Lower Bounds for Semi-adaptive Data Structures via Corruption

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Abstract
In a dynamic data structure problem we wish to maintain an encoding of some data in memory, in such a way that we may efficiently carry out a sequence of queries and updates to the data. A long-standing open problem in this area is to prove an unconditional polynomial lower bound of a trade-off between the update time and the query time of an adaptive dynamic data structure computing some explicit function. Ko and Weinstein provided such lower bound for a restricted class of semi-adaptive data structures, which compute the Disjointness function. There, the data are subsets \( x_1, \ldots, x_k \) and \( y \) of \( \{1, \ldots, n\} \), the updates can modify \( y \) (by inserting and removing elements), and the queries are an index \( i \in \{1, \ldots, k\} \) (query \( i \) should answer whether \( x_i \) and \( y \) are disjoint, i.e., it should compute the Disjointness function applied to \( (x_i, y) \)). The semi-adaptiveness places a restriction in how the data structure can be accessed in order to answer a query. We generalize the lower bound of Ko and Weinstein to work not just for the Disjointness, but for any function having high complexity under the smooth corruption bound.

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1 Introduction
A suitable setting to study data structures is the cell probe model [22]. Here we think of the memory divided into registers, or cells, where each cell can carry \( w \) bits, and we measure efficiency by counting the number of memory accesses, or probes, needed for each query — the query time \( t_q \) and each update — the update time \( t_u \). The main goal of this line of research is to understand the inherent trade-off between \( w \), \( t_q \) and \( t_u \), for various interesting problems. Specifically, one would like to show lower bounds on \( t = \max\{t_q, t_u\} \) for reasonable choices of \( w \) (which is typically logarithmic in the size of the data).

The first lower bound for this setting was proven by Fredman and Saks [9], which proved \( t = \Omega(\log n / \log \log n) \) for various problems. These lower bounds were successively improved [10,11,13,14], and we are now able to show that certain problems with non-Boolean
queries require \( t = \Omega(\log n / \log \log n)^2 \), and certain problems with Boolean queries require \( t = \Omega((\log n / \log \log n)^{3/2}) \).

The major unsolved question in this area is to prove a polynomial lower bound on \( t \). For example, consider the dynamic reachability problem, where we wish to maintain a directed \( n \)-vertex graph in memory, under edge insertions and deletions, while being able to answer reachability queries (“is vertex \( i \) connected to vertex \( j \)?”). Is it true that any scheme for the dynamic reachability problem requires \( t = \omega(n^\delta) \), for some constant \( \delta > 0 \)? Indeed, such a lower bound is known under various complexity-theoretic assumptions; the question is whether such a lower bound may be proven unconditionally.

In an influential paper [19], Mihai Pătrașcu proposed an approach to this unsolved question. He defines a data structure problem, called the multiphase problem. Let us represent partial functions \( f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\} \) as total functions \( f' : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,\ast\} \) where \( f'(x,y) = \ast \) if \( f(x,y) \) is not defined. Then associated with a partial Boolean function \( f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,\ast\} \), and a natural number \( k \geq 1 \), we may define a corresponding multiphase problem of \( f \) as the following dynamic process:

**Phase I - Initialization.** We are given \( k \) inputs \( x_1, \ldots, x_k \in \{0,1\}^n \), and are allowed to preprocess this input in time \( nk \cdot t_p \).

**Phase II - Update.** We are then given another input \( y \in \{0,1\}^n \), and we have time \( n \cdot t_u \) to read and update the memory locations from the data structure constructed in Phase I.

**Phase III - Query.** Finally, we are given a query \( i \in [k] \), we have time \( t_q \) to answer the question whether \( f(x_i,y) = 1 \). If \( f(x_i,y) \) is not defined, the answer can be arbitrary.

Typically we will have \( k = \text{poly}(n) \). Let us be more precise, and consider randomized solutions to the above problem.

**Definition 1 (Scheme for the multiphase problem of \( f \)).** Let \( f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,\ast\} \) be a partial Boolean function. A scheme for the multiphase problem of \( f \) with preprocessing time \( t_p \), update time \( t_u \) and query time \( t_q \) is a triple \( D = (E, \{U_y\}_{y \in \{0,1\}^n}, \{Q_i\}_{i \in [k]} \) where:

- \( E : (\{0,1\}^n)^k \rightarrow (\{0,1\}^w)^k \) maps the input \( x \) to the memory contents \( E(x) \), where each of the \( s \) memory locations holds \( w \) bits. \( E \) must be computed in time \( nk \cdot t_p \).

- For each \( y \in \{0,1\}^n \), \( U_y : (\{0,1\}^{w^s} \rightarrow (\{0,1\}^w)^u \) is a decision-tree of depth \( \leq n \cdot t_u \), which reads \( E(x) \) and produces a sequence \( U_y(E(x)) \) of \( u \) updates.

- For each \( i \in [k] \), \( Q_i : (\{0,1\}^{w^s})^k \times (\{0,1\}^w)^u \rightarrow \{0,1\} \) is a decision-tree of depth \( \leq t_q \).

- For all \( x \in (\{0,1\}^n)^k \), \( y \in \{0,1\}^n \), and \( i \in [k] \),

\[
f(x_i,y) \neq \ast \implies Q_i(E(x),U_y(E(x))) = f(x_i,y).
\]

In a randomized scheme for the multiphase problem of \( f \), each \( U_y \) and \( Q_i \) are distributions over decision trees, and it must hold that for all \( x \in (\{0,1\}^n)^k \), \( y \in \{0,1\}^n \), and \( i \in [k] \),

\[
f(x_i,y) \neq \ast \implies \Pr_{U_y} \left[ Q_i(E(x),U_y(E(x))) = f(x_i,y) \right] \geq 1 - \varepsilon.
\]

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1. See [17]. Strictly speaking, these conditional lower bounds only work if the preprocessing time, which is the time taken to encode the data into memory, is also bounded. But we will ignore this distinction.
2. In the usual way of defining the update phase, we have a read/write decision-tree \( U_y \) which changes the very same cells that it reads. But when \( w = \Omega(\log s) \), this can be seen to be equivalent, up to constant factors, to the definition we present here, where we have a decision-tree \( U_y \) that writes the updates on a separate location. In order to simulate a scheme that uses a read/write decision-tree, we may use a hash table with \( O(1) \) worst-case lookup time, such as cuckoo hashing. Then we have a read-only decision-tree \( U_y(E(x)) \) whose output is the hash table containing all the \( i \in [s] \) which were updated by \( U_y(E(x)) \), associated with their final value in the execution of \( U_y(E(x)) \).
3. All our results will hold even if \( Q_i \) is allowed to depend arbitrarily on \( x_i \). This makes for a less natural model, however, so we omit this from the definitions.
The value \( \varepsilon \) is called the error probability of the scheme.

Pătraşcu [19] considered this problem where \( f = \text{DISJ} \) is the Disjointness function:

\[
\text{DISJ}(x, y) = \begin{cases} 
0 & \text{if there exists } i \in [n] \text{ such that } x_i = y_i = 1 \\
1 & \text{otherwise}
\end{cases}
\]

He conjectured that any scheme for the multiphase problem of \( \text{DISJ} \) must have \( \max\{t_p, t_u, \ell_3\} \geq n^{\delta} \) for some constant \( \delta > 0 \).

Pătraşcu shows that such lower bounds on the multiphase problem for \( \text{DISJ} \) would imply polynomial lower bounds for various dynamic data structure problems. For example, such lower bounds would imply that dynamic reachability requires \( t = \Omega(n^{\delta}) \).

Finally, Pătraşcu then defined a 3-player Number-On-Forehead (NOF) communication game, such that lower bounds on this game imply matching lower bounds for the multiphase problem. The game associated with a function \( f : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\} \) is as follows:

1. Alice is given \( x_1, \ldots, x_k \in \{0, 1\}^n \) and \( i \in [k] \), Bob gets \( y \in \{0, 1\}^n \) and \( i \in [k] \) and Charlie gets \( x_1, \ldots, x_k \) and \( y \).
2. Charlie sends a private message of \( \ell_1 \) bits to Bob and then he is silent.
3. Alice and Bob communicate \( \ell_2 \) bits and want to compute \( f(x_i, y) \).

Pătraşcu [19] conjectured that if \( \ell_1 = o(k) \), then \( \ell_2 \) has to be bigger than the communication complexity of \( f \). However, this conjecture turned out to be false. The randomized communication complexity of \( \text{DISJ} \) is \( \Omega(n) \) [20, 11, 3], but Chattopadhyay et al. [7] construct a protocol for \( f = \text{DISJ} \) where both \( \ell_1, \ell_2 = O(\sqrt{n} \cdot \log n) \).

So the above communication model is more powerful than it appears at first glance.

However, a recent paper by Ko and Weinstein [12] succeeds in proving lower bounds for a simpler version of the multiphase problem, which translate to lower bounds for a restricted class of dynamic data structure schemes. They manage to prove a lower bound of \( \Omega(\sqrt{n}) \) for the simpler version of the multiphase problem which is associated with the Disjointness function \( f = \text{DISJ} \). The main contribution of our paper is to generalize their lower bound to any function \( f \) which has large complexity according to the smooth corruption bound, under a product distribution. Disjointness is such a function [2], but so is the Inner Product, Gap Orthogonality, and Gap Hamming Distance [21].

Our proof method is significantly different: Ko and Weinstein use information complexity to derive their lower-bound (similar to [3, 4]), whereas we construct a large nearly-monochromatic rectangle. Our proof is reminiscent of [6], but via a more direct bucketing argument. We furthermore show that this lower bound holds also for randomized schemes, and not just for deterministic schemes.

### 1.1 Semi-adaptive Multiphase Problem

Let us provide rigorous definitions.

**Definition 2** (Semi-adaptive random data structure [12]). Let \( f : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1, *\} \) be a partial function. A scheme \( D = (E, \{U_y\}_{y \in \{0, 1\}^n}, \{Q_i\}_{i \in [k]}) \) for the multiphase problem

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The conjecture remains that if \( \ell_1 = o(k) \), then \( \ell_2 \) has to be larger than the maximum distributional communication complexity of \( f \) under a product distribution. This is \( \tilde{\Omega}(\sqrt{n}) \) for Disjointness [2].
of $f$ is called semi-adaptive if any path on the decision-tree $Q_i : (\{0,1\}^w)^k \times (\{0,1\}^w)^m \rightarrow \{0,1\}$ first queries the first part of the input (the $E(x)$ part), and then queries the second part of the input (the $U(E(x))$ part). If $D$ is randomized, then this property must hold for every randomized choice of $Q_i$.

We point out that the reading of the cells in each part is completely adaptive. The restriction is only that the data structure can not read cells of $E(x)$ if it already started to read cells of $U(E(x))$. Ko and Weinstein state their result for deterministic data structures, i.e., $\varepsilon = 0$ thus the data structure always returns the correct answer.

▶ Theorem 3 (Ko and Weinstein [12]). Let $k \geq \Omega(n)$. Any semi-adaptive deterministic data structure that solves the multiphase problem of the $\text{DISJ}$ function, must have either $t_0 \cdot n \geq \Omega(k/w)$ or $t_0 \geq \Omega(\sqrt{n/w})$.

To prove the lower bound they reduce the semi-adaptive data structure into a low correlation random process.

▶ Theorem 4 (Ko and Weinstein [12]). Let $x_1, \ldots, x_k$ be random variables over $\{0,1\}^n$ and each of them is independently distributed according to the same distribution $\mu_1$ and let $y$ be a random variable over $\{0,1\}^n$ distributed according to $\mu_2$ (independently of $x_1, \ldots, x_k$). Let $D$ be a randomized semi-adaptive scheme for the multiphase problem for a partial function $f : [0,1]^n \times [0,1]^n \rightarrow [0,1,\ast]$ with error probability bounded by $\varepsilon$. Then, for any $p \leq o(k)$ there is a random variable $z \in \{0,1\}^m$ and $i \in [k]$ such that:

1. $\Pr[f(x_i, y) \neq \ast, z_m \neq f(x_i, y)] \leq \varepsilon$.
2. $I(x_i : y \mid z) \leq t_q \cdot w + o(t_q \cdot w)$.
3. $I(y : z) \leq t_q \cdot w$.
4. $I(x_i : y \mid z) \leq O(\frac{\log n \cdot w}{p})$.

Ko and Weinstein [12] proved Theorem 4 for the deterministic schemes for the $\text{DISJ}$ function and in the case where $\mu_1 = \mu_2$. However, their proof actually works for any (partial) function $f$ and for any two, possibly distinct distributions $\mu_1$ and $\mu_2$. Moreover, their proof also works for randomized schemes. The resulting statement for randomized schemes for any function $f$ is what we have given above. To complete the proof of their lower bound, Ko and Weinstein proved that if we set $p$ (and $k$) large enough so that $I(x_i : y \mid z) \leq o(1)$ then such random variable $z$ can not exist when $f$ is the $\text{DISJ}$ function. It is this second step which we generalize.

Let $f : X \times Y \rightarrow \{0,1\}$ be a function and $\mu$ be a distribution over $X \times Y$. A set $R \subseteq X \times Y$ is a rectangle if there exists sets $A \subseteq X$ and $B \subseteq Y$ such that $R = A \times B$. For $b \in \{0,1\}$ and $0 \leq \rho \leq 1$, we say the rectangle $R$ is $\rho$-almost $b$-monochromatic for $f$ under $\mu$ if $\mu(R \cap f^{-1}(1-b)) \leq \rho \cdot \mu(R \cap f^{-1}(b))$. We say the distribution $\mu$ is a product distribution if there are two independent distribution $\mu_1$ over $X$ and $\mu_2$ over $Y$ such that $\mu(x,y) = \mu_1(x) \times \mu_2(y)$. For $0 \leq \alpha \leq \frac{1}{2}$, the distribution $\mu$ is $\alpha$-balanced according to $f$ if $\mu(f^{-1}(0)), \mu(f^{-1}(1)) \geq \alpha$. We will prove that the existence of a random variable $z$ given by Theorem 4 implies that, for any $b \in \{0,1\}$, any balanced product distribution $\mu$ and any function $g$ which is “close” to $f$, there is a large (according to $\mu$) $\rho$-almost $b$-monochromatic rectangle for $g$ in terms of $t_q$. This technique is known as smooth corruption bound [3, 5] or smooth rectangle bound [10]. We denote the smooth corruption bound of $f$ as $\text{scb}^\mu_\lambda$. Informally, $\text{scb}^\mu_\lambda(f) \geq s$ if there is $b \in \{0,1\}$ and a partial function $g : X \times Y \rightarrow \{0,1,\ast\}$ which is close to $f$ such that any $\rho$-almost $b$-monochromatic rectangle $R \subseteq X \times Y$ for $g$ has

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5 “Closeness” is measured by the parameter $\lambda \in \mathbb{R}$, see Section 2.1 for the formal definition.
size (under $\mu$) at most $2^{-n}$. We will define smooth corruption bound formally in the next section. Thus, if we use Theorem 4 as a black box we generalize Theorem 3 for any function of large corruption bound.

**Theorem 5 (Main Result).** Let $\lambda, \bar{\epsilon}, \bar{\alpha} \geq 0$ such that $\alpha \geq 2\epsilon$ for $\epsilon = \bar{\epsilon} + \lambda$, $\alpha = \bar{\alpha} - \lambda$. Let $\mu$ be a product distribution over $\{0,1\}^n \times \{0,1\}^n$ such that $\mu$ is $\bar{\alpha}$-balanced according to a partial function $f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,*\}$. Any semi-adaptive randomized scheme for the multiphase problem of $f$, with error probability bounded by $\bar{\epsilon}$, must have either $t_q \cdot n \geq \Omega(k/w)$, or

$$t_q \cdot w \geq \Omega\left(\alpha \cdot \text{scb}_\mu^{O(c/\alpha)}(f)\right).$$

We point out that $\Omega$ and $O$ in the bound given above hide absolute constants independent of $\alpha, \epsilon$ and $\lambda$.

As a consequence of our main result, and of previously-known bounds on corruption, we will be able to show new lower-bounds of $t_q = \Omega(\frac{k}{w})$ against semi-adaptive schemes for the multiphase problem of the Inner Product, Gap Orthogonality and Gap Hamming Distance functions (where the gap is $\sqrt{n}$). These lower-bounds hold assuming that $t_q = \omega(\frac{k}{nw})$. They follow from the small discrepancy of the Inner Product, and from a bound shown by Sherstov on the corruption of the Gap Orthogonality following by a reduction to the Gap Hamming Distance. This result also gives an alternative proof of the same lower-bound proven by Ko and Weinstein for the Disjointness function, of $t_q = \Omega(\frac{k^2}{nw})$. This follows from the bound on corruption of Disjointness under a product distribution, shown by Babai et al.

The paper is organized as follows. In Section 2 we give important notation, and the basic definitions from information theory and communication complexity. The proof of Theorem 5 appears in Section 3. The various applications appear in Section 4.

## 2 Preliminaries

We use a notational scheme where sets are denoted by uppercase letters, such as $X$ and $Y$, elements of the sets are denoted by the same lowercase letters, such as $x \in X$ and $y \in Y$, and random variables are denoted by the same lowercase boldface letters, such as $x$ and $y$. We will use lowercase greek letters, such as $\mu$, to denote distributions. If $\mu$ is a distribution over a product set, such as $X \times Y \times Z$, and $(x, y, z) \in X \times Y \times Z$, then $\mu(x, y, z)$ is the probability of seeing $(x, y, z)$ under $\mu$. We will sometimes denote $\mu$ by $\mu(x, y, z)$, using non-italicized lowercase letters corresponding to $X \times Y \times Z$. This allows us to use the notation $\mu(x)$ and $\mu(y)$ to denote the $x$- and $y$-marginals of $\mu$, for example; then if we use the same notation with italicized lowercase letters, we get the marginal probabilities, i.e., for each $x \in X$ and $y \in Y$

$$\mu(x) = \sum_{y, z} \mu(x, y, z) \quad \mu(y) = \sum_{z} \mu(x, y, z).$$

If $y \in Y$, then we will also use the notation $\mu(x \mid y)$ to denote the $x$-marginal of $\mu$ conditioned seeing the specific value $y$. Then for each $x \in X$ and $y \in Y$, we have

$$\mu(x \mid y) = \sum_{z} \mu(x, y, z).$$

We will also write $(x, y, z) \sim \mu$ to mean that $(x, y, z)$ are random variables chosen according to the distribution $\mu(x, y, z)$, i.e., for all $(x, y, z) \in X \times Y \times Z$, $\Pr[x = x, y = y, z = z] = \mu(x, y, z)$.
Naturally if \( W \subseteq X \times Y \times Z \), then \( \mu(A) = \sum_{(x,y,z) \in A} \mu(x,y,z) \). We let \( \text{supp}(\mu) \) denote the support of \( \mu \), i.e., the set of \((x, y, z)\) with \( \mu(x, y, z) > 0 \).

We now formally define the smooth corruption bound and related measures from communication complexity, and refer the book by Kushilevitz and Nisan [13] for more details. At the end of this section we provide necessary notions of information theory which are used in the paper, and for more details on these we refer to the book by Cover and Thomas [5].

2.1 Rectangle Measures

Let \( f : X \times Y \to \{0, 1, *\} \) be a partial function and \( \mu(x,y) \) be a distribution over \( X \times Y \). We say that \( f \) is \( \lambda \)-close to a function \( g : X \times Y \to \{0, 1\} \) under \( \mu \) if

\[
\Pr_{(x,y) \sim \mu}[f(x,y) \neq g(x,y)] \leq \lambda.
\]

Let

\[
\mathcal{R}_\rho^b = \{ R \subseteq X \times Y \text{ rectangle } | \mu(R \cap f^{-1}(1-b)) \leq \rho \cdot \mu(R \cap f^{-1}(b)) \}
\]

be the set of \( \rho \)-almost \( b \)-monochromatic rectangles for \( f \) under \( \mu \). The complexity measure \( \text{mono} \) quantifies how large almost \( b \)-monochromatic rectangles can be [5]:

\[
\text{mono}_\rho^b(f) = \min_{b \in \{0, 1\}} \max_{R \in \mathcal{R}_\rho^b} \mu(R)
\]

Using \( \text{mono} \) we can define the corruption bound of a function as \( \text{cb}_\rho^b(f) = \log \frac{1}{\text{mono}_\rho^b(f)} \) and the smooth corruption bound as

\[
\text{scb}_\rho^b(f) = \max_{\lambda : \lambda \text{-close to } f \text{ under } \mu} \text{cb}_\rho^b(g).
\]

Thus, if \( \text{scb}_\rho^b(f) \geq s \) then there is a \( b \in \{0,1\} \) and a function \( g \) which \( \lambda \)-close to \( f \) under \( \mu \) such that for any \( \rho \)-almost \( b \)-monochromatic rectangle for \( g \) under \( \mu \) it holds that \( \mu(R) \leq 2^{-s} \).

The notion \( \text{mono}_\rho^b \) is related to the discrepancy of a function:

\[
\text{disc}_\mu(f) = \max_{R : \text{rectangle of } X \times Y} \left| \mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1)) \right|.
\]

It is easy to see that for a total function \( f \) holds that \( \text{disc}_\mu(f) \geq (1 - 2\rho) \cdot \text{mono}_\rho^b(f) \) for any \( \rho \). Thus, Theorem [5] will give us lower bounds also for functions of small discrepancy.

2.2 Information Theory

We define several measures from information theory. If \( \mu'(z), \mu(z) \) are two distributions such that \( \text{supp}(\mu') \subseteq \text{supp}(\mu) \), then the Kullback-Leibler divergence of \( \mu' \) from \( \mu \) is

\[
D_{KL}(\mu' \parallel \mu) = \sum_z \mu'(z) \log \frac{\mu'(z)}{\mu(z)}.
\]

With Kullback-Leibler divergence we can define the mutual information, which measures how close (according to KL divergence) is a joint distribution to the product of its marginals. If we have two random variables \((x, y) \sim \mu(x,y)\), then we define their mutual information to be

\[
I(x : y) = D_{KL}(\mu(x,y) \parallel \mu(x) \times \mu(y)) = \mathbb{E}_{y \sim \mu(y)} \left[ D_{KL}(\mu(x \mid y) \parallel \mu(x)) \right].
\]
If we have three random variables \((x, y, z) \sim \mu(x, y, z)\), then the mutual information of \(x\) and \(y\) conditioned by \(z\) is

\[
I(x : y | z) = \mathbb{E}_{z \sim \mu(z)} \left[ I(x : y | z = z) \right] = \mathbb{E}_{z \sim \mu(z)} \left[ \text{D}_{\text{KL}}(\mu(x, y | z) \| \mu(x | z) \times \mu(y | z)) \right]
\]

We present several facts about the mutual information, the proofs can be found in the book of Cover and Thomas [5].

> **Fact 6** (Chain Rule). For any random variables \(x_1, x_2, y\) and \(z\) holds that

\[
I(x_1 x_2 : y | z) = I(x_1 : y | z) + I(x_2 : y | z, x_1).
\]

Since mutual information is never negative, we have the following corollary.

> **Corollary 7.** For any random variables \(x, y\) and \(z\) holds that \(I(x : y) \leq I(x : y z)\).

The \(\ell_1\)-distance between two distributions is defined as

\[
\|\mu'(z) - \mu(z)\|_1 = \sum_z |\mu'(z) - \mu(z)|.
\]

There is a relation between \(\ell_1\)-distance and Kullback-Leibler divergence.

> **Fact 8** (Pinsker’s Inequality). For any two distributions \(\mu'(z)\) and \(\mu(z)\), we have

\[
\|\mu'(z) - \mu(z)\|_1 \leq \sqrt{2 \cdot \text{D}_{\text{KL}}(\mu'(z) \| \mu(z))}
\]

### 3 The Proof of Theorem 5

Let \(f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,\ast\}\) be a partial function. Suppose there is a semi-adaptive random scheme \(D\) for the multiphase problem of \(f\) with error probability bounded by \(\epsilon\) such that \(t_a \cdot n \leq o(k/w)\). Let \(\mu(x, y) = \mu_1(x) \times \mu_2(y)\) be a product distribution over \(\{0,1\}^n \times \{0,1\}^n\), such that \(\mu(x, y)\) is \(\tilde{\alpha}\)-balanced according to \(f\). Let \(b \in \{0,1\}\) and \(g : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,\ast\}\) be a partial function which is \(\lambda\)-close to \(f\) under \(\mu\). We will prove there is a large almost \(b\)-monochromatic rectangle for \(g\).

Let \(x_1, \ldots, x_k\) be independent random variables each of them distributed according to \(\mu_1\) and \(y\) be an independent random variable distributed according to \(\mu_2\). Let the random variable \(z \in \{0,1\}^m\) and the index \(i \in [k]\) be given by Theorem 3 applied to the random variables \(x_1, \ldots, x_k, y\) and the function \(f\). For simplicity we denote \(x = x_i\).

We will denote the joint distribution of \((x_1, \ldots, x_k, y, z)\) by \(\mu(x_1, \ldots, x_k, y, z)\). Note that here the notation is consistent, in the sense that \(\mu(x_i, y) = \mu_1(x_i) \times \mu_2(y)\) for all \(i \in [k]\). We will then need to keep in mind that \(\mu(z)\) is the \(z\)-marginal of the joint distribution of \((x_1, \ldots, x_k, y, z)\).

By \(f(x, y) \neq \ast\) \(z_m\) we denote the event that the random variable \(z_m\) gives us the wrong answer on an input from the support of \(f\), i.e. \(f(x, y) \neq \ast\) and \(f(x, y) \neq z_m\) hold simultaneously. By Theorem 3 we know that \(\Pr \left[ f(x, y) \neq \ast \right] \leq \epsilon\). Since \(f\) and \(g\) are \(\lambda\)-close under \(\mu\), we have that \(\mu\) is still balanced according to \(g\) and \(g(x, y) \neq \ast\) \(z_m\) with small probability, as stated in the next observation.

> **Observation 9.** Let \(\alpha = \tilde{\alpha} - \lambda\) and \(\epsilon = \tilde{\epsilon} + \lambda\). For the function \(g\) it holds that

1. The distribution \(\mu(x, y)\) is \(\alpha\)-balanced according to \(g\).
2. \(\Pr \left[ g(x, y) \neq \ast \right] \leq \epsilon\).
Proof. Let \( b' \in \{0, 1\} \). We will bound \( \mu(g^{-1}(b')) \).

\[
\tilde{\alpha} \leq \Pr[f(x, y) = b'] = \Pr[f(x, y) = b', f(x, y) = g(x, y)] + \Pr[f(x, y) = b', f(x, y) \neq g(x, y)] \\
\leq \Pr[g(x, y) = b'] + \lambda.
\]

Thus, by rearranging we get \( \mu(g^{-1}(b')) \geq \tilde{\alpha} - \lambda = \alpha \). The proof of the second bound is similar:

\[
\Pr[g(x, y) \neq z_m] = \Pr[f(x, y) \neq z_m, f(x, y) = g(x, y)] + \Pr[g(x, y) \neq z_m, f(x, y) \neq g(x, y)] \leq \tilde{\alpha} + \lambda = \varepsilon.
\]

Let \( c \) be the bound on \( I(x : y z) \) and \( I(y : z) \) given by Theorem 4. Since \( I(x : z) \leq I(x : y z) \), we have \( I(x : z), I(y : z) \leq t_q \cdot w + o(t_q \cdot w) = c \). We will prove that if we assume that \( t_q \cdot n < o(k/w) \) and we choose \( p \) large enough (\( p \) of Theorem 4) then we can find a rectangle \( R \subseteq X \times Y \) such that \( R = O(\varepsilon/\alpha) \)-almost\-b-monochromatic for \( g \) and \( \mu(R) \geq \frac{1}{2^c} \) for \( c' = O(\frac{t_q \cdot w}{\alpha}) \). Thus, we have \( \text{mono}_{\mu}^{O(\varepsilon/\alpha)}(g) \geq 2^{-c'} \) and consequently

\[
\text{scb}_{\mu}^{O(\varepsilon/\alpha), \lambda}(f) \leq O \left( \frac{t_q \cdot w}{\alpha} \right).
\]

By rearranging, we get the bound from Theorem 5.

Let us sketch the proof of how we can find such a rectangle \( R \). We will first fix the random variable \( z \) to \( z \) such that \( x \) and \( y \) are not very correlated conditioned on \( z = z \), i.e., the joint distribution \( \mu(x, y \mid z) \) is very similar to the product distribution of the marginals \( \mu(x \mid z) \times \mu(y \mid z) \). Moreover, we will pick \( z \) in such a way the probability of error \( \Pr[g(x, y) \neq z_m \mid z = z] \) is still small. Then, since \( \mu(x, y \mid z) \) is close to \( \mu(x,z) \times \mu(y \mid z) \), the probability of error under the latter distribution will be small as well, i.e., if \( (x', y') \sim \mu(x \mid z) \times \mu(y \mid z) \), then \( \Pr[g(x', y') \neq z_m] \) will also be small. Finally, we will find subsets \( A \subseteq \text{supp}(\mu(x \mid z)), B \subseteq \text{supp}(\mu(y \mid z)) \) of large mass (under the original distributions \( \mu_1 \) and \( \mu_2 \)), while keeping the probability of error on the rectangle \( R = A \times B \) sufficiently small.

Let us then proceed to implement this plan. Let \( \beta = \alpha - \varepsilon \). We will show that \( \beta \) is a lower bound for the probability that \( z_m = \) \( b \) given by Theorem 4, i.e., \( I(x : y \mid z) \leq \gamma = O \left( \frac{t_q \cdot w}{p} \right) \).

\textbf{Lemma 10. There exists } z \text{ in } Z \text{ such that}

1. \( z_m = b \).
2. \( I(x : y \mid z) = z = \frac{5}{3} \cdot \gamma \).
3. \( D_{\text{KL}}(\mu(x \mid z) || \mu(x)), D_{\text{KL}}(\mu(y \mid z) || \mu(y)) \leq \frac{6}{3} \cdot c \).
4. \( \Pr[g(x, y) \neq z_m \mid z = z] \leq \frac{5}{3} \cdot \varepsilon \).

\textbf{Proof.} Note that

\[
\alpha \leq \Pr[g(x, y) = b] = \Pr[g(x, y) = b, z_m = b] + \Pr[g(x, y) = b, z_m \neq b] \leq \Pr[z_m = b] + \varepsilon.
\]

Thus, by rearranging we get \( \Pr[z_m = b] \geq \alpha - \varepsilon = \beta \). By expanding the information \( I(x : y \mid z) \) we find

\[
\gamma \geq I(x : y \mid z) = \mathbb{E}_{z \sim \mu(x)} \left[ I(x : y \mid z = z) \right]
\]
and by the Markov inequality we get that
\[
\Pr_{z \sim \mu(z)} \left[ I(x : y \mid z = z) \geq \frac{5}{\beta} \cdot \gamma \right] \leq \frac{\beta}{5}.
\]
Similarly, for the information \( I(x : z) \):
\[
c \geq I(x y : z) \geq I(x : z) = \mathbb{E}_{z \sim \mu(z)} \left[ D_{\text{KL}}(\mu(x \mid z) \parallel \mu(x)) \right]
\]
and so
\[
\Pr_{z \sim \mu(z)} \left[ D_{\text{KL}}(\mu(x \mid z) \parallel \mu(x)) \geq \frac{5}{\beta} \cdot e \right] \leq \frac{\beta}{5}.
\]
The bound for \( I(y : z) \) is analogous. Let \( e_z = \Pr_{\mu} [g(x, y) \neq \ast z_m \mid z = z] \). Then,
\[
e \geq \Pr_{\mu} [g(x, y) \neq \ast z_m] = \sum_{z \in \mathcal{Z}} \mu(z) \cdot e_z = \mathbb{E}_{z \sim \mu(z)} [e_z]
\]
\[
\Pr_{z \sim \mu(z)} \left[ e_z \geq \frac{5}{\beta} \cdot e \right] \leq \frac{\beta}{5}.
\]
Thus, by a union bound we may infer the existence of the sought \( z \in \mathcal{Z} \).

Let us now fix \( z \in \mathcal{Z} \) from the previous lemma. Let \( \mu_z(x, y) = \mu(x, y \mid z) \) be the distribution \( \mu(x, y) \) conditioned on \( z = z \), and let \( \mu'_z(x, y) = \mu(x \mid z) \times \mu(y \mid z) \) be the product of its marginals. Let \( S \) be the support of \( \mu_z(x, y) \), and let \( S_x \) and \( S_y \) be the supports of \( \mu'_z(x) \) and \( \mu'_z(y) \), respectively, i.e., \( S_x \) and \( S_y \) are the projections of \( S \) into \( X \) and \( Y \).

Then Pinsker’s inequality will give us that \( \mu_z \) and \( \mu'_z \) are very close. Let \( \delta = \sqrt{\frac{10}{\beta}} \cdot \gamma \).

\[\begin{align*}
\textbf{Lemma 11.} & \quad \|\mu_z(x, y) - \mu'_z(x, y)\|_1 \leq \delta \\
\textbf{Proof.} & \quad \text{Indeed, by Pinsker’s inequality,} \\
& \quad \|\mu_z(x, y) - \mu'_z(x, y)\|_1 \leq \sqrt{2 \cdot D_{\text{KL}}(\mu_z(x, y) \parallel \mu'_z(x, y)).}
\end{align*}\]

The right-hand side is \( \sqrt{2 \cdot D_{\text{KL}}(\mu(x, y \mid z) \parallel \mu(x \mid z) \times \mu(y \mid z))} \), which by definition of mutual information equals \( \sqrt{2 \cdot I(x : y \mid z = z)} \), and by Lemma 10 this is \( \leq \sqrt{\frac{10}{\beta}} \cdot \gamma = \delta \).

For the sake of reasoning, let \((x', y') \sim \mu'_z(x, y)\) be random variables chosen according to \( \mu'_z \). Let \( \epsilon' = \frac{5}{\beta} \cdot \epsilon + \delta \). It then follows from Lemma 10 and Lemma 11 that:

\[\begin{align*}
\textbf{Lemma 12.} & \quad \Pr [g(x', y') \neq \ast z_m] \leq \epsilon'.
\end{align*}\]

\[\begin{align*}
\textbf{Proof.} & \quad \text{We prove that} \\
& \quad \left| \Pr [g(x, y) \neq \ast z_m \mid z = z] - \Pr [g(x', y') \neq \ast z_m] \right| \leq \delta.
\end{align*}\]

Since \( \Pr [g(x, y) \neq \ast z_m \mid z = z] \leq \frac{5}{\beta} \cdot \epsilon \) by Lemma 10 the lemma follows. Let
\[B = \{(x, y) \in S_x \times S_y : g(x, y) \neq z_m, g(x, y) \neq \ast \} .\]
Thus, we have the following.

\[
\left| \Pr [g(x, y) \neq z_m \mid z = z] - \Pr [g(x', y') \neq z_m] \right| \\
= \left| \sum_{(x, y) \in B} \mu_z(x, y) - \mu'_z(x, y) \right| \\
\leq \sum_{(x, y) \in B} \left| \mu_z(x, y) - \mu'_z(x, y) \right| \leq \delta \quad \text{by triangle inequality and Lemma 11}
\]

Let \( c' = \frac{5}{\delta} \cdot c \). We will prove the ratio between \( \mu'_z(x') \) and \( \mu(x') \) is larger than \( 2^{O(c')} \) with only small probability (when \( x' \sim \mu'_z(x) \)). The same holds for \( \mu'_z(y') \) and \( \mu(y') \).

\begin{lemma}
\[ \Pr \left[ \mu'_z(x') \geq 2^{6c'} \cdot \mu(x') \right], \Pr \left[ \mu'_z(y') \geq 2^{6c'} \cdot \mu(y') \right] \leq \frac{1}{2}. \]
\end{lemma}

\begin{proof}
We prove the lemma for \( \mu'_z(x') \), the proof for \( \mu'_z(y') \) is analogous. By Lemma 10 we know that \( D_{KL}(\mu(x) \mid z) = D_{KL}(\mu_z(x) \mid \mu(x)) = D_{KL}(\mu'_z(x) \mid \mu(x)) \leq c' \). We expand the Kullback-Leibler divergence:

\[
c' \geq D_{KL}(\mu'_z(x) \mid \mu(x)) = \sum_{x \in S_x} \mu'_z(x) \log \frac{\mu'_z(x)}{\mu(x)} = \mathbb{E} \left[ \log \frac{\mu'_z(x')}{\mu(x')} \right],
\]

and then use the Markov inequality:

\[
\Pr \left[ \mu'_z(x') \geq 2^{6c'} \cdot \mu(x') \right] = \Pr \left[ \log \frac{\mu'_z(x')}{\mu(x')} \geq 6c' \right] \leq \frac{1}{6}. \quad \Box
\]

We now split \( S_x \) and \( S_y \) into buckets \( C^x_1 \) and \( C^y_1 \) (for \( \ell \geq 1 \)), where the \( \ell \)-th buckets are

\[
C^x_\ell = \left\{ x \in S_x \mid \frac{(\ell - 1) \cdot \mu(x)}{2c'} < \mu'_z(x) \leq \frac{\ell \cdot \mu(x)}{2c'} \right\},
\]

\[
C^y_\ell = \left\{ y \in S_y \mid \frac{(\ell - 1) \cdot \mu(y)}{2c'} < \mu'_z(y) \leq \frac{\ell \cdot \mu(y)}{2c'} \right\}.
\]

In a bucket \( C^x_\ell \) there are elements of \( S_x \) such that their probability under \( \mu'_z(x) \) is approximately \( \frac{1}{2c} \)-times bigger than their probability under \( \mu(x) \). By Lemma 13 it holds that with high probability the elements \( x \in S_x, y \in S_y \) are in the buckets \( C^x_\ell \) and \( C^y_\ell \) for \( \ell \leq 2^{7c'} \). Thus, if we find a bucket \( C^x_\ell \) for \( \ell_1 \leq 2^{7c'} \) which has probability at least \( \frac{1}{2^{2c'} \ell_1} \) under \( \mu'_z(x) \), then it has also probability at least \( \frac{1}{2^{2c'} \ell_2} \) under \( \mu(x) \). The same holds also for buckets \( C^y_\ell \). In the next lemma we will show that there are buckets \( C^x_{\ell_1} \) and \( C^y_{\ell_2} \) of large probability under \( \mu'_z \) such that the probability of error on \( C^x_{\ell_1} \times C^y_{\ell_2} \) is still small.

\begin{lemma}
There exist buckets \( C^x_{\ell_1} \) and \( C^y_{\ell_2} \) such that
\begin{enumerate}
\item \( 1 < \ell_1, \ell_2 \leq 2^{7c'} \).
\item \( \Pr [x' \in C^x_{\ell_1}] \cdot \Pr [y' \in C^y_{\ell_2}] \geq \frac{1}{2^{2c'} \ell_1} \).
\item \( \Pr [g(x', y') \neq z_m, (x', y') \in C^x_{\ell_1} \times C^y_{\ell_2}] \leq 6c' \cdot \Pr [(x', y') \in C^x_{\ell_1} \times C^y_{\ell_2}] \).
\end{enumerate}
\end{lemma}

\begin{proof}
We prove that \( \ell_1, \ell_2 \) exist via the probabilistic method. Let \( \ell_1 \) and \( \ell_2 \) be the buckets of \( x' \) and \( y' \), respectively. Thus \( \Pr [\ell_1 = \ell] = \Pr [x' \in C^x_{\ell}] \) and \( \Pr [\ell_2 = \ell] = \Pr [y' \in C^y_{\ell}] \).

Let \( B_1, B_2 \subseteq L' = \{1, \ldots, 2^{7c'}\} \) be sets of indices of small probability, i.e., for \( i \in \{1, 2\} \)

\[
B_i = \left\{ \ell \in L' \mid \Pr [\ell = \ell] \leq \frac{1}{6 \cdot 2^{7c'}} \right\}.
\]
We will prove that with high probability we have $2^{7c'} \geq \ell_1 > 1$ and $\ell_1 \not\in B_1$. The proof for $\ell_2$ is analogous.

$$\Pr[\ell_1 = 1] = \Pr[x' \in C_y^\ell] = \sum_{x \in C_y^\ell} \mu'_x(x) \leq \sum_{x \in C_y^\ell} \frac{\mu(x)}{2^{c'}} \leq \frac{1}{2^{c'}}$$

By Lemma 13 we get $\Pr[\ell_1 > 2^{7c'}] = \Pr[\mu'_x(x') \geq 2^{6c'} \cdot \mu(x')] \leq \frac{1}{6}$. There is only small probability that $\ell_1$ is in $B_1$.

$$\Pr[\ell_1 \in B_1] = \sum_{\ell \in B_1} \Pr[\ell_1 = \ell] \leq \frac{|L'|}{6 \cdot 2^{7c'}} = \frac{1}{6}$$

Thus, we have that $\ell_1 \in B_1$ or $\ell_1 = 1$ or $\ell_1 > 2^{7c'}$ with probability at most $\frac{2}{3} + \frac{2}{3}$.

By Lemma 12 we have that $\Pr[g(x', y') \neq z_m] \leq \epsilon'$. By expanding the probability and by Markov inequality we will now get the last inequality for $C_{\ell_1}^x$ and $C_{\ell_2}^y$. Let

$$e(\ell_1, \ell_2) = \Pr[g(x', y') \neq z_m \mid x' \in C_{\ell_1}^x, y' \in C_{\ell_2}^y].$$

We will prove there is $\ell_1$ and $\ell_2$ such that $e(\ell_1, \ell_2) \leq 6\epsilon'$. This is equivalent to the third bound of the lemma. We have $\epsilon' \geq \Pr[g(x', y') \neq z_m] = \mathbb{E}[e(\ell_1, \ell_2)]$ and thus, by Markov, $\Pr[e(\ell_1, \ell_2) \leq 6\epsilon'] \leq \frac{1}{6}$. By a union bound we conclude that there must exist $1 < \ell_1, \ell_2 \leq 2^{7c'}$ such that $\Pr[\ell_1 = \ell_1, \ell_2 = \ell_2] \geq \frac{1}{6 \cdot 2^{7c'}}$ and $e(\ell_1, \ell_2) \leq 6\epsilon'$.

As a corollary we will prove that the rectangle $C_{\ell_1}^x \times C_{\ell_2}^y$ (given by the previous lemma) is a good rectangle under the original distribution $\mu$.

**Corollary 15.** There exists a rectangle $R \subseteq S_x \times S_y$ such that

1. $\Pr[(x, y) \in R] \geq \frac{1}{6 \cdot 2^{13c'}}$.
2. $\Pr[g(x, y) \neq z_m, (x, y) \in R] \leq 24\epsilon' \cdot \Pr[(x, y) \in R]$.

**Proof.** Let $R = C_{\ell_1}^x \times C_{\ell_2}^y$ where $C_{\ell_1}^x$ and $C_{\ell_2}^y$ are buckets given by Lemma 14. By Lemma 14 we get

$$\frac{1}{6 \cdot 2^{7c'}} \leq \Pr[x' \in C_{\ell_1}^x] = \sum_{x \in C_{\ell_1}^x} \mu'_x(x) \leq \sum_{x \in C_{\ell_1}^x} \frac{\ell_1 \cdot \mu(x)}{2^{c'}} = \Pr[x \in C_{\ell_1}^x] \cdot \frac{\ell_1}{2^{c'}}.$$

By rearranging we get

$$\Pr[x \in C_{\ell_1}^x] \geq \frac{2^{c'}}{6 \ell_1 \cdot 2^{7c'}} \geq \frac{1}{6 \cdot 2^{13c'}}.$$

The bound for $\Pr[y \in C_{\ell_2}^y]$ is analogous, thus we have $\Pr[(x, y) \in R] \geq \frac{1}{36 \cdot 2^{13c'}}$. (Here and below, we crucially use the fact that $x, y$ are given by a product distribution.) Now we prove the second bound for $R$. Let $B = \{(x, y) \in R : g(x, y) \neq z_m, g(x, y) \neq *\}$.

$$6\epsilon' \cdot \Pr[(x, y) \in R] \cdot \frac{\ell_1 \ell_2}{2^{2c'}} \geq 6\epsilon' \cdot \Pr[(x', y') \in R] \geq \Pr[(x', y') \in B] \quad \text{by definition of buckets}$$

$$\geq \Pr[(x', y') \in B] \geq \Pr[(x, y) \in B] \cdot \frac{(\ell_1 - 1)(\ell_2 - 1)}{2^{2c'}} \quad \text{by definition of buckets}$$

Thus, by rearranging we get

$$\Pr[(x, y) \in B] \leq 6\epsilon' \cdot \Pr[(x, y) \in R] \cdot \frac{\ell_1 \ell_2}{(\ell_1 - 1)(\ell_2 - 1)} \leq 24\epsilon' \cdot \Pr[(x, y) \in R],$$

as $\frac{\ell_1 \ell_2}{(\ell_1 - 1)(\ell_2 - 1)} \leq 4$ for $\ell_1, \ell_2 > 1$ by Lemma 14. □
Proof of Theorem 5. Suppose that \( t_u \cdot n \leq o(k/w) \). Let \( R \) be the rectangle given by Corollary 15. It holds that the rectangle \( R \) is \( 24\varepsilon' \)-almost \( b \)-monochromatic for \( g \) under \( \mu \).

Therefore, for the function \( g \) holds that

\[
\text{mono}^{24\varepsilon'}_{\mu}(g) \geq \Pr[(x, y) \in R] \geq \frac{1}{36 \cdot 2^{26\varepsilon'}}.
\]

We need to argue that \( \varepsilon' \) is \( O(\varepsilon/\alpha) \). By definition,

\[
\varepsilon' = \frac{5}{\alpha - \epsilon} \cdot \epsilon + \delta.
\]

We recall that

\[
\delta = O\left(\sqrt{\frac{t_u \cdot n \cdot w}{p}}\right) \leq \sqrt{\frac{o(k)}{p}}.
\]

Thus, we can set \( p \) to be large enough so that \( \delta \) be smaller than arbitrary constant and still \( p \leq o(k) \). By the assumption we have \( 2\varepsilon < \alpha \). Thus, \( \frac{\varepsilon}{\alpha - \varepsilon} \leq \frac{\varepsilon}{\alpha} \) and we conclude that \( \varepsilon' \) is \( O(\varepsilon/\alpha) \).

Since \( c' = O\left(\frac{t_u \cdot n \cdot w}{\alpha (1 - \varepsilon)}\right) \), we get the result by rearranging Inequality (1). ▶

4 Applications

In this section we apply Theorem 5 to derive lower bounds for several explicit functions – Inner Product (\( \text{IP} \)), Disjointness (\( \text{DISJ} \)), Gap Orthogonality (\( \text{ORT} \)) and Gap Hamming Distance (\( \text{GHD} \)):

\[
\text{IP}(x, y) = \sum_{i \in [n]} x_i \cdot y_i \mod 2,
\]

\[
\text{GHD}_n(x, y) = \begin{cases} 1 & \text{if } \Delta_H(x, y) \geq \frac{n}{2} + \sqrt{n}, \\ 0 & \text{if } \Delta_H(x, y) \leq \frac{n}{2} - \sqrt{n}. \end{cases}
\]

The function \( \Delta_H \) is the Hamming Distance of two strings, i.e., \( \Delta_H(x, y) \) is a number of indices \( i \in [n] \) such that \( x_i \neq y_i \). For \( \text{IP}_z(x, y) = \sum_{i \in [n]} (-1)^{x_i + y_i} \) we define

\[
\text{ORT}_{n,d}(x, y) = \begin{cases} 1 & \text{if } |\text{IP}_z(x, y)| \geq 2d \cdot \sqrt{n} \\ 0 & \text{if } |\text{IP}_z(x, y)| \leq d \cdot \sqrt{n}. \end{cases}
\]

The standard value for \( d \) is 1, thus we denote \( \text{ORT}_n = \text{ORT}_{n,1} \). Note that \( \Delta_H(x, y) = \frac{n - \text{IP}_z(x, y)}{2} \) and \( \text{IP}_z(x, y) \) is the Inner Product of \( x', y' \) over \( \mathbb{R} \) where \( x' \) and \( y' \) arise from \( x \) and \( y \) by replacing 0 by 1 and 1 by \(-1\). We present previous results with bounds for measures of interest under hard distributions.

\begin{itemize}
\item \textbf{Theorem 16 (\cite{13}).} Let \( \mu_1 \) be a uniform distribution on \( \{0, 1\}^n \times \{0, 1\}^n \). Then,

\[
\text{disc}_{\mu_1}(\text{IP}) \leq \frac{1}{2^n/2}.
\]

\item \textbf{Theorem 17 (Babai et al. \cite{2}).} Let \( \rho < 1/100 \) and \( \mu_2 \) be a a uniform distribution over \( S \times S \), where \( S \) consists of \( n \)-bit strings containing exactly \( \sqrt{n} \) 1’s. Then,

\[
\text{mono}^{(\rho)}_{\mu_2}(\text{DISJ}) \leq \frac{1}{2^{\Omega(\sqrt{n})}}.
\]
\end{itemize}
Sherstov [21] provided a lower bound of communication complexity of GHD by lower bound of corruption bound of $\text{ORT}_{n, \frac{1}{8}}$ following by reduction to GHD.

**Theorem 18 (Sherstov [21]).** Let $\rho > 0$ be sufficiently small and $\mu_3$ be a uniform distribution over $\{0, 1\}^n \times \{0, 1\}^n$. Then,

$$\text{cb}^\rho_\mu (\text{ORT}_{n, \frac{1}{8}}) \geq \rho \cdot n.$$  

By this theorem and Theorem 5 we get a lower bound for data structures for $\text{ORT}_{n, \frac{1}{8}}$. By reductions used by Sherstov [21] we also get a lower bounds for $\text{ORT}$ and GHD.

\[
\begin{align*}
\text{ORT}_{n, \frac{1}{8}}(x, y) &= \text{ORT}_{64n}(x^{64}, y^{64}) \\
\text{ORT}_n(x, y) &= \text{GHD}_{10n+15\sqrt{n}}(x^{10^{15}\sqrt{n}}, y^{10^{15}\sqrt{n}}) \\
&\quad \land \neg \text{GHD}_{10n+15\sqrt{n}}(x^{10^{15}\sqrt{n}}, y^{10^{15}\sqrt{n}})
\end{align*}
\]

Where $s^i$ denote $i$ copies of $s$ concatenated together. Let $D$ be a semi-adaptive random scheme for the multiphase problem of the presented functions with sufficiently small error probability. By the theorems presented in this section and by Theorem 5 we can derive the following lower bounds for $t_q \cdot w$, assuming that $t_u \cdot n \leq o(k/w)$.

| Function $f$ | Ballancedness of the hard distribution | Lower bound of $t_q \cdot w$ |
|-------------|--------------------------------------|-----------------------------|
| IP          | $\frac{1}{2}$                        | $\Omega(n)$                |
| DISJ        | $\sim \frac{1}{3}$                  | $\Omega(\sqrt{n})$         |
| ORT$_n$     | $\Theta(1)$                         | $\Omega(n)$                |
| GHD$_n$     | N/A (lower bound is via reduction)   | $\Omega(n)$                |

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