Constructing processes with prescribed mixing coefficients

Leonid (Aryeh) Kontorovich
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel

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Abstract

The rate at which dependencies between future and past observations decay in a random process may be quantified in terms of mixing coefficients. The latter in turn appear in strong laws of large numbers and concentration of measure results for dependent random variables. Questions regarding what rates are possible for various notions of mixing have been posed since the 1960’s, and have important implications for some open problems in the theory of strong mixing conditions.

This paper deals with \(\eta\)-mixing, a notion defined in [Kontorovich and Ramanan], which is closely related to \(\phi\)-mixing. We show that there exist measures on finite sequences with essentially arbitrary \(\eta\)-mixing coefficients, as well as processes with arbitrarily slow mixing rates.

1 Introduction

1.1 Preliminaries

Strong mixing conditions deal with quantifying the decaying dependence between blocks of random variables in a stochastic process. These have been traditionally used to establish strong laws of large numbers for non-independent processes. Bradley [4, 5, 6] is an encyclopedic source on the matter; see also his survey paper [3]. In [6, Chapter 26], Bradley traces the early research on mixing rates to Volkonskiı̆ and Rozanov [19] and gives a comprehensive account of the progress since then.

Our interest in strong mixing was motivated by the desire for concentration of measure bounds for non-independent random sequences. Given the excellent survey papers and monographs dealing with concentration of measure (in particular, [14], [15], and [18]), we will give only the briefest summary here.

Suppose \(\Omega\) is a finite\(^1\) set and let \(\mu\) be an arbitrary (nonproduct) probability measure on \(\Omega^n\). We proceed to define a type of strong mixing used throughout this note. For \(1 \leq i < j \leq n\) and \(x \in \Omega^i\), let

\[\mathcal{L}(X_j^n | X_i^i = x)\]

\(^1\)The results hold verbatim for countable sets, and extend naturally to \(\mathbb{R}\) under mild assumptions; see [11, 12].
be the distribution of $X^n_j \equiv (X_j, \ldots, X_n)$ conditioned on $X^n_i = x$. For $y \in \Omega^{i-1}$ and $w, w' \in \Omega$, define
\[ \eta_{ij}(y, w, w') = \| \mathcal{L}(X^n_j | X^n_i = yw) - \mathcal{L}(X^n_j | X^n_i = yw') \|_{TV}, \]
where $\| \cdot \|_{TV} \equiv \frac{1}{2} \| \cdot \|_1$ is the total variation norm; likewise, define
\[ \bar{\eta}_{ij} = \max_{y \in \Omega^{i-1}, w, w' \in \Omega} \eta_{ij}(y, w, w'). \]

This notion of mixing is by no means new; it can be traced (at least implicitly) to Marton’s work [16] and is quite explicit in Samson [17] and Chazottes et al. [7]. We are not aware of a standardized term for this type of mixing, and have referred to it as $\eta$-mixing in previous work [13]. It was observed in [17] that the $\phi$-mixing coefficients bound the $\eta$-mixing ones:
\[ \bar{\eta}_{ij} \leq 2\phi_{j-i}, \]
and conjectured in [11] that
\[ \frac{1}{2} \sum_{i=1}^{n-1} \phi_i \leq 1 + \max_{1 \leq i < n} \left[ \sum_{j=i+1}^{n} \bar{\eta}_{ij} \right]; \]
the latter remains open.

In all instances, $\eta$-mixing has come up in the context of concentration of measure. In particular, define $\Gamma$ and $\Delta$ to be upper-triangular $n \times n$ matrices, with $\Gamma_{ii} = \Delta_{ii} = 1$ and
\[ \Gamma_{ij} = \sqrt{\bar{\eta}_{ij}}, \quad \Delta_{ij} = \bar{\eta}_{ij} \]
for $1 \leq i < j \leq n$.

Samson [17] proved that any distribution $\mu$ on $[0,1]^n$ and any convex $f : [0,1]^n \to \mathbb{R}$ with $\|f\|_{Lip} \leq 1$ (with respect to $\ell_2$) satisfy
\[ \mu \{ |f - \mu f| > t \} \leq 2 \exp \left( - \frac{t^2}{2 \| \Gamma \|^2_2} \right) \]
where $\| \Gamma \|^2_2$ is the $\ell_2$ operator norm.

Chazottes et al. [7] and independently, the author with K. Ramanan [13] showed that any distribution $\mu$ on $\Omega^n$ and any $f : \Omega^n \to \mathbb{R}$ with $\|f\|_{Lip} \leq n^{-1/2}$ (with respect to the Hamming metric) satisfy
\[ \mu \{ |f - \mu f| > t \} \leq 2 \exp \left( - \frac{t^2}{2 \| \Delta \|^2_{\infty}} \right) \]
where $\| \Delta \|_{\infty}$ is the $\ell_{\infty}$ operator norm ($\| \Delta \|_{\infty}$ may be replaced by $\| \Delta \|_2$ and [7] achieves a better constant in the exponent).

Results of type (4) and (5) are known as concentration of measure inequalities; broadly, they assert that any “sufficiently continuous” function is tightly concentrated about its mean. Such bounds have a remarkable range of applications, spanning abstract fields such as asymptotic Banach
space theory [1, 18] as well as more practical ones such as randomized algorithms [8] and machine learning [2]. Strong laws of large numbers are readily obtained from concentration bounds [12].

Having motivated the study of mixing and measure concentration, let us turn to the behavior of the \( \eta \)-mixing coefficients. It is immediate from the construction that \( \bar{\eta}_{ij} \) is an upper-triangular \( n \times n \) matrix satisfying

\[
\text{(P1)} \quad \bar{\eta}_{ij} = 0 \text{ for } i \geq j \\
\text{(P2)} \quad 0 \leq \bar{\eta}_{ij} \leq 1 \text{ for } 1 \leq i < j \leq n.
\]

It is also simple to show (as we shall do below in Lemma 2.1) that

\[
\text{(P3)} \quad \bar{\eta}_{j_2} \leq \bar{\eta}_{j_1} \text{ for } i < j_1 < j_2.
\]

1.2 Main results

A natural question (first posed in [11]) is whether the conditions (P1)-(P3) completely characterize the possible \( (\bar{\eta}_{ij}) \) matrices, or if there are some other constraints that the \( \eta \)-mixing coefficients must satisfy. The main technical result of this note is Theorem 2.7, which resolves this question in the affirmative. Thus, for any “valid” (i.e., satisfying (P1)-(P3)) \( n \times n \) matrix \( H = (h_{ij}) \), there is a finite set \( \Omega \) and a probability measure \( \mu \) on \( \Omega^n \) such that \( \bar{\eta}_{ij}(\mu) = h_{ij} \) for \( 1 \leq i < j \leq n \).

More broadly, it is of interest to characterize the possible mixing rates that various processes may have. Chapter 26 of [6] deals with this question and gives several intricate constructions of random processes having prescribed mixing rates, under various types of strong mixing. Following the work of Kesten and O’Brien [10], it emerged that essentially arbitrary mixing rates are possible for various mixing notions. Thus it is not surprising that the same holds true for \( \eta \)-mixing; this is an easy consequence of our main result (Corollary 2.9).

Along the way, we collect various other observations regarding the \( \eta \)-mixing coefficients – some of which are auxiliary in proving our main results, and others may be of independent interest.

1.3 Notation

We use the indicator variable \( \mathbb{1}_{\{\cdot\}} \) to assign 0-1 truth values to the predicate in \( \{\cdot\} \).

Random variables are capitalized \( (X) \), specified sequences are written in lowercase \( (x \in \Omega^n) \), the shorthand \( X^j_i = (X_i, \ldots, X_j) \) is used for all sequences, and sequence concatenation is denoted multiplicatively: \( x^j_{i_1} x^k_{j+1} = x^k_i \). Sums will range over the entire space of the summation variable; thus \( \sum x^j_i f(x^j_i) \) stands for

\[
\sum_{x^j_i \in \Omega^j_i} f(x^j_i),
\]

where \( \Omega^j_i \) is just \( \Omega^{j-i+1} \), re-indexed for convenience. For \( y \in \Omega^i_1 \) and \( x \in \Omega^n_j \), we will write \( \mu(x \mid y) \) as a shorthand for \( \mu \left\{ X^n_j = x \mid X^i_1 = y \right\} \); no confusion should arise.

The total variation norm of a signed measure \( \nu \) on \( \Omega^n \) (i.e., vector \( \nu \in \mathbb{R}^{\Omega^n} \)) is defined by

\[
\|\nu\|_{TV} = \frac{1}{2} \|\nu\|_1 = \frac{1}{2} \sum_{x \in \Omega^n} |\nu(x)|
\]
(the factor of 1/2 is not entirely standard). Unless otherwise stated, \(\Omega\) is a finite set. Whenever we wish to be explicit about the dependence of \(\eta_{ij}\) and \(\bar{\eta}_{ij}\) on a given measure \(\mu\), we will write \(\eta_{ij}(\mu; y, w, w')\) and \(\bar{\eta}_{ij}(\mu)\), respectively.

## 2 Constructions and proofs

Let us begin with an easy verification that (P3) holds for all \((\bar{\eta}_{ij})\):

**Lemma 2.1.** Let \((\bar{\eta}_{ij})_{1 \leq i < j \leq n}\) be the \(\eta\)-mixing matrix associated with a probability measure \(\mu\) on \(\Omega^n\). Then, for all \(1 \leq i < j \leq n\), we have

\[
\bar{\eta}_{ij} \leq \bar{\eta}_{j1}.
\]

**Proof.** Fix \(1 \leq i < j \leq n\) and \(y \in \Omega_i^{i-1}, w, w' \in \Omega_i^j\). Then

\[
\eta_{ij}(y, w, w') = \frac{1}{2} \sum_{x \in \Omega_j^j} \left| \mu(x | yw) - \mu(x | yw') \right|
\]

\[
= \frac{1}{2} \sum_{x \in \Omega_j^j} \left( \sum_{u \in \Omega_j^{j-1}} \left| \mu(ux | yw) - \mu(ux | yw') \right| \right)
\]

\[
\leq \frac{1}{2} \sum_{x \in \Omega_j^j} \sum_{u \in \Omega_j^{j-1}} \left| \mu(ux | yw) - \mu(ux | yw') \right|
\]

\[
= \frac{1}{2} \sum_{z \in \Omega_j^j} \left| \mu(z | yw) - \mu(z | yw') \right|
\]

\[
= \eta_{j1}(y, w, w').
\]

\(\square\)

Next, we establish a simple continuity property of \(\bar{\eta}_{ij}\):

**Lemma 2.2.** Suppose \(\Omega\) is a finite set and let \(\mathcal{P}_+(\Omega)\) be the set of all strictly positive probability measures \(\mu\) on \(\Omega^n\) (i.e., \(\mu(x) > 0\) for all \(x \in \Omega^n\)). Endow \(\mathcal{P}_+(\Omega)\) with the metric \(\|\cdot\|_{TV}\). Then, for all \(1 \leq i < j \leq n\), the functional \(\bar{\eta}_{ij} : \mathcal{P}_+(\Omega) \to \mathbb{R}\) is continuous with respect to \(\|\cdot\|_{TV}\).

**Proof.** The continuity of \(\eta_{ij}(y, w, w') : \mu \mapsto \mathbb{R}\) for fixed \(y \in \Omega_i^{i-1}, w, w' \in \Omega_j^j\) follows immediately from Lemma 5.4.1 of [11]. The claim follows since continuity is preserved under finite maxima. \(\square\)

**Remark 2.3.** Continuity breaks down on the boundary of \(\mathcal{P}_+(\Omega)\); see Section 5.4 of [11] for an example.

Our construction of a measure with the desired mixing coefficients will proceed in stages, the final object being composed of intermediate ones. The building blocks will be measures of a particular simple form. For \(1 \leq k < n\), let \(h \in \mathbb{R}_{k+1}^+\) be a vector of length \(n - k\), satisfying

\[
0 \leq h_{j+1} \leq \ldots \leq h_j \leq 1
\]

for \(k < j < n\); any such \(h\) will be called a valid \(k\)th row. We say that the measure \(\mu\) on \(\Omega^n\) is pure \(k\)th row (with respect to \(h\)) if its \(\eta\)-mixing matrix \((\bar{\eta}_{ij})_{1 \leq i < j \leq n}\) satisfies

\[
\bar{\eta}_{ij} = \mathbf{1}_{\{i=k\}} h_j.
\]

Our first technical result is the existence of arbitrary pure \(k\)th row measures:
Lemma 2.4. Fix $1 \leq k < n$ and suppose $h \in \mathbb{R}^n_{k+1}$ is a valid $k^{th}$ row vector. Then there exists a measure $\mu$ on $\{0,1\}^n$ which is pure $k^{th}$ row with respect to $h$.

Proof. The proof will proceed by algorithmic construction. Let a valid $k^{th}$ row vector $h \in \mathbb{R}^n_{k+1}$ be given. Initialize $\mu^{(n+1)}$ to be the uniform measure:

$$\mu^{(n+1)}(x) = 2^{-n}, \quad x \in \{0,1\}^n.$$ 

For $v \in [0,1]$, define the measure $\mu^{(n,v)}$ on $\{0,1\}^n$ by

$$\mu^{(n,v)}(x) = \alpha_n(v)[v\mathbb{1}_{x_k=x_n}]\mu^{(n+1)}(x) + (1-v)\mathbb{1}_{x_k\neq x_n}\mu^{(n+1)}(x),$$

where $\alpha_n(v)$ is the normalization constant ensuring that $\sum_x \mu^{(n,v)}(x) = 1$, and define $f_n : [0,1] \to [0,1]$ by

$$f_n(v) = \eta_{kn}(\mu^{(n,v)}).$$

Lemma 2.2 assures the continuity of $f$ and it is straightforward to verify that $f_n(0) = f_n(1) = 1$ and $f_n(1/2) = 0$. Thus, there exists a $v^* \in [0,1]$ such that $f_n(v^*) = h_n$; define the new measure $\mu^{(n)}$ by

$$\mu^{(n)}(x) = \mu^{(n,v^*)}(x). \quad (6)$$

Similarly, for $v \in [0,1]$, define

$$\mu^{(n-1,v)}(x) = \alpha_{n-1}(v)[v\mathbb{1}_{x_k=x_{n-1}}]\mu^{(n)}(x) + (1-v)\mathbb{1}_{x_k\neq x_{n-1}}\mu^{(n)}(x), \quad x \in \{0,1\}^n$$

(where $\alpha_{n-1}(v)$ is again the appropriate normalization constant) and define $f_{n-1} : [0,1] \to [0,1]$ by

$$f_{n-1}(v) = \eta_{k,n-1}(\mu^{(n-1,v)}).$$

Again, it is easily seen that $f_{n-1}(0) = f_{n-1}(1) = 1$ and $f_{n-1}(1/2) = h_n$, so by continuity there is a $v^* \in [0,1]$ for which $f_{n-1}(v^*) = h_{n-1}$; so we may define the new measure

$$\mu^{(n-1)}(x) = \mu^{(n-1,v^*)}(x). \quad (7)$$

By construction, we have $\eta_{k,n-1}(\mu^{(n-1)}) = h_{n-1}$; we claim that additionally,

$$\eta_{k,n}(\mu^{(n-1)}) = h_n \quad (8)$$

(in other words, the second modification in (7) did not “ruin” the effects of the first modification in (6)). The claim in (8) holds because in fact for all $y \in \{0,1\}^k$ and $x \in \{0,1\}$, we have

$$\mu^{(n)} \left\{ X_n = x \mid X^k_1 = y \right\} = \mu^{(n-1)} \left\{ X_n = x \mid X^k_1 = y \right\}; \quad (9)$$

the latter fact is straightforward (though somewhat tedious) to verify.

We may now proceed by induction. Let $\mu^{(t)}$ be defined, for $k+1 < t \leq n$. Define, for $v \in [0,1]$,

$$\mu^{(t-1,v)}(x) = \alpha_{t-1}(v)[v\mathbb{1}_{x_k=x_{t-1}}]\mu^{(t)}(x) + (1-v)\mathbb{1}_{x_k\neq x_{t-1}}\mu^{(t)}(x), \quad x \in \{0,1\}^n.$$
and let \( f_{t-1} : [0, 1] \to [0, 1] \) be
\[
f_{t-1}(v) = \bar{\eta}_{k,t-1}(\mu^{(t-1),v}).
\]
Choose \( v^* \in [0, 1] \) so that \( f_{t-1}(v^*) = h_{t-1} \) and define the new measure
\[
\mu^{(t-1)} = \mu^{(t-1),v^*}.
\]
Again, a straightforward calculation gives
\[
\mu^{(t)} \left\{ X^n_i = x \mid X_1^k = y \right\} = \mu^{(t-1)} \left\{ X^n_i = x \mid X_1^k = y \right\}
\]
for all \( y \in \{0, 1\}^k \) and all \( x \in \{0, 1\}^{n-t+1} \), which ensures that
\[
\bar{\eta}_{k,t-1}(\mu^{(t-1)}), \bar{\eta}_{k,t}(\mu^{(t-1)}), \ldots, \bar{\eta}_{k,n}(\mu^{(t-1)})
\]
all have the right values. The process terminates when we have constructed \( \mu^{(k+1)} \); this is our desired pure \( k^{th} \) row measure with respect to \( h \). It remains to verify that \( \bar{\eta}_{ij}(\mu^{(k+1)}) = 0 \) for \( i \neq k \), but this is almost immediate.

Remark 2.5. The “backwards” order of constructing the measures \( \mu^{(t)} \) with \( t = n, n-1, \ldots, k+1 \) is essential. A construction in the “forward” order fails precisely because (10) no longer holds. The reader is invited to verify that the marginals of the constructed measure \( \mu = \mu^{(k+1)} \) are identical, with \( \mu \{ X_i = 0 \} = \mu \{ X_i = 1 \} = 1/2 \) for \( 1 \leq i \leq n \).

Next we turn to product measures. There are (at least) two natural ways to form products of probability measures; we shall refer to them as series and parallel. Let \( \mathcal{X}, \mathcal{Y} \) be finite sets and \( m, n \in \mathbb{N} \). If \( \mu \) is a measure on \( \mathcal{X}^m \) and \( \nu \) a measure on \( \mathcal{X}^n \), we define their series product, denoted by \( \mu \oplus \nu \), to be the following measure on \( \mathcal{X}^{m+n} \):
\[
(\mu \oplus \nu)(z) = \mu(x)\nu(y), \quad z = xy \in \mathcal{X}^{m+n}, x \in \mathcal{X}^m, y \in \mathcal{X}^n.
\] (11)

If \( \mu \) is a measure on \( \mathcal{X}^n \) and \( \nu \) a measure on \( \mathcal{Y}^n \), we define their parallel product, denoted by \( \mu \otimes \nu \), to be the following measure on \( (\mathcal{X} \times \mathcal{Y})^n \):
\[
(\mu \otimes \nu)(z) = \mu(x)\nu(y), \quad z = (x, y) \in (\mathcal{X} \times \mathcal{Y})^n.
\]

As our main construction will involve parallel products of measures, the following simple result is useful.

Lemma 2.6. Let \( \mu \) and \( \nu \) be probability measures on \( \mathcal{X}^m \) and \( \mathcal{Y}^n \), respectively, and let \( \bar{\eta}_{ij}(\mu), \bar{\eta}_{ij}(\nu) \) and \( \bar{\eta}_{ij}(\mu \otimes \nu) \) be the corresponding \( \eta \)-mixing matrices. Then we have
\[
\max \{ \bar{\eta}_{ij}(\mu), \bar{\eta}_{ij}(\nu) \} \leq \bar{\eta}_{ij}(\mu \otimes \nu) \leq \bar{\eta}_{ij}(\mu) + \bar{\eta}_{ij}(\nu)
\]
for all \( 1 \leq i < j \leq n \).
Proof. Fix $i < j$. Throughout this proof, $x$ will denote sequences over $\mathcal{X}$, $y$ sequences over $\mathcal{Y}$, and $z = (x, y)$ over $\mathcal{X} \times \mathcal{Y}$. Pick arbitrary $z_1^{-1} = (x_1^{-1}, y_1^{-1})$ and $z_i = (x_i, y_i)$, $z'_i = (x'_i, y'_i)$. Then we expand

$$
\eta_{ij}(\mu \otimes \nu; z_1^{-1}, z_i, z'_i) = \left\| (\mu \otimes \nu)(\cdot | z_1^{-1}z_i) - (\mu \otimes \nu)(\cdot | z_1^{-1}z'_i) \right\|_{TV}
$$

(13)

$$
= \frac{1}{2} \sum_{z_n} \left| (\mu \otimes \nu)(z^n | z_1^{-1}z_i) - (\mu \otimes \nu)(z^n | z_1^{-1}z'_i) \right|
$$

$$
= \frac{1}{2} \sum_{x^n_j} \sum_{y^n_j} \left| \mu(x^n_j | x_1^{-1}x_i) \nu(y^n_j | y_1^{-1}y_i) - \mu(x^n_j | x_1^{-1}x'_i) \nu(y^n_j | y_1^{-1}y'_i) \right|
$$

$$
\geq \frac{1}{2} \sum_{x^n_j} \sum_{y^n_j} \left[ \mu(x^n_j | x_1^{-1}x_i) \nu(y^n_j | y_1^{-1}y_i) - \mu(x^n_j | x_1^{-1}x'_i) \nu(y^n_j | y_1^{-1}y'_i) \right]
$$

$$
= \frac{1}{2} \sum_{x^n_j} \left| \mu(x^n_j | x_1^{-1}x_i) - \mu(x^n_j | x_1^{-1}x'_i) \right|
$$

$$
= \eta_{ij}(\mu; x_1^{-1}, x_i, x'_i).
$$

Exchanging the roles of $x$ and $y$ yields the lower bound in (12). To obtain the upper bound, we apply the $||-||_{TV}$ tensorization property (see Lemma 2.2.5 in [11]) to (13):

$$
\left\| (\mu \otimes \nu)(\cdot | z_1^{-1}z_i) - (\mu \otimes \nu)(\cdot | z_1^{-1}z'_i) \right\|_{TV} \leq \left\| (\mu \cdot | x_1^{-1}x_i) - (\mu \cdot | x_1^{-1}x'_i) \right\|_{TV} + \left\| (\nu \cdot | y_1^{-1}y_i) - (\nu \cdot | y_1^{-1}y'_i) \right\|_{TV} - \left\| (\mu \cdot | x_1^{-1}x_i) - (\nu \cdot | x_1^{-1}x'_i) \right\|_{TV} \left\| (\nu \cdot | y_1^{-1}y_i) - (\nu \cdot | y_1^{-1}y'_i) \right\|_{TV}
$$

which yields the desired bound. \qed

The interested reader may consult Lemma 3.2.1 of [11] for some observations regarding the behavior of $\eta$-mixing coefficients under series products.

We are now ready to prove the main result of this note.

**Theorem 2.7.** Let $H = (h_{ij})$ be any $n \times n$ matrix satisfying (P1), (P2) and (P3). Then there exists a finite set $\Omega$ and a probability measure $\mu$ on $\Omega^n$ such that

$$
\bar{h}_{ij}(\mu) = h_{ij}
$$

for $1 \leq i < j \leq n$.

**Proof.** For $k = 1, \ldots, n - 1$, let $h^{(k)} \in \mathbb{R}^n_{k+1}$ be the vector $(h_{k,k+1}, h_{k,k+2}, \ldots, h_{k,n})$ – i.e., the nonzero entries of the $k^{th}$ row of $H$. Then Lemma 2.4 provides a measure $\mu^{(k)}$ on $\{0,1\}^n$ which is pure $k^{th}$ row with respect to $h^{(k)}$. Let $\mu$ be the (parallel) product of these pure $k^{th}$ row measures:

$$
\mu = \mu^{(1)} \otimes \mu^{(2)} \otimes \ldots \mu^{(n-1)};
$$

note that $\mu$ is a measure on $\Omega^n$, where $\Omega = \{0,1\}^{n-1}$. By definition of pure $k^{th}$ row measures and by Lemma 2.6, we have that (14) holds. \qed
Remark 2.8. Our construction requires an exponential state space, $|\Omega| = 2^{n-1}$. Are there analogous constructions using fewer states? In Section 5.7 of [11] we constructed a measure $\mu$ on $\{0, 1\}^n$ satisfying (14) for the special case where the rows of $H$ are constant: $h_{i,i+1} = h_{i,i+2} = \ldots = h_{i,n}$; it seems unlikely that the general case is achievable with a constant number of states.

Up to this point, we have been discussing the $\eta$-mixing coefficients of probability measures on finite sequences. This notion extends quite naturally to random processes – i.e., probability measures $\mu$ on $\Omega^\mathbb{N}$. Let $\mu_n$ be the marginal distribution of $X_1^n$ and denote by $\bar{\eta}_{ij}^{(n)}$ the $\eta$-mixing matrix of $\mu_n$. It is straightforward to verify that in general, $\bar{\eta}_{ij}^{(n)}$ depends on $n$ and that

$$\bar{\eta}_{ij}^{(n)} \leq \bar{\eta}_{ij}^{(n+1)}$$

for $1 \leq i < j \leq n$. Let $\Delta_n(\mu)$ be the $n \times n$ matrix $\Delta$ corresponding to $\mu_n$, as defined in (3). Recall that the $\ell_\infty$ operator norm of a nonnegative matrix is its maximal row sum. Thus we can define the $\eta$-mixing rate of the process $\mu$ as the function $R_\mu : \mathbb{N} \to \mathbb{R}$:

$$R_\mu(n) = \|\Delta_n(\mu)\|_\infty.$$ 

It’s clear that (i) $R_\mu$ is nondecreasing and (ii) $1 \leq R_\mu(n) \leq n$; any function satisfying these properties will be called a valid rate function.

Corollary 2.9. Let $r : \mathbb{N} \to \mathbb{N}$ be a valid rate function. Then there is a set $\Omega = \{0, 1\}^\mathbb{N}$ and a measure $\mu$ on $\Omega^\mathbb{N}$ such that

$$\limsup_{n \to \infty} \frac{R_\mu(n)}{r(n)} = 1.$$  

(15)

Proof. We begin with the simple observation that if $r$ is a valid rate function then for all $k \geq 1$ and all $0 < \varepsilon < 1$, there is an $n = n(k, \varepsilon) > k$ and an $h = h(k, \varepsilon) \in [0, 1]$ such that

$$1 - \varepsilon \leq \frac{h(k, \varepsilon)(n - k)}{r(n)} \leq 1.$$  

(16)

Let $1 > \varepsilon_1 > \varepsilon_2 > \ldots > 0$ be a sequence decreasing to 0. Pick a $k \geq 1$ and let $n(k) = n(k, \varepsilon_k)$ and $h(k) = h(k, \varepsilon_k)$, as stipulated in (16). Define $h^{(k)} \in \mathbb{R}_{k+1}^n$ by

$$h^{(k)}_j = h(k), \quad k < j \leq n(k),$$

and let $\mu^{(k)}$ be the measure on $\{0, 1\}^{n(k)}$ which is pure $k$th row with respect to $h^{(k)}$, as constructed in Lemma 2.4. Let $\beta$ be the symmetric Bernoulli measure on $\{0, 1\}$ (i.e., $\beta(0) = \beta(1) = 1/2$) and define the measure $\tilde{\mu}^{(k)}$ on $\{0, 1\}^\mathbb{N}$ by

$$\tilde{\mu}^{(k)} = \mu^{(k)} \oplus \beta \oplus \beta \oplus \ldots$$

where the operation $\oplus$ is defined in (11). In this way, we have obtained a countable collection of measures $\{\tilde{\mu}^{(k)} : k = 1, 2, \ldots\}$ on $\{0, 1\}^\mathbb{N}$; note that by construction, we have for each $k$

$$1 - \varepsilon_k \leq \frac{\|\Delta_{n(k)}(\tilde{\mu}^{(k)})\|_\infty}{r(n(k))} \leq 1.$$  

(17)
Now let $\mu$ be the measure on $\left(\{0,1\}^\mathbb{N}\right)^\mathbb{N}$ obtained by taking the (parallel) product of all the $\hat{\mu}^{(k)}$’s:

$$\mu = \hat{\mu}^{(1)} \otimes \hat{\mu}^{(2)} \otimes \ldots$$

(the $\otimes$ operator is defined in (12)). It remains to verify that $\mu$ is a well-defined probability measure on $\Omega^\mathbb{N}$, $\Omega = \{0,1\}^\mathbb{N}$ by applying the Ionescu Tulcea theorem ([9, Theorem 6.17]), and that (17) continues to hold when $\hat{\mu}^{(k)}$ is replaced with $\mu$ – the latter is straightforward. \footnote{To accommodate infinite state spaces, the max in (2) needs to be replaced with sup.}

Remark 2.10. Our construction required an uncountable state space, $\Omega = \{0,1\}^\mathbb{N}$. Are analogous constructions possible with smaller $\Omega$? Is there a construction achieving (15) with $\lim$ in place of $\limsup$?

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