The Hadron-quark Crossover in Neutron Star within Gaussian Process Regression Method

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Abstract

The equations of state of the neutron star at the hadron-quark crossover region are interpolated with the Gaussian process regression (GPR) method, which can reduce the randomness of present interpolation schemes. The relativistic mean-field (RMF) model and Nambu–Jona-Lasinio (NJL) model are employed to describe the hadronic phase and quark phase, respectively. In the RMF model, the coupling term between $\omega$ and $\rho$ mesons is considered to control the density-dependent behaviors of symmetry energy, i.e., the slope of symmetry energy $L$. Furthermore, the vector interaction between quarks is included in the NJL model to obtain the additional repulsive contributions. Their coupling strengths and the crossover windows are discussed in the present framework under the constraints on the neutron star from gravitational-wave detections, massive neutron star measurements, mass–radius simultaneous observation of the NICER Collaboration, and the neutron skin thickness of $^{208}$Pb from PREX-II. It is found that the slope of symmetry energy, $L$, should be around $50–90$ MeV and the crossover window is $(0.3, 0.6)\text{ fm}^{-3}$ with these observables. Furthermore, the uncertainties of neutron star masses and radii in the hadron-quark crossover regions are also predicted by the GPR method.

Unified Astronomy Thesaurus concepts: Neutron stars (1108); Nuclear astrophysics (1129)

1. Introduction

Great achievements in the observations of the compact star have been obtained recently from two aspects, which have inspired many new developments in the relevant subjects. The gravitational-wave (GW) signal from the binary neutron star (BNS) merger offers a great opportunity to probe the inner structures of the neutron star (NS), which was first detected by the LIGO and Virgo Scientific Collaborations (LVC) in 2017, i.e., the GW170817 event (Abbott et al. 2017a, 2017b). Its electromagnetic counterpart was quickly found on 2 days and the era of multimessenger astronomy began. The analysis of the GW170817 event extracted the upper limit of tidal effects in BNS, which provides a new observable of the NS besides its radius and mass (Abbott et al. 2018). The two NS radii from GW170817 were estimated to be $R_1 = 11.9^{+1.4}_{-1.2}$ and $R_2 = 11.9^{+1.4}_{-1.4}$ km at the 90% credible level if the equation of state (EoS) supports NSs with masses larger than $1.97M_\odot$. Another detection of a compact binary coalescence, GW190814, reported by LVC involves a $22.2–24.3M_\odot$ black hole and a compact object with a mass of $2.50–2.67M_\odot$ (Abbott et al. 2020). In the additional information of the EoS, the secondary component of the binary was inconclusive, which might be either the heaviest NS or the lightest black hole ever discovered. Over the past 3 years, significant efforts have been devoted to studying the secondary object, which could be a rapidly rotating NS with exotic degrees of freedom (Li et al. 2020; Most et al. 2020; Demircik et al. 2021; Dexheimer et al. 2021), the most massive NS (Huang et al. 2020; Tan et al. 2020), the fastest pulsar (Zhang & Li 2020; Zhou et al. 2021), or a binary black hole merger (Fattoyev et al. 2020; Sedrakian et al. 2020; Tews et al. 2020).

Another achievement in probing the properties of the compact star is the observation of the X-ray emission from several hot spots of an NS surface (Bogdanov et al. 2019). In 2019, the Neutron star Interior Composition Explorer (NICER) Collaboration reported an accurate measurement of the mass and radius of PSR J0030+0451, a mass of $1.44^{+0.12}_{-0.13}M_\odot$ with a radius of $13.02^{+1.24}_{-1.06}$ km (Miller et al. 2019) and a mass of $1.34^{+0.15}_{-0.16}M_\odot$ with a radius of $12.71^{+1.34}_{-1.19}$ km (Riley et al. 2019) by two independent groups. Recently, the radius of another pulsar, PSR J0740+6620, with mass 2.08$^{+0.07}_{-0.06}M_\odot$ (Cromartie et al. 2020; Fonseca et al. 2021) was reported by two independent groups (Miller et al. 2021; Riley et al. 2021) based on NICER and X-ray Multi-Mirror (XMM-Newton) observations. The inferred radius of this massive NS is constrained to $12.39^{+1.30}_{-0.99}$ km for the mass $2.072^{+0.067}_{-0.066}M_\odot$ by Riley et al. (2021) and $13.7^{+2.6}_{-1.5}$ km for the mass $2.08M_\odot$ by Miller et al. (2021) at the 68% credible level.

The EoS of super-dense matter is an essential input and completely indispensable for understanding the processes in heavy-ion collision experiments, core-collapse supernovae, and the properties of the NS, like mass, radius, and tidal deformability. The present observations from the NS require that the EoS of the nuclear matter must be moderately soft at relatively low densities to obtain a smaller radius and must be stiff enough at high densities to generate a heavier mass. Besides the constraints from the universe, nuclear experiments in the terrestrial laboratory provided stringent constraints on the EoS around the nuclear saturation densities, such as the saturation density, binding energy per nucleon, incompressibility, and symmetry energy. The density dependence of symmetry energy will determine the behavior of neutron-rich EoS in the high-density region. However, the slope of symmetry energy at nuclear saturation density is still not well understood.

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determined. Recently, the Lead Radius Experiment (PREX-II) at the Jefferson Lab updated the value of the neutron skin thickness of $^{208}$Pb as $R_{\text{skin}}^{208} = 0.283 \pm 0.071$ fm (Adhikari et al. 2021), which was originally reported by PREX-I (Abrahamyan et al. 2012). The neutron skin thickness has a strong linear correlation with the slope of symmetry energy, $L$. The strong correlation between $L$ and $R_{\text{skin}}^{208}$, and between $L$ and symmetry energy, $E_{\text{sym}}$, led to the limits at saturation density, $E_{\text{sym}} = 38.1 \pm 4.7$ MeV and $L(\rho_0) = 106 \pm 37$ MeV (Reed et al. 2021). In addition, the recent SrRIT experiment at RIKEN suggested the slope of the symmetry energy to be $42 \leq L \leq 117$ MeV by using ratios of the charged pion spectra measured at high transverse momenta (Estee et al. 2021). This rather large value of $L$ from experimental data, as well as the constraint on $L$ from astrophysics, greatly challenge our understanding of nuclear many-body methods.

Many attempts have been made to obtain the EoS of super-dense nuclear matter in an NS under different nuclear many-body theoretical frameworks. The density functional theory (DFT; Vautherin & Brink 1972; Shen et al. 1998; Douchin & Haensel 2001; Shen 2002; Long et al. 2006, 2007; Sun et al. 2008; Dutra 2012; Bao et al. 2014a; Bao & Shen 2014b; Dutra et al. 2014) can effectively determine the nucleon-nucleon interaction by fitting the ground state properties of finite nuclei or the empirical saturation properties of infinite nuclear matter. It has been widely used to investigate various properties of finite nuclei and infinite nuclear matter and has achieved many successes in the fields of nuclear physics and astrophysics. Walecka proposed the first available version of covariant density functional theory (CDFT) based on the Hartree approximation (Walecka 1974), namely, the $\sigma - \omega$ model, also known as the relativistic mean-field (RMF) model. Then the additions of $\rho$ meson, nonlinear terms of $\sigma$ and $\omega$ mesons, and the coupling terms of the $\rho$ meson with $\sigma$ and $\omega$ mesons improved this model and better described the nuclear many-body systems (Boguta & Bodner 1977; Serot 1979; Sugahara & Toki 1994; Horowitz & Piekarewicz 2001).

With the density increasing, many non-nucleonic degrees of freedom, like $\Delta$ resonances (Glendenning & Moszkowski 1991; Drago et al. 2014a, 2014b; Zhu et al. 2016; Li et al. 2018), meson condensation (Barshay et al. 1973; Baym 1973; Pandharipande et al. 1995; Glendenning & Schaffner-Bielich 1999; Li et al. 2006), and hyperons (Ambartsumyan & Saakyan 1960; Glendenning 1985; Schiffrin & Mishustin 1996; Shen 2002; Weber 2005; Weissenborn et al. 2012; Katayama & Saito 2015) may appear in the inner core of an NS from $(2-3)\hbar\omega(\rho_0 \approx 0.16$ fm$^{-3})$. The presence of hyperons will soften the EoS of NS matter, leading to a reduction in the maximum mass, which may be incompatible with the existence of stars with masses larger than $2M_\odot$. In the phase diagram of dense matter (Fukushima & Hatsuda 2011), the hadronic phase (HP) plays a denominated role at low temperature and low chemical potential, while at high temperature or large chemical potential, the dense matter will be deconfined as quark-gluon plasma (QGP) in which the fundamental degrees of freedom are quarks and gluons. It has been proposed that in dense matter the quarks will gradually emerge as the density increases, leading to a hadron-quark phase (QP) transition (Baym 1979; Celik et al. 1980; Glendenning 1992; Satz 1998). The possible astrophysical role of quarks in NSs has been discussed frequently since the proposal of the quark models (Baym & Chin 1976; Buballa 2005).

In the description of the QP, the phenomenological models of interacting quarks (Nambu & Jona-Lasinio 1961a, 1961b; Nakazato et al. 2008; Sagert et al. 2009) should be adopted to describe the inner core of the NS since it is still very difficult to directly use quantum chromodynamics (QCD) theory except at very high baryon chemical potentials, i.e., $\mu_B \geq (3-6)$ GeV, or high baryon densities, i.e., $\rho_B \geq (10-100)\rho_0$ (Freedman & McLerran 1977, 1978; Fraga et al. 2001). In this paper, a three-flavor Nambu–Jona-Lasinio (NJL) model with vector repulsion (Hatsuda & Kunihiro 1994; Vogl & Weise 1991; Rehberg et al. 1996; Buballa 2005; Wu & Shen 2017) is adopted, which was first proposed by Nambu & Jona-Lasinio (1961a, 1961b). The NJL model can successfully describe the dynamical chiral symmetry breaking and the generation of constituent quark masses. It has been widely used to study the role of quark degree of freedom in compact stars (Schertler et al. 1999; Blaschke et al. 2005; Lawley et al. 2006; Klähn et al. 2007; Masuda et al. 2013a, 2013b; Orsaria et al. 2014; Chu et al. 2015). Meanwhile, the vector repulsive interaction between quarks is taken into account since it can temper the growth of vector density and smooth out the chiral restoration (Kitazawa et al. 2003; Bratovic et al. 2013). In some works, diquark condensation is considered in NJL model when performing the interpolation (Aryiyan et al. 2021; Ivanetskyi & Blaschke 2022a, 2022b).

Different constructions have been employed to describe the hadron-QP transition in an NS, such as the Maxwell (Rosenhauer & Staub 1991; Klähn et al. 2007; Agrawal 2010) or Gibbs construction (Schertler et al. 1999; Li et al. 2009; Logoteta et al. 2013; Orsaria et al. 2013; Wu & Shen 2017; Wu et al. 2018; Ju et al. 2021a, 2021b) for describing the first-order transition and interpolation constructions for converting the transition to be a crossover type (Masuda et al. 2013a, 2013b; Kojo et al. 2015), where the hadronic and quark descriptions do not overlap. Once the behaviors of NS matter at low and high densities are known, the EoS of the intermediate region where the phase transition occurs can be parameterized using an interpolating method to produce a massive NS. Masuda et al. (2013a, 2013b) investigated two different interpolation constructions, pressure, and energy density interpolated as hyperbolic functions of baryon density, respectively. Hell & Weise (2014) interpolated pressure as a function of energy density using a similar function and Kojo et al. (2015) interpolated the pressure as a polynomial function of baryon chemical potential. However, these interpolation constructions are highly dependent on interpolation functions, leading to great uncertainty in EoSs obtained by interpolation functions.

Therefore, in this work, we try to find an interpolation method that can reduce the uncertainty of interpolated values and give the magnitude of the uncertainties. The Gaussian process regression (GPR) method from the statistics provides a good scheme to predict the unknown data with the training database and can give the uncertainty of the predictions (Rasmussen 1996; Kuss & Rasmussen 2005; Rasmussen & Williams 2006; Schulz et al. 2018). Furthermore, we would like to determine the values of $L$ and the crossover windows with the latest observables from GW detection, NICER, and neutron skin thickness from PREX-II. In the HP, $\omega$ meson and $\rho$ meson coupling terms are included in the RMF model, i.e., the IUFSU parameter set (Fattoyev & Piekarewicz 2010; Fattoyev et al. 2010). The coupling constants relevant to the vector-isoscalar meson, $\rho$, can be manipulated to generate different $L$ (Dutra 2012) at saturation density with fixed symmetry energy at subsaturation density (Horowitz & Piekarewicz 2001; Zhang & Chen 2013; Bao et al. 2014a).

This paper is organized as follows. In Section 2, we briefly describe the RMF formalism, the NJL model, and the GPR
method. In Section 3, the properties of nuclear matter and the NS are presented and discussed. Finally, a summary and conclusion will be given in Section 4.

2. EoS

2.1. HP

We adopt the RMF model to describe the hadronic matter in a low-density region. In the RMF model (Walecka 1974; Müller & Serot 1996; Horowitz & Piekarewicz 2001; Bao et al. 2014a), the nucleons interact with each other by exchanging various mesons, including the isoscalar meson(σ), vector-isovector meson(ω), and vector-isoscalar meson(ρ). The interaction between the ω and ρ mesons is involved in a full Lagrangian,

$$\mathcal{L}_{\text{RMF}} = \sum_{i=n,p} \bar{\psi}_i \left[ i \gamma^\mu \partial_\mu - M_i^* - \gamma^0 \left( g_\omega \omega_i^\dagger + g_\rho \rho_i^\dagger \right) \right] \psi_i$$

where $W_{\mu\nu}$ and $\tilde{W}_{\mu\nu}$ are the antisymmetry tensor fields of ω and ρ mesons. Within the mean-field approximation, the meson fields are treated as classical fields, $\langle \sigma \rangle = \sigma$, $\langle \omega_i \rangle = \omega$, $\langle \rho_i \rangle = \rho$. Together with the Euler–Lagrange equations, the equations of motion for nucleons and mesons are given by

$$\left[ i \gamma^\mu \partial_\mu - M_i^* - \gamma^0 \left( g_\omega \omega_i^\dagger + g_\rho \rho_i^\dagger \right) \right] \psi_i = 0,$$

$$m_i^* \sigma + g_2 \sigma^2 + g_3 \sigma^3 = g_\sigma (\rho_i^* + \rho_i)$$

$$m_i^* \omega + c_3 \omega^2 + 2 \Lambda_\omega g^\sigma_2 \omega^2 \omega^2 = g_\omega (\rho_i^* + \rho_i)$$

$$m_i^* \rho + 2 \Lambda_\rho g^\sigma_2 \omega^2 \rho^2 = \frac{g_\rho}{2} (\rho_i^* - \rho_i)$$

where $\rho_i^*$, $\rho_i$ (i = n, p) are the scalar and vector densities of species i, respectively. They are generated by the expectation value of nucleon fields. $M_i^* = M_i - g_i^* \sigma$ is the effective nucleon mass.

The hadronic matter in an NS, which contains nucleons and leptons, should satisfy the charge neutrality, $\rho_\mu = \rho_\nu + \rho_{\bar{\nu}}$, and β equilibrium, $\mu_\mu = \mu_\nu - \mu_{\bar{\nu}}$, $\mu_\nu = \mu_{\bar{\nu}}$. The chemical potentials of nucleons and leptons can be derived from the thermodynamics equations at zero temperature,

$$\mu_i = \sqrt{k_F^2 + M_i^{*2}} + g_\omega \omega_i^\dagger + \frac{g_\rho}{2} \rho_i^\dagger \rho_i, \quad i = n, p$$

$$\mu_l = \sqrt{k_F^2 + m_l^{*2}}, \quad l = e, \mu,$$

where $k_F$ is the Fermi momentum, which is related to the vector density by $\rho_i = k_F^3/3\pi^2$. With the energy-momentum tensor in a uniform system, the total energy density and pressure of the hadronic matter can be written as

$$\varepsilon_{\text{HP}} = \frac{1}{2} \sum_{i=n,p} \int_0^{k_F} \sqrt{k_F^2 + M_i^{*2}} k^2 dk$$

$$+ \frac{1}{2} m_\omega^2 \omega_i^\dagger \omega_i + \frac{1}{3} \frac{g_2}{3} \sigma^3 + \frac{1}{4} \frac{g_3}{4} \sigma^4$$

$$+ \frac{1}{2} m_\rho^2 \rho_i^\dagger \rho_i + \frac{3}{4} \frac{c_3}{c_3} \omega^4 + \frac{1}{2} \frac{g_\omega}{2} \rho^2 + 3 \Lambda_\omega (g^\sigma_2 \omega^2) (g^\rho_2 \rho^2).$$

$$P_{\text{HP}} = \frac{1}{2} \sum_{i=n,p} \int_0^{k_F} \frac{k^4 dk}{\sqrt{k_F^2 + M_i^{*2}}}$$

$$- \frac{1}{2} m_\omega^2 \omega_i^\dagger \omega_i - \frac{1}{3} \frac{g_2}{3} \sigma^3 - \frac{1}{4} \frac{g_3}{4} \sigma^4$$

$$+ \frac{1}{2} m_\rho^2 \rho_i^\dagger \rho_i + \frac{1}{4} \frac{c_3}{c_3} \omega^4 + \frac{1}{2} \frac{g_\omega}{2} \rho^2 + \Lambda_\omega (g^\sigma_2 \omega^2) (g^\rho_2 \rho^2).$$

2.2. QP

In the description of quark matter, a three-flavor NJL model is adopted (Hatsuda & Kunihiro 1994; Masuda et al. 2013b). The Lagrangian density is

$$\mathcal{L}_{\text{NJL}} = \bar{q} (i \gamma^\mu \partial_\mu - m) q + G_S \sum_{i=0}^{8} \left| \left( \bar{q} \lambda_i q \right) \right|^2$$

$$+ \left( \bar{q} \gamma_5 \lambda_5 q \right)^2 - G_V (\gamma^\mu q) \gamma^\mu q$$

$$- K \left( \text{det} \left( \bar{q} (1 + \gamma_5) q \right) - \text{det} \left( \bar{q} (1 - \gamma_5) q \right) \right),$$

where q is the quark field with three flavors and three colors together with the current quark mass matrix $m = \text{diag}(m_u, m_d, m_s)$. The term related to $G_S$ is a chiral symmetric four-quark interaction, where $\lambda_5$ are the Gell–Mann matrices with $\lambda_0 = \sqrt{2/3} I$. The term proportional to $G_V$ introduces additional vector and axial-vector interactions to produce universal repulsion between quarks (Kitazawa et al. 2003; Bratovic et al. 2013; Masuda et al. 2013a), which plays an important role in describing massive stars. The last term related to the coefficient K corresponds to the six-quark Kobayashi–Maskawa–‘t Hooft interaction.

Within mean-field approximation, the non-diagonal components of the condensates in a flavor space can be ignored. The constituent quark masses $m_i^*$ (i = u, d, s) can be generated self-consistently through the gap equations,

$$m_i^* = m_i - 4G_S \sigma_i + 2K \sigma_i \sigma_s,$$

where $\sigma_i = \langle \bar{q}_i q_i \rangle$ is the quark condensation in i flavor. The pressure can be evaluated from the thermodynamics potential, $P = - \Omega = \sum_i \mu_i n_i - TS - \varepsilon$, where $\varepsilon$ is the energy density, T is the temperature, S is the entropy density, and $\mu_i$ is the effective chemical potential of quarks, which can be expressed as

$$\mu_i = \mu_i^* + 2G_V \sum_{i=u,d,s} n_i.$$
Here, \( n_i = (q_i^+ q_i) \) is the quark number density. The pressure and the energy density of the QP from the NJL model are

\[
P_{\text{NJL}} = -2G_S (\sigma_0^2 + \sigma_d^2 + \sigma_s^2) + G_V (n_u + n_d + n_s)^2 + 4K\sigma_d \sigma_s + \frac{3}{\pi^2} \sum_{i=u,d,s} \int_{k_i}^{\Lambda} k^2 dk \sqrt{k^2 + m_i^2}
\]

\[
\varepsilon = 2G_S (\sigma_0^2 + \sigma_d^2 + \sigma_s^2) - G_V (n_u + n_d + n_s)^2 - \frac{3}{\pi^2} \sum_{i=u,d,s} \int_{k_i}^{\Lambda} k^2 dk \sqrt{k^2 + m_i^2}
\]

(9)

where \( \Omega_{\text{vac}} \) is introduced to ensure that the pressure and energy density in vacuum are zero,

\[
\Omega_{\text{vac}} = \sum_{i=u,d,s} \left[ -\frac{3}{\pi^2} \int_{k_i}^{\Lambda} k^2 dk \sqrt{k^2 + m_i^2} \right] + 2G_S (\sigma_0^2 + \sigma_d^2 + \sigma_s^2) - 4K\sigma_d \sigma_s.
\]

(10)

For the quark matter consisting of a neutral mixture of quarks (\( u, d, s \)) and leptons (\( e, \mu \)), the \( \beta \) equilibrium and the charge neutrality conditions should also be considered,

\[
\mu_s = \mu_d = \mu_u + \mu_e, \quad \mu_\mu = \mu_e.
\]

(11)

\[
\frac{2}{3} n_u - \frac{1}{3} (n_d + n_s) - n_e - n_\mu = 0.
\]

(12)

The total energy density and pressure of QP including the contributions from both quarks and leptons are given by

\[
\varepsilon_{\text{QP}} = \varepsilon_{\text{NJL}} + \sum_{i=u,d,s} \int_{k_i}^{\Lambda} k^2 dk \sqrt{k^2 + m_i^2},
\]

(13)

\[
P_{\text{QP}} = \sum_{i=u,d,s,e,\mu} n_i \mu_i - \varepsilon_{\text{QP}}.
\]

(14)

### 2.3. Hadron-quark Crossover

To describe the hadron-quark-matter crossover in the NS, we need to construct an interpolation for \( P(\rho_0) \) so that the EoSs between hadron and quark matters can be smoothly connected. The crossover region is characterized by the lower density, \( \rho_L \), and upper density, \( \rho_U \). The EoSs of hadronic and quark matter are only used at the \( \rho_L < \rho_H < \rho_U \) regions, respectively. In the density range \( \rho_L < \rho_H < \rho_U \), the GPR method is adopted to perform the \( P - \rho_H \) interpolation instead of using the pure hadronic matter EoS or the quark matter EoS.

Before introducing the GPR method, we present a conventional univariate (one-dimensional) continuous probability distribution, Gaussian distribution, whose probability density function at random real variable \( x \) in a set \( X \) is

\[
p(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right].
\]

(15)

The parameter \( \mu \) is the mean value of the distribution, while \( \sigma \) is the variance. \( \sqrt{2\pi\sigma^2} \) is the normalizations constant to ensure the integration of this distribution to be one. The expression, \( X \sim N(\mu, \sigma^2) \) denotes that \( p(X = x) = N(x\mid \mu, \sigma^2) \) and we can say that the variable in the set \( X \) follows a Gaussian distribution. However, a system is usually described by more than one feature variable, \( (x_1, \ldots, x_n) \), that are correlated to each other. A multivariate Gaussian distribution should be used to model the variables altogether. The probability density function with an \( n \)-dimensional random vector \( x = (x_1, \ldots, x_n)^T \) can be extended as

\[
N(x\mid \mu, \Sigma) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right],
\]

(16)

where \( \mu = (\mu_1, \ldots, \mu_n)^T \) represents the mean vector of \( x \) and \( \Sigma \) is an \( n \times n \) covariance matrix. The \( \Sigma \) is defined to be a symmetric, positive-definite matrix with \( \Sigma_{ij} = \text{cov}(x_i, x_j) \) for the \( (i, j) \) element. Equation (16) can be represented as \( X \sim \mathcal{N}(\mu, \Sigma) \).

A Gaussian process (GP) defined by Rasmussen is a collection of random variables, any finite numbers of which have consistent joint Gaussian distributions (Rasmussen 1996; Rasmussen & Williams 2006). More specifically, the multivariate Gaussian distribution describes the behavior of a finite random vector, while a GP is a stochastic one defined over a continuum of values, which can be treated as a function. In other words, in this process, the function \( f(x) \) can be treated like a very long vector. It is fully defined by a mean function \( m(x) \) and covariance function, \( k(x, x') \), which respectively describe the mean value in the process at any point, and the covariance at any two points,

\[
f(x) \sim GP(m(x), k(x, x')).
\]

(17)

For any finite set of points, the regression function modeled by a multivariate Gaussian distribution is given as \( p(f\mid X) = \mathcal{N}(f\mid \mu, K) \), which can be denoted by

\[
f = (f(x_1), f(x_2), \ldots, f(x_n))^T \sim \mathcal{N}(\mu, K),
\]

(18)

where \( K \) is the covariance matrix with elements \( K_{ij} = k(x_i, x_j) \) and \( \mu = (m(x_1), \ldots, m(x_n))^T \) is the mean value vector. To clearly distinguish the GP and Gaussian distribution, we use \( m \) and \( k \) in the former and \( \mu \) and \( K \) in the latter.

The covariance function, which is also called the kernel of the GP, can represent some form of distance or similarity (Murphy 2012). If the inputs \( x_j \) and \( x_j' \) are close to each other, we generally expect that \( f(x_j) \) and \( f(x_j') \) will be close as well. This measure of similarity is embedded in the covariance function. There have been significant works on constructing kernels and analyzing their properties. The popular squared exponential (SE) covariance function is widely used and has the form

\[
k_{\text{SE}}(x, x') = s_f^2 \exp \left[ -\frac{|x - x'|^2}{2l^2} \right].
\]

(19)

The length-scale hyperparameter \( l \) determines how wiggly the functions are. The signal variance parameter \( s_f^2 \) can be regarded as the amplitude, which controls the variability of sample functions from the mean function. The other available options...
for kernels, like the rational quadratic kernel, Matérn kernel, and $\gamma$-exponential kernel are shown in Rasmussen & Williams (2006) and Murphy (2012).

Although a function can be treated as an infinite vector, we only have to make predictions for finite points. Suppose a training set $X = (x_1, \cdots, x_m)^T$ and the function outputs $y = (f(x_1), \cdots, f(x_m))^T$ with $y \sim N(\mu_y, K_{yy})$. Given a test set $X^* = (x_1^*, \cdots, x_n^*)^T$, the regression function outputs can be predicted by a multivariate Gaussian, $f \sim N(\mu_f, K_{ff})$. According to the definition of the GP, previous observations $y$ and the function values $f$ follow a joint or multivariate normal distribution, which can be written as

$$
\begin{pmatrix}
y \\
f
\end{pmatrix} \sim N
\begin{pmatrix}
\mu_y \\
\mu_f
\end{pmatrix},
\begin{pmatrix}
K_{yy} & K_{yf} \\
K_{fy} & K_{ff}
\end{pmatrix}
$$

(20)

where $K_{yy} = k(X, X)$ is $n \times n$, $K_{ff} = k(X^*, X^*)$ is $m \times m$, $K_{yf} = k(X^*, X)$ is $m \times n$, and $K_{fy} = k(X^*, X^*)$ is $m \times m$. The joint probability distribution over $y$ and $f$ is $p(y, f|X, X^*) = N(f|\mu_f, \Sigma_f)$ over the unknown outputs $f$. Using the standard results, then we can obtain

$$
\begin{align*}
\mu_{f|y} &= \mu_f + K_{fy}^{T}K_{yy}^{-1}(y - \mu_y), \\
\Sigma_{f|y} &= K_{ff} - K_{fy}^{T}K_{yy}^{-1}K_{fy},
\end{align*}
$$

(21)

Furthermore, noise is often taken into account in the GPR models. If there is an additive independent Gaussian noise, $\epsilon$ with variance $\sigma^2$, the output $y'$ of a function $f$ at input $X$ can be written as

$$
y' = f(x) + \epsilon, \quad \epsilon \sim N(0, \sigma^2).
$$

(22)

The prior on the noisy observations becomes $\text{cov}(y') = K_{yy} + \sigma^2I$. Then the joint distribution of the noisy observations and the function values at the testing point becomes

$$
\begin{pmatrix}
y' \\
f
\end{pmatrix} \sim N
\begin{pmatrix}
0 \\
\mu_f + \sigma^2I^{-1}K_{fy}
\end{pmatrix},
\begin{pmatrix}
K_{yy} + \sigma^2I & K_{yf} \\
K_{fy} & K_{ff} - \sigma^2I^{-1}K_{fy}
\end{pmatrix}
$$

(23)

where the mean function is zero for notational simplicity. Hence, the posterior predictive equation in GPR is

$$
\begin{align*}
f|y', X, X^* &\sim N(K_{fy}^{T}(K_{yy} + \sigma^2I)^{-1}y', \\
&K_{ff} - K_{fy}^{T}(K_{yy} + \sigma^2I)^{-1}K_{fy}).
\end{align*}
$$

(24)

The chosen kernel can decide the predictive performance of the GP. In this work, we choose the SE kernel in Equation (19) for the noisy observations. The kernel function contains hyperparameters such as the length scale, signal variance, and noise variance, which need to be inferred from the known data. The inferences for all hyperparameters should be obtained by computing the probability of the known data instead of figuring out the contributions of hyperparameters. A common approach is to estimate them by taking the maximum marginal likelihood.

Given the hyperparameters $\theta = l, s_j, \sigma_r$, and inputs $(X, y')$, from Equation (24), the negative log marginal likelihood can be written as

$$
\log p(y'|\theta, X) = - \frac{1}{2} y'^T K_{y'}^{-1} y' - \frac{1}{2} \log |K_{y'}| - \frac{n}{2} \log(2\pi), \\
\theta = l, s_j, \sigma_r.
$$

(25)

where $K_{y'} = K_{yy} + \sigma^2I$ is the covariance matrix for the noisy targets $y'$. The first term in the above equation measures the data fit, the second term is a model complexity term, and the last term is a normalization constant. Now the partial derivatives of negative log marginal likelihood with respect to the hyperparameters can be obtained

$$
\frac{\partial}{\partial \theta} \log p(y'|\theta, X) = \frac{1}{2} y'^T K_{y'}^{-1} \frac{\partial K_{y'}}{\partial \theta} K_{y'}^{-1} y' - \frac{1}{2} tr(K_{y'}^{-1} \frac{\partial K_{y'}}{\partial \theta}). \\
\theta = l, s_j, \sigma_r = 0.
$$

(26)

Finally, the outputs $f$ will be well defined after solving this equation to get the hyperparameters $l, s_j$, and $\sigma_r$.

When interpolating the pressure $P$ as a function of baryon density, $\rho_B$, in the hadron-quark crossover region of the NS we can treat the set $(\rho_B, P)$ in hadronic and QPs as observations, where $P = f(\rho_B)$. The prior distribution on the noisy observations of a zero-mean GP is $P \sim N(0, k(\rho_B, \rho_B) + \sigma^2I)$ with the SE kernel. The target point to be predicted is the pressure $P^*$ at the corresponding $\rho_B^*$ in the crossover density region. Then the joint distribution of training observations $P$ and predictive targets $P^*$ will be obtained from Equation (23). Finally, we can use the conditioning and marginalization as Equations (24)–(26) to find the best hyperparameters and figure out the mean values and the covariance, where the mean values are the pressures in the hadron-quark crossover section that we expect. In addition, the 95% confidence interval can be estimated by the standard deviation, which can be calculated through the diagonal element of the covariance as shown in Equation (21) or Equation (24) to generate the uncertainties of the interpolation method.

### 3. Results and Discussion

For the HP of the NS, the RMF model with the IUFSU parameter set is employed. Its original parameter set and corresponding saturation properties of symmetric nuclear matter are given in Tables 1 and 2, respectively (Fattoyev et al. 2010). There is an additional coupling term between the $\omega$ and $\rho$ mesons compared to the other RMF parameter sets, $\Lambda_6$, which will influence the density dependence of the symmetry energy. The slope of the symmetry energy from the IUFSU model is just 47.165 MeV. It is much smaller than those from the NL3 or TM1 parameter set, which are around 110 MeV.

Around the saturation density of the symmetry nuclear matter, $\rho_0$, the symmetry energy, $E_{\text{sym}}$, can be expanded in a Taylor series as a function of baryon vector density,
\[ \rho_B = \rho_n + \rho_p, \]

\[ E_{\text{sym}}(\rho_B) = E_{\text{sym}}(\rho_0) + \frac{L}{3} \left( \frac{\rho_B - \rho_0}{\rho_0} \right)^2 + \cdots, \tag{27} \]

where the slope of the symmetry energy is \( L = 3\rho_0 \left[ \partial E_{\text{sym}}(\rho)/\partial \rho \right]_{\rho=\rho_0}. \) It characterizes the density dependence of \( E_{\text{sym}} \) and is linearly correlated with the neutron skin thickness \( R_{\text{skin}}^{208} \text{Pb} \). However, the uncertainty in the present measurement, such as PREX experiments, (Abrahamyan et al. 2012; Adhikari et al. 2021), prevents us from inferring the slope of the symmetry energy. In order to explore the influence of \( L \) on neutron-rich systems, several new parameter sets based on the original IUFSU set are developed, which can keep the isoscalar properties of nuclear matter and finite nuclei from the IUFSU set, and change their isovector properties by controlling the strengths of \( g_\rho \) and \( \Lambda_v \) (Bao et al. 2014a).

The parameters, \( g_\rho \) and \( \Lambda_v \), based on the IUFSU model with \( L \) from 50 MeV to 110 MeV are listed in Table 3 (Bao et al. 2014a; Hu et al. 2020). The parameters \( g_\rho \) and \( \Lambda_v \) are adjusted, where \( L \) is a given value at saturation density and \( E_{\text{sym}} \) is fixed at the nuclear subsaturation density of \( \rho = 0.11 \text{ fm}^{-3} \). These \( L \) satisfy the recent constraints, \( 42 \leq L \leq 117 \text{ MeV} \) from the SrRIT Collaboration (Esteé et al. 2021) and \( L = 106 \pm 37 \text{ MeV} \) obtained by using the strong correlations between \( R_{\text{skin}}^{208} \) and \( L \) within the covariant energy density functional from PREX-II (Reed et al. 2021). Other parameters are completely the same as those in the original IUFSU set given in Table 1. The symmetry energies at the saturation point and the corresponding \( R_{\text{skin}}^{208} \) from these parameter sets also satisfy the constraints from PREX-II, \( E_{\text{sym}}(\rho_0) = 38.1 \pm 4.7 \text{ MeV} \) (Reed et al. 2021), and \( R_{\text{skin}}^{208} = 0.283 \pm 0.071 \text{ fm} \) (Adhikari et al. 2021).

### Table 2

| \( \rho_0 \text{[fm}^{-3}] \) | \( E_0 \text{[MeV]} \) | \( K \text{[MeV]} \) | \( E_{\text{sym}} \text{[MeV]} \) | \( L \text{[MeV]} \) |
|-----------------|--------|--------|----------------|--------|
| IUFSU           | 0.155  | −16.397| 230.749        | 31.336 | 47.165 |

**Note.** \( E_0, K, E_{\text{sym}} \) and \( L \) are the energy per nucleon, incompressibility, symmetry energy, and the slope of symmetry energy at saturation density \( \rho_0 \), respectively.

### Table 3

Parameters \( g_\rho \) and \( \Lambda_v \) Generated From the IUFSU Model for Different Slope \( L \) at Saturation Density \( \rho_0 \) with Fixed Symmetry \( E_{\text{sym}} = 26.78 \text{ MeV} \) at \( \rho = 0.11 \text{ fm}^{-3} \)

| \( \rho_0 \text{[MeV]} \) | \( g_\rho \) | \( \Lambda_v \) | \( E_{\text{sym}}(\rho_0) \text{[MeV]} \) | \( R_{\text{skin}}^{208} \text{[fm]} \) |
|-----------------|--------|--------|----------------|--------|
| 50.0            | 12.8202| 0.0420 | 31.68          | 0.1739 |
| 70.0            | 10.3150| 0.0220 | 33.94          | 0.2278 |
| 90.0            | 9.3559 | 0.0098 | 35.74          | 0.2571 |
| 110.0           | 8.8192 | 0.0011 | 37.27          | 0.2770 |

**Note.** \( E_{\text{sym}}(\rho_0) \) and \( R_{\text{skin}}^{208} \) are the symmetry energy at saturation density, and the neutron skin thickness of \( ^{208}\text{Pb} \), respectively (Bao et al. 2014a).

### Figure 1

The interpolated pressure between the IUFSU model with \( L = 70 \) in the HP and the NJL model with \( G_V = G_S \) in the QP with the GPR method.

### Table 4

NS Properties for the IUFSU Model with Different \( L \)

| \( L \text{[MeV]} \) | \( M_{\text{max}}/M_e \) | \( R_{\text{max}} \text{[km]} \) | \( \rho_{\text{max}} \text{[fm}^{-3}] \) | \( R_{\text{i}} \text{[km]} \) | \( \rho_{1.4} \text{[fm}^{-3}] \) | \( \Lambda_{1.4} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 50.0            | 1.9387          | 11.1548         | 12.0408         | 13.4080         | 0.4330          | 512             |
| 70.0            | 1.9365          | 11.2419         | 1.0418          | 13.0800         | 0.4285          | 532             |
| 90.0            | 1.9447          | 11.4126         | 1.0192          | 13.5400         | 0.4158          | 604             |
| 110.0           | 1.9853          | 11.7106         | 0.9651          | 13.8622         | 0.3862          | 735             |

Considering an NS consisting of the pure hadronic matter, its properties, such as the maximum masses, \( M_{\text{max}} \), the corresponding radius, \( R_{\text{max}} \), the central density, \( \rho_{\text{max}} \), the radius of a 1.4\( M_e \) NS, \( R_{1.4} \), the density of a 1.4\( M_e \) NS, \( \rho_{1.4} \), and dimensionless tidal deformability of a 1.4\( M_e \) NS, \( \Lambda_{1.4} \), can be derived by solving the Tolman–Oppenheimer–Volkoff (TOV) equation (Tolman 1939; Oppenheimer & Volkoff 1939) with the EoS of NS matter. These properties of NSs obtained from the above IUFSU parameter sets with different \( L \) are shown in Table 4. We can find that the maximum masses of the NSs from these sets are not sensitive to \( L \) and are all less than 2\( M_e \). On the other hand, with the increase of \( L \), the radii corresponding to the maximum mass are from 11.15–11.71 km, while the radii of 1.4\( M_e \) change in the range of 12.40–13.54 km. The central densities are around 1.0 fm\(^{-3}\). They become about 0.38–0.43 fm\(^{-3}\) for the 1.4\( M_e \) NS (Hu et al. 2020).

For the QP, the HK parameter set of the SU(3) NJL model was adopted (Hatsuda & Kunihiro 1994) with \( \Lambda = 631.4 \text{ MeV} \), \( G_A^2 = 1.835 \), \( G_D^2 = 92.9 \), \( m_{ud} = 5.5 \text{ MeV} \), and \( m_s = 135.7 \text{ MeV} \), where \( \Lambda \) is the three-momentum cutoff. In the SU(3) NJL model, the vector coupling \( G_V \) has not been well determined. It has a similar magnitude to the scalar coupling scale \( G_S \sim G_V \sim 2G_S \) (Bratovic et al. 2013; Masuda et al. 2013b) in order to explain the lattice results on the curvature of the linear chiral restoration at zero density. The studies on the QCD phase diagram suggest that it could be comparable to or even larger than \( G_S \) (Lourenço 2012). Therefore, its value is chosen as \( G_V = 1.0, 1.4, \) and 1.8\( G_S \) in this work to study its influences on quark matter since the vector repulsion plays an important role in stiffening the
EoS. With the increase of $G_V$, stiffer EoSs can be obtained and generate massive compact stars.

In the hadron-quark crossover region, the interpolation method is not unique. At least, there are three choices to connect the hadronic and QPs for the crossover EoSs, which are performed in the $P - \rho_B$, $\varepsilon - \rho_B$, or $P - \mu_B$ plane (Masuda et al. 2013a; Kojo et al. 2015; Baym et al. 2018). In this work, we use the $P - \rho_B$ combination, while the QP is described by the conventional NJL model including the vector coupling term, $G_V$. Actually, we previously attempted the other two combinations of $P - \mu_B$ and $\varepsilon - \rho_B$ before. However, the results were not reasonable for the EoSs of massive NSs with larger vector coupling strengths $G_V$. Therefore, the $P - \rho_B$ interpolation is considered in the present work. An example of $P - \rho_B$ interpolation between the HP and QP with the GPR method is shown in Figure 1. The density range of the interpolation region, which is also called the crossover window, is from $0.3$–$0.6$ fm$^{-3}$. The EoS of the HP is provided from the IUFSU model with $L = 70$ MeV, while that of the QP is generated by the NJL model with $G_V = G_S$. The solid curve is the interpolation result with the GPR method, and the shaded area represents the 95% confidence interval of the interpolated pressure. The corresponding energy density, $\varepsilon(\rho_B)$, can be obtained by integrating the thermodynamic relation $\partial \varepsilon / \rho_B = (P + \varepsilon) / \rho_B$ numerically with a given initial value, $\varepsilon_L$, following the idea in Masuda et al. (2013a, 2013b). This method guarantees thermodynamic consistency in the present calculation.

When modeling phase transitions in NSs, the conventional choice for the apparent density of the hadron-quark crossover is
$\rho_L = (2-3)\rho_0$, below which the hadronic phase exists. Similarly, the deconfinement density of nucleons should happen in the high-density region. The reasonable choice for the closure density of the hadron-quark crossover is around $\rho_U \sim (4-7)\rho_0$ (Baym 1979; Celik et al. 1980). To investigate the effects of the crossover window between HPs and QPs on the EoSs and NS structure, the crossover windows are fixed as $(\rho_L, \rho_U) = (0.3, 0.6), (0.3, 0.8), (0.4, 0.6)$, and $(0.4, 0.8)$fm$^{-3}$ in this work. In addition, the hyperonic star including the hyperons with different nonlinear and density-dependent models have been investigated in the framework of the RMF model (Huang et al. 2022), which include the IUFSU parameter set. The threshold density of the first hyperon for the IUFSU model is about 0.38 fm$^{-3}$. The inclusion of hyperons will soften the EoS of the HP. When the hadron-quark crossover phase starts from $\rho_L = 0.3$ fm$^{-3}$, the hyperon cannot appear in the HP, while the $\rho_U$ is changed to 0.4 fm$^{-3}$, the hyperon only can exist between 0.38 and 0.40 fm$^{-3}$. It may slightly make the HP EoS softer. Therefore, the hyperon effect on the crossover EoS and the corresponding neutron structure can be negligible.

In Figure 2, the pressure as a function of energy density is plotted in the case of $P - \rho_B$ interpolation with the GPR method between the IUFSU model with $L = 50, 70, 90$, and 110 MeV, and the NJL model with $G_V = 1.0, 1.4$, and 1.8$G_S$. The crossover windows are chosen to be $(\rho_L, \rho_U) = (0.3, 0.6), (0.3, 0.8), (0.4, 0.6)$, and $(0.4, 0.8)$fm$^{-3}$, and can be inferred from the density range of the shaded areas. The EoSs of the NS crust as the nonuniform matter were calculated with the corresponding IUFSU parameterizations in the framework of the self-consistent Thomas–Fermi approximation (Bao & Shen 2015). In the core region of an NS, the EoSs of the uniform matter in the low-density region are mainly determined.
by different \( L \) in the HP. When hadron-quark crossover appears, the pressure and energy density are strongly dependent on the crossover windows and the vector coupling strength of quarks. A larger \( G_V \) can generate a stiffer EoS. Furthermore, the uncertainties in the interpolation process can also be obtained. The shaded areas are 95\% confidence intervals. The uncertainty increases rapidly with pressure increasing and it is about 20\% for \( G_V = 1.8 G_S \) to the pressure at the upper limit of the crossover window, \( \rho_U \).

The speed of sound, \( c_s^2 = \frac{\sqrt{dP/d\epsilon}}{\epsilon} \), in NS matter is one of the measurements to quantify the stiffness of EoS. In Figure 3, \( c_s^2 \) of the interpolated EoS with different \( L \) and \( G_V \) as a function of density are plotted. A discontinuity of \( c_s^2 \) at the lower limit density, \( \rho_L \), is caused by the occurrence of hadron-quark crossover. It steeply increases at the beginning crossover region and approaches a maximum value around 1.0 at \( \rho = 0.6 \text{ fm}^{-3} \) and then reduces to around 0.6 at the high-density region. Furthermore, it is found that the vector strength between quarks cannot be too strong, otherwise, the speed of sound of NS matter may exceed 1.0, which will violate the causality. When the crossover window is fixed, the slope of symmetry energy has a few influences on the \( c_s \). Its magnitude will be obviously changed with the crossover window at a fixed \( L \). Since \( c_s \) largely increases as \( \rho_L \) increases, the hadron-quark crossover in an NS cannot occur later than baryon density, 0.4 \( \text{fm}^{-3} \) with a stiffer EoS, which is consistent with recent Bayesian analysis under the assumption of first-order hadron-QP transition and the join constraints from NICER and GW170817 (Li et al. 2021). Recently, we also explored the structured hadron-quark mixed phase with the energy minimization method, Gibbs.

Figure 4. Radius vs. NS mass with the EoS from the GPR interpolation method. The parameter sets in the HP and QP, and the crossover windows are the same as in Figure 2.
construction, and Maxwell one, where the onset density of the quark is strongly dependent on the quark interactions (Ju et al. 2021a, 2021b).

The properties of the NS, such as the mass–radius relation, can be provided after substituting the EoS to a TOV equation. In Figure 4, the mass–radius relations are shown within the above EoSs to describe the hadron-quark crossover with the GPR method. In addition, various constraints from the observables of massive NSs, PSR J1614-2230 (1.928 ± 0.017M⊙) (Demorest et al. 2010; Antoniadis et al. 2013), PSR J0348+0432 (2.01 ± 0.04M⊙) (Antoniadis et al. 2013), the mass and radius simultaneous measurement of PSR J0740+6620 (a mass of 2.072$^{+0.067}_{-0.066}$M⊙ with a radius 12.39^{+1.30}_{-1.24}$ km; Riley et al. 2021) and PSR J0030+0451 (a mass of 1.34^{+0.11}_{-0.10}$M⊙ with a radius 12.71^{+1.41}_{-0.10}$ km; Riley et al. 2019) are shown. The radius of the NS at 1.4M⊙ extracted from GW170817, $R_{1.4} = 11.9 ± 1.4$ km (Abbott et al. 2018) is also considered.

With the introduction of the hadron-quark crossover, the present EoSs can easily describe the 2M⊙ massive NS, while they are just around 1.95M⊙ with the pure hadronic matter from the IUFSU parameter sets as shown before. The maximum mass of the NS is strongly dependent on $\rho_U$ of the crossover window and $G_V$ of quark interaction. It can be heavier than 2.55M⊙ with $G_V = 1.8G_S$ due to the stiffer EoS. Therefore, the secondary compact object in GW190814 may be an NS containing the hadron-quark crossover. If the pure QP appears later than 0.6 fm$^{-3}$, the maximum mass of the NS in the mean value will be less than 2.5M⊙ since the hadronic

Figure 5. Tidal deformability as a function of NS mass with GPR interpolation. The parameter sets in the HP and QP, and the crossover windows are the same as in Figure 2.
matter can provide more contributions to the compositions of the NS. The radius of the NS is very sensitive to the slope of symmetry energy $\Lambda$. It will become larger with the $L$ increasing not only at $1.4M_\odot$ but also at $2M_\odot$. The cases of $L = 110 \, \text{MeV}$ are almost excluded by the constrains of GW detection, the GW170817 event. The hadron-quark crossover can appear in the lower-mass NS around $1.0M_\odot$ with $\rho_L = 0.3 \, \text{fm}^{-3}$, while it occurs near or even above $1.4M_\odot$ with $\rho_L = 0.4 \, \text{fm}^{-3}$. The uncertainty of the GPR method will take the $1.6\% \sim 6.4\%$ errors on the radii of an NS at $2M_\odot$. Once the uncertainty is considered, more EoSs can generate a heavier NS, whose mass is larger than $2.5M_\odot$.

The dimensionless tidal deformability, $\Lambda$ of NS, is extracted by the GW detection in BNS mergers, which can denote the deformation of an NS in an external gravitational field. $\Lambda$ is related to the mass, the radius of an NS, with the definition $\Lambda = (2/3)k_2[(c^2/G)(R/M)^2]$, where $k_2$ is the second Love number (Hinderer 2008; Hinderer et al. 2010) and $R$ is the stellar radius, and $M$ the stellar mass. The GW event of the BNS merger, GW170817, provides the constraint on $\Lambda$ at $M_{1.4}$ with $\Lambda_{1.4} = 190^{+120}_{-100}$ (Abbott et al. 2018). In Figure 5, the $\Lambda$ as a function of NS mass from the EoSs with the GPR interpolation is shown. In the NS consisting of pure hadronic matter, $\Lambda$ is very sensitive to $L$ and there is a strong linear relationship between them (Hu et al. 2020). In the present framework including the hadron-quark crossover, the dimensionless tidal deformability is dependent on not only the $L$ but also the crossover window. The $\Lambda$ is around 475–550 at $L = 50 \, \text{MeV}$, and around 540–550 at $L = 70 \, \text{MeV}$, which satisfies the measurement of GW170817. The larger slope of symmetry energy can generate a bigger $\Lambda$, which will increase $30\% \sim 40\%$ from $L = 50 \, \text{MeV}$ to $L = 110 \, \text{MeV}$ at the same crossover window. If the hadron-quark crossover appears earlier, the tidal deformability with the uncertainty will reduce for the light NS.

Properties of NSs, i.e., the maximum mass ($M_{\text{max}}$), the central density ($\rho_c$), the radius ($R_{1.4}$), and dimensionless tidal deformability ($\Lambda_{1.4}$) at $M_{1.4}$ with a different slope of symmetry energy $\Lambda$ for HP, different vector coupling strength $G_V$ for QP, and various crossover windows are summarized in Tables 5 and 6. The maximum masses of NSs from present EoSs are around $2.20 \sim 2.56M_\odot$. The corresponding radii are $11.00 \sim 12.09 \, \text{km}$. The radii at $1.4M_\odot$ are in the range between $12.58$ and $13.58 \, \text{km}$. All of them are obviously influenced by the crossover windows. If the mean speeds of the sound of these EoSs exceed the speed of light, they are shown in the last column.

### 4. Summary and Perspectives

The interpolated EoS to describe the hadron-quark crossover in the NS was calculated with the GPR method by connecting the hadronic matter at low density and quark matter at high density. The hadronic matter was described by the RMF model, where the isoscalar parameters were fixed and the isovector ones were readjusted to study the symmetry energy effects. The NJL model was employed to study the quark matter. A vector interaction between quarks was introduced to generate an additional repulsive contribution.

### Table 5

| $\Lambda_{1.4}$ | $\nu^2$ | $\Lambda_{1.4}$ | $\nu^2$ |
|-----------------|----------|-----------------|----------|
|                 |          |                 |          |

Note. The crossover windows are chosen to be $(\rho_L, \rho_U) = (0.3, 0.6)$, and $(0.3, 0.8) \, \text{fm}^{-3}$. 

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The effects of the slope of symmetry energy, crossover windows, and vector interaction of the NJL model on the EoSs and properties of NS were systematically investigated. A larger slope of symmetry energy generates a bigger NS radius and properties of NS were clearly shown in the GPR method in comparison with the hadron-quark crossover. The crossover picture avoids the complicated mechanism of phase transition and provides a way to generate a massive NS. It is very hard to estimate the uncertainties of results and has arbitrariness in the previous interpolation methods. The GPR method as a nonparametric Bayesian approach can calculate the uncertainty of the predictions with the training data. Therefore, the EoSs are generated by the GPR method and present conclusions are independent of interpolation methods. The python code for the EoSs of hadron-quark crossover with the GPR method can be found on GitHub3 and a version (v1.1.1) deposited on Zenodo: doi: 10.5281/zenodo.6802604.

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Note. The crossover windows are chosen to be $(\rho_c, \rho_v) = (0.4, 0.6)$, and $(0.4, 0.8)$fm$^{-3}$.

| $(\rho_c, \rho_v$) = (0.4, 0.6)fm$^{-3}$ | $L = 50$ MeV | $L = 70$ MeV | $L = 90$ MeV | $L = 110$ MeV |
|----------------------------------|--------------|--------------|--------------|---------------|
| $G_V = 1.0G_S$ | $G_V = 1.4G_S$ | $G_V = 1.8G_S$ | $G_V = 1.0G_S$ | $G_V = 1.4G_S$ | $G_V = 1.8G_S$ |
| $M_{max}$ (M$_\odot$) | 2.2941 | 2.4336 | 2.5437 | 2.2903 | 2.4279 | 2.5338 |
| $R_{max}$ (km) | 11.4886 | 11.5908 | 11.6991 | 11.5793 | 11.6816 | 11.7601 |
| $\rho_{max}$ (fm$^{-3}$) | 0.9134 | 0.8797 | 0.8420 | 0.9141 | 0.8705 | 0.8422 |
| $R_{1.4}$ (km) | 12.4105 | 12.4125 | 12.4155 | 12.7457 | 12.7485 | 12.7452 |
| $\rho_{1.4}$ (fm$^{-3}$) | 0.4202 | 0.4125 | 0.4126 | 0.4152 | 0.4116 | 0.4094 |
| $\Lambda_{1.4}$ | $>c^2$ | $>c^2$ | $>c^2$ | $>c^2$ | $>c^2$ | $>c^2$ |

This table shows the properties of NSs obtained from the GPR interpolation between the IUFSU sets for hadronic matter with $L = 50$, 70, 90, and 110 MeV, and the NJL models for quark matter with $G_V = 1.0$, 1.4, and 1.8G$_S$.
