Modified logarithmic Sobolev inequalities in null curvature

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Abstract

We present a logarithmic Sobolev inequality adapted to a log-concave measure. Assume that $\Phi$ is a symmetric convex function on $\mathbb{R}$ satisfying $(1 + \varepsilon)\Phi(x) \leq x\Phi'(x) \leq (2 - \varepsilon)\Phi(x)$ for $x \geq 0$ large enough and with $\varepsilon \in [0, 1/2]$. We prove that the probability measure on $\mathbb{R}$ $\mu_\Phi(dx) = e^{-\Phi(x)/Z_\Phi}dx$ satisfies a modified and adapted logarithmic Sobolev inequality: there exist three constant $A, B, D > 0$ such that for all smooth $f > 0$,

$$\text{Ent}_{\mu_\Phi}(f^2) \leq A \int H_\Phi \left( \frac{f'}{f} \right)^2 d\mu_\Phi,$$

with $H_\Phi(x) = \begin{cases} \Phi^*(Bx) & \text{if } |x| \geq D, \\ x^2 & \text{if } |x| \leq D. \end{cases}$

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1 Introduction

A probability measure $\mu$ on $\mathbb{R}^n$ satisfies a logarithmic Sobolev inequality if there exists $C \geq 0$ such that, for every smooth enough functions $f$ on $\mathbb{R}^n$,

$$\text{Ent}_\mu(f^2) \leq C \int |\nabla f|^2 d\mu,$$

(1)

where

$$\text{Ent}_\mu(f^2) := \int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu$$

and where $|\nabla f|$ is the Euclidean length of the gradient $\nabla f$ of $f$.

Gross in [Gro75] defines this inequality and shows that the canonical Gaussian measure with density $(2\pi)^{-n/2}e^{-|x|^2/2}$ with respect to the Lebesgue measure on $\mathbb{R}^n$ is the basic example of measure $\mu$ satisfying (1) with the optimal constant $C = 2$. Since then, many results have presented measures satisfying an such inequality, among them the famous Bakry-Émery $\Gamma_2$-criterion, that we recall now in our particular case. Let $\mu(dx) = \exp(-f(x))dx$, a probability measure on $\mathbb{R}^n$ and assume that there exists $\lambda > 0$ such that,

$$\forall x \in \mathbb{R}^n, \quad \text{Hess}(f(x)) \succeq \lambda \text{Id},$$

(2)

in the sense of symmetric matrix. Then Bakry and Émery prove that $\mu$ is satisfying inequality (1) with a optimal constant $0 \leq C \leq 2/\lambda$. We refer to [BÉ85, Bak94] for the $\Gamma_2$-criterion and to [ABC+00, Led99] for a review on logarithmic Sobolev inequality.

The interest of this paper is to give a logarithmic Sobolev inequality when the probability measure $\mu$ on $\mathbb{R}$ does’nt satisfies (1) but it is still log-concave function which mean that $f''(x) \geq 0$.

An answer can be given for the following measure: Let $\alpha \geq 1$ and define the probability measure $\mu_\alpha$ on $\mathbb{R}$ by

$$\mu_\alpha(dx) = \frac{1}{Z_\alpha} e^{-|x|^\alpha} dx,$$

(3)
where \( Z_\alpha = \int e^{-|x|^\alpha} \, dx \).

The authors prove, in [GGM05], that for \( 1 < \alpha < 2 \), the measure \( \mu_\alpha \) satisfies the following inequalities, for all smooth function such that \( f \geq 0 \) and \( \int f^2 \, d\mu_\alpha = 1 \),

\[
\text{Ent}_{\mu_\alpha} (f^2) \leq A \text{Var}_{\mu_\alpha} (f) + B \int_{f \geq 2} |f'|^\beta \, f^2 \, d\mu_\alpha,
\]

(4)

where \( A \) and \( B \) are some constants and

\[
\text{Var}_{\mu_\alpha} (f) := \int f^2 \, d\mu_\alpha - \left( \int f \, d\mu_\alpha \right)^2.
\]

It is well-known that the probability measure \( \mu_\alpha \) satisfies (still for \( \alpha \geq 1 \)) a Poincaré inequality (or spectral gap inequality) which is for every smooth enough function \( f \),

\[
\text{Var}_{\mu_\alpha} (f) \leq C \int |\nabla f|^2 \, d\mu_\alpha,
\]

(5)

where \( 0 < C < \infty \).

Then using (5) and (4) we get that \( \mu_\alpha \) satisfies also this modified logarithmic Sobolev inequality for all smooth and positive function \( f \),

\[
\text{Ent}_\mu (f^2) \leq C \int H_{a,\alpha} \left( \frac{f'}{f} \right) f^2 \, d\mu,
\]

(6)

here and in the whole paper the convention that \( 0 \cdot \infty = 0 \) is assumed, otherwise stated where \( a \) and \( C \) are positive constants and

\[
H_{a,\alpha} (x) = \begin{cases} 
  x^2 & \text{if } |x| < a, \\
  |x|^\beta & \text{if } |x| \geq a,
\end{cases}
\]

with \( 1/\alpha + 1/\beta = 1 \). The last version of logarithmic Sobolev inequality admits a \( n \) dimensional version, for all smooth function \( f \) on \( \mathbb{R}^n \),

\[
\text{Ent}_{\mu_\alpha^\otimes n} (f^2) \leq C \int H_{a,\alpha} \left( \frac{\nabla f}{f} \right) f^2 \, d\mu_\alpha^\otimes n,
\]

(7)

where by definition we have taken

\[
H_{a,\alpha} \left( \frac{\nabla f}{f} \right) := \sum_{i=1}^n H_{a,\alpha} \left( \frac{\partial_i f}{f} \right).
\]

(8)

Note that Bobkov and Ledoux give in [BL97] a corresponding result for the critical (exponential) case, when \( \alpha = 1 \).

Our main purpose here will be to establish the generalization of inequalities (4), (6) and (7) when the measure on \( \mathbb{R} \) is only a log-concave measure between \( e^{-|x|} \) and \( e^{-x^2} \). More precisely, let \( \Phi \) be a \( C^2 \) convex function on \( \mathbb{R} \). Suppose for simplicity that \( \Phi \) is symmetric. We assume that \( \Phi \) satisfies the following property, there exists \( M > 0 \) and \( 0 < \varepsilon \leq 1/2 \) such that \( \Phi(M) > 0 \) and

\[
\forall x \geq M, \quad (1 + \varepsilon) \Phi(x) \leq x \Phi'(x) \leq (2 - \varepsilon) \Phi(x)
\]

(\( H \))

We assume during the article that the function \( \Phi \) on \( \mathbb{R} \) is satisfying hypothesis (\( H \)).

**Remark 1.1** The assumption (\( H \)) implies that there exists \( m_1, m_2 > 0 \) such that

\[
\forall x \geq M, \quad m_1 x^{1/(1-\varepsilon)} \leq \Phi(x) \leq m_2 x^{2-\varepsilon}.
\]

This remark explains how, under the hypothesis (\( H \)), the function \( \Phi \) is between \( e^{-|x|} \) and \( e^{-x^2} \).
Due to the remark 1.1, $\int e^{-\Phi(x)} \, dx < \infty$. Then we define the probability measure $\mu_\Phi$ on $\mathbb{R}$ by

$$
\mu_\Phi(dx) = \frac{1}{Z_\Phi} e^{-\Phi(x)} \, dx,
$$

where $Z_\Phi = \int e^{-\Phi(x)} \, dx$.

The main result of this article is the following theorem:

**Theorem 1.2** Let $\Phi$ satisfying the property (H) then there exists constants $A, A', B, D, \kappa \geq 0$ such that for any smooth functions $f \geq 0$ satisfying $\int f^2 \, d\mu_\Phi = 1$ we have

$$
\text{Ent}_{\mu_\Phi}(f^2) \leq A \text{Var}_{\mu_\Phi}(f) + A' \int_{f^2 \geq \kappa} H_\Phi\left(\frac{f}{f'}\right) f^2 \, d\mu_\Phi,  \tag{9}
$$

where

$$
H_\Phi(x) = \begin{cases} 
\Phi^*(Bx) & \text{if } |x| > D, \\
x^2 & \text{if } |x| \leq D,
\end{cases}
$$

(10)

where $\Phi^*$ is the Legendre-Fenchel transform of $\Phi$, $\Phi^*(x) := \sup_{y \in \mathbb{R}} \{x \cdot y - \Phi(y)\}$.

It is well known that the measure $\mu_\Phi$ satisfies a Poincaré inequality (inequality (5) for the measure $\mu_\Phi$, see for example Chapter 6 of [ABC+00]). Then we obtain the following corollary:

**Corollary 1.3** Let $\Phi$ satisfying the property (H) then there exists $A, B, D \geq 0$ such that for any smooth functions $f > 0$ we have

$$
\text{Ent}_{\mu_\Phi}(f^2) \leq A \int H_\Phi\left(\frac{f}{f'}\right) f^2 \, d\mu_\Phi,  \tag{11}
$$

where $H_\Phi$ is defined on (10).

In [GGM05] we investigate some particular example, where $\Phi(x) = |x|^\alpha \log \beta |x|$, for $\alpha \in ]1, 2[$ and $\beta \in \mathbb{R}$. Theorem 1.2 gives the result in the general case.

**Definition 1.4** Let $\mu$ a probability measure on $\mathbb{R}^n$. We said that $\mu$ satisfies a Logarithmic Sobolev Inequality (LSI) of function $H_\Phi$ (defined on (10)) if there exists $A \geq 0$ such that for any smooth functions $f > 0$ we have

$$
\text{Ent}_{\mu_\Phi}(f^2) \leq A \int H_\Phi\left(\frac{\nabla f}{f'}\right) f^2 \, d\mu_\Phi,  \tag{LSI}
$$

(11)

where $H_{a, \alpha}(\nabla f)$ is defined on (8).

The LSI of function $H_\Phi$ is the $n$-dimensional version of inequality (11).

In Section 2 we will give the proof of Theorem 1.2. It is an adaptation of particular case studied in [GGM05] but it is more technical and complicated. The proof is cut into two parts, Proposition 2.4 and 2.9. In Subsection 2.1, we will describe the case where the entropy is large and in Subsection 2.2 we will study the other case, when the entropy is small. The two cases are very different as we can see in the next section but they are connected to the Hardy’s inequality, that we will point out now.

Let $\mu, \nu$ be Borel measures on $\mathbb{R}^+$. Then the best constant $A$ so that every smooth function $f$ satisfies

$$
\int_0^\infty (f(x) - f(0))^2 \, d\mu(x) \leq A \int_0^\infty f^2 \, d\nu  \tag{12}
$$

is finite if and only if

$$
B = \sup_{x > 0} \left\{ \mu([x, \infty)) \int_0^x \left( \frac{d\nu}{dt} \right)^{-1} \, dt \right\}  \tag{13}
$$
is finite, where $\nu^{ac}$ is the absolutely continuous part of $\nu$ with respect to $\mu$. Moreover, we have (even if $A$ or $B$ is infinite),

$$B \leq A \leq 4B.$$ 

One can see for example [BG99, ABC+00] for a review in this domain.

In Section 3 we will explain some classical properties of this particular logarithmic Sobolev inequality. We explain briefly how, as in the classical logarithmic Sobolev inequality of Gross,

- The $LSI$ of function $H_\Phi$ satisfies the tensorisation and the perturbation properties.
- The $LSI$ of function $H_\Phi$ implies also Poincaré inequality.

The last application proposed is the concentration property for probability measure satisfying inequality (11). We obtain Hoeffding’s type inequality, assume that a measure $\mu$ on $\mathbb{R}$ satisfies inequality (11) and let $f$ be a Lipschitz function on $\mathbb{R}$ with $\|f\|_{\text{Lip}} \leq 1$. Then we get, for some constants $A,B,D \geq 0$ independent of the dimension $n$,

$$\mathbb{P}\left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) - \mu(f) \right) > \lambda \right) \leq \begin{cases} 2 \exp\left(-nA\Phi(B\lambda)\right) & \text{if } \lambda \geq D, \\ 2 \exp\left(-nA\lambda^2\right) & \text{if } 0 \leq \lambda \leq D, \end{cases}$$

or equivalently,

$$\mathbb{P}\left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(X_k) - \mu(f) \right) > \lambda \right) \leq \begin{cases} 2 \exp\left(-nA\Phi\left(B\frac{\lambda}{\sqrt{n}}\right)\right) & \text{if } \lambda \geq D\sqrt{n}, \\ 2 \exp\left(-A\lambda^2\right) & \text{if } 0 \leq \lambda \leq D\sqrt{n}. \end{cases}$$

Inequality (15) is interesting because for large enough $n$ we find the Gaussian concentration, this is natural due to the convergence of $\frac{1}{\sqrt{n}}(\sum_{k=1}^{n} f(X_k) - \mu(f))$ to the Gaussian. This result is not a new one, Talagrand explains it in [Tal95], see also [Led01] for a large review on this topic.

Note to finish the introduction that Barthe, Cattiaux and Roberto [BCR05] are studying the same sort of log-concave measure. They prove also functional inequalities with an other point of view, namely Beckner type inequalities or $\Phi$-Sobolev inequalities, in particular one of their results is concentration inequalities for the same measure $\mu_\Phi$. Let us also mention that the first author in [Gen05], via Prekopa-Leindler inequality, recovers partly our large entropy result.

2 Proof of logarithmic Sobolev inequality (Theorem 1.2)

Before explaining the proof of Theorem 1.2 we give a lemma for classical properties satisfied by the function $\Phi$.

**Lemma 2.1** Assume that $\Phi$ satisfies assumption (H) then there exists $C \geq 0$ such that for large enough $x \geq 0$,

$$x^2 \leq C\Phi^*(x),$$

$$\varepsilon\Phi(\Phi'^{-1}(x)) \leq \Phi^*(x) \leq (1-\varepsilon)\Phi(\Phi'^{-1}(x)),$$

$$\frac{1}{C}\Phi'^{-1}(x) \leq \frac{\Phi^*(x)}{x} \leq C\Phi'^{-1}(x).$$

The proof of Lemma 2.1 is an easy consequence of the property (H).

For this we will note by smooth function a locally absolutely continuous function on $\mathbb{R}$. This is the regularity needed for the use of Hardy inequality in our case.
2.1 Large entropy

The proof of LSI for large entropy is based on the next lemma, we give a LSI saturate on the left.

Lemma 2.2 Let $h$ defined as follows

$$h(x) = \begin{cases} \frac{1}{2|x|^2} & \text{if } |x| < M \\ x & \text{if } |x| \geq M. \end{cases}$$  \hspace{1cm} (19)

Then there exists $C_h \geq 0$ such that for every smooth function $g$ we have

$$\text{Ent}_{\mu_\Phi}(g^2) \leq C_h \int g^2 h d\mu_\Phi. \hspace{1cm} (20)$$

Proof

We use Theorem 3 of [BR03] which is a refinement of the criterion of a Bobkov-Götze theorem (see Theorem 5.3 of [BG99]).

The constant $C_h$ satisfies $\max(b_+, b_-) \leq C_h \leq \max(B_-, B_+)$ where

$$b_+ = \sup_{x \geq 0} \mu_\Phi([x, +\infty[) \log \left(1 + \frac{1}{2\mu_\Phi([x, +\infty[)} \int_0^x Z_\Phi \frac{e^{\Phi(t)}}{h(t)} dt, \right),$$

$$b_- = \sup_{x \leq 0} \mu_\Phi([-\infty, x]) \log \left(1 + \frac{1}{2\mu_\Phi([-\infty, x])} \int_x^0 Z_\Phi \frac{e^{\Phi(t)}}{h(t)} dt, \right),$$

$$B_+ = \sup_{x \geq 0} \mu_\Phi([x, +\infty[) \log \left(1 + \frac{x^2}{\mu_\Phi([x, +\infty[)} \int_0^x Z_\Phi \frac{e^{\Phi(t)}}{h(t)} dt, \right),$$

$$B_- = \sup_{x \leq 0} \mu_\Phi([-\infty, x]) \log \left(1 + \frac{x^2}{\mu_\Phi([-\infty, x])} \int_x^0 Z_\Phi \frac{e^{\Phi(t)}}{h(t)} dt. \right).$$

An easy approximation proves that for large positive $x$

$$\mu_\Phi([x, \infty[) = \int_x^\infty \frac{1}{Z_\Phi} e^{-\Phi(t)} dt \sim_{\infty} \frac{1}{Z_\Phi \Phi'(x)} e^{-\Phi(x)}, \hspace{1cm} (21)$$

and

$$\int_0^x Z_\Phi \frac{e^{\Phi(t)}}{h(t)} dt \sim_{\infty} \frac{Z_\Phi}{h(x) \Phi'(x)} e^{\Phi(x)},$$

and one may prove similar behaviors for negative $x$.

Then, there is $K$ such that for $x \geq M$,

$$\mu_\Phi([x, +\infty[) \log \left(1 + \frac{1}{2\mu_\Phi([x, +\infty[)} \int_0^x Z_\Phi \frac{e^{\Phi(t)}}{h(t)} dt \leq K \frac{\Phi(x)}{\Phi'(x)^2 h(x)} = K \left(\frac{\Phi(x)}{x \Phi'(x)}\right)^2.$$  \hspace{1cm} (22)

The right hand term is bounded by the assumption (H).

A simple calculation then yields that constants $b_+, b_-, B_+$ and $B_-$ are finite and the lemma is proved. \hfill \triangleright

Remark 2.3 Note that this lemma can be proved in a more general case, when $\Phi$ does not satisfy hypothesis (H). In [BL00] the authors prove this result for the symmetric exponential measure.
Proposition 2.4 There exists $A, B, D, A' > 0$ such that for any functions $f \geq 0$ satisfying

$$\int f^2 d\mu_\Phi = 1 \text{ and } \Ent_{\mu_\Phi}(f^2) \geq 1$$

we have

$$\Ent_{\mu_\Phi}(f^2) \leq A'\Var_{\mu_\Phi}(f) + A \int_{f \geq 2} H_\Phi\left(\frac{f'}{f}\right) d\mu_\Phi,$$

where

$$H_\Phi(x) = \begin{cases} \Phi'(Bx) & \text{if } |x| \geq D, \\ x^2 & \text{if } |x| \leq D. \end{cases}$$

As we will see in the proof, $A'$ does not depend on the function $\Phi$.

Proof of Proposition 2.4

Let $f \geq 0$ satisfying $\int f^2 d\mu_\Phi = 1$.

A careful study of the function

$$x \to -x^2 \log x^2 + 5(x - 1)^2 + x^2 - 1 + (x - 2)_+^2 \log(x - 2)_+^2$$

proves that for every $x \geq 0$

$$x^2 \log x^2 \leq 5(x - 1)^2 + x^2 - 1 + (x - 2)_+^2 \log(x - 2)_+^2.$$

We know that $\int (f - 1)^2 d\mu_\Phi \leq 2\Var_{\mu_\Phi}(f)$, recalling that $\int f^2 d\mu_\Phi = 1$ and $f \geq 0$,

$$\int f^2 \log f^2 d\mu_\Phi \leq 5 \int (f - 1)^2 d\mu_\Phi + \int (f^2 - 1) d\mu_\Phi + \int (f - 2)_+^2 \log(f - 2)_+^2 d\mu_\Phi \leq 10\Var_{\mu_\Phi}(f) + \int (f - 2)_+^2 \log(f - 2)_+^2 d\mu_\Phi.$$

Since $\int f^2 d\mu_\Phi = 1$, one can easily prove that

$$\int (f - 2)_+^2 d\mu_\Phi \leq 1,$$

then $\int (f - 2)_+^2 \log(f - 2)_+^2 d\mu_\Phi \leq \Ent_{\mu_\Phi}((f - 2)_+^2)$, and

$$\Ent_{\mu_\Phi}(f^2) \leq 10\Var_{\mu_\Phi}(f) + \Ent_{\mu_\Phi}((f - 2)_+^2).$$

Hardy’s inequality of Lemma 2.2 with $g = (f - 2)_+$ gives

$$\Ent_{\mu_\Phi}((f - 2)_+^2) \leq C_h \int (f - 2)_+^2 h d\mu_\Phi = C_h \int_{f \geq 2} f h d\mu_\Phi. \tag{23}$$

Due to the assumption (H), the function $h(x) = x^2/\Phi(x)$, is increasing on $[M, \infty]$ and

$$\lim_{x \to \infty} h(x) = \infty.$$

We can assume that $\Phi(M) > 0$. We note $m = h(M) > 0$ Let us define the function $\tau$ as follow

$$\tau(x) = \begin{cases} x\Phi(h^{-1}(m))/(8C_h m) & \text{if } 0 \leq x \leq m \\ \Phi(h^{-1}(x))/(8C_h) & \text{if } x \geq m \end{cases} \tag{24}$$

For all $x \geq M$, we have $\tau(h(x)) = \Phi(x)/(8C_h)$ and then, an easy calculus gives that $\tau$ is increasing on $[0, \infty[$.
Let $u > 0$,

\[
C_h \int_{f \geq 2} f^2 h d\mu \Phi = C_h \int_{f \geq 2} u \left( \frac{f'}{f} \right)^2 \frac{h}{u} f^2 d\mu \Phi \\
\leq C_h \int_{f \geq 2} \tau^* \left\{ u \left( \frac{f'}{f} \right)^2 \right\} f^2 d\mu \Phi + \int_{f \geq 2} C_h \tau \left( \frac{h}{u} \right) f^2 d\mu \Phi.
\]

For every function $f$ such that $\int f^2 d\mu = 1$ and for every measurable function $g$ such that $\int f^2 g d\mu$ exists we get

\[
\int f^2 g d\mu \leq \text{Ent}_{\mu^g_f} (f^2) + \log \int e^g d\mu \Phi.
\]

Indeed, this inequality is also true for all function $g \geq 0$ even if the above integrals are infinite. This inequality is also true for all function $g \geq 0$ even integrals are infinite.

We apply the previous inequality with $g = 4C_h \tau \left( \frac{h}{u} \right)$ and we obtain

\[
\int f^2 \tau \left( \frac{h}{u} \right) f^2 d\mu \Phi \leq \frac{1}{4} \int 4C_h \tau \left( \frac{h}{u} \right) f^2 d\mu \Phi \leq \frac{1}{4} \left( \text{Ent}_{\mu^g_f} (f^2) + \log \int e^{4C_h \tau \left( \frac{h}{u} \right)} d\mu \Phi \right).
\]

If $u = 1$ we have, by construction, $\int e^{4C_h \tau \left( \frac{h}{u} \right)} d\mu \Phi < \infty$, then we get

\[
\lim_{u \to \infty} \int e^{4C_h \tau \left( \frac{h}{u} \right)} d\mu \Phi = 1.
\]

Then, by the bounded convergence theorem, there exists $u_0$ such that $\int e^{4C_h \tau \left( \frac{h}{u_0} \right)} d\mu \Phi \leq e$. Thus we have

\[
\text{Ent}_{\mu^g_f} ((f - 2)^2) \leq C_h \int_{f \geq 2} \tau^* \left\{ u_0 \left( \frac{f'}{f} \right)^2 \right\} f^2 d\mu \Phi + \frac{1}{4} \text{Ent}_{\mu^g_f} (f^2) + \frac{1}{4}
\]

$\text{Ent}_{\mu^g_f} (f^2) \geq 1$, implies

\[
\text{Ent}_{\mu^g_f} (f^2) \leq 20 \text{Var}_{\mu^g_f} (f) + 2C_h \int_{f \geq 2} \tau^* \left\{ u_0 \left( \frac{f'}{f} \right)^2 \right\} f^2 d\mu \Phi.
\]

Then Lemma 2.5 gives the proof of inequality (22). $\triangleright$

**Lemma 2.5** There exist constants $A, B, C, D \geq 0$ such that

\[
\forall x \geq 0, \quad \tau^* (x^2) \leq \begin{cases} 
A \Phi^* (Cx) & \text{if } x \geq D, \\
Bx^2 & \text{if } x \leq D.
\end{cases}
\]

**Proof**

Let $x > 0$,

\[
\tau^* (x) = \sup_{y \geq 0} \{ xy - \tau (y) \}.
\]

Let $m = h(M) > 0$, then

\[
\tau^* (x) = \max \left\{ \sup_{y \leq m} \{ xy - \tau (y) \}, \sup_{y \geq m} \{ xy - \tau (y) \} \right\}
\]

\[
\leq \sup_{y \leq m} \{ xy - \tau (y) \} + \sup_{y \geq m} \{ xy - \tau (y) \}.
\]
We have \( \sup_{y \in [0,m]} \{ xy - \tau(y) \} \leq xm \), because \( \tau \) is positive. Then the definition of \( \tau \) implies that

\[
\sup_{y \geq m} \{ xy - \tau(y) \} = \sup_{y \geq M} \left\{ x \frac{y^2}{\Phi(y)} - \frac{\Phi(y)}{8C_h} \right\}.
\]

Let define \( \psi_x(y) = xy^2/\Phi(y) - \Phi(y)/(8C_h) \) for \( y \geq M \). We have

\[
\psi_x'(y) = xy \frac{2\Phi(y) - y\Phi'(y)}{\Phi^2(y)} - \frac{\Phi'(y)}{8C_h}.
\]

Due to the property (H), there is \( D > 0 \) such that

\[
\forall x \geq D, \sup_{y \geq m} \{ xy - \tau(y) \} = x \frac{y^2}{\Phi(y)} - \frac{\Phi(y)}{8C_h},
\]

where \( y_x \geq m \) satisfies

\[
x = \frac{1}{8C_h} \frac{\Phi(y_x)\Phi^2(y_x)}{y_x(2\Phi(y_x) - y_x\Phi'(y_x))}.
\]

The assumption (H) implies that

\[
\epsilon y_x \Phi'(y_x) \leq 2\Phi(y_x) - y_x\Phi'(y_x) \leq \frac{1 - \epsilon}{1 + \epsilon} y_x \Phi'(y_x),
\]

then

\[
\frac{1}{8C_h(1 - \epsilon)(2 - \epsilon)} \Phi^2(y_x) \leq x \leq \frac{1}{8C_h(1 + \epsilon)} \Phi^2(y_x).
\]

Equation (25) gives,

\[
y_x \leq \Phi^{-1}(\sqrt{C}\epsilon)
\]

where \( C > 0 \). Then we get

\[
\forall x \geq D, \quad \sup_{y \geq m} \{ xy - \tau(y) \} \leq \frac{(2 - \epsilon)^2}{8C_h\epsilon(1 + \epsilon)} \Phi(\Phi^{-1}(\sqrt{C}\epsilon)).
\]

We get with the assumption (H),

\[
\forall x \geq D, \quad \sup_{y \geq m} \{ xy - \tau(y) \} \leq \frac{(2 - \epsilon)^2}{8C_h\epsilon(1 + \epsilon)} \Phi(\Phi^{-1}(\sqrt{C}\epsilon)).
\]

We obtain, using inequality (17) of Lemma 2.1,

\[
\forall x \geq D, \quad \sup_{y \geq m} \{ xy - \tau(y) \} \leq \frac{1}{8C_h\epsilon^2(1 + \epsilon)} \Phi^*(\sqrt{C}\epsilon).
\]

then,

\[
\forall x \geq D, \quad \tau^*(x) \leq xm + K\Phi^*(\sqrt{C}\epsilon).
\]

Using inequality (16) of Lemma 2.1 we get

\[
\forall x \geq D, \quad \tau^*(x) \leq K'\Phi^*(\sqrt{C}\epsilon),
\]

for some \( K' \geq 0 \).

On the other hand, the function \( \tau \) is non-negative and satisfy \( \tau(0) = 0 \) then \( \tau^*(0) = 0 \). \( \tau^* \) is also a convex function, then there exists \( m' \) such that

\[
\forall x \in [0, D], \quad \tau^*(x) \leq xm',
\]

which proves the lemma. \( \triangleright \)
Corollary 2.6 For any smooth function $f > 0$ on $\mathbb{R}$ satisfying

$$\int f^2 d\mu = 1, \text{ and } \text{Ent}_{\mu_f}(f^2) \geq 1,$$

we have

$$\text{Ent}_{\mu_f}(f^2) \leq C \int H_\Phi \left( \frac{f'}{f} \right) f^2 d\mu,$$

where

$$H_\Phi(x) = \begin{cases} \Phi^\ast(Bx) & \text{if } |x| \geq D \\ x^2 & \text{if } |x| \leq D, \end{cases}$$

and $B, D \geq 0$.

Proof

Due to the property (H) the measure $\mu_\Phi$ satisfies a Spectral Gap inequality,

$$\text{Var}_{\mu_\Phi}(f) \leq C_{SG} \int f'^2 d\mu,$$

with $C_{SG} \geq 0$. We apply inequality (22) to get the result. $\triangleright$

2.2 Small entropy

Lemma 2.7 Let $\lambda > 0$ and define the function $\psi$ by

$$\psi(x) = \{(\Phi^\ast)^{-1}(\lambda \log x)\}^2.$$

Then for all $\lambda > 0$ there exists $A_\lambda > 0$ such that the function $\psi$ is well defined, positive, increasing, concave on $[A_\lambda, \infty[$ and satisfies $\psi(A_\lambda) \geq 1$.

Proof

Let $\lambda > 0$ be fixed. Classical property of the Legendre-Frenchel transform implies that $\Phi^\ast$ is convex. Due to the property (H), $(\Phi^\ast)^{-1}(\lambda \log x)$ is well defined for $x \geq M_1$ with $M_1 > 0$. Then we get on $[M_1, \infty[$,

$$\psi'(x) = 2g'(\lambda \log x)g(\lambda \log x) \frac{\lambda}{x},$$

and

$$\psi''(x) = 2g(\lambda \log x) \frac{\lambda^2}{x^2} \left( g''(\lambda \log x) - \frac{g'(\lambda \log x)}{\lambda} + \frac{g'^2(\lambda \log x)}{g(\lambda \log x)} \right),$$

where, for simplicity, we have noted $g = (\Phi^\ast)^{-1}$.

For $x$ large enough $g$ is non-negative and increasing and then $\psi$ is increasing on $[M_2, \infty[$, with $M_2 \geq 0$.

An easy estimation gives that as $x$ goes to infinity,

$$\frac{g'(x)}{g(x)} = o_\infty(1), \quad (26)$$

then since $(\Phi^\ast)^{-1}$ is concave, for all large enough $x$, $\psi''(x) \leq 0$. Then one can find $A_\lambda > 0$ such that properties on the Lemma 2.7 are true. $\triangleright$

The proof of LSI for small entropy is based on the next lemma, we give a LSI saturate on the right.
Lemma 2.8 There exists $\lambda > 0$ which depends on the function $\Phi$ such that if we note by $A_\lambda$ the constant of Lemma 2.7 we get for all $g$ defined on $[T, \infty]$ with $T \in [T_1, T_2]$ for some fixed $T_1, T_2$, and verifying that

$$g(T) = \sqrt{A_\lambda}, \ g \geq \sqrt{A_\lambda} \text{ and } \int_T^\infty g^2 d\mu_\alpha \leq 2A_\lambda + 2.$$ 

Then we get

$$\int_T^\infty (g - \sqrt{A_\lambda})^2 \psi(g^2) d\mu \leq C_1 \int_{[T, \infty]} g^2 d\mu,$$

(27)

where $\psi$ is defined on Lemma 2.7.

The constant $C_1$ depend on $\Phi$ and $\lambda$ but does not depend on the value of $T \in [T_1, T_2]$.

Proof

\[<\text{ Let use Hardy’s inequality as explained in the introduction. We have } g(T) = A_\lambda. \text{ We apply inequality (12) on } [T, \infty] \text{ with the function } (g - \sqrt{A_\lambda})_+ \text{ and the following measures } d\mu = \psi(g^2) d\mu_\Phi \text{ and } \nu = \mu_\Phi. \]

Then the constant $C$ in inequality (27) is finite if and only if

$$B = \sup_{x \geq T} \int_T^x e^{\Phi(t)} dt \int_x^\infty \psi(g^2) d\mu,$$

is finite.

By Lemma 2.7, $\psi$ is concave on $[A_\lambda, \infty]$ then by Jensen inequality, for all $x \geq T$ we get

$$\int_x^\infty \psi(g^2) d\mu_\Phi \leq \mu_\Phi([x, \infty]) \psi \left( \frac{\int_x^\infty g^2 d\mu_\Phi}{\mu_\Phi([x, \infty])} \right).$$

Then we have

$$B \leq \sup_{x > T_1} \left\{ \int_T^x e^{\Phi(t)} dt \mu_\Phi([x, \infty]) \psi \left( \frac{\int_x^\infty g^2 d\mu_\Phi}{\mu_\Phi([x, \infty])} \right) \right\}

(28)$$

Due to the property (H) there exists $K > 1$ such that

$$\Phi'(x)e^{\Phi(x)} \leq e^{K\Phi(x)},

(29)$$

and

$$\int_T^x e^{\Phi(t)} dt \leq \frac{K e^{\Phi(x)}}{\Phi'(x)}, \ \int_x^\infty e^{-\Phi(t)} dt \leq \frac{e^{-\Phi(x)}}{\Phi'(x)},$$

for large enough $x$. By (29) we get also for large enough $x$ that

$$e^{-K\Phi(x)} \leq \int_x^\infty e^{-\Phi(t)} dt.$$

Then for large enough $x$, uniformly in the previous $g$, one have

$$\int_T^x e^{\Phi(t)} dt \mu_\Phi([x, \infty]) \psi \left( \frac{\int_x^\infty g^2 d\mu_\Phi}{\mu_\Phi([x, \infty])} \right) \leq \frac{K}{\Phi'(x)} \psi \left( \frac{\int_x^\infty g^2 d\mu_\Phi}{K e^{K\Phi(x)}} \right).$$

For $x$ large enough,

$$\frac{\int_x^\infty g^2 d\mu_\Phi}{K} \leq 1.$$
Then, by definition of $\psi$, for large enough $x$,
\[
\int_{T_1}^x e^{\Phi(t)} dt_{\mu_{\Phi}}([x, \infty[) \psi \left( \int_x^\infty g^2 d\mu_{\Phi} \right) \leq K \left( \Phi^{-1}(\lambda K \Phi(x)) \right)^2.
\]
There is also $C_\epsilon$ such that, for $x$ large enough
\[
\Phi^*(x) \leq \Phi'(\Phi^{-1}(C_\epsilon x)),
\]
as one can see from equation (17).
Then one can choose $\lambda = 1/(KC_\epsilon)$ and the lemma is proved. Note that $\lambda$ depends only on the function $\Phi$.
The constant $B$ on (28) is bounded by $K$ which does’nt depend on $T$ on $[T_1, T_2]$.

**Proposition 2.9** There exists $A, A', B, D > 0$ such that for any functions $f \geq 0$ satisfying
\[
\int f^2 d\mu_\alpha = 1 \text{ and } \mathbf{Ent}_{\mu_\alpha}(f^2) \leq 1
\]
we have
\[
\mathbf{Ent}_{\mu_{\Phi}}(f^2) \leq A \mathbf{Var}_{\mu_\alpha}(f) + A' \int_{f^2 \geq A_\lambda} H \left( \frac{f'}{f} \right) d\mu_{\Phi},
\]
where
\[
H(x) = \begin{cases} \Phi^*(Bx) & \text{if } |x| \geq D, \\ x^2 & \text{if } |x| \leq D. \end{cases}
\]

**Proof**
\[\triangleleft\]
Let $f \geq 0$ satisfying $\int f^2 d\mu_\alpha = 1$.
We can assume that $A_\lambda \geq 2$. A careful study of the function
\[
x \rightarrow -x^2 \log x^2 + A(x - 1)^2 + x^2 - 1 + (x - \sqrt{A_\lambda})^2 \log(x - \sqrt{A_\lambda})^2
\]
proves that there exists $A$ such that for every $x \in \mathbb{R}^+$
\[
x^2 \log x^2 \leq A(x - 1)^2 + x^2 - 1 + (x - \sqrt{A_\lambda})^2 \log(x - \sqrt{A_\lambda})^2.
\]
Then we get
\[
\mathbf{Ent}_{\mu_\alpha}(f^2) = \int f^2 \log f^2 d\mu_\alpha \leq A \mathbf{Var}_{\mu_\alpha}(f) + \int \left( f - \sqrt{A_\lambda} \right)^2 \log f^2 d\mu_\alpha,
\]
where $\sqrt{A_\lambda}$ is defined as in Lemmas 2.7 and 2.8.

Fix $\lambda$ as in Lemma 2.8. We define the function $K$ on $[A_\lambda, \infty[$ by
\[
K(x) = \frac{\log x^2}{\psi(x^2)},
\]
where $A_\lambda$ is defined on Lemma 2.8.
Let now define $T_1 < T_2$ such that
\[
\mu_{\Phi}([\infty, T_1]) = \frac{3}{8}, \mu_{\Phi}([T_1, T_2]) = \frac{1}{4} \text{ and } \mu_{\Phi}([T_2, +\infty[) = \frac{3}{8}.
\]
Since $\int f^2 d\mu_{\Phi} = 1$ there exists $T \in [T_1, T_2]$ such that $f(T) \leq A_\lambda$.

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Let us define $g$ on $[T_1, \infty]$ as follow

$$g = \sqrt{A_\lambda} + \left( f - \sqrt{A_\lambda} \right)_+ K(f) \text{ on } [T, \infty].$$

Function $g$ satisfies $g(T) = \sqrt{A_\lambda}$ and $g(x) \geq \sqrt{A_\lambda}$ for all $x \geq T$.

Then we have

$$\int_T^\infty g^2 \, d\mu_F \leq \int_{T_1}^\infty g^2 \, d\mu_F \leq 2A_\lambda + 2 \int_{[T_1, \infty] \cap \{ f^2 \geq A_\lambda \}} f^2 K^2(f) \, d\mu_F \leq 2A_\lambda + 2 \int_{[T_1, \infty]} f^2 \log(f^2) \, d\mu_F \leq 2A_\lambda + 2, \quad (31)$$

where we are using the growth of $\psi$ on $[A_\lambda, \infty]$ and $\psi(A_\lambda) \geq 1$.

Assumptions on Lemma 2.8 are satisfied, we obtain by inequality (27)

$$\int_T^\infty (g - \sqrt{A_\lambda})_+^2 \psi(g^2) \, d\mu_F \leq C_1 \int_{[T, \infty] \cap \{ f^2 \geq A_\lambda \}} g^2 \, d\mu_F.$$

Let us compare the various terms now.

Due to the property (H), $K$ is lower bounded on $[\sqrt{A_\lambda}, \infty]$ by $\alpha \geq 1$ (maybe for $A_\lambda$ larger), then we get firstly

$$\sqrt{A_\lambda} + \left( f - \sqrt{A_\lambda} \right)_+ K(f) \geq \sqrt{A_\lambda} + \left( f - \sqrt{A_\lambda} \right)_+ \alpha \geq f \text{ on } \{ f^2 \geq A_\lambda \}.$$

Then

$$\left(g - \sqrt{A_\lambda}\right)_+^2 \psi(g^2) = \left(f - \sqrt{A_\lambda}\right)_+^2 K(f)^2 \psi\left(\sqrt{A_\lambda} + \left(f - \sqrt{A_\lambda}\right)_+ K(f)\right)^2 \geq \left(f - \sqrt{A_\lambda}\right)_+^2 K(f)^2 \psi(f^2) = \left(f - \sqrt{A_\lambda}\right)_+^2 \log f^2,$$

by the definition of $K$, then we obtain

$$\int_T^\infty (f - \sqrt{A_\lambda})_+^2 \log f^2 \, d\mu_F \leq \int_T^\infty (g - \sqrt{A_\lambda})_+^2 \psi(g^2) \, d\mu_F. \quad (32)$$

Secondly we have on $\{ f \geq \sqrt{A_\lambda} \}$

$$g' = f'K(f) + \left(f - \sqrt{A_\lambda}\right)_+ f'K'(f)$$

$$= f'K(f) \left(1 + \left(f - \sqrt{A_\lambda}\right)_+ \frac{K'(f)}{K(f)}\right).$$

But we have for $x \geq \sqrt{A_\lambda}$

$$\left|1 + (x - \sqrt{A_\lambda}) \frac{K'(x)}{K(x)}\right| \leq 1 + x \left|\frac{K'(x)}{K(x)}\right| \leq 1 + \frac{\lambda g'(\lambda 2 \log x)}{x (\lambda 2 \log x)},$$

where $g(x) = \Phi^{-1}(x)$. Using Lemma 2.7 and the estimation (26) we obtain that there exists $C > 0$ such that for all $x \geq \sqrt{A_\lambda}$,

$$\left|1 + (x - \sqrt{A_\lambda}) \frac{K'(x)}{K(x)}\right| \leq C$$
Lemma 2.10 There exists $u_0 > 0$ such that, for all $x \geq A_\lambda$ we have

$$\tau_2 \left( \frac{K^2(x)}{u_0} \right) \leq \frac{1}{2} \log x.$$
Proof

Let $\kappa = 2\lambda/(1 - \epsilon)$.

For all $x \geq M$, where $M$ is defined on equation (24), we have

$$\tau_2(h(x)) = \frac{\Phi(x)}{\kappa},$$

$$\tau_2\left(\frac{x^2}{\Phi(x)}\right) = \frac{\Phi(x)}{\kappa}.$$

$\tau_2$ is increasing, then due to the property (H) we have for $x \geq M$

$$\tau_2\left((1 + \epsilon)^2 \frac{\Phi(x)}{\Phi'(x)^2}\right) \leq \frac{\Phi(x)}{\kappa}.$$

Using now inequality (17) one has

$$\frac{1}{\Phi'(x)} \geq \frac{1}{\Phi^*((1 - \epsilon)\Phi(x))},$$

then for all $x \geq M$,

$$\tau_2\left((1 + \epsilon)^2 \frac{\Phi(x)}{\Phi^*((1 - \epsilon)\Phi(x))^2}\right) \leq \frac{\Phi(x)}{\kappa}.$$

Take now $z = (1 - \epsilon)\Phi(x)$,

$$\tau_2\left((1 + \epsilon)^2 \frac{z}{(1 - \epsilon)\Phi^*(z)^2}\right) \leq \frac{z}{(1 - \epsilon)\kappa},$$

to finish take $x = \exp\left(\frac{4z}{(1 - \epsilon)\kappa}\right)$ to obtain

$$\tau_2\left((1 + \epsilon)^2 \frac{\log x^2}{\Phi^*\left(\frac{(1 - \epsilon)\kappa}{2}\log x^2\right)^2}\right) \leq \frac{1}{2}\log x^2.$$

Recall that $\lambda = (1 - \epsilon)\kappa/2$ and let take $u_0 = 1/((1 + \epsilon)^2\kappa)$, to obtain the result for $x \geq C$, where $C$ is a constant depending on $\Phi$.

If we have $A_\lambda < C$, one can change the value of $u_0$ to obtain also the results on $[A_\lambda, C]$. ⊲

Proof of Theorem 1.2

Let give the proof of the theorem we need to give an other result like Proposition 2.4. By the same argument as in Proposition 2.4 one can also prove that there exists $A, A', B, D > 0$ such that for any functions $f \geq 0$ satisfying

$$\int f^2 d\mu_\alpha = 1 \text{ and } \text{Ent}_{\mu_\alpha}(f^2) \geq 1$$

we have for some $C'(A_\lambda), C(A_\lambda)$

$$\text{Ent}_{\mu_\Phi}(f^2) \leq C'(A_\lambda)\text{Var}_{\mu_\Phi}(f) + C(A_\lambda)\int_{f^2 \geq A_\lambda} H\left(\frac{f'}{f}\right) d\mu_\Phi,$$

(35)

where $H_\Phi$ is defined on (10) and $A_\lambda$ on the Proposition 2.9. To introduce $A_\lambda$, we just have to change constants in the inequality.

Then the proof of the theorem is a simple consequence of (35) and Proposition 2.9. ⊲
3 Classical properties and applications

Let us give here properties inherited directly from the methodology known for classical logarithmic Sobolev inequalities.

**Proposition 3.1**
1. This property is known under the name of tensorisation.

Let \( \mu_1 \) and \( \mu_2 \) two probability measures on \( \mathbb{R}^{n_1} \) and \( \mathbb{R}^{n_2} \). Suppose that \( \mu_1 \) (resp. \( \mu_2 \)) satisfies the \( a_{LSI} \) with function \( H_\Phi \) and constant \( A_1 \) (resp. with constant \( A_2 \)) then the probability \( \mu_1 \otimes \mu_2 \) on \( \mathbb{R}^{n_1+n_2} \), satisfies a \( a_{LSI} \) with function \( H_\Phi \) and constant \( \max \{ A_1, A_2 \} \).

2. This property is known under the name of perturbation.

Let \( \mu \) a measure on \( \mathbb{R}^n \) a \( a_{LSI} \) with function \( H_\Phi \) and constant \( A \). Let \( h \) a bounded function on \( \mathbb{R}^n \) and defined \( \tilde{\mu} \) as

\[
d\tilde{\mu} = \frac{e^h}{Z} d\mu,
\]

where \( Z = \int e^h d\mu \).

Then the measure \( \tilde{\mu} \) satisfies a \( a_{LSI} \) with function \( H_\Phi \) and the constant \( D = Ae^{2\text{osc}(h)} \), where \( \text{osc}(h) = \sup(h) - \inf(h) \).

3. Link between \( a_{LSI} \) of function \( H_\Phi \) with Poincaré inequality.

Let \( \mu \) a measure on \( \mathbb{R}^n \). If \( \mu \) satisfies a \( a_{LSI} \) with function \( H_\Phi \) and constant \( A \), then \( \mu \) satisfies a Poincaré inequality with the constant \( A \). Let us recall that \( \mu \) satisfies a Poincaré inequality with constant \( A \) if

\[
\text{Var}_\mu(f) \leq A \int |\nabla f|^2 d\mu,
\]

for all smooth function \( f \).

**Proof**

One can find the details of the proof of the properties of tensorisation and perturbation and the implication of the Poincaré inequality in chapters 1 and 3 of [ABC +00] (Section 1.2.6., Theorem 3.2.1 and Theorem 3.4.3).

**Proposition 3.2**

Assume that the probability measure \( \mu \) on \( \mathbb{R} \) satisfies a \( a_{LSI} \) with function \( H_\Phi \) and constant \( A \). Then there exists constants \( B,C,D \geq 0 \), independent of \( n \) such that: if \( F \) is a function on \( \mathbb{R}^n \) such that \( \forall i, \|\partial_i F\|_\infty \leq \zeta \), then we get for \( \lambda \geq 0 \),

\[
\mu^\otimes(n)(|F - \mu^\otimes_n(F)| \geq \lambda) \leq \begin{cases} 
2 \exp\left( -nB\Phi\left( \frac{\lambda}{n\zeta} \right) \right) & \text{if } \lambda > nD\zeta, \\
2 \exp\left( -B\frac{\lambda^2}{n\zeta^2} \right) & \text{if } 0 \leq \lambda \leq nD\zeta.
\end{cases} \quad (36)
\]

**Proof**

Let us first present the proof when \( n = 1 \). Assume, without loss of generality, that \( \int F d\mu = 0 \). Due to the homogeneous property of (36) on can suppose that \( \zeta = 1 \).

Let us recall briefly Herbst’s argument (see Chapter 7 [ABC +00] for more details). Denote \( \psi(t) = \int e^{tf} d\mu \), and remark that \( a_{LSI} \) of function \( H_\Phi \) applied to \( f^2 = e^{tF} \), using basic properties of \( H_\Phi \), yields to

\[
t\psi'(t) - \psi(t) \log \psi(t) \leq AH_\Phi\left( \frac{t}{2} \right) \psi(t) \quad (37)
\]

which, denoting \( K(t) = (1/t) \log \psi(t) \), entails

\[
K'(t) \leq \frac{A}{t^2} H_\Phi\left( \frac{t}{2} \right).
\]
Then, integrating, and using $K(0) = \int Fd\mu = 0$, we obtain
\[ \psi(t) \leq \exp \left( At \int_0^t \frac{1}{s^2} H_{\Phi} \left( \frac{s}{2} \right) ds \right). \tag{38} \]

Then we get using Markov inequality
\[ \mu(|F - \mu(F)| \geq \lambda) \leq 2 \exp \left( \min_{t \geq 0} \left\{ At \int_0^t \frac{1}{s^2} H_{\Phi} \left( \frac{s}{2} \right) ds = \lambda t \right\} \right). \]

Let note, for $t > 0$,
\[ G(t) = At \int_0^t \frac{1}{s^2} H_{\Phi} \left( \frac{s}{2} \right) ds - \lambda t. \]

An easy study proves that $G$ admits a single minimum on $\mathbb{R}^+$ (except maybe if $\lambda = 0$). Then due to the definition of $H_{\Phi}$ we get that
\[ \min_{t \geq 0} \{ G(t) \} = -\frac{\lambda^2}{A}, \quad \text{if } \lambda \leq AD. \]

Assume now that $\lambda \geq AD$ then we obtain after derivation
\[ \min_{t \geq 0} \{ G(t) \} = -A \Phi^\ast \left( t_0 \frac{B}{2} \right), \quad \text{with } \lambda t_0 = \lambda t_0 = At_0 \int_0^{t_0} \frac{1}{s^2} H_{\Phi} \left( \frac{s}{2} \right) ds + AH_{\Phi} \left( t_0 \frac{1}{2} \right). \tag{39} \]

We first prove that there exists $C > 0$ such that for all $t_0$ large enough
\[ t_0 \int_0^{t_0} \frac{1}{s^2} H_{\Phi} \left( \frac{s}{2} \right) ds \leq CH_{\Phi} \left( t_0 \frac{1}{2} \right). \tag{40} \]

For $\kappa \geq 0$ large enough and $t_0 \geq \kappa$ we get using then inequality (17) we get
\[ t_0 \int_0^{t_0} \frac{1}{s^2} H_{\Phi} \left( \frac{s}{2} \right) ds \leq Ct_0 \int_0^{t_0} \frac{1}{s^2} \Phi \left( \Phi^{-1} \left( \frac{s}{2} \right) \right) ds, \]

with $C > 0$. Then by a change of variables and integration by parts, for large enough $t_0$,
\[ t_0 \int_0^{t_0} \frac{1}{s^2} \Phi \left( \Phi^{-1} \left( \frac{s}{2} \right) \right) ds = \frac{t_0}{2} \int_{\Phi^{-1} \left( \frac{s}{2} \right)}^{\Phi^{-1} \left( \frac{t_0}{2} \right)} \Phi(u) \Phi''(u) du \]
\[ \leq \frac{t_0}{2} \Phi(\Phi^{-1} \left( \frac{t_0}{2} \right)) \Phi''(\Phi^{-1} \left( \frac{t_0}{2} \right)) + \frac{t_0}{2} \Phi^{-1} \left( \frac{t_0}{2} \right) \]
\[ \leq Ct_0 \Phi^{-1} \left( \frac{t_0}{2} \right), \]

for some other $C > 0$. Then we get, using inequality (18), for $t_0$ large enough,
\[ t_0 \int_0^{t_0} \frac{1}{s^2} H_{\Phi} \left( \frac{s}{2} \right) ds \leq Ct_0 \Phi^{-1} \left( \frac{t_0}{2} \right) \leq C' \Phi^\ast \left( \frac{t_0}{2} \right). \]

for some constant $C' > 0$ and for $t_0$ large enough and inequality (40) is proved. By (40) and (39) one get for $t_0$ large enough,
\[ \lambda t_0 \leq A' \Phi^\ast \left( \frac{t_0}{2} \right), \]

for some constant $A' > 0$. But, using inequality (18) we get then
\[ \Phi'(A\lambda) \leq Ct_0, \]
\[ \min_{t \geq 0} \{ G(t) \} \leq -A \Phi^\ast \left( B\Phi'(C\lambda) \right) \leq -A \Phi^\ast \left( \Phi'(C\lambda) \right), \]

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if $\lambda$ is large enough and for some other constants $A, B, C, C' \geq 0$. Using inequality (17), we obtain the result in dimension 1.

For the $n$-dimensional extension, use the tensorisation property of LSI of function $H_\Phi$ and

$$\sum_{i=1}^{n} H_\Phi \left( \frac{t}{2} \partial_i F \right) \leq n H_\Phi \left( \frac{t}{2} \right).$$

Then we can use the case of dimension 1 with the constant $A$ replaced by $An$.

**Remark 3.3** Let us present a simple application of the preceding proposition to deviation inequality of the empirical mean of a function. Consider the real valued function $f$, with $|f'| \leq 1$. Let apply Proposition 3.2 with the two functions

$$F(x_1, ..., x_n) = \frac{1}{n} \sum_{k=1}^{n} f(x_k) \quad \text{and} \quad F(x_1, ..., x_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(x_i).$$

We obtain then

$$\mathbb{P} \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) - \mu(f) > \lambda \right) \leq \begin{cases} 
2 \exp \left( -n A \Phi \left( B \lambda \right) \right) & \text{if } \lambda \geq D, \\
2 \exp \left( -n A \lambda^2 \right) & \text{if } 0 \leq \lambda \leq D,
\end{cases}$$

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(X_k) - \mu(f) > \lambda \right) \leq \begin{cases} 
2 \exp \left( -n A \Phi \left( \frac{B \lambda}{\sqrt{n}} \right) \right) & \text{if } \lambda \geq D \sqrt{n}, \\
2 \exp \left( -A \lambda^2 \right) & \text{if } 0 \leq \lambda \leq D \sqrt{n}.
\end{cases}$$

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