Numerical calculation of the natural vibration frequencies of thermo-elastic rods taking into account the effect of the temperature gradient

O G Kikvidze¹, G G Sakhvadze² and G Zh Sakhvadze²

¹ Akaki Tsereteli State University, Kutaisi, Georgia
² Mechanical Engineering Research Institute of Russian Academy of Sciences (IMASH RAN), Moscow, Russia

Abstract. Elastic rod elements are considered, the axial line of which in the natural state is a flat curve and remains a flat curve after loading. The following assumptions are used: the normal cross-sections, which are flat before deformation, remain flat and normal to the deformed thermo-elastic axial line after deformation; the temperature field is stationary and non-uniform. Nonlinear equations of motion of the rod are obtained, taking into account the rotational inertia of cross-section and the extensibility of the axial line.

Keywords: thermo-elastic rod, dynamic loading, temperature gradient, vibration frequency

1. Introduction
The use of new light metallic materials in building structures makes it relevant to study the effect of the temperature factor on the strength and stiffness within the limits of elasticity, both under static and dynamic loads. The definition of natural vibration frequencies of an elastic rod, taking into account the axial line extension, is considered in the papers [1-3]. Non-uniform temperature distribution at small temperature change can have a significant impact on the frequency of free vibrations of the rod. The use of numerical calculation methods makes it possible to solve the nonlinear problem and automates calculations for various anchoring conditions and complex rod geometry.

2. Problem statement
The aim of the presented article is to develop a method of numerical calculation of the dynamics of rod elements of structures under thermo-mechanical loading. Such calculations are necessary when designing light metallic materials in building structures and other related industries, and today such a method of calculation is not found in publications.

3. Theory
Let us consider the rod vibrations in the plane in which the axial line is located. We shall investigate the rod motion relative to the natural (unloaded) state. The temperature field is stationary and varies only along the height of the cross-section in the bending plane. In this formulation the static equations at large displacements are obtained in the paper [4].

Let us consider the rod element that has a forward velocity relative to the y and z axes and an angular velocity θ̇ relative to the x axis. In its original state we denote the radius of curvature of the...
axial line by $r_0$, and the angle of inclination of the tangent to the $z$ axis by $\theta_0$. In general, the rod element can be affected by distributed and concentrated forces and moments that are time-variant. In the Cartesian reference system, the forces and moments that affect the rod element when it moves in the bending plane are shown in Fig. 1.

**Figure 1.** Illustration to the derivation of the equation of the rod motion.

Using the d’Alambert principle and the static equations [4] and taking into account the forces and moments of inertia, the differential equations of motion will take form

\[
\rho A(l) \frac{\partial^2 v}{\partial t^2} = \frac{\partial R}{\partial l} + q_y, \\
\rho A(l) \frac{\partial^2 w}{\partial t^2} = \frac{\partial H}{\partial l} + q_z, \\
\frac{\partial}{\partial l} (I_x \theta) = \frac{\partial M}{\partial l} + m + R \cos \theta - H \sin \theta,
\]

where $\rho$ is the density of the material; $A$ is the cross-sectional area; $v$, $w$ are the displacements in the direction of the $y$ and $z$ axes, respectively; $I_x^0$ is the physical moment of inertia of the unit length rod element. For the principal axes of the cross-section $I_x^0=I_y$; $I_x$ is the geometrical moment of inertia of the cross-section, $M$ is the bending moment; $R$, $H$ are the components of the internal force vector; $q_y$, $q_z$ are the components of the distributed external force vector; $m$ is the intensity of the external bending moment; $l$ is the arc length of the deformed thermo-elastic line.

For the kinematics of deformation, the relations obtained in the paper [4] are valid. By replacing the usual derivatives in them with partial derivatives, we obtain

\[
\frac{\partial v}{\partial l_0} = (1 + \varepsilon_0) \sin \theta - \sin \theta_0, \\
\frac{\partial w}{\partial l_0} = (1 + \varepsilon_0) \cos \theta - \cos \theta_0, \\
\frac{\partial \theta}{\partial l_0} = \frac{1 + \varepsilon_0}{r_0} + k_x,
\]

where $\varepsilon_0$ is the deformation of the thermo-elastic line; $l_0$ is the arc length of the non-deformed thermo-elastic line; $k_x$ characterizes the curvature change.

The $\varepsilon_0$ and $k_x$ values are determined by the formulas [4]

\[
\varepsilon_0 = \frac{N}{A^*} + \left[ \int \varepsilon^T EdA \right] / A^*, \quad k_x = \frac{M}{I_x^*} + \left[ \int \varepsilon^T y EdA \right] / I_x^*,
\]

where $A^* = \int EdA$ is the generalized area; $I_x^* = \int y^2 EdA$ is the generalized moment of inertia; $E=E(T)$ is the material elasticity modulus; $T=T(y)$ is the temperature; $\varepsilon^T$ is the temperature deformation.
The normal force $N$ in the cross-section is \[ N = H \cos \theta + R \sin \theta \]

According to the formulas (3), we shall represent the deformation $\epsilon_0$ and curvature $k_x$ as a sum of components from the corresponding force factors and temperature

\[ \epsilon_0 = \epsilon_0^N + \epsilon_0^T, \quad k_x = k_x^M + k_x^T, \quad \epsilon_0^N = \frac{N}{A}, \quad k_x^M = \frac{M}{I_x}, \]

\[ \epsilon_0^T = \frac{1}{A} \int e^T E dA, \quad k_x^T = \frac{1}{I_x} \int e^T y E dA. \]

For straight rods $r_0 \to \infty$, $\theta_0 = 0$ and equations (1) and (2) take the following form:

\[ \rho A(z) \frac{\partial^2 v}{\partial t^2} = \frac{\partial R}{\partial z} + q_y, \]
\[ \rho A(z) \frac{\partial^2 w}{\partial t^2} = \frac{\partial M}{\partial z} + m + R \cos \theta, \quad (5) \]
\[ \rho I_x(z) \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial M}{\partial z} + R, \quad \frac{\partial \theta}{\partial z} = k_x. \]

3.1. Small free vibrations of a straight rod

When designing elastic rod elements operating in dynamic modes, it is necessary to determine the spectrum of frequencies (more specifically, the first few frequencies) depending on the anchoring conditions and the static stress-strain state. The frequencies are determined from the equations of small free vibrations of the rod relative to its natural state or relative to the state of equilibrium.

We shall obtain the equations of small free vibrations of the rod, assuming that the inner forces and displacements arising during the vibrations are small. The components of the displacement and force vectors ($v, w, \theta, R, H, M$) are values of the first order of smallness, therefore their products are neglected in the derivation of the equations of motion. Let us set the external loads equal to zero $q_y = q_z = 0$, $m = 0$.

In the case of small displacements $\sin \theta \approx \theta, \cos \theta \approx 1, \frac{1}{1+\epsilon_0} \approx 1 - \epsilon_0$ and we disregard the summand of the second order of smallness $\epsilon_0 k_x$. Consequently, the nonlinear equations (4) are simplified and take the form [4]:

\[ \rho A(z) \frac{\partial^2 v}{\partial t^2} = \frac{\partial R}{\partial z}, \quad \frac{\partial v}{\partial z} = \theta, \]
\[ \rho I_x(z) \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial M}{\partial z} + R, \quad \frac{\partial \theta}{\partial z} = k_x. \]

The equations for an elastic rod without taking into account the axis deformation and temperature changes are listed in [5].

3.2. Numerical calculation method
To determine the frequencies of free vibrations, we present the force and kinematic factors in the form:

\[ v(z, t) = v^*(z) e^{i\omega t}, \quad k_x(z, t) = k_x^*(z) e^{i\omega t}, \]

\[ w(z, t) = w^*(z, t) e^{i\omega t}, \quad R(z, t) = R^*(z, t) e^{i\omega t}, \]

\[ \theta(z, t) = \theta^*(z) e^{i\omega t}, \quad H(z, t) = H^*(z) e^{i\omega t}, \]

\[ \varepsilon_0(z, t) = \varepsilon_0^*(z) e^{i\omega t}, \quad M(z, t) = M^*(z) e^{i\omega t}. \]

From the system of equations (5) taking into account formulas (6), we obtain ordinary differential equations:

\[ \frac{d v^*}{dz} = -\rho A(z) \omega^2 v^*, \quad \frac{d R^*}{dz} = -\rho I_x \omega^2 \theta^* - R^*, \]

\[ \frac{d R^*}{dz} = -\rho A(z) \omega^2 \theta^*, \quad \frac{d w^*}{dz} = \varepsilon_0^*(z), \quad \frac{d M^*}{dz} = -\rho I_x \omega^2 \theta^* - R^*, \]

\[ \frac{d w^*}{dz} = \varepsilon_0^*(z), \quad \frac{d M^*}{dz} = -\rho I_x \omega^2 \theta^* - R^*, \quad \frac{d \theta^*}{dz} = k_x^*. \]

To integrate the system of differential equations, (7) boundary conditions are necessary that reflect the anchoring of the rod ends. For the numerical solution of the problem we introduce dimensionless quantities [4]:

\[ \lambda_1 = \frac{A_0}{\rho_0 A_0 L^2}, \quad \lambda_2 = \frac{\rho I_x}{\rho_0 A_0 L^2}, \quad \lambda_3 = \frac{E I_x}{E_0 I_{x0}}, \quad \lambda_4 = \frac{E I_x}{E_0 I_{x0}}, \]

\[ \lambda_5 = \frac{E I_x}{E_0 I_{x0}}, \quad \lambda_6 = \frac{E I_x}{E_0 I_{x0}}, \quad \lambda_7 = \frac{E I_x}{E_0 I_{x0}}. \]

\[ \lambda_8 = \frac{E I_x}{E_0 I_{x0}}, \quad \lambda_9 = \frac{E I_x}{E_0 I_{x0}}. \]

\[ \Omega^2 = \frac{\rho_0 A_0}{E_0 I_{x0}} \frac{\omega^2 L^4}{T_0}, \]

where: \( T_0, E_0 \) is the room temperature and the corresponding modulus of the elasticity of the material, \( \alpha \) is the coefficient of thermal expansion, \( L \) is the beam length, \( A_0, I_{x0} \) is the area and the moment of inertia of the cross-section at the origin of the coordinates.

In dimensionless values, the system of equations (7) has the form:

\[ \frac{d R^*}{dz} = -\lambda_1 \Omega^2 \varphi^*, \quad \frac{d v^*}{dz} = \theta^*, \]

\[ \frac{d R^*}{dz} = -\lambda_2 \Omega^2 \varphi^*, \quad \frac{d w^*}{dz} = \lambda_3 H + \varepsilon_0^*, \]

\[ \frac{d M^*}{dz} = -\lambda_3 \Omega^2 \theta - R^*, \quad \frac{d \theta^*}{dz} = \lambda_4 \lambda_5 + \frac{k_x^*}{L}. \]

The system of ordinary differential equations (8) can be written in vector-matrix form. To do this, let us introduce designations:

\[ x_1 = \varphi, \quad x_2 = \varphi^*, \quad x_3 = \theta, x_4 = \theta^*, \quad x_5 = H, \quad x_6 = M. \]

Consequently, the system of equations (8) will take the form:

\[ \frac{d X}{dz} = B X + C, \]

where:

\[ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ \varepsilon_0^* \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ -\lambda_1 \Omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 \Omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 \Omega^2 - 1 & 0 & 0 \end{pmatrix}. \]
There are six boundary conditions on both ends of the vibrating rod, from which relations between
the constants of the general solution and the frequency equation (the eigenvalue problem) can be
obtained. Thus, the forms of free vibrations and their frequencies will be established.

On the computational side, the eigenvalue problems are very similar to boundary problems, to
solve which we use the shooting method. The difference lies in the shooting not only on the missing
left boundary conditions, but also on the required eigenvalues.

The mathematical editor Mathcad uses the \textit{sbval} and \textit{bvalfit} functions to solve the eigenvalue
problems. The system of equations (8) is supplemented by a differential equation with a boundary
condition:

$$\frac{dx_7}{d\bar{z}} = 0, \quad x_7(0) = A,$$

where: $x_7 = \Omega^2$, $A$ is the shooting parameter.

4. Obtained simulation results and discussion

Calculations have been performed for a free cantilever beam of rectangular cross-section. The
temperature in the cross-section changes according to the square function

$$T = T_0 + (T_1 - T_0)(\frac{y}{b})^2,$$

where $b$ is the height of the cross-section.

The boundary conditions have the form $x_1(0) = 0; x_2(0) = 0; x_3(0) = 0; x_4(1) = 0; x_5(1) = 0; x_6(1) = 0$. Figures 2 and 3 show the results of the calculation. For strength calculations, the first
natural frequency is essential [5]. According to the calculation, it equals to 6.2, which is about 11%
less than for an elastic rod without taking into account the axial deformation [6].

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure2.png}
\caption{The dependence of the dimensionless displacement components on the
dimensionless coordinate $\bar{Z}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure3.png}
\caption{The dependence of the dimensionless components of the internal force vector on the
dimensionless coordinate $\bar{Z}$.}
\end{figure}

Designations: 1 - $s^4 \equiv \Theta \cdot 10^5$,
2 - $s^2 \equiv \bar{\nu} \cdot 10^5$, 3 - $s^3 \equiv \bar{W} \cdot 10^5$.

The cantilever has been calculated when a kinematic perturbation, varying according to the cosine
law, acts in the cross-section of the anchorage. The temperature deformations are not taken into
account in the calculations. The dimensionless peak value of the kinematic perturbation is assumed to
be 0.001. The boundary conditions have the form: $x_1(0) = 0.001; x_2(0) = 0; x_3(0) = 0; x_4(1) = 0; x_5(1) = 0; x_6(1) = 0.$
Figure 4 shows the calculation results.

**Figure 4.** The dependence of dimensionless displacement components on the \( Z \) coordinate.

Designations: 1 - \( s^3 \equiv \overline{w} \cdot 10^8 \), 2 - \( s^4 \equiv 0 \cdot 10^{10} \).

**Figure 5.** The dependence of dimensionless vibration frequency \( \Omega \) on the temperature \( T_1 \) (\( ^0 \mathrm{C} \)).

Designations: 1 - \( \Omega = \frac{\Omega}{10^3} \), 2 - \( \Omega_1 = \frac{\Omega_1}{10^3} \).

By numerical calculation, the value of eigenvalue \( \Omega = 0.339 \) is obtained; the cross-sectional force and the bending value are actually equal to zero.

Figure 5 shows the graphs of change in the dimensionless vibration frequency when the rod is non-uniformly heated in the cross-section, when the temperature on the one side of the rod is constant and equal to \( 20^\circ \mathrm{C} \), and on the other side it changes and when the rod is uniformly heated. The boundary conditions are: \( x_1(0) = 0; x_2(0) = 0; x_3(0) = 0; x_4(1) = 0; x_5(1) = 0; x_6(1) = 0. \)

5. Conclusion

The most general nonlinear equations of the rod motion in the plane under thermo-mechanical loading have been obtained, taking into account the rotational inertia of cross-section and the deformation of the thermo-elastic axial line. The numerical calculation method of natural vibration frequencies is developed in the mathematical editor Mathcad. The results of calculations show that if the rod is uniformly heated, the first natural vibration frequency decreases with increasing temperature. It has been found that with non-uniform heating in the cross-section, the temperature gradient significantly affects the value of the vibration frequency. It has also been found that the vibration frequency in general is three times greater than with a uniform temperature distribution. Calculations have shown that the difference between the vibration frequencies increases with an increase in the temperature gradient in the cross-section. With the help of the developed method we can also determine the resonance zones during thermo-metrical loading of rod elements of structures or buildings, which are usually calculated according to the rod model.

6. References

[1] Valle J, Fernandez D, Madrenas J 2019 *Int J Mech Sci* **153-154** pp 380–390
[2] Bokaian A J 1990 *Sound Vib* **142** (3) pp 481–498
[3] Li X, Tang A, Xi L J 2013 *Constr Steel Res* **80** pp 15–22
[4] Kikvidze O 2003 *Problems of mechanical engineering and machine reliability* **1** pp 49–53
[5] Kiselev V A 1980 *Structural mechanics. Dynamics and stability of structures* (Moscow: Stroyizdat)
[6] Timoshenko S P, Yang D H, Weaver W 1985 *Fluctuations in engineering* (Moscow: Mashinostroenie) (in Russian)