Von Neumann’s inequality for commuting operator-valued multishifts

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Abstract. Recently, Hartz proved that every commuting contractive classical multishift with non-zero weights satisfies the matrix-version of von Neumann’s inequality. We show that this result does not extend to the class of commuting operator-valued multishifts with invertible operator weights. In particular, we show that the tensor product $A \otimes B$ satisfies the von Neumann’s inequality if and only if $A$ satisfies the von Neumann’s inequality, where $A$ is a contractive $d$-tuple of $n \times n$ matrices and $B$ is a contractive $d$-tuple of operators satisfying the matrix-version of von Neumann’s inequality with the Taylor spectrum containing the unit $d$-torus. We also exhibit several families of operator-valued multishifts for which the von Neumann’s inequality always holds.

1. Introduction

The celebrated von Neumann’s inequality [27] says that if $T$ is a contraction on a Hilbert space $\mathcal{H}$, then $\|p(T)\| \leq \sup_{|z|<1} |p(z)|$ for every polynomial $p$. Generalizing this result, Sz.-Nagy [23] proved that every contraction has a unitary dilation. Later Ando [2] (see also [24]) extended this result and showed that every pair of commuting contractions dilates to a pair of commuting unitaries, and hence, every pair of commuting contractions satisfies the von Neumann’s inequality. Thus it is natural to ask whether the von Neumann’s inequality holds for a $d$-tuple of commuting contractions, $d \geq 3$. Surprisingly, it fails for $d \geq 3$. In fact, Varopoulos, in [25], showed that there exists big enough $d$ for which the von Neumann’s inequality fails for a $d$-tuple of commuting contractions. In the addendum of the same paper, he together with Kaijser and independently Crabb and Davie [4] gave examples of three commuting contractions which do not satisfy the von Neumann’s inequality. Since then it has been one of the peculiar topics in operator theory. In [22], Question 36], Shields asked whether a $d$-tuple of commuting contractive weighted shifts (in other words, contractive classical multishift) satisfies the von Neumann’s inequality. This question was attributed to Lubin and was explicitly mentioned in [15]. Recently, Hartz [12] answered this question affirmatively and proved the following result:

**Theorem 1.1.** [12, Theorem 1.1] Let $T = (T_1, \ldots, T_d)$ be a contractive classical multishift with non-zero weights. Then $T$ dilates to a $d$-tuple of commuting unitaries.
In view of this, it is natural to ask whether the above result extends to the class of commuting operator-valued multishifts with invertible operator weights. The purpose of this note is to study the von Neumann’s inequality for commuting operator-valued multishifts. A key tool in this study is the following characterization for the tensor product of two $d$-tuples of commuting contractions to satisfy the von Neumann’s inequality.

**Theorem 1.2.** Let $n, d$ be positive integers, $A = (A_1, \ldots, A_d)$ be a commuting $d$-tuple of $n \times n$ contractive matrices and $B = (B_1, \ldots, B_d)$ be a $d$-tuple of commuting contractions on a Hilbert space $\mathcal{H}$. Suppose that $B$ satisfies the matrix-version of von Neumann’s inequality and the unit $d$-torus $\mathbb{T}^d$ is contained in the Taylor spectrum $\sigma(B)$ of $B$. Then $A \otimes B = (A_1 \otimes B_1, \ldots, A_d \otimes B_d)$ satisfies the von Neumann’s inequality if and only if $A$ satisfies the von Neumann’s inequality.

Using this characterization, we prove that if $A = (A_1, \ldots, A_d)$ is a $d$-tuple of commuting $n \times n$ contractive matrices and $T = (T_1, \ldots, T_d)$ is a commuting operator-valued multishift on $L^2_{\mathbb{C}^d}(\mathbb{N}^d)$ with operator weights given by $A^{(j)}_{\alpha} = A_j$ for all $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$, then $T$ satisfies the von Neumann’s inequality if and only if $A$ satisfies the von Neumann’s inequality. This readily yields a family of operator-valued multishifts with invertible operator weights which do not satisfy the von Neumann’s inequality. This is in contrast with Theorem 1.1. We conclude this paper with a concrete example of a commuting operator-valued multishift with invertible operator weights which does not satisfy the von Neumann’s inequality. This example is motivated by the one which Kaïsjer and Varopoulos gave to disprove the von Neumann’s inequality for 3-tuple of commuting contractions. We refer the reader to [19, 11, 23, 13, 21] for recent developments related to the von Neumann’s inequality.

We set below the notations used in posterior sections. For a set $X$ and a positive integer $d$, $X^d$ stands for the $d$-fold Cartesian product of $X$. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the set of nonnegative integers, set of integers, the field of real numbers and the field of complex numbers, respectively. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we set $|\alpha| := \sum_{j=1}^d \alpha_j$. For $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, the complex conjugate $\overline{w} \in \mathbb{C}^d$ of $w$ is given by $(\overline{w}_1, \ldots, \overline{w}_d)$, while $w^\alpha$ denotes the complex number $\prod_{j=1}^d w_j^{\alpha_j}$. The symbol $\mathbb{D}^d$ is reserved for the open unit polydisc in $\mathbb{C}^d$ centered at the origin whereas $\overline{\mathbb{D}}^d$ stands for the closed unit polydisc in $\mathbb{C}^d$. Let $\mathcal{H}$ be a complex Hilbert space. If $F$ is a subset of $\mathcal{H}$, the closed linear span of $F$ is denoted by $\overline{\{x : x \in F\}}$. For a positive integer $m$, the orthogonal direct sum of $m$ copies of $\mathcal{H}$ is denoted by $\mathcal{H}^{(m)}$. Let $\mathcal{B}(\mathcal{H})$ denote the unital Banach algebra of bounded linear operators on $\mathcal{H}$. The multiplicative identity $I$ of $\mathcal{B}(\mathcal{H})$ is sometimes denoted by $I_\mathcal{H}$. The norm on $\mathcal{H}$ is denoted by $\| \cdot \|_\mathcal{H}$ and whenever there is no confusion likely, we remove the subscript $\mathcal{H}$ from $\| \cdot \|_\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, then $T^*$ denotes the Hilbert space adjoint of $T$ and $\rho(T)$ denotes the spectral radius of $T$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a contraction if $\|T\| \leq 1$. By a commuting $d$-tuple $T = (T_1, \ldots, T_d)$ in $\mathcal{B}(\mathcal{H})$, we mean a collection of commuting operators $T_1, \ldots, T_d$ in $\mathcal{B}(\mathcal{H})$ and $T$ is said to be contractive if $T_j$ is a contraction for each $j = 1, \ldots, d$. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we understand $T^\alpha$ as the operator $T_{\alpha_1} \cdots T_{\alpha_d}$, where we adhere to the convention that $A^0 = I_\mathcal{H}$ for $A \in \mathcal{B}(\mathcal{H})$. A $d$-tuple $T = (T_1, \ldots, T_d)$ of commuting contractions on $\mathcal{H}$ is said to satisfy the matrix-version of von Neumann’s inequality if for every positive integer $m$,

$$
\| (p_{i,j}(T))_{1 \leq i,j \leq m} \|_{\mathcal{B}(\mathcal{H}^{(m)})} \leq \sup_{z \in \mathbb{D}^d} \| (p_{i,j}(z))_{1 \leq i,j \leq m} \|_{\mathcal{B}(\mathbb{C}^m)}, \quad p_{i,j} \in \mathbb{C}[z_1, \ldots, z_d],
$$

where we adhere to the convention that $A^0 = I_\mathcal{H}$ for $A \in \mathcal{B}(\mathcal{H})$. A $d$-tuple $T = (T_1, \ldots, T_d)$ of commuting contractions on $\mathcal{H}$ is said to satisfy the matrix-version of von Neumann’s inequality if for every positive integer $m$,
where $\mathbb{C}[z_1, \ldots, z_d]$ denotes the ring of polynomials over $\mathbb{C}$ in $d$ complex variables $z_1, \ldots, z_d$.

2. Von Neumann’s inequality for tensor product of tuples

In this section, we prove Theorem 1.2 and to this end, we need the following lemma.

**Lemma 2.1.** Let $d$ be a positive integer and $B = (B_1, \ldots, B_d)$ be a $d$-tuple of commuting contractions on a Hilbert space $\mathcal{H}$. If the unit $d$-torus $T^d$ is contained in the Taylor spectrum $\sigma(B)$ of $B$, then for a polynomial $p \in \mathbb{C}[z_1, \ldots, z_d]$, the spectral radius $r(p(B))$ of the operator $p(B)$ is given by

$$ r(p(B)) = \sup_{z \in B^d} |p(z)|. $$

**Proof.** Let $p \in \mathbb{C}[z_1, \ldots, z_d]$ be a polynomial. Then by the spectral mapping theorem [5], we get $p(\sigma(B)) = \sigma(p(B))$. Hence

$$ r(p(B)) = \sup_{\lambda \in \sigma(p(B))} |\lambda| = \sup_{\lambda \in \sigma(B)} |\lambda| = \sup_{z \in \sigma(B)} |p(z)|. $$

Since $T^d \subseteq \sigma(B) \subseteq \mathbb{B}^d$, it follows that $\sup_{z \in \sigma(B)} |p(z)| = \sup_{z \in B^d} |p(z)|$. This gives the desired conclusion. \hfill \Box

We are now ready to prove the Theorem 1.2.

**Proof of Theorem 1.2.** Let $k \in \mathbb{N}$ and

$$ p(z) = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C}, \quad z \in \mathbb{C}^d, $$

be a scalar-valued polynomial. Consider the matrix valued polynomial $p_A(z)$ given by

$$ p_A(z) = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} a_\alpha A^\alpha z^\alpha, \quad z \in \mathbb{C}^d. $$

Then note that

$$ p_A(B) = p(A \otimes B). \quad (1) $$

Since $B = (B_1, \ldots, B_d)$ satisfies the matrix version of von Neumann’s inequality, it follows that

$$ \|p(A \otimes B)\|_{\mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})} \leq \|p_A(B)\|_{\mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})} \leq \sup_{z \in B^d} \|p_A(z)\|_{\mathcal{B}(\mathbb{C}^n)}. \quad (2) $$

Further, for $x, y \in \mathbb{C}^n$ with $\|x\| = \|y\| = 1$, define the scalar-valued polynomial

$$ p_{A,x,y}(z) = \langle p_A(z)x, y \rangle = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} a_\alpha (A^\alpha x, y) z^\alpha, \quad z \in \mathbb{C}^d. $$
Then
\[
\|p_{A,x,y}(B)\|_{B(H)} = \left\| \sum_{\alpha \in \mathbb{N}^d, |\alpha| < k} a_\alpha \langle A^\alpha x, y \rangle_{C^\alpha} B^\alpha \right\|_{B(H)}
\]
\[
= \sup_{v, w \in H, \|v\| = \|w\| = 1} \left| \left\langle \sum_{\alpha \in \mathbb{N}^d, |\alpha| < k} a_\alpha \langle A^\alpha x, y \rangle_{C^\alpha} B^\alpha v, w \right\rangle_H \right|
\]
\[
= \sup_{v, w \in H, \|v\| = \|w\| = 1} \left| \sum_{\alpha \in \mathbb{N}^d, |\alpha| < k} a_\alpha \langle A^\alpha x, y \rangle_{C^\alpha} \langle B^\alpha v, w \rangle_H \right|
\]
\[
\leq \sup_{v, w \in H, \|v\| = \|w\| = 1} \left| \left\langle p_A(B)(x \otimes v), y \otimes w \right\rangle_{C^n \otimes H} \right|
\]
Taking supremum over all \(x, y \in C^n\) with \(\|x\| = \|y\| = 1\), we get
\[
\sup_{\|x\| = \|y\| = 1} \|p_{A,x,y}(B)\|_{B(H)} \leq \|p_A(B)\|_{B(C^n \otimes H)}. \tag{3}
\]
Since for each \(x, y \in C^n\), \(p_{A,x,y}(z)\) is a scalar-valued polynomial and \(B\) satisfies the (matrix-version of) von Neumann's inequality, we must have
\[
\|p_{A,x,y}(B)\|_{B(H)} \leq \sup_{z \in \mathbb{D}^d} |p_{A,x,y}(z)|. \tag{4}
\]
Also, from Lemma 2.1 we get
\[
\|p_{A,x,y}(B)\|_{B(H)} \geq r(p_{A,x,y}(B)) = \sup_{z \in \mathbb{D}^d} |p_{A,x,y}(z)|. \tag{5}
\]
By (4) and (5), we get
\[
\|p_{A,x,y}(B)\|_{B(H)} = \sup_{z \in \mathbb{D}^d} |p_{A,x,y}(z)| \quad x, y \in C^n. \tag{6}
\]
Now observe that for fixed \(z \in \mathbb{D}^d\),
\[
\|p_A(z)\|_{B(C^n)} = \sup_{\|x\| = \|y\| = 1} |p_{A,x,y}(z)| \leq \sup_{\|x\| = \|y\| = 1} \|p_{A,x,y}(B)\|_{B(H)} \leq \|p_A(B)\|_{B(C^n \otimes H)} \leq \sup_{z \in \mathbb{D}^d} \|p_A(z)\|_{B(C^n)}.
\]
Taking supremum over \(z \in \mathbb{D}^d\) on the left most term of the above inequality, we get \(\|p_A(B)\|_{B(C^n \otimes H)} = \sup_{z \in \mathbb{D}^d} \|p_A(z)\|_{B(C^n)}\). Thus
\[
\|p(A)\|_{B(C^n)} = \left\| \sum_{\alpha \in \mathbb{N}^d, |\alpha| < k} a_\alpha A^\alpha \right\|_{B(C^n)} = \|p_A(1)\|_{B(C^n)} \leq \sup_{z \in \mathbb{D}^d} \|p_A(z)\|_{B(C^n)}
\]
\[
= \|p_A(B)\|_{B(C^n \otimes H)} \leq \|p(A \otimes B)\|_{B(C^n \otimes H)}.
\]
Now, if \(A \otimes B\) satisfies the von Neumann’s inequality, then it is immediate from above that \(A\) also satisfies the von Neumann’s inequality.

Conversely, assume that \(A\) satisfies the von Neumann’s inequality. Let \(k \in \mathbb{N}\) and
\[
p(z) = \sum_{\alpha \in \mathbb{N}^d, |\alpha| < k} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C}, \quad z \in \mathbb{D}^d,
\]
be a scalar-valued polynomial. For fixed $z \in \mathbb{D}^d$, consider the scalar-valued polynomial $p_z(w)$ given by

$$p_z(w) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha w^\alpha \quad w \in \mathbb{C}^d.$$  

Since $A$ satisfies the von Neumann’s inequality,

$$\|p_z(A)\|_{\mathcal{B}(\mathcal{C}^n)} = \left\| \sum_{\alpha \in \mathbb{N}^d} a_\alpha A^\alpha z^\alpha \right\|_{\mathcal{B}(\mathcal{C}^n)} \leq \sup_{w \in \mathbb{D}^d} |p_z(w)|.$$  

(7)

Observe that $p_B(A) = p(A \otimes B)$ and since $B = (B_1, \ldots, B_d)$ satisfies matrix version of the von Neumann’s inequality, it follows from (7) that

$$\|p(A \otimes B)\|_{\mathcal{B}(\mathcal{C}^n \otimes \mathcal{H})} = \|p_B(A)\|_{\mathcal{B}(\mathcal{C}^n \otimes \mathcal{H})} \leq \sup_{z \in \mathbb{D}^d} \|p(A)\|_{\mathcal{B}(\mathcal{C}^n)} \leq \sup_{z \in \mathbb{D}^d} \sup_{w \in \mathbb{D}^d} |p_z(w)| = \sup_{z \in \mathbb{D}^d} |p(z)|.$$  

This completes the proof of the theorem. \qed

3. Operator-valued multishift and the von Neumann’s inequality

This section is devoted to the study of the von Neumann’s inequality for commuting operator-valued multishifts. Before exhibiting a family of commuting contractive operator-valued multishifts with invertible operator weights which do not satisfy the von Neumann’s inequality, we briefly recall the notion of operator-valued multishift. The notion of operator-valued unilateral weighted shift was introduced by Lambert in [16] and was studied considerably thereafter (see [17, 14] for related study). We refer to its several variable generalization as the operator-valued multishift. It seems that the notion of operator-valued multishift was not formally introduced and systematically studied earlier but it appeared at several places in the literature, see for instance [6, 20, 3]. We now proceed towards the formal definition of operator-valued multishift.

Let $d$ be a positive integer and \{ $H_\alpha : \alpha \in \mathbb{N}^d$ \} be a multisequence of complex separable Hilbert spaces. Let $\mathcal{H} = \oplus_{\alpha \in \mathbb{N}^d} H_\alpha$ be the orthogonal direct sum of $H_\alpha$, $\alpha \in \mathbb{N}^d$. Then $\mathcal{H}$ is a Hilbert space with respect to the following inner product:

$$\langle x, y \rangle_\mathcal{H} = \sum_{\alpha \in \mathbb{N}^d} \langle x_\alpha, y_\alpha \rangle_{H_\alpha}, \quad x = \oplus_{\alpha \in \mathbb{N}^d} x_\alpha, \quad y = \oplus_{\alpha \in \mathbb{N}^d} y_\alpha \in \mathcal{H}.$$  

If $H_\alpha = H$ for all $\alpha \in \mathbb{N}^d$, then we denote $\mathcal{H} = \oplus_{\alpha \in \mathbb{N}^d} H$ by $\ell^2(\mathbb{N}^d)$. Let \{ $A^{(j)}_\alpha : \alpha \in \mathbb{N}^d, j = 1, \ldots, d$ \} be a multisequence of bounded linear operators $A^{(j)}_\alpha : H_\alpha \to H_\alpha + \varepsilon_j$. An operator-valued multishift $T$ on $\mathcal{H} = \oplus_{\alpha \in \mathbb{N}^d} H_\alpha$ with operator weights \{ $A^{(j)}_\alpha : \alpha \in \mathbb{N}^d, j = 1, \ldots, d$ \} is a $d$-tuple of operators $T_1, \ldots, T_d$ in $\mathcal{H}$ defined by

$$\mathcal{D}(T_j) := \left\{ x = \oplus_{\alpha \in \mathbb{N}^d} x_\alpha \in \mathcal{H} : \sum_{\alpha \in \mathbb{N}^d} \| A^{(j)}_\alpha x_\alpha \|^2 < \infty \right\},$$  

$$T_j(\oplus_{\alpha \in \mathbb{N}^d} x_\alpha) := \oplus_{\alpha \in \mathbb{N}^d} A^{(j)}_{\alpha - \varepsilon_j} x_{\alpha - \varepsilon_j}, \quad x = \oplus_{\alpha \in \mathbb{N}^d} x_\alpha \in \mathcal{D}(T_j), \quad j = 1, \ldots, d,$$

where $\varepsilon_j$ is the $d$-tuple in $\mathbb{N}^d$ with 1 in the $j$th place and zeros elsewhere. For each $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$, if $\alpha_j = 0$, then we interpret $A^{(j)}_{\alpha - \varepsilon_j}$ to be a zero operator and $x_{\alpha - \varepsilon_j}$ as a zero vector. Note that each $T_j$, $j = 1, \ldots, d$, is a densely defined linear operator in $\mathcal{H}$.

Proposition 3.1. Let $d$ be a positive integer and $T = (T_1, \ldots, T_d)$ be an operator-valued multishift on $\mathcal{H} = \oplus_{\alpha \in \mathbb{N}^d} H_\alpha$ with operator weights \{ $A^{(j)}_\alpha : \alpha \in \mathbb{N}^d, j = 1, \ldots, d$ \}. Then the following statements hold:
(i) For \( j = 1, \ldots, d \), \( T_j \) is bounded if and only if
\[
\sup_{\alpha \in \mathbb{N}^d} \| A_{\alpha}^{(j)} \| < \infty.
\] (8)

(ii) For \( i, j = 1, \ldots, d \), \( T_i \) commutes with \( T_j \) if and only if
\[
A_{\alpha + \varepsilon_j}^{(i)} A_{\alpha}^{(j)} = A_{\alpha + \varepsilon_j}^{(j)} A_{\alpha}^{(i)} \quad \text{for all } \alpha \in \mathbb{N}^d.
\] (9)

Proof. To see (i), let \( x = \oplus_{\alpha \in \mathbb{N}^d} x_{\alpha} \in \mathcal{H} \) and \( j = 1, \ldots, d \). Then
\[
\| T_j x \|^2 = \sum_{\alpha \in \mathbb{N}^d} \| A_{\alpha - \varepsilon_j}^{(j)} x_{\alpha - \varepsilon_j} \|^2 = \sum_{\alpha \in \mathbb{N}^d} \| A_{\alpha}^{(j)} x_{\alpha} \|^2 \leq \sum_{\alpha \in \mathbb{N}^d} \| A_{\alpha}^{(j)} \|^2 \| x_{\alpha} \|^2.
\]
Thus
\[
\| T_j \| \leq \sup_{\alpha \in \mathbb{N}^d} \| A_{\alpha}^{(j)} \|.
\] (10)

Further, let \( \beta \in \mathbb{N}^d \) and \( h \in H_{\beta} \) be such that \( \| h \| = 1 \). Consider \( x = \oplus_{\alpha \in \mathbb{N}^d} x_{\alpha} \in \mathcal{H} \) such that
\[
x_{\alpha} = \begin{cases} h & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.}
\end{cases}
\]

It follows that \( \| T_j \| \geq \| T_j x \| = \| A_{\beta}^{(j)} h \| \) and hence, \( \| T_j \| \geq \| A_{\beta}^{(j)} \| \). Since \( \beta \in \mathbb{N}^d \) is arbitrary, it follows that
\[
\| T_j \| \geq \sup_{\alpha \in \mathbb{N}^d} \| A_{\alpha}^{(j)} \|.
\] (11)

From (10) and (11) we get that
\[
\| T_j \| = \sup_{\alpha \in \mathbb{N}^d} \| A_{\alpha}^{(j)} \|.
\] (12)

The desired conclusion in (i) now follows from (12).

Let \( x = \oplus_{\alpha \in \mathbb{N}^d} x_{\alpha} \in \mathcal{H} \) and \( i, j = 1, \ldots, d \). Then
\[
T_j T_i x = T_j \left( \oplus_{\alpha \in \mathbb{N}^d} A_{\alpha - \varepsilon_j}^{(i)} x_{\alpha - \varepsilon_j} \right) = \oplus_{\alpha \in \mathbb{N}^d} A_{\alpha - \varepsilon_j}^{(j)} A_{\alpha - \varepsilon_i - \varepsilon_j}^{(i)} x_{\alpha - \varepsilon_i - \varepsilon_j}.
\]

Similarly,
\[
T_i T_j x = \oplus_{\alpha \in \mathbb{N}^d} A_{\alpha - \varepsilon_i}^{(j)} A_{\alpha - \varepsilon_j - \varepsilon_i}^{(i)} x_{\alpha - \varepsilon_j - \varepsilon_i}.
\]
Thus \( T_j T_i = T_i T_j \) if and only if
\[
A_{\alpha - \varepsilon_j}^{(j)} A_{\alpha - \varepsilon_i - \varepsilon_j}^{(i)} = A_{\alpha - \varepsilon_i}^{(j)} A_{\alpha - \varepsilon_j - \varepsilon_i}^{(i)}, \quad \alpha \in \mathbb{N}^d.
\]

After re-indexing \( \alpha - \varepsilon_i - \varepsilon_j \) by \( \alpha \), we obtain (ii). \( \square \)

Let \( T = (T_1, \ldots, T_d) \) be an operator-valued multishift on \( \mathcal{H} = \oplus_{\alpha \in \mathbb{N}^d} H_{\alpha} \) with operator weights \( \{ A_{\alpha}^{(j)} : \alpha \in \mathbb{N}^d, \ j = 1, \ldots, d \} \). We refer to \( T \) as a commuting operator-valued multishift if the operator weights satisfy (5) and (6). Let us see how the class of classical multishifts is contained in that of operator-valued multishifts.

Let \( \{ w_{\alpha}^{(j)} : \alpha \in \mathbb{N}^d, \ j = 1, \ldots, d \} \) be a multisequence of non-zero complex numbers such that \( \sup_{\alpha \in \mathbb{N}^d} \| w_{\alpha}^{(j)} \| < \infty \) and \( w_{\alpha + \varepsilon_i}^{(j)} = w_{\alpha}^{(j)} w_{\varepsilon_i}^{(j)} \) for all \( \alpha \in \mathbb{N}^d \), \( i, j = 1, \ldots, d \). Let \( \mathcal{H} = \ell^2(\mathbb{N}^d) \). Set \( A_{\alpha}^{(j)} := w_{\alpha}^{(j)} I_{\mathbb{C}} \) for all \( \alpha \in \mathbb{N}^d \) and \( j = 1, \ldots, d \). Then the commuting operator-valued multishift \( T = (T_1, \ldots, T_d) \) with operator weights \( \{ A_{\alpha}^{(j)} : \alpha \in \mathbb{N}^d, \ j = 1, \ldots, d \} \) is commonly known as classical multishift which was introduced in (15). Note that \( \ell^2(\mathbb{N}^d) \) is nothing but \( \ell^2(^d\mathbb{N}) \) and hence, in future, we shall write \( \ell^2(^d\mathbb{N}) \) in place of \( \ell^2(\mathbb{N}^d) \).

We produce several families of commuting operator-valued multishifts for which the von Neumann’s inequality always holds. We begin with the following lemma which generalizes (16) Corollary 3.2.

LEMMA 3.2. Let $d$ be a positive integer and $H$ be a complex separable Hilbert space. Let $T = (T_1, \ldots, T_d)$ and $S = (S_1, \ldots, S_d)$ be two commuting operator-valued multishifts on $\ell^2_{H}(\mathbb{N}^d)$ with respective unitary operator weights $\{A_{\alpha}^{(j)} : \alpha \in \mathbb{N}^d, j = 1, \ldots, d\}$ and $\{\tilde{A}_{\alpha}^{(j)} : \alpha \in \mathbb{N}^d, j = 1, \ldots, d\}$. Then $T$ and $S$ are unitarily equivalent.

PROOF. Let $U_0 = I_H$ and $U_{\alpha+j} = \tilde{A}_{\alpha}^{(j)} U_{\alpha} A_{\alpha}^{(j)*}$ for all $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$. Since $\tilde{A}_{\alpha}^{(j)}$ and $A_{\alpha}^{(j)}$ are unitary operators, it is clear that each $U_{\alpha}$ is a unitary operator on $H$. Set $U := \oplus_{\alpha \in \mathbb{N}^d} U_{\alpha}$. To show that $U$ is a unitary operator on $\ell^2_{H}(\mathbb{N}^d)$, we only need to show that $U$ is well defined. To this end, first note that if $|\alpha| \leq 1$, then $U_{\alpha}$ is well defined. Hence suppose that $\alpha \in \mathbb{N}^d$ is such that $|\alpha| \geq 2$. Let $\alpha = \beta + \epsilon_j = \gamma + \epsilon_k = \delta + \epsilon_j + \epsilon_k$ for some $\beta, \gamma, \delta \in \mathbb{N}^d$ and $j, k \in \{1, \ldots, d\}$. Now using (9), we get

$$
U_{\beta+\epsilon_j} = \tilde{A}_{\beta}^{(j)} U_{\beta} A_{\beta}^{(j)*} = \tilde{A}_{\delta+\epsilon_j}^{(k)} U_{\delta+\epsilon_j} A_{\delta+\epsilon_j}^{(k)*} = \tilde{A}_{\gamma}^{(k)} U_{\gamma} A_{\gamma}^{(k)*} = U_{\gamma+\epsilon_k}.
$$

Thus $U$ is well defined. It is a routine verification to show that $UT_j U^* = S_j$ for all $j = 1, \ldots, d$. This completes the proof. \hfill $\square$

PROPOSITION 3.3. Let $d$ be a positive integer and $H$ be a complex separable Hilbert space. Let $T = (T_1, \ldots, T_d)$ be a commuting operator-valued multishift on $\ell^2_{H}(\mathbb{N}^d)$ with unitary operator weights $\{A_{\alpha}^{(j)} \in \mathcal{B}(H) : \alpha \in \mathbb{N}^d, j = 1, \ldots, d\}$. Then $T$ satisfies the von Neumann’s inequality.

PROOF. It follows from the preceding lemma that $T$ is unitarily equivalent to the operator-valued multishift on $\ell^2_{H}(\mathbb{N}^d)$ with operator weights being the identity operator on $H$. In other words, $T$ is unitarily equivalent to the unweighted multishift on $\ell^2_{H}(\mathbb{N}^d)$. Since the unweighted multishift on $\ell^2_{H}(\mathbb{N}^d)$ dilates to the (unweighted) bilateral multishift on $\ell^2_{H}(\mathbb{Z}^d)$, which is a $d$-tuple of commuting unitary operators, it follows that $T$ satisfies the von Neumann’s inequality. \hfill $\square$

Let us see another family of commuting operator-valued multishifts for which the von Neumann’s inequality always holds.

PROPOSITION 3.4. Let $n, d$ be positive integers and $T = (T_1, \ldots, T_d)$ be a commuting operator-valued multishift on $\ell^2_{\mathcal{C}^n}(\mathbb{N}^d)$ with operator weights given by

$$
A_{\alpha}^{(j)} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & w_{\alpha}^{(j)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & w_{n,\alpha}^{(j)}
\end{pmatrix}
$$

for all $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$.

Then $T$ is unitarily equivalent to $W_1 \oplus \cdots \oplus W_n$, where $W_k, k = 1, \ldots, n$, is the classical multishift on $\ell^2(\mathbb{N}^d)$ with weights $\{w_{k,\alpha}^{(j)} : \alpha \in \mathbb{N}^d, j = 1, \ldots, d\}$.

PROOF. Let $\{e_1, \ldots, e_n\}$ be an orthonormal set of vectors in $\mathbb{C}^n$. For $k = 1, \ldots, n$, define

$$
e_{\alpha,k} = \oplus_{\beta \in \mathbb{N}^d} x_{\beta} \text{ where } x_{\beta} = 0 \text{ if } \beta \neq \alpha \text{ and } x_{\alpha} = e_k.
$$

For each $k = 1, \ldots, n$, set

$$
\mathcal{M}_k := \bigvee \{e_{\alpha,k} : \alpha \in \mathbb{N}^d\}.
$$

Then $\mathcal{M}_k$ is a reducing subspace of each $T_j$ and $(T_1 \mid_{\mathcal{M}_k}, \ldots, T_d \mid_{\mathcal{M}_k})$ is unitarily equivalent to a
classical multishift $W_k$ with weights $\{w_{k,\alpha}^{(j)} : \alpha \in \mathbb{N}^d, j = 1, \ldots, d\}$ for $k = 1, \ldots, n$. This completes the proof. □

As the von Neumann’s inequality respects the direct sum, the following corollary immediately follows from Theorem 1.1 and the preceding proposition.

**Corollary 3.5.** Let $n, d$ be positive integers and $T = (T_1, \ldots, T_d)$ be a commuting contractive operator-valued multishift on $\ell^2_{\mathbb{C}}(\mathbb{N}^d)$ with operator weights being invertible $n \times n$ diagonal matrices. Then $T$ satisfies the von Neumann’s inequality.

The following proposition facilitates us to produce a class of operator-valued multishifts for which the von Neumann’s inequality does not hold.

**Proposition 3.6.** Let $n, d$ be positive integers and $T = (T_1, \ldots, T_d)$ be the commuting operator-valued multishift on $\ell^2_{\mathbb{C}}(\mathbb{N}^d)$ with operator weights given by $A_\alpha^{(j)} = A_j$ for all $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$, where $A = (A_1, \ldots, A_d)$ is a $d$-tuple of commuting $n \times n$ contractive matrices. Then $T$ satisfies the von Neumann’s inequality if and only if $A$ satisfies the von Neumann’s inequality.

**Proof.** Observe that the Hilbert space $\ell^2_{\mathbb{C}}(\mathbb{N}^d)$ can be realized as $\mathbb{C}^n \otimes \ell^2(\mathbb{N}^d)$. Hence it is not difficult to see that

\[ T_j = A_j \otimes S_j \text{ for all } j = 1, \ldots, d, \]

where $S = (S_1, \ldots, S_d)$ is the unweighted multishift on $\ell^2(\mathbb{N}^d)$. Note that $S$ is unitarily equivalent to the $d$-tuple of operators of multiplication by the coordinate functions on the Hardy space of the polydisc $\mathbb{D}^d$. Hence it follows that the Taylor spectrum of $S$ is $\mathbb{D}^d$. Now the desired conclusion is immediate from Theorem 1.2. □

**Corollary 3.7.** Let $d$ be a positive integer and $A = (A_1, \ldots, A_d)$ be a $d$-tuple of commuting $2 \times 2$ or $3 \times 3$ contractive matrices. Let $T = (T_1, \ldots, T_d)$ be the commuting operator-valued multishift on $\ell^2_{\mathbb{C}}(\mathbb{N}^d)$ ($n = 2$ or $3$) with operator weights given by $A_\alpha^{(j)} = A_j$ for all $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$. Then $T$ satisfies the von Neumann’s inequality.

**Proof.** The fact that any $d$-tuple of commuting $2 \times 2$ contractive matrices satisfies the von Neumann’s inequality was established in [7] while that for a $d$-tuple of commuting $3 \times 3$ contractive matrices was proved in [11]. Now rest of the proof is immediate from the preceding proposition. □

We are now ready to give the example which we mentioned in the beginning of this text. Our example is motivated from the one given by Kaijser and Varopoulos [25] to disprove the von Neumann’s inequality for 3-tuple of commuting contractions.

**Example 3.8.** Let $c \in (0, 1/(6 + \sqrt{30}))$. Following [10] Definition 2.5] (see also [26]), consider the Varopoulos operators on $\mathbb{C}^4$ given by

\[ V_j = \begin{pmatrix} 0 & x_j & y_j & 0 \\ 0 & 0 & 0 & x_j \\ 0 & 0 & 0 & y_j \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad j = 1, 2, 3, \]

where $x_j, y_j \in \mathbb{R}$ and $x_j^2 + y_j^2 = (1-c)^2$ for each $j = 1, 2, 3$. Set $A_j := cI + V_j$ for all $j = 1, 2, 3$. Since $(V_1, V_2, V_3)$ is a commuting tuple, it follows that $A = (A_1, A_2, A_3)$ is also a commuting 3-tuple of invertible matrices. Moreover, $\|A_j\| \leq c + \|V_j\| = 1$ for all $j = 1, 2, 3$. Let $T = (T_1, T_2, T_3)$ be the commuting operator-valued multishift
on $\ell_2^3(\mathbb{N}^d)$ with operator weights given by $A^{(j)}_\alpha = A_j$ for all $\alpha \in \mathbb{N}^d$ and $j = 1, 2, 3$. Now, consider the Varopoulos-Kaijser polynomial $p_\nu(z) := z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_2z_3 - 2z_3z_1$.

It is shown in [25] (see also [13, 10]) that $\sup_{z \in \mathbb{D}^3} |p_\nu(z)| = 5$. Further, we observe that $p_\nu(A_1, A_2, A_3)$ is given by

$$\begin{pmatrix}
3 \sum_{j,k=1}^3 c^2 a_{jk} \sum_{j,k=1}^3 c a_{jk} (x_j + x_k) & 3 \sum_{j,k=1}^3 c a_{jk} (y_j + y_k) & 3 \sum_{j,k=1}^3 a_{jk} (X_j, X_k) \\
0 & 3 \sum_{j,k=1}^3 c^2 a_{jk} & 3 \sum_{j,k=1}^3 c a_{jk} (x_j + x_k) \\
0 & 0 & 3 \sum_{j,k=1}^3 c^2 a_{jk} & 3 \sum_{j,k=1}^3 c a_{jk} (y_j + y_k) \\
0 & 0 & 0 & 3 \sum_{j,k=1}^3 c^2 a_{jk}
\end{pmatrix},$$

where

$$(a_{jk}) := \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

and $X_j = (x_j, y_j)$ for $j = 1, 2, 3$. From above it can be concluded that

$$\|p_\nu(A_1, A_2, A_3)\| \geq \left| \sum_{j,k=1}^3 a_{jk} (X_j, X_k) \right| = 3(1 - c)^2 - 2(\langle X_1, X_2 \rangle + \langle X_2, X_3 \rangle + \langle X_3, X_1 \rangle).$$

Following [8, Lemma 2.18], it can be shown that the right hand side of the above inequality achieves its maximum value at $X_1 = (1 - c)(1, 0)$, $X_2 = (1 - c)(-1/2, -\sqrt{3}/2)$ and $X_3 = (1 - c)(-1/2, \sqrt{3}/2)$ and therefore

$$\|p_\nu(A_1, A_2, A_3)\| \geq 6(1 - c)^2.$$

Thus using this and the facts that $c < 1/(6 + \sqrt{30})$ and $\sup_{z \in \mathbb{D}^3} |p_\nu(z)| = 5$, we conclude that $\|p_\nu(A_1, A_2, A_3)\| > \sup_{z \in \mathbb{D}^3} |p_\nu(z)|$. Now from Proposition [8, 4] we deduce that the operator-valued multishift $T = (T_1, T_2, T_3)$ does not satisfy the von Neumann’s inequality.

**Acknowledgment.** The authors are grateful to Gadadhar Misra and Sameer Chavan for several helpful suggestions, constant support and careful proof reading of the manuscript. Further, we express our gratitude to the faculty and the administration of Department of Mathematics and Statistics, IIT Kanpur and Department of Mathematics, IISc Bangalore for their warm hospitality during the preparation of this paper.

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