Are Lagrangian stochastic models at odds with statistical theories of relative dispersion?

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Abstract

In an article on statistical modelling of turbulent relative dispersion, Franzese & Cassiani (2007, p. 402) commented on Lagrangian stochastic models and reported some concern about the consistency between statistical and stochastic modelling of turbulent dispersion. In this short article, comparison of the two approaches is performed. As far as the dependence of models from turbulence constants is concerned, the two theoretical approaches are found to be in perfect agreement eliminating every possible concern.

1 Introduction

In an article on statistical theory of relative dispersion, Franzese & Cassiani (2007) (hereinafter FC) found that within their approximations, the Richardson constant \( C_r \) is expressed by

\[
C_r = 6\alpha C_0 ,
\]

(1)

where \( C_0 \) is the, supposedly universal, Kolmogorov constant of the second-order Lagrangian structure function and \( \alpha \) is expressed by

\[
\alpha = \frac{C_L}{2} \left[ \left( 1 + \frac{4}{3C_L} \right)^{1/2} - 1 \right] ,
\]

(2)

where \( C_L \) is a measure of the ratio between “a length scale of the energy containing eddies” (FC) and \( \sigma_u T_L \), \( \sigma_u \) being the r.m.s of the turbulent velocity and \( T_L \) the Lagrangian integral time scale. \( C_L \) is then determined by some closure assumption and turns out to be 8/3. Using arguments based on their definition of scales, FC conclude that:

“The anomalous inverse relation between \( C_0 \) and \( C_r \) observed in stochastic Lagrangian models […] arises from the violation of (4.11)
Increasing $C_0$ with a fixed $C_k$ determines a spurious increase in $C_\sigma$, namely, the proportion between Eulerian and Lagrangian scales is altered, with an overestimated value of $L_E$. In such conditions, the particles separate at a slower rate because the fraction of energy used for the separation process is underestimated.

It is not clear whether the FC claim is that Lagrangian stochastic models (LSM) are incorrect while statistical theories, by contrast, display the "correct" features. Nevertheless, a clarification on the connection between these two approaches can be worthwhile in the light of the doubt generated by the lack of clarity of the FC article on this particular aspect. In fact the comparison of Lagrangian statistical and stochastic models was not the main purpose of FC. Thus, in the present article the opportunity is taken to clarify the problem in the framework defined by Maurizi et al. (2004) (hereinafter MPT).

2 A reminder and a critical analysis of MPT results

It is useful here to recall the basics of the MPT work. The idea was to study some general properties of the WM models for relative turbulent dispersion. The frame is that of Kolmogorov (1941) theory (hereinafter K41) and the approach is the Well Mixed (WM) condition (Thomson, 1987, 1990). The consideration on which the MPT analysis is based, is the fact that Eulerian and Lagrangian properties co-exist in the description of relative dispersion and that their typical scales (Eulerian length and Lagrangian time scales) can be used to highlight model properties.

From K41 it is known that in homogeneous isotropic turbulence, the second-order longitudinal Eulerian structure function $S_E^{(2)}$ in the inertial sub-range is:

$$S_E^{(2)} = \langle ([u(x + \Delta r) - u(x)] \cdot \Delta r(\Delta r)^{-1})^2 \rangle = C_k(\varepsilon \Delta r)^{2/3}$$

for $\eta \ll \Delta r \equiv ||\Delta r|| \ll L_E$, where $\varepsilon$ is the turbulent kinetic energy dissipation rate, $\eta$ is the Kolmogorov microscale and $L_E$ is the Eulerian integral length scale. Using the relationship between second-order structure function and correlation coefficient $S^{(2)}(\Delta) = 2\sigma_u^2[1 - R(\Delta)]$ (where $\Delta$ is either $\Delta r$ or $\Delta t$ for Eulerian and Lagrangian formulation, respectively) it turns out that in the inertial subrange

$$R_E(\Delta r) = 1 - \frac{S_E^{(2)}}{2\sigma_u^2} = 1 - \frac{C_k(\varepsilon \Delta r)^{2/3}}{2\sigma_u^2}. \tag{4}$$

Equation (4) can be used as a definition for a length scale

$$\lambda_E = \left(\frac{2}{C_k}\right)^{3/2} \frac{\sigma_u^3}{\varepsilon}. \tag{5}$$

\footnote{Actually, an equation similar to Eq. (10) in the present paper.}

\footnote{The same as $\lambda_E$ in this context.}
It is straightforward to follow the same procedure in the Lagrangian frame. In fact, on dimensional grounds it is known that (Monin & Yaglom, 1975) at the leading order in $\Delta t$,

$$S_L^{(2)}(t) \equiv \langle [v_i(t + \Delta t) - v_i(t)]^2 \rangle = C_0(\varepsilon \Delta t)$$

in which $v_i(t) = u_i(X(t), t)$ is the Lagrangian velocity, i.e., the Eulerian velocity at the particle position $X(t)$. Equation (6) is valid for $\tau_\eta \ll \Delta t \ll T_L$, $\tau_\eta$ being the Kolmogorov time scale. This gives for the Lagrangian autocorrelation function:

$$R_L(\Delta t) = 1 - \frac{C_0(\varepsilon \Delta t)}{2\sigma_u^2}$$

that can be used as a definition for a time scale

$$\tau_L = \frac{2\sigma_u^2}{C_0 \varepsilon}.$$  

Equation (8) represents the Lagrangian counterpart of Equation (5) and corresponds to the known relationship given by Tennekes (1982).

It can be observed here that the above definitions link the scales to their corresponding constants: $C_k$ and $C_0$ for the Eulerian length and Lagrangian time scales, respectively. It is worth pointing out that the presence of redundant scales (length, time, velocity) is not surprising considering that they are in fact the scales of the independent ingredients of a LSM: the Lagrangian structure function enters for compatibility with small scale behaviour (Thomson, 1987); the two-point Eulerian structure function is imposed by the WM condition (Thomson, 1990) through the Eulerian probability density function (pdf) of the flow velocity, and the kinetic energy is not directly connected to $\lambda E \tau_{L}^{-1}$ and therefore it is another (independent) parameter of the Eulerian pdf. This redundancy can be regarded as the manifestation of the competing role of Eulerian and Lagrangian scales in relative dispersion. “Real” turbulence does not display any variability of the constants because Eulerian and Lagrangian properties are both uniquely determined by dynamical equations. Note also that LSM theory is valid for infinite Reynolds number $Re$ and therefore no variations of constants can be attributed to variations in $Re$. However, varying constants is possible in models where Eulerian and Lagrangian properties are imposed as “phenomenological” model constraints.

The above defined scales can be used to render non-dimensional the Fokker-Planck equation for the probability density function of the process $p = p(u, x; t)$, where $u \equiv (u^{(1)}, u^{(2)})$ and $x \equiv (x^{(1)}, x^{(2)})$ (with superscript referring to particle 1 and 2):

$$\frac{\partial}{\partial t} p + \beta \frac{\partial}{\partial x_i}(u_i p) + \frac{\partial}{\partial u_i}(a_i p) = \frac{\partial^2}{\partial u_i \partial u_i} p$$

where all the quantities involved are non-dimensional: $t \to \tau_L t$, $x_i \to \lambda_E x_i$, $u_i \to \sigma_u u_i$, $p \to \sigma_u^{-3} p$ and $a_i \to \sigma_u \tau_L^{-1} a_i$. With these scalings, the constant $\beta$
is the sole remaining parameter of Equation (9) and is expressed by
\[ \beta = \frac{\sigma_u \tau_L}{\lambda_E} \equiv \left( \frac{C_k^3}{2C_0} \right)^{1/2} \] (10)

which is a non-dimensional combination of the above-defined scales and can be recognised to be a possible definition for the quantity commonly known as Lagrangian-to-Eulerian scale ratio. It can be observed that the alternative choice \( a_i \to \sigma_u^2 \lambda_E^{-1} a_i \) for the drift term scaling is still possible but, while changing the form of Equation (9), would not affect its dependence on \( \beta \) as the unique parameter.

The results of MPT are worth a comment. The arguments used are, in general, not sufficient to state that “any” solution of Equation (9) depends solely on \( \beta \) because, in fact, the drift term \( a_i \) results from the application of the WM condition:
\[ a_i = C_0 \varepsilon \frac{\partial}{\partial u_i} \log P_E + \frac{\Phi_i}{P_E}, \] (11)

where
\[ \frac{\partial}{\partial u_i} \Phi_i = \frac{\partial}{\partial t} P_E + \frac{\partial}{\partial x_i} P_E, \] (12)

and in general can depend also on other parameters via \( P_E \) and/or via the assumptions made to remove the indeterminacy intrinsic to the WM condition.

Nevertheless, it can be demonstrated that for Gaussian \( P_E \) (Thomson, 1990; Borgas & Sawford, 1994), the non-dimensional \( a_i \) actually depends solely on \( \beta \) confirming the MPT statement for this class of models. In fact, considering the general class of of Gaussian models presented by Borgas & Sawford (1994), the (dimensional) drift term always has a form of the type
\[ a_i = \sigma_u \tau_L^{-1} A_i + \sigma_u^2 \lambda_E^{-1} B_i \]
from which \( \sigma_u^{-1} \tau_L a_i = A_i + \beta B_i \) with \( A_i \) and \( B_i \) non-dimensional and independent of \( C_k \) and \( C_0 \).

Full 3-dimensional (3D) solutions are presently beyond reach. However, quasi–one-dimensional (Q1D) approach (Kurbanmuradov, 1997) makes it possible to estimate to what extent the MPT conclusions are valid also for departures of \( P_E \) from Gaussianity. In Kurbanmuradov (1997) a systematic study of the behaviour of the Q1D model in response to variation of non-Gaussian properties was carried out. The result was that the differences observed as a result of variations of \( C_0 \) (i.e., \( \beta \)) are much larger than those that result from departure from Gaussianity. In addition, Kurbanmuradov (1997) also noticed that non-Gaussian and Gaussian Q1D models behave qualitatively the same.

It can be inferred that, at least in the Q1D frame, \( \beta \) is still the driving parameter of the non-dimensional Fokker-Plank solutions, with non-Gaussianity playing a minor role. In other words, the drift term \( a \) can be expressed as
\[ f(\beta, G) = f_0(\beta) + O(G) \]
with \( G \) being the parameter driving the non-Gaussianity of \( P_E \).

Another property that is expected to play a role so as to introduce a further parameter, is rotation (Sawford, 1999) which is related to the non-uniqueness problem. However, investigating also on the consequences of this aspect is beyond the scope of the present work.
3 Connection between Lagrangian Statistical and Stochastic approaches

In terms of the T90 theory, the validity of MPT results, although rigorously true only for Gaussian (but still approximately true for non-Gaussian $P_E$), means that once the non-uniqueness problem is resolved (by selecting one of the infinitely many solutions of the WM conditioned problem) results of Equation (9) depend solely on $\beta$.

Considering the Richardson law

$$\left\langle \Delta X_i^2 \right\rangle = C_r \varepsilon t^3 \tag{13}$$

reducing it to non-dimensional form, taking into account that $\Delta X_i$ is a Lagrangian quantity, i.e., $\left\langle \Delta X_i^2 \right\rangle \to \sigma_u \tau_L^{-1} \left\langle \Delta X_i^2 \right\rangle$ [Maurizi et al., 2006], it turns out that

$$\left\langle \Delta X_i^2 \right\rangle = 2 C_r^* t^3 \tag{14}$$

where all the variables are non-dimensional and $C_r^* = C_r C_0^{-1}$ is the normalised Richardson coefficient.

This result was used to analyse and to arrange systematically data from literature. Figure reports results from different LSM both Gaussian (Borgas & Sawford, 1994) and non-Gaussian (Kurbanmuradov, 1997). The regularity of the behaviour of $C_r^*$ with varying $\beta$ is striking. All the models are in agreement with the diffusion limit (Borgas & Sawford, 1994) and their rate of growth with $\beta$ is also monotonic. Moreover, as anticipated above, departures from Gaussianity do not modify the general picture given by MPT.

A direct consequence of Equation (14) is that if $\left\langle \Delta X_i^2 \right\rangle$ is solution of a given WM model it will depend on $\beta$ only so as $C_r^*$, and consequently

$$C_r = F(\beta) C_0 \tag{15}$$

where the functional form of $F(\beta) = 0.5(\Delta x^2) t^{-3}$ depends only on the assumption made to close the non-uniqueness problem. It is clear now that Equation (11) and Equation (15) are equivalent when viewed in the frame of the scaling described so far proving the qualitative consistency of the Lagrangian statistical and stochastic (WM) approaches.

While it is clear that there is a dependence of the WM solution on $\beta$, it is not evident if FC. In fact, finding a dependence of $\alpha$ on $\beta$ in the FC theory development is not straightforward because a closure assumption is used before its explicit definition so that the dependence of $\alpha$ on $\beta$ is hidden.

In their Section 5, FC state that

$$T_y / \tau_L = (3\alpha/4)^{-1/3} \tag{16}$$

where $T_y$ is defined as the time at which the cloud of marked particles reaches the dimension at which it starts to behave diffusively. However, Equation (16)
is a consequence of the closure assumption made in their subsequent Section 7. Thus before introducing the assumption that then leads to $C_L = 8/3$, the actual expression of $C_L$ reads:

$$C_L = \frac{4}{3} \left[ \left( \frac{\tau_L}{T_y} + 1 \right)^2 - 1 \right]^{-1} \quad (17)$$

and consequently, from Equation (19), the following expression holds:

$$\alpha = \frac{2}{3} \left( \tau_L/T_y \right)^2 \left( \tau_L/T_y + 2 \right) \quad (18)$$

It can be argued that $T_y$ must be a function of the Lagrangian-to-Eulerian scale ratio $\beta$. Evidence for this dependence comes from asymptotic behaviour in the ideal limiting cases: in the limit of infinite spatial correlation ($\beta \to 0$), particles do not separate so that $T_y \to \infty$, while for vanishing spatial correlation ($\beta \to \infty$) the two-particles are independent since the beginning so that $T_y \to \tau_L$.

Although the above arguments clearly indicate that $T_y/\tau_L$ must be a function of $\beta$, it is impossible to proceed in this direction without further assumptions. However, any arbitrary assumption can be avoided noting that, being defined as the ratio between a measure “of a length scale of the energy containing eddies” and $\sigma^2 T_y$, $C_L$ turns out to be proportional to $\beta^{-2}$. This consideration forces to recognise that the quantity $6C_\sigma$ in FC is inessential and can be set to unit. In fact it is the ratio between two quantities both proportional to $\langle u^2 \rangle \varepsilon^{-2}$.

Substituting the relationship between $C_L$ and $\beta$ in Equation (2), gives

$$\alpha = \frac{\gamma}{2\beta^2} \left[ \left( 1 + \frac{4\beta^2}{3\gamma} \right)^{1/2} - 1 \right]^3 \quad (19)$$

with $\gamma$ to be determined. Equation (19) can be used to exploit the exact dependence of Equation (11) from $\beta$. Using the FC values: $\alpha = (18\sqrt{6} - 44)/6$ and $\beta = 0.44$, it turns out that $\gamma \simeq 0.53$. The curve representing FC model in terms of $g^*$ as a function of $\beta$ is reported in Figure 1 for comparison with LSM results.

It can be observed that Equation (19) for $\beta \to 0$, shows the same power law dependence ($\beta^4$) as the BS94 limit for Gaussian LSM, while having a very different coefficient as can be appreciated in Figure 1. In fact, using the BS94 limit as constraint for Equation (19), gives $\gamma \simeq 0.25$ which, in terms of the FC closure, means that instead of $T_y \simeq 2.22\tau_L$ one should use $T_y \simeq 1.38\tau_L$. With this “BS94 compliant” closure, for $\beta = 0.44$, it results that $g^* \simeq 0.31$ which is more than three times larger that the value given by FC $g^* = 0.09$. This highlight a strong sensitivity of FC results to the value selected for the closure. The resulting curve is reported in Figure 1.

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4This is the value to be used in the exact Ornstein-Uhlenbeck solution to obtain the FC value $C_L = 8/3$ that, in turn, is obtained by using $T_y = 2$ with an approximate solution.
Figure 1: Normalised Richardson coefficient $C_r^*$ as a function of the Lagrangian-to-Eulerian scale ratio $\beta$. Symbols are as follows: open square, Borgas & Sawford (1994, model 4.2a); open diamonds, Borgas & Sawford (1994, model 7.6 with $\varphi = -0.4$); open triangle, Borgas & Sawford (1994, model 4.3); small reverse full triangle, Kurbanmuradov (1997) for different departures from Gaussianity. Continuous line is the diffusion limit (Borgas & Sawford, 1994). Line dotted with full circles is Equation (1) (FC) with $C_L = 0.53\beta^{-2}$ and line dotted with open circles is the same equation with $C_L = 0.25\beta^{-2}$, the value for consistency with BS94 as $\beta \to 0$. 
4 Conclusions

The analysis performed shown that the proportionality between $C_r$ and $C_0$ is common to both Lagrangian statistical and stochastic derived models once, using inertial sub-range scaling, the Lagrangian-to-Eulerian scale ratio $\beta$ is recognised as driving parameter, and then kept constant. There is no intrinsic violation of this scale ratio in LSM in that both $C_K$ and $C_0$ can be varied independently.

It was also shown that the connections between the two approaches is even more intimate in that also the results of FC model formally depend on $\beta$. In addition, FC model was shown to depend strongly on the value adopted for the transition time from ballistic to diffusive regime which is a rather poorly definable quantity.

In view of the results presented here, the consistency between Lagrangian theories is not surprising at all because the ingredients used for both approaches are the same and both rely on K41. Moreover, being the results of FC derived from a pure statistical theory à la Batchelor (1952), its success and the consistency with the LSM, enhances the validity of the latter rather than invalidates it.

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