Sums of finitely many distinct reciprocals

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Dedicated to Erdős Pál 1913-1996

Abstract

\( \mathcal{F} \) denotes the family of all finite nonempty \( S \subseteq \mathbb{N} := \{1, 2, \ldots, \} \), and \( \mathcal{F}(X) := \mathcal{F} \cap \{S : S \subseteq X\} \) when \( X \subseteq \mathbb{N} \). This paper treats the function \( \sigma : \mathcal{F} \to \mathbb{Q}^+ \) given by \( \sigma : S \mapsto \sigma S := \sum \{1/x : x \in S\} \), and the function \( \delta : \mathcal{F} \to \mathbb{N} \) defined by \( \sigma S = \nu S/\delta S \) where the integers \( \nu S \) and \( \delta S \) are coprime.

**Theorem 1.1.** For each \( r \in \mathbb{Q}^+ \), there exists an infinite pairwise disjoint subfamily \( \mathcal{H}_r \subseteq \mathcal{F} \) such that \( r = \sigma S \) for all \( S \in \mathcal{H}_r \).

**Theorem 1.2.** Let \( X \) be a pairwise coprime set of positive integers. Then \( \sigma | \mathcal{F}(X) \) and \( \delta | \mathcal{F}(X) \) are injections. Also, \( \sigma C \in \mathbb{N} \) for \( C \in \mathcal{F}(X) \) only if \( C = \{1\} \).

1 Introduction

\( \mathcal{F} \) denotes the family of all nonempty finite subsets \( S \subseteq \mathbb{N} := \{1, 2, 3, \ldots, \} \), and \( \mathcal{I} \) denotes its subfamily of finite intervals \( \{m, n\} := \{m, m+1, \ldots, n-1, n\} \) of consecutive integers. The set of positive rational numbers is written \( \mathbb{Q}^+ \). The present paper is devoted principally to the function \( \sigma : \mathcal{F} \to \mathbb{Q}^+ \) defined by

\[
\sigma : S \mapsto \sigma S \quad \text{where} \quad \sigma S := \sum_{x \in S} \frac{1}{x}.
\]

For \( r \in \mathbb{Q}^+ \), the expression \( \mathcal{F}_r \) denotes the family of all finite \( S \subseteq \mathbb{N} \) for which \( r = \sigma S \).

**Theorem 1.1.** For each \( r \in \mathbb{Q}^+ \), there exists an infinite pairwise disjoint subfamily \( \mathcal{H}_r \subseteq \mathcal{F}_r \).

The function \( \sigma \) induces two other functions, \( \nu : \mathcal{F} \to \mathbb{N} \) and \( \delta : \mathcal{F} \to \mathbb{N} \), via the fact that for each \( S \in \mathcal{F} \) there is a unique coprime pair \( (\nu S, \delta S) \) of positive integers for which \( \sigma S = \nu S/\delta S \). We discuss both \( \delta \) and \( \sigma \).

When \( X \subseteq \mathbb{N} \) then \( \mathcal{F}(X) := \mathcal{F} \cap \{S : S \subseteq X\} \). Thus, e.g., \( \mathcal{F}(\mathbb{N}) = \mathcal{F} \).

**Theorem 1.2.** Let \( X \) be a pairwise coprime subset of \( \mathbb{N} \). Then \( \sigma | \mathcal{F}(X) \) and \( \delta | \mathcal{F}(X) \) are injections. Also, \( \sigma C \in \mathbb{N} \) for \( C \in \mathcal{F}(X) \) only if \( C = \{1\} \).

Our work grew from our interest in the set \( \sigma [\mathcal{I}] \) of “harmonic rationals”, by which people mean the numbers that occur as sums of finite segments of the harmonic series

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j-1} + \frac{1}{j} + \frac{1}{j+1} + \cdots = \infty.
\]

It is well known and easy to see that \( \sigma [\mathcal{I}] \) is dense in \( \mathbb{R}^+ \), but \( \sigma [\mathcal{I}] \neq \mathbb{Q}^+ \) is true as well. Indeed, L. Theisinger [11] proved in 1915 that \( \sigma [1, n] \in \mathbb{N} \) only if \( n = 1 \). In 1918 J. Kürschák [7] proved that \( \sigma [m, n] \in \mathbb{N} \) only if \( m = n = 1 \). The latter fact is recalled, for instance, as Exercise 3 on Page 7 of [1].

Other natural subfamilies of \( S \in \mathcal{F} \) for which \( \sigma S \notin \mathbb{N} \) were noted later. P. Erdős [4], also Page 157 of [6], extended the Theisinger-Kürschák theorem to the finite segments of an arithmetic series:

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If \( d \geq 1 \), and if either \( m > 1 \) or \( k > 1 \), then \( \sum_{j=0}^{k-1} \frac{1}{m + dj} \notin \mathbb{N} \).

Erdős’ result was carried further by H. Belbachir and A. Khelladi \[2\]:

For \( \{ a_0, a_1, \ldots, a_{k-1} \} \subseteq \mathbb{N} \), if \( d \geq 1 \), and if either \( m > 1 \) or \( k > 1 \), then \( \sum_{j=0}^{k-1} \frac{1}{(m + dj)^n} \notin \mathbb{N} \).

According to Erdős \[4\], looking beyond sums of distinct reciprocals R. Obláth showed that the sum of Obláth fails to be an integer provided only that his \( a_i \) are odd whenever \( i \) is even.

Every Theisinger-Kürschák sort of result we mentioned specifies a case where \( \sigma S \notin \mathbb{N} \). Eventually we branched off into a side topic, which led to our rediscovering a result published \[5\] in 1946:

**Theorem (Erdős-Niven)** The function \( \sigma|I \) is injective.

Our reinvention of this Erdős-Niven wheel resulted in machinery that provoked us to consider an analogous surjectivity question; to wit: Is Range(\( \sigma \)) = \( \mathbb{Q}^+ \)? Theorem \[1.1\] answers this in the affirmative.

We prove Theorem \[1.1\] in \S 2 and Theorem \[1.2\] in \S 3. In \S 4 we look again at a serendipitous gift.

## 2 Surjectivity

The following equality holds for all complex numbers \( z \notin \{-1, 0\} \). Its utility earns it the name, Vital Identity:

\[
\frac{1}{z} = \frac{1}{z+1} + \frac{1}{z(z+1)},
\]

The Vital Identity serves as our main tool for proving Theorem \[1.1\] by giving us that \( \sigma \{ n \} = \sigma \{ n+1, n(n+1) \} \) for all \( n \in \mathbb{N} \). This fact can be usefully restated as \( \sigma \{ n \} = \sigma \{ \circ n, \star n \} \), where \( \circ : \mathbb{N} \to \mathbb{N} \) and \( \star : \mathbb{N} \to \mathbb{N} \) are strictly increasing functions defined by \( \circ : n \mapsto n + 1 \) and by \( \star : n \mapsto n + (n+1) \).

Each word \( w \) in the alphabet \( \{ \circ, \star \} \) expresses a string of function compositions engendering a strictly increasing function \( w : \mathbb{N} \to \mathbb{N} \). Context will tell us when the word \( w \) is to be treated as an injection.

An easy induction on \( k \geq 1 \) shows that the integer \( \star^k n \) has at least \( k + 1 \) distinct prime factors if \( n \geq 2 \).

Nobody will doubt that the tribe \( \mathcal{F}/\sigma := \{ \mathcal{F}_r : r \in \mathbb{Q}^+ \} \) is an infinite partition of the family \( \mathcal{F} \). So, our only substantive task is to show, for \( r \in \mathbb{Q}^+ \), that there is an infinite pairwise disjoint subfamily \( \mathcal{H}_r \subseteq \mathcal{F}_r \), whence the family \( \mathcal{F}_r \) itself is infinite.

There are infinitely many pairs \( (a, b) \in \mathbb{N}^2 \) for which \( r = a/b \). For the sake of convenience, we will choose and fix \( b \geq 2 \) in order to avoid unessential issues due to the fact that \( \circ 1 = \circ 1 \). We then begin by constructing an infinite pairwise disjoint subfamily \( \mathcal{G}_{1/b} \subseteq \mathcal{F} \) for which \( 1/b = \sigma S \) whenever \( S \in \mathcal{G}_{1/b} \).

The expression \( \mathbf{W} \) denotes the set of all finite words \( w \) in the letters \( \circ \) and \( \star \). The length of the word \( w \) is written \( |w| \). When interpreted as a function, the word of length zero is the identity permutation on \( \mathbb{N} \).

For \( k \geq 0 \) we define \( \mathbf{W}_k \) to be the set of all \( w \in \mathbf{W} \) with \( |w| = k \), and \( \mathbf{W}_k n \) denotes the multiset of all integers \( w n \) for \( w \in \mathbf{W}_k \). Similarly, \( \mathbf{W}_n \) denotes the multiset of all \( w n \) with \( w \in \mathbf{W} \).

We will need to deal with the fact that \( \mathbf{W}n = \mathbf{v}n \) can happen while \( w \) and \( v \) are distinct as words.

**Paradigm Examples:** When \( w := \circ \star^2 \circ^3 \) then \( \mathbf{W}n = \circ \star^2 \circ^3 n = \circ \star^2 (n + 3) = \circ \star (n + 3)(n + 4) = (n + 3)(n + 4)((n + 3)(n + 4) + 1) = (n + 3)(n + 4)((n + 3)(n + 4) + 1) + 1 \). Thus \( \mathbf{W}1 = 421 \). Incidentally, \( |w| = |\circ \star^2 \circ^3| = 6 \). So \( w \in \mathbf{W}_{6} \) and \( 421 \in \mathbf{W}_{6} \).

Although \( \circ^4 \) and \( \star \) are distinct as words, \( \circ^4 2 = 6 = \star 2 \). However, notice that \( |\circ^4| = 4 \neq 1 = |\star| \).

It is useful to write each word \( w \in \mathbf{W} \) in the format \( w = \circ^j_1 \circ^j_2 \circ^j_3 \cdots \circ^j_n \circ^j_h \) where \( j_i \geq 0 \), since the roles played by the basic components \( \circ \) and \( \star \) in our story will differ. Of course then \( |w| = r + \sum_{i=0}^{r} j_i \).

The following lemma is obvious.

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1 Many semigroups would write \( \mathbf{W} \) as \( \{ \circ, \star \}^* \). It is the free monoid on the letters \( \circ \) and \( \star \).
Lemma 2.1. Let $b \leq m < *m \leq n < *(m + 1)$. Then $wb = n$ if and only if either $w = o^{n-b}$ or there is an integer $k \in [b, m] := \{b, b+1, \ldots, m\}$ and a word $u$ such that $ub = k$ and such that $w = o^{n-k} * u$.

Lemma 2.2. Let $n > b$. Then $o^{n-b}b = n$. But if $o^{n-b}b \neq w$ while $wb = n$ then $|w| < n - b$.

Proof. Obviously $o^{n-b}b = n$ and $|o^{n-b}b| = n - b$. It is also clear that if $*$ is a letter in the word $w$ then fewer than $n - b$ compositional steps are needed to reach $n$, and so $|w| < n - b$.

Lemma 2.3. Let $b \leq k < k' \leq m < *m \leq n < *(m + 1)$. Let $wb = n = w'b$ where $w = o^{n-k} * u$ with $ub = k$, and where $w' = o^{n-k'} * u'$ with $u'b = k'$. Then $|w| > |w'|$.

Proof. Since $|w| = n - *k + 1 + |u|$ and $|w'| = n - *k' + 1 + |u'|$, it suffices to prove $-k + |u| > -k' + |u'|$; i.e., that $*k' - *k > |u'| - |u|$. But $k \leq k' - 1$, and $k' - b$ is by Lemma 2.2 the length of the longest word $v$ for which $vb = k'$. So $*k' - *k = k'(k' + 1) - k(k + 1) \geq k'(k' + 1) - (k' - 1)k = 2k' > k' - b \geq |u'| - |u|$. □

Theorem 2.4. Let $wb = n = w'b$ with $w \neq w'$. Then $|w| \neq |w'|$.

Proof. We will argue by induction on $n \geq b$. For $n = b$ the theorem is obvious. So pick $n > b$. Suppose for every $s \in [b, n-1]$ that, if $v \neq v'$ are words with $vb = s = v'b$, then $|v| \neq |v'|$. By Lemma 2.1 we can write $w = o^{n-k}u$ and $w' = o^{n-k'}u'$ where $ub = k$ and $u'b = k'$. Without loss of generality $b \leq k \leq k' < n$.

Case: $k = k'$. Then $|w| - |w'| = |u'| - |u|$. By the inductive hypothesis, $|u'| = |u|$ if and only if $u' = u$. But $u' = u$ if and only if $w' = w$ in the present Case. So $w' = w$ if and only if $|w'| = |w|$ in this Case.

Case: $k < k'$. Then $|w| > |w'|$ by Lemma 2.3. So in this Case too, if $w' \neq w$ then $|w'| \neq |w|$. □

Corollary 2.5. For every integer $k \geq 0$, the multiset $W_k b$ is a simple set. So $|W_k b| = |W_k| = 2^k$.

Lemma 2.6. If $k$ is a nonnegative integer then $\sigma(W_k b) = 1/b$.

Proof. We induce on $k \geq 0$. Basis Step: $\sigma(W_0 b) = \sigma(b) = 1/b$. Inductive Step: Pick $k \geq 0$, and suppose that $\sigma(W_k b) = 1/b$. The family $E := \{o^v * v : v \in W_k b\}$ is pairwise disjoint, and $\bigcup E = W_{k+1} b$. But the Vital Identity implies that $\sigma(o^v * v b) = \sigma(v b)$ for every $v \in W$. Hence $\sigma(W_{k+1} b) = \sigma(W_k b) = 1/b$. □

Corollary 2.7. Let $b \geq 2$. Then there exists an infinite pairwise disjoint family $G_{1/b} \subseteq F$ such that $\sigma S = 1/b$ for every $S \in G_{1/b}$.

Proof. Let $k_1 := 0$. Pick $j \geq 0$, and suppose for each $i \in [0, j]$ that the integer $k_i$ has been chosen so that $k_1 < k_2 < \cdots < k_j$ and such that the family $\{W_{k_i} b, W_{k_i} b, \ldots W_{k_i} b\} \subseteq F$ is pairwise disjoint. Notice for each integer $t \geq 2$ that $o^t b = \min W_t b < \max W_t b = o^t b$. So $W_t b \cap \{W_k b : i \in [1, j]\} = \emptyset$ if $t > \bar{k}_j b$. We therefore can define $k_{j+1} := 1 + \bar{k}_j b$ with the assurance that then the family $\{W_k b : i \in [1, j+1]\} \subseteq F$ is pairwise disjoint. Define $G_{1/b} := \{W_k b : i \in \mathbb{N}\}$. The corollary follows by Corollary 2.6 and Lemma 2.6. □

It is now easy to finish establishing Theorem 1.1

Proof. We diminish clutter by writing $B_i := W_i b$ for the $W_i b$ in our proof of Corollary 2.7, and letting $G_{1/b} := \{B_1, B_2, \ldots\}$ be as promised by Corollary 2.7. Recalling that $r = a/b$ and that $\sigma B_i = 1/b$ for every $i$, we partition the family $G_{1/b}$ into a tribe of $a$-membered subfamilies; e.g., this tribe could be $\{C_1, C_2, \ldots\}$ where $C_1 := \{B_1, B_2, \ldots, B_{a-1}\}$, $C_2 := \{B_{a1}, B_{a1+1}, \ldots, B_{2a}\}$, $C_3 := \{B_{2a+1}, B_{2a+2}, \ldots, B_{3a}\}$, \ldots Let $D_k := \bigcup C_k$ for each $k \in \mathbb{N}$, and define an infinite pairwise disjoint subfamily $\mathcal{H}_{a/b} := \{D_1, D_2, \ldots\} \subseteq F$. Since obviously $\sigma D_i = a/b = r$ for every $i \in \mathbb{N}$, Theorem 1.1 is established. □

Reviewing the argument above, we notice that there are infinitely many ways to partition the family $F$ into a tribe of $a$-membered subfamilies, and thus to obtain alternative tribes of $a$-membered families whose unions comprise the membership of other candidates to the title $\mathcal{H}_{a/b}$ besides the family to which we have given that name. That is to say, the Vital Identity confers on us an ability to produce infinitely many subfamilies of $F$, any of which could legitimately be called $\mathcal{H}_{a/b}$. Of course all of these candidates are subfamilies of $F := \{S : r = \sigma S\} \cap F$, given, as we are, that $r = a/b$. Moreover, there are further complexities which might encumber attempts towards a full specification of the family $F$. We now glance at a few of them.

First, other, undiscovered, algorithms may yield members of $F$ which the Vital Identity cannot provide.
Second, all of the families $F_{a/b}$ remarked in the preceding paragraph utilized only the expression of the rational $r$ as its fractional form, $a/b$ for a specific pair $(a, b)$ with $b \geq 2$. But $r = a'/b'$ for infinitely many $⟨a', b'⟩ \in \mathbb{N} \times \mathbb{N}$. Each of these $⟨a', b'⟩$ provides additional families of finite sets $S \subseteq \mathbb{N}$ for which $r = \sigma S$.

Third, there are other procedures, besides the one elaborated in the propositions proved above, whereby for $s \in \mathbb{Q}^+$ the Vital Identity uncovers infinite subfamilies of $F_s$. One such of these alternative procedures involves the generation – from an arbitrary “seed” $A_1 \in F$ – of an infinite sequence $⟨A_i⟩_{i=1}^\infty$ of multisets whose terms we take pains to make simple. Indeed, we arrange for $⟨A_i⟩_{i=1}^\infty$ to be a sequence in $F_{\sigma A_1}$.

It is our guess that each such sequence has an infinite subsequence, the family of whose terms is pairwise disjoint. If our guess gets verified, then a duplication of the final portion of our proof above of Theorem [1.1] will provide another route to that theorem, via a different class of pairwise disjoint subfamilies of $F_s$.

Anyway, these sequences of sets are sufficiently interesting to justify our briefly laying the groundwork for their future study. Moreover, they do give us new infinite subfamilies of $F_s$.

Call an integer $r_i$ replaceable for $A_i$ iff $r_i$ is the least element $x \in A_i$ such that $\{\sigma x, ⋆ x\} \cap A_i = \emptyset$. Plainly each $A_i$ contains exactly one replaceable element. The recursion that generates $⟨A_i⟩_{i=1}^\infty$ is given by $A_{i+1} := (A_i \setminus \{r_i\}) \cup \{σr_i, ⋆ r_i\}$. Of course $|A_{i+1}| = |A_i| + 1$. By the Vital Identity, $σA_i = σA_1$ for all $i \in \mathbb{N}$. We refer to $⟨A_i⟩_{i=1}^\infty$ as the $σ$-sequence from seed $A_1$. It is obvious that each such sequence is infinite.

**Example.** To identify the replaceable element $r_1$ for $A_1 := \{3, 4, 5, 10, 12, 30\}$ we work upward from $\min A_i$. We see that $r_1 \neq 3$ because $\{3, ⋆ 3\} = \{4, 12\} \subseteq A_1$, and $r_1 \neq 4$ because $\langle 4 \rangle = 5 \in A_1$, and $r_1 \neq 5$ because $\langle 5 \rangle = 30 \in A_i$. So $r_1 = 10$, since $\langle 10, ⋆ 10 \rangle = \{11, 110\}$ while $\{11, 110\} \cap A_1 = \emptyset$. The fact that $\langle 12, ⋆ 12 \rangle = \{13, 156\}$ while $\{13, 156\} \cap A_1 = \emptyset$ nominates 12 as a candidate for $r_1$; but 12 loses the election to 10 since $10 < 12$. Candidate 30 gets even fewer votes than 12 got. So $A_{i+1} = \{3, 4, 5, 11, 12, 30, 110\}$.

If $r_1 = \min A_i$, and if $r_i$ is replaceable, then we say that $r_i$ is doomed in $A_i$. Since our sequence-generating recursion never introduces into $A_{i+1}$ an integer smaller than $\min A_i$, the sequence $⟨A_i⟩_{i=1}^\infty$ is nondecreasing. If $d$ is doomed in $A_i$ then $d = \min A_i < \min A_{i+1}$, and $d \notin A_j$ for all $j > i$. If $\min A_i$ is not doomed in $A_i$ then surely $\min A_{i+1} = \min A_i$. Clearly, our guess above is equivalent to our surmise that $\lim_{i \to \infty} \min A_i = \infty$.

En route to our guess that every $σ$-sequence $⟨A_i⟩_{i=1}^\infty$ has an infinite pairwise disjoint subsequence $⟨A_{i,j}⟩_{i=1}^\infty$, we experimented with the recursion operating from several different seed sets. We report on the nondefinitive results we got with the seed $A_1 := \{2\}$. The first six terms of this sequence are: $A_{1,1} = A_1 = \{2\}; A_{1,2} = A_2 = \{3, 6\}; A_3 = \{4, 6, 12\}; A_4 = \{5, 6, 12, 20\}; A_{1,5} = A_5 = \{5, 7, 12, 20, 42\}; A_6 = \{6, 7, 12, 20, 30, 42\}$. In order to reach a secure $A_{1,j}$ we must have that $\min A_{1,j} > \max (A_{1,j} \cup \{A_{1,j-1} \cup A_{1,j-2}\}) = \max \{2, 3, 5, 6, 7, 12, 20, 42\}$. Since the first term of $⟨A_{i,j}⟩$ with $\min A_{1,j} = 7$ is $A_{27}$, we expect hours of pen work before $A_{1,j}$ is reached. A kid with a computer would help. But, even Hal may drag its electronic feet before giving us, say... $A_{1,j,cb}$.

How fast does the integer sequence $⟨j_i⟩ = \langle 1, 2, 5, \ldots \rangle$ increase? We believe the sequence is infinite. Is it?

**Problem.** Considered in these three lights, an exhaustive treatment of the full family $F_r$ remains at issue. One would like to recognize all of the $S \in F$ for which it must happen that $r = \sigma S$. We do not have this information even in the restricted case that $r \in \mathbb{N}$. Indeed, it would be germane to know this for $r = 1$.

**Conjecture.** When $r = a/b \in \mathbb{Q}^+$ with $a$ and $b \geq 1$ coprime and when $c \in \mathbb{N}$, then there is a partition $U_{(r,c)} \subseteq F \cap [cb, \infty)$ such that $\sigma S = r$ for every $S \in U_{(r,c)}$. This would strengthen Theorem [1.1].

### 3 Injectivity revisited

Lower case Greek letters always denote functions of a set variable, except where those symbols may be highjacked to designate numerical values assumed by such functions. For instance, the numbers $\sigma X$ and $\sigma X'$ may be abbreviated to $\sigma$ and $\sigma'$, respectively, when context obviates ambiguity.

Recall that the function $\sigma : F \to \mathbb{Q}^+$ induces two other functions, $\nu : F \to \mathbb{N}$ and $δ : F \to \mathbb{N}$, via the fact that $\sigma X = \nu X/δ X$ for a unique coprime pair $\nu X$ and $δ X$ of positive integers.

The least common multiple $µ X$ of the integers in $X$ is useful for our project, since $µ X/\nu X$ is an integer for each $x \in X$, and so $σ X, µ X$ is an integer. Thus the equality $\sigma = σ µ/µ$ provides an easy presentation of $σ X$ as a fraction of integers. Of course the lowest terms reduction of the fraction $σ µ/µ$ is $ν/δ$.
We write \( m \mid n \) to state that \( m \) divides \( n \). For \( \{m, n\} \subseteq \mathbb{N} \) and \( v \geq 0 \), the expression \( m^v \mid n \) is read “\( m^v \) exactly divides \( n \)”, and means that both \( m^v \mid n \) and \( m^{v+1} \) does not divide \( n \).

We evoke two classic results, both of which are proved in [4]. The first was conjectured by J. Bertrand but established by P. Chebyshev. The second, due to J. J. Sylvester [10], extends the first.

The following fact is known either as Bertrand’s Postulate or as Chebyshev’s Theorem.

**Theorem 3.1. (Chebyshev)** If \( n \geq 2 \) then there is a prime \( p \) such that \( n < p < 2n \).

It serves our purposes to state a slightly offbeat version of Sylvester’s Theorem:

**Theorem 3.2. (Sylvester)** If \( n < 2m \) then there is a prime \( p > n - m \) for which \( p \mid \mu[m, n] \).

One relevant corollary of Sylvester and Chebyshev is that if \( m < n \) then there is a prime \( p > n - m \) such that \( p \mid \mu[m, n] \) for some \( v \in \mathbb{N} \), and \( p^v \mid x \) for exactly one \( x \in [m, n] \). This idea has legs:

**Definition 1.** For \( X \in \mathbb{F} \), when \( v \in \mathbb{N} \) and \( p \) is a prime integer, we call \( p^v \) a sylvester power for \( X \) if \( p^v \mid \mu X \) while \( p^v \mid x \) for exactly one \( x \in X \). The expression \( S(X) \) denotes the set of all sylvester powers for \( X \).

If \( p^v \mid \mu X \) while \( p^v > \max X - \min X \), then surely \( p^v \in S(X) \). We proceed to set the stage.

**Example.** \([1000, 1004] = \{1000, 1001, 1002, 1003, 1004\} = \{2^2 \cdot 5^3, 7 \cdot 11 \cdot 13, 2 \cdot 3 \cdot 167, 17 \cdot 59, 2^2 \cdot 251\} \). Thus, the set of sylvester powers of this interval is \( S[1000, 1004] = \{2^2, 5^3, 7, 11, 13, 17, 59, 167, 251\} \). The sylvester powers of an interval can be numerous.

If \( 1 < m < n \) then \( |S[m, n]| \geq 2 \), and indeed \( 2^v \in S[m, n] \) for some \( v \in \mathbb{N} \). The latter fact comes from

**Lemma 3.3.** If \( 2^v \mid \mu[m, n] \) then there is exactly one \( x \in [m, n] \) with \( 2^v \mid x \), and moreover \( \{|m, n|\} < 2^{v+1} \).

**Proof.** The lemma is immediate for \( m = n \). So let \( n > m \). There exists \( v \in \mathbb{N} \) with \( 2^v \mid \mu[m, n] \). So \( 2^v \mid x \) for some \( x \in [m, n] \), whence \( x = 2^v a \) for an odd integer \( a \). Then \( x + 2^v = 2^v(a + 1) \) is the smallest multiple of \( 2^v \) larger than \( x \). Since \( a + 1 \) is even, we see that \( 2^{v+1}(x + 2^v) \). Thus \( x + 2^v > n \) since otherwise \( 2^{v+1}\mid \mu[m, n] \). Similarly \( x - 2^v < m \) since \( 2^{v+1}\mid(x - 2^v) \). So \( x \) is the only multiple of \( 2^v \) in \([m, n]\). It follows also that \( 2^{v+1} = (x + 2^v) - (x - 2^v) \geq n - m + 2 > n - m + 1 = |\{m, n\}| \), and therefore \( 2^{v+1} > |\{m, n\}| \) as alleged. \( \square \)

**Lemma 3.4.** For \( X \in \mathbb{F} \) and \( p \) prime and \( v \in \mathbb{N} \), let \( p^v \in S(X) \). Then \( p^v \mid \delta X \).

**Proof.** Recall that \( \sigma X = \sigma X \mu X / \mu X = \nu X / \delta X \) where \( \nu X \) and \( \delta X \) are coprime. Since \( p^v \) is sylvester for \( X \), there is a unique multiple of \( p^v \) in \( X \). Then \( p^v(\mu X / z) \) for all \( z \in X \setminus \{x\} \), but \( \mu X / x \) is coprime to \( p \). Therefore \( \sigma X \mu X = \sum \{\mu X / t : t \in X\} \) is coprime to \( p \). So \( \nu X \) is coprime to \( p \). The lemma follows. \( \square \)

**Theorem 3.5.** For \( \{X, Y\} \subseteq \mathbb{F} \) and \( v \in \mathbb{N} \), let \( p^v \in S(X) \setminus S(Y) \) with \( p^v > \max Y - \min Y \). Then \( \delta X \neq \delta Y \), and so \( \sigma X \neq \sigma Y \).

**Proof.** There exists \( u \geq 0 \) with \( p^u \mid \mu Y \). If \( u = v \) then the size of \( p^v \) implies that \( p^v \in S(Y) \), contrary to hypothesis. So \( u \neq v \).

Let \( u > v \). Then \( p^u \in S(Y) \). It is given that \( p^v \in S(X) \), thus \( \delta X \neq \delta Y \) by Lemma 3.4.

Now instead let \( u < v \). Then, either \( p^u \in S(Y) \) whence \( \delta X \neq \delta Y \) by Lemma 3.4 or there exist two distinct \( y \in Y \) for which \( 2^v \mid y \), in which event \( p^v \mid \delta Y \) for some \( t \in [0, u] \), and again \( \delta X \neq \delta Y \) since \( v > t \). \( \square \)

Lemmas 3.3 and 3.4 immediately establish the classic and already cited following result.

**Theorem 3.6. (Theisinger-Kürschák)** If \( \sigma[m, n] \) is an integer then \( m = n = 1 \).

The notion of a sylvester power suggests a way of strengthening the Erdős-Niven theorem. The number of quadruples \( m < n < m' < n' \) for which \( S[m, n] = S[m', n'] \) seems to be finite. The only such quadruples of which we are aware are the two giving us \( S[4, 7] = \{2^2, 3, 5, 7\} = S[20, 21] \) and \( S[5, 7] = \{2, 3, 5, 7\} = S[14, 15] \). A consequence would be that almost always \( \delta[m, n] \neq \delta[m', n'] \) when \( 1 < m < n < m' < n' \).

We hope that a modification of a proof of Sylvester’s Theorem could establish our

**Conjecture.** If \( 1 < m < n < m' < n' \) and if \( n - m \leq n' - m' \) then \( S[m, n] \neq S[m', n'] \).
For each divergent subseries $1/x := \sum_{i=1}^{\infty} 1/x_i$ of the harmonic series, if $\mathcal{I}(x)$ is the family of finite segments of $x := (x_i)_{i=1}^{\infty}$, then $(\sigma X : X \in \mathcal{I}(x))$ is dense in $\mathbb{R}^+$. For which such $x$ is $\sigma[\mathcal{I}(x)]$ injective?

The prime reciprocals series $1/p := 1/2 + 1/3 + 1/5 + 1/7 + \cdots$ diverges. Let $\mathbb{P} := \{p_1 < p_2 < p_3 < \cdots\}$ be the set of all primes. Are $\sigma[F(\mathbb{P})]$ and $\delta[F(\mathbb{P})]$ injective? In general, if $1/d := \sum_{i=1}^{\infty} 1/d_i$ is a divergent harmonic-series subseries with $D := \{d_i : i \in \mathbb{N}\}$ pairwise coprime, then must $\delta[F(D)]$ and $\sigma[F(D)]$ be injective? Our Theorem 1.2 answer such questions affirmatively. So we now prove that theorem.

Proof. Let $A$ and $B$ be distinct nonempty finite subsets of the pairwise coprime set $\mathbb{X} \subseteq \mathbb{N}$. Then without loss of generality there exist $a \in A \setminus B$ and a prime $p$ which divides $a$ but which is coprime to every $y \in (A \cup B) \setminus \{a\}$. Then $p|\delta A$ but $\neg(p|\delta B)$. Therefore $\delta A \neq \delta B$, and so $\sigma A \neq \sigma B$. As for the theorem’s final claim, if $C$ is a finite nonempty subset of $X$ then $\sigma C \in \mathbb{N} \iff \delta C = 1 \iff C = \{1\} \iff \sigma C = 1$.

$$\nu[F(X)]$$ can fail to be injective: $\nu\{n\} = 1, \nu\{3, 13\} = 16 = \nu\{5, 11\}, \nu\{5, 13\} = 18 = \nu\{7, 11\}, \ldots$ It is a symmetric function with $\nu C \geq (|C| \cdot \prod(C)/(\max C)^{|C|}$ for each $C \in F(X)$. Recalling the list $p_1 < p_2 < \cdots$ of all primes, we infer that $\nu\{p_1, p_2, \ldots, p_k\} > k^p_1 + p_2 \cdots p_k$ for all $k \geq 2$, and that $p_1 > p_k$ if $p_1 \nu\{p_1, p_2, \ldots, p_k\}$.

It is reasonable to ask: $\nu[F(X)] = ?$. For what nonsingletons, $C \neq D$ in $F(X)$, does $\nu C = \nu D$ hold?

Like the functions $\star^k$, we shall glance in the next section, the function $\nu$ is potentially useful as a hunter of prime integers. For, if $q_1, \ldots, q_k$ are any $k$ distinct primes, and if $e_1, \ldots, e_k$ is any $k$-length sequence of positive integers then $\nu\{q_1^{e_1}, q_2^{e_2}, \ldots, q_k^{e_k}\}$ is coprime to each of these $q_i$.

4 Stars

Let $\mathbb{P}_b$ be the set of all primes $p$ which divide $\star^j b$ for some integer $j \geq 0$. We have seen that $|\mathbb{P}_b| = \mathbb{N}_0$. Notice that $\mathbb{P}_b = \mathbb{P}_{\nu\mu}$ for all $j \geq 0$. For which $b$, if any, does it happen that $\mathbb{P}_b = \mathbb{P}$?

Theorem 4.1. Let $p \in \mathbb{P}_b$, let $i := i(b,p) \geq 0$ be the least integer with $p|\star^j b$, and hence with $p^n\parallel \star^j b$ for some $n := n(b,p) \geq 1$. Then $p^n\parallel \star^j b$ for every $j > i$.

Proof. Let $p^n\parallel \star^j b$. Since $\star^{j+1} b := \star^j b + 1$ and $\star^j b$ is coprime to $\star^j b + 1$, we see that $p^n\parallel \star^{j+1} b$.

Let $p_1 < p_2 < p_3 < \cdots$ be the primes $2 < 3 < 5 < \cdots$ in their usual order. For $p \in \mathbb{P} \setminus \mathbb{P}_b$ we define $n(b,p) := 0$. Each $b \in \mathbb{N}$ induces a sequence $s(b) := (n(b,p_j))_{j=1}^{\infty}$ of nonnegative integers with $p_j^{n(b,p_j)}\parallel \star^b$ for all sufficiently large $v$. Moreover, as observed above, $n(b,p_j) \geq 1$ for infinitely many $j$.

Clearly $s(b) = s(\star^k b)$ for every $k \in \mathbb{N}$. However, it is easy to see that the set $\{s(b) : b \in \mathbb{N}\}$ is infinite.

5 Coda

We are aware of no endorsements for the subject we treated. Of course, not every idea that survives and eventually shows itself to be profitable is introduced by somebody prominent or even identifiable by name.

We apologize for any failure of ours to cite relevant prior work by others; all such omissions are inadvertent. Our Vital Identity could have been used centuries ago. Was it?

As always, friends and family have supported the conception birth and development of our works.²

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²Arthur Tuminaro goaded one of us into initiating the study this paper came to embody. Its growth profited from our conversations with Bob Cotton, Jacqueline Grace, David Hobby, and Rob Sulman. James Ruffo caught one of our early errors.
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