Sample Paths in Wavelet Theory

Palle E. T. Jorgensen *,1

Department of Mathematics, The University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, U.S.A.

Abstract

We consider a class of convergence questions for infinite products that arise in wavelet theory when the wavelet filters are more singular than is traditionally built into the assumptions. We establish pointwise convergence properties for the absolute square of the scaling functions. Our proofs are based on probabilistic tools.

Key words: scaling identity, quadrature filter, low-pass, Kolmogorov, sampling, pointwise convergence

2000 MSC: 42C40, 42A16, 43A65, 42A65

1 Introduction. Wavelets

We will be primarily interested in the infinite products that typically occur in wavelet analysis [12, Ch. 5]; see also [7]. As is well known, the traditional approach to wavelet-scaling functions is based on analysis of wavelet filters which are periodic functions in one or more variables, and which have some degree of regularity (say, Lipschitz at low frequencies). In addition, the filter functions must satisfy a variant of the following two features: (i) some quadrature condition, and (ii) some low-pass condition. But conditions (i)–(ii) are known to be incompatible with several new classes of examples. For these wider classes of wavelets, we show that a new uncertainty principle for wavelet filters is at play. Reflecting this uncertainty principle for wavelet filters, we note that recent papers on wavelet sets, and on other frequency-localized wavelets (see,
e.g., [3,4,10,11]), necessitate the consideration of wavelet filters which are defined only a.e., and for which conditions (i)–(ii) are more subtle. As a result, the standard deterministic methods then do not immediately apply.

In this paper, we use a random walk model (see [13,44]) in a new analysis of wavelets, but for a class of wavelets where the filters are more singular than is the case in the standard references [12,20,26,29,38,39,46]. Our results apply also to iterative algorithms for wavelet packets [47,48]. In these problems solutions are constructed from an algorithm which relates a certain function \( \varphi(t) \), \( t \in \mathbb{R} \), called a scaling function, to its scaled version \( \varphi(Nt) \) where \( N \) is a fixed integer, 2 or more, or an expansive integral \( d \) by \( d \) matrix \( A \) if \( t \) is in \( \mathbb{R}^d \).

For \( d = 1 \), the formula which determines \( \varphi \) is

\[
\varphi(t) = N \sum_{k \in \mathbb{Z}} a_k \varphi(Nt - k), \quad t \in \mathbb{R}.
\]

It is called the scaling identity. Consider the filter function \( m(x) = \sum_{k \in \mathbb{Z}} a_k e^{-i2\pi kx} \) for \( z = \exp(-i2\pi x) \) in the circle \( \mathbb{T} \), or the \( d \)-torus \( \mathbb{T}^d \). It is traditional to impose some degree of regularity, e.g., Lipschitz, or low-pass for \( x = 0 \); but we shall not do this. Instead we shall work with a probabilistic notion which serves as a minimal restriction on the filter \( m(x) \) guaranteeing the existence of \( L^2 \) solutions to the scaling identity (1.1). It is not at all clear a priori that this identity should have \( L^2 \) solutions, or even solutions that are given by some kind of convergent algorithm. The coefficients in (1.1) are called masking coefficients, filters, or filter coefficients. For special values of the coefficients, it turns out that there is a normalized \( L^2 \) solution \( \varphi \), and that pointwise convergence of a good approximation can be established in a meaningful way. Moreover, we show that the convergence behavior is dictated by properties of an associated transfer operator \( R \), or Ruelle operator.

A main point in this paper is our suggestion of a different and more versatile approach to how we impose low-pass conditions on the basic wavelet filters. Our approach is probabilistic, and it is based directly on the Ruelle operator \( R = R_W \), and on a certain Perron–Frobenius eigenfunction \( h \) for \( R \) (see Theorem 5.2). Our work is motivated in part by recent considerations of frequency-localized wavelets, as pioneered in for example in the papers [2,4,5,10,11]. These papers suggest an interesting context for wavelets that are necessarily frequency-localized. But at the same time, it is evident from this work that the corresponding wavelet filters will then typically not satisfy the low-pass properties that are otherwise known to hold for more traditional time- (or space-) localized wavelets.

The expression on the right-hand side in equation (1.1) is also called a subdivision because of the function values \( \varphi(Nt - k) \). An iteration of the operations on the right hand side in (1.1) on some initial function is called a cascade.
This approximation is one approach to the function \( \varphi \), and the other is the infinite product formula (3.5) for the Fourier transform of \( \varphi \).

The relation between \( \varphi \) and its scaled refinement is well understood when we pass from the time domain to the frequency domain, via the Fourier transform. In that case the relation is multiplicative, and involves a certain periodic matrix function \( m \), called the low-pass filter. For further details, see formulas (3.2) and (3.4) below.

The study of the filters \( m \) is part of signal processing (see, e.g., [31, 45]). But by a ‘mathematical miracle’, they have become one of the most useful tools in wavelet constructions; and at the same time, they have pointed to a host of exciting applications of wavelet mathematics; see, e.g., [27]. To get a path space measure for some of the wavelet problems, we use the quadrature mirror properties (see (2.3) below), or their generalizations, which are assumed for the filter function \( m \), also called a frequency response function. It is periodic, in one or several variables. In \( \mathbb{R}^d \), there is a variety of choices of a period lattice for the problems at hand.

For a number of applications (see especially [4, 5, 10, 11]), we must consider vector versions of the scaling identity (1.1); and in that case, the solution \( \varphi \), the scaling function, will be viewed as a vector-valued function, i.e., a function from \( \mathbb{R}^d \) into some Hilbert space, typically finite-dimensional. In that case, the coefficients \( a_k \) in (1.1) will be matrix-valued; and the product on the right-hand side in (1.1) will be matrix acting on vector. Following [4], we then say that the initial resolution subspace \( V_0 \) in \( L^2(\mathbb{R}^d) \) has multiplicity. In the more traditional multiresolution analysis (MRA) approach to wavelets, \( \varphi \) is a scalar function, and \( V_0 \) is the closed span of the \( \mathbb{Z} \)-translates of \( \varphi \). The case of multiplicity is called the generalized MRA (GMRA) approach. The present considerations deal with the issue of passing from the filter function (possibly matrix-valued) to the scaling function \( \varphi \). However, the formulation of our ideas in the scalar case may easily be modified to the matrix/vector case; and for the sake of simplicity, our technical discussion below will be presented in the scalar case. We leave to the reader the spelling out of the generalization to GMRAs.

The fact that there are solutions in \( L^2(\mathbb{R}^d) \) is not at all obvious; see [12]. In application to images, the subspace \( V_0 \) (where \( V_0 := \text{cl span}\{ \varphi(\cdot - k) \mid k \in \mathbb{Z}^d \} \)) may represent a certain resolution, and hence there is a choice involved, but we know by standard theory, see, e.g., [12], that under appropriate conditions, such choices are possible. As a result there are extremely useful, and computationally efficient, wavelet bases in \( L^2(\mathbb{R}^d) \). A resolution subspace \( V_0 \) within \( L^2(\mathbb{R}^d) \) can be chosen to be arbitrarily fine: Finer resolutions correspond to larger subspaces.
As noted for example in [8,18], a variant of the scaling equation is also used in computer graphics: there data is successively subdivided and the refined level of data is related to the previous level by prescribed masking coefficients. The latter coefficients in turn induce generating functions which are direct analogues of wavelet filters.

One reason for the computational efficiency of wavelets lies in the fact that wavelet coefficients in wavelet expansions for functions in \( V_0 \) may be computed using matrix iteration, rather than by a direct computation of inner products: the latter would involve integration over \( \mathbb{R}^d \), and hence be computationally inefficient, if feasible at all. The deeper reason for why we can compute wavelet coefficients using matrix iteration is an important connection to the subband filtering method from signal/image processing involving digital filters, down-sampling and up-sampling. In this setting filters may be realized as functions \( m_0 \) on a \( d \)-torus, e.g., quadrature mirror filters.

As emphasized for example in [34] and [9], because of down-sampling, the matrix iteration involved in the computation of wavelet coefficients involves so-called slanted Toeplitz matrices \( F \) from signal processing. The slanted matrices \( F \) are immediately available; they simply record the numbers (masking coefficients) from the \( \varphi \)-scaling equation. These matrices further have the computationally attractive property that the iterated powers \( F^k \) become successively more sparse as \( k \) increases, i.e., the matrix representation of \( F^k \) has mostly zeros, and the non-zero terms have an especially attractive geometric configuration. In fact subband signal processing yields a finite family, \( F, G, \) etc., of such slanted matrices; for example (with \( L \) for “low frequency” and \( H \) for “high frequency”):

\[
\varphi = \sum_k P_k \varphi(2 \cdot k), \quad \psi = \sum_k Q_k \varphi(2 \cdot k),
\]

\[
\begin{cases}
F: & y^L_n = \frac{1}{\sqrt{2}} \sum_k P_{k-2n} x_k, \\
G: & y^H_n = \frac{1}{\sqrt{2}} \sum_k Q_{k-2n} x_k.
\end{cases}
\]

The wavelet coefficients at scaling level \( k \) of a numerical signal \( s \) from \( V_0 \) are then simply the coordinates of \( GF^k s \). By this we mean that a signal in \( V_0 \) is represented by a vector \( s \) via a fixed choice of scaling function; see [12,8,31].

Given some choice of scaling function \( \varphi \), then under suitable conditions (e.g., orthogonality or \textit{a priori} frame estimates) we may define the operator \( W: V_0 \to \ell^2 \) which links the resolution subspace \( V_0 \) with the sequence space \( \ell^2 \) of signals \( s = (s_k) \) as follows:

\[
W \left( \sum_{k \in \mathbb{Z}^d} s_k \varphi(\cdot - k) \right) = (s_k).
\]
Here $W$ becomes a well defined linear operator, $W: V_0 \rightarrow \ell^2$.

Then the matrix product $GF^k$ is applied to $s$; and the matrices $GF^k$ get more slanted as $k$ increases.

Our approach begins with the observation that the computational feature of this engineering device can be said to begin with an endomorphism $r_A$ of the $d$-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$; an endomorphism which results from simply passing matrix multiplication by $A$ on $\mathbb{R}^d$ to the quotient by $\mathbb{Z}^d$. It is then immediate that the inverse images $r_A^{-1}(x)$ are finite for all $x$ in $\mathbb{T}^d$, in fact $\#r_A^{-1}(x) = |\det A|$. From this we recover the scaling identity, and we note that the wavelet scaling equation is a special case of a more general identity known in computational fractal theory and in symbolic dynamics [32]. We show that wavelet algorithms and harmonic analysis naturally generalize to affine iterated function systems. Moreover, in this general context, we are able to build the ambient Hilbert spaces for a variety of dynamical systems which arise from the iterated dynamics of endomorphisms of compact spaces [15].

As a consequence, the fact that the ambient Hilbert space in the traditional wavelet setting is the more familiar $L^2(\mathbb{R}^d)$ is merely an artifact of the choice of filters $m_0$. As we further show, by enlarging the class of admissible filters, there is a variety of other ambient Hilbert spaces possible with corresponding wavelet expansions: the most notable are those which arise from iterated function systems (IFS) of fractal type, for example for the middle-third Cantor set, and scaling by $3$.

With examples, with theorems, and with graphics, we hope to bring these threads to light in this little book: The journey from wavelets to fractals via signal processing.

More generally, there is a variety of other natural dynamical settings (affine IFSs) that invite the same computational approach.

The two most striking examples which admit such a harmonic analysis are perhaps complex dynamics and subshifts. Both will be worked out in detail. In the first case, consider a given rational function $r(z)$ of one complex variable. We then get an endomorphism $r$ acting on an associated Julia set $X$ in the complex plane $\mathbb{C}$ as follows: This endomorphism $r: X \rightarrow X$ results by restriction to $X$ [6]. (Details: Recall that $X$ is by definition the complement of the points in $\mathbb{C}$ where the sequence of iterations $r^n$ is a normal family. Specifically, the Fatou set $F$ of $r(z)$ is the largest open set in $\mathbb{C}$ where $r^n$ is a normal sequence of functions, and we let $X$ be the complement of $F$. Here $r^n$ denotes the $n$'th iteration of the rational function $r(z)$.) The induced endomorphism $r$ of $X$ is then simply the restriction to $X$ of $r(z)$. If $r$ then denotes the resulting endomorphism, $r: X \rightarrow X$, it is known [16] that $\#r^{-1}(x) = \text{degree of } r$, for every $x$ in $X$ (except for a finite set of singular points).
In the second case, for a particular one-sided subshift, we may take $X$ as the corresponding state space, and again we have a naturally induced finite-to-one endomorphism of $X$ of geometric and computational significance.

But in the general framework, there is not a natural candidate for the ambient Hilbert space. That is good in one sense, as it means that the subband filters $m_0$ which are feasible will constitute a richer family of functions on $X$.

In all cases, the analysis is governed by a random-walk model with successive iterations where probabilities are assigned on the finite sets $\#r^{-1}(x)$ and are given by the function $W := |m_0|^2$. This leads to a transfer operator $R_W$ which has features in common with the classical operator considered first by Perron and Frobenius for positive matrices, in particular it has a Perron–Frobenius eigenvalue, and positive Perron–Frobenius eigenvectors, one on the right, a function, and one on the left, a measure; see [42]. This Perron–Frobenius measure, also sometimes called the Ruelle measure, is an essential ingredient for our construction of an ambient Hilbert space. All of this, we show, applies to a variety of examples, and as we show, has the more traditional wavelet setup as a special case, in fact the special case when the Ruelle measure on $\mathbb{T}^d$ is the Dirac mass corresponding to the point 0 in $\mathbb{T}^d$ (additive notation) representing zero frequency in the signal processing setup.

There are two more ingredients entering in our construction of the ambient Hilbert space: a path-space measure governed by the $W$-probabilities, and certain finite cycles for the endomorphism $r$. For each $x$ in $X$, we consider paths by infinite iterated tracing back with $r^{-1}$ and recursively assigning probabilities with $W$. Hence we get a measure $P_x$ on a space of paths for each $x$. These measures are in turn integrated in $x$ using the Ruelle measure on $X$. The resulting measure will now define the inner product in the ambient Hilbert space.

Since the first question is to decide when $\varphi$ is in $L^2$, we iterate and get an infinite matrix product involving the matrix $W := m^*m$. Since $W$ is positive semidefinite, we may create a positive path measure of a random walk starting at $x$ in some period interval. In several dimensions, $x$ starts in a fundamental domain $D$ for some fixed lattice, for example $\mathbb{Z}^d$. The paths starting at $x$ arise by iteration of the inverse branches of $x \rightarrow Ax \mod \mathbb{Z}^d$. There are $N = |\det A|$ distinct branches. These $N$ branches may be viewed as endomorphisms of $D$.

In this paper, we will consider pointwise convergence of the infinite product (3.5) below for the Fourier transform $\hat{\varphi}$ of the scaling function $\varphi$. But the traditional low-pass/regularity considerations for the filter $m(x)$ are more general. Once pointwise convergence (Theorem 5.2) is established, then the quadrature property for $m(x)$ will imply that the scaling function $\varphi$ is automatically in $L^2(\mathbb{R})$. 

6
We construct our random walks in a general framework which includes both wavelets, wavelet packets, and some of the other more classical problems. We further show how some of the classical questions may be phrased and solved with the use of path space measures.

It is interesting to contrast our proposed approach with the more traditional one used in wavelet analysis: see, e.g., [12]. Traditionally, some kind of Lipschitz or Dini regularity condition must be assumed for the filter. Then the corresponding infinite product may be made precise, and we can turn to the question of when the wavelet generators are in $L^2(\mathbb{R}^d)$; see, e.g., [30] and [15]. As it turns out, both of these issues have natural formulations, and solutions, in terms of the path space measures. And the results allow a wider generality. In a variety of wavelet questions for band-limited wavelets, the regularity conditions just aren’t satisfied for the filters $m$ that are dictated by the setting and the applications; see, e.g., [3].

2 Path space

A well tested tool in analysis, and in mathematical physics, centers around the application of path-space measures. This tool is used in attacking a variety of singular convergence, or approximation, problems. We will adopt this viewpoint in our study of wavelet approximations. Traditionally, the setting for wavelet questions has included assumptions concerning continuity, or some kind of differentiability. In contrast, we shall work almost entirely in the measurable category. One advantage of our present approach is that we stay in the measurable category when addressing problems from multiresolution analysis (MRA). Earlier work on the use of probability in wavelets includes that of R.F. Gundy et al.; see [14,17,19,21,22,23,24,25].

Our present viewpoint is more general than [14], and it starts with the random walks naturally associated with a measure space $(X,\mathcal{B})$ and a given measurable onto map $\sigma: X \to X$ such that $\#\sigma^{-1}(\{x\}) = N$ for all $x \in X$, where $N$, $2 \leq N < \infty$, is fixed. Iteration of the branches

\[ \sigma^{-1}(\{x\}) := \{ y \in X \mid \sigma(y) = x \} \]

then yields a combinatorial tree. If $\omega = (\omega_1, \omega_2, \ldots) \in \Omega := \{0, 1, \ldots, N - 1\}^N$, an associated path may be thought of as an infinite extension of the finite walks

\[ \tau_{\omega_n} \cdots \tau_{\omega_2} \tau_{\omega_1} x \]
starting at $x$, where $(\tau_i), \ i = 0, 1, \ldots, N - 1,$ is a system of inverses, i.e., where
\[
\sigma \circ \tau_i = \text{id}_X, \quad 0 \leq i < N. \tag{2.2}
\]
If $W: X \to [0, 1]$ is a given measurable function such that
\[
\sum_{y: \sigma(y) = x} W(y) = 1, \tag{2.3}
\]
then an associated measure $P_x$ on $\Omega$ may be defined as follows. Suppose some function $f \in C(\Omega)$ depends only on a finite number of coordinates, say $\omega_1, \ldots, \omega_n$, then set
\[
\int_{\Omega} f \, dP_x = \sum_{(\omega_1, \ldots, \omega_n)} f(\omega_1, \ldots, \omega_n) W(\tau_{\omega_1} x) W(\tau_{\omega_2} \tau_{\omega_1} x) \cdots W(\tau_{\omega_n} \cdots \tau_{\omega_1} x). \tag{2.4}
\]
Extensions of this formula to $\Omega$ can be done in a number of ways: see the cited references.

A special feature of this construction, which will be explored in the present monograph, is that of attractive convergence properties for infinite products of the form
\[
\prod_{n, \omega_1, \ldots, \omega_n} W(\tau_{\omega_1} x) \cdots W(\tau_{\omega_n} \cdots \tau_{\omega_1} x) \tag{2.5}
\]
over certain subsets of $\Omega$. As it turns out, these infinite products are determined by the measures $(P_x)_{x \in X}$, and by the Ruelle transition operator
\[
(R_W g)(x) := \sum_{y: \sigma(y) = x} W(y) g(y), \quad g \in L^\infty(X). \tag{2.6}
\]
The operator $R$ in (2.6) is called the transition operator, the Ruelle operator, or the Ruelle–Perron–Frobenius operator, and it will play a major role in what follows.

Many problems in dynamics are governed by transition probabilities $W$, and $P_x$, and by an associated transition operator $R_W$ as in (2.6). Wavelet theory is a case in point, and we show that fundamental convergence questions for wavelets, and properties of the solutions to (1.1), depend on the positive solutions $h$ to the eigenvalue problem $R_W(h) = h$. In Theorem 5.2, we show that there is a particular solution $h$, see (3.11), which determines the issue of pointwise convergence of the infinite product. We refer to the discussion around (3.11)–(3.12) below. The solutions $h$ are called harmonic, and the function $h$ in (3.11) is a special harmonic function which will play a central role in our main result, Theorem 5.2 below.
3 Multiresolutions

The multiresolution approach to wavelets involves functions on \( \mathbb{R} \). It begins with the fixed-point problem

\[
\varphi(t) = N \sum_{k \in \mathbb{Z}} a_k \varphi(Nt - k), \quad t \in \mathbb{R},
\]

where a given sequence \((a_k)_{k \in \mathbb{Z}}\) is chosen with special filtering properties, e.g., quadrature mirror filters; see [15,20,26,29,38,39,46]. The equation (3.1) is called the scaling identity, and the \( a_k \)'s the response coefficients, or the masking coefficients. Introducing the Fourier series

\[
m(x) = \sum_{k \in \mathbb{Z}} a_k e^{-i2\pi kx}
\]

and Fourier transform

\[
\hat{\varphi}(x) = \int_{\mathbb{R}} e^{-i2\pi tx} \varphi(t) \, dt,
\]

we get the relation

\[
\hat{\varphi}(x) = m\left(\frac{x}{N}\right) \hat{\varphi}\left(\frac{x}{N}\right), \quad x \in \mathbb{R},
\]

which suggests a closer inspection of the infinite products

\[
\prod_{n=1}^{\infty} m\left(\frac{x}{N^n}\right).
\]

Since we shall want solutions \( \varphi \) to (3.1) which are in \( L^2(\mathbb{R}) \), (3.4)–(3.5) suggest the corresponding convergence questions for the function \( W := |m|^2 \).

When \((P_x)_{x \in [0,1]}\) is the family of measures on \( \Omega \) corresponding to \( W = |m|^2 \), then the formal infinite product

\[
|\hat{\varphi}(x)|^2 = \prod_{n=1}^{\infty} W\left(\frac{x}{N^n}\right)
\]

is \( P_x (\{(0,0,0,\ldots)\}) \), i.e., the measure of the singleton \((0,0,\ldots)\) (an infinite string of zeroes) in \( \{0,\ldots,N-1\}^N \). Our main theorem (Theorem 5.2) gives a necessary and sufficient condition for the pointwise convergence of (3.6) in a rather general context. We note that because of assumption (2.3), \( \varphi \) will automatically be in \( L^2(\mathbb{R}) \), once pointwise convergence is established. There is a natural way (based on Euclid’s algorithm) of embedding \( \mathbb{Z} \) into

\[
\Omega = \{0,\ldots,N-1\}^N \times \{0,\ldots,N-1\}^N
\]
such that
\[ P_x(Z) = \sum_{k \in \mathbb{Z}} |\hat{\phi}(x + k)|^2. \] (3.8)

Even though the measures \( P_x \) (Section 6) are defined \textit{a priori} on the over-countable probability space, the surprise is that they are in fact supported only on a fixed thin (countable) subset of \( \Omega \).

\textbf{Remark 3.1.} Note that, in general, it is not at all clear that the measures \( (P_x), x \in X \), on \( \Omega \) should even have atoms. Typically, they don’t! But if atoms exist, i.e., when there are points \( \omega \in \Omega \) such that \( P_x(\{\omega\}) > 0 \), we note that this yields convergence of an associated infinite product. Let \( N_0 := \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\} \). Using Euclid, and the \( N \)-adic expansion
\[ k = i_1 + i_2 N + \cdots + i_n N^{n-1} \quad \text{for} \ k \in \mathbb{N}_0, \] (3.9)
we see that the points
\[ \omega(k) = (i_1, \ldots, i_n, 0, 0, 0, \ldots) \quad \text{\( \infty \) string of zeroes} \]
represent a copy of \( \mathbb{N}_0 \) sitting in \( \Omega \). With the identification \( k \leftrightarrow \omega(k) \), we set
\[ P_x(N_0) = \sum_{k=0}^{\infty} P_x(\{\omega(k)\}). \] (3.10)

But in general, this function \( P_x(N_0) \) might be zero. Our first observation about
\[ h(x) := P_x(N_0), \quad x \in X, \] (3.11)
is that it solves the eigenvalue problem
\[ R_W h = h. \] (3.12)

We say that \( h \) is a minimal harmonic function relative to \( R_W \). Note that in general \( h = 0 \) may happen!

\textbf{Remark 3.2.} Let \( X, B, \sigma, \tau_0, \ldots, \tau_{N-1} \), and \( W \) be as described above, and let \( (P_x)_{x \in X} \) be the corresponding transition probabilities. Let
\[ 0 = (0, 0, 0, \ldots) \in \Omega = \{0, 1, \ldots, N-1\}^N. \]

While in general, often \( P_x(\{0\}) = 0 \), the case when \( P_x(\{0\}) > 0 \) is important. The condition \( P_x(\{0\}) > 0 \) is a way of making precise sense of the infinite product
\[ P_x(\{0\}) = \prod_{n=1}^{\infty} W(\tau_0^n x). \] (3.13)
If, for example, \( \lim_{n \to \infty} W(\tau_0^n x) < 1 \), then it is immediate from (3.13) that \( P_x(\{0\}) = 0 \).

Suppose \( P_x(\{0\}) > 0 \). Then it follows that
\[
P_x(\{(i_1, \ldots, i_n, 0)\}) = W(\tau_{i_1} x) \cdots W(\tau_{i_n} x) P_{\tau_{i_n} \cdots \tau_{i_1}}(\{0\}).
\] (3.14)

Using (3.9), we shall identify \( k \in \mathbb{N}_0 \) with the point \( \omega(k) \in \Omega \), and write \( P_x(\{k\}) \) for the expression in (3.14).

An important question for dyadic wavelets in \( L^2(\mathbb{R}) \) is the issue of when these wavelets form orthonormal bases (ONB’s). A dyadic wavelet function \( \psi \in L^2(\mathbb{R}) \) generates an ONB if the double-indexed family
\[
\{ 2^{n/2} \psi(2^n t - k) \mid n, k \in \mathbb{Z} \}
\] (3.15)
satisfies (i) and (ii) below:

(i) \( \int_{\mathbb{R}} \overline{\psi_{n,k}(t)} \psi_{m,l}(t) \, dt = \delta_{n,m} \delta_{k,l} \), with
\[
\psi_{n,k}(t) := 2^{n/2} \psi(2^n t - k),
\] (3.16)

and

(ii) the closed linear span of \( \{ \psi_{n,k} \mid n, k \in \mathbb{Z} \} \) is \( L^2(\mathbb{R}) \).

In our analysis of the scaling identity
\[
\varphi(t) = 2 \sum_{k \in \mathbb{Z}} a_k \varphi(2t - k)
\] (3.17)
(a special case of (3.1)), we will be looking at two functions \( \varphi \) and \( \psi \); the second one may be taken to be
\[
\psi(t) = 2 \sum_{k \in \mathbb{Z}} (-1)^{k+1} \bar{a}_{1-k} \varphi(2t - k).
\] (3.18)

This analysis is the approach to wavelets which goes under the name of multiresolution analysis. The function \( \psi \) which is used in (3.15)–(3.16) is the solution to (3.18). The two standing conditions which are placed on the numbers \( (a_k)_{k \in \mathbb{Z}} \), called masking coefficients, are
\[
\sum_{k \in \mathbb{Z}} \bar{a}_k a_{k+2n} = \frac{1}{2} \delta_{0,n}, \quad n \in \mathbb{Z},
\] (3.19)
and
\[
\sum_{k \in \mathbb{Z}} a_k = 1.
\] (3.20)
These conditions in themselves do not imply orthonormality in (3.16), but only the following much weaker property:

$$\sum_{n,k \in \mathbb{Z}} |\langle \psi_{n,k} | f \rangle|^2 = \|f\|^2 = \int_{\mathbb{R}} |f(t)|^2 \, dt, \quad f \in L^2(\mathbb{R}).$$  \hspace{1cm} (3.21)

A system of functions \((\psi_{n,k})\) satisfying (3.21) is called a Parseval frame, or a normalized tight frame.

Given (3.19)-(3.20), it turns out that the ONB property for the wavelet is equivalent to either one of the following two conditions for the normalized scaling function \(\varphi\):

$$\int_{\mathbb{R}} \varphi(t) \varphi(t-k) \, dt = \delta_{0,k},$$  \hspace{1cm} (3.22)

or

$$\|\varphi\|_{L^2(\mathbb{R})} = 1.$$  \hspace{1cm} (3.23)

4 Sampling

In this section and the next, we study an intriguing relationship between the following three problems:

(1) When does the scaling identity (3.1) have \(L^2\) solutions?
(2) How may the transition probabilities \(P_x\) be used in sampling certain functions at the points \(x+k\) as \(k\) runs over a set of integers?
(3) When is the infinite product (3.5) pointwise convergent?

In the wavelet applications, \(X = [0,1]\), and the system \(\sigma, \tau_0, \ldots, \tau_{N-1}\) is as follows:

$$\begin{cases}
\sigma(x) = Nx \mod 1, \\
\tau_j(x) = \frac{x+j}{N}, \quad j = 0,1,\ldots,N-1.
\end{cases}$$  \hspace{1cm} (4.1)

**Lemma 4.1.** Setting \(F(x) := P_x(\{0\})\), and

$$k = i_1 + i_2 N + \cdots + i_n N^{n-1} \quad (\in \mathbb{N}_0),$$  \hspace{1cm} (4.2)

we get the formula

$$P_x(\{k\}) = F(x+k),$$  \hspace{1cm} (4.3)

where we have identified \(k\) with the point \(\omega(k) := (i_1,\ldots,i_n,0)\) in \(\Omega\).

**Proof.** To see this, identify functions on \([0,1]\) with 1-periodic functions on \(\mathbb{R}\), and note that the second formula in (4.1) yields \(\tau_{i_n} \cdots \tau_{i_1}(x) = (x+k)/N^n\)
where \( k \) is given by (4.2). Hence, if \( 1 \leq s \leq n \), then
\[
W(\tau_i \cdots \tau_1(x)) = \frac{x + i_1N + \cdots + i_sN^{s-1}}{N^s} = \frac{x + k}{N^s}.
\]

It follows that (4.3) is really just a rewrite of (3.14). The right-hand side of (3.14) yields
\[
W\left(\frac{x + k}{N}\right) \cdots W\left(\frac{x + k}{N^n}\right) F\left(\frac{x + k}{N^n}\right) = F(x + k),
\]
which is the desired conclusion.

**Remark 4.2.** To extend \( P_x(\cdot) \) from \( \mathbb{N}_0 \) to \( \mathbb{Z} \), recall that \( k \in \mathbb{N}_0 \) is identified with the singleton \( \omega(k) = (i_1, \ldots, i_n, 0) \) via (4.2). Now, if \( -N^n \leq k < 0 \), then set
\[
P_x(\{k\}) := P_x\left(\{\omega\left(N^{n+1} + k\right)\}\right).
\]

To help the reader gain some intuitive feeling for the conclusion in Lemma 4.1, observe that the right-hand side of (4.3) represents a sampling of the function \( F \) at the integral translates on \( \mathbb{R} \), starting at \( x \), i.e., \( x + k \). Obviously, different subsets of \( \Omega \) would yield different sets of sampling points for \( F \), including nonuniformly distributed sampling points; see [1].

Starting with Shannon [43], the theory of sampling has emerged as a significant tool in signal processing; see, e.g., the beautifully written survey [1] as well as the references cited therein. Thus, in a general context, our formula (4.3) offers a probabilistic prescription for sampling of functions on the real line, and at the same time it stresses the ‘random’ feature of sampling.

## 5 A convergence theorem for infinite products

We now show how this viewpoint from sampling theory is closely related to some fundamental properties of the measures \( P_x \). In particular, our Theorem 5.2 gives a necessary and sufficient condition for pointwise convergence of the infinite product (3.5), or more generally (2.5), with the condition for convergence stated in terms of the harmonic function \( h \) of (3.11). The relationship between \( h \) and the measure family \( P_x \) is studied more systematically below.

Let \( A \subset \mathbb{Z} \). Returning to (4.3), we set
\[
P_x(A) := \sum_{k \in A} P_x(\{k\}) = \sum_{k \in A} F(x + k).
\]

13
As in Lemma 4.1, the number \( N, \ N \geq 2, \) and the function \( W \) are given. The measures \( P_x \) are constructed from these data using (2.4), and we have the two functions
\[
F(x) := P_x(\{ (0, 0, 0, \ldots) \}) \tag{5.2}
\]
and
\[
h(x) := P_x(\mathbb{Z}), \quad x \in \mathbb{R}. \tag{5.3}
\]
Finally, for \( k \in \mathbb{N} \), set
\[
N^k \mathbb{Z} := \{ N^k j \mid j \in \mathbb{Z} \}. \tag{5.4}
\]
Using (4.5), we see that \( N^k \mathbb{Z} \) is represented in \( \Omega = \{0, 1, \ldots, N - 1\}^\mathbb{N} \) as
\[
(0, \ldots, 0, \omega_1, \omega_2, \ldots, 0, 0, 0, \ldots) \tag{5.5}
\]
An infinite string of zeroes will be denoted \( 0^\infty \).

**Lemma 5.1.** Let \( N, W, F, P_x, \) and \( h \) be as described above; see (5.2)–(5.3). Then \( h \) satisfies the following cocycle identity:
\[
h(x) W(x) = P_{Nx}(N\mathbb{Z}), \quad x \in \mathbb{R}. \tag{5.6}
\]
In particular, if \( h(x) \equiv 1 \) a.e. \( x \in \mathbb{R} \), then we recover the function \( W \) from the transition probabilities \( P_x \), a.e. \( x \in \mathbb{R} \).

**Proof.** We calculate the left-hand side in (5.6), using the earlier equations:
\[
h(x) W(x) = W(x) \sum_{j \in \mathbb{Z}} F(x + j)
\]
\[
= \sum_{j \in \mathbb{Z}} W(x + j) F(x + j)
\]
\[
= \sum_{j \in \mathbb{Z}} F(N(x + j))
\]
\[
= \sum_{j \in \mathbb{Z}} F(Nx + Nj)
\]
\[
= \sum_{j \in \mathbb{Z}} P_{Nx}(\{Nj\})
\]
\[
= P_{Nx}(N\mathbb{Z}).
\]
This is the desired identity (5.6), and the proof is completed. \( \square \)

**Theorem 5.2.** Let \( N, W, F, P_x, \) and \( h \) be as described above. Let \( x \in \mathbb{R} \), and suppose that \( P_x(\{0\}) > 0 \). Then the following two conditions are equivalent.
(a) The limit on the right-hand side below exists, and

\[ F(x) = \lim_{n \to \infty} \prod_{k=1}^{n} W\left( \frac{x}{N^k} \right). \]

(b) The limit on the left-hand side below exists, and

\[ \lim_{n \to \infty} h\left( \frac{x}{N^n} \right) = 1. \]

Proof. (a) \(\Rightarrow\) (b). An iteration of the identity (5.6) in Lemma 5.1 above yields

\[ P_x(N^k \mathbb{Z}) = \left( \prod_{j=1}^{k} W\left( \frac{x}{N^j} \right) \right) h\left( \frac{x}{N^k} \right). \] (5.7)

Using (5.5), and working in \(\Omega\), we find

\[ \bigcap_{k \in \mathbb{N}} N^k \mathbb{Z} = \{0\}. \] (5.8)

An application of a standard result in measure theory [41, Theorem 1.19(e), p. 16] now yields existence of the following limit:

\[ \lim_{k \to \infty} P_x(N^k \mathbb{Z}) = P_x(\{0\}) = F(x). \] (5.9)

Since (a) is assumed, and \( F(x) > 0 \), we conclude that the limit \( h\left( x/N^k \right) \), for \( k \to \infty \), must exist as well, and further that

\[ F(x) = F(x) \lim_{k \to \infty} h\left( \frac{x}{N^k} \right). \]

Using again \( F(x) > 0 \), we finally conclude that (b) holds.

(b) \(\Rightarrow\) (a). Recall that formula (5.7) holds in general. If (b) is assumed, we then conclude that the limit \( \prod_{j=1}^{k} W\left( x/N^j \right) \) exists as \( k \to \infty \). The limit on the left-hand side in (5.7) exists and is \( F(x) \). As a result, we get

\[ F(x) = \lim_{k \to \infty} \prod_{j=1}^{k} W\left( \frac{x}{N^j} \right) \lim_{k \to \infty} h\left( \frac{x}{N^k} \right) = \lim_{k \to \infty} \prod_{j=1}^{k} W\left( \frac{x}{N^j} \right), \]

which is (a).

\[ \square \]

6 Ruelle’s wavelet transition operator

The next definitions (Definitions 6.1) and the lemma (Lemma 6.2) give us the precise details behind the two critical notions from Theorem 5.2 above; i.e.,
the cocycles, and the random walk measures $P_x$. An existence and uniqueness theorem for $P_x$ is then presented in Section 8.

**Definitions 6.1.**

(a) The *Ruelle operator* $R = R_W$ is defined by

$$ (Rf)(x) = \sum_{y \in X, \sigma(y) = x} W(y) f(y), \quad x \in X, \ f \in L^\infty(X), \quad (6.1) $$

and maps $L^\infty(X)$ into itself.

(b) Let $\Omega$ be the compact Cartesian product

$$ \Omega = \mathbb{Z}_N^N = \{0, \ldots, N-1\}^N = \prod_{1}^{\infty} \{0, \ldots, N-1\}. \quad (6.2) $$

(c) A bounded measurable function $V: X \times \Omega \to \mathbb{C}$ is said to be a *cocycle* if

$$ V(x, (\omega_1, \omega_2, \ldots)) = V(\tau_{\omega_1}(x), (\omega_2, \omega_3, \ldots)) \quad (6.3) $$

for all $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$.

(d) A function $h: X \to \mathbb{C}$ is said to be *harmonic*, or $R_W$-*harmonic*, if

$$ R_W h = h. \quad (6.4) $$

(e) Let $n \in \mathbb{N}$, and let $i_1, \ldots, i_n \in \mathbb{Z}_N$. Then the subset

$$ A(i_1, \ldots, i_n) := \{ w \in \Omega \mid \omega_1 = i_1, \ldots, \omega_n = i_n \} \quad (6.5) $$

is called a *cylinder set*.

We shall use the following correspondence (see [33] and [17] for details and proofs) between the cocycles $V$ from (c) and the harmonic functions $h$ from (d): If $V$ is given as in (c), then the function $h(x) := P_x[V(x, \cdot)]$ has the properties in (d). Conversely, for every $h$ satisfying (d), including (6.4), there is a martingale limit which lets us recover a cocycle $V$, $P_x$ a.e., such that $h(x) := P_x[V(x, \cdot)]$. In the present discussion, we are interested in the cocycles $V$ which arise as indicator functions $\chi_S$ for certain cyclic and invariant subsets $S$ of $\Omega$. We refer to Section 8 below.

**Existence of the measures $P_x$**

The cylinder sets generate the topology of $\Omega$, and its Borel sigma-algebra. In determining Radon measures on $\Omega$, it is therefore convenient to first specify them on cylinder sets. This approach was initiated by Kolmogorov [35]; see also Nelson [40]. Recall that $\Omega$ is compact in the Tychonoff topology, and
that we may use the Stone–Weierstraß theorem on $C(\Omega) = \text{the algebra of all continuous functions on } \Omega$.

**Lemma 6.2.** Let $X, B, \mu, \sigma, \tau_0, \ldots, \tau_{N-1}$, and $W$ be given as described above. We make the following more restrictive assumption on $W$:

$$
\sum_{y \in X, \sigma(y) = x} W(y) = 1 \quad \text{a.e. } x \in X. \quad (6.6)
$$

Then for every $x \in X$ there is a unique positive Radon probability measure $P_x$ on $\Omega$ such that

$$
P_x(A(i_1, \ldots, i_n)) = W(\tau_{i_1} x) W(\tau_{i_2} \tau_{i_1} x) \cdots W(\tau_{i_n} \cdots \tau_{i_1} x). \quad (6.7)
$$

The main fact about the Ruelle operator

$$
R_W f(x) = \sum_i W(\tau_i x) f(\tau_i x)
$$

is this [17]: Under suitable conditions on $W$, there is a unique probability measure (Ruelle measure $\nu$) satisfying

$$
\nu \cdot R_W = \nu,
$$

and a unique continuous minimal eigenfunction $h_{\text{min}}$,

$$
R_W h_{\text{min}} = h_{\text{min}}.
$$

This function $h_{\text{min}}$ is minimal in the ordered convex set

$$
\left\{ h \in C(X) \mid 0 \leq h \leq 1, R_W h = h, \int_X h \, d\nu = 1 \right\},
$$

and moreover

$$
h_{\text{min}}(x) = P_x(\mathbb{Z}).
$$

**Remark 6.3.** It turns out that the general case $\sum_y \cdots \leq 1$ may be reduced to (6.6). So (6.6) is not really a restriction.

**Proof of Lemma 6.2.** If $P$ is a Radon measure on $\Omega$, we set

$$
P[f] := \int_{\Omega} f(\omega) \, dP(\omega) \quad \text{for all } f \in C(\Omega). \quad (6.8)
$$

Set

$$
C_{\text{fin}}(\Omega) = \{ f \in C(\Omega) \mid \exists n \text{ such that } f(\omega) = f(\omega_1, \ldots, \omega_n), \}
$$

i.e., $f$ depends only on the first $n$ coordinates in $\Omega$. \quad (6.9)
Increasing $n$ in definition (6.9), we get an ascending nest of subalgebras of $C(\Omega)$,

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots \subset \mathfrak{A}_n \subset \mathfrak{A}_{n+1} \subset \cdots,$$  \hspace{1cm} (6.10)

with

$$\bigcup_{n=1}^{\infty} \mathfrak{A}_n = C(\Omega),$$

where $\bar{}$ stands for norm-closure. We set

$$C_{\text{fin}}(\Omega) := \bigcup_{n=1}^{\infty} \mathfrak{A}_n.$$  

An immediate application of Stone–Weierstraß shows that $C_{\text{fin}}(\Omega)$ is uniformly dense in $C(\Omega)$. Let $x \in X$, and $f \in C_{\text{fin}}(\Omega)$. Suppose

$$f(\omega) = f(\omega_1, \ldots, \omega_n),$$

and set

$$P_x[f] = \sum_{(\omega_1, \ldots, \omega_n) \in \mathbb{Z}_+^n} W(\tau_{\omega_1}x) \cdots W(\tau_{\omega_n} \cdots \tau_{\omega_1}x) f(\omega_1, \ldots, \omega_n).$$  \hspace{1cm} (6.11)

Note that, if there is some $h \in L^\infty(X)$ such that

$$f(\omega_1, \ldots, \omega_n) = h(\tau_{\omega_n} \cdots \tau_{\omega_1}x),$$

then

$$P_x[f] = \sum_{(\omega_1, \ldots, \omega_n)} W(\tau_{\omega_1}x) \cdots W(\tau_{\omega_n} \cdots \tau_{\omega_1}x) h(\tau_{\omega_n} \cdots \tau_{\omega_1}x)$$

$$= \sum_{y \in X, \sigma^n y = x} W(\sigma^{n-1}y) \cdots W(\sigma y) W(y) h(y)$$

$$= (R^m W h)(x).$$  \hspace{1cm} (6.12)

We now show that $P_x[f]$ is well-defined. This is the Kolmogorov consistency: we must check that the number $P_x[f]$ is the same when some $f \in \mathfrak{A}_n (\subset \mathfrak{A}_{n+1})$ is viewed also as an element in $\mathfrak{A}_{n+1}$. Then

$$f(\omega) = f(\omega_1, \ldots, \omega_n)$$

$$= f(\omega_1, \ldots, \omega_n, \omega_{n+1}),$$
and
\[
P_x \left[ f_{n+1} \right] = \sum_{\omega_1, \ldots, \omega_{n+1}} W (\tau_{\omega_1} x) \cdots W (\tau_{\omega_{n+1}} x) f (\omega_1, \ldots, \omega_{n+1}) \\
= \sum_{\omega_1, \ldots, \omega_n} W (\tau_{\omega_1} x) \cdots W (\tau_{\omega_n} x) \\
\cdot \left( \sum_{\omega_{n+1}} W (\tau_{\omega_{n+1}} \tau_{\omega_n} \cdots \tau_{\omega_1} x) \right) f (\omega_1, \ldots, \omega_n) \\
= \sum_{\omega_1, \ldots, \omega_n} W (\tau_{\omega_1} x) \cdots W (\tau_{\omega_n} x) f (\omega_1, \ldots, \omega_n) \\
= P_x \left[ f_n \right],
\]
as claimed.

The consistency conditions may be stated differently in terms of conditional probabilities: for \( f \in C (\Omega) \), set
\[
P_x^{(n)} [ f ] = P_x [ f \mid \mathcal{A}_n ] \tag{6.13}
= \sum_{(\omega_1, \ldots, \omega_n)} W (\tau_{\omega_1} x) \cdots W (\tau_{\omega_n} x) f (\omega_1, \ldots, \omega_n).
\]

We proved that
\[
P_x^{(n)} [ f ] = P_x^{(n+1)} [ f ] \quad \text{for all } f \in \mathcal{A}_n.
\]

Using now the theorems of Stone–Weierstraß and Riesz, we get the existence of the measure \( P_x \) on \( \Omega \). It is clear that it has the desired properties. In particular, the property (6.7) results from applying (6.11) to the function
\[
f (\omega) := \delta_{i_1, \omega_1} \cdots \delta_{i_n, \omega_n}, \quad \omega \in \Omega, \tag{6.14}
\]
when the point \((i_1, \ldots, i_n)\) is fixed.

These functions, in turn, span a dense subalgebra in \( C (\Omega) \) (by Stone–Weierstraß), so \( P_x \) is determined uniquely by (6.7).

\[\square\]

7 Transition measures

We now compute the Markov transition measures \( P_x \), as \( x \) varies over the set \( X \). The construction of \( P_x \), and the probability space \( \Omega \), begins with a sequence of measures \( P_x^{(n)} \), \( n = 1, 2, \ldots \), corresponding to finite paths of length \( n \). Then we show that the infinite path-space measure, Lemma 6.2, results from an application of the Kolmogorov extension principle. Our technical analysis
involves a certain transition operator $R$ which generalizes Lawton’s wavelet transition operator; see [36,37].

We already mentioned how the measures $P_x$ serve to prescribe sampling of functions on the real line, Lemma 4.1. However, a deeper understanding of this sampling viewpoint is facilitated by the introduction of the Perron–Frobenius–Ruelle operator $R$, Definitions 6.1, and an associated family of harmonic functions. For these harmonic functions, we note in Section 6 that there is a crucial analogue from classical harmonic analysis of Fatou-boundary function in the present discrete context. As noted after Definition 6.1 below, the boundary value functions take the form of certain cocycles, and the function $h$ from (3.11) corresponds to a special $\mathbb{N}_0$-cocycle. The existence of these cocycles is based on a martingale convergence theorem; see [33, Theorem 2.7.1] and [17] for additional details.

A fundamental tightness condition for the random walk model is introduced. The transition probabilities $P_x$ live in a universal probability space $\Omega$, but the essential convergence questions for the infinite products depend on the the $P_x$’s being supported on a certain copy of $\mathbb{N}_0$ (the natural numbers), or of $\mathbb{Z}$ (the integers). This refers to the natural embedded of $\mathbb{Z}$ in $\Omega$ which we described above.

8 Kolmogorov’s consistency condition

For general reference, we now make explicit the extension principle of Kolmogorov [35] in its function-theoretic form.

Lemma 8.1. (Kolmogorov) Let $N \geq 2$ be fixed, and let

$$\Omega = \{0, 1, \ldots, N - 1\}^N.$$  

For $n = 1, 2, \ldots$, let

$$P^{(n)} : \mathfrak{A}_n \to \mathbb{C}$$

be a sequence of linear functionals such that (i)–(iii) hold:

(i) $P^{(n)} [ \mathbb{1} ] = 1$, where $\mathbb{1}$ denotes the constant function 1 on $\Omega$,

(ii) $f \in \mathfrak{A}_n$, $f \geq 0$ pointwise $\implies P^{(n)} [ f ] \geq 0$,

and

(iii) $P^{(n)} [ f ] = P^{(n+1)} [ f ]$ for all $f \in \mathfrak{A}_n$. 

20
Then there is a unique Borel probability measure $P$ on $\Omega$ such that
\[ P[f] = P^{(n)}[f], \quad f \in \mathfrak{A}_n. \] (8.1)

Specifically, for $P$, we have the implication
\[ f \in C(\Omega), \ f \geq 0 \text{ pointwise} \implies P[f] \geq 0. \] (8.2)

**Remark 8.2.** Here we have identified positive linear functionals $P$ on $C(\Omega)$ with the corresponding Radon measures $\tilde{P}$ on $\Omega$, i.e.,
\[ P[f] = \int_{\Omega} f \, d\tilde{P}. \] (8.3)

This identification $P \leftrightarrow \tilde{P}$ is based on an implicit application of Riesz’s theorem; see [41, Chapter 1].

**Proof of Lemma 8.1.** The proof of Kolmogorov’s extension result may be given several forms, but we note that the argument we used above (in a special case), based on an application of the Stone–Weierstraß theorem, also works in general. \hfill \Box

The probability space $\Omega$

We note that the probability space $\Omega$ itself carries mappings $\sigma$ and $\tau_i, \ i = 0, 1, \ldots, N - 1$. Stressing the $\Omega$-dependence, we write
\[ \sigma^\Omega(\omega) = (\omega_2, \omega_3, \ldots) \quad \text{and} \quad \tau_i^\Omega(\omega) = (i, \omega_1, \omega_2, \ldots) \] (8.4)
for $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$.

**Remark 8.3.** The connection between the cylinder sets in (6.5) and the iterated function systems (IFS) $(X, \sigma, \tau_0, \ldots, \tau_{N-1})$ may be spelled out as follows: the cylinder sets in $\Omega$ generate the sigma-algebra of measurable subsets of $\Omega$, and similarly the subsets $\tau_{i_1} \cdots \tau_{i_n}(X) \subset X$ generate a sigma-algebra of measurable subsets of $X$. When nothing further is specified, these will be the sigma-algebras which we refer to when discussing the measurable functions on $\Omega$ and $X$. In particular, we will denote by $M(\Omega)$ and $M(X)$ the respective algebras of all bounded measurable functions on $\Omega$, respectively $X$.

Note that if $X = [0, 1]$, and if
\[ \tau_i x = \frac{x + i}{N}, \quad 0 \leq i \leq N - 1, \]
then we recover the familiar $N$-adic subintervals:

$$\tau_{i_1} \cdots \tau_{i_n}(X) = \left[ \frac{i_1}{N} + \cdots + \frac{i_n}{N^n}, \frac{i_1}{N} + \cdots + \frac{i_n}{N^n} + \frac{1}{N^n} \right]. \quad (8.5)$$

**Lemma 8.4.** There is a unique mapping $\rho : M(\Omega) \to M(X)$ which satisfies

$$\rho(fg) = \rho(f)\rho(g) \quad (8.6)$$

and

$$\rho\left(\chi_{A(i_1,\ldots,i_n)}\right) = \chi_{\tau_{i_1} \cdots \tau_{i_n}(X)}. \quad (8.7)$$

The mapping $\rho$ is an isomorphism of $M(\Omega)$ onto $M(X)$.

**Proof.** Recalling (6.14), we note that

$$\chi_{A(i_1,\ldots,i_n)}(\omega) = \delta_{i_1,\omega_1} \cdots \delta_{i_n,\omega_n}, \quad \omega \in \Omega.$$

As a result,

$$\chi_{A(i_1,\ldots,i_n)}\chi_{A(j_1,\ldots,j_n)} = \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}\chi_{A(i_1,\ldots,i_n)}. \quad (8.8)$$

We then define $\rho$ first on $\mathfrak{A}_n$ by

$$\rho\left(\sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n}\chi_{A(i_1,\ldots,i_n)}\right) = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n}\chi_{\tau_{i_1} \cdots \tau_{i_n}(X)},$$

where $a_{i_1,\ldots,i_n} \in \mathbb{C}$, and note that (8.6) is satisfied.

It is easy to check that the extension of $\rho$ from $\mathfrak{A}_n$ to $\mathfrak{A}_{n+1}$ is consistent. The final extension from $\bigcup_n \mathfrak{A}_n$ to $M(\Omega)$ is done by Kolmogorov’s lemma, and it can be checked that $\rho$ has the properties stated in the conclusion of the present lemma.

**Theorem 8.5.** Let $X$, $W$, and $N$ be as described in the beginning of this chapter, and let $\{P_x \mid x \in X\}$ be the process obtained in the conclusion of Lemma 6.2. Then

$$\sum_{i=0}^{N-1} W(\tau_i x) P_{\tau_i x} [f(i, \cdot)] = P_x [f] \quad \text{for all } f \in C(\Omega). \quad (8.9)$$

Moreover, equation (8.9) determines $(P_x)$ uniquely.

**Remark 8.6.** Stated informally, formula (8.9) is an assertion about the random walk: it says that if the walk starts at $x$, then with probability one, it makes a transition to one of the $N$ points $\tau_0(x), \ldots, \tau_{N-1}(x)$. The probability of the move $x \mapsto \tau_i x$ is $W(\tau_i x)$. Recall (6.6) asserts that $\sum_i W(\tau_i x) = 1$. 

22
Proof of Theorem 8.5. It follows from (6.11) and the arguments in the proof of Lemma 6.2 that it is enough to verify (8.9) for $f \in C_{\text{fin}}(\Omega)$, or for $f \in \mathfrak{A}_n$. Let $f \in \mathfrak{A}_n$. Then

$$\sum_i W(\tau_ix) P_{\tau_ix} [f(i, \cdot)]$$

$$= \sum_i \sum_{\omega_1, \ldots, \omega_n} W(\tau_ix) W(\tau_{\omega_1} \tau_ix) \cdots W(\tau_{\omega_n} \cdots \tau_{\omega_1} \tau_ix) f(i, \omega_1, \ldots, \omega_n)$$

$$= \text{by (6.11)} P_x[f],$$

which is the desired conclusion. Recall $C_{\text{fin}}(\Omega)$ is norm-dense in $C(\Omega)$.

Note that the formula (8.9) generalizes the familiar notion of selfsimilarity for measures introduced by Hutchinson in [28]. In fact, (8.9) may be restated as

$$\sum_{i=0}^{N-1} W(\tau_ix) P_{\tau_ix} \circ (\tau_i^\Omega)^{-1} = P_x. \quad (8.10)$$

The uniqueness part of the theorem follows then from Lemma 8.1. \qed

Acknowledgements. The author is pleased to acknowledge numerous constructive discussions about infinite products with Professors Dorin Dutkay and Richard Gundy; with Akram Aldroubi, John Benedetto and other members of our NSF-Focused Research Group (FRG). And we thank Brian Treadway for outstanding typesetting and helpful suggestions.

References

[1] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev. 43 (2001), 585–620.

[2] L. Baggett, A. Carey, W. Moran, and P. Ohring, General existence theorems for orthonormal wavelets, an abstract approach, Publ. Res. Inst. Math. Sci. 31 (1995), no. 1, 95–111.

[3] L.W. Baggett, P.E.T. Jorgensen, K.D. Merrill, and J.A. Packer, An analogue of Bratteli-Jorgensen loop group actions for GMRA’s, Wavelets, Frames, and Operator Theory (College Park, MD, 2003) (C. Heil, P.E.T. Jorgensen, and D.R. Larson, eds.), Contemp. Math., vol. 345, American Mathematical Society, Providence, 2004, pp. 11–25.

[4] L.W. Baggett, H.A. Medina, and K.D. Merrill, Generalized multi-resolution analyses and a construction procedure for all wavelet sets in $\mathbb{R}^n$, J. Fourier Anal. Appl. 5 (1999), 563–573.
[5] L.W. Baggett and K.D. Merrill, *Abstract harmonic analysis and wavelets in $\mathbb{R}^n$*, The Functional and Harmonic Analysis of Wavelets and Frames (San Antonio, 1999) (L.W. Baggett and D.R. Larson, eds.), Contemp. Math., vol. 247, American Mathematical Society, Providence, 1999, pp. 17–27.

[6] A.F. Beardon, *Iteration of Rational Functions: Complex Analytic Dynamical Systems*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991.

[7] J.J. Benedetto and M. Leon, *The construction of single wavelets in D-dimensions*, J. Geom. Anal. **11** (2001), no. 1, 1–15.

[8] O. Bratteli and P.E.T. Jorgensen, *Wavelets through a Looking Glass: The World of the Spectrum*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2002.

[9] C. Brislawn, *Fingerprints go digital*, Notices Amer. Math. Soc. **42** (1995), 1278–1283.

[10] X. Dai and D.R. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc. **134** (1998), no. 640.

[11] X. Dai, D.R. Larson, and D.M. Speegle, *Wavelet sets in $\mathbb{R}^n$*, J. Fourier Anal. Appl. **3** (1997), 451–456.

[12] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, SIAM, Philadelphia, 1992.

[13] P. Diaconis and D. Freedman, *Iterated random functions*, SIAM Rev. **41** (1999), 45–76, graphical supplement 77–82.

[14] V. Dobrić, R. Gundy, and P. Hitczenko, *Characterizations of orthonormal scale functions: A probabilistic approach*, J. Geom. Anal. **10** (2000), 417–434.

[15] D.E. Dutkay and P.E.T. Jorgensen, *Wavelets on fractals*, preprint, University of Iowa, 2003, [http://arXiv.org/abs/math.CA/0305443](http://arXiv.org/abs/math.CA/0305443), to appear in Rev. Mat. Iberoamericana.

[16] D.E. Dutkay and P.E.T. Jorgensen, *Martingales, endomorphisms, and covariant systems of operators in Hilbert space*, preprint, University of Iowa, 2004, [http://arXiv.org/abs/math.CA/0407330](http://arXiv.org/abs/math.CA/0407330), submitted to Int. Math. Res. Not.

[17] D.E. Dutkay and P.E.T. Jorgensen, *Hilbert spaces of martingales supporting certain substitution-dynamical systems*, Conform. Geom. Dyn. **9** (2005), 24–45.

[18] D.E. Dutkay and P.E.T. Jorgensen, *Iterated function systems, Ruelle operators, and invariant projective measures*, preprint, University of Iowa, 2005, [http://arxiv.org/abs/math.DS/0501077](http://arxiv.org/abs/math.DS/0501077), submitted to Math. Comp.

[19] K.J. Falconer, *Wavelet transforms and order-two densities of fractals*, J. Statist. Phys. **67** (1992), 781–793.

[20] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2001.
[21] R.F. Gundy, *Martingale theory and pointwise convergence of certain orthogonal series*, Trans. Amer. Math. Soc. **124** (1966), 228–248.

[22] R.F. Gundy, *Two remarks concerning wavelets: Cohen’s criterion for low-pass filters and Meyer’s theorem on linear independence*, The Functional and Harmonic Analysis of Wavelets and Frames (San Antonio, 1999) (L.W. Baggett and D.R. Larson, eds.), Contemp. Math., vol. 247, American Mathematical Society, Providence, 1999, pp. 249–258.

[23] R.F. Gundy, *Low-pass filters, martingales, and multiresolution analyses*, Appl. Comput. Harmon. Anal. **9** (2000), 204–219.

[24] R.F. Gundy, *Wavelets and probability*, preprint, Rutgers University, material presented during the author’s lecture at the workshop “Wavelets and Applications”, Barcelona, Spain, July 1–6, 2002, [http://www.imub.ub.es/wavelets/Gundy.pdf](http://www.imub.ub.es/wavelets/Gundy.pdf)

[25] R.F. Gundy and K. Kazarian, *Stopping times and local convergence for spline wavelet expansions*, SIAM J. Math. Anal. **31** (2000), 561–573.

[26] E. Hernández and G. Weiss, *A First Course on Wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Florida, 1996.

[27] C. Heil, P.E.T. Jorgensen, and D.R. Larson (eds.), *Wavelets, Frames, and Operator Theory: Papers from the Focused Research Group Workshop held at the University of Maryland, College Park, MD, January 15–21, 2003*, Contemp. Math., vol. 345, American Mathematical Society, Providence, 2004.

[28] J.E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[29] S. Jaffard, Y. Meyer, and R.D. Ryan, *Wavelets: Tools for Science & Technology*, revised ed., SIAM, Philadelphia, 2001.

[30] P.E.T. Jorgensen, *Ruelle Operators: Functions Which Are Harmonic with Respect to a Transfer Operator*, Mem. Amer. Math. Soc. **152** (2001), no. 720.

[31] P.E.T. Jorgensen, *Matrix factorizations, algorithms, wavelets*, Notices Amer. Math. Soc. **50** (2003), 880–894, [http://www.ams.org/notices/200308/200308-toc.html](http://www.ams.org/notices/200308/200308-toc.html)

[32] P.E.T. Jorgensen, *Iterated function systems, representations, and Hilbert space*, Internat. J. Math. **15** (2004), 813–832.

[33] P.E.T. Jorgensen, *Analysis and Probability*, monograph manuscript, 2005.

[34] P.E.T. Jorgensen, *Measures in wavelet decompositions*, Adv. Appl. Math. **23** (2005), 561–590.

[35] A.N. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer-Verlag, Berlin–New York, 1977, reprint of the 1933 original; English translation: *Foundations of the Theory of Probability*, Chelsea, 1950.
[36] W.M. Lawton, *Necessary and sufficient conditions for constructing orthonormal wavelet bases*, J. Math. Phys. **32** (1991), 57–61.

[37] W.M. Lawton, *Multiresolution properties of the wavelet Galerkin operator*, J. Math. Phys. **32** (1991), 1440–1443.

[38] S.G. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, Orlando–San Diego, 1998.

[39] Y. Meyer and R. Coifman, *Wavelets: Calderón-Zygmund and Multilinear Operators*, Cambridge Studies in Advanced Mathematics, vol. 48, Cambridge University Press, Cambridge, 1997, Translated from the 1990 and 1991 French originals by David Salinger.

[40] E. Nelson, *Topics in Dynamics, I: Flows*, Mathematical Notes, Princeton University Press, Princeton, NJ, 1969.

[41] W. Rudin, *Real and Complex Analysis*, third ed., McGraw-Hill, New York, 1987.

[42] D. Ruelle, *The thermodynamic formalism for expanding maps*, Comm. Math. Phys. **125** (1989), 239–262.

[43] C.E. Shannon, *Communication in the presence of noise*, Proc. Inst. Radio Engineers **37** (1949), 10–21.

[44] F. Spitzer, *Principles of Random Walk*, second ed., Graduate Texts in Mathematics, vol. 34, Springer-Verlag, New York, 1976, first softcover printing, 2001.

[45] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, Massachusetts, 1996.

[46] D.F. Walnut, *An Introduction to Wavelet Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2002.

[47] M.V. Wickerhauser, *Best-adapted wavelet packet bases*, Different Perspectives on Wavelets (San Antonio, TX, 1993) (I. Daubechies, ed.), Proc. Sympos. Appl. Math., vol. 47, American Mathematical Society, Providence, 1993, pp. 155–171.

[48] M.V. Wickerhauser, *Adapted Wavelet Analysis from Theory to Software*, IEEE Press, New York, A.K. Peters, Wellesley, MA, 1994.