QUANTUM SUPERGROUPS OF $GL(n|m)$ TYPE: DIFFERENTIAL FORMS, KOSZUL COMPLEXES AND BEREZINIAN

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Abstract. We introduce and study the Koszul complex for a Hecke $R$-matrix. Its cohomologies, called the Berezinian, are used to define quantum superdeterminant for a Hecke $R$-matrix. Their behaviour with respect to Hecke sum of $R$-matrices is studied. Given a Hecke $R$-matrix in $n$-dimensional vector space, we construct a Hecke $R$-matrix in $2n$-dimensional vector space commuting with a differential. The notion of a quantum differential supergroup is derived. Its algebra of functions is a differential coquasitriangular Hopf algebra, having the usual algebra of differential forms as a quotient. Examples of superdeterminants related to these algebras are calculated. Several remarks about Woronowicz’s theory are made.

0.1. Short description of the paper.

0.1.1. In this paper we will be concerned with differential Hopf algebras generated by sets of matrix elements. We start (Section 1) by giving a construction of such algebras, generalising the construction [31, 27] of a bialgebra generated by a single set of matrix elements (without differential) as a universally coacting bialgebra preserving several algebras generated by a set of coordinates. In our generalisation the data are morphisms in the category of graded differential complexes.

0.1.2. Given a Hecke $R$-matrix for a vector space $V$, we construct in this paper another Hecke $R$-matrix $\mathcal{R}$ for the space $W = V \oplus V$ equipped with the differential $d = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the grading $\sigma : W \to W$, $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The matrix $\mathcal{R}$ is distinguished by the property

$$\mathcal{R}(d \otimes 1 + \sigma \otimes d) = (d \otimes 1 + \sigma \otimes d)\mathcal{R}.$$ 

0.1.3. The algebra $H$ of functions on the quantum supergroup constructed from $\mathcal{R}$ is a $\mathbb{Z}$-graded differential coquasitriangular Hopf algebra (Section 2). In brief, it defines a differential quantum supergroup. A quotient of $H$ is the $\mathbb{Z}_{\geq 0}$-graded differential Hopf algebra $\Omega$ of differential forms defined via $R$ in [25, 26, 30, 35].

The classical version ($q = 1$) of this construction is: take a vector space $V$, add to it another copy of it with the opposite parity and consider the general linear supergroup of the $\mathbb{Z}/2$-graded space so obtained.

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0.1.4. We introduce Koszul complexes for Hecke $R$-matrices in Section 3. They are $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-graded algebras with two differentials, $D$ of degree $(1,1)$ and $D'$ of degree $(-1,-1)$. We calculate their anticommutator, called the Laplacian. The cohomology space of $D$ is called the Berezinian. It generalizes the determinant, coinciding in the even case with the highest exterior power of $V$. The behaviour of Koszul complexes and Berezinians with respect to the Hecke sum [16] is described: we prove that the Berezinian of a Hecke sum is the tensor product of Berezinians.

0.1.5. The Berezinian is used to define the quantum superdeterminant in Section 4. We calculate the superdeterminant in several examples. In particular, in the algebra $\Omega$ of differential forms on the standard quantum $GL(n|m)$ the superdeterminant equals 1. This confirms the idea that there are no central group-like elements in $\Omega$, explaining why it has not been possible to construct a differential calculus on special linear groups with the same dimension as the classical case.

0.1.6. We make several remarks on Woronowicz’s theory [32] in Section 5. In particular, each first order differential calculus is extended to a differential Hopf algebra.

0.2. Notations and conventions. $k$ denotes a field of characteristic 0. In this paper a Hopf algebra means a $k$-bialgebra with an invertible antipode. Associative comultiplication is denoted by $\Delta x = x_{(1)} \otimes x_{(2)}$, the counit by $\varepsilon$, the antipode in a Hopf algebra by $\gamma$. If $H$ is a Hopf algebra, $H^{\text{op}}$ denotes the same coalgebra $H$ with opposite multiplication, $H_{\text{op}}$ denotes the same algebra $H$ with the opposite comultiplication.

When $X$ is a graded vector space, $\hat{a}$ denotes the degree of a homogeneous element $x_a \in X$.

The braiding in a braided tensor category $\mathcal{C}$ [7] (e.g. in the category of representations of a quasitriangular Hopf algebra) is denoted by $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$, where $X, Y \in \text{Ob} \, \mathcal{C}$. By definition, the maps $1_{V^{\otimes k-1}} \otimes c_{V,V} \otimes 1_{V^{\otimes n-k-1}} : V^{\otimes n} \rightarrow V^{\otimes n}$ obey the braid group relations. Given $\sigma \in \mathfrak{S}_n$, we denote by $(c)_\sigma$ and $(c^{-1})_\sigma$ the maps $V^{\otimes n} \rightarrow V^{\otimes n}$ coming from the liftings of $\sigma$ to the elements of the braid group representing reduced expressions of $\sigma$. The first case is a word in the generators $1 \otimes c \otimes 1$, the second is a word in $1 \otimes c^{-1} \otimes 1$.

We often use tangles to describe maps constructed from an $R$-matrix and pairings. In the conventions of [13] we denote
diagram{c: X \otimes Y \rightarrow Y \otimes X}
diagram{c^{-1}: X \otimes Y \rightarrow Y \otimes X}
0.3. Preliminaries. We recall some definitions from [10] and some results from
[12].

**Definition 0.1.** Let \( T : A \otimes B \to B \otimes A \), \( a_i \otimes b_j \mapsto T^{kl}_{ij} b_k \otimes a_l \), be a linear map,
written in terms of bases \((a_i),(b_j)\) of the finite-dimensional vector spaces \( A, B \). Let \( A^*, B^* \) be the spaces of linear functionals on \( A, B \) with the dual bases \((a^i),(b^j)\).

Define linear maps \( T^\sharp, T^\flat \) as

\[
T^\sharp : B \otimes A^* \to A^* \otimes B, \quad b_i \otimes a^j \mapsto T^{ji}_{kl} a^k \otimes b_l^k,
\]

\[
T^\flat : B^* \otimes A \to A \otimes B^*, \quad b^i \otimes a_j \mapsto T^{ki}_{lj} a^k \otimes b^l i.
\]

These maps are characterised by the properties

\[
(\text{ev} \otimes 1) (1 \otimes T^\sharp) = (1 \otimes \text{ev}) (T \otimes 1) : A \otimes B \otimes A^* \to B
\]

\[(1 \otimes \text{ev}) (T^\flat \otimes 1) = (\text{ev} \otimes 1) (1 \otimes T) : B^* \otimes A \otimes B \to A,
\]

where \( \text{ev} : A \otimes A^* \to k \) and \( \text{ev} : B^* \otimes B \to k \) are the usual pairings given by evaluation.

The operations \( \sharp \) and \( \flat \) are inverse to each other, and \( T^{\sharp\sharp\sharp\sharp} = T \).

Note that if \( T^\sharp \) is invertible, \( 1 \otimes T \) can be cancelled from an expression like \((U \otimes 1)(1 \otimes T)(S \otimes 1)\) where \( S : A \otimes C \to C \otimes A \) and \( U : C \otimes B \to D \). To be precise, suppose

\[
(U \otimes 1)(1 \otimes T)(S \otimes 1) = W : A \otimes C \otimes B \to D \otimes A
\]

\[
a_i \otimes c_j \otimes b_k \mapsto w_{ijkl} m_i d_k \otimes a_n,
\]

where \((c_i)\) and \((d_i)\) are bases of \( C \) and \( D \), and define

\[
W^\sharp : C \otimes B \otimes A^* \to A^* \otimes D \quad \text{by} \quad b_j \otimes c_k \otimes a^m \mapsto w_{ijk} m_i a^j \otimes d_l.
\]

Then

\[
(1 \otimes U)(S^\sharp \otimes 1) = W^\sharp (1 \otimes T^\sharp^{-1}) : C \otimes A^* \otimes B \to A^* \otimes B \otimes C.
\]

**Definition 0.2.** Let \( \mathcal{C} \) be an abelian monoidal category with an exact monoidal functor to \( k\text{-vect} \), e.g. \( \mathcal{C} = H\text{-mod} \) for some Hopf algebra \( H \). Let \( X \in \text{Ob}\mathcal{C} \). An object \( X^\vee \in \mathcal{C} \) with a non-degenerate pairing and co-pairing

\[
\text{ev} : X \otimes X^\vee \to k, \quad \text{coev} : k \to X^\vee \otimes X
\]
is called a right dual of \( X \) if \( \text{ev}, \text{coev} \in \text{Mor}\mathcal{C} \) are compatible in the standard sense (see e.g. [7]). Similarly, a left dual is an object \( ^\vee X \in \mathcal{C} \) with

\[
\text{ev} : ^\vee X \otimes X \to k, \quad \text{coev} : k \to X \otimes ^\vee X.
\]
When $X$ is an $H$-module, $X^\vee$ and $^\vee X$ are two different $H$-module structures on the space of linear functionals $X^*$. The category is called rigid if each object has left and right duals.

The following theorem was proven in the symmetric monoidal case in [10].

**Theorem 0.1** ([12]). Let $C$ be a rigid braided abelian monoidal category with an exact monoidal functor to $k$-vect, e.g. $C = H$-mod for some quasitriangular Hopf algebra $H$. Denote the braiding for some $A, B \in C$ by

$$R = c_{A,B} : A \otimes B \to B \otimes A.$$  

Then the following braiding isomorphisms are uniquely determined by $R$:

- $c_{A^\vee,B^\vee} = (R^{-1})^\vee : A \otimes B^\vee \to B^\vee \otimes A,$
- $c_{A^\vee,B} = (R^\vee)^{-1} : A^\vee \otimes B \to B \otimes A^\vee,$
- $c_{A,B^\vee} = R^\vee\#: A \otimes B^\vee \to B^\vee \otimes A^\vee,$
- $c_{A,B} = (R^\#)^{-1} : A \otimes B \to B \otimes A,$
- $c_{A^\vee,B} = R^\#: \vee A \otimes B \to B \otimes \vee A,$
- $c_{A,B^\vee} = R^{\#\#} : \vee A \otimes B \to \vee B \otimes \vee A.$

**Theorem 0.2** ([12]). If $R$ is an $R$-matrix, i.e. a map $R : V \otimes V \to V \otimes V$ for finite-dimensional $V$ satisfying the Yang-Baxter relation (braid relation)

$$\label{eq:Yang-Baxter} (R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R) : V \otimes V \otimes V \to V \otimes V \otimes V,$$

with invertible $R$ and $R^\#$, then $R^{-1\#}, R^\#$, $R^\#, R^{-1\#}, R^{\#\#}$ are also invertible.

**Proof.** We regard $V^*$ as a right dual of $V$, with the pairing $ev : V \otimes V^* \to k$ given by the usual evaluation, which we also write as $ev(x \otimes \xi) = \langle x, \xi \rangle$ for $x \in V, \xi \in V^*$. We define a pairing between $V \otimes V$ and $V^* \otimes V^*$ by

$$\langle x \otimes y, \xi \otimes \eta \rangle = \langle x, \eta \rangle \langle y, \xi \rangle.$$  

Then $R^{\#\#}$ is the transpose of $R$ with respect to this pairing, which is non-degenerate, so $R^{\#\#}$ is invertible if $R$ is, and $R^\# = (R^\#)^{\#\#}$ is invertible if $R^\#$ is. Since $R^\# = R^{\#\#}$ while $R^{-1\#} = R^{-1\#\#\#} = (R^{\#\#})^{-1\#}$ and $R^{\#\#}$ satisfies the same conditions as $R$, it remains only to prove that $R^{-1\#}$ is invertible.

According to Theorem 0.1, $R^{\#\#^{-1}} : V^* \otimes V \to V \otimes V^*$ is a braiding isomorphism, and we can regard $V^*$ as a left dual of $V$ with the pairing $lev : V^* \otimes V \to k$ given by $lev = ev \circ R^{\#\#^{-1}}$. The braid relation (0.6) in $V \otimes V \otimes V$ implies

$$\label{eq:braid} (R^{\#\#^{-1}} \otimes 1)(1 \otimes R)(R^{-1\#} \otimes 1) = (1 \otimes R^{\#\#^{-1}})(R \otimes 1)(1 \otimes R^{\#\#^{-1}}) : V \otimes V^* \otimes V \to V \otimes V^* \otimes V.$$  

Multiplying on the left by $ev \odot 1$ and using (0.1) with $T = R^{-1}$,

$$\label{eq:lev} (lev \odot 1)(1 \otimes R)(R^{-1} \odot 1) = 1 \otimes lev : V \otimes V^* \otimes V \to V.$$  

Using (0.5) with $T = R, S = R^{-1\#}$ and $U = lev$, so that $W = 1 \otimes lev$ and $W^\# = lev \odot 1$,

$$\label{eq:lev1} (1 \otimes lev)(R^{-1\#\#} \odot 1) = (lev \odot 1)(1 \otimes R^{\#\#^{-1}}) : V^* \otimes V^* \otimes V \to V^*.$$
Let $N \subset V$ be the null space of $\text{lev}$, so that $\text{lev}(V^* \otimes N) = 0$. Then $V^* \otimes V^* \otimes N$ is annihilated by the left-hand side of the above equation, so from the right-hand side we obtain $R^{q-1}(V^* \otimes N) \subseteq N \otimes V^*$. Since $R^{q-1}$ is non-singular, we have equality. Hence

$$\text{ev}(N \otimes V^*) = \text{ev} \circ R^{q-1}(V^* \otimes N) = \text{lev}(V^* \otimes N) = 0.$$ 

But $\text{ev}$ is a non-degenerate pairing, so $N = 0$; hence $\text{lev}$ is a non-degenerate pairing.

Now return to (0.7) and, by tensoring on the right with $V^*$ and applying $\text{ev}$, write it in the form

$$\text{ev}(\text{lev} \otimes 1 \otimes 1)(1 \otimes R \otimes 1)(R^{-1q} \otimes 1 \otimes 1) = \text{ev}(1 \otimes \text{lev} \otimes 1) : V \otimes V^* \otimes V \otimes V^* \rightarrow k.$$ 

Since $\text{ev}$ and $\text{lev}$ are non-degenerate, the right-hand side is a non-degenerate bilinear form on $V \otimes V^*$ and therefore $R^{-1q}$ is non-singular. \hfill\square

**Definition 0.3.** A Hecke $\tilde{R}$-matrix is an $\tilde{R}$-matrix $R$ satisfying the quadratic equation

$$(R - q)(R + q^{-1}) = 0$$

for some $q \in k^*$, $q^2 \neq -1$, and such that $R^q$ is invertible.

## 1. A construction for graded Hopf algebras

1.1. **Universally coacting bialgebras.** Let $\mathcal{O}$ be a collection of finite-dimensional $\mathbb{Z}/2 \times \mathbb{Z}$-graded vector spaces, and let $\mathcal{M} = \{f : X_{i_1} \otimes \ldots \otimes X_{i_k} \rightarrow X_{j_1} \otimes \ldots \otimes X_{j_m}, \}$ $(X_{i_1}, X_{j_m} \in \mathcal{O})$ be a family of linear maps, preserving degree or changing it by $(1, 0)$.

Let $X_0 = k$ be a chosen one dimensional space of degree $(0,0)$.

Consider a category $\mathcal{H}$ of $\mathbb{Z}/2 \times \mathbb{Z}$-bialgebras $H$ together with grading-preserving coactions $(\delta_X : X \rightarrow X \otimes H)_{X \in \mathcal{O}}$ such that

(a) the coaction on $X_0$ is given by $\delta_0 : k \rightarrow k \otimes H, 1 \mapsto 1 \otimes 1$;

(b) any $f \in \mathcal{M}$ is a $H$-comodule homomorphism.

The morphisms $(H, \delta_X)_{X \in \mathcal{O}} \rightarrow (H', \delta'_X)_{X \in \mathcal{O}}$ are $\mathbb{Z}/2 \times \mathbb{Z}$-graded bialgebra maps $g : H \rightarrow H'$ such that

$$\delta'_X = (X \xrightarrow{\delta_X} X \otimes H \xrightarrow{1 \otimes g} X \otimes H'). \quad (1.1)$$

**Theorem 1.1** (Bialgebra construction). There exists a universal coacting bialgebra $(H, \delta_X) = \text{initial object of the category } \mathcal{H}$. That is, for any bialgebra $(H', \delta'_X) \in \mathcal{H}$ there is exactly one bialgebra map $g : H \rightarrow H'$ with the property (1.1).

**Proof.** Choose a graded basis $(x_a)$ in each space $X \in \mathcal{O}$. Introduce a $\mathbb{Z}/2 \times \mathbb{Z}$-graded coalgebra $C = \bigoplus_{X \in \mathcal{O}} (\text{End}_k X)^* = k\{t_X^a_b\}$ with coaction

$$X \rightarrow X \otimes C, \quad x_b \mapsto x_a \otimes t_X^a_b$$
on each $X_i$. Then the tensor algebra $T(C)$ is a $\mathbb{Z}/2 \times \mathbb{Z}$-graded bialgebra [9] coacting on each $X_i$. With each map

$$f : X^1 \otimes \ldots \otimes X^k \rightarrow Y^1 \otimes \ldots \otimes Y^m \in \mathcal{M},$$
n $X^i \in \mathcal{O}, Y^j \in \mathcal{O}$, is associated a subspace $\text{Rel}(f) \subset T(C)$ as follows. Choosing graded bases $(x_a^i) \subset X^i, (y_b^j) \subset Y^j$ and writing corresponding matrix elements as
t_{X^e_a}, t_{Y^f_d}$ we define this $\mathbb{Z}/2 \times \mathbb{Z}$-graded subspace as
\[
\text{Rel}(f) = k \left\{ (-1)^{\sum_i (a_i - c_i)} f_{c_1 \ldots c_k} x^{a_1} \cdots x^{a_k} - (-1)^{\sum_i (b_i - a_i)} d_i y^{b_1} \cdots y^{b_m} f_{a_1 \ldots a_k} \right\},
\]
where $\hat{g} \in \mathbb{Z}/2 \times \mathbb{Z}$ denotes the degree of a basic vector indexed by $g$ (the product $\hat{g}\hat{h}$ is the inner product $(\mathbb{Z}/2 \times \mathbb{Z}) \times (\mathbb{Z}/2 \times \mathbb{Z}) \to \mathbb{Z}/2)$. A more readable form for Rel$(f)$ will be given later (Theorem 1.4).

The bialgebra $T(C)$ is a universal bialgebra coacting on each $X_i$. In particular, for any $H' \in \mathcal{H}$ there is unique bialgebra map $h : T(C) \to H'$, preserving coactions in the sense of (1.1). For any such $H'$, we get $h(\text{Rel}(f)) = 0$ and this equation is equivalent to the condition that $f$ is a $H'$-comodule morphism.

Define now an algebra
\[
H = T(C)/(t_{k^1} - 1, \text{Rel}(f))_{f \in \mathcal{M}}.
\]
As the subspaces $k\{t_{k^1} - 1\}$ and Rel$(f)$ are coideals of $T(C)$ for any $f \in \mathcal{M}$, the algebra $H$ is a bialgebra. By the above consideration $H$ belongs to $\mathcal{H}$ and is an initial object.

This theorem is due essentially to Takeuchi [31] (except for the grading) where the case $\mathcal{O} = \{X\}$, $\mathcal{M} = \{P_k : X \otimes X \to X \otimes X\}$ is considered ($P_k$ are projections and $\sum P_k = 1$).

1.2. Hopf algebra case. Let $\mathcal{O}$ and $\mathcal{M}$ be as in Section 1.1. Now we discuss the question when the bialgebra $H$ constructed in Theorem 1.1 is a Hopf algebra.

**Theorem 1.2** (Hopf algebra construction). Assume that for each $X \in \mathcal{O}$ there exist compositions of tensor monomials in maps from $\mathcal{M}$, i.e.
\[
f = 1 \otimes f_1 \otimes 1 \cdots 1 \otimes f_n \otimes 1 : X \otimes (Y_1 \otimes \ldots \otimes Y_k) \to k,
g = 1 \otimes g_1 \otimes 1 \cdots 1 \otimes g_m \otimes 1 : (W_1 \otimes \ldots \otimes W_l) \otimes X \to k,
\]
with $f_i \in \mathcal{M}$, which are pairings, non-degenerate in the argument $X$. Assume also existence of compositions of tensor monomials in maps from $\mathcal{M}$
\[
h = 1 \otimes h_1 \otimes 1 \cdots 1 \otimes h_a \otimes 1 : (U_1 \otimes \ldots \otimes U_c) \otimes X,
\]
\[
j = 1 \otimes j_1 \otimes 1 \cdots 1 \otimes j_b \otimes 1 : X \otimes (V_1 \otimes \ldots \otimes V_d),
\]
which are non-degenerate in $X$ in the sense that the induced linear maps $(U_1 \otimes \ldots \otimes U_c)^\vee \to X$ and $(V_1 \otimes \ldots \otimes V_d)^\vee \to X$ are surjective. Then the universal bialgebra from Theorem 1.1 has an invertible antipode.

**Proof.** Extend $\mathcal{O}$ and $\mathcal{M}$, adding for each $X$ new spaces $A, B$ to $\mathcal{O}$ and new maps
\[
Y_1 \otimes \ldots \otimes Y_k \to Y_1 \otimes \ldots \otimes Y_k/\text{Ann}_{\text{right}} f \equiv A, \quad X \otimes A \to k,
\]
\[
W_1 \otimes \ldots \otimes W_l \to W_1 \otimes \ldots \otimes W_l/\text{Ann}_{\text{left}} g \equiv B, \quad B \otimes X \to k.
\]
It is easy to see that the resulting bialgebra $H$ will not change. Also we add minimal subspaces together with inclusions
\[
C \hookrightarrow U_1 \otimes \ldots \otimes U_c, \quad D \hookrightarrow V_1 \otimes \ldots \otimes V_d
\]
such that $h$ and $j$ factorize through
\[ k \mapsto C \otimes X, \quad k \mapsto X \otimes D \]
(and we add also these maps to $\mathcal{M}$). Again the algebra $H$ will not change.

But now for each $X \in \mathcal{O}$ there are $X_1^\vee, X_2^\vee, X_3^\vee, X_4^\vee \in \mathcal{O}$ together with non-degenerate pairings and co-pairings
\[ \text{ev} : X \otimes X_1^\vee \to k, \quad \text{ev} : X_2^\vee \otimes X \to k, \]
\[ \text{coev} : k \to X_3^\vee \otimes X, \quad \text{coev} : k \to X \otimes X_4^\vee. \]
The theorem follows from a proposition, proven in [12] in the even case and easily generalized to the $\mathbb{Z}/2$-graded case:

**Proposition 1.3.** Let data $\mathcal{O}, \mathcal{M}$ be such that for each $X \in \mathcal{O}$ there are $X_1^\vee, X_2^\vee, X_3^\vee, X_4^\vee \in \mathcal{O}$ as above. Then the universal bialgebra $H$ has an invertible antipode $\gamma$. Denote the coaction of $H$ by $x_i \mapsto x_j t_{X_i^j}, \ x^k \mapsto x^j \otimes t_{X^i}, m \mapsto n x \otimes t_{X^m}$, where $(x_i), (x_k), (m x)$ are bases of $X, X^\vee = X_1^\vee$ or $X_3^\vee, \ ^\vee X = X_2^\vee$ or $X_4^\vee$. Then the antipode and its inverse satisfy
\[ \gamma(t_{X^i}) = (-1)^{j(i-j)} t_{X^j}, \]
\[ \gamma^{-1}(t_{X^i}) = (-1)^{j(i-j)} t_{X^j}. \]

\[ \square \]

### 1.3. Bialgebras in rigid monoidal categories.

The ideas of Section 1.1 can be applied not only to vector spaces, or graded vector spaces, or to vector spaces with additional structures. It can be generalized to symmetric (or even braided) closed monoidal categories. See e.g. [15, 22, 34]. Having in mind applications to quantum differential calculus, we describe one such scheme following [13, 14].

Let $\mathcal{V}$ be a noetherian abelian symmetric monoidal rigid category. (We can even assume it braided instead of symmetric, but this generalization will not be used here. See [14] for applications in the braided case.) Let $k = \text{End}_\mathcal{V} I$ be a field, where $I \in \text{Ob} \mathcal{V}$ is a unit object; then $\mathcal{V}$ is $k$-linear. Consider the category $\hat{\mathcal{V}} = \text{ind-} \mathcal{V}$ of $k$-linear left exact functors $\mathcal{V}^{\text{op}} \to k\text{-Vect}$. It is known that $\hat{\mathcal{V}}$ is a $k$-linear abelian symmetric monoidal closed category. The category $\mathcal{V}$ can be viewed as a full subcategory of $\hat{\mathcal{V}}$ via the representing functor $h : \mathcal{V} \to \hat{\mathcal{V}}, \ X \mapsto h_X, \ h_X(Y) = \text{Hom}_\mathcal{V}(Y, X)$.

Take a family $\mathcal{O}$ of $\mathbb{Z}/2$-graded objects of $\mathcal{V}$ and a family $\mathcal{M} = \{ f : X_{i_1} \otimes \ldots \otimes X_{i_k} \to X_{j_1} \otimes \ldots \otimes X_{j_m} \}$ of morphisms of $\mathcal{V}$, preserving the grading or changing it by 1. Denote by $\mathcal{C}$ the subcategory of $\mathcal{V}$ consisting of all tensor products $X_{i_1} \otimes \ldots \otimes X_{i_k}$ with all tensor polynomials of $f \in \mathcal{M}$ as morphisms. $\mathcal{C}$ is a monoidal subcategory of $\mathcal{V}$. The bifunctor $B : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{V}, \ X \times Y \mapsto X^\vee \otimes Y$ has a coend $H = \int^{Y \in \mathcal{C}} Y^\vee \otimes Y \in \hat{\mathcal{V}}$ which can be found from the exact sequence in $\hat{\mathcal{V}}$
\[ \bigoplus_{f : Y \to Z \in \text{Mor} \mathcal{C}} Z^\vee \otimes Y \xrightarrow{1 \otimes f - f \otimes 1} \bigoplus_{Y \in \text{Ob} \mathcal{C}} Y^\vee \otimes Y \to H \to 0. \]

Here $f^\dagger : Z^\vee \to Y^\vee \in \mathcal{V}$ is the dual morphism to $f : Y \to Z \in \mathcal{V}$. As could be expected from [15, 14], the object $H \in \hat{\mathcal{V}}$ turns out to be a superbialgebra in $\hat{\mathcal{V}}$. 
Theorem 1.4. (a) Let \( Z = \bigoplus_{Y \in \Ob C} Y^\vee \otimes Y \in \hat{\mathcal{V}} \) with the operations of comultiplication

\[
\Delta : \bigoplus_{Y \in \Ob C} Y^\vee \otimes Y = \bigoplus_{Y \in \Ob C} Y^\vee \otimes I \otimes Y \xrightarrow{1 \otimes \text{coev}_Y \otimes 1} \bigoplus_{Y \in \Ob C} Y^\vee \otimes Y \otimes Y^\vee \otimes Y \subset \bigoplus_{Y \in \Ob C} Y^\vee \otimes Y \otimes (\bigoplus_{Y \in \Ob C} Y^\vee \otimes Y);
\]

unit

\[
\varepsilon : \bigoplus_{Y \in \Ob C} Y^\vee \otimes Y \xrightarrow{\text{ev}} \bigoplus_{Y \in \Ob C} Y \xrightarrow{\sum} I;
\]

multiplication

\[
m : (\bigoplus_{X \in \Ob C} X^\vee \otimes X) \otimes (\bigoplus_{Y \in \Ob C} Y^\vee \otimes Y) = \bigoplus_{X,Y} X^\vee \otimes X \otimes Y \otimes Y \xrightarrow{1 \otimes S \otimes 1} \bigoplus_{X,Y} (X \otimes Y)^\vee \otimes (X \otimes Y) \to \bigoplus_{Z} Z^\vee \otimes Z,
\]

where \( \sigma : Y^\vee \otimes X^\vee \otimes X \to Y^\vee \otimes X^\vee \otimes X \) equals \((-1)^{(k-j)}\) when restricted to a homogeneous component \( Y_i^\vee \otimes X_j^\vee \otimes X_k \) with \( i, j, k \in \mathbb{Z}/2 \), and \( S \) is the symmetry; and unit

\[
\eta : I \cong I^\vee \otimes I \hookrightarrow \bigoplus_{Y} Y^\vee \otimes Y.
\]

Then \( Z \) is a superalgebra in \( \hat{\mathcal{V}} \).

(b) \( \sum_f \text{Im}(1 \otimes f - f^t \otimes 1) \) is a bi-ideal in \( \bigoplus_{Y} Y^\vee \otimes Y \), therefore \( H \) is a superbialgebra in \( \hat{\mathcal{V}} \).

(c) The morphisms

\[
\delta_X : X = I \otimes X \xrightarrow{\text{coev}_X \otimes 1} X \otimes X^\vee \otimes X \hookrightarrow X \otimes (\bigoplus_{Y} Y^\vee \otimes Y) \to X \otimes H
\]

are coactions of \( H \) on \( X \in \mathcal{O} \).

(d) The bialgebra \( H \) together with the coactions \( \delta_X \) is universal in a sense similar to that of Section 1.1.

The proof is straightforward. New in this theorem in comparison with, say, [14] is the \( \mathbb{Z}/2 \)-grading. It is valid also for braided categories \( \mathcal{V} \).

Example 1.1. The case of \( \mathcal{V} = \mathbb{Z}\text{-grad-vect} \), the category of \( \mathbb{Z} \)-graded finite dimensional vector spaces with grading-preserving linear maps, is already considered in Section 1.1. Here \( \hat{\mathcal{V}} = \mathbb{Z}\text{-grad-Vect} \). The symmetry is given by \( S(x \otimes y) = (-1)^{p(x)p(y)} y \otimes x \) on homogeneous vectors \( x, y \), where \( p \) is the degree. The projection map \( j : \bigoplus_{Y} Y^\vee \otimes Y \to H \) onto the universal algebra from Section 1.1 is given by \( y^a \otimes y_b \mapsto t^a_{\mathcal{M}} \). The statement that \( \sum_f \text{Im}(1 \otimes f - f^t \otimes 1) = \text{Ker} j \) is equivalent to the statement from Theorem 1.1 that \( \text{Rel}(f), f \in \mathcal{M} \), generate the ideal of relations of \( H \).

Theorem 1.2 holds also in this setting with maps replaced by morphisms.

Theorem 1.5 (Hopf algebra construction). Assume that for each \( X \in \mathcal{O} \) there exist compositions of tensor monomials in morphisms from \( \mathcal{M} \)

\[
f = 1 \otimes f_1 \otimes 1 \cdots 1 \otimes f_n \otimes 1 : X \otimes (Y_1 \otimes \ldots \otimes Y_k) \to k,
\]

\[
g = 1 \otimes g_1 \otimes 1 \cdots 1 \otimes g_m \otimes 1 : (W_1 \otimes \ldots \otimes W_i) \otimes X \to k,
\]
which are pairings, non-degenerate in the argument \(X\). Assume also existence of compositions of tensor monomials in maps from \(\mathcal{M}\)
\[
h = 1 \otimes h_1 \otimes 1 \cdots 1 \otimes h_a \otimes 1 : k \to (U_1 \otimes \ldots \otimes U_c) \otimes X,
\]
\[
j = 1 \otimes j_1 \otimes 1 \cdots 1 \otimes j_b \otimes 1 : k \to X \otimes (V_1 \otimes \ldots \otimes V_d),
\]
which are non-degenerate in \(X\) in the sense that the induced morphisms \((U_1 \otimes \ldots \otimes U_c)^\vee \to X\) and \((V_1 \otimes \ldots \otimes V_d)^\vee \to X\) are epimorphisms. Then the universal bialgebra \(H \in \hat{\mathcal{V}}\) has an invertible antipode.

**Proof.** Repeating the proof of Theorem 1.2, we can assume that \(\mathcal{C}\) is closed under duality \(-^\vee\). Then we define an antiendomorphism
\[
\gamma = (-1)^{j(i+j)} : \bigoplus_{Y,i,j} Y_i^\vee \otimes Y_j \to \bigoplus_{Y^\vee,i,j} Y_j^{\vee\vee} \otimes Y_i^\vee.
\]
It preserves the ideal of relations of \(H\) and the antipode of \(H\) comes as its quotient. \(\square\)

1.4. **Differential bialgebra case.** Some methods of constructing differential bialgebras were described by Maltsiniotis [18, 19], Manin [21] and one of us [28]. We present here a general framework for such constructions. The results of the previous section are applied to the category \(\mathcal{V} = \mathcal{D}\) of \(\mathbb{Z}\)-graded differential finite-dimensional vector spaces, \((V, d : V \to V), V = \bigoplus_{i \in \mathbb{Z}} V_i, d^2 = 0\). The differential \(d\) has degree 1. The category \(\hat{\mathcal{V}} = \hat{\mathcal{D}}\) consists of all \(\mathbb{Z}\)-graded differential vector spaces and their grading-preserving linear maps commuting with the differential.

So, our data are a family of differential \(\mathbb{Z}/2 \times \mathbb{Z}\)-graded spaces \(\mathcal{O}\) with differentials of degree \((0,1)\) and a family of linear maps \(\mathcal{M} = \{f\}\) of degree \((0,0)\) or \((1,0)\) commuting with the differential. The output of Theorem 1.4 is a \(\mathbb{Z}/2 \times \mathbb{Z}\)-graded bialgebra \(H\) in the category \(\hat{\mathcal{D}}\), that is, a differential \(\mathbb{Z}/2 \times \mathbb{Z}\)-graded bialgebra.

We have a forgetful functor \(\Phi : \mathcal{D} \to \mathbb{Z}\text{-grad-vect}\) forgetting the differential. The constructions of Section 1.1 with \(\mathcal{V}' = \mathbb{Z}\text{-grad-vect}\) and above with \(\mathcal{V} = \mathcal{D}\) give bialgebras \(H'\) and \(H\). They are identified, \(\Phi(H) = H'\). To give explicitly the differential in the bialgebra \(H'\) making it into \(H\), consider the coalgebra \(C = \bigoplus_{X \in \mathcal{O}} \text{Comat} X = \bigoplus_{X \in \mathcal{O}} X^\vee \otimes X\). It has a unique map \(d : C \to C\) of degree \((0,1)\) which makes the diagram
\[
X \xrightarrow{\delta_X} X \otimes C \quad \downarrow d \quad \downarrow d \otimes 1 + \sigma \otimes d
\]
commute. In our conventions \(\sigma : X \to X\) is the map whose eigenvectors are the homogeneous elements \(x\) of \(X\), the eigenvalue associated with such an element being
\((-\hat{x})^x = (-1)^{x(x-1)}\). Indeed, \(d : C \to C\) is given by
\[
dt x_i \bigg|_a = (-1)^i (t_{x_i} f_c d^c_a - d^f y_t x_i b),
\]
where \(d^c_a\) denotes the matrix of the map \(d : X_i \to X_i, x_a \mapsto x_c d^c_a\), in a chosen basis.

**Proposition 1.6.** The map \(d : C \to C\) is a codifferential of degree \((0, 1) \in \mathbb{Z}/2 \times \mathbb{Z}\), that is, \(d\) is a graded coderivation

\[
\begin{array}{c c}
C & C \otimes C \\
\downarrow d & \downarrow d \otimes 1 + \sigma \otimes d \\
C & C \otimes C
\end{array}
\]

and \(d^2 = 0\).

**Proof.** Straightforward. \(\square\)

1.4.1. **Construction for differential bialgebras.** The differential bialgebra \(H\) constructed in Section 1.3 usually does not correspond to a geometric type of object. However, using previous results we can construct more realistic examples.

Assume given the following data (comp. [28, 30]). Let \(X_1^*, X_2^*, \ldots, X_r^*\) be \(\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\)-graded algebras (where \(\bullet\) denotes the \(\mathbb{Z}_{\geq 0}\)-degree) with multiplication \(m_u : X_u^* \otimes X_v^* \to X_u^*\), \(1 \leq u \leq r\), generated by finite-dimensional spaces \(X_u^1\) and \(X_u^0 = k\). Let \(d_u : X_u^* \to X_u^*\) be differentials of degree \((0, 1, 0)\). Assume bijective linear maps \(f_u : X_u^1 \to X_u^1 + 1\) of degree \(\deg f_u = (p_u, 0) \in \mathbb{Z}/2 \times \mathbb{Z}\) are given for \(1 \leq u \leq r - 1\) such that \(f_u d_u = d_{u+1} f_u\).

These data are transformed to the family \(\mathcal{O} = \{X_i^1\}_{i \geq 0, 1 \leq u \leq r}\) of differential \(k\)-spaces and a family of maps \(\mathcal{M} = \{m_u : X_u^1 \otimes X_v^1 \to X_u^1 + 1, f_u : X_u^1 \to X_u^1 + 1 \mid i, j \geq 0, 1 \leq u \leq r, 1 \leq v < r\}\) of degree \((*, 0) \in \mathbb{Z}/2 \times \mathbb{Z}\). We have already proved

**Theorem 1.7** (Differential bialgebra construction). The bialgebra \(H\) constructed from these data in Theorem 1.1 is a differential bialgebra.

**Remark 1.1.** We can always reduce such data to the case of algebras \(Y_1^*, Y_2^*, \ldots, Y_r^*\) generated by the same space \(Y_u^1 = Y^1\) with \(f_u = \text{id}_{Y_1}\). For instance, having \(X_1^*, X_2^*, \ldots, X_r^*\) and \(f_1 : X_1^1 \to X_1^1\), we define \(Y_1^* = X_1^*\), and \((Y_2^*, \hat{m}_2)\) generated by \(Y_2^1 = X_1^1\) is defined as a graded algebra “isomorphic” to \(X_2^*\) via a bijective map \(f \equiv f^* : Y_2^* \to X_2^*\), such that \(f^1 = f_1\) and \(\deg f^k = (kp_1, 0) \in \mathbb{Z}/2 \times \mathbb{Z}\) (recall that \(\deg f_1 = (p_1, 0)\)). Precisely we require the diagram

\[
\begin{array}{c c c c}
Y_2^* \otimes Y_2^* & \xrightarrow{\hat{m}_2} & Y_2^* \\
| & f \otimes f & | & f \\
X_2^* \otimes X_2^* & \xrightarrow{m_2} & X_2^*
\end{array}
\]

to commute, where as usual \((f \otimes f^*)(y \otimes y') = (-1)^{f \otimes f^*} f(y) \otimes f^*(y')\). The map \(f\) will be an isomorphism of graded algebras only if \(p_1 = 0\).

All such \((Y_2^*, \hat{m}_2)\) are naturally isomorphic as graded associative algebras. One can verify that \(Y_2^1 = k \oplus X_1^1 \oplus X_2^1 \oplus X_3^1 \oplus X_2^1 \oplus \ldots\) (with appropriately changed grading if \(p_1 = 1\)) satisfies these conditions.
1.4.2. **Differential forms on quantum semigroups.** Let $X_1^\bullet, X_2^\bullet, \ldots, X_r^\bullet$ be $\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$-graded algebras with additional structures as in Section 1.4.1. Then Theorem 1.7 gives a $\mathbb{Z}/2 \times \mathbb{Z}$-graded differential bialgebra $H$.

**Theorem 1.8.** Let $H^{<0}$ denote $\oplus_{a \in \mathbb{Z}/2, b < 0} H^{a,b} \subset H$. Then

$$\Omega = H/(H^{<0}, dH^{<0})$$

is a $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded differential bialgebra. The algebra $\Omega$ is a universal bialgebra in the category of differential $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded bialgebras coacting on $X_1^\bullet, X_2^\bullet, \ldots, X_r^\bullet$.

**Proof.** The subspace $H^{<0} \oplus dH^{-1} \subset H$ is $d$-invariant (here $H^{-1} \overset{\text{def}}{=} H^{0,-1} \oplus H^{1,-1}$). Also this subspace is a coideal. Hence, $\Omega = H/(H^{<0} \oplus dH^{-1})$ is a differential bialgebra.

Having some $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded bialgebra $f \in \mathcal{H}$ coacting on $X_1^\bullet, X_2^\bullet, \ldots, X_r^\bullet$ we find by Theorem 1.1 the morphism of bialgebras $\phi : H \rightarrow F$. If, in addition, $F$ is a differential bialgebra, differentially coacting on $X_1^\bullet, X_2^\bullet, \ldots, X_r^\bullet$, we get that $\phi$ is a differential bialgebra morphism. Since $H^{<0} \subset \ker \phi$, this implies $dH^{<0} \subset \ker \phi$, whence the theorem is proven. \hfill \Box

**Definition 1.2.** We call $\Omega$ from Theorem 1.8 the algebra of differential forms on a quantum (super)semigroup, corresponding to the function algebra $\Omega^0 \overset{\text{def}}{=} \Omega^{0,0} \oplus \Omega^{1,0}$.

This terminology is justified by examples.

2. **Examples of differential quantum supergroups**

2.1. **Hecke $R$-matrices as construction data.** Let $V$ be a $\mathbb{Z}/2$-graded vector space with a graded basis $(x_i), 1 \leq i \leq n$ and let $R : V \otimes V \rightarrow V \otimes V, \ x_i \otimes x_j \mapsto R_{ij}^{kl} x_k \otimes x_l$ be a solution to the Yang-Baxter equation. We assume that $R$ preserves the grading and is diagonalizable with two eigenvalues $q$ and $-q^{-1}, q \neq \pm i$. Denote by $dV$ another copy of $V$ with a basis $(dx_i)$. Set $X^1 = V \oplus dV$ and

$$X^\bullet = T^\bullet(X^1)/(R(x_i \otimes x_j) - q x_i \otimes x_j, R(dx_i \otimes x_j) - q^{-1} dx_i \otimes x_j, R(dx_i \otimes dx_j) + q^{-1} dx_i \otimes dx_j)$$

It is understood that $R$ is extended to

$$R : dV \otimes V \rightarrow V \otimes dV, \ dx_i \otimes x_j \mapsto R_{ij}^{kl} x_k \otimes dx_l$$

$$R : dV \otimes dV \rightarrow dV \otimes dV, \ dx_i \otimes dx_j \mapsto R_{ij}^{kl} dx_k \otimes dx_l$$

Make $X^\bullet$ into a $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-graded algebra by setting

$$\deg x_i = (p(x_i), 0, 1), \quad \deg dx_i = (p(x_i), 1, 1)$$

where $p$ is the $\mathbb{Z}/2$-degree in $V$. Introduce a differential in $X^\bullet$ by $d(x_i) = dx_i$.

Let $\Xi$ be a copy of $V$ with the opposite parity, that is, with a basis graded by $(\xi_i), p(\xi_i) = p(x_i) + 1$. One more copy of $\Xi$ denoted $d\Xi$ has a basis $(d\xi_i)$. Consider the algebra $Y$ generated by $Y^1 = \Xi \oplus d\Xi$

$$Y^\bullet = T^\bullet(Y^1)/(R(\xi_i \otimes \xi_j) + q^{-1} \xi_i \otimes \xi_j, R(d\xi_i \otimes \xi_j) + q d\xi_i \otimes \xi_j, R(d\xi_i \otimes d\xi_j) - q d\xi_i \otimes \xi_j)$$

with degree

$$\deg \xi_i = (p(x_i) + 1, 0, 1), \quad \deg d\xi_i = (p(x_i) + 1, 1, 1),$$
and differential \( d(\xi_i) = d\xi_i \).

The map \( J : X^1 \to Y^1, \ i \mapsto \xi_i, \ dx_i \mapsto d\xi_i \), of degree \((1,0) \in \mathbb{Z}/2 \times \mathbb{Z}\) commutes with the differential.

The data \((X^\bullet,Y^\bullet,J)\) determine a differential bialgebra \( H \) as Theorem 1.7 claims. To find its structure explicitly we analyze the generators and relations proposed in the theorem. The matrices \( T \) and \( T' \) made of matrix elements of the spaces \( X^1 \) and \( Y^1 \) can be decomposed in blocks
\[
T = \begin{pmatrix} t & r \\ p & s \end{pmatrix}, \quad T' = \begin{pmatrix} t' & r' \\ p' & s' \end{pmatrix},
\]
so the coaction on \( X^1 \) and \( Y^1 \) is
\[
\delta x_i = x_j \otimes t^j_i + dx_j \otimes p^j_i, \\
\delta dx_i = x_j \otimes r^j_i + dx_j \otimes s^j_i, \\
\delta \xi_i = \xi_j \otimes t^{ij}_i + d\xi_j \otimes p^{ij}_i, \\
\delta d\xi_i = \xi_j \otimes r^{ij}_i + d\xi_j \otimes s^{ij}_i.
\]

The relations corresponding to \( J \) say precisely that \( T' = T \) in \( H \). Clearly, the entries of \( T \) generate the algebra \( H \). The relations \( \text{Rel}(m : X^\bullet \otimes X^\bullet \to X^\bullet) \) coincide with those asserting that \( \text{Ker}(m : X^\bullet \otimes X^\bullet \to X^\bullet) \) is an \( H \)-subcomodule. A particular requirement is that
\[
\text{Ker}(m_{11} : X^1 \otimes X^1 \to X^2) = \{ R(x \otimes y) - q x \otimes y, R(dx \otimes y) - q^{-1} dx \otimes y, R(dx \otimes dy) + q^{-1} dx \otimes dy \}
\]
is an \( H \)-subcomodule. Similarly
\[
\text{Ker}(m'_{11} : Y^1 \otimes Y^1 \to Y^2) = \{ R(\xi \otimes \eta) + q^{-1} \xi \otimes \eta, R(d\xi \otimes \eta) + q d\xi \otimes \eta, R(d\xi \otimes d\eta) - q d\xi \otimes d\eta \}
\]
is an \( H \)-subcomodule. The graded isomorphism of \( H \)-comodules \( J \otimes J : X^1 \otimes X^1 \to Y^1 \otimes Y^1 \) sends the subspace
\[
K' = \{ R(x \otimes y) + q^{-1} x \otimes y, R(dx \otimes y) + q dx \otimes y, R(dx \otimes dy) - q dx \otimes dy \}
\]
to \( \text{Ker} m'_{11} \), whence \( K' \) is an \( H \)-subcomodule.

Therefore, we have to impose on \( H \) relations equivalent to the fact that \( \text{Ker} m_{11} \cap X^1 \otimes X^1 \) are \( H \)-subcomodules. Remark that
\[
\text{Ker} m_{11} \oplus K' = X^1 \otimes X^1.
\]

Indeed, introduce a grading-preserving linear map
\[
\mathcal{R} = \begin{pmatrix} R & 0 & 0 & 0 \\ 0 & 0 & R^{-1} & 0 \\ 0 & R & q - q^{-1} & 0 \\ 0 & 0 & 0 & -R^{-1} \end{pmatrix}
\]
\[
\mathcal{R} : X^1 \otimes X^1 \to X^1 \otimes X^1 = V \otimes V \oplus V \otimes dV \oplus dV \otimes V \oplus dV \otimes dV
\]
(The matrix acts on a vector from the right.) It satisfies the same quadratic equation as \( R \)
\[
(\mathcal{R} - q)(\mathcal{R} + q^{-1}) = 0.
\]
Its eigenspace with eigenvalue $q$ (resp. $-q^{-1}$) is $K'$ (resp. $\text{Ker } m_{11}$).

We conclude that the equations to impose express the fact that $\mathcal{R}$ is an automorphism of the $H$-comodule $X^1 \otimes X^1$. Define accordingly

$$H = k\langle T^i_j \rangle /((-1)^{(i-k)}R^{kl}_{ij}T^i_j - (-1)^{(i-k)}R^{kl}_{ij}T^i_j)$$

(2.1)

The standard assertion is that $H$ is a bialgebra with the canonical coaction on $X^1, Y^1$. It coacts also on $X^\bullet = T^\bullet (X^1)/\text{Ker } m_{11}, Y^\bullet = T^\bullet (Y^1)/\text{Ker } m_{11}'$ since the ideal of relations is a $H$-subcomodule by construction. The bialgebra $H$ was built so that any other bialgebra from $\mathcal{H}$ is an epimorphic image of it, hence it is universal.

The bialgebra $H$ has generators $t^j_i, r^j_i, p^j_i, s^j_i$ of degree

- $\deg t^j_i = (p(x_i) - p(x_j), 0)$,
- $\deg r^j_i = (p(x_i) - p(x_j), 1)$,
- $\deg p^j_i = (p(x_i) - p(x_j), -1)$,
- $\deg s^j_i = (p(x_i) - p(x_j), 0)$.

If $V$ is even (all $p(x_i) = 0$), there is a presentation

$$Rt t_s t = t t R, \quad Rs t s = s s R$$

$$r s = Rr s R, \quad p s = Rp s R$$

$$p s = R s p R, \quad R p = R p R$$

in the usual notation, with $R$ standing for the matrix $(R_{ij}^k)$ and $t_1$, for example, standing for the Kronecker product matrix $t \otimes 1 = (t_{ij}^k \delta_j^k)$. In the general case the same presentation holds with additional minus signs from (2.1).

It follows by Theorem 1.7 that $H$ is a differential bialgebra with the differential

$$dt = r, \quad dr = 0, \quad dp = t - s, \quad ds = r.$$  

(2.2)

(2.3)

In Theorem 1.8 the bialgebra $\Omega$ is constructed. Clearly, in our example it is generated by $t^j_i$ and $r^j_i = dt^j_i$ and has the relations of $H$ plus the identifications $p = 0, s = t$. So the defining relations of $\Omega$ are

$$R t t = t t R \quad (2.4)$$

$$d t t = R t d t R \quad (2.5)$$

$$d t t = - R d t d t R \quad (2.6)$$

We recognize in $\Omega$ the algebra of differential forms related to $R$ [25, 26, 30, 35].
2.2. The supersemigroup. A straightforward check proves

**Proposition 2.1.** \( R \) satisfies the Yang–Baxter equation (in braid form).

Therefore the bialgebra \( H \) can be interpreted as the quantized algebra of functions on a supersemigroup. If we interpret the algebra \( k\langle t \rangle/(R t_1 t_2 - t_3 t_2 R) \) as the algebra of functions on quantum \( Mat(n|m) \) related to \( R \), then \( H \) will be the algebra of functions on \( Mat(n + m \mid n + m) \) related to \( R \).

We can continue the procedure of constructing new solutions to the Yang–Baxter equation starting with \( R \) instead of \( R \).

2.3. The supergroup. Now we use the category \( V \) of \( \mathbb{Z}/2 \times \mathbb{Z} \)-graded differential vector spaces with the differential of degree \((0, 1) \in \mathbb{Z}/2 \times \mathbb{Z}\). To make the differential bialgebra from Section 1.4.1 into a Hopf algebra, we have to add the dual vector space \( Z \) to the given \( X = V \oplus dV \) with \( R : X \otimes X \to X \otimes X \). The subspaces \( V \) and \( dV \) have degrees 0 and 1 \( \in \mathbb{Z} \). We define \( Z = W \oplus U \), where \( W = V^\vee \) has degree 0 \( \in \mathbb{Z} \) and the basis \( (w_i) \) dual to \( (x_i) \), and \( U = (dV)^\vee \) has degree \(-1 \in \mathbb{Z}\) and the basis \( (u_i) \) dual to \( (dx_i) \). The differential \( d : Z \to Z \), \( dw^i = 0 \), \( du^i = -w^i \) makes

\[
\text{ev} : X \otimes Z \to k, \quad \text{coev} : k \to Z \otimes X
\]

into morphisms of \( V \). Also \( R \) is a morphism of \( V \) since it commutes with \( d : X \otimes X \to X \otimes X \), \( d(x \otimes y) = dx \otimes y + \sigma(x) \otimes dy \).

Assume now that \( R^\sharp \) is invertible, so \( R \) is Hecke. By Theorem 0.2 \( R^{-1} : W \otimes W^\vee \to W^\vee \otimes W \) is also invertible. A simple calculation gives

\[
R^\sharp = \begin{pmatrix}
R^\sharp & 0 & 0 & -q(q^{-1})^{1/\gamma}_\gamma \\
0 & 0 & R^\sharp & 0 \\
0 & R^{-1} & 0 & 0 \\
0 & 0 & 0 & -R^{-1} & 0
\end{pmatrix} : X \otimes X^\vee \to X^\vee \otimes X,
\]

where \( \gamma_{ijkl} = 1^\sharp \delta_{ik} \delta_{jl} \). Therefore, \( R^\sharp \) is also invertible and \( R \) is Hecke. By Theorem 0.1 \( R^\sharp \) and \( R^{\sharp \sharp} \) are morphisms, and in particular commute with the differential \( d \).

Thus, we can consider the following data in \( V \): the two objects \( O = \{X, Z\} \) and the family \( M \) of morphisms

\[
\begin{align*}
R : X \otimes X & \to X \otimes X, \\
R^\sharp : X \otimes Z & \to Z \otimes X, \\
R^{\sharp \sharp} : Z \otimes Z & \to Z \otimes Z, \\
\text{ev} : X \otimes Z & \to k, \\
\text{coev} : k & \to Z \otimes X.
\end{align*}
\]

These data are closed under duality in the sense of Theorem 1.5. The missing pairing and co-pairing are constructed as

\[
Z \otimes X \xrightarrow{R^{-1}} X \otimes Z \xrightarrow{\text{ev}} k, \quad (2.7)
\]

\[
k \xrightarrow{\text{coev}} Z \otimes X \xrightarrow{R^{-1}} X \otimes Z. \quad (2.8)
\]
They are non-degenerate as follows from the general theory [12]. Therefore the
bialgebra $H$ constructed from the data $\mathcal{O}, \mathcal{M}$ is a differential Hopf algebra. It is
generated by matrix elements $T^i_j$ and $\bar{T}^i_j$ of $X$ and $Z$. The relations corresponding
to $\mathcal{M}$ are

\begin{align}
(-1)^{\bar{i}(\bar{i}-\bar{k})} R^{\bar{m} \bar{n}}_{\bar{k} \bar{i}} T^k_i T^j_j &= (-1)^{\bar{s}(\bar{m}-\bar{n})} T^p_m T^s_n R^{m n}_{i j}, \\
(-1)^{\bar{i}(\bar{i}-\bar{k})} R^{i \bar{k}}_{p \bar{k}} T^k_i \bar{T}^j_j &= (-1)^{\bar{s}(\bar{m}-\bar{n})} T^m_p T^n_s R^{i j}_{m n}, \\
(-1)^{\bar{k}(\bar{i}-\bar{j})} R^{i j}_{s p} T^k_i \bar{T}^j_j &= \delta^j_i, \\
(-1)^{\bar{j}(\bar{k}-\bar{j})} \bar{T}^k_i T^j_k &= \delta^j_i.
\end{align}

We know that the antipode $\gamma$ is invertible and the last two equations give, in par-
ticular,

$$\gamma(\bar{T}^j_j) = (-1)^{\bar{j}(\bar{i}-\bar{j})} T^j_j.$$  

From (2.7), (2.8) one can find also $\gamma(T^i_j)$.

Divide the matrix $\bar{T}$ into four submatrices

$$\bar{T} = \begin{pmatrix} \bar{t} & \bar{r} \\ \bar{p} & \bar{s} \end{pmatrix}$$

similarly to $T$. The degrees are

\begin{align}
\deg \bar{t}^j_i &= (p(x_i) - p(x_j), 0), & \deg \bar{r}^j_i &= (p(x_i) - p(x_j), -1), \\
\deg \bar{p}^j_i &= (p(x_i) - p(x_j), 1), & \deg \bar{s}^j_i &= (p(x_i) - p(x_j), 0).
\end{align}

The differential is given by (2.2), (2.3) and

\begin{align}
d\bar{t} &= \bar{p}, & d\bar{r} &= \bar{s} - \bar{t}, \\
d\bar{p} &= 0, & d\bar{s} &= \bar{p}.
\end{align}

**Remark 2.1.** The use of $R^\sharp$ and $R^{\sharp\sharp}$ is dictated by Theorem 0.1.

**2.3.1. Differential forms on quantum $GL(n)$.** The algebra

$$\Omega = H/(H^{<0}, dH^{<0}) = H/(p, t - s, \bar{r}, \bar{t} - \bar{s})$$

is the algebra of differential forms on a quantum (super)group $GL(n)$ related to $R$.
In the even case it is generated by $t, r, \bar{t}$, because $\bar{p} = -\bar{t} \cdot r^t \cdot \bar{t}$. The defining relations are (2.4)–(2.6) and

\begin{align}
R^\sharp r_1 \bar{t}_2 &= \bar{t}_1 r_2 R^{-\sharp} \\
R^{\sharp\sharp} \bar{t}_1 \bar{t}_2 &= \bar{t}_1 \bar{t}_2 R^{\sharp\sharp} \\
t^t \bar{t} &= 1 = \bar{t} t^t
\end{align}

In the general case there are similar relations with additional signs.
3. Koszul complexes

3.1. The Berezinian. In this section we temporarily forget about differentials \(d\) and consider arbitrary Hecke \(\hat{R}\)-matrices. Recall that with each Hecke \(\hat{R}\)-matrix \(R : V \otimes V \to V \otimes V\) two \(\mathbb{Z}_{\geq 0}\)-graded algebras are associated. The symmetric algebra \(S^\bullet(V)\) is the quotient of \(T^\bullet(V)\) by relations
\[
R(x \otimes y) = qx \otimes y, \quad x, y \in V
\]
and the external algebra \(\Lambda^\bullet(V)\) has relations
\[
R(x \otimes y) = -q^{-1}x \otimes y, \quad x, y \in V.
\]

Our definition of a Koszul complex is a \(q\)-deformation of [11], where most results of this section were obtained for symmetric monoidal categories \((q = 1)\). Also the following definition is a braided version of one of the Koszul complexes for quadratic algebras [20]. The Koszul complex for a Hecke symmetry possesses more structure; namely, it will have two differentials.

**Definition 3.1.** The **Koszul complex** of \(V\) is the \(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\)-graded algebra \(K^{\bullet\bullet}(V)\), quotient of \(T^{\bullet\bullet}(V \oplus V^\vee)\) by relations
\[
R(v \otimes \bar{v}) = -q^{-1}v \otimes \bar{v} \quad \text{for} \ v, \bar{v} \in V,
\]
\[
R^1(v \otimes v') = qv \otimes v' \quad \text{for} \ v \in V, v' \in V^\vee,
\]
\[
R^{\bullet\bullet}(v' \otimes \bar{v}') = qv' \otimes \bar{v}' \quad \text{for} \ v', \bar{v}' \in V^\vee.
\]

The middle commutation relation can be written also as
\[
v' \otimes v = q_{V^\vee, V}(v' \otimes v) \quad \text{for} \ v \in V, v' \in V^\vee.
\] (3.1)

Using the diamond lemma [1] we decompose \(K^{\bullet\bullet}(V)\) into the tensor product of a symmetric and an external algebra. Precisely, the maps
\[
S^m(V^\vee) \otimes \Lambda^n(V) \to K^{0m}(V) \otimes K^{n0}(V) \xrightarrow{m} K^{nm}(V),
\] (3.2)
\[
\Lambda^n(V) \otimes S^m(V^\vee) \to K^{n0}(V) \otimes K^{0m}(V) \xrightarrow{m} K^{nm}(V)
\]
are isomorphisms of \(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\)-graded vector spaces.

Let \(\{v_i \mid i = 1, \ldots, n\}\) be a basis for \(V\) and \(\{v^i \mid i = 1, \ldots, n\}\) be the dual basis for \(V^\vee\). In the following we use the summation convention on indices which occur once in an upper and once in a lower position.

**Proposition 3.1.** The element \(d = v^i v_i = \text{coev}(1) \in K^{1,1}(V) \simeq V^\vee \otimes V\) obeys the following commutation relations in \(K(V)\)
\[
dv = -q^2vd \quad \text{for} \ v \in V,
\]
\[
dv' = v'd \quad \text{for} \ v' \in V^\vee.
\]

**Proof.** Using graphical notation
\[
d = v^i \overset{i}{\bigtriangleup} v_i
we have
\[ dv = v^i v_i v = -qv^i(-q^{-1}v_i v) = -qv^i R(v_i \otimes v) = \]
\[ = -q \begin{array}{c}
  v^i \\
  v_i \\
  v \\
  v \\
  v_i \\
\end{array} = -q^2 \begin{array}{c}
  v^i \\
  v_i \\
  v \\
  v_i \\
  v \end{array} = -q^2 v_i = 0, \]
and
\[ v' d = v' v^i v_i = q^{-1}(qv' v^i) v_i = q^{-1} R^\sharp (v' \otimes v^i) v_i = \]
\[ = q^{-1} \begin{array}{c}
  v' \\
  v_i \\
  v' \end{array} = \begin{array}{c}
  v^i \\
  v_i \\
  v_i' \\
\end{array} = v_i' = dv'. \]

Corollary 3.2. $d^2 = 0$

Indeed,
\[ d^2 = dv' v_i = v' dv_i = -q^2 v^i d v_i = -q^2 d^2. \]

Definition 3.2. A differential of degree $(1,1)$ $D = D_V : K(V) \rightarrow K(V)$ in the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-graded vector space $K^{**}(V)$ is defined by left multiplication by $d$. Its $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-graded space of cohomologies
\[ \text{Ber}(V) = H^*(K(V), D) \]
is called the Berezinian of $V$.

3.2. The Hecke sum. The $\check{R}$-matrix $\mathcal{R}$ from Section 2.1 is a special element of a more general family of $\check{R}$-matrices studied by Majid and Markl [16].

Theorem 3.3. (a) Let $R : V \otimes V \rightarrow V \otimes V$, $R' : U \otimes U \rightarrow U \otimes U$ be two diagonalizable $\check{R}$-matrices with the eigenvalues $q$ and $-q^{-1}$, and let $Q : U \otimes V \rightarrow V \otimes U$ be a bijective linear map satisfying the equations
\[ (1 \otimes Q)(Q \otimes 1)(1 \otimes R) = (R \otimes 1)(Q \otimes 1)(1 \otimes Q), \]
\[ (Q \otimes 1)(1 \otimes Q)(R' \otimes 1) = (1 \otimes R')(Q \otimes 1)(1 \otimes Q). \]
Then the linear map $\mathcal{R} : X \otimes X \rightarrow X \otimes X$, $X = V \oplus U$, given by
\[ \mathcal{R} = \begin{pmatrix}
  R & 0 & 0 & 0 \\
  0 & 0 & Q^{-1} & 0 \\
  0 & Q & q-q^{-1} & 0 \\
  0 & 0 & 0 & R' \\
\end{pmatrix} \]
is a diagonalizable $\check{R}$-matrix with eigenvalues $q$ and $-q^{-1}$.

(b) If in addition $R^\sharp$, $R'^\sharp$, $Q^\sharp$, $Q^{-1}\sharp$ are all invertible, then also $R^\sharp\sharp$, $R^{-1}\sharp$ are.
Proof. (a) Reduces to a theorem from [16] and can be checked straightforwardly.
(b) The matrix
\[ \mathcal{R}^z = \begin{pmatrix} R^z & 0 & 0 & (q - q^{-1})^{[\frac{1}{2}]} \\ 0 & 0 & Q^z & 0 \\ 0 & Q^{-1}z & 0 & 0 \\ 0 & 0 & 0 & R^z \end{pmatrix}, \] (3.3)
where \( \frac{1}{2} = (V \otimes V \xrightarrow{ev} k \xrightarrow{coev} U \otimes U), \frac{1}{2}^{jk} = \delta^j_k \delta^k_l \), is obviously invertible. By Theorem 0.2 \( \mathcal{R}^{-1z} \) is also invertible. \( \square \)

The space \( X \) equipped with \( \mathcal{R} \) is denoted \( V \oplus_{\mathcal{Q}} U \) and is called the Hecke sum.

3.3. Koszul complex of a Hecke sum. Let us analyze in detail the structure of the algebra \( K^{\bullet\bullet}(V \oplus_{\mathcal{Q}} U) \). The algebra \( \Lambda^*(V \oplus_{\mathcal{Q}} U) \) is a quotient of \( T^*(V \oplus U) \) by the relations
\[
\begin{align*}
R(v \otimes \bar{v}) &= -q^{-1}v \otimes \bar{v} \quad \text{for } v, \bar{v} \in V, \\
Q(u \otimes v) &= -qu \otimes v \quad \text{for } v \in V, u \in U, \\
R'(u \otimes \bar{u}) &= -q^{-1}u \otimes \bar{u} \quad \text{for } u, \bar{u} \in U.
\end{align*}
\]
Since
\[
\mathcal{R}^{\sharp\sharp} = \begin{pmatrix} R^{\sharp\sharp} & 0 & 0 & 0 \\ 0 & q - q^{-1} & Q^{-1\sharp\sharp} & 0 \\ 0 & Q^{\sharp\sharp} & 0 & 0 \\ 0 & 0 & 0 & R^{\sharp\sharp} \end{pmatrix},
\]
the algebra \( \mathcal{S}^*((V \oplus_{\mathcal{Q}} U)^{\vee}) \) is a quotient of \( T^*(V^{\vee} \oplus U^{\vee}) \) by the relations
\[
\begin{align*}
R^{\sharp\sharp}(v' \otimes \bar{v}') &= qv' \otimes \bar{v}' \quad \text{for } v', \bar{v}' \in V^{\vee}, \\
Q^{\sharp\sharp}(u' \otimes v') &= qu' \otimes v' \quad \text{for } v' \in V^{\vee}, u' \in U^{\vee}, \\
R^{\sharp\sharp}(u' \otimes \bar{u}') &= qu' \otimes \bar{u}' \quad \text{for } u', \bar{u}' \in U^{\vee}.
\end{align*}
\]
The commutation relations between \( x \in V \oplus_{\mathcal{Q}} U \) and \( x' \in (V \oplus_{\mathcal{Q}} U)^{\vee} \) in \( K^{\bullet\bullet}(V \oplus_{\mathcal{Q}} U) \) are
\[
\mathcal{R}^z(x \otimes x') = qx \otimes x'.
\]
In particular, (3.3) implies that \( K^{\bullet\bullet}(U) \) is a subalgebra of \( K^{\bullet\bullet}(V \oplus_{\mathcal{Q}} U) \), but \( K^{\bullet\bullet}(V) \) is not. In fact, the following commutation relation between \( v \in V \) and \( v' \in V^{\vee} \) holds in \( K^{\bullet\bullet}(V \oplus_{\mathcal{Q}} U) \)
\[
R^z(v \otimes v') + (q - q^{-1})\langle v, v' \rangle d_U = qv \otimes v',
\]
where \( d_U = u^j u_j \).

The factorization property (3.2) applied to \( V \oplus_{\mathcal{Q}} U \) together with factorization properties of \( \mathcal{S}^*((V \oplus_{\mathcal{Q}} U)^{\vee}) \) and \( \Lambda^*(V \oplus_{\mathcal{Q}} U) \) imply that
\[
\mathcal{S}(V^{\vee}) \otimes \Lambda(V) \otimes K(U) \rightarrow K(V \oplus_{\mathcal{Q}} U) \xrightarrow{\mathcal{M}} K(V \oplus_{\mathcal{Q}} U)
\] (3.4)
is an isomorphism of graded vector spaces. Proposition 3.1 shows that (3.2) is an isomorphism of complexes if the differential in $S^* (V^\vee) \otimes \Lambda^* (V)$ is the insertion of $d_V = v^i \otimes v_i$ in the middle:

$$D(y^1 \ldots y^m \otimes y_1 \ldots y_n) = y^1 \ldots y^m v^i \otimes v_i y_1 \ldots y_n. \tag{3.5}$$

Composing (3.2) and (3.4) we get an isomorphism of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-graded spaces

$$\phi : K^{**} (V) \otimes K^{**} (U) \to S^* (V^\vee) \otimes \Lambda^* (V) \otimes K^{**} (U) \to K^{**} (V \oplus_Q U).$$

**Theorem 3.4.** (a) The map $\phi$ is an isomorphism of complexes, that is,

$$D_{V \oplus_Q U} \circ \phi = \phi (D_V \otimes 1 + (-1)^{n} \otimes D_U) : K^{n,m} (V) \otimes K^{k,l} (U) \to K^{n+k+1,m+l+1} (V \oplus_Q U).$$

(b) The isomorphism $\phi^{-1}$ induces an isomorphism

$$\text{Ber} (V \oplus_Q U) \simeq \text{Ber} (V) \otimes \text{Ber} (U).$$

**Proof.** (a) From the commutation relations in $K (V \oplus_Q U)$

$$Q(u \otimes v) = -qu \otimes v, \quad Q^\vee (v \otimes u') = qv \otimes u'$$

we find for $v \in V$, $u \in U$, $u' \in U^\vee$

$$u \otimes v = -q^{-1} Q(u \otimes v), \quad u' \otimes v = qQ^\vee (u' \otimes v).$$

Hence,

$$d_V u = u' v, \quad v = -q^{-1} u Q(u \otimes v) = -(Q^\vee \otimes 1)(u \otimes Q (u \otimes v)) = -vu' u = -v d_V.$$

Using Proposition 3.1 for $X = V \oplus_Q U$, the above formula and (3.5) we compute for arbitrary $v_1' \in V^\vee$, $y_0 \in V$, $u_i' \in U^\vee$, $z_i \in U$

$$D_X (v_1' \ldots v_m y_1 \ldots y_n u_1' \ldots u_i' z_1 \ldots z_k) =$$

$$= d_X v_1' \ldots v_m y_1 \ldots y_n u_1' \ldots u_i' z_1 \ldots z_k$$

$$= v_1' \ldots v_m d_X y_1 \ldots y_n u_1' \ldots u_i' z_1 \ldots z_k$$

$$= v_1' \ldots v_m (d_V + d_U) y_1 \ldots y_n u_1' \ldots u_i' z_1 \ldots z_k$$

$$= v_1' \ldots v_m d_V y_1 \ldots y_n u_1' \ldots u_i' z_1 \ldots z_k$$

$$+ (-1)^n v_1' \ldots v_m y_1 \ldots y_n d_U u_1' \ldots u_i' z_1 \ldots z_k$$

$$= D_X (v_1' \ldots v_m y_1 \ldots y_n) u_1' \ldots u_i' z_1 \ldots z_k$$

$$+ (-1)^n v_1' \ldots v_m y_1 \ldots y_n D_U (u_1' \ldots u_i' z_1 \ldots z_k).$$

(b) Follows from (a) by K"unneth’s theorem \cite{8}.

### 3.4. Dual Koszul complex.

Consider now the Koszul complex $K^{**} (V^\vee)$ of the left dual $V^\vee$, which is generated by $V^\vee \oplus V$ and has the relations

$$R^p (v \otimes \bar{v}) = -q^{-1} v \otimes \bar{v} \quad \text{for } v, \bar{v} \in V^\vee,$$

$$R^p (v \otimes \bar{v}) = q' v \otimes \bar{v} \quad \text{for } v \in V, v' \in V^\vee,$$

$$R (v \otimes \bar{v}) = q v \otimes \bar{v} \quad \text{for } v, \bar{v} \in V.$$

The middle commutation relation is

$$v \otimes v' = q c_{V^\vee V} (v \otimes v').$$
It has a special element
\[ d' = c_{V \vee V}(v_i \otimes i^*v) = q^{-1}(v_i)i^*v = q^{-1}d_{V \vee V} \in K^{1,1}(\vee V). \]

We shall be interested in the differential \( D_{V \vee V} \) given by the right multiplication by \( d' \) (it differs from \( D_{V \vee V} \) by a power of \( q \) with a sign depending on the grading, as Proposition 3.1 shows).

We want to extend the pairing \( \langle \cdot, \cdot \rangle = ev : \vee V \otimes V \to k \) to a pairing between Koszul complexes of \( \vee V \) and \( V \). The natural pairings \( T^n(\vee V) \otimes T^n(V) \to k \), \( T^n(V) \otimes T^n(\vee V) \to k \) are also denoted \( \langle \cdot, \cdot \rangle \), \( \langle y_n \ldots y_1, z_1 \ldots z_n \rangle = \langle y_1, z_1 \rangle \ldots \langle y_n, z_n \rangle \). We use Jimbo's symmetrizer and antisymmetrizer \[ 6\]

interpreted as a pairing \( \pi \) coincides with
\[ \langle \cdot, \cdot \rangle \]
induces a pairing satisfying \( \text{Sym}^m \)
\[ \text{Sym}_m = \sum_{\sigma \in \mathfrak{S}_m} q^{l(\sigma)-m(m-1)/2} R_{\sigma} \]
\[ = \sum_{\sigma \in \mathfrak{S}_m} q^{m(m-1)/2-l(\sigma)}(R^{-1})_{\sigma} : T^m(X) \to T^m(X), \] (3.6)
\[ \text{Ant}_n = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{l(\sigma)}q^{n(n-1)/2-l(\sigma)} R_{\sigma} \]
\[ = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{l(\sigma)}q^{l(\sigma)-n(n-1)/2}(R^{-1})_{\sigma} : T^n(X) \to T^n(X), \] (3.7)

satisfying \( (\text{Sym}_m)^2 = [m]_q!\text{Sym}_m, (\text{Ant}_n)^2 = [n]_q!\text{Ant}_n \).

The pairing
\[ T^n(\vee V) \otimes T^m(V) \otimes T^m(\vee V) \otimes T^n(V) \to k \]
a \otimes b \otimes c \otimes f \longleftrightarrow \langle a, \text{Ant}_n(f) \rangle \langle \text{Sym}_m(b), c \rangle = \langle \text{Ant}_n(a), f \rangle \langle b, \text{Sym}_m(c) \rangle

vanishes on ideals of relations of symmetric and exterior algebras. Therefore, it induces the pairing
\[ \pi : \Lambda^n(\vee V) \otimes S^m(V) \otimes S^m(\vee V) \otimes \Lambda^n(V) \to k \]
or
\[ \pi : K^{n,m}(\vee V) \otimes K^{n,m}(V) \to k. \] (3.8)

When \( q \) is not a root of unity this pairing is non-degenerate by the theory of Iwahori–Hecke algebras. In this case the natural map \( \text{Im Sym}_m \to S^m(V) \) is an isomorphism and \( \text{Sym}_m \) is proportional to a projection. When \( q \) is a root of unity the pairing is usually degenerate.

Similarly, the pairing
\[ T^m(V) \otimes T^n(\vee V) \otimes T^n(V) \otimes T^m(\vee V) \to k \]
b \otimes a \otimes f \otimes c \longleftrightarrow \langle a, \text{Ant}_n(f) \rangle \langle \text{Sym}_m(b), c \rangle

induces a pairing
\[ S^m(V) \otimes \Lambda^n(\vee V) \otimes \Lambda^n(V) \otimes S^m(\vee V) \to k. \]

Interpreted as a pairing
\[ K^{n,m}(\vee V) \otimes K^{n,m}(V) \to k \]
it coincides with \( \pi \) from (3.8). Indeed, using the commutation relations (3.1) and
\[ 'v \otimes v = q^{-1}(e_{V \vee V})^{-1}(v \otimes v) \]
for \( v \in V \), we can write both pairings as quotients of equal maps

\[
T^n(V) \otimes T^m(V) \otimes T^m(V) \otimes T^n(V) \to k
\]

Proposition 3.5. (a) The map

\[
D_{\mathcal{V}}^R = \sum_{k=1}^{m} \sum_{l=1}^{n} (-1)^{n-l} q^{k+l-m-1}
\]

composed with the projection \( T^{n-1}(V) \otimes T^{m-1}(V) \to \Lambda^{n-1}(V) \otimes S^{m-1}(V) \) factorizes through a unique map

\[
D_{\mathcal{V}}^R : \Lambda^n(V) \otimes S^m(V) \to \Lambda^{n-1}(V) \otimes S^{m-1}(V).
\]

(b) The map \( D_{\mathcal{V}}^R : K^{n,m}(V) \to K^{n-1,m-1}(V) \) so constructed is a transpose of \( D_{\mathcal{V}} \) in the following sense

\[
\pi(D_{\mathcal{V}}^R(a), b) = \pi(a, D_{\mathcal{V}}(b))
\]

for \( a \in K^{n,m}(V) \), \( b \in K^{n-1,m-1}(V) \).

(c) \((D_{\mathcal{V}}^R)^2 = 0 : K^{n,m}(V) \to K^{n-2,m-2}(V)\).

Proof. (a) Using the identity \( R - R^{-1} - q + q^{-1} = 0 \) one can show that the ideal of relations \([T^n(V) \cap (\text{Im}(R^b + q^{-1}))] \otimes T^m(V)\) goes to \([T^{n-1}(V) \cap (\text{Im}(R^b + q^{-1}))] \otimes T^{m-1}(V)\), that is to itself, and \(T^n(V) \otimes [(\text{Im}(R - q)) \cap T^m(V)] \) goes to \(T^{n-1}(V) \otimes [(\text{Im}(R - q)) \cap T^{m-1}(V)]\).

(b) Equation (3.10) follows from the equation

\[
\left( T^n(V) \otimes T^m(V) \xrightarrow{D_{\mathcal{V}}^R} T^{n-1}(V) \otimes T^{m-1}(V) \right)
\]

\[\xrightarrow{\text{Ant}_{n-1} \otimes \text{Sym}_{m-1}} T^{n-1}(V) \otimes T^{m-1}(V) \]

\[
= \left( T^n(V) \otimes T^m(V) \xrightarrow{\text{Ant}_n \otimes \text{Sym}_m} T^n(V) \otimes T^m(V) \right)
\]

\[\xrightarrow{1 \otimes \text{ev} \otimes 1} T^{n-1}(V) \otimes T^{m-1}(V) \]
Substituting for $\text{Sym}_m$ and $\text{Ant}_n$ in the right hand side their expressions (3.6) and (3.7), we reorder the sum, picking up such $\sigma \in \mathfrak{S}_m$ that $\sigma(k) = 1$ and such $\sigma \in \mathfrak{S}_n$ that $\sigma(l) = 1$. The sum in the left hand side with fixed $k, l$ equals the $(k, l)^{th}$ summand of $D^R_{n-1}$ multiplied by $\text{Ant}_{n-1} \otimes \text{Sym}_{m-1}$.

(c) $D_V$ is a differential; this explains why $D^R_{V}$ is. The proof consists of straightforward calculation. □

**Proposition 3.6.** (a) The map

$$
\sum_{k=1}^{m} \sum_{l=1}^{n} (-1)^{l-1} q^{k+l-n-1} \sum_{k=1}^{m} \sum_{l=1}^{n} \begin{array}{cc}
1 & k \\
& & m \\
1 & l & n
\end{array}
$$

: $T^m(V^\vee) \otimes T^n(V) \to T^{m-1}(V^\vee) \otimes T^{n-1}(V) \to S^{m-1}(V^\vee) \otimes \Lambda^{n-1}(V)$ (3.11)

factorizes through a unique map

$$D'_V : S^m(V^\vee) \otimes \Lambda^n(V) \to S^{m-1}(V^\vee) \otimes \Lambda^{n-1}(V).$$

(b) The constructed map $D'_V : K^{n,m}(V) \to K^{n-1,m-1}(V)$ is a transpose of $D^R_V$ in the following sense:

$$\pi(D^R_V(a), b) = \pi(a, D'(b))$$

for $a \in K^{n-1,m-1}(V)$, $b \in K^{n,m}(V)$.

(c) $D^2 = 0$.

The proof is similar to that of Proposition 3.5.

**Proposition 3.7.** The differentials are related as follows:

$$D_V = (-1)^n q^{2n+1} D^R_V : K^{n,m}(V) \to K^{n+1,m+1}(V)$$

$$D^R_V = (-1)^{n+1} q^{2n-1} D'_V : K^{n,m}(V) \to K^{n-1,m-1}(V)$$

**Proof.** The first formula follows from Proposition 3.1. To get the second formula we represent $K^{n,m}(V)$ as $S^m(V^\vee) \otimes \Lambda^n(V)$. Then $D^R_V$ equals

$$S^m(V^\vee) \otimes \Lambda^n(V) \xrightarrow{q^{n(m,n)}} \Lambda^n(V) \otimes S^m(V^\vee) \xrightarrow{D^R_V} \Lambda^{n-1}(V) \otimes S^{m-1}(V^\vee) \xrightarrow{q^{-(n-1)(m-1)}} \Lambda^{n-1}(V) \otimes S^{m-1}(V^\vee)$$

where the middle arrow is given by (3.9). Drawing this map, we reduce it to a factor map of

$$\sum_{k=1}^{m} \sum_{l=1}^{n} (-1)^{n-l} q^{k+l+n-2} \sum_{k=1}^{m} \sum_{l=1}^{n} \begin{array}{cc}
1 & k \\
& & m \\
1 & l & n
\end{array}
$$

: $T^m(V^\vee) \otimes T^n(V) \to T^{m-1}(V^\vee) \otimes T^{n-1}(V) \to S^{m-1}(V^\vee) \otimes \Lambda^{n-1}(V)$. 

This differs from (3.11) by a power of $q$ with a sign. □

3.5. Differentials on a Hecke sum. It is possible to express the differentials $D'^R$, $D'$, $D_R$ on a Hecke sum in terms of those on summands.

Theorem 3.8. The maps

$$\psi: \Lambda^p(\vee U) \otimes S^r(U) \otimes K^{n,m}(\vee V) \to K^{\bullet \bullet}(\vee (V \oplus Q U)) \otimes^3 m \to K^{p+n,r+m}(\vee (V \oplus Q U))$$

combine into an isomorphism of graded vector spaces. The differential $D'^R$ satisfies

$$D'^R_{\vee X} \circ \psi = \psi \circ (q^{p+r} \otimes D'^R_{\vee V} + D'^R_{\vee U} \otimes (-1)^{n+q+m})$$

$$: K^{p,r}(\vee U) \otimes K^{n,m}(\vee V) \to K^{p+n-1,r+m-1}(\vee X).$$

Proof. We have to compute the following expression for $D'^R_{\vee X}$:

$$\Lambda^p(\vee U) \otimes S^r(U) \otimes \Lambda^n(\vee V) \otimes S^m(V)$$

$$\downarrow \Lambda^p(\vee U) \otimes \Lambda^n(\vee V) \otimes S^r(U) \otimes S^m(V)$$

$$\downarrow D'^R_{\vee X}$$

$$\Lambda^{p-1}(\vee U) \otimes \Lambda^n(\vee V) \otimes S^{r-1}(U) \otimes S^m(V) + \oplus \Lambda^p(\vee U) \otimes \Lambda^{n-1}(\vee V) \otimes S^r(U) \otimes S^{m-1}(V) + \ldots$$

$$\downarrow \oplus \Lambda^p(\vee U) \otimes \Lambda^{n-1}(\vee V) \otimes S^r(U) \otimes S^{m-1}(V) + \ldots$$

where we have singled out two summands from four (or even more) possible. In fact these two suffice, as calculation shows:

$$D'^R_{\vee X} = \sum_{a=1}^{p} \sum_{b=1}^{r} (-1)^{p+n-a} q^{b+a-r-m-1+rn-n(r-1)}$$
+ \sum_{b=1}^{r} \sum_{c=1}^{n} (-1)^{n-c} q^{b+p+c-r-m-1+rn-(n-1)(r-1)}

+ \sum_{c=1}^{n} \sum_{d=1}^{m} (-1)^{n-c} q^{d+p+c-m-1+rn-(n-1)r}

+ \sum_{a=1}^{p} \sum_{d=1}^{m} (-1)^{n+p-a} q^{d+a-m-1+rn-ar}

Here all crossings are interpreted via $\mathcal{R}$ which reduces to $R$ and $R'$ in the first and the third sum, giving $D_{U}^{R} \otimes (-1)^{n}q^{n-m}$ and $q^{p+r} \otimes D_{V}^{R}$ respectively. The second sum vanishes, but the fourth does not. Indeed, using the equation

$$\mathcal{R} = Q \oplus (q - q^{-1}) : U \otimes V \to V \otimes U \oplus U \otimes V$$

we find the fourth sum

$$\sum_{a=1}^{p} \sum_{d=1}^{m} (-1)^{n+p-a} q^{d+a-m-1} =$$
\[
(q - q^{-1}) \sum_{a=1}^{p} \sum_{d=1}^{m} \sum_{t=1}^{r} \sum_{u=1}^{n} (-1)^{n+p-a} q^{d+a-m-1} = (q - q^{-1}) D^R_U \otimes (-1)^n q^{-m} \sum_{d=1}^{m} q^{2d-1} = D^R_U \otimes (-1)^n q^{-m} (q^{2m} - 1).
\]

Summing up everything we get
\[
D^R_X = D^R_U \otimes (-1)^n q^{-m} + q^{p+r} \otimes D^R_V + D^R_U \otimes (-1)^n q^{-m} (q^{2m} - 1)
= D^R_U \otimes (-1)^n q^{n+m} + q^{p+r} \otimes D^R_V,
\]

omitting \(\psi\). □

The differential \(D^R_X\) is a transpose of \(D_X\), so it seems that Theorems 3.4 and 3.8 contradict each other. The following Proposition shows that there is no contradiction, and, moreover, Theorem 3.8 can be deduced from Theorem 3.4 and vice versa on the assumption that \(q\) is not a root of unity. We preferred to give independent proofs, which work for any \(q\).

**Proposition 3.9.** Let \(x \in K^{n,m}(V), y \in K^{p,r}(U), z \in K^{p,r}(\U), w \in K^{n,m}(\U)\). Then
\[
\pi_X(\psi(z \otimes w), \phi(x \otimes y)) = q^{n+p+m} \pi_V(w, x) \pi_U(z, y).
\]

**Proof.** Let \(a \in T^m(V^\U), b \in T^n(V), x = ab, c \in T^r(U^\U), d \in T^p(U), y = cd, e \in T^p(V^\U), f \in T^r(U), z = ef, g \in T^n(V^\U), h \in T^m(V), w = gh\). Then
\[
\pi_V(w, x) \pi_U(z, y) = \pi_V(gh, ab) \pi_U(e, cd)
= \langle h, \text{Sym}_m a \rangle \langle g, \text{Ant}_a b \rangle \langle f, \text{Sym}_r c \rangle \langle e, \text{Ant}_p d \rangle.
\]
Denote
\[ \tilde{c} \otimes \tilde{b} := (c^{-1})_{\tau(n,r)}(b \otimes c) \in T^r(U^\vee) \otimes T^n(V), \]
\[ \tilde{g} \otimes \tilde{f} := c_{\tau(n,r)}(f \otimes g) \in T^n(V) \otimes T^r(U). \]
Then
\[ \phi(x \otimes y) = \phi(ab \otimes cd) = q^{-nr} a \tilde{c} \tilde{b} d, \]
\[ \psi(z \otimes w) = \psi(ef \otimes gh) = q^{rn} e \tilde{g} \tilde{f} h, \]
and
\[ \pi_X(\psi(z \otimes w), \phi(x \otimes y)) = \langle e \otimes \tilde{g}, \langle \text{Sym}_{r+m}(\tilde{f} \otimes h), a \otimes \tilde{c} \rangle \text{Ant}_{n+p}(\tilde{b} \otimes d) \rangle. \]

Introduce symmetric and antisymmetric shuffling operators
\[ \text{Symshf}_{r,m} = \sum_{\sigma \in \text{Shuffles}_{r,m}} q^{rm-l(\sigma)}(\mathcal{R}^{-1})_{\sigma}, \]
\[ \text{Antshf}_{n,p} = \sum_{\sigma \in \text{Shuffles}_{n,p}} (-1)^{l(\sigma)} q^{np-l(\sigma)}\mathcal{R}_\sigma. \]
Then we have
\[ \text{Sym}_{r+m} = \text{Symshf}_{r,m} \circ \text{Sym}_r \otimes \text{Sym}_m, \]
\[ \text{Ant}_{n+p} = \text{Antshf}_{n,p} \circ \text{Ant}_n \otimes \text{Ant}_p. \]
Plugging this into (3.12), we see that only one term with \( \sigma = 1 \) contributes from each shuffling operator. Therefore
\[ \pi_X(\psi(z \otimes w), \phi(x \otimes y)) = q^{np+mr} \langle e \otimes \tilde{g}, \langle \text{Sym}_r \tilde{f} \otimes \text{Sym}_m h, a \otimes \tilde{c} \rangle \text{Ant}_n \tilde{b} \otimes \text{Ant}_p d \rangle \]
\[ = q^{np+mr} \langle \text{Sym}_m h, a \rangle \langle \tilde{g} \otimes \text{Sym}_r \tilde{f}, \tilde{c} \otimes \text{Ant}_n \tilde{b} \rangle \langle e, \text{Ant}_p d \rangle \]
\[ = q^{np+mr} \langle h, \text{Sym}_m a \rangle \langle \text{Sym}_r f \otimes g, \text{Ant}_n b \otimes c \rangle \langle e, \text{Ant}_p d \rangle \]
\[ = q^{np+mr} \pi_V(w, x) \pi_U(z, y). \]
Here we used the equation
\[ q^n \]
\[ \begin{array}{c}
\text{Sym}_r \\
\text{Ant}_n = \\
\text{Sym}_r \\
\text{Ant}_n
\end{array} \]
The result of this proposition can be stated as $\psi = q^{np + mrt} \phi^{-1}$, and using this one can prove the equivalence of Theorems 3.4 and 3.8, when the pairings $\pi$ are not degenerate, that is when $q$ is not a root of unity.

3.6. 8-dimensions. The number which is called 8-dimension here was studied in [17] under the name of rank. We believe that our term is less confusing.

**Definition 3.3.** The 8-dimension of $V$ is the number

$$\dim_8 V = (k \xrightarrow{\text{coev}} V^\vee \otimes V \xrightarrow{\zeta} V \otimes V^\vee \xrightarrow{\text{ev}} k) \equiv \begin{array}{c} V \\ \bigcirc \bigcirc \\ V \end{array}$$

The properties of 8-dimension for Hecke $R$-matrices are summarized in the following

**Proposition 3.10.** (a) The addition formula for 8-dimensions

$$\dim_8 (V \oplus Q U) = \dim_8 V + \dim_8 U - (q - q^{-1}) \dim_8 V \dim_8 U.$$

(b) The 8-dimension is related to the automorphism $V \rightarrow V$

$$\nu_V^{-2} \equiv \begin{array}{c} V \\ \bigcirc \bigcirc \\ V \end{array} = 1 - (q - q^{-1}) \dim_8 V : V \rightarrow V.$$

(c) $\dim_8 V^\vee = \dim_8 V$.

**Proof.** (a) By definition

$$\dim_8 (V \oplus Q U) = (k \xrightarrow{\text{coev}} (V \oplus Q U)^\vee \otimes (V \oplus Q U) \xrightarrow{R^{z-1}} (V \oplus Q U) \otimes (V \oplus Q U)^\vee \xrightarrow{\text{ev}} k).$$

By (3.3) we get

$$R^{z-1} = \begin{pmatrix} R^{z-1} & 0 & 0 & A \\ 0 & 0 & Q^{-1} & 0 \\ 0 & Q^{z-1} & 0 & 0 \\ 0 & 0 & 0 & R^{z-1} \end{pmatrix} , \quad (3.13)$$

where

$$A = (q^{-1} - q) \left( V^\vee \otimes V \xrightarrow{R^{z-1}} V \otimes V^\vee \xrightarrow{\text{ev}} k \xrightarrow{\text{coev}} U^\vee \otimes U \xrightarrow{R^{z-1}} U \otimes U^\vee \right).$$
Plugging this into the definition,
\[
\dim_8(V \oplus Q U) = (k \xrightarrow{\text{coev}} V^\vee \otimes V \xrightarrow{R_{V}^{-1}} V \otimes V^\vee \xrightarrow{\text{ev}} k) \\
+ (q^{-1} - q)(k \xrightarrow{\text{coev}} V^\vee \otimes V \xrightarrow{R_{V}^{-1}} V \otimes V^\vee \xrightarrow{\text{ev}} k) \\
+ (k \xrightarrow{\text{coev}} U^\vee \otimes U \xrightarrow{R_{U}^{-1}} U \otimes U^\vee \xrightarrow{\text{ev}} k) \\
= \dim_8 V + (q^{-1} - q) \dim_8 V \dim_8 U + \dim_8 U.
\]

(b) Follows from the identity
\[
\begin{array}{c}
V \\
V
\end{array}
+ (q - q^{-1})
\begin{array}{c}
V \\
V
\end{array}
= \begin{array}{c}
V \\
V
\end{array}
= \begin{array}{c}
V \\
V
\end{array}
= \dim_8 V.
\]

(c) We have
\[
\dim_8 V = \begin{array}{c}
V \\
V
\end{array} = \begin{array}{c}
V \\
V
\end{array} = \begin{array}{c}
V \\
V
\end{array} = \dim_8 V.
\]

\[\square\]

**Theorem 3.11.** The differential $D^{R}_X$ satisfies
\[
D^{R}_X \circ \psi = \psi \circ (\nu_U^{-2} q^{-2r} \otimes D^{R}_V + D^{R}_U \otimes (-1)^n q^{-2n})
\colon K^{p,r}(V^\vee) \otimes K^{n,m}(V^\vee) \to K^{p+n+1,r+m+1}(V^\vee X).
\]

**Proof.** The proof is quite analogous to that of Theorem 3.4. The only new point is the decomposition of $d'_{X}$ into a $U$-part and a $V$-part:
\[
d'_{X} = c_{V^\vee V}(v_i \otimes i' v + u_i \otimes i' u) = R_{V}^{b-1}(v_i \otimes i' v + u_i \otimes i' u)
= R_{V}^{b-1}(v_i \otimes i' v) + R_{V}^{b-1}(u_i \otimes i' u) + (q^{-1} - q) \dim_8 V \dim_8 U R_{V}^{b-1}(v_i \otimes i' v)
= R_{V}^{b-1}(u_i \otimes i' u) + (1 + (q^{-1} - q) \dim_8 V) R_{V}^{b-1}(v_i \otimes i' v)
= d'_{V} + \nu_U^{-2} d'_{V}
\]

Here we used the explicit form of
\[
\mathcal{R}^{b-1} = \begin{pmatrix}
R_{V}^{b-1} & 0 & 0 & 0 \\
0 & 0 & Q_{V}^{-1} & 0 \\
0 & Q_{V}^{-1} & 0 & 0 \\
F & 0 & 0 & R_{V}^{b-1}
\end{pmatrix},
\]

where
\[
F = (q^{-1} - q)(U \otimes V \xrightarrow{R_{V}^{b-1}} V^\vee \otimes U \xrightarrow{\text{ev}} k \xrightarrow{\text{coev}} V \otimes V \xrightarrow{R_{V}^{b-1}} V \otimes V).
\]
So the factor \( \nu_U^{-2} \) appears in the statement. \( \square \)

**Theorem 3.12.** (a) The differential \( D' \) satisfies

\[
D'_{V \oplus U} \circ \phi = \phi \circ (D'_V \otimes q^{m-n} \otimes D'_U)
\]

: \( K^{n,m}(V) \otimes K^{p,r}(U) \rightarrow K^{n+p-1,m+r-1}(V \oplus U). \)

(b) The map induced by \( \phi \)

\[
H^*(K(V), D'_V) \otimes H^*(K(U), D'_U) \rightarrow H^*(K(V \oplus U), D'_{V \oplus U})
\]

is an isomorphism.

**Proof.** (a) The proof is similar to that of Theorem 3.8. When calculating \( D'_X \) one uses the identity

\[
(V \otimes V \xrightarrow{c_{U,V,X}} X \otimes X^\vee \xrightarrow{\ev} k) = \nu_U^{-2} (V \otimes V \xrightarrow{c_{U,V,X}} V \otimes V^\vee \xrightarrow{\ev} k).
\]

It is proven using (3.13)

\[
(V \otimes V \xrightarrow{R_{V,U}} X \otimes X^\vee \xrightarrow{\ev} k) =
\]

\[
= (V \otimes V \xrightarrow{R_{V,U}} V \otimes V^\vee \xrightarrow{\ev} k)
\]

\[
+ (q^{-1} - q)(V \otimes V \xrightarrow{R_{V,U}} V \otimes V^\vee \xrightarrow{\ev} k \xrightarrow{\coev} U^\vee \otimes U \xrightarrow{R_{V,U}} U \otimes U^\vee \xrightarrow{\ev} k)
\]

\[
= (V \otimes V \xrightarrow{R_{V,U}} V \otimes V^\vee \xrightarrow{\ev} k)(1 + (q^{-1} - q) \dim U)
\]

\[
= \nu_U^{-2} (V \otimes V \xrightarrow{R_{V,U}} V \otimes V^\vee \xrightarrow{\ev} k).
\]

(b) Renormalize the differentials introducing

\[
\bar{D}_V = \nu_U^{-2} q^{-n-m} D'_{V}, \quad K^{n,m}(V) \rightarrow K^{n-1,m-1}(V),
\]

\[
\bar{D}_U = q^{-r-p} D'_U, \quad K^{p,r}(U) \rightarrow K^{p+1,r+1}(U),
\]

\[
\bar{D}_X = q^{-p+n-m} \phi^{-1} D'_X \phi : K^{m,n}(V) \otimes K^{p,r}(U) \rightarrow K^{**}(V) \otimes K^{**}(U).
\]

These are also differentials and \( H^*(K(V), \bar{D}_V) = H^*(K(V), D'_V), H^*(K(U), \bar{D}_U) = H^*(K(U), D'_U), H^*(K(V) \otimes K(U), \bar{D}_X) \simeq H^*(K(X), D'_X) \). But \( \bar{D}_X = \bar{D}_V \otimes 1 + (-1)^n \otimes \bar{D}_U \) and the Künneth theorem says that the tensor product of the two first spaces is isomorphic to the third one. \( \square \)

When \( q \) is not a root of unity, Theorems 3.11 and 3.12 can be deduced from each other, using duality and Proposition 3.9.

### 3.7. The Laplacian.

**Definition 3.4.** The anticommutator

\[
L = D'D + DD' : K^{m,n}(V) \rightarrow K^{m,n}(V)
\]

is called the *Laplacian for the Koszul bidifferential complex.*

Theorems 3.4 and 3.12 have a straightforward
Corollary 3.13. The Laplacian for a Hecke sum can be calculated on $K^{n,m}(V) \otimes K^{p,r}(U)$ as

$$L_X = L_V \otimes q^{p-r}V^{-2}_U + q^{m-n} \otimes L_U.$$ 

The Laplacian is calculated for an arbitrary Hecke $\hat{R}$-matrix in

Theorem 3.14. The Laplacian in $K(V)$ is multiplication by the number

$$L|_{K^{m,n}(V)} = q^{n-m} \dim_8 V + [m - n]_q.$$ 

Proof. We compute the lifting of $L$ to the space $T^m(V^\vee) \otimes T^n(V)$:

$$L = D'D + DD'$$

$$= q^{-n-m}$$

$$+ \sum_{l=1}^{n} (-1)^l q^{l-n-m}$$

$$+ \sum_{k=1}^{m} q^{k-n-m}$$

$$+ \sum_{1 \leq k \leq m \atop 1 \leq l \leq n} (-1)^l q^{k+l-n-m}$$

$$+ \sum_{1 \leq k \leq m \atop 1 \leq l \leq n} (-1)^{l-1} q^{k+l-n-m}$$

$$= q^{-n-m} \dim_8 V + \sum_{l=1}^{n} (-1)^l q^{l-n-m} V^{-2}_V$$


\[+ \sum_{k=1}^{m} q^{k-n-m} \]

When this map projects to \( S^m(V^\vee) \otimes \Lambda^n(V) \) the braiding can be replaced by its eigenvalue \( q \) or \( -q^{-1} \)

\[
L|_{K^{n,m}(V)} = q^{-n-m} \dim_S V - \nu_V^{-2} \sum_{l=1}^{n} q^{2l-1-n-m} + \sum_{k=1}^{m} q^{2k-1-n-m}
\]

\[
= q^{-n-m} \left\{ \dim_S V + (-1 + (q - q^{-1}) \dim_S V)(q^{2n} - 1)/(q - q^{-1})
\right. \\
\left. + (q^{2m} - 1)/(q - q^{-1}) \right\}
\]

\[
= q^{-n-m} \left\{ q^{2n} \dim_S V + (q^{2m} - q^{2n})/(q - q^{-1}) \right\}
\]

\[
= q^{n-m} \dim_S V + [m - n]_q.
\]

Assume that the cohomology space \( H^{n,m}(K(V), D) \subset \text{Ber} V \) is non-trivial. Since the restriction of \( L \) to \( H^{n,m}(K(V), D) \) vanishes, this can happen only if \( L|_{K^{n,m}(V)} = q^{-n-m} \dim_S V + [m - n]_q = 0 \). So we find

\[
\dim_S V = q^{m-n} [n - m]_q,
\]

\[
\nu_V^{-2} = q^{2(n-m)}.
\]

If, additionally, the category is a ribbon category with the ribbon twist \( \nu_V = q^{n-m} : V \to V \) (square root of \( \nu_V^2 \)) we find a categorical dimension of \( V \) as

\[
\dim_C V \equiv \quad \nu = [n - m]_q.
\]

We conclude that 8-dimension is a sort of categorical dimension (a \( q \)-integer) multiplied by a power of \( q \).

\textbf{Remark 3.1.} The construction of the Berezinian applies to an object \( V \) in an abelian \( k \)-linear braided rigid monoidal category \( C \) with the braiding \( B = c_{V,V} : V \otimes V \to V \otimes V \), satisfying \( (B - q)(B + q^{-1}) = 0 \). As an example take a \( \mathbb{C} \)-linear category \( C_1 \)
generated over $\mathbb{C}$ by tangles with two colours denoted $V$ and $V^\vee$ with the relations
\[
\begin{array}{cccccc}
V & V & V & V & V & V \\
\xymatrix{& & & & & \ar@{-}[l] }
\end{array} = (q - q^{-1})
\]
where $\alpha \in \mathbb{C} - 0$ is a parameter (see e.g. [13]). Denote by $\mathcal{C}$ its Karoubi envelope, whose objects are idempotents of $C_1$. The category $\mathcal{C}$ is a $\mathbb{C}$-linear braided rigid monoidal category. For generic values of the parameter $q$ (all except a countable number) this category is semisimple abelian. The above discussion shows that if $\alpha$ is not a power of $q^2$ the cohomology $\text{Ber} V$ vanishes. Therefore non-vanishing of the Berezinian implies a sort of integrality condition on structure constants of $\text{End} \mathcal{C}$.

**Conjecture.** Let $\mathcal{C}$ be a $k$-linear abelian braided rigid monoidal category with $\text{End}_\mathcal{C}I = k$, $\dim_k \text{Hom}_\mathcal{C}(A, B) < \infty$, generated by an object $V$ and its dual $V^\vee$ such that $(c_{V,V} - q)(c_{V,V} + q^{-1}) = 0$. If $\text{Ber} V$ is not null, it is an invertible object, and the pair $(n, m) \in \mathbb{Z}^2$ such that $\text{Ber} V = H^{n,m}(K(V), D)$ is called the superdimension of $V$, $\text{sdim} V = n|m$.

The object $V$ is called even if there exists $n \in \mathbb{Z}_{>0}$ such that $\Lambda^n(V)$ is invertible and $\Lambda^{n+1}(V) = 0$. The object $V$ is called odd if there exists $m \in \mathbb{Z}_{>0}$ such that $\mathcal{S}^m(V)$ is invertible and $\mathcal{S}^{m+1}(V) = 0$.

**Remark 3.2.** If $\mathcal{C}$ consists of $k$-vector spaces (it is equipped with a faithful exact monoidal functor $\mathcal{C} \to k\text{-vect}$), the number $n$ such that $\Lambda^n(V) \neq 0$, $\Lambda^{n+1}(V) = 0$, is not necessarily $\dim_k V$. There are examples constructed by Gurevich [4, 5] in which $n < \dim_k V$.

**Proposition 3.15.** Let $q$ be not a root of unity and let $\mathcal{C}$ be as in the above conjecture.

(a) Assume that $V \in \text{Ob} \mathcal{C}$ is even, $\Lambda^{n+1}(V) = 0$, $\Lambda^n(V) \neq 0$. Then $\text{Ber} V = \Lambda^n(V)$ and $\text{sdim} V = n|0$.

(b) Assume that $V \in \text{Ob} \mathcal{C}$ is odd, $\mathcal{S}^{n+1}(V) = 0$, $\mathcal{S}^n(V) \neq 0$. Then also $\mathcal{S}^{m+1}(V^\vee) = 0$, $\mathcal{S}^m(V^\vee)$ is invertible and $\text{Ber} V = \mathcal{S}^m(V^\vee)$, $\text{sdim} V = 0|m$.

**Proof.** (a) We have $K^{n,0}(V) = \Lambda^n(V)$ and $K^{n+1,1}(V) = 0$. Thus $K^{n,0}(V) \subset \text{Ker} D$ and $\text{Im} D \cap K^{n,0}(V) = 0$, wherefore $\Lambda^n(V) \subset \text{Ber} V$. This implies $\dim_k V = q^{-n}[n]q = (1 - q^{-2n})/(q - q^{-1})$. Since $q$ is not a root of unity, Theorem 3.14 claims that $L|_{K^{a,b}(V)} \neq 0$ if $a - b \neq n$. Therefore the subcomplexes $(K^{a,b}(V))_{a-b=p}$ are
acyclic except \( p = n \). The last subcomplex has only one non-zero term \( \Lambda^n(V) \) and the assertion follows.

(b) Similarly.

In many examples the vector space \( V \) with \((R-q)(R+q^{-1}) = 0\) can be represented in the form

\[
V = V_1 \oplus Q_1 V_2 \oplus Q_2 \cdots \oplus Q_{k-1} V_k
\]

with some order in which the operations \( \oplus Q_i \) are performed and any space \( V_i \) is even or odd. If \( q \) is not a root of unity, from the above Proposition and Theorem 3.4 we deduce that \( \text{Ber} V \) is one dimensional.

The case of a root \( q \) of unity does not follow from the above, but at least we know that if \( V \) is one-dimensional, so is \( \text{Ber} V \cong V \) or \( V^\vee \). Hence, \( V \) given by (3.15) with \( \dim k V_i = 1 \) has one-dimensional \( \text{Ber} V \). In particular, it is true for the vector representation \( V \) of \( U_q(\mathfrak{sl}(n|m)) \), \( \text{sdim} V = n|m \).

3.7.1. Using the renormalized differentials \( \bar{D} \) from (3.14) we can write Theorem 3.14 as

\[
D_V \bar{D}_V + \bar{D}_V D_V = \nu^{-2} q^{2(n-m)} q^{2(m-n)} - \nu^{-2}_q : K^{n,m}(V) \to K^{n,m}(V),
\]

\[
D_U \bar{D}_U + \bar{D}_U D_U = \nu^{-2} q^{2(r-p)} - \nu^{-2}_q : K^{p,r}(U) \to K^{p,r}(U).
\]

Write \( K^{\bullet\bullet}(V) \) as a direct sum of subcomplexes \( \oplus_{k \in \mathbb{Z}} C_V^\bullet(k) \) with \( C_V^\bullet(k) = K^{n,n-k}(V) \). Then

\[
K(X) = \oplus_{k,l \in \mathbb{Z}} C_V^\bullet(k) \otimes C_U^\bullet(l).
\]

Clearly, \( H^*(C_V^\bullet(k), D_V) = 0 \), \( H^*(C_U^\bullet(l), D_U) = 0 \) unless \( q^{2k} = \nu^2 \), and \( H^*(C_V^\bullet(k), D_U) = 0 \), \( H^*(C_U^\bullet(l), D_U) = 0 \) unless \( q^{2l} = \nu^2 \). When \( q^{2k} = \nu^2 \) (resp. \( q^{2l} = \nu^2 \)) the differentials \( D_V, D_V \) (resp. \( D_U, D_U \)) anticommute.

**Theorem 3.16.** The following conditions are equivalent for any \( V \) with a Hecke \( \bar{R} \)-matrix:

(i) \( K^{\bullet\bullet}(V) = I_{st} \oplus M \) is a direct sum of bidifferential subcomplexes \( M \) and \( I_{st} \) for some \( s, t \in \mathbb{Z}_{\geq 0} \), where \( H^*(M, D_V) = 0 \) and \( I_{st}^{n,m} = \mathbb{C} \) if \( n = s, m = t \) and 0 otherwise.

(ii) All the complexes \( C_V^\bullet(k) \) with \( q^{2k} = \nu^2 \) are \( D_V \)-acyclic, except one which decomposes as \( I_s \oplus N \) for some \( s \in \mathbb{Z}_{\geq 0} \), where \( H^*(N, D_V) = 0 \) and \( I_s^n = \mathbb{C} \) if \( n = s \) and 0 otherwise.

(iii) The natural embedding \( i \) and projection \( j \) in

\[
\frac{\text{Ker} D_V \cap \text{Ker} D_V'}{\text{Ker} D_V \cap \text{Ker} D_V'} = i \frac{\text{Ker} D_V \cap \text{Ker} D_V'}{\text{Im} D_V \cap \text{Ker} D_U'} \quad j \frac{\text{Ker} D_V \cap \text{Ker} D_V'}{\text{Im} D_V \cap \text{Ker} D_U'}
\]

are isomorphisms of one dimensional spaces.

**Proof.** (i) \( \iff \) (ii): Clear from the above reasoning.

(i) \( \Rightarrow \) (iii): The restriction of \( i \) (resp. \( j \)) to the subcomplex \( M \) is an embedding into (resp. surjection from) the zero space.
The proof of (iii) ⇒ (ii) is omitted. It follows by a classification theorem for indecomposable bidifferential complexes with anticommuting \( D, \bar{D} \), which hopefully will be published elsewhere. We do not use this implication in this paper.

**Proposition 3.17.** If \( V \) and \( U \) satisfies conditions (i) or (ii) of the above theorem, then so does \( V \oplus_Q U \).

**Proof.** The category of bidifferential complexes decomposable as in (i) is closed under tensor multiplication by Künnett’s theorem.

**Conjecture.** Any Hecke \( \hat{R} \)-matrix satisfies the conditions (i)-(iii) of Theorem 3.16.

3.7.2. **Duality for Berezinians.** Assume that \( q \) is not a root of unity, so that the pairing \( \pi : \mathcal{K}_{n,m}^0(V) \otimes \mathcal{K}_{n,m}^0(V) \to k \) is non-degenerate. Since \( D^R_{\pi} \) is the transpose of \( D \), the pairing

\[
(\text{Ker } D^R_{\pi}/ \text{Im } D^R_{\pi}) \otimes (\text{Ker } D_{\pi}/ \text{Im } D_{\pi}) \to k
\]

is non-degenerate. Hence, \( H^*(K^0(V), D_{\pi}) = H^*(D^R_{\pi}) \) is naturally a dual space to \( \text{Ber } V = H^*(K(V), D_{\pi}) \). This duality breaks down when \( q \) is a root of unity.

We shall denote

\[
\text{Ber'} V = \frac{\text{Ker } D_{\pi} \cap \text{Ker } D'_{\pi}}{\text{Ker } D_{\pi} \cap \text{Ker } D'_{\pi} \cap (\text{Im } D_{\pi} + \text{Im } D'_{\pi})}
\]

For any \( q \) \( \pi \) induces a pairing

\[
\frac{\text{Im } D^R_{\pi} + \text{Im } D^R_{\pi} + (\text{Ker } D'^R_{\pi} \cap \text{Ker } D^R_{\pi})}{\text{Im } D^R_{\pi} + \text{Im } D^R_{\pi}} \otimes \text{Ber'} V \to k,
\]

which is non-degenerate if \( q \) is not a root of unity. By Proposition 3.7 the left differentials \( D_{\pi}, D'_{\pi} \) can be used above instead of the right ones \( D^R_{\pi}, D'^R_{\pi} \). The first multiplicand can be written also as

\[
\frac{\text{Ker } D'_{\pi} \cap \text{Ker } D_{\pi}}{\text{Im } D'_{\pi} + \text{Im } D_{\pi} \cap (\text{Ker } D'_{\pi} \cap \text{Ker } D_{\pi})} = \text{Ber'} V.
\]

Therefore, we obtain a pairing

\[
\pi : \text{Ber'} V \otimes \text{Ber'} \to k,
\]

non-degenerate if \( q \) is not a root of unity.

**Conjecture.** The above pairing is always non-degenerate.

Theorems 3.4, 3.8, 3.11, 3.12 imply the existence of external products, coherent with \( i \)'s and \( j \)'s from Theorem 3.16(iii),

\[
\text{Ber'} V \otimes \text{Ber'} U \to \text{Ber'} (V \oplus_Q U), \quad [\omega_V] \otimes [\omega_U] \mapsto [\phi(\omega_V \otimes \omega_U)],
\]

\[
\text{Ber'} V \otimes \text{Ber'} V \to \text{Ber'} V (V \oplus_Q U), \quad [\omega_V] \otimes [\omega_U] \mapsto [\psi(\omega_V \otimes \omega_U)],
\]

where

\[
\omega_V \in \mathcal{K}_{n,m}^0(V) \cap \text{Ker } D_{\pi} \cap \text{Ker } D'_{\pi}, \quad (3.17)
\]

\[
\omega_U \in \mathcal{K}_{p,r}^0(U) \cap \text{Ker } D_{\pi} \cap \text{Ker } D'_{\pi}, \quad (3.18)
\]

\[
\omega_V \in \mathcal{K}_{p,r}^0(V) \cap \text{Ker } D_{\pi} \cap \text{Ker } D'_{\pi}, \quad (3.19)
\]

\[
\omega_V \in \mathcal{K}_{n,m}^0(V) \cap \text{Ker } D_{\pi} \cap \text{Ker } D'_{\pi} V \quad (3.20)
\]
Proposition 3.9 shows that
\[ \pi_X([\psi(\omega \nu \cup \omega \nu)], [\phi(\omega \nu \cup \omega \nu)]) = q^{np+mr} \pi_V([\omega \nu], [\omega \nu]) \pi_U([\omega \nu], [\omega \nu]) \]  
(3.21)

This gives some evidence in favour of the conjecture.

4. Calculation of Berezinians

4.1. The quantum superdeterminant. We calculate Berezinians in several examples and show that these spaces are one dimensional. Let a non-zero vector \( \omega \in \text{Ber} V \) constitute a basis. The coaction \( \delta : \text{Ber} V \rightarrow \text{Ber} V \otimes H \), \( \omega \mapsto \omega \otimes \tau \) determines a group-like element \( \tau \in H \), where \( H \) is the Hopf superalgebra determined by (2.9)–(2.13). Thus \( \varepsilon(\tau) = 1 \), \( \Delta(\tau) = \tau \otimes \tau \). The element \( \tau \) can be called \( \text{Ber} T \), the Berezinian of \( (T^i_j) \), or the quantum superdeterminant \( \text{sdet}_q T \).

The commutation properties of \( \tau \) are determined by the braiding properties of \( \omega \). Introduce linear bijections \( \alpha : V \rightarrow V \), \( v_i \mapsto v_j \lambda^j_i \), and \( \beta : V \rightarrow V \), \( v_i \mapsto v_j \beta^j_i \), by
\[ c(v \otimes \omega) = \omega \otimes \alpha(v), \]
\[ c(\omega \otimes v) = \beta(v) \otimes \omega. \]

**Proposition 4.1.** (a) The maps \( \alpha, \beta \) are symmetries of \( R : V \otimes V \rightarrow V \otimes V \) in the following sense
\[ (\alpha \otimes \alpha)R = R(\alpha \otimes \alpha), \quad (\beta \otimes \beta)R = R(\beta \otimes \beta). \]

(b) The maps \( \alpha \beta, \beta \alpha \) are automorphisms of the \( H \)-comodule \( V \).

(c) The element \( \omega \) has a well defined degree \( p(\omega) \) and
\[ \tau T^i_j \tau^{-1} = \sum_{i,k} (-1)^{p(\omega)p(v_i) - p(v_k)} \alpha^k_l T^l_k \alpha^{-1^j_i} = \sum_{i,k} (-1)^{p(\omega)p(v_i) - p(v_k)} \beta^{-1^j_i} T^k_l \beta^i_l \]
\[ \tau T^i_j \tau^{-1} = \sum_{i,k} (-1)^{p(\omega)p(v_i) - p(v_k)} \alpha^{-1^k_i} T^k_l \alpha^{j_i} = \sum_{i,k} (-1)^{p(\omega)p(v_i) - p(v_k)} \beta^k_i T^{-1^j_i} \beta^i_i \]

**Proof.** (a) Follows from the Yang–Baxter equation applied to \( V \otimes V \otimes \text{Ber} V \) and to \( \text{Ber} V \otimes V \otimes V \).

(b) Indeed, \( c^2 = 1 \otimes \beta \alpha : \text{Ber} V \otimes V \rightarrow \text{Ber} V \otimes V \) and \( c^2 = \alpha \beta \otimes 1 : V \otimes \text{Ber} V \rightarrow V \otimes \text{Ber} V \) are automorphisms.

(c) \( \text{Ber} V \) is a \( \mathbb{Z}/2 \)- or \( \mathbb{Z}/2 \times \mathbb{Z} \)-graded space. The formulae express the fact that \( c : V \otimes \text{Ber} V \rightarrow \text{Ber} V \otimes V \), \( c : \text{Ber} V \otimes V \rightarrow V \otimes \text{Ber} V \) are morphisms.

**Corollary 4.2.** If for some \( a, b \in k^× \)
\[ \alpha(v_i) = (-1)^{p(v_i)p(\omega)} av_i \]
(4.1)
or
\[ \beta(v_i) = (-1)^{p(v_i)p(\omega)} bv_i, \]
(4.2)
then \( \tau = \text{Ber} T \) is a central element.

In the case of (4.1) or (4.2), the Hopf algebra \( H' = H/(\tau - 1) \), referred to as a semispecial quantum linear group, is big enough. In the particular case of a special quantum linear group, when \( a = b = 1, p(\omega) = 0 \) (resp. \( p(\omega) = 1 \)), the algebra \( H' = H/(\tau - 1) \) (resp. \( H' = H/(\tau^2 - 1) \)) is a coquasitriangular Hopf algebra. Here we can add to the defining system of morphisms an isomorphism to the trivial comodule \( \text{Ber} V \simeq k \) (resp. \( (\text{Ber} V)^{\otimes 2} \simeq k \)).

4.2. One dimensional example. Take \( V = \mathbb{C}^{1|0} \) with \( R = q \). Let \( v \in V \), \( v \neq 0 \) and let \( w \in V^\vee \) be its dual vector. \( K(V) = \mathbb{C}[w, v]/(v^2) \) contains the element \( d = vw \). The differentials are given by

\[
D(w^k) = w^{k+1}v, \quad D(w^kv) = 0, \\
D'(w^k) = 0, \quad D'(w^kv) = q^{-1}[k]qv^{k-1}.
\]

The Laplacian is

\[
L|_{K^0, m} = q^{-1}[m + 1]q, \quad L|_{K^1, m} = q^{-1}[m]q.
\]

The cohomologies \( H^*(K(V), D') \) are infinite dimensional if \( q \) is a root of unity and one dimensional otherwise. The Berezinian \( \text{Ber} V \simeq \text{Ber} V' \) is always one dimensional and \( \omega_V = v \in K^{1,0}(V), v \in \text{Ker} D \cap \text{Ker} D' \), is a cocycle giving its basis. Remark that \( K(V) = K^{1,0}(V) \oplus M, M = \bigoplus_{(n,m) \notin (1,0)} K^{n,m}(V) \) is a decomposition into bidifferential subcomplexes and \( M \) is \( D \)-acyclic. A basis of \( \text{Ber} V \simeq \text{Ber} V' \) is given by the dual vector \( \omega_V = \tau v \in V = K^{1,0}(V') \) and \( \pi(\omega_{V'}, \omega_V) = 1 \), so the conjectures are verified in this example.

4.3. Standard \( GL(n|m) \) \( R \)-matrices. The standard \( R \)-matrices for \( GL(n|m) \) are obtained by iterating the following construction.

**Proposition 4.3.** Let \( R : V \otimes V \to V \otimes V \) be a Hecke \( \tilde{R} \)-matrix in a \( \mathbb{Z}/2 \)-graded space \( V \). Let \( \phi : V \to V \) be a symmetry of \( R \), a bijective linear map of degree 0, satisfying \( \phi \otimes \phi \circ R = R \circ \phi \otimes \phi \). Consider a one dimensional \( \mathbb{Z}/2 \)-graded vector space \( U \) with an \( \tilde{R} \)-matrix \( R' : U \otimes U \to U \otimes U \), such that either \( U \) is even and \( R' = q \), or \( U \) is odd and \( R' = -q^{-1} \). The map \( Q : U \otimes V \to V \otimes U \), \( Q(u \otimes v) = \phi(v) \otimes u \), satisfies the conditions of Theorem 3.3, defining a Hecke \( \tilde{R} \)-matrix in \( V \otimes Q U \).

**Proof.** Clear. \( \square \)

We start with one dimensional space \( V_1 \), add up one dimensional spaces \( V_i \) and obtain \( V = V_1 \oplus q_1 V_2 \oplus q_2 \cdots \oplus q_{k-1} V_k \), using diagonal matrices

\[
\phi_i : V_1 \oplus V_2 \oplus \cdots \oplus V_i \to V_1 \oplus V_2 \oplus \cdots \oplus V_i, \quad \phi_i (v_j) = q_{i+1,j} v_j,
\]

which are symmetries of \( \tilde{R} \)-matrices on \( V_1 \oplus q_1 V_2 \oplus q_2 \cdots \oplus q_{i+1} V_i \). Here \( q_{ab} \in k^\times \), \( a > b \) are parameters. We introduce additional parameters \( q_{ab}, a \leq b \) satisfying

\[
q_{ab}q_{ba} = 1 \quad \text{if } a \neq b, \\
q_{ii} = \begin{cases} q & \text{if } v_i \text{ is even,} \\ -q^{-1} & \text{if } v_i \text{ is odd.}
\end{cases}
\]
The resulting $\hat{R}$-matrix in the $\mathbb{Z}/2$-graded space $V$ is described by
\[
\hat{R}(v_i \otimes v_j) = q_{ij} v_j \otimes v_i \quad \text{for } i \leq j,
\hat{R}(v_i \otimes v_j) = q_{ij} v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j \quad \text{for } i > j.
\]
The set of indices is divided in two parts: even indices $\{e_1 < e_2 < \cdots < e_n\}$ and odd indices $\{o_1 < o_2 < \cdots < o_m\}$.
Theorem 3.4 and Proposition 3.17 say that Ber $V \simeq$ Ber$'V$ is one dimensional and has a basic vector
\[
\omega_V = v^{o_1} \cdots v^{o_m} v_{e_1} \cdots v_{e_n} \in \text{Ber}
\]
(including the case of roots of unity). The ordering of this product can be changed due to commutation relations based on the expression for $\hat{R}$.
\[
\hat{R}^\delta(v_i \otimes v_j) = q_{ji} v^j \otimes v_i \quad \text{for } i \neq j,
\hat{R}^\delta(v_j \otimes v^j) = q_{jj} v^j \otimes v_j + (q - q^{-1}) \sum_{i > j} v_i \otimes v_i.
\]
The element
\[
\omega_{\vee V} = e_{n} v \cdots e_{1} v v_{o_{m}} \cdots v_{o_{1}} \in K(\vee V)
\]
gives a basis of Ber $\vee V \simeq$ Ber$'\vee V$. Lifting these elements to
\[
\tilde{\omega}_V = v^{o_1} \otimes \cdots \otimes v^{o_m} \otimes v_{e_1} \otimes \cdots \otimes v_{e_n} \in T^n(V^\vee) \otimes T^n(V),
\]
\[
\tilde{\omega}_{\vee V} = e_{n} v \otimes \cdots \otimes e_{1} v \otimes v_{o_{m}} \otimes \cdots \otimes v_{o_{1}} \in T^n(\vee V) \otimes T^n(V),
\]
we find by (3.21)
\[
\pi(\tilde{\omega}_{\vee V}, \tilde{\omega}_{\vee V}) = q^{n(n-1)/2 + m(m-1)/2}.
\]
Therefore, $\pi(\tilde{\omega}_{\vee V}, -)$ is a nontrivial linear functional on Ker $D_V$, vanishing on Im $D_V$. So we conclude that
\[
\delta(\omega_V) = \omega_V \otimes \frac{1}{\pi(\tilde{\omega}_{\vee V}, \tilde{\omega}_{\vee V})} \pi(\tilde{\omega}_{\vee V}, \tilde{\omega}_{\vee V}(0))\tilde{\omega}_{\vee V}(1)
\]
\[
= \omega_V \otimes q^{-n(n-1)/2 - m(m-1)/2} \sum_{a,b} \text{(sign)} \langle \text{Ant}(e_{n} v \otimes \cdots \otimes e_{1} v) \otimes \text{Sym}(v_{o_{m}} \otimes \cdots \otimes v_{o_{1}}),
\]
\[
\sum_{\mu \in S_m} q^{-l(\mu)} \langle (R^{-1})_{\mu}(v_{o_{m}} \otimes \cdots \otimes v_{o_{1}}), v^{a_1} \otimes \cdots \otimes v^{a_m} \rangle \tilde{t}_{a_1}^{o_1} \cdots \tilde{t}_{a_m}^{o_m} t_{e_1}^{b_1} \cdots t_{e_n}^{b_n}
\]
\[
= \omega_V \otimes \sum_{a,b} \text{(sign)} \sum_{\mu \in S_m} q^{-l(\mu)} \langle (R^{-1})_{\mu}(v_{o_{m}} \otimes \cdots \otimes v_{o_{1}}), v^{a_1} \otimes \cdots \otimes v^{a_m} \rangle \tilde{t}_{a_1}^{o_1} \cdots \tilde{t}_{a_m}^{o_m}
\]
\[
\times \sum_{\lambda \in \mathbb{S}_n} (-1)^{l(\lambda)} q^{-l(\lambda)} \langle (R^{b_1})_{\lambda}(v_{b_1} \otimes \cdots \otimes v_{b_n}), v_{e_1} \otimes \cdots \otimes v_{e_n} \rangle t_{b_1}^{e_1} \cdots t_{b_n}^{e_n}
\]
\[
= \omega_V \otimes \sum_{\mu \in S_m} q^{-l(\mu)} \left( \prod_{1 \leq i < j \leq m} q_{\mu(i), \mu(j)}^{-1} \tilde{t}_{o_{\mu(1)}}^{o_1} \cdots \tilde{t}_{o_{\mu(m)}}^{o_m} \right)
\]
\[
\times \sum_{\lambda \in \mathbb{S}_n} (-1)^{l(\lambda)} q^{-l(\lambda)} \left( \prod_{1 \leq i < j \leq n} q_{\lambda(i), \lambda(j)}^{-1} t_{e_{\lambda(1)}}^{e_1} \cdots t_{e_{\lambda(n)}}^{e_n} \right)
\]
\[
\equiv \omega_V \otimes \det_q \tilde{t}_{\text{odd}} \det_q t_{\text{even}}
\]
\[ \equiv \omega_V \otimes \text{sdet}_q t. \]  

(4.3)

Here \((\text{sign})\) comes from the graded coproduct and it disappears in the final answer because only even elements \(\tilde{t}_{\alpha_i}^{\alpha_i}\) and \(\tilde{t}_{e_{ij}}^{e_{ij}}\) contribute to the final formula. When \(m = 0\) the superdeterminant coincides with the usual quantum determinant \(\text{det}_q t\) from [24].

In the formulae

\[
c(v \otimes \omega_V) = \omega_V \otimes \frac{1}{\pi(\omega_V \otimes \omega_V)} (\pi \otimes 1)(1 \otimes c)(\tilde{\omega}_V \otimes v \otimes \tilde{\omega}_V) \quad (4.4)
\]

\[
c(\omega_V \otimes v) = \omega_V \otimes \frac{1}{\pi(\omega_V \otimes \omega_V)} (1 \otimes \pi)(c \otimes 1)(\tilde{\omega}_V \otimes v \otimes \tilde{\omega}_V) \quad (4.5)
\]

the contribution of terms proportional to \((q - q^{-1})\) is nil. Therefore we find

\[
\alpha(v_i) = q_{ie_1} \ldots q_{ie_n} q_{io_1}^{1} \ldots q_{io_m}^{-1} v_i, \\
\beta(v_i) = q_{e_{i1}} \ldots q_{e_{in}} q_{o_{i1}}^{1} \ldots q_{o_{im}}^{-1} v_i.
\]

Their composition

\[
\alpha \beta(v_i) = \prod_{j=1}^{n} q_{ie_j} \prod_{k=1}^{m} q_{io_k}^{-1} \prod_{j=1}^{n} q_{e_{ij}} \prod_{k=1}^{m} q_{o_{ki}}^{-1} v_i = q^2 v_i
\]

because

\[
q_{ij} q_{ji} = \begin{cases} 
1 & \text{if } i \neq j, \\
q^2 & \text{if } i = j \text{ is even}, \\
q^{-2} & \text{if } i = j \text{ is odd}.
\end{cases}
\]

If for some constant \(a\) and all \(i\)

\[
q_{ie_1} \ldots q_{ie_n} q_{io_1}^{1} \ldots q_{io_m}^{-1} = (-1)^{mp(v_i)} a
\]

we have semispecial linear group and \(\tau\) is a central element. If \(n \neq m\), we can rescale \(R\), multiplying it by \(q^{1/(m-n)}\). Then \(\alpha\) and \(\beta\) multiply by \(q^{-1}\) and their product becomes 1. So if additionally \(a = q\), we have a special linear group. When \(n = m\) rescaling will not help to construct a special linear group.

4.4. Berezinians for differential supergroups obtained from standard quantum \(GL(n|m)\). We note that provided \(q\) is not a sixth root of unity, the algebra \(H\) of functions on the differential quantum supergroup has the same growth properties as the corresponding supercommutative algebra: specifically, it has a linear basis consisting of all alphabetically ordered monomials in \(p^j, r^j, s^j, t^j\) with the powers of \(p\) and \(r\) not exceeding 1 ([27], Theorem 3). Again Theorem 3.4 tells us that \(\text{Ber} X \simeq \text{Ber}' X\) is one dimensional. The basis \((v_i) = (dv_i) \subset U = dV\) has the changed degree \(p(u_i) = (p(v_i), 1) \in \mathbb{Z}/2 \times \mathbb{Z}\); hence a basic vector of \(\text{Ber} U\) is similar to that of \(\text{Ber} V\) with swapped even and odd indices

\[
\omega_U = u^{e_1} \ldots u^{e_n} u_{o_1} \ldots u_{o_m} \in \text{Ber} U.
\]

This gives a basic vector of \(\text{Ber} X\)

\[
\omega_X = v^{e_1} \ldots v^{o_m} v_{e_1} \ldots v_{e_n} u^{e_1} \ldots u^{e_n} u_{o_1} \ldots u_{o_m} \in \text{Ker} D_X \cap \text{Ker} D_X'.
\]

(4.6)
Using the commutation relations in $K(X)$

$$qv_i u^j = q_{ij} u^j v_i$$

if $i \neq j$,

$$qv_j u^j = q_{jj} u^j v_j + (q - q^{-1}) \sum_{i > j} u^i v_i$$

we represent this cocycle in another form

$$\tilde{\omega}_X = v_{o_1} \ldots v_{o_m} \left( \sum_{k_i, l_i \geq e_i} c_{k_1 \ldots k_n}^{l_1 \ldots l_n} u^{k_1} \ldots u^{k_l} v_{i_1} \ldots v_{i_n} \right) u_{o_m} \ldots u_{o_1}.$$  

An element

$$\omega_\vee X = o_1 u \ldots o_m u e_1 \ldots u e_n v \ldots v o_m \ldots v o_1 \in K(\vee X)$$

or in another form

$$\tilde{\omega}_\vee X = o_1 u \ldots o_m u \left( \sum_{r_1, s_1 \leq e_i} b_{r_1 \ldots r_n}^{s_1 \ldots s_n} (p_1 u) \ldots (p_1 u) u_{r_1} \ldots u_{r_n} \right) v_{o_m} \ldots v_{o_1}$$

is a cocycle from $\text{Ker} D_{\vee X} \cap \text{Ker} D'_{\vee X}$ giving a basis of $\text{Ber}'_{\vee X} \simeq \text{Ber}_X$. The pairing between these cocycles is found from (3.21)

$$\pi(\omega_\vee X, \omega_X) = q^{(n+m)(n+m-1)}.$$

The superdeterminant $\tau_X$ is

$$\tau_X = q^{-(n+m)(n+m-1)} \pi(\omega_\vee X, \omega_X(0)) \omega_X(1).$$

The explicit formula is rather complicated. At least $\tau_X$ is grouplike and $d\tau_X = 0$.

Lift $\omega_X$ to an element of the tensor product

$$\tilde{\omega}_X \in T^m(V^\vee) \otimes T^n(V) \otimes T^m(U^\vee) \otimes T^m(U)$$

inserting $\otimes$ between elements of (4.6). We know in principle how to calculate $c(x \otimes \tilde{\omega}_X)$, $c(\tilde{\omega}_X \otimes x)$ for $x \in X$. To find $c(x \otimes [\omega_X])$ and $c([\omega_X] \otimes x)$ we use (4.4) and (4.5) applied to $X$. All terms proportional to $(q - q^{-1})$ do not contribute to

$$\alpha_X(x) = q^{-(n+m)(n+m-1)} (\pi \otimes 1)(1 \otimes c)(\omega_\vee X \otimes x \otimes \tilde{\omega}_X)$$

and

$$\beta_X(x) = q^{-(n+m)(n+m-1)} (1 \otimes \pi)(c \otimes 1)(\tilde{\omega}_X \otimes x \otimes \omega_X) \vee).$$

The principal terms give

$$\alpha_X(v_i) = q_{ie_1} \ldots q_{ie_n} q_{i_1o_1}^{-1} \ldots q_{i_mo_m}^{-1} q_{o_1i_1}^{-1} \ldots q_{o_mi_i}^{-1} q_{e_1i} \ldots q_{e_ni} v_i$$

$$= \prod_{k=1}^{n} q^{2h_{e_k}} \prod_{i=1}^{m} q^{2h_{i_1 o_1}} v_i$$

$$= q^2 v_i$$

and similarly

$$\alpha_X(u_i) = (-1)^{n+m} q^2 u_i,$$

$$\beta_X(v_i) = v_i,$$

$$\beta_X(u_i) = (-1)^{n+m} u_i.$$
The $\mathbb{Z}/2 \times \mathbb{Z}$-gradings of the elements involved are
\[
\begin{align*}
\deg(v_i) &= (p(v_i), 0), \\
\deg(u_i) &= (p(v_i), 1), \\
\deg(\omega_X) &= (0, m - n);
\end{align*}
\]

hence
\[
\begin{align*}
(-1)^{\deg(\omega_X) \deg(v_i)} &= 1, \\
(-1)^{\deg(\omega_X) \deg(u_i)} &= (-1)^{n + m}.
\end{align*}
\]

Finally,
\[
\begin{align*}
\alpha(x) &= (-1)^{\deg(\omega_X) \deg(x)} q^2 x, \\
\beta(x) &= (-1)^{\deg(\omega_X) \deg(x)} x,
\end{align*}
\]

so $\tau_X$ is a central element and we are in a semispecial situation. The quotient $H/(\tau_X - 1)$ is a differential Hopf algebra which is no longer coquasitriangular.

4.4.1. The superdeterminant for differential forms. Considering another quotient of $H$, the algebra of differential forms $\Omega$, we can ask about the image of $\tau_X$ in $\Omega$. Unlike $\tau_X$ its image can be calculated by similar reasoning to the proof of Proposition 3.9.

We get in $\Omega$
\[
\begin{align*}
\tau_X &= q^{-n(n-1)-m(m-1)} (\text{sign}) \langle \text{Ant}_{m} (a_1 u \otimes \ldots \otimes a_m u) \otimes \text{Sym}(u_{e_1} \otimes \ldots \otimes u_{e_n}) \otimes \\
&\quad \otimes \text{Ant}_m (e_1 v \otimes \ldots \otimes e_1 v) \otimes \text{Sym}(v_{o_1} \otimes \ldots \otimes v_{o_1}), \\
&\quad \sum_{a,b,c,d} v_1^{a_1} \otimes \ldots \otimes v_1^{a_m} \otimes v_{b_1} \otimes \ldots \otimes v_{b_n} \otimes u_{c_1} \otimes u_{c_n} \otimes \ldots \otimes u_{d_1} \\
&\quad \equiv \det_q \bar{t}_{\text{odd}} \det_q t_{\text{even}} \det_q \bar{t}_{\text{even}} \det_q t_{\text{odd}}
\end{align*}
\]

because
\[
\begin{align*}
\delta([\omega_V]) &= [\omega_V] \otimes \text{sdet}_q t \quad &\text{by (4.3)} \\
\delta([\omega_{V^*}]) &= [\omega_{V^*}] \otimes \text{sdet}_q \bar{t} \quad &\text{similarly} \\
\text{Ber } V \otimes \text{Ber}(V^*) &\simeq k \in C \quad &\text{by (3.16)}
\end{align*}
\]
imply
\[
\delta([\omega_V \otimes [\omega_{V^\vee}]) = [\omega_V] \otimes [\omega_{V^\vee}] \otimes \text{sdet}_q t \otimes \text{sdet}_q \bar{t}
\]
and, finally,
\[
\text{sdet}_q t \otimes \text{sdet}_q \bar{t} = 1.
\]

Therefore, for the standard \( R \)-matrix of \( GL(n|m) \) type there are epimorphisms of differential Hopf algebras
\[
H \to H/(\tau_X - 1) \to \Omega
\]
and the image of the superdeterminant \( \tau_X \) is 1 in \( \Omega \). So it is unreasonable to look for an \( SL \)-version of the algebra \( \Omega \) of differential forms on quantum \( GL(n|m) \), because it is, in a sense, already of \( SL \) type!

5. Hopf bimodules

5.1. Recollection of basic facts. There is a well known definition of a Hopf module over a Hopf algebra. We recall it in a \( \mathbb{Z}/2 \)-graded version.

A left Hopf module \((M, a_L, \delta_L)\) over a \( \mathbb{Z}/2 \)-graded Hopf algebra \( F \) is a left \( F \)-module \((M, a_L : F \otimes M \to M)\) with a coaction \( \delta_L : M \to F \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}\) such that
\[
\delta_L(fm) = (-1)^{\hat{x}(2)m_{(-1)}} f_{(1)}m_{(-1)} \otimes f_{(2)}m_{(0)}.
\]
Here \( \hat{x} \in \mathbb{Z}/2 \) is the parity of a homogeneous element \( x \). Similarly a right Hopf module \((M, a_R, \delta_R)\) is defined with \( a_R : M \otimes F \to M \) and \( \delta_R : M \to M \otimes F, m \mapsto m_{(0)} \otimes m_{(1)}\). The mixed notions \((M, a_L, a_R, \delta_L, \delta_R)\) are required to satisfy
\[
\delta_R(fm) = (-1)^{\hat{x}(2)m_{(0)}} f_{(1)}m_{(0)} \otimes f_{(2)}m_{(1)}
\]
and
\[
\delta_L(mf) = (-1)^{\hat{x}(1)m_{(-1)}} f_{(1)}m_{(-1)} \otimes m_{(0)}f_{(2)}
\]
respectively.

The results of this section were also independently obtained by Schauenburg [23] in greater generality.

Definition 5.1. A Hopf \( F \)-bimodule \((M, a_L, a_R, \delta_L, \delta_R)\) is a vector space \( M \) with left and right actions of \( F \) and left and right coactions of \( F \) such that \((M, a_L, \delta_L), (M, a_L, \delta_R), (M, a_R, \delta_L), (M, a_R, \delta_R)\) are Hopf modules, \((M, a_L, a_R)\) is an \( F \)-bimodule (i.e. \( (fm)g = f(mg) \)) and \((M, \delta_L, \delta_R)\) is an \( F \)-bicomodule (i.e. \( 1 \otimes \delta_R)\delta_L = (\delta_L \otimes 1)\delta_R \)). Schauenburg calls such \( M \) two-sided two-cosided Hopf modules [23].

Examples 5.2. 1) The Hopf algebra \( F \) is a Hopf \( F \)-bimodule, when equipped with regular actions and coactions.

2) Differential forms of the first order \( \Omega^1 \) (see Section 2.3.1) make a Hopf module over the algebra of functions \( F = \Omega^0 \) with coactions determined by the decomposition \( \Delta = \delta_L \otimes \delta_R : \Omega^1 \to \Omega^0 \otimes \Omega^1 \oplus \Omega^1 \otimes \Omega^0 \).

3) The same for an arbitrary \( \mathbb{Z}/2 \times \mathbb{Z}_{\geq 0} \)-graded differential Hopf algebra.
Let $M$ be a left Hopf $F$-module and let 
\[ M' = \{ m \in M \mid \delta_L(m) = 1 \otimes m \}. \]
Then $M$ is isomorphic to a direct sum of $\dim M'$ copies of the regular Hopf module, $M \cong F \otimes M'$ (see e.g. [29]). This gives an equivalence between the category of left Hopf $F$-modules and the category of vector spaces.

Let $M$ be a Hopf $F$-bimodule. Then $\delta_R(M') \subset M' \otimes F$ by the bicomodule property, so $M'$ is a right $F$-comodule. It can be viewed also as a left $F^*$-module. The right coadjoint action of $F$
\[ m \prec f = (-1)^{\hat{m}(1)} \gamma(f(1)) m f(2) \]
preserves $M'$, so $M'$ is a right $F$-module.

**Theorem 5.1** (See also Schauenburg [23]). The module $M'$ satisfies
\[ \delta_R(n \prec g) = (-1)^{\hat{g}(2)} \hat{g}(1) + \hat{g}(2) n(0) \prec g(2) \otimes \gamma(g(1)) n(1) g(3) \]  
(5.1)
for any $n \in M'$, $g \in F$.

Given a right $F$-module and $F$-comodule $(N, \prec, \delta_R)$ which satisfies (5.1), we make $M = F \otimes N$ into a Hopf bimodule by setting
\[ g.(f \otimes n) = (gf) \otimes n, \]  
(5.2)
\[ (f \otimes n).g = (-1)^{\hat{g}(1)} (fg(1)) \otimes (n \prec g(2)), \]  
(5.3)
\[ \delta_L(f \otimes n) = f(1) \otimes (f(2) \otimes n), \]  
(5.4)
\[ \delta_R(f \otimes n) = (-1)^{\hat{f}(0)} (f(1) \otimes n(0)) \otimes (f(2) n(1)). \]  
(5.5)
We have $M' = k \otimes N \cong N$, and every Hopf bimodule $M$ can be constructed in this way.

**Corollary 5.2.** The abelian category of Hopf $F$-bimodules is equivalent to the abelian category $\mathcal{DF}$ of right $F$-modules and right $F$-comodules satisfying (5.1).

The last category was introduced by Yetter [33], who called such $N$ right crossed $F$-bimodules.

**Proof of Theorem 5.1.** Clearly, $a_L, a_R, \delta_L, \delta_R$ are uniquely determined by $\delta_R|_{M'}$ and $\prec|_{M'}$ via (5.2)–(5.5). On the other hand, for any right crossed $F$-bimodule $N$ these formulae define operations such that $(M, a_L, \delta_L), (M, a_L, \delta_R), (M, a_R, \delta_L)$ are Hopf bimodules, $(M, a_L, a_R)$ is a bimodule, and $(M, \delta_L, \delta_R)$ is a bicomodule. Only the Hopf relation between $a_R$ and $\delta_R$ is left. It is equivalent to the identity (5.1). Indeed, to deduce (5.1) we remark that
\[ \delta_R((-1)^{\hat{g}(1)} (\gamma(g(1)) \otimes n).g(2)) = \delta_R(1 \otimes (n \prec g)) = 1 \otimes \delta_R(n \prec g) \]
is equal in a Hopf bimodule to
\[ (-1)^{\hat{g}(1)} \delta_R(\gamma(g(1)) \otimes n).g(2) \otimes g(3) = 
\]
\[ = (-1)^{\hat{g}(1)} \hat{g}(1) + \hat{g}(2) \gamma(g(1)) \otimes n(0) \otimes g(2) \otimes \gamma(g(0)) n(1) g(3) 
\]
\[ = (-1)^{\hat{g}(1)} \hat{g}(1) + \hat{g}(2) \gamma(g(1)) \otimes n(0) \otimes g(2) \otimes \gamma(g(1)) n(1) g(3). \]

Vice versa, (5.1) implies the Hopf relation between $a_R$ and $\delta_R$. \qed
Remark 5.1. When the pairing $\langle \cdot, \cdot \rangle : F^\circ \otimes F \to k$ of $F$ with its dual Hopf algebra $F^\circ$ is non-degenerate, the category of finite dimensional left modules over the Drinfeld double $D(F^\circ)$ [3] coincides with the subcategory of finite dimensional objects of $\mathcal{D}Y_F$. The double $D(F^\circ)$ is defined as a $\mathbb{Z}/2$-graded Hopf algebra generated by its Hopf subalgebras $F^\circ$ and $F^{op}$ with the commutation relations

$$(-1)^{\hat{x}(1)\hat{y}} x(1)y(2) x(2) = (-1)^{\hat{x}(1)\hat{y}} y(1) x(1) y(2)$$

for any $x \in F^{op}$ and $y \in F^\circ$.

**Remark 5.2.** If $F$ is coquasitriangular, which means the existence of a Hopf pairing $\rho : F \otimes F^{op} \to k$ with the property

$$(-1)^{\hat{f}(1)} f(2) \rho(f(2), g(2)) = (-1)^{\hat{f}(2)} \hat{g}(1) \rho(f(1), g(1)) f(2) g(2)$$

for $f, g \in F$, the category comod-$F$ is embedded in $\mathcal{D}Y_F$. The right action of $F$ on a right $F$-comodule $N$ is chosen as

$$n.f = n(0) \rho(n(1), f)$$

for $n \in N$, $f \in F$. Dually, we can say that some representations of the double $D(F^\circ)$ come from representations of the quasitriangular algebra $F^\circ$.

### 5.1.1. Tensor product of Hopf bimodules

Given two Hopf $F$-bimodules $M$ and $N$ we make their tensor product $M \otimes_F N$ into a Hopf bimodule by setting

$$f.(m \otimes_F n) = (f.m) \otimes_F n,$$

$$(m \otimes_F n).f = m \otimes_F (n.f),$$

$$\delta_L(m \otimes_F n) = (-1)^{\hat{m}(0)\hat{n}(-1)} m(-1)n(-1) \otimes (m(0) \otimes_F n(0)),$$

$$\delta_R(m \otimes_F n) = (-1)^{\hat{m}(1)\hat{n}(0)} (m(0) \otimes_F n(0)) \otimes m(1)n(1).$$

All necessary checks are left to the reader. Since $N$ is a free left $F$-module, we have an isomorphism of left $F$-modules $M \otimes_F N \cong M \otimes N^l$. This implies that the map

$$M^l \otimes N^l \to (M \otimes_F N)^l$$

is an isomorphism. Indeed, for $n \in N^l$ clearly $m \otimes n \in (M \otimes_F N)^l$ iff $m \in M^l$. The right $F$-module and comodule structure induced on $M^l \otimes N^l$ by this isomorphism is

$$\delta_R(m \otimes n) = (-1)^{\hat{m}(1)\hat{n}(0)} (m(0) \otimes n(0)) \otimes m(1)n(1),$$

$$(m \otimes n) \lhd f = (-1)^{\hat{m}(1)\hat{n}(1)} \gamma(f(1))(m \otimes_F n)f(2)$$

$$= (-1)^{\hat{m}(1)\hat{n}(1)} \gamma(f(1))m \otimes_F n f(2)$$

$$= (-1)^{\hat{m}(1)+\hat{n}(0)} \gamma(f(0)+f(1)+f(2)) \gamma(f(0))m f(1) \otimes_F \gamma(f(2)) n f(3)$$

$$= (-1)^{\hat{m}(1)} (m \lhd f(1)) \otimes (n \lhd f(2)).$$

Therefore we can strengthen Corollary 5.2.

**Proposition 5.3** (See also Schauenburg [23]). The category of Hopf $F$-bimodules is tensor equivalent to the category $\mathcal{D}Y_F$ with the tensor product determined by (5.6) and (5.7) (for $D(F^\circ)$-modules this is the usual tensor product).

**Corollary 5.4.** The category of Hopf $F$-bimodules is a braided tensor category.
The braiding was discovered by Woronowicz [32]. The explanation is quite simple: $\mathcal{D}Y_F$ is braided with the braiding [33]

$$c : X \otimes Y \to Y \otimes X, \quad x \otimes y \mapsto (-1)^{x y(0)} y(0) \otimes x \triangleleft y(1)$$

where $x \in X$, $y \in Y$, $X, Y \in \mathcal{D}Y_F$. On $D(F^o)$-modules this braiding is $P R$, where $R$ is the universal $R$-matrix of the double [3]. The induced braiding for Hopf $F$-bimodules is

$$c : M \otimes_F N \to N \otimes_F M, \quad f m \otimes_F n \mapsto (-1)^{m n(0)} f n(0) \otimes_F m \triangleleft n(1),$$

where $m \in M^I$, $n \in N^I$, $f \in F$. Another presentation of the braiding is

$$c(m \otimes n) = (-1)^{(m_{(-1)} + \hat{m}(0)) (\hat{n}(0) + \hat{n}(1))} m(-2) n(0) \gamma(\hat{n}(1)) \otimes \gamma(m_{(-1)}) n(0) n(2)$$

for $m \in M$, $n \in N$.

5.2. Differential Hopf algebras determined by Hopf bimodules.

**Example 5.3.** The algebra $F$ considered as a regular right $F$-module equipped with the right coadjoint coaction

$$\nabla f = (-1)^{f(1)} f(2) \otimes \gamma(f(1)) f(3)$$

becomes itself a right crossed bimodule. The subobject $K = \operatorname{Ker}(\varepsilon : F \to k)$ is also in $\mathcal{D}Y_F$.

From the results of Woronowicz [32] one can conclude that a bicovariant first order differential calculus is precisely a Hopf bimodule $M$ together with an epimorphism $\omega : K \to M^I \in \mathcal{D}Y_F$. The differential $d : F \to M$ is recovered from $\omega$ as $d f = f(1) \omega(f(2) - \varepsilon(f(2)))$. If $d : F \to M$ is given, we construct a map $\omega : F \to M^I$, $\omega(f) = \gamma(f(1)) d f(2)$, whose restriction to $K$ is a morphism from $\mathcal{D}Y_F$.

Suppose that such $d : F \to M$ constitute a part of $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded differential Hopf algebra $A$, so $A^0 = F$, $A^1 = M$. The Cartan–Maurer formula tells us that

$$d \omega(f) = (d \gamma(f(1))) d f(2)$$

$$= -\gamma(f(1)) (d f(2)) \gamma(f(3)) d f(4)$$

$$= -\omega(f(1)) \omega(f(2))$$

for any $f \in F$.

**Theorem 5.5.** Let $M$ be a Hopf $F$-bimodule, and let $\omega : K \to M^I$ be an epimorphism in $\mathcal{D}Y_F$. There exists a universal $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded algebra $E^* \in F$-Hopf-bimod, with a differential which is a bicomodule map, generated by $E^0 = F$, $E^1 = M$. It is

$$E^* = E^*_F(M) = T^* F(M)/\langle \omega(a(1)) \otimes \omega(a(2)) \rangle_{a \in J},$$

where $J = \operatorname{Ker}(\omega : K \to M^I)$. Moreover, $E^*_F(M)$ is a $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded differential Hopf $k$-algebra.
Proof. If we drop the differential, the algebra $T^{*}_F(M)$ will be the universal $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded algebra and a Hopf bimodule such that $T^0_F(M) = F$, $T^1_F(M) = M$. When the differential is considered, (5.9) shows that $\omega(a_{(1)})\omega(a_{(2)}) = 0$ in $E$ for $a \in J$. On the other hand, the ideal $(\omega(a_{(1)}) \otimes \omega(a_{(2)}))_{a \in J}$ is a Hopf subbimodule, and one can check that the differential $d : E^0 \to E^1$ extends to the whole of $E^*$ uniquely.

The algebra $T^{*}_F(M)$ has a comultiplication $\Delta : T_F(M) \to T_F(M) \otimes T_F(M)$ coinciding with $\Delta : F \to F \otimes F$ and $\Delta = \delta_L \oplus \delta_R : M \to F \otimes M \oplus M \otimes F$ in the lowest degrees. It extends to all $T^*_F(M)$ making $T^*_F(M)$ into a Hopf algebra with the antipode $\gamma$, satisfying $\gamma(df) = d\gamma(f)$ for $f \in F$. The ideal $(\omega(a_{(1)}) \otimes \omega(a_{(2)}))_{a \in J}$ is a $\gamma$-invariant coideal, therefore $E^*$ is a Hopf algebra.

Corollary 5.6. $E^*_F(M)$ is a universal $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded differential Hopf algebra generated by $E^*_F(M) = F$, $E^1_F(M) = M$.

Remark 5.3. This theorem should be compared with a result of Brzeziński [2]. With the same assumptions he proves the existence of a graded differential Hopf algebra $(M^\wedge, d)$ generated by $F$ and $M$, namely

$$M^\wedge = T^*_F(M)/(\text{Ker}(\sigma - 1)),$$

where $\sigma = c : M^I \otimes M^I \to M^I \otimes M^I$ is the braiding (5.8). By Theorem 5.5 $M^\wedge$ is a quotient of $E_F(M)$ and, indeed, one can check straightforwardly that

$$\{\omega(a_{(1)}) \otimes \omega(a_{(2)}) \mid a \in J\} \subset \text{Ker}(\sigma - 1).$$

Indeed,

$$(\omega \otimes \omega)\nabla = (1 - \sigma)(\omega \otimes \omega)\Delta$$

and $\nabla(J) \subset J \otimes F$. In the Hecke case both algebras coincide (see discussion in Section 5.3).

The universal differential calculus [32] is worth mentioning as a particular case. This is the Hopf bimodule $U = \text{Ker}(m : F \otimes F \to F)$ with the operations

$$f.(g \otimes h) = fg \otimes h,$$

$$(g \otimes h).f = g \otimes hf,$$

$$\delta_L(g \otimes h) = (-1)^{g(0)h(-1)}g(-1)h(-1) \otimes (g(0) \otimes h(0)),$$

$$\delta_R(g \otimes h) = (-1)^{g(1)h(0)}(g(0) \otimes h(0)) \otimes g(1)h(1)$$

and the differential $d : F \to U$, $df = 1 \otimes f - f \otimes 1$. The map $\omega' : K \to U$, $b \mapsto \gamma(b_{(1)}) \otimes b_{(2)}$ is an embedding and $\omega'(K) = U^I$. Therefore, $J = 0$ and $E^*_F(U) = T^*_F(U)$.

Proposition 5.7. Let $E^*$ be a $\mathbb{Z}/2 \times \mathbb{Z}_{\geq 0}$-graded algebra and a Hopf $F$-bimodule with a differential which is a bicomodule map, generated by $E^0 = F$, $E^1 = M$. If a system of defining relations of $E^*$ is obtained by differentiating a system of defining relations of the bimodule $M$, the algebra $E^*$ is isomorphic to $E^*_F(M)$. 

Proof. Cover \( M \) by the universal Hopf bimodule \( U \) as in the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & J & \longrightarrow & K & \longrightarrow M^t & \longrightarrow 0 \\
\downarrow & & \downarrow \omega' & & \downarrow & & \downarrow \\
0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow M & \longrightarrow 0
\end{array}
\]

The upper exact sequence is in \( \mathcal{D}Y_F \) and \( N^t = J \).

A system of defining relations of \( M \) means a collection \( \{r_i\} \subset N \subset U \) such that \( F\{r_i\}F = N \). Differentiating it we get a system of relations of \( E^* \{ (p \otimes p)(dr_i) \} \subset M \otimes_F M \), where \( dr_i \in U \otimes_F U \). We have to prove that

\[ F\{(p \otimes p)(dr_i)\}F = F\{\omega(b_{(1)}) \otimes \omega(b_{(2)}) \mid b \in J\}F. \]

The \( \subset \) inclusion. Since \( r_i \in N \), it can be represented as \( r_i = \sum_k a^k_i \omega'(b_k) \), where \( a^k_i \in F \) and \( (b_k) \) is a \( k \)-basis of \( J \). Hence,

\[
dr_i = \sum_k da^k_i \omega'(b_k) + \sum_k a^k_i d \omega'(b_k)
\]

\[
= \sum_k da^k_i \omega'(b_k) - \sum_k a^k_i \omega'(b_{k(1)}) \omega'(b_{k(2)})
\]

\[
\in U \otimes_F N + F\{\omega'(b_{(1)}) \otimes_F \omega'(b_{(2)}) \mid b \in J\}
\]

which implies

\[ (p \otimes p)(dr_i) \in F\{\omega(b_{(1)}) \otimes \omega(b_{(2)}) \mid b \in J\}. \]

The \( \supset \) inclusion. Represent an arbitrary element of \( N^t \) in the form \( \omega'(b) = \sum_i f_ir_ig_i \), where \( f_i, g_i \in F \), \( b \in J \). Then

\[
-\omega'(b_{(1)}) \otimes_F \omega'(b_{(2)}) = d\omega'(b) = d(f_ir_ig_i)
\]

\[
= \sum_i (df_i)r_ig_i + \sum_i f_i(dr_ig_i) - \sum_i f_irdg_i
\]

\[
\in U \otimes_F N + F\{dr_i\}F + N \otimes_F U
\]

which implies

\[ \omega(b_{(1)}) \otimes \omega(b_{(2)}) \in F\{(p \otimes p)(dr_i)\}F. \]

Together with Theorem 5.5 this proposition states that given a bimodule with a differential \( d : F \to M \) one constructs a graded differential Hopf algebra \( E^*_F(M) \) simply by differentiating the relations of \( M \). However, this method is too universal to single out interesting cases.

5.3. The quantum \( GL(n|m) \) case. The algebra of differential forms \( \Omega \) constructed from an arbitrary Hecke \( \tilde{R} \)-matrix satisfies the hypotheses of Proposition 5.7. Therefore \( \Omega^* \cong E^*_F(\Omega^t) \) are isomorphic \( \mathbb{Z}/2 \times \mathbb{Z}_{\geq 0} \)-graded differential Hopf algebras. It is possible to find the kernel \( J \) explicitly. We will do it in the purely even case; the general case differs only by signs.
Let $R : V \otimes V \to V \otimes V$ be a Hecke $\hat{R}$-matrix, and let $t^a_b = t_{\nu^a_b}$, $t^b_a = t_{\nu^b_a}$ be matrix elements from $\Omega^*$. Multiply equation (2.4), or
\[(dt^a_b) t^c_f = R^{ac}_{gh} t^g_i dt^h_j R^{ij}_{bf},\]
by $\gamma(t^c_d) \gamma(t^d_a) = t^k_c t^l_a$ on the left:
\[\gamma(t^c_d) \gamma(t^d_a)(dt^a_b) t^c_f = R^{ac}_{gh} t^k_d t^l_i dt^h_j R^{ij}_{bf}.\]
Applying (2.15) in the form
\[R^{ac}_{gh} t^l_i = t^c_m t^p_n R^{lm}_{ip},\]
we get
\[\gamma(t^l_m) dt^l_f - \gamma(t^c_d) \gamma(t^d_a) dt^a_b = \gamma(t^c_d) t^c_m t^p_n dt^h_j R^{lm}_{ip} R^{ij}_{bf}.\]
This simplifies to
\[\omega(t^l_m t^k_d) - \delta^l_m \omega(t^k_d) = R^{lk}_{ip} R^{ij}_{bj} \omega(t^p_j),\]
so we have
\[t^l_{mb} t^k_d - \delta^l_m t^k_d - R^{lk}_{ip} R^{ij}_{bj} t^p_j + R^{lk}_{ij} R^{ij}_{bf} \in J.\]  
(5.10)
This equation is equivalent to (2.4) modulo the other relations. Similarly, equation (2.15) is equivalent to the relation
\[\tilde{t}^h_i t^k_d - \delta^h_i t^k_d - R^{1gh}_{ac} R^{1mc}_{kl} t^l_p + R^{1gh}_{ac} R^{1ac}_{kl} \in J.\]  
(5.11)
Equation (2.16) is equivalent to the relation
\[\tilde{t}^h_i + u^2_k t^k_d - R^{1jh}_{ig} t^l_j - (1 + \nu^{-1}_V) \delta^h_i \in J,\]  
(5.12)
where $u^2_k = \sum_c R^{i-1cc}_{ik}$, and also to the relation
\[t^a_b + u^2_m k R^{al}_{bk} t^m_l - \delta^a_b (1 + \nu_V^2) \in J,\]  
(5.13)
where $u^2_m = \sum_c R^{1bc-1cc}_{mk}$. The relations (5.10)–(5.12) make a complete list of relations of $M = F \otimes K/J$. Therefore, the right ideal in $F$ generated by (5.10)–(5.12) coincides with $J$.

In calculating $M_l$ we can use the following remarks. There is an isomorphism
\[\frac{\{T(t^a_{b}, \tilde{t}^b_{a})\}_{a,b}}{(5.10), (5.11), (5.12)} T(t^a_{b}, \tilde{t}^b_{a})_{a,b} \overset{\sim}{\longrightarrow} k\{1, t^a_{b}\}_{a,b}.\]
Indeed, any word in $t, \tilde{t}$ starting with $tt \ldots$ can be shortened using (5.10), a word starting with $t\tilde{t} \ldots$ shortens by (5.11), and (5.12), (5.13) reduce $\tilde{t}t \ldots$ and $t\tilde{t} \ldots$ to previous cases. One can show that the ideal of relations of $F$ projects to 0 by $j$. It is sufficient to check that $j$ projects (5.10), $t^a_{b}(5.10)$, (5.11), $\tilde{t}^a_b(5.11)$, (5.12), $t^a_{b}(5.12)$ to 0. This implies that $\dim M_l = (\dim V)^2$ and $M_l$ is spanned by $\omega(t^a_{b})$.

Relations in the differential algebra of left invariant differential forms, which is an algebra in the category $\mathcal{D} \mathcal{Y} F$ corresponding to $\Omega$, are found by Tsygan [30]. They can be also obtained in the form $\omega(b_{(1)})\omega(b_{(2)})$, where $b$ is given by (5.10):
\[\omega(t^k_c) R^{lc}_{bza} \omega(t^z_f) + R^{lk}_{ip} \omega(t^p_j) R^{ij}_{xyz} \omega(t^z_y) R^{zy}_{bf} = 0.\]  
(5.14)
Equations (5.11), (5.12) also give some relations which follow from the above due to the identification (5.12)
\[ \omega(t^g h) = -u_{-1}^2 \omega(t^j_k) R^{-1j} \omega(t^k_j). \]

The algebra \( \Omega \) coincides in the Hecke case with the differential graded Hopf algebra \( \mathcal{M} = T^\gamma_\mathcal{P}(M)/\ker(\sigma - 1) \) constructed by Brzeziński [2] after Woronowicz’s ideas [32]. To prove this we have to show that the set of relations (5.14) coincides with \( \ker(\sigma - 1) \). By definition (5.8) the braiding \( \sigma \) is
\[ \sigma(\omega(t^a_b) \otimes \omega(t^c_f)) = \omega(t^a_b) \otimes \omega((t^a_b - \delta^a_b) \gamma(t^c_g) t^h_f). \]
The substitution \( \gamma(t^j_g) = (u_1^2)^{-1} t^{i} j u_1^2 c \) together with relations (5.10)–(5.13) reduces this expression to
\[ R^{-1ij}_{bc} R^{-1ia}_{lg} R_{kp} R_{jf} \omega(t^g_h) \otimes \omega(t^p_m). \]
This formula as well as the identity
\[ \sigma(R_{jk}^{ib} \omega(t^a_b) \otimes \omega(t^c_f)) = R_{jf}^{km} R_{kp}^{ib} \omega(t^a_b) \otimes \omega(t^p_m) R^{-1ia}_{lg} \]
were obtained by Sudbery [26]. In the basis
\[ X_{ij}^{ia} = R_{ij}^{ib} \omega(t^a_b) \otimes \omega(t^c_f) \]
of \( ML \otimes ML \) the braiding is expressed as
\[ \sigma(X_{ij}^{ia}) = R_{jk}^{km} X_{km}^{ia} R^{-1ia}_{lg}. \]

The relations (5.14) form the subspace
\[ I = \text{span}\{ X_{ij}^{lk} + R_{ip}^{lk} X_{xy}^{ip} R_{bj}^{xy} \} \]
which is contained in
\[ \ker(\sigma - 1) = \{ \text{tr}(AX) = A_{ia}^{ij} X_{ij}^{ia} \mid RA = AR \} \]
by Remark 5.3. All matrices \( A \) commuting with \( R \) have the form \( P_+ B P_+ + P_- C P_- \), where \( R = qP_+ - q^{-1} P_- \) is the spectral decomposition. Since \( \text{tr}(A(X + RXR)) = (1 + q^{\pm 2}) \text{tr}(AX) \) if \( A = P_+ B P_\pm \), we conclude that \( \ker(\sigma - 1) \subset I \), proving the claim.

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