Persistence of periodic and homoclinic orbits, first integrals and commutative vector fields in dynamical systems

Shoya Motonaga* and Kazuyuki Yagasaki

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Yoshida-Honnachi, Sakyo-ku, Kyoto 606-8501, Japan
E-mail: mnaga@amp.i.kyoto-u.ac.jp and yagasaki@amp.i.kyoto-u.ac.jp

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Abstract
We study persistence of periodic and homoclinic orbits, first integrals and commutative vector fields in dynamical systems depending on a small parameter $\varepsilon > 0$ and give several necessary conditions for their persistence. Here we treat homoclinic orbits not only to equilibria but also to periodic orbits. We also discuss some relationships of these results with the standard subharmonic and homoclinic Melnikov methods for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems, and with another version of the homoclinic Melnikov method for autonomous perturbations of multi-degree-of-freedom Hamiltonian systems. In particular, we show that a first integral which converges to the Hamiltonian or another first integral as the perturbation tends to zero does not exist near the unperturbed periodic or homoclinic orbits in the perturbed systems if the subharmonic or homoclinic Melnikov functions are not identically zero on connected open sets. We illustrate our theory for four examples: the periodically forced Duffing oscillator, two identical pendula coupled with a harmonic oscillator, a periodically forced rigid body and a three-mode truncation of a buckled beam.

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*Author to whom any correspondence should be addressed.
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1. Introduction

Let $\mathcal{M}$ be an $n$-dimensional paracompact oriented $C^4$ real manifold for $n \geq 2$. Here we require its paracompactness and orientedness for defining integrals on $\mathcal{M}$. Consider dynamical systems of the form

$$\dot{x} = X_\varepsilon(x), \quad x \in \mathcal{M},$$

(1.1)

where $\varepsilon$ is a small parameter such that $0 < \varepsilon \ll 1$, and the vector field $X_\varepsilon$ is $C^3$ with respect to $x$ and $\varepsilon$. Let $X_\varepsilon(x) = X_0(x) + \varepsilon X_1(x) + O(\varepsilon^2)$ for $\varepsilon > 0$ sufficiently small. The system (1.1) becomes

$$\dot{x} = X_0(x)$$

(1.2)

when $\varepsilon = 0$, and it is regarded as a perturbation of (1.2). Assume that the unperturbed system (1.2) has a periodic or homoclinic orbit and a first integral or commutative vector field. Here we are mainly interested in their persistence in (1.1) for $\varepsilon > 0$ sufficiently small.

Bogoyavlenskij [6] extended a concept of Liouville integrality [3, 20], which is defined for Hamiltonian systems, and proposed a definition of integrability for general systems. For (1.1), its integrability means that there exist $k$ ($\geq 1$) commutative vector fields containing $X_\varepsilon$ and $n - k$ ($\geq 0$) first integrals for them such that the vector fields and first integrals are, respectively, linearly and functionally independent over a dense open set in $\mathcal{M}$. For integrable systems in this meaning, we have a statement similar to the Liouville–Arnold theorem for Hamiltonian systems (e.g., section 49 in chapter 10 of [3]): the flow on a level set of the first integrals is diffeomorphically conjugate to a linear flow on the $k$-dimensional torus $T^k$ if the level set of the first integrals is a $k$-dimensional compact manifold (see proposition 2 of [6]). Thus, the existence of first integrals and commutative vector fields is closely related to integrability of (1.1).

Even if the unperturbed system (1.2) is integrable, the perturbed system (1.1) is generally believed to be nonintegrable for $\varepsilon > 0$ small. For example, when the system (1.1) is analytic and Hamiltonian for $\varepsilon \geq 0$, a famous result of Poincaré [23] says that its analytic Liouville integrability does not persist for $\varepsilon > 0$ under some generic assumptions. This means that not only first integrals independent of the Hamiltonian but also (Hamiltonian) vector fields commutative with $X_0$ do not persist in general. See also [16] for a more general result on nonexistence of first integrals, which was extended to non-Hamiltonian systems in [17]. Moreover, Morales-Ruiz [21] studied time-periodic Hamiltonian perturbations of single-degree-of-freedom Hamiltonian systems with homoclinic orbits, and showed a relationship between their nonintegrability and a version due to Ziglin [35] of the Melnikov method [19] by taking the time $t$ and small parameter $\varepsilon$ as state variables. Here the Melnikov method enables us to detect transversal self-intersection of complex separatrices of periodic orbits unlike the standard version [12, 19, 25]. More concretely, under some restrictive conditions, he essentially proved that they are meromorphically nonintegrable in the Bogoyavlenskij sense if the Melnikov functions are not identically zero, when a generalization due to Ayoul and Zung [5] of the Morales-Ramis theory [20, 22], which provides a sufficient condition for nonintegrability of dynamical systems, is applied. See section 4.1 below for more details. On the other hand, to the authors’ knowledge, the persistence of first integrals and commutative vector fields, especially when the unperturbed system (1.2) is nonintegrable, in non-Hamiltonian systems has attracted little attention.

In this paper, we give several necessary conditions for persistence of periodic or homoclinic orbits, first integrals or commutative vector fields in (1.1). In particular, we treat homoclinic
orbits not only to equilibria but also to periodic orbits. Moreover, we see that persistence of periodic or homoclinic orbits and first integrals or commutative vector fields near them have the same necessary conditions (cf theorems 2.1–2.4, 3.5, 3.8, 3.10 and 3.12). This indicates close relationships between the dynamics and geometry of the perturbed systems. We also discuss some relationships of these results with the standard subharmonic and homoclinic Melnikov methods [12, 19, 25, 28] for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems as in [21], and with another version of the homoclinic Melnikov method due to Wiggins [24] for autonomous Hamiltonian perturbations of multi-degree-of-freedom integrable Hamiltonian systems. The subharmonic Melnikov method provides a sufficient condition for persistence of periodic orbits in the perturbed system: if the subharmonic Melnikov functions have a simple zero, then such orbits persist. For the latter homoclinic Melnikov method, we restrict ourselves to the case in which the unperturbed systems have invariant manifolds consisting of periodic orbits to which there exist homoclinic orbits since only such a situation can be treated in our result, although the technique was developed for more general systems. These versions of the Melnikov methods are described shortly in section 4 below. In particular, we show that a first integral which converges to the Hamiltonian or another first integral as \( \epsilon \to 0 \) does not exist near the unperturbed periodic or homoclinic orbits in the perturbed system for \( \epsilon > 0 \) sufficiently small if the subharmonic or homoclinic Melnikov functions are not identically zero on connected open sets. We illustrate our theory for four examples: the periodically forced Duffing oscillator [12, 25], two identical pendula coupled with a harmonic oscillator, a periodically forced rigid body [34] and a three-mode truncation of a buckled beam [30]. The persistence of first integrals is discussed in the first and second examples, the persistence of a first integral and periodic orbits in the third one and the persistence of commutative vector fields in the fourth one.

The outline of this paper is as follows: in sections 2 and 3, we present our main results for first integrals and commutative vector fields, respectively, as well as for both of periodic and homoclinic orbits. For the reader’s convenience, in appendix A, we collect basic notions and facts on connections of vector bundles and linear differential equations as auxiliary materials for section 3. In section 4, we describe some relationships of the main results with the subharmonic and/or homoclinic Melnikov methods when the unperturbed system (1.2) is a single- or multi-degree-of-freedom Hamiltonian system. Finally, we give the four examples to illustrate our theory in section 5.

2. First integrals

In this section, we discuss persistence of periodic and homoclinic orbits and first integrals for (1.1). In the discussion here, less smoothness of \( \mathcal{M} \) and \( X \), is needed: \( \mathcal{M} \) and \( X \), are \( C^3 \) and \( C^2 \), respectively.

2.1. Periodic orbits

We begin with a case in which the unperturbed system (1.2) has a periodic orbit in (1.2). We make the following assumptions on (1.2):

(A1) There exists a \( T \)-periodic orbit \( \gamma(t) \) for some constant \( T > 0 \) in (1.2);

(A2) There exists a non-constant \( C^3 \) first integral \( F(x) \) of (1.2), i.e.,

\[
dF(X^0) = 0,
\]

near \( \Gamma = \{ \gamma(t) | t \in [0, T] \} \).

\[7576\]
Define
\[ I_{F, \gamma} := \int_0^T dF(X^1)(\gamma(t)) \, dt. \] (2.1)

We state our main results for persistence of periodic orbits and first integrals.

**Theorem 2.1.** Assume that (A1) and (A2) hold. If the perturbed system (1.1) has a $T$-periodic orbit $\gamma_\varepsilon$ depending $C^2$-smoothly on $\varepsilon$ such that $T_0 = T$ and $\gamma_0 = \gamma$, then the integral $I_{F, \gamma}$ must be zero.

**Proof.** Assume that (A1) and (A2) hold and the system (1.1) has a periodic orbit $\gamma_\varepsilon = \gamma + O(\varepsilon)$ for $\varepsilon > 0$. Since $\gamma_\varepsilon$ is a $T_\varepsilon$-periodic orbit in (1.1), we compute
\[ \int_0^{T_\varepsilon} dF(X^1)(\gamma_\varepsilon(t)) \, dt = F(\gamma_\varepsilon(T_\varepsilon)) - F(\gamma_\varepsilon(0)) = 0. \]

On the other hand, since $F$ is a first integral of $X^0$, we have $dF(X^0) = 0$, so that
\[ dF(X^1)(\gamma_\varepsilon(t)) = \varepsilon \, dF(X^1)(\gamma(t)) + O(\varepsilon^2). \]

Since $T_\varepsilon = T + O(\varepsilon)$, we see by the above two equations that
\[ \int_0^{T_\varepsilon} dF(X^1)(\gamma(t)) \, dt = \varepsilon \int_0^T dF(X^1)(\gamma(t)) \, dt + O(\varepsilon^2) = \varepsilon I_{F, \gamma} + O(\varepsilon^2) = 0. \]

Thus, we obtain $I_{F, \gamma} = 0$. \hfill $\square$

**Theorem 2.2.** Assume that (A1) and (A2) hold. If the perturbed system (1.1) has a $C^3$ first integral $F_{\varepsilon}(x)$ depending $C^2$-smoothly on $\varepsilon$ near $\Gamma$ such that $F_0(x) = F(x)$, then the integral $I_{F, \gamma}$ must be zero.

**Proof.** Assume that (A1) and (A2) hold and the system has a first integral $F_\varepsilon = F + \varepsilon F^1 + O(\varepsilon^2)$ near $\Gamma$. Since $\gamma$ is a $T$-periodic orbit in (1.2), we have
\[ \int_0^T dF_\varepsilon(X^0)(\gamma(t)) \, dt = F_\varepsilon(\gamma(T)) - F_\varepsilon(\gamma(0)) = 0. \] (2.2)

On the other hand, since $dF_\varepsilon(X_\varepsilon) = 0$ and
\[ dF_\varepsilon(X^0) = dF_\varepsilon(X^0) + \varepsilon \, dF_\varepsilon(X^1) + O(\varepsilon^2), \]
we have
\[ dF_\varepsilon(X^0) = -\varepsilon \, dF(X^1) + O(\varepsilon^2) \] (2.3)

near $\Gamma$. From (2.2) and (2.3) we obtain
\[ \int_0^T dF_\varepsilon(X^0)(\gamma(t)) \, dt = -\varepsilon \int_0^T dF_\varepsilon(X^1)(\gamma(t)) \, dt + O(\varepsilon^2) = -\varepsilon I_{F, \gamma} + O(\varepsilon^2) = 0, \]

which yields the desired result. \hfill $\square$

Theorems 2.1 and 2.2 mean that if $I_{F, \gamma} \neq 0$, then neither the periodic orbit $\gamma$ nor first integral $F$ persists in (1.1) for $\varepsilon > 0$. 7577
2.2. Homoclinic orbits

We next consider a case in which the unperturbed system (1.2) has a homoclinic orbit to an equilibrium or to a periodic orbit in (1.2). Instead of (A1) and (A2), we assume the following on (1.2):

(A1') There exists a homoclinic orbit $\gamma^h(t)$ to a $T$-periodic orbit $\gamma^p(t)$ in (1.2);

(A2') There exists a non-constant $C^3$ first integral $F(x)$ of (1.2) near $\Gamma^h = \{\gamma^h(t) | t \in \mathbb{R}\} \cup \Gamma^p$, where $\Gamma^p = \{\gamma^p(t) | t \in [0, T]\}$.

In assumption (A1'), $\gamma^p$ may be an equilibrium. As seen below we have statements similar to theorems 2.1 and 2.2 in this case but another idea is needed for their proofs since the situation is not simple when $\gamma^h(t)$ is a homoclinic orbit to a periodic orbit.

We first define an integral which plays a similar role as $\tilde{I}_{F, \gamma}$ in section 2.1 (see equation (2.1)). Let $\gamma^h$ be not an equilibrium. Choose a point $x_0 = \gamma^p(0)$ and take an $(n-1)$-dimensional hypersurface $\Sigma$ as the Poincaré section such that $\gamma^p$ intersects $\Sigma$ transversely at $x_0$. Restricting $\Sigma$ to a sufficiently small neighborhood of $x_0$ if necessary, we can assume that $\gamma^h(t)$ intersects $\Sigma$ transversely infinitely many times, say at $T_j \in \mathbb{R}$ with $T_{j-1} < T_j$, $j \in \mathbb{Z}$, such that $\lim_{j \to \pm \infty} T_j = -\infty$ and $\lim_{j \to \pm \infty} T_j = +\infty$, since it converges to $\gamma^p(t)$. In particular,

$$\lim_{j \to \pm \infty} \gamma^h(T_j) = x_0.$$

See figure 1. So we formally define

$$\tilde{I}_{F, \gamma} := \lim_{k \to +\infty} \int_{T_k}^{T_k} dF(X^1)(\gamma^h(t))dt.$$  (2.4)

If $\gamma^p$ is an equilibrium, then equation (2.4) is reduced to

$$\tilde{I}_{F, \gamma} = \int_{-\infty}^{+\infty} dF(X^1)(\gamma^h(t))dt$$  (2.5)

by taking any sequence $\{T_j\}_{j=-\infty}^{\infty}$ such that $\lim_{j \to \pm \infty} T_j = \pm \infty$. 

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**Figure 1.** Poincaré section $\Sigma$. 

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We now state our main results for persistence of homoclinic orbits and first integrals.

**Theorem 2.3.** Assume that \((A1')\) and \((A2')\) hold and that there exists a periodic orbit \(\gamma^p\) depending \(C^1\)-smoothly on \(\varepsilon\) in \((1.1)\) such that \(\gamma^p_0 = \gamma^p\). If the perturbed system \((1.1)\) has a homoclinic orbit \(\gamma^h_{\varepsilon}\) to \(\gamma^p_{\varepsilon}\) depending \(C^2\)-smoothly on \(\varepsilon\) such that \(\gamma^h_0 = \gamma^h\), then the limit in the right-hand side of \((2.4)\) exists and \(\mathcal{F}_{\varepsilon, h} = 0\).

**Proof.** Assume that the hypotheses of this theorem hold, \(\gamma^p\) is not an equilibrium but periodic orbit, and the system \((1.1)\) has a homoclinic orbit \(\gamma^h_{\varepsilon} = \gamma^h + O(\varepsilon)\) to a periodic orbit \(\gamma^p_{\varepsilon} = \gamma^p + O(\varepsilon)\). For \(\varepsilon > 0\) sufficiently small, the periodic orbit \(\gamma^p_{\varepsilon}\) intersects the Poincaré section \(\Sigma\) transversely, say at \(t = 0\). Similarly, \(\gamma^h_{\varepsilon}\) intersects \(\Sigma\) transversely infinitely many times, say at \(T_j^+ \in \mathbb{R}\) with \(T^+_{j+1} < T^+_j\), \(j \in \mathbb{Z}\), such that \(\lim_{j \to \pm \infty} T_j^+ = \pm \infty\). Moreover,

\[
\lim_{j \to \pm \infty} \gamma^h(T_j^+) = x_\varepsilon := \gamma^p(0).
\]

We easily see that

\[
\lim_{k \to \pm \infty} \int_{T_k^+}^{T_k^+ + T^+} dF(X_\varepsilon)(\gamma^h_\varepsilon(t))dt = \lim_{k \to \pm \infty} \left( F(\gamma^h_\varepsilon(T_k^+)) - F(\gamma^h_\varepsilon(T_k^+)) \right) = 0. \tag{2.6}
\]

Introduce a metric in a neighborhood of \(x_0\) using the standard Euclidean one for the local coordinates. For \(\delta > 0\) sufficiently small, let \(k > 0\) be an integer such that \(\gamma^h(T_{\pm j})\) lie in a \(\delta\)-neighborhood of \(x_0\) for \(j > k\). We can choose \(\varepsilon > 0\) sufficiently small such that on \([T_k^+, T_k^-]\]

\[
\gamma^h_\varepsilon(t) = \gamma^h_\varepsilon(t) + O(\varepsilon),
\]

which yields \(T_j^+ = T_j + O(\varepsilon)\) for \(|j| \leq k\) and

\[
dF(X_\varepsilon)(\gamma^h_\varepsilon(t)) = \varepsilon dF(X^1)(\gamma^h(t)) + O(\varepsilon^2)
\]

since \(dF(X^0) = 0\). Hence,

\[
\int_{T_k^+}^{T_k^+ + T^+} dF(X_\varepsilon)(\gamma^h_\varepsilon(t))dt = \varepsilon \int_{T_k^+}^{T_k^+ + T^+} dF(X^1)(\gamma^h(t))dt + O(\varepsilon^2). \tag{2.7}
\]

Taking \(\delta \to 0\), we have \(T_{\pm k}^\varepsilon \to \pm \infty\), so that by \((2.6)\) and \((2.7)\) the limit in the right-hand side of \((2.4)\) exists and it must be zero. \(\square\)

**Theorem 2.4.** Assume that \((A1')\) and \((A2')\) hold. If the perturbed system \((1.1)\) has a \(C^1\) first integral \(F_\varepsilon\) depending \(C^2\)-smoothly on \(\varepsilon\) near \(1^h\) such that \(F_0 = F\), then the limit in the right-hand side of \((2.4)\) exists and \(\mathcal{F}_{\varepsilon, h} = 0\).

**Proof.** Assume that the hypotheses of the theorem hold, \(\gamma^p\) is not an equilibrium but periodic orbit, and the system \((1.1)\) has a first integral \(F_\varepsilon = F + \varepsilon F^1 + O(\varepsilon^2)\) near \(1^h\). We compute

\[
\lim_{k \to \pm \infty} \int_{T_k}^{T_k + T^+} dF_\varepsilon(X^0)(\gamma^h_\varepsilon(t))dt = \lim_{k \to \pm \infty} \left( F_\varepsilon(\gamma^h_\varepsilon(T_k)) - F_\varepsilon(\gamma^h_\varepsilon(T_k)) \right) = 0. \tag{2.8}
\]

On the other hand, by \((2.3)\)

\[
\int_{T_k}^{T_k + T^+} dF(X^0)(\gamma^h_\varepsilon(t))dt = -\varepsilon \int_{T_k}^{T_k + T^+} dF(X^1)(\gamma^h_\varepsilon(t))dt + O(\varepsilon^2). \tag{2.9}
\]
As in the proof of theorem 2.2, it follows from (2.8) and (2.9) that the limit in the right-hand side of (2.4) exists and it must be zero.

\[ \square \]

\textbf{Remark 2.5.}

(a) In the proofs of theorems 2.3 and 2.4, when $\gamma^h$ is an equilibrium, we only have to choose any strictly monotonically increasing and diverging sequences \( \{ T_j^\varepsilon \}_{j=-\infty}^{\infty} \) such that \( T_j^\varepsilon = T_j + O(\varepsilon) \), \( j \in \mathbb{Z} \), and to apply the same arguments.

(b) In theorem 2.3, if the periodic orbit (or equilibrium) $\gamma^p$ is hyperbolic, then the condition on existence of $\gamma^p$ is not needed since such a periodic orbit (or equilibrium) necessarily exists.

Theorems 2.3 and 2.4 mean that if $\mathcal{J}_{F,\gamma^h} \neq 0$, then neither the homoclinic orbit $\gamma^h$ nor first integral $F$ persists in (1.1) for $\varepsilon > 0$.

\section{Commutative vector fields}

In this section, we discuss persistence of periodic and homoclinic orbits and commutative vector fields for (1.1).

\subsection{Variational and adjoint variational equations}

Before stating the main results, we give some preliminary results on variational and adjoint variational equations. A similar treatment in a complex setting are found in [4, 11, 22]. For the reader’s convenience, some auxiliary materials are provided in appendix A.

Let $\mathcal{M}$ be an $n$-dimensional paracompact oriented $C^3$ real manifold as in section 2. Let $X$ be a $C^2$ vector field on $\mathcal{M}$ and let $\Gamma \phi$ be an integral curve given by a non-stationary solution $x = \phi(t)$ to the associated differential equation

\[ \dot{x} = X(x), \]  

(3.1)

The immersion $i : \Gamma \phi \to \mathcal{M}$ induces a subbundle $T_{\Gamma \phi} := i^* T \mathcal{M}$ of the vector bundle $T \mathcal{M}$, where $i^*$ represents the pullback of $i$. Let $s : \Gamma \phi \to T_{\Gamma \phi}$ be a $C^1$ section of $T_{\Gamma \phi}$. We define the variational equation (VE) of $X$ along $\Gamma \phi$ as

\[ \nabla s := dt \otimes L_X Y |_{\Gamma \phi} = 0, \]  

(3.2)

where ‘$\otimes$’ represents the tensor product, $Y$ is any $C^1$ vector field extension of $s$ to $\mathcal{M}$, $L_X$ represents the Lie derivative along $X$, and ‘$dt \otimes$’ is frequently omitted in references. Here $\nabla$ is a connection of $T_{\Gamma \phi}$, and $s$ is a horizontal section of $\nabla$ if it satisfies the VE (3.2) (see appendix A.1). Locally, equation (3.2) is expressed as

\[ \frac{d \Xi}{dt} = \frac{\partial X}{\partial s}(\phi(t)) \Xi, \quad \Xi = (\Xi_1, \ldots, \Xi_n)^T, \]  

(3.3)

in the frame $\left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$ associated with the coordinates $(x_1, \ldots, x_n)$, where the superscript ‘$T$’ represents the transpose operator and

\[ s = \sum_{j=1}^n \Xi_j \frac{\partial}{\partial x_j}. \]
See appendix A.2.1 for the derivation of (3.3).
Let \( T^\ast_{\Gamma_\phi} \) be the dual bundle of \( T_{\Gamma_\phi} \), and let \( \alpha : \Gamma_\phi \to T^\ast_{\Gamma_\phi} \) be a \( C^1 \) section of \( T^\ast_{\Gamma_\phi} \).

**Lemma 3.1.** The dual connection \( \nabla^\ast \) of \( \nabla \) (see appendix A.1) is given by

\[
\nabla^\ast \alpha = dt \otimes L_X \omega |_{\Gamma_\phi},
\]

where \( \omega : \mathcal{M} \to T^\ast \mathcal{M} \) is any \( C^1 \) differential one-form extension of \( \alpha \).

**Proof.** Let \( s \) be a section of \( T_{\Gamma_\phi} \) and let \( Y \) be its vector field extension as above. The Lie derivative \( L_X \) satisfies

\[
L_X \langle Y, \omega \rangle = \langle L_X Y, \omega \rangle + \langle Y, L_X \omega \rangle,
\]

which yields

\[
\frac{\partial}{\partial t} \otimes \langle Y, \omega \rangle |_{\Gamma_\phi} = \langle \nabla s, \alpha \rangle + \langle s, \frac{\partial}{\partial t} \otimes L_X \omega |_{\Gamma_\phi} \rangle
\]

when restricted to \( \Gamma_\phi \), where \( \langle \cdot, \cdot \rangle \) denotes the natural pairing by the duality. On the other hand, since \( \Gamma_\phi \) is an integral curve of \( X \), we have

\[
d(s, \alpha) = \frac{d}{dt} \langle Y, \omega \rangle |_{\Gamma_\phi} = \frac{\partial}{\partial t} X(\langle Y, \omega \rangle) |_{\Gamma_\phi} = \frac{\partial}{\partial t} \otimes L_X \langle Y, \omega \rangle |_{\Gamma_\phi}.
\]

By definition, we obtain (3.4). \( \square \)

We call

\[
\nabla^\ast \alpha = 0
\]

the adjoint variational equation (AVE) of \( X \) along \( \Gamma_\phi \). Thus, \( \alpha \) is a horizontal section of \( \nabla^\ast \) if it satisfies the AVE (3.5). Locally, equation (3.5) is expressed as

\[
\frac{d\eta}{dt} = - \left( \frac{\partial X}{\partial x}(\phi(t)) \right)^T \eta, \quad \eta = (\eta_1, \ldots, \eta_n)^T,
\]

in the frame \((dx_1, \ldots, dx_n)\), where

\[
\alpha = \sum_{j=1}^{n} \eta_j dx_j.
\]

See appendix A.2.2 for the derivation of (3.6).

**Lemma 3.2.**

(i) If \( X \) has a first integral \( F \), then the following hold:

(iia) The section \( \alpha = dF |_{\Gamma_\phi} \) of \( T^\ast_{\Gamma_\phi} \) satisfies the AVE (3.5) of \( X \) along \( \Gamma_\phi \);

(iib) \( \langle s, dF |_{\Gamma_\phi} \rangle \) is a first integral of the VE (3.2) of \( X \) along \( \Gamma_\phi \), i.e.,

\[
d(s, dF |_{\Gamma_\phi}) = 0
\]

if the section \( s \) of \( T_{\Gamma_\phi} \) satisfies (3.2).
(ii) If $X$ has a commutative vector field $Z$, i.e.,

$$ [X, Z] = 0, $$

where $[\cdot, \cdot]$ denotes the Lie bracket, then the following hold:

(iia) The section $s = Z|_{\Gamma}$ of $T_{\Gamma}$ satisfies the VE (3.2);

(iib) $\langle Z|_{\Gamma}, \alpha \rangle$ is a first integral of the AVE (3.5), i.e.,

$$ d\langle Z|_{\Gamma}, \alpha \rangle = 0 $$

if the section $\alpha$ of $T^{\ast}_{\Gamma}$ satisfies (3.5).

**Proof.** Let $s$ and $\alpha$ satisfy the VE (3.2) and AVE (3.5), respectively. Since

$$ d\langle s, \alpha \rangle = \langle \nabla s, \alpha \rangle + s\langle \nabla^\ast \alpha \rangle = \langle 0, \alpha \rangle + \langle s, 0 \rangle = 0, $$

we see that $\langle s, \alpha \rangle$ is a constant. Hence, parts (ib) and (iib) immediately follow from (iia) and (ia), respectively.

Now we show (ia) and (iia). If $X$ has a first integral $F$, then by Cartan’s formula (see, e.g., theorem 4.2.3 of [18]) we have

$$ L_X dF = d(i_X(dF)) + i_X(d^2F) = 0, $$

where $i_X$ denotes the interior product of $X$. This yields (ia) when restricted to $\Gamma$. If $X$ has a commutative vector field $Z$, then we obtain (iia) since $L_XZ|_{\Gamma} = [X, Z]|_{\Gamma} = 0$. □

Similar results to lemma 3.2 for symplectic connections can be proven by using the musical isomorphism of symplectic forms (see lemma 4.1 of [22] and chapter 4 of [23]).

### 3.2. Periodic orbits

We turn to the issue of persistence of periodic orbits and commutative vector fields in (1.1). Instead of (A2), we assume the following on (1.2):

(A3) The unperturbed system (1.2) has a $C^3$ commutative vector field $Z$, i.e.,

$$ [X^0, Z] = 0, $$

near $\Gamma$, such that it is linearly independent of $X^0$.

**Lemma 3.3.** Under assumption (A1), the connection $\nabla^\ast$ of $T_{\Gamma}$ has a nontrivial horizontal section $\omega : \Gamma \rightarrow T_{\Gamma}^\ast$, i.e., $\omega$ satisfies the AVE (3.5) of $X^0$ along $\Gamma$.

**Proof.** Let $\psi^t$ denote the flow of $X^0$ and let $x_0 = \gamma(0) \in \Gamma$. For any $p \in \Gamma$, there exists a unique time $t_p \in [0, T)$ such that $\gamma(t_p) = \psi^{t_p}(x_0) = p$. Define $\theta : \Gamma \rightarrow [0, T)$ by $\theta(p) := t_p$. Since $\theta \circ \gamma = \text{id}$, we have

$$ \frac{d}{dt} \theta(\gamma(t)) = 1, \quad (3.7) $$

where $\text{id}$ represents the identity map. On the other hand, by the tubular neighborhood theorem (e.g., theorem 5.2 in chapter 4 of [14]), there is a neighborhood $\mathcal{N}(\Gamma)$ of $\Gamma$ which is diffeomorphic to the normal bundle $N_{\Gamma}$ of $\Gamma$ in $\mathcal{M}$. Let $f : \mathcal{N}(\Gamma) \rightarrow N_{\Gamma}$ be the diffeomorphism, and
let \( \pi : N \Gamma \to \Gamma \) be the natural projection. Define a map \( \Theta : \mathcal{N}(\Gamma) \to \mathbb{R} \) by \( \Theta := f^* \pi^* \theta \). Since \( f|_{\Gamma} = \text{id} \) and \( \pi|_{\Gamma} = \text{id} \), we have
\[
\Theta|_{\Gamma} = \theta. \tag{3.8}
\]
Using (3.7) and (3.8), we show that for \( x \in \Gamma \)
\[
(i_X d\Theta)_x = (L_X \Theta)_x = \lim_{t \to 0} \frac{\Theta(\psi^t(x)) - \Theta(x)}{t} = \lim_{t \to 0} \frac{\theta(\psi^t(x)) - \theta(x)}{t} = 1,
\]
which yields
\[
L_X(d\Theta)|_{\Gamma} = (i_X d^2 \Theta + d(i_X d\Theta)|_{\Gamma}) = d((i_X d\Theta)|_{\Gamma}) = 0
\]
by Cartan’s formula. Hence, by lemma 3.1 we see that \( \omega = d\Theta|_{\Gamma} \) is a nontrivial horizontal section of \( \nabla^* \) since \( \theta \) is not a constant.

\[\square\]

**Remark 3.4.**

(a) Let \( \mathcal{M} = \mathbb{R}^n \). We see that \( \dot{\gamma}(t) \) is a periodic solution to the VE (3.3) and consequently its Floquet exponents (see, e.g., section 2.4 of [10]) include one. Hence, the AVE (3.6) possesses one as its Floquet exponent and consequently it has a periodic solution, which provides a horizontal section of \( \nabla^* \) as guaranteed by lemma 3.3.

(b) Assume that (A1) and (A2) hold. From lemma 3.2 (ia) we see that \( dF|_{\Gamma} \) is a horizontal section of \( \nabla^* \).

Let \( \omega \) be a horizontal section of \( \nabla^* \) as stated in lemma 3.3, and define the integral
\[
\mathcal{J}_{\omega,Z,\gamma} := \int_0^T \omega([X^1, Z])\gamma(t) dt. \tag{3.9}
\]

We now state our result on persistence of commutative vector fields.

**Theorem 3.5.** Assume that (A1) and (A3) hold. If the perturbed system (1.1) has a \( C^3 \) commutative vector field \( Z_\varepsilon \) depending \( C^2 \)-smoothly on \( \varepsilon \) near \( \Gamma \) such that \( Z_0 = Z \), then the integral \( \mathcal{J}_{\omega,Z,\gamma} \) must be zero.

For the proof of theorems 3.5 we use the cotangent lift trick [5], and rewrite (1.1) as a Hamiltonian system. In this situation the persistence of commutative vector fields of (1.1) is reduced to that of first integrals of the lifted Hamiltonian system. We first explain the trick in a general setting, following [5]. See chapter 5 of [18] for necessary information on Hamiltonian mechanics.

Let \( T^*\mathcal{M} \) be the cotangent bundle of \( \mathcal{M} \) and let \( \pi : T^*\mathcal{M} \to \mathcal{M} \) be the natural projection. Define a differential one-form \( \lambda : T^*\mathcal{M} \to T^*(T^*\mathcal{M}) \), which is often called a (Poincaré-) Liouville form, as
\[
\lambda_z = p(d\pi_z(\cdot)),
\]
where \( z = (x, p) \in T^*\mathcal{M} \). Letting \( \Omega_0 = d\lambda \), we have a symplectic manifold \( (T^*\mathcal{M}, \Omega_0) \). In the local coordinates \( (x_1, \ldots, x_n, p_1, \ldots, p_n) \), \( \lambda \) and \( \Omega_0 \) are written as
\[
\lambda = \sum_{k=1}^n p_k \, dx_k \quad \text{and} \quad \Omega_0 = \sum_{k=1}^n dp_k \wedge dx_k,
\]
respectively.
Let $X$ be a smooth vector field on $\mathcal{M}$, and define a function $h_X : T^*\mathcal{M} \to \mathbb{R}$ as

$$h_X(x, p) = \langle p, X(x) \rangle,$$

where $(x, p) \in T^*\mathcal{M}$. Then the Hamiltonian vector field $\dot{X}$ with the Hamiltonian $h_X$ on the symplectic manifold $(T^*\mathcal{M}, \Omega_0)$ is called the cotangent lift of $X$. Note that the smoothness of $\dot{X}$ is less by one than that of $X$. In the local coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$, the vector field $\dot{X}$ is expressed as

$$\frac{dx}{dt} = X(x) \left( = \frac{\partial h_X}{\partial p} \right), \quad \frac{dp}{dt} = -\left( \frac{\partial X(x)}{\partial x} \right)^T p \left( = -\frac{\partial h_X}{\partial x} \right),$$

the second equation of which has the same form as the AVE (3.6) when $x = \phi(t)$.

**Lemma 3.6.** For any vector fields $X$ and $Z$ on $\mathcal{M}$ we have

$$\{h_X, h_Z\} = h_{[X,Z]}$$

(see equation (3.10)), where $\{\cdot, \cdot\}$ denotes the Poisson bracket for the symplectic form $\Omega_0$.

**Proof.** In the local coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$, we write

$$X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}, \quad Z = \sum_{j=1}^n Z_j \frac{\partial}{\partial x_j}, \quad p = \sum_{i=1}^n p_i \, dx^i.$$

We compute

$$\{h_X, h_Z\} = \{\langle p, X(x) \rangle, \langle p, Z(x) \rangle\} = \left\{ \sum_{i=1}^n p_i X_i(x), \sum_{j=1}^n p_j Z_j(x) \right\} = \sum_{k=1}^n \left( X_k \sum_{i=1}^n p_i \frac{\partial Z_i}{\partial x_k} - Z_k \sum_{j=1}^n p_j \frac{\partial X_j}{\partial x_k} \right) = \sum_{i=1}^n \sum_{k=1}^n \left( X_k \frac{\partial Z_i}{\partial x_k} - Z_k \frac{\partial X_i}{\partial x_k} \right) = \langle p, [X, Z] \rangle = h_{[X,Z]},$$

which yields the desired result. \hfill \square

We also need the following fact, which was used in the proof of proposition 2 of [5].

**Lemma 3.7.** If $Z$ is a commutative vector field of $X$, then $h_Z$ is a first integral for the cotangent lift $\dot{X}$ of $X$.

**Proof.** It follows from lemma 3.6 that $d h_Z(\dot{X}) = \{h_X, h_Z\} = h_{[X,Z]}$. Hence, $d h_Z(\dot{X}) = 0$ if $[X,Z] = 0$. \hfill \square

We are now in a position to prove theorem 3.5.

**Proof of Theorem 3.5.** Assume that the hypotheses of the theorem hold. Let $\dot{X}_0$ be the cotangent lift of $X_0$. By lemma 3.3 there exists a section $\omega$ of $T^*_\Gamma$ satisfying the AVE (3.5) of $X^0$ along $\Gamma$ and $\tilde{\gamma}(t) = (\gamma(t), \omega(t))$ is a $T$-periodic solution for $\dot{X}_0$. Moreover, by lemma 3.7 $\dot{X}_0$ has a first integral $h_Z$. 

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Suppose that the system (1.1) has a commutative vector field \( Z_\varepsilon = Z + O(\varepsilon) \) near \( \Gamma \). Then by lemma 3.7, \( h_\varepsilon = h_Z + O(\varepsilon) \) is a first integral of \( \hat{X}_\varepsilon \) near \( \hat{\Gamma} = \{ \hat{\gamma}(t) | t \in [0, T] \} \). Using lemma 3.6, we compute

\[
\mathcal{J}_{h_\varepsilon, \hat{\gamma}} = \int_0^T dh_\varepsilon(X)(\hat{\gamma}(t))dt = \int_0^T \{ h_{X^1}, h_Z \}(\hat{\gamma}(t))dt = \int_0^T \{ \omega_i, [X^1, Z] \} \gamma_0 dt = \int_0^T \omega([X^1, Z]) \gamma_0 dt = \mathcal{J}_{\omega, Z, \gamma}
\]

for \( \hat{X}_\varepsilon \). We apply theorem 2.2 to complete the proof. \( \square \)

Theorem 3.5 means that if \( \mathcal{J}_{\omega, Z, \gamma} \neq 0 \), then the commutative vector field \( Z \) does not persist in (1.1) for \( \varepsilon > 0 \).

As in the proof of theorem 3.5, we see that if \( \gamma_0(t) \) is a \( T_\varepsilon \)-periodic orbit in (1.1), then by lemma 3.3 there exists a section \( \omega_\varepsilon = \omega + O(\varepsilon) \) of \( T^* \hat{\gamma} \) satisfying the AVE (3.5) of \( \gamma_\varepsilon \) along \( \Gamma_\varepsilon \) and \( \hat{\gamma}(t) = (\gamma_\varepsilon(t), \omega_\varepsilon, \gamma_0(t)) \) is a \( T_\varepsilon \)-periodic orbit for the cotangent lift \( \hat{X}_\varepsilon \) of \( X_\varepsilon \), where \( \Gamma_\varepsilon = \{ \gamma_\varepsilon(t) | t \in [0, T_\varepsilon] \} \). Here the section \( \omega_\varepsilon \) of \( T^* \hat{\gamma} \) satisfies the AVE (3.5) of \( X^0 \) along \( \Gamma \). Applying theorem 2.1 to \( \hat{X}_\varepsilon \), we obtain the following result on persistence of periodic orbits.

**Theorem 3.8.** Assume that (A1) and (A3) hold. If the perturbed system (1.1) has a \( T_\varepsilon \)-periodic orbit \( \gamma_\varepsilon \) depending \( C^2 \)-smoothly on \( \varepsilon \) such that \( T_0 = T \) and \( \gamma_0 = \gamma \), then the integral \( \mathcal{J}_{\omega, Z, \gamma} \) is zero for some section \( \omega \) of \( T^* \hat{\gamma} \) satisfying the AVE (3.5) of \( X^0 \) along \( \Gamma \).

Theorem 3.8 means that if \( \mathcal{J}_{\omega, Z, \gamma} \neq 0 \) for any horizontal section \( \omega \) of \( \nabla^* \), then the periodic orbit \( \gamma \) does not persist in (1.1) for \( \varepsilon > 0 \).

### 3.3. Homoclinic orbits

We next discuss the persistence of homoclinic orbits and commutative vector fields in (1.1). Instead of (A3) we assume the following on (1.2):

(A3') The unperturbed system (1.2) has a \( C^3 \) commutative vector field \( Z \) near \( \Gamma^0 \), such that it is linearly independent of \( X^0 \).

In the proof of lemma 3.3, we did not essentially use the fact that \( \gamma(t) \) is periodic. So we prove the following lemma similarly.

**Lemma 3.9.** Under assumption (A1'), the connection \( \nabla^* \) of \( T^* \hat{\gamma} \) has a nontrivial horizontal section \( \omega^h : \Gamma^h \to T^* \Pi^h \), i.e., \( \omega^h \) satisfies the AVE (3.5) of \( X^0 \) along \( \Gamma^h \) where \( \Gamma^h = \{ \gamma^h(t) | t \in \mathbb{R} \} \).

Let \( \omega^h \) be such a horizontal section of \( \nabla^* \) as stated in lemma 3.9, and define

\[
\mathcal{J}_{\omega^h, Z, \gamma^h} := \lim_{k \to +\infty} \int_{T^*_h} \omega^{h}([X^1, Z]),_{\gamma^h(t)}dt,
\]

where the sequence \( \{ T^*_j \}_{j=-\infty}^{\infty} \) is taken as in (2.4). If \( \gamma^h \) is an equilibrium, then equation (3.13) is reduced to

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\[ \mathcal{J}_{\omega^h Z, \gamma} = \int_{-\infty}^{\infty} \omega^h([X^1, Z])_{\gamma(t)} dt \]  

(3.14)

like (2.5).

**Theorem 3.10.** Assume that (A1') and (A3') hold. If the perturbed system (1.1) has a $C^3$ commutative vector field $Z_\varepsilon$ depending $C^3$-smoothly on $\varepsilon$ near $\Gamma^h$ such that $Z_0 = Z$, then the limit in the right-hand side of (3.13) exists and $\mathcal{J}_{\omega^h Z, \gamma} = 0$.

**Proof.** If $\gamma^h$ is a periodic orbit, then by lemma 3.3 there exists a horizontal section $\omega^h$ of $T^*_\varepsilon$ satisfying the AVE (3.5) of $X^0$ along $\Gamma^h$ and $\gamma^h(t, \omega^h_{\gamma^h(t)})$ is a periodic orbit for the cotangent lift $\hat{X}^0$ of $X^0$. Similarly, by assumptions (A1') and lemma 3.9, we have a homoclinic orbit $\gamma^h(t, \omega^h_{\gamma^h(t)})$ to the periodic orbit $(\gamma^h(t, \omega^h_{\gamma^h(t)}))$ for $\hat{X}^0$, where $\omega^h_\gamma$ is a horizontal section of $\nabla^\gamma$ for $T^*_\varepsilon$. By applying theorem 2.4 to the cotangent lift $\hat{X}_\varepsilon$ of $X_\varepsilon$, the rest of the proof is done similarly as in theorem 3.5. \hfill \Box

Theorem 3.10 means that if $\mathcal{J}_{\omega^h Z, \gamma} \neq 0$, then the commutative vector field $Z$ does not persist in (1.1) for $\varepsilon > 0$.

**Remark 3.11.** Using theorems 2.2, 2.4, 3.5 and 3.10, we can determine whether given first integrals and commutative vector fields do not persist in (1.1) but there still exist a sufficient number of first integrals and commutative vector fields depending smoothly on the parameter $\varepsilon$. For example, the unperturbed system (1.2) may have different first integrals and commutative vector fields which persist. So we have to overcome this difficulty to extend the results of Poincaré [23] and Kozlov [16, 17] and obtain a sufficient condition for such nonintegrability of the perturbed systems.

As in the proof of theorem 3.10, if $\gamma^h = \gamma^h + O(\varepsilon)$ is a $T_\varepsilon$-periodic orbit with $T_\varepsilon = T + O(\varepsilon)$ and $\gamma^h = \gamma^h + O(\varepsilon)$ is a homoclinic orbit to $\gamma^h$ in (1.1), then by lemmas 3.3 and 3.9 there exist horizontal sections $\omega_\gamma^h = \omega^h + O(\varepsilon)$ and $\omega^h_\gamma = \omega^h + O(\varepsilon)$ of $\nabla^\gamma$ for $T^*_\varepsilon = \{\gamma^h(t)|t \in [0, T_\varepsilon]\}$ and $T^*_\varepsilon = \{\gamma^h(t)|t \in [0, T_\varepsilon]\}$, respectively, so that for the cotangent lift $\hat{X}_\varepsilon$ of $X_\varepsilon$, $\gamma^h(t, \omega^h_{\gamma^h(t)})$ is a periodic orbit to which $(\gamma^h(t), \omega^h_{\gamma^h(t)})$ is a homoclinic orbit. Here the section $\omega^h_\gamma$ of $\nabla^\gamma$ satisfies the AVE (3.5) along $\Gamma^h$. Applying theorem 2.1 to $\hat{X}_\varepsilon$, we obtain the following.

**Theorem 3.12.** Assume that (A1') and (A3') hold and that there exists a periodic orbit $\gamma^h$ depending $C^3$-smoothly on $\varepsilon$ such that $\gamma^h_0 = \gamma^h$. If the perturbed system (1.1) has a homoclinic orbit $\gamma^h_\varepsilon$ depending $C^3$-smoothly on $\varepsilon$ in (1.1) such that $\gamma^h_0 = \gamma^h$, then the limit in the right-hand side of (3.13) exists and $\mathcal{J}_{\omega^h Z, \gamma} = 0$ for some section $\omega$ of $T^*_\varepsilon$ satisfying the AVE (3.5) along $\Gamma^h$.

Theorem 3.12 means that if $\mathcal{J}_{\omega^h Z, \gamma} \neq 0$ for any horizontal section $\omega^h$ of $\nabla^\gamma$ for $\Gamma^h$, then the homoclinic orbit $\gamma^h$ does not persist in (1.1) for $\varepsilon > 0$.

4. Some relationships with the Melnikov methods

In this section, we discuss some relationships of the main results in sections 2 and 3 with the standard subharmonic and homoclinic Melnikov methods [12, 19, 25, 28], which provide sufficient conditions for persistence of periodic and homoclinic orbits, respectively, in time-periodic perturbations of single-degree-of-freedom Hamiltonian systems, and with another version of the homoclinic Melnikov method due to Wiggins [24] for autonomous perturbations of multi-degree-of-freedom integrable Hamiltonian systems.
4.1. Standard Melnikov methods

We first review the standard Melnikov methods for subharmonic and homoclinic orbits. See [12, 25, 28] for more details.

We consider systems of the form

\[
\dot{x} = J_2DH(x) + \varepsilon g(x, t), \quad x \in \mathbb{R}^2, \tag{4.1}
\]

where \(\varepsilon\) is a small parameter as in the previous sections, \(H : \mathbb{R}^2 \rightarrow \mathbb{R}\) and \(g : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2\) are, respectively, \(C^3\) and \(C^2\) in \(x\), \(g(x, t)\) is \(T\)-periodic in \(t\) with \(T > 0\) a constant, and \(J_2, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\).

When \(\varepsilon = 0\), equation (4.1) becomes a single-degree-of-freedom Hamiltonian system with the Hamiltonian \(H(x)\),

\[
\dot{x} = J_2DH(x). \tag{4.2}
\]

Let \(\theta = t \mod T\) so that \(\theta \in S^1_T\), where \(S^1_T = \mathbb{R}/T\mathbb{Z}\). We rewrite (4.1) as an autonomous system,

\[
\dot{x} = J_2DH(x) + \varepsilon g(x, \theta), \quad \dot{\theta} = 1. \tag{4.3}
\]

We begin with the subharmonic Melnikov method [12, 25, 28], and make the following assumption:

\((M)\) The unperturbed system (4.2) possesses a one-parameter family of periodic orbits \(q^\alpha(t)\) with period \(T^\alpha\), \(\alpha \in (\alpha_1, \alpha_2)\), for some \(\alpha_1 < \alpha_2\).

Fix the value of \(\alpha \in (\alpha_1, \alpha_2)\) such that

\[
\frac{lT^\alpha}{mT} \tag{4.4}
\]

for some relatively prime integers \(l, m > 0\). When \(\varepsilon = 0\), equation (4.3) has a one-parameter family of \(mT\)-periodic orbits \((x, \theta) = (q^\alpha(t - \tau), t), \tau \in [0, T]\). Note that \((x, \theta) = (q^\alpha(t - \tau - jT), t)\) represents the same periodic orbit in the phase space \(\mathbb{R}^2 \times S^1_T\) for \(j = 0, 1, \ldots, m - 1\). Define the subharmonic Melnikov function as

\[
M^{m/l}(\tau) := \int_0^{mT} DH(q^\alpha(t)) \cdot g(q^\alpha(t), t + \tau)dt, \tag{4.5}
\]

where the dot ‘\(\cdot\)’ represents the standard inner product in \(\mathbb{R}^2\). We have the following (see [12, 25, 28] for the proof).

**Theorem 4.1.** If the subharmonic Melnikov function \(M^{m/l}(\tau)\) has a simple zero at \(\tau = \tau_0 \in S^1_T\), then for \(\varepsilon > 0\) sufficiently small equation (4.3) has a periodic orbit of period \(mT\) near the unperturbed periodic orbit \((x, \theta) = (q^\alpha(t - \tau_0), t)\) satisfying (4.4).

Theorem 4.1 means that the periodic orbit \((x, \theta) = (q^\alpha(t - \tau_0), t)\) persists in (4.3) for \(\varepsilon > 0\) sufficiently small if \(M^{m/l}(\tau)\) has a simple zero at \(\tau = \tau_0\). The stability of the perturbed periodic orbit can be also determined easily [28]. Moreover, several bifurcations of the periodic orbits were discussed in [28, 31, 32].
We next review the homoclinic Melnikov method [12, 19, 25] and assume the following instead of (M):

(M') The unperturbed system (4.2) possesses a hyperbolic saddle point $p$ connected to itself by a homoclinic orbit $q^p(t)$.

When $\varepsilon = 0$, equation (4.3) has a hyperbolic $T$-periodic orbit $(x, \theta) = (p, t)$ with a one-parameter family of homoclinic orbits $(x, \theta) = (q^p(t - \tau), t)$, $\tau \in \mathbb{S}_T^1$. Note that $(x, \theta) = (q^p(t - \tau - jT), t)$ represents the same homoclinic orbit in the phase space $\mathbb{R}^2 \times \mathbb{S}_T^1$ for $j = 0, 1, \ldots , m - 1$. We easily show that there exists a hyperbolic periodic orbit near $(x, \theta) = (p, t)$ (see [12, 25] for the proof). Define the homoclinic Melnikov function as

$$M(\tau) := \int_{-\infty}^{\infty} DH(q^p(t)) \cdot g(q^p(t), t + \tau) dt$$

We have the following (see [12, 19, 25] for the proof).

**Theorem 4.2.** If the homoclinic Melnikov function $M(\tau)$ has a simple zero at $\tau = \tau_0 \in \mathbb{S}_T^1$, then for $\varepsilon > 0$ sufficiently small equation (4.3) has a transverse homoclinic orbit to the hyperbolic periodic orbit near $(x, \theta) = (q^p(t - \tau_0), t)$.

Theorem 4.2 means that the homoclinic orbit $(x, \theta) = (q^p(t - \tau_0), t)$ persists in (4.3) for $\varepsilon > 0$ sufficiently small if $M(\tau)$ has a simple zero at $\tau = \tau_0$. By the Smale–Birkhoff theorem [12, 25], the existence of transverse homoclinic orbits to hyperbolic periodic orbits implies that chaotic motions occur in (4.3), i.e., in (4.1).

We now describe some relationships of our results on persistence of first integrals with the standard Melnikov methods for (4.3), which has the Hamiltonian $H(x)$ is a first integral when $\varepsilon = 0$. We first state the relationship for the subharmonic Melnikov method.

**Theorem 4.3.** Suppose that assumption (M) and the resonance condition $lT^m = mt$ hold for $l, m > 0$ relatively prime integers. If equation (4.3) has a $C^3$ first integral $F_\varepsilon(x, t) = H(x) + O(\varepsilon)$ depending $C^2$-smoothly on $\varepsilon$ in a neighborhood of

$$\Gamma_{\tau_0}^\alpha = \{(q^p(t - \tau_0), t) | t \in [0, mT]\}$$

with $\tau_0 \in \mathbb{S}_T^1$, then there exists a connected open set $\Pi \subset \mathbb{S}_T^1$ such that $\tau_0 \in \Pi$ and the subharmonic Melnikov function $M^{m/l}(\tau)$ is zero on $\Pi$.

**Proof.** Assume that the hypotheses of the theorem hold and $F_\varepsilon$ is a first integral of (4.3) such that $F_0 = H$. Then $\gamma^{m/l}(t) = (q^l(t - \tau), t)$ is an $mT$-periodic orbit in (4.3) with $\varepsilon = 0$ for any $\tau \in [0, T)$. Letting $F = H$, we write the integral (2.1) as

$$\mathcal{F}^{m/l}_{H, \gamma^{m/l}} = \int_{0}^{mT} DH(q^l(t - \tau)) \cdot g(q^l(t - \tau), t) dt$$

$$= \int_{0}^{mT} DH(q^l(t)) \cdot g(q^l(t), t + \tau) dt,$$

which coincides with $M^{m/l}(\tau)$. We choose a connected open set $\Pi \subset \mathbb{S}_T^1$ such that the neighborhood of $\Gamma_{\tau_0}^\alpha$ contains $\bigcup_{\tau \in \Pi} \Gamma_{\tau}^\alpha$. Applying theorem 2.2 to the unperturbed periodic orbit $\gamma^{m/l}_{\tau}$ for $\tau \in \Pi$, we obtain the desired result.

Theorem 4.3 means that if there exists a connected open set $\Pi \subset \mathbb{S}_T^1$ such that $M^{m/l}(\tau)$ is not identically zero on $\Pi$, then the first integral $H$ does not persist near $\bigcup_{\tau \in \Pi} \Gamma_{\tau}^\alpha$ in (4.3) for $\varepsilon > 0$.  


Remark 4.4. Under the hypotheses of theorem 4.3 the following hold:

(a) It follows from theorem 2.1 that if the periodic orbit \((x, \theta) = (q^0(t - \tau), t)\) persists in (4.3), then \(M^{m/l}(\tau) = 0\);
(b) If equation (4.3) has such a first integral near \(\bigcup_{\tau \in [1]} \Gamma_{h_{a0}}\), then \(M^{m/l}(\tau)\) is identically zero on \(S^1_{\tau}\);
(c) If \(H, g\) are analytic and equation (4.3) has such a first integral near \(\tau_0\) with some \(\tau_0 \in S^1_{\tau}\), then \(M^{m/l}(\tau)\) is identically zero on \(S^1_{\tau}\).

The statement of part (a) consists with theorem 4.1. Part (c) follows from the identity theorem (e.g., theorem 3.2.6 of [1]) since \(M^{m/l}(\tau)\) is also analytic.

Similarly, we have the following result for the homoclinic Melnikov method.

Theorem 4.5. Suppose that assumption \((M')\) holds. If equation (4.3) has a \(C^3\) first integral \(F_\varepsilon(x, t) = H(x) + O(\varepsilon)\) depending \(C^2\)-smoothly on \(\varepsilon\) in a neighborhood of

\[\Gamma_{h_{\tau_0}} = \{(q^h(t - \tau_0), t) | t \in \mathbb{R}\}\]

with \(\tau_0 \in S^1_{\tau}\), then there exists a connected open set \(\Pi \subset S^1_{\tau}\) such that \(\tau_0 \in \Pi\) and the homoclinic Melnikov function \(M(\tau)\) is zero on \(\Pi\).

Proof. Assume that \((M')\) holds. Then in (4.3) with \(\varepsilon = 0\), \((p, t)\) represents a periodic orbit, to which \(\tau^h_0(t) = (q^h(t - \tau), t)\) is a homoclinic orbit, for any \(\tau \in [0, T]\). We take the Poincaré section \(\Sigma = \{(x, \theta) \in \mathbb{R}^2 \times S^1_{\tau} | \theta = 0\}\) and set \(T_j = jT\), \(j \in \mathbb{Z}\). Letting \(F = H\), we write the integral in (2.4) as

\[\int_{-\beta}^{\beta} DH(q^h(t - \tau)) \cdot g(q^h(t - \tau), t) dt = \int_{-\beta}^{\beta} DH(q^h(t)) \cdot g(q^h(t), t + \tau) dt,\]

which converges to \(M(\tau)\) as \(j \to \infty\). We choose a connected open set \(\Pi \subset S^1_{\tau}\) such that the neighborhood of \(\Gamma_{h_{\tau_0}}^h\) contains \(\bigcup_{\tau \in \Pi} \Gamma_{h_{\tau_0}}^h\). Applying theorem 2.4 to the unperturbed homoclinic orbit \(\tau^h_{\tau_0}\) for \(\tau \in \Pi\), we obtain the desired result. \(\square\)

Theorem 4.5 means that if there exists a connected open set \(\Pi \subset S^1_{\tau}\) such that \(M(\tau) \neq 0\) on \(\Pi\), then the first integral \(H\) does not persist near \(\bigcup_{\tau \in \Pi} \Gamma_{h_{\tau_0}}^h\) in (4.3) for \(\varepsilon > 0\).

Remark 4.6. Under the hypotheses of theorem 4.5 the following hold, as in remark 4.4:

(a) It follows from theorem 2.3 that if the homoclinic orbit \((x, \theta) = (q^0(t - \tau), t)\) persists in (4.3), then \(M(\tau) = 0\);
(b) If equation (4.3) has such a first integral near \(\bigcup_{\tau \in [1]} \Gamma_{h_{\tau_0}}\), then \(M(\tau)\) is identically zero on \(S^1_{\tau}\);
(c) If \(H, g\) are analytic and equation (4.3) has such a first integral near \(\tau_0\) with some \(\tau_0 \in S_{\tau}\), then \(M(\tau)\) is identically zero on \(S^1_{\tau}\).

The statement of part (a) consists with theorem 4.2.
4.2. Another version of the homoclinic Melnikov method

We next consider \((m + 1)\)-degree-of-freedom Hamiltonian systems of the form

\[
\dot{x} = J_{2m} D_2 H^1(x, I, \theta) + \varepsilon J_{2m} D_1 H^1(x, I, \theta), \\
\dot{\theta} = D_1 H^0(x, I) + \varepsilon D_1 H^1(x, I, \theta),
\]

(4.7)

for which \(H_0(x, I, \theta) = H^0(x, I) + \varepsilon H^1(x, I, \theta)\) is the Hamiltonian, where \(m \geq 1\) is an integer, \(V \subset \mathbb{R}\) is an open interval, \(H^0(x, I), H^1(x, I, \theta)\) are \(C^3\) in \((x, I, \theta)\), and \(J_{2m}\) is the \(2m \times 2m\) symplectic matrix given by

\[
J_{2m} = \begin{pmatrix}
0 & id_m \\
-did_m & 0
\end{pmatrix},
\]

where \(id_m\) is the \(m \times m\) identity matrix. When \(\varepsilon = 0\), equation (4.7) becomes

\[
\dot{x} = J_{2m} D_2 H^0(x, I), \quad \dot{i} = 0, \quad \dot{\theta} = D_1 H^0(x, I).
\]

(4.8)

Note that \(I\) and \(\theta\) are scalar variables. We assume the following on the unperturbed system (4.8):

(W1) For each \(I \in V\), the first equation is has \(m\) \(C^3\) first integrals \(F_j(x, I)\), \(j = 1, \ldots, m\), with \(F_1(x, I) = H^0(x, I)\) such that \(D_j F_j(x, I)\), \(j = 1, \ldots, m\), are linearly independent except at equilibria and they are in involution, i.e., \(\{F_1(x, I), F_j(x, I)\} := D_j F_j(x, I) \cdot J_{2m} D_2 F_j(x, I) = 0, i, j = 1, \ldots, m\).

(W2) For each \(I \in V\) the first equation has a hyperbolic equilibrium \(x'\) and an \((m - 1)\)-parameter family of homoclinic orbits \(q'(t; \alpha)\), \(\alpha \in V \subset \mathbb{R}^{m-1}\), to \(x'\), where \(x'\) and \(q'(t; \alpha)\) depend \(C^2\)-smoothly on \(I\) and \(\alpha\), and \(V\) is an connected open in \(\mathbb{R}^{m-1}\).

(W3) \(D_1 H^0(q'(t; \alpha), I) > 0\) for \((I, \alpha) \in V \times V\).

Obviously, \(I\) is a first integral of (4.8) as well as \(F_j(x, I)\), \(j = 1, \ldots, m\), so that the Hamiltonian system (4.7) is Liouville integrable [3, 20]. Thus, equation (4.7) is a special case in a class of systems called ‘system III’ in chapter 4 of [24], in which very wide classes of systems containing more general Hamiltonian systems, especially having multiple action and angular variables such as the scalar variables \(I\) and \(\theta\) in (4.7), were discussed.

In (4.8) \(\mathcal{N}_0 = \{(x', I, \theta) | I \in V, \theta \in S^1_{2\pi}\}\) is a two-dimensional normally hyperbolic invariant manifold with boundary whose stable and unstable manifolds coincide along the homoclinic manifold

\[
\mathcal{F}_b = \{(q'(t; \alpha, I, \theta) | I \in V, \alpha \in V, \theta \in S^1_{2\pi}\}.
\]

Here ‘normal hyperbolicity’ means that the expansive and contraction rates of the flow generated by (4.8) normal to \(\mathcal{N}_0\) dominate those tangent to \(\mathcal{N}_0\). Note that \((x, I, \theta) = (x^b, I_0, dH^0(x^b, I_0)t + \theta_0)\) represents a periodic orbit on \(\mathcal{N}_0\) for \((I_0, \theta_0) \in V \times S^1_{2\pi}\). Using the invariant manifold theory [26], we show that for \(|\varepsilon| > 0\) sufficiently small equation (4.7) also has a two-dimensional normally hyperbolic invariant manifold \(\mathcal{N}_\varepsilon\) near \(\mathcal{N}_0\) and its stable and unstable manifolds are close to those of \(\mathcal{N}_0\). Moreover, the invariant manifold \(\mathcal{N}_\varepsilon\) consists of periodic orbits \(\gamma^p_j\), which are given as intersections between \(\mathcal{N}_\varepsilon\) and the level sets \(H(x, I, \theta) = \text{const.}\) since \(D_1 H^0(x', I) \geq 0\) by (W3), near \(\gamma^p_j = \{(x', I, \theta) | \theta \in S^1_{2\pi}\}\) for \(I \in V\). Note that \(\mathcal{N}_\varepsilon\) can be invariant by taking two periodic orbits as its boundary as in proposition 2.1 of [29].
Let \( \theta = \theta^\prime(t; \alpha) \) denote the solution to
\[
\dot{\theta} = D_t H^0(q^\prime(t; \alpha), I)
\]
with \( \theta(0) = 0 \), i.e.,
\[
\theta^\prime(t; \alpha) = \int_0^t D_t H^0(q^\prime(t; \alpha), I) \, dt.
\]

Then \( \gamma^P_{I,\theta_0}(t) = (q^\prime(t; \alpha), I, \theta^\prime(t; \alpha) + \theta_0) \) is a homoclinic orbit to the periodic orbit \( \gamma^P_I \) in (4.8) for any \( \theta_0 \in S^1_\pi \). Let \( \{T^J_{\alpha}\}_{j=\infty}^{\infty} \) be a sequence for \((I, \alpha) \in V \times \tilde{V}\) such that
\[
\theta^\prime(T^J_{\alpha}; \alpha) = 0, \quad j \in \mathbb{Z}, \quad \text{and} \quad \lim_{j \to \pm\infty} T^J_{\alpha} = \pm \infty. \tag{4.9}
\]

By assumption (W3) there exists such an sequence \( \{T^J_{\alpha}\}_{j=\infty}^{\infty} \). Define the \textit{Melnikov functions} for (4.7) as
\[
\tilde{M}^I_{\alpha}(\theta_0, \alpha) = \lim_{j \to \infty} \int_{T^J_{\alpha} - j}^{T^J_{\alpha} + j} D_t H^1(q^\prime(t; \alpha), I, \theta^\prime(t; \alpha) + \theta_0) \, dt \tag{4.10}
\]
and
\[
\tilde{M}^I_{\alpha}(\theta_0, \alpha) = \lim_{j \to \infty} \int_{T^J_{\alpha} - j}^{T^J_{\alpha} + j} \left( D_x F_k(q^\prime(t; \alpha), I) \cdot J_{2m} D_x H^1(q^\prime(t; \alpha), I, \theta^\prime(t; \alpha) + \theta_0) \right) \, dt \tag{4.11}
\]
for \( k = 2, \ldots, m \). Note that the definitions of the Melnikov functions \( \tilde{M}^I_{\alpha}, k \geq 2 \), are different from the original ones of [24]. We call \( M^I = (\tilde{M}^I_{\alpha}, \ldots, \tilde{M}^I_m) \) the \textit{Melnikov vector}. From theorem 4.1.19 of [24] we obtain the following result for (4.7).

**Theorem 4.7.** Suppose that assumptions (W1)–(W3) hold. If for some \( I \in V \)

(a) \( M^I(\theta, \alpha) = 0 \);

(b) \( \det DM^I(\theta, \alpha) \neq 0 \)

at \((\theta, \alpha) = (\theta_0, \alpha_0)\), then the \((m+1)\)-dimensional stable and unstable manifolds \( W^s(\gamma^P_{I,\epsilon}) \) and \( W^u(\gamma^P_{I,\epsilon}) \) intersect transversely near \((x, I, \theta) = (q^\prime(0; \alpha_0), I, \theta_0)\) on the level set of \( H, (\gamma^P_{I,\epsilon}) \).

**Proof.** Assume that the hypotheses of theorem 4.7 hold. Let \( \tilde{M}^I_1(\theta, \alpha) = \tilde{M}^I_1(\theta, \alpha) \) and
\[
\tilde{M}^I_k(\theta, \alpha) = \tilde{M}^I_k(\theta, \alpha) + D_t F_k(x_t, I) \tilde{M}^I_1(\theta, \alpha), \quad k = 2, \ldots, m,
\]
and let \( \tilde{M}^I = (\tilde{M}^I_1, \ldots, \tilde{M}^I_m) \), which is the original Melnikov vector defined in [24] for (4.7). Note that in [24], although a time sequence does not appear in the formulas (4.1.84) and (4.1.85) or (4.1.101) and (4.1.102), such conditional convergences of the integrals as (4.10) and (4.11) are implicitly assumed (see his arguments on system III in part iii) of section 4.1.4 of [24]). We see that if \( M^I(\theta, \alpha) \) satisfies conditions (a) and (b) at \((\theta, \alpha) = (\theta_0, \alpha_0)\), then \( \tilde{M}^I(\theta_0, \alpha_0) = 0 \) and \( \det DM^I(\theta_0, \alpha_0) \neq 0 \), since \( \det DM^I(\theta_0, \alpha_0) = \det DM^I(\theta_0, \alpha_0) \), which follows from
\[ D\tilde{M}'(\theta_0, \alpha_0) = \begin{pmatrix} D_\theta M'_1(\theta_0, \alpha_0), D_\alpha M'_1(\theta_0, \alpha_0) \\ D_\theta M'_2(\theta_0, \alpha_0), D_\alpha M'_2(\theta_0, \alpha_0) \\ \vdots \\ D_\theta M'_m(\theta_0, \alpha_0), D_\alpha M'_m(\theta_0, \alpha_0) \end{pmatrix} + \begin{pmatrix} 0 \\ D_1F_2(x_j, I)(D_\theta M'_1(\theta_0, \alpha_0), D_\alpha M'_1(\theta_0, \alpha_0)) \\ \vdots \\ D_1F_m(x_j, I)(D_\theta M'_1(\theta_0, \alpha_0), D_\alpha M'_1(\theta_0, \alpha_0)) \end{pmatrix}. \]

We obtain the desired result from theorem 4.1.19 of [24].

**Remark 4.8.** The Melnikov vector \( \tilde{M}'(\theta, \alpha) \) does not depend on the choice of time sequence \( \{ T_j^{(\alpha)} \}_{j=\infty} \). Actually, letting \( \{ T_j^{(\alpha)} \}_{j=\infty} \) be a different time sequence satisfying

\[ \theta_j(T_j^{(\alpha)}; \alpha) = \hat{\theta}_0, \quad j \in \mathbb{Z}, \quad \text{and} \quad \lim_{j \to \pm\infty} T_j^{(\alpha)} = \pm\infty \]

instead of (4.9), we have

\[ \tilde{M}'_1(\theta, \alpha) := \lim_{j \to \pm\infty} \int_{T_{j-1}^{(\alpha)}}^{T_j^{(\alpha)}} D_\theta H^1(q(t; \alpha), I, \theta(t; \alpha) + \theta) dt \]

\[ = M'_1(\theta, \alpha) + \lim_{j \to \pm\infty} \left( \int_{T_{j-1}^{(\alpha)}}^{T_j^{(\alpha)}} D_\theta H^1(q(t; \alpha), I, \theta(t; \alpha) + \theta) dt \right) \]

\[ = M'_1(\theta, \alpha) \]

since

\[ \lim_{t \to \pm\infty} D_\theta H^1(q(t; \alpha), I, \theta) = D_\theta H^1(x^1, I, \theta), \]

\[ \lim_{t \to \pm\infty} D_\theta H^0(q(t; \alpha), I) = D_\theta H^0(x^1, I) \]

and

\[ \left( \int_{T_{j-1}^{(\alpha)}}^{T_j^{(\alpha)}} + \int_{T_{j-1}^{(\alpha)}}^{T_j^{(\alpha)}} \right) D_\theta H^1(x^1, I, \theta(t; \alpha) + \theta) D_\theta H^0(q(t; \alpha), I) dt \]

\[ = \left( \int_{T_{j-1}^{(\alpha)}}^{T_j^{(\alpha)}} + \int_{T_{j-1}^{(\alpha)}}^{T_j^{(\alpha)}} \right) \frac{d}{dt} H^1(x^1, I, \theta(t; \alpha) + \theta) dt = 0. \]

Similarly, we show

\[ \tilde{M}'_k(\theta, \alpha) := \lim_{j \to \pm\infty} \int_{T_{j-1}^{(\alpha)}}^{T_j^{(\alpha)}} (D_1F_k(q(t; \alpha), I) \cdot J_{2m}D_\theta H^1(q(t; \alpha), I, \theta(t; \alpha) + \theta) \]

\[ - D_1F_k(q(t; \alpha), I)D_\theta H^1(q(t; \alpha), I, \theta(t; \alpha) + \theta) dt \]

\[ = \tilde{M}'_k(\theta, \alpha), \quad k = 2, \ldots, m. \]
Theorem 4.7 means that the homoclinic orbit \( \gamma_{h,0,\alpha_0}^b(t) \) persists in (4.7) for \( \varepsilon > 0 \) sufficiently small if the Melnikov vector \( M^1(\theta, \alpha) \) satisfies its hypotheses. By the Smale–Birkhoff theorem [12, 25], such transverse intersection between the stable and unstable manifolds of periodic orbits implies that chaotic motions occur in (4.7).

We now describe a relationship of our results on persistence of first integrals with the homoclinic Melnikov methods for (4.7), in which the Hamiltonian \( H_r(x, I, \theta) \) is always a persisting first integral. We have the following result.

**Theorem 4.9.** Suppose that assumptions (W1)–(W3) hold. If the Hamiltonian system (4.7) has a \( C^1 \) first integral \( F_{k,\varepsilon}(x, I, \theta) = F_k(x, I) + O(\varepsilon) \) (resp. \( F_{m+1,\varepsilon}(x, I, \theta) = I + O(\varepsilon) \)) depending \( C^2 \) smoothly on \( \varepsilon \) in a neighborhood of

\[
\tilde{\Gamma}_{h,0,\alpha_0}^b = \{(q^b_t(t; \alpha_0), I_0, \theta^b_t(t; \alpha_0))| t \in \mathbb{R}\}
\]

for some \( k = 2, \ldots, m \), then there exists a connected open set \( \Pi \subset V \times S^1_{2\pi} \times \tilde{V} \) such that \( (I_0, \theta_0, \alpha_0) \in \Pi \) and the Melnikov function \( \tilde{M}_1^1(\theta, \alpha) = 0 \) (resp. \( \tilde{M}_1^1(\theta, \alpha) = 0 \)) on \( \tilde{\Pi} \).

**Proof.** Assume that (W1)–(W3) hold. We choose the Poincaré section \( \Sigma = \{(x, I, \theta) \in \mathbb{R}^{2m} \times V \times S^1_{2\pi} \} \) and take \( T_j = T_j^{\alpha_0} \), \( j \in \mathbb{Z} \) (cf equation (4.9)). Letting \( F = F_k \) for \( k = 2, \ldots, m \) (resp. \( F = I \)), we write the integral in (2.4) as

\[
\int_{T_j}^{T_{j+1}} \left( D_x F_k(q^t(\theta; \alpha), I) \cdot J_{2m} D_x H^1(q^t(\theta; \alpha), I, \theta^t(\theta; \alpha) + \theta_0) \\
- D_x F_k(q^t(\theta; \alpha), I) D_\theta H^1(q^t(\theta; \alpha), I, \theta^t(\theta; \alpha) + \theta_0) \right) dt
\]

for the homoclinic orbit \( \gamma_{h,0,\alpha_0}^b(t) \). We choose a connected open set \( \tilde{\Pi} \subset V \times S^1_{2\pi} \times \tilde{V} \) such that the neighborhood of \( \tilde{\Gamma}_{h,0,\alpha_0}^b \) contains \( \bigcup_{(I, \theta, \alpha) \in \Pi} \tilde{\Gamma}_{h,0,\alpha_0}^b \). Applying theorem 2.4 to the unperturbed homoclinic orbit \( \gamma_{h,0,\alpha}^b(t) \) for \( (I, \theta, \alpha) \in \Pi \), we obtain the desired result. \( \square \)

**Remark 4.10.** Under the hypotheses of theorem 4.9 the following hold as in remarks 4.4 and 4.6:

(a) It follows from theorem 2.3 that if the homoclinic orbit \( \gamma_{h,0,\alpha_0}^b(t) \) persists in (4.7), then \( \tilde{M}^b(\theta_0, \alpha_0) = 0 \);

(b) If equation (4.7) has such a first integral near \( \tilde{\Gamma}^b \), then the corresponding Melnikov function is identically zero on \( V \times S^1_{2\pi} \times \tilde{V} \);

(c) If \( H^0, H^1 \) are analytic and equation (4.7) has such a first integral except for \( H_r \) near \( \tilde{\Gamma}_{h,0,\alpha_0}^b \) with some \( (I_0, \theta_0, \alpha_0) \in \Pi \), then \( \tilde{M}^1(\theta) = 0 \) is identically zero on \( \tilde{\Gamma}^b \).

The statement of part (a) consists with theorem 4.7.

5. Examples

We now illustrate the above theory for four examples: the periodically forced Duffing oscillator [12, 25, 27, 28], two identical pendula coupled with a harmonic oscillator, a periodically forced rigid body [34] and a three-mode truncation of a buckled beam [30].
5.1. Periodically forced Duffing oscillator

We first consider the periodically forced Duffing oscillator

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = ax_1 - x_1^3 + \varepsilon(\beta \cos \omega t - \delta x_2), \]  

(5.1)

where \( x_1, x_2 \in \mathbb{R}, \quad a = 1 \) or \(-1, \) and \( \beta, \delta, \omega \) are positive constants. Equation (5.1) becomes a single-degree-of-freedom Hamiltonian system with the Hamiltonian

\[ H = -\frac{1}{2}ax_2^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \]  

(5.2)

for \( \varepsilon = 0 \) and it is a special case of (4.1). See figure 2 for the phase portraits of (5.1) with \( \varepsilon = 0. \)

We begin with the case of \( a = 1. \) When \( \varepsilon = 0, \) in the phase plane there exist a pair of homoclinic orbits \( q_h^\pm(t) = (\pm \sqrt{2} \text{sech } t, \mp \sqrt{2} \text{sech } t \text{ tanh } t), \) a pair of one-parameter families of periodic orbits

\[ q_k^\pm(t) = \left( \pm \frac{\sqrt{2}}{\sqrt{2} - k^2} \text{dn} \left( \frac{t}{\sqrt{2} - k^2} \right), \right. \]

\[ \left. \mp \frac{\sqrt{2}k^2}{2 - k^2} \text{sn} \left( \frac{t}{\sqrt{2} - k^2} \right) \text{cn} \left( \frac{t}{\sqrt{2} - k^2} \right) \right), \quad k \in (0, 1), \]

inside each of them, and a one-parameter periodic orbits

\[ \tilde{q}_k(t) = \left( \frac{\sqrt{2}k}{\sqrt{2k^2 - 1}} \text{cn} \left( \frac{t}{\sqrt{2k^2 - 1}} \right), \right. \]

\[ \left. - \frac{\sqrt{2}k}{2k^2 - 1} \text{sn} \left( \frac{t}{\sqrt{2k^2 - 1}} \right) \text{dn} \left( \frac{t}{\sqrt{2k^2 - 1}} \right) \right), \quad k \in \left(1/\sqrt{2}, 1\right). \]
outside of them, as shown in figure 2(a), where $sn$, $cn$ and $dn$ represent the Jacobi elliptic functions with the elliptic modulus $k$. See [9] for general information on elliptic functions. The periods of $q_k^\pm(t)$ and $\tilde{q}_k^\pm(t)$ are given by $T^k = 2K(k)\sqrt{2 - k^2}$ and $\tilde{T}^k = 4K(k)\sqrt{2k^2 - 1}$, respectively, where $K(k)$ is the complete elliptic integral of the first kind. See also [12, 25].

Assume that the resonance conditions

$$I_T^k = \frac{2\pi m}{\omega}, \quad \text{i.e.,} \quad \omega = \frac{2\pi m}{2K(k)\sqrt{2 - k^2}},$$

and

$$I_{\tilde{T}}^k = \frac{2\pi m}{\omega}, \quad \text{i.e.,} \quad \omega = \frac{2\pi m}{4K(k)\sqrt{2k^2 - 1}},$$

hold for $q_k^\pm(t)$ and $\tilde{q}_k^\pm(t)$, respectively, with $l, m > 0$ relatively prime integers. We compute the subharmonic Melnikov function (4.5) for $q_k^\pm(t)$ and $\tilde{q}_k^\pm(t)$ as

$$M^\text{sub}_\pm(\tau) = -\delta J_1(k, l) \pm \beta J_2(k, m, l) \sin \tau$$

and

$$\tilde{M}^\text{sub}_\pm(\tau) = -\delta \tilde{J}_1(k, l) + \beta \tilde{J}_2(k, m, l) \sin \tau,$$

respectively, where

$$J_1(k, l) = \frac{4l[(2 - k^2)E(k) - 2k^2K(k)]}{3(2 - k^2)^{3/2}},$$

$$J_2(k, m, l) = \begin{cases} 
\sqrt{2\pi \omega} \text{sech} \left( \frac{m\pi K(k')}{K(k)} \right) & \text{(for } l = 1); \\
0 & \text{(for } l \neq 1), 
\end{cases}$$

$$\tilde{J}_1(k, l) = \frac{8l[(2k^2 - 1)E(k) + K^2(k)]}{3(2k^2 - 1)^{3/2}},$$

$$\tilde{J}_2(k, m, l) = \begin{cases} 
2\sqrt{2\pi \omega} \text{sech} \left( \frac{m\pi K(k')}{2K(k)} \right) & \text{(for } l = 1 \text{ and } m \text{ odd);} \\
0 & \text{(for } l \neq 1 \text{ or } m \text{ even).} 
\end{cases}$$

Here $E(k)$ is the complete elliptic integral of the second kind and $K' = \sqrt{1 - k^2}$ is the complementary elliptic modulus. We see that the subharmonic Melnikov functions $M^\text{sub}_\pm(\tau)$ and $\tilde{M}^\text{sub}_\pm(\tau)$ are not identically zero on any connected open set in $S^1_T$. We also compute the homoclinic Melnikov function (4.6) for $q_k^\pm(t)$ as

$$M_\pm(\tau) = -\frac{4}{3} \delta \pm \sqrt{2\pi \omega} \beta \text{csch} \left( \frac{\pi \omega}{2} \right) \sin \tau,$$

which is not identically zero on any connected open set in $S^1_T$. See [12, 25] for the computations of the Melnikov functions.
Let
\[
R = \{ k \in (0, 1) | k \text{ satisfies (5.3) for } m, l \in \mathbb{N} \},
\]
\[
\tilde{R} = \{ k \in \left(1/\sqrt{2}, 1 \right) | k \text{ satisfies (5.4) for } m, l \in \mathbb{N} \},
\]
and let
\[
S^k_\pm = \{ (x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1_T | x = q_k^\pm(t) \},
\]
\[
\tilde{S}^k = \{ (x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1_T | x = \tilde{q}^k(t) \},
\]
\[
S^h_\pm = \{ (x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1_T | x = q^h_\pm(t) \}.\]

Applying theorems 4.3 and 4.5, we obtain the following. 

**Proposition 5.1.** The first integral (5.2) does not persist near \( S^k_\pm \) for \( k \in R, \tilde{S}^k \) for \( k \in \tilde{R} \), and \( S^h_\pm \) in (5.1) with \( a = 1 \) for \( \varepsilon > 0 \).

**Remark 5.2.** When \( \beta > 0 \) but \( \delta = 0 \), so that equation (5.1) is Hamiltonian, the statement of proposition 5.1 still holds near \( S^k_\pm \) for \( k \in R_1, \tilde{S}^k \) for \( k \in \tilde{R}_0 \), and \( S^h_\pm \), where
\[
R_1 = \{ k \in (0, 1) | k \text{ satisfies (5.3) with } l = 1 \},
\]
\[
\tilde{R}_0 = \{ k \in \left(1/\sqrt{2}, 1 \right) | k \text{ satisfies (5.4) with } l = 1 \text{ and } m \text{ odd} \}.\]

We turn to the case of \( a = -1 \). When \( \varepsilon = 0 \), in the phase plane there exists a one-parameter family of periodic orbits
\[
\gamma^k(t) = \left( \frac{\sqrt{2}k}{\sqrt{1 - 2k^2}} \operatorname{cn} \left( \frac{t}{\sqrt{1 - 2k^2}} \right), -\frac{\sqrt{2}k}{1 - 2k^2} \operatorname{sn} \left( \frac{t}{\sqrt{1 - 2k^2}} \right) \operatorname{dn} \left( \frac{t}{\sqrt{1 - 2k^2}} \right) \right), \quad k \in (0, 1/\sqrt{2}),
\]
as shown in figure 2(b), and their period is given by \( \tilde{T}^k = 4K(k)\sqrt{1 - 2k^2} \). See [27, 28]. Assume that the resonance condition
\[
\Omega^k = \frac{2\pi m}{\omega}, \quad \text{i.e.,} \quad \omega = \frac{\pi m}{2IK(k)\sqrt{1 - 2k^2}} \tag{5.5}
\]
holds for \( l, m > 0 \) relatively prime integers. We compute the subharmonic Melnikov function (4.5) for \( \gamma^k(t) \) as
\[
\tilde{M}^{m//l}(\tau) = -\delta J_1(k, l) \pm \beta J_2(k, m, l) \sin \tau,
\]
where
\[
J_1(k, l) = \frac{8\left[(2k^2 - 1)E(k) + k^2K(k)\right]}{3(1 - 2k^2)^{3/2}},
\]
\[
J_2(k, m, l) = \begin{cases} \sqrt{2}\pi^m \text{sech} \left( \frac{\pi mK(k')}{2K(k)} \right) & \text{for } l = 1 \text{ and } m \text{ odd;} \\ 0 & \text{for } l \neq 1 \text{ or } m \text{ even.} \end{cases}
\]
See [27, 28] for the computations of the Melnikov function. Thus, the Melnikov function $M_{m/l}(\tau)$ is not identically zero on any connected open set in $S^1_1$.

Let

$$\hat{R} = \left\{ k \in \left(0, 1/\sqrt{2}\right) \mid k \text{ satisfies (5.5) for } m, l \in \mathbb{N} \right\}$$

and let

$$\hat{S}^k = \{(x, \theta) \in \mathbb{R}^2 \times S^1_1 \mid x = \gamma^k(t)\}.$$ 

Applying theorem 4.3, we obtain the following.

**Proposition 5.3.** The first integral (5.2) does not persist near $\hat{S}^k$ for $k \in \hat{R}$ in (5.1) with $a = -1$ for $\varepsilon > 0$.

**Remark 5.4.** When $\beta > 0$ but $\delta = 0$, i.e., equation (5.1) is Hamiltonian, the statement of proposition 5.7 still holds near $\hat{S}^k$ for $k \in \hat{R}_o$, where

$$\hat{R}_o = \left\{ k \in \left(0, 1/\sqrt{2}\right) \mid k \text{ satisfies (5.5) with } l = 1 \text{ and } m \text{ odd} \right\}.$$ 

5.2. Two pendula coupled with a harmonic oscillator

We next consider the three-degree-of-freedom Hamiltonian system

$$\begin{align*}
\dot{x}_1 &= x_3, & \dot{x}_3 &= -\sin x_1 - \varepsilon y_1 \sin x_1, \\
\dot{x}_2 &= x_4, & \dot{x}_4 &= -\sin x_2 - \varepsilon y_1 \sin x_2, \\
\dot{y}_1 &= y_2, & \dot{y}_2 &= -\omega_0^2 y_1 + \varepsilon (\cos x_1 + \cos x_2)
\end{align*}$$

(5.6)

with the Hamiltonian

$$H = -\cos x_1 - \cos x_2 + \frac{1}{2}(x_3^2 + x_4^2 + \omega_0^2 y_1^2 + y_2^2) - \varepsilon y_1 (\cos x_1 + \cos x_2),$$

where $x_1, x_2 \in S^1_2, x_3, x_4, y_1, y_2 \in \mathbb{R}$ and $\omega_0$ is a positive constant. The system (5.6) represents non-dimensionalized equations of motion for two identical pendula coupled with a harmonic oscillator shown in figure 3. Here the gravitational force acts downwards, and the spring $K$ generates a restoring force $Ky_1$, where $y_1$ is the displacement of the mass $M = m$ from the pivot of the pendula. Linear restoring forces with a spring constant of $O(\varepsilon)$ and zero natural length also occur between the two masses $m$ and the mass $M$. In particular, $\omega_0^2 = K\ell/Mg + O(\varepsilon)$, where $g$ is the gravitational acceleration and $\ell$ is the length from the pivot to the mass $m$.

Introduce the action-angle coordinates $(I, \theta) \in \mathbb{R}_+ \times S^1_2$ such that

$$y_1 = \sqrt{\frac{2I}{\omega_0}} \sin \theta, \quad y_2 = \sqrt{2\omega_0 \ell} \cos \theta.$$
and rewrite (5.6) as
\[
\begin{align*}
\dot{x}_1 &= x_3, \\
\dot{x}_3 &= -\sin x_1 - \epsilon \sqrt{2I/\omega_0} \sin \theta \sin x_1, \\
\dot{x}_2 &= x_4, \\
\dot{x}_4 &= -\sin x_2 - \epsilon \sqrt{2I/\omega_0} \sin \theta \sin x_2, \\
\dot{I} &= \epsilon \sqrt{2I/\omega_0} \cos \theta (\cos x_1 + \cos x_2), \\
\dot{\theta} &= \omega_0 - \epsilon \frac{\sin \theta}{\sqrt{2I/\omega_0}} (\cos x_1 + \cos x_2),
\end{align*}
\] (5.7)
which has the form (4.7) with
\[
\begin{align*}
H^0(x, I) &= -\cos x_1 - \cos x_2 + \frac{1}{2} (x_3^2 + x_4^2) + \omega_0 I, \\
H^1(x, I, \theta) &= -\sqrt{2I/\omega_0} \sin \theta (\cos x_1 + \cos x_2),
\end{align*}
\]
where \( \mathbb{R}_+ \) denotes the set of nonnegative real numbers. When \( \epsilon = 0 \), the \( x \)-component of (5.7) has a first integral
\[
F_2(x, I) = -\cos x_1 + \frac{1}{2} x_3^2
\]
and a hyperbolic equilibrium \( x^f = (\pi, \pi, 0, 0) \) to which there exist four one-parameter families of homoclinic orbits
\[
\begin{align*}
q^{t}_{\pm, +}(t; \alpha) &= (\pm 2 \arcsin(\tanh t), 2 \arcsin(\tanh(t + \alpha)), \pm 2 \sech t, 2 \sech(t + \alpha)), \\
q^{t}_{\pm, -}(t; \alpha) &= (\pm 2 \arcsin(\tanh t), -2 \arcsin(\tanh(t + \alpha)), \pm 2 \sech t, -2 \sech(t + \alpha)),
\end{align*}
\]
where \( \alpha \in \mathbb{R} \). Thus, assumptions (W1)–(W3) hold with \( m = 2 \).
We compute (4.11) for the homoclinic orbits \( (x, I, \theta) = (q^{t}_{\pm, \pm}(t; \alpha), I, \omega_0 t + \theta_0) \) as
\[ M_2^j(\theta_0, \alpha) = -\sqrt{\frac{2T}{\omega_0}} \int_{-\infty}^{\infty} 2 \sin(\omega_0 t + \theta_0) \text{sech} t \sin(2 \arcsin(\tanh t)) dt \]
\[ = -4 \sqrt{\frac{2T}{\omega_0}} \cos \theta_0 \int_{-\infty}^{\infty} \text{sech}^2 t \tanh t \sin \omega_0 t dt \]
\[ = -\pi \sqrt{8 \omega_0^2} \text{csch} \left( \frac{\pi \omega_0}{2} \right) \cos \theta_0. \]

On the other hand, letting \( \{T_j^{l,\alpha} \}_{j=-\infty}^{\infty} \) be a time sequence satisfying (4.9), we write the integral in (4.10) as
\[ -\sqrt{\frac{2T}{\omega_0}} \int_{T_j^{l,\alpha}}^{T_j^{l+\alpha,\alpha}} \cos(\omega_0 t + \theta_0) \cos(2 \arcsin(\tanh t)) dt + \cos(2 \arcsin(\tanh(t + \alpha))) dt \]
\[ = -\sqrt{\frac{2T}{\omega_0}} \left( \cos \theta_0 \int_{T_j^{l,\alpha}}^{T_j^{l+\alpha,\alpha}} (1 - 2 \tanh^2 t) \cos \omega_0 t dt + \sin \theta_0 \int_{T_j^{l,\alpha}}^{T_j^{l+\alpha,\alpha}} (1 - 2 \tanh^2 t) \sin \omega_0 t dt - \sin(\theta_0 - \alpha \omega) \int_{T_j^{l,\alpha}}^{T_j^{l+\alpha,\alpha}} (1 - 2 \tanh^2 t) \sin \omega_0 t dt \right). \]

Since
\[ \lim_{j \to \pm \infty} \omega_0 T_j^{l,\alpha} = 0 \mod 2\pi, \]
we have
\[ \lim_{j \to \infty} \int_{T_j^{l,\alpha}}^{T_j^{l+\alpha,\alpha}} (1 - 2 \tanh^2 t) \cos \omega_0 t dt = 2\pi \omega_0 \text{csch} \left( \frac{\pi \omega_0}{2} \right) \]
and
\[ \lim_{j \to \infty} \int_{T_j^{l,\alpha}}^{T_j^{l+\alpha,\alpha}} (1 - 2 \tanh^2 t) \sin \omega_0 t dt = 0. \]
Hence, we obtain
\[ M'_1(\theta_0, \alpha) = -\sqrt{8} \Omega \omega_0^2 \sin \left( \frac{\pi \omega_0}{2} \right) \left( \cos \theta_0 + \cos(\theta_0 - \omega_0 \alpha) \right). \]

We see that \( M'_1(\theta_0, \alpha), k = 1, 2, \) are not identically zero on any connected open set in \( \mathbb{R}_+ \times S_{2\pi}^1 \times \mathbb{R}. \) Applying theorem 4.9, we obtain the following.

**Proposition 5.5.** The first integrals \( F_2(x, I) \) and \( I \) do not persist near
\[ \Gamma^h_+ = \Gamma^h_+ \cup \Gamma^h_+ \cup \Gamma^h_{-+} \cup \Gamma^h_- \]
in (5.7) for \( \varepsilon > 0, \) where
\[ \Gamma^h_{\pm, \pm} = \{(q^h_{\pm, \pm}(t; \alpha), I, \theta)| t \in \mathbb{R}, I \in \mathbb{R}_+, \theta \in S_{2\pi}^1 \}. \]

**Remark 5.6.** We have
\[
\det DM'_1(\theta, \alpha) = -4\pi^2 \omega_0^2 \sin \left( \frac{\pi \omega_0}{2} \right) \sin \theta_0 \sin(\theta_0 - \omega_0 \alpha).
\]

Hence, if \( M'_1(\theta, \alpha) = 0, \) then \( \det DM'_1(\theta, \alpha) \neq 0. \) From theorem 4.7 we see that the stable and unstable manifolds of the perturbed periodic orbit near \( \gamma^p_0 = \{(x^f, I, \theta)| \theta \in S_{2\pi}^1 \} \) intersect transversely on its level set for \( \varepsilon > 0 \) sufficiently small.

### 5.3. Periodically forced rigid body

We next consider the three-dimensional system
\[
\begin{align*}
\omega_1 &= \frac{I_2 - I_3}{I_1} \omega_3 \omega_1 - \frac{I_0}{I_1} \Omega \omega_2 + \frac{\ell b}{I_1} V_1, \\
\omega_2 &= \frac{I_3 - I_1}{I_2} \omega_3 \omega_1 + \frac{I_0}{I_2} \Omega \omega_1 + \frac{\ell b}{I_2} V_2, \\
\omega_3 &= \frac{I_1 - I_2}{I_3} \omega_1 \omega_2 + \frac{\ell d}{I_3} V_3,
\end{align*}
\]
which provides a mathematical model for a quadrotor helicopter shown in figure 4. In the model, \( \omega_j \in \mathbb{R} \) and \( I_j > 0, j = 1, 2, 3, \) respectively, denote the angular velocities and moments of inertia about the quadrotor’s principal axes, \( \ell \) represents the length from the center of mass to the rotational axis of the rotor, and \( I_0, b \) and \( d, \) respectively, represent the rotor’s moment of inertia about the rotational axis, thrust factor and drag factor. Moreover,
\[ \Omega = \Omega_2 + \Omega_4 - \Omega_1 - \Omega_3 \]
and
\[ V_1 = \Omega_2^2 - \Omega_2^2, \quad V_2 = \Omega_3^2 - \Omega_1^2, \quad V_3 = \Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2, \]
where \( \Omega_j \) is the angular velocity of the \( j \)th rotor for \( j = 1–4. \) See [8, 13] for the derivation of (5.8). In particular, the quadrotor can hover only if
\[ \Omega_j = \Omega_0 := \frac{1}{2} \sqrt{\frac{m_0 g}{b}}, \quad j = 1–4, \]
where \( m_0 \) and \( g \) are, respectively, the quadrotor’s mass and gravitational acceleration.
Let $T > 0$ be a constant, and let $\Omega_j = \Omega_0 + \varepsilon \Delta \Omega_j(t)$, where $\Delta \Omega_j(t)$ is a $T$-periodic function, for $j = 1–4$. This corresponds to a situation in which the quadrotor is subjected to periodic perturbations when hovering. Let

$$v_1(t) = \Delta \Omega_4(t) - \Delta \Omega_2(t), \quad v_2(t) = \Delta \Omega_3(t) - \Delta \Omega_1(t),$$

$$v_3(t) = \Delta \Omega_4(t) + \Delta \Omega_2(t) - \Delta \Omega_3(t) - \Delta \Omega_1(t)$$

and

$$\beta_0 = I_0 \Omega_0, \quad \beta_1 = \beta_2 = 2\ell b \Omega_0^2, \quad \beta_3 = 2\ell d \Omega_0^2.$$

Equation (5.8) is written as

$$\dot{\omega}_1 = \frac{I_2 - I_1}{I_1} \omega_2 \omega_3 + \varepsilon \left( \frac{-\beta_0}{I_1} v_3(t) \omega_2 + \frac{\beta_1}{I_1} v_1(t) \right) + O(\varepsilon^2),$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2} \omega_3 \omega_1 + \varepsilon \left( \frac{\beta_0}{I_2} v_3(t) \omega_1 + \frac{\beta_2}{I_2} v_2(t) \right) + O(\varepsilon^2),$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3} \omega_1 \omega_2 + \varepsilon \frac{\beta_3}{I_3} v_3(t) + O(\varepsilon^2),$$

in which chaotic motions were discussed in [34] when $\beta_1 = 0$ and $v_2(t) = v_3(t) = \sin \nu t$ with $\nu > 0$ a constant. When $\varepsilon = 0$, equation (5.9) has a first integral

$$F(\omega) = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

and nonhyperbolic equilibria at

$$p_{1\pm}(c_1) = (\pm c_1, 0, 0), \quad p_{2\pm}(c_2) = (0, \pm c_2, 0), \quad p_{3\pm}(c) = (0, 0, \pm c_3)$$

on the level set $F(\omega) = c > 0$, where $c_j = \sqrt{2c/I_j}, j = 1, 2, 3$. The first integral $F(\omega)$ corresponds to the (Hamiltonian) energy of the unperturbed rigid body.

Let $X_\varepsilon(\omega, t) = X^0(\omega) + \varepsilon X^1(\omega, t) + O(\varepsilon^2)$ denote the non-autonomous vector field of (5.9) and define the corresponding autonomous vector field $\tilde{X}_\varepsilon(\omega, \theta) = \tilde{X}^0(\omega, \theta) + \varepsilon \tilde{X}^1(\omega, \theta) + O(\varepsilon^2)$ on $\mathbb{R}^3 \times S^1$ like (4.3), where

$$\tilde{X}^0(\omega, \theta) = \left( X^0(\omega) \right)_1, \quad \tilde{X}^1(\omega, \theta) = \left( X^1(\omega, \theta) \right)_1.$$
The unperturbed vector field $\tilde{X}(\omega, \theta)$ has six one-parameter families of nonhyperbolic periodic orbits $\gamma_{j\pm, c}(t) = (p_{j\pm}(c), t)$, $j = 1, 2, 3$. We compute the integral (2.1) as

$$\mathcal{J}_{F, \gamma_{j\pm, c}} = \int_0^T dF(\tilde{X}(p_{j\pm}(c), t))dt = \pm c_j \beta_j \int_0^T v_j(t)dt, \quad j = 1, 2, 3,$$

and apply theorems 2.1 and 2.2 to obtain the following.

**Proposition 5.7.** For $j = 1, 2, 3$, if

$$\beta_j \int_0^T v_j(t) \neq 0,$$

then the periodic orbit $\gamma_{j\pm, c}(t)$ does not persist for any $c_j > 0$ and the first integral $F(\omega)$ does not persist near

$$\{(p_{j\pm}(c), \theta) \in \mathbb{R} \mid c_j > 0, \theta \in S^1_T\} \cup \{(p_{j\pm}(c), \theta) \in \mathbb{R} \mid c_j > 0, \theta \in S^1_T\}$$

in (5.9).

**Remark 5.8.**

(a) In [34], when $\beta_1 = 0$ and $v_2(t) = v_3(t) = \sin \nu t$ with $\nu > 0$ a constant, it was shown that the periodic orbits $\gamma_{2\pm, c}$ persist for $c_2 > 0$ if and only if $\beta_0 = 0$ or $\beta_3 = 0$ (see proposition 2 of [34]).

(b) The unperturbed vector field $X(\omega)$ has another first integral

$$\tilde{F}(\omega) = (I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2),$$

which corresponds to the angular momentum of the rigid body. We compute the integral (2.1) as

$$\mathcal{J}_{\tilde{F}, \gamma} = \int_0^T d\tilde{F}(\tilde{X}(p_{\pm}(c), t))dt = \pm 2 c_j I_j \beta_j \int_0^T v_j(t)dt, \quad j = 1, 2, 3,$$

so that the same statement as proposition 5.7 holds for $\tilde{F}(\omega)$.

### 5.4. Three-mode truncation of a buckled beam

Finally, we consider the six-dimensional autonomous system

\begin{align}
\dot{x}_1 &= x_4, & \dot{x}_4 &= x_1 - (x_1^3 + \beta_1 x_2^2 + \beta_2 x_3^2)x_1, \\
\dot{x}_2 &= x_5, & \dot{x}_5 &= -\omega_1^2 x_2 - \beta_1 (x_1^2 + \beta_1 x_2^2 + \beta_2 x_3^2)x_2, \\
\dot{x}_3 &= x_6, & \dot{x}_6 &= -\omega_2^2 x_3 - \beta_2 (x_1^2 + \beta_1 x_2^2 + \beta_2 x_3^2)x_3,
\end{align}

(5.10)

which represents a three-mode truncation of a buckled beam shown in figure 5, where $x_j \in \mathbb{R}$, $j = 1\ldots 6$, and $\omega_j, \beta_j > 0$, $j = 1, 2$, are constants such that $\omega_1 < \omega_2$. See [30] for the details on the model. In (5.10) there is a saddle-center equilibrium at $(x_1, \ldots, x_6) = (0, \ldots, 0)$ and it has a homoclinic orbit. It was also shown in [33] that for almost all pairs of $\beta_1, \beta_2 > 0$ the system (5.10) exhibits chaotic motions and it is nonintegrable.
Figure 5. Buckled beam. The variables \( u \) and \( P \) represent the deflection and compressive force, respectively. The length of the beam when \( u \equiv 0 \) is non-dimensionalized to the unity.

Let \( x_j = \sqrt{\varepsilon} y_j, \ j = 1–6, \) with the small parameter \( \varepsilon. \) We rewrite (5.10) as

\[
\begin{align*}
\dot{y}_1 &= y_4, & \dot{y}_4 &= y_1 - \varepsilon (y_1^2 + \beta_1 y_2^2 + \beta_2 y_3^2) y_1, \\
\dot{y}_2 &= y_5, & \dot{y}_5 &= -\omega_1^2 y_2 - \varepsilon \beta_1 (y_1^2 + \beta_1 y_2^2 + \beta_2 y_3^2) y_2, \\
\dot{y}_3 &= y_6, & \dot{y}_6 &= -\omega_2^2 y_3 - \varepsilon \beta_2 (y_1^2 + \beta_1 y_2^2 + \beta_2 y_3^2) y_3,
\end{align*}
\]

(5.11)

which is regarded as a perturbation of a linear system. When \( \varepsilon = 0, \) equation (5.11) has two one-parameter families of periodic orbits

\[
\begin{align*}
\gamma_{1,c}(t) &= (0, c \sin \omega_1 t, 0, 0, c \omega_1 \cos \omega_1 t, 0), \\
\gamma_{2,c}(t) &= (0, 0, c \sin \omega_2 t, 0, 0, c \omega_2 \cos \omega_2 t)
\end{align*}
\]

for \( c > 0, \) three first integrals

\[
F_1(y) = -y_1^2 + y_4^2, \quad F_2(y) = \omega_1^2 y_2^2 + y_5^2, \quad F_3(y) = \omega_2^2 y_3^2 + y_6^2,
\]

and six commutative vector fields

\[
\begin{align*}
Z_1 &= (y_1, 0, 0, y_4, 0, 0), & Z_2 &= (y_4, 0, 0, y_1, 0, 0), \\
Z_3 &= (y_5, 0, 0, y_2, 0, 0), & Z_4 &= (0, y_5, 0, 0, -\omega_1^2 y_2, 0), \\
Z_5 &= (0, y_6, 0, 0, y_3, 0), & Z_6 &= (0, 0, y_6, 0, 0, -\omega_2^2 y_3).
\end{align*}
\]

Moreover, the AVE of (5.11) with \( \varepsilon = 0 \) is given by

\[
\begin{align*}
\dot{\eta}_1 &= -\eta_4, & \dot{\eta}_4 &= \omega_1^2 \eta_5, & \dot{\eta}_5 &= \omega_2^2 \eta_6, \\
\dot{\eta}_2 &= -\eta_1, & \dot{\eta}_3 &= \omega_2^2 \eta_6, & \dot{\eta}_6 &= -\eta_3,
\end{align*}
\]

which has four linearly independent periodic solutions

\[
\begin{align*}
\tilde{\gamma}_1(t) &= (0, \omega_1 \sin \omega_1 t, 0, 0, \cos \omega_1 t, 0), \\
\tilde{\gamma}_2(t) &= (0, \omega_1 \cos \omega_1 t, 0, 0, -\sin \omega_1 t, 0),
\end{align*}
\]

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\[ \tilde{\gamma}_3(t) = (0, 0, \omega_2 \sin \omega_2 t, 0, 0, \cos \omega_2 t), \]
\[ \tilde{\gamma}_4(t) = (0, 0, \omega_2 \cos \omega_2 t, 0, 0, -\sin \omega_2 t) \]

and two linearly independent unbounded solutions. We compute (2.1) and (3.9) as
\[ \mathcal{J}_{F_j, \gamma_{\ell, c}} = \int_0^{2\pi/\omega_{\ell}} dF_j(X_j(t)) \gamma_{\ell, c}(t) dt = 0, \quad j = 1, 2, 3 \quad \text{and} \quad \ell = 1, 2, \]
and
\[ \mathcal{J}_{\tilde{\gamma}_j Z_k, \gamma_{\ell, c}} = \int_0^{2\pi/\omega_{\ell}} \tilde{\gamma}_j(t) \cdot [X_1, Z_k] \gamma_{\ell, c}(t) dt \]
\[ = \begin{cases} \frac{3}{2} \pi \beta_1^2 c^3 & \text{if } (j, k, \ell) = (2, 3, 1); \\ \frac{3}{2} \pi \beta_2^2 c^3 & \text{if } (j, k, \ell) = (4, 5, 2); \\ 0 & \text{otherwise}, \end{cases} \quad (5.12) \]

where \( X_1 \) represents the \( O(\varepsilon) \)-terms of the vector field in (5.11). In (5.12), the subscript \( j \) is allowed to take 1 or 2 for \( \ell = 1 \), and 3 or 4 for \( \ell = 2 \). Theorems 2.1, 2.2 and 3.8 give no meaningful information on persistence of periodic orbits and first integrals, but application of theorem 3.5 yields the following.

**Proposition 5.9.** The commutative vector fields \( Z_3 \) and \( Z_5 \) do not persist near the \( (y_2, y_3) \)- and \( (y_3, y_6) \)-planes, respectively, in (5.11). Moreover, in (5.10), near the origin, there is no commutative vector field which has the linear term
\[ \tilde{Z}_3 = (0, x_2, 0, 0, x_5, 0) \quad \text{or} \quad \tilde{Z}_5 = (0, 0, x_3, 0, 0, x_6). \]

**Proof.** The first part immediately follows from application of theorem 3.5. The second part is easily proven since a vector field having such a linear term for (5.10) is transformed to \( Z_3 + O(\varepsilon) \) or \( Z_5 + O(\varepsilon) \) for (5.11). \( \square \)

**Remark 5.10.**

(a) By the Lyapunov center theorem (e.g., theorem 5.6.7 of [2]), there exist two families of periodic orbits in (5.10) if \( \omega_2 / \omega_1, \omega_1 / \omega_2 \notin \mathbb{Z} \). Hence, the periodic orbits \( \gamma_{j, c}, j = 1, 2, \) persist in (5.11) for such values of \( \omega_j, j = 1, 2, \) at least.

(b) As shown in [33], the Hamiltonian system (5.10) is nonintegrable for almost all pairs of \( \beta_j, j = 1, 2 \). Hence, the three first integrals \( F_j(y), j = 1, 2, 3 \), do not persist in (5.11) for such values of \( \beta_j, j = 1, 2, \) at least.

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Appendix A. Some auxiliary materials for section 3

In this appendix, we provide some prerequisites for section 3: basic notions and facts on connections of vector bundles and linear differential equations. Similar materials are found in [11, 15, 22]. See, e.g. [7], for necessary information on vector bundles.

A.1. Connections and horizontal sections

We begin with connections of vector bundles and their horizontal sections. Henceforth $M$ represents a $C^1$ $m$-dimensional manifold for $m \in \mathbb{N}$, and $E$ represents a $C^1$ vector bundle of rank $r$ over $M$ with a projection $\pi : E \to M$ for some $m, r \in \mathbb{N}$. Let $C(M)$ be a set of all $C^1 \mathbb{R}$-valued functions on $M$ and let $C(M, E)$ be a set of all $C^1$ sections of $E$. Let $T^* M$ be the cotangent bundle of $M$. Note that $T^* M \otimes E$ is also a $C^1$ vector bundle. We first give basic definitions.

**Definition A.1.** An $\mathbb{R}$-linear map 
\[ \nabla : C(M, E) \to C(M, T^* M \otimes E) \]

is called a connection of the vector bundle $E$ if 
\[ \nabla(fs) = df \otimes s + f \nabla s \quad (A.1) \]

for any $f \in C(M)$ and $s \in C(M, E)$. A section $s \in C(M, E)$ is said to be horizontal for the connection $\nabla$ if $\nabla s = 0$.

Let $U \subset M$ be an open neighborhood and let $\{e_j\}_{j=1}^r$ be a frame on $U$, so that any section $s \in C(M, E)$ is expressed as
\[ s = \sum_{j=1}^r s^j e_j \quad (A.2) \]
on $U$ for some $s^j \in C(M)$ for $j = 1, \ldots, r$.

**Definition A.2.** For each $i = 1, \ldots, r$ we can write
\[ \nabla e_i = \sum_j \theta^i_j \otimes e_j \quad (A.3) \]

where $\theta^i_j : M \to T^* M$, $j = 1, \ldots, r$. The $r \times r$ matrix $\theta = (\theta^i_j)$ is called the connection form of $\nabla$ on $U$ in the frame $\{e_j\}_{j=1}^r$.

Let $s \in C(M, E)$. Using (A.1) and (A.3), we compute
\[ \nabla s = \sum_{j=1}^r \nabla (s^j e_j) = \sum_{j=1}^r (ds^j \otimes e_j + s^j \nabla e_j) \]
\[ = \sum_{i=1}^r ds^i \otimes e_i + \sum_{i=1}^r s^i \left( \sum_{j=1}^r \theta^i_j \otimes e_j \right) = \sum_{i=1}^r \left( ds^i + \sum_{j=1}^r s^j \theta^i_j \right) \otimes e_i. \]

Hence, the condition for the section $s$ to be horizontal, $\nabla s = 0$, is equivalent to
\[ ds^i + \sum_{j=1}^r s^j \theta^i_j = 0, \quad i = 1, \ldots, r, \quad (A.4) \]
on $U$. 

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Definition A.3. Let $E^*$ be the dual bundle of $E$. A connection $\nabla^*$ of $E^*$ given by
\[
d(s, \alpha) = (\nabla s, \alpha) + (s, \nabla^* \alpha)
\] (A.5)
for any $s \in C(M, E)$ and $\alpha \in C(M, E^*)$ is called a dual connection of $\nabla$.

Let $\{e^j\}_{j=1}^r$ be the dual frame for the frame $\{e_j\}_{j=1}^r$, i.e.,
\[
\langle e_i, e^j \rangle = \delta_{ij}, \quad i, j = 1, \ldots, r,
\] (A.6)
where $\delta_{ij}$ is Kronecker’s delta. We have the following relation between connections and their dual connections.

Proposition A.4. Let $\theta = (\theta^j_i)$ be the connection form of a connection $\nabla$ on $U$. Then the connection form $\theta^* = (-\theta^i_j)$ of the dual connection $\nabla^*$ is given by $\theta^* = -\theta_i^j$ on $U$.

Proof. Using (A.5) and (A.6), we compute
\[
0 = d\langle e_i, e^j \rangle = \langle \nabla e_i, e^j \rangle + \langle e_i, \nabla^* e^j \rangle.
\]
Since by (A.3) and (A.6)
\[
\langle \nabla e_i, e^j \rangle = \left( \sum_{k=1}^r \theta^j_k \otimes e_k, e^j \right) = \theta^j_i = \left( e_i, \sum_{k=1}^r \theta^j_k \otimes e^k \right),
\]
we obtain
\[
\left( e_i, \sum_{k=1}^r \theta^j_k \otimes e^k + \nabla^* e^j \right) = 0, \quad i, j = 1, \ldots, r.
\]
Hence,
\[
\nabla^* e^j = - \sum_{k=1}^r \theta^j_k \otimes e^k
\]
for $j = 1, \ldots, r$. □

A.2. Connections and linear differential equations

Let $m = 1$ and assume that the one-dimensional manifold $M$ is paracompact and connected. We will see below that a connection of the vector bundle $E$ defines a linear differential equation and horizontal sections of the connection correspond to solutions to the differential equations.

Take an open neighborhood $U \subset M$ and its local coordinate $t \in \mathbb{R}$. Let $\nabla$ be a connection and let $s \in C(M, E)$ be a horizontal section of $\nabla$ given by (A.2). We write the connection form $\theta = (\theta^j_i)$ as
\[
\theta^j_i = a_{ij}(t)dt
\]
for some $a_{ij}(t) \in C(M)$. Then equation (A.4) is expressed as
\[
ds^i + \sum_{j=1}^r a_{ij}(t)s^j dt = 0, \quad i = 1, \ldots, r.
\] (A.7)

Let $A(t) = (A_{ij}(t))$ be an $r \times r$ matrix with $A_{ij}(t) := - a_{ij}(t)$ and let $\hat{s}(t) = (s^1(t), \ldots, s^r(t))^T$. From (A.7) we obtain a linear differential equation
\[
\frac{d}{dt} \hat{s}(t) = A(t) \hat{s}(t). \tag{A.8}
\]

Thus, the relation $\nabla \hat{s} = 0$ is locally represented by a linear differential equation. Below we apply the above argument to the VE (3.2) and AVE (3.5) to derive (3.3) and (3.6), respectively.

**A.2.1. Derivation of (3.3).** We consider the VE (3.2) and set $M = \Gamma$ and $E = T_\Gamma$ with $r = n$.

Choose the frame \( \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \) and write

\[
X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}
\]

locally. We compute

\[
\nabla \frac{\partial}{\partial x_i} = dt \otimes L_X \left( \frac{\partial}{\partial x_i} \right) \bigg|_\Gamma = dt \otimes \left( \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \right) \bigg|_\Gamma
\]

\[
= dt \otimes \left( - \sum_{j=1}^{n} \frac{\partial X_j}{\partial x_i} \frac{\partial}{\partial x_j} \right) \bigg|_\Gamma = - \sum_{j=1}^{n} \frac{\partial X_j}{\partial x_i} \phi(t) dt \otimes \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n,
\]

so that

\[
\theta^i_j = - \frac{\partial X_j}{\partial x_i} (\phi(t)) dt, \quad \text{i.e.,} \quad A_{ij}(t) = \frac{\partial X_j}{\partial x_i}(\phi(t)), \quad i, j = 1, \ldots, n. \tag{A.9}
\]

This yields (3.3) along with (A.8).

**A.2.2. Derivation of (3.6).** We next consider the AVE (3.5) in the setting of appendix A.2.1. Choose the frame \( (dx_1, \ldots, dx_n) \). Using proposition A.4 and (A.9), we obtain

\[
\theta^i_j = \frac{\partial X_j}{\partial x_i}(\phi(t)) \quad \text{i.e.,} \quad A_{ij}(t) = - \frac{\partial X_j}{\partial x_i}(\phi(t)), \quad i, j = 1, \ldots, n.
\]

This yields (3.6) along with (A.8).

**ORCID iDs**

Shoya Motonaga © https://orcid.org/0000-0001-7761-1609

Kazuyuki Yagasaki © https://orcid.org/0000-0001-9901-248X

**References**

[1] Ablowitz M J and Fokas A S 2003 Complex Variables: Introduction and Applications 2nd edn (Cambridge: Cambridge University Press)

[2] Abraham R and Marsden J E 1978 Foundations of Mechanics 2nd edn (New York: Benjamin-Cummings)

[3] Arnold V I 1989 Mathematical Methods of Classical Mechanics 2nd edn (New York: Springer)

[4] Audin M 2008 Hamiltonian Systems and Their Integrability (Providence, RI: American Mathematical Society)

[5] Ayoul M and Zung N T 2010 Galoisian Obstructions to non-Hamiltonian integrability C. R. Math. Acad. Sci. Paris 348 1323–6
[6] Bogoyavlenskij O I 1998 Extended integrability and bi-Hamiltonian systems Commun. Math. Phys. 196 19–51
[7] Bott R and Tu L W 1982 Differential Forms in Algebraic Topology (Berlin: Springer)
[8] Bouabdallah S, Murriero P and Siegwart R 2004 Design and control of an indoor micro quadrotor IEEE Int. Conf. Robotics and Automation (ICRA ’04) (New Orleans, LA 26 April–1 May 2004) pp 4393–8
[9] Byrd P F and Friedman M D 1954 Handbook of Elliptic Integrals for Engineers and Physicists (Berlin: Springer)
[10] Chicone C 2006 Ordinary Differential Equations with Applications 2nd edn (New York: Springer)
[11] Churchill R C and Rod D L 1988 Geometrical aspects of Ziglin’s nonintegrability theorem for complex Hamiltonian systems J. Differ. Equ. 76 91–114
[12] Guckenheimer J and Holmes P 1983 Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (New York: Springer)
[13] Hamel T, Mahony R, Lozano R and Ostriowski J 2002 Dynamic modelling and configuration stabilization for an X4-flyer IFAC Proc. Vol. 35 217–22
[14] Hirsch M W 1976 Differential Topology (New York: Springer)
[15] Ilyashenko Y and Yakovenko S 2008 Lectures on Analytic Differential Equations (Providence, RI: American Mathematical Society)
[16] Kozlov V V 1983 Integrability and non-integrability in Hamiltonian mechanics Russ. Math. Surv. 38 1–76
[17] Kozlov V V 1996 Symmetries, Topology and Resonances in Hamiltonian Mechanics (Berlin: Springer)
[18] Marsden J E and Ratiu T S 1999 Introduction to Mechanics and Symmetry 2nd edn (New York: Springer)
[19] Melnikov V K 1963 On the stability of the center for time periodic perturbations Trans. Moscow Math. Soc. 12 1–56
[20] Morales-Ruiz J J 1999 Differential Galois Theory and Non-Integrability of Hamiltonian Systems (Basel: Birkhäuser)
[21] Morales-Ruiz J J 2002 A note on a connection between the Poincaré–Arnold–Melnikov integral and the Picard–Vessiot theory Differential Galois Theory Polish Acad. Sci. Inst. Math. vol 58 ed T Crespo and Z Hajto (Banach Center Publications) pp 165–75
[22] Morales-Ruiz J J and Ramis J P 2001 Galoisian obstructions to integrability of Hamiltonian systems Methods Appl. Anal. 8 33–96
[23] Poincaré H 1892 New Methods of Celestial Mechanic vol 1 (New York: AIP)
[24] Wiggins S 1988 Global Bifurcations and Chaos: Analytical Methods (New York: Springer)
[25] Wiggins S 1990 Introduction to Applied Nonlinear Dynamical Systems and Chaos (New York: Springer)
[26] Wiggins S 1994 Normally Hyperbolic Invariant Manifolds in Dynamical Systems (New York: Springer)
[27] Yagasaki K 1994 Homoclinic motions and chaos in the quasiperiodically forced van der Pol–Duffing oscillator with single well potential Proc. R. Soc. A 445 597–617
[28] Yagasaki K 1996 The Melnikov theory for subharmonics and their bifurcations in forced oscillations SIAM J. Appl. Math. 56 1720–65
[29] Yagasaki K 2000 Horseshoes in two-degree-of-freedom Hamiltonian systems with saddle-centers Arch. Ratton. Mech. Anal. 154 275–96
[30] Yagasaki K 2001 Homoclinic and heteroclinic behavior in an infinite-degree-of-freedom Hamiltonian system: chaotic free vibrations of an undamped, buckled beam Phys. Lett. A 285 55–62
[31] Yagasaki K 2002 Melnikov’s method and codimension-two bifurcations in forced oscillations J. Differ. Equ. 185 1–24
[32] Yagasaki K 2003 Degenerate resonances in forced oscillators Discrete Continuous Dyn. Syst. - Ser. B 3 423–38
[33] Yagasaki K 2005 Homoclinic and heteroclinic orbits to invariant tori in multi-degree-of-freedom Hamiltonian systems with saddle-centres Nonlinearity 18 1331–50
[34] Yagasaki K 2018 Heteroclinic transition motions in periodic perturbations of conservative systems with an application to forced rigid body dynamics Regul. Chaotic Dyn. 23 438–57
[35] Ziglin S L 1981 Self-intersection of the complex separatrices and the nonexistence of the integrals in the Hamiltonian systems with one-and-half degrees of freedom J. Appl. Math. Mech. 45 411–3