A LITTLE MORE ON THE ZERO-DIVISOR GRAPH AND THE
ANNIHILATING-IDEAL GRAPH OF A REDUCED RING

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ABSTRACT. We have tried to translate some graph properties of $\mathcal{A}G(R)$ and $\Gamma(R)$ to the topological properties of Zariski topology. We prove that $\text{Rad}(\Gamma(R))$ and $\text{Rad}(\mathcal{A}G(R))$ are equal and they are equal to 3, if and only if the zero ideal of $R$ is an anti fixed-place ideal, if and only if $\text{Min}(R)$ does not have any isolated point, if and only if $\Gamma(R)$ is triangulated, if and only if $\mathcal{A}G(R)$ is triangulated. Also, we show that if the zero ideal of a ring $R$ is a fixed-place ideal, then $dt(\mathcal{A}G(R)) = |B(R)|$ and also if in addition $|\text{Min}(R)| > 2$, then $dt(\mathcal{A}G(R)) = |B(R)|$. Finally, it has been shown that $dt(\mathcal{A}G(R))$ is finite, if and only if $dt(\mathcal{A}G(R))$ is finite; if and only if $\text{Min}(R)$ is finite.

1. INTRODUCTION

Let $R$ be a commutative ring with unity. By $\text{Spec}(R)$ we mean the set of all prime ideals of $R$. A semi-prime ideal means an ideal which is an intersection of prime ideals. $R$ is called a reduced ring, if the zero ideal of $R$ is semi-prime. Through this paper $R$ is the commutative unitary reduced ring. For each ideal $I$ of $R$ and each subset $S$ of $R$, we denote the ideal $\{x \in R : Sx \subseteq I\}$ by $(I : S)$. When $I = \{0\}$ we write $\text{Ann}(S)$ instead of $(\{0\} : S)$ and call it the annihilator of $S$. Also we write $\text{Ann}(a)$ instead of $\text{Ann}(\{a\})$. A prime ideal $P$ is said to be a minimal prime ideal over an ideal $I$ if there are not any prime ideal strictly contained in $P$ that contains $I$. By $\text{Min}(I)$ we mean the set of all minimal prime ideals over $I$. We use $\text{Min}(R)$ instead of $\text{Min}(\{0\})$. A prime ideal $P$ is called a Bourbaki associated prime divisor of an ideal $I$ if $(I : x) = P$, for some $x \in R$. We denote the set of all Bourbaki associated prime divisors of an ideal $I$ by $B(I)$. It is easy to see that $B(I) \subseteq \text{Min}(I)$, for any ideal $I$ of a ring $R$. We use $B(R)$ instead of $B(\{0\})$. Let $I$ be a semi-prime ideal, $P \subseteq \text{Min}(I)$ is called irredundant with respect to $I$ if $I \neq \bigcap_{P \neq P \in \text{Min}(I)} P$. If $I$ is equal to the intersection of all irredundant ideals with respect to $I$, then we call it a fixed-place ideal, exactly, by [6, Theorem 2.1], we have $I = \bigcap B(I)$. If $B(I) = \emptyset$, then $I$ is called an anti-fixed place ideal. We use $B(R)$ instead of $B(\{0\})$. For more information about the fixed-place ideals and anti-fixed-place ideals, see [1, 2].

Let $G = (V(G), E(G))$ be an undirected graph. A vertex is called a pendant vertex if it is adjacent to just one vertex. For each pair of vertices $u$ and $v$ in $V(G)$, the length of the shortest path between $u$ and $v$, is denoted by $d(u, v)$, is called the distance between $u$ and $v$. The eccentricity of a vertex $u$ of $G$ is denoted by $\text{ecc}(u)$ and is defined to be maximum of $\{d(u, v) : v \in G\}$. The minimum of $\{\text{ecc}(u) : u \in G\}$, denoted by $\text{Rad}(G)$, is called the radius of $G$. We say $G$ is

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retraction is a homomorphism, if each vertex of \( G \) is vertex of some triangle. Two vertices \( u \) and \( v \) are called orthogonal, if \( u \) and \( v \) are adjacent and there are not any vertex which is adjacent to the both vertices \( u \) and \( v \). A graph homomorphism \( \varphi \) from a graph \( G = \langle V(G), E(G) \rangle \) to a graph \( H = \langle V(H), E(H) \rangle \), is a map from \( V(G) \) to \( V(H) \) that \( \{u, v\} \in E(G) \) implies \( \{f(u), f(v)\} \in E(H) \), for all pairs of vertices \( u, v \in V(G) \). A retraction is a homomorphism \( \varphi \) from a graph \( G \) to a subgraph \( H \) of \( G \) such that \( \varphi(v) = v \), for each vertex \( v \in V(H) \). In this case the subgraph \( H \) is called a retract of \( G \). A subset \( D \) of vertex of a graph is called a dominating set if every vertex of graph is either in \( D \) or adjacent to some vertex of \( D \). Also, a total dominating set of a graph is a family \( S \) of vertex of graph such that every vertex is adjacent to some vertex of \( S \). The dominating number and total dominating number of a graph is the minimum cardinality of dominating set and total dominating set of graph, respectively. We denote the dominating number and total dominating number of a graph \( G \) by \( dt(G) \) and \( dt_t(G) \), respectively. For every \( u, v \in V(G) \), we denote the length of the shortest cycle containing \( u \) and \( v \) by \( gi(u, v) \).

Suppose \( I \) and \( a \) are an ideal and element of \( R \), respectively. If \( \text{Ann}(I) \neq \{0\} \), then \( I \) is called annihilating-ideal and if \( \text{Ann}(a) \neq \{0\} \), then \( a \) is called a zero-divisor element. Let \( \mathbb{A}(R)^* \) be the family of all non-zero annihilating-ideals and \( Z(R)^* \) be the family of all non-zero zero-divisor element of \( R \). \( \mathbb{A}G(R) \) is a graph with the vertices \( \mathbb{A}(R)^* \), and two distinct vertices \( I \) and \( J \) are adjacent, if \( IJ = \{0\} \). Also, \( \Gamma(R) \) is a graph with vertices \( Z(R)^* \), and two distinct vertices \( a \) and \( b \) are adjacent, if \( ab = 0 \). \( \mathbb{A}G(R) \) and \( \Gamma(R) \) are called the annihilating-ideal graph and the zero-divisor graph of \( R \), respectively.

Through this paper, all \( Y \subseteq \text{Spec}(R) \) is considered by Zariski topology; i.e., by assuming as a base for the closed sets of \( Y \), the sets \( h_Y(a) \) where \( h_Y(a) = \{P \in Y : a \in P\} \). Hence, the closed sets of \( Y \) are of the form \( h_Y(I) = \bigcap_{a \in I} h_Y(a) = \{P \in Y : I \subseteq P\} \), for some ideal \( I \) in \( R \). Also, we set \( h_Y(I) = Y \setminus h_Y(I) \). When \( Y = \text{Min}(R) \) we write \( h_m \) instead of \( h_Y \). A point \( P \in \text{Spec}(R) \) is called a quasi-isolated point, if \( P \) is an isolated point of \( \text{Min}(R) \). By \[24\] Theorem 2.3 and Corollary 2.4, the space \( \text{Min}(R) \) is a Hausdorff space in which \( \{h_m(a) : a \in R\} \) is base of clopen sets.

In this research, \( C(X) \) denotes the ring of all real-valued continuous functions on a Tychonoff space \( X \) and we abbreviate \( \mathbb{A}(C(X))^* \) and \( \mathbb{A}G(C(X)) \) by \( \mathbb{A}(X)^* \) and \( \mathbb{A}G(X) \), respectively. We denote the set of all isolated point of \( X \), by \( I(X) \). A space \( X \) is called almost discrete, if \( \overline{I(X)} = X \).

The reader is referred to \[14\] \[31\] \[32\] \[23\] \[24\] for undefined terms and notations.

The researchers tried to define a graph illustration for some kind of mathematical aspects. For example \[31\] in the lattice literature, \[12\] in the measure literature, \[16\] in topology literature and \[13\] in the linear algebra. The study of translating graph properties to algebraic properties is an interesting subject for mathematicians. The introducing and studying of the concept of zero-divisor graph of a commutative is started in \[15\]. In this article the author let all elements of the commutative ring be vertices of the this graph. In \[11\], it has been studied the zero-divisor graph whose vertices are the non-zero zero-divisor elements. Studying of this graph has been continued in several articles; see \[25\] \[10\] \[4\] \[5\] \[29\] \[30\]. Also, First the annihilating-ideal graph has been introduced and studied in \[19\] and then it has been studied in several articles; see \[20\] \[9\] \[2\] \[27\] \[22\] \[28\].

In the rest of this section we give a retract of the annihilating graph. Section 2, devoted to translating the graph properties of these graphs to Zariski topology.
Also, we note an impossible assumption in [30]. In Section 3, by obtained tools in Section 2, we characterize the radius of \( \Gamma(R) \), \( \mathcal{A}\mathcal{G}(R) \), \( \Gamma(X) \) and \( \mathcal{A}\mathcal{G}(X) \) and show that \( \text{Rad}(\Gamma(R)) \) and \( \text{Rad}(\mathcal{A}\mathcal{G}(R)) \) are equal and they are equal to 3, if and only if the zero ideal of \( R \) is an anti fixed-place ideal, if and only if \( \text{Min}(R) \) does not have any isolated point, if and only if \( \Gamma(R) \) is triangulated, if and only if \( \mathcal{A}\mathcal{G}(R) \) is triangulated. In the last section, the domination number of the annihilating-ideal graph has been studied. In this section we show that \( |B_{\Gamma}()| \leq \text{dt}(\mathcal{A}\mathcal{G}(R)) \). Also, we note a mistake of [28] and we characterize the domination of a ring in which the zero ideal is a fixed-place ideal, if and only if \( \text{Min}(R) \) is finite; if and only if \( \text{Min}(R) \) is finite; if and only if \( \text{Min}(R) \) is finite; if and only if \( \text{Min}(R) \) is finite.

For each subset \( S \) of \( R \) let \( P_{S} \) be the intersection of all minimal prime ideals containing \( S \). An ideal \( I \) in \( R \) is said to be strongly \( z^{\circ} \)-ideal (or briefly \( sz^{\circ} \)-ideal) if \( P_{F} \subseteq I \), for every finite subset \( F \) of \( I \). Since the intersection of every family of strong \( z^{\circ} \)-ideals is a strong \( z^{\circ} \)-ideal, the smallest strong \( z^{\circ} \)-ideal containing an ideal \( I \) exists, and we denote this by \( I_{sz^{\circ}} \). For more details about the strong \( z^{\circ} \)-ideals, see [26, 8, 17].

**Lemma 1.1.** Let \( I \) and \( J \) be ideals of \( R \). \( I \) is adjacent to \( J \), if and only if \( I_{sz^{\circ}} \) is adjacent to \( J_{sz^{\circ}} \).

**Proof.** \( \Rightarrow \). Suppose that \( a \in I_{sz^{\circ}} \) and \( b \in J_{sz^{\circ}} \), then, by [17] Proposition 7.5, finite subsets \( F \) of \( I \) and \( G \) of \( J \) exist such that \( h_{m}(G) \subseteq h_{m}(a) \) and \( h_{m}(H) \subseteq h_{m}(b) \). Since \( I \) is adjacent to \( J \), \( IJ = \{0\} \), so \( GH = \{0\} \), this implies that \( \text{Min}(R) = h_{m}(GH) = h_{m}(G) \cup h_{m}(H) \subseteq h_{m}(a) \cup h_{m}(b) = h_{m}(ab) \), thus \( h_{m}(ab) = \text{Min}(R) \), hence \( ab \in kh_{m}(ab) = \{0\} \), and therefore \( ab = 0 \). This shows that \( I_{sz^{\circ}}J_{sz^{\circ}} = \{0\} \) and therefore \( I_{sz^{\circ}} \) is adjacent to \( J_{sz^{\circ}} \).

\( \Leftarrow \). It is clear. \( \Box \)

**Proposition 1.2.** The family of all \( sz^{\circ} \)-ideals of \( \mathcal{A}(R)^{\ast} \) is a retract of \( \mathcal{A}\mathcal{G}(R) \).

**Proof.** Suppose that \( I \in \mathcal{A}(R)^{\ast} \), so \( J \in \mathcal{A}(R)^{\ast} \) exists such that \( IJ = \{0\} \). By Lemma 1.1 \( I_{sz^{\circ}} \) is adjacent to \( J_{sz^{\circ}} \). Since \( 0 \neq I \subseteq I_{sz^{\circ}} \subseteq \text{Ann}(J_{sz^{\circ}}) \subseteq \text{Ann}(J) \neq X \), \( I_{sz^{\circ}} \in \mathcal{A}(R)^{\ast} \). This shows that the map \( \varphi \) from \( \mathcal{A}(R)^{\ast} \) to the family of all \( sz^{\circ} \)-ideals of \( \mathcal{A}(R)^{\ast} \), defined by \( \varphi(I) = I_{sz^{\circ}} \) is a retraction and therefore the family of all \( sz^{\circ} \)-ideals of \( \mathcal{A}(R)^{\ast} \) is a retract of \( \mathcal{A}\mathcal{G}(R) \). \( \Box \)

2. Zariski topology

In this section we give Zariski topological characterization of elements of \( \Gamma(R) \) and \( \mathcal{A}\mathcal{G}(R) \), then we characterize the adjacency, distance, orthogonality, eccentricity and triangulation of vertices of these graphs. Also, it has been shown that \( \text{Rad}(\Gamma(R)), \text{Rad}(\mathcal{A}\mathcal{G}(R)) > 1 \).

**Proposition 2.1.** Let \( Y \subseteq \text{Spec}(R) \) and \( \bigcap Y = \{0\} \). If \( a \) is an element and \( I \) is an ideal of \( R \), then

(a) \( a = 0 \), if and only if \( h_{Y}(a) = Y \).

(b) \( \text{Ann}(a) \neq 0 \), if and only if \( h_{Y}(a) \neq Y \).

(c) \( I = \{0\} \), if and only if \( h_{Y}(I) = Y \).

(d) \( I \) is an annihilating-ideal, if and only if \( h_{Y}(I) \neq Y \).
Proof. (a) and (c). Since $\bigcap Y = \{0\}$, they are clear.
(b) Since $\text{Ann}(a) = h\ell'_a(a)$, $\text{Ann}(a) \neq \{0\}$ if and only if $kh\ell'_a(a) \neq \{0\}$; and it is equivalent to say that $kh\ell'_a(a) \neq Y$, because $\bigcap Y = \{0\}$, and therefore it is equivalent to $h\ell'_a(I) \neq Y$.
(d). The proof is analogously similar to the proof part (b). $\square$

Lemma 2.2. Let $Y \subseteq \text{Spec}(R)$ and $\bigcap Y = \{0\}$.

(a) For each $a, b \in Z(R)^*$, $a$ is adjacent to $b$, if and only if $h\ell'_a(a) \cap h\ell'_b(b) = \emptyset$.
(b) For each $I, J \in \kappa(R)^*$, $I$ is adjacent to $J$, if and only if $h\ell'_I(I) \cap h\ell'_J(J) = \emptyset$.

Proof. It is evident. $\square$

In [30 Proposition 2.2], the concept of distance in $\Gamma(R)$ has been characterized by the Zariski topology on $\text{Spec}(R)$. In the following proposition we generalize this characterization by every reduced family of prime ideals and also we characterize the concept of distance in $\kappa(\mathcal{G})$.

Proposition 2.3. Let $I, J \in \kappa(R)^*$, $a, b \in Z(R)^*$, $Y \subseteq \text{Spec}(R)$ and $\bigcap Y = \{0\}$. Then

(a) $d(a, b) = 1$, if and only if $h\ell'_a(a) \cap h\ell'_b(b) = \emptyset$.
(b) $d(a, b) = 2$, if and only if $h\ell'_a(a) \cap h\ell'_b(b) \neq \emptyset$ and $h\ell'_a(a) \cup h\ell'_b(b)$ is not dense in $Y$.
(c) $d(a, b) = 3$, if and only if $h\ell'_a(a) \cap h\ell'_b(b) \neq \emptyset$ and $h\ell'_a(a) \cup h\ell'_b(b)$ is dense in $Y$.
(d) $d(I, J) = 1$, if and only if $h\ell'_I(I) \cap h\ell'_J(J) = \emptyset$.
(e) $d(I, J) = 2$, if and only if $h\ell'_I(I) \cap h\ell'_J(J) \neq \emptyset$ and $h\ell'_I(I) \cup h\ell'_J(J)$ is not dense in $Y$.
(f) $d(I, J) = 3$, if and only if $h\ell'_I(I) \cap h\ell'_J(J) \neq \emptyset$ and $h\ell'_I(I) \cup h\ell'_J(J)$ is dense in $Y$.

Proof. (a) and (d). They are clear, by Lemma 2.2.
(b $\Rightarrow$). By Lemma 2.2, $h\ell'_a(a) \cap h\ell'_b(b) \neq \emptyset$. By the assumption, there is an ideal $c \in Z(R)^*$, such that $c$ is adjacent to the both vertices $a$ and $b$. Now Lemma 2.2 implies that $h\ell'_a(a) \cap h\ell'_c(c) = h\ell'_a(a) \cap h\ell'_b(b) \subseteq h\ell'_a(a) \cup h\ell'_b(b) \subseteq h\ell'_c(c)$ ($\ast$)

Since $c \neq 0$, by Proposition 2.1, $h\ell_{Y}(c) \neq Y$, and since $h\ell_{Y}(c)$ is closed, ($\ast$) follows that $h\ell'_a(a) \cup h\ell'_b(b)$ is not dense in $Y$.
(b $\Leftarrow$). By part (a), $d(a, b) > 1$. Since $\{h\ell'_a(c) : c \in R\}$ is a base for Zariski topology, by the assumption, there is some $c \in R$ such that $h\ell'_a(a) \cup h\ell'_b(b) \subseteq h\ell'_c(c) \subseteq Y$, so $h\ell'_a(a) \cap h\ell'_c(c) = h\ell'_a(a) \cap h\ell'_b(b) = \emptyset$, $Y \neq h\ell_{Y}(a)$ and $h\ell'_c(c) \neq Y$, thus $c \in Z(R)^*$ and $c$ is adjacent to the both vertices $a$ and $b$, hence $d(a, b) = 2$.
(c). It deduces from parts (a), (b) and [11 Theorem 2.2].
(e). By this fact that $\{h\ell'_K(K) : K$ is an ideal of $R\}$ is a base for Zariski topology, it is similar to part (b)
(f). It concludes from parts (d), (e) and [10 Theorem 7.1]. $\square$

Theorem 2.4. Let $I, J \in \kappa(R)^*$, $a, b \in Z(R)^*$, $Y \subseteq \text{Spec}(R)$ and $\bigcap Y = \{0\}$. Then

(a) Two vertices $I$ and $J$ are orthogonal, if and only if $h\ell'_I(I) \cap h\ell'_J(J) = \emptyset$ and $h\ell'_I(I) \cup h\ell'_J(J) = Y$. 


(b) Two vertices a and b are orthogonal, if and only if \( h^*_Y(a) \cap h^*_Y(b) = \emptyset \) and \( h^*_Y(a) \cup h^*_Y(b) = Y \).

Proof. (a \Rightarrow). By the assumption and Lemma 2.2, I is adjacent to J, so \( h^*_Y(I) \cap h^*_Y(J) = \emptyset \). If \( h^*_Y(I) \cup h^*_Y(J) \neq Y \), since \( \{ h^*_Y(K) : K \) is an ideal of \( R \} \) is a base for Zariski topology, it follows that there is some ideal \( K \) of \( R \) such that \( h^*_Y(K) \cap [h^*_Y(I) \cup h^*_Y(J)] = \emptyset \), so \( h^*_Y(K) \cap h^*_Y(I) = h^*_Y(K) \cap h^*_Y(J) = \emptyset \), \( h^*_Y(K) \neq Y \) and \( h^*_Y(K) \neq Y \), thus \( K \in A(R)^* \), by Proposition 2.1 and \( K \) is adjacent to the both vertices \( I \) and \( J \), by Lemma 2.2 which contradicts the assumption, hence \( h^*_Y(I) \cup h^*_Y(J) = Y \).

(a \Leftarrow). By the assumption and Lemma 2.2, \( h^*_Y(I) \cap h^*_Y(J) = \emptyset \). On contrary, suppose that there is an \( K \in A(R)^* \), such that \( K \) is adjacent to the both vertices \( I \) and \( J \), then \( h^*_Y(K) \cap [h^*_Y(I) \cup h^*_Y(J)] = \emptyset \), by Lemma 2.2. Since \( K \in A(R)^* \), by Proposition 2.1 \( h^*_Y(K) \neq \emptyset \), and therefore \( h^*_Y(I) \cup h^*_Y(J) \neq Y \), which contradicts the assumption.

(b). By this fact \( \{ h^*_Y(c) : c \in R \} \) is a base for Zariski topology, it is similar to part (a). □

Suppose that \( \bigcap Y = \{ 0 \} \). Since for every \( I \in A(R)^* \), I and \( \text{Ann}(I) \) are orthogonal, the above theorem implies that \( h^*_Y(I) \cap h^*_Y(\text{Ann}(I)) = Y \). Similarly, for every \( a \in Z(R)^* \) and \( b \in \text{Ann}(a) \), we have \( h^*_Y(a) \cup h^*_Y(b) = Y \).

Theorem 2.5. Suppose that \( I \in A(R)^* \), \( a \in Z(R)^* \), \( Y \subseteq \text{Min}(R) \) and \( \bigcap Y = \{ 0 \} \).

Then

(a) For every \( I \in A(R)^* \), ecc(I) \( > 1 \).
(b) ecc(I) = 2, if and only if \( h^*_Y(I) \) is singleton.
(c) ecc(I) = 3, if and only if \( h^*_Y(I) \) is not singleton.
(d) For every \( a \in Z(R)^* \), ecc(a) \( > 1 \).
(e) ecc(a) = 2, if and only if \( h^*_Y(a) \) is singleton.
(f) ecc(a) = 3, if and only if \( h^*_Y(a) \) is not singleton.

Proof. Since \( R \) is not an integral domain and \( \bigcap Y = \{ 0 \} \), it follows that \( |Y| \geq 2 \).

(c \Rightarrow). By the assumption there is some \( J \in A(R)^* \) such that \( d(I, J) = 3 \). Lemma 2.2 implies that \( h^*_Y(I) \cap h^*_Y(J) = \emptyset \) and \( h^*_Y(I) \cup h^*_Y(J) = Y \). On contrary, suppose that \( h^*_Y(I) \) is singleton, then \( h^*_Y(I) \subseteq h^*_Y(J) \) and therefore \( h^*_Y(J) = h^*_Y(I) \cup h^*_Y(J) = Y \), so \( J \notin A(R)^* \), by Lemma 2.2 which is a contradiction.

(c \Leftarrow). By the assumption, there are distinct prime ideals \( P \) and \( Q \) in \( h^*_Y(I) \). Since \( Y \subseteq \text{Min}(R) \) is Hausdorff and \( \{ h^*_Y(K) : K \) is an ideal of \( R \} \) is a base for \( Y \), there are ideals \( J \) and \( K \) such that \( h^*_Y(J), h^*_Y(K) \subseteq h^*_Y(I) \), \( P \in h^*_Y(J) \), \( Q \in h^*_Y(K) \) and \( h^*_Y(J) \cap h^*_Y(K) = \emptyset \). Thus

\[
\begin{align*}
h^*_Y(J + \text{Ann}(I)) \cap h^*_Y(K) & = [h^*_Y(J) \cup h^*_Y(\text{Ann}(I))] \cap h^*_Y(K) \\
& \subseteq [h^*_Y(J) \cap h^*_Y(K)] \cup [h^*_Y(\text{Ann}(I)) \cap h^*_Y(K)] = \emptyset.
\end{align*}
\]

Hence \( h^*_Y(J + \text{Ann}(I)) \neq Y \) and \( h^*_Y(J + \text{Ann}(I)) \neq Y \), so \( J + \text{Ann}(I) \in A(R)^* \). Since \( h^*_Y(I) \cap h^*_Y(J + \text{Ann}(I)) \supseteq h^*_Y(I) \cap h^*_Y(J) = h^*_Y(I) \neq \emptyset \) and

\[
\begin{align*}
h^*_Y(I) \cap h^*_Y(J + \text{Ann}(I)) & \supseteq h^*_Y(I) \cap h^*_Y(J) = h^*_Y(I) \neq \emptyset
\end{align*}
\]

and

\[
\begin{align*}
h^*_Y(I) \cap h^*_Y(J + \text{Ann}(I)) & \supseteq h^*_Y(I) \cap h^*_Y(J) = h^*_Y(I) \neq \emptyset
\end{align*}
\]
by Proposition 2.3, \(d(I, J + \text{Ann}(I)) = 3\) and therefore \(\text{ecc}(I) = 3\), by [19] Theorem 7.1.

(a) Suppose that there is some \(I \in \mathcal{H}(R)\) such that \(\text{ecc}(I) = 1\). By part (c), \(h_Y^c(I)\) is singleton, so there is some \(P \in Y\), such that \(h_Y^c(I) = \{P\}\), thus \(\text{Ann}(I) = P\), hence \(\{0\} \neq I \subseteq \text{Ann}(P)\). Since \(\text{ecc}(I) = 1\), \(I\) is adjacent to \(\text{Ann}(P)\), consequently \(I \text{Ann}(P) = \{0\}\), this implies that for every \(a \in I\), \(a^2 \in I \text{Ann}(P) = \{0\}\), and therefore \(a^2 = 0\). Since \(R\) is reduced, \(a = 0\), and consequently \(I = \{0\}\), which is a contradiction.

(b) By parts (a), (c) and [19] Theorem 7.1, it is clear.

The proof of (d), (e) and (f) are similar to parts (a), (b) and (c), respectively. □

The following corollary is an immediate consequence of the above theorem.

**Corollary 2.6.** \(\text{Rad} \Gamma(R) > 1\) and \(\text{Rad} \mathcal{A} \mathcal{G}(R) > 1\).

**Proposition 2.7.** Let \(a \in Z(R)^*\), \(I \in \mathcal{H}(R)^*\), \(Y \subseteq \text{Min}(R)\) and \(\bigcap Y = \{0\}\). Then

(a) \(a\) is a vertex of a triangle, if and only if \(h_Y(a)\) is not singleton.

(b) \(I\) is a vertex of a triangle, if and only if \(h_Y(I)^c\) is not singleton.

**Proof.** (a \(\Rightarrow\)). By the assumption, there are vertices \(b, c \in \mathcal{H}(R)^*\) such that \(a, b\) and \(c\) are pairwise vertices which are adjacent together. Thus \(h_Y^c(a), h_Y^c(b)\) and \(h_Y^c(c)\) are pairwise disjoint nonempty sets, by Theorem 2.2 and Proposition 2.1, hence \(h_Y^c(b) \cup h_Y^c(c) \subseteq h_Y(a)\) and \(|h_Y^c(b) \cup h_Y^c(c)| \geq 2\), since \(h_Y^c(b) \cup h_Y^c(c)\) is open, it follows that \(h_Y(a)\) is not singleton.

(a \(\Leftarrow\)). Suppose that \(P\) and \(Q\) are distinct elements of \(h_Y(a)\). Since \(Y \subseteq \text{Min}(R)\) is Hausdorff, \(h_Y(a)\) is open and \(\{h_Y^c(x) : x \in R\}\) is a base for \(Y\), there are \(b, c \in R\) such that \(P \in h_Y^c(b) \subseteq h_Y(a)\), \(Q \in h_Y^c(c) \subseteq h_Y(a)\) and \(h_Y^c(b) \cap h_Y^c(c) = \emptyset\), so \(h_Y^c(a), h_Y^c(b)\) and \(h_Y^c(c)\) are pairwise disjoint nonempty sets which are not dense in \(Y\). Now Proposition 2.1 implies that \(b, c \in \mathcal{H}(R)^*\) and Theorem 2.2 concludes that \(a, b\) and \(c\) are pairwise vertices which are adjacent together, hence \(a\) is a vertex of a triangle.

(b). It is similar to part (a). □

**Proposition 2.8.** Suppose that \(a, b \in Z(R)^*\) are not pendant vertices, \(Y \subseteq \text{Min}(R)\) and \(\bigcap Y = \{0\}\). Then

(a) \(h_Y^c(a) \cap h_Y^c(b) = \emptyset\) and \(h_Y^c(a) \cup h_Y^c(b) \neq Y\), if and only if \(\text{gi}(a, b) = 3\).

(b) If \(2 \notin Z(R)\), \(h_Y^c(a) \cap h_Y^c(b) = \emptyset\) and \(h_Y^c(a) \cup h_Y^c(b) = Y\), then \(\text{gi}(a, b) = 4\).

(c) Suppose that \(h_Y^c(a) \cap h_Y^c(b) \neq \emptyset\). Then \(h_Y^c(a) \cup h_Y^c(b) \neq Y\), if and only if \(\text{gi}(a, b) = 4\).

(d) Suppose that \(2 \notin Z(R)\) and \(h_Y^c(a) \cap h_Y^c(b) \neq \emptyset\). Then \(h_Y^c(a) \cup h_Y^c(b) = Y\), if and only if \(\text{gi}(a, b) = 6\).

**Proof.** By Proposition 2.1 and Lemma 2.2, it has a similar proof to [30] Theorem 3.4. □

**Theorem 2.9.** Suppose that \(I, J \in \mathcal{H}(R)^*\) and they are not pendant vertices. The following statements hold.

(a) \(h_Y^c(I) \cap h_Y^c(J) = \emptyset\) and \(h_Y^c(I) \cup h_Y^c(J) \neq Y\), if and only if \(\text{gi}(I, J) = 3\).

(b) If \(h_Y^c(I) \cap h_Y^c(J) = \emptyset\) and \(h_Y^c(I) \cup h_Y^c(J) = Y\), then \(\text{gi}(I, J) = 4\).

(c) If \(h_Y^c(I) \cap h_Y^c(J) \neq \emptyset\) and \(h_Y^c(I) = h_Y^c(J)\), then \(\text{gi}(I, J) = 4\).
(d) If \( h^c_I \cap h^c_J \neq \emptyset \) and \( h^c_I \neq h^c_J \) and \( Y \setminus h^c_I \cup h^c_J \) is not singleton, then \( gi(I, J) = 4 \).

(e) If \( h^c_I \cap h^c_J \neq \emptyset \), \( h^c_I \neq h^c_J \) and \( Y \setminus h^c_I \cup h^c_J \) is singleton, then \( 4 \leq gi(I, J) \leq 5 \).

(f) If \( gi(I, J) = 5 \), then \( h^c_I \cap h^c_J \neq \emptyset \), \( h^c_I \neq h^c_J \) and \( Y \setminus h^c_I \cup h^c_J \) is singleton.

**Proof.** (a \( \Rightarrow \)). By Lemma 2.2, \( I \) is adjacent to \( J \) and by Theorem 2.4 \( I \) and \( J \) are not orthogonal. Thus \( gi(I, J) = 3 \).

(a \( \Leftarrow \)). Then \( I \) is adjacent to \( J \) and the vertices \( I \) and \( J \) are not orthogonal, so by Lemma 2.2 we have \( h^c_I \cap h^c_J = \emptyset \) and by Proposition 2.3, \( h^c_I \cap h^c_J = Y \).

(b). By the assumption \( IJ = \{0\} \), and we can see easily that \( h^c_I \cap h^c_J = \emptyset \), so \( h^c_I \cap h^c_J \cap h^c_J = \emptyset \). Now Lemma 2.2 concludes that \( Ann(I)Ann(J) = \{0\} \). Since \( I \) and \( J \) are not pendant vertices, there are \( I_1, J_1 \in \mathcal{A}(X)^* \) such that \( I \) is adjacent to \( I_1 \neq J \) and \( I \) is adjacent to \( J_1 \neq I \), so \( I_1J_1 = \{0\} \) and \( I \subseteq Ann(I) \) and \( J \subseteq Ann(J) \), hence \( I_1J_1 \subseteq Ann(I)Ann(J) = \{0\} \) and therefore \( I_1J_1 = \{0\} \).

Consequently, \( I \) is adjacent to \( J \), \( J \) is adjacent to \( J_1 \), \( J_1 \) is adjacent to \( I_1 \) and \( I_1 \) is adjacent to \( I \), they imply that \( gi(I, J) = 4 \).

(c). We can conclude from the assumption and part (a) that \( gi(I, J) \geq 4 \). Clearly \( Ann(I), Ann(J) \in \mathcal{A}(R)^* \). Since \( h^c_I \cap h^c_J = \emptyset \), it follows that \( h^c_I \cap h^c_J \subseteq h^c_I \cap h^c_J \cap h^c_J = \emptyset \), so, by Lemma 2.2 \( IAnn(J) = \{0\} \). Similarly, we can show that \( JAnn(I) = \{0\} \). If \( Ann(I) \neq Ann(J) \), then \( I \) is adjacent to \( Ann(J) \), \( J \) is adjacent to \( Ann(I) \) and \( I \) and \( J \) are adjacent to \( I \) and \( J \) respectively. Now we suppose that \( Ann(I) = Ann(J) \). Since \( I \) is adjacent to \( Ann(I) \) and \( I \) is not a pendant vertex, it follows there is some vertex \( I_1 \in \mathcal{A}(X)^* \) distinct from \( Ann(I) \) such that \( I \) is adjacent to \( I_1 \), then \( I_1I = \{0\} \), so \( I_1 \subseteq Ann(I) = Ann(J) \) and therefore \( I_1J = \{0\} \). Consequently, \( I \) is adjacent to \( Ann(I) \), \( Ann(J) \) is adjacent to \( J \), \( J \) is adjacent to \( I_1 \) and \( I_1 \) is adjacent to \( I \) and \( J \).

(d). We can conclude from the assumption and part (a) that \( gi(I, J) \geq 4 \). Since \( \{h^c_K \cup h^c_J \} \) is an ideal of \( R \) is a base for \( Y \), \( Y \) is Hausdorff and \( Y \setminus h^c_I \cap h^c_J \) is not singleton, it follows that there are two distinct ideals \( K_1 \) and \( K_2 \) such that \( h^c_K \cap h^c_J = \emptyset \) and \( h^c_K \cap h^c_J = \emptyset \) and \( h^c_K \cap h^c_J = \emptyset \). Hence \( h^c_K \cap h^c_K = h^c_K \cap h^c_K = h^c_K \cap h^c_K = \emptyset \). Then, by Theorem 2.1 \( K_1, K_2 \in \mathcal{A}(R)^* \), and by Lemma 2.2 \( I \) is adjacent to \( K_1 \), \( K_1 \) is adjacent to \( J \), \( J \) is adjacent to \( K_2 \) and \( K_2 \) is adjacent to \( I \). Consequently, \( gi(I, J) = 4 \).

(e). By part (a), \( gi(I, J) \geq 4 \). Since \( Y \setminus h^c_I \cup h^c_J \neq Y \) and \( \{h^c_K \} \) is an ideal of \( R \), it follows that there is some ideal \( K \) of \( R \) such that \( h^c_K \cap h^c_J = \emptyset \) and \( h^c_K \cap h^c_J = \emptyset \). By Theorem 2.1 \( K_1 \in \mathcal{A}(R)^* \) and Lemma 2.2 concludes that \( K_1 \) is adjacent to the both vertices \( I \) and \( J \). If there is an \( K_2 \) distinct from \( K_1 \) such that \( h^c_K \cap h^c_J = \emptyset \), then \( K_2 \) also is adjacent to the both vertices \( I \) and \( J \). Thus \( gi(I, J) = 4 \). Now suppose that \( h^c_K = h^c_K \) implies that \( K = K_1 \). If \( h^c_K \subseteq h^c_K \), then \( h^c_K \subseteq h^c_K \), so \( Y \setminus h^c_K \cup h^c_K = Y \setminus h^c_K \) and therefore, by the assumption, \( Y \setminus h^c_K \) is singleton. Since \( J \) is not a pendant vertex, there is some vertex \( K_2 \) such that \( K_2 \) is adjacent to \( J \), thus, by Lemma 2.2 \( h^c_K \setminus h^c_J = \emptyset \), so \( h^c_K \cap h^c_J = \emptyset \), thus \( h^c_K \subseteq Y \setminus h^c_J \).
By Theorem 2.1, $h_Y(K_2) \neq \emptyset$ and therefore $h_Y(K_2) = Y \setminus h_Y(J)$. Similarly, we can show that $h_Y(K_1) = Y \setminus h_Y(J)$, hence $h_Y(K_1) = h_Y(K_2)$, which is a contradiction. Hence $h_Y(I) \subsetneq h_Y(J)$. Similarly one can show $h_Y(J) \subsetneq h_Y(I)$, thus $h_Y(I) \setminus h_Y(J)$ and $h_Y(J) \setminus h_Y(I)$ are disjoint nonempty open sets. Since \{h_Y(K) : K is an ideal of R\} is a base for Y, there are distinct ideals K_2 and K_3 such that $h_Y(K_2) \subseteq h_Y(I) \setminus h_Y(J)$ and $h_Y(K_3) \subseteq h_Y(I) \setminus h_Y(J)$. Consequently, $h_Y(J) \cap h_Y(K_2) = h_Y(K_2) \cap h_Y(K_3) = h_Y(K_3) \cap h_Y(I) = \emptyset$. By Theorem 2.1, we have $K_2, K_3 \in \mathbb{A}(R)^*$ and Lemma 2.2 concludes that I is adjacent to $K_1, K_3$ is adjacent to $K_2, K_2$ is adjacent to $K_3$ and $K_3$ is adjacent to I, and therefore $gi(I, J) \leq 5$.

(f). It is clear, by parts (a)-(e). \hfill \Box

Suppose that $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, I = \{0\} \times \mathbb{Z} \times \mathbb{Z} \times \{0\}, J = \mathbb{Z} \times \{0\} \times \mathbb{Z} \times \{0\}, R' = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, I' = \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\}, J = \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}$. Then the both pair vertices I, J $\in \mathbb{A}(R)^*$ and $I', J' \in \mathbb{A}(R')^*$ satisfy in the conditions of part (e) of the above theorem but it is seen readily that $gi(I, J) = 4$ and $gi(I', J') = 5$.

Now we can conclude the following corollary from the above theorem and [19, Corollary 4.2].

**Corollary 2.10.** If for some $I, J \in \mathbb{A}(R)^*$, we have $gi(I, J) = 5$, then the following equivalent conditions hold

(a) $\text{Min}(R)$ has an isolated point.
(b) $B(R) \neq \emptyset$.

3. Radius and Triangulation

This section is has been devoted to study the radius of the triangulation of $\Gamma(R)$ and $\mathbb{A}G(R)$. We show that the concept of the anti fixed-place ideal plays the main role in this studying.

**Theorem 3.1.** The following statement are equivalent.

(a) $\text{Rad} \Gamma(R) = 3$.
(b) $\text{Rad} \mathbb{A}G(R) = 3$.
(c) The zero ideal of $R$ is an anti fixed-place ideal.
(d) The $\text{Min}(R)$ does not have any isolated point.

**Proof.** (a) $\Rightarrow$ (b). Suppose that $\text{Rad} \mathbb{A}G(R) \neq 3$, then, by Corollary 2.6 and 19, there is some $I \in \mathbb{A}(R)^*$ such that $\text{ec}(I) = 2$, hence, Theorem 2.5 there is some $P \in \text{Min}(R)$ such that $h_m^c(I) = \{P\}$, thus $\text{Ann}(I) = P$. Set $0 \neq a \in I$, then $\emptyset \neq h_m^c(a) \subseteq h_m^c(I) = \{P\}$, so $h_m^c(a) = \{P\}$ and therefore $\text{ec}(a) = 2$, by Theorem 2.3. Consequently, $\text{Rad} \Gamma(R) \neq 3$.

(b) $\Rightarrow$ (c). Suppose the zero ideal of $R$ is not an anti fixed-place ideal, then there is an affiliated prime ideal $P$, hence $a \in Z(R)^*$ exists such that $\text{Ann}(a) = P$, this implies that $\langle a \rangle \in \mathbb{A}(R)^*$ and $h_m^c(\langle a \rangle) = h_m^c(a) = \{P\}$ and therefore $\text{Rad} \mathbb{A}G(R) \neq 3$, by Theorem 2.5.

(c) $\Rightarrow$ (a). Suppose that $\text{Rad} \Gamma(R) \neq 3$, then, Corollary 2.6 and 19, there is some $a \in Z(R)^*$ such that $\text{ec}(a) = 2$, hence, by Theorem 2.6 there is some $P \in \text{Min}(R)$ such that $h_m^c(a) = \{P\}$, thus $\text{Ann}(a) = P$, hence $P$ is affiliated prime ideal, so $P \in B(R) \neq \emptyset$ and therefore the zero ideal of $R$ is not an anti fixed-place ideal.

(c) $\Leftrightarrow$ (d). It implies from [19, Corollary 4.3]. \hfill \Box
The following corollary is an immediate consequence of the above theorem and Corollary 2.6.

**Corollary 3.2.** The following statement are equivalent.

(a) \( \text{Rad} \Gamma(R) = 2 \).

(b) \( \text{Rad} \mathcal{A}G(R) = 2 \).

(c) The zero ideal of \( R \) is not an anti fixed-place ideal.

(d) The \( \text{Min}(R) \) has an isolated point.

Now we can conclude the following corollary from the above theorem and corollary.

**Corollary 3.3.** \( \text{Rad} \Gamma(R) = \text{Rad} \mathcal{A}G(R) \).

**Corollary 3.4.** Suppose that \( X \) is a Tychonoff topological space. Then

\[
\text{Rad} \Gamma(X) = \text{Rad} \mathcal{A}G(X) = \begin{cases} 
2 & \text{If } X \text{ has an isolated point.} \\
3 & \text{If } X \text{ does not have any isolated point.}
\end{cases}
\]

**Proof.** It conclude from [6, Corollary 5.4], Theorem 3.1 and Corollary 3.2. \( \square \)

**Theorem 3.5.** The following statements are equivalent.

(a) The zero ideal of \( R \) is an anti fixed-place ideal.

(b) \( \Gamma(R) \) is triangulated.

(c) \( \text{Min}(R) \) does not have any isolated point.

**Proof.** (a) \( \Rightarrow \) (b). Suppose that \( \Gamma(R) \) is not triangulated, then \( a \in Z(R)^* \) exists such that \( a \) is not a vertex of any triangle, so by Proposition 2.7, \( h_m(a) \) is singleton, hence there is a \( P \in \text{Min}(R) \) such that \( h_m(a) = \{P\} \). Since \( h_m(a) \) is open and \( \{h_m(x) : x \in R\} \) is base for \( Y \), there is some \( b \in R \) such that \( P \in h_m^c(b) \subseteq h_m(a) = \{P\} \), thus \( h_m^c(b) = \{P\} \) and therefore \( \text{Ann}(b) = P \). It shows that \( P \) is affiliated prime ideal, hence \( P \in \mathcal{B}(R) \neq \emptyset \) and consequently the zero ideal is not an anti fixed-place ideal.

(b) \( \Rightarrow \) (c). By [30, Theorem 3.1], \( \text{Spec}(R) \) does not have any quasi-isolated point, i.e., \( \text{Min}(R) \) does not have any isolated point.

(c) \( \Rightarrow \) (a). It concludes from [6, Corollary 4.3]. \( \square \)

**Theorem 3.6.** The following statements are equivalent.

(a) The zero ideal of \( R \) is an anti fixed-place ideal.

(b) \( \mathcal{A}G(R) \) is triangulated.

(c) \( \text{Min}(R) \) does not have any isolated point.

**Proof.** (a) \( \Rightarrow \) (b). It is similar to proof of the part (a) \( \Rightarrow \) (b) of the previous theorem.

(b) \( \Rightarrow \) (a). Suppose that the zero ideal of \( R \) is not an anti fixed-place ideal. Then \( P \in \mathcal{B}(R) \neq \emptyset \) exists, hence \( P \) is a affiliated prime ideal, so there is some \( a \in R \) such that \( \text{Ann}(a) = P \), thus \( h_m^c(a) = \{P\} \). This implies that \( \{P\} \) is open in \( \text{Min}(R) \), therefore \( h_m^c(P) = \text{Min}(R) \setminus \{P\} \) is closed and consequently \( h_m^c(P) = \text{Min}(R) \setminus \{P\} \). Thus \( h_m(P)^c = h_m^c(P) \subseteq \text{Min}(R) \setminus \{P\} \). Now Proposition 2.7 concludes that \( P \) is not a vertex of any triangle and therefore \( \mathcal{A}G(R) \) is not triangulated.

(a) \( \Leftrightarrow \) (c). It is clear, by [6, Corollary 4.3]. \( \square \)
In the [30] Corollary 3.3, it has been asserted that “Let \( R \) be a reduced ring and let \( \text{Spec}(R) \) be finite. Then \( \Gamma(R) \) is a triangulated graph if and only if \( \text{Spec}(R) \) has no isolated points.”. If \( \text{Spec}(R) \) is finite, then \( \text{Min}(R) \) is finite, so the zero ideal of \( R \) is fixed-place and therefore it is not anti fixed-place, hence by the above theorem \( \Gamma(R) \) is not triangulated. Hence the assumption “\( \Gamma(R) \) is a triangulated graph” in this assertion is impossible.

Now we can conclude the following corollary from the above theorems.

**Corollary 3.7.** \( \Gamma(R) \) is triangulated, if and only if \( \mathcal{A} \mathcal{G}(R) \) is triangulated.

Now we can conclude easily from Theorem 3.5 and [6, Corollary 5.4], that \( \Gamma(X) \) is triangulated, if and only if \( X \) does not have any isolated point. This fact has been shown in [16, Theorem 4.5].

If \( \text{Min}(R) \) is finite, then the zero ideal of \( R \) is fixed-place and therefore it is not anti fixed-place, hence, by Corollary 3.2, \( \text{Rad}\Gamma(R) = \text{Rad}\mathcal{A} \mathcal{G}(R) = 2 \).

4. Domination number

The main purpose of this section is studying of domination number of \( \mathcal{A} \mathcal{G}(R) \) and then \( \mathcal{A} \mathcal{G}(X) \). In this studying, we employ the Bourbaki associated prime divisor of the zero ideal and the fixed-place ideal notion.

**Lemma 4.1.** Let \( I \) be an ideal in \( \mathcal{A}(R)^* \). The following statements are equivalent.

(a) \( I \) is prime.

(b) \( I \) is a maximal element of \( \mathcal{A}(R)^* \).

(c) \( I \) is a Bourbaki associated prime divisor of the zero ideal of \( R \).

**Proof.** (a) \( \Rightarrow \) (b). Suppose that \( I \subseteq J \) and \( J \in \mathcal{A}(R)^* \), thus \( 0 \neq a \in \text{Ann}(J) \) exists. Since \( R \) is a reduced ring, \( a \notin J \), then \( a \notin I \) and \( aJ \subseteq I \), thus \( J \subseteq I \), hence \( I = J \). Consequently, \( I \) is a maximal element of \( \mathcal{A}(R)^* \).

(b) \( \Rightarrow \) (c). Since \( I \in \mathcal{A}(R)^* \), there is some \( 0 \neq a \in R \) such that \( \text{Ann}(a) = I \). Suppose that \( xy \in I \) and \( x \notin I \), then \( I = \text{Ann}(a) \subseteq \text{Ann}(ax) \), so \( y \in \text{Ann}(ax) \subseteq \text{Ann}(a) = I \), by the maximality of \( I \), hence \( I \) is prime, and therefore \( I \) is a Bourbaki associated prime divisor of the zero ideal.

(c) \( \Rightarrow \) (a). It is clear. \( \square \)

**Proposition 4.2.** The following statements hold.

(a) Suppose that \( I \in \mathcal{A}(R)^* \). \( I \) is contained in some maximal element of \( \mathcal{A}(R)^* \), if and only if \( \text{Min}(I) \cap \mathcal{B}(R) \neq \emptyset \).

(b) Every element of \( \mathcal{A}(R)^* \) is contained in some maximal element of \( \mathcal{A}(R)^* \), if and only if the zero ideal of \( R \) is a fixed-place ideal.

(c) \( \mathcal{A}(R)^* \) does not have any maximal element, if and only if the zero ideal of \( R \) is an anti fixed-place ideal.
Proof. \( (a \implies) \). By Lemma \[4.1 \] \( P \in \mathcal{B}(R) \) exists such that \( I \subseteq P \), since \( P \in \text{Min}(R) \), it follows that \( P \in \text{Min}(I) \) and therefore \( P \in \mathcal{B}(R) \cap \text{Min}(R) \neq \emptyset \).

\( (a \iff) \). It is clear, by Lemma \[4.1 \]

\( (b \implies) \). On contrary, suppose that \( \cap_{P \in \mathcal{B}(R)} P \neq \{0\} \), so there is some \( 0 \neq a \in \cap_{P \in \mathcal{B}(R)} P \). Then

\[
\text{Ann}(a) = (0 : a) = \left( \bigcap_{P \in \text{Min}(R)} P : a \right) = \bigcap_{a \notin P \in \text{Min}(R)} P
\]

By the assumption, there is some \( P_0 \in \mathcal{B}(R) \) such that \( \text{Ann}(a) \subseteq P_0 \), then \( \cap_{a \notin P \in \text{Min}(R)} P \subseteq P_0 \), and therefore

\[
\Rightarrow \{0\} = \left( \bigcap_{P \neq P \in \text{Min}(R)} P \right) \cap P_0 = \bigcap_{P_0 \neq P \in \text{Min}(R)} P
\]

which is a contradiction.

\( (b \iff) \). By the assumption, \( \cap_{P \in \mathcal{B}(R)} P = \{0\} \). So

\[
\text{Ann}(I) = (0 : I) = \left( \bigcap_{P \in \mathcal{B}(R)} P : I \right) = \bigcap_{P \in \mathcal{B}(R)} (P : I) = \bigcap_{I \notin P \in \mathcal{B}(R)} P
\]

Hence \( P \in \mathcal{B}(R) \) exists such that \( I \subseteq P \) and thus, by Lemma \[4.1 \], it completes the proof.

\( (c) \). It is evident, by Lemma \[4.1 \]. \( \square \)

In the proof of \[23 \] Theorem 2.2] It has been asserted that “By Zo rn’s Lemma, it is clear that if \( \mathbb{A}(R)^* \neq \emptyset \), then \( \mathbb{A}(R)^* \) has a maximal element”. But by the above proposition, we know that if the zero ideal of a ring \( R \) is anti fixed-place, then \( \mathbb{A}(R)^* \) does not have any maximal element. For example, since \( \mathbb{R} \) does not have any isolated point, by \[11 \] Corollary 5.4, the zero ideal of \( C(\mathbb{R}) \) is an anti fixed-place ideal and therefore \( \mathcal{B}(C(\mathbb{R})) = \emptyset \). In this case, \( M = \mathcal{B}(C(\mathbb{R})) = \emptyset \), so \[23 \] Theorem 2.2] is not true in general.

Theorem 4.3. For each ring \( R \),

(\( a \)) \( |\mathcal{B}(R)| \leq \text{dt}_t(\mathbb{A}\mathcal{G}(R)) \).

(\( b \)) If \( |\text{Min}(R)| > 2 \), then \( |\mathcal{B}(R)| \leq \text{dt}_t(\mathbb{A}\mathcal{G}(R)) \).

Proof. \( (a) \). Suppose that \( D \) is a total dominating set of \( \mathbb{A}\mathcal{G}(R) \). For each \( P \in \mathcal{B}(R) \), there is some \( I_P \in D \), such that \( I_P \) is adjacent to \( P \), so \( PI_P = \{0\} \), thus \( P \subseteq \text{Ann}(I_P) \), hence \( P = \text{Ann}(I_P) \), by Lemma \[4.1 \]. Now suppose that \( I_P = I_Q \), for some \( P, Q \in \mathcal{B}(R) \), then \( P = \text{Ann}(I_P) = \text{Ann}(I_Q) = Q \) and thus the map \( P \sim I_P \) is one-to-one. This implies that \( |\mathcal{B}(R)| \leq |D| \) and consequently \( |\mathcal{B}(R)| \leq \text{dt}_t(\mathbb{A}\mathcal{G}(R)) \).

\( (b) \). Let \( D \) be a dominating set. For each \( P \in \mathcal{B}(R) \), if \( P \in D \), then we set \( K_P = P \) and if \( P \notin D \), there is some \( K_P \in D \) such that \( K_P \) is adjacent to \( P \). Suppose that \( K_P = K_Q \), for some \( P, Q \in \mathcal{B}(R) \). If \( P, Q \in D \), then \( P = K_P = K_Q = Q \). If \( P, Q \notin D \), then \( P \) and \( Q \) are adjacent to \( K_P \) and \( K_Q \), respectively, so \( PK_P = QK_Q = \{0\} \), thus \( P \subseteq \text{Ann}(K_P) \) and \( Q \subseteq \text{Ann}(K_Q) \) and therefore \( P = \text{Ann}(K_P) = \text{Ann}(K_Q) = Q \), by Lemma \[4.1 \] Finally, without loss of generality, we assume \( P \in D \) and \( Q \notin D \), then \( P = K_P \) and \( K_Q \) is adjacent to \( Q \), so \( P \) is adjacent to \( Q \) and thus \( PQ = \{0\} \). Hence for each \( P' \in \text{Min}(R) \), \( PQ = \{0\} \subseteq P' \).
and therefore either $P \subseteq P'$ or $Q \subseteq P'$, so, by Lemma 4.1 either $P = P'$ or $Q = P'$. This implies that $|\text{Min}(R)| \leq 2$, which contradicts the assumption. Consequently, the map $P \mapsto K_P$ is one-to-one and thus $|B(R)| \leq \text{dt}(\mathcal{A}\mathcal{G}(R))$. 

\textbf{Theorem 4.4.} If the zero ideal of $R$ is a fixed-place ideal, then

(a) $\text{dt}_t(\mathcal{A}\mathcal{G}(R)) = |B(R)|$.

(b) If $|\text{Min}(R)| > 2$, then $\text{dt}_t(\mathcal{A}\mathcal{G}(R)) = |B(R)|$.

\textbf{Proof.} (a). By the above theorem it is sufficient to show that $\text{dt}_t(\mathcal{A}\mathcal{G}(R)) \leq |B(R)|$. For every $P \in B(R)$, pick $a_P \in R$, such that $\text{Ann}(a_P) = P$. For each $K \in \mathcal{A}(R)^*$, by the assumption and Proposition 4.2, there is some $P \in B(R)$ such that $K \subseteq P = \text{Ann}(a_P)$, so $Ra_P K = \{0\}$ and therefore $K$ is adjacent to $Ra_P$. This implies that $\{Ra_P : P \in B(R)\}$ is a dominating set and consequently, $\text{dt}_t(\mathcal{A}\mathcal{G}(R)) \leq |B(R)|$.

(b). By the fact that $\text{dt}(\mathcal{A}\mathcal{G}(R)) \leq \text{dt}_t(\mathcal{A}\mathcal{G}(R))$, it follows from (a) and the above theorem.

We know that if $\text{Min}(R)$ is finite, then the zero ideal of $R$ is a fixed-place ideal and $\text{Min}(R) = B(R)$. Thus [27, Theorem 2.4 and Theorem 2.5] and [25, Theorem 2.4 and Theorem 2.5] are immediate consequences of the above theorem. Also, we can conclude the following corollary from the above theorem and [6, Theorems 5.2 and 5.5].

\textbf{Corollary 4.5.} Suppose $X$ is an almost discrete space. Then

(a) $\text{dt}_t(\mathcal{A}\mathcal{G}(X)) = |I(X)|$.

(b) If $|X| > 2$, then $\text{dt}_t(\mathcal{A}\mathcal{G}(X)) = |I(X)|$.

\textbf{Theorem 4.6.} If the zero ideal of a ring $R$ is not a fixed-place ideal, then $\text{dt}(\mathcal{A}\mathcal{G}(R))$ and $\text{dt}_t(\mathcal{A}\mathcal{G}(R))$ are infinite.

\textbf{Proof.} Suppose that $D$ is a dominating set of $\mathcal{A}\mathcal{G}(R)$. By Proposition 4.2, there is some $J_1 \in \mathcal{A}(R)^*$ which is not contained in a maximal element of $\mathcal{A}(R)^*$. If $J_1 \in D$, then we set $I_1 = K_1 = J_1$. If $J_1 \notin D$, there is some vertex $I_1 \in D$ which is adjacent to $J_1$, then $J_1 I_1 = \{0\}$, so $J_1 \subseteq \text{Ann}(I_1)$, in this case we set $K_1 = \text{Ann}(I_1)$. Since $J_1$ is not contained in a maximal element of $\mathcal{A}(R)^*$ and $J_1 \subseteq K_1$, there is some $J_2 \in \mathcal{A}(R)^*$ such that $K_1 \subseteq J_2$, similarly we can find $K_2 \in \mathcal{A}(R)^*$ in which either $I_2 = K_2 \subseteq D$ or $K_2 = \text{Ann}(I_2)$, for some $I_2 \in D$. By induction, we have the following

$$J_1 \subseteq K_1 \subset J_2 \subseteq K_2 \subset \ldots \subset J_n \subseteq K_n \subset \ldots$$

Now suppose that $n \neq m$, then $K_n \neq K_m$. Without loss of generality, we assume $n < m$, hence we have four cases

case 1: If $I_n = K_n$ and $I_m = K_m$, then it is evident that $I_n \neq I_m$.

case 2: If $K_n = \text{Ann}(I_n)$ and $K_m = \text{Ann}(I_m)$, so it is clear that $I_n \neq I_m$.

case 3: If $K_n = I_n$ and $K_m = \text{Ann}(I_m)$, then $I_n \subset \text{Ann}(I_m)$, so $I_n I_m = \{0\}$, hence $I_n \neq I_m$, because otherwise, $I_n^2 = \{0\}$ and therefore $I_n = \{0\}$, which is a contradiction.

case 4: If $K_n = \text{Ann}(I_n)$ and $K_m = I_m$, then $\text{Ann}(I_n) \subset I_m$, so $\text{Ann}(I_m) \subset \text{Ann}(\text{Ann}(I_n))$, hence $I_n \neq I_m$, because otherwise, similar to case 3, $\text{Ann}(I_n) = \{0\}$, which is a contradiction.

Since $\{I_n : n \in \mathbb{N}\} \subseteq D$, it follows that $D$ is infinite and consequently $\text{dt}(\mathcal{A}\mathcal{G}(R))$ is infinite. Hence $\text{dt}_t(\mathcal{A}\mathcal{G}(R))$ is finite, by this fact that $\text{dt}(\mathcal{A}\mathcal{G}(R)) \leq \text{dt}_t(\mathcal{A}\mathcal{G}(R))$. 

\qed
Now by the above theorem, $dt_t(\mathcal{A}\mathcal{G}(C(R)))$ and $dt_t(\mathcal{A}\mathcal{G}(C(R)))$ are infinite, so the inequality in Theorem 4.3 can be proper.

**Corollary 4.7.** The following statements are equivalent

(a) $dt_t(\mathcal{A}\mathcal{G}(R))$ is finite
(b) $dt_t(\mathcal{A}\mathcal{G}(R))$ is finite
(c) $\text{Min}(R)$ is finite

**Proof.** It follows immediately from Theorems 4.4 and 4.6 and this fact that if $\text{Min}(R)$ is finite, then the zero ideal is a fixed-place ideal. $\square$

Finally in the following proposition we generalize [28, Theorem 2.3] to the infinite version.

**Proposition 4.8.** For each reduced ring $R$, we have $dt_t(\Gamma(R)) \leq dt_t(\mathcal{A}\mathcal{G}(R))$.

**Proof.** Suppose that $D$ is a total dominating set of $dt_t(\mathcal{A}\mathcal{G}(R))$. So for each $I \in P$, there is some $0 \neq a_I \in I$. For every $a \in R$, there is some $I \in D$ such that $I$ is adjacent to $Ra$ in $\mathcal{A}\mathcal{G}(R)$, thus $RaI = \{0\}$, hence $a_I = 0$ and therefore $a_I$ is adjacent to $a$ in $\Gamma(R)$. Consequently, $\{a_I : I \in D\}$ is a total dominating set of $\Gamma(R)$ and this implies that $dt_t(\Gamma(R)) \leq dt_t(\mathcal{A}\mathcal{G}(R))$. $\square$

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