Abstract

For systems of one-component interacting oscillators on the d-dimensional lattice, \( d > 1 \), whose potential energy besides a large nearest-neighbour (n-n) ferromagnetic translation-invariant quadratic term contains small non-nearest-neighbour translation invariant term, an existence of a ferromagnetic long-range order for two valued lattice spins, equal to a sign of oscillator variables, is established for sufficiently large magnitude \( g \) of the n-n interaction with the help of the Peierls type contour bound. The Ruelle superstability bound is used for a derivation of the contour bound.

1 Introduction and main result

Let’s consider the system of one-dimensional oscillators on the \( d \)-dimensional lattice \( \mathbb{Z}^d \), with the potential energy (on a set \( \Lambda \) with the finite cardinality \( |\Lambda| \))

\[
U(q_\Lambda) = \sum_{x \in \Lambda} (u(q_x) - 2dgq_x^2) + g \sum_{|x-y|=1, x,y \in \Lambda} (q_x - q_y)^2 + U'(q_\Lambda), \quad g \geq 1
\]

(1.1)

Here \( q_x \) the oscillator coordinate taking value in \( \mathbb{R} \), \( q_X = (q_x, x \in X) \), the one-particle potential (external field) \( u \) is a bounded below even polynomial having a degree \( degu = 2n \), \( U' \) is an even translation invariant function such that \( U \) satisfies the superstability and regularity conditions, \( |x| \) is the Euclidean norm of the integer valued vector \( x \), \( d > 1 \).

Let \(<\cdot>_\Lambda, <>\) denote the Gibbs classical or quantum average for the system confined to \( \Lambda \) and the system in the thermodynamic limit, i.e. \( \Lambda = \mathbb{Z}^d \), respectively.

For classical systems

\[
< F_X >_\Lambda = Z^{-1}_\Lambda \int F_X(q_\Lambda) e^{-\beta U(q_\Lambda)} dq_\Lambda, \quad Z_\Lambda = \int e^{-\beta U(q_\Lambda)} dq_\Lambda.
\]

If \( \hat{F}_X \) is the operator of a multiplication by the function \( F_X(q_X) \) then the quantum average is given by

\[
< F_X >_\Lambda = Z^{-1}_\Lambda \text{Tr}(\hat{F}_X e^{-\beta H_\Lambda}), \quad Z_\Lambda = \text{Tr}(e^{-\beta H_\Lambda})
\]

where \( H_\Lambda = -\frac{1}{2m} \sum_{x \in \Lambda} \partial^2_x + U(q_\Lambda) \), and \( \partial_x \) is the partial derivative in \( q_x \).

The corner stone of proving an existence of lro, using generalized Peierls argument, is the following contour bound

\[
< \prod_{<x,x'> \in \Gamma} \chi_x^+ \chi_{x'}^- >_\Lambda \leq e^{-E|\Gamma|}, \quad (1.2)
\]

where \( \Gamma \) is a set of nearest neighbours, \( |\Gamma| \) is the number of them in it,
\[ \chi^+_a =\chi((0,\infty))(q_x), \quad \chi^-_a =\chi((\infty,0))(q_x), \]

\( \chi_{(a,b)} \) is the characteristic function of the open interval \((a,b)\).

The bound (1.2) was earlier derived in [BF], [FL] for several classes of classical ferromagnetic systems or classical systems with the nearest-neighbour pair interaction (see also, [Si], [SH], [BW]). If one puts \( s_x = \text{sign}(q_x) \), then taking into account that 
\[ \chi^+(-) = \frac{1}{2}[1 + (-)s_x] \] one obtains
\[ 4 < \chi^+_x\chi^-_y >_\Lambda = 1+ <s_x> - <s_y> - <s_x s_y> \Lambda . \]

Since the systems are invariant under the transformation of changing signs of the oscillator variables we have
\[ <s_x s_y> \Lambda = 1 - 4 <\chi^+_x\chi^-_y> \Lambda . \]

Now in order to prove the ferromagnetic long-range order for the spins \( s_x \) one has to show that the average in the rhs in the equality is strictly less than \( \frac{1}{4} \). This can be proved with the aid of the following lemma ([GJS], [FL])

**LEMMA 1.1**

If the bound (1.2) holds, \( d > 1 \), and \( e^{-E} \) is sufficiently small then there exist positive numbers \( a, a' \) such that
\[ <\chi^+_x\chi^-_y> \leq a'e^{-aE}. \tag{1.3} \]

So, if one shows that \( E \) can be made arbitrary large while increasing \( g \) or \( \beta \), then the lro for the above spins will be proved.

In this paper we prove the ferromagnetic lro for the systems, in which interaction is neither ferromagnetic nor n-n, but essentially ferromagnetic for sufficiently large \( g \) (see Remark 5). We establish (1.2) for the simplest polynomial \( u(q) = \eta q^{2n} \) with the help of the Ruelle superstability bound [RI] and show that \( E \) in (1.2) is positive and growing for growing \( g \), or more precisely
\[ E = e_0 - 2^{-1}\ln (8\pi^{-1} e_0) - E_0, \quad e_0 = [g^n 2d(\eta n)^{-1}]^{\frac{1}{2n-2}}, \tag{1.4} \]

where \( E_0 \) depends on \( g, \beta \) and is found from the superstability bound (for the rescaled and translated correlation functions). \( E_0 \) is bounded in \( g \) for classical systems and grows slowly for quantum systems (see Lemma 1.2).

The proposed technique is based on the precise knowledge how the constant, defining \( E_0 \) in the superstability bound, depends on the potential energy (Theorem 2.1). It is inspired by the technique proposed in [AKR] for quantum ferromagnetic systems, which by rescaling of the oscillator variables, can be reduced to the above systems with the pair quadratic infinite-range interaction

\[ U' = \sum_{x,y \in \Lambda} C_{x-y}(q_x - q_y)^2, \quad u(q) = \eta q^4. \]

In this paper a complicated version of (1.2) is proposed and the lro is proved for Gibbs loop path system associated to the quantum system via FK formula and unit spin which are signs of the averaged Wiener path. A small parameter, appearing in the potential energy, determining a depth of the symmetric wells of the external potential is not associated with the magnitude of n-n interaction in it.
Our approach stresses the necessity to consider a large magnitude of n-n interaction which determines the depth of the symmetric wells of the external potential $u(q) - 2dq^2$ ($e_0, -e_0$ are the only its real minima).

Proofs of an existence of an order parameter for ferromagnetic quantum oscillator systems with n-n interaction, which are based on the reflection positivity, can be found in [KP], [BK] (see also [DLS]). Vanishing of the order parameter in the quantum limit (mass is vanishing) is established in [VZ].

**THEOREM 1.1**

Let the potential energy of the one-component oscillator classical or quantum system is given by (1.1), $u(q) = \eta q^{2n}$. Let, also, $U'$ be a translation invariant and an even function such that the condition of superstability and regularity hold for it

$$U'(q_{\Lambda}) \geq - \sum_{x \in \Lambda} [Bg'||q_x|^l + B'], \quad 2 \leq l < 2n,$$

$$|W'(q_{X_1,q_{X_2}})| = |U'(q_{X_1 \cup X_2}) - U'(q_{X_1}) - U'(q_{X_2})| \leq \frac{g'}{2} \sum_{x,y \in X_2} \Psi'_x(|q_x|^l + |q_y|^l), \quad \Psi'_y \geq 0,$$

where $l' < \frac{l}{2}, l > 2; l' \leq \frac{l}{2}, l = 2$ for non-negative $U'$ in the second inequality and $l' \leq 0$ in both inequalities if $U'$ is non-positive; $B, B', \Psi'_x$ are non-negative constants, $||\Psi'||_1 = \sum_x \Psi'_x < \infty$ and the summation is performed over $Z^d$.

Then there is the ferromagnetic lro in classical and quantum systems for the spins $s_x$ for sufficiently large $g : g \gg 1$, i.e. $< s_x s_y > \gg 0$.

Since $s_x$ are scale invariant and their average is not changed after rescaling of oscillator variables, we can deal with the rescaled by $g^{-\frac{1}{2}}$ variables and the potential energy $U_g$

$$U_g(q_{\Lambda}) = \sum_{x \in \Lambda} u_0^g(q_x) + \sum_{|x-y|=1, x,y \in \Lambda} (q_x - q_y)^2 + U'(g^{-\frac{1}{2}}q_{\Lambda}), \quad (1.5)$$

where

$$u_0^g(q) = g^{-n}\eta q^{2n} - 2dq^2.$$

The correlation functions or reduced density matrices generated by $U_g$ will be denoted by $\rho_g$.

The main idea of the proof originates from the inequality

$$< \prod_{<x,x'> \in \Gamma} \chi^+_x \chi^-_{x'} >_\Lambda \leq (e'e_0)^{|\Gamma|} e^{-e_0|\Gamma|} < e^{Q_{g,r}} >_\Lambda, \quad (1.6)$$

where $e'$ is a positive constant, $e_0$ is a growing function of $g$, the expectation value is determined by $\rho_g$ and

$$Q_{g,r}(q_{\Lambda}) = \sum_{<x,x'> \in \Gamma} Q_g(q_x,q_y), \quad Q_g(q_x,q_y) = \frac{1}{e_0} \{(q_x - q_y)^2 + 4(|q_x^2 - e_0^2| + |q_y^2 - e_0^2|)\}.$$

Here we used the inequality

$$\chi^+(q_x)\chi^-(q_x') \leq (e'e_0)^{\frac{1}{2}} e^{-e_0} \exp\{Q_g(q_x,q_y)\}. \quad (1.7)$$
Theorem 1.1 will be proved if we prove the following lemma.

**LEMMA 1.2**
Let the conditions of Theorem 1.1 be satisfied. Let, also, \( e_0 \) be given by (1.4). Then there exists a function \( E_0(g) \) on the interval \([1, \infty)\) such that

\[
< e^{Q_{g,v}} > \leq e^{[\Gamma]E_0}.
\]

(1.8)

For the classical systems \( E_0 \) is a bounded function on the interval \([1, \infty)\).

For the quantum systems if

\[
k(g) = (1 + e^{-\sqrt{m}g})^{-1}(1 - e^{-\sqrt{m}g}) - \frac{20}{3} e_0^{-1} \sqrt{\frac{g}{m}} > 0
\]

then there exists a bounded continuous functions \( E_*(g) \) on \([1, \infty)\) such that

\[
E_0 \leq \frac{1}{2} \left[ \ln \frac{gm}{k(g)} - \ln (1 - e^{-2\sqrt{m}g}) \right] + \sqrt{\frac{g}{m}} (\frac{64}{9}k(g) - \beta) + E_*(g).
\]

(1.10)

Lemmas 1.1, 1.2, i.e. (1.3) and (1.10) prove Theorem 1.1 since \( e_0 \) grows faster than \( \sqrt{g} \).

Function \( E_* \) in the Lemma is defined by (3.9-11),(3.14).

In the second and third sections we’ll give the proof this lemma for classical and quantum systems, respectively. Proofs of Lemma 1.1 and (1.7) are standard and will not be give here (see [GJS],[FL],[AKR]).

2 Lemma 1.2 via superstability argument. Classical systems.

For classical systems with the rescaled potential energy

\[
< F_X >_\Lambda = Z^{-1}_\Lambda \int F_X(q_X)e^{-\beta U_g(q_\Lambda)}dq_\Lambda = \int F_X(q_X)\rho^\Lambda(q_X)dq_X,
\]

\[
\rho^\Lambda(g_X) = Z^{-1}_\Lambda \int e^{-\beta U_g(q_\Lambda)}dq_\Lambda \chi_X, \quad Z_\Lambda = \int e^{-\beta U_g(q_\Lambda)}dq_\Lambda.
\]

Here the integration is performed over \( R_{|\Lambda|} \) and \( \rho^\Lambda \) are the correlation functions. By \( \rho_g \) we’ll denote the correlation functions in the thermodynamic limit.

Changing the variables \( q_x \to q_x - e_0 \), in the integral in the right-hand-side of (1.6) and using the translation invariance of the Lebesque measure we obtain

\[
< e^{Q_{g,v}} >= \int \rho_g(q_{\Gamma} + e_0) \exp\{Q_{g,\Gamma}(q_{\Gamma} + e_0)\}dq_{\Gamma}, \quad q_{\Gamma} = (q_x, q_y; < x, y > \in \Gamma), \quad (2.1)
\]

\[
Q_{g,\Gamma}(q_{\Gamma} + e_0) \leq \sum_{<x,x'> \in \Gamma} \left\{ \frac{10}{3}e_0(q_x^2 + q_x'^2) + \frac{8}{3}|q_x| + |q_x'| \right\}, \quad q_X + e_0 = (q_x + e_0, x \in X).
\]

The polynomial \( Q \) becomes bounded in \( g \) if it is translated by \( e_0 \). As a result, we have to prove that the correlation functions, translated by \( e_0 \), in the limit of growing \( g \) satisfy the usual superstability bound.

It is not difficult to check that if \( e_0 \) is given by (1.4) then
\[ u^0_g(q) = 2dn^{-1}[e_0^{-2n+2}q^{2n} - nq^2]. \]

From this we immediately see that
\[ v^0_g(q + e_0) = p_g(q) + bq^2 - b', \quad b = 2dn^{-1}(2n(n - 1) - n), \quad b' = 2d^{n-1}n e_0^2, \]
where \( p_g \) is a bounded below polynomial in \( e_0^{-1} \) and \( q \) (the linear term proportional to \( e_0 \) is absent in it)
\[ p_g(q) = 2dn^{-1}\sum_{s=3}^{2n} s!(2n - s)! q^s e_0^{2-s}. \]

Now we have to establish the accurate superstability and regularity conditions for the translated by \( e_0 \) potential energy.

The superstability bound is given by
\[ U_g(qX + e_0) \geq \sum_{x \in X} \tilde{u}_g(q_x) - |X|B_g, \quad B_g = b' + B', \quad (2.2) \]
where
\[ \tilde{u}_g(q) = (v^0_g(q + e_0) + b') - Bg^{-\frac{1}{2}+\nu'}|q|^\nu. \]

For a non-negative \( U' \)
\[ \tilde{u}_g(q) = u^0_g(q + e_0) + b'. \]

Let's put
\[ U_{sg}(qX) = U_g(qX + e_0) - \sum_{x \in \Lambda} u_{*g}(q_x) + |X|B_g, \quad (2.3) \]
\[ u_{*g} = \tilde{u}_g - v_g, \quad v_g(q) = q^2 + g^{-\frac{1}{2}+\nu'}|q|^\nu. \]

\( B_g \) diverges if \( g \) tends to infinity since \( b' \). We can add \( |\Lambda|B_g \) to the potential energy since the expression for the correlation functions is not changed after this.

Then the following superstability condition holds
\[ U_{*}(qX) \geq \sum_{x \in X} v_{g}(q_x). \quad (2.4) \]

The regularity condition, also, holds
\[ |W_{*g}(qX_1; qX_2)| = |U_{*g}(qX_1 \cup X_2) - U_{*g}(qX_1) - U_{*g}(qX_2)| \leq \]
\[ \leq \frac{1}{2} \sum_{x \in X_1, y \in X_2} \Psi_{|x-y|}[v_{g}(q_x) + v_{g}(q_y)], \quad X_1 \cap X_2 = \emptyset, \quad (2.5) \]
where \( \Psi_{|x|} = 2\delta_{|x|,1} + \Psi'_{|x|} \).

Applying \( |X| - 1 \) times the regularity condition the following important condition is also derived
\[ U_{*g}(qX) \leq \sum_{x \in X} \tilde{U}_g(q_x), \quad \tilde{U}_g(q) = U_{*g}(q) + \||\Psi||_2 v_{g}(q). \quad (2.6) \]

From the definition of the functions determining \( \tilde{U}_g \), taking into account that \( U_g(q) = u^0_g(q) \), we derive
\[ \tilde{U}_g(q) = B' + (1 + ||\Psi||_1)q^2 + (1 + B + ||\Psi||_1)g^{-\frac{1}{2}+r'}|q|^l. \]

Let’s put
\[ \rho_\Lambda^A(q_\Lambda) = \exp\{\beta \sum_{x \in A} u_{sg}(q_x)\} \rho_g^A(q_\Lambda + e_0). \]

Then \( \rho_\Lambda^A \) are expressed in terms of \( U_{sg} \) after adding to \( U_g \) the large in \( g \) terms independent of oscillator variables
\[ \rho_\Lambda^A(q_x) = Z_{\Lambda}^{-1} \int e^{-\beta U_{sg}(q_\Lambda)} \mu_{sg}(dq_\Lambda \backslash X), \quad Z_{\Lambda} = \int e^{-\beta U_{sg}(q_\Lambda)} \mu_s(dq_\Lambda), \quad (2.7) \]
where
\[ \mu_{sg}(dq_Y) = \exp\{-\beta \sum_{x \in Y} u_{sg}(q_x)\} dq_Y. \]

As a result of the superstability and regularity conditions for \( U_{sg} \) the following theorem is true [R].

**THEOREM 2.1**

Let the condition (2.4-5) hold for a positive polynomial \( v_g \) and the function \( u_{sg} \) be such that the measure \( \mu_{sg} \) is finite. Then for arbitrary \( 0 < 3\varepsilon < 1, \ r > 0 \) for the correlation functions defined by (2.7) the following (superstability) bound is valid
\[ \rho_\Lambda^A(q_x) \leq \exp\{-\sum_{x \in X} [\beta(1 - 3\varepsilon)v_g(q_x) - c_0(I_{r,u_{sg}}, I_{u_{sg}})]\}, \quad (2.8) \]
where \( c \) is a positive continuous monotonous growing at infinity function,
\[ I_{r,u} = e^{-\frac{1}{2}\beta||\Psi||_1 v_g(r)} I_{0u}, \quad I_{0u} = \int_{|q| \leq r} \exp\{-\beta[\tilde{U}_g + u(q)]\} dq, \]
\[ I_u = \int \exp\{-\beta[(1 - 3\varepsilon)v_g(q) + u(q)]\} dq. \]

We formulated the Ruelle result in a more explicit form in order to trace the dependence in \( g \) in all the terms.

(2.1) and theorem 2.1 yields
\[ < e^{Q_{g,r}} > \leq e^{||\Gamma||E_0}, \quad E_0 = E^0 + e_*(g), \quad e_*(g) = c_0(I_{r,u_{sg}}, I_{u_{sg}}), \quad (2.9) \]
\[ E^0 = 2 \ln \int \exp\{-\beta(1 - 3\varepsilon)v_g(q) - \beta u_{sg}(q) + \frac{10}{3\epsilon_0} q^2 + \frac{8}{3} |q|\} dq. \]

As a result, (1.2) holds with \( E \) given by (1.4). From the conditions of the Theorem 1.1 it follows that \( E^0 \) and \( e_* \) exist in the limit of vanishing \( g^{-1} \). Here we have to rely on the following significant equalities
\[ \lim_{g^{-1} \to 0} (a_g^0(q + e_0) + b') = bq^2, \quad \lim_{g^{-1} \to 0} v_g(q) = kq^2, \quad b \geq 4d, \]
where \( k = 1 \) or \( k = 2 \). From the inequalities (\( |q| \leq r \))
\[ \tilde{U}_g(q) \leq B' + (1 + ||\Psi||_1)r^2 + (1 + B + ||\Psi||_1)g^{-\frac{1}{2}+r'} |q|^l, \quad (2.10) \]
$$u_{*g}(q) \leq \bar{u}_g(q) + \nu_g(q) \leq p_g(r) + br^2 + v_g(r) + Bg^{-\frac{t}{2} + \nu r}$$

(2.11)

it follows that

$$(I^{-1}_{0u*_{g}})^{-1} \leq e^{-\beta p_g^+ (r)},$$

(2.12)

$$p_g^+ (r) = p_g(r) + B' + (2 + ||\Psi||_1)r^2 + (2 + B + ||\Psi||_1)g^{-\frac{t}{2} + \nu r}.$$  

(2.13)

Polynomial $p_g^+ (r)$ is uniformly bounded in $g$

$$p_g^+ (r) \leq p^0 (r) + B' + (2 + ||\Psi||_1)r^2 + (2 + B + ||\Psi||_1)r^t = \bar{p}(r),$$

(2.14)

$$p^0 (r) = 2dn^{-1} \sum_{s=3}^{2n} \frac{s!(2n-s)!}{n!} r^s.$$  

So, $e_*(g)$ is a bounded function.

Classical part of Lemma (1.2) is proved. Application of Lemma 1.1 completes the proof of

Theorem 1.1 for classical systems.

3 Lemma 1.2 via superstability bound. Quantum systems.

For quantum rescaled systems the Gibbs average of the operator $\hat{F}_X$ of multiplication by the

the function $F_X(q_X)$ is determined by the reduced density matrices (RDMs) $\rho^A_g(q_X|q_X)$,

$$< F_X >_A = Z_A^{-1} Tr(\hat{F}_X e^{-\beta H^A_g}) = Z_A^{-1} (\sqrt{g})^{[N]} \int F_X(q_X) e^{-\beta H^A_g} (q_X; q) dqX = \int F_X(q_X) \rho^A_g(q_X|q_X) dqX,$$

(3.1)

$$\rho^A_g(q_X|q_X) = (\sqrt{g})^{[N]} \int \rho^A_g(\omega_X) P_{q_X|q_X} (dw_X), \quad \rho^A_g(\omega_X) = Z_A^{-1} \int e^{-U_0(\omega_X)} P_0 (d\omega_X),$$

(3.2)

where $\omega = (q, w) \in \Omega^* = \mathbb{R} \times \Omega$, $\Omega$ is the probability space of Wiener paths($w \in \Omega$), $P^t_{q,q'} (dw)$ is the Wiener (conditional)measure concentrated on paths, starting from $q$ and arriving in $q'$ at the time $t$, $P_0 (d\omega) = \sqrt{g} dq P_{q,q}^{g\beta} (dw),$

$$U_g(\omega_X) = g^{-1} \int_0^\beta U_g(w_X(t)) dt = \int_0^\beta U_g(w_X(gt)) dt$$

In deriving the formulas we applied the Feymann-Kac formula to the kernel $e^{-\beta H^A_g (\sqrt{g^{-1}} q; \sqrt{g^{-1}} q')} \hat{F}_X (\sqrt{g^{-1}} q; \sqrt{g^{-1}} q')$ of the operator $e^{-\beta H^A_g}$ and the relation

$$\int P^t_{q' = g^{-1}q, \sqrt{g^{-1}} q'} (dw) f(w(t_1), ..., w(t_n)) = \sqrt{g} \int P^t_{q' = q} (dw) f(\sqrt{g^{-1}} w(t_1)), ..., \sqrt{g^{-1}} w(t_n)),$$

(3.3)

which follows from

$$\exp \{ t \partial^2 \} (\sqrt{g^{-1}} q; \sqrt{g^{-1}} q') = (4\pi t)^{-\frac{1}{2}} \exp \{- |q - q'|^2 \} = \sqrt{g} \exp \{tg \partial^2 \} (q; q').$$

The rescaled Hamiltonian is given by
\[ H_\Lambda^g = g\left(-\frac{1}{2m} \sum_{x \in \Lambda} \partial_{z}^2 + g^{-1}U_g(q_\Lambda)\right) \]

In order to prove Lemma 1.2 one has to estimate \( \rho_g^\Lambda(q_X + e_0|q_X + e_0) \).

From the translation invariance of the conditional Wiener measure and the measure \( P_0 \) it follows that

\[ \rho_g^\Lambda(q_X + e_0|q_X + e_0) = (\sqrt{g})^{|X|} \int \rho_g^\Lambda(\omega_X + e_0) P_{q_X X}^{g\beta}(dw_X), \]

\[ \rho_g^\Lambda(\omega_X + e_0) = Z_\Lambda^{-1} \int e^{-U_g(\omega_X + e_0)} P_0(d\omega_X|X), \quad Z_\Lambda = \int e^{-U_g(\omega_X + e_0)} P_0(d\omega_X), \]

where \( \omega + e_0 = w(t) + e_0, w(0) = q, t \in [0, t] \).

As a result

\[ < e^{Q_{g, r}} >_\Lambda = (\sqrt{g})^{|X|} \int e^{Q_{g, r}(q_\Lambda + e_0)} \rho_g^\Lambda(\omega_X + e_0) dq P_{q_\Lambda q_\Lambda}^{g\beta}(dw_\Gamma), \]

It is evident that Lemma 1.2 can be proved now with the help of the analogue of the superstability bound for \( \rho_g^\Lambda(\omega_X + e_0) \) which was proved by Park [P].

In order to prove the analogue of Theorem 2.1 one has to derive the superstability and regularity conditions for \( U_{*g}(\omega_\Lambda) = U_g(\omega_X + e_0) \). But now it is easy since in the previous section we established them for \( U_g(q_\Lambda + e_0) \).

So, let by \( u_{sg}(\omega), v_g(\omega), U_g(\omega) \) be denoted the corresponding functions, depending on \( w(gt) \), being integrated by \( dt \) on the interval \([0, \beta]\) and

\[ U_{sg}(\omega_\Lambda) = \int_{0}^{\beta} U_{sg}(w_\Lambda(gt)) dt. \]

where \( U_{sg}(w_\Lambda(gt)) \) is defined by (2.3)(instead of \( w \) may be written). Then

\[ U_*(\omega_X) \geq \sum_{x \in X} v_g(\omega_x). \quad (3.4) \]

\[ |W_{sg}(\omega_X; \omega_X)| = |U_{sg}(\omega_{X \cup X_2}) - U_{sg}(\omega_X) - U_{sg}(q_{X_2})| \leq \]

\[ \leq \frac{1}{2} \sum_{x \in X_1, y \in X_2} \Psi_{x-y}[v_g(\omega_x) + v_g(\omega_y)], \quad X_1 \cap X_2 = \emptyset, \quad (3.5) \]

\[ U_*(\omega_X) \leq \sum_{x \in X} U_{sg}(\omega_x). \]

Let’s put

\[ \rho_{sg}^\Lambda(\omega_\Lambda) = \exp\left\{ \sum_{x \in X} u_{sg}(\omega_x) \right\} \rho_g^\Lambda(\omega_\Lambda + e_0). \]

Hence

\[ \rho_{sg}^\Lambda(\omega_X) = Z_{*\Lambda}^{-1} \int e^{-U_{sg}(\omega_\Lambda)} P_{0}(d\omega_{\Lambda|X}), \quad Z_{*\Lambda} = \int e^{-U_{sg}(\omega_\Lambda)} P_{0}(d\omega_{\Lambda}), \quad (3.6) \]

where

\[ P_{0}(d\omega_\Gamma) = \exp\left\{ - \sum_{x \in Y} u_{sg}(\omega_x) \right\} P_{0}(d\omega_\Gamma). \]

As a result of (3.3-5) the following theorem is true.
THEOREM 3.1

Let the condition \((3.4-5)\) hold for a positive polynomial \(v_g(q)\) and the functional \(u_{sg}\) be such that the measure \(P_0\) is finite. Then for arbitrary \(0 < 3\varepsilon < 1, r > 0\) for the correlation functions defined by \((3.6)\) the following (superstability) bound is valid

\[
\rho_g^\lambda(\omega_x) \leq \exp\left\{ -\sum_{x \in I} [(1-3\varepsilon)v_g(\omega_x) - c_0(I_{r,usg}, I_{usg})]\right\}, \tag{3.7}
\]

where is a positive continuous monotonous growing at infinity function,

\[
I_{r,u} = e^{-\frac{1}{2}\beta||\phi||_1 I_{0u}}, \quad \bar{v}_{g,r} = ess \sup_{\omega \in \Omega_r^*} v_g(\omega),
\]

\[
I_{0usg} = \int_{\Omega_r^*} \exp\left\{ -[\bar{U}_g(\omega) + u_{sg}(\omega)]\right\} P_0(d\omega), \quad I_{usg} = \int \exp\left\{ -(1-3\varepsilon)v_g(\omega) + u_{sg}(\omega)\right\} P_0(d\omega),
\]

where \(\Omega_r^* = \{\omega \in \Omega^*: |w(t)| \leq r\}\).

The proof of this theorem does not essentially differs from the proof of Theorem 2.1.

Proof of Lemma 2.1.

Theorem 3.1 yields the following equalities

\[
E^0 = 2 \ln \sqrt{g} \int \exp\left\{ -\beta[1 - 3\varepsilon]v_g(w) - u_{sg}(w)\right\} + \frac{10}{3c_0} q^2 + \frac{8}{3} |q| \right\} P_{q,q}^{g\beta}(dw) dq
\]

\[
e_*(g) \right\}
\]

\[
I_0 = \int \exp\left\{ -2\beta[(1-3\varepsilon)v_g(w) + u_{sg}(w)] + \frac{1}{2} w_g^2(\beta)\right\} P_{q,q}^{g\beta}(dw) dq
\]

where \(w_g^2(\beta) = \int_0^1 w^2(gt)dt\).

From the FK formula it follows that \(I_0\) is the trace of the kernel of the exponent of perturbed generator of the Wiener process. So, the Golden-Thompson inequality \(Tr(e^{A+B}) \leq Tr(e^{A}e^B)\) yields

\[
gI_0 \leq \sqrt{gm}(2\pi\beta)^{-\frac{1}{2}} \int \exp\left\{ -2\beta[(1-3\varepsilon)v_g(q) + u_{sg}(q) + \frac{q^2}{4}]\right\} dq = \sqrt{gm}I_0^-. \tag{3.9}
\]

Here we took into account that

\[
\exp\left\{ t\partial^2\right\}(q; q') = (4\pi t)^{-\frac{1}{2}} \exp\left\{ -\frac{q - q'}{4t}\right\}.
\]

\(I_0\) is finite since \(b \geq 4d\).
For \( I_0 \) after the rescaling \( q = (m^{-1}g)^{\frac{1}{2}} \tilde{q} \) we have (\( \tilde{q}^2 \) is the operator of multiplication by \( \tilde{q}^2 \))

\[
I_0 = \left( \frac{g}{m} \right)^{\frac{1}{2}} \int \exp \{-g\beta(-2m)^{-1}\partial^2 + \frac{1}{2g}\tilde{q}^2\} \{((m^{-1}g)^{\frac{1}{2}}q, (m^{-1}g)^{\frac{1}{2}}q) \times \exp \{3^{-1}(20e_0^{-1}\sqrt{m^{-1}gq^2} + 8(m^{-1}g)^{\frac{1}{2}}|q|)\} dq.
\]

From (3.3) it follows that

\[
\left( \frac{g}{m} \right)^{\frac{1}{2}} \exp \{-g\beta(-2m)^{-1}\partial^2 + \frac{1}{2g}\tilde{q}^2\} \{((m^{-1}g)^{\frac{1}{2}}q, (m^{-1}g)^{\frac{1}{2}}q') =
\]

\[
e^{-2^{-1}\sqrt{g\beta}} \exp \{-\sqrt{\frac{g\beta}{m}}(-\partial^2 + \tilde{q}^2 - 1)\}(q, q') =
\]

\[
e^{-2^{-1}\sqrt{g\beta}} \exp \{-\frac{q^2}{2} + \frac{q'^2}{2}\} \exp \{-\sqrt{\frac{g\beta}{m}}(-\frac{1}{2}\partial^2 + \tilde{q}\partial)\}(q, q') =
\]

\[
\sqrt{\pi^{-1}}(1 - e^{-2\sqrt{g\beta}})\frac{1}{2} \exp \{-\frac{q^2}{2} + \frac{q'^2}{2} - (1 - e^{-2\sqrt{g\beta}})^{-1}(q' - e^{-\sqrt{g\beta}}q)^2 - 2^{-1}\sqrt{\frac{g\beta}{m}}\}.
\]

Here we used the relation

\[
\frac{1}{2}(-\partial^2 + \tilde{q}^2 - 1) = e^{-\sqrt{\frac{g\beta}{m}}}[-\frac{1}{2}\tilde{q}^2 + \tilde{q}\partial]\tilde{q}^2
\]

and the well-known formula for the density of the transition probability for the Ornstein-Uhlenbeck process. Hence

\[
I_0 = \sqrt{\pi^{-1}}e^{-\sqrt{\frac{g\beta}{m}}}(1 - e^{-2\sqrt{g\beta}})^{-\frac{1}{2}} \int \exp \{3^{-1}(20e_0^{-1}\sqrt{g\beta}\tilde{q}^2 + 8(\frac{g}{m})^{\frac{1}{2}}|q|)\} \times \exp \{-(1 + e^{-\sqrt{g\beta}})^{-1}(1 - e^{-\sqrt{g\beta}}q^2)\} dq.
\]

As a result

\[
I_0 = (1 - e^{-2\sqrt{g\beta}})^{-\frac{1}{2}}k(g)^{-\frac{1}{2}} \exp \{\frac{64}{9}k(g)^{-1}\sqrt{\frac{g}{m}}\}, \tag{3.10}
\]

\[
e^{E_0} \leq \sqrt{m}e^{-\sqrt{\frac{g\beta}{m}}k(g)^{-1}}(1 - e^{-2\sqrt{g\beta}})^{-\frac{1}{2}}k(g)^{-\frac{1}{2}}I_0, \tag{3.11}
\]

where \( k(g) \) is given by (1.9).

Applying the Golden-Thompson inequality we obtain

\[
I_{u_{\alpha}} \leq \sqrt{m}I_0. \tag{3.12}
\]

Repeating (2.10-11), using the equality

\[
\sqrt{g} \int P_{\alpha 0}(dw) = \sqrt{(2\pi\beta)^{-1}m},
\]

we derive, also, the analogue of (2.12) for the quantum case

\[
(I_{u_{\alpha_{0}}})^{-1} \leq \sqrt{2\pi\beta m^{-1}r^{-1}}e^{e_{p^{+}}(r)} \tag{3.13}
\]

As a result

Combining all these bounds we see that \( e_+(g) \) is bounded and (1.10) holds with
\[ E_*(g) = \ln I_0^- + e_*(g). \] (3.14)

Lemma 1.2 is proved. Theorem 1.1 is proved with an aid of Lemma 1.1 and (1.7).

**REMARKS.**

1. If one cancels the boundary term \( g \sum_{x \in \partial \Lambda} q_x^2 \) then (1.1) is reduced to
   \[ U(q_\Lambda) = \sum_{x \in \Lambda} u(q_x) - g \sum_{|x-y|=1, x,y \in \Lambda} q_x q_y + U'(q_\Lambda). \]
   where \( \partial \Lambda \) is the boundary of \( \Lambda \). If \( U' > 0 \) then the systems considered in [BF] can be recovered.

   Surprisingly the proposed technique does not respect this boundary term since it creates an obstruction for obtaining the bound from above (2,6) for the rescaled and translated potential energy which guarantees uniform boundedness in \( g \) of \( I^{-1} \) and \( \sum e_x \).

2. If one cancels the boundary term \( \sum_{x \in \partial \Lambda} u(q_x, q_x) \) then (1.1) is equal to
   \[ U(q_X) = \sum_{x,y \in X, |x-y|=1} u(q_x, q_y) + U'(q_X) \]
   \[ u(q_x, q_y) = (4d)^{-1}(u(q_x) + u(q_y)) - q_x q_y \]

   If \( U' \) is expressed in the same form as the first term in the rhs of the last equality then systems which are dealt with in [FL] can be recovered. Theorem (1.1) can be proved for such the potential energy taking into account in a special way a contribution of the boundary term to the superstability, regularity conditions and (2.6) for the rescaled and translated potential energy (see [S]).

3. Theorem 1.1 proves an existence of a phase transition for the case \( U' \) is expressed through a pair (special) potential, since it is known that in this case in the high-temperature phase there is an exponential decrease of correlations [K].

4. The magnitude of n-n interaction plays an exceptional role in the proposed approach since vanishing of it automatically implies vanishing of the spin two-point function for n-n sites. This means that \( E \) in (1.2) has to depend on the magnitude of n-n interaction, tending to zero together with it. So, one ought, always, to rescale by the magnitude (in an appropriate power) all the variables, when starting to derive the Peierls type contour bound using (1.7) with \( e_0 \) depending on it.

5. Essentially ferromagnetic interaction may be characterized by the property that the ferromagnetic configuration, consisting of the coordinate \( e_0 \) (minimum of a one-particle potential) at each lattice site, is more favorable than the associated antiferromagnetic (staggered) configuration, consisting of the coordinate \( -e_0 \) at the odd sublattice and \( e_0 \) at the even sublattice for sufficiently large \( g \), i.e. the potential energy on the former configuration is less than on the latter. This property follows from the superstability condition for the rescaled \( U'_g \) in the formulation of Theorem 1.1 and the fact that the growth in \( g \) of \( g^{-s} e_0^2 s \), \( s < n \), is more slow than \( e_0^2 \). In other words, the ferromagnetic n-n part of the potential energy suppresses antiferromagnetic ground states for sufficiently large \( g \).

6. If one puts
   \[ U'(q_\Lambda) = \sum_{x,y \in \Lambda} C_{x-y}(q_x - q_y)^2, \quad |C_{x-y}| \leq C_{|x-y|}, \quad ||C^0||_1 < \infty, \]
   where \( ||C^0||_1 \) does not depend on \( g \) then the conditions of Theorem 1.1 are satisfied.

7. The proof of Theorem 1.1 for classical systems and more general polynomial potentials \( u \) can be found in [S]
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REFERENCES
[AKR] S.Albeverio, O.Kondratiev, O.Rebenko, Journ.Stat.Phys., 92, 5/6, p.1137, 1998.
[BF] J.Bricmont, J.-R.Fontaine, Journ.Stat.Phys., 26, N4, p.745, 1981.
[BK] V.Barbulyak, Yu.Kondratiev, Reports Nat.Acad.Sci. of Ukraine, 10, p.19, 1991.
[BW] C.Borgs, R.Waxler, Commun.math.phys.,126, p.683, 1990.
[DLS] F.Dyson, E.Lieb, B.Simon, J.Stat.Phys.,18, p.335,1978.
[FL] J.Frohlich, E.Lieb, Comm.Math.Phys., 60, p.233, 1978.
[GJS] J.Glimm, A.Jaffe, T.Spencer, Commun.Math.Phys.,45, p.203.(1975).
[KP] B.Khoruzhenko, L.Pastur, Teor.Mat.Fiz.,73, p.1094, 1987.
[P] Y.Park, J.Korean Math.Soc., 23, p.43, 1985.
[K] H.Kunz, Comm.math.phys.,59, p.53, 1978.
[R] D.Ruelle, Comm.Math.phys., 18,p,127, 1970.
[Sh] S.Shlosman, Uspehi Math.Sci., v.41, N3(249), p.69, 1986 (in Russian).
[Si] Ya.G. Sinai, Phase transitions. Rigorous results. Moscow. Nauka. 1980 (there is the English translation).
[S] W.Skrypnik, J.Phys.A.(to be published)
[VZ] A.Verbeure, V.Zagrebnov, No-go theorem for quantum structural phase transitions. Preprint-KUL-TP-95/05.