Circular-arc graphs are intersection graphs of arcs on the circle. The aim of our work is to present a polynomial time algorithm testing whether two circular-arc graphs are isomorphic. To accomplish our task we construct decomposition trees, which are the structures representing all normalized intersection models of circular-arc graphs. Normalized models reflect the neighbourhood relation in circular-arc graphs and can be seen as their canonical representations; in particular, every intersection model can be easily transformed into a normalized one.

Our work adapts and appropriately extends the previous work on the similar topic done by Hsu [SIAM J. Comput. 24(3), 411–439, (1995)]. In his work, Hsu developed decomposition trees representing all normalized models of circular-arc graphs. However due to the counterexample given in [Discrete Math. Theor. Comput. Sci., 15(1), 157–182, 2013], his decomposition trees can not be used by algorithms testing isomorphism of circular-arc graphs.

1. Introduction

Circular-arc graphs are intersection graphs of arcs on the circle. Circular-arc graphs generalize interval graphs, which are the intersection graphs of intervals on a real line. The problems for circular-arc graphs tend to be harder than for their interval counterparts. A good example illustrating our remark is the problem of compiling the lists of minimal forbidden induced subgraphs for these classes of graphs. For interval graphs such a list was completed by Lekkerkerker and Boland already in the 1960s [18] but for circular-arc graphs, despite a flurry of research [1, 10, 11, 16, 20, 26, 27], it is still unknown. We refer the readers to the survey papers [7, 21], where the state of research on the structural properties of circular-arc graphs is outlined.

The situation is different for the recognition problem. The first linear time algorithm for the recognition of interval graphs was given by Booth and Lueker [2] in the 1970s. A few years later, the first polynomial time algorithm for the recognition of circular-arc graphs was given by Tucker [28]. The complexity of this algorithm has been subsequently improved in [9, 14]. Currently, there are known at least two linear-time algorithms recognizing circular-arc graphs [15, 22].
Using the concept of $PQ$-trees, Booth and Lueker [2] gave a linear time algorithm testing isomorphism of interval graphs already in the 1970s. On the other hand, the isomorphism problem for circular-arc graphs has been open for almost 40 years. There are known linear algorithms solving the isomorphism problem on proper circular-arc graphs [6, 19]. The isomorphism problem can be solved in linear time [6] and logarithmic space [17] in the class of Helly circular-arc graphs. The partial results for the general case have been given in [3]. Only recently, the first polynomial time-algorithm for the isomorphism problem for circular-arc graphs was announced by Nedela, Ponomarenko, and Zeman [24].

In the paper [14], Hsu claimed a theorem describing the structure of all normalized representations of circular-arc graphs and a polynomial time algorithm for the isomorphism problem. In his work Hsu developed decomposition trees, which are structures that represent all conformal models of a circular-arc graph. Based on his decomposition trees, Hsu proposed the algorithm testing isomorphism of circular-arc graphs. However, the Hsu’s algorithm was proven to be incorrect and a counterexample for its correctness was constructed by Curtis, Lin, McConnell, Nussbaum, Soulignac, Spinrad, and Szwarcfiter in [6]. In particular, decomposition trees invented by Hsu can not be used to test whether two circular-arc graphs are isomorphic.

1.1. Our work. We adapt and extend Hsu’s ideas appropriately and we construct refined decomposition trees representing all normalized models of a circular-arc graph. In our work we exploit the ideas invented by Spinrad [25], which enabled him to characterize all circular-arc graphs with clique cover two in terms of appropriately chosen two-dimensional posets. We extend those ideas to the whole class of circular-arc graphs and plug them appropriately to the Hsu’s framework. Eventually, we develop a decomposition tree representing all normalized models of circular-arc graphs. Given such decomposition trees, we develop a simple polynomial time algorithm for the isomorphism problem on circular-arc graphs. We do not attempt to optimize time complexity of our algorithm. It seems that the refined decomposition trees (which can be constructed in a linear time by known recognition algorithms) satisfy some properties that might be used to develop a fast algorithm for the isomorphism problem on circular-arc graphs.

Our paper is organized as follows:

- In Section 2 we define normalized models of circular-arc graphs. We describe Hsu’s approach to the isomorphism problem on circular-arc graphs, and we provide a counterexample to his algorithm constructed in [6]. In this section we also compare our work to Hsu’s work.
- In Section 3 we introduce notation used throughout the paper.
- In Section 4 we describe all tools required to prove our results, including join decomposition of circle graphs, modular decomposition, and transitive orientations of graphs.
In Section 5 we describe a decomposition tree that keeps a track of all normalized models of a circular-arc graph.
In Section 6 we present a polynomial time algorithm for the isomorphism problem on circular-arc graphs.

2. Normalized models of circular-arc graphs

A circular-arc model \( \psi \) of a graph \( G = (V, E) \) is a collection of arcs \( \{ \psi(u) : u \in V \} \) of a given circle \( C \) such that \( \psi(u) \) intersects \( \psi(v) \) if and only if \( uv \in E \), \( u, v \in V \). A graph \( G \) is circular-arc if \( G \) admits a circular-arc model.

Let \( a \) and \( b \) be two arcs in \( C \). We say that:

- \( a \) and \( b \) are disjoint if \( a \cap b = \emptyset \),
- \( a \) contains \( b \) if \( b \subset a \),
- \( a \) is contained in \( b \) if \( a \subset b \),
- \( a \) and \( b \) cover the circle if \( a \cup b = C \),
- \( a \) and \( b \) overlap, otherwise.

Suppose that \( G = (V, E) \) is a graph with no universal and no twins. Suppose that \( |V| = n \). Following the ideas from [14, 28], the intersection matrix of \( G \) is an \( n \times n \) matrix \( M_G \), of which rows and columns correspond to the vertices of \( G \) and whose entries \( M_G[u, v] \) are defined such that:

\[
M_G[u, v] = \begin{cases} 
  \text{di} & \text{if } uv \notin E, \\
  \text{cs} & \text{if } uv \in E \text{ and } N[u] \subseteq N[v], \\
  \text{cd} & \text{if } uv \in E \text{ and } N[v] \subseteq N[u], \\
  \text{cc} & \text{if } \text{foreach } w \in N(v) \cap N(u), N[w] \subset N[v], \text{ and} \\
  \text{ov} & \text{foreach } w \in N(u) \cap N(v), N[w] \subset N[u], \\
  \text{otherwise}. & 
\end{cases}
\]

Note that for every \( u, v \in V \) the matrix \( M_G \) satisfies the following properties:

- \( M_G[u, v] = \text{ov} \) iff \( M_G[v, u] = \text{ov} \), \( M_G[u, v] = \text{cc} \) iff \( M_G[v, u] = \text{cc} \), \( M_G[u, v] = \text{cs} \) iff \( M_G[v, u] = \text{cd} \).

The intersection matrix \( M_G \) encodes the relative relation between the neighborhoods of the vertices in the graph \( G \). If \( G \) is a circular-arc graph, the entries of \( M_G \) reflect the relative position of arcs in so-called normalized models of \( G \).

**Definition 2.1.** A circular-arc model \( \phi \) of \( G \) is normalized if for every pair \((u, v)\) of distinct vertices of \( G \) the following conditions are satisfied:

1. if \( u \text{ di } v \), then \( \psi(u) \) and \( \psi(v) \) are disjoint,
2. if \( u \text{ cs } v \), then \( \psi(u) \) contains \( \psi(v) \),
3. if \( u \text{ cd } v \), then \( \psi(u) \) is contained in \( \psi(v) \),
4. if \( u \text{ cc } v \), then \( \psi(u) \) and \( \psi(v) \) cover the circle,
5. if \( u \text{ ov } v \), then \( \psi(u) \) and \( \psi(v) \) overlap.
Note that every circular-arc model $\psi$ of $G$ fulfills (1) and (5), but it might not satisfy (2), (3), or (4). However, every circular-arc model $\psi$ of $G$ can be turned into a normalized model by carrying out a normalization procedure on $\psi$ – whenever there are adjacent vertices $u, v$ in $G$ violating (2), (3) or (4), perform the following transformation on $\psi$:

- if $u$ cs $v$ but $\psi(v)$ is not contained in $\psi(u)$, pick the endpoint of $\psi(u)$ contained in $\psi(v)$ and pull it outside $\psi(u)$ so as $\psi(u)$ contains $\psi(v)$,
- if $u$ cd $v$ but $\psi(u)$ is not contained in $\psi(v)$, pick the endpoint of $\psi(v)$ contained in $\psi(u)$ and pull it outside $\psi(v)$ so as $\psi(u)$ is contained in $\psi(v)$,
- if $u$ cc $v$ but $\psi(u)$ and $\psi(v)$ do not cover the circle, pick the endpoint of $\psi(u)$ from outside $\psi(v)$ and the endpoint of $\psi(v)$ from outside $\psi(u)$ and pull these endpoints towards each other until they pass somewhere on the circle $C$.

The above transformation keeps $\psi$ a model of $G$ and if performed in an appropriate order, it eventually lead to a normalized model of $G$ – see [14, 28] for more details.

**Theorem 2.2** ([14, 28]). Suppose $G$ is a graph with no twins and no universal vertices. Then, $G$ is circular-arc graph if and only if $G$ has a normalized model.

Our goal is to describe the structure representing all normalized models of a circular-arc graph $G$. To achieve our goal, we follow the approach taken by Hsu [14]. We transform a circular-arc graph $G$ into an associated circle graph $G_c$. Then, we are looking for some special circle models of $G_c$, called conformal, which are in one-to-one correspondence with normalized models of $G$. After that, we describe the structure of all conformal models of $G_c$, thus obtaining a description of all normalized models of $G$. Similarly to Hsu’s work, to achieve our goals we exploit a join decomposition of $G_c$, a structure describing all circle models of $G_c$, and a modular decomposition of $G_c$, a structure that appears to be appropriate to characterize all conformal models of $G_c$.

**Definition 2.3** ([14]). Let $G$ be a graph with no twins and no universal vertices. The graph $G_c$ associated with $G$ has $V(G)$ as the set of its vertices and the set

$$\{uv \in E(G) : M_G[u, v] = ov\}$$

as the set of its edges.

There is a natural straightening procedure that transforms normalized models $\psi$ of $G$ into oriented circle models $\phi$ of $G_c$: it converts every arc $\psi(v)$ into an oriented chord $\phi(v)$ such that: $\phi(v)$ has the same endpoint as $\psi(v)$ and $\phi(v)$ has $\psi(v)$ on its left side when we traverse $\phi(v)$ from its tail to its head. Clearly, for every $u, v \in V(G)$, the oriented chords $\phi(v)$ and $\phi(u)$ intersect if and only if $\psi(u)$ and $\psi(v)$ overlap. Hence, a non-oriented $\phi$ is a circle model of $G_c$.

**Lemma 2.4.** If $G$ is a circular-arc graph, then $G_c$ is a circle graph.
However, the converse operation is not always possible – we can not revert any circle model of $G_c$ into a normalized circular-arc model of $G$. We associate with every vertex $v \in V(G)$ two sets, $left(v)$ and $right(v)$:
\[
left(v) = \{ u \in V(G) : \text{ } u \text{ di } v \text{ or } u \text{ cc } v \},
\]
\[
right(v) = \{ u \in V(G) : \text{ } u \text{ di } v \text{ or } u \text{ cs } v \}.
\]

If $\phi$ is an oriented circle model of $G_c$, obtained by the straightening of $\psi$, the oriented chords $\phi(u)$ for $u \in left(v)$ lie on the left side of $\psi(v)$ and the oriented chords $\phi(u)$ for $u \in right(v)$ lie on right side of $\psi(v)$, for every $v \in V(G)$.

**Definition 2.5.** An oriented circle model $\phi$ of $G_c$ is conformal if for every $u, v \in V(G)$:

- $u \in left(v)$ iff $\phi(u)$ lies on the left side of $\phi(v)$,
- $u \in right(v)$ iff $\phi(u)$ lies on the right side of $\phi(v)$.

So, the straightening procedure transforms normalized models of $G$ into conformal models of $G_c$.

On the other hand, a *bending procedure* transforms conformal models $\phi$ of $G_c$ into normalized models $\psi$ of $G$. Given an oriented chord $\phi(v)$ we let $\psi(v)$ to be an arc with the same endpoints as $\phi(v)$ and placed such that $\psi(v)$ is on the left side of $\phi(v)$. Note that bending procedure transforms conformal models $\psi$ of $G_c$ into normalized models of $G$. Indeed, suppose $(u, v)$ is a pair of distinct vertices in $G$.

By the construction of $\psi$:

- if $u \text{ di } v$, then $u \in right(v)$ and $v \in right(u)$, and hence $\psi(u)$ and $\psi(v)$ are disjoint,
- if $u \text{ cs } v$, then $u \in right(v)$ and $v \in left(u)$, and hence $\psi(v)$ contains $\psi(v)$,
- if $u \text{ cd } v$, then $u \in left(v)$ and $v \in right(u)$, and hence $\psi(u)$ and $\psi(v)$ are disjoint,
- if $u \text{ cc } v$, then $u \in left(v)$ and $v \in left(u)$, and hence $\psi(u)$ and $\psi(v)$ cover the circle,
- if $u \text{ ov } v$, then $\psi(u)$ and $\psi(v)$ overlap.

So, $(u, v)$ satisfies the conditions (2.5). [1]-[5], and hence $\psi$ is a normalized model of $G$. Summing up, we get the following theorems.

**Theorem 2.6.** Let $G$ be a graph with no twins and no universal vertices. Let $G_c$ be a graph associated with a graph $G$. Then, $G$ is a circular-arc graph if and only if $G_c$ is a circle graph and $G_c$ admits a conformal model.

**Theorem 2.7.** Let $G$ be a circular-arc graph with no twins and no universal vertices. There is a one-to-one correspondence between normalized models of $G$ and conformal models of $G_c$.

2.1. Hsu’s approach. The straightening procedure and the bending procedure were introduced by Hsu [14]. However, his straightening procedure does not orient the chords in $\phi$. Thus, bending procedure needs to be performed more carefully,
but still can be uniquely determined if \( G \) has no universal vertices. In particular, Hsu proved Lemma 2.4 and Theorems and 2.7. The main difference between our works lies in the definition of conformal models of \( G_c \). In Hsu’s work, conformal models are non-oriented. In fact, in [14] the following definition is assumed: a non-oriented circle model \( \phi \) of \( G_c \) is conformal if for every vertex \( u \) of \( G \) the chords associated with vertices in \( \text{left}(v) \) are on one side of \( \phi(v) \) and those associated with vertices in \( \text{right}(v) \) are on the other side of \( \phi(v) \). Such a definition has one drawback: given a conformal model \( \phi \) of \( G_c \) there might exist two non-isomorphic circular-arc graphs \( G \) and \( G' \) such that \( G_c = G'_c \) and \( \phi \) is conformal for both \( G \) and \( G' \) – see Figures 1–3. This observation was noted by Curtis, Lin, McConnell, Nussbaum, Soulignac, Spinrad, and Szwarcfiter [6] and resulted in the construction of a counterexample to the Hsu’s isomorphism algorithm. As stated in [6], “The origin of the mistake in Hsu’s algorithm is the statement: To test the isomorphism between two circular-arc graphs \( G \) and \( G' \), it suffices to test whether there exists isomorphism conformal models for \( G \) and \( G' \),” which is not true due to the example shown in Figures 1–3.

**Figure 1.** Graphs \( G \) and \( G' \) are not isomorphic as they have different number of edges.

[Diagram of two graphs showing different numbers of edges]

**Figure 2.** Normalized models of \( G \) and \( G' \).

2.2. **Our approach.** The isomorphism algorithm constructed in our work tests whether there exists a bijection \( \alpha \) between conformal models \( \phi \) and \( \phi' \) of \( G_c \) and \( G'_c \), respectively, that preserves the orientations of the chords, that is, that satisfies the following properties for every \( (u, v) \in V \times V \):
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Figure 3. From left to right: non-oriented conformal model of $G$ and $G'$ according to the Hsu’s definition, oriented conformal models of $G$ and $G'$ according to our definition.

- $\phi(u)$ is on the left side of $\phi(v)$ iff $\alpha(\phi(u))$ is on the left side of $\alpha(\phi(v))$,
- $\phi(u)$ is on the right side of $\phi(v)$ iff $\alpha(\phi(u))$ is on the right side of $\alpha(\phi(v))$.

Hence, for every $u, v \in V$ the bijection $\alpha$ behaves such that:

- $\phi(u)$ and $\phi(v)$ are disjoint iff $\alpha(\phi(u))$ and $\alpha(\phi(v))$ are disjoint,
- $\phi(u)$ contains $\phi(v)$ iff $\alpha(\phi(u))$ contains $\alpha(\phi(v))$,
- $\phi(u)$ is contained in $\phi(v)$ iff $\alpha(\phi(u))$ is contained in $\alpha(\phi(v))$,
- $\phi(u)$ and $\phi(v)$ cover the circle iff $\alpha(\phi(u))$ and $\alpha(\phi(v))$ cover the circle,

This shows that $\phi$ and $\alpha(\phi)$ indeed represent the isomorphic circular-arc graphs.

Our goal is to develop a structure that represents all oriented conformal models of $G_c$. We follow the framework proposed by Hsu, but our definitions and proofs differ from those proposed in [14]. As we have already mentioned, we exploit the ideas invented by Spinrad [25] that allow us to characterize some parts of circular-arc graphs in terms of two-dimensional orders. Eventually, we develop a decomposition tree representing all oriented conformal models of $G_c$.

3. Preliminaries

Suppose $G$ is a graph with no twins and no universal vertices. Suppose $\psi$ is a normalized model of $G$ and $\phi$ is a conformal model of $G_c$ associated with $\psi$. Conformal model $\phi$ is represented by means of a circular word $\tau$ over the set of letters $V^* = \{v^0, v^1 : V(G)\}$. The circular word $\tau$ is obtained from the model $\phi$ as follows. We choose a point $P$ on the circle $C$ and then we traverse $C$ in the clockwise order: if we pass the tail of the chord $\phi(v)$ we append the letter $v^0$ to $\tau$ and when we pass the head of the chord $\phi(v)$ we append the letter $v^1$ to $\tau$. When we encounter $P$ again, we make the word $\tau$ circular. We write $\phi \equiv \tau$ to denote that $\tau$ is a word representation of $\phi$. We consider two conformal models $\phi_1$ and $\phi_2$ of $G_c$ as equivalent, written $\phi_1 \equiv \phi_2$, if the word representations of $\phi_1$ and $\phi_2$ are equal. Usually we use the same symbol to denote a conformal model of $G_c$ and its word representation. Figure 4 shows a circular-arc graph $G = (V, E)$, where $V = \{v_1, \ldots, v_6\}$ and $E = \{v_ivi_{i+1} : i \in [5]\} \cup \{v_6v_1\}$, its normalized model $\psi$, and a conformal model $\phi$ of $G_c$ associated with $\psi$. 
Let $\psi$ be a circular-arc model of $G$. Let $L$ be any line in the plane and let $\psi^R$ be the reflection of $\psi$ over $L$. Clearly, $\psi^R$ is also a circular-arc model of $G$. Now, suppose $\phi$ and $\phi^R$ are conformal models of $G$ associated with $\psi$ and $\psi^R$, respectively, represented by means of circular words $\phi$ and $\phi^R$ on $V^*$. Note that $\phi^R$ is obtained from $\phi$ as follows: we traverse the circular word $\phi$ in the anti-clockwise order and we replace $v^0$ by $v^1$ and $v^1$ by $v^0$, for every $v \in V$. The conformal model $\phi^R$ obtained this way is called the reflection of $\phi$. Note the following relation between $\phi$ and $\phi^R$: for every $uv \in E(G_c)$ the circular circular word $u^0v^0u^1v^1$ appears in $\phi$ iff the circular word $u^0v^1u^1v^0$ appears in $\phi^R$.

Let $\phi$ be a conformal model of $G_c$. We assume that the left side of $\phi(u)$ is the part of the circle on the left side of $\phi(u)$ when we traverse $\phi(u)$ from $\phi(u^0)$ to $\phi(u^1)$. The right side of $\phi(u)$ is the opposite side of the left side of $\phi(u)$.

Let $G = (V, E)$ be a circular-arc graph with no twins and no universal vertices and let $\phi$ be a conformal model of $G_c$. The elements of $V$ are called letters, the elements of $V^* = \{v^0, v^1 : v \in V\}$ are called labeled letters. Given a set $A' \subset V^*$, we denote by $\phi|A'$ either a circular word $\phi$ restricted to the extended letter from $A'$ or a set of all maximum contiguous subwords of the circular word $\phi$ containing all the extended letters from $A'$. Usually the meaning of $\phi|A'$ is clear from the context; otherwise, we say explicitly whether $\phi|A'$ is a circular word or a set of contiguous subwords of $\phi$. We say $\phi|A'$ forms $k$ contiguous subwords in $\phi$ if the set $\phi|A'$ has exactly $k$ elements. If $k = 1$, we say that $\phi|A'$ is a contiguous subword of the circular word $\phi$. Let $A$ be a subset of $V$. By $A^*$ we denote the set $\{v^0, v^1 : v \in A\}$. We abbreviate and we denote $\phi|A^*$ by $\phi|A$. Let $u'$ and $v'$ be labeled letters in a circular word $\phi$. We say that a labeled letter $u'$ is between $u'$ and $v'$ in $\phi$ if we pass $u'$ when we traverse $\phi$ from $\phi(u')$ to $\phi(v')$ in the clockwise order.

![Figure 4](image_url)

**Figure 4.** Circular-arc graph $G$, its normalized representation $\psi$, and associated conformal model $\phi$.

To become familiar with our notation consider an example given in Figure 4. Consider the following model $\psi$ of $G$: the arcs of $\psi$ occur in the order $\psi(v_1), \psi(v_2), \psi(v_3), \psi(v_4), \psi(v_5), \psi(v_6)$ when we traverse $C$ in the clockwise order starting from the point in $\psi(v_1) \setminus \bigcup_{j=1}^5 \psi(v_j)$. A conformal model $\phi$ associated with $\psi$ has the
form
\[ \phi \equiv v_2^0 v_1^1 v_3^0 v_2^1 v_4^0 v_3^1 v_5^0 v_6^0 v_5^1 v_4^1 v_1^0 v_6^1. \]

For \( A' = \{v_1^0, v_1^1, v_6^0, v_6^1\} \), \( \phi|A' \) equals to the circular word \( v_6^0 v_1^1 v_6^0 v_1^0 \) if \( \phi|A' \) is treated as the circular subword of \( \phi \) or equals to the set of words \( \{v_6^0, v_1^0, v_6^1, v_1^1\} \) if \( \phi|A' \) is treated as the set of all contiguous subwords in \( \phi \). The circular word \( \phi|A \), where \( A = \{u_1, u_3\} \), is equal to the circular word \( \phi|A' \). The labeled letters \( v_6^0, v_5^1, v_1^0, v_6^1, v_2^0, v_1^0, v_3^0 \) are between \( v_4^1 \) and \( v_2^1 \) in \( \phi \) and the labeled letters \( v_4^0, v_3^1, v_5^0 \) are between \( v_3^1 \) and \( v_1^1 \) in \( \phi \).

Let \( A \subset V \) and let \( A' \subset A^* \). If \( A' \) is such that \( A \) contains exactly one labeled letter \( v' \in \{v^0, v^1\} \) for every \( v \in A \), then \( A' \) is called a labeled copy of \( A \). We say that a word \( \tau \) is a labeled permutation of \( A \) if the set containing all the labeled letters from \( \tau \) is a labeled copy of \( A \). If \( A' \) is a labeled copy of \( A \) (or \( \tau \) is a labeled permutation of \( A \)), by \( u^* \) we denote the labeled letter \( u^j \in \{u^0, u^1\} \) such that \( u^j \in A' (u_j \in \tau) \), for \( u \in A \).

A circle model of a graph \( G_c = (V_c, E_c) \) is a collection of chords \( \{\psi(u) : u \in V\} \) of a given circle \( C \) such that \( \psi(u) \) intersects \( \psi(v) \) if and only if \( uv \in E_c, u, v \in V \). A graph \( G_c \) is a circle graph if \( G_c \) admits a circle model. The word representation \( \tau \) of \( \psi \) is obtained similarly to the word representation of a conformal model of a circular graph, except that we append \( v \) to \( \tau \) whenever we pass the end of the chord \( \psi(v) \) for \( v \in V \). We write \( \psi \equiv \tau \) if \( \tau \) is a word representation of \( \psi \). Two circle models \( \psi_1 \) and \( \psi_2 \) of \( G_c \) as equivalent, written \( \psi_1 \equiv \psi_2 \), if their word representations are equal. The reflection \( \psi^R \) of a circle model \( \psi \) of \( G_c \) is defined analogously.

We use similar notations for circle graphs \( G_c \) and their circle models as for conformal models \( \phi \).

Suppose \( G = (V, E) \) is a graph with no twins and no universal vertices and \( G_c \) is the graph associated with \( G \). We denote \( G_c \) by \( (V, \sim) \), where \( \sim \) is the set of all edges in \( G_c \). We denote the complement \( \overline{G}_c \) of \( G_c \) by \( (V, \|) \). If \( U \) is a subset of \( V \), by \( (U, \sim) \) we denote the subgraph of \( G_c \) induced by \( U \) and by \( (U, \|) \) we mean the subgraph of \( \overline{G}_c \) induced by \( U \). For two sets \( U_1, U_2 \subseteq V \), we write \( U_1 \sim U_2 \) (\( U_1 \| U_2 \)) if \( u_1 \sim u_2 \) (\( u_1 \| u_2 \), respectively) for every \( u_1 \in U_1 \) and \( u_2 \in U_2 \).

In the rest of the paper we will require an analogue of Theorem 2.7 extended on induced subgraphs of \( G \) and \( G_c \).

**Definition 3.1.** Suppose \( U \) is a non-empty subset of \( V(G) \). A circular-arc model \( \psi \) of \( G[U] \) is normalized if every pair of vertices \( (u, v) \) from \( U \) satisfies conditions 2.7.1-5.

Note that \( (u, v) \) satisfy conditions 2.4.1-5 with respect to \( M_G \), not with respect to \( M_{G[U]} \). In particular, for any non-empty subset \( U \) of \( V(G) \), if \( \psi \) is a normalized model of \( G \), then \( \psi \) restricted to \( U \) is a normalized model of \( G[U] \).

**Definition 3.2.** Let \( U \) be a non-empty subset of \( V(G) \). An oriented circle model \( \phi \) of \( (U, \sim) \) is conformal if for every \( v \in U \) the oriented chords \( \phi(u) \) for \( u \in \text{left}(v) \cap U \)
lie on the left side of \( \phi(v) \) and the chords \( \phi(u) \) for \( u \in \text{right}(v) \cap U \) lie on the right side of \( \phi(v) \).

**Theorem 3.3.** Let \( G \) be a circular-arc graph and let \( U \) be a non-empty subset of \( V(G) \). There is a one-to-one correspondence between normalized models of \( G[U] \) and conformal models of \( (U, \sim) \).

Note that the above theorem does not hold if we assume Hsu’s definition of conformal models.

4. **Tools**

4.1. **The structure of all representations of a circle graph.** The description of the structure of all circle models of circle graphs, presented in this subsection, is taken from the article by Chaplick, Fulek, and Klavík [4]. The concept of split decomposition is due to Cunningham [5], Theorem 4.1 is due to Gabor, Supowit, and Hsu [12], relation \( \sim \) is due to Chaplick, Fulek, and Klavík [4], which were inspired by Naji [23] (see [1] for more details), maximal splits are due to Chaplick, Fulek, and Klavík [4].

Suppose \( G_c \) is a connected circle graph. A **split** in \( G_c \) is a tuple \((A, \alpha(A), B, \alpha(B))\), where:

- The sets \( A, B, \alpha(A), \alpha(B) \) form a partition of \( V(G_c) \),
- We have \( A \neq \emptyset \) and \( B \neq \emptyset \), but possibly \( \alpha(A) = \emptyset \) or \( \alpha(B) = \emptyset \),
- We have \( A \sim B \),
- We have \( \alpha(A) \parallel (B \cup \alpha(B)) \) and \( \alpha(B) \parallel (A \cup \alpha(A)) \).

Since \( G_c \) is connected, \((A, \alpha(A), B, \alpha(B))\) is determined uniquely by \( A \) and \( B \). Hence, without loosing any information, we say \((A, \alpha(A), B, \alpha(B))\) is just the split between \( A \) and \( B \), and we denote \((A, \alpha(A), B, \alpha(B))\) simply by \((A, B)\).

A split \((A, B)\) is **non-trivial** if \(|A \cup \alpha(A)| \geq 2\) and \(|B \cup \alpha(B)| \geq 2\); otherwise \((A, B)\) is **trivial**.

**Theorem 4.1 ([12]).** If \( G_c \) has no non-trivial split, \( G_c \) has only two circle models, one being the reflection of the other.

On the other hand, if \( G_c \) has non-trivial splits, \( G_c \) may have many non-equivalent circle models.

A split in \( G_c \) between \( A \) and \( B \) is **maximal** if there is no split in \( G_c \) between \( A' \) and \( B' \), where \( A' \) and \( B' \) are such that \( A \subseteq A' \), \( B \subseteq B' \), and \( |A| < |A'| \) or \(|B| < |B'| \). Lemma 1 of [4] provides the following characterization of maximal splits in \( G_c \): a split between \( A \) and \( B \) is maximal if and only if there exists no connected component \( C \) in the graph \((\alpha(A), \sim)\) such that for every vertex \( u \in C \) either \( u \sim A \) or \( u \parallel A \), and similarly for \( \alpha(B) \) and \( B \). This observation allows us to present the algorithm for computing a maximal split in \( G_c \) (see [4] for more details):

- start with any non-trivial split between \( A \) and \( B \),
• while there exists \( C \) as described in Lemma 1 of \([4]\): if \( C \subseteq \alpha(A) \) set \( A = A \) and \( B = B \cup C' \), and if \( C \subseteq \alpha(B) \), set \( B = B \) and \( A = A \cup C' \), where \( C' \) is the set of these vertices in \( C \) that are adjacent to \( A \) and adjacent to \( B \), respectively,
• return \((A, B)\).

Suppose \((A, B)\) is a maximal split in \( G_c \) produced by the above algorithm. Note that \((A, B)\) might be trivial. Lemma 2 of \([4]\) proves the following property: if \((A, B)\) is a trivial maximal split with \(|A| = \{a\}\) and \( \alpha(A) = \emptyset \), then \( a \) is an articulation vertex in \( G_c \), i.e., \( G_c \setminus a \) is disconnected.

4.2. Structure of circle models of \( G_c \) with respect to a non-trivial maximal split \((A, B)\). Suppose \( G_c \) has a non-trivial maximal split \((A, B)\). Let \( C = A \cup B \). Let \( \sim \) be the smallest equivalence relation on \( C \) containing the pairs:

- \( u \sim v \) if \( uv \notin E(G_c) \),
- \( u \sim v \) if \( u, v \) are connected by a path in \( G_c \) with the inner vertices in \( \alpha(A) \cup \alpha(B) \).

Suppose \( C_1, \ldots, C_k \) are the equivalence classes of \( \sim \). Note that every \( C_i \) is contained either in \( A \) or in \( B \), for every \( i \in [k] \). Hence, \( k \geq 2 \). Note also that for every connected component \( D \) of the graph \((V \setminus C, \sim)\) there is a unique index \( i(D) \in [k] \) such that for every \( u \in D \) and every \( v \in C \), we have \( v \in C_{i(D)} \) if \( u \sim v \). Let \( \alpha(C_1), \ldots, \alpha(C_k) \) be a partition of the set \( V \setminus C \) such that \( \alpha(C_i) \) contains all connected components \( D \) of \((V \setminus C, \sim)\) such that \( i(D) = i \). Let \( G_i \) be a graph obtained from \( G_c \) by contracting the vertices of \( V \setminus (C_i \cup \alpha(C_i)) \) into a single vertex \( v_i \). Thus, \( G_i \) is such that \( V(G_i) = C_i \cup \alpha(C_i) \cup \{v_i\}, v_i v \in E(G_i) \) for every \( v \in C_i \), \( v_i v \notin E(G_i) \) for every \( v \in \alpha(C_i) \), \( uv \in E(G_i) \) if \( u \sim v \) for every \( u, v \in C_i \cup \alpha(C_i) \). Note that every circle model of \( G_i \) has the form \( v_i \tau_i v_i \tau_i' \), where for every \( v \in C_i \) both \( \tau \) and \( \tau' \) contain the letter \( v \) exactly once and for every \( v \in \alpha(C_i) \) both the letters \( v \) are contained either in \( \tau \) or in \( \tau' \). The next theorem describes the relationship between the set of all circle models of \( G_c \) and the set of all circle models of \( G_i \).

**Theorem 4.2** (Proposition 1 from \([4]\)). The following statements hold:

1. If \( v_i \tau_i v_i \tau_i' \) is a circle model of \( G_i \) for \( i \in [k] \) and \( i_1, \ldots, i_k \) is a permutation of the set \([k]\), then

\[
\tau \equiv \tau_{i_1} \cdots \tau_{i_k} \tau_{i_k'} \cdots \tau_{i_1}'
\]

is a circle model of \( G_c \).

2. If \( \tau \) is a circle model of \( G_c \), then

\[
\tau \equiv \tau_{i_1} \cdots \tau_{i_k} \tau_{i_1}' \cdots \tau_{i_k}'
\]

where \( i_1, \ldots, i_k \) is a permutation of \([k]\) and \( v_{i_j} \tau_{i_j} v_{i_j} \tau_{i_j}' \) is a circle model of \( G_{i_j} \), for \( j \in [k] \).
4.3. Structure of circle models of $G_c$ with respect to a trivial maximal split $(A, B)$. Suppose $(A, B)$ is a trivial maximal split in $G_c$. Without loss of generality we assume that $A = \{a\}$ and $\alpha(A) = \emptyset$. We recall that $a$ is an articulation of $G_c$ by Lemma 2 in [4]. Suppose that $D_1, \ldots, D_k$ are connected components of $G_c \setminus a$, for some $k \geq 2$. Let $C_i = \{v \in D_i : v \sim a\}$ and $\alpha(C_i) = \{v \in D_i : c \parallel a\}$. Let $G_i$ be the restriction of $G_c$ to the set $\{a\} \cup C_i \cup \alpha(C_i)$, i.e., $G_i = (\{a\} \cup C_i \cup \alpha(C_i), \sim)$. Note that every circle model of $G_i$ has the form $a\tau_ia\tau_i'$, where for every $v \in C_i$ the letter $v$ is contained exactly once in both $\tau_i$ and $\tau_i'$ and for every $v \in \alpha(G_i)$ both the letters $v$ are contained either in $\tau_i$ or in $\tau_i'$. The next theorem describes the relation between the set of all circle models of $G_c$ and the set of all circle models of $G_i$.

Theorem 4.3 (Proposition 2 in [4]). The following statements hold:

1. If $a\tau_ia\tau_i'$ is a circle model of $G_i$ for $i \in [k]$ and $i_1, \ldots, i_k$ is a permutation of $[k]$, then

$$\tau \equiv a\tau_{i_1} \ldots \tau_{i_k} a\tau_{i_k}' \ldots \tau_{i_1}'$$

is a circle model of $G_c$.

2. If $\tau$ is a circle model of $G_c$, then

$$\tau \equiv a\tau_{i_1} \ldots \tau_{i_k} a\tau_{i_k}' \ldots \tau_{i_1}'$$

where $i_1, \ldots, i_k$ is a permutation of $[k]$ and $a\tau_{i_j}a\tau_{i_j}'$ is a circle model of $G_{i_j}$ for $j \in [k]$.

Note that the above theorem is valid for any vertex $a$ in $G_c$ such that $a$ is an articulation point in $G_c$.

4.4. Modular decomposition of $G_c$. Let $G = (V, E)$ be a graph with no twins and no universal vertices. Let $G_c$ be a graph associated with $G$.

A non-empty set $M \subseteq V$ is a module in $G_c$ if $x \sim M$ or $x \parallel M$ for every $x \in V \setminus M$. The singleton sets and the whole $V$ are the trivial modules of $G_c$. The graph $(U, \sim)$ is prime if $(U, \sim)$ has no modules other than the trivial ones.

A module $M$ of $G_c$ is strong if $M \subseteq N$, $N \subseteq M$, or $M \cap N = \emptyset$ for every other module $N$ in $G_c$. In particular, two strong modules are either nested or disjoint. The modular decomposition of $G_c$, denoted by $\mathcal{M}(G_c)$, is the family of all strong modules of $G_c$. The modular decomposition $\mathcal{M}(G_c)$ ordered by inclusion forms a tree in which $V$ is the root, the children of a strong module $M$ are the maximal proper subsets of $M$ from $\mathcal{M}(G_c)$, and the leaves are the singleton modules $\{x\}$ for $x \in V$.

The children of a non-singleton module $M \in \mathcal{M}(P)$ form a partition of $M$. A module $M \in \mathcal{M}(G_c)$ is serial if $M_1 \sim M_2$ for any two children $M_1$ and $M_2$, parallel if $M_1 \parallel M_2$ for any two children $M_1$ and $M_2$, and prime otherwise. Equivalently, $M \in \mathcal{M}$ is serial if $(M, \sim)$ is disconnected, parallel if $(M, \sim)$ is disconnected, and prime if both $(M, \sim)$ and $(M, \parallel)$ are connected.
4.5. Permutation subgraphs in $G_c$ and the structure of its permutation models. Let $G = (V, E)$ be a graph with no twins and no universal vertices. Let $G_c$ be a graph associated with $G$ and let $U \subset V$. The graph $(U, \sim)$ is a permutation subgraph of $G_c$ if there exists a pair $(\tau^0, \tau^1)$, where $\tau^0$ and $\tau^1$ are permutations of $U$, such that

$$x \sim y \iff x \text{ appears before } y \text{ in both } \tau^0 \text{ and } \tau^1, \text{ or } y \text{ appears before } x \text{ in both } \tau^0 \text{ and } \tau^1.$$

If this is the case, $(\tau^0, \tau^1)$ is called a permutation model of $(U, \sim)$. Note a difference between our definition and the usual definition for permutation graphs, in which $x \sim y$ iff the relative position of $x$ and $y$ is mixed in $\tau^0$ and $\tau^1$. An explanation for our approach is hidden in Claim 4.5.

Note that whenever $(U, \sim)$ is a permutation subgraph of $G_c$ with a permutation model $(\tau^0, \tau^1)$, then $(U, ||)$ is a permutation subgraph of $\overline{G_c}$ with permutation models $(\tau^0_R, \tau^1)$ and $(\tau^0, \tau^1_R)$, where $\tau^m$ is the reverse of the word $\tau^m$, for $m \in \{0, 1\}$.

**Definition 4.4.** A module $U$ of $G_c$ is proper if there is a vertex $x \in V \setminus U$ such that $x \sim U$.

The next claim shows that every proper module $U$ in $G_c$ induces a permutation subgraph in $G_c$.

**Claim 4.5.** Suppose $U$ is a proper module in $G_c$ such that $x \sim U$ for some $x \in V \setminus U$. Then, for any circle model $\psi$ of $G_c$,

$$\psi((U \cup \{x\}) \equiv x\tau x\tau',$$

where $(\tau, \tau')$ and $(\tau', \tau)$ are permutation models of $(M, \sim)$. In particular, $(U, \sim)$ is a permutation subgraph of $G_c$.

**Proof.** Let $\psi$ be a circle model of $G_c$. Note that every chord $\psi(u)$ for $u \in U$ has its endpoints on different sides of the chord $\psi(x)$. Thus, $\psi((M \cup \{x\}) \equiv x\tau x\tau'$, where $\tau$ and $\tau'$ are permutations of $U$. Clearly, for every $u, v \in U$, $\psi(u)$ intersect $\psi(v)$ iff either $u$ appears before $v$ in both $\tau$ and $\tau'$ or $v$ appears before $u$ in both $\tau$ and $\tau'$. In particular, both $(\tau, \tau')$ and $(\tau', \tau)$ are permutation models of $(U, \sim)$. $\square$

Let $U$ be a proper module in $G_c$. Now, our goal is to describe all permutation models of $(U, \sim)$. To accomplish our task we use the modular decomposition of $(U, \sim)$. Note that the modular decomposition of $\mathcal{M}(U, \sim)$ is associated with $\mathcal{M}(G_c)$ by the following equation:

$$\mathcal{M}(U, \sim) = \{ M \in \mathcal{M}(G_c) : M \subseteq U \} \cup \{ U \}.$$

An orientation $(U, \prec)$ of $(U, \sim)$ is a binary relation on $U$ such that

$$u \prec v \iff \text{ either } u \prec v \text{ or } v \prec u, \text{ for } u, v \in U.$$

In other words, an orientation $(U, \prec)$ arises by orienting every edge $u \sim v$ of $(U, \sim)$ from $u$ to $v$ (denoted $u \prec v$) or from $v$ to $u$ (denoted $v \prec u$). An orientation $(U, \prec)$ of $(U, \sim)$ is transitive if $\prec$ is a transitive relation on $U$. 

Theorem 4.6 ([13]). If $M_1, M_2 \in \mathcal{M}(U, \sim)$ are such that $M_1 \sim M_2$, then every transitive orientation $(U, \prec)$ satisfies either $M_1 \prec M_2$ or $M_2 \prec M_1$.

Let $M$ be a strong module in $\mathcal{M}(U, \sim)$. The edge relation $\sim$ restricted to the edges between vertices in different children of $M$ is denoted by $\sim_M$. If $x \sim y$, then $x \sim_M y$ for exactly one module $M \in \mathcal{M}(U, \sim)$. Hence, the set $\{ \sim_M : M \in \mathcal{M}(U, \sim) \}$ forms a partition of the edge set $\sim$ of the graph $(U, \sim)$.

Theorem 4.7 ([13]). There is a one-to-one correspondence between the set of transitive orientations $(U, \prec)$ of $(U, \sim)$ and the families

$$\{(M, \prec_M) : M \in \mathcal{M}(U, \prec) \text{ and } \prec_M \text{ is a transitive orientation of } (M, \sim_M)\}$$

given by $x < y \iff x \prec_M y$, where $M$ is the module in $\mathcal{M}$ such that $x \sim_M y$.

The above theorem asserts that every transitive orientation of $(U, \sim)$ restricted to the edges of the graph $(M, \sim_M)$ induces a transitive orientation of $(M, \sim_M)$, for every $M \in \mathcal{M}(U, \sim)$. On the other hand, every transitive orientation of $(U, \sim)$ can be obtained by independent transitive orientation of $(M, \sim_M)$, for $M \in \mathcal{M}(U, \sim)$. Gallai [13] characterized all possible transitive orientation of strong modules $(M, \sim_M)$.

Theorem 4.8 ([13]). Let $M$ be a prime module in $\mathcal{M}(U, \sim)$. Then, $(M, \sim_M)$ has two transitive orientations, one being the reverse of the other.

A parallel module $(M, \sim_M)$ has exactly one (empty) transitive orientation. The transitive orientations of serial modules $(M, \sim)$ correspond to the total orderings of its children, that is, every transitive orientation of $(M, \sim_M)$ is of the form $M_{i_1} \prec \ldots \prec M_{i_k}$, where $i_1 \ldots i_k$ is a permutation of $[k]$ and $M_1, \ldots, M_k$ are the children of $M$ in $\mathcal{M}(U, \sim)$.

Since $(U, \sim)$ is a proper module in $G_c$, then $(U, \sim)$ admits a permutation model $(\tau^0, \tau^1)$. Note that $(\tau^0, \tau^1)$ orients transitively the graphs $(U, \sim)$ and $(U, \parallel)$ such that

\[
\begin{align*}
  x \prec y & \iff \ x \text{ occurs before } y \text{ in } \tau^0 \text{ and } x \sim y, \\
  x \prec y & \iff \ x \text{ occurs before } y \text{ in } \tau^0 \text{ and } x \parallel y.
\end{align*}
\]

(1)

That is, the order in $(U, \prec)$ and $(U, \prec)$ is consistent with the order in $\tau^0$. On the other hand, given transitive orientations $\prec$ and $\prec$ of $(U, \sim)$ and $(U, \parallel)$, respectively, one can construct a permutation model $(\tau^0, \tau^1)$ of $(U, \sim)$ such that

\[
\begin{align*}
  x \text{ occurs before } y \text{ in } \tau^0 & \iff \ x \prec y \text{ or } x \prec y, \\
  x \text{ occurs before } y \text{ in } \tau^1 & \iff \ x \preceq y \text{ or } y \prec x.
\end{align*}
\]

(2)

Theorem 4.9 ([8]). Let $(U, \sim)$ be a proper submodule of $G_c$. There is a one-to-one correspondence between permutation models $(\tau^0, \tau^1)$ of $(U, \sim)$ and the pairs $(\prec, \prec)$ of transitive orientations of $(U, \parallel)$ and $(U, \sim)$, respectively, established by equations (1) and (2).
4.6. Modular decomposition of $\mathcal{M}(G_c)$ and circle models of $G_c$. In this subsection we describe properties of circle models of $G_c$ with respect to the modular decomposition of the graph $G_c$.

The purpose of the next lemmas is to describe the restrictions of circle models of $G_c$ to the modules $M$ from $\mathcal{M}(G_c)$. For this purpose, for a given module $M \in \mathcal{M}(G_c)$, we define

\[
N[M] = \{x \in V \setminus M : x \sim M\},
\]
\[
C[M] = \text{the connected component of } G_c \text{ containing the module } M.
\]

Note that $C[M]$ is not defined in the case when $M = V$ and $G_c$ is disconnected.

**Claim 4.10.** Suppose $M$ is a proper prime or a proper parallel module in $\mathcal{M}(G_c)$. For any circle model $\psi$ of $G_c$ we have

\[
\psi|\left(M \cup N[M]\right) \equiv \pi \tau \pi' \tau',
\]

where $(\tau, \tau')$ is a permutation model of $(M, \sim)$ and $\pi, \pi'$ are permutations of $N[M]$. In other words, $\psi|M$ forms two contiguous subwords in the circular word $\psi|\left(M \cup N[M]\right)$.

**Proof.** Since $M$ is proper, we can pick $x \in V \setminus M$ such that $x \sim M$. Orient the chord $\psi(x)$ arbitrarily. By Claim 4.5 we have that

\[
\psi|\left(M \cup \{x\}\right) \equiv x^0 \tau x^1 \tau',
\]

where $(\tau, \tau')$ is a permutation model of $(M, \sim)$. We need to show that

\[
\psi|\left(M \cup N[M]\right) \equiv \pi \tau \pi' \tau',
\]

where $\pi, \pi'$ are some permutations of $N[M]$.

Fix $z \in N[M]$ such that $z \neq x$. Suppose for a contradiction that the chord $\psi(z)$ has one of its ends between the ends of the chords corresponding to the letters of $\tau$. That is, suppose that $x^0 \tau_1 z \tau_2 x^1$ is a subword of the circular word $\psi|\left(M \cup \{x, z\}\right)$, where $\tau_1$ and $\tau_2$ are non-empty words such that $\tau_1 \tau_2 = \tau$. Now, consider a partition of $M$ into two sets, $M_1$ and $M_2$:

\[
M_1 = \{u \in M : u \in \tau_1\} \quad \text{and} \quad M_2 = \{u \in M : u \in \tau_2\}.
\]

Since every $\psi(u)$ for $u \in M$ must intersect $\psi(z)$, we have that $x^i \tau'_1 z \tau'_2 x^0$ is a subword of $\psi|\left(M \cup \{z, t\}\right)$, where $\tau'_i$ is a permutation of the letters in $\tau_i$ for every $i \in [2]$. It means, in particular, that every two chords $\psi(u_1)$ and $\psi(u_2)$ for $u_1 \in M_1$ and $u_2 \in M_2$ intersect. So, we have $M_1 \sim M_2$, which contradicts that $M$ is a prime or a parallel module in $\mathcal{M}(G_c)$. \hfill $\square$

**Lemma 4.11.** Suppose $M$ is a proper prime module in $\mathcal{M}(G_c)$. Suppose that $\psi$ is a circle model of $G_c$. Then, $\psi|M$ forms two consecutive subwords in the circular word $\psi|C[M]$.
Proof. By Claim 4.10, $\psi((M \cup N[M])) \equiv \tau \pi \pi' \tau'$, where $\pi, \pi'$ are permutations of $N[M]$. Clearly, $N[M] \subset C[M]$. To complete the proof of the lemma it suffices to show that either $\psi((M \cup \{v\})) \equiv \nu \nu \nu \nu'$ or $\psi((M \cup \{v\})) \equiv \tau \nu \nu \nu'$ for every $v \in C[M] \setminus N[M]$. Assume otherwise. Since $(C[M], \sim)$ is connected and since $M$ is a module in $(C[M], \sim)$, there is $u \in C[M] \setminus N[M]$ such that

\[ \psi((M \cup \{u\})) \equiv \tau_1 u \tau_2 \tau_2' u \tau_1', \]

where $\tau_1, \tau_2$ and $\tau_1', \tau_2'$ are such that $\tau_1 \tau_2 = \tau$ and $\tau_2' \tau_1' = \tau'$, both $\tau_1, \tau_2$ are non-empty or both $\tau_1', \tau_2'$ are non-empty. Since $u \parallel M$, we conclude that $\tau_i$ is a permutation of $\tau_i'$ for $i \in [2]$. Hence, the sets

\[ M_1 = \{ w \in M : w \in \tau_1 \} \quad \text{and} \quad M_2 = \{ w \in M : w \in \tau_2 \} \]

are such that $M_1 \neq \emptyset$, $M_2 \neq \emptyset$, and $M_1 \parallel M_2$. So, $(M, \sim)$ is not connected, which contradicts the fact that $M$ is a prime module in $\mathcal{M}(G_c)$.

Lemma 4.12. Suppose $M$ is a proper parallel module in $\mathcal{M}(G_c)$ with children $M_1, \ldots, M_k$. Then, any circle model $\psi$ of $G_c$ satisfies

\[ \psi|M \equiv \tau_{i_1} \ldots \tau_{i_k} \tau'_{i_k} \ldots \tau'_{i_1}, \]

where $(i_1, \ldots, i_k)$ is a permutation of $[k]$ and $(\tau_{i_j}, \tau'_{i_j})$ is a permutation model of $(M_{i_j}, \sim)$ for every $j \in [k]$. Moreover, for every $j \in [k]$ the set $\psi|M_{i_j}$ consists of two contiguous subwords, $\tau_{i_j}$ and $\tau'_{i_j}$, in the circular word $\psi|C[M]$.

Proof. Since $M$ is proper, we can pick $x \in V \setminus M$ such that $x \sim M$. Since $M$ is parallel, $M_i$ is either serial or prime. In particular, $(M_i, \sim)$ is connected for every $i \in [k]$. Since $x$ is an articulation point in $(M \cup \{x\}, \sim)$, by Theorem 4.13 we have that

\[ \psi((M \cup \{x\})) \equiv x \tau_{i_1} \ldots \tau_{i_k} x \tau^{'}_{i_k} \ldots \tau^{'}_{i_1}, \]

where $(i_1, \ldots, i_k)$ is a permutation of $[k]$ and $(\tau_{i_j}, \tau^{'}_{i_j})$ is a permutation model of $(M_{i_j}, \sim)$ for every $j \in [k]$. Since $M$ is proper, by Claim 4.10 we have that

\[ \psi((M \cup \{x\})) \equiv \pi \tau_{i_1} \ldots \tau_{i_k} \pi^{'} \tau^{'}_{i_1} \ldots \tau^{'}_{i_k}, \]

where $\pi$ and $\pi'$ are permutations of $N[M]$. Using arguments similar to those used in the previous lemma, we prove that $\tau_{i_j}$ and $\tau^{'}_{i_j}$ form two contiguous subwords in the circular word $\psi|C[M]$.

Lemma 4.13. Suppose $M$ is a serial module in $\mathcal{M}(G_c)$ with children $M_1, \ldots, M_k$. Then, any circle model $\psi$ of $G_c$ satisfies

\[ \psi|M \equiv \tau_{i_1} \ldots \tau_{i_k} \tau'_{i_k} \ldots \tau'_{i_1}, \]

where $(i_1, \ldots, i_k)$ is a circular permutation of $[k]$ and $(\tau_{i_j}, \tau'_{i_j})$ is a permutation model of $(M_{i_j}, \sim)$ for every $j \in [k]$. Moreover, for every $j \in [k]$ the set $\psi|M_{i_j}$ consists of two contiguous subwords, $\tau_{i_j}$ and $\tau'_{i_j}$, in the circular word $\psi|C[M]$.
Proof. Since $M$ is serial, $M_i$ is either prime or parallel. Moreover, since $x \sim M_i$ for every $x \in M \setminus M_i$, $M_i$ is proper. From Theorem 4.10 we deduce that
\[
\psi[M] \equiv \tau_{11} \ldots \tau_{ik} \tau_{i1}^j \ldots \tau_{ik}^j,
\]
where $(i_1, \ldots, i_k)$ is a circular permutation of $[k]$ and $(\tau_{ij}, \tau_{ij}^j)$ is a permutation model of $(M_{ij}, \sim)$ for every $j \in [k]$. Now, it remains to prove that $\tau_{ij}$ and $\tau_{ij}^j$ are two contiguous subwords in $\psi[C[M]$. From Claim 4.10 applied to every child of $M$, for every $x \in N[M]$ we must have
\[
\psi(M \cup \{x\}) \equiv \tau_{i1} \ldots \tau_{ij} x \tau_{ij+1} \ldots \tau_{ik} \tau_{i1}^j \ldots \tau_{ij}^j x \tau_{ij+1} \ldots \tau_{ik}^j
\]
for some $j \in [k]$. Assume that $\psi[M_{ij}]$ does not form two contiguous subwords in $\psi[C[M]$ for some $j \in [k]$. By the above observation and by the connectivity of $(C[M], \sim)$, there is $u \in C[M] \setminus N[M]$ such that $u \parallel M$ and $\psi[M_{ij}] \equiv \tau_{ij} u \tau_{ij}^j u \tau_{ij}^j$, where $\tau_1, \tau_2$, and $\tau_1^i, \tau_2^i$ are such that $\tau_1 \tau_2 = \tau_{ij}$ and $\tau_2^i \tau_1^i = \tau_{ij}^j$, and both $\tau_1, \tau_2$ or both $\tau_1^i, \tau_2^i$ are non-empty. Since $u \parallel M_{ij}$, $\tau_1$ is a permutation of $\tau_i$ for $i \in [2]$. However, $\psi(u)$ must intersect every chord from $\psi(M \setminus M_{ij})$, which contradicts $u \parallel M$. \qed

5. The structure of all conformal models of $G_c$

In this section we describe the structure of all conformal models of $G_c$. We split our work into subsections that cover the following cases:

- Subsection 5.2 $\overline{G_c}$ is disconnected, which corresponds to the case when $V$ is a serial module in $\mathcal{M}(G_c)$.
- Subsection 5.3 $\overline{G_c}$ and $G_c$ are connected, which corresponds to the case when $V$ is an improper prime module in $\mathcal{M}(G_c)$.
- Subsection 5.4 $\overline{G_c}$ is connected, which corresponds to the case when $V$ is an improper parallel module in $\mathcal{M}(G_c)$.

Moreover, in the subsequent subsections we describe all conformal models of the subgraph of $G_c$ induced by $M$, where $M$ is a serial, an improper prime, and an improper parallel module in $\mathcal{M}(G_c)$, respectively. We start with Subsection 5.1 in which we examine the restrictions of conformal models of $G_c$ to proper prime and proper parallel modules in $\mathcal{M}(G_c)$.

5.1. Proper prime and proper parallel modules of $G_c$. The ideas presented in this subsection form a natural extension of Spinrad’s work done for circular-arc graphs with clique cover two [25].

Suppose $M$ is a proper prime or a proper parallel module in $\mathcal{M}(G_c)$. Suppose $C[M]$ is the connected component containing $M$. Pick a vertex $r$ in $M$ and call it a representant of $M$.

Let $\phi$ be a conformal model of $G_c$ and let $\phi'$ be the restriction of $\phi$ to $C[M]$. By Lemmas 4.11 and 4.12 $\phi'|M$ forms two contiguous subwords, $\tau_{\phi}^{0}$ and $\tau_{\phi}^{1}$, in the circular word $\phi'$. Suppose $\tau_{\phi}^{0}$ and $\tau_{\phi}^{1}$ are indexed such that $\tau_{\phi}^{j}$ contains the letter
Suppose also that $M^0_{\phi}$ and $M^1_{\phi}$ are the sets of the letters from the words $\tau^0_\phi$ and $\tau^1_\phi$, respectively. Note that $M^0_{\phi}$ and $M^1_{\phi}$ are labeled copies of $M$ and $\{M^0_{\phi}, M^1_{\phi}\}$ forms a partition of $M^*$. Note that the pair $(\tau^0_\phi, \tau^1_\phi)$ is an oriented permutation model of $(M, \sim)$. The non-oriented permutation model $(\tau^0_\phi, \tau^1_\phi)$ corresponds, according to Theorem 4.9, to a pair of transitive orientations $(<^0_\phi, <^1_\phi)$ of $(M, ||)$ and $(M, \sim)$, respectively. It occurs that the transitive orientation $<^0_\phi$ and the sets $M^0_{\phi}, M^1_{\phi}$ are independent on the choice of a conformal model $\phi$ of $G_c$, which is proven in the next claim.

**Claim 5.1.** Suppose $M$ is a proper prime or a proper parallel module in $\mathcal{M}(G_c)$ with a fixed representant $r \in M$. There is a transitive orientation $(M, <^0_M)$ of $(M, ||)$, there are labeled copies $M^0$ and $M^1$ of $M$ forming a partition of $M^*$, such that

$$(<^0_\phi, M^0_{\phi}, M^1_{\phi}) = (<^0_M, M^0, M^1) \text{ for every conformal model } \phi \text{ of } G_c.$$  

**Proof.** Algorithm [1] shows how to compute the triple $(<^0_M, M^0, M^1)$. We use the following abbreviations to capture the relations between the orientations of the chords $\phi(u)$ and the membership of $u^0, u^1$ in the sets $M^0$ and $M^1$:

- $\phi(u)$ is oriented from $M^0_{\phi}$ to $M^1_{\phi}$ (from $M^1_{\phi}$ to $M^0_{\phi}$) if $u^0 \in M^0_{\phi}$ and $u^1 \in M^1_{\phi}$ ($u^0 \in M^1_{\phi}$ and $u^1 \in M^0_{\phi}$, respectively),
- $u$ is oriented from $M^0$ to $M^1$ (from $M^1$ to $M^0$) if $u^0 \in M^0$ and $u^1 \in M^1$ ($u^0 \in M^1$ and $u^1 \in M^0$, respectively).

The instruction of Algorithm [1] orient $u$ from $M^0$ to $M^1$, means add $u^0$ to $M^0$ and add $u^1$ to $M^1$. The correctness of the algorithm is asserted by the following observations that hold for any conformal model $\phi$ of $G_c$. Let $\phi$ be any conformal model of $G_c$. The chord $\phi(r)$ is oriented from $M^0_{\phi}$ to $M^1_{\phi}$. Let $v, u \in M$ be such that $v || u$. If the chord $\phi(u)$ is oriented from $M^0_{\phi}$ to $M^1_{\phi}$, then:

- if $v \in left(u)$ then $v$ appears before $u$ in $\tau^0_\phi$ and hence $v <^0_\phi u$ holds,
  - if $u \in right(v)$ then $\phi(v)$ is oriented from $M^0$ to $M^1$,
  - if $u \in left(v)$ then $\phi(v)$ is oriented from $M^1$ to $M^0$,
- if $v \in right(u)$ then $v$ appears after $u$ in $\tau^0_\phi$ and hence $u <^0_\phi v$ holds,
  - if $u \in left(v)$ then $\phi(v)$ is oriented from $M^0$ to $M^1$,
  - if $u \in right(v)$ then $\phi(v)$ id oriented from $M^1$ to $M^0$.

We proceed similarly for the case when $u^0 \in M^1$ and $u^0 \in M^1$. \hfill \Box

The tuple $M^0 = (<^0_M, M^0, M^1)$ is called the orientation of $M$. Given the orientation $M^0$ of $M$, we define admissible models for $M^0$, which are oriented models of $(M, \sim)$ that may appear as the restrictions of conformal models of $G_c$ to the sets $M^0$ and $M^1$. 

Algorithm 1: Computing \(<_{M}^{0}, M^{0}, M^{1}\)> for a proper prime/parallel M

\begin{verbatim}
input : a proper prime or a proper parallel module M represented by r
output : \(<_{M}^{0}, M^{0}, M^{1}\>)
1 \(<_{M}^{0} := \emptyset;\)
2 orient r from \(M^{0}\) to \(M^{1}\);
3 bfs\((r, M) := \) breath-first search ordering of \((M, ||)\) starting in \(r;\)
4 \foreach \(u \in M\) such that \(u \parallel v\) and \(u < v\) in \(bfs(r, M)\) do
5   if \(u\) is oriented from \(M^{0}\) to \(M^{1}\) then
6     if \(v \in right(u)\) then
7       add \((u, v)\) to \(<_{M}^{0}\);
8       if \(u \in left(v)\) then orient \(v\) from \(M^{0}\) to \(M^{1}\);
9       if \(u \in right(v)\) then orient \(v\) from \(M^{1}\) to \(M^{0}\);
10      if \(v \in left(u)\) then
11        add \((v, u)\) to \(<_{M}^{0}\);
12        if \(u \in left(v)\) then orient \(v\) from \(M^{1}\) to \(M^{0}\);
13        if \(u \in right(v)\) then orient \(v\) from \(M^{0}\) to \(M^{1}\);
14     if \(u\) is oriented from \(M^{1}\) to \(M^{0}\) then
15       if \(v \in left(u)\) then
16         add \((u, v)\) to \(<_{M}^{0}\);
17         if \(u \in right(v)\) then orient \(v\) from \(M^{1}\) to \(M^{0}\);
18         if \(u \in left(v)\) then orient \(v\) from \(M^{0}\) to \(M^{1}\);
19       if \(v \in right(u)\) then
20         add \((v, u)\) to \(<_{M}^{0}\);
21         if \(u \in left(v)\) then orient \(v\) from \(M^{1}\) to \(M^{0}\);
22         if \(u \in right(v)\) then orient \(v\) from \(M^{0}\) to \(M^{1}\);
23     end
24 end
\end{verbatim}

Definition 5.2. Let \(M\) be a proper prime or a proper parallel module in \(\mathcal{M}(G_{c})\) and let \(M^{0} = \left(<_{M}^{0}, M^{0}, M^{1}\right)\) be the orientation of \(M\). A pair \((\tau^{0}, \tau^{1})\) is an admissible model for \(M^{0}\) if:

- \(\tau^{0}\) is a permutation of \(M^{0}\),
- \(\tau^{1}\) is a permutation of \(M^{1}\),
- \((\tau^{0}, \tau^{1})\) is an oriented permutation model of \((M, \sim)\) that corresponds to the pair \((<, \prec)\) of transitive orientations of \((M, ||)\) and \((M, \sim)\), respectively, where \(< = <_{M}^{0}\) and \(\prec\) is some transitive orientation of \((M, \sim)\).

Hence, \((\phi'|M^{0}, \phi'|M^{1})\) is an admissible model for \(M^{0}\), where \(\phi'\) is the restriction of a conformal model \(\phi\) of \(G_{c}\) to \(C[M]\). Note that \((\phi'|M^{1}, \phi'|M^{0})\) is also an oriented directed model of \((M, \sim)\), \(\phi'|M^{1}\) is a permutation of \(M_{1}\), \(\phi'|M^{0}\) is a permutation
of $M^0$, and $(\phi'|M^1, \phi'|M^0)$ corresponds to the transitive orientations $(<^1_\phi, <^0_\phi)$, where $<^0_\phi$ is the reverse of $<^1_\phi$. In particular, $(\phi'|M^1, \phi'|M^0)$ is admissible for $(<^1_M, M^1, M^0)$, where $<^1_M$ is the reverse of $<^0_M$. We denote $(<^1_M, M^1, M^0)$ by $\mathbb{M}^1$ and we call $\mathbb{M}^1$ as the dual orientation to $\mathbb{M}^0$. Given the above definition, we can summarize the results of this subsection with the following lemma.

**Lemma 5.3.** Suppose $\phi$ is a conformal model of $G_c$, $M$ is a proper prime or a proper parallel module in $\mathcal{M}(G_c)$, and $C[M]$ is the connected component of $G_c$ containing $M$. Then, the pair $(\phi'|M^0, \phi'|M^1)$ is an admissible model for $\mathbb{M}^0$ and the pair $(\phi'|M^1, \phi'|M^0)$ is an admissible model for $\mathbb{M}^1$, where $\phi'$ is the restriction of a conformal model $\phi$ to $C[M]$.

### 5.2. Conformal models of serial modules

Suppose $M$ is a serial module in $G_c$ with children $M_1, \ldots, M_k$. Since $M$ is serial, every $M_i$ is a proper prime or a proper parallel module in $\mathcal{M}(G_c)$. Suppose that a representant $r_i$ of $M_i$ is fixed and $\mathbb{M}^0_i = (<^0_{M_i}, M^1_i, M^0_i)$ and $\mathbb{M}^1_i = (<^1_{M_i}, M^0_i, M^1_i)$ are the orientations of $M_i$. Having in mind Lemmas 4.13 and 5.3 one can obtain a theorem describing all conformal models of $(M, \sim)$. In fact, the following theorem can be seen as a direct application of Spinrad’s results [25] to Hsu’s framework [14].

**Theorem 5.4.** Suppose $M$ is a serial module in $G_c$ with children $M_1, \ldots, M_k$. Every conformal model $\phi$ of $(M, \sim)$ has the form

$$\phi \equiv \tau'_{i_1} \ldots \tau'_{i_k} \tau''_{i_1} \ldots \tau''_{i_k}$$

where $(i_1, \ldots, i_k)$ is a permutation of $[k]$ and for every $j \in [k]$ the pair $(\tau'_{i_j}, \tau''_{i_j})$ is an admissible model of some orientation $\mathbb{M}'_{i_j}$ of $M_{i_j}$.

On the other hand, for every permutation $(i_1, \ldots, i_k)$ of $[k]$, every orientation $\mathbb{M}'_{i_j}$ of $M_{i_j}$, and every admissible model $(\tau'_{i_j}, \tau''_{i_j})$ for $\mathbb{M}'_{i_j} \in \{\mathbb{M}^0_{i_j}, \mathbb{M}^1_{i_j}\}$, a circular word

$$\phi \equiv \tau'_{i_1} \ldots \tau'_{i_k} \tau''_{i_1} \ldots \tau''_{i_k}$$

is a conformal model of $G_c[M]$.

The above theorem can be used to characterize all conformal models of $G_c$ in the case when the graph $\overline{G_c} = (V, \parallel)$ is disconnected. Indeed, in this case $V$ is serial in $(V, \sim)$ and Theorem 5.4 applies.

Theorem 5.4 can be used to give the recognition algorithm for circular-arc graphs $G$ for which $\overline{G_c}$ is disconnected.

**Theorem 5.5 ([14], [25]).** Suppose $G$ is a graph with no twins and no universal vertices. Suppose the graph $G_c$ associated with $G$ is such that $\overline{G_c}$ is disconnected. The following statements are equivalent:

1. $G$ is a circular-arc graph.
2. For every child $M$ of $V$ in $\mathcal{M}(G_c)$ the orientation $(M, <^0_M, M^0, M^1)$ admits an admissible model.
For every child $M$ of $V$ in $\mathcal{M}(G_c)$ the relation $(M, \lessdot_M^0)$ computed by Algorithm 1 is a two-dimensional poset, which means that $(M, \lessdot_M^0)$ is a transitive orientation of $(M, \|)$ and $(M, \sim)$ is transitively orientable.

5.3. Conformal models of improper prime modules. Suppose $G$ is a circular-arc graph with no twins and no universal vertices. Suppose $M$ is an improper prime module in $G_c$. It means that either $M \subseteq V$ and $M$ is a connected component in $G_c$ or $M = V$ and both $G_c$ and $\overline{G_c}$ are connected (corresponding to the case where $V$ is a prime module in $\mathcal{M}(G_c)$). In any case, $M$ is a connected component of $G_c$.

Let $M_1, \ldots, M_k$ be the children of the module $M$. Let $U$ be a $k$-element set containing one element from every $M_i$, $i \in [k]$. Clearly, $(U, \sim)$ contains no non-trivial modules and hence the graph $(U, \sim)$ is prime. The next lemma is crucial for our work. We mention here that an analogous lemma was stated by Hsu [14], however, we prove it again using quite different ideas as we work with conformal models defined in a different way.

**Lemma 5.6.** The graph $(U, \sim)$ has exactly two conformal models, one being the reflection of the other.

*Proof.* Since $G$ is circular-arc graph, $G_c$ has at least two conformal models, from which one is the reflection of the other. Since the restriction of any conformal model of $G_c$ to the set $U$ is conformal for $(U, \sim)$, the graph $(U, \sim)$ has at least two conformal models, from which one is the reflection of the other. Our goal is to prove that these models are the only conformal models of $(U, \sim)$.

We prove the lemma by induction on the number of vertices in $(U, \sim)$. The smallest prime graph has 4 vertices. The only prime graph with 4 vertices is the path $P_4$. One can easily check that $P_4$ has two conformal models, one being the reflection of the other. This proves the base of the induction.

Suppose that $(U, \sim)$ has at least 5 vertices. If $(U, \sim)$ has no splits, by Theorem [17], the graph $(U, \sim)$ has two circle models, $\phi$ and $\phi^R$, where $\phi^R$ is the reflection of $\phi$. Note that the chords in $\phi$ can be oriented in a unique way to get a conformal model of $(U, \sim)$. Indeed, as $(U, \sim)$ is prime, for every vertex $v \in U$ there is $u \in U$ such that $u \parallel v$. Now, the orientation of the chord $\phi(v)$ depends on whether $u \in \text{left}(v)$ or $u \in \text{right}(v)$: if $u \in \text{left}(v)$ ($u \in \text{right}(v)$) we orient $\phi(v)$ such that the oriented chord $\phi(v)$ has $\phi(u)$ on its left (right, respectively) side. Similarly, $\phi^R$ can be oriented in a unique way to get a conformal model of $(U, \sim)$. Clearly, the oriented conformal models $\phi$ and $\phi^R$ are the only possible conformal models for $(U, \sim)$ and $\phi^R$ is the reflection of $\phi$.

Suppose $(U, \sim)$ has a non-trivial split. For this case the general idea of the proof is as follows. We take a maximal split in $(U, \sim)$ and then, using a structure induced by this split, we divide $(U, \sim)$ into so-called probes. A probe is a special proper induced subgraph of $(U, \sim)$ which, as we shall prove, has a unique, up to reflection,
Definition 5.7. A probe in \((U, \sim)\) is a quadruple \((y, x, X, \alpha(X))\) that satisfies the following properties:

1. \(x \neq y, X \neq \emptyset, \alpha(X) \neq \emptyset\), the sets \(\{y, x\}, X, \alpha(X)\) are pairwise disjoint, and the set \(P = \{x, y\} \cup X \cup \alpha(X)\) is a proper subset of \(U\),
2. \(y \sim x, y \parallel X \cup \alpha(X), x \sim X, x \parallel \alpha(X)\), and the graph \((P, \sim)\) is connected,
3. for every \(z \in U \setminus P\), either \(z \parallel (X \cup \alpha(X))\), or \(z \sim X\) and \(z \parallel \alpha(X)\), or \(z \sim (X \cup \alpha(X))\).

Claim 5.8. Let \((y, x, X, \alpha(X))\) be a probe in \((U, \sim)\), let \(P = \{x, y\} \cup X \cup \alpha(X)\). Then, \((P, \sim)\) has a unique, up to the reflection, conformal model.

Proof. Let \(Z = \{z \in X : z\) has only one neighbour in the graph \((P, \sim)\}\). Note that the only neighbor of \(z \in Z\) is the vertex \(x\). Note also that \(|Z| \leq 1\) as otherwise \(Z\) would be a non-trivial module in \((U, \sim)\) by property (3). We claim that:

- If \(|Z| = 1\), then \(\{y\} \cup Z\) is the only non-trivial module in \((P, \sim)\).
- If \(Z = \emptyset\), then \((P, \sim)\) has no non-trivial modules.

Suppose \(M\) is a non-trivial module in \((P, \sim)\). We consider four cases depending on the intersection of \(M\) with the set \(\{y, x\}\).

Suppose \(M \cap \{y, x\} = \emptyset\). Since \(x \notin M\) and since \(x \sim X\) and \(x \parallel \alpha(X)\), we must have either \(M \subseteq X\) or \(M \subseteq \alpha(X)\). Then, by property (3) of \(P\), every \(u \in U \setminus P\) satisfies either \(u \sim M\) or \(u \parallel M\). So, \(M\) is also a non-trivial module in \((U, \sim)\), which contradicts the assumption of the lemma.

Suppose \(M \cap \{y, x\} = \{y, x\}\). Since \(X \sim x\) and \(X \parallel y\), we must have \(X \subseteq M\). Since \((P, \sim)\) is connected, we need to have \(\alpha(X) \subseteq M\) as otherwise we would find a vertex \(u \in P \setminus M\) such that \(u\) is adjacent to some vertex in \(M\) and is non-adjacent to some vertex in \(M\). So, \(M = P\), which contradicts that \(M\) is a non-trivial module in \((P, \sim)\).

Suppose \(M \cap \{y, x\} = \{x\}\). Since \(y \sim x\) and \(y \parallel X \cup \alpha(X)\), we must have \(M \cap (X \cup \alpha(X)) = \emptyset\). It follows that \(M\) is trivial in \((P, \sim)\), a contradiction.

Suppose \(M \cap \{y, x\} = \{y\}\). Note that \(M \cap \alpha(X) = \emptyset\). Otherwise, \(x\) from outside \(M\) is adjacent to \(y\) in \(P\) and non-adjacent to a vertex in \(M \cap \alpha(X))\), which can not be the case. Let \(M_X = M \cap X\). If \(M_X = \emptyset\), then \(M = \{y\}\) and \(M\) is trivial. So, we must have \(M_X \neq \emptyset\). Notice that for every vertex \(t \in (X \cup \alpha(X)) \setminus M_X\) we have that \(t \parallel M_X\). Otherwise, \(t\) from outside \(M\) would have a neighbor in \(M\) and the non-neighbor \(y\) in \(M\). If \(|M_X| \geq 2\), then by property (3) of \(P\), \(M_X\) would be a non-trivial module in \((U, \sim)\), which can not be the case. So, \(M\) might be a non-trivial module of \((P, \sim)\) only when \(|M_X| = 1\), i.e., when \(M_X = \{z\}\) for some \(z \in X\). In this case, \(z\) is adjacent only to the vertex \(x\) in \((P, \sim)\), which shows \(Z = \{z\}\). So, we must have \(M = \{y, z\}\), which completes the proof of our subclaim.

Now, we show that \((P, \sim)\) has a unique, up to reflection, conformal model.
Suppose $Z = \emptyset$. As we have shown, the graph $(P, \sim)$ contains no non-trivial modules. Since $P$ has strictly fewer vertices than $(U, \sim)$, from the inductive hypothesis we get that $(P, \sim)$ has a unique, up to reflection, conformal model.

Suppose $Z = \{z\}$. Then $\{y, z\}$ is the only non-trivial module in $(P, \sim)$. Thus, the graph $(P \setminus \{z\}, \sim)$ is prime. By the inductive hypothesis, $(P \setminus \{z\}, \sim)$ has exactly two conformal models, $\phi$ and $\phi^R$, where $\phi^R$ is the reflection of $\phi$. Note that the vertex $x$ is an articulation point in the graph $(P \setminus \{z\}, \sim)$. Suppose that $(P \setminus \{z\}, \sim)$ has exactly $k$ connected components, $\phi$, such that the vertex $z$ is an articulation point in the graph $(\phi \setminus \{\phi\}, \sim)$ for some $k \geq 2$. Note that $D_i = \{y\}$ for some $i \in [k]$. By Theorem 4.3, $\phi \equiv x^0\tau_i x^1\tau'_i \tau''_i$ for $\tau\in \{x\}$ and $x^0\tau_i x^1\tau'_{i,j}$ is a conformal model of $(\{x\} \cup D_{ii}, \sim)$ for $j \in [k]$. We show that there is a unique extension of $\phi$ by the oriented chord $\phi(z)$ such that the extended $\phi$ is conformal for $(P, \sim)$. Clearly, the extended $\phi$ must be of the form:

$$\phi \equiv x^0\tau_i \cdots \tau_i z' \tau_{i+1} \cdots \tau_i x^1\tau'_i \cdots \tau''_i \tau'_{i+1} \cdots \tau'_i$$

for some $l \in \{0, \ldots, k\}$, where $z'$ and $z''$ are such that $\{z', z''\} = \{a^0, a^1\}$. For every $i \in [k]$ pick a vertex $a_i$ in the component $D_i$ such that $x \sim a_i$. Note that $\phi(z)$ must be on the left side of $\phi(a_i)$ if $z \in \text{left}(a_i)$ and $\phi(z)$ must be on the right side of $\phi(a_i)$ if $z \in \text{right}(a_i)$. Hence, the place in $\phi$ (or equivalently, the index $l$) for the chord $\phi(z)$ is uniquely determined. The orientation of $\phi(z)$ depends on whether $y \in \text{left}(z)$ or $y \in \text{right}(z)$. 

Suppose $(U, \sim)$ has a non-trivial split. We use the algorithm given in Section 4.3 to compute a maximal split $(A, B)$ in $(U, \sim)$. Depending on whether $(A, B)$ is trivial or not, we assume the following notation:

- if $(A, B)$ is non-trivial, we assume that $C_1, \ldots, C_k$ and $\alpha(C_1), \ldots, \alpha(C_k)$ are such as defined in Subsection 4.2.
- if $(A, B)$ is trivial, we assume that $A = \{a\}$ and that $C_1, \ldots, C_k$ and $\alpha(C_1), \ldots, \alpha(C_k)$ are such as defined in Subsection 4.3.

We partition the set $[k]$ into two subsets: $I_1, I_2$, such that:

- $i \in I_1$ if $|C_i \cup \alpha(C_i)| = 1$,
- $i \in I_2$ if $|C_i \cup \alpha(C_i)| \geq 2$.

Note that $|I_1| \leq 1$ as otherwise $\bigcup_{i \in I_1} C_i$ would be a non-trivial module in $(U, \sim)$. Without loss of generality we assume $C_1, \ldots, C_k$ are enumerated such that $I_1 = \{k\}$ if $I_1 \neq \emptyset$. Let $i \in I_2$. Note that $\alpha(C_i) \neq \emptyset$ as otherwise $C_i$ would be a non-trivial module in $(U, \sim)$. Moreover, since $(U, \sim)$ is connected, some vertex in $C_i$ is adjacent to some vertex in $\alpha(C_i)$. Hence, for every $i \in [k]$ we can pick two vertices $a_i, b_i \in C_i \cup \alpha(C_i)$ such that:

- $a_i \subseteq C_i$, $b_i \subseteq \alpha(C_i)$, and $a_i \sim b_i$ if $i \in I_2$,
- $a_i = b_i$, where $\{a_i\}$ is the only vertex in $A_i$, if $i \in I_1$. 
We split the proof into two cases, depending on whether or not the following condition is satisfied:

\[(*)\] 

For every \(i \in I_2\) there exist \(x, y \in U \setminus (C_i \cup \alpha(C_i))\) such that 

\((x, y, C_i, \alpha(C_i))\) is a probe in \((U, \sim)\).

We claim that condition \((*)\) is not satisfied only when \((A, B)\) is a trivial split, \(k = 2\), and \(|C_2 \cup \alpha(C_2)| = 1\). Let \(i \in I_2\). Suppose \(k \geq 3\). If \((A, B)\) is non-trivial, the set \(C_i \cup \alpha(C_i)\) can be extended to a probe by the vertices \(a_j, b_j\), where \(j\) is any index in \(I_2\) different than \(i\). If \((A, B)\) is trivial, the set \(C_i \cup \alpha(C_i)\) can be extended to a probe by the vertices \(a, a_j\), where \(j\) is any index in \([k]\) different than \(i\). Suppose \(k = 2\) and suppose \((A, B)\) is non-trivial. Note that \(|C_j \cup \alpha(C_j)| \geq 3\) for every \(j \in [2]\). Otherwise, the only vertex \(a_j\) in \(C_j\) is adjacent to the only vertex \(b_j \in \alpha(C_j)\), and hence the split \((A, B)\) is not maximal. Hence, the set \(C_i \cup \alpha(C_i)\) can be extended to a probe by the vertices \(a, a_j\), where \(j\) is the index in \([2]\) different than \(i\). If \((A, B)\) is trivial and \(|C_2 \cup \alpha(C_2)| \geq 2\), then the set \(C_i \cup \alpha(C_i)\) can be extended to a probe by the vertices \(a, a_j\), where \(j\) is the index in \([2]\) different than \(i\). So, the only case when condition \((*)\) is not satisfied is when \((A, B)\) is a trivial split, \(k = 2\), \(|C_2| = 1\), and \(|\alpha(C_2)| = 0\).

Suppose \((*)\) is satisfied. Let

\[R_i = \begin{cases} 
\{a_1, a_2, \ldots, a_i, b_i\} & \text{if } (A, B) \text{ is non-trivial,} \\
\{a, a_1, b_1, \ldots, a_i, b_i\} & \text{if } (A, B) \text{ is trivial,}
\end{cases}\]

let

\[S = \begin{cases} 
\{a_1, a_2\} & \text{if } (A, B) \text{ is non-trivial,} \\
\{a, a_1\} & \text{if } (A, B) \text{ is trivial,}
\end{cases}\]

and let \(R = R_k\). Eventually, let

\[\phi^0_R \equiv a_0^0a_1^0a_2^0a_3^0 \quad \text{and} \quad \phi^1_R \equiv a_0^0a_2^1a_1^1a_2^0 \quad \text{if } (A, B) \text{ is non-trivial,}\]

\[\phi^2_R \equiv a_0^0a_2^0a_1^0a_2^1 \quad \text{and} \quad \phi^3_R \equiv a_0^1a_2^0a_1^0 \quad \text{if } (A, B) \text{ is trivial.}\]

In any case, \(\phi^0_R\) is the reflection of \(\phi^1_R\) and any conformal model \(\phi\) of \((U, \sim)\) extends either \(\phi^0_R\) or \(\phi^1_R\). We show that:

- For every conformal model \(\phi^j_R\) of \((R, \sim)\) extending \(\phi^1_R\) there is at most one conformal model \(\phi^j\) of \((U, \sim)\) such that \(\phi^j|R \equiv \phi^j_R\), for every \(j \in [2]\).
- There is a unique conformal model \(\phi^j_R\) of \((R, \sim)\) such that \(\phi^j_R|S = \phi^1_S\), for \(j \in \{0, 1\}\).

Clearly, \(\phi^0_R\) must be the reflection of \(\phi^1_R\). \(\phi^j_R\) must have one extension \(\phi^1\) to a conformal model of \((U, \sim)\) for every \(j \in [2]\), and \(\phi^1\) must be the reflection of \(\phi^0\).

This will show the lemma for the case when condition \((*)\) is satisfied.

Let \(i \in I_2\). Suppose that \(\phi^0\) is a unique conformal model of \((P_i, \sim)\) that extends \(\phi^0_S\), where \(P_i = \{y, x\} \cup C_i \cup \alpha(C_i)\) is a probe in \((U, \sim)\) for some \(x, y \in U \setminus (C_i \cup \alpha(C_i))\). Since the restriction of every conformal model to the set \(P_i\) is conformal, we must have \(\phi^0|P_i \equiv \phi^0\) for every conformal model \(\phi^0\) of \((U, \sim)\) extending \(\phi^0_S\). Hence, we have \(\phi^0|(P_i \setminus y) \equiv \phi^0|P_i \setminus y\). Assume that \(\phi^0|P_i \setminus y \equiv R x' \pi x'' \pi'\).
where \( \{x', x''\} = \{x^0, x^1\} \) and \( \pi_i, \pi'_i \) are chosen such that both the labeled letters of \( b_i \) appear in \( \pi_i \). Note that

\[
either b_0^0a_0^0b_1^1, \ or \ b_1^1a_0^0b_1^0, \ or \ b_1^0a_1^0b_1^1, \ or \ b_1^1a_1^0b_1^0 \ is \ a \ subword \ of \ \pi_i.
\]

Having in mind Theorems 4.2 and 4.3, we conclude that every conformal model \( \phi \) of \((U, \sim)\) extending \( \phi_S^0 \) must be of the form:

\[
(**) \quad \phi \equiv \begin{cases}
\tau_{i_1} \ldots \tau_{i_k} \tau'_{i_1} \ldots \tau'_{i_k} & \text{if } (A, B) \text{ is non-trivial}, \\
\pi^0 \tau_{i_1} \ldots \tau_{i_k} \pi^1 \tau'_{i_1} \ldots \tau'_{i_k} & \text{if } (A, B) \text{ is trivial},
\end{cases}
\]

where \( i_1, \ldots, i_k \) is a permutation of the set \([k]\) and

- \( \{\tau_{i_j}, \tau'_{i_j}\} = \{a_0^0, \ a_1^0\} \), for \( i_j \in I_1 \),
- \( (\tau_{i_j}, \tau'_{i_j}) = (\pi_{i_j}, \pi'_{i_j}) \) or \( (\tau_{i_j}, \tau'_{i_j}) = (\pi'_{i_j}, \pi_{i_j}) \), for \( i_j \in I_2 \).

It means, in particular, that for every conformal model \( \phi_R^0 \) of \((R, \sim)\) extending \( \phi_S^0 \) there exists at most one conformal model \( \phi \) of \((U, \sim)\) that extends \( \phi_R^0 \). We prove similarly that for every conformal model \( \phi_R^0 \) of \((R, \sim)\) extending \( \phi_S^0 \) there is at most one conformal model \( \phi^1 \) such that \( \phi^1|R \equiv \phi_R^0 \).

Now, we show that there is a unique conformal model \( \phi \) of \((R, \sim)\) that extends \( \phi_S^0 \). Suppose \((A, B)\) is non-trivial. We claim that for every \( i \in [2, k] \) there is a unique conformal model \( \phi \) of \((R_i, \sim)\) extending \( \phi_S^0 \). To prove the claim for \( i = 2 \) we need to show that there is a unique extension \( \phi \equiv \phi_S^0 \) by the chords \( \phi(b_1) \) and \( \phi(b_2) \). The chord \( \phi(b_1) \) must be placed such that it intersect one of the ends of \( \phi(a_1) \). If \( b_1 \in left(a_2) \) \( b_1 \in right(a_2) \), \( \phi(b_1) \) must intersect the end of \( \phi(a_1) \) that is on the left (right, respectively) side of \( \phi(a_2) \). The orientation of \( \phi(b_2) \) can be decided based on whether \( a_1 \in left(b_2) \) or whether \( a_1 \in right(b_2) \). We show similarly that the placement and the orientation of \( \phi(b_2) \) are uniquely determined.

Suppose \( \phi \) is a unique conformal model of \((R_{k-1}, \sim)\) extending \( \phi_S^0 \). We show that there is a unique extension of \( \phi \) by the chords \( \phi(a_i) \) and \( \phi(b_i) \). Note that \( \{\{a_1, \ldots, a_{i-1}\}, \sim\} \) is a clique in \((R_k, \sim)\), and hence the chords \( \{\phi(a_1), \ldots, \phi(a_{i-1})\} \) are pairwise intersecting. There are \((i - 1)\) possible placements for the non-oriented chord \( \phi(a_i) \). Every such a placement determines uniquely the partition of \( \{b_1, \ldots, b_{i-1}\} \) into two sets \( A \) and \( B \) such that the chords from \( \phi(A) \) are on one side of the non-oriented chord \( \phi(a_i) \) and the chords from \( \phi(B) \) are on the opposite side of \( \phi(a_i) \). Note that the partitions \( \{A, B\} \) corresponding to different placements of \( \phi(a_i) \) are different: shifting the chord \( \phi(a_i) \) by one chord \( \phi(a_j) \) moves \( b_j \) either from \( A \) to \( B \) or from \( B \) to \( A \). Hence, to keep \( \phi \) conformal, only one placement for \( \phi(a_i) \) can be compatible with the partition

\[
\{\{b_1, \ldots, b_{i-1}\} \cap left(a_i), \{b_1, \ldots, b_{i-1}\} \cap right(a_i)\}.
\]

The orientation of \( \phi(a_i) \) can be decided based on whether \( b_1 \in left(a_i) \) or whether \( b_1 \in right(a_i) \). With a similar ideas to those used earlier, we show that there is a unique extension of \( \phi \) by the chord \( \phi(b_i) \) if we want to keep \( \phi \) conformal.

The case when the split \((A, B)\) is trivial is handled with similar ideas. This completes the proof of the lemma for the case when condition [4] is satisfied.
Now consider the case when condition (i) is not satisfied. This happens when \( k = 2 \) and \(|C_2 \cup \alpha(C_2)| = 1\). That is, we have \( C_2 = \{a_2\} \) and \( \alpha(C_2) = \emptyset \). If \(|C_1 \cup \alpha(C_1)| = 2\), then \( C_1 = \{a_1\} \), \( \alpha(C_1) = \{b_1\} \), and \( U = \{a_1, b_1, a, a_2\} \). In this case \((U, \sim)\) induces \( P_4 \) in \( G_c \), which has a unique, up to reflection, conformal model.

For the remaining of the proof we assume that \(|C_1 \cup \alpha(C_1)| \geq 3\). We consider two cases depending on whether or not \((\{a\} \cup C_1 \cup \alpha(C_1), \sim)\) is prime.

Suppose \((\{a\} \cup C_1 \cup \alpha(C_1), \sim)\) is prime. By the inductive hypothesis, \((\{a\} \cup C_1 \cup \alpha(C_1), \sim)\) has two conformal models, \( \phi \) and its reflection \( \phi^R \). Suppose that \( \phi \equiv a^0 \pi a^1 \pi' \) for some \( \pi, \pi' \). By Theorem 4.3, there are two extensions of \( \phi \) by the chord \( \phi(a_2) \) that lead to a circle model of \((U, \sim)\): \( \phi^1 \equiv a^0 a_2 \pi a^1 \pi' a_2 \) or \( \phi^2 \equiv a^0 \pi a_2 a^1 a_2 \pi' \). Depending on whether \( a_2 \in \text{left}(a_1) \) or \( a_2 \in \text{right}(a_1) \), only one among them can be extended to a conformal model of \((U, \sim)\). The orientation of \( \phi(a_2) \) can be decided based on whether \( a_1 \in \text{left}(a_2) \) or whether \( a_1 \in \text{right}(a_2) \).

Suppose \((\{a\} \cup C_1 \cup \alpha(C_1), \sim)\) has a non-trivial module \( M \). Observe that \( a \in M \). Otherwise, we would have either \( M \subseteq C_1 \) or \( M \subseteq \alpha(C_1) \). In both these cases, \( M \) would be a trivial module of \((U, \sim)\), a contradiction. Note also that \( M \cap C_1 \neq \emptyset \).

Suppose that \( M \cap C_1 = \emptyset \). Clearly, \( M \cap \alpha(C_1) \neq \emptyset \) as \( M \) is a non-trivial module in \((\{a\} \cup C_1 \cup \alpha(C_1), \sim)\). Note that \((M \cap \alpha(C_1)) \sim C_1 \) as otherwise there would be a vertex \( x \in C_1 \) from outside \( M \) that is adjacent to the vertex \( a \) in \( M \) and non-adjacent to a vertex in \((M \cap \alpha(C_1)) \). Moreover, since \( \alpha(C_1) \parallel a \), any vertex from \( \alpha(C_1) \setminus M \) can not be adjacent to a vertex in \( M \cap \alpha(C_1) \). Hence, the split \((A, B)\) is not maximal as the set \( A = \{a\} \) can be extended by \( \alpha(C_1) \cap M \), which can not be the case. Eventually, observe that \( C_1 \cap M = \emptyset \). Otherwise, since \((\{a\} \cup C_1) \subset M \) and \( a \parallel \alpha(C_1) \), one can show by the connectivity of \((U, \sim)\) that \( M = \{a\} \cup C_1 \cup \alpha(C_1) \). Hence, \( M \) would be trivial in \((\{a\} \cup C_1 \cup \alpha(C_1), \sim)\), which is not the case.

For the remaining part of the proof, assume that \( M_1 = M \cap C_1 \) and \( M_2 = C_1 \setminus M \). As we have shown, \( M_1 \neq \emptyset \) and \( M_2 \neq \emptyset \). Note that every connected component \((D, \sim)\) in \((\alpha(C_1), \sim)\) satisfies \( D \subset M \) or \( D \cap M = \emptyset \). We split the connected components \((D, \sim)\) of \((\alpha(C_1), \sim)\) into two groups \( \alpha(M_1) \) and \( \alpha(M_2) \): \( D \subset \alpha(M_1) \) if there is an edge between a vertex in \( D \) and a vertex in \( M_1 \); otherwise \( D \subset \alpha(M_2) \).

Note that for every component \( D \subset \alpha(M_2) \) we have \( D \parallel M \) and there is a vertex in \( D \) adjacent to a vertex in \( M_2 \) as \((U, \sim)\) is connected. Observe that \( D \subset M \) if \( D \subset \alpha(M_1) \). Eventually, note that \( M_2 \sim (M_1 \cup \alpha(M_1)) \) as otherwise there would be a vertex \( y \in M_2 \) from outside \( M \) adjacent to \( a \) in \( M \) and not adjacent to some vertex in \( M \). Summing up, we have that:

- \( M_1 \neq \emptyset \), \( M_2 \neq \emptyset \), and \( \alpha(M_1) \neq \emptyset \) or \( \alpha(M_2) \neq \emptyset \),
- \( M_2 \sim (M_1 \cup \alpha(M_1)) \),
- \( \alpha(M_2) \parallel (M_1 \cup \alpha(M_1)) \),

Note that not necessarily \( \alpha(M_1), \alpha(M_2) = (\alpha(C_1) \cap M, \alpha(C_1) \setminus M) \). It might happen that there is a unique singleton component \((D, \sim)\) in \((\alpha(C_1), \sim)\) such that \( D \subset \alpha(M_2) \) and \( D \subset M \) (in this case \( D \parallel M_1 \) and \( D \sim M_2 \), and hence \( D \) is a
singleton as otherwise $D$ would be a non-trivial module in $(U, \sim)$. For any other component $D$, $D \in \alpha(M_1)$ iff $D \subset M$. Having in mind the above conditions, note that:

- $(a_2, a, M_1, \alpha(M_1))$ is a probe in $(U, \sim)$ if $\alpha(M_1) \neq \emptyset$,
- $(a_2, a, M_2, \alpha(M_2))$ is a probe in $(U, \sim)$ if $\alpha(M_2) \neq \emptyset$.

If $\alpha(M_i) = \emptyset$ then $|M_i| = 1$ as otherwise $M_i$ would be a non-trivial module in $(U, \sim)$.

Let $S = \{a, a_2\}$. We show that there is a unique conformal model $\phi$ of $U$ that extends $\phi_S \equiv a^0a_2^1a_0^2$. Suppose $\alpha(M_1) \neq \emptyset$. Let $P_i = \{a_2, a\} \cup M_i \cup \alpha(M_i)$ for $i \in [2]$. By Claim 5.8, $(P_1, \sim)$ has a unique conformal model $\phi_1$ extending $\phi_S$, which is of the form either $\phi_1 \equiv a^0a_2^1a_0^1\pi_1^0\pi_0^2a^0_2$ or $\phi_1 \equiv a^0a_2^1a_0^1\pi_0^1\pi_0^2a^0_2$. Suppose the first case, that is,

$$\phi_1 \equiv a^0a_2^1\pi_0^1a^1\pi_0^2a^0_2.$$

Since any conformal model $\phi$ of $(U, \sim)$ extending $\phi_S$ must also extend $\phi_1$, $\phi$ must be of the form:

$$\phi \equiv a^0a_2^1\pi_0^1a^1\pi_0^2a_2^0,$$

where for every $u \in C_1$ both words $\tau'_\phi$ and $\tau''_\phi$ contain exactly one labeled letter of $u$ and for every $u \in \alpha(C_1)$ both the labeled letters of $u$ are either in $\tau'_\phi$ or $\tau''_\phi$. Hence, a conformal model $\phi_2$ of $(P_2, \sim)$ extending $\phi_S$ is uniquely determined and has the form

$$\phi_2 \equiv a^0a_2^1\pi_2^1a^1\pi_2^2a_2^0,$$

where the uniqueness of $\pi_2^1, \pi_2^2$ follows from (3) and from Claim 5.8 if $\alpha(M_2) \neq \emptyset$ and from $|M_2| = 1$ if $\alpha(M_2) = \emptyset$.

Claim 5.9. Let $\phi$ be a conformal model of $(U, \sim)$ extending $\phi_S$. Then:

1. $\pi'_1$ and $\pi'_2$ are subwords of $\tau'_\phi$ and $|\pi'_1| + |\pi'_2| = |\tau'_\phi|$,
2. $\pi''_1$ and $\pi''_2$ are subwords of $\tau''_\phi$ and $|\pi''_1| + |\pi''_2| = |\tau''_\phi|$.
3. for every $u \in \alpha(M_2)$ and every $v \in \alpha(M_1)$, either $\phi(u)$ and $\phi(v)$ are on the opposite site of $\phi(a)$, or there are on the same side of $\phi(a)$ and then the chord $\phi(v)$ has the chords $\phi(u)$ and $\phi(a)$ on the opposite sides.

Proof. The first two statements are obvious. Suppose $\phi(u)$ and $\phi(v)$ are on the same side of $\phi(a)$, but the chord $\phi(v)$ has the chords $\phi(u)$ and $\phi(a)$ on the opposite side. Then, $\phi(u)$ must intersect some chord from $\phi(P_1)$ as $(P_1, \sim)$ is connected. However, this is not possible as $u \parallel P_1$. 

Our goal is to show that there is unique way to compose the words $\pi'_1$ and $\pi'_2$ and the words $\pi''_1$ and $\pi''_2$ to get a conformal model of $(U, \sim)$.

Suppose that there are two non-equivalent models $\phi_1$ and $\phi_2$ extending $\phi_S$. We say that $x \in M_1 \cup \alpha(M_1)$ and $y \in M_2 \cup \alpha(M_2)$ are mixed in $\tau'_\phi_1$ and $\tau'_\phi_2$ if $\tau'_\phi_1 \{x', y'\} \neq \tau'_\phi_2 \{x', y'\}$ for some $x' \in \{x^0, x^1\}$ and $y' \in \{y^0, y^1\}$. That is, $x'$ and
y' are mixed in $\tau_{\phi_1}'$ and $\tau_{\phi_2}'$ if they occur in $\tau_{\phi_1}''$ and $\tau_{\phi_2}''$ in different order. We introduce the notion of being mixed in $\tau_{\phi_1}''$ and $\tau_{\phi_2}''$ similarly. Clearly, if $\phi_1$ and $\phi_2$ are non-equivalent, there are vertices $x \in M_1 \cup \alpha(M_1)$ and $y \in M_2 \cup \alpha(M_2)$ such that $x$ and $y$ are mixed either in $\tau_{\phi_1}'$ and $\tau_{\phi_2}'$ or in $\tau_{\phi_1}''$ and $\tau_{\phi_2}''$. Suppose $x \in M_1 \cup \alpha(M_1)$ and $y \in M_2 \cup \alpha(M_2)$ are mixed in $\tau_{\phi_1}'$ and $\tau_{\phi_2}'$. We claim that $x \in M_1$ and $y \in M_2$.

We can not have $x \in M_1$ and $y \in \alpha(M_2)$ as in any conformal model $\phi$ of $(U, \sim)$ in the form $**$ the chord $\phi(y)$ is always on the right side of $\phi(x)$ if $y \in \text{right}(x)$ or is always on the left side of $\phi(x)$ if $y \in \text{left}(x)$. We can not have $x \in \alpha(M_1)$ and $y \in M_2$ as in every conformal model of $(U, \sim)$ in the form $***$ the chord $\phi(x)$ intersects $\phi(y)$. Finally, by Claim 5.3 we can not have $x \in \alpha(M_1)$ and $y \in \alpha(M_2)$. It means that $x \in M_1$ and $y \in M_2$ and hence $x \sim y$. However, it also means that $x$ and $y$ are mixed in $\tau_{\phi_1}''$ and $\tau_{\phi_2}''$. Hence, from now we abbreviate and we say that $x$ and $y$ are mixed if $x$ and $y$ are mixed in $\tau_{\phi_1}'$ and $\tau_{\phi_2}'$ and in $\tau_{\phi_1}''$ and $\tau_{\phi_2}''$. Now, we claim that for every $x \in M_1$ and $y \in M_2$:

\begin{equation}
\{x, y\} \sim \alpha(M_1) \text{ and } \{x, y\} \parallel \alpha(M_2) \text{ if } x \text{ and } y \text{ are mixed.}
\end{equation}

We prove $\{x, y\} \sim \alpha(M_1)$. Clearly, $y \sim \alpha(M_1)$ and $x \parallel \alpha(M_2)$ by the properties of $P_1$ and $P_2$. Suppose there is $v \in \alpha(M_2)$ such that $v \parallel x$. If this is the case, the relative position of $\phi(x)$ and $\phi(v)$ is the same in any conformal model $\phi$ of $(U, \sim)$ in the form $***$. Since $\phi_i(v)$ intersects $\phi_i(v)$ for every $i \in [2]$, $x$ and $y$ can not be mixed. The second statement of (4) is proved with similar ideas. Note that $(C_1, \sim)$ is a permutation subgraph of $(U, \sim)$ as $a \sim C_1$. Hence, if $x$ is mixed with $y$ and $x \parallel z$ for some $z \in M_1$, then $z$ is also mixed with $y$. Similarly, if $x$ is mixed with $y$ and $t \parallel y$ for some $t \in M_2$, then $t$ is mixed with $x$. Now, let

$$W = \bigcup \{\{z, t\} : z \text{ and } t \text{ are mixed}\},$$

that is, $W$ contains all elements from $C_1$ that are mixed with some other element from $C_1$. Note that $W$ contains at least two elements as there are at least two elements from $C_1$ that are mixed. Moreover, $W \subset C_1 \subseteq U$. Note that $W \sim (C_1 \sim W)$ by the observation noted above. Now, by (4) we deduce that $W$ is a non-trivial module in $(U, \sim)$, which can not be the case.

Consider the remaining case $\alpha(M_2) \neq \emptyset$ and $\alpha(M_1) = \emptyset$. In this setting we have $|M_1| = 1$. Suppose $M_1 = \{x\}$. We show that there is a unique extension of $\phi_2 \equiv \phi_2'' = \phi_2' = \phi_2''' = a^0$ by a chord $\phi(x)$ to a conformal model of $(U, \sim)$. We introduce the mixing relation analogously to the previous case. Using similar ideas, if $u \in \alpha(C_2)$, $u$ and $x$ can not be mixed. So, $x$ can be mixed only with the elements in $M_2$. Let

$$W = \bigcup \{\{x, y\} : x \text{ and } y \text{ are mixed}\}.$$

Clearly, if there are two non-equivalent models $\phi_1$ and $\phi_2$, then $x$ is mixing with some element $y \in M_2$. Then $\{x, y\} \subset W \subset C_1$. Using similar ideas as previously, one can check that $W$ is a non-trivial module in $(U, \sim)$. \hfill \square
Suppose \( \phi_0^1 \) and \( \phi_1^1 \) are two conformal models of \((U, \sim)\) described by Lemma 5.10. Inspired by Hsu [14], we are going to define a consistent decomposition of \(M\), the usefulness of which will be highlighted in the upcoming Lemma 5.11. For every \(i \in [k]\) we introduce an equivalence relation \(K\) in the set \(M_i\). Depending on the type of \(M_i\), the relation \(K\) is defined as follows:

- If \(M_i\) is prime, then \(vKv'\) for every \(v, v' \in M\).
- If \(M_i\) is serial, then
  \[
  vKv' \iff \{\text{left}(v) \cap (U \setminus M_i), \text{right}(v) \cap (U \setminus M_i)\} = \\
  \{\text{left}(v') \cap (U \setminus M_i), \text{right}(v') \cap (U \setminus M_i)\}.
  \]
- If \(M_i\) is parallel, then
  \[
  vKv' \iff \text{either } \{v, v'\} \subseteq \text{left}(u) \text{ or } \{v, v'\} \subseteq \text{right}(u),
  \]
  for every \(u \in U \setminus M_i\).

Suppose \(K(M_i)\) is a set of equivalence classes of \(K\)-relation in the module \(M_i\). The set \(K(M_i)\) is called a consistent decomposition of \(M_i\) and the set \(K(M) = \bigcup_{i=1}^{k} K(M_i)\) is called a consistent decomposition of \(M\). The elements of \(K(M_i)\) and \(K(M)\) are called the consistent submodules of \(M_i\) and \(M\), respectively.

**Claim 5.10.** Suppose \(M_i\) is a child of \(M\) in \(\mathcal{M}(G_c)\). Then:

1. if \(M_i\) is prime, then \(K(M_i) = \{M_i\}\),
2. if \(M_i\) is serial or parallel, then every consistent submodule of \(M_i\) is a union of some children of \(M_i\) in \(\mathcal{M}(G_c)\).

In particular, every consistent submodule of \(M_i\) is a submodule of \(M_i\).

**Proof.** Note that \(M_i\) is proper as \(M_i\) is a child of a prime module \(M\). Also, recall that \(M\) is the connected component of \(G_c\) containing \(M_i\). Now, statement (1) follows from Lemma 4.11 applied to the proper prime module \(M_i\) and the connected component \(M\) containing \(M_i\).

Statement (2) follow from Lemma 4.12 (Lemma 4.13) applied to the proper parallel (proper serial, respectively) module \(M_i\) and the connected component \(M\) of \(G_c\) containing \(M_i\).

Suppose \(K_1, \ldots, K_n\) is a consistent decomposition of \(M\). Recall that the set \(U\) contains one element from every child of \(M\). So, \(|K_i \cap U| \leq 1\) for every \(i \in [n]\). A set \(S = \{s_1, \ldots, s_n\}\) is a skeleton of \(M\) if \(U \subseteq S\) and \(|K_i \cap S| = \{s_i\}\) for every \(i \in [n]\). That is, a skeleton of \(M\) is a superset of \(U\) containing one element from every consistent submodule \(K_i\) of \(M\).

**Lemma 5.11.** Suppose \(K_1, \ldots, K_n\) is a consistent decomposition of \(M\) and \(S = \{s_1, \ldots, s_n\}\) is a skeleton of \(M\). Then:

1. The graph \((S, \sim)\) has exactly two conformal models, \(\phi_0^S\) and \(\phi_1^S\), one being the reflection of the other.
2. For every conformal model \(\phi\) of \((M, \sim)\) and every \(i \in [n]\), the words in \(\phi|K_i\) form two contiguous subwords in the circular word \(\phi\). 


Proof. Without loss of generality we assume that \( U = \{ s_1, \ldots, s_{|U|} \} \). Let \( m \in \{0,1\} \). Our proof is based on the following claim.

For every \( j = \{|U|, \ldots, n\} \) there is a unique conformal model \( \phi^m_j \) of \( (\{s_1, \ldots, s_j\}, \sim) \) such that \( \phi^m_j|U = \phi^m_U \).

Clearly, statement \( \text{(1)} \) follows from the claim for \( j = n \).

We prove our claim by induction on \( j \). For \( j = |U| \) the claim is trivially satisfied. Suppose \( \text{(5)} \) holds for \( j = l - 1 \) for some \( l > |U| \). Our goal is to prove \( \text{(5)} \) for \( j = l \).

From the inductive hypothesis, there is a unique extension \( \phi^m_{l-1} \) of \( \phi^m_U \) on the set \( \{s_1, \ldots, s_{l-1}\} \). Suppose for a contradiction that there are two non-equivalent conformal models of \( (\{s_1, \ldots, s_l\}, \sim) \) extending \( \phi^m_{l-1} \). That is, suppose there are two different placements of the chord \( \phi(s_l) \) that lead to non-equivalent conformal models of \( (\{s_1, \ldots, s_l\}, \sim) \). Equivalently, there is a circular word \( \phi \) extending \( \phi^m_{l-1} \) by the letters \( x^0, x^1, y^0, y^1 \) such that both \( \phi' \equiv \phi|\{s_1, \ldots, s_{l-1}, x\} \) and \( \phi'' \equiv \phi|\{s_1, \ldots, s_{l-1}, y\} \) are two non-equivalent conformal models of \( (\{s_1, \ldots, s_l\}, \sim) \) if we replace \( x^0 \) by \( s^0_1 \) and \( x^1 \) by \( s^1_1 \) in \( \phi' \) and \( y^0 \) by \( s^0_1 \) and \( y^1 \) by \( s^1_1 \) in \( \phi'' \). Note that for every \( s \in \{s_1, \ldots, s_{l-1}\} \) the circular word \( \phi \) satisfies the following properties:

- if \( s \in \text{left}(s_l) \), then \( \phi(s) \) must be on the left side of \( \phi(x) \) and \( \phi(y) \).
- if \( s \in \text{right}(s_l) \), then \( \phi(s) \) must be on the right side of \( \phi(x) \) and \( \phi(y) \).
- if \( s \sim s_l \), then \( \phi(s) \) must intersect both \( \phi(x) \) and \( \phi(y) \).

We consider two cases depending on whether the chords \( \phi(x) \) and \( \phi(y) \) intersect in \( \phi \).

Suppose that \( \phi(x) \) and \( \phi(y) \) do not intersect. Suppose \( \phi(y) \) is on the right side of \( \phi(x) \) and \( \phi(x) \) is on the left side of \( \phi(y) \). Since \( \phi' \) and \( \phi'' \) are non-equivalent, there is \( s^* \) in \( \{s_1, \ldots, s_{l-1}\}^* \) such that \( \phi(s^*) \) is on the right side of \( \phi(x) \) and on the left side of \( \phi(y) \). Since the chord \( \phi(s) \) can not intersect both \( \phi(x) \) and \( \phi(y) \), we have \( s \parallel s_l \). But then \( \phi(s) \) is on the right side of \( \phi(x) \) and on the left side of \( \phi(y) \), which contradicts the properties of \( \phi \) listed above.

Suppose that \( \phi(y) \) is on the right side of \( \phi(x) \) and \( \phi(x) \) is on the right side of \( \phi(y) \). Let \( s \in \{s_1, \ldots, s_{l-1}\} \) be such that \( s \parallel s_l \). The chord \( \phi(s) \) must lie on the right side of \( \phi(x) \) and the right side of \( \phi(y) \). Any other placement of \( \phi(s) \) will contradict one of the properties of \( \phi \) listed above. Furthermore, \( \phi(s) \) can not have \( \phi(x) \) and \( \phi(y) \) on its different sides. Hence, \( \phi(s) \) has both its ends either between \( \phi(x^1) \) and \( \phi(y^0) \) or between \( \phi(y^1) \) and \( \phi(x^0) \). Note that \( s \) and \( s_l \) belong to different children of \( M \). Otherwise, the chord \( \phi(u) \) for any \( u \in U \setminus M \) such that \( u \sim M \) can not intersect \( \phi(s), \phi(x), \) and \( \phi(y) \) at the same time. Thus, there is a path \( P \) in the graph \( (\{s_1, \ldots, s_l\}, \sim) \) joining \( s \) and \( s_l \) such that all inner vertices of \( P \) are in \( U \). Then, there must be a vertex \( s' \in P \) such that \( \phi(s') \) has \( \phi(x) \) and \( \phi(y) \) on different sides, which contradicts the properties of \( \phi \). The remaining cases are proved similarly.

Suppose that \( \phi(x) \) and \( \phi(y) \) intersect. Without loss of generality we assume that \( \phi|\{x,y\} = x^0y^1x^1y^0 \). First, note that for every \( s \in \{s_1, \ldots, s_{l-1}\} \) the chord \( \phi(s) \)
can not have both its ends between \( \phi(x^0) \) and \( \phi(y^0) \). Otherwise, \( \phi(s) \) would be on the left side of \( \phi(x) \) and on the right side of \( \phi(y) \), which would contradict the properties of \( \phi \). For the same reason, \( \phi(s) \) can not have both its ends between \( \phi(x^1) \) and \( \phi(y^1) \). Let \( S' \) be a set of all \( s \in \{s_1, \ldots, s_{i-1}\} \) such that \( \phi(s) \) has one end between \( \phi(x^0) \) and \( \phi(y^0) \) and the other end between \( \phi(x^1) \) and \( \phi(y^1) \). Clearly, \( S' \neq \emptyset \) as \( \phi' \) and \( \phi'' \) are not equivalent. Observe that \( S' \cup \{s_i\} \) is a proper module in \( \{s_1, \ldots, s_i, \sim\} \). Indeed, for every \( t \in \{s_1, \ldots, s_{i-1}\} \setminus S' \) the chord \( \phi(t) \) has either both ends between \( \phi(y^0) \) and \( \phi(x^1) \) or between \( \phi(y^1) \) and \( \phi(x^0) \), or has one end between \( \phi(y^0) \) and \( \phi(x^1) \) and the second between \( \phi(y^1) \) and \( \phi(x^0) \). In any case, either \( t \sim (S' \cup \{s_i\}) \) or \( t \parallel (S' \cup \{s_i\}) \). Since the sets \( M_1, \ldots, M_k \) restricted to \( \{s_1, \ldots, s_i\} \) form a partition of \( \{s_1, \ldots, s_i, \sim\} \) into \( k \) maximal modules in \( \{s_1, \ldots, s_i, \sim\} \), we imply that \( (S' \cup \{s_i\}) \subseteq M_i \) for some \( i \in [k] \). In particular, \( M_i \) must be a serial child of \( M \) as \( s_i \sim S' \). Since for every \( u \in U \) such that \( u \parallel M \), the chord \( \phi(u) \) has both its ends between \( \phi(y^0) \) and \( \phi(x^1) \) or between \( \phi(y^1) \) and \( \phi(x^0) \), we have \( s_1Ks' \) for every \( s' \in S' \). However, it can not be the case as \( S \) contains exactly one element from every equivalence class of \( K \)-relation in \( M \).

To prove statement (2) assume that \( \phi \) is a conformal model of \( (M, \sim) \).

Suppose \( K_j = M_i \), where \( M_i \) is a prime child of \( M \). Then, statement (2) follows from Lemma 4.11 applied to the module \( M_i \) contained in the connected component \( M \) of \( G_c \).

Suppose \( K_j \) is a consistent submodule of \( M_i \), where \( M_i \) is a serial child of \( M \). Since \( M \) is prime, there is \( x \in M \setminus M_i \) such that \( x \sim M_i \). Suppose that \( x \in M_{i'} \) for some \( i' \in [k] \) different than \( i \). By Claim 4.5, \( \phi(K_j \cup \{x\}) = x^0x^1x^2x^3 \), where \( (x^0, x^1, x^2, x^3) \) is an oriented permutation model of \( (K_j, \sim) \). Denote by \( l^0 \) and \( l^3 \) the first and the last letter from \( K_j \), respectively, if we traverse \( \phi \) from \( \phi(x^0) \) to \( \phi(x^1) \). Similarly, denote by \( r^0 \) and \( r^3 \) the first and the last letter from \( K_j \), respectively, if we traverse \( \phi \) from \( \phi(x^1) \) to \( \phi(x^0) \). We claim that there is \( u \in U \setminus M_i \) such that \( \phi(u) \) has both its ends either between \( \phi(l^2) \) and \( \phi(r^0) \) or between \( \phi(r^3) \) and \( \phi(l^0) \). Assume otherwise. Let \( T \) be the set of all \( t \in M \) such that \( \phi(t) \) has one end between \( \phi(l^2) \) and \( \phi(r^0) \) and the second end between \( \phi(r^3) \) and \( \phi(l^0) \). Note that \( T \neq \emptyset \) as \( x \in T \). We claim that that \( M_i \cup T \) is a module in \( (M, \sim) \). Indeed, for every \( v \in M \setminus (M_i \cup T) \) the chord \( \phi(v) \) has both its ends either between \( \phi(l^0) \) and \( \phi(l^2) \) or between \( \phi(r^3) \) and \( \phi(r^0) \). In particular, we have \( u \parallel (M_i \cup T) \), which proves that \( M_i \cup T \) is a module in \( (M, \sim) \). Since \( M \) is prime, there is \( v \in M \setminus M_i \) such that \( u \parallel M_i \). In particular, \( v \) is not in \( T \), \( M_i \cup T \subsetneq M \), and hence \( (M_i \cup T) \) is a non-trivial module in \( (M, \sim) \). However, this contradicts the fact that \( M_i \) is a maximal non-trivial module in \( (M, \sim) \). This completes our claim, that is, there is \( u \in U \setminus M_i \) such that \( \phi(u) \) has both its ends either between \( \phi(l^2) \) and \( \phi(r^0) \) or between \( \phi(r^3) \) and \( \phi(l^0) \).

Now, suppose that \( \phi(K_j) \) does not form two contiguous subwords in \( \phi(M) \). That is, there is \( y \in M \setminus K_j \) such that \( \phi(y) \) has an end between \( \phi(l^0) \) and \( \phi(l^2) \) or between \( \phi(r^3) \) and \( \phi(r^0) \). Assume that \( \phi(y^0) \) is between \( \phi(l^0) \) and \( \phi(l^2) \) – the other
case is handled analogously. The end \( \phi(y^0) \) splits \( K_j \) into two sets:

\[
K_j' = \{ v \in K_j : \phi(v) \text{ has an end between } \phi(l^0) \text{ and } \phi(y^0) \}, \quad \text{and} \\
K_j'' = \{ v \in K_j : \phi(v) \text{ has an end between } \phi(y^0) \text{ and } \phi(l^3) \}.
\]

By Lemma 4.13, \( K_j' \) and \( K_j'' \) are the unions of some children of \( M_i \). Denote by \( l^1 \) and \( l^2 \) the last and the first labeled letter from the sets \( K_j' \) and \( K_j'' \), respectively, if we traverse \( \phi \) from \( \phi(x^0) \) to \( \phi(x^1) \). Similarly, denote by \( r^1 \) and \( r^2 \) the last and the first labeled letter from the sets \( K_j' \) and \( K_j'' \), respectively, if we traverse \( \phi \) from \( \phi(x^1) \) to \( \phi(x^0) \). Assume that \( y \parallel K_j \). Then, \( y \) is not in \( M_i \), as \( M_i \) is serial. Suppose that \( y \in M_i \) for some \( l \neq [k] \) different than \( i \). Let \( P \) be a shortest path in \( (M_i, \sim) \) that joins \( y \) and \( M_i \) with all the inner vertices in \( U \). Let \( v \) be a neighbor of \( y \) in \( P \). Clearly, the chord \( \phi(v) \) either has both ends between \( \phi(l^1) \) and \( \phi(l^2) \) or has one end between \( \phi(l^1) \) and \( \phi(l^2) \) and the second between \( \phi(r^1) \) and \( \phi(r^2) \). In any case, every chord from \( \phi(M_i) \) has both ends between \( \phi(l^1) \) and \( \phi(l^2) \) or both ends between \( \phi(r^1) \) and \( \phi(r^2) \). Let \( u' \in U \) be such that \( u' \in M_i \). Now, note that \( u, u' \in U \setminus M_i \), witness that \( (v', v'') \notin K \) for every \( v' \in K_j' \) and every \( v'' \in K_j'' \), which contradicts that \( K_j \) is an equivalence class of \( K \)-relation in \( M_i \). Assume that \( y \sim K_j \). Then \( \phi(y^1) \) must be between \( \phi(r^1) \) and \( \phi(r^2) \). If \( y \in M_i \), then \( yK \) has both ends between \( \phi(l^1) \) and \( \phi(l^2) \) if \( y \in K_j \), which contradicts that \( K_j \) is an equivalence class of \( K \)-relation in \( M_i \). So, \( y \notin M_i \). Then, using an analogous argument as for the existence of \( u \), we show that there is \( u' \in U \setminus M_i \) such that \( u' \parallel M_i \) and \( \phi(u') \) has both ends either between \( \phi(l^1) \) and \( \phi(l^2) \) or between \( \phi(r^1) \) and \( \phi(r^2) \). Consequently, \( u \) and \( u' \) witness that \( (v', v'') \notin K \) for every \( v' \in K_j' \) and every \( v'' \in K_j'' \) — a contradiction.

Suppose \( K_j \) is a consistent submodule of a parallel module \( M_i \). Since \( M_i \) is prime, there is \( x \in M_i \), such that \( x \sim M_i \). From Claim 4.5, \( \phi((K_j \cup \{ x \})) \equiv x^0 \tau x^1 \tau' \), where \((\tau, \tau')\) is an oriented permutation model of \((K_j, \sim)\). Let \( l^0 \) and \( l^3 \) be the first and the last letter from \( K_j' \) if we traverse \( \phi \) from \( x^0 \) to \( x^1 \). Similarly, let \( r^0 \) and \( r^3 \) be the first and the last letter from \( K_j'' \) if we traverse \( \phi \) from \( x^1 \) to \( x^0 \). Suppose statement 2 does not hold. That is, there is \( y \in (M_i \setminus K_j) \) such that \( \phi(y) \) has one of its ends between \( \phi(l^0) \) and \( \phi(l^3) \) or between \( \phi(r^0) \) and \( \phi(r^3) \). Suppose that \( \phi(u^0) \) lies between \( \phi(l^0) \) and \( \phi(l^3) \); the other cases are handled analogously. Split \( K_j \) into two subsets, \( K_j' \) and \( K_j'' \), where

\[
K_j' = \{ v \in K_j : \phi(v) \text{ has an end between } \phi(l^0) \text{ and } \phi(y^0) \}, \quad \text{and} \\
K_j'' = \{ v \in K_j : \phi(v) \text{ has an end between } \phi(y^0) \text{ and } \phi(l^3) \}.
\]

By Lemma 4.12, \( K_j' \) and \( K_j'' \) are the unions of some children of \( M_i \). Denote by \( l^1 \) and \( l^2 \) the last and the first labeled letter in \( K_j' \) and \( K_j'' \), respectively, if we traverse \( \phi \) from \( \phi(l^0) \) to \( \phi(l^3) \). Similarly, denote by \( r^1 \) and \( r^2 \) the last and the first labeled letter in \( K_j'' \) and \( K_j' \), respectively, if we traverse \( \phi \) from \( \phi(r^0) \) to \( \phi(r^3) \). Since \( \phi(y) \) can not intersect the chords from \( \phi(K_j') \) and from \( \phi(K_j'') \) at the same time, we have \( y \parallel K_j \). Suppose \( \phi(y) \) has both ends between \( \phi(l^1) \) and \( \phi(l^2) \). Then we have \( y \notin M_i \) as \( \phi(y) \) does not intersect \( \phi(x) \). Again, using the idea of the shortest path between
Let $\phi$ be any conformal model of $(M, \sim)$. By Lemma 5.6, we have $\phi|S = \phi^m_S$ for some $m \in \mathbb{Z}$.

Let $K_i$ be a consistent submodule of $M$ for some $i \in [n]$. Suppose $\tau_{i,\phi}^0$ and $\tau_{i,\phi}^1$ are two contiguous subwords of $\phi$ in $|K_i|$ enumerated such that $\tau_{i,\phi}^0$ contains the letter $s_j^i$ for $j \in \{0, 1\}$ — see Lemma 5.11. Let $K_{i,\phi}^j$ be a set of the letters contained in $\tau_{i,\phi}^j$ for $j \in \{0, 1\}$. Note that $K_{i,\phi}^0$, $K_{i,\phi}^1$ are labeled copies of $K_i$ and $\{K_{i,\phi}^0, K_{i,\phi}^1\}$ forms a partition of $K_i$. Clearly, $(\tau_{i,\phi}^0, \tau_{i,\phi}^1)$ is an oriented permutation model of $(K_i, \sim)$. Let $(\prec_{K_i,\phi}^0, \prec_{K_i,\phi}^1)$ be transitive orientations of $(K_i, \parallel)$ and $(K_i, \sim)$, respectively, corresponding to the non-oriented permutation model $(\tau_{i,\phi}^0, \tau_{i,\phi}^1)$ of $(K_i, \sim)$.

Finally, let $\pi_\phi$ be a circular permutations of $K_{i,\phi}^0, K_{i,\phi}^1, \ldots, K_{n,\phi}^0, K_{n,\phi}^1$ that arises from $\phi_M$ by replacing every contiguous word $\phi_M|K_{i,\phi}^j$ by the set $K_{i,\phi}^j$. It occurs that the sets $K_{i,\phi}^j$ and the transitive orientations $(\prec_{i,\phi}^0)$ of $(K_i, \parallel)$ do not depend on the choice of a conformal model $\phi$ of $G_c$. Moreover, $\pi_\phi$ may take only two values depending on whether $\phi|S \equiv \phi^0_S$ or $\phi|S \equiv \phi^1_S$.

**Claim 5.12.** Suppose $M$ is an improper prime module in $\mathcal{M}(G_c)$, $K_1, \ldots, K_n$ is a consistent decomposition of $M$, and $S$ is a skeleton of $M$. For every $i \in [n]$ there are labeled copies $K_i^0$ and $K_i^1$ of $K_i$ forming a partition of $K_i$ and a transitive orientation $<_{K_i}^0$ of $(K_i, \parallel)$ such that

$$(<_{K_i,\phi}^0, K_{i,\phi}^0, K_{i,\phi}^1) = (<_{K_i,\phi}^0, K_i^0, K_i^1)$$

for every conformal model $\phi$ of $(M, \sim)$ and every $i \in [n]$.

Moreover, there are circular permutations $\pi^0(M), \pi^1(M)$ of $\{K_1^0, K_1^1, \ldots, K_n^0, K_n^1 \}$ such that

$$\pi_\phi = \begin{cases} 
\pi^0(M) & \text{if } \phi|S = \phi^0_S \\
\pi^1(M) & \text{if } \phi|S = \phi^1_S
\end{cases}$$

for every conformal model $\phi$ of $(M, \sim)$.

Moreover, $\pi^0(M)$ is the reflection of $\pi^1(M)$.
Algorithm 2: Computing the orientations of improper prime module $M$

**input**: $K_1, \ldots, K_n, S$ – a consistent decomposition and a skeleton of $M$, $\phi_S^0, \phi_S^1$ – two conformal models of $(S, \sim)$

**output**: $(<^0_{K_i}, K_0^i, K_1^i)$ for every $i \in [n]$, $\pi^0(M)$, $\pi^1(M)$

1. for $i = 1$ to $n$ do
2.  pick $s_j \in \{s_1, \ldots, s_n\}$ such that $s_j \sim K_i$;
3.  orient $s_i$ from $K_0^i$ to $K_1^i$;
4.  foreach $v \in K_i$ do
5.   if the vertices $v$ and $s_i$ have the vertex $s_j$ on the same side then
6.     orient $v$ from $K_0^i$ to $K_1^i$
7.   else
8.     orient $v$ from $K_1^i$ to $K_0^i$
9.  end
10. $<^0_{K_i} = \emptyset$;
11. foreach $(u, v) \in K \times K$ such that $u \parallel v$ do
12.   if $u \in \text{right}(v)$ then
13.     if $v^0 \in K_0^i$ then add $(v, u)$ to $<^0_{K_i}$;
14.     else add $(u, v)$ to $<^0_{K_i}$;
15.   else
16.     if $v^0 \in K_1^i$ then add $(v, u)$ to $<^0_{K_i}$;
17.     else add $(v, u)$ to $<^0_{K_i}$;
18.   end
19. end
20. foreach $m \in \{0, 1\}$ do
21.   $\pi^m(M)$ – a circular order of $\{K_0^1, K_1^1, \ldots, K_0^n, K_1^n\}$ arisen from $\phi_S^m$ by replacing every letter $s_i^j$ by the set $K_i^j$;
22. end

**Proof.** Algorithm 2 shows how to compute $(<^0_{K_i}, K_0^i, K_1^i)$ for every $i \in [n]$ and $\pi^m(M)$ for every $m \in \{0, 1\}$. Let $\phi$ be any conformal model of $G_c$. By Lemma 5.11 [2], all the chords from $\phi(K_i)$ are on the same side of $\phi(s_j)$, where $s_j \in S$ is such that $s_j \parallel K_i$. Hence, the orientations of every edge $\phi(v)$ for $v \in K_i$ can be determined based on whether $v$ and $s_i$ have the vertex $s_j$ on the same side in the way as it is stated by Algorithm 2. That $<^0_{i,\phi}$ agrees with $<^0_i$ is obvious. The last part of the claim follows by Lemma 5.11. □

The elements of the set $\{K_0^0, K_1^0, \ldots, K_0^n, K_1^n\}$ are called slots of $M$, $\pi^0(M), \pi^1(M)$ are called circular permutations of the slots of the module $M$. Moreover, with every slot $K_i^j \in \pi^m(M)$ we associate an orientation $\pi_i^0 = (<^0_i, K_0^i, K_1^i)$ and with
every slot $K_i^1 \in \pi^m(M)$ we associate a dual orientation $\mathbb{K}_i^1 = (\prec_i^1, K_i^1, K_i^0)$, where $\prec_i^1$ is the reverse of $\prec_i^0$. Note that the two orientations of $K_i$ are the same for $\pi^0(M)$ and $\pi^1(M)$. Furthermore, we extend the notion of admissible model on the consistent submodules of $M$. The admissible models $(\tau^0, \tau^1)$ for an orientation $\mathbb{K}_i^0$ are described as in Definition 5.12, except that the orientation $M^0_i$ is replaced by $\mathbb{K}_i^0$. The admissible models for $M^1_i$ are defined by symmetry. One can check that for every model $\phi$ of $(M, \sim)$, $(\phi|K^0_i, \phi|K^1_i)$ is an admissible model for $\mathbb{K}_i^0$ and $(\phi|K^1_i, \phi|K^0_i)$ is an admissible model for $\mathbb{K}_i^1$.

Definition 5.13. Let $M$ be an improper prime module in $\mathcal{M}(G_c)$ and let $\pi^m(M)$ for some $m \in \{0, 1\}$ be a circular permutation of the slots of $M$. A circular word $\phi$ on the set $M^*$ is admissible for $\pi^m(M)$ if $\phi$ arises from $\pi^m(M)$ by exchanging every slot $K_i^1$ by a permutation $\tau_i^1$, where $\tau_i^0, \tau_i^1$ are such that $(\tau_i^0, \tau_i^1)$ is an admissible model of $\mathbb{K}_i^0$ for every $i \in [n]$.

The next theorem provides a description of all conformal models of $(M, \sim)$.

Theorem 5.14. Suppose $G$ is a circular-arc graph with no twins and no universal vertices. Suppose $M$ is an improper prime module in $\mathcal{M}(G_c)$. A circular word $\phi$ is a conformal model of $(M, \sim)$ if and only if $\phi_M$ is an admissible model of $\pi^m(M)$ for some $m \in \{0, 1\}$.

Proof. Suppose $\phi$ is a conformal model of $(M, \sim)$. By Claim 5.12 we get $\pi_\phi = \pi^m(M)$ for some $m \in \{0, 1\}$ and $(\phi|K^0_i, \phi|K^1_i)$ is admissible for $\mathbb{K}_i^0$ for every $i \in [n]$. Thus, $\phi$ is admissible for $\pi^m(M)$.

Suppose $\phi$ is an admissible model for $\pi^m(M)$ for some $m \in \{0, 1\}$. Since $G$ is a circular-arc graph, $G_c$ has a conformal model. Hence, $(M, \sim)$ has a conformal model, say $\phi'$. Since the reflection of a conformal model is also conformal, we may assume that $\phi|S \equiv \phi'|S$. Now, we start with $\phi'$ and for every $i \in [n]$ we replace the words $(\phi'|K^0_i, \phi'|K^1_i)$ in $\phi'$ by the words $(\phi|K^0_i, \phi|K^1_i)$, respectively. Finally, we obtain an oriented model $\phi$ of $G_c$. Since $\prec_i^{0, \phi'} \equiv \prec_i^{0, \phi}$, one can easily check that after every transformation the chords $\phi(v)$ and $\phi'(v)$ have on its both sides the chords representing exactly the same sets of vertices. This proves that $\phi$ is also conformal.

The above theorem characterizes all conformal models of $G_c$ when $G_c$ and $\overline{G}_c$ are connected. Indeed, in this case $V$ is an improper prime module in $\mathcal{M}(G_c)$ and hence Theorem 5.14 applies.

The results of this section can be also used to provide the recognition algorithm for circular-arc graphs $G$ such that both $G_c$ and $\overline{G}_c$ are connected.

Theorem 5.15. Suppose $G$ is a graph with no twins and no universal vertices such that $G_c$ and $\overline{G}_c$ are connected. The following statements are equivalent:

1. $G$ is a circular-arc graph.
2. $G_c$ admits an admissible model $\phi$ and $\phi$ is conformal.
5.4. Conformal representations of improper parallel modules. The results from the first part of this section were inspired by Hsu’s work \cite{Hsu}. In particular, $T_{NM}$ tree and its properties were described in \cite{Hsu}. Moreover, admissible models for prime children of $V$ are partially inspired by \cite{Hsu}.

So far, we have not described conformal models of $G_e$ in only one case: when the graph $G_e$ is disconnected. This corresponds to the case where $V$ is an improper parallel module in $\mathcal{M}(G_e)$.

Suppose $G$ is a circular-arc graphs with no twins and no universal vertices such that $G_e$ is disconnected. Denote the children of $V$ in $\mathcal{M}(G_e)$ by $\mathcal{M}(V)$. Note that every module $M$ in $\mathcal{M}(V)$ is either an improper prime or an improper serial module in $\mathcal{M}(G_e)$. Observe that for every module $M \in \mathcal{M}(V)$ and every $v \in V \setminus M$ we have either $M \subseteq \text{left}(v)$ or $M \subseteq \text{right}(v)$.

Let $M$ be a module in $\mathcal{M}(V)$ and let $v \in V \setminus M$. We say that $M$ is on the left side of $v$ if $M \subseteq \text{left}(v)$ and $M$ is on the right side of $v$ if $M \subseteq \text{right}(v)$. Let $v$ be a vertex and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two disjoint sets of modules from $\mathcal{M}(V)$ such that $v \notin \bigcup \mathcal{M}_1 \cup \bigcup \mathcal{M}_2$. We say that $v$ separates $\mathcal{M}_1$ and $\mathcal{M}_2$ if either $\bigcup \mathcal{M}_1 \subseteq \text{left}(v)$ and $\bigcup \mathcal{M}_2 \subseteq \text{right}(v)$ or $\bigcup \mathcal{M}_2 \subseteq \text{left}(v)$ and $\bigcup \mathcal{M}_1 \subseteq \text{right}(v)$. We say that $v$ separates $\mathcal{M}_1$ and $\mathcal{M}_2$ if $v$ separates $\{M_1\}$ and $\{M_2\}$. We use an analogous notation with the obvious meaning for conformal models $\phi$ of $G_e$.

Two modules $M_1, M_2 \in \mathcal{M}(V)$ are non-separated if there is no vertex $v \in V \setminus (M_1 \cup M_2)$ that separates $M_1$ and $M_2$. A set $N \subseteq \mathcal{M}(V)$ is a node in $G_e$ if $N$ is a maximal subset of pairwise non-separated modules from $\mathcal{M}(V)$.

Let $T_{NM}$ be a bipartite graph containing all the modules in $\mathcal{M}(V)$ and all the nodes in $G_e$ with an edge between a module $M$ in $\mathcal{M}(V)$ and a node $N$ in $G_e$ iff $M \in N$. Let $N_T[M]$ denotes the neighbours of a module $M$ in $T_{NM}$ and let $N_T[N]$ denotes the neighbors of a node $N$ in $T_{NM}$.

**Claim 5.16.** The following statements hold:

1. For every module $M \in T_{NM}$ and every two nodes $N_1, N_2 \in N_T[M]$ there is a vertex $v \in M$ that separates the modules from $N_1 \setminus \{M\}$ and the modules from $N_2 \setminus \{M\}$.

2. The bipartite graph $T_{NM}$ is a tree. All leaves of $T_{NM}$ are in the set $\mathcal{M}(V)$.

**Proof.** Let $M$ be a module in $T_{NM}$ and let $N_1, N_2$ be two different nodes adjacent to $M$ in $T_{NM}$. Since $N_1, N_2$ are different maximal subsets of pairwise non-separated modules from $\mathcal{M}(V)$, there is a module $M_1 \in N_1 \setminus N_2$ and a module $M_2 \in N_2 \setminus N_1$ such that $M_1$ and $M_2$ are separated by some $v \in V \setminus (M_1 \cup M_2)$. Suppose $M_1 \subseteq \text{left}(v)$ and $M_2 \subseteq \text{right}(v)$. Suppose $v \notin M$. Then, depending on whether $M \subseteq \text{right}(v)$ or $M \subseteq \text{left}(v)$, either $M$ and $M_1$ are separated by $v$ or $M$ and $M_2$ are separated by $v$. Hence, either $N_1$ or $N_2$ is not a node in $G_e$. So, $v \in M$. That the modules from $N_1 \setminus \{M\}$ are on the left side of $v$ and the modules from $N_2 \setminus \{M\}$ are on the right side of $v$ is trivial. The other case is proved analogously. This proves \cite{Hsu}.
Now, we will show that $T_{NM}$ is a tree. First we prove that $T_{NM}$ contains no cycles. Suppose that $M_1N_1\ldots M_kN_k$ for some $k \geq 2$ is a cycle in $T_{NM}$. Since $N_1, N_k$ are neighbors of $M_1$ in $T_{NM}$, there is $v \in M_1$ that separates the modules from $N_1 \setminus \{M_1\}$ and the modules from $N_k \setminus \{M_1\}$. So, $v$ separates $M_k$ and $M_2$. So, $M_k$ and $M_2$ can not be in a common node of $G_c$, and hence $k \geq 3$. Since $M_2$ and $M_k$ are separated by $v$, there is $i \in [2, k - 1]$ such that $M_i$ and $M_{i+1}$ are separated by $v$ as well. So, $M_i$ and $M_{i+1}$ are not contained in a common node of $G_c$. But this can not happen as $N_i$ contains both $M_i$ and $M_{i+1}$. This shows that $T_{NM}$ is a forest.

To show that $T_{NM}$ has all leaves in the set $\mathcal{M}(V)$ it suffices to show that every node in $G_c$ has at least two elements. Suppose for a contradiction that $\{M\}$ is a node in $T_{NM}$, where $M$ is a module in $\mathcal{M}(V)$. Let $\phi$ be a conformal model of $G_c$. Pick a vertex $u \in V \setminus M$ and a vertex $v \in M$ such that $u^* \in \{v^0, v^1\}$ and $v^* \in \{u^0, u^1\}$. Then, note that $M$ and the module from $\mathcal{M}(V)$ containing $u$ are non-separated, which contradicts the fact that $\{M\}$ is a node in $G_c$. It remains to show the connectivity of $T_{NM}$. Suppose $T_{NM}$ is a forest and suppose $\phi$ is any conformal model of $G_c$. Then, there are vertices $u$ and $v$ such that $u^*$ and $v^*$ are consecutive in $\phi$ for some $u^* \in \{v^0, v^1\}$ and $v^* \in \{u^0, u^1\}$, where $u \in M_u$, $v \in M_v$, and $M_u$ and $M_v$ are in different connected components of $T_{NM}$. However, by the choice of $u$ and $v$ there is no vertex in $M \setminus (M_u \cup M_v)$ that separates $M_u$ and $M_v$. So, $M_u$ and $M_v$ are non-separated, and hence they are connected in $T_{NM}$ by a node containing $M_u$ and $M_v$, a contradiction. \hfill \Box

Let $M$ be a module and let $N$ be a node in $T_{NM}$ such that $M$ is adjacent to $N$ in $T_{NM}$. Let $T_{NM} \setminus M$ be a forest obtained from $T_{NM}$ by deleting the module $M$ and let $V_{T\setminus M}(N)$ be the union of the modules from the connected component of $T_{NM} \setminus M$ containing the node $N$. Similarly, let $T_{NM} \setminus N$ be a forest obtained from $T_{NM}$ by deleting the node $N$ and let $V_{T\setminus N}(M)$ be the union of the modules from the connected component of $T_{NM} \setminus N$ containing the module $M$.

**Claim 5.17.** Suppose $M$ is a module in $T_{NM}$ and $v$ is a vertex in $M$. For every node $N \in N_T[M]$, either $V_{T\setminus M}(N) \subseteq left(v)$ or $V_{T\setminus M}(N) \subseteq right(v)$.

**Proof.** Let $F_N$ be a connected component of $T_{NM} \setminus M$ containing $N$. Suppose there are two modules $M_1, M_2 \in F_N$ such that $M_1 \subseteq left(v)$ and $M_2 \subseteq right(v)$. So, there is a path $P$ between $M_1$ and $M_2$ in $F_N$. Hence, there are three consecutive vertices $M_1'N'M_2'$ in the path $P$ such that $M_1' \subseteq left(v)$ and $M_2' \subseteq right(v)$. Hence, $M_1'$ and $M_2'$ are separated by $v$, which contradicts $M_1'$, $M_2' \subseteq N'$.

The above claim allows us to extend the notions of being on the left/right side of $v \in M$ for the subsets $V_{T\setminus M}(N)$ for $N \in N_T[M]$. We extend the notion $v \in M$ separates $V_{T\setminus M}(N')$ and $V_{T\setminus M}(N'')$ for $N', N'' \in N_T[M]$ similarly.

**Claim 5.18.** Let $M$ be a module in $T_{NM}$, $N$ be a node in $T_{NM}$, and $\phi$ be a conformal model of $G_c$. Then:
(1) For every node \( N' \in N_T[M] \) the set \( \phi|V_{T\setminus M}(N') \) contains a single contiguous subword of \( \phi \). Moreover, for every two different nodes \( N', N'' \in N_T[M] \) there is \( v \in M \) such that \( \phi(v) \) separates \( \phi|V_{T\setminus M}(N') \) and \( \phi|V_{T\setminus M}(N'') \).

(2) For every module \( M' \in N_T[N] \) the set \( \phi|V_{T\setminus N}(M') \) contains a single contiguous subword of \( \phi \).

Proof. Statement (1) follows from Claim 5.18 and Claim 5.16.

Since \( N \) is a maximal subset of \( M(V) \) containing pairwise unseparated modules, \( \phi|M'' \) is a contiguous word in the circular word \( \phi|(\bigcup N) \) for every \( M'' \in N_T[N] \).

Now, applying Claim 5.18 for the module \( \phi \), we deduce that \( \phi|V_{T\setminus N}(M') \) is a contiguous subword of \( \phi \).

Let \( \phi \) be a conformal model of \( G_c \), \( M \) be a module in \( T_{NM} \), and \( N \) be a node in \( T_{NM} \). For every \( N' \in N_T[M] \) we replace the contiguous subword \( \phi|V_{T\setminus M}(N') \) in \( \phi \) by the letter \( N' \). We denote the circular word arisen this way by \( \phi|(M \cup N_T[M]) \).

Similarly, for every \( M' \in N_T[N] \) we replace the contiguous subword \( \phi|V_{T\setminus N}(M') \) in \( \phi \) by the letter \( M' \). We denote the circular word arisen this way by \( \phi|N_T[N] \).

Note that \( \phi|N_T[N] \) is a circular permutation of the modules from the set \( N_T[N] \).

Fix a prime module \( M \) in \( T_{NM} \). Suppose \( \phi \) is any conformal model of \( G_c \). Denote by \( \phi' \) the circular word \( \phi|(M \cup N_T[M]) \). Clearly, \( \phi|M \) is a conformal model of \( (M, \sim) \). Let \( S \) be a slot in \( \pi_{\phi|M}(M) \). Let \( \tau_{\phi'}(S) \) be the smallest contiguous subword of \( \phi' \) containing all the letters from \( S \) and containing no letter from other slots of \( \pi_{\phi|M} \). Clearly, \( \tau_{\phi'}(S)|S \) is a permutation of \( S \). From Claim 5.18(1) we deduce that \( \tau_{\phi'} \) has the form

\[
\tau_{\phi'}(S) = \tau_{\phi'}^1 N_{\phi'}^1 \tau_{\phi'}^2 N_{\phi'}^2 \ldots \tau_{\phi'}^{l-1} N_{\phi'}^{l-1} \tau_{\phi'}^l \text{ for some } l \geq 1,
\]

where \( N_{\phi'}^1, \ldots, N_{\phi'}^{l-1} \) is a sequence of all nodes from \( N_T[M] \) occuring in \( \tau_{\phi'}(S) \) and \( \tau_{\phi'}^1, \ldots, \tau_{\phi'}^l \) are non-empty words satisfying \( \tau_{\phi'}^1 \ldots \tau_{\phi'}^l = \tau_{\phi'}(S)|S \). Next, let

\[
p_{\phi'}(S) = (S_{\phi'}^1, N_{\phi'}^1, S_{\phi'}^2, N_{\phi'}^2, \ldots, S_{\phi'}^{l-1}, N_{\phi'}^{l-1}, S_{\phi'}^l),
\]

where \( S_{\phi'}^i \) is the set containing all labeled letters from the word \( \tau_{\phi'}^i \) for \( i \in [l] \). In particular, note that \( (S_{\phi'}^1, \ldots, S_{\phi'}^l) \) is an ordered partition of the slot \( S \). Now, let \( (S, T) \) be two consecutive slots in \( \pi_{\phi|M}(M) \). Denote by \( p_{\phi'}(S, T) \) the set of all nodes from \( N_T[M] \) that appear between \( \tau_{\phi'}(S) \) and \( \tau_{\phi'}(T) \) in the circular word \( \phi' \).

Claim 5.18(1) proves that the set \( p_{\phi'}(S, T) \) is either empty or contains exactly one node. Similarly to the previous sections, it occurs that \( p_{\phi'}(S) \) and \( p_{\phi'}(S, T) \) do not depend on the choice of \( \phi \) provided we choose from a set of conformal models \( \phi \) for which the circular order of the slots \( \pi_{\phi|M} \) of \( M \) is the same.

Claim 5.19. Let \( m \in \{0, 1\} \) and let \( M \) be a prime module in \( T_{NM} \).

(1) For every slot \( S \) in \( \pi^m(M) \) there exists a sequence

\[
p_m(S) = (S_m^1, N_m^1, S_m^2, \ldots, S_m^{l-1}, N_m^{l-1}, S_m^l)
\]
where \((S^1_m, \ldots, S^l_m)\) is an ordered partition of \(S\) and \(N^1_m, \ldots, N^{l-1}_m\) are nodes from \(N_T[M]\) such that
\[
p_m(S) = p_{\phi(M \cup N_T[M])}(S)
\]
for every conformal model of \(G_e\) such that \(\pi_{\phi[M]} = \pi^m(M)\).

(2) For every two consecutive slots \((S, T)\) in \(\pi^m(M)\) there exists a set \(p_m(S, T) \subset N_T(M)\) such that
\[
p_m(S, T) = p_{\phi(M \cup N_T[M])}(S, T)
\]
for every conformal model \(\phi\) of \(G_e\) such that \(\pi_{\phi[M]} = \pi^m(M)\).

Proof. Roughly speaking, the algorithm that computes \(p_m(S)\) works as follows. We start with the circular orientation \(\pi^m(M)\) and for every node \(N \in N_T[M]\) we find a slot \(S\) in \(\pi^m(S)\) or two consecutive slots \((S, T)\) in \(\pi^m(S, T)\) where we need to place the node \(N\), basing on whether \(V_{T \setminus M}(N)\) is on the left or on the right side of \(v\), for every \(v \in M\).

From the perspective of the slot \(S\), the algorithm works as follows. Let \(K\) be a consistent submodule of \(M\) associated with the slot \(S\). Let \(v\) be a vertex in \(M \setminus K\) such that \(v \sim K\); such a vertex exists as \(M\) is prime. Let \(L\) be a consistent submodule of \(M\) containing \(v\). Assume that \(v\) is oriented from \(L^0\) to \(L^1\) and \(S\) appears between \(L^0\) and \(L^1\) in \(\pi^m(M)\) (for the other cases the algorithm works similarly). The algorithm starts with \(p_m(S) = (S)\). Then, for every node \(N \in N_T[M]\) such that \(V_{T \setminus N} \subset left(v)\) the algorithm computes the sets \(S_1\) and \(S_2\), where
\[
S_1 = \{ s^0 \in S : \text{the vertex } s \text{ has } N \text{ on its left side} \} \cup \{ s^1 \in S : \text{the vertex } s \text{ has } N \text{ on its right side} \}
\]
and
\[
S_2 = S \setminus S_1.
\]
If \(S_1 = \emptyset\) or \(S_2 = \emptyset\), the algorithm does nothing. Otherwise, the algorithm refines \(p_m(S)\) by \((S_1, N, S_2)\), which means that it finds the unique set \(S' \in p_m(S)\) such that \(S' \cap S_1 \neq \emptyset\) and \(S' \cap S_2 \neq \emptyset\) and then it replaces \(S'\) in \(p_m(S)\) by the triple \((S' \cap S_1, N, S' \cap S_2)\).

The correctness of the algorithm is asserted by the following observations. Let \(\phi\) be any conformal model of \(G_e\) such that \(\pi_{\phi[M]} = \pi^m(M)\). Let \(\phi' \equiv \phi|M \cup N_T[M]\). Note that \(\phi(S)\) are on the left side of \(\phi'(v)\). If \(V_{T \setminus M}(N) \subset right(v)\), then \(\phi'(N)\) is on the right side of \(\phi'(v)\) and hence \(N\) can not appear in \(p_{\phi'}(S)\). If \(V_{T \setminus M}(N) \subset left(v)\), then \(N\) may occur in \(p_{\phi'}(S)\). Note that \(S_1\) and \(S_2\) are defined such that the letters from \(\phi'(S_1)\) appear before \(\phi'(N)\) and the letters from \(\phi'(S_2)\) appear after \(\phi'(N)\) when we traverse \(\phi'\) from \(\phi(v^0)\) to \(\phi(v^1)\). If \(S_1 = \emptyset\) or \(S_2 = \emptyset\), then \(N\) does not appear in \(T_{\phi}(S)\). Otherwise, the refinement of \(p_m(S)\) by \((S_1, S_2)\) asserts that \(p_m(S)\) agrees with \(p_{\phi'}(S)\). For every two consecutive slots from \(\pi^m(M)\) the sets \(p_m(S, T)\) can be computed similarly. \(\square\)
For every slot \( S \) in \( \pi^m(M) \), \( p_m(S) \) is called a pattern of the slot \( S \) in \( \pi^m(M) \), \((N_1, \ldots, N_{l-1})\) is called a sequence of nodes of the slot \( S \) in \( \pi^m(M) \), and \((S_1, \ldots, S_l)\) is called an ordered partition of the slot \( S \) in \( \pi^m(M) \). For every two consecutive slots \((S, T)\) in \( \pi^m(M) \), \( p_m(S, T) \) is called a set of nodes between \( S \) and \( T \) in \( \pi^m(S) \).

Suppose \( K_1, \ldots, K_n \) is a consistent decomposition of \( M \). We recall that every circular order of the slots \( \pi^m(M) \) of \( M \) keeps for the slots \( K_0^0, K_1^1 \) the orientations \( K_0^0 = (\prec_{K_0}, K_0^1, K_1^1) \) and \( K_1^0 = (\prec_{K_0}, K_1^1, K_0^1) \). Let \( \text{inside}_m(K_i) \) be the set of all nodes from \( N_T[M] \) that appear either in \( p_m(K_0^0) \) or in \( p_m(K_1^1) \). That is, \( \text{inside}_m(N_i) \) contains all nodes from \( N_T[M] \) that appear either in \( \tau_{\phi}(K_0^0) \) or in \( \tau_{\phi}(K_1^1) \), where \( \phi \equiv \phi | M \cup N_T[M] \) and \( \phi \) is a conformal model of \( G_G \) such that \( \phi | M \equiv \phi^m(M) \). Note that for every \( N \in N_T[M] \), \( N \) appears in \( p_0(K_0^0) \) iff \( N \) appears in \( p_1(K_1^1) \), where \( K_0^0, K_1^1 \) are such that \( (K_0^0, K_1^1) \). So, \( \text{inside}_m(K_i) \) is independent on \( m \) and we can set \( \text{inside}(M) = \text{inside}_0(K_0^0) = \text{inside}_0(K_1^1) \). Now, we enrich the orientations of the slots \( K_0^0 \) and \( K_1^1 \) in \( \pi^m(M) \) by the patterns \( p_m(K_0^0) \) and \( p_m(K_1^1) \), obtaining two extended orientations of \( K_i \) for every \( m \in \{0, 1\} \):

\[
\mathcal{K}_{i,m}^0 = (\prec_{K_0}, K_0^0, K_1^1, p_m(K_0^0), p_m(K_1^1)) \quad \text{and} \quad \mathcal{K}_{i,m}^1 = (\prec_{K_0}, K_1^1, K_0^0, p_m(K_0^0), p_m(K_1^1)).
\]

Note that the orientations of \( K_0^0 \) and \( K_1^1 \) in \( \pi^0(M) \) and \( \pi^1(M) \) are different if \( \text{inside}(K_i) \neq \emptyset \) (precisely, the patterns in the orientations of \( K_0^0 \) and \( K_1^1 \) differ when we change \( m \)).

**Definition 5.20.** Let \( K_i \) be a consistent submodule of \( M \) and let \( \mathcal{K}_{i,m}^0, \mathcal{K}_{i,m}^1 \) be an extended orientation of \( K_i \). A pair \( (\tau^0, \tau^1) \), where \( \tau^0 \) and \( \tau^1 \) are words containing all the labeled letters from \( K_0^0 \cup K_1^1 \) and all the nodes from \( \text{inside}(K_i) \) is an extended admissible model for \( \mathcal{K}_{i,m}^0 \) if

- \( (\tau^0 | K_0^0, \tau^1 | K_1^1) \) is an admissible model for \( (\prec_{K_i}, K_0^0, K_1^1) \),
- the order of the nodes in \( \tau^1 \) equals to the order of the nodes in \( p_m(K_1^1) \), for every \( j \in \{0, 1\} \),
- the ordered partition of \( K_0^0 \) arisen from \( \tau^1 \) equals to the ordered partition of \( K_1^1 \) in \( p_m(K_1^1) \), for every \( j \in \{0, 1\} \).

The previous definition was formulated so as to get the following claim.

**Claim 5.21.** Let \( M \) be a prime module in \( T_{NM} \). Let \( \phi \) be a conformal model of \( G_G \) such that \( \phi | M \equiv \pi^m(M) \) and let \( \phi' \equiv \phi | M \cup N_T[M] \). Then:

- \( (\tau_{\phi'}(K_0^0), \tau_{\phi'}(K_1^1)) \) is an extended admissible model for \( \mathcal{K}_0^m \),
- \( (\tau_{\phi'}(K_1^1), \tau_{\phi'}(K_0^0)) \) is an extended admissible model for \( \mathcal{K}_1^m \).

**Definition 5.22.** Let \( M \) be a prime module in \( T_{NM} \) and let \( m \in \{0, 1\} \). A circular word \( \phi \) on the set of letters \( M^* \cup N_T[M] \) is an extended admissible model for \( \pi^m(M) \) if

- \( \pi(\phi | M) = \pi^m(M) \),
- for every slot \( K_0^0 \) in \( \pi^m(M) \) the pair \( (\tau_{\phi'}(K_0^0), \tau_{\phi'}(K_1^1)) \) is an admissible model for \( \mathcal{K}_i^m \),
for every two consecutive slots \((K'_i, K''_i)\) in \(\pi^m(M)\), \(\pi_\phi(K'_i, K''_i) = p_m(K'_i, K''_i)\).

A circular word \(\phi'\) on the set of letters \(M^* \cup N_T[M]\) is an extended admissible model for \((M, \sim)\) if \(\phi'\) is extended admissible model for \(\pi^m(M)\) for some \(m \in \{0, 1\}\).

Again, the previous definition was formulated so as to get the following claim.

**Claim 5.23.** If \(\phi\) is a conformal model of \(G_c\) such that \(\pi_{\phi[M]} = \pi^m(M)\) for a prime module \(M\) in \(T_{NM}\), then \(\phi([M \cup N[M]])\) is an extended admissible model for \(\pi^m(M)\).

Note that the reflection of an admissible model for \(\pi^0(M)\) is an admissible model for \(\pi^1(M)\). Using this observation a curious reader may verify the relation between \(p_0(K'_0), p_0(K''_0)\) and \(p_1(K'_0), p_1(K''_0)\).

Now, suppose that \(M\) is a serial module in \(T_{NM}\). Suppose \(N_1, \ldots, N_k\) are the children of \(M\) in \(\mathcal{M}(G_c)\). Recall that every \(N_i\) is a proper prime or a proper parallel module in \(\mathcal{M}(G_c)\). In particular, \(N_i\) has two orientations, and hence the set \(\text{inside}(N_i)\) is defined.

To achieve an analogous description of serial modules as we obtained for prime ones, we first partition \(M\) into consistent submodules. If \(\text{inside}(N_i) \neq \emptyset\), we denote \(N_i\) as a consistent submodule of \(M\). In the set of the remaining elements in \(M\), that is, in the set \(\bigcup\{N_i : i \in [k]\}\) and \(\text{inside}(N_i) = \emptyset\) we introduce an equivalence relation \(K\) defined such that

\[
uKv \iff \{\text{left}(u) \cap (V \setminus M), \text{right}(u) \cap (V \setminus M)\} = \\
\{\text{left}(v) \cap (V \setminus M), \text{right}(v) \cap (V \setminus M)\}.
\]

The equivalence classes of \(K\) are the remaining consistent submodules of \(M\). Note that \(uKv\) for every \(u, v \in N_i\) such that \(\text{inside}(N_i) = \emptyset\). Thus, every consistent submodule of \(M\) is the union of some children of \(M\). The set of all consistent submodules of \(M\) forms a partition of \(M\), called a consistent decomposition of the serial module \(M\). Note that it might happen that \(M\) has only one consistent submodule. This take place when \(\text{inside}(N_i) = \emptyset\) for every child \(N_i\) of \(M\) and when every conformal model \(\phi\) of \(G_c\) is of the form

\[
\phi([M \cup N_T[M]]) \equiv \tau N \tau' N' \quad \text{or} \quad \phi([M \cup N_T[M]]) \equiv \tau N \tau',
\]

where \(N, N'\) are the only nodes in \(N_T[M]\) and \(\tau, \tau'\) are labeled permutations of \(M\).

Suppose \(K_1, \ldots, K_n\) is a consistent decomposition of \(M\). A skeleton of \(M\) is a subset \(\{s_1, \ldots, s_n\}\) of \(M\) such that \(s_i \in K_i\) for every \(i \in [n]\). The next lemma can be seen as an analogue of Lemma 5.11

**Lemma 5.24.** Suppose \(M\) is a serial module in \(\mathcal{M}(V)\), \(K_1, \ldots, K_n\) is a consistent decomposition of \(M\), and \(S = \{s_1, \ldots, s_n\}\) is a skeleton of \(M\). Then:

1. There exist two conformal models \(\phi_S^0\) and \(\phi_S^1\) of \((S, \sim)\), one being the reflection of the other, such that for every conformal model \(\phi\) of \(G_c\) we have \(\phi[S] \equiv \phi_S^m\) for some \(m \in \{0, 1\}\).
2. If \(n \geq 2\), for every conformal model \(\phi\) of \(G_c\) and every \(i \in [n]\), the set \(\phi[K_i]\) consists of two contiguous subwords of the circular word \(\phi[M]\).
Proof. Suppose \( K_1, \ldots, K_n \) are enumerated such that \( i < j \) if \( \text{inside}(K_i) \neq \emptyset \) and \( \text{inside}(K_j) = \emptyset \). Let \( S_i = \{s_1, \ldots, s_i\} \). Note that \( (S_i, \sim) \) is a clique for every \( i \in [n] \). We claim that for every \( i \in [2, n] \) there exist two models of \( (S_i, \sim) \), say \( \phi_{S_i}^0 \) and its reflection \( \phi_{S_i}^1 \) such that for every conformal model \( \phi \) of \( G_c \), either
\[
\phi|S_i \equiv \phi_{S_i}^0 \quad \text{or} \quad \phi|S_i \equiv \phi_{S_i}^1.
\]
Then (1) follows from the claim for \( i = n \). We prove the claim by induction on \( i \).

Note that \( (S_2, \sim) \) has two conformal models,
\[
\phi_{S_2}^0 \equiv s_1^0 s_2^0 s_1^1 s_2^1 \quad \text{and} \quad \phi_{S_2}^1 \equiv s_1^1 s_2^0 s_1^0 s_2^0,
\]
and \( \phi_{S_2}^0 \) is the reflection of \( \phi_{S_2}^1 \). So, the claim holds for \( i = 2 \). Suppose the claim holds for \( i = j - 1 \) for some \( j \in [3, n] \). To show the claim for \( i = j \) it suffices to show that there is a unique extension \( \phi_{S_j}^0 \) of \( \phi_{S_{j-1}}^0 \) on the set \( S_j \), such that \( \phi|S_j \equiv \phi_{S_j}^0 \) holds for every conformal model \( \phi \) of \( G_c \) such that \( \phi|S_{j-1} \equiv \phi_{S_{j-1}}^0 \).

Suppose for a contradiction that there are two conformal models of \( G_c \), \( \phi \) and \( \phi' \), such that \( \phi|S_{j-1} \equiv \phi'|S_{j-1} \equiv \phi_{S_{j-1}}^0 \) and \( \phi|S_j \neq \phi'|S_j \). That is, the chords \( \phi(s_j) \) and \( \phi'(s_j) \) extend \( \phi_{S_{j-1}}^0 \) into two non-equivalent models. Hence, there are two different vertices \( s_l, s_k \in S_{j-1} \) such that
\[
\phi|\{s_l, s_k\} \equiv s_l^0 s_k^0 s_l^1 s_k^1, \\
\phi'|\{s_l, s_k, s_j\} \equiv s_l^1 s_k^0 s_l^0 s_k^1 s_j^1, \\
\phi'|\{s_l, s_k, s_j\} \equiv s_k^0 s_l^1 s_j^0 s_k^1 s_j^0 s_k^1 s_j^1.
\]

where \( s_l^i, s_k^i, s_j^i, s_j^{**} \) are such that \( \{s_l^i, s_k^i\} = \{s_j, s_j^{**}\} = \{s_j^0, s_j^1\} \). First, suppose that
\[
\phi|\{s_l, s_k, s_j\} \equiv s_l^0 s_k^0 s_j^0 s_k^1 s_l^1 s_k^1 s_j^1.
\]
Let \( \phi_M \) and \( \phi'_M \) be equal \( \phi|(M \cup N_T(M)) \) and \( \phi'|(M \cup N_T(M)) \), respectively. Suppose that \( \text{inside}(K_j) \neq \emptyset \). Let \( N \in N_T[M] \) be a node such that \( N \in \text{inside}(K_j) \).

Consider the position of \( \phi_M(N) \) and \( \phi'_M(N) \) relatively to the chords \( \phi_M(s_j), \phi_M(s_k) \) in \( \phi_M \) and \( \phi'_M(s_k), \phi'_M(s_j) \) in \( \phi'_M \). Note that in \( \phi_M \) the point \( \phi_M(N) \) is either on the left side of both \( \phi_M(s_k) \) and \( \phi_M(s_j) \) or on the right side of both \( \phi_M(s_k) \) and \( \phi_M(s_j) \). However, in \( \phi'_M \) the point \( \phi'_M(N) \) is either on the right side of \( \phi'_M(s_k) \) and the left side of \( \phi'_M(s_j) \) or on the left side of \( \phi'_M(s_k) \) and the right side of \( \phi'_M(s_j) \). However, this is not possible as \( \phi \) and \( \phi' \) are conformal models of \( G_c \). So, we must have \( \text{inside}(K_j) = \emptyset \). For the same reason we also have \( \text{inside}(K_j) = \emptyset \). To complete the proof in this case, we show that for every \( N \in N_T[M], V_{T\backslash M}(N) \subseteq \text{left}(s_l) \) iff \( V_{T\backslash M}(N) \subseteq \text{left}(s_j) \). If this is the case, then \( s_j K s_l \), which contradicts that \( s_j \) and \( s_l \) are from two different equivalence classes of \( K \)-relation. Suppose that \( V_{T\backslash M}(N) \subseteq \text{left}(s_l) \) and \( V_{T\backslash M}(N) \subseteq \text{right}(s_j) \). Then \( \phi_M(N) \) is between \( \phi_M(s_j) \) and \( \phi_M(s_l) \) in \( \phi_M \) and between \( \phi'_M(s_l) \) and \( \phi'_M(s_j) \) in \( \phi' \). Thus, \( \phi_M(N) \) is on the right side of \( \phi_M(s_l) \) and on the left side of \( \phi'_M(s_l) \). This can not be the case. The second implication is proved analogously. All the remaining cases corresponding to different orientations of \( \phi(s_j) \) and \( \phi'(s_j) \) are proven in a similar way.
Let $\phi$ be any conformal model of $G_c$ and let $\phi_M = \phi|(M \cup N_T(M))$. Statement (2) obviously holds if $K_i$ is a child of $M$ in $\mathcal{M}(G_c)$. So, suppose $K_i$ is the union of at least two children of $M$ in $\mathcal{M}(G_c)$. Since $n \geq 2$, there is $s_j \in S$ such that $s_j \sim K_i$. Denote by $l^0$ and $l^3$ the first and the last labeled letter from $K_i$, respectively, if we traverse $\phi_M$ from $s^0_i$ to $s^1_i$. Denote by $r^0$ and $r^3$ the first and the last labeled letter from $K_i$, respectively, if we traverse $\phi_M$ from $s^1_i$ to $s^0_i$. To show statement (2) suppose for a contrary that there is $v \in M \setminus K_i$ such that $\phi(v)$ has an end, say $\phi(v')$, between $\phi(l^0)$ and $\phi(l^3)$. Suppose $v''$ is such that $\{v', v''\} = \{v^0, v^1\}$. Note that for every child $M'$ of $M$ such that $M' \subseteq K_i$, inside($M') = \emptyset$. Thus, there are children $M_1, M_2$ of $M$ that satisfy the following properties. If we traverse from $\phi_M(l^0)$ to $\phi_M(l^3)$ then we encounter an end of every chord from $\phi(M_1)$ first, then $\phi(v')$, and then an end of every chord from $\phi(M_2)$. Since $M$ is serial, by Lemma 4.13 if we traverse from $\phi_M(r^0)$ to $\phi_M(r^3)$ then we encounter an end of every chord from $\phi(M_2)$ first, then $\phi(v'')$, and then an end of every chord from $\phi(M_1)$. Denote by $l^1$ and $l^2$ the last labeled letter from $M_1$ and the first labeled letter from $M_2$ if we traverse $\phi_M$ from $\phi_M(l^0)$ to $\phi_M(l^3)$. Similarly, denote by $r^1$ and $r^2$ the last labeled letter from $M_2$ and the first labeled letter from $M_1$ if we traverse $\phi_M$ from $\phi_M(r^0)$ to $\phi_M(r^3)$. First, note that there is a node $N' \in N_T[M]$ such that $\phi_M(N')$ is either between $\phi_M(l^1)$ or $\phi_M(l^2)$ or between $\phi_M(r^1)$ and $\phi_M(r^2)$. Otherwise, we have that $\nu K M_1$ and $\nu K M_2$, which contradicts with $v \notin K_i$. However, we can not have a node from $N_T[M]$ between $\phi(l^1)$ and $\phi(l^2)$ and between $\phi(r^1)$ and $\phi(r^2)$, as then $M_i$ is not in $K$-relation with $M_2$. So, suppose $N' \in N_T[M]$ is between $\phi(l^1)$ and $\phi(l^2)$ and between $\phi(r^1)$ and $\phi(r^2)$ there is no node from $N_T[M]$. Furthermore, note that there is a node $N''$ between $\phi_M(l^3)$ and $\phi_M(r^0)$ or between $\phi_M(r^3)$ and $\phi_M(l^0)$ as otherwise $s_j$ would be in $K$-relation with both $M_1$ and $M_2$. In any case, $N'$ and $N''$ prove that $M_1$ and $M_2$ are not in $K$-relation, a contradiction.

Let $\phi'$ be a conformal model of $G_c$ and let $\phi$ be the restriction of $\phi'$ to $M$. Because of the analogy between Lemmas 5.11 and Lemma 5.21 for every serial child $M$ of $V$ we can introduce the same parameters as for prime children of $V$. The only difference is when $M$ is a serial module that has exactly one consistent submodule $M$. In this case $\phi'$ takes one of the form given by (3), and consequently $\tau^0_{i, \phi'}$ and $\tau^1_{i, \phi'}$ are defined such that they are equal to $\tau$ and $\tau'$, where $\tau^0_{i, \phi'}$ contains $s^0_i$ and $\tau^1_{i, \phi'}$ contains $s^1_i$. In particular, for serial children of $V$ we obtain an analogue of Claim 5.12.

Claim 5.25. Suppose $M$ is a serial module in $T_{NM}$, $K_1, \ldots, K_n$ is a consistent decomposition of $M$, and $S$ is a skeleton of $M$. For every $i \in [n]$ there are labeled copies $K^0_{i, \phi}$ and $K^1_{i, \phi}$ of $K_i$ forming a partition of $K^+_i$ and a transitive orientation $<^0_{K_i}$ of $(K_i, \|)$ such that

$$(<^0_{K_i, \phi}, K^0_{i, \phi}, K^1_{i, \phi}) = (<^0_{K_i}, K^0_i, K^1_i)$$

for every conformal model $\phi$ of $G_c$ and every $i \in [n]$.
Moreover, there are circular permutations $\pi^0(M), \pi^1(M)$ of $\{K^0_1, K^1_0, \ldots, K^0_n, K^1_n\}$ such that

$$\pi_{\phi|M} = \begin{cases} 
\pi^0(M) & \text{if } \phi(S) = \phi^0_S \\
\pi^1(M) & \text{if } \phi(S) = \phi^1_S
\end{cases} \quad \text{for every conformal model } \phi \text{ of } G_c.$$

Moreover, $\pi^0(M)$ is the reflection of $\pi^1(M)$.

**Proof.** We use Algorithm 2 to compute all the invariants of $M$ with only one difference: instead the vertex $s_j$ we choose any vertex $v$ from $V \setminus M$. □

Given Claim 5.25 we define analogously the pattern $p_m(S)$ for every slot $S$ in $\pi^m(M)$ and the sets $p^m(S, T)$ for every two consecutive slots $(S, T)$ in $\pi^m(M)$. With similar ideas we get an analogue of Claim 5.19. Eventually, we extend the notions of extended orientations for every slots $K^0_i, K^1_i$ of a consistent submodule $K_i$ of $M$, the notions of extended admissible models for $\mathbb{K}^0_i$ and $\mathbb{K}^1_i$ for $M$, and the notions of extended admissible models for $(M, \sim)$.

Finally, we are ready to provide a theorem describing all conformal models of $G_c$ in the case when the graph $G_c$ is disconnected. In the next theorem we will be using two properties of an extended admissible model $\phi^M$ of $(M, \sim)$, where $M$ is a module in $T_{NM}$:

\begin{enumerate}
  \item[(C1)] $\phi^M|M$ is a conformal model of $(M, \sim)$, and
  \item[(C2)] $\phi^M(N')$ is on the left side of $\phi(u) \iff V_{T \setminus M}(N') \subset left(u)$,
  $\phi^M(N')$ is on the right side of $\phi(u) \iff V_{T \setminus M}(N') \subset right(u)$,
\end{enumerate}

for every $N \in N_T[M]$ and every $v \in M$.

**Theorem 5.26.** Suppose $G$ is a circular-arc graph with no twins and no universal vertices such that $G_c$ is disconnected. Then, for every module $M$ in $T_{NM}$ the circular word $\phi|(M \cup N_T[M])$ is an extended admissible model of $(M, \sim)$ and for every node $N \in T_{NM}$ the circular word $\phi|N_T[N]$ is a circular permutation of $N_T[N]$.

On the other hand, given an extended admissible model $\phi^M$ for every module $M \in T_{NM}$ and a circular permutation $\pi(N)$ of $N_T[N]$ for every node $N \in T_{NM}$, there is a conformal model $\phi$ of $G_c$ such that

\begin{equation}
(\ast) \quad \phi|(M \cup N_T[M]) \equiv \phi^M \quad \text{for every module } M \text{ in } T_{NM}, \quad \phi|(N \cup N_T[N]) \equiv \pi(N) \quad \text{for every node } N \text{ in } T_{NM}.
\end{equation}

**Proof.** The first part of the proof results from the considerations carried out in this subsection.

Fix any module $R$ in $T_{NM}$ as the root of $T_{NM}$. Let $A$ be a vertex in the rooted tree $T_{NM}$. By $V_T(A) \subset V$ we denote the set of all vertices contained in $A$ and in all modules descending $A$ in $T_{NM}$. By $B$ we denote the parent of $A$ in $T_{NM}$ or empty word if $A = R$. We proceed $T_{NM}$ bottom-up and for any vertex $A$ in the rooted tree $T_{NM}$ we construct a word $\phi_A$ consisting of all the labeled letters from $V_T^*(A)$ such that the circular word $\phi_A \equiv \phi_A^0 B$ satisfies the following properties:
(1) For every \( u, v \) in \( V_T(A) \), \( \phi_A(u) \) is on the left side of \( \phi_A(v) \) if \( u \in left(v) \).

(2) For every \( u, v \) in \( V_T(A) \), \( \phi_A(u) \) is on the right side of \( \phi_A(v) \) if \( u \in right(v) \).

(3) For every \( u, v \) in \( V_T(A) \), \( \phi_A(u) \) intersects \( \phi_A(v) \) if \( u \sim v \).

(4) For every \( v \) in \( V_T(A) \), \( \phi(B) \) is on the left side of \( \phi(v) \) if \( V_{T \setminus A}(B) \subset left(v) \).

(5) For every \( v \) in \( V_T(A) \), \( \phi(B) \) is on the right side of \( \phi(v) \) if \( V_{T \setminus A}(B) \subset right(v) \).

If we construct such a word \( \phi'_A \) for every vertex \( A \) in \( T_{NM} \), then \( \phi_R \), where \( R \) is the root of \( T_{NM} \), is a conformal model of \( G_c \). Moreover, from the construction of \( \phi'_A \), it will be clear that \( \phi_R \) satisfies the desired properties (in particular, \( \phi_R \) is such that \( \phi'_A \) is a contiguous subword for every \( A \in T_{NM} \)).

Suppose \( A \) is a leaf in \( T_{NM} \). So, \( A \) is a module in \( T_{NM} \). Let \( \phi^A \) be an admissible model of \( (A, \sim) \). Then, \( \phi^A \equiv \tau^AB \), where \( B \) is the parent of \( A \) in \( T_{NM} \). We set \( \phi'_A = \tau^A \) and we claim that \( \phi'_A \) satisfies properties (1)–(5). By property (C1) of \( \phi^A \) we deduce that \( \phi_A \) satisfies \( (1) - (3) \). By property (C2) of \( \phi^A \) we deduce that \( \phi_A \) satisfies \( (1) - (5) \).

Now, suppose that \( A \) is a node. Suppose \( A_1, \ldots, A_k \) are the children of \( A \) in \( T_{NM} \). Suppose \( \phi'_A \) is the word that has been constructed for \( A_i \). Suppose \( A_1, \ldots, A_k \) are enumerated such that \( \pi(A) \equiv BA_1 \ldots A_k \), where \( \pi(A) \) is a circular permutation of \( N_T[A] \) associated with the node \( A \). Let \( \phi'_A = \phi'_A_1 \ldots \phi'_A_k \). We claim that \( \phi_A \equiv B\phi'_A \) satisfies conditions (1)–(5). Suppose \( u, v \) in \( M \). If \( uv \) in \( V_T(A_i) \) for some \( i \in [k] \), (1)–(3) are satisfied by \( \phi_A \) as they are satisfied by \( \phi_{A_i} \). Suppose \( u \in A_i \) and \( v \in A_j \). But then, \( u \in V_{T \setminus A_j}(A) \). In particular, \( u \in left(v) \) yields \( V_{T \setminus A_j}(A) \subset left(v) \) by property (C2) of \( \phi^A \). Hence, in the word \( \phi_{A_j} \), the letter \( A_j \) is on the left side of \( \phi_{A_j}(v) \). Since \( \phi_A \) arises from \( \phi_{A_i} \) by substituting the letter \( A \) by the word including \( u^i, u^j \), we deduce that \( \phi(u) \) is on the left side of \( \phi_A(v) \). We prove similarly that \( u \in right(v) \) yields \( \phi(u) \) is on the right side of \( \phi_A(u) \). Now, let us prove that \( \phi_A \) satisfies (1). Let \( v \) in \( A_j \). We have \( V_{T \setminus A_j}(A) \subset left(v) \) if \( V_{T \setminus A_j}(B) \subset left(v) \) for every \( v \) in \( V_T(A) \). Suppose \( v \) in \( A_j \). Hence, \( \phi_{A_j}(A) \) is on the left side of \( \phi_{A_j}(v) \). But the word \( \phi_A \) arises from \( \phi_{A_i} \) by substituting the letter \( A \) by the word containing \( B \), so we have \( \phi_A(B) \) is on the left side of \( \phi_A(v) \). Property (5) is proved similarly.

Suppose \( A \) is a module in \( T_{NM} \) with children \( A_1, \ldots, A_k \). Let \( \phi^A \) be a an extended admissible model of \( (A, \sim) \) associated with \( A \). In the word \( \phi^A \) we exchange the letter \( A_i \) by the word \( \phi'_A \), for every \( i \in [k] \). We denote the obtained word by \( \phi_A \). We set \( \phi'_A \) such that \( \phi_A = \phi'_A B \). Now, we claim that \( \phi_A \) satisfies (1)–(5). If \( u, v \in A \), properties (1)–(3) follow by property (C1) of the conformal model \( \phi^A \) of \( (A, \sim) \). If \( v \in A \) and \( u \in A_i \), properties (1)–(2) follow by property (C2) of \( \phi^A \). If \( v \in A_i \), the proof of (1)–(2) is the same as in the previous case (now \( u \) might belong to \( A \), but \( \phi_A(u) \) is on the left side of \( \phi_A(v) \) if \( \phi_{A_i}(A) \) is in the left side of \( \phi_{A_i}(v) \)). It remains to prove (1)–(5). If \( v \in A \), properties (1)–(5) follow by property (C2) of \( \phi^A \). If \( v \in A_j \) for some \( j \in [k] \), properties (1)–(5) are proved similarly to the previous case.
Now, it is clear from the construction of $\phi_R$ that
\[
\phi_R(M \cup N_T[M]) \equiv \phi^m \quad \text{for every module } M \text{ in } T_{NM}, \\
\phi_R(N \cup N_T[N]) \equiv \pi(N) \quad \text{for every node } N \text{ in } T_{NM}.
\]

The result of this section can be used to provide the recognition algorithm for circular-arc graphs for the case when the input graph $G$ is such that $G_c$ is disconnected. For this purpose we need a short definition: an extended model $\phi^m$ of a module $M \in T_{NM}$ is conformal if $\phi^m$ satisfies properties $\{C_1\}$ and $\{C_2\}$.

**Theorem 5.27.** Let $G$ be a graph with no twins and no universal vertices. Suppose that $G$ is such that $G_c$ is disconnected. The following statements are equivalent:

1. $G$ is a circular-arc graph.
2. For every child $M$ of $V$ in $M(G_c)$: $(M, \sim)$ admits an extended admissible model $\phi^M$ and $\phi^M$ is conformal.

6. ISOMORPHISM OF CIRCULAR-ARC GRAPHS

Two graphs $G$ and $H$ are isomorphic if there exists a bijection $\alpha : V(G) \to V(H)$ such that $uv \in E(G)$ iff $\alpha(u)\alpha(v) \in E(H)$. Our goal is to provide an algorithm that tests whether two circular-arc graphs $G$ and $H$ given on the input are isomorphic.

Note that every isomorphism between $G$ and $H$ maps universal vertices in $G$ into universal vertices in $H$. So, if $G$ and $H$ have different number of universal vertices, then $G$ and $H$ are not isomorphic. Otherwise, $G$ and $H$ are isomorphic iff $G'$ and $H'$ are isomorphic, where $G'$ and $H'$ are obtained from $G$ and $H$ by deleting all universal vertices in $G$ and $H$, respectively. So, in the rest of the paper we assume $G$ and $H$ have no twins.

Suppose $G$ and $H$ are circular-arc graphs with no universal vertices. For every vertex $v \in V(G)$ in $G$ we define the set of its twins in $G$:

$$T_G(v) = \{ w \in V(G) : N_G[v] = N_G[w] \},$$

where $N_G[v]$ is the closed neighbourhood of $v$ in $G$. Clearly, $\{T_G(v) : v \in V\}$ forms a partition of the set $V(G)$. Let $V'$ be a set containing a representant from every set $\{T_G(v) : v \in G\}$. Let $G'$ be a graph induced by the set $V'$ in $G$ and let $m_G(v)$ denotes the number of elements in the set $T_G(v)$, for every $v' \in V'$. Note that $G'$ has no universal vertices and no twins. We call $(G', m_G)$ a circular-arc graph with multiplicities or simply a circular-arc graph. A pair $(H', m_H)$ for the graph $H$ is defined analogously. We say $(G', m_G)$ and $(H', m_H)$ are isomorphic if there is an isomorphism $\alpha'$ from $G'$ to $H'$ that preserves multiplicities, that is, that satisfies $m_G(v) = m_H(\alpha'(v))$ for every for every $v \in V(G')$.

**Claim 6.1.** $G$ and $H$ are isomorphic if and only if $(G', m_G)$ and $(H', m_H)$ are isomorphic.
**Proof.** Let \( \alpha \) be an isomorphism between \( G \) and \( H \). Note that for every \( v \in V(G') \) \( \alpha \) maps every twin \( u \) of \( v \) in \( G \) into a twin \( \alpha(u) \) of \( \alpha(v) \) in \( H \). Hence, \( \{\alpha(w) : w \in T_G[v]\} = T_H[\alpha(v)] \). In other words, \( \alpha \) maps every set \( T_G[v] \) into a set \( T_H[\alpha(v)] \). So, \( \alpha|V' \) is an isomorphism between \( (G', m_G) \) and \( (H', m_H) \). The converse implication is trivial. \( \square \)

In the rest of this section we assume that \( (G, m_G) \) and \( (H, m_H) \) are given on the input, where \( G \) and \( H \) are circular-arc graphs with no universal vertices and no twins. By \( G_c \) and \( H_c \) we denote the graphs associated with \( G \) and \( H \), respectively.

**Theorem 6.2.** Let \( (G, m_G) \) and \( (H, m_H) \) be two circular-arc graphs and let \( \alpha \) be a bijection from \( V(G) \) to \( V(H) \). Then \( \alpha : V(G) \rightarrow V(H) \) is an isomorphism from \( (G, m_G) \) to \( (H, m_H) \) iff \( \alpha \) preserves multiplicities and for every pair \( (u, v) \in V(G) \times V(G) \):

1. \( u \in \text{left}(v) \) if and only if \( \alpha(u) \in \text{left}(\alpha(v)) \).
2. \( u \in \text{right}(v) \) if and only if \( \alpha(v) \in \text{right}(\alpha(v)) \).

**Proof.** It is clear that every isomorphism \( \alpha \) between \( (G, m_G) \) and \( (H, m_H) \) preserves multiplicities and satisfies (1) and (2).

Suppose \( u, v \in V(G) \). Suppose \( G_c \) and \( H_c \) are associated with \( G \) and \( H \), respectively. Suppose that \( u \) and \( v \) overlap in \( G_c \), which is equivalent to \( u \notin (\text{left}(v) \cup \text{right}(v)) \). Hence, by (1) and (2), we get \( \alpha(u) \notin (\text{left}(\alpha(v)) \cup \text{right}(\alpha(v))) \), which yields that \( \alpha(u) \) overlaps \( \alpha(v) \) in \( H_c \). Further, by (1) and (2) we have that:

- \( u \in \text{left}(v) \) and \( v \in \text{left}(v) \) iff \( \alpha(u) \in \text{left} \alpha(v) \) and \( \alpha(v) \in \text{left} \alpha(v) \),
- \( u \in \text{left}(v) \) and \( v \in \text{right}(v) \) iff \( \alpha(u) \in \text{left} \alpha(v) \) and \( \alpha(v) \in \text{right} \alpha(v) \),
- \( u \in \text{right}(v) \) and \( v \in \text{left}(v) \) iff \( \alpha(u) \in \text{right} \alpha(v) \) and \( \alpha(v) \in \text{left} \alpha(v) \),
- \( u \in \text{right}(v) \) and \( v \in \text{right}(v) \) iff \( \alpha(u) \in \text{right} \alpha(v) \) and \( \alpha(v) \in \text{right} \alpha(v) \).

Hence, \( uv \in E(G) \) iff \( \alpha(u) \alpha(v) \in E(H) \). \( \square \)

If \( \alpha : V(G) \rightarrow V(H) \) satisfies the conditions from the previous lemma, then \( \alpha \) is said to be an isomorphism between \( (G_c, m_G) \) and \( (H_c, m_h) \).

Suppose \( \alpha \) is a conformal model of \( G_c \). The image of \( \phi \) by \( \alpha \), denoted \( \alpha(\phi) \), is a circular word on \( V^*(H) \) that arises from \( \phi \) by exchanging the labeled letter \( u^0 \) by \( \alpha^0(u) \) and the labeled letter \( u^1 \) by \( \alpha^1(u) \), for every \( u \in V^*(G) \). Since \( \alpha \) is an isomorphism between \( (G_c, m_G) \) and \( (H_c, m_H) \), \( \alpha(\phi) \) is a conformal model of \( (H_c, m_H) \). We extend the notion of the image of \( \tau \) by \( \alpha \), denoted \( \alpha(\tau) \), on words \( \tau \) consisting of letters from \( V^*(G) \).

### 6.1. Extended admissible models of oriented modules.

Suppose \( K \) is a consistent submodule of \( M \) and suppose

\[ \mathbb{K}_m' = (K, \ll_K, K', K''_m, p_m(K'), p_m(K'')) \]

is an extended orientation of \( K \).
The goal of this subsection is to characterize the structure of all extended admissible models of $K_m'$. By Theorem 4.13, the admissible models of $(K, \prec_K, K', K'')$ are in the correspondence with the transitive orientations $(\prec, \prec)$ of the graphs $(K, ||)$ and $(K, \sim)$, where $\prec = \prec_K$. So, we have the correspondence between the admissible models of $(K, \prec_K, K', K'')$ and the transitive orientations of $(K, \sim)$. However, not every transitive orientation $\prec$ of $(K, \sim)$ gives back an extended admissible model of $K_m'$. Again, using Theorem 4.13, one can check the correspondence between the extended admissible models for $K_m'$ and the set of transitive orientations $(K, \prec)$ of $(K, \sim)$ that satisfy for every $x, y \in K$ the following condition:

$$(7) \quad \text{If } x \sim y \text{ and } x \text{ is before } y \text{ in } p_m(K'), \text{ then } x \prec y.$$ 

The abbreviation $x$ is before $y$ in $p_m(K')$ means that $x^*$ and $y^*$ are in different subsets of the ordered partition associated with $p_m(K')$ and the set containing $x^*$ is before the set containing $y^*$ in $p_m(K')$.

Our goal is to characterize all transitive orientations $(K, \prec)$ of $(K, \sim)$ that satisfy condition (7). We call such orientations admissible for $K_m'$.

Recall from Subsection 4.5 that every transitive orientation $(K, \prec)$ of $(K, \sim)$ can be obtained by an independent transitive orientation of the edges of every strong module in $\mathcal{M}(K, \sim)$ – see Theorem 4.7. Let $A$ be a strong module in $\mathcal{M}(K, \sim)$ with the children $A_1, \ldots, A_k$. Recall that $(A, \sim_A)$ is obtained by restricting $(A, \sim)$ to the edges between the children of $A$.

Suppose $A$ is a prime module in $\mathcal{M}(K, \sim)$. By Theorem 4.16, $(A, \sim_A)$ has two transitive orientations, $\prec^0_A$ and $\prec^1_A$, one being the reverse of the other. An orientation $\prec^i_A$ of $(A, \sim)$ is admissible for $K_m'$ if $\prec^i_A$ satisfies conditions (7) for every two vertices $x, y$ from different children of $A$. In particular, if $\text{inside}(A) = \emptyset$ then both $\prec^0_A$ and $\prec^1_A$ are admissible. Otherwise, only one among $\prec^0_A$ and $\prec^1_A$ is admissible for $K_m'$. Indeed, since $\text{inside}(A) \neq \emptyset$, there are $x, y \in A$ such that $x$ is before $y$ in $p_m(K')$ for some $K'' \in \{K', K''\}$. Since $(A, \sim_A)$ is connected, there are $x', y' \in A$ such that $x' \sim_A y'$ and $x'$ is before $y'$ in $p(K'')$. It means that the orientation $(A, \prec^i_A)$ that satisfies $y \prec_A x'$ is not admissible for $K_m'$.

Now, suppose $A$ is a serial module in $\mathcal{M}(K, \sim)$. The transitive orientations of $(A, \sim_A)$ are in one-to-one correspondence with the linear orders of $A_1, \ldots, A_k$. That is, every transitive orientation of $(A, \sim_A)$ is of the form $A_{i_1} \prec \ldots \prec A_{i_k}$, where $i_1, \ldots, i_k$ is a permutation of $[k]$. Now, we define $\prec^w_A$ relation on the set $\{A_1, \ldots, A_k\}$, where

$$(8) \quad A_i \prec^w_A A_j \iff \text{there is } x \in A_i \text{ and } y \in A_j \text{ such that } x \text{ appears before } y \text{ in } p_m(K') \text{ for some } K'' \in \{K', K''\}.$$ 

We claim that $(\{A_1, \ldots, A_k\}, \prec^w_A)$ is a partial order. Suppose $A_i, A_j, A_l$ are such that $A_i \prec^w_A A_j \prec^w_A A_l$. It means that there are $x \in A_i$, $y, y' \in A_j$, and $z \in A_l$ such that $x$ is before $y$ in either $p_m(K')$ or $p_m(K'')$ and $y'$ is before $z$ in either $p_m(K')$ or $p_m(K'')$. Since $x \sim \{y, y'\}$ as $A$ is serial, we deduce that $x$ is before $y$ in
either \( p_m(K') \) or \( p_m(K'') \). This proves that \( \{A_1, \ldots, A_k\}, \prec_{\pi}^{\omega} \) is a partial order. Furthermore, we claim that \( \{A_1, \ldots, A_k\}, \prec_{\pi}^{\omega} \) is a weak order, which means that \( \{A_1, \ldots, A_k\}, \prec_{\pi}^{\omega} \) can be partitioned into a set of antichains such that every two of them induce a complete bipartite poset in \( \{A_1, \ldots, A_k\}, \prec_{\pi}^{\omega} \). One can check that \( \{A_1, \ldots, A_k\}, \prec_{\pi}^{\omega} \) is a weak order iff it does not have three elements \( A_i, A_j, A_l \) such that \( A_j \prec_P A_l \) and \( A_i \) is incomparable to \( A_j \) and \( A_l \). Suppose there exists \( A_i, A_j, A_l \) with such properties. It means that there are \( x \in A_j \) and \( y \in A_l \) such that \( x \) is before \( y \) in \( p(K''') \) for some \( K''' \in \{K', K''\} \). Let \( z \) be any vertex from \( A_i \). Clearly, \( z \) is either before \( y \) in \( p_m(K''') \) or \( z \) is after \( x \) in \( p_m(K''') \). So, \( A_i \) is \( \prec_{\pi}^{\omega} \)-comparable to \( A_j \) or \( A_l \). An orientation \( A_i, A_j, A_l \) of \( (A, \prec_A) \) is admissible for \( \mathbb{K}' \) if \( \prec_A \) extends \( \prec_{\pi}^{\omega} \). The result of this subsection are summarized by the following claim.

**Claim 6.3.** There is a one-to-one correspondence between the set of admissible orientations \((K, \prec) \) for \( \mathbb{K}' \) and the families

\[
\{(A, \prec_A) : A \in \mathcal{M}(K, \sim) \text{ and } \prec_A \text{ is an admissible orientation of } (A, \sim_A) \text{ for } \mathbb{K}' \}
\]

given by \( x \prec y \iff x \prec_A y \), where \( A \) is the module in \( \mathcal{M}(K, \sim) \) such that \( x \sim_A y \).

### 6.2. Local isomorphisms between oriented modules.

Suppose that

\[
\mathbb{M} = (M, <_M, M^0, M^1, p(M^0), p(M^1)) \text{ and } \mathbb{N} = (N, <_N, N^0, N^1, p(N^0), p(N^1))
\]

are two extended orientations of some consistent modules \( M \) and \( N \) from \( \mathcal{M}(G_c) \) and \( \mathcal{M}(H_c) \), respectively. Let \( u \in M \). We say that \( u \) occurs in \( \mathbb{M} \) at position \((i, j)\), written \( \text{pos}(u) = (i, j) \) if \( u \) is in the \( i \)-th subset of the ordered partition \( p(M^0) \) and in the \( j \)-th subset of the ordered partition \( p(M^0) \). The position of \( u \in N \) is defined analogously.

**Definition 6.4.** Let \( A \) be a strong module in \( \mathcal{M}(M, \sim) \) and \( B \) be a strong module in \( \mathcal{M}(N, \sim) \). We say that \( A \) and \( B \) are locally isomorphic (in \( \mathbb{M} \) and \( \mathbb{N} \)) if there is a bijection \( \alpha : A \to B \) that satisfies for every \( u, v \in M \):

1. \( u<_M v \iff \alpha(u) <_N \alpha(v) \),

and for every \( u \in M \):

2. \( u \) is oriented from \( M^0 \) to \( M^1 \) iff \( \alpha(u) \) is oriented from \( N^0 \) to \( N^1 \),
3. \( \text{pos}(u) = \text{pos}(\alpha(u)) \) and \( m_G(u) = m_H(\alpha(u)) \),

We say that \( \mathbb{M} \) and \( \mathbb{N} \) are locally isomorphic if they are locally isomorphic on \( M \) and \( N \).

If \( \alpha \) is a local isomorphism between \( \mathbb{M} \) and \( \mathbb{N} \), then \( |p(M^0)| = |p(N^0)| \), and \( |p(M^1)| = |p(N^1)| \). Then, we extend naturally \( \alpha \) on the set \( M \cup \text{inside}(M) \): \( \alpha \) map the the consecutive nodes in \( p(M^0) \) into the consecutive nodes in \( p(N^0) \) and the the consecutive nodes in \( p(M^1) \) into the consecutive nodes in \( p(N^1) \).

Let \( \alpha \) be a local isomorphism between \( \mathbb{M} \) and \( \mathbb{N} \) and let \( (\tau^0, \tau^1) \) be an extended admissible model of \( \mathbb{N} \). Note that the image \( (\alpha(\tau^0), \alpha(\tau^1)) \) of \( (\tau^0, \tau^1) \) by \( \alpha \) is an extended admissible model of \( \mathbb{N} \).
Let $A$ be a strong module in $\mathcal{M}(A, \sim)$. By $\text{height}(A)$ we denote the height of $A$ in the modular decomposition tree $\mathcal{M}(M, \sim)$, that is, the length of the longest path from $A$ to a leaf in $\mathcal{M}(M, \sim)$. We define $\text{height}(B)$ for every $B \in \mathcal{M}(N, \sim)$ similarly. Clearly, if $\text{height}(M) \neq \text{height}(N)$, then $\mathcal{M}$ and $\mathcal{N}$ can not be locally isomorphic.

Now, we provide an algorithm that tests whether $\mathcal{M}$ and $\mathcal{N}$ are locally isomorphic. The algorithm traverses the trees $\mathcal{M}(M, \sim)$ and $\mathcal{M}(N, \sim)$ in the bottom-up order and for every $A \in \mathcal{M}(M, \sim)$ and every $B \in \mathcal{M}(N, \sim)$ such that $\text{height}(A) = \text{height}(B)$ it checks whether $A$ and $B$ are locally isomorphic. If this is the case, the algorithm computes a local isomorphism $\alpha'_{AB}$ between $A$ and $B$. On the other hand, we show that if there is a local isomorphism $\alpha$ between $\mathcal{M}$ and $\mathcal{N}$, then the algorithm denotes $A$ and $\alpha(A)$ as locally isomorphic, where $A$ is any strong module in $\mathcal{M}(M, \sim)$. Hence, the algorithm accepts $\mathcal{M}$ and $\mathcal{N}$ iff they are locally isomorphic.

We distinguish the following cases: both $A, B$ are leaves, both $A, B$ are parallel, both $A, B$ are prime, and both $A, B$ are serial. In the remaining cases, $A$ and $B$ can not be locally isomorphic.

**Case 1:** $A$ is a leaf in $\mathcal{M}(M, \sim)$ and $B$ is a leaf in $\mathcal{M}(N, \sim)$. Let $\{u\} = A$ and $\{v\} = B$. The algorithm denotes $A$ and $B$ as locally isomorphic if and only if the following three conditions hold: $m_G(u) = m_H(v)$, $u^0 \in M^0$ iff $v^0 \in M^0$, and $\text{pos}(u) = \text{pos}(v)$. If this is the case, the algorithm sets $\alpha'_{AB}(u) = v$.

Suppose $A$ and $B$ are denoted as locally isomorphic. One can easily check that $\alpha'_{AB}$ satisfies properties (1)–(3), and hence $\alpha'_{AB}$ is a local isomorphism between $A$ and $B$.

On the other hand, if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic by $\alpha$, $\{\alpha(u)\}$ must be a leaf in $\mathcal{M}(N, \sim)$, and $m_G(u) = m_H(\alpha(u))$, $u^0 \in M^0$ iff $\alpha^0(u) \in M^0$, and $\text{pos}(u) = \text{pos}(\alpha(u))$. Hence, the algorithm denotes $A = \{u\}$ and $\alpha(A) = \{\alpha(u)\}$ as locally isomorphic.

Now, we assume that $A$ and $B$ are not leaves. Clearly, if $A$ and $B$ have different number of children or have different types, then $A$ and $B$ can not be locally isomorphic. We suppose $A_1, \ldots, A_k$ are the children of $A$ in $\mathcal{M}(M, \sim)$ enumerated such that $\tau^0|A_i$ appears before $\tau^0|A_j$ for every $i < j$, where $\tau^0, \tau^1$ is a fixed extended admissible model for $\mathcal{M}$.

**Case 2:** $A$ and $B$ are parallel. Let $B_1, \ldots, B_k$ be the children of $B$ enumerated according to $<_N$, that is, $B_1 <_N \ldots <_N B_k$. The algorithm denotes $A$ and $B$ as locally isomorphic if and only if for every $i \in [k]$ the sets $A_i$ and $B_i$ have been denoted as locally isomorphic. If this is the case, for every $i \in [k]$ and every $u \in A_i$ the algorithm sets $\alpha'_{AB}(u) = v$, where $v \in B_i$ is such that $\alpha'_{A_i,B_i}(u) = v$.

Suppose $A$ and $B$ are denoted as locally isomorphic. Note that $\alpha'_{AB}$ satisfies properties (2)–(4) as the corresponding properties are satisfied by $\alpha'_{A_i,B_i}$ for every $i \in [k]$. Suppose $u_1, u_2$ in $A$ are such that $u_1 <_M u_2$. If $u_1, u_2 \in A_i$ for some $i \in [k]$, then $\alpha'_{AB}(u_1) <_N \alpha'_{AB}(u_2)$. Indeed, $\alpha'_{AB}(u_1) <_N \alpha'_{AB}(u_2)$ iff $\alpha'_{A_i,B_i}(u_1) <_N \alpha'_{A_i,B_i}(u_2)$.
\[\alpha'_{A,B}(u_2)\] by definition of \(\alpha'_A\) and \(\alpha'_B\) as the corresponding properties are satisfied. If \(u_1 \in A_i\) and \(u_2 \in A_j\) for some \(i < j\), then \(\alpha'_{AB}(u_1) <_N \alpha'_{AB}(u_2)\) as \(\alpha_{AB}(u_1) \in B_i, \alpha_{AB}(u_2) \in B_j,\) and \(B_i <_N B_j\). This proves \(\alpha'_{AB}\) satisfies (I).

On the other hand, suppose \(\mathcal{M}\) and \(\mathcal{N}\) are locally isomorphic by \(\alpha\). From inductive hypothesis, for every \(i \in [k]\) the sets \(A_i\) and \(\alpha(A_i)\) have been marked as locally isomorphic. Since \(\alpha(A_1) <_N \ldots <_N \alpha(A_k)\) by (I), the algorithm denotes \(A\) and \(\alpha(A)\) as locally isomorphic.

**Case 3:** \(A\) and \(B\) are prime. For every (at most two) admissible orientations \((B, \prec_B)\) of \((B, \sim_B)\) the algorithm does the following. First, it computes the order \(B_1, \ldots, B_k\) of the children of \(B\) in which \(B_i\) is before \(B_j\) iff \(B_i <_N B_j\) or \(B_i \prec_B B_j\).

The algorithm denotes \(A\) and \(B\) as locally isomorphic if for every \(i \in [k]\) the sets \(A_i\) and \(B_i\) have been denoted as locally isomorphic. If this is the case, for every \(i \in [k]\) and every \(u \in A_i\) the algorithm sets \(\alpha'_{AB}(u) = v\), where \(v \in B_i\) is such that \(\alpha'_{A,B_i}(u) = v\).

Suppose \(A\) and \(B\) have been marked as locally isomorphic at the time when an admissible orientation \(\prec_B\) of \((B, \sim_B)\) was processed. Note that \(\alpha'_{AB}\) satisfies properties (2) - (4) as the corresponding properties are satisfied by \(\alpha'_{A,B_i}\) for every \(i \in [k]\). Suppose \(u_1, u_2 \in M\) are such that \(u_1 < u_2\). If \(u_1, u_2 \in A_i, \) for some \(i \in [k]\), then \(\alpha'_{AB}(u_1) < \alpha'_{AB}(u_2)\) as \(\alpha'_{A,B_i}(u_1) < \alpha'_{A,B_i}(u_2)\). Suppose \(u_1 \in A_i\) and \(u_2 \in A_j\).

Then \(A_i \parallel A_j\). Note that the numbering of \(B_1, \ldots, B_k\) asserts that \(A_i\) is before \(A_j\) in \(A_1, \ldots, A_k\) iff \(B_i <_N B_j\). This proves (I).

On the other hand, suppose that \(\mathcal{M}\) and \(\mathcal{N}\) are locally isomorphic by \(\alpha\). Since \(\alpha\) is a local isomorphism between \(\mathcal{M}\) and \(\mathcal{N}\), the image \((\alpha|\tau^0), \alpha(\tau^1)\) of \((\tau^0, \tau^1)\) is an extended admissible model of \(\mathcal{N}\). Then, \((\alpha(\tau^0), \alpha(\tau^1))\) corresponds to transitive orientations \((<_N, \prec)\) of \((N, \mid)\) and \((N, \sim)\), respectively, where \(\prec\) is an admissible orientation for \(\mathcal{N}\). Let \(\prec\) be the restriction of \(\prec\) to the edges of \((\alpha(N), \sim_{\alpha(N)})\).

Now, when processing the admissible orientation \((A, \prec)\) for \(\mathcal{N}\), the algorithm orders the children of \(\alpha(A)\) in the order \(\alpha(A_1), \ldots, \alpha(A_k)\). Since for every \(i \in [k]\) the sets \(A_i\) and \(\alpha(A_i)\) have been marked as locally isomorphic, the algorithm will denote \(A\) and \(\alpha(A)\) as locally isomorphic.

**Case 4:** \(A\) and \(B\) are serial. Let \((A, \prec_A)\) and \((B, \prec_B)\) be the weak orders associated with the serial modules \(A\) and \(B\) defined by (5). For every child \(A'\) of \(A\) denote by \(\text{height}(A')\) the height of \(A'\) in \((A, \prec_A)\), that is, the size of the chain in \((A, \prec_A)\) which has \(A'\) as its largest element. Define \(\text{height}(B')\) for every child \(B'\) of \(B\) similarly. Let \((A, B, \mathcal{E}_{AB})\) be a bipartite graph, where:

- \(A\) is the set of the children \(A'\) of \(A\);
- \(B\) is the set of the children \(B'\) of \(B\);
- \(\mathcal{E}_{AB}\) is the set of all pairs \((A', B')\) such that \(A'\) and \(B'\) have been denoted as locally isomorphic and \(\text{height}(A') = \text{height}(B')\).

The algorithm marks \(A\) and \(B\) as locally isomorphic iff there is a perfect matching \(\mathcal{M}\) in the bipartite graph \((A, B, \mathcal{E}_{AB})\). If such a matching exists, for every \((A', B') \in \mathcal{E}_{AB}\), the algorithm marks \(A'\) and \(B'\) as locally isomorphic.
\( M \) and every \( u \in A' \) the algorithm sets \( \alpha'_{AB}(u) = v \), where \( v \in B' \) is such that \( \alpha'_{AB}(u) = v \).

Suppose \( A \) and \( B \) have been marked as locally isomorphic. Since for every \( u_1, u_2 \in A, u_1 <_M u_2 \) yields \( u_1, u_2 \in A' \) for some child \( A' \) of \( A \) as \( A \) is serial, we deduce that \( \alpha'_{AB} \) satisfies \( (\mathcal{N}) \) as \( \alpha'_{A',B'} \) satisfies \( (\mathcal{N}) \) for every \( (A', B') \in \mathcal{M} \).

On the other hand, suppose \( \mathcal{M} \) and \( \mathcal{N} \) are locally isomorphic by \( \alpha \). Since \( \alpha \) preserves positions, we deduce that \( \text{height}(A_i) \) in \( (A, \prec^\omega_\alpha) \) equals \( \text{height}(\alpha(A_i)) \) in \( (\alpha(A), \prec^\omega_{\alpha(\alpha)}) \). In particular, for every \( A' \) the pair \( (A', \alpha(A')) \) is in the edge set of the bipartite graph \( (A, B, E_{AB}) \). In particular, \( \{(A', \alpha(A')) : A' \in \mathcal{A}\} \) establishes a perfect matching in \( (A, B, E_{AB}) \). Hence, the algorithm denotes \( A \) and \( \alpha(A) \) as locally isomorphic.

### 6.3. Isomorphisms between modules with slots.

Suppose \( M \) and \( N \) are two modules in \( G_c \) and \( H_c \) for which the circular permutations of the slots \( \pi^0(M), \pi^1(M) \) and \( \pi^0(N), \pi^1(N) \) are defined. We fix an orientation \( \pi(M) \in \{\pi^0(M), \pi^1(M)\} \) and an orientation \( \pi(N) \in \{\pi^0(N), \pi^1(N)\} \). We choose a slot \( K' \) in \( \pi(M) \) and a slot \( N' \) in \( \pi(N) \) — we say \( \pi(M) \) is pinned in \( K' \) and \( \pi(N) \) is pinned in \( N' \). Let \( K'_1 \equiv K'_2 \equiv \ldots \equiv K'_{2n} \) be the order of the slots in \( \pi(M) \) if we traverse \( \pi(M) \) in the clockwise order starting from \( K' = K'_1 \). Let \( K_i \) be the consistent submodule of \( M \) associated with \( K'_i \), for every \( i \in [2n] \). That is, for every consistent submodule \( K \) of \( M \) there are different \( i \) and \( j \) such that \( K = K_i = K_j \). Let \( K'_1 = (K_i, <_{K'_i}, K'_i, K''_i, p(K'_i), p(K''_i)) \) be an (extended) orientation of \( K_i \), where \( p(K'_i), p(K''_i) \) are appropriate for the chosen \( \pi(M) \). We assume \( p(K'_1) = (K'_1) \) and \( p(K''_i) = (K''_i) \) if \( K_i \) has no extended orientation. We introduce similar notation for the slots of \( N \).

**Definition 6.5.** Suppose \( M, \pi(M), \pi(N), K', L' \) are as given above. We say \( \pi(M) \) pinned in \( K' \) is isomorphic to \( \pi(N) \) pinned in \( L' \) if there is a bijection \( \alpha \) from \( M \cup N_{T_G}[M] \) to \( N \cup N_{T_R}[N] \) such that:

- \( \alpha|K_i \) establishes a local isomorphism between \( K'_i \) and \( L'_i \), for every \( i \in [2n] \),
- \( \alpha|N_{T_G}(M) \) is a bijection from \( N_{T_G}(M) \) to \( N_{T_R}(N) \) and \( \alpha \) maps every node in \( p(K'_i, K''_{i+1}) \) into a node in \( p(L'_i, L'_{i+1}) \), cyclically.

Note that there may exist many isomorphisms between \( \pi(M) \) and \( \pi(N) \) pinned in \( K' \) and \( L' \), respectively. However, their restrictions to the set \( N_{T_G}[M] \) are unique and are determined by the patterns of the slots in \( \pi(M) \) and \( \pi(N) \).

Clearly, an isomorphism between \( \pi(M) \) and \( \pi(N) \) can be computed by the algorithm presented in the previous section.

**Claim 6.6.** Let \( \alpha : M \cup N_{T_G}[M] \rightarrow N \cup N_{T_R}[N] \) be an isomorphism between \( \pi(M) \) and \( \pi(N) \) pinned in \( K' \) and \( L' \), respectively. Then \( \alpha \) satisfies the following properties for every \( u, v \in M \) and every \( N' \in N_{T_G}[M] \):

1. \( u \in \text{left}(v) \iff \alpha(u) \in \text{left}(\alpha(v)) \),
2. \( u \in \text{right}(v) \iff \alpha(u) \in \text{right}(\alpha(v)) \),
3. \( N' \in \text{left}(v) \iff \alpha(N') \in \text{left}(\alpha(v)) \).
In particular, if $\phi^M$ is an extended admissible model of $(M, \sim)$ for $\pi(M)$, then the image $\alpha(\phi^M)$ is an extended admissible model of $(N, \sim)$ for $\pi(N)$.

Proof. If $u$ and $v$ belong to the same consistent submodule of $M$, then $u \in left(v)$ iff $\alpha(u) \in left(\alpha(v))$ as $\alpha|K_i$ is a local isomorphism on $K_i$ and $\alpha(K_i)$. Suppose $u$ and $v$ belong to two different submodules of $M$. For every $u' \in \{u^0, u^1\}$ let $i(u')$ be such that $u' \in K'_{i(u')}$. Define $i(v')$ similarly. Since $u \in left(v)$, we have

$$
\pi(M)\{K'_{i(u^0)}, K'_{i(u^1)}, K'_{i(u^0)}, K'_{i(u^1)}\} \equiv K'_{i(u^0)}K'_{i(u^0)}K'_{i(u^1)}K'_{i(u^1)}
$$

for some $\{u', u''\} = \{u^0, u^1\}$. However, we have $\alpha^0(v) \in L_i(u^0)$, $\alpha^1(v) \in L_i(u^1)$, $\alpha^0(u) \in L_i(u^0)$, $\alpha^1(u) \in L_i(u^1)$ as local isomorphisms preserve orientations of the vertices, and hence

$$
\pi(N)\{L'_{i(u^0)}, L'_{i(u^1)}, L'_{i(u^0)}, L'_{i(u^1)}\} \equiv L'_{i(u^0)}L'_{i(u^0)}L'_{i(u^1)}L'_{i(u^1)}.
$$

This proves $u \in left(v)$ iff $\alpha(u) \in left(\alpha(v))$. The remaining statements of the claim are proved analogously.

Now, we are ready to provide an isomorphism algorithm testing whether two circular graphs $(G, m_G)$ and $(H, m_H)$ are isomorphic. As in the previous sections, we consider three cases:

- $V(G_c)$ and $V(H_c)$ are serial.
- $V(G_c)$ and $V(H_c)$ are prime.
- $V(G_c)$ and $V(H_c)$ are parallel.

In the remaining cases, $(G, m_G)$ and $(H, m_H)$ can not be isomorphic.

6.4. $V(G_c)$ and $V(H_c)$ are serial. Suppose $(G, m_G)$ and $(H, m_H)$ are two circular-arc graphs such that both $V(G_c)$ and $V(H_c)$ are serial. Suppose that:

- $M_1, \ldots, M_k$ are the children of $V(G_c)$ in $M(G_c)$,
- $N_1, \ldots, N_k$ are the children of $V(H_c)$ in $M(H_c)$,
- $M_i^0 = (M_i, <^0_i, M_i^0, M_i^1)$ and $M_i^1 = (M_i, <^1_i, M_i^1, M_i^0)$ are the orientations of $M_i$, for $i \in [k]$,
- $N_i^0 = (N_i, <^0_i, N_i^0, N_i^1)$ and $N_i^1 = (N_i, <^1_i, N_i^1, N_i^0)$ are the orientations of $N_i$, for $i \in [k]$.

See Section 5.3 how to compute the orientations for $M_i$ and $N_i$.

The algorithm constructs a bipartite graph $G_{MN}$ between the modules $\{M_1, \ldots, M_k\}$ and $\{N_1, \ldots, N_k\}$, where there is an edge in $G_{MN}$ between $M_i$ and $N_j$ iff there are $M_i^0 \in \{M_i^0, M_i^1\}$ and $N_j^0 \in \{N_j^0, N_j^1\}$ such that $M_i^0$ and $N_j^0$ are locally isomorphic. The algorithm accepts $(G, m_G)$ and $(H, m_H)$ iff there is a perfect matching $\mathcal{M}$ in $G_{MN}$.

We claim that the algorithm accepts $(G, m_G)$ and $(H, m_H)$ iff $(G, m_G)$ and $(H, m_H)$ are isomorphic. Suppose $\alpha$ is an isomorphism between $(G, m_G)$ and $(H, m_H)$. Clearly, $\alpha(M_1), \ldots, \alpha(M_k)$ is a permutation of $N_1, \ldots, N_k$. By Theorem 5.3, the image $N'_i$ of $M_i^0$ by $\alpha$ satisfies either $N'_i = N_i^0$ or $N'_i = N_i^1$. Now, one can easily verify that $M_i^0 = (M_i, <^0_i, M_i^0, M_i^1)$ is locally isomorphic with $N_i^0 = (N_i, <^0_i, N_i^0, N_i^1)$.
(N'_i, <'_i, N''_i). So, (M_i, α(M_i)) is an edge of G_{MN} and \{(M_i, α(M_i)) : i ∈ [k]\} is a perfect matching in G_{MN}. So, the algorithm accepts (G, m_G) and (H, m_H).

Now, suppose that the algorithm accepts (G, m_G) and (H, m_H). Suppose M is a matching between M_1, ..., M_k and N_1, ..., N_k in G_{MN}. Without loss of generality suppose that the children of V(H_c) are enumerated such that (M_i, N_i) ∈ M for every i ∈ [k]. Let α_i be a local isomorphism between M'_i and N'_i, where M'_i ∈ \{M'_0, M'_1\} and N'_i ∈ \{N'_0, N'_1\}. Let α : V(G_c) → V(H_c) be a mapping such that α|M_i = α_i. Now, given that α_i is a local isomorphism between M'_i and N'_i, one can easily verify that α is an isomorphism between (G, m_G) and (H, m_H).

6.5. V(G_c) and V(H_c) are prime. Suppose (G, m_G) and (H, m_H) are two circular-are graphs such that both V(G_c) and V(H_c) are prime. Suppose that π^0(V(G_c)), π^1(V(G_c)) and π^0(V(H_c)), π^1(V(H_c)) are circular permutations of the slots of V(G_c) and V(H_c), respectively.

The algorithm iterates over circular orders π(V(G)) in \{π^0(V(G)), π^1(V(G))\} and π(V(H)) in \{π^0(V(H)), π^1(V(H))\}. For every pair (π(V(G)), π(V(H))) the algorithm fixes a slot K' in π(V(G)). Next, it iterates over all slots L' in π(V(H)) and checks whether π(V(G)) and π(V(H)) pinned in K' and L' are isomorphic. It accepts (G, m_G) and (H, m_H) iff for some choice of π(V(G)), π(V(H)), and L' the algorithm finds out that π(V(G)) and π(V(H)) pinned in K' and L' are isomorphic.

If the algorithm accepts (G, m_G) and (H, m_H), then (G, m_G) and (H, m_H) are isomorphic, which follows immediately from Claim 6.4. Suppose α is an isomorphism between (G_c, m_G) and (H_c, m_H). Let φ be a conformal model of G_c. The image α(φ) of φ by α is a conformal model of H_c. Let π(V(G)) = π_φ(V(G)), π(V(H)) = π_φ(V(H)), K' be a slot chosen by the algorithm at the time when it processes π(V(G)) and π(V(H)), and let α(K') = \{α(u') : u' ∈ K'\} be the image of the slot K'. Now, note that the algorithm accepts (G, m_G) and (H, m_H) when it processes π(V(G)), π(V(H)), and α(K').

6.6. V(G_c) and V(H_c) are parallel. Suppose that:

- T_G and T_H are T_{NM} trees for G_c and H_c, respectively,
- M_1, ..., M_k are the children of V(G_c) and K_1, ..., K_k are the children of V(H_c).

Suppose T_G is rooted in R, where R is any leaf module in T_G. For every module (node) A in the rooted tree T_G, by V_{T_G}[A] we denote the union of all modules descending A in T_G, including A. We introduce the similar notation for every module (node) B in the rooted T_H.

Let φ be a conformal model of G. Let α be an isomorphism between G_c and H_c and let α(φ) be the image of φ by α. Note that α maps every module M in T_G into a module α(M) in H_c. In particular, the root R of T_G is mapped into a leaf module α(R) in T_H. Let α(R) be the root of T_H. Note that the children of every module M (node N) are mapped into the children of α(M) (α(N), respectively. By Claim 5.1, for every node/module A in T_G, φ|V_{T_G}[A] is a contiguous subword
of $\phi$. We denote this subword by $\phi_A'$ (note Theorem 5.26) where $\phi_A'$ is constructed from extended admissible models $\phi[M \cup N_T[M]]$ for modules $M \in T_G$ and from circular permutations $\phi[N \cup N_T[N]]$ for nodes $N$ in $T_G$). Similarly, the image $\alpha(\phi_A')$ of $\phi_A'$ by $\alpha$ is a contiguous subword of $\alpha(\phi)$.

Now, we present the algorithm that tests whether $(G, m_G)$ and $(H, m_H)$ are isomorphic. The algorithm picks a leaf module $R$ in $T_G$ arbitrarily and sets $R$ as the root of $T_G$. Next, the algorithm iterates over all leaves $R'$ of $T_H$, sets $R'$ as the root of $T_H$, and does the following. It traverses the trees $T_G$ and $T_H$ bottom-up and for every two nodes $A \in T_G$ and $B \in T_H$ or two modules $A \in T_G$ and $B \in T_H$ it tests whether there is a bijection $\alpha : V_{T_G}[A] \to V_{T_H}[B]$ that satisfies the following conditions:

- $m_G(u) = m_H(\alpha(u))$ for every $u \in V_{T_G}[A]$,
- the image of $\phi_A'$ by $\alpha$ is a contiguous subword of some conformal model of $H_c$.

Such a mapping $\alpha$ is called an isomorphism between $A$ and $B$ in the rooted trees $T_G$ and $T_H$.

The algorithm accepts $(G, m_G)$ and $(H, m_H)$ if and only if there is a leaf module $R'$ in $T_H$ such that $R$ and $R'$ are isomorphic in the trees $T_G$ and $T_H$ rooted in $R$ and $R'$, respectively.

Suppose the algorithm accepts $(G, m_G)$ and $(H, m_H)$. Note that an isomorphism $\alpha$ between $R$ and $R'$ in the rooted trees $T_G$ and $T_H$ maps bijectively the vertices from $V(G_c)$ into $V(H_c)$, preserves multiplicities, and the image of $\phi_R$ by $\alpha$ is a conformal model of $H_c$. In particular, it means that $\alpha$ is an isomorphism between $G_c$ and $H_c$.

On the other hand, since $\alpha$ is an isomorphism between $G_c$ and $H_c$, $\alpha$ is an isomorphism between $R$ and $\alpha(R)$ in the trees $T_G$ and $T_H$ rooted in $R$ and $\alpha(R)$, and hence the algorithm accepts $(G, m_G)$ and $(H, m_H)$.

It remains to show how the algorithm tests whether $A$ and $B$ are isomorphic in the rooted trees $T_G$ and $T_H$. Whenever the algorithm denotes $A$ and $B$ as isomorphic, it constructs an isomorphism $\alpha'_{AB}$ between $A$ and $B$. We also show that the algorithm denotes $A$ and $\alpha(A)$ as isomorphic.

Suppose that $A$ and $B$ are nodes in $T_G$ and $T_H$, respectively. Suppose $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ are the children of $A$ in $T_G$ and $B_1, \ldots, B_l$ are the children of $B$ in $T_H$. We define a bipartite graph $G_{AB}$ on $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_l\}$ with the edge between $A_i$ and $A_j$ if $A_i$ and $B_j$ have been denoted as isomorphic. The algorithm denotes $A$ and $B$ as isomorphic iff there is a perfect matching $M$ between $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_l\}$ in $G_{AB}$. If this is the case, for every $(A', B') \in M$ and every $u \in A'$ the algorithm sets $\alpha'_{AB}(u) = v$, where $v \in B'$ is such that $\alpha'_{A', B'}(u) = v$.

Suppose the algorithm denotes $A$ and $B$ as isomorphic. Clearly, $\alpha'_{AB}$ preserves multiplicities. By Theorem 5.26, $\phi_A' = \phi'_{A_1} \cdot \ldots \cdot \phi'_{A_k}$ for some permutation $A_1', \ldots, A_k'$ of $A_1, \ldots, A_k$. Suppose $B_1', \ldots, B_l'$ is a permutation of the children of $B$ such that $(A_i', B_j') \in M$. Hence, the image $\tau_{B_j'}$ of $\phi'_{A_i}$ by $\alpha'_{A_i', B_j'}$ is a contiguous subword
of some conformal model of $H_c$, for every $i \in [k]$. By Theorem 5.20, the image $\tau_{B'_1} \ldots \tau_{B'_k}$ of $\phi'_A$ by $\alpha'_{AB}$ is a contiguous subword of a conformal model of $H_c$. So, $\alpha'_{AB}$ is an isomorphism between $A$ and $B$.

Suppose $G_c$ and $H_c$ are isomorphic. For every child $A'$ of $A$ algorithm denoted $A'$ and $\alpha(A')$ as isomorphic. Note that $\alpha(A_1), \ldots, \alpha(A_k)$ is the set of all the children of $B$. Hence, $\{(A_i, \alpha(A_i)) : i \in [k]\}$ is a perfect matching in $G_{AB}$. Hence, the algorithm denotes $A$ and $\alpha(A)$ as isomorphic in $T_G$ and $T_H$.

Suppose $A$ and $B$ are leaves in $T_G$ and $T_H$, respectively. In particular, $A$ and $B$ are modules in $T_G$ and $T_H$. Let $p(A)$ be the parent node of $A$ in $T_G$. Let $p(B)$ be the parent of $B$ in $T_H$. Note that $\phi|(A \cup N_{T_G}[A])$ is an extended admissible model of $(A, \sim)$.

Let $\pi_{\phi,A}(A)$ be the circular order of the slots in $\phi|A$ with patterns in $\phi|(A \cup N_{T_G}[A])$. Let $S'$ be a slot in $\pi_{\phi,A}(A)$ such that $p(A) \in p_0(S')$ or $p(A) \in p_0(S', S'')$ for some two consecutive slots $(S', S'')$ in $\pi_\phi(A)$. Let $\pi^0(B)$ and $\pi^1(B)$ be two circular orders of the slots of $B$ in $H_c$. For every $\pi(B) \in \{\pi^0(B), \pi^1(B)\}$ the algorithm does the following. First, it chooses a slot $T'$ in $\pi(B)$ such that either $p(B) \in p(T')$ or $p(B) \in p(T', T'')$, where $T''$ is such that $(T', T'')$ are consecutive slots in $\pi(B)$. Next, the algorithm checks whether $\pi_{\phi,A}(A)$ and $\pi(B)$ pinned in $S'$ and $T'$ are isomorphic. Suppose it is the case and $\alpha_{AB}$ is an isomorphism between $\pi_{\phi,A}(A)$ and $\pi(B)$ pinned in $S'$ and $T'$. Note that $\alpha_{AB}$ needs to map $p(A)$ into $p(B)$. The algorithm denotes $A$ and $B$ as isomorphic, and returns $\alpha_{AB}|A$ as an isomorphism $\alpha'_{AB}$ between $A$ and $B$.

Suppose $A$ and $B$ are denoted as isomorphic. Note that $A \phi'_A$ is an extended admissible model of $(A, \sim)$. Note that $\alpha_{AB}$ satisfies Claim 6.6 and hence $\alpha_{AB}(A \phi_A)$ is an extended admissible model of $(B, \sim)$. In particular, by Theorem 5.20, the image of the word $\phi'_A$ by $\alpha'_{AB}$ can be extended to a conformal model of $H_c$. This shows that $\alpha'_{AB}$ is a local isomorphism between $A$ and $B$.

Suppose $G_c$ and $H_c$ are isomorphic. Clearly, $\alpha(A)$ is a leaf in $T_H$. Consider the order of the slots of $\alpha(A)$ in the admissible model $\alpha(\phi)|\alpha(A)$ of $(\alpha(A), \sim)$. Note that when the algorithm processes the circular order of the slots $\pi_{\alpha(\phi)|\alpha(A)}(\alpha(A))$, it will denote $A$ and $\alpha(A)$ as isomorphic.

Eventually, suppose $A$ and $B$ are modules in $T_G$ and $T_H$, respectively. In this case the algorithm is similar as for the leaves, except one difference. Suppose $\alpha_{AB} : A \cup N_{T_G}[A] \to B \cup N_{T_G}[B]$ is an isomorphism between $\pi_{\phi,A}(A)$ and $\pi(B)$ pinned in $S'$ and $T'$. First, the algorithm checks whether $\alpha_{AB}$ maps the parent $p(M)$ of $A$ into the parent $p(B)$ of $B$ (if the parents exist). Moreover, for every node $N \in N_{T_G}[A]$ different than $p(A)$, the algorithm checks whether $N$ and $\alpha_{AB}(N)$ have been denoted as isomorphic. If this is the case, the algorithm accepts $A$ and $B$. The mapping $\alpha'_{AB}$ is defined as follows. If $u \in A$ we set $\alpha'_{AB}(u) = v$, where $v \in B$ is such that $\alpha_{AB}(u) = v$ and for every child $N$ of $A$ and every $u \in V_{T_G}[N]$ the algorithm sets $\alpha_{AB}(u) = v$, where $v \in V_{T_H}(\alpha_{AB}(N))$ is such that $\alpha'_{N_{\alpha_{AB}(N)}}(u) = v$.

Suppose $A$ and $B$ are denoted as isomorphic. Let $\phi^A \equiv A \phi_A|(A \cup N_{T_G}[A])$. Note that $\phi^A$ is an extended admissible model for $A$. By Claim 6.6 the image of $\phi^A$ by
$\alpha_{AB}$ is an extended admissible model for $B$. Now, replace in $\alpha_{AB}(\phi^A)$ every child $N'$ of $B$ different than $p(B)$ by the word $\alpha'_{AB}(\phi'_N)$, where $N$ in $N_{Tc}[A]$ is such that $\alpha_{AB}(N) = N'$. Note that we obtained the word $p(B)\alpha'_{AB}(\phi'_A)$. Since $\alpha'_{AB}(\phi'_N)$ is a subword of a conformal model of $H_c$, $\alpha(\phi^A)$ is an admissible model of $B$, we deduce by Theorem 5.26 that $\alpha'_{AB}(\phi'_A)$ is a contiguous subword of some conformal model of $H_c$.

Suppose $G_c$ and $H_c$ are isomorphic. We prove that the algorithm will denote $A$ and $\alpha(A)$ as isomorphic in a similar way as for the leaves.

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