DIFFERENTIAL TOPOLOGY OF GAUSSIAN RANDOM FIELDS

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ABSTRACT. Motivated by numerous questions in random geometry, given a smooth manifold $M$, we approach a systematic study of the differential topology of Gaussian Random Fields (GRF) $X : M \to \mathbb{R}^k$, i.e. random variables with values in $C^\infty(M, \mathbb{R}^k)$ inducing on it a Gaussian measure. We endow the set of GRFs with the narrow topology and we first prove some preliminary results relating the convergence in the Whitney $C^\infty$ topology of the covariance structure of $X$ and the random variable $X \in C^\infty(M, \mathbb{R}^k)$ itself. When dealing with a convergent family $\{X_d\}_{d \in \mathbb{N}}$ of GRFs, these results allow to compute the limit probabilities of a family of events in terms of the probability distribution of the limit GRF.

We complement this study by proving two important technical tools: the first is an infinite dimensional, probabilistic version of the Parametric Transversality Theorem, which ensures that, under some conditions, a GRF is almost surely transversal to any given submanifold of the jet space; the second is a generalization of the Kac-Rice formula for transversal intersections, which allows to count the cardinality of the transversal preimage of a manifold under a random map. (This result is formulated in a very general framework, allowing to consider the case of the preimage under the jet map of a random field of a submanifold of the jet space.)

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1. INTRODUCTION

1.1. **Gaussian random maps.** Let $M$ be a smooth $m$-dimensional manifold (possibly with boundary). We denote by $E^r = C^r(M, \mathbb{R}^k)$ the space of differentiable maps endowed with the weak Whitney topology, where $r \in \mathbb{N} \cup \{\infty\}$, and we call $\mathcal{P}(E^r)$ the set of probability measures on $C^r(M, \mathbb{R}^k)$, endowed with the narrow topology.

In this paper we are interested in a special subset of $\mathcal{P}(E^r)$, namely the set $\mathcal{G}(E^r)$ of **Gaussian measures**: these are probability measures with the property that for every finite set of points $p_1, \ldots, p_j \in M$ the evaluation map $\varphi : C^r(M, \mathbb{R}^k) \to \mathbb{R}^{jk}$ at these points induces (by pushforward) a Gaussian measure on $\mathbb{R}^{jk}$. \(^1\) We denote by $\mathcal{G}^r(M, \mathbb{R}^k)$ the set of $C^r$ **gaussian random fields** (GRF) i.e. random variables with values in $E^r$ that induce a Gaussian measure (see Definition 11 below).

\(^1\)This definition can be proved to be equivalent to that of a Gaussian measure on the topological vector space $E^r$.
Remark 1. One can define a Gaussian random section of a vector bundle $E \to M$ in an analogous way (the evaluation map here takes values in the finite dimensional vector space $E_{p_1} \oplus \cdots \oplus E_{p_n}$). We choose to discuss only the case of trivial vector bundles to avoid a complicated notation, besides any vector bundle can be linearly embedded in a trivial one, so that any Gaussian random section can be viewed as a Gaussian random field. For this reason the results we are going to present regarding GRFs are true, mutatis mutandis, for Gaussian random sections of general vector bundles.

We have the following sequence of continuous injections:
\[
\mathcal{G}(E^\infty) \subset \cdots \subset \mathcal{G}(E^r) \subset \cdots \subset \mathcal{G}(E^0) \subset \mathcal{P}(E^0),
\]
with the topologies induced by the inclusion $\mathcal{G}(E^r) \subset \mathcal{P}(E^r)$ as a closed subset.

A Gaussian random field $X$ induces itself a Gaussian measure on $\mathcal{C}^r(M, \mathbb{R}^k)$, measure that we denote by $[X]$. Two fields are called equivalent if they induce the same measures. A Gaussian measure $\mu = [X] \in \mathcal{G}(E^r)$ gives rise to a differentiable function $K_\mu \in \mathcal{C}^r(M \times M, \mathbb{R}^{k \times k})$ called the covariance function and defined for $p, q \in M$ by:
\[
K_\mu(p, q) = \mathbb{E}(X(p)X(q)^T) = \int_{E^r} f(p)f(q)^T d\mu(f).
\]
Equivalent fields give rise to the same covariance function, and to every covariance function there corresponds a unique (up to equivalence) Gaussian field. Our first theorem clarifies the relation between convergence in $\mathcal{G}(E^r)$ of Gaussian measures with respect to the narrow topology and convergence in $\mathcal{C}^r(M \times M, \mathbb{R}^{k \times k})$, with the weak Whitney topology, of the corresponding covariance functions.

Theorem 2 (Measure-Covariance). The natural map
\[
(1.1) \quad K^r : \mathcal{G}(E^r) \to \mathcal{C}^r(M \times M, \mathbb{R}^{k \times k}),
\]
given by $K_\mu : \mu \mapsto K_\mu$, is injective and continuous for all $r \in \mathbb{N} \cup \{\infty\}$; when $r = \infty$ this map is also a closed topological embedding.

We observe at this point that the condition $r = \infty$ in the second part of the statement of Theorem 2 is necessary: as Example 31 and Theorem 32 show, it is possible to build a family of $\mathcal{C}^r$ ($r \neq \infty$) GRFs with covariance structures which are $\mathcal{C}^r$ converging but such that the family of GRFs does not converge narrowly to the GRF corresponding to the limit covariance.

Theorem 2 is especially useful when one has to deal with a family of Gaussian fields depending on some parameters, as it allows to infer asymptotic properties of probabilities on $\mathcal{C}^\infty(M, \mathbb{R}^k)$ from the convergence of the covariance functions (notice that this “implication” goes the opposite way of the arrow in (1.1)).

Theorem 3 (Limit Probabilities). Let $\{X_d\}_{d \in \mathbb{N}} \subset \mathcal{G}^\infty(M, \mathbb{R}^k)$ be a sequence of Gaussian fields such that the sequence $\{K_d\}_{d \in \mathbb{N}}$ of the associated covariance functions converges to $K$ in $\mathcal{C}^\infty(M \times M, \mathbb{R}^{k \times k})$. Then there exists $X \in \mathcal{G}^\infty(M, \mathbb{R}^k)$ with $K_X = K$ such that for every Borel set $A \subset E^\infty$ we have
\[
(1.2) \quad P(X \in \text{int}(A)) \leq \liminf_{d \to \infty} P(X_d \in A) \leq \limsup_{d \to \infty} P(X_d \in \overline{A}) \leq P(X \in \overline{A}).
\]
Remark 4. The notion of narrow convergence of a family \( \{X_d\}_{d \in \mathbb{N}} \) of GRFs corresponds to the notion of convergence in law of random elements in a topological space and it regards just the probability measures \([X_d]\). By Skorohod Theorem (see [3, Theorem 6.7]) this notion corresponds to almost sure convergence up to equivalence of GRFs. In case one is interested in the almost sure convergence or in the convergence in probability of a particular sequence of GRFs one should be aware that those two notions take into account also the joint probabilities, for example convergence in probability is equivalent to narrow convergence of the couple \((X_d, X) \Rightarrow (X, X)\) (Theorem 65 in Appendix A).

1.2. The representation of a gaussian random field. The previous Theorem 3 raises two natural questions:

(1) when is the leftmost probability in (1.2) strictly positive?

(2) For which sets \(A \subset E^\infty\) do we have equality of all the terms in (1.2)?

To answer the first question we need to introduce a few auxiliary concepts. First, given a gaussian field \(X = (X^1, \ldots, X^k) \in G^r(M, \mathbb{R}^k)\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we consider the Hilbert space \(\Gamma_X\) defined by:

\[
\Gamma_X = \text{span}\{X^j(p), p \in M, j = 1, \ldots, k\}\big|_{L^2(\Omega, \mathcal{F}, \mathbb{P})}.
\]

There is a linear and continuous injection \(\rho_X : \Gamma_X \to E^r\) given by:

\[
\rho_X(\gamma) = E(X(\cdot)\gamma) = \left(\langle X^1(\cdot), \gamma \rangle_{L^2(\Omega, \mathcal{F}, \mathbb{P})}, \ldots, \langle X^k(\cdot), \gamma \rangle_{L^2(\Omega, \mathcal{F}, \mathbb{P})}\right)
\]

(the fact that \(\rho_X\) is a continuous injection is proved in Proposition 33).

The image of \(\rho_X\) coincides with the Cameron-Martin space of the measure \([X]\) and we denote it by \(H_X\).

In this context next two theorems are well-known in the general theory of abstract Wiener spaces. However here we present and give a proof of these results adapted to our language, with the scope of making the exposition more complete and self-contained. (The connection between our definitions and the general abstract ones follows from the discussion in Appendix B.) The following theorem, for instance, is a corollary of the classical Ito-Nisio Theorem, adapted to our setting, and gives a standard way of representing a gaussian random field.

**Theorem 5** (Representation theorem). Let \(X \in G^r(M, \mathbb{R}^k)\). For every Hilbert orthonormal basis \(\{h_n\}_{n \in \mathbb{N}}\) of \(H_X\), there exists a sequence \(\{\xi_n\}_{n \in \mathbb{N}}\) of independent, standard gaussians such that the series \(\sum_{n \in \mathbb{N}} \xi_n h_n\) converges\(^2\) in \(E^r\) to \(X\) almost surely.

To finally address question (1) above, we recall also the concept of support of a gaussian random field \(X \in G^r(M, \mathbb{R}^k)\) defined as:

\[
supp(X) = \{f \in E^r \text{ such that } \mathbb{P}(U) > 0 \text{ for every neighborhood } U \text{ of } f\}.
\]

If we have a representation of \(X\) as in Theorem 5, next result gives a description of its support.

\(^2\)Given a sequence \(\{x_n\}_{n \in \mathbb{N}} \subset E\), the sentence “the series \(\sum_{n \in \mathbb{N}} x_n\) converges in \(E\) to \(x\)” means that \(s_N = \sum_{n \leq N} x_n\) converges in \(E\) to \(x\) as \(N \to \infty\).
Theorem 6 (The support of a gaussian random map). Let \( X \in \mathcal{G}^r(M, \mathbb{R}^k) \). Let \( \{f_n\}_{n \in \mathbb{N}} \subset E^r \) and consider a sequence \( \{\xi_n\}_{n \in \mathbb{N}} \) of independent, standard gaussians. Assume that the series \( \sum_{n \in \mathbb{N}} \xi_n f_n \) converges in \( E^r \) to \( X \) almost surely. Then
\[
\text{supp}(X) = \text{span}\{f_n\}_{n \in \mathbb{N}}^{C^r(M, \mathbb{R}^k)}.
\]

In particular, the support of a gaussian measure is always a closed vector subspace.

1.3. Differential topology from the random point of view. Addressing question (2) above, let us observe that the probabilities in (1.2) are equal iff \( \mathbb{P}(\partial A) = 0 \), and the study of this condition naturally leads us to the world of Differential Topology.

When studying smooth maps, most relevant sets are given imposing some conditions on their jets (this is what happens, for example, when studying a given singularity class). For example, let us take for \( A \subset E^\infty \) in Theorem 3 an open set defined by a condition on the \( r \)-th jet of \( X \):
\[
A = \{ f \in E^\infty \text{ such that } j^r_p f \subseteq J^r(M, \mathbb{R}^k) \text{ for all } x \in M \}.
\]

Observe that if \( V \) is an open set with smooth boundary \( \partial V \), then there is no map \( f \in \partial A \) satisfying \( j^r_p f \pitchfork \partial V \). This is a frequent situation, indeed in most cases, the boundary of \( A \) consists of functions whose jet is not transversal to a given submanifold \( W \subset J^r(M, \mathbb{R}^k) \), and then the problem of having the equality in (1.2) reduces to \( \mathbb{P}(j^r X \pitchfork W) = 1 \). Motivated by this, we prove the following.

Theorem 7. Let \( X \in \mathcal{G}^\infty(M, \mathbb{R}^k) \) and denote \( F = \text{supp}(X) \). Let \( r \in \mathbb{N} \). Assume that for every \( p \in M \) we have
\[
\text{supp}(j^r_p X) = J^r_p(M, \mathbb{R}^k) \tag{1.3}
\]
Then for any submanifold \( W \subset J^r(M, \mathbb{R}^k) \), we have \( \mathbb{P}(j^r X \pitchfork W) = 1 \).

Let us explain condition (1.3) better. Given \( X \in \mathcal{G}^r(M, \mathbb{R}^k) \) and \( p \in M \) one can consider the random vector \( j^r_p X \in J^r(M, \mathbb{R}^k) \): this is a Gaussian variable and (1.3) is the condition that the support of this gaussian variable is the whole \( J^r_p(M, \mathbb{R}^k) \). For example, if the support of a \( C^r \)-gaussian field \( X \) equals the whole \( E^r \), then for every \( W \subset J^r(M, \mathbb{R}^k) \) we have \( X \pitchfork W \) with probability one.

We will actually prove Theorem 7 as a corollary of the following more general theorem, that gives an infinite dimensional version of the classical Parametric Transversality Theorem (see Theorem 41).

Theorem 8 (Probabilistic transversality). Let \( X \in \mathcal{G}^\infty(M, \mathbb{R}^k) \) and denote \( F = \text{supp}(X) \). Let \( P, N \) be smooth manifolds and \( W \subset N \) a submanifold. Assume that \( \Phi : P \times F \rightarrow N \) is a smooth map such that \( \Phi \pitchfork W \). Then
\[
\mathbb{P}\{\phi(X) \pitchfork W\} = 1,
\]
where \( \phi_p(f) = \Phi(p, f) \).
1.4. A generalized counting formula for the jet map. In this context it is natural to ask, when $M$ is $m$-dimensional, $X : M \to \mathbb{R}^k$ is a GRF and $W \subseteq J^r(M, \mathbb{R}^k)$ is a submanifold of codimension $m$ to which $X$ is almost surely transversal, for the expected cardinality of $j^rX^{-1}(W)$, i.e. the expected number of points where the $r$-th jet of $X$ belongs to $W$. To answer this question, we conclude our study by proving the following generalization of the Kac-Rice Theorem. (Our feeling is that the proof of this version is both more general and simpler than the standard proofs.)

We will consider submanifolds $W \subseteq J^r(M, \mathbb{R}^k)$ which are “nice enough” in the sense of Definition 57 (which ensures the volume of $W$ does not grow too fast) and Definition 59 (which captures the notion of a submanifold of the jet space which is constant on fibers of the source map); we call such $W \subseteq J^r(M, \mathbb{R}^k)$ a good submanifold. Here are some examples of good submanifolds:

1. $W = M \times \{0\} \subseteq J^0(M, \mathbb{R}^m)$ and $j^0f^{-1}(W)$ is the zero set of $f$;
2. $W = \{j^1 = 0\} \subseteq J^1(M, \mathbb{R})$ and $j^1f^{-1}(W)$ is the set of critical points of $f$;
3. $W = \{\mathrm{rk}(j^1) = \ell\} \subseteq J^1(M, \mathbb{R}^k)$ and the set $j^1f^{-1}(W)$ consists of the points where the rank of the differential of $f$ is $\ell$ (when $(k-\ell)(m-\ell) = m$ this is a zero-dimensional manifold).

**Theorem 9** (Generalized Kac-Rice counting formula). Let $W \subseteq J^r(M, \mathbb{R}^k)$ be a good smooth submanifold of codimension $m = \dim M$. Denote by $VM \to M$ the density bundle (note: $VM = M \times \mathbb{R}$ when $M$ is orientable). There exists a universal measurable bundle map

$$
\rho_W : J^{r+1, r+1}_+ (M \times M, \mathbb{R}^{k \times k})|_M \to VM
$$

with the property that for any $X \in G^{r+1}_+ (M, \mathbb{R}^k)$ such that $j^{r+1}_pX$ is non degenerate for all $p \in M$, for every Borel set $A \subset M$ the following formula holds:

$$
\mathbb{E}\# (j^rX^{-1}(W) \cap A) = \int_A \rho_W(j^{r+1, r+1}_p X)dp.
$$

(See section 7, Corollary 61 for a more detailed statement.)

2. Preliminaries

2.1. Space of smooth functions. Let $M$ be a smooth manifold of dimension $m$. We will always implicitly assume that $M$ is Hausdorff and second countable, possibly with boundary. Let $k \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{+\infty\}$. We will refer at the set of $C^r$ functions

$$
E^r = C^r(M, \mathbb{R}^k)
$$

as a topological space with the weak Whitney topology as in [8,10]. Let $Q : D \hookrightarrow M$ be an embedding of a compact set $D \subset \mathbb{R}^n$, we define for any $f \in C^r(M, \mathbb{R}^k)$,

$$
\|f\|_{Q,r} = \sup \{ |\partial_\alpha (f \circ Q)(x)| : \alpha \in \mathbb{N}^m, \ |\alpha| \leq r, \ x \in \text{int}(D) \}.
$$

Then for $r \in \mathbb{N}$ finite, the weak topology on $C^r(M, \mathbb{R}^k)$ is defined by the family of seminorms $\{\|\cdot\|_{Q,r}\}$, while the topology on $C^\infty(M, \mathbb{R}^k)$ is defined by the whole family $\{\|\cdot\|_{Q,r}\}_{Q,r}$. We recall that for any $r \in \mathbb{N} \cup \{\infty\}$, the topological space $C^r(M, \mathbb{R}^k)$ is a
Polish space: it is separable and homeomorphic to a complete metric space. We will also need to consider the space $C^{r,r}(M \times M, \mathbb{R}^k)$ consisting of those functions that, in any chart, have continuous partial derivatives of order at least $r$ with respect to both the product variables. The topology on this space is defined by the seminorms

$$
\|f\|_{Q,(r,r)} = \sup \left\{ |\partial_{(\alpha, \beta)}(f \circ Q)(x,y)| : \alpha, \beta \in \mathbb{N}^m, |\alpha|, |\beta| \leq r, x, y \in \text{int}(D) \right\},
$$

where now $Q(x, y) = (Q_1(x), Q_2(x)) \in M \times M$ and $Q_1, Q_2$ are embeddings.

**Lemma 10.** Let $f, f_n \in C^r(M, \mathbb{R}^k)$. $f_n \to f$ in $C^r(M, \mathbb{R}^k)$ if and only if for any convergent sequence $p_n \to p$ in $M$,

$$
\textstyle j_{p_n}^* f_n \to j_p^* f \quad \text{in} \quad J^r(M, \mathbb{R}^k).
$$

**Proof.** See [8, Chapter 2, Section 4].

It follows that given an open cover $\{U_\ell\}_{\ell \in L}$ of $M$, the restriction maps define a topological embedding $C^r(M, \mathbb{R}^k) \to \prod_{\ell \in L} C^r(U_\ell, \mathbb{R}^k)$, indeed any converging sequence $p_n \to p$ belongs to some $U_\ell$ eventually. In particular suppose that $Q_\ell : \mathbb{D} \to M$ are a countable family of embeddings of the unit $m$-disk $\mathbb{D}$ such that $\text{int}(Q_\ell(\mathbb{D})) = U_\ell$ is a covering of $M^3$. Then the maps $Q_\ell^* : f \mapsto f \circ Q_\ell$ define a topological embedding

$$
(2.1) \quad \{Q_\ell^*\}_\ell : C^r(M, \mathbb{R}^k) \hookrightarrow \left( C^r(\mathbb{D}, \mathbb{R}^k) \right)^L
$$

We refer to the book [8] for more details on topologies on spaces of differentiable functions.

2.2. **Gaussian Random Fields.** Most of the material in this section, can be found in the book [1] and in the paper [10]; we develop the language in a slightly different way so that it suits our point of view focused on measure theory.

Recall that a real random variable $\gamma$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be Gaussian if there are real numbers $\mu \in \mathbb{R}$ and $\sigma \geq 0$, such that $\gamma \sim N(\mu, \sigma^2)$, meaning that it induces the $N(\mu, \sigma^2)$ measure on the real numbers, which is $\delta_\mu$ if $\sigma = 0$, and for $\sigma \geq 0$ it has density

$$
\rho(t) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(t-\mu)^2}{2\sigma^2}}.
$$

In this paper, unless otherwise specified, all gaussian variables and vectors are meant to be centered, namely with $\mu = 0$.

A (centered) gaussian random vector $\xi$ in $\mathbb{R}^k$ is a random variable on $\mathbb{R}^k$ s.t. for any covector $\lambda \in (\mathbb{R}^k)^*$, the real random variable $\lambda^T \xi$ is (centered) gaussian. In this case we write $\xi \sim N(0, K)$ where $K = \mathbb{E}[\xi \xi^T]$ is the so called covariance matrix. If $\xi$ is

\[\text{This is always possible in a smooth manifold without boundary, by definition, and it is still true if the manifold has boundary: if } p \in \partial M, \text{ take an embedding of the unit disk } Q : \mathbb{D} \to M \text{ such that } Q(\partial \mathbb{D}) \text{ intersects } \partial M \text{ in an open neighbourhood of } p, \text{ then the interior of } Q(\mathbb{D}), \text{ viewed as a subset of } M, \text{ contains } p.\]
a gaussian random vector in $\mathbb{R}^k$, there is a random vector $\gamma \sim N(0, 1_j)$ in $\mathbb{R}^j$ and an injective $k \times j$ matrix $A$ s.t.

$$\xi = A\gamma.$$ 

In this case $K = AA^T$ and the support of $\xi$ is the image of $A$, that is

$$\text{supp}(\xi) = \{ p \in \mathbb{R}^k : \mathbb{P}\{U_p \} > 0 \text{ for all neighborhoods } U_p \ni p \} = \text{Im}A,$$

indeed $\xi \in \text{Im}A$ with $\mathbb{P} = 1$. If $A$ is invertible, $\xi$ is said to be nondegenerate, this happens if and only if $\det K \neq 0$, if and only if $\text{supp}(\xi) = \mathbb{R}^n$ if and only if the probability induced by $\xi$ admits a density, which is given by the formula

$$(2.2) \quad \mathbb{P}\{\xi \in U\} = \frac{1}{(2\pi)^{\frac{n}{2}} \det K} \int_U e^{-\frac{1}{2}W^T K^{-1} W} dW^n.$$ 

**Definition 11** (Gaussian Random Field). Let $M$ be a smooth manifold. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space. An $\mathbb{R}^k$-valued *Random Field* (RF) on $M$ is a measurable map

$$X : \Omega \to (\mathbb{R}^k)^M,$$

with respect to the product $\sigma-$algebra on the codomain. An $\mathbb{R}$-valued RF is called a *Random Function*.

Let $r \in \mathbb{N} \cup \{\infty\}$. We say that $X$ is a $C^r$ field, if $X_\omega \in C^r(M, \mathbb{R}^k)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. We say that $X$ is a *Gaussian Random Field* (GRF) if for any finite collection of points $p_1, \ldots, p_j \in M$, the random vector in $\mathbb{R}^{jk}$ defined by $(X(p_1), \ldots, X(p_j))$ is gaussian. We denote by $\mathcal{G}^r(M, \mathbb{R}^k)$ the set of $C^r$ gaussian fields.

When dealing with random fields $X : \Omega \to (\mathbb{R}^k)^M$, we will most often use the shortened notation of omitting the dependence from the variable $\omega$. In this way $X : M \to \mathbb{R}^k$ is a *random map*, i.e. a random element\(^4\) of $(\mathbb{R}^k)^M$.

**Remark 12.** In the above definition, the sentence:

“$X_\omega \in C^r(M, \mathbb{R}^k)$ for $\mathbb{P}$-almost every $\omega \in \Omega$”

means that the set $\{ \omega \in \Omega : X_\omega \in C^r(M, \mathbb{R}^k) \}$ contains a measurable set $\Omega_0$ which has probability one. We make this remark because the subset $C^r(M, \mathbb{R}^k)$ doesn’t belong to the product $\sigma-$algebra of $(\mathbb{R}^k)^M$.

**Lemma 13.** For all $r \in \mathbb{N} \cup \{+\infty\}$ the Borel $\sigma$-algebra $\mathcal{B}\left(C^r(M, \mathbb{R}^k)\right)$ is generated by the sets

$$\{ f \in C^r(M, \mathbb{R}^k) : f(p) \in A \}$$

with $p \in M$ and $A \subset \mathbb{R}^k$ open.

**Proof.** See [10, p. 43,44]. \(\square\)

\(^4\)Recall that, given a measurable space $(S, \mathcal{A})$, a measurable map from a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ to $S$ is also called a *Random Element* of $S$ (see [3]). Random variables and random vectors are random elements of $\mathbb{R}$ and $\mathbb{R}^k$, respectively.
As a consequence we have that the Borel σ-algebra $\mathcal{B}(\mathcal{C}^r(M, \mathbb{R}^k))$ is the restriction to $\mathcal{C}^r(M, \mathbb{R}^k)$ of the product σ-algebra of $(\mathbb{R}^k)^M$. It follows that $X$ is a $\mathcal{C}^r$ RF on $M$ if and only if it is $\mathbb{P}$—almost surely equal to a random element of $\mathcal{C}^r(M, \mathbb{R}^k)$.

A second consequence is that if $X$ is a $\mathcal{C}^r$ RF, then the associated map $\tilde{X} : \Omega \times M \to \mathbb{R}^k$ is measurable, being the composition $e \circ (X \times \text{id})$, where $e : \mathcal{C}^r(M, \mathbb{R}^k) \times M \to \mathbb{R}^k$ is the continuous map defined by $e(f, p) = f(p)$.

**Lemma 14.** $\mathcal{C}^r(M, \mathbb{R}^k)$ is a Borel subset of $\mathcal{C}^0(M, \mathbb{R}^k)$, for all $r \in \mathbb{N} \cup \{+\infty\}$.

*Proof.* See [10, p. 43,44].

If $X$ is a $\mathcal{C}^r$-RF, then it induces a probability measure $X_*\mathbb{P}$ on $\mathcal{C}^r(M, \mathbb{R}^k)$, or equivalently (because of Lemma 14) a probability measure on $\mathcal{C}^0(M, \mathbb{R}^k)$ supported on $\mathcal{C}^r(M, \mathbb{R}^k)$. We say that two RFs are equivalent if they induce the same measure; note that this can happen also if they are defined on different probability spaces.

It is easy to see that every probability measure $\mu$ on $\mathcal{C}^r(M, \mathbb{R}^k)$ is induced by some RF (just take $\Omega = \mathcal{C}^r(M, \mathbb{R}^k)$, $\mu = \mathbb{P}$ and the identity field). This means that the study of $\mathcal{C}^r$ random fields up to equivalence corresponds to the study of Borel probability measures on $\mathcal{C}^r(M, \mathbb{R}^k)$.

Note that, as a consequence of Lemma 13, a Borel measure $\mu$ on $\mathcal{C}^r(M, \mathbb{R}^k)$ is uniquely determined by its finite dimensional distributions, which are the measures induced on $\mathbb{R}^{kj}$ by evaluation on $j$ points.

We will write $\mu = [X]$ to say that the probability measure $\mu$ is induced by a random field $X$. In particular we define Gaussian Measures on $\mathcal{C}^r(M, \mathbb{R}^k)$ to be those measures that are induced by a $\mathcal{C}^r$-GRF, equivalently we give the following measure-theoretic definition.

**Definition 15** (Gaussian measure). Let $M$ be a smooth manifold and let $r \in \mathbb{N} \cup \{\infty\}$, $k \in \mathbb{N}$. A Gaussian Measure on $\mathcal{C}^r(M, \mathbb{R}^k)$ is a probability measure on the topological (Polish) space $\mathcal{C}^r(M, \mathbb{R}^k)$, with the property that for any finite set of points $p_1, \ldots p_j \in M$, the measure induced on $\mathbb{R}^{jk}$ by the map $f \mapsto (f(p_1), \ldots, f(p_j))$ is gaussian (centered and possibly degenerate). We denote by $\mathcal{G}(E^r)$ the set of gaussian probability measures on $E^r = \mathcal{C}^r(M, \mathbb{R}^k)$.

**Remark 16.** In general a Gaussian measure on a topological vector space $W$ is defined as a Borel measure on $W$ such that all the elements in $W^*$ are Gaussian random variables. In the case $W = \mathcal{C}^r(M, \mathbb{R}^k)$, this is equivalent to Definition 15, because the set of functionals $f \mapsto (f(p_1), \ldots, f(p_j))$ is dense in the topological dual $W^*$ (see Theorem 72 of Appendix A) and an almost sure limit of gaussian variables is gaussian.

2.3. **The topology of Random Fields.** We denote by $\mathcal{P}(E^r)$, the set of all Borel probability measures on $E^r$. We shall endow the space $\mathcal{P}(E^r)$ with the narrow topology, defined as follows. Let $\mathcal{C}_b(E^r)$ be the Banach space of all bounded continuous functions from $E^r$ to $\mathbb{R}$. 
Definition 17 (Narrow topology). The narrow topology on $\mathcal{P}(E^r)$ is defined as the coarsest topology such that for every $\varphi \in \mathcal{C}_b(E^r)$ the map $E\varphi : \mathcal{P}(E^r) \to \mathbb{R}$ given by:

$$\text{ev}_\varphi : \mathcal{P} \mapsto \int_{E^r} \varphi \, d\mathcal{P}$$

is continuous.

In other words, the narrow topology is the topology induced by the weak-$*$ topology of $\mathcal{C}_b(E^r)^*$, via the inclusion $\mathcal{P}(E^r) \hookrightarrow \mathcal{C}_b(E^r)^*$, $\mathcal{P} \mapsto E^r\{\cdot\}$.

Remark 18. The narrow topology is also classically referred to as the weak topology (see [11] or [3]). We avoid the latter terminology to prevent confusion with the topology induced by the weak topology of $\mathcal{C}_b(E^r)^*$, which is strictly finer. Indeed if a sequence of probability measures $\mu_n$ converges to a probability measure $\mu$ in the weak topology of $\mathcal{C}_b(E^r)^*$, then for any measurable set $A \in E^r$, it holds $\lim_{n \to \infty} \mu_n(A) = \mu(A)$. This is a strictly stronger condition than narrow convergence by Portmanteau's theorem.

From the point of view of random fields, $[X_n] \Rightarrow [X] \in \mathcal{P}(E^r)$, if and only if

$$\lim_{n \to \infty} \mathbb{E}\{\varphi(X_n)\} = \mathbb{E}\{\varphi(X)\} \quad \forall \varphi \in \mathcal{C}_b(E^r)$$

and in this case we will simply write $X_n \Rightarrow X$. This notion of convergence of random variables is called convergence in law or in distribution.

Note that $C^r$ narrow convergence implies $C^s$ narrow convergence, for every $s \leq r$, but not vice versa. Indeed there are continuous injections

$$\mathcal{C}(E^\infty) \subset \cdots \subset \mathcal{C}(E^r) \subset \cdots \subset \mathcal{C}(E^0) \subset \mathcal{P}(E^0).$$

Proposition 19. $\mathcal{C}(E^r)$ is closed in $\mathcal{P}(E^r)$.

Proof. Let $X_n \in \mathcal{C}(M, \mathbb{R}^k)$ s.t. $X_n \Rightarrow X \in \mathcal{P}(E^r)$. Then for any $p_1, \ldots, p_j \in M$ we have

$$(X_n(p_1), \ldots, X_n(p_j)) \Rightarrow (X(p_1), \ldots, X(p_j))$$

in $\mathcal{P}(\mathbb{R}^j)$. Therefore the latter is a gaussian random vector and $[X] \in \mathcal{C}(E^r)$. \qed

We recall the following useful fact relating properties of the topology of $E$ to properties of the narrow topology on $\mathcal{P}(E)$; for the proof the reader is referred to [11, p. 42-46].

Proposition 20. The following properties are true:

1. $E$ is separable and metrizable if and only if $\mathcal{P}(E)$ is separable and metrizable.

   In this case, the map $E \hookrightarrow \mathcal{P}(E)$, defined by $f \mapsto \delta_f$, is a closed topological embedding and the convex hull of its image is dense in $\mathcal{P}(E)$.

2. $E$ is compact if and only if $\mathcal{P}(E)$ is compact.

3. $E$ is Polish if and only if $\mathcal{P}(E)$ is Polish.
The following corollary will be useful for us.

**Corollary 21.** Let $E_1$ and $E_2$ be two separable metric spaces. Let $\pi: E_1 \to E_2$ be continuous. Then the induced map $\pi_*: \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ is continuous. If moreover $\pi$ is a topological embedding, then $\pi_*$ is a topological embedding as well.

*Proof.* If $\pi$ is continuous, then for any bounded and continuous real function $\varphi \in C_b(E_2)$, the composition $\varphi \circ \pi$ is in $C_b(E_1)$. Hence the function $\int_{E_1} (\varphi \circ \pi): \mathcal{P}(E_1) \to \mathbb{R}$ defined as $\mathbb{P} \mapsto \int_{E_1} (\varphi \circ \pi) d\mathbb{P}$ is continuous. Observe that for any $\mathbb{P} \in \mathcal{P}(E_1)$

$$
\int_{E_1} (\varphi \circ \pi) d\mathbb{P} = \int_{E_2} \varphi d(\pi_*\mathbb{P}) = \left(\int_{E_2} \varphi\right) \circ \pi_*(\mathbb{P}),
$$

thus the composition $(\int_{E_1} \varphi) \circ \pi_*: \mathcal{P}(E_1) \to \mathbb{R}$ is continuous for any $\varphi \in C_b(E_2)$. From the definition of the topology on $\mathcal{P}(E_2)$, it follows that $\pi_*$ is continuous.

Assume now that $\pi$ is a topological embedding. This is equivalent to say that any open set $U \subset E_1$ is of the form $\pi^{-1}(V)$ for some open subset $V \subset E_2$, thus the same thing is true for Borel sets. It follows that $\pi_*$ is injective, indeed if two probability measures $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(E_1)$, have equal induced measures $\pi_*\mathbb{P}_1 = \pi_*\mathbb{P}_2$, then

$$
\mathbb{P}_1\{\pi^{-1}(V)\} = \mathbb{P}_2\{\pi^{-1}(V)\}
$$

for any Borel subset $V \subset E_2$, thus $\mathbb{P}_1\{U\} = \mathbb{P}_2\{U\}$ for any Borel subset $U \subset E_1$, hence $\mathbb{P}_1 = \mathbb{P}_2$.

It remains to prove that $\pi_*^{-1}$ is continuous on the image of $\pi_*$. Let $\mathbb{P}_n \in \mathcal{P}(E_1)$ be such that $\pi_*\mathbb{P}_n \Rightarrow \pi_*\mathbb{P}_0$. Let $U \subset E_1$ be open, then there is some $V \subset E_2$ open such that $\pi^{-1}(V) = U$ and, by Portmanteau’s theorem (see [11, p. 40]), we get

$$
\liminf_n \mathbb{P}_n\{U\} = \liminf_n \pi_*\mathbb{P}_n\{V\} \geq \pi_*\mathbb{P}_0\{V\} = \mathbb{P}_0\{U\}.
$$

This implies that $\mathbb{P}_n \Rightarrow \mathbb{P}_0$. We conclude using point (1) of Proposition 20, and the fact that sequential continuity is equivalent to continuity on metric spaces. □

**Example 22.** Let $\phi: M \to N$ be a $\mathcal{C}^r$ maps between smooth manifolds, then the map $\phi^*: \mathcal{C}^r(N,W) \to \mathcal{C}^r(M,W)$ defined as $\phi^*(f) = f \circ \phi$ is continuous, therefore the induced map between the spaces of probabilities, which we still denote as $\phi^*$, is continuous. The same holds for the map $\phi_*: \mathcal{C}^r(W,M) \to \mathcal{C}^r(W,N)$, such that $\phi_*(f) = \phi \circ f$.

### 2.4. The covariance function.

Given a gaussian random vector $\xi$, it is clear by equation (2.2) that the corresponding measure $\mu_\xi$ on $\mathbb{R}^m$ is determined by the covariance matrix $K = \mathbb{E}\{\xi^T\}$. Similarly, if $X \in \mathcal{G}^r(M,\mathbb{R}^k)$, then $[X]$ is a measure on $\mathcal{C}^0(M,\mathbb{R}^k)$ and it is uniquely determined by its finite dimensional distributions, which are the gaussian measures induced on $\mathbb{R}^{kj}$ by evaluation on $j$ points. It follows that $[X]$ is uniquely determined by the collection of all the covariances of the evaluations on two points in $M$, which we call covariance function.

**Definition 23** (covariance function). Given $X \in \mathcal{G}^r(M,\mathbb{R}^k)$, we define its covariance function as:

$$
K_X: M \times M \to \mathbb{R}^{k \times k}
$$
\[ K_X(p,q) = \mathbb{E}\{X(p)X(q)^T\}. \]

The function \( K_X \) is symmetric: \( K_X(p,q)^T = K_X(q,p) \) and non-negative definite, which means that for any \( p_1, \ldots, p_j \in M \) and \( \lambda_1, \ldots, \lambda_j \in \mathbb{R}^k \),
\[ \sum_{i=1}^j \lambda_i^2 K_X(p_i, p_i) \lambda_j \geq 0. \]

If \( X \) is a gaussian random function on \( M \), defined on a probability space \((\Omega, \mathcal{G}, \mathbb{P})\), then it is also defined a map
\[ \gamma_X : M \to L^2(\Omega, \mathcal{G}, \mathbb{P})^k \]
such that \( \gamma_X(p) = X(p) \).

The fact that \( X \) is gaussian is equivalent to say that span\{\( \gamma_X(M) \)\} is a gaussian subspace of \( L^2(\Omega, \mathcal{G}, \mathbb{P})^k \), namely a vector subspace whose elements are gaussian random vectors. Next proposition from [10] will be instrumental for us.

**Proposition 24** (Lemma A.3 from [10]). Let \( X \in \mathcal{G}^r(M, \mathbb{R}^k) \), then the map \( \gamma_X : M \to L^2(\Omega, \mathcal{G}, \mathbb{P})^k \) is \( C^r \). Moreover if \( x, y \) are any two coordinate charts on \( M \), then
\[ \mathbb{E}\left\{ \partial_\alpha X(x)(\partial_\beta X(y))^T \right\} = \partial_{(\alpha,\beta)} K_X(x,y). \]
for any multi-indices \(|\alpha|, |\beta| \leq r\).

We prove now a simple Lemma that will be needed in the following. Given a differentiable map \( f \in C^r(M, \mathbb{R}^k) \) with \( r \geq 1 \), and a smooth vector field \( v \) on \( M \), we denote by \( vf \) the derivative of \( f \) in the direction of \( v \).

**Lemma 25.** Let \( X \in \mathcal{G}^r(M, \mathbb{R}^k) \) and let \( v \) be a smooth vector field on \( M \). Then \( vX \in \mathcal{G}^{r-1}(M, \mathbb{R}^k) \). (Notice that, as a consequence, the \( r \)-jet of a \( C^r \) GRF is also a GRF.)

**Proof.** Since \( X \in C^r(M, \mathbb{R}^k) \) almost surely, then \( vX \in C^{r-1}(M, \mathbb{R}^k) \) almost surely, thus \( vX \) defines a probability measure supported on \( C^{r-1}(M, \mathbb{R}^k) \). To prove that it is a gaussian measure, note that \( vX(p) \) is either a \( \mathcal{N}(0,0) \) gaussian, if \( v_p = 0 \), or an almost sure limit of gaussian vectors, indeed passing to a coordinate chart \( x^1, \ldots, x^m \) centered at \( p \) s.t. \( v_p = \frac{\partial}{\partial x^2} \), we have
\[ vX(p) = \lim_{t \to 0} \frac{X(t,0,\ldots,0) - X(0,0,\ldots,0)}{t} \text{ a.s.} \]
therefore it is gaussian. The analogous argument can be applied when we consider a finite number of points in \( M \).

\[ \Box \]

### 2.5. A gaussian inequality.

The scope of this section is to prove Theorem 28, which contains a key technical inequality. This result follows from a general inequality valid for GRFs, not necessarily continuous.

We define for all \( \varepsilon > 0 \) the quantity \( N(\varepsilon) \), to be the minimum number of \( L^2 \)-balls of radius \( \varepsilon \) needed to cover \( \gamma_X(M) \). This number is always finite if \( \gamma_X(M) \) is relatively compact in \( L^2 \). We will need the following Theorem from [1].
**Theorem 26** (Theorem 1.3.3 from [1]). Let \( \gamma_X(M) \) be compact in \( L^2(\Omega, \mathcal{G}, \mathbb{P}) \). Let \( \Delta_X = \text{diam}(\gamma_X(M)) \). There exists a universal constant \( C \) such that

\[
\mathbb{E}\left\{ \sup_{x \in M} X(t) \right\} \leq C \int_0^{\Delta_X} \sqrt{\ln N(\varepsilon)} d\varepsilon
\]

As a corollary, in our setting we can derive the following.

**Lemma 27.** Let \( X \in G^1(M, \mathbb{R}) \) and consider an embedding \( Q : D \hookrightarrow M \) of a compact disk \( D \subset \mathbb{R}^m \). There is a constant \( C_Q > 0 \) such that

\[
\mathbb{E}\{\|X\|_{Q,0}\} \leq C_Q \sqrt{\|K_X\|_{Q \times Q,1}}
\]

**Proof.** It is not restrictive to assume that \( M = D \) and \( Q = \text{id} \). Notice that since the map \( \gamma_X \) is continuous, by Proposition 24, it follows that \( \gamma_X(D) \) is compact in \( L^2(\Omega, \mathcal{G}, \mathbb{P}) \), so that we can apply Theorem 26 to get that

\[
\mathbb{E}\{\|X\|_{D,0}\} \leq 2C \int_0^{\Delta_X} \sqrt{\ln N(\varepsilon)} d\varepsilon
\]

Moreover, for any \( q, p \in D \), we have that

\[
\|X(p) - X(q)\|_{L^2}^2 = K(p,p) + K(q,q) - 2K(p,q) \\
\leq |K(p,p) - K(p,q)| + |K(q,q) - K(p,q)| \\
\leq 2 \sup_{x,y \in D} \left| \frac{\partial K}{\partial x}(x,y) \right| |p - q|,
\]

where \( K = K_X \). Thus, denoting \( \Lambda^2 = 2\|K\|_{Q \times Q,1} \), we obtain that

\[
(2.3) \quad \|X(p) - X(q)\|_{L^2} \leq \Lambda|q - p|^{\frac{1}{2}}.
\]

Let now \( \tilde{N}(\rho) \) be the minimum number of standard balls in \( \mathbb{R}^m \) with radius \( \rho \), required to cover \( D \). A consequence of (2.3) is that every ball of radius \( \rho \) in \( D \) is contained in the preimage via \( \gamma_X \) of a ball of radius \( \Lambda \rho^{\frac{1}{2}} \) in \( L^2 \), therefore \( N(\varepsilon) \leq \tilde{N}\left(\frac{\varepsilon^2}{\Lambda^2}\right) \) and \( \Delta_X \leq \Lambda \sqrt{R} \), where \( R \) is the diameter of \( D \), so that

\[
\mathbb{E}\{\|X\|_{D,0}\} \leq 2C \int_0^{\Delta_X} \sqrt{\ln \tilde{N}\left(\frac{\varepsilon^2}{\Lambda^2}\right)} d\varepsilon \leq 2C \Lambda \int_0^{\sqrt{R}} \sqrt{\ln \tilde{N}(s^2)} ds.
\]

Now since \( D \subset \mathbb{R}^m \), there is a constant \( c_m \) such that \( \tilde{N}(\rho) \leq c_m \left(\frac{R}{\rho}\right)^m \), therefore

\[
\int_0^{\sqrt{R}} \sqrt{\ln \tilde{N}(s^2)} ds \leq \int_0^{\sqrt{R}} \sqrt{\ln c_m \left(\frac{R}{s^2}\right)^m} ds < \infty,
\]

and we conclude. \( \square \)

We are now able to prove the required gaussian inequality.
**Theorem 28.** Let \( X \in \mathcal{G}^r(M, \mathbb{R}^k) \) and consider an embedding \( Q : D \hookrightarrow M \) of a compact disk \( D \subset \mathbb{R}^m \). Then

\[
\mathbb{E}\{\|X\|_{Q,r-1}\} \leq C \sqrt{\|K_X\|_{Q \times Q,(r,r)}},
\]

where \( C \) is a constant depending only on \( Q, r \) and \( k \).

**Proof.** A repeated application of Lemma 25 proves that \( \partial_\alpha X^i \) is gaussian, so that we can use Lemma 27 as follows.

\[
\mathbb{E}\{\|X\|_{Q,r-1}\} \leq \sum_{|\alpha| < r, i \leq k} C_Q \sqrt{\|\partial_\alpha X^i\|_{Q \times Q,1}} = \sum_{|\alpha| < r, i \leq k} C_Q \sqrt{\|\partial_{(\alpha,\alpha)} K_X^{i,i}\|_{Q \times Q,1}} \leq C(Q,r,k) \sqrt{\|K_X\|_{Q \times Q,(r,r)}}.
\]

\( \square \)

3. **Proof of Theorem 2 and Theorem 3**

3.1. **Proof that \( K^r \) is injective and continuous.** We already noted that \( K_X \) determines \([X]\), and this is equivalent to say that \( K^0 \) is injective. It follows that \( K^r \) is injective for every \( r \), since \( K^r \) is just the restriction of \( K^0 \) to \( \mathcal{G}(E^r) \).

Let us prove continuity. Since both the domain and the codomain are metrizable topological spaces, it will be sufficient to prove sequential continuity. Let \( \mu_n \Rightarrow \mu \in \mathcal{G}(E^r) \). Let \( X \in \mathcal{G}^r(M, \mathbb{R}^k) \) be a GRF such that \( \mu = [X] \) and for every \( n \in \mathbb{N} \) let \( X_n \in \mathcal{G}^r(M, \mathbb{R}^k) \) be such that \( \mu_n = [X_n] \). By Skorohod’s representation Theorem (see [3, Theorem 6.7]) we can assume that the \( X_n \) are GRFs defined on a common probability space \((\Omega, \mathcal{G}, \mathbb{P})\) and that \( X_n \rightarrow X \) almost surely in the topological space \( C^r(M, \mathbb{R}^k) \).

To prove \( C^{r,r} \) convergence of \( K_n = K_{X_n} \) to \( K = K_X \), it is sufficient (and necessary) to show that given coordinate charts \((x, y)\) on \( M \times M \), a sequence \((x_n, y_n) \rightarrow (x_0, y_0)\), a couple of indices \(|\alpha|, |\beta| \leq r \) and two indices \( i, j \in \{1, \ldots, k\} \), then

\[
(3.1) \quad \partial_{(\alpha,\beta)} K_n^{i,j}(x_n, y_n) \rightarrow \partial_{(\alpha,\beta)} K^{i,j}(x_0, y_0).
\]

Let \( \gamma_n = \partial_\alpha X^{i}_n(x_n) \) and \( \xi_n = \partial_\beta X^{j}_n(y_n) \). By Lemma 25, these two random vectors are gaussian; moreover \( \gamma_n \rightarrow \gamma \) and \( \xi_n \rightarrow \xi \) almost surely. It follows that the convergence holds also in \( L^2(\Omega, \mathcal{G}, \mathbb{P}) \), so that

\[
\mathbb{E}\{\gamma_n \xi_n\} \rightarrow \mathbb{E}\{\gamma \xi\},
\]

which is exactly (3.1).
3.2. Relative compactness. As we will see in Theorem 32 the map $K_r$ is not proper when $r$ is finite. However, we have the following partial result.

**Theorem 29.** Let $r \in \mathbb{N}$ and consider $[X_n] \in \mathcal{G}(E^{r+2})$ be such that for every $Q : D \hookrightarrow M$ embedding of a compact set $D \subset \mathbb{R}^m$, 

$$\sup_n \|K_{X_n}\|_{Q \times Q, (r+2, r+2)} < \infty.$$ 

Then the sequence $\{[X_n]\}_{n \in \mathbb{N}}$ is relatively compact in $\mathcal{G}(E^r)$.

An analogous result holds also when $r = \infty$.

**Theorem 30.** Let $[X_n] \in \mathcal{G}(E^\infty)$ be such that for every $Q : D \hookrightarrow M$ embedding of a compact set $D \subset \mathbb{R}^m$ and every $r \in \mathbb{N}$:

$$\sup_n \|K_{X_n}\|_{Q \times Q, r} < \infty.$$ 

Then the sequence $\{[X_n]\}_{n \in \mathbb{N}}$ is relatively compact in $\mathcal{G}(E^\infty)$.

Before proving this Theorem, recall that $C^r(M, \mathbb{R}^k)$ has the product topology with respect to the countable family of maps $\{Q^r_\ell\}_{\ell \in \mathbb{N}}$, defined like in (2.1). It follows that a subset $A \subset C^r(M, \mathbb{R}^k)$ is relatively compact if and only if $Q^r_\ell A \subset C^r(D, \mathbb{R}^k)$ is relatively compact for all $\ell$. In particular, if $r < \infty$, given constants $A_\ell > 0$, the set

$$\mathcal{A}^r = \left\{ f \in C^r(M, \mathbb{R}^k) : \|f\|_{Q_\ell, r+1} \leq A_\ell \forall \ell \right\}$$

is compact in $C^r(M, \mathbb{R}^k)$. Similarly, given $A^\infty_\ell > 0$ for all $r, \ell \in \mathbb{N}$, the set

$$\mathcal{A}^\infty = \left\{ f \in C^\infty(M, \mathbb{R}^k) : \|f\|_{Q_\ell, r} \leq A^\infty_\ell \forall r, \ell \right\}$$

is compact in $C^\infty(M, \mathbb{R}^k)$. An important fact to note here is that every compact set in $C^\infty(M, \mathbb{R}^k)$ is contained in a set of the form $\mathcal{A}^\infty$, while the analogous fact is not true when $r$ is finite.

The proof of Theorem 29 is essentially the same than that of Theorem 30, hence we give only the latter.

**Proof of theorem 30.** By Prohorov’s Theorem (see [3, Theorem 5.2]), it is sufficient to prove that $\{[X_n]\}_{n}$ is tight in $\mathcal{G}(E^\infty)$, i.e. that if for every $\varepsilon > 0$ there is a compact set $\mathcal{A} \subset E^\infty$, such that $\mathbb{P}(X_n \not\in \mathcal{A}) \geq 1 - \varepsilon$ for any $n \in \mathbb{N}$.

Fix $\varepsilon > 0$, and let $Q_\ell$ as above. By Theorem 28 we have the inequality

$$\mathbb{E} \{\|X_n\|_{Q_\ell, r}\} \leq C^r_\ell \sup_n \|K_{X_n}\|_{Q_\ell \times Q_{2r+2}, 2r+2} \leq B^r_\ell$$

for some positive constants $B^r_\ell, C^r_\ell > 0$. By assumption, the constants $B^r_\ell$ exist finite. Define $A^r_\ell = (B^r_\ell)^{-1}2^{r+\ell+2}$ and consider the compact set

$$\mathcal{A} = \left\{ f \in C^\infty(M, \mathbb{R}^k) : \|f\|_{Q_\ell, r} \leq \frac{1}{\varepsilon} A^r_\ell \forall r, \ell \right\}.$$
By subadditivity and Markov’s inequality we have that for all $n \in \mathbb{N}$:

$$
\mathbb{P}\{X_n \notin \mathcal{A}\} \leq \sum_{r,\ell \in \mathbb{N}} \mathbb{P}\left\{\|X_n\|_{Q,r} > \frac{1}{\varepsilon} A_r^\ell\right\}
\leq \sum_{r,\ell \in \mathbb{N}} \frac{B_r^\ell}{A_r^\ell} \varepsilon
= \sum_{r,\ell \in \mathbb{N}} 2^{-(r+\ell+2)} \varepsilon = \varepsilon.
$$

We conclude that $\{[X_n]\}_n$ is tight. \hfill \Box

### 3.3. Proof that $\mathcal{K}^\infty$ is a closed topological embedding.

We already know that $\mathcal{K}^\infty$ is injective and continuous. To prove that it is a closed topological embedding it is sufficient to show that $\mathcal{K}^\infty$ is proper: both $\mathcal{G}(E^\infty) \subset \mathcal{P}(E^\infty)$ and $E^\infty$ are metrizable spaces, and a proper map between metrizable spaces is closed.

Let $A \subset C^\infty(E^\infty)$ be a compact set; then for any $Q : D \hookrightarrow M$ embedding of a compact subset $D \subset \mathbb{R}^m$ and for every $r \in \mathbb{N}$, it holds

$$
\sup_{K \in \mathcal{A}} \|K\|_{Q \times Q,r} < \infty.
$$

Therefore Theorem 30 implies that the closed set $(\mathcal{K}^\infty)^{-1}(\mathcal{A})$ is also relatively compact, hence compact in $\mathcal{G}(E^\infty)$.

### 3.4. Proof of Theorem 3.

By Theorem 2, if $K_d \xrightarrow{C^\infty} K$ then $\mu_d \Rightarrow \mu$. Observe also that, by definition for every $A \subset E^\infty$:

$$
\mathbb{P}(X \in A) = \mu(A) \quad \text{and} \quad \mathbb{P}(X_d \in A) = \mu_d(A).
$$

Consequently (1.2) follows from Portmanteau’s theorem (see [3, Theorem 2.1]).

### 3.5. Addendum: a “counter-theorem”.

It would not be difficult to improve Theorem 28 in order to control $\mathbb{E}\{\|X\|_{Q,r}\}$ with a $(r+\alpha, r+\alpha)$ Holder norm of the covariance function, if the latter is finite for some $\alpha \in (0, 1)$. But it is impossible to get such an estimate with $\alpha = 0$, as the following example shows.

**Example 31.** Let $D \subset \mathbb{R}^m$ compact with non empty interior. We now construct a sequence of smooth GRFs $X_n \in \mathcal{G}^0(D, \mathbb{R})$, with $\|K_{X_n}\|_{D,0} \to 0$, such that

$$
\liminf_{n \to \infty} \mathbb{E}\{\|X_n\|_{Q,0}\} \geq 1.
$$

Let $I_1^{(n)}, \ldots, I_{n_2}^{(n)}$ be disjoint open sets in $D$, containing points $x_1^{(n)}, \ldots, x_{n_2}^{(n)}$. Let $\varphi_1^{(n)}, \ldots, \varphi_{n_2}^{(n)}$ be smooth functions $\varphi_i^{(n)} : D \to [0,1]$ such that $\varphi_i^{(n)}$ is supported in $I_i^{(n)}$ and $\varphi_i^{(n)}(x_i^{(n)}) = 1$ (see Figure 1). Let $\gamma_i$ be a countable family of independent standard
gaussian random variables. Let $a_n \in \mathbb{R}$ be the real number such that $\mathbb{P}\{|\gamma| > a_n\} = \frac{1}{n}$, for any $\gamma \sim \mathcal{N}(0, 1)$, hence $a_n \to +\infty$. Define

$$X_n = \frac{1}{a_n} \sum_{i=1}^{n^2} \gamma_i \varphi_i^{(n)}(x) \in \mathcal{G}^{0}(D, \mathbb{R}).$$

Then $K_{X_n}(x, y) = \frac{1}{a_n} \varphi_i^{(n)}(x) \varphi_j^{(n)}(y)$ for some $i = i_x, j = j_x$, thus $\|K_{X_n}\|_{D, 0} \to 0$.

We can now estimate the probability that the $C^0$-norm of $X_n$ is small by

$$\mathbb{P}\{\|X_n\|_{D, 0} < 1\} \leq \mathbb{P}\left\{ \max_{i=1, \ldots, n^2} |X_n(x_i^{(n)})| < 1 \right\} = \mathbb{P}\{|\gamma| < a_n\}^{n^2} = \left(1 - \frac{1}{n}\right)^{n^2} \xrightarrow{n \to \infty} 0.$$ 

Consequently, by Markov’s inequality

$$\lim \inf_{n \to \infty} \mathbb{E}\{\|X_n\|_{D, 0}\} \geq \lim_{n \to \infty} \mathbb{P}\{\|X_n\|_{D, 0} \geq 1\} = 1. \tag{3.2}$$

Note that the function $K(x, y) = 0$ is the covariance function of a GRF $X_0$, which corresponds to the probability measure $\delta_0 \in \mathcal{G}(E^0)$ concentrated on the zero function $0 \in C^0(D, \mathbb{R})$. Since $\mathbb{P}\{\|X_0\|_{D, 0} \geq 1\} = 0$, equation (3.2) proves also that $[X_n]$ does not converge to $[X_0]$ in $\mathcal{G}(E^0)$, even if $K_{X_n} \to K_{X_0}$ in $C^0(D, \mathbb{R})$.

The previous Example 31 can be generalized to prove the following result, which shows the condition $r = \infty$ in the second part of the statement of Theorem 2 is necessary.

**Theorem 32.** If $r$ is finite, the map $(\mathcal{K}^r)^{-1}$ is not continuous.

**Proof.** Let $X_n \in \mathcal{G}^0(D, \mathbb{R})$ as in Example 31. Since $X_n$ is a sum of functions with compact support, we can as well consider $X_n$ as a random element of $C^0(\mathbb{R}, \mathbb{R})$. So that $K_{X_n} \to 0$ in $C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, because their support is contained in $D \times D$, but $X_n \neq 0$. 

\[\text{Figure 1. The function } \varphi_i^{(n)} \text{ from Example 31 is supported on the interval } I_i^{(n)} = (y_i^{(n)}, z_i^{(n)}) \text{ and takes value 1 at } x_i^{(n)}.\]
Let now \( Y_n \) be the GRF defined as
\[
Y_n(\cdot) = \int_c^c \cdots \int_c^c X_n(s_1) ds_1 \ldots ds_r
\]
for some \( c \notin D \). Then \( Y_n \in \mathcal{G}(\mathbb{R}, \mathbb{R}) \), and \( \frac{d^r}{dx^r} Y_n = X_n \). Moreover
\[
\frac{d^{2r}}{dx^r dy^r} K_{Y_n} = K_{X_n} \to 0
\]
in \( C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and \( K_{Y_n} = 0 \) in a neighbourhood of \((c, c)\), therefore \( K_{Y_n} \to 0 \) in \( C^{r,r}(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \).

Define a function \( \rho : \mathbb{R}^{m-1} \to [0, 1] \) supported inside the unit ball and such that \( \rho(0) = 1 \). Let \( j : \mathbb{R}^m \to M \) be any embedding. Denoting \((t, x) \in \mathbb{R} \times \mathbb{R}^{m-1} = \mathbb{R}^m \), define the transformation \( T : C_c^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(M, \mathbb{R}^k) \) such that \( y \mapsto z = Ty \), where
\[
z(j(t, x)) = \rho(x)y(t)v
\]
for some \( v \in \mathbb{R}^k \). Since \( \rho \) has compact support, \( T \) is continuous; moreover \( Y_n \in C_c^\infty(\mathbb{R}, \mathbb{R}) \) almost surely, so that \( Z_n = Ty_n \) is a well defined GRF of class \( C^r \) on \( M \). Thanks to the continuity of \( T \), we have that \( K_{Z_n} \to 0 \) in \( C^{r,r}(M, \mathbb{R}^k) \times \mathbb{R}^k) \), but \( Z_n \not\Rightarrow 0 \) in \( \mathcal{G}(M, \mathbb{R}^k) \) because \( Z_n \circ j(\cdot, 0) = X_n \).

4. Proof of Theorem 5

The easiest example of GRF is a field of the type \( X = \xi_1 X_1 + \cdots + \xi_n X_n \), where for \( i = 1, \ldots, n \) each \( \xi_i \) a real gaussian variables, \( X_i \in C^r(M, \mathbb{R}^k) \) and the \( \xi_i \) are independent. A slightly more general example is a series
\[
X = \sum_{n=0}^{\infty} \xi_n X_n
\]
which is narrowly convergent (i.e. such that \( s_n = \sum_{j \leq n} \xi_j X_j \Rightarrow X \) for some \( X \in \mathcal{G}(M, \mathbb{R}^k) \)).

The scope of this section is to prove that in a sense every GRF is of this form – in the case \( r = 0, k = 1 \) and \( M \subset \mathbb{R}^m \), this result is well described in [10, Section A.5] (Such decomposition result holds for a very general class of Gaussian measures, see for example [4, p. 112]).

Given a gaussian random field \( X \in \mathcal{G}(M, \mathbb{R}^k) \), recall the definition of the Hilbert space:
\[
\Gamma_X = \overline{\text{span}\{X^j(p), p \in M\}}^{L^2(\Omega, \mathcal{G}, \mathbb{P})}.
\]
There is a linear map \( \rho_X : \Gamma_X \to E^r \) given by:
\[
\rho_X(\gamma) = \mathbb{E}(X(\cdot)\gamma) = \left( (X^1(\cdot), \gamma)_{L^2(\Omega, \mathcal{G}, \mathbb{P})}, \ldots, (X^k(\cdot), \gamma)_{L^2(\Omega, \mathcal{G}, \mathbb{P})} \right)
\]

**Proposition 33.** The map \( \rho_X \) is a linear, continuous injection.
Proof. Linearity is evident.

To prove injectivity, let $\gamma \in \Gamma_X$ and assume that $\rho_X(\gamma) = 0$. Then $\langle \gamma, X^j(p) \rangle_{L^2} = 0$ for all $p \in M$ and $j = 1, \ldots, k$, so that $\gamma \in \Gamma_X^\perp$, thus $\gamma = 0$.

By linearity it is sufficient to check continuity at $\gamma = 0$. Let $Q : D \hookrightarrow M$ be the embedding of a compact set $D \subset \mathbb{R}^m$. If $r$ is finite, we have

$$\|\rho_X(\gamma)\|_{Q,r} = \sup_{|\alpha| \leq r, x \in D} |\mathbb{E}\{\partial_\alpha (X \circ Q)(x)\gamma\}|$$

$$\leq \sup_{|\alpha| \leq r, x \in D} \mathbb{E}\{|\partial_\alpha (X \circ Q)(x)|^2\}^{\frac{1}{2}} \|\gamma\|_{L^2}$$

$$= \sup_{|\alpha| \leq r, x \in D} \left( \sum_{j=1}^k \partial_{(\alpha,\alpha)}(K^{j,j}_X \circ Q)(x, x) \right)^{\frac{1}{2}} \|\gamma\|_{L^2}$$

$$\leq \left( k \|K_X\|_{Q \times Q, (r,r)} \right)^{\frac{1}{2}} \|\gamma\|_{L^2}.$$ 

Therefore $\lim_{\gamma \to 0} \|\rho_X(\gamma)\|_{Q,r} = 0$ for every $Q$, hence $\rho_X$ is continuous. For the case $r = \infty$, it is sufficient to note that continuity with respect to $E^r$ for every $r$, implies continuity with respect to $E^\infty$. □

4.1. The Cameron-Martin's space. We denote by $\mathcal{H}_X \subset C^r(M, \mathbb{R}^k)$ the image of the map $\rho_X$. It contains all functions $h^j_p = \rho_X(X^j(p))$ of the form

$$h^j_p(q) = \begin{pmatrix} K_X(q,p)^{1j} \\ \vdots \\ K_X(q,p)^{kj} \end{pmatrix}.$$  

for some $p \in M$ and $j \in \{1, \ldots, k\}$.

Moreover $\mathcal{H}_X$ carries a Hilbert structure induced by the map $\rho_X$, which makes it isomorphic to $\Gamma_X$. It follows that $\mathcal{H}_X$ is the Hilbert completion of $\operatorname{span}\{h^j_p : p \in M, j = 1, \ldots, k\}$, endowed with the scalar product

$$\langle h^j_p, h^l_q \rangle_{\mathcal{H}_X} \doteq \left\langle X^j(p), X^l(q) \right\rangle_{L^2} = K^{j,l}_X(p,q).$$

Therefore the Hilbert space $\mathcal{H}_X$ depends only on $K_X$, or equivalently on $[X]$. This space called Cameron-Martin’s space of $[X]$ (see [4] for more details).

Note that $\mathcal{H}_X$ is separable, since $M$ is, hence it has a countable Hilbert orthonormal basis $\{h_n\}_{n \in \mathbb{N}}$, corresponding via $\rho_X$ to a Hilbert orthonormal basis $\{\xi_n\}_{n \in \mathbb{N}}$ in $\Gamma_X$. This means that for any $p$ and $j$, one has $h^j_n(p) = \langle X^j(p), \xi_n \rangle$, namely that $h^j_n(p)$ is precisely the $n^{th}$ coordinate of $X^j(p)$ with respect to the basis $\{\xi_n\}_{n \in \mathbb{N}}$. In other words:

$$X(p) = \lim_{n \to \infty} \sum_{m \leq n} \xi_m h^j_m(p),$$
where the limit is taken in $L^2(\Omega, \mathcal{S}, \mathbb{P})$. In particular, since $L^2$ convergence of random variables implies convergence in probability:

$$
\lim_{{n \to \infty}} \mathbb{P} \left\{ \left| \sum_{{m>n}} \xi_m h_m(p) \right| > \varepsilon \right\} = 0.
$$

4.2. A convergence criterion. We deduce now a useful convergence criterion for a random series. It essentially follows from Ito-Nisio theorem, which we recall for the reader’s convenience.

**Theorem 34** (Ito-Nisio). Let $E$ be a separable real Banach space. Let $M \subset E^*$ be such that the family of sets of the form $\{f \in E \ | \ \langle p, f \rangle \in A\}$, with $A \in \mathcal{B}(\mathbb{R})$, generates the Borel $\sigma$-algebra of $E$. Let $\{x_n\}_{n \in \mathbb{N}}$ be independent symmetric random elements of $E$, define

$$X_n = \sum_{{m \leq n}} x_m.$$

Then the following statements are equivalent:

1. $X_n$ converges almost surely;
2. $\{X_n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(E)$;
3. There is a random variable $X$ with values in $E$ such that $\langle p, X_n \rangle \to \langle p, X \rangle$ in probability for all $p \in M$.

**Remark 35.** In the original paper [9], the theorem is stated with the hypothesis that $M = E^*$, but the same proof still works in the slightly weaker assumptions of Theorem 34.

**Theorem 36.** Let $x_n \in \mathcal{G}^r(M, \mathbb{R}^k)$ with the $x_n$ independent GRFs and consider the GRF

$$X_n = \sum_{{j \leq n}} x_j.$$

The following conditions are equivalent.

1. $X_n$ converges in $\mathcal{C}^r(M, \mathbb{R}^k)$ almost surely.
2. Denoting by $\mu_n$ the measure associated to $X_n$, we have that $\{\mu_n\}_{n \in \mathbb{N}}$ is relatively compact in $\mathcal{F}(E^r)$.
3. There is a random field $X$ such that for all $p \in M$ the sequence $\{X_n(p)\}_{n \in \mathbb{N}}$ converges in probability to $X(p)$.

**Proof.** We prove both that (1) $\iff$ (2) and (3) $\iff$ (1). We repeatedly use the fact that a.s. convergence implies convergence in probability, which in turn implies convergence in distribution (narrow convergence).

(1) $\Rightarrow$ (2) This descends directly from the fact that almost sure convergence implies narrow convergence.

(1) $\Rightarrow$ (3) This step is also clear, since the almost sure convergence of $X_n$ to some random field $X$ implies that for any $p \in M$ the sequence of random vectors $X_n(p)$ converges to $X(p)$ almost surely and hence also in probability.
(3) ⇒ (1) and (2) ⇒ (1) Let $Q_\ell : D \hookrightarrow M$, be a countable family of embeddings of the compact disk, as in (2.1). Note that if $\{X_n\}_n$ is tight in $G^r(M, \mathbb{R}^k)$ (i.e. $\mu_n$ is tight in $G(E)$), then $\{X_n \circ Q_\ell\}_n$ is tight in $G^r(D, \mathbb{R}^k)$. Moreover, if $X_n \circ Q_\ell \to X \circ Q_\ell$ almost surely in $C^r(D, \mathbb{R}^k)$, for every $i \in \mathbb{N}$, then $X_n \to X$ almost surely in $C^r(M, \mathbb{R}^k)$. Therefore it is sufficient to prove the theorem in the case $M = D$. For analogous reasons, we can assume that $r$ is finite.

The topological vector space $E = C^r(D, \mathbb{R}^k)$ has the topology of a separable real Banach space, with norm

$$\| \cdot \|_E = \| \cdot \|_{ID, r}.$$ 

Since the $\sigma$-algebra $\mathcal{B}(C^r(M, \mathbb{R}^k))$ is generated by sets of the form $\{ f : f(p) \in A \}$, where $p \in M$ and $A \subset \mathbb{R}^k$ is open and since gaussian variables are symmetric, we can conclude applying the Ito-Nisio Theorem 34 to the sequence $X_n$ of random elements of $E^r$.

4.3. **Proof of Theorem 5.** Let $\{h_n\}_{n \in \mathbb{N}}$ be a Hilbert orthonormal basis for $\mathcal{H}_X$ and set $\xi_n = \rho_X^{-1}(h_n)$ (it is a family of independent, real gaussian variables). Arguing as after equation (4.2) we get that for every $p \in M$ and $j = 1, \ldots, k$ we have convergence in probability for the series:

$$X(p) = \lim_{n \to \infty} \sum_{m \leq n} h_m(p) \xi_n,$$

so that the a.s. convergence of the series in $C^r(M, \mathbb{R}^k)$ follows from point (1) of Theorem 36.

4.4. **A useful corollary.** A useful corollary of the Theorem 5 is the following lemma, which establishes when it is possible to split a GRF in a sum of independent ones.

**Lemma 37 (Splitting lemma).** Let $X \in G^r(M, \mathbb{R}^k)$. Let $V \subset \mathcal{H}_X$ be a vector subspace closed in the Hilbert topology. Then there are $Y, Z \in G^r(M, \mathbb{R}^k)$ such that $Y$ and $Z$ are independent and $Y + Z = X$; moreover $\mathcal{H}_Y = V$, $\mathcal{H}_Z = V^\perp$ with the same Hilbert product.

**Proof.** Choose an orthonormal basis $c_n$ of $V$ and an orthonormal basis $b_n$ of $V^\perp$. Then, by Theorem 5, we can write $X = \sum_n \xi_n^1 c_n + \xi_n^2 b_n$, with $\xi_n^i$ independent standard real gaussian variables. Define $Y = \sum_n \gamma_n^1 c_n$ and $Z = \sum_n \gamma_n^2 b_n$.

To prove that $\mathcal{H}_Y = V$, recall that $\mathcal{H}_X \cong \Gamma_X = \overline{\text{span} \{ \xi_n^1 : i, n \}^L}^{L^2(\Omega, \mathcal{F}, \mathbb{P})}$ as Hilbert spaces, via the isomorphism given by $\rho_X$. Note that since $c_n$ form an orthonormal system, we have

$$\Gamma_Y = \overline{\text{span} \left\{ \sum_n \xi_n^1 c_n(x) : x \in M \right\}}^{L^2(\Omega, \mathcal{F}, \mathbb{P})} = \overline{\text{span} \{ \xi_n^1 : i, n \}^{L^2(\Omega, \mathcal{F}, \mathbb{P})}},$$

so that $\rho_X \mid \Gamma_Y = \rho_Y$. Therefore

$$\mathcal{H}_Y = \rho_Y(\Gamma_Y) = \rho_X \left( \overline{\text{span} \{ \gamma_n^1 : i, n \}^{L^2(\Omega, \mathcal{F}, \mathbb{P})}} \right) = V.$$
and the Hilbert product on $\mathcal{H}_Y$ is the restriction of that of $\mathcal{H}_X$. With the same argument we get that $\mathcal{H}_Z = V^\perp$. □

5. Proof of Theorem 6

Note that, by definition, the support of $X \in \mathcal{G}^r(M, \mathbb{R}^k)$ has the property that if it intersects an open set $U$, then $\mathbb{P}\{X \in U\} > 0$. The following proposition guarantees that the converse is also true, namely if $\mathbb{P}\{X \in U\} > 0$, then $U \cap \text{supp}(X) \neq \emptyset$.

**Proposition 38.** The support of $X \in \mathcal{G}^r(M, \mathbb{R}^k)$ is the smallest closed set $C \subset E^r$ such that $\mathbb{P}\{X \in C\} = 1$.

**Proof.** By definition we can write the complement of $\text{supp}(X)$ as

$$(\text{supp}(X))^c = \bigcup \{U \subset \mathcal{C}^r(M, \mathbb{R}^k) \text{ open such that } \mathbb{P}\{X \in U\} = 0\}.$$}

Consequently $\text{supp}(X)$ equals the intersection of all closed sets $C \subset \mathcal{C}^r(M, \mathbb{R}^k)$ such that $\mathbb{P}\{X \in C\} = 1$, hence it is closed. Since $\mathcal{C}^r(M, \mathbb{R}^k)$ is second countable the union and the intersection above can be taken over a countable family, hence $\mathbb{P}\{X \in \text{supp}(X)\} = 1$. □

**Remark 39.** Assume that $X_n \Rightarrow X \in \mathcal{G}^r(M, \mathbb{R}^k)$, then for any open set $U \subset \mathcal{C}^r(M, \mathbb{R}^k)$ such that $U \cap \text{supp}(X) \neq \emptyset$, there is a constant $p_U = \mathbb{P}\{X \in U\} > 0$ such that for every $\epsilon > 0$ and $n$ big enough, one has

$$\mathbb{P}\{X_n \in U\} \geq p_U - \epsilon.$$}

In particular, it implies that

$$\text{supp}(X) \subset \bigcup_{n_0} \bigcap_{n \geq n_0} \text{supp}(X_n) = \liminf_{n \to \infty} \text{supp}(X_N).$$}

5.1. Proof of Theorem 6. We start by observing that $X \in \overline{\text{span}\{f_n\}_n}$ with $\mathbb{P} = 1$, therefore:

$$\text{supp}(X) \subset \overline{\text{span}\{f_n\}_{n \in \mathbb{N}}} \cap \mathcal{C}^r(M, \mathbb{R}^k).$$}

Let now $c = \sum_{n=0}^{N_0} a_n f_n$ and let $U_c \subset \mathcal{C}^r(M, \mathbb{R}^k)$ be an open neighbourhood of $c$ of the form

$$U_c = \left\{ f \in \mathcal{C}^r(M, \mathbb{R}^k) : \|f - c\|_{Q,r} < \varepsilon \right\}.$$}

for some embedding $Q$.

Denote by $S_N = \sum_{n \leq N} \xi_n f_n$. Observe that if $N \geq N_0$, then $S_N - c \in \text{span}\{f_1 \ldots f_N\}$, which is a finite dimensional vector space, hence there is a constant $A_N > 0$ such that $\|\sum_{n=0}^{N} a_n f_n\|_{Q,r} \leq A_N \max \{|a_0| \ldots |a_N|\}$. By the convergence in probability of $S_N$ to
\(X\), there is \(N > N_0\) big enough such that \(\mathbb{P}\{\|X - S_N\|_{Q,r} \geq \frac{\varepsilon}{2}\} < \frac{1}{2}\), so that, setting \(a_n = 0\) for \(n > N_0\), we have:

\[
\mathbb{P}\{X \in U_c\} \geq \mathbb{P}\{\|X - S_N\|_{Q,r} < \frac{\varepsilon}{2}, \|S_N - c\|_{Q,r} < \frac{\varepsilon}{2}\}
\]
\[
\geq \mathbb{P}\left\{\|S_N - c\|_{Q,r} < \frac{\varepsilon}{2}\right\} \frac{1}{2}
\]
\[
\geq \left( \prod_{n=0}^{N} \mathbb{P}\{\xi_n - a_n| < \frac{\varepsilon}{2A_N}\}\right) \frac{1}{2}
\]
\[
> 0.
\]

Every open neighbourhood of \(c\) in \(C^r(M, \mathbb{R}^k)\) contains a subset of the form of \(U_c\), therefore \(c \in \text{supp}(X)\). Since \(\text{supp}(X)\) is closed, we conclude.

**Remark 40** (The support via the covariance). Let \(X \in \mathcal{G}^r(M, \mathbb{R}^k)\), then we can also describe the support of \(X\) in terms of its covariance:

\[
\text{supp}(X) = \overline{H_X} = \text{span}\{h^{ij}_p : p \in M, j = 0 \ldots k\}.
\]

Where \(h^{ij}_p\) is defined as in (4.1). In particular, note that the support of a GRF is always a vector space, thus any neighbourhood of 0 has positive probability.

### 6. Proof of Theorem 7

**6.1. Transversality.** We want to prove some results analogous to Thom’s Transversality Theorem (see \([8, \text{Section 3, Theorem 2.8}]\)) in our probabilistic setting. We first recall the so called parametric transversality theorem. Let \(f: M \to N\) be a smooth map, \(W \subset N\) a submanifold and \(K \subset M\) be any subset. Then we say that \(f\) is transverse to \(W\) on \(K\) and write \(f - \langle K, W\rangle\), if and only if for every \(x \in K \cap f^{-1}(W)\) we have:

\[
df_x(T_xM) + T_{f(x)}W = T_{f(x)}N.
\]

We recall the following classical tool, usually called the Parametric Transversality Theorem.

**Theorem 41** (Section 3, Theorem 2.7 from \([8]\)). Let \(g: P \times F \to N\) be a smooth map between smooth manifolds of finite dimension. Let \(W \subset N\) be a smooth submanifold and \(K \subset M\) be any subset. If \(g - \langle K \times F, W\rangle\), then \(g(\cdot, f) - \langle K, W\rangle\) for almost every \(f \in F\).

In our context we prove the following infinite-dimensional, probabilistic version of Theorem 41.

**Theorem 42.** Let \(F \subset E^r\) such that \(F = \text{supp}(X)\) for some \(X \in \mathcal{G}^\infty(M, \mathbb{R}^k)\). Let \(P, N\) be smooth manifolds and \(W \subset N\) a submanifold. Assume that \(\Phi: P \times F \to N\) is a “smooth”\(^5\) map such that \(\Phi - \langle \cdot, W\rangle\). Then

\[
\mathbb{P}\{\Phi(\cdot, X) - \langle \cdot, W\}\} = 1.
\]

\(^5\)Here by “smooth” we mean that:

(1) the map \(\Phi\) is smooth when restricted to finite dimensional subspaces;

(2) the linear map \((p, f, v) \mapsto D_{(p, f)}\Phi v = D_{(p, f)}(\Phi|_{\text{span}(f, v)}) v\) is continuous in all its arguments.
A particular case in which we can apply Theorem 42 is when \( P = M, N = J^r = J^r(M, \mathbb{R}^k), r = \infty \) and \( \Phi \) is the jet-evaluation map
\[
j^r : M \times E^\infty \to J^r, \quad (p, f) \mapsto j^r_{pf} f.
\]
It is not difficult to show that this map is “smooth” in the sense of the statement of Theorem 42.

**Proof.** (In order to simplify notations we denote by \( \phi(X) \) the map \( p \mapsto \Phi(p, X) \).)

First we show that we can assume \( W \) to be compact (possibly with boundary). Indeed let \( W = \cup_{k \in \mathbb{N}} W_k \), such that \( W_k \) is compact. Then \( \Phi|_{P \times F} \mid W_k \) for any \( k \), and
\[
P\{\phi(X) \mid_W W \} \geq 1 - \sum_{k \in \mathbb{N}} (1 - P\{\phi(X) \mid_W W_k \}) .
\]

Moreover we claim that it is sufficient to prove the following weaker statement.

\((\ast)\) For all \( p \in P \) and \( x \in F \) there are neighbourhoods \( Q_p \) of \( p \) in \( P \) and \( N_x \) of \( x \) in \( E^r \) such that:
\[
P\{\phi(X) \mid_{Q_p} W | X \in N_x \} = \frac{P\{\{\phi(X) \mid_{Q_p} W \} \cap \{X \in N_x \} \}}{P\{\{X \in N_x \} \} = 1} .
\]

Assume that \((\ast)\) is true, then there exists a countable open cover of \( P \times F \) of the form \( Q_k \times N_i \) such that \( P\{\phi(X) \mid_{Q_k} W | X \in N_i \} = 1 \), equivalently \( P\{\phi(X) \mid_{Q_k} W, X \in N_i \} = 0 \). Thus
\[
P\{\phi(X) \mid W \} \leq \sum_{k,l} P\{\phi(X) \mid_{Q_k} W, X \in N_l \} = 0,
\]
so the claim is true.

Let’s prove \((\ast)\). Let \( p \in P \) and \( x \in F \). Since \( W \subset N \) is closed, if \( \phi_p(x) \notin W \), then \( \phi_q(x) \notin W \) for all \( q \) in a compact neighbourhood \( Q \) of \( p \) and \( x \) in some neighbourhood \( N_x \) of \( x \) in \( E^r \), so that, in particular \( P\{\phi(X) \mid_{Q} W | X \in N_x \} = 1 \). Assume that \( \phi_p(x) = \theta \in W \), then by hypothesis we have that
\[
D(p, x) \Phi(T_p P + F) + T_\theta W = T_\theta J^r,
\]
hence there is a finite dimensional space \( F_0 = \text{span}\{f_1, \ldots, f_a\} \subset F \) such that
\[
D(p, x) \Phi(T_p P + F_0) + T_\theta W = T_\theta N.
\]
Note that \( F_0 = T_x F_x \), where \( F_x = x + \text{span}\{f_1, \ldots, f_a\} \). Therefore \( \Phi|_{P \times F} \mid_{Q \times D_\varepsilon} W \) (here we are in a finite dimensional setting), moreover there are a compact neighbourhood \( p \in Q \subset P \) and \( \varepsilon > 0 \) such that
\[
(6.1) \quad \Phi|_{P \times F} \mid_{Q \times D_\varepsilon} W .
\]
where \( D_\varepsilon = D_\varepsilon(x, f) = \{x + f u : u \in \mathbb{R}^a, |u| \leq \varepsilon\} \). Observe that the set of \((a+1)\)-tuples \((x, f) = (x, f_1, \ldots, f_a) \in F \times F^a \) for which \((6.1)\) holds (with fixed \( \varepsilon \)), form an open set, indeed the map
\[
\tau: F \times F^a \to C^\infty(P \times \mathbb{R}^a, N), \quad \tau(x, f) : (p, u) \mapsto \Phi(p, x + fu)
\]
is continuous and the set $\Theta = \{ \theta : \theta \in Q \times D_\varepsilon \}$ is open in the codomain because $Q \times D_\varepsilon$ is compact and $W$ is closed (check [8, p. 74]); therefore

$$\tau^{-1}(\Theta) = \{(x, f) \in F \times F^a : (6.1) \text{ holds}\}$$

is open. It follows that there is an open neighbourhood $V_\varepsilon$ of $x$ and an $h \in (H_X)^a$ such that (6.1) holds with $(\tilde{x}, h)$ for any $\tilde{x} \in V_\varepsilon$.

Define $\Lambda = \{ e \in E^r : \phi(e) \in Q \}$. By Theorem 41 we get that if $\tilde{x} \in V_\varepsilon$, then $\phi(\tilde{x} + hu) \in \Lambda$, equivalently $(\tilde{x} + hu) \in \Lambda$, for almost every $|u| \leq \varepsilon$. Using Fubini-Tonelli, we have

$$0 = \int_{V_\varepsilon} \left( \int_{D_\varepsilon} \mathbb{P}\{\tilde{x} + hu \notin \Lambda\} du \right) d[X](\tilde{x}) = \int_{D_\varepsilon} \mathbb{P}\{X + hu \notin \Lambda, X \in V_\varepsilon\} du.$$

hence $\mathbb{P}\{X + hu \in (V_\varepsilon + hu) \setminus \Lambda\} = 0$ for almost every $|u| \leq \varepsilon$. Let $u$ be also so small that $x \in V_\varepsilon + hu$, then, taking $N_x = V_\varepsilon + hu$, we have that $\mathbb{P}\{X + hu \in N_x \setminus \Lambda\} = 0$. Since $hu \in H_X$, the Cameron-Martin theorem (see [7, Theorem 4.24]) implies that $[X]$ is absolutely continuous with respect to $[X + hu]$ and consequently $\mathbb{P}\{X \in N_x \setminus \Lambda\} = 0$. In other words this says that $\mathbb{P}\{\phi(X) \in Q \} | X \in N_x \} = 1$ and (*) is proved. $\square$

We give know a criteria to check the validity of the hypothesis of Theorem 42, without necessarily knowing the support of $X$. Before that, let’s observe that the canonical map $J_p^r \rightarrow M$ is a smooth vector bundle over $M$ with fiber $J_p^r$, so that $T_\theta J_p^r$ is canonically identified with $J_p^r$ itself, for all $\theta \in J_p^r$.

**Proposition 43.** Let $X \in C^r(M, \mathbb{R}^k)$ and $F = \text{supp}(X)$. Let $W \subset J_p^r$ be a smooth manifold and fix a point $p \in M$. Consider the following conditions.

(A) $j_p^r|_{M \times F} \in W$ along $\{p\} \times F$;
(B) the vector space $\text{supp}(j_p^r X)$ is transversal to $(T_\theta W \cap T_{\theta} J_p^r)$ in $J_p^r$, for all $\theta \in j_p^r(F) \cap W$;
(C) $\text{supp}(j_p^r X) = J_p^r$;
(D) Given a chart of $M$ around $p$, the matrix

$$(6.2) \quad (\partial_{(\alpha, \beta)} K_X(p, p))|_{|\alpha|, |\beta| \leq r}$$

has maximal rank.

Then

$$(D) \iff (C), \quad (C) \implies (B) \quad \text{and} \quad (B) \implies (A).$$

**Proof.** $(B) \implies (A)$ Let $f \in F$ such that $\theta = j_p^r f \in W$. Under the identification $T_\theta J_p^r = J_p^r$, mentioned above, we have

$$D_{(p, f)} j_p^r(0, g) = \frac{d}{dt} j_p^r(f + tg) = j_p^r g,$$
so that \( D_{(p,f)}j^r(T_pF) = j_p^r(F) = \text{supp}(j_p^rX). \) Then for all \((p,f) \in (j^r)^{-1}(W) \cap M \times F,\) we have
\[
D_{(p,f)}j^r(T_{(p,f)}M \times F) + T_bW \supset D_{(p,f)}j^r(T_pM) + \text{supp}(j_p^rX) + T_bW \cap J_p^r = \]
\[
= D_{(p,f)}j^r(T_pM) + J_p^r = \]
\[
= T_bJ^r.
\]

The last equality follows from the fact that the map \( j^r f \) is a section of the bundle \( J^r \rightarrow M. \)

\((C) \implies (B)) \) Obvious.

\((D) \iff (C)) \) Any chart around \( p \) defines a linear isomorphism
\[
J_p^r \rightarrow \mathbb{R}^{\{\alpha: |\alpha| \leq r\}}, \quad j_p^r f \mapsto (\partial_a f(p))_\alpha.
\]

With this coordinate system, the covariance matrix of the gaussian random vector \( j_p^r X, \)

is exactly the one in (6.2), hence the \( j_p^r f \) is nondegenerate and the result follows. \(\square\)

Given \( X \in G^\infty(M, \mathbb{R}^k), \) we can also consider it as an element of \( G^r(M, \mathbb{R}^k) \) such that \( P\{X \in C^\infty(M, \mathbb{R}^k)\} = 1. \) We use the notation \( \text{supp}_{C^r}(X) \subset E^r \) to denote the support of the latter, namely
\[
\text{supp}_{C^r}(X) = \mathcal{H}_X^{C^r}.
\]

**Corollary 44.** Let \( X \in G^\infty(M, \mathbb{R}^k), \) such that \( \text{supp}_{C^r}(X) = C^r(M, \mathbb{R}^k). \) Then for every submanifold \( W \subset J^r(M, \mathbb{R}^k), \) one has
\[
P\{j^r X \not\subset W\} = 1.
\]

**Proof.** Clearly \( X \) satisfies for every \( p \in M \) condition \( (C) \) of the proposition above, hence the hypothesis of Theorem 42 are satisfied for every \( W. \) \(\square\)

### 7. Kac-Rice formula for transversal intersections

Let \( X : M \rightarrow N \) be a random \( C^1 \) map and let \( W \subset N \) be a submanifold of codimension \( m. \) Assume that \( X \not\subset W \) and \( X|_{\partial M} \not\subset W \) with \( P = 1, \) then \( X^{-1}(W) \) is a random discrete subset of \( M \setminus \partial M. \) If now \( U \subset M \) is a relatively compact open set, then the number
\[
\#_{X \in W}(U) = \#(X^{-1}(W) \cap U)
\]
is almost surely finite and next Lemma 45 tells that this number is a measurable function and that we can take its mean value:
\[
\mathbb{E}\#_{X \in W}(U) = \mathbb{E}\{\#(X^{-1}(W) \cap U)\}.
\]

**Lemma 45.** Let \( X \) be as above. For any \( A \in \mathcal{B}(M), \) the function \( \#_{X \in W}(A) \) is measurable and the set function
\[
\mathbb{E}\#_{X \in W} : \mathcal{B}(M) \rightarrow [0, +\infty] \quad A \mapsto \mathbb{E}\#_{X \in W}(A)
\]
is a Borel measure on \( M. \)
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Proof. Let \( \mathcal{D} \) be the family of subsets \( A \subset M \) such that the function \( \#_{X \in W}(A) \) is measurable. The family \( \mathcal{D} \) contains every relatively compact open set in \( M \) and it is closed under intersection, hence it is enough to prove that it is also a Dynkin class to conclude, by the Monotone Class Theorem (see [6, p. 3]) that \( \mathcal{D} \supset \mathcal{B}(M) \): in fact \( \mathcal{D} \) is a \( \sigma \)-algebra that contains the \( \sigma \)-algebra generated by the relatively compact open sets, which is the Borel \( \sigma \)-algebra.

Actually, it is more convenient to consider a countable increasing family of relatively compact open subsets \( M_i \) such that \( \bigcup_i M_i = M \) and work with the class \( \mathcal{D}_i = \{ A \in \mathcal{D} : A \subset M_i \} \), because \( \#_{X \in W}(A) \) is almost surely finite for any \( A \subset M_i \).

By previous considerations, \( M_i \in \mathcal{D}_i \). If \( A, B \in \mathcal{D}_i \) and \( A \subset B \), then \( B \setminus A \in \mathcal{D}_i \), because since \( \#_{X \in W}(B) \) is almost surely finite, we can write \( \#_{X \in W}(B) = \#_{X \in W}(B) - \#_{X \in W}(A) \). Suppose that \( A_k \in \mathcal{D} \) is increasing, then

\[
\#_{X \in W}(\bigcup_k A_k) = \lim_k \#_{X \in W}(A_k),
\]

thus \( A = \bigcup_k A_k \in \mathcal{D} \) because \( \#_{X \in W}(A) \) is the pointwise limit of measurable functions, thus in particular if \( A_k \in \mathcal{D}_i \), then \( A \in \mathcal{D}_i \). It follows that \( \mathcal{D}_i \) is indeed a Dynkin class, hence \( \mathcal{D} \supset \mathcal{D}_i \supset \mathcal{B}(M_i) \). Now let \( A \in \mathcal{B}(M) \), then \( A \) is the union of the increasing sequence \( A \setminus M_i \) and since \( A \setminus M_i \in \mathcal{B}(M_i) \subset \mathcal{D} \), we can use again the formula in (7.1), to conclude that \( A \in \mathcal{D} \).

Clearly \( \mathbb{E}\#_{X \in W} \) is finitely additive and \( \mathbb{E}\#_{X \in W}(\emptyset) = 0 \), therefore to prove that \( \mathbb{E}\#_{X \in W} \) is a measure it is enough to show that it is continuous from below. This can be done just by taking the mean value in (7.1), since the sequence \( \#_{X \in W}(A_i) \) is increasing. \[\square\]

The scope of this chapter is to show that, under some additional assumptions on \( X \), the measure \( \mathbb{E}\#_{X \in W} \) obtained in this way is absolutely continuous (essentially “with respect to the Lebesgue measure”, in a sense that will be specified below) and the density, when \( M = \mathbb{R}^m \) and \( N = \mathbb{R}^k \), is

\[
\rho(p) = \int_W \mathbb{E} \left\{ |\det(N(w)^T d_u X)| \bigg| X(u) = w \right\} q_{X(u)}(w) d\Sigma(w),
\]

where \( d\Sigma \) is the volume density of \( W \), \( q_{X(u)}(\cdot) \) is the density of the random vector \( X(u) \) and \( N(w) \) is a matrix whose columns form an orthonormal basis of \( T_w W^\perp \). In the case \( W = \{0\} \), one recovers the celebrated Kac-Rice Formula:

\[
\mathbb{E}\{X = 0\} = \int_M \mathbb{E} \left\{ |\det d_u X| \bigg| X(u) = 0 \right\} q_{X(u)}(0) du.
\]

To do so we are going to follow the scheme of the proof of the standard formula: first we derive a deterministic counting formula, then take the mean value and play with the

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\[6\]Let \( M \) be a nonempty set; a Dynkin class \( \mathcal{D} \) is a collection of subsets of \( M \) such that:

1. \( M \in \mathcal{D} \);
2. if \( A, B \in \mathcal{D} \) and \( A \subset B \), then \( B \setminus A \in \mathcal{D} \);
3. given a family of sets \( \{A_k\}_{k \in \mathbb{N}} \) with \( A_k \in \mathcal{D} \) and \( A_k \subset A_{k+1} \), then \( \bigcup_k A_k \in \mathcal{D} \).

The Monotone Class Theorem (see [6, p. 3]) says that if a family \( \mathcal{D} \) is a Dynkin class which contains a family \( \mathcal{P} \) closed by intersection, then it contains also the \( \sigma \)-algebra generated by \( \mathcal{P} \).
order of integration. However, we will make use of Lebesgue’s differentiation theorem to avoid some of the technicalities usually needed, making the proof simpler also in the standard case. After that we are going to focus on the case of Gaussian Random Fields with the purpose of analyze the dependence of this formula on the covariance function.

7.1. Densities. Let $M$ be a smooth manifold. The density bundle $VM$ is the vector bundle:

$$VM = \wedge^m(T^*M) \otimes L$$

where $L$ is the orientation bundle (see [5, Section 7]); $VM$ is a smooth real line bundle and the fiber can be identified canonically with

$$V_pM = \{\rho: (T_pM)^m \to \mathbb{R}; \rho = \pm|\omega|, \text{ for some } \omega \in \wedge^mT_p^*M\}.$$

Given a set of coordinates $x^1, \ldots, x^m$ on $U$, we denote

$$dx = dx^1 \ldots dx^m = |dx^1 \wedge \cdots \wedge dx^m|.$$

Using the language of [5], $dx$ is the section $(dx^1 \wedge \cdots \wedge dx_m) \otimes e_U$, where $e_U$ is the section of $L|U$ which in the trivialization induced by the chart $(x_1, \ldots, x_m)$ corresponds to “1”.

We have, by definition, that

$$dx = \left|\det \left(\frac{\partial x}{\partial y}\right)\right| dy.$$

for any other set of coordinates $y^1, \ldots, y^m$. It follows that an atlas for $M$ with transition functions $g_{a,b}$ defines a trivializing atlas for the vector bundle $VM$, with transition functions for the fibers given by $|\det(Dg_{a,b})|$.

The sections of $VM$ are called densities; we define $\mathcal{D}(M)$ to be the space of smooth densities and by $\mathcal{D}_c(M)$ the subset of the compactly supported ones.

From the formula for transition functions it is clear that there is a canonical linear function

$$\int_M: \mathcal{D}_c(M) \to \mathbb{R} \quad \int_M \rho = \int_M \rho(p)dp.$$

We can define the modulus of a density through the continuous map $|\cdot|: VM \to VM$ defined locally by the identity $|(\omega(x)dx)| = |\omega(x)|dx$. With this in mind it is clear that we can define the spaces of measurable sections $L(M)$, $L^+(M)$, $L^1(M)$, $L^1_{loc}(M)$ and extend the integral $\int_M$ to $L^1(M)$.

**Definition 46.** We say that a real Radon measure (see [2]) $\mu$ on $M$ is absolutely continuous if $\mu(A) = 0$ on any zero measure subset $A \subset M$. In other words $\mu$ is absolutely continuous if and only if $\varphi_*(\mu|_U)$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^m$ for any chart $\varphi: U \subset M \to \mathbb{R}^m$.

In this language, the Radon-Nikodym Theorem takes the following form.

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7Here we mean *measurable* with respect to the Lebesgue measure of some coordinate chart and not just for the Borel $\sigma$–algebra. This notion makes sense because measurability is a local property and it is invariant under diffeomorphisms.
**Theorem 47.** Let $\mu$ be an absolutely continuous real Radon measure on $M$. Then there is a density $\rho \in L^1_{\text{loc}}(M)$ such that
\[
\mu(A) = \int_A \rho
\]
for all $A \in \mathcal{B}(A)$.

**Remark 48.** Note that the line bundle $VM$ is canonically oriented: the positive sections are those $\rho$ for which $\rho = |\rho|$, so that it is isomorphic to the trivial line bundle, but a crucial fact is that the isomorphism is not canonical, indeed it depends on the choice of some nonvanishing section, for instance one can take the volume density of a riemannian metric on $M$, that’s why in general one cannot integrate functions directly. It is easy to see that $M$ is orientable if and only if the bundle $\wedge^m T^*M$ is trivial as well and the choice of an orientation on $M$ corresponds exactly to the choice of an isomorphism $VM \cong \wedge^m T^*M$. Once this choice is made, it is possible to integrate differential forms of degree $m$ by considering them as densities. In general it is always possible to integrate the modulus of an $m$–form, which is a continuous density defined by the continuous map
\[
|\cdot| : \wedge^m T^*M \to VM \quad \alpha \mapsto |\alpha|
\]
so that locally $|\omega(x)dx^1 \wedge \cdots \wedge dx^m| = |\omega(x)|dx$.

**7.2. Deterministic Formula.** Next Lemma 49 is instrumental and essentially says that given a submanifold $W \hookrightarrow N$ of codimension $m$, possibly with boundary, there exists a subset $H \subset N$, obtained as a locally finite union of closed submanifolds $H_i \subset N$ of codimension $\geq 1$ transverse to $W$ and a smooth submersion $\varphi : N \setminus H \to \mathbb{R}^m$ such that $W \setminus H = \varphi^{-1}(0)$.

Let $1_B : \mathbb{R}^m \to \mathbb{R}$ be the characteristic function of the open unit ball $B = B_1 \subset \mathbb{R}^m$ and let $b_\varepsilon$ be defined as
\[
b_\varepsilon(t) = \frac{1}{\text{vol}(B)} 1_B \left( \frac{t}{\varepsilon} \right) \frac{1}{\varepsilon^m},
\]
so that supp\((b_\varepsilon) \subset B_\varepsilon\) and \(\int b_\varepsilon(t)dt = 1\) (here \(dt = dt_1 \wedge \cdots \wedge dt_m\)). In the setting of Lemma 49 above, we can consider a measurable \(m\)-form

\[(7.2) \quad \eta_\varepsilon = \varphi^*(b_\varepsilon(t)dt)\]

in \(N\), which as shown in the following Proposition, acts as a discontinuous version of the Poincarè dual with “compact vertical support” of \(W\) (in the sense of \([5]\)).

**Proposition 50.** Let \(W \subset N\) be a closed submanifold of codimension \(m\). Let \(H = \cup_i H_i\), \(\varphi\) and \(\eta_\varepsilon\) as above. Let \(M\) be a compact manifold of dimension \(m\) and let \(f: M \rightarrow N\) be a smooth map such that \(f \cap W\), and \(f^{-1}(W) \subset (M \setminus \partial M) \setminus f^{-1}(H)\). Then for all \(\varepsilon \leq \bar{\varepsilon}\) small enough,

\[
#\{f \in W\} = \int_M |f^*\eta_\varepsilon|.
\]

**Proof.** From the transversality assumptions, it follows that \(f^{-1}(W)\) is a discrete set contained in the open manifold \(M_0 = (M \setminus \partial M) \setminus f^{-1}(H)\). Since \(\varphi \circ f\vert_{M_0}\) has 0 as a regular value, there is an \(\varepsilon > 0\) small enough, such that for all \(\varepsilon \leq \bar{\varepsilon}\), the set \(f^{-1}\varphi^{-1}(B_\varepsilon)\) is a disjoint union of neighbourhoods \(U_p\) of the points in \(\{p \in M: f(p) \in W\}\) and the restriction of \(\varphi \circ f\) to \(U_p\) is a diffeomorphism onto its image \(B_\varepsilon\). Therefore we have

\[
#\{f \in W\} = \sum_{f(p) \in W} 1
= \sum_{f(p) \in W} \int_{B_\varepsilon} b_\varepsilon(t)dt
= \sum_{f(p) \in W} \int_{U_p} |(\varphi \circ f)^*(b_\varepsilon(t)dt)|
= \int_M |f^*\eta_\varepsilon|.
\]

\[\square\]

### 7.3. General abstract formula.

Let \(W \subset N\), \(H = \cup_i H_i\), and \(\varphi: N \setminus H \rightarrow \mathbb{R}^m\) as in Lemma 49 and let \(\eta_\varepsilon\) as in (7.2). Suppose that \(X: M \rightarrow N\) is a \(C^1\) random map, i.e. a random element of \(C^1(M, N)\) endowed with the weak Whitney topology. Let \(p \in M\) such that \(\mathbb{P}\{X(p) \in H\} = 0\), then \(\|X^*\eta_\varepsilon(p)\|\) is a well defined random element of the vector space \(V_pM\) and we can take its mean value \(\mathbb{E}\{\|X^*\eta_\varepsilon(p)\|\} \in V_pM\). If this holds for almost every \(p \in M\), then the density \(\mathbb{E}\{\|X^*\eta_\varepsilon\|\}\) is defined almost everywhere and it is measurable. To prove this, observe that the map \((f, p) \mapsto g(f, p) = |f^*\eta_\varepsilon(p)|\) is defined and continuous on \(C^1(M, N) \times M\;\setminus\;L\), where \(L = \{(f, p): f(p) \in H\}\) and by Tonelli’s Theorem, for any absolutely continuous Radon measure \(\mu = \int \rho\), we have

\[
\int_N d[X] \otimes \mu = \int_M \mathbb{P}\{X(p) \in H\}\rho = 0,
\]
so that $N$ has measure zero thus $g$ is well defined almost everywhere. By Tonelli’s theorem, the formula
\[
E\{|X^*\eta_\varepsilon(p)|\} = \int g(f,p)d[X](f)
\]
defines an element of $L^+(M)$ (a positive measurable density, possibly taking infinite values).

**Remark 51.** Note that $P\{X(p) \in H\} = 0$ for almost every $p \in M$ if and only if $X^{-1}(H)$ is almost surely a measure zero set in $M$, indeed
\[
\int_M P\{X(p) \in H\} \rho = E\{\mu(X^{-1}(H))\}
\]
for any $\mu = \int \rho$. In particular if $X \not\subset H$ with $P = 1$, it follows that $X^{-1}(H)$ is almost surely a countable union of codimension-one submanifolds, hence of measure zero, thus $E\{|X^*\eta_\varepsilon(p)|\}$ is well defined up to almost everywhere equivalence.

**Theorem 52.** Let $W \subset N$ be a closed submanifold. Let $X: M \to N$ be a random map that satisfies the following hypothesis,

1. $X \in C^1(M,N)$ almost surely\(^8\).
2. $P\{X \not\subset S \text{ and } X|_{\partial M} \not\subset S\} = 1$ for any submanifold $S \subset N$.
3. $\exists \rho \in L^1_{loc}(M)$ such that
\[
\rho = \lim_{\varepsilon \to 0^+} E\{|X^*\eta_\varepsilon|\} \quad \text{a.e. and in } L^1_{loc}(M),
\]

where $\eta_\varepsilon$ is constructed as in (7.2).

Then $E\#_{X \in W}$ is an absolutely continuous Radon measure with density $\rho_{X \in W} = \rho$, equivalently for any Borel set $A \subset M$ we have
\[
E\#_{X \in W}(A) = \int_A \rho_{X \in W}.
\]

**Remark 53.** Hypothesis (2) of Theorem 52 is quite stronger than what is needed in the proof. Indeed it is enough to require that $X \not\subset W$ almost surely and that one can do the construction of $H$ as in Lemma 49 in such a way that, with probability one, $X^{-1}(W) \subset (M \setminus \partial M) \setminus X^{-1}(H)$ and $X^{-1}(H)$ has zero measure in $M$.

\(^8\)Equivalently, $X$ is a random element of $C^1(M,N)$. 
Proof. Let $\Omega \subset M$ be a relatively compact open set, then we have
\[
E^\#_{X \in W}(\Omega) = E^\#(\{f \in W\} \cap \Omega)
\]
\[
= E \left\{ \lim_{\varepsilon \to 0} \int_\Omega |X^* \eta_\varepsilon| \right\} \quad \text{(by Proposition 50)}
\]
\[
\leq \liminf_{\varepsilon \to 0} E \left\{ \int_\Omega |X^* \eta_\varepsilon| \right\} \quad \text{(Fatou)}
\]
\[
= \liminf_{\varepsilon \to 0} \int_\Omega E \{|X^* \eta_\varepsilon|\} \quad \text{(Tonelli)}
\]
\[
= \int_\Omega \rho,
\]
the last equality because of $L^1_{loc}$ convergence.

It follows that the measure $E^\#_{X \in W}$ is absolutely continuous with respect to the Radon measure $\nu = \int \rho$: indeed let $A \in B(M)$ such that $\nu(A) = 0$, then $A$ can be splitted as a countable union of relatively compact Borel sets $\{A_i\}_{i \in \mathbb{N}}$ and
\[
E^\#_{X \in W}(A_i) \leq \inf_{\Omega \supset A_i} E^\#_{X \in W}(\Omega) \leq \inf_{\Omega \supset A_i} \int_\Omega \rho = \nu(A_i) = 0,
\]
thus $E^\#_{X \in W}(A) = 0$. It follows that there exists a measurable function $g: M \to [0,1]$, such that $E^\#_{X \in W} = \int g \rho$.

To end the proof it is sufficient to prove that $g(p) \geq 1$ for almost every $p \in M$. Let $B_\delta$ be the preimage of the ball of radius $\delta$ with respect to some coordinate chart centered at $p \in M \setminus \partial M$. Let us consider the set
\[
A_\delta = \{ f \in C^1(M, N): f(B_\delta) \subset N \setminus H \text{ and } \varphi \circ f|_{B_\delta} \text{ is an embedding} \}.
\]
It is easy to see that $A_\delta$ is a Borel set in $C^1(M, N)$, moreover for any $f \in A_\delta$ and every $\varepsilon > 0$ we have that
\[
\#_{f \in W}(B_\delta) \geq 1 \geq \int_{\varphi(f(B_\delta))} b_\varepsilon(t) dt = \int_{B_\delta} |f^* \eta_\varepsilon|.
\]
Therefore
\[
\lim_{\delta \to 0} \frac{E^\#_{X \in W}(B_\delta)}{\mathcal{L}^m(B_\delta(p))} \geq \lim_{\delta \to 0} \frac{\mathbb{P}\{X \in A_\delta\}}{\mathcal{L}^m(B_\delta)} \geq \lim_{\delta \to 0} \frac{1}{\mathcal{L}^m(B_\delta)} E \left\{ \left( \int_{B_\delta} |X^* \eta_\varepsilon| \right) 1_{A_\delta}(X) \right\}
\]
\[
= \lim_{\delta \to 0} E \left\{ \left( \frac{1}{\mathcal{L}^m(B_\delta)} \int_{B_\delta} |X^* \eta_\varepsilon| \right) 1_{A_\delta}(X) \right\}
\]
\[
\geq E \left\{ \left( \liminf_{\delta \to 0} \frac{1}{\mathcal{L}^m(B_\delta)} \int_{B_\delta} |X^* \eta_\varepsilon| \right) 1_{A_\delta}(X) \right\}
\]
\[
= E \left\{ (|X^* \eta_\varepsilon|) 1_{\cup_{\delta > 0} A_\delta}(X) \right\} = (*)
\]
where in the last step we used the fact that $|X^* \eta_\varepsilon|$ is continuous at $p$, almost surely. Note that since this computation is on a single chart, we are identifying densities with...
functions: \( \rho(u)du \cong \rho(u) \) and
\[
|X^*\eta_\varepsilon(u)| \cong |\det d_u(\varphi \circ f)|b_\varepsilon(\varphi \circ f(u)).
\]
The set \( A = \bigcup_{\delta > 0} A_\delta \) is precisely
\[
A = \{ f \in C^1(M, N): f(p) \in N \setminus H \text{ and } d_p(\varphi \circ f) \text{ is an isomorphism} \}
\]
so that, by construction, if \( f \notin A \), then \( f^*\eta_\varepsilon(p) = 0 \), hence
\[
(*) = \mathbb{E} \{|X^*\eta_\varepsilon(p)|\}.
\]
Sending \( \varepsilon \to 0 \) and using Lebesgue’s differentiation theorem we obtain
\[
g(p)\rho(p) = \lim_{\delta \to 0} \frac{\mathbb{E}\#_{X \in W}(B_\delta)}{\mathcal{L}^m(B_\delta(p))} \geq \rho(p).
\]
for almost every \( p \in M \) and the proof is finished. \( \square \)

If the hypotheses of the previous theorem are satisfied, one is left with the problem of computing the limit \( \rho = \lim_{\varepsilon \to 0} \mathbb{E}\{|X^*\eta_\varepsilon|\} \), which under suitable assumptions on the pointwise distributions \( \{j^1_pX\} \) can be often computed with the formula (7.3) given below
(for instance if they admits a continuous density that is integrable enough).
We are going to focus in particular on the case of a nondegenerate Gaussian Random Field.

7.4. The Gaussian case. In this last paragraph, we will prove that in the case when \( X \in G^1(M, \mathbb{R}^k) \) satisfies a certain non degeneracy hypothesis, we can compute explicitly the density \( \rho \) in Theorem 52 and the result depends pointwise and smoothly on the covariance function of \( j^1_pX \).

**Lemma 54.** Let \( W \subset N = \mathbb{R}^k \) be compact, \( \varphi, H = \cup_i H_i \) as in Lemma 49 and let \( \eta_\varepsilon \) as in (7.2). Let \( X \in G^1(M, \mathbb{R}^k) \) such that \( j^1_pX \) is non degenerate.

Then \( (\varepsilon, p) \mapsto \mathbb{E}\{|X^*\eta_\varepsilon|(p)| \) is a continuous function on \( [0, +\infty) \times M \).

Let \( (u, x, J) \in U \times \mathbb{R}^k \times \mathbb{R}^{m \times k} \) be local coordinates on \( J^1(M, \mathbb{R}^k) \) and let \( \rho_u \) be the probability density of \( j^1_pX \) in these coordinates. The density \( \rho = \rho_{X \in W} \) from Theorem 52 is given by:

\[
\rho_{X \in W}(p) = \lim_{\varepsilon \to 0} \frac{\mathbb{E}\{|X^*\eta_\varepsilon|(p)|}{\mathbb{E}\#_{X \in W}(B_\delta)} = du \int_W \int_{\mathbb{R}^{m \times k}} |\det(N(w)^T J)|\rho_u(w, J)dJd\Sigma(w)
\]

where \( d\Sigma \) is the volume density of \( W \) induced by the euclidean structure of \( \mathbb{R}^k \) and \( N(w) \) is any \( k \times m \) matrix whose columns \( n_1(w) \ldots n_m(w) \) form an orthonormal basis of \( T_w W^\perp \).

**Proof.** Without loss of generality we can restrict to the case \( M = U \subset \mathbb{R}^m \). Note that the function \( \rho_u(x, J) > 0 \) is smooth in each variable and

\[
\rho_u(x, J) \leq c_1(u) \exp \left(-c_2(u)(\|x\|^2 + \|J\|^2)\right)
\]
for a couple of continuous functions $c_1, c_2 : U \to (0, +\infty)$. We have

\[
\mathbb{E}\{|X^\ast \eta_t|(u)\} = \mathbb{E}\{|X^\ast \phi^t (b \varepsilon(t) dt)\}
\]
\[
= \int_{J^u_{\mathbb{R}}(M, \mathbb{R}^k)} |\text{det}(d_x \phi J)| b \varepsilon(\phi(x)) J^1_u |\phi(x), J\rangle dx
\]
\[
= \int_{\mathbb{R}^k} b \varepsilon(\phi(x)) \left( \int_{\mathbb{R}^{m \times k}} |\text{det}(d_x \phi J)| \rho_u(x, J) J \right) dx = (*)
\]

For $\varepsilon > 0$ small enough, we can take coordinates $(y, t)$ on the open set $\phi^{-1}(B \varepsilon)$ such that $t = \phi(x(y, t))$, so that

\[(*) = \int_{B \varepsilon} b \varepsilon(t) e(u, t) dt.
\]

Where

\[e(u, t) = \int_Y \left( \int_{\mathbb{R}^{m \times k}} |\text{det}\left( \frac{\partial t}{\partial x}(y, t) J \right)| \rho_u(x(y, t), J) J \right) \left| \text{det}\left( \frac{\partial x}{\partial y} \frac{\partial x}{\partial t}(y, t) \right) \right| dy.
\]

It is easy to check that, differentiating the equation $t = \phi(x(y, t))$, one obtains the following identity:

\[
\left| \text{det}\left( \frac{\partial t}{\partial x}(y, t) J \right) \text{det}\left( \frac{\partial x}{\partial y} \frac{\partial x}{\partial t}(y, t) \right) \right| = \left| \text{det}\left( N(y, t)^T J \right) \text{det}\left( \frac{\partial x}{\partial y}(y, t)^T \frac{\partial x}{\partial t}(y, t) \right) \right|^{\frac{1}{2}}
\]

where $N(y, t)$ is a matrix whose columns form an orthonormal basis of $T_{x(y, t)} \phi^{-1}(t)^\perp$. By definition, the volume density of $\phi^{-1}(t)$ is

\[d \Sigma_{\phi^{-1}(t)}(x(y, t)) = \text{det}\left( \frac{\partial x}{\partial y}(y, t)^T \frac{\partial x}{\partial t}(y, t) \right)^{\frac{1}{2}} dy,
\]

hence

\[e(u, t) = \int_Y \left( \int_{\mathbb{R}^{m \times k}} |\text{det}(N(y, t)^T J)| \rho_u(x(y, t), J) J \right) dJ \ d \Sigma_{\phi^{-1}(t)}(x(y, t)) dt.
\]

It follows from the inequality (7.4) and the compactness of $Y$ that $e(u, t)$ is finite for each $(u, t)$ and $e(u, t)$ is a continuous function of $u \in U$ and $t \in B \varepsilon$. Moreover

\[e(u, 0) = \int_Y \left( \int_{\mathbb{R}^{m \times k}} |\text{det}(N(y, 0)^T J)| \rho_u(x(y, 0), J) J \right) dJ \ d \Sigma_{\phi^{-1}(0)}(x((y, 0))
\]
\[= \int_{W} \left( \int_{\mathbb{R}^{m \times k}} |\text{det}(N(w)^T J)| \rho_u(w, J) J \right) dJ \ d \Sigma_{W}(w).
\]
It remains to prove that \( \mathbb{E}\{|X^*\eta_\varepsilon|(u)\} = \int_{B_\varepsilon} b_\varepsilon(t)e(u, t)dt \) is continuous at \((u, 0)\) and takes the value \( e(u, 0) \). To do so, make the change of coordinates \( t \to \varepsilon t \), so that

\[
\lim_{\varepsilon \to 0, u \to \bar{u}} \mathbb{E}\{|X^*\eta_\varepsilon|(u)\} = \lim_{\varepsilon \to 0, u \to \bar{u}} \int_{B_\varepsilon} b_0\left(\frac{t}{\varepsilon}\right)e(u, \varepsilon t)\frac{dt}{\varepsilon^m} = \lim_{\varepsilon \to 0, u \to \bar{u}} \int_{B_1} b_0(t)e(u, \varepsilon t)dt = \int_{B_1} b_0(t)e(\bar{u}, 0)dt = e(\bar{u}, 0).
\]

□

The expression on the right hand side of (7.3) is independent on the chosen chart (this can be proved directly, but it also follows from Lemma 54 since the left hand side of (7.3) is an intrinsic object) and depends smoothly on \( p \), thus defines a smooth density \( \rho_{X\in W} \in \mathcal{D}(M) \).

Observe now that \( \rho_x(x, J) \) depends smoothly on the covariance matrix \([K_{j_p^p X}]\) on the open sets where \( j_p^p X \) is nondegenerate, therefore \( \rho_{X \in W}(p) = \rho_W(p, K_{j_p^p X}) \) for some smooth function \( \rho_W \). More precisely, let us make the identification

\[
K_{j_p^p X} = \mathbb{E}\{\partial_\alpha X(p)\partial_\beta X(p)\}_{|\alpha|,|\beta|\leq 1} = (\partial_{[\alpha, \beta]} K_X(p, p))_{|\alpha|,|\beta|\leq 1} = j_{p,p}^{1,1} K_X
\]

and let us consider the subset \( J_{++}^{1,1}(M \times M, \mathbb{R}^{k \times k})|_M \) of the bigraded jet bundle \( J^{1,1}(M \times M, \mathbb{R}^{k \times k}) \) consisting of all jets \( j_{p,p}^{1,1} K \) that are positive definite i.e. those that under the identification in (7.5), represent the covariance of some nondegenerate Gaussian random vector in \( J_+^{1}(M, \mathbb{R}^k) \). Then we have a well defined smooth bundle map

\[
\rho_W: J_{++}^{1,1}(M \times M, \mathbb{R}^{k \times k})|_M \to VM
\]

\[
\rho_W(j_{p,p}^{1,1} K_X) = \rho_{X \in W}(p)
\]

for any \( W \subset \mathbb{R}^k \) compact.

**Definition 55.** Let \( U, W \subset E \) be vector spaces and let \( J: U \to E \) be a linear map. Then we define the map

\[
J \circ J: VE \to VW \otimes VU
\]

\[
\rho \mapsto \rho(J(u_1), \ldots, J(u_m), w_1, \ldots, w_l)|u_1^* \land \cdots \land u_m^*| \otimes |w_1^* \land \cdots \land w_l^*|
\]

where \( u_1, \ldots, u_m \) and \( w_1, \ldots, w_l \) are respectively a basis for \( U \) and for \( W \) with \( u_1^*, \ldots, u_m^* \) and \( w_1^*, \ldots, w_l^* \) as dual basis.

**Theorem 56.** Let \( W \subset \mathbb{R}^k \) be a smooth submanifold of codimension \( m = \dim M \). There exists a universal measurable bundle map

\[
\rho_W: J_{++}^{1,1}(M \times M, \mathbb{R}^{k \times k})|_M \to VM \cup \{\infty\}
\]
Now let \( W \) be a compact manifold. We have that in this case, arguing as in the discussion before, the same Theorem 52, uniformly on compact sets. Therefore we can conclude applying Theorem 52. Note also that the hypotheses (1) of Theorem 52 are satisfied and by Lemma 54 we have that

\[
\rho_W(j_{p,p}^{1,1}K_X) = \lim_{\varepsilon \to 0^+} \mathbb{E}\{|X^*\eta_\varepsilon|(p)\}
\]

uniformly on compact sets. Therefore we can conclude applying Theorem 52. Note also that in this case, arguing as in the discussion before the same Theorem 52, \( \rho_W \) is smooth.

Let \( S \subset \mathbb{R}^k \) be a submanifold of codimension larger than \( m + 1 \). Since \( \mathbb{P}\{X^{-1}(S) = \emptyset\} = 0 \), then \( \mathbb{E}\#X \in W = \mathbb{E}\#X \in W \setminus S \), moreover \( \rho_{W \setminus S} = \rho_W \) because \( W \cap S \) has certainly measure zero in \( W \), so that, in particular \( \rho_{W \setminus \partial W} = \rho_W \).

Now let \( W \subset \mathbb{R}^k \) be any smooth submanifold, then \( W = \bigcup_{R \in \mathbb{N}} W_R \) for a countable family of compact manifolds \( W_R \) intersecting only on their boundary, so that

\[
\mathbb{E}\#X \in W(A) = \sum_{R \in \mathbb{N}} \mathbb{E}\#X \in W_R(A)
\]

\[
= \sum_{R \in \mathbb{N}} \int_A \rho_{W_R}(j_{p,p}^{1,1}K_X)
\]

\[
= \int_A \sum_{R \in \mathbb{N}} \rho_{W_R}(j_{p,p}^{1,1}K_X)
\]

Now \( \rho_{W_R} \) is smooth, thus the sum \( \sum_{R \in \mathbb{N}} \rho_{W_R} = \rho_W \) is measurable and

\[
\sum_{R \in \mathbb{N}} \rho_{W_R}(j_{p,p}^{1,1}K_X) = \sum_{R \in \mathbb{N}} \int_{W_R} \mathbb{E}\{|\det (N(w)^T d_p X)| \mid X(p) = w\} q_X(p)(w) d\Sigma(w) =
\]

\[
= \int_{W} \mathbb{E}\{|\det (N(w)^T d_p X)| \mid X(p) = w\} q_X(p)(w) d\Sigma(w) =
\]

\[
= \rho_W(j_{p,p}^{1,1}K_X).
\]
It remains to prove the intrinsic formula. Consider the map
\[
d_pX : T_pM \to \mathbb{R}^k \supset T_wW,
\]
with \(X(p) = w\). Then \(d_pX \colon V_w\mathbb{R}^k \to V_wW \otimes V_pM\) acts on \(\rho_{X(p)}(w) = q_{X(u)}(w) d\Sigma(w) \in V_w\mathbb{R}^k\) as follows. Consider a coordinate system \(u\) near \(p\), let \(g(y)\) be a parametrization of \(W\) in a neighbourhood of \(w\), let \(x\) denote the euclidean coordinates on \(\mathbb{R}^k\) and let \(\tilde{N}\) be an orthonormal basis of \(T_wW\). Then
\[
d_pX \cdot \rho_{X(p)}(w) = J \cdot dx \ q_{X(u)}(w)
\]
\[
= |\det \left( J \frac{\partial g}{\partial y} \right) | \ dy \otimes du \ q_{X(u)}(w)
\]
\[
= |\det \left( N^T \left( J \frac{\partial g}{\partial y} \right) \tilde{N} \right) | \ dy \otimes du \ q_{X(u)}(w)
\]
\[
= |\det(N(w)^T J) | q_{X(p)}(w) \det \left( \frac{\partial g^T \partial g}{\partial y} \right)^{1/2} \ dy \otimes du
\]
\[
= |\det(N(w)^T d_uX) | q_{X(p)}(w) d\Sigma(w) \otimes du.
\]
Moreover since \(d_pX\cdot\) is a linear map, then its (conditional) mean value is a linear map as well: for any \(\rho \in V_w\mathbb{R}^k\), we have
\[
\mathbb{E}\{d_pX \cdot X(p) = w]\rho = \mathbb{E}\{d_pX \cdot \rho | X(p) = w\} \in V_wW \otimes V_pM.
\]
Integrating the above expression over \(W\) gives a density element in \(V_pM\), so that
\[
\int_W \mathbb{E} \left\{ d_uX \left| X(u) = w \right. \right\} \rho_{X(u)}(w) = \rho_W(J_{1,1}^{1,1}K_X)
\]
\[
= \int_W \mathbb{E} \left\{ |\det(N(w)^T d_uX) | \left| X(u) = w \right. \right\} q_{X(u)}(w) d\Sigma(w).
\]
\[\square\]

We now give a rough criterium to establish for which submanifolds \(W \subset \mathbb{R}^k\) the map \(\rho_W\) is finite and smooth.

**Definition 57.** Let \(W \subset \mathbb{R}^k\). We say that \(W\) has subexponential growth if for any \(\delta > 0\),
\[
\text{vol}(W \cap B_{R+1} \setminus B_R) = o\left(e^{\delta R^2}\right)
\]
as \(R \to +\infty\).

**Theorem 58.** Let \(W \subset \mathbb{R}^k\) be a smooth submanifold of codimension \(m = \dim M\) with subexponential growth. Then the Kac-Rice density builder of \(W\)
\[
\rho_W : J_+^{1,1}(M \times M, \mathbb{R}^{k \times k})|_M \to V M
\]
is finite and smooth.
Proof. It is sufficient to prove it in the case when \( M = U \subset \mathbb{R}^k \) is a compact domain. Then, under the identification made in (7.5) the map \( \rho_W \) takes the following form.

\[
\rho_W : U \times \{ K \in \mathbb{R}^{(m+1)k \times (m+1)k} : \det K \neq 0 \} \to VM
\]

Define the compact submanifold \( W_R = W \cap \mathbb{D}_{R+1} \setminus B_R \), so that \( \rho_{W_R} \) is finite and smooth. Moreover we already showed in the proof of Theorem 56 that \( \rho_W = \sum_{R \in \mathbb{N}} \rho_{W_R} \). To conclude the proof we need to show that for any derivative \( \partial = \frac{\partial}{\partial x^a \partial K^b} \), the series \( \sum_{R \in \mathbb{N}} \rho_{W_R} \) converges uniformly on compact sets.

It is easy to show, by induction on the order of differentiation, that

\[
\partial \left( \frac{1}{(2\pi)^{\frac{k+1}{2}} \det(K)^{\frac{1}{2}}} \exp \left[ -\frac{1}{2} \left( \frac{x}{J} \right)^T K \left( \frac{x}{J} \right) \right] \right) = \exp \left[ -\frac{1}{2} \left( \frac{x}{J} \right)^T K \left( \frac{x}{J} \right) \right] P_0(K, x, J)
\]

where \( P_0 \) is a polynomial in \( x \) and \( J \), whose coefficients are smooth functions of \( K \). Therefore for all \( u, K \) in a compact set \( \Omega \), we have an estimate

\[
|\partial \rho_{W_R}(u, K)| \leq \int_{W_R} \int_{\mathbb{R}^{k \times m}} \det \left( N(w)^T J \right) c_1 |x|^p |J|^q e^{-c_2 (|x|^2 + |J|^2)} dJd\Sigma(w) \leq \int_{W_R} \left( \int_{\mathbb{R}^{k \times m}} c_3 |J|^{(q+m)} e^{-c_2 |J|^2} dJ \right) (R + 1)^p e^{-c_2 R^2} d\Sigma(w) \leq c_4 R^p e^{-c_2 R^2} \text{vol}(W_R);
\]

hence

\[
\sum_{R \in \mathbb{N}} \max_{(u, K) \in \Omega} |\partial \rho_{W_R}(u, K)| \leq c_4 \sum_{R \in \mathbb{N}} R^p e^{-c_2 R^2} \text{vol}(W_R) < \infty
\]

because \( \text{vol}(W_R) = o \left( e^{-c_2 R^2} \right) \). \( \square \)

7.5. Higher jets. In this subsection we extend the result of Theorem 58 to the case of \( j^r X \) for some \( X \in \mathcal{G}^r(M, \mathbb{R}^k) \). We are going to focus on a very particular case of submanifolds \( W \subset j^r(M, \mathbb{R}^k) \): those that can be locally trivialized in unison with the bundle.

Definition 59. Let \( E \to M \) be a vector bundle of rank \( k \). We say that a submanifold \( W \subset E \) is linearly fibered if for every point \( p \in M \) there is a neighbourhood \( p \in U \subset M \) and a trivialization map \( \chi : E|_U \to U \times \mathbb{R}^k \) such that \( \chi(W \cap E|_U) = U \times W_0 \) for some \( W_0 \subset \mathbb{R}^k \) smooth submanifold.

The notion of subexponential growth given in Definition 57 is invariant under linear isomorphisms, therefore it makes sense in any vector space. So if \( V \) is a vector space and \( W \subset V \) is a smooth submanifold, we say that \( W \) has subexponential growth if it has this property in a (and hence any) linear coordinate system.
Corollary 61. Let \( W \subset J^r(M, \mathbb{R}^k) \) be a linearly fibered smooth submanifold of codimenson \( m = \dim M \). There exists a universal measurable bundle map

\[
\rho_W : J_{+}^{r+1, r+1} (M \times M, \mathbb{R}^k) |_M \rightarrow VM \cup \{+\infty\}
\]

with the property that for any \( X \in \mathcal{G}^{r+1} (M, \mathbb{R}^k) \) such that \( j^{r+1}_p X \) is non degenerate for all \( p \in M \), the following formula holds true for any Borel set \( A \subset M \).

\[
\mathbb{E} \# j^r X \in W(A) = \int_A \rho_W (j^{r+1, r+1}_p K_X) dp \in \mathbb{R} \cup \{+\infty\}
\]

The formula, written in the intrinsic notation, is

\[
\rho_{j^r X \in W}(p) = \rho_W (j^{r+1, r+1}_p K_X) = \int_{W_p} \mathbb{E} \left\{ \frac{dp(j^r X)}{T_w W} \bigg| j^r_p X = w \right\} \rho_{j^r_p X}(w).
\]

If Moreover \( W_p = W \cap J^r_p (M, \mathbb{R}^k) \) has subexponential growth, then \( \rho_W \) is finite and smooth.

Proof. It is sufficient to prove the theorem locally, thus we can assume that \( M = U \subset \mathbb{R}^m \)
so that \( J^r(U, \mathbb{R}^k) \cong U \times \mathbb{R}^N \) for some \( N \) big enough. Moreover we can assume that \( W = U \times W_0 \) for some \( W_0 \subset \mathbb{R}^N \) submanifold.

We get from Theorem 56, the measurable map

\[
\rho_{W_0} : J_{+}^{1, 1} (U \times U, \mathbb{R}^{N \times N}) |_U \rightarrow VU \cup \{+\infty\}
\]

with the property that for any \( Y \in \mathcal{G}^1 (U, \mathbb{R}^N) \) with nondegenerate first jet,

\[
\mathbb{E} \# Y \in W_0 = \int_{W_0} \rho_{W} (j^{1, 1}_p K_Y),
\]

with

\[
\rho_{W} (j^{1, 1}_u K_Y) = du \int_W \mathbb{E} \left\{ |\det(N (w)^T du Y)| \bigg| X(u) = w \right\} \mathbb{P}_{Y(u)}(w) \Sigma(w) = \int_{W_0} \mathbb{E} \left\{ \frac{du \hat{Y}}{T_u W} \bigg| Y(u) = w \right\} \rho_{Y(u)}(w) = \int_{W_0} \mathbb{E} \left\{ \frac{du \hat{Y}}{T_u W} \bigg| Y(u) = w \right\} \rho_{Y(u)}(w).
\]
where \( \hat{Y}(u) = (u, Y(u)) \). Now if \( X \in \mathcal{G}^{r+1}(U, \mathbb{R}^k) \) and \( \hat{Y} = j^r X \in \mathcal{G}^1(U, \mathbb{R}^N) \) is such that \( j^{r+1} X \cong j^1(j^r X) \) is non degenerate, after making an identification similar to that of (7.5):

\[
I: J^{r+1,r+1}_+(U \times U, \mathbb{R}^{k \times k}) \mid_U \to J^{1,1}_+(U \times U, \mathbb{R}^{N \times N}) \mid_U
\]

such that

\[
j^{r+1,r+1} K X \mapsto j^{1,1} K j^r X,
\]

the expression above becomes

\[
\mathbb{E}\# j^r X \in W_0 = \int \rho_{W_0} \circ I (j^{r+1,r+1} K X),
\]

so that we end up with a map

\[
\rho_W = \rho_{W_0} \circ I: J^{r+1,r+1}_+(U \times U, \mathbb{R}^{k \times k}) \mid_U \to VU \cup \{+\infty\}.
\]

In the case when \( W_p \) has subexponential growth, finiteness and smoothness of \( \rho_W \) follow from Theorem 58.

\[\square\]

**Appendix A. A criterium for smooth convergence**

**A.1. Convergence in a Banach space.** In this section we prove a useful criterion for the narrow convergence of a family of gaussian random fields. We begin with the following proposition.

**Proposition 62.** Let \((F, \| \cdot \|)\) be a Banach space. Let \( \{p_{d,a}\}_{a,d \in \mathbb{N}} \subset F \) be a sequence such that

\[
\sum_{a=0}^{\infty} \sup_d \|p_{d,a}\| < \infty
\]

and such that \( \text{span}\{p_{d,a} : d \in \mathbb{N}\} \) is finite dimensional for all \( a \in \mathbb{N} \). Then for every \( d \), the series \( \sum_{a=0}^{\infty} p_{d,a} = f_d \) converges in \( F \) and the family \( \{f_d\}_d \) is relatively compact in \( F \).

**Proof.** The existence of \( f_d \) for any fixed \( d \) is granted by the completeness of \( F \) which implies that every absolutely convergent series is convergent in \( F \).

To prove relative compactness we recall that a bounded set \( S \) in a Banach space is relatively compact if and only if for every \( \epsilon > 0 \) there exists a finite dimensional space \( A_\epsilon \subset F \) such that:

\[
S \subset B_\epsilon(A_\epsilon),
\]

where \( B_\epsilon(A_\epsilon) = \{x \in F : d(x, A_\epsilon) \leq \epsilon\} \). Observe first that the sequence \( S = \{f_d\}_d \) is bounded, in fact contained in the ball of radius \( R = \sum_{a=0}^{\infty} \sup_d \|p_{d,a}\| \). Given now \( \epsilon > 0 \), then there is \( a_\epsilon \in \mathbb{N} \) such that \( \sum_{a=a_\epsilon}^{\infty} \sup_d \|p_{d,a}\| \leq \epsilon \) and if we define

\[
A_\epsilon = \text{span}\{p_{d,a} : d \in \mathbb{N}, a \leq a_\epsilon\}
\]

we see that every element of \( \{f_d\}_{d \in \mathbb{N}} \) is at distance at most \( \epsilon \) from \( A_\epsilon \) and the conclusion follows. \[\square\]
\textbf{Theorem 63.} Let $p_{d,a} \in C^r(M, \mathbb{R}^k)$, with $r \in \mathbb{N} \cup \{+\infty\}$. Assume that:

1. for every $Q: D \hookrightarrow M$ embedding of a compact set $D \subset \mathbb{R}^m$ and every finite $s \leq r$
   \[
   \sum_{a=0}^{\infty} \sup_d \|p_{d,a}\|_{Q, s} < \infty;
   \]
2. $\text{span}\{p_{d,a}: d \in \mathbb{N}\}$ is finite dimensional for all $a \in \mathbb{N}$.

Then for every $d$, the series $\sum_{a=0}^{\infty} p_{d,a}$ converges to some $f_d \in C^r(M, \mathbb{R}^k)$ and the family $\{f_d\}_d$ is relatively compact.

\textbf{Proof.} Observe that the topology in $C^\infty(M, \mathbb{R}^k)$ is the product topology with respect to the inclusions $i_r: C^\infty(M, \mathbb{R}^k) \subset C^r(M, \mathbb{R}^k)$, therefore convergence and relative compactness in $C^\infty(M, \mathbb{R}^k)$ are equivalent, respectively, to convergence and relative compactness in $C^r(M, \mathbb{R}^k)$ for every $r$. For this reason it is sufficient to prove the theorem in the case $r < \infty$.

Let $Q_i: \mathbb{D} \hookrightarrow M$ be a countable family of embeddings as in (2.1). Then the collection of functions $Q_i^t p_{d,a} \in C^r(D_i, \mathbb{R}^k)$ satisfies the hypothesis of Proposition 62, thus there exists $f_d^i \in C^r(D_i, \mathbb{R}^k)$ such that $\sum_{a=0}^{\infty} p_{d,a} \circ Q_i = f_d^i$. By glueing together the functions $f_d^i$s we can define a function $f_d: M \to \mathbb{R}^k$ which must be of class $C^r$ on $Q_i(\text{int}(D_i))$, therefore $f_d \in C^r(M, \mathbb{R}^k)$.

Since $\|\sum_{a=0}^{A} p_{d,a} - f_d\|_{Q_i, r} \to 0$ for each $i$, it follows that the series $\sum_{a=0}^{\infty} p_{d,a}$ actually converges to $f_d$ in $C^r(M, \mathbb{R}^k)$. Finally, the relative compactness of $\{f_d\}_d$ in $C^r(M, \mathbb{R}^k)$ follows from relative compactness of $\{Q_i^t f_d\}_d$ for each $i$. 

\hfill $\square$
A.2. Convergence of gaussian random series. Next theorem combines the previous results to give a criterion for narrow convergence of gaussian fields.

**Theorem 64.** Let \( \{x_{a,d}\}_{a,d \in \mathbb{N}} \) be a family of independent, smooth gaussian fields, such that:

1. for all \( r \in \mathbb{N} \) and for every \( Q : D \to M \) embedding of a compact set \( D \subset \mathbb{R}^m \):
   \[
   \sum_{a=0}^{\infty} \sup_d \|K_{x_{d,a}}\|_{Q,r} < \infty;
   \]

2. \( \text{span}\{K_{x_{d,a}} : d \in \mathbb{N}\} \) is finite dimensional for every \( a \).

Then for any fixed \( d \), the series \( f_d = \sum_{a=0}^{\infty} x_{d,a} \) is almost surely convergent in \( C^\infty(M, \mathbb{R}^k) \) (in particular it defines a smooth random field). Moreover the sequence of random fields \( f_d = \sum_{a=0}^{\infty} x_{d,a} \) is relatively compact in \( G^\infty(M, \mathbb{R}^k) \).

**Proof.** Define \( f_d^A = \sum_{a=0}^{A} x_{d,a} \). Note that \( K_{f_d^A} = \sum_{a=0}^{A} K_{x_{d,a}} \) because of the independence assumption. By hypothesis and Theorem 63 we know that the sequence \( \{K_{f_d^A}\}_A \) is relatively compact in \( C^\infty(M \times M, \mathbb{R}^{k \times k}) \), therefore by Theorem 2 we get that \( \{f_d^A\}_A \) is a relatively compact sequence in \( G^\infty(M, \mathbb{R}^k) \). We can then apply Theorem 36 to deduce that \( f_d^A \) converges in \( C^\infty(M, \mathbb{R}^k) \) almost surely to \( f_d \).

The covariance function of \( f_d \) is \( K_{f_d} = \sum_{a=0}^{\infty} K_{x_{d,a}} \), which by hypothesis form a relative compact sequence in \( C^\infty(M \times M, \mathbb{R}^{k \times k}) \), thus we can conclude, using again Theorem 2, that the sequence \( \{f_d\}_d \) is relative compact in \( G^\infty(M, \mathbb{R}^k) \). \(\square\)

A.3. Other notions of convergence. Next theorem describes the subtle difference between the notion of convergence in probability of random fields and that of narrow convergence.

**Theorem 65.** Let \( X_d, X \in G^r(M, \mathbb{R}^k) \). The sequence \( X_d \) converges to \( X \) in probability if and only if \( (X_d, X) \Rightarrow (X, X) \).

**Proof.** First, note that if \( X_d \to X \) in probability, then \( (X_d, X) \to (X, X) \) in probability and therefore \( (X_d, X) \Rightarrow (X, X) \).

Let \( d \) be any metric on \( C^r(M, \mathbb{R}^k) \). Since \( d \) is a continuous function, if \( (X_d, X) \Rightarrow (X, X) \) then \( d(X_d, X) \Rightarrow 0 \), which is equivalent to convergence in probability. \(\square\)

Note that in the case \( r = \infty \), it is possible to combine Theorem 2 with Theorem 65 to check convergence in probability only if the joint distribution of \( (X_d, X) \) is gaussian, which is not true in general.

The next theorem gives a criteria for almost sure convergence of random fields, in general. It is not clear how to exploit the fact that we are in the gaussian case to make it simpler.

**Theorem 66.** Let \( X_{d,j}, X_{\infty,j} \in G^r(M, \mathbb{R}^k) \). Assume that

i. \( X_{d,j} \to X_{\infty,j} \) in \( C^r(M, \mathbb{R}^k) \) almost surely for all \( j \in \mathbb{N} \).
ii. \( |X_{j,d}|_{Q,s} \leq a_{j,Q,s} \) almost surely and \( \sum_{j \in \mathbb{N}} \mathbb{E}\{a_{j,Q,s}\} < \infty \) for all \( Q \) and \( s \leq r \) finite. Then, with probability one, the series \( \sum_{j \in \mathbb{N}} X_{d,j} = S_d \) converges for all \( d \in \mathbb{N} \cup \{\infty\} \) and \( S_{d,j} \to S_{\infty,j} \) in \( C^r(M, \mathbb{R}^k) \).

**Proof.** Consider a countable family \( \mathcal{Q} \) of embeddings as in (2.1). Fix \( d \in \mathbb{N} \cup \{\infty\} \). Note that

\[
\mathbb{E}\left\{ \sum_{j} \|X_{d,j}\|_{Q,r} \right\} \leq \sum_{j} \mathbb{E}a_{j} < \infty
\]

thus \( \sum_{j} \|X_{d,j}\|_{Q,r} < \infty \) almost surely, for all \( Q \in \mathcal{Q} \) and \( s \leq r \) finite. It follows by Theorem 63 that, with probability one, the series \( \sum_{j \in \mathbb{N}} X_{d,j} = S_d \) converges for all \( d \in \mathbb{N} \cup \{\infty\} \).

Now with probability one, we get that the sequence \( \|X_{d,j} - X_{\infty,j}\|_{Q,s} \) satisfies the hypothesis of the dominated convergence Theorem for series, for all \( Q \in \mathcal{Q} \) and \( s \leq r \) finite, therefore

\[
\lim_{d \to \infty} \sum_{j} \|X_{d,j} - X_{\infty,j}\|_{Q,s} = 0
\]

It follows that (within the same event of probability one) \( \|S_d - S\|_{Q,s} \to 0 \) for all \( Q \in \mathcal{Q} \) and all \( s \leq r \) finite, in other words: \( S_d \to S \) in \( C^r(M, \mathbb{R}^k) \).

**Appendix B.** The dual of \( E^r \)

Let \( E^r = C^r(M, \mathbb{R}^k) \). Let \((E^r)^*\) be the set of all linear and continuous functions \( T : E^r \to \mathbb{R} \), endowed with the weak-* topology, namely the topology induced by the inclusion \((E^r)^* \subset \mathbb{R}^{E^r}\), when the latter is given the product topology.

**Remark 67.** When \( M \) is an open subset \( M \subset \mathbb{R}^m \) and \( k = 1 \), the elements of \((E^\infty)^*\) are exactly the distributions with compact support.

**Lemma 68.** Let \( T \in (E^r)^* \). There exists a finite set \( \mathcal{Q} \) of embeddings \( Q : \mathbb{D} \to M \), a constant \( C > 0 \) and a finite natural number \( s \leq r \), such that

\[
|T(f)| \leq C \max_{Q \in \mathcal{Q}} \|f\|_{Q,s}
\]

for all \( f \in E^r \). As a consequence, denoting \( K = \bigcup_{Q \in \mathcal{Q}} Q(\mathbb{D}) \), there is a unique \( \hat{T} \in (C^s(K, \mathbb{R}^k))^* \) such that \( T(f) = \hat{T}(f|_K) \) for all \( f \in E^r \).

**Remark 69.** Denote by \( \Omega = \text{int}(K) \subset M \). The space \( C^s(K, \mathbb{R}^k) \) is well defined whenever \( K = \Omega \) and is homeomorphic to the image of the restriction map

\[
C^s(M, \mathbb{R}^k) \to C^s(\Omega, \mathbb{R}^k), \quad f \mapsto f|_\Omega.
\]

Moreover it is a Banach space with the norm (depending on \( \mathcal{Q} \))

\[
(B.1) \quad \|f\|_{K,s} = \max_{Q \in \mathcal{Q}} \|f\|_{Q,s}.
\]

Note that in this case \( \|f\|_{K,s} \) depends only on \( f|_\Omega \).
Proof. Let $Q_n : \mathbb{D} \hookrightarrow M$ be a countable family of embeddings such that $g_N \to 0$ in $E^r$ if and only if $\|g_N\|_{Q_n,s} \to 0$ for all $n \in \mathbb{N}$ and $s \leq r$.

Assume that for all $N \in \mathbb{N}$ there is a function $f_N \in E^r$, such that
\begin{equation}
|T(f_N)| > N \max_{n \leq N} \|f_N\|_{Q_n,N}. \tag{B.2}
\end{equation}
Then the sequence
\[ g_N = \frac{f_N}{N \max_{n \leq N} \|f_N\|_{Q_n,N}} \]
converges to 0 in $E^r$, indeed $\|g_N\|_{Q_n,s} \leq \frac{1}{N} N \max_{n \leq N} \|f_N\|_{Q_n,N}$. Therefore by the continuity of $T$, we get that $T(g_N) \to 0$, but $|T(g_N)| > 1$ according to (B.2), so we get a contradiction. It follows that there exists $N$ such that
\[ |T(f)| \leq N \max_{n \leq N} \|f\|_{Q,n} \]
for all $f \in E^r$.

Note that $\Omega \supset Q(int(\mathbb{D}))$, thus if $p \in K \backslash \Omega$, then $p \in Q(\partial \mathbb{D})$ for some $Q \in \mathcal{Q}$ and therefore $p \in Q(int(\mathbb{D})) \subset \Omega$. This proves that $K = \Omega$.

Let $f, g \in E^r$ be such that $f|_\Omega = g|_\Omega$, then
\[ |T(f) - T(g)| = |T(f - g)| \leq C \max_{Q \in \mathcal{Q}} \|f - g\|_{Q,s} = C \|f\|_{K} - g\|_{K,s} = 0. \]
It follows that the function $L : C^r(K, \mathbb{R}^k) \to \mathbb{R}$ such that $L(f) = T(f)$ for all $f \in E^r$, is well defined and continuous with respect to the norm $\| \cdot \|_{K,s}$. Since $C^r(K, \mathbb{R}^k)$ is dense in $C^\infty(K, \mathbb{R}^k)$, there is a unique way to extend $L$ to a bounded linear functional on $C^\infty(K, \mathbb{R}^k)$, that we call $\hat{T}$. \hfill \Box

We recall the following classical theorem from functional analysis (see [2, Theorem 1.54]), which we can use to give a more explicit description of $(E^r)^\ast$.

**Theorem 70** (Riesz’s representation theorem). Let $K$ be a compact metrizable space. Let $\mathcal{M}(K)$ be the Banach space of Radon measures on $K$ (on a compact set they all are finite Borel signed measures), endowed with the total variation norm. Then the map
\[ \mathcal{M}(K) \to (C(K))^\ast, \quad \mu \mapsto \int_K (\cdot) d\mu \]
is a linear isometry of Banach spaces.

**Theorem 71.** Let $\mathcal{M}'_{\text{loc}}$ be the set of all $T \in (E^r)^\ast$ of the form
\[ T(f) = \int_{\mathbb{D}} \partial_\alpha (f^j \circ Q) d\mu, \]
for some embedding $Q : \mathbb{D} \hookrightarrow M$, some finite multiindex $|\alpha| \leq r$, some $j \in \{1, \ldots, k\}$ and some $\mu \in \mathcal{M}(\mathbb{D})$. Then $(E^r)^\ast = \text{span}\{\mathcal{M}'_{\text{loc}}\}$. 
Proof. Let \( T \in (E')^* \) and let \( Q, s, K, C \) and \( \hat{T} \) defined as in lemma 68. Consider the topological space
\[
D = \mathbb{D} \times Q \times \{ \alpha \in \mathbb{N}^m : |\alpha| \leq s \} \times \{1, \ldots, k\}.
\]
\( D \) is a finite union of disjoint copies of the closed disk, therefore it is compact and metrizable. There is a continuous linear embedding with closed image
\[
\mathcal{J}^s : C^s(K, \mathbb{R}^k) \hookrightarrow C(D), \quad \mathcal{J}^s f(u, Q, \alpha, j) = \partial_\alpha (f^j \circ Q)(u).
\]
Indeed \( \|\mathcal{J}^s f\|_{C(D)} \leq \|f\|_{K,s} \leq \sqrt{k} \|\mathcal{J}^s f\|_{C(D)} \), if \( \|\cdot\|_{K,s} \) is defined as in (B.1). By identifying \( C^s(K, \mathbb{R}^k) \) with its image under \( \mathcal{J}^s \), we can extend \( \hat{T} \) to the whole \( C(D) \), using Hahn-Banach theorem and the extension can thus be represented by a Radon measure \( \mu \in \mathcal{M}(\mathbb{D}) \).

Denote by \( \mu_{Q,\alpha,j} \in \mathcal{M}(\mathbb{D}) \) the restriction of \( \mu \) to the connected component \( \mathbb{D} \times \{Q\} \times \{\alpha\} \times \{j\} \). Let \( T_{Q,\alpha,j} \) be the element of \( \mathcal{M}'_{loc} \) associated with \( Q, \alpha, j \) and \( \mu_{Q,\alpha,j} \). Then we have
\[
T(f) = \hat{T}(f|_K) = \int_D \mathcal{J}^s f d\mu = \sum_{Q \in \mathcal{Q}, |\alpha| \leq s, j=1,\ldots,k} \int_{D \times \{Q\} \times \{\alpha\} \times \{j\}} \mathcal{J}^s f d\mu = \sum_{Q \in \mathcal{Q}, |\alpha| \leq s, j=1,\ldots,k} \int_D \partial_\alpha (f^j \circ Q) d\mu_{Q,\alpha,j} = \sum_{Q \in \mathcal{Q}, |\alpha| \leq s, j=1,\ldots,k} T_{Q,\alpha,j}(f),
\]
for all \( f \in E^r \). Therefore \( T \in \text{span}\{\mathcal{M}'_{loc}\} \). \( \square \)

The manifold \( M \) is topologically embedded in \((E')^*\), via the natural association \( p \mapsto \delta_p \). We denote by \( \delta_M \subset (E')^* \) the image of the latter map (it is a closed subset).

**Corollary 72.** \((E')^* = \overline{\text{span}\{\delta_M\}}\).

*Proof.* It is sufficient to prove that \( \mathcal{M}'_{loc} \subset \overline{\text{span}\{\delta_M\}} \), moreover we can clearly restrict to the case \( M = \mathbb{D} \) and \( Q = \text{id} \).

Observe that any functional of the type \( \delta_p \circ \partial_\alpha \) belongs to \( \overline{\text{span}\{\delta_M\}} \). This can be proved by induction on the order of differentiation \( |\alpha| \): if \( |\alpha| = 0 \) there is nothing to prove, otherwise we have
\[
\delta_u \circ \frac{\partial}{\partial u^i} \circ \partial_\alpha = \lim_{n \to \infty} n \left( \delta_{u + \frac{1}{n} e^i} \circ \partial_\alpha - \delta_u \circ \partial_\alpha \right) \in \overline{\text{span}\{\delta_M\}}.
\]
Note also that any \( T^r \in \mathcal{M}'_{loc} \) is of the form \( T^0 \circ \partial_\alpha \) for some \( T^0 \in \mathcal{M}'_{loc} \) and \( |\alpha| \leq r \) and, along with the previous consideration, this implies that it is sufficient to prove the theorem in the case for \( r = 0 \) and we can conclude with the following lemma
Lemma 73. Let $K$ be a compact metric space. The subspace span$\{\delta_K\}$ is sequentially dense (and therefore dense) in $\mathcal{M}(K)$, with respect to the weak-* topology on $\mathcal{M}(K) = \mathcal{C}(K)^*$.

Let $\mu$ be a Radon measure on $K$. Define for any $n \in \mathbb{N}$ a partition $\{A^n_i\}_{i \in I_n}$ of $K$ in Borel subsets of diameter smaller that $\frac{1}{n}$ and let $a^n_i \in A^n_i$. Define
\[
t_n = \sum_{i \in I_n} \mu(A^n_i) \delta_{a^n_i}.
\]
Given $f \in \mathcal{C}(K)$, we have
\[
|\int_K f d\mu - t_n(f)| \leq \sum_{i \in I_n} \int_{A^n_i} |f - f(a^n_i)| d|\mu|
\]
\[
\leq |\mu|(K) \sup_{|x-y| \leq \frac{1}{n}} |f(x) - f(y)|.
\]
By the Heine-Cantor theorem, $f$ is uniformly continuous on $K$, hence the last term in (B.3) goes to zero as $n \to \infty$. Therefore for every $f \in \mathcal{C}(K)$ we have that $t_n(f) \to \int_K f d\mu$, equivalently $t_n \to \mu$ in the weak-* topology. □

We conclude with a nice fact concerning the case $r = \infty$.

Proposition 74. Let $T \in \mathcal{M}_{loc}^{\infty}$. Then the associated measure $\mu$ can be assumed to be of the form $\rho du$ for some $\rho \in L^{\infty}(\Omega)$.

Proof. Let $T_0$ be associated with $Q$, $\alpha$, $\mu$. It is not restrictive to assume $M = \mathbb{D}$ and $Q = id$.

Let us consider the linear functional $T$ on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ given by
\[
T(\varphi) = T_0(\varphi|_{\mathbb{D}}) = \int_{\mathbb{D}} \partial_\alpha \varphi d\mu
\]
Note that for all $\varphi \in \mathcal{C}_c^\infty$
\[
\max_{\mathbb{R}^n} \|\partial_\alpha f\| \leq \int_{\mathbb{R}^n} |\partial_{\alpha+e} f| du,
\]
where $e = (1, \ldots, 1)$. Define $V \subset L^1$ as $V = \{\partial_{\alpha+e} \varphi : \varphi \in \mathcal{C}_c^\infty\}$, and let $\lambda : V \to \mathbb{R}$ defined by $\lambda(\partial_{\alpha+e} \varphi) = T(\varphi)$. Then $\lambda$ is a (well defined) linear and bounded functional on $(V, \| \cdot \|_{L^1})$, since
\[
|\lambda(\partial_{\alpha+e} \varphi)| = |T(\varphi)|
\]
\[
= |\int_{\mathbb{D}} \partial_\alpha \varphi d\mu|
\]
\[
\leq |\mu|(\mathbb{D}) \max_{\Omega} \|\partial_\alpha \varphi\|
\]
\[
\leq |\mu|(\mathbb{D}) \|\partial_{\alpha+e} \varphi\|_{L^1}.
\]
The Hahn-Banach theorem, implies that $\lambda$ can be extended to a continuous linear functional $\Lambda$ on the whole space $L^1(\Omega)$ and hence it can be represented by a $\rho \in L^{\infty}(\Omega) = L^1(\Omega)^*$. 


In particular for all $\varphi \in C^\infty_c$ we have that
\[ T(\varphi) = \lambda(\partial_\alpha + e \varphi) = \int_{\mathbb{R}^n} \partial_\alpha \varphi pdu. \]
\[ \square \]

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