_0|\tilde{T}_0|^2 AD(\epsilon_F)/\Delta_0 \) is a dimensionless parameter that describes the coupling strength at the interface. This is our main result.

The spin susceptibility without the vertex correction can be obtained by taking the first-order term with respect to \( \Lambda_j \):

\[ \chi^R(0, \omega) \sim \frac{\hbar\omega D(\epsilon_F)}{2\pi} \left[ \frac{2\Lambda_0^R - \Lambda_1^R - \Lambda_2^R}{\hbar\omega} \right] - D(\epsilon_F) \]

\[ \sim \frac{\hbar\omega D(\epsilon_F)}{2\pi} \int_0^{2\pi} \frac{d\varphi}{\hbar\omega + i\Gamma} \left[ \frac{1}{2} \left[ 1 + \cos^2(\phi(\varphi) - \theta) \right] + \frac{1}{\hbar\omega + 2\hbar\epsilon_{\text{eff}}(\varphi) + i\Gamma} \right] - D(\epsilon_F). \] (56)

The imaginary part of \( \chi(0, i\omega_n) \) reproduces the result of Ref. 27. Using this expression, the shift in the FMR frequency and the modulation of the Gilbert damping without the vertex correction are obtained as

\[ \frac{\delta\omega_{nv}}{\omega_0} = \alpha_{G,0} \text{Re} F_{nv}(\omega_0), \] (57)

\[ \delta\alpha_G = -\alpha_{G,0} \text{Im} F_{nv}(\omega_0), \] (58)

\[ F_{nv}(\omega) = \frac{\Delta_0}{2\pi\Gamma} \left[ 2\Lambda_1^R - \Lambda_1^R - \Lambda_2^R \right] - \frac{\Delta_0}{\pi\hbar\omega}. \] (59)

IV. MODULATION OF THE GILBERT DAMPING

First, we show the result for the modulation of the Gilbert damping, \( \delta\alpha_G \), for \( \alpha/\beta = 0, 1, \) and 3 and discuss the effect of the vertex correction by comparing it with the result without the vertex correction in Sec. IV A. Next, we discuss the strong enhancement of the Gilbert damping near \( \alpha/\beta = 1 \). Sec. IV B.

A. Effect of vertex corrections

First, let us discuss the case of \( \alpha/\beta = 0 \), i.e., the case when only the Dresselhaus spin-orbit interaction exists. Figure 3 (a) shows the effective Zeeman field \( \hbar\epsilon_{\text{eff}} \) along the Fermi surface. Figure 3 (b) and (c) show the modulations of the Gilbert damping without and with the vertex correction. The horizontal axes of Fig. 3 (b) and (c) denote the resonant frequency \( \omega_0 = h\gamma_{\text{dc}} \) in the FMR experiment. Note that the modulation of the Gilbert damping, \( \delta\alpha_G \), is independent of \( \theta \), i.e., the azimuth angle of \( \langle S \rangle \). The four curves in Fig. 3 (b) and (c) correspond to \( \Gamma/\Delta_0 = 0.1, 0.2, 0.5, \) and 1.0. We find that these two graphs have a common qualitative feature: the modulation of the Gilbert damping has two peaks at \( \omega_0 = 0 \) and \( \omega_0 = 2\Delta_0 \) and their widths become larger as \( \Gamma \) increases. The peak at \( \omega_0 = 0 \) corresponds to elastic spin-flip of conduction electrons induced by the transverse magnetic field via the exchange bias of the FI, while the peak at \( \omega_0 = 2\Delta_0 \) is induced by spin excitation of conduction electrons due to magnon absorption. In the case of \( \alpha/\beta = 0 \), the vertex correction changes the modulation of the Gilbert damping moderately (compare Fig. 3 (c) with (b)). The widths of the two peaks at \( \omega_0 = 0 \) and \( \omega = 2\Delta_0 \) become narrower when the vertex correction is taken into account (see Appendix B for the analytic expressions).

The case of \( \alpha/\beta = 1 \) is special because the effective Zeeman field \( \hbar\epsilon_{\text{eff}} \) always points in the direction of \( (-1, 1) \) or \((1, -1)\), as shown in Fig. 3 (d). The amplitude of \( \hbar\epsilon_{\text{eff}} \) depends on the angle of the wavenumber of the conduction electrons, \( \varphi \),

\[ \hbar\epsilon_{\text{eff}}(\varphi) = 2\Delta_0|\sin(\varphi + \pi/4)|, \] (60)
FIG. 3. (Left panels) Effective Zeeman field $h_{\text{eff}}$ on the Fermi surface. (Middle panels) Modulation of the Gilbert damping, $\delta \alpha_{G}^0$, without vertex correction. (Right panels) Modulation of the Gilbert damping with vertex correction, $\delta \alpha_{G}$. In the middle and right panels, the modulation of the Gilbert damping is plotted as a function of the FMR frequency, $\omega_0 = \gamma h_{dc}$. The spin-orbit interactions are as follows. (a), (b), (c): $\alpha/\beta = 0$. (d), (e), (f): $\alpha/\beta = 1$. (g), (h), (i): $\alpha/\beta = 3$.

and varies in the range of $0 \leq 2h_{\text{eff}} \leq 4\Delta_0$. Figures 3(e) and (f) show the modulation of the Gilbert damping without and with the vertex correction for $\Gamma/\Delta = 0.5$. The five curves correspond to five different angles of $\langle S \rangle$, $\theta = -\pi/4, -\pi/8, 0, \pi/8$, and $\pi/4$. The most remarkable feature revealed by comparing Fig. 3(f) with (e) is that the peak at $\omega_0 = 0$ disappears if the vertex correction is taken into account (see Appendix C for the analytic expressions). In the subsequent section, we will show that $\delta \alpha_{G}(\omega)$ has a delta-function-like singularity at $\omega = 0$ for $\alpha/\beta = 1$ due to the spin conservation law along the direction of $h_{\text{eff}}$.

In the case of $\alpha/\beta = 3$, the direction of the effective Zeeman field $h_{\text{eff}}$ varies along the Fermi surface (Fig. 3(g)). Figures 3(h) and (i) show the modulation of the Gilbert damping without and with the vertex correction for $\Gamma/\Delta = 0.5$. For $\alpha/\beta = 3$, a peak at $\omega_0 = 0$ appears even when the vertex correction is taken into account. The broad structure in the range of $4\Delta_0 \leq \hbar \omega_0 \leq 8\Delta_0$ is caused by the magnon absorption process where its range reflects the distribution of the spin-splitting energy $2h_{\text{eff}}$ along the Fermi surface. By comparing Fig. 3(h) and (i), we find that the vertex correction changes the result only moderately as in the case of $\alpha/\beta = 0$; the peak structure at $\omega_0 = 0$ becomes sharper when the vertex correction is taken into account while the broad structure is slightly enhanced.
FIG. 4. Modulation of the Gilbert damping calculated for \(\alpha/\beta = 1.1\) (a) without the vertex correction and (b) with the vertex correction. The horizontal axis is the FMR frequency \(\omega_0\) and the five curves correspond to five different angles of \(\langle S \rangle\), i.e., \(\theta = -\pi/4, -\pi/8, 0, \pi/8,\) and \(\pi/4\). (c) Enlarged plot of the modulations of the Gilbert damping as a function of the FMR frequency \(\omega_0\). The angle of \(\langle S \rangle\) is fixed as \(\theta = \pi/4\) and the three curves correspond to \(\alpha/\beta = 1.03, 1.05,\) and 1.1. In all the plots, we have chosen \(\Gamma/\Delta_0 = 0.5\).

B. Strong enhancement of the Gilbert damping

Here, we examine the strong enhancement of the Gilbert damping for \(\alpha/\beta \approx 1\). As explained in Sec. II A, the spin component in the direction of the azimuth angle \(3\pi/4\) in the \(xy\) plane is exactly conserved at \(\alpha/\beta = 1\) (see also Fig. 3 (d)). When the value of \(\alpha/\beta\) is shifted slightly from 1, the spin conservation law is broken but the spin relaxation becomes remarkably slow. To see this effect, we show the modulation of the Gilbert damping without and with the vertex correction for \(\alpha/\beta = 1.1\) in Fig. 4 (a) and (b), respectively. The five curves correspond to five different azimuth angles of \(\langle S \rangle\), and the energy broadening is set as \(\Gamma/\Delta_0 = 0.5\). Fig. 4 (a) and (b) indicate that the Gilbert damping is strongly enhanced at \(\omega_0 = 0\) only when the vertex correction is taken into account. This is the main result of our work.

Figure 4 (c) plots the modulation of the Gilbert damping with the vertex correction for \(\Gamma/\Delta_0 = 0.5\) and \(\theta = \pi/4\), the latter of which corresponds to the case of the strongest enhancement at \(\omega_0 = 0\). The three curves correspond to \(\alpha/\beta = 1.03, 1.05,\) and 1.1. As the ratio of \(\alpha/\beta\) approaches 1, the peak height at \(\omega_0 = 0\) gets larger. For \(\alpha/\beta \approx 1\), \(\delta\alpha_G\) is calculated approximately as

\[
\frac{\delta\alpha_G}{\alpha_G,0} \approx \frac{\Delta_0}{2\pi} \frac{\Gamma_s}{(\hbar\omega_0)^2 + \Gamma_s^2} \sin^2\left(\theta + \frac{\pi}{4}\right),
\]

(61)

\[
\Gamma_s = \frac{2}{\Gamma} \int_0^{\pi/2} d\phi \frac{\langle h_x + h_y \rangle^2}{2\pi (1 + 2h_{eff}/\Gamma)^2},
\]

(62)

where \(\Gamma_s\) gives the peak width in Fig. 4 (b) and (c) (see Appendix D for a detailed derivation). For \(\alpha/\beta = 1 + \delta\) \((\delta \ll 1)\), \(\Gamma_s\) is proportional to \(\delta^2\) and approaches zero in the limit of \(\delta \to 0\). This indicates that \(\Gamma_s\) corresponds to the spin relaxation rate due to a small breakdown of the spin conservation law away from the special point of \(\alpha/\beta = 1\). Note that the peak height of \(\delta\alpha_G\) at \(\omega_0 = 0\) diverges at \(\alpha/\beta = 1\). This indicates that for \(\alpha/\beta = 1\), \(\delta\alpha_G(\omega_0)\) has a delta-function like singularity at \(\omega_0 = 0\), which is not drawn in Fig. 3 (f).

FIG. 5. Modulation of the Gilbert damping as a function of \(\alpha/\beta\). The four curves correspond to \(h\omega_0/\Delta_0 = 0.005, 0.01, 0.02,\) and 0.05. We have taken the vertex correction into account and have chosen \(\Gamma/\Delta_0 = 0.5\).

which is not drawn in Fig. 3 (f).

Figure 5 plots the modulation of the Gilbert damping for \(\Gamma/\Delta_0 = 0.5\) and \(\theta = \pi/4\) as a function of \(\alpha/\beta\). The four curves correspond to \(h\omega_0/\Delta_0 = 0.005, 0.01, 0.02,\) and 0.05, respectively. This figure indicates that when we fix the resonant frequency \(\omega_0\) and vary the ratio of \(\alpha/\beta\), the Gilbert damping is strongly enhanced when \(\alpha/\beta\) is slightly smaller or larger than 1. We expect that this enhancement of the Gilbert damping is strong enough to be observed experimentally.

V. SHIFT IN THE FMR FREQUENCY

Next, we discuss the shift in the FMR frequency when the vertex correction is taken into account. The contour plots in Fig. 6 (a), (b), and (c) for \(\alpha/\beta = 0, 1,\) and 3
FIG. 6. (Upper panels) Modulations of the Gilbert damping, $\delta \alpha_G/\alpha_G,0$ for (a) $\alpha/\beta = 0$, (b) $\alpha/\beta = 1$, and (c) $\alpha/\beta = 3$. (Lower panels) Shifts in the FMR frequency, $\delta \omega/\omega,0$, for (d) $\alpha/\beta = 0$, (e) $\alpha/\beta = 1$, and (f) $\alpha/\beta = 3$. The horizontal axes are the FMR frequency, $\omega_0 = \gamma h_{dc}$, while the vertical axes show the azimuth angle of the spontaneous spin polarization, $\theta$, in the FI. In all the plots, we have considered vertex corrections and have chosen $\Gamma/\Delta_0 = 0.5$. In (c) and (e), there are regions in which the values exceed the upper limits of the color bar located in the right side of each plot; the maximum value is about 0.65 in (c) and about 10 in (e) (see also Fig. 7). In addition, the graph (b) cannot express a delta-function-like singularity at $\omega_0 = 0$ (see the main text).

summarize the modulation of the Gilbert damping, $\delta \alpha_G$. These plots have the same features as in Fig. 3 (c), (f), and (i). Fig. 6 (d), (e), and (f) plot the shift in the FMR frequency $\delta \omega/\omega_0$ with contour plots for $\alpha/\beta = 0$, 1, and 3. By comparing Fig. 6 (a), (b), and (c) with (d), (e), and (f), we find that some of the qualitative features of the FMR frequency shift are common to those of the modulation of the Gilbert damping, $\delta \alpha_G$: (i) they depend on $\theta$ for $\alpha/\beta > 0$, while they do not depend on $\theta$ for $\alpha/\beta = 0$, (ii) the structure at $\omega_0 = 0$ due to elastic spin-flip appears, and (iii) the structure within a finite range of frequencies due to magnon absorption appears. We can also see a few differences between $\delta \omega/\omega_0$ and $\delta \alpha_G$. For example, $\delta \omega_0/\omega_0$ has a dip-and-peak structure at $h \omega_0/\Delta_0 = 2$ where $\delta \alpha_G$ has only a peak. Related to this feature, $\delta \omega_0/\omega_0$ has a tail that decays more slowly than that for $\delta \alpha_G$. The most remarkable difference is that $\delta \omega_0/\omega_0$ diverges at $\omega_0 = 0$ for $\alpha/\beta = 1$ except for $\theta = 3\pi/4, 7\pi/4$, reflecting the delta-function-like singularity of $\delta \alpha_G$ at $\omega_0 = 0$. These features are reasonable because $\delta \omega_0/\omega_0$ and $\delta \alpha_G$ are determined by the real and imaginary parts of the retarded spin susceptibility, are related to each other through the Kramers-Kronig conversion.

The main panel of Fig. 7 shows the frequency shift $\delta \omega/\omega_0$ for $\alpha/\beta = 1.1$ as a function of the resonant frequency $\omega_0$. The five curves correspond to $\theta = -\pi/4, -\pi/8, 0, \pi/8$, and $\pi/4$. Although the frequency shift appears to diverge in the limit of $\omega_0 \rightarrow 0$ in the scale of the main panel, it actually grows to a finite value and then goes to zero as $\omega_0$ approaches zero (see the inset of Fig. 7). For $\alpha/\beta = 1 + \delta$ ($\delta \ll 1$), the frequency shift
is calculated approximately as
\[
\frac{\Delta \omega_0}{\alpha G_0 \omega_0} \simeq \frac{\Delta_0}{2\pi} \frac{\hbar \omega_0}{(i\omega_0)^2 + \Gamma_s^2} \sin^2 \left( \frac{\theta + \pi}{4} \right),
\]
where \( \Gamma_s \) is the spin relaxation rate defined in Eq. (62) (see Appendix D for the detailed derivation). We expect that this strong enhancement of the frequency shift near \( \alpha/\beta = 1 \) can be observed experimentally.

**VI. SUMMARY**

We theoretically investigated spin pumping into a two-dimensional electron gas (2DEG) with a textured effective Zeeman field caused by Rashba- and Dresselhaus-type spin-orbit interactions. We expressed the change in the peak position and the linewidth in a ferromagnetic resonance (FMR) experiment that is induced by the 2DEG within a second-order perturbation with respect to the interfacial exchange coupling by taking the vertex correction into account. We found that, for almost all of the parameters, the vertex correction modifies the modulation of the Gilbert damping only moderately most of the parameters, the vertex correction modifies the Gilbert damping only moderately. However, we found that the Gilbert damping is strongly enhanced at low frequencies when the Rashba- and Dresselhaus-type spin-orbit interactions are chosen to be almost equal but slightly different. This feature appears only when the vertex correction is taken into account and is considered to originate from the slow spin relaxation related to the spin conservation law that holds when the two spin-orbit interactions completely match. A similar enhancement was found for the frequency shift of the FMR. We expect that this remarkable enhancement can be observed experimentally.

Our work provides a theoretical foundation for spin pumping into two-dimensional electrons with a spin-textured Zeeman field on the Fermi surface. Although we have treated a specific model for two-dimensional electron systems with both the Rashba- and Dresselhaus spin-orbit interactions, our formulation and results will be helpful for describing spin pumping into general two-dimensional electron systems such as surface states and atomic layer compounds.

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**Appendix A: Derivation of Eq. (51)**

In this appendix, we give a detailed derivation of Eq. (51) from Eq. (50). First, we modify Eq. (50) as
\[
\chi(0, i\omega_n) = \frac{1}{\delta A} \sum_{\nu} \sum_{\nu'} \left[ \nu \nu' \sin 2(\phi - \theta) \right] \frac{X_j}{i\hbar \omega_m - E_k' + i\Gamma/2 \text{sgn}(\omega_m)} \times \frac{1}{i\hbar \omega_n + i\hbar \omega_n - E_k' + i\Gamma/2 \text{sgn}(\omega_m + \omega_n)},
\]
and \((X_1, X_2, X_3) = (X, Y, Z)\). A standard procedure based on the residue integral enables us to express the sum \(I_{\nu\nu', j}\) for \(\omega_n > 0\) as a complex integral on the contour \(C\) shown in Fig. 8(a). This contour can be modified into a sum of the four contours, \(C_i\) \((i = 1, 2, 3, 4)\), shown in Fig. 8(b). Accordingly, \(I_{\nu\nu', j}\) is written as
\[
I_{\nu\nu', j} = \sum_{i=1}^{4} x_{\nu\nu', j}^{C_i},
\]

**FIG. 8. Schematic picture of the change in the contour integral. (a) The original contour. (b) The modified contour.**
Using formula (43), we replace the sum over $\nu$ by using the integral variable to $X$ Here, we have used the fact that $X_j(z, \omega_n)$ is independent of $z$ for $0 < \text{Im} z < \omega_n$ from Eq. (44) and have defined its value as $\tilde{X}_j(\omega_n)$ ($j = 1, 2, 3$). From Eqs. (40)-(42), $\tilde{X}_j(\omega_n)$ are calculated as

$$\tilde{X}_1(\omega_n) = \frac{1}{(1 - \lambda_0(\omega_n))^2 - \lambda_2(\omega_n)^2 - \lambda_3(\omega_n)^2},$$

$$\tilde{X}_2(\omega_n) = \frac{1 - \lambda_0(\omega_n) - \tilde{\Lambda}_2(\omega_n)}{(1 - \lambda_0(\omega_n))^2 - \lambda_2(\omega_n)^2 - \lambda_3(\omega_n)^2},$$

$$\tilde{X}_3(\omega_n) = \frac{i}{1 - \lambda_0(\omega_n) + \tilde{\Lambda}_1(\omega_n)}.$$ 

By changing the integral variable to $E' = E - E_k^\nu$ in the first term and to $E'' = -(E - E_k^\nu)$ in the second term in Eq. (A1), we obtain

$$\int_{-\infty}^{\infty} d\xi (f(E') + f(E'') - f(E - E_k^\nu)) = 2E' + E_k^\nu - E_k^\nu'.$$

Then, by performing the $E'$-integral, we obtain

$$\int_{-\infty}^{\infty} d\xi (\tilde{T}_{\nu\nu'}^C, \tilde{T}_{\nu\nu'}^C) = \frac{i\hbar\omega_n}{E_k^\nu - E_k^\nu' + i\Delta_n + i\Gamma}.$$ 

Next, let us consider the contribution from $C_1$ and $C_4$. On these two contours, $X_j(z, \omega_n)$ is independent of $z$ and its value is defined by $X_j'(\omega_n)$ ($j = 1, 2, 3$). Because $X_j(z, \omega_n)$ ($j = 0, 1, 2, 3$) becomes zero for $\text{Im} z < 0$ or $\omega_n < \text{Im} z$ from Eq. (44), $X_j'(\omega_n)$ are given as

$$X_1'(\omega_n) = 0, \quad X_2'(\omega_n) = 1, \quad X_3'(\omega_n) = i.$$ 

A similar calculation to that of $C_2$ and $C_3$ yields

$$\int_{-\infty}^{\infty} d\xi (\tilde{T}_{\nu\nu'}^C, \tilde{T}_{\nu\nu'}^C) = -\tilde{X}_j'(\omega_n).$$

By substituting these results into Eq. (A1), we obtain

$$\chi(0, \omega_n) = D(\epsilon_F) \sum_{\nu, \nu'} \int_0^{2\pi} \frac{d\varphi}{2\pi} \left[ \nu \nu' \sin 2(\phi - \theta) \tilde{x}_1(\omega_n) \right. 
+ \left. (1 - \nu \nu' \cos 2(\phi - \theta))(1 - i \tilde{x}_2(\omega_n)) \right].$$

Finally, Eq. (51) is derived by substituting the expressions for $\tilde{x}_j(\omega_n)$ and by analytic continuation $\omega_n \to \omega + i\delta$.

Appendix B: Analytic Expression for $\alpha/\beta = 0$

In this appendix, we derive analytic expressions of the modulation of the Gilbert damping when $\alpha/\beta = 0$ to see quantitative effect of taking the vertex correction is taken into account. For $\alpha/\beta = 0$, the spin-splitting energy $2h_{\text{eff}}$ ($= 2\Delta_0$) is constant along the Fermi surface, and $\tilde{\Lambda}_3^R(\omega)$ ($j = 0, 1, 2, 3$) is simplified as

$$\tilde{\Lambda}_3^R(\omega) = \frac{i\Gamma}{4\Delta_0} \sum_{\nu, \nu'} \frac{1}{\hbar\omega/\Delta_0 + (\nu - \nu') + i\Gamma/\Delta_0},$$

$$\tilde{\Lambda}_3^R(\omega) = \frac{i\Gamma}{4\Delta_0} \sum_{\nu, \nu'} \frac{\nu \nu'}{\hbar\omega/\Delta_0 + (\nu - \nu') + i\Gamma/\Delta_0},$$

$$\tilde{\Lambda}_3^R(\omega) = \tilde{\Lambda}_3^R(\omega) = 0.$$ 

Then, we obtain the modulation of the Gilbert damping with the vertex corrections,

$$\frac{\delta\Omega_G}{\Omega_{G,0}} \approx \frac{\Delta_0}{2\pi \Gamma} \left\{ \frac{\tilde{\Lambda}_3^R(\omega_0)}{1 - \tilde{\Lambda}_3^R(\omega_0) + \tilde{\Lambda}_3^R(\omega_0)} \right\}.$$ 

The modulation of the Gilbert damping without the vertex correction is obtained by considering only the first-order term with respect to $\tilde{\Lambda}_3^R(\omega_0)$,

$$\frac{\delta\Omega_G^\text{G}}{\Omega_{G,0}} \approx \frac{\Delta_0}{2\pi \Gamma} \left\{ 2\tilde{\Lambda}_3^R(\omega_0) - \tilde{\Lambda}_3^R(\omega_0) \right\}.$$
damping can be analytically calculated as
\[
\frac{\delta \alpha_G}{\alpha_{G,0}} \approx \frac{\Delta_0}{4\pi} \cdot \frac{\Gamma/2}{(\hbar \omega_0)^2 + (\Gamma/2)^2},
\]
(B6)
\[
\frac{\delta \alpha_{G,0}^{\nu}}{\alpha_{G,0}} \approx \frac{\Delta_0}{4\pi} \cdot \frac{\Gamma}{(\hbar \omega_0)^2 + \Gamma^2}.
\]
(B7)

This indicates that the peak width is halved by taking the vertex correction is taken into account, which is consistent with the results shown in Fig. 3(b) and (c).

In a similar way, we can evaluate the modulation of the Gilbert damping near the peak at \(\omega_0 = 2\Delta_0/h\) as
\[
\frac{\delta \alpha_G}{\alpha_{G,0}} \approx \frac{\Delta_0}{4\pi} \cdot \frac{3\Gamma/4}{(\hbar \omega_0 - 2\Delta_0)^2 + (3\Gamma/4)^2} + \frac{\Gamma/2}{(\hbar \omega_0 - 2\Delta_0)^2 + (\Gamma/2)^2},
\]
(B8)
\[
\frac{\delta \alpha_{G,0}^{\nu}}{\alpha_{G,0}} \approx \frac{\Delta_0}{4\pi} \cdot \frac{3\Gamma/2}{(\hbar \omega_0 - 2\Delta_0)^2 + \Gamma^2}.
\]
(B9)

As well, for the peak at \(\omega_0 = 2\Delta_0/h\), the peak width becomes smaller when the vertex correction is taken into account. This observation is consistent with the results shown in Fig. 3(b) and (c). We note that for a finite value of \(\Gamma\) a sum of Eqs. (B6) and (B8) (Eqs. (B7) and (B9)) gives a better analytic form which fits the numerical result with (without) the vertex correction.

Finally, we note that the same analytical expressions for \(\delta \alpha_G\) and \(\delta \alpha_{G,0}^{\nu}\) can be obtained for the case of \(\beta = 0\), i.e., when only the Rashba spin-orbit interaction exists.

**Appendix C: Analytic Expression for \(\alpha/\beta = 1\)**

In this appendix, we derive analytic expressions of the modulation of the Gilbert damping when \(\alpha/\beta = 1\). In this case, the effective Zeeman field is parallel to the \((-1, 1, 0)\) direction and its amplitude is given as
\[
h_{eff}(\varphi) = 2\Delta_0 |\sin(\varphi + \pi/4)|,
\]
(C1)

Then, \(\Lambda_j^R(\omega)\) \((j = 0, 1, 2, 3)\) becomes
\[
\Lambda_j^R(\omega) = \frac{i\Gamma}{4\Delta_0} \sum_{\nu\nu'} J_{\nu\nu'},
\]
(C2)
\[
\Lambda_j^I(\omega) = \frac{-i\Gamma}{4\Delta_0} \sum_{\nu\nu'} \nu\nu' J_{\nu\nu'},
\]
(C3)
\[
\Lambda_2^R(\omega) = -\sin 2\theta \Lambda_1^R(\omega),
\]
(C4)
\[
\Lambda_3^R(\omega) = -\cos 2\theta \Lambda_1^R(\omega)
\]
(C5)

where
\[
J_{\nu\nu'}(\omega) \equiv \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{\Delta_0}{\hbar \omega + (\nu - \nu')h_{eff}(\varphi) + i\Gamma}.
\]
(C6)

In the case of \(\theta = \pi/4\), the modulation of the Gilbert damping with the vertex correction is expressed as
\[
\frac{\delta \alpha_G}{\alpha_{G,0}} = \frac{\Delta_0}{2\pi\Gamma} \Re \left[ -2 + \frac{1}{1 - \Lambda_0^R(\omega_0) + \Lambda_1^R(\omega_0)} \right] + \frac{1}{1 - \Lambda_0^R(\omega_0) - \Lambda_1^R(\omega_0)}
\]
(C7)

The third term of the above equation is calculated as
\[
\frac{1}{1 - \Lambda_0^R(\omega_0)} = \frac{1}{1 - \frac{\Gamma}{\hbar \omega_0} + i\Gamma} = \frac{\hbar \omega_0 + i\Gamma}{\hbar \omega_0}.
\]
(C8)

This indicates that the expansion with respect to \(\Lambda_j^R\) cannot be allowed for \(\omega_0 \ll \Gamma\). This is why the modulation without the vertex correction, which is obtained by taking from the first-order term of \(\Lambda_j^R\) in Eq. (C7) as
\[
\delta \alpha_{G,0}^{\nu} = \frac{\Delta_0}{2\pi\Gamma} \Re \left[ 2\Lambda_1^R(\omega_0) \right],
\]
(C9)
gives a different result near \(\omega_0 \approx 0\). Actually, for \(\theta = \pi/4\), \(\delta \alpha_G\) and \(\delta \alpha_{G,0}^{\nu}\) are calculated as
\[
\frac{\delta \alpha_G}{\alpha_{G,0}} = \frac{\Delta_0}{2\pi\Gamma} \Re \left[ i\frac{\Gamma}{\Delta_0} (J_{+++} + J_{++-}) \right],
\]
(C10)
\[
\delta \alpha_{G,0}^{\nu} = \frac{1}{4\pi} \Re \left[ i\left( J_{++-} + J_{++-} + J_{++} + J_{--} \right) \right].
\]
(C11)

Note that Eq. (C10) is not valid for \(\omega_0 = 0\). As indicated from the absence of \(J_{++}\) and \(J_{--}\), the graph of \(\delta \alpha_G(\omega_0)\) has no peak at zero frequency even though \(\delta \alpha_{G,0}^{\nu}(\omega_0)\) has a peak there. This observation is consistent with Fig. 3(e) and (f).

In the case of \(\theta = -\pi/4\), the modulations of the Gilbert damping with and without the vertex correction are
\[
\frac{\delta \alpha_G}{\alpha_{G,0}} = \frac{\Delta_0}{\pi\Gamma} \Re \left[ i\frac{\Gamma}{\Delta_0} (J_{++-} + J_{++-}) \right],
\]
(C12)
\[
\delta \alpha_{G,0}^{\nu} = \frac{1}{2\pi} \Re \left[ i(J_{++-} + J_{++-}) \right].
\]
(C13)

Note that \(\delta \alpha_{G,0}^{\nu}\) is obtained by taking the first-order term in Eq. (C12). As indicated by the absence of the terms, \(J_{++} \text{ and } J_{--}\), neither \(\delta \alpha_G\) nor \(\delta \alpha_{G,0}^{\nu}\) has any structure around \(\omega_0 = 0\). It can be checked that these two expressions give almost the same result when \(\Gamma \lesssim \Delta_0\), which is consistent with Fig. 3(e) and (f). Note as well that \(\delta \alpha_G\) is just doubled compared with the result for \(\theta = \pi/4\) in Eq. (C10).

**Appendix D: Approximate Expressions near \(\alpha/\beta = 1\)**

In this appendix, we derive the approximate expressions Eqs. (61) and (63) for \(\alpha/\beta = 1 + \delta\ (\delta \ll 1)\) and
\[ \omega \simeq 0. \] For \( \alpha/\beta = 1 + \delta \) (\( \delta \ll 1 \)), we can approximate

\begin{align*}
\cos 2(\phi - \theta) & \simeq \sin 2\theta \left(1 + \frac{h_x + h_y}{h_{\text{eff}}^2}\right), \\
\sin 2(\phi - \theta) & \simeq \cos 2\theta \left(1 + \frac{h_x + h_y}{h_{\text{eff}}^2}\right).
\end{align*}

\[ \text{(D1)} \]

\[ \text{(D2)} \]

Then, we obtain

\begin{align*}
\tilde{\Lambda}_2^R & \approx X \sin 2\theta, \\
\tilde{\Lambda}_3^R & \approx X \cos 2\theta, \\
X & = \frac{\Gamma}{4} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\nu\nu'} \frac{\nu\nu'}{\omega} \left(1 + \frac{(h_x + h_y)^2}{h_{\text{eff}}^2}\right).
\end{align*}

\[ \text{(D3)} \]

\[ \text{(D4)} \]

\[ \text{(D5)} \]

in the low-frequency region. Here, the contribution of the second term of the bracket in Eq. (55) does not have a singularity at \( \omega_0 = 0 \) because \( \tilde{\Lambda}_2^R \) and \( \tilde{\Lambda}_3^R \) do not depend on the effective Zeeman field \( h_{\text{eff}} \). Therefore, the singularity comes from the first term of the bracket in Eq. (55) and we can approximate \( F(\omega) \) as

\[ \begin{align*}
F(\omega) & \simeq \frac{\Delta_0}{2\pi i} \int \frac{d\phi}{2\pi} \left[ \tilde{\Lambda}_2^R (1 - \tilde{\Lambda}_3^R) - \tilde{\Lambda}_3^R (1 - \tilde{\Lambda}_2^R) + (\tilde{\Lambda}_3^R)^2 \right] \\
& \quad - \frac{\Delta_0}{\pi \hbar \omega} \\
& \quad - \frac{\Delta_0}{\pi \hbar \omega}.
\end{align*} \]

\[ \text{(D6)} \]

Finally, using the equation,

\[ 1 - \tilde{\Lambda}_0^R + X = \Gamma X + i \hbar \omega + O(\omega^2), \]

\[ \text{(D7)} \]

we find that the third term in the bracket in Eq. (D6) is divergent at \( \omega = 0 \) in the limit of \( \delta \to 0 \) since the denominator vanishes. By substituting Eq. (D7) into Eq. (D6), the most singular part is calculated as

\[ F(\omega) \simeq \frac{\Delta_0}{2\pi i} \int \frac{d\phi}{2\pi} \left[ \frac{\sin^2(\theta + \pi/4)}{\Gamma \pm i \hbar \omega} - \frac{\Delta_0}{\pi \hbar \omega}. \right] \]

\[ \text{(D8)} \]

Using Eqs. (53) and (54), it is straightforward to obtain Eqs. (61) and (63).
The result for the case of $\beta = 0$, i.e., the case when only the Rashba spin-orbit interaction exists, is the same as the case of $\alpha = 0$. 