A proposal for exploring quantum theory in curved spacetime in lab

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Gravity curves spacetime. In regions where the de Broglie wavelength is very small compared to the curvature of spacetime, the wave equations in flat spacetime can be generalized to curved spacetime. The validity of the formulation when the de Broglie wavelength becomes comparable to the curvature is an open question. To test these formulations experimentally, huge energy of the order of the Planck mass would be required. Euclideanized spacetime is used to model thermal systems. In this work, an equivalence between spatial variation of temperature in thermal baths and curvature of Euclideanized spacetime is propounded. The variation in temperature is recast as a variation in the metric. The Dirac equation is then solved in this curved Euclideanized spacetime. The curvature in Euclideanized spacetime can be obtained in Chromodynamic or Electromagnetic scale energies. If this equivalence is correct, it could provide a platform to experimentally explore quantum mechanics in curved spacetime.

Keywords : Quantum theory, Curved spacetime, Dirac Equation, Temperature, Equivalence
PACS numbers : 04.62.+v, 04.60.-m, 04.20.Gz

I. INTRODUCTION

Quantum theory in curved spacetime is a topic of intense research. In fact, it has led to many interesting phenomena like the Hawking radiation [1] and Unruh effect [2]. However, testing the quantum theory in curved spacetime is a formidable challenge due to the high energies required to curve spacetime. The regime where the de Broglie wavelength becomes comparable to the spacetime curvature is an even more interesting regime, as the current physics is expected to break down. However, experimental exploration in this regime is an even more formidable challenge, as the energies involved are in the Planck scale.

On a different front, Euclideanized spacetime is often used to model thermal systems. The temperature is modeled as inverse of imaginary time. In this work, the equivalence between temperature and Euclideanized spacetime is taken a step further, by modeling the temperature variation in space as curvature in Euclidean spacetime. It may be clarified that a spatial variation in temperature does not curve the original Minkowski spacetime. It is the mathematical equivalence between the spatial variation of temperature and the curvature of the corresponding Euclideanized spacetime that is being propounded here. If this equivalence is indeed true, then experimental results in a system of a thermal bath, with temperature gradient, can be used to explore the formulation of quantum mechanics in curved spacetime.

The thermal bath can either be a Quantum Chromodynamic (QCD) plasma or a Quantum Electrodynam (QED) plasma. If the thermal bath were to be a QCD plasma, namely the Quark Gluon Plasma (QGP), then the energies involved would be in the GeV scale and the distances involved would be in femtometer (fm) scale. These are realistic systems that are being produced at the Large Hadron Collider (LHC) and Relativistic Heavy Ion Collider (RHIC) [3]. If however the thermal bath were to be a QED plasma, then the energies reduce further to keV or even tens of ev [4], and the distances involved would be in nanometer (nm) or picometer (pm) range. A QED plasma is composed of electrons, ions and neutral atoms. There are typically two types of QED plasma, a) thermal plasma, which is in local thermal equilibrium and b) non-thermal or cold plasma [5]. In a thermal plasma, all the species constituting the plasma are at the same or similar temperature [6].

The rest of the paper is as follows. In Sec. II, we first look at the Wilson Polyakov loop correlator which determines the potential between two particles in a thermal bath. This is used to motivate the equivalence between temperature gradients and curvature of spacetime. In Sec. III we solve the Dirac equation in the Euclidean curved spacetime for a massive spinor. In Sec. IV the numerical work performed for solving the Dirac equation is presented. Finally, in Sec. V we look at the conclusions.

II. THE EQUIVALENCE

Let us first consider a scenario where a heavy Quark and anti-Quark are situated in a thermal bath, having a temperature gradient. The Wilson-Polyakov loop that has been used to model the scenario of the temperature gradient is depicted in Fig. I The relation of the Wilson-Polyakov loop correlator to the potential between the Quark and anti-Quark could be modeled as [6]:

\[ < P(0)P(L) > = e^{-V_{QQ} \beta}, \]

(1)

where, \( \beta = 1/T \) and \( T \) is the temperature of the system. \( V_{QQ} \) is the potential between the Quark and anti-Quark.
dependence can be seen as a result of curvature in the
space, with a “constant” curvature. If time were to depend on the spatial location, then the Euclideanized spacetime should be curved.

This gives rise to an equivalence principle, that variation in temperature can be recast as a variation in the metric. If this is indeed true, it is possible to mimic curved Euclideanized spacetime by resorting to spatial variation of temperature in a thermal bath.

Since spatial temperature variation is required, the thermal bath (either the QCD or QED thermal bath), would be in a state of local thermal equilibrium instead of a full-fledged thermal equilibrium. It seems that QGP system can be described by ideal hydrodynamics \[7\], which implies that the QCD plasma would be in local thermal equilibrium. As described in Sec. \[6\] the QED plasma can be either a cold plasma or a thermal plasma. The thermal plasma is hotter, and is in a state of local thermal equilibrium \[6\]. The cold plasma is however, a non-equilibrium thermal state. The formalism presented here, would then apply mainly to the QGP or the QED thermal plasma, which is in a state of local thermal equilibrium.

Thus we see that by drawing upon the equivalence between temperature gradients and Euclideanized spacetime curvature, it is possible to study curved spacetime at energies as low as keV scale. Temperature gradients in GeV scale are already produced in the lab at the LHC and RHIC. However, these are QCD systems in the strong coupling regime, and hence difficult to model analytically. QED system in the keV range may present an excellent opportunity to study experimentally, curved Euclideanized spacetime, and as well as model it analytically to a high degree of accuracy.

III. DIRAC SPINOR

A candidate Euclideanized metric representing spatial temperature variation is:

\[
ds^2 = g_{00}(z)dt^2 - dx^2 - dy^2 - dz^2,
\]

with \(g_{00}(z) = V(z)\), and \(t = i\beta\).

The Ricci scalar curvature for the metric in Eq. \[2\] is given by

\[
R = \frac{2VV'' - V'^2}{2V^2},
\]

where the ‘ \(\prime\) refers to differentiation w.r.t. \(z\).

The Dirac equation in curved space time is given by \[8-10\]

\[
i\gamma^a e^a_\mu D_\mu \psi - m \psi = 0
\]

where,

- \(D_\mu = \partial_\mu - \frac{i}{2} \omega^a_\mu \sigma_{ab}\),
- \(\omega^a_\mu\) is the spin connection,
\( \sigma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b] \), and \( \gamma_a \) being the Dirac matrices in flat spacetime,

- \( e_\mu^\nu \) is the vierbein.

In the notation used, \( \mu, \nu, \) etc., are indices in curved spacetime, and \( a, b, \) etc., are indices in flat spacetime.

We now solve the Dirac equation given by Eq. 4

The coordinate axes used are \((i\beta, x, y, z)\). The term, \( \frac{-i}{c} \sigma_{\mu\nu} \sigma_{\nu\lambda} \) for the metric in Eq. 2 evaluates to \( \frac{1}{8} \gamma_0 \gamma^3 \), with \( \gamma^0, \gamma^3 \) being the Dirac matrices in flat space, and \( f(z) = \frac{-2V'(z)}{V(z)} \). We take the solution to be of the form

\[
\psi = u(z) \exp(-ig_0 p^0 \beta),
\]

where the spinor \( u(z) \) is given by

\[
u(z) = \begin{pmatrix} (\zeta_0(z) + i\eta_0(z))
\end{pmatrix},
\]

with, \( \zeta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), i.e. \( \zeta_0 \) indicates a spin up or spin down particle. Substituting in Eq. 4 and evaluating both the real and imaginary parts to 0, we get:

\[
-k_2 \zeta_1 - s \zeta_1' - m \eta_2 = 0,
\]

\[
k_2 \eta_1 + sn \eta_1' - m \zeta_2 = 0,
\]

\[
-k_1 \zeta_2 + s \zeta_2' - m \eta_1 = 0,
\]

\[
k_1 \eta_2 - sn \eta_2' - m \zeta_1 = 0,
\]

where,

- \( s \) is the eigenvalue of the Pauli \( \sigma^3 \) matrix. Explicitly, \( s = 1 \), if \( \zeta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( s = -1 \) if \( \zeta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Thus, it’s value is twice the spin of the spinor particle.

- \( k_1 = \frac{C - f(z)s/8}{\sqrt{V(z)}} \).

- \( k_2 = \frac{C + f(z)s/8}{\sqrt{V(z)}} \).

In coming up with the equations in Eq. 7 we have taken \( p^0 = \frac{C}{\sqrt{V(z)}} \), where, \( C = \frac{(2n+1)\pi}{\beta_T(0)}, \) \( n = 0, 1, 2, \ldots; \) \( \beta_T(z) \) is the inverse temperature. In flat spacetime, with \( V(z) = V_0 = \text{constant}, \) \( \frac{C}{V_0} \) would be the Matsubara frequency. In fact, without loss of generality, one can take \( V_0 = 1 \) and \( V(0) = 1 \) (by absorbing the scaling factor into \( \beta \)), which gives, \( C = \frac{(2n+1)\pi}{\beta(0)} \). In this article, \( C \) itself is sometimes referred to as Matsubara frequency. This choice of \( p^0 \), keeps the exponent in Eq. 3 independent of \( z \), and is also seen to solve the Dirac equation. The anti-periodicity of \( \psi \) is discussed at the end of the current section, Sec. III.

A simple algebraic manipulation of the equations in Eq. 7 results in:

\[
\zeta_1' + s(k_2 - k_1) \zeta_1 + (-k_1 k_2 + sk_2^2 - m^2) \zeta_1 = 0,
\]

\[
\eta_1' + s(k_2 - k_1) \eta_1 + (-k_1 k_2 + sk_2^2 - m^2) \eta_1 = 0,
\]

\[
\zeta_2' + s(k_2 - k_1) \zeta_2 + (-k_1 k_2 - sk_1^2 - m^2) \zeta_2 = 0,
\]

\[
\eta_2' + s(k_2 - k_1) \eta_2 + (-k_1 k_2 - sk_1^2 - m^2) \eta_2 = 0.
\]

We first solve Eq. 8 in flat space i.e. when the temperature gradient is 0. This would provide a baseline \( \zeta_1^{\text{flat}}, \eta_1^{\text{flat}}, \) etc. When, the gradient \( V'(z) = 0 \), and \( V(z) = V_0 \), the equations in Eq. 8 simplify to:

\[
\zeta_1'' + \left( -\frac{C^2}{V_0} - m^2 \right) \zeta_1 = 0,
\]

\[
\eta_1'' + \left( -\frac{C^2}{V_0} - m^2 \right) \eta_1 = 0,
\]

\[
\zeta_2'' + \left( -\frac{C^2}{V_0} - m^2 \right) \zeta_2 = 0,
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\]

whose solution is given by \( \zeta_1^{\text{flat}} = A_1 \exp \left( -\sqrt{\frac{C^2}{V_0} + m^2} z \right) \) and \( \eta_2^{\text{flat}} = A_2 \exp \left( \sqrt{\frac{C^2}{V_0} + m^2} z \right) \). For any plasma of sufficient width, \( A_2 \ll A_1 \). So, we ignore \( A_2 \), and take \( A_1 = 1 \). Thus, we get:

\[
\zeta_1^{\text{flat}} = \exp \left( -\sqrt{\frac{C^2}{V_0} + m^2} z \right).
\]

It is easily seen that the summation over the Matsubara frequencies, i.e., \( \sum_n \zeta_1(z) \) converges for \( z \neq 0 \). The same solution is obtained for \( \eta_1^{\text{flat}}, \eta_2^{\text{flat}}, \) and \( \eta_2^{\text{flat}}, \) as the equations in Eq. 8 are the same. Substituting in Eq. 4 and evaluating it at \( z = 0 \), we get a relation between the initial conditions, \( \eta_1(0), \zeta_1(0), \eta_2(0) \) and \( \zeta_2(0) \):

\[
\zeta_1(0) = \frac{k_1^{\text{flat}}(0) + sk_0 \eta_2(0)}{m},
\]

\[
\eta_1(0) = \frac{-k_1^{\text{flat}}(0) - sk_0 \zeta_2(0)}{m},
\]

\[
\zeta_2(0) = \frac{k_2^{\text{flat}}(0) - sk_0 \eta_1(0)}{m},
\]

\[
\eta_2(0) = \frac{-k_2^{\text{flat}}(0) + sk_0 \zeta_1(0)}{m},
\]

where \( k_0 = \sqrt{\frac{C^2}{V_0} + m^2} \). The superscript "flat" refers to the values of \( k_2(z) \) and \( k_1(z) \) in flat spacetime, which happens when temperature gradient is zero. In fact, \( k_2^{\text{flat}}(z) = k_1^{\text{flat}}(z) = \frac{C}{V_0} \), as \( f(z) = 0 \). The initial conditions for \( \zeta_1(0), \eta_1(0), \zeta_2(0) \) and \( \eta_2(0) \) are maintained the same when going over from flat spacetime to curved spacetime.

To understand the spinor behavior, in the given curved space, we specifically look at \( \zeta_1 \) in detail. From Eqs. 8 one can infer that the other wavefunctions, \( \eta_1, \zeta_2 \) and \( \eta_2 \) have same or similar behavior. When the gradient becomes non-zero, let the modified spinor solution be \( \zeta_1 = \zeta_1^{\text{flat}} + \Delta \zeta_1 \). Expand \( \Delta \zeta_1 \) as a polynomial, i.e., \( \Delta \zeta_1 = \theta_1 z + \frac{\theta_2}{2} z^2 + \ldots \). Substituting in Eq. 7 expanding the other terms in Taylor series and equating the \( 0^{th} \)
power of $z$ to 0, we get,

$$\theta_1 = \frac{-f(0)}{8\sqrt{V_0}} \zeta_1(0). \quad (12)$$

Substituting $\theta_1$ in the equation for $\zeta_1(= \zeta_1^{\text{flat}} + \Delta \zeta_1)$ in Eq. 8 and evaluating the expression at $z = 0$ and making use of the initial conditions from Eq. 11 we get after some algebra,

$$\theta_2 = \left[ \frac{f(0)}{4\sqrt{V_0}} \left( \sqrt{\frac{C^2}{V_0} + m^2 + \frac{f(0)}{8\sqrt{V_0}} - sk_2'(0) \right) \right] \zeta_1(0). \quad (13)$$

It is to be noted that $\theta_1$ is essentially the first derivative $\Delta \zeta_1''(0)$, and $\theta_2$, the second derivative $\Delta \zeta_1''$. $\theta_1$ is seen to be purely a function of $f(z)$ i.e. spacetime curvature, and independent of other parameters like spin, ‘s’, the Matsubara frequency, ‘C’, and mass, ‘m’. Thus, to a first order, the deviation from the baseline curve, $\zeta_1^{\text{flat}}$ is dependent only on the spin connection, which in turn depends upon the curvature of the space. $\theta_2$ on the other hand is dependent on $s$, $C$ and $m$. Thus, the deviation to the spinor solution from $\zeta_1^{\text{flat}}$, due to the curvature in the given Euclideanized spacetime, depends to a second order on the spin, $s$, and mass, $m$.

We now explore the anti-periodicity of the fermionic wavefunction $\psi$ around the topological cylinder at all points in space. As discussed earlier, let $C = (2n+1)\pi$. Then, $\exp \left\{ -ig_{00}(z)\beta(z) \right\} = \exp \left\{ -i(2n+1)\pi \right\}$ at $\beta = \beta_T(0)$, which is independent of $z$. In other words, the anti-periodicity of the fermion wavefunction is retained at all values of $z$. This also determines, that, $\beta = \beta_T(0)$ would be the value used in R.H.S. of Eq. 11. We note that the inverse temperature at a point $z$ is given by "proper" $\beta = \beta_p = \sqrt{g_{00}(z)}\beta$. A "proper" Matsubara frequency could then be given as, $\omega_p = \sqrt{g_{00}}\beta = \frac{C}{\sqrt{V(z)}}$.

**IV. NUMERICAL ANALYSIS**

For the purpose of numerical analysis, we take $V(z) = \frac{1}{(1+\alpha z)^2}$, with $\alpha = 0.09$. The mass of an electron has been taken as 0.511 MeV [11]. The temperature taken (at $z = 0$) was $\frac{1}{\beta_T(0)} = 1$keV. Equation 8 was solved numerically for $\zeta_1(z)$, using the ode45 function of octave software. A linearly spaced grid of ten thousand points was chosen for simulation and ranged from $z_0 = 0$ to $z_1 = 10$MeV$^{-1} \approx 1.973$pm. The initial conditions for simulations were estimated as: $\zeta_1^{\text{sim}}(z) \approx \zeta_1^{\text{flat}}(z)$, and $\zeta_1^{\text{sim}}'(z) \approx \zeta_1^{\text{flat}'}(z)$. In order to finally compare with $\zeta_1^{\text{flat}}$, we scale the simulated $\zeta_1^{\text{sim}}$ as

$$\zeta_1(z) = \zeta_1^{\text{sim}}(z) \frac{\zeta_1^{\text{flat}}(0)}{\zeta_1^{\text{sim}}(0)}.$$

This ensures that $\zeta_1(0) = \zeta_1^{\text{flat}}(0)$. It is possible to perform the above scaling, because, from Eq. 8 it is evident that, if $\zeta_1(z)$ is a solution, then any scaled version of $\zeta_1(z)$ is also a solution. The plot of $\zeta_1(z)$ and $\zeta_1^{\text{flat}}(z)$ is shown in Fig. 2. The Ricci scalar curvature, $R(z)$, is plotted in Fig. 3.
V. CONCLUSION

Starting with the help of a Polyakov loop correlator, we have propounded a mathematical equivalence between temperature gradient and curvature of Euclideanized spacetime. The variation in temperature is recast as a variation in the metric. The Dirac equation for a massive fermion is then solved in the resulting Euclideanized curved spacetime. The solution indicates an exponentially decaying wavefunction in the thermal bath. The spacetime curvature leads to modification in the decay rate.

It may be possible to stretch the mathematical equivalence to regions where a characteristic length in the system becomes comparable to that of the curvature of the Euclideanized spacetime. Define the quantity $q_1 = \left| \frac{\zeta'}{\zeta_1} \right| = \left| \frac{\eta'}{\eta_1} \right|$. Then $q_1$ can be interpreted as the imaginary momentum. Similarly, one can have $q_2 = \left| \frac{\zeta'}{\zeta_2} \right| = \left| \frac{\eta'}{\eta_2} \right|$. In Quantum systems in curved spacetime, an interesting region would be when the de Broglie wavelength $> \frac{1}{\sqrt{R}}$, where $R$ = Ricci scalar curvature. In a similar analogy, a region of interest could be when $\frac{1}{\eta_1} \geq \frac{1}{\sqrt{R}}$ and $\frac{1}{\eta_2} \geq \frac{1}{\sqrt{R}}$; or alternatively, when $q_1 \leq \sqrt{R}$ and $q_2 \leq \sqrt{R}$. This can happen when

- The mass $m$ is very small.
- The Matsubara frequency $C$ is small, or alternatively, one has a large $\beta_T$, and one is looking at low Matsubara frequencies. These conditions then boil down to having low mean temperature but relatively large temperature gradient. Large temperature gradients will lead to large $R$.

Thus regions where $q_i \gg \sqrt{R}$ (i = 1, 2) and $q_i \leq \sqrt{R}$, would be two regions where one might want to explore the Dirac equation in curved space. However, local thermal equilibrium may break down for large $R$, rendering the region $q_i \leq \sqrt{R}$, difficult to explore.

It is seen that the length scales shown in the Fig. 2, are in picometer range, which would still be formidable to reproduce in the lab. But at least the scales are not in the Planck scale. Future research may be able to come up with scenarios, where the distances involved are in nanometer range instead of picometer range.

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