ON DECOMPOSITION OF $\theta_2^{2n}(\tau)$ AS THE SUM OF EISENSTEIN SERIES AND CUSP FORMS

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ABSTRACT. Based on the values of the Weierstrass elliptic function $\wp(z|\tau)$ at $z = \pi\tau/2$, $(\pi + \pi\tau)/2, (\pi + \pi\tau)/4, (\pi + 2\pi\tau)/4$ and the theory of modular forms on the arithmetic group $\Gamma_0(2)$, we decompose $\theta_2^{2n}(\tau)$ as sum of Eisenstein series and cusp forms. Using the recurrence relation of $\wp(2n)(z|\tau)$, we provide an algorithm to determine the exact form of these cusp forms.

1. INTRODUCTION

We will adopt the definitions of theta functions given in [9, Chapter 21]. The reader’s familiarity with the basic properties of Weierstrass elliptic function and the theta functions is assumed. Let $\theta_j(z|\tau), j = 1, 2, 3, 4,$ denote the Jacobi’s theta functions and, for brevity, $\theta_j(\tau) = \theta_j(0|\tau)$. It is easy to find that

$$\theta_2(\tau) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} = 2q^{1/4}(q^2; q^2)_\infty (-q^2; q^2)_\infty.$$

Here and later we use the standard $q$-series notation and $q = \exp(\pi i \tau)$ with $\tau \in \mathbb{H} = \{\tau \mid \tau = x + iy \text{ and } y > 0\}$, where $\mathbb{H}$ denote the upper half plane:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),$$

for all positive integers $n$.

In [8] Sun made an interesting connection between the Wallis’ formula and $\theta_2^2(\tau)$. His observation is as follows. By (1.1), we have

$$\prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{4n-2})(1 - q^{4n+2})} = (1 - q^2) \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{4n-2})^2} = \frac{1}{4}(1 - q^2)q^{-1/2}\theta_2^2(\tau).$$

From the well-known Wallis’ formula

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}$$

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and observe that
$$\lim_{q \to 1} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{4n-2})(1 - q^{4n+2})} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n - 1)(2n + 1)},$$
we have
$$\frac{\pi}{2} = \lim_{q \to 1} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{4n-2})(1 - q^{4n+2})} = \lim_{q \to 1} \frac{1}{4} (1 - q^2)q^{-1/2}\theta_2^2(\tau).$$
Thus, Sun regarded
$$\prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{4n-1})(1 - q^{4n+1})}$$
as a $q$-analogue of the Wallis’ formula. And Sun further listed two identities:
$$\prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^{2n-1}}^4 = \sum_{k=0}^{\infty} \frac{q^k(1 + q^{2k+1})}{(1 - q^{2k+1})^2},$$
$$\prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^{2n-1}}^8 = \sum_{k=0}^{\infty} \frac{q^{2k}(1 + 4q^{2k+1} + q^{4k+2})}{(1 - q^{2k+1})^4}.$$
Recently, based on results in [1], Goswami [5] generalized Sun’s results by obtaining a representation of
$$\prod_{n=1}^{\infty} \left(1 - q^{2n} \right)^{4k}$$
as the sum of Eisenstein series and cusp forms. In particular, for $k = 3$, he proved
$$\sum_{k=0}^{\infty} \frac{q^k(1 + q^{2k+1})P_4(q^{2k+1})}{(1 - q^{2k+1})^6} = 256q \prod_{n=1}^{\infty} \left(1 - q^{2n} \right)^{12} + (q;q)^{12};$$
where $P_4(x) = x^4 + 236x^3 + 1446x^2 + 236x + 1$.

It seems worthwhile pointing out certain important distinctions between our work and Goswami’s. He considered exclusively the decomposition $\theta_2^{4k}(q)$ as the sums of Eisenstein series and cusp forms for $k = 1, 2...$; whereas we deal with $\theta_2^{2k}(q)$ in Corollary 4.2, Corollary 4.5 and Corollary 5.3. In addition, he showed that $\theta_2^{4k}(q)$ (with $q = e^{2\pi i \tau}$) are modular forms of weight $2k$ with respect to the arithmetic group $\Gamma_0(4)$ for $k = 1, 2...$. In our work, with $q = e^{\pi i \tau}$, we obtain a refinement of Goswami’s results by showing that $\theta_2^{8k}(\tau)$ and $\theta_2^{8k-4}(\tau)$ are modular forms of weight $4k$ and $4k - 2$ with respect to the arithmetic group $\Gamma_0(2)$ and $\Gamma(2)$ for $k = 1, 2...$, respectively in Section 4. We remark that the arithmetic groups $\Gamma(2)$ and $\Gamma_0(4)$ are isomorphic.
Based on the the corollaries established in Section 3, we shall provide different and self-contained proofs of Goswami’s results, and re-state them in slightly different forms as Corollary 7.3 and Corollary 7.7. Furthermore, we prove that the palindromic feature of the coefficients of the polynomials exhibited above: $x + 1, x^2 + 4x + 1, x^4 + 236x^3 + 1446x^2 + 236x + 1$ in the cases of $\theta_2^{4k}(q)$ for $k = 1, 2$ and 3. In fact, they hold in general for all the polynomials associated with $\theta_2^{k\pm 2}(q)$ ($k = 1, 2\ldots$). The corresponding problem for the cases $\theta_2^{8k\pm 4}(q)$ is considerably more complicated than that of $\theta_2^{8k\pm 4}(q)$. Not only are the Eisenstein series involved modular forms with multipliers, but also they are somewhat peculiar combinations of two Eisenstein series.

For the purpose of motivation, we begin with the following list of representations of $\theta_2^{2k}(\tau)$ as the sums of Eisenstein series and the cusp forms.

$$
\theta_2^{2}(\tau) = 4q^{1/2}\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1 - q^{2n+1}},
$$

$$
\theta_2^{4}(\tau) = 16\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1 - q^{4n+2}},
$$

$$
\theta_2^{6}(\tau) = 4q^{1/2}\sum_{n=0}^{\infty} (2n+1)^2 \left( \frac{q^n}{1 + q^{2n+1}} - \frac{(-1)^n q^n}{1 - q^{2n+1}} \right),
$$

$$
\theta_2^{8}(\tau) = 2^8\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1 - q^{4n}},
$$

$$
5\theta_2^{10}(\tau) = 4q^{1/2}\sum_{n=0}^{\infty} (2n+1)^4 \left( \frac{q^n}{1 + q^{2n+1}} + \frac{(-1)^n q^n}{1 - q^{2n+1}} \right) - 8q^{1/2} \left( \frac{q^2; q^2}{(q^4; q^4)} \right)_{\infty},
$$

$$
\theta_2^{12}(\tau) = 16\sum_{n=0}^{\infty} \frac{(2n+1)^5 q^{2n+1}}{1 - q^{4n+2}} - 16q(q^2; q^2)_{\infty}^{12},
$$

$$
61\theta_2^{14}(\tau) = 4q^{1/2}\sum_{n=0}^{\infty} (2n+1)^6 \left( \frac{q^n}{1 + q^{2n+1}} - \frac{(-1)^n q^n}{1 - q^{2n+1}} \right) - 91 \times 2^6 q^{3/2}(q^2; q^2)_{\infty}^{10}(q^4; q^4)_{\infty}^{4},
$$

$$
17\theta_2^{16}(\tau) = 2^{13}\sum_{n=1}^{\infty} \frac{n^7 q^{2n}}{1 - q^{4n}} - 2^{13} q(q; q)_{\infty}^8 (q^2; q^2)_{\infty}^8,
$$

$$
1385\theta_2^{18}(\tau) = 4q^{1/2}\sum_{n=0}^{\infty} (2n+1)^8 \left( \frac{q^n}{1 + q^{2n+1}} + \frac{(-1)^n q^n}{1 - q^{2n+1}} \right)
- q^{1/2} \left( \frac{q^2; q^2}{(q^4; q^4)} \right)_{\infty}^{30} - 763 \times 2^9 q^{5/2}(q^2; q^2)_{\infty}^6 (q^4; q^4)_{\infty}^{12},
$$

$$
(q^2; q^2)_{\infty} = \prod_{n=1}^{\infty} (1 - q^n).$$
\[
\theta_2^{20}(\tau) = 16 \sum_{n=0}^{\infty} \frac{(2n+1)^9 q^{2n+1}}{1 - q^{4n+2}} - 16 q \frac{(q^2; q^2)^{28}}{(q^4; q^4)^8} \\
- 77 \times 2^{12} q^3 (q^2; q^2)^4 (q^4; q^4)^{16},
\]

\[
50521 \theta_2^{22}(\tau) = 4q^{1/2} \sum_{n=0}^{\infty} (2n+1)^{10} \left( \frac{q^n}{1 + q^{2n+1}} - \frac{(-1)^n q^n}{1 - q^{2n+1}} \right) \\
- 138677 \times 2^{14} q^{7/2} (q^2; q^2)^{2} (q^4; q^4)^{20} - 7381 \times 2^6 q^{3/2} \frac{(q^2; q^2)^{36}}{(q^4; q^4)^4},
\]

\[
691 \theta_2^{24}(\tau) = 2^{16} \sum_{n=1}^{\infty} \frac{n^{11} q^{2n}}{1 - q^{4n}} - 2^{16} q(q; q)^{24} - 259 \times 2^{19} q^2 (q^2; q^2)^{24}.
\]

We observe that, among these examples, the Eisenstein series part of \( \theta_2^{2k}(\tau) \) can be characterized in terms of the congruence class modulo 8 to which \( 2k \) belongs. For example, if \( 2k \equiv 4 \mod 8 \), the Eisenstein series is, modulo the constant multiple, of the form

\[
\sum_{n=0}^{\infty} \frac{(2n+1)^{k-1} q^{2n+1}}{1 - q^{4n+2}}.
\]

We now describe the setting and methods for establishing the identities of this paper. Let

\[
\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, c \equiv 0 \pmod{2} \right\}
\]

and

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.
\]

For \( \tau \in \mathbb{H} \) and \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2) \), define \( \sigma \tau = \frac{a \tau + b}{c \tau + d} \). Let

\[
G_0(2) = \{ \sigma \tau \mid \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2) \}.
\]

Then

\[
T(\tau) = \tau + 1 \quad \text{and} \quad S\tau = \frac{\tau}{-2\tau + 1}
\]

are generators for \( G_0(2) \). A choice of fundamental domain of \( G_0(2) \) is

\[
\Omega_0(2) = \{ \tau \mid 0 \leq x \leq 1 \text{ and } |\tau - 1/2| \geq 1/2 \}
\]

with the boundary properly identified. There are two cusp points: 0 and \( \infty \).
Using the facts:
\[ \theta_2(\tau + 1) = e^{\pi i/4} \theta_2(\tau), \quad \theta_2(-1/\tau) = \sqrt{-i\tau} \theta_4(\tau) \quad \text{and} \quad \theta_4(-1/\tau) = \sqrt{-i\tau} \theta_2(\tau), \]
it follows that
\[ \theta_2^2(\tau + 1) = i\theta_2^2(\tau) \quad \text{and} \quad \theta_2^2\left(\frac{\tau}{-2\tau + 1}\right) = (-2\tau + 1)\theta_2^2(\tau). \]

In general, for \( \sigma = \left(\begin{smallarray}{cc} a & b \\ c & d \end{smallarray}\right) \in \Gamma_0(2) \), we have

\[ \theta_2^2\left(\frac{a\tau + b}{c\tau + d}\right) = \psi(\sigma)(c\tau + d)\theta_2^2(\tau), \tag{1.3} \]

where
\[ \psi(\sigma) = \begin{cases} -ie^{\frac{\pi i}{2}(bd+1)} & \text{if } c \equiv 2 \pmod{4}, \\
e^{\frac{\pi i}{2}(bd+d-1)} & \text{if } c \equiv 0 \pmod{4}, \end{cases} \]
and the details is given in Theorem 7.9. Thus, in general,

\[ \theta_2^{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = \psi^k(\sigma)(c\tau + d)^k\theta_2^{2k}(\tau) \]

and, in particular,

\[ \theta_2^{8k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{4k}\theta_2^{8k}(\tau). \]

Let \( k \) be a non-negative integer and let \( M_k(\Gamma_0(2); \psi) \) denote the space of functions \( f \) on \( \mathbb{H} \) satisfying (see [2, p. 78]):

1. \( f \) is analytic on \( \mathbb{H} \);
2. \( f \) is analytic at all cusps of \( \Gamma_0(2) \);
3. \( f \left(\frac{a\tau + b}{c\tau + d}\right) = \psi(\sigma)(c\tau + d)^k f(\tau); \)

where \( \sigma = \left(\begin{smallarray}{cc} a & b \\ c & d \end{smallarray}\right) \in \Gamma_0(2). \)

Each \( M_k(\Gamma_0(2); \psi) \) can be decomposed further as:

\[ M_k(\Gamma_0(2); \psi) = E_k(\Gamma_0(2); \psi) \oplus S_k(\Gamma_0(2); \psi); \]

where \( E_k(\Gamma_0(2); \psi) \) and \( S_k(\Gamma_0(2); \psi) \) are the vector spaces of the cusp forms and Eisenstein series of weight \( k \). Thus,

\[ \theta_2^{2k}(\tau) \in M_k(\Gamma_0(2); \psi^k). \]
We note that if \( k \equiv 0 \pmod{4} \), then \( \psi^k \equiv 1 \). We denote \( S_k(\Gamma_0(2)) = S_k(\Gamma_0(2); 1) \), \( E_k(\Gamma_0(2)) = E_k(\Gamma_0(2); 1) \) and \( M_k(\Gamma_0(2)) = M_k(\Gamma_0(2); 1) \).

Let \( \wp(z|\tau) \) denote the Weierstrass elliptic function with the period lattice \( \Lambda \) generated by \( \pi \) and \( \pi\tau \). We say \( z \) is a point of order \( N \) if \( Nz \in \Lambda \). We will be concerned only with the values of the Weierstrass elliptic function \( \wp(z|\tau) \) at certain points of orders 2 and 4.

In Section 2, we will list the key identities of the theta functions, the formulas expressing \( \wp(z|\tau) \) in terms of the theta functions, and the values of \( \wp(z|\tau) \) along with its derivatives at certain points of orders 2 and 4. In Section 3, we derive the formulas expressing \( \wp(n)(z|\tau) \) in terms of the Eisenstein series and \( q \)-series expansions. In Sections 4 and 5, we decompose \( \theta_{2k}^2(\tau) \) into the sum of Eisenstein series and cusp forms. This is accomplished by constructing Eisenstein series which matches the values of \( \theta_{2k}^2(\tau) \) at the cusp points of \( \Omega_0(2) \) using the Weierstrass elliptic function. In Section 6, we derive a recurrence relation expressing the value of \( \wp(n)(z|\tau) \) in terms of \( \wp(k)(z|\tau) \), \( k < n \). Appealing to the recurrence relation, we derive the identities listed at the beginning of this section. Some miscellaneous results of independent interest and a list of identities are mentioned in Sections 7 and 8.

2. Preliminaries

We list the needed identities of the theta functions and the Weierstrass elliptic function.

(A) From [9], we find

\[
\begin{align*}
\theta_2^4(\tau) &= \theta_3^4(\tau) - \theta_4^4(\tau), \\
\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) &= 2q^{1/4}(q^2; q^2)_\infty^3,
\end{align*}
\]

\[
\theta_2(\tau)\theta_3(\tau) = \frac{1}{2}\theta_2^2(\tau/2), \\
\theta_3(\tau)\theta_4(\tau) = \theta_4^2(2\tau),
\]

and

\[
\lim_{\tau \to 0} \tau \theta_2^2(\tau) = i.
\]

(B) From [2, Theorem 1.12, Theorem 1.14, Theorem 1.18], we have

\[
(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),
\]

where

\[
\begin{align*}
g_2 &= \frac{4}{3} \left( 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1 - q^{2n}} \right), \\
g_3 &= \frac{8}{27} \left( 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^{2n}}{1 - q^{2n}} \right), \\
e_1 &= \wp \left( \frac{\pi}{2} | \tau \right) = \frac{1}{3} \left( \theta_3^4(\tau) + \theta_4^4(\tau) \right), \\
e_2 &= \wp \left( \frac{\pi \tau}{2} | \tau \right) = -\frac{1}{3} \left( \theta_2^4(\tau) + \theta_3^4(\tau) \right), \\
e_3 &= \wp \left( \frac{\pi + \pi \tau}{2} | \tau \right) = \frac{1}{3} \left( \theta_2^4(\tau) - \theta_4^4(\tau) \right);
\end{align*}
\]
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From [2, p. 10], we find

$$\wp(z|\tau) = \frac{1}{z^2} + \sum_{m,n=-\infty \atop (m,n)\neq(0,0)}^{\infty} \frac{1}{(z + m\pi + n\pi\tau)^2} - \frac{1}{(m\pi + n\pi\tau)^2},$$

(2.2)

$$\wp^{(k)}(z|\tau) = (k+1)! \sum_{m,n=-\infty}^{\infty} \frac{1}{(z + m\pi + n\pi\tau)^{k+2}},$$

for all positive integers $k$.

(C) The connection between the theta function $\theta_1(z|\tau)$ and $\wp(z|\tau)$ is given by

$$\wp(z|\tau) = -\left(\frac{\theta_1'}{\theta_1}\right)'(z|\tau) - \frac{E_2(\tau)}{3},$$

where

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}}.$$

(D) We recall that [9, p. 489]

$$\frac{\theta_1'}{\theta_1}(z|\tau) = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin 2nz,$$

$$\frac{\theta_4'}{\theta_4}(z|\tau) = 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \sin 2nz,$$

$$\frac{\theta_1'}{\theta_1}(z + \frac{\pi\tau}{2}|\tau) = \frac{\theta_1'}{\theta_4}(z|\tau) - i.$$

Thus

(2.3)

$$\wp(z + \frac{\pi\tau}{2}|\tau) = -\left(\frac{\theta_1'}{\theta_4}\right)'(z|\tau) - \frac{E_2(\tau)}{3}.$$

Setting $z = 0$ and $z = \frac{\pi}{2}$ in the above equation, respectively, we derive [3, 6]

$$\wp\left(\frac{\pi\tau}{2}|\tau\right) = -\frac{1}{3} \left(1 + 24 \sum_{n=0}^{\infty} \frac{nq^n}{1 + q^n}\right) = -\frac{1}{3} (\theta_2^4(\tau) + \theta_3^4(\tau)), $$

(2.4)

$$\wp\left(\frac{\pi + \pi\tau}{2}|\tau\right) - \wp\left(\frac{\pi + \pi\tau}{4}|\tau\right) = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{(2n + 1)q^{2n+1}}{1 - q^{4n+2}} = -\theta_2^4(\tau).$$

(E) From [6], we find

$$\wp\left(\frac{\pi + 2\pi\tau}{4}|\tau\right) = -\frac{1}{3} (\theta_3^4(2\tau) - 5\theta_2^4(2\tau)).$$


\[ \wp' \left( \frac{\pi + 2\pi \tau}{4} \right) = 4\theta_2^2(2\tau)\theta_4^4(2\tau), \]
\[ \wp'' \left( \frac{\pi + 2\pi \tau}{4} \right) = -2^4\theta_2^4(2\tau)\theta_4^4(2\tau). \]

Replacing \( \tau \) by \( \tau + 1/2 \) in the above equations, we have
\[ \wp \left( \frac{\pi + \pi \tau}{2} \right) = \frac{1}{3}(\theta_4^4(2\tau) + 5\theta_2^4(2\tau)), \]
\[ \wp' \left( \frac{\pi + \pi \tau}{2} \right) = 4i\theta_2^3(2\tau)\theta_4^4(2\tau), \]
\[ \wp'' \left( \frac{\pi + \pi \tau}{2} \right) = 2^4\theta_4^4(2\tau)\theta_4^4(2\tau). \]

(F) Since \( g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = \frac{4}{3}(\theta_3^6(\tau) + \theta_2^6(\tau) - \theta_1^6(\tau)\theta_4^6(\tau)) \) and \( \wp'' = 6\wp^2 - \frac{1}{2}g_2 \), we obtain
\[ \wp'' \left( \frac{\pi}{2} \right) = 6\wp^2 \left( \frac{\pi}{2} \right) - \frac{1}{2}g_2 = 2\theta_2^4(\tau)\theta_4^4(\tau), \]
\[ \wp'' \left( \frac{\pi \tau}{2} \right) = 6\wp^2 \left( \frac{\pi \tau}{2} \right) - \frac{1}{2}g_2 = 2\theta_2^4(\tau)\theta_4^4(\tau) = \frac{1}{8}\theta_2^4(\tau/2), \]
\[ \wp'' \left( \frac{\pi + \pi \tau}{2} \right) = 6\wp^2 \left( \frac{\pi + \pi \tau}{2} \right) - \frac{1}{2}g_2 = -2\theta_2^4(\tau)\theta_4^4(\tau). \]

3. Some Eisenstein series derived from the Weierstrass Elliptic function

We now construct Eisenstein series which will play a vital role in the decomposition of \( \theta_2^{2n}(\tau) \) into sums of Eisenstein series and cusp forms.

Let

\[ \chi_2(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \]

and \( \chi(n) = \sin \left( \frac{n\pi}{2} \right) \). Define

\[ \sigma_{k,\chi}(n) := \sum_{d|n} d^k \chi(d) = \sum_{d|n} d^k \sin \left( \frac{d\pi}{2} \right). \]

Differentiating (2.3) repeatedly, we derive

**Lemma 3.1.** For all integers \( k \geq 1 \), we have

\[ \wp^{(2k)} \left( z + \frac{\pi \tau}{2} \right) = (-1)^{k+1}2^{2k+3} \sum_{n=1}^{\infty} \frac{n^{2k+1}q^n}{1-q^{2n}} \cos 2nz, \]
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(3.2) $\varphi^{(2k-1)}\left(z + \frac{\pi \tau}{2}\right) = (-1)^{k+1}2^{2k+2} \sum_{n=1}^{\infty} \frac{n^{2k}q^n}{1 - q^{2n}} \sin 2nz.$

Taking $z = 0$ and $z = \pi/2$ in (3.1), respectively, and combining with (2.2), then

Theorem 3.2. For all integers $k \geq 1$, we have

(3.3) $\varphi^{(2k)}\left(\frac{\pi \tau}{2}\right) = \frac{(2k + 1)!}{\pi^{2k+2}} \sum_{m,n=-\infty}^{\infty} \frac{\chi_2(n)}{(m + n \frac{\pi}{2})^{2k+2}}$

(3.4) $= (-1)^{k+1}2^{2k+3} \sum_{n=1}^{\infty} \frac{n^{2k+1}q^n}{1 - q^{2n}}.$

(3.5) $\varphi^{(2k)}\left(\frac{\pi \tau}{2}\right) - \varphi^{(2k)}\left(\frac{\pi + \pi \tau}{2}\right) = \frac{2^{2k+2}(2k + 1)!}{\pi^{2k+2}} \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m \chi_2(n)}{(m + n \tau)^{2k+2}}$

(3.6) $= (-1)^{k+1}2^{2k+4} \sum_{n=0}^{\infty} \frac{(2n + 1)^{2k+1}q^{2n+1}}{1 - q^{4n+2}}.$

Theorem 3.3. For all integers $k \geq 1$, we have

(3.7) $\varphi^{(2k-1)}\left(\frac{\pi + 2\pi \tau}{4}\right) - i \varphi^{(2k-1)}\left(\frac{\pi + \pi \tau}{2}\right) = \frac{2^{4k+2}(2k)!}{\pi^{2k+1}} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (4n + 2)\tau)^{2k+1}} - \frac{i}{(4m + 2n + 2 + (4n + 2)\tau)^{2k+1}}$

(3.8) $= (-1)^k2^{2k+2} \sum_{n=0}^{\infty} (2n + 1)^{2k} \left(\frac{(-1)^n q^{2n+1}}{1 - q^{4n+2}} + \frac{q^{2n+1}}{1 + q^{4n+2}}\right)$

(3.9) $= (-1)^{k+1}2^{2k+3} \sum_{n=0}^{\infty} \sigma_{2k,\chi}(4n + 1)q^{4n+1};$

(3.10) $\varphi^{(2k-1)}\left(\frac{\pi + 2\pi \tau}{4}\right) + i \varphi^{(2k-1)}\left(\frac{\pi + \pi \tau}{2}\right) = \frac{2^{4k+2}(2k)!}{\pi^{2k+1}} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (4n + 2)\tau)^{2k+1}} + \frac{i}{(4m + 2n + 2 + (4n + 2)\tau)^{2k+1}}$

(3.11) $= (-1)^k2^{2k+2} \sum_{n=0}^{\infty} (2n + 1)^{2k} \left(\frac{(-1)^n q^{2n+1}}{1 - q^{4n+2}} - \frac{q^{2n+1}}{1 + q^{4n+2}}\right).$
\[(3.12) \quad = (-1)^{k+1} 2^{2k+3} \sum_{n=0}^{\infty} \sigma_{2k,\chi}(4n+3)q^{4n+3}.\]

**Proof.** From (2.2) and Lemma 3.1,
\[
\psi^{(2k-1)} \left( \frac{\pi + 2\pi \tau}{4} \right) \\
= (-1)^k 2^{2k+2} \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)^{2k} q^{2n+1}}{1 - q^{4n+2}} \\
= 2^{4k+2}(2k)! \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (4n + 2)^2)\tau}^{2k+1}, \\
- i\psi^{(2k-1)} \left( \frac{\pi + \pi \tau}{2} \mid \frac{2\tau + 1}{2} \right) \\
= (-1)^k 2^{2k+2} \sum_{n=1}^{\infty} \frac{(2n+1)^{2k} q^{2n+1}}{1 + q^{4n+2}} \\
= \frac{2^{4k+2}(2k)!}{\pi^{2k+1}} \sum_{m,n=-\infty}^{\infty} \frac{-i}{(4m + 2n + 2 + (4n + 2)\tau)^{2k+1}}.
\]

The identities (3.7), (3.8), (3.10) and (3.11) follow readily.

For (3.9) and (3.12), by expanding the summands into geometric series and inverting the order of summation, we readily find that
\[
\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^k q^{2n+1}}{1 - q^{4n+2}} \\
= \sum_{n=1}^{\infty} n^k q^n \sin \frac{n\pi}{2} \\
= \sum_{n=1,2|m}^{\infty} n^k q^n \sin \frac{n\pi}{2} \\
= \sum_{n=0}^{\infty} \sigma_{k,\chi}(2n+1) q^{2n+1}.
\]

Letting \( z = \frac{\pi}{4} \) in (3.2), we have
\[
\psi^{(2k-1)} \left( \frac{\pi + 2\pi \tau}{4} \right) \\
= (-1)^{k+1} 2^{2k+2} \sum_{n=1}^{\infty} \frac{n^{2k} q^n}{1 - q^{2n}} \sin \frac{n\pi}{2}.
\]
ON DECOMPOSITION OF $\theta_2^{2k}(\tau)$ AS THE SUM OF EISENSTEIN SERIES AND CUSP FORMS

$$= (-1)^{k+1}2^{2k+2} \sum_{n=0}^{\infty} \sigma_{2k,\chi}(2n + 1)q^{2n+1}. $$

Replacing $\tau$ by $\tau + 1/2$, we have

$$= (-1)^k2^{2k+2} \sum_{n=1}^{\infty} \frac{i^{n+1}n^{2k}q^n}{1 + (-1)^{n+1}q^{2n}} \sin \frac{n\pi}{2} $$

$$= (-1)^{k+1}2^{2k+2} \sum_{n=0}^{\infty} (-1)^n \sigma_{2k,\chi}(2n + 1)q^{2n+1}. $$

Combining the above two equations, we get the desired identities. □

For later use, we define, for integer $k \geq 1$,

$$M_{4k}(\tau; \chi_2) := \sum_{m,n=-\infty}^{\infty} \frac{\chi_2(n)}{(m + n\tau)^{4k}}, $$

$$M_{4k+2}^*(\tau; \chi_2) := \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m \chi_2(n)}{(m + n\tau)^{4k+2}}, $$

$$M_{2k+1}(\tau; \chi) := \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (2n + 1)\tau)^{2k+1}} + \sum_{m,n=-\infty}^{\infty} \frac{(-1)^{k+1}i}{(4m + 2n + 2 + (2n + 1)\tau)^{2k+1}}. $$

It is interesting to point out that $M_{2k+1}(\tau; \chi)$ is the difference of Eisenstein series constructed from elliptic functions of different moduli, $\tau, \tau + \frac{1}{2}$, evaluated at points of order 4.

Recall

$$\lim_{\tau \to 0} \tau \theta_2^2(\tau) = i, \quad \lim_{\tau \to \infty} \tau \theta_2^2(\tau) = 0, $$

$$\theta_2^{2k}(\tau + 1) = i^{k} \theta_2^{2k}(\tau), \quad \theta_2^{2k} \left( \frac{\tau}{-2\tau + 1} \right) = (-2\tau + 1)^{k} \theta_2^{2k}(\tau). $$

To decompose $\theta_2^{2k}(\tau)$ into the sum of Eisenstein series and cusp forms, we will show, in the following sections, that the behavior of these Eisenstein series under the transformations:

$$T(\tau) = \tau + 1 \quad \text{and} \quad S(\tau) = \frac{\tau}{-2\tau + 1}$$
matches precisely with that of \( \theta_{2k}^{2k}(\tau) \) under the same transformations. We recall that \( \Gamma_0(2) \), in the fundamental domain \( \Omega_0(2) \), has two cusp points with 0 and \( \infty \). Since both \( \theta_{2k}^{2k}(\tau) \) and these Eisenstein series vanish as \( \tau \to \infty \), we will scale these Eisenstein series to match the values of the corresponding \( \theta_{2k}^{2k}(\tau) \) at 0. This allows us to decompose \( \theta_{2k}^{2k}(\tau) \) into the sum of Eisenstein series and cusp forms.

4. REPRESENTATIONS OF \( \theta_{2k}^{8k}(\tau) \) AND \( \theta_{2k}^{8k-4}(\tau) \)

We begin by recalling [4, p. 35 (21)], for all positive integers \( k \),

\[
\zeta(2k) = (-1)^{k+1}2^{2k-1}n^{2k}B_{2k}(2k)!,
\]

where \( B_{2k} \) denotes the \( 2k \)-th Bernoulli number.

4.1. Representations of \( \theta_{2k}^{8k}(\tau) \). Recall

\[
\chi_2(n) = \begin{cases} 
0 & \text{if } n \text{ is even} \\
1 & \text{if } n \text{ is odd.}
\end{cases}
\]

**Theorem 4.1.** For all positive integers \( k \), we obtain

\[
\theta_{2k}^{8k}(\tau) = \frac{(4k)!}{\pi^{4k}(1 - 2^{4k})B_{4k}} \sum_{m,n=-\infty}^{\infty} \frac{\chi_2(n)}{(m + n\tau)^{4k}} + T_{4k}(\tau);
\]

where \( T_{4k}(\tau) \in S_{4k}(\Gamma_0(2)) \).

**Proof.** It is easy to verify that, by checking the generators of \( \Gamma_0(2) \), \( \theta_{2k}^{8k}(\tau) \) and

\[
M_{4k}(\tau; \chi_2) := \sum_{m,n=-\infty}^{\infty} \frac{\chi_2(n)}{(m + n\tau)^{4k}}
\]

are modular of weight 4k with respect to the arithmetic group \( \Gamma_0(2) \) which has fundamental domain \( \Omega_0(2) \) with cusps at 0 and \( \infty \).

We now consider the values of \( M_{4k}(\tau; \chi_2) \) and \( \theta_{2k}^{8k}(\tau) \) at the cusps. Clearly, as \( \tau \to \infty \),

\[
M_{4k}(\tau; \chi_2) \to 0 \quad \text{and} \quad \theta_{2k}^{8k}(\tau) \to 0.
\]

We need to choose an appropriate constant \( A_{4k} \), such that as \( \tau \to 0 \),

\[
\lim_{\tau \to 0} \tau^{4k} \left( \theta_{2k}^{8k}(\tau) - A_{4k}M_{4k}(\tau; \chi_2) \right) = 0.
\]

This will imply that

\[
\theta_{2k}^{8k}(\tau) = A_{4k}M_{4k}(\tau; \chi_2) + T_{4k}(\tau);
\]

where \( T_{4k}(\tau) \in S_k(\Gamma_0(2)) \).

We compute the values of \( A_{4k} \).
From (2.1) and (4.3),
\[ 1 = \lim_{\tau \to 0} \tau^{4k} \theta_2(\tau) = A_{4k} \lim_{\tau \to 0} \tau^{4k} M_{4k}(\tau; \chi_2). \]

We note
\[
\sum_{m,n=-\infty}^{\infty} \frac{\chi_2(n)}{(m+n\tau)^{4k}} = \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+(2n+1)\tau)^{4k}} = \frac{1}{\tau^{4k}} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^{4k}} + \sum_{m,n=-\infty}^{\infty} \frac{1}{m+(2n+1)\tau)^{4k}}.
\]

Next, we observe
\[
\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^{4k}} = 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{4k}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{4k}} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^{4k}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{4k}} - 2 \sum_{n=1}^{\infty} \frac{1}{n^{4k}} = (2 - 2^{1-4k}) \zeta(4k).
\]

Then,
\[
\lim_{\tau \to 0} \tau^{4k} \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+(2n+1)\tau)^{4k}} = \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^{4k}} + \lim_{\tau \to 0} \tau^{4k} \sum_{m,n=-\infty}^{\infty} \frac{1}{m+(2n+1)\tau)^{4k}} = (2 - 2^{1-4k}) \zeta(4k) + \lim_{\tau \to 0} \sum_{m,n=-\infty}^{\infty} \frac{1}{m/\tau + (2n+1)^{4k}}.
\]

\[ (4.5) \]

From (4.4) and (4.5), we obtain
\[ (4.6) \]
\[ A_{4k} = \frac{(4k)!}{(1 - 2^{4k})\pi^{4k} B_{4k}}. \]

Substituting (4.6) into the equation (4.3), we complete the proof of Theorem 4.1.
Corollary 4.2. For all positive integers \( k \), we have
\[
\theta_2^{8k}(\tau) = \frac{2^{4k+3}k}{(1-2^{4k})B_{4k}} \sum_{n=1}^{\infty} \frac{n^{4k-1}q^{2n}}{1-q^{4n}} + T_{4k}(\tau).
\]

Proof. From (3.3) and (3.4), we find
\[
\frac{(4k-1)!}{\pi^{4k}} \sum_{m,n=-\infty}^{\infty} \chi_2(n) \frac{1}{(m+n\tau)^{4k}}
= \frac{(4k-1)!}{\pi^{4k}} \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+(2n+1)\tau)^{4k}}
= \varphi^{(4k-2)}(\pi \tau | 2\tau)
\]
\[
-2^{4k+1} \sum_{n=1}^{\infty} \frac{n^{4k-1}q^{2n}}{1-q^{4n}}.
\]
(4.7)

From (4.7) and Theorem 4.1, we get the desired identity. \( \square \)

4.2. Representations of \( \theta_2^{8k+4}(\tau) \).

Theorem 4.3. For all positive integers \( k \), we have
\[
\theta_2^{8k+4}(\tau) = \frac{(4k+2)!}{(1-2^{4k+2})\pi^{4k+2}B_{4k+2}} \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m\chi_2(n)}{(m+n\tau)^{4k+2}} + T_{4k+2}(\tau);
\]
where \( T_{4k+2}(\tau) \in S_{4k+2}(\Gamma_0(2); \psi^2) \).

Proof. Let
\[
M^*_{4k+2}(\tau; \chi_2) = \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m\chi_2(n)}{(m+n\tau)^{4k+2}}.
\]
We need to show
\[
M^*_{4k+2}(\tau+1; \chi_2) = -M^*_{4k+2}(\tau; \chi_2),
M^*_{4k+2} \left( \frac{\tau}{-2\tau + 1}; \chi_2 \right) = (-2\tau + 1)^{4k+2} M^*_{4k+2}(\tau; \chi_2).
\]
However, we will establish the following stronger lemma.

Lemma 4.4. Let \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2) \). We have
\[
M^*_{4k+2} \left( \frac{a\tau + b}{c\tau + d}; \chi_2 \right) = (-1)^b (c\tau + d)^{4k+2} M^*_{4k+2}(\tau; \chi_2).
\]
Proof. Since $c \equiv 0 \pmod{2}$ and $ad \equiv 1 \pmod{2}$, thus $a \equiv d \equiv 1 \pmod{2}$,
\[
\chi_2(mc + na) = \chi_2(n)\chi_2(a) = \chi_2(n),
\]
and, since $\chi_2(n) = 0$ if $n$ is even,
\[
(-1)^{md+nb} = (-1)^m(-1)^nb,
\]

Then,
\[
\begin{align*}
M^*_{4k+2} \left( \frac{a\tau + b}{c\tau + d}; \chi_2 \right) & = (c\tau + d)^{4k+2} \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m \chi_2(n)}{((md + nb) + (mc + na)\tau)^{4k+2}} \\
& = (-1)^b(c\tau + d)^{4k+2} \sum_{m,n=-\infty}^{\infty} \frac{(-1)^{md+nb} \chi_2(mc + na)}{((md + nb) + (mc + na)\tau)^{4k+2}} \\
& = (-1)^b(c\tau + d)^{4k+2}M^*_{4k+2}(\tau; \chi_2).
\end{align*}
\]

This establishes the desired identity. □

We need choose an appropriate constant $A_{4k+2}$, such that as $\tau \to 0$,
\[
\lim_{\tau \to 0} \tau^{4k+2} \left( \theta_2^{8k+4}(\tau) - A_{4k+2}M_{4k+2}(\tau; \chi_2) \right) = 0. 
\tag{4.9}
\]

Next, we compute the values of $A_{4k+2}$. From (2.1) and (4.9), we have
\[
-1 = \lim_{\tau \to 0} \tau^{4k+2}\theta_2^{8k+4}(\tau) = A_{4k+2} \lim_{\tau \to 0} \tau^{4k+2}M^*_{4k+2}(\tau; \chi_2). 
\tag{4.10}
\]

Then,
\[
\lim_{\tau \to 0} \tau^{4k+2} \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m \chi_2(n)}{(m + n\tau)^{4k+2}} \\
= \sum_{n=-\infty}^{\infty} \frac{1}{(2n + 1)^{4k+2}} + \lim_{\tau \to 0} \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m \chi_2(n)}{(m/\tau + n)^{4k+2}} \\
= (2 - 2^{-4k-1})\zeta(4k + 2). 
\tag{4.11}
\]

Combining (4.10), (4.11) and (4.1), we obtain
\[
A_{4k+2} = \frac{(4k + 2)!}{(1 - 2^{4k+2})\pi^{4k+2}B_{4k+2}}
\]
and this yields the desired conclusion. □
**Corollary 4.5.** For non-negative integers $k$, we find

$$\theta_2^{8k+4}(\tau) = \frac{-8(2k+1)}{(1-2^{4k+2})B_{4k+2}} \sum_{n=0}^{\infty} \frac{(2n+1)^{4k+1}q^{2n+1}}{1-q^{4n+2}} + T_{4k+2}(\tau).$$

**Proof.** From (3.5) and (3.6), we have

$$\frac{(4k+1)!}{\pi^{4k+2}} \sum_{m,n=-\infty}^{\infty} \frac{(-1)^m \chi_2(n)}{(m+n\tau)^{4k+2}} = \left( -\psi^{(4k)}(\frac{\pi \tau}{2}) + \psi^{(4k)}(\frac{\pi + \pi \tau}{2}) \right).$$

(4.12)

Combining (2.4), (4.12) and (4.8), we obtain the desired identity. □

We remark that it is easy to see that $\theta_2^{8k+4}(\tau)$ and $M_{4k+2}(\tau; \chi_2)$ are modular of weight $4k+2$ with respect to the arithmetic group $\Gamma(2)$ which has fundamental domain $\Omega(2)$ with cusps at $0, 1$ and $\infty$;

$$\Omega(2) = \{ \tau : 0 \leq x \leq 2, |\tau - 1/2| \geq 1/2 \ and \ |\tau - 3/2| \geq 1/2 \}$$

is a fundamental domain of $\Gamma(2)$.

### 5. Representations of $\theta_2^{8k+2}(\tau)$ and $\theta_2^{8k-2}(\tau)$

Define

$$L(k) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k}.$$ 

We recall [1, p.42 Eq:(16)], for all integers $k \geq 0$,

$$L(2k+1) = \frac{(-1)^k \pi^{2k+1} E_{2k}}{2^{2k+2}(2k)!},$$

(5.1)

where $E_{2k}$ denotes the Euler’s number. We note $(-1)^k E_{2k} > 0$.

Let

$$M_{2k+1}(\tau; \chi) = \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m+1+(2n+1)\tau)^{2k+1}} + \sum_{m,n=-\infty}^{\infty} \frac{(-1)^{k+1} i}{(4m+2n+2+(2n+1)\tau)^{2k+1}}.$$
Lemma 5.1. For all integers \( k \geq 1 \), we have
\[
M_{4k+1}(\tau + 1; \chi) = iM_{4k+1}(\tau; \chi),
\]
\[
M_{4k+1} \left( \frac{\tau}{-2\tau + 1}; \chi \right) = (-2\tau + 1)^{4k+1} M_{4k+1}(\tau; \chi),
\]
\[
M_{4k-1}(\tau + 1; \chi) = -iM_{4k-1}(\tau; \chi),
\]
\[
M_{4k-1} \left( \frac{\tau}{-2\tau + 1}; \chi \right) = (-2\tau + 1)^{4k-1} M_{4k-1}(\tau; \chi).
\]

Proof. Let
\[
A(\tau) = \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (2n + 1)\tau)^{4k+1}},
\]
\[
B(\tau) = \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 2n + 2 + (2n + 1)\tau)^{4k+1}}.
\]

It is easy to check that
\[
A(\tau + 1) = B(\tau) \quad \text{and} \quad B(\tau + 1) = -A(\tau).
\]
That is, \( M_{4k+1}(\tau + 1; \chi) = A(\tau + 1) + iB(\tau + 1) = B(\tau) + iA(\tau) = iM_{4k+1}(\tau; \chi) \).

Next,
\[
A \left( \frac{\tau}{-2\tau + 1} \right) = (-2\tau + 1)^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (2n - 8m - 1)\tau)^{4k+1}}
\]
\[
= (-2\tau + 1)^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (2n - 8m - 1)\tau)^{4k+1}}
\]
\[
= (-2\tau + 1)^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (2n + 1)\tau)^{4k+1}}
\]
\[
= (-2\tau + 1)^{4k+1} A(\tau),
\]
and
\[
B \left( \frac{\tau}{-2\tau + 1} \right) = (-2\tau + 1)^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 2n + 2 + (-2n - 8m - 3)\tau)^{4k+1}}
\]
\[
= (-2\tau + 1)^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 2n + 2 + (-2n - 8m - 3)\tau)^{4k+1}}
\]
\[
= (-2\tau + 1)^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 2n + 2 + (2n + 1)\tau)^{4k+1}}
\]
\[ (-2\tau + 1)^{4k+1} B(\tau). \]

That is,

\[
M_{4k+1} \left( \frac{\tau}{-2\tau + 1}; \chi \right) = (-2\tau + 1)^{4k+1} A(\tau) - i (-2\tau + 1)^{4k+1} B(\tau) = (-2\tau + 1)^{4k+1} M_{4k+1}(\tau; \chi).
\]

The proofs of the remaining identities are identical, we omit them. \( \square \)

Theorem 5.2. For all integers \( k \geq 1 \), we obtain

\[
\theta^{8k+2}_2(\tau + 1) = \frac{2(4k)!}{\pi^{4k+1} E_{4k}} M_{4k+1}(\tau; \chi) + T_{4k+1}(\tau), \tag{5.2}
\]

\[
\theta^{8k-2}_2(\tau) = \frac{2(4k-2)!}{\pi^{4k-1} E_{4k-2}} M_{4k-1}(\tau; \chi) + T_{4k-1}(\tau), \tag{5.3}
\]

where \( T_{4k+1}(\tau) \in S_{4k+1}(\Gamma_0(2); \psi) \) and \( T_{4k-1}(\tau) \in S_{4k-1}(\Gamma_0(2); \psi^{-1}) \).

Proof. Recalling the facts:

\[
\theta^{8k+2}_2(\tau + 1) = i \theta^{8k+2}_2(\tau) \quad \text{and} \quad \theta^{8k+2}_2 \left( \frac{\tau}{-2\tau + 1} \right) = (-2\tau + 1)^{4k+1} \theta^{8k+2}_2(\tau),
\]

we will choose an appropriate constant \( A_{4k+1} \), such that as \( \tau \to 0 \),

\[
\lim_{\tau \to 0} \tau^{4k+1} \left( \theta^{8k+2}_2(\tau) - A_{4k+1} M_{4k+1}(\tau; \chi) \right) = 0. \tag{5.4}
\]

Then

\[
\theta^{8k+2}_2(\tau) = A_{4k+1} M_{4k+1}(\tau; \chi) + T_{4k+1}(\tau), \tag{5.5}
\]

where \( T_{4k+1}(\tau) \in S_{4k+1}(\Gamma_0(2); \psi) \).

We now compute the values of \( A_{4k+1} \).

From (2.1) and (5.4),

\[
\frac{i}{2^{4k+1}} = \lim_{\tau \to 0} \tau^{4k+1} \theta^{8k+2}_2(2\tau) = A_{4k+1} \lim_{\tau \to 0} \tau^{4k+1} M_{4k+1}(2\tau; \chi).
\]

Then,

\[
\lim_{\tau \to 0} \tau^{4k+1} M_{4k+1}(2\tau; \chi) = \lim_{\tau \to 0} \tau^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 1 + (4n + 2)\tau)^{4k+1}}
\]
ON DECOMPOSITION OF $\theta_{2n}^2(\tau)$ AS THE SUM OF EISENSTEIN SERIES AND CUSP FORMS

$- i \lim_{\tau \to 0} \tau^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 2n + 2 + (4n + 2)\tau)^{4k+1}}.$

The limit of the first sum is 0, to evaluate the second sum, we re-write it as

$\tau^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 2n + 2 + (4n + 2)\tau)^{4k+1}}$

and we note, as $\tau \to 0$, the second sum goes to 0 and the first sum becomes

$\tau^{4k+1} \sum_{m,n=-\infty}^{\infty} \frac{1}{(4m + 2n + 2 + (4n + 2)\tau)^{4k+1}} = - \sum_{m=-\infty}^{\infty} \frac{1}{(4m + 1)^{4k+1}} = - \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)^{4k+1}}.$

Letting $\tau \to 0$, we obtain

$(5.6) \quad \frac{1}{A_{4k+1}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m + 1/2)^{4k+1}}.$

Substituting (5.6) into the equation (5.5), we complete the proof of the (5.2) and (5.3) is identical, we omit it.

**Corollary 5.3.** For all positive integers $k$, we have

$(5.7) \quad \theta_{2}^{8k+2}(\tau) = - \frac{8}{E_{4k}} q^{1/2} \sum_{n=0}^{\infty} \sigma_{4k,\chi}(4n + 1)q^{2n} + T_{4k+1}(\tau)$

$$= \frac{4}{E_{4k}} q^{1/2} \sum_{n=0}^{\infty} (2n + 1)^{4k} \left( \frac{(-1)^n q^n}{1 - q^{2n+1}} + \frac{q^n}{1 + q^{2n+1}} \right) + T_{4k+1}(\tau),$$

$(5.8) \quad \theta_{2}^{8k-2}(\tau) = \frac{8}{E_{4k-2}} q^{1/2} \sum_{n=0}^{\infty} \sigma_{4k-2,\chi}(4n + 3)q^{2n+1} + T_{4k-1}(\tau)$

$$= - \frac{4}{E_{4k-2}} q^{1/2} \sum_{n=0}^{\infty} (2n + 1)^{4k} \left( \frac{(-1)^n q^n}{1 - q^{2n+1}} - \frac{q^n}{1 + q^{2n+1}} \right) + T_{4k-1}(\tau).$$
Proof. From (3.9), we find

\[(4k)! \frac{4}{\pi} M_{4k+1}(2\tau; \chi)\]

\[= \wp^{(4k-1)} \left( \frac{\pi + 2\pi \tau}{4} \right) - i\wp^{(4k-1)} \left( \frac{\pi + \pi \tau}{2} \right) |\tau + 1/2| \]

\[= -2^{4k+3} \sum_{n=0}^{\infty} \sigma_{4k,\chi}(4n+1)q^{4n+1} \]

\[= 2^{4k+2} \sum_{n=0}^{\infty} (2n+1)^{4k} \left( \frac{(-1)^n q^{2n+1}}{1 - q^{4n+2}} + \frac{q^{2n+1}}{1 + q^{4n+2}} \right).\]

Substituting (5.9) into (5.2), we get the desired identity (5.7) and so is (5.8). \(\Box\)

6. A Recurrence relation for the Weierstrass elliptic function

The main goal of this section is to determine the cusp forms appeared in the previous section using an algorithm based on a recurrence relation for \(\wp^{(2n)}(z|\tau)\).

It follows, from (F), that

\[\wp'' = 6\wp^2 - \frac{g_2}{2};\]

and from the Leibniz’s rule for differentiation, we have, for all positive integers \(n\),

\[\wp^{(n+2)} = 6 \sum_{k=0}^{n} \binom{n}{k} \wp^{(n-k)} \wp^{(k)} .\]

We conclude from (2.3) that

\[\wp^{(2n-1)} \left( \frac{\pi \tau}{2} |\tau \right) = 0,\]

for all positive integers \(n\).

This, together with (6.1), gives the following recurrence relation.

Lemma 6.1. For all positive integers \(n\), we have

\[\wp^{(2n+2)} \left( \frac{\pi \tau}{2} |\tau \right) = 6 \sum_{k=0}^{n} \binom{2n}{2k} \wp^{(2n-2k)} \left( \frac{\pi \tau}{2} |\tau \right) \wp^{(2k)} \left( \frac{\pi \tau}{2} |\tau \right).\]

Thus,

\[\wp^{(4)} \left( \frac{\pi \tau}{2} |\tau \right) = 12\wp \left( \frac{\pi \tau}{2} |\tau \right) \wp'' \left( \frac{\pi \tau}{2} |\tau \right),\]

\[\wp^{(6)} \left( \frac{\pi \tau}{2} |\tau \right) = 12 \left( \wp \left( \frac{\pi \tau}{2} |\tau \right) \wp^{(4)} \left( \frac{\pi \tau}{2} |\tau \right) + 3(\wp'' \left( \frac{\pi \tau}{2} |\tau \right))^2 \right).\]
\[
\varphi^{(8)} \left( \frac{\pi \tau}{2} | \tau \right) = 12 \left( \varphi \left( \frac{\pi \tau}{2} | \tau \right) \varphi^{(6)} \left( \frac{\pi \tau}{2} | \tau \right) + 15 \varphi'' \left( \frac{\pi \tau}{2} | \tau \right) \varphi^{(4)} \left( \frac{\pi \tau}{2} | \tau \right) \right)
\]
\[
= 12^3 \left( \varphi'' \left( \frac{\pi \tau}{2} | \tau \right) \varphi \left( \frac{\pi \tau}{2} | \tau \right) + \frac{3}{2} (\varphi'' \left( \frac{\pi \tau}{2} | \tau \right))^2 \varphi \left( \frac{\pi \tau}{2} | \tau \right) \right),
\]
and
\[
\varphi^{(10)} \left( \frac{\pi \tau}{2} | \tau \right) = 12 \left( \varphi \left( \frac{\pi \tau}{2} | \tau \right) \varphi^{(8)} \left( \frac{\pi \tau}{2} | \tau \right) + 28 \varphi'' \left( \frac{\pi \tau}{2} | \tau \right) \varphi^{(6)} \left( \frac{\pi \tau}{2} | \tau \right) + 35 \varphi'' \left( \frac{\pi \tau}{2} | \tau \right) \varphi^{(4)} \left( \frac{\pi \tau}{2} | \tau \right) \right)
\]
\[
= 12^3 \left( 12 \varphi'' \left( \frac{\pi \tau}{2} | \tau \right) \varphi' \left( \frac{\pi \tau}{2} | \tau \right) + 81 (\varphi \left( \frac{\pi \tau}{2} | \tau \right) \varphi'' \left( \frac{\pi \tau}{2} | \tau \right))^2 + 7 (\varphi'' \left( \frac{\pi \tau}{2} | \tau \right))^3 \right).
\]
It is clear that we can express \( \varphi^{(2k)} \left( \frac{\pi \tau}{2} | \tau \right) \) as polynomial of \( \varphi \left( \frac{\pi \tau}{2} | \tau \right) \) and \( \varphi'' \left( \frac{\pi \tau}{2} | \tau \right) \):
\[
\varphi^{(2k)} \left( \frac{\pi \tau}{2} | \tau \right) = P_{2k} \left( \varphi \left( \frac{\pi \tau}{2} | \tau \right), \varphi'' \left( \frac{\pi \tau}{2} | \tau \right) \right);
\]
where \( P_{2k}(x, y) \) is polynomial in \( x \) and \( y \).

Taking \( z = 0 \) in \([3.1]\), we have
\[
\varphi^{(2k)} \left( \frac{\pi \tau}{2} | \tau \right) = (-1)^{k+1} 2^{2k+3} \sum_{n=1}^{\infty} \frac{n^{2k+1} q^n}{1 - q^{2n}}.
\]

Recall, from (B) and (F),
\[
\varphi \left( \frac{\pi \tau}{2} | \tau \right) = -\frac{1}{3} \left( \theta_2^4(\tau) + \theta_4^4(\tau) \right),
\]
\[
\varphi'' \left( \frac{\pi \tau}{2} | \tau \right) = 2 \theta_2^4(\tau) \theta_4^4(\tau) = \frac{1}{8} \theta_2^8(\tau/2),
\]
we derive
\[
(-1)^{k+1} 2^{2k+3} \sum_{n=1}^{\infty} \frac{n^{2k+1} q^n}{1 - q^{2n}} = P_{2k} \left( -\frac{1}{3} (\theta_2^4(\tau) + \theta_4^4(\tau)), 2 \theta_2^4(\tau) \theta_3^4(\tau) \right).
\]
For \( k = 1 \), replacing \( \tau \) by \( 2\tau \), we obtain from \([6.6]\),
\[
\theta_2^8(\tau) = 2^8 \sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1 - q^{4n}}.
\]
We now establish:
\[
691 \theta_2^{24}(\tau) = 2^{16} \sum_{n=1}^{\infty} \frac{n^{11} q^n}{1 - q^{2n}} - 2^{16} q(q; q)_\infty^{24} - 259 \times 2^{19} q^2(q^2; q^2)_\infty^{24}.
\]
Proof. From (6.5) and appealing to the elementary facts of theta functions:

\begin{align*}
\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) &= 2q^{1/4}(q^2; q^2)^3, \\
\theta_2(\tau)\theta_3(\tau)\theta_1(\tau) &= 2q^{1/4}(q; q^6)\phi_2, \\
(\theta_2^4(\tau) + \theta_3^4(\tau))^2 &= (\theta_2^4(\tau) - \theta_3^4(\tau))^2 + 4\theta_2^4(\tau)\theta_3^4(\tau) = \theta_3^8(\tau) + 4\theta_2^4(\tau)\theta_3^4(\tau), \\
(\theta_2^4(\tau) + \theta_3^4(\tau))^4 &= \theta_4^{16}(\tau) + 8\theta_2^4(\tau)\theta_3^4(\tau)\theta_1^8(\tau) + 16\theta_2^8(\tau)\theta_3^8(\tau),
\end{align*}

we have

\begin{align*}
2^{13} \sum_{n=1}^{\infty} \frac{n^{11}q^n}{1 - q^{2n}} &= \psi^{(10)} \left( \frac{\pi \tau}{2} \right) \\
&= 12^3 \left\{ 12(2\theta_2^4(\tau)\theta_3^4(\tau)) \times \frac{1}{3^4}(\theta_2^4(\tau) + \theta_3^4(\tau))^4 \\
&\quad + 81(2\theta_2^4(\tau)\theta_3^4(\tau))^2 \times \frac{1}{3^2}(\theta_2^4(\tau) + \theta_3^4(\tau))^2 + 7(2\theta_2^4(\tau)\theta_3^4(\tau))^3 \right\} \\
&= 2^8 \left\{ 1382(\theta_2(\tau)\theta_3(\tau))^{12} + 259(\theta_2(\tau)\theta_3(\tau)\theta_4(\tau))^8 + 2(\theta_2(\tau)\theta_3(\tau)\theta_4^4(\tau))^4 \right\} \\
&= 691 \times 2^{-3}\theta_2^{24}(\tau/2) + 2^{16} \times 259q^2(q^2; q^2)^{24} + 2^{13}q(q; q)^{24}.
\end{align*}

Replacing \( \tau \) by \( 2\tau \) and rearranging the terms, we obtain the stated identity. \( \square \)

Next, we prove

\[ \theta_2^{12}(\tau) = 16 \sum_{n=0}^{\infty} \frac{(2n + 1)^5q^{2n+1}}{1 - q^{4n+2}} - 16q(q^2; q^2)^{12}. \]

Proof. From (6.2),

\begin{align*}
-2^7 \sum_{n=1}^{\infty} \frac{n^5q^n}{1 - q^{2n}} &= \psi^{(4)} \left( \frac{\pi \tau}{2} \right) \\
&= -8(\theta_2^4(\tau) + \theta_3^4(\tau)) \left( \theta_2^4(\tau)\theta_3^4(\tau) \right).
\end{align*}

Then

\[ 16 \sum_{n=1}^{\infty} \frac{n^5q^n}{1 - q^{2n}} = \theta_2^4(\tau)\theta_3^4(\tau) \left( \theta_2^4(\tau) + \theta_2^4(\tau) \right) \]

and replacing \( \tau \) by \( \tau + 1 \), we obtain

\[ 16 \sum_{n=1}^{\infty} \frac{(-1)^n n^5q^n}{1 - q^{2n}} = -\theta_2^4(\tau)\theta_3^4(\tau)(\theta_2^4(\tau) - \theta_2^4(\tau)). \]
Subtracting the above identities,

\[
32 \sum_{n=1}^{\infty} \frac{(2n+1)^3 q^{2n+1}}{1 - q^{4n+2}} = \theta_2^4(\tau)(\theta_3^4(\tau) + \theta_4^4(\tau)) + \theta_2^8(\tau)(\theta_3^4(\tau) - \theta_4^4(\tau))
\]

\[
- \theta_2^4(\tau)(\theta_3^8(\tau) + \theta_4^8(\tau)) + \theta_2^{12}(\tau)
\]

\[
= \theta_2^4(\tau)(\theta_3^4(\tau) - \theta_4^4(\tau))^2 + 2\theta_2^4(\tau)\theta_3^4(\tau)\theta_4^1(\tau) + \theta_2^{12}(\tau)
\]

\[
= 2\theta_2^{12}(\tau) + 2\theta_2^4(\tau)\theta_3^4(\tau)\theta_4^1(\tau)
\]

\[
= 2\theta_2^{12}(\tau) + 32q(q^2; q^2)_{12}.^\infty.
\]

This establishes the desired identity. □

We now consider the cases of \(\theta_2^6(\tau), \theta_2^{10}(\tau)\) and \(\theta_2^{14}(\tau)\). Their formulas are derived from Theorem 3.3 with \(k = 1, k = 2\) and \(k = 3\).

We begin with \(\theta_2^6(\tau)\).

Recall, from (E),

\[
\wp'\left(\frac{\pi + 2\pi \tau}{4}|\tau\right) = 4\theta_2^2(2\tau)\theta_4^4(2\tau).
\]

Taking \(k = 1\) and \(z = 0\) in (3.2), we have

\[
\wp'\left(\frac{\pi + 2\pi \tau}{4}|\tau\right) = 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 - q^{2n}} \sin \frac{n\pi}{2}
\]

or, after replacing \(\tau\) by \(\tau/2\),

\[
\theta_2^2(\tau)\theta_4^4(\tau) = 4q^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)^2 q^n}{1 - q^{2n+1}}.
\]

Replacing \(q\) by \(-q\) (or equivalently \(\tau\) by \(\tau + 1\)), we obtain

\[
\theta_2^2(\tau)\theta_4^4(\tau) = 4q^{1/2} \sum_{n=0}^{\infty} \frac{(2n+1)^2 q^n}{1 + q^{2n+1}}.
\]

Thus

\[
\theta_2^6(\tau) = \theta_2^2(\tau)(\theta_3^4(\tau) - \theta_4^4(\tau)) = 4q^{1/2} \sum_{n=0}^{\infty} (2n+1)^2 \left( \frac{q^n}{1 + q^{2n+1}} - \frac{(-1)^n q^n}{1 - q^{2n+1}} \right).
\]
Notably,

\[ \wp' \left( \frac{\pi + \pi \tau}{4} \right) + i \wp' \left( \frac{2\pi + \pi \tau}{4} \right) = -4 \theta_2^6(\tau). \]

Next we consider \( \theta_2^{10}(\tau) \) and \( \theta_2^{14}(\tau) \).

From (E), we have

\[ \wp \left( \frac{\pi + 2\pi \tau}{4} \right) = -\frac{1}{3} \left( \theta_3^4(2\tau) - 5\theta_2^4(2\tau) \right), \]
\[ \wp' \left( \frac{\pi + 2\pi \tau}{4} \right) = 4\theta_2^2(2\tau)\theta_4^4(2\tau). \]

And the recurrence relation (6.1) with \( n = 1 \) and \( n = 3 \), we have

\[ \wp^{(3)} \left( \frac{\pi + 2\pi \tau}{4} \right) = 12 \wp \left( \frac{\pi + 2\pi \tau}{4} \right) \wp' \left( \frac{\pi + 2\pi \tau}{4} \right) \]
\[ = 12 \left( -\frac{1}{3} \left( \theta_3^4(2\tau) - 5\theta_2^4(2\tau) \right) \right) \left( 4\theta_2^2(2\tau)\theta_4^4(2\tau) \right) \]
\[ = 16\theta_2^2(2\tau)\theta_4^4(2\tau) \left( 5\theta_2^4(2\tau) - \theta_3^4(2\tau) \right) \]

and

\[ \wp^{(5)} \left( \frac{\pi + 2\pi \tau}{4} \right) \]
\[ = -6 \sum_{k=0}^{3} \binom{3}{k} \wp^{(3-k)} \left( \frac{\pi + 2\pi \tau}{4} \right) \wp^{(k)} \left( \frac{\pi + 2\pi \tau}{4} \right) \]
\[ = 12 \left( \wp \left( \frac{\pi + 2\pi \tau}{4} \right) \wp^{(3)} \left( \frac{\pi + 2\pi \tau}{4} \right) + 3\wp' \left( \frac{\pi + 2\pi \tau}{4} \right) \wp'' \left( \frac{\pi + 2\pi \tau}{4} \right) \right) \]
\[ = -2304\theta_2^6(2\tau)\theta_3^8(2\tau) + 64\theta_2^2(2\tau)\theta_4^4(2\tau) \left( 5\theta_2^4(2\tau) - \theta_3^4(2\tau) \right)^2. \]

Replacing \( \tau \) by \( \tau + 1/2 \), we have

\[ \wp^{(3)} \left( \frac{\pi + \pi \tau}{2} \right) = 16i\theta_2^2(2\tau)\theta_3^4(2\tau) \left( -5\theta_2^4(2\tau) - \theta_3^4(2\tau) \right), \]
\[ \wp^{(5)} \left( \frac{\pi + \pi \tau}{2} \right) = -2304i^3\theta_2^6(2\tau)\theta_3^8(2\tau) + 64i\theta_2^2(2\tau)\theta_4^4(2\tau) \left( -5\theta_2^4(2\tau) - \theta_3^4(2\tau) \right)^2. \]
ON DECOMPOSITION OF $\theta_2^4(\tau)$ AS THE SUM OF EISENSTEIN SERIES AND CUSP FORMS

Then

\[
\wp^{(3)} \left( \frac{\pi + 2\pi \tau}{4} \mid \tau \right) - i\wp^{(3)} \left( \frac{\pi + \pi \tau}{2} \mid \frac{2\tau + 1}{2} \right) \\
= 16\theta_2^2(2\tau)\theta_4^4(2\tau)(5\theta_3^4(2\tau) - \theta_3^4(2\tau)) - 16i^2\theta_2^2(2\tau)\theta_4^4(2\tau)(-5\theta_2^4(2\tau) - \theta_4^4(2\tau)) \\
= -32\theta_2^2(2\tau)\theta_4^4(2\tau) + 80\theta_6^6(2\tau)\theta_4^4(2\tau) - 80\theta_2^6(2\tau)\theta_3^4(2\tau) \\
= -32\theta_2^2(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau) - 80\theta_2^6(2\tau)
\]

and

\[
\wp^{(5)} \left( \frac{\pi + 2\pi \tau}{4} \mid \tau \right) + i\wp^{(5)} \left( \frac{\pi + \pi \tau}{2} \mid \frac{2\tau + 1}{2} \right) \\
= -2304\theta_2^6(2\tau)\theta_6^8(2\tau) + 64\theta_3^6(2\tau)\theta_4^8(2\tau)(5\theta_1^4(2\tau) - \theta_3^4(2\tau))^2 \\
- 2304i^2\theta_2^6(2\tau)\theta_6^8(2\tau) + 64i^2\theta_3^6(2\tau)\theta_4^8(2\tau)(-5\theta_2^4(2\tau) - \theta_4^4(2\tau))^2 \\
= -2^6 \times 61\theta_2^14(2\tau) - 2^6 \times 91\theta_6^6(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau).
\]

Hence,

\[
80\theta_2^{10}(2\tau) = -\wp^{(3)} \left( \frac{\pi + 2\pi \tau}{4} \mid \tau \right) + i\wp^{(3)} \left( \frac{\pi + \pi \tau}{2} \mid \frac{2\tau + 1}{2} \right) \\
- 32\theta_2^2(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau), \\
2^6 \times 61\theta_2^{14}(2\tau) = -\wp^{(5)} \left( \frac{\pi + 2\pi \tau}{4} \mid \tau \right) - i\wp^{(5)} \left( \frac{\pi + \pi \tau}{2} \mid \frac{2\tau + 1}{2} \right) \\
- 2^6 \times 91\theta_2^6(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau).
\]

The desired identities follow from (3.8), (3.11) and Jacobi’s product identities for theta functions.

To complete the proofs of the identities listed in Section 1, additional identities derived from the recurrence relation of $\wp(z|\tau)$ are provided in Section 7.

7. Comments

We start with a set of elementary lemmas which might be of independent interest in themselves.

**Lemma 7.1.** Suppose

\[
\frac{p_n(x)}{(1-x)^n} = \sum_{k=0}^{\infty} k^{n-1} x^k.
\]

Then,

\[
p_1(x) = 1,
\]
\[ p_{n+1}(x) = nx p_n(x) + x(1-x)p'_n(x). \]  

Thus, \( p_2(x) = x, p_3(x) = x^2 + x \) and \( p_4(x) = x^3 + 4x^2 + x. \)

**Proof.** We note that 
\[
x \frac{d}{dx} \frac{p_n(x)}{(1-x)^n} = \sum_{k=0}^{\infty} k^n x^k = \frac{p_{n+1}(x)}{(1-x)^{n+1}}
\]
and from which we derive the recurrence relation.

Using the recurrence relation (7.1), we claim that the coefficients of \( p_n(x) \) are palindromic.

**Lemma 7.2.** Suppose
\[ p_n(x) = a_{n-1,n}x^{n-1} + a_{n-2,n}x^{n-2} + \ldots + a_{2,n}x^2 + a_{1,n}x. \]
Then \( a_{n-1,n} = a_{1,n} = 1 \) and
\[ a_{j,n} = a_{n-j,n}, \]
for \( j = 1, 2, \ldots, n-1. \)

**Proof.** We will prove by induction.

Suppose
\[ p_n(x) = x^{n-1} + a_{n-2,n}x^{n-2} + \ldots + a_{2,n}x^2 + x. \]
Assume the coefficients of \( p_n \) are palindromic.
Then \( a_{n-1,n} = a_{1,n} = 1 \) and
\[ a_{j,n} = a_{n-j,n}, \]
for \( j = 1, 2, \ldots, n-1. \)

Let
\[ p_{n+1}(x) = x^n + a_{n-1,n+1}x^{n-1} + \ldots + a_{2,n+1}x^2 + x. \]
From (7.1), for \( m = 1, 2, \ldots, n, \)
\[ a_{m,n+1} = (n + 1 - m)a_{m-1,n} + ma_{m,n}. \]
Then, from (7.2),
\[ a_{n+1-m,n+1} = ma_{n-m,n} + (n + 1 - m)a_{n+1-m,n} = a_{m,n+1}. \]
This establishes the claim.
In fact, we have

**Lemma 7.3.** Let

\[
p_n(x) = x^{n-1} + a_{n-2,n}x^{n-2} + \ldots + a_{2,n}x^2 + x.
\]

Then, for \(1 \leq m \leq n - 1\),

\[
a_{m,n} = \sum_{j=0}^{m} (-1)^j \binom{n}{j} (m - j)^{n-1}.
\]

**Proof.**

\[
p_n(x) = (1 - x)^n \sum_{k=0}^{\infty} k^{n-1} x^k
\]

\[
= \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} x^j \sum_{k=0}^{\infty} k^{n-1} x^k
\]

\[
= \sum_{m=1}^{n-1} x^m \sum_{j=0}^{m} (-1)^j \binom{n}{j} (m - j)^{n-1}
\]

\[
+ \sum_{m=n}^{\infty} x^m \sum_{j=0}^{n} (-1)^j \binom{n}{j} (m - j)^{n-1}
\]

\[
= \sum_{m=1}^{n-1} x^m \sum_{j=0}^{m} (-1)^j \binom{n}{j} (m - j)^{n-1}.
\]

This gives the desired result. \(\square\)

Moreover, as bonuses, we also derive additional identities:

\[
\sum_{k=0}^{n-1} (-1)^j \binom{n}{j} (n - 1 - j)^{n-1} = 1
\]

and for all \(m \geq n\),

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} (m - j)^{n-1} = 0.
\]

**Lemma 7.4.** For \(|q| < 1\), we have

\[
\sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^{2n}} = \sum_{n=0}^{\infty} \frac{p_k(q^{2n+1})}{(1 - q^{2n+1})^k}.
\]
Proof. Since
\[
\frac{q^n}{1 - q^{2n}} = \sum_{j=0}^{\infty} q^{(2j+1)n},
\]
then
\[
\sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1 - q^{2n}} = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n^{k-1} q^{(2j+1)n}
\]
\[
= \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{(2j+1)n}
\]
\[
= \sum_{j=0}^{\infty} \frac{p_k(q^{2j+1})}{(1 - q^{2j+1})^k}.
\]
□

From Corollary 4.2.

**Corollary 7.5.** For integer \( k \geq 1 \), we find
\[
\theta_{2k}^{8k}(\tau) = \frac{2^{4k+3} k}{(1 - 2^k)B_4k} \sum_{n=1}^{\infty} \frac{p_{4k}(q^{4n+2})}{(1 - q^{4n+2})^{4k}} + T_4k(\tau).
\]

Similarly, let
\[
\frac{P_n(x)}{(1 - x)^n} = \sum_{k=0}^{\infty} (2k + 1)^{n-1} x^k.
\]

We note
\[
(2x \frac{d}{dx} + 1) \frac{P_n(x)}{(1 - x)^n} = \sum_{k=0}^{\infty} (2k + 1)^n x^k = \frac{P_{n+1}(x)}{(1 - x)^{n+1}}
\]
and from which we derive

**Lemma 7.6.** Suppose
\[
\frac{P_n(x)}{(1 - x)^n} = \sum_{k=0}^{\infty} (2k + 1)^{n-1} x^k.
\]

Then
\[
P_1(x) = 1 \quad \text{and} \quad P_{n+1}(x) = ((2n - 1)x + 1)P_n(x) + 2x(1 - x)P_n'(x);
\]
the coefficients of $P_n(x)$ are palindromic and
\[
\sum_{n=1}^\infty \frac{(2n+1)^{k-1}q^n}{1-q^{2n}} = \sum_{n=0}^\infty \frac{P_{2n+1}(q)}{(1-q^{2n+1})^k}.
\]

We can re-write Corollary 4.5 as

**Corollary 7.7.** For integer $k \geq 0$, we have
\[
\theta_{8k+4}(\tau) = -8(2k+1)\sum_{n=0}^\infty \frac{P_{4k+2}(q^{2n+1})}{(1-q^{2n+1})^{4k+2}} + T_{4k+2}(\tau).
\]

To give the details of (1.3), we need recall Dedekind’s transformation formula for $\eta(\tau)$ in [7, p. 10].

**Lemma 7.8.** Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \). We have
\[
(1) \text{If } d > 0 \text{ and } d \text{ is odd,}
\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right) e^{(d(b-c)+ac(1-d^2)+3d-3)\pi i/12 \sqrt{ct+d} \eta(\tau)},
\]

(2) If $c > 0$ and $c$ is odd,
\[
\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{d}{c}\right) e^{(c(a-d)+bd(1-c^2)-3c+3)\pi i/12 \sqrt{-i(c\tau+d)} \eta(\tau)},
\]

**Theorem 7.9.** Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \). We have
\[
(1) \text{ when } c \equiv 2 \pmod{4},
\theta_2\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{d}{c}\right) \sqrt{-i(c\tau+d)} e^{\frac{\pi i}{4}(bd+1)} \theta_2(\tau);
\]

(2) when $c \equiv 0 \pmod{4}$,
\[
\theta_2\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right) \sqrt{ct+d} e^{\frac{\pi i}{4}(bd+d-1)} \theta_2(\tau).
\]

Especially,
\[
\theta_2^2\left(\frac{a\tau + b}{c\tau + d}\right) = \psi(\sigma)(ct+d)\theta_2^2(\tau),
\]

where
\[
\psi(\sigma) = \begin{cases} 
-ie^{\frac{\pi i}{4}(bd+1)} & \text{if } c \equiv 2 \pmod{4}, \\
e^{\frac{\pi i}{4}(bd+d-1)} & \text{if } c \equiv 0 \pmod{4}.
\end{cases}
\]
Proof. Consider a transformation \( \tau \to (a\tau + b)/(c\tau + d) \)

\[
\frac{\eta^2(2\tau)}{\eta(\tau)} \to \frac{\eta^2 \left( \frac{a\tau + b}{c\tau + d} \right)}{\eta \left( \frac{a\tau + b}{c\tau + d} \right)}.
\]

Now when \( c \equiv 2 \) (mod 4), and hence, by (7.4)

\[
\theta_2 \left( \frac{a\tau + b}{c\tau + d} \right) = \left\{ \left( \frac{d}{c/2} \right) \sqrt{-i(c/2 \cdot 2\tau + d)} e^{(c/2(a+d)+2bd(1-c^2/4)-3/2c+3)i/12} \eta(2\tau) \right\}^2
\]

\[
= \frac{(d/2) \sqrt{-i(c\tau + d)} e^{(c(a+d)+bd(1-c^2)-3c+3)i/12} \eta(\tau)}{(d/2) \sqrt{-i(c\tau + d)} e^{(c(a+d)+bd(1-c^2)-3c+3)i/12} \eta(\tau)}
\]

\[
= 2 \left( \frac{d}{c} \right) \sqrt{-i(c\tau + d)} e^{\frac{\pi i}{4}(bd+1)} \eta^2(2\tau) \eta(\tau)
\]

\[
= \left( \frac{d}{c} \right) \sqrt{-i(c\tau + d)} e^{\frac{\pi i}{4}(bd+1)} \theta_2(\tau),
\]

Next when \( c \equiv 0 \) (mod 4), and hence, by (7.5)

\[
\theta_2 \left( \frac{a\tau + b}{c\tau + d} \right) = \left\{ \left( \frac{c/2}{d} \right) \sqrt{(c/2 \cdot 2\tau + d)} e^{(d(2b-c/2)+ac/2(1-d^2)+3d-3)i/12} \eta(2\tau) \right\}^2
\]

\[
= \frac{(c/2) \sqrt{(c\tau + d)} e^{(d(b-c)+ac(1-d^2)+3d-3)i/12} \eta(\tau)}{(c/2) \sqrt{(c\tau + d)} e^{(d(b-c)+ac(1-d^2)+3d-3)i/12} \eta(\tau)}
\]

\[
= 2 \left( \frac{c}{d} \right) \sqrt{c\tau + d} e^{\frac{\pi i}{4}(bd+1)} \eta^2(2\tau) \eta(\tau)
\]

\[
= \left( \frac{c}{d} \right) \sqrt{c\tau + d} e^{\frac{\pi i}{4}(bd+1)} \theta_2(\tau).
\]

We get the desired identities. \( \square \)

8. A List of Identities

For the convenience of the reader, we provide the following set of identities which are derived from the recurrence relation of \( \wp(z|\tau) \) mentioned in Section 6 and are used to establish the identities of \( \theta_2^{2n}(\tau) \), for \( n = 2, 3, 4, \ldots, 12 \), mentioned in Section 1

\[
2^4 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^{2n}} = \theta_2^4(\tau) \theta_4^4(\tau) = 2^{-4} \theta_4^8(\tau/2),
\]
From Theorem 3.3,

\[ 2^n \sum_{n=0}^{\infty} \frac{(2n+1)^3 q^{2n+1}}{1-q^{4n+2}} = \theta_2^4(\tau) \left( \theta_3^4(\tau) + \theta_4^4(\tau) \right) ; \]

\[ 2^4 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^{2n}} = \theta_2^4(\tau) \theta_3^4(\tau) \left( \theta_2^4(\tau) + \theta_3^4(\tau) \right) ; \]

\[ 2^n \sum_{n=0}^{\infty} \frac{(2n+1)^5 q^{2n+1}}{1-q^{4n+2}} = \theta_2^8(\tau) \left( 2\theta_2^4(\tau) + 2\theta_3^4(\tau) \theta_4^4(\tau) \right) \]

\[ = 2\theta_2^{12}(\tau) + 2\theta_2^4(\tau) \theta_3^4(\tau) \theta_4^4(\tau) ; \]

\[ 2^n \sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^{2n}} = \theta_2^4(\tau) \theta_3^4(\tau) \left( 2\theta_2^8(\tau) + 17\theta_2^4(\tau) \theta_3^4(\tau) \right) \]

\[ = \theta_2^4(\tau) \theta_3^4(\tau) \theta_4^8(\tau) + 17\theta_2^8(\tau) \theta_3^4(\tau) ; \]

\[ 2^n \sum_{n=0}^{\infty} \frac{(2n+1)^7 q^{2n+1}}{1-q^{4n+2}} = \theta_2^4(\tau) \left( \theta_3^4(\tau) + \theta_4^4(\tau) \right) \left( 17\theta_2^8(\tau) + 2\theta_4^4(\tau) \theta_4^4(\tau) \right) ; \]

\[ 2^n \sum_{n=1}^{\infty} \frac{n^9 q^n}{1-q^{2n}} = \theta_2^4(\tau) \theta_3^4(\tau) \left( \theta_2^4(2\tau) + \theta_3^4(\tau) \right) \left( \theta_2^8(\tau) + 29\theta_2^4(\tau) \theta_3^4(\tau) + \theta_3^8(\tau) \right) , \]

\[ 2^n \sum_{n=0}^{\infty} \frac{(2n+1)^9 q^{2n+1}}{1-q^{4n+2}} = 62\theta_2^{20}(\tau) + 154\theta_2^{12}(\tau) \theta_3^4(\tau) + 2\theta_2^4(\tau) \theta_4^8(\tau) ; \]

\[ 2^n \sum_{n=1}^{\infty} \frac{n^{11} q^n}{1-q^{2n}} = 4\theta_2^4(\tau) \theta_3^4(\tau) \theta_4^{16}(\tau) + 259\theta_2^8(\tau) \theta_3^8(\tau) \theta_4^8(\tau) + 1382\theta_2^{12}(\tau) \theta_3^4(\tau), \]

\[ 2^n \sum_{n=0}^{\infty} \frac{(2n+1)^{11} q^{2n+1}}{1-q^{4n+2}} = \theta_2^4(\tau) \left( \theta_3^4(2\tau) + \theta_4^4(\tau) \right) \]

\[ \times \left( 1383\theta_2^{16}(\tau) + 1131\theta_2^8(\tau) \theta_3^4(\tau) \theta_4^4(\tau) + 2\theta_2^{12}(\tau) \theta_4^4(\tau) \right) . \]

From Theorem 3.3,

\[ \sum_{n=0}^{\infty} (2n+1)^{2k} \left( \frac{q^{2n+1}}{1+q^{2n+1}} + \frac{(-1)^n q^{2n+1}}{1-q^{4n+2}} \right) = -2 \sum_{n=0}^{\infty} \sigma_{2k,\chi}(4n+1)q^{4n+1}, \]

\[ \sum_{n=0}^{\infty} (2n+1)^{2k} \left( \frac{q^{2n+1}}{1+q^{2n+1}} - \frac{(-1)^n q^{2n+1}}{1-q^{4n+2}} \right) = -2 \sum_{n=0}^{\infty} \sigma_{2k,\chi}(4n+3)q^{4n+3} . \]
Instead of using the Eisenstein series, alternatively, we have

\[
2^3 \sum_{n=0}^{\infty} \sigma_{2,\chi}(4n + 1)q^{4n+1} = \theta_2^2(2\tau)(\theta_3^4(2\tau) + \theta_4^4(2\tau)),
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{2,\chi}(4n + 3)q^{4n+3} = -\theta_2^6(2\tau);
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{4,\chi}(4n + 1)q^{4n+1} = 5\theta_2^{10}(2\tau) + 2\theta_2^3(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau),
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{4,\chi}(4n + 3)q^{4n+3} = -5\theta_2^6(2\tau)(\theta_3^4(2\tau) + \theta_4^4(2\tau));
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{6,\chi}(4n + 1)q^{4n+1} = \theta_2^2(2\tau)(61\theta_2^8(2\tau) + \theta_3^4(2\tau)\theta_4^4(2\tau))(\theta_3^4(2\tau) + \theta_4^4(2\tau)),
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{6,\chi}(4n + 3)q^{4n+3} = -61\theta_2^{14}(2\tau) - 91\theta_2^6(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau);
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{8,\chi}(4n + 1)q^{4n+1} = 1385\theta_2^{18}(2\tau) + 3052\theta_2^{10}(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau) + 2\theta_2^3(2\tau)\theta_3^8(2\tau)\theta_4^8(2\tau),
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{8,\chi}(4n + 3)q^{4n+3} = -\theta_2^2(2\tau) (\theta_3^4(2\tau) + \theta_4^4(2\tau)) (1385\theta_2^{12}(2\tau) + 410\theta_2^4(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau));
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{10,\chi}(4n + 1)q^{4n+1} = \theta_2^2(2\tau) (\theta_3^4(2\tau) + \theta_4^4(2\tau)) (50521\theta_2^{16}(2\tau) + 38147\theta_2^8(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau) + \theta_3^8(2\tau)\theta_4^8(2\tau));
\]

\[
2^3 \sum_{n=0}^{\infty} \sigma_{10,\chi}(4n + 3)q^{4n+3} = - (50521\theta_2^{22}(2\tau) + 138677\theta_2^{14}(2\tau)\theta_3^4(2\tau)\theta_4^4(2\tau) + 7381\theta_2^6(2\tau)\theta_3^8(2\tau)\theta_4^8(2\tau));
\]

\[
\theta_2^6(2\tau) = -8 \sum_{n=0}^{\infty} \sigma_{2,\chi}(4n + 3)q^{4n+3},
\]

\[
5\theta_2^{10}(2\tau) = 8 \sum_{n=0}^{\infty} \sigma_{4,\chi}(4n + 1)q^{4n+1} - 8\frac{(q^4; q^4)_\infty^{14}}{(q^8; q^8)_\infty^4},
\]
61\theta_2^{14}(2\tau) = -8 \sum_{n=0}^{\infty} \sigma_{6,\chi}(4n+3)q^{4n+3} - 91 \times 2^6 q^3(q^4; q^4)_{10}\infty(q^8; q^8)_{\infty}^4,

1385\theta_2^{18}(2\tau) = 8 \sum_{n=0}^{\infty} \sigma_{8,\chi}(4n+1)q^{4n+1} - q\frac{(q^4; q^4)_{12}^{30}}{(q^8; q^8)_{\infty}^{12}} - 763 \times 2^9 q^5(q^4; q^4)_6^{(q^8; q^8)_{\infty}^{12}}.

50521\theta_2^{22}(2\tau) = -8 \sum_{n=0}^{\infty} \sigma_{10,\chi}(4n+3)q^{4n+3} - 138677 \times 2^{14} q^7(q^4; q^4)_2^{2(q^8; q^8)_{\infty}^{20}} - 7381 \times 2^6 q^3(q^4; q^4)_{12}^{26}(q^8; q^8)_{\infty}^{4}.

Lastly, for \theta_2^2(q), we recall [9, p. 511]

cdu = \frac{2\pi}{K\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+\frac{1}{2}} \cos(2n+1)z}{1 - q^{2n+1}}

or equivalently,

\theta_2(q)\theta_3(q) \frac{\theta_2(z|q)}{\theta_3(z|q)} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+\frac{1}{2}} \cos(2n+1)z}{1 - q^{2n+1}}.

Set z = 0 in the above equation, we have

\theta_2^2(q) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+\frac{1}{2}}}{1 - q^{2n+1}}.

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